WOBBLY TYPE LOCUS ON FANO MANIFOLDS OF PICARD NUMBER ONE AND DRINFELD’S CONJECTURE

SARBESWAR PAL

Abstract. Let $X$ be a Fano manifold of Picard number one. In this article, we will introduce a notion of wobbly type points in $X$ and show that locus of wobbly type points forms a divisor under certain hypothesis. For the case of moduli space of stable vector bundles on curves, we will show that the wobbly type locus and the usual wobbly locus coincide which gives a proof of Drinfeld’s conjecture which claims that the wobbly locus is pure of co-dimension one.

1. Introduction

Let $X$ be smooth projective Fano variety of Picard number one over the field of complex numbers. There are two main motivations behind this article.

First one is a folklore that says that if $X$ is different from the projective space, then any surjective endomorphism $X \to X$ must be bijective. The conjecture was proved for homogeneous spaces in [7], for hypersurfaces of the projective space in [1] and for Fano manifolds containing a rational curve with trivial normal bundle in [6]; the last work solves the Conjecture in case $\dim(X) = 3$. Recently, Hwang and Nakayama suggested in [15] a strategy to this conjecture. Recall that for a surjective endomorphism $f : X \to X$ a reduced divisor $D$ is called a completely invariant divisor if $(f^*D)_{\text{red}} = D$.

Then Hwang and Nakayama have given the following criterion:

Theorem 1.1 (Hwang-Nakayama). Let $X$ be a Fano manifold of Picard number one different from the projective space. If an endomorphism of $X$ is étale outside a completely invariant divisor, it is bijective.

The second motivation is the Drinfeld’s conjecture.

Let $C$ be a smooth projective complex curve of genus $g \geq 2$ and let $K_C$ be its canonical line bundle. Fixing a line bundle $\delta$ of degree $d$ we consider the coarse moduli space $\mathcal{M}_{\mathcal{C}}(r, \delta)$ parameterizing semi-stable rank-$r$ vector bundles of fixed determinant of degree $d$ over $C$. We recall that a vector bundle $E$ is called very stable if $E$ has no non-zero nilpotent Higgs field $\phi \in H^0(X, \text{End}(E) \otimes K)$. A stable vector bundle is called wobbly if it is
not very stable. G. Laumon [10, Proposition 3.5] proved, assuming \( g \geq 2 \), that a very stable vector bundle is stable and that the locus of very stable bundles is a non-empty open subset of \( \mathcal{M}_C(r, \delta) \). Hence the locus of wobbly bundles is a closed subset \( \mathcal{W}_C \subset \mathcal{M}_C(r, \delta) \).

**Conjecture 1.2** (Drinfeld). [10, Remarque 3.6 (ii)] \( \mathcal{W}_C \) is of pure codimension 1.

The term “wobbly” was introduced in the paper [9]. The conjecture have been proved completely for rank 2 case by current author of this article and jointly with C. Pauly in [11] and [12]. Note that if \( r \) and \( d \) are co-prime to each other the \( \mathcal{M}_C(r, \delta) \) is smooth. It is also known that in the co-prime case \( \mathcal{M}_C(r, \delta) \) is a Fano manifold of Picard number one.

In view of Theorem 1.1, to attack the folklore conjecture, the first strategy is to construct a divisor \( D \) which may serve the purpose of the Theorem 1.1. In this article, we will construct a divisor \( D \) which remains invariant under any non-constant endomorphism of \( X \). The main idea for that is to introduce the notion of wobbly type points on a Fano manifold \( X \) which are basically points in \( X \) lying on some minimal rational curve which is not free. The name wobbly came from the notion of wobbly bundles in the moduli space of stable vector bundles over curves. Then we will show that under certain hypothesis, the locus of wobbly type points form a divisor on \( X \) which remains invariant for any non constant surjective endomorphism of \( X \).

At the moment we are unable to show that the divisor of wobbly type points is completely invariant. But we believe that it is completely invariant and any surjective endomorphism of \( X \) is étale outside the wobbly type locus. Hopefully, in our future work we shall come up with the proof of the desired facts. However, our construction has another nice application which gives a proof of the Drinfeld’s conjecture.

**Organization of the paper:**
In section 2 we will recall some basic facts which we need in the subsequent sections.

In section 3 we will formally define the notion of wobbly type points in \( X \) and we will show that the locus of wobbly type points in \( X \) forms a divisor \( Y \subset X \) under certain hypothesis. More precisely we will prove the following theorem.

**Theorem 1.3.** Let \( X \) be a Fano manifold of dimension \( n \geq 3 \) such that \( c_2(X) \) is given by a two cycle \( \lambda c_1^2 + [Z] \), where \([Z]\) is a two cycle such that pullback to any surface is non-negative and \( \lambda \geq \frac{1}{2} \). Then the locus of non-free minimal rational curves in \( X \), i.e., the locus of wobbly type points \( Y \) is an invariant divisor for any endomorphism of \( X \).
In section 4, we will show that the invariant divisor constructed in theorem 1.3 coincides with the wobbly locus when $X = \mathcal{M}_C(r, \delta)$. More precisely we will prove the following theorem.

**Theorem 1.4.** A stable vector bundle $E \in \mathcal{M}_C(r, \delta)$ is wobbly if and only if it passes through a minimal rational curve which is not free.

This settles the Drinfeld’s conjecture.

2. Preliminaries

In this subsection we will recall some basic definitions and facts about Fano manifold which are mostly taken from [14]. Let $X$ be a Fano manifold of dimension $n$. A parametrized rational curve in $X$ is a morphism $\mathbb{P}^1 \to X$ which is birational over its image. We will not distinguished parametrized rational curves from its image $f(\mathbb{P}^1)$ and we call $f$, a rational curve in $X$.

Recall that any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. Thus given a rational curve $f : \mathbb{P}^1 \to X$, the pull back of the tangent bundle $T_X$ splits as

$$f^*T_X = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus ... \oplus \mathcal{O}(a_n), a_1 \geq a_2 \geq ... \geq a_n.$$  

The rational curve $f$ is said to be free if all the integers $a_1, a_2, ..., a_n$ in the above equation are non-negative.

From now onwards, we will assume $X$ is a Fano manifold of dimension $n$ with Picard number 1.

It is known that through every point of $X$, there is a rational curve in $X$. It is also known that for a general point $x \in X$, there is a free rational curve through $x$.

The degree of $f^*K_X^{-1}$ will be called the anti-canonical degree of the curve $f : \mathbb{P}^1 \to X$. A component $K$, of the Hilbert scheme of rational curves in $X$ is said to dominating component if the evaluation map $\mathbb{P}^1 \times K \to X$ is dominant. A minimal rational component is a dominating component in the Chow space of $X$ of minimal degree, whose members are called minimal rational curves. A rational curve $f$ is said to be standard if

$$T_X|_{f(\mathbb{P}^1)} := f^*T_X \simeq \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{d-2} \oplus [\mathcal{O}]^{n-d+1},$$

where $d$ is the anticanonical degree of $f$.

**Theorem 2.1.** A generic member of a minimal rational component is standard.

**Proof.** See [14, Theorem 1.2] \hfill \Box

**Definition 2.2.** The direct summand, $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^{d-2}$ of $f^*T_X$ of a standard minimal rational curve is said to be the strictly positive part of $f^*T_X$.

Choose a minimal rational component $K$. For a general point $x \in X$, let $K_x$ be the normalization of the Chow space of members of $K$ through $x$. Then we have...
Theorem 2.3. \( K_x \) is a union of finitely many smooth algebraic varieties of dimension \( d - 2 \).

Proof. See [14, Theorem 1.3] \( \square \)

Remark 2.4. Note that by theorem 2.3, the minimal rational curves in \( K \) cover a Zariski dense open subset of \( X \). Again as by theorem 2.1, a generic member of \( K \) is standard, the standard minimal curves in \( K \) cover a Zariski dense open subset of \( X \).

Proposition 2.5. Let \( X \) be a smooth quasi-projective variety defined over a field \( k \) and let \( f : \mathbb{P}^1 \rightarrow X \) be a rational curve.

(a) If \( f \) is free, the evaluation map

\[
ev : \mathbb{P}^1 \times \text{Mor}(\mathbb{P}^1, X) \rightarrow X
\]

is smooth at all points of \( \mathbb{P}^1 \times \{[f]\} \).

(b) If there is a scheme \( M \) with a point \( m \in M \) and a morphism \( e : \mathbb{P}^1 \times M \rightarrow X \) such that \( e|_{\mathbb{P}^1 \times m} = f \) and the tangent map to \( e \) is surjective at some point of \( \mathbb{P}^1 \times m \), the curve \( f \) is free.

Proof. See [5, Proposition 2.12] \( \square \)

2.1. Vector bundles on ruled surface. Let \( S = C \times \mathbb{P}^1 \) be a ruled surface over a smooth projective curve \( C \) and \( p_1 : S \rightarrow C \) be the projection. Then any torsion free sheaf \( E \) on the ruled surface \( S \), its restriction to a generic fibre \( p_1^{-1}(t) = S_t \) has the form

\[
E|_{S_t} = \bigoplus_{i=1}^{n} \mathcal{O}_{S_t}(\alpha_i)^{\oplus r_i}, \alpha_1 > \alpha_2 > ... > \alpha_n.
\]

The \( \alpha = (\alpha_1^{\oplus r_1}, ..., \alpha_n^{\oplus r_n}) \) is called the generic splitting type of \( E \). Any such \( E \) admits a relative Harder-Narasimhan filtration

\[
0 = E_0 \subset E_1 \subset ... \subset E_n = E
\]

of which the quotient sheaves \( F_i = E_i/E_{i-1} \) are torsion free with generic splitting type \( (\alpha_i^{\oplus r_i}) \) respectively. Then it is easy to see that

\[
2c_2(E) = 2 \sum_{i=1}^{n} c_2(F_i) + 2 \sum_{i=1}^{n} c_1(E_{i-1})c_1(F_i) = 2 \sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - \sum_{i=1}^{n} c_1(F_i)^2.
\]

Lemma 2.6. Any torsion free sheaf \( E \) of rank \( r \) on a ruled surface, with generic splitting type \( (0^{\oplus r}) \), must have \( c_2(E) \geq 0 \).

Proof. See [17, Lemma 2.1]. \( \square \)
Definition 2.7. A rank $r$ vector bundle $E$ on a ruled surface $X \times \mathbb{P}^1$ with generic splitting type $0^{\oplus r}$ is said to have a jumping line $S_t = p^{-1}_1(t)$ at $t \in X$ if

$$E|_{S_t} = \bigoplus_{i=1}^n \mathcal{O}_{S_t}(\alpha_i)^{\oplus r_i}, \alpha_1 > ... > \alpha_n$$

with the type $(\alpha_1^{\oplus r_1}, ..., \alpha_n^{\oplus r_n})$ different from $(0^{\oplus r})$.

Let $E$ be a rank $r$ vector bundle on a ruled surface $S = X \times \mathbb{P}^1$, with generic splitting type $0^{\oplus r}$ and $S_t$ be a jumping line. Then we can perform the elementary transformation on $E$ along $S_t$, by taking $F$ to be the kernel of the surjective homomorphism $\varphi : E \to E|_{S_t} \to \mathcal{O}_{S_t}(\alpha_n)^{\oplus r_n}$. Then clearly, we have the following exact sequence

$$(2.4) \quad 0 \to F \to E \to \mathcal{O}_{S_t}(\alpha_n)^{\oplus r_n} \to 0.$$ 

Lemma 2.8. $c_1(F) = c_1(E) - r_n S_t$ and $c_2(F) = c_2(E) + r_n \alpha_n$.

Proof. By the exact sequence $(2.4)$, the computation is straightforward. \qed

Lemma 2.9. If $c_2(E) \geq 1$ and $E$ has generic splitting type $(0^{\oplus r})$, then $E$ has at least one jumping line $S_t$ and at any jumping line $S_t$, with splitting type $(\alpha_1^{\oplus r_1}, ..., \alpha_n^{\oplus r_n})$, one must have $\alpha_n < 0$.

The proof can be found in [17, Lemma 2.4]. For completeness we include it here.

Proof. The bundle $E$ has at least one jumping line. Otherwise, $E$ will be a pullback of a vector bundle over $X$, which is impossible as $c_2(E) \geq 1$ and $\alpha_n < 0$ follows from Lemma 2.8. \qed

2.2. Families of Vector bundles on $\mathbb{P}^1$. Let $V$ be a family of vector bundles on $\mathbb{P}^1$. Suppose that the family contains a vector bundle $V_0$ with a given splitting type. Then the expected co-dimension of the locus of vector bundles in $V$ with that splitting type is $h^1(\text{End}(V_0))$ (see [2, Lemma 2.4] and [3]). We have the following Lemma due to Coskun and Riedl [3].

Lemma 2.10. Let $A$ be a vector bundle on $\mathbb{P}^1$ and consider the family $F$ of vector bundles given by surjective kernels of homomorphisms in $\text{Hom}(A, \mathcal{O}_{\mathbb{P}^1}(k))$. Then the locus of bundles in $F$ with the same splitting type as $V$ has co-dimension at least $h^1(\text{End}(V)) - h^1(A \otimes V^*)$.

Proof. Briefly, the Kodaira-Spence map $\kappa : \text{Hom}(A, \mathcal{O}_{\mathbb{P}^1}(k)) \to \text{Ext}^1(V, V)$ naturally factors into

$$\text{Hom}(A, \mathcal{O}_{\mathbb{P}^1}(k)) \to^\mu \text{Hom}(V, \mathcal{O}_{\mathbb{P}^1}(k)) \to^\nu \text{Ext}^1(V, V),$$

where $\mu$ and $\nu$ are the natural morphisms that occur when applying $\text{Hom}(-, \mathcal{O}_{\mathbb{P}^1}(k))$ and $\text{Hom}(V, -)$ to the sequence $0 \to V \to A \to \mathcal{O}_{\mathbb{P}^1}(k) \to 0$, respectively. Since $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k)) = 0$, the morphism $\mu$ is surjective and the rank
of the morphism $\nu$ is at least $h^1(\text{End}(V)) - h^1(A \otimes V^*)$. Hence, the codimension of the locus of bundles with the given splitting type in the family is the same as the codimension in the versal deformation space (see [2, Lemma 2.4] and [1]).

\section{Proof of theorem \ref{thm:main}}

Let $K$ be a minimal rational component of anti-canonical degree $d$ as in section 2. $K$ can be obtained by taking quotient of corresponding component of $\text{Mor}_d(\mathbb{P}^1, X)$, where $\text{Mor}_d(\mathbb{P}^1, X)$ is the variety of degree $d$ morphisms $\mathbb{P}^1 \to X$ with respect to $-K_X$, by 3–dimensional automorphism group of $\mathbb{P}^1$. Let $f : \mathbb{P}^1 \to X$ be a general element in $K$, which is standard. Then $h^1(f^*(TX)) = 0$, where $TX$ is the tangent bundle of $X$. Hence $K$ is smooth at $f$ and it has dimension $= h^0(f^*(TX)) - 3 = n + d - 3$.

On the other hand, by Proposition 10, in [19], there is a subvariety $K' \subset K$ of dimension $n - 1$, such that restriction of the evaluation map to $\mathbb{P}^1 \times K'$ is dominant. In particular, the evaluation map is generically finite. By [5, Remark 2.6], $K'$ is smooth in codimension 2 and a general curve is complete.

Thus a general curve in $K'$ do not intersect the singular locus. Hence a general curve in $K'$ is smooth and complete.

\begin{definition}
A minimal rational curve in $K$ is called special rational curve, if $a_n < 0$, in equation \ref{eq:an}.
\end{definition}

\begin{definition}
A point in $x \in X$ is said to be wobbly type if there is a special rational curve in $X$ containing $x$ and the set of wobby type points in $X$ is said to be wobbly type locus in $X$.
\end{definition}

\begin{remark}
Note the the locus of wobbly type points need not be non-empty. For example, if the tangent bundle of $X$ is nef, then locus of wobby type points is empty.
\end{remark}

\begin{theorem}
Let $X$ be a Fano manifold of dimension $n \geq 3$ such that $c_2(X)$ is given by a two cycle $\lambda c_1^2 + [Z]$, where $[Z]$ is a two cycle such that the pull back to any smooth surface is non-negative and $\lambda \geq \frac{1}{2}$. Then the set of special rational curves form a divisor in $K'$.
\end{theorem}

\begin{proof}
Note that the curves in $K'$, cover $K'$. Thus we get a collection of smooth projective curves $\{D_t\}$, which cover at least a dense open subset of $K'$. Also we have a collection of morphisms $\varphi_t : \mathbb{P}^1 \times D_t \to X$. Since a general element of $K'$ is free, each $D_t$ in the collection contains a standard minimal rational curve. Thus the vector bundle $\varphi_t^*TX$ has generic splitting type $(2, 1^{\oplus d-2}, 0^{\oplus n-d+1})$. Thus one has a relative Harder-Narasimhan filtration of the form

\begin{equation}
0 = E_0 \subset E_1 \subset E_2 \subset E_3 = \varphi_t^*(TX),
\end{equation}

of which the quotient sheaves $F_1 = E_1$ has splitting type $(2), F_2 = E_2/E_1$ has generic splitting type $(1^{\oplus (d-2)})$ and $F_3 = E_3/E_2$ has generic splitting...
type \((0^{\oplus(n-d+1)})\).
Then it is easy to see that
\[
(3.2) \quad 2c_2(\varphi_1^*TX) = 2 \sum_{i=1}^{3} c_2(F_i) + c_1(\varphi_1^*TX)^2 - \sum_{i=1}^{3} c_1(F_i)^2
\]
Note that
\[
c_1(\varphi_1^*(TX)) = p_1^*O_{\mathbb{P}^1}(d) + p_2^*O_D(\tilde{d})
\]
for some positive integer \(\tilde{d}\), where \(p_1, p_2\) denote the projection maps to the first and second factor respectively. Let \(F'_2 := F_2 \otimes p_1^*O_{\mathbb{P}^1}(1)\). Note that
\[
c_1(F_1) = p_2^*O_{D_1}(d_1) + p_1^*O_{\mathbb{P}^1}(2),
\]
c_1(F_2) = \(p_2^*O_{D_1}(d_2) + p_1^*O_{\mathbb{P}^1}(d - 2)\), and \(c_1(F_3) = p_2^*O_{D_1}(d_3)\) for some integers \(d_1, d_2, d_3\) and we have \(d_1 + d_2 + d_3 = \tilde{d}\). Thus we have,
\[
(3.3) \quad c_2(\varphi_1^*TX) = \sum_{i=1}^{3} c_2(F_i) + \frac{1}{2}c_1(\varphi_1^*TX)^2 - 2 \sum_{i=1}^{3} c_1(F_i)^2
\]
\[
= c_2(F'_2) \otimes p_1^*(O_{\mathbb{P}^1}(1)) + c_2(F_3) + d\tilde{d} - 2d_1 - d_2(d - 2)
\]
\[
= c_2(F'_2) + (d - 3)c_1(F'_2), p_1^*(O_{\mathbb{P}^1}(1)) + c_2(F_3) + d\tilde{d} - 2d_1 - d_2(d - 2)
\]
\[
= c_2(F'_2) + c_2(F_3) + d\tilde{d} - 2d_1 - d_2.
\]
Now \(c_2(\varphi_1^*(TX)) = \varphi_1^*(c_2(TX)) = \varphi_1^*(\lambda c_1^2 + [Z])\) which is \(\geq \frac{1}{2} \varphi_1^*(c_1^2) = d\tilde{d}\) as \(\lambda \geq \frac{1}{2}\) and pull back of \([Z]\) to \(\mathbb{P}^1 \times D_1\) is non-negative. Thus \(c_2(F'_2) + c_2(F_3)\) is strictly positive, i.e., \(c_2(F'_2) + c_2(F_3) \geq 1\).
In other words, at least one of \(c_2(F'_2)\) or \(c_2(F_3)\) is strictly positive. Hence there is an non-empty open set \(U\) in the family of curves such that for any \(t \in U\) on \(\mathbb{P}^1 \times D_t\), \(c_2(F'_2) \geq 1\) or \(c_2(F_3) \geq 1\) or both \(c_2(F'_2)\) and \(c_2(F_3)\) are strictly positive. Now we consider these possibilities case by case.

Case I: \(c_2(F_3) \geq 1\) on \(\mathbb{P}^1 \times D_t\) for all \(t \in U\):
Note that \(F_3\) has generic splitting type \((0^{\oplus(n-d+1)})\). Thus by Lemma 2.9, \(F_3\) has at least one jumping line with splitting type \((a_1, ..., a_{n-d+1})\) where \(a_{n-d+1} < 0\). Thus for every general point \(t\) in the family of curves, there is an effective divisor in the curve \(D_t\), such that the minimal rational curves associated to the points in the support of the divisor are special. Note that being the complement of the set of free rational curves in \(K'\), the set special rational curves is closed in \(K'\). Let \(V\) be the closed subset of special curves in \(K'\). As the collection of curves \(\{D_t\}\) cover a dense open subset of \(K'\), and each curve \(D_t\) in the collection intersects \(V\) in a divisor in \(D_t\), \(V\) is a divisor in \(K'\).

Case II: \(c_2(F'_2) \geq 1\) on \(\mathbb{P}^1 \times D_t\) for all \(t \in U\):
Note that \(F'_2\) has generic splitting type \((0^{\oplus d-2})\). Thus by Lemma 2.9, \(F'_2\) has at least one jumping line with splitting type \((a_1, ..., a_{d-2})\) where \(a_{d-2} < 0\). If there is a non-empty open set \(U'\) in \(U\) such that for \(t \in U', a_{d-2} < -1\), then \(F_2 = F'_2 \otimes p_1^*(O_{\mathbb{P}^1}(1))\) will not be free. Then as in Case I, one can conclude the theorem.
Let us assume that for a general element \(t \in U, a_{d-2} = -1\) and \(c_2(F_3) = 0\). If
q \in D_1$ be a point corresponding to the jumping line of $F'_2$, then the minimal rational curve corresponding to it is free but non-standard. Thus we will get a divisor in $\mathcal{K}'$ consisting of free rational curves which are not standard. Note the versal deformation space of a general point in $\mathcal{K}'$ can be thought as an open subset in $\mathbb{P}(\text{Hom}(A, \mathcal{O}_{\mathbb{P}^1})) \cap \mathcal{K}'$, where $A = [\mathcal{O}(-1)]^d - 2 \oplus [\mathcal{O}]^{n-d}$. Thus Using Lemma 2.10 one can easily see that the subvariety of $\mathcal{K}'$ consisting non-standard, free rational curves has co-dimension at least 2, a contradiction.

Let us assume that $X$ is a Fano manifold of dimension $n \geq 3$ and satisfies the hypothesis of Theorem 3.4. Let $Y'$ be the divisor in $\mathcal{K}'$ consisting special rational curves. Since $\mathcal{K}'$ is smooth away from a co-dimension 2 subvariety, a general element of $Y'$ is smooth in $\mathcal{K}$. Thus every direct summand of $f^*TX$ for a general $f \in Y'$ has degree at least $-1$. Then as in the previous theorem the versal deformation space of a general point in $\mathcal{K}'$ can be thought as an open subset in $\mathbb{P}(\text{Hom}(A, \mathcal{O}_{\mathbb{P}^1})) \cap \mathcal{K}'$, where $A = [\mathcal{O}(-1)]^d - 2 \oplus [\mathcal{O}]^{n-d}$. Thus Using Lemma 2.10 one can easily see that the subvariety of $\mathcal{K}'$ consisting of rational curves such that the co-normal bundle has at least two direct summand of positive degree, has co-dimension at least 2. Since $Y'$ is a divisor, for a general element $f \in Y'$, $f^*TX$ has exactly one direct summand of degree $-1$. Thus $\mathcal{K}$ contains an element $f : \mathbb{P}^1 \to X$ such that $f^*TX$ contains exactly one direct summand of degree $\leq -1$. Let $V$ be the subvariety of $\mathcal{K}$ consisting special rational curves. Then a general element of each component of $V$ has the property that the restriction of $TX$ to the curve has exactly one direct summand of degree $-1$. Hence it fills a divisor in $X$ under the evaluation map. Let $Y$ be the union of such divisors of $X$, where the union is taken over the dominating minimal rational components in $\text{Mor}(\mathbb{P}^1, X)$. Since there are only finitely many minimal rational components, $Y$ is a divisor in $X$.

**Remark 3.5.** Let $X^\text{free} = X \setminus Y$. Then clearly $X^\text{free}$ is an open dense subset. Using Proposition 2.5 one can easily see that any minimal rational curve whose image intersects $X^\text{free}$ is free.

**Proof of Theorem 1.3:** Let $\varphi : X \to X$ be a finite surjective endomorphism. Let $p \in Y$ and $f : \mathbb{P}^1 \to X$ be a special rational curve containing $p$. Let $g : \mathbb{P}^1 \to f^* X \to \varphi^* X$ be the composition. If possible, let $\varphi(p) \in X^\text{free}$. If $g$ is birational to its image, then $g$ has to be a minimal rational curve intersecting $X^\text{free}$, hence free, a contradiction as $f$ is special. Thus $g$ is not birational to its image. Hence $g$ factors through $g : \mathbb{P}^1 \to h^* \mathbb{P}^1 \to h' \to X$, where $h$ is a morphism of degree $m$ and $h'$ is birational to its image. Here $m$ is the degree of $\varphi$. Therefore, $h' : \mathbb{P}^1 \to X$ is a minimal rational curve passing through $\varphi(p)$, hence free. Note that if $E$ is any vector bundle on $X$, then $\mu(\varphi^* E) = m \mu(E)$, where $m$ is the degree of $\varphi$. Thus the Harder-Narasimhan filtration of $(\varphi \circ f)^* TX$ is the pull back of the Harder-Narasimhan filtration.
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of $f^*TX$. Thus the slope of each quotient in the filtration is $m$-times of the slope of the corresponding quotient of the filtration of $f^*TX$.

Now $\varphi \circ f = h' \circ h$. Thus if $h'$ is free then $f$ is free, a contradiction. Thus $\varphi(Y) \subset Y$.

4. Wobbly locus on moduli of vector bundles on curves

Let $C$ be a smooth projective curve of genus $g \geq 5$ over complex numbers. Let $M_C(r, \delta)$ be the moduli space of semi-stable vector bundles on $C$ of rank $r$ with fixed determinant $\delta$ of degree $d$. We will assume that $r$ and $d$ are co-prime to each other. In this situation, it is known that semistable bundles coincide with stable bundles and $M_C(r, \delta)$ is a smooth projective variety. It is also known that $M_C(r, \delta)$ is a Fano manifold of Picard number 1.

From now onwards, we will write $M$ instead of $M_C(r, \delta)$.

Let $E$ be the universal bundle on $M \times C$ which is known to exist. Then the tangent bundle $T_M$ can be identified with $R_1\pi^*(\text{Ad}(E))$, where $\text{Ad}(E)$ denotes the bundle of trace free endomorphisms of $E$ and $\pi: M \times C \to M$ be the projection morphism. For a stable bundle $E$ on $C$, $H^0(C, \text{Ad}(E)) = 0$, thus $\pi^*(\text{ad}(E)) = 0$. Also we have $R^i\pi^*(\text{Ad}(E)) = 0$ for all $i \geq 2$. Thus $\pi^!(\text{Ad}(E)) = -T_M$.

Now using Grothendieck-Riemann-Roch, we have $c_2(T_M) = \frac{1}{2}(c_1^2(T_M) + \pi^*c_3(\text{Ad}(E)))$.

Thus $M$ satisfies the hypothesis of the theorem 3.4. Hence subset $Y$ constructed in theorem 1.3 is a divisor in $M$.

**Definition 4.1.** A vector bundle $E$ is called very stable if $E$ does not admit any nonzero Higgs field. The complement of very stable bundles in $M$ is known as wobbly locus.

**Theorem 4.2.** A stable vector bundle $E \in M$ is wobbly if and only if there is a special rational curve $l$ in $M$ containing $E$. In particular, the wobbly locus coincides with the divisor $Y$ constructed in theorem 1.3.

**Proof.** Let $l$ be a special rational curve in $M$. Let $h: T^*M \to W$ be the restriction of the Hitchin map to the co-tangent bundle of $M$, where $W$ is the Hitchin space which is known to be affine space of dimension $n$, where $n$ is the dimension of $M$. Let $h|_l$ be the restriction of $h$ to $T^*M|_l$. Since $l$ is special, $T^*M|_l$ contains a line subbundle $O_{\mathbb{P}^1}(m)$ with $m \geq 1$. Thus there is sheaf inclusion $O_{\mathbb{P}^1}(1) \to T^*M|_l$. On the other hand, it is known that the total space of $O_{\mathbb{P}^1}(1)$ is isomorphic to $\mathbb{P}^2 \setminus \{p\}$, where $p \in \mathbb{P}^2$ is a point. Now by Hartogs theorem any regular function on $\mathbb{P}^2 \setminus \{p\}$, can be extended to $\mathbb{P}^2$ and hence constant. Thus the composition of the sheaf inclusion $O_{\mathbb{P}^1}(1) \to T^*M|_l$ with the Hitchin map is zero. Thus the restriction of the Hitchin map to the cotangent space $T^*_EM \to W$ has positive dimensional fiber for every $E \in l$. Thus $E$ is not very stable bundle [20 Corollary 1.2].

Conversely, let $E$ be a wobbly bundle. Let $K_E$ be the variety of minimal rational curve passing through $E$. Following the notations as in [13], let
\( \hat{S} \) be the closure of the union of the smooth rational curves in \( T^*M \) given by \( H^0(l, T^*M) \) as \( l \) varies over \( K \). Let \( S \subset \mathbb{P}(T^*M) \) be the corresponding projective variety. Let \( S_E \) be the intersection \( S \cap \mathbb{P}(T^*_E M) \).

Let \( D \) be the subset of the Hitchin space \( \mathcal{W} \) such that the corresponding spectral curves are singular. It is known that \( D \) is irreducible hypersurface in \( \mathcal{W} \) [21, Corollary 1.5 and Remark 1.7] and \( h^{-1}(D) = S \) [13, Theorem 4.4]. Then one can see that

\[
    h^{-1}(D) \cap \mathbb{P}(T^*_E M) = S_E = \{ [s(E)] : s \in \cup l \in K_E \mathbb{P}(H^0(l, T^*M)) \}.
\]

Since \( E \) is wobbly, there is a non-zero vector \( v \in T^*_E M \) such that \( h(v) = 0 \).

On the other hand, since \( 0 \in D \), there is rational curve \( l \in K_E \) and a section \( s \in H^0(l, T^*M) \) such that \( v \in [s(E)] \). In particular \( h(s(E)) = 0 \). If \( T^*_M |_l \) contain a line subbundle of degree \( \geq 1 \), then we are done.

Let us assume that \( T^*_M |_l \) does not contain any line subbundle of degree \( \geq 1 \). Then \( s \) is nowhere vanishing section and \( s(l) \) is a rational curve in \( T^*_M \) not intersecting the zero section of \( T^*_M \). Since \( h \) is a morphism with affine target space, \( h(s(l)) = 0 \). Thus all the bundles in \( l \) are wobbly.

Let \( ev : \mathbb{P}^1 \times K_E \to M \) be the evaluation map. Since \( h^1(l, T^*M) = 0 \), for all \( l \in K_E \), \( (p_2)_*(ev^*T^*M) \) is a vector bundle on \( K_E \) of rank \( h^0(l, T^*M) = n - d + 1 \).

Claim: The vector bundle \( (p_2)_*(ev^*T^*M) \) is trivial.

Proof of the claim: Note that we have a relative Harder-Narasimhan filtration of the form

\[
    0 = E_0 \subset E_1 \subset E_2 \subset E_3 = ev^*T^*M,
\]

of which the quotient sheaves \( F_1 = E_1 \) has constant splitting type \( (0^{n-d+1}) \).

Thus \( F_1 \simeq p_2^*F \) for some vector bundle \( F \) of rank \( n - d + 1 \) on \( K_E \). On the other hand, since \( ev(\{0\} \times K_E) = E \), all the Chern classes of \( F_1 \) vanish and \( ev^*T^*_M |_{\{0\} \times K_E} \) is trivial. Now \( F_1 |_{\{0\} \times K_E} \subset ev^*T^*_M |_{\{0\} \times K_E} \). Since a subbundle of a trivial bundle which has all the Chern classes are zero is necessarily trivial, \( F \) is trivial. Hence \( F_1 \) is trivial. We also have that \( H^0(ev^*T^*M) = H^0(F_1) \). Thus \( (p_2)_*(ev^*T^*M) \) has \( n - d + 1 \) no-flipping sections. Therefore, \( (p_2)_*(ev^*T^*M) \) is trivial.

We have a natural morphism \( \Psi : \mathbb{P}((p_2)_*(ev^*T^*M)) \to S_E \), which takes \( (l, [s]) \) to \( [s(E)] \). Let \( \bar{s} \) be a section of \( (p_2)_*(ev^*T^*M) \) passing through \( (l, [s]) \), where \( l \) and \( s \) as earlier. Then \( h(\Psi(\bar{s}(K_E))) \) is constant. Since \( (l, [s]) \in \bar{s}(K_E) \) and \( h(\Psi((l, [s]))) = 0 \), every point in the locus of \( K_E \) is wobbly.

Let \( M_E \) be the locus of \( K_E \) and \( E_1 \) is a general point of \( M_E \). Then similarly \( M_{E_1} \) is contained in the wobbly locus. Since \( M \) is minimally rationally connected, that is, through any two points there is a chain of minimal rational curves connecting them, continuing the process, one can fill \( M_{\text{free}} \) by wobbly bundles, which is a contradiction, as the wobbly locus is a proper closed subset.
Then we have the following theorem which was conjectured by Drinfeld follows immediately.

**Theorem 4.3.** The wobbly locus is pure of co-dimension one.

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Email address: sarbeswar11@gmail.com, spal@iisertvm.ac.in

IISER Thiruvananthapuram, Maruthamala P. O., Kerala 695551