A Discourse on the Benney Equation

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March 28, 2022

\textbf{Abstract}

It is shown that the Benney system is equivalent to a one body classical dynamical problem in which the interaction potential is a functional of the action function. The general solution of this equation is responsible for the general solution of the Benney system. Some particular solutions of the selfconsistent Hamilton-Jacobi equation arising in this investigation are presented in explicit form.
1 Introduction

We use the form of the Benney [1] system from [2], where it is presented (‘in its original form’) as:

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \left( \int_0^y \frac{\partial v}{\partial y} \right) \frac{\partial v}{\partial x} + \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( \int_0^h dy v \right) = 0 \quad (1)
\]

where the unknowns are the functions \( v \equiv v(t; x, y) \), \( h \equiv h(t, x) \).

This paper is not aimed at the problems of symmetry and numerous recurrence relations which follow from the Benney system (1) and are described in [1, 2, 3, 4, 5, 6, 7] but directly to the problem of construction of the general solution to it. The history and literature on Benney system reader can be found in [2].

The general solution in our approach to the problem can provisionally be divided into two steps. In the first we propose a parametric representation of the solution of the Benney system (in implicit form) in terms of an assumed distribution function. In the second step we propose a self-consistent equation, which this function should satisfy. The general solution of this equation is responsible for the general solution of the Benney system.

Thus the goal of this paper is to demonstrate that the Benney system can be presented in the form of one body classical dynamic Hamilton-Jacobi equation, with the potential expressed nonlocally in terms of the action function. We discuss also the connection of this problem with an ordinary differential equation of the second order. A particular solution of (1), depending on two arbitrary functions each of one argument is constructed. However the general solution of this equation must depend on one arbitrary function of two independent arguments and one function of one argument (the initial values of the functions \( v, h \)). Such a solution is unknown for us at this moment.

The paper is organized in the following way. In section 2 we consider in detail the case with \( h = 0 \) and explain the strategy of the calculations, which are also applicable to the general case without any considerable change. In sections 3 and 4 the general case is considered and shown that the Benney system is equivalent to a single non-linear integro-differentiation equation, which in turn is equivalent to a Hamilton-Jacobi one, with the potential in the form of a nonlocal functional of the second order derivatives of the action function. In terms of the general solution (not only of its first integral!) of the self-consistent Hamilton-Jacobi equation the general solution of the
Benney system (1) is represented in implicit form. Two different particular solutions of this equation with functional arbitrariness of two arbitrary single argument functions are also presented here. The connection of the self-consistent Hamilton-Jacobi equation with an ordinary differential equation of the second order and the ways of obtaining its general solution are discussed in section 5. In section 6 we represent our results in the form of a theorem. Section 7 is devoted to a discussion of the results of the paper and discussion of possible perspectives for future investigation. In the Appendix the reader will find the connection of the solution of the main equation of the second subsection of section 3 with the Hamilton-Jacobi equation in a field of force depending only upon the time variable.

2 Preliminary manipulations

This section contains detailed calculations leading to a parametrical representation of the general solution of the Benney system in implicit form under the simplest assumption that \( h = 0 \) in (1). The reader for whom the result is more interesting than its derivation can begin reading directly the parametrical representation of the solution (29), which is checked independently in the next few lines.

Let us define \( u(t, x, y) \equiv \int_y^0 dy' v(t, x, y') \) and resolve the second equation of the system (1) as

\[
h = H_x(t, x), \quad u(t; x, H_x) = -H_t
\]

After such a substitution, the system (1) takes the form:

\[
u_{t,y} + u_y u_{xy} - u_x u_{yy} + H_{xx} = 0, \quad u(t; x, H_x) = -H_t, \quad u(t; x, 0) = 0
\]

To have an experience in working with the last system of equations let us at first restrict ourselves to the particular solution for which \( H = \text{Const} \) (really this is an inessential restriction for the parametrical representation of the solution as can be seen from the results of the section 3).

Thus we have to solve the single equation for the unknown function \( u \) with the boundary condition:

\[
u_{t,y} + u_y u_{xy} - u_x u_{yy} = 0, \quad u(t; x, 0) = 0
\]
Let us present the solution of (4) in the form \( \phi(u; t, x, y) = \text{const} \), considering \( \phi \) as an unknown function with four arguments. Direct calculations of the first and second order derivatives using the rules of differentiation of implicit functions leads to the result:

\[
\begin{align*}
\text{Det}_3 \begin{pmatrix}
0 & \phi_u & \phi_t \\
\phi_u & \phi_{uu} & \phi_{ut} \\
\phi_y & \phi_{uy} & \phi_{yt}
\end{pmatrix} + \text{Det}_3 \begin{pmatrix}
0 & \phi_y & \phi_x \\
\phi_y & \phi_{yy} & \phi_{yx} \\
\phi_u & \phi_{uy} & \phi_{ux}
\end{pmatrix} = 0
\end{align*}
\] (5)

The following notation will be used:

\[
\alpha = \frac{\phi_t}{\phi_u}, \quad \beta = \frac{\phi_x}{\phi_u}, \quad \lambda = \frac{\phi_y}{\phi_u}
\] (6)

\[
\dot{A} = \frac{\partial}{\partial y} - \lambda \frac{\partial}{\partial u}, \quad \dot{B} = \frac{\partial}{\partial x} - \beta \frac{\partial}{\partial u}, \quad \dot{C} = \frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial u}
\]

The system of equations, which the functions \( \alpha, \beta, \lambda \) satisfy (as a consequence of their definition only) takes the form:

\[
[\dot{A}, \dot{B}] = [\dot{B}, \dot{C}] = [\dot{C}, \dot{A}] = 0
\] (7)

In addition directly from (5) the main equation for them, which can be written in two equivalent forms, follows:

\[
\{ \phi, \lambda \}_{tu} = \{ \phi, \lambda \}_{xy}, \quad \{ \phi, \alpha \}_{yu} = \lambda^2 \{ \phi, \frac{\beta}{\lambda} \}_{yu}
\] (8)

where \( \{X, Y\}_{a,b} \equiv X_aY_b - X_bY_a \). Considering \( \alpha, \beta \) as functions depending upon four independent arguments \( \phi, \lambda, x, t \), we obtain for the last equation of (8):

\[
(\lambda_y - \lambda \lambda_u)(\alpha \lambda - \lambda^2 \frac{\beta}{\lambda}) = 0
\] (9)

and rewrite (4) as:

\[
(\lambda_t - \alpha \lambda_u) \beta \lambda + \beta_t = \alpha_x + (\lambda_x - \beta \lambda_u) \alpha \lambda
\] (10)

\[
(\lambda_x - \beta \lambda_u) = (\lambda_y - \lambda \lambda_u) \beta \lambda, \quad (\lambda_y - \lambda \lambda_u) \alpha \lambda = (\lambda_t - \alpha \lambda_u)
\]

Combining the first equation of (10) with the last two we conclude that:

\[
\alpha_x = \beta_t, \quad \alpha = \Theta_t, \quad \beta = \Theta_x
\] (11)
and after substitution of (33) into (9) we arrive at a linear equation for a single function $\Theta$:

$$-\Theta_x + \lambda \Theta_{x,\lambda} = \Theta_{t,\lambda}$$  \hspace{1cm} (12)

The most straightforward way to obtain its solution is by differentiation with respect to the argument $\lambda$. This leads it to:

$$\left(\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x}\right)\Theta_{\lambda,\lambda} = 0$$

The general solution of the last equation is obvious:

$$\Theta = \int^\lambda d\lambda' (\lambda - \lambda') G(x + \lambda' t, \lambda'; \phi) + \lambda K(x, t; \phi) + L(x, t; \phi)$$  \hspace{1cm} (13)

In what follows we will denote the ”moments” of the function $G$ by the corresponding upper index:

$$G^s(x, t; \phi, \lambda) = \int^\lambda d\lambda'(\lambda - \lambda') G(x + \lambda' t, \lambda'; \phi)$$

In this notation $\alpha$ and $\beta$ functions take the form:

$$\alpha = (F_{tt} - G^1_t) - \lambda(F_{xt} - G^0_x), \quad \beta = (F_{xt} - G^1_x) - \lambda(F_{xx} - G^0_x)$$

Substituting $\alpha, \beta, \lambda$ via derivatives of the function $\phi$ from (8) we come to the following system of equations of the first order:

$$\phi_x = (F_{xt} - G^1_x) \phi_u - (F_{xx} - G^0_x) \phi_y, \quad \phi_t = (F_{tt} - G^1_t) \phi_u - (F_{xt} - G^0_t) \phi_y$$  \hspace{1cm} (14)

Up to now we have used only one equation from the system (10) which together with the second equation (8) leads to (2). Two remaining equations (10) have as their direct corollary the following system:

$$\lambda_x = (F_{xt} - G^1_x) \lambda_u - (F_{xx} - G^0_x) \lambda_y, \quad \lambda_t = (F_{tt} - G^1_t) \lambda_u - (F_{tx} - G^0_t) \lambda_y$$  \hspace{1cm} (15)
The systems (14) and (15) are selfconsistent in the sense of equality of the second mixed derivatives and our nearest goal now is to extract the consequences, which follow from them.

Let us consider in (14) the function $\phi \equiv \theta(x,t,\lambda;u) = \psi(x,t,\lambda;y)$. Keeping in mind (15) we can transform (14) to two forms:

$$\theta_x = (F_{xt} - G^1_x)\theta_u, \quad \theta_t = (F_{tt} - G^1_t)\theta_u,$$

$$\psi_x = -(F_{xx} - G^0_x)\psi_y, \quad \psi_t = -(F_{xt} - G^0_t)\psi_y$$

with the obvious general solution:

$$\phi = \theta = P(u + F_t - G^1, \lambda) = Q(y - F_x + G^0, \lambda) \quad (16)$$

where $P$ and $Q$ are arbitrary functions of their two arguments. Resolving (16) with respect to the first arguments of $P$ and $Q$ functions, we obtain a system of two functional equations, which the functions $\phi$ and $\lambda$ satisfy:

$$u + F_t - G^1 = R(\lambda, \phi), \quad y - F_x + G^0 = S(\lambda, \phi) \quad (17)$$

Here $R$ and $S$ are arbitrary functions of two arguments (inverses of $P, Q$).

We recall that $F = F(x,t,\phi), \ G^0(x,t,\lambda,\phi), \ G^1(x,t,\lambda,\phi)$ and for this reason (17) is a system of functional equations which determine the functions $\lambda, \phi$ implicitly. By the rules of differentiation of implicit functions it may be seen that the functions $\lambda$ and $\phi$ defined by (16), (17) satisfy the systems (14) and (13).

To show this and to find the additional condition which connects the functions $R, S$ (arbitrary up to now) let us calculate derivatives of the functions $\lambda, \phi$. For this the following abriviations will be used

$$A = R_\phi + G^1_\phi - F_{t,\phi}, \quad B = R_\lambda + G^1_\lambda, \quad C = S_\phi - G^0_\phi + F_{x,\phi}, \quad D = S_\lambda - G^0_\lambda$$

and the linear system of equations from which the corresponding derivatives are determined takes the form:

$$F_{tt} - G^1_t = A\phi_t + B\lambda_t, \quad -F_{xt} + G^0_t = C\phi_t + D\lambda_t,$$

$$F_{tx} - G^1_x = A\phi_x + B\lambda_x, \quad -F_{xx} + G^0_x = C\phi_x + D\lambda_x,$$
\[ 1 = A\phi_u + B\lambda_u, \quad 0 = C\phi_u + D\lambda_u, \]
\[ 0 = A\phi_y + B\lambda_y, \quad 1 = C\phi_y + D\lambda_y. \]

One can verify that the linear systems (14) and (15) are satisfied and for \( \lambda \) we obtain:
\[ \lambda = \frac{\phi_y}{\phi_u} = -\frac{B}{D} = -\frac{R_\lambda + G_1^1}{S_\lambda - G_0^0} \]

From the last expression, keeping in mind the definitions of \( G^0 \) and \( G^1 \) functions it immediately follows that:
\[ \lambda S_\lambda = -R_\lambda \]

The last equation may be resolved in terms of only one arbitrary function \( T \) of two arguments \( \lambda \) and \( \phi \):
\[ R = \lambda T_\lambda - T, \quad S = -T_\lambda \]

To come back to the solution of the initial system (3) and express \( u \) as a function of the arguments \( (x,t,y) \) it is necessary to put \( \phi = \text{constant} \) in all formulae above. After such a substitution the functions \( G \) and \( F \) reduce to functions of two independent arguments and \( T \) to only one, \( \lambda \).

Substituting the last expressions into (17) and recalling the boundary condition \( (H = \text{constant}) \) from we obtain:
\[ u + F_t - G^1 = \lambda T_\lambda - T, \quad y - F_x + G^0 = -T_\lambda, \quad u(x,t,0) = 0 \quad (18) \]

It is not difficult to check that the function \( T \) can be included in \( G \) by the of substitution:
\[ G(x_1, x_2) \rightarrow G(x_1, x_2) + T_{x_2,x_2}(x_2), \]

which is equivalent to equating \( T \) in (18) to zero.

Thus finally (18) takes the form:
\[ u + F_t - G^1 = 0, \quad y - F_x + G^0 = 0, \quad u(x,t,0) = 0 \quad (19) \]

Two first equations define parametrically (via \( \lambda = -u_y = -v \) \( u \) as function of its three arguments. The third one can be considered as a boundary condition, but really this is an equation determining two two dimensional.
functions $F(x,t), \lambda(x,t,0) \equiv \nu$. Indeed, substituting $u = y = 0$ into (19) we obtain:

$$F_t - G^1(x,t,\nu) = 0, \quad -F_x + G^0(x,t,\nu) = 0 \quad (20)$$

Comparing the second mixed partial derivatives of the function $F$ and keeping in mind the definition of the "moments" $G^1, G^0$ we obtain finally:

$$(\nu_t - \nu \nu_x)G(x + \nu t, \nu) = 0 \quad (21)$$

It is easy to check, that the solution of Monge equation:

$$\nu_t - \nu \nu_x = 0$$

is implicitly defined by the equation

$$G(x + \nu t, \nu) = g = \text{constant}$$

Below we rewrite (18) including the function $F$ in the lower limit of the integrals and emphasise the functional dependence of all functions involved:

$$u(t,x,y) - \int_{\nu(t,x)}^{-v(t,x,y)} d\lambda G(x + \lambda t, \lambda) = 0 \quad y + \int_{\nu(t,x)}^{-v(t,x,y)} d\lambda G(x + \lambda t, \lambda) = 0$$

$$(22)$$

The equations (22) define implicitly the general solution of the Benney system in the case $h = 0$, depending on one function $G$ of two independent arguments (which is in connected with the initial value of the function $v$).

3 The general case of the Benney equation

Now we consider the general case of the Benney system (1). All steps of the calculations will be the same as the previous with the additional terms connected with $H_{xx}$ in (3).

Equation (4) now reads:

$$\text{Det}_3 \begin{pmatrix} 0 & \phi_y & \phi_x \\ \phi_y & \phi_{yu} & \phi_{yt} \\ \phi_y & \phi_{uy} & \phi_{yt} \end{pmatrix} + \text{Det}_3 \begin{pmatrix} 0 & \phi_y & \phi_x \\ \phi_y & \phi_{yy} & \phi_{yx} \\ \phi_y & \phi_{uy} & \phi_{yx} \end{pmatrix} + H_{xx} \phi_u^3 = 0 \quad (23)$$
The definitions of the functions $\alpha$, $\beta$, $\lambda$ do not change and the equations which follow from (7), (10) are preserved in form.

As a corollary of (23) the modified main equation (9) is:

$$ (\lambda y - \lambda \lambda_u)(\alpha_\lambda - \lambda^2(\frac{\beta}{\lambda})_\lambda) + H_{xx} = 0 \quad (24) $$

The equation for the function $\Theta$ takes the form:

$$ (\lambda y - \lambda \lambda_u)(\Theta_x - \lambda \Theta_{x,\lambda} + \Theta_{t,\lambda}) + H_{xx} = 0 \quad (25) $$

Further calculations word for word repeat the same story of the previous section. The crucial point is the understanding that an arbitrary function of four arguments $\Theta \equiv \Theta(t, x, \lambda, \phi)$ may be presented in the form:

$$ \Theta = \int d\lambda'(\lambda - \lambda')G(t, x, \lambda'; \phi) + F(t, x, \phi) + \lambda \Phi(t, x, \phi) \quad (26) $$

where $G = \Theta_{\lambda,\lambda}$. Indeed (26) is one possible form a Taylor series for the function $\Theta$ with respect to one of its independent arguments. The form (24) doesn’t have any connection with equation which the function $G$ satisfy.

All formulae between (14) and (20) preserve their form with the obvious exchange $-F_x \rightarrow \Phi, F_t \rightarrow F$ and lead to the final result; instead of (20):

$$ u + F - G^1 = 0, \quad y + \Phi + G^0 = 0, \quad u(x, t, 0) = 0, \quad u(x, t, H_x) = -H_t, \quad (27) $$

where $F, \Phi$ are two arbitrary functions of three arguments $(t, x, \phi)$.

After differentiation of the second equation (27) with respect to operator $\frac{\partial}{\partial y} - \lambda \frac{\partial}{\partial \alpha}$ we obtain:

$$ (\lambda_y - \lambda \lambda_u) = -\frac{1}{\Theta_{\lambda,\lambda}} $$

Substituting the last expression into (23) and subsequent differentiation of the result with the respect to the argument $\lambda$ we obtain an equation for function $G \equiv \Theta_{\lambda,\lambda}$:

$$ -G_t + \lambda G_x - H_{xx}G_\lambda = 0 \quad (28) $$

The last equation is exactly the equation for the distribution function for a one body classical dynamical system with Hamiltonian function $\frac{1}{2}\lambda^2 + H_x$.
Indeed, the condition of conservation of some quantity along the trajectory can be written as:

$$\rho_t = \{H, \rho\}_{\lambda,x}$$

This is exactly the equation for the function $G$ above. (In the quantum case it is the equation for the density matrix with the exchange of Poisson brackets with commutators).

Now let us include functions $F, \Phi$ from (27) into the first boundary condition and rewrite (27) in the form:

$$u - \int_{-v}^{v} d\lambda \lambda G(t, x, \lambda) = 0 \quad y + \int_{-v}^{v} d\lambda G(t, x, \lambda) = 0 \quad (29)$$

where $\nu$ is a function of two arguments $t, x$. It is obvious that to satisfy the first boundary condition $\nu = -v(t, x, 0)$. \[1\]

The condition on the second boundary leads to the additional restriction:

$$H_t + \int_{-v}^{v} d\lambda \lambda G(t, x, \lambda) = 0 \quad H_x + \int_{-v}^{v} d\lambda G(t, x, \lambda) = 0 \quad (30)$$

($\mu = -v(x, t, -H_x)$).

Now let us check by direct calculation the conditions under which functions $u(t, x, y)$, $H(t, x)$ defined by the (27) and (30) are a solution of the Benney equation (3).

In the notation used above the Benney equation may be rewritten as ($u_y = v = -\lambda$):

$$v_t + vv_x - u_x v_y + H_{xx} = 0$$

After differentiation of the second equation (29) with respect to $t$ we find:

$$-v_t G(t, x, -v) - \nu_t G(t, x, \nu) + \int_{-v}^{v} d\lambda G_t(t, x, \lambda) = 0$$

Substituting the derivatives of $G_t$ from the equation for it (28) we obtain finally:

$$-v_t = -\frac{\int_{-v}^{v} d\lambda \lambda G_x(t, x, \lambda) - H_{xx}(G(-v) - G(\nu)) - \nu_t G(\nu)}{G(-v)}$$

\[1\] We remind the reader that in all formulae below it is necessary to put $\phi \equiv \text{constant}$, as explained in the previous section.
By the same technique we have:

\[ u_x v_y = \frac{\int_{v}^{-} d\lambda \lambda G_x(t, x, \lambda) + (v v_x G(-v) - \nu v_x G(\nu))}{G(-v)} \]

or

\[ -(v v_x - u_x v_y) = \frac{\int_{v}^{-} d\lambda \lambda G_x(t, x, \lambda) - \nu v_x G(\nu)}{G(-v)} \]

After summation of these results we obtain the equality:

\[ v_t + v v_x - u_x v_y = -H_{xx} + \frac{G(\nu)}{G(-v)}(v_t - \nu v_x - H_{xx}) \]

and conclude that in order to satisfy the Benney equation \( \nu \) function must be the solution of the equation:

\[ \nu_t - \nu v_x - H_{xx} = 0 \]

The condition of equality of the second mixed partial derivatives with respect to \( t, x \) of the function \( H \) follows from (30) and leads to the conclusion that the function \( \mu \) satisfies the same equation as \( \nu \) and the condition of selfconsistency of the whole construction may be written in the form:

\[ H_{xx} = \nu_t - \nu v_x = \mu_t - \mu v_x, \quad H_x = \int_{\nu}^{\mu} d\lambda G(t, x, \lambda) \quad (31) \]

where \( G \) satisfies (28).

Considering \( \lambda \) in equation (28) as an unknown function of three independent arguments \((x, t, g)\) and defining:

\[ g = G(x, t, \lambda) \]

we arrive at the equation, which this function satisfies:

\[ \lambda_t - \lambda \lambda_x - H_{xx} = 0 \quad (32) \]

from which it is follows that:

\[ \nu = \lambda(x, t, g_1), \quad \mu = \lambda(x, t, g_2) \]
After exchanging integration variables under the integral sign in (30) we amalgamate the Benney equation with the corresponding boundary conditions into a single (seemingly very strange) integro-differential equation:

$$\lambda_t - \lambda \lambda_x - \int_{g_1}^{g_2} g dg \lambda_{g,x} = 0$$  

(33)

The main function $G$ from (28) may be reconstructed implicitly via the solution of the last equation:

$$\lambda = \lambda(x, t, G)$$

### 3.1 The simplest particular solution of the Benney system

In this subsection we would like to demonstrate the selfconsistency of the previous construction by the example of the "simplest" solution of Benney system. This also will give us the possibility of understanding which class of functions is typical for this problem.

Of course, distribution function $G = -\frac{1}{2A}$ = constant satisfies (28) with an arbitrary choice of the potential function $H_x$. Thus as a corollary of (30) we have ($G = -\frac{1}{2A}$):

$$H_x = \frac{1}{2A}(\mu - \nu), \quad H_t = \frac{1}{4A}(\mu^2 - \nu^2)$$

From (29) we obtain:

$$v = -2Ay - \nu, \quad u = -Ay^2 - \nu y, \quad h = H_x = \frac{1}{2A}(\mu - \nu)$$  

(34)

By direct computation one can become convinced that (34) is really the solution of the Benney system if $\mu, \nu$ functions satisfy (35), which in the case under consideration takes the form

$$\mu_t - \mu \mu_x = \nu_t - \nu \nu_x = \frac{1}{2A}(\mu - \nu)_x$$  

(35)

For linearization of the last system let us exchange the dependent and independent variables. In other words let us consider $(t, x)$ as a functions of
the pair of variables \((\mu, \nu)\). Corresponding formulae for this transformation are obvious:

\[
\begin{align*}
t &= t(\mu, \nu) \\
x &= x(\mu, \nu)
\end{align*}
\]

\[
\begin{align*}
0 &= t_\nu \nu_t + t_\mu \mu_t \\
1 &= x_\nu \nu_x + x_\mu \mu_x
\end{align*}
\]

After solving of the last equations and substitution of the result into \((??)\) we obtain the first order linear system of equations:

\[
(x + \nu t)_\mu = -(x + \mu t)_\nu = \frac{1}{2A} (t_\mu + t_\nu)
\]

From the last system we conclude that

\[
(x + \nu t) = -\theta_\nu, \quad (x + \mu t) = \theta_\mu,
\]

\[
t = \frac{\theta_\nu + \theta_\mu}{\mu - \nu}, \quad x = -\frac{\mu \theta_\nu + \nu \theta_\mu}{\mu - \nu} \equiv \frac{1}{2} (\theta_R - \frac{\Sigma}{R} \theta_\Sigma), \quad (36)
\]

and obtain the equation, which the function \(\theta\) satisfy. We present it in terms of the variables \(\Sigma = \frac{1}{2}(\mu + \nu), R = \frac{1}{2}(\mu - \nu)\) in which it takes the most simple form

\[
(1 - \frac{1}{AR}) \theta_{\Sigma, \Sigma} = \theta_{R, R}
\]

This is a linear equation of second order with the separable variables. It allows us to obtain its general solution depending on two arbitrary functions, each of one argument. Equations \((34)\) have to be reversed and variables \(\mu, \nu\) may be expressed as functions of the initial variables \(x, t\).

Formulae \((34)\) give a particular solution of the Benney system depending on two arbitrary functions each of one argument. Of course, this is not the general solution of this system which must depend on one function of two arguments (the initial value \(v(x, y, 0)\)) and one function of one argument (the initial value \(h(x, 0)\)).

The example considered shows that it is impossible to expect some simple analytical expression for the general solution of the Benney system.
3.2 Particular solution of main equation for \( \lambda \) function

Let us seek the solution of main equation (33) in the form:

\[ \lambda = xA(t, g) + B(t, g) \]

We have in consequence:

\[ (xA_t + B_t) - A(xA + B) + \left( \int_{g_1}^{g_2} dgg(xA_g + B_g) \right)_x = 0 \] (37)

Equating coefficients at zero and unity, decreasing \( x \) to zero and solving the system of equations arising for the functions \( A, B \) we obtain finally:

\[ \lambda = -\frac{x + \Phi(t) + V(g)}{t + U(g)} + \Phi_t, \quad \Phi = \int_{g_1}^{g_2} dggU_g \ln(t + U(g)) \] (38)

where \( U(g), V(g) \) are arbitrary functions.

Resolving (implicitly) the equation \( \lambda = \lambda(x, t, G) \) and substituting the result into (29) we come to the solution of the Benney system depending on two arbitrary functions \( U, V \), each of one argument. Of course this solution is of a different kind to that in the previous subsection. In this case the distribution function \( G \) is different from a constant. In some cases under a definite choice of the form \( U, V \) functions all calculations may be done explicitly and solution of Benney system can be presented by analytical expressions.

In the Appendix the reader can find another way to obtain the solution of the present subsection, connected with the ordinary differential equation of the second order (see section 5).

4 One body classical mechanical problem with self-consistent potential of interaction

Equation (33) has the form of two dimensional conservation law. It allows us to solve this equation in the form:

\[ \lambda = -S_x, \quad \frac{\lambda^2}{2} + \int_{g_1}^{g_2} gdg\lambda_g = -S_t \]
After eliminating the function $\lambda$ we pass to the one body Hamilton-Jacobi equation:

$$S_t + \frac{S^2}{2} - \int_{g_1}^{g_2} g dg S_{g,x} \equiv S_t + \frac{S^2}{2} + V(x,t) = 0, \quad (39)$$

where the potential function $V$ is expressed nonlocally via the derivatives of action function $S$.

We would like to point out here that a situation of this kind is not a new one. It is sufficient to remember the famous non-linear Schrödinger equation, where the potential of interaction coincides with the square of the modulus of the wave function. In the problem considered here the action function plays the same role (classical limit of the wave function). The potential function is expressed in a non-local way in terms of the derivatives of this $S$.

5 Connection with a second order ordinary differential equation

After differentiation of the main equation (33) with respect to the argument $g$ (keeping in mind that the integral term of equation doesn’t depend upon $g$), we obtain:

$$(\lambda_g)_t = (\lambda\lambda_g)_x, \quad \lambda_g = f_x, \quad \lambda\lambda_g = f_t$$

where the function $f$ satisfies in turn the equation arising after eliminating the function $\lambda$ from the previous system:

$$\left(\frac{f_t}{f_x}\right)_g = f_x$$

Considering in the last equation $x \equiv X$ as an unknown function of the independent arguments $(t, f, g)$ we have in consequence:

$$f = f(X, t, g), \quad 1 = f_x X_f, \quad 0 = f_t + f_x X_t, \quad 0 = f_g + f_x X_g \quad (40)$$

and as a corollary the equation, which the function $X$ satisfies:

$$X_g X_{t,f} - X_f X_{t,g} = 1 \quad (41)$$

Differentiation of the last equation with respect to the argument $t$ leads to:

$$X_{ttf} X_g - X_{ttg} X_f = 0$$

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This means that the function $X_{tt}$ depends on the arguments $(g, f)$ only via the function $X$ and thus we have

$$X_{tt} = Q(X, t) \quad (42)$$

In other words, from (41) it follows that function $X$ is the solution of an ordinary differential equation of the second order (42) with respect to the argument $t$.

Let us present the general solution of this equation in the form:

$$X = X(c_1, c_2; t)$$

and fix the choice of integration $c_1, c_2$ constants by the condition $^2$:

$$X_{c_1}X_{t,c_2} - X_{c_2}X_{t,c_1} = 1$$

Then $c_1, c_2$ as functions of the arguments $(f, g)$ thanks to (41) are connected by a canonical transformation with the generating function $W(c_1, g)$:

$$c_2 = W(c_1, c_1, g), \quad f = W_g(c_1, g)$$

Collecting all these results we are able to rewrite the main equation (33) in the form of a functional integral equation determining the unknown function $X$:

$$x = X(W_{c_1}, c_1, t), \quad X_{c_1}X_{t,c_2} - X_{c_2}X_{t,c_1} = 1,$$

$$X_{tt}(W_{c_1}, c_1, t) - \left( \int_{g_1}^{g_2} dgW_g \right)_{xx} = 0, \quad c_2 = W_{c_1}(c_1, g) \quad (43)$$

where, as was mentioned above, $W(c_1, g)$ is the generating function of the canonical transformation.

In (43) the unknowns are both $X$ and $W$ and we have no idea at this moment of any constructive solution of this system, except by guesswork. In the Appendix we show that the choice of the second order ordinary differential equation in the form $X_{tt} = Q(t)$ corresponds to a solution of subsection 2 in the section 3.

$^2$Let us recall that these constants are defined up to a general covariant transformation. This circumstance always allows us to satisfy the condition below.
6 The main Theorem

The results of the previous sections may be summarised in the following Theorem.

Each solution of nonlinear integro-differential equation

\[ \lambda_t - \lambda \lambda_x - \int_{g_1}^{g_2} dg g \lambda_{g,x} = 0 \]

is connected (implicitly) with the solution of Benney system

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \left( \int_0^y dy \frac{\partial v}{\partial y} \right) \frac{\partial v}{\partial y} + \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( \int_0^h dy v \right) = 0, \]

via the following formulae

\[ u(x, y, t) - \int_{v(x,t)}^{-\nu(y,x,t)} d\lambda \lambda G(t, x, \lambda) = 0, \quad y + \int_{\nu(x,t)}^{-\nu(y,x,t)} d\lambda G(t, x, \lambda) = 0 \]

\[ u(x, y, t) = \int_0^y dy' v(y', x, t), \quad h = \int_{\nu}^{\mu} d\lambda G(t, x, \lambda) \equiv \int_{g_1}^{g_2} dg g \lambda_g \]

where

\[ \mu = \lambda(t, x, g_2), \quad \nu = \lambda(t, x, g_1), \quad \lambda = \lambda(t, x, G) \]

7 Outlook

The main result of the paper consists in the Theorem of the previous section, giving a solution of the Benney system parametrically in implicit form in terms of a distribution function and a selfconsistent equation (33), which this function must satisfy. If it is possible to find the general solution of the last equation in the future the problem of the construction of the general solution of Benney system would be solved with the help of formulae (30).

The most interesting questions about possible representations of the results in the form available for experimental application (the origin of the Benney system is the hydrodynamical problem of surfaces waves [1]) are out of the framework of the present paper. To encompass these questions will be possible after more detailed investigation of the main equation (33) and the discovery of ways of its regular integration (maybe on the level of the computer computations).
All results of the present paper are obtained on intuitive background and demand for the their rigorous foundation more powerfull mathematical methods. The authors can guess that they connected with the better known investigation of the inner symmetry group of Benney system. We hope to come back to these interesting questions in our future publications.

8 Acknowledgements.

Authors thanks D.B.Fairlie, S.F.Luzanov and I.D.Pleshakov for discussion of the results and big help in the preparation the manuscript.

9 Appendix

In this Appendix we would try to widen the solution of the second subsection and obtain it on a more systematic background. With this purpose let us consider more precisely and evaluate the integral in (43).

\[ f_{xx} = \left( \frac{1}{X_f} \right)_x = -\frac{X_{ff}}{X_f^2} f_x = \frac{1}{2} \left( \frac{1}{X_f^2} \right)_f = -\frac{1}{2} \left( \frac{X_g}{X_f} \right)_t, \quad f \quad (44) \]

In the process of the evaluation above we have used equations connected with derivatives of the functions \( f \) and \( X \). In (44) the derivatives with respect to the argument \( t \) is only a partial one. To have the possibility of taking it out of the integration sign it is necessary to increase it to a total derivative. This is achieved by the following obvious manipulation:

\[ \left( \frac{X_g}{X_f} \right)_{t,f} = \frac{d}{dt} \left( \frac{X_g}{X_f} \right)_f - \left( \frac{X_g}{X_f} \right)_{f,f} f_t \]

Let us consider the possibility that the second term of the last equality in its turn is the total derivatives with respect to the argument \( t \). It is obvious that for this it is sufficient assume additionally:

\[ \frac{X_g}{X_f} = U(f,g) + a(t,g)f + b(t,g) \quad (45) \]

( but not the \( t \) variable).
Under the assumptions above we can once integrate (43) with the result:

\[ X_t + \frac{1}{2} \int_{g_1}^{g_2} dgga(t, g) = P(g, f), \quad X = \Phi(t) + P(g, f)t + Q(g, f) \quad (46) \]

from equations (41) and (43) we obtain additionally:

\[ X_f = (-a_t f + b_t)^{-\frac{1}{2}}, \quad X_g = (-a_t f + b_t)^{-\frac{1}{2}}(U + a f + b) \quad (47) \]

The most simple way to resolve the last system of equations is to consider as a first step the condition of equality of the second mixed partial derivatives and compare one of the (selfconsistent) equations (47) with (46).

The condition of selfconsistency of the second mixed derivatives reads as:

\[ a_t g f + b_t g = a_t(U + a f + b) - 2(U_f + a)(a_t f + b_t) \quad (48) \]

In next three lines is presented the result of consequent differentiation of (48) with the respect to the argument \( f \):

\[ a_t g = -a_t(U_f + a) - 2U_{ff}(a_t f + b_t) \]

\[ 0 = 3a_t\frac{U_{ff}}{U_{fff}} + 2(a_t f + b_t) \]

\[ 0 = a_t(3(\frac{U_{ff}}{U_{fff}})_f + 2) \]

Going in the reverse direction we obtain in consequence (all big letters are arbitrary functions of argument \( g \) only):

\[ U_{ff} = B(A-2f)^{-\frac{3}{2}}, \quad U_f = B(A-2f)^{-\frac{3}{2}} + D, \quad U = -B(A-2f)^{\frac{3}{2}} + Df + E \]

\[ a_t g = -(D + a)a_t, \quad 2b + Aa = F \]

The last equation for the function \( a \) after integration once with respect to the argument \( t \) may go over to the form \((a + D = z)\) of a Riccati equation (now with respect to the argument \( g \)):

\[ z_g + \frac{z^2}{2} = \nu_g + \frac{\nu^2}{2} \]
In the last equation we have represented the arbitrary function arising in a special form allowing us to integrate the Riccati equation with the final result:

\[ a + D = -\frac{\nu_{gg}}{\nu_g} + \frac{2\nu_g}{\Phi(t) + \nu} \]

After these preliminary calculations we are able to compare the first equation (47) with (46). For \((a_t f + b_t)\) from the results above we have:

\[(a_t f + b_t) = (A - 2f) \frac{\nu_g \Phi_t}{(\Phi + \nu)^2}\]

or

\[(X_f)^2 = (P_f t + Q_f)^2 = \frac{(\Phi(t) + \nu)^2}{(2f - A)\nu_g \Phi_t}\]

All other computations are obvious and lead finally precisely to the solution of the second subsection in section 3.

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