A COMPLETELY INTEGRABLE SYSTEM ON $G_2$ COADJOINT ORBITS

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ABSTRACT. We construct a Gelfand-Zeitlin system on a one-parameter family of $G_2$ coadjoint orbits that are multiplicity-free Hamiltonian $SU(3)$-spaces. Using this system we prove a lower bound for the Gromov width of these orbits. This lower bound agrees with the known upper bound.

1. INTRODUCTION

Gelfand-Zeitlin systems are completely integrable systems constructed from maximal non-Abelian Hamiltonian symmetries of a symplectic manifold. The classic examples of Gelfand-Zeitlin systems were constructed on $SU(n)$ and $SO(n)$ coadjoint orbits [6]. These systems were employed by [16] to prove tight lower bounds for the Gromov width of $SU(n)$ and $SO(n)$ coadjoint orbits that, together with the upper bounds obtained in [4], compute the Gromov width. Tight lower bounds were also recently proven for $Sp(n)$ coadjoint orbits using integrable systems constructed from toric degenerations [8].

Motivated by the work of [16], and a desire for more examples of Gelfand-Zeitlin systems, this paper describes a Gelfand-Zeitlin system on a one-parameter family of $G_2$ coadjoint orbits that are multiplicity-free $SU(3)$-spaces. As complex manifolds, these coadjoint orbits are degree 18, 5 dimensional projective subvarieties of $\mathbb{CP}^{13}$ [1]. Using the convexity theorem of [13] we are able to compute the image of the Gelfand-Zeitlin system – a 5-dimensional convex polytope – without undue difficulty and this image classifies an open dense submanifold of the coadjoint orbit up to symplectomorphism. Combining the lower bound methods employed recently in [16], and the upper bounds due to [4], we can then prove

**Theorem 1.** If $O_\lambda$ is the $G_2$ coadjoint orbit through $(0, \lambda)$ as in Figure 1 equipped with the Kostant-Kirillov-Souriau symplectic form $\omega_\lambda$, then

$$GWidth(O_\lambda, \omega_\lambda) = \frac{\lambda}{\sqrt{3}}.$$  

The other one-parameter family of 5 (complex) dimensional $G_2$ coadjoint orbits, corresponding to the short simple root, are quadrics in $\mathbb{CP}^6$ that are homogeneous spaces for $SO(7, \mathbb{C})$. As such, they are $SO(8)$ coadjoint orbits and carry Gelfand-Zeitlin systems already described in [16] that, combined with the upper bound of [4], give the precise Gromov width of these orbits.

One may also deduce the Gromov width of both these families of 5 dimensional $G_2$ coadjoint orbits from the recent work of [14], since they have second Betti number $b_2 = 1$. The lower bound in the main theorem of op. cit. follows from work of [8, 12] that constructs completely integrable torus actions on projective varieties from toric degenerations. In comparison, the virtue of this work is that the momentum map and polytope have a very straightforward descriptions.

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Root system of $G_2$

Positive Weyl chamber of $G_2$

Figure 1

2. Gelfand-Zeitlin systems and multiplicity-free spaces

Introduced by [6], Gelfand-Zeitlin systems can be viewed as examples of the following general construction, a detailed exposition of which can be found in [13].

Let $G$ be a compact, connected Lie group and let $(M, \omega, \Phi, G)$ be a connected Hamiltonian $G$-space. Given a chain of subgroups $H_1 \leq \cdots \leq H_n = G$ with corresponding subalgebras $h_k = \text{Lie}(H_k)$, there are dual projection maps $p_k : h^*_k \to h^*_k$. Fixing closed positive Weyl chambers, $t^*_k, + \subset h^*_k$, we can construct the map

$$\Lambda = (s, s \circ p_{n-1}, \ldots, s \circ p_1 \circ \cdots \circ p_{n-1}).$$

where $s(\xi)$ is defined as the unique element of $(H_k \cdot \xi) \cap t^*_{k,+}$. The restriction of the map to the open, dense submanifold $U = (\Lambda \circ \Phi)^{-1}(\mathcal{T}^{\text{int}})$ is a smooth momentum map for a Hamiltonian torus action. This construction is referred to as Thimm’s trick or the method of Thimm, and one may say that the map $\Lambda \circ \Phi$ is constructed by Thimm’s trick, in reference to the paper [18].

Recall that a Hamiltonian $G$-space is multiplicity-free if for all $\xi \in g^*$ the reduced space $\Phi^{-1}(\xi)/G_\xi$ is a point. An effective Hamiltonian torus action is completely integrable if the dimension of the torus is half the real dimension of the manifold. If $M$ is a multiplicity-free $G$-space, and for each $1 \leq k < n$, every $H_{k+1}$-coadjoint orbit is a multiplicity-free $H_k$-space, the Hamiltonian torus action induced on $U$ is completely integrable [7]. Following Guillemin and Sternberg’s terminology in [6], we refer to such a map (and the associated densely defined torus action) as a Gelfand-Zeitlin system.

\[1\] To be more precise, one should replace each of the closed chambers $t^*_{k,+}$ in (2) with the closure of the principal face of $t^*_{k,+}$ corresponding to the induced Hamiltonian $H_k$-actions on $M$.

\[2\] One should note that there are multiple spellings of Zeitlin in the literature, including Tsetlin and Cetlin. Works by Kostant-Wallach, Kogan-Miller, and Guillemin-Sternberg respectively each choose a different spelling.
**Example 4.** Let \( \mathcal{O} \) be a \( U(k) \) coadjoint orbit, and define the subgroup
\[
H_{k-1} = \{ \text{diag} (A, 1) : A \in U(k-1) \} \cong U(k-1).
\]
Guillemin and Sternberg showed that \( \mathcal{O} \) is a multiplicity-free \( U(k-1) \)-space. Thus if \( M \) is a multiplicity-free \( SU(n) \)-space, then the map \( \Lambda \circ \Phi \) constructed on \( M \) by Thimm’s trick for the chain of subgroups
\[
U(1) \leq U(2) \leq \cdots \leq U(n-1) \leq SU(n)
\]
is a Gelfand-Zeitlin system.

If \( \Lambda \circ \Phi : M \to T \) is a map constructed by Thimm’s trick, then the submanifold \( U \) is connected. Using this fact and the convexity theorem for proper torus momentum maps \(^2\) one can prove

**Theorem 5.** \(^3\) Let \( M \) be a connected symplectic manifold and suppose \( \Lambda \circ \Phi : M \to T \) is a map constructed by Thimm’s trick. If \( \Lambda \circ \Phi \) is a proper map, then \( \Lambda \circ \Phi(M) \) is convex and the fibres of \( \Lambda \circ \Phi \) are connected.

3. A Gelfand-Zeitlin system on \( \mathcal{O}_\lambda \)

In [20], Chris Woodward gives the following example.

**Example 6.** Consider the exceptional Lie group \( G_2 \) with maximal torus \( T \) and let \( t \) be the corresponding subalgebra. Fix the standard identifications \( t \cong t^* \cong \mathbb{R}^2 \) and let
\[
S = \left\{ \alpha_1 = \left( -\frac{3}{2}, \frac{\sqrt{3}}{2} \right), \alpha_2 = (1, 0) \right\} \subset \mathbb{R}^2
\]
be the standard choice of simple roots (see Figure 1A). The positive Weyl chamber \( t^{*}_{G_2, +} \) given by this choice of simple roots is shaded in Figure 1B. There is an embedding of \( SU(3) \) in \( G_2 \) with maximal torus \( T \) corresponding to the long roots of \( G_2 \), as shown in Figure 2A.

Coadjoint \( G_2 \)-orbits are parameterized by points in \( t^*_{G_2, +} \). In particular, we write \( \mathcal{O}_\lambda \) for the \( G_2 \) coadjoint orbit through the point \( (0, \lambda) \) on the vertical edge of \( t^*_{G_2, +} \), equipped with its Kostant-Kirillov-Souriau symplectic form \( \omega_\lambda \). The following facts are explained in [20].

1. The coadjoint orbit \( \mathcal{O}_\lambda \cong G_2/(SU(2) \times U(1)) \) has real dimension 10.
2. The \( SU(3) \) subgroup acts locally freely on \( \mathcal{O}_\lambda \). Since \( \dim SU(3) + \text{rank } SU(3) = 10 \), \( \mathcal{O}_\lambda \) is a multiplicity free \( SU(3) \)-space.
3. The Kirwan polytope for the \( SU(3) \)-action can be understood using the Heckman formula of [5] and is the triangle illustrated in Figure 2B. In coordinates, the three vertices of this triangle are
\[
(0, \lambda), \left( \frac{\lambda}{2\sqrt{3}}, \frac{\lambda}{2} \right), \left( -\frac{\lambda}{2\sqrt{3}}, \frac{\lambda}{2} \right).
\]
In terms of the roots of \( G_2 \) the vertices are
\[
\frac{\lambda(\alpha_4 - \alpha_5)}{\sqrt{3}}, \frac{-\lambda \alpha_5}{\sqrt{3}}, \frac{\lambda \alpha_4}{\sqrt{3}}.
\]
Following the previous section, consider the chain of groups
\[
U(1) \leq U(2) \leq SU(3)
\]
where we choose the embeddings
\[
U(1) \cong \{ \text{diag} (a, 1) : a \in U(1) \} \leq U(2), U(2) \cong \left\{ \frac{1}{\det(A)} \text{diag} (A, 1) : A \in U(2) \right\} \leq SU(3).
\]
Since $O_\lambda$ is a multiplicity-free $SU(3)$-space, the map
\begin{equation}
\Lambda \circ \Phi : O_\lambda \to t_{SU(3)}^* \times t_{U(2)}^* \times t_{U(1)}^*
\end{equation}
is a Gelfand-Zeitlin system. For the remainder of this paper $\Lambda \circ \Phi$ will denote this specific Gelfand-Zeitlin system.

4. THE GELFAND-ZEITLIN POLYTOPE OF $O_\lambda$

In this section we compute the image of the Gelfand-Zeitlin system on $O_\lambda$. In order to do this we compute the vertices of the image, a list of bounding inequalities, and apply Theorem 5.

Fix the standard identifications
\begin{align*}
    u(n) &\cong \{ X \in M_n \mathbb{C} : X = \overline{X} \} \\
    t_{U(n)}^* &\cong u^*(n) \cong \{ \text{diag}(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R} \} \cong \mathbb{R}^n \\
    L_{U(n)}^* &= \text{Hom}(L_{U(n)}^*, \mathbb{Z}) \cong \{ \text{diag}(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{Z} \} \cong \mathbb{Z}^n \\
    t_{U(n),+}^* &\cong \{ \text{diag}(x_1, \ldots, x_n) \in t_u(n) : x_1 \leq \cdots \leq x_n \}
\end{align*}

From the previous section we also have the identifications
\begin{align*}
    t_{SU(3)}^* &= \mathbb{R}^2 \\
    L_{SU(3)}^* &= \mathbb{Z} \left\langle \alpha_2 = (1, 0), \alpha_4 = \left( \frac{-1}{2}, \frac{\sqrt{3}}{2} \right) \right\rangle.
\end{align*}

With these identifications the Gelfand-Zeitlin system (7) is a map to $T \subseteq \mathbb{R}^5 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^1 \cong t_{SU(3)}^* \times t_{U(2)}^* \times t_{U(1)}^*$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$, where
\begin{align*}
    T &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \frac{\pm 3x_1}{2} + \frac{\sqrt{3}x_2}{2} \geq 0 \text{ and } x_3 \leq x_4 \right\} \\
    &\cong t_{SU(3),+}^* \times t_{U(2),+}^* \times t_{U(1),+}^*
\end{align*}

Further, in these coordinates the weight lattice $L^* = \text{Hom}(L, \mathbb{Z})$ is
\begin{align*}
    L^* &= L_{SU(3)}^* \times L_{U(2)}^* \times L_{U(1)}^* \\
    &\cong \mathbb{Z} \left\langle (1, 0, 0, 0, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0 \right), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1) \right\rangle.
\end{align*}
Table 1. Vertices of the Gelfand-Zeitlin polytope $\Delta_\lambda$

| $(x_1, x_2, x_3, x_4, x_5)$ | $(0, \lambda, 0, 0, 0)$ |
|-----------------------------|-------------------------|
| $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, 0, \frac{-\lambda}{\sqrt{3}})$ | $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}})$ |
| $(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, 0)$ | $(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}})$ |
| $(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}})$ | $(0, \lambda, 0, \frac{\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}})$ |
| $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}})$ | $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}})$ |
| $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}})$ | $(0, \lambda, \frac{-\lambda}{\sqrt{3}}, \frac{-\lambda}{\sqrt{3}}, \frac{\lambda}{\sqrt{3}})$ |

The Kirwan polytope of the previous section gives us inequalities

$$\frac{-\sqrt{3}x_1}{2} + \frac{x_2}{2} \leq \frac{\lambda}{2},$$
$$\frac{\sqrt{3}x_1}{2} + \frac{x_2}{2} \leq \frac{\lambda}{2},$$
$$\frac{\lambda}{2} \leq x_2.$$

(9)

From the interlacing inequalities for eigenvalues of Hermitian matrices, one derives

$$\frac{-x_1}{3} - \frac{x_2}{\sqrt{3}} \leq x_3 \leq \frac{2x_1}{3} \leq x_4 \leq \frac{-x_1}{3} + \frac{x_2}{\sqrt{3}}$$
$$x_3 \leq x_5 \leq x_4.$$

(10)

One can find elements of $\Phi(\mathcal{O}_\lambda)$ that make each of the inequalities strict, so the image of $\Lambda \circ \Phi$ contains the points listed in Table 1. By Theorem 5 and the inequalities (9) and (10) we have

**Proposition 11.** The image of the Gelfand-Zeitlin system $\Lambda \circ \Phi : \mathcal{O}_\lambda \to \mathbb{R}^5$ is the convex hull of the 13 points listed in Table 1. This convex hull, which we will refer to as $\Delta_\lambda$, is a 5-dimensional convex polytope.

5. The Gromov width of $\mathcal{O}_\lambda$

The *Gromov width* of a symplectic manifold is defined as

$$\text{GWidth}(M, \omega) = \sup \{ \pi r^2 > 0 : \exists \text{ a symplectic embedding } B^{2n}(r) \to M \}$$

where $B^{2n}(r)$ is the open ball of radius $r$ in $\mathbb{R}^{2n}$ equipped with the standard symplectic structure. Coadjoint orbits have the Kostant-Kirillov-Souriau symplectic structure and
their Gromov width has been a subject of much study, including work on complex Grassmannians by [11, 15] and later other families of coadjoint orbits [21, 16, 18, 8, 14]. In this section we follow the same approach as [16] to compute a lower bound for the Gromov width of $O_\lambda$ from the polytope $\Delta_\lambda$.

Suppose a connected symplectic manifold $(M, \omega)$ is equipped with a completely integrable action of a torus $T$ generated by a momentum map $\mu : M \to \mathfrak{t}^*$. If there is an open, convex set $C \subset \mathfrak{t}^*$ such that $\mu(M) \subset C$ and the map $\mu : M \to C$ is proper, then $M$ is classified up to $T$-equivariant symplectomorphism by its image, $\mu(M)$ ([10], Proposition 6.5). In particular, the open submanifold $\mu^{-1}(\mu(M)^{\text{int}})$, with the restricted symplectic structure, is symplectomorphic to $\mu(M)^{\text{int}} \times T$ with the symplectic structure $\omega$ such that $\mu_!: \mu(M)^{\text{int}} \times T \to \mathfrak{t}^*$ is a momentum map for the torus action $t \cdot (\xi, s) = (\xi, ts)$. Thus, to obtain a lower bound on the Gromov width of $M$, it is sufficient to obtain a lower bound on the Gromov width of $\mu(M)^{\text{int}} \times T$. A lower bound on the Gromov width of $\mu(M)^{\text{int}} \times T$ can be extracted from the the combinatorics of $\mu(M)$, as we will now explain.

Define $\Delta^n(l) = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : 0 < y_1, \ldots, y_n, \text{ and } y_1 + \cdots + y_n < l\}$, the standard open simplex of size $l$. A version of the following Lemma was proven in [19] for $n = 2$. Proofs for arbitrary $n$ can be found in [16, 17].

Recall that if $(V_i, L_i)$, $i = 1, 2$, are vector spaces $V_i$ together with full rank lattices $L_i$, then an integral affine map $B : V_1 \to V_2$ is a map of the form $B = A + v$ where $A : V_1 \to V_2$ is a linear map that sends a $\mathbb{Z}$-basis for $L_1$ to a $\mathbb{Z}$-basis for $L_2$ and $v \in V_2$.

**Lemma 12.** Let $T$ be a compact torus of dimension $n$ and let $S$ be an open subset of $\mathfrak{t}^*$. If there is an integral affine map $B : (\mathbb{R}^n, \mathbb{Z}^n) \to (\mathfrak{t}^*, \mathbb{Z}^*)$ such that $B(\Delta^n(l)) \subset S$, then $l \leq G\text{Width}(S \times T, \omega)$.

If a line segment in $(\mathbb{R}^n, L)$ can be translated to a scalar multiple of a primitive vector in the lattice $L$, then that scalar is the integral affine length of the line segment. A vertex $v$ of a convex polytope $P \subset \mathbb{R}^n$ is smooth if the edges of $P$ incident to $v$ are spanned by a $\mathbb{Z}$-basis for $L$ (after translation by $v$). If $v$ is a smooth vertex and the minimum integral affine length of the edges incident to $v$ is $l$, then there exists an integral affine map $B$ such that $B(\Delta^n(l)) \subset P^{\text{int}}$.

The other half of Theorem 1 follows from

**Theorem 13.** [4] If $O_\lambda$ is the $G_2$ coadjoint orbit through $(0, \lambda)$ as in Figure 7 then $G\text{Width}(O_\lambda, \omega_\lambda) \leq \frac{\lambda}{\sqrt{3}}$.

We are now equipped to prove the main theorem.

**Proof of Theorem 1.1.** Recall that the pre-image of $T^{\text{int}}$ under $\Lambda \circ \Phi$ is a connected open dense submanifold $U \subset O_\lambda$. The restriction $\Lambda \circ \Phi : U \to C = T^{\text{int}}$ is a proper map to an open convex set that generates a completely integrable Hamiltonian torus action. Thus by [10], $(U, \omega_\lambda)$ is the unique non-compact toric manifold with moment polytope $\Lambda \circ \Phi(U) = \Delta_\lambda \cap C \subset (\mathbb{R}^5, \mathbb{Z}^*)$.

It is straightforward to check from Table 1 that the minimum integral affine length of an edge of $\Delta_\lambda$ (with respect to the weight lattice $L^*$ in Equation 8) is $\frac{\lambda}{\sqrt{3}}$. Any vertex of $\Delta_\lambda$ contained in $C$ (of which there are several) is smooth, so by the discussion following Lemma 12...
there exists an integral affine map \( B : (\mathbb{R}^5, \mathbb{Z}^5) \to (\mathbb{R}^5, L^*) \) that embeds \( \Delta^\text{int}_\lambda \) into \( \Delta^\text{int}_\lambda \). Thus by Lemma \([12]\) and the discussion preceding it we have shown that

\[
\frac{\lambda}{\sqrt{3}} \leq \text{GWidth}(\Delta^\text{int}_\lambda \times T, \omega) \leq \text{GWidth}(\mathcal{O}_\lambda, \omega_\lambda).
\]

Combining this with the upper bound of \([4]\) completes the proof. \( \square \)

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