Mathematical Models of Synchronous Reluctance Motor Considering Iron Loss and Cross-Magnetic Saturation with Reciprocity Relation of Mutual Inductance

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1. Introduction

A synchronous reluctance motor (SynRM) is one of brushless AC synchronous motors, having unique features (robustness, low cost and low environmental load) since the rotor is composed only of iron core. In addition to traditional studies such as reduction of electrical loss and improvement of torque density, practical researches aimed at expanding industry applications such as power factor improvement and torque pulsation reduction have also progressed in recent years (1)–(3).

Many rotor structures have been proposed for SynRM. Among them, flux barrier type SynRMs are superior in manufacturing and have a high saliency ratio that produces a large torque. Meanwhile the circumference of the rotor surface of the machine is uniformly short-circuited by iron core. As a result, d- and q-axes inductances (Ld and Lq) change depending on both d- and q-axes currents (id and iq) since the rotor surface has a magnetic path common to both axes. This phenomenon is often called cross-magnetic saturation.

Some studies have attempted to derive a mathematical model that can accurately model the dynamic characteristics of SynRM having the cross-magnetic saturation effect. In Ref. (4), it has been pointed out that it is necessary to satisfy the reciprocity relation of mutual inductance. As an example, an approximation function of \( L_d(id, iq) \) and \( L_q(id, iq) \) by a power series that can take cross-magnetic saturation into consideration has been shown. However, approximation accuracy for machines such as flux barrier type SynRM having strong magnetic saturation characteristics has not been discussed. In Ref. (5), an accurate inductance approximation function of \( L_d(id, iq) \) and \( L_q(id, iq) \) has been proposed. This approximation function is continuous and differentiable over the entire current range (from negative infinity to positive infinity). However, it is difficult to derive a theoretical inverse inductance function that calculates \( i_d \) and \( i_q \) from d- and q-axes flux linkages \( \psi_d \) and \( \psi_q \). In Ref. (6), a simple but accurate mathematical model of SynRM based on inverse inductance expression (7)–(8) has been discussed, and an improved expression of \( L_d^{-1}(\psi_d, \psi_q) \) and \( L_q^{-1}(\psi_d, \psi_q) \) that calculate \( i_d \) and \( i_q \) has been proposed. However, the reversibility between the d- and q-axes inverse inductance functions and the d- and q-axes inductance functions is not guaranteed. In Ref. (9), an approximation function of \( \psi_d(id, iq) \) and \( \psi_q(id, iq) \) has been presented. This approximation function is based on a simple fractional formula, and it is possible to derive a theoretical expression of the inverse function with satisfying the reversibility. However, as well as Ref. (4), approximation accuracy for machines such as flux barrier type SynRM having strong magnetic saturation characteristics has not been discussed, and this approximation function is not continuous and differentiable over the entire current range. In addition, the approximation functions in Refs. (5)–(9) do not guarantee the reciprocity relation.

To address this situation, the authors have proposed a new inductance approximation function of \( L_d(id, iq) \) and \( L_q(id, iq) \). Upgrading the inductance approximation function in Ref. (5), the authors have derived a new approximation inductance function of \( L_d(id, iq) \) and \( L_q(id, iq) \). This function has excellent features, such as continuous and differentiable over the entire current range.
when taking into account cross-magnetic saturation and satisfactory reciprocity relation of the mutual inductance. In Ref. (10), a strict state equation of SynRM based on the proposed approximation function and a vector control simulation block diagram have been shown. In addition, a method to calculate \( \dot{i}_d \) and \( \dot{i}_q \) by an equivalent inverse function that is based on repeated calculation and has the reversibility to the proposed inductance approximation function.

In this paper, the effectiveness of considering the reciprocity relation of the mutual inductance has been verified. The detailed derivation process of the strict mathematical model of SynRM considering both the iron loss and cross magnetic saturation is clarified (11). Moreover, a simplified mathematical model that can reduce amount of calculation by employing the equivalent inverse function is shown.

The proposed method is implemented on a 1.1 kW flux barrier type vector-controlled SynRM. The validity and impact of considering the iron loss and cross magnetic saturation is clarified (11). Moreover, a simplified mathematical model that can reduce amount of calculation by employing the equivalent inverse function is shown.

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2. Inductance Approximation Function of D- and Q-axes Inductances

2.1 Properties of the tested SynRM

Fig. 1 shows a cross section of the tested machine’s rotor. The rating values are summarized in Table 1. The stator has a star-connected cross section of the tested machine’s rotor. The rating values are also another-axis currents except for large current range. One can see from the data that d- and q-axes inductances vary with not only self-axis but also another-axis currents except for large current range.

2.2 Approximation inductance function of D- and Q-axes inductances

The reciprocity relation is the nature that partial derivative of the flux linkage of the self-axis due to the current of the other axis is equal to each axis, which is written as follows (40).

\[
\frac{\partial \psi_i}{\partial i} = \frac{\partial \psi_i}{\partial i}
\]

2.2.1 Conventional approximation inductance function

The conventional inductance approximation function having a high approximation accuracy, being continuous and differentiable for entire range of currents, and considering cross-magnetic saturation is written as follows (10).

\[
L_d(i_d, i_q) = A_d + \frac{B_d}{i_d^2 + C_d i_d^2 + D_d}\left(1 - \frac{1}{C_d i_d^2 + 1}\right)
\]

\[
L_q(i_d, i_q) = A_d + \frac{B_d}{i_q^2 + C_d i_q^2 + D_d}\left(1 - \frac{1}{C_d i_q^2 + 1}\right)
\]

Here, \( A_d, B_d, C_d \), and \( D_d \) (c: 0, 1, 2, 3, d or q) are the constants.

2.2.2 Proposed approximation inductance function

The proposed inductance approximation function satisfying the reciprocity relation, having high approximation accuracy, being continuous and differentiable for entire range of currents, and considering cross-magnetic saturation is written as follows.

\[
L_d(i_d, i_q) = A_d + \frac{B_d}{i_d^2 + C_d i_d^2 + D_d}\left(1 - \frac{1}{C_d i_d^2 + 1}\right)
\]

\[
L_q(i_d, i_q) = A_d + \frac{B_d}{i_q^2 + C_d i_q^2 + D_d}\left(1 - \frac{1}{C_d i_q^2 + 1}\right)
\]

Applying Eqs. (2) and (3) to Eq. (1), Eqs. (4) and (5) can be obtained. One can see that Eq. (4) is not the same as Eq. (5). This demonstrates that the conventional approximation inductance function does not theoretically guarantee physical consistency including reciprocity relation.

\[
\frac{\partial \psi_i}{\partial i} = \frac{\partial \psi_i}{\partial i}
\]

Table 1. Rating values of the tested SynRM.

| Item               | Value | Unit |
|--------------------|-------|------|
| Rated output       | 1.1   | kW   |
| Rated voltage      | 178   | V (rms) |
| Rated current      | 6.3   | A (rms) |
| No. of poles       | 4     |      |
| Maximum speed      | 2,200 | rpm  |

Fig. 1. Rotor structure of the tested flux-barrier type SynRM.
Using the result of Table 2, the authors calculated $L_d - i_d$ curves for some $i_q$ and $L_q - i_q$ curves for some $i_d$. The curves of Fig. 2(a) show the results. It is confirmed that Eqs. (6) and (7) can correctly express the tendency of the measured inductance values for the entire current range except when $i_d$ is extremely small. Note that, in the actual drive of the test machine, the magnetic flux is stably established in the entire load range when $i_d$ is about 3A or more. Of course, it is desirable that the calculation accuracy is high even in the range where $i_d$ is extremely small. However, it was difficult to improve the accuracy over the entire current range. To realize this, it is considered that an inductance approximation function having a complicated structure with a higher order is required. Therefore, the authors identified the constant of the inductance approximation function so that the accuracy of the inductance difference $(L_d - L_q)$ in the region where $i_d$ is 3A or more was improved.

As shown in Fig. 2(a), the authors omitted to use the values of the measured $L_d$ around $i_d$=4A, $i_d$=3, 6, 9A in identifying the constants $A_0$, $B_0$, $C_0$, and $D_0$ in Eqs. (6) and (7). However, it was not a problem. Fig. 2(b) shows that the calculated $L_d - i_d$ curve when $i_d$=4A can express the tendency of the measured inductance.

### 2.4 Verification of the reciprocity relation of mutual inductance

Next, the authors verified the impact of considering the reciprocity relation.

Eq. (1) implies the following equation (4). Here, $M_{dq}$ and $M_{qd}$ are the mutual inductances that should be equal.

$$M_{dq} = \frac{\partial \psi_q}{\partial i_d} = \frac{\partial (L_q i_d i_q)}{\partial i_d}$$

$$M_{qd} = \frac{\partial \psi_q}{\partial i_q} = \frac{\partial (L_q i_d i_q)}{\partial i_q}$$

Figs. 3 and 4 show the curves calculated when the conventional Eqs. (2) and (3) in Ref. (5) are substituted for Eqs. (10) and (11). It
Eqs. (10) and (11). It can be naturally confirmed that calculated when the proposed Eqs. (6) and (7) are substituted for loss effect by means of the equivalent iron loss resistance of SynRM considering the iron loss and cross-magnetic saturation similar to Ref. (5), the authors derive a strict mathematical model. Using Eqs. (6) and (7), this suggests that the conventional approximation inductance function does not guarantee the reciprocity relation of mutual inductance.

Figs. 5 and 6 show \( M_{ib} - i_b \) and \( M_{id} - i_d \) curves calculated when the proposed Eqs. (6) and (7) are substituted for Eqs. (10) and (11). It can be naturally confirmed that \( M_{ib} - i_b \) in Fig. 5(a) and \( M_{id} - i_d \) in Fig. 5(b) are equal, and \( M_{ib} - i_b \) in Fig. 6(a) and \( M_{id} - i_d \) in Fig. 6(b) are also equal.

3. Mathematical Models of SynRM

3.1 Strict mathematical model

Adopting a process similar to Ref. (5), the authors derive a strict mathematical model of SynRM considering the iron loss and cross-magnetic saturation using Eqs. (6) and (7).

Fig. 7 shows a circuit model of SynRM considering the iron loss effect by means of the equivalent iron loss resistance \( R_m \). The relation between the currents and flux linkages is written as

\[
\begin{bmatrix}
i_d \\
i_q \\
i_{ms} \\
i_{mq}
\end{bmatrix} =
\begin{bmatrix}
i_{ms} \\
i_{mq} \\
i_{ms} \\
i_{mq}
\end{bmatrix}
+ \frac{1}{R_m}
\begin{bmatrix}
\psi_{sd} \\
\psi_{sq} \\
\psi_{sm} \\
\psi_{sq}
\end{bmatrix}
\tag{12}
\]

where

\[
\begin{bmatrix}
\psi_{sd} \\
\psi_{sq} \\
\psi_{sm} \\
\psi_{sq}
\end{bmatrix} =
\begin{bmatrix}
L_s & M_{sm} & M_{sq} & i_{ms} \\
M_{sm} & L_s & M_{sq} & i_{ms} \\
M_{sm} & M_{sq} & L_s & i_{ms} \\
M_{sm} & M_{sq} & M_{sq} & i_{ms}
\end{bmatrix}
\tag{13}
\]

\[
L_s = l_a + L_m - L_d \cos(2\theta_r + 2\pi / 3) \tag{14}
\]

\[
L_s = l_a + L_m - L_d \cos(2\theta_r - 2\pi / 3) \tag{15}
\]

\[
M_{sm} = -L_m / 2 - L_d \cos(2\theta_r + 2\pi / 3) \tag{16}
\]

\[
M_{mq} = -L_m / 2 - L_d \cos(2\theta_r) \tag{17}
\]

\[
M_{sm} = -L_m / 2 - L_d \cos(2\theta_r + 2\pi / 3) \tag{18}
\]

Here, \( P \) is differential operator \((=d/dt)\), \( \theta_r \) is the electrical rotor angle, \( L_s, L_m, \) and \( L_d \) are the self-inductances, \( M_{sm}, M_{mq}, \) and \( M_{sd} \) are the mutual-inductances, \( l_a \) is the armature leakage inductance, \( L_{oq} \) and \( L_{oq} \) are the constants.

When the transformation matrix

\[
[c] = \sqrt{2 \over 3} \begin{bmatrix}
\cos(\theta_r - 2\pi / 3) & \cos(\theta_r) & \cos(\theta_r + 2\pi / 3)
\end{bmatrix}
\tag{20}
\]

is multiplied from the left side to both sides of Eq. (12), the current equations in the rotational d-q frame can be obtained as follows.

\[
i_d = i_{ad} + \frac{v_{ms}}{R_m} \tag{21}
\]

\[
i_q = i_{aq} + \frac{P(v_{ms} - \omega_a L_m (i_{ms}, i_{ms}) i_{ms}) / R_m}{R_m} \tag{22}
\]

Here, \( i_{ad} \) and \( i_{aq} \) are the d- and q-axes load currents. \( \omega_a \) is the electrical rotor angular frequency.

\[
\theta_r = \omega_a t + \phi_a \tag{23}
\]

Fig. 7. Circuit model of SynRM.
Note that the authors assume that $L_d$ and $L_q$ vary with $i_{md}$ and $i_{mq}$, and $R_m$ depends on only $\omega$. In case of the tested SynRM, $R_m$ can be expressed as

$$R_m = A_m + B_m / (\omega_m^2 + D_m).$$

(23)

Here, $A_m$, $B_m$, and $D_m$ are 600, $-42,687,100$ and $80,717$, respectively. These values can be determined by no-load test results in some rotational speeds.

In the stator circuit,

$$v_s = R_s i_s + v_{sat}$$

(24)

and

$$v_r = R_r i_r + v_{nr}.$$  

(25)

Here, $R_s$ is the stator (armature) resistance.

Eqs. (21) and (22) are substituted for Eqs. (24) and (25) with eliminating $v_{sat}$ and $v_{nrq}$. Then, the following incomplete state equation can be obtained.

$$P(L_s (i_{sat}, i_{sq}) v_{sat}) = -(R_s R_m / (R_s + R_m)) v_{sat}$$

$$+ \omega_m L_s (i_{sat}, i_{sq}) v_{sat} + (R_s / (R_s + R_m)) v_{sat}$$

(26)

$$P(L_q (i_{sat}, i_{sq}) v_{sat}) = -(R_s R_m / (R_s + R_m)) v_{sat}$$

$$- \omega_m L_q (i_{sat}, i_{sq}) v_{sat} + (R_s / (R_s + R_m)) v_{sat}$$

(27)

To derive a complete state equation whose state variable is $[i_{ms}, i_{mq}]^T$, it is necessary to strictly separate $P(i_{ms})$ and $P(i_{mq})$ from the left sides of Eqs. (26) and (27). Using Eqs. (6) and (7), the left sides of Eqs. (26) and (27) can be expressed as follows.

$$P(L_s (i_{sat}, i_{sq}) v_{sat}) = P(L_{20} (i_{sat}, i_{sq}))$$

$$- P(L_{21} (i_{sat}, i_{sq}) L_{22} (i_{sat}, i_{sq}) - L_{21} (i_{sat}, i_{sq}) L_{22} (i_{sat}, i_{sq}) P(L_{22} (i_{sat}, i_{sq})))$$

$$= K_s P(i_{sat}) + K_s P(i_{sat})$$

(28)

$$P(L_q (i_{sat}, i_{sq}) v_{sat}) = P(L_{20} (i_{sat}, i_{sq}))$$

$$- P(L_{21} (i_{sat}, i_{sq}) L_{22} (i_{sat}, i_{sq}) - L_{21} (i_{sat}, i_{sq}) L_{22} (i_{sat}, i_{sq}) P(L_{22} (i_{sat}, i_{sq})))$$

$$= K_q P(i_{sat}) + K_q P(i_{sat})$$

(29)

Here,

$$K_1 = L_{20} (i_{sat}, i_{sq}) - B_m P_{x_0} / (X_s (i_{sat})))^2$$

$$- L_{21} (i_{sat}, i_{sq}) C_s P_{x_0} / (X_s (i_{sat})))^2$$

$$L_{22} (i_{sat}, i_{sq})$$

(30)

$$K_2 = -2 L_{21} (i_{sat}, i_{sq}) C_s P_{x_0} / (X_s (i_{sat})))^2$$

(31)

$$K_3 = -2 L_{22} (i_{sat}, i_{sq}) C_s P_{x_0} / (X_s (i_{sat})))^2$$

(32)

$$K_4 = L_{20} (i_{sat}, i_{sq}) - B_m P_{x_0} / (X_s (i_{sat})))^2$$

$$- L_{21} (i_{sat}, i_{sq}) C_s P_{x_0} / (X_s (i_{sat})))^2$$

$$L_{22} (i_{sat}, i_{sq})$$

(33)

$$P_{x_0} = 4 i_{qs}^2 + 2 C_s i_{qs}$$

(34)

Finally, by substituting Eqs. (28) and (29) for Eqs. (26) and (27), the following non-linear state equation of SynRM can be derived.

$$P(i_{sat}) A + B v_s$$

(38)

Here,

$$A = [K_1 \ K_3]^T - [R_s / (R_s + R_m) \ - \omega_m L_s (i_{sat}, i_{sq}) - R_s / (R_s + R_m)]$$

(39)

$$B = \frac{R_r}{R_s + R_m} [K_1 \ K_3]^T.$$  

(40)

The output equation for calculating stator current and torque equation are respectively expressed as follows.

$$i_{sat} = \frac{R_s}{R_s + R_m} [i_{sat}] + \frac{1}{R_s + R_m} v_s$$

(41)

$$T_s = N_p \alpha (L_s \ i_{sat} - L_q \ i_{sat}) / i_{sat}$$

(42)

Here, $N_{pp}$ is the number of pole pairs.

### 3.2 Simplified mathematical model employing equivalent inverse inductance function

D- and q-axes inverse inductance $L_d^{-1}$ and $L_q^{-1}$ are defined as follows.

$$i_{sat} = L_d^{-1} \psi_d$$

(43)

$$i_{sq} = L_q^{-1} \psi_q$$

(44)

In general, theoretical equations of $L_d^{-1} (\psi_d, \psi_q)$ and $L_q^{-1} (\psi_d, \psi_q)$ with the reversibility are derived by solving the following system of equations for $i_{ms}$ and $i_{mq}$ and dividing the solutions by $\psi_d$ and $\psi_q$, respectively.

$$\psi_d = L_d \ i_{sat} \ i_{sq}$$

(45)

$$\psi_q = L_q \ i_{sat} \ i_{sq}$$

(46)

In case of the proposed Eqs. (6) and (7), however, it is difficult to obtain the theoretical equations of $L_d^{-1} (\psi_d, \psi_q)$ and $L_q^{-1} (\psi_d, \psi_q)$ because Eqs. (6) and (7) are too complicated to solve Eq. (45). Thus, the authors propose an equivalent approach to achieve d- and q-axes inverse inductance functions based on a simple repeated algorithm, described as follows.

$$i_{sat} (K) = L_d^{-1} (K-1) \psi_d (K-1)$$

$$i_{sq} (K) = L_q^{-1} (K-1) \psi_q (K-1)$$

(47)
\[
\begin{align*}
L_{d}\mathbf{K}^{-1}(K) &= \frac{1}{L_{d}} \left( i_{d}(K), i_{m}(K) \right) \\
L_{q}\mathbf{K}^{-1}(K) &= \frac{1}{L_{q}} \left( i_{q}(K), i_{m}(K) \right)
\end{align*}
\] (47)

Here, \( K \) is the sampling point. \( L_{d}\mathbf{K}^{-1} \) and \( L_{q}\mathbf{K}^{-1} \) converge to the true values completely by repeating the calculation of Eqs. (46) and (47) several times. Meanwhile, in actual simulations, it is considered that it is sufficient to calculate Eqs. (46) and (47) only once for each sampling period.

Employing \( L_{d}\mathbf{K}^{-1} \) and \( L_{q}\mathbf{K}^{-1} \), the authors show a simplified but accurate mathematical model of SynRM. Eqs. (21), (22), (24), and (25) can be rewritten as follows.

\[
\begin{align*}
i_{d} &= i_{d}\mathbf{K} + \left( P \left( v_{d}\right) - \alpha_{o} v_{d} \right) / R_{a} \\
i_{q} &= i_{q}\mathbf{K} + \left( P \left( v_{q}\right) + \alpha_{o} v_{q} \right) / R_{a} \\
v_{d} &= R_{d} i_{d} + R_{a} \left( i_{d} - i_{d}\mathbf{K} \right) \\
v_{q} &= R_{q} i_{q} + R_{a} \left( i_{q} - i_{q}\mathbf{K} \right)
\end{align*}
\] (48)(49)(50)(51)

By substituting Eqs. (43)-(44) into Eqs. (48)-(51) and eliminating \( i_{d}\) and \( i_{q} \), the simplified state equation can be obtained as follows.

\[
\begin{bmatrix}
p \left( v_{d}\right) \\
p \left( v_{q}\right)
\end{bmatrix}
= 
\begin{bmatrix}
-R_{R_{a}} L_{d}\mathbf{K}^{-1} & \alpha_{o} R_{a} \\
\alpha_{o} & -R_{R_{a}} L_{q}\mathbf{K}^{-1} \left( R_{a} + R_{q} \right)
\end{bmatrix}
\begin{bmatrix}
v_{d} \\
v_{q}
\end{bmatrix}
+ 
\begin{bmatrix}
R_{a} \\
R_{a} \left( R_{a} + R_{q} \right)
\end{bmatrix}
\begin{bmatrix}
v_{d}\mathbf{K} \\
v_{q}\mathbf{K}
\end{bmatrix}
\] (52)

The output equation and torque equation are written as

\[
\begin{align*}
\begin{bmatrix}
i_{d} \\
i_{q}
\end{bmatrix}
&= \frac{R_{a}}{R_{a} + R_{s}} \begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
l_{d}^{-1} - L_{q}^{-1} \\
l_{q}^{-1}
\end{bmatrix}
\begin{bmatrix}
v_{d} \\
v_{q}
\end{bmatrix}
\end{align*}
\] (53)

\[
T_e = N_{p} v_{d} \left(l_{d}^{-1} - L_{q}^{-1}\right).
\] (54)

4. Implementation Example

4.1 Simulation Condition

Fig. 8 shows a simulation block diagram of a vector-controlled SynRM. The figure on the right side is the motor part utilizing the proposed strict mathematical model of SynRM described in Section 3.1. The figure on the left side is the controller part in which the conventional current vector control system is mounted.

Fig. 9 shows a simulation block diagram of a vector-controlled SynRM in which the proposed simplified mathematical model using the equivalent inverse inductance function described in Section 3.2 is employed. Note that the difference between Figs. 8 and 9 is only the configuration of the mathematical model of SynRM. It is also found that, in Fig. 9, calculating Eqs. (6)-(7), (30)-(37) and (39)-(40) that are needed in Fig. 8 are eliminated by employing Eqs. (46) and (47).

In Figs. 8 and 9, the cutoff angular frequencies of current and speed feedback systems are 1,000rad/s and 20rad/s, respectively. \( R_{s} \) is 1.04Ω. The d-axis current command \( i_{d} \) is 5A. The maximum \( i_{q} \) value is 6A in the \( i_{q} \) limiter. The nominal values of d- and q-axes inductances \( L_{dn} \) and \( L_{qn} \), mounted on the controller, are respectively 0.065H and 0.01H so that these values are set to be close to \( L_{d}(i_{d},0) \) when \( i_{d} \) is 5A and \( L_{q}(0,i_{q}) \) when \( i_{q} \) is 6A, respectively. The moment of inertia \( J_{m} \) is 0.0208kg-m². The viscous damping factor \( B_{m} \) is 0.00268kg-m²/s.
4.2 Experimental setup The authors carried out actual driving tests to verify the proposed mathematical models of SynRM. Fig. 10 shows the experimental setup. A 12kVA three-phase precision linear power amplifier is employed to investigate basic characteristics without time-domain voltage harmonics caused by an inverter. In a digital signal processor, a vector control system that is the same as Figs. 8 and 9 is installed. The current and speed control periods are 0.1ms and 1ms respectively.

4.3 Verification of the proposed strict mathematical model Based on the block diagram of Fig. 8, the authors carried out the transient performance simulations of the tested SynRM. Fig. 11 shows the simulation results of a rotational speed step response when the rotational speed command value changes from 750r/min to 1,500r/min under $i_d^*$ is kept in 5A. Fig. 11(a) shows the results when both the iron loss and cross-saturation are neglected; $R_w$ is infinite and $L_d$ and $L_q$ are treated as constants ($L_{dn}$ and $L_{qn}$ respectively) in the mathematical model of SynRM in Fig. 8. Fig. 11(b) shows the results when the iron loss is considered using Eq. (23) but the cross-magnetic saturation is neglected. Fig. 11(c) shows the results when iron loss is neglected but the cross-magnetic saturation is considered. Fig. 11(d) shows the results when both the iron loss and cross-magnetic saturation are considered just as shown in Fig. 8. In each figure, an experimental result of the speed step response performed under exactly the same experimental conditions as the simulation is also shown.

One can see that the simulated transient responses of Fig. 11(d) are more accurate than those of Figs. 11(a), 11(b) and 11(c), which demonstrates the validity of the proposed strict mathematical

![Fig. 10. Experimental setup.](image)

![Fig. 11. Experimental and Simulation results of a speed step response employing the proposed strict mathematical model of SynRM.](image)
model of SynRM. In Fig. 11(d), both the speed setting time and the average value of the limited torque accurately simulate the experimental results. Meanwhile, in Figs. 11(a), 11(b) and 11(c), they do not accurately reproduce the experimental values even though both $i_d$ and $i_q$ during the $i_d^*$ limiter works are correctly controlled to 5A and 6A respectively.

Note that the torque ripple is observed in the experimental result, but is not seen in the simulation results. Major components of the ripple are 25Hz and 100Hz at 1,500r/min. So, the causes of the ripple are considered to be slight distortion of shaft connection and magnetic unbalance which are not taken into account in the mathematical model of SynRM.

Since $R_m$ is relatively large in the test machine ($R_m=362\Omega$ at 1,500r/min), $i_d$ and $i_q$ expressed by Eqs. (21) and (22) are respectively close to $i_{md}$ and $i_{mq}$ in the steady state. Namely, in the torque calculation of the test machine, the sensitivity of the inductance difference (affected by cross-magnetic saturation) becomes higher than that of the iron loss resistance. In addition, it is also observed from Figs. 11(b) and 11(c) that considering cross-magnetic saturation is more important than considering iron loss in order to improve the simulation accuracy in torque of the test machine.

The iron loss was about one-third of the copper loss at the rated operating point of the tested machine, so it is considered that test machine.

4.4 Verification of the proposed simple mathematical model Based on Fig. 9 using the equivalent inverse inductance function, the authors carried out transient performance simulations of the tested SynRM.

Fig. 12 shows the simulation results of the rotational speed step response when the rotational speed command value changes from 750r/min to 1,500r/min under $i_d^*$ is kept in 5A. The experimental result shown in Fig. 12 is the same as that in Fig. 11. From the figures, it can be found that the simulation results of Figs. 12(a), 12(b), 12(c) and 12(d) correspond to those of Figs. 11(a), 11(b), 11(c) and 12(d), respectively. This demonstrates the validity of the proposed simple mathematical model of SynRM using the equivalent inverse inductance function. It is also confirmed that it is sufficient to calculate Eqs. (46) and (47) only once for each sampling period in simulating the driving performance of a vector controlled SynRM.

4.5 Convergence performance of the equivalent inverse inductance function In Fig. 9, Eqs. (46) and (47) construct a nonlinear discrete feedback system. The stability of the system is of interest, but it has been difficult to prove. So, instead of a strict stability proof of the system, the authors used a repetition algorithm shown in Fig. 13 and verified convergence performances of the proposed inverse inductance function when d- and q-axes step flux linkages were inputted.

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![Figure 11](image1.png)

![Figure 12](image2.png)
Fig. 13. Algorithm to verify the convergence performance of the equivalent inverse inductance function.

Figs. 14, 15 and 16 show the results, which simulate convergence performances respectively when \( \psi_d \) changes, when \( \psi_q \) does, and when both \( \psi_d \) and \( \psi_q \) do under the current phase angle \((\tan^{-1}(|i_{md}|/|i_{md}|))\) is 45 degrees. From these figures, it is confirmed that the system converges after several repetitions regardless of the polarity of the currents.

5. Conclusion

Accurate mathematical models of SynRM that can take iron loss and cross-magnetic saturation with satisfying the reciprocity relations for the mutual inductances were presented. Comparing simulation and experimental results on a vector controlled SynRM demonstrated the validity of the proposed mathematical models.
The results are summarized as follows.

1) A unique approximation function for the d- and q-axes inductances for a SynRM having strong magnetic saturation characteristics was proposed.

2) The proposed inductance approximation function has the features, such as continuous and differentiable over the entire current range when taking into account cross-magnetic saturation and satisfactory reciprocity relation of mutual inductance. In addition, it is possible to derive an equivalent inverse inductance function.

3) Based on the proposed inductance approximation function a mathematical model of SynRM derived with strictly paying attention to the effect of differential operator \( P \) on flux linkages \( (P(L_{d(i)w(i)q(i)}m_{w})) \) was proposed.

4) The proposed strict mathematical model in 3) mentioned above can accurately simulate the driving performance of a vector-controlled SynRM, and quantitatively evaluate the impact of considering the iron loss and cross-magnetic saturation.

5) A mathematical model of SynRM simplified by employing the equivalent inverse inductance function was proposed.

6) The proposed simple mathematical model in 5) mentioned above can simulate the driving performance of a vector control SynRM with the same accuracy as the proposed strict mathematical model.

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