A DIRECT DECOMPOSITION OF 3-STRIP TABLEAUX

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ABSTRACT. Baryshnikov and Romik derived combinatorial identities for the numbers of the m-strip tableaux. This generalized the classical André’s theorem for the number of up-down permutations. They asked for a bijective proof for the enumeration of 3-strip tableaux. In this paper we count the 3-strip tableaux by decomposition. The key decomposition, in combination of the recursions for tangent numbers \( E_{2n-1} \) and Bernoulli numbers \( B_{2n} \), provides a semi-bijective proof for the enumeration of 3-strip tableaux.

1. Introduction

A permutation \( w = a_1a_2 \cdots a_n \in S_n \) is called an up-down permutation if \( a_1 < a_2 > a_3 < a_4 > \cdots \). It is well-known that the number of up-down permutations of \([n]\) is Euler number \( E_n \), whose exponential generating function is

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.
\]

(1.1)

This is also called André’s theorem. Sometimes \( E_{2n} \) is called a secant number and \( E_{2n+1} \) a tangent number. Up-down permutations can be viewed as a special case of a standard Young tableau. For instance, the permutation \( \sigma = 132546 \) is an up-down permutation, which can be identified as the tableau below.

Formally speaking, we adopt the notations from \([1]\). An integer partition is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) are integers. We identify each partition \( \lambda \) with its Young diagram and speak of them interchangeably. Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), the Young diagram of shape \( \lambda \) is a left-justified array of \( \lambda_1 + \lambda_2 + \cdots + \lambda_n \) boxes with \( \lambda_1 \) in the first row, \( \lambda_2 \) in the second row, and so on. A skew Young diagram is the difference \( \lambda/\mu \) of two Young diagrams where \( \mu \subset \lambda \). If \( \lambda/\mu \) is a skew Young diagram, a standard Young tableau of shape \( \lambda/\mu \) is a filling of the boxes of \( \lambda/\mu \) with the integers \( 1, 2, \ldots, |\lambda/\mu| \) that is increasing along rows and columns, where \( |\lambda/\mu| \) is the number of boxes of shape \( \lambda/\mu \) and is called the size of shape \( \lambda/\mu \). Given any skew shape \( \lambda/\mu \) of size \( n \), let \( f^{\lambda/\mu} \) denote the number of standard Young tableaux of shape \( \lambda/\mu \), i.e., the number of ways to put \( 1, 2, \ldots, n \) into the squares of the diagram of \( \lambda/\mu \), each number \( 1, 2, \ldots, n \) occurring exactly once, so that the rows and columns are increasing. Given a standard Young tableau, we can form the reading word of the tableau by reading the last row from left to right, then the next-to-last row, and so on. The reading word of the above tableau is exactly the permutation \( \sigma = 132546 \).

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Up-down permutations of $[n]$ are in simple bijection with standard Young tableaux of shape

$$\theta_n = (m + 1, m, m - 1, \ldots, 3, 2)/(m - 1, m - 2, \ldots, 1, 0)$$

when $n = 2m$ is even, or

$$\theta_n = (m, m, m - 1, m - 2, \ldots, 3, 2)/(m - 1, m - 2, \ldots, 1, 0)$$

when $n = 2m - 1$ is odd. Clearly this bijection converts each standard Young tableau of shape $\theta_n$ into an up-down permutation via its reading word. By “thickening” the shape $\theta_n$, Baryshnikov and Romik generalized the classical enumeration formula (1.1) for up-down permutations [1]. They introduced the $m$-strip tableaux and enumerated the $m$-strip tableaux by using transfer operators, but the computations become more complicated as $m$ increases. The standard Young tableaux of shape $\theta_n$ are exactly 2-strip tableaux, which are counted by the Euler number $E_n$. For the particular case $m = 3$, there are three different shapes of 3-strip tableaux, denoted by $\sigma_{3n-2}, \sigma_{3n-1}, \sigma_{3n}$, respectively. Let $\sigma_{3n-2}$ be the Young diagram of shape

$$(m, m - 1, m - 2, \ldots, 3, 2)/(m - 2, m - 3, \ldots, 1, 0)$$

that contains $3n - 2$ boxes when $n \geq 3$, and $\sigma_1$ be the Young diagram of shape $(1)$, $\sigma_4$ be the Young diagram of shape $(2, 2)$. Let $\sigma_{3n-1}$ be the Young diagram of shape

$$(m, m, m - 1, m - 2, \ldots, 3, 2)/(m - 1, m - 2, \ldots, 1, 0)$$

that contains $3n - 1$ boxes when $n \geq 3$, and $\sigma_2$ be the Young diagram of shape $(1, 1)$, $\sigma_5$ be the Young diagram of shape $(2, 2, 2)/(1, 0, 0)$. Let furthermore $\sigma_{3n}$ be the Young diagram of shape

$$(m, m, m - 1, \ldots, 3, 2, 1)/(m - 1, m - 2, \ldots, 1, 0, 0)$$

that contains $3n$ boxes when $n \geq 3$, and $\sigma_3$ be the Young diagram of shape $(1, 1, 1)$, $\sigma_6$ be the Young diagram of shape $(2, 2, 2, 2)/(1, 0, 0, 0)$. Below we show three standard Young tableaux of shape $\sigma_7, \sigma_8, \sigma_9$, from left to right, respectively.

\[
\sigma_7 = \begin{bmatrix}
3 & 6 \\
1 & 4 & 7 \\
2 & 5 \\
\end{bmatrix} \\
\sigma_8 = \begin{bmatrix}
6 \\
3 & 7 \\
1 & 4 & 8 \\
2 & 5 \\
\end{bmatrix} \\
\sigma_9 = \begin{bmatrix}
7 \\
4 & 8 \\
1 & 5 & 9 \\
2 & 6 \\
3 \\
\end{bmatrix}
\]

Baryshnikov and Romik proved

**Theorem 1** ([1], 3-strip tableaux).

\begin{align*}
(1.2) & \quad f^{\sigma_{3n-2}} = \frac{(3n - 2)!E_{2n-1}}{(2n - 1)!2^{2n-2}}, \\
(1.3) & \quad f^{\sigma_{3n-1}} = \frac{(3n - 1)!E_{2n-1}}{(2n - 1)!2^{2n-1}}, \\
(1.4) & \quad f^{\sigma_{3n}} = \frac{(3n)!(2^{2n-1} - 1)E_{2n-1}}{(2n - 1)!2^{2n-1}(2^{2n} - 1)}.
\end{align*}

Baryshnikov and Romik [1] asked for a bijective proof of Theorem 1, namely to prove Theorem 1 by directly relating 3-strip tableaux to up-down permutations in some combinatorial way. Here we prove Theorem 1 by decomposing 3-strip tableaux. The key decomposition offers an inductive way to relate 3-strip tableaux and up-down permutations, but finding a purely bijective proof of Theorem 1 still remains open.

While it is easy to bijectively prove $f^{\sigma_1} = E_1$ and $2^2 f^{\sigma_4} = 4E_3$, a framework to bijectively prove $2^{2n-2} f^{\sigma_{3n-2}} = (3n - 2)!/(2n - 1)!E_{2n-1}$ in general has met with
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frustratingly little progress. It seems to me, in order to prove eq. (1.2), we have to first decompose the up-down permutation of size $2n - 1$ into small pieces, i.e., the up-down permutations of size 3 or 1, then start inserting $(n - 1)$ distinct integers from $[3n - 2]$ successively. That motivated us to consider the decomposition of 3-strip tableaux.

After Baryshnikov and Romik published their results [1], Stanley [3] also generalized shape $\theta_r$ by introducing the skew partition $\sigma(a, b, c, n)$. $\sigma(a, b, c, n)$ is defined to be the skew partition whose Young diagram has $a$ squares in the first row, $b$ squares in the other nonempty rows, and $n$ rows in total. Moreover, each row begins $c - 1$ columns to the left of the row above, with $b \geq c$. Although the standard Young tableaux of shape $\sigma(a, b, c, n)$ is neither a subset nor a superset of the $m$-strip tableaux, the skew partition $\sigma(2, 3, 2, n)$ (resp. $\sigma(3, 3, 2, n)$) after the transposition of rows to columns, is exactly $\sigma_{3n-1}$ (resp. $\sigma_{3n}$). It follows that $f^{\sigma(2,3,2,n)} = f^{\sigma_{3n-1}}$ and $f^{\sigma(3,3,2,n)} = f^{\sigma_{3n}}$.

Stanley [3] derived the generating functions for the standard Young tableaux of shape $\sigma(a, b, c, n)$ by analyzing the determinant in the Aitken formula. For any integer partition $\lambda$, let $\ell(\lambda)$ be the length of $\lambda$. If $\ell(\lambda) \leq m$ and $\mu \subseteq \lambda$, the Aitken formula asserts that

$$f^{\lambda/\mu} = n! \det \left[ \frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^m.$$  

The Aitken formula can be obtained by applying the exponential specialization on the Jacobi-Trudi identity, see [2] [3]. For the special cases $f^{\sigma(2,3,2,n)} = f^{\sigma_{3n-1}}$ and $f^{\sigma(3,3,2,n)} = f^{\sigma_{3n}}$, the exponential generating functions for 3-strip tableaux of shape $\sigma_{3n-1}$ and $\sigma_{3n}$ are [3]

$$\sum_{n \geq 1} \frac{f^{\sigma_{3n-1}}}{(3n - 1)!} x^{2n} = \sum_{n \geq 1} \frac{(-1)^{n-1} x^{2n}}{(2n)!} \sum_{n \geq 0} \frac{(-1)^n x^{2n}}{(2n + 1)!} = x \tan \left( \frac{x}{2} \right)$$  

$$\sum_{n \geq 1} \frac{f^{\sigma_{3n}}}{(3n)!} x^{2n} = \sum_{n \geq 1} \frac{(-1)^{n-1} x^{2n}}{(2n + 1)!} \sum_{n \geq 0} \frac{(-1)^n x^{2n}}{(2n + 1)!} = \frac{x}{\sin(x) - 1}$$  

In the next two sections we will prove the generating functions for the 3-strip tableaux of shape $\sigma_{3n-1}$ for $i = 0, 1, 2$ by decomposing the 3-strip tableaux.

2. THE KEY DECOMPOSITION

In what follows, we represent each standard Young tableau by a natural labeling on its corresponding poset. For $i = 0, 1, 2$, let $P_{\sigma_{3n-1}}$ be the poset whose elements are the squares of the Young diagram of shape $\sigma_{3n-1}$, with $t$ covering $s$ if $t$ lies directly to the right or directly below $s$, with no squares in between. In this way, each Young diagram of shape $\sigma_{3n-i}$ can be represented by the Hasse diagram of $P_{\sigma_{3n-i}}$ and we will speak of them interchangeably. A natural labeling of $P_{\sigma_{3n-i}}$ is an order-preserving bijection $\eta : P_{\sigma_{3n-i}} \to [3n - i]$, i.e., a natural labeling $\eta$ is a bijection such that $\eta(x) \leq \eta(y)$ for every $x, y \in P_{\sigma_{3n-i}}$ and $x \leq y$. Sometimes a natural labeling of $P_{\sigma_{3n-i}}$ is also called a linear extension of $P_{\sigma_{3n-i}}$. The number of linear extensions of $P_{\sigma_{3n-i}}$ is denoted $e(P_{\sigma_{3n-i}})$. Then we have $f^{\sigma_{3n-i}} = e(P_{\sigma_{3n-i}})$ because each 3-strip tableau of shape $\sigma_{3n-i}$ can be identified as a natural labeling of the poset $P_{\sigma_{3n-i}}$ for $i = 0, 1, 2$, see [2]. As an example, every 3-strip tableau of
is represented by a natural labeling of its corresponding poset

from left to right. Let $\tau_{3n}$ be the Young diagram of shape $(m, m, m - 1, m - 2, \ldots, 3)/(m - 1, m - 2, \ldots, 1, 0)$ that contains $3n$ boxes when $n \geq 2$, and $\tau_3$ be the Young diagram of shape $(2, 2)/(1, 0)$. For instance, a standard Young tableau of shape $\tau_6$ is represented as

We say a standard Young tableau $T$ of shape $\sigma_{3n-i}$ filled with integers $m_1, \ldots, m_{3n-i}$ if the corresponding natural labeling of $P_{\sigma_{3n-i}}$ is an order-preserving bijection $\eta : P_{\sigma_{3n-i}} \rightarrow \{m_1, \ldots, m_{3n-i}\}$. If $m_i = i$ for all $i$, then $T$ is the usual standard Young tableau of shape $\sigma_{3n-i}$. Let $P_{\sigma_{3n-i}}^d$ be the dual poset of $P_{\sigma_{3n-i}}$ and $\sigma_{3n-i}^d$ be the Hasse diagram of dual poset $P_{\sigma_{3n-i}}^d$, then the shape $\sigma_{3n-i}^d$ is obtained by flipping the shape $\sigma_{3n-i}$ upside down and therefore $f^{\sigma_{3n-i}} = f^{\sigma_{3n-i}^d}$. We will use this fact to prove two simple but important observations in Lemma 2.

**Lemma 2.** The numbers $f^{\sigma_{3n-2}}, f^{\sigma_{3n-1}}, f^{\sigma_{3n}}$ and $f^{\tau_{3n}}$ satisfy

\begin{align}
(3n - 1) f^{\sigma_{3n-2}} &= 2 f^{\sigma_{3n-1}} \\
(3n) f^{\sigma_{3n-1}} &= f^{\sigma_{3n}} + f^{\tau_{3n}}.
\end{align}

**Proof.** Given a pair $(T, i)$ where $i \in [3n - 1]$ and $T$ is a standard Young tableau of shape $\sigma_{3n-2}$ filled with integers from $[3n - 1] \setminus \{i\}$. Suppose $\omega(T) = a_1 a_2 \cdots a_{3n-2}$ is the reading word of tableau $T$, then the $3$-strip tableau $T$, after omitting the labels in between, is

where $a_j \in [3n - 1]$ for all $j$. If $i < a_{3n-2}$, then we put $i$ below $a_{3n-2}$ such that $a_{3n-2}$ covers $i$. Graphically, we obtain
which is a natural labeling on the poset $P_{\sigma_{3n-1}}$. If $i > a_{3n-2}$, then we put $i$ above $a_{3n-2}$ such that $i$ covers $a_{3n-2}$ and we obtain

which is a natural labeling on the dual poset $P_{\sigma_{3n-1}}$. It follows that $(3n-1)f^{\sigma_{3n-2}} = f^{\sigma_{3n-1}} + f^{\sigma_{3n-1}} = 2f^{\sigma_{3n-1}}$, i.e., eq. (2.1) follows. Next we prove eq. (2.2). Given a pair $(T_1, i)$ where $i \in [3n]$ and $T_1$ is a standard Young tableau of shape $\sigma_{3n-1}$ filled with integers from $[3n] - \{i\}$. Suppose $\omega(T_1) = b_1 b_2 \cdots b_{3n-1}$ is the reading word of tableau $T_1$, if $i < b_1$, then we obtain a tableau of shape $\tau_{3n}$ by putting $i$ below $b_1$ such that $b_1$ covers $i$. Otherwise if $i > b_1$, then we obtain a tableau of shape $\sigma_{3n}$ by putting $i$ above $b_1$ such that $i$ covers $b_1$. This implies eq. (2.2). □

From Lemma 2 we find in order to enumerate the 3-strip tableaux of shape $\sigma_{3n-i}$ for $i = 0, 1, 2$, it suffices to enumerate the standard Young tableaux of shape $\sigma_{3n-2}$ and $\tau_{3n}$. In the following we shall introduce the way to decompose each standard Young tableau of shape $\sigma_{3n-2}$ and $\tau_{3n}$, which gives a combinatorial proof of

**Theorem 3.** For $n \geq 2$, the numbers $f^{\sigma_{3n-2}}$ and $f^{\tau_{3n}}$ satisfy

\begin{align}
(2.3) \quad f^{\sigma_{3n-2}} &= \frac{1}{2n-1} \sum_{i=1}^{n-1} \left( \frac{3n-2}{3i-1} \right) f^{\sigma_{3n-3i-1}} \\
(2.4) \quad f^{\tau_{3n}} &= \frac{1}{2n+1} \sum_{i=1}^{n-1} \left( \frac{3n}{3i} \right) f^{\tau_{3n-3i}}.
\end{align}

**Proof.** For every element $a$ in the standard Young tableau $T$ of shape $\sigma_{3n-2}$, there are at most two elements that cover $a$. We call an element $b$ the left (resp. right) parent of $a$ if $b$ covers $a$ and $b$ is to the left (resp. right) of $a$, denoted by $p_{1,T}(a)$ (resp. $p_{2,T}(a)$). If $a$ has no right (resp. left) parent, then we assume $p_{2,T}(a) = +\infty$ (resp. $p_{1,T}(a) = +\infty$).

We next define a reflection $\gamma$ that maps each standard Young tableau $T$ into its mirror image $\gamma(T)$, with respect to a vertical line outside $T$. Below we show the tableau $T$ and its mirror image $\gamma(T)$.

We say the elements $x_1, x_2, x_3$ are on the bottom row of $T$. It is clear the set of standard Young tableaux of shape $\sigma_{3n-2}$ (resp. $\tau_{3n}$) is closed under the reflection $\gamma$. We use $\min(T)$ to denote the minimal element of tableau $T$. Let $T_{\sigma_{3n-2}}^{x_{1j}}$ (resp. $T_{\tau_{3n}}^{x_{1j}}$) be the set of standard Young tableau $T$ filled with integers $x_1, \ldots, x_{3n-2}$ satisfying

1. the left parent of $\min(T)$ is greater (resp. smaller) than the right parent of $\min(T)$, i.e., $p_{1,T}(\min(T)) > p_{2,T}(\min(T))$ (resp. $p_{1,T}(\min(T)) < p_{2,T}(\min(T))$),
2. $\min(T)$ is the $j$-th element from left to right of the bottom row of $T$.
Then from the reflection $\gamma$ we can easily see
\[
|\mathcal{T}_{\sigma_{3n-2}^{1,j}}| = |\mathcal{T}_{\sigma_{3n-2}^{2,n-j}}| \quad \text{and} \quad \sum_{j=1}^{n-1} |\mathcal{T}_{\sigma_{3n-2}^{1,j}}| = \frac{1}{3} f^{\sigma_{3n-2}}.
\]
Given a pair $(T, i)$ where $i \in [3n - 1]$ and $T \in \mathcal{T}_{\sigma_{3n-2}^{1,j}}$ is filled with integers from $[3n - 1] - \{i\}$, then we can represent $T$ as below

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  x_1 & & c & > & a & x_2 \\
 & & \text{min}(T) & & & \\
  x_3 & & & x_8 & & x_9 \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]

where $c = p_{1,T}(\min(T)) > a = p_{2,T}(\min(T))$. Let furthermore $\sigma_{3n-2}^{1,j}$ be the shape of $T$ represented by the Hasse diagram

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  \vdots & & \text{min} & & & \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]

where we use $\square$ to emphasize the location of the minimal element of shape $\sigma_{3n-2}^{1,j}$ is fixed by the assumption on $T$. We will construct the bijection $(T, i) \mapsto g(T)$ by comparing the values of $i$ and $a = p_{2,T}(\min(T))$. If $i > a$, then we let $i$ cover $a$ in the new tableau $g(T)$. In this case we notice $\min(T) = \min(g(T)) = 1$ and the tableau $g(T)$ can be represented as

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  \vdots & & \square & \vdots & \vdots \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]

where $c > a$. Let $\sigma_{1}^{i}$ be the shape of $g(T)$ under the condition $i > a$, represented by

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  \vdots & & \text{min} & & & \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]

where the location of the minimal element is fixed by $T$. If $i < a$, then we let $a$ cover $i$ in the new tableau $g(T)$ and the tableau $g(T)$ can be represented as

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  \vdots & & \text{min}(T) \square & \vdots & \vdots \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]

where $c > a$. Let $\sigma_{2}^{i}$ be the shape of $g(T)$ under the condition $i < a$, represented by

\[\begin{array}{cccccc}
  \vdots & \vdots & & \vdots & \vdots \\
  \vdots & & \text{min} & & & \\
  \vdots & \vdots & & \vdots & \vdots \\
\end{array}\]
where only two elements in $\Box$ can be the minimal element. We shall next prove
\begin{equation}
(2.5)
\sum_{j=1}^{n-1} f^{\sigma_j} = (n - \frac{1}{2}) f^{\sigma_{3n-2}}.
\end{equation}

We first observe that for the tableau $g(T)$ of shape $\sigma_2^i$, $\min(g(T))$ is either equal to $\min(T)$ or equal to $i$. If $i < \min(T)$, then $i = \min(g(T)) = 1$ and $\min(T) = 2$.

By removing $i$ and replacing every element $j$ by $j - 1$, we obtain a standard Young tableau filled with integers $1, 2, \ldots, 3n - 2$ from $T_{\sigma_2^{i-1}}$. By summing over all the possible locations of the minimal element of $T$, we get
\begin{equation}
\sum_{j=1}^{n-1} f^{\sigma_{3n-2}^{j-1}} = \frac{1}{2} f^{\sigma_{3n-2}}.
\end{equation}

If $i > \min(T)$, then $\min(g(T)) = \min(T) = 1$ and $c > a > i$. We remove $\min(T)$ from the tableau $g(T)$ and connect $i$ with $c$, next we replace every element $j$ by $j - 1$, which gives us

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram of a standard Young tableau $T$ with colored elements.}
\end{figure}

where $i' = i - 1$, $c' = c - 1$, $a' = a - 1$, $x_m' = x_m - 1$ and $i'$ is not necessary to be the minimal element 1. We color the element $i'$ in the above tableau. Therefore the tableau $g(T)$ under the condition $i > \min(T)$, is uniquely corresponding to a tableau of shape $\sigma_{3n-2}$, having the $j$-th element $i'$ from the bottom row colored, whose left parent is larger than its right parent. By summing over all the possible values of $j$, we need to count the number of standard Young tableau $T$ of shape $\sigma_{3n-2}$ having a colored element $i'$ from the bottom row and $p_{1,T}(i') > p_{2,T}(i')$. This number is $\frac{1}{2} (n - 1) f^{\sigma_{3n-2}}$ since there are $(n - 1)$ ways to choose the colored element from the bottom row and by reflection $\gamma$, this number is equal to the number of standard Young tableau $T'$ of shape $\sigma_{3n-2}$ having a colored element $m'$ from the bottom row, such that $p_{1,T'}(m') < p_{2,T'}(m')$. In sum, after discussing two cases $i < \min(T)$ and $i > \min(T)$, we have proved
\begin{equation}
\sum_{j=1}^{n-1} f^{\sigma_2} = \frac{1}{2} n f^{\sigma_{3n-2}}
\end{equation}

and eq. (2.5) follows immediately since inserting $i$ to $T$ for $T \in T_{\sigma_2^{i-1}}$ gives us
\begin{equation}
f^{\sigma_i} + f^{\sigma_2} = (3n - 1) f^{\sigma_{3n-2}},
\end{equation}

from which it follows
\begin{equation}
\sum_{j=1}^{n-1} (f^{\sigma_i} + f^{\sigma_2}) = \frac{1}{2} (3n - 1) f^{\sigma_{3n-2}}.
\end{equation}

Consequently it remains to prove
\begin{equation}
(2.6)
f^{\sigma_1} = \frac{1}{2} \left( \frac{3n}{3i} \right) f^{\sigma_{3n-1}} f^{\sigma_{3n-3i-1}}.
\end{equation}

For a given tableau $g(T)$ of shape $\sigma_1^i$, we have $\min(T) = \min(g(T)) = 1$. By removing 1 from the tableau $g(T)$ and replacing every label $m$ by $m - 1$, we obtain a standard Young tableau $T_2$. 

Together with the exponential generating function for $E_f$, immediately have the generating function of $f$ from these generating functions, Theorem 1 is proved. More precisely, let

$$f(x) = \sum_{n \geq 1} \frac{f^{\sigma_{3n-1}}(3n-2)!}{x^{2n-1}}, \quad g(x) = \sum_{n \geq 1} \frac{f^{\sigma_{3n}}(3n-1)!}{x^{2n-1}},$$

then from eq. (2.1) we have $f(x) = 2g(x)$. Furthermore, eq. (2.3) is equivalent to $f'(x) = 1 + g(x)^2$ where $f(0) = 0$. This leads to a unique solution, $g(x) = \tan(x/2)$. Together with the exponential generating function for $E_{2n-1}$, the formula for $f^{\sigma_{3n-2}}$ and $f^{\sigma_{3n-1}}$, eq. (1.2) and eq. (1.3) are proved. Similarly, let

$$h(x) = \sum_{n \geq 1} \frac{f^{\sigma_{3n}}(3n)!}{x^{2n-1}},$$

then eq. (2.4) is equivalent to $-xh'(x) = -x + 2h(x) - xh^2(x)$ where $h(0) = 0$. This yields a unique solution

$$h(x) = -\frac{1}{\tan x} + \frac{1}{x},$$

and consequently in view of eq. (2.2), the exponential generating function for $f^{\sigma_{3n}}$ is equal to $g(x) - h(x)$, thus eq. (1.4) follows. By considering the expansion of $x/\sin(x)$, we finally obtain the coefficients $f^{\sigma_{3n}}$, i.e., eq (1.4). Alternatively, we can obtain the expression of $f^{\sigma_{3n-2}}$ and $f^{\sigma_{3n}}$ by using the recursion of tangent numbers $E_{2n-1}$ and Bernoulli numbers $B_{2n}$. The Bernoulli numbers $B_{2n}$ are integers defined from the Euler number $E_{2n-1}$ by the relation:

$$B_{2n} = \frac{nE_{2n-1}}{2^{2n-1}(2^{2n-1} - 1)} \quad \text{where} \quad n \geq 1.$$
After verifying the initial conditions that $f^{\sigma_1} = E_1$, $2^2 f^{\sigma_4} = 4 E_3$, $1!(2^2 - 1)f^{\tau_3} = 3! E_1$ and $3!(2^4 - 1)f^{\tau_6} = 6! E_3$, we can inductively prove eq. (1.2) and

$$f^{\tau_{3n}} = \frac{(3n)! E_{2n-1}}{(2n-1)!(2^{2n} - 1)}.$$ 

In view of Lemma 2, we can further obtain the expression of $f^{\sigma_{3n-1}}$ and $f^{\tau_{3n}}$. Henceforth the proof of Theorem 1 is complete.

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