DEPENDENT DELAY STABILITY CHARACTERIZATION FOR A POLYNOMIAL T-S CARBON DIOXIDE MODEL

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ABSTRACT. By extending some linear time delay systems stability techniques, this paper, focuses on continuous time delay nonlinear systems (TDNS) dependent delay stability conditions. First, by using the Takagi Sugeno Fuzzy Modeling, a novel relaxed dependent delay stability conditions involving uncommon free matrices, are addressed in Linear Matrix Inequalities (LMI). Then, as application a Nonlinear Carbon Dioxide Model is used and rewritten by a change of coordinate to the interior equilibrium point. Next, by using the non-linearity sector method the model is transformed to a corresponding Fuzzy Takagi Sugeno (TS) multi-model. Also, the maximum delay margin to which the model is stable, is identified. Finally, to prove the analytic results a numerical simulation is also performed and compared to other methods.

1. Introduction. During these last years the greenhouse effect and the pollution have been one of the most awful outcomes of the carbon dioxide increase, due to the forest deforestation and human activities. In addition, the intensification interest for the environment respect, leads governments and researchers to investigate the high level of $CO_2$. Most studies has concluded a direct link between 3 parameters, $CO_2$ level, human population and forest biomass [1, 4, 9, 14, 25, 37]. In particular, [37] represents a first mathematical model, where its stability conditions is discussed and how severe deforestation may lead to destabilize environment. Also, in [25] a new model is proposed which involve a human population variable, where it is shown that the proposed model is aimed to unstableness whenever the deforestation rate exceeds a critical value. In [26] a variant model of [25], exposed the impact of the delayed reforestation program, where the need to develop a convenient way to study the nonlinear mathematical models stated in the cited references and others.

The aim of this paper is to study the stability conditions of a delayed TS fuzzy model extracted from a carbon dioxide nonlinear system to study its wide-ranging behavior, by taking into account all the cited above parameters.
The fuzzy logic origin lies in the fuzzy set theory developed by Lotfi Zadeh in 1965 [39]. It was used later to model systems based on human reasoning [20]. In the literature, we can distinguish two fuzzy representation categories, “Mamdani” type, based knowledge and “Takagi Sugeno” type, based model. The general structure of a fuzzy model is composed of rules set in the form “If Premise then Consequence”. The utmost difference between Mamdani and Takagi Sugeno presentation is inside the consequence part of the “If-Then” rules. As the Consequence part of TS model uses mathematical acronyms, that of Mamdani uses a linguistic abbreviation, more comprehensive and closer to human being decision making. In addition, more the system complexity is increasing, less the fuzzy model Mamdani type use is interesting and might be hard to implement, due to the huge number of interactions to be upkeep in each fuzzy rule. Also, it is proven that Takagi-Sugeno model is a Mamdani Type generalization [20, 30].

Also, the great benefit of a fuzzy representation remains in the transformation of a given nonlinear system to a several linear time invariant sub-models, using extremum values of the so-called premise variables, weighting by the membership functions. In addition, due to its equivalency, it’s more convenient to study a given non-linear system with its correspondent Takagi-Sugeno model [24, 36]. Another reason is the problematic use of the nonlinear model in both control and observation [2, 24]. But, T-S models are configurable following the designer strategy. And, the stability conditions, through TS modeling, give a balance between complexity calculation and conservativeness. At last, the TS conditions may sometimes have a huge numerical calculation, but with good premise variables choice, the designers may decrease significantly the numerical resolution [31].

The document is organized as follows. In the second section we begin with formulating the problem and stating some preliminaries. In section 3 we present the developed technique. As application a $CO_2$ nonlinear model is stated and transformed to its equivalent Takagi-Sugeno model in Section 4. In Section 5, numerical results and discussions are presented. At the end, the paper is finished with some concluding remarks and perspectives.

2. Problem statement and preliminaries.

2.1. Problem statement. The T-S fuzzy modeling method is introduced to give an equivalent way to present a smooth nonlinear system. The main idea is to partition the nonlinear system into several local linear subsystems. And, the overall nonlinear behavior of the system can be obtained by blending these subsystems through nonlinear fuzzy membership functions $h_i$. In that way, we can inspire from analytical methods used for linear systems, to extend some to nonlinear systems.

In the following, let’s consider an unforced nonlinear system with state-delay which could be presented by the following Takagi-Sugeno Fuzzy system:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(A_i x(t) + A_d x(t - \tau)), \quad \forall \ t \geq 0$$

$$x(t) = \phi(t), \quad \forall \ t \in [-\tau, 0]$$

(1)

Where $\phi(t)$ is the initial condition, $x(t) \in \mathbb{R}^n$ is the state variable, $r$ is rules number and $\tau$ is a varying time delay respecting the conditions below:

$$\tau \in [0, \bar{\tau}]$$

$$\dot{\tau} \in [0, \bar{d}] \subset [0, 1]$$

(2)
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and:

$$0 \leq h_i \leq 1, \forall i \in 1...r$$

$$\sum_{i=1}^{r} h_i = 1$$  \hspace{1cm} (3)

2.2. Preliminaries. The presence of the delay, an unstableness’s factor, lead to a complex global stability for the delayed nonlinear systems, even more complicated than the delay free case, and it shares the same use of sufficient conditions with the free delay systems which represents a conservative way to analyze a system stability [6, 36], however by optimizing the upper bound terms researchers try to improve results and minimize this conservativeness effect. Also, the stability conditions are split to delay dependent [13, 23, 36] or delay independent techniques [13, 18, 36]. In case of delay free systems the stability relies on Lyapunov theory, while nonlinear TDS use in most cases, either Lyapunov–Krasovskii functional [13, 33] or Lyapunov–Razumikhin method [5, 13].

Independent delay conditions. Independent delay conditions are the simplest way to investigate a system stability, it relies on resolving LMIs without involving delay terms, which means theoretically, if such conditions exists, that the system will be stable for any delay value [13, 18, 36], but practically such conditions are hard and sometimes even impossible to find.

Dependent delay conditions. In recent years several researchers diagnosed, the delay dependent stability conditions to overcome the conservativeness of the independent delay conditions, but also to specify the maximum delay margin to which the nonlinear system with delay will remain stable. In most cases researchers relies on model transformation using the Leibniz–Newton formula on the term $x_t(\tau)$, adding slack variables, developing some cross-term bounding techniques (Jensen’s Inequality, Wirtinger’s Inequality, Bassel and Legendre Inequalities,...)[13, 32, 33].

Remark 1. In the following equations (*) denotes the off-diagonal block element’s transpose.

Theorem 2.1. [29] Assuming that an uncertain time-invariant delay lies in $[0, \bar{h}]$, i.e., $h \in [0, \bar{h}]$. Then if there exist matrices $P > 0, Q > 0, X > 0$, and $Z$ such that:

$$\begin{pmatrix}
\Lambda & -Z^T A_d & -Z^T A_d^T X & \bar{h}(Z^T + P) \\
* & -Q & A^T_d A^T_d X & 0 \\
* & * & -X & 0 \\
* & * & * & -X
\end{pmatrix} < 0$$  \hspace{1cm} (4)

Then the unforced linear TDS is asymptotically stable.

Where $\Lambda = (A + A_d)T P + P(A + A_d) + Z^T A_d + A^T_d Z + Q$

Lemma 2.2. [8] For any positive matrix $Z > 0$, and firing probability $h_i$ we have the inequality:

$$\chi^T \sum_{j=1}^{r} \sum_{i=1}^{r} h_i h_j \begin{pmatrix} A^T_d ZA_i & A^T_d ZA_j \\ A^T_d ZA_i & A^T_d ZA_j \end{pmatrix} \chi \leq \chi^T \sum_{i=1}^{r} h_i \begin{pmatrix} A^T_d ZA_i & A^T_d ZA_d \\ A^T_d ZA_i & A^T_d ZA_d \end{pmatrix} \chi$$  \hspace{1cm} (5)

Where $\chi^T = [x^T \; x_1^T]$
3. Main results. The aim of this section is to extend the [29]'s work developed for linear time delay systems to Takagi Sugeno Fuzzy systems. Besides, substituting common matrices by uncommon ones as a novelty to get relaxed conditions.

**Theorem 3.1.** If there exist matrices $P > 0, Q > 0, Z_i > 0$, and $S_i$ such that:

\[
\begin{pmatrix}
P(A_i + A_{d,i}) + (A_i^T + A_{d,i}^T)P + S_i^T A_{d,i} + A_{d,i}^T S_i + Q & -S_i^T A_{d,i} & \bar{\tau}(S_i^T + P) & A_{d,i}^T A_d^T Z \\
-S_i^T A_{d,i} & -(1-d)Q & 0 & A_{d,i}^T A_d^T Z \\
\bar{\tau}(S_i + P) & 0 & -Z_i & 0 \\
ZA_dA_i & ZA_dA_i & 0 & -Z
\end{pmatrix} < 0
\]

(6)

For every:

\[
i \in 1...r \\
\tau \in [0, \bar{\tau}] \\
\bar{\tau} \in [0, d] \subset [0, 1]
\]

(7)

Where:

\[
A_d^T = (A_{d,1}^T \ldots A_{d,r}^T) \\
Z = \text{diag}\{Z_1, \ldots, Z_r\} \text{ a diagonal block matrix}
\]

(8)

Then the unforced TS-TDS Eq.1 is asymptotically stable for any time-delay $\tau$ verifying Eq.7.

**Proof.** Let’s take the unforced TS-TDS

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(A_i x(t) + A_{d,i} x(t-\tau)), \quad \forall \ t \geq 0
\]

(9)

\[
x(t) = \phi(t), \quad \forall \ t \in [-\tau, 0]
\]

And a Lyapunov Krasovskii Functional candidate:

\[
V(x(t - \tau)) = V_1 + V_2 + V_3
\]

(10)

Where:

\[
V_1 = x^T(t)P x(t)
\]

\[
V_2 = \sum_{k=1}^{r} \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{x}(\alpha) A_{d,i}^T X_k A_{d,i} \dot{x}(\alpha) d\alpha d\beta
\]

(11)

\[
V_3 = \int_{t-\tau}^{t} x^T(s)Q x(s) ds
\]

And $\tau \in [0, \bar{\tau}]$

It can be shown that : $\sigma_1 \parallel x(t) \parallel \leq V \leq \sigma_2 \parallel x_t \parallel_w$ Where: $\parallel x_t \parallel_w = \max_{0 \leq \tau \leq \bar{\tau}} (\parallel x(t-\tau) \parallel + \sqrt{\int_{-\tau}^{0} |\dot{x}(s)|^2 ds})$ and $\sigma_1, \sigma_2$ are two positive real scalars [8, 13], which yields to:

\[
\dot{V}_1 = 2x^T(t)P \dot{x}(t)
\]

\[
= \sum_{i=1}^{r} h_i(2x^T(t)PA_i x(t) + 2x^T(t)PA_{d,i} x(t-\tau))
\]

(12)
Next by using the Leibniz-Newton Formula [29, 13]:

\[ x(t - \tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s) ds \]  

(13)

The Equation Eq.12 could be rewritten as:

\[ \dot{V}_1 = \sum_{i=1}^{r} h_i (x^T(t)(P(A_i + A_{d_i}) + (A^T_i + A^T_{d_i})P)x(t) 
- 2x^T(t)PA_{d_i} \int_{t-\tau}^{t} \dot{x}(s) ds \]  

(14)

Remark 2. [29] For any given vectors \( u, v_i \), any positive matrix \( X_j \) and any matrix \( M_j \), we have:

\[ -2 \int_{\Omega} u^T(s)v_i(s)ds \leq \int_{\Omega} \chi^T \left( X_j (M^T_j X_j + I)X_j^{-1}(X_j M_j + I) \right) \chi ds \]  

(15)

Where: \( \chi = \left( \begin{array}{c} u(s) \\ v_i(s) \end{array} \right) \)

Remark 3. Without loss of generality and in order to simplify the demonstration we will limit in the following, the use of Eq.15 to the case where \( i = j \).

With:

\[ u(s) = P \dot{x}(t) \]
\[ v_i(s) = A^T_{d_i} \dot{x}(s) \]  

(16)

We get the inequality:

\[ \dot{V}_1 \leq \sum_{i=1}^{r} h_i (x^T(t)(P(A_i + A_{d_i}) + (A^T_i + A^T_{d_i})P)x(t) 
+ \tau x^T P(M^T_i X_i + I)X_i^{-1}(X_i M_i + I)P \dot{x}(t) 
+ 2x^T(t)PM^T_i X_i A_{d_i} \int_{t-\tau}^{t} \dot{x}(s) ds 
+ \int_{t-\tau}^{t} \dot{x}^T(s) A^T_{d_i} X_i A_{d_i} \dot{x}(s) ds \]  

(17)

\[ \dot{V}_1 \leq \sum_{i=1}^{r} h_i (x^T(t)(P(A_i + A_{d_i}) + (A^T_i + A^T_{d_i})P 
+ \tau P(M^T_i X_i + I)X_i^{-1}(X_i M_i + I)P + PM^T_i X_i A_{d_i} + A^T_{d_i} X_i M_i P)x(t) 
- x^T(t)PM^T_i X_i A_{d_i} \dot{x}(t - \tau) - x^T(t - \tau) A^T_{d_i} X_i M_i P \dot{x}(t) 
+ \int_{t-\tau}^{t} \dot{x}^T(s) A^T_{d_i} X_i A_{d_i} \dot{x}(s) ds \]  

Since \( \forall i, h_i \leq 1 \), Eq.18 will be:

\[ \dot{V}_1 \leq \sum_{i=1}^{r} h_i (x^T(t)(P(A_i + A_{d_i}) + (A^T_i + A^T_{d_i})P 
+ \tau P(M^T_i X_i + I)X_i^{-1}(X_i M_i + I)P + PM^T_i X_i A_{d_i} + A^T_{d_i} X_i M_i P)x(t) 
+ \tau P(M^T_i X_i + I)X_i^{-1}(X_i M_i + I)P + PM^T_i X_i A_{d_i} + A^T_{d_i} X_i M_i P)x(t) \]
Where:

\[ \chi = \bar{S} \]

Using Lemma 1 one gets:

\[ \dot{V}_1 \leq \chi^T \sum_{i=1}^{r} h_i \begin{pmatrix} P(A_i + A_{d_i}) + (A_i^T + A_{d_i}^T)P + PM^T X_i A_{d_i} + A_{d_i}^T X_i M P & -PM^T X_i A_{d_i} \\ PM^T X_i A_{d_i} + A_{d_i}^T X_i M P & 0 \end{pmatrix} \chi \]

where \( \chi^T = [x^T(t)x^T] \)

\[ \dot{V}_2 = \sum_{k=1}^{r} \bar{\tau} \dot{x}^T(s) A_{d_k}^T X_k A_{d_k} \dot{x}(s) - \sum_{k=1}^{r} \int_{t-\bar{\tau}}^{t} \dot{x}^T(s) A_{d_k}^T X_k A_{d_k} \dot{x}(s) ds \]

\[ \dot{V}_3 = x^T(t)Qx(t) - (1 - \bar{\tau})x^T(t - \tau)Qx(t - \tau) \]

\[ \dot{V}_3 \leq \chi^T \begin{pmatrix} Q & 0 \\ 0 & -(1 - d)Q \end{pmatrix} \chi \]

Where \( \chi^T = [x^T(t)x^T] \) and 0 ≤ \( \tau \) < d < 1

Combining Eq. 20, Eq.24 and Eq.26 with the variable’s change \( S_i = X_i M_i P \) and \( Z_i = \bar{\tau} X_i \) we get:

\[ \dot{V} \leq \sum_{i=1}^{r} h_i \chi^T \Lambda \chi \]

where:

\[ \Lambda = \begin{pmatrix} P(A_i + A_{d_i}) + (A_i^T + A_{d_i}^T)P + S_i^T A_{d_i} + \bar{\tau}^2 (S_i^T + P)Z_i^{-1} (S_i + P) + A_{d_i}^T S_i + Q & -S_i^T A_{d_i} \\ -A_{d_i}^T S_i & -(1 - d)Q \end{pmatrix} \]

By the application of Lyapunov Krasovskii Theorem [13] and the Schur Complement to \( \Lambda \) [3] in two steps, it can be shown that:

\[ \dot{V} \leq -\sigma_3 \| x(t) \| \]
Corollary 1. If there exist matrices \( P > 0, Q > 0, V > 0, \) and \( W_i \) such that:

\[
\begin{pmatrix}
\Lambda & -W_i^T A_d & A_d^T A_d^T V & \tau(W_i^T + P) \\
* & -(1 - d)Q & A_d^T A_d^T V & 0 \\
* & * & -V & 0 \\
* & * & * & -V
\end{pmatrix} < 0, \quad \forall \, i \, \text{in} \, 1...r \tag{31}
\]

Where: \( \Lambda = (A_i + A_d)^T P + P(A_i + A_d) + W_i^T A_d + A_d^T W_i + Q \) Then the unforced TS-TDS Eq.1 with common \( A_d \) is asymptotically stable for any time-delay \( \tau \in [0, \tilde{\tau}] \) and \( 0 \leq \tilde{\tau} < d < 1 \)

Proof. Let's take the unforced TS-TDS

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(A_i x(t) + A_d x(t - \tau)), \quad \forall \, t \geq 0
\]
\[
x(t) = \phi(t), \quad \forall \, t \in [-\tau, 0]
\]

It is easy to see that in case of common \( A_d \), i.e. \( A_d_i = A_d \), the LK Functional \( V_2 \) in Eq.11 will be reduced to:

\[
V_2' = \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{x}(\alpha) A_d^T X A_d \dot{x}(\alpha) d\alpha d\beta 
\]

While the other functionals \( V_3 \) and \( V_4 \) keep the same form as in Eq.11.

By using the Leibniz-Newton equation Eq.13, and Eq.15 with a common matrix \( X \) where:

\[
u_i(s) = v(s) = v(s) = A_d^T \dot{x}(s)
\]

The derivative of the functional \( V_1 \) will be:

\[
\dot{V}_1 \leq \sum_{i=1}^{r} h_i(x^T(t)(P(A_i + A_d) + (A_i^T + A_d^T)P
\]
\[
+ \tau P(M_i^T X + I) X^{-1} (X M_i + I) P + P M_i^T X A_d + A_d^T X M_i P)x(t) - x^T(t) P M_i^T X A_d x(t - \tau) - x^T(t - \tau) A_d^T X M_i P x(t)
\]
\[
+ \int_{t-\tau}^{t} \dot{x}^T(s) A_d^T X A_d \dot{x}(s) ds
\]

Remark 4. The use of uncommon matrix \( X_i \) is possible but in that case the functional \( V_2 \) should keep the form used in Eq.11, and some modifications to the actual demonstration will be necessary.
Knowing that: \( \sum_{i=1}^{r} h_i \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds = \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds \), Eq.35 will be reduced to:

\[
\dot{V}_1 \leq \sum_{i=1}^{r} h_i [x^T(t)(P(A_i + A_d) + (A_i^T + A_d^T)P + \tau P(M_i^T X + I)X_i^{-1}(X M_i + I) + P M_i^T X A_d + A_d^T X M_i P) x(t) \\
- x^T(t) P M_i^T X A_d x(t) - x^T(t) - x^T(t - \tau) A_d^T X M_i P x(t)] \\
+ \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds
\]

Which bring us to:

\[
\dot{V}_1 \leq \chi^T \sum_{i=1}^{r} h_i \begin{pmatrix}
P(A_i + A_d) + (A_i^T + A_d^T)P + P M_i^T X A_d + A_d^T X M_i P - P M_i^T X A_d \\
\tau P(M_i^T X + I)X_i^{-1}(X M_i + I) + P M_i^T X A_d + A_d^T X M_i P \\
0
\end{pmatrix} \chi \\
+ \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds
\]

Where: \( \chi^T = [x^T(t)x_d^T] \)

\[
\dot{V}_2 = \dot{x}^T(s)A_d^T X A_d \dot{x}(s) - \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds
\]

\[
\dot{V}_2 \leq \sum_{i=1}^{r} h_i \chi^T \dot{x}^T(s) \begin{pmatrix}
A_i^T A_i^T X A_d A_i & A_i^T A_d^T X A_d \chi \\
A_d^T A_i^T X A_d A_i & A_d^T A_d^T X A_d \chi
\end{pmatrix} \chi \\
- \int_{t-\tau}^{t} \dot{x}(s)A_d^T X A_d \dot{x}(s) ds
\]

\[
\dot{V}_3 = x^T(t)Q x(t) - (1 - \tau) x^T(t - \tau) Q x(t - \tau)
\]

\[
\dot{V}_3 \leq \chi^T \begin{pmatrix}
Q & 0 \\
0 & (1 - d)Q
\end{pmatrix} \chi
\]

Where \( \chi^T = [x^T(t)x_d^T] \) and \( 0 < \tau < d < 1 \)

Combining Eq.37, Eq.39 and Eq.41 with the variable’s change \( S_i = X M_i P \) and \( Z = \tau X \) we get:

\[
\dot{V} \leq \sum_{i=1}^{r} h_i \chi^T A \chi
\]

Where:

\[
A = \begin{pmatrix}
P(A_i + A_d) + (A_i^T + A_d^T)P + S_i^T A_d + S_i^T \tau (S_i^T + P) Z_i^{-1}(S_i + P) + A_d^T S_i + Q & -S_i^T A_d \\
-A_d^T S_i & -A_d^T S_i
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
A_i^T A_i^T Z A_d A_i & A_i^T A_d^T Z A_d A_d \\
A_d^T A_i^T Z A_d A_i & A_d^T A_d^T Z A_d A_d
\end{pmatrix}
\]

By the application of Lyapunov Krasovskii Theorem [13] and the Schur Complement to \( A \) [3] in two steps, it can be shown that :

\[
\dot{V}^* \leq -\sigma_3 \| x(t) \|
\]
Where $\sigma'_3$ is a positive real scalar, wherever the inequality below holds $\forall i \in 1...r$:

$$
\begin{pmatrix}
P(A_i + A_d) + (A_i^T + A_d^T)P \\
+S_i^T A_d + A_d^T S_i + Q \\
-A_d^T S_i \\
Z A_d A_i \\
\hat{\tau}(S_i + P) \\
\end{pmatrix}
\begin{pmatrix}
P(A_i + A_d) + (A_i^T + A_d^T)P \\
+S_i^T A_d + A_d^T S_i + Q \\
-A_d^T S_i \\
Z A_d A_i \\
\hat{\tau}(S_i + P) \\
\end{pmatrix}
\begin{pmatrix}
S_i^T A_d A_i \\
A_d^T Z \\
0 \\
0 \\
\end{pmatrix}
< 0
$$

(45)

4. Application.

4.1. Carbon dioxide model. The work [25], investigated the stability property of the positive equilibrium for Lotka Volterra model, describing the interaction between atmospheric carbon dioxide concentration, human population and forest biomass. Also, by constructing a suitable Lyapunov function, a set of sufficient conditions which ensure the global asymptotic stability of the positive equilibrium was obtained [10, 25]. By removing the elements representing the deforestation effects, and replacing them by reforestation elements a variant model [26] is proposed. Also, it defines a delay as the time gap between the reforestation needs assessment and the beginning of the reforestation program [26].

In the following we will represents the characteristic of [26]'s model, which we are going to use as application of the developed method above.

$$
\dot{X} =
\begin{pmatrix}
Q_0 + \lambda N - \alpha C - \lambda_1 CF \\
\mu F(1 - \frac{F}{F_M}) - \phi NF + \zeta RF \\
\gamma(M - F(t - \tau)) - \delta_0 R
\end{pmatrix}
$$

(46)

where:

$$
\dot{X} =
\begin{pmatrix}
\dot{C} \\
\dot{N} \\
\dot{F} \\
\dot{R}
\end{pmatrix}
$$

(47)

And:

- $C(t)$ : Atmospheric Carbon Dioxide level. (ppm)
- $N(t)$ : Human population. (person)
- $F(t)$ : Forest biomass. (ton)
- $R(t)$ : Reforestation Measurement. (dollar)
- $Q_0$ : Natural atmospheric Carbon Dioxide elevation rate. (ppm.year$^{-1}$)
- $\lambda$ : Anthropogenic atmospheric Carbon Dioxide elevation rate coefficient. (ppm.[person.year]$^{-1}$)
- $\alpha$ : Natural atmospheric Carbon Dioxide depletion rate coefficient year$^{-1}$
- $\lambda_1$ : Atmospheric Carbon Dioxide depletion rate coefficient due to forest biomass (year$^{-1}$)
- $s$ : Intrinsic Human population growth rate. (year$^{-1}$)
- $L$ : Human population carrying capacity. (person)
- $\theta$ : Human population depletion rate coefficient due to Carbon Dioxide. (ppm.year$^{-1}$)
- $\pi$ : Human population growth ratio due to forest biomass. (person.ton$^{-1}$)
- $\phi$ : Deforestation rate coefficient. (person.year$^{-1}$)
**µ** : Intrinsic Forest biomass growth rate. \(\text{year}^{-1}\)

**M** : Forest biomass carrying capacity. \(\text{ton}\)

**ζ** : Forest biomass growth ratio due to reforestation effort \(\text{dollar.year}^{-1}\)

**γ** : Reforestation efforts implementation rate coefficient \(\text{dollar.}(\text{ton.year})^{-1}\)

**τ** : Forest biomass measurement and reforestation efforts implementation time gap \(\text{year}\)

**δ₀** : Reforestation efforts declination rate coefficient \(\text{year}^{-1}\)

By combining all the possibilities to get an equilibrium point \(\dot{X} = 0\), we could find four possible equilibria points [26]:

- **E₁** = \(\left(\frac{Q_0}{\alpha}, 0, 0, M, 0\right)\) which is always feasible.
- **E₂** = \(\left(\frac{s(sQ_0 + \alpha L)}{(s\alpha + \theta\lambda L)}, N, 0, \frac{\gamma M}{\delta_0}\right)\) which is feasible if:
  \[s - \frac{\theta Q_0}{\alpha} > 0\]
- **E₃** = \(\left(\frac{Q_0}{\alpha}, 0, M, 0\right)\) which is always feasible.
- **E₄** = \(\left(C^*, N^*, F^*, R^*\right)\) which is feasible if:
  \[\mu + \frac{s\gamma M}{\delta_0} - \phi L\frac{(s\alpha - \theta Q_0)}{(s\alpha + \theta\lambda L)} > 0\] and \(s - \frac{\theta Q_0}{\alpha + \lambda M} + \pi\phi M > 0\)

In this paper, we are interested in the positive interior equilibria point \(E₄\) due to its relevant position which meets with the actual world context where all values in \(E₄\) are strictly positives.

All variables and parameters in Eq.46 are positives, and more precisely \(C(t) > 0\) and \(F(\vartheta) ≥ 0\) for \(\vartheta ∈ [-\tau, 0]\), in addition the functional region of Eq.46 is reduced to \([ε, C_m][0, N_m][0, F_m][0, R_m] \subset \mathbb{R}^4\) [25, 26], where \(ε\) is a small strict positive real number and:

\[
C_m = \frac{Q_0 + \lambda N_m}{\alpha}
\]

\[
N_m = L(1 + \frac{\pi\phi}{s} F_m)
\]

\[
F_m = M
\]

\[
R_m = \frac{\gamma M}{\delta_0}
\]

(48)

4.2. **Takagi Sugeno transformation.** To find a Takagi Sugeno model, the system Eq.46 should be transformed to a zero equilibrium model by taking the change of coordinate: \(x = X - E₄\), to get the form:

\[
\dot{x}(t) = Jx(t) + J_\tau x(t - \tau) + g(x), \quad \forall \ t ≥ 0
\]

\[
x(t) = \phi(t), \quad \forall \ t ∈ [-\tau, 0]
\]

(49)

Where \(J\) and \(J_\tau\) represents the Eq.46 linearization according to the original vector \(x(t)\) and the delayed \(x(t - \tau)\) near to the equilibrium point \(E₄\), and \(g(x)\) represents the nonlinear conduct of Eq.46 while the model get away from the equilibrium point \(E₄\).

Where:

\[
J = \begin{pmatrix}
-(\alpha + \lambda_1 F^*) & \frac{\lambda}{\alpha} & -\lambda_1 C^* & 0 \\
-\theta N^* & -\frac{\theta N^*}{L} & \pi\phi N^* & 0 \\
0 & -\phi F^* & -\frac{\mu F^*}{M} & \zeta F^* \\
0 & 0 & 0 & -\delta_0
\end{pmatrix}
\]

(50)
DEPENDENT DELAY STABILITY FOR A POLYNOMIAL T-S $CO_2$ MODEL

$$J_\tau = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\gamma & 0
\end{pmatrix}$$  \hfill (51)

$$g(x) = \begin{pmatrix}
-\lambda_1 x_1 x_3 \\
-sx_2^2 - \theta x_1 x_2 + \pi x_2 x_3 \\
-\frac{sx_2^2}{M} - \phi x_2 x_3 + \xi x_3 x_4 \\
0
\end{pmatrix}$$  \hfill (52)

Next, \(g(x)\) vector should be expressed in \(A(x)x\) form, and it is useful to mention that many arrangements could be generated, since different designers may lead to different fuzzy design models [20], but a good design choice will bring a less rules which is helpful for the controller and/or the observer design [6, 36]. For the system Eq.49, we chose to pick up a configuration involving just two state variables which will reduce the rules number to \(2^2 = 4\) rules [10]:

$$A = \begin{pmatrix}
-\lambda_1 x_3 & 0 & 0 & 0 \\
-\theta x_2 & -\frac{sx_2^2}{M} & \pi x_2 & 0 \\
0 & -\phi x_3 & -\frac{sx_2^2}{M} & \xi x_3 \\
0 & 0 & 0 & 0
\end{pmatrix}$$  \hfill (53)

The whole system will be of the form:

\[
\dot{x}(t) = Ax(t) + A_\tau x_\tau(t), \quad \forall \ t \geq 0
\]

\[
x(t) = \phi(t), \quad \forall \ t \in [-\tau, 0]
\]

Where:

\[
x_\tau(t) = x(t - \tau)
\]

\[
A = J + A
\]

\[
A_\tau = J_\tau
\]

Next, the nonlinear system Eq.54 will be transformed to several LTI sub-models by using extremum values of the so called premise variables which are chosen to be \(z_1 = x_2, z_2 = x_3\), and the firing probability weighting \(h_i(z(t))\) [20, 34, 36].

According to Non-linearity sector [6, 24, 36] each premise variable could be written as:

\[
z_i = Max_{z_i}.M_{i1} + min_{z_i}.M_{i2}
\]

Where:

\[
z_i \in [min_{z_i}, Max_{z_i}]
\]

\[
M_{i1} + M_{i2} = 1
\]

Which yields:

\[
M_{i1} = \frac{z_i - min_{z_i}}{Max_{z_i} - min_{z_i}}
\]

\[
M_{i2} = \frac{Max_{z_i} - z_i}{Max_{z_i} - min_{z_i}}
\]

The Takagi Sugeno multi model of Eq.54 will be then described as:

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(A_i x(t) + A_\tau x_\tau(t)), \quad \forall \ t \geq 0
\]

\[
x(t) = \phi(t), \quad \forall \ t \in [-\tau, 0]
\]
Where:
\[ h_i = \frac{w_i}{\sum_{j=1}^{n} w_i} \]
\[ w_i = \prod_{j=1}^{n} M_{jk} \]  

(60)

With: \( n \) is the number of premise variables and:
\[ j \in \{1..n\} \]
\[ k_j \in \{1, 2\} \]

(61)

In case of Eq. 53, and assuming:
\[ z_1 \in [\epsilon_N - N^*, N_m - N^*] \]
\[ z_2 \in [\epsilon_F - F^*, F_m - F^*] \]

(62)

Where: \( \epsilon_N \succeq 0 \) and \( \epsilon_F \succeq 0 \) are two real positive numbers to be chosen, we got:
\[ M_{11} = \frac{z_1 - (\epsilon_N - N^*)}{N_m - \epsilon_N} \]
\[ M_{12} = \frac{N_m - N^* - z_1}{N_m - \epsilon_N} \]
\[ M_{21} = \frac{z_2 - (\epsilon_F - F^*)}{F_m - \epsilon_F} \]
\[ M_{22} = \frac{F_m - F^* - z_2}{F_m - \epsilon_F} \]

(63)

Which lead us to:
\[ w_1 = M_{11}.M_{21} \]
\[ w_2 = M_{11}.M_{22} \]
\[ w_3 = M_{12}.M_{21} \]
\[ w_4 = M_{12}.M_{22} \]

(64)

and:
\[ A_1 = A_{MM} = \begin{pmatrix} -\alpha + \lambda_1 F_m & \lambda & -\lambda_1 C^* & 0 \\ -\theta N_m & -\frac{s_N}{L} & \pi \phi N_m & 0 \\ 0 & -\phi F_m & -\frac{\mu F_m}{M} & \zeta F_m \\ 0 & 0 & 0 & -\delta_0 \end{pmatrix} \]

\[ A_2 = A_{Mm} = \begin{pmatrix} -\alpha + \lambda_1 \epsilon_F & \lambda & -\lambda_1 C^* & 0 \\ -\theta \epsilon_N & -\frac{s_N}{L} & \pi \phi \epsilon N_m & 0 \\ 0 & -\phi \epsilon F_m & -\frac{\mu \epsilon F_m}{M} & \zeta \epsilon F_m \\ 0 & 0 & 0 & -\delta_0 \end{pmatrix} \]

\[ A_3 = A_{mM} = \begin{pmatrix} -\alpha + \lambda_1 F_m & \lambda & -\lambda_1 C^* & 0 \\ -\theta \epsilon_N & -\frac{s_N}{L} & \pi \phi \epsilon N_m & 0 \\ 0 & -\phi F_m & -\frac{\mu F_m}{M} & \zeta F_m \\ 0 & 0 & 0 & -\delta_0 \end{pmatrix} \]

\[ A_4 = A_{mm} = \begin{pmatrix} -\alpha + \lambda_1 \epsilon_F & \lambda & -\lambda_1 C^* & 0 \\ -\theta \epsilon_N & -\frac{s_N}{L} & \pi \phi \epsilon N_m & 0 \\ 0 & -\phi \epsilon F_m & -\frac{\mu \epsilon F_m}{M} & \zeta \epsilon F_m \\ 0 & 0 & 0 & -\delta_0 \end{pmatrix} \]

(65)
5. Numerical simulation and discussion. To show the reliability of the proposed method on the carbon dioxide model, and in the same time to have a comparative basis. We chose to use, in a first try, the numerical values showed in Table 1.

![Table 1. Model Parameter Values](image)

It corresponds to the case of India proposed in [26]. And we could find the equilibrium points and the maximal values for the model variables Carbon dioxide, human population, forest biomass and reforestation measurement as:

\[
X_m = 10^6 \times [0.0007; 0.0100; 0.7500; 3.0000] \\
X_c = 10^5 \times [0.0054; 0.0984; 7.4933; 0.0268]
\]

(66)

In addition, by taking \( \epsilon_N = \epsilon_F = 1 \), the matrices \( A_i \) will be then given as:

\[
A_1 = A_{MM} = \begin{pmatrix}
-0.0196 & 0.0006 & -2.61 \times 10^{-6} & 0 \\
-0.0100 & -0.0320 & 2.84 \times 10^{-7} & 0 \\
0 & -0.5325 & -0.0130 & 1.9500 \\
0 & 0 & 0 & -0.0002
\end{pmatrix}
\]

\[
A_2 = A_{Mm} = \begin{pmatrix}
-0.0160 & 0.0006 & -2.61 \times 10^{-6} & 0 \\
-0.0100 & -0.0320 & 2.84 \times 10^{-7} & 0 \\
0 & -7.10 \times 10^{-7} & -1.73 \times 10^{-8} & 2.60 \times 10^{-6} \\
0 & 0 & 0 & -0.0002
\end{pmatrix}
\]

(67)

\[
A_3 = A_{mM} = \begin{pmatrix}
-0.0196 & 0.0006 & -2.61 \times 10^{-6} & 0 \\
-10^{-6} & -3.2 \times 10^{-6} & 2.84 \times 10^{-11} & 0 \\
0 & -0.5325 & -0.0130 & 1.9500 \\
0 & 0 & 0 & -0.0002
\end{pmatrix}
\]

\[
A_4 = A_{mm} = \begin{pmatrix}
-0.0160 & 0.0006 & -2.61 \times 10^{-6} & 0 \\
-10^{-6} & -3.2 \times 10^{-6} & 2.84 \times 10^{-11} & 0 \\
0 & -7.10 \times 10^{-7} & -1.73 \times 10^{-8} & 2.60 \times 10^{-6} \\
0 & 0 & 0 & -0.0002
\end{pmatrix}
\]

First, by performing a simulation for different delays \( \tau \), we could notice in Fig. 1 that the model is converging toward its equilibrium point for \( \tau = 1, 3 \) and 7 years, while that of \( \tau = 9 \) years is diverging. And we can conclude that the maximal delay margin that ensure stability is between 7 and 9 years.

Also, we could notice in Fig. 2, that the initial condition value has an impact on the convergence aspect of the model for the same delay value. In some conditions the system may not converge even for small delays.

Then, in order to study the system global stability, we used the LMI solver Mosek [38] coupled to the interface Yalmip [22]. The solver calculates the maximal delay margin ensuring the feasibility of the given conditions (LMIs) of the technique. The calculations are done on a functional space (The TS-TDS model is built
Figure 1. States evolution in accordance to time in years for \(\tau = 1, 3, 7\) and 9 years with initial value \(x_0 = -0.1 X_e\)

Figure 2. States evolution in accordance to time in years for \(\tau = 7\) years for different initial conditions

assuming that \(z_1 \in [N_e - N_m; N_m - N_e]\), \(z_2 \in [F_e - F_m; F_m - F_e]\) and \(d = 0\). As a result, we got the conditions (31) feasible till a maximal delay of 8.555 years with matrices:

\[
P = 10^3 \begin{pmatrix}
4.8702 & 0.1449 & 0.0002 & -0.0132 \\
0.1449 & 4.1268 & -0.0000 & 0.0028 \\
0.0002 & -0.0000 & 0.0000 & -0.0000 \\
-0.0132 & 0.0028 & -0.0000 & 0.0004 \\
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
101.102 & 24.5020 & 0.0045 & -0.2568 \\
24.5020 & 137.9615 & 0.0006 & 0.0406 \\
0.0045 & 0.0006 & 0.0000 & -0.0000 \\
-0.2568 & 0.0406 & -0.0000 & 0.0007 \\
\end{pmatrix}
\]
In addition, a comparison with other methods under the same conditions is performed to evaluate its conservativeness. The LMIs are feasible for all the methods except the independent delay approaches [5, 36], and the results are stated in Table 2 showing that our approach gives a lesser result by 16.5 days than the theoretical value 8.6 years [26].

| Method | Th. 58 [36] | Th. 6.1 [2] | Cor. 1 [19] | Th. 1 [11] | Our Approach |
|--------|-------------|-------------|-------------|-------------|---------------|
| $\bar{\tau}_{\text{max}}$ | Infeasible | 8.561 | 8.591 | 8.595 | 8.555 |

**Table 2.** Maximal Delay Margin for Different Methods in years

Finally, to underline the Forest Biomass Lower Boundary (FBLB) negative impact on $\bar{\tau}$, we made simulations for different ratios $\vartheta$ where $FBLB = -\vartheta F_e$ (i.e. The TS-TDS model is built assuming that $z_1 \in [-N_e; N_m - N_e]$, $z_2 \in [-\vartheta * F_e; F_m - F_e]$ and $d = 0$), the results state that beyond a critical value ($-0.44 F_e$) all the methods, will become infeasible [11]. The result was expected knowing that the woodland is a basic $\text{CO}_2$ storage, and an initial situation $x_0$ where that reserve is lesser than a critical point will destabilize the system even if the delay margin is small.

6. **Conclusion.** In this paper, a new delay dependent stability conditions was proved and addressed in linear matrix inequalities. Its effectiveness was verified through application on a carbon dioxide model with a numerical application on India case. Some conservativeness is noticed in comparison to other dependent delay methods. Also, the study on this environmental model showed using examples the effect of the delayed reforestation on the carbon dioxide model and precisely forest biomass lower boundary’s negative effect on stability or the time taken to reach it. Finally, the simulation carried out shows that, the chosen model with the used data has an intrinsic huge stability time (around 300 years in case of $\tau = 2$ years). As a perspective it will be interesting to rerun the simulations on estimated parameters for the Moroccan case. Besides, the transformation into a forced nonlinear system to stabilize the model around the chosen equilibrium point by a controller, in order to reduce the stabilization time to an acceptable and managing value.

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REFERENCES

[1] J. M. Albertine, W. J. Manning, M. DaCosta, K. A. Stinson, M. L. Muilenberg and C. A. Rogers, Projected carbon dioxide to increase grass pollen and allergen exposure despite higher ozone levels, *PLoS ONE*, **9** (2014), 1–6.

[2] A. Benzaouia and A. El Hajjaji, *Advanced Takagi-Sugeno Fuzzy Systems: Delay and Saturation*, Vol.8, Studies in Systems, Decision and Control, 2014.

[3] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matriz Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics SIAM Philadelphia, SIAM Philadelphia, 1994.

[4] J.-C. Calvet, A.-L. Gibelin, J.-L. Roujean, E. Martin, P. Le Moigne, H. Douville and J. Noilhan, Past and future scenarios of the effect of carbon dioxide on plant growth and transpiration for three vegetation types of southwestern France, *Atmos. Chem. Phys. Discuss.*, **8** (2008), 397–406.

[5] Y.-Y. Cao and P. M. Frank, Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi-Sugeno fuzzy models, *Fuzzy Sets Syst.*, **124** (2001), 213–229.

[6] M. Chadli, D. Maquin and J. Ragot, Stability and Stabilisability of Continuous Takagi-Sugeno Systems, Journées Doctorales d’Automatique Sep 2001 Toulouse France, 2001.

[7] M. Chadli, D. Maquin and J. Ragot, Static Output Feedback for Takaki-Sugeno Systems: An LMI Approach, Proceeding of the 10th Mediterranean conference on control and automation-MED2002, Lisbon, Portugal, 2002.

[8] B. Chen and X. Liu, Delay-dependent robust H control for T-S fuzzy systems with time delay, *IEEE Transactions on Fuzzy Systems*, **13** (2005), 544–556.

[9] S. Devi and R. P. Mishra, Preservation of the forestry biomass and control of increasing atmospheric CO$_2$ using concept of reserved forestry biomass, *Int. J. Appl. Comput. Math.*, **6** (2020), Paper No. 17, 26 pp.

[10] A. Elmajidi, H. Elmazoudi, J. Elalami and N. Elalami, Carbon dioxide stability by a fuzzy takagi sugeno model, *Proceeding of the 4th Journée Scientifique d’Analyse des Systemes et Traitement de l’Information, Rabat Morocco*, (2017), pp.cdrom.

[11] A. Elmajidi, E. El Mazoudi, J. Elalami and N. Elalami, New delay dependent stability condition for a carbon dioxide takagi sugeno model, *Proceedings of the 6th International Conference on Wireless Technologies, Embedded, and Intelligent Systems (WITS2020)*, 2020.

[12] A. Elmajidi, E. El Mazoudi, J. Elalami and N. Elalami, A fuzzy logic control of a polynomial carbon dioxide model, *Ecology, Environment and Conservation*, **25** (2019), 876–887.

[13] E. Fridman, Tutorial on Lyapunov-based methods for time-delay systems, *Eur. J. Control*, **20** (2014), 271–283.

[14] T. J. Goreau, Control of atmospheric carbon dioxide, *Glob. Environ. Change*, **2** (1992), 5–11.

[15] E. Jarlebring, Computing the stability region in delay-space of a TDS using polynomial eigenproblems, *IFAC Proceedings Volumes*, **39** (2006), 296–301.

[16] H. K. Khalil, *Nonlinear Systems*, 3$^{rd}$Ed, Prentice Hall, Inc., 2002.

[17] K. Kim, J. Jho and W. Kwon, Design of T-S(Takagi-Sugeno) fuzzy control systems under the bound on the output energy, *Automation and Systems Engineering*, **1** (1999).

[18] H. A. Kruthika, A. D. Mahindrakar and R. Pasumarthy, Stability analysis of nonlinear time-delayed systems with application to biological models, *Int. J. Appl. Math. Comput. Sci.*, **27** (2017), 91–103.

[19] C. Li, H. Wang and X. Liao, Delay-dependent robust stability of uncertain fuzzy systems with time-varying delays, *IEEE Proc.-Control Theory Appl.*, **151** (2004), 417–421.

[20] J. H. Lilly, *Fuzzy Control and Identification*, Ed, John Wiley & Sons, Inc, 2010.

[21] C. Lin, Q.-G. Wang, T. H. Lee and Y. He, LMI Approach to Analysis and Control of Takagi-Sugeno Fuzzy Systems with Time Delay, Vol. 351, Lecture Notes in Control and Information Sciences, 2007.

[22] J. Löfberg, *YALMIP: A Toolbox for Modeling and Optimization in Matlab*, In Proceedings of the CACSD Conference, 2004.

[23] Y. Manai, M. Benrejeb and P. Borne, New Approach of stability for time-delay Takagi-Sugeno fuzzy system based on fuzzy weighting-dependent lyapunov functionals, *Applied Mathematics*, **2** (2011), 1339–1345.

[24] A. Maria Nagy, Analyse et synthese de multimodeles pour le diagnostic: Application a une station d’epuration, Ph.D Thesis, France. [https://tel.archives-ouvertes.fr](https://tel.archives-ouvertes.fr), 2010.
[25] A. K. Misra and M. Verma, A mathematical model to study the dynamics of carbon dioxide gas in the atmosphere, Appl. Math. Comput., 219 (2013), 8595–8609.

[26] A. K. Misra, M. Verma and E. Venturino, Modeling the control of atmospheric carbon dioxide through reforestation: effect of time delay, Model. Earth Syst. Environ., 1 (2015).

[27] Y. S. Moon, P. Park, W. H. Kwon and Y. S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems, Internat. J. Control, 74 (2001), 1447–1455.

[28] A.-T. Nguyen, K. Tanaka, A. Dequidt and M. Dambrine, Static output feedback design for a class of constrained Takagi-Sugeno fuzzy systems, J. Franklin Inst., 354 (2017), 2856–2870.

[29] P. G. Park, A delay-dependent stability criterion for systems with uncertain time-invariant delays, IEEE Trans. Automat. Control, 44 (1999), 876–877.

[30] T. J. Ross, Fuzzy Logic with Engineering Application, 3rd Ed, John Wiley & Sons, Inc, 2010.

[31] A. Sala, On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems, Annual Reviews in Control, 33 (2009), 48–58.

[32] A. Seuret and F. Gouaisbaut, On the use of the Wirtinger inequalities for time-delay systems, Proceedings of the 10-th IFAC Workshop on Time Delay Systems The International Federation of Automatic Control Northeastern University Boston USA, (2012), pp.cdrom.

[33] A. Seuret, F. Gouaisbaut and L. Baudouin, Overview of lyapunov methods for time-delay systems, LAAS-CNRS, n 16308 (2016), hal-01369516.

[34] T. Takagi and M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, Readings in Fuzzy Sets for Intelligent Systems, (1993), 387–403.

[35] K. Tanaka and M. Sugeno, Stability analysis and design of fuzzy control systems, Fuzzy Sets and Systems, 45 (1992), 135–156.

[36] K. Tanaka and H. O. Wang, Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach, 1st Ed, John Wiley & Sons, Inc, 2001.

[37] K. Tennakone, Stability of the biomass-carbon dioxide equilibrium in the atmosphere: Mathematical model, Applied Mathematics and Computation, 35 (1990), 125–130.

[38] MOSEK modeling cookbook, 3.2.2 (2020).

[39] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (1965), 338–353.

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