Dynamic Dilatonic Domain Walls

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Abstract

Motivated by the “universe as a brane” idea, we investigate the motion of a \((D-2)\)-brane (or domain wall) that couples to bulk matter. Usually one would expect the spacetime outside such a wall to be time dependent however we show that in certain cases it can be static, with consistency of the Israel equations yielding relationships between the bulk metric and matter that can be used as ansätze to solve the Einstein equations. As a concrete model we study a domain wall coupled to a bulk dilaton with Liouville potentials for the dilaton both in the bulk and on the wall. The bulk solutions we find are all singular. Some have black hole or cosmological horizons, beyond which our solutions describe domain walls moving in time dependent bulks. A significant period of world volume inflation occurs if the potential on the wall is not too steep; in some cases the bulk also inflates (with the wall comoving) while in others the wall moves relative to a non-inflating bulk. We apply our method to obtain cosmological solutions of Hořava-Witten theory compactified on a Calabi-Yau space.

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1 Introduction

A domain wall in a $D$-dimensional spacetime is an extended object with $D-2$ spatial dimensions which partitions the spacetime into different domains. In cosmology, the different domains might correspond to different vacua of some Higgs field. We will use the term “domain wall” loosely to refer to any $(D-2)$-brane moving in $D$ dimensions.

When the gravitational back-reaction of a domain wall is included, the global causal structure of the resulting spacetime is usually modified. In the simplest models, one considers domain walls which do not couple to any bulk fields with the domain wall world volume governed by a Nambu-Goto action

$$S_{DW} = -\mu \int_{\Sigma} d^{D-1}x \sqrt{-h}$$

(1.1)

where $\mu$ is a parameter which corresponds to the tension or energy density of the wall, $\Sigma$ denotes the world volume swept out by the wall and $h$ denotes the determinant of the world volume metric. More precisely, the stress-energy tensor $T_{MN}$ of the domain wall takes the simple form

$$T_{MN} = \mu \delta(x) \text{diag}(1, -1, \ldots, -1, 0),$$

(1.2)

where $x$ is the direction transverse to the wall. This stress-energy tensor is a distributional source for the Einstein equations: the metric is at most $C^0$ as we move across such a “thin wall” of energy density. One approach to understanding the gravitational field of such an object is to divide the spacetime into smooth domains, where each domain is bounded by a domain wall. If we vary the metric to obtain the equations of motion then we obtain the well-known Israel matching conditions:\footnote{Our convention is that the normal to domain wall points into the bulk on both sides, so we take a sum over the sides of the wall rather than a difference.}

$$\{K_{MN} - Kh_{MN}\} = 8\pi G\mu h_{MN},$$

(1.3)

where $h_{MN}$ is the induced metric on the domain wall, $K_{MN}$ its extrinsic curvature and $K = h^{MN}K_{MN}$. The extrinsic curvature is typically discontinuous across a domain wall; the curly brackets denote summation over each side of the wall\footnote{Our convention is that the normal to domain wall points into the bulk on both sides, so we take a sum over the sides of the wall rather than a difference.}. We present a careful derivation of the Israel conditions from the Einstein-Hilbert action in section 2.
The Israel conditions are often used to study the motion of a domain wall in a static bulk spacetime. They are satisfied by seeking totally umbilic \((K_{MN} \propto h_{MN})\) surfaces in the bulk, slicing along such a surface and then gluing to another such bulk. For example, domain walls between domains corresponding to different vacua of a Higgs field can be obtained by taking the bulk spacetime to be flat \([2, 3, 4]\) or false vacuum decay can be studied by taking the two bulk portions to have differing cosmological constants \([2, 5]\).

One can have a domain wall enclosing a bubble of true vacuum in a sea of false vacuum. The Israel conditions reduce to two equations, one giving the velocity of the domain wall and the other its acceleration. Consistency of these equations follows from the bulk Einstein equations.

If the domain wall action is of the Nambu-Goto form considered above then the energy density on the wall is fixed so there can be no transfer of energy between the bulk matter and the wall. However it is possible to consider more general situations in which matter is localised to the wall and energy can flow on or off the wall. In this case, the right hand side of equation (1.3) is replaced by the energy-momentum tensor of the wall matter. For simple static bulk solutions the Israel conditions reduce to two equations, one relating the velocity of the wall to its energy density and the other relating its acceleration to its energy density and pressure. There is also an equation of state for the matter on the wall. Thus there are three equations for three quantities (position, energy density and pressure) so one would expect a solution to exist without further restrictions on the bulk spacetime.

This changes if the domain wall couples to matter in the bulk. For example, in string theory a brane or domain wall will usually be coupled to a dilaton \(\phi\) - a bulk scalar field which measures the scale or deformation properties of some internal manifold. The simple Nambu-Goto form for the world volume action is replaced by an action of the form

\[
S_{DW} = -\int_{\Sigma} d^{D-1}x \sqrt{-h} \hat{V}(\phi),
\]

(1.4)

where the wall tension \(\hat{V}\) depends on the value of the dilaton on the wall. If the domain wall moves through regions of varying dilaton then energy will flow on or off the wall. If one attempts to find solutions for the domain wall motion as before then one encounters a problem. Once again there are three equations for three quantities but now the energy density and pressure on the wall are specified by its position in the bulk. Hence there is no guarantee that a solution will exist for the motion of the wall in a static bulk spacetime:
in general the motion of the domain wall will make the bulk time dependent. Studying dynamics in this model is hard because the bulk metric will depend on two parameters (time and distance from the wall) so the Einstein equations become more difficult to solve.

In this paper we investigate the circumstances under which it is possible for a domain wall coupled to bulk matter to move in a static bulk. Starting from a static ansatz for the bulk metric we find in section 4 that consistency of the Israel equations requires that the bulk metric and matter be related in a certain way. This can be used to find solutions of the bulk Einstein equations. We concentrate on the example of a bulk dilaton with Liouville potentials in the bulk and on the domain wall. This model can be motivated by massive supergravity theories and p-brane world volume actions. Domain walls in such models have been extensively studied for static bulk spacetimes [6, 7] (see [8] for a review), especially in the supersymmetric case [3, 4]. However all of these solutions assume a constant dilaton on the domain wall whereas in our solutions the dilaton evolves in time on the wall.

Our bulk solutions are described in section 5. All are singular but in some cases the singularity is hidden behind a horizon. Some have cosmological horizons beyond which the metric becomes time dependent. The motion of the domain wall can be followed across the horizon so some of our solutions describe the evolution of a domain wall in a cosmological background.

There has been recent interest in cosmological models in which our universe is viewed as a brane moving in a higher dimensional spacetime, possibly with a very low fundamental Planck scale and consequently large extra dimensions [10]. A topic of particular interest in this scenario is how inflation occurs on the brane. In [11] it was concluded that for a viable model the bulk must be non-static during inflation. However in this model inflation was assumed to be driven by a scalar field restricted to the brane world volume and the possibility of the energy density on the brane coming from a bulk field was not discussed. In [12] it was described how inflation can occur if one of a stack of branes is displaced from the others. In the case of a stack of D-branes, the energy density that drives world volume inflation arises from the energy of open strings stretched between the branes. This energy can be viewed as coming from the non-zero expectation value of a world volume Higgs field. Inflation was discussed in [13] where the bulk spacetime was assumed to be non-static but the gravitational back reaction of the domain wall was treated in an approximate manner that didn’t correspond to a wall
localized in the extra dimensions. In [14] solutions were given for localized domain walls with inflation driven by matter living on the wall. Our model can be used to study inflation driven by energy density on the domain wall coming from a bulk field, taking full account of the gravitational back reaction of the wall.

We find in section 6 that power law inflation can occur on the domain wall provided that the potential on the world volume is not too steep. In some cases inflation occurs because the domain wall is comoving with an inflationary bulk while in others it occurs because the domain wall moves relative to the bulk.

In section 7 we show that for some values of the parameters our model can be obtained by dimensional reduction of a Nambu-Goto domain wall moving in a bulk spacetime with a cosmological constant. The world volumes of these domain walls undergo exponential inflation, which gives the power law inflation described above in the dimensionally reduced theory.

Cosmological solutions of Hořava-Witten theory, the strongly coupled limit of the $E_8 \times E_8$ heterotic superstring theory, have been recently discussed in [15, 16, 17]. The orbifold fixed planes in this theory can be viewed as domain walls so our method is well suited to finding new solutions. We discuss these in section 8.

In section 9 we discuss our conclusions and speculate on possible generalisations and applications of our work to cosmology and the dynamics of branes in string theory.

2 The Israel Matching Conditions

This section consists of a simple derivation of the Israel conditions [1] for matching the metric across a domain wall. The reader familiar with this derivation is advised to skip to the next section.

Let $M$ be a $D$ dimensional manifold containing a domain wall $\Sigma$, which splits $M$ into two parts, $M_\pm$. We demand that the metric be continuous everywhere and that the derivatives of the metric be continuous everywhere except on $\Sigma$. We shall denote the two sides of $\Sigma$ as $\Sigma_\pm$. 
Varying the Einstein-Hilbert action in $M_{\pm}$ gives

$$\delta S_{EH} = -\frac{1}{2} \int_{\Sigma_{\pm}} d^{D-1}x \sqrt{-h} g^{MN} n^P (\nabla_M \delta g_{NP} - \nabla_P \delta g_{MN}), \quad (2.1)$$

where $n_M$ is the unit normal pointing into $M_{\pm}$ and the induced metric on $\Sigma_{\pm}$ is given by the tangential components of the projection tensor $h_{MN} = g_{MN} - n_M n_N$. Note that the quantity in brackets vanishes when contracted with $n^M n^N n^P$ so we can replace $g^{MN}$ by $h^{MN}$ to get

$$\delta S_{EH} = -\frac{1}{2} \int_{\partial M} d^{D-1}x \sqrt{-h} h^{MN} n^P (\nabla_M \delta g_{NP} - \nabla_P \delta g_{MN}). \quad (2.2)$$

This expression contains a normal derivative of the metric variation, which we are allowing to be discontinuous across $\Sigma$, so the contributions from the two bulk regions will not necessarily cancel. Therefore it is necessary to include a Gibbons-Hawking boundary term \[18\] to cancel this term. On each side of the domain wall we include a contribution to the action

$$S_{GH} = -\int_{\Sigma_{\pm}} d^{D-1}x \sqrt{-h} K, \quad (2.3)$$

where $K$ is the trace of the extrinsic curvature of $\Sigma_{\pm}$ i.e. $K = h^{MN} K_{MN}$, $K_{MN} = h^P_M h^Q_N \nabla_P n_Q$.

We now have to compute the variation of this new term. This is slightly complicated by the fact that $n_M$ depends on the metric because it is normalised to unit length. When the metric is varied, the change in $n_M$ is

$$\delta n_M = \frac{1}{2} n_M n^P n^Q \delta g_{PQ}. \quad (2.4)$$

The variation in $K$ is

$$\delta K = -K^{MN} \delta g_{MN} - h^{MN} n^P (\nabla_M \delta g_{NP} - \frac{1}{2} \nabla_P \delta g_{MN}) + \frac{1}{2} K n^P n^Q \delta g_{PQ}, \quad (2.5)$$

and the variation of the Gibbons-Hawking term is

$$\delta S_{GH} = -\int_{\Sigma_{\pm}} d^{D-1}x \sqrt{-h} (\delta K + \frac{1}{2} K h^{MN} \delta g_{MN}). \quad (2.6)$$

\[2\]We use units in which $8\pi G = 1$, a positive signature metric and a curvature convention for which de Sitter space has a positive Ricci scalar.
Note that $\delta K$ contains a term $\frac{1}{2}h^{MN}n^P\nabla_P\delta g_{MN}$, which cancels the corresponding normal derivative in the variation of the Einstein-Hilbert action. The total variation is
\begin{equation}
\delta S_{EH} + \delta S_{GH} = \int_{\Sigma_\pm} d^{D-1}x \sqrt{-h} \left[ \frac{1}{2} h^{MN} n^P \nabla_M \delta g_{NP} + K^{MN} \delta g_{MN} - \frac{1}{2} K n^M n^N \delta g_{MN} - \frac{1}{2} K h^{MN} \delta g_{MN} \right]. \tag{2.7}
\end{equation}

To proceed further we need the following simple result for a vector field $X^M$ tangential to $\Sigma_\pm$
\begin{equation}
\nabla_M X^M = h^{MN} \nabla_M X_N + n^M n^N \nabla_M X_N = \tilde{\nabla}_M X^M - X^M n^N \nabla_N X_M, \tag{2.8}
\end{equation}
where $\tilde{\nabla}$ is the covariant derivative associated with the induced metric on $\Sigma_\pm$. Using this and the definition of $K_{MN}$ gives
\begin{equation}
h^{MN} n^P \nabla_M \delta g_{NP} = \nabla_M (h^{MN} n^P \delta g_{NP}) - \delta g_{NP} \nabla_M (h^{MN} n^P) \tag{2.9}
= \nabla_M (h^{MN} n^P \delta g_{NP}) + K n^M n^N \delta g_{MN} - K^{MN} \delta g_{MN}.
\end{equation}
Finally this can be substituted into (2.7) and the total derivative term integrated away to give
\begin{equation}
\delta S_{EH} + \delta S_{GH} = \frac{1}{2} \int_{\Sigma_\pm} d^{D-1}x \sqrt{-h} \left( K^{MN} - Kh^{MN} \right) \delta g_{MN}. \tag{2.10}
\end{equation}

Note that $K_{MN}$ can be discontinuous across $\Sigma$ so the contributions from $\Sigma_\pm$ need not cancel.

If the domain wall has action
\begin{equation}
S_{DW} = \int_{\Sigma} d^{D-1}x \sqrt{-h} L_{DW} \tag{2.11}
\end{equation}
then its variation is
\begin{equation}
\delta S_{DW} = \int_{\Sigma} d^{D-1}x \sqrt{-h} \delta h^{MN} \delta g_{MN}, \tag{2.12}
\end{equation}
where we have used the fact that $t^{MN} \equiv 2 \frac{\delta S_{DW}}{\delta h^{MN}}$ is tangential to the domain wall. The variation of the total action $S = S_{EH} + S_{GH} + S_{DW}$ gives the Israel conditions
\begin{equation}
\{ K_{MN} - Kh^{MN} \} = -t_{MN}, \tag{2.13}
\end{equation}
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where the curly brackets denote summation over both sides of $\Sigma$.

The stress energy tensor $t^{MN}$ on the domain wall is not necessarily conserved because energy can flow between the domain wall and the bulk. This can be seen by taking the divergence of the Israel equations:

$$\nabla_M t^{MN} = - \{\nabla_M K^{MN} - h^{MN} \nabla_M K\}.$$  \hfill (2.14)

The right hand side can be evaluated using Codacci’s equation \cite{19}, giving

$$\nabla_M t^{MN} = - \{h^{NM} R_{MPn}^P\} = - \{h^{NM} T_{MPn}^P\},$$  \hfill (2.15)

where $T^{MN}$ is the bulk energy momentum tensor and we have made use of the bulk Einstein equation. This equation describes conservation of energy when it moves from the bulk to the boundary or vice versa.

### 3 The Equations of Motion

The example that we shall be focussing on is Einstein gravity with a scalar field (dilaton) in the bulk and a domain wall that couples to the bulk dilaton. The action is

$$S = \int_M d^D x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right) + S_{DW},$$  \hfill (3.1)

where

$$S_{DW} = - \int_\Sigma d^{D-1} x \sqrt{-h} \{K\} + \hat{V}(\phi).$$  \hfill (3.2)

Note that we have absorbed the Gibbons-Hawking terms into the domain wall action. Once again, curly brackets denote summation over both sides of the wall. The bulk Einstein equation is

$$R_{MN} = \partial_M \phi \partial_N \phi + \frac{2}{D-2} V(\phi) g_{MN}.$$  \hfill (3.3)

Varying the scalar field gives

$$\int_M d^D x \sqrt{-g} \left( \nabla^2 \phi - \frac{dV}{d\phi} \right) \delta \phi + \int_\Sigma d^{D-1} x \sqrt{-h} \left( \{n. \partial \phi\} - \frac{d\hat{V}}{d\phi} \right) \delta \phi = 0$$  \hfill (3.4)
because \( n \) points \( \text{into} \) the bulk. This yields the equation of motion

\[
\nabla^2 \phi = \frac{dV}{d\phi} \quad (3.5)
\]

with a boundary condition at the domain wall

\[
\{ n \cdot \partial \phi \} = \frac{d\dot{V}}{d\phi}. \quad (3.6)
\]

This ensures that the energy conservation equation 2.15 is satisfied.

We also need the Israel equations, which can be written

\[
\{ K_{MN} \} = -\frac{1}{D-2} \dot{V}(\phi) h_{MN}. \quad (3.7)
\]

## 4 Domain Wall Motion in a Static Background

We shall seek solutions in which the bulk spacetime is symmetric under reflection in the domain wall, hence the extrinsic curvatures on each side of the wall are the same and the Israel equations become

\[
K_{MN} = -\frac{1}{2(D-2)} \dot{V}(\phi) h_{MN}. \quad (4.1)
\]

In general there is no reason to assume that a moving domain wall will give rise to a static bulk spacetime. However it is sometimes possible that this can occur. Consider a bulk metric

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + R(r)^2 d\Omega_k^2, \quad (4.2)
\]

where \( d\Omega_k^2 \) is the line element on a \( D - 2 \) dimensional space of constant curvature with metric \( \bar{g}_{mn} \) and Ricci tensor \( \bar{R}_{mn} = k(D-3)\bar{g}_{mn} \) with \( k \in \{-1, 0, 1\} \). We shall also include a dilaton \( \phi(r) \).

Let the position of the domain wall be \( r = r(t) \) with the above metric valid on the \( r < r(t) \) parts of surfaces of constant \( t \) and its reflection valid on the \( r > r(t) \) parts. The unit normal pointing into \( r < r(t) \) is

\[
n_M = \frac{\sqrt{AB}}{\sqrt{A - B r^2}} (r, -1, 0, \ldots, 0) \quad (4.3)
\]
where $\dot{r} = \frac{dr}{dt}$. The proper velocity of the domain wall is

$$u^M = \frac{1}{\sqrt{A - B \dot{r}^2}} (1, \dot{r}, 0, \ldots, 0). \quad (4.4)$$

The extrinsic curvature is $K_{MN} = h^P_M h^Q_N \nabla_P n_Q$ where $h_{MN} = g_{MN} - n_M n_N$. We shall compute it in the basis given by $u_M, n_M, e^{(1)}_M, \ldots, e^{(D-2)}_M$ where the $e^{(i)}_M$ are an orthonormal basis for the $D - 2$ dimensional spatial sections. In this basis, the $ij$ components of the extrinsic curvature are

$$K_{ij} = -\frac{\sqrt{AB}}{B} \frac{R'}{R} \frac{1}{\sqrt{A - B \dot{r}^2}} h_{ij}, \quad (4.5)$$

where a prime denotes a derivative with respect to $r$. The $K_{00}$ component can be calculated as described in [20]. Orthogonality of $u^M$ and $n_M$ gives

$$K_{00} = K_{MN} u^M u^N = u^M u^N \nabla_M n_N = -n_N u^M \nabla_M u_N = -n_M A^M, \quad (4.6)$$

where $A^M$ is the proper acceleration of the wall. Note $u_M A^M = 0$ so $A^M = \hat{A} n_M$ for some $\hat{A}$. Hence $K_{00} = -\hat{A}$. To compute $\hat{A}$ we can exploit the existence of a timelike Killing vector $k = \frac{\partial}{\partial \tau}$. Let $\tau$ denote proper time along the wall. Then

$$\frac{d}{d\tau} (k^M u_M) = u^M u^N \nabla_M k_N + k_M A^M = k_M A^M = k^M n_M \hat{A}, \quad (4.7)$$

where we have used Killing’s equation. Putting these results together gives

$$K_{00} = \frac{1}{\tau} \sqrt{AB} \frac{d}{dt} \left( \frac{A}{\sqrt{A - B \dot{r}^2}} \right). \quad (4.8)$$

(If $\dot{r} = 0$ then $K_{00} = \frac{\hat{A}'}{2A\sqrt{B}}$, which agrees with the $\dot{r} \to 0$ limit of (4.8).)

Having computed the extrinsic curvature we can now substitute into the Israel equations. The $ij$ component gives

$$\frac{R'}{R} = \frac{\dot{V}(\phi)}{2(D-2) \sqrt{A - B \dot{r}^2}}, \quad (4.9)$$

which can now be used to eliminate $\sqrt{A - B \dot{r}^2}$ from the 00 component. This yields

$$\left( \frac{R'}{R} \right)^{-1} \frac{d}{dt} \left( \frac{R'}{R} \right) = \frac{\dot{V}(\phi) \sqrt{AB}'}{\dot{V}(\phi) \sqrt{AB}} - \frac{R'}{R}. \quad (4.10)$$
This equation has to hold at every point visited by the domain wall. Thus unless the domain wall remains at fixed $r$ (i.e. $\dot{r} \equiv 0$) then it has to hold over a range of $r$ and can therefore be integrated, giving

$$R' = C\hat{V}(\phi)\sqrt{AB}, \quad (4.11)$$

where $C$ is a constant.

We can now turn to the boundary condition on the dilaton. Equation (3.6) can be simplified using equation (4.9) and reflection symmetry to give

$$\frac{d\phi}{dR} = -\frac{D - 2}{R} \frac{1}{\hat{V}} \frac{d\hat{V}}{d\phi}, \quad (4.12)$$

which, if the wall visits a range of $R$, can be solved (in principle) to yield $\phi$ as a function of $R$ without specifying the bulk potential.

Hence demanding that the domain wall be non-static in a static bulk gives conditions relating the bulk metric and dilaton. In the next section we shall use these conditions as ansätze to solve the bulk Einstein equations.

## 5 The Bulk Metric

It is convenient to adopt the gauge $A(r) = B(r)^{-1} = U(r)$ for the metric. The field equations are then

$$\frac{R''}{R} = -\frac{1}{D-2} \phi^2, \quad (5.1)$$

$$-\frac{D-2}{4} \frac{1}{R^{D-2}} (U'R^{D-2})' = V, \quad (5.2)$$

$$-\frac{1}{2R^{D-2}} (U(R^{D-2})')' + \frac{k(D-2)(D-3)}{2R^2} = V, \quad (5.3)$$

$$\frac{1}{R^{D-2}} (R^{D-2}U'\phi)' = \frac{dV}{d\phi}. \quad (5.4)$$

The ansätze (4.11) and (4.12) can be employed to seek solutions of these equations. They ensure that equation (5.1) is satisfied. To proceed further it is necessary to specify the domain wall potential. We shall specialise to the case of a Liouville potential:

$$\hat{V}(\phi) = \hat{V}_0 e^{\alpha\phi}. \quad (5.5)$$
Equations 4.11 and 4.12 can be solved simultaneously to give
\[ \phi(r) = \phi_* - \frac{\alpha(D - 2)}{\alpha^2(D - 2) + 1} \log r, \]
\[ R(r) = (\alpha^2(D - 2) + 1)C\hat{V}_0 e^{\alpha\phi_* r^{\frac{1}{\alpha^2(D - 2) + 1}}}, \]
where \( \phi_* \) is a constant of integration. (A second constant of integration can be set to zero by shifting the range of \( r \).)

To make further progress it is necessary to specify the bulk potential. We shall assume that this is also of Liouville type:
\[ V(\phi) = V_0 e^{\beta\phi}. \]

Then equation 5.3 can be solved for \( U(r) \). Substituting into equations 5.2 and 5.4 yields constraints on the parameters. There are three types of solutions.

Type I solutions have \( \alpha = \beta = 0 \), so the potential becomes a cosmological constant. The solution has constant dilaton \( \phi = \phi_0 \). After rescaling \( t \) and changing variable from \( r \) to \( R \), the metric can be written
\[ ds^2 = -U(R)dt^2 + U(R)^{-1}dR^2 + R^2d\Omega^2_k, \]
with
\[ U(R) = k - 2MR^{-(D-3)} - \frac{2V_0}{(D - 1)(D - 2)}R^2, \]
where \( M \) is a constant. If \( M = 0 \) then this is simply the metric of de Sitter, Minkowski or anti-de Sitter spacetime according to the sign of \( V_0 \). \( U(R) \) is sketched in figure 1 for \( M \neq 0 \). It is easy to read off the horizon structure of the solutions from these diagrams.
When $V_0 > 0, M > 0$ there are two possibilities. If $k = +1$ and

$$\left[\frac{2V_0}{(D-2)(D-3)}\right]^{D-3} [(D-1)M]^2 < 1 \quad (5.11)$$

(this corresponds to the dotted line in the first graph of figure [1]) then the solution is simply Schwarzschild-de Sitter, which has black hole and cosmological horizons given by the two zeros of $U(R)$. If $k \neq +1$ or equation (5.11) is not satisfied (corresponding to the solid line) then the solution is nowhere static ($R$ is a time coordinate) and there is a cosmological singularity at $R = 0$. At late times the metric approaches that of de Sitter space.

If $V_0 > 0, M < 0$ then there is a timelike naked singularity at $R = 0$. There is also a cosmological horizon (with geometry determined by $k$) beyond which the metric is asymptotically de Sitter.

For $V_0 < 0, M > 0$, there is a singularity at $R = 0$ surrounded by an event horizon beyond which the metric is asymptotically anti-de Sitter. If $k = +1$ this is the Schwarzschild-anti de Sitter solution. If $k = 0$ or $k = -1$ then it describes a “topological” black hole with a flat or hyperbolic event horizon. This can be made compact by making identifications however this would also make the spatial sections of the domain wall compact.

When $V_0 < 0, M < 0$, there are two possibilities. If $k = -1$ and

$$\left[\frac{2|V_0|}{(D-2)(D-3)}\right]^{D-3} [(D-1)|M|^2 < 1 \quad (5.12)$$

(corresponding to the dotted line in the final graph of figure [1]) then the metric describes a topological black hole in an asymptotically anti-de Sitter space, while if $k \neq -1$ or equation (5.12) is not satisfied then it describes a timelike naked singularity in an asymptotically anti-de Sitter space.

Type II solutions have $\alpha = \beta/2$, $k = 0$ and

$$U(r) = (1 + b^2)^2 r^{\frac{2}{1+b^2}} \left(-2Mr - \frac{\rho_0 - \rho_1}{1+b^2} - \frac{2\Lambda}{D-1-b^2}\right), \quad (5.13)$$

$$R(r) = r^{\frac{1}{1+b^2}}, \quad (5.14)$$

$$\phi(r) = \sqrt{D-2} \left(\phi_0 - \frac{b}{1+b^2} \log r\right), \quad (5.15)$$
where $M$ and $\phi_0$ are constants of integration and

$$b = \frac{1}{2} \beta \sqrt{D - 2},$$  \hspace{1cm} (5.16)

$$\Lambda = \frac{V_0 e^{2b\phi_0}}{D - 2}. \hspace{1cm} (5.17)$$

The function $U(r)$ is sketched in figure 2. These solutions (and their charged generalizations) were derived for $D = 4$ in [21] using the ansatz $R \propto r^N$. The solutions with $M = 0$ for arbitrary $D$ were derived in [9]. Note that when $\alpha = \beta/2$, the theory becomes scale invariant in the sense that a constant scale transformation $g_{MN} \rightarrow \Omega^2 g_{MN}$, $\phi \rightarrow \phi - \frac{2}{\beta} \log \Omega$ simply multiplies the action by a constant: $S \rightarrow \Omega^{D-2} S$. This means that the equations of motion are invariant under such a transformation.

All of the type II solutions are singular at $r = 0$. For some the singularity is timelike and for others it is spacelike. There is at most one horizon, which is like a black hole horizon for some solutions and like a cosmological horizon for others. The asymptotic (large $r$) behaviour of the solutions depends on the value of $b^2$. 

Figure 2: $U(r)$ for the Type II solutions.
If $b^2 < D - 1$ then the asymptotic behaviour of the metric is determined by the sign of $V_0$. If $V_0 > 0$ then $r$ is a time coordinate. By rescaling $t$ and the flat spatial sections and changing variable $r \rightarrow T(r)$, the metric can be written as a FRW universe with flat spatial sections:

$$ds^2 \sim -dT^2 + T^{\frac{2}{b^2}} d\mathbf{x}^2, \quad (5.18)$$

which is inflating if $b^2 < 1$. If $V_0 < 0$ then $r$ is a spatial coordinate. By rescaling the other coordinates and changing variable $r \rightarrow \rho(r)$, one can put the metric into the form

$$ds^2 \sim d\rho^2 + \rho^{\frac{2}{b^2}}(-dt^2 + d\mathbf{x}^2), \quad (5.19)$$

which resembles the metric on anti-de Sitter space written in horospherical coordinates. Note that when $M > 0$ these solutions have a black hole type horizon and can be interpreted as black $(D - 2)$ brane solutions, generalizing some of the supersymmetric solutions given in [9].

When $b^2 > D - 1$ the asymptotic behaviour of the metric is determined by $M$. If $M > 0$ then $r$ is a time coordinate and the metric has anisotropic spatial sections and resembles a Kasner solution

$$ds^2 \sim -dT^2 + T^{\frac{2b^2 - (D - 3)}{b^2 + D - 1}} dt^2 + T^{\frac{4}{b^2 + D - 1}} d\mathbf{x}^2, \quad (5.20)$$

where we have performed a transformation $r \rightarrow T(r)$ and rescaled the other coordinates. The $t$ dimension expands faster than the other three but none of the spatial dimensions inflate. If $M < 0$ then $r$ is a spatial coordinate and $t$ a time coordinate and the metric is the same as above with the signs of the first two terms changed. When $V_0 < 0$ (still with $M < 0$), the metric has a black hole type horizon and hence can be interpreted as a topological black hole with the curious property that the “mass” $M$ determines the asymptotic structure.

Type III solutions have $\alpha = \frac{2}{b(D - 2)}$. The metric is given by

$$U(r) = (1 + b^2)^2 r^{\frac{2}{1 + \beta^2}} \left( -2Mr^{\frac{1+b^2}{1+b^2}} - \frac{2\Lambda}{(1 + b^2(D - 3))} \right), \quad (5.21)$$

$$R(r) = \gamma r^{\frac{2}{1 + \beta^2}}, \quad (5.22)$$
where

$$\gamma = \left( \frac{(D-3)}{2k\Lambda(1-b^2)} \right)^{\frac{1}{2}} \quad (5.23)$$

and \(\phi(r)\) is the same as for the type II solutions. \(b\) and \(\Lambda\) are defined as above and \(k\) is given by the sign of \(\Lambda(1-b^2)\). (If \(b^2 = 1\) then only the type II solution exists.) The function \(U(r)\) is sketched in figure 3. These solutions (and their charged generalizations) were derived in [22] for \(k = +1\) and in [21] for \(k = -1, D = 4\) by making the ansatz \(r \propto r^N\). The solutions are all singular at \(r = 0\). Some have a horizon of black hole or cosmological type. Their asymptotic (large \(r\)) behaviour is determined by the sign of \(V_0\). If \(V_0 > 0\) then by rescaling \(t\) and performing a change of variable \(r \rightarrow T(r)\) the metric can be written in an anisotropic cosmological form

$$ds^2 \sim -dT^2 + T^2 dt^2 + \frac{(D-3)b^4}{|1-b^2|(1+b^2(D-3))} T^2 d\Omega_2^k, \quad (5.24)$$

with \(k\) given by the sign of \(1-b^2\). When \(b^2 < 1\) the spatial sections have cylindrical topology and the axial \((t)\) dimension inflates but when \(b^2 > 1\) the \(t\) dimension grows more slowly than the other spatial dimensions.

If \(V_0 < 0\) then \(r\) is a spatial coordinate and the asymptotic solution is the same as the above with the signs of the first two terms changed i.e.

$$ds^2 \sim -\rho^2 dt^2 + d\rho^2 + \frac{(D-3)b^4}{|1-b^2|(1+b^2(D-3))} \rho^2 d\Omega_2^k, \quad (5.25)$$

Figure 3: \(U(r)\) for the Type III solutions. The value of \(k\) in the second row is the same as in the first row when \(b^2 < 1\) and minus this when \(b^2 > 1\).
$k$ is given by the sign of $b^2 - 1$.

Note that only the $V_0 < 0, M > 0$ solutions are of black hole type. Of these, the $b^2 < 1$ solutions have hyperbolic spatial sections and the $b^2 > 1$ solutions have round spatial sections.

We have obtained solutions to the bulk field equations with non-trivial dilaton by restricting $\alpha$ to take one of two values determined by $\beta$. For each value of $\beta$ there are two solutions, one with $k = 0$ and the other with $k = \pm 1$.

We refer the reader to references [21, 22] for analyses of the horizon structure and thermodynamic properties of these solutions.

6 Analysis of Domain Wall Motion

The trajectory of the domain wall is determined by the Israel equation 4.9. For the gauge used in the previous section, this reduces to

$$\frac{R'}{R} = \frac{\dot{V}(\phi)}{2(D - 2)\sqrt{\left(\frac{dr}{d\tau}\right)^2 + U}},$$

(6.1)

where $\tau$ is proper time on the domain wall world volume. There are two subtleties that we have to deal with before studying solutions of this equation. The first is that the solutions of the previous section all have $R' > 0$, so the above equation appears to rule out the possibility of $\dot{V} < 0$. However reversing the direction of the normal to the domain wall reverses the sign on the right hand side. Equivalently, if $\dot{V} < 0$ then the bulk solutions derived above are valid on the $r > r(t)$ parts of the surfaces of constant $t$ rather than the $r < r(t)$ parts. The geometrical interpretation of this is that if one approaches a domain wall along its normal then the spatial sections grow if the wall has positive energy density but decrease if it has negative energy density.

The second subtlety arises when $U < 0$. If horizons are present then $U$ is positive in some (static) region but can be negative in other regions. These can be dealt with in the standard way by introducing Eddington-Finkelstein coordinates [19]. The analysis of section 4 can be repeated in these coordinates, reproducing equations 6.1 and 4.11, which demonstrates their validity when the domain wall crosses a horizon. If $U$ is negative everywhere then $r$ becomes the time coordinate and it is convenient to reverse the direction of
the normal so that the bulk solution is valid on the \( t < t(r) \) parts of surfaces of constant \( r \). Equations 6.1 and 1.1 then become valid in this case too.

Equation 6.1 can be written

\[
\frac{1}{2} \left( \frac{dR}{d\tau} \right)^2 + F(R) = 0 \tag{6.2}
\]

where \( \tau \) is proper time on the domain wall i.e. the induced metric on the domain wall is

\[
ds^2 = -d\tau^2 + R(\tau)^2 d\Omega_k^2. \tag{6.3}
\]

This is the metric of a FRW universe. Equation 6.2 determines the evolution of the scale factor \( R(\tau) \) and is simply the equation for a particle of unit mass and zero energy rolling in a potential \( F(R) \). The potential \( F(R) \) is given by

\[
F(R) = \frac{1}{2} UR'^2 - \frac{1}{8(D-2)^2} \dot{V}^2 R^2. \tag{6.4}
\]

Clearly solutions only exist when \( F(R) \leq 0 \). This is automatic if \( U < 0 \) i.e. if \( r \) is a time coordinate. Inflationary solutions are of particular interest. Inflation on the domain wall is defined by \( \frac{dF}{dR} < 0 \).

In the next subsections we shall compute \( F(R) \) for the solutions found in the previous section.

### 6.1 Type I Solutions

These have

\[
F(R) = \frac{k}{2} - MR^{-(D-3)} - \dot{\Lambda} R^2, \tag{6.5}
\]

where the effective cosmological constant on the domain wall is

\[
\dot{\Lambda} = \frac{1}{D-2} \left[ \frac{V_0}{D-1} + \frac{\dot{V}_0^2}{8(D-2)} \right]. \tag{6.6}
\]

There are several cases for which these solutions have been extensively studied. Consider first the case \( M = 0 \). The bulk spacetime is simply de Sitter, Minkowski or anti-de Sitter space depending on the sign of \( V_0 \). The position of the domain wall is given by solving equation 6.2. For \( \dot{\Lambda} > 0 \), \( R \) increases exponentially, corresponding to a de Sitter solution on the domain wall world volume. When \( \dot{V}_0 > 0 \), the total bulk spacetime is given by matching the bulk
Figure 4: $F(R)$ for the Type I solutions. Dotted lines indicate alternative behaviour.

solution for $R < R(\tau)$ to a copy of itself across the domain wall at $R = R(\tau)$. When $\dot{V}_0 < 0$, the $R > R(\tau)$ part of the bulk is matched to a copy of itself across the domain wall. An example is the Vilenkin-Ipser-Sikivie domain wall [2, 3, 4], which has Minkowski space in the bulk and spherical spatial sections. For $\dot{V}_0 > 0$ the solution corresponds to gluing the (flat) interior of a de Sitter hyperboloid embedded in Minkowski space to a copy of itself while for $\dot{V}_0 < 0$ the exterior of the hyperboloid must be used.

If $\dot{\Lambda} < 0$ then a non-trivial solution only exists for the case of open spatial sections. The domain wall can expand to a maximum value of $R$ and then recollapse to $R = 0$. The world volume and bulk solutions are both anti-de Sitter.

When $\dot{\Lambda} = 0$ the domain wall vacuum energy exactly cancels the effect of the (negative) bulk cosmological constant (as far as the motion of the wall is concerned) and the world volume metric is flat. Note that $F \leq 0$ requires $k = 0$ or $k = -1$. In these cases the domain wall is simply a horosphere of the bulk anti-de Sitter space.

When $M \neq 0$ there is a singularity at $R = 0$. If the domain wall has positive energy density (i.e. $\dot{V}_0 > 0$) then the relevant part of the bulk spacetime is $R < R(\tau)$, which contains the singularity. If it has negative energy density then the relevant part is $R > R(\tau)$, which is non-singular unless the wall reaches $R = 0$.

From any solution for the domain wall motion one can generate another by time reversal. In what follows we shall only discuss one member of each pair of time reversal related solutions.

$F(R)$ is plotted in figure 4. The qualitative behaviour of the wall is easily read off from this figure. The four graphs correspond to the following cases.

Case i) $\dot{\Lambda} > 0, M < 0$. There are two possibilities. The first (given by
the dotted line in the first graph of figure [4] is \( k = +1 \) with

\[
(D - 1) \left( \frac{2\hat{\Lambda}}{D - 3} \right)^{(D-3)} |M|^2 < 1,
\]

which corresponds to a Schwarzschild-de Sitter/anti-de Sitter bulk solution. In this case the wall can either expand out of the white hole region of the bulk spacetime and recollapse into the black hole region, or it can collapse from infinity, stop outside the black hole and re-expand to infinity.

If \( k \neq +1 \) or equation (6.7) is not satisfied then the domain wall emerges from a white hole or cosmological singularity and expands to infinity.

Case ii) \( \hat{\Lambda} > 0, M < 0 \). The bulk has a timelike naked singularity. The domain wall collapses from infinity but is stopped by the repulsive singularity. It then re-expands to infinity.

Case iii) \( \hat{\Lambda} < 0, M > 0 \). The bulk solution is anti-de Sitter space with a black hole or topological black hole. The domain wall expands out of the white hole region and the negative cosmological constant overwhelms the energy density of the domain wall, causing it to recollapse into the black hole.

Case iv) \( \hat{\Lambda} < 0, M < 0, k = -1 \) and

\[
(D - 1) \left( \frac{2|\hat{\Lambda}|}{D - 3} \right)^{(D-1)} |M|^2 < 1,
\]

The domain wall starts at a finite distance from a naked singularity, which repels it, causing it to accelerate away. This expansion is halted by the bulk (negative) cosmological constant and recollapse occurs. The cycle then repeats. Thus the world volume of the domain wall describes an open universe that undergoes a brief period of inflation.

Note that the cases with \( \hat{\Lambda} > 0 \) all expand to infinity. This expansion is accelerating so the world volume undergoes inflation. It is easy to see that the world volume solution approaches de Sitter space at late times.

The cases \( k = 0, -1 \) with \( V_0 < 0 \) have been studied recently by Mann [23], who was interested in pair creation of charged black holes with arbitrary event horizon topology. His bulk configurations are slightly more general than our Type I solutions because he allows for a single \( U(1) \) charge in the bulk. He studied the equations of motion for a domain wall in these backgrounds because he was using the domain wall mechanism [24] to pair create the black holes.
6.2 Type II Solutions

\[ F(R) = -R^{2(1-b^2)} (MR^{-(D-1-b^2)} + \hat{\Lambda}) , \]  

(6.9)

where

\[ \hat{\Lambda} = \frac{e^{2b\phi_0}}{D-2} \left( \frac{V_0}{D-1-b^2} + \frac{\dot{V}_0^2}{8(D-2)} \right) . \]  

(6.10)

$F(R)$ is sketched in figure 5. Four classes of behaviour are apparent.

Class i) $F(R) > 0$ everywhere. No solutions exist.

Class ii) $F(R) < 0$ everywhere. These all have $\hat{\Lambda} > 0$ and $M > 0$ and describe a domain wall that emerges out of a singularity at $R = 0$ and expands forever. The singularity may be a timelike naked singularity, a cosmological singularity or a white hole type singularity hidden behind an event horizon.

Class iii) $F(R)$ positive for small $R$ and negative for large $R$. These solutions describe a domain wall collapsing from infinity to a minimum size (where $F(R) = 0$) and then re-expanding to infinity. The collapse is halted by a repulsive timelike naked singularity in the bulk at $r = 0$. 

Figure 5: $F(R)$ for the Type II solutions.
Class iv) $F(R)$ negative for small $R$ and positive for large $R$. These solutions describe either a domain wall expanding out of a timelike naked singularity at $R = 0$ and then recollapsing into it, or expanding out of a white hole type singularity and collapsing into a black hole type singularity.

It is interesting to ask how the solutions that expand forever behave at late times. If $b^2 < D - 1$ then the domain wall can only expand forever if $\hat{\Lambda} > 0$. The scale factor at late times behaves as $\tau^{\frac{1}{b^2-1}}$, which is inflationary if $b^2 < 1$. When $V_0 > 0$ this is the same behaviour as the scale factor of the $D - 1$ dimensional spatial sections of the bulk (see equation (5.18)) so the domain wall is simply comoving with these spatial sections. If $V_0 < 0$ then the bulk metric is static and the expansion is driven by the energy density of the domain wall. In the coordinates of equation (5.19), the position of the domain wall is given by $\rho = \tau$ and $t = \sqrt{b^2-1} \tau^{1-b^2}$. When $b^2 < 1$, $t$ approaches a constant at late proper time while if $b^2 > 1$ then $t$ becomes large at late proper time.

When $b^2 > D - 1$, the large $R$ behaviour of both the domain wall and the bulk is given by the sign of $M$. If $M < 0$ then the domain wall cannot expand indefinitely. If $M > 0$ then at late times, $R$ is proportional to $\tau^{\frac{1}{b^2-1}}$. Comparing this with the behaviour of the bulk (equation (5.24)) we see that the domain wall once again sits at a fixed position relative to the expanding bulk spatial sections and the bulk expands fastest in the direction transverse to the domain wall.

There are two solutions which give a finite period of inflation. These are the $\hat{\Lambda} > 0, M < 0, 1 < b^2 < D - 1$ solution and the $\hat{\Lambda} < 0, M > 0, b^2 > D - 1$ solution. In both of these the domain wall collapses from infinity, gets repelled by a timelike naked singularity and then expands. Inflation occurs when the expansion starts. In both cases the scale factor increases by a factor of

$$\left( \frac{D - 3 + b^2}{2(b^2 - 1)} \right)^{\frac{1}{b^2-1}}$$

which is cosmologically negligible unless $b^2$ is exponentially close to 1. Note that this expression is independent of $M$, a consequence of scale invariance.
For the type III solutions, let $a^2 = \frac{1}{D-1}$, $b^2 < \frac{1}{D-1}$, and $b^2 > 1$. We have

$$F(R) = -\frac{(D-3)b^4}{2k(1-b^2)(1+b^2(D-3))} - M\gamma^2 b^4 \left(\frac{R}{\gamma}\right)^{-(D-3+\frac{1}{D})} - \frac{\dot{V}_0}{\gamma^2} e^{\frac{2\omega}{\gamma^2}} \left(\frac{R}{\gamma}\right)^{-2\left(\frac{1}{D}-1\right)}.$$  

This is sketched in figure 6. The behaviour can be divided into five classes.

Class i) $F(R)$ is negative everywhere. In these solutions the domain wall expands out of a cosmological singularity, timelike naked singularity or white hole singularity and expands forever.

Class ii) $F(R)$ is negative for small $R$ and positive for large $R$. The domain wall either expands out of a timelike naked singularity and recollapses into the same singularity, or it expands out of a white hole singularity and collapses into a black hole singularity.

Class iii) $F(R)$ is positive for small $R$ and negative for large $R$. The domain wall collapses from infinity to a finite size and then re-expands to

Figure 6: $F(R)$ for the Type III solutions. Dashed lines indicate asymptotes. Dotted lines indicate alternative behaviour.

### 6.3 Type III Solutions

For the type III solutions, let $a^2 = \frac{1}{D-1}$, $b^2 < \frac{1}{D-1}$, and $b^2 > 1$. We have

$$F(R) = -\frac{(D-3)b^4}{2k(1-b^2)(1+b^2(D-3))} - M\gamma^2 b^4 \left(\frac{R}{\gamma}\right)^{-(D-3+\frac{1}{D})} - \frac{\dot{V}_0}{\gamma^2} e^{\frac{2\omega}{\gamma^2}} \left(\frac{R}{\gamma}\right)^{-2\left(\frac{1}{D}-1\right)}.$$  

This is sketched in figure 6. The behaviour can be divided into five classes.

Class i) $F(R)$ is negative everywhere. In these solutions the domain wall expands out of a cosmological singularity, timelike naked singularity or white hole singularity and expands forever.

Class ii) $F(R)$ is negative for small $R$ and positive for large $R$. The domain wall either expands out of a timelike naked singularity and recollapses into the same singularity, or it expands out of a white hole singularity and collapses into a black hole singularity.

Class iii) $F(R)$ is positive for small $R$ and negative for large $R$. The domain wall collapses from infinity to a finite size and then re-expands to
infinity. In most cases the collapse is halted by a timelike naked singularity. The exception is if \( b^2 > 1 \), \( V_0 < 0 \) and \( M \) is positive but less than a calculable upper bound (corresponding to the solid line in the relevant graph of figure 3). Then the turning point occurs outside the horizon of a spherical black hole.

Class iv) \( F(R) \) is positive for a finite range of \( R \). This only occurs when \( \frac{1}{D-1} < b^2 < 1 \), \( V_0 < 0 \) and \( M \) is negative but greater than a calculable lower bound (corresponding to the solid line in the relevant graph of figure 3). The domain wall is repelled by a naked timelike singularity, inflates for a brief period, decelerates to a halt, recollapses and then repeats this cycle. The world volume describes an open ‘bouncing’ universe.

The solutions which expand to infinity exhibit late time behaviour of two different types. When \( b^2 < 1 \) and \( V_0 > 0 \), the first term in \( F \) is dominant at late times. This term can be thought of as the term arising from the bulk curvature. All of these solutions have spherical spatial sections and at late times the scale factor grows linearly with proper time

\[
R(\tau) \sim \left[ \frac{(D-3)b^4}{(1-b^2)(1+b^2(D-3))} \right]^{\frac{1}{2}} \tau. \tag{6.13}
\]

If we compare this with the asymptotic behaviour of the bulk metric, given by equation 5.24, it is clear that the domain wall is simply comoving with the bulk i.e. it lies at fixed \( t \). Recall that the topology of the bulk spatial sections is cylindrical. The domain wall remains at a fixed position on the axis of this cylinder as it expands. Note that inflation occurs in the bulk in the direction transverse to the domain wall.

The second type of behaviour occurs for \( b^2 > 1 \), when the domain wall energy density becomes dominant at late times. These solutions undergo power law inflation on the world volume with the scale factor growing proportionally to \( \tau^{b^2} \). When \( V_0 > 0 \), the coordinates of the domain wall with respect to the bulk spacetime (give by equation 5.24) are \( T \propto \tau^{b^2} \) and \( t \propto \tau^{-b^2-1} \), so the wall moves relative to the bulk spatial sections. When \( V_0 < 0 \), the bulk spacetime is given by equation 5.25 and the position of the domain wall by \( \rho \propto \tau^{b^2} \) and \( t \propto \tau^{-b^2-1} \).

It is also possible to have world volume inflation when \( b^2 < 1 \). Both of the solutions with \( \frac{1}{D-1} < b^2 < 1 \) and \( M < 0 \) have finite period of inflation caused by the repulsive effect of a timelike singularity. In the \( V_0 < 0 \) case,
the amount of inflation is negligible. In the $V_0 > 0$ case, a significant amount of inflation only occurs if the dimensionless quantity

$$\frac{|M|V(\phi_0)}{\dot{V}(\phi_0)^2}$$

is exponentially large, corresponding to a very strong singularity or very small boundary potential.

If $b^2 < \frac{1}{D-1}$, $V_0 > 0$ and $M < 0$ then an infinite amount of inflation is possible. However this inflation dies out, with the expansion approaching constant velocity at late times. A significant period of rapid inflation would require exponential tuning as above.

7 Dimensional Reduction

Liouville potentials typically arise from dimensional reduction. To see which couplings can arise in this manner, consider a model in $D + n$ dimensions with action

$$S = \frac{1}{8\pi G} \int d^{D+n} \tilde{x} \sqrt{-\tilde{g}} (\frac{1}{2} \tilde{R} - V_0) - \frac{1}{8\pi G} \int _{\Sigma} d^{D+n-1} \tilde{x} \sqrt{-\tilde{h}} (\{\tilde{K}\} + \dot{\tilde{V}}_0),$$

(7.1)

where bars denote $(D + n)$-dimensional quantities. The domain wall has a simple Nambu-Goto action with tension $\dot{V}_0$. We can dimensionally reduce this using the ansatz

$$ds^2 = e^{2A(x)} g_{MN}(x) dx^M dx^N + e^{2B(x)} g_{mn}(y) dy^m dy^m,$$

(7.2)

where $g_{MN}$ and $G_{mn}$ are the metrics on $D$-dimensional spacetime and a $n$-dimensional internal Einstein space respectively and $B(x) = -(\frac{D-2}{n})A(x)$ (to obtain the reduced action in the Einstein frame). The bulk action reduces to

$$S_{\text{bulk}} = \int d^D x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 - \frac{\beta_1}{2} \nabla^2 \phi - V_0 e^{\beta_1 \phi} + \frac{1}{2} R_n e^{\beta_2 \phi} \right),$$

(7.3)

where $R_n$ is the Ricci scalar of the $n$-dimensional space,

$$\beta_1 = 2 \left( \frac{n}{(D-2)(D+n-2)} \right)^{\frac{1}{2}}, \quad \beta_2 = 2 \left( \frac{(D+n-2)}{n(D-2)} \right)^{\frac{1}{2}}$$

(7.4)
and $\phi(x) = 2A(x)/\beta_1$. We have chosen units so that $8\pi G = 1$, where $G$ is the $D$ dimensional Newton constant. Note that if we define the parameters $b_i = \frac{1}{2} \beta_i \sqrt{D-2}$ as in section then we have $b_1 = 1/b_2$.

If the normal to the domain wall points only in directions corresponding to the $D$-dimensional space then the domain wall action reduces to

$$S_{DW} = - \int d^{D-1}x \sqrt{-h} \left( \left\{ K + \frac{\beta_1}{2} n.\partial \phi \right\} + \dot{V}_0 e^{\alpha \phi} \right), \quad (7.5)$$

where $\alpha = \frac{\beta_1}{2} = \frac{2}{\beta_2(D-2)}$. The total derivative in the bulk action cancels the normal derivative in the domain wall action (recall that $n$ points into the bulk). The bulk action has a sum of Liouville potentials. The boundary action has a single Liouville potential. If either $R_\alpha = 0$ or $V_0 = 0$ then the model reduces to the one we have considered in previous sections and admits solutions of type II or type III respectively.

This implies that some of our type II (with $b^2 < 1$) and type III solutions (with $b^2 > 1$) can be oxidized to higher dimensional models in which the domain wall and bulk actions are much simpler. Oxidation of a type II solution yields a type I solution in $(D+n)$ dimensions with a non-vanishing bulk cosmological constant and spatial sections that are products of $D$-dimensional flat space with a $n$-dimensional Ricci flat space. Oxidation of a type III solution yields a type I solution with vanishing bulk cosmological constant and spatial sections that are products of a $D$-dimensional sphere (hyperboloid) with a positively (negatively) curved $n$-dimensional Einstein space.

In terms of the higher dimensional theory, the components of the Einstein equations on the internal space give rise to the scalar equation of motion of the reduced theory. The components of the Israel conditions on the internal space give equation describing the jump in the scalar field at the domain wall.

Note that the scale transformations of the type II solutions obtained by dimensional reduction leave the $(D+n)$ dimensional metric invariant provided one assumes that the metric on the (Ricci flat) internal space scales in a suitable way. This symmetry arises as a result of the dimensional reduction ansatz. (The cosmological constant and domain wall tension break the scale invariance of the higher dimensional action.)

The results of this section suggest that it would be possible to generalize our method to deal with a bulk potential consisting of a sum of Liouville potentials with parameters $b = \frac{1}{\sqrt{D-2}}$ and $1/b$. In fact it is straightforward
to show that for a Liouville potential on the wall, this is the most general bulk potential for which our method will work\footnote{Actually there is a special limiting case, namely $\alpha^2 = 1/(D - 2)$ (i.e. $b^2 = 1$), for which the bulk potential takes the form $(V_0 + V_1 \phi)e^{\pm \sqrt{b} \phi}$.}. The bulk solutions in this case all have $k = \pm 1$. The $k = +1$ solutions were constructed in \cite{22}.

\section{Hořava-Witten Cosmology}

An interesting scenario for which our method can be used to find solutions is that of strongly coupled $E_8 \times E_8$ heterotic string theory, which has been identified by Hořava and Witten with M-theory compactified on a $S^1/Z_2$ orbifold with $E_8$ gauge fields living on each orbifold fixed plane \cite{23, 24}. This can be compactified to five dimensions on a Calabi-Yau space \cite{27}. Matching the predicted values for the four dimensional gravitational and GUT couplings leads one to the conclusion that the orbifold is an order of magnitude larger than the Calabi-Yau space \cite{21, 28}. Our universe is identified with one of the orbifold fixed planes. The other fixed plane describes a “shadow” universe that interacts with our own only via bulk fields.

Lukas et al have shown that this five dimensional theory admits a supersymmetric solution describing a pair of domain walls (the orbifold fixed planes) \cite{29}. Simple cosmological solutions have also been found by separating variables in the bulk spacetime \cite{16, 17}. We can apply the methods of the previous sections to find further solutions.

The model of Lukas et al has $\alpha = \frac{\sqrt{2}}{2} = -\sqrt{2}$, $V_0 = \frac{a^2}{6}$ and $\hat{V}_0 = \mp \sqrt{2}a$, where $a$ is a constant related to the number of units of four-form flux on the internal Calabi-Yau space, and the two sign choices refer to domain walls of negative and positive tension, which we shall call $M_1$ and $M_2$ respectively. Note that this theory cannot be obtained by dimensional reduction of a theory with a cosmological constant. The theory is scale invariant, so a type II ($k = 0$) solution exists. The bulk solution is described by

\begin{equation}
U(r) = 49r^{\frac{5}{2}} \left( \frac{a^2}{18} e^{-\sqrt{6} \phi_0} - 2Mr^{\frac{3}{2}} \right),
\end{equation}

\begin{equation}
R(r) = r^{\frac{5}{2}},
\end{equation}

\begin{equation}
\phi(r) = \sqrt{3} \left( \phi_0 + \frac{\sqrt{6}}{7} \log r \right).
\end{equation}
The effective potential for the domain walls simplifies considerably:

\[ F(R) = -MR^{-8}, \quad (8.4) \]

from which we see that \( M \geq 0 \) is necessary for domain wall solutions to exist in this bulk. If \( M = 0 \) then \( F \equiv 0 \) and the domain walls can be put anywhere and will remain static. The bulk solution has a timelike naked singularity at \( r = 0 \). The spatial sections must decrease towards \( M_1 \) and increase towards \( M_2 \) so if the former is at \( r = r_1 > 0 \) then we must put the latter at \( r = r_2 > r_1 \). The bulk spacetime that is left after imposing reflection symmetry in each domain wall is \( r_1 < r < r_2 \), which is non-singular. This solution is simply the supersymmetric domain wall solution of Lukas et al.

When \( M > 0 \) there is a timelike naked singularity at \( R = 0 \) and a cosmological horizon at \( R = \frac{1}{6}|a|\sqrt{M e^{\frac{\sqrt{6}\phi}{2}}} \). Static domain wall solutions are no longer possible. The position of \( M_i \) is given by

\[ R_i(\tau) = \left( 5\sqrt{2M(\tau_i \pm \tau)} \right)^{\frac{1}{5}}, \quad (8.5) \]

where the \( \tau_i \) are constants. The scale factors on each domain wall grow very slowly with the velocity approaching zero at late times. There are two choices of sign for each domain wall, leading to a total of four solutions. In an obvious notation these can be classified as followed:

\( (++) \) solution. The + sign occurs for both walls so they are both moving outwards. We need \( \tau_1 < \tau_2 \) for \( M_1 \) to occur at the smaller value of \( R \). This domain wall becomes singular at a finite proper time in its past. At late times the separation between the two domain walls tends to zero as \( \tau^{-\frac{4}{5}} \). The \( (--) \) solution is simply the time reversed version of this.

\( (-+) \) solution. \( M_1 \) has \( R \) decreasing and \( M_2 \) has \( R \) increasing. They must have been coincident at some time in the past then moved apart with one falling into the singularity. The \( (+- \) solution is the time reverse of this, which has \( M_1 \) emerging from the singularity then colliding with \( M_2 \) which is moving towards the singularity.

Since the distance between the universe and the shadow universe gives the string coupling, \( (-+) \) solutions describe the “pair creation” of a universe-shadow universe pair from a region of very weak string coupling. Likewise, the \( (+-) \) solutions describe the annihilation of such a pair. It is unclear how we can assign a rate, or probability, for such processes.
9 Conclusions

If a domain wall couples to a bulk matter field then one would expect the bulk spacetime to be time dependent. We have investigated the conditions under which it is possible to have such a domain wall moving in a static bulk spacetime with a dilaton. For the case of Liouville potentials in the bulk and on the wall we have found that if the bulk and boundary exponents are related in a certain ways then solutions can be found.

The bulk solutions we have found are all singular and have at most one horizon of black hole or cosmological type. Moving across such a horizon takes one to a non-static region. This allowed us to study dilatonic domain walls moving in a time dependent bulk. The behaviour of the domain walls is qualitatively similar to the constant dilaton solutions (type I) in certain respects i.e. they can fall into the singularity or expand forever. However in other respects our solutions are different. For example, all of the type I solutions that expand to infinity undergo inflation whereas we have found new solutions that decelerate to a constant velocity at infinity.

There has been recent interest in cosmological models in which our universe is viewed as a brane moving in a higher dimensional spacetime \cite{10}. In light of this, we have concentrated on the issue of inflation on the domain wall world volume. If the domain wall inflates then it must either move in the bulk or the bulk must be inflating too. Most recent work has concentrated on the latter possibility. We have solutions describing both cases. We believe that our model is of interest because the inflaton couples to the domain wall but is not restricted to its world volume. For the type II solutions, power law inflation occurs when $b^2 < 1$ and for the type III solutions when $b^2 > 1$. In the former case the bulk is also inflating (if $V_0 > 0$) while in the latter it is not. The critical value for inflation in both cases can be expressed as $\alpha^2 < \frac{1}{D-2}$, so it appears that it is the domain wall coupling rather than the bulk coupling that dictates whether inflation occurs. (Note that the critical value for inflation in the bulk is $\beta^2 < \frac{1}{D-2}$.) These inflationary solutions are similar to the Vilenkin-Ipser-Sikivie domain wall \cite{2,3,4} (which undergoes exponential inflation) in the respect that inflation occurs because the domain wall energy dominates bulk effects. However, in our solutions the energy density of the domain wall tends to zero at late times but inflation continues. (A similar effect occurs in bulk inflation from an exponential potential.)

We have shown that dimensional reduction of a theory consisting of Ein-
stein gravity with a cosmological constant and domain walls with Nambu-Goto type actions gives a theory of the type we have considered in this paper. Interestingly the dimensionally reduced theories always have \( \alpha^2 < \frac{1}{D-2} \), so domain wall inflation arises naturally from dimensional reduction. This is presumably because the domain wall spacetime in the higher dimensional theory is inflating. However exponential inflation in the higher dimensional theory becomes power law inflation in the reduced theory.

Our method can be used to find new cosmological solutions of Hořava-Witten theory. These solutions do not appear very phenomenologically interesting. In order to obtain inflation in this model it seems necessary to include matter restricted to the world volume of the domain walls [14].

We would like to finish by mentioning possible generalizations and applications of our work. It would be interesting to see if our method could be extended to find solutions with different potentials in the bulk or on the boundary. An obvious generalization would be to investigate further the domain wall motion when the bulk dilaton potential is a sum of Liouville potentials with parameters \( b \) and \( 1/b \), as obtained from dimensional reduction of type I solutions. Other possible generalizations could include putting charge in the bulk (such solutions were described in [21, 22]) or including matter fields restricted to the domain wall world volume.

Another obvious way of generalizing our work would be to relax the assumption of reflection symmetry in the domain wall. One could also allow the bulk potential to vary discontinuously across the wall, as studied for the case of a cosmological constant in [2, 5]. Such a scenario arises in type IIA string theory when D8 branes are considered. Romans [30] found a “massive” generalization of the type IIA supergravity theory. His theory contains a Liouville type potential with an arbitrary (positive) coefficient. This was reformulated in [31] in terms of a ten form field strength of a nine form potential that the D8 branes of type IIA string theory couple to. The expectation value of this ten form jumps across a D8-brane. This can be viewed as a jump in the coefficient of a bulk Liouville potential. Our method might be useful in obtaining solutions describing dynamic D8 branes in this theory.

The main question concerning our solutions is whether they are stable. In other words, if the domain wall is perturbed then will the perturbation remain small or will it grow and change the bulk metric, for example converting a static metric to a time-dependent one? If they are stable then it would be interesting to examine how perturbations behave in this model, especially as
viewed by an inhabitant of the domain wall world volume.

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