Measurement contextuality is implied by macroscopic realism

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(Dated: August 23, 2011)

Ontological theories of quantum mechanics provide a realistic description of single systems by means of well-defined quantities conditioning the measurement outcomes. In order to be complete, they should also fulfil the minimal condition of macroscopic realism. Under the assumption of outcome determinism and for Hilbert space dimension greater than two, they were all proved to be contextual for projective measurements. In recent years a generalized concept of non-contextuality was introduced that applies also to the case of outcome indeterminism and unsharp measurements. It was pointed out that the Beltrametti-Bugajski model is an example of measurement non-contextual indeterminist theory. Here we provide a simple proof that this model is the only one with such a feature for projective measurements and Hilbert space dimension greater than two. In other words, there is no extension of quantum theory providing more accurate predictions of outcomes and simultaneously preserving the minimal labelling of events through projective operators. As a corollary, non-contextuality for projective measurements implies non-contextuality for unsharp measurements. By noting that the condition of macroscopic realism requires an extension of quantum theory, unless a breaking of unitarity is invoked, we arrive at the conclusion that the only way to solve the measurement problem in the framework of an ontological theory is relaxing the hypothesis of measurement non-contextuality in its generalized sense.

PACS numbers: 03.65.Ta, 02.50.Cw

I. INTRODUCTION

Quantum mechanics provides an operationally complete and minimal description of states, transformations and events. Procedures that cannot be statistically discriminated are described in the quantum formalism by the same mathematical objects. This is for example the case of the quantum state, which contains only the statistically significant information about the preparation protocol. The same holds for events of a von Neumann measurement, which are labelled by projective operators $\hat{E}_k$. This description is operationally complete since the probability $p(\hat{E}_k)$ of an event $\hat{E}_k$ does not depend on any supplementary information, such as the complete set of events $\{\hat{E}_1, \hat{E}_2, \ldots\}$ being measured. It is also minimal because events labelled with different projectors have different statistical weights for some preparation state. For reasons of economy, one could desire to preserve the same minimal description also in ontological theories of quantum mechanics, traditionally known as hidden variable theories.

Originally, these theories were designed for the purpose of introducing determinism and realism in quantum mechanics [1]. Whereas in the standard interpretation the state represents the overall statistically significant information about the preparation procedure, an ontological theory provides a realistic description of the actual state of affairs of a single system. Employing the recent terminology [2], we call these states ontic states in order to distinguish them from the quantum states. In the framework of an ontological theory they condition the probabilities of events. The theory is said outcome deterministic if the ontic state determines with certainty the result of a measurement, that is, if the conditional probabilities of events are zero or one. For example this is the case of the de Broglie-Bohm theory, where the ontic state is identified with the quantum mechanical wave-function and additional variables describing the actual positions of the particles.

Additionally, an ontological theory should also satisfy the minimal condition of macroscopic realism [3] in order to be complete, that is, if there are two or more macroscopically distinct states available to a macroscopic system, then the ontological theory must attribute one of these states to the system. In general a theory with outcome determinism, such as the de Broglie-Bohm theory, fulfills this criterion. The Beltrametti-Bugajski model [4] is a counter-example of ontological model for measurements that does not provide a macroscopic realist description, unless the postulate of wave-function collapse of the standard interpretation is retained in the model.

Since in the quantum formalism an event is labelled by the projective operator $\hat{E}_k$, it seems natural to keep this structure also in the ontological theory. However the Kochen-Specker theorem establishes that this is impossible for Hilbert space dimensions greater than 2 and outcome determinism [3]. The occurrence of the event $\hat{E}_k$ for a given ontic state depends in general on the whole set of projectors involved in the measurement. This feature is called contextuality. In recent years a generalized concept of non-contextuality was introduced that allows for outcome indeterminism and apply to unsharp mea-
measurements \cite{6}, that it, positive operator valued measurements (POVM). Furthermore this definition applies also to states and transformations. Essentially, an ontological theory is non-contextual for states, transformations and measurement events if the probability distributions, the transition rates and the conditional probabilities associated respectively with states, transformations and events preserve the same minimal labelling of quantum mechanics. In Ref. \cite{6} it was shown that non-contextuality for state preparation is incompatible with quantum statistics. This can be seen also as a consequence of the fact that the probability distribution associated with a pure quantum state $|\psi\rangle$ cannot be a quadratic function of $|\psi\rangle$ \cite{2}. However there is not a similar constraint for measurements \cite{6}. Indeed the Beltrametti-Bugajski model \cite{4} is a simple example of indeterminist theory that is measurement non-contextual.

In this article we report a simple proof that the Beltrametti-Bugajski model is essentially the only theory that is non-contextual for projective measurements. In other words, we prove that no extension of quantum theory exists providing more accurate predictions of outcomes and simultaneously preserving the minimal labelling of events through projective operators. These findings are closely related to a recent result reported by Colbeck and Renner \cite{8}. They conclude that every extension of quantum theory with improved predictions of outcomes is incompatible with the hypothesis that the measurement parameters in a region of spacetime, $\Omega$, can be chosen to be statistically independent of any variable that does not lie in the future lightcone of $\Omega$. This hypothesis implies a particular condition of non-contextuality for measurements.

As a corollary of our result, we show that non-contextuality for unsharp measurements is implied by non-contextuality for sharp measurements (the projective ones). Finally, we arrive at the conclusion that the only way to solve the measurement problem in the framework of an ontological theory and fulfill the minimal condition of macroscopic realism without breaking the unitarity of evolutions is relaxing the generalized condition of measurement non-contextuality. In Sec. \textbf{III} we introduce the general properties that an ontological theory has to satisfy and prove the theorem of uniqueness considering only projective measurements. In Sec. \textbf{III} the discussion is extended to the case of positive operator valued measurements. In Sec. \textbf{IV} we discuss the results and their relation to the question tackled in Ref. \cite{8}. First, we show that any extension of quantum theory cannot be accomplished by preserving measurement non-contextuality. Then, by noting that such an extension is necessary in order to introduce macroscopic realism, we infer that measurement contextuality is implied by macroscopic realism. The conclusions are drawn in the last section. The main message of this paper is showing that macroscopic realism can turn to be a very useful ingredient for establishing general properties that an ontological theory must satisfy.

\section{Non-contextuality for Projective Measurements}

In an ontological theory a quantum state $|\psi\rangle$ is associated with a probability distribution on an ontological space whose elements are denoted by $X$. In general, the probability distribution could depend on the context of the preparation $|\psi\rangle$ \cite{2}, thus we label it with an additional variable $\eta$ and define the map

$$|\psi\rangle \rightarrow \{\rho(X|\psi,\eta)\} \quad (1)$$

that associates quantum states with sets of distributions. If there is no dependence on $\eta$, the ontological model is said to be non-contextual for preparation of pure states. Later on it will be shown that this kind of non-contextuality is implied by measurement non-contextuality. The probability distributions satisfy the conditions

$$\rho(X|\psi,\eta) \geq 0, \quad (2)$$

$$\int dX \rho(X|\psi,\eta) = 1. \quad (3)$$

In the quantum formalism, a measurement is associated with a set of commuting projectors $\{\hat{E}_1, \hat{E}_2, \ldots\}$ representing the events. Each complete set of events satisfies the relation

$$\sum_k \hat{E}_k = \hat{1}. \quad (4)$$

In the ontological model the probability $P(\hat{E}_k|X)$ of an event $\hat{E}_k$ is conditioned by the ontic state $X$. It satisfies the inequalities

$$0 \leq P(\hat{E}_k|X) \leq 1 \quad (5)$$

and, for each complete set $\{\hat{E}_k\}$ of commuting projectors, the identity

$$\sum_k P(\hat{E}_k|X) = 1. \quad (6)$$

We have explicitly employed the hypothesis of measurement non-contextuality by assuming that the probability of an event $\hat{E}_k$ does not depend on the whole set of projective operators. The projector $\hat{E}_1$ can for example be an element of the set $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \ldots\}$ or $\{\hat{E}_1', \hat{E}_2', \hat{E}_3', \ldots\}$, but the conditional probability does not depend on this change of context. Gleason’s theorem states that a probability distribution satisfying properties \cite{5,6,6} has the form

$$P(\hat{E}_k|X) = \text{Tr}[\hat{E}_k \hat{\rho}(X)] \quad (7)$$

for some Hermitian operator $\hat{\rho}(X)$, provided that the Hilbert space has dimension 3 or greater \cite{4}. The ontological theory is equivalent to quantum mechanics if

$$\int dX P(\hat{E}_k|X) \rho(X|\psi,\eta) = \langle \psi|\hat{E}_k|\psi\rangle. \quad (8)$$
The Beltrametti-Bugajski model trivially satisfies properties \((23)\). The space of ontic states is the projective Hilbert space and the probability distribution associated with a quantum state \(|\psi\rangle\) is

\[
\rho(X|\psi) = \delta(X - \psi),
\]

(9)

\(\psi\) being the ray of \(|\psi\rangle\). The conditional probability for an event \(\hat{E}_k\) given the ontic state \(X\) is

\[
P(\hat{E}_k|X) = \langle X|\hat{E}_k|X\rangle.
\]

(10)

It is simple to prove that any ontological theory that is non-contextual for projective measurement is essentially equivalent to the Beltrametti-Bugajski model.

**Theorem 1.** Given a quantum system with a Hilbert space dimension greater than 2, any associated ontological theory that is non-contextual for projective measurements is equivalent to the Beltrametti-Bugajski theory after a suitable coarse graining of the ontological space.

**Proof:** Since the ontological theory is non-contextual, then any complete set of projective measurements is associated with conditional probabilities satisfying properties \((5)\). Gleason’s theorem \((2)\) implies that there exists a trace-one Hermitian operator \(\hat{\rho}(X)\) such that

\[
P(\hat{E}_k|X) = \text{Tr}[\hat{E}_k\hat{\rho}(X)],
\]

(11)

for any set \(\{\hat{E}_k\}\) of projective measurements. Let us consider the event \(\hat{E}_\psi = |\psi\rangle\langle\psi|\). If \(X\) is in the support of \(\rho(X|\psi, \eta)\) for some context \(\eta\), then the conditional probability \(P(\hat{E}_\psi|X)\) is equal to 1. Indeed we have from Eq. \((3)\) that

\[
\int dX P(\hat{E}_\psi|X)\rho(X|\psi, \eta) = 1
\]

(12)

Because of properties \((235)\) and Eq. \((12)\),

\[
\rho(X|\psi, \eta) \neq 0 \Rightarrow P(\hat{E}_\psi|X) = 1.
\]

(13)

This implication is intuitively obvious, if a system is prepared in a quantum state \(|\psi\rangle\) and its ontic state is \(X\), then the probability of obtaining the state \(|\psi\rangle\) given \(X\) is 1. Thus, since \(\hat{\rho}(X)\) is a positive trace-one Hermitian operator, we have from Eqs. \((11)\) \((13)\) that

\[
\rho(X|\psi, \eta) \neq 0 \Rightarrow \hat{\rho}(X) = |\psi\rangle\langle\psi|.
\]

(14)

Equation \((14)\) says that if \(X\) is in the support of the distribution \(\rho(X|\psi, \eta)\) then the Hermitian operator \(\hat{\rho}(X)\) is equal to the pure density operator \(|\psi\rangle\langle\psi|\). As a consequence, the support of two probability distributions associated with different rays are not overlapping. Furthermore, the conditional probabilities, given by Eq. \((11)\), are constant on the support of each probability distribution \(\rho(X|\psi, \eta)\), that is, they are not sensitive to the fine structure of the distributions inside the support. This allows us to perform a coarse graining of the ontological space, dividing it in equivalent classes \(\hat{X}\) and associating each class with a ray \(\psi\),

\[
\hat{X} \leftrightarrow \psi.
\]

(15)

The state \(\hat{X}\) is the union of the supports of the distributions \(\rho(X|\psi, \eta)\) with \(\psi\) fixed and \(\eta\) spanning every possible value. The coarse grained probability distribution on \(X\) is

\[
\rho(\hat{X}|\psi) = \delta(\hat{X} - \psi)
\]

(16)

and does not depend on \(\eta\). It is obtained by integrating the original distribution on the associated equivalent class. The conditional probability given \(\hat{X}\) is

\[
P(\hat{E}_k|\hat{X}) = \langle \hat{X}|\hat{E}_k|\hat{X}\rangle.
\]

(17)

The probability distribution \((16)\) and the conditional probability \((17)\) correspond to the Beltrametti-Bugajski model. \(\square\)

Thus, any ontological theory that preserves the minimal labelling for events of quantum mechanics necessarily coincides with the Beltrametti-Bugajski model. In particular, it is \(\psi\)-ontic \((10)\), a \(\psi\)-ontic theory being an ontological theory of quantum mechanics that associates two different quantum states with non-overlapping probability distributions. In other words, the ontic state contains the full information on the quantum state, which therefore represents some element of reality. This property was inferred in Ref. \([11]\) by using the hypothesis of POVM non-contextuality, which will be discussed in the next section. The Beltrametti-Bugajski model is clearly \(\psi\)-ontic because of the delta shape of the distribution defined by Eq. \((9)\). Conversely, in a \(\psi\)-epistemic theory, only probability distributions associated with orthogonal states have disjoint supports. Thus, it is not possible to infer the quantum state by the knowledge of the ontic state. In such a theory the quantum state does not represent an element of reality, but it contains a mere statistical information about the actual ontic state. An example of \(\psi\)-epistemic model is the Kochen-Specker (KS) model for a qubit \((3)\). The ontological space is the set of Bloch vectors \(\vec{v}\). If a quantum state is represented by a Bloch vector \(\vec{b}\), the associated probability distribution is, up to a normalization constant,

\[
\rho(\vec{v}|\vec{b}) = \theta(\vec{v} \cdot \vec{b})\vec{v} \cdot \vec{b},
\]

(18)

where \(\theta(x)\) is the Heaviside function. The support of this probability distribution is a hemisphere. It is clear that two probability distributions \(\rho(\vec{v}|\vec{b})\) and \(\rho(\vec{v}|\vec{b}')\) are not overlapping only if \(\vec{b}\) and \(\vec{b}'\) are anti-parallel, corresponding to orthogonal quantum states. By labelling an event with a Bloch vector \(\vec{c}\), the conditional probability of \(\vec{c}\) given the ontic state \(\vec{v}\) is, in the KS model,

\[
P(\vec{c}|\vec{v}) = \theta(\vec{c} \cdot \vec{v}).
\]

(19)

This probability does not have the structure given by Eq. \((11)\) implied by the Gleason theorem for a Hilbert space dimension greater than 2.
There exist other models for qubit satisfying the conditions \(23\) but inequivalent to the Beltrametti-Bugajski model, such as the Bell’s model \(12\) and a one-dimensional model recently reported in Refs. \(13, 14\).

III. NON-CONTEXTUALITY FOR UNSHARP MEASUREMENTS

The generalized concept of non-contextuality can be also applied to unsharp measurements, that is, positive operator valued measurements (POVM). They can be physically implemented by means of a projective measurement on the system and an ancilla. A POVM is defined by a set of positive operators \(\{Q_k\}\) satisfying the condition

\[
\sum_k Q_k = \mathbb{1}.
\]  

(20)

Each event is associated with an operator \(\hat{Q}_k\). An ontological theory is non-contextual for unsharp measurements if the conditional probability for an event \(\hat{Q}_k\) given an ontic state \(X\) does not depend on the whole set of positive operators \(\{\hat{Q}_1, \hat{Q}_2, \ldots\}\) and, more in general, on the physical implementation of the measurement. It is a trivial task to prove the following.

**Lemma 1.** The Beltrametti-Bugajski theory is POVM non-contextual.

Proof: A POVM measurement is made on a system \(A\) by performing a measurement of the projectors \(\hat{E}_k\) on \(A\) and an ancilla \(B\). Let \(A\) be in a pure state. The overall state is \(\hat{\rho} = \hat{\rho}_A \hat{\rho}_B\), where \(\hat{\rho}_A = |\psi_A\rangle\langle\psi_A|\).

The positive operators \(\hat{Q}_k\) are given by the equation

\[
\hat{Q}_k = \text{Tr}[\hat{\rho}_B \hat{E}_k].
\]

(21)

The probability of event \(\hat{Q}_k\) is equal to the probability of event \(\hat{E}_k\), that is,

\[
p(\hat{Q}_k) = \text{Tr}_A[\hat{Q}_k \hat{\rho}_A] = \text{Tr}_{AB} [\hat{E}_k \hat{\rho}_A \hat{\rho}_B] = \tilde{p}(\hat{E}_k)
\]

(22)

At the ontological level, the probability of \(\hat{Q}_k\) is given by

\[
p(\hat{Q}_k) = \tilde{p}(\hat{E}_k) = \int P(\hat{E}_k|X_A, X_B) \delta(X_A - \psi_A) \rho_B(X_B) dX_A dX_B,
\]

(23)

where the probability distribution \(\rho_B\) satisfies the relation

\[
\hat{\rho}_B = \int |X\rangle\langle X| \rho_B(X) dX.
\]

(24)

Note that this equation, because of the preparation contextuality for mixed states \(6\), is not necessarily invertible and many probability distributions can correspond to the same density operator \(\hat{\rho}_B\). For example, the quantum state \(\frac{1}{\sqrt{2}} |\uparrow\rangle + |\downarrow\rangle\langle\downarrow|\) is associated with any probability distribution of the form \(\frac{1}{2} \delta(X_B - \psi_1) + \frac{1}{2} \delta(X_B - \psi_2)\), where \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are two generic orthogonal states. Integrating in \(X_B\), Eq. \(23\) becomes by means of Eqs. \(17, 24\)

\[
p(\hat{Q}_k) = \int \text{Tr}||X_A\rangle\langle X_A|\hat{\rho}_B \hat{E}_k|\delta(X_A - \psi_A) dX_A.
\]

(25)

Tracing away the system \(B\) and using Eq. \(21\), we obtain that the probability of \(\hat{Q}_k\) is given by

\[
p(\hat{Q}_k) = \int P(\hat{Q}_k|X_A) \delta(X_A - \psi_A) dX_A,
\]

(26)

where

\[
P(\hat{Q}_k|X) = \text{Tr}_A(\hat{Q}_k|X\rangle\langle X|),
\]

(27)

and the lemma is proved. \(\Box\)

Thus, at the ontological level the conditional probability of an event depends only on the associated positive operator. This labelling does not have memory of the particular projectors \(\hat{E}_k\) and ancilla state \(\rho_B(X)\) that were used. Thus, the Beltrametti-Bugajski model is POVM non-contextual. The proved theorem and lemma imply the following.

**Corollary:** The projective measurement non-contextuality implies POVM non-contextuality for Hilbert dimensions greater than 2.

If we require the POVM non-contextuality for a qubit as an additional hypothesis, it is easy to show, by using the generalized Gleason theorem in Ref. \(15\), that the statement of the theorem proved in the previous section holds in any dimension. Indeed the models by Kochen and Specker, Bell and the recent one reported in Refs. \(13, 14\) are all contextual for unsharp measurements. Apart from the Bell model, this is also a direct consequence of the fact that non-contextuality for POVM implies \(\psi\)-onticity, as proved in Ref. \(11\). Indeed neither the Kochen-Specker model or that in Refs. \(13, 14\) are \(\psi\)-ontic.

IV. EXTENDED QUANTUM THEORY AND MACROSCOPIC REALISM

A. Extended quantum theory

In the Copenhagen interpretation, the quantum state is not anything more than a mathematical tool for evaluating probabilities. It merely represents the information about the preparation procedure of systems. Conversely, an ontological interpretation is designed to provide a realistic description of systems through well-defined classical variables. In this section we take a step back and try to compromise between the two interpretations. The technical results do not differ from those in Sec. III but their interpretation is different and makes our results more closely related to recent findings \(8\).

Instead of replacing the quantum state with ontic states, we can preserve it as a mathematical tool for de-
scribing preparation procedures and enrich the description by some additional information, provided by classical variables. It is important to stress that \( |\psi\rangle \) is not supposed to describe some real entity, such as in de Broglie-Bohm mechanics and Beltrametti-Bugajs\( \kappa \)ki model. In this framework, an ensemble of systems, identically prepared in a pure state \( |\psi\rangle \) within the context \( \eta \), is described by the preparation protocol \( \{ |\psi\rangle, \eta \} \) and a probability distribution, \( \rho(Y|\psi, \eta) \), of a classical variable \( Y \), that is,

\[
\text{preparation} \rightarrow \{ |\psi\rangle, \eta, \rho(Y|\psi, \eta) \}. \tag{28}
\]

The probability of an outcome \( \hat{E}_k \) is conditioned by the classical variable \( Y \) and depends also on the overall preparation procedure \( \{ |\psi\rangle, \eta \} \). Additionally, for the moment we assume that it also depends on the measurement context, specified by a parameter, \( \tau \). We indicate this conditional probability with \( P(\hat{E}_k|Y, \tau, \psi, \eta) \). Thus,

\[
\text{event probability} \rightarrow P(\hat{E}_k|Y, \tau, \psi, \eta). \tag{29}
\]

The model is equivalent to quantum theory if equation

\[
\int dY P(\hat{E}_k|Y, \tau, \psi, \eta) \rho(Y|\psi, \eta) = \langle \psi|\hat{E}_k|\psi \rangle \tag{30}
\]

is satisfied. The overall statements (28-30) define an extended quantum theory, where quantum information is supplemented by classical information. The extension is said \textit{non-trivial} if the conditional probability \( P(\hat{E}_k|Y, \tau, \psi, \eta) \) is not constant on the support of \( \rho(Y|\psi, \eta) \). It provides a fully ontological description if \( P(\hat{E}_k|Y, \tau, \psi, \eta) \) does not depend on \( \{ \psi, \eta \} \). The Beltrametti-Bugajs\( \kappa \)ki model is an example of trivial extension where the classical variables replace completely \( \psi \). In practice, \( \psi \) is reinterpreted as a physical field, but no further information is introduced that provides a more accurate predictions of outcomes. Just as quantum theory is formally identical to the Beltrametti-Bugajs\( \kappa \)ki model and differs only in the interpretation, any trivial extension of quantum theory is formally identical to the Beltrametti-Bugajs\( \kappa \)ki model. In general, an extended quantum theory is formally identical to a \( \psi \)-ontic theory where \( \{ \psi, Y \} \) are the ontological variables, with the only difference that the quantum state \( |\psi\rangle \) is interpreted as a container of information about the preparation procedure. In other words, an extended theory is a mixture of an operational and ontological description.

The Kochen-Specker theorem establishes that there is no deterministic ontological theory that is non-contextual for measurements. However, this result does not rule out the possibility of supplementing the quantum state with some amount of classical information and simultaneously preserving the minimal labelling of events through projective operators (without \( \tau \)). So in principle, we could have a non-trivial extended quantum theory defined by statements (28-30) and satisfying the condition

\[
P(\hat{E}_k|Y, \tau, \psi, \eta) = P(\hat{E}_k|Y, \psi, \eta). \tag{31}
\]

With a slight abuse of terminology, we call this condition \textit{non-contextuality} for measurements. This expanded definition is justified by the following.

**Lemma 2.** If there is a (non-trivial) extended quantum theory that is non-contextual for measurements, then there is a (non-trivial) ontological theory with the same property. The converse is also true.

By non-trivial ontological theory we mean a theory that is essentially different from the Beltrametti-Bugajs\( \kappa \)ki model. The proof of this lemma is very simple. Indeed, every extended theory generates an ontological theory with \( \langle \psi, Y \rangle \) (and possibly the preparation context \( \eta \)) as ontological variables. Furthermore, if the extended theory is non-trivial, then the generated ontological theory is non-trivial. It is obvious that the property of non-contextuality is preserved in this change of interpretation. The converse is also true, since a (non-trivial) ontological theory is a (non-trivial) extended quantum theory where the conditional probabilities for events do not depend on \( \psi \).

By using the results in Sec. II it is easy to prove the following.

**Theorem 2.** Quantum theory cannot be extended in a non-trivial way by preserving the minimal labelling for events of quantum mechanics. In other words, there is not a non-trivial extended quantum theory that is non-contextual for measurements.

In spite of the different interpretation, this theorem is a rephrasing of the theorem 1 proved in Sec. II. Indeed, as previously said, an extended theory is formally identical to an ontological theory, where \( \{ \psi, Y \} \) are the ontological variables. We have proved in Sec. II that a such theory is essentially equivalent to the Beltrametti-Bugajs\( \kappa \)ki model under the hypothesis of non-contextuality for measurements, that is, \( Y \) does not introduce any improvement in the prediction of outcomes. Indeed, as stated by lemma 2, if there was a non-contextual and non-trivial extended theory, then there would be a non-contextual ontological theory essentially different from the Beltrametti-Bugajs\( \kappa \)ki model.

A similar theorem was proved in Ref. [8], where the authors used a hypothesis of statistical independence between the setting parameters of a measurement and the variables that do not lie in the future light cone of those parameters (\textit{free choice hypothesis}). By means of this hypothesis they derived a relation similar to Eq. (31), which directly implies that quantum mechanics cannot be extended. This was initially accomplished in the case of maximally entangled states and subsequently generalized by means of the hypothesis (called QMb) that every quantum process is unitary. Our proof has the advantage of simplicity, resting upon the well-established and powerful Gleason theorem [9]. Furthermore it does not need any additional hypothesis on the dynamics, such as QMb.
B. Macroscopic realism

It is interesting to note that macroscopic realism requires that some amount of classical information has to be supplied to the quantum state, provided that the quantum alternatives have reached some level of "macroscopicness". In order to be complete, a theory should contain the description of this classical information in its formalism. Suppose for example that a microscopic quantum system, $A$, is in the superposition $|1\rangle + |−1\rangle$ and interacts with a macroscopic device, $D$, which is initially in the state $|\varphi_0\rangle$. After the interaction the overall quantum system evolves towards the entangled state

$$|\Psi\rangle = |1\rangle|\varphi_1\rangle + |−1\rangle|\varphi_{−1}\rangle,$$

where $|\varphi_{±1}\rangle$ are two macroscopically distinct states, for example corresponding to different positions of a pointer. The state $|\Psi\rangle$ contains the information about the initial preparation of the system $A+D$ and its subsequent evolution. Macroscopic realism imposes that this information has to be supplemented with some classical information indicating the actual macroscopic state of the device. The classical information can be stored in a binary variable, $n = ±1$, where $±1$ correspond to the macroscopic states $|\varphi_{±1}\rangle$. In this extended description, the overall information on the system $A+D$ is given by the pair

$$(|\Psi\rangle, n)$$

and, additionally, other parameters describing the context. In general, the outcome of every external observation performed on $A+D$ has a probability that depends on both $|\Psi\rangle$ and $n$. Indeed, the probability of finding the device in the state $|\varphi_{±}\rangle$ has to be equal to 0 or 1, provided that $n = ±1$ or $n = ±1$. In other words, the outcome is completely determined for the measurement of the "pointer" state and the extension cannot be trivial. Thus, macroscopic realism imposes that quantum theory has to be extended in a non-trivial way with some amount of classical information and this cannot be accomplished without giving up non-contextuality for measurement. In this inference we have implicitly used the additional hypothesis of unitarity for evolutions, as discussed in the following paragraph.

It is worthwhile to note that in practise it is not possible to perform every kind of measurements on a macroscopic system and the events that can be actually observed are not in general affected by the replacement of the superposition of $|1\rangle|\varphi_1\rangle$ and $|−1\rangle|\varphi_{−1}\rangle$ with their mixture. This property is called decoherence and provides a justification to the quantum state reduction, which is one of the postulates of quantum theory. It implies a loss of information, for which the states $(|\Psi\rangle, ±1)$ are indistinguishable from the states $(|±1\rangle|\varphi_{±1}\rangle, ±1)$. The reduction postulate is particularly relevant in this context because the quantum state collapse into macroscopic distinct states would make unnecessary the addition of classical information and would remove the contradiction between non-contextuality for measurements and macroscopic realism. In our opinion, the quantum state collapse is not more fundamental than the loss of information in statistical mechanics and in fact the detailed full information is provided by the state $(|\Psi\rangle, ±1)$. Indeed, in principle nothing forbids one to perform a measurement that is able to distinguish a superposition of states from their mixture. This is particularly true if the device is mesoscopic. Promoting the decoherence to the rank of fundamental principle would raise the issue of deciding what is the level of "macroscopicness" and decoherence above which a superposition is replaced by a mixture.

We conclude this section by noting that an argument that uses decoherence as a fundamental principle for explaining macroscopic realism implicitly seems to require a weakening of causality principle. Indeed, according to this explanation, one system is in a defined macroscopic system because never in the future there will be an emerging property revealing interference between macroscopically distinct states. This point is made more explicit in the consistent histories approach to quantum mechanics [16], where the condition for consistency involves the whole temporal history. The weakening of causality principle was also suggested in recent papers [13, 14, 17] as a solution of the problem of the exponential growth of resources that are required for specifying an ontic state. However, the challenge of undermining the causality principle is beyond the purpose of this paper.

V. CONCLUSIONS

We have proved that the Beltrametti-Bugajski model is in practise the only model that is non-contextual for projective measurements and Hilbert space dimension $N$ greater than 2. Equivalently, it is not possible to extend non-trivially quantum theory without giving up non-contextuality for measurements. As a corollary, we have shown that POVM non-contextuality for $N > 2$ is implied by sharp measurement non-contextuality. The theorem can be generalized to a qubit by employing the additional hypothesis of POVM non-contextuality for $N = 2$.

The Beltrametti-Bugajski model is the simplest example of ontological theory of a measurement process. However it cannot be considered a completely realistic theory. In particular, since it still needs to invoke a measurement made by something external to the system, in fact does not solve the measurement problem without retaining the postulate of wave-function collapse required in the standard interpretation. An exhaustive realistic model should at least satisfy a criterion of macroscopic realism, attributing for example a sufficiently well defined value to the position of macroscopic objects that does not depend on a possible external observation. This is the case of the de Broglie-Bohm mechanics, where the wave-function is supplied by additional variables describing the positions of all the particles compounding a system. The Beltrametti-Bugajski model fails to sat-
satisfy the criterion of macroscopic realism. Suppose for example that the position of an object is in the superposition of two macroscopically separated values. The Beltrametti-Bugajski model, being in fact a rephrasing of the quantum mechanics, does not give any description of the actual position of that object, unless a breaking of unitarity is invoked through the quantum state collapse. Indeed, any non-trivial extension of the theory acted to introduce macroscopic realism without breaking unitarity would automatically make the theory measurement contextual. Thus, the only way to solve the measurement problem in the framework of an ontological theory is relaxing the hypothesis of measurement non-contextuality in the generalized sense introduced in Ref. [6].

We conclude noting that macroscopic realism implies a certain degree of outcome determinism, as discussed in Sec. IV. Given a macroscopic system, there should be always a set of macroscopic states such that the system is in one of them. The result of a possible measurement, acted to know that state, would be completely determined by the hidden variable state of the system. It is interesting to observe that the criterion of macroscopic realism is rarely taken in consideration in the study of ontological models. We have shown that it imposes some constraints and is very useful to deduce general properties that an ontological theory must satisfy, such as measurement contextuality.

Acknowledgments

A. M. acknowledges useful discussions with Robert W. Spekkens, Brian Morris and Roger Colbeck. This work was partially supported by the National Natural Science Foundation of China under Grant No.10775175. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI.

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