Deeply inelastic scattering structure functions on a hybrid quantum computer

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We outline a strategy to compute deeply inelastic scattering structure functions using a hybrid quantum computer. Our approach takes advantage of the representation of the fermion determinant in the QCD path integral as a quantum mechanical path integral over 0+1-dimensional fermionic and bosonic worldlines. The proper time evolution of these worldlines can be determined on a quantum computer. While extremely challenging in general, the problem simplifies in the Regge limit of QCD, where the interaction of the worldlines with gauge fields is strongly localized in proper time and the corresponding quantum circuits can be written down. As a first application, we employ the Color Glass Condensate effective theory to construct the quantum algorithm for a simple dipole model of the $F_2$ structure function. We outline further how this computation scales up in complexity and extends in scope to other real-time correlation functions.

Introduction. From its early role in the discovery of partons and asymptotic freedom, deeply inelastic scattering (DIS) of electrons off protons and heavier nuclei has been a fundamental tool in studying the quark-gluon structure of matter \[1\]-\[5\]. The simplest DIS quantities are the inclusive structure functions $F_2$ and $F_L$ which, respectively, provide information on the sum of the quark and antiquark distributions, and the gluon distribution, in the target proton or nucleus \[6\]-\[7\].

Computing structure functions from first principles is an outstanding problem in Quantum Chromodynamics (QCD) because they are proportional to nucleon/nuclear matrix elements of products of electromagnetic currents that are light-like separated in Minkowski spacetime. In contrast, Monte Carlo computations in lattice QCD are robust in Euclidean spacetime. The operator product expansion (OPE) \[8\] allows one to compute moments of structure functions in lattice QCD, but such computations are restricted to low moments \[9\]. There are interesting alternative developments whereby so-called quasi-pseudo-pdfs are computed on the lattice \[10\]-\[13\]; excellent reviews of the status of “classical” computation of structure functions can be found in \[14\]-\[15\].

Given the formidable challenge, it is worthwhile to ask whether simulations on a quantum computer can overcome the limitations of classical Monte Carlo approaches. An appropriate analogy is that of the sign problem at finite baryon chemical potential in QCD \[16\]-\[17\]. In this Noisy-Intermediate-Scale-Quantum (NISQ) era \[18\], a path forward is to identify simple but scalable problems that are tractable and to explore their implementation on quantum devices \[19\]-\[25\].

We will outline in this letter a strategy whereby progress can be made by expressing the fermion determinant in the QCD effective action as a quantum mechanical “worldline” path integral \[26\]-\[39\] over fermionic and bosonic variables \[40\]. After introducing the essential elements of this worldline formalism, we will apply it to discuss the quantum computation of $F_2$ in an effective field theory approach to the high energy “Regge” limit of QCD. We will then identify the quantum circuits needed for a simple computation. The latter part of this letter will discuss the gradual scaling of this computation in complexity, and its expansion in scope, to address scattering problems that are classically challenging in both perturbative and nonperturbative QCD.

Worldline representation of the effective action. The Euclidean QCD+QED effective action can be expressed as

$$\Gamma_{\text{E}}[A;\omega] = \frac{1}{2} \text{Tr} \log \left( D^2 - \frac{i}{4} F_{\mu\nu} [\gamma^\mu, \gamma^\nu] \right),$$

where the $A_\mu$ represent the QCD gauge fields, $\omega^\mu$ the QED photon field, with $g$ and $\epsilon$ the respective $SU(N_c)$ and $U(1)$ gauge couplings. Further, we have $D^2 = D_\mu D^\mu$, with the covariant derivative $D_\mu = \partial_\mu - ig A_\mu - ieD_\mu = \frac{i}{g} [D_\mu, D^\mu]$ is the field-strength tensor, and the operator trace is over position, as well as color and spin degrees of freedom \[41\]. The logarithm in Eq. \[1\] can be written in integral form and a heat-kernel regularization of this expression \[26\] allows one to evaluate the functional trace in a coherent state basis for the coordinate $\ddot{x}_\mu(x) = x_\mu(x)$ and the momentum $\hat{\mathbf{p}}_\mu(p) = p_\mu(p)$, giving $\text{tr}_x(\mathcal{O}) = \int d^4x \langle x|\mathcal{O}|x \rangle = \int d^4p \langle p|\mathcal{O}|p \rangle$. The trace over the Dirac matrix structure may be expressed in a coherent state basis of fermionic creation-annihilation operators \[26\]-\[28\]

$$\hat{a}_1^\dagger = \frac{1}{2} (\gamma_1 + i\gamma_3), \quad \hat{a}_1 = \frac{1}{2} (\gamma_1 - i\gamma_3)$$
$$\hat{a}_2^\dagger = \frac{1}{2} (\gamma_2 + i\gamma_4), \quad \hat{a}_2 = \frac{1}{2} (\gamma_2 - i\gamma_4)$$

(2)

of the Clifford algebra of Euclidean Dirac matrices, satisfying $[\gamma_\mu, \gamma_\nu]_+= 2\delta_{\mu\nu}$ ($\mu = 1, 2, 3, 4$). The fermionic coherent states are defined by

$$\hat{a}_i|\theta\rangle = \theta_i|\theta\rangle, \quad \hat{a}_i^\dagger|\theta^*\rangle = \theta_i^*|\theta^*\rangle,$$

(3)
where $\theta_i, \theta_i'$, with $i = 1, 2$, are Grassmann variables. The spinor trace in this coherent state basis is given by the functional integral $\text{tr}_c \mathcal{O} \equiv \int d^2 \theta (-\theta) \mathcal{O}(\theta)$ with the normalization $\text{tr}_c \mathcal{I} = 4$. The Majorana representation of these is

$$
\psi_1 = \frac{1}{\sqrt{2}} (\theta^* + \theta_1), \quad \psi_2 = \frac{1}{\sqrt{2}} (\theta^* - \theta_1),
$$

$$
\psi_3 = -\frac{i}{\sqrt{2}} (\theta^* - \theta_1), \quad \psi_4 = -\frac{i}{\sqrt{2}} (\theta^* + \theta_1).
$$

Likewise, as outlined in Appendix B, the trace for SU(3) color may be expressed as the Grassmann integral

$$
\text{tr}_c \mathcal{O} = \int d^3 \lambda (-\lambda) \mathcal{O}(\lambda) \text{ by introducing the coherent states, } c_\alpha(\lambda) = \lambda_\alpha |\lambda\rangle, c_\alpha^*(\lambda^*) = \lambda_\alpha^* |\lambda^*\rangle, \text{ where } \alpha = 1, 2, 3.
$$

However for the purposes of this discussion, we shall keep the trace over color explicit. Employing these coherent states, and analytically continuing to Minkowski space ($-i \Gamma \rightarrow i \Gamma$), one obtains the quantum mechanical path integral

$$
\Gamma[A;a] = -\frac{i}{2} \int_0^\infty \frac{dt}{T} \text{tr}_c \int P \int d\phi \cdot e^{iS[A,a]},
$$

with the action

$$
S[A;a] = \int dt \left( p_\mu \dot{\phi}^\mu + \frac{i}{2} \bar{\psi} \gamma^\mu \phi^\mu - H[A;a] \right),
$$

and the worldline Hamiltonian given by

$$
H[A;a] = P^2 + ig \psi \gamma^\mu F_{\mu\nu}[A] \psi^\nu + i \epsilon \psi \gamma^\mu F_{\mu\nu}[a] \psi^\nu.
$$

In the inclusive DIS process $\ell(\ell') + N(P) \rightarrow \ell(\ell') + X$, the cross-section for the interaction between the lepton ($\ell$) and the hadron ($N$) can be factorized into the convolution of the lepton tensor $L_{\mu\nu}$, corresponding to the exchange of a virtual photon $\gamma^*$ with spacelike four-momentum $q = l - l' \equiv (q^- \equiv q^-, q^x, 0, 0)$, and the hadron tensor $W_{\mu\nu}$, representing the interaction of the virtual photon with the parton constituents of the hadron $H$. The latter is given by the imaginary part of the forward Compton amplitude:

$$
W_{\mu\nu}(q, P, S) = \text{Im} \int d^4 x \text{e}^{i q \cdot x} \left( P[S] T \hat{J}^\mu(x) \hat{J}^\nu(0) \right) |P, S>,
$$

where $\text{T}$ denotes time ordering of the bilinear product of electromagnetic current operators $\hat{J}^\mu = \bar{\psi} \gamma^\mu \psi$ of quark fields $\Psi$ in the hadron state with four-momentum $P$ and spin $S$. Structure functions extracted from $W_{\mu\nu}(q, P, S)$ are expressed in terms of the Lorentz scalars $Q^2 = -q^2 > 0$ and $x_{Bj} = Q^2/(2P \cdot q)$. In particular, $F_2(x_{Bj}, Q^2)$ is obtained from the projection $F_2 \equiv \Pi^{\mu\nu} W_{\mu\nu}$, with

$$
\Pi^{\mu\nu} = \frac{3P\rho}{4a} \left[ \frac{\mu\rho - \rho^\mu}{3} \right], \quad a = P \cdot q/(2x_{Bj}) + M^2, \quad \text{and } M \text{ the hadron mass}.
$$

In lattice gauge theory, direct evaluation of the time ordered product on the r.h.s of Eq. (B) requires computing the ratio of four-point and two point Euclidean correlators and subsequent analytic continuation from Euclidean to Minkowski space. In Appendices A and B we outline a first principles computation of $W_{\mu\nu}$ in the worldline formalism which illustrates the complexity of the problem. We will return to these issues shortly.

We will argue here that significant progress towards quantum computation of structure functions can be made in Regge kinematics, corresponding to $Q^2 = \text{fixed}$, and $x_{Bj} \approx Q^2/s \rightarrow 0$, where the squared center-of-mass energy $s \approx 2P^2 + q^2$. In this limit, a Born-approximation separation of scales appears between fast ($x_{Bj} \sim 1$) and slow ($x_{Bj} \ll 1$) degrees of freedom, allowing the former to be described as static color sources and the latter as dynamical gauge fields coupled to the sources.

This argument is quantified in the QCD Condensate effective field theory (CGC EFT), wherein an emergent scale proportional to the density of color sources grows through a Wilsonian renormalization group evolution of the separation between sources and fields with decreasing $x_{Bj}$. In Regge asymptotics, this scale is larger than intrinsic nonperturbative QCD scales and therefore the hadron tensor in the CGC EFT can be written as $W_{\mu\nu}(q, P, S) = \frac{P^+}{\pi e^2} \text{Im} \int d^4 X \int dX^- \int \int d^4 x \times e^{i q \cdot x} e^{-i k \cdot (X^- + \frac{d}{2})} e^{-ik' \cdot (X^- - \frac{d}{2})}

$$
\times \int \mathcal{D}[\rho] \mathcal{W}[\rho] \mathcal{W}[k', A] e^{iT[A] + iS[A,a]},
$$

where $\int_k = \int d^4 k/(2\pi)^4$. We work in lightcone coordinates $x^\pm \equiv (x^0 \pm x^3)/\sqrt{2}, P^\pm \equiv (P^0 \pm P^3)/\sqrt{2}$, assuming a right moving hadron with large $P^+$. In Eq. (B), $S[A,a] = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + J \cdot A$ [61]. Here $J_{\mu\nu} = \delta^\mu\nu \rho(x^-, x_\perp)$, where the static large $x_{Bj}$ color source density $\rho(x^-, x_\perp)$ ($c = 1, \ldots, 8$) has support limited to $\Delta x^- = 1/P^+ \text{ and } W_{\mu\nu}[k, \rho] = \text{a gauge invariant weight functional representing the nonperturbative distribution of these sources.}$ The polarization tensor $\tilde{\Gamma}^{\mu\nu}$ is given by

$$
\tilde{\Gamma}^{\mu\nu}[k, k'] = \int d^4 z d^4 z' \frac{i \delta \Gamma[A;\alpha]}{\delta \alpha_\mu(z) \delta \alpha_\nu(z')} |_{z=0} e^{i k \cdot z + i k' \cdot z'},
$$

where $\Gamma[A;\alpha]$ is the QCD+QED effective action with the worldline representation given in Eq. (5).

We will now discuss the computation of $F_2$ on a quantum computer. The simplest problem we can address in 3+1-dimensions is to determine the quantum algorithm for the static “shock wave” solution $A_B^\mu = (0, A_B^\perp, 0, 0)$ to the Yang-Mills equations in the CGC EFT where...
\[ A_{\gamma}^i(x) = \bar{\rho}^i(x) \delta(x^{-}) \], with \( \rho^i(x, x^-) \approx \bar{\rho}(x) \delta(x^-) \).

In this background, the worldline effective action can be written as
\[
\Gamma[A_{\alpha}, a] = -\frac{i}{2} tr_c \int_0^T dT \int d^4 x d^2 \theta (x, -\theta) e^{-i \hat{H} [A_{\alpha}, a] T} |x, \theta \rangle \langle x, \theta |,
\]
where the Hamiltonian operator \( \hat{H} \) is obtained by quantizing Eq. (7). While consistent quantization requires that we eliminate states with indefinite metric from the physical subspace of the theory [62], it will not be relevant for the consideration of \( F_2 \) we consider here. We will return to this important issue later in the text and at length in Appendix A.

Computing the hadron tensor Eq. (9) from Eq. (10) and Eq. (11) facilitates the simplest possible hybrid quantum computation where only the spinor trace is simulated on a digital quantum computer. Details of this computation starting from Eq. (11) are given in Appendix [7] with the result given by
\[
F_2(q, P) = \frac{\sigma Q^2}{2 \pi e^2} \int |D[\rho]| W[\rho] \int \sum_{x, z} |\Psi_{L,T}(x, z, x, z)|^2 \times D(x) i \int d^2 \theta (-\theta) \left[ \Omega_{L,T}(x, z, x, z) \right] |\theta \rangle.
\]
Here \( \int_{x, z} = \int d^2 x d^2 z \), \( \sum_{x, z} \) is the sum over the photon polarization and quark flavors, \( \sigma = \int d x X_L \) is the transverse radius of the hadron/nucleus and \( |\Psi_{L,T}(x, z)|^2 \) denotes the modulus squared of the wave function of a virtual photon with longitudinal (L) or transverse (T) polarization to split into a quark-antiquark “dipole”.
\[
|\Psi_L(z, x)|^2 = \frac{e_x^2 e^2}{2 \pi^3} Q^2 z \cdot (1 - z)^2 K^2_0(\Delta x),
\]
\[
|\Psi_T(z, x)|^2 = \frac{e_y^2 e^2}{8 \pi^3} \left[ \left( z^2 + (1 - z)^2 \right) K^2_1(\Delta x) + m^2_0 K^2_0(\Delta x) \right],
\]
where \( e_x \) is the fractional charge of a quark of flavor \( f \), \( Q_f^2 \equiv z(1-z)Q^2 + m^2 \) and \( K_{\mu} \) are modified Bessel functions of the second kind. The dipole-hadron cross-section is given by \( D(x, z) = -\frac{3}{\Delta x} tr_c (1 - U(x) U^d (0)) \).

In Eq. (12), \( \Omega_{L,T} \) are given by
\[
\Omega_L(z, x, x) = \frac{1}{2z(1-z)} \left\{ - \frac{3}{4} \left[ (2z - 1) + 2 \phi^- \phi^+ \right] \right. \times \left[ (2z - 1) - 2 \phi^- \phi^+ \right] - \phi^+ \phi^- \phi^+ \phi^- - \phi^+ \phi^- \phi^+ \phi^- - z(1-z) + \frac{3}{4} \left. \right\},
\]
where \( \Omega_T(z, x, x) = 1 \) is trivial and need not be quantum computed. The interaction with the shock wave is represented by
\[
U(x, x) = \exp \left[ -ig \int_{-\Delta x/2}^{\Delta x/2} dx \, A_{\gamma}^i(x, x) \right],
\]
where \( \Delta x = 1/P^+/0 \) is the width of the shock wave in the high energy limit while \( 2z - 1 \pm 2 \phi^- \phi^+ \) and \( \phi^+ \phi^- \) are the photon vertex insertions described further below. Eqs. (12)-(15) are derived in Appendix A. We note finally that we set \( \exp(i \Gamma) \) in Eq. (9) to unity, which is valid to leading order in the coupling.

Worldline Algorithm for the dipole model. The trace in Eq. (12) of Eq. (15) can be determined on a quantum computer. We quantize \( \psi^\mu \rightarrow \psi^\mu = \gamma^\mu \gamma^\mu / \sqrt{2} \), where \( \gamma^\mu \) are the Dirac matrices in Minkowski spacetime satisfying \( [\gamma^\mu, \gamma^\nu] = 2 g_{\mu \nu} \) with \( (+, -, - , - ) \) signature and \( \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). We then replace \( \gamma^{\mu} = \hat{b}_1^2 - b_1) / \sqrt{2} \), \( \psi^3 = (\hat{b}_2^3 - b_2) / \sqrt{2} \) and \( \psi^2 = -i (\hat{b}_2^3 - b_2) / \sqrt{2} \), where \( \hat{b}_i \), \( b_i \) are fermion creation and annihilation operators satisfying \( [\hat{b}_i^\dagger, b_j] = \delta_{ij} \). Further, we define lightcone operators \( \psi^+ = \hat{b}_1^3 \) and \( \psi^- = -b_1^3 \).

Performing a Jordan-Wigner transformation \( \hat{b}_1^3 = (\sigma^x - i \sigma^y) / 2 \otimes I \), \( \hat{b}_2^3 = (\sigma^x + i \sigma^y) / 2 \otimes I \), \( \hat{b}_2^3 = \sigma^z \otimes (\sigma^x - i \sigma^y) / 2 \) and \( \hat{b}_2^3 = \sigma^z \otimes (\sigma^x + i \sigma^y) / 2 \) we can write the individual terms in Eq. (15) and Eq. (16) as
\[
\psi^+ \psi^- = -\frac{1}{2} [\sigma^x + \sigma^y] \otimes I ,
\]
\[
\psi^1 \psi^1 = -\frac{1}{2 \sqrt{2}} [\sigma^x \pm i \sigma^y] \otimes \sigma^x ,
\]
\[
\psi^2 \psi^2 = -\frac{1}{2 \sqrt{2}} [\sigma^x \mp i \sigma^y] \otimes \sigma^y ,
\]
allowing us to express the shock wave operator and the photon vertex terms as quantum circuits of 2-qubit operations involving tensor products of the Pauli spin operators and the unit operator.

One can evaluate the spin traces in Eq. (12) with the quantum circuit
\[
\begin{array}{c}
\hat{\rho}_c = |0\rangle \langle 0 | \quad \hat{H} \\
\hat{\rho}_n = \mathbb{I}_n / 2^n \quad \Omega_{L,T}
\end{array}
\]
\[
\begin{array}{c}
\hat{\rho}_c = |0\rangle \langle 0 | \quad \text{an auxiliary control qubit} \quad \hat{\rho}_c = |0\rangle \langle 0 | \quad \text{initially, which combine to form the density matrix} \quad \hat{\rho} = \rho_c \otimes \rho_n \quad \text{valid for any unitary qubit} \quad \Omega_{L,T}
\end{array}
\]
\[
\begin{array}{c}
\text{the controlled-} \Omega_{L,T} \text{ gate}, \quad \text{the measurement of the Pauli operators} \quad \sigma^x \text{ and} \quad \sigma^y \text{. The action of the latter on the control qubit yields the real and imaginary part of} \quad \text{Tr}[\Omega_{L,T}], \quad \text{respectively} \quad [63]. \quad \text{The controlled-} \Omega_{L,T} \text{ gate can be straightforwardly constructed and is given in Appendix A. This completes the quantum algorithm to measure the worldline trace in Eq. (12).}
\end{array}
\]
Expanding in complexity and scope. The toy problem we outlined has the virtue that analytical results for $F_2$ are known for specific choices of $W[\rho]$ and therefore provide a benchmark to test our quantum algorithm. It can however be expanded significantly in complexity. To appreciate this, we note that the r.h.s of Eq. (3) is the “in-in” matrix element of a real-time correlation function, where

$$|P, S\rangle = \hat{U}_{0, -\infty}(\hat{\Phi}_{P, S}|0\rangle),$$

represents the state of the hadron (specifically a proton) before its interaction with the virtual photon. Here $\hat{U}(t, t') \equiv \exp(-i\hat{H}(t-t'))$ where $\hat{H}$ is the QCD Hamiltonian. The operator $\hat{\Phi}_{P, S}$ creates a “valence” quark and gluon state with the proton’s quantum numbers from the non-interacting vacuum at past infinity, $\hat{\rho}_{\text{init}} \equiv \hat{\Phi}_{P, S}|0\rangle\langle\hat{\Phi}_{P, S}|0\rangle^\dagger$; the proton is the result of its subsequent evolution with the QCD Hamiltonian.

In the worldline formalism, as outlined in Appendix A we can write Eq. (8) as

$$W^{\mu\nu} = \frac{1}{\pi e^2} \text{Im} \int d^4z \, e^{iqz} \sum_{n=0}^{\infty} \frac{t^{n+4}}{n!} \int [\prod_{k=1}^{n+4} d^4x \delta_{2k}^{(4)} \delta_{2k}^{(4)}] \int dA_1 dA_2 \, \text{tr}_c \langle x_1, -\theta_1, A_1|\hat{\rho}_{\text{init}}|x_2, \theta_2, A_2\rangle \times \langle x_2, \theta_2, A_2|\hat{U}_{-\infty, z}(\hat{\Phi}_{P, S}(z), \hat{U}_{\infty, z}, \theta_1, A_1)\rangle,$$

where $[x, \theta, A] = [x, \theta]|A\rangle$ and $[x, \theta] = \prod_{k=1}^{2} |x_k, \theta_k\rangle$ and we identified worldline $\tau$ with physical time $\tau$. The evolution operator is $\hat{U}_{(t, t')} \equiv \exp\{-i\hat{H}(t-t')\}$ with $\hat{H} = \hat{H}_{\text{YM}} + \hat{A}^{k, n}$. The Yang-Mills Hamiltonian in temporal-axial gauge $\hat{H}_{\text{YM}} = \int d^3x \frac{1}{2} \hat{E}_a^2(x) + \frac{1}{4} \hat{B}_a^2(x)^2$, where $\hat{E}_a$, $\hat{B}_a$ are chromo-electric and chromo-magnetic field operators. The coordinate-fixed Hamiltonian of the $k$-th worldline $\hat{H}^k$ is

$$\hat{H}^k = \frac{1}{2g^2_k}(\hat{P}^2_k + ig\hat{V}^k \hat{F}_{\mu\nu}[A(\hat{x}_k)]\hat{V}^k + ig\hat{\mu} \hat{F}_{\mu\nu}[a(\hat{x}_k)]\hat{\mu}^k),$$

and $J^\mu_{(4)}(z) = -\sum_{k=1}^{\infty} (\delta H^k/\delta a_\mu(z))_{a=0}$ is the worldline electromagnetic current operator. It is important to note that $\hat{\rho}_0$ and $\hat{\rho}_0$ are not dynamical operators and are removed from the physical Hilbert space by Dirac and mass-shell constraints, as discussed in Appendix A.

Also, as further discussed in Appendices A and B the initial density matrix at past infinity $\hat{\rho}_{\text{init}} = \hat{\rho}_{\text{YM}} \otimes \hat{\rho}_f \otimes \hat{\rho}_T \otimes \hat{I}_n$, where $\hat{\rho}_{\text{YM}} = |0\rangle\langle 0|$ is the noninteracting Yang-Mills vacuum, $\hat{\rho}_f$ contains initial conditions for $k = 1, 2, 3$ valence quarks, $\hat{\rho}_f$ is the quark-antiquark dipole $\hat{\rho}_T = \hat{I}$ in the polarization tensor $\Gamma^{\mu\nu}$, and $\hat{I}_n$ is a unit matrix representing the other $k = 5, \ldots, n$ “sea-quark” Fock-states. Eq. (20) is the master formula for computing structure functions as the initial value problem $\partial_t \hat{\rho} = \hat{I}_n[\hat{H}, \hat{\rho}]$ with the initial condition $\hat{\rho}_{\text{init}}$ followed by the measurement of electromagnetic field operators.

The simpler expression for $W^{\mu\nu}$ in Eq. (20) is obtained from Eq. (20) because of the separation of time scales between large and small $x_B$ modes we alluded to previously. Valence quarks $(k = 1, 2, 3)$ and large $x_B$ partons become quasi-static color sources and therefore the tensor product representing their density matrix can be replaced with the weight functional $W[\rho]$ [67]. Further, in Eq. (19), only the polarization tensor $(k = 4)$, representing the virtual photon splitting into a quark-antiquark pair is computed explicitly and the path integral over gauge fields is greatly simplified in the CGC EFT by performing the weak coupling expansion around $A_{\mu}$. Within this EFT framework itself one can make progress by extending the quantum computation to more nontrivial background fields, an example being the sub-eikonal corrections to the shock wave fields [68-70]. Another significant extension is to include higher order terms in the weak coupling expansion of the QCD effective action. The expansion of $\exp(i\Gamma)$ in Eq. (9) includes fermion loops at each order in perturbation theory; these can be simulated by increasing the number of worldline qubits as described previously. Their coupling to gluons and the inclusion of gluon loops can be generated by “gluon worldlines” represented in the effective action as point particle Grassmann bilinears [71].

One can extend the scope of our worldline approach to quantum compute not just structure functions but in principle multi-leg and multi-loop scattering amplitudes [72]. The diagrammatic expansion of Feynman amplitudes in perturbation theory exhibit factorial growth in the number of diagrams at each loop order [22, 73]. In sharp contrast, such computations on a quantum computer with fermionic and bosonic worldline variables would require resources that only scale polynomially in the number of qubits [74]. It is essential for any significant extension beyond the simplest toy problem discussed here that the bosonic worldline variables $\hat{x}$ and $\hat{p}$ be quantum simulated. An appealing strategy [75, 76] is to map $\hat{x}$ and $\hat{p}$ on to the low energy space of a harmonic oscillator (HO) basis. This discrete bosonic basis can be approximated in a binary representation [77] by qubits whose number grows linearly with the dimension of the truncated HO basis [78]. One may also realize bosonic systems using vibrational states in trapped-ion systems [79, 81] or (analog) cold atom systems [82, 83]. The construction of the quantum circuits in full generality for the worldline Hamiltonian in Eq. (11) will be discussed in a forthcoming paper [84].

It is much more nontrivial to go beyond perturbation theory for the gauge fields either in Hamiltonian operator [85] or Euclidean formulations [86] on the lattice. This is a subtle problem and its implementation on analog and digital quantum devices is deservedly a subject of much attention [87, 88], prominent examples being quantum link models [89] and the matrix product state for-
malism\cite{91,96}. It will interesting to investigate how one may integrate these important developments with the worldline approach outlined here.

We note finally that this phase space worldline formalism permits a semi-classical Moyal expansion\cite{97} to construct Wigner functions\cite{39}. These are accessible in DIS\cite{98,99} and therefore allow one to probe systematically, at higher orders in $\hbar$, parton entanglement in QCD at high energies\cite{100,101,102}.

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Appendix A: Proton structure from worldlines - general formalism

The proton state defined by Eq.\ (19) is encoded in a partition function $Z = \text{Tr}(\hat{\rho}_{PS})$, where $\hat{\rho}_{PS} = [P,S]/[P,S]$ is the proton’s density matrix. It is related to the initial state $\hat{\rho}_{\text{init}} \equiv \hat{\Phi}_{PS}(0)\hat{\Phi}_{PS}(0)$ by the QCD evolution operator and may be expressed as the Schwinger-Keldysh path integral\cite{103,104}.

$$Z = \text{Tr}[\hat{\rho}_{PS}] = \text{Tr} \left[ \hat{U}_{(0,-\infty)}\hat{\rho}_{\text{init}}\hat{U}^\dagger_{(-\infty,0)} \right] = \int dA_1 dA_2$$

$$\times \int d\Psi_1 d\Psi_2 \langle A_1, \Psi_1 | \hat{\rho}_{\text{init}} | A_2, \Psi_2 \rangle \int_A D\Psi_1 \int_D \Psi_1 \Psi_2 e^{iS_C},$$

(A1)

where $S_C = \int d^4x \left[ -\frac{1}{2} \left( F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi \right) \right]$ is the QCD action which has support on the Keldysh double time contour $C$ depicted in Fig. (1). We abbreviated $D_{\mu} = \partial_{\mu} - igA_{\mu}$, $dA_{1/2} = \prod_x dA_{1/2}(x)$, where $A_{1/2}(x) = A_1(x/2)$, $DA = \prod_x dA_{1/2}(x,t)$ and likewise for the fermionic integrals.

In Eq. (A1), $\langle A_1, \Psi_1 | \hat{\rho}_{A_2} | A_2, \Psi_2 \rangle$ are matrix elements of an initial density matrix of non-interacting quarks and gluons at $t \to -\infty$, $\hat{\rho}_{\text{init}} = \hat{\rho}_{YM} \otimes \hat{\rho}_V$, where $\hat{\rho}_{YM} = |0\rangle\langle 0|$ is the Yang-Mills vacuum and $\hat{\rho}_V$ is a three valence quark state with the proton’s quantum numbers. Aiming at performing the fermionic path integral, we may write

$$Z = \int dA_1 dA_2 \langle A_1 | \hat{\rho}_{YM} | A_2 \rangle \int_D \Psi_1 \hat{Z}_f[A] \text{ exp } \{iS_C^{YM}\},$$

(A2)

where

$$Z_f[A] \equiv \int d\Psi_1 d\Psi_2 \langle \Psi_1 | \hat{\rho}_V | \Psi_2 \rangle \int_D \Psi_1 \text{ exp } \{iS_C^{YM}\},$$

(A3)

with $iS_C^{YM} = \frac{1}{2} \int d^4z c^\dagger \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi$ and $S_C^{YM} = \int d^4x \left( \bar{\Psi} \gamma^\mu F_{\mu\nu} \Psi \right)$.

Before considering the more complicated case of a baryon with three quarks, we consider first that of a single valence quark. We will also ignore the flavor part of this single valence quark state. Matrix elements for momentum, spin and color of its initial density matrix $\hat{\rho}_q = | p, s, c \rangle \langle p, s, c |$ are given by

$$\langle \Psi_1 | \hat{\rho}_q | \Psi_2 \rangle = \langle \Psi_1 | p, s, c \rangle \langle p, s, c | \Psi_2 \rangle = 2E_p \int d^3x \int d^3x_2 [u_{p,s,c}^\dagger \Psi(x_1)] \langle \Psi^\dagger(x_2) u_{p,s,c} | e^{ip(x_1-x_2)},$$

(A4)

where $u_{p,s,c} \equiv u_{p,s} \otimes \psi_c$ represents the quark spinor and color wave function, and $E_p \approx |p|$ for light quarks. The QCD path integral Eq.\ (A3) may be written in terms of this initial condition as

$$Z_f[A] \equiv \int d\Psi_1 d\Psi_2 2E_p \int d^3x_1 d^3x_2 \left[ u_{p,s,c}^\dagger \Psi(x_1) \right]$$

$$\times \left[ \Psi^\dagger(x_2) \cdot u_{p,s,c} \right] e^{ip(x_1-x_2)} \int_D \Psi_1 \Psi_2 \exp \{iS_C^{YM}\}$$

$$= 2E_p \int d^3x_1 d^3x_2 e^{ip(x_1-x_2)} \left[ u_{p,s,c}^\dagger \gamma_j u_{p,s,c} \right]$$

$$\times \frac{\delta^2}{\partial J_1(x_1) \partial J_2(x_2)} \left[ \int \Psi_1^\dagger \Psi_2 \exp \{iS_C^{YM}\} \right] |_{J=J_0}$$

$$= \text{det} \left[ -i\hat{G}^{-1} \right] 2E_p \int d^3x_1 d^3x_2 e^{ip(x_1-x_2)}$$

$$\times \text{Tr} \left[ u_{p,s,c}^\dagger u_{p,s,c}^\dagger \gamma_0 \hat{G}(x_2,x_1) \right],$$

(A5)
where \[ \int D\Psi' D\bar{\Psi} = \int d\Psi_1 d\Psi_2 \int D^2 \Psi \bar{D} \Psi \] and \( \hat{G} \equiv (i\partial[A] - m)^{-1} \) is the quark propagator. We used that \( \hat{G}(x_1, x_2) \gamma^0 = \gamma^0 \hat{G}(x_2, x_1) \) at equal times \( t_1 = t_2 \) in the last equality of Eq. (A5). The trace in Eq. (A5) and the indices \( i, j \) are over spin and color.

Employing the worldline representation of the fermion determinant, \( \det(-i\hat{G}^{-1}) = \exp(i\Gamma[A]) \), yields

\[
\Gamma[A] = \text{Tr} \log(-i\hat{G}^{-1}) = \text{tr}_c \int \mathcal{D}x \mathcal{D}p \int \mathcal{D}\theta \mathcal{D}\theta^* \int \frac{\mathcal{D}c \mathcal{D}x}{\text{Vol}} \exp \left\{ i \int \mathcal{D}x \bar{\psi} \gamma^\mu \left( i \gamma^\nu \partial_\nu - H[x, p, \theta, \theta^*] \right) \psi \right\},
\]

(A6)

where \( i = 1, 2 \) and the Hamiltonian given by

\[
H[x, p, \theta, \theta^*; A] = \frac{\epsilon}{2} \left( p^2 + ig \psi^\mu F_{\mu\nu}[A] \psi^\nu \right) - i \frac{\epsilon}{2} P_i \psi^i,
\]

(A7)

with \( \psi^0 = (\theta^*_1 - \theta_1)/\sqrt{2}, \psi^3 = (\theta^*_1 + \theta_1)/\sqrt{2}, \psi^1 = (\theta^*_2 + \theta_2)/\sqrt{2} \) and \( \psi^2 = -(i\theta^*_2 - \theta_2)/\sqrt{2} \) in Minkowski spacetime. It is convenient to keep the \( \theta^*_1, \theta^*_2 \), instead of the Majorana representation \( \psi^\mu \) when we give explicit expression of the valence quark initial density matrix in Appendix [3].

Naive quantization of \( \psi^0 \rightarrow \psi^0 \) as in Eq. (7) leads to states with indefinite metric. To consistently quantize the Dirac theory we therefore need to restrict the path integral measure \( D\psi \equiv i\mathcal{D}\theta \mathcal{D}\theta^* \) in Eq. (A6) to the physical subspace of the Hamiltonian in Eq. (A7). The Dirac constraint defining this subspace is implemented via an (anticommuting) Lagrange multiplier variable \( \chi(\tau) \) and the mass-shell constraint via the commuting Lagrange multiplier, the “einbein” \( \epsilon(\tau) \). Note that in this more general real-time formulation the \( dT/T \) integral in Eq. (5) is replaced by the integral over \( \epsilon(\tau) \). For details of this “BRST fixing” of a gauge symmetry related to worldline reparametrization, we refer the reader to the discussion in [34][39]. In this formulation, “Vol” denotes the volume of the gauge group [34].

The Dirac and mass shell constraints can be solved on the operator level to eliminate \( \psi^0 \) and \( p^0 \). These are given by \( \psi^0 = \bar{p}^0 \psi^i/p^0 \), where \( p^0 = \pm |p| \) (at leading order in the coupling \( g \)) acting on states of definite three-momentum \( p \). Solutions of the constraint equation at higher order in \( g \) are given in section III of [37].

Defining,

\[
\langle x_1, -\theta_1|\hat{\rho}_{\bar{c}}|x_2, \theta_2 \rangle \equiv \langle -\theta_1|u_{p,s}u^\dagger_{p,s}\gamma^0|\theta_2 \rangle \Phi_c \Phi^\dagger_c \times 2E_p \delta(x^0_1(\tau = 0) - t_0) \delta(x^0_2(\tau = T) - t_0) e^{ip_\mu(x^\mu_1 - x^\mu_2)},
\]

(A8)

where \( t_0 \rightarrow -\infty \), and using Schwinger’s proper time representation of the quark propagator in Eq. (A5) we write

\[
2E_p \int dx_1 dx_2 \int d^2 \theta_1 d^2 \theta_2 (x_1, -\theta_1)|\hat{\rho}_{\bar{c}}|x_2, \theta_2 \rangle \times \langle x_2 | \hat{\rho}_{\bar{c}}|x_1, \theta_1 \rangle \exp \left\{ i \int \mathcal{D}x \bar{\psi} \gamma^\mu \left( i \gamma^\nu \partial_\nu - H[x, p, \theta, \theta^*] \right) \psi \right\},
\]

(A9)

with this and Eq. (A6), we can formally express the partition function of the proton with three valence quarks as

\[
Z = \text{tr}_c \int dA_1 dA_2 \int \left\{ \prod_{k=1}^3 d^4 x_k d^2 \theta_k d^2 \theta^*_k \right\}
\times \langle A_1|\hat{\rho}_{\bar{c}}|A_2 \rangle \langle x_1, -\theta_1|\hat{\rho}_{\bar{c}}|x_2, \theta_2 \rangle
\times \int A_2 \left\{ \prod_{k=1}^3 \int \mathcal{D}x_k \mathcal{D}p_k \int \mathcal{D}\theta_k \mathcal{D}\theta^*_k \frac{\mathcal{D}c \mathcal{D}x}{\text{Vol}} \right\}
\times \exp \left\{ i S^\text{YM}_C + i \sum_{k=1}^3 S^\text{YM}_k \right\} \exp \left\{ i\Gamma[A] \right\},
\]

(A10)

where the fermion worldline action is

\[
S^C = \int d\tau \left\{ x^{\mu}_k \delta^{\mu\nu} - i \frac{\epsilon}{2} \delta^{k\nu} \theta^i_k \theta^* \hat{\epsilon}_i - H \right\},
\]

(A11)

and the Hamiltonian is given by Eq. (A7). Here \( i = 1, 2 \) label the components of the Grassmann variables defined in Eq. (3), while \( k \) labels the valence quarks. The expression for the three valence quark initial density matrix for this worldline path integral including spin, color and flavor is given in Appendix [3]. For the sake of a compact notation, we omit henceforth writing \( \mathcal{D}c \mathcal{D}x \) explicitly and shall instead consider it to be part of the worldline path integral measure.

Upon (gauge-)fixing the worldline parametrization, by identifying worldline time \( \tau \in [0, T] \) with physical time \( x^{0}_{1/2} \) for upper and lower Keldysh contours,

\[
x^{0}_1 \equiv x^{0}(\tau) = t_0 + \tau, \quad \tau \leq T/2,
\]

\[
x^{0}_2 \equiv x^{0}(\tau) = t_0 + (T - \tau), \quad \tau > T/2,
\]

(A12)
Eq. [A10] becomes a Schwinger-Keldysh path integral in physical time $x^0$ for the time evolution of bosonic and Grassmann worldline variables as well as Yang-Mills fields. In this coordinate-fixed formulation, we set the einbein parameter $\epsilon = 1/p^0$ and employ the temporal-axial gauge $A^0 = 0$.

The hadron tensor in Eq. [5] is defined by an “in-in” matrix element of time-ordered electromagnetic currents. Introducing (auxiliary) electromagnetic fields $a_\mu(x)$, we can relate this matrix element to the proton partition function Eq. [A10],

$$
\langle P, S | T \tilde{j}^\mu(x) \tilde{j}^\nu(0) | P, S \rangle = \text{Tr} \left[ \hat{\rho}_{P,S} T \{ \tilde{j}^\mu(x) \tilde{j}^\nu(0) \} \right]
$$

$$= \frac{\delta^2 Z}{i \partial a_\mu(z) i \partial a_\nu(0)}. \quad (A13)
$$

Using the definition

$$
\left[ x^k \prod_{k=1}^3 \int D x^k D p^k \frac{\theta^k}{\theta^k t} \int D \theta^k D \theta^{* -k} \right] \exp \left\{ i \sum_{k=1}^3 S_k^c \right\}
$$

$$= \langle x_2, \theta_2 | \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] | \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] | x_1, \theta_1 \rangle, \quad (A14)
$$

where the worldline time evolution operator for three valence quarks is

$$\hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \equiv \exp \left\{ - i \sum_{k=1}^3 \hat{H}_k[A; a](t - t') \right\}, \quad (A15)
$$

we can write the hadron tensor as

$$W^{\mu\nu}(q, P, S) = \frac{1}{\pi \epsilon^2} \text{Im} \text{tr}_c i^3 \int d^4 x e^{iqz} \int dA_1 dA_2 \times \int \left\{ \prod_{k=1}^3 d^4 x_d^2 d^4 \theta^k d^2 \theta^k \right\} \langle A_1 | \hat{\rho}_{YM}[A_2] \rangle \times \langle x_1, -\theta_1 | \hat{H}_2[x_2, \theta_2] | A_2 \rangle \times \langle \{ x_2, \theta_2 | \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] | x_1, \theta_1 \} \times \exp \left\{ i S_c^{YM}[A] + i \Gamma[A; a] \right\} \bigg|_{t=0} \bigg|_{t' = 0} \bigg|^2 \right\}.
$$

Here, $\Gamma[A; a]$ is given in Eq. [A6]. The time ordering in this expression is such that one current operator insertion is on the upper (representing the amplitude), the other on the lower (representing the conjugate amplitude) Keldysh contour [199].

The derivative in Eq. (A16) yields two terms,

$$W^{\mu\nu}(q, P) = \frac{1}{\pi \epsilon^2} \text{Im} \text{tr}_c i^3 \int d^4 x e^{iqz} \int dA_1 dA_2 \langle A_1 | \hat{\rho}_A[A_2] \rangle \times \int \left\{ \prod_{k=1}^3 d^4 x_d^2 d^4 \theta^k d^2 \theta^k \right\} \langle x_1, -\theta_1 | \hat{H}_2[x_2, \theta_2] | A_2 \rangle \times \langle \{ x_2, \theta_2 | \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] | x_1, \theta_1 \} \times \exp \left\{ i S_c^{YM}[A] + i \Gamma[A; a] \right\} \bigg|_{t=0} \bigg|_{t' = 0} \bigg|^2 \right\}.
$$

The first term is the valence quark contribution where the derivative acts on any of the valence quarks,

$$\frac{\delta}{i \partial a_\mu(z)} U_{(+\infty, -\infty)}^{(3)}[A; a] \equiv U_{(+\infty, -\infty)}^{(3)}[A; a] \hat{j}_\mu^{(3)}(z) U_{(+\infty, -\infty)}^{(3)}[A; a] \equiv \sum_{k=1}^n \hat{j}_k^{(3)}(z) \right\},
$$

as depicted at the top of Fig. (3). Here, $\hat{j}_k^{(3)}(z) = \sum_{j=1}^n \hat{j}_j^{(3)}(z)$ can be explicitly computed by varying the worldline Hamiltonian Eq. (21),

$$\hat{j}_k^{(3)}(z) = \frac{e}{\hbar_k} \left[ \hat{p}_k + i \hat{\psi}_k \hat{\psi}^*_k \right] \delta^{(3)}(z - \hat{x}_k(z^0)). \quad (A19)
$$

The second term in Eq. (A17), where the derivative acts on the exponential $\exp\{i \Gamma[A; a]\}$, yields the photon polarization tensor,

$$i \Gamma^{\mu\nu}[A](z, 0) \equiv \text{tr}_c i \int d^4 x d^2 \theta(x, -\theta) \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \times \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \bigg|_{x, \theta} \bigg|_{x, \theta} \bigg|^2.
$$

In the dipole picture, this term may be understood as the virtual photon fluctuating into a quark-antiquark pair which subsequently interacts with the color field of the target (see bottom figure of Fig. (3)). This term provides by far the dominant contribution to $F_2$ in the high energy limit of the CGC EFT with the first term suppressed by $x_{Bj}$ as $x_{Bj} \to 0$.

To bring Eq. (A17) into a form useful for quantum simulation, we insert a complete set of states into Eq. (A20) and write

$$i \Gamma^{\mu\nu}[A](z, 0) = \text{tr}_c i \int d^4 x_1 d^4 x_2 \int d^2 \theta_1 d^2 \theta_2 \times \langle x_1, -\theta_1 | \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \hat{U}_{(+\infty, -\infty)}^{(3)}[A; a] \bigg|_{x_1, \theta_1} \bigg|_{x_2, \theta_2} \bigg|^2 \right\}.
$$

FIG. 3. Top: photon interaction with valence quarks. Bottom: Interaction with quark-antiquark pairs created from the color field of the proton. The red-dashed line denotes the imaginary part taken via Cutkosky rules or equivalently the separation of the amplitude and the conjugate amplitude in this process.

The first term is the valence quark contribution where the derivative acts any of the valence quarks,
We then perform a loop expansion of \( \exp(i\Gamma[A]) = \sum_{n=0}^{\infty}(i\Gamma[A])^n/n! \), allowing us to express the hadron tensor Eq. (A17) as

\[
W^{\mu\nu}(q, P, S) = \sum_{n=0}^{\infty} \frac{1}{n!} W^{\mu\nu}_{(n)}(q, P, S),
\]

where the \( n \)-quark loop contribution is given by

\[
W^{\mu\nu}_{(n)} = \frac{1}{\pi \alpha^2} \text{Im} \text{tr} \int d^4z \, e^{iq \cdot z} \int \left[ \prod_{k=1}^{4+n} d^4x_k d^4x_\ell d^3\theta^k d^2\theta^\ell \right] \times i^{4+n} \times dA_1 dA_2 \langle A_1| \hat{\rho}_A |A_2 \rangle \langle x_1, -\theta_1| \hat{\rho}_V \otimes \hat{\rho}_T \otimes \mathbb{I}_n |x_2, \theta_2 \rangle \times (x_2, \theta_2, A_2) \hat{U}_{(-\infty,z)}(\hat{\mu}_j(z) \hat{U}_{(z,\infty)}(x_2, \theta_2) \times \hat{U}_{(\infty,0)} \hat{U}_{(0,-\infty)} |x_1, \theta_1, A_1 \rangle.
\]

The worldline and Yang-Mills evolution operator is

\[
\hat{U}(t, t') = \exp\{-i\hat{H}(t - t')\} + \sum_{k=1}^{4+n} \hat{H}^k
\]

where \( \hat{H} = \hat{H}_{\text{YM}} + \hat{H}_{\text{color}} \) and the Yang-Mills Hamiltonian in terms of Grassmann coherent states,

\[
\hat{H}_{\text{YM}} = \int d^3x \left( \frac{1}{2} \hat{E}^a(x)^2 + \frac{1}{2} \hat{B}^a(x)^2 \right),
\]

where \( \hat{E}^a(x) \) and \( \hat{B}^a(x) = e^{ijk} \hat{E}^{a,ijk}(x)/2 \) are the chromo-electric and chromo-magnetic field operators respectively.

In the CGC EFT, as noted in the main text, this expression simplifies considerably to the one in Eq. (9), whereby the contribution of the valence quarks and large \( x_B \) partons is absorbed into the weight functional \( W[\rho] \). Further, for the dipole model computation depicted in Fig. (4), it is sufficient to keep only the \( n = 0 \) term in Eq. (A22).

**Appendix B: Worldline representation of the proton’s spin, flavor and color valence structure**

In this Appendix, we discuss the representation of the proton’s initial density matrix composed of the valence quarks, including color, spin and flavor. The valence wave function of a proton polarized along its momentum \( (S = 1/2) \) is a direct product of the a color-singlet and a combined spin/flavor part \( \hat{\Phi}_{P,S}|0\rangle = |\text{color}\rangle \otimes |\text{spin/flavor}\rangle \).

\[
|\text{color}\rangle = \frac{1}{\sqrt{6}} \sum_{ijk} e^{ijk}|i,j,k\rangle.
\]

We outline first the worldline representation of the spin part. The single quark spin density matrix with spin \( s = \pm 1/2 \) along the three-momentum direction \( p \) can be written as

\[
u_{P,S}^\pm|\psi_P^\pm⟩ = \frac{1}{2}(1 + 2s\gamma_5)\rho_\gamma^0 |0⟩
\]

Using the representation of the Clifford algebra of the Lorentz group and the representation of the chirality matrix in terms of fermion creation and annihilation operators \( \hat{b}_i^\dagger, \hat{b}_i \),

\[
\hat{\gamma}_5 = -(1)\sum_{j=1}^2 \hat{b}_j^\dagger \hat{b}_j = -\sum_{j=1}^2 (1 - 2\hat{b}_j^\dagger \hat{b}_j),
\]

and assuming a right-moving valence quark with large \( p^+ \approx \sqrt{2}|p| \), we obtain

\[
\nu_{P,S}^\pm|\psi_P^\pm⟩ = |p⟩ \left[ 1 - 2s \sum_{j=1}^2 (1 - 2\hat{b}_j^\dagger \hat{b}_j) \right] (1 - \hat{b}_1^\dagger \hat{b}_1).
\]

To write the color structure of the worldline density matrix and the effective action, we first introduce the representation of the color trace \( \text{tr}_c \) of an operator \( \hat{O} \) in terms of Grassmann coherent states,

\[
\text{tr}_c \hat{O} = \int d^3\lambda \langle -\lambda | \hat{O} |\lambda \rangle.
\]

More generally, the trace over a path ordered color matrix exponential in this formalism is

\[
\text{tr}_c \mathcal{P} \exp \left[ i \int_0^T d\tau M(\tau) \right] = \int D\phi \int D\lambda \lambda^\dagger D\lambda
\]

\[
\times e^{i\phi(\lambda^\dagger \lambda + \frac{2}{3n})} \exp \left[ i \int_0^T d\tau (i\lambda^\dagger \frac{d\lambda}{d\tau} - \lambda^\dagger M\lambda) \right].
\]
where the Grassmann variables \( \lambda_i, \lambda_i^* \), with \( i = 1, 2, 3 \) being Wigner-Weyl symbols of fermionic operators \( \hat{c}_i, \hat{c}_i^\dagger \),
\[
\hat{c}_i(\lambda) = \lambda_i(\lambda), \quad \hat{c}_i^\dagger(\lambda^*) = \lambda_i^*(\lambda^*), \tag{B9}
\]
similar to Eqs. (23) for spin. In Eq. (33), \( \phi \) is the Lagrange multiplier implementing the constraint restricting the fermion creation and annihilation operators to act on a finite dimensional representation of \( SU(3) \). The color-matrix valued coordinate fixed worldline Hamiltonian in Eq. (21) may therefore be generalized to
\[
\hat{H}^k = \frac{1}{2 p_k^0} \left( \hat{P}_k^2 + i g \hat{\psi}^a_k \gamma^k F^a_{\mu\nu} [A(\hat{x}_k)] T^a_{bc} c^b \hat{\psi}^c_k \right)
+ i g \hat{\psi}^a_k F^a_{\mu\nu} [a(\hat{x}_k)] \hat{\psi}^\dagger_k, \tag{B10}
\]
Here, \( \hat{P}_\mu = p_\mu - i g \hat{c}^a_i \mu^a(x) \hat{c}_i - i e a_\mu(x) \) and \( n = 3 \) is the dimension of the matrix \( M(\tau) \).

A single quark color matrix can be written in terms of the unit matrix \( \mathbb{I}_3 \) and the \( SU(3) \) generators \( t^a \) (in fundamental representation) as \( \hat{\rho}_{\text{color}} = \mathbb{I}_3/3 + \sum_{a=1}^{8} \phi^a t^a \), where \( \phi^a \) are c-number coefficients. In the worldline formulation, where color traces are expressed through Grassmann coherent states, this may equivalently written as
\[
\hat{\rho}_{\text{color}} = 1 + \frac{2 a_{abc}}{A_R d^2 t^a_{ij} t^b_{kl} t^c_{mn} c_i^j c_k l^c_m c_n} + 2 \phi^a t^a_{ij} c_i^j , \tag{B11}
\]
as derived in (39).

In the (r,g,b) basis the coefficients \( \phi^a \) are
\[
\phi^a = \left\{ \begin{array}{ll}
\delta^{a3} + \frac{1}{\sqrt{3}} \delta^{a8} & \text{(red)} \\
-\delta^{a3} + \frac{1}{\sqrt{3}} \delta^{a8} & \text{(green)} \\
-\frac{1}{\sqrt{3}} \delta^{a8} & \text{(blue)}
\end{array} \right. \tag{B12}
\]

The color-flavor-spin density matrix of a baryon, containing three valence quarks is written as the product of a symmetric spin-flavor Eq. (17) and an antisymmetric color-singlet part Eq. (15). Employing the Jordan-Wigner transformation, as in the spinor case, the color density matrix may be written as a three-qubit quantum circuit.

Appendix C: Quantum Computation of \( F_2 \) in the Color Glass Condensate effective field theory

We will outline here the key elements in the hybrid computation of the structure function \( F_2 \), starting from the worldline representation of the effective action in the shock wave background field \( A_{cl} \), given by
\[
\Gamma[A_{cl}, a] = -\frac{i}{2} \text{Tr} \int_0^\infty \frac{dT}{T} \int d^4 x \int d^2 \theta \sum_{i,j=1}^N \langle x, -\theta | \hat{S}_{i+1,j-1}(a) \rangle e^{-i H_{[A_{cl}, a]} T} \hat{S}_{i+1,j-1}(a) | x, \theta \rangle. \tag{C1}
\]

where the worldline Hamiltonian \( \hat{H}[A_{cl}, a] \) is defined by Eq. (7) and depends on both the external electromagnetic field \( a_\mu(x) \) and background gluon field of the target \( A_\mu(x) \). In the Regge limit of QCD, the imaginary part of the polarization tensor \( \Gamma^{\mu\nu} \) in Eq. (10) has the physical interpretation of the virtual photon (emitted by the electron) splitting into a color singlet \( q\bar{q} \) “dipole” long before its interaction with the target. This interaction, which has the structure of a shock wave, is instantaneous and localized on the world-line at two arbitrary instants \( \tau \) and \( \tau' \) in proper time.

The quantum computation of the trace over the fermionic degrees of freedom in \( \Gamma \), given by the integral \( \int d^2 \theta \), is our principal objective here. As discussed in Appendix B, the trace over color can also be expressed as a quantum mechanical path integral in the worldline action. For the purposes of this computation, to keep things simple, we will keep the trace over color explicit.

We first divide the worldline effective action into \( N \) segments of size \( \delta \tau \). Since the interaction \( A_{cl} \) is localized, at most two segments of the worldline can interact with the background field. As a result, we get
\[
\Gamma[A_{cl}, a] = -\frac{i}{2} \text{Tr} \int_0^\infty \frac{dT}{T} \int d^4 x \int d^2 \theta \sum_{i,j=1}^N \langle x, -\theta | \hat{S}_{i+1,j-1}(a) \rangle e^{-i H_{[A_{cl}, a]} T} \hat{S}_{i+1,j-1}(a) | x, \theta \rangle. \tag{C2}
\]
Here \( \hat{H}_n[A_{cl}, a = 0] \) is the Hamiltonian of segment number \( n \) where the interaction with the shock wave background field is located and where the photon field \( a_\mu(x) = 0 \). Further, \( \hat{S}_{m,n}(a) \) denotes segments where the converse is true; the worldline only interacts with the external photon field:
\[
\hat{S}_{m,n}(a) = e^{-i H_n[A=0,a] \delta \tau} ... e^{-i H_{n-1}[A=0,a] \delta \tau} \tag{C3}
\]
In general, there could be a segment of the worldline containing both the shock wave field \( A_{cl} \) and the photon field \( a_\mu \) defined by the general operator \( \exp(-i H_n[A, a] \delta \tau) \).
Examples of these segments are shown in Fig. (5)(c) and (d). Indeed, there are several contributions of this type in Eq. (10) and one has to sum over all of these contributions. However, in the computation of $F_2$, there is a cancellation between terms with the structure of Figs. (5) and (14). As a result, only the diagrams in Fig. (14) and Fig. (6) contribute. To discuss the contribution of Fig. (14) to Eq. (C2), one first dresses the shock wave interaction segment (labeled $r$ below) with two momentum states $p_1$ and $p_2$ using the completeness relation

$$\int d^4p_{1,2} \langle p_{1,2} | = 1, \tag{C4}$$

yielding

$$\langle p_1 | e^{-i\hat{H}|A=0,a|\delta r} | p_2 \rangle = ie\langle p_1 | (\hat{p}_a + \hat{a}_p - 2i\hat{\psi}^\mu \hat{\psi}^\nu \partial_\mu \partial_\nu) \delta r | p_2 \rangle. \tag{C5}$$

The r.h.s here is obtained by expanding the evolution operator in Eq. (C2) up to linear terms in the photon field $a_{\nu}(x)$. (Since there are no contact terms in Fig. (6), it is safe to omit quadratic terms in the expansion).

Next inserting first a complete set of coordinate states, and subsequently taking the Fourier transformation of the first derivative of Eq. (C5) with respect to the external photon field, we obtain the relation

$$\int d^4z \frac{\delta}{\delta a_\mu(z)} \langle p_1 | e^{-i\hat{H}|A=0,a|\delta r} | p_2 \rangle e^{ikz} \tag{C6}$$

$$= ie \int d^4z \int d^4x \frac{\delta}{\delta a_\mu(z)} \langle p_1 | (p_1 a(x) + a(x)p_2 - 2i\hat{\psi}^\rho \hat{\psi}^\sigma \partial_\rho a_\sigma(x)) \langle x | p_2 \rangle e^{ikz} \delta r \tag{C6}$$

$$= ie \int d^4x \langle p_1 | [ (p_1^\mu + p_2^\mu) - 2\hat{\psi}^\rho \hat{\psi}^\sigma k_\rho ] \langle x | p_2 \rangle e^{ikz} \delta r,$$

which gives the structure of the interaction segment of the worldline with the external photon of momentum $k$.

Similarly, one can derive the form of the interaction segment between the worldline and the shock wave:

$$\langle p_1 | e^{-i\hat{H}|A,a=0|\delta r} | p_2 \rangle = - \int d^4x \int \frac{dk^+}{2\pi} e^{-ik^+x^-}$$

$$\times \langle p_1 | [ (p_1^\mu + p_2^\mu) + 2i\hat{\psi}^\rho \hat{\psi}^\sigma \partial_\rho] \times e^{-ig \int A^\mu_{\perp}(z^-)dz^-} \langle x | p_2 \rangle e^{ikz} \delta r, \tag{C7}$$

where the phase in $A^\mu_{\perp}$ denotes multiple scattering off the background field. Substituting the explicit form of the interaction vertices Eqs. (C6) and (C7) into Eq. (C2), yields for the contribution from Fig. (14).

$$W_{\mu\nu}(a, P) = -\frac{P^+ + P^-}{16\pi q^2} \int d^2x_\perp \int \frac{d^2k_\perp}{(2\pi)^3} e^{ik_\perp x_\perp} \frac{1}{(p_1^2 + m^2 + z(1-z)Q^2)} \{ (p_1 - k)^2 + m^2 + z(1-z)Q^2 \}$$

$$\times \left\{ i \int d^2\theta_0 \left[ (2p^\perp \cdot k^\perp - K^\perp_0 (Q^2 x_\perp) - 2 \hat{\psi}^\rho \hat{\psi}^\sigma k_\rho \right] e^{-i \int A^\mu_{\perp}(z^-)dz^-} \right\} \delta r,$$

where we integrated over intermediate coordinates and momenta, as well as over the positions of the interaction segments on the worldline. Note that we introduced the variable $z = p^- / q^-$. The diagram in Fig. (6)(b) has a similar structure allowing one to compute $W_{\mu\nu}(a, P)$. Further technical details of the worldline computation of this “dipole model” can be found in Ref. [60].

To compute $F_2$, we need to sum both contributions and consider the projection

$$F_2 = \frac{Q^2}{4P \cdot q} \left[ \frac{3Q^2}{(P \cdot q)^2} P^\mu P^\nu - q^\mu q^\nu \right] W_{\mu\nu}, \tag{C9}$$

where $W_{\mu\nu} = W_{\mu\nu}^{(a)} + W_{\mu\nu}^{(b)}$. After integrating over transverse momenta $p_\perp$ and $k_\perp$ and summing over flavors $f$, we get our final result

$$F_2(q, P) = \frac{Q^2 \sigma N_c}{16\pi^3} \sum_f \frac{e^2}{f} \int d^4z \int dx_\perp x_\perp \left[ 1 - \frac{1}{N_c} \text{tr}_c U(x_\perp) U^\dagger(0_\perp) \right]$$

$$\times I \int d^2\theta_0 \left[ - 3Q^2 z(1-z)[(2z - 1) + 2 \hat{\psi}^\rho \hat{\psi}^\sigma][2z - 1] \hat{\psi}^\rho \hat{\psi}^\sigma K_0^2(Q_0^2 x_\perp) + 2\{z^2 + (1-z)^2\} Q_0^2 K_0^2(Q_0^2 x_\perp) - 4z(1-z)Q^2 \hat{\psi}^\rho \hat{\psi}^\sigma \hat{\psi}^\delta \hat{\psi}^\sigma K_0^2(Q_0^2 x_\perp) - 4z(1-z)Q^2 \hat{\psi}^\rho \hat{\psi}^\sigma \hat{\psi}^\delta \hat{\psi}^\sigma K_0^2(Q_0^2 x_\perp) + (1-z)(4z^2 - 4z + 3)Q^2 K_1^2(Q_1^2 x_\perp) + 2m_f^2 K_0^2(Q_0^2 x_\perp) \right\} \theta_0$$

where $Q_0^2 = z(1-z)Q^2 + m_f^2$ and $U(x_\perp) = \exp \left\{ -ig \int_{-x_\perp/2}^{x_\perp/2} dz^- A^\mu(x^-, x_\perp) \right\}$

and is the result quoted in Eq. (10). These $x_\perp$ indepen-
dent terms stem from the diagram in Fig. 3b and ensure the UV finiteness of the expression at this order. Finally combining the terms containing the MacDonald functions $K_{0,l}$ into the well-known $\gamma^* \rightarrow \bar{q}q$ wave functions given in Eqs. (13-14), we arrive at our final expression for $F_2$ quoted in Eq. (12). As noted earlier, $F_L$ can be obtained by taking a different kinematic projection of $W^{\mu\nu}$.

**Appendix D: Quantum Circuit for the worldline computation of $F_2$**

In this Appendix, we present details of the quantum circuits required for the worldline computation of $F_2$ in Eq. (12). To compute the trace in Eq. (12) we employ the circuit in Eq. (18). Here the $n=2$ circuit qubits are initially in a mixed state with density matrix $\rho_2 = I_2/2^2$, while an additional control qubit (in a pure state $|0\rangle$) is used. The combined density matrix of control and circuit qubits after employing Eq. (18) is

$$\rho_{2+c} = \frac{1}{2^2} \left( \mathbb{1}_2 \bigotimes \Omega_{L,T}^\dagger \mathbb{1}_2 \right),$$

where measurement of $\sigma^x$ and $\sigma^y$ on the control qubit yields

$$\langle \sigma^x \rangle = \text{Tr}[\sigma^x \hat{\rho}] = \frac{1}{2^2} \text{Re}[\text{Tr} \Omega_{L,T}],$$

$$\langle \sigma^y \rangle = \text{Tr}[\sigma^y \hat{\rho}] = -\frac{1}{2^2} \text{Im}[\text{Tr} \Omega_{L,T}].$$

A crucial ingredient in Eq. (18) is the controlled gate $C(\Omega_{L,T}) = \left( \begin{array}{cc} \mathbb{1}_2 & 0 \\ 0 & \Omega_{L,T} \end{array} \right)$. Since $\Omega_{L,T}$ is decomposed into more fundamental gates $\Omega_{L,T} = \prod_i G_i$, where the $G_i$ stand for the Hadamard gate $H = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)$ and the phase gate $S = \left( \begin{array}{cc} 1 & 0 \\ 0 & i \end{array} \right)$, we can construct $C(\Omega_{L,T})$ from the controlled gates of its constituents [77],

$$C(\Omega_{L,T}) = \prod_i C(G_i).$$

This allows us to write the control circuit of Eq. (17).

We can also use the fact that $C(\sigma^z)$ is a standard gate available on present and future hardware to write $C(\sigma^x)$ and $C(\sigma^y)$,

$$\sigma^x = H \sigma^z H, \quad \sigma^y = SH \sigma^z HS^\dagger,$$

using

$$SH \sigma^z HS^\dagger = \sigma^y,$$

$$H \sigma^z H = \sigma^x,$$

where $\{SH \} = \{ S \bigotimes H \}$. Implementations of controlled Hadamard- and phase-gates can be found for example in [77, 108].

A significant extension of our simple toy problem is the quantization of the bosonic and fermionic worldline variables in Minkowski space and their real-time Hamiltonian evolution. We will discuss the Hamiltonian quantization of the worldline theory, including that of the bosonic $\hat{x}$ and $\hat{p}$, in a forthcoming paper [84]. While quantum simulating the bosonic part of the worldline Hamiltonian will require a large number of qubits, we have shown here that the fermionic spin (and color) parts are still fairly simple, requiring two (plus three for color) qubits per worldline.

Unitary time evolution of the consistently quantized worldline Hamiltonian operator $\hat{H}$ discussed in Appendix A can be realized by Suzuki-Trotterization at $O(\delta \tau)$,

$$\hat{U}[\tau] = \lim_{N \to \infty} \left( \prod_{i=1}^N e^{-i\hat{H}(\tau_i)\delta \tau} \right)^N$$

For $\delta \tau = \tau/N$ and sufficiently small $\delta \tau$ one may further write each term in Eq. (D8)

$$e^{-i\hat{H}(\tau_i)\delta \tau} = \prod_k e^{-i\hat{H}_k(\tau_i)\delta \tau} + O(\delta \tau^2),$$

where $\hat{H} = \sum_k \hat{H}_k$, with each term a tensor decomposition of Pauli matrices with real coefficients acting on $(2+3)N_{WL} + N_B$ qubits. Here $(2+3)N_{WL}$ is the number of qubits realizing the fermionic color and spin degrees of freedom, and $N_B$ is the number of qubits for the bosonic $\hat{x}$ and $\hat{p}$, scaling linearly with the volume of the physical subspace for $(x^\mu, \mu^\nu)$.

All terms in Eq. (D9) can be expressed via (n-qubit) $z$-rotations utilizing the standard identities Eq. (D6). To avoid writing large circuits, we illustrate our strategy for the smallest nontrivial number of qubits $n = 2$. In this case, we can write all possible matrix exponentials of tensor products of $\sigma^x$ and $\sigma^y$ operators as

$$e^{i\sigma^x \otimes \sigma^x} = [H \otimes H] e^{i\sigma^x \otimes \sigma^x} [H \otimes H],$$

$$e^{i\sigma^y \otimes \sigma^y} = [SH \otimes SH] e^{i\sigma^y \otimes \sigma^y} [HS^\dagger \otimes HS^\dagger],$$

$$e^{i\sigma^x \otimes \sigma^y} = [H \otimes SH] e^{i\sigma^x \otimes \sigma^y} [H \otimes HS^\dagger],$$

$$e^{i\sigma^y \otimes \sigma^x} = [SH \otimes H] e^{i\sigma^y \otimes \sigma^x} [HS^\dagger \otimes H],$$

or alternatively in circuit notation

$$\begin{align*}
\begin{array}{c}
\sigma^x \otimes \sigma^x = H \begin{array}{c}
\sigma^x \otimes \sigma^x \end{array}
\end{array}
\end{align*},$$

$$\begin{align*}
\begin{array}{c}
\sigma^y \otimes \sigma^y = \begin{array}{c}
\sigma^y \otimes \sigma^y
\end{array}
\end{array}
\end{align*},$$

$$\begin{align*}
\begin{array}{c}
\sigma^x \otimes \sigma^y = \begin{array}{c}
\sigma^x \otimes \sigma^y
\end{array}
\end{array}
\end{align*},$$

$$\begin{align*}
\begin{array}{c}
\sigma^y \otimes \sigma^x = \begin{array}{c}
\sigma^y \otimes \sigma^x
\end{array}
\end{array}
\end{align*}.$$
The remaining circuit uses the CNOT gate \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) for the exponential of \( \sigma^z \) operators \( \exp \left( i \sigma^z \otimes \cdots \otimes \sigma^z \right) \).

\[
e^{i \sigma^z \otimes \cdots \otimes \sigma^z} = \begin{pmatrix} SH \otimes \cdots \otimes H \end{pmatrix} \cdot e^{i \sigma^z \otimes \cdots \otimes \sigma^z} \cdot \begin{pmatrix} SH \otimes \cdots \otimes H \end{pmatrix}^{-1}, \quad \text{(D17)}
\]

of \( n \) qubits is found in textbooks \cite{77}.

\[
n \left\{ e^{i \sigma^z \otimes \cdots \otimes \sigma^z} \right\} = \begin{pmatrix} 0 \rangle \otimes \cdots \otimes \begin{pmatrix} 0 \rangle \otimes e^{i \sigma^z} \end{pmatrix} \rangle, \quad \text{(D18)}
\]

and uses one ancilla qubit. Finally, using \( \exp (i \sigma^z \otimes \cdots \otimes \sigma^z) = I \otimes \cdots \otimes \exp (i \sigma^z \otimes \cdots) \), we have outlined the gate sequences required to implement unitary time evolution of the worldline Hamiltonian.

\[\text{References}\]

1. M. Breidenbach, J. I. Friedman, H. W. Kendall, E. D. Bloom, D. H. Coward, H. C. DeStaebler, J. Drees, L. W. Mo, and R. E. Taylor, Phys. Rev. Lett. 23, 935 (1969).
2. J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
3. J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).
4. D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973).
5. J. Blumlein, Prog. Part. Nucl. Phys. 69, 28 (2013), arXiv:1208.6087 [hep-ph].
6. A. M. Cooper-Sarkar, R. C. E. Devenish, and A. De Roeck, Int. J. Mod. Phys. A13, 3885 (1998), arXiv:hep-ph/9712031 [hep-ph].
7. M. Arneodo, Phys. Rept. 240, 301 (1994).
8. G. F. Sterman, An Introduction to quantum field theory (Cambridge University Press, 1993).
9. F. Winter, W. Detmold, A. S. Gambhir, K. Orginos, M. J. Savage, P. E. Shanahan, and M. L. Wagan, Phys. Rev. D96, 094512 (2017), arXiv:1709.00395 [hep-lat].
10. X. Ji, Phys. Rev. Lett. 110, 262002 (2013), arXiv:1305.1539 [hep-lat].
11. C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, K. Jansen, F. Steffens, and C. Wiese, Phys. Rev. D92, 014502 (2015), arXiv:1504.07455 [hep-lat].
12. J.-W. Chen, S. D. Cohen, X. Ji, H.-W. Lin, and J.-H. Zhang, Nucl. Phys. B911, 246 (2016), arXiv:1603.06664 [hep-ph].
13. A. Radyushkin, Phys. Lett. B767, 314 (2017), arXiv:1612.05170 [hep-ph].
14. H.-W. Lin et al., Prog. Part. Nucl. Phys. 100, 107 (2018), arXiv:1711.07916 [hep-ph].
15. W. Detmold, R. G. Edwards, J. J. Dudek, M. Engelhardt, H.-W. Lin, S. Meinl, K. Orginos, and P. Shanahan (USQCD), (2019), arXiv:1904.09512 [hep-lat].
16. G. Ortiz, J. E. Gubernatis, E. Knill, and R. Laflamme, Phys. Rev. A64, 022319 (2001), [Erratum: Phys. Rev. A65,029902(2002)], arXiv:cond-mat/0012334 [cond-mat].
17. A. Alexandrou, P. F. Bedaque, H. Lamm, and S. Lawrence (NuQS), (2019), arXiv:1903.06577 [hep-lat].
18. J. Preskill, Quantum 2, 79 (2018).
19. S. Lloyd, Science 273, 1073 (1996).
20. J. I. Cirac and P. Zoller, Nature Physics 8, 264 (2012).
21. P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Reports on Progress in Physics 75, 082401 (2012).
22. S. P. Jordan, K. S. M. Lee, and J. Preskill, Quant. Inf. Comupt.14,1014 (2014) (2011), arXiv:1112.4833 [hep-th].
23. J. Carlson, D. Dean, M. Hjorth-Jensen, D. Kaplan, J. Preskill, K. Roche, M. Savage, and M. Troyer, Institute For Nuclear Theory Report 18-008 (2018).
24. J. Preskill, Proceedings, 36th International Symposium on Lattice Field Theory (Lattice 2018): East Lansing, MI, United States, July 22-28, 2018. PoS LATTICE2018, 024 (2018) arXiv:1811.10085 [hep-lat].
25. H. Lamm, S. Lawrence, and Y. Yamauuchi (NuQS), (2019), arXiv:1903.08807 [hep-lat].
26. M. J. Strassler, Nucl. Phys. B385, 145 (1992), arXiv:hep-ph/9205205 [hep-ph].
27. E. D’Hoker and D. G. Gagne, Nucl. Phys. B467, 297 (1996) arXiv:hep-th/9512080 [hep-th].
28. E. D’Hoker and D. G. Gagne, Nucl. Phys. B467, 272 (1996) arXiv:hep-th/9508131 [hep-th].
29. M. Mondragon, L. Nellen, M. G. Schmidt, and C. Schubert, Phys. Lett. B351, 200 (1995) arXiv:hep-th/9502125 [hep-th].
30. M. Mondragon, L. Nellen, M. G. Schmidt, and C. Schubert, Phys. Lett. B366, 212 (1996), arXiv:hep-th/9510036 [hep-th].
31. A. Hernandez, T. Konstandin, and M. G. Schmidt, Nucl. Phys. B812, 290 (2009) arXiv:0810.4092 [hep-ph].
32. J. Jalilian-Marian, S. Jeon, R. Venugopalan, and J. Wirstam, Phys. Rev. D62, 045020 (2000), arXiv:hep-ph/9910299 [hep-ph].
33. C. Schubert, Phys. Rept. 355, 73 (2001) arXiv:hep-th/0010136 [hep-th].
34. F. Bastianelli and P. van Nieuwenhuizen, Path integrals and anomalies in curved space Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2006).
35. O. Corradini and C. Schubert (2015) arXiv:1512.08694 [hep-th].
36. F. Bastianelli, R. Bonezzi, O. Corradini, and E. Latini, JHEP 10, 098 (2013) arXiv:1309.1608 [hep-th].
37. N. Mueller and R. Venugopalan, Phys. Rev. D96, 016023 (2017) arXiv:1702.01233 [hep-ph].
38. N. Mueller and R. Venugopalan, Phys. Rev. D97.
We will consider only light fermions and for simplicity will not include a mass term explicitly in most of the computations.

Y. Ohnuki and T. Kashiwa, Prog. Theor. Phys. 60, 548 (1978).

A. Barducci, R. Casalbuoni, and L. Lusanna, Nucl. Phys. B180, 141 (1981).

M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory (Addison-Wesley, Reading, USA, 1995).

Similar expressions for $F_1$ and $g_1, g_2$ can be found in [113].

K.-F. Liu and S.-J. Dong, Phys. Rev. Lett. 72, 1790 (1994), arXiv:hep-ph/9306299 [hep-ph].

U. Aglietti, M. Ciuchini, G. Corbo, E. Franco, G. Martinelli, and L. Silvestrini, Phys. Lett. B143, 411 (1989), arXiv:hep-ph/9804416 [hep-ph].

L. D. McLerran and R. Venugopalan, Phys. Rev. D96, 094513 (2017), arXiv:1704.08993 [hep-lat].

J. Liang, T. Draper, K.-F. Liu, A. Rothkopf, and Y.-B. Yang, (2019), arXiv:1906.05312 [hep-ph].

Interestingly, these asymptotics are the most daunting for classical computing approaches since the OPE breaks down at small $x_B$, [116].

L. D. McLerran and R. Venugopalan, Phys. Rev. D49, 2233 (1994), arXiv:hep-ph/9309289 [hep-ph].

L. D. McLerran and R. Venugopalan, Phys. Rev. D49, 3532 (1994), arXiv:hep-ph/9311205 [hep-ph].

L. D. McLerran and R. Venugopalan, Phys. Rev. D50, 2225 (1994), arXiv:hep-ph/9402335 [hep-ph].

F. Gels, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, Ann. Rev. Nucl. Part. Sci. 60, 463 (2010), arXiv:1002.0333 [hep-ph].

J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, Phys. Rev. D59, 014014 (1998), arXiv:hep-ph/9706377 [hep-ph].

J. Jalilian-Marian, A. Kovner, and H. Weigert, Phys. Rev. D59, 014015 (1998), arXiv:hep-ph/9709432 [hep-ph].

E. Iancu, A. Leonidov, and L. D. McLerran, Nucl. Phys. A692, 583 (2001) arXiv:hep-ph/0011241 [hep-ph].

E. Ferreiro, E. Iancu, A. Leonidov, and L. McLerran, Nucl. Phys. A703, 489 (2002) arXiv:hep-ph/0109115 [hep-ph].

L. D. McLerran and R. Venugopalan, Phys. Rev. D59, 094002 (1999) arXiv:hep-ph/9809427 [hep-ph].

A. Tarasov and R. Venugopalan, (2019), arXiv:1903.11624 [hep-ph].

Note that the term $J \cdot A$ can also be written in a gauge invariant generalization [117], but for the problem of interest here, this simpler form will suffice.

F. A. Berezin and M. S. Marinov, Annals Phys. 104, 336 (1977).

E. Knill and R. Laflamme, Physical Review Letters 81, 5672 (1998).

A. Datta, S. T. Flammia, and C. M. Caves, Physical Review A 72, 042316 (2005).

D. Shepherd, arXiv preprint quant-ph/0608132 (2006).

We set on the upper Keldysh contour $x_+^2 \equiv x^2(\tau) = t_0 + \tau$ for $\tau \leq T/2$ and on the lower Keldysh contour $x_-^2 \equiv x^2(\tau) = t_0 + (T - \tau)$ for $\tau > T/2$.

S. Jeon and R. Venugopalan, Phys. Rev. D70, 105012 (2004) arXiv:hep-ph/0406169 [hep-ph].

T. Altinoluk, N. Armesto, G. Beuf, M. Martinez, and C. A. Salgado, JHEP 07, 068 (2014) arXiv:1404.2219 [hep-ph].

T. Altinoluk, N. Armesto, G. Beuf, A. Mascosco, JHEP 01, 114 (2016) arXiv:1505.01400 [hep-ph].

P. Agostini, T. Altinoluk, and N. Armesto, Eur. Phys. J. C79, 600 (2019) arXiv:1902.04483 [hep-ph].

M. Reuter, M. G. Schmidt, and C. Schubert, Annals Phys. 259, 313 (1997) arXiv:hep-th/9610191 [hep-th].

We note that worldline methods have been extensively used in such computations [26] [118–122]; it would be interesting to explore computing the nontrivial color and spinor traces therein utilizing the worldline quantum circuits discussed in Appendix D.

S. P. Jordan, K. S. M. Lee, and J. Preskill, Science 336, 1130 (2012), arXiv:1111.3633 [quant-ph].

An important issue to address in this context is the renormalization of infinities at each loop order. These have been discussed extensively in [95] [123–126].

A. Macridin, P. Spentzouris, J. Amundson, and R. Harnik, Phys. Rev. Lett. 121, 110504 (2018) arXiv:1802.07341 [quant-ph].

A. Macridin, P. Spentzouris, J. Amundson, and R. Harnik, Phys. Rev. A98, 042312 (2018) arXiv:1805.09298 [quant-ph].

M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2010).

L. A. Wu and D. A. Lidar, J. Math. Phys. 43, 4506 (2002) arXiv:quant-ph/0109078 [quant-ph].

R. Blatt and C. F. Roos, Nature Physics 8, 277 (2012).

J. Casanova, L. Lamata, I. Egusquiza, R. Gerritsma, C. F. Roos, J. J. García-Ripoll, and E. Solano, Physical review letters 107, 260501 (2011).

L. Lamata, A. Mezzacapo, J. Casanova, and E. Solano, EPJ Quantum Technology 1, 9 (2014).

I. Bloch, J. Dalibard, and S. Nascimbene, Nature Physics 8, 267 (2012).

J. Berges, “Scaling up quantum simulations,” (2019).

N. Mueller, A. Tarasov, and R. Venugopalan, in preparation.

J. B. Kogut and L. Susskind, Phys. Rev. D11, 395 (1975).

K. G. Wilson, Phys. Rev. D10, 2445 (1974), [319(1974)].

U.-J. Wiese, Annalen Phys. 525, 777 (2013) arXiv:1305.1602 [quant-ph].

E. A. Martinez, C. A. Muschik, P. Schindler, D. Nigg, A. Erhard, M. Heyl, P. Hauke, M. Dalmonte, T. Monz, P. Zoller, et al., Nature 534, 516 (2016).

N. Klco, E. Dumitrescu, A. McCaskey, T. Morris, R. Pooser, M. Sanz, E. Solano, P. Lougovski, and M. Savage, arXiv preprint arXiv:1803.03326 (2018).

A. Alexandru, P. F. Bedaque, S. Harmalkar, H. Lamm, S. Lawrence, and N. C. Warrington (NuQS), (2019), arXiv:1906.11213 [hep-lat].

T. Byrnes, P. Sriganes, R. J. Bursill, and C. J. Hamer, Phys. Rev. D66, 013002 (2002) arXiv:hep-lat/0202014 [hep-lat].

T. Sugihara, Journal of High Energy Physics 2005, 022
Note the volume $\text{Vol}$ of the gauge group in Eq. (A9) differs from that in Eq. (A6), due to the absence of zero modes present for periodic boundary conditions.

$N_B$ is the product of the number of bosonic modes (the phase space volume) times the number of qubits $n$ required to realize one mode. Because there is no Pauli exclusion principle, this phase space can contain an arbitrary number of bosonic worldlines. However, discretization of a single bosonic mode via binary representation of $n$ qubits is accurate up to $O(n_B/n)$, where $n_B$ is the occupation number of that mode.

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