We consider a new class of instantons in context of quantum field theory of a scalar field coupled with a chaotic background source field. We show how the instanton associated to the quantum tunneling from a metastable false to the true vacuum will be corrected by an exponential enhancement factor. Possible implications are discussed.

I. INTRODUCTION

As is known, an instanton describes a quantum process of tunneling through a barrier. In non-relativistic quantum mechanics, the barrier is usually localized in the space, \( V(r) \). On the other hand, in quantum field theory, the tunneling process is more subtle because it happens in the field internal space: a spin 0 particle, with quantum field operator \( \phi \), can tunnel from a false vacuum to the global minimum of its interaction potential \( V(\phi) \). This implies that not all gauge theories have a stable vacuum. In cosmology, a quantum tunneling can lead to the nucleation of a Bubble or a new Baby Universe from the Mother Universe [1–3]. As known, the tunneling probability can be enhanced in presence of a thermal bath. The most relevant theoretical example is in context of the baryogenesis in the Early Universe, where \( B + L \) violating transitions (\( B - L \) preserving), mediated by Standard Model sphalerons, are enhanced by thermal background fluctuations while strongly suppressed in laboratory [4, 5]. The possibility of a long-lived metastable vacuum was also largely studied in many models with dynamical supersymmetry breaking [6].

On the other hand, a simple class of chaotic instantons in non-relativistic 1 + 1 quantum mechanics was studied in Refs. [7, 8]. In particular, the wave function of a particle can be chaotized by a series of kicks inside a potential. The chaotic fluctuations can assist the quantum tunneling through the wall. So that, it is conceivable that a subtler analogous process can be considered in context of quantum field theories. Previous studies and definitions of chaos in quantum field theory can be found in Refs. [9–11]. However, chaos instantons in quantum field theory were never suggested and calculated in literature. On the other hand, studies of possible crossroads among chaos theory and field theories can deserve some surprising effects. For instance, in our previous papers, we have shown that Sinai billiards of singular geometries can scatter and chaotize quantum wave functions [13–16]. A new quantum decoherence effect induced by the non-trivial space-time topology emerged with possible implications in black hole physics and cosmic strings web.

In this paper, we will show how a new class of chaos assisted instantons can be found in quantum field theory of a scalar coupled to a chaotic background source field. We will show that the quantum tunneling probability is exponentially enhanced by chaotic fluctuations. Our calculations will be considered in semiclassical regime, treating the interactions of the scalar field with the background source as a perturbation.

II. CHAOTIC INSTANTON SOLUTION

Let us consider the

\[
V(\phi) = V_0 + a V_{\text{pert}}
\]

(1)

\[
V_{\text{pert}} = \varphi \Sigma = \varphi \sigma_0 \sum \cos(2\pi n k_\mu x^\mu / X)
\]

(2)

where we define \( \Sigma \) chaotic background source field, decomposed as an Fourier series of space-time plane waves with \( X \) a characteristic scales with dimensions \( [X] = M^{-1} \), \( [\sigma_0] = M^3 \) and \( V_0 \) is a generic double-well potential shown in Fig.1, \( a \) is a small dimensionless coupling constant with dimension \( [a] = M^0 \), while \( [\sigma_0] = M^3 \). This interaction describes a particle in the double-well potential coupled with a chaotic background field. With \( V_{\text{pert}} = 0 \), the quantum tunneling process is described by the usual Coleman-De Luccia instanton [1–3]. However,
the perturbation forms a stochastic energy layer, and the tunneling probability is described by chaotic instantons. Assuming that the chaotic corrections can be treated as perturbation, the chaotic instanton has a form

$$\varphi = \varphi_0 + \delta \varphi$$  \hspace{1cm} (3)

and they are solutions of the Euclidean equations of motion

$$-\Box_E \varphi + V_0'(\varphi) + a \Sigma = 0$$  \hspace{1cm} (4)

with

$$-\Box_E \varphi_0 + V_0'(\varphi_0) = 0$$  \hspace{1cm} (5)

leading to the

$$-\Box_E \delta \varphi + \delta V_0' + a \Sigma = 0$$  \hspace{1cm} (6)

$$\delta V_0' = V_0'(\varphi_0 + \delta \varphi) - V_0(\varphi_0) = 3\lambda \varphi^3 \delta \varphi - 2m^2 \varphi \delta \varphi + O(\delta \varphi)$$

On the other hand, \(\Sigma\) can be rewritten in term of a Dirac comb in space-time \[12\]

$$\sigma_0 \sum \cos(2\pi n k_x x / X) = \sigma_0 N' \sum_{n'}^{\infty} \delta(x - n'X)$$  \hspace{1cm} (7)

where \(N' = V_D\) and \(V_D = X^D\).

Incidently, let us note that setting \(V' - m^2 = 0\), the equation can be solved with the method of Green’s functions and the superposition principle:

$$\varphi(y_E) = \sum_{n'} \varphi_{n'}(y_E)$$  \hspace{1cm} (8)

$$\varphi_{n'}(y_E) = \int d^D x_E G_{n'}(y_E, x_E) \delta(y_E - x_E - n'X_E)$$  \hspace{1cm} (9)

$$(-\Box_E - m^2)G_{n'}(y_E, x_E) = -4\pi \delta(y_E - x_E - nX_E)$$  \hspace{1cm} (10)

The retarded Green’s function is

$$G_{n'}^r(y_E, x_E) = \theta(y_E^0 - x_E^0 - n'X_E^0) [\delta(z_{n'}) - m \sqrt{-2z_{n'}} J_1(m \sqrt{-2z_{n'}})]$$  \hspace{1cm} (11)

while the advanced Green’s function is

$$G_{n'}^a(y_E, x_E) = \theta(-y_E^0 - x_E^0 - n'X_E^0) [\delta(z_{n'}) - m \sqrt{-2z_{n'}} J_1(m \sqrt{-2z_{n'}})]$$  \hspace{1cm} (12)

with

$$z_{n'} = \frac{1}{2} \nu_{E_1}^{ab} (y_E^a - x_E^a - nX_E^a)(y_E^b - x_E^b - n'X_E^b)$$

and \(J_1\) the first order Bessel function. So that, \(\varphi\) is a noisy superposition of harmonics \(\varphi_{n'}\).

In \(D = 4\), the euclidean action is

$$S_E = 4\pi^2 \int_0^\infty d\rho_0 \frac{1}{2} \rho_0^2 + V_0(\varphi) + a\varphi \Sigma(\rho) \]$$  \hspace{1cm} (13)

with a resulting normalized factor

$$K = \frac{S_E^2}{4\pi^2} \int \frac{-\Box_E + V''(\varphi_0)}{-\Box_E + V''(\varphi)}$$  \hspace{1cm} (14)

while the formal tunneling rate is

$$\frac{\Gamma}{V} = Ke^{-S_E}$$  \hspace{1cm} (15)

which can be rewritten as

$$\frac{\Gamma}{V} = \frac{S_E^2}{4\pi^2} e^{-(\Gamma[\varphi] - \Gamma[\varphi_0])}$$  \hspace{1cm} (16)

where

$$\Gamma[\varphi] = S_E[\varphi] - \frac{1}{2} \text{Re} \text{tr} \log(-\Box_E + V''(\varphi))$$  \hspace{1cm} (17)

and the equation of motion is

$$\varphi'' + \frac{3}{\rho} \varphi' = \frac{dV_0}{d\varphi} + a\Sigma$$  \hspace{1cm} (18)

in which, as done by Coleman and De Luccia, we neglect the term \(\frac{3}{\rho} \varphi'\) in Eq.\[19\] i.e. physically motivated by a thin-wall approximation:

$$\varphi'' \approx \frac{dV_0}{d\varphi} + a\Sigma$$  \hspace{1cm} (19)

Let us note that in WKB approximation \(\lambda_{\text{Compton}}^C >> X\), we can perform a continuous spectrum approximation, and the background source is reduced to a constant \(\Sigma = \sigma_0 N' \sum_{n'} \delta \to \sigma_0 \) where \(\sum_{n'} \to \int dt' \ldots\) and \(\int dt' \delta = 1/N'\). This term becomes formally analogous to a driven constant force in an anharmonic oscillator. This corresponds to an energy density scale \(a\sigma_0 \delta \epsilon_C\), where \(\delta \epsilon_C\) is the chaotic energy layer, measured by the bottom of the false minima. The chaotic layer is \(\delta \epsilon_C = 2\pi r_0^{-1}\), where \(r_0\) is the radial constant related to \(X\) as \(X^4 = \pi^2 r_0^4/2\). Now, let us suppose that the background \(\Sigma\) is localized in space-time volume with radius \(R \leq \rho\), i.e. \(R = f \rho\) with \(0 \leq f \leq 1\). Physically, it will positively contribute to the formation of a bubble in this space-time radius. Outside the wall \(\phi = \phi_+\) the euclidean action is

$$\hat{S}_{E,\phi = \phi_+} = 0$$  \hspace{1cm} (20)
where inside the wall $\phi = \phi_-$

$$\tilde{S}_{E,(\phi=\phi_-)} = -\frac{\pi^2}{2} \rho^4 \epsilon + \pi^3 f^4 \rho^4 a \sigma_0 r_0^{-1}$$

(21)

where

$$\epsilon = V_0(\phi_+) - V_0(\phi_-)$$

Within the wall:

$$\tilde{S}_{E,(\phi_-<\phi<\phi_+)} = 2\pi^2 \rho^3 \int d\rho [V_0(\phi) - V_0(\phi_+)]$$

(22)

So that, the total $\tilde{S}_E$ is

$$\tilde{S}_E = -\frac{\pi^2}{2} \rho^4 \epsilon + 2\pi^2 \rho^3 s_1 + \pi^3 f^4 \rho^4 a \sigma_0 r_0^{-1}$$

(23)

which is stationary if

$$\tilde{\rho} = 3s_1/(\epsilon - 2\pi a f^4 \sigma_0 r_0^{-1})$$

The expression is well defined for $2\pi a f^4 \sigma_0 r_0^{-1} < \epsilon$. Finally, the tunneling amplitude is

$$\frac{1}{V} \Gamma \simeq K e^{S_0 + \pi^3 f^4 \rho^4 a \sigma_0 r_0^{-1}}$$

(24)

So that, as we can see, an exponential enhancement factor with respect to the standard instanton.

III. CONCLUSIONS AND REMARKS

In this paper, we have considered the problem of a scalar field in a false vacuum coupled to a chaotic background source field. The transition amplitude gets an extra exponential correction with respect to the result obtained by Coleman et al. [1][3]. This is the first example of chaos assisted instantonic solution in quantum field theory. However, it is conceivable that the solution suggested in this paper is only one particular example in a large class of possible solutions of various quantum field theories. For instance, it could be interesting to study possible chaos instantons in gauge theories. Perhaps, chaos gauge instantons could provide important insights in QCD asymptotic freedom, confinement and nuclear physics decays. On the other hand, in context of early Universe, chaos assisted enhancements can induce new first order phase transitions. As mentioned above, we have not include gravity in our considerations. Coleman and De Luccia demonstrated that the inclusion of gravity implies that the transition from a false vacuum generates a Bubble [3]. Now, let us comment on possible existence of chaotic solitons. In fact a general correspondence among instantons in gauge theories with solitons in higher dimension is known in literature. G. Dvali, H. B. Nielsen and N. Tetradis have shown a correspondence among the t’Hooft-Polyakov monopole and a gauge instanton solution in the lower dimensional theory localized on the domain walls [17]. Other examples and generalizations of the instanton/monopole correspondence were studied in literature [18][21]. So that, if the scalar field considered above is an Higgs field, it can source the presence of monopoles, vortices or cosmic strings in context of GUT theories. For example monopoles, kinks, vortices and domain walls in false vacua were studied in [27][30]: topological solitons can tunnel with a finite probability mediated by a gauge instantons (of a lower dimensional gauge theory). So that, we suggest that in presence of a chaotic source, new solitons could correspond to chaotic instantons. These considerations could be extended in context of open string theories, where instantons correspond to Euclidean D-branes wrapping the internal Calabi-Yau compactification (see [22][26] for general reviews on these aspects). Finally, it is possible that in presence of a primordial chaotic background electroweak gauge instantons or sphalerons get an extra enhancement factor higher then the thermally induced one. On the other hand, the chaotic enhancement effects could induce strong violations of $B - L$, if associated exotic instantons were assisted by a chaotic background [31][33].

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