Field Theory of Mesoscopic Fluctuations in Superconductor/Normal-Metal Systems

Alexander Altland, B D Simons, and D Taras-Semchuk

Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK

Thermodynamic and transport properties of normal disordered conductors are strongly influenced by the proximity of a superconductor. A cooperation between mesoscopic coherence and Andreev scattering of particles from the superconductor generates new types of interference phenomena. We introduce a field theoretic approach capable of exploring both averaged properties and mesoscopic fluctuations of superconductor/normal-metal systems. As an example the method is applied to the study of the level statistics of a SNS-junction.

Physical properties of both superconductors and mesoscopic normal metals are governed by mechanisms of macroscopic quantum coherence. Their interplay in hybrid systems comprised of a superconductor adjacent to a normal metal gives rise to qualitatively new phenomena (see Ref. [1] for a review): Aspects of the superconducting characteristics are imparted on the behaviour of electrons in the normal region. This phenomenon, known as the “proximity effect”, manifests itself in a) the mean (disorder averaged) properties of SN-systems being substantially different from those of normal metals and b) various types of mesoscopic fluctuations which not only tend to be larger than in the pure N-case but also can be of qualitatively different physical origin. Although powerful quasi-classical methods, based largely on the pioneering work of Eilenberger [2] and Usadel [3], have been developed to analyse the manifestations of the proximity effect in average characteristics of SN-systems, far less is known about the physics of mesoscopic fluctuations: While the quasi-classical approach is not tailored to an analysis of fluctuations, standard diagrammatic techniques [4] used in the study of N-mesoscopic fluctuations can often not be applied due to the essentially non-perturbative influence of the fully established proximity effect. Important progress was made recently by extending the scattering formulation of transport in N-mesoscopic systems to the SN-case [5]. This approach has proven powerful in the study of various transport fluctuation phenomena but is not applicable to the study of fluctuations on a local and truly microscopic level.

In the present Letter we introduce a general framework that combines key elements of the quasi-classical approach with more recent methods developed in N-mesoscopic physics into a unified approach. As a result we obtain a formalism that can be applied to the general analysis of mesoscopic fluctuations superimposed on a proximity effect influenced mean background. In order to demonstrate the practical use of the approach we will consider the example of spectral fluctuations as a typical representative of a mesoscopic phenomenon. The density of states (DoS) of N-mesoscopic systems exhibits quantum fluctuations around its disorder averaged mean value which can be described in terms of various types of universal statistics. The analogous question in the SN-case – What types of statistics govern the disorder induced fluctuation behaviour of the proximity effect influenced DoS? – has not been answered so far. Although space limitations prevent us from discussing this problem in detail, our main result, the emergence of some kind of modified Wigner Dyson statistics [5], will be derived below.

To be specific we consider the geometry shown in Fig. 1, where $\varphi = \varphi_1 - \varphi_2$ represents the relative phase of the complex order parameters of the two superconductors. It is well known [6] that even the mean DoS of the SNS-system exhibits non-trivial behavior which is difficult to describe within standard perturbative schemes: States which fall within the superconducting gap, $\Delta$, are confined to the normal metal. The proximity effect then further induces a minigap in the DoS of the normal region around the Fermi-energy, $\epsilon_F$, whose size of $O(\epsilon_F - D_n / L^2)$ depends sensitively on $\varphi$ ($D_n$ is the diffusion constant and $\hbar = 1$ throughout). To analyse the fluctuation behavior of the DoS, $\nu(\epsilon)$, around its disorder averaged background, $\langle \nu(\epsilon) \rangle$, we will consider the two-point correlation function [7].

$$R_2(\epsilon, \omega) = \langle \nu(\epsilon) \rangle^{-2} \langle \nu(\epsilon + \omega)\nu(\epsilon - \omega/2) \rangle_c.$$  

The starting point of our analysis is the Gor'kov equation for the matrix advanced/retarded ($a/r$) Green function [8]

$$G^{r,a}_c = \begin{pmatrix} G^{r,a}_{\nu} & F^{r,a} \\ F^{r,a}_{\nu} & G^{r,a}_{\nu} \end{pmatrix},$$

where

\*\*\*
\[
\epsilon_\pm = \epsilon \pm i0, \quad \tilde{A} \text{ is the vector potential of an external magnetic field, } \Delta = \Delta \sigma^h \exp(-i\sigma^h), \quad \text{represents the (spatially dependent) complex order parameter with phase } \varphi, \text{ and Pauli matrices } \sigma^h \text{ operate in the Nambu or particle/hole (ph) space.}
\]

\[
\epsilon_\pm = \epsilon \pm i0, \quad \tilde{A} \text{ is the vector potential of an external magnetic field, } \Delta = \Delta \sigma^h \exp(-i\sigma^h), \quad \text{represents the (spatially dependent) complex order parameter with phase } \varphi, \text{ and Pauli matrices } \sigma^h \text{ operate in the Nambu or particle/hole (ph) space.}
\]

The impurity potential in the N-region is taken to be Gaussian \( \delta \)-correlated with zero mean and correlation \( \langle V(\vec{r})V(\vec{r}') \rangle = \delta^d(\vec{r} - \vec{r}')/(2\pi\nu\tau) \), where \( \nu \) denotes the DoS of the bulk normal metal at \( \epsilon_F \), and \( \tau \) represents the mean free scattering time. In the following the complex order parameter in the S-region is imposed and not obtained self-consistently \([8]\). Where the S- and N-region are distinct (as in the SNS junction), the bulk DoS, \( \nu_{n,s} \) and scattering time, \( \tau_{n,s} \) will be chosen independently.

Traditionally the impurity averaged Green function \([8]\) is computed within a quasi-classical approximation, i.e. the Schrödinger equation \([8]\) is reduced to an effective transport equation, the Eilenberger equation \([8]\), which in the dirty limit simplifies further to the diffusive Usadel equation \([8]\). Here we develop a field theoretical formulation that integrates concepts of the quasi-classical formalism into a more general framework allowing for the computation of disorder averaged products of Green functions, a necessary requirement for the calculation of correlation functions such as \( G_{\alpha\beta}(\vec{r},\vec{r}') \). The basic strategy will be to start from a (microscopically derived) generating functional whose points of stationary phase obey the Usadel equation. By investigating fluctuations around this quasi-classical limit, correlations between the different Green functions will be explored. In the following we formulate this program in more detail.

As in the pure N-case, ensemble averaged products of advanced and retarded Gorkov Green functions can be described in terms of generating functionals of nonlinear \( \sigma \)-model type \([9]\) (see Ref. \([8]\) for a review on the \( \sigma \)-model analysis of Green functions in N-mesoscopic physics). In the dirty limit, \( (\epsilon, \Delta) < \tau^{-1} < \epsilon_F \), the generalization of the supersymmetric N-type \( \sigma \)-model \([8]\) reads

\[
\int_{\mathcal{Q}^2=\mathbb{I}} DQ(\cdots) e^{-S[Q]},
\]

\[
S[Q] = -\frac{\pi\nu}{8} \text{str} \left[ D(\tilde{D}Q)^2 + 4iQ(\Delta + \epsilon + \frac{\omega^+}{2}\sigma^3\sigma^3) \sigma^h \right],
\]

where \( \tilde{D} = \tilde{\partial} - i(\epsilon/c)[\tilde{A}_\phi \sigma^3 \otimes \sigma^h,] \) represents a covariant derivative, \( \tilde{A}_\phi = \tilde{A} + c/(2\epsilon) \tilde{\partial} \phi \) accounts for both the external field and the phase of the order parameter, \( \Delta = \Delta \sigma^h \), the Pauli matrices \( \sigma^h \), \( \sigma^3 \) and \( \sigma^3 \otimes \sigma^h \) operate in fermion/boson, time-reversal and ar-blocks respectively \([8]\). The symbol \( D \) stands for a space dependent diffusion constant which may take separate values, denoted as \( D_{n,s} \), in the N and S regions. Although specific pre-exponential source terms (denoted by ellipses in Eq. \([8]\)) must be chosen according to any given correlation function (such as \( G_{\alpha\beta}(\vec{r},\vec{r}') \)), their precise form does not influence the analysis below and we therefore refer to Ref. \([8]\) for their detailed structure. The integration in \([8]\) extends over a 16 × 16-dimensional matrix field \( Q = T^{-1} \sigma^h \otimes \sigma^3 T \), whose symmetries are identical with those of the conventional \( \sigma \)-model \([8]\).

The expression \([8]\) differs in two respects from the \( \sigma \)-model for N-systems: i) the appearance of a ph-space associated with the 2 × 2-matrix structure of the Gorkov Green function, and ii) the presence of the order parameter \( \Delta \). Whereas i) can be accounted for by a doubling of the matrix dimension of the field \( Q \), ii) calls for more substantial modifications: For \( \Delta \neq 0 \), standard perturbative schemes for the evaluation of the functional \([8]\) fail \([1]\), an indication of the fact that the superconductor influences the properties of the normal metal heavily. Under these conditions a more efficient approach is first to subject the action to a mean field analysis and then to consider fluctuations around a newly defined — and generally space dependent — stationary field configuration. A variation of the action \([8]\) with respect to \( Q \), subject to the constraint \( Q^2 = \mathbb{I} \), generates a non-linear equation for the saddle-point,

\[
D_{\tilde{\partial}}(\tilde{D}Q) + \left[ \tilde{Q}, \Delta \sigma^h - i(\epsilon + \omega^+\sigma^3)\sigma^h \right] = 0.
\]

Current conservation implies the boundary condition \([9]\), \( \sigma_n Q \partial_x|_{x_s} = \sigma_x Q \partial_x|_{x_s} \), where \( \sigma_{n,s} = e^2\nu_{n,s}D_{n,s} \) denotes the conductivity and \( \partial_x|_{x_n(x_s)} \) is a normal derivative at the N(S)-side of the interface. An inspection of \([8]\) shows that only the particle/hole components of the matrix field \( Q \) are coupled by the saddle point equation. It is thus sensible to make a block diagonal ansatz \( Q = \text{bdig} (q_+, q_-) \), where the eight dimensional retarded, \( q_+ \), and advanced, \( q_- \) subblocks are diagonal in both time reversal and boson/fermion space. Noting that the saddle point configuration \( -i\pi\nu q_\pm \) of the nonlinear \( \sigma \)-model is associated with the impurity averaged retarded/advanced Green function \([8]\), we identify Eq. \([8]\) as the Usadel equation. The general connection between the \( \sigma \)-model formalism and the quasi-classical approach has first been noticed in \([8]\).
below.

So far our analysis has been for SN-systems of a general geometry. Specializing the discussion to the particular SNS-junction shown in Fig. (1), we set $\Delta(\vec{r}) = \Delta \Theta(|x| - L/2)$ constant inside the superconductor ($\Delta \gg E_c$), and zero in the normal region, with a phase $\pi/2 + \text{sgn}(x) \varphi/2$. The saddle-point equation depends sensitively on both the presence or absence of an external magnetic field and the phase difference between the order parameters. Taking the external field to be zero, it is convenient to focus on two extreme cases: (i) $\varphi = 0$ (orthogonal symmetry), and (ii) $\varphi \gg 1/\sqrt{\gamma}$ (unitary symmetry). Here $g = E_c/\bar{d} \gg 1$ denotes the dimensionless conductance and $d$ represents the bulk single-particle level spacing of the normal metal.

The *disorder averaged local DoS* can be obtained from the analytical solution of the Usadel-saddle point equation $[6,14–17]$ as $\nu(\vec{r}) = \nu \text{Re}[q_{12}(\vec{r})]$. Its space/energy dependence is shown in Fig. 2 for the case (i), zero phase difference, and a particular value of the material parameter $\gamma = \nu_n \sqrt{D_n/\nu_s \sqrt{D_s}}$. The most striking feature of the average DoS is the appearance of a spatially constant minigap in the N-region. The gap attains its maximum width $E_c$ at $\varphi = 0$ and shrinks to 0 as $\varphi$ approaches $\pi [15]$. Within the superconductor, $\nu(E_c < \epsilon \ll \Delta)$ decays exponentially on a scale set by the bulk coherence length $\xi = (D_s/2\Delta)^{1/2}$.

Consider the saddle point equation $[6]$ in the simple case $\omega = \varphi = 0$. Obviously, as it commutes with all matrices $\sigma_{\text{ph}}^i$, any spatially constant rotation $T$ of type (b) gives rise to another solution. In other words, the (b)-fluctuations represent Goldstone modes with an action that vanishes in the limit of spatial constancy and $\omega \to 0$. Since any $T$ diagonal in ph-space inevitably has to couple between advanced and retarded indices $[6]$, these modes lead to *correlations between advanced and retarded Green functions* (and thereby to mesoscopic fluctuations) which become progressively more pronounced as $\omega$ approaches zero.

In the limit of small frequencies $\omega < E_c$, the *ergodic regime*, the global zero mode $Q_0 = T_0^{-1} q T_0$, $[T_0, \sigma_{\text{ph}}^i] = 0$, $T_0(\vec{r}) = \text{const.}$, plays a unique role: Whereas fluctuations with non-vanishing spatial dependence give rise to contributions to the action of $O(g \gg 1)$ $[6]$, this mode couples only to the frequency difference $\omega$. Restricting attention to the pure zero mode contribution, we obtain the effective action

$$S_0[Q_0] = -i \frac{\omega}{2} \text{str} [Q_0 \sigma_{\text{ph}}^i],$$

where $\bar{d}(\epsilon) = (\int \nu(\epsilon))^{-1}$ denotes the average level spacing and the ph-degrees of freedom have been traced out. From this result it follows $[6]$ that, in the ergodic regime, the spectral statistics of an SNS system is governed by Wigner-Dyson fluctuations $[5,19]$ of (i) orthogonal or (ii) unitary symmetry superimposed upon an energetically non-uniform mean DoS. Furthermore, a comparison of Eq. (6) with the analogous action for N-systems $[6]$ shows the correlations to depend on an average level spacing that is effectively *halved*. This reflects the strong “hybridization” of levels at energies $\sim \epsilon_F \pm \epsilon$ induced by Andreev scattering at the SN-interface.

In further contrast to N-systems, the range over which Wigner-Dyson statistics apply turns out to be greatly diminished by non-universal fluctuations. To prove the last statement, fluctuations of type (c), coupling between advanced/retarded and particle/hole components simultaneously, have to be taken into account. Due to their complexity, the detailed analysis of the (c)-type fluctuations is cumbersome and will be deferred to a forthcoming study.
ing publication [17]. Here we restrict ourselves to a brief and qualitative discussion of the principal effect of these fluctuations on the universal zero mode action [10].

As can be seen from the general structure of the action (4), (c)-type fluctuations in the vicinity of the mini-gap are generally `massive' (governed by an action which is at least of order $\epsilon/d \gtrsim g \gg 1$). It is thus permissible to treat these fluctuations in a simple Gaussian approximation. As a result we arrive at an effective action which is non-perturbative in the above zero mode configuration and quadratic in the (c)-type perturbations. Integrating over these fields in a spirit similar to the analysis performed in Ref. [21] we obtain the modified zero mode action

$$S[Q_0] = S_0[Q_0] - \frac{\kappa(\epsilon)}{g} \left( \frac{\omega}{d(\epsilon)} \right)^2 \text{str} \left[ \sigma^a_3, Q_0 \right]^2, \quad (7)$$

where $\kappa(\epsilon) \sim O(1)$ denotes a constant dependent on the sample geometry [17]. Eq. (7) has a structure equivalent to that found in the study of universal parametric correlation functions and explicit expressions for $R_2$ for both orthogonal and unitary ensembles can be deduced from Ref. [21]. Qualitatively, the additional contribution in (7) counteracts the zero-mode fluctuations for non-vanishing frequencies $\omega$. Already for energy separations $\omega/d(\epsilon) \sim \sqrt{g}$, the zero-mode integration is largely suppressed which manifests in an exponential vanishing of the level correlations on these scales. This is in contrast to the pure N-case where the Wigner-Dyson regime (prevailing up to frequencies $\omega \sim E_c$) is succeeded by other forms of algebraically decaying spectral statistics in the high frequency domain $\omega > E_c$ [22].

In conclusion a general framework has been developed in which the interplay of mesoscopic quantum coherence phenomena and the proximity effect can be explored. An investigation of the spectral statistics of an SNS geometry revealed that level correlations are Wigner-Dyson distributed with strong non-universal corrections at large energy scales. Finally, we remark that for quantum structures in which transport is not diffusive but ballistic and boundary scattering is irregular, a ballistic $\sigma$-model involving the classical Poisson bracket can be derived [23]. In this case, the saddle-point condition recovers the Eilenberger equation of transport [2].

We are indebted to Boris Altshuler, Anton Andreev, Konstantin Efetov, Dima Khmel’nitskii, Valodya Falko and Martin Zirnbauer for useful discussions. One of us (DT-S) acknowledges the financial support of the EPSRC. The hospitality of the ITP in Santa Barbara and the Lorentz Center in Leiden are gratefully acknowledged. This research was supported in part by the National Science Foundation under Grant No. PHY94-07194.

[1] C. W. J. Beenakker, Rev. Mod. Phys. 69, 53 (1997); C. J. Lambert and R. Raimondi, cond-mat/9708056.
[2] G. Eilenberger, Z. Phys. 182, 427 (1965); ibid. 214, 195 (1968).
[3] K. D. Usadel, Phys. Rev. Lett. 25, 507 (1970).
[4] B. L. Altshuler and B. Z. Spivak, Zh. Eksp. Teor. Fiz. 92, 609 (1987) [Sov. Phys. JETP 65, 343 (1987)].
[5] M. L. Mehta, Random Matrices (Academic, New York, 1991).
[6] For a review, see Nonequilibrium Superconductivity, ed. V. L. Ginzburg, Nova Science Publications, 1988.
[7] $R_2(\epsilon, \omega)$ for the particular gapless ($\varphi = \pi$), zero-dimensional ($\omega < E_c$) case has been analysed previously in A. Altland and M. R. Zirnbauer, Phys. Rev. Lett. 76, 3420 (1996); K. M. Frahm, P. W. Brouwer, J. A. Melsen and C. W. J. Beenakker, Phys. Rev. Lett. 76, 2981 (1996).
[8] In general, the proximity of a N-metal leads to a renormalization of the superconducting order parameter in the boundary region. However, we do not expect this phenomenon to have an essential impact on our results.
[9] K. B. Efetov, Supersymmetry in Disorder and Chaos, Cambridge University Press, New York (1997).
[10] R. Oppermann, Nuclear Phys. B 280, 753 (1987).
[11] For finite $\Delta$, a perturbative expansion of $T = e^{W}$ in powers of $W$ leads to contributions that are linear in $W$. These linear vertices generally have to be summed up to infinite order which is difficult if not impossible.
[12] B. A. Muzykantskii and D. E. Khmel’nitskii, Phys. Rev. B 51, 5840 (1995).
[13] M. Yu. Kupriyanov and V.F.Lukichev, Zh. Eksp. Teor. Fiz. 94, 139 (1988) [Sov. Phys. JETP 67, 1163 (1988)].
[14] W. Belzig, C. Bruder and G. Schoen, Phys. Rev. B 54, 9443 (1996).
[15] F. Zhou, P. Charlat, B. Spivak and B. Pannetier, cond-mat/9707056.
[16] Note that the solution of Eq. (3) does not in general belong to the standard saddle-point manifold of the $\sigma$-model. It can however be accessed by a deformation of the integration contour [13].
[17] A. Altland, B. D. Simons, D. Taras-Semchuk, to be published.
[18] V. I. Fal’ko and K. B. Efetov, Europhys. Lett. 32, 627 (1995); A. D. Mirlin, Phys. Rev. B 53, 1186 (1996).
[19] With reference to the specific correlation function $R_2$, we remark that only massive fluctuations in the $\phi$-sector contribute to connected correlators of the form $\langle G_{\phi}^A G_{\phi}^A \rangle$ allowing such terms to be neglected.
[20] V. E. Kravtsov and A.D. Mirlin, JETP Lett. 60, 65 (1994).
[21] B. D. Simons and B. L. Altshuler, Phys. Rev. Lett. 70, 4063 (1993).
[22] B.L.Altshuler and B.I.Shklovskii, Sov. Phys. JETP 64, 127 (1986).
[23] B. A. Muzykantskii and D. E. Khmel’nitskii, JETP Lett. 62, 76 (1995); A. V. Andreev et al., Phys. Rev. Lett. 76, 3947 (1996).