Energy and directional signatures
for plane quantized gravity waves

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Abstract

Solutions are constructed to the quantum constraints for planar
gavity (fields dependent on z and t only) in the Ashtekar complex
connection formalism. A number of operators are constructed and
applied to the solutions. These include the familiar ADM energy and
area operators, as well as new operators sensitive to directionality
(z+ct vs. z-ct dependence). The directionality operators are quan-
tum analogs of the classical constraints proposed for unidirection-
al plane waves by Bondi, Pirani, and Robinson (BPR). It is argued
that the quantum BPR constraints will predict unidirectionality reli-
ably only for solutions which are semiclassical in a certain sense.
The ADM energy and area operators are likely to have imaginary
eigenvalues, unless one either shifts to a real connection, or allows
the connection to occur other than in a holonomy. In classical the-
ory, the area can evolve to zero. A quantum mechanical mechanism
is proposed which would prevent this collapse.

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I Introduction: Classical Radiation Criteria

The connection-triad variables introduced by Ashtekar have sim-
plified the constraint equations of quantum gravity; further, these

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variables suggest that in the future we may be able to reformulate gravity in terms of non-local holonomies rather than local field operators \[\text{(2, 3, 4)}\]. However, the new variables are unfamiliar, and it is not always clear what they mean physically and geometrically. In particular, it is not clear what operators or structures correspond to gravity waves. Although the quantum constraint equations are much simpler in the new variables, and solutions to these equations have been found \[\text{(2, 3, 4)}\], it is not clear whether any of these solutions contain gravitational radiation.

This is the fifth of a series of papers which search for operator signatures for gravitational radiation by applying the Ashtekar formalism to the problem of plane gravitational waves. Paper I in the series \[\text{(4)}\] constructed classical constants of the motion for the plane wave case, using the more familiar geometrodynamics rather than Ashtekar connection dynamics. Papers II and III switched to connection dynamics and proposed solutions to the quantum constraints \[\text{(3, 4)}\]. The constraints annihilate the solutions of II except at boundary points, and annihilate the solutions of III everywhere. Paper IV constructs an operator \(L_z\) which measures total intrinsic spin around the z axis \[\text{(10)}\]. The present paper proposes operator signatures which are sensitive to the directionality of gravitational radiation \((z-ct \text{ vs. } z+ct \text{ dependence})\) and applies those operators (as well as the spin, energy, and area operators) to the solutions constructed in II and III.

It is not easy to detect the presence of radiation, even when the problem is formulated classically, using the more familiar metric variables. One would like to define gravitational radiation using an energy criterion \((\text{as: radiation is a means of transporting energy through empty space ...})\). Gravitational energy is notoriously difficult to define, however, since there is no first-order-in-derivatives-of-the-metric quantity which is a tensor. Accordingly, in the period 1960-1970 several authors developed an algebraic criterion involving transverse components of a second-order quantity, the Weyl tensor \([11, 12, 13, 14]\). To use the criterion, one needs to know which directions are “transverse”; hence the criterion is most useful when the direction of propagation is clear from the symmetry: e.g. radial propagation (for spherical symmetry) or z-axis propagation (for...
planar symmetry, the case studied in the present paper). The planar metrics considered here \[13, 16\] admit two null vectors \(k\) and \(l\) which have the right hypersurface orthogonality properties to be the propagation vectors for right-moving (\(k\)) and left-moving (\(l\)) gravitational waves along the \(z\) axis, so that the propagation direction is especially easy to identify. The Weyl criterion is derived and discussed in Appendix D.

A second, more group-theoretical criterion was developed by Bondi, Pirani, and Robinson (BPR) \[17\]. It is applicable when the plane wave is unidirectional, that is, when the wave is either right-moving (depending only on \(z-ct\)) or left-moving (depending only on \(z+ct\)). The unidirectional case is especially intriguing. It is relatively simple, since no scattering occurs \[18, 19\]. Nevertheless the full complexity of gravity is already present; the unidirectional case is not simply waves propagating in an inert background. In particular, no one has been able to cast the Hamiltonian into a free-field form in terms of variables \((\pi_i, q^j)\) which commute in the canonical \([\pi_i, q^j] = -i\hbar\delta^j_i\) manner characteristic of non-interacting, non-gravitational theories. Also, the BPR criterion for unidirectional radiation requires three amplitudes to vanish, rather than the two one would naively expect from counting the two polarizations associated with unidirectional radiation. I shall argue that the remaining vanishing amplitude represents a constraint on the background geometry, a constraint which must be satisfied in order for the waves to propagate without backscattering off the background. The BPR group is derived and discussed in section II.

Note that one criterion, that based on the Weyl tensor, is relegated to an appendix; while the BPR criterion is discussed in the body of the paper. I have done this primarily because the BPR amplitudes are much simpler than the Weyl amplitudes. It is not possible to ignore the Weyl amplitudes completely, however; they are central to the literature of the 60’s. Further, expressions which appear complex at one time may appear simple at a later time. At one time the traditional scalar constraint was thought to be too complex because it contained a factor of \(1/\sqrt{g}\). Then Thiemann proposed a regularization of this constraint which actually requires that factor \[20\]. Similarly, the present ”complexity” of the Weyl
amplitudes may disappear once a quantum regularization is constructed.

It is of some interest to reexpress the two classical criteria, Weyl and BPR, in the Ashtekar language, even if one does not go on to consider the quantum case. However, one would really like to construct from each classical criterion a corresponding quantum operator. I construct such operators in section III and argue that these operators are reliable only when acting on wavefunctionals for states which are semiclassical, in a sense to be defined in section III.

In section IV, I construct additional solutions to the quantum constraints. In section V, I apply the BPR quantum operators to the wavefunctional solutions obtained in papers II-III, as well as the new solutions constructed in section IV. Also in section V, I apply the ADM energy operator to the solutions, as well as the area operator and the operator $L_Z$ for total intrinsic spin.

There are three appendices. Two of the appendices (A and C) cover calculational details and the details of the Weyl criterion not used in the body of the paper. Appendix B considers the ADM energy. There is a modest surprise here: normally the ADM energy is considered to be given by the surface term in the Hamiltonian; but in the quantum case it is possible for the volume term to contribute also.

My notation is typical of papers based upon the Hamiltonian approach with concomitant 3 + 1 splitup. Upper case indices $A, B, \ldots, I, J, K, \ldots$ denote local Lorentz indices ("internal" SU(2) indices) ranging over $X, Y, Z$ only. Lower case indices $a, b, \ldots, i, j, \ldots$ are also three-dimensional and denote global coordinates on the three-manifold. Occasionally the formula will contain a field with a superscript $(4)$, in which case the local Lorentz indices range over $X, Y, Z, T$ and the global indices are similarly four-dimensional; or a $(2)$, in which case the local indices range over $X, Y$ (and global indices over $x, y$) only. The $(2)$ and $(4)$ are also used in conjunction with determinants; e. g., $g$ is the usual 3x3 spatial determinant, while $(2)e$ denotes the determinant of the 2x2 $X, Y$ subblock of the triad matrix $e^A_a$. I use Levi- Civita symbols of various dimensions: $\epsilon_{TXYZ} = \epsilon_{XYZ} = \epsilon_{XY} = +1$. The basic variables of the Ashtekar approach are an inverse densitized triad $\tilde{E}^a_A$ and a complex SU(2)
connection $A_a^A$.

$$\tilde{E}_a^a = e\epsilon_a^a;$$  \hspace{1cm} (1)

$$[\tilde{E}_a^a, A_b^B] = \hbar\delta(x - x')\delta_A^B\delta_a^b.$$ \hspace{1cm} (2)

The planar symmetry (two spacelike commuting Killing vectors, $\partial_x$ and $\partial_y$ in appropriate coordinates) allows Husain and Smolin [21] to solve and eliminate four constraints (the x and y vector constraint and the X and Y Gauss constraint) and correspondingly eliminate four pairs of $(\tilde{E}_a^a, A_a^A)$ components. The 3x3 $\tilde{E}_a^a$ matrix then assumes a block diagonal form, with one 1x1 subblock occupied by $E_z^a$ plus one 2x2 subblock which contains all the “transverse” $E_a^a$, that is, those with $a = x, y$ and $A = X, Y$. The 3x3 matrix of connections $A_a^A$ assumes a similar block diagonal form. None of the surviving fields depends on $x$ or $y$.

The local Lorentz indices are vector rather than spinor; strictly speaking the internal symmetry is O(3) rather than SU(2), gauge-fixed to O(2) rather than U(1). Often it is convenient to shift to transverse fields which are eigenstates of the surviving gauge invariance O(2):

$$\tilde{E}_a^\pm = [\tilde{E}_a^x \pm i\tilde{E}_a^y]/\sqrt{2},$$ \hspace{1cm} (3)

where $a = x, y$; and similarly for $A_a^\pm$.

In papers I-III I use the letter H to denote a constraint (scalar, vector, or Gauss). In the present paper I adopt what is becoming a more common convention in the literature and use the letter C to denote a constraint, while reserving the letter H for the Hamiltonian. The quantity denoted $C_S$ in the present paper is identical to the constraint denoted $H_S$ in papers II-III. This convention underscores the fact that every gravitational theory has constraints, but not every gravitational theory has a Hamiltonian.

In three spatial dimensions it is usual to place the boundary surface at spatial infinity. Bringing the surface at infinity in to finite points is a major change, because at infinity the metric goes over to flat space, and flat space is a considerable simplification. In the present case (effectively one dimensional because of the planar symmetry) the space does not become flat at $z$ goes to infinity, and nothing is lost by considering an arbitrary location for the boundary surface. The “surface” in one dimension is of course just two points.
The two endpoints of a segment of the $z$ axis. The notation $z_b$ denotes either the left or right boundary point $z_l$ or $z_r$, $z_l \leq z \leq z_r$. The result that the space does not become flat as $z$ goes to infinity was established in paper II. Note that this result agrees with one’s intuition from Newtonian gravity, where the potential in one spatial dimension due to a bounded source does not fall off, but grows as $z$ at large $z$.

If a certain solution does not satisfy the Gauss constraint (or other constraint) at the boundary, this does not mean that necessarily there is something wrong with the solution. In classical theory the solutions satisfy the constraints everywhere. In quantum theory, however, when the constraints are imposed after quantization, in the Dirac manner, it is only necessary that the smeared constraint annihilate the solution:

$$\int dz \delta N(z) C(z) \psi = 0. \quad (4)$$

The expression $\delta N C$ generates a small change $\delta N$ in the Lagrange multiplier $N$. If $N$ obeys a boundary condition of the form $N \rightarrow$ constant at boundaries $z_b$, then eq. (4) must respect this boundary condition, which means

$$\delta N(z_b) = 0. \quad (5)$$

Eq. (4) implies that $C(z_b)\psi$ does not have to vanish. A statement that ”this solution does not obey the constraint at the boundaries” does not mean necessarily that the solution is flawed.

II Bondi-Pirani-Robinson Symmetry

Bondi, Pirani, and Robinson (BPR) argue that the metric of a unidirectional plane gravitational wave should be invariant under a five-parameter group of symmetries. Their argument proceeds essentially as follows. First they point out that a plane electromagnetic wave moving in the $+z$ direction is invariant under a five parameter group. (Besides the obvious $\partial_x$, $\partial_y$, and $\partial_t$ symmetries, there are two ”null rotations” which rotate the $v = (t + z)/\sqrt{2}$ direction into the $x$ or $y$ direction.) Then for gravitational plane waves they construct five Killing vectors which have the same Lie algebra as the
corresponding Killing vectors for the electromagnetic case. (More precisely they construct ten Killing vectors, one set of five for ct+z waves, and a similar set of five for ct-z waves.)

This section constructs the five ct-z vectors, then imposes the usual symmetry requirement that the Lie derivative of the basic Ashtekar fields must vanish in the direction of the Killing vectors. In this way one finds that the fields must obey certain constraints; the section closes with a discussion of the physical meaning of these constraints in the classical theory.

It is convenient to do the proofs in a gauge which has been simplified as much as possible using the \( \partial_v \) symmetry, then afterwards transform the results to a general gauge. The plane wave metrics we consider here possess two hypersurface orthogonal null vectors; if the two hypersurfaces are labeled \( u = \) constant and \( v = \) constant (\( u \) and \( v = (ct \pm z)/\sqrt{2} \)), then one can always transform the metric to a conformally flat form in the (z,t) sector by using \( u \) and \( v \) as coordinates [16]:

\[
\text{ds}^2 = -2dudvf(u,v) + \sum_{ab} g^{ab} dx^a dx^b. \tag{6}
\]

The sums over \( a, b, c, \ldots \) extend over \( x, y \) only. If one now invokes the symmetry under \( \partial_v \), then \( f(u,v) \) depends on \( u \) only, and one can remove the function \( f \) by transforming to a new \( u \) coordinate. In this gauge, Rosen gauge [22], the metric is (not just conformally flat, but) flat in the (z,t) sector and non-trivial only in the (x,y) sector.

\[
\text{ds}^2 = -2dudv + \sum_{ab} g^{ab} dx^a dx^b. \tag{7}
\]

In Rosen gauge, the Killing vectors are \( \partial_x, \partial_y, \partial_v, \) and

\[
\xi^{(c)\lambda} = x^c \delta^\lambda_v + \int^u g^{cd}(u') du' \delta^\lambda_4. \tag{8}
\]

The constraints imposed by the first three \( \partial_x, \partial_y, \) and \( \partial_v \) Killing vectors are satisfied already, because of the choice of gauge. I now work out the constraints which the last two Killing vectors, eq. (8), impose on the Ashtekar variables (in Rosen gauge first, than in a general gauge). I summarize the highlights of the calculation in this section, and move the algebraic details to appendix A.
It is necessary to calculate the symmetry constraints on the four-dimensional tetrads and Ashtekar connections first, since the Killing vectors are intrinsically four-dimensional, then carry out a 3+1 decomposition to obtain the constraints on the usual three-dimensional densitized triad and connection. At the four-dimensional level, the three local Lorentz boosts have been gauge-fixed by demanding that three of the tetrads vanish:

\[ e^I_M = 0, M = \text{space}. \]  \hspace{1cm} (9)

The gauge condition of eq. (9) is the standard choice, used with all metrics [23]. In addition, for the special case of the plane wave metric, the gauge-fixing of the XY Gauss constraint and xy spatial diffeomorphism constraints imply that four more tetrads vanish [21].

\[ e^X_X = e^Y_Y = e^Z_Z = e^Z_Y = 0. \]  \hspace{1cm} (10)

At the four-dimensional level, the requirement of vanishing Lie derivative in the direction of the Killing vector gives

\[ 0 = \xi^\lambda \partial_\lambda e_\alpha^I - \partial_\beta \xi^\alpha e_\beta^I - L^I e_\alpha^I; \]  \hspace{1cm} (11)

\[ 0 = \xi^\lambda \partial_\lambda (4) A_{\alpha}^{I} + \partial_\alpha \xi^\lambda (4) A_{\lambda}^{I} + \mathcal{L}^{I}_{-} (4) A_{\alpha}^{I} + \mathcal{L}^{I}_{+} (4) A_{\lambda}^{I} - \partial_\alpha \mathcal{L}^{I}. \]  \hspace{1cm} (12)

These equations are not quite the usual Lie derivatives because of the the \( L \) and \( \mathcal{L} \) terms. \( L \) and \( \mathcal{L} \) are local Lorentz transformations. If \( \xi = \partial_\lambda \), \( \partial_\gamma \), or \( \partial_\nu \), no \( L \) or \( \mathcal{L} \) is required. If \( \xi = \xi^{}(c) \), one of the two Killing vectors defined at eq. (8), then a Lorentz transformation \( L \) is required in eq. (11); otherwise the symmetry destroys the gauge conditions, eqs. (8) and (10). Then for consistency, \( (4) A \) in eq. (12) must undergo the same Lorentz transformation; since \( (4) A \) is self-dual, the Lorentz transformation \( \mathcal{L} \) in eq. (12) must be the self-dual version of the Lorentz transformation \( L \):

\[ 2 \mathcal{L}^{I} = L^{I} + i \delta^{I}_{-} (e^{I}_{MN} / 2 \epsilon_{TXYZ}) L^{MN}. \]  \hspace{1cm} (13)

The phase \( \delta / \epsilon_{TXYZ} = \pm 1 \) is the duality eigenvalue which determines whether the theory is self-dual or anti-self-dual. Because I include the extra factor of \( \epsilon_{TXYZ} \), eq. (13) contains two factors of\( \epsilon \), so is independent of one’s choice of phase for this quantity. After the
four-dimensional theory is rewritten in 3+1 form, all results will depend only on $\delta$. In the body of the paper I choose $\delta = +1$, but appendix A indicates what happens for the opposite choice $\delta = -1$.

It is a straightforward matter to determine the Lorentz transformation $L$ which will preserve the gauge conditions of eqs. (9) and (10), then solve eqs. (11) and (12). This is done in appendix A. From eq. (A23) in that appendix, eqs. (11) and (12) imply the following constraints on the connection $A$.

\[
0 = A_a^-; \quad 0 = -A_a^+ + 2 \text{"Re"} A_a^+,
\]
(right-moving) \hspace{1cm} (14)

where $a = x,y$ only. The connection $A$ is now the usual 3+1 connection, not the four-dimensional connection $(4)A$. Re $A_a^X$ without the quotes is the usual real part, containing no factors of $i$, while

\[
\text{“Re”}A_a^+ \equiv (\text{Re}A_a^X + i\text{Re}A_a^Y)/\sqrt{2}.
\]

(15)

“Re” $A$ contains a factor of $i$, because of the $i$ in the definition of the $(X \pm iY)$ $O(2)$ eigenstates, and is no longer real. If one writes out the “Re” and “Im” parts of $A_a^\pm$ it is easy to see that the two constraints of eq. (14) are just the complex conjugates of each other. To obtain the constraints for left-moving waves, interchange + and - in eq. (14).

Eq. (14) can be interpreted physically by using the classical equations of motion to prove theorems about the spin behavior of the BPR fields. Again, the required calculations are done in an appendix (appendix B); and this section summarizes the main conclusions.

To interpret the spin content of the four amplitudes which vanish, it is better to work with the four combinations $E_a^A A_a^- \text{ and } E_b^B [-A_b^+ + 2 \text{“Re”} A_b^+]$. (One can always recover the original four amplitudes from these four, because one can always invert the 2x2 matrix formed from the transverse components of the densitized triad, $E_b^B$ with $B = X,Y$ and $b = x,y$.) Eq. (B4) of appendix B expresses the total spin angular momentum $L_Z$ of the gravitational wave in terms of these combinations.

\[
L_Z = i \int dz \{e_+^y e_+^x \tilde{E}_-^a [A_a^- + (A_a^- - 2\text{Re}A_a^-)]
\]

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\[ + e^y e^{-z} \tilde{E}_z^a \{ A^+_a + (A^+_a - 2 \text{Re} A^+_a) \} - (x \leftrightarrow y), \]  

(16)

where \( e_{Aa} \) and \( e^a_A \) are triad and inverse triad fields, respectively. Out of the four possible combinations \( \tilde{E}^a_A A^+_a \) and \( \tilde{E}^a_B [-A^+_a + 2 \text{ "Re" } A^+_a] \), only two combinations \( \tilde{E}^a_A A^-_a \) and \( \tilde{E}^a_A [-A^+_a + 2 \text{ "Re" } A^+_a] \) contribute to the spin angular momentum. Since these are the two amplitudes with \( O(2) \) helicity \( \pm 2 \) in the local Lorentz frame, it is natural to interpret \( \tilde{E}^a_A A^\pm_a \) as an amplitude for a wave having helicity \( \pm 2 \). Both helicity \( \pm 2 \) combinations must vanish in order to eliminate the two polarizations moving in the \( ct + z \) direction.

Two more, helicity zero combinations, \( \tilde{E}^a_A A^-_a \) and \( \tilde{E}^a_A [-A^+_a + 2 \text{ "Re" } A^+_a] \), also contain the fields of eq. (14). How does one interpret these helicity zero amplitudes?

Using the Gauss constraint plus the classical equations of motion in a conformally flat gauge, one can prove that these two constraints collapse to a single constraint, eq. (B10) of Appendix B.

\[ 0 = (\partial_t + \partial_z) \tilde{E}_Z^z. \]  

(17)

\( \tilde{E}_Z^z = e^z_a e = (2)e \), where \( (2)e \) is the determinant of the 2x2 transverse sector of the triad matrix, a scalar function of \( (z,t) \). This function would seem to characterize the background geometry, rather than the wave. In general relativity, however, “background” and wave are inseparable, in the sense that the “background” is not inert. The wave will scatter off the background, in general, unless it obeys the constraint given by eq. (17).

This completes the survey of the constraints predicted by BPR symmetry, and their physical interpretation in the classical context. In the next section these classical expressions are promoted to quantum operators.

### III The Transition from Classical to Quantum Criterion

This section lists four issues which arise when converting a classical expression into a quantum criterion. I summarize and discuss each issue, then show the application to the BPR criteria.
A  Factor ordering and regularization

I choose a factor ordering which is natural and simple within the complex connection formalism. The ordering is “functional derivatives to the right” [24]. That is, I quantize in a standard manner, by replacing one half the fields by functional derivatives,

\[ \tilde{E}_Z^a \rightarrow -\hbar \delta / \delta A_z^Z; \]
\[ A_a^A \rightarrow +\hbar \delta / \delta \tilde{E}_A^a, \]

(for \( a = x,y \) and \( A = X,Y \)),

\( (18) \)

and then order the functional derivatives to the right in every operator or constraint. Regularization, if needed, is via point splitting [21]. This approach has the virtue of consistency, since I have used it in two previous papers on quantization of plane waves.

B  Semiclassicality

An example from QED will be helpful in explaining what is meant by semiclassicality. The BPR criteria are essentially field strengths for waves moving in a given direction with a given polarization. An analogous quantity from flat space QED is

\[ F \equiv F^\mu^\nu m_\mu k_\nu \]

Here \( F^\mu^\nu \) is the self-dual QED field strength and \( (k,l,m,\bar{m}) \) is the usual flat space null tetrad: \( k \) and \( l \) are null vectors with space components along \( \pm z \); \( m \) and \( \bar{m} \) are transverse polarization vectors along \( (\hat{x} \pm i\hat{y})/\sqrt{2} \). Classically, the criterion for absence of radiation along \( k \) with polarization \( \bar{m} \) is \( F = 0 \). The corresponding quantum criterion for absence of radiation, obtained after replacing classical \( A \) fields by quantum operators \( \hat{A} \) is not

\[ \hat{F}_\psi = 0, \text{ (wrong)} \quad (20) \]

but rather

\[ \langle \text{seicl} | \hat{F} | \text{seicl} \rangle = 0. \quad (21) \]

Since the QED field strength \( \hat{F} \) contains creation as well as annihilation operators, it cannot annihilate any state, and eq. (20) is too
strong. The classical statement $\mathcal{F} = 0$ merely implies the existence of a corresponding semiclassical state such that eq. (21) holds. I have deliberately used the term “semiclassical” rather than “coherent’ to describe the state in eq. (21), because the latter term conventionally denotes a state which is an eigenfunction of the annihilation operator, and annihilation operators usually are not available in quantum gravity. While annihilation operators may not exist, certainly semiclassical states will, because of the correspondence principle.

Clearly the criterion eq. (21) is more difficult to apply than eq. (20), but let us survey the damage; the situation may not be hopeless. To define “semiclassical” without invoking coherent states or annihilation operators, one can study

$$\hat{F}|\text{semicl}\rangle = f(z)|\text{semicl}\rangle + |\text{remainder}\rangle,$$

(22)

where $\hat{F}$ stands for a typical BPR field. $|\text{semicl}\rangle$ is normed to unity, although $|\text{remainder}\rangle$ need not be. I define semiclassical, not by requiring $f(z)$ to be large, but rather by requiring the $|\text{remainder}\rangle$ to be small. If I require $f(z)$ to be large (perhaps reasoning that “classical” means large quantum numbers) then I exclude the vanishing amplitude case, $f(z) = 0$, where $|\text{semicl}\rangle$ is the vacuum with respect to a given radiation mode. The vacuum is a well-defined state, classically, and one expects it to have a quantum analog. For $|\text{semicl}\rangle$ to approximate a classical state, therefore, it is not necessary that $f(z)$ be large; only that the fluctuations away from this state be small. These fluctuations are measured by

$$\langle \hat{F} \hat{F} \rangle - \langle \hat{F} \rangle \langle \hat{F} \rangle = \langle \text{remainder} | \text{remainder} \rangle \leq (l_p/l)^2/l^2.$$

(23)

Here the fluctuations are assumed to be small compared to the size of typical matrix elements $\langle f|\hat{F}|i\rangle \approx l_p/l^2$ one gets when $|i\rangle$ and $|f\rangle$ are few-graviton states and $\hat{F}$ is a canonical degree of freedom in the linearized theory. $l_p$ is the Planck length, and $l$ is a typical length or wavelength. (To modify this discussion for the Weyl fields of appendix D, which are classical dimension $1/l^2$, replace $l_p/l^2$ by $l_p/l^3$. From now on $\langle \rangle$ is understood to indicate an average over a semiclassical state, unless explicitly indicated otherwise.)

One might ask what is meant by a typical length $l$, in a quantized and diffeomorphism invariant theory where no background metric is
available. In such a theory, even in the absence of a background metric, length, area, and volume operators can be defined \([25, 26, 20]\), and the eigenvalues of these geometric operators are dimensionless functions of spins \(j_i\), times factors of \(l_p\) to give the correct dimension. The spins label the irreducible representations of SU(2) associated with each holonomy (if the wavefunctional is in connection representation, a product of holonomies) or associated with each edge of a spin network (if the wavefunctional is a spin network state). Thus one expects \(l = l_0(j_i)l_p\), \(l_0\) dimensionless and \(\gg 1\). (In the planar case, SU(2) is gauge-fixed to O(2) and presumably the SU(2) eigenvalues \(j\) will be replaced by the O(2) eigenvalues \(m = \text{spin angular momentum component along } z\).) Evidently, then, to check that eq. (23) is satisfied, one should apply the geometric operators to the state first, in order to estimate \(l\). One also needs the measure in Hilbert space; but in favorable situations one might be able to tell that \(|\text{remainder}\rangle\) is small simply by inspection of eq. (22).

Since the criterion of eq. (23) is an inequality, it cannot be used to draw sharp distinctions between states. For example, in the linearized limit, if \(|N\rangle\) denotes an eigenstate of the number operator having \(N\) quanta of a given polarization and direction, then \(f(z)\) will be zero, while the norm eq. (23) will be order \(N (l_p/l)^2/l^2\). There will be uncertainties in estimating \(l\), so that the criterion cannot distinguish sharply between the vacuum state and a number eigenstate having small occupation number \(N\).

C Non-polynomiality

The BPR operators, eq. (4), occur in complex conjugate pairs, and one member of the BPR pair involves \(\text{Re } A^A_a\), which is a known, but non-polynomial function of \(\tilde{E}^a_A\). In particular, \(\text{Re } A^A_a\) contains factors of \(1/(2)\tilde{E}\), where \(\tilde{E}\) is the 2x2 determinant formed from the \(\tilde{E}^a_A\) with internal indices \(A = X,Y\) and global indices \(a = x,y\). I have dealt with a similar operator, \(1/\tilde{E}^2_Z\), in a previous paper; but would just as soon not do so here.

One can use the fact that the BPR constraints come in complex conjugate pairs, plus semiclassicality, to prove the following theorem (and then one uses the theorem to avoid dealing with the non-
polynomiality). Theorem:

\[ \langle -A_a^+ + 2\text{Re}A_a^+ \rangle = \langle A_a^- \rangle^*, \quad (24) \]

and similarly for the other BPR pair. This result is just what one would expect from the corresponding result for expectation values of complex operators in ordinary quantum mechanics (for instance \( \langle p+iq \rangle^* = \langle p-iq \rangle \)) except that here the basic operators are not Hermitean, so that the proof is slightly longer. Proof: Expand out the \( A_a^\pm \) operators using

\[
A_a^\pm = \frac{(A_a^X \pm iA_a^Y)}{\sqrt{2}} = \frac{\hbar(\delta/\delta \tilde{E}_a^X \pm i\delta/\delta \tilde{E}_a^Y)}{\sqrt{2}}. \quad (25)
\]

Integrate by parts the functional derivative on the left side of eq. (24), using

\[
\int \mu \psi^* \hbar \delta \psi / \delta \tilde{E}_a^A = \int \mu [(-\hbar)\delta / \delta \tilde{E}_a^A \psi^*] \psi + \int \mu \psi^* \text{Re} A_a^A \psi. \quad (26)
\]

Here \( \psi \) is the semiclassical state and \( \int \mu \) is the measure, a path integral over the fields in \( \psi \). \( \mu \) need not be known in detail, except that it enforces the reality condition in eq. (26) (via \( (-\hbar)\delta \mu / \delta \tilde{E}_a^A = 2\text{Re} A_a^A \mu \)). Also, \( \mu \) must be real (\( \mu^* = \mu \)) in order for norms to be real. If one inserts the relations eq. (26) on the left in eq. (24) and carries out the complex conjugation (using \( \mu^* = \mu \)); the result is the right-hand side of eq. (24).

\[ \blacksquare \]

D Kinematic vs. physical operators

I continue with the comparison of the BPR operators to field strengths in QED. In QED, operators representing observables should involve only true, rather than gauge degrees of freedom, therefore should commute weakly with the smeared Gauss constraint. Similarly, in quantum gravity, observables should commute with the constraints. The BPR operators do not do this.

An operator will be said to be kinematic if it commutes only with the Gauss and diffeomorphism constraints; and physical if it commutes with all the constraints, including the scalar constraint. It is desirable for an operator to commute with as many of the Hamiltonian constraints as possible, since the operator is then presumably...
independent of one’s choice of arbitrary coordinates. Physical operators are best of all, but even a diffeomorphism and Gauss invariant operator can furnish valuable information about the physical meaning of a state. The volume, length, and area operators, for example, are only diffeomorphism and Gauss invariant.

I shall show below that the BPR amplitudes for the unidirectional case can be modified so as to make them kinematical. I shall refer to the modified amplitudes as the unidirectional BPR amplitudes, since they continue to refer to excitations moving in a single direction. (That is, it is not necessary to combine a right-moving operator with a left-moving operator to get a kinematical operator.) Further, in both the scattering and unidirectional cases, it is possible to construct additional combinations of the BPR operators which are actually physical, rather than kinematic only; but (except in the linearized limit) these physical combinations no longer refer to excitations moving in a single direction.

From the remarks at the end of section I, in the planar symmetry case only three constraints $\hat{C}_i$ survive gauge fixing: the generator of spatial diffeomorphisms along $z$, the Gauss constraint for internal rotations around $Z$, and the scalar constraint. Consider first $\hat{C}_z$, the diffeomorphism constraint. The BPR fields are local, scalar functions of $(z, t)$, of density weight zero, and therefore do not commute with $\hat{C}_z$. Evidently the BPR fields have unacceptable dependence on the arbitrary coordinate label $z$. The simplest remedy is to get rid of $z$ by integrating the BPR amplitude $\hat{F}$, over $z$, following the prescription that observables in quantum gravity should be non-local; but the density weight of $\hat{F}$ must be unity rather than zero in order for integration to produce a diffeomorphism invariant quantity. One would like a density weight of unity for another reason: operators which are not density weight unity often turn out to be background-metric-dependent when regulated \[27\].

There are two ways to modify the BPR operators to make them density weight unity. (Either) replace

$$A^\pm_a \rightarrow \tilde{E}^\pm_a A^\pm_a;$$
$$-A^\pm_a + 2 \text{Re}A^\mp_a \rightarrow \tilde{E}^\pm_a [-A^\mp_a + 2 \text{Re}A^\mp_a]; a = x, y \text{ only;}$$

(27)
(or) replace

\[ A_b^B \rightarrow \tilde{E}_B^a A_b^B, \]

\[ -A_b^B + 2\text{Re}A_b^B \rightarrow \tilde{E}_B^b[-A_b^B + 2\text{Re}A_b^B]; a,b = x,y \text{ only}; \]

(physical) (28)

Classically, both these transformations are invertible. (Recall that the \( \tilde{E}_a^A \) matrix is block diagonal, with one 2x2 subblock containing \( a = x,y \), and \( A = X,Y \text{ or } +,- \) only.) Both sets of new operators contain the same information as the old, therefore; and I can adopt either set as the new quantum BRS criterion (after integration over \( dz \) and inclusion of appropriate holonomies to secure gauge invariance; see below). The first set is easier to interpret, since each operator contains only \( A_a^+ \) or only \( A_a^- \) but not both, hence the name “unidirectional”: each operator refers to either left- or right-moving excitations, as do the original BPR operators. Each operator in the second set is a mix of left- and right-moving operators. The first set, though not gauge invariant, can be made kinematic by sandwiching between appropriate holonomies and integrating over \( z \). The second set needs only an integration over \( z \) to make it (not only kinematic, but also) physical. I now verify the foregoing two statements in detail.

The first set can be made Gauss invariant by sandwiching each operator between holonomies. Since the internal rotations around \( X \) and \( Y \) are fixed, and only rotations around \( Z \) remain, the gauge group is \( U(1) \) rather than \( SU(2) \). The irreducible representations are one-dimensional and labeled by a single integer or half-integer, the spin along \( Z \). In eq. (27) some of the new unidirectional BPR fields have spin \( \pm + \pm' = 0 \), and are already Gauss invariant, while others have spin \( \pm + \pm' = \pm2 \); only the latter fields are non-invariant and need to be sandwiched between holonomies, for example

\[ .\tilde{E}_a^a A_a^+ \rightarrow \int dz M(z_\ell, z) \tilde{E}_a^a(z) S_- A_a^+ S_- \times M(z, z_l) M(z_l, z_r) \]

(unidirectional), (29)
where $M$ is a holonomy along the $z$ axis,

$$M(z_2, z_1) = \exp[i \int_{z_1}^{z_2} S_z A_z^Z(z') dz']. \quad (30)$$

$S_{\pm}, S_z$ are the usual Hermitian $SU(2)$ generators. In eq. (29) the integration over $z$ has been added to enforce spatial diffeomorphism invariance. If $M(z_2, z_1)$ is pictured as a flux line running from $z_1$ to $z_2$, then eq. (29) contains two parallel flux lines forming a hairpin contour, with the open end at the right-hand boundary $z_r$: reading from left to right, each flux line starts at $z_1$, runs to $E_+^a A_a^+ +$, then to $z_1$, the closed end of the hairpin; then the flux line turns around and returns to $z_r$. The flux line must be pictured as “open” at the $z_r$ end, because the $S_{-} S_{-}$ in the middle of the contour changes the eigenvalue of the $S_z$ in the holonomies. For the ingoing holonomy $M(z_r, z)$, $S_z$ in eq. (30) is evaluated at one eigenvalue, $S_z = m_z$; while for the return holonomy $M(z_{r}, z_1)$, $S_z$ is evaluated at a different eigenvalue $S_z = m_z + 2$. At the open end of the flux line, the $m_z$ values do not match, and there is no possibility of taking a trace.

The operators of eq. (29) are gauge invariant, despite the absence of a trace and the presence of open flux lines at $z = z_r$. (This is the point in the argument where one uses the assumption of unidirectionality.) The wave is travelling toward $z_r$ but has not yet reached that point; therefore the boundary condition at $z_r$ is flat space. Internal Gauss rotations must respect this boundary condition, which means the triad at $z_r$ cannot be rotated. Technically, the Gauss constraint $C_G$ must be smeared, $\int \Lambda(z) C_G(z) dz$; and the smeared constraint commutes with the operators in eq. (29) at $z = z_r$, because $\lim_{z \to z_r} \Lambda(z) = 0$.

It is straightforward to rewrite the holonomies as exponentials to the $m_z$ or $m_z + 2$ power, and convince oneself that the value of the constant $m_z$ does not matter. Also one can use any $(2j+1) \times (2j+1)$ dimensional representation of the matrices $S_{-}, S_z$, and the value of $j$ does not matter: changing $j$ merely changes the operators in eq. (29) by constants.

One can also check that the criterion of eq. (28) is physical, after integration over $dz$.

$$G^a_b = \int dz E^a_A A^B_b. \quad (31)$$
(Husain and Smolin \[21\] use the notation $K^a_b$ for these operators, but it is perhaps better to reserve the letter $K$ for extrinsic curvature.)

For the most part the check is a straightforward computation of the commutator between the criterion and the scalar constraint; and the proof has been done already by Husain and Smolin. I will discuss only the one step which involves a slight subtlety, an integration by parts. (This step is trivial in the case studied by Husain and Smolin, since their $z$ axis has the topology of a circle and there are no surface terms.)

One starts from the integrated scalar constraint
\[
\int dz' \delta \tilde{N}(z') \hat{C}_S(z'),
\]
where
\[
\hat{C}_S = \tilde{E}_C \tilde{E}_D^{\alpha \beta} \varepsilon_{\alpha \beta}^{\gamma \delta} A^M A^N \varepsilon_{\gamma \delta}^{\mu \nu} \varepsilon_{\mu \nu} A^P / 4 + \varepsilon_{CD} \tilde{E}_C F^{D}_{zc} \tilde{E}_Z^{zc}.
\] (32)

$\varepsilon_{cd}$ is the two dimensional constant Levi Civita symbol ($c,d = x,y$ or $X,Y$ depending on context), and $F$ is the field strength
\[
F^D_{zc} = (\partial_z^D - \varepsilon_{DE} z^E) A^E_c.
\] (33)

$\delta \tilde{N}$ is a small change in the densitized lapse. Since it must leave invariant the coordinate system at the boundaries, it vanishes at boundaries. In II-III I used a scalar constraint which equals eq. (32) divided by a factor of $\tilde{E}_Z^{zc}$, but here I use eq. (32) itself, for simplicity, because factors of $\tilde{E}_Z^{zc}$ do not matter: the operators of eq. (28) commute with $\tilde{E}_Z^{zc}$. The canonical commutator is
\[
[A^A, \tilde{E}_B^{B}] = \hbar \delta^A_B \delta(z - z'),
\] (34)
and one must use this to evaluate
\[
[ \int dz \tilde{E}_B^A, \int dz' \delta \tilde{N}(z') \hat{C}_S(z')].
\] (35)

$\hat{C}_S$, eq. (32), is the sum of two terms, and only the commutator with the second term involves an integration by parts. The second term is proportional to the field strength $F^D_{zc}$, so that the commutator has the form
\[
\int dz dz' \delta \tilde{N}(z') \cdots [\tilde{E}_B^A(z), \tilde{E}_C^C(z')] F^D_{zc}(z')
\]
\[
= \int dz dz' \delta \tilde{N}(z') \cdots \{ \tilde{E}_B^C(z), F^D_{zc}(z') A^A_B + \tilde{E}_B^A A^A_B [\tilde{E}_C^C(z), F^D_{zc}(z')] \}
\]

18
\[
= \int dzdz' \delta \mathcal{N}(z') \quad \cdots \quad \hbar \{ \tilde{E}_C^a(z')[-\partial_z \delta(z - z')A^D_B(z) + \epsilon_{DB}A^Z_zA^B_B] \\
+ \tilde{E}_C^a[\delta(z - z')]F_{2b}^D \}
\]

\[
= \int dzdz' \delta \mathcal{N}(z') \quad \cdots \quad \hbar \{ \tilde{E}_C^a(z')[-F_{2b}^D \delta(z - z')] \\
+ \tilde{E}_C^a[\delta(z - z')]F_{2b}^D \} + \text{ST}
\]

\[
= 0 + \text{ST}, \quad (36)
\]

where \( \text{ST} \) denotes the surface term which arises on converting \(-\partial_z \delta(z - z')\) to \(+\partial_z \delta(z - z')\) and integrating by parts with respect to \(z\).

\[
\text{ST} = \int dz' \delta \mathcal{N}(z') \cdots \{ \tilde{E}_C^a(z')[\delta(z - z')A^D_B(z)]_{z=z_l}^{z=z_t} \}
\]

\[
= 0. \quad (37)
\]

At a first glance one might think that surface terms vanish only when integrating by parts with respect to \(z'\), since the \( \delta \mathcal{N} \) depends on \(z'\), not \(z\). Because of the \( \delta(z - z') \), however, \( \delta \mathcal{N}(z') \) kills surface terms arising from integration by parts with respect to either variable, \(z\) or \(z'\). \( \Box \)

The discussion just given proves that the first constraint \( \int \tilde{E}_B^aA^B_B \) in eq. (28) is physical, but says nothing about its complex conjugate partner \( \int \tilde{E}_B^a[-A^B_B + 2\text{Re}A^B_B] \). In the previous section and in part B of the present section I remarked that the BPR constraints come in complex conjugate pairs, for example,

\[
A^-_a = 0; \quad -A^+_a + 2\text{Re}A^+_a = 0. \quad (38)
\]

The second member of the pair is much harder to work with, because of the non-polynomiality of the \( \text{Re}A^B_a \) factor. This, plus the complex conjugate pairing, suggests that an indirect approach might save labor: do the calculation for the first constraint, then take the Hermitian conjugate to obtain the corresponding result for the harder constraint. For example, commute the first constraint in eq. (28) with \( \dot{C}_i \), then take the Hermitian conjugate of the commutator to obtain the commutator of the second constraint with \( \dot{C}_i \). This indirect approach works when \( \dot{C}_i \) is the Gauss or diffeomorphism constraint, because these two constraints are self-adjoint.
However, the scalar constraint is not self-adjoint for my factor ordering, or indeed, for any of the factor orderings most often encountered in practice when working with a complex connection. Therefore one cannot simply take a Hermitean conjugate to prove that the second constraint in eq. (28) is physical; one must directly work out the commutator of the second constraint with $\hat{C}_S$. I now do this.

Proof that the second constraint of eq. (28) is physical: since the sum of the two constraints of eq. (28) is proportional to $\tilde{E}_B^b \Re A^B_a$, one need only prove that this expression (or more precisely its integral over $z$) commutes with the scalar constraint. The $\Re A^B_a$ factor is non-polynomial; but fortunately the calculation is not involved, because $\tilde{E}_B^b \Re A^B_a$ turns out to be a total derivative. Therefore its integral is a surface term, and the surface term is physical, because $\hat{C}_S$ is smeared by a $\delta N$ which vanishes at the surfaces. The following formulas may be used to prove that $\tilde{E}_B^b \Re A^B_a$ is a total derivative:

$$\Re A^B_a = \epsilon^B_{BJ} \omega^Z_J, \quad \omega^I_J = \left[\Omega_{i[ja]} + \Omega_{j[ia]} - \Omega_{a[ij]}\right] e^I e^J; \quad \Omega_{i[ja]} = e_M [\partial_j e^M_a - \partial_a e^M_j]/2.$$  

(39)

The first line is the reality condition for the transverse connections $A^B_a$. The Lorentz connection $\omega^I_J$ is four-dimensional; $e^M_i$ is the tetrad, and $e_M^i$ is its inverse. The standard Lorentz gauge fixing condition, $e^Z = 0$, may be used to simplify the sums over $i$ in the definition of $\omega^I_J$, when $I = Z$; one gets

$$\omega^Z_I = -\partial_z g_{ij} e^Z_j e^I_j/2; \quad \tilde{E}_B^b \Re A^B_a = e^b e^B_{BJ} \omega^Z_J = -\partial_j g_{aj} e^b_j/2.$$  

(40)

which is a total derivative as required. □

It is fortunate that both constraints in eq. (28) are physical; otherwise the scalar constraint would treat the two polarizations of gravitational waves differently, since the two constraints are needed in eq. (28) because there are two polarizations. It would not be surprising if the two polarizations are treated differently in the wave-functional, since that quantity is not directly observable. In fact the
two polarizations are treated differently, at least in the wavefunctional for the linearized limit \cite{28}. However, it would be surprising if the two polarizations have different behaviour under commutation with the scalar constraint. That would lead one to question the usual choices of factor ordering, since this ordering causes the scalar constraint to be non-Hermitean, and a Hermitean constrain would treat the two polarizations symmetrically. In section VI, I discuss indirect evidence that the factor ordering should be changed; but I have no direct evidence that the ordering is incorrect.

(This paragraph probably can be skipped on a first reading.) In the plane wave case one has special \((z,t)\) coordinates, call them \((z',t')\), which parameterize the hypersurfaces \(z' \pm ct' = \text{const.}\) One can therefore envisage solving the problem of meaningless \(z\) coordinate in another way, by a transformation of coordinates to the more meaningful coordinate \(z'\). One might still integrate over \(dz\), but with a factor of \(\partial z'/\partial z\) inserted in the integrand. (This factor would give the desired density weight of one.) I have perhaps not given this option the attention it deserves. Note, however, it would require a great deal of attention. Information about the special surfaces \(z' = \text{const.}\) is encoded in the metric, so that the factor \(\partial z'/\partial z\) is not just a function, but a functional of the metric components (and a highly non-polynomial function at that!) This factor becomes an operator in the quantum case, and it is a highly non-trivial question how one regulates and factor orders this operator. This is the problem one runs into if one shifts to \(z'\) at the quantum level. If one tries to shift at the classical level, then one must fix the \(C_z\) gauge and replace Poisson brackets by Dirac brackets which respect the gauge-fixing condition. The classical Dirac brackets between the surviving observables tend to be messy, and so far I have found no quantum representation.

(The following three paragraphs contain material which at first glance may seem to be of only historical interest, but will be needed later in section V.) I interpret the BPR criteria in a semiclassical sense, as \(<\hat{F}> = 0\) rather than \(\hat{F}\psi = 0\). Yet the Hamiltonian constraints are always imposed strongly, as \(\hat{C}_i\psi = 0\), even though (in the linearized limit, at least) the \(\hat{C}_i\) are sums of creation and destruction operators, like the BPR field strengths. Why this differ-
ence in treatment? This same question was posed and answered in a different context, Lorentz gauge QED, many years ago, and it is worthwhile to take a moment here to review that discussion \cite{29}. In Lorentz gauge QED, the analog of the $C_i$ is (the usual Gauss constraint, plus) the four-divergence $\partial A = 0$. The analog of the strong requirement $\hat{C}_i \psi = 0$ would be $\partial \hat{A} \psi = 0$; and the analog of the semiclassical requirement $<\hat{C}_i> = 0$ would be $\partial \hat{A}^+ \psi = 0$, where the superscript $+$ denotes positive frequency components. (Since a splitup into positive and negative frequencies is available in QED, there is no need to introduce a semiclassical average.)

Both constraints, the strong and the positive frequency/semiclassical, are used in the Lorentz gauge literature. Authors who employ the positive frequency constraint tend to treat the “unphysical” part of the Hilbert space with more respect. (Remember that the Lorentz gauge condition is designed to eliminate the effects of the unphysical, longitudinal and timelike components.) Heitler \cite{30} is a typical proponent of this approach: he gives a very careful treatment of the unphysical sector, including a full discussion of the Gupta-Bleuler formalism. The payoff is that dot products over the full Hilbert space are well-defined, including dot products of longitudinal and timelike photons. Authors who employ the stronger constraint \cite{31} pay a price: it is possible to find states which are annihilated by both the annihilation and the creation parts of $\partial \hat{A}$, but these states are not normalizable in the unphysical sector \cite{29}. This result is not particularly surprising: since the creation operators in $\partial \hat{A}$ create a state with one more timelike or longitudinal photon, $\psi$ must be a sum over an unbounded, infinite number of longitudinal and timelike occupation numbers. This infinite sum leads to the divergence in the norm. The authors who use the strong constraint are well aware of this difficulty, and they circumvent it by requiring that the dot product in Hilbert space be taken over physical excitations only.

Returning to the gravitational case, one can now see why the strong criterion will work for the $C_i$, but not for the BPR operators. For the moment, imagine the gravitational theory to be linearized, so that the analogy to QED is strongest. The creation operators for the BPR operators create physical quanta, not unphysical. If I impose the strong criterion, I get states which have an infinite norm in the
physical sector. There is no way of avoiding this by restricting the measure at a later step. If I now pass from the linearized to the full theory, there is no reason why the strong criterion should suddenly become applicable. I must use the semiclassical criterion, which is justified using the correspondence principle.

IV Additional Solutions

In the previous section I derived quantum BPR operators. In the present section I construct new solutions to the constraints. In the next section I apply the BPR, ADM energy, and \( L_Z \) operators to the solutions constructed in paper II-III and this section.

I start from the solutions considered in III. These are strings of transverse \( \tilde{E}_A^a \) operators, ordered along the \( z \) axis, and separated by holonomies:

\[
\psi = \prod_{i=1}^{n} \int_{z_0}^{z_{i+1}} dz_i \Theta(z_{i+1} - z_i) M(z_{i+1}, z_i) \tilde{E}_A^a(z_i) S_A \Theta(z_1 - z_0) ]M(z_0, z_{n+1}).
\]  

(41)

The \( M \) are holonomies along \( z \),

\[
M(z_{i+1}, z_i) = \exp[i \int_{z_i}^{z_{i+1}} A^X(z') S_Z dz'],
\]  

(42)

and the \( S_M \) are the usual Hermitean SU(2) generators. These can be \( 2j+1 \) dimensional; they need not be Pauli matrices. The \( \Theta \) functions in eq. (41) are Heaviside step functions which path-order the integrations, \( z_0 \leq z_1 \leq \cdots \leq z_{n+1} \). For this section only, the boundary points \( z_i \) and \( z_{i+1} \) are relabeled \( z_0 \) and \( z_{n+1} \). Although the metric is not flat at the boundaries, it can be taken as conformally flat at boundaries, with any radiation present confined to a wave packet near the origin [8].

Since the full SU(2) invariance has been gauge fixed to O(2), it is convenient to use basis fields introduced at eq. (43), fields which are irreducible representations of O(2). These are one-dimensional, labeled by the eigenvalue of \( S_Z \), e. g.,

\[
\tilde{E}_A^a = (\tilde{E}_X^a \pm i \tilde{E}_Y^a) / \sqrt{2};
\]

(43)

\[
\tilde{E}_A^a S_A = \tilde{E}_A^a S_- \text{ or } \tilde{E}_A^a S_+.
\]
Because the irreducible representations are one-dimensional, there is no need to sum over both values of $A_1 = \pm$ in eq. (41), in order to obtain a Gauss-invariant expression; nor is it necessary to take the trace in that equation. However, one must be sure to have an equal number of $S_+$ and $S_-$ matrices in the chain, in order to form a closed loop of flux with no open ends violating Gauss invariance. That is, if one visualizes each holonomy $M(z_{i+1}, z_i)$ as a flux line along $z$ from $z_i$ to $z_{i+1}$, then the factor in the square bracket, eq. (41), may be visualized as a flux line from $z_0$ to $z_{n+1}$. The line varies in thickness (varies in $S_Z$ eigenvalue) because of the $S_\pm$ operators encountered along the way, but the final $S_Z$ value at $z_{n+1}$ must equal the initial $S_Z$ value at $z_0$ (there must be an equal number of $S_+$ and $S_-$ matrices in the chain). Then the final holonomy in eq. (41), $M(z_0, z_{n+1})$, can join the two ends at $z_0$ and $z_{n+1}$ and turn the open flux line into a closed flux loop. As shown in III, the wavefunctional of eq. (41) can be made to satisfy all the constraints, by suitable choice of the $a_i$ and $A_i$.

In order to obtain a larger, kinematical space, as well as more physical solutions, one can simplify (!) eq. (41) by dropping all $\Theta$ functions. This removes the path ordering, or (in visual terms) this allows flux lines to double back on themselves.

$$\psi_{\text{kin}} = \prod_{i=1}^{n} \int_{z_0}^{z_{n+1}} dz_i M(z_{i+1}, z_i) \tilde{E}_{A_i}(z_i) S_{A_i} M(z_0, z_{n+1}).$$

Again, the expression is Gauss invariant even if $A_1 = \pm$ is not summed over; and no trace is needed.

To check the spatial diffeomorphism and scalar constraints, one must first obtain these constraints from the Hamiltonian, written out in an $O(2)$ eigenbasis:

$$H_T = N'[i^{(2)}\tilde{E} \left( \tilde{E}_Z^2 \right)^{-1} \epsilon_{ab} A_+^a A_-^b + \sum_{\pm}(\pm i)\tilde{E}_\pm^b F_{zb}^\mp]
\quad + iN^z \sum_{\pm} \tilde{E}_\pm^b F_{zb}^\mp
\quad - iN_G[\partial_z \tilde{E}_Z - \sum_{\pm}(\pm i)\tilde{E}_\pm^a A_\pm^a] + ST
\equiv N'C_S + N^z C_z + N_G C_G + ST,$$  

(45)
where
\[ F_{z_b}^\tau = [\partial_z \mp iA^Z_z]A_{z_b}^\tau. \] (46)

\( \tilde{E} \) is the determinant of the 2x2 transverse subblock of the matrix \( \tilde{E}_A^a \). ST denotes surface terms (terms evaluated at the two endpoints on the z axis, \( z_0 \) and \( z_{n+1} \)). The detailed form of these terms is worked out in II but will not be needed here. The primed lapse \( N' \) equals the usual lapse \( N \) multiplied by a factor of \( \tilde{E}_z^z \), and correspondingly the scalar constraint \( C_S \) is the usual constraint divided by \( \tilde{E}_z^z \). As shown in II, this renormalization leads to a much simpler constraint algebra, but again the details of this will not be relevant here. The system is quantized by replacing transverse \( A^A_a \) and \( \tilde{E}_z^z \) by functional derivatives, these being the fields conjugate to the fields in \( \psi \).

\[ A^\pm_a \rightarrow \hbar\delta/\delta\tilde{E}_a^a; \]
\[ \tilde{E}_z^z \rightarrow -\hbar\delta/\delta A_z^z. \] (47)

The operator ordering (already adopted in eq. (41)) is functional derivatives to the right. The first term in eq. (45) contains an inverse operator \( (\tilde{E}_z^z)^{-1} \); this is well-defined provided \( \tilde{E}_z^Z_M \) never vanishes, that is, provided the \( S_z \) in eq. (42) never has the eigenvalue zero.

The physics must be invariant under small changes \( \delta N \) in the lapse and shift, so that the constraint should be written as

\[ 0 = \int dz[\delta N'C_S + \delta N^z C_z]\psi_{\text{kin}}; \] (48)
\[ 0 = \delta N'(z_b) = \delta N^z(z_b), \] (49)

where \( z_b \) is either boundary point, \( z_0 \) or \( z_{n+1} \). Eq. (49) guarantees that the boundary conditions at \( z_b \) are left unchanged by the transformation of eq. (48).

Both the \( C_S \) and \( C_z \) constraints in eq. (48) contain terms proportional to \( F_{z_b}^\tau \), the field strength defined at eq. (46). When a typical term of this type acts on \( \psi \), the result is (up to constants)

\[ \int dz\delta N(z)\tilde{E}_z^a F_{z_b}^-[\psi_{\text{kin}}] \]
\[ = \int dz\delta N(z)\tilde{E}_z^a F_{z_b}^-[\cdots \int dz_i M\tilde{E}_z^a S_- M \cdots ] \]
\[ = \int dz \delta N \sum_i \tilde{E}_+^{a_i} \left[ \cdots \int dz_i M(\partial z_i - iA^Z_\perp) \delta(z - z_i) S \cdot M \cdots \right] \]
\[ = \cdots \left[ \cdots \int dz_i (z - z_i)(\partial z_i - iA^Z_\perp) M S \cdot M \cdots \right] \]
\[ = \cdots \left[ \cdots \int dz_i (z - z_i)(-iM[S_\perp S - ]MA^Z_\perp - iA^Z_\perp MS \cdot M) \cdots \right] \]
\[ = 0. \]

(50)

On the third line I have changed the \( \partial z \) to \( \partial z_i \) and integrated by parts with respect to \( z_i \). The surface terms at \( z_i = z_b \) vanish because the \( \delta N(z) \delta(z - z_i) \) yields a factor of \( \delta N(z_b) \), which vanishes at boundaries. Thus \( \psi_{\text{kin}} \) is annihilated by all constraint terms containing field strengths \( F_{Z\perp} \). This is already enough to prove that the spatial diffeomorphism constraint \( C_z \) annihilates \( \psi_{\text{kin}} \), hence \( \psi_{\text{kin}} \) is at least part of a kinematical basis, if not a physical basis.

The state would be physical if it were also annihilated by the first term in \( C_S \), which I call \( C_E \) because of the \( \tilde{E} \) factor which it contains. (This term also happens to be proportional to a field strength, namely \( F_{Z\perp} \).) The following is a sufficient condition for \( \psi_{\text{kin}} \) to be physical: \( C_E \) annihilates the state if it contains only two out of the four transverse fields:

(either) \( \tilde{E}_+^x \) and \( \tilde{E}_-^x \) only;
(or) \( \tilde{E}_+^y \) and \( \tilde{E}_-^y \) only.

(51)

If \( \psi \) contains \( \tilde{E}_+^x \) and \( \tilde{E}_-^x \) only, for instance, their product \( \tilde{E}_+^x \tilde{E}_-^x \) is not contained in the triad determinant \( \tilde{E} \tilde{E} \); therefore the connection determinant \( \epsilon_{ab} A^+_a A^-_b \) in \( C_E \) will necessarily annihilate any wavefunctional containing only \( \tilde{E}_+^x \) and \( \tilde{E}_-^x \). Note the wavefunctional must have an equal number of \( S_\perp \) and \( S_\parallel \) operators in the chain.

One can generate additional physical states, starting from those described by eq. (44) and eq. (51), by applying the operators \( G_y^x \) or \( G_x^y \), where the \( G^a_b \) are integrals over \( z \) of the weight one objects constructed at eq. (28):

\[ G^a_b = \int_{z_0}^{z_{n+1}} dz \tilde{E}_A^a A^A_b. \]

(52)
As shown in section III, the operators $G^a_b$ commute with $H$; they are physical. Hence application of $m$ factors of $G^x_y$ to a functional $\psi[\hat{E}^x_{+}, \hat{E}^x_{-}]$ replaces $m$ $x$ superscripts in the chain by $y$ superscripts, but leaves $\psi$ physical. The operators $G^a_b$ change the total intrinsic spin of the wavefunctional; for a discussion of this point, see section VI.

I have labeled the generators $S_{A_i}$ in eq. (41) using two quantum numbers, $j$ and $m$ (more precisely, initial or final $m$, if the generator $S_{A_i}$ is one which changes $m$). Once the $SU(2)$ symmetry is broken to $O(2)$, however, the $j$ quantum number loses significance. I could replace the $S_{A_i}$ by any other matrix with the same $m$ (and $\Delta m$) but different $j$, and $\psi$ would change only by a constant factor.

Even though the $j$ has no physical significance, it is mathematically convenient to use $S_{A_i}$ having definite $j$. One can then employ the familiar commutation relations of the $S_{A_i}$ in calculations. Also, the planar state $\psi$ is presumably a limit of some three-dimensional state for which the label $j$ has meaning. The fact that states of different $j$ are equivalent in the planar limit presumably means that the correspondence between three-dimensional and planar states is many to one.

V Application to Solutions

In this section I study the solutions constructed so far by applying several operators to them: the modified BPR operators (from section III), the ADM energy operator (from paper III and appendix C), the area operator $\hat{E}^x_z$ for areas in the $xy$ plane (from paper III), and the operator $L^z$ giving the total spin angular momentum around the $z$ axis (from paper IV and appendix B). All the solutions (those constructed in papers II-III as well as the new solutions constructed in section IV) have the form of strings of transverse $\hat{E}^a_{\pm}(z_i)S_{\pm}^a$ operators separated by holonomies $M(z_{i+1},z_i)$. The solutions in papers II-III have additional step functions $\theta(z_{i+1} - z_i)$ which path order the integrations over the $z_i$, but the $\theta$ factors will play a minor role in the considerations of the present section.

Consider first the ADM energy operator. Often this is identified with the surface term in the Hamiltonian, but, as discussed in
appendix C, the volume term can also contribute. In the present case the volume term typically does contribute, but its only effect is to double the size of the surface term, and I will ignore volume contributions. The surface term is, from eq. (C4),

$$H_{st} = -\epsilon_{MN}\tilde{E}_b^b A^N \bigg|_{z_l}$$

$$= i\hbar\left[\tilde{E}_b^b \delta/\delta \tilde{E}_b^b - \tilde{E}_b^b \delta/\delta \tilde{E}_b^b \bigg|_{z_l}\right]$$

(53)

When this operator acts upon a factor of $\tilde{E}_b^b (z_i)dz_i$ in the wavefunctional, it gives $\mp i\hbar \tilde{E}_b^b dz_i$ times a factor of $\delta(z_i - z_r)$ or $\delta(z_i - z_l)$. Obviously none of the solutions is an eigenfunction of the ADM energy, since the $\delta$ function deletes one integration $dz_i$. One could perhaps construct an eigenfunction by summing over an infinite number of solutions, each containing one more $dz_i$ integration. Each additional integration should be multiplied by an additional factor of $i$, to cancel the $i$ in eq. (53) and make the eigenvalue real. Investigation of such sums is beyond the scope of the present paper. Without a measure one does not know whether such a sum converges to a normalizable result.

If the Gauss constraint,

$$C_G = -i[\partial_z \tilde{E}_Z^Z - \epsilon_{MN}\tilde{E}_M^a A^N_a],$$

(54)

vanishes at the boundaries, it can be used to simplify the ADM surface term to

$$H_{st} = -\tilde{E}_Z^Z \bigg|_{z_l}$$

$$= i\hbar \left(\delta/\delta A^Z_{Z} \bigg|_{z_l}\right).$$

(55)

Unwanted factors of $i$ are now more of a problem. $A^Z_{Z}$ occurs only in holonomies, where it is always multiplied by (a real matrix $S_Z$ times) a factor of $i$, and the $\tilde{E}_Z^Z$ will bring down this factor of $i$. Except for the solutions constructed in paper II, there is always at least one boundary where the Gauss constraint holds, so that factors of $i$ will be a generic problem. Of course one could eliminate the problem by discarding the holonomy structure, but this is a solution almost as unattractive as the problem.

The operator $\tilde{E}_Z^Z$ occurs in the ADM energy, while $\tilde{E}_Z^Z$ itself is the area operator for areas in the xy plane. ($\tilde{E}_Z^Z = ee_Z^Z = (2)e.$)
Therefore the area operator also has pure imaginary eigenvalues, a situation already noted in Appendix D of paper II; see also DePietri and Rovelli [32].

Even if one for the moment ignores the factors of i, there is another problem with the area operator: at any boundary where the Gauss constraint is satisfied, the area operator will always give zero. If the Gauss constraint is satisfied, say at the left boundary \( z_l \), then there is no net flux exiting at \( z_l \). One can regroup the holonomies until there are no \( M(z_i, z_l) \), only \( M(z_l, z_i) \); or until every \( M(z_i, z_l) \) is paired with an \( M(z_l, z_j) \) to give \( M(z_i, z_l) \) \( M(z_l, z_j) = M(z_i, z_j) \). Either way, there is no holonomy depending on \( A^a_Z(z_l) \), and the area operator \( \tilde{E}^a_Z(z_l) \) gives zero.

Next consider the action of the BPR operators. The solutions given in II (and some of those considered in section IV) contain either \( \tilde{E}^a_+ \) operators or \( \tilde{E}^a_- \) operators, but not both. A wavefunctional which contains only \( \tilde{E}^a_- \) operators (for example) will be annihilated by \( A^-_a = \hbar \delta / \delta \tilde{E}^a_+ \), even before any semiclassical average is taken:

\[
A^-_a \psi[\tilde{E}^a_-] = 0. \tag{56}
\]

In classical theory, \( A^-_a = 0 \) is a signal that the solution is pure left-moving; and from this one might expect that \( \psi[\tilde{E}^a_-] \) is unidirectional. Condition eq. (56) is too strong, however. As discussed at eq. (21) of section III, one expects at most a vanishing semiclassical average \( \langle A^-_a \rangle = 0 \). In fact from the remarks on Lorentz gauge QED in the concluding paragraphs of section III, eq. (56) implies that \( \psi \) is probably not a normalizable state!

The solutions of paper III and most of those from section IV contain both \( \tilde{E}^a_+ \) and \( \tilde{E}^a_- \), hence are not annihilated by any BPR operator. One cannot conclude that these solutions are infinite norm, therefore. However, they do suffer from the problems described previously in this section, those associated with the ADM energy and area operators.

VI Conclusions

Papers II-III proposed new solutions to the constraints, and section IV of the present paper proposes still more solutions. However, the
investigations of section V have demonstrated that these solutions are less than satisfying in several respects.

This outcome is perhaps not surprising. In earlier work, simplicity was the primary criterion for choosing the method of quantization (polarization and factor ordering), as well as the primary criterion for constructing solutions. Simplicity is the criterion one uses when information is scarce, however. In the present paper several operators of physical significance were available, and the theory and its solutions can be held to a standard more demanding than simplicity, the standard of a reasonable physical interpretation. As a result, solutions are called into doubt; but this happens for a reason which is fundamentally positive: more is known about how to interpret and understand the solutions.

The difficulties with imaginary eigenvalues encountered in section V seem to require fundamental revisions in the theory, since the difficulties are closely linked to the complex nature of the connection. As discussed in section V, the operator $\tilde{E}_z$, present in both the area operator and the ADM energy operator, has complex eigenvalues because of the $i$ in the holonomies, $\exp[iA^n_{\cdots}]$. Getting rid of the $i$ entails dropping or modifying the holonomic structure, not a pleasant prospect. Recently Thiemann [20] has proposed an alternative formalism based upon a real connection. Thiemann’s alternative is motivated primarily by issues of regularization, but has the desirable side effect of producing real eigenvalues for the area operator.

It is a little harder to see how switching to a real connection will cure the problem of zero eigenvalues of the area operator at boundaries, also uncovered in section V. The zero eigenvalues occur only at boundaries where Gauss invariance is satisfied. Since $\tilde{E}_z$ is a Gauss invariant, at first sight any connection between area and Gauss invariance seems strange. The connection is indirect, via the structure of the complex connection $A^Z_z = i \text{Im} A^Z_z + \text{Re} A^Z_z$. $\text{Im} A^Z_z$ is the part which does not commute with $\tilde{E}^Z_z$; therefore its presence in the wavefunctional gives rise to non zero area. $\text{Re} A^Z_z$ is the part which transforms like a connection under Gauss rotations; therefore it is needed in the wavefunctional for gauge invariance. The notions of area and gauge invariance are linked only because $\text{Im} A$ and $\text{Re} A$ are linked together to form a single (complex) connection. At
a boundary where Gauss invariance is satisfied, if there is no net flux exiting, then there is no dependence on the connection, hence no area. In the Thiemann scheme one still joins Im $A$ and Re $A$ together to form a single (real) connection; in fact the Thiemann connection is just the Ashtekar connection without the factor of $i$. The real and imaginary parts of the connection are separated when constructing the Thiemann constraints. Must they be separated when constructing the wavefunctional as well? If the answer is yes, the wavefunctional would not be purely a product of holonomies.

Before doing anything as drastic as dropping the holonomic structure, it is a good idea to investigate what happens to the zero area argument when it is extrapolated from the planar case to the full, three-space-dimensional case. The argument that Gauss invariance leads to zero area depends on properties of the wavefunctional at boundaries, and the behaviour at boundaries changes markedly with spatial dimension.

In the full three-dimensional case, the smearing function for the Gauss constraint must vanish at spatial infinity, so that there is no need for the Gauss constraint to annihilate the wavefunctional there. As a result, net flux may pass through the boundary at infinity, and there is no difficulty obtaining finite area at the boundary, even when the wavefunctional is purely a product of holonomies. This suggests that the zero area may be a problem which occurs only in the planar limit.

In fact the problem may not exist even in the planar limit, if one takes this limit correctly. Imagine the generic three dimensional flux configuration which is well approximated by planar symmetry: near the origin, the flux lines corresponding to holonomies containing $A^Z_a \ dx^a$ are finite in cross section and well-collimated along the z axis. The planar wavefunctionals constructed in papers II-III contain factors of $S_{\pm}$, presumably relics of Clebsch-Gordan coefficients coupling $A^Z_a \ dx^a$ holonomies to holonomies containing $A^X_a \ dx^a$ and $A^Y_a \ dx^a$. The latter are represented by flux lines lying in planes $z = constant$. Although $A^Z_a \ dx^a$ flux lines are well collimated near the origin, they must diverge far out along $z$ into the past or future. A sketch of the $A^Z_a \ dx^a$ flux lines resembles a drawing depicting radial geodesics near a wormhole: the flux lines come in from radial infinity, pass
through a narrow “throat” oriented along $z$, then diverge once more to radial infinity. (The wavefronts perpendicular to these rays are constructed from $A^X_a\, dx^a$ and $A^Y_a\, dx^a$ holonomies.) Alternatively, the $A^Z_a\, dx^a$ flux lines at infinity may not exit through the surface at infinity, but may loop back and close on themselves, resembling the flux lines of a solenoid in magnetostatics. Either outcome is allowed by the boundary conditions on the Gauss smearing function at infinity, and for either behavior at infinity, the behavior at the throat is the same. If one takes a cross section through two points $z_l$ and $z_r > z_l$ at the throat, one finds net $A^Z_a\, dx^a$ flux through both boundary points $z_l$ and $z_r$. If this picture of the three dimensional flux is correct, then in the planar limit one should not impose Gauss invariance at the boundaries. Planar solutions would resemble the “open flux” solutions studied in paper II. If there is $A^Z_a\, dx^a$ flux through the boundaries, the zero area problem at boundaries disappears.

Even though the $A^Z_a\, dx^a$ flux lines now extend throughout the entire range $z_l \leq z \leq z_r$, one can still construct localized wave packets. In paper II I reviewed the geometrodynamical treatment of the planar problem, and introduced the Szekeres scalar fields $B$, $W$, and $A$. (It is a little easier to work out the boundary conditions for $B$, $W$, and $A$, rather than work directly with the Ashtekar fields; as shown in II, the boundary conditions on $B$, $W$, and $A$ then imply corresponding boundary conditions on the Ashtekar variables.) When only the field $A$ is present, the forces on a cloud of test particles are isotropic in the transverse $(xy)$ direction; the elliptical distortions characteristic of gravitational waves appear only when the $B$ and $W$ fields are non-zero. (In a more covariant language, the components of the Weyl tensor which give rise to transverse deviations of geodesics are present only when $B$ and $W$ are non-zero.) One can impose wave packet boundary conditions on $B$ and $W$ (equivalently, on transverse components of the Weyl tensor), requiring these quantities to vanish at the boundaries $z_b$; but this requirement tells us nothing about the behavior of $A$ at the boundaries. The variable $A$ determines geometrical quantities such as areas: the Ashtekar area operator $\tilde{E}_Z^a$ is just $\exp(A)$. Perhaps one has a “wave packet”, but not a “geometry packet”. One may require localized, wave packet behavior for $B$ and $W$, but not for the more geometrical quantity $A$. 

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To summarize, there are two possible solutions to the zero area problem. The first splits the connection, abandoning the strict holonomic form for the wavefunctional. The second allows the Gauss constraint to be non-zero at boundaries and in effect assumes that the “open flux” boundary conditions used in paper II are generic. Further information and thought is needed before one can decide between these two alternatives.

Apart from the zero area difficulty, there is another reason why the transition from three to one space dimension needs more attention. Ultimately one would like to use the planar case as a guide to the behavior of radiation in the full, three dimensional case. In the full case, one expects the connections to occur in holonomies, so as to preserve gauge invariance. In the planar case, the gauge fixing allows the transverse connections to occur outside of holonomies. Two key radiative properties of the planar solutions, their directionality and spin, are associated with the transverse sector, which least resembles the three-dimensional case. It will probably be necessary to recast the transverse sector in a more holonomic language, in order to understand more clearly what happens on passing to the full theory.

For the moment let us overlook any possible difficulties with zero area and suppose that one shifts to a real connection, in order to eliminate the problem with imaginary eigenvalues. One can ask whether the solutions constructed in papers II-III and section IV are likely to survive the shift to a real connection formalism. The present solutions may not survive, if the factor ordering is changed; and it is easy to imagine a reason why one might want to change the factor ordering. The scalar constraint usually must be taken to be non-Hermitean, in a complex connection formalism; whereas with a real connection one may wish to factor order so as to make the constraint Hermitean.

It may be helpful to comment briefly on why the scalar constraint is difficult to make Hermitean in a complex connection formalism. The usual recipe for constructing a self-adjoint operator is to factor order it, and then, if the operator is not self-adjoint, form the average $(C + C^\dagger) / 2$. $C^\dagger$ is $C$, with the order of all operators reversed and the connections $A$ replaced by $A^\dagger$; in turn the $A^\dagger$ are replaced by $-A$.
+ 2 \text{Re } A. This last step introduces the unwanted non-polynomial expressions \text{Re } A into the $C^\dagger$ term. In the case of the Gauss and spatial diffeomorphism constraints, identities may be used to eliminate the \text{Re } A contributions, and $C^\dagger$ is well-behaved, in fact identical to $C$. In the case of the scalar constraint, the unwanted \text{Re } A terms do not go away. Within a real connection framework, the $A^\dagger$ is just $A$, and the traditional $(C + C^\dagger)/2$ recipe is easier to implement.

Although the present solutions may not survive as exact solutions, they may constitute approximate solutions to the new, modified constraints, solutions valid in the limit $\hbar \to 0$. This would happen because the new and old scalar constraints presumably will differ only by a reordering of factors, hence will differ by terms of order $\hbar$. Also, the $G^a_b$ operators, defined at eq. (31), were shown to be constants of the motion by Husain and Smolin [21]. These operators are likely to remain constants of the motion, in any transition to a new operator ordering, because of the close connection between the $G^a_b$ and total spin [10].

Even though the complex connection formalism may not be appropriate for the dynamics, this formalism is the natural one to use when constructing a criterion for the presence of radiation. Note that the BPR operators were derived in section II using only symmetry considerations; no assumptions were made about the factor ordering or dynamics. Even if one dropped the Ashtekar connection and used the Thiemann connection, one would have to reintroduce the Ashtekar connection in order to express the results of section II succinctly! Anyone familiar with the classical results on radiative criteria will not be surprised at this: much of that work is most conveniently expressed using the language of complex connections. (See for example the work on the Weyl tensor quoted in section I and Appendix D.)

My original intention in constructing the BPR operators was to elucidate the spin and directional character (that is, left vs. right-moving character) of radiative quantum states. Since these operators turned out to be semiclassical in character, and no measure is yet available, it is perhaps early days to make a final judgment on the efficacy of the BPR operators. However, it is probably true that any unidirectional operator (BPR or other) will be at most kinematical;
that is, it will commute with the Gauss and diffeomorphism constraints, but not the scalar constraint. Conversely, any operator which is physical (commutes with all constraints) will not be unidirectional; i.e. it will mix BPR operators of opposite directionality. The concept of unidirectionality is well defined in classical theory; see section II and appendix B. In quantum theory, however, every unidirectional wave must traverse a vacuum filled with zero point fluctuations which are moving in both directions. Presumably this is the intuitive reason why the modified BPR operators constructed in section III are either unidirectional or physical, but not both. Consistent with this interpretation of unidirectionality as primarily a classical concept, it is possible to prove that the unidirectional BPR operators are physical in classical theory; but factor ordering problems prevent the proof from going through in the quantum case. Presumably any criterion for unidirectionality will have to be at most a semiclassical one.

Classically, exact plane wave solutions are known in which the area operator $\tilde{E}_Z = (2)e$ evolves to zero \textsuperscript{16}. In fact this collapse behavior appears to be generic; solutions which do not collapse are rare and are unstable under small perturbations \textsuperscript{33}. The zero area cannot be removed by a change of coordinates, since typically there is an accompanying singularity in a scalar polynomial quadratic in components of the Weyl curvature tensor. One can ask whether a quantum-mechanical effect might prevent this collapse. It is not possible to answer this question definitively within the present context, because the area operator has imaginary eigenvalues. Nevertheless, one can see the outlines of a possible quantum solution which would avoid a collapse. The quantum area operator $\tilde{E}_Z$ acts on holonomies $\exp(i \int A_Z \, S Z \, dz)$; so long as $S_Z$ is not allowed to assume the value zero, $\tilde{E}_Z$ cannot have the eigenvalue zero. In the solutions constructed in papers II-III and section IV, the $S_Z$ value in each holonomy does not evolve dynamically, therefore remains non-zero if chosen to be non-zero initially. It remains to be seen whether this happy state of affairs will persist to a new formalism with real eigenvalues for the area operator.
A Details of the BPR calculation

This appendix solves eqs. (11) and (12) for the connection and tetrads obeying BPR symmetry, the invariance group for unidirectional plane gravitational waves. When setting up a complex connection formalism, it is necessary to choose three phases: when defining the Lagrangian at the four-dimensional level, one must choose the duality phase $\delta$ and the phase of $\epsilon_{TXYZ}$ (see for example eq. (13)); and an additional phase comes in when rewriting the four-dimensional formalism in 3+1 canonical form. These phases are explained in appendix A of II, and I use the same phase choices here as in that paper. I begin at the four-dimensional level by solving eq. (11) for the Lorentz transformation $L$.

$$L_{I'J} = -\partial_\beta \xi^\alpha e_{\alpha'I'} e^\beta_{J}.$$  \hfill (A1)

I have dropped a $\xi^{(c)} \partial_\lambda$ term which is zero because $\xi^{(c)}$, eq. (8), is a linear combination of $\partial_x$, $\partial_y$, and $\partial_v$, all of which annihilate $e^{\alpha}_I$. In Rosen gauge, from eq. (8),

$$\partial_\beta \xi^\alpha = \delta^\alpha_{\beta} g^{c\alpha} - \delta^\alpha_{\beta} g^{u\alpha}.$$  \hfill (A2)

Therefore

$$L_{I'J} = -e^{c}_{I'} e^{u}_{I} + e^{u}_{I'} e^{c}_{I}.$$  \hfill (A3)

$L$ is antisymmetric, as it should be. Eq. (11) determines $L$, eq. (A3), but otherwise imposes no new constraints on the tetrads beyond those already imposed by $\partial_x$, $\partial_y$, and $\partial_v = 0$.

Next consider eq. (12). In principle, one should be able to determine all the constraints on the $A^A$ by solving these equations directly, but they are awkward, and it is easier to adopt an indirect approach. Given the tetrads, compute the Lorentz connection $\omega^{IJ}_a$; then compute $(4)A$, which is just the self-dual version of $\omega$. In this way one finds that many components of $(4)A$ are identically zero. When this information is inserted into eq. (12), that equation reduces to the trivial statement $0 = 0$, for most values of the indices; for a small number of index values the equation is non-trivial and can be solved with moderate effort.

The equation relating $\omega$ to the tetrads is

$$\omega_{ija} \equiv e_{ij} e_{ja} \omega^{IJ}_a$$
\(-g_{ij,i} + g_{ai,j} + e_{jK} \partial_a e^K_i)/2. \quad (A4)

From this equation, at least one of i, j, or a must be u, since derivatives with respect to x,y,v are zero. Further, the tetrad matrix in Rosen gauge is 2x2 block diagonal, with the zt to ZT block containing constants only. This implies that at most one of i, j, or a must be u. After stripping off the (block diagonal) tetrads, one finds that the only non-zero \( \omega \) are

\[ \omega_{XY}^u, \omega_{VA}^u, \quad (A5) \]

where a = x,y only and A = X,Y only. Now compute \((4)A\) from \(\omega\), using

\[ 2(4)A^U = \omega^U + i\delta(\epsilon^{IJ}/2\epsilon_{TXYZ})\omega^M, \quad (A6) \]

where the duality eigenvalue \(\delta_{TXYZ}\) equals +1, given my conventions \(\delta = \epsilon_{TXYZ} == 1\). The only non-zero \((4)A\) components are

\[ (4)A^X, (4)A^T, (4)A^V. \quad (A7) \]

At first glance one might think this list is too short; there should be non-zero \((4)A^U\) as well, because the \(\epsilon^{IJ}_{MN}/2\epsilon_{TXYZ}\) term in eq. (A6) will map the VA indices on \(\omega_{VA}^u\) into UB (B = X,Y or \pm). Duality maps VX into VY, not UY, however, because of the identities

\[ \epsilon_{TXYZ} = \epsilon_{UVXY} = \epsilon_{VX}, \quad (A8) \]

e tc., where the last, mixed index tensor with two V’s is the one which occurs in duality relations. This explains why there are no \((4)A^U\); to see what happened to \((4)A^V\), note

\[ 2(4)A^V = \omega^V + i\delta(\epsilon^{V}_{MN}/2\epsilon_{TXYZ})\omega^M, \]
\[ = \omega^V + i\delta(\epsilon^V /\epsilon_{TXYZ})\omega^V \]
\[ = \omega^V (1 \pm \delta). \quad (A9) \]

For my phase choice \(\delta = +1\), \((4)A^V\) vanishes.

From eq. (A7), there is no need to consider \(\alpha = v\) in eq. (12). I consider first \(\alpha = a = x,y\). The first term in eq. (12) is a linear combination of \(\partial_x, \partial_y\), and \(\partial_v\), which vanishes for any \(\alpha\). The second
term vanishes from eq. (A2) and the absence of any $λ = v$ component of $(4)_A$. Then eq. (12) collapses to

$$0 = 0 + 0 + \mathcal{L}_I^I \, (4)_A^J + \mathcal{L}_J^J \, (4)_A^{IJ} - 0.$$  \hfill (A10)

From eqs. (A3) and (13), the only non-zero elements of $L$ are

$$\mathcal{L}_{UX} = i\mathcal{L}_{UY} = (e^c_X + ie^c_Y)/2 = e^c_e/\sqrt{2};$$  \hfill (A11)

or

$$\mathcal{L}_{U^+} = e^c_e; \mathcal{L}_{U^-} = 0.$$  \hfill (A12)

Inserting this and eq. (A7) into eq. (A10), one finds that all the $\alpha = a = x, y$ equations are trivially $0 = 0$.

Finally, consider $\alpha = u$. The first term in eq. (10) vanishes as before. The only $IJ$ index pair which does not give $0 = 0$ is $IJ = VA, A = X, Y$ only, which gives

$$0 = 0 + \partial_u \xi^\lambda \, (4)_A^V + \mathcal{L}_Y^V \, (4)_A^V + \mathcal{L}_U^A \, (4)_A^{JV} - \partial_u \mathcal{L}^\lambda \, (4)_A^V.$$  \hfill (A13)

Use eq. (A2) to simplify the first term; use eq. (A12) to simplify the remaining terms and to show that the $A = -$ equation is trivial. Then

$$0 = 0 + g^{ca} A^v_{u} - 2e^c_+ A^{-+}_{u} + \partial_u e^c;$$

$$= g^{ca} A^v_{u} - 2e^c_B A^{B+}_{u} + \partial_u e^c.$$  \hfill (A14)

I have used the duality relation $A^V^U = -iA^X^Y$, and $A^{-+} = iA^{XY}$. On the second line, recall that a “plus’ index always pairs with a “minus’ index to form the two-dimensional dot product: $e^c_B A^{B+} = e^c_X A^{-+} + 0$. The $A_u$ field in eq. (A14) is a linear combination of $A_z$ and $A_t$ fields, and the $A_t$ fields are non-dynamical Lagrange multipliers for the Gauss constraints. I therefore eliminate the $A_u$ field in order to obtain a constraint on the dynamical field $A^v_{u}$. From duality and eqs. (A4) and (A5) for $\omega$,}

$$2e^{(4)}_a B^a = e_{ab} [\omega^B + i(\delta/2\epsilon_{TXYZ})\epsilon^{B+}_{MN}\omega^M_N]$$

$$= [e^{j+}\omega^a + 0]$$

$$= e^{j+} [e_{JK} \partial_u e^K]/2.$$  \hfill (A15)
I solve eq. (A14) for $A^V_A$ and insert eq. (A15).

\[
(4) A^V_a^+ = e^{j[x]} \frac{\partial u_a^e}{2} - g_{ca} \partial u_c^e \\
= e^{j[x]} [2e_{JK} \partial u_a^e_K - \partial u g_{ja}]/2 - \partial u e_a^e + \partial u g_{ca} e^c _+ \\
= \partial u g_{ca} e^c _+ /2. \tag{A16}
\]

The right-hand side of eq. (A16) is proportional to the part of $A^V_a^+$ which contains no $i\delta$ factor.

\[
2 \text{“Re”}(4) A^V_a^+ = (\omega_a^{VX} + i\omega_a^{VY})/\sqrt{2} \\
= e^{jV} (e^{Xj} + ie^{Yj})\omega_{ja}/\sqrt{2} \\
= e^{+j}[\partial u g_{aj}]/2. \tag{A17}
\]

Therefore

\[
0 = -(4) A^V_a^+ + 2 \text{“Re”}(4) A^V_a^+ \\
= (4) A^V_a^+, \text{ for } \delta = -1. \tag{A18}
\]

The second line means that the first line is the four dimensional connection computed with the opposite choice for the duality eigenvalue, $\delta = -1$ rather than $\delta = +1$.

This result is very easy to transform from Rosen to a general gauge in the $z,t$ sector, since the “minus” and “a” indices in the $x,y$ sector remain invariant under such a transformation. In order to maintain the gauge conditions eqs. (11) and (12) on the tetrads, it is necessary to combine any four-dimensional diffeomorphism $(z,t) \rightarrow (z',t')$ with a Lorentz transformation, as at eq. (11). A short calculation shows that a very simple Lorentz transformation $L_T^T = \partial t'/\partial z$ will maintain all the gauge conditions of eqs. (11) and (12). For transforming $(4)A$ one needs the corresponding $L$; the only non-zero matrix elements will be $L^T_Z$, $L^X_Y$, or equivalently $L^V_V$, $L^V_Y$, and $L^\pm_T$. Thus the coordinate transformation to the general gauge amounts to a Lorentz transformation which multiplies eq. (A18) by an overall factor.

\[
(4) A_{a}^{V+} = L_{V}^{V} L_{a}^{+} (4) A_{a}^{V+}, \text{ for } \delta = -1, \tag{A19}
\]

where every $L$ and every $(4)A$ is to be calculated using the $\delta = -1$ convention. From eq. (A19), the quantity in eq. (A18) vanishes.
in every gauge. Similarly, from eq. (A7), the following quantities vanish in every gauge:

\[
0 = -(4)A_{a}^{U+} + 2 \text{Re}''(4)A_{a}^{U+} \\
= (4)A_{a}^{V-} \\
= (4)A_{a}^{U-}.
\] (A20)

The eqs. (A20) and (A18) may be rewritten as

\[
0 = (4)A_{a}^{T+} - 2\text{Re}'(4)A_{a}^{T+} \\
= (4)A_{a}^{Z+} - 2\text{Re}'(4)A_{a}^{Z+} \\
= (4)A_{a}^{T-} \\
= (4)A_{a}^{Z-},
\] (A21)

where every connection in eq. (A21) is evaluated using the \(\delta = +1\) convention. For the opposite duality convention, \(\delta = -1\), exchange (+ ↔ −) everywhere in eq. (A21). For \(\delta = +1\) but left-moving rather than right moving waves, again exchange (+ ↔ −) in eq. (A21).

So far the calculation has been carried out entirely at the four-dimensional level. The four-dimensional connection \(A_{a}^{4}\) is related to the usual 3+1 connection \(A\) by the following equation from Appendix A of II.

\[
A_{a}^{S} = -\epsilon_{MNS}^{(4)}A_{a}^{MN}.
\] (A22)

Then the four-dimensional eq. (A21) implies the 3+1 dimensional equations

\[
0 = A_{a}^{-}; \\
0 = -A_{a}^{+} + 2\text{Re}''A_{a}^{+}.
\] (A23)

Again, exchange (+ ↔ −) for the opposite duality convention or left-moving waves.

**B  Kinematics of the \(A_{a}^{\pm}\) fields: spin**

From paper IV, the integral

\[
L_{Z} = i \int dz[\bar{E}_{y}^y(A_{x}^{I} - \text{Re}A_{x}^{I}) - (x \leftrightarrow y)]
\] (B1)
gives the total spin angular momentum of the wave, and is a constant of the motion \([10]\). The integral is over the entire wave packet, that is from \(z_l\) to \(z_r\). As in II, the fields and Weyl tensor components which produce transverse displacements of test particles are assumed to vanish at the boundaries, with support only in the region \(z_l < z < z_r\). It is at first sight surprising that any conserved quantity associated with the Lorentz group should be given by a volume integral (integral over \(z\)) rather than by a surface term (term evaluated at the endpoints \(z_l\) and \(z_r\)). However, in the one-dimensional planar case, the extensive gauge-fixing in the x,y plane removes all gauge freedom, except for rigid rotations around \(z\), and the x,y sector of the theory resembles special relativity rather than general relativity.

This appendix rewrites the integrand of \(L_Z\) in terms of the unidirectional fields defined at eq. (27), in order to understand the spin content of the latter. Introduce triads \(e^a_i\) and inverse triads \(e_i^a\), and write the integrand of \(L_Z\) as

\[
\tilde{E}^i y \Im A^j_x - (x \leftrightarrow y) = [(e^i_j e_{Kx} - (x \leftrightarrow y))\tilde{E}^a_i \Im A^j_a]
= [(e^i_j e_{Kx} + e^i_j e_{Kx} - (x \leftrightarrow y))\tilde{E}^a_i \Im A^j_a](B2)
\]

On the last line the term antisymmetric in \(J,K\) is proportional to \(\tilde{E}^a_i \Im A^j_a \epsilon_{JK}\). This expression is part of the Gauss constraint \(\partial_z \tilde{E}^a_z + \tilde{E}^a_i A^j_a \epsilon_{JK} = 0\), which implies \(\tilde{E}^a_i \Im A^j_a \epsilon_{JK} = 0\). Hence the term antisymmetric in \(J,K\) can be dropped. The term symmetric in \(J,K\) can be expanded in O(2) eigenstates, keeping in mind that every + index must be contracted with a - index. The \(J \neq K\) terms are proportional to

\[
e^y_i e_{-x} + e^{-y}_i e_{+x} = e^y_i e_{Bx} = \delta^y_x = 0.
\]

Hence these terms can be dropped also. The surviving terms are products of tensors with \(J = K\), therefore helicity \(\pm 2\) in the local Lorentz frame, a reassuring result.

\[
L_Z = -\int dz [e^y_i e_{+x} \tilde{E}^a_i \Im A^+_a + e^{-y}_i e_{-x} \tilde{E}^a_i \Im A^-_a - (x \leftrightarrow y)]
= \int dz \{e^y_i e_{+x} \tilde{E}^a_i [A^-_a + (A^-_a - 2 \, \text{"Re"} A^-_a)\}
\]
\[e^y e^{-x} E^n_+ [A^n_+ + (A^n_+ - 2 \text{ Re } A^n_+)] - (x \leftrightarrow y). \quad (B4)\]

On the last line I have written \(L_Z\) in terms of the unidirectional BPR fields introduced at eq. (27). This expression for gravitational spin angular momentum, possesses the same coordinate times momentum structure as the corresponding expression for electromagnetic spin angular momentum:

\[\bar{L}_{\text{em}} = \frac{(1/4\pi)}{\int d^3x [\vec{E} \times \vec{A}], \quad (B5)\]

That is, one can interpret the unidirectional quantities \(\vec{E} \ A\) and \(\vec{E} (A - 2 \text{ Re } A) \vec{E}\) in eq. (B4) as momenta associated with waves of definite helicity. This parallel with QED does not extend too far: these “momenta’ have nothing like free-field commutation relations with each other, or with the triad “coordinates”.

It is now clear why one wants the two combinations \(\vec{E}^a_+ A^+_a\) and \((-A^-_a + 2 \text{ Re } A^-_a) \vec{E}^-_a\) to vanish: these two constraints remove left-moving helicity \(\pm 2\) contributions from \(L_Z\). Why must the remaining, helicity zero combinations vanish? The helicity zero combinations are \(\vec{E}^a_- A^+_a\) and \((-A^-_a + 2 \text{ Re } A^-_a) \vec{E}^a_+\). They are complex conjugates of each other, so that by adding and subtracting them from each other one gets pure imaginary and pure real constraints

\[0 = \vec{E}^a_B (A^B_a - \text{ Re } A^B_a) - i \varepsilon_{AB} \text{ Re } A^B_a \vec{E}^a_B; \quad (B6)\]
\[0 = -i \varepsilon_{AB} (A^A_a - \text{ Re } A^A_a) \vec{E}^a_B + \vec{E}^a_B \text{ Re } A^B_a. \quad (B7)\]

Now consider the classical equation of motion

\[0 = -i \vec{E}^a_B \frac{\delta}{\delta A^B_a} - \delta H/\delta A^z; \quad (B8)\]

On the second line I use the Hamiltonian of eq. (13). I also use the unidirectionality assumption (for the first time in this section; \(L_Z\) is the spin operator also for the scattering case) and evaluate the Hamiltonian in Rosen (or at least conformally flat) gauge. The metric in a general gauge has the form

\[ds^2 = [(-(N')^2 + (N^x)^2) dt^2 + 2N^x dz dt + dz^2] g_{zz} + x, y \text{ sector}, \quad (B9)\]
so that to obtain conformal gauge, one must take $N^z = 0$ and $N' = 1$ in the Hamiltonian, where $N^z$ is the shift and $N'$ is the renormalized lapse defined following eq. (45). From the real part of eq. (B8), the $\tilde{E} \text{ Re } A$ term in eq. (B7) vanishes. The rest of this equation is just the imaginary part of the Gauss constraint, $\partial_z \tilde{E}_Z^Z + \epsilon_{AB} A^A \tilde{E}_B^Z = 0$, and vanishes also. This leaves eq. (B9). The $\epsilon \text{ Re } A \tilde{E}$ term may be simplified using the real part of the Gauss constraint; the $\tilde{E} (A - \text{ Re } A)$ term may be simplified using the imaginary part of the equation of motion, eq. (B8). The result is simply

$$0 = -i(\partial_t + \partial_z)\tilde{E}_Z^z, \quad (B10)$$

in any conformally flat gauge. This equation is discussed further at eq. (17) of section II.

C The ADM energy

It is a worthwhile exercise to express the ADM energy in terms of BPR operators. In the usual three-space dimensional case, the Hamiltonian expressed in terms of the original ADM variables is the sum of a volume integral plus a surface term $H_{st}$ [34, 35],

$$H = \int d^3x[N\eta_{sc} + N^iC_i] + H_{st}. \quad (C1)$$

In the classical theory, the ADM energy is just the surface term, since the constraints in the volume term must vanish everywhere when the solution obeys the classical equations of motion. Often one says that in the quantum case the ADM energy is just the surface term also, but this is not quite right, as we shall see in a minute.

In the planar, one-space dimensional case, the expression for the Hamiltonian in terms of Ashtekar variables looks superficially much the same as eq. (C1), [8],

$$\int dz[N'\eta_{sc} + N^2C_z + N_GC_G] + H_{st}, \quad (C2)$$

except for the additional Gauss constraint, and the prime on $N'$. (The prime means I have renormalized the usual Ashtekar lapse
by absorbing a factor of $\tilde{E}_Z$ into the lapse, as explained in II.) In both one and three spatial dimensions, one might be tempted to drop the volume terms, in the quantum mechanical case, because the constraints $C_i$ are required to annihilate the wavefunctional. However, the statement that the scalar constraint (say) annihilates the wavefunctional means, not $C_{sc}\psi = 0$, or even $\int dz C_{sc}\psi = 0$, but rather
\[
\int dz\delta N' C_{sc}\psi = 0,
\]
(C3)
where $\delta N'$ is a small change in the lapse. The arbitrary change $\delta N'$ must preserve the boundary condition at spatial infinity, $N' \to 1$. Hence $\delta N' \to 0$ there. On the other hand, when $C_{sc}$ occurs in the Hamiltonian of eq. (C2), it is multiplied by $N'$ rather than $\delta N'$. There is no need for $N'$ to vanish at the boundaries (in fact it becomes unity there). Now suppose the evaluation of the action of $C_{sc}$ on $\psi$ requires an integration by parts with respect to $z$. In eq. (C3), when the constraint acts upon $\psi$, surface terms at $z = z_b$ will vanish because of the boundary condition on $\delta N'$. When $H$ acts upon $\psi$, however, the $C_{sc}$ in $H$ is smeared by $N'$ rather than $\delta N'$; the former does not vanish at boundaries. Consequently the volume term can contribute to the ADM energy in the quantum case. In the planar case both the Gauss and scalar constraints in the volume term can contribute to the ADM energy; neither $N'$ nor $N_G$ is required to vanish at the boundaries. In the usual 3+1 dimensional case with flat space boundary conditions at infinity, only the scalar constraint in the volume term can contribute; the remaining constraints are smeared by $N_i$ which are required to vanish at spatial infinity.

For the plane wave case, the surface terms in the Hamiltonian were computed in section 4 of II.

\[
H_{st} = -\epsilon_{MN}\tilde{E}_M^b A_b^N_{z_i}^{z_i} \\
= i[\tilde{E}_-^b A_0^b - \tilde{E}_+^b A_b^b] \\
= i\hbar[\delta/\delta \tilde{E}_-^b - \tilde{E}_+^b \delta/\delta \tilde{E}_+^b]_{z_i}^{z_i} \\
= \text{(C4)}
\]

To simplify the boundary term quoted in II, I have invoked the boundary conditions $N^z \to 0$, $N' \to 1$ on the shift and renormalized lapse. Evidently the ADM energy contains the BPR operators which are sensitive to the long-range scalar potential, which suggests that
these operators may play a role even in the presence of waves which are not unidirectional.

The operator of eq. (C1) gives a finite result when applied to the solutions of II-III; there is no need to renormalize. However, the solutions are not eigenfunctions of this operator. A typical solution involves \( n \) integrations \( dz_i \) over the locations of the \( n \) \( \mathbf{E}_\Lambda^a(z_i) \) operators contained in the wavefunctional, and the ADM operator acts as a "lowering operator" , removing one integration. Hence an eigenfunction would have to be an infinite sum over wavefunctionals of all possible values of \( n \). It is beyond the scope of this paper to investigate the finiteness of the norm of such a sum.

D The transverse Weyl criterion

1. Classification of Weyl tensors.

The Weyl tensor is the part of the Reimann tensor which can be non-zero even in empty space, and certain of its components induce transverse vibrations when inserted into the equation of geodesic deviation [36]. It is therefore a natural object to work with when constructing a criterion for the presence of radiation [37]. The construction proceeds in two steps. The first step is a straightforward mathematical problem: classify Weyl tensors using their algebraic properties. In the second step, one uses physical arguments to determine the Weyl class(es) most closely associated with radiation.

To begin with the mathematical problem, there are 10 independent real components of the Weyl tensor, and from these one can construct 5 independent complex components which have simple duality properties.

\[
\mathcal{C}_{abcd} = \left[ \mathcal{C}_{abcd} + i(\delta/2\epsilon_{TXYZ})\epsilon_{abmn}C_{cd}^{nm} \right]/2; \quad (D1)
\]

\[
\mathcal{C}_{abcd} = i(\delta/2\epsilon_{TXYZ})\epsilon_{abmn}C_{cd}^{mn}. \quad (D2)
\]

Lower case Roman indices \( a, b, c, \ldots \) are global; upper case Roman indices \( A, B, C, \ldots \) are local Lorentz. \( \epsilon_{abmn} \) is the totally antisymmetric global tensor, while \( \epsilon_{TXYZ} \) is the corresponding local Lorentz quantity, the Levi-Civita constant tensor. The duality eigenvalue is \( \delta/\epsilon_{TXYZ} = \pm 1 \). There is another Levi-Civita tensor hidden in the
\( \epsilon_{abmn} \),

\[ \epsilon_{abmn} = e^A \epsilon^B \epsilon^C \epsilon^D \epsilon_{ABCD}; \]

therefore \( \epsilon_{TXYZ} \) and its associated sign convention drop out after the conversion to Ashtekar variables and the 3 + 1 splitup. The final 3+1 Hamiltonian contains only the phase \( \delta \). My convention is \( \delta = +1 \), but in this paper all results are stated in a manner which facilitates a conversion to the opposite convention. Of course the combinations with simple duality properties also have simple transformation properties in the local Lorentz frame, which is why one chooses to work with \( C \) rather than \( C \), when attempting a classification.

Petrov was the first to classify Weyl tensors by their algebraic properties \([11]\); but for present purposes the equivalent classification scheme due to Debever \([12, 13]\) is more convenient. A null vector \( k \) is said to be a principal null vector (Debever vector) of \( C \) if

\[ k[aC]_{mn}[cC]k^mk^n = 0; k_ak^a = 0. \quad (D3) \]

Debever proved that a Weyl tensor can have up to four distinct principal null vectors, and he classified Weyl tensors by the number of degeneracies among these vectors. If \([1111]\) denotes the Weyl tensors which have 4 distinct Debever vectors, \([211]\) the Weyl tensors with two vectors degenerate and the rest distinct, etc., then the five classes are \([1111]\), \([211]\), \([22]\), \([31]\), and \([4]\). (The corresponding five Petrov classes are I, II, D, III, and N, respectively.)

Now suppose \( k \) is a Debever vector obtained by solving eq. \((D3)\). Make it one leg of a null tetrad \( k, l, m, \bar{m} \). Choose the \( Z \) axis of a local free fall frame so that \( k \) and \( l \) have spatial components along \( \pm Z \), while \( m \) and \( \bar{m} \) are transverse.

\[ -k_al^a = m\bar{m}^a = 1; \]
\[ k_ak^a = l_a^a = m_\bar{m}^a \]
\[ = k_\bar{m}^a = l_\bar{m}^a = 0. \quad (D4) \]

C may be expanded in this basis, and (not surprisingly) one gets five possible terms.

\[ C_{abcd} = C_1 V_{ab} V_{cd} \]

46
\begin{align*}
+C_2(V_{ab}M_{cd} + M_{ab}V_{cd}) \\
+C_3(M_{ab}M_{cd} - U_{ab}V_{cd} - V_{ab}U_{cd}) \\
+C_4(U_{ab}M_{cd} + M_{ab}U_{cd}) \\
+C_5U_{ab}U_{cd},
\end{align*} 

(D5)

where

\begin{align*}
V_{ab} &= 2k_{[a}m_{b]}; \\
U_{ab} &= 2l_{[a}\bar{m}_{b]}; \\
M_{ab} &= 2k_{[a}h_{b]} + 2m_{[a}\bar{m}_{b]}.
\end{align*} 

(D6)

The five combinations in eq. (D5) are the only ones allowed by the duality convention \( \delta = +1 \). The expansion for the opposite duality convention may be obtained from eqs. (D5) and (D6) by interchanging \( m \) and \( \bar{m} \) in eq. (D6). Eq. (D5) is essentially the expansion given by Szekeres (36), after a relabeling of the basis vectors \((k, l, m, \bar{m}) \to (k, -m, t, \bar{t})\). Since the expansion treats \( k \) and \( l \) quite symmetrically, it is valid also for the case that \( l \), rather than \( k \) is the principal null vector.

At this point one turns from the mathematical to the physical: which Petrov/Debever class(es), or which term(s) in eq. (D5), are most closely associated with radiation? Consider first which of the five tensors in eq. (D5) distorts a cloud of test particles in the manner expected for gravitational radiation. Szekeres finds that only the \( C_1 \) and \( C_5 \) terms produce the transverse displacements in the XY plane characteristic of gravitational radiation in the linearized theory. \( C_2 \) and \( C_4 \) produce longitudinal displacements in the XZ or YZ planes. \( C_3 \) produces a Coulomb (or tidal force) displacement: \( C_3 \) distorts a sphere of particles into an ellipsoid of revolution with axis along \( Z \).

These facts suggest that the \( C_1 \) and \( C_5 \) terms signal the presence of radiation.

There is another set of arguments which suggest that the \( C_1 \) and \( C_5 \) terms are closely associated with the presence of radiation. If the Weyl tensor contains \textit{only} a \( C_1 \) or \( C_5 \) term, the tensor is type N. (A type N tensor with \( k \) as principal null vector contains only a \( C_1 \) term; a type N tensor with \( l \) as principal null vector contains only a \( C_5 \) term.) Type N is closely associated with radiation. Along characteristic curves, when the metric is discontinuous, the discontinuity in the
Weyl tensor is type N. In the linearized theory, the tensor associated with unidirectional gravitational radiation is type N. At large distances from bounded sources, the surviving components of the Weyl tensor are type N ("peeling theorem").

Although the $C_1$ and $C_5$ terms are closely associated with type N, it would be better to call these terms transverse Weyl components, rather than type N components, since a tensor which is not type N can nevertheless contain $C_1$ or $C_5$ terms. A [22] (type II) field contains $C_1$ plus some admixture of longitudinal component, while [1111] (type I) contains $C_1$, $C_5$, and $C_3$ terms. In a theory as nonlinear as general relativity, one can expect that a collision between $C_1$ and $C_5$ transverse waves will produce some $C_3$ (Coulomb) component, and the tensor will be type I rather than type N. In asymptotic regions, after the transverse wave has "outrun" its Coulomb companion, presumably the tensor will revert to type N; but in general one should be looking for $C_1$ and $C_5$, rather than type N or any other specific Petrov class. One should describe this radiation criterion as the transverse Weyl criterion, rather than the type N criterion.

I now construct operators which project out the $C_1$ and $C_5$ terms.

\[
\begin{align*}
C_1 &= C_{abcd}^{ab\bar{m}b\bar{c}m^d}; \\
C_5 &= C_{abcd}^{a\bar{k}b\bar{c}m^d}; \\
C_3 &= -C_{abcd}^{ab\bar{m}b\bar{k}c^d}.
\end{align*}
\]

For completeness I have included also the expression for the pure Coulomb component $C_3$. The plane wave case has no longitudinal components; $C_2$ and $C_4$ vanish identically.

2. Transition to Ashtekar variables.

At this point I specialize to the case of plane waves along the Z axis. By definition, the plane wave metric has two hypersurface orthogonal null vectors which may serve as normals to right- and left-moving wavefronts $U = (cT-Z)/\sqrt{2} = \text{const.}$ and $V = (cT+Z)/\sqrt{2} = \text{const.}$ I identify these normals with $k$ (right-moving) and $l$ (left-moving), so that a small change in the wave phase will look like

\[
\begin{align*}
k_a dx^a &= (-dT + dZ)/\sqrt{2} = -dU; \\
l_a dx^a &= (-dT - dZ)/\sqrt{2} = -dV.
\end{align*}
\]
Hypersurface orthogonality demands $k_a dx^a \propto dU$, etc.; the normalization conditions force the constants of proportionality to be as shown in eq. (D8); and the overall phase of $k_a$ and $l_a$ is fixed by the requirement that $k^0$ and $l^0$ be positive, i.e. future-pointing.) Lower case $x$ denotes a global coordinate; upper case ($T, Z, U, V$) denotes a coordinate in a local Lorentz frame. From the expression for the (inverse) tetrads, $e^A_a dx^b = dX^A$, $k$ and $l$ may be identified with the tetrads

$$
k^a = -e^{aU} = +e^a_V; \quad l^a = e^a_U. \quad (D9)
$$

Similarly,

$$
m^a = e^a_{\bar{V}}; \quad \bar{m}^a = e^a_{\bar{U}}. \quad (D10)
$$

Evidently the quantities $C_i$ are (global scalars and) tensors in a Local Lorentz frame.

The eq. (D8) places quite a strong restriction on the null basis, going beyond what is required to maintain the normalization eq. (D4). The choice eq. (D9) certainly is not unique. For example, $k_a$ remains null if it is rescaled by an arbitrary function. (Simultaneously $l_a$ must be rescaled by the inverse function, in order to maintain the normalization condition $-k_a l_a = 1$). Similarly $m$ and $\bar{m}$ may be rescaled. The choice eqs. (D9) and (D10) facilitates calculations and leads to highly symmetric formulas for $C_1$ and $C_5$. (In this basis, $C_1$ is just $C_5$ with some plus and minus indices interchanged; see eq. (D20) below). For further discussion of the effect of choice of basis, see the remarks following eq. (D20).

Conversion to the Ashtekar language is straightforward. $C$ is essentially the (four dimensional) Ashtekar field strength, since the $C$ tensor is self-dual, and in empty space the Weyl tensor is the full Riemann tensor.

$$
(4)^{AB C_{\alpha \beta \gamma}}_{\delta \epsilon \delta} = e^{Aa} e^{Bb} C_{\alpha \beta \gamma}^{ab cd}. \quad (D11)
$$

The four-dimensional field strengths $(4)^F_{\alpha \beta \gamma}^{\delta \epsilon}$ may be replaced by 3+1 quantities $F$ by using standard formulas.

$$
(4)^{TM}_{\alpha \beta \gamma} = -i \epsilon \delta F_{\alpha \beta \gamma}^{M} / 2; \\
(4)^{MN}_{\alpha \beta \gamma} = \sigma \epsilon \delta \delta F_{\alpha \beta \gamma}^{M} / 2. \quad (D12)
$$
M, N, S = space only. σ is a new phase which appears at the 3+1 reduction step. I choose \( \sigma = -1 \), for reasons explained in Appendix A of II. This phase ( unlike \( \delta \) ) merely changes the overall sign of the \( C_1 \), and I shall not keep track of the \( \sigma \) dependence in the future.

Useful corollaries of eq. (D12) are

\[
\begin{align*}
(4) F^\pm_{cd} &= (i/\sqrt{2})F^\pm_{cd}[(\delta \pm 1)]/2; \\
(4) F^{U\pm}_{cd} &= (i/\sqrt{2})F^{U\pm}_{cd}[(\delta \mp 1)]/2. \quad (D13)
\end{align*}
\]

If tetrads eqs. (D9) and (D10) and field strengths eq. (D13) are inserted into eq. (D7) for \( C_1 \), the result for \( \delta = +1 \) is

\[
C_1 = i[-F^{+}_{cd}e^c_T + F^{+}_{cd}e^c_Z]/2. \quad (D14)
\]

(For \( \delta = -1 \) replace + by -.) For any metric, typically the Lorentz boosts are gauge-fixed by demanding that three of the tetrads vanish:

\[
e^t_M = 0, M = \text{space}. \quad (D15)
\]

For the special case of the plane wave metric, the gauge-fixing of the XY Gauss constraint and xy spatial diffeomorphism constraints imply that four more tetrads vanish.

\[
e^x_X = e^y_Y = e^z_Z = e^t_T = 0. \quad (D16)
\]

The tetrad matrix reduces to two 2x2 subblocks which link x, y to X, Y (or ±) and z, t to Z, T. Therefore the first term in eq. (D14) (and only the first term) contains an \( F^{+}_{td} \) term, \( d = x \) or \( y \), with unacceptable time derivatives of the "coordinate" \( A^+_d \). I eliminate this term using the classical equations of motion, which are

\[
(4) F^{AB}_{cd}e^c_A = 0, \quad (D17)
\]

or after 3+1 splitup, and setting \( B = + \),

\[
0 = (4) F^{A+}_{cd}e^c_A = (4) F^{−+}_{cd}e^c_− + (4) F^{U+}_{cd}e^c_U + (4) F^{V+}_{cd}e^c_V \\
= F^{+}_{cd}e^c_V \\
= F^{+}_{cd}e^c_T + F^{+}_{cd}e^c_Z. \quad (D18)
\]

On the second line the \( (4) F^{U+} \) term vanishes because of eq. (D13), and the \( (4) F^{+}_{cd}e^c_− \) may be dropped because at the next step the entire
term will be contracted with $e^d_+$. When eq. (D18) is inserted into eq. (D14), the result is

$$C_1 = i F^+_{ed} e^d_+ e^d_+.$$  \hspace{1cm} (D19)

This is not quite Ashtekar form, because the triads must be densitized. Also, the $c$ index can equal $z$ only, because of the gauge conditions eqs. (D15) and (D16). The final result is (for $C_5$ and $C_3$ also, since they are calculated similarly)

$$C_1 = i F^+_{zd} E^d_+/\langle 2 \rangle E_+ ;$$

$$C_5 = i F^-_{zd} E^-_+/\langle 2 \rangle E_+ ;$$

$$C_3 = F^Z_{xy}/2 E^Z_z.$$  \hspace{1cm} (D20)

$\langle 2 \rangle E_+$ is the determinant of the 2x2 XY subblock of the tetrad matrix. The results for $\delta = -1$ are the same, except for overall phases, and interchange of + and - everywhere.

In the case that the wave is unidirectional, the results eq. (D20) are consistent with the BPR constraints. For example, if the wave is right-moving, then the principal vector is $k$, associated with the tensor $C_1$. From eq. (14), $A^-_a$ vanishes, implying that ($C_1$ is finite, while) $C_5$ vanishes.

The $C_i$ of eq. (D20) were calculated in a specific basis; in the language of section III, they are not even kinematic, much less physical. In particular the factors of $\langle 2 \rangle E_+$ in eq. (D20) are basis dependent. For example, suppose one shifts from the tetrad basis, eq. (D9), to a basis in which $k^a$ is affinely parameterized. (In the tetrad basis one has $k^b k_{ab} = \lambda k_a$, $\lambda \neq 0$.) Then the $\langle 2 \rangle E_+$ factor in $C_1$ disappears, replaced by a factor of $E^Z_z$. One could continue to rescale $C_1$ in this manner, until it became density weight unity; then it could be sandwiched between holonomies and integrated over $z$ to make it kinematic. Presumably one would have to tolerate some degree of non-polynomiality in the final result.

Although it would not be hard to make the transverse operator $C_1$ kinematical, it is unlikely that $C_1$ could be made physical. Gravitational radiation is closely identified with transversality only in the linearized theory. In the full classical theory, scattering of two transverse waves produces a $C_3$ Coulomb component \[16\].
Presumably, then, the commutator of any purely transverse operator with the Hamiltonian will not be especially simple, even in the classical theory. This is one reason why the main body of the paper concentrates on the BPR operators, rather than the Weyl tensor. Although it is unlikely that any transverse criterion could be made physical, a kinematical criterion for transversality should be both feasible and useful.

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