ON THE EXISTENCE OF EXTREMALS FOR MOSER TYPE INEQUALITIES IN GAUSS SPACE

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Abstract. The existence of an extremal in an exponential Sobolev type inequality, with optimal constant, in Gauss space is established. A key step in the proof is an augmented version of the relevant inequality, which, by contrast, fails for a parallel classical inequality by Moser in the Euclidean space.

How to cite this paper

This paper has been accepted to International Mathematics Research Notices and the final publication is available at

https://doi.org/10.1093/imrn/rnaa165.

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1. Introduction and main results

A celebrated result by Moser [45], dealing with the borderline case of the Sobolev embedding theorem [48, 54, 56], asserts that if \( \Omega \) is an open set in \( \mathbb{R}^n \) with finite Lebesgue measure \( |\Omega| \), then

\[
\sup_u \int_{\Omega} \exp^{n'}(\alpha_n |u|) \, dx < \infty,
\]

where the supremum is extended over all functions \( u \in W^{1,n}_0(\Omega) \) satisfying the constraint

\[
\int_{\Omega} |\nabla u|^n \, dx \leq 1.
\]

Here, \( \exp^{\beta}(t) = e^{\beta t} \) for \( \beta > 0 \) and \( t \geq 0 \), and \( \alpha_n = n\omega_n^{1/n} \), where \( \omega_n \) denotes the Lebesgue measure of the unit ball in \( \mathbb{R}^n \). Moreover, the constant \( \alpha_n \) in equation (1.1) is sharp, in the sense that the supremum is infinite if \( \alpha_n \) is replaced by any larger constant.

An additional remarkable feature of inequality (1.1) is that, for any set \( \Omega \) as above, the supremum is, in fact, a maximum. Namely, there exists a function \( u \) at which the supremum is attained. The first contribution in this connection is [16], where the case when \( \Omega \) is a ball is considered. The result for arbitrary domains is established in [27] for \( n = 2 \), and in [42] for any \( n \geq 2 \). However, the supremum in (1.1) and its extremals are still unknown, even in a ball. A radially symmetric extremal is shown to exist in this special domain, but the existence of additional non-symmetric extremals is not excluded.

Notice that, by contrast, extremals are absent in the classical Sobolev inequality for \( W^{1,p}_0(\Omega) \), for \( 1 < p < n \), with optimal exponent and constant, in any domain \( \Omega \neq \mathbb{R}^n \) – see, for example, [52, Chapter I, Section 4.7].

A major difficulty in the proof of the existence of an extremal in (1.1) is related to the lack of compactness of the embedding

\[
W^{1,n}_0(\Omega) \to \exp L^{n'}(\Omega).
\]

Here, given \( \beta > 0 \), we denote by \( \exp L^\beta(\Omega) \) the Orlicz space associated with a Young function equivalent to \( \exp^\beta(t) \) near infinity. To be more specific, sequences \( \{u_k\} \subset W^{1,n}_0(\Omega) \) such that \( \int_{\Omega} |\nabla u_k|^n \, dx \leq 1 \)

2020 Mathematics Subject Classification. 46E35, 28C20.

Key words and phrases. Gaussian Sobolev inequalities; Gauss measure; Exponential inequalities; Existence of extremals; Moser type inequalities; Orlicz spaces.
need not enjoy the property that the sequence \( \{ \exp^n (\alpha_n |u_k|) \} \) be uniformly integrable. This prevents one from applying the classical direct methods of the calculus of variations to pass to the limit in \( \int_{Q} \exp^n (\alpha_n |u_k|) \, dx \) as \( k \to \infty \). The proof of the existence of extremals thus calls for the use of concentration-compactness techniques.

Moser’s inequality has inspired a number of investigations on sharp exponential inequalities associated with borderline embeddings of Sobolev type. The contributions [1, 3–5, 8, 14, 17–19, 24, 28–31, 35–41, 43, 46, 50, 55] just supply a taste of this rich line of research.

Unconventional counterparts of Moser’s inequality in Gauss space have recently been offered in [21]. Recall that the Gauss space \((\mathbb{R}^n, \gamma_n)\) is \(\mathbb{R}^n\) endowed with the Gauss probability measure \(\gamma_n\) obeying
\[
d\gamma_n(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}} \, dx \quad \text{for} \ x \in \mathbb{R}^n.
\]

One version of the Gaussian exponential inequalities in question tells us that, given any \(\beta \in (0, 2]\) and \(M > 1\),
\[
\sup_u \int_{\mathbb{R}^n} \exp^{\frac{2\beta}{2+\beta}} (\kappa_\beta |u|) \, d\gamma_n < \infty,
\]
where the supremum is extended over all weakly differentiable functions \(u: \mathbb{R}^n \to \mathbb{R}\) fulfilling the gradient bound
\[
\int_{\mathbb{R}^n} \Exp^{\beta}(|\nabla u|) \, d\gamma_n \leq M
\]
and the normalization condition
\[
m(u) = 0,
\]
see [21, Theorem 1.1]. Here,
\[
\kappa_\beta = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\beta},
\]
m\((u)\) stands for either the mean value \(m(u)\), or the median \(\text{med}(u)\) of \(u\) over \((\mathbb{R}^n, \gamma_n)\) (see Section 2 for a precise definition), and the function \(\Exp^{\beta}\) is the convex envelope of \(\exp^{\beta}\), which obviously agrees with \(\exp^{\beta}\) near infinity for every \(\beta > 0\), and globally if \(\beta \geq 1\).

The constant \(\kappa_\beta\) is sharp in (1.7), in an even stronger sense than \(\alpha_n\) in (1.1). Indeed, if \(\kappa_\beta\) is replaced by any larger constant, then for every \(M > 1\) there exists a function \(u\), fulfilling conditions (1.5) and (1.6), such that
\[
\int_{\mathbb{R}^n} \exp^{\frac{2\beta}{2+\beta}} (\kappa_\beta |u|) \, d\gamma_n = \infty.
\]
Another diversity between the Euclidean and the Gaussian inequalities is in that the value of \(M\) in (1.5) is surprisingly irrelevant, whereas the value 1 in (1.2) is critical.

Inequality (1.4) provides us with quantitative information on the Gaussian Sobolev embedding
\[
W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n, \gamma_n),
\]
where \(W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n)\) denotes the Sobolev space built upon the Orlicz space \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\). Embedding (1.8) is a parallel of (1.3) in Gauss space. Notice that the spaces \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\) and \(\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n, \gamma_n)\) appearing in (1.8) are optimal [22, Proposition 4.4 (iii)]. In particular, in contrast with (1.3), the degree of integrability of a function \(u\) can be weaker than that of its gradient \(|\nabla u|\) in Gauss space. This is due to the decay of the measure \(\gamma_n\) near infinity.

Let us point out that embedding (1.8) does hold, in fact, for every \(\beta > 0\), as shown in [22]. Yet, if \(\beta > 2\), the validity of inequality (1.4) turns out to be sensitive to the value of \(M\): the supremum is still finite provided that \(M\) is small enough, whereas it becomes infinite if \(M\) is too large. This phenomenon is pointed out in [21, Theorem 1.1]. The exact threshold for \(M\) can only be characterized in a very implicit form, which does not allow for its identification. It is therefore unclear for which constants \(M\) problem (1.4)–(1.6) is meaningful.
Embeddings (1.8) continue, at a scale of exponential integrability, the family of classical Gaussian embeddings
\[
W^{1,p}(\mathbb{R}^n, \gamma_n) \to L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n),
\]
for \( p \in [1, \infty) \), where \( L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n) \) denotes the Orlicz space built upon any Young function equivalent to \( t^p(\log t)^{\frac{p}{2}} \) near infinity. They have their roots in the seminal paper by Gross [34], dealing with the case \( p = 2 \). The extension to \( p \neq 2 \) can be found in [2]. Further developments and related results are the subject of a vast literature, including [2, 6, 7, 10, 11, 13, 15, 23, 26, 32, 44, 47, 49].

The purpose of the present paper is to investigate the existence of an extremal in the Gaussian exponential inequality (1.4). We give an affirmative answer to this question, thus providing an analogue of the existence result for Moser’s inequality in the Euclidean space. Moreover, any possible maximizer \( u \) of (1.4) is shown to be necessarily a one-variable function. By contrast, as mentioned above, any (yet qualitative) characterization of extremals seems to be missing for Moser’s inequality.

**Theorem 1.1 [Existence of maximizers].** Let \( \beta \in (0, 2] \) and \( M > 1 \). Then the supremum in (1.4) is attained. Moreover, the level sets of any extremal function \( u \) are half-spaces; namely, there exist an increasing function \( h : \mathbb{R} \to \mathbb{R} \) and \( \xi \in \mathbb{R}^n \) such that
\[
(1.10) \quad u(x) = h(x \cdot \xi) \text{ for a.e. } x \in \mathbb{R}^n.
\]
Here, the dot “\( \cdot \)” stands for scalar product in \( \mathbb{R}^n \).

A key novelty in the proof of Theorem 1.1 is that, unlike the Euclidean case, the maximization problem for the Gaussian inequality (1.4) can be attacked via the direct methods of the calculus of variations. This is possible thanks to an improvement, of independent interest, of inequality (1.4). Despite the fact that, like (1.3), embedding (1.8) is non-compact, we are able to show that a uniform integrability property, somewhat stronger than (1.4), holds under the same constraints (1.5) and (1.6). Specifically, we prove that if \( \varphi : [0, \infty) \to [0, \infty) \) is an increasing function that diverges to \( \infty \) as \( t \to \infty \) with a sufficiently mild growth, then
\[
(1.11) \quad \sup_u \int_{\mathbb{R}^n} \exp \frac{2\beta}{t^p} (k\beta |u|) \varphi(|u|) \, d\gamma_n < \infty,
\]
where the supremum is extended over all weakly differentiable functions \( u : \mathbb{R}^n \to \mathbb{R} \) satisfying conditions (1.5) and (1.6). This is the content of our second main result, where an essentially optimal growth condition on \( \varphi \) for (1.11) to hold is exhibited.

**Theorem 1.2 [Improved integrability].** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a continuous increasing function. Assume that either \( \beta \in (0, 2) \) and
\[
(1.12) \quad \int_{\mathbb{R}^n} t^{-\frac{4}{4-\beta^2}} \varphi(t) \, dt < \infty,
\]
or \( \beta = 2 \) and
\[
(1.13) \quad \int_{\mathbb{R}^n} e^{-\frac{1}{8} (\log t)^2} \varphi(t) \, dt < \infty
\]
Then, inequality (1.11) holds for any choice of \( M > 1 \) in condition (1.5). Conversely, if \( \beta \in (0, 2) \) and
\[
(1.14) \quad \int_{\mathbb{R}^n} t^{-\frac{4+\beta}{4-\beta^2}} \varphi(t) \, dt = \infty,
\]
or \( \beta = 2 \) and
\[
(1.15) \quad \int_{\mathbb{R}^n} e^{-\frac{1+\beta}{8} (\log t)^2} \varphi(t) \, dt = \infty
\]
for some $\varepsilon > 0$, then inequality (1.11) fails for any choice of $M > 1$ in (1.5), since there exists a function $u$ satisfying conditions (1.5) and (1.6), and such that

\begin{equation}
(1.16) \quad \int_{\mathbb{R}^n} \exp\left(\kappa_{\beta} |u|^{2+\beta} \right)^{\frac{2\beta}{2+\beta}} \varphi(|u|) \, d\gamma_n = \infty.
\end{equation}

**Remark 1.3.** The conclusion of Theorem 1.2 does not contradict the fact that $\exp L^{2+\beta}(\mathbb{R}^n, \gamma_n)$ is the optimal Orlicz target for embeddings of $W^1 L^\beta(\mathbb{R}^n, \gamma_n)$. Indeed, any Young function equivalent to $\exp L^{2+\beta}(\kappa t) \varphi(t)$ near infinity, where $\kappa$ is any positive constant and $\varphi$ fulfills condition (1.12) or (1.13), is equivalent (in the sense of Young functions) to $\exp L^{2+\beta}(t)$ itself near infinity. Hence, the Orlicz space associated with any Young function of this kind agrees with $\exp L^{2+\beta}(\mathbb{R}^n, \gamma_n)$, up to equivalent norms.

**Remark 1.4.** Inequality (1.4) still holds if the integral condition (1.5) is replaced by the Luxemburg norm constraint

\begin{equation}
(1.17) \quad \|\nabla u\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1,
\end{equation}

where $B$ is any Young function such that

\[
N_1 e^{t\beta} \leq B(t) \leq N_2 e^{t\beta} \quad \text{for } t > t_0,
\]

for some $N_2 > N_1 > 0$ and $t_0 > 0$ [21, Theorem 3.1 and Remark 3.5]. The existence of maximizers in (1.4) under conditions (1.17) and (1.6) can be established via an argument completely analogous to that of the proof of Theorem 1.1. In particular, such an argument relies upon an improved version of the result of [21, Theorem 3.1] in the spirit of Theorem 1.2, which tells us that inequality (1.11) holds, when the supremum is extended over all functions $u$ fulfilling conditions (1.17) and (1.6), provided that $\varphi$ is any function as in Theorem 1.2.

2. Ehrhard symmetrization and ensuing inequalities

The point of departure of our approach are some rearrangement inequalities for the gradient of Sobolev functions on Gauss space. These inequalities in their turn rely upon the isoperimetric inequality that links the Gauss measure of a set $E \subset \mathbb{R}^n$ to its Gauss perimeter. Recall that the Gauss perimeter $P_{\gamma_n}(E)$ of $E$ can be defined as

\[
P_{\gamma_n}(E) = \mathcal{H}^{n-1}_{\gamma_n}(\partial^M E),
\]

where we have set

\[
d\mathcal{H}^{n-1}_{\gamma_n}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^{n-1}(x),
\]

$\partial^M E$ denotes the essential boundary of $E$ and $\mathcal{H}^{n-1}$ is the $(n - 1)$-dimensional Hausdorff measure. The Gaussian isoperimetric inequality asserts that half-spaces minimize Gauss perimeter among all measurable subsets of $\mathbb{R}^n$ with prescribed Gauss measure [12, 53]. Also, as shown in [15], half-spaces are the only minimizers. Note that

\begin{equation}
(2.1) \quad \gamma_n(\{x \in \mathbb{R}^n : x_1 \geq t\}) = \Phi(t) \quad \text{for } t \in \mathbb{R},
\end{equation}

where $x = (x_1, \ldots, x_n)$ and $\Phi : \mathbb{R} \to (0, 1)$ is the function defined as

\begin{equation}
(2.2) \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{\tau^2}{2}} \, d\tau \quad \text{for } t \in \mathbb{R}.
\end{equation}

Moreover,

\[
P_{\gamma_n}(\{x \in \mathbb{R}^n : x_1 \geq t\}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \text{for } t \in \mathbb{R}.
\]

Therefore, on defining the function $I : [0, 1] \to [0, \infty)$ as

\begin{equation}
(2.3) \quad I(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^{-1}(\sqrt{s})^2}{2}} \quad \text{for } s \in (0, 1),
\end{equation}

and \( I(0) = I(1) = 0 \), the Gaussian isoperimetric inequality takes the analytic form
\[ I(\gamma_n(E)) \leq P_{\gamma_n}(E) \]
for every measurable set \( E \subset \mathbb{R}^n \), with equality if and only if \( E \) agrees with a half-space (up to a set of measure zero). The function \( I \) is accordingly called the isoperimetric function (or isoperimetric profile) of Gauss space. Note that \( I \) is symmetric about \( \frac{1}{2} \), namely
\[ (2.5) \quad I(s) = I(1-s) \quad \text{for } s \in [0,1]. \]

Also,
\[ (2.6) \quad -\Phi'(t) = I(\Phi(t)) \quad \text{for } t \in \mathbb{R}, \]
where ‘\( \prime \)’ denotes differentiation. An Ehrhard symmetral of a function \( u \in \mathcal{M}(\mathbb{R}^n, \gamma_n) \) is a function which is equimeasurable with \( u \) and whose level sets are half-spaces. Here, \( \mathcal{M}(\mathbb{R}^n, \gamma_n) \) denotes the space of measurable functions on \( \mathbb{R}^n \) with respect to \( \gamma_n \). A parallel notation will be employed for the space of measurable functions on more general measure spaces. Recall that two functions, possibly defined on different measure spaces, are called equimeasurable if all their level sets have like measures.

The signed decreasing rearrangement \( u^\circ: (0,1) \to \mathbb{R} \) of a function \( u \in \mathcal{M}(\mathbb{R}^n, \gamma_n) \) is defined as
\[ u^\circ(s) = \inf\{ t \in \mathbb{R} : \gamma_n(\{u > t\}) \leq s \} \quad \text{for } s \in (0,1). \]
The functions \( u \) and \( u^\circ \) are equimeasurable. Thus, the mean value of \( u \) satisfies
\[ \text{mv}(u) = \int_{\mathbb{R}^n} u \, d\gamma_n = \int_0^1 u^\circ \, ds. \]

Moreover, the median of \( u \) can be defined as
\[ \text{med}(u) = u^\circ(\frac{1}{2}). \]

Thanks to equation (2.1), the function \( u^\bullet: \mathbb{R}^n \to \mathbb{R} \), defined as
\[ u^\bullet(x) = u^\circ(\Phi(x_1)) \quad \text{for } x \in \mathbb{R}^n, \]
is an Ehrhard symmetral of \( u \). Owing to the equimeasurability of \( u, u^\circ, u^\bullet \), one has that
\[ (2.7) \quad \int_0^1 A(|u^\circ|) \, ds = \int_{\mathbb{R}^n} A(|u^\bullet|) \, d\gamma_n = \int_{\mathbb{R}^n} A(|u|) \, d\gamma_n \]
for every increasing function \( A: [0, \infty) \to [0, \infty] \).

A Pólya–Szegő principle on the decrease of gradient integrals under Ehrhard symmetrization holds if the integrand \( A: [0, \infty) \to [0, \infty] \) is a Young function, namely a convex function vanishing at the origin. This result is established in [25], and is recalled in the next proposition, where, in addition, a characterization of the cases of equality is offered. Such a characterization is needed in view of the piece of information on extremals given in Theorem 1.1. A proof seems not to be available in the literature, and is provided below.

**Proposition 2.1.** Let \( A \) be a Young function. Assume that \( u \) is a weakly differentiable function in \( \mathbb{R}^n \) such that
\[ (2.8) \quad \int_{\mathbb{R}^n} A(|\nabla u|) \, d\gamma_n < \infty. \]

Then the function \( u^\circ \) is locally absolutely continuous in \((0,1)\), the function \( u^\bullet \) is weakly differentiable in \( \mathbb{R}^n \), and
\[ (2.9) \quad \int_0^1 A(-u^{\circ\prime}(s)I(s)) \, ds = \int_{\mathbb{R}^n} A(|\nabla u^\bullet|) \, d\gamma_n \leq \int_{\mathbb{R}^n} A(|\nabla u|) \, d\gamma_n. \]

Moreover, if \( A \) is strictly positive in \((0, \infty)\) and finite-valued, and equality holds in the inequality in (2.9), then all level sets of \( u \) are half-spaces, namely there exists \( \xi \in \mathbb{R}^n \), with \( |\xi| = 1 \), such that
\[ (2.10) \quad u(x) = u^\circ(\Phi(x \cdot \xi)) \quad \text{for a.e. } x \in \mathbb{R}^n. \]
Proof. The fact that \( u^* \) is locally absolutely continuous in \((0,1)\), and hence the function \( u^* \) is weakly differentiable in \( \mathbb{R}^n \), goes back to [25] – see also [22, Lemma 3.3]. Assume now that \( u \) fulfills (2.8). The coarea formula tells us that
\[
(2.11) \quad \int_{\mathbb{R}^n} f |\nabla u| \, d\gamma_n = \int_{-\infty}^{\infty} \int_{\{u=t\}} f \, dH_{\gamma_n}^{-1} \, dt
\]
for every Borel function \( f: \mathbb{R}^n \to \mathbb{R} \), provided that a representative of \( u \) is suitably chosen. In particular, the choice of \( f = \chi_{\{\nabla u = 0\}} \) in equation (2.11) tells us that
\[
(2.12) \quad H_{\gamma_n}^{-1}(\{\nabla u = 0\} \cap \{u = t\}) = 0 \quad \text{for a.e.} \ t \in \mathbb{R}.
\]
Next, set
\[
\mu(t) = \gamma_n(\{u > t\}) \quad \text{for} \ t \in \mathbb{R}.
\]
We claim that
\[
(2.13) \quad \frac{H_{\gamma_n}^{-1}(\{u^* = t\})}{|\nabla u^*|_{\{u^* = t\}}} = -\mu'(t) \geq \int_{\{u = t\}} \frac{1}{|\nabla u|} \, dH_{\gamma_n}^{-1} \quad \text{for a.e.} \ t \in \mathbb{R},
\]
where \( |\nabla u^*|_{\{u^* = t\}} \) denotes the (constant) value of \( |\nabla u^*| \) restricted to the set \( \{u^* = t\} \). Indeed, the inequality in (2.13) follows from the fact that
\[
\mu(t) = \int_{\{u > t\}} (\chi_{\{\nabla u = 0\}} - \chi_{\{\nabla u = 0\}}) \, d\gamma_n
\]

\[
= \int_t^{\infty} \int_{\{u = \tau\}} \frac{\chi_{\{\nabla u = 0\}}}{|\nabla u|} \, dH_{\gamma_n}^{-1} \, d\tau + \gamma_n(\{u > t\} \cap \{\nabla u = 0\})
\]

\[
= \int_t^{\infty} \int_{\{u = \tau\}} \frac{1}{|\nabla u|} \, dH_{\gamma_n}^{-1} \, d\tau + \gamma_n(\{u > t\} \cap \{\nabla u = 0\}) \quad \text{for} \ t \in \mathbb{R},
\]
where the second equality holds by formula (2.11) and the third one owing to equation (2.12). In order to verify the equality in equation (2.13), let us set
\[
D_{u^*} = \{x \in \mathbb{R}^n : u^* \text{ is approximately differentiable at } x\}
\]
and
\[
D_{u^0} = \{s \in (0,1) : u^0 \text{ is approximately differentiable at } s\}.
\]
Recall that
\[
(2.14) \quad \gamma_n(\mathbb{R}^n \setminus D_{u^*}) = 0 \quad \text{and} \quad \|(0,1) \setminus D_{u^0}\| = 0,
\]
since \( u^* \) and \( u^0 \) are weakly differentiable functions. Here, vertical bars \( | \cdot | \) stand for the Lebesgue measure. In the remaining part of this proof, let \( \nabla u^* \) and \( u^0 \) denote the approximate differentials of \( u^* \) and \( u^0 \), respectively. Set
\[
D_{u^*}^+ = \{x \in \mathbb{R}^n : \nabla u^*(x) \text{ exists and } \nabla u^*(x) \neq 0\}, \quad D_{u^*}^0 = \{x \in \mathbb{R}^n : \nabla u^*(x) \text{ exists and } \nabla u^*(x) = 0\},
\]
and
\[
D_{u^0}^+ = \{s \in (0,1) : u^0'(s) \text{ exists and } u^0'(s) \neq 0\}, \quad D_{u^0}^0 = \{s \in (0,1) : u^0'(s) \text{ exists and } u^0'(s) = 0\}.
\]
A Gaussian version of [20, Lemma 3.2], with analogous proof, tells us that the equality in equation (2.12) holds for a.e. \( t \in u^*(D_{u^0}^+), \) and that \( \mu'(t) = 0 \) for a.e. \( t \notin u^*(D_{u^0}^+) \). Also, one can verify that \( u^* \) is approximately differentiable at \( x \) if and only if \( u^0 \) is approximately differentiable at \( x_1 \), and that
\[
\nabla u^0(x) = -u^0'((\Phi(x_1)))\Phi(x_1)(1,0,\ldots,0).
\]
Hence, one infers that \( u^*(D_{u^0}^0) = u^0(D_{u^0}^0), u^*(D_{u^0}^+) = u^0(D_{u^0}^+) \) and \( u^*(\mathbb{R}^n \setminus D_{u^*}) = u^0((0,1) \setminus D_{u^0}) \). By [20, Lemma 2.4], \( u^0(D_{u^0}^0) = 0 \). Also, \( u^0((0,1) \setminus D_{u^0}) = 0 \), owing to the second equation in (2.14) and to the fact that absolutely continuous functions map sets of measure zero into sets of measure zero. Altogether, we infer that the equality holds in (2.13) for a.e. \( t \in \mathbb{R} \).
Next, the following chain holds:

\[
\int_{\{u = t\}} \frac{A(|\nabla u|)}{|\nabla u|} d\mathcal{H}^{n-1}_{\gamma_n} \geq A \left( \frac{\int_{\{u=t\}} |\nabla u| d\mathcal{H}^{n-1}_{\gamma_n}}{\int_{\{u=t\}} |\nabla u| d\mathcal{H}^{n-1}_{\gamma_n}} \right) \int_{\{u = t\}} \frac{1}{|\nabla u|} d\mathcal{H}^{n-1}_{\gamma_n}
\]

\[
= A \left( \frac{\mathcal{H}^{n-1}_{\gamma_n}(\{u = t\})}{\int_{\{u=t\}} |\nabla u| d\mathcal{H}^{n-1}_{\gamma_n}} \right) \int_{\{u = t\}} \frac{1}{|\nabla u|} d\mathcal{H}^{n-1}_{\gamma_n}
\]

\[
\geq A \left( \frac{\mathcal{H}^{n-1}_{\gamma_n}(\{u = t\})}{\mathcal{H}^{n-1}_{\gamma_n}(\{u^* = t\})} \right) \frac{|\nabla u^*|_{(u^* = t)}}{|\nabla u^*|_{(u^* = t)}}
\]

\[
\geq A \left( \frac{|\nabla u^*|_{(u^* = t)}}{|\nabla u^*|_{(u^* = t)}} \right) \mathcal{H}^{n-1}_{\gamma_n}(\{u^* = t\})
\]

\[
= \int_{\{u^* = t\}} \frac{A(|\nabla u^*|)}{|\nabla u^*|} d\mathcal{H}^{n-1}_{\gamma_n} \quad \text{for a.e. } t \in \mathbb{R}.
\]

Observe that the first inequality in (2.15) holds by Jensen’s inequality, the second one by inequality (2.13) and the fact that the function \(A(t)/t\) is non-decreasing, and the third one since, owing to the isoperimetric inequality in Gauss space (2.4),

\[
\mathcal{H}^{n-1}_{\gamma_n}(\{u = t\}) = \mathcal{P}_{\gamma_n}(\{u > t\}) \geq \mathcal{P}_{\gamma_n}(\{u^* > t\}) = \mathcal{H}^{n-1}_{\gamma_n}(\{u^* = t\}) \quad \text{for a.e. } t \in \mathbb{R}.
\]

An integration of inequality (2.15) in \(t\) over \(\mathbb{R}\) and the use of coarea formula (2.11) yield

\[
\int_{\mathbb{R}^n} A(|\nabla u|) \, d\gamma_n = \int_{-\infty}^{\infty} \int_{\{u = t\}} A(|\nabla u|) \, d\mathcal{H}^{n-1}_{\gamma_n} \, dt
\]

\[
\geq \int_{-\infty}^{\infty} \int_{\{u^* = t\}} A(|\nabla u^*|) \, d\mathcal{H}^{n-1}_{\gamma_n} \, dt = \int_{\mathbb{R}^n} A(|\nabla u^*|) \, d\gamma_n,
\]

whence equation (2.9) follows.

As far as the equality case is concerned, if the inequality in (2.9) holds as equality, then the same is true in (2.17). If \(A\) is strictly positive in \((0, \infty)\) and finite-valued, then it is strictly increasing. An inspection of the proof of inequality (2.17) then reveals that, in particular, equality must hold in inequality (2.16) for a.e. \(t \in \mathbb{R}\). The characterization of extremal sets in the Gaussian isoperimetric inequality implies that \(\{u > t\}\) is a half-space for a.e. \(t \in \mathbb{R}\). Since the level sets of any function are nested, this implies that the level sets of \(u\) are half-spaces with parallel boundaries. Hence, equation (2.10) follows. \(\square\)

3. Proof of Theorem 1.2

Our proof of the enhanced version of inequality (1.4), contained in Theorem 1.2, rests upon a precise estimate for the asymptotic behaviour of a norm in an Orlicz space depending on the functions \(\Phi\) and \(I\) introduced in (2.2) and (2.3). We begin by recalling a few facts from the theory of Young functions and Orlicz spaces that are needed in deriving this estimate.

Let \((\mathcal{R}, \nu)\) be a non-atomic probability space that, in what follows, will be either \(\mathbb{R}^n\) endowed with the Gauss measure \(\gamma_n\), or \((0, 1)\) endowed with the Lebesgue measure (in which case the measure will be omitted in the notation). The Orlicz space \(L^A(\mathcal{R}, \nu)\) built upon a Young function \(A\) is defined as

\[
L^A(\mathcal{R}, \nu) = \left\{ \phi \in \mathcal{M}(\mathcal{R}, \nu) : \int_{\mathcal{R}} A\left(\frac{|\phi|}{\lambda}\right) \, d\nu < \infty \text{ for some } \lambda > 0 \right\}.
\]

The space \(L^A(\mathcal{R}, \nu)\) is a Banach space equipped with the Luxemburg norm given by

\[
\|\phi\|_{L^A(\mathcal{R}, \nu)} = \inf \left\{ \lambda > 0 : \int_{\mathcal{R}} A\left(\frac{|\phi|}{\lambda}\right) \, d\nu \leq 1 \right\}
\]
for \( \phi \in L^A(\mathcal{R}, \nu) \). One has that \( L^A(\mathcal{R}, \nu) = L^B(\mathcal{R}, \nu) \) (up to equivalent norms) if and only if \( A \) and \( B \) are Young functions equivalent near infinity, in the sense that \( A(c_1 t) \leq B(t) \leq A(c_2 t) \) for some positive constants \( c_1 \) and \( c_2 \), and for sufficiently large \( t \). Recall that

\[
L^\infty(\mathcal{R}, \nu) \to L^A(\mathcal{R}, \nu) \to L^1(\mathcal{R}, \nu)
\]

for every Young function \( A \). The Orlicz norm \( \| \cdot \|_{L^A(\mathcal{R}, \nu)} \), given by

\[
\| \phi \|_{L^A(\mathcal{R}, \nu)} = \sup \left\{ \int_{\mathcal{R}} \phi \psi \, d\nu : \int_{\mathcal{R}} \tilde{A}(|\psi|) \, d\nu \leq 1 \right\}
\]

for \( \phi \in L^A(\mathcal{R}, \nu) \), is equivalent to the Luxemburg norm. Here, \( \tilde{A} : [0, \infty) \to [0, \infty] \) denotes the Young conjugate of \( A \), defined as

\[
\tilde{A}(t) = \sup\{ \tau t - A(\tau) : \tau \geq 0 \}
\]

for \( t \geq 0 \), which is also a Young function. Notice that, if \( a : [0, \infty) \to [0, \infty] \) is the non-decreasing left-continuous function such that

\[
A(t) = \int_0^t a(\tau) \, d\tau \quad \text{for } t \geq 0,
\]

then \( \tilde{A} \) admits the representation formula

\[
\tilde{A}(t) = \int_0^t a^{-1}(\tau) \, d\tau \quad \text{for } t \geq 0,
\]

where \( a^{-1} \) denotes the (generalized) left-continuous inverse of \( a \). By the very definition of Young conjugate, one has that

\[
(3.1) \quad t \tau \leq A(t) + \tilde{A}(\tau) \quad \text{for } t, \tau \geq 0.
\]

Moreover, equality holds in (3.1) if and only if either \( t = a^{-1}(\tau) \) or \( \tau = a(t) \). If \( \phi \in L^A(\mathcal{R}, \nu) \) and \( E \) is a \( \nu \)-measurable subset of \( \mathcal{R} \), we use the abridged notation

\[
\| \phi \|_{L^A(E, \nu)} = \| \phi \chi_E \|_{L^A(\mathcal{R}, \nu)} \quad \text{and} \quad \| \phi \|_{L^A(E, \nu)} = \| \phi \chi_E \|_{L^A(\mathcal{R}, \nu)}.
\]

In particular, one has that

\[
(3.2) \quad \| 1 \|_{L^A(E, \nu)} = \nu(E) \tilde{A}^{-1}(1/\nu(E)).
\]

Here, \( \tilde{A}^{-1} \) denotes the (generalized) right-continuous inverse of \( \tilde{A} \). A sharp form of the Hölder inequality in Orlicz spaces tells us that

\[
(3.3) \quad \int_{\mathcal{R}} \phi \psi \, d\nu \leq \| \phi \|_{L^A(\mathcal{R}, \nu)} \| \psi \|_{L^{\tilde{A}}(\mathcal{R}, \nu)}
\]

for every \( \phi \in L^A(\mathcal{R}, \nu) \) and \( \psi \in L^{\tilde{A}}(\mathcal{R}, \nu) \).

The next lemma provides us with a uniform bound for the integral in (1.11) for functions obeying (1.5). Such a bound involves a function \( F_B \), which is in its turn associated with a Young function \( B \) obeying

\[
(3.4) \quad B(t) = Ne^{t^\beta} \quad \text{for } t > t_0
\]

for some \( N > 0 \) and \( t_0 > 0 \). The function \( F_B : (0, \infty) \to (0, \infty) \) is defined as

\[
(3.5) \quad F_B(t) = \frac{1}{|t|} \left| \frac{1}{t} \right|_{L^\tilde{B}(\Phi(t), \frac{1}{2})} + \frac{\sqrt{2\pi}}{2} B^{-1}(1) \quad \text{for } t > 0.
\]

**Lemma 3.1.** Let \( \beta > 0 \), \( \kappa > 0 \) and \( M > 1 \). Assume that the function \( \varphi : [0, \infty) \to [0, \infty) \) is continuous and such that the function \( \exp^{\frac{2\beta}{1+\beta}}(\kappa t)\varphi(t) \) is non-decreasing. Then there exists a Young function \( B \) of the form (3.4) such that

\[
(3.6) \quad \int_{\mathbb{R}^n} \exp^{\frac{2\beta}{1+\beta}}(\kappa |u|) \varphi(|u|) \, d\gamma_n \leq \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{\int_0^{\infty} F_B(t) \, dt} dt.
\]
for every weakly differentiable function \( u \) in \( \mathbb{R}^n \) satisfying conditions (1.5) and (1.6).

**Proof.** Choose \( t_0 \) so large that \( \text{Exp}^\beta(t) = \text{exp}^\beta(t) \) for \( t \geq t_0 \), and define the Young function \( A : [0, \infty) \to [0, \infty) \) as

\[
A(t) = \begin{cases} \frac{t}{t_0} \text{Exp}^\beta(t_0) & \text{for } t \in [0, t_0) \\ \text{Exp}^\beta(t) & \text{for } t \in [t_0, \infty). \end{cases}
\]

Set \( N = 1/(M + \text{Exp}^\beta(t_0)) \) and let \( B = NA \). Then \( B \) is a Young function. If \( u \) is a function as in the statement, then

\[
\int_{\mathbb{R}^n} B(|\nabla u|) \, d\gamma_n \leq N \int_{\{|\nabla u| \geq t_0\}} \text{Exp}^\beta(|\nabla u|) \, d\gamma_n + N \int_{\{|\nabla u| < t_0\}} \text{Exp}^\beta(t_0) \, d\gamma_n \leq N(M + \text{Exp}^\beta(t_0)) = 1.
\]

Hence, by the very definition of Luxemburg norm, \( \|\nabla u\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1 \). Inequality (3.6) is thus a consequence of [21, Lemma 4.5].

The behaviour near infinity of the function \( F_B \), defined by (3.5), is described in Lemma 3.7 below. This is the key estimate to which we alluded above. Its proof requires several expansions and bounds related to the functions entering the definition of \( F_B \). They are the content of some preliminary lemmas, whose statements and proofs need some notation related to expansions of functions that are fixed hereafter.

Given a function \( F \) defined in some neighbourhood of infinity, and \( k \in \mathbb{N} \), we write

\[
F(t) = \mathcal{E}_1(t) + \cdots + \mathcal{E}_k(t) + \cdots \quad \text{as } t \to \infty
\]

to denote that

\[
\lim_{t \to \infty} \frac{F(t)}{\mathcal{E}_1(t)} = 1 \quad \text{if } k = 1, \quad \text{and } \quad \lim_{t \to \infty} \frac{F(t) - \left[ \mathcal{E}_1(t) + \cdots + \mathcal{E}_j(t) \right]}{\mathcal{E}_{j+1}(t)} = 1 \quad \text{for } 1 \leq j \leq k - 1 \text{ otherwise}.
\]

Clearly, if equation (3.7) holds, then

\[
F(t)^\sigma = \mathcal{E}_1(t)^\sigma + \sigma \mathcal{E}_1(t)^{\sigma-1} \mathcal{E}_2(t) + \cdots \quad \text{as } t \to \infty
\]

for every \( \sigma > 0 \). Also,

\[
F(t)^\sigma = \mathcal{E}_1(t)^\sigma + \sigma \mathcal{E}_1(t)^{\sigma-1} \mathcal{E}_2(t) + \sigma \mathcal{E}_1(t)^{\sigma-1} \mathcal{E}_2(t) \left( \frac{\mathcal{E}_3(t)}{\mathcal{E}_2(t)} + \frac{\sigma - 1}{2} \frac{\mathcal{E}_2(t)}{\mathcal{E}_1(t)} \right) + \cdots \quad \text{as } t \to \infty.
\]

The next three lemmas are stated without proofs. They can be obtained by simple considerations, via L'Hôpital’s rule.

**Lemma 3.2.** Assume that either \( \sigma \in (-1, \infty) \) and \( d \geq 1 \), or \( \sigma \in (-\infty, -1] \) and \( d > 1 \). Let \( \Psi_\sigma : (d, \infty) \to [0, \infty) \) be the function defined as

\[
\Psi_\sigma(t) = \int_d^t (\tau^2 - 1)^\sigma \, d\tau \quad \text{for } t > d.
\]

Then

\[
\Psi_\sigma(t) = \begin{cases} 
\frac{c + \cdots}{\log t + c + \cdots} & \text{if } \sigma \in (-\infty, -\frac{1}{2}) \\
\frac{+c + \cdots}{-\frac{1}{2} \log t + c + \cdots} & \text{if } \sigma = -\frac{1}{2} \\
\frac{1}{2\sigma + 1} \int_0^{2\sigma+1} \left\{ +c + \cdots \right. & \text{if } \sigma \in (-\frac{1}{2}, \frac{1}{2}) \\
\frac{1}{2\sigma - 1} \int_0^{2\sigma-1} \left\{ -\frac{\sigma}{2\sigma - 1} \left[ +c + \cdots \right. \right. & \text{if } \sigma \in (\frac{1}{2}, \frac{3}{2}) \\
\frac{3}{8} \log t + c + \cdots & \text{if } \sigma = \frac{3}{2} \\
\frac{1}{2} \frac{\sigma(\sigma-1)}{2(\sigma-3)} \int_0^{2\sigma-3} + \cdots & \text{if } \sigma \in (\frac{3}{2}, \infty)
\end{cases}
\]

where \( c \) denotes a constant, possibly different at different occurrences, depending on \( \sigma \) and \( d \).
Lemma 3.3. Assume that either $\sigma \in (-1, \infty)$ and $d \geq 1$, or $\sigma \in (-\infty, -1]$ and $d > 1$. Let $\mathcal{Y}_\sigma \colon (a, \infty) \to [0, \infty)$ be the function defined as

$$\mathcal{Y}_\sigma(t) = \int_a^t (\tau^2 - 1)^\sigma \log(\tau^2 - 1) \, d\tau \quad \text{for } t > d.$$  

Then

$$\mathcal{Y}_\sigma(t) = \begin{cases} 
  c + \cdots & \text{if } \sigma \in (-\infty, -\frac{1}{2}) \\
  (\log t)^2 + \cdots & \text{if } \sigma = -\frac{1}{2} \\
  \frac{2}{2\sigma + 1} t^{2\sigma + 1} \log t - \frac{2}{(2\sigma + 1)^2} t^{2\sigma + 1} + \cdots & \text{if } \sigma \in (-\frac{1}{2}, \infty) 
\end{cases}$$

where $c$ is a constant depending on $\sigma$ and $d$.

Lemma 3.4. Let $B$ be a Young function obeying (3.4) for some $\beta > 0$ and $N > 0$, and let $b \colon [0, \infty) \to [0, \infty)$ be the left-continuous function such that

$$B(t) = \int_0^t b(\tau) \, d\tau \quad \text{for } t > 0.$$  

Then

$$b^{-1}(t) = (\log t)^{\frac{1}{2}} + \frac{1 - \beta}{\beta^2} (\log t)^{\frac{1}{2} - 1} \log \log t - \frac{\log N \beta}{\beta} (\log t)^{\frac{1}{2} - 1} + \cdots \quad \text{as } t \to \infty.$$  

Two crucial steps in view of the proof of Lemma 3.7 are enucleated in Lemma 3.5 and 3.6.

Lemma 3.5. Let $\Phi$ be the function defined by (2.2). Let $B$ be any finite-valued Young function of the form (3.12). Then

$$\mathcal{Y}_\sigma(t) = \int_0^t B\left(\frac{1}{\mathcal{Y}_\sigma(s)}\right) \, ds \quad \text{for } t > 0,$$

where $\lambda_t$ is the unique positive number such that

$$\int_0^t B\left(\lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(s)}}\right) e^{-\frac{1}{\mathcal{Y}_\sigma(s)}} \, ds = \sqrt{2\pi}.$$  

Proof. Fix $t \in (0, \infty)$. By the definition of Orlicz norm,

$$\left\| \frac{1}{t} \right\|_{L^{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}} = \sup \left\{ \int_0^t \mathcal{Y}_\sigma(s) \, ds : \int_0^t B(\mathcal{Y}_\sigma(s)) \, ds \leq 1 \right\}$$

The change of variables $s = \Phi(\tau)$ and $f = g(\Phi)$ in both integrals in (3.15) yield, via (2.6),

$$\left\| \frac{1}{t} \right\|_{L^{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}} = \sup \left\{ \int_0^t f(\tau) \, d\tau : \int_0^t B(f(\tau)) e^{-\frac{1}{\mathcal{Y}_\sigma(\Phi(t))}} \, d\tau \leq \sqrt{2\pi} \right\}.$$  

Given any function $f \in \mathcal{M}(0, t)$ such that $f \geq 0$ and any $\lambda_t > 0$, from Young’s inequality (3.1) we infer that

$$\int_0^t f(\tau) \, d\tau = \int_0^t \frac{f(\tau)}{\lambda_t} \lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}} e^{-\frac{1}{\mathcal{Y}_\sigma(\Phi(t))}} \, d\tau \leq \int_0^t B\left(\frac{f(\tau)}{\lambda_t}\right) e^{-\frac{1}{\mathcal{Y}_\sigma(\Phi(t))}} \, d\tau + \int_0^t \tilde{B}\left(\lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t))}}\right) e^{-\frac{1}{\mathcal{Y}_\sigma(\Phi(t))}} \, d\tau.$$  

Define the function $f_t \colon [0, t] \to [0, \infty)$ as

$$f_t(\tau) = \lambda_t b^{-1}\left(\lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}}\right) \quad \text{for } \tau \in [0, t].$$

By the case of equality in Young’s inequality (3.1),

$$f_t(\tau) e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}} = \lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}} \frac{f_t(\tau)}{\lambda_t} + \tilde{B}\left(\lambda_t e^{\frac{1}{\mathcal{Y}_\sigma(\Phi(t), \frac{1}{2})}}\right) \quad \text{for } \tau \in [0, t].$$
Now, assume that $\lambda_t$ obeys (3.14), namely

\[(3.18) \quad \int_0^t B\left(\frac{f(t)}{\lambda_t}\right) e^{-\frac{\tau^2}{2}} d\tau = \sqrt{2\pi}.
\]

Observe that $\lambda_t$ is uniquely defined for each $t \in (0, \infty)$ by the monotonicity of the function $B \circ b^{-1}$. Combining (3.17) with (3.18), one obtains that

\[(3.19) \quad \int_0^t f_t(\tau) d\tau = \sqrt{2\pi} + \int_0^t \tilde{B}(\lambda_t e^{\frac{\tau^2}{2}}) e^{-\frac{\tau^2}{2}} d\tau,
\]

whence, via (3.16) and (3.19),

\[
\sup \left\{ \int_0^t f(\tau) d\tau : \int_0^t B\left(\frac{f(\tau)}{\lambda_t}\right) e^{-\frac{\tau^2}{2}} d\tau \leq \sqrt{2\pi} \right\} = \sqrt{2\pi} + \int_0^t \tilde{B}(\lambda_t e^{\frac{\tau^2}{2}}) e^{-\frac{\tau^2}{2}} d\tau = \int_0^t f_t(\tau) d\tau.
\]

Therefore,

\[
\left\| \frac{1}{t} \right\|_{L^\beta\left(\psi(t)^{\frac{1}{2}}\right)} = \sup \left\{ \int_0^t \frac{f(\tau)}{\lambda_t} d\tau : \int_0^t B\left(\frac{f(\tau)}{\lambda_t}\right) e^{-\frac{\tau^2}{2}} d\tau \leq \sqrt{2\pi} \right\}
\]

\[
= \frac{1}{\lambda_t} \int_0^t f_t(\tau) d\tau = \int_0^t b^{-1}(\lambda_t e^{\frac{\tau^2}{2}}) d\tau,
\]

and (3.13) follows. \(\square\)

**Lemma 3.6.** Let $B$ be a Young function of the form (3.4) for some $\beta \in (0, 2]$ and $N > 0$. Assume that $\lambda_t$ satisfies (3.14). Then

\[(3.20) \quad \lambda_t = c_\beta \begin{cases} t^{1-\frac{\beta}{2}} + \cdots & \text{if } \beta \in (0, 2) \\ \frac{1}{\log t} + \cdots & \text{if } \beta = 2 \end{cases} \quad \text{as } t \to \infty,
\]

where

\[
c_\beta = 2^{\frac{1}{\beta} - 1} \sqrt{\pi} \begin{cases} 2 - \beta & \text{if } \beta \in (0, 2) \\ 0 & \text{if } \beta = 2 \end{cases}
\]

**Proof.** One has that

\[
B(t) = Ne^{\beta t} = \frac{1}{\beta} t^{1-\beta} b(t)
\]

for large $t$. Hence, by Lemma 3.4 and equation (3.8),

\[(3.21) \quad B(b^{-1}(t)) = \frac{1}{\beta} t((\log t)^{\frac{1}{\beta} - 1} + \frac{1-\beta}{\beta^2} t((\log t)^{\frac{1}{\beta} - 2} \log t + \cdots) \quad \text{as } t \to \infty.
\]

If $\beta \neq 1$, the second addend on the right-hand side of equation (3.21) is strictly positive. Consequently, there exists $t_0 \in (1, \infty)$ such that

\[(3.22) \quad B(b^{-1}(t)) \geq \frac{t}{\beta} ((\log t)^{\frac{1}{\beta} - 1} \quad \text{for } t > t_0.
\]

If $\beta = 1$, then $B = b$ near infinity, and (3.22) holds, as equality, as well.

Now, note that $\lambda_t$ is a decreasing function of $t$, by the monotonicity of the function $B \circ b^{-1}$. We claim that

\[(3.23) \quad \lim_{t \to \infty} \lambda_t = 0.
\]

To see this, assume, by contradiction, that $\lim_{t \to \infty} \lambda_t = \lambda$ for some $\lambda > 0$. Choose $\tau_0 > 0$ so that $\lambda e^{\frac{\tau^2}{2}} \geq t_0$ for $\tau > \tau_0$. Then, letting $t \to \infty$ in (3.14), yields, by Fatou’s lemma,

\[
\sqrt{2\pi} \geq \int_0^\infty B\left(b^{-1}(\lambda e^{\frac{\tau^2}{2}})\right) e^{-\frac{\tau^2}{2}} d\tau \geq \int_{\tau_0}^\infty B\left(b^{-1}(\lambda e^{\frac{\tau^2}{2}})\right) e^{-\frac{\tau^2}{2}} d\tau \geq \frac{\lambda}{\beta} \int_{\tau_0}^\infty \left(\frac{\tau^2}{2} + \log \lambda\right)^{\frac{1}{\beta} - 1} d\tau.
\]
This is impossible, since the last integral diverges, inasmuch as \( 2/\beta - 2 \geq -1 \). Equation (3.23) is therefore established.

Now, fix \( t \geq t_0 \) so large that \( \lambda t < 1 \), and set
\[
(3.24) \quad \tau(t) = \sqrt{2 \log \frac{t_0}{\lambda t}}.
\]

If \( t \) is such that
\[
(3.25) \quad \tau(t) < t,
\]
then, owing to equation (3.22),
\[
\sqrt{2\pi} = \int_0^t B \left( b^{-1} \left( \lambda t e^{\frac{\tau^2}{2}} \right) \right) e^{\frac{-\tau^2}{2}} d\tau
\]
\[
\geq \int_{\tau(t)}^t B \left( b^{-1} \left( \lambda t e^{\frac{\tau^2}{2}} \right) \right) e^{\frac{-\tau^2}{2}} d\tau \geq \frac{\lambda}{\beta} \int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda t \right)^{\frac{1}{\beta} - 1} d\tau.
\]

Hence, by the change of variables \( \tau = r\sigma(t) \), where
\[
(3.27) \quad \sigma(t) = \sqrt{2 \log \frac{1}{\lambda t}},
\]
one obtains that
\[
\int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda t \right)^{\frac{1}{\beta} - 1} d\tau = \int_{\tau(t)}^t \left( \frac{\tau^2}{2} - \frac{\sigma(t)^2}{2} \right)^{\frac{1}{\beta} - 1} d\tau
\]
\[
= 2^{1 - \frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} - 1} \int_{\tau(t)/\sigma(t)}^{t/\sigma(t)} (r^2 - 1)^{\frac{1}{\beta} - 1} dr
\]
\[
= 2^{1 - \frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} - 1} \left[ \Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) - \Psi_{\frac{1}{\beta} - 1} \left( \frac{\tau(t)}{\sigma(t)} \right) \right].
\]

Here, \( \Psi_{\frac{1}{\beta} - 1} \) denotes the function defined as in (3.10), with \( d = 1 \). If, in addition to (3.25), we assume that
\[
(3.29) \quad \lim_{t \to \infty} \frac{t}{\sigma(t)} = \infty,
\]
then, since \( \lim_{t \to \infty} \tau(t)/\sigma(t) = 1 \), Lemma 3.2 entails that
\[
(3.30) \quad \int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda t \right)^{\frac{1}{\beta} - 1} d\tau = 2^{1 - \frac{1}{\beta}} \left\{ \frac{\beta}{2 - \beta} t^{\frac{2}{\beta} - 1} + \cdots \text{ if } \beta \in (0, 2) \right. \left. \log t - \log \sigma(t) + \cdots \right. \text{ if } \beta = 2
\]
as \( t \to \infty \).

Next, fix \( \varepsilon > 0 \) and observe that equation (3.21) implies that
\[
(3.31) \quad B(b^{-1}(t)) \leq \frac{1 + \varepsilon}{\beta} t (\log t)^{\frac{1}{\beta} - 1} \quad \text{for } t > t_0.
\]

We may assume, without loss of generality, that inequality (3.31) holds with the same \( t_0 \) as in (3.22), by choosing a larger value of \( t_0 \), if necessary. Since \( B \) is a Young function, we have that \( B(t) \leq t b(t) \) for \( t > 0 \), whence
\[
B(b^{-1}(t)) \leq t b^{-1}(t) \quad \text{for } t > 0.
\]
Therefore, if \( \tau(t) < t \), then
\[
\sqrt{2\pi} = \int_0^{\tau(t)} B \left( b^{-1} \left( \lambda_t e^{\tau^2} \right) \right) e^{-\tau^2} \, d\tau + \int_{\tau(t)}^t B \left( b^{-1} \left( \lambda_t e^{\tau^2} \right) \right) e^{-\tau^2} \, d\tau
\leq \lambda_t \int_0^{\tau(t)} b^{-1} \left( \lambda_t e^{\tau^2} \right) \, d\tau + \frac{1+\varepsilon}{\beta} \lambda_t \int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda_t \right)^{\frac{1}{2}} - 1 \, d\tau
\]
(3.32)
\[
\leq \lambda_t \tau(t) b^{-1} \left( \lambda_t e^{\tau_0^2} \right) + \frac{1+\varepsilon}{\beta} \lambda_t \int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda_t \right)^{\frac{1}{2}} - 1 \, d\tau
= \lambda_t \tau(t) b^{-1}(t_0) + \frac{1+\varepsilon}{\beta} \lambda_t \int_{\tau(t)}^t \left( \frac{\tau^2}{2} + \log \lambda_t \right)^{\frac{1}{2}} - 1 \, d\tau
\]
for \( t \geq t_0 \), whereas, if \( \tau(t) \geq t \), then
\[
\sqrt{2\pi} \leq \int_0^{\tau(t)} B \left( b^{-1} \left( \lambda_t e^{\tau^2} \right) \right) e^{-\tau^2} \, d\tau \leq \lambda_t \tau(t) b^{-1}(t_0) \quad \text{for } t \geq t_0.
\]
For later use, observe also that, by the very definition (3.24) of \( \tau(t) \),
\[
\lim_{t \to \infty} \lambda_t \tau(t) = 0.
\]
Let \( \beta \in (0, 2) \). In order to prove (3.30), we have to show that
\[
\lim_{t \to \infty} \frac{\lambda_t}{c_\beta t^{1-\frac{2}{\beta}}} = 1.
\]
Assume, by contradiction, that (3.35) fails. Then there exist \( \delta > 0 \) and a sequence \( \{t_k\} \) such that \( \lim_{k \to \infty} t_k = \infty \), and either
\[
\lambda_k \geq (1+\delta)c_\beta t_k^{1-\frac{2}{\beta}}, \quad \text{or}
\]
(3.37)
\[
\lambda_k \leq (1-\delta)c_\beta t_k^{1-\frac{2}{\beta}}
\]
for \( k \in \mathbb{N} \). Firstly, suppose that (3.36) is in force. Observe that the sequences \( \{\tau(t_k)\} \) and \( \{\sigma(t_k)\} \) defined as in (3.24) and (3.27), respectively, satisfy
\[
\sigma(t_k) < \tau(t_k) \leq \sqrt{2 \left( \frac{2}{\beta} - 1 \right)} \log t_k + 2 \log \frac{t_0}{(1+\delta)c_\beta}.
\]
Equation (3.38) ensures that condition (3.25) is fulfilled with \( t = t_k \) for large \( k \), namely
\[
\tau(t_k) < t_k,
\]
and that the limit in (3.29) holds when evaluated on the sequence \( \{t_k\} \), i.e.
\[
\lim_{k \to \infty} \frac{t_k}{\sigma(t_k)} = \infty.
\]
Hence, owing to equations (3.26), (3.30) and (3.38),
\[
\sqrt{2\pi} \geq \frac{\lambda_{t_k}}{\beta} \int_{\tau(t_k)}^{t_k} \left( \frac{\tau^2}{2} + \log \lambda_t \right)^{\frac{1}{2}} - 1 \, d\tau = \frac{\lambda_{t_k}}{\beta} 2^{1-\frac{2}{\beta}} \frac{\beta}{2-\beta} t_k^{\frac{2}{\beta}} - 1 + \cdots \geq (1+\delta)c_\beta \frac{2^{1-\frac{2}{\beta}}}{2-\beta} + \cdots = (1+\delta)\sqrt{2\pi} + \cdots
\]
for large \( k \), a contradiction.
Secondly, suppose that (3.37) holds. Then inequality (3.39) holds, provided that $k$ is large enough, otherwise, by (3.33) (on taking a subsequence, if necessary),

$$2\pi \leq \lambda_k \left( \frac{\lambda_k}{\tau(t_k)} \right) b^{-1}(t_0),$$

which contradicts (3.34). If there exist a subsequence of $\{t_k\}$, still denoted by $\{t_k\}$, and a constant $c \in [1, \infty)$ such that $t_k/\tau(t_k) \to c$, then, by (3.28) and (3.32),

$$\sqrt{2\pi} \leq \lambda_k \left( \frac{\lambda_k}{\tau(t_k)} \right) b^{-1}(t_0) + \frac{1 + \varepsilon}{\beta} \lambda_k \int_{\tau(t_k)}^{t_k} \left( \frac{\tau^2}{2} + \lambda_k \right) \frac{1}{\frac{\beta}{2}} \mathrm{d}\tau$$

$$= \lambda_k \left( \frac{\lambda_k}{\tau(t_k)} \right) b^{-1}(t_0) + \frac{1 + \varepsilon}{\beta} 2 \left( 1 - \frac{1}{\beta} \right) \varepsilon e^{-\frac{\varepsilon(t_k)^2}{\beta}} \frac{\sigma(t_k)}{\frac{\beta}{2} - 1} \left[ \Psi \left( \frac{\theta}{\beta} - 1 \right) \left( \frac{\tau(t_k)}{\sigma(t_k)} \right) \right]$$

for large $k$. Clearly,

$$\lim_{k \to \infty} \left[ \left( \frac{t_k}{\sigma(t_k)} \right) - \Psi \left( \frac{\theta}{\beta} - 1 \right) \left( \frac{\tau(t_k)}{\sigma(t_k)} \right) \right] = \Psi \left( \frac{\theta}{\beta} - 1 \right)(c),$$

since the right-hand side of (3.42) tends to zero as $k \to \infty$, a contradiction.

It remains to consider the case when, up to subsequences, $t_k/\tau(t_k) \to \infty$. This implies that equation (3.40) holds as well. Thus, by (3.32) and (3.30),

$$\sqrt{2\pi} \leq \lambda_k \left( \frac{\lambda_k}{\tau(t_k)} \right) b^{-1}(t_0) + \frac{1 + \varepsilon}{\beta} 2 \left( 1 - \frac{1}{\beta} \right) \varepsilon e^{-\frac{\varepsilon(t_k)^2}{\beta}} \frac{\sigma(t_k)}{\frac{\beta}{2} - 1} \left[ \Psi \left( \frac{\theta}{\beta} - 1 \right) \left( \frac{\tau(t_k)}{\sigma(t_k)} \right) \right]$$

whence, owing to (3.37),

$$\sqrt{2\pi} \leq \lambda_k \left( \frac{\lambda_k}{\tau(t_k)} \right) b^{-1}(t_0) + \left( 1 + \varepsilon \right) \left( 1 - \delta \right) \sqrt{2\pi} + \cdots$$

as $k \to \infty$.

This again leads to a contradiction, provided that $\varepsilon$ is chosen so small that $1 + \varepsilon < \frac{1}{\tau \sqrt{\pi}}$, since the first addend on the right-hand side of equation (3.43) tends to 0 as $k \to \infty$, thanks to (3.34).

Assume now that $\beta = 2$. In order to prove (3.20), we need to show that

$$\lim_{t \to \infty} \frac{\lambda_t}{\log t} = 1.$$

Suppose, by contradiction, that (3.44) fails. Then there exist $\delta \in (0, 1)$ and a sequence $\{t_k\}$ such that $t_k = \infty$ and either

$$\lambda_k \geq \left( 1 + \delta \right) \frac{c_2}{\log t_k}$$

or

$$\lambda_k \leq \left( 1 - \delta \right) \frac{c_2}{\log t_k}$$

for $k \in \mathbb{N}$. Assume that (3.45) is satisfied. Observe that $\tau(t_k)$ and $\sigma(t_k)$ obey

$$\sigma(t_k) < \tau(t_k) \leq \sqrt{2 \log \log t_k + 2 \log \left( \frac{t_k}{1 + \delta} \right) c_2}$$

for large $k$. Hence, $\tau(t_k) < t_k$ for large $k$, and $t_k/\sigma(t_k) \to \infty$ and $\log t_k/\log \sigma(t_k) \to \infty$ as $k \to \infty$. Consequently, equations (3.26), (3.30) and (3.45) imply that

$$\sqrt{2\pi} \geq \frac{\lambda_k}{2} \sqrt{2 \log \left( \log t_k - \log \sigma(t_k) \right)} + \cdots = \frac{1}{\sqrt{2}} \lambda_k \log t_k + \cdots \geq (1 + \delta) \sqrt{2\pi} + \cdots$$

as $k \to \infty$, a contradiction.

Suppose next that (3.46) is in force. Notice that inequalities (3.41) and (3.42) also hold if $\beta = 2$. The same argument as in the case when $\beta \in (0, 2)$ then rules out all cases where there does not exist a subsequence of $\{t_k\}$, called again $\{t_k\}$, such that

$$\lim_{k \to \infty} \frac{t_k}{\tau(t_k)} = \infty.$$
We can thereby assume that such a subsequence exists. Thanks to the fact that \( \sigma(t_k) > 1 \) if \( k \) large enough, we infer from (3.30) and (3.32) that
\[
\sqrt{2\pi} \leq \lambda_{t_k} \tau(t_k) b^{-1}(t_0) + \frac{1 + \varepsilon}{2} \sqrt{2\lambda_{t_k} (\log t_k - \log \sigma(t_k))} + \cdots \\
\leq \lambda_{t_k} \tau(t_k) b^{-1}(t_0) + \frac{1 + \varepsilon}{2} \sqrt{2\lambda_{t_k} \log t_k} + \cdots \\
= \lambda_{t_k} \tau(t_k) b^{-1}(t_0) + (1 + \varepsilon)(1 - \delta)\sqrt{2\pi} + \cdots \quad \text{as} \quad k \to \infty.
\]
This is again a contradiction, if \( \varepsilon \) is chosen in such a way that \( 1 + \varepsilon < \frac{1}{1 - \delta} \), since the first addend on the rightmost side approaches zero as \( k \to \infty \). \( \square \)

**Lemma 3.7.** Let \( B \) be a Young function of the form (3.4) for some \( \beta \in (0, 2] \) and \( N > 0 \), and let \( \Phi \) and \( I \) be the functions given by (2.2) and (2.3), respectively. Then
\[
\left\| \frac{1}{I} \right\|_{\tilde{L}^{\beta}(\Phi(t), \frac{1}{t})} = 2^{-\frac{1}{\beta}} \frac{\beta}{2 + \beta} \frac{2^{\frac{1}{\beta} + 1}}{\frac{1}{\beta} + 1} \left\{ \begin{array}{ll}
- \frac{2}{2 - \beta} \frac{\tau^2}{2} - \log t + c_{\beta,N} \frac{\tau^{2 - 1}}{2} + \cdots & \text{if} \quad \beta \in (0, 2) \\
\frac{1}{2} (\log t)^2 - \log t \log t + \cdots & \text{if} \quad \beta = 2
\end{array} \right.
\]
as \( t \to \infty \), where \( c_{\beta,N} \) is a constant depending on \( \beta \) and \( N \).

**Proof.** Fix \( \varepsilon \in (0, 1) \). By Lemma 3.4, there exists \( \tau_0 > 0 \) such that
\[
\begin{align*}
(3.47) & \quad b^{-1}(\tau) \leq (1 + \varepsilon)(\log \tau)^{\frac{1}{\beta}}, \\
(3.48) & \quad b^{-1}(\tau) \leq (\log \tau)^{\frac{1}{\beta}} + \frac{1 - \beta}{\beta^2} (\log \tau)^{\frac{1}{\beta} - 1} \log \log \tau + K^+_{\varepsilon}(\log \tau)^{\frac{1}{\beta} - 1} \\
(3.49) & \quad b^{-1}(\tau) \geq (\log \tau)^{\frac{1}{\beta}} + \frac{1 - \beta}{\beta^2} (\log \tau)^{\frac{1}{\beta} - 1} \log \log \tau + K^-_{\varepsilon}(\log \tau)^{\frac{1}{\beta} - 1}
\end{align*}
\]
for \( \tau > \tau_0 \), where
\[
K^+_{\varepsilon} = - \frac{\log \beta N}{\beta} + \varepsilon \quad \text{and} \quad K^-_{\varepsilon} = - \frac{\log \beta N}{\beta} - \varepsilon.
\]
Lemma 3.6 tells us that the function \( t \mapsto \lambda_t, t > 0 \), defined in Lemma 3.5, decreases to zero as \( t \to \infty \). Hence, there exists \( t_0 > 0 \) such that \( \lambda_t < 1 \) for \( t > t_0 \). Let \( \sigma(t) \) be the function defined by (3.27) for \( t > t_0 \). Then \( \sigma(t) \) is increasing on \((t_0, \infty)\) and
\[
(3.50) \quad \lambda_t = e^{-\sigma(t)^2} \quad \text{for} \quad t > t_0.
\]
Lemma 3.6 also tells us that
\[
(3.51) \quad \sigma(t) = \begin{cases} 
\sqrt{2\left(\frac{2}{\beta} - 1\right)} \log t + \cdots & \text{if} \quad \beta \in (0, 2) \\
\sqrt{2} \log \log t + \cdots & \text{if} \quad \beta = 2
\end{cases} \quad \text{as} \quad t \to \infty.
\]
Consequently,
\[
(3.52) \quad \lim_{t \to \infty} \frac{t}{\sigma(t)} = \infty.
\]
Next, choose \( C > 1 \) fulfilling the inequality
\[
(3.53) \quad e^{(C^2 - 1) \frac{\sigma(t)^2}{2}} > \tau_0 \quad \text{for} \quad t > t_0.
\]
Thanks to (3.52), there exists \( t_1 > t_0 \) such that \( t > C\sigma(t) \) for \( t > t_1 \). On setting
\[
(3.54) \quad I_1(t) = \int_0^{C\sigma(t)} b^{-1}(\lambda_t \frac{t^2}{2}) \, d\tau
\]
and
\begin{equation}
I_2(t) = \int_{C\sigma(t)}^t b^{-1}\left(\lambda t e^{\frac{\tau^2}{2}}\right) d\tau
\end{equation}
for \(t > t_1\), equation (3.13) can be rewritten as
\begin{equation}
\left\| \frac{1}{T} L \beta(\Phi(t),\frac{1}{T}) \right\| = I_1(t) + I_2(t) \quad \text{for } t > t_1.
\end{equation}

We begin with an estimate for \(I_1\). By (3.54) and the monotonicity of \(b^{-1}\),
\begin{equation}
I_1(t) \leq C\sigma(t) b^{-1}\left(\lambda t e^{\frac{C^2 - 1}{2} \frac{\sigma(t)^2}{2}}\right) \quad \text{for } t > t_1.
\end{equation}

By equation (3.50), this inequality ensures that
\begin{equation}
I_1(t) \leq C\sigma(t) b^{-1}\left(e^{(C^2 - 1)\frac{\sigma(t)^2}{2}}\right) \quad \text{for } t > t_1.
\end{equation}

Thus, by (3.57), (3.53) and (3.47),
\begin{equation}
I_1(t) \leq K\sigma(t)^{\frac{1}{\beta} + 1} \quad \text{for } t > t_1,
\end{equation}
where
\begin{equation}
K = (1 + \varepsilon)C \left(\frac{C^2 - 1}{2}\right)^{\frac{1}{2}}.
\end{equation}
Therefore, from (3.51) and (3.58) we obtain that
\begin{equation}
0 \leq I_1(t) \leq K^2 \left(\frac{1}{2} + \frac{1}{\beta}\right) \left(\frac{2}{\beta} - 1\right) \log t + \cdots \quad \text{if } \beta \in (0, 2) \quad \text{as } t \to \infty.
\end{equation}

We now deal with \(I_2\). Equations (3.50) and (3.55) entail that \(I_2\) can be expressed as
\begin{equation}
I_2(t) = \int_{C\sigma(t)}^t b^{-1}\left(e^{\frac{\tau^2 - \sigma(t)^2}{2}}\right) d\tau \quad \text{for } t > t_1.
\end{equation}

From inequality (3.53) one has that
\begin{equation}
e^{\frac{\tau^2 - \sigma(t)^2}{2}} > \tau_0 \quad \text{for } \tau > C\sigma(t),
\end{equation}
provided that \(t > t_1\). Set
\begin{equation}
I_{21}(t) = \int_{C\sigma(t)}^t \left(\frac{\tau^2}{2} - \frac{\sigma(t)^2}{2}\right)^{\frac{1}{\beta} - 1} d\tau,
\end{equation}
\begin{equation}
I_{22}(t) = \int_{C\sigma(t)}^t \left(\frac{\tau^2}{2} - \frac{\sigma(t)^2}{2}\right)^{\frac{1}{\beta} - 1} \log \left(\frac{\tau^2}{2} - \frac{\sigma(t)^2}{2}\right) d\tau
\end{equation}
and
\begin{equation}
I_{23}(t) = \int_{C\sigma(t)}^t \left(\frac{\tau^2}{2} - \frac{\sigma(t)^2}{2}\right)^{\frac{1}{\beta} - 1} d\tau,
\end{equation}
for \(t > t_1\). By inequality (3.48),
\begin{equation}
I_2(t) \leq I_{21}(t) + \frac{1 - \beta}{\beta^2} I_{22}(t) + K^+ \epsilon I_{23}(t)
\end{equation}
and, by (3.49)
\begin{equation}
I_2(t) \geq I_{21}(t) + \frac{1 - \beta}{\beta^2} I_{22}(t) + K^- \epsilon I_{23}(t)
\end{equation}
for $t > t_1$.

Let us focus on $I_{21}$. By a change of variables,

$$ I_{21}(t) = \frac{\sigma(t)^2}{2} r^2 - \frac{\sigma(t)^2}{2} \int_C \left( t - \sigma(t) \right)^{2-\frac{2}{\beta}} \frac{2}{\beta} \sigma(t)^{\frac{2}{\beta} + 1} \int_C \frac{t}{\sigma(t)} \left( r^2 - 1 \right)^{\frac{1}{\beta}} dr 

= 2^{-\frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} + 1} \Psi_{\frac{1}{\beta}} \left( \frac{t}{\sigma(t)} \right) \quad \text{for } t > t_1,
$$

where $\Psi_{\frac{1}{\beta}}$ is defined as in (3.10), with $d = C$. Thanks to equation (3.52), we may use Lemma 3.2 to infer what follows: if $\beta \in (0, 2)$, then

$$ \Psi_{\frac{1}{\beta}} \left( \frac{t}{\sigma(t)} \right) = \frac{\beta}{2 + \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{\beta} + 1} - \frac{1}{2 - \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{\beta} - 1} + \begin{cases} \frac{1}{\beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{\beta} - 3} & \text{if } \beta \in (0, \frac{2}{3}) \\ \frac{3}{8} \log \frac{t}{\sigma(t)} + \cdots & \text{if } \beta = \frac{2}{3} \\ c + \cdots & \text{if } \beta \in (\frac{2}{3}, 2) \end{cases} 

\text{as } t \to \infty, \text{ whence}

$$ I_{21}(t) = 2^{-\frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} + 1} - 2^{-\frac{1}{\beta}} \frac{1}{2 - \beta} \sigma(t)^2 + 2^{-\frac{1}{\beta}} \begin{cases} \frac{1}{\beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{\beta} - 3} & \text{if } \beta \in (0, \frac{2}{3}) \\ \frac{3}{8} \sigma(t)^{\frac{2}{\beta} + 1} \log t + \cdots & \text{if } \beta = \frac{2}{3} \\ c \sigma(t)^{\frac{2}{\beta} + 1} + \cdots & \text{if } \beta \in (\frac{2}{3}, 2) \end{cases} 

\text{as } t \to \infty; \text{ if } \beta = 2, \text{ then}

$$ \Psi_{\frac{1}{\beta}} \left( \frac{t}{\sigma(t)} \right) = \frac{1}{2} \left( \frac{t}{\sigma(t)} \right)^2 - \frac{1}{2} \log \frac{t}{\sigma(t)} + c + \cdots \quad \text{as } t \to \infty, 

\text{whence}

$$ I_{21}(t) = \frac{1}{2 \sqrt{2}} t^2 - \frac{1}{2 \sqrt{2}} \sigma(t)^2 \log t + \frac{1}{2 \sqrt{2}} \sigma(t)^2 \log \sigma(t) + \cdots \quad \text{as } t \to \infty.

\text{Also, coupling equation (3.27) with Lemma 3.6 enables us to deduce that}

$$ \sigma(t)^2 = \begin{cases} 2 \left( \frac{2}{3} - 1 \right) \log t - 2 \log c_3 + \cdots & \text{if } \beta \in (0, 2) \\ 2 \log \log t - 2 \log c_2 + \cdots & \text{if } \beta = 2 \end{cases} \quad \text{as } t \to \infty, 

\text{whence}

$$ I_{21}(t) = 2^{-\frac{1}{\beta}} \frac{\beta}{2 + \beta} t^{\frac{2}{\beta} + 1} + 2^{-\frac{1}{\beta}} \begin{cases} -\frac{2}{\beta} t^{\frac{2}{\beta} - 1} \log t + \frac{2}{2 - \beta} \log c_3 t^{\frac{2}{\beta} - 1} + \cdots & \text{if } \beta \in (0, 2) \\ -\log t \log \log t + \log c_2 \log t + \cdots & \text{if } \beta = 2 \end{cases} 

\text{as } t \to \infty. \text{ Let us next deal with } I_{22}(t). \text{ We have that}

$$ I_{22}(t) = \int_{C \sigma(t)}^{t} \left( \frac{t^2}{2} - \frac{\sigma(t)^2}{2} \right)^{\frac{1}{\beta} - 1} \log \left( \frac{t^2}{2} - \frac{\sigma(t)^2}{2} \right) d\tau

= 2^{1 - \frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} - 1} \left[ \log \frac{\sigma(t)^2}{2} \Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) + \Upsilon_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) \right] = I_{221}(t) + I_{222}(t) 

\text{for } t > t_1, \text{ where the functions } \Psi_{\frac{1}{\beta} - 1} \text{ and } \Upsilon_{\frac{1}{\beta} - 1} \text{ are defined as in (3.10) and (3.11), with } d = C, \text{ and where we have set}

$$ I_{221}(t) = 2^{1 - \frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} - 1} \log \frac{\sigma(t)^2}{2} \Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right), 

\text{and}

$$ I_{222}(t) = 2^{1 - \frac{1}{\beta}} \sigma(t)^{\frac{2}{\beta} - 1} \Upsilon_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right).
for \( t > t_1 \). As far as \( I_{221} \) is concerned, thanks to Lemma 3.2,

\[
\Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) = -\frac{\beta}{2 - \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{3} - 1} - \frac{\beta - 1}{3\beta t^{\frac{2}{3}}} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{3} - 3} + \cdots \quad \text{if} \quad \beta \in (0, \frac{3}{2})
\]

\[
\frac{1}{2} \log t + \cdots \quad \text{if} \quad \beta = \frac{2}{3}
\]

\[
\frac{c + \cdots}{\beta - 1} \quad \text{as} \quad t \to \infty.
\]

Hence, if \( \beta \in (0, 2) \), then

\[
\sigma(t)^{\frac{2}{3} - 1} \Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) = -\frac{\beta}{2 - \beta} t^{\frac{2}{3} - 1} - \frac{\beta - 1}{3\beta t^{\frac{2}{3}}} \sigma(t)^{\frac{2}{3} - 3} + \cdots \quad \text{if} \quad \beta \in (0, \frac{2}{3})
\]

\[
\frac{1}{2} \log t + \cdots \quad \text{if} \quad \beta = \frac{2}{3}
\]

\[
\frac{c + \cdots}{\beta - 1} \quad \text{as} \quad t \to \infty.
\]

and, if \( \beta = 2 \), then

\[
\Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) = \log t - \log \sigma(t) + c + \cdots \quad \text{as} \quad t \to \infty.
\]

Note that \( \sigma(t)^{\frac{2}{3} - 1} = 1 \) in the latter case. Equation (3.27) and Lemma 3.6 imply that

\[
\log \frac{\sigma(t)^{2}}{2} = \begin{cases} 
\log \log t + \log \left( \frac{2}{\beta} - 1 \right) + \cdots & \text{if} \quad \beta \in (0, 2) \\
\log \log \log t - \log \sigma(t)^{2} + \cdots & \text{if} \quad \beta = 2
\end{cases}
\quad \text{as} \quad t \to \infty.
\]

Furthermore, if \( \beta = 2 \), then

\[
\log \sigma(t) = \frac{1}{2} \log \log \log t + \frac{1}{2} \log 2 + \cdots \quad \text{as} \quad t \to \infty.
\]

Therefore,

\[
I_{221}(t) = \begin{cases} 
2^{-\frac{1}{2}} \frac{2\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log \log t + 2^{-\frac{1}{2}} \frac{2\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log \left( \frac{2}{\beta} - 1 \right) + \cdots & \text{if} \quad \beta \in (0, 2) \\
\sqrt{2} \log \log t \log \log t - \frac{\sqrt{2}}{t} \left( \log \log \log t \right)^{2} + \cdots & \text{if} \quad \beta = 2
\end{cases}
\quad \text{as} \quad t \to \infty.
\]

as \( t \to \infty \). The behaviour of the term \( I_{222}(t) \) can be determined as follows. Owing to Lemma 3.3,

\[
\Psi_{\frac{1}{\beta} - 1} \left( \frac{t}{\sigma(t)} \right) = \begin{cases} 
\frac{2\beta}{2 - \beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{3} - 1} \log \frac{t}{\sigma(t)} - \frac{2\beta^{2}}{(2 - \beta)^{2}} \left( \frac{t}{\sigma(t)} \right)^{\frac{2}{3} - 1} + \cdots & \text{if} \quad \beta \in (0, 2) \\
\log \frac{t}{\sigma(t)} + \cdots & \text{if} \quad \beta = 2
\end{cases}
\quad \text{as} \quad t \to \infty.
\]

As a consequence, if \( \beta \in (0, 2) \), then

\[
I_{222}(t) = 2^{-\frac{1}{2}} \frac{4\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log t - 2^{-\frac{1}{2}} \frac{4\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log \sigma(t) - 2^{-\frac{1}{2}} \frac{4\beta^{2}}{(2 - \beta)^{2}} t^{\frac{2}{3} - 1} + \cdots \quad \text{as} \quad t \to \infty,
\]

whereas, if \( \beta = 2 \), then

\[
I_{222}(t) = \sqrt{2} \left( \log t \right)^{2} - 2 \sqrt{2} \log t \log \sigma(t) + \cdots \quad \text{as} \quad t \to \infty.
\]

Inasmuch as

\[
\log \sigma(t) = \begin{cases} 
\frac{1}{2} \log \log t + \frac{1}{2} \log \left( 2 \left( \frac{2}{\beta} - 1 \right) \right) + \cdots & \text{if} \quad \beta \in (0, 2) \\
\frac{1}{2} \log \log t + \frac{1}{2} \log 2 + \cdots & \text{if} \quad \beta = 2
\end{cases}
\quad \text{as} \quad t \to \infty,
\]

we conclude that

\[
I_{222}(t) = \begin{cases} 
2^{-\frac{1}{2}} \frac{4\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log t - 2^{-\frac{1}{2}} \frac{2\beta^{2}}{2 - \beta} t^{\frac{2}{3} - 1} \log t \\
2^{-\frac{1}{2}} \frac{2\beta}{2 - \beta} t^{\frac{2}{3} - 1} \log \left( \frac{2}{\beta} - 1 \right) + \log 2 + \frac{2\beta}{2 - \beta} \log \sigma(t) - 2^{-\frac{1}{2}} \frac{4\beta}{(2 - \beta)^{2}} t^{\frac{2}{3} - 1} + \cdots & \text{if} \quad \beta \in (0, 2) \\
\sqrt{2} \log \left( \log t \right)^{2} - 2 \sqrt{2} \log t \log \log t - \sqrt{2} \log \log t + \cdots & \text{if} \quad \beta = 2
\end{cases}
\quad \text{as} \quad t \to \infty.
\]

(3.72)
Combining equations (3.70), (3.71) and (3.72) tells us that
\[(3.73) \quad I_{22}(t) = \begin{cases} 
2^{-\frac{1}{4}} \frac{4\beta}{2-\beta} t^{\frac{3}{4} - 1} \log t - 2^{-\frac{1}{4}} \frac{2\beta}{2-\beta} \left( \frac{2\beta}{2-\beta} + \log 2 \right) t^{\frac{3}{4} - 1} + \cdots & \text{if } \beta \in (0, 2) \\
\sqrt{2}(\log t)^2 - \sqrt{2} \log 2 \log t + \cdots & \text{if } \beta = 2
\end{cases} \quad \text{as } t \to \infty.
\]
We finally turn our attention on the term $I_{23}(t)$. Since
\[
I_{23}(t) = \int_C (t-2)^2 (\sigma(t)^2)^{\frac{1}{2} - 1} d\tau = 2^{1-\frac{1}{4}} \sigma(t)^{\frac{3}{4} - 1} \Psi \left( \frac{t}{\sigma(t)} \right) \quad \text{for } t > t_1,
\]
from Lemma 3.2 one can infer that
\[
I_{23}(t) = 2^{1-\frac{1}{4}} \sigma(t)^{\frac{3}{4} - 1} \left\{ \frac{\beta}{2-\beta} \left( \frac{t}{\sigma(t)} \right)^{\frac{3}{4} - 1} + \cdots \right\} \quad \text{if } \beta \in (0, 2) \quad \text{as } t \to \infty.
\]
Hence,
\[
(3.74) \quad I_{23}(t) = \begin{cases} 
2^{-\frac{1}{4}} \frac{4\beta}{2-\beta} t^{\frac{3}{4} - 1} + \cdots & \text{if } \beta \in (0, 2) \\
\sqrt{2} \log t + \cdots & \text{if } \beta = 2
\end{cases} \quad \text{as } t \to \infty.
\]
Altogether, if $\beta \in (0, 2)$, then, by (3.65), (3.69), (3.73) and (3.74),
\[
(3.75) \quad I_2(t) \leq 2^{-\frac{1}{4}} \frac{1}{2+\beta} t^{\frac{3}{4} + 1} - 2^{-\frac{1}{4}} \frac{4\beta}{2-\beta} t^{\frac{3}{4} - 1} \log t \left( \frac{2}{\beta} - \frac{1-\beta}{2-\beta} \frac{4\beta}{2-\beta} \right) + 2^{-\frac{1}{4}} \frac{2}{2-\beta} t^{\frac{3}{4} - 1} \log t \left( \frac{2}{\beta} - \frac{1-\beta}{2-\beta} \frac{4\beta}{2-\beta} \right) + \cdots \quad \text{as } t \to \infty,
\]
and, by making use of (3.66) instead of (3.65),
\[
(3.76) \quad I_2(t) \leq 2^{-\frac{1}{4}} \frac{1}{2+\beta} t^{\frac{3}{4} + 1} - 2^{-\frac{1}{4}} \frac{4\beta}{2-\beta} t^{\frac{3}{4} - 1} \log t \left( \frac{2}{\beta} - \frac{1-\beta}{2-\beta} \frac{4\beta}{2-\beta} \right) + 2^{-\frac{1}{4}} \frac{2}{2-\beta} t^{\frac{3}{4} - 1} \log t \left( \frac{2}{\beta} - \frac{1-\beta}{2-\beta} \frac{4\beta}{2-\beta} \right) + \cdots \quad \text{as } t \to \infty.
\]
Therefore, owing to the arbitrariness of \( \varepsilon \),
\[
(3.77) \quad I_2(t) = 2^{-\frac{1}{4}} \frac{1}{2+\beta} t^{\frac{3}{4} + 1} - 2^{-\frac{1}{4}} \frac{4\beta}{2-\beta} t^{\frac{3}{4} - 1} \log t + 2^{-\frac{1}{4}} c_{\beta,N} t^{\frac{3}{4} - 1} + \cdots \quad \text{as } t \to \infty
\]
for a suitable constant $c_{\beta,N} \in \mathbb{R}$.

Similarly, if $\beta = 2$, then
\[
(3.78) \quad I_2(t) = \frac{1}{2} t^2 - \frac{1}{\sqrt{2}} \log t \log t + \frac{1}{\sqrt{2}} \log c_2 \log t + \cdots \quad \text{as } t \to \infty.
\]
Owing to equation (3.59), formulas (3.77) and (3.78) continue to hold with $I_2(t)$ replaced by $I_1(t) + I_2(t)$ on the left-hand side. Hence, if $\beta \in (0, 2)$, then by (3.56),
\[
\left\| \frac{1}{T} \right\|_{L^\beta(\Phi(t)^\frac{1}{2})} = 2^{-\frac{1}{4}} \frac{\beta}{2+\beta} t^{\frac{3}{4} + 1} - 2^{-\frac{1}{4}} \frac{2}{2-\beta} t^{\frac{3}{4} - 1} \log t + 2^{-\frac{1}{4}} c_{\beta,N} t^{\frac{3}{4} - 1} + \cdots \quad \text{as } t \to \infty,
\]
and, if $\beta = 2$, then
\[
\left\| \frac{1}{T} \right\|_{L^\beta(\Phi(t)^\frac{1}{2})} = \frac{1}{2} t^2 - \frac{\sqrt{2}}{4} \log t^2 - \frac{1}{\sqrt{2}} \log t \log t + \cdots \quad \text{as } t \to \infty.
\]
The proof is complete. \(\square\)
Lemma 3.7, coupled with equation (3.9), immediately implies the following result.

**Lemma 3.8.** Let $B$ be a Young function of the form (3.4) for some $\beta \in (0, 2)$ and $N > 0$. Let $F_B$ be the function defined by (3.5), and let $\kappa_\beta$ be the constant given by (1.7). Then

$$
[k_\beta F_B(t)]^\beta_2 = \frac{t^2}{2} + \begin{cases} 
- \frac{2}{\beta} \log t + c_{\beta,N} + \cdots & \text{if } \beta \in (0, 2) \\
- \frac{1}{2} (\log t)^2 - \log t \log t + \cdots & \text{if } \beta = 2
\end{cases}
$$

as $t \to \infty$,

where $c_{\beta,N}$ is the constant appearing in Lemma 3.7.

We are now in a position to accomplish the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Assume that $u$ is a weakly differentiable function satisfying conditions (1.5) and (1.6). Then, by Lemma 3.1, there exists a Young function $B$ of the form (3.4) such that

$$
(3.79) \quad \int_{\mathbb{R}^n} \exp^{\frac{\beta}{2-\beta}} k_\beta |u| \varphi(|u|) \, d\gamma_n \leq \sqrt{\frac{2}{\pi}} \int_0^\infty e^{[\kappa_\beta F_B(t)] \frac{\beta}{2-\beta} - \frac{t^2}{2}} \varphi(F_B(t)) \, dt.
$$

Thanks to Lemma 3.8,

$$
(3.80) \quad [k_\beta F_B(t)]^\beta_2 - \frac{t^2}{2} = \begin{cases} 
- \frac{2}{\beta} \log t + c_{\beta,N} + \cdots & \text{if } \beta \in (0, 2) \\
- \frac{1}{2} (\log t)^2 - \log t \log t + \cdots & \text{if } \beta = 2
\end{cases}
$$

as $t \to \infty$.

Moreover, from Lemma 3.7 and the definition of $F_B$, one can deduce that

$$
(3.81) \quad F_B(t) \leq \mu_\beta t^{\frac{\beta}{2} + 1} + \cdots \quad \text{near infinity},
$$

where $\mu_\beta = 2^{\frac{1}{2}} t^\beta_{\beta-2}$.

Assume first that $\beta \in (0, 2)$. The integral on the right hand side of (3.79) converges if

$$
(3.82) \quad \int_0^\infty e^{- \frac{2}{\beta} \log t} \varphi(F_B(t)) \, dt < \infty.
$$

Since $\varphi$ is an increasing function, equation (3.81) implies that

$$
\varphi(F_B(t)) \leq \varphi(\mu_\beta t^{\frac{\beta}{2} + 1}) \quad \text{for large } t.
$$

Therefore, equation (3.82) holds provided that

$$
\int_0^\infty e^{- \frac{\beta}{\beta - 2} \log t} \varphi(\mu_\beta t^{\frac{\beta}{2} + 1}) \, dt < \infty.
$$

The convergence of the last integral follows from assumption (1.12).

Assume next that $\beta = 2$. Owing to equation (3.80), given any $\delta > 0$, one has that

$$
[k_\beta F_B(t)]^\beta_2 - \frac{t^2}{2} \leq - \frac{1}{2} (\log t)^2 + (\delta - 1) \log t \log t \quad \text{for large } t.
$$

Hence, in the light of inequality (3.81), the integral on the right-hand side of (3.79) converges if there exists $\delta > 0$ such that

$$
\int_0^\infty e^{- \frac{1}{2} (\log t)^2 + (\delta - 1) \log t \log t} \varphi(\mu_2 t^2) \, dt < \infty.
$$

The last inequality follows from assumption (1.13), via the change of variables $s = \mu_2 t^2$. The validity of inequality (1.11) is thus established for both $\beta \in (0, 2)$ and $\beta = 2$, under the respective assumptions.

It remains to prove the sharpness of these assumptions. Let $\tau_0 > 0$ and let $f : (\tau_0, \infty) \to (0, \infty)$ be a locally integrable function. Define the function $u : \mathbb{R}^n \to \mathbb{R}$ as

$$
u(x) = \text{sgn} x_1 \begin{cases} 
0 & \text{for } |x_1| < \tau_0 \\
\int_{\tau_0}^{|x_1|} f(\tau) \, d\tau & \text{for } |x_1| \geq \tau_0
\end{cases}
$$
Then $u$ is weakly differentiable, $\text{med}(u) = \text{mv}(u) = 0$ and
\[
|\nabla u(x)| = \begin{cases} 
0 & \text{for } |x_1| < \tau_0 \\
 f(|x_1|) & \text{for a.e. } x \text{ such that } |x_1| > \tau_0.
\end{cases}
\]

Fix $t_0 > 0$ so large that $\text{Exp}^\beta(t) = e^{t\beta}$ for $t > t_0$, and then choose $\tau_0 > 0$ such that
\[
\frac{e^{\frac{\tau}{2}}}{\tau (\log \tau)^2} \geq \text{Exp}^\beta(t_0) \quad \text{for } \tau > \tau_0.
\]

Set
\[
(3.83) \quad f(\tau) = \text{Exp}^{-1}\left(\frac{e^{\frac{\tau}{2}}}{\tau (\log \tau)^2}\right) \quad \text{for } \tau > \tau_0,
\]

where $E^{-1}$ denotes the inverse of the function $\text{Exp}^\beta$ on $(t_0, \infty)$. Thus,
\[
\int_{\mathbb{R}^n} \text{Exp}^\beta(|\nabla u|) d\gamma_n = 2 \int_0^{t_0} d\gamma_1 + \frac{2}{\sqrt{2\pi}} \int_{t_0}^\infty \text{Exp}^\beta(f(\tau)) e^{-\frac{\tau}{2}} d\tau \leq 1 + \frac{2}{\sqrt{2\pi}} \int_{t_0}^\infty \frac{d\tau}{\tau (\log \tau)^2}.
\]

Since $M > 1$ and the last integral converges, we may assume, on increasing $\tau_0$, if necessary, that
\[
\int_{\mathbb{R}^n} \text{Exp}^\beta(|\nabla u|) d\gamma_n \leq M.
\]

As far as the integral in (1.16) is concerned, we have that
\[
(3.84) \quad \int_{\mathbb{R}^n} \exp^{\frac{2\beta}{1+\beta}} (\kappa_\beta |u|) \varphi(|u|) d\gamma_n \geq \frac{2}{\sqrt{2\pi}} \int_{t_0}^\infty \exp \left\{ \left( \frac{\tau}{2} \frac{\beta}{t} - \frac{t^2}{2} \right) \right\} \varphi \left( \int_{t_0}^t f(\tau) d\tau \right) dt.
\]

Since $E^{-1}(\tau) = (\log \tau)^\frac{1}{\beta}$ if $\tau \geq \text{Exp}^\beta(t_0)$, the function $f$ defined by equation (3.83) takes the form
\[
f(\tau) = \left( \frac{\log \frac{e^{\frac{\tau}{2}}}{\tau (\log \tau)^2}}{\tau (\log \tau)^2} \right)^\frac{1}{\beta} = \left( \frac{\tau^2}{2} - \log \tau - 2 \log \log \tau \right)^\frac{1}{\beta} \quad \text{for } \tau > \tau_0.
\]

Routine computations resting upon L'Hôpital's rule then tell us that
\[
\int_{t_0}^t f(\tau) d\tau = \frac{1}{2} \ln \frac{t^2}{2 + \beta} \left( \frac{\beta}{2} + \frac{t}{\beta} \right) + 2 \frac{1}{\beta} \left\{ \begin{array}{ll}
\frac{(-2t^2)^{\frac{1}{2} + \frac{1}{\beta} - 1} \log t + \frac{4}{2 - \beta} \frac{\beta}{2} \log t}{-\frac{t}{2} (\log t)^2 + 2 \log t \log \log t + \cdots}
\end{array} \right. \quad \text{if } \beta \in (0, 2)
\]
as $t \to \infty$. Consequently, by formula (3.8),
\[
\left( \kappa_\beta \int_{t_0}^t f(\tau) d\tau \right)^\frac{2\beta}{1+\beta} - \frac{t^2}{2} = \left\{ \begin{array}{ll}
\frac{(-2t^2)^{\frac{1}{2} + \frac{1}{\beta} - 1} \log t + \frac{4}{2 - \beta} \frac{\beta}{2} \log t}{-\frac{t}{2} (\log t)^2 + 2 \log t \log \log t + \cdots}
\end{array} \right. \quad \text{if } \beta \in (0, 2)
\]
as $t \to \infty$.

Now, there exists $\tau_1 > \tau_0$ such that
\[
(3.85) \quad \int_{t_0}^t f(\tau) d\tau \geq \frac{\mu_\beta}{2} t^{\frac{2}{1+\beta} + 1} \quad \text{for } t > \tau_1,
\]

where we have set $\mu_\beta = 2^{-\frac{1}{\beta}} \frac{\beta}{2 + \beta}$, and, simultaneously,
\[
(3.86) \quad \left( \kappa_\beta \int_{t_0}^t f(\tau) d\tau \right)^\frac{2\beta}{1+\beta} - \frac{t^2}{2} \geq -g(t) \quad \text{for } t > \tau_1,
\]
where we have set
\[
g(t) = \begin{cases} 
\frac{2}{\beta} \log t + \frac{8}{\beta^2} \log \log t & \text{if } \beta \in (0, 2) \\
\frac{1}{2} (\log t)^2 + 4 \log t \log \log t & \text{if } \beta = 2
\end{cases}
\]
for large \( t \). Therefore, since \( \varphi \) is an increasing function, from inequality (3.84) we can deduce that
\[
\int_{\mathbb{R}^n} \exp^{\frac{2 \beta}{\pi}} (|\kappa_\beta| |u|) \varphi(|u|) \, d\gamma_n \geq \frac{2}{\sqrt{2 \pi}} \int_{\tau_1}^{\infty} e^{-g(t)} \varphi \left( \frac{H_\beta}{2} t^{\frac{2}{\beta} + 1} \right) dt.
\]
Observe that, in deriving inequality (3.87), we have exploited the lower bounds (3.85), (3.86) and replaced the lower limit of integration \( \tau_0 \) by \( \tau_1 \).

First, assume that \( \beta \in (0, 2) \). The integral on the right-hand side of (3.87) diverges if
\[
\int_{\tau_1}^{\infty} t^{-\frac{2}{\beta} + \frac{8}{\beta^2} \frac{\mu}{\gamma_\beta}} \varphi(t) \, dt = \infty.
\]
By a change of variables, equation (3.88) is equivalent to
\[
\int_{\tau_1}^{\infty} t^{-\frac{4}{\pi} - \frac{8}{\pi^2} \frac{\mu}{\gamma_\beta}} \varphi(t) \, dt = \infty.
\]
Thus, equation (1.16) follows via (3.87), by assumption (1.14).

Next, suppose that \( \beta = 2 \). The integral on the right-hand side of (3.87) diverges if
\[
\int_{\tau_1}^{\infty} e^{-\frac{1}{2} (\log t)^2 - 4 \log t \log \log t} \varphi \left( \frac{H_2}{2} t^2 \right) dt = \infty.
\]
A change of variables again shows that equation (3.89) certainly holds provided that
\[
\int_{\tau_1}^{\infty} e^{-\frac{1}{4} (\log t)^2 - \frac{5}{4} \log t \log \log t} \varphi(t) \, dt = \infty.
\]
Equation (1.16) now follows from a combination of (3.87) and (1.15).

4. Proof of Theorem 1.1

Besides Theorem 1.2, some classical results of functional analysis come into play in our proof of Theorem 1.1. They are recalled below, in a form suitable for our applications.

**Theorem A [Equi-integrability (de la Vallée-Poussin)].** Let \((\mathcal{R}, \nu)\) be a probability space. Then a sequence \(\{u_k\} \subset L^1(\mathcal{R}, \nu)\) is equi-integrable if and only if there exists a convex function \(\Psi: [0, \infty) \to [0, \infty)\), satisfying \(\lim_{t \to \infty} \Psi(t)/t = \infty\), such that
\[
\sup_k \int_{\mathcal{R}} \Psi(|u_k|) \, d\nu < \infty.
\]

**Theorem B [Convergence in L^1 (Vitali)].** Let \((\mathcal{R}, \nu)\) be a probability space and let \(\{u_k\}\) be a sequence in \(L^1(\mathcal{R}, \nu)\). Then \(\{u_k\}\) converges to a function \(u \in L^1(\mathcal{R}, \nu)\) if and only if \(\{u_k\}\) is equi-integrable and converges to \(u\) in measure.

The following result is a consequence of [33, Section 1.2, Theorem 8].

**Theorem C [Semicontinuity (Serrin)].** Let \(g: \mathbb{R}^n \to [0, \infty)\) be a convex function. Then the functional defined as
\[
\int_{\mathbb{R}^n} g(\nabla u) \, d\gamma_n
\]
for a weakly differentiable function \(u: \mathbb{R}^n \to \mathbb{R}\) is sequentially lower semicontinuous with respect to weak* convergence in \(W^{1,1}(\mathbb{R}^n, \gamma_n)\).

An application of the Banach Alaoglu theorem to the space \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\) yields the next theorem. Note that the space \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\) is the dual of the separable space \(L(\log L)^{\frac{1}{\beta}}(\mathbb{R}^n, \gamma_n)\).
Theorem D [Weak* compactness in $\exp L^2(\mathbb{R}^n, \gamma_n)$ (Banach–Alaoglu)]. Assume that $\{u_k\}$ is a bounded sequence in $\exp L^2(\mathbb{R}^n, \gamma_n)$. Then there exist $u \in \exp L^2(\mathbb{R}^n, \gamma_n)$ and a subsequence $\{u_{k_i}\}$ such that $u_{k_i} \to u$ in the weak* topology of $\exp L^2(\mathbb{R}^n, \gamma_n)$.

The last preliminary result, concerning convergence of medians, is contained in Lemma 4.1.

Lemma 4.1. Assume that $u \in W^{1,1}(\mathbb{R}^n, \gamma_n)$ and that $\{u_k\}$ is a sequence in $W^{1,1}(\mathbb{R}^n, \gamma_n)$ such that $u_k \to u$ in $L^1(\mathbb{R}^n, \gamma_n)$. Then there is a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$\lim_{k \to \infty} m(u_k) = m(u).$$

Here, $m(u)$ denotes either the mean value or the median of $u$.

Proof. The claim is trivial if $m(\cdot)$ is the mean value. Let us consider the case when $m(\cdot)$ is the median. As recalled in Proposition 2.1, the functions $u_k^\ell$, for $k \in \mathbb{N}$, and $u^\ell$ are (locally absolutely) continuous, since $u_k$ and $u$ are Sobolev functions. As a consequence of [9, Chapter 3, Theorem 7.4], the signed decreasing rearrangement is a contraction from $L^1(\mathbb{R}^n, \gamma_n)$ in $L^1(0,1)$. Therefore, since $u_k \to u$ in $L^1(\mathbb{R}^n, \gamma_n)$, one also has that $u_k^\ell \to u^\ell$ in $L^1(0,1)$. Thus, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$\lim_{k \to \infty} u_k^\ell = u^\ell \quad \text{a.e. in } (0,1).$$

We claim that (4.2) holds, in fact, everywhere in $(0,1)$. To verify this claim, assume, by contradiction, that (4.2) is violated for some $s_0 \in (0,1)$, namely there exists $\varepsilon > 0$ such that

$$|u_k^\ell (s_0) - u^\ell (s_0)| > \varepsilon$$

for some subsequence $\{u_k^\ell\}$. Hence, either the set $U = \{\ell : u_k^\ell (s_0) > u^\ell (s_0) + \varepsilon\}$ or the set $L = \{\ell : u_k^\ell (s_0) < u^\ell (s_0) - \varepsilon\}$ is infinite. Assume, for instance, that $U$ is infinite, the proof in the case when $L$ is infinite being analogous. By the continuity of $u^\ell$, there exists $\delta > 0$ such that $|u^\ell (s) - u^\ell (s_0)| < \varepsilon/2$ for every $s \in (0,1)$ obeying $|s - s_0| < \delta$. Since equation (4.2) holds for a.e. $s \in (0,1)$, there exists $r_0 \in (s_0 - \delta, s_0)$ such that $u_k^\ell (r_0) \to u^\ell (r_0)$ as $\ell \to \infty$. On the other hand,

$$u^\ell (r_0) + \frac{\varepsilon}{2} < u^\ell (s_0) + \varepsilon < u_k^\ell (s_0) \leq u_k^\ell (r_0) \quad \text{for } \ell \in U$$

since the functions $u_k^\ell$ are non-increasing. This contradicts the convergence of the sequence $\{u_k^\ell (r_0)\}$ to $u^\ell (r_0)$. Equation (4.1) follows from our claim, since

$$\lim_{k \to \infty} \text{med}(u_k) = \lim_{k \to \infty} u_k^\ell (\frac{1}{2}) = u^\ell (\frac{1}{2}) = \text{med}(u).$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_k\} \subset W^1 \exp L^2(\mathbb{R}^n, \gamma_n)$ be a maximizing sequence in (1.4), namely $m(u_k) = 0$ and

$$\int_{\mathbb{R}^n} \text{Exp}^\beta (|\nabla u_k|) \, d\gamma_n \leq M$$

for $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \exp^{2\beta} (|\kappa_{\beta} u_k|) \, d\gamma_n = S,$$

where $S$ denotes the supremum in (1.4).

Since $W^1 \exp L^2(\mathbb{R}^n, \gamma_n)$ is compactly embedded into $L^1(\mathbb{R}^n, \gamma_n)$ (see e.g. [51, Theorem 7.3]), there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $u_k \to u$ in $L^1(\mathbb{R}^n, \gamma_n)$ and

$$u_k \to u \quad \text{a.e. in } \mathbb{R}^n.$$

Also, by Lemma 4.1, we may assume that $m(u_k) \to m(u)$, whence we infer that $m(u) = 0$.

By inequality (4.3) and Theorem D, there exists a measurable function $V : \mathbb{R}^n \to \mathbb{R}^n$, with $|V| \in \exp L^2(\mathbb{R}^n, \gamma_n)$, and a subsequence of $\{\nabla u_k\}$, still denoted by $\{\nabla u_k\}$, such that $\nabla u_k \to V$ in the weak* topology of $\exp L^2(\mathbb{R}^n, \gamma_n)$. By the definition of weak gradient, we have that $u$ is weakly differentiable,
\( \nabla u = V, \ u \in W^1 \exp L^\beta(\mathbb{R}^n, \gamma_n), \) and \( \nabla u_k \to \nabla u \) in the weak* topology of \( \exp L^\beta(\mathbb{R}^n, \gamma_n) \). In particular, \( \nabla u_k \to \nabla u \) in the weak* topology of \( L^1(\mathbb{R}^n, \gamma_n) \). Consequently, owing to Theorem C and inequality (4.3),

\[
\int_{\mathbb{R}^n} \exp^\beta(|\nabla u|) \, d\gamma_n \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \exp^\beta(|\nabla u_k|) \, d\gamma_n \leq M.
\]

Our next task is to show that

\[
\int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u|) \, d\gamma_n = S.
\]

Thanks to Theorem 1.2, there exist a continuously increasing function \( \varphi : [0, \infty) \to [0, \infty) \) satisfying \( \lim_{t \to \infty} \varphi(t) = \infty \) and a constant \( C \) such that

\[
\int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u_k|) \varphi(|u_k|) \, d\gamma_n \leq C
\]

for \( k \in \mathbb{N} \). Define the function \( \Theta : [0, \infty) \to [0, \infty) \) as

\[
\Theta(t) = t \varphi \left( \frac{1}{\kappa_\beta} (\log t)^{\frac{1}{2} + \frac{1}{\beta}} \right) \quad \text{for} \ t \geq 1
\]

and \( \Theta(t) = 0 \) for \( t \in [0, 1) \). Then the function \( \Theta(t)/t \) is non-decreasing and \( \lim_{t \to \infty} \Theta(t)/t = \infty \). Finally, define the function \( \Psi : [0, \infty) \to [0, \infty) \) by

\[
\Psi(t) = \int_0^t \frac{\Theta(\tau)}{\tau} \, d\tau \quad \text{for} \ t \geq 0.
\]

Then \( \Psi \) is a Young function such that \( \lim_{t \to \infty} \Psi(t)/t = \infty \) and \( \Psi(t) \leq \Theta(t) \) for \( t \in [0, \infty) \). Thus, owing to (4.6),

\[
\sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \Psi \left( \exp^{2\beta \beta} (\kappa_\beta |u_k|) \right) \, d\gamma_n \leq \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \Theta \left( \exp^{2\beta \beta} (\kappa_\beta |u_k|) \right) \, d\gamma_n
\]

\[
= \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u_k|) \varphi(|u_k|) \, d\gamma_n \leq C.
\]

As a consequence of Theorem A, the sequence of functions \( \left\{ \exp^{2\beta \beta} (\kappa_\beta |u_k|) \right\} \) is equi-integrable in \( L^1(\mathbb{R}^n, \gamma_n) \). Also, by (4.4), this sequence converges to \( \exp^{2\beta \beta} (\kappa_\beta |u|) \) a.e. in \( \mathbb{R}^n \), and hence it converges in measure to the same function. From Theorem B, we deduce that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u_k|) \, d\gamma_n = \int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u|) \, d\gamma_n,
\]

whence equation (4.5) follows. This shows that \( u \) is actually a maximizer for (1.4).

It remains to show that if \( u \) is a maximizer, then it has necessarily the form (1.10). Assume, by contradiction, that this is not the case. Then \( u \neq u^* \) (even up to rotations about 0). Hence, owing to Proposition 2.1, applied with \( A(t) = \exp^\beta(t) - \exp^\beta(0) \),

\[
\int_{\mathbb{R}^n} \exp^\beta(|\nabla u^*|) \, d\gamma_n < \int_{\mathbb{R}^n} \exp^\beta(|\nabla u|) \, d\gamma_n \leq M.
\]

Moreover,

\[
\int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u^*|) \, d\gamma_n = \int_{\mathbb{R}^n} \exp^{2\beta \beta} (\kappa_\beta |u|) \, d\gamma_n.
\]

Therefore, the supremum in (1.4) is also attained at \( u^* \). We have that

\[
u^*(x) = \phi(x_1) \quad \text{for} \ x \in \mathbb{R}^n,
\]

for some non-decreasing locally absolutely continuous function \( \phi : \mathbb{R} \to \mathbb{R} \). Note that

\[
\int_{\mathbb{R}^n} \exp^\beta(|\nabla u^*|) \, d\gamma_n = \int_{\mathbb{R}^n} \exp^\beta \left( \phi' \right) \, d\gamma_1.
\]
and (4.10) \[ \int_{\mathbb{R}^n} \exp^{\frac{2\beta}{n}} (\kappa_\beta |u^*|) \, d\gamma_n = \int_{\mathbb{R}} \exp^{\frac{2\beta}{n}} (\kappa_\beta |\phi|) \, d\gamma_1. \]

By equations (4.7) and (4.9),
\[ \int_{\mathbb{R}} \Exp^\beta (\phi') \, d\gamma_1 < M \]
and, thanks to (4.8) and (4.10),
\[ \int_{\mathbb{R}} \exp^{\frac{2\beta}{n}} (\kappa_\beta |\phi|) \, d\gamma_1 = \int_{\mathbb{R}^n} \exp^{\frac{2\beta}{n}} (\kappa_\beta |u|) \, d\gamma_n. \]

Define, for \( \lambda > 1 \), the function \( \eta_\lambda: \mathbb{R} \to [0, \infty) \) as
\[ \eta_\lambda(t) = (\phi'(t) + \log \lambda)^\frac{1}{\beta} \quad \text{for} \ t \in \mathbb{R}. \]
Assume first that \( m(u) \) stands for \( \text{med}(u) \) in (1.4). Then \( \text{med}(\phi) = \text{med}(u) = 0 \), and hence \( \phi(0) = 0 \). Also, \( \phi(t) \geq 0 \) if \( t \geq 0 \) and \( \phi(t) \leq 0 \) if \( t \leq 0 \). Define, for \( \lambda > 0 \), the function \( \psi_\lambda: \mathbb{R} \to \mathbb{R} \) as
\[ \psi_\lambda(t) = \int_0^t \eta_\lambda(\tau) \, d\tau \quad \text{for} \ t \in \mathbb{R}, \]
and the function \( v_\lambda: \mathbb{R}^n \to \mathbb{R} \) as
\[ v_\lambda(x) = \psi_\lambda(x_1) \quad \text{for} \ x \in \mathbb{R}^n. \]
One has that \( \text{med}(v_\lambda) = \psi_\lambda(0) = 0 \). Next, let \( t_0 \geq 0 \) be such that
\[ \Exp^\beta (t) = \begin{cases} \frac{t_0}{\beta} (e^{t_0} - 1) + 1 & \text{for} \ t \in [0, t_0) \\ e^{t_0} & \text{for} \ t \in [t_0, \infty) \end{cases}. \]

Of course, \( t_0 = 0 \) if \( \beta \geq 1 \). Moreover,
\[ \lim_{\lambda \to 1^+} \int_{\mathbb{R}^n} \Exp^\beta (|\nabla v_\lambda|) \, d\gamma_n = \lim_{\lambda \to 1^+} \int_{\mathbb{R}} \Exp^\beta (\eta_\lambda) \, d\gamma_1 \]
\[ = \lim_{\lambda \to 1^+} \left( \frac{t_0}{\beta} \int_{\{\eta_\lambda < t_0\}} \eta_\lambda \, d\gamma_1 + \int_{\{\eta_\lambda < t_0\}} \eta_\lambda \, d\gamma_1 + \int_{\{\eta_\lambda \geq t_0\}} e^{\gamma_\lambda} \, d\gamma_1 \right) \]
\[ = \frac{e^{t_0} - 1}{t_0} \int_{\{\phi' < t_0\}} \phi' \, d\gamma_1 + \int_{\{\phi' < t_0\}} d\gamma_1 + \int_{\{\phi' \geq t_0\}} e^{(\phi')^{\beta}} \, d\gamma_1 \]
\[ = \int_{\mathbb{R}^n} \Exp^\beta (\phi') \, d\gamma_1, \]
where the third equality holds thanks to the dominated convergence theorem. By equations (4.11) and (4.15),
\[ \int_{\mathbb{R}^n} \Exp^\beta (|\nabla v_\lambda|) \, d\gamma_n < M \]
provided that \( \lambda \) is sufficiently close to 1. On the other hand, since \( \eta_\lambda(t) > \phi'(t) \geq 0 \) in \( \mathbb{R} \), one has that \( |\psi_\lambda(t)| > |\phi(t)| \) for \( t \in \mathbb{R} \setminus \{0\} \), and hence
\[ \int_{\mathbb{R}^n} e^{(\kappa_\beta |v_\lambda|)^\frac{2\beta}{n}} \, d\gamma_n = \int_{\mathbb{R}} e^{(\kappa_\beta |\psi_\lambda|)^\frac{2\beta}{n}} \, d\gamma_1 > \int_{\mathbb{R}} e^{(\kappa_\beta |\phi|)^\frac{2\beta}{n}} \, d\gamma_1 = \int_{\mathbb{R}^n} e^{(\kappa_\beta |u|)^\frac{2\beta}{n}} \, d\gamma_n \]
thanks to (4.12). Altogether, the maximizing property of \( u \) is contradicted.

Next, suppose that \( m(u) \) stands for \( \text{mv}(u) \) in (1.4). Since \( \phi \) is continuous, there exists \( t_0 \in \mathbb{R} \) such that \( \phi(t_0) = \text{mv}(u) = 0 \). Let \( \eta_\lambda \) be as in (4.13), let \( \psi_\lambda \) be defined by
\[ \psi_\lambda(t) = \int_{t_0}^t \eta_\lambda(\tau) \, d\tau \quad \text{for} \ t \in \mathbb{R}, \]
and let $v_\lambda$ be the function associated with $\psi_\lambda$ as in (4.14). Since equation (4.15) continues to hold, one can choose $\lambda > 1$ such that equation (4.16) is fulfilled. If $mv(\psi_\lambda) = 0$, then we obtain a contradiction as above. Assume that, instead, $mv(\psi_\lambda) \neq 0$, say $mv(\psi_\lambda) > 0$, to fix ideas. Therefore,
\[
\int_{t_0}^{\infty} \psi_\lambda \, d\gamma_1 > -\int_{-\infty}^{t_0} \psi_\lambda \, d\gamma_1.
\]
Consequently, there exists $\theta \in (1, \lambda)$ such that, on defining $\eta_{\lambda, \theta}: \mathbb{R} \to [0, \infty)$ as
\[
\eta_{\lambda, \theta}(t) = \begin{cases} 
(\phi'(t)^\beta + \log \theta)^{\frac{1}{\beta}} & \text{if } t \geq t_0 \\
(\phi'(t)^\beta + \log \lambda)^{\frac{1}{\beta}} & \text{if } t < t_0,
\end{cases}
\]
and $\psi_{\lambda, \theta}: \mathbb{R} \to \mathbb{R}$ as
\[
\psi_{\lambda, \theta}(t) = \int_{t_0}^{t} \eta_{\lambda, \theta}(\tau) \, d\tau \quad \text{for } t \in \mathbb{R},
\]
one has that $mv(\psi_{\lambda, \theta}) = 0$. If the function $v_{\lambda, \theta}$ is defined as in (4.14), with $\psi_\lambda$ replaced by $\psi_{\lambda, \theta}$, then equations (4.16) and (4.17) still hold, with $v_\lambda$ and $\psi_\lambda$ replaced by $v_{\lambda, \theta}$ and $\psi_{\lambda, \theta}$. The maximizing property of $u$ is thus contradicted also in this case. \(\square\)

**Acknowledgment.** We wish to thank the referee for their careful reading of the paper, and for their valuable comments.

**Compliance with Ethical Standards**

**Funding.** This research was partly funded by:

(i) Research Project 201758MTR2 of the Italian Ministry of University and Research (MIUR) Prin 2017 “Direct and inverse problems for partial differential equations: theoretical aspects and applications”;

(ii) GNAMPA of the Italian INdAM – National Institute of High Mathematics (grant number not available);

(iii) Grant P201-18-00580S of the Czech Science Foundation.

**Conflict of Interest.** The authors declare that they have no conflict of interest.

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