ON EXISTENCE OF EXTREMIZERS FOR THE TOMAS-STEIN INEQUALITY FOR $S^1$

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ABSTRACT. The Tomas-Stein inequality or the adjoint Fourier restriction inequality for the sphere $S^1$ states that the mapping $f \mapsto \hat{f}_\sigma$ is bounded from $L^2(S^1)$ to $L^6(\mathbb{R}^2)$. We prove that there exists an extremizer for this inequality. We also prove that any extremizer satisfies $|f(-x)| = |f(x)|$ for almost every $x \in S^1$.

1. Introduction

The Tomas-Stein inequality or the adjoint Fourier restriction inequality for the sphere $S^1$ asserts that

\begin{equation}
\|\hat{f}_\sigma\|_{L^6(\mathbb{R}^2)} \leq R\|f\|_{L^2(S^1,\sigma)}
\end{equation}

where the constant $R > 0$ is defined to be the optimal constant

\begin{equation}
R := \sup\{\|\hat{f}_\sigma\|_{L^6(\mathbb{R}^2)} : \|f\|_{L^2(S^1,\sigma)} = 1\},
\end{equation}

and $\sigma$ denotes the surface measure on the unit sphere $S^1$, and the Fourier transform is defined by

\begin{equation}
\hat{f}(\xi) := \int e^{-i\xi \cdot x} f(x) dx.
\end{equation}

Definition 1.1. A function $f \in L^2(S^1)$ is said to be an extremizer or an extremal for (1) if $f \neq 0$ a. e., and

\begin{equation}
\|\hat{f}_\sigma\|_{L^6(\mathbb{R}^2)} = R\|f\|_{L^2(S^1)}.
\end{equation}

An extremizing sequence for the inequality (1) is a sequence $\{f_\nu\} \in L^2(S^1)$ satisfies $\|f_\nu\|_{L^2(S^1)} = 1$ and $\lim_{\nu \to \infty} \|\hat{f}_\nu\|_{L^6(\mathbb{R}^2)} = R$. An extremizing sequence is said to be pre-compact if any subsequence has a sub-subsequence which is Cauchy in $L^2(S^1)$.

This paper is devoted to studying the existence of extremals for this basic inequality and to characterizing some properties of extremizers. The main result is the following

**Theorem 1.2.** There exists an extremal function for (1).

Moreover we show that the extremizers enjoy the following symmetry.

**Theorem 1.3.** Every extremizer satisfies $|f(-x)| = |f(x)|$ for almost every $x \in S^1$. 

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In [12], for the adjoint Fourier restriction inequality for the sphere $S^2$, we prove the existence of extremals by showing that any extremizing sequence of nonnegative functions is precompact in $L^2(S^2)$; and the extremals satisfy $|f(-x)| = |f(x)|$ for almost every $x \in S^2$. We are also able to prove that constants are local extremals. Furthermore in [13], we show that nonnegative extremizers are indeed smooth, and completely characterize complex extremals: Any complex extremizer is of the form $e^{ix\xi}f(x)$ for some nonnegative extremizer $f$ and some $\xi \in \mathbb{R}^3$, and if $\{f_\nu\}$ is a complex extremizing sequence then there exists $\{\xi_\nu\}$ such that $\{e^{-ix\xi_\nu}f_\nu\}$ is precompact. Recently, in [16], Foschi proves that constant functions are extremizers for the two dimensional sphere. In [14], Fanelli, Vega and Visciglia consider similar questions and establish existence for a family of non-endpoint Fourier restriction operators.

In the context of the adjoint Fourier restriction inequality for the paraboloid (or the Strichartz inequality for the Schrödinger equation), Kunze [18] proves the existence of extremals when the spatial dimension is one by a concentration-compactness argument. Foschi [15] proves that Gaussian functions are explicit extremals in spatial dimensions one and two by two successive applications of the Cauchy-Schwarz inequalities; independently Hundertmark and Zharnitsky [17] obtain similar results. Bennett, Bez, Carbery and Hundertmark [2] show that Gaussians are extremizers from the perspective of the heat-flow deformation method [3, 4]. For non-$L^2$ adjoint Fourier restriction inequality for paraboloids, Christ and Quilodrán [11] show that Gaussians are rarely extremizers by studying the corresponding Euler-Lagrange equations. In higher dimensions, the existence result of extremizers is known, which is achieved by using the tool of profile decompositions by the author [25]. In the context of a convolution inequality with the surface measures of the paraboloids, the existence of quasi-extremals and extremals was studied by Christ [8, 9].

The main results Theorem 1.2 and Theorem 1.3 are proven by following the framework designed in the paper [12]. To be more precise, the first part of the analysis in this paper, Step 1 to Step 4, follows similarly as in [12] to obtain a nonnegative extremizing sequence $f_\nu$ which is “even upper normalized” with respect to a sequence of caps $C_\nu$. In the second part of the analysis, i.e., at the last Step 5, we develop a profile decomposition for the adjoint sphere restriction operator in the spirit of [5, 7]. When $f$ is supported on sufficiently small caps on the sphere, we approximate $\hat{f}\sigma$ by linear Schrödinger waves. This idea appeared previously in [10] where the authors approximate the Airy wave at high frequency by a Schrödinger wave.

The analysis in this paper can be viewed as a manifestation of the the concentration-compactness approach developed in a series of works by Lions [19, 20, 21, 22], which is however adapted to our case to cope with the nonlocal characteristics of the adjoint Fourier restriction operator.

We present the outline and results in detail in Section 2.

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2. Outline of the proof and definitions

This section consists of notations, definitions and statements of some intermediate results which are not repeated subsequently. We start with several definitions. Let \( \sigma_P \) be the canonical measure on the parabola \( P = \{ (x, y) \in \mathbb{R}^2 : y = \frac{1}{2}|x|^2 \} \), and set

\[
(5) \quad \mathcal{R}_P := \sup \frac{\| \hat{f} \sigma_P \|_{L^6(\mathbb{R}^2)}}{\| f \|_{L^2(P, \sigma_P)}},
\]

\[
(6) \quad S := \sup \frac{\| f \sigma * f \sigma * f \sigma \|_{L^2(\mathbb{R}^3)}^{1/3}}{\| f \|_{L^2(S^1, \sigma)}},
\]

\[
(7) \quad \mathcal{P} := \sup \frac{\| f \sigma_P * f \sigma_P * f \sigma_P \|_{L^2(\mathbb{R}^3)}^{1/3}}{\| f \|_{L^2(P, \sigma_P)}}.
\]

Note that by Plancherel's theorem, \( \mathcal{R} = 2\pi S \), \( \mathcal{R}_P = 2\pi \mathcal{P} \). There holds that \( |f \sigma * f \sigma * f \sigma| \leq |f \sigma * f |\sigma * f |\sigma| \). If \( f \) is an extremizer to (1), so if \( |f| \). This applies to any extremizing sequence \( \{ f_\nu \} \). Thus in order to prove the existence of extremizers, we will restrict our attention to nonnegative functions and nonnegative extremizing sequences.

**Step 1, A strict comparison.** By a dilation argument, we see that the sharp constants for the adjoint Fourier restriction inequalities for the sphere and the paraboloid satisfy, \( \mathcal{R} \geq \mathcal{R}_P \), where \( \mathcal{R}_P \) is defined in (5). This reasoning appears previously in [12]. Indeed, we take an extremizer for the paraboloid, which is known as Gaussians from [15] and [17], dilate it so that it is essentially supported in a sufficiently small set of the paraboloid; we paste the extremizer onto the sphere in an obvious way and then osculate the sphere by the parabolic scaling \( (x', x_d) \to (\lambda x', \lambda^2 x_d) \) where \( x = (x', x_d) \in \mathbb{R}^d \) and \( \lambda > 0 \). In the limits, we see that the relation \( \mathcal{R} \geq \mathcal{R}_P \) holds. So there arises the most severe obstruction to the existence of extremizers that, for an extremizing sequence \( \{ f_\nu \} \) satisfying \( \| f_\nu \|_2 = 1 \), any subsequential weak limit of \( |f_\nu|^2 \) could conceivably converge to a Dirac mass at a point of \( S^1 \). If it were the case, then \( \mathcal{R} = \mathcal{R}_P \). To rule out this scenery, an essential step is to prove \( \mathcal{R} > \mathcal{R}_P \). Because any extremal enjoys a symmetry \( |f(x)| = |f(-x)| \), there is a possibility that the extremizing sequence might converge weakly to a linear combination of two Dirac masses at antipodal points of \( S^1 \). To rule it out, one needs a strict comparison \( S > (5/2)^{1/6} \mathcal{P} \), which is achieved by using a perturbation argument, which we sketch in Appendix A.

**Proposition 2.1.**

\[
(8) \quad \mathcal{R} > (5/2)^{1/6} \mathcal{R}_P.
\]

**Step 2, Antipodal symmetrization.** We will show that “extremals” to (1) enjoy a symmetry \( |f(-x)| = |f(x)| \).

**Definition 2.2.** A complex-valued function \( f \in L^2(S^1) \) is said to be even if \( \overline{f(-x)} = f(x) \) for almost every \( x \in S^1 \). For nonnegative functions, this condition is simplified to \( f(-x) = f(x) \).
**Definition 2.3.** Let \( f \) be nonnegative \( L^2(S^1) \) function. The antipodally symmetric rearrangement \( f_* \) is the unique non-negative element of \( L^2(S^1) \) which satisfies
\[
\begin{align*}
(9) \quad & f_*(x) = f_*(-x), \quad \text{for all } x \in S^1, \\
(10) \quad & f_*(x)^2 + f_*(-x)^2 = f(x)^2 + f(-x)^2, \quad \text{for all } x \in S^1.
\end{align*}
\]
In other words, \( f_* = \sqrt{\frac{f(x)^2 + f(-x)^2}{2}} \) and \( \|f_*\|_2 = \|f\|_2 \).

**Proposition 2.4.** For any nonnegative function \( f \in L^2(S^1) \),
\[
(11) \quad \|f \sigma \ast f \sigma \ast f \sigma\|_2 \leq \|f \sigma \ast f_* \ast f_* \sigma\|_2,
\]
with strict inequality if and only if \( f = f_* \) almost everywhere. Consequently any extremizer for the inequality (1) satisfies \( |f(-x)| = |f(x)| \) for almost every \( x \in S^1 \).

The analogue for \( S^2 \) is established in [12]. We remark that Foschi [16] has provided a much shorter proof for that by using the Cauchy-Schwarz inequality.

**Step 3, A refinement of Tomas-Stein’s inequality.** Similarly as in [12], we define what caps mean on \( S^1 \).

**Definition 2.5.** The cap \( C = C(z, r) \) with center \( z \in S^1 \) and radius \( r \in (0, 1] \) is the set of all points \( y \in S^1 \) which lie in the same hemisphere as \( z \) and are centered at \( z \), and which satisfy \( \pi_{H_z}(y) < r \), where the subspace \( H_z \subset \mathbb{R}^2 \) is the orthogonal complement of \( z \) and \( \pi_{H_z} \) denotes the orthogonal projection onto \( H_z \).

In [12], the refinement of Tomas-Stein’s inequality for \( S^2 \) developed by Bourgain [6] and Moyua, Vargas and Vega [23] provides some useful information on the near-extremals for the adjoint Fourier restriction inequality for \( S^2 \): Any near extremal can be decomposed into a major part, which obey some upper bound in the point-wise sense, and a lower \( L^2 \)-norm bound, plus an error term. For \( S^1 \), we have the following refinement

**Lemma 2.6.** For \( f \in L^2(S^1) \). There exists \( \alpha \in (0, 1) \) such that
\[
(12) \quad \|\widehat{f \sigma}\|_6 \leq \left( \sup_C \frac{1}{|C|^{1/2}} \int_C |f| |d\sigma| \right)^\alpha \|f\|_{L^2(S^1)}^{1-\alpha},
\]
where \( C \) denotes a cap on \( S^1 \).

We establish this lemma by using the bilinear restriction estimates for functions on \( S^1 \) whose supports are “transverse”, i.e., the unit normals to each set are separated by an angle \( > 0 \). The argument is similar to that for [1, Theorem 1.3].

As a consequence of the refinement in Lemma 2.6, we have

**Proposition 2.7.** For any \( \delta > 0 \) there exists \( C_\delta < \infty \) and \( \eta_\delta > 0 \) with the following properties. If \( f \in L^2(S^1) \) satisfies \( \|f \sigma\|_6 \geq \delta |R| \|f\|_2 \), then there exists a decomposition...
\begin{align*}
f = g + h \text{ and a cap } C \text{ satisfying that} \\
0 &\leq |g|, |h| \leq |f|, \\
g, h &\text{ have disjoint supports,} \\
|g(x)| &\leq C_\delta \|f\|_2 |C|^{-1/2} \chi_C(x), \forall x, \\
\|g\|_2 &\geq \eta_\delta \|f\|_2.
\end{align*}

Here both \(C_\delta^{-1}\) and \(\eta_\delta\) are proportional to \(\delta^{O(1)}\). If \(f \geq 0\), \(g\) and \(h\) can be chosen such that \(g, h \geq 0\) almost everywhere.

**Step 4, Upper even normalized w.r.t the caps.** As in [12], we introduce the notion of rescaling maps that pull back functions on \(S^1\) to \(\mathbb{R}\) and obtain some preliminary control on the near-extremals.

**Definition 2.8** (Rescaling map \(\phi_C\)). Let \(B \subset \mathbb{R}\) denote the unit ball. To any cap of radius \(\leq 1\) is associated a rescaling map \(\phi_C : B \rightarrow C\). For \(z = (0,1)\), \(\phi_C(y) = (ry, \sqrt{1 - r^2y^2})\). For general \(z\), define \(\psi_z(y) = r^{-1}L(\pi(y))\) where \(\pi\) is again the orthogonal projection onto \(H_z\), \(L : H_z \rightarrow \mathbb{R}\) is an arbitrary linear isometry and \(\phi_C = \psi_z^{-1}\).

For a cap \(C = C(z, r)\), we remark that \(\phi_C\) is naturally extended to defined on the set \(\{y : |y| < r\}\).

**Definition 2.9** (Pullbacks). Define the pullbacks \(\phi^*_C = r^{1/2}(f \circ \phi_C)(y)\) where \(r\) is the radius of the cap \(C\).

**Remark 2.10.** This definition of pullbacks makes sense if \(f\) is supported in the cap of radius 1 concentric with \(C\).

**Definition 2.11** (Upper normalized w.r.t. caps and balls). Let \(\Theta : [1, \infty) \rightarrow (0, \infty)\) satisfy \(\Theta(R) \rightarrow 0\) as \(R \rightarrow \infty\). A function \(f \in L^2(S^1)\) is said to be upper normalized with respect to a cap \(C = C(z, r) \subset S^1\) of radius \(r\) and center \(z\) if the following hold

\begin{align*}
\|f\|_2 &\leq C < \infty, \\
\int_{|f| > Rr^{-1/2}} |f|^2 d\sigma(x) &\leq \Theta(R), \forall R \geq 1, \\
\int_{|x - z| \geq Rr} |f|^2 d\sigma(x) &\leq \Theta(R), \forall R \geq 1.
\end{align*}

An even function \(f\) is said to be upper even-normalized with respect to \(\Theta\), and \(C\) if \(f\) can be decomposed into \(f = f_+ + f_-\) where \(f_-(x) = f_+(-x)\), and \(f_+\) is upper normalized with respect to \(\Theta\) and \(C\). A function \(f \in L^2(\mathbb{R})\) is said to be upper normalized with respect to the unit ball in \(\mathbb{R}\) if

\begin{align*}
\|f\|_2 &\leq C < \infty, \\
\int_{|f| > R} |f|^2 dx &\leq \Theta(R), \forall R \geq 1, \\
\int_{|x| \geq R} |f|^2 dx &\leq \Theta(R), \forall R \geq 1.
\end{align*}
Definition 2.12 (Near-extremal). A nonzero function $f \in L^2(S^1)$ is said to be $\delta$-nearly extremal for the inequality (1) if
\begin{equation}
\|f \sigma \ast f \sigma \ast f \sigma\|_2 \geq (1 - \delta)^3 S^3 \|f\|_2^3.
\end{equation}

The following proposition provides a preliminary decomposition for nearly extremals.

Proposition 2.13. There exists a function $\Theta : [1, \infty) \rightarrow (0, \infty)$ satisfying $\Theta(R) \rightarrow 0$ as $R \rightarrow \infty$ with the following property. For any $\varepsilon > 0$, there exists $\delta > 0$ such that any nonnegative even functions $f \in L^2(S^1)$ satisfying $\|f\|_2 = 1$ which is a $\delta$-nearly extremal may be decomposed as $f = F + G$, where $F$ and $G$ are even and nonnegative with disjoint supports, $\|G\|_2 \leq \varepsilon$ and there exists a cap $C$ such that $F$ is upper even-normalized with respect to $C$.

It follows from two crucial facts: the refinement of Tomas-Stein’s inequality for $S^1$ in Proposition 2.7, and a geometric fact that “distant caps interact weakly” we will establish in Section 4. The latter asserts, roughly speaking, that $\|\chi C \sigma \ast \chi C \sigma \ast \chi C \sigma\|_2 \ll |C||C'|^{1/2}$ unless the caps $C$ and $C'$ have comparable radii and nearby centers.

Step 5, Ruling out small caps and existence of extremals. In [10], the authors observe that the linear Airy evolution at high frequency is well approximated by a linear Schrödinger evolution, which is used in [24] to establish the linear profile decomposition for the Airy equation. In this paper, we have observed that a similar phenomena occurs for $\tilde{f} \sigma$ when $f$ is supported on a very small cap. More precisely, given any extremizing sequence $\{f_\nu\}$ which is upper even normalized with respect to caps $C_\nu$ with radii $r_\nu \rightarrow 0$, $\tilde{f}_\nu \sigma$ can be written as a superposition of “orthogonal” linear Schrödinger waves, plus a small error term. In this case, there follows that
\begin{equation}
\mathcal{R} \leq (5/2)^{1/6} \mathcal{R}_P.
\end{equation}

But it is a contradiction to the strict inequality that $\mathcal{R} > (5/2)^{1/6} \mathcal{R}_P$. Thus $\inf_\nu r_\nu > 0$. Then for “large caps”, one can indeed prove $f_\nu$ is precompact, which leads to the existence of extremals for (1).

Proposition 2.14. Let $\{f_\nu\} \subset L^2(S^1)$ be an extremizing sequence for the inequality (1) satisfying $\|f_\nu\|_2 = 1$ and $|f(-x)| = |f(x)|$ for a.e. $x \in S^1$. Suppose that each $|f_\nu|$ is upper even-normalized with respect to a cap $C_\nu \cup (-C_\nu)$ where $C_\nu = C(z_\nu, r_\nu)$, with constants uniform in $\nu$. Then
\begin{equation}
\inf_\nu r_\nu > 0.
\end{equation}

In this case, an extremal for (1) is obtained.

The strict comparison on the optimal constants for Tomas-Stein’s inequalities for the sphere and the paraboloid is essential to obtain existence of extremals to Tomas-Stein’s inequality for the sphere. In high dimensions, it is the lack of the strict comparison on the optimal constants and the algebraic property of the even integer 6 that prevent us from obtaining the existence of extremals.

Step 1 is established in the Appendix. We present Step 2 through Step 5 in what follows.
3. Step 2. Antipodal symmetrization

In this section, we will prove the functional \( \|f \star f \star f\|_2^2 / \|f\|_2^6 \) is non-decreasing under the antipodal symmetrization defined in Definition 2.2.

Proof of Proposition 2.4. For \( f \geq 0 \),

\[
\|f \star f \star f\|_2^2 = \int f(a_1) \times \cdots \times f(a_6)d\lambda(a_1, \cdots, a_6)
\]

for a certain non-negative measure \( \lambda \) which is supported by the set

\[
\{(a_1, \cdots, a_6) \in (\mathbb{R}^2)^6 : a_1 + a_2 + a_3 = a_4 + a_5 + a_6\},
\]

and which is invariant under the following transformations

\[
\begin{align*}
(a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_4, a_5, a_6, a_1, a_2, a_3), \\
(a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_{\tau(1)}, a_{\tau(2)}, a_{\tau(3)}, a_4, a_5, a_6), \\
(a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_1, a_2, -a_3, -a_4, a_5, a_6), \\
(a_1, a_2, a_3, a_4, a_5, a_6) & \mapsto (a_1, -a_4, -a_5, -a_2, a_3, a_6),
\end{align*}
\]

where \( \tau \in S^3 \), the permutation group of order 3. We denote by \( G \) the finite group of symmetries of \((\mathbb{R}^2)^6\) generated by these symmetries. The cardinality of \( G \) is \( 2 \times 6! \) since there holds a short exact sequence

\[
1 \mapsto \{\pm 1\} \mapsto G \mapsto S^6 \mapsto 1.
\]

Note that in order for a sequence \((a_1, a_2, a_3, a_4, a_5, a_6)\) of fixed order to satisfy the requirement (25), the only way is to add “−” sign. Hence from basic algebra, there holds that \( |G/\{\pm 1\}| = |S^6| = 6! \); thus \( |G| = 2 \times 6! \) follows.

By the orbit of a point we mean its image under \( G \); by a generic point we mean one whose orbit has cardinality \( 2 \times 6! \). In (24), it suffices to integrate only over all generic 6-tuples \((a_1, \cdots, a_6)\) satisfying (25), since they form a set of full \( \lambda \)-measure.

To the orbit \( O \) we associate the functions

\[
\mathcal{F}(O) = \sum_{(a_1, \cdots, a_6) \in O} f(a_1) \times \cdots \times f(a_6),
\]

\[
\mathcal{F}_*(O) = \sum_{(a_1, \cdots, a_6) \in O} f_*(a_1) \times \cdots \times f_*(a_6)
\]

Let \( \Omega \) denote the set of all orbits of generic points. We can write

\[
\|f \star f \star f\|_2^2 = \int_{\Omega} \mathcal{F}(O)d\mu(O),
\]

\[
\|f_\ast \star f_\ast \star f_\ast\|_2^2 = \int_{\Omega} \mathcal{F}_*(O)d\mu(O)
\]
for a certain nonnegative measure $\mu$. Therefore it suffices to prove that for any generic orbit $O$,

$$
(30) \sum_{(a_1, \ldots, a_6) \in O} f(a_1) \times \cdots \times f(a_6) \leq \sum_{(a_1, \ldots, a_6) \in O} f_*(a_1) \times \cdots \times f_*(a_6).
$$

Fix any generic orbit 6-tuple $(a_1, \ldots, a_6)$ satisfying (25), we prove (30) for its orbit. By homogeneity, we may assume that $f(a_1)^2 + f(-a_1)^2 = 1$ and that the same holds simultaneously for $a_i$ for $i = 2, \ldots, 6$. Thus we may write

$$
(31) \begin{align*}
    f(a_1) &= \cos(\theta_1), f(a_2) = \cos(\theta_2), \ldots, f(a_6) = \cos(\theta_6), \\
    f(-a_1) &= \sin(\theta_1), f(-a_2) = \sin(\theta_2), \ldots, f(-a_6) = \sin(\theta_6)
\end{align*}
$$

for $\theta_i \in [0, \pi/2]$ for $i = 1, \ldots, 6$. Thus by definition

$$
(32) f_* = 2^{-1/2}.
$$

Before writing out $\sum_{(a_1, \ldots, a_6) \in O} f(a_1) \times \cdots \times f(a_6)$ for a generic orbit of $(a_1', \ldots, a_6')$, we note that there will be $2 \times 6!$ summands, which can be organized into 20 terms. Here 20 comes out because $20 = \frac{2 \times 6!}{2 \times 3! \times 3!}$. Below we will write out a long formula,

$$
\begin{align*}
    \frac{1}{2 \times 3! \times 3!} \sum_{(a_1, \ldots, a_6) \in O} f(a_1) \times \cdots \times f(a_6) \\
    &= \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \cos(\theta_4) \cos(\theta_5) \cos(\theta_6) \\
    &\quad + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4) \sin(\theta_5) \sin(\theta_6) \\
    &\quad + \left( \begin{array}{cc}
    \cos(\theta_1) \cos(\theta_2) \sin(\theta_3) & \sin(\theta_4) \cos(\theta_5) \cos(\theta_6) \\
    \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) & \cos(\theta_4) \sin(\theta_5) \cos(\theta_6) \\
    \sin(\theta_1) \cos(\theta_2) \cos(\theta_3) & \cos(\theta_4) \cos(\theta_5) \sin(\theta_6) \\
    \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) & \sin(\theta_4) \sin(\theta_5) \cos(\theta_6) \\
    \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) & \sin(\theta_4) \cos(\theta_5) \sin(\theta_6) \\
    \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) & \cos(\theta_4) \sin(\theta_5) \sin(\theta_6)
\end{array} \right) \\
&=: \Gamma(\theta_1, \ldots, \theta_6).
\end{align*}
$$

We will organize terms according to $\cos(\theta_5) \cos(\theta_6), \sin(\theta_5) \sin(\theta_6), \sin(\theta_5) \cos(\theta_6)$ and $\cos(\theta_5) \sin(\theta_6)$ and rewrite $\Gamma$ as

$$
(34) \begin{align*}
    \Gamma &= \cos(\theta_5) \cos(\theta_6) A + \sin(\theta_5) \sin(\theta_6) B + \left( \sin(\theta_5) \cos(\theta_6) + \cos(\theta_5) \sin(\theta_6) \right) C \\
    &= \cos(\theta_5) \cos(\theta_6) A + \sin(\theta_5) \sin(\theta_6) B + \sin(\theta_5 + \theta_6) C
\end{align*}
$$

where $A$ and $B$ contain 4 terms, respectively, and $C$ contains 6 terms. For the sake of simplicity, we will not explicitly write out $A$, $B$ and $C$; we will do so whenever it is necessary.

We aim to show that

$$
(35) \max_{(\theta_1, \ldots, \theta_6) \in [0, \pi/2]^6} \Gamma = \Gamma(\pi/4, \ldots, \pi/4) = \frac{5}{2}.
$$
This maximal value of $\Gamma$ matches values taken by
\[
\frac{1}{2 \times 3! \times 3!} \sum_{(a_1, \ldots, a_6)} f_*(a_1) \times \cdots \times f_*(a_6) = \frac{20}{8} = \frac{5}{2}.
\]

Suppose that $(\alpha_1, \ldots, \alpha_6)$ is a critical point of $\Gamma$, then at this point, there holds that
\[
\frac{d\Gamma}{d\theta_5} = \frac{d\Gamma}{d\theta_6} = 0,
\]
which implies that, at this critical point,
\[
0 = \frac{d\Gamma}{d\theta_5} - \frac{d\Gamma}{d\theta_6}.
\]
We expand the right hand side of (37) out to see
\[
\sin(\alpha_5 - \alpha_6)(A + B) = 0.
\]
We will show that $A + B|_{(\alpha_1, \ldots, \alpha_6)} > 0$; otherwise, since every term in $A + B$ is nonnegative, that $A + B = 0$ will imply that every term is actually zero. This further implies that
\[
\Gamma(\alpha_1, \cdots, \alpha_6) < \frac{5}{2},
\]
which contradicts to the definition that $(\alpha_1, \cdots, \alpha_6)$ is a critical point. Indeed, We writes out
\[
A = \cos(\theta_1) \cos(\theta_2) \cos(\theta_3 - \theta_4) + \sin(\theta_1 + \theta_2) \cos(\theta_3) \sin(\theta_4),
\]
\[
B = \sin(\theta_1) \sin(\theta_2) \cos(\theta_3 - \theta_4) + \sin(\theta_1 + \theta_2) \sin(\theta_3) \cos(\theta_4).
\]
Hence
\[
0 = A + B = \cos(\theta_1 - \theta_2) \cos(\theta_3 - \theta_4) + \sin(\theta_1 + \theta_2) \sin(\theta_3 + \theta_4).
\]
This implies that
\[
\begin{align*}
\cos(\theta_1 - \theta_2) \cos(\theta_3 - \theta_4) &= 0, \\
\sin(\theta_1 + \theta_2) \sin(\theta_3 + \theta_4) &= 0.
\end{align*}
\]
So we have the following four combinations,
\[
\begin{align*}
\theta_1 - \theta_2 &= \pi/2, & \theta_1 - \theta_2 &= -\pi/2, & \theta_1 + \theta_2 &= 0, \text{ or } \pi, & \theta_1 + \theta_2 &= 0, \text{ or } \pi, \\
\theta_3 + \theta_4 &= 0, \text{ or } \pi. & \theta_3 + \theta_4 &= 0, \text{ or } \pi. & \theta_3 - \theta_4 &= -\pi/2. & \theta_3 - \theta_4 &= \pi/2.
\end{align*}
\]
In all cases, we can show that
\[
\Gamma < \frac{5}{2}.
\]
For instance, assume the first instance with
\[
\theta_1 - \theta_2 = \pi/2, \theta_3 + \theta_4 = 0.
\]
Then by the fact that all $\theta_i \in [0, \pi/2]$, we have
\[
\theta_1 = \pi/2, \theta_2 = \theta_3 = \theta_4 = 0.
\]
Then $\Gamma$ is simplified to
\[ \Gamma = \sin(\theta_5 + \theta_6) \leq 1 < 5/2. \]

Thus (38) forces that
\[ \theta_5 = \theta_6. \] (44)

By symmetry it immediately follows that
\[ \theta_4 = \theta_5 = \theta_6 := \beta, \]
\[ \theta_1 = \theta_2 = \theta_3 := \alpha \] (45)

Observing that by symmetry we may exchange, say, $f(a_3) = \cos(\theta_3) = \cos(\alpha)$ and $f(-a_4) = \sin(\theta_4) = \sin(\beta) = \cos(\frac{\pi}{2} - \beta)$ in $\Gamma$, we conclude that
\[ \alpha + \beta = \pi/2. \] (46)

Combining (45) and (46), we see that
\[ \Gamma(\alpha_1, \cdots, \alpha_6) = 20 \cos^3(\alpha) \cos^3(\beta) = \frac{20}{8} \left(2 \sin(\alpha) \cos(\alpha)\right)^3 = \frac{5}{2} \sin^3(2\alpha) \leq \frac{5}{2} \]
with “=” if and only if $\alpha = \frac{\pi}{4}$. Hence the only choice for critical points is
\[ \alpha = \beta = \frac{\pi}{4}. \] (48)

To conclude, we have established the claim (35). Hence (30) follows. Therefore the proof of Theorem (2.4) is complete. \qed

4. Step 3. A refined estimate and a geometric fact

In this section we first establish the refinement of Tomas-Stein inequality for $S^1$ in Lemma 2.6, which easily implies Proposition 2.7, see e.g. [7, 24]. In the end of this section, we establish a geometric fact that “distant caps interact weakly”.

We aim to prove the following

**Proposition 4.1.** For $f \in L^2(S^1)$. There exists $\alpha \in (0, 1)$ such that
\[ \|\hat{f}\sigma\|_6 \leq \left(\sup_C \frac{1}{|C|^{1/2}} \int_C |f| d\sigma\right)^\alpha \|f\|_{L^2(S^1)}^{1-\alpha}, \]
where $C$ denotes a cap on $S^1$.

We recall two lemmas: The first is on the bilinear restriction estimates for functions whose supports are transverse.

**Lemma 4.2.** Let $f, g \in L^2(S^1)$ and assume that $f$ and $g$ are supported on the caps $C_1$ and $C_2$, which are separated by $2^j$ for $j \leq 0$. Then
\[ \|\hat{f}\sigma \hat{g}\sigma\|_2 \leq C 2^{-j/2} \|f\|_2 \|g\|_2. \] (49)
The proof of this lemma follows from Cauchy-Schwarz’s inequality and the geometric estimate \(\|\sigma * \sigma'\|_\infty \leq C2^{-j}\) where \(\sigma\) and \(\sigma'\) denote the surface measures supported on the two caps on the sphere \(S^1\) which are separated by an angle \(\geq 2^j, j \leq 0\).

The second is on the “almost orthogonality” for functions which have disjoint supports in the Fourier space, see for instance [28, Lemma 6.1].

**Lemma 4.3.** Let \(\{R_k\}\) be a collection of rectangles and \(c > 0\) such that the dilates \((1+c)R_k\) are almost disjoint (i.e., \(\sum_k \chi_{(1+c)R_k} \leq C < \infty\)), and suppose that \(\{f_k\}\) is a collection of functions supported by \(\{R_k\}\). Then for all \(1 \leq p \leq \infty\),

\[
\left\| \sum_k \widehat{f}_k \right\|_p \lesssim \left( \sum_k \|\widehat{f}_k\|_{p_0}^p \right)^{1/p_0},
\]

where \(p_0 = \min\{p, p/(p-1)\}\).

**Proof of Proposition 4.1.** For \(j \leq 0\), we partition \(S^1\) into a union of caps \(C_k^j\) of length \(\sim 2^j\). For a given cap \(C_k^j\), we partition it into two equal caps \(C_{m}^{j-1}\) and \(C_{n}^{j-1}\) of length \(2^{j-1}\); we say \(C_k^j\) is the “parent” of \(C_{m}^{j-1}\) and \(C_{n}^{j-1}\). We define a relation \(C_{k_1}^j \sim C_{k_2}^j\) if they are not adjoint but their parents are adjoint. Then

\[
\hat{f} \hat{\sigma} \hat{f} \sigma = \sum_j \sum_k \sum_{l: C_k^j \sim C_l^j} \hat{f}_k \sigma \hat{f}_l \sigma,
\]

where \(f_k := f \chi_{C_k^j}\) and \(f_l := f \chi_{C_l^j}\). We write \(\sum_j \sum_{(k,l)} := \sum_j \sum_k \sum_{l: C_k^j \sim C_l^j}\).

By Lemma 4.2,

\[
\|\hat{f}_k \sigma \hat{f}_l \sigma\|_2 \leq C 2^{-j/2} \|f_k\|_2 \|f_l\|_2.
\]

On the other hand,

\[
\|\hat{f}_k \sigma \hat{f}_l \sigma\|_\infty \leq C \|f_k\|_1 \|f_l\|_1.
\]

Then by interpolation,

\[
\|\hat{f}_k \sigma \hat{f}_l \sigma\|_3 \leq C 2^{-j/2} \|f_k\|_{3/2} \|f_l\|_{3/2}.
\]

From (51) and (52), and Lemma 4.3,

\[
\|\hat{f} \sigma\|_6^2 = \|\hat{f} \sigma \hat{f} \sigma\|_3 = \left\| \sum_j \sum_{(k,l)} \hat{f}_k \sigma \hat{f}_l \sigma\|_3 \right\|_3
\]

\[
\leq \left( \sum_j \sum_{(k,l)} \|\hat{f}_k \sigma \hat{f}_l \sigma\|_3^{3/2} \right)^{2/3}
\]

\[
\leq \left( \sum_j \sum_{(k,l)} 2^{-j/2} \|f_k\|_{3/2}^{3/2} \|f_l\|_{3/2}^{3/2} \right)^{2/3}
\]

\[
\leq \left( \sum_j \sum_k 2^{-j/2} \|f_k\|_{3/2}^{3/2} \right)^{2/3}.
\]
to pass to the last inequality, we have used that for given $k$, there are at most $O(1)$'s $C_l \sim C_{l,k}$.

By interpolation, for any $3/2 < p < 2$,

$$2^{-j/2} \| f_k \|_{3/2}^3 \leq 2^{-j/2} \| f_k \|_1^{3(1-\theta)} \| f_k \|_p^{3\theta} \leq (2^{-j/2} \| f_k \|_1)^{3(1-\theta)} 2^{-j(-1+3\theta/2)} \| f_k \|_p^{3\theta},$$

where $\theta = p/3(p - 1)$. So if normalizing $\| f \|_2 = 1$ and taking $\alpha = 1 - \theta$, it suffices to show that

$$\sum_j \sum_k 2^{-j(3\theta/2 - 1)} \| f_k \|_p^{3\theta} \lesssim 1.$$

We decompose $f_k = f_k \chi_{|f| \leq 2^{-j/2}} + f_k \chi_{|f| > 2^{-j/2}} =: f_k^- + f_k^+$. For $f_k^-$,

$$\sum_j \sum_k 2^{-j(3\theta/2 - 1)} \| f_k^- \|_p^{3\theta} \lesssim 1.$$

Indeed, we apply Hölder’s inequality with $(3\theta/p, 3\theta/(3\theta - p))$,

$$\sum_k \| f_k^- \|_p^{3\theta} \leq \sum_k \int_{C_k} |f|^{3\theta/(2j)} \leq \int |f|^{3\theta(2j)}^{3\theta/p - p}.$$

Since $3\theta/2 - 1 - \frac{3\theta - p}{p} < 0$ as $p < 2$,

$$\sum_j \sum_k 2^{-j(3\theta/2 - 1)} \| f_k^- \|_p^{3\theta} \leq \int \sum_{j \leq 0: |f| < 2^{-j/2}} |f|^{3\theta} \sum_{j \leq 0: |f| < 2^{-j/2}} 2^{-j(3\theta/2 - 1 - \frac{3\theta - p}{p})}$$

$$= C \int |f|^{3\theta} \sum_{j \leq 0: |f| < 2^{-j/2}} 2^{3\theta(\frac{j}{p} - \frac{1}{2})}$$

$$\leq C \int |f|^{3\theta} \times \int |f|^{3\theta - \frac{6\theta}{p}} \leq C \int |f|^{6\theta - \frac{6\theta}{p}}$$

$$= C \int |f|^2 < \infty.$$

For $f_k^+$, we estimate it as follows: as $p < 2$ and $3\theta = p'$, then $3\theta/p = p'/p > 1$; then

$$\sum_{j,k} 2^{-j(\frac{3\theta}{p} - 1)} \| f_k^+ \|_p^{3\theta} = \sum_{j,k} 2^{-j(\frac{p'}{2} - 1)} \| f_k^+ \|_p^{p'}$$

$$\leq \left( \sum_{j,k} 2^{-jp(\frac{1}{2} - \frac{1}{p})} \| f_k^+ \|_p^{p'} \right)^{p'/p} = \left( \sum_{j,k} 2^{-j\frac{p'-p}{2}} \| f_k^+ \|_p^{p'} \right)^{p'/p}$$

$$= \left( \sum_j 2^{-j\frac{2-p}{2}} \int_{|f| > 2^{-j/2}} |f|^p \right)^{p'/p} = \left( \int |f|^p \sum_{j \leq 0: |f| > 2^{-j/2}} 2^{-j(2-p)} \right)^{p'/p}$$

$$\leq C \left( \int |f|^2 \right)^{p'/p} \leq C.$$

Hence Proposition 4.1 follows. \qed
Then Proposition 2.7 follows by a similar argument as in [12, 24].

Now we turn to show that “distant caps interact weakly”. We start with defining the distance between caps. The distance $\rho$ between two given caps $C(z, r)$ and $C'(z', r')$ is

$$\frac{r}{r'} + \frac{r'}{r} + \frac{|z - z'|}{r}.$$  

For any metric space $(X, \rho)$ and any equivalence relation $\equiv$ on $X$, recall from the basic algebra the function $\rho([x], [y]) = \inf_{x' \in [x], y' \in [y]} \rho(x', y')$ is a metric on the set of equivalence classes $X/\equiv$. Let $\mathcal{M}$ be the set of all caps $C \subset S^1$ modulo the equivalence relation $C \equiv -C$, where $-C = \{-x, x \in C\}$. Then a metric on $\mathcal{M}$ can be defined in the following way.

**Definition 4.4.** For any two caps $C, C' \in S^1$,

$$\rho([C], [C']) = \min(\rho(C, C'), \rho(-C, C')),$$  

where $[C]$ denotes the equivalence class $[C] = \{C, -C\} \in \mathcal{M}$.

We will write $\rho(C, C') = \rho([C], [C'])$.

**Lemma 4.5.** For any $\varepsilon > 0$ there exists $\rho < \infty$ such that

$$\|\chi_C \sigma \star \chi_{C'} \sigma\|_{L^2} \leq \varepsilon |C||C'|^{1/2}$$

whenever

$$\rho(C, C') > \rho.$$  

**Proof.** Set

$$f = |C|^{-1/2} \chi_C \leq C r^{-1/2} \chi_C, \quad \tilde{f} = |\tilde{C}|^{-1/2} \chi_{\tilde{C}} \leq C \tilde{r}^{-1/2} \chi_{\tilde{C}}.$$  

Assume $\tilde{r} \leq r$ and $\tilde{r} \ll 1$ and consider the case where no points are nearly antipodals. The case where the points are antipodals can be reduced to the previous case by using the identity

$$(f_1 \sigma \star f_2 \sigma \star f_3 \sigma, f_4 \sigma \star f_5 \sigma \star f_6 \sigma) = (\tilde{f}_1 \sigma \star \tilde{f}_2 \sigma \star \tilde{f}_3 \sigma, \tilde{f}_4 \sigma \star \tilde{f}_5 \sigma \star \tilde{f}_6 \sigma)$$

for any non-negative functions $f_j \in L^2(S^1)$, $j = 1, \cdots, 6$. Let $\varepsilon$ be given. It suffices to show, if $\rho(C, C') > \rho$,

$$\|\chi_C \sigma \star \chi_{C'} \sigma\|_{L^{3/2}} \leq \varepsilon |C|^{1/2} |C'|^{1/2}.$$  

**Case I.** Assume $r \sim \tilde{r}$ and $|x - \tilde{x}| \geq 10r$. Observe that

$$\|f \sigma \star \tilde{f} \sigma\|_{\infty} \leq (r\tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|},$$

and $f \sigma \star \tilde{f} \sigma$ is supported in a rectangle with width $r$ and height $r|x - \tilde{x}|$. Then

$$\|f \sigma \star \tilde{f} \sigma\|_{L^{3/2}(\mathbb{R}^2)} \leq (r\tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|} (r^2 |x - \tilde{x}|)^{2/3} = C \left(\frac{r}{|x - \tilde{x}|}\right)^{1/3} \ll 1.$$
**Case II.** Assume \( \tilde{r} \ll r \) and \( |x - \tilde{x}| \geq 10r \). Still there holds

\[
\|(f \sigma * \tilde{f} \sigma)\|_{\infty} \leq (r \tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|},
\]

but \( f \sigma * \tilde{f} \sigma \) is supported in a tube with base length \( r \) and width \( \tilde{r}|x - \tilde{x}| \). Then

\[
\|(f \sigma * \tilde{f} \sigma)\|_{L^{3/2}(\mathbb{R}^2)} \leq (r \tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|} (r \tilde{r}|x - \tilde{x}|)^{2/3} \leq C \frac{(r \tilde{r})^{1/6}}{|x - \tilde{x}|^{1/3}} \leq C(\tilde{r}/r)^{1/6} \ll 1.
\]

**Case III.** Assume that \( \tilde{r} \ll r \) and \( |x - \tilde{x}| \leq 10r \). The analysis in Case II almost applies if \( |x - \tilde{x}| \geq cr \) for some universal constant \( c > 0 \), while it breaks down when their centers are very close. To cope with this difficulty, we employ the trick in [12]: we replace \( f \) by its restriction \( F \) to the complement of the cap \( \tilde{C}^* \) centered at \( \tilde{x} \) of radius \( 10r^{3/4}/r^{1/4} \). Then

\[
\|f - F\|_2 \leq Cr^{-1/2}(r^{3/4}/r^{1/4})^{1/2} \leq C(\tilde{r}/r)^{1/8} \ll 1.
\]

Then we observe that

\[
\|(F \sigma * \tilde{f} \sigma)\|_{\infty} \leq (r \tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|},
\]

but \( F \sigma * \tilde{f} \sigma \) is supported in a tube with base length \( r \) and width \( \tilde{r}|x - \tilde{x}| \), whose area is less than \( Cr\tilde{r}|x - \tilde{x}| \). Then

\[
\|(F \sigma * \tilde{f} \sigma)\|_{L^{3/2}(\mathbb{R}^2)} \leq (r \tilde{r})^{-1/2} \frac{C}{|x - \tilde{x}|} (r \tilde{r}|x - \tilde{x}|)^{2/3}
\]

\[
\leq C \frac{(r \tilde{r})^{1/6}}{|x - \tilde{x}|^{1/3}} \leq C \frac{(r \tilde{r})^{1/6}}{(r^{3/4}/r^{1/4})^{1/3}} \leq C(\tilde{r}/r)^{1/12} \ll 1.
\]

\(\square\)

5. Step 4. On near-extremals: Proposition 2.13.

In this section, we aim to prove Proposition 2.13, which roughly speaking states that any nearly extremal to the inequality satisfies some appropriately scaled upper bounds relative to some cap up to a small error in \( L^2 \). As remarked in [12], its proof is largely a formal argument relying on two inputs, Lemma 2.7 and Lemma 4.5 which are already established in the previous step. We begin with

**Lemma 5.1.** Let \( f = g + h \in L^2(S^1) \). Suppose that \( \langle g, h \rangle = 0 \) and \( g \neq 0 \), and that \( f \) is a \( \delta \)-nearly extremal for some \( \delta \in (0, 1/4) \). Then

\[
\frac{\|h\|_2}{\|f\|_2} \leq C \max \left( \frac{\|h \sigma * h \sigma * h \sigma\|_2^{1/3}}{\|h\|_2}, \delta^{1/2} \right),
\]

where \( 0 < C < \infty \) is a constant independent of \( g \) and \( h \).

The proof of this lemma is similar to [12, Lemma 7.1] and will be omitted.
5.2. A decomposition algorithm. Let \( f \in L^2(S^1) \) be a nonnegative function with positive norm. The same algorithm in [12] applies:

\[
f = \sum_{0 \leq k \leq \nu} f_k + G_{\nu+1}, \quad \nu = 0, 1, \ldots
\]

with the following properties.

- \( G_0 := f \) and \( \varepsilon_0 = 1/2 \). The inputs for Step \( \nu \) are a nonnegative function \( G_\nu \in L^2(S^1) \) and a positive number \( \varepsilon_\nu \). The outputs are functions \( f_\nu \) and \( G_{\nu+1} \) and nonnegative numbers \( \varepsilon_\nu^* \) and \( \varepsilon_{\nu+1}^* \).
  - If \( \|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2 = 0 \), then \( G_\nu = 0 \) almost everywhere. The algorithm then terminates and we define \( \varepsilon_\nu^* = 0 \), \( f_\nu = 0 \), and \( G_\nu = 0 \), \( f_\mu = 0 \) and \( \varepsilon_\mu = 0 \) for \( \mu > \nu \).
  - If \( 0 < \|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2 < \varepsilon_\nu^* S^3 \|f\|_2^3 \), we replace \( \varepsilon_\nu \) by \( \varepsilon_\nu/2 \), and repeat until the first time that \( \|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2 \geq \varepsilon_\nu^* S^3 \|f\|_2^3 \). Define \( \varepsilon_\nu^* \) to be this value of \( \varepsilon_\nu \). Then

\[
(\varepsilon_\nu^* S^3 \|f\|_2^3) \leq \|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2 \leq 8(\varepsilon_\nu^* S^3 \|f\|_2^3).
\]

Then an application of Lemma 2.7 yields a decomposition for \( G_\nu \): namely we obtain a cap \( C_\nu \) and \( G_\nu = f_\nu + G_{\nu+1} \), where \( f_\nu \) and \( G_{\nu+1} \) satisfy the properties listed in that Lemma. We remark that the constants \( C_\nu \leq C(\varepsilon_\nu^*)^{-O(1)} \) and \( \eta_\nu \geq C(\varepsilon_\nu^*)^{O(1)} \). Define \( \varepsilon_{\nu+1} = \varepsilon_\nu^* \) and move on to the next step \( \nu + 1 \).

- If \( f \) is even, then \( f_\nu \) may likewise be chosen to be even.
- If the algorithm terminates at some finite step \( \nu \), then we have a finite decomposition \( f = \sum_{0 \leq k \leq \nu} f_k \) and \( \varepsilon_k^* = 0 \) for \( k \geq \nu + 1 \).
- If the algorithm never terminates, then \( \nu_\nu^* \to 0 \) and \( \sum_{0 \leq k \leq N} f_\nu \to f \) in \( L^2(S^1) \) as \( N \to \infty \).

The algorithm yields some useful information when the decomposition algorithm is applied to nearly extremals. This is a consequence of Lemma 2.7 and Lemma 5.1.

**Lemma 5.3.** There exists a continuous function \( \theta : (0, 1] \to (0, \infty) \) such that for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for any nonnegative \( \delta \)-nearly extremal \( f \) with \( \|f\|_{L^2(S^1)} = 1 \), then

\[
\|f_\nu\|_2 \geq \theta(\|G_\nu\|_2)
\]

for any index \( \nu \) such that \( \|G_\nu\|_2 \geq \varepsilon \).

**Proof.** Let \( C \) be the exact constant appearing in Lemma 5.1. Given \( \varepsilon > 0 \). We choose \( \delta > 0 \) such that \( C\delta^{1/2} < \varepsilon/2 \). Then the second alternative in Lemma 5.1 yields that

\[
\|G_\nu\|_2 \leq C\left(\frac{\|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2^{1/3}}{\|G_\nu\|_2}\right)^3,
\]

that is to say,

\[
\|G_\sigma \sigma \ast G_\sigma \sigma \ast G_\sigma \sigma\|_2 \geq \left(\frac{1}{C^3}\right)^{1/3} \|G_\nu\|_2^3.
\]
Then an application of Lemma 2.7 yields that, there exists a function \( \theta : (0, 1] \to (0, \infty) \) such that
\[
\|f_\nu\|_2 \geq \eta \|G_\nu\|_2 =: \theta(\|G_\nu\|_2)
\]
where \( \eta \) is as in Lemma 2.7 and \( \theta(x) \) can be regarded as \( O(x^{O(1)}) \).
\[\square\]

Moreover, if \( f \) is nearly extremal, the norms of \( f_\nu \) and \( G_\nu \) enjoy upper bounds independent of \( f \) for all except for large \( \nu \).

**Lemma 5.4.** There exists a sequence of positive constants \( \gamma_\nu \to 0 \) and a function \( N : (0, \frac{1}{2}] \to \mathbb{Z}^+ \) satisfying \( N(\delta) \to \infty \) as \( \delta \to 0 \) such that for any nonnegative \( f \in L^2(S^1) \), if \( f \) is \( \delta \)-nearly extremal then the quantities \( \varepsilon_*^\nu \) obtained when the decomposition algorithm is applied to \( f \) satisfy
\[
\begin{align*}
(70) & \quad \|G_\nu\|_2 \leq \gamma_\nu \|f\|_2 \text{ for all } \nu \leq N(\delta), \\
(71) & \quad \varepsilon_*^\nu \leq \gamma_\nu \text{ for all } \nu \leq N(\delta), \\
(72) & \quad \|f_\nu\|_2 \leq \gamma_\nu \|f\|_2 \text{ for all } \nu \leq N(\delta).
\end{align*}
\]

**Proof.** The proof will be similar to that in [12, Lemma 8.3]. We normalize \( \|f\|_2 = 1 \). Since \( \|f_\nu\|_2 \leq \|G_\nu\|_2 \), and (69) yields that
\[
(\varepsilon_*^\nu)^3 \mathbf{S}^3 \leq \|G_\nu \sigma * G_\nu \sigma * G_\nu \sigma\|_2 \leq \mathbf{S}^3 \|G_\nu\|_2^3,
\]
we see that (70) will imply both (71) and (72). Hence we focus on proving (70). Let \( \eta \) be the function appearing in Lemma 2.7 and we know that \( \eta(\delta) = O(\delta^{O(1)}) \).

We first choose \( \gamma_\nu \) such that it tends to zero so slowly that
\[
\nu \gamma_\nu^2 \eta^2 (c_0 \gamma_\nu^3) > 2 \text{ for all } \nu,
\]
where \( c_0 \) will be clear below; it is possible since \( \eta(\delta) \) is in form of \( O(\delta^{O(1)}) \). Then given \( \delta > 0 \), we choose \( N(\delta) \) to be the least \( \nu \) such that
\[
\gamma_\nu \leq C \delta^{1/2},
\]
where \( C \) is the exact constant appearing in Lemma 5.1; we see that \( N(\delta) \to \infty \) as \( \delta \to 0 \) and that \( \gamma_\nu > C \delta^{1/2} \) for all \( \nu \leq N(\delta) \). Obviously the choices of \( \gamma_\nu \) and \( N \) are independent of \( f \).

Now let \( f \) and \( \delta > 0 \) be given. If there were \( \nu \) such that \( \nu \leq N(\delta) \) and \( \|G_\nu\|_2 \geq \gamma_\nu \), then
\[
\|G_\nu\|_2 \geq \gamma_\nu > C \delta^{1/2}.
\]
Then Lemma 5.1 yields that
\[
\|G_\nu\|_2 \leq \frac{C \|G_\nu \sigma * G_\nu \sigma * G_\nu \sigma\|_2}{\|G_\nu\|_2}.
\]
In other words,
\[
\|G_\nu \sigma * G_\nu \sigma * G_\nu \sigma\|_2 \geq (c_0 \|G_\nu\|_2^3) \|G_\nu\|_2^3 \geq (c_0 \gamma_\nu^3) \|G_\nu\|_2^3.
\]
where \( c_0 = C^{-1} \). Then Lemma 2.7 yields that
\[
\|f_\nu\|_2^2 \geq \eta^2 (c_0 \gamma_\nu^3) \|G_\nu\|_2^2 \geq \eta^2 (c_0 \gamma_\nu^3) \gamma_\nu^2.
\]
Since \( \|G_\mu\|_2 \geq \|G_\nu\|_2 \) for \( \mu \leq \nu \), there also holds that \( \|G_\mu\|_2 > C\delta^{1/2} \) for \( \mu \leq \nu \). So one may repeat the procedure above and find that
\[
\|f_\mu\|_2^2 \geq \eta^2(c_0\gamma_\nu^3)\gamma_\nu^2 \text{ for } \mu \leq \nu.
\]
On the other hand, we have \( \sum_{0 \leq \mu \leq \nu} \|f_\mu\|_2^2 \leq \|f\|_2^2 = 1 \), which gives
\[
\sum_{\mu \leq \nu} \eta^2(c_0\gamma_\nu^3)\gamma_\nu^2 \leq 1, \Rightarrow \nu\gamma_\nu^2\eta^2(c_0\gamma_\nu^3) \leq 1.
\]
This is a contradiction to the choice of \( \gamma_\nu \) in (73). So we finish the proof of this lemma. \( \square \)

5.5. A geometric property of the decomposition. In the previous subsection, based on the single analytic fact Lemma 2.7, we have established that the \( L^2 \)-norms of \( f_\nu \) and \( G_\nu \) obtained when the decomposition algorithm is applied to nearly extremals \( f \) satisfy some uniform upper bounds as in Lemma 5.3 and Lemma 5.4. On the other hand, in Lemma 4.5, we have proved that “distant caps interact weakly”; this will provide us some additional information on near extremals. We first recall a lemma in [12, Lemma 9.1], which follows easily from the pigeonhole principle.

**Lemma 5.6.** In any metric space, for any \( N \) and \( r \), any finite set \( S \) of cardinality \( N \) and diameter equal to \( r \) may be partitioned into two disjoint non-empty subsets \( S = S_1 \cup S_2 \) such that the distance of \( S_1 \) and \( S_2 \) is no less than \( r/2N \). Moreover, given two points \( s_1, s_2 \in S \) satisfying distance \( (s_1, s_2) = r \), this partition may be constructed such that \( s_1 \in S_1 \) and \( s_2 \in S_2 \).

As a consequence, we have

**Lemma 5.7.** For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) and \( 0 < \lambda < \infty \) such that for any nonnegative \( \delta \)-nearly extremal \( f \), the summands \( f_\nu \) obtained from the decomposition algorithm and the associated caps \( C_\nu \) satisfy
\[
\theta(C_j, C_k) \leq \lambda, \text{ whenever } \|f_\nu\|_2 \geq \varepsilon \|f\|_2 \text{ and } \|f_k\|_2 \geq \varepsilon \|f\|_2.
\]

**Proof.** We follow the proof from [12]. We normalize \( \|f\|_2 = 1 \) and let \( \varepsilon \) be given; also suppose that \( \|f_{\delta_{\ell_0}}\|_2, \|f_{k_0}\|_2 \geq \varepsilon \). Let \( N \) be the smallest integer such that \( \|G_{N+1}\|_2 < \varepsilon^3 \). This choice of \( N \) may depend on \( f \) but it will not affect our final choice of \( \lambda \). Since \( \|f_\nu\|_2 \leq \|G_\nu\|_2 \) and \( \|G_\nu\|_2 \) is a non-increasing function of \( \nu \), we see that \( j_0, k_0 \leq N \). Moreover Lemma 5.4 yields that there exists a \( M_\varepsilon \) which depends only on \( \varepsilon \) such that \( N \leq M_\varepsilon \). If we choose \( \delta \) to be sufficiently small but depending on \( \varepsilon \), we see that Proposition 2.7 yields that \( f_\nu \leq \theta(\varepsilon)\|C_\nu\|^{-1/2} \chi_{C_\nu \cup -C_\nu} \), where \( \theta \) is a continuous, strictly positive function on \( (0,1) \). For those \( \nu \leq N \), we have \( \|G_\nu\|_2 \geq \varepsilon^3 \).

Now let \( 0 < \lambda < \infty \) to be a large quantity to be specified. It suffices to show that if \( \delta(\varepsilon) \) is sufficiently small, an assumption that \( \theta(C_j, C_k) > \lambda \) would lead to a contradiction that \( f \) is a \( \delta \)-nearly extremal.
We apply the previous Lemma 5.6 to see that
\[ F = F_1 + F_2 := \sum_{\nu \in S_1} f_{\nu} + \sum_{\nu \in S_2} f_{\nu}, \]
where \([0, N] = S_1 \cup S_2, j_0 \in S_1\) and \(k_0 \in S_2\); also we have \(\varrho(C_j, C_k) \geq \frac{\lambda}{2N} \geq \frac{\lambda}{2M_{\varepsilon}}\) for all \(j \in S_1, k \in S_2\). For \(i, j \in \{1, 2\}\) and \(i \neq j\),
\[
\|F_i \sigma * F_j \|_2 \leq \sum_{j \in S_1, k \in S_2} \|f_j \sigma * f_j \sigma * f_k \|_2 \leq M_{\varepsilon}^3 \gamma(\frac{\lambda}{2M_{\varepsilon}}) \theta^3(\varepsilon),
\]
where \(\gamma(t) \to 0\) as \(t \to \infty\) as in Lemma 4.5. Therefore,
\[
\begin{align*}
&\|F \sigma * F \sigma * F \sigma\|_2^2 \\
&\leq \|F_1 \sigma * F_1 \sigma * F_1 \sigma\|_2^2 + \|F_2 \sigma * F_2 \sigma * F_2 \sigma\|_2^2 + \sum_{i \neq j, i \in \{1, 2\}} \|F_i \sigma * F_i \sigma * F_j \sigma\|_2 \|f\|_2^3 \\
&\leq S^6 (\|F_1\|_2^2 + \|F_2\|_2^2) + CM_{\varepsilon}^3 \gamma(\frac{\lambda}{2M_{\varepsilon}}) \theta^3(\varepsilon) \\
&\leq S^6 (\|F_1\|_2^2 + \|F_2\|_2^2) \max\{\|F_1\|_2^4, \|F_2\|_2^4\} + CM_{\varepsilon}^3 \gamma(\frac{\lambda}{2M_{\varepsilon}}) \theta^3(\varepsilon) \\
&\leq S^6 (1 - \varepsilon^4) + CM_{\varepsilon}^3 \gamma(\frac{\lambda}{2M_{\varepsilon}}) \theta^3(\varepsilon),
\end{align*}
\]
where we have used \(\|F_1\|_2 \geq \varepsilon\) and \(\|F_2\|_2 \geq \varepsilon\) in passing to the last inequality. On the other hand,
\[
(1 - \delta)^6 S^6 \leq \|f \sigma * f \sigma * f \sigma\|_2^2 \leq \|F \sigma * F \sigma * F \sigma\|_2^2 + C\|f\|_2^6 \|f - F\|_2^2 \\
\leq \|F \sigma * F \sigma * F \sigma\|_2^2 + C\varepsilon^6.
\]
So from (75) and (76), we see that
\[
(1 - \delta)^6 S^6 \leq C\varepsilon^6 + S^6 (1 - \varepsilon^4) + CM_{\varepsilon}^3 \gamma(\frac{\lambda}{2M_{\varepsilon}}) \theta^3(\varepsilon).
\]
Recall that \(\gamma(t) \to 0\) as \(t \to \infty\). So given \(\varepsilon > 0\) which is small, if we chose a sufficiently small \(\delta = \delta(\varepsilon)\), then (77) will result in a contradiction if \(\lambda\) is allowed to be sufficiently large. Hence the conclusion of the lemma follows. \(\square\)

5.8. Upper bounds for extremizing sequences. In this subsection, we prove Proposition 2.13.

Lemma 5.9. There exists a function \(\Theta : [1, \infty) \to (0, \infty)\) satisfying \(\lim_{R \to \infty} \Theta(R) = 0\) such that the following holds: given any \(\varepsilon > 0\) and \(\bar{R} > 0\), there exists \(\delta > 0\) such that for any nonnegative \(\delta\)-nearly extremal \(f\) with \(\|f\|_2 = 1\), we have a decomposition
\[
f = F + G
\]
where \(F\) and \(G\) are even and nonnegative with disjoint supports. Moreover this decomposition satisfies \(\|G\|_2 \leq \varepsilon\) and there exists a cap \(C = C(z, r)\) such that for any \(R \in [1, \bar{R}]\), we
have

\[ (78) \quad \int_{\min\{|x+z|,|x-z|\} \geq Rr} F^2(x) d\sigma(x) \leq \Theta(R), \]

\[ (79) \quad \int_{\{F(x) \geq Rr^{-1/2}\}} F^2(x) d\sigma(x) \leq \Theta(R). \]

Let us postpone the proof of this lemma; now we prove Proposition 2.13 by using it.

**Proof of Proposition 2.13 from Lemma 5.9.** Let \( \varepsilon \) and \( f \) be given. We assume that the \( \Theta \) given by Lemma 5.9 is a continuous, strictly decreasing function. Define \( \bar{R} = \bar{R}(\varepsilon) \) by the equation \( \Theta(\bar{R}) = (\varepsilon/2)^2 \). Let \( C = C(z,r) \) and \( \delta = \delta(\varepsilon, \bar{R}(\varepsilon)) \) along with \( F \) and \( G \) satisfy the conclusions in Lemma 5.9. We re-define

\[ (80) \quad f = ((1 - \chi) F + (\chi F + G) =: \tilde{F} + \tilde{G}, \]

where \( \chi(x) = 1 \) if \( \min\{|x-z|,|x+z|\} \geq \bar{R}r \) or \( F(x) \geq \bar{R}r^{-1/2} \). Then it easily follows that

\[ (81) \quad \|\tilde{G}\|_2 \leq \|G\|_2 + \|\chi F\|_2 \leq \varepsilon + 2 \times \varepsilon/2 = 2\varepsilon. \]

On the other hand, we also have

\[ (82) \quad \int_{\min\{|x+z|,|x-z|\} \geq Rr} \tilde{F}^2(x) d\sigma(x) \leq \Theta(R), \]

\[ \int_{\{F(x) \geq Rr^{-1/2}\}} \tilde{F}^2(x) d\sigma(x) \leq \Theta(R). \]

Indeed, when \( R \leq \bar{R} \), \( \tilde{F} \leq F \),

\[ (83) \quad \int_{\min\{|x+z|,|x-z|\} \geq Rr} \tilde{F}^2(x) d\sigma(x) \leq \int_{\min\{|x+z|,|x-z|\} \geq Rr} F^2(x) d\sigma(x) \leq \Theta(R), \]

\[ \int_{\{F(x) \geq Rr^{-1/2}\}} \tilde{F}^2(x) d\sigma(x) \leq \int_{\{F(x) \geq Rr^{-1/2}\}} F^2(x) d\sigma(x) \leq \Theta(R); \]

when \( R \geq \bar{R} \), from the support information of \( \chi \),

\[ (84) \quad \int_{\min\{|x+z|,|x-z|\} \geq Rr} \tilde{F}^2(x) d\sigma(x) \leq \int_{\min\{|x+z|,|x-z|\} \geq Rr} (1 - \chi) F^2(x) d\sigma(x) = 0, \]

\[ \int_{\{F(x) \geq Rr^{-1/2}\}} \tilde{F}^2(x) d\sigma(x) \leq \int_{\{F(x) \geq Rr^{-1/2}\}} (1 - \chi) F^2(x) d\sigma(x) = 0. \]

Hence the proof of Proposition 2.13 is complete if we assume Lemma 5.9. \( \square \)

We are left with proving Lemma 5.9.

**Proof of Lemma 5.9.** Let \( \varepsilon > 0 \) and \( f \geq 0 \) be given with \( \|f\|_2 = 1 \) and also \( R \in [1, \bar{R}]. \) Let \( \{f_\nu, G_\nu\} \) be the pairs obtained from the decomposition algorithm. Choose \( \delta = \delta(\varepsilon) \)
sufficiently small and $M = M(\varepsilon)$ sufficiently large such that
\begin{equation}
\|G_{M+1}\|_2 \leq \varepsilon/2,
\end{equation}
\begin{equation}
f_\nu, G_\nu \text{ satisfy the conclusions in Lemma 5.4 for all } \nu \leq M.
\end{equation}
Set $F = \sum_{0 \leq \nu \leq M} f_\nu$. Then $\|f - F\|_2 = \|G_{M+1}\|_2 \leq \varepsilon/2$. Let $\eta : [1, \infty) \rightarrow (0, \infty)$ be a function to be chosen in the end of the proof satisfying that $\eta(t) \rightarrow 0$ as $t \rightarrow \infty$. This function will not depend on $\bar{R}$.

Let $A(\eta) := \inf\{\nu : \|f_\nu\|_2 < \eta\}$. Then set $N := \min\{M, A(\eta)\}$. Clearly from the upper bound on $A(\eta)$, $N$ is majorized by a quantity depending only on $\eta$ by Lemma 5.4. Set $\mathcal{F} = \mathcal{F}_N := \sum_{0 \leq \nu \leq N} f_\nu$. Then it follows from Lemma 5.4 that
\begin{equation}
\|F - \mathcal{F}\|_2 \leq \gamma(\eta),
\end{equation}
where $\gamma(\eta) \rightarrow 0$ as $\eta \rightarrow 0$. The function $\eta$ is independent of $\varepsilon$ and $\bar{R}$.

Let $\mathcal{C}_0 = \mathcal{C}_0(z_0, r_0)$ be the cap associated to $f_0$ in the decomposition of $f$, and $\mathcal{C}_0$ will be the desired cap in Lemma 5.9. Then we need to find a function $\Theta$ to guarantee that both (78) and (79) hold; in this process, we need to choose a suitable function $\eta$. Suppose that the functions $R \mapsto \eta(R)$ and $R \mapsto \Theta(R)$ are chosen such that
\begin{equation}
\eta(R) \rightarrow 0, \text{ as } R \rightarrow \infty,
\end{equation}
\begin{equation}
\gamma(\eta(R)) \leq \Theta(R), \text{ for all } R.
\end{equation}
Then by (86), $F - \mathcal{F}$ satisfies the desired estimates (78) and (79). Then it suffices to show that
\begin{equation}
\mathcal{F}(x) = 0, \text{ whenever } \min\{|x-z_0|, |x-z_0| \geq Rr_0,
\end{equation}
\begin{equation}
\|\mathcal{F}\|_\infty \leq Rr_0^{-1/2}.
\end{equation}
Before proving (88), we recall several facts. Firstly each summand $f_k \leq C(\eta)|\mathcal{C}_k|^{-1/2} \chi_{\mathcal{C}_k \cup -\mathcal{C}_k}$ where $C(\eta) < \infty$ depends only on $\eta$, and $f_k$ is supported by $\mathcal{C}_k \cup -\mathcal{C}_k$. Moreover, for all $k \leq N$, $\|f_k\|_2 \geq \eta$ by the definition of $N$. Then an application of Lemma 5.7 implies that there exists a function $\eta \mapsto \lambda(\eta)$ such that, if $\delta$ is sufficiently small as a function of $\eta$, we have $\varrho(\mathcal{C}_k, \mathcal{C}_0) \leq \lambda(\eta)$ for all $k \leq N$. This is needed for $\eta = \eta(R)$ for all $R$ in the compact set $[1, \bar{R}]$ so that $\delta$ can be chosen as a function of $\bar{R}$ alone. Hence $\delta$ may be chosen as a function of $\bar{R}$ in addition to the previous dependence on $\varepsilon$.

We are ready to prove (88). Given $x \in S^1$ with $\min\{|x-z_0|, |x+z_0| \geq Rr_0$, either $f_k(x) = 0$ or $\mathcal{C}_k$ has radius $\geq \frac{1}{2} Rr_0$, or the center $z_k$ of $\mathcal{C}_k$ satisfies that $\min\{|z_k+z_0|, |z_k-z_0| \geq \frac{1}{4} Rr_0$. In the latter two cases, there always holds that $\varrho(\mathcal{C}_k, \mathcal{C}_0) \geq CR$. So
\begin{equation}
R \leq CA(\eta(R)).
\end{equation}
This is a contradiction to the choice of $\eta$ if $\eta(R) \rightarrow 0$ slowly enough as $R \rightarrow \infty$. Then we have $\mathcal{F}(x) \equiv 0$ when $\min\{|x+z|, |x-z| \geq Rr_0$. With the choice of $\eta$, $\Theta$ can be defined by
\begin{equation}
\Theta(R) := \gamma(\eta(R)).
\end{equation}
Then (78) holds for all $R \in [1, \bar{R}]$. 

Next we prove (89). We claim that \( \|F\|_\infty \leq Rr_0^{-1/2} \) if \( R \) is taken sufficiently large as a function of \( \eta \). Indeed, because the summands \( f_k \) have pairwise disjoint supports, it suffices to control \( \max_{k \leq N} \|f_k\|_\infty \). For this, Lemma 2.7 implies that

\[
\|f_k\|_\infty \leq C(\eta)r_k^{-1/2}, \quad C(\eta) = O(\eta^{-O(1)}).
\]

If \( \eta(R) \) is chosen to go to zero sufficiently slowly to ensure that \( C(\eta(R))\lambda(\eta(R)) < R \) for all \( k \leq N \), then (79) holds provided that \( \Theta \) is defined as in (91). Indeed, given any \( k \leq N \), \( \|f_k\|_\infty \leq Rr_0^{-1/2} \) would follow if \( C(\eta(R))r_k^{-1/2} \leq Rr_0^{-1/2} \); then it reduces to show that

\[
C(\eta(R))\lambda(\eta(R)) \leq R \quad \text{(92)}
\]

However (92) is guaranteed if we choose \( \eta(R) \to 0 \) sufficiently slow as \( R \to \infty \).

Finally \( \eta \) must be chosen to tend to zero slowly enough to satisfy the requirements in (90) and (92). With this choice of \( \eta \), the proof of Lemma 5.9 is complete. \( \square \)

6. Step 5. Ruling out small caps and existence of extremals

This step aims to establish the Proposition 2.14. We split the proof into 3 subsections, 6.1, 6.5 and 6.6. In Subsection 6.1, we prove two propositions, one on the decomposition of \( f_\nu \sigma \) into “profiles”, and the other on orthogonality of such profiles. Then in Subsection 6.5, we rule out the “small caps” case where \( \lim_{\nu \to \infty} r_\nu = 0 \) with the additional information that \( R > (5/2)^{1/6}R_\nu \). Then we are left with “large caps” case, i.e., where \( \inf_\nu r_\nu > 0 \). In Subsection 6.6, we show that an extremal is obtained for (1).

Let \( \{f_\nu\} \) be an even nonnegative extremizing sequence, uniformly upper even normalized with respect to the caps \( \{C_\nu \cup (-C_\nu)\} \). Without loss of generality, we may assume that \( C_\nu \) is supported on the upper hemisphere of \( S^1 \), \( S^1_+ := \{y \in S^1 : y \cdot (0, 1) > 0\} \). The sequence \( \{f_\nu\} \) satisfies that \( \|f_\nu\|_{L^2(S^1)} = 1 \). Suppose that \( \inf_\nu r_\nu = 0 \). Then up to a subsequence, we may assume that \( \lim_{\nu \to \infty} r_\nu = 0 \).

Decompose

\[
2^{1/2}f_\nu(x) = f_\nu^+(x) + f_\nu^+(-x) + f_\nu^b(x),
\]

where \( f_\nu^+ \) is real, \( f_\nu^+ \) is supported on \( C(z_\nu, r_\nu^{1/2}) \), and \( \|f_\nu^b\|_{L^2} \to 0 \) as \( \nu \to \infty \).

Set

\[
g_\nu := \phi_\nu^+(f_\nu^+) = r_\nu^{1/2}f_\nu^+(\phi_\nu)/((1 - r_\nu^2y^2)^{1/2}),
\]
where \( \phi^*_\nu \) is the rescaling map associated to \( \mathcal{C}_\nu \cup (-\mathcal{C}_\nu) \). It is not hard to see that \( g_\nu \) is upper normalized with respect to the unit ball \( \mathbb{B} \subset \mathbb{R} \), i.e.,

\[
\begin{align*}
g_\nu &\geq 0, \\
\|g_\nu\|_2 &\to 1, \text{ as } \nu \to \infty, \\
\int_{|x| \geq R} |g_\nu|^2 dx &\leq \Theta(R), \forall R \geq 1, \\
\int_{g_\nu \geq R} |g_\nu|^2 dx &\leq \Theta(R), \forall R \geq 1, \\
\Theta(R) &\to 0, \text{ as } R \to \infty.
\end{align*}
\]

(93)

Since \( f_\nu \) is even and \( \|f_\nu\|_{L^2} = 1 \), we have \( \|f^+_\nu\|_{L^2} \to 1 \) as \( \nu \to \infty \). Moreover the function \( F_\nu := f^+_\nu(x) + f^+_\nu(-x) \) satisfies

\[
\frac{\|F_\nu \ast F_\nu \ast F_\nu \sigma\|_{L^6}^2}{\|F_\nu\|_{L^6}^6} = \frac{20}{8} \frac{\|f^+_\nu \sigma \ast f^+_\nu \sigma \ast f^+_\nu \sigma\|_{L^2}^2}{\|f^+_\nu\|_{L^2}^6}.
\]

(94)

So we have

\[
\limsup_{\nu \to \infty} \|f_\nu \sigma \ast f_\nu \sigma \ast f_\nu \sigma\|_{L^2}^2 = \frac{5}{2} \limsup_{\nu \to \infty} \|f^+_\nu \sigma \ast f^+_\nu \sigma \ast f^+_\nu \sigma\|_{L^2}^2.
\]

We will show that the right hand side of (94) is less than \( \frac{5}{2} \text{P}^6 \) by developing a profile decomposition for \( f^+_\nu \). For simplicity of notations, in the subsection 6.1 below, we will write \( f_\nu \) as \( f^+_\nu \).

6.1. **Key propositions.** Recall that we write \( f^+_\nu \) as \( f_\nu \) and \( f^+_\nu \) is supported on \( \mathcal{C}(z_\nu, r_{\nu}^{1/2}) \). By rotation invariance, we may assume that \( z_\nu = (0,1) \) for all \( \nu \). The decomposition for \( \tilde{f}_\nu \sigma \) is motivated by the rescaling relation,

\[
\begin{align*}
|\tilde{f}_\nu \sigma(x,t)| &= \left| \int_{\mathcal{C}_\nu} e^{i(x,t) \cdot \xi} f_\nu(\xi) d\sigma(\xi) \right| \\
&= \left| \int_{|y| \leq 1/2} e^{i(x,t) \cdot \sqrt{1-y^2} f_\nu(y, \sqrt{1-y^2})} \frac{dy}{\sqrt{1-y^2}} \right| \\
&= r_{\nu}^{1/2} \int e^{ir_{\nu}x y + it \sqrt{1-r_{\nu}^2 y^2}} f_{\nu}(r_{\nu} y, \sqrt{1-r_{\nu}^2 y^2}) \cdot \frac{dy}{\sqrt{1-r_{\nu}^2 y^2}} \\
&= r_{\nu}^{1/2} \int e^{i r_{\nu} y \cdot \frac{y^2}{2}} e^{i r_{\nu} t \cdot \left( \sqrt{1-r_{\nu}^2 y^2} \frac{y^2}{2} + \frac{y^2}{2} \right)} \frac{1}{(1-r_{\nu}^2 y^2)^{1/2}} \frac{r_{\nu}^{1/2} f_{\nu}(r_{\nu} y, \sqrt{1-r_{\nu}^2 y^2})}{(1-r_{\nu}^2 y^2)^{1/2}} \cdot dy \\
&= r_{\nu}^{1/2} \frac{r_{\nu}^2 t \Delta}{2} (h_{\nu}(r_{\nu}^2 t, y) g_\nu(y)) (r_{\nu} x),
\end{align*}
\]

(95)
where

\[ h_\nu(t, y) := e^{it \left( \sqrt{1 - r^2 y^2} - 1 + \frac{|y|^2}{2} \right)} . \]

So a decomposition for \( \widehat{f_\nu} \sigma \) immediately follows once we have a decomposition for \( \{g_\nu\} \).

Set \( h_\nu^{-1} = 1/h_\nu \).

**Proposition 6.2.** Let \( \{g_\nu\} \) and \( \{h_\nu\} \) be defined as above. Then there exists a sequence \( (x^k_\nu, t^k_\nu) \in \mathbb{R}^2 \) and \( e^l_\nu \in \mathcal{L}^2(\mathbb{R}) \) such that

\[ g_\nu(y) = \sum_{j=1}^l e^{\frac{it_j y^2}{2}} e^{-ix_j y} h_\nu^{-1}(t_j_\nu, y) \phi^j + e^l_\nu(y) \]

with the following properties: The parameters \( \{(x^k_\nu, t^k_\nu)\} \) satisfy, for \( k \neq j \),

\[ |x^k_\nu - x^j_\nu| + |t^k_\nu - t^j_\nu| \to \infty , \text{ as } \nu \to \infty . \]

For each \( l \geq 1 \),

\[ \|f_\nu\|_{\mathcal{L}^2(S^1)}^2 = \sum_{j=1}^l \|\phi^j\|_2^2 + \|e^l_\nu\|_2^2 \text{ as } \nu \to \infty . \]

The function \( e^l_\nu \) satisfies

\[ \limsup_{l \to \infty} \limsup_{\nu \to \infty} \left\| r_\nu^{1/2} e^{\frac{it_j y^2}{2}} \left[ h_\nu(r_\nu^2 t, y) \frac{e^l_\nu}{(1 - r_\nu^2 y^2)^{1/4}} \right](r_\nu x) \right\|_{\mathcal{L}^2_{\tau, x}(\mathbb{R}^2)} = 0 . \]

where

\[ e^{it \Delta} f(x) = \int e^{ixy - \frac{it y^2}{2}} f(y) dy . \]

**Proposition 6.3** (Orthogonality). Let \( \{(x^j_\nu, t^j_\nu)\} \) be as above and set

\[ G^k_\nu := e^{\frac{it_k y^2}{2}} e^{-ix_k y} h_\nu^{-1}(t^k_\nu, y) \phi^k , \]

\[ G^j_\nu := e^{\frac{it_j y^2}{2}} e^{-ix_j y} h_\nu^{-1}(t^j_\nu, y) \phi^j . \]

Then for \( k \neq j \),

\[ \lim_{\nu \to \infty} \left\| \left( r_\nu^{1/2} e^{\frac{it_j y^2}{2}} \left[ h_\nu(r_\nu^2 t, y) \frac{G^k_\nu}{(1 - r_\nu^2 y^2)^{1/4}} \right](r_\nu x) \right) \left( r_\nu^{1/2} e^{\frac{it_j y^2}{2}} \left[ h_\nu(r_\nu^2 t, y) \frac{G^j_\nu}{(1 - r_\nu^2 y^2)^{1/4}} \right](r_\nu x) \right) \right\|_{\mathcal{L}^2_{\tau, x}(\mathbb{R}^2)} = 0 . \]

We state a useful lemma on a localized linear restriction estimate, which will be used in the proof of Proposition 6.2.

**Lemma 6.4.** Let \( 4 < q < 6 \) and \( h_\nu \) be defined as in (96). Assume that \( \lim_{\nu \to \infty} r_\nu = 0 \). Then if \( \|(1 - r_\nu^2 y^2)^{1/4} f\| \leq M \) for some \( M > 0 \) and for all \( |y| \leq R \) and all sufficiently large \( \nu \),

\[ \|e^{it \Delta} h_\nu(t, y) \frac{f(y)}{(1 - r_\nu^2 y^2)^{1/4}}\|_{\mathcal{L}^q_{\tau, x}} \leq CM , \text{ uniformly in sufficiently large } \nu , \]
where the constant may depend on $R$, but not on $\nu$.

Proof. Choose $r_\nu$ sufficiently small such that $B(0, R) \subset \{|y| \leq \frac{1}{2} r_\nu^{-1}\}$. We write

$$e^{\frac{it}{r_\nu^2}} \int_0^1 f(y) \frac{1}{1 - r_\nu^2 y^2} \frac{dy}{1 - |y|^2} = \int e^{i r_\nu y + \frac{it}{r_\nu^2} \sqrt{1 - |y|^2}} f(r_\nu^{-1} y) \frac{dy}{1 - |y|^2} \frac{dy}{1 - |y|^2}.$$

Then

$$\left| \int e^{i r_\nu y + i \frac{it}{r_\nu^2} \sqrt{1 - |y|^2}} f(r_\nu^{-1} y) (1 - |y|^2)^{\frac{1}{2}} \frac{dy}{\sqrt{1 - |y|^2}} \right|_{L^1(x)} = r_\nu^{-1 + \frac{3}{q}} \left| \int_{S^1} e^{i r_\nu y + i t \sqrt{1 - |y|^2}} f(r_\nu^{-1} y) (1 - |y|^2)^{\frac{1}{2}} d\sigma \right|_{L^1(x)} \leq r_\nu^{-1 + \frac{3}{q}} \| f(r_\nu^{-1} y) (1 - y^2)^{1/4} \|_{L^p(\sigma, +)},$$

where $p$ satisfies $\frac{3}{q} = 1 - \frac{1}{p}$, $p < 2$, and $L^p(\sigma, +)$ is understood as integrating over $S^1$; we have also regarded $f(y)$ as a function on the upper hemisphere $S^1 := \{ z \in S^1 : z \cdot (0, 1) > 0 \}$.

Then continuing the above inequality, we have

$$r_\nu^{-1 + \frac{3}{q}} \left( \int |f(r_\nu^{-1} y) (1 - y^2)^{1/2} \frac{dy}{\sqrt{1 - y^2}} \right)^{1/p} \leq r_\nu^{-1 + \frac{3}{q} + \frac{1}{p}} \left( \int \frac{|f(y) (1 - r_\nu^2 y^2)^{1/2} \frac{dy}{\sqrt{1 - r_\nu^2 y^2}}}{\sqrt{1 - y^2}} \right)^{1/p} \leq CMR^{1/p},$$

for all sufficiently large $\nu$. This finishes the proof of Lemma 6.4.

Now we will first prove Proposition of 6.2, and then Proposition 6.3.

The proof of Proposition 6.2. We split the proof into two steps.

Step 1. For $(x, t) \in \mathbb{R}^2$, we define

$$T_\nu(g)(y) = e^{\frac{-it}{r^2}} e^{i x \nu} h_\nu(t, y) g(y);$$

analogously $T_\nu^i$ for $(x, t^i)$ for $i \geq 1$, and $T_\nu^{-1}(g)(y) = e^{\frac{-it}{r^2}} e^{i x \nu} h_\nu^{-1}(t, y) g(y)$. Let $P^0$ denote the sequence $\{g_\nu\}_{\nu \geq 1}$. Then we define the set

$$W(P^0) = \{ w - \lim_{\nu \to \infty} T_\nu(P^0_\nu)(y) \} \in L^2(\mathbb{R}) : (x, t, \nu) \in \mathbb{R}^2,$$
where \( w - \lim f_* \) denotes a weak limit of \( \{f_*\} \) in \( L^2 \). Define the blow-up criterion associated to \( \mathcal{W}(P^0) \):

\[
\mu(P^0) := \sup\{\|\phi\|_{L^2(\mathbb{R})} : \phi \in \mathcal{W}(P^0)\}.
\]

Then for any \( \phi \in \mathcal{W}(P^0) \),

\[
\|\phi\|_{L^2} \leq \limsup_{\nu \to \infty} \|T_{\nu}(g_\nu)\|_{L^2} = \limsup_{\nu \to \infty} \frac{r_*^{1/2} f_* (r_* y, \sqrt{1 - r_*^2 y^2})}{(1 - |r_* y|^2)^{1/4}} = \limsup_{\nu \to \infty} \|f_*\|_{L^2(\sigma, +)},
\]

where the integral in \( L^2(\sigma, +) \) should be understood as integrating over the upper hemisphere.

If \( \mu(P^0) = 0 \), then we set \( l = 0 \), and \( c_0 = g_* \) for all \( \nu \geq 1 \). Otherwise, \( \mu(P^0) > 0 \), then up to a subsequence, there exists nontrivial \( \phi^1 \in L^2 \) and \( (x_*^1, t_*^1)_{\nu \geq 1} \) such that

\[
(104) \quad \phi^1 = w - \lim_{\nu \to \infty} T_{\nu}^1(P^0_\nu)(y),
\]

\[
(105) \quad \|\phi^1\|_2 \geq \frac{1}{2} \mu(P^0).
\]

Let \( P^1 \) denote the sequence \( \{g_* - (T_{\nu}^1)^{-1}(\phi^1)(y)\}_{\nu \geq 1} \) and set

\[
e_0^1 := g_* - (T_{\nu}^1)^{-1}(\phi^1)(y).
\]

It is not hard to see that

\[
(106) \quad w - \lim_{\nu \to \infty} T_{\nu}^1(P^1_\nu) = 0,
\]

\[
(107) \quad \|f_*\|_{L^2(S^1)}^2 - \|\phi^1\|_2^2 = \|e_0^1\|_2^2,
\]

as \( \nu \to \infty \).

For \( P^1 = \{g_* - (T_{\nu}^1)^{-1}(\phi^1)(y)\}_{\nu \geq 1} \), we iteratively consider the set

\[
\mathcal{W}(P^1) = \{w - \lim_{\nu \to \infty} T_{\nu}(P^1_\nu) : (t_*^1, x_*^1) \in \mathbb{R}^2\}.
\]

Then we test whether \( \mu(P^1) > 0 \): if \( \mu(P^1) = 0 \), then the algorithm stops. If not, then up to a subsequence, there exists nontrivial \( \phi^2 \in L^2 \) and \( (x_*^2, t_*^2)_{\nu \geq 1} \) such that

\[
(108) \quad \phi^2 = w - \lim_{\nu \to \infty} T_{\nu}^2(P^1_\nu)(y),
\]

\[
(109) \quad \|\phi^2\|_2 \geq \frac{1}{2} \mu(P^1).
\]

By a similar consideration as in (106) and (107), if setting \( P^2 = \{P^1_\nu - (T_{\nu}^2)^{-1}(\phi^2)\} \) and assuming (98), then

\[
(108) \quad w - \lim_{\nu \to \infty} T_{\nu}^2(P^2_\nu) = 0,
\]

\[
\|f_*\|_2^2 - \sum_{j=1}^{2} \|\phi^j\|_2^2 = \|e_0^2\|_2^2,
\]

as \( \nu \to \infty \),

where

\[
e_0^2 := g_* - (T_{\nu}^1)^{-1}\phi^1 - (T_{\nu}^2)^{-1}\phi^2.
\]
The orthogonality in the $L^2$ norm above needs an input, namely, (98). Otherwise, up to a subsequence we may assume that

$$|t^2_\nu - t^1_\nu| + |x^2_\nu - x^1_\nu| \to c, \text{ as } \nu \to \infty,$$

for some $0 \leq c < \infty$. In this case, the dominated convergence theorem gives, up to a subsequence,

$$T^2_\nu(T^1_\nu)^{-1} \text{ converges strongly in } L^2.$$

This will imply that

(110) \quad $T^2_\nu(P^1_\nu) \to 0$, weakly in $L^2$,

as $T^1_\nu(P^1_\nu) \to 0$ weakly in $L^2$ and the following relation holds,

$$T^2_\nu(P^1_\nu) = T^2_\nu(T^1_\nu)^{-1}(T^1_\nu(P^1_\nu)).$$

But the claim in (110) is a contradiction to the existence of nontrivial $\phi^2$. So (98) holds.

Iterating this argument, a diagonalization process produces a family of pairwise orthogonal sequences $(x^j_\nu, t^j_\nu)$ and $\phi^j$ satisfying (97), (98) and (99). Since $\sum_j \|\phi^j\|_2^2 \leq \sup_\nu \|f_\nu\|_2^2 < \infty$ and $\mu(P^{l+1}) \leq 2\|\phi^l\|_2$, we have

(111) \quad $\mu(P^l) \to 0$, as $l \to \infty$.

To conclude this step, we deduce some information on $e^l_\nu$. Firstly, the orthogonality condition (98) implies that, for any $\psi \in L^\infty$, the orthogonality (98) implies that, for each $l \geq 1$,

(112) \quad $\|g^\nu\psi\|_2^2 = \sum_{j=1}^l \|\phi^j\psi\|_2^2 + \|e^l_\nu\psi\|_2^2$

as $\nu \to \infty$. In particular, this holds for $\psi \in S$, the Schwartz class on $\mathbb{R}$.

Let $R \gg 1$. Define a set

$$E = \{y \in \mathbb{R} : |y| \leq R \text{ and } |g^\nu(y)| \leq R \text{ for all sufficiently large } \nu\}.$$

Then (112) implies that, for any $l \geq 1$,

$$\limsup_{\nu \to \infty} \|e^l_\nu 1_E\|_{L^\infty} \leq CR$$

for some $C > 0$. This further implies that,

$$\limsup_{\nu \to \infty} \|(1 - r^2_\nu y^2)^{1/4} e^l_\nu 1_E\|_{L^\infty} \leq CR.$$

**Step 2.** At this step, we show that the localized restriction estimate $L^\infty \to L^q_{t,x}$ for some $q < 6$ in Lemma 6.4, together with the information that $\lim \mu(P^l) = 0$, will imply (100). To do it, by scaling, the norm on the left hand side of (100) is equivalent to

$$\left\| \int e^{i y y + it} \frac{\sqrt{1-r^2_\nu y^2}}{r^2} e^l_\nu(y) \frac{dy}{(1 - r^2_\nu y^2)^{1/4}} \right\|_{L^q_{t,x}}.$$
For each $R \gg 1$, recall the definition of the set $E$. We split
\[ e^l_\nu = e^l_\nu 1_E + e^l_\nu (1 - 1_E). \]
Since the following operator is uniformly bounded from $L^2(\mathbb{R})$ to $L^6_t(\mathbb{R} \times \mathbb{R})$,
\[ \phi \mapsto \int e^{ixy + it\sqrt{1 - r^2y^2^{-1}}} \phi(y) \frac{(1 - 1_E)}{(1 - r^2y^2)^{1/4}} dy, \]
and $f_\nu$ is upper normalized with respect to $C(z_\nu, r_\nu)$, up to a subsequence, we may conclude that by (112)
\[ \left\| \int e^{ixy + it\sqrt{1 - r^2y^2^{-1}}} e^l_\nu(y)(1 - 1_E) \frac{(1 - 1_E)}{(1 - r^2y^2)^{1/4}} dy \right\|_{L^6_t} \leq \left\| e^l_\nu(y)(1 - 1_E) \right\|_2 \leq C \left\| g_\nu(1 - 1_E) \right\|_2 \leq \Theta(R) \]
as $\nu \to \infty$. So we may restrict our attention to $e^l_\nu$ on $E$. By the discussion at the end of \textbf{Step 1}, we may assume that, for all $l \geq 1$,
\[ \limsup_{\nu \to \infty} \left\| (1 - r^2y^2)^{1/2} e^l_\nu 1_E \right\|_\infty \leq CR. \]
Then by Lemma 6.4,
\[ \limsup_{\nu \to \infty} \left\| e^{\frac{itl}{2}} (h_\nu(t) \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}}) \right\|_{L^6_t} \leq C \]
for some $C > 0$ independent of $\nu$ and $l$. Then by the interpolation, establishing (100) is reduced to
\[ \limsup_{l \to \infty} \limsup_{\nu \to \infty} \left\| e^{\frac{itl}{2}} (h_\nu(t) \frac{e^l_\nu(y)}{(1 - r^2y^2)^{1/4}}) \right\|_{L^\infty_t} = 0. \]
This will follow from the fact that $\mu(P^l) \to 0$ as $l \to \infty$. Indeed, there exists $(x^l_\nu, t^l_\nu)$ such that, up to a subsequence,
\[ \left\| e^{\frac{itl}{2}} (h_\nu(t^l_\nu) \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}}) (x^l_\nu) \right\|_{L^\infty_t} \sim \left\| e^{\frac{itl}{2}} (h_\nu(t)) \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}} \right\|_{L^\infty_t}. \]
On the other hand, since $e^l_\nu$ is compactly supported,
\[ e^{-\frac{itl}{2}y^2} e^{ixl_y} h_\nu(t^l_\nu, y) \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}} = e^{-\frac{itl}{2}y^2} e^{ixl_y} h_\nu(t^l_\nu, y) e^l_\nu, \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}} \phi_R(y) \]
for some suitable bump function $\phi_R$ adapted to the ball $B(0, R)$; taking integration in $y$ on both sides, we have
\[ \left\| e^{\frac{itl}{2}} (h_\nu(t_\nu, y)) \frac{e^l_\nu 1_E}{(1 - r^2y^2)^{1/4}} (x^l_\nu) \right\|_{L^6_t} \leq \left\| e^{\frac{itl}{2}} e^{ixl_y} h_\nu(t^l_\nu, y) e^l_\nu, \frac{\phi_R 1_E}{(1 - r^2y^2)^{1/4}} \right\|_{L^6_t}. \]
Since $P^l = \{ e^l_\nu \}_{\nu \geq 1}$, by the definition of $\mu(P^l)$,
\[ \text{LHS (115)} \leq \mu(P^l) \left\| \frac{\phi_R 1_E}{(1 - r^2y^2)^{1/4}} \right\|_{L^2} \leq \mu(P^l) \left\| \phi_R 1_E \right\|_{L^2} \to 0, \text{ as } l \to \infty, \]
since $\mu(P^l) \to 0$ as $l \to \infty$. This finishes the proof of (100).

Therefore the proof of Proposition 6.2 is complete. \qed
Next we show that (98) implies the orthogonality result (103) in Proposition 6.3.

The proof of Proposition 6.3. To begin, we may assume that $\phi^j$ and $\phi^k$ are smooth functions with compact supports. Also we recall that

$$e^{\frac{it}{\Delta}} \left( h_\nu(t, y) \frac{G^k_\nu}{(1 - r^2_\nu y^2)^{1/4}} \right) = \int e^{i(x - x^\nu_t - (t - t^\nu_y)^2)} e^{i(t - t^\nu_y)} \left( \frac{\sqrt{1 - r^2_\nu y^2 - 1}}{r^2_\nu} + \frac{y^2}{2} \right) \frac{\phi^k(y)}{(1 - r^2_\nu y^2)^{1/4}} dy.$$

Likewise for $e^{\frac{it}{\Delta}} \left( h_\nu(t, y) \frac{G^j_\nu}{(1 - r^2_\nu y^2)^{1/4}} \right)$. Then by a change of variables, we need to show

$$\left\| e^{\frac{it}{\Delta}} \left( e^{i(t - t^\nu_y)} \left( \frac{\sqrt{1 - r^2_\nu y^2 - 1}}{r^2_\nu} + \frac{y^2}{2} \right) \frac{\phi^j}{(1 - r^2_\nu y^2)^{1/4}} \right) (x - (x^j - x^k)) \right\| \to 0$$

as $\nu$ goes to infinity.

For a large $N \gg 1$, set

$$\Omega_N := \{(t, x) : |t| + |x| \leq N\}, \quad \Omega_{N, \nu} = \Omega_N - (t^\nu_t - t^\nu_y, x^\nu_t - x^\nu_y).$$

We first claim that, for $\Omega = \Omega_N$ or $\Omega_{N, \nu}$,

$$\int_{\Omega} \left| e^{\frac{it}{\Delta}} \left( e^{i(t - t^\nu_y)} \left( \frac{\sqrt{1 - r^2_\nu y^2 - 1}}{r^2_\nu} + \frac{y^2}{2} \right) \frac{\phi^j}{(1 - r^2_\nu y^2)^{1/4}} \right) (x - (x^j - x^k)) \right|^3 dxdt \to 0$$

as $N$ goes to infinity uniformly in $\nu$. Here $\Omega^c := \mathbb{R}^2 \setminus \Omega$.

We write

$$e^{\frac{it}{\Delta}} \left( e^{i(t - t^\nu_y)} \left( \frac{\sqrt{1 - r^2_\nu y^2 - 1}}{r^2_\nu} + \frac{y^2}{2} \right) \frac{\phi^k}{(1 - r^2_\nu y^2)^{1/4}} \right) (x) = \int e^{i(xy + it \sqrt{1 - r^2_\nu y^2 - 1}} \frac{\phi^k(y)}{(1 - r^2_\nu y^2)^{1/4}} dy.$$

For $y$ in a compact set in $\mathbb{R}$ and all sufficiently small $r_\nu > 0$, we have

$$\left| \frac{\partial^2}{r_\nu^2} \left( \frac{\sqrt{1 - |r_\nu y|^2 - 1}}{r_\nu^2} \right) \right| \sim 1/4, \text{ uniformly in all sufficiently large } \nu.$$

We state three important estimates uniformly in all sufficiently large $\nu$. The first is by the stationary phase estimate [26, p.334]:

$$\left| e^{\frac{it}{\Delta}} \left( e^{i(t - t^\nu_y)} \left( \frac{\sqrt{1 - r^2_\nu y^2 - 1}}{r^2_\nu} + \frac{y^2}{2} \right) \frac{\phi^k}{(1 - r^2_\nu y^2)^{1/4}} \right) (x) \right| \leq C_{\phi^k} |t|^{-1/2}.$$
Secondly by integration by parts, if $|x| \geq C|t|$ for a large constant $C > 0$ depending on the size of the compact support of $\phi^k$, for all sufficiently large $\nu$,

$$
\left| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \leq C_\phi |x|^{-1}. 
\right.
$$

(122)

Thirdly there always holds a trivial bound, for all $x, t$,

$$
\left| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \leq C_\phi. 
\right.
$$

(123)

Here all constants $C_\phi$ depends on the function $\phi^k$ but independent of $\nu$. We are now ready to prove (119) when $\Omega = \Omega_N$; the case where $\Omega = \Omega_{N, \nu}$ is similar and so will be omitted. By the Cauchy-Schwarz inequality,

$$
\text{LHS of (119)} \leq C \left\| e^{it\left(\frac{(t_j^\nu - t_{j-1}^\nu)}{2} \Delta \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^j}{(1-r^2y^2)^{1/4}} \right)}(x - (x_j^\nu - x_{j-1}^\nu) \right) \right)} \right\|_{L^6(\mathbb{R}^2)}
\times \left\| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \right\|_{L^6(\mathbb{R}^2)}
\leq C \left\| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \right\|_{L^6(\mathbb{R}^2)}
\times \left\| e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^j}{(1-r^2y^2)^{1/4}} \right)}(x - (x_j^\nu - x_{j-1}^\nu) \right) \right\|_{L^6(\mathbb{R}^2)}.
$$

(124)

The first term is bounded by the Tomas-Stein inequality and a change of variables. For the second term, by using estimates (121), (122) and (123), we see that

$$
\left\| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \right\|_{L^6(\mathbb{R}^2)} \to 0, \text{ as } N \to \infty, \text{ uniform in } \nu.
$$

(125)

Therefore we have established (119). To finish the proof (118), we need to show that, for a fixed $N \gg 1$,

$$
\int_{\Omega_N \cap \Omega_{N, \nu}} e^{it\left(\frac{(t_j^\nu - t_{j-1}^\nu)}{2} \Delta \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^j}{(1-r^2y^2)^{1/4}} \right)}(x - (x_j^\nu - x_{j-1}^\nu) \right) \right)} \times
\left\| e^{\frac{\nu\Delta}{2}} \left( e^{it\left(\frac{\sqrt{1-r^2y^2-1} + 2}{r^2} \frac{\phi^k}{(1-r^2y^2)^{1/4}} \right)}(x) \right) \right\|_{L^6(\mathbb{R}^2)} \to 0
$$

(126)

as $\nu$ goes to infinity. It actually holds as

$$
\text{measure}(\Omega_N \cap \Omega_{N, \nu}) \to 0, \text{ when } \lim_{\nu \to \infty} |t_j^\nu - t_{j-1}^\nu| + |x_j^\nu - x_{j-1}^\nu| = \infty,
$$
and we can apply $L^\infty_{t,x}$-bounds to both integrals, which are controlled as $\phi^j$ and $\phi^k$ are assumed to be bounded and compactly supported. Therefore the proof of (103) is complete. \hfill $\square$

6.5. **Ruling out small caps.** By the discussion at the beginning of Section 6, we aim to show that

(127) \[ \lim_{\nu \to \infty} \|\overline{f^+_\nu} \sigma\|_6^6 \leq \mathcal{R}_P^6, \]

which leads to $\mathcal{R} \leq (5/2)^{1/6} \mathcal{R}_P$. However, it is a contradiction to the strict inequality in Proposition 2.1.

By Propositions 6.2 and 6.3,

\[
\lim_{\nu \to \infty} \|\overline{f^+_\nu} \sigma\|_6^6 \leq \sum_{j=1}^{\infty} \lim_{\nu \to \infty} \|e^{it\Delta} (h_\nu(t,y) \frac{G^j_\nu}{(1-r^2_{\nu}y^2)^{1/4}})\|_6^6
\]

\[
= \sum_{j=1}^{\infty} \lim_{\nu \to \infty} \left\| \int e^{i(x-x')2} e^{i\frac{r^2}{2}} \frac{\phi^j(y)}{(1-r^2_{\nu}y^2)^{1/4}} dy \right\|_6^6
\]

(128)

\[
= \sum_{j=1}^{\infty} \|e^{it\Delta} \phi^j\|_6^6 \leq \mathcal{R}_P^6 \sum_j \|\phi^j\|_2^6
\]

\[
\leq \mathcal{R}_P^6 \left( \sum_j \|\phi^j\|_2^2 \right)^3
\]

\[
\leq \mathcal{R}_P^6 \lim_{\nu \to \infty} \|f^+_\nu\|_2^6 = \mathcal{R}_P^6.
\]

This proves (127). Here we have used

\[
\lim_{\nu \to \infty} \left\| \int e^{ixy} e^{-\frac{\nu^2}{2}} \frac{\phi^j(y)}{(1-r^2_{\nu}y^2)^{1/4}} dy - e^{it\Delta} \phi^j(x) \right\|_6^6 = 0.
\]

This follows from the stationary phase analysis and the dominated convergence theorem. So far we have proved that the first half of Proposition 2.14 that $\inf \nu r_\nu > 0$.

6.6. **Big caps; existence of extremals.** In this section we aim to prove the second half of Proposition 2.14: There exists an extremal function for the Tomas-Stein inequality (1). The proof is similar to the process of ruling out small caps above. Let $\{f_\nu\}$ be an extremizing sequence of nonnegative functions supported on the whole sphere and even upper normalized with respect to caps $C_\nu \cup (-C_\nu)$. We have proved that $\inf \nu r_\nu > 0$. Then up to a subsequence, the uniform upper normalization means simply that $\|f_\nu\|_{L^2(S^1)} \leq 1$, and there exists a function $\Theta$ independent of $\nu$ and satisfying that $\Theta(R) \to 0$ as $R \to \infty$. 
such that
\[ \int_{|f_{\nu}(x)| > R} |f_{\nu}(x)|^2 d\sigma(x) \leq \Theta(R) \]
for all \( \nu \). The radii no longer enter into the discussion.

We denote \( f_{\nu}^\pm \) the restrictions of \( f_{\nu} \) to the upper hemisphere \( S^1_+ \) and the lower. Then we see that \( f_{\nu}^+(x) = f_{\nu}^-(x) \), for \( x \in S^1_+ \), and \( \|f_{\nu}\|_2^2 = 2\|f_{\nu}^\pm\|_2^2 \). Moreover, by a simple change of variables,

\[
(129) \quad \overline{f_{\nu}}^\sigma(t, x) = \overline{f_{\nu}^+}(t, -x) = \overline{f_{\nu}^+}(t, x).
\]
Then
\[
(130) \quad \overline{f_{\nu}}^\sigma(t, x) = \overline{f_{\nu}^+}(t, x) + \overline{f_{\nu}^+}(t, x) = 2\Re \overline{f_{\nu}^+}(t, x),
\]
where \( \Re f \) denotes the real part of \( f \).

Write
\[
\overline{f_{\nu}^+}(x, t) = \int_{S^1_+} e^{i(t,x)\cdot z} f_{\nu}^+(z)d\sigma(z)
= \int e^{i(x,y\cdot t)\cdot \sqrt{1-y^2}} f_{\nu}^+(y, \sqrt{1-y^2}) \frac{d\nu}{\sqrt{1-y^2}}.
\]

Similar to Propositions 6.2 and 6.3 in the subsection 6.1, we will develop a profile decomposition for \( f_{\nu}^+(y, \sqrt{1-y^2})/(1-y^2)^{1/4} \).

**Proposition 6.7.** Let \( \{f_{\nu}^+\} \) be defined above. Then there exists a sequence \( (x_{\nu}^k, t_{\nu}^k) \in \mathbb{R}^2 \) and \( e_{\nu}^1 \in L^2(\mathbb{R}) \) such that

\[
(131) \quad \frac{f_{\nu}^+(y)}{(1-y^2)^{1/4}} = \sum_{j=1}^l e^{-i\nu^j y - i t_{\nu}^j \sqrt{1-y^2} \phi^j(y)} + e_{\nu}^1(y)
\]
with the following properties: The parameters \( \{(x_{\nu}^k, t_{\nu}^k)\} \) satisfy, for \( k \neq j \),

\[
(132) \quad |x_{\nu}^k - x_{\nu}^j| + |t_{\nu}^k - t_{\nu}^j| \to \infty, \ \text{as} \ \nu \to \infty.
\]
For each \( l \geq 1 \),

\[
(133) \quad \|f_{\nu}^+\|_{L^4(S^1)}^2 = \sum_{j=1}^l \|\phi^j\|_2^2 + \|e_{\nu}^1\|_2^2, \ \text{as} \ \nu \to \infty.
\]
The function \( e_{\nu}^1 \) satisfies, if \( E_{\nu}^1 = (1-y^2)^{1/4}e_{\nu}^1 \)

\[
(134) \quad \limsup_{l \to \infty} \limsup_{\nu \to \infty} \left\| \overline{E_{\nu}^1}^\sigma \right\|_{L^2_{x,t}(\mathbb{R}^2)} = 0.
\]

**Proposition 6.8 (Orthogonality).** Let \( \{(x_{\nu}^j, t_{\nu}^j)\} \) be as above and set

\[
G_{\nu}^k := e^{-i\nu^k y - i t_{\nu}^k \sqrt{1-y^2} \frac{1}{2} \phi^k},
\]

\[
G_{\nu}^j := e^{-i\nu^j y - i t_{\nu}^j \sqrt{1-y^2} \frac{1}{2} \phi^j}.
\]
Then for \( k \neq j \),

\[
\lim_{\nu \to \infty} \left\| \frac{G^k_\nu \sigma}{1} \right\|_{L^6_1(R^2)} = 0.
\]

The proofs are similar and so we omit the details. Now we are ready to prove the existence of extremals for (1).

\[
\mathcal{R}^6 = \lim_{\nu \to \infty} \left\| f^\nu \sigma \right\|_6^6 = 2^6 \lim_{l \to \infty} \sup_{\nu \to \infty} \left\| \mathcal{R} \left\{ \sum_{j=1}^l G^j_\nu \sigma + E^j_\nu \sigma \right\} \right\|_6^6
\]

\[
\leq 2^6 \lim_{l \to \infty} \sup_{\nu \to \infty} \left\| \mathcal{R} \sum_{j=1}^l G^j_\nu \sigma \right\|_6^6
\]

\[
= \sum_j \left\| \int e^{i(x,t) \cdot z} (1 - y^2)^{1/4} [\phi^j(y)1_{S^1_+}(z) + \tilde{\phi}^j(y)1_{-S^1_+}(z)] d\sigma(y) \right\|_6^6
\]

\[
\leq \mathcal{R}^6 \sum_j \left( (1 - y^2)^{1/4} [\phi^j(y)1_{S^1_+}(z) + \tilde{\phi}^j(y)1_{-S^1_+}(z)] \right) \left\| L^{2}(S^1, \sigma) \right\|_6^6
\]

\[
\leq \mathcal{R}^6 \left( 2 \left\| L^2(f^\nu \sigma) \right\|_2^2 \right)^3 \leq \mathcal{R}^6 \left( \sum_j 2 \left\| L^2(f^j \sigma) \right\|_2^2 \right)^3
\]

\[
\leq \mathcal{R}^6 (2 \left\| f^\nu \sigma \right\|_2^2)^3 = \mathcal{R}^6 \left\| f^\nu \sigma \right\|_6^6 = \mathcal{R}^6.
\]

where \( z = (y, \cdot) \in S^1 \), and \( 1_{S^1_+} \) and \( 1_{-S^1_+} \) denotes the indicator functions of the upper and the lower hemispheres of \( S^1 \), respectively.

Then \( \mathcal{R}^6 = \mathcal{R}^6 \) forces all the inequalities above to be equal. On the other hand, because \( 2 \sum_j \left\| \phi^j \right\|_2^2 \leq \left\| f^\nu \sigma \right\|_2^{2}(S^1, \sigma) = 1 \), there will be only one \( j \) left from the sharpness of embedding of \( \ell^3 \) into \( \ell^1 \). Thus for this \( j \), there exists an extremal

\[
\left( \phi(y)1_{S^1_+} + \tilde{\phi}^j(y)1_{-S^1_+} \right) (1 - y^2)^{1/4}.
\]

This completes the proof of Proposition 2.14 and hence Theorem 1.2.

**Appendix A. A strict comparison, \( S > (5/2)^{1/6} P \).**

In this section, we aim to establish the strict comparison inequality in Proposition 2.1 by using a similar perturbation argument in [12, Section 17] on

\[
\left\| f^\epsilon \sigma \right\|_6^6 / \left\| f^\epsilon \sigma \right\|_2^2,
\]

where \( f^\epsilon \) is defined in (141).

We list several definitions.

\[
(139) \quad (y, (1 - |y|^2)^{1/2}) = (y, 1 - \frac{1}{2} |y|^2 - \frac{1}{8} |y|^4 + O(|y|^6)),
\]
\[(140)\]
\[d\sigma_\varepsilon := (1 + \frac{1}{2}|y|^2 + O(|y|^4)) \, dy,\]

\[(141)\]
\[f_\varepsilon(z) := \varepsilon^{-1/4} e^{(z^2-1)/\varepsilon} \chi_{|z_1| \leq \frac{1}{\varepsilon} \chi_{z_2 > 0}}.\]

\[u_\varepsilon(t, x) := \int_{S^1} f_\varepsilon(z) e^{-i(x, t) \cdot z} \, d\sigma(z)\]
\[= \varepsilon^{-1/4} e^{-it} \int_{\mathbb{R}} e^{-\frac{1}{2}|y|^2 - \frac{1}{8}|y|^4 + O(|y|^6)} \varepsilon^{-1} \times\]
\[\times e^{-ix \cdot y} e^{-it} \left(\frac{1}{2}|y|^2 + \varepsilon^2 O(|y|^4)\right) (1 + \frac{1}{2}|y|^2 + O(|y|^4)) \chi_{|y| \leq 1/2} \, dy\]
\[= \varepsilon^{-1/4} e^{-it} \int_{\mathbb{R}} e^{-i x^2 y^2} e^{-i t} \left(\frac{1}{2}|y|^2 + \varepsilon^2 O(|y|^4)\right) \times\]
\[\times (1 + \frac{\varepsilon}{2}|y|^2 + O(\varepsilon^2 |y|^4)) \chi_{|y| \leq 1/2} \varepsilon^{1/2} \, dy,\]

where a change of variables is applied in passing to the last inequality. We continue to set
\[v_\varepsilon(t, x) := \varepsilon^{-1/4} u_\varepsilon(\varepsilon^{-1} t, \varepsilon^{-1/2} x)^\text{as}\]

\[(143)\]
\[= e^{-i x^2} \int_{\mathbb{R}} e^{-i x^2 y^2} \varepsilon^{-1} \left(\frac{1}{2}|y|^2 + \varepsilon^2 O(|y|^4)\right) \times\]
\[\times (1 + \frac{\varepsilon}{2}|y|^2 + O(\varepsilon^2 |y|^4)) \chi_{\varepsilon^{1/2} y} \, dy,\]

\[(144)\]
\[w_\varepsilon(t, x) := \int_{\mathbb{R}} e^{-i x^2 y^2} \varepsilon^{-1} \left(\frac{1}{2}|y|^2 + \varepsilon^2 O(|y|^4)\right) (1 + \frac{\varepsilon}{2}|y|^2) \, dy.\]

\[(145)\]
\[g_\varepsilon(y) := e^{-\frac{1}{2}|y|^2 - \frac{\varepsilon}{8}|y|^4},\]

\[(146)\]
\[d\sigma_\varepsilon(y) = (1 + \frac{\varepsilon}{2}|y|^2) \, dy.\]

Note that \(1 - it \to 1 + it\) when passing \(v_\varepsilon\) to \(w_\varepsilon\), which amounts to a complex conjugation.

Then we see that
\[(147)\]
\[\|v_\varepsilon\|_{L^6(\mathbb{R}^2)} = \|u_\varepsilon\|_{L^6(\mathbb{R}^2)},\]

\[(148)\]
\[\|w_\varepsilon\|_{L^6(\mathbb{R}^2)} = \|v_\varepsilon\|_{L^6(\mathbb{R}^2)} + O(\varepsilon^2) = \|u_\varepsilon\|_{L^6(\mathbb{R}^2)} + O(\varepsilon^2), \text{ as } \varepsilon \to 0^+.\]

\[(149)\]
\[\|f_\varepsilon\|_{L^2(S^1, \sigma)}^2 = \|g_\varepsilon\|_{L^2(\mathbb{R}, \sigma_\varepsilon)}^2 + O(\varepsilon^2), \text{ as } \varepsilon \to 0^+.\]

We consider the functional
\[(150)\]
\[\Psi(\varepsilon) = \log \frac{\|u_\varepsilon\|_{L^6(\mathbb{R}^2)}}{\|f_\varepsilon\|_{L^6(\mathbb{R}^2)}},\]

which is initially defined for \(\varepsilon > 0\) and extends continuously and differentially to \(\varepsilon = 0\).

The derivative is
\[(151)\]
\[\partial_\varepsilon|_{\varepsilon=0} \Psi(\varepsilon) = \frac{\partial_\varepsilon|_{\varepsilon=0} \|w_\varepsilon\|_{L^6(\mathbb{R}^2)}^6}{\|w_0\|_{L^6(\mathbb{R}^2)}^6} - 3 \frac{\partial_\varepsilon|_{\varepsilon=0} \|g_\varepsilon\|_{L^2(\mathbb{R}, \sigma_\varepsilon)}^2}{\|g_0\|_{L^2}^2}.\]
We observe that,
\[
\Psi(0) = \log(R_P^6).
\]
We will calculate that

**Lemma A.1.**
\[
\partial_{\varepsilon}|_{\varepsilon=0} \Psi(\varepsilon) > 0.
\]

**Proof.**
\[
\partial_{\varepsilon}|_{\varepsilon=0} w_\varepsilon = \left[\frac{1}{2}(1 + it)\partial_t^2 + i\partial_t\right] \int_{\mathbb{R}} e^{-ixy - \frac{1+it}{2}|y|^2} dy
\]
\[
= \left[\frac{1}{2}(1 + it)\partial_t^2 + i\partial_t\right] c_0(1 + it)^{-1/2} e^{-\frac{|x|^2}{2(1+it)}}
\]
\[
= \left[\frac{1}{2}(1 + it)\partial_t^2 + i\partial_t\right] w_0(t, x),
\]
where $c_0 > 0$ is some universal constant and $w_0 := c_0(1 + it)^{-1/2} e^{-\frac{|x|^2}{2(1+it)}}$. Define
\[
\phi(t, x) := -\frac{1}{2}|x|^2(1 + it)^{-1} - \frac{1}{2}\log(1 + it).
\]
Then
\[
w_0(t, x) = c_0(1 + it)^{-1/2} e^{-\frac{|x|^2}{2(1+it)}} = c_0e^\phi.
\]
Continuing the computation in (154),
\[
\left[\frac{1 + it}{2} (\phi_t^2 + \phi_{tt}) + i\phi_t\right] w_0.
\]
We compute
\[
\phi_t = \frac{i}{2}|x|^2(1 + it)^{-2} - \frac{i}{2}(1 + it)^{-1},
\]
\[
\phi_{tt} = |x|^2(1 + it)^{-3} - \frac{1}{2}(1 + it)^{-2},
\]
\[
\phi_t^2 = -\frac{1}{4}|x|^4(1 + it)^{-4} + \frac{1}{2}|x|^2(1 + it)^{-3} - \frac{1}{4}(1 + it)^{-2}.
\]
Thus
\[
\phi_t^2 + \phi_{tt} = -\frac{1}{4}|x|^4(1 + it)^{-4} + \frac{3}{2}|x|^2(1 + it)^{-3} - \frac{3}{4}(1 + it)^{-2}.
\]
Then
\[
\frac{1 + it}{2} (\phi_t^2 + \phi_{tt}) + i\phi_t =
\]
\[
= -\frac{1}{8}|x|^4(1 + it)^{-3} + \frac{1}{4}|x|^2(1 + it)^{-2} + \frac{1}{8}(1 + it)^{-1}.
\]
Taking the real part in (162), we have
\[
\Re\left[\frac{1+it}{2}(\phi_t^2 + \phi_{tt}) + i\phi_t\right]
\]
\[
= -\frac{1}{8}|x|^4(1 + t^2)^{-3}(1 - 3t^2) + \frac{1}{4}|x|^2(1 + t^2)^{-2}(1 - t^2) + \frac{1}{8}(1 + t^2)^{-1}.
\]
Since
\[
\partial_\varepsilon \|w_\varepsilon\|_6 = \partial_\varepsilon \int |w_\varepsilon|^6 = \partial_\varepsilon \int (w_\varepsilon \overline{w_\varepsilon})^3 = 6 \int |w_\varepsilon|^6 \Re\left(\frac{\partial_\varepsilon w_\varepsilon}{w_\varepsilon}\right),
\]
we have
\[
\partial_\varepsilon|_{\varepsilon=0}\|w_\varepsilon\|_6 = 6 \int \Re\left[\frac{1+it}{2}(\phi_t^2 + \phi_{tt}) + i\phi_t\right] |w_0|^6 dt
\]
\[
= c_6^0 \int \left[ -\frac{3}{4}|x|^4(1 + t^2)^{-3}(1 - 3t^2) + \frac{3}{2}|x|^2(1 + t^2)^{-2}(1 - t^2) + \frac{3}{4}(1 + t^2)^{-1} \times
\]
\[
(1 + t^2)^{-3/2}e^{-\frac{3|x|^2}{1+it}} dt
\]
\[
= c_6^0 \int \left[ -\frac{3}{4}|x|^4(1 - 3t^2) + \frac{3}{2}|x|^2(1 - t^2) + \frac{3}{4}(1 + t^2)^{-2} e^{-\frac{3|x|^2}{1+it}} dt
\]
\[
= c_6^0 \int \left( -\frac{3}{4}|x|^4 - \frac{3\pi}{12\sqrt{3}} + \frac{3}{2|t|} - \frac{3\pi}{4\sqrt{3}} \right) (1 + t^2)^{-2} dt,
\]
where we have used that
\[
\int e^{-3|x|^2} dx = \frac{\sqrt{\pi}}{\sqrt{3}}, \quad \int |x|^2 e^{-3|x|^2} dx = \frac{\sqrt{\pi}}{6\sqrt{3}}, \quad \text{and} \quad \int |x|^4 e^{-3|x|^2} dx = \frac{\sqrt{\pi}}{12\sqrt{3}}.
\]
Hence we continue (165),
\[
= c_6^0 \left( -\frac{\sqrt{\pi}}{16\sqrt{3}} \int (t^2 - 3)(1 + t^2)^{-2} dt + \frac{3\sqrt{\pi}}{4\sqrt{3}} \int (1 + t^2)^{-2} dt \right)
\]
\[
= c_6^0 \frac{\sqrt{\pi}}{16\sqrt{3}} \left( -\int t^2(1 + t^2)^{-2} dt + 15 \int (1 + t^2)^{-2} dt \right)
\]
\[
= c_6^0 \frac{\sqrt{\pi}}{16\sqrt{3}}
\]
where we have used that
\[
\int t^2(1 + t^2)^{-2} dt = \pi/2, \quad \text{and} \quad \int (1 + t^2)^{-2} dt = \pi/2.
\]
To conclude so far, we obtain,
\[
\partial_\varepsilon|_{\varepsilon=0}\|w_\varepsilon\|_6 = c_6^0 \frac{\sqrt{\pi}}{16\sqrt{3}}.
\]
On the other hand,
\[
\|w_0\|_6 = c_6^0 \int \left(1 + t^2\right)^{-3/2} e^{-\frac{3|x|^2}{1+it}} dt = c_6^0 \frac{\pi\sqrt{\pi}}{2\sqrt{3}}.
\]
Therefore we conclude that
\[
\frac{\partial \varepsilon |_{\varepsilon=0}}{\|w_0\|_6^2} \|w_0\|_6^6 = \frac{7}{8}.
\]
We are left with computing \(3 \frac{\partial \varepsilon |_{\varepsilon=0}}{\|g_0\|_2^2} | g_0\|_2^2 \):
\[
\frac{\partial \varepsilon |_{\varepsilon=0}}{\|g_0\|_2^2} = \int g_0 \left( -\frac{1}{4} y^4 + \frac{1}{2} y^2 \right) e^{-y^2} dy
\]
\[
= \frac{1}{4} \times \frac{3 \sqrt{\pi}}{4} + \frac{1}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{16}.
\]
\[
\|g_0\|_2^2 = \int e^{-y^2} dy = \sqrt{\pi}.
\]
Note that in the first inequality we have used that
\[
\int y^4 e^{-y^2} dy = \frac{3 \sqrt{\pi}}{4}, \quad \text{and} \quad \int y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.
\]
So we have
\[
\frac{3 \partial \varepsilon |_{\varepsilon=0}}{\|g_0\|_2^2} \|g_0\|_2^2 = \frac{3}{16}.
\]
Combining (151), (168) and (170), we see that
\[
\partial \varepsilon |_{\varepsilon=0} \Psi(\varepsilon) = \frac{7}{8} - \frac{3}{16} = \frac{11}{16} > 0,
\]
which establishes the claim in Lemma A.1.

Then the following symmetry consideration completes the proof of Proposition 2.1: Let \( f_\varepsilon \)
be defined in (141), and \( f_\varepsilon(x) := f_\varepsilon(-x), \) \( F_\varepsilon := (f_\varepsilon + \tilde{f}_\varepsilon)/\sqrt{2}. \) Then \( \|F_\varepsilon\|_2 = \|f_\varepsilon\|_2 \) and
\[
\frac{\|F_\varepsilon \sigma \ast F_\varepsilon \sigma \ast F_\varepsilon \sigma\|_2}{\|F_\varepsilon\|_2^3} \geq (5/2)^{1/2} \frac{\|f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma\|_2}{\|f_\varepsilon\|_2^3}.
\]
We focus on proving (172).
\[
\|F_\varepsilon \sigma \ast F_\varepsilon \sigma \ast F_\varepsilon \sigma\|_2 = 2^{-3/2} (f_\varepsilon \sigma + \tilde{f}_\varepsilon \sigma) \ast (f_\varepsilon \sigma + \tilde{f}_\varepsilon \sigma) \ast (f_\varepsilon \sigma + \tilde{f}_\varepsilon \sigma).
\]
Because of the identity (59),
\[
\langle f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma, f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma \rangle = \langle f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast \tilde{f}_\varepsilon \sigma, \tilde{f}_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma \rangle
\]
\[
= \langle f_\varepsilon \sigma \ast \tilde{f}_\varepsilon \sigma \ast f_\varepsilon \sigma, \tilde{f}_\varepsilon \sigma \ast f_\varepsilon \sigma \ast \tilde{f}_\varepsilon \sigma \rangle
\]
\[
= \langle \tilde{f}_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma, \tilde{f}_\varepsilon \sigma \ast \tilde{f}_\varepsilon \sigma \ast \tilde{f}_\varepsilon \sigma \rangle.
\]
So by nonnegativity, we see that
\[
\frac{\|F_\varepsilon \sigma \ast F_\varepsilon \sigma \ast F_\varepsilon \sigma\|_2^2}{\|f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma\|_2^2} \geq 2^{-3}(1 + 9 + 9 + 1) \|f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma\|_2^2 = (5/2) \|f_\varepsilon \sigma \ast f_\varepsilon \sigma \ast f_\varepsilon \sigma\|_2^2,
\]
which yields (172).
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