A stage-structured predator–prey model with distributed maturation delay and harvesting

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ABSTRACT
A stage-structured predator–prey system with distributed maturation delay and harvesting is investigated. General birth and death functions are used. The local stability of each feasible equilibria is discussed. By using the persistence theory, it is proven that the system is permanent if the coexistence equilibrium exists. By using Lyapunov functional and LaSalle invariant principle, it is shown that the trivial equilibrium is globally stable when the other equilibria are not feasible, and that the boundary equilibrium is globally stable if the coexistence equilibrium does not exist. Finally, sufficient conditions are derived for the global stability of the coexistence equilibrium.

1. Introduction
The construction and study of models for the population dynamics of predator–prey interactions have remained an important area in theoretical ecology since the famous Lotka–Volterra equations. A most crucial element in these models is the functional response, the function that describes the number of prey consumed per predator per unit time for given quantities of prey and predator. In classical models of Lotka–Volterra type, it is assumed that all individuals of a single species have largely similar capabilities to hunt or reproduce. This assumption seems not be realistic for most, if not all, animals and insects. In the real world, there are many species whose individuals have a life history that takes them through two stages, immature and mature, where immatures are raised by their parents, and the rate they attack at prey and the reproductive rate can be ignored (see, for example, [8, 10, 15, 17, 18]).

Beginning with the pioneering work of Aiello and Freedman [1] in 1990s, who proposed and studied a single species model with stage structure and discrete delay, stage-structured models with discrete delay have attracted much attention in recent decades, (see, for example, [2, 4, 6, 8–10, 13, 14]). An important assumption behind those works is that all individuals belonging to the same species take the identical amount of time to become mature, which seems biologically unreasonable since individuals in a population do not necessarily always mature at the same age.
In view of such stage-structured predator–prey model, Gourley and Kuang [11] formulated a general and robust predator–prey model with stage structure with constant maturation time delay and performed a systematic mathematical and computational study. They have shown that there is a window in maturation time-delay parameter that generates sustainable oscillatory dynamics. Thus, they study the following model

\[
\begin{align*}
  x'(t) &= rx(t) \left(1 - \frac{x(t)}{k}\right) - ay(t)p(x(t)), \\
  y'(t) &= ba e^{-\gamma \tau} y(t - \tau) p(x(t - \tau)) - dy(t), \\
  j'(t) &= by(t)p(x(t)) - b e^{-\gamma \tau} y(t - \tau) p(x(t - \tau)) - \gamma j(t) \\
  x(\theta), y(\theta) &\geq 0 \text{ are continuous on } -\tau \leq t \leq 0, \\
  x(0), y(0), j(0) &> 0,
\end{align*}
\]  

where \(x(t)\) and \(y(t)\) represent prey and mature predator densities, respectively. \(j(t)\) denotes the immature or juvenile predator density. Here, \(r\) is the specific growth rate of the prey and \(k\) is its carrying capacity. The parameters \(b\) and \(d\) are the adult predators’ birth and death rates, respectively. In addition, the juveniles suffer a mortality rate of \(\gamma\) (the through-stage death rate) and take \(\tau\) units of time to mature. The function \(p(x)\) is the adult predators’ functional response and it is assumed to be differentiable and satisfy

\[
p(0) = 0, \quad p(x) \text{ is strictly increasing, } \frac{p(x)}{x} \text{ bounded for all } x \geq 0.
\]

An alternative of stage-structured models with discrete delays is to use a distributed delay term allowing for a distribution of maturation times, weighted by a probability density function. Therefore, the stage-structured models of distributed delay type is a generalization of those corresponding model of discrete delay type. There are many different stage-structured models with distributed delay in the literature (see, for example, [3, 5, 7, 11, 15, 16]).

Al-Omari and Gourley [7] proposed a general model for a single species with stage structure in which the maturation delay is modelled as a distribution. The model takes the form

\[
\begin{align*}
  u'_i(t) &= b(u_m(t)) - \gamma u_i(t) - \int_0^\infty b(u_m(t - s)) f(s) e^{-\gamma s} \, ds, \\
  u'_m(t) &= \int_0^\infty b(u_m(t - s)) f(s) e^{-\gamma s} \, ds - d(u_m(t)),
\end{align*}
\]

where \(\int_0^\infty f(s) \, ds = 1\) and \(f \geq 0\), because \(f\) is a probability density function. In this system, \(u_i\) and \(u_m\) denote the numbers of immature (juvenile) and mature (adult) members of the population, respectively, and \(b(\cdot)\) and \(d(\cdot)\) are called the birth and death functions and satisfy certain assumptions. The parameter \(\gamma\) measures the juvenile death rate. They considered various aspects of the above system including positivity and also found that the dynamics of the model depends largely on the qualitative form of the birth function, which depends on the total number of adults. They also derived a reaction–diffusion extension of the model, where they found that it again depends largely on the qualitative form of the birth function.
Motivated by the above works, the objective of this paper is to adapt and study system (1) by introducing the distributed maturation time delay into predator populations and linear harvesting of the immature and mature predator populations and also of prey population. Also, we use general birth and death functions for the mature predator populations. Therefore, we consider the following model:

\[
x'(t) = rx(t) \left(1 - \frac{x(t)}{k}\right) - b(y(t))p(x(t)) - h_1x(t),
\]

\[
y'(t) = \int_0^\infty f(s) e^{-(\gamma + h_3)s}b(y(t-s))p(x(t-s))\,ds - d(y(t)) - h_2y(t),
\]

\[
y_j'(t) = b(y(t))p(x(t)) - \int_0^\infty f(s) e^{-(\gamma + h_3)s}b(y(t-s))p(x(t-s))\,ds - \gamma y_j(t) - h_3y_j(t).
\]

Here, the terms \(h_1, h_2,\) and \(h_3\) are the harvesting efforts of the prey, mature and immature populations, respectively. Note that the term \(e^{-\gamma s}\) in system (2) is replaced by the term \(e^{-(\gamma + h_3)s}\) in system (3), because \(e^{-(\gamma + h_3)s}\) is the probability of an individual born at time \(t - s\) still being alive and is not harvested at time \(t.\) This follows from the assumption that the death and harvesting of immature populations are following a linear law given by the terms \(-\gamma y_j\) and \(-h_3y_j.\)

Notice that, by the third equation of system (3), we have

\[
y_j(t) = \int_{-\infty}^t \left(\int_{t-s}^\infty f(\xi)\,d\xi\right) b(y(s))p(x(s)) e^{-(\gamma + h_3)(t-s)}\,ds,
\]

that is, \(y_j(t)\) is completely determined by \(b(y(t))\) and \(p(x(t)).\) Therefore, the dynamics of model (3) are determined by the first two equations. Now, both \(r\) and \(k\) of system (3) can be easily scaled off by appropriate rescaling of time and the \(x\) variable. Accordingly, system (3) becomes

\[
x'(t) = x(t)(1 - x(t)) - b(y(t))p(x(t)) - h_1x(t),
\]

\[
y'(t) = \int_0^\infty f(s) e^{-(\gamma + h_3)s}b(y(t-s))p(x(t-s))\,ds - d(y(t)) - h_2y(t).
\]

For initial data of system (5), we assume that

\[
x(t) = \phi_1(t) \geq 0, \quad y(t) = \phi_2(t) \geq 0 \quad \text{for} \quad -\infty < t \leq 0 \quad \text{with} \quad x(0), y(0) > 0.
\]

It is straightforward to show that the solutions of system (5), subject to (6), satisfy \(x(t), y(t) > 0\) on \((0, \infty).\) This fact is important for both the modelling and the analysis.

In the next sections, we shall prove theorems for the case when the kernels \(f(s)\) have compact support, that is, \(f(s) = 0\) for all \(s \geq \tau,\) for some \(\tau > 0,\) and normalized \(\int_0^\tau f(s)\,ds = 1.\) This implies that no individual ever takes longer than \(\tau\) units of time to
mature. In this case system (5) becomes, where \( \eta = \gamma + h_3 \),
\[
\begin{align*}
x'(t) &= x(t)(1 - x(t)) - b(y(t))p(x(t)) - h_1x(t), \\
y'(t) &= \int_0^t f(s) e^{-\eta s} b(y(t-s))p(x(t-s)) \, ds - d(y(t)) - h_2y(t)
\end{align*}
\]
\( (7) \)

Also, in the rest of this paper, we need the following assumptions:

(H1): \( b(0) = d(0) = 0 \) and \( b(a) \) is strictly increasing for all \( a > 0 \).

(H2): \( p(0) = 0 \), \( p(x) \) is strictly increasing \( p(x)/x \) bounded for all \( x \geq 0 \).

Now, we will prove a theorem giving that the solutions of Equation (7) are eventually bounded for all time.

**Theorem 1.** If \( \eta < h_2 \) holds, then every solution \((x(t), y(t))\) of system (7) is uniformly bounded.

**Proof.** We first prove that \( x(t) \) is eventually bounded. By positivity of solutions
\[
x'(t) \leq x(1 - x).
\]
By comparison, \( x(t) \leq u(t) \) where \( u(t) \) satisfies
\[
u'(t) = u(t)(1 - u(t)), \quad u(0) = x(0) > 0.
\]
But \( \lim_{t \to \infty} u(t) = 1 \). Therefore
\[
\lim_{t \to \infty} \sup_{t \to \infty} x(t) \leq \lim_{t \to \infty} u(t) = 1,
\]
and so \( x(t) \) is uniformly bounded.

Let
\[
V(t) = x(t) + y(t) + \int_0^t f(s) \int_{t-s}^t e^{-\eta(t-\xi)} b(y(\xi))p(x(\xi)) \, d\xi \, ds.
\]
By positivity of solutions, all terms in \( V \) are positive. Now
\[
\begin{align*}
V'(t) &= x(1 - x) - b(y(t))p(x(t)) - h_1x(t) + \int_0^t f(s) e^{-\eta s} b(y(t-s))p(x(t-s)) \, ds \\
&\quad - d(y(t)) - h_2y(t) - \eta \int_0^t f(s) \int_{t-s}^t e^{-\eta(t-\xi)} b(y(\xi))p(x(\xi)) \, d\xi \, ds \\
&\quad + b(y(t))p(x(t)) \int_0^t f(s) \, ds - \int_0^t f(s) e^{-\eta s} b(y(t-s))p(x(t-s)) \, ds \\
&\leq x(1 - x) - h_1x(t) - \eta \int_0^t f(s) \int_{t-s}^t e^{-\eta(t-\xi)} b(y(\xi))p(x(\xi)) \, d\xi \, ds - h_2y(t) \\
&= x(1 - x) - h_1x(t) - h_2y(t) - \eta(V - x(t) - y(t)),
\end{align*}
\]
so that
\[ V' + \eta V \leq x(1 - x) - h_1 x(t) + \eta x(t) + (\eta - h_2)y(t) \]
\[ \leq x(1 - x) - h_1 x(t) + \eta x(t), \]

but since \( x(t) \) is uniformly bounded, there exists a positive constant \( M \) such that
\[ V' + \eta V \leq M < \infty. \]

Hence
\[ V \leq \frac{M}{\eta} \left( V(0) - \frac{M}{\eta} \right) e^{-\eta t}. \]

Thus, \( V \) is uniformly bounded. Since the individual terms in the expression for \( V \) are all positive, we deduce that \( y \) is uniformly bounded.

We believe that a boundedness result probably holds even if \( \eta > h_2 \). Because if \( y \) got very large the effect on \( x \) would be inhibitive, and then low values of \( x \) would, in turn, reduce \( y \).

2. **Equilibria and their feasibility**

The equilibria of our model are determined by setting \( x' = y' = 0 \) in system (7) and solving the resulting algebraic equations.

If \( y = 0 \), then \( x(1 - h_1 - x) = 0 \), so that \( x = 0 \) or \( x = 1 - h_1 \). This gives two equilibria the origin \( E_0 = (0, 0) \) and \( E_1 = (1 - h_1, 0) \). But it is clear that if \( h_1 > 1 \), then \( E_0 = (0, 0) \) is the only nonnegative equilibrium of system (7). If \( y \neq 0 \), then a further equilibrium \( E_2 = (\bar{x}, \bar{y}) \) may be possible. The \( \bar{x} \) component of such an equilibrium would need to satisfy
\[ b(\bar{y})p(\bar{x}) \int_0^\tau f(s) e^{-\eta s} ds = d(\bar{y}) + h_2 \bar{y}, \tag{8} \]
and we also need \( \bar{x} \leq 1 - h_1 \), because otherwise \( \bar{y} \) would be negative. From the assumptions \( H2 \), one easily sees that the condition for an ecologically relevant equilibrium of the form \( E_2 = (\bar{x}, \bar{y}) \) (distinct from \( E_1 = (1 - h_1, 0) \)) is
\[ R(a) = \frac{b'(a)p(1 - h_1) \int_0^\tau f(s) e^{-\eta s} ds}{d'(a) + h_2} > 1 \quad \text{for all } a > 0. \tag{9} \]

The \( \bar{y} \) component of the equilibrium is found from
\[ b(\bar{y}) = \frac{\bar{x}(1 - h_1 - \bar{x})}{p(\bar{x})}. \tag{10} \]

Note that condition (9) makes clear how the interior equilibrium \( E_2 \), if feasible, depends on the immatures’ mortality rate, \( \gamma \), or the harvesting effort of the immature, \( h_3 \), or mature predator populations, \( h_2 \), and it depends on the maturation delay as represented by the probability density function \( f(s) \) and the other parameters. It is automatically satisfied if the parameters \( \gamma \) or \( h_3 \) or \( h_2 \) is small enough.
3. Global stability of equilibria

In this section, we shall prove theorems on the global stability of the equilibria $E_0$, $E_1$ and $E_2$.

3.1. Global stability of $E_0$

We shall show that when the equilibria $E_1$ and $E_2$ do not exist, then $E_0$ is globally asymptotically stable.

Theorem 2. Assume the initial data (6) and $h_1 > 1$ hold. Then $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$.

Proof. Consider the functional

$$V(t) = x(t) + y(t) + \int_0^t \int_{t-s}^t f(s) e^{-\eta(t-\xi)} b(y(\xi)) p(x(\xi)) d\xi \, ds.$$  

Note that $V(x, y) \geq 0$ and $V(x, y) = 0$ if and only if $x = 0$ and $y = 0$. Then for $t$ sufficiently large,

$$V'(t) = x(1-x) - h_1 x(t) - d(y(t)) - h_2 y(t)$$

$$- \eta \int_0^t \int_{t-s}^t e^{-\eta(t-\xi)} b(y(\xi)) p(x(\xi)) d\xi \, ds \leq x(1 - h_1) < 0.$$  

A direct application of the well-known Lyapunov–LaSalle theorem [14] (Kuang [13, Theorem 2.5.3]) shows that $\lim_{t \to \infty} x(t) = 0$ and $\lim_{t \to \infty} y(t) = 0$. The proof of the theorem is complete.

Therefore, our results suggest that the trivial equilibrium is globally stable when the other equilibria are not feasible. That is, we obtained that, if the harvesting effort of the prey population, $h_1$, is larger than one, then the prey and predator population will go to extinction.

3.2. Global stability of $E_1$

In this section, we prove a theorem on the global stability of the equilibrium $E_1$ when infant mortality of the predator or harvesting effort of the infant or adult predator is very high, i.e. when $R(0) < 1$. That is, the boundary equilibrium is globally stable if the coexistence equilibrium does not exist.

Theorem 3. Assume the initial data (6), $h_1 < 1$ and $R(0) < 1$ hold. Then $(x(t), y(t)) \to (1 - h_1, 0)$ as $t \to \infty$.

Proof. Since $R(0) < 1$, there exists $\varepsilon > 0$ such that $b'(0)p(1-h_1+\varepsilon) \int_0^\tau f(s) e^{-(\gamma+h_3)s} \, ds < d'(0) + h_2$. Also, by positivity of solutions, $x'(t) \leq x(t)(1 - h_1 - x(t))$. This implies
that \( \limsup_{t \to \infty} x(t) \leq 1 - h_1 \) and therefore, there exists \( T > 0 \) such that \( x(t) \leq 1 - h_1 + \varepsilon \) for \( t \geq T \). Then, for \( t \geq T + \tau \)

\[
y'(t) = \int_{0}^{T} f(s) e^{-ns}b(y(t - s))p(x(t - s)) \, ds - d(y(t)) - h_2 y(t)
\]

\[
\leq p(1 - h_1 + \varepsilon) \int_{0}^{T} f(s) e^{-ns}b(y(t - s)) \, ds - d(y(t)) - h_2 y(t).
\]

By comparison, \( y(t) \) is bounded above by the solution \( u(t) \) of

\[
u'(t) = p(1 - h_1 + \varepsilon) \int_{0}^{T} f(s) e^{-ns}b(u(t - s)) \, ds - d(u(t)) - h_2 u(t),
\]

with, for \( s \leq 0 \), \( u(t) = \max\{y(s), s \in [-\tau, 0]\} > 0 \). Then since \( R(0) < 1 \), it follows from [3, Theorem 3] that \( u(t) \to 0 \). Hence \( y(t) \to 0 \).

Next, we shall show that \( x(t) \to 1 - h_1 \) as \( t \to \infty \). Let \( \varepsilon > 0 \), then since \( y \to 0 \) there exists \( T_1 > 0 \) such that \( -\varepsilon \leq y(t) \leq \varepsilon \) when \( t \geq T_1 \). Then, for \( t \geq T_1 \)

\[
x'(t) \geq x(1 - x) - p(x)b(\varepsilon) - h_1 x,
\]

which means that \( x(t) \geq \tilde{x}(t) \) where \( \tilde{x}(t) \) satisfied

\[
\tilde{x}'(t) = \tilde{x}(1 - \tilde{x}) - p(\tilde{x})b(\varepsilon) - h_1 \tilde{x}, \quad t > T_1 \quad \text{with} \quad \tilde{x}(T_1) = x(T_1) > 0.
\]

From \( H2 \), for sufficiently small \( \varepsilon \), \( \tilde{x}(t) \to \tilde{x}^\ast \) as \( t \to \infty \), where \( \tilde{x}^\ast \) is the positive root of the right-hand side of the last differential equation. Accordingly,

\[
\liminf_{t \to \infty} x(t) \geq \lim_{t \to \infty} \tilde{x}(t) = \tilde{x}^\ast.
\]

Letting \( \varepsilon \to 0 \), we conclude that

\[
\liminf_{t \to \infty} x(t) \geq 1 - h_1.
\]

We already know that \( \limsup_{t \to \infty} x(t) \leq 1 - h_1 \). Consequently, \( \lim_{t \to \infty} x(t) = 1 - h_1 \). This completes the proof of the theorem.

### 3.3. Global stability of \( E_2 \)

Finally, to prove the stability of \( E_2 \) we need the following lemma based on Hale and Waltman’s famous persistence theory [12]. Consider a complete metric space \( X \) with metric \( d \). \( T \) is a continuous semiflow on \( X \), that is, a continuous mapping \( T : [0, \infty) \times X \to X \) with the following properties:

\[
T_{t_1}T_{t_2} = T_{t_1 + t_2}, \quad t, s \geq 0; \quad T_0(x) = x, \quad x \in X.
\]

Here, \( T_t \) denotes the mapping from \( X \) to \( X \) given by \( T_t(x) = T(t, x) \). The distance \( d(x, y) \) of a point \( x \in X \) from a subset \( Y \) of \( X \) is defined by

\[
d(x, y) = \inf_{y \in Y} d(x, y).
\]

Recall that the positive orbit \( \gamma^+(x) \) through \( x \) is defined as \( \gamma^+(x) = \bigcup_{t \geq 0} \{T(t, x)\} \), and its \( \omega \)-limit set is \( \omega(x) = \bigcap_{t \geq 0} \bigcup_{t \geq s} T(t)x \). Define \( W^s(A) \) the stable set of a compact invariant
set $A$ as

$$W^s(A) = \{x : x \in X, \omega(x) \neq \phi, \omega(x) \subset A\},$$

and the particular invariant sets of interest are $\tilde{A}_\beta = \bigcup_{x \in A_\beta} \omega(x)$.

(H3): Assume that $X$ is the closure of open set $X^0$ and $\partial X^0$ is nonempty and is the boundary of $X^0$. Moreover, the $C^0$-semigroup $T$ on $X$ satisfies

$$T(t) : X^0 \to X^0, \quad T(t) : \partial X^0 \to \partial X^0.$$  \tag{11}

**Lemma 1** [12]. Assume that $T(t)$ satisfies (H3) and the following conditions

(i) there exists a $t_0 > 0$ such that $T(t)$ is compact for $t > t_0$;
(ii) $T(t)$ is point dissipative in $X$;
(iii) $\tilde{A}_\beta$ is isolated and has acyclic covering $\tilde{M} = \{M_1, M_2, \ldots, M_n\}$;
(iv) $W^s(M_i) \cap X^0 = \phi$.

Then, $X^0$ is a uniform repeller with respect to $\partial X^0$, that is, there exists an $\varepsilon > 0$ such that

$$\liminf_{t \to \infty} d(T(t)x, X^0) \geq \varepsilon,$$

for any $x \in \partial X^0$.

**Theorem 4.** Assume that $h_1 < 1$ and $R(\bar{y}) > 1$ hold, then system (7) is uniformly persistent.

**Proof.** We begin by showing that the boundary planes of $R^2^+_\tau = \{(x, y) : x \geq 0, y \geq 0\}$ repel the positive solutions to system (7) uniformly. Let $C^+([-\tau, 0], R^2^+_\tau)$ denote the space of continuous functions mapping $[-\tau, 0]$ into $R^2^+_\tau$. We choose

$$C_1 = \{(\phi_1, \phi_2) \in C^+([-\tau, 0], R^2^+_\tau) : \phi_1(\theta) \equiv 0, \theta \in [-\tau, 0]\},$$

$$C_2 = \{(\phi_1, \phi_2) \in C^+([-\tau, 0], R^2^+_\tau) : \phi_1(\theta) > 0, \phi_2(\theta) \equiv 0, \theta \in [-\tau, 0]\}.$$  

Denote $C = C_1 \cup C_2, X = C^+([-\tau, 0], R^2^+_\tau)$ and $X^0 = \text{Int } C^+([-\tau, 0], R^2^+_\tau)$, then $C = \partial X^0$. Clearly, $C \cap X^0 = \phi$.

We verify below that the conditions of Lemma 1 are satisfied. We have from the positivity of the solutions that $X^0$ and $\partial X^0$ are invariant, and thus (H3) holds. Moreover, by Theorem 1, assumptions H1 and H2, $i = 1, 2$, conditions (i) and (ii) of Lemma 1 are satisfied. Note that system (7) has two constant solutions in $C = \partial X^0$, $\tilde{E}_0 = (0, 0) \in C_1$, $\tilde{E}_1 = (1 - h_1, 0) \in C_2$. It is clear that if $(x(t), y(t))$ is a solution of system (7), then we have $x^t((\phi_1, \phi_2) \in C_1 \equiv 0$, then we get $x(t) = 0$, $c_2(\partial_1, \partial_2) \equiv 0$ for all $t \geq 0$, using the second equation of (7), we get $y(t) = -d(y(t)) - h_2y(t) \leq 0$, hence all points in $C_1$ approaches $\tilde{E}_0$, i.e. $C_1 = W^s(\tilde{E}_0)$. Similarly, we have all points in $C_2$ approach $\tilde{E}_1$, that is, $C_2 = W^s(\tilde{E}_1)$. Hence, $\tilde{A}_\beta = \tilde{E}_0 \cup \tilde{E}_2$ and clearly it is isolated. Noting that $C_1 \cap C_2 = \phi$, this means that $\tilde{A}_\beta$ is acyclic, satisfying condition (iii) of Lemma 1.

Now, we show that $W^s(\tilde{E}_i) \cap X^0 = \phi, i = 1, 2$. Since Theorem 1 indicates that $W^s(\tilde{E}_0) \cap X^0 = \phi$, we only need to prove $W^s(\tilde{E}_1) \cap X^0 = \phi$. Assume on the contrary, that is, $W^s(\tilde{E}_1) \cap X^0 \neq \phi$, then there exists a positive solution $(x(t), y(t))$ of system (7) with
lim_{t \to \infty} (x(t), y(t)) = (1 - h_1, 0). Note that, since y(t) → 0 for sufficiently small ε there exists T > 0 such that
\[ x(t) > 1 - h_1 - \varepsilon, \quad y(t) < \varepsilon \quad \text{for all } t \geq T, \]
and also, since R(\bar{y}) > 1, then
\[ b'(\bar{y})p(1 - h_1 - \varepsilon) \int_0^\tau f(s) e^{-\eta s} ds > d'(\bar{y}) + h_2. \]  \tag{12}

By the second equation of Equation (7), we have
\[ y'(t) = \int_0^\infty f(s) e^{-\eta s} b(y(t - s))p(x(t - s)) ds - d(y(t)) - h_2y(t) \]
\[ > p(1 - h_1 - \varepsilon) \int_0^\infty f(s) e^{-\eta s} b(y(t - s)) ds - d(y(t)) - h_2y(t) \quad \text{for all } t \geq T + \tau. \]

Consider the equation
\[ w'(t) = p(1 - h_1 - \varepsilon) \int_0^\tau b(w(t - s)) f(s) e^{-\eta s} ds - d(w(t)) - h_2 w(t), \]  \tag{13}
\[ w(t) = \max y(s), \quad s \in [-\tau, 0]. \]

It is easy to obtain that w(t) > 0, t > 0 for Equation (13). Now, let w* denotes a positive equilibrium of Equation (13). Denote v(t) = w(t) - w*, thus Equation (13) takes the form
\[ v'(t) = b'(w^*)p(1 - h_1 - \varepsilon) \int_0^\tau f(s) e^{-\eta s} v(t - s) ds - d'(w^*)v(t) - h_2v(t). \]

Therefore, by using Equation (12) and [3, Theorem 4] we will have lim_{t \to \infty} v(t) = \infty, then lim_{t \to \infty} w(t) = \infty for all solution of Equation (13). Thus, this leads to lim_{t \to \infty} y(t) = \infty. Contradiction the boundedness of y(t). Then, W^s(\bar{E}_2) \cap X^0 = \phi holds. Thus, all conditions of Lemma 1 are satisfied. Therefore, there exists an \varepsilon_0 > 0 such that
\[ \lim \inf_{t \to \infty} u_m(t) \geq \varepsilon_0, \quad \lim \inf_{t \to \infty} v_m(t) \geq \varepsilon_0. \]  \tag{14}

Thus, system (7) is uniformly persistent. The proof is complete. Now it follows from [12, Theorem 3.2] that the following theorem holds.

**Theorem 5.** Assume that the initial data (6), h_1 < 1 and R(\bar{y}) > 1 hold, then the equilibrium E_2 is globally asymptotically stable.

Thus, the interior equilibrium is globally stable, that is the predator coexists with the prey permanently, if and only if \[ b'(\bar{y})p(1 - h_1) \int_0^\tau f(s) e^{-\eta s} ds > d'(\bar{y}) + h_2, \] which is automatically satisfied if the immature species has small mortality rates \gamma, and/or little harvesting effort among the immature and mature species \gamma_j, \gamma as measured by h_3 and h_2, respectively, leading to the permanence of the predator population. Also, we can emphasize that the dynamics of our model depends heavily on the qualitative form of the birth and death functions, which depend on the total number of adults, as well as on the maturation time delay of the predator population as represented by the probability density function f(s).
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