The MacWilliams identity for $m$-spotty weight enumerator over
\[ F_2 + uF_2 + \cdots + u^{m-1}F_2 \]

MinJia Shi

School of Mathematical Sciences of Anhui University, 230601 Hefei, Anhui, China

Abstract

Past few years have seen an extensive use of RAM chips with wide I/O data (e.g., 16, 32, 64 bits) in computer memory systems. These chips are highly vulnerable to a special type of byte error, called an $m$-spotty byte error, which can be effectively detected or corrected using byte error-control codes. The MacWilliams identity provides the relationship between the weight distribution of a code and that of its dual. The main purpose of this paper is to present a version of the MacWilliams identity for $m$-spotty weight enumerators over $F_2 + uF_2 + \cdots + u^{m-1}F_2$ (shortly $R_{u,m,2}$).

keywords: Byte error-control codes; $m$-spotty byte error; MacWilliams identity

1 Introduction

The error control codes have a significant role in improving reliability in communications and computer memory system [1]. For the past few years, there has been an increased usage of high-density RAM chips with wide I/O data, called a byte, in computer memory systems. These chips are highly vulnerable to multiple random bit errors when exposed to strong electromagnetic waves, radio-active particles or high-energy cosmic rays. To overcome this, a new type of byte error known as spotty byte error has been introduced in which the error occurs at random $t$ or fewer bits within a $b$-bit byte [13], if more than one spotty byte error occur within a $b$-bit byte, then it is known as a multiple spotty byte error or $m$-spotty byte error [12]. To determine the error-detecting and error-correcting capabilities of a code, some special types of polynomials, called weight enumerators, are studied.

One of the most celebrated results in the coding theory is the MacWilliams identity that describes how the weight enumerator of a linear code and the weight enumerator of the dual code relate to each other. This identity has found widespread application in the coding theory [2]. Recently various weight enumerators with respect to $m$-spotty weight have been introduced and studied. Suzuki et al. [12] defined Hamming weight enumerator for binary byte error-control codes, and proved a MacWilliams identity for it. M. Özen and V. Siap [3] and I. Siap [9] extended this result to arbitrary finite fields and to the ring $F_2 + uF_2$ with $u^2 = 0$, respectively. I. Siap [10] defined $m$-spotty Lee weight and $m$-spotty Lee weight enumerator of byte error-control codes over $Z_4$ and derived a MacWilliams identity. A. Sharma and A. K. Sharma introduced joint $m$-spotty weight enumerators of two byte error-control codes over the ring of integers modulo $l$ and over arbitrary finite fields with respect to $m$-spotty Hamming weight [7], $m$-spotty Lee weight [5] and $r$-fold joint $m$-spotty weight [6]. M. Özen and V. Siap [4] proved a MacWilliams identity for the $m$-spotty RT weight enumerators of binary codes, which was generalized to the case of arbitrary finite fields and ring $Z_l$ by M. J. Shi [8]. In this paper, we will consider the MacWilliams identity for $m$-spotty weight enumerators of linear codes over $R_{u,m,2}$, which generalizes the result of [9]. The organization of this paper is as follows: Section 2 provides definitions of $m$-spotty weight and some basic knowledge about $R_{u,m,2}$. Section 3 presents the MacWilliams identity for $m$-spotty

* This research is partially supported by NNSF of China (61202068, 11126174), Talents youth Fund of Anhui Province Universities (2012SQRL020ZD), E-mail address: smjwcl.good@163.com (Min-Jia Shi).

2000 AMS Mathematics Subject Classification: 94B05, 94B20
weight, and Section 4 illustrates the weight distribution of the \(m\)-spotty byte error control code with an example.

2 Preliminaries

Consider the finite commutative ring \(R_{u,m,2} = \mathbb{F}_2[u]/(u^m) = \mathbb{F}_2 + u\mathbb{F}_2 + \cdots + u^{m-1}\mathbb{F}_2\) with \(u^m = 0\), when \(m = 1\), the ring \(\mathbb{F}_2\) is a field, when \(m = 2\), the ring \(R_{u,2,2} = \mathbb{F}_2 + u\mathbb{F}_2\). In the rest of this paper, we assume that \(m \geq 3\) is a positive integer. Any linear code \(C\) over \(R_{u,m,2}\) is permutation equivalent to a code with generator matrix:

\[
G_1 = \begin{pmatrix}
I_{k_1} & A_{11} & A_{12} & A_{13} & \cdots & A_{1,m-1} & A_{1m} \\
0 & uI_{k_2} & uA_{22} & uA_{23} & \cdots & uA_{2,m-1} & uA_{2m} \\
0 & 0 & u^2I_{k_3} & u^2A_{33} & \cdots & u^2A_{3,m-1} & u^2A_{3m} \\
0 & 0 & 0 & 0 & \cdots & 0 & u^{m-1}I_{k_m} & u^{m-1}A_{km}
\end{pmatrix},
\]

where \(I_{k_i}\) is a nonnegative integer, and the columns are grouped into blocks of length \(k_1, k_2, \cdots, k_m\), \(n - k\) where \(k = \sum_{i=1}^{m} k_i\), \(k\) being the number of rows of \(G_1\). Any code \(C\) has a generator matrix in the standard form, and the parameters \(k, k_2, \cdots, k_m\) are the same for any generator matrix \(G_1\) in the standard form for \(C\). Each codeword of \(C\) can be expressed in the form \((x_1, x_2, \cdots, x_m)G_1\) where each \(x_i\) is a vector of length \(k_i\) with components in \(R_{u,m-i+1,p}\). Thus, \(C\) has \(2^s\) codewords where \(s = \sum_{i=1}^{m} (m - i + 1)k_i\).

**Definition 2.1** (see [13]). A spotty byte error is defined as \(t\) or fewer bits errors within a \(b\)-bit byte, where \(1 \leq t \leq b\). When none of the bits in a byte is in error, we say that no spotty byte error has occurred.

We can define the \(m\)-spotty weight and the \(m\)-spotty distance over \(R_{u,m,2}\) as follows.

**Definition 2.2.** Let \(e \in R_{u,m,2}\) be an error vector and \(e_i \in R_{u,m,2}^b\) be the \(i\)-th byte of \(e\), where \(N = nb\) and \(1 \leq i \leq n\). The number of \(t/b\)-errors in \(e\), denoted by \(w_M(e)\), and called \(m\)-spotty weight is defined as

\[
w_M(e) = \sum_{i=1}^{n} \left\lfloor \frac{w_M(e_i)}{t} \right\rfloor,
\]

where \(\lfloor x \rfloor\) denotes the smallest integer not less than \(x\). If \(t = 1\), this weight, defined by \(w_M\), is equal to the Hamming weight. In a similar way, we define the \(m\)-spotty distance of two codewords \(u\) and \(v\) as

\[
d_M = \sum_{i=0}^{n} \left\lfloor \frac{d_M(u_i,v_i)}{t} \right\rfloor.
\]

Further, it is also straightforward to show that this distance is a metric in \(R_{u,m,2}^N\).

Hereinafter, codes will be taken to be of length \(N\) where \(N\) is a multiple of byte length \(b\), i.e. \(N = nb\).

Let \(e = (c_1, c_2, \cdots, c_N)\) and \(v = (v_1, v_2, \cdots, v_N)\) be two elements of \(R_{u,m,2}^N\). The inner product of \(e\) and \(v\), denoted by \((e, v)\), is defined as follows:

\[
(c, v) = \sum_{i=1}^{n} (c_i, v_i) = \sum_{i=1}^{n} \left( \sum_{j=1}^{b} c_{i,j}v_{i,j} \right).
\]

Here, \((c_i, v_i) = \sum_{j=1}^{b} c_{i,j}v_{i,j}\) denotes the inner product of \(c_i\) and \(v_i\), respectively.
Let $C$ be a linear code over $R_{u,m,2}^N$. The set $C^\perp = \{v \in R_{u,m,2}^N | (u,v) = 0, \text{ for all } u \in C\}$ is also a linear code over $R_{u,m,2}$ and it is called the dual code of $C$.

3 The MacWilliams identity

Every element $x \in R_{u,m,2}$ can be written uniquely as $x = r_0(x) + ur_1(x) + \cdots + u^{m-1}r_{m-1}(x)$, where $r_i(x) \in \mathbb{F}_2$, $i = 0, 1, \cdots, m - 1$. It is easy to check that $x$ is a unit if and only if $r_0(x) = 1$. Hence there are

$$\left(\frac{m-1}{1}\right) + \left(\frac{m-1}{2}\right) + \cdots + \left(\frac{m-1}{m-1}\right) = 2^{m-1}$$

units in $R_{u,m,2}^* = R_{u,m,2} \setminus \{0\}$. The number of zero divisors in $R_{u,m,2}^*$ is equal to $2^{m-1} - 1$.

In $R_{u,m,2}$, there exists the chain of ideals:

$$R_{u,m,2} = \langle 1 \rangle \supset \langle u \rangle \supset \langle u^2 \rangle \supset \cdots \supset \langle u^{m-1} \rangle \supset \langle u^m \rangle = 0.$$ 

**Definition 3.1.** Define sets $A$ and $B$ as follows: we first divide the elements of $R_{u,m,2}$ into two equal parts such that

(i) 0 and 1 always belong to the set $A$;

(ii) both parts have the same number of zero divisors (Here, we consider element 0 as a “zero divisor”) and they split each ideal as well as all the units;

(iii) For all $a, b \in A$, then $a + b \in A$, i.e. the set $A$ is closed with respect to addition;

(iv) For all $a, b \in B$, then $a + b \in A$;

(v) For all $a \in A, b \in B$, then $a + b \in B$.

Moreover, sets $A$ and $B$ are uniquely determined in the above Definition.

**Example 3.1.** Consider the ring $R_{u,4,2} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$. In $R_{u,4,2}$, there exists the chain of ideals: $R_{u,4,2} = \langle 1 \rangle \supset \langle u \rangle \supset \langle u^2 \rangle \supset \langle u^3 \rangle \supset \langle 0 \rangle$, where $\langle u \rangle = \{0, u, u^2, u^3, u + u^2, u + u^3, u^2 + u^3, u + u^2 + u^3\}$, $\langle u^2 \rangle = \{0, u^2, u^3, u^2 + u^3\}$ and $\langle u^3 \rangle = \{0, u^3\}$. Hence, according to Definition 3.1, we can obtain

$$A = \{0, 1, u, 1 + u, u^2, u + u^2, 1 + u^2, 1 + u + u^2\}$$

and

$$B = \{u^3, u + u^3, u^2 + u^3, u + u^2 + u^3, 1 + u^3, 1 + u + u^3, 1 + u^2 + u^3, 1 + u + u^2 + u^3\}.$$ 

**Definition 3.2.** We define the character $\chi$:

$$\chi(a) = \begin{cases} 1, & \text{if } a \in A; \\ -1, & \text{if } a \in B, \end{cases}$$

where sets $A$ and $B$ are defined as in Definition 3.1. We note that $\chi$ is a nontrivial character, i.e. $\chi$ is not the identity map on the nonzero ideals of $R_{u,m,2}$. Moreover, for all $a, b \in R_{u,m,2}$, we have $\chi(a + b) = \chi(a) \cdot \chi(b)$.

**Definition 3.3.** Let $v = (v_1, v_2, \cdots, v_b) \in R^b$. Then the support of $v$ is defined by $\text{supp}(v) = \{i | v_i \neq 0\}$ and the complement of $\text{supp}(v)$ is denoted $\overline{\text{supp}(v)}$. 

#### 3
Definition 3.4. Let \( c = (c_1, c_2, \ldots, c_b) \in R^b \) and define
\[
S_k(c) = \{ v \in R^b | \text{supp}(v) \subseteq \text{supp}(c) \text{ and } k = |\text{supp}(v)| \}
\]
\[
\overline{S}_k(c) = \{ v \in R^b | \text{supp}(v) \subseteq \text{supp}(c) \text{ and } k = |\text{supp}(v)| \}.
\]

In order to prove our main theorem, we should first introduce the following lemmas.

Lemma 3.1. Let \( H \neq 0 \) be an ideal of \( R_{u,m,2} \). Then
\[
\sum_{a \in H} \chi(a) = 0.
\]

Proof. We can obtain the result readily by using the definition of character \( \chi \) in (1).

Lemma 3.2. Let \( a \in R_{u,m,2} \). Then
\[
\sum_{r \in R_{u,m,2}} \chi(ar) = \begin{cases} 
2^m, & \text{if } a = 0, \\
0, & \text{if } a \neq 0.
\end{cases}
\]

(2)

Proof. If \( a = 0 \), then clearly \( \chi(ar) = 1 \) for all \( r \in R_{u,m,2} \) and hence the result follows. Otherwise, if \( a \) is a unit, then elements \( ar \), for all \( r \in R_{u,m,2} \), run over all elements of \( R_{u,m,2} \), which forms a trivial ideal \( R_{u,m,2} \). If \( a(\neq 0) \) is a zero divisor, then elements \( ar \), for all \( r \in R_{u,m,2} \), form a proper ideal of \( R_{u,m,2} \). Hence, according to Lemma 3.1, if \( a \neq 0 \), we have \( \sum_{r \in R} \chi(ar) = 0 \).

Lemma 3.3. Let \( v = (v_1, v_2, \ldots, v_b) \in R^b_{u,m,2} \), with \( w(c) = j \neq 0 \) and \( k \in \{1, 2, \ldots, j\} \). Then we have
\[
\sum_{0 \leq w(v) \leq k} \chi((c,v)) = 0.
\]

Proof. The proof is similar to [9], so we omit it here.

Lemma 3.4. Let \( c = (c_1, c_2, \ldots, c_b) \in R^b_{u,m,2} \) and \( w(c) \neq 0 \). For all \( k \) positive integers, we let \( I_k = \{i_1, i_2, \ldots, i_k\} \subseteq \text{supp}(c) \) and \( I_0 = \phi \). Then we have
\[
\sum_{\substack{v \in R^b_{u,m,2} \\
\text{supp}(v) = I_k}} \chi((c,v)) = (-1)^k.
\]

Proof. For \( k = 0 \) i.e. \( I_0 = \phi \), we have \( \sum_{v \in R^b_{u,m,2} \atop \text{supp}(v) = I_0} \chi((c,v)) = \sum_{w(v)=0} \chi(0) = 1 \). For \( k = 1 \), according to Lemma 3.3, we have
\[
\sum_{v \in R^b_{u,m,2} \atop \text{supp}(v) = I_1} \chi((c,v)) = \sum_{v_{i_1} \in R^b_{u,m,2}} \chi(c_{i_1}v_{i_1}) = \sum_{v_{i_1} \in R_{u,m,2}} \chi(c_{i_1}v_{i_1}) - 1 = -1.
\]
For $k = 2$, according to Lemma 3.3, we have
\[
\sum_{v \in \mathcal{R}_{u,m,2}^h \text{ supp}(v) \subseteq I_2} \chi((c, v)) = \sum_{i_1, i_2 \in I_2 \atop v_{i_1}, v_{i_2} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1} + c_2 v_{i_2}) = \sum_{i_1, i_2 \in I_2 \atop v_{i_1}, v_{i_2} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1} + c_2 v_{i_2}) - \sum_{i_1, i_2 \in I_2 \atop v_{i_1}, v_{i_2} \in \mathcal{R}_{u,m,2}} \chi(c_3 v_{i_1} + c_4 v_{i_2}) = 0 - 1 - 2(\sum_{v_{i_1} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1}) - 1) = 1.
\]
Now, we assume that the identity holds true for $k = r \geq 3$, i.e., $\sum_{v \in \mathcal{R}_{u,m,2}^h \text{ supp}(v) \subseteq I_r} \chi((c, v)) = (-1)^r$. For $k \geq r + 1$, suppose supp(v) = \{i_1, i_2, \ldots, i_r+1\}. Then we have
\[
\sum_{v \in \mathcal{R}_{u,m,2}^h \text{ supp}(v) \subseteq I_{r+1}} \chi((c, v)) = \sum_{i_1, i_2, \ldots, i_{r+1} \in I_{r+1} \atop v_{i_1}, v_{i_2}, \ldots, v_{i_{r+1}} \in \mathcal{R}_{u,m,2}} \chi(\sum_{j=1}^{r+1} c_j v_{i_j}) = \sum_{i_1, i_2, \ldots, i_{r+1} \in I_{r+1} \atop v_{i_1}, v_{i_2}, \ldots, v_{i_{r+1}} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1}) - \binom{r+1}{1} \sum_{i_1, i_2, \ldots, i_{r+1} \in I_{r+1} \atop v_{i_1}, v_{i_2}, \ldots, v_{i_{r+1}} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1}) - \cdot \cdot \cdot - \binom{r+1}{r} \sum_{i_1, i_2, \ldots, i_{r+1} \in I_{r+1} \atop v_{i_1}, v_{i_2}, \ldots, v_{i_{r+1}} \in \mathcal{R}_{u,m,2}} \chi(c_1 v_{i_1}) - 1 = 0 - \binom{r+1}{1}(-1)^r - \binom{r+1}{2}(-1)^{r-1} - \cdot \cdot \cdot - \binom{r+1}{r-1}(-1)^1 - 1 = (-1)^{r+1}.
\]

In the last line of equation above, we set $a = 1, b = -1$ in the following equation:
\[
(a + b)^{r+1} = \sum_{i=1}^{r+1} \binom{r+1}{i} a^i b^{r+1-i}.
\]

**Corollary 3.1.** Let $c = (c_1, c_2, \ldots, c_9) \in \mathcal{R}_{u,m,2}^h$ and $w(c) = j \neq 0$. For all $0 \leq k \leq j$, we have
\[
\sum_{v \in \mathcal{S}_k(c)} \chi((c, v)) = (-1)^k \binom{j}{k}.
\]

**Proof.** According to Definition 3.3 and Lemma 3.4, we get
\[
\sum_{v \in \mathcal{S}_k(c)} \chi((c, v)) = \sum_{I_k \subseteq \text{supp}(c)} \sum_{\text{supp}(v) = I_k} \chi((c, v)) = \sum_{I_k \subseteq \text{supp}(c)} (-1)^k = (-1)^k \binom{j}{k}.
\]

**Lemma 3.5.** Let $c = (c_1, c_2, \ldots, c_9) \in \mathcal{R}_{u,m,2}^h$ and $w(c) = j \neq 0$. For all $0 \leq k \leq j$, we have
\[
\sum_{v \in \mathcal{S}_k(c)} \chi((c, v)) = (2^m - 1)^k \binom{b-j}{k}.
\]
Proof. Since \( v \in S_k(c) \) with \( \supp(v) \subseteq \supp(c) \), we have \( \chi((c, v)) = 1 \). Further, since \( k = |\supp(v)| \), there are \( \binom{b-j}{k} \) ways of choosing a subset of size \( k \) from the complement of support of \( c \) of size \( k \). For each subset of size \( k \), the sum of the \( j \) th power of \( 2 \) equals \( 2^j \). Hence, the result follows.

According to Lemma 3.5 and Corollary 3.1, we have the following corollary.

**Corollary 3.2.** Let \( c = (c_1, c_2, \ldots, c_b) \in R_{u,m}^b \) and \( w(c) = j \), \( 0 \leq j_1 \leq j \) and \( 0 \leq j_2 \leq b - j \). We define

\[
S_{j_1, j_2}(c) = \{ v \in R_{u,m}^b | j_1 = |\supp(v) \cap \supp(c)| \text{ and } j_2 = |\supp(v) \cap \supp(c)| \}.
\]

Then we can obtain

\[
\sum_{v \in S_{j_1, j_2}(c)} \chi((c, v)) = (-1)^{j_1} (2^m - 1)^{j_2} \binom{j}{j_1} \binom{b-j}{j_2}.
\]

The proof of the following two lemmas are similar to those of Lemma 2.7 and Lemma 2.8 in [9], so we omit it here.

**Lemma 3.6.** Let \( c = (c_1, c_2, \ldots, c_b) \in R_{u,m}^b \) and \( w(c) = j \). Then

\[
\sum_{v \in R^b} \chi((c, v)) z^{[w_M(v)/t]} = \sum_{j_1=0}^{b-j} \sum_{j_2=0}^{2^m-1} (-1)^{j_1} (2^m - 1)^{j_2} \binom{j}{j_1} \binom{b-j}{j_2} z^{(j_1+j_2)/t}.
\]

The following lemma plays an important role in deriving the MacWilliams identity for \( m \)-spotty weight.

**Lemma 3.7.** Let \( C \) be a linear code of length \( nb \) over \( R_{u,m,2} \) and \( C^\perp \) its dual code and

\[
\hat{f}(u) = \sum_{v \in R^b} \chi((c, v)) f(v).
\]

Then

\[
\sum_{u \in C^\perp} = \frac{1}{|C|} \sum_{u \in C} \hat{f}(u).
\]

Let \( \alpha_j = \# \{ i : w(c_i) = j, 1 \leq i \leq n \} \). That is, \( \alpha_j \) is the number of bytes having Hamming weight \( j \), \( 0 \leq j \leq b \), in a codeword. The summation of \( \alpha_0, \alpha_1, \ldots, \alpha_b \) is equal to the code length in bytes, that is \( \sum_{j=0}^{b} \alpha_j = n \). The Hamming weight distribution vector \( \alpha_0, \alpha_1, \ldots, \alpha_b \) is determined uniquely for the codeword \( c \). Then, the \( m \)-spotty weight of the codeword \( c \) is expressed as \( w_M(c) = \sum_{j=0}^{b} \alpha_j \cdot j/t \). Let \( A(\alpha_0, \alpha_1, \ldots, \alpha_b) \) be the number of codewords with Hamming weight distribution vector \( \alpha_0, \alpha_1, \ldots, \alpha_b \). For example, Let \( c = (0u0 u^20u^3 1uw1 000 u10) \) is a codeword with byte \( b = 3 \). Then the Hamming weight distribution vector of the codeword is \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 = (1, 1, 2, 1) \). Therefore, \( A_{(1,1,2,1)} \) is the number of codewords with Hamming weight distribution vector \( (1,1,2,1) \).

We are now ready to define the \( m \)-spotty weight enumerator of a byte error control code over \( R_{u,m,2} \).

**Definition 3.5.** The weight enumerator for \( m \)-spotty byte error control code \( C \) is defined as

\[
W(z) = \sum_{c \in C} z^{w_M(c)}.
\]
By using the parameter \( A_{(\alpha_0, \alpha_1, \ldots, \alpha_b)} \), which denotes the number of codewords with Hamming weight distribution vector \( \{\alpha_0, \alpha_1, \ldots, \alpha_b\} \), \( W(z) \) can be expressed as follows:

\[
W(z) = \sum_{(\alpha_0, \ldots, \alpha_b)} A_{(\alpha_0, \ldots, \alpha_b)} \prod_{j=0}^b (z^{[j/t]})^\alpha_j.
\]

**Theorem 3.1.** Let \( C \) be a code over \( R_{a,m,2} \). The relation between the \( m \)-spotty \( t/b \)-weight enumerators of \( C \) and its dual is given by

\[
W^\perp(z) = \sum_{(\alpha_0, \ldots, \alpha_b)} A^\perp_{(\alpha_0, \ldots, \alpha_b)} \prod_{j=0}^b (z^{[j/t]})^\alpha_j = \frac{1}{|C|} \sum_{(\alpha_0, \ldots, \alpha_b)} A_{(\alpha_0, \ldots, \alpha_b)} \prod_{j=0}^b (F_j^{(b,m)}(z))^{\alpha_j},
\]

where \( F_j^{(b,m)}(z) = \sum_{j_1=0}^{b-j} \sum_{j_2=0}^{b-j} (-1)^{j_1} (2^m - 1)^{j_2} \binom{j}{j_1} \binom{j}{j_2} z^{[(j_1 + j_2)/t]} \).

**Proof.** Let \( f(v) \) in Lemma 3.7 be considered as \( f(v) = z^{w_M(v)} \). Then the function \( \tilde{f}(c) \) is calculated as follows:

\[
\tilde{f}(c) = \sum_{v \in R_{a,m,2}^b} \chi\{c, v\} z^{w_M(v)}
= \sum_{v_1 \in R_{a,m,2}^b} \sum_{v_2 \in R_{a,m,2}^b} \cdots \sum_{v_n \in R_{a,m,2}^b} \chi\{c_1, v_1\} \chi\{c_2, v_2\} \cdots \chi\{c_n, v_n\} \prod_{i=1}^n z^{w_H(v_i)/t}
= \prod_{i=1}^n \left( \sum_{v_i \in R_{a,m,2}^b} \chi\{c_i, v_i\} z^{w_H(v_i)/t} \right),
\]

By applying Lemma 3.6, we have,

\[
\tilde{f}(c) = \prod_{i=1}^n \left( \sum_{j_1=0}^{k_i} \sum_{j_2=0}^{b-k_i} (-1)^{j_1} (2^m - 1)^{j_2} \binom{k_i}{j_1} \binom{b-k_i}{j_2} z^{[(j_1 + j_2)/t]} \right),
\]

where \( k_i = w(c_i) \). Thus

\[
\tilde{f}(c) = \prod_{j=0}^b \left( \sum_{j_1=0}^j \sum_{j_2=0}^{b-j} (-1)^{j_1} (2^m - 1)^{j_2} \binom{j}{j_1} \binom{b-j}{j_2} z^{[(j_1 + j_2)/t]} \right)^{\omega_j(c)},
\]

where \( \omega_j(c) = |\{i | w(c_i) = j\}| \).

\[
\tilde{f}(c) = \prod_{i=1}^n \left( \sum_{j_1=0}^j \sum_{j_2=0}^{b-j} (-1)^{j_1} (2^m - 1)^{j_2} \binom{j}{j_1} \binom{b-j}{j_2} z^{[(j_1 + j_2)/t]} \right)^{\omega_j(c)}.
\]

After rearranging the summations on both sides according to the weight distribution vectors of codewords in \( C^\perp \) and \( C \) respectively, we have the result

\[
W^\perp(z) = \sum_{(\alpha_0, \ldots, \alpha_b)} A^\perp_{(\alpha_0, \ldots, \alpha_b)} \prod_{j=0}^b (z^{[j/t]})^\alpha_j = \frac{1}{|C|} \sum_{(\alpha_0, \ldots, \alpha_b)} A_{(\alpha_0, \ldots, \alpha_b)} \prod_{j=0}^b (F_j^{(b,m)}(z))^{\alpha_j}.
\]
4 Example

Let

\[ G = \begin{pmatrix}
1 & 0 & 0 & u + u^2 & 0 & 0 \\
0 & u & 0 & u^2 & 0 & u^3 \\
0 & 0 & u^2 & 0 & u & 0 \\
0 & 0 & u^3 & 0 & 0 & u
\end{pmatrix} \]

be the generator matrix of a linear code \( C \) over \( R_{u,4,2} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2 \) of length 6. \( C \) has \( 2^3 = 512 \) codewords. The dual code of \( C \) is a linear code of length 6 over \( R_{u,4,2} \) and it has \( 2^{15} = 32768 \) codewords.

The Hamming weight distribution vectors of the codewords of \( C \), the number of codewords, and polynomials \( F_j^{(b,m)}(z) \) for \( b = 3 \) and \( t = 2 \) are shown in Tables 1 and 2 for the necessary computations to apply the main theorem.

### Table 1

| \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) | number |
|------------------------------------------|--------|
| (2, 0, 0, 0)                            | 1      |
| (0, 2, 0, 0)                            | 18     |
| (0, 0, 2, 0)                            | 88     |
| (0, 0, 0, 2)                            | 104    |
| (1, 1, 0, 0)                            | 3      |
| (1, 0, 1, 0)                            | 7      |
| (1, 0, 0, 1)                            | 5      |
| (0, 1, 1, 0)                            | 72     |
| (0, 1, 0, 1)                            | 58     |
| (0, 0, 1, 1)                            | 156    |

### Table 2

Polynomials \( V_j^{(3,4)} \) for \( t = 2 \) and \( b = 3 \).

| \( F_j^{(3,4)}(z) \) | polynomial |
|----------------------|------------|
| \( F_0^{(3,4)}(z) \) | \( 1 + 720z + 3375z^2 \) |
| \( F_1^{(3,4)}(z) \) | \( 1 + 224z - 225z^2 \) |
| \( F_2^{(3,4)}(z) \) | \( 1 - 16z + 15z^2 \) |
| \( F_3^{(3,4)}(z) \) | \( 1 - z^2 \) |

According to the expression of \( W(z) \) and Table 1, we obtain the \( m \)-spotty weight enumerator of \( C \) as

\[ W(z) = 1 + 10z + 183z^2 + 214z^3 + 104z^4. \]

By applying Theorem 3.1 and Table 2, we obtain

\[
W^+(z) = \frac{1}{|C|} \sum_{(\alpha_0,\alpha_1,\alpha_2,\alpha_3) = 2} A_{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)} \prod_{j=0}^{3} (F_j(z))^{\alpha_j} \\
= \frac{1}{512} \left[(F_0^{(3,4)}(z))^2 + 18(F_1^{(3,4)}(z))^2 + 88(F_2^{(3,4)}(z))^2 + 104(F_3^{(3,4)}(z))^2 + 3F_0^{(3,4)}(z)F_1^{(3,4)}(z)
+ 7F_0^{(3,4)}(z)F_2^{(3,4)}(z) + 5F_0^{(3,4)}(z)F_3^{(3,4)}(z) + 72F_1^{(3,4)}(z)F_2^{(3,4)}(z)
+ 156F_2^{(3,4)}(z)F_3^{(3,4)}(z) + 58F_1^{(3,4)}(z)F_3^{(3,4)}(z)\right] \\
= 1 + 85z + 3153z^2 + 9707z^3 + 19822z^4.
\]

5 Conclusion
This paper has presented the MacWilliams identity for $m$-spotty weight enumerators of the $m$-spotty byte error control codes. This provides the relation between the $m$-spotty weight enumerator of the code and that of the dual code. Also, the indicated identity includes the MacWilliams identity over $\mathbb{F}_2 + u\mathbb{F}_2$ in [9] as a special case.

References

[1] Fujiwara E.: Code design for dependable system, Theory and practical application, Wiley & Sons, Inc., 2006.

[2] MacWilliams F. J., Sloane N. J.: The Theory of error-correcting codes, North-Holland Publishing Company, Amsterdam, 1978.

[3] Özen M., Siap V.: The MacWilliams identity for $m$-spotty weight enumerators of linear codes over finite fields, Computers and Mathematics with Applications, Vol.61, No. 4, 1000-1004, (2011).

[4] Özen M., Siap V.: The MacWilliams identity for $m$-spotty Rosenbloom-Tsfasman weight enumerator, Journal of the Franklin Institute, http://dx.doi.org/10.1016/j.jfranklin.2012.06.002, (2012).

[5] Sharma A., Sharma A. K.: Sharma, On some new $m$-spotty Lee weight enumerators, Des., Codes Cryptogr., Doi 10.1007/s10623-012-9725-z, (2012).

[6] Sharma A., Sharma A. K.: On MacWilliams type identities for $r$-fold joint $m$-spotty weight enumerators, Discret. Math., Vol.312, No. 22, 3316-3327, 2012.

[7] Sharma A., Sharma A. K.: MacWilliams type identities for some new $m$-spotty weight enumerators, IEEE Trans. Inform. Theory, Vol. 58, No. 6, 3912-3924, (2012).

[8] Shi M. j.: The MacWilliams identity for $m$-spotty Rosenbloom-Tsfasman weight enumerator over finite fields and ring $\mathbb{Z}_l$, communicated for publication.

[9] Siap I.: An identity between the $m$-spotty weight enumerators of a linear code and its dual, Turkish Journal of Mathematics, doi:10.3906/mat-1103-55, (2012).

[10] Siap I.: MacWilliams identity for $m$-spotty Lee weight enumerators, Appl. Math. Lett., Vol.23, Issue 1, 13-16, (2010).

[11] Suzuki K., Kashiyama T., Fujiwara E.: MacWilliams identity for $m$-spotty weight enumerator, ISIT 2007, Nice, France, 31-35, (2007).

[12] Suzuki K., Kashiyama T., Fujiwara E.: A general class of $m$-spotty weight enumerator, IEICE-Trans. Fundam., Vol. E90-A, No.7, 1418-1427, (2007).

[13] Umanesan G., Fujiwara E.: A class of random multiple bits in a byte error correcting and single byte error detecting (St/bEC-SbED) codes, IEEE Trans. on Comput., Vol. 52, No.7, 835-847, (2003).