Characteristics classes of $\text{SL}(N, \mathbb{C})$-bundles and quantum dynamical elliptic R-matrices

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Abstract

We discuss quantum dynamical elliptic R-matrices related to arbitrary complex simple Lie group $G$. They generalize the known vertex and dynamical R-matrices and play an intermediate role between these two types. The R-matrices are defined by the corresponding characteristic classes describing the underlying vector bundles. The latter are related to elements of the center $Z(G)$ of $G$. While the known dynamical R-matrices are related to the bundles with trivial characteristic classes, the Baxter–Belavin–Drinfeld–Sklyanin vertex R-matrix corresponds to the generator of the center $Z_N$ of $\text{SL}(N)$. We construct the R-matrices related to $\text{SL}(N)$-bundles with an arbitrary characteristic class explicitly and discuss the corresponding IRF models.

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1. Introduction

The quantum dynamical R-matrices and the quantum dynamical Yang–Baxter (QDYB) equation they satisfied, were introduced by G Felder [1, 2], while without the spectral parameter these structures appeared earlier [3–5]. The classical version of the QDYB equation is the classical dynamical Yang–Baxter equation and its solution is the classical dynamical r-matrix (CDRM). The classification of the CDRMs with the elliptic spectral parameter was proposed in [6] and then generalized in [26, 28, 22] (see also [29]).

The standard elliptic quantum R-matrix does not depend on dynamical variables [10, 7] (see also [8, 9]). It is defined only for the group $G = \text{SL}(N, \mathbb{C})$. We will refer to it as Baxter–Belavin–Drinfeld–Sklyanin R-matrix. The quantum R-matrix and the Yang–Baxter equation are the key tools for the quantum inverse scattering method [11–17]. In particular, they define the commutation relations in the vertex-type models and the corresponding Sklyanin-type algebras [18–20, 64].

On the other hand, Felder’s R-matrix depends on additional dynamical variable $u \in \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g} = \text{Lie} (G)$. It is related to the IRF models of statistical mechanics [2].
can be explained as a monopole solution of the Bogomolny equation [85].

are spectrally dual to Gaudin models. In terms of a gauge field theory the Hecke transformation limit of the integrable chains. As is known from the recent papers [86], the integrable chains nonautonomous Zhukovsky–Volterra gyrostat [79–82]. The field (1+1) generalizations of the Hitchin systems [73–78] on elliptic curves the modification relates the Painlevé VI equation and Hitchin–Nekrasov (Gaudin) models are discussed in [52, 83, 84]. They describe the continuous Ruijsenaars type [55–62] with the elliptic Euler–Arnold tops [63, 52, 64]. In the theory a singular gauge transformation on the Lax matrices and relates the models of Calogero–

| $\lambda$ | $\exp (\frac{2\pi i}{N})$ | $\exp (\frac{2\pi i}{N}p) , N = pl$ |
|---|---|---|
| $\lambda$ | Felder case | Baxter–Belavin–Drinfeld–Sklyanin case |

Here we consider an intermediate situation. In our approach it arises when $G$ is a simple Lie group with non-trivial centers, i.e. when $G$ is a classical group or $E_6$ and $E_7$. In these cases the dynamical parameter belongs to some subalgebra $h^0$ of the Cartan subalgebra $h$ of the Lie algebra $g$. At the classical level the problem was investigated from a different point of view in [26, 28] and in our papers [21–24]. In particular, we constructed there the classical elliptic $r$-matrices.

The quantum version of the classical construction [6, 26] was proposed in [25, 27]. The construction of the quantum $R$-matrices is more elaborate. It depends essentially on the representation space $V$. We suggest that upon the appropriate choice of $V$ the $R$-matrices can be constructed in the general situation as in the case $h^0 = h$ and $G = SL(N, \mathbb{C})$. We demonstrate it explicitly for $R$-matrices in $SL(N, \mathbb{C})$ case with elliptic dependence of the spectral parameter. If $N$ is a prime number then there are only two types of $R$-matrices. The first one is the Baxter–Belavin–Drinfeld–Sklyanin vertex $R$-matrix [4, 7] and the second type is the Felder dynamical $R$-matrix [2]. But if $N = pl$ ($p \neq 1, N$), then there exist new types of $R$-matrices. Their construction is the main result of this paper.

While different universal structures related to the Yang–Baxter equations are well studied for arbitrary simple Lie group in trigonometric and rational cases [30–38], the elliptic solution of the QDYB equation with spectral parameter (2.13) is known only in the $SL(N, \mathbb{C})$ case [40–44].

In the $A_{N−1}$ case, the center of $G = SL(N, \mathbb{C})$ is the cyclic group $\mu_N = \mathbb{Z}/N\mathbb{Z}$. Represent elements of $\mu_N$ as $\exp \frac{2\pi i j}{N}, j = 0 \ldots , N − 1$. Then Felder’s case corresponds to $j = 0$ while Baxter–Belavin–Drinfeld–Sklyanin’s one appears from $j = 1$. The intermediate situation takes place when $j = p > 1$ and $N = pl$. In this case dim $h = \text{g.c.d}(j, N) = p > 1$.

The purpose of the paper is to construct explicitly the quantum elliptic dynamical $R$-matrix in the intermediate case. The answer is given by the theorem 3.1 (section 3.4). It is shown that the suggested $R$-matrix satisfies the QDYB equation (2.13) with $u \in C \subset h_0$ (2.11). The result is schematically presented in table 1. The last column is the case of our interest.

The classical integrable system corresponding to the intermediate case is the system of interacting elliptic tops [21]. Our goal is to quantize its classical $r$-matrix.

It should be mentioned that the dynamical and non-dynamical elliptic $R$-matrices are related by the dynamical twist [46, 47][ see also [48–51]]. This twist was interpreted as a modification of bundle (or Hecke transformation) in [52]. At the classical level it acts by a singular gauge transformation on the Lax matrices and relates the models of Calogero–Ruijsenaars type [55–62] with the elliptic Euler–Arnold tops [63, 52, 64]. In the theory of integrable models of statistical mechanics this Hecke transformation defines a passage from the so-called IRF type models [65, 66] to the vertex-type models [46, 47, 67–70]. In the isomonodromic deformation problem [71, 72] corresponding to the Hitchin systems [73–78] on elliptic curves the modification relates the Painlevé VI equation and nonautonomous Zhukovsky–Volterra gyrostat [79–82]. The field (1+1) generalizations of the Hitchin–Nekrasov (Gaudin) models are discussed in [52, 83, 84]. They describe the continuous limit of the integrable chains. As is known from the recent papers [86], the integrable chains are spectrally dual to Gaudin models. In terms of a gauge field theory the Hecke transformation can be explained as a monopole solution of the Bogomolny equation [85].
The paper is organized as follows: in the following section we review construction of bundles over elliptic curves and define classical and quantum elliptic $R$-matrices, related to these bundles. In section 3 we, first, recall the known quantum $R$-matrices corresponding to the first and to the second columns in table 1. Then the quantum $R$-matrix for the intermediate case is suggested (3.26) and the QDYB equation is verified (theorem 3.1). Finally, we discuss possible applications of the obtained solution of the QDYB equation to IRF models.

2. Characteristic classes of bundles over elliptic curves and $R$-matrices

2.1. Characteristic classes of bundles over elliptic curves

Let $G$ be a complex simple Lie group with a non-trivial center $Z(G)$. A universal cover $\tilde{G}$ of $G$ in all cases apart from $G_2$, $F_4$ and $E_8$ has a non-trivial center $Z(\tilde{G})$. The center can be determined in terms of the co-weight $P^\vee$ and the co-root $Q^\vee$ lattices in the Cartan subalgebra $h$ of the Lie algebra $\text{Lie}(G) = \mathfrak{g}$ [89]. Namely, $Z(\tilde{G}) \sim P^\vee / Q^\vee$. The center $Z(G)$ is a cyclic group except for the case $g = D_{2n}$. In the latter case the group $\tilde{G} = \text{Spin}_{2n}(\mathbb{C})$ has a non-trivial center $Z(\text{Spin}_{2n}) = (\mu_1^\times \times \mu_2^\times)$, $\mu_2 = \mathbb{Z} / 2\mathbb{Z}$. If $Z(\tilde{G})$ is cyclic, then there exists a fundamental co-weight $\sigma^\vee \in P^\vee$ generating $Z(\tilde{G})$. It means that $\text{ord}(Z(\tilde{G})) \sigma^\vee \in Q^\vee$.

The adjoint group is the quotient $G^{ad} = \tilde{G} / Z(\tilde{G})$. For the cases $A_{n-1}$ (when $n = pl$ is non-prime) and $D_l$, the center $Z(\tilde{G})$ has non-trivial subgroups $\mathbb{Z}_l \sim \mu_l = \mathbb{Z} / l\mathbb{Z}$. Assume that $(p, l)$ are co-prime. There exists the quotient-groups

$$G_1 = \tilde{G} / \mathbb{Z}_l, \quad G_p = G_1 / \mathbb{Z}_p, \quad G^{ad} = G_1 / Z(G_1),$$

where $Z(G_1)$ is the center of $G_1$ and $Z(G_1) \sim \mu_p = Z(\tilde{G}) / \mathbb{Z}_l$. The group $\mathbb{Z}_l$ is generated by the co-weight $\sigma^\vee$ such that $j \sigma^\vee \notin Q^\vee$ for $0 < j < l$ and $l \sigma^\vee \in Q^\vee$.

Let $N = \text{ord}(Z(\tilde{G}))$. Then we come to the diagram

In what follows we consider as $G$ the group $G_1$. If $l = 1$ or $l = \text{Ord}(Z(\tilde{G})) G$ coincides with $\tilde{G}$ or with $G^{ad}$. Let $E_G$ is a principle $G$-bundle over an elliptic curve $\Sigma_r = \mathbb{C} / (\mathbb{Z} + r \mathbb{Z})$. For a $G$-module $V$ we define a holomorphic $G$-bundle $E = E_G \times_G V$ (or simply $E_G$) over $\Sigma_r$. The bundle $E_G$ has the space of sections $\Gamma(E_G) = \{ s \}$, where $s$ takes values in $V$. The bundle $E_G$ is defined by transition matrices of its sections around the fundamental cycles. Then sections of $E_G(V)$ assume the quasi-periodicities:

$$s(z + 1) = Q(z)s(z), \quad s(z + \tau) = \Lambda(z)s(z),$$

where $Q(z), \Lambda(z) \in \text{End}(V)$. Then $Q(z), \Lambda(z)$ satisfy the following equation:

$$Q(z + \tau)\Lambda(z)Q(z)^{-1}\Lambda^{-1}(z + 1) = Id.$$  

4 The center $Z(\text{Spin}_{2n})$ is generated by the co-weights corresponding to the left and the right spinors.
It follows from \([93]\) that it is possible to choose the constant transition operators. Then we
come to the equation

\[
Q\Lambda Q^{-1}\Lambda^{-1} = Id.
\]  

(2.4)

Solutions of this equation are defined up to conjugations from the moduli space of \(E_G\)-bundles
over \(\Sigma_r\). We can modify (2.4) as

\[
Q\Lambda Q^{-1}\Lambda^{-1} = \xi Id,
\]

where \(\xi\) is a generator of the center \(\mathcal{Z}(\mathcal{G})\). In this case \((Q, \Lambda)\) are the clutching operators for
\(G^{ad}\)-bundles, but not for \(\mathcal{G}\)-bundles, and \(\xi\) plays the role of obstruction to lift the \(G^{ad}\)-bundle
to the \(\mathcal{G}\)-bundle.

Let \(p = \text{ord}(\mathcal{Z}(\mathcal{G}))/l\). Consider a bundle with the space of sections defined by the
quasi-periodicities:

\[
s(z + 1) = Qs(z), \quad s(z + \tau) = \Lambda_ps(z),
\]

such that

\[
Q\Lambda_p Q^{-1}\Lambda_p^{-1} = \xi^p Id,
\]

(2.6)

where \(\xi^p\) generates \(\mathbb{Z}_l\). It means that \(\xi^p\) is an obstruction to lift the \(G_l\)-bundle to the \(\mathcal{G}\)-bundle.

The obstructions can be formulated in terms of the cohomology of \(\Sigma_r\). Namely, the first
cohomology \(H^1(\Sigma_r, G(\mathcal{O}_{\Sigma_r}))\) of \(\Sigma_r\) with coefficients in analytic sheaves \(G(\mathcal{O}_{\Sigma_r})\) defines the
moduli space \(\mathcal{M}(G, \Sigma_r)\) of holomorphic \(G\)-bundles over \(\Sigma_r\). Using (2.1) we write three exact
sequences:

\[
1 \to \mathcal{Z}(\mathcal{G}) \to \mathcal{G}(\mathcal{O}_{\Sigma_r}) \to G^{ad}(\mathcal{O}_{\Sigma_r}) \to 1,
\]

\[
1 \to \mathcal{Z}(G_l) \to G_l(\mathcal{O}_{\Sigma_r}) \to G^{ad}_l(\mathcal{O}_{\Sigma_r}) \to 1,
\]

\[
1 \to \mathcal{Z}(G_l) \to G_l(\mathcal{O}_{\Sigma_r}) \to G^{ad}_l(\mathcal{O}_{\Sigma_r}) \to 1.
\]

Then we come to the long exact sequences:

\[
\to H^1(\Sigma_r, \mathcal{G}(\mathcal{O}_{\Sigma_r})) \to H^1(\Sigma_r, G^{ad}(\mathcal{O}_{\Sigma_r})) \to H^2(\Sigma_r, \mathcal{Z}(\mathcal{G})) \sim \mathcal{Z}(\mathcal{G}) \to 0,
\]

(2.7)

\[
\to H^1(\Sigma_r, \mathcal{G}(\mathcal{O}_{\Sigma_r})) \to H^1(\Sigma_r, G_l(\mathcal{O}_{\Sigma_r})) \to H^2(\Sigma_r, \mathcal{Z}(G_l)) \sim \mathcal{Z}(G_l) \to 0,
\]

(2.8)

\[
\to H^1(\Sigma_r, G_l(\mathcal{O}_{\Sigma_r})) \to H^1(\Sigma_r, G^{ad}_l(\mathcal{O}_{\Sigma_r})) \to H^2(\Sigma_r, \mathcal{Z}(G_l)) \sim \mathcal{Z}(G_l) \to 0.
\]

(2.9)

The elements from \(H^2\) are obstructions to lift bundles, namely

\[
\xi (E_{G^{ad}}) \in H^2(\Sigma_r, \mathcal{Z}(\mathcal{G})) \text{ – obstructions to lift } E_{G^{ad}} \text{-bundle to } E_{\mathcal{G}} \text{-bundle},
\]

\[
\xi (E_{G_l}) \in H^2(\Sigma_r, \mathcal{Z}(G_l)) \text{ – obstructions to lift } E_{G_l} \text{-bundle to } E_{\mathcal{G}} \text{-bundle},
\]

\[
\xi^p (E_{G^{ad}_l}) \in H^2(\Sigma_r, \mathcal{Z}(G_l)) \text{ – obstructions to lift } E_{G^{ad}_l} \text{-bundle to } E_{\mathcal{G}} \text{-bundle}.
\]

**Definition 2.1.** Images of \(H^1(\Sigma_r, G(\mathcal{O}_{\Sigma_r}))\) in \(H^2(\Sigma_r, \mathcal{Z})\) are called the characteristic classes
\(\xi(\mathcal{E}_G)\) of \(G\)-bundles.

It was proved in \([22]\) that for generic bundles the solution of equation (2.6) can be written as

\[
Q = e(\kappa), \quad \Lambda_p = e(u)\Lambda_0, \quad \Lambda_p^0 = Id,
\]

(2.10)

\[
\kappa = e(\rho/\hbar), \quad \rho = \frac{1}{2}\sum_{\alpha^\vee > 0} \alpha^\vee, \quad h - \text{Coxeter number},
\]

where \(\{\alpha^\vee\}\) are co-roots of \(\mathfrak{g}\). The element \(\Lambda_0\) is uniquely defined by the element from \(\mathcal{Z}(G)\)
which is an element of the Weyl group \(W(\mathfrak{h})\) (it acts as a symmetry of the extended Dynkin
diagram [89]). Moreover, \( u \in \text{Ker}(\Lambda_0 - Id) \) and \( u \) belongs to Cartan subalgebra \( \mathfrak{h}_0 \subset \mathfrak{h} \), where \( \mathfrak{h} \) is a Cartan subalgebra containing \( \mathcal{Q} \). \( u \) plays the role of a parameter in the moduli space \( \mathcal{M}_G \) of holomorphic \( G \)-bundles over \( \Sigma_r \). The subalgebra \( \mathfrak{h}_0 \) is a Cartan subalgebra of invariant subalgebra \( \mathfrak{g}_0 \subset \mathfrak{g} \) [22]. There exists a basis \( \Pi^\vee \) in \( \mathfrak{h}_0 \) such that \( \Pi \) is a system of simple roots for \( \mathfrak{g}_0 \). See [22] for the list of these subalgebras. If \( p = N \), we come to the trivial bundles. In this case \( \Lambda_0 = Id, \mathfrak{h}_0 = \mathfrak{h} \) and \( \mathfrak{g}_0 = \mathfrak{g} \).

Let \( \mathcal{Q}^u \) be the co-root lattice in \( \mathfrak{h}_0 \) generated by \( \Pi^\vee \) and \( \mathcal{P}^\vee \) is the co-weight lattice. Then a big cell in the moduli space \( \mathcal{M}_G \) of the \( G \)-bundles can be identified with the fundamental domains \( C^\vee \) of the affine Weyl group \( W \ltimes (\tau \Gamma^\vee \oplus \Gamma^\vee) \), where \( \Gamma \) is a sublattice \( \mathcal{Q}^u \subseteq \Gamma \subseteq \mathcal{P}^\vee \) in \( \mathfrak{h}_0 \):

\[
u \in C^\vee = \mathfrak{h}_0/(W \ltimes (\tau \Gamma^\vee \oplus \Gamma^\vee)). \tag{2.11}
\]

### 2.2. Elliptic \( R \)-matrices

A general form of the (modified) QDYB equation related to a simple complex Lie group \( G \) has the following form. Let \( h \) be a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} = \text{Lie} G \). We identify \( \mathfrak{h} \) with its dual space \( \mathfrak{h}^* \) by means of the Killing form. Consider finite-dimensional \( \tilde{G} \)-modulus \( V_j \), \( j = 1, 2, 3 \) and let \( V = \bigoplus_{\mu \in C} V[\mu] \) be the weight decomposition. Let \( z \in \mathbb{C} \) be the spectral parameter and \( u \in \mathfrak{h}_0 \). The quantum elliptic dynamical \( R \)-matrix \( R(u, z) \) is the map \( \mathfrak{h}_0 \times \mathbb{C} \rightarrow \text{End}(V_j \otimes V_k), \) \( (j, k = 1, 2, 3), j \neq k \), depending on the Planck constant \( \hbar \). \( R \) satisfies the following conditions.

- \( R \) has fixed quasi-periodicities with respect to the lattice \( \mathbb{Z} + r \mathbb{Z} \subset \mathbb{C} \). Let \( \mathcal{Q} \) and \( \Lambda \) be some fixed elements of \( G \) (transition functions) satisfying (2.6). Then

\[
R(u, z + 1) = \text{Ad}_{\mathcal{Q}} R(u, z), \quad R(u, z + \tau) = \text{Ad}_{\Lambda(u)} R(u, z). \tag{2.12}
\]

where the adjoint operators (2.10) act on the first factor \( \text{End}(V_j \otimes V_k) \). It means that \( R(u, z) \) is a section of a bundle \( \text{End}(V_j \otimes V_k) \) over the elliptic curve \( \Sigma_r = \mathbb{C}/(\mathbb{Z} + r \mathbb{Z}) \). The bundle has the characteristic class defined by \( \zeta^u \) (2.6). It is an obstruction to lift the \( \text{End} \)-bundle, considered as the \( G \)-bundle, to the \( \tilde{G} \)-bundle.\(^5\) The dynamical parameter \( u \) plays the role of the tangent vector to the moduli space of the bundle.

- \( R \) satisfies the QDYB equation in \( \text{Aut}(V_1 \otimes V_2 \otimes V_3) \)

\[
R^{12}(u - he^{(1)}, z_{12})R^{13}(u + he^{(2)}, z_{13})R^{23}(u - he^{(1)}, z_{23}) = R^{23}(u + he^{(1)}, z_{23})
\times R^{12}(u + he^{(2)}, z_{12})R^{13}(u + he^{(3)}, z_{13}) (z_{ik} = z_i - z_k), \tag{2.13}
\]

where \( e^{(j)} = (e_1, \ldots, e_l) \) is a basis in \( \mathfrak{h}_0 \) (for example, \( [\mathfrak{h}_0, \alpha] \subset \Pi \) (2.17)), and the superscript \( (j) \) means the action of \( \mathfrak{h}_0 \) on \( V_j \). The shift of the dynamical parameter \( u - he^{(j)} \) means, for example, that \( R^{12}_{V_j V_k}(u - he^{(j)}) \) acts on the tensor product \( v_1 \otimes v_2 \otimes v_3 \) as \( R^{12}_{V_j V_k}(u - (\mu, e^{(3)})h) \) for \( v_3 \in V_3[\mu] \).

- The unitarity condition

\[
R^{12}(u, z_{12})R^{21}(u, z_{21}) \sim \text{Id}_{V_1 \otimes V_j}. \tag{2.14}
\]

- The weight zero condition

\[
[X^1 + X^2, R^{12}(u, z_{12})] = 0, \quad \forall X \in \mathfrak{h}_0. \tag{2.15}
\]

\(^5\) There is an apparent inconsistency of this condition since some \( V \) are only \( \tilde{G} \) modules, but not \( G \) modules. But to define the \( R \)-matrix we consider the modules \( \text{End} V \) and they are \( G \) modules and even \( G^w \) modules for any \( V \). In particular, the Baxter–Belavin–Drinfeld–Sklyanin \( R \)-matrix considered below corresponds to the non-trivial \( \text{PSL}(N, \mathbb{C}) = \text{End}(V_0) \)-bundle, and it is defined for the \( N \)-vector representation \( V_N \).
The quasi-classical limit $\hbar \to 0$

$$R(u, z) = \frac{1}{\hbar} \text{Id} \otimes \text{Id} + r(u, z) + O(\hbar),$$ (2.16)

where $r(u, z)$ is defined below the CDRM. In this sense $R(u, z)$ is a quantization of the CDRM $r(u, z)$. In particular cases we obtain in this way the classical non-dynamical elliptic Belavin–Drinfeld $r$-matrix [9], or classical dynamical elliptic $r$-matrix [87, 88]. The latter types of $r$-matrices were classified in [6, 27].

These conditions do not define the $R$-matrix uniquely. There are additional transformations corresponding to shifts along the dynamical parameter. We will not discuss this issue here.

Let us focus now on the classical $r$-matrix in (2.16). We define the CDRM following [22]. In the general case they have the following form. To define the $r$-matrices for arbitrary characteristic classes let us define the special basis (the general sine basis (GS basis)) in the corresponding to shifts along the dynamical parameter. We will not discuss this issue here.

6 For $A_n$ and $E_6$ root systems it is convenient to choose canonical bases in $\mathfrak{h} \oplus \mathbb{C}$. 
where functions \( \varphi_{\alpha}(z) \) are defined in (A.34) and \( \{\tilde{h}_\alpha^0\} \) is the dual basis to \( \{h_\alpha\} \) in the invariant Cartan subalgebra \( h_0 \).

### 3. Quantum R-matrices related to SL\((N, \mathbb{C})\)

We apply the general construction of the quantum R-matrix to the case \( G = \text{SL}(N, \mathbb{C}) \) and \( V \) is the standard vector representation. We pass from the GS basis to the tensor basis (3.13) and write in this basis the transition matrices \( Q \) and \( \Lambda \).

#### 3.1. The moduli space of SL\((N, \mathbb{C})\)-bundles over elliptic curves

The dynamical parameter \( u \) belongs to the moduli space of vector bundles over elliptic curves. We describe it here. We identify \( h^* \) and \( h \subset \text{sl}(N, \mathbb{C}) \) by means of the standard metric on \( \mathbb{C}^N \.

The roots and co-roots for \( \text{sl}(N, \mathbb{C}) \) coincide and, therefore, the co-root lattice \( Q^\vee \) coincides with the root lattice \( Q \). Let \( \{e_j\} \) be the standard basis in \( \mathbb{C}^N \). Then

\[
Q = \left\{ \sum m_j e_j | m_j \in \mathbb{Z}, \sum m_j = 0 \right\},
\]

generated by the simple roots \( \Pi = \{\alpha_k\} = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{N-1} = e_{N-1} - e_N\} \).

The fundamental weights \( \varpi_k, (k = 1, \ldots, N-1) \), dual to the basis of simple roots \( \Pi^\vee \sim \Pi \), are given by

\[
\varpi_j = e_1 + \cdots + e_j - \frac{j}{N} \sum_{l=1}^N e_l, \quad j = 1, \ldots, N-1 \quad \text{(3.2)}
\]

In a similar way we identify the fundamental weights and the fundamental co-weights. They generate the weight (co-weight) lattice

\[
P \subset h, P = \left\{ \sum_{i} n_i \varpi_i | n_i \in \mathbb{Z} \right\}, \quad \text{or} \quad P = \left\{ \sum_{j=1}^N m_j e_j, m_j \in \mathbb{Z}, m_j - m_k \in \mathbb{Z} \right\} \quad \text{(3.3)}
\]

The quotient-group \( P/Q \) is isomorphic to the center \( Z(\text{SL}(N, \mathbb{C})) \sim \mu_N \). It is generated by \( \zeta = \exp 2\pi i \varpi_1 \).

For the trivial bundles corresponding to Felder’s R-matrix we have few moduli spaces \( C_l \) (2.11), corresponding to a choice of the sublattice \( P_l \). If \( N \) is a prime number, then we have two options only. Let

\[
C^+ = \{ u \in \mathbb{C}^N | \Re u_1 \geq \Re u_2 \geq \ldots \geq \Re u_N \}
\]

be a positive Weyl chamber. Then similar to the lattice \( P_l \) in (2.11) we may take \( Q \) (\( l = 1 \))(3.1) or \( P \) (\( l = N \))(3.3). Then we come to the two types of alcoves

\[
C^+ = \left\{ u \in C^+ | u_j \sim u_j + n_j + m_j, n_j, m_j \in \mathbb{Z}, \sum n_j = \sum m_j = 0 \right\}, \quad \text{(3.4)}
\]

\[
C^N = \left\{ u \in C^N | n_j, m_j \in \mathbb{Z}, n_j - n_k \in \mathbb{Z}, m_j - m_k \in \mathbb{Z} \right\} \quad \text{(3.5)}
\]

\( C^+ \) is the moduli space of \( \text{SL}(N, \mathbb{C}) \)-bundles, while \( C^N \) is the moduli space of trivial \( \text{PSL}(N, \mathbb{C}) \)-bundles (i.e \( \text{PSL}(N, \mathbb{C}) \)-bundles that can be lifted to \( \text{SL}(N, \mathbb{C}) \)-bundles).
If $N$ is not prime, then there are other sublattices of the weight lattice. For example, if $N = pl$ there are

$$C^l = \left\{ u \in C^l \mid n_j, m_j \in \frac{1}{l} \mathbb{Z}, n_j - n_k \in \mathbb{Z}, m_j - m_k \in \mathbb{Z} \right\},$$

$$C^p = \left\{ u \in C^l \mid n_j, m_j \in \frac{1}{p} \mathbb{Z}, n_j - n_k \in \mathbb{Z}, m_j - m_k \in \mathbb{Z} \right\},$$

in addition to $C^1$ and $C^N$. They are the moduli space of trivial $G_p = \text{SL}(N, \mathbb{C})/\mu_p^l$- and $\text{GL}_l = \text{SL}(N, \mathbb{C})/\mu_l^l$-bundles. In general, the number of different moduli spaces corresponds to the number of prime factors of $N$.

Consider the non-trivial bundles with transition matrices satisfying (3.8), where $\zeta$ can be represented as

$$\zeta = e(\sigma_p),$$

$$\sigma_p = \left( \frac{N-p}{N}, \ldots, \frac{N-p}{N}, -\frac{p}{N}, \ldots, -\frac{p}{N} \right) = \left( \frac{l-1}{l}, \ldots, \frac{l-1}{l}, -\frac{1}{l}, \ldots, -\frac{1}{l} \right), \quad (l\sigma_p \in Q).$$

It is a generator of the group $\mu_l^l$, and in this way it is the obstruction to lift the $G_l = \text{SL}(N, \mathbb{C})/\mu_l^l$-bundle to the $\text{SL}(N, \mathbb{C})$-bundle (see (2.8)). We consider the root $Q_l$ and the weight $P_l$ lattices in $\tilde{h}_0$ (3.9). In the canonical basis $e_j = E_{jj}$ ($j = 1, \ldots, p$), they have the form (3.1) and (3.3). In particular,

$$P_l = \left\{ \gamma = \sum_{j=1}^{p} n_j e_j, n_j \in \frac{1}{p} \mathbb{Z}, \sum_{j=1}^{p} n_j = 0, n_j - n_k \in \mathbb{Z} \right\}.$$

It is an invariant sublattice of $P$.

If $p$ is a prime number, then, similar to (3.4) and (3.5) we have two types of moduli spaces

$$C^{l,1}: u_j \sim u_j + \tau m_j + n_j, \quad n_j, m_j \in \mathbb{Z}, \sum_{j=1}^{p} n_j = \sum_{j=1}^{p} m_j = 0, \quad (3.6)$$

and for

$$C^{l,p}: u_j \sim u_j + \tau m_j + n_j, \quad n_j, m_j \in \frac{1}{p} \mathbb{Z},$$

$$\sum_{j=1}^{p} n_j = \sum_{j=1}^{p} m_j = 0, \quad n_j - n_k \in \mathbb{Z}, m_j - m_k \in \mathbb{Z}. \quad (3.7)$$

If $p$ is non-prime we have additional types of moduli spaces as given above for the trivial bundles.

3.2. Description of the adjoint bundles and the model of interacting tops

In [21], the classical integrable models corresponding to $\text{SL}(N, \mathbb{C})$-bundles with non-trivial characteristic classes were studied. Let us recall the results. We consider the $\text{SL}(N, \mathbb{C})$-bundles with

$$N = lp, \quad l, p \in \mathbb{Z}$$

defined by its multiplicators (2.2) with the center element

$$\zeta = \exp \left( \frac{2\pi i}{p} \right) = \exp \left( \frac{2\pi i}{l} \right) \in \mathbb{Z}/N\mathbb{Z}.$$
from the condition (2.6). The multiplicators can be written explicitly in terms of \( SL(N, \mathbb{C}) \)-valued generators of the finite Heisenberg group (A.26) and (A.27):

\[
\mathcal{Q} \Lambda^p \mathcal{Q}^{-1} \Lambda^{-p} = \exp \left( \frac{2\pi i}{N} \right) \text{Id}, \quad \mathcal{Q}, \Lambda \in SL(N, \mathbb{C}).
\] (3.8)

The dimension of the moduli space of these bundles equals \( \text{g.c.d.}(N, p) = p \) [90–92]. Indeed, it is easy to see that the following Cartan element of the Lie algebra \( u \in \mathfrak{h} \subset \mathfrak{sl}(N, \mathbb{C}) \) commutes with both \( \mathcal{Q} \) and \( \Lambda^p \):

\[
u = \text{diag}(u_1, \ldots, u_p, u_1, \ldots, u_p, \ldots, u_1, \ldots, u_p) = \bigoplus_{j=1}^p u_{p \times p}, \quad \left( \sum_{j=1}^p u_j = 0 \right).
\] (3.9)

It was shown in [21] that there exists such a number matrix \( S \) (combination of permutations) that

\[
\begin{align*}
S \nu S^{-1} &= \bigoplus_{j=1}^p u_j \text{Id}_{p \times p}, \\
\mathcal{Q} S \Lambda S^{-1} &= \bigoplus_{j=1}^p \mathcal{Q} \Lambda_{p \times p}, \\
S \Lambda \nu S^{-1} &= \bigoplus_{j=1}^p \nu \Lambda_{p \times p}.
\end{align*}
\] (3.10)

The latter means that any section \( L(z) \in \Gamma(\text{End}(E_{SL(N, \mathbb{C})}(V))) \) has the following quasi-periodicity properties:

\[
\begin{align*}
L_{ij}(z + 1) &= e \left( \frac{I - I}{N} \right) Q_{p \times p} L_{ij}(z) Q_{p \times p}^{-1}, \\
L_{ij}(z + \tau) &= e(-u_I) \Lambda_{p \times p} L_{ij}(z) \Lambda_{p \times p}^{-1} e(u_I).
\end{align*}
\] (3.11)

The factor \( e \left( \frac{I - I}{N} \right) \) can be removed by

\[
L_{ij}(z) \rightarrow L_{ij}(z) e \left( -\frac{I - I}{N} \right),
\]

\[
u \rightarrow u_I - I \frac{\tau}{N}.
\]

Finally, the boundary conditions are of the form

\[
\begin{align*}
L_{ij}(z + 1) &= Q_{p \times p} L_{ij}(z) Q_{p \times p}^{-1}, \\
L_{ij}(z + \tau) &= e(-u_I) \Lambda_{p \times p} L_{ij}(z) \Lambda_{p \times p}^{-1} e(u_I).
\end{align*}
\] (3.12)

Therefore, it is natural to use the following basis:

\[
E_{ij}^n = E_{ij} \otimes T_n \in \mathfrak{gl}(N, \mathbb{C}), \quad E_{ij} \in \mathfrak{gl}(p, \mathbb{C}), \quad T_n \in \mathfrak{gl}(l, \mathbb{C})
\] (3.13)

in the Lie algebra \( \mathfrak{gl}(N, \mathbb{C}) \), where \( E_{ij} \) is a standard matrix basis in \( \mathfrak{gl}(p, \mathbb{C}) \) (generated by the fundamental representation of \( \mathfrak{gl}(p, \mathbb{C}) \)) and \( T_n \) is the basis of \( \mathfrak{gl}(l, \mathbb{C}) \) defined in (A.29)–(A.32). The basis (3.13) will be used in section 3.4 to construct the quantum \( R \)-matrix.

The described types of bundles were used in [21] in order to construct the ‘models of interacting tops’.
3.3. Baxter–Belavin–Drinfeld–Sklyanin and Felder quantum R-matrices

As we said, there exist two extreme cases in the description of the R-matrices. The first case is the vertex R-matrices [10, 7]. These R-matrices correspond to the SL(N, \mathbb{C})-bundles with the characteristic classes \( \zeta = \exp \frac{2\pi i}{N} \), where \( k \) and \( N \) are co-prime. In this case \( \mathfrak{h}_0 = \{0\} \). These are the so-called non-dynamical R-matrices. Another case corresponds to the SL(N, \mathbb{C})-bundles with the trivial characteristic classes \( \zeta = 1 \) [1, 25]. We first consider the R-matrices for these two cases. The R-matrices satisfy the non-dynamical or dynamical Yang–Baxter equations correspondingly. The latter equations are

\[
R^{12}(z - w)R^{13}(z)R^{23}(w) = R^{23}(w)R^{13}(z)R^{12}(z - w),
\]

\[
R^{12}(u, z - w)R^{13}(u - h^2, z)R^{23}(u, w) = R^{23}(u - h^e, w)R^{13}(u, z)R^{12}(u - h^e, z - w).
\]

3.3.1. Non-dynamical case: Baxter–Belavin–Drinfeld–Sklyanin vertex R-matrix. The elliptic non-dynamical R-matrix is related to SL(N, \mathbb{C}). It is defined in the basis (A.29) as follows:

\[
R^{12}(z) = \sum_{a \in \Gamma_N} \varphi_a(z, \omega_a + \hbar)T_a \otimes T_{-a}
\]

**Proposition 3.1.** The Baxter–Belavin–Drinfeld–Sklyanin R-matrix (3.16) satisfies the quantum Yang–Baxter equation (3.14) [10, 7].

**Proof.** Consider a basic component of the tensor product \( T_a \otimes T_{-a} \otimes T_{b} \):

\[
\sum_{c \in \Gamma_N} \kappa_{0,0,k_a,k_{2a+2b},c}\varphi_{a-c}(\omega_a - \omega_c + \hbar, z - w)\varphi_c(\omega_c + \hbar, z)\varphi_{-b-c}(-\omega_b - \omega_c + \hbar, w)
\]

Making the shift \( c \rightarrow -c - b \) in the lhs, we have the following expression for (lhs)—(rhs):

\[
\sum_{c \in \Gamma_N} \kappa_{0,0,k_{2a},\kappa_c,2a+2b}\varphi_{a+b+c}(\omega_a + \omega_b + \omega_c + \hbar, z - w)\varphi_{-b-c}(-\omega_b - \omega_c + \hbar, z)\varphi_c(\omega_c + \hbar, w)
\]

\[
- \varphi_{a-c}(\omega_a - \omega_c + \hbar, z - w)\varphi_c(\omega_c + \hbar, z)\varphi_{-b-c}(-\omega_b - \omega_c + \hbar, w)).
\]

Using (A.22) in the case \( a + b \neq 0 \) mod \( \Gamma_N \) we obtain

\[
\varphi_a(\omega_a + 2\hbar, z)\varphi_{-a-b}(-\omega_a - \omega_b, w)\sum_{c \in \Gamma_N} \kappa_{0,0,k_{2a},\kappa_c,2a+2b}(E_1(\omega_a + \omega_b + \omega_c + \hbar)
\]

\[
- E_1(\omega_a - \omega_c + \hbar) + E_1(-\omega_b - \omega_c + \hbar) - E_1(\omega_c + \hbar)) = 0.
\]

Indeed, \( \kappa_{0,0,k_{2a},\kappa_c,2a+2b} \) is invariant under the substitution \( c \rightarrow c - a - b \). Then making this shift in the first \((E_1(\omega_a + \omega_b + \omega_c + \hbar))\) and the third \((E_1(-\omega_b - \omega_c + \hbar))\) terms one can see that the whole sum vanishes.

In the case \( a + b = 0 \) mod \( \Gamma_N \) it follows from (A.24) for (lhs)—(rhs) that

\[
\varphi_a(\omega_a + 2\hbar, z) \left( \frac{N}{2\pi i} \right)^3 \sum_{c \in \Gamma_N} (E_2(\omega_c + \hbar) - E_2(\omega_a - \omega_c + \hbar)) = 0,
\]

where the normalization factor \( \frac{N}{2\pi i} = \kappa(0, 0) \) appears from (A.30). \( \square \)
3.3.2. Dynamical case: Felder R-matrix. Felder’s R-matrix is defined as follows [1]:

\[ R_{12}(u, z) = \sum_{i,j} r_{ij}(u, z) E_{ij} \otimes E_{ji} + \sum_{\mu \neq v} \rho_{\mu\nu} E_{\mu\nu} \otimes E_{\nu\mu}, \]  

(3.18)

where

\[ r_{ij}(u, z) = \phi(u_{ij} + \delta_{ij} h, z), \quad \rho_{ij} = \phi(-u_{ij}, h), \quad u_{ij} = u_i - u_j \]

and

\[ R_{13}(z, u - \hbar h(2)_1) = \sum_{m,n,s} t_{mn}(z) E_{mn} \otimes \tilde{E}_{zs} \otimes E_{nm} + \sum_{\gamma \neq \xi} \tilde{\rho}_{\gamma\xi} E_{\gamma\gamma} \otimes \tilde{E}_{\xi\xi} \otimes E_{\xi\xi}. \]

We use ‘check’ to indicate the possible shift of the argument of \( R_{13} \) by \(-\hbar h(2)_1\):

\[ \sum_{m,n,s} t_{mn}(z) E_{mn} \otimes \tilde{E}_{zs} \otimes E_{nm} = \sum_{m,n,s} \phi(u_{mn} + \delta_{mn} h - h\delta_{mn} + h\delta_{ns}, z) E_{mn} \otimes \tilde{E}_{zs} \otimes E_{nm} \]

\[ \times \sum_{\gamma \neq \xi, s} \tilde{\rho}_{\gamma\xi} E_{\gamma\gamma} \otimes \tilde{E}_{\xi\xi} \otimes E_{\xi\xi} = \sum_{\gamma \neq \xi, s} \phi(-u_{\gamma\xi} + \delta_{\gamma\xi} h - \delta_{\xi\xi} h, h) E_{\gamma\gamma} \otimes \tilde{E}_{\xi\xi} \otimes E_{\xi\xi}. \]  

(3.19)

Proposition 3.2. Felder’s R-matrix (3.18) satisfies the QDYB equation (3.15).

We omit here the proof of this proposition since it is contained as a particular case of more general structure which will be discussed in section 3.4.

3.3.3. Classical limits. Let us also take the classical limits of the quantum R-matrices (3.16) and (3.18):

\[ r^{BD}_{12}(z, w) = \lim_{h \to 0} \left( R^{BD}_{12}(z, w) - \frac{1}{\hbar} \otimes 1 \right) \]

\[ = E_1(z - w) 1 \otimes 1 + \sum_{\alpha \in \Gamma} \phi_{\alpha}(z - w, \omega_\alpha) T_\alpha \otimes T_{-\alpha}. \]

(3.20)

Note that the summation is taken over \( \Gamma_N \) (A.28). This r-matrix satisfies the classical Yang–Baxter equation:

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \]

(3.21)

For the dynamical r-matrix we have

\[ r^{\ast}_{12}(z, w) = \lim_{h \to 0} \left( R^{\ast}_{12}(z, w) - \frac{1}{\hbar} 1 \otimes 1 \right) \]

\[ = E_1(z - w) \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i,j} \phi(z - w, u_{ij}) E_{ij} \otimes E_{ji} - \sum_{i,j} E_{ij}(u_{ij}) E_{ii} \otimes E_{jj}. \]  

(3.22)

The modified classical Yang–Baxter equation

\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] + D^{(1)}_b r_{23} - D^{(2)}_b r_{13} + D^{(3)}_b r_{12} = 0. \]

(3.23)

In the standard basis the operator \( D^{(1)}_b \) is written as follows:

\[ D^{(1)}_b = \sum_{i=1}^{N} E_{ii} \otimes 1 \otimes 1 \partial_{u_i}. \]

It should be mentioned that the r-matrix (3.22) without the last sum \( \sum_{\gamma \neq \xi} \) also satisfies (3.23). The reason is that it can be removed by the dynamical twist (see e.g. [44] or [24]).

Note that both r-matrices \( r^{BD} \) (3.20) and \( r^{\ast} \) (3.22) are particular cases of the general form (2.20).
3.4. General quantum R-matrices for SL(N, C)-bundles

Let us consider the basis (3.13) in GL(N, C) ≃ GL(p, C) × GL(l, C), where N = lp, l, p ∈ ℤ:

\[ E_{ij}^a = E_{ij} \otimes T_a, \quad E_{ij} \in \text{gl}(p, \mathbb{C}), \quad T_a \in \text{gl}(l, \mathbb{C}), \]

(3.24)

where \( E_{ij} \) is a standard matrix basis in the fundamental representation of GL(p, C) and \( T_a \) is the basis of GL(l, C) defined in (A.29). From (A.30) it follows that:

\[ E_{ij}^a E_{kl}^b = \kappa_{a,b} \delta_{ij} E_{jl}^{a+b}. \]

(3.25)

Now let us introduce the following R-matrix:

\[ R_{12}(u, z) = \sum_{i,j}^{p} \sum_{a \in V_1} r_{ij}^a(u, z) E_{ij}^a \otimes E_{ji}^{-a} + \sum_{\mu, \nu}^{p} \rho_{\mu \nu}^0 E_{\mu \nu}^0 \otimes E_{\nu \mu}^0, \]

(3.26)

where

\[ r_{ij}^a(u, z) \equiv r_{ij}^a(z) = \phi_{-a}(-u_{ij} - \delta_{ij}h, z), \quad \rho_{ij}^0 = \phi(-u_{ij}, l h), \]

\[ u_{ij} = u_i - u_j, \quad \omega_a = \frac{a_1 + \tau a_2}{l}. \]

In particular, if \( p = 1 \) (\( l = N \)) then (3.26) coincides with (3.16). For \( p = N \) (\( l = 1 \)) we come to (3.18). In this way the elliptic R-matrix (3.26) unifies the dynamical and non-dynamical cases.

Consider dependence of \( R(u, z) \) on \( z \) and \( u \). The shifts of \( z \) yield (see (A.37))

\[ R(u, z + \tau) = \lambda_{p \times p} R(u, z) \lambda_{p \times p}^{-1}, \quad R(u, z + \tau) = e_1(u + h) \Lambda_{p \times p} R(u, z) \Lambda_{p \times p}^{-1} e_1(-u - h), \]

where \( \lambda_{p \times p} \) and \( \Lambda_{p \times p} \) act on the second factor of the basis (3.24), and the adjoint transformation by \( e_1(u + h) \) acts on the first factor as \( E_{ij} \rightarrow e_1(u_i - u_j + \delta_{ij} h) E_{ij} \). These conditions define the characteristic class of bundles.

Consider quasi-periodicities of \( R(u, z) \) (\( u \in \hat{G}_0 \)) with respect to shifts of the weight lattice \( P_1 \) or the root lattice \( Q_1 \) in \( \hat{G}_0 \) (see (3.6) and (3.7)). Let \( \gamma = (m_1, \ldots, m_p) \in P_1 \), or \( Q_1 \) and \( \gamma_p(y, z) = e(\gamma z) = \text{diag}(e(m_1 z), \ldots, e(m_p z)) \). It follows from (3.26) and (A.38) that

\[ R(u + \gamma, z) = R(u, z), \quad R(u + \gamma, z) = (\gamma_q^{-1}(y, z) \otimes \text{Id}_l) R(u, z) (\gamma_p(y, z) \otimes \text{Id}_l). \]

It means that \( R(u, z) \) is a section of the trivial bundle over the moduli spaces (3.6) and (3.7).

Now we prove the main result of the paper:

**Theorem 3.1.** The R-matrix (3.26) satisfies the QDYB equation (3.15).

**Proof.** By analogy with the notation \( \gamma \) from (3.19) we use an acute accent \( \gamma \) for the indication of possible shift of the argument of \( R_{23} \) (in the rhs of (3.15)) and a tilde \( \tilde{\gamma} \) for \( R_{12} \) (in the rhs of (3.15)). Let us write down equation (3.15) explicitly:

\[ \text{lhs} = \sum_{ij} r_{ij}^{\tilde{\gamma} - e} (z - w) r_{mz}^e (z) r_{kl}^{-b} (w) E_{ij}^{a-e} E_{mn}^{c-e} \otimes E_{ji}^{b-e} E_{kl}^{b-e} \otimes E_{mn}^{c-e} E_{kl}^{e+c} \]

\[ + r_{ij}^{\tilde{\gamma} - e} (z - w) r_{mz}^e (z) r_{kl}^{-b} (w) E_{ij}^{a-e} E_{mn}^{c-e} \otimes E_{ji}^{b-e} E_{kl}^{b-e} \otimes E_{mn}^{c-e} E_{kl}^{e+c} \]

\[ + r_{ij}^{\tilde{\gamma} - e} (z - w) r_{mz}^e (z) r_{kl}^{-b} (w) E_{ij}^{a-e} E_{mn}^{c-e} \otimes E_{ji}^{b-e} E_{kl}^{b-e} \otimes E_{mn}^{c-e} E_{kl}^{e+c} \]

\[ + r_{ij}^{\tilde{\gamma} - e} (z - w) r_{mz}^e (z) r_{kl}^{-b} (w) E_{ij}^{a-e} E_{mn}^{c-e} \otimes E_{ji}^{b-e} E_{kl}^{b-e} \otimes E_{mn}^{c-e} E_{kl}^{e+c} \]

\[ + r_{ij}^{\tilde{\gamma} - e} (z - w) r_{mz}^e (z) r_{kl}^{-b} (w) E_{ij}^{a-e} E_{mn}^{c-e} \otimes E_{ji}^{b-e} E_{kl}^{b-e} \otimes E_{mn}^{c-e} E_{kl}^{e+c} \]
\[ + \rho^0_{ij}(z-w) \hat{r}^{a,c}_{ij}(z-w) E^a_{\mu\nu} E^c_{\alpha\beta} \\]
The notation $\delta_{ui}$ here (for example, in the first line of (3.30)) means that the corresponding $\tilde{r}_{ij}$ has the shift of the argument $u_{kj}$ by $-h$ if $k = m$ and by $+h$ when $j = m$.

Making the change of the summation variable $c \rightarrow -c - b$ in the lhs (3.29) we obtain the same factor $\kappa_{0,0,k,b,K_2 + 2b,c} \rightarrow \kappa_{0,0,k,b,K_2 + 2b,c}$ for both sides due to (A.31):

**lhs** = \[ \sum_{\kappa_{0,0,k,b,K_2 + 2b,c}} \delta_{ui} r_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) E_{nm} \otimes E_{kn} \otimes E_{kn}^b + \delta_{ui,c} \kappa_{0,0,k,b,K_2 + 2b,c} \delta_{ki} r_{ij}^{a+b+c} (z - w) \tilde{r}_{ji}^{b-c} (z) E_{ij} \otimes E_{ij} \otimes E_{ij}^b + \delta_{ui,c} \kappa_{0,0,k,b,K_2 + 2b,c} \delta_{ki} r_{ij}^{a+b+c} (z - w) \tilde{r}_{ji}^{b-c} (z) E_{ij} \otimes E_{ij} \otimes E_{ij}^b \]

**rhs** = \[ \sum_{\kappa_{0,0,k,b,K_2 + 2b,c}} \delta_{ui} \delta_{mn} \delta_{kn} \tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) E_{mj} \otimes E_{kj} \otimes E_{kj}^b + \delta_{ui,c} \kappa_{0,0,k,b,K_2 + 2b,c} \delta_{mn} \delta_{kn} \tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) E_{mn} \otimes E_{mj} \otimes E_{kj}^b + \delta_{ui,c} \kappa_{0,0,k,b,K_2 + 2b,c} \delta_{mn} \delta_{kn} \tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) E_{mn} \otimes E_{mj} \otimes E_{kj}^b \]

A careful check shows that the equality (3.31) and (3.32) holds. The general idea of the verification is the following: if $a \neq -b$, the proof is similar to the one given for the non-dynamical case (3.16) and if $a = -b$, the equality is achieved by the *' $\rho$-term* in the $R$-matrix via the summation of (A.42) over $c$.

Let us demonstrate the verification for some concrete cases:

\[ E_{ij}^a \otimes E_{kj}^{a+b} \otimes E_{kn}^b. \]

\[ \text{lhs} = \sum_{c \in T_j} \kappa_{0,0,k,b,K_2 + 2b,c} \]

\[ (\tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) + \delta_{ij} \delta_{a+b+c} \kappa_{0,0,k,b,K_2 + 2b,c} \tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (z) \tilde{r}_{j}^{b-c} (w) ) \]

\[ = \text{rhs} = \sum_{c \in T_j} \kappa_{0,0,k,b,K_2 + 2b,c} \]

\[ (\tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (w) + \delta_{ij} \delta_{a+b+c} \kappa_{0,0,k,b,K_2 + 2b,c} \tilde{r}_{ij}^{a+b+c} (z - w) \tilde{r}_{j}^{b-c} (z) \tilde{r}_{j}^{b-c} (w) ). \]

9 Here and elsewhere we imply unequal lower indices while the upper may be dependent, e.g. $a = -b$ or $a = 0$ or $b = 0$. Note also that the summation will be taken only over $b$. 

14
Index $s$ in the lhs of (3.33) is responsible for the possible shift of argument in $\tilde{r}$. In this case we can see that it does not match the corresponding arguments. The same holds for indices $t$ and $q$. Thus there are no shifts in this case. Now combining the first terms from both sides we obtain:

$$\sum_{c \in \Gamma} \kappa_{0,0}^{\kappa,\kappa_2,2a+2b}(\varphi_{a+b+c}(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c}) \varphi_{-b-c}(z, u_{j} - \omega_{b} - \omega_{c}) \varphi_{w}(w, u_{d} + \omega_{c})$$

$$= \delta_{a-b-c}(z-w, u_{j} + \omega_{a} - \omega_{c}) \varphi_{a-b-c}(z, u_{j} + \omega_{a} - \omega_{b} - \omega_{c}))$$

$$\delta_{a-b-c}(\tilde{r}_{ij}^c(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c}) \tilde{r}_{ij}^{-c}(z-w, u_{j} + \omega_{a} + \omega_{b} + \omega_{c}))$$

(3.34)

Let us examine the lhs of (3.34). Due to (A.42) it simplifies to:

1. For $a \neq -b \mod 2N^2$:

$$\text{lhs}(3.34) = \varphi_{a}(z, \omega_{a} + u_{j}) \varphi_{a-b}(w, -\omega_{a} - \omega_{b}) \sum_{c \in \Gamma} \kappa_{0,0}^{\kappa,\kappa_2,2a+2b}(E_{1}(u_{d} + \omega_{a} + \omega_{b} + \omega_{c})$$

$$E_{1}(u_{j} + \omega_{a} - \omega_{b} - \omega_{c}) + E_{1}(u_{d} + \omega_{a} + \omega_{b} - \omega_{c})) = 0$$

exactly as in (3.17).

2. For $a = -b \mod 2N^2$:

$$\delta_{a-b-c}^{3}(\tilde{r}_{ij}^c(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c}))$$

$$\delta_{a-b-c}^{3}(\tilde{r}_{ij}^{-c}(z-w, u_{j} + \omega_{a} + \omega_{b} + \omega_{c}))$$

(3.35)

$$E_{i}^{a} \otimes E_{i}^{a-b} \otimes E_{j}^{b} :$$

$$\text{lhs} = \sum_{c \in \Gamma} \kappa_{0,0}^{\kappa,\kappa_2,2a+2b}(\varphi_{a+b+c}(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c} + \hbar)$$

$$\times \varphi_{-b-c}(z, u_{j} - \omega_{b} - \omega_{c} + \hbar) \varphi_{w}(w, u_{d} + \hbar)$$

$$= \delta_{a-b-c}^{3}(\tilde{r}_{ij}^c(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c} + \hbar) \tilde{r}_{ij}^{-c}(z-w, u_{j} - \omega_{b} - \omega_{c} + \hbar))$$

(3.37)

$$E_{i}^{a} \otimes E_{i}^{a-b} \otimes E_{j}^{b} :$$

$$\delta_{a-b-c}^{3}(\tilde{r}_{ij}^c(z-w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c} - (z, u_{j} - \omega_{b} - \omega_{c}) \varphi_{w}(w, u_{d} + \omega_{c} + \hbar))$$

$$\delta_{a-b-c}^{3}(\tilde{r}_{ij}^{-c}(z-w, u_{j} - \omega_{b} - \omega_{c}) \varphi_{w}(w, u_{d} + \omega_{a} + \omega_{b} + \omega_{c} + \hbar))$$

(3.38)

or

$$\varphi_{a+b}(z-w, u_{d} + \omega_{a} + \omega_{b}) \varphi_{-b}(z, u_{j} - \omega_{b}) + \varphi_{a}(z, u_{j} + \omega_{a}) \varphi_{-a-b}(w, u_{d} - \omega_{a} - \omega_{b})$$

$$\varphi_{a+b}(z-w, u_{d} + \omega_{a} + \omega_{b}) \varphi_{-b}(z, u_{j} - \omega_{b}) + \varphi_{a}(z, u_{j} + \omega_{a}) \varphi_{-a-b}(w, u_{d} - \omega_{a} - \omega_{b})$$

(3.39)
3.5. Trigonometric and rational limits

We can calculate the trigonometric limit $\Im \tau \to +\infty$ of the elliptic $R$-matrix (3.26) using (A.11) and (A.35)

$$R^{\text{trig}}(u, z) = \sum_{i,j} \sum_{\alpha \in A_i} r_{ij}^\alpha(u, z) E_{ij}^\alpha \otimes E_{ji}^{-\alpha} + \sum_{\alpha \neq \beta} \rho_{ij}^{\alpha \beta} E_{ij}^\alpha \otimes E_{ji}^{-\beta},$$

$$r_{ij}^\alpha(u, z) = \begin{cases} \cot \pi z + \cot \pi \left( u_{ij} + \frac{a_1}{N} + \delta_{ij} \hbar \right) & a_2 = 0, \\ e \left( \frac{a_2}{N} + 1 \right) \sin^{-1} \pi z & a_2 \neq 0. \end{cases}$$

$$\rho_{ij}^{\alpha \beta} = \frac{\sin \pi (l \hbar - l_{ij})}{\sin \pi (l \hbar) \sin \pi (l_{ij})}, \quad u_{ij} = u_i - u_j.$$  

Going to the rational limit we find

$$r_{ij}^\alpha(u, z) = \begin{cases} \frac{1}{\pi z} + \frac{1}{\pi \left( u_{ij} + \frac{a_1}{N} + \delta_{ij} \hbar \right)} & a_2 = 0, \\ \frac{1}{\pi z} & a_2 \neq 0. \end{cases}$$

$$\rho_{ij}^{\alpha \beta} = \frac{1}{\pi l \hbar} + \frac{1}{\pi l_{ij}}.$$  

For the elliptic $R$-matrices and related models there exists another trigonometric and rational limit [96–98]. This construction can be generalized to $\text{SL}(N, \mathbb{C})$ elliptic matrix. The same approach can be applied in our case as well but we will not develop this issue here.

4. Dynamical $R$-matrices and integrable systems

4.1. IRF models

Following [1] we construct the Boltzmann weights of the interaction-round-the-face models starting with the quantum $R$-matrices described above. Let $\mu \in \mathfrak{h}^*$ be a weight of $(\mu \in \mathfrak{h}^*)$ of the vector representation of $\text{sl}(N, \mathbb{C})$ in $V$. In other words we have $N$ weights

$$P_V = \left\{ \mu_j = \frac{1}{N}(-1, \ldots, -1, N - 1, -1, \ldots, -1) : (j = 1, \ldots, N) \right\},$$

where $N - 1$ stays on the $j$ place. Let $V[\mu_j]$ be the corresponding component of the space $V$, and $E[\mu_j] : V \to V[\mu_j]$ is a projection. In our case all $V[\mu_j]$ are one-dimensional.

Define the local states $a, b, c, d \in \mathfrak{h}^*$ of the IRF model $b-a = \mu_4, c-b = \mu_3, d-c = \mu_2, \quad d-a = \mu_1$, where all weights from $P_V$ (4.1), satisfy the equality $\mu_1 + \mu_2 = \mu_3 + \mu_4$.

Define the map $W(a, b, c, d) : V[\mu_4] \otimes V[\mu_2] \to V[\mu_4] \otimes V[\mu_2]$ by means of the $R$-matrix

$$W(a, b, c, d, z, u) = E[c - b] \otimes E[b - a] R(u + ha + hc, z) V[d - a] \otimes V[c - d].$$

In fact $W(a + \hat{u}, b + \hat{u}, c + \hat{u}, d + \hat{u}, z - 2\hat{u})$ is independent on $\hat{u}$. In this way we can define the Boltzmann weights of the IRF model as $W(a, b, c, d, z) = W(a, b, c, d, 0, z)$. The partition function of the IRF model takes the form

\[ \text{We will also use notation } \delta_{\alpha, \beta} \rho_{ij}^{\alpha \beta} = \rho_{ij}^{\beta \beta} \text{ in order to keep uniformity in formulae.} \]
\[ Z = \sum_{\text{lattice}} W(a_{ij}, a_{i,j+1}, a_{i-1,j+1}, a_{i-1,j-1}). \]

If \( R \) satisfies QDYB equation (3.15), then \( W \) obeys star-triangle relations [2]

\[
\sum_g W^{12}(b, c, d, g, z_{12})W^{13}(a, b, g, f, z_{13})W^{23}(f, g, d, c, z_{23})
= \sum_g W^{23}(a, b, c, g, z_{23})W^{13}(g, c, d, e, z_{13})W^{12}(a, g, e, f, z_{12})
\]
on \( V[f - a] \otimes V[e - f] \otimes V[d - e] \) providing the integrability of the model.

### 4.2. Elliptic quantum groups

Let \( R \) be a solution of QDYB. Define the quantum Lax operator \( L(\hat{v}, u, \hat{S}, z) \) as a map of \( \mathfrak{h}_0 \times \mathbb{C} \) to \( \text{Aut}(V) \). Here \( \hat{S} \) are generators of \( \text{SL}(N, \mathbb{C}) \) acting on the module \( V \), and \( [\hat{v}_j, u_k] = \hbar \delta_{jk} \).

The Lax operator satisfies the equation

\[
R^{12}(z - w, u + \hbar e^3)L^1(u - \hbar e^2, z)L^2(u + \hbar e^1, w)
= L^2(u + \hbar e^3, w)L^1(u - \hbar e^2)R^{12}(u - \hbar e^3 u, z - w),
\]

where \( L^1 = L \otimes \text{Id}, L^2 = \text{Id} \otimes L. \)

Assume that \( L(\hat{v}, u, z, \hat{S}) \) satisfies the quasi-periodicity conditions (2.12) and the weight zero condition (2.15). The relation (4.3) defines the quadratic algebra with respect to \( u \) and \( \hat{S} \).

This algebra is the Felder elliptic quantum group [2] for the trivial bundles and \( R(3.18) \). In the case \( R(3.16) \) we come to the Sklyanin–Feigin–Odesski algebras [18–20, 64].

In the classical limit, \( L = L(\hat{v}, u, \hat{S}, z) \) becomes the classical Lax operator for interacting tops described in section 3.1. The angular momentum variable \( S \) belongs to the co-adjoint \( \text{SL}(p, \mathbb{C}) \) orbit corresponding after quantization to the representation \( V \). In this way we obtain the quadratic Poisson algebras

\[
\{L^1(u, S), L^2(u, S)\} = [r(u), L^1(u, S), L^2(u, S)]
\]
defining the Poisson structure on the phase space of interacting tops.

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Appendix. Elliptic functions

A.1. Basic definitions and properties

Let \( q = \exp(2\pi i \tau) \), where \( \tau \) is the modular parameter of the elliptic curve \( E_\tau \). The basic element is the theta function:

\[
\vartheta(z|\tau) = q^{\frac{1}{2}} \exp \left( \frac{i\pi}{4} (e^{iz} - e^{-iz}) \right) \prod_{n=1}^{\infty} \left( 1 - q^n \right) \left( 1 - q^n e^{2i\pi z} \right) \left( 1 - q^n e^{-2i\pi z} \right).
\]  
(A.1)

The Eisenstein functions

\[
E_1(z|\tau) = \partial_z \log \vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z,
\]  
(A.2)

where

\[
\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau + n)^2} = \frac{24}{2\pi^4} \eta'(\tau) \eta(\tau),
\]  
(A.3)

and \( \eta(\tau) = q^{\frac{1}{24}} \prod_{m>0} (1 - q^m) \) is the Dedekind function;

\[
E_2(z|\tau) = -\partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1.
\]  
(A.4)

The higher Eisenstein functions

\[
E_j(z) = \frac{(-1)^j}{(j-1)!} \partial_z^{j-2} E_2(z) \quad (j > 2).
\]  
(A.5)

Relation to the Weierstrass functions

\[
\xi(z, \tau) = E_1(z, \tau) + 2\eta_1(\tau) z,
\]  
(A.6)

\[
\wp(z, \tau) = E_2(z, \tau) - 2\eta_1(\tau).
\]  
(A.7)

The next important function is

\[
\phi(u, z) = \frac{\vartheta(u + z) \vartheta(0)}{\vartheta(u) \vartheta(z)},
\]  
(A.8)

\[
\phi(u, z) = \phi(z, u), \quad \phi(-u, -z) = -\phi(u, z).
\]  
(A.9)

It has a pole at \( z = 0 \) and

\[
\phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2} (E_2^2(u) - \wp(u)) + \cdots.
\]  
(A.10)

Trigonometric limit for \( \phi(u, z) \) follows from (A.1),

\[
\lim_{\gamma \to \infty} \phi(u, z) = \frac{\sin \pi (z + u)}{\sin \pi z \sin \pi u}.
\]  
(A.11)

Let \( f(u, z) = \partial_\gamma \phi(u, z) \). Then

\[
f(u, z) = \phi(u, z) (E_1(u + z) - E_1(u)).
\]  
(A.12)

Heat equation

\[
\partial_t \phi(u, w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u, w) = 0.
\]  
(A.13)
Quasi-periodicity

\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{1/2}e^{-2\pi i\tau}\vartheta(z),
\]
(A.14)

\[
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i,
\]
(A.15)

\[
E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z),
\]
(A.16)

\[
\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i\tau}\phi(u, z);
\]
(A.17)

\[
f(u, z + 1) = f(u, z), \quad f(u, z + \tau) = e^{-2\pi i\tau}f(u, z) - 2\pi i\phi(u, z).
\]
(A.18)

The Fay three-section formula:

\[
\phi(u_1, z_1)\phi(u_2, z_2) = \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0.
\]
(A.19)

A particular case of this formula is the Calogero functional equation

\[
\phi(u, z)\partial_u\phi(v, z) - \phi(v, z)\partial_u\phi(u, z) = (E_2(u) - E_2(v))\phi(u + v, z),
\]
(A.20)

\[
\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u).
\]
(A.21)

Another important relation is

\[
\phi(v, z - w)\phi(u_1 - v, z)\phi(u_2 + v, w) - \phi(u_1 - u_2 - v, z - w)\phi(u_2 + v, z)\phi(u_1 - v, w)
\]
\[= \phi(u_1, z)\phi(u_2, w)f(u_1, u_2, v),
\]
(A.22)

where

\[
f(u_1, u_2, v) = E_1(v) - E_1(u_1 - u_2 - v) + E_1(u_1 - v) - E_1(u_2 + v).
\]
(A.23)

Taking limit \(u_2 \to 0\) in (A.22) we obtain

\[
\phi(v, z - w)\phi(u_1 - v, z)\phi(v, w) - \phi(u_1 - v, z - w)\phi(v, z)\phi(u_1 - v, w)
\]
\[= \phi(u_1, z)(E_2(v) - E_2(u_1 - v)),
\]
(A.24)

which is equivalent to (A.20) due to (A.12).

**Theta functions with characteristics.** For \(a, b \in \mathbb{Q}\) by definition:

\[
\theta\left[\begin{array}{c}
 a \\
 b
\end{array}\right]\left( z, \tau \right) = \sum_{j \in \mathbb{Z}} e\left( \frac{(j + a)^2 \tau}{2} + (j + a)(z + b) \right).
\]

In particular, the function \(\vartheta\) (A.1) is a theta function with characteristics:

\[
\vartheta(\frac{1}{2}, \tau) = \theta\left[\begin{array}{c}
 \frac{1}{2} \\
 \frac{1}{2}
\end{array}\right](x, \tau).
\]
(A.25)

**Properties:**

\[
\theta\left[\begin{array}{c}
 a \\
 b
\end{array}\right]\left( z + 1, \tau \right) = e(a)\theta\left[\begin{array}{c}
 a \\
 b
\end{array}\right]\left( z, \tau \right),
\]

\[
\theta\left[\begin{array}{c}
 a \\
 b
\end{array}\right]\left( z + a\tau, \tau \right) = e\left( -a^2\frac{\tau}{2} - a'(z + b) \right)\theta\left[\begin{array}{c}
 a' + a \\
 b
\end{array}\right]\left( z, \tau \right),
\]

\[
\theta\left[\begin{array}{c}
 a + j \\
 b
\end{array}\right]\left( z, \tau \right) = \theta\left[\begin{array}{c}
 a \\
 b
\end{array}\right]\left( z, \tau \right), \quad j \in \mathbb{Z}.
\]
A.2. Lie Group GL\((N, \mathbb{C})\) and elliptic functions

Introduce the notation (see [95])
\[
e_\alpha(z) = \exp \left( \frac{2\pi i}{N} z \right)
\]
and two matrices
\[
Q = \text{diag}(e_1(1), \ldots, e_N(m), \ldots, 1)
\]
\[
\Lambda = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
We have \(Q\Lambda = \exp\left(-\left(\frac{2\pi i}{N}\right)\Lambda\right)Q\). Let
\[
\Gamma_N = \mathbb{Z}^2_N = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \quad \tilde{\Gamma}_N = \mathbb{Z}^2_N / N \mathbb{Z}
\]
be the two-dimensional lattice of order \(N^2\) and \(N^2 - 1\) correspondingly. The matrices
\[
Q^a A^a, \quad a = (a_1, a_2) \in \mathbb{Z}^2_N\n\]
generate a basis in the group GL\((N, \mathbb{C})\), while \(Q^a A^a, \quad a = (a_1, a_2) \in \mathbb{Z}^2_N\)
generate a basis in the Lie algebra gl\((N, \mathbb{C})\). Consider the projective representation of \(\mathbb{Z}^2_N\) in GL\((N, \mathbb{C})\),
\[
a \to T_a = \frac{N}{2\pi i} e^{\frac{N}{2}\alpha_a (a_1, a_2)},
\]
\[
T_a T_b = \kappa_{a,b} T_{a+b}, \quad \kappa_{a,b} = \frac{N}{2\pi i} e_{\alpha_a} \left(-\frac{a \times b}{2}\right). \tag{A.30}
\]
Let us mention some simple properties of \(\kappa\):
\[
\kappa_{a,b} = \left(\frac{N}{2\pi i}\right)^2, \quad \kappa_{a,c} \kappa_{b,c} = \frac{N}{2\pi i} \kappa_{a+b,c}, \quad \kappa_{a,a} = \frac{N}{2\pi i}. \tag{A.31}
\]
Note that \(\kappa_{a,b}\) can be interpreted as a non-trivial two-co-cycle in \(H^2(\mathbb{Z}^2_N, \mathbb{Z}_{2N})\). It follows from (A.30) that
\[
[T_a, T_\beta] = C(\alpha, \beta) T_{a+\beta}, \tag{A.32}
\]
where \(C(\alpha, \beta) = \frac{N}{2} \sin \frac{\pi}{2} (\alpha \times \beta)\) are the structure constants of gl\((N, \mathbb{C})\).

Deformed elliptic functions
\[
\phi_a(\eta, z) = e_\alpha(\eta, z) \phi(\omega_a + \eta, z), \quad \omega_a = \frac{a_1 + a_2}{N}, \quad a \in \mathbb{Z}_N^2, \quad \eta \in \Sigma^\tau, \tag{A.33}
\]
and
\[
\phi^m(\eta, z) = e(\kappa, \beta) \phi \left(\eta + \frac{a_1}{N}, \frac{m}{t}, z\right). \tag{A.34}
\]

Trigonometric limit for \(\phi_a(\eta, z)\) (see (A.11))
\[
\lim_{t \to +\infty} \phi_a(\eta, z) = \begin{cases} 
\cot \pi z + \cot \pi \left(\eta + \frac{a_1}{N}\right) \quad & a_2 = 0, \\
\exp \left(\frac{a_2}{N} + \frac{1}{2}\right) z \sin^{-1} \pi z \quad & a_2 \neq 0.
\end{cases} \tag{A.35}
\]

It follows from (A.17) that \(\phi_a(\eta, z)\) is well defined on \(\mathbb{Z}_N^2\):
\[
\phi_{a+c}(\eta, z) = \phi_a(\eta, z), \quad \text{for } c_{1,2} \in \mathbb{Z} \text{ mod } N \tag{A.36}
\]
\[ \varphi_{\eta, z} = \varphi_{\eta}(z, z + 1) = e_{N}(m)\varphi_{a}(\eta, z), \quad \varphi_{\eta}(\eta, z + 1) = e_{N}(N\eta)\varphi_{a}(\eta, z); \quad (A.37) \]

\[ \varphi_{\varphi_{a}(\eta, z) + 1} = e_{N}(m)\varphi_{a}(\eta, z), \quad \varphi_{\varphi_{a}(\eta, z + 1)} = e_{N}(N\eta)\varphi_{a}(\eta, z). \quad (A.38) \]

For \( \varphi_{a}^{m}(u, z) \) (A.34) we have

\[ \varphi_{a}^{m}(u, z + 1) = e((\nu, \beta))\varphi_{a}^{m}(u, z), \quad \varphi_{a}^{m}(u, z + \tau) = e - (u, \beta) - \frac{m}{\tau}\varphi_{a}^{m}(u, z). \quad (A.39) \]

The following formulae can be proved directly by checking the structure of poles and quasi-periodic properties:

\[ \sum_{a \in \mathbb{Z}_{N}^{(2)}} E_{z}(\varphi, \varphi_{b}) = N^{2}E_{z}(N\varphi). \quad (A.40) \]

By analogy with (A.19) and (A.22)-(A.24) we have

\[ \varphi_{a+b}(z - w, u + \omega_{a} + \omega_{b})\varphi_{a-b}(z, v - \omega_{b}) + \varphi_{a}(z, u + v + \omega_{a})\varphi_{a-b}(w, -u - \omega_{a} - \omega_{b}) \]

\[ = \varphi_{a}(z - w, u + v + \omega_{a})\varphi_{a-b}(w, v - \omega_{b}) \quad (A.41) \]

\[ \varphi_{a+b+c}(z - w, u + \omega_{a} + \omega_{b} + \omega_{c})\varphi_{a-c}(z, v - \omega_{b} - \omega_{c})\varphi_{c}(w, u + \omega_{c}) \]

\[ - \varphi_{a-c}(z - w, v + \omega_{a} - \omega_{c})\varphi_{c}(z, u + \omega_{c})\varphi_{b-c}(w, v - \omega_{b} - \omega_{c}) \quad (A.42) \]

\[ = \begin{cases} \varphi_{a+b+c}(z - w, u + v)(\varphi_{a}(z, u + \omega_{a} + \omega_{b} + \omega_{c}) - E_{z}(u + \omega_{a} + \omega_{b} + \omega_{c}) - E_{z}(v + \omega_{a} + \omega_{b} + \omega_{c})) \\ + E_{z}(u + \omega_{a} + \omega_{b} + \omega_{c}) - E_{z}(u + \omega_{a} + \omega_{c})), \text{ if } a + b \neq 0 \mod \mathbb{Z}_{N}^{(2)}, \end{cases} \]

\[ \begin{cases} \varphi_{a}(z, u + v)(E_{z}(u + \omega_{a}) - E_{z}(v + \omega_{a} + \omega_{b})), \text{ if } a + b = 0 \mod \mathbb{Z}_{N}^{(2)}. \end{cases} \]

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