ABSORBING ANGLES, STEINER MINIMAL TREES, AND ANTIPODALITY

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Abstract. We give a new proof that a star \{op_i : i = 1, \ldots, k\} in a normed plane is a Steiner minimal tree of its vertices \{o, p_1, \ldots, p_k\} if and only if all distances between the normalizations \frac{1}{\|p_i\}}{p_i} equal 2. CL-spaces include the mixed \ell_1 and \ell_\infty sum of finitely many copies of \mathbb{R}.

Keywords: Steiner minimal tree, absorbing angle, antipodality, face antipodality, Minkowski geometry.

1. Introduction

1.1. Minkowski geometry. Let \mathcal{M}^d denote a d-dimensional normed space (or Minkowski space) with origin o, i.e., \mathbb{R}^d equipped with a norm \|\cdot\|. We call an \mathcal{M}^2 a Minkowski plane. Denote the unit ball by \mathcal{B} := \{x \in \mathbb{R}^d : \|x\| \leq 1\}. The dual \mathcal{M}_d^* of \mathcal{M}^d is \mathbb{R}^d equipped with the dual norm

\|x\|_* := \max_{\|y\| \leq 1} \langle x, y \rangle,

where \langle \cdot, \cdot \rangle denotes the inner product on \mathbb{R}^d. The dual unit ball \mathcal{B}_* := \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \forall y \in \mathcal{B}\} is also known as the polar body of \mathcal{B}.

For example, the d-dimensional Minkowski spaces \ell_1^d and \ell_\infty^d are duals of each other, where \ell_1^d has the norm \|\langle x_1, \ldots, x_d \rangle\|_1 := \sum_{i=1}^d |x_i| and \ell_\infty^d has the norm \|\langle x_1, \ldots, x_d \rangle\|_\infty := \max\{|x_i| : i = 1, \ldots, d\}.

A vector \mathcal{B} \in \mathcal{M}^d is dual to \mathcal{B} \in \mathcal{M}^d, x \neq o, if \|x\|_* = 1 and \langle x, x \rangle = \|x\|, i.e., \mathcal{B} is a dual unit vector that attains its norm at \mathcal{B}. In this case the hyperplane \{x \in \mathbb{R}^d : \langle x, x \rangle = 1\} supports the unit ball at \frac{1}{\|x\|}x. Any hyperplane supporting the unit ball at \frac{1}{\|x\|}x is given in this way by some \mathcal{B}, dual to \mathcal{B}. A unit vector \mathcal{B} \in \mathcal{M}^d is a regular direction if there is only one hyperplane that supports \mathcal{B} at \mathcal{B}. Note that the norm function \mathcal{B}(x) := \|x\| is differentiable at \mathcal{B} \neq o if and only if \frac{\partial}{\partial \mathcal{B}} p is a regular direction, and then the gradient \nabla \mathcal{B}(p) is the unique vector in \mathcal{M}_d^* dual to \mathcal{B}.

The exposed face of the unit ball \mathcal{B} defined by a unit vector \mathcal{B} \in \mathcal{M}_d^* is

\{\mathcal{B} : \langle a, \mathcal{B} \rangle = 1\}.

Similarly, a unit vector \mathcal{B} \in \mathcal{M}^d defines an exposed face \{\mathcal{B} \} of \mathcal{B}. If \mathcal{B} is a polytope then all faces are exposed, and each face \mathcal{F} of \mathcal{B} corresponds to a face \mathcal{F}_* of \mathcal{B} as follows:

\mathcal{F} := \{a : \langle a, \mathcal{B} \rangle = 1 \forall \mathcal{B} \}.

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1.2. Trees. Let $S \subset \mathcal{M}^d$ be a finite, non-empty set of points. A spanning tree $T$ of $S$ is an acyclic connected graph with vertex set $S$. Denote its edge set by $E(T)$. A Steiner tree $T$ of $S$ is a spanning tree of some finite $V \subset \mathcal{M}^d$ such that $S \subseteq V$ and such that the degree of each vertex in $V \setminus S$ is at least 3. The vertices in $S$ are the terminals of the Steiner tree $T$, and the vertices in $V \setminus S$ the Steiner points of $T$. The length of a tree $T$ in $\mathcal{M}^d$ is

$$\ell(T) := \sum_{xy \in E(T)} \|x - y\|.$$  

A Steiner minimal tree (SMT) of $S$ is a Steiner tree of $S$ of smallest length. The requirement that Steiner points have degree at least 3 is for technical convenience, since Steiner points of degree at most 2 can easily be eliminated using the triangle inequality without making the tree longer. It is easily seen that the number of Steiner points is at most $\#S - 2$. It then follows by a simple compactness argument that any non-empty finite $S$ has a SMT.

A star with center $s$ is a tree in which the vertex $s$ is joined to all other vertices. If $s \in S$ has neighbors $s_1, \ldots, s_k \in V$ in some SMT, then clearly the star joining $s$ to each $s_i$, $i = 1, \ldots, k$, is a SMT of $\{s, s_1, \ldots, s_k\}$. Thus, to characterize the neighborhoods of terminals in SMTs, it is sufficient to characterize SMTs which are stars with the center a terminal. This is the intent of Theorem 1 below.

1.3. Angles. An angle $\angle x_1x_0x_2$ in $\mathcal{M}^d$ is absorbing if the function

$$x \mapsto \|x - x_0\| + \|x - x_1\| + \|x - x_2\|$$

attains its minimum at $x_0$. Thus $\angle x_1x_0x_2$ is absorbing if and only if the star $\{x_0, x_1, x_2\}$ is a SMT of $\{x_0, x_1, x_2\}$.

**Lemma 1.** Let $a$ and $b$ be unit vectors in $\mathcal{M}^d$. Then the following are equivalent:

1. $\langle a, b \rangle$ is absorbing.
2. There exist unit vectors $a^*$ and $b^*$ in $\mathcal{M}^d$ such that $\langle a^*, a \rangle = \langle b^*, b \rangle = 1$ and $\|a^* + b^*\| \leq 1$.
3. The exposed faces $[a]^*$ and $[-b]^* = -[b]^*$ of the dual unit ball are at distance $\leq 1$.

**Proof.** $[1] \iff [2]$ is part of Lemma 5.4 in [5]. $[2] \iff [3]$ is trivial. \qed

In particular, this is a property of the angle alone:

**Corollary 1.** If $y_i$ is a point on the ray $x_0x_i$, $i = 1, 2$, then $\{x_0, x_1, x_2\}$ is a SMT of $\{x_0, x_1, x_2\}$ if and only if $\{x_0, y_1, x_0, y_2\}$ is a SMT of $\{x_0, y_1, y_2\}$. [5] Proposition 3.3.

Furthermore, an angle containing an absorbing angle is itself absorbing.

See also Proposition 3.3 and Lemma 5.4 of [5].

1.4. The planar case. We are now able to formulate the first result. In any Minkowski space, all angles made by two incident edges of a SMT are clearly absorbing. (For angles in a minimal spanning tree even more is true [4].) Remarkably, as shown in [8], for a Minkowski plane the condition that all angles are absorbing is also sufficient for a star to be a SMT of its vertices.

**Theorem 1.** Let $p_1, \ldots, p_k \neq o$ be points in a Minkowski plane $\mathcal{M}^2$. Then the star joining each $p_i$ to $o$ is a SMT of $\{o, p_1, \ldots, p_k\}$ if (and only if) all angles $\angle p_iop_j$, $i \neq j$, are absorbing.

This result is used in [8] to show that the maximum degree of a vertex in a SMT in a Minkowski plane is 6, with equality only if the unit ball is an affine regular hexagon; for all other planes the maximum is 4 if there exist supplementary absorbing angles, and 3 otherwise. The proof given in [8] employs a long case analysis. The new proof presented in Section 2 is more conceptual.
1.5. **Antipodality and higher dimensions.** Theorem 1 does not hold anymore in Minkowski spaces of dimension at least 3. For example, let the unit ball be the projection of a \((d+1)\)-cube along a diagonal. (When \(d = 3\), this is the rhombic dodecahedron.) In this Minkowski space, the star joining \(o\) to all \(2^{d+1} - 2\) vertices of the unit ball is not a SMT of these vertices if \(d \geq 3\), despite all the angles being absorbing [9]. However, Theorem 1 extends to both \(\ell_1^d\) and \(\ell_\infty^d\).

Our second result is a generalization of this fact. We first introduce some more notions, involving antipodality. Two boundary points of the unit ball \(B\) are **antipodal** if there exist distinct parallel hyperplanes supporting the two points. Equivalently, unit vectors \(a\) and \(b\) are antipodal if and only if \(||a - b|| = 2\).

**Lemma 2.** If \(a\) and \(b\) are antipodal unit vectors in a Minkowski space, then \(\angle aob\) is an absorbing angle.

**Proof.** Since the union of the segments \(oa\) and \(ob\) form a shortest path from \(a\) to \(b\), these two segments form a SMT of \(\{o, a, b\}\), hence \(\angle aob\) is absorbing.

The converse of the above lemma is not necessarily true, as the Euclidean norm shows. We call the unit ball of a Minkowski space **Steiner antipodal** if two points \(a\) and \(b\) on the boundary of the unit ball are antipodal whenever \(\angle aob\) is absorbing.

**Theorem 2.** Consider the following properties of a set \(\{p_1, \ldots, p_k\}\) of unit vectors in a Minkowski space \(M^d\).

\[
\begin{align*}
(4) & \quad \text{All angles } \angle p_i o p_j \text{ are absorbing.} \\
(5) & \quad \text{All distances } ||p_i - p_j|| = 2. \\
(6) & \quad \text{The star } \bigcup_{i=1}^{k}[o, p_i] \text{ is a SMT of } \{p_1, \ldots, p_k\}. \\
(7) & \quad \text{The star } \bigcup_{i=1}^{k}[o, p_i] \text{ is a SMT of } \{o, p_1, \ldots, p_k\}. 
\end{align*}
\]

Then the implications \(5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 4\) hold. Furthermore, \(4\) to \(7\) are equivalent if, and only if, the norm is Steiner antipodal.

**Proof.** \(6 \Rightarrow 7\) is true, since all Steiner trees of \(\{o, p_1, \ldots, p_k\}\) are also Steiner trees of \(\{p_1, \ldots, p_k\}\), by considering \(o\) to be a Steiner point.

\(7 \Rightarrow 4\) holds, since if any star \([o, p_i] \cup [o, p_j]\) can be shortened, it would also shorten the star \(\bigcup_{i=1}^{k}[o, p_i]\).

The implication \(4 \Rightarrow 5\) is equivalent to the definition of Steiner antipodality.

This leaves \(5 \Rightarrow 6\). Note that the given star is a Steiner tree of length \(k\). It is sufficient to show that all Steiner trees have length \(\geq k\). However, note that the open unit balls centered at the \(p_i\) are pairwise disjoint, since \(||p_i - p_j|| = 2\) for distinct \(i \neq j\). Any Steiner tree will have to join each \(p_i\) to the boundary of the unit ball with centre \(p_i\). The part of the Steiner tree inside this ball must therefore have length at least 1. It follows that the length of any Steiner tree must be at least \(k\). \(\square\)

In order to apply this result, we need a characterization of Steiner antipodal norms in terms of duality.

**Proposition 1.** The following are equivalent in any Minkowski space \(M^d\):

\[
\begin{align*}
(8) & \quad \text{The norm is Steiner antipodal.} \\
(9) & \quad \text{The unit ball is a polytope and any two disjoint faces of} \\
& \quad \text{the dual unit ball are at distance } > 1.
\end{align*}
\]
Proof. (8) $\Rightarrow$ (9) is immediate from the definition of Steiner antipodality and Lemma 1. (8) $\Rightarrow$ (9) follows upon noting that if a convex body is not a polytope, then there are disjoint exposed faces that are arbitrarily close to each other.

A Minkowski space is a CL-space if for every maximal proper face $F$ of the unit ball $B$ we have $B = \text{conv}(F \cup (-F))$. It is easily seen from finite dimensionality that the unit ball of a CL-space is a polytope. CL-spaces were introduced by R. E. Fullerton (see [7]), although the notion has been studied before by Hanner [1], who proved that the unit balls of CL-spaces are $\{0, 1\}$-polytopes. McGregor [6] showed that CL-spaces are exactly those spaces with numerical index 1. What is important for our purposes is that CL-spaces turn out to be Steiner antipodal.

Hanner [1] identified an important subclass of CL-spaces, namely those that can be built up from the one-dimensional space $\mathbb{R}$ using $\ell_1$-sums and $\ell_\infty$-sums. For two Minkowski spaces $\mathcal{M}$ and $\mathcal{N}$ of dimension $d$ and $e$ we define their $\ell_1$-sum $\mathcal{M} \oplus_1 \mathcal{N}$ and $\ell_\infty$-sum $\mathcal{M} \oplus_\infty \mathcal{N}$ to be the Minkowski spaces on $\mathbb{R}^{d+e}$ with norms $\|x, y\|_1 = \|x\| + \|y\|$ and $\|x, y\|_\infty = \max\{\|x\|, \|y\|\}$. Note that the unit ball of $\mathcal{M} \oplus_1 \mathcal{N}$ is the convex hull of the unit ball of $\mathcal{M}$ when embedded as $\mathcal{M} \oplus \{0\}$ and the unit ball of $\mathcal{N}$ when embedded as $\{0\} \oplus \mathcal{N}$. The unit ball of $\mathcal{M} \oplus_\infty \mathcal{N}$ is the Cartesian product of the unit balls of $\mathcal{M}$ and $\mathcal{N}$. The unit balls of these spaces are called Hanner polytopes. We thus introduce the name Hanner space for these spaces. For more information see [1] [3] [2] [7].

We summarize the above discussion as follows.

**Proposition 2.** All Hanner spaces are CL-spaces. All CL-spaces are Steiner antipodal.

**Proof.** It is clear and well-known that Hanner spaces are CL-spaces (see, e.g., [7]).

It is also well-known that the dual of a CL-space is a CL-space as well [6]. To prove the second part of the proposition, it is by Proposition 1 sufficient to show that any two disjoint faces $F$ and $G$ of the unit ball $B$ are at distance $> 1$. Suppose that $F$ is contained in the facet $F'$. Then all vertices of $B$ disjoint from $F$ must lie in the opposite facet $-F'$. It follows that $G \subseteq -F'$, and $F$ and $G$ are therefore at distance 2. $\square$

2. Proof of Theorem 1

**Lemma 3.** Let $\ell$ be a line passing through a Steiner point $s$ of a SMT $T$ in a Minkowski plane $\mathcal{M}^2$. Assume that $\ell$ is parallel to a regular direction. Then $T$ has edges incident to $s$ in both open half planes bounded by $\ell$.

**Proof.** Without any assumption on $\ell$, the edges incident to $s$ cannot all lie in the same open half plane bounded by $\ell$. Indeed, such a tree can be shortened as follows (Fig. 1). Let some line $m$ intersect the interior of each edge $sp_i$, $i = 1, \ldots, k$, in $q_i$, say. Remove edges $sp_1$, $sp_k$, and $sq_i$, $i = 2, \ldots, k - 1$, and add edges $p_1q_2$, $q_iq_{i+1}$, $2 \leq i \leq k - 2$, and $q_{k-1}p_k$, to obtain a new Steiner tree $T'$, without the Steiner point $s$, but with new Steiner points $q_i$, $2 \leq i \leq k - 1$. By the triangle inequality, $\ell(T) - \ell(T') \geq \sum_{i=2}^{k-1} \|q_i\| > 0$, contradicting the minimality of $T$.

We now assume that $\ell$ is parallel to a regular direction. It is sufficient to show that $sp_1$ and $sp_k$ cannot be opposite edges both on $\ell$, with all other $sp_i$, $2 \leq i \leq k - 1$, on the same side of $\ell$ (Fig. 2). Let $s_2$ be a variable point on $sp_2$ with $\|s_2 - s\|$ small. Denote the intersection of
Define the measure of all Steiner points of \( o \). Moreover, has to be the star with centre \( o \) which is negative if the union of the segments contained in the (closed) angle. Choose \( \tilde{\eta} \).  

\[ \text{Proof of Theorem 1.} \]

\[ \forall o,p \] all Steiner points in the interiors of the angles \( o,p \) and \( k \) and introduces new Steiner points \( s_2,\ldots,s_{k-1} \). Denoting the new tree by \( T' \), it follows that the length changes by

\[ \ell(T') - \ell(T) = \|s_2 - p_1\| + \|s_2 - p_k\| - \|s - p_1\| - \|s - p_k\| - \sum_{i=2}^{k-1} \|s - s_i\| \]

\[ \leq \|(s - p_1) + (s_2 - s)\| - \|s - p_1\| + \|(s - p_k) + (s_2 - s)\| - \|s - p_k\| - \|s_2 - s\| . \]

Since \( s - p_1 \) and \( s - p_k \) are parallel to a regular direction, the norm is differentiable at both points, i.e.,

\[ \lim_{s_2 \to s} \frac{\|(s - p_1) + (s_2 - s)\| - \|s - p_1\|}{\|s_2 - s\|} = \langle u_s, s_2 - s \rangle \]

and

\[ \lim_{s_2 \to s} \frac{\|(s - p_k) + (s_2 - s)\| - \|s - p_k\|}{\|s_2 - s\|} = \langle -u_s, s_2 - s \rangle . \]

(Since \( s - p_1 \) and \( s - p_k \) are in opposite directions, their duals are opposite in sign.) It follows that

\[ \ell(T') - \ell(T) \leq \langle u_s, s_2 - s \rangle + \langle -u_s, s_2 - s \rangle + o(\|s_2 - s\|) - \|s_2 - s\| \]

which is negative if \( \|s_2 - s\| \) is sufficiently small. Then \( \ell(T') < \ell(T) \), a contradiction. \( \Box \)

**Proof of Theorem 2.** Without loss of generality, the segments \( op_1,\ldots,op_k \) are ordered around \( o \). Assume that all angles \( o,p_1p_2 \) are absorbing. We start off with an arbitrary SMT of \( \{o,p_1,\ldots,p_k\} \) and modify it in two steps without increasing the length. In Step 1 we eliminate all Steiner points in the interiors of the angles \( o,p_1p_{i+1} \). In Step 2 we eliminate all edges between vertices on different rays \( \overrightarrow{op_i} \). The edges of the final SMT are then all contained in the the union of the segments \( op_i, i = 1,\ldots,k \). This tree cannot have Steiner points, and so has to be the star with centre \( o \). This concludes the proof.

**Step 1:** For each angle \( o,p_1p_{i+1} \) (where we let \( k + 1 \equiv 0 \)), choose a regular direction \( r_i \) not contained in the (closed) angle. Choose \( \tilde{r}_i \in \overrightarrow{op_i} \) and \( \tilde{q}_i \in \overrightarrow{op_{i+1}} \) such that \( \tilde{r}_i\tilde{q}_i \) is parallel to \( r_i \) (Fig. 3). For each point \( s \) in the interior of \( o,p_1p_2 \), write \( s = \alpha p + \beta q \) (uniquely, and then, moreover, \( \alpha,\beta > 0 \)) and define the measure of \( s \) to be

\[ |s| := \alpha + \beta . \]

Define the measure \( |T| \) of any Steiner tree \( T \) of \( \{o,p_1,\ldots,p_k\} \) to be the sum of the measures of all Steiner points of \( T \) not on any ray \( \overrightarrow{op_i} \). Let

\[ \mu = \inf \{|T| : T \text{ is a SMT of } \{o,p_1,\ldots,p_k\} \} . \]
Let $T_n$ be a sequence of SMTs of $\{o, p_1, \ldots, p_k\}$ with $\lim_{n \to \infty} |T_n| = \mu$. Since there are only finitely many combinatorial types of Steiner trees on a set of $k+1$ points, we may, by passing to a subsequence, assume without loss of generality that all $T_n$ have the same combinatorial type with Steiner points $s_1^{(n)}, \ldots, s_m^{(n)}$, say. By taking further subsequences, we may assume that each sequence of Steiner points converge, say $s_i^{(n)} \to s_i$, $i = 1, \ldots, m$. In the limit we obtain a Steiner tree $T_0$ with $\ell(T_0) = \lim_{n \to \infty} \ell(T_n)$, hence $T_0$ is a SMT. Also, $|T_0| \leq \lim_{n \to \infty} |T_n|$, since the measure of a Steiner point is continuous in the interior of an angle, hence $\lim_{n \to \infty} |s_i^{(n)}| = |s_i|$ if $s_i$ is still in the interior of the same angle, otherwise $\lim_{n \to \infty} |s_i^{(n)}| \geq |s_i| = 0$ if $s_i$ is on one of the rays $\overrightarrow{op_j}$. Therefore, $|T_0| = \mu$. It remains to show that $\mu = 0$, since this will imply that $T_0$ does not have any Steiner point in the interior of an angle.

Suppose that $\mu > 0$. We obtain a contradiction by constructing a SMT $T'$ with $|T'| < \mu$. Let $s$ be a Steiner point of $T_0$ in the interior of $\angle p_1o p_{k+1}$, say (Fig. 3(a)). Without loss of generality there is no point of $T_0$ in the translated angle $s + \angle p_1op_{k+1}$, since such a point is necessarily another Steiner point $s'$ and we may then repeatedly choose a new Steiner point $s''$ in $s' + \angle p_1op_{k+1}$, until this procedure halts.

Let $\ell$ be the line through $s$ parallel to $r_i$. The points on $\ell$ in the interior of $\angle p_1op_{k+1}$ all have the same measure, and the points on the same side of $\ell$ as $o$ have smaller measure. By Lemma 3 there is an edge $sx_1$ incident to $s$ on the same side of $\ell$ as $o$. There are at least two more edges $sx_2$ and $sx_3$. Since not all edges are in an open half plane bounded by a line through $s$, we may choose $x_2$ and $x_3$ such that the angle $\angle x_2sx_3$ contains the translated angle $s + \angle p_1op_{k+1}$ in its interior (with $s$ excluded). It follows that there is a point $s'$ on $sx_1$ sufficiently close to $s$ such that $\angle x_2s'x_3$ contains the translate $s' + \angle p_1op_2$, and so is still absorbing (Figure 4(b)). We may therefore replace the edges $sx_2$, $sx_3$ and $ss'$ by $s'x_2$ and $s'x_3$ without lengthening $T_0$, to obtain a new SMT $T'$. However, $|s'| < |s|$, hence $|T'| < |T| = \mu$, which gives the required contradiction.
Step 2: Note that for any absorbing angle $\angle p(opj)$,
\[
\|p_i - o\| + \|p_j - o\| + \|o - p_j\| \leq \|p_i - p_j\| + \|p_j - p_j\| + \|o - p_j\|,
\]
i.e., $\|p_i - p_j\| \geq \|p_i\|$.

Suppose that the SMT $T$ has an edge between two points on different segments, say between $q_i$ on $op_i$ and $q_j$ on $op_j$. Without loss of generality, the unique path in $T$ from $o$ to $q_i$ passes through $q_j$ (otherwise interchange $q_i$ and $q_j$). Since $\angle q_i o q_j$ is absorbing, $\|q_i - q_j\| \geq \|q_i\|$. We can then replace the edge $q_i q_j$ by $o q_j$, without losing connectivity and without lengthening $T$. This process may be repeated until all edges are on the segments $op_i$, which finishes Step 2.

\[\square\]

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