Active matter refers to systems comprising entities with the ability to generate directed motion perpetually [1,6]. Interactions among these constituent units can cause the spontaneous emergence of collective behavior that includes collective motion [7,10], turbulent patterns [11,19], and motility-induced phase separation [20,23]. To study a many-body systems of this kind a key objective is to derive a set of equations that can capture the coarse-grained behavior of the system from the corresponding microscopic dynamics. A simple and yet successful method to achieve this task is to first decide on a set of hydrodynamic variables, and then write down the most general possible set of equations of motion (EOM) based on an expansion of these variables and their spatial derivatives, where the only constraints are the conservation laws and symmetries of the underlying microscopic dynamics. Two classic examples are the hydrodynamic theories of thermal fluids and momentum non-conserving active fluids, where the hydrodynamics variables are the density and velocity fields, whose dynamics are described by the Navier-Stokes [24] and Toner-Tu equations [8,9], respectively. To incorporate fluctuations into the EOM, Gaussian noise terms are typically added, which are justified by the central limit theorem as it reflects the fact that summing independent random variables is a one-sided positive Lévy stable noise with characteristic function $e^{ik\eta(t)} = e^{-\langle \xi \rangle}$, where $0 < \alpha < 1$ and $\langle \cdot \rangle$ denotes the averaging over $\eta$. The step size process $\eta$ is a white Gaussian noise of variance $\sigma$, i.e., $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\sigma^2(t-t')$ with $\langle \cdot \rangle$ denoting the averaging over $\xi$. The step size process $\eta$ is a white Gaussian noise of variance $\sigma$, i.e., $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\sigma^2(t-t')$ with $\langle \cdot \rangle$ denoting the averaging over $\xi$. The corresponding probability distribution has power-law tails $\propto \eta^{-(\alpha+1)}$. We note that for distributions with exponent $\alpha \geq 1$ the model is effectively equivalent to self-propelled dynamics with constant particle velocity [7].
To derive hydrodynamic EOM for this model, we apply the BBGKY hierarchical formalism \[53\] together with the simplest closure procedure to the hierarchical equations. Namely, we derive the Fokker-Planck equation for the single-particle distribution \( P(\mathbf{r}, \theta, t) \) by approximating the two-particle distribution \( P_2(\mathbf{r}, \theta, \mathbf{r}', \theta', t) \approx P(\mathbf{r}, \theta, t)P(\mathbf{r}', \theta, t) \) \[42\]. This approximation is known to be suitable for diluted systems with long-range interactions such as plasmas \[44\]. Nonetheless, we believe that this also constitutes a good approximation in our model because the Lévy step size statistics can effectively render the binary interactions among the active particles long-ranged. Using this closure and taking \( d \) to be infinitesimal in the hydrodynamic limits, we obtain the Fokker-Planck equation for the single-particle distribution \[42\]:

\[
\left( \frac{\partial}{\partial t} + D_{n(\theta)}^\alpha - \sigma \frac{\partial^2}{\partial \theta^2} - \frac{\partial}{\partial \theta} M[P] \right) P = 0 ,
\]

where \( M[P] \) is a functional of \( P \) of the form \( -\gamma \int_\pi^0 \sin(\theta') P(\mathbf{r}, \theta', t) d\theta' \) \[40\], \[43\], \[50\], and \( D_{n(\theta)}^\alpha \) is the directional fractional derivative defined as \[51\], \[52\]

\[
D_{n(\theta)}^\alpha P = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty [1 - T_{\zeta n(\theta)}]^\alpha \frac{d\zeta}{\zeta^{1+\alpha}} ,
\]

with the translation operator \( T_{\zeta n(\theta)} P(\mathbf{r}, \theta, t) = P(\mathbf{r} + \zeta \mathbf{n}(\theta), \theta, t) \),

which is \([-i(\mathbf{k} \cdot \mathbf{n}(\theta))]^\alpha \hat{P}(\mathbf{k}, \theta, t) \) in spatial Fourier transform.

With Eq. (3) at our disposal, we can construct systematically the hydrodynamic description of the microscopic model \[1\] in terms of the density field \( \rho(\mathbf{r}, t) = \int_\pi^0 P(\mathbf{r}, \theta', t) d\theta' \), the director field \( \mathbf{p}(\mathbf{r}, t) = \int_\pi^0 \mathbf{n}(\theta) P(\mathbf{r}, \theta, t) d\theta \) and the nematic tensor \( \mathbf{Q}(\mathbf{r}, t) = \int_\pi^0 \mathbf{n}(\theta) \mathbf{n}(\theta) - \frac{1}{2} \mathbf{1} \) \( P(\mathbf{r}, \theta, t) d\theta \), with \( \mathbf{1} \) being the 2 \times 2 identity matrix. These fields are related to the lower order modes \( f_0, f_1 \) and \( f_2 \), respectively, of the angular Fourier transform of \( P \) \[40\], \[43\], \[50\], which is defined as \( P(\mathbf{r}, \theta, t) = (2\pi)^{-1} \sum_{m \in \mathbb{Z}} f_m(\mathbf{r}, t)e^{-im\theta} \), where \( f_m = \int_0^\pi e^{im\theta} P(\mathbf{r}, \theta, t) d\theta \). The temporal evolution equations of the modes \( f_m \) can now be obtained from Eq. (3); in spatially Fourier transformed space, these are given by

\[
\left( \frac{\partial}{\partial t} + \Upsilon_0 k^\alpha + \sigma m^2 \right) \hat{f}_m + \frac{\gamma m}{2} \left( \hat{f}_{m+1} + \hat{f}_{m-1} \right) \hat{f}_1 = -k^\alpha \sum_{m'=1}^\infty i m' \Upsilon_m' \left( e^{-im'\phi} \hat{f}_{m+m'} + e^{im'\phi} \hat{f}_{m-m'} \right) ,
\]

where \( \ast \) denotes the convolution of two functions, i.e., \( \hat{f}_j(\mathbf{k}, t) + \hat{f}_j(\mathbf{k}, t) = \int_{-\infty}^\infty d^2k' \hat{f}_j(\mathbf{k}-\mathbf{k}', t) \hat{f}_j(\mathbf{k}', t) \), \( \phi \) specifies the direction of the Fourier variable, i.e., \( \mathbf{k} \equiv k \mathbf{n}(\phi) \) with \( k \equiv |\mathbf{k}| \) \[53\], and we have defined the following \( \alpha \)-dependent real and positive coefficients \[42\]

\[
\Upsilon_m(\alpha) = \frac{(i)^m}{2\pi} \int_0^\infty e^{i\theta' m} (-i \cos \theta')^\alpha d\theta' .
\]

Similar to ordinary active fluids, the hydrodynamic behaviour of the system around the onset of collective motion is dictated only by the lower order modes \[40\], \[43\], \[50\] because all the higher-order modes are suppressed by the “mass” term \( \propto \sigma m^2 \) and by the homogeneous term \( \propto \Upsilon_0 k^\alpha \) resulting from the fractional operator \[1\]. We therefore adopt the same approximation strategy used in ordinary active matter, i.e., we assume that \( f_m \approx 0 \) for \( m \geq 3 \) and \( \partial_\theta f_2 \approx 0 \). The equation for \( f_2 \) then becomes algebraic and can be solved. Using this exact result and re-expressing the equations for \( f_0 \) and \( f_1 \) in terms of the density, director, and nematic fields, we obtain the hydrodynamic EOM, to order \( k^\alpha \), \[42\]

\[
(\partial_t + \Upsilon_0 \mathbf{D}^\alpha) \rho = 2\Upsilon_1 \mathbb{I}^{1-\alpha} (\nabla \cdot \mathbf{p}) - \lambda_2 \mathbb{I}^{2-\alpha} (\nabla \cdot \mathbf{Q} \_0) ,
\]

\[
(\partial_t + \Upsilon_0 \mathbf{D}^\alpha) \mathbf{p} = [\kappa_0 \gamma - \xi |\mathbf{p}|^2] \mathbf{p} + \Upsilon_1 \mathbb{I}^{1-\alpha} \nabla \rho + \lambda_1 \mathbb{I}^{1-\alpha} \mathbf{Q} \_0 - \Upsilon_2 \mathbb{I}^{2-\alpha} \mathbb{I} \times \mathbf{p} + \lambda_2 \mathbb{I}^{3-\alpha} \mathbf{Q} \_1 - \mathbf{Q} \_p ,
\]

where \( \mathbb{I}^\beta \) is the fractional Riesz integral defined by its Fourier transform \( \mathbb{I}^\beta h(\mathbf{r}) \rightarrow k^{-\beta} \hat{h}(\mathbf{k}) \) (for \( \text{Re} \beta > 0 \) and \( h \) a test function), and \( \mathbb{D}^\beta \) is the corresponding fractional derivative whose Fourier representation is \( \mathbb{D}^\beta h(\mathbf{r}) \rightarrow k^{-\beta} \hat{h}(\mathbf{k}) \) \[51\], \[52\]. The operators \( \mathbb{I}^\beta \) and \( \mathbb{D}^\beta \) are the inverses of each other. In Eqs (5) and (6), we have also defined the following non-linear \((\mathbf{r}, t)\)-dependent vector fields

\[
\mathbf{Q}_0 = 2[(\mathbf{p} \cdot \nabla) \mathbf{p} + (\nabla \cdot \mathbf{p}) \mathbf{p} - |\mathbf{p}|^2] ,
\]

\[
\mathbf{Q}_1 = (\nabla \cdot \mathbf{p}) \mathbb{I} \mathbf{p} + (\mathbf{p} \times \nabla) \times \mathbb{I} \mathbf{p} + 3[(\mathbf{w} \times \nabla) \mathbf{p} + (\mathbf{w} \times \mathbf{p}) \cdot (\mathbf{w} \times \nabla) \mathbf{w}) ,
\]

where \( \mathbf{w}(\mathbf{r}, t) = \nabla \times \mathbf{p} \) is the vorticity field. The nematic tensor \( \mathbf{Q} \) is specified by its components \( Q_{xx} = Q_1 = -Q_{yy} \) and \( Q_{xy} = Q_2 = Q_{yx} \), which are

\[
Q_1 = -B_0 \mathbb{I} \mathbf{p} \mathbf{p}_j + \frac{\lambda_1}{2} \mathbb{I}^{1-\alpha} T_{ij} \mathbf{p}_j - \frac{\lambda_2}{4} \mathbb{I}^{2-\alpha} T_{ij} \partial_j \rho + \frac{\lambda_3}{4} \mathbb{I}^{3-\alpha} T_{ij} \mathbf{Q}_1 \mathbf{p}_j - B_1 \mathbb{I}^{4-\alpha} T_{ij} \mathbf{Q}_{ij} .
\]
where \(i, j = 1, 2\) and \(p_1\) and \(p_2\) are the components of \(\mathbf{p}\) along the \(x\) and \(y\) axis, respectively. In the above, we have also defined the tensor function \(M_{ij} \equiv p_1 \delta_{ij} + p_2 \epsilon_{ij}\) and the differential operator \(\delta_{ij} \equiv \partial_x \delta_{ij} + \partial_y \epsilon_{ij}\), where \(\delta_{ij}\) is the Kronecker symbol and \(\epsilon_{ij}\) is the generator of counter-clockwise rotations, with \(\epsilon_{11} = 0 = \epsilon_{22}\) and \(\epsilon_{21} = 1 = -\epsilon_{12}\). Furthermore, we have introduced the density-dependent quantity \(\kappa_0(\rho) \equiv -\sigma + (\gamma \rho)/2\) and the coefficients

\[
\lambda_1 = \frac{\gamma Y_1}{4\sigma}, \quad \lambda_2 = \frac{\gamma Y_2}{2\sigma}, \quad \lambda_3 = \frac{\gamma Y_3}{2\sigma}, \quad (13)
\]

\[
B_0 = \frac{\gamma^2 T_0}{32\sigma^2}, \quad B_1 = \frac{\gamma^2 T_4}{16\sigma^2}, \quad \xi = \frac{\gamma^2}{8\sigma}. \quad (14)
\]

The hydrodynamic EOM \(8\) and \(9\) are our first main result and may be viewed as the counterpart of the Toner-Tu equations for ALM. We note that, for \(\alpha = 1\), we have \(\Upsilon_i = 0\) for all \(i \neq 1\) and \(\Upsilon_1 = -1/2\) \(\\(8\\), so that \(8\) and \(9\) are reduced to a version of the Toner-Tu equations \(8\) \(9\) \(56\) without the ordinary viscosity term, which is absent because of the truncation in the wavenumber \(k\).

At the mean-field level, our EOM predict the same phase behaviour of the Toner-Tu equations, because all terms involving (integral or fractional) spatial derivatives can be ignored. Equations \(8\) and \(9\) thus indicate (a) the disordered phase with \(\mathbf{p} = 0\) for \(\sigma \geq \sigma_i\); and (b) the ordered phase with \(\mathbf{p}^* = \sqrt{8\sigma \kappa_0(\rho^*)/\gamma^2} \mathbf{e}_1\), where \(\mathbf{e}_1\) denotes the arbitrary direction of spontaneous symmetry breaking for \(\sigma < \sigma_i\). The density dependent threshold rotational noise strength is \(\sigma_i(\rho^*) \equiv \gamma \rho^*/2\).

The mean-field description is known to be inadequate at the onset of collective motion for ordinary active matter due to the emergence of the characteristic banding instability, which manifests itself as a longitudinal instability (i.e., along the direction of collective motion), with the \(k \to 0\) mode being the most unstable. Strikingly, this is no longer true in ALM. To demonstrate this, we perturb the hydrodynamic fields \(\rho\) and \(\mathbf{p}\) as \(\rho^* + \delta \rho e^{st+iq\cdot r}\) and \(\mathbf{p}^* + \delta \mathbf{p}_0 e^{st+iq\cdot r}\), respectively, and solve \(8\) and \(9\) at the linear level in \(\delta \rho_0\) and \(\mathbf{p}_0\). Eliminating them then yields the dispersion relation with solutions \(s_{\pm}(q)\) such that \(\Re s_+ > \Re s_-\). The spatially homogeneous phases predicted by the mean-field theory are only stable if \(\Re s_+ < 0\). Similar to ordinary active fluids, we find that the disordered phase is stable against perturbations.

FIG. 2. Finite size scaling analysis of the microscopic model of active Lévy matter. We run 300 independent numerical simulations of the Langevin equations \(1\) in a square box of linear length \(L = 48, 56, 64, 72\) with periodic boundary conditions. Other parameters are \(\rho = 2, \gamma = 0.25\) and \(\alpha = 1/2\) (\(\alpha = 1\) for Vicsek type dynamics). The model is initialized in the homogeneously disordered phase and updated for \(2 \times 10^6\) time sweeps, which are sufficient for equilibration \(12\). (a) Time averaged polar order parameter \(\varphi\) vs. the rotational noise strength \(\sigma\). (b) Snapshots of the system configuration for ordinary active matter in the disordered phase (left) and in the unstable region at the onset of the transition (right). (c) Similar snapshots for active Lévy matter at the critical point (left) and in the ordered phase (right). (d) Binder cumulant \(U^{(L)}_4\) vs. the distance from the critical noise strength \(\tau\). Inset: Quadratic fits of Binder cumulant curves around the critical point. In panels (a) and (d) error bars are plotted denoting 1 s.e.m.; in the inset of panel (d) we also plot 95% confidence bands on the best fit lines.
Fit results

(a) Gradient of the Binder cumulant at criticality $\partial U^{(L)}_4/\partial \tau|_{\tau=0} \approx 0$ vs. the system size $L$.

(b) Susceptibility at criticality $\chi_L$ vs. $L$.

(c) Correlation time of the polar order parameter $\xi_L$ at criticality vs. $L$.

FIG. 3. Numerical estimation of static and dynamic critical exponents for the order-disorder transition in active Lévy matter.

Model parameters are $\rho = 2$, $\gamma = 0.25$ and $\alpha = 1/2$. The simulation data (in logarithmic scales) are fitted with the linear function $mL + c$ by using a weighted least squares method. Error bars are plotted denoting 1 s.e.m.; 95% confidence bands on fit parameters are also shown. (a) Gradient of the Binder cumulant at criticality $\partial U^{(L)}_4/\partial \tau|_{\tau=0}$ vs. the system size $L$. (b) Susceptibility at criticality $\chi_L$ vs. $L$. (c) Correlation time of the polar order parameter $\xi_L$ at criticality vs. $L$.

in all directions [42]. However, for the ordered phase, in the hydrodynamic limit, we obtain [42]

\begin{align}
\text{Re}(s_-) &\approx -2\kappa_0(\rho^*) + O(q^a), \\
\text{Re}(s_+) &\approx -2\kappa_0(\rho^*) + 2T_2q^a + O(q^{2a}).
\end{align}

The first eigenvalue, $s_-$, always has a negative real part and describes the fast relaxation of small perturbations from $p^*$. For normal active fluids, the real part of the second eigenvalue, $s_+$, is always positive as the system approaches the onset of collective motion [45], indicating the banding instability that renders the transition first-order [55,58]. For ALM, instead, this eigenvalue has a negative real part because $T_0 > 2T_2$ for all $0 < \alpha < 1$. Therefore, the banding instability is made stable by the Lévy motion of the active particles. Furthermore, we can demonstrate that transversal perturbations (i.e., orthogonal to the direction of collective motion) are also suppressed [42]. As a result, in ALM, the ordered phase always remains stable at the onset of collective motion in the hydrodynamic limit, thus rendering the order-disorder phase transition potentially critical. These predictions represent the second main result of our work.

We now investigate the critical behavior predicted by our theory with numerical simulations. We focus here on the particular value $\alpha = 1/2$ [42]. We first note that the criticality of the phase transition manifests in the smooth approach of the time averaged polar order parameter $\varphi$ to zero as the rotational noise is increased, as opposed to the abrupt jump found in ordinary active matter, which is characteristic of a first-order transition (Fig. 2b). Snapshots of the system configuration also confirm these predictions (Figs. 2b and 2c). We now characterize the critical exponents of the phase transition by performing a finite size scaling analysis [59]. Specifically, we first estimate the asymptotic critical noise strength at fixed density, $\sigma_*$, by identifying the crossing of the Binder cumulant $U^{(L)}_4 \equiv \langle \varphi^2 \rangle_L/\langle \varphi^4 \rangle_L$ for different system sizes $L$ (Fig. 2d). Here, $\langle \rangle_L$ denotes the ensemble average taken at finite system size $L$. We estimate: $\sigma_* = 0.239816(2)$ (error is 1 s.e.m.). Using the scaling relation $\partial U^{(L)}_4/\partial \tau|_{\tau=0} \propto L^{1/\nu}$, where $\tau = -1 + \sigma/\sigma_*$, we can estimate directly the static exponent $\nu$ (Fig. 3a). In addition, using the scaling relations for the polar order parameter at criticality $\varphi \propto L^{-\beta/\nu}$ and that for the susceptibility $\chi_L \propto L^{\gamma/\nu}$, with $\chi_L \equiv L^2(\langle \varphi^2 \rangle_L - \langle \varphi \rangle^2_L)$, we can estimate the ratios of static critical exponents $\beta/\nu$ and $\gamma/\nu$ (Fig. 3b). To estimate the dynamic exponent $z$, we use the scaling relation for the correlation time of the polar order parameter $\xi_L \propto L^z$ (Fig. 3c). The estimates obtained are summarized in Table I. The numerical characterization of the critical properties of the order-disorder transition is our third main result. Intriguingly, these estimates for $\alpha = 1/2$ seem to indicate that the static properties of the critical transition in ALM (for the particular $\alpha$ considered here) belongs to the same universality class of the equilibrium model in two spatial dimensions with long-range interaction energy $\propto 1/|r_{ij}|^{2+2\alpha}$ and $n$-component order parameter [61,62]. Elucidating analytically this connection by employing dynamic renormalization group methods, similarly to what was accomplished for ordinary active matter in incompressible conditions [63,65], is an interesting open problem that we aim to elucidate in future investigations.

In this Letter, we derived the first hydrodynamic description of ALM, which generalizes the conventional theory of active fluids by incorporating Lévy stable distributed fluctuations with diverging variance. We then revealed that, unlike ordinary polar active matter, the order-disorder transition in ALM is critical and estimated the corresponding critical exponents numerically. Our work highlights the novel physics exhibited by ac-

| $\nu$ | $\beta/\nu$ | $\gamma/\nu$ | $z$ |
|-------|-----------|-----------|------|
| 1.00999(3) | 0.497(1) | 0.996(4) | 0.991835(4) |

TABLE I. Numerical estimates of the critical exponents for $\alpha = 1/2$. Errors are expressed as 1 s.e.m.
tive matter models integrating both anomalous diffusive motility and inter-particle interactions. Interesting future directions include the investigation of the effects of Lévy displacement dynamics on collective phenomena other than collective motion such as active turbulence \cite{11, 19} and motility-induced phase separation \cite{22, 23}, as well as their relevance to biological systems \cite{66}.

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[35] B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables* (Addison-Wesley, Cambridge, United States, 1954).

[36] R. Grossmann and F. Peruani, and M. Bär “Superdiffusion, large-scale synchronization, and topological defects,” Phys. Rev. E 93, 040102(R) (2016). Here the authors studied the synchronization of phase oscillators à la Kuramoto performing Lévy flight dynamics. As the phase variables do not relate to the direction of the oscillators, the oscillators can not reproduce collective motion.

[37] G. Estrada-Rodriguez and H. Gimperlein, “Interacting particles with Lévy strategies: Limits of transport equations for swarm robotic systems,” arXiv:1807.10124v3 (2018). Here the authors studied a hydrodynamic formulation of a system of interacting active particles with Lévy walk dynamics. However, the only hydrodynamic variable considered is the density and the resulting theory disallows collective motion.

[38] Benoit B Mandelbrot, *The fractal geometry of nature*, Vol. 2 (WH freeman New York, 1982).

[39] Barry D Hughes, Michael F Shlesinger, and Elliott W Montroll, “Random walks with self-similar clusters,” Proc. Natl. Acad. Sci. 78, 3287–3291 (1981).

[40] F. Peruani, A. Deutsch, and M. Bär, “A mean-field theory for self-propelled particles interacting by velocity alignment mechanisms,” Eur. Phys. J. Spec. Top. 157, 111–122 (2008).

[41] D. Applebaum, *Lévy processes and stochastic calculus* (Cambridge university press, Cambridge, England, 2014).

[42] A. Cairoli and C. F. Lee, “Active Lévy matter: Anomalous Diffusion, Hydrodynamics and Linear Stability,” The accompanying long paper.

[43] K. Huang, *Statistical Mechanics*, 2nd ed. (John Wiley & Sons, New York, USA, 1987).

[44] A. A. Vlasov, “The vibrational properties of an electron gas,” Phys.-Uspekhi 10, 721–733 (1968).

[45] E. Bertin, M. Droz, and G. Grégoire, “Boltzmann and hydrodynamic description for self-propelled particles,” Phys. Rev. E 74, 022101 (2006).

[46] A. Baskaran and M. C. Marchetti, “Hydrodynamics of self-propelled hard rods,” Phys. Rev. E 77, 011920 (2008).

[47] E. Bertin, M. Droz, and G. Grégoire, “Hydrodynamic equations for self-propelled particles: microscopic derivation and stability analysis,” J. Phys. A 42, 445001 (2009).

[48] C. F. Lee, “Fluctuation-induced collective motion: A single-particle density analysis,” Phys. Rev. E 81, 031125 (2010).

[49] A. Peshkov, E. Bertin, F. Ginelli, and H. Chaté, “Boltzmann-Ginzburg-Landau approach for continuous descriptions of generic Vicsek-like models,” Eur. Phys. J. Spec. Top. 223, 1315–1344 (2014).

[50] E. Bertin, “Theoretical approaches to the steady-state statistical physics of interacting dissipative units,” J. Phys. A 50, 083001 (2017).

[51] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and Applications*, Vol. 157 (Gordon and Breach, Yverdon, 1993).

[52] Vasily E Tarasov, *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media* (Springer Science & Business Media, 2011).

[53] J. P. Taylor-King, R. Klages, S. Fedotov, and R. A. Van Gorder, “Fractional diffusion equation for an n-dimensional correlated Lévy walk,” Phys. Rev. E 94, 012104 (2016).

[54] J. Toner, “Reanalysis of the hydrodynamic theory of fluid, polar-ordered flocks,” Phys. Rev. E 86, 031918 (2012).

[55] G. Grégoire and H. Chaté, “Onset of collective and cohesive motion,” Phys. Rev. Lett. 92, 025702 (2004).

[56] H. Chaté, F. Ginelli, G. Grégoire, and F. Raynaud, “Collective motion of self-propelled particles interacting without cohesion,” Phys. Rev. E 77, 046113 (2008).

[57] A. P. Solon and J. Tailleur, “Revisiting the flocking transition using active spins,” Phys. Rev. Lett. 111, 078101 (2013).

[58] A. P. Solon and J. Tailleur, “Flocking with discrete symmetry: The two-dimensional active ising model,” Phys. Rev. E 92, 042119 (2015).

[59] F. Ginelli and H. Chaté, “Relevance of metric-free interactions in flocking phenomena,” Phys. Rev. Lett. 105, 168103 (2010).

[60] D. P. Landau and K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge University Press, Cambridge, England, 2014).

[61] M. E. Fisher, S.-K Ma, and B. G. Nickel, “Critical Exponents for Long-Range Interactions,” Phys. Rev. Lett. 29, 917-920 (1972).

[62] M. Suzuki, “Critical Exponents for Long-Range Interactions I,” Prog. Theor. Phys. 49, 424-441 (1973).

[63] L. Chen, J. Toner, and C. F. Lee, “Critical phenomenon of the order–disorder transition in incompressible active fluids,” New J. Phys. 17, 042002 (2015).

[64] L. Chen, C. F. Lee, and John Toner, “Mapping two-dimensional polar active fluids to two-dimensional soap and one-dimensional sandblasting,” Nat. Commun. 7 (2016).

[65] L. Chen, C. F. Lee, and J. Toner, “Incompressible polar active fluids in the moving phase in dimensions d > 2,” New J. Phys. 20, 113035 (2018).

[66] V. Zaburdaev, S. Denisov, and J. Klafter, “Lévy walks,” Rev. Mod. Phys. 87, 483 (2015).