Uniqueness Results for Free Boundary Minimal Hypersurfaces in Conformally Euclidean Balls and Annular Domains

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Abstract
In this paper, we prove that a flat free boundary minimal n-disk, \( n \geq 3 \), in the unit Euclidean ball \( \mathbb{B}^{n+1} \) is the unique compact free boundary minimal hypersurface whose squared norm of the second fundamental form is less than either \( \frac{n^2}{4} \) or \( \frac{(n-2)^2}{4|x|^2} \). Moreover, we prove analogous results for compact free boundary minimal hypersurfaces in annular domains or balls with a conformally Euclidean metric.

Keywords
Minimal hypersurfaces · Free boundary surfaces · Gap curvature

Mathematics Subject Classification 53A10 · 49Q10 · 35P15

1 Introduction

Free boundary minimal submanifolds are an important branch of Differential Geometry and have received much attention. A classical result due to Nitsche [11] is the following: Let \( D^2 \subset \mathbb{R}^3 \) be a proper branched minimal immersion with free boundary on the standard unit sphere, where \( D^2 \) is a disk. Then \( D^2 \) is a flat disk. There has been much work extending this result in many different directions. For instance, Fraser–Schoen [6] showed an extension of Nitsche’s Theorem for surfaces with arbi-
trary codimension. However, there is no complete version of Nitsche’s theorem for submanifolds with high dimension. From another side, as an application of Nitsche’s Theorem and Gauss–Bonnet Theorem we get the following: Let $\Sigma \subset \mathbb{R}^3$ be a compact free boundary minimal surface immersed in the standard unit Euclidean ball such that $|A|^2 \leq 4$, where $|A|^2$ denote the squared norm of the second fundamental form of $\Sigma$. Then $\Sigma$ is a flat disk. In order to see that, we use the Gauss–Bonnet Theorem to obtain

$$2\pi(2 - 2g - r) = \int_{\Sigma} \left( 2 - \frac{|A|^2}{2} \right) d\Sigma \geq 0,$$

where $g$ denotes the genus of $\Sigma$ and $r$ the number of its boundary components. Hence, either $g = 0$ and $r = 1$, or $g = 0$ and $r = 2$. If $g = 0$ and $r = 1$, it follows from Nitsche’s Theorem that $\Sigma$ is a flat disk. If $g = 0$ and $r = 2$, then $|A|^2 = 4$. Applying Simon’s equation $\Delta |A|^2 = -2|A|^4 + 4|\nabla A|^2$, which is valid for minimal surfaces in the 3-dimensional Euclidean space, we get a contradiction. In [4], the authors proved the same result considering high codimensional surfaces $\Sigma \subset \mathbb{B}^{2+k}$. We point out that the constant $c = 4$ is not sharp for that result. With a convergence argument, and using the result above with the inequality $|A|^2 \leq 4$, we can slightly improve that constant: there exists a positive $\varepsilon_0$ such that the only free boundary minimal surface $\Sigma \subset \mathbb{B}^3$ satisfying $|A|^2 \leq 4 + \varepsilon_0$ is the flat disk. In fact, if that is false, for each $\varepsilon > 0$ such that $\varepsilon \to 0$ there exists a free boundary minimal surface $\Sigma_\varepsilon$ which is not totally geodesic and

$$|A|^2 \leq 4 + \varepsilon.$$

As the second fundamental form is uniformly bounded, there exists a free boundary minimal surface $\Sigma$ such that, up to a subsequence, $\Sigma_\varepsilon \to \Sigma$ and the second fundamental form of $\Sigma$ satisfy $|A| \leq 4$. Hence, $\Sigma$ is a flat disk. As $\Sigma_\varepsilon \to \Sigma$, we obtain that $\Sigma_\varepsilon$ is topologically a disk, for $\varepsilon$ small enough. Then, from the Nitsche–Fraser–Schoen’s uniqueness Theorem, we obtain a contradiction.

Another application of Nitsche–Fraser–Schoen’s Theorem is the following: Let $\Sigma \subset \mathbb{R}^{2+k}$ be a compact free boundary minimal surface immersed in the standard unit Euclidean ball such that $\int_{\Sigma} |A|^2 d\sigma \leq 4\pi$, then $\Sigma$ is a flat disk. In fact, assume that $\Sigma$ has genus $g$ and $r$ boundary components. It follows from the Gauss–Bonnet Theorem that

$$2\pi(2 - 2g - r) = \int_{\Sigma} \left( 2 - \frac{|A|^2}{2} \right) d\Sigma \geq 2|\Sigma| - 2\pi.$$

The result follows from the fact that $|\Sigma| \geq \pi$ and equality holds only for the flat disk (see, for instance, Brendle [3], and Fraser–Schoen [9]), since this implies that $g = 0$ and $r = 1$. Note that this also shows that there exists no compact free boundary minimal surface in $\mathbb{B}^{2+k}$ with $\int_{\Sigma} |A|^2 d\Sigma = 4\pi$. As a direct consequence of this we have that: if $\Sigma$ is not totally geodesic, then

$$||A||^2_{\infty} |\Sigma| > 4\pi(2g + r - 1).$$
Therefore, the set of free boundary minimal surfaces satisfying $||A||^2_{\infty} |\Sigma| \leq C$, for some positive constant $C$, is a compact set (see [7]). Again, with a convergence argument, we can slightly improve this: there exists a positive $\varepsilon_0$ such that the only free boundary minimal surface $\Sigma \subset \mathbb{B}^{2+k}$ satisfying $\int_{\Sigma} |A|^2 d\Sigma \leq 4\pi + \varepsilon_0$ is the flat disk.

In the unit sphere, we also have similar results: Let $\Sigma \subset \mathbb{B}^{2+k}$ be a free boundary minimal surface, where $\mathbb{B}^{2+k}$ is a geodesic ball contained in a hemisphere of $\mathbb{S}^{2+k}$. If either $|A|^2 \leq 2$ or $\int_{\Sigma} |A|^2 d\Sigma \leq 2 |\mathbb{B}^2|$, where $|\mathbb{B}^2|$ is the area of a geodesic disk with the same radius as $\mathbb{B}^{2+k}$, then $\Sigma$ is a totally geodesic disk. The proof follows the same lines of the above discussion and the results in [10].

Our first result is on that question concerning the extension of those results to high dimension:

**Theorem 1.1** Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a compact free boundary minimal hypersurface immersed in $(n+1)$-dimensional unit Euclidean ball with $n \geq 3$. Assume that either the inequality

$$|A|^2 \leq \frac{n^2}{4} \text{ or } |x|^2 |A|^2 \leq \frac{(n-2)^2}{4}$$

is satisfied on $\Sigma^n$. Then, $\Sigma^n$ is a totally geodesic disk $\mathbb{D}^n$ passing through the center of the ball.

In [1], L. Ambrozio and I. Nunes showed that if a compact free boundary minimal surface $\Sigma$ in the unit Euclidean 3-ball satisfies the condition $|A|^2 \langle x, N(x) \rangle^2 \leq 2$, where $\langle x, N(x) \rangle$ denote the support function, then $\Sigma$ is a flat equatorial disk or the critical catenoid. In a certain sense, the Theorem 1.1 with the condition $|x|^2 |A|^2 \leq \frac{(n-2)^2}{4}$ bears some resemblance with the Ambrozio–Nunes’s result, because both indicates a kind of curvature gap phenomenon.

Very recently, Cavalcanti et al. [4], using a different method, has been able to prove topological results for high codimensional free boundary submanifolds in the unit Euclidean ball considering an explicit upper bound for the number $||\Phi||^2 := |A|^2 - n|\tilde{H}|^2$, where $\tilde{H}$ is the mean curvature vector.

As Nitsche’s Theorem has also a valid version in some Euclidean conformal spaces, we can ask about the validity of a version of the Theorem 1.1 for a class of Euclidean conformal spaces as in [12].

Let $(\mathbb{B}_n^{p+1}, \bar{g})$, $n \geq 3$, be an Euclidean ball with radius $r$ centered at origin with a conformal metric $\bar{g} = e^{2h} \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean metric and $h(x) = u(|x|^2)$ with $u : [0, r^2) \rightarrow \mathbb{R}$ being a smooth function. We will write $\bar{0}$ to denote the center of $\mathbb{B}_n^{p+1}$. Note that if $s \leq r$ is the Euclidean distance from a point $x \in \mathbb{B}_n^{p+1}$ to $\bar{0}$, then the distance $\bar{s}$, from $x$ to $\bar{0}$ with respect to the conformal metric $\bar{g}$, is given by

$$\bar{s} = s I(s), \quad \text{where } I(s) = \int_0^1 e^{u(t^2 s^2)} dt. \quad (1.1)$$

Thus, given $(\mathbb{B}_r^{n+1}, \bar{g})$ as above, we will write $\bar{r}$ to denote the radius of $\mathbb{B}_r^{n+1}$ with respect to metric $\bar{g}$.
Now, suppose that the function \( u \) satisfies the conditions
\[
\begin{align*}
& (c.1) \quad u''(|x|^2) - u'(|x|^2)^2 \leq 0 \quad \text{and} \\
& (c.2) \quad -(\bar{\kappa} u''(|x|^2) |\bar{x}|^2 - u'(|x|^2) \leq 0.
\end{align*}
\]
Then, we can check that \( (\mathbb{B}^{n+1}_r, \bar{g}) \) has nonpositive sectional curvature (see Sect. 2). For those spaces, we obtain the following gap results.

**Theorem 1.2** Let \( \Sigma^n \subset (\mathbb{B}^{n+1}_r, \bar{g}) \) be an immersed compact free boundary minimal hypersurface, where \( \bar{g} = e^{2u(|x|^2)} \langle \cdot, \cdot \rangle \) and \( u \) satisfies the conditions (c.1) and (c.2). Assume that the inequality
\[
|A|^2 \leq \frac{n^2}{4r^2}
\]
is satisfied on \( \Sigma^n \). Then, \( \Sigma^n \) is a totally geodesic disk \( \mathbb{D}^n \) passing through the center of the ball.

**Theorem 1.3** Let \( \Sigma^n \subset (\mathbb{B}^{n+1}_r, \bar{g}) \) be an immersed compact free boundary minimal hypersurface, where \( \bar{g} = e^{2u(|x|^2)} \langle \cdot, \cdot \rangle \) and \( u \) satisfies the conditions (c.1) and (c.2). Assume that the inequality
\[
|A|^2 \leq \frac{(n-2)^2}{4\bar{s}^2}
\]
is satisfied on \( \Sigma^n \setminus \bar{0} \), where \( \bar{s} = \bar{s}(p) \) is given by (1.1). Then, \( \Sigma^n \) is a totally geodesic disk \( \mathbb{D}^n \) passing through the center of the ball.

Consider \( r_1 < r_2 \) and let \( \mathcal{A}(r_1, r_2) := \mathbb{B}^{n+1}_r \setminus \mathbb{B}^{n+1}_{r_1} \) be the \((n+1)\)-dimensional annulus. In this context, we obtain the following results.

**Theorem 1.4** Let \( \Sigma^n \subset (\mathcal{A}(r_1, r_2), \langle \cdot, \cdot \rangle) \) be a free boundary minimal hypersurface immersed in Euclidean \((n+1)\)-dimensional annulus. Assume that the inequality
\[
|A|^2 \leq \frac{n^2}{4r_2^2}
\]
is satisfied on \( \Sigma^n \).

(i) If \( r_1^2 = \frac{4(n-1)}{n^2} r_2^2 \), then \( \Sigma^n \) is a totally geodesic annulus.

(ii) If \( n \geq 3 \) and \( r_1^2 = \frac{4(n-1)}{n^2} r_2^2 \), then either \( \Sigma^n \) is a totally geodesic annulus or \( \Sigma^n \) is a truncated cone in such way that \( \partial \Sigma^n = \mathbb{T}_{r_1} \cup \mathbb{T}_{r_2} \), where \( \mathbb{T}_{r_i} \subset S^n_{r_i} \) is a Clifford Torus.

In [15], S. H. Park and J. Pyo showed: “A compact embedded free boundary minimal annulus in a 3-dimensional annulus \( \mathcal{A}(r_1, r_2) \) meeting each sphere of \( \partial \mathcal{A} \) along the boundary curves is flat”. It is a natural question to ask if the uniqueness result above can be extended to higher dimensions. In this sense, the Theorem 1.4 can be viewed as a generalization of the Park–Pyo’s result to high dimension in the following way:
we does not require any assumption in the topology of \(\Sigma\) or in the localization of its boundary, we only require a controlled second fundamental form.

For our next result, we define

\[
m_0 = \sup \left\{ e^{2u(|x|^2)} ; x \in A(r_1, r_2) \right\}.
\]

**Theorem 1.5** Let \(\Sigma^n \subset (A(r_1, r_2), \bar{g})\) be a free boundary minimal hypersurface immersed in \((n + 1)\)-dimensional annulus with a metric \(\bar{g} = e^{2u(|x|^2)} \langle \cdot, \cdot \rangle\) and \(u\) satisfying the conditions (c.1) and (c.2). Assume that the inequality

\[
|A|^2 \leq \frac{n^2}{4r_2^2}
\]

is satisfied on \(\Sigma^n\).

(i) If \(r_2^2 < \frac{4(n - 1)}{n^2} \left( \frac{I(r_2)^2}{m_0} \right) r_2^2\), then \(\Sigma^n\) is a totally geodesic annulus.

(ii) If \(r_2^2 = \frac{4(n - 1)}{n^2} \left( \frac{I(r_2)^2}{m_0} \right) r_2^2\), then either \(\Sigma^n\) is a totally geodesic annulus or \(\Sigma^n\) is a truncated cone in such way that \(\partial \Sigma^n = T_{r_1} \cup T_{r_2}\), where \(T_{r_i} \subset S^n_{r_i}\) is a Clifford torus.

### 2 Preliminaries and Proofs of the Main Results

The next theorem was the motivation for establishing the results of this paper. We will state it in the context that is of interest for us, that is, for minimal hypersurfaces with boundary. The general case can be found in [2]. Recall that a manifold \(\tilde{M}\) is said be a Hadamard space if it is complete, simply-connected and has nonpositive sectional curvature.

**Theorem 2.1** (Mirandola–Batista–Vitório, [2]) Let \(\Sigma^n\) be a compact minimal hypersurface in a Hadamard Space \(\tilde{M}^{n+1}\). Let \(\bar{r} = d_{\tilde{M}}(\cdot, \xi)\) be the distance in \(\tilde{M}\) from a fixed point \(\xi \in \tilde{M}\). Consider \(1 \leq p < \infty\) and \(-\infty < \gamma < n\). Then, for all function \(0 \leq \psi \in C^1(\Sigma)\) it holds,

\[
\frac{(n - \gamma)^2}{p^p} \int_{\Sigma} \frac{\psi^p}{\bar{r}^\gamma} + \frac{\gamma(n - \gamma)^{p-1}}{p^{p-1}} \int_{\Sigma} \frac{\psi^p}{\bar{r}^\gamma} \left\| \bar{\nabla} \psi \right\|_2^2 \leq \int_{\Sigma} \frac{|\nabla^n \psi|^p}{\bar{r}^{\gamma-p}} + \frac{(n - \gamma)^{p-1}}{p^{p-1}} \int_{\partial \Sigma} \frac{\psi^p}{\bar{r}^{\gamma-1}} \left\langle \bar{\nabla} \psi, \nu \right\rangle,
\]

furthermore, if \(p > 1\) then equality occur if and only if \(\psi \equiv 0\) on \(\Sigma^n\).

For the particular case which \(0 \leq \gamma < n\) and \(p = 2\), the inequality (2.2) becomes

\[
\frac{(n - \gamma)^2}{4} \int_{\Sigma} \frac{\psi^2}{\bar{r}^\gamma} \leq \int_{\Sigma} |\nabla^n \psi|_2^2 \bar{r}^{(2-\gamma)} + \frac{(n - \gamma)^2}{2} \int_{\partial \Sigma} \frac{\psi^2}{\bar{r}^{\gamma-1}} \left\langle \bar{\nabla} \psi, \nu \right\rangle.
\]
As we considered previously, let \((\mathbb{B}_{r}^{n+1}, \bar{g})\), \(n \geq 3\), be a Euclidean ball with radius \(r\) and centered at origin with a conformal metric \(\bar{g} = e^{2h} \langle \cdot, \cdot \rangle\), where \(h(x) = u(|x|^2)\) and \(u : [0, r^2) \to \mathbb{R}\) being a smooth function.

We will establish conditions on the function \(u\) so that \((\mathbb{B}_{r}^{n+1}, \bar{g})\) has nonpositive sectional curvature where the previous theorem can be applied.

### 2.1 Basic Relationship Between Geometries of the \((\mathbb{B}_{r}^{n+1}, \bar{g})\) and \((\mathbb{B}_{r}^{n+1}, \langle \cdot, \cdot \rangle)\)

We will always consider canonical coordinates on \(\mathbb{B}_{r}^{n+1}\) and \(\bar{x}\) will denote the vector field which associate to each point \(x = (x_1, \ldots, x_{n+1})\) the vector \(\bar{x} = \sum x_i \partial_i\). Under conformal change on Riemannian metric, we have the following classical formulae (see [16]).

**Proposition 2.1** (Conformal change formulas) Let \(\nabla\) and \(\bar{\nabla}\) be a Riemannian connection of \((M^{n+1}, g)\) and \((\mathbb{B}_{r}^{n+1}, \bar{g})\) respectively, where \(\bar{g} = e^{2h}g\) for some smooth function \(h : M \to \mathbb{R}\). Then, for smooth vector fields \(X, Y, Z \in X(M)\) we have:

1. \(\bar{\nabla}_Y X = \nabla_Y X + Y(h)X + X(h)Y - g(X, Y)\nabla h\)
2. \(\bar{R}(Y, Z)X = R(Y, Z)X + g(Y, X)\nabla_Y \nabla h - g(Z, X)\nabla_Y \nabla h - \{(\text{Hess } h)(Z, X) - X(h)Z(h) - X(g(\nabla h, \nabla h)g(X, Z))\} Y + \{(\text{Hess } h)(Y, X) - Y(h)X(h) + g(\nabla h, \nabla h)g(Y, X)\} Z, + \{(Y(h)g(Z, X) - Z(h)g(Y, X))\} \nabla h\)
3. \(\bar{\text{Ric}}(Z, X) = \text{Ric}(Z, X) - (n-1)(\text{Hess } h)(Z, X) + (n-1)Z(h)X(h) - \left\{ \Delta h + (n-1)g(\nabla h, \nabla h) \right\} g(Z, X),\)
4. \(\bar{R}_{\bar{g}} = e^{-2h}\left[R_g - 2n\Delta h - n(n-1)g(\nabla h, \nabla h)\right].\)

where \(\nabla h\), (Hess \(h\)) and \(\Delta h\) are calculated with respect the metric \(g\).

**Lemma 2.1** Let \((\mathbb{B}_{r}^{n+1}, \langle \cdot, \cdot \rangle)\) be the unit Euclidean ball. Then, the Hessian of the function \(h(x) = u(|x|^2)\) is given by

\[
\text{(Hess } h)(x)(Y, Z) = 4u''(|x|^2)\langle \bar{x}, Y \rangle \langle \bar{x}, Z \rangle + 2u'(|x|^2)\langle Y, Z \rangle.
\]

**Proof** Note that the gradients vectors of the functions \(h\) and \(u'(|x|^2)\) are given by \(g\text{grad}_\cdot h = 2u'(|x|^2)\bar{x}\) and \(g\text{grad}_\cdot u'(|x|^2) = 2u''(|x|^2)\bar{x}\), respectively. Denote by \(\bar{\nabla}\) the Riemannian connection of \((\mathbb{B}_{r}^{n+1}, \langle \cdot, \cdot \rangle)\). Thus, we have

\[
\text{(Hess } h)(x)(Y, Z) = \langle \bar{\nabla}_Y g\text{grad}_\cdot h, Z \rangle = 2\left\langle \bar{\nabla}_Y u'(|x|^2)\bar{x}, Z \right\rangle
\]

\[
= 2\left\langle Y(u'(|x|^2))\bar{x} + u'(|x|^2)\bar{\nabla}_Y \bar{x}, Z \right\rangle
\]

\[
= 4u''(|x|^2)\langle \bar{x}, Y \rangle \langle \bar{x}, Z \rangle + 2u'(|x|^2)\langle Y, Z \rangle.
\]
Lemma 2.2  Let \((\mathbb{R}^{n+1}, \bar{g})\) be the Euclidean ball with radius \(r\) and conformal metric 
\(\bar{g} = e^{2h}(\cdot, \cdot)\), where \(h(x) = u(|x|^2)\). Suppose that function \(u\) satisfies \(u''(|x|^2) - u'(|x|^2)^2 \leq 0\). If \(\bar{N}\) is an unit vector in the tangent space \(T_x\mathbb{R}^{n+1}\), then

\[
\bar{\text{Ric}}(\bar{N}, \bar{N})(x) \leq -4ne^{-2h}[u''(|x|^2) |x|^2 + u'(|x|^2)].
\]

**Proof**  Note that if \(\bar{N}\) satisfy \(\bar{g}(\bar{N}, \bar{N}) = 1\), then the vector \(N = e^h\bar{N}\) satisfies \((N, N) = 1\). It follows from the conformal change formulae,

\[
\bar{\text{Ric}}(\bar{N}, \bar{N}) = e^{-2h}\text{Ric}(N, N)
\]

\[
e^{-2h}\left\{ \text{Ric}_{\mathbb{R}^{n+1}}(N, N) - (n - 1)(\text{Hess} h)(N, N) + (n - 1)N(h)^2
\right. \\
\left. - \{\Delta h + (n - 1) |\nabla h|^2\} (N, N) \right\}
\]

\[
e^{-2h}\left\{ -4(n - 1)u''(|x|^2) \langle \bar{x}, N \rangle^2 - 2(n - 1)u'(|x|^2)
\right. \\
\left. + 4(n - 1)u'(|x|^2)^2 \langle \bar{x}, N \rangle^2
\right. \\
\left. - 4u''(|x|^2) |\bar{x}|^2 - 2(n + 1)u'(|x|^2) - 4(n - 1)u'(|x|^2)^2 |\bar{x}|^2 \right\}
\]

\[
e^{-2h}\left\{ 4(n - 1)[u'(|x|^2)^2 - u''(|x|^2)] \langle \bar{x}, N \rangle^2 - 4nu'(|x|^2)
\right. \\
\left. - 4u''(|x|^2) |\bar{x}|^2 - 4(n - 1)u'(|x|^2)^2 |\bar{x}|^2 \right\}
\]

\[\overset{(*)}{\leq} e^{-2h}\left\{ 4(n - 1)[u'(|x|^2)^2 - u''(|x|^2)] |x|^2 - 4nu'(|x|^2)
\right. \\
\left. - 4u''(|x|^2) |\bar{x}|^2 - 4(n - 1)u'(|x|^2)^2 |\bar{x}|^2 \right\}
\]

\[= -4ne^{-2h}\left\{ u''(|x|^2) |\bar{x}|^2 + u'(|x|^2) \right\}.\]

\(\square\)

Lemma 2.3  Consider \((\mathbb{R}^{n+1}, \bar{g})\). Suppose that the function \(u\) satisfies the following conditions:

(i) \(u''(|x|^2) - u'(|x|^2)^2 \leq 0\),
(ii) \(-u''(|x|^2) |\bar{x}|^2 - u'(|x|^2)^2 \leq 0\).

Then, \((\mathbb{R}^{n+1}, \bar{g})\) has nonpositive sectional curvature.

**Proof**  Let \(\pi\) be a tangent plane at point \(x \in (\mathbb{R}^{n+1}, \bar{g})\) and let \(\bar{K}(x)(\pi)\) be the sectional curvature with respect to \(\pi\). Consider \(\{\bar{E}_1, \bar{E}_2\}\) a base to \(\pi\) with \(\bar{g}(\bar{E}_i, \bar{E}_j) = \delta_{ij}\). Thus, for \(E_i = e^h\bar{E}_i\) we obtain \(\langle E_i, E_j \rangle = \delta_{ij}\). By Lemma 2.2 and from the formulae in the conformal change for the curvature obtained in Proposition 2.1, it follows that

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\[ \tilde{K}(x)(\pi) = \tilde{g}(\tilde{R}(\tilde{E}_1, \tilde{E}_2)\tilde{E}_2, \tilde{E}_1) \]
\[ = e^{-2h} \langle \tilde{R}(E_1, E_2) E_2, E_1 \rangle \]
\[ = e^{-2h} \left\{ \langle R(E_1, E_2) E_2, E_1 \rangle + \langle E_1, E_2 \rangle \langle \nabla E_2 \nabla h, E_1 \rangle \right. \]
\[ - \langle E_2, E_2 \rangle \langle \nabla E_1 \nabla h, E_1 \rangle \]
\[ - \left\{ (\text{Hess } h)(E_2, E_2) - E_2(h)^2 + |\nabla h|^2 \langle E_2, E_2 \rangle \right\} \langle E_1, E_1 \rangle \]
\[ + \left\{ (\text{Hess } h)(E_1, E_2) - E_1(h) E_2(h) + |\nabla h|^2 \langle E_1, E_2 \rangle \right\} \langle E_2, E_1 \rangle \]
\[ + \left\{ E_1(h) (E_2, E_2) - E_2(h) \langle E_1, E_2 \rangle \right\} \langle \nabla h, E_1 \rangle \right\} \]
\[ = e^{-2h} \left\{ -(\text{Hess } h)(E_1, E_1) - (\text{Hess } h)(E_2, E_2) \right. \]
\[ + E_2(h)^2 - |\nabla h|^2 + E_1(h)^2 \right\} \]
\[ = e^{-2h} \left\{ -4u''(|x|^2) \langle \tilde{x}, E_1 \rangle^2 - 2u'(|x|^2) - 4u''(|x|^2) \langle \tilde{x}, E_2 \rangle^2 \right. \]
\[ - 2u'(|x|^2)^2 + 4u'(|x|^2)^2 \langle \tilde{x}, E_1 \rangle^2 + 4u'(|x|^2)^2 \langle \tilde{x}, E_2 \rangle^2 - 4u'(|x|^2)^2 |\tilde{x}|^2 \left\} \right. \]
\[ = 4e^{-2h} \left\{ u'(|x|^2)^2 - u''(|x|^2) \right\} \left( \langle \tilde{x}, E_1 \rangle^2 + \langle \tilde{x}, E_2 \rangle^2 \right) \]
\[ - u'(|x|^2)^2 - u''(|x|^2)^2 |\tilde{x}|^2 \left\}. \right. \]

Note that,
\[ \langle \tilde{x}, E_1 \rangle^2 + \langle \tilde{x}, E_2 \rangle^2 \leq |\tilde{x}|^2. \]

To see this, we observe that the left hand side of the above inequality computes the squared length of the orthogonal projection of the vector \( \tilde{x} \) (with respect to \( \langle \cdot, \cdot \rangle \) onto the plane \( \pi \).

Now, from the condition (i) follows that
\[ \left[ u'(|x|^2)^2 - u''(|x|^2) \right] \left( \langle \tilde{x}, E_1 \rangle^2 + \langle \tilde{x}, E_2 \rangle^2 \right) \leq \left[ u'(|x|^2)^2 - u''(|x|^2) \right] |\tilde{x}|^2 \]
and from the condition (ii), we finally obtain that
\[ \tilde{K}(x)(\pi) \leq 4e^{-2h} \left\{ u'(|x|^2)^2 - u''(|x|^2) \right\} |\tilde{x}|^2 - u'(|x|^2)^2 - u''(|x|^2)^2 |\tilde{x}|^2 \]
\[ = 4e^{-2h} \left\{ -u''(|x|^2) |\tilde{x}|^2 - u'(|x|^2)^2 \right\} \]
\[ \leq 0. \]

\( \Box \)

**Example 2.1** Consider \( u_1 : [0, \infty) \rightarrow \mathbb{R} \) given by \( u_1 \equiv 0 \) and \( u_2 : [0, 1) \rightarrow \mathbb{R} \) given by \( u_2(t) = \ln \left( \frac{2}{1-t} \right) \). The functions \( u_1 \) and \( u_2 \) satisfy the conditions (i) and (ii) in previous Lemma. In fact, \( M_1 := (B^\infty_{\infty}, \tilde{g}) \) for \( \tilde{g} = e^{2u_1(|x|^2)} \langle \cdot, \cdot \rangle \) is the Euclidean.
space $\mathbb{R}^{n+1}$ and $M_2 := (\mathbb{B}^{n+1}, \bar{g})$ for $\bar{g} = e^{2u_2(|x|^2)} \langle \cdot, \cdot \rangle$ is the Hyperbolic Space $\mathbb{H}^{n+1}$. Both are Hadamard spaces.

**Example 2.2** Consider $u : [0, \infty) \to \mathbb{R}$ given by $u(t) = \frac{t}{4n}$. The conditions (i) and (ii) are satisfied for all $t$. Therefore, $\left(\mathbb{B}^{n+1}_R, e^{\frac{|u|^2}{4n}} \langle \cdot, \cdot \rangle\right)$ with $R = \infty$ is a Hadamard space.

### 3 GAP Results for $|A|^2$ of Free Boundary Minimal Hypersurfaces in Euclidean Conformal Ball

From now on, we consider the manifold $(\mathbb{B}^{n+1}_r, \bar{g})$, where $\bar{g} = e^{2u(|x|^2)} \langle \cdot, \cdot \rangle$ for a function $u$ which satisfies the conditions (i) and (ii) in Lemma 2.3. Thus, $(\mathbb{B}^{n+1}_r, \bar{g})$ has nonpositive sectional curvature.

**Lemma 3.1** Let $\Sigma^n \subset (\mathbb{B}^{n+1}_r, \bar{g})$ be an immersed compact minimal hypersurface with boundary $\partial \Sigma^n$. Consider the operator $L := \Delta + |A|^2 + q$ on $\Sigma^n$, where $q : \Sigma^n \to \mathbb{R}$ is a nonpositive smooth function. Assume that the inequality

$$|A|^2 \leq \frac{n^2}{4\bar{s}^2}$$

is satisfied on $\Sigma^n$. Then, the first eigenvalue $\lambda_1$ of the operator $L$ to problem

$$\begin{cases}
\Delta v + (|A|^2 + q)v = -\lambda v & \text{on } \Sigma^n, \\
v = 0 & \text{on } \partial \Sigma^n,
\end{cases}$$

is strictly positive. Consequently, if $w$ satisfy $L[w] = 0$ and $w = 0$ on $\Sigma^n$, then $w \equiv 0$.

**Proof** Suppose by contradiction that $\lambda_1 \leq 0$. Let $v_1$ such that

$$\begin{cases}
\Delta v_1 + (|A|^2 + q)v_1 = -\lambda_1 v_1 & \text{on } \Sigma^n, \\
v_1 = 0 & \text{on } \partial \Sigma^n.
\end{cases}$$

We can assume that $v_1$ is positive, since the first eigenvalue of problem (*) is simple. It follows from integration by parts, the inequality (3.4) and $\lambda_1 + q \leq 0$, that

$$\int_{\Sigma} |\nabla v_1|^2 = \int_{\Sigma} -v_1 \Delta v_1 d\Sigma = \int_{\Sigma} (\lambda_1 + |A|^2 + q)v_1^2 \leq \int_{\Sigma} |A|^2 v_1^2 \leq \frac{n^2}{4\bar{s}^2} \int_{\Sigma} v_1^2. \quad (3.5)$$

Considering $\bar{s}$ the distance from a point $p \in \Sigma^n$ to point $\bar{0}$ in $(\mathbb{B}^{n+1}_r, \bar{g})$, we have $\bar{s} \leq \bar{r}$. Since $v_1 \equiv 0$ on $\partial \Sigma$, the inequality (2.3) for $\gamma = 0$ gives us

$$\frac{n^2}{4} \int_{\Sigma} v_1^2 \leq \int_{\Sigma} |\nabla v_1|^2 \bar{s}^2 \leq \bar{r}^2 \int_{\Sigma} |\nabla v_1|^2. \quad (3.6)$$
Hence, it follows from the estimate (3.5) that

\[
\frac{n^2}{4} \int_{\Sigma} v_1^2 \leq \tilde{r}^2 \frac{n^2}{4\tilde{r}^2} \int_{\Sigma} v_1^2 = \frac{n^2}{4} \int_{\Sigma} v_1^2.
\]

This means that equality occurs in (3.6) and by Theorem 2.1 the function \( v_1 \) should be a null function, a contradiction. Therefore, \( \lambda_1 > 0 \).

**Remark 3.1** If we consider the condition

\[
|A|^2 \leq \frac{(n-2)^2}{4\tilde{s}^2} \quad \text{in} \quad \Sigma^n \setminus \tilde{0}
\]

(3.7)

instead of (3.4), where \( \tilde{s} = \tilde{s}(x) \) denotes the distance in \((\mathbb{B}_r^{n+1}, \tilde{g})\) from a point \( x \in \Sigma^n \) at point \( \tilde{0} \), and using \( \gamma = 2 \), we obtain the same result: \( \lambda_1 \) is strictly positive and consequently the solution \( L[v] = 0 \) for the problem (*) above should be the null function.

Note that the inequality (3.4) imposes a more rigid constraint on the total length of \( |A|^2 \), whereas the inequality (3.7) imposes a control in the growth of \( |A|^2 \) when the point \( p \in \Sigma^n \) becomes arbitrarily next to \( \tilde{0} \).

**Lemma 3.2** Let \( \Sigma^n \subset (\mathbb{B}_r^{n+1}, \tilde{g}) \) be an immersed compact free boundary minimal hypersurface. Consider the function \( v = \tilde{g}(\tilde{x}, \tilde{N}) \) defined on \( \Sigma^n \) where \( \tilde{N} \) denotes the normal vector field to \( \Sigma^n \). Then, the function \( v \) is solution to problem:

\[
\begin{cases}
L[v] = 0 & \text{in} \quad \Sigma^n, \\
v = 0 & \text{on} \quad \partial \Sigma^n,
\end{cases}
\]

where \( L[v] = \Delta v + |A|^2 v + qv \) and,

\[
q = \tilde{\text{Ric}}(\tilde{N}, \tilde{N}) + 4ne^{-2h}[u''(|x|^2)|x|^2 + u'(|x|^2)].
\]

**Proof** Since the vector field \( \tilde{x} \) on \( \mathbb{B}_r^{n+1} \) is conformal, that is, the Lie derivative satisfies

\[
\mathcal{L}_{\tilde{x}} \tilde{g} = 2\sigma \tilde{g},
\]

where \( \sigma(x) = 1 + 2u'(|x|^2)|x|^2 \), the Proposition 2.1 of [14] ensures that

\[
\Delta v + |A|^2 v + \tilde{\text{Ric}}(\tilde{N}, \tilde{N})v = -n\tilde{N}(\sigma).
\]

(3.8)

The gradient of the function \( \sigma \) with respect to the Euclidean metric is given by,

\[
g_{\tilde{x}\tilde{a} \tilde{d}}(\sigma) = 4[u''(|x|^2)|x|^2 + u'(|x|^2)]\tilde{x}.
\]

As the gradient of \( \sigma \) with respect to the metric \( \tilde{g} \) is given by \( \tilde{\nabla}\sigma = e^{-2h}g_{\tilde{x}\tilde{a} \tilde{d}}(\sigma) \), we have,

\[
\tilde{\nabla}\sigma = 4e^{-2h}[u''(|x|^2)|x|^2 + u'(|x|^2)]\tilde{x}.
\]
Therefore,

\[ \tilde{N}(\sigma) = \tilde{g}(\tilde{\nabla} \sigma, \tilde{N}) = 4e^{-2h}[u''(|x|^2) |x|^2 + u'(|x|^2)] \tilde{g}(\tilde{x}, \tilde{N}). \]  

(3.9)

Replacing (3.9) on (3.8), we have

\[ \Delta v + |A|^2 v + \tilde{\text{Ric}}(\tilde{N}, \tilde{N})v + 4ne^{-2h}[u''(|x|^2) |x|^2 + u'(|x|^2)]v = 0, \]

as desired. Finally, \( v \equiv 0 \) on \( \partial \Sigma^n \) follows from the free boundary condition. \[ \square \]

**Theorem 3.1** Let \( \Sigma^n \subset (\mathbb{B}_{r}^{n+1}, \bar{g}) \) be an immersed compact free boundary minimal hypersurface. Assume that the inequality

\[ |A|^2 \leq \frac{n^2}{4r^2} \]  

(3.10)

is satisfied on \( \Sigma^n \). Then, \( \Sigma^n \) is a totally geodesic disk through the origin.

**Proof** Consider the function \( v = \bar{g}(\tilde{x}, \tilde{N}) \) defined on \( \Sigma^n \), where \( \tilde{N} \) is the normal vector to \( \Sigma^n \). Consider also the operator \( L := \Delta + |A|^2 + q \), where

\[ q = \tilde{\text{Ric}}(\tilde{N}, \tilde{N}) + 4ne^{-2h}[u''(|x|^2) |x|^2 + u'(|x|^2)]. \]

The Lemma 2.2 ensures that \( q \leq 0 \) and by hypothesis (3.10), the Lemma 3.1 ensure that the first eigenvalue of the operator \( L := \Delta + |A|^2 + q \) for the problem \( L[v_1] = -\lambda_1 v_1 \), with \( v_1 = 0 \) on the boundary \( \partial \Sigma \), is strictly positive. As in this case we have \( L[v] = 0 \), it follows that \( v \equiv 0 \) in \( \Sigma^n \). Thus, \( 0 \equiv v = \bar{g}(\tilde{x}, \tilde{N}) = e^{2h} \langle \tilde{x}, \tilde{N} \rangle \), and therefore \( \langle \tilde{x}, \tilde{N} \rangle \equiv 0 \). Let \( k_i \) be the principal curvatures of \( \Sigma^n \subset (\mathbb{B}_{r}^{n+1}, \langle \cdot, \cdot \rangle) \) and let \( \bar{k}_i \) be the principal curvatures of \( \Sigma^n \subset (\mathbb{B}_{r}^{n+1}, \bar{g}) \), where \( i = 1, \ldots, n \). We have that \( k_i \) and \( \bar{k}_i \) are related by equation (see Lemma 10.1.1, [13]),

\[ \bar{k}_i = \frac{1}{e^h} \left( k_i - 2u'(|\tilde{x}|^2) \langle \tilde{x}, N \rangle \right), \quad i = 1, \ldots, n, \]

where \( N = e^h \tilde{N} \) denotes the normal vector to \( \Sigma \) with respect to the metric \( \langle \cdot, \cdot \rangle \). Since \( \Sigma^n \) is minimal in the metric \( \bar{g} \), we must have

\[ 0 = \sum_{i=1}^{n} \bar{k}_i = \sum_{i=1}^{n} \frac{1}{e^h} \left( k_i - 2u'(|\tilde{x}|^2) \langle \tilde{x}, N \rangle \right) = \sum_{i=1}^{n} \frac{1}{e^h} \left( k_i - 2u'(|\tilde{x}|^2)e^h \langle \tilde{x}, \tilde{N} \rangle \right) = \sum_{i=1}^{n} \frac{1}{e^h} k_i, \]

that is, \( \sum_{i=1}^{n} k_i = 0 \) and therefore \( \Sigma^n \) is also a minimal hypersurface in \( (\mathbb{B}_{r}^{n+1}, \langle \cdot, \cdot \rangle) \) satisfying \( \langle \tilde{x}, N \rangle \equiv 0 \). Thus, \( \Sigma^n \) is a totally geodesic disk passing through the origin. To see this, we can use the same arguments as in [8] (Theorem 3.1 or Lemma 6.4). \[ \square \]
Based on Remark 3.1, we also have the following theorem, whose proof is analogous to the proof of above result.

**Theorem 3.2** Let \( \Sigma^n \subset (\mathbb{R}^{n+1}, \bar{g}) \) be an immersed compact free boundary minimal hypersurface. Assume that the condition

\[
|A|^2 \leq \frac{(n-2)^2}{4s^2} \quad (3.11)
\]

is satisfied on \( \Sigma^n \setminus \tilde{0}, \) where \( s = \tilde{s}(x) \) is the distance from a point \( x \in \Sigma^n \) to point \( \tilde{0} \) with respect to the metric \( \bar{g}. \) Then, \( \Sigma^n \) is a totally geodesic disk through the origin.

### 4 GAP Results for \( |A|^2 \) of Free Boundary Minimal Hypersurfaces in Annular Domain

For \( r_1 < r_2 < r, \) define the \((n+1)\)-dimensional annulus \( A(r_1, r_2) := \mathbb{R}^{n+1}_{r_2} \setminus \mathbb{R}^{n+1}_{r_1}. \)

Now, consider the manifold \((A(r_1, r_2), \bar{g}),\) where \( \bar{g} = e^{2h} \langle \cdot, \cdot \rangle \) with \( h(x) = u(|x|^2) \) and \( u : [0, r^2) \to \mathbb{R} \) a smooth function satisfying the conditions of Lemma 2.3. When we want to emphasize that the metric \( \bar{g} \) coincides with the canonical metric (that is, for \( u \equiv 0 \), we write \((A(r_1, r_2), \langle \cdot, \cdot \rangle)\) instead of \((A(r_1, r_2), \bar{g}),\) unless otherwise stated.

**Theorem 4.1** Let \( \Sigma^n \subset (A(r_1, r_2), \bar{g}) \) be an immersed compact free boundary minimal hypersurface. Assume that for all points \( p \in \Sigma^n \)

\[
|A|^2 \leq \frac{n^2}{4r^2} \quad (4.12)
\]

Then, \( \Sigma^n \) is tangent to position vector field \( \bar{x} \) and furthermore, the boundary \( \partial \Sigma^n \) intersects the two connected components of the boundary \( \partial A(r_1, r_2) = S^n_{r_1} \cup S^n_{r_2}. \)

**Proof** In analogous way to the proof of the Theorem 3.1, we conclude that the function \( v = \bar{g}(\bar{x}, \bar{N}) \) must be identically null on \( \Sigma^n, \) what means that \( \Sigma^n \) is tangent to vector field \( \bar{x}. \) Thus, the boundary \( \partial \Sigma^n \) necessarily intersects each one of connected components of the boundary \( \partial A(r_1, r_2), \) otherwise, \( \partial \Sigma^n \) would be in only one connected component of \( \partial A(r_1, r_2) \) and the function \( f(x) = |x|^2 \) restricted to \( \Sigma^n \) would have a local minimum or a local maximum at some interior point \( x_0 \in \Sigma^n, \) \( x_0 \neq \tilde{0}. \) We have that \( \nabla f(x_0) = e^{-2h} \bar{x}_0, \) where \( \nabla f \) denotes the gradient of \( f \) in \((A(r_1, r_2), \bar{g}). \) As \( x_0 \) is a critical point of \( f \) restrict to \( \Sigma^n, \) the gradient vector \( \nabla f(x_0) \) must be in the direction of the unit normal vector \( \bar{N} \) to \( \Sigma^n \) at \( x_0. \) Thus, we have \( 0 \neq \bar{g}(\nabla f(x_0), \bar{N}) = e^{-2h} \bar{g}(\bar{x}_0, \bar{N}), \) which does not occur since \( v \equiv 0. \)

The above theorem says that a minimal hypersurface \( \Sigma^n \subset (A(r_1, r_2), \bar{g}) \) with free boundary satisfying the condition (4.12), necessarily has its support function \( v = \bar{g}(\bar{x}, \bar{N}) \) identically null, and furthermore, the boundary \( \partial \Sigma^n \) should intersect each one of the connected components of \( \partial A. \) For \( n \geq 3, \) the Euclidean space \( \mathbb{R}^{n+1} \) admits minimal hypersurfaces which are not totally geodesic, as some specific types of...
For our purposes we consider cones, whose portion which intersects the annulus \((\mathcal{A}(r_1, r_2), \langle \cdot, \cdot \rangle)\) forms a minimal hypersurface with free boundary. These minimal cones have a singularity at the origin \(0\), so we have \(|A|^2(p)\) arbitrarily large when \(p\) approaches \(0\). Thus, it is natural to expect that some condition between the radius \(r_1\) and \(r_2\) can be necessary to characterize minimal hypersurfaces \(\Sigma^n \subset (\mathcal{A}(r_1, r_2), \bar{g})\) satisfying \((4.12)\) as being totally geodesic. This is what we will do in the next section.

### 4.1 The Case of a Euclidean Annulus

For our purposes we consider \(\Gamma^{n-1}\) a closed orientable hypersurface in \(\mathbb{S}^n\). Let \(C_\Gamma := \{\lambda y; y \in \Gamma^{n-1}, \lambda \in (0, \infty)\}\) be a cone in \(\mathbb{R}^{n+1}\) with vertex at origin. We will refer to \(C_\Gamma\) as the cone over \(\Gamma\) that intersects the sphere \(\mathbb{S}^n\) along \(\Gamma\). Note that the support function \(v = \langle \bar{x}, N \rangle\) is such that \(v \equiv 0\) on \(C_\Gamma\), and if \(\Sigma^n \subset \mathbb{R}^{n+1}\) is a hypersurface which has the support function identically null then \(\Sigma^n\) is contained in some cone \(C_\Lambda\) for some hypersurface \(\Lambda^{n-1} \subset \mathbb{S}^n\). The next Lemma is a standard fact about minimal cones and its proof is omitted.

**Lemma 4.1** The cone \(C_\Gamma\) is a minimal hypersurface in \(\mathbb{R}^{n+1}\) if, and only if, \(\Gamma\) is a minimal hypersurface in \(\mathbb{S}^n\).

Below, we have an example of non-trivial minimal hypersurface in \(\mathbb{R}^{n+1}\) tangent to vector field \(\bar{x}\).

**Example 4.1** For \(n \geq 3\), consider the Clifford torus \(\mathbb{T}_{m,n} := S^m_{\lambda_1} \times S^{(n-1) - m}_{\lambda_2}\) where \(\lambda_1 = \sqrt{\frac{m}{n-1}}, \lambda_2 = \sqrt{\frac{(n-1) - m}{n-1}}\) and \(1 \leq m \leq n - 2\). Since \(\mathbb{T}_{m,n}\) is a minimal hypersurface in \(\mathbb{S}^n\), the cone \(C_{\mathbb{T}_{m,n}} = \{\lambda y; y \in \mathbb{T}_{m,n}, \lambda \in (0, \infty)\}\) is a minimal hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\).

Given \(\lambda > 0\), consider the hypersurface \(\Gamma_\lambda := C_\Gamma \cap \mathbb{S}_\lambda^n\), where \(\mathbb{S}_\lambda^n\) denotes the sphere of radius \(\lambda\). Note that \(\Gamma_\lambda\) is obtained from \(\Gamma^{n-1} \subset \mathbb{S}^n\) by a homothety, that is, every point \(x_0 \in \Gamma_\lambda\) it is such that there is a point \(p \in \Gamma^{n-1}\) such that \(x_0 = \lambda p\). Let \(|A|^2\) and \(|A_\lambda|^2\) denoting the square of the second fundamental form of \(C_\Gamma\) as a hypersurface of \(\mathbb{R}^{n+1}\) and of \(\Gamma_\lambda\) as a hypersurface of \(\mathbb{S}^n\), respectively.

**Lemma 4.2** Consider \(\lambda > 0\), let \(\Gamma_{\lambda}\) be a closed hypersurface in \(\mathbb{S}_\lambda^n\) and let \(C_\Gamma\) be the cone over \(\Gamma_\lambda\). We have,

\[
|A|^2(q) = |A_\lambda|^2(q)
\]

for all \(q \in C_\Gamma \cap \Gamma_\lambda\).

**Proof** The proof follows from the observation that the radial direction at point \(q \in C_\Gamma\) is a principal direction of the cone \(C_\Gamma\) with principal curvature zero. \(\square\)

**Lemma 4.3** Let \(\Gamma^{n-1} \subset \mathbb{S}^n\) be a closed hypersurface and let \(C_{\Gamma}\) be the cone over \(\Gamma^{n-1}\). Given \(q \in C_{\Gamma}\), define \(\lambda = \|q\|\) and \(\Gamma_\lambda = \Gamma_{\lambda} \cap \mathbb{S}_{\lambda}^n\). For \(p = \frac{1}{\|q\|}q \in \mathbb{S}^n\), we have

\[
|A_\lambda|^2(q) = \frac{1}{\|q\|^2} |A_1|^2(p),
\]

\((4.13)\)
where $A_\lambda$ and $A_1$ denote the second fundamental form of $\Gamma_\lambda$ as submanifold of $\mathbb{S}_\lambda^n$ and $\Gamma^{n-1}$ as submanifold of $\mathbb{S}^n$, respectively.

**Remark 4.1** As a consequence of the identity (4.13), we have that the cone $C_\Gamma$ over $\Gamma^{n-1}$ is a totally geodesic minimal hypersurface in $\mathbb{R}^{n+1}$ if and only if $\Gamma^{n-1}$ is a totally geodesic minimal hypersurface in $\mathbb{S}^n$.

By lemmas (4.2) and (4.3), we have that if $C_\Gamma$ is a cone in $\mathbb{R}^{n+1}$ over some hypersurface in $\mathbb{S}^n$ and $q \in C_\Gamma$ then, for $p = \frac{1}{|q|}q$ we have,

$$|A|^2 (q) = \frac{1}{|q|^2} |A_1|^2 (p),$$  \hspace{1cm} (4.14)

where $|A|^2$ denotes the square of second fundamental form of $C_\Gamma$.

This says that the square of the second fundamental form of $C_\Gamma$ as hypersurface of $\mathbb{R}^{n+1}$ when calculated on $\Gamma_\lambda = C_\Gamma \cap \mathbb{S}_\lambda^n$ for some $\lambda$, can be compared to the square of the second fundamental form of a hypersurface $\Gamma \subset \mathbb{S}^n$ obtained by a homothety $\Gamma = \lambda \Gamma_\lambda$. In view of the Lemma 4.1, this comparison becomes useful in our context if the cone $C_\Gamma$ is a minimal hypersurface $\mathbb{R}^{n+1}$ due to the following theorem.

**Theorem 4.2** (Chern–do Carmo–Kobayashi, [5]) Let $\Gamma^{n-1}$ be a closed minimal hypersurface in the unit sphere $\mathbb{S}^n$. Assume that its second fundamental form $A_1$ satisfies,

$$|A_1|^2 \leq n - 1.$$

Then,

(1) $|A_1|^2 \equiv 0$ and $\Gamma^{n-1}$ is an equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$,

(2) or $|A_1|^2 \equiv n - 1$ and $\Gamma^{n-1}$ is one of Clifford torus $\mathbb{T}_{n,m}$.

**Example 4.2** For $n \geq 3$, let $C_T$ be a minimal cone in $\mathbb{R}^{n+1}$ over the Clifford torus $\mathbb{T}_{m,n} \subset \mathbb{S}^n$. For $r_2 = 1$ and $\gamma_1 = \frac{4(n-1)}{n^2}$, consider the annulus $A(r_1, 1)$ and let $C_T(r_1, 1) = C_T \cap A(r_1, 1)$ be a portion of $C_T$ inside $A(r_1, 1)$. We will refer to $C_T(r_1, 1)$ as a truncated cone over $\mathbb{T}_{m,n}$. In view of the Eq. (4.14), for any point $q \in C_T(r_1, 1)$ we have,

$$|A|^2 (q) = \frac{1}{|q|^2} |A_1|^2 (p).$$

But, by the above theorem, we have $|A_1|^2 \equiv n - 1$. Then,

$$|A|^2 (q) = \frac{n - 1}{|q|^2} \leq \frac{n - 1}{r_1^2} = \frac{n^2}{4},$$

that is,

$$|A|^2 (q) \leq \frac{n^2}{4} \quad \forall q \in C_T(r_1, 1).$$
The preceding example says that condition $|A|^2(q) \leq \frac{n^2}{4r_2^2}$ without any other assumption is not sufficient to characterize a free boundary minimal hypersurface on $\mathcal{A}(r_1, r_2)$ as being totally geodesic like was done on Theorem 3.1. For this case, we need an additional hypothesis which concerns about a distancing condition between $r_1$ and $r_2$ as will be clear in Corollary 4.1. But before that, we will see a slightly more general case which will be useful to study a free boundary minimal hypersurfaces on $(\mathcal{A}(r_1, r_2), \bar{g})$.

**Proposition 4.1** Let $\Gamma \subset S^n$ be a closed minimal hypersurface and let $C_\Gamma$ be a minimal cone in $\mathbb{R}^{n+1}$ over $\Gamma$. Consider $C_\Gamma(r_1, r_2) = C_\Gamma \cap \mathcal{A}(r_1, r_2)$ the truncated cone inside of the annulus $\mathcal{A}(r_1, r_2)$. Assume that for some constant $a_0$ the condition

$$|A|^2(q) \leq \frac{n^2}{4r_2^2}a_0 \quad (4.15)$$

is satisfied for all $q \in C_\Gamma(r_1, r_2)$.

(i) If

$$r_1^2 < \frac{4(n-1)}{n^2a_0} r_2^2, \quad (4.16)$$

then $\Gamma$ is totally geodesic.

(ii) If

$$r_1^2 = \frac{4(n-1)}{n^2a_0} r_2^2 \quad (4.17)$$

and the equality (4.15) occur at some point $q \in C_\Gamma(r_1, r_2)$, we have that $\Gamma$ is a Clifford Torus $\mathbb{T}_{m,n}$.

**Proof** Let $|A|^2$ and $|A_1|^2$ be the square of the second fundamental form of $C_\Gamma \subset \mathbb{R}^{n+1}$ and $\Gamma \subset S^n$ respectively. For each $q \in C_\Gamma(r_1, r_2)$, the Eq. (4.14) provides,

$$|A_1|^2(p) = |A|^2(q) |\tilde{q}|^2, \quad (4.18)$$

where $p = \frac{1}{|q|} q \in \Gamma$. Thus, by hypothesis (4.15) and the condition on the rays $r_1$ and $r_2$ we have

$$|A_1|^2(p) = |A|^2(q) |\tilde{q}|^2 \leq \frac{n^2}{4r_2^2}a_0 |\tilde{q}|^2 \leq \frac{n-1}{r_1^2} |\tilde{q}|^2,$$

is that,

$$|A_1|^2(p) \leq \frac{n-1}{r_1^2} |q|^2.$$
The inequality above is true for all \( q \in C_{r}(r_{1}, r_{2}) \) and \( p \in \Gamma \) such that \( p = \frac{1}{|q|}q \) and \( r_{1} \leq |q| \leq r_{2} \), in particular, for \( |q| = r_{1} \) we have

\[
|A_{1}|^{2}(p) \leq n - 1
\]

(4.19)

for all \( p \in \Gamma \). If the condition (4.16) is satisfied, the inequality above is strict and by Theorem 4.2, follows that \( \Gamma \subset S^{n} \) is a totally geodesic hypersurface.

Now, assume that the condition (4.17) is satisfied and suppose that the equality is achieved in (4.15) at some point \( q_{0} \in C_{r} \). Thus, the equality is achieved in (4.19) at \( p = \frac{1}{|q_{0}|}q_{0} \). Again, the Theorem 4.2 ensures that \( \Gamma \subset S^{n} \) is a Clifford torus.

**Corollary 4.1** (Theorem 4.1) Let \( \Sigma^{n} \subset (\mathcal{A}(r_{1}, r_{2}), \langle \cdot, \cdot \rangle) \) be a compact-free boundary minimal hypersurface immersed in a \((n + 1)\)-dimensional Euclidean annulus. Assume that the inequality

\[
|A|^{2} \leq \frac{n^{2}}{4r^{2}}
\]

(4.20)

is satisfied on \( \Sigma^{n} \).

(i) If \( r_{1}^{2} < \frac{4(n-1)}{n^{2}} r^{2} \), then \( \Sigma^{n} \) is a totally geodesic annulus.

(ii) If \( r_{1}^{2} = \frac{4(n-1)}{n^{2}} r^{2} \), then either \( \Sigma^{n} \) is a totally geodesic annulus or \( \Sigma^{n} \) is a truncated cone in such way that \( \partial \Sigma^{n} = \mathbb{T}_{r_{1}} \cup \mathbb{T}_{r_{2}} \), where \( \mathbb{T}_{r} \subset S^{n}_{r} \) is a Clifford torus.

**Proof** By Theorem 4.1 the support function \( v = \langle \vec{x}, N \rangle \) is identically null on \( \Sigma^{n} \) and the boundary \( \partial \Sigma^{n} \) intersect the two connected components of \( \partial \mathcal{A}(r_{1}, r_{2}) \). In summary, we have that \( \Sigma^{n} \) is a subset of a minimal cone in \( \mathbb{R}^{n+1} \). The conclusions follows from Proposition 4.1 by choose of \( a_{0} = 1 \). \( \square \)

**4.2 The Case of Annulus Conformal to Euclidean Annulus**

Let \( \Sigma^{n} \subset (\mathcal{A}(r_{1}, r_{2}), \tilde{g}) \) be an immersed compact free boundary minimal hypersurface, where the metric \( \tilde{g} \) is not necessarily Euclidean. We remember that if \( p \in \Sigma^{n} \) has Euclidean distance at \( \vec{0} \) given by \( s = |p| \), then the distance from \( p \) to \( \vec{0} \) with respect to the metric \( \tilde{g} = e^{2u(|x|^{2})} \langle \cdot, \cdot \rangle \) is given by

\[
\tilde{s} = sI(s),
\]

(4.21)

where \( I(s) = \int_{0}^{1} e^{u(t^{2}s^{2})} dt \). For our next purposes, we define

\[
m_{0} = \sup\{e^{2u(|x|^{2})}, x \in \mathcal{A}(r_{1}, r_{2})\}.
\]
Corollary 4.2  (Theorem 4.1) Let \( \Sigma^n \subset (\mathcal{A}(r_1, r_2), \bar{g}) \) be an immersed free boundary minimal hypersurface. Assume that the inequality

\[
|A|^2 \leq \frac{n^2}{4r_2^2}
\]

is satisfied on \( \Sigma^n \).

(i) If \( r_1^2 < \frac{4(n-1)}{n^2} \left( \frac{I(r_2)^2}{m_0} \right) r_2^2 \), then \( \Sigma^n \) is a totally geodesic annulus.

(ii) If \( r_1^2 = \frac{4(n-1)}{n^2} \left( \frac{I(r_2)^2}{m_0} \right) r_2^2 \), then either \( \Sigma^n \) is a totally geodesic annulus or \( \Sigma^n \) is a truncated cone in such way that \( \partial \Sigma^n = \mathbb{T}_{r_1} \cup \mathbb{T}_{r_2} \), where \( \mathbb{T}_{r_i} \subset \mathbb{S}^n_{r_i} \) is a Clifford Torus.

Proof By Theorem 4.1, the support function \( v = \bar{g}(\bar{x}, \bar{N}) = e^{2h} |\bar{x}, \bar{N}| \) is identically null on \( \Sigma^n \) and the boundary \( \partial \Sigma \) intersects the two connected components of \( \partial (\mathcal{A}(r_1, r_2), \bar{g}) \). We let \( \Sigma^n_{\delta} \) denote the hypersurface \( \Sigma^n \) with the geometry induced by Euclidean metric. Thus, the second fundamental form of \( \Sigma^n_{\delta} \) will be denoted by \( A_{\delta} \).

The principal curvatures \( \bar{k}_i \) and \( k_i \) of \( \Sigma^n \) and \( \Sigma^n_{\delta} \) respectively, are related by the equation,

\[
\bar{k}_i = \frac{1}{e^h} k_i - 2u'(|x|^2) \langle \bar{x}, N \rangle,
\]

where \( N \) denotes the normal vector to \( \Sigma^n_{\delta} \) given by \( N = e^h \bar{N} \). Since \( v \equiv 0 \), we have \( \langle \bar{x}, N \rangle \equiv 0 \), and by Eq. (4.23) we conclude that \( \Sigma^n \) is a free boundary minimal hypersurface in \( (\mathcal{A}(r_1, r_2), \langle \cdot, \cdot \rangle) \). More precisely, \( \Sigma^n \) is minimal truncated cone \( C_{\Gamma} \). Moreover,

\[
\bar{k}_i = \frac{1}{e^h} k_i \Rightarrow e^{2h} |A|^2 = |A_{\delta}|^2
\]

and by definition of \( m_0 \), we have

\[
|A_{\delta}|^2 = |A|^2 e^{2h} \leq |A|^2 m_0.
\]

By hypothesis (4.22), follows that

\[
|A_{\delta}|^2 \leq |A|^2 m_0 \leq \frac{n^2}{4r_2^2} m_0 \leq \frac{n^2}{4r_2^2} \frac{r_2^2}{I(r_2)^2} m_0 \leq \frac{n^2}{4r_2^2} \frac{m_0}{I(r_2)^2} = \frac{n^2}{4r_2^2} a_0.
\]

where \( a_0 = \frac{m_0}{I(r_2)^2} \). Thus,

\[
|A_{\delta}|^2 \leq \frac{n^2}{4r_2^2} a_0.
\]
Now, we observe that $\Sigma^n_\delta \subset (A(r_1, r_2), \langle \cdot, \cdot \rangle)$ is a minimal hypersurface satisfying the condition (4.24) and the results in i) and ii) follows from Proposition 4.1. \hfill \Box

**Example 4.3** Let $\mathbb{H}^{n+1} = (\mathbb{B}_1^{n+1}, \tilde{g})$ be the hyperbolic space, where $\tilde{g} = e^{2h} \langle \cdot, \cdot \rangle$ with $h(x) = u(|x|^2) = \ln \left( \frac{2}{1 + |x|^2} \right)$. Consider $r_1 < r_2 < 1$ and $(A(r_1, r_2), \tilde{g}) \subset \mathbb{H}^{n+1}$. By definition of $m_0$ and by (4.21) we have

$$\frac{I(r_2)^2}{m_0} = \frac{4(\tanh^{-1}(r_2))^2}{r_2^2} \frac{(1 - r_2^2)^2}{4} = \left( \frac{\tanh^{-1}(r_2)(1 - r_2^2)}{r_2} \right)^2.$$

Let $\Sigma^n \subset (A(r_1, r_2), \tilde{g})$ be an immersed free boundary minimal hypersurface. Assume that the inequality

$$|A|^2 \leq \frac{n^2}{4r_2^2}$$

is satisfied on $\Sigma^n$.

(i) If $r_1^2 < \frac{4(n - 1)}{n^2}(\tanh^{-1}(r_2)(1 - r_2^2))^2$, then $\Sigma^n$ is a totally geodesic annulus.

(ii) If $r_1^2 = \frac{4(n - 1)}{n^2}(\tanh^{-1}(r_2)(1 - r_2^2))^2$, then either $\Sigma^n$ is a totally geodesic annulus or $\Sigma^n$ is a truncated cone in such way that $\partial \Sigma^n = \mathbb{T}_1 \cup \mathbb{T}_2$, where $\mathbb{T}_i \subset S^n_{r_i}$ is a Clifford Torus.

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