Typical Gaussian quantum information

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Abstract
We investigate different geometries and invariant measures on the space of mixed Gaussian quantum states. We show that when the global purity of the state is held fixed, these measures coincide and it is possible, within this constraint, to define a unique notion of volume on the space of mixed Gaussian states. We then use the so defined measure to study typical non-classical correlations of two mode mixed Gaussian quantum states, in particular entanglement and steerability. We show that under the purity constraint alone, typical values for symplectic invariants can be computed very elegantly, irrespectively of the non-compactness of the underlying state space. Then we consider finite volumes by constraining the purity and energy of the Gaussian state and compute typical values of quantum correlations numerically.

Keywords: geometry of quantum states, quantum information theory, Gaussian states, quantum nonlocality

(Some figures may appear in colour only in the online journal)

1. Introduction

Typical properties of quantum states are interesting, for example, for the characterization of entanglement [1]. There is hope that the understanding of quantum correlations in mixed multipartite systems can be furthered by looking at properties averaged over all states [2]. This will also be in the focus of this article, where, for example, we aim to answer the fundamental question of what the fraction of separable states compared to all physical states is [3, 4]. In addition, the averaging approach has relevant applications in the study of the emergence of thermodynamic behavior in many-particle quantum systems [5]. Interestingly, it has been suggested that most processes in generic systems lead to fast equilibration of typical states [6, 7].

Here, typical means that one considers a uniform distribution in the space of all states. A geometry on the state space has to be fixed, which then gives rise to a unitarily invariant
volume element. For pure states there is a unique unbiased measure which emerges from the Haar measure on the unitary group [8], whereas for mixed states such a unique measure in general does not exist [9]. Investigations of typical properties of mixed states of quantum systems were first pioneered in [3, 4]. Most results so far are for systems with finite dimensional Hilbert space.

We will focus our investigations on the geometry and typical properties of mixed Gaussian states, which form a subspace of continuous variable quantum states. Gaussian states are important for two main reasons, they can be created and manipulated experimentally using linear mode evolution [10, 11] and are completely characterized by a finite number of parameters, the first and the second moments of canonical position- and momentum operators [12]. Gaussian states are best represented by their (positive) Gaussian Wigner function [13]. They are the cornerstone for continuous variable quantum information processing [14, 15] and can be experimentally characterized using both homodyning- and single-photon detection techniques [16, 17].

For pure Gaussian states, a unique unbiased measure using the invariant Haar measure on the symplectic group has been constructed in [18]. For mixed Gaussian states, different invariant measures have been constructed in [19] using the Hilbert–Schmidt metric and in [20] using ideas from information geometry. Even though Gaussian states are easy to characterize, the state space is not compact, which is related to the possibility of having arbitrarily squeezed states [18]. Therefore, the constructed measures are not normalizable unless some further restrictions are made such as fixing the energy of the state [18, 21].

In this work we study three very differently motivated measures on mixed Gaussian states and show analytically that when the purity is fixed, these three measures are equivalent up to a constant. For finite dimensional systems, in contrast, the equivalence does not hold [4]. The observation allows us to propose a unique measure for mixed Gaussian states with fixed purity. Finally, we apply our results in order to study typical quantum correlations in bipartite Gaussian states. As an example we show in figure 1(a) the proportion of entangled two-mode Gaussian states as a function of local purities $\mu_A, \mu_B$ for a fixed global purity $\mu$. The single hatched area in the figure was identified in [22] to be a so called coexistence region, where it is not possible to discriminate whether a state is entangled or separable by purity measurements alone. In figure 1(b) the proportion of entangled states along a cut $\mu_A = \mu_B$ through the physical domain is shown while the global purity $\mu = 0.5$. Interestingly we see that in the coexistence region the proportion of entangled states decreases linearly with $\mu_A/\mu_B$ and reaches zero in the separable region. The boundary of the physical domain is at $\mu_A/\mu_B = \sqrt{\mu} \approx 0.71$. With a measure at hand we can even go beyond the results presented in figure 1 and characterize also the amount of typical entanglement.

The outline of the article is the following. To start, we collect the main properties of Gaussian states and introduce the relevant symplectic invariants in section 2. Then, in section 3 we review different classes of quantum correlations such as entanglement and steerability and their quantitative measures for Gaussian states. In section 4 we construct three different measures for mixed Gaussian states and show that they are equivalent when the global purity is fixed. In section 5 we discuss typical quantum correlations of two mode Gaussian states. To deal with the issue of non-compactness we apply two different strategies. First the typical entanglement as a function of marginal purities and with fixed global purity, as in figure 1, is discussed. Then we examine compact subspaces of mixed Gaussian states with fixed global purity given by constraining the energy of the states. Lastly, in section 6 we conclude. We have collected many of the technical details and computations in appendices A–C.
A continuous variable (CV) system is a quantum system with Hilbert space $H_i$ isomorphic to $L^2(\mathbb{R})$. The description of an $N$-mode CV system is based on the Hilbert space $H = \bigotimes_{i=1}^{N} H_i$. In this space the canonical position- and momentum operators
\[ \hat{q}_i = \hat{a}_i + \hat{a}_i^\dagger, \quad \hat{p}_i = i(\hat{a}_i^\dagger - \hat{a}_i), \] (1)
are related to the creation and annihilation operators acting on Fock states in the usual manner, with commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ and $[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]$. We follow the standard conventions in this field leading to a commutation relation $[\hat{q}_k, \hat{p}_l] = 2i\delta_{kl}$. For notational convenience we group together the canonical operators in a single vector
\[ \hat{R} = (\hat{q}_1, \hat{p}_1, ..., \hat{q}_N, \hat{p}_N), \quad [\hat{R}_i, \hat{R}_j^\dagger] = 2i\Omega_{ij}, \quad \Omega = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \] (2)
where $\Omega$ is called the symplectic matrix. A CV quantum state given by a density matrix $\hat{\rho}$ can be represented in phase space by the Wigner function
\[ W(x) = \int_{\mathbb{R}^{2N}} \frac{d^{2N}\xi}{(2\pi)^{2N}} e^{-i\xi^T \Omega x} \text{tr} \left\{ \hat{D}(\xi) \hat{\rho} \hat{D}(\xi)^\dagger \right\}, \quad \hat{D}(\xi) = e^{iR^T \Omega \xi}, \] (3)
with $x \in \mathbb{R}^{2N}$ and $\hat{D}(\xi)$ is the Weyl- or shift operator. A state is called Gaussian iff it has a Gaussian Wigner function
\[ W(x) = \frac{1}{\pi^{N/2} \det \Sigma} \exp \left( -\frac{1}{2}(x - l)^T \Sigma^{-1} (x - l) \right). \] (4)

The quantities $l_i = \text{tr} \left\{ \hat{R} \right\}$ and $\Sigma_i = \frac{1}{2} \text{tr} \left\{ \hat{\rho} \left( \hat{R}_i \hat{R}_i^\dagger + \hat{R}_i^\dagger \hat{R}_i \right) \right\} - l_i l_i$ are called the displacement vector and the covariance matrix, respectively. The displacement of any Gaussian state.

2. Continuous variable systems and Gaussian quantum states

A continuous variable (CV) system is a quantum system with Hilbert space $H_i$ isomorphic to $L^2(\mathbb{R})$. The description of an $N$-mode CV system is based on the Hilbert space $H = \bigotimes_{i=1}^{N} H_i$. In this space the canonical position- and momentum operators
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are related to the creation and annihilation operators acting on Fock states in the usual manner, with commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ and $[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]$. We follow the standard conventions in this field leading to a commutation relation $[\hat{q}_k, \hat{p}_l] = 2i\delta_{kl}$. For notational convenience we group together the canonical operators in a single vector
\[ \hat{R} = (\hat{q}_1, \hat{p}_1, ..., \hat{q}_N, \hat{p}_N), \quad [\hat{R}_i, \hat{R}_j^\dagger] = 2i\Omega_{ij}, \quad \Omega = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \] (2)
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Figure 1. (a) Domain of separable states (double hatched), entangled states (dark blue) and coexistence of entangled and separable states (single hatched) for two mode Gaussian states. The shading indicates the proportion of entangled states as a function of marginal purities $\mu_A$ and $\mu_B$ when the global purity $\mu = 0.5$. In the unhatched white region no physical states exist. (b) Proportion of entangled states in the physical domain when $\mu_A = \mu_B$ and $\mu = 0.5$.
can be brought to zero by action of a local shift operator. Therefore $I$ contains no information about the correlations between modes and will be set to zero in the following. An $N$-mode Gaussian state is then fully characterized by its covariance matrix (CM). In fact, any real symmetric matrix $\Sigma$ that satisfies

$$\Sigma + i\Omega \geq 0,$$

is a CM of a Gaussian quantum state. Unitary operations $U_G$ generated by self-adjoint operators that are quadratic in the canonical operators preserve Gaussianity of a state. If $\Sigma$ and $I$ are the covariance matrix and displacement vector of the Gaussian state $\hat{\rho}$ and $\hat{\rho} \mapsto \hat{\rho}' = U_G \hat{\rho} U_G^\dagger$, then

$$\Sigma \mapsto \Sigma' = S \Sigma S^T, \quad I \mapsto I' = SI,$$

where $S \in \text{Sp}(2N)$ is a symplectic transformation $S^T \Omega S = \Omega$ [23]. Gaussian unitaries can be used to diagonalize a Gaussian density operator: for any covariance matrix $\Sigma$ there exists a symplectic transformation $S$ such that $S^T \Sigma S$ is diagonal, with each diagonal entry appearing twice (Williamson form [24])

$$S^T \Sigma S = D = \bigoplus_{i=1}^{N} \begin{pmatrix} \nu_i & 0 \\ 0 & \nu_i \end{pmatrix}, \quad \nu_i \geq 1.$$  

The $N$ values $\nu_i$ are called the symplectic eigenvalues of $\Sigma$. They characterize a Gaussian state up to unitary transformations, and are thus equivalent to the eigenvalues of a density operator $\hat{\rho}$. The density operator corresponding to a diagonal covariance matrix $D$ is a tensor product of thermal states. This state is pure if and only if all $\nu_i$ are equal to one, that is if it is the vacuum state. The purity of a Gaussian quantum state is given by the inverse of the product of the symplectic eigenvalues

$$\mu(\Sigma) = \frac{1}{\sqrt{\det \Sigma}} = \prod_{k=1}^{N} \frac{1}{\nu_k}.$$  

As for a symplectic matrix $\det S = 1$, the purity is invariant under symplectic transformations of $\Sigma$. Another symplectic invariant relevant for this article is the seralian [22]

$$\Delta(\Sigma) = \sum_{k=1}^{N} \nu_k^2.$$  

If the total state of a $N$-mode CV system composed of system $A$ and $B$ consisting of $N_A$ and $N_B = N - N_A$ modes is Gaussian, then also the reduced states are Gaussian and the corresponding CMs $\Sigma_A$ and $\Sigma_B$ are two diagonal blocks in the total CM

$$\Sigma = \begin{pmatrix} \Sigma_A & C \\ C^T & \Sigma_B \end{pmatrix}.$$  

The marginal purities $\mu_A = \frac{1}{\sqrt{\det \Sigma_A}}$ and $\mu_B = \frac{1}{\sqrt{\det \Sigma_B}}$ are the purities of the reduced states. A symplectic transformation is called local if $S = S_A \oplus S_B$ with $S_A \in \text{Sp}(2N_A)$, $S_B \in \text{Sp}(2N_B)$. These transformations correspond to local unitary operations and do not change non-local correlations between $A$ and $B$ such as entanglement.

Following [18], we also define the ‘energy’ of a CV state with respect to the Hamiltonian

$$\hat{H} = \sum_i \frac{1}{2} (\hat{q}_i^2 + \hat{p}_i^2).$$
Its expectation value for a Gaussian state is
\[
E = \frac{1}{2} \text{tr} \{ \hat{\rho} \hat{H} \} = \frac{1}{2} \text{tr} \{ \Sigma \} + \frac{1}{2} \mu^T.
\] (12)

One should rather think of \( E \) being proportional to the number of excitations in the state \( \hat{\rho} \).

3. Entanglement and steerability of two mode Gaussian states

In this section necessary measures to quantify entanglement and also quantum steering are introduced.

3.1. Entanglement

For \( 1 \times N \)-mode Gaussian states the positive partial transpose (PPT) or Peres–Horodecki criterion, introduced for CV systems by Simon [25] is a necessary and sufficient condition for separability. The transpose of a density matrix \( \hat{\rho} \) corresponds to a mirror reflection in phase space, which means that the sign of the momentum flips. The non-unitary partial transpose operation corresponds to an inversion of the momenta of one party only. For the \( 1 \times 1 \)-mode Gaussian case with the partial transpose applied to the second mode, this is written as \( R \mapsto \Lambda R = \tilde{R} = (q_1, p_1, q_2, -p_2)^T \) with \( \Lambda = \text{diag}(1, 1, 1, -1) \). A Gaussian state with CM \( \Sigma \) is separable iff the CM after partial transposition satisfies the bona fide condition (5) i.e. is the CM of a physical state.

The logarithmic negativity
\[
E_N = \max \left\{ 0, -\log_2 \tilde{\nu} \right\},
\] (13)

with \( 2\tilde{\nu}^2 = \tilde{\Delta} \pm \sqrt{\tilde{\Delta}^2 - 4/\mu_2^2} \) and \( \tilde{\Delta} = 2/\mu_A + 2/\mu_B - \Delta \) is a quantitative measure for entanglement as it measures the degree of violation of the PPT criterion [26].

3.2. Steerability

In the hierarchy of quantum correlations, steering is a distinct class for general quantum states. It is stronger than entanglement but weaker than non-locality, and it is inherently asymmetric [27, 28]. In a typical steering scenario there are two parties, Alice and Bob, who share a quantum state. Alice has some fixed set of measurements, described by a set of positive operator valued measures that she can perform locally and Bob can do local state tomography. If the state is \( A \to B \) steerable, Alice can then, by measuring her local observables, steer Bobs state to such state assemblages that they cannot be described by any local hidden state model. Similarly, the state is \( B \to A \) steerable if the roles of Bob and Alice are interchanged.

An operational criterion for Gaussian \( A \to B \) steering is comparable to the bona fide condition in equation (5), requires only Gaussian measurements to be made, and leads to a feasible expression in terms of the covariance matrix of the joint state [27]. A \( 1 \times 1 \)-mode Gaussian state is \( A \to B \) steerable iff \( \mu > \mu_A \) is satisfied [29]. Again, by interchanging the roles of Alice and Bob, the criterion for \( B \to A \) steerability is obtained. We call a state steerable if it is \( A \to B \) or \( B \to A \) steerable.

A quantitative measure for \( A \to B \) steering is [29]
\[
G^{A \to B}(\Sigma) = \max \left\{ 0, \ln \left( \frac{\mu}{\mu_A} \right) \right\}.
\] (14)
as well as $G^{B\rightarrow A}$ will never exceed the logarithmic negativity (13). The steering measure (14) quantifies the violation of the $1 \times 1$-mode Gaussian $A \rightarrow B$ steering criteria. We call the maximum of $G^{A\rightarrow B} (\Sigma)$ and $G^{B\rightarrow A} (\Sigma)$ the steerability

$$G(\Sigma) = \max \left\{ 0, \ln \left( \frac{\mu}{\mu_A} \right), \ln \left( \frac{\mu}{\mu_B} \right) \right\}.$$  \hspace{1cm} (15)

4. Invariant measures for mixed Gaussian states

In the space of pure quantum states there exists a unique notion of volume given by the invariant measure of the unitary group. In particular any pure quantum state can be written as a unitary transformation of a fixed pure state $|\psi\rangle = U|\psi_0\rangle$. Thus the invariant measure (Haar measure) on the unitary group gives a non-biased measure for pure quantum states [9]. The same holds of course for pure Gaussian states: since the covariance matrix of a pure Gaussian state can be written as $\Sigma = S^T S$, the invariant measure on the symplectic group $Sp(2N)$ is a unique measure in the space of pure Gaussian quantum states. This has been studied in [18].

Considering mixed states, there no longer exists a unique invariant measure, since there is the additional non-unitary freedom in the eigenvalues of the density matrix [30], or equivalently the symplectic eigenvalues in the case of Gaussian states. The volume element of any invariant measure for mixed Gaussian states can then be written as [19]

$$dV = P(\nu_1, ..., \nu_N) d\mu_N(S) \prod_{i=1}^N d\nu_i$$  \hspace{1cm} (16)

where $d\mu_N(S)$ denotes the invariant measure on the symplectic group $Sp(2N)$ and $P$ is a probability density of the eigenvalues $\nu_i$. In the following we compare three very differently motivated measures by exploiting this decomposition.

4.1. Comparison of invariant measures

4.1.1. Hilbert–Schmidt. The Hilbert–Schmidt measure is a natural and easily computable measure on the space of operators acting on $\mathcal{H}$ induced by the unitarily invariant Hilbert–Schmidt metric $d_{\text{HS}}^2 = \text{tr}(d\rho^2)$. Confining this to the manifold of Gaussian quantum states gives a metric on the space of admissible covariance matrices $\Sigma$ [19]

$$d_{\text{HS}}^2 = \frac{1}{16\sqrt{\text{det} \Sigma}} \left( \text{tr}(\Sigma^{-1} d\Sigma) \right)^2 + 2\text{tr}((\Sigma^{-1} d\Sigma)^2).$$  \hspace{1cm} (17)

Expressing covariance matrices by the symplectic eigenvalue decomposition $\Sigma = S^T D S$ the volume element of the induced measure can, up to a constant, be written as

$$dV_{\text{HS}} = P_{\text{HS}}(\nu_1, ..., \nu_N) d\mu_N(S) \prod_{i=1}^N d\nu_i,$$  \hspace{1cm} (18)

$$P_{\text{HS}}(\nu_1, ..., \nu_N) = \left( \prod_{k=1}^N \nu_k \right)^{-N(N+\frac{1}{2})+1} \prod_{l>m=1}^N (\nu_l^2 - \nu_m^2)^2.$$  \hspace{1cm} (19)

Detailed computations and the derivation of equations (18) and (19) can be found in [19].
4.1.2. Fisher–Rao. The fact that Gaussian CV quantum states have a positive Wigner function everywhere allows to borrow ideas from classical information geometry and apply them, at least formally, to quantum systems in the Gaussian domain. One such idea is to use the Fisher–Rao metric, which is a metric in the space of probability distributions [31], as a metric for the space of Gaussian quantum states. We would like to stress that Gaussian quantum states are not classical states, and the similarity of formalism between classical and quantum phase space distributions should not be pushed too far, since quantum mechanics is a fundamentally non-commutative theory.

Following [20], the Fisher–Rao metric can be expressed with covariance matrices as

\[
d_{\text{FR}}^2 = \frac{1}{2} \text{tr}((\Sigma^{-1} d\Sigma)^2).
\]

(20)

Note that a similar term appears also in equation (17). Thus to express the measure in terms of symplectic eigenvalues, large parts of the results from [19] for the derivation of the Hilbert–Schmidt measure can be utilized. The resulting expression is

\[
dV_{\text{FR}} = P_{\text{FR}}(\nu_1, \ldots, \nu_N) \prod_{i=1}^{N} d\nu_i,
\]

(21)

\[
P_{\text{FR}}(\nu_1, \ldots, \nu_N) = \left( \prod_{k=1}^{N} \nu_k \right)^{-2N+1} \prod_{\ell > m=1}^{N} (\nu_{\ell}^2 - \nu_{m}^2)^2.
\]

(22)

To be more self contained, we have included the derivation of equations (21) and (22) in appendix A.

4.1.3. Reduced states of pure Gaussian states. A practical scheme to sample \(N\)-dimensional mixed states from the Hilbert–Schmidt measure is to sample \(N^2\)-dimensional pure states from the Haar measure and partially trace over \(N\) degrees of freedom. This method has been applied to qubits and finite dimensional systems, see for example [9, 32]. Here, we consider sampling pure Gaussian states from the Haar measure, introduced in [18], with doubled mode number \(2N\) and partially trace out \(N\) modes. In order to obtain a useful representation for this measure, we write the CV of the \(2N\) mode pure state in the form \(\Sigma = S^T \sigma S\), with \(S = S_A \oplus S_B\), where \(S_A, S_B \in \text{Sp}(2N)\) and [13]

\[
\sigma = \begin{pmatrix} D & C \\ C & D \end{pmatrix}, \quad D = \bigoplus_{i=1}^{N} \begin{pmatrix} \nu_i & 0 \\ 0 & \nu_i \end{pmatrix}, \quad C = \bigoplus_{i=1}^{N} \begin{pmatrix} \sqrt{\nu_i^2 - 1} & 0 \\ 0 & -\sqrt{\nu_i^2 - 1} \end{pmatrix}.
\]

(23)

The symplectic eigenvalues of the \(N\)-mode subsystems \(A\) and \(B\) are denoted with \(\nu_i\). The volume element can be found in [18] and it reads

\[
d\mu_{2N}(S) = P_{2N}(\nu_1, \ldots, \nu_N) \prod_{i=1}^{N} d\nu_i,
\]

\[
P_{2N}(\nu_1, \ldots, \nu_N) = \left( \prod_{k=1}^{N} \nu_k \right)^2 \prod_{\ell > m=1}^{N} (\nu_{\ell}^2 - \nu_{m}^2)^2.
\]

(24)

Tracing out \(N\) modes then just corresponds to integrating over one of the local symplectic groups giving rise merely to a constant factor. The density of symplectic eigenvalues \(P_{2N}\) then defines an invariant measure for mixed \(N\)-mode Gaussian states.

7
4.2. Unique fixed purity measure

We observe that the probability densities $P_{\text{HS}}$, $P_{\text{FR}}$ and $P_{\text{2N}}$ over the symplectic eigenvalues $\nu_1, \ldots, \nu_N$ only differ by a prefactor, which is a power of the purity. Therefore, if we consider Gaussian states with a fixed purity then all of the three measures are identical up to a constant. Even though all measures are invariant measures this statement is non-trivial for $N > 1$ and purities $\mu < 1$. The observation strongly suggests to consider Gaussian states of fixed purity and the measure

$$dV_\mu = P_\mu(\nu_1, \ldots, \nu_N) d\mu_\text{2N}(S) \prod_{i=1}^{N} d\nu_i,$$

$$P_\mu(\nu_1, \ldots, \nu_N) = \delta\left(\mu - \prod_{k=1}^{N} \frac{1}{\nu_k}\right) \prod_{l>m=1}^{N} (\nu_l^2 - \nu_m^2)^2. \quad (25)$$

We have shown that when the global purity is fixed, three very different measures on the set of mixed Gaussian states are equivalent up to a constant. Similar studies for finite dimensional systems were done in [4]. There, the numerical data sampled from different invariant measures conditioned on purity showed close but not perfect agreement, in contrast to our analytical findings for Gaussian states.

5. Typical quantum correlations of two mode Gaussian states

The proposed volume element (25) allows us to study in detail the typical correlation properties of two mode Gaussian states. We now attempt to compute the typical values of such quantum correlations. As mentioned, the three measures will provide equivalent statistical information for fixed purity. Without loss of generality we choose to construct the volume element from the Hilbert–Schmidt measure (B.6). Up to local symplectic transformations, irrelevant for non-local correlations, any two mode Gaussian state is completely characterized by the purity $\mu$ of the state, the two marginal purities $\mu_A, \mu_B$ of the one mode subsystems, and the seralian $\Delta$ [11, 13]. It is therefore advantageous to express the Hilbert–Schmidt volume element in these variables. Following the calculations in appendix B we find the simple expression

$$dV_\text{HS} = \sqrt{\frac{3}{512}} \frac{\mu^7}{\mu_A \mu_B} d\mu_A d\mu_B d\mu d\Delta d\mu_1(S_A) d\mu_1(S_B), \quad (26)$$

where $d\mu_1(S_A) d\mu_1(S_B)$ is the invariant measure of the local symplectic transformations. Even after constraining the purity to a fixed value when computing the volume of Gaussian states there appear two divergences related to arbitrarily strong squeezing. Firstly, the volume of the non-compact local symplectic group is infinite due to single-mode squeezing. Secondly, unbounded two-mode squeezing allows for arbitrarily small marginal purities. To circumvent this problem, further restrictions on the states considered have to be made. We propose two different strategies: either fixing the marginal purities or fixing the energy of the states.

5.1. Purity constrained typical quantum correlations

The global and marginal purities of Gaussian states are experimentally accessible [16, 17], and knowledge thereof may already be sufficient to decide whether a mixed Gaussian state is entangled or not [11, 13]. With a measure at hand we can now quantify the typical amount of
entanglement expected for given purities. This way we can also characterize a region where purity measurements alone cannot determine if a state is entangled or not, further referred to as the coexistence region. In particular, the typical value of, for instance, the logarithmic negativity $E_N$ for Gaussian states of fixed purities $\mu, \mu_A, \mu_B$ is given by

$$\langle E_N \rangle_{\mu, \mu_A, \mu_B} = \frac{\int_{\Sigma + i \Omega \geq 0} d\Delta d\mu_1 (S_A) d\mu_1 (S_B) E_N (\mu, \mu_A, \mu_B, \Delta)}{\int_{\Sigma + i \Omega \geq 0} d\Delta d\mu_1 (S_A) d\mu_1 (S_B)},$$

(27)

The bona fide condition $\Sigma + i \Omega \geq 0$ indicates that the domain of integration is the set of admissible covariance matrices belonging to physical Gaussian states. Note that divergent contributions corresponding to the volume of the local symplectic group cancel because $E_N$ is a local symplectic invariant. Thus, even though the volumes of the non-compact subspaces are infinite, their ratio is still finite so that typical values such as (27) can be computed. $\Delta_{\text{min}}/\Delta_{\text{max}}$ are the limits for the seralian resulting from the bona fide condition. The integral can be solved analytically. In figure 2 we provide the results for three different values of $\mu$. Naturally high purities and small marginal purities correspond to a large amount of entanglement. In the double hatched region only separable states exist and thus the typical value of $E_N$ is zero. The single hatched area is the before mentioned coexistence region, and the shaded area without hatching corresponds to values of marginal purities where all states are entangled. For marginal purity values in the white and unhatched area no physical states exist. In figure 3 we show a cut along $\mu_A = \mu_B$ trough the domain of physical states for three different values of $\mu$. In each case the the average entanglement decreases in non-linear fashion for increasing marginal purity. As the marginal purity increases, average entanglement decreases to zero in the separable domain and ceases to be well defined for marginal purities that are in the unphysical domain $\mu_A/\mu_B > \sqrt{\pi}$. We stress that if we would plot again the proportion of entangled states for a fixed global purity as in figure 1(b) we would see linear behavior for any linear cut trough the coexistence region.

We do not investigate typical steering under the purity constraint, since the local and global purities alone already completely determine the steerability of a Gaussian state [29].

5.2. Energy constrained typical quantum correlations

While the volumes of purity constrained subspaces of the last section were still infinite themselves (while their ratio was finite) we here consider finite volumes. A physically well motivated constraint that leads to a compact domain is to consider Gaussian state of fixed purity and fixed energy. An energy restriction for pure states has also been introduced in [18, 21]. The volume element we consider is

$$dV_{\mu, E} = \delta \left( E - \frac{1}{2} \text{tr} \Sigma \right) \frac{\sqrt{3}}{512} \frac{\mu^2}{\mu_A \mu_B} d\mu_A d\mu_B d\Delta d\mu_1 (S_A) d\mu_1 (S_B),$$

(28)

with $E$ from (12). This is more involved than fixing marginal purities since the energy of a Gaussian state is not a local symplectic invariant, i.e. it depends not only on the marginal purities but also on the amount of local squeezing. Of particular interest are again mean values of functions depending on the local symplectic invariants $\mu, \mu_A, \mu_B$ and $\Delta$, such as the logarithmic negativity and the steerability.
The relevant integrals can be simplified by carrying out the integration over the local symplectic groups, see appendix C. In particular we obtain

$$\langle f(\mu, \mu_A, \mu_B, \Delta) \rangle |_{\mu,E} = \frac{\int_{\Sigma^+ + \Omega \geq 0} dV_{\mu,E} f(\mu, \mu_A, \mu_B, \Delta)}{\int_{\Sigma^+ + \Omega \geq 0} dV_{\mu,E}}. \quad (29)$$

The relevant integrals can be simplified by carrying out the integration over the local symplectic groups, see appendix C. In particular we obtain

$$\int_{\Sigma^+ + \Omega \geq 0} dV_{\mu,E} f(\mu, \mu_A, \mu_B, \Delta)$$

$$= K'' \int_{\sigma + \Omega \geq 0} d\mu_B d\Delta \frac{\mu^3}{\mu_A^2 \mu_B^2} \left( E - \left( \frac{1}{\mu_A} + \frac{1}{\mu_B} \right) \right) \Theta \left( E - \left( \frac{1}{\mu_A} + \frac{1}{\mu_B} \right) \right) f(\mu, \mu_A, \mu_B, \Delta) \quad (30)$$

where $K''$ is a constant. To explicitly compute the integrals we first analytically solve the $\Delta$-integral and then treat the remaining two-dimensional integral over the marginal purities numerically with an iterative and adaptive Monte Carlo method [33, 34].
In figure 4 the proportion of entangled and steerable states, as well as typical values for the logarithmic negativity and steerability, are displayed for four different energies as functions of the purity. Each curve starts at a different point which is the minimal possible purity for the given energy. Higher values of the purity allow for energy to be distributed to squeezing which can generate entanglement between the two modes. As a result all curves are monotonically increasing. Since all steerable states are entangled the proportion of steerable states is always smaller than the proportion of entangled states.

The results for pure states $\mu = 1$ (big dots) are computed using the invariant measure on the symplectic group, from [18]. Almost all pure states are entangled and steerable, thus the curves in figures 4(a) and (c) reach one at $\mu = 1$.

6. Conclusions and outlook

In this article we have shown that three different unbiased measures for mixed Gaussian states are equivalent when constrained on the states with fixed purity. This result is somewhat surprising. The rigorous equivalence observed in the Gaussian case is at variance with numerical results obtained for finite dimensional systems in [4], where different measures were close to each other but not equivalent. We then proposed a unique unbiased measure for fixed purity Gaussian states.
With this result a volume element suitable to compute typical correlation properties of two mode \((1 \times 1)\) mixed Gaussian states is constructed. We first investigated whether a typical state with given global and marginal purities is entangled or separable and then quantified the typical amount of entanglement. In this situation a region of coexistence exists, where it is not possible to discriminate whether a state is entangled or separable by purity measurements alone. Using the unique invariant measure we were able to compute the typical entanglement in the whole state space, allowing us to also characterize the coexistence region.

A second way to resolve the problems arising from integrating over a non-compact state space is to consider compact subspaces by fixing the energy of the state. This is a physically well motivated restriction that has been suggested by others before. For high purities our results converge to the typical pure state values in the Haar invariant measure of the symplectic group.

In the future, we will use our results to study generic properties of Gaussian channels via the Choi–Jamiolkowski isomorphism \([35, 36]\) and via probe states with limited resources \([37]\) and compare the two approaches. Also our results could be used for channel discrimination tasks that require optimization over the probe states \([38]\), or even to characterize the distinguishability of CV channels. In the probe state approach, there is naturally a limited amount of resources available for the experimenter, such as states with limited energy. This underlines the usefulness of the energy constraint used in this work. On the other hand, the constraints on the global and local purities, even if the volumes are infinite, lead to rather generic and elegant results for typical values of symplectic invariants.

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Appendix A. Volume element in Fisher–Rao metric

We use the strategy of \([19]\) to derive the Fisher–Rao volume element explicitly for the symplectic eigenvalue decomposition \(\Sigma = S^TDS\). The infinitesimal shift in \(\Sigma\) can be written as \(d\Sigma = S^T(dD + dHD + DdH)S\), where \(H\) is a special Hamiltonian matrix as defined in \([19]\)

\[
H = \begin{pmatrix}
H^{(11)} & \cdots & H^{(1N)} \\
\vdots & \ddots & \vdots \\
H^{(N1)} & \cdots & H^{(NN)}
\end{pmatrix}, \\
H^{(ij)} = \begin{pmatrix} X_{ij} & Y_{ij} \\ Z_{ij} & -X_{ji} \end{pmatrix},
\]

(A.1)

with \(N \times N\) matrices \(X, Y = Y^T, Z = Z^T\) and \(Z\) having vanishing diagonal \(Z_{ii} = 0^1\). The line element of the Fisher Rao metric for Gaussian states is

\[
ds_{FR}^2 = \frac{1}{2} \text{tr}((\Sigma^{-1}d\Sigma)^2) = \frac{1}{2} \text{tr}(D^{-1}dD)^2 + \text{tr}(dH)^2 + \text{tr}(D^{-1}dH^TDdH).
\]

(A.2)

The single terms give explicitly

\(^1\)Note the difference in ordering of position and momentum compared to the reference.
\[ \text{tr}(D^{-1}dD)^2 = 2 \sum_{i=1}^{N} \frac{d\nu_i^2}{\nu_i^2}, \quad (A.3) \]

\[ \text{tr}(dH)^2 = 2 \sum_{i,j=1}^{N} dX_i dX_j + 2 \sum_{i,j=1}^{N} (dY^j_i dZ^j_i + dZ^j_i dY^j_i), \quad (A.4) \]

\[ \text{tr}(dHD^{-1}dH^T D) = \sum_{i,j=1}^{N} dX^j_i \left( \frac{\nu^j_i + \nu^j_i}{\nu^j_i} \right) + \sum_{i,j=1}^{N} (dY^j_i dZ^j_i + dZ^j_i dY^j_i) \]

\[ + \sum_{i=1}^{N} dY^2_i. \quad (A.5) \]

Overall, the distance element is

\[
d_{\text{FR}}^2 = 2 \left( \begin{array}{c} d\nu_1 \\ \vdots \\ d\nu_N \\ \nu_N^{-2} \\ \nu_N^{-2} \\ \vdots \\ \nu_N^{-2} \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} \nu_1^{-2} \\ \vdots \\ \nu_N^{-2} \end{array} \right) \left( \begin{array}{c} d\nu_1 \\ \vdots \\ d\nu_N \\ dX_{NN} \end{array} \right)

+ \sum_{i>j=1}^{N} \left( dX^j_i \right)^T \left( \frac{\nu^j_i + \nu^j_i}{\nu^j_i} \right) \left( \frac{\nu^j_i + \nu^j_i}{\nu^j_i} \right) \left( \begin{array}{c} dX^j_i \\ dX^j_j \end{array} \right) + \left( \begin{array}{c} dY^j_i \\ dY^j_j \end{array} \right)

+ \sum_{i>j=1}^{N} \left( dY^j_i \right)^T \left( \frac{\nu^j_i + \nu^j_i}{\nu^j_i} \right) \left( \frac{\nu^j_i + \nu^j_i}{\nu^j_i} \right) \left( \begin{array}{c} dY^j_i \\ dY^j_j \end{array} \right). \quad (A.6) \]

One can read the explicit form of the metric tensor. The measure \( \sqrt{\det g} \) turns out to be

\[
\sqrt{\det g} = \sqrt{\det \left( \frac{\delta_{ij}}{\nu_i \nu_j} \right) \prod_{l>m=1}^{N} \left( \frac{\nu_l \nu_m}{\nu_l \nu_m} \right)^2 - 4}

= \sqrt{\frac{1}{\prod_{k=1}^{N} \nu_k^2} \det \left( \frac{\delta_{ij}}{\nu_i \nu_j} \right) \prod_{k=1}^{N} \frac{1}{\nu_k} \prod_{l>m=1}^{N} \left( \nu_l^2 \nu_m^2 \right)^2}

= \left( \prod_{k=1}^{N} \frac{1}{\nu_k} \right)^{2N-1} \prod_{l>m=1}^{N} \left( \nu_l^2 - \nu_m^2 \right)^2. \quad (A.7) \]

**Appendix B. Two-mode Hilbert–Schmidt volume element**

Any two mode covariance matrix can be written in the standard form [13]

\[
\Sigma = S \sigma S^T, \quad \sigma = \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}, \quad (B.1) \]
with the local symplectic transformation $S = S_A \oplus S_B$, and $S_A, S_B \in \text{Sp}(2)$. Thus we may write
\begin{equation}
    d\Sigma = S^T (d\sigma + dH^T \sigma + \sigma dH) S,
\end{equation}
where $1 + dH$ is an infinitesimal local symplectic transformation, i.e. $H$ is a Hamiltonian matrix with
\begin{equation}
    H = \begin{pmatrix} x_A & y_A \\ z_A & -x_A \end{pmatrix} \oplus \begin{pmatrix} x_B & y_B \\ z_B & -x_B \end{pmatrix}.
\end{equation}
We can now insert this parametrization in the expression for the Hilbert–Schmidt line element
\begin{equation}
    (17)
\end{equation}
to compute the metric tensor. One obtains the volume element
\begin{equation}
    dV_{HS} = \sqrt{3} \frac{a^2 b^2 (c_+^2 - c_-^2)}{256 (c_+^2 - ab)^2 (c_-^2 - ab)^2} da db dc_+ dc_- dx_A dy_A dz_A dz_B.
\end{equation}
The invariant measure on the symplectic group is given by
\begin{equation}
    d\mu_1(S_A) = d\mu_1(S_B) = d\mu_A d\mu_B d\mu d\Delta d\mu_1(S_A) d\mu_1(S_B).
\end{equation}

### Appendix C. Energy constraint

The single mode symplectic operation $S_A \in \text{Sp}(2)$ can be written as
\begin{equation}
    S_A = O' W O, \quad O', O \in \text{SO}(2), \quad W = \begin{pmatrix} w & 0 \\ 0 & 1/w \end{pmatrix}, \quad w \geq 1.
\end{equation}
Using this, the invariant measure over the symplectic group can be decomposed in a compact part corresponding to rotations and a non-compact part corresponding to single mode squeezing \cite{18}
\begin{equation}
    d\mu_1(S_A) = K d\lambda_A d\mu_1(O) d\mu_1(O'), \quad \lambda_A = \frac{1}{2} (w^2 + 1/w^2),
\end{equation}
where $d\mu_1(O)$ is an invariant measure over $\text{SO}(2)$ and $K$ is a normalization factor. The energy of a single mode covariance matrix, written as $\Sigma_{A/B} = \mu_A^{-1} S_A^T S_A$ is then
\begin{equation}
    E_A = \frac{1}{2} \text{tr} \Sigma_A = \frac{\lambda_A}{\mu_A}.
\end{equation}
The energy of the two mode CM $\Sigma$ is given by the sum of the energies of the single mode subsystems $A$ and $B$
\begin{equation}
    E = \frac{1}{2} \text{tr} \Sigma = \frac{\lambda_A}{\mu_A} + \frac{\lambda_B}{\mu_B}.
\end{equation}
We can carry out the integrals over the local symplectic groups respecting an energy constraint
\[
\int_{\Sigma + \Omega \geq 0} dV_{\mu, E} (\mu, \mu_A, \mu_B, \Delta) = K' \int_{\sigma + \Omega \geq 0} d\mu_A d\mu_B d\Delta \frac{\mu^2}{\mu_A^2 \mu_B^2} f(\mu, \mu_A, \mu_B, \Delta) \\
\times \int_{1}^{\infty} d\lambda_A d\lambda_B \delta \left( E - \left( \frac{\lambda_A}{\mu_A} + \frac{\lambda_B}{\mu_B} \right) \right) \int d\mu (O_A) d\mu (O'_A) d\mu (O_B) d\mu (O'_B) \\
= K'' \int_{\sigma + \Omega \geq 0} d\mu_A d\mu_B d\Delta \frac{\mu^2}{\mu_A^2 \mu_B^2} f(\mu, \mu_A, \mu_B, \Delta) \int_{1}^{\infty} d\lambda_A d\lambda_B \delta \left( E - \left( \frac{\lambda_A}{\mu_A} + \frac{\lambda_B}{\mu_B} \right) \right).
\]

Computing the \( \lambda \)-integrals over the delta function gives

\[
\int_{\Sigma + \Omega \geq 0} dV_{\mu, E} (\mu, \mu_A, \mu_B, \Delta) = K'' \int_{\sigma + \Omega \geq 0} d\mu_A d\mu_B d\Delta \frac{\mu^2}{\mu_A^2 \mu_B^2} \left( E - \left( \frac{1}{\mu_A} + \frac{1}{\mu_B} \right) \right) \Theta \left( E - \left( \frac{1}{\mu_A} + \frac{1}{\mu_B} \right) \right) f(\mu, \mu_A, \mu_B, \Delta).
\]

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