THE GEOMETRY AT INFINITY OF A HYPERBOLIC
RIEMANN SURFACE OF INFINITE TYPE

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ABSTRACT. We study geodesics on planar Riemann surfaces of
infinite type having a single infinite end. Of particular interest
is the class of geodesics that go out the infinite end in a most
efficient manner. We investigate properties of these geodesics and
relate them to the structure of the boundary of a Dirichlet polygon
for a Fuchsian group representing the surface.

1. INTRODUCTION

A flute surface $S$ is most simply described as a connected domain
in the complex plane, for which all but one of the components in the
complement of $S$ is isolated from the others. Flute surfaces were first
considered by Basmajian [3] as examples of the simplest sort of hy-
perbolic Riemann surface of infinite type. A flute surface has a single
infinite end. The presence of such an infinite end, even one of this
simple sort, allows for many different possibilities for the geometry of
the surface, which have no parallels in the theory of finite surfaces.
In this paper, our main concern is with the behavior of the geometry
associated to the infinite end of the surface, as described by the special
classes of infinite critical and subcritical geodesic rays. These are geo-
desic rays that head either directly, or almost directly, out the infinite
end of the surface. In the theory of Fuchsian groups, these types of
rays are related to the existence of Dirichlet and Garnett points in the
limit set of a Fuchsian group representing the surface [5, 7, 8].

Our approach is to employ a sequence of cut and paste operations to
construct flute surfaces with complex end structure, where the building
blocks are the simple untwisted flutes surfaces studied in [3, 5]. We re-
fer to the surfaces constructed in this way as quilted flute surfaces. The
geometry out the infinite end of a quilted flute surface is considerably
more complex than the end geometry of an untwisted flute. Neverthe-
less, we demonstrate that there are certain fundamental similarities.

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The boundary at infinity of a Dirichlet polygon for a Fuchsian group may be regarded as one measure of the complexity of the end geometry of the surface represented by the group. Applying our construction of quilted surfaces we show that, up to a sparse set, one can exercise surprising control over the boundary at infinity of a Dirichlet polygon for a quilted surface group. To this end we prove

**Theorem 1.** Let $K$ be a compact subset of $\mathbb{R}$. There is a Fuchsian group $G$ representing a quilted flute surface and a point $\tilde{p} \in \mathbb{H}^2$ so that the boundary at infinity of the Dirichlet polygon for $G$ centered at $\tilde{p}$ consists of the union of the set $K$ and a countable set of isolated parabolic fixed points of $G$.

The paper is structured as follows. We begin in Section 2 with a review of some facts about flute surfaces. In Section 3 we construct quilted flute surfaces and derive some of their basic properties. In Section 4 we investigate a special class of canonical critical rays that go out the infinite end of a quilted flute surface. The complexity of the set of such rays is one measure of the complexity of the surface. Before moving on to probe more results about quilted surfaces we must turn to the hyperbolic plane. In Section 5 we prove several lemmas, crucial to the proofs in later sections. In Section 6 we develop an intrinsic version of Theorem 1 for quilted surfaces and show how this can be used to to prove Theorem 1. In the last section we look at the finer structure of the set of infinite critical and subcritical geodesics rays. The main result of Section 7 is that there is an underlying geodesic scaffolding, heading out the infinite end, which all infinite critical and subcritical geodesic rays must approach asymptotically. This is a broad generalization of a similar result about untwisted flutes, that appeared in [5]. In obtaining these results we employ a number of lemmas. Several of these are results about plane hyperbolic geometry which are interesting in their own right.

## 2. Basic properties of flute surfaces

The main reference for the material in this section is [5].

Define an *end* $E$ of a manifold $M$ as follows. Let $K_1 \subset K_2 \subset \ldots \subset M$ be a nested sequence of compact subsets of $M$ so that $\bigcup_{i=1}^{\infty} K_i = M$. An end $E$ is a sequence of connected components $\{E_i\}$ in the complement of $K_i$ so that $E_{i+1} \subset E_i$. This definition can be made independent of the given exhaustion $\{K_i\}$. A ray $\sigma$ is said to *go out the end* $E$, if for each integer $i > 0$ all but a compact segment of $\sigma$ belongs to $E_i$. 
Let $S$ be a hyperbolic surface. An end $E$ of $S$ is called a puncture if there is a subset $D$ of $S$ which is conformally equivalent to the punctured disc $\{z \mid 0 < |z| < 1\}$ and for $i$ large, $E_i \subset D$. Similarly, we call $E$ a hole if there is a subset $D$ of $S$ which is conformally equivalent to an annulus $\{z \mid 1 < |z| < r\}$ for some $r > 1$ and for $i$ large, $E_i \subset D$. $E$ is a finite end if it is either a puncture or a hole; otherwise it is an infinite end.

An end $E = \{E_i\}$ is said to be of the second kind if $S$ contains a half-plane $P$ and for all $i > 0$, $E_i \cap P \neq \emptyset$. If an end is not of the second kind then it is of the first kind. A puncture is of the first kind and a hole is of the second kind.

Let $S$ denote the infinite cylinder $C = S^1 \times (0, \infty)$, with the set of points $\{(1, n) \mid n \in \mathbb{N}\}$ deleted, and define the space $F$ of isometry classes of complete metrics of constant curvature $-1$, that is, hyperbolic metrics on the surface $S$. Define an involution $r : S \to S$ by $r(e^{i\theta}, t) = (e^{-i\theta}, t)$. Let $F_0 \subset F$ be the set of isometry classes in $F$ for which there exists a representative surface on which $r$ is an isometry. Henceforth we shall treat elements of $F$ and $F_0$ as hyperbolic surfaces and suppose, in the latter case, that $r$ is an isometry. A surface in $F$ is called a flute; one in $F_0$ is called an untwisted flute. An explicit construction of flutes is given in [3]. A flute has one infinite end and has a finite end associated to each of the deleted points $(1, n)$ and to the ideal boundary $S^1 \times \{0\}$. Note that each of these ends can be of the first or the second kind, depending on the hyperbolic metric we have chosen. The proposition below, whose proof is given after we develop some notation, shows that this definition of a flute surface is consistent with the definition given in the introduction.

**Proposition 1.** $S$ is a flute surface if and only if $S$ is conformally equivalent to a connected domain in $\mathbb{C}$ with a single infinite end.

Let $S$ be a hyperbolic surface and $\sigma : [0, \infty) \to S$ a geodesic ray. Here and henceforth, all geodesics are parameterized by arc length. Define the function $\Delta_\sigma(t) = t - d_S(\sigma(0), \sigma(t))$, where $d_S$ denotes distance as defined by the hyperbolic metric on $S$. The ray $\sigma$ is then said to be horocyclic, critical, or subcritical if $\Delta_\sigma$ is respectively, unbounded, zero, or nonzero but bounded. A critical ray may be said to travel directly out an end of $S$, and a subcritical ray may be said to travel almost directly out an end. It is known (see, [3, 7]) that critical rays are simple and subcritical rays are simple beyond some point.

Let $E = \{E_i\}$ be an end of a flute $F$. It is known from [5] that for any $p \in F$ there is a critical ray with initial point $p$ that goes out the end $E$. When the end $E$ is a finite end, critical and subcritical rays that
go out the end $E$ are called \textit{finite} critical or subcritical rays; otherwise, if the end $E$ is an infinite end, they are called \textit{infinite}. Furthermore, a critical or a subcritical ray always goes out some end of $F$. We shall primarily be interested in the infinite critical and subcritical rays on $F$.

Given a flute surface $F \in \mathcal{F}$, and an integer $n \geq 0$, let $\alpha_n$ denote the simple closed geodesic on $F$ in the free homotopy class of the curve

$$t \to (e^{it}, n + \frac{1}{2}), 0 \leq t \leq 2\pi.$$ 

We shall refer to a geodesic $\alpha_n$ as a \textit{dividing loop} on $F$. Note that these are well defined with the possible exception of $\alpha_0$, which exists only if the end corresponding to the ideal boundary $S^1 \times \{0\}$ is a hole. In what follows we shall always take this end to be a hole so that $\alpha_0$ does exist.

Let $\beta^* = \{(-1, t) \mid t \in (0, \infty)\}$ and for integers $n \geq 0$ let $\gamma_n = \{(1, t) \mid n < t < n + 1\}$. Suppose in addition that $F$ is an untwisted flute. Then $\beta^*$ and $\gamma_n$ are geodesics since they are fixed by the isometry $r$. We shall refer to the geodesics $\gamma_n$ as the $\gamma$-curves of $F$. For each $n$ the geodesics $\beta^*$ and $\gamma_n$ are both orthogonal to the dividing loop $\alpha_n$. Also, for any point $p$ on $\beta^*$ the geodesic ray beginning at $p$ going out the infinite end of $F$ along $\beta^*$ is a critical ray. We shall refer to $\beta^*$ as the canonical Dirichlet geodesic on $F$ and assume it to be oriented out the infinite end, with $\beta^*(0) \in \alpha_0$.

\textbf{Proof of Proposition} Suppose $S$ is a domain in the complex plane with a single infinite end. We refer to the connected components in the complement of $S$ as complementary components. Let $\Delta_\infty$ denote the complementary component corresponding to the infinite end of $S$. $S$ is endowed with the unique hyperbolic metric in its conformal equivalence class. We shall define a sequence of simple closed geodesics on $S$ so that each component in the complement of this set of geodesics on $S$ is a triply connected domain, referred to as a pair of pants, (see [2]).

Let $\Delta$ be a complementary component not containing the point at infinity. Define the distance between $\Delta$ and $\Delta_\infty$ as the infimum of the (Euclidean) distances between points in $\Delta$ and $\Delta_\infty$ and denote this distance by $d(\Delta, \Delta_\infty)$. It is possible to index the complementary components not containing the point at infinity by $\mathbb{N}$, so that $d(\Delta_i, \Delta_\infty) \geq d(\Delta_{i+1}, \Delta_\infty)$. Let $\overline{\alpha}_1$ be a simple closed curve on $S$ that divides $\mathbb{C}$ into two pieces, one of which contains only the two complementary components $\Delta_1$ and the component containing the point at infinity. Let $\alpha_1$ be the geodesic in the free homotopy class of $\overline{\alpha}_1$ on $S$. Suppose the geodesics $\alpha_1, \ldots, \alpha_n$ have been defined. Let $\overline{\alpha}_{n+1}$ be a
simple closed curve, disjoint from $\alpha_n$, so that the region of $C$ bounded by $\alpha_n$ and $\pi_{n+1}$ contains the single complementary component $\Delta_{n+1}$. Let $\alpha_{n+1}$ be the geodesic freely homotopic to $\pi_{n+1}$ on $S$.

Let $Q$ be the set of geodesics $\alpha_i, i \in \mathbb{N}$ defined above. Let $P_i$ denote the connected component of $S \setminus Q$ whose boundary meets the boundary of $\Delta_i$. Each of the $P_i$ is a pair of pants. Now $S$ can be reconstructed from the sets $P_i$ by ‘gluing’ $P_i$ to $P_{i+1}$ along their common geodesic boundary $\alpha_i$ to get a flute surface, as in [3].

To prove the converse, we simply observe that every closed curve on a flute surface $F$ divides. It is known, (see [1]), that $F$ is then conformally equivalent to a domain in the plane. Since $F$ has a single infinite end, the proof is complete.

\[\square\]

3. **Gluing untwisted flutes**

Flute surfaces on which the asymptotic geometry displays more diverse behavior than that exhibited by untwisted flutes can be constructed by gluing together untwisted flutes of the first kind that have been sliced open along their canonical Dirichlet geodesics. We shall describe a way to perform the construction to allow for infinitely many gluings along a superstructure of scaffolding curves defined by choosing a closed subset of an oriented circle of a given circumference.

3.1. **The finite steps.** Let $A$ denote the hyperbolic cylinder with the oriented simple closed geodesic $\alpha_0$ dividing $A$ into subsets $A^+$ and $A^-$, where $A^+$ is to the right of $\alpha_0$. $A$ is completely determined by the length of $\alpha_0$, which we denote by $a = |\alpha_0|$. Let $C$ be a closed subset of $\alpha_0$ and let $p$ be a distinguished point on $\alpha_0$. The complement of $C$ in $\alpha_0$ is a countable union of open geodesic segments which we refer to as intervals. Order the intervals lexicographically in terms of length (larger lengths precede smaller lengths) and oriented distance from $p$, to get a sequence of oriented intervals $\{I_i\}_{i=1}^l$, where $l$, which may be infinity, is the number of components in the complement of $C$ on $\alpha_0$. Henceforth, we assume that $2 \leq l \leq \infty$. Label the endpoints of $I_i$, $e_{i1}$ and $e_{i2}$, where $I_i$ is oriented from $e_{i1}$ to $e_{i2}$. Through each point $e_{ij}$ there is a unique biinfinite geodesic $e_{ij}$ orthogonal to $\alpha_0$ and oriented in the direction of the end $A^+$, with $e_{ij}(0) = e_{ij}$. Except in the case where two geodesics with different names coincide, these geodesics are pairwise disjoint and the two ends of each $e_{ij}$ go out the two ends of the cylinder $A$. We shall refer to the the union of the geodesics $e_{ij}$ and the intervals $I_i$ as the **scaffolding**. Note that the scaffolding is completely determined
by the choice of the point \( p \), the orientation on \( \alpha_0 \) the length \( |\alpha_0| \) and the set \( C \).

Let \( E_i \) denote the hyperbolic strip in \( A \) bounded by the geodesics \( \epsilon_1^i \) and \( \epsilon_2^i \) and containing the interval \( I_i \). The scaffolding structure will serve as a foundation for the construction in which the hyperbolic strip \( E_i \) shall be removed and then replaced by an untwisted flute surface which has been sliced open along its canonical Dirichlet geodesic. Consequently, in addition to the foundational information provided, the construction also requires a description of the flute surfaces that are to be glued in. For each \( i \in \mathbb{N} \), let \( F_i \) be an untwisted flute. We shall henceforth assume that the infinite end on each of the untwisted flutes \( F_i \) is of the 1st kind. One can provide sufficient conditions which guarantee this; for example, if the dividing loops grow sufficiently slowly, then the end is of the 1st kind.

Denote the sequence of dividing geodesics on \( F_i \) by \( \alpha_{i,k} \) where \( k = 0, 1, 2, \ldots \). We suppose that the length \( |\alpha_{i,0}| \) of \( \alpha_{i,0} \) is equal to the length \( |I_i| \) of the interval \( I_i \). On the flute \( F_i \) orient the canonical Dirichlet geodesic \( \beta_i^* \) in the direction of the infinite end of \( F_i \) and so that \( \beta_i^*(0) \in \alpha_{i,0} \). Orient the geodesic \( \alpha_{i,0} \) on \( F_i \) so that the infinite end is to its right. Cut open \( F_i \) along \( \beta_i^* \) to get the complete hyperbolic surface \( F_i^* \) with boundary. \( F_i^* \) has the two boundary geodesics \( \beta_1^i \) and \( \beta_2^i \), where the cut open \( \alpha_{i,0} \) is oriented from the the point \( \beta_1^i \cap \alpha_{i,0} \) to \( \beta_2^i \cap \alpha_{i,0} \). Each \( \beta_i^j \) inherits a parameterization from \( \beta_i^* \). By re-identifying the boundary geodesics \( \beta_1^i \) and \( \beta_2^i \) so that \( \beta_1^i(t) \) is glued to \( \beta_2^i(t) \), for \( t \in \mathbb{R} \), the starting flute \( F_i \) will be reproduced.

On \( F_i^* \) we shall continue to refer to the cut open dividing loops \( \alpha_{i,k} \) by the same names. Observe that the involution \( r \) of \( F_i \) induces an isometric involution \( r^* \) of \( F_i^* \), interchanging \( \beta_1^i(t) \) and \( \beta_2^i(t) \) for \( t \in \mathbb{R} \), fixing the \( \gamma \) curves and mapping each \( \alpha_{i,k} \) onto itself.

With the data consisting of \( A, p, C \) and the surfaces \( F_i \), we define a sequence of flute surfaces \( S_n \), constructed by a succession of single replacements of the type described at the beginning of the section, as follows. Set \( S_0 = A \). Suppose that the hyperbolic surface \( S_i \) has been defined for some \( i \geq 0 \), and has the property that \( S_i \) contains an isometric copy of the original scaffolding, so that the union of the scaffolding and the strips \( E_k \) with \( k > i \) embeds isometrically in \( A \). (Here objects with the same name are identified by the embedding.)

In order to construct \( S_{i+1} \) begin by removing the interior of the strip \( E_{i+1} \) from \( S_i \). Insert \( F^*_i \) in place of \( E_{i+1} \) by identifying the two pairs of geodesics \( \epsilon_1^{i+1} \) and \( \epsilon_2^{i+1} \) and \( \beta_1^{i+1}, j = 1, 2 \), so that \( \epsilon_j^{i+1}(t) \) is identified with \( \beta_j^{i+1}(t) \) for \( t \in \mathbb{R} \). On \( S_{i+1} \) the identified pair \( \epsilon_i^{i+1} \) and \( \beta_i^{i+1} \) shall be denoted
by $\beta_{i+1}^j$, which inherits their parameterization. The surface $S_{i+1}$ has a hyperbolic metric defined locally at every point and is easily seen to be complete. Therefore $S_{i+1}$ is a hyperbolic surface. Furthermore, it naturally contains a copy of $S_i \setminus E_{i+1}$, as well as a copy of $F^*_{i+1}$.

The set of all the geodesics $\beta_m^j$ on $S_n$ where $m = 1, \ldots, n$ and $j = 1, 2$ shall be denoted by $B_n$.

**Proposition 2.** The surfaces $S_n$ are all flute surfaces.

**Proof.** First observe that if two planar surfaces are glued together in the plane then the result is a planar surface. Each flute $F_i$, as well as its cut open relative $F^*_i$, is planar. By induction it follows that each of the surfaces $S_n$ is planar.

Next we argue by induction that each $S_n$ has a single infinite end. Suppose this is true for $S_i$. Let $K^*_j$ be a compact exhaustion of $S_i$ with the property that $I_{i+1} \subseteq K^*_j$ for each $j$ and similarly choose a compact exhaustion $K^\#_j$ of $F^*_{i+1}$ so that $\alpha_{i+1,0} \subseteq K^\#_j$ for each $j$. The $K^*_j$ correspond to the sets on $S_i \setminus E_{i+1}$, also called $K^*_j$, which give a compact exhaustion of $S_i \setminus E_{i+1}$. Similarly, the $K^\#_j$ correspond to sets on $F^*_{i+1}$, also called $K^\#_j$, which induce a compact exhaustion of $F^*_{i+1}$. Then $K_j = K^\#_j \cup K^*_j$ is a compact exhaustion of $S_{i+1}$. Let $E_j$ be a nested sequence of complementary components of the sequence $\{K_j\}$. Then one of the following three possibilities holds: for $j$ sufficiently large, either $E_j \subset S_i \setminus E_{i+1}$ or $E_j \subset F^*_{i+1}$, or $E_j$ has non-empty intersection with both $S_i \setminus E_{i+1}$ and $F^*_{i+1}$ for all $j$. In the first two cases the corresponding end must be finite. In the last case it is possible that the sets $E_j$ are all cylinders lying to the left of $\alpha_0$ and consequently, the corresponding end is finite. The remaining case occurs when each $E_j$ is a union of open sets on $S_i \setminus E_{i+1}$ and $F^*_{i+1}$ belonging to the infinite end of each of those surfaces. In only this last case is it possible for the corresponding end to be infinite. Thus $S_{i+1}$ has a single infinite end and $S_n$ is a flute surface for all integers $n \geq 0$. □

3.2. **Geometric convergence of flute surfaces.** Here we show that it is possible to define a flute surface as a limit of the finite constructions described in the previous section. We begin with the Fuchsian groups representing the surfaces and show that there is a way to normalize the groups so that one representing a surface $S_n$ contains the group for the surface $S_m$ when $n > m$. This leads to the definition of the surface $S_\infty$. Then, on the level of the intrinsic geometry, we prove that it is possible to view an arbitrarily large chunk of $S_\infty$ as the isometric images of a large chunk of $S_n$, for $n$ sufficiently large. This last fact leads to a proof...
that $S_\infty$ is a flute surface. These results will allow us to assume, in the
proofs of several of the theorems that follow, that we are working on
one of the surfaces $S_n$, rather than on the more complex surface $S_\infty$.

**Lemma 1.** There exists a nested sequence of Fuchsian groups $\Gamma_0 \subset \Gamma_1 \subset \ldots$, so that for each integer $n \geq 0$, $\mathbb{H}^2/\Gamma_n = S_n$. Furthermore, the image of the imaginary axis $I \subset \mathbb{H}^2$ covers the geodesic $\alpha_0$ on $S_n$ and the left half-plane $H^-$ covers the annular region $A^-$ on $S_n$.

**Proof.** Given $|\alpha_0| = a$, let $\Gamma_0$ be the group generated by the Möbius transformation $g_0(z) = e^{a}z$. Observe that the imaginary axis, denoted by $I$, projects to the geodesic $\alpha_0$ on $A = \mathbb{H}^2/\Gamma_0$ and the left and right half-planes project, respectively, to the subannuli $A^-$ and $A^+$.

Define the sequence of Fuchsian groups recursively. Suppose the
$\Gamma_n$ have been defined. Set $E_{n+1}^+ = E_{n+1} \cap A^+$. Remove the subsurface $E_{n+1}^+$ from $S_n$ to get the surface $S'_{n+1}$. A loop on $S'_{n+1}$ is homotopically trivial on $S_n$ if and only if it is homotopically trivial on $S_n$. Therefore, the preimage $\tilde{S}'_{n+1}$ of $S'_{n+1}$ under the covering projection $\pi_n : \mathbb{H}^2 \to S_n$ is connected and simply connected, $\Gamma_n$-invariant set and the restriction $\pi_n : S'_{n+1} \to S'_{n+1}$ is the universal covering.

Suppose $G_{n+1}$ is a Fuchsian group representing $S_{n+1}$. Let $F_{n+1}^+ = F_{n+1}^* \cap A^+$. Then remove $F_{n+1}^*$ from $S_{n+1}$ to get the surface $S_{n+1}^+$. Let $\tilde{S}_{n+1}^+$ be a connected preimage of $S_{n+1}^+$ under the covering projection $\pi_{n+1} : \mathbb{H}^2 \to S_{n+1}$. Then $\tilde{S}_{n+1}^+$ is simply connected and the restriction $\pi_{n+1} : S_{n+1}^+ \to S_{n+1}^+$ is the universal covering. Let $G_{n+1}^+$ be the stabilizer of $\tilde{S}_{n+1}^+$ in $G_{n+1}$.

By the construction, $S_{n+1}^+$ is the image of the imaginary axis $I \subset \mathbb{H}^2$. Since $\tilde{S}_{n+1}^+$ and $\tilde{S}_{n+1}^+$ are subsets of hyperbolic space, $\tilde{\varphi}$ must be the restriction of a Möbius transformation and $\Gamma_n = \tilde{\varphi}^{-1}G_{n+1}\tilde{\varphi}$. It follows also that $\tilde{\varphi}^{-1}G_{n+1}\tilde{\varphi}$ is a Fuchsian group representing $S_{n+1}$ and $G_{n+1} \supset \Gamma_n$.

Note that $\varphi$ maps the geodesic $\alpha_0$ on $S'_{n}$ to its counterpart on $S_{n+1}^+$ and identifies the corresponding copies of $A^-$. Thus $\tilde{\varphi}$ will map $I$ to a geodesic in $\tilde{S}_{n+1}^+$ that covers $\alpha_0$ and takes the left half-plane $H^-$ in $\mathbb{H}^2$ to a corresponding hyperbolic half-plane in $\tilde{S}_{n+1}^+$ that covers $A^-$. It follows that for all integers $n \geq 0$, $H^-$ is precisely invariant in $\Gamma_n$ under $\Gamma_0$; that is, $h(H^-) \cap H^- \neq \emptyset$ if and only if $h \in \Gamma_0$. Thus, $H^-$ projects to $A^-$ and $I$ projects to $\alpha_0$ on each $S_n$. 

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We are now in a position to define the limiting surface $S_\infty$. First, define $\Gamma_\infty = \bigcup_{n=0}^{\infty} \Gamma_n$, where $2 < l \leq \infty$ is the number of components in the complement of $C$ on $\alpha_0$. None of the $\Gamma_n$ contain elliptic elements, and therefore the same must be true of $\Gamma_\infty$. Since, for $n > 0$, $\Gamma_n$ is non-elementary, it follows that $\Gamma_\infty$ is a Fuchsian group. Define $S_\infty = \mathbb{H}^2 / \Gamma_\infty$. Let $\pi_\infty \to S_\infty$ denote the quotient map. Set the notation $S_\infty = S(\alpha_0, p, C, \{F_i\})$ to emphasize the fact that the orientation on $\alpha_0$, the point $p$, the length $a = |\alpha_0|$, the closed set $C$, and the sequence of untwisted flutes $F_i$ together define the surface.

As usual $B(q, r) \subset \mathbb{H}^2$ shall denote the set of points of distance less than $r$ from $q$ and we let $B^c(q, r)$ denote its closure. Let $i$ be the imaginary value $\sqrt{-1}$. Without loss of generality, we can suppose that $i$ projects to the point $p$ on $A$. Consider the translates of the closed ball $B^c(i, r)$ by the transformations in $\Gamma_n$. Given $r > 0$ there is a value $M_r \in \mathbb{N}$ so that for $M_r \leq n \leq \infty$ and $g \in \Gamma_n$, if $g(B^c(i, r)) \cap B^c(i, r) \neq \emptyset$ then $g \in \Gamma_{M_r}$. This must be so since, in a Fuchsian group, only finitely many $\Gamma_\infty$-translates of the compact set $B^c(i, r)$ may intersect $B^c(i, r)$.

Thus, for $n \geq M_r$, the projections $\pi_n(B^c(i, r))$ are isometrically equivalent in the most natural way. Note $\pi_n(B^c(i, r))$ is connected and since the set of transformations that create intersections is finite, the complement of $\pi_n(B^c(i, r))$ in $S_n$ has $k < \infty$ components. Let $U_{n,1}, \ldots, U_{n,k-1}$ denote the complementary components of $\pi_n(B^c(i, r))$ in $S_n$ that do not contain the infinite end. Call these the finite components. For $n \geq M_r$, define $S_n(r) = \bigcup_{j=1}^{k-1} U_{n,j} \cup \pi_n(B^c(i, r))$

Let $p_n \in S_n$ be the projection $\pi_n(i)$ and similarly set $\alpha_0^n = \pi_n(I)$. We have just given distinct names to the point $p$ and the geodesic $\alpha_0$. Intrinsically, for $i \in \mathbb{N}$, $S_n(r)$ is the closed ball of radius $r$ about $p_n$ in $S_n$, with the finite complementary components added on. It is not yet clear that this will work for $S_\infty$, since we have not proved that $S_\infty$ is a flute.

The next proposition asserts a kind of strong geometric convergence of the surfaces $S_n$ to $S_\infty$.

**Proposition 3.** Given $r > e^2$ there is a number $N_r \in \mathbb{N}$ so that for $m > n \geq N_r$ there exists an isometric bijection $\varphi_{n,m} : S_n(r) \to S_m(r)$ with $\varphi_{n,m}(\alpha_0^n) = \alpha_0^m$ and $\varphi_{n,m}(p_n) = p_m$. Furthermore, for $n \geq N_r$ there exists an isometric embedding $\varphi_n : S_n(r) \to S_\infty$ with $\varphi_n(\alpha_0^n) = \alpha_0^\infty = \pi_\infty(\alpha_0)$ and $\varphi_n(p_n) = p_\infty = \pi_\infty(p)$.

A collar of width $2R$ about a simple closed geodesic $\beta$ on a hyperbolic surface $S$, written $C_S(\beta, R)$, is the set of points on $S$ of distance less than $R$ from $\beta$. The classical Collar Lemma implies that given $R > 0$ there is an number $L > 0$ so that if $|\beta| < L$ then $C_S(\beta, R)$ is isometric
to the collar \( C_A(\alpha, R) \), where \( A \) is an annulus with core geodesic of length \( |\alpha| = |\beta| \). More precisely, \( C_S(\beta, R) \) is isometric to the collar \( C_A(\alpha, R) \) if \( \sinh R = (2 \sinh |\beta|/2)^{-1} \), where \( |\alpha| = |\beta| \), [6].

One can analogously define the collars \( C_{E_j}(I_j, R) \) and \( C_{F_j}(\alpha_{j,0}, R) \) about the geodesic segments on the surfaces \( E_j \) and \( F_j \), respectively. Observe that identifying the boundary geodesics \( e^1_j \) and \( e^2_j \) of \( E \) will produce an annulus. The Collar Lemma, applied to this annulus and the flute \( F_j \), implies that given \( R > 0 \) there is a \( J \in \mathbb{N} \) so that if \( j > J \) then \( C_{E_j}(I_j, R) \) is isometrically equivalent to \( C_{F_j}(\alpha_{j,0}, R) \) by an isometry taking \( \alpha_{j,0} \) to \( I_j \).

**Proof.** The proposition is trivial if the number of complementary components, \( l \), of \( C \) in \( \alpha_0 \) is finite. We therefore suppose that \( l = \infty \).

We normalize so that the surfaces are represented by Fuchsian groups, as in Lemma[1]. Following up on the comments preceding the statement of the proposition, given \( r > 0 \) there is an \( M_r \in \mathbb{N} \) so that if \( g \in \Gamma_n \) for \( M_r \geq n \geq \infty \), and \( g(\mathbb{B}(i, r)) \cap \mathbb{B}(i, r) \neq \emptyset \), then \( g \in \Gamma_{M_r} \). The closed ball \( \mathbb{B}(i, r) \) projects to a set \( \mathbb{B}_n^c(r) \subset S_n \). Since \( r > e^{\frac{2}{c}}_M \), \( \mathbb{B}(i, r) \) contains a segment of \( I \) which projects to \( \alpha_0 \). Then for \( M_r \leq n < m \leq \infty \) there is an isometric bijection \( \varphi_{n,m} : \mathbb{B}_n^c(r) \to \mathbb{B}_m^c(r) \), which maps \( \alpha_0^m \) to \( \alpha_0^n \) and \( p_n \) to \( p_m \). We suppose that the finite complementary components \( U_{n,j} \) and \( U_{m,j} \) share the corresponding boundaries on \( \mathbb{B}_n^c(r) \) and \( \mathbb{B}_m^c(r) \), respectively. We shall prove that there is a \( N_r \geq M_r \) so that for any \( n \geq N_r \), the process of creating \( S_{n+1} \) from \( S_n \) by excising \( E_{n+1} \) and gluing in \( F_{n+1}^* \), does not change the subsurface \( S_n(r) \).

Suppose \( n > M_r \). Since \( \mathbb{B}_n^c(r) \) is compact, by the Collar Lemma there must exist a value \( R > 0 \) and an integer \( J_r \geq M_r \) so that for \( j \geq J_r \), \( \mathbb{B}_n^c(r) \cap E_j \subset C_{E_{j}}(I_j, R) \). Consequently, the region \( E_j \setminus C_{E_{j}}(I_j, R) \) must lie in the infinite component in the complement of \( \mathbb{B}_n^c(r) \).

Also, by the Collar Lemma we may choose \( N_r \geq J_r \) so that for \( j \geq N_r \) the collar \( C_{F_j}(\alpha_{j,0}, R) \) on \( F_j^* \) is isometrically equivalent to \( C_{E_j}(I_j, R) \). Thus, for \( n \geq N_r \), \( S_{n+1} \) can be constructed from \( S_n \) by replacing \( E_{n+1} \setminus C_{E_{n+1}}(I_{n+1}, R) \) by \( F_{n+1}^* \setminus C_{F_{n+1}}(\alpha_{n+1,0}, R) \). Consequently, \( S_{n+1} \) differs from \( S_n \) only in the infinite component in the complement of \( B_{n+1}^c(r) \simeq B_n^c(r) \). The first part of the proposition follows.

Let \( \hat{S}(r) \) be the connected \( \pi_{N_r} \)-preimage of \( S_{N_r}(r) \), which contains \( i \) and let \( \Gamma \subset \Gamma_{N_r} \) denote the stabilizer of \( \hat{S}(r) \) in \( \Gamma_{N_r} \). By what we have shown, it follows that \( \Gamma \) is the stabilizer of \( \hat{S}(r) \) in \( \Gamma_n \) for all \( n \geq N_r \) and therefore in \( \Gamma_\infty \), as well. The final statement of the proposition follows. \( \square \)
Define $S_\infty(r) = \varphi_n(S_n(r))$ for some $n > N_r$. The sequence of closed balls $\pi_\infty(B^*(i,k))$, for $k \in \mathbb{N}$, is a compact exhaustion of $S_\infty$. Associated to this exhaustion there is an infinite end $E$ with $E_k = S_\infty \setminus S_\infty(k)$. It follows from the proposition that every other end is finite. Thus we have

**Corollary 1.** $S_\infty$ is a flute.

To simplify notation, we shall henceforth dispense with superscripts and refer to the geodesic $\alpha_0$ and the point $p$ on the surfaces $S_n$ and $S_\infty$.

### 4. A special class of critical rays on $S_\infty$

Given a point $c \in C$, let $\sigma_c$ denote the geodesic ray on $S_\infty = S(\alpha_0, p, C, \{F_i\})$ beginning at $c$, orthogonal to $\alpha_0$ and oriented out the infinite end of $S_\infty$. If $c$ lies on the boundary of one of the intervals $I_j$ in the complement of $C$, then the ray $\sigma_c$ will be contained in the scaffolding geodesic on $S_\infty$ passing through $c$.

**Theorem 2.** For each $c \in C$, $\sigma_c$ is an infinite critical ray.

We begin by proving several lemmas, which, along with the lemmas of the next section, shall be of use throughout the paper. Theorem 2 will follow as an immediate corollary of Lemma 4 in which we prove something slightly stronger about the ray $\sigma_c$, namely, subarcs of $\sigma_c$ realize the distance between any point on $\sigma_c \setminus \{\sigma_c(0)\}$, and the curve $\alpha_0$.

Many of the lemmas in this and the next section will share the common setup stated below, which will be invoked repeatedly.

Let $S^*$ be either the closed set $F^*$, an untwisted flute cut open along its canonical Dirichlet ray $\beta^*$, or the hyperbolic strip $E^* \subset \mathbb{H}^2$ bounded by two parallel, non asymptotic geodesics. Let $\alpha_0$ denote the first, cut-open dividing loop on $F^*$, or the common orthogonal between the boundary geodesics on $E^*$. The geodesics $\beta_1$ and $\beta_2$ on the boundary of $S^*$ are parameterized so that $\beta_1(0)$ and $\beta_2(0)$ are the end points of $\alpha_0$ and the geodesics $\beta_i(t)$, $i = 1, 2$, go out the same end of the surface as $t \to \infty$; in the case $S^* = F^*$, this end is the infinite end of $F^*$.

Let $F$ be the untwisted flute with canonical Dirichlet ray $\beta^*$ which, when cut open along $\beta^*$, produces $F^*$. Note that $\beta_1$ and $\beta_2$ inherit their parametrization from $\beta^*$ and that $\alpha_0$ is orthogonal to both of these boundary geodesics. Let the union of the $\gamma$-curves of $F$ be denoted by $\Gamma$. The curves in $\Gamma$ along with $\beta^*$ comprise the fixed-point set of the canonical isometric involution $r : F \to F$. The involution $r$ of $F$ induces an isometric involution $r^* : F^* \to F^*$ that interchanges the boundary.
curves $\beta_1$ and $\beta_2$. The fixed-point set of $r^*$ is now the set of curves (corresponding to) $\Gamma$. Moreover, as $r^*$ is an isometry, $r^*(\beta_1(t)) = \beta_2(t')$ if and only if $t = t'$.

**Lemma 2.** Let $P$ be a point in $E^*$ that does not lie on $\alpha_0$. Then there is a unique point $Q$ on $\alpha_0$ such that the geodesic segment $\delta$ joining $P$ to $Q$ realizes the distance $d(P, \alpha_0)$. Further, $\delta$ meets $\alpha_0$ at a right angle. In particular, if $P$ lies on $\beta_1$ (respectively, $\beta_2$) then $Q$ lies on $\beta_1$ (respectively, $\beta_2$).

**Proof.** Let $\alpha$ be the full, bi-infinite geodesic containing $\alpha_0$, and let $Q$ be the point on $\alpha$ for which the length of the geodesic arc $\delta$, joining $P$ to $Q$, is a minimum. From elementary geometry, the angle at $Q$ where $\delta$ meets $\alpha$ must be a right angle. Suppose $Q$ lies outside $E^*$, that is, $Q$ is a point on $\alpha \setminus \alpha_0$. Then $\delta$ must cross a boundary curve of $E^*$, without loss of generality, say $\beta_1$, where $\beta_1$ meets $\alpha$ at a point $R$. Then the triangle, $\Delta PQR$, formed by segments $PQ$, $QR$, and $RP$ has a right angle at $Q$ and the angle at $R$ is greater than $\pi/2$. This is impossible. Therefore, $Q$ lies on $\alpha_0$. \qed

Let $\beta$ be one of the two boundary curves of $F^*$.

**Lemma 3.** Suppose $\alpha : [0, \tau^*] \to F^*$ is a geodesic which does not lie on $\beta$, with $\alpha(0) \in \alpha_0$ and $\alpha(\tau^*) = \beta(t^*)$ for some $t^* > 0$. Then $t^* < \tau^*$.

**Proof.** Consider the region $\Delta$ on $F^*$ bounded by $\alpha$, the arc $\beta([0, t^*])$ and the arc of $\alpha_0$ from $\alpha(0)$ to $\beta(0)$. If $\Delta$ is simply connected then it is a hyperbolic triangle. In that case, since $\beta$ is orthogonal to $\alpha_0$ it is a right triangle with hypotenuse $\alpha$. It follows that $t^* < \tau^*$.

Suppose now that $\Delta$ is not simply connected. The $\gamma$-curves of $F^*$ divide the surface into simply connected subsurfaces $S_1$ and $S_2$. Suppose $\beta \subset S_1$. Use the involution $r^*$ to reflect each arc of $\alpha$ in $S_2$ to an arc on $S_1$. The arcs of $\alpha$ and $r^*(\alpha)$ on $S_1$ form a piecewise geodesic $\overline{\alpha}$, which inherits its parametrization from $\alpha$. The geodesic $\alpha'$ from $\overline{\alpha}(0)$ to $\alpha(\tau^*) = \beta(t^*)$ on $S_1$ has length less than $\alpha$. Applying the first case to $\alpha'$ shows that its length is greater than $t^*$. Therefore, $|\alpha| = \tau^* > |\alpha'| > t^*$. The lemma is proved. \qed

Theorem 2 is an immediate consequence of the following lemma. Let $S$ be one of $S_\infty$ or $S_n$ for some $n \in \mathbb{N}$.

**Lemma 4.** Let $c \in C$ and $t > 0$. Then the geodesic arc $\sigma_c([0, t])$ is the unique arc that realizes the distance between $\sigma_c(t)$ and $\alpha_0$. In particular, if $\delta : [0, \tau] \to S$ is a geodesic joining a point $Q \neq \sigma_c(0)$ on $\alpha_0$ to $\sigma_c(t)$, then $\tau > t$. 


Proof. Since $S$ is a complete, convex hyperbolic manifold, there is a
geodesic arc from $\sigma_c(t)$ to some point on $\alpha_0$ whose length is $d(\sigma_c(t), \alpha_0)$. 
Suppose that $Q$ is a point on $\alpha_0$ and that $\delta : [0, \tau] \to S$ is a geodesic arc
joining $Q = \delta(0)$ to $\sigma_c(t)$.

Suppose, for the moment, that $S = S_\infty$. Since the geodesic arcs
$\delta([0, \tau]), \sigma_c([0, t])$ and $\alpha_0$ are compact, there is some $r > 0$ for which all
of these arcs lie in $B_\infty^c(r) \subset S_\infty^c(r)$. Thus, there is a positive integer $N_r$
that, without loss of generality, we are working on the surface $S_n$ for
some $n \geq N_r$. If, on the other hand, $S = S_n$ for some $n \in \mathbb{N}$ then
we are again working on a surface $S_n$. In either case we may assume
that we are working on a surface $S_n$ for some fixed $n \in \mathbb{N}$.

There are two cases to consider. First we shall show that if $Q \neq
\sigma_c(0)$, then $\tau > t$. In the second case we shall show that if $Q = \sigma_c(0),$
but $\delta$ is distinct from $\sigma_c([0, t])$, then $\tau > t$. The conclusion will be
that $\sigma_c([0, t])$ is the unique geodesic arc from $\sigma_c(t)$ to $\alpha_0$ realizing the
distance from $\sigma_c(t)$ to $\alpha_0$.

Case I. Recall that $B_n$ is the set consisting of the boundary geodesics
of the cut-open untwisted flutes, $F_i^*, i = 1, \ldots, n$, in $S_n$. Let $\beta$
de designate the full (bi-infinite) geodesic in $S_n$ that contains $\sigma_c$
so that for $t \geq 0$, $\sigma_c(t) = \beta(t)$. Let $B_n^+ = B_n \cup \{\beta\}$.
Then there is a largest point $\tau_0 \in [0, \tau]$ such that $\delta((0, \tau_0)) \cap B_n^+ = \emptyset$.
Let $\beta_0$ be the bi-infinite geodesic in $B_n^+$ for which there is a $t_0 > 0$
such that $\delta(\tau_0) = \beta_0(t_0)$. (It is possible that
$\tau_0 = \tau$ and that $\beta_0 = \beta$.) The arcs $\beta_0([0, t_0])$ and $\delta((0, \tau_0))$
lie in exactly one cut-open untwisted flute or simply connected hyperbolic strip.
It then follows respectively from either Lemma 3 or Lemma 2 that $\tau_0 > t_0$.

Suppose that $\tau_0 \neq \tau$. Then the piecewise geodesic $\beta_0([0, t_0])$
followed by $\delta([\tau_0, \tau])$ is shorter than $\delta([0, \tau])$, i.e., $t_0 + (\tau - \tau_0) =
(t_0 - \tau_0) + \tau < \tau$.
In this case there must be a smooth geodesic arc even shorter than
this piecewise geodesic joining $\beta_0(0)$ to $\sigma_c(t)$. If $\tau_0 = \tau$ then $\beta_0 = \beta$,
$t = \tau_0$ and we have $t < \tau$ and $\beta([0, t]) = \sigma_c([0, t])$ is shorter than
$\delta([0, \tau])$. In either case, there is a shorter path from $\sigma_c(t)$ to $\alpha_0$ than $\delta$.
We have shown that if $Q \neq \sigma_c(0)$ is a point on $\alpha_0$, then a geodesic arc
$\delta : [0, \tau] \to S$ joining $Q$ to $\sigma_c(t)$ does not realize the distance
d$(\sigma_c(t), \alpha_0)$.

Case II. Suppose that $\delta : [0, \tau] \to S$ joins $Q = \delta(0) = \sigma_c(0)$ to
$A = \sigma_c(t)$, and suppose that $\delta$ is distinct from $\sigma_c([0, t])$.

Since $\delta$ is distinct from the arc $\sigma_c([0, t])$, the angle at which $\delta$ meets
$\alpha_0$ at $Q$ can not be a right angle. It follows that, by moving a small
distance from $Q = \sigma_c(0)$ along $\delta$ to a point $B$, there is a point $E$ on $\alpha_0$
so that arcs $QB, BE$, and $EQ$ form a right triangle, with right angle
at $E$, and hypotenuse $QB$. It then follows that the piecewise geodesic
from $A$ to $B$ along $\delta$, followed by $BE$, is shorter than $\delta$. Therefore $\delta$

does not realize the distance from $A = \sigma_c(t)$ to $\alpha_0$.

It follows that the arc $\sigma_c([0,t])$ must be the unique geodesic arc
realizing the distance from $\sigma_c(t)$ to $\alpha_0$. In particular, as in Case I, it
follows that if $Q \neq \sigma_c(0)$ is a point on $\alpha_0$ and $\delta : [0,\tau] \to S_{\infty}$ joins $Q$
to $\sigma_c(t)$, then $\tau > t$. \qed

The lemma above shows that $\sigma_c$ is a critical ray on all of the surfaces
$S_n$, $n \in \mathbb{N}$, and on $S_{\infty}$.

The last two lemmas of this section will be of use throughout the
rest of the paper. The first is a generalization of Lemma 4. Again, let
$S = S_{\infty}$ or $S = S_n$ for some $n \in \mathbb{N}$.

**Lemma 5.** Let $\delta : [\tau, \tau'] \to S$ be a geodesic arc for which there are not
necessarily distinct points $c$ and $c'$ in $C$ such that $\delta(\tau) \in \sigma_c$ and $\delta(\tau') \in
\sigma_{c'}$. Suppose that $\delta$ is not a subarc of $\sigma_c$ or $\sigma_{c'}$ and let $t$ and $t'$ be reals
so that $\delta(\tau) = \sigma_c(t)$ and $\delta(\tau') = \sigma_{c'}(t')$. Then, $\tau' - \tau > |t' - t| \geq t' - t$.

**Proof.** The lemma holds if $t = t'$ so assume, without loss of generality,
that $t < t'$. Note that $|\delta| \geq d(\delta(\tau), \delta(\tau')) = d(\sigma_c(t), \sigma_{c'}(t'))$, and as $\delta$
is not a subarc of either $\sigma_c$ or $\sigma_{c'}$, the piecewise geodesic formed by
the segments $\sigma_c([0,t])$ followed by $\delta([\tau, \tau'])$, is not smooth. It follows
that there is a shorter, smooth geodesic $\delta'$ joining the points $\sigma_c(0)$ and
$\sigma_{c'}(t')$. Noting as observed above, that

$$\tau' - \tau = |\delta| \geq d(\delta(\tau), \delta(\tau')) = d(\sigma_c(t), \sigma_{c'}(t'))$$

and employing Lemma 4 we now have,

$$t + (\tau' - \tau) = d(\sigma_c(0), \sigma_c(t)) + |\delta|$$

$$\geq d(\sigma_c(0), \sigma_c(t)) + d(\sigma_c(t), \sigma_{c'}(t'))$$

$$> |\delta'|$$

$$\geq d(\sigma_c(0), \sigma_{c'}(t'))$$

$$> d(\sigma_{c'}(t'), \alpha_0)$$

$$= d(\sigma_c(0), \sigma_{c'}(t'))$$

$$= t'.$$

Therefore, $\tau' - \tau > t' - t = |t' - t|$. \qed

Let $S^*$ be either $F^*$ or $E^*$ as defined above. Let $\beta$ be one of $\beta_1$ or
$\beta_2$. 


Lemma 6. Let \( \delta : [\tau_1, \tau_2] \to S^* \) be a non-trivial geodesic arc where \( \delta \) is not a subarc of either \( \beta_1 \) or \( \beta_2 \) and for some some \( t_1, t_2 > 0 \), \( \delta(\tau_1) = \beta_1(t_1), \) and \( \delta(\tau_2) = \beta(t_2) \). Then \( \tau_2 - \tau_1 > |t_2 - t_1| \).

Proof. Though there are direct proofs, for efficiency, we proceed by using the results we have already obtained. Note that if \( S^* \) is a cut-open untwisted flute, we may reglue \( S^* \) along \( \beta_1 \) and \( \beta_2 \) and obtain an untwisted flute on which we may apply Lemma 5 to obtain the result. Suppose \( S^* \) is a simply connected hyperbolic strip. We proceed in a fashion identical to the proof of Lemma 5. We employ Lemma 2. Assume, without loss of generality that \( t_2 > t_1 \), and observe that the piecewise geodesic formed by \( \beta([0, t_1]) \) followed by \( \delta([\tau_1, \tau_2]) \) is not smooth. Arguing exactly as before, we have

\[
\begin{align*}
t_1 + (\tau_2 - \tau_1) &= d(\beta_1(0), \beta_1(t_1)) + (\tau_2 - \tau_1) \\
&\geq d(\alpha_0, \beta_1(t_1)) + d(\delta(\tau_1), \delta(\tau_2)) \\
&> d(\alpha_0, \beta_2(t_2)) = d(\beta_2(0), \beta_2(t_2)) \\
&= t_2.
\end{align*}
\]

It follows \( \tau_2 - \tau_1 > t_2 - t_1 = |t_2 - t_1| \). \( \square \)

5. Additional lemmas from hyperbolic geometry

The lemmas in this section will be crucial in what follows and again involve the application of basic geometry in the hyperbolic plane. In Lemmas 9 and 10 we derive properties of geodesic arcs on cut-open, untwisted flute surfaces.

Lemma 7. Let \( A, B \) and \( E \) be vertices of a hyperbolic triangle where there is a right angle at \( E \). Let \( a, b \) and \( e \) be the lengths of the sides opposite \( A, B \) and \( E \) respectively. Then,

\[
0 < \log \left( \frac{k^2 + 1}{2k} \right) < e - b < \log k,
\]

where \( a = \log k \).

Note that for \( a > 0 \) we have \( \log \cosh(a) = \log \left( \frac{k^2 + 1}{2k} \right) \), where \( a = \log k \). Therefore, the inequality may also be written in the form, \( 0 < \log \cosh(a) < e - b < a \).

Proof. The proof requires little more than a computation. By an appropriate isometry we may take \( E = i, B = ki, k > 1 \) and \( A = s + ti \), where \( s^2 + t^2 = 1 \) and \( s, t \in (0, 1) \). By using the standard formula, \( \rho(z, w) = \log \left[ \frac{|z-w|+|z-w|}{|z-w|-|z-w|} \right] \), for the hyperbolic distance between points \( z \) and \( w \) in \( \mathbb{H}^2 \), an elementary computation shows that
a = \log k\) and that \(f(k, t) := e - b = \log \left[\frac{1+k^2+\sqrt{1+k^4+2k^2(1-2t^2)}}{2k(1+\sqrt{1-t^2})}\right].\) Elementary multivariate calculus and some elementary algebra show that
\[
\frac{\partial f}{\partial t} = \frac{1}{t} \left[\frac{1}{\sqrt{1-t^2}} - \frac{1+k^2}{\sqrt{1+k^4+2k^2(1-2t^2)}}\right] > 0 \text{ for } 0 < t < 1.
\]
Therefore, as a function of \(t\), \(f\) has no critical points and is increasing. It follows that the minimum occurs at \(t = 0\) and is \(\log(\frac{k^2+1}{2k})\). The maximum value occurs at \(t = 1\) and is \(\log k\). The former value is approached in the limit as \(b \to \infty\) and the latter value is approached as \(b \to 0\).

\[\square\]

**Lemma 8.** Let \(\gamma\) be a line in the hyperbolic plane, \(B\) a point not lying on \(\gamma\). Let \(E\) be a point on \(\gamma\) such that segment \(BE\) is perpendicular to \(\gamma\). Suppose that \(A\) and \(C\) are points on \(\gamma\) (not necessarily distinct from each other or from \(E\)). Let \(k > 1\) be such that \(|BE| = \log k\). Then \(|AB| + |BC| > |AC| + 2\log(\frac{k^2+1}{2k})\).

**Proof.** First note that \(|AB| > |AE| + \log(\frac{k^2+1}{2k})\). If \(A\) and \(E\) are distinct then this is a direct consequence of Lemma 7. If \(A\) and \(E\) coincide, then \(|AE| = 0\), and \(|AB| = |EB| = \log k\) so the inequality amounts to the observation that if \(k > 1\) then \(\log k > \log(\frac{k^2+1}{2k})\). Similarly we have the inequality \(|BC| > |CE| + \log(\frac{k^2+1}{2k})\). Next, observe that no matter what configuration the points \(A, E\) and \(C\) have on \(\gamma\), \(|AE| + |EC| \geq |AC|\). Thus \(|AB| + |BC| > |AE| + |CE| + 2\log(\frac{k^2+1}{2k}) \geq |AC| + 2\log(\frac{k^2+1}{2k})\).

\[\square\]

Recall the standard setup from early in Section 4 where \(F\) is an untwisted flute which, when cut along \(\beta^*\), produces \(F^*\). Let \(R^*\) be the quotient, \(F^*/(r^*)\), and \(\pi : F^* \longrightarrow R^*\) be the canonical projection. \(R^*\) may be regarded as a region in the hyperbolic plane with geodesic and ideal boundary. The geodesic boundary consists of the images of \(\beta_1\), \(\beta_2\) and the \(\gamma\)-curves of \(\Gamma\) under \(\pi\). We set \(\beta = \pi(\beta_1) = \pi(\beta_2)\) and, since there is little risk of confusion, we let \(\Gamma\) denote the union of the \(\gamma\)-curves, or their projections, on \(F, F^*\) and \(R^*\).

Let \(\beta'\) be one of \(\beta_1\) or \(\beta_2\) and let \(\delta : [\tau_1, \tau_2] \to F^*\) be a non-trivial geodesic arc, not contained in \(\beta_1\), with \(\delta(\tau_1) \in \beta_1\) and \(\delta(\tau_2) \in \beta'\). Since \([\tau_1, \tau_2]\) is a closed set, there must be a point \(\lambda_0 \in [\tau_1, \tau_2]\) such that \(d(\delta(\lambda_0), (\beta_1 \cup \beta_2)) = d(\delta(\lambda), (\beta_1 \cup \beta_2))\) for all \(\lambda \in [\tau_1, \tau_2]\).

**Lemma 9.** Let \(\delta : [\tau_1, \tau_2] \longrightarrow F, \beta_1\) and \(\beta_2\) be as above. Then there is a point \(\lambda_0 \in [\tau_1, \tau_2]\) such that \(d(\delta(\lambda_0), (\beta_1 \cup \beta_2)) \geq d(\delta(\lambda), (\beta_1 \cup \beta_2))\) for all \(\lambda \in [\tau_1, \tau_2]\), and further \(\delta(\lambda_0)\) lies on \(\Gamma\), the fixed axis of \(r^*\).
Proof. It only remains to show that \( \delta(\lambda_0) \) lies on \( \Gamma \). Let \( B' = \delta(\lambda) \) for some \( \lambda \in (\tau_1, \tau_2) \). Suppose that \( B' \) does not lie on a \( \gamma \)-curve. Let \( B = \pi(B') \) and let \( \delta^* = \pi(\delta) \), which is a piecewise geodesic arc on \( R^* \).

There are two cases to consider. In the first case \( B \) lies on a geodesic segment of \( \delta^* \) joining a point \( A \) on \( \beta \) to a point \( D \) in \( \Gamma \). The point \( D \) lies on a geodesic \( \gamma \subset \Gamma \) which is disjoint from \( \beta \). It follows from elementary hyperbolic geometry that \( B \) is closer to \( \beta \) than \( D \); to wit:

Let \( C \) be the point on \( \beta \) such that segment \( DC \) is perpendicular to \( \beta \). If \( C = A \) then, clearly, \(|AB| < |AD|\) since \( B \) lies between \( A \) and \( D \). If \( C \neq A \) then consider the hyperbolic triangle \( \Delta ADC \). The point \( B \) lies on side \( AD \) and must be closer to the line through segment \( AC \) than the opposite vertex \( D \).

Now we turn to the second case, in which \( B \) lies on a geodesic segment \( \alpha \) of \( \delta^* \) joining two points on \( \Gamma \). The full geodesic containing the arc \( \alpha \) meets two of the \( \gamma \)-curves on the boundary of \( R^* \) and therefore \( \alpha \) must be disjoint from \( \beta \). Let \( A \) and \( C \) be points on \( \alpha \) so that \( B \) lies between \( A \) and \( C \). It is an elementary fact in hyperbolic geometry that one of \( A \) or \( C \) is farther from \( \beta \) than \( B \).

It follows that \( \delta(\lambda_0) \) must lie on \( \Gamma \), the fixed axis of \( r^* \).

Consider again the situation set up in the paragraph proceeding Lemma 9.

**Lemma 10.** Let \( \delta : [\tau_1, \tau_2] \to F^* \) be a geodesic arc joining \( \beta_1 \) and \( \beta' \) as above. Let reals \( t_1 \) and \( t_2 \) be chosen so that \( \delta(t_1) = \beta_1(t_1) \) and \( \delta(t_2) = \beta'(t_2) \). The arc \( \delta \) crosses \( \Gamma \) at a point \( \delta(\lambda_0) \), for some \( \lambda_0 \in [\tau_1, \tau_2] \). Let \( k > 1 \) be chosen so that \( \log k = d(\delta(\lambda_0), (\beta_1 \cup \beta_2)) \). Then

\[
\tau_2 - \tau_1 > |t_2 - t_1| + 2 \log \left[ \frac{k^2 + 1}{2k} \right].
\]

Proof. Let \( R^* \) be the planar quotient surface \( F^*/\langle r^* \rangle \). We shall employ Lemma 8 where \( \gamma \) in the lemma, corresponds to the boundary geodesic \( \beta = \pi(\beta_1) = \pi(\beta') \), \( B = \pi(\delta) \) is the projection into \( R^* \) of the point at which \( \delta \) crosses \( \Gamma \), \( A = \pi(\beta_1(t_1)) = \beta(t_1) \), and \( C = \pi(\beta'(t_2)) = \beta(t_2) \).

The length of \( \delta \) is equal to the length of the piecewise geodesic \( \delta^* := \pi(\delta) \). Suppose \( \delta \) crosses \( \Gamma \) exactly once. Then

\[
\tau_2 - \tau_1 = |\delta| = |\delta^*| = |AB| + |BC|.
\]

On the other hand, if \( \delta \) crosses \( \Gamma \) more than once then

\[
\tau_2 - \tau_1 = |\delta| = |\delta^*| \geq |AB| + |BC|.
\]

In either case, \( \tau_2 - \tau_1 \geq |AB| + |BC| \).
Let $E$ be the point on $\beta$ such that segment $BE$ is perpendicular to $\beta$ and note that $|BE| = d(\delta^*(\lambda_0), \beta) = \log k$. The hypotheses for Lemma 8 are satisfied and therefore, $\tau_2 - \tau_1 \geq |AB| + |BC| > |AC| + 2\log \left(\frac{k^{\frac{3}{2}} + 1}{2k}\right) = |t_2 - t_1| + 2\log \left(\frac{k^2 + 1}{2k}\right)$. 

\[ \Box \]

We will need yet another result from hyperbolic geometry.

**Lemma 11.** Let $l_1$ and $l_2$ be asymptotic geodesics in $\mathbb{H}^2$ which both limit at a point $Q \in \partial \mathbb{H}^2$. Suppose that $B \in l_1$ and $A \in l_2$ are points so that the arc $AB$ is orthogonal to $l_1$. Further suppose that $D$ is a point on $l_1$ between $B$ and $Q$ and $C$ is a point on $l_2$ between $A$ and $Q$ so that the arc $CD$ is also orthogonal to $l_1$. Then $|AC| + |CD| < |AB| + |BD|$.

Observe that this lemma is about a special sort of quadrilateral, $BACD$, where the sides $BD$ and $AC$ lie on parallel, asymptotic geodesics.

**Proof.** Let $m$ denote the full geodesic in $\mathbb{H}^2$ containing the arc $AB$. Without loss of generality we may take $m$ the semicircle passing through $-1$, $i$ and $+1$; $B = i$; $A$ is the point $x_0 + iy_0$ for some $x_0, 0 < x_0 < 1$; and $l_2$ is the line $Rez = x_0$. Let $C$ be the point $x_0 + iy_1$, $y_1 > y_0$. Then $D$ is the point $ri$ where $x_0^2 + y_1^2 = r^2$. Using the standard formula, $\rho(z, w) = \log \left(\frac{|z - w|}{|z - w| - |z - w|}\right)$, for the distance between two points, $z, w$, in the hyperbolic plane, an entirely elementary calculation shows that: $|BD| = \log r$; $|AB| = \log \left(\frac{1 + x_0}{y_0}\right)$; $|AC| = \log \left(\frac{2}{y_0}\right)$; and $|CD| = \log \left(\frac{2 + x_0}{y_0}\right)$. Therefore,

$$|AB| + |BD| - (|AC| + |CD|) = \log(1 + x_0) + \log \left(\frac{r}{r + x_0}\right).$$

For fixed $x_0$, the right hand side of the equation is a strictly increasing function of $r$. Moreover, when $r = 1$ the right hand side above has value 0. Therefore, $|AC| + |CD| < |AB| + |BD|$.

\[ \Box \]

6. Specifying the boundary of a Dirichlet polygon

6.1. **Definitions and a theorem.** Let $G$ be a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ and representing the surface $S = \mathbb{H}^2 / G$. Given $\tilde{p} \in \mathbb{H}^2$, one defines the Dirichlet polygon $D(\tilde{p}, G) = D$ of $G$ centered at $\tilde{p}$ to be the set of all points $q \in \mathbb{H}^2$ so that $d(\tilde{p}, q) \leq d(g(\tilde{p}), q)$ for all $g \in G \setminus \{id\}$.

Let $p$ be the projection of $\tilde{p}$ to $S$ and let $\sigma$ be a critical ray on $S$ with $\sigma(0) = p$. Then $\sigma$ lifts to a geodesic ray $\bar{\sigma}$ beginning at $\tilde{p}$, which is entirely contained in $D$. Recall that the closure of a set $X$ is written
$X^c$ and $\hat{C}$ denotes the Riemann sphere. Let $\partial_{\infty} D$ be that part of the boundary of $D^c \subset \hat{C}$ lying at infinity; that is, in the extended real line $\hat{\mathbb{R}}$. Then we have $\lim_{t \to \infty} \tilde{\sigma}(t) \in \partial_{\infty} D$. Conversely, if $q$ is a point in $\partial_{\infty} D$, then the ray $\tilde{\sigma}$ beginning at $\tilde{p}$ and limiting at $q$ projects to a critical ray on $S$. These observations follow easily from the definitions of a critical ray and a Dirichlet polygon. Clearly, there is an intimate relationship between critical rays and the boundary points of Dirichlet polygons.

We show how quilted surfaces can be used to prove the following

**Theorem 3.** Given any compact set $C^* \subset \mathbb{R}$, which is not an interval, there exists a Fuchsian group $\Gamma_{\infty}$, representing a quilted surface $S_{\infty}$, and a point $\tilde{p} \in \mathbb{H}^2$ so that $\partial_{\infty} D(\tilde{p}, \Gamma_{\infty})$ consists of the union of $C^*$, a countable set of isolated parabolic fixed points of $\Gamma_{\infty}$ and an interval.

The interval in the theorem comes from the finite end of the surface $S_{\infty}$ associated with the annulus $A^-$. If we were to cut out $A^-$ and glue a twice punctured disc into its place, the boundary of the Dirichlet polygon would not contain the interval. We shall do exactly this at the end of the section, to prove Theorem 1.

The idea of the proof of Theorem 3 is to begin with an annulus $A$ with a dividing geodesic $\alpha_0$ of length $a \leq 1$ and to choose a closed set $C$ on $\alpha_0$ that corresponds to the set $C^*$. We then choose appropriate untwisted flute surfaces $F_i$ to define the surface $S_{\infty} = S(\alpha_0, p, C, \{F_i\})$ so that for a fixed point $p \in \alpha_0$ and for all, but possibly one value of, $c \in C$ there exists a unique critical ray beginning at $p$ which is asymptotic to $\sigma_c$ and, furthermore, these are all of the infinite critical rays on $S_{\infty}$. This surface will be uniformized by a Fuchsian group for which the theorem holds.

6.2. **Working on quilted surfaces.** We begin by constructing quilted surfaces like those described above. Suppose $|\alpha_0| = a$ and $C \subset \alpha_0$ is a closed set. Let $\log k = R$ and define $\kappa(R) = 2 \log \left( \frac{k^2 + 1}{2k} \right) = 2 \log \cosh(R)$. Let $k > 1$ be a value for which $\kappa(R) = a + 1$. For each $i \in \mathbb{N}$, let $F_i$ be an untwisted flute on which all of the finite ends, except the end associated to the geodesic $\alpha_{i,0}$, are punctures. Further suppose that for $j > 0$ the dividing curves $\alpha_{i,j}$ have length $R$. It follows easily from this choice, that each of the $F_i$ has an infinite end of the first kind. We shall maintain this stipulation on the structure of the $F_i$ for the remainder of the section. Also, $S_{\infty} = S(\alpha_0, p, C, \{F_i\})$ shall be a surface constructed with these assumptions. Observe that with our choice of flutes $F_i$, the surface $S_{\infty}$ is the same, independent of the choice of the point $p$. 
Fix $p \in \alpha_0$. Given $c \in C$ define the piecewise geodesic rays $\overrightarrow{c}$ and $\overleftarrow{c}$. The ray $\overrightarrow{c}$ is formed by the shorter arc of $\alpha_0$ from $p$ to $c$, followed by $\sigma_c$. $\overleftarrow{c}$ is formed in a similar fashion using the longer arc of $\alpha_0$ from $p$ to $c$. In the special case where the point $c$ is half-way around $\alpha_0$ from $p$, the two arcs of $\alpha_0$ are of equal length and we make an arbitrary choice of which is $\overrightarrow{c}$ and which is $\overleftarrow{c}$. Let $\delta_c^s$ and $\delta_c^l$ be the geodesic rays beginning at $p$ which are asymptotically homotopic to $\overrightarrow{c}$ and $\overleftarrow{c}$, respectively. By this we mean that, for example, there are lifts of $\overrightarrow{c}$ and $\overleftarrow{c}$ to $H^2$ which have the same initial point and the same endpoint at infinity. Observe that both $\delta_c^s$ and $\delta_c^l$ are asymptotic to the critical ray $\sigma_c$.

**Proposition 4.** For each $c \in C$ one of $\delta_c^s$ or $\delta_c^l$ is a critical ray. If $\delta$ is an infinite critical ray on $S_\infty$ beginning at $p$, then for some $c \in C$, $\delta = \delta_c^s$ or $\delta = \delta_c^l$. Moreover, there can be at most one value of $c$ for which both rays are critical rays.

As a consequence of the proof of Theorem 3 we will be able to conclude that $\delta_c^s$ is always critical.

We begin with some notation and prove a lemma. Let $F^*$ be a cut-open, untwisted flute bounded by geodesics $\beta$ and $\beta'$. Suppose $\epsilon$ and $\delta$ are geodesics with their initial points on $\beta$ and their terminal points on $\beta'$. We shall say that $\epsilon$ is $\beta$-homotopic to $\delta$ if $\epsilon$ is homotopic to $\delta$ by a homotopy that does not move the initial and terminal points of $\epsilon$ off the geodesics $\beta$ and $\beta'$. Observe that if $\epsilon$ is $\beta$-homotopic to a dividing curve $\alpha_j$ on $F^*$ then it crosses a single $\gamma$-curve on $F^*$; that is, it crosses the unique $\gamma$-curve which is orthogonal to $\alpha_j$.

On the same surface $F^*$, recall that $\beta$ and $\beta'$ are parameterized so that $\beta(0), \beta'(0) \in \alpha_0$ and both $\beta$ and $\beta'$ go out the infinite end as $t \to \infty$. Also, the canonical involution $\tau^*$ fixes the $\gamma$-curves, leaves invariant the dividing curves and interchanges $\beta$ and $\beta'$. Let $\overline{c}$ denote one of the geodesics $\overrightarrow{c}$ or $\overleftarrow{c}$.

**Lemma 12.** Let $\alpha : [0, \tau^*] \to S_\infty$ be a geodesic with $\alpha(0) = p$ and $\alpha(\tau^*) = \overline{c}(t^*)$ for some $t^* \in [0, \infty)$. Suppose there is an arc of $\alpha$ that is $\beta$-homotopic to a dividing curve $\alpha_i \neq \alpha_0$ on the cut-open, untwisted flute $F^* \subset S_\infty$. Then $\tau^* > t^* + 1$.

**Proof.** By hypothesis there is a cut-open flute $F^*$ bounded by geodesics $\beta$ and $\beta'$ and a connected segment of $\alpha \cap F^*$ that is $\beta$-homotopic to $\alpha_i \neq \alpha_0$. Let $0 \leq \tau < \tau'$, and $t, t' \geq 0$ and $\beta$, $\beta'$ be such that $\alpha([\tau, \tau'])$ is $\beta$-homotopic to $\alpha_i$, $\alpha(\tau) = \beta(t)$, and $\alpha(\tau') = \beta'(t')$. Since $|\alpha_i| = R$,
by Lemma \[ \tau' - \tau > |t' - t| + \kappa(R) \geq t' - t + a + 1 \]. It also follows from Lemma 4 that \( \tau > t \).

Let \( t_c \) be the length of the subarc of \( \overline{\delta}_c \) that runs along \( \alpha_o \) and joins \( p \) to \( c \). The length of the scaffolding curve lying along \( \delta_c \) from the point \( c \) on \( \alpha_0 \) to \( \alpha(\tau^*) \) is then \( t^* - t_c \). Note also that \( a - t_c > 0 \). Suppose \( \tau^* \neq \tau' \). Then by Lemma 5, \( \tau^* - \tau' > |t^* - t_c - t'| > t^* - t_c - t' \). Putting this all together we have,

\[
\tau^* = (\tau^* - \tau') + (\tau' - \tau) + \tau
\]
\[
> (t^* - t_c - t') + (t' - t + a + 1) + t
\]
\[
= t^* + (a - t_c) + 1 > t^* + 1.
\]

If \( \tau^* = \tau' \) we also have \( t^* - t_c = t' \). Then, similar to the above, we have \( \tau^* = (\tau^* - \tau) + \tau > (t^* - t_c - t + a + 1) + t = t^* + (a - t_c) + 1 > t^* + 1 \). Therefore, in either case, \( \tau^* > t^* + 1 \). \( \square \)

Let \( \delta_c \) denote one of the geodesics \( \delta_c^* \) or \( \delta_c^I \) and denote the other one by \( \delta_c' \).

**Proof of Proposition 4.** To begin we prove the first statement of the proposition. Let \( \alpha : [0, u^*] \to S_\infty \) be a geodesic with \( \alpha(0) = p \) and \( \alpha(u^*) = \delta_c(s^*) \) for some \( u^*, s^* \in [0, \infty) \). We will prove that one of \( \delta_c \) or \( \delta_c' \) is critical by showing that for any such geodesic \( \alpha \), either \( s^* \leq u^* \) or \( \delta_c \) is critical. We argue by contradiction. Suppose there exists a geodesic \( \alpha \), as above, with \( u^* < s^* \). We may further assume that \( \alpha \) realizes the distance between \( p \) and \( \alpha(u^*) \) and is therefore a minimal length geodesic between its endpoints. In particular, \( \alpha \) is simple. As described earlier, there is no loss of generality in assuming that \( S_\infty \) is one of the finitely glued surfaces \( S_n \).

The first case to consider is where \( \alpha \) contains a subarc which is \( \beta \)-homotopic to a dividing curve \( \alpha_{i,j} \) with \( j \neq 0 \) on a cut-open subflute \( F_i^* \). Choose a minimal length geodesic \( \mu \) from \( \delta_c(s^*) \) to \( \overline{\delta}_c \), where \( \mu(0) = \delta_c(s^*) \) and \( \mu(\tau^*) = \overline{\delta}_c(t_s) \) for values \( t^*, \tau^* \in [0, \infty) \). Define the piecewise geodesic arcs \( \alpha^* \) and \( \delta_c^* \) by adjoining the arc \( \mu \) to \( \alpha \) and to the arc of \( \delta_c \), between \( p \) and \( \delta_c(s^*) \), respectively. In particular,

\[
(1) \quad \alpha^*(u) = \begin{cases} 
\alpha(u) & 0 \leq u \leq u^* \\
\mu(u - u^*) & u^* \leq u \leq \tau^* + u^*
\end{cases}
\]

and

\[
\delta_c^*(s) = \begin{cases} 
\delta_c(s) & 0 \leq s \leq s^* \\
\mu(s - s^*) & s^* \leq s \leq \tau^* + s^*
\end{cases}
\]

Together, \( \delta^* \) and \( \overline{\delta}_c \) bound a hyperbolic quadrilateral that satisfies the hypothesis of Lemma 11. It follows that \( \tau^* + s^* < t^* \).
Let $\rho$ be the geodesic homotopic to $\alpha^*$ relative to endpoints. $\rho$ crosses the same $\gamma$-curves as $\alpha$ on subflutes. Therefore, $\rho$ contains a subarc which is $\beta$-homotopic to $\alpha_{i,j}$ on $F_i^*$. Then employing the above inequality and Lemma 12, we have

$$\tau^* + u^* = |\alpha^*| > |\rho| > t^* + 1 > \tau^* + s^* + 1.$$

Thus, $u^* > s^*$, which gives a contradiction.

Now consider the case where no subarc of $\alpha$ is $\beta$-homotopic to an arc $\alpha_{i,j}$ with $j \neq 0$ on a cut-open subflute $F_i^*$. Then, in order, $\alpha$ crosses the curves $\beta_1, \beta_2, \ldots, \beta_m \in B_n^+$. Recall that $B_n^+ = B_n \cup \{\beta\}$ where $\beta$ is the full geodesic containing $\sigma_c$.

First we suppose that two consecutive curves $\beta_j$ and $\beta_{j+1}$ are equal. Then there is a cut-open, untwisted subflute $F_j^*$, one of whose boundary geodesics is $\beta_j$, so that $\alpha \cap F_j^*$ contains an arc $\epsilon$ both of whose endpoints lie on $\gamma$-curves of $F_j^*$.

Apply the canonical involution $r^*$ to get the geodesic arc $r^*(\epsilon)$. Since the $\gamma$-curves of $F_j^*$ are fixed by $r^*$, $\epsilon$ and $r^*(\epsilon)$ share the same endpoints. Thus we may replace the geodesic segment $\epsilon$ of $\alpha$ by its reflection $r^*(\epsilon)$. This results in a new piecewise geodesic joining $p$ to $\alpha(u^*)$. Taking the geodesic freely homotopic to this curve relative to endpoints gives a geodesic $\alpha'$ which is shorter than $\alpha$. This contradicts the assumption that the length of $\alpha$ is the distance between its endpoints. We may then suppose that all of the curves $\beta_j$ are distinct. Possibly abusing notation, let $F_j^*$ denote the cut-open flute surface bounded by $\beta_j$ and $\beta_{j+1}$.

We would like to show that $\rho = \alpha \cap F_j^*$ is $\beta$-homotopic to $\alpha_{j,0}$ for each cut-open subflute $F_j^*$, $j = 1, \ldots, m - 1$. If not, then by earlier considerations, it cannot be $\beta$-homotopic to $\alpha_{j,k}$ for any $k \in \mathbb{N}$. Consequently, $\rho$ must contain at least 3 intersections with $\gamma$-curves on some $F_m^*$ and there will be an arc $\epsilon$ of $\rho$ whose endpoints lie on distinct $\gamma$-curves of $F_m^*$. As above, replace the arc $\epsilon$ of $\alpha$ by its reflection $r^*(\epsilon)$. Taking the geodesic homotopic to this piecewise geodesic path results in a shorter geodesic between the endpoints of $\alpha$, which is a contradiction. We may now suppose that for each of the cut open flutes $F_j^*$, $\alpha \cap F_j^*$ is $\beta$-homotopic to $\alpha_{j,0}$.

If none of the $\beta_j$ contains the ray $\sigma_c$ then $\alpha$ will cross exactly the same curves in $B_n$ as the geodesic arc of $\delta_c$ from $p$ to $\delta_c(s^*) = \alpha(u^*)$. For this to happen $\alpha$ must be homotopic to the arc of $\delta_c$ and consequently it will actually coincide with that arc. Therefore $s^* = u^*$.

If one of the geodesics $\beta_j$ does contain the ray $\sigma_c$ then it must be the last one, $\beta_m$. Now consider the ray $\delta'_c$. Either it is critical or there
is a geodesic $\alpha' : [0, u'] \to S_\infty$ with $\alpha'(0) = p$, $\alpha(u') = \delta'_c(s')$ for some $u', s' \in [0, \infty)$ and $u' < s'$.

Adjust the choice of $n > N_r$ so that $B^c_r(r) \supset \alpha'$ as well. The first part of the proof will be completed by showing that both $\alpha$ and $\alpha'$ cannot realize the distance between their endpoints. As a consequence of the preceding arguments, if $\alpha'$ realizes the distance between its endpoints then, as with $\alpha$, $\alpha'$ will cross each geodesic in $B^+_c$ at most once and the last geodesic in $B^+_c$ that $\alpha'$ crosses must contain the ray $\sigma_c$. Since $\delta_c$ and $\delta'_c$ limit at $\sigma_c$ from opposite sides, the geodesics $\alpha$ and $\alpha'$ must intersect; that is, $\alpha(d) = \alpha'(e)$ for some $d < u'$ and $e < u'$. Without loss of generality suppose that $d \leq e$. Then consider the piecewise geodesic arc which is $\alpha$ from $p$ to $\alpha(d)$ followed by $\alpha'$ from $\alpha(d) = \alpha'(e)$ to $\alpha'(u')$. The geodesic freely homotopic to this arc relative to endpoints, goes from $p$ to $\alpha'(u')$ but is shorter than $\alpha'$, showing that $\alpha'$ does not realize the distance between its endpoints. That completes the proof that at least one of $\delta'_c$ or $\delta'_d$ is critical.

The second point to the proposition is that the only possible infinite critical rays beginning at $p$ are the rays $\delta'_c$ or $\delta'_d$ for $c \in C$. Suppose, to the contrary, that there is an infinite critical ray $\alpha$ distinct from the above ones. All critical rays beginning at $p$ intersect only at $p$. Consequently, there would be values $c_1$ and $c_2$ in $C$ bounding an interval on $\alpha_0$ so the $\alpha$ would lie in the region on $S_\infty$ bounded by $\delta_{c_1}$ and $\delta_{c_2}$.

Then there is a cut-open flute subsurface $F^*$ bounded by $\beta_1 \supset \sigma_{c_1}$ and $\beta_2 \supset \sigma_{c_2}$. Let $\alpha_i, i \geq 0$ denote the dividing curves on $F^*$ and let $\gamma_i, i \geq 0$ be the $\gamma$-curves on $F^*$ where $\gamma_i \cap \alpha_j \neq \emptyset$ if and only if $i = j$. Throughout the paper we have assumed that on all flutes $F_i$, the infinite end is of the first kind. It follows from [5], that a geodesic ray going out the infinite end of $F$ must either cross the $\gamma$-curves or be asymptotic to the canonical Dirichlet ray $\beta^*$. Therefore, in order for $\alpha$ to be distinct from $\delta_{c_1}$ and $\delta_{c_2}$, it must intersect at least one of the $\gamma$-curves other than $\gamma_0$. If $\alpha$ were to intersect two of the $\gamma$-curves then, as before, we could use the involution to produce a shorter geodesic between $p$ and a point on $\alpha$. Since $\alpha$ is critical, this is not possible. Thus $\alpha$ would have to meet exactly one curve $\gamma_k$ for $k > 0$ and then, beyond that, would be asymptotic to either $\sigma_{c_1}$ or $\sigma_{c_2}$. Without loss of generality we take it to be the former and write $c_1 = c$. Then $\alpha$ would also be eventually asymptotic to $\delta_c$.

Since $\alpha$ is asymptotic to $\delta_c$, which is itself asymptotic to $\overline{\delta}_c$, we can find values $t^*, s^*, u^* \in [0, \infty)$ so that the following inequalities are satisfied by the distances to $\delta_c$: $d(\alpha(u^*), \overline{\delta}_c(t^*)) < 1/3$ and $d(\delta_c(s^*), \overline{\delta}_c(t^*)) <
Let $\mu$ be the geodesic from $\alpha(u^*) = \mu(0)$ to $\delta_c(t^*) = \mu(\tau^*)$ and let $\nu$ be the geodesic from $\delta_c(s^*) = \nu(0)$ to $\delta_c(t^*) = \nu(\zeta^*)$.

Define $\alpha^*$ as in Equation (1). Also, define the piecewise geodesic

$$
\delta_c^*(s) = \begin{cases} 
\delta_c(s) & 0 \leq s \leq s^* \\
\nu(s - s^*) & s^* \leq s \leq s^* + \zeta^* \\
\mu^{-1}(s - s^* - \zeta^*) & s^* + \zeta^* \leq s \leq s^* + \zeta^* + \tau^*, 
\end{cases}
$$

Where $\mu^{-1}$ is the geodesic $\mu$ traversed in the opposite direction.

Let $\rho$ and $\delta^*$ denote the geodesics homotopic to $\alpha^*$ and $\delta_c^*$, respectively. Note that $\delta^*$ is a geodesic from $\alpha(0)$ to $\alpha(u^*)$. As a consequence of Lemma 12 applied to $\rho$, $u^* + \tau^* > |\rho| > t^* + 1$. By the definition of $\delta_c^*$, and considering its restriction to the interval $[0, s^* + \zeta^*]$, we have $s^* < t^* + 1/3$. Then

$$
uu^* + \tau^* > |\rho| > t^* + 1 > s^* + \zeta^* + \tau^* > |\delta^*|$$

which shows that $\alpha$ cannot be critical. That completes the proof of the second statement of the proposition.

Now we need to see that there is at most one $c \in C$ for which $\delta_c$ and $\delta'_c$ are both critical. If not, and there were a second $c' \in C$ so that $\delta'_{c'}$ and $\delta'_{c}$ are critical, then two of the four critical rays would have to intersect at a point other than $p$. But since critical ray cannot intersect, this is impossible.

Observe that by considering the ideal triangle with sides $\alpha_0$, $\delta_c$ and $\delta'_c$, one can show that both $\delta_c$ and $\delta'_c$ are critical if the distance from $c$ to $p$ along $\alpha_0$ is $a/2$.

We shall also need the following lemma in our proof of Theorem 3.

**Lemma 13.** Let $E$ be a finite end of the first kind on a surface $S$ and let $p$ be a point on $S$. Then there exist only finitely many critical rays beginning at $p$ going out the end $E$.

**Proof.** Choose a number $M > 0$ so that in the complement of the ball $B(p, M)$ there is a component $V$ which is a punctured disc containing the end $E$. Let $m$ denote the length of the boundary of $B(p, M)$.

Suppose $\alpha$ is a geodesic ray going out the end $E$ that intersects a component $U \not\subset V$ in the complement of $B(p, M + m)$. Then there is a subarc of $\alpha$ which intersects $U$ and has its endpoints in $\partial B(p, M)$. This arc of $\alpha$ has length greater than $m$. Therefore, replacing this arc by a curve in $\partial B(p, M)$ joining its endpoints, results in a shorter curve from $p$ to any point of $\alpha$ lying beyond the arc. This shows that $\alpha$ cannot be a critical ray.
Thus each critical ray from $p$ out $E$ must lie in $V \cup B(p, M + m)$. The end $E$ is of the first kind. Therefore, there cannot exist two geodesic rays beginning at $p$, that go out $E$ and bound a simply connected region on $S$. Since $B(p, M + m)$ has finitely many complementary components, any set of simple disjoint geodesic rays beginning at $p$ and going out $E$ must be finite. In particular, there can only be finitely many critical rays. \hfill $\square$

6.3. The Dirichlet polygon. Let $A$ be the hyperbolic cylinder with oriented dividing geodesic $\alpha_0$ of length $a \leq 1$. $A$ can be uniformized by a Fuchsian group $\Gamma_0$, generated by the transformation $g(z) = e^a z$. The covering projection $\pi : \mathbb{H} \to A$ takes the imaginary axis $I$, oriented from $0$ to $\infty$, to the oriented geodesic $\alpha_0$. With this setup, the left half-plane $H^-$ covers the subannulus $A^-$ and the right half-plane covers the subannulus $A^+$. We fix the point $\bar{p} = i e^{\frac{\pi}{2}} \in I$ and let $p = \pi(\bar{p}) \in \alpha_0$.

Given a compact set $C^* \subset \mathbb{R}$, we shall define a projection of $C^*$ to a closed set $C$ on $\alpha_0$, which shall be used to construct a quilted surface. There is a Möbius transformation $\varphi(z) = Az + B$, $A, B \in \mathbb{R}, A > 0$, taking $C^*$ into the interval $[1, \alpha]$ so that $1, \alpha \in \varphi(C^*)$.

Given $x \neq 0$, let $\psi(x)$ denote the hyperbolic geodesic in $\mathbb{H}^2$ with endpoints $x$ and $-x$. Define the map $\Lambda : C^* \to I$, that takes $c^* \in C^*$ to the point $I \cap \psi(\varphi(c^*))$. Define $\bar{C} = \Lambda(C^*)$. Then $\pi \circ \Lambda : C^* \to \alpha_0$ defines a map which is one-to-one, except for identifying the endpoints of $C^*$. Define $C = \pi \circ \Lambda(C^*)$.

As in Section 3.1 $C$ defines a sequence of oriented intervals $\{I_i\}$ on $\alpha_0$. Choose flute surfaces $F_i$ so that for $j > 0, |\alpha_{i,j}| = R$, the value defined at the beginning of Section 6.2 and so that all finite ends, except for $A^-$, are punctures. Then the quilted surface $S_{\infty} = S(\alpha_0, p, C, \{F_i\})$ satisfies the hypotheses of the previous section and therefore, Proposition 4 holds.

Let $\{S_n\}$ be the collection of surfaces converging to $S_{\infty}$ and let $\Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_{\infty}$ be the associated sequence of Fuchsian groups where $\mathbb{H}^2/\Gamma_n = S_n$ as in Lemma 1. It follows from Lemma 1 and Proposition 3 that for $0 \leq k \leq \infty$, the covering maps $\pi_k : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma_k = S_k$ maps $I$ to $\alpha_0$, $\bar{C}$ to $C$ and $\bar{p} = i e^{\frac{\pi}{2}}$ to $p$.

Let $\Gamma_{\infty}$ be the Fuchsian group $\varphi^{-1}(\Gamma_\infty)\varphi$ and let $\bar{p}^* = \varphi^{-1}(\bar{p})$. Recall that $D(\bar{p}, \Gamma_{\infty})$ is the Dirichlet polygon of $\Gamma_{\infty}$ centered at $\bar{p}$ and $\partial_{\infty} D(\bar{p}, \Gamma_{\infty})$ is its boundary at infinity.

Recall from Proposition 4 that one of $\delta_c^e$ or $\delta_c^l$ is critical. Let $\delta_c$ be one of these rays that is critical.

Theorem 3 is an immediate consequence of the following theorem.
Theorem 4. $\partial_\infty D(\tilde{p}, \Gamma_\infty^*)$ consists of the union of the set $C^*$, the interval $\varphi^{-1}([-e^a, -1])$ and a countable set of isolated parabolic fixed points of $\Gamma_\infty^*$.

Proof. Since $\varphi^{-1}(\partial_\infty D(\tilde{p}, \Gamma_\infty)) = \partial_\infty D(\tilde{p}, \Gamma_\infty^*)$, it suffices to prove that $\partial_\infty D(\tilde{p}, \Gamma_\infty)$ consists of the union of the set $\varphi(C^*)$, the interval $[-e^a, -1]$ and a countable set of isolated parabolic fixed points of $\Gamma_\infty$. So without loss of generality we suppose $\varphi(z) = z$.

Since $\tilde{p}$ is chosen to be the hyperbolic midpoint of the arc of $I$ with endpoints $i$ and $ie^a$, the Dirichlet polygon $D(\tilde{p}, \Gamma_0)$ is bounded by the two geodesics $\psi(1)$ and $\psi(e^a)$. Thus $\partial_\infty D(\tilde{p}, \Gamma_0) = [-e^a, -1] \cup [1, e^a]$. It follows from Lemma 1 that for each of the subgroups $\Gamma_k$, $k \in \mathbb{N} \cup \{\infty\}$ the left half-plane is precisely invariant under the subgroup $\Gamma_0 \subset \Gamma_k$; that is, $g(H^-) = H^-$, for $g \in \Gamma_0$ and $g(H^-) \cap H^- = \emptyset$ for $g \in \Gamma_k \setminus \Gamma_0$.

Thus the boundary at infinity of $D(\tilde{p}, \Gamma_k) \cap H^-$ is $[-e^a, -1]$, for all $k \in \mathbb{N} \cup \{\infty\}$.

As observed earlier, there is an intimate relationship between points on the boundary at infinity of the Dirichlet polygon at $\tilde{p}$ for $\Gamma_\infty$ and the critical rays on $S_\infty$ beginning at $p$. In particular, $q \in \partial_\infty D(\tilde{p}, \Gamma_\infty)$ if and only if there is a critical ray $\alpha$ on $S_\infty$ beginning at $p$ and a lift $\tilde{\alpha}$ to $\mathbb{H}^2$ beginning at $\tilde{p}$ so that $\lim_{t \to \infty} \tilde{\alpha}(t) = q$. In order to simplify notation we shall refer to $q$ as the endpoint of the ray $\tilde{\alpha}$.

For example, the points of $[-e^a, -1] \subset \partial_\infty D(\tilde{p}, \Gamma_\infty)$ are the endpoints of lifts of critical rays beginning at $p$ that go out the finite end on the subsurface $\Lambda^-$. For each $c \in C$, $\delta_c$ lifts to a geodesic ray $\tilde{\delta}_c$, beginning at $\tilde{p}$. Since $\delta_c$ is a critical ray, the endpoint $q$ of $\tilde{\delta}_c$ lies in the boundary of $D(\tilde{p}, \Gamma_\infty)$. Let $Q$ denote the set of all endpoints of the lifts of rays $\tilde{\delta}_c$ for $c \in C$. We show that $Q = C^*$.

Observe that the region $\Omega \subset S_\infty$ bounded by the geodesic rays $\delta_c$ and $\tilde{\delta}_c$ is simply connected. Chose the lift $\tilde{\Omega}$ of $\Omega$ to $\mathbb{H}^2$ so that $\delta_c$ lifts to $\tilde{\delta}_c$. Then the boundary of $\tilde{\Omega}$, $\partial \tilde{\Omega}$, contains a lift of the arc of $\alpha_0 \subset \tilde{\delta}_c$ passing through $\tilde{p}$ whose length is less than or equal to $\frac{\pi}{2}$. This lift must then be an arc of $I$ lying between $i$ and $ie^a$. Similarly, $\sigma_c \subset \partial \tilde{\Omega}$ lifts to a geodesic ray $\tilde{\sigma}_c \subset \partial \tilde{\Omega}$, which is orthogonal to the lift $I$ of $\alpha_0$. The geodesic rays $\tilde{\delta}_c$ and $\tilde{\sigma}_c$ are asymptotic and therefore share the same endpoint $q$. It follows that $\tilde{\sigma}_c(0) = \tilde{c} \in C$, and then $q = \Lambda^{-1}(\tilde{c}) \in C^*$.

Conversely, it follows by the same considerations that, if $c^* \in C^*$ and $c = \pi \circ \Lambda(c^*)$, then the lift $\tilde{\delta}_c$ of $\delta_c$ beginning at $\tilde{p}$ has endpoint $c^*$. Thus $C^* = Q$.

We next show that if $q \in \partial_\infty D(\tilde{p}, \Gamma_\infty)$ is not in $[-e^a, -1]$ or $C^*$ then $q$ is an isolated parabolic fixed point of $\Gamma_\infty$. Suppose $\tilde{\alpha} : [0, \infty) \to$
\[ H^2 \text{ is a geodesic ray with initial point } \tilde{p} \text{ and endpoint } q. \text{ Then } \tilde{\alpha} \text{ projects to a critical ray } \alpha \text{ on } S_\infty. \text{ If } \alpha \text{ goes out the infinite end of } S_\infty, \text{ then by Proposition } \frac{4}{4} \alpha = \delta_c \text{ for some } c \in C \text{ and then, by the earlier arguments, } q \in C^*. \text{ Furthermore, if } \alpha \text{ goes out the end of } S_\infty \text{ corresponding to the annular region } A^-, \text{ then we have seen that } q \in [-e^a, -1]. \text{ The only remaining possibility is that } \alpha \text{ goes out a finite end } E \text{ of } S_\infty, \text{ corresponding to a puncture. In that case } q \text{ must be a parabolic fixed point of } \Gamma_\infty. \]

It is well known that if \( q \) is a parabolic fixed point of \( \Gamma_\infty \) in \( \mathbb{R} \), then there is a horocycle (open disc) \( N \) in \( H^2 \), tangent to \( \mathbb{R} \) at \( q \) so that any geodesic ray in \( H^2 \) that intersects \( N \) projects to a self-intersecting geodesic on \( S_\infty \). Since critical rays are simple, no lift of a critical ray on \( S_\infty \) beginning at \( p \) to one beginning at \( \tilde{p} \) can intersect \( N \). Consequently, there is a neighborhood of \( q \) in \( \mathbb{R} \) that does not contain other boundary points of \( D(\tilde{p}, \Gamma_\infty) \).

Finally, since \( \Gamma_\infty \) is countable, there can be only countably many parabolic fixed points. That completes the proof. \( \square \)

### 6.4. The proof of Theorem 1

Choose \( a < 1 \) so that \( |\alpha_0| = a \) and \( \sinh^{-1}[(\sinh \frac{|\alpha_0|}{2})^{-1}] > a \). Then the collar neighborhood \( C_{S_\infty}(\alpha_0, a) \) of \( \alpha_0 \) of width \( 2a \) is an embedded annulus in \( S_\infty = S(\alpha_0, p, C, \{F_i\}) \). There is a unique hyperbolic, twice punctured disc \( D \), for which the boundary geodesic \( \beta \) has length \( a \). As above, the Collar Lemma also tells us that \( \beta \) has an embedded collar neighborhood of width \( 2a \) in \( D \).

On \( S_\infty \) remove the annular region \( A^- \) to the left of \( \alpha_0 \), to get the surface \( \overline{S}_\infty \) with geodesic boundary \( \alpha_0 \). Similarly, on \( D \) remove the annular region in the complement of \( \beta \) on \( D \) to get a twice punctured disc \( \overline{D} \) with geodesic boundary \( \beta \). Choose a point \( q \) on \( \beta \) so that the unique minimal length geodesic arc on \( \overline{D} \), separating the punctures, with both endpoints on \( \beta \) begins at \( q \).

We can now glue the surfaces \( \overline{S}_\infty \) and \( \overline{D} \) together by identifying \( \beta \) and \( \alpha_0 \) so that \( p \) and \( q \) are matched by the identification. The resulting surface \( M_\infty \) is a hyperbolic surface with infinitely many ends: one is an infinite end and the rest are punctures. Let \( G \) be the Fuchsian group representing \( M_\infty \), which may be chosen so that \( G \supset \Gamma_\infty \).

The theorem will be proved by showing that \( \partial_\infty D(\tilde{p}, G) \) consists of the set \( \varphi(K) \), where \( \varphi \) is as in the proof of Theorem 1 and a set of isolated parabolic fixed points of \( G \). Reverting to familiar notation, let \( C^* = K \) and, without loss of generality, we suppose that \( \varphi(z) = z \). We consider the problem intrinsically on \( M_\infty \). Since \( \overline{S}_\infty \) sits naturally inside \( M_\infty \), the geodesic rays \( \delta_c \) are all defined on \( M_\infty \). We shall extend
Proposition 4 to apply to the surfaces $M_\infty$, by showing that the only infinite critical rays on $M_\infty$ are still the $\delta_c$ for $c \in C$.

Let us see how the above will suffice to prove the theorem. First, observe that on the new surface there is no finite end of the second kind. As a result, every point in $\partial_\infty D(\tilde{p}, G)$ is the endpoint of a lift of an infinite critical ray or, as we have seen from earlier arguments, an isolated parabolic fixed point of $G$. As a consequence of the extended version of Proposition 4, the only lifts of infinite critical rays beginning at $\tilde{p}$ are the $\tilde{\delta}_c$ for $c \in C$, and their endpoints comprise exactly the set $C^*$.

It remains for us to prove, what we call, the extended proposition. This is done by showing that if $\alpha$ is a critical ray on $M_\infty$, then $\alpha \subset S_\infty \subset M_\infty$. But if $\alpha \subset S_\infty \subset S_\infty$ is an infinite critical ray, then by Proposition 4, $\alpha$ is one of the rays $\delta_c$, $c \in C$. Suppose $\alpha: [0, u^*] \rightarrow M_\infty$ is a geodesic ray with $\alpha(0) = p$, $\alpha \not\subset S_\infty$, and $\alpha$ goes out the infinite end of $M_\infty$. Then $\alpha$ will have non-empty intersection with the interior of the surface $\overline{D} \subset M_\infty$ but must eventually lie on $\overline{S}_\infty$. In order for this to occur, there would need to be values $u_1, u_2$, $0 \leq u_1 < u_2$ so that $\alpha([u_1, u_2]) \subset \overline{D}$ and $\alpha(u_1), \alpha(u_2) \in \alpha_0$.

It is now possible to construct a piecewise geodesic $\overline{\alpha}$ lying entirely on $\overline{S}_\infty$, which is strictly shorter than $\alpha$ between the same endpoints. The geodesic $\overline{\alpha}$ is made by following an arc of $\alpha$ from $\alpha(0)$ to $\alpha(u_1)$, then following an arc of $\alpha_0$ from $\alpha(u_1)$ to $\alpha(u_2)$ and finally, following the arc of $\alpha$ from $\alpha(u_2)$ to $\alpha(u^*)$. Since the arc $\alpha([u_1, u_2])$ crosses half the collar $C_{M_\infty}(\alpha_0, a)$ twice in the interior of $\overline{D}$, its length must be greater than $2a$. But the arc of $\alpha_0$ replacing it has length less than $a$. Thus $\alpha$ cannot be critical. That completes the proof of Theorem 4.

7. Critical and subcritical rays get close to $\Sigma$

Consider the set $\Sigma = \{\sigma_c | c \in C\}$. Let $\epsilon > 0$ be given and let $N(\Sigma, \epsilon)$ be the $\epsilon$-neighborhood of $\Sigma$. We are interested in showing that the infinite critical and subcritical rays eventually lie in an $\epsilon$-neighborhood of $\Sigma$.

**Theorem 5.** Let $\sigma: [0, \infty) \rightarrow S_\infty$ be an infinite critical or subcritical ray on $S_\infty$. Given $\epsilon > 0$ there is a positive real $t_\epsilon$ so that if $t > t_\epsilon$, then $\sigma(t) \in N(\Sigma, \epsilon)$.

Let $\sigma: [0, \infty) \rightarrow S_\infty$ be a geodesic ray that goes out the infinite end of $S_\infty$. We employ a construction that will enable us to examine a sequence, $\{\lambda_i\}$, of points on $\sigma$ that lie as far as possible from $\Sigma$, in particular, farther than a given positive constant $\epsilon$. The proof of
Theorem 5 will amount to showing that the hypothesis that \( \{\lambda_i\} \) is an infinite sequence leads to the conclusion that \( \sigma \) is neither critical nor subcritical.

Let \( \epsilon > 0 \) be given. Define \( f : [0, \infty) \longrightarrow [0, \infty) \) by \( f(\lambda) = d(\sigma(\lambda), \Sigma) \). Clearly, \( f \) is continuous, so the set \( E = f^{-1}((0, \infty)) \) is a countable union of disjoint, open, connected components. Let \( \{V_i \mid i \in I'\} \), where \( I' \) is some countable index set, be the collection of these open intervals, and let \( I \subset I' \) be the set of indices for which \( V_i \cap f^{-1}((\epsilon, \infty)) \) is nonempty. Let \( I_b \subset I \) denote the set of \( i \in I \) for which \( V_i \) is bounded. For each \( i \in I_b \), let \( d_i = \max_{\lambda \in V_i} f(\lambda) \), and choose exactly one point \( \lambda_i \in f^{-1}(d_i) \cap V_i \). Clearly, for each \( i \in I_b \), \( \sigma(\lambda_i) \) is a choice of a point on \( \sigma \) that is farthest from \( \Sigma \) for all points \( \sigma(\lambda), \lambda \in V_i \).

Given \( \lambda, \mu \in L := \{\lambda_i : i \in I_b\} \), note that if the set \([\lambda, \mu] \cap L \) has at least \( m \) elements, then \( \mu - \lambda > 2(m - 1)\epsilon \). As \( \mu - \lambda \) is the length of the segment of \( \sigma \) that goes from \( \sigma(\lambda) \) to \( \sigma(\mu) \), it follows that for all \( \lambda, \mu \in L \), \( \lambda < \mu, [\lambda, \mu] \cap L \) is a finite set.

Assume for the remainder of this discussion that \( L \) is infinite. Given the remark above, we may now (re-)order \( L \) into a strictly increasing sequence \( \{\lambda_i\}_{i=1}^{\infty} \) where \( \lambda_i \to \infty \) as \( i \to \infty \). Evidently, the hypothesis that \( L \) is an infinite set implies that each of the (reordered) intervals \( V_i \) has finite length and therefore, that \( I_b = I \).

For each \( i \in \mathbb{N} \) let \( \tau_{2i-1} \) and \( \tau_{2i} \) be, respectively, the left and right endpoints of the interval \( V_i \). Note that for all \( i \), \( \sigma(\tau_{2i-1}) \) and \( \sigma(\tau_{2i}) \) lie on scaffolding geodesics in \( \Sigma \). Thus, for each \( i \in \mathbb{N} \), there are, not necessarily distinct, scaffolding curves \( \beta_{2i-1} \) and \( \beta_{2i} \), as well as positive reals \( t_{2i-1} \) and \( t_{2i} \), such that \( \beta_{2i-1}(t_{2i-1}) = \sigma(\tau_{2i-1}) \) and \( \beta_{2i}(t_{2i}) = \sigma(\tau_{2i}) \). Observe that \( \beta_{2i-1}(t_{2i-1}) = \sigma(\tau_{2i-1}) \) and \( \beta_{2i}(t_{2i}) = \sigma(\tau_{2i}) \) are the endpoints of the smallest connected segment of \( \sigma \) that contains \( \sigma(\lambda_i) \), and has endpoints lying in \( \Sigma \). In other words, the segment \( \sigma(\tau_{2i-1}, \tau_{2i}) \) of \( \sigma \), which includes the point \( \sigma(\lambda_i) \), has empty intersection with \( \Sigma \). Since each point of \( S_\infty \) lies either on \( \Sigma \) or in the interior of a cut-open subflute, it follows that \( \sigma(\tau_{2i-1}, \tau_{2i}) \) lies on a cut-open subflute, denoted \( F_i^* \). Now, \( \beta_{2i-1} \) is one of the boundary scaffolding geodesics of \( F_i^* \); let \( \beta_i \) be the other and observe that either \( \beta_i = \beta_{2i-1} \) or \( \beta_i = \beta_{2i} \). Note that, from the construction, for each \( i \in \mathbb{N} \), we have \( \sigma : [\tau_{2i-1}, \tau_{2i}] \rightarrow F_i^* \), \( \sigma(\tau_{2i-1}), \sigma(\tau_{2i}) \in \beta_{2i-1} \cup \beta_i = \partial F_i^* \), and \( \lambda_i \in [\tau_{2i-1}, \tau_{2i}] \) such that \( d(\sigma(\lambda_i), (\beta_{2i-1} \cup \beta_i)) \geq d(\sigma(\lambda), (\beta_{2i-1} \cup \beta_i)) \) for all \( \lambda \in [\tau_{2i-1}, \tau_{2i}] \), that is, \( \sigma(\lambda_i) \) realizes the maximal value for \( d(\sigma(\lambda), \beta_{2i-1} \cup \beta_i) \) among all points on \( \sigma([\tau_{2i-1}, \tau_{2i}]) \). By Lemma 4, \( \sigma(\lambda_i) \) must lie on a \( \gamma \)-curve of...
Lemma 14. Let $\epsilon > 0$ be given. With definitions as above, there is a positive constant $\kappa(\epsilon)$ such that $\tau_{2i} - \tau_{2i-1} > |t_{2i} - t_{2i-1}| + 2\log \left( \frac{k^2 + 1}{2k} \right)$.

Generally, for all $i$, $\tau_{i+1} - \tau_i > |t_{i+1} - t_i|$; in particular, $\tau_{2i+1} - \tau_{2i} > |t_{2i+1} - t_{2i}|$.

Proof of Theorem 5

Let $\epsilon > 0$ be given. For the geodesic ray $\sigma$, construct the collection $V = \{ V_i \mid i \in I \}$ as defined above. Recall that each for each $V_i$, $\sigma(V_i)$ lies in the cut-open, untwisted flute $F_i^*$ and $V_i$ contains a point $\lambda_i$ for which $d(\sigma(\lambda_i), \Sigma) > \epsilon$. Further, any point $t > 0$ for which $d(\sigma(t), \Sigma) > \epsilon$ must lie in some $V_i$. Therefore, if the collection $V$ is a finite collection of sets, and each $V_i$ in the collection is bounded, then the conclusion of the theorem is true, namely, there is a $t_\epsilon > 0$ such that, $\sigma(t) \in N(\Sigma, \epsilon)$ for $t > t_\epsilon$.

We note first that $V_i$ cannot be unbounded. In that case, since $\sigma(V_i)$ lies in exactly one cut-open, untwisted flute, $F_i^*$, the curve $\sigma$ would, after a point, lie entirely in the flute $F_i^*$. But then by the hypothesis that the each flute $F_i$ is of the first kind, the results in [3] can be used to deduce that $\sigma$ must eventually lie arbitrarily close to the scaffolding curves bounding this subflute, and therefore must eventually lie in $N(\Sigma, \epsilon)$.

We now know the sets $V_i$ in the collection $V$ are each bounded. It remains to show that the collection is finite. Suppose that $V$ is an infinite collection. We will show that in this case, $\sigma$ is neither critical nor subcritical. Employing the earlier construction, there are infinite sequences of scaffolding curves $\beta_i$, $\beta_i'$ and infinite sequences $\lambda_i$, $\tau_i$, $t_i \in [0, \infty]$, $i \in \mathbb{N}$ as above, where for each $i$, $d(\sigma(\lambda_i), \Sigma) > \epsilon$; in particular, $d(\sigma(\lambda_i), (\beta_{2i-1} \cup \beta_i')) > \epsilon$. By Lemma 14 there is a positive constant $\kappa(\epsilon)$ such that, for each $i \in \mathbb{N}$,

$$\tau_{2i} - \tau_{2i-1} > |t_{2i} - t_{2i-1}| + \kappa(\epsilon);$$
and for all $i$, 
\[ \tau_{2i+1} - \tau_{2i} > t_{2i+1} - t_{2i}. \]

Choose $N \in \mathbb{N}$ such that $N\kappa(\epsilon) > |\alpha_0| + d(\sigma(0), \alpha_0) + |t_1 - \tau_1| + m$, where $m$ is an arbitrarily chosen positive real. The length of the curve $\sigma([0, \tau_{2N}])$ is $\tau_{2N}$. From Lemma 14 we have,

\[
\begin{align*}
\tau_{2N} - \tau_1 &= \sum_{i=1}^{2N} (\tau_{i+1} - \tau_i) \\
&= \sum_{i=1}^{N} (\tau_{2i} - \tau_{2i-1}) + \sum_{i=1}^{N-1} (\tau_{2i+1} - \tau_{2i}) \\
&> \sum_{i=1}^{N} (|t_{2i} - t_{2i-1}| + \kappa(\epsilon)) + \sum_{i=1}^{N-1} |t_{2i+1} - t_{2i}| \\
&\geq N\kappa(\epsilon) + \sum_{i=1}^{N} (t_{2i} - t_{2i-1}) + \sum_{i=1}^{N-1} (t_{2i+1} - t_{2i}) \\
&= N\kappa(\epsilon) + \sum_{i=1}^{2N-1} (t_{i+1} - t_i) \\
&= N\kappa(\epsilon) + t_{2N} - t_1 \\
&> |\alpha_0| + d(\sigma(0), \alpha_0) + |t_1 - \tau_1| + m + t_{2N} - t_1.
\end{align*}
\]

Therefore,
\[
\tau_{2N} > |\alpha_0| + d(\sigma(0), \alpha_0) + |t_1 - \tau_1| - (t_1 - \tau_1) + t_{2N} + m \\
\geq |\alpha_0| + d(\sigma(0), \alpha_0) + t_{2N} + m.
\]

Note that the piecewise curve beginning at $\sigma(0)$, proceeding along a minimal length geodesic to a point on $\alpha_0$, then along $\alpha_0$ to the scaffolding curve $\beta_{2N}(0)$, along $\beta_{2N}$ to $\beta_{2N}(t_{2N})$, has length less than $|\alpha_0| + d(\sigma(0), \alpha_0) + t_{2N}$. It follows from the inequality above that the length of the curve $\sigma([0, \tau_{2N}])$ is larger than this piecewise curve by at least $m$, where $m$ was arbitrary. Therefore, it follows that $\sigma$ can be neither critical nor subcritical. \[\square\]

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