SIGN-CHANGING SOLUTIONS FOR SOME NONHOMOGENEOUS NONLOCAL CRITICAL ELLIPTIC PROBLEMS

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Abstract. We construct multiple sign-changing solutions for the nonhomogeneous nonlocal equation
\[
(-\Delta_{\Omega})^s u = |u|^{4 - \frac{4}{N} - 2s} u + \varepsilon f(x) \quad \text{in} \ \Omega,
\]
under zero Dirichlet boundary conditions in a bounded domain \(\Omega\) in \(\mathbb{R}^N\), \(N > 4s, s \in (0, 1]\), with \(f \in L^\infty(\Omega), f \geq 0 \) and \(f \neq 0\). Here, \(\varepsilon > 0\) is a small parameter, and \((-\Delta_{\Omega})^s\) represents a type of nonlocal operator sometimes called the spectral fractional Laplacian. We show that the number of sign-changing solutions goes to infinity as \(\varepsilon \to 0\) when it is assumed that \(\Omega\) and \(f\) have certain smoothness and possess certain symmetries, and we are also able to establish accurately the contribution of the nonhomogeneous term in the found solutions. Our proof relies on the Lyapunov-Schmidt reduction method.

1. Introduction. This paper addresses the existence of multiple sign-changing solutions to the problem
\[
\begin{cases}
(-\Delta_{\Omega})^s u = |u|^{p-1} u + \varepsilon f(x) & \text{in} \ \Omega, \\
\quad u = 0 & \text{on} \ \partial\Omega,
\end{cases}
\]
where \(\Omega\) is a bounded and smooth domain in \(\mathbb{R}^N\), \(N > 4s, s \in (0, 1]\), \(p := \frac{N + 2s}{N - 2s}\), \(f \in L^\infty(\Omega)\) is a nonhomogeneous perturbation, \(f \geq 0\) and \(f \neq 0\), and \(\varepsilon > 0\) is a small parameter. Here, \((-\Delta_{\Omega})^s\) represents a nonlocal operator defined as follows. Let \(\{\mu_k\}_{k=1}^\infty\) be the set of eigenvalues of \(-\Delta\) in \(\Omega\) with zero Dirichlet boundary conditions, and let \(\{\varphi_k\}_{k=1}^\infty\) be the corresponding set of normalized eigenfunctions in \(L^2(\Omega)\). Then, it is verified that \(0 < \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_k \to +\infty\) as \(k \to +\infty\), \(\int_{\Omega} \varphi_j \varphi_k \, dx = \delta_{jk}\), where \(\delta_{jk}\) is the Kronecker delta and
\[
\begin{cases}
-\Delta \varphi_k = \mu_k \varphi_k & \text{in} \ \Omega, \\
\varphi_k = 0 & \text{on} \ \partial\Omega.
\end{cases}
\]
The operator \((-\Delta_{\Omega})^s\), for \(s \in (0, 1]\), is defined by
\[
(-\Delta_{\Omega})^s u := \sum_{k=1}^\infty \mu_k^s b_k \varphi_k \quad \text{for all} \ u \in C_c^\infty(\Omega),
\]

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where \( u = \sum_{k=1}^{\infty} b_k \phi_k \) and \( b_k = \int_{\Omega} u \phi_k \, dx \). This operator can be extended by density for \( u \) in the Hilbert space
\[
\mathcal{H}^s(\Omega) := \left\{ u \in L^2(\Omega) \mid \sum_{k=1}^{\infty} \mu_k b_k^2 < +\infty \right\}
\]
and it realizes an isomorphism between \( \mathcal{H}^s(\Omega) \) and its dual \( (\mathcal{H}^s(\Omega))^* \). The operator \((-\Delta_\Omega)^s\) has been referred to as the spectral fractional Laplacian.

Nonlocal operators such as the fractional Laplacian arise in nonlinear diffusion problems in plasma physics, cell models in biology, and stock simulations in mathematical finance; see, for instance, [19, 26, 2], respectively. All of these phenomena involve Lévy processes if we address the jump processes of particles, cells and price indices. See also [18] and the references therein, which cover other interesting applications of the fractional Laplacian operator.

The existence of positive solutions for certain nonlinear problems involving the root square spectral fractional Laplacian operator, i.e. for \( s = \frac{1}{2} \), was first studied by Cabré and Tan [5]; see also [8, 23]. The existence results for critical problems in a wide range of \( s \) for the spectral fractional Laplacian have been studied by Barrios et al. [3] and Tan [24]. Shang et al. [22] and Colorado et al. [13] recently studied independently the problem (1) and proved the existence of positive solutions in the spirit of work by Tarantello [25], who studied the analogue problem to (1) for the Laplacian operator; see also [4, 21, 7, 1] and the references therein.

As far as we know, there are no results known for \( s \in (0, 1) \) on sign-changing solutions of (1). Hence, the next step in studying (1) at the critical exponent concerns the existence of sign-changing solutions, such as in the case of \( s = 1 \), in which multiple sign-changing solutions of the corresponding problem involving the Laplacian operator were found by Clapp et al. [12] and recently by Musso [20]. Specifically, when \( s = 1 \), it was proven in [12] that a large number of nonradial solutions to problem (1) appear when \( \varepsilon \to 0 \), provided that the domain \( \Omega \) and the nonhomogeneity \( f \) satisfy certain symmetry conditions, while in [20], such symmetry conditions on \( \Omega \) and \( f \) were left aside. In both works, the main tool used was the Lyapunov-Schmidt reduction method. In [12], the solutions were constructed by taking advantage of the symmetries on \( \Omega \) and \( f \), while in [20], the solutions were constructed by making a fine analysis to adjust certain parameters to control the size of the solution away from certain points in \( \Omega \) over which the concentration phenomenon is produced. Unfortunately, neither of these two works explicitly established the contribution of nonhomogeneity \( f \) in the found solutions.

In this way, it is interesting to look for multiple sign-changing solutions for (1), with \( s \in (0, 1) \), and to establish with precision the contribution of the nonhomogeneity \( f \). The key in our reasoning is to modify the reduction procedure, taking into account the following function \( \gamma \) defined by
\[
\gamma(x) := \int_{\Omega} f(x)G(x, y) \, dy, \quad x \in \Omega,
\]
where \( G \) is the Green’s function of the operator \((-\Delta_\Omega)^s\) under zero Dirichlet boundary conditions. That is, \( \gamma \) solves the problem
\[
\begin{cases}
(-\Delta_\Omega)^s \gamma = f(x) & \text{in } \Omega, \\
\gamma = 0 & \text{on } \partial \Omega.
\end{cases}
\]
To establish our main result, it is also convenient to recall that all positive solutions of the equation

\[ (-\Delta)^s u = u^p \]  

in \( \mathbb{R}^N \),

where \((-\Delta)^s\) is the classical fractional Laplacian defined on all of \( \mathbb{R}^N \), are given by the functions \( W_{\lambda,\xi} \), defined by

\[ W_{\lambda,\xi}(x) := \kappa_0(N, s) \left( \frac{\lambda}{\lambda^2 + |x - \xi|^2} \right)^{N - 2s}, \quad x \in \mathbb{R}^N, \]

where \( \kappa_0(N, s) > 0 \) is a suitable constant, \( \xi \in \mathbb{R}^N \) and \( \lambda > 0 \); see \([10]\). Then, for our purpose we consider here the projection of the function \( W_{\lambda,\xi} \) onto \( H^s(\Omega) \), that we denote by \( V_{\lambda,\xi} \).

We now impose certain assumptions on the domain \( \Omega \) and the nonhomogeneous term \( f \). We write \( x = (\zeta, x_3, \ldots, x_N) = (\zeta, x') \) for each point in \( \mathbb{R}^N = \mathbb{C} \times \mathbb{R}^{N - 2} \), and we assume that \( \Omega \) satisfies the following properties:

\( \) (H1) \( \Omega \) is a bounded and \( C^2 \)-domain in \( \mathbb{R}^N \), \( N > 4s \) with \( s \in (0, 1] \), such that \( 0 \in \Omega \).

\( \) (H2) If \( (\zeta, x') \in \Omega \), then \( e^{i\theta} \zeta, x' \) \in \( \Omega \) for all \( \theta \in [0, 2\pi] \).

\( \) (H3) If \( (\zeta, x_3, \ldots, x_i, \ldots, x_N) \in \Omega \), then \( (\zeta, x_3, \ldots, -x_i, \ldots, x_N) \in \Omega \) for each \( i = 3, \ldots, N \).

For the nonhomogeneous perturbation \( f \), we assume that

\( \) (H4) \( f \in C^{1+\alpha}(\overline{\Omega}) \) with \( \alpha > 2s, \ f \geq 0 \) and \( f \neq 0 \).

\( \) (H5) \( f = f(|\xi|, x') \) in \( \Omega \), and \( f \) is even in the variable \( x_i \) for each \( i = 3, \ldots, N \).

Since we consider that \( f \) verifies (H4), then \( \gamma \in C^{1+\alpha - 2s}(\overline{\Omega}) \), see for example \([17]\).

Specifically, our aim is to find solutions to (1) that exhibit spikes at the vertices of a regular polygon, and in which can observe explicitly the contribution of the nonhomogeneous term. More precisely, our main result is the following.

**Theorem 1.1.** Assume that \( \Omega \) satisfies (H1), (H2), and (H3) and that \( f \) satisfies (H4) and (H5). Then, there exists \( k_0 = k_0(\Omega) \) such that for each \( k \in \mathbb{N} \) that verifies \( k \geq k_0 \), the following holds: if \( \varepsilon_n \) is any sequence of positive numbers such that \( \varepsilon_n \to 0 \) as \( n \to +\infty \), then, up to subsequences, solutions \( u_{\varepsilon_n} \) of (1) exist for each sufficiently small \( \varepsilon = \varepsilon_n \) that are given by

\[ u_{\varepsilon}(x) = V_{\lambda_{\varepsilon,0}}(x) - \sum_{j=1}^{k} V_{\lambda_{\varepsilon,\rho Q_{jk}}} + \varepsilon\gamma(x) + \theta_{\varepsilon}(x), \quad x \in \Omega, \]

where the remainder term \( \theta_{\varepsilon} \) satisfies \( \theta_{\varepsilon} \to 0 \) uniformly in \( \Omega \) as \( \varepsilon \to 0 \) and \( \lambda_{\varepsilon}^\pm = \varepsilon \frac{2\pm j}{2\mp j} \lambda^\pm \) for certain positive numbers \( \lambda^+ \) and \( \lambda^- \). Here, \( \rho \) is a positive number, and

\[ Q_{jk} = (e^{2\pi j / n}, 0) \quad \text{for each} \ j = 1, \ldots, k. \]

The proof of Theorem 1.1 is based on the Lyapunov-Schmidt reduction method following the approach devised by Del Pino, Felmer and Musso \([16]\) in combination with certain analytical tools for radial extremal solutions of some nonlocal semilinear equations and bubbling solutions for some nonlocal elliptic equations under zero Dirichlet boundary conditions that were recently developed in \([9]\) and \([15]\), respectively. Our results here can be seen as a nonlocal version of the corresponding results by Clapp, Del Pino and Musso \([12]\, Theorem 1.1\), where the Laplacian operator was considered. Due to our approach, we can now explicitly see the contribution of the nonhomogeneous term on the solutions found, which improves the result in \([12]\), even for the case of \( s = 1 \).
This paper is organized as follows. In Section 2, we introduce the framework to formulate our problem, study the associated linear theory and show the finite dimensional reduction scheme used. Here, the nondegeneracy result found in [14] plays a key role in applying a contraction mapping argument that facilitates our approach. In Section 3, we calculate the reduced energy connecting this problem with the reduction of the variational problem to that of finding critical points for a function which comes from the expansion of the full energy. Finally, in Section 4, we prove Theorem 1.1 by using degree theory that takes advantage of the symmetry hypothesis on Ω and f.

2. Finite-dimensional reduction. Let Ω be a bounded and smooth domain in $\mathbb{R}^N$, $N > 4s$ with $s \in (0,1]$. To study the problem (1), we introduce the corresponding extension problem, which will allow for us to solve problem (1) by using variational methods.

2.1. Workframe. It is well known that the extension by Caffarelli and Silvestre [6] provides a powerful tool to address problems involving nonlocal operators modeled on the fractional Laplacian. Since this point of view facilitates subsequent calculations, it is convenient to describe an extension to the case of bounded domains such as that devised by Cabré and Tan [5] in the case $s = \frac{1}{2}$ (see [9] for the case of $s \in (0,1)$).

Let $\mathcal{C} := \Omega \times (0,\infty)$ and $\partial_L \mathcal{C} := \partial \Omega \times [0,\infty)$, and consider the space $H^s_{0,L}(\mathcal{C})$ that is the completion of the space

$$C^s_{0,L}(\mathcal{C}) := \left\{ \tilde{U} \in C^\infty(\overline{\mathcal{C}}) | \tilde{U}(\cdot,0) = U \text{ on } \Omega, \tilde{U} = 0 \text{ on } \partial_L \mathcal{C} \right\},$$

with respect to the norm

$$\|\tilde{U}\|_C := \int_C t^{1-2s}|\nabla \tilde{U}|^2 \, dx \, dt. \quad (5)$$

It is known that $H^s_{0,L}(\mathcal{C})$ is a Hilbert space endowed with the inner product

$$\langle \tilde{U}, \tilde{V} \rangle_C = \int_C t^{1-2s} \nabla \tilde{U} \cdot \nabla \tilde{V} \, dx \, dt.$$ 

Since we are assuming that Ω is a bounded and smooth domain, then the space $\mathcal{H}^s(\Omega)$ can be characterized as

$$\mathcal{H}^s(\Omega) = \text{tr}_\Omega \left( H^s_{0,L}(\mathcal{C}) \right).$$

In other words, if we denote by $\mathcal{D}^s(\mathbb{R}^{N+1}_+)$ the completion of $C^\infty_0(\mathbb{R}^{N+1}_+)$ with respect to the norm $\| \cdot \|_{\mathbb{R}^{N+1}_+}$, then for a bounded function $U \in \mathcal{H}^s(\Omega)$, there exists a unique extension:

$$\tilde{U} := \text{Ext}(U) \in \mathcal{D}^s(\mathbb{R}^{N+1}_+)$$

that satisfies

$$\begin{cases} \text{div}(t^{1-2s} \nabla \tilde{U}) = 0 & \text{in } \mathcal{C}, \\ \tilde{U} = 0 & \text{on } \partial_L \mathcal{C}, \\ \tilde{U}(\cdot,0) = U & \text{on } \Omega. \end{cases}$$

Thus, the nonlocal operator that we study here is realized by

$$(-\Delta_\Omega)^s U(x) = \partial_s^\nu \tilde{U}(x,0) := -\kappa_s \lim_{t \to 0} t^{1-2s} \frac{\partial}{\partial t} \tilde{U}(x,t),$$

where $\kappa_s$ is a constant depending on s.
2.2. **Ansatz.** For notational convenience, we henceforth drop the subindex $n$ from $\varepsilon_n$ and define $p := \frac{N+2s}{N-2s}$.

To construct the predicted solutions in Theorem 1.1, it is convenient to rescale the domain $\Omega$ by letting

$$\Omega_\varepsilon := \varepsilon^{-\frac{2s}{N-2s}} \Omega,$$

and to introduce the change in variables

$$v(x) := -\varepsilon u(\varepsilon^{-\frac{2s}{N-2s}} x), \quad x \in \Omega_\varepsilon.$$ 

Then, $u \in H^s(\Omega)$ is a solution to problem (1) if and only if $v \in H^s(\Omega_\varepsilon)$ solves the problem

$$\begin{cases}
(\Delta_{\Omega_\varepsilon})^s v - |v|^{p-1}v - \varepsilon^{p+1} \tilde{f}(x) & \text{in } \Omega_\varepsilon, \\
v = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}$$

where $\tilde{f}(x) = f(\varepsilon^{-\frac{2s}{N-2s}} x)$, $x \in \Omega_\varepsilon$.

Since $\Omega_\varepsilon \uparrow \mathbb{R}^N$ and $\varepsilon^{p+1} \tilde{f} \to 0$ a.e. as $\varepsilon \to 0$, then for any sufficiently small $\varepsilon > 0$ it is reasonable to link solutions of (6) with solutions of the equation

$$(\Delta)^s u = u^p \quad \text{in } \mathbb{R}^N,$$

whose positive solutions are all given by the functions $W_{\lambda, \xi}$ defined in (4), where $\xi \in \mathbb{R}^N$ and $\lambda > 0$.

Let us now consider the numbers

$$\lambda_j > 0 \quad \text{for } j = 1, \ldots, k+1,$$

and for every $(k+1)$-tuple of points $\xi = (\xi_1, \ldots, \xi_{k+1}) \in \Omega^{k+1}$, let us define

$$\xi_i' = \varepsilon^{-\frac{2s}{N-2s}} \xi_i \in \Omega_\varepsilon \quad \text{and} \quad \xi' = (\xi_1', \ldots, \xi_k', \xi_{k+1}', \xi_{k+2}') \in \Omega_\varepsilon^{k+1}.$$

To find nontrivial solutions of (6), it is natural to look for solutions in the class of functions that satisfy the symmetries of $\Omega_\varepsilon$ and $\tilde{f}$ according to hypotheses (H2)-(H3) and (H5), respectively, which to a first approximation appear as

$$v \sim \left( \sum_{i=1}^k W_{\lambda_i, \xi_i'} \right) - W_{\lambda_{k+1}, \xi_{k+1}}.$$

for appropriate choices of points $\xi_i'$ and parameters $\lambda_i$. To take into account the boundary condition in problem (6), a better approximation is then given by the projections of the functions $W_{\lambda_i, \xi_i'}$ onto $H^s(\Omega_\varepsilon)$. More precisely, we define by $V_{\lambda_i, \xi_i'}$ the unique solution of the problem

$$\begin{cases}
(\Delta_{\Omega_\varepsilon})^s V_{\lambda_i, \xi_i'} - |V_{\lambda_i, \xi_i'}|^{p-1}V_{\lambda_i, \xi_i'} = 0 & \text{in } \Omega_\varepsilon, \\
V_{\lambda_i, \xi_i'} = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}$$

which implies that its extension $\tilde{V}_{\lambda_i, \xi_i'}$ satisfies the problem

$$\begin{cases}
\text{div}(t^{1-2s} \nabla \tilde{V}_{\lambda_i, \xi_i'}) = 0 & \text{in } \mathcal{C}_\varepsilon, \\
\tilde{V}_{\lambda_i, \xi_i'} = 0 & \text{on } \partial \mathcal{C}_\varepsilon, \\
\partial^\nu_t \tilde{V}_{\lambda_i, \xi_i'}(\cdot, 0) = |W_{\lambda_i, \xi_i'}|^p & \text{on } \Omega_\varepsilon,
\end{cases}$$

where $\mathcal{C}_\varepsilon = \Omega_\varepsilon \times (0, \infty)$ and $\partial \mathcal{C}_\varepsilon := \partial \Omega_\varepsilon \times [0, \infty)$. Moreover, the functions $V_{\lambda_i, \xi_i'}$ can be expressed as

$$V_{\lambda_i, \xi_i'} = \tilde{W}_{\lambda_i, \xi_i'} - \Phi_{\lambda_i, \xi_i'} \quad \text{in } \Omega_\varepsilon,$$
where \( \Phi_{\lambda_i, \xi_i} \) is the trace on \( \Omega_\varepsilon \) of the unique solution \( \tilde{\Phi}_{\lambda_i, \xi_i} \) of the problem:

\[
\begin{cases}
\text{div}(t^{1-2s} \nabla \tilde{\Phi}_{\lambda_i, \xi_i}) = 0 \quad \text{in} \ C_\varepsilon, \\
\tilde{\Phi}_{\lambda_i, \xi_i} = W_{\lambda_i, \xi_i} \quad \text{on} \ \partial_t C_\varepsilon, \\
\partial_n \tilde{\Phi}_{\lambda_i, \xi_i}(\cdot, 0) = 0 \quad \text{on} \ \Omega_\varepsilon.
\end{cases}
\]

For notational convenience, we let

\[
V_i := V_{\lambda_i, \xi_i}, \quad V^+ := \sum_{i=1}^k V_i, \quad V^- := V_{\lambda_{k+1}, \xi_{k+1}} \quad \text{and} \quad V := V^+ - V^-.
\]

We then look for a solution \( U \) of (6) of the form

\[
U(x) = V(x) + \psi(x) + \phi(x), \quad x \in \Omega_\varepsilon,
\]

where \( \psi \) is a term we define to make explicit the contribution of nonhomogeneity of the equation in (6) and where \( \phi \) represents a lower-order term. Specifically, we choose \( \psi \) as the solution to the problem

\[
\begin{cases}
(\Delta_{\Omega_\varepsilon})^s \psi = -\varepsilon_p \bar{f} \quad \text{in} \ \Omega_\varepsilon, \\
\psi = 0 \quad \text{on} \ \partial \Omega_\varepsilon.
\end{cases}
\]

By convenience, we introduce the function \( W := W_{1,0} \) (this is the function \( W_{\lambda, \xi} \) with \( \lambda = 1 \), and \( \xi = 0 \)). Since any bounded solution \( \vartheta \) of the equation

\[
(-\Delta)^s \vartheta = pW_{p-1} \vartheta \quad \text{in} \ \mathbb{R}^N
\]

belongs to span \( \{ \frac{\partial W_{\lambda, \xi}}{\partial \lambda}, \frac{\partial W_{\lambda, \xi}}{\partial \xi} \}_{i=1,2,...,N} \) (see [14]), it is convenient to consider the following functions defined for all \( x \in \Omega_\varepsilon \). Let \( \Psi_{ij} \) be the \( H^s(\Omega_\varepsilon) \)-projection of the function \( W_{ij} \), where

\[
W_{ij} := \frac{\partial W_i}{\partial x_j}, \quad j = 1, \ldots, N, \quad W_{i0} := \frac{\partial W_i}{\partial \lambda_i} = (x - \xi_i) \cdot \nabla W_i + (N - 2s)W_i,
\]

with \( W_i := W_{\lambda, \xi_i} \) from here onwards. Then, \( \Psi_{ij} \) satisfies the problem:

\[
\begin{cases}
(\Delta_{\Omega_\varepsilon})^s \psi_{ij} = pW_{i}^{p-1} \psi_{ij} \quad \text{in} \ \Omega_\varepsilon, \\
\psi_{ij} = 0 \quad \text{on} \ \partial \Omega_\varepsilon.
\end{cases}
\]

Continuing our reasoning, we seek a function \( \phi \) such that for certain numbers \( \lambda_i \) and points \( \xi_i \), a function \( U \) as in (8) becomes a solution to the problem:

\[
\begin{cases}
(-\Delta_{\Omega_\varepsilon})^s U = |U|^{p-1} U - \varepsilon_p \bar{f}(x) + \sum_{i=1}^{k+1} \sum_{j=0}^N c_{ij} V_i^{p-1} \psi_{ij} \quad \text{in} \ \Omega_\varepsilon, \\
\phi = 0 \quad \text{on} \ \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} V_i^{p-1} \psi_{ij} \phi = 0 \quad \text{for all} \ i, j,
\end{cases}
\]

for certain constants \( c_{ij} \).
2.3. Linear problem. The first step to solve (10) involves the following linear problem: given $h \in C^\alpha(\overline{\Omega}_\varepsilon)$, find a function $\phi$ and constants $c_{ij}$ such that

\[
\begin{cases}
(-\Delta_{\Omega_\varepsilon})^s \phi - p|V|^{p-1}\phi = h + \sum_{i=1}^{k+1} \sum_{j=0}^N c_{ij} V^{p-1}_i \Psi_{ij} & \text{in } \Omega_\varepsilon, \\
\phi = 0 & \text{on } \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} V^{p-1}_i \Psi_{ij} \phi = 0 & \text{for all } i,j,
\end{cases}
\]  
(11)

for certain constants $c_{ij}$. Let us denote the linear operator $M_\varepsilon$ associated with (11) by

\[M_\varepsilon(\phi) := (-\Delta_{\Omega_\varepsilon})^s \phi - p|V|^{p-1}\phi\]
under the aforementioned orthogonality condition. To solve problem (10), we rewrite the first equation in (10) as

\[M_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k+1} \sum_{j=0}^N c_{ij} V^{p-1}_i \Psi_{ij},\]
where the nonlinear term is

\[N_\varepsilon(\phi) := |V + \psi + \phi|^{p-1}(V + \psi + \phi) - |V + \psi|^{p-1}(V + \psi) - p|V + \psi|^{p-1}\phi + p(|V + \psi|^{p-1} - |V|^{p-1})\phi,\]
and the error term is given by

\[R_\varepsilon := \left(|V|^{p-1}V - \left(\sum_{i=1}^{k} W_i^P - W_{k+1}^P\right)\right) + (|V + \psi|^{p-1}(V + \psi) - |V|^{p-1}V).\]

(14)

Now, it is useful to introduce convenient norms that depend on $\xi' \in \Omega^{k+1}_\varepsilon$. For a function $\varphi$ defined in $\Omega_\varepsilon$, we let

\[\|\varphi\|_\sigma := \sup_{x \in \Omega_\varepsilon} \left(\sum_{j=1}^{k+1} \left(1 + |x - \xi_j|^2\right)^{-\frac{N-2s}{2}}\right)^{-\frac{\sigma}{2}} \varphi(x),\]
(15)
for some $\sigma > 0$.

Before continuing, we make the following choice of points and parameters belonging to the set:

\[\mathcal{M}_\delta := \{\{\xi, \lambda\} : \text{dist}(\xi_i, \partial \Omega) > \delta; |\xi_i - \xi_j| > \delta \text{ if } i \neq j; \delta < \lambda_i < \delta^{-1}\},\]
(16)

where $\xi = (\xi_1, \ldots, \xi_{k+1}) \in \mathbb{R}^{(k+1)N}$ and $\lambda = (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{R}^{k+1}$.

To invert the operator $M_\varepsilon$ defined in (12) satisfying the orthogonality conditions given in (11), we introduce the spaces $L^\infty_\sigma(\Omega_\varepsilon)$ and $L^2_\sigma(\Omega_\varepsilon)$ to be the spaces of functions defined in $\Omega_\varepsilon$, endowed with the finite norm $\|\cdot\|_\sigma$ and $\|\cdot\|_{\sigma_2}$, respectively, where these norms correspond to the norm given in (15) when we replace $\sigma$ with $\sigma_1$ or $\sigma_2$, respectively. It follows that the operator $M_\varepsilon$ is uniformly invertible with respect to the aforementioned weighted $L^\infty_\sigma(\Omega_\varepsilon)$-spaces for all sufficiently small $\varepsilon$, up to subsequences. This fact is established in the next proposition, whose proof is a slight modification of the corresponding result obtained in [16] in a similar way to [15].

**Proposition 1.** Let $\sigma_1 = \frac{2s}{N-2s}$, and let $\sigma_2 = \frac{4s}{N-2s}$, both for all $N > 4s$. Assume that $(\xi, \lambda) \in \mathcal{M}_\delta$, with $\mathcal{M}_\delta$ defined as in (16). Then, there are numbers $\varepsilon_0 > 0$, $C > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\overline{\Omega}_\varepsilon)$, problem (11) has a unique
solution \( \phi = M^{-1}_\varepsilon(h) \). Furthermore, the map \((\xi, \lambda, h) \rightarrow M^{-1}_\varepsilon(h) = \phi\) is of class \(C^1\) and satisfies

\[
\|\phi\|_{\sigma_1} \leq C \|h\|_{\sigma_2} \quad \text{and} \quad \|\nabla_{\xi, \lambda} \phi\|_{\sigma_1} \leq C \|h\|_{\sigma_2}.
\]

Finally, the following estimate on the numbers \(c_{ij}\) holds:

\[
|c_{ij}| \leq C \|h\|_{\sigma_2}.
\]

We henceforth denote by \(C\) a generic constant that is independent of \(\varepsilon\) and of the particular \((\xi, \lambda) \in \mathcal{M}_\delta\), with \(\mathcal{M}_\delta\) defined as in (16).

2.4. Finite-dimensional reduction. We now return to the nonlinear problem. We previously proved the following estimates.

**Lemma 2.1.** Let \(\phi \in L^\infty_{\sigma_1}(\Omega_\varepsilon)\) satisfying \(\|\phi\|_{\sigma_1} < 1\). There exist \(\varepsilon_0 > 0\) and \(C > 0\) such that for each \(0 < \varepsilon < \varepsilon_0\), the following estimates hold:

\[
\|N_\varepsilon(\phi)\|_{\sigma_2} \leq \begin{cases} 
C(\|\phi\|_{\sigma_1} + \varepsilon \frac{2(N-4s)}{N-2s})\|\phi\|_{\sigma_1} & \text{if } 4s < N \leq 6s, \\
C(\varepsilon^{2(p-2)}\|\phi\|_{\sigma_1} + \varepsilon^{p-1 + \varepsilon} \frac{(p-1)(3N-10s)}{2(N-2s)}\|\nabla_{\xi, \lambda} \phi\|_{\sigma_1}) & \text{if } N > 6s,
\end{cases}
\]

and

\[
\|\nabla_{\xi, \lambda} N_\varepsilon(\phi)\|_{\sigma_2} \leq \begin{cases} 
C(\varepsilon^{2(N-4s+1)} \frac{2(N-2s)}{N-2s} \|\phi\|_{\sigma_1} + \varepsilon \frac{2(N-2s)}{N-2s} \|\nabla_{\xi, \lambda} \phi\|_{\sigma_1}) & \text{if } 4s < N \leq 6s, \\
C(\varepsilon^{2(p-2)} \frac{2(N-2s)}{N-2s} + \varepsilon \frac{1}{N-2s} \frac{(N-5s+1)}{(N-2s)} \|\nabla_{\xi, \lambda} \phi\|_{\sigma_1}) & \text{if } N > 6s,
\end{cases}
\]

where \(N_\varepsilon(\phi)\) is given by (13).

**Proof.** Note that

\[
|N_\varepsilon(\phi)| \leq p(p-1)|V + \phi + t_1 \phi|^{p-2} |\phi|^2 + p(p-1)|V + \bar{t}_2 \psi|^{p-2} |\psi|^2,
\]

with \(t_1 \in (0, 1)\). If \(4s < N \leq 6s\), then \(p - 2 = \frac{6s-N}{N-2s} \geq 0\), and we deduce that

\[
\left| \left( \sum_{i=1}^{k+1} W_i \right)^{-\frac{4s}{2s}} N_\varepsilon(\phi) \right| \leq C(\|\phi\|_{\sigma_1}^2 + \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1}),
\]

since \(|V + \phi + t_1 \phi|\) and \(|V + \bar{t}_2 \psi|^{p-2}|\psi|\) are bounded.

In the case of \(N > 6s\), one has \(p - 2 = \frac{6s-N}{N-2s} < 0\). It is then convenient to separate the course of study into two regions. First, we consider the region where \(x \in \Omega_\varepsilon\) is such that \(\text{dist}(x, \partial \Omega_\varepsilon) \geq \varepsilon^{-\frac{2s}{N-2s}}\). Hence, if \(|\psi| + |\phi| > \frac{1}{2}|V|\), it follows that

\[
\left| \left( \sum_{i=1}^{k+1} W_i \right)^{-\frac{4s}{2s}} |V + \phi + t_1 \phi|^{p-2} |\phi|^2 \right| \leq C \varepsilon^{2(p-2)} \left( \sum_{i=1}^{k+1} W_i \right)^{-\frac{2s}{2s}} \phi^2 = C \varepsilon^{2(p-2)} \|\phi\|_{\sigma_1}^2,
\]

and

\[
\left| \left( \sum_{i=1}^{k+1} W_i \right)^{-\frac{4s}{2s}} |V + \bar{t}_2 \psi|^{p-2} |\psi| |\phi| \right| \leq C \varepsilon^{2(p-2)} \left( \sum_{i=1}^{k+1} W_i \right)^{-\frac{4s}{2s}} \phi \psi = C \varepsilon^{2(p-2)} \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1}.
\]
Then,
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} N_\varepsilon(\phi) \leq C\varepsilon^{2(2-p)} \left( \|\phi\|_{\sigma_1}^2 + \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1} \right).
\]

On the other hand, if \(|\psi| + |\phi| \leq \frac{1}{2}|V|\), it follows that
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} |V + \psi + \tilde{t}_1 \phi|^{p-2} |\phi|^2 \leq C|W|^{\frac{2(p-1)}{N-2} - \frac{4(2-p)}{N-2}} \left( \sum_{i=1}^{k+1} W_i \right)^{\frac{2(p-1)}{N-2}} |\phi|^p
\]
\[
= C\varepsilon^{\frac{4(2-p)}{N-2}(p-2)} \|\phi\|_{\sigma_1}^p.
\]

and
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} |V + \tilde{t}_2 \psi|^{p-2} |\psi||\phi| \leq C|W|^{\frac{2(p-1)}{N-2} - \frac{4(2-p)}{N-2}} \left( \sum_{i=1}^{k+1} W_i \right)^{\frac{2(p-1)}{N-2}} \|\phi||\psi|^{p-1}
\]
\[
= C\varepsilon^{\frac{2(p-1)}{N-2}(p-2)} \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1}^{p-1}.
\]

Then,
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} N_\varepsilon(\phi) \leq C\varepsilon^{\frac{4(2-p)}{N-2}(p-2)} \left( \|\phi\|_{\sigma_1}^2 + \varepsilon \frac{2(2-p)}{N-2} \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1} \right).
\]

We now consider the region where \(x \in \Omega_x\) is such that \(\text{dist}(x, \partial \Omega_x) < \varepsilon^{-\frac{2}{N-2}} \delta\). In this case, straightforward calculations lead again to
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} N_\varepsilon(\phi) \leq C\varepsilon^{\frac{4(2-p)}{N-2}(p-2)} \left( \|\phi\|_{\sigma_1}^p + \varepsilon \frac{4(2-p)}{N-2} \|\phi\|_{\sigma_1} \|\psi\|_{\sigma_1}^{p-1} \right).
\]

Observe now that \(\varepsilon^{2\gamma}(x) = -\psi(\varepsilon^{-\frac{2}{N-2}} x)\) for all \(x \in \Omega\), being \(\gamma\) the solution of (3). Hence, since \(\gamma\) is bounded it follows that \(\|\psi\|_{\infty} = O(\varepsilon^2)\). By combining the previous estimates and by noticing that \(\|\psi\|_{\sigma_1} = O(\varepsilon^{-\frac{2}{N-2}})\), we get (17). For obtaining (18), we only observe that
\[
\left| \frac{\partial N_\varepsilon}{\partial \xi_{ij}} (\phi) \right| \leq p(p-1)|V + \psi + \tilde{t}_1 \psi|^{p-2} \left| \frac{\partial V}{\partial \xi_{ij}} \right| |\phi|
\]
\[
+ p(p-1)|V + \tilde{t}_2 \psi|^{p-2} \left| \frac{\partial \phi}{\partial \xi_{ij}} \right| |\psi| + p(p-1)|V|^{p-2} \left| \frac{\partial V}{\partial \xi_{ij}} \right| |\phi|
\]
for some \(\tilde{t}_1, \tilde{t}_2 \in (0, 1)\); and similarly when we derivate respect to \(\lambda_i\). Then, proceeding as before, the proof is completed.

Now, observe that
\[
|V + \psi|^{p-1}(V + \psi) - |V|^{p-1}V = p|V + \tilde{t}_1 \psi|^{p-1}\psi
\]
for some \(\tilde{t} \in (0, 1)\). In addition, by taking into account that \(W_{\lambda_i, \xi_i}(x) - V_i(x) = C\varepsilon^2 + o(\varepsilon^2)\) for \(|x - \xi_i| < \varepsilon^{-\frac{2}{N-2}} \delta\) and \(\delta < \lambda_i < \delta^{-1}\), it is straightforward to confirm that
\[
\left| \sum_{i=1}^{k+1} W_i \right|^{\frac{4(2-p)}{N-2}} \left( |V|^{p-1} V - \left( \sum_{i=1}^{k} W_i^p - W_{k+1}^p \right) \right) \leq C\varepsilon^2.
\]
On the complement of these regions
\[ \left| V^{p-1} V - \left( \sum_{i=1}^{k} W_i^p - W_{k+1}^p \right) \right| \leq C \varepsilon^{2p}, \]
we obtain
\[ \| R_\varepsilon \|_{\sigma_2} \leq C (\varepsilon^2 + \| \psi \|_\infty), \quad (19) \]
where \( R_\varepsilon \) is given by (14). Then, \( \| R_\varepsilon \|_{\sigma_2} \leq C \varepsilon^2 \). Besides, since
\[ \left| \sum_{i=1}^{k+1} W_i \right| \frac{-\Delta \phi}{\partial \xi} (|V+\xi\phi|^p \phi) = (p-1) \left| \sum_{i=1}^{k+1} W_i \right| \frac{-\Delta \phi}{\partial \xi} (|V+\xi\phi|^p \phi), \]
and similarly when we derivate respect to \( \lambda \), we obtain
\[ \| \nabla \xi \lambda R_\varepsilon \|_{\sigma_2} \leq C \varepsilon^2. \]
Furthermore, if we let \( \eta_0 = M_\varepsilon^{-1}(R_\varepsilon) \), it is clear that problem (10) can be transformed into
\[ \begin{cases} (-\Delta_\varepsilon)^s \phi - p|V|^{p-1} \phi = N_\varepsilon(\eta_0 + \phi) + \sum_{i=1}^{k+1} \sum_{j=0}^{N} c_{ij} V_i^{p-1} \Psi_{ij} & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{in } \partial \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} V_i^{p-1} \Psi_{ij} \phi \, dx = 0 & \text{for all } i, j, \end{cases} \quad (20) \]
for certain constants \( c_{ij} \), depending on \( \varepsilon \).

The next result shows the existence of solutions for (20) (or equivalently, for (10)).

**Proposition 2.** Let \( (\xi, \lambda) \in \mathcal{M}_\delta \), with \( \mathcal{M}_\delta \) defined as in (16). Then, \( C > 0 \) exists such that for all sufficiently small \( \varepsilon > 0 \), a unique solution \( \phi = \tilde{\phi}(\xi, \lambda) \) to problem (20) exists, having the form \( \phi = \tilde{\phi} - \eta_0 \), with \( \eta_0 = M_\varepsilon^{-1}(R_\varepsilon) \). Moreover, the map \( (\xi, \lambda) \mapsto \tilde{\phi}(\xi, \lambda) \) is of \( C^1 \)-class for the \( \| \cdot \|_{\sigma_1} \)-norm and satisfies
\[ \| \tilde{\phi} \|_{\sigma_1} \leq C \varepsilon^2 \quad \text{and} \quad \| \nabla \xi \lambda \tilde{\phi} \|_{\sigma_1} \leq C \varepsilon^2. \]

**Proof.** Observe that \( \phi \) is a solution of (10) if and only if \( \phi = M_\varepsilon^{-1}(N_\varepsilon(\phi - \eta_0) + R_\varepsilon) =: S_\varepsilon(\phi) \). Consider the set \( \mathcal{E}_r = \{ \phi \in \mathcal{H}^s(\Omega_\varepsilon) \mid \| \phi \|_{\sigma_1} \leq r \varepsilon^2 \} \), with \( r > 0 \) to be chosen later. Note that if we can show that \( S_\varepsilon \) is a contraction, then there is a fixed point in \( \mathcal{E}_r \) for \( S_\varepsilon \), which is equivalent to solving (10). From estimates (17) and (19), we obtain that
\[ \| S_\varepsilon(\phi) \|_{\sigma_2} \leq C \| N_\varepsilon(\phi) + R_\varepsilon \|_{\sigma_2} \leq r \varepsilon^2 \]
for all small \( \varepsilon \) provided that \( r \) is chosen to be sufficiently large but independent of \( \varepsilon \). It is straightforward to confirm that \( S_\varepsilon \) maps \( \mathcal{E}_r \) into itself for this choice of \( r \), implying that \( S_\varepsilon \) is a contraction mapping in \( \mathcal{E}_r \), which follows from the fact that \( N_\varepsilon \) defines a contraction in the \( \| \cdot \|_{\sigma_1} \)-norm. Therefore, there exists a unique fixed point in the set \( \mathcal{E}_r \).

Concerning the differentiability properties of the function \( \phi(\xi, \lambda) \), we write
\[ B(\xi, \lambda, \phi) := \phi - M_\varepsilon^{-1}(N_\varepsilon(\phi) + R_\varepsilon). \]
Moreover, we have \( B(\xi, \lambda, \phi) = 0 \). We write
\[ \nabla_{\phi} B(\xi, \lambda, \phi)[\theta] = \theta - M_\varepsilon^{-1}(\theta \nabla_{\phi}(N_\varepsilon(\phi))) =: \theta + K_1(\theta). \]
It is easily verified that the following estimate holds: \( \|K_1(\theta)\|_{\sigma_1} \leq C \varepsilon \|\theta\|_{\sigma_1} \). In this way, the linear operator \( \nabla_\xi B(\xi, \lambda, \phi) \) is invertible for all small \( \varepsilon \), with uniformly bounded inverse, in \( C_{\sigma_1} \), the Banach space of continuous functions in \( \Omega_\varepsilon \) with bounded \( \| \cdot \|_{\sigma_1} \)-norm. The operator also depends continuously on its parameters. Let us differentiate with respect to \( \xi \). We have
\[
\nabla_\xi B(\xi, \lambda, \phi) = - \langle \nabla_\xi (M_\varepsilon^{-1}) \rangle (N_\varepsilon(\phi) + R_\varepsilon) - M_\varepsilon^{-1}(\nabla_\xi N_\varepsilon(\phi) + \nabla_\xi R_\varepsilon),
\]
where all the previous expressions depend continuously on their parameters. Hence, the implicit function theorem yields that \( \phi = \phi(\xi, \lambda) \) is a \( C^1 \) function within \( C_{\sigma_1} \). Moreover, we have
\[
\nabla_\xi \phi = - (\nabla_\phi B(\xi, \lambda, \phi))^{-1} \nabla_\xi B(\xi, \lambda, \phi),
\]
and thus,
\[
\| \nabla_\xi \phi \|_{\sigma_1} \leq C(\|N_\varepsilon(\phi)\|_{\sigma_2} + \|R_\varepsilon\|_{\sigma_2} + \|\nabla_\xi N_\varepsilon(\phi)\|_{\sigma_2} + \|\nabla_\xi R_\varepsilon\|_{\sigma_2}) \leq C \varepsilon^2.
\]

Analogous arguments present the differentiability with respect to \( \lambda \). Therefore, the proof of Proposition 2 is completed. \( \square \)

3. Variational reduction. In this section, we continuously use the notation introduced in the previous sections.

3.1. First approximation. The energy functional associated with (6) is given by
\[
J_\varepsilon(U) = \frac{\kappa_\varepsilon}{2} \int_{\mathcal{C}_\varepsilon} t^{1-2s}|\nabla \tilde{U}|^2 \, dt - \frac{1}{p+1} \int_{\Omega_\varepsilon} |U|^p \, dx + \varepsilon p+1 \int_{\Omega_\varepsilon} fU \, dx. \tag{21}
\]
Let
\[
J_{\varepsilon,0}(V) = \frac{\kappa_\varepsilon}{2} \sum_{i=1}^{k+1} \int_{\mathcal{C}_\varepsilon} t^{1-2s}|\nabla \tilde{V}_i|^2 \, dx - \sum_{i=1}^{k+1} |V_i|^{p+1} \int_{\Omega_\varepsilon} \, dx
\]
\[
+ \frac{\kappa_\varepsilon}{2} \sum_{i \neq j, j \neq k+1} \int_{\mathcal{C}_\varepsilon} t^{1-2s} \nabla \tilde{V}_i \nabla \tilde{V}_j \, dx - \kappa_\varepsilon \sum_{i \neq k+1} \int_{\mathcal{C}_\varepsilon} t^{1-2s} \nabla \tilde{V}_i \nabla \tilde{V}_{k+1} \, dx dt \tag{22}
\]
\[
- \frac{1}{p+1} \int_{\Omega_\varepsilon} \left( |V|^p + 1 - \sum_{i=1}^{k} |V_i|^{p+1} \right) \, dx.
\]

We choose points and parameters belonging to the set \( \mathcal{M}_4 \) defined in (16). The advantage of this constraint on points and parameters is the validity of an expansion of \( J_{\varepsilon}(U) \) in terms of Green’s function and of its regular part of the spectral fractional Laplacian on \( \Omega \) under zero Dirichlet boundary conditions. Specifically, we denote by \( G(\cdot, y) \) the Green’s function of the operator \( (-\Delta_\Omega)^s \) under zero Dirichlet boundary conditions, namely, the solution of
\[
\begin{cases}
(-\Delta_\Omega)^s G(x, y) = \delta_0(x-y), & x \in \Omega, \\
G(x, y) = 0, & x \in \partial \Omega,
\end{cases}
\]
where \( \delta_0(x) \) denotes the Dirac mass at the origin, and by \( H(x, y) \), its regular part, namely,
\[
H(x, y) = \Gamma(x-y) - G(x, y) \quad \text{for all } (x, y) \in \Omega \times \Omega.
\]
Here, \( \Gamma \) is the fundamental solution of the fractional Laplacian: \( \Gamma(x) := b_{N,s}|x|^{2s-N} \), where \( b_{N,s} \) is a suitable constant.
According to some estimates in [11] and [15], it is known that given $\xi_i \in \Omega$ and $\lambda_i > 0$, the following estimates hold:
\[
\begin{aligned}
\Phi_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x) &= \beta_N \lambda_i \frac{N-2s}{2} H(x, \xi_i) \varepsilon^2 + o(\varepsilon^2), \\
\nabla \xi_i \Phi_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x) &= \beta_N \lambda_i \frac{N-2s}{2} \nabla \xi_i H(x, \xi_i) \varepsilon^{2(N-\frac{2s}{N-2s})} + o(\varepsilon^{2(N-\frac{2s}{N-2s})}), \\
\left(\frac{\partial}{\partial \lambda_i} \Phi_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x)\right) &= \left(\frac{N-2s}{2}\right) \beta_N \lambda_i \frac{N-2s}{2} H(x, \xi_i) \varepsilon^2 + o(\varepsilon^2),
\end{aligned}
\]
uniformly for $x \in \Omega$; and
\[
\begin{aligned}
\nabla \xi_i \Psi_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x) &= \beta_N \lambda_i \frac{N-2s}{2} \nabla \xi_i H(x, \xi_i) \varepsilon^{2(N-\frac{2s}{N-2s})} + o(\varepsilon^{2(N-\frac{2s}{N-2s})}), \\
\left(\frac{\partial}{\partial \lambda_i} \Psi_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x)\right) &= \left(\frac{N-2s}{2}\right) \beta_N \lambda_i \frac{N-2s}{2} H(x, \xi_i) \varepsilon^2 + o(\varepsilon^2),
\end{aligned}
\]
uniformly for $x$ in each compact subset of $\Omega \setminus \{\xi_i\}$, where
\[
\beta_N = \int_{R^N} W^p \, dx.
\]
In fact, we have that
\[
\begin{aligned}
\widetilde{W}_{\lambda_i, \xi_i}(\varepsilon^{-\frac{2}{N-2s}} x, t) &= \int_{R^N} \tilde{\Gamma}(x - y, t) W_{\lambda_i, \xi_i}(y) \, dy \\
&= \beta_N \lambda_i \frac{N-2s}{2} \tilde{\Gamma}(x - \xi_i, t) \varepsilon^2 + o(\varepsilon^2),
\end{aligned}
\]
uniformly for $(x, t) \in \partial \mathcal{C}$. Moreover, the following estimates also hold uniformly in the $C^1$-sense respect to the points $\xi_i$ and parameters $\Lambda_i$ such that $(\xi, \Lambda) \in \mathcal{M}_\delta$:
\[
\kappa_s \int_{C_\varepsilon} t^{1-2s} |\nabla V_i|^2 \, dx - \kappa_s \int_{R^N} t^{1-2s} |\nabla \widetilde{W}|^2 \, dx \, dt
\]
\[
= \beta_N \lambda_i \frac{N-2s}{2} \lambda_j \frac{N-2s}{2} G(\xi_i, \xi_j) \varepsilon^2 + o(\max\{\lambda_i, \lambda_j\}^{-N-2s}) + o(\varepsilon^2)
\]
and
\[
\kappa_s \int_{C_\varepsilon} t^{1-2s} |\nabla V_i \nabla \widetilde{V}_j| \, dx \, dt
\]
\[
= \beta_N \lambda_i \frac{N-2s}{2} \lambda_j \frac{N-2s}{2} G(\xi_i, \xi_j) \varepsilon^2 + o(\varepsilon^2).
\]
It follows that
\[
\int_{\Omega} |V_i|^{p+1} \, dx = \int_{R^N} |W|^{p+1} \, dx - (p + 1) \beta_N \lambda_i \frac{N-2s}{2} \lambda_j \frac{N-2s}{2} H(\xi_i, \xi_j) \varepsilon^2 + o(\varepsilon^2)
\]
and
\[
\frac{1}{p+1} \int_{\Omega} \left( \sum_{i=1}^{k} \left| V_i \right|^{p+1} \right) \, dx = \beta_N \lambda_i \frac{N-2s}{2} \lambda_j \frac{N-2s}{2} G(\xi_i, \xi_j) \varepsilon^2 + o(\varepsilon^2).
\]
We focus on the asymptotic expansion for the energy functional $J_\varepsilon$ at the function $V$, assuming that $(\xi, \Lambda) \in \mathcal{M}_\delta$, with $\mathcal{M}_\delta$ defined as in (16). By convenience, we adjust the constraint set $\mathcal{M}_\delta$ defined in (16). We let
\[
\mathcal{M}_\delta = \{ (\xi, \Lambda) : (\xi, \Lambda) \in \mathcal{M}_\delta \text{ with } \lambda_i = (\beta_i^{-1})^{\frac{2}{N-2s}} \text{ for all } i \},
\]
where \( \Lambda = (\Lambda_1, \ldots, \Lambda_{k+1}) \in \mathbb{R}^{k+1} \) and \( \beta_N \) is given in (25). Then,
\[
(\xi, \lambda) \in M_\delta \text{ if and only if } (\xi, \Lambda) \in M^*_\delta.
\]

**Proposition 3.** Assume that \( f \) satisfies \((H_4)\). Given a small \( \delta > 0 \), the following expansions hold
\[
J_\varepsilon(V) = (k+1)S_N + \varepsilon^2 \Theta_k(\xi, \Lambda) + o(\varepsilon^2) \quad (31)
\]
and
\[
\nabla_{\xi, \Lambda} J_\varepsilon(V) = \varepsilon^2 \nabla_{\xi, \Lambda} \Theta_k(\xi, \Lambda) + o(\varepsilon^2) \quad (32)
\]
both uniformly with respect to \( (\xi, \Lambda) \in M^*_\delta \), with \( M^*_\delta \) defined as in (30). Here, the constant \( S_N \) is given by
\[
S_N = \frac{\kappa s}{2} \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla \tilde{W}|^2 \, dx \, dt - \frac{1}{p+1} \int_{\mathbb{R}^N} W^{p+1} \, dx, \quad (33)
\]
and the function \( \Theta_k \) is defined by
\[
\Theta_k(\xi, \Lambda) = \frac{1}{2} \left( \sum_{j=1}^{k+1} H(\xi_j, \xi_j) \Lambda_j^2 - \sum_{i, j \neq k+1} G(\xi_i, \xi_j) \Lambda_i \Lambda_j \right)
+ 2 \sum_{i \neq k+1} G(\xi_i, \xi_{k+1}) \Lambda_i \Lambda_{k+1}) \right) + \sum_{j=1}^k \gamma(\xi_j) \Lambda_j - \gamma(\xi_{k+1}) \Lambda_{k+1}. \quad (34)
\]

**Proof.** Note that we can write
\[
J_\varepsilon(V) = J_{\varepsilon,0}(V) + \varepsilon^{p+1} \int_{\Omega_\varepsilon} f(x) V(x) \, dx. \quad (35)
\]

where \( J_\varepsilon \) and \( J_{\varepsilon,0} \) are given by (21) and (22), respectively. From (23), (24), (26)–(29), we have
\[
J_{\varepsilon,0}(V) = (k+1)S_N + \frac{1}{2} \left( \sum_{j=1}^{k+1} H(\xi_j, \xi_j) \Lambda_j^2 - \sum_{i, j \neq k+1} G(\xi_i, \xi_j) \Lambda_i \Lambda_j, \right.
+ 2 \left. \sum_{i \neq k+1} G(\xi_{k+1}, \xi_i) \Lambda_i \Lambda_{k+1} \right) \varepsilon^2 + o(\varepsilon^2), \quad (36)
\]
uniformly in the \( C^1 \)-sense with respect to the points \( \xi_i \) and parameters \( \lambda_i > 0 \) such that \( (\xi, \lambda) \) satisfies (16).
On the other hand, taking into account that away from \( x = \xi_i \),
\[
V_i = V_{\lambda_i, \xi_i}(\varepsilon^2 \pi^{2s/(p+2)} x) = \beta_N \lambda_i^{\frac{N+2s}{p+2}} G(x, \xi_i) \varepsilon^2 + o(\varepsilon^2), \quad (37)
\]
uniformly on each compact subset of \( \Omega \) and considering the fact that \( \gamma \) satisfies (2),
then a direct computation yields.
\[ \varepsilon^{p+1} \int_{\Omega_{r}} \tilde{f} V_{i} dx = \varepsilon^{p+1} \sum_{i=1}^{k} \int_{\Omega_{r}} f V_{i} dx - \varepsilon^{p+1} \int_{\Omega_{r}} f V_{k+1} dx \]

\[ = \sum_{i=1}^{k} \int_{\Omega} f(x) V_{i}(\varepsilon^{-\frac{2}{p+1}} x) dx - \int_{\Omega} f(x) V_{k+1}(\varepsilon^{-\frac{2}{p+1}} x) dx \]

\[ = \varepsilon^{2} \sum_{i=1}^{k} \beta_{N} \lambda_{i}^{\frac{N-2}{2}} \int_{\Omega} f(x) G(x, \xi_{i}) dx \]

\[ - \varepsilon^{2} \beta_{N} \lambda_{k+1}^{\frac{N-2}{2}} \int_{\Omega} f(x) G(x, \xi_{k+1}) dx + o(\varepsilon^{2}) \]

\[ = \left( \sum_{i=1}^{k} \gamma(\xi_{i}) \lambda_{i} - \gamma(\xi_{k+1}) \lambda_{k+1} \right) \varepsilon^{2} + o(\varepsilon^{2}). \]

Hence, the claim (31) follows from estimates (35)–(38). Finally, observe that \( \gamma \in C^{1+\alpha-2s}(\Omega) \) with \( \alpha - 2s > 0 \), and that the estimates (35)–(38) are uniform in the \( C^{1} \)-sense respect to the points \( \xi_{i} \) and parameters \( \Lambda_{i} \) such that \( (\xi, \Lambda) \in \mathcal{M}_{\delta}^{*} \). Then (32) holds, which completes the proof. \( \square \)

### 3.2. Reduction of the variational problem

In the rest of this paper, we consider \( (\xi, \Lambda) \in \mathcal{M}_{\delta}^{*} \), with \( \mathcal{M}_{\delta}^{*} \) defined as in (30), and the function \( \phi := \phi(\xi, \Lambda) \) given by Proposition 2, which is the only solution of problem (20), equivalent to solving (10). According to the previous calculations, note that the condition \( c_{i} = 0 \) in (20), for all \( i = 1, 2, \ldots, k \), is equivalent to

\[ U := V + \psi + \phi(\xi, \Lambda) \]

being a solution of problem (6), where \( V \) is as in (7) and \( \psi \) solves (9). Therefore, we need to find points \( (\xi, \Lambda) \) so that the system \( c_{i}(\xi, \Lambda) = 0 \) for all \( i = 1, 2, \ldots, k \) has a solution. This system turns out to be equivalent to a variational problem, as stated below. Consider

\[ J_{\varepsilon}(\xi, \Lambda) := J_{\varepsilon}(V + \psi + \phi(\xi, \Lambda)), \tag{39} \]

where \( J_{\varepsilon} \) is the functional given in (21). Hence, we obtain the following by means of standard arguments.

**Lemma 3.1.** The function \( U = V + \psi + \phi(\xi, \Lambda) \) is a solution of (6) if and only if \( (\xi, \Lambda) \) is a critical point of \( J_{\varepsilon} \).

**Proof.** If we assume that \( U = V + \psi + \phi(\xi, \Lambda) \) is a solution of (6), then for each \( i = 1, 2, \ldots, N, j = 1, \ldots, k + 1 \), it follows that

\[ \nabla_{\xi, \Lambda} J_{\varepsilon}(U) \left[ \frac{\partial U}{\partial \xi_{ij}} \right] = 0. \]

In other words,

\[ \frac{\partial J_{\varepsilon}}{\partial \xi_{ij}}(\xi, \Lambda) = \kappa_{s} \int_{\Omega_{r}} t^{1-2s} \nabla \tilde{U} \nabla \left( \frac{\partial U}{\partial \xi_{ij}} \right) dt dx - \int_{\Omega_{r}} |U|^{p-1} U \frac{\partial U}{\partial \xi_{ij}} + \varepsilon^{p+1} \int_{\Omega_{r}} f(x) \frac{\partial U}{\partial \xi_{ij}} \]

\[ = \int_{\Omega_{r}} \left( (-\Delta_{\Omega_{r}})^{s} U - |U|^{p-1} U + \varepsilon^{p+1} f(x) \right) \frac{\partial U}{\partial \xi_{ij}} = 0 \]

for all \( i, j \). Similarly, we obtain

\[ \frac{\partial J_{\varepsilon}}{\partial \Lambda_{i}}(\xi, \Lambda) = 0 \]
for all \(i\), so that \((\xi, \Lambda)\) is a critical point of \(\mathcal{J}_\varepsilon\).

On the other hand, if \((\xi, \Lambda)\) is a critical point of \(\mathcal{J}_\varepsilon\), then

\[
\nabla_{\xi} J_\varepsilon(U) \left[ \frac{\partial U}{\partial \xi_{ij}} \right] = 0 \quad \text{for all } i, j, \quad \text{and} \quad \nabla_{\Lambda} J_\varepsilon(U) \left[ \frac{\partial U}{\partial \Lambda} \right] = 0 \quad \text{for all } i. \tag{40}
\]

Since

\[
\frac{\partial U}{\partial \xi_{ij}} = \Psi_{ij} + o(1) \quad \text{for all } i, j, \quad \text{and} \quad \frac{\partial U}{\partial \Lambda} = \Psi_{i0} + o(1) \quad \text{for all } i,
\]

we obtain

\[
\nabla_{\xi} J_\varepsilon(U) [\Psi_{ij} + o(1)] = 0 \quad \text{for all } i, j, \quad \text{and} \quad \nabla_{\Lambda} J_\varepsilon(U) [\Psi_{i0} + o(1)] = 0 \quad \text{for all } i,
\]

where \(o(1) \to 0\) in the \(\| \cdot \|_{\sigma_1}\)-norm, since we have also seen that \(\|\nabla_{(\xi,\Lambda)} \phi\|_{\sigma_1} = o(1)\).

By definition of \(\phi\), we have that

\[
\nabla_{\xi, \Lambda} J_\varepsilon(U) [\phi] = 0 \quad \text{for all } \phi \in H^s(\Omega_{\varepsilon}) \text{ satisfying the orthogonality condition}
\]

\[
\int_{\Omega_{\varepsilon}} V_{l}^{p-1} \Psi_{ij} \phi = 0 \quad \text{for all } i, j.
\]

Note that for a given function \(\theta\) in the span \(\{\Psi_{l\ell}\}_{l=1,\ldots,k+1; \ell=0,\ldots,N}\), we can find constants \(b_{ij}\) such that \(\theta - \sum_{ij} b_{ij} \Psi_{ij}\) satisfies the system

\[
\int_{\Omega_{\varepsilon}} V_{l}^{p-1} \Psi_{ij} \theta = \sum_{ij} b_{ij} \int_{\Omega_{\varepsilon}} V_{l}^{p-1} \Psi_{ij} \Psi_{ij}
\]

for all \(l, \ell\), which has a uniformly invertible associated matrix. In particular,

\[b_{ij} = O(\|\theta\|_{\sigma_1}).\]

Then, (40) is equivalent to

\[
\nabla_{\xi} J_\varepsilon(U) [\Psi_{ij} + o(1)] = 0 \quad \text{for all } i, j, \quad \text{and} \quad \nabla_{\Lambda} J_\varepsilon(U) [\Psi_{i0} + o(1)] = 0 \quad \text{for all } i,
\]

where \(\theta \in \text{span} \{\Psi_{l\ell}\}_{l=1,\ldots,k+1; \ell=0,\ldots,N}\). In this way, we obtain

\[
\nabla_{\xi} J_\varepsilon(U) [\Psi_{ij}] = 0 \quad \text{for all } i, j, \quad \text{and} \quad \nabla_{\Lambda} J_\varepsilon(U) [\Psi_{i0}] = 0 \quad \text{for all } i.
\]

By the definition of \(c_{ij}\), we can conclude that \(c_{ij} = 0\) for all \(i, j\), indicating that \(U\) solves (6).

### 3.3. Expansion of the full energy.

The next step is to validate an expansion for \(\mathcal{J}_\varepsilon\), which is crucial for finding its critical points. For convenience, we introduce the constant

\[
c_f = \int_{\Omega} f(x) \gamma(x) \, dx. \tag{41}
\]

**Proposition 4.** Under the assumptions of Proposition 3, the expansions

\[
\mathcal{J}_\varepsilon(\xi, \Lambda) = (k + 1) S_N + \varepsilon^2 (\Theta_k(\xi, \Lambda) + c_f) + o(\varepsilon^2), \tag{42}
\]

and

\[
\nabla_{\xi, \Lambda} \mathcal{J}_\varepsilon(\xi, \Lambda) = \varepsilon^2 \nabla_{\xi, \Lambda} \Theta_k(\xi, \Lambda) + o(\varepsilon^2) \tag{43}
\]

hold uniformly with respect to \((\xi, \Lambda) \in M_\delta^*, \text{ with } M_\delta^* \text{ defined as in (30)}, \text{ where } \Theta_k\]

is uniformly bounded independently of \(\varepsilon\). Here, \(S_N\) and \(\Theta_k\) are given by (33) and (34), respectively.
Proof. The first step to achieve our goal is to prove that

$$J_\varepsilon(\xi, \Lambda) - J_\varepsilon(V + \psi) = o(\varepsilon^2)$$

(44)

and

$$\nabla_{\varepsilon,i}(J_\varepsilon(\xi, \Lambda) - J_\varepsilon(V + \psi)) = o(\varepsilon^2).$$

(45)

Let us set $\phi = \phi(\xi, \Lambda)$, and note that

$$J_\varepsilon(\xi, \Lambda) - J_\varepsilon(V + \psi) = -\int_0^1 \tau D^2 J_\varepsilon(V + \psi + \tau \phi)[\phi, \phi]d\tau$$

$$= \int_0^1 \tau \left( \kappa_s \int_{C_\varepsilon} t^{1-2s}|\nabla \phi|^2 - \int_{\Omega_s} p|V + \psi + \tau \phi|^{p-1} \phi^2 \right) d\tau$$

$$= \int_{\Omega_s} (N_\varepsilon(\phi) + R_\varepsilon) \phi$$

$$- \int_0^1 \tau \left( \int_{\Omega_s} p(|V + \psi + \tau \phi|^{p-1} - |V|^{p-1}) \phi^2 \right) d\tau.$$

Differentiating with respect to $\xi$ variables, we obtain

$$\nabla_{\varepsilon,i}(J_\varepsilon(\xi, \Lambda) - J_\varepsilon(V + \psi)) = \int_{\Omega_s} \nabla_{\varepsilon,i}((N_\varepsilon(\phi) + R_\varepsilon) \phi)$$

$$- \int_0^1 \tau \left( \int_{\Omega_s} p\nabla_{\varepsilon,i}(|V + \psi + \tau \phi|^{p-1} - |V|^{p-1}) \phi^2 \right) d\tau,$$

and since $\|\phi\|_{\sigma_1} + \|\psi\|_{\sigma_1} + \|\nabla_{\varepsilon,i} \phi\|_{\sigma_1} + \|\nabla_{\varepsilon,i} \phi\|_{\sigma_1} + \|N_\varepsilon(\phi)\|_{\sigma_2} + \|R_\varepsilon\|_{\sigma_2} = O(\varepsilon^2)$,

the validity of (44) and (45) is confirmed.

The next step is to prove that

$$J_\varepsilon(V + \psi) - J_\varepsilon(V) = \varepsilon^2 c_f + o(\varepsilon^2),$$

(46)

where $c_f$ is given by (41), and

$$\nabla_{\varepsilon,i}(J_\varepsilon(V + \psi) - J_\varepsilon(V)) = o(\varepsilon^2).$$

(47)

Note that

$$J_\varepsilon(V + \psi) - J_\varepsilon(V) = \int_0^1 \tau \left( \kappa_s \int_{C_\varepsilon} t^{1-2s}|\nabla \psi|^2 - \int_{\Omega_s} p|V + \tau \psi|^{p-1} \psi^2 \right) d\tau.$$

In addition, from (41), we have that

$$\int_0^1 \tau \left( \kappa_s \int_{C_\varepsilon} t^{1-2s}|\nabla \phi|^2 \right) d\tau = -\varepsilon^{p+1} \int_{\Omega_s} \bar{f}\psi = \varepsilon^2 \int \bar{f}\gamma = \varepsilon^2 c_f$$

and since $\|\psi\|_{\infty} = O(\varepsilon^{p+1})$, we obtain

$$\left| \int_0^1 \tau \left( \int_{\Omega_s} p|V + \tau \psi|^{p-1} \psi^2 \right) d\tau \right| = o(\varepsilon^2).$$

Furthermore, note that

$$\nabla_{\varepsilon,i}(J_\varepsilon(V + \psi) - J_\varepsilon(V)) = \int_0^1 \tau \left( \int_{\Omega_s} p\nabla_{\varepsilon,i}(|V + \tau \psi|^{p-2} \psi^2) \right) d\tau$$

and since $\|\psi\|_{\infty} \leq O(\varepsilon^{p+1})$, it is straightforward to confirm that

$$\nabla_{\varepsilon,i}(J_\varepsilon(V + \psi) - J_\varepsilon(V)) = o(\varepsilon^2).$$
It follows that the validity of the estimates (46) and (47) is verified. Similarly, we validate the results for the differentiability with respect to $\Lambda$. Therefore, the estimates (42) and (43) hold, and the proof is completed.

4. **Existence of the predicted solutions.** We are now in a position to prove Theorem 1.1. We proceed by adapting the arguments used to complete the proof in [12, Theorem 1.1].

Due to the assumptions on $\Omega$ and $f$ given in (H1)-(H5), we look for critical points of $\mathcal{J}_\varepsilon$ of the special form

$$
\xi_i = \rho Q_i, \quad \xi_{k+1} = 0, \quad \Lambda_i = \Lambda_0, \quad \forall i = 1, \ldots, k.
$$

Let us set

$$
\mathcal{I}_\varepsilon(\rho, \Lambda_0, \Lambda_{k+1}) = \mathcal{J}_\varepsilon\left(\rho(Q_1, \ldots, Q_k, 0), (\Lambda_0(1, \ldots, 1), \Lambda_{k+1})\right),
$$

(48)

where $\mathcal{J}_\varepsilon$ is given by (39). We have the following result.

**Lemma 4.1.** Under the assumptions of Theorem 1.1, if $(\rho, \Lambda_0, \Lambda_{k+1})$ is a critical point of $\mathcal{I}_\varepsilon$, then $(\xi, \Lambda) = \left(\rho(Q_1, \ldots, Q_k, 0), (\Lambda_0(1, \ldots, 1), \Lambda_{k+1})\right)$ is a critical point of $\mathcal{J}_\varepsilon$.

**Proof.** Note that the functions $V$ and $\bar{f}$ are even with respect to each of the variables $x_3, \ldots, x_N$ in $\Omega_\varepsilon$ and are invariant under rotations in the plane spanned by the first two coordinates. Since the function $\psi$ solves the problem 9, it is clear that it has the same properties as functions $V$ and $\bar{f}$. Moreover, since the function $\phi = \phi(\xi, \Lambda)$ given by Proposition 2 solves (10), it is also evident that $\phi(\xi, \Lambda)(x_1, \ldots, x_N)$ shares the same properties with $V$, $\psi$ and $\bar{f}$. Therefore, since (10) is uniquely solvable, $c_{ij} = 0$ automatically for all $1 \leq i \leq k$ and $2 \leq j \leq N$ and $c_{k+1,1} = 0$.

Consequently, only the term

$$
\sum_{j=1}^k c_{j1} V_j^{p-1} \Psi_{j1} + \sum_{j=1}^{k+1} c_{j0} V_j^{p-1} \Psi_{j0}
$$

appears in the right-hand side of the first equation in (10).

Using again the invariance under rotations in the $(y_1, y_2)$ plane of $\phi$, the previous summation reduces to

$$
\sum_{j=1}^k (\alpha_1 V_j^{p-1} \bar{\Psi}_j + \alpha_2 V_j^{p-1} \Psi_{j0}) + \alpha_3 V_{k+1}^{p-1} \Psi_{(k+1)0}
$$

where

$$
\bar{\Psi}_j(x) = \Psi_{j1}(x) \cos \left(\frac{2\pi j}{k}\right) + \Psi_{j2}(x) \sin \left(\frac{2\pi j}{k}\right), \quad x \in \Omega_\varepsilon,
$$

and $\alpha_i = \alpha_i(\rho, \Lambda_0, \Lambda_{k+1})$, $i = 1, 2, 3$. Therefore, finding the critical points of $\mathcal{J}_\varepsilon$ of the form $(\xi, \Lambda) = \left(\rho(Q_1, \ldots, Q_k, 0), (\Lambda_0(1, \ldots, 1), \Lambda_{k+1})\right)$ reduces to solving $\alpha_i(\rho, \Lambda_0, \Lambda_{k+1}) = 0$ for $i = 1, 2, 3$.

On the other hand, these relations are equivalent to saying that $(\rho, \Lambda_0, \Lambda_{k+1})$ is a critical point of $\mathcal{I}_\varepsilon$. In fact, observe first that

$$
\frac{\partial}{\partial \rho}(V + \psi + \phi) = \sum_{i=1}^k \sum_{j=1}^N \Psi_{ij} + o(1),
$$
\[
\frac{\partial}{\partial \Lambda_0} (V + \psi + \phi) = \sum_{i=1}^{k} \Psi_i + o(1)
\]
and
\[
\frac{\partial}{\partial \Lambda_{k+1}} (V + \psi + \phi) = \Psi_{(k+1)0} + o(1).
\]
where \(o(1) \to 0\) as \(\varepsilon \to 0\), and note that \(\nabla I_{\varepsilon} = 0\) is equivalent to
\[
\nabla J_{\varepsilon}(V + \psi + \phi) = \left[ \frac{\partial}{\partial \rho} (V + \psi + \phi) \right] = \nabla J_{\varepsilon}(V + \psi + \phi) = 0.
\]
By using the functions that belong to the space spanned by the \(\Psi_{ij}\)'s and a function \(\vartheta\) that satisfies \(\int_{\Omega_\varepsilon} V^p \vartheta = 0\) for all \(i, j\), we deduce from (49) that
\[
3 \sum_{i=1}^{3} (\delta_{ij} + o(1)) \alpha_i = 0 \quad \text{for} \quad j = 1, 2, 3.
\]
Thus, \(\alpha_1 = \alpha_2 = \alpha_3 = 0\).

**Proof of Theorem 1.1.** By considering (48) and (39) and from Proposition 4, we can rewrite \(I_{\varepsilon}\) by means of the quadratic form:
\[
I_{\varepsilon}(\rho, \Lambda_0, \Lambda_{k+1}) = \left( k + 1 \right) S_N + \varepsilon^2 \left( \Theta_k(\rho, \Lambda_0, \Lambda_{k+1}) + c_f \right) + o(\varepsilon^2),
\]
where
\[
\Theta_k(\rho, \Lambda_0, \Lambda_{k+1}) = \Lambda_0^2 \left( H(\rho Q_1, \rho Q_1) - \sum_{j \neq 1} G(\rho Q_1, \rho Q_j) \right) + 2\Lambda_{k+1} \Lambda_0 G(0, \rho Q_j) + k^{-1} \Lambda_{k+1}^2 H(0, 0) + \gamma(\rho Q_1) \Lambda_0 + k^{-1} \gamma(0) \Lambda_{k+1}.
\]
If \(\nabla_{(\Lambda_0, \Lambda_{k+1})} \Theta_k(\hat{\Lambda}_0, \hat{\Lambda}_{k+1}) = 0\), then we have
\[
\Theta_k(\hat{\Lambda}_0, \hat{\Lambda}_{k+1}) = \frac{1}{\det M_k(\rho)} \left( H(0, 0) \gamma^2(\rho Q_1) + 2G(0, \rho Q_1) \gamma(\rho Q_1) \gamma(0) \right.
\]
\[
+ \left. \left( H(\rho Q_1, \rho Q_1) - \sum_{j \neq 1} G(\rho Q_1, \rho Q_j) \right) \gamma^2(0) \right),
\]
where
\[
M_k(\rho) = \begin{bmatrix}
    H(\rho Q_1, \rho Q_1) - \sum_{j \neq 1} G(\rho Q_1, \rho Q_j) & G(0, \rho Q_1) \\
    G(0, \rho Q_1) & \frac{1}{k} H(0, 0)
\end{bmatrix}.
\]
Consider now
\[
\tilde{\Theta}_k(\rho) = \Theta_k|_{\nabla_{(\Lambda_0, \Lambda_{k+1})} \Theta_k = 0}(\rho).
\]
Straightforward computation yields
\[
\tilde{\Theta}_k(\rho) = -\frac{1}{2 \det M_k(\rho)} \theta_k(\rho),
\]
The previous remarks, together with (53) and (54), imply (52).

The key observation to show that \( \Theta_k \) has an admissible critical point for any sufficiently large \( k \) is the following: there exists \( \hat{\rho} > 0 \), \( k_0 \in \mathbb{N} \) such that

\[
\theta_k(\rho) < 0 \quad \text{for all} \quad \rho \in [0, \hat{\rho}], \quad \text{for all} \quad k \geq k_0.
\]  

In fact, observe that for \( \rho \to 0 \), \( H(\rho Q_1, \rho Q_1), \gamma(\rho Q_1) \) are bounded quantities. Moreover, from the properties of the Green function, there exists a \( \delta > 0 \) such that for \( 0 < \rho < \delta \) and \( j \neq 1 \), we have

\[
G(\rho Q_1, \rho Q_j) \geq \frac{b_{N,s}}{\rho^{N-2s}|Q_1 - Q_k|^{N-2s}} - O(1) \quad \text{and} \quad G(\rho Q_1, 0) \leq \frac{b_{N,s}}{\rho^{N-2s}} + O(1), \tag{53}
\]

where \( O(1) \) denotes quantities that are uniformly bounded and positive in \([0, \delta]\).

Hence, we obtain

\[
\frac{1}{k} \sum_{j \neq 1} G(\rho Q_1, \rho Q_j) \geq \frac{1}{k} \sum_{j \neq 1} \left( \frac{b_{N,s}}{\rho^{N-2s}|Q_1 - Q_k|^{N-2s}} - O(1) \right) 
\]

\[
\geq \frac{1}{k} \sum_{j \neq 1} \left( \frac{b_{N,s}}{(2\pi j)^{N-2s} \rho^{N-2s}} - O(1) \right) 
\]

\[
\geq \begin{cases} 
\frac{b_{N,s}}{(2\pi)^N \rho^{N-2s}} \frac{k^{N-2s-1} - 1}{N - 2s - 1} - O(1) & \text{if } N > 2s + 1, \\
\frac{b_{N,s}}{(2\pi)^N \rho^{N-2s}} \log k - O(1) & \text{if } N = 2s + 1, \\
\frac{b_{N,s}}{(2\pi)^N \rho^{N-2s}} k^{2s-N} - O(1) & \text{if } N < 2s + 1.
\end{cases} \tag{54}
\]

The previous remarks, together with (53) and (54), imply (52).

A direct computation indicates that \( \theta_k(\rho) < 0 \). We claim that for \( 0 \leq \rho \leq \hat{\rho} \) and \( k \geq k_0 \),

\[
\det M_k(\rho) < 0. \tag{55}
\]

Using the properties of Green’s function and its regular part, one easily finds that for any \( k \),

\[
\lim_{\rho \to 0^+} \det M_k(\rho) = -\infty \quad \text{and} \quad \lim_{\rho \to \hat{\rho}^+} \det M_k(\rho) = +\infty.
\]

Then, for any \( k \), there exists \( \hat{\rho}_k, 0 < \hat{\rho}_k < R \) with the property that

\[
\det M_k(\rho) < 0 \quad \text{for} \quad 0 < \rho < \hat{\rho}_k, \quad \text{and} \quad \det M_k(\hat{\rho}_k) = 0. \tag{56}
\]

Consequently, \( \hat{\rho} < \hat{\rho}_k \) for any sufficiently large \( k \), and straightforward computation gives

\[
\theta_k(\hat{\rho}_k) > 0. \tag{57}
\]

We now have the tools to show that \( \tilde{\Theta}_k(\rho) \) has a minimum in \((0, \hat{\rho}_k)\), with a negative value, for any sufficiently large \( k \). In fact, for sufficiently large \( k \), (50), (51), (52) and (55) imply that

\[
\lim_{\rho \to 0^+} \tilde{\Theta}_k(\rho) = 0 \quad \text{and} \quad \tilde{\Theta}_k(\rho) < 0 \quad \text{for} \quad \rho \sim 0^+;
\]

on the other hand, (50), (51), (56) and (57) yield

\[
\lim_{\rho \to \hat{\rho}_k^+} \tilde{\Theta}_k(\rho) = +\infty.
\]
Call $c_k$ and $\rho_k$, respectively, the minimum value and the minimum point of $\tilde{\Theta}_k$ in $(0, \rho_k)$, that is,

$$c_k = \tilde{\Theta}_k(\rho_k) = \min_{\rho \in (0, \rho_k)} \tilde{\Theta}_k(\rho) < 0.$$ 

We can then conclude that $(\rho_k, \Lambda_0(\rho_k), \Lambda_{k+1}(\rho_k))$ is a critical point for $\tilde{\Theta}_k$. We conclude that this critical point is admissible.

We also have that $\Lambda_1(\hat{\rho}_k) > 0$ and $\det M_k(\hat{\rho}_k) < 0$.

On the other hand, since $\det M_k(\rho_k) < 0$ and $\tilde{\Theta}_k(\rho_k) < 0$, we have $\Theta_k(\rho_k) < 0$. Hence,

$$\Lambda_{k+1}(\rho_k) > -\frac{1}{\det M_k(\rho_k)} \left( G(0, \rho_k Q_1) \gamma(\rho_k Q_1) + H(0, 0) \frac{\gamma^2(\rho_k)}{\gamma(0)} \right) > 0.$$ 

To conclude the proof, we show that this critical point persists under small $C^1$ perturbations. In fact, we have that $\mathcal{I}_c(\rho, \Lambda_0, \Lambda_{k+1})$ has itself a critical point $(\rho_k^\epsilon, \hat{\Lambda}_k^\epsilon, \Lambda_{k+1}^\epsilon)$ close to $(\rho_k, \Lambda_0(\rho_k), \Lambda_{k+1}(\rho_k))$.

Let $a > 0$, and define

$$B_a = (\rho_k - a, \rho_k + a) \times (\Lambda_0(\rho_k) - a, \Lambda_0(\rho_k) + a) \times (\Lambda_{k+1}(\rho_k) - a, \Lambda_{k+1}(\rho_k) + a).$$

Since $\rho_k$ is a nondegenerate minimum of $\Theta_k$ and from the definition of the function $\Theta_k$, we can choose $a$ sufficiently small so that the following relations hold:

$$\frac{\partial}{\partial \rho} \Theta_k(\rho - a, \Lambda_0, \Lambda_{k+1}(\rho) - a) < 0, \quad \frac{\partial}{\partial \rho} \Theta_k(\rho + a, \Lambda_0, \Lambda_{k+1} + a) > 0$$

for all $(\Lambda_0, \Lambda_{k+1}) \in [\Lambda_0(\rho_k) - a, \Lambda_0(\rho_k) + a] \times [\Lambda_{k+1}(\rho_k) - a, \Lambda_{k+1}(\rho_k) + a]$,

$$\frac{\partial}{\partial \Lambda_k} \Theta_k(\rho, \Lambda_0, \Lambda_{k+1}(\rho) - a) < 0, \quad \frac{\partial}{\partial \Lambda_k} \Theta_k(\rho, \Lambda_0, \Lambda_{k+1}(\rho) + a) > 0$$

for all $(\rho, \Lambda_0) \in [\rho_k - a, \rho_k + a] \times [\Lambda_0(\rho_k) - a, \Lambda_0(\rho_k) + a]$, and

$$\frac{\partial}{\partial \Lambda_0} \Theta_k(\rho, \Lambda_0(\rho_k) - a, \Lambda_{k+1}) > 0, \quad \frac{\partial}{\partial \Lambda_0} \Theta_k(\rho, \Lambda_0(\rho_k) + a, \Lambda_{k+1}) < 0$$

for all $(\rho, \Lambda_{k+1}) \in [\rho_k - a, \rho_k + a] \times [\Lambda_{k+1}(\rho_k) - a, \Lambda_{k+1}(\rho_k) + a]$. It follows that the local degree $\deg(\nabla \Theta_k, B_a, 0)$ is well defined and different from zero.

Hence, for all sufficiently small $\varepsilon > 0$, we also obtain $\deg(\nabla \mathcal{I}_c, B_a, 0) \neq 0$. This operation shows the existence of a critical point for $\mathcal{I}_c$ and concludes the proof of Theorem 1.1 based on Lemma 3.1 and Lemma 4.1. Indeed, given $x \in \Omega$, if $(\xi, \Lambda) = (\rho(Q_1, \ldots, Q_k), 0, \Lambda_0(1, \ldots, 1), \Lambda_{k+1})$ is the critical point of $\mathcal{I}_c$ and if we consider

$$\hat{V}(x) = -\varepsilon^{-1} V(\varepsilon^{-\frac{1}{2}d} x), \quad \hat{\phi}(x) = -\varepsilon^{-1} \phi(\xi, \Lambda)(\varepsilon^{-\frac{1}{2}d} x)$$

and

$$\hat{\psi}(x) = -\varepsilon^{-1} \psi(\varepsilon^{-\frac{1}{2}d} x),$$

we then return to our original variables, concluding that the function

$$u_\varepsilon(x) = -\hat{V}(x) - \hat{\phi}(x) - \hat{\psi}(x), \quad x \in \Omega,$$

(where $\hat{\psi} = -\varepsilon \gamma$) is a solution of (1), which completes the proof. \qed
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