An implementation of an efficient direct Fourier transform of polygonal areas and volumes

Brian B. Maranville
NIST Center for Neutron Research
100 Bureau Drive, Gaithersburg MD 20899 USA

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Calculations of the Fourier transform of a constant quantity over an area or volume defined by polygons (connected vertices) are often useful in modeling wave scattering, or in fourier-space filtering of real-space vector-based volumes and area projections. If the system is discretized onto a regular array, Fast Fourier techniques can speed up the resulting calculations but if high spatial resolution is required the initial step of discretization can limit performance; at other times the discretized methods result in unacceptable artifacts in the resulting transform. An alternative approach is to calculate the full Fourier integral transform of a polygonal area as a sum over the vertices, which has previously been derived in the literature using the divergence theorem to reduce the problem from a 3-dimensional to line integrals over the perimeter of the polygon surface elements, and converted to a sum over the straight segments of that contour. We demonstrate a software implementation of this algorithm and show that it can provide accurate approximations of the Fourier transform of real shapes with faster convergence than a block-based (voxel) discretization.

Keywords: Fourier transform, polygon, scattering

I. INTRODUCTION

In calculations of scattering of neutrons, electrons, x-rays, etc. in the weak limit (the first Born approximation) it is useful to be able to quickly calculate the Fourier transform (FT) of a constant quantity such as the scattering power of a material over a well-defined volume.

The two most frequently used approaches to this problem are to use a combination of known analytical transforms of shapes to create models or to discretize the volume on a regular, rectangular grid. Both these strategies offer great speed in the calculation, as the analytic solution is typically a small number of functions added together, while the gridded volume approach can take advantage of the inherent efficiency of Fast Fourier (FFT) methods.

There are however cases where it is not practical to construct a model from simple shapes (the union of spheres example in this paper); at other times the discretization itself onto a rectangular grid is time-consuming or error-prone. If a shape of interest has surfaces that are either very curved or slightly misaligned with the coordinate axes, the gridded representation will contain large flat surface segments along the one of the underlying coordinates, resulting in artifacts in the FT that can dominate the calculation.

It was first shown by Laue and later expanded by others that using the divergence theorem, it is possible to directly calculate the Fourier transform of a volume element defined by a polygonal surface mesh. Now this type of surface parametrization is regularly used in 3-dimensional graphics (openGL) as well as in finite-element modeling programs such as the micromagnetic modeling program NMAG, the surface-meshing program netgen, and other pde solvers.

II. DIVERGENCE THEOREM METHOD IN 2 DIMENSIONS

As was shown earlier in, one can approach the Fourier integral of polygonal volumes by beginning with the divergence theorem in 2 dimensions:

\[
\int_A (\nabla \cdot \mathbf{F}) \, dA = \oint_C \mathbf{F} \cdot \hat{n} \, ds
\]

where \( \mathbf{F} \) is an arbitrary function of \( (x, y) \), \( C \) is the perimeter of the area (traversed counter-clockwise,) and \( \hat{n} \) is the unit vector normal to that perimeter and pointing outward from the defined area. This can be used to evaluate the FT \( A \) integral by constructing \( \mathbf{F} \) as follows:

\[
\mathbf{F} = \left( \begin{array}{c} e^{iQ_x x + iQ_y y} \\ e^{iQ_x x + iQ_y y} \end{array} \right)
\]

One can verify that \( \nabla \cdot \mathbf{F} = e^{i(Q_x x + Q_y y)} \), which is the original integrand. Then for a perimeter line segment \( S \) (again, traversing \( C \) counter-clockwise) with start point \( \mathbf{p}_0 \equiv (x_0, y_0) \) and end point \( \mathbf{p}_1 \equiv (x_1, y_1) \), the normal is

\[
\hat{n}_S = \left( \frac{(y_1 - y_0)/d}{(x_1 - x_0)/d} \right)
\]

where \( d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \) is the length of the segment. The RHS of Eq. for that segment becomes
\[ \int_{r_i}^{r_{i+1}} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = -\left( \frac{\mathbf{Q} \cdot \hat{\mathbf{n}}}{Q^2} \right) \left[ e^{iQ \cdot \mathbf{r}} \right] \left( \frac{\mathbf{Q} \cdot \mathbf{r}_{i+1} - e^{iQ \cdot \mathbf{r}_i}}{Q^2} \right) \]

where \( \Delta_x = x_1 - x_0 \), \( \Delta_y = y_1 - y_0 \). We can rewrite Eq. 5 more generically for any segment

\[ \int_{r_i}^{r_{i+1}} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = -\left( \frac{\mathbf{Q} \cdot \hat{\mathbf{n}}}{Q^2} \right) \left( e^{iQ \cdot \mathbf{r}_{i+1}} - e^{iQ \cdot \mathbf{r}_i} \right) \]

Where \( \mathbf{d}_i = \mathbf{r}_{i+1} - \mathbf{r}_i \). Factoring out \( e^{iQ \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)} \) and noting that \( \hat{\mathbf{n}} \) is defined to be perpendicular to \( \mathbf{d}_i \), one gets

\[ \int_{r_i}^{r_{i+1}} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = -\frac{Q \cdot \hat{\mathbf{n}}}{Q^2} \left( e^{iQ \cdot \mathbf{r}_i} \cdot \sin \left( \frac{Q \cdot \mathbf{d}_i}{Q^2} \right) \right) \]

where \( \theta \) is the angle between \( \mathbf{Q} \) and \( \mathbf{d}_i \).

It is interesting to note that for a closed polygon the exponential \( e^{iQ \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)} \) will show up exactly twice in the sum, once as the \( (x_0, y_0) \) term and once as the \( (x_1, y_1) \) term. Thus the sum can be recast as a sum over all the vertices instead of all the segments, with a weighting that depends on the difference of the tangent of the angle between \( \mathbf{Q} \) and the segment for the two segments meeting at that point.

The form of Eq. 5 is preferred though, because in the limit of \( \mathbf{Q} \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i) \rightarrow 0 \) the quantity in parentheses at the right will not diverge.

### III. DIVERGENCE THEOREM METHOD IN 3 DIMENSIONS

Using the same logic to extend to three dimensions, the integral becomes

\[ \int V \mathbf{F} \cdot d\mathbf{V} = \int_{A_m} \mathbf{F} \cdot \hat{\mathbf{n}}_m \, dA \]

over a volume \( V \) with a polygonal surface (made up of connected planar polygons) such as a tetrahedron or cube. The three-dimensional divergence theorem can be stated as

\[ \int V (\nabla \cdot \mathbf{F}) \, dV = \int_{S} \mathbf{F} \cdot \hat{\mathbf{n}}_S \, dS \]

* Correspondence email address: brian.maranville@nist.gov

Because there are a finite number of planar polygons that make up the surface, the integral can be decomposed to a sum of integrals over those regions.

\[ \int_{S} \mathbf{F} \cdot \hat{\mathbf{n}}_S \, dS = \sum_m \int_{A_m} \mathbf{F} \cdot \hat{\mathbf{n}}_m \, dA \]

Where \( m \) is the number of polygons on the surface, and \( \hat{\mathbf{n}}_m \) is the surface normal for area \( A_m \).

Again we define

\[ \mathbf{F} = -iQ e^{iQ \cdot \mathbf{r}} \]

and again, \( \nabla \cdot \mathbf{F} = e^{iQ \cdot \mathbf{r}} \), the original integrand. For a given surface polygon,

\[ \int_A \mathbf{F} \cdot \hat{\mathbf{n}} \, dA = \int_{A_{\perp \hat{n}}} \left( \frac{-iQ}{Q^2} \right) e^{iQ \cdot \mathbf{r}} \, dA \]

Where \( Q_{\perp} \) is the component of \( \mathbf{Q} \) along \( \hat{\mathbf{n}} \). Since the area being integrated over is by definition perpendicular to the surface normal, we can pull out that component of the integrand as a constant

\[ \int_A \mathbf{F} \cdot \hat{\mathbf{n}} \, dA = \left( \frac{-iQ_{\perp}}{Q^2} \right) e^{iQ_{\perp} \cdot \mathbf{r}} \int_{A_{\perp \hat{n}}} dA \]

where \( Q_{\perp}, r_{\perp} \) are the components perpendicular to \( \hat{\mathbf{n}} \) (in the integration plane.)

Now the volume integral is reduced to a sum of area integrals of the same type as was treated in the previous section. Inserting Eq. 5 into Eq. 9, we get an expression like the one found in Eq. 10.80 of [4]:

\[ \int_A \mathbf{F} \cdot \hat{\mathbf{n}}_m \, dA = \int_{A_{\perp \hat{n}}} \left( \frac{-iQ_{\perp}}{Q^2} \right) e^{iQ_{\perp} \cdot \mathbf{r}} \, dA \]
\[ \text{FT}_V = \sum_m I_m = \sum_m \frac{-iQ_m}{Q^2} e^{iQ_m r_i} \sum_{t=0}^{\nu_m} \frac{-Q_{\perp m} \cdot (\hat{n}_m \times (r_{t+1} - r_t))}{Q_{\perp m}^2} \left( \frac{e^{iQ_{\perp m} r_{t+1}} - e^{iQ_{\perp m} r_t}}{Q_{\perp m} \cdot (r_{t+1} - r_t)} \right) \] (14)

where the last sum is a counterclockwise sum over vertices \( r_i \) in the contour \( C_m \) defining area element \( A_m \), and the notation for counting vertices along \( C_m \) is understood to wrap around, so that if there are \( \nu_m \) vertices defining \( C_m \) then \( r_{\nu_m+1} \equiv r_0 \). Also, \( ||m, \perp m|| \) refer to components parallel and perpendicular to the the surface normal \( \hat{n}_m \) of area element \( A_m \).

IV. APPLICATION TO REAL GEOMETRIES

A. Transform of a rectangle

As an example of a 2-d system, let’s take a rectangle of length \( a \) along the \( x \) axis and \( b \) along the \( y \) axis, as seen in Fig. 1 The definite integral in this case is separable in \( x \) and \( y \), and we get

\[ \iiint_{\text{Rectangle}} e^{iQ \cdot r} \, dA = \int_0^a e^{iQ_y x} \, dx \int_0^b e^{iQ_x y} \, dy \] (15)

If we travel counterclockwise around the rectangle we get these vertices

\[ r_0 = (0, 0) \]
\[ r_1 = (a, 0) \]
\[ r_2 = (a, b) \]
\[ r_3 = (0, b) \] (16)

and putting them into Eq. 14

\[ \oint_C \mathbf{F} \cdot \hat{n} \, ds = \frac{1}{Q^2} \frac{Q_x a}{Q_x Q_y} \left[ e^{iQ_x a} - e^{i00} \right] \\
+ \frac{1}{Q^2} \frac{Q_y b}{Q_x Q_y} \left[ e^{iQ_y b} - e^{iQ_x a} \right] \\
+ \frac{1}{Q^2} Q_x a \left[ e^{iQ_y b} - e^{i(Q_x a + Q_y b)} \right] \\
+ \frac{1}{Q^2} Q_y b \left[ e^{i00} - e^{iQ_x a} \right] \] (17)

and a bit of algebra yields

\[ \text{FT}_{\text{Rect}} = \frac{1}{Q^2} \frac{Q_x^2 + Q_y^2}{Q_x Q_y} \left[ e^{iQ_x b} + e^{iQ_y a} - e^{i(Q_x a + Q_y b)} - 1 \right] \] (18)

which is equivalent to Eq. 15

B. Rectangular prism

Now extending to the third dimension, we take a rectangular prism of length \( a \) along the \( x \) axis, \( b \) along the \( y \) axis and \( c \) along the \( z \) axis. The definite integral is then

\[ \iiint_{\text{Prism}} e^{iQ \cdot r} \, dV = \frac{i(Q_y a - 1)(e^{iQ_y b} - 1)(e^{iQ_x c} - 1)}{Q_x Q_y Q_z} \] (19)

Breaking it up into the 6 surfaces that compose the boundary of this region, we get from Eq. 19 the following for the top surface \( (z = c) \):

\[ I_{z=c} = \frac{iQ_x e^{iQ_x c}}{Q^2} \left( e^{iQ_x a} - 1 \right)(e^{iQ_y b} - 1) \] (20)

while for the bottom surface \( (z = 0) \) we get

\[ I_{z=0} = \frac{-iQ_x (e^{iQ_x a} - 1)(e^{iQ_y b} - 1)}{Q^2} \] (21)

Adding those two pieces (and multiplying by \( Q_z \) on the top and bottom) we get

\[ I_{z=c} + I_{z=0} = \frac{iQ_x}{Q^2} \left( e^{iQ_x a} - 1 \right)(e^{iQ_y b} - 1)(e^{iQ_x c} - 1) \] (22)

and clearly the sum of all six interfaces will give the same answer as in Eq. 19 above.

C. Sphere

The Fourier transform of a sphere is known analytically as well, so we can compare the discretization methods for
Figure 2: Voxelization of the unit sphere, with voxel dimension = 0.2. This results in 522 non-zero voxels.

Figure 3: Surface meshing of the unit sphere. The mesh consists of 120 triangles connecting 83 nodes.

Figure 4: FT$_V$ of the unit sphere (along $Q_x$) as calculated analytically (solid green line), with the grid voxelization (blue plus symbols) and with the polygon approximation to the surface (red open circles).

The absolute value of FT for this polygonal shell in the $Q_x, Q_y$ plane is plotted as a heatmap in Fig. 6 with the color scale on the right.

V. CONCLUSIONS

We have shown that by using a direct calculation of the Fourier transform of a constant function over a volume defined by a connected-polygon surface, a reasonable approximation to the FT of a sphere can be achieved from a discretization to just 120 surface triangles. This is compared to a approximation of the same shape using a much larger number of regularly-gridded volume elements (voxels). The same technique is shown to be straightforward to apply to complicated shapes that are hard to calculate the analytic FT for directly, such as the

\[ FT_V(Q) = \frac{4\pi}{3} \Gamma(5/2) \frac{J_{3/2}(Q)}{(Q/2)^{3/2}} \text{ (sphere) } \] (23)

where $\Gamma$ is the gamma function, and $J_{3/2}$ is a half-integer Bessel function of the first kind.

Also, the sphere was voxelized using the open-source program \textit{binvox} and the results can be seen in Fig. 2.

Finally, a surface mesh as rendered by the open-source meshing program \textit{netgen} is seen in Fig. 3. The meshing input radius was adjusted so that the volume of the resulting polygon matches that of the unit sphere.

The calculation of FT from each of these methods is presented in Fig. 4 calculated along the $Q_x$-axis (the artifacts from gridding are likely to be most apparent along

D. Union of spheres

Consider a volume defined by the union of two spheres with radius $r = 1.0$ and centers at $x = -0.6$ and $x = 0.6$. This is an example of a volume for which it is difficult to calculate the FT analytically, as it would require subtracting the FT of the lens-shaped overlap from the sum of the two sphere FT functions. A surface mesh of this geometry can be seen in Fig. 5 (again rendered by \textit{netgen}).

The absolute value of FT for this polygonal shell in the $Q_x, Q_y$ plane is plotted as a heatmap in Fig. 6 with the color scale on the right.
Figure 5: Surface meshing of the volume created by the union of two spheres. The mesh consists of 160 triangles connecting 83 nodes. No refinement of the mesh was performed beyond the initial meshing.

Figure 6: $Q_x, Q_y$ projection of the absolute value of the Fourier transform of the union of two spheres. (Calculated at $Q_z = 0$)

overlapping spheres. Artifacts from the conversion to a rectilinear basis are avoided, and this should provide a more straightforward method of calculating the FT from volumes where the surface mesh is already known, such as can be extracted from tomography, microscopy or other real-space probes, without requiring the additional (computationally intensive) process of voxelization.

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[1] S. R. Kline, Journal of Applied Crystallography 39, 895 (2006)
[2] "Sans model function documentation," (2009).
[3] M. v. Laue, Annalen der Physik 418, 55 (1936)
[4] R. W. James, The Optical Principles of the Diffraction of X-rays (G. Bell and Sons, Ltd., London, 1967) pp. 548–555.
[5] T. Fischbacher, M. Franchin, G. Bordignon, and H. Fangohr, Magnetics, IEEE Transactions on 43, 2896 (2007)
[6] J. Schöberl, Comput. Visual Sci. 1, 41 (1997)
[7] J. Schöberl, "Netgen mesh generator," (2014).
[8] M. Kazhdan, in Proceedings of the third Eurographics symposium on Geometry processing (Eurographics Association, 2005) p. 73.
[9] F. S. Nooruddin and G. Turk, IEEE Transactions on Visualization and Computer Graphics 9, 191 (2003)
[10] P. Min, "[binvox] 3d mesh voxelizer," (2014).
Appendix A: Support files and code

1. Geometry definition for union of two spheres

Using the geometry file format for *netgen*:

```plaintext
algebraic3d

solid main = sphere (-0.6, 0, 0; 1);
solid second = sphere (0.6, 0, 0; 1);

solid combined = main or second;
tlo combined;
```

2. Python code for calculations

This includes a function for reading in a surface mesh file that can be written from *netgen*, as well as a demo of the main function for calculating the Fourier transform of the example in section [IV D]

```python
import numpy as np
eps = 1e-30

def read_surface_meshfile(fileobj):
    f = fileobj
    if f.readline().strip() != "surfacemesh":
        print("not a surface mesh file")
    numpoints = int(f.readline().strip())
    points = np.empty((numpoints, 3), dtype="float")
    for i in range(numpoints):
        points[i] = np.array(f.readline().strip().split(), dtype="float")
    numelements = int(f.readline().strip())
    elements = np.empty((numelements, 3), dtype="int")
    for j in range(numelements):
        elements[j] = np.array(f.readline().strip().split(), dtype="int")
        # indexing of points in the geometry file begins with 1,
        # but it begins with 0 in numpy ndarray, so need to subtract 1
        # from every point in element index array:
        return {'points': points, 'elements': elements - 1}

def resolve_coords(points, elements, wrap=True):
    # convert elements with point ids to lists of coords
    # if wrap is True, add point[0] to the end of each list
    el = elements.copy()
    if wrap:
        el = np.concatenate((el, el[:,:1]), axis=1)
    return (points[el]).copy()

def get_normal_vec(p):
    v1 = p[:,1] - p[:,0]
    v2 = p[:,2] - p[:,0]
    normal = np.cross(v1, v2)
    normal = normal * 1.0 / np.sqrt(np.sum(normal*normal, axis=1))[:,None]
    return normal

def fourier_vec(qx, qy, qz, p):
    """ calculate the fourier transform of the volume bounded by the elements
    in the list p""
    # 
    # ******************************************
```

---

Note: The code provided is a snippet from a larger source, and the full code is not fully transcribed here due to its size and complexity. The code snippets have been formatted to maintain readability and preserve the structure of the original text.
# Inputs:
# qx, qy and qz should have 3 dimensions each but they can be sparse, 
# i.e. qx.shape can be (4,1,1) when qy.shape is (1, 12, 1) etc. 
# in which case they will be broadcast.

# p is an array of elements, which are themselves an array of points 
# describing a counterclockwise trip around the border of the element 
# (as seen from the outside of the volume) where a point is the array 
# [x,y,z]

##########################################################################
# Outputs: 
# result is an array covering all qx, qy and qz of the fourier transform 
# of the volume enclosed by the elements in p 
##########################################################################

normal = get_normal_vec(p)
# now do dot product
dotx = qx[:,:,:,None] * normal[:,0] 
doty = qy[:,:,:,None] * normal[:,1] 
dotz = qz[:,:,:,None] * normal[:,2] 
Qn_length = dotx + doty + dotz
Qnx = Qn_length * normal[None, None, None, :,0] 
Qny = Qn_length * normal[None, None, None, :,1] 
Qnz = Qn_length * normal[None, None, None, :,2] 
Qpx = qx[:,:,:,None] - Qnx 
Qpy = qy[:,:,:,None] - Qny 
Qpz = qz[:,:,:,None] - Qnz 
Qp = np.concatenate((Qpx[...,None], Qpy[...,None], Qpz[...,None]), axis=-1) 

Qsq = qx**2 + qy**2 + qz**2 
Qpsq = np.sum(Qp * Qp, axis=-1)

## Note: p[:,0] is the first point in the element (for all elements)
rn_length = np.sum(normal * p[:,0], axis=1)
subsum = np.zeros_like(Qn_length, dtype="complex")

## Here is equation 17 from the attached publication for a single element 
# (including the for loop which sums over the vertices)
result = (1j * Qn_length / Qsq[:,:,:,None]) 
    * np.exp(1j * Qn_length * rn_length[None, None, None, :]) 
        * np.exp(1j * np.sum(Qp*p[None, None, None,:,i+1],axis=-1)-eps/2.0) 
        * np.exp(1j*np.sum(Qp*p[None, None, None,:,i], axis=-1)-eps/2.0))
    - np.exp(1j*np.sum(Qp*p[None, None, None,:,i], axis=-1)-eps/2.0)) 
    / (np.sum(Qp*(p[None, None, None, :, i+1] - p[None, None, None, :, i]), axis=-1) + 1j * eps)
subsum += (sub1 * sub2 * sub3)

return result * subsum

def demo():
    from pylab import figure, xlabel, ylabel, title, imshow, colorbar, show
    import StringIO

    qx = np.linspace(-10, 10, 40) + eps
qy = np.linspace(-10, 10, 40) + eps
qz = np.linspace(0, 0, 1) + eps
qx, qy, qz = np.meshgrid(qx, qy, qz, sparse=True)
extent = (qx.min(), qx.max(), qy.min(), qy.max())

surf = read_surface_meshfile(StringIO.StringIO(twospheres_surfacemesh))
pp = resolve_coords(surf['points'], surf['elements'])
ft = fourier_vec(qx, qy, qz, pp)
fts = np.sum(ft[:, :, 0], axis=-1)
fig1 = figure()

imshow(abs(fts), extent=extent, aspect=1)
title('Abs(FT)')
xlabel('Q_x', size='large')
ylabel('Q_y', size='large')
colorbar()

fig2 = figure()

imshow(fts.imag, extent=extent, aspect=1)
title('Imaginary FT')
xlabel('Q_x', size='large')
ylabel('Q_y', size='large')
colorbar()
show()
|       |       |       |       |
|-------|-------|-------|-------|
| -0.80336 | -0.978812 | -0.0239454 |
| -1.19796 | -0.0347093 | -0.800775 |
| -1.18082 | -0.758163 | -0.296374 |
| -1.23593 | 0.456043 | -0.622585 |
| -1.20125 | 0.760906 | -0.243974 |
| -1.276 | 0.676061 | 0.293208 |
| -1.22172 | 0.230192 | 0.748648 |
| -1.10981 | -0.519442 | -0.685767 |
| -1.1967 | -0.780973 | 0.184463 |
| -1.47888 | 0.0876796 | -0.468915 |
| -1.38991 | -0.362765 | 0.494419 |
| -1.50034 | 0.409012 | -0.148656 |
| -1.52632 | 0.184822 | 0.328273 |
| -1.43381 | -0.350022 | -0.426898 |
| -1.45293 | -0.521864 | -0.0129389 |
| -1.59324 | -0.11427 | -0.0324898 |
| 0.353411 | -0.922424 | -0.2972 |
| 0.35295 | -0.576474 | -0.778873 |
| 0.359646 | -0.00259245 | -0.970682 |
| 0.365788 | 0.577679 | -0.781941 |
| 0.354885 | 0.929085 | -0.276983 |
| 0.350682 | 0.915547 | 0.315627 |
| 0.356403 | 0.564491 | 0.788676 |
| 0.360185 | -0.0185137 | 0.970642 |
| 0.354104 | -0.571601 | 0.782821 |
| 0.355115 | -0.92195 | 0.300067 |
| 0.778156 | -0.806435 | -0.563846 |
| 0.777153 | -0.329327 | -0.927448 |
| 0.807472 | 0.308424 | -0.928348 |
| 0.815371 | 0.808817 | -0.547202 |
| 0.775977 | 0.93647 | 0.0383495 |
| 0.772101 | 0.779013 | 0.602926 |
| 0.781753 | 0.290957 | 0.939314 |
| 0.789647 | -0.311115 | 0.931258 |
| 0.779065 | -0.781899 | 0.597134 |
| 0.783485 | -0.982953 | -0.0116784 |
| 1.13571 | -0.519537 | -0.665658 |
| 1.21105 | -0.0477391 | -0.790149 |
| 1.21305 | 0.486838 | -0.62222 |
| 1.19736 | 0.784525 | -0.166363 |
| 1.13036 | 0.790032 | 0.307512 |
| 1.19753 | 0.460595 | 0.656358 |
| 1.15168 | -0.0067317 | 0.834029 |
| 1.23508 | -0.439877 | 0.63497 |
| 1.2143 | -0.753463 | -0.23437 |
| 1.18111 | -0.771965 | 0.257649 |
| 1.47664 | -0.31298 | -0.365446 |
| 1.4629 | 0.448967 | -0.232032 |
| 1.44925 | 0.479567 | 0.220866 |
| 1.48355 | 0.124405 | 0.468166 |
| 1.47929 | 0.115775 | -0.461995 |
| 1.4961 | -0.418884 | 0.146737 |
| 1.5993 | 0.0340127 | -0.0153453 |
| 0.0209034 | -0.00574919 | -0.0184872 |

| 2 | 1 | 11 |
| 3 | 2 | 12 |
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| 5 | 4 | 14 |
| 6 | 5 | 15 |
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| 8 |18 | 19 |
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|10 |19 | 20 |
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|13 |12 | 21 |
|12 |11 | 22 |
|14 |13 | 23 |
|15 |14 | 24 |
|16 |15 | 25 |
|17 |16 | 26 |
|18 |17 | 27 |
|19 |18 | 28 |
|20 |19 | 29 |
|11 |20 | 30 |
|21 |20 | 29 |
|29 |19 | 28 |
|28 |18 | 27 |
|27 |17 | 26 |
|26 |16 | 25 |
|25 |15 | 24 |
|24 |14 | 23 |
|23 |13 | 22 |
|22 |11 | 30 |
|12 |22 | 21 |
|23 |21 | 31 |
|22 |30 | 32 |
|24 |23 | 33 |
|23 |31 | 33 |
|25 |24 | 34 |
|24 |33 | 34 |
|26 |25 | 35 |
|25 |34 | 35 |
|27 |26 | 36 |
|26 |35 | 36 |
|28 |27 | 36 |
|29 |28 | 40 |
|21 |22 | 37 |
|31 |21 | 37 |
|22 |32 | 37 |
|30 |29 | 38 |
|32 |30 | 38 |
|29 |40 | 38 |
|28 |36 | 40 |
|33 |31 | 39 |
|76 |82 | 81 |
```python
if __name__ == '__main__':
    demo()
```