Geodesic string condensation from symmetric tensor gauge theory: 
a unifying framework of holographic toy models

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In this work we reason that there is a universal picture for several different holographic toy model constructions, and a gravity-like bulk field theory that gives rise it. First, we observe that the perfect tensor-networks and hyperbolic fracton models are both equivalent to the even distribution of bit-threads on geodesics in the AdS space. Such picture is also a natural “leading-order” approximation to the holographic entanglement properties. Then, we argue that the rank-2 U(1) theory with linearized diffeomorphism as its gauge symmetry, also known as a case of Lifshitz gravity, is the bulk field theory behind such picture. The Gauss’ laws and spatial curvature require the electric field lines along the geodesics to be the fundamental dynamical variables, which lead to geodesic string condensation. These results provide an intuitive way to understand the entanglement structure of gravity in AdS/CFT.

INTRODUCTION

Modern physics has entered an exciting era of intertwined influences between quantum many-body systems, quantum gravity, and quantum information. Many of these interdisciplinary conversations revolve around the holographic principle [1, 2] and anti-de Sitter/conformal field theory (AdS/CFT) correspondence [3, 4]. As a conjectured duality between quantum gravity in (d+1)-dimensional asymptotically AdS spacetime and a d-dimensional CFT on its boundary, the holographic principle is a profound insight of quantum gravity, and also acts as a powerful tool for condensed matter problems [5].

In 2006, Ryu and Takayanagi conjectured that the entanglement entropy of a boundary CFT segment is measured by the area of the corresponding extremal covering surface in the AdS geometry [6, 7]. This conjecture reveals the intimate relation between entanglement and geometry in quantum gravity. Following the insight of Swingle [8], various tensor-network holographic toy models were built [8–14], and they uncover the quantum-informational correcting feature of holographic entanglement.

On the condensed matter hemisphere, the fracton states of matter were studied intensively in recent years [15–23]. The gapless versions of fracton states, namely rank-2 U(1) [R2-U1] gauge theories [21, 22, 24, 25], were found to be different cases of Lifshitz gravity [26, 27]. The charge excitations dubbed “fractons” also show gravitational attractions [28]. Inspired by these discoveries, a toy fracton model in AdS space was studied and shown to satisfy holographic properties in a similar fashion as the holographic tensor-networks [29, 30].

Some important questions remain unanswered despite these progress. First, what is the connection between the different holographic toy models? Is there a universal picture behind them? Furthermore, how can we derive its continuous limit from a bulk field theory, and how is the bulk theory related to gravity? This is in particular intriguing for the perfect tensor models, since they are clever constructions directly based on holographic entanglement properties, but their correspondence to any concrete bulk field theory is still unknown. Although the hyperbolic fracton model is a spin model in the bulk, it is still far from a field theory that shows satisfactory resemblance to gravity.

In this work, we advance our understanding of holographic toy models by addressing these two questions. First, we point out that there is a universal picture behind different constructions of holographic toy models: a homogeneous and isotropic distribution of bit-threads (or up to some lattice discretization). We then show that the traceful, vector charged R2-U1, a theory with linearized diffeomorphism as its gauge symmetry, gives rise to this continuous bit-thread picture. We reason that in the presence of spatial curvature, the gauge symmetry, and the consequent Gauss’ laws, only allow electric field lines along a geodesics to be the fundamental dynamical variables (magnetic field). Any loop configurations like in conventional U(1) theory are forbidden. Hence the entanglement structure is determined by the “geodesic string condensation,” which is exactly the continuous bit-thread picture. As such, we establish the connection between the holographic toy models and a concrete gravity-like bulk field theory, and shows how entanglement structure emerges from linearized diffeomorphism.

BIT-THREAD TYPE HOLOGRAPHIC TOY MODELS AS A UNIVERSAL PICTURE

This work is motivated by recent progress on toy models of holography. They are based on two different constructions: the hyperbolic fracton model [29, 30] and the perfect tensor-networks [8–14]. But we observe that they
belong to a universal picture: They are equivalent to bit-threads arranged on a tessellation of the hyperbolic disk. A bit-thread is a line with entangled qubits (or more generally, we can consider any quantum/classical degrees of freedom) at its two ends [31–33]. A flow of the bit-threads in the AdS space, when saturating minimal covering surface of a boundary subregion, gives the correct entanglement entropy described by the Ryu-Takayanagi formula [RT-formula] [6, 7]

$$S_A = \frac{\text{area}(\gamma_A)}{4G_N}. \quad (1)$$

The hyperbolic fracton model is dual to the eight-vertex model defined on the edges of the pentagon tesselation (Fig. 1) [30]. At low temperature, the eight-vertex model becomes a web of independent one-dimensional chains with ferromagnetic couplings. Each chain is then a classical bit-thread with its two ends correlated.

Recent work by Jahn et al. [34, 35] show that the perfect tensor-network can be described by Majorana modes via Jordan-Wigner transformation. In the Mjorana fermion language, the tensor-network state becomes a collection of Majorana dimers. Each dimer locates at the two ends of a geodesic on the tensor tessellation, which is the bit-thread.

Following these observations, a universal picture of the holographic toy model emerges: by arranging the bit-threads in the AdS space homogeneously and isotropically in the continuous limit, or on a regular tessellation in the discrete case, the simplest toy model of holography can be constructed.

Such toy models capture the RT-formula for entanglement entropy of any connected boundary subregion. Instead of adjusting the bit-thread flow to saturate the target covering surface like in the original proposal [31], the bit-threads in this picture are in a fixed configuration, but the RT-formula is satisfied due to their even distribution in the bulk. The bulk information is defined in the dual model in both the hyperbolic fracton model and the perfect tensor-networks. Its reconstruction obeys the Rindler reconstruction rule [36, 37], again when the boundary subregion is connected.

It can be viewed as a “leading order” approximation of the entanglement structure of AdS/CFT. Built upon a collection of two-body entangled qubits/bits only, it naturally fails to capture the finer entanglement structure of genuine gravitational AdS/CFT.

For example, the entanglement spectrum of a boundary subregion is always flat, thus the nth-Rényi entanglement entropy

$$S_n(\rho_A) = \frac{1}{1-n} \log \text{Tr} \rho_A^n \quad (2)$$

has no n-dependence, while in AdS/CFT the n-dependence is non-trivial [38]. Also, such models deviate from the RT-formula when the boundary subregion has multiple disconnected components. This deviation is due to the bit-threads connecting different components of the boundary subregion, which is discussed in Ref. [30].

In Table. I, we have summarized the comparison between genuine AdS/CFT, the bit-thread type toy models, and for completeness also the holographic random tensor-networks proposed by Yang et al. [11–13]. The random tensor-network satisfies RT-formula for arbitrary boundary subregion, and does not belong to the universal picture proposed here.

These observations lead to the question this work addresses: what is the bulk field theory that gives rise to the bit-thread type of holographic toy models? These toy models capture the RT-formula and Rindler reconstruction at “leading order,” but fails at “higher order,” or the finer entanglement structure. Thus, the reasonable speculation is that such bulk theory cannot be the full-fledged general relativity, but it has to share certain essential features of gravity or is a special limit/case of it.

We will show that, the bulk theory that describes the bit-thread type toy models is the Lifshitz gravity [26, 39] in the high energy theory literature, or the traceful, vector charged rank-2 U(1) gauge theory [21, 22, 24, 25, 27, 40] in the condensed matter physics literature. As a special case of general relativity, its gauge symmetry is the spatial part of the linearized diffeomorphism. As we will elaborate, the consequence of such gauge symmetry is that the electric field lines can only travel along the geodesics in AdS space, instead of forming local loops like in conventional gauge theory. Hence, the entanglement structure is the continuous bit-thread distribution.
This restricts the movement of a vector charge $\rho$ to higher dimensions. The R2-U1 theory has gauge symmetry

$$A^{ij} \rightarrow A^{ij} + \partial^i \lambda^j + \partial^j \lambda^i.$$ (3)

Taking $A^{ij}$ as the perturbation of the metric $A^{ij} = h^{ij} - \delta^{ij}$ in general relativity, the transformation is the linearized limit of diffeomorphism.

The gauge symmetry corresponds to the Gauss’ laws of the electric field at low energy. In this case, the electric field is a rank-2 symmetric tensor

$$E^{ij} = E^{ji}.$$ (4)

The Gauss’ laws imposed on the electric field is

$$\partial_i E^{ij} = 0.$$ (5)

We take both Eq.(4) and (5) to be the Gauss’ laws at the low energy sector of the theory.

The charge for such diffeomorphism-like gauge theory is a vector, defined as

$$\rho^i = \partial_j E^{ij}.$$ (6)

Beside the total vector charge conservation, the symmetric condition imposes an additional conservation law

$$\int dv \, \rho \times \mathbf{x}^k = \int dv \, \epsilon^{kij} \partial_i E^{ij} = - \int dv \, \epsilon^{kij} E^{ij} = 0.$$ (7)

This restricts the movement of a vector charge $\rho$. The charge $\rho$ can only move in the direction of itself. It has crucial consequences in the entanglement structure, as we shall see.

Finally, in the flat space, the magnetic field is the simplest gauge symmetry-invariant term,

$$B = \epsilon^{ai} \epsilon^{bj} \nabla_a \nabla_b A_{ij}.$$ (8)

And the Hamiltonian is

$$\mathcal{H}_{\text{R2-U1-flat}} = U E^{ij} E^{ij} + t B^2.$$ (9)

### RANK-2 U(1) THEORY AND ITS FLAT-SPACE DYNAMICS

Let us first quickly review the traceful, vector-charged version of R2-U1 theory [21, 22, 24, 25]. Here we work in two-dimensional space, but the physics naturally extends to higher dimensions.

The R2-U1 theory has gauge symmetry

$$A^{ij} \rightarrow A^{ij} + \partial^i \lambda^j + \partial^j \lambda^i.$$ (3)

Taking $A^{ij}$ as the perturbation of the metric $A^{ij} = h^{ij} - \delta^{ij}$ in general relativity, the transformation is the linearized limit of diffeomorphism.

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### ENTANGLEMENT STRUCTURE FROM GAUGE SYMMETRY: CONVENTIONAL U(1) AS AN EXAMPLE

Before examining the entanglement structure of the R2-U1 in AdS space, let us first review the topological entanglement entropy in the conventional U(1) gauge theory from the condensed matter point-of-view [41–44]. It is the string-net condensation picture proposed by Levin and Wen [41]. This helps to understand the logical chain of how gauge symmetry determines the entanglement structure.

The gauge symmetry for conventional U(1) theory

$$A^i \rightarrow A^i + \partial^i \lambda$$ (10)

as our starting point determines the Gauss’ law to be electric charge conservation

$$\partial_i E^i = 0$$ (11)

At low energy, the operations of gauge field $A^x$, $A^y$ are to construct microscopic electric fields, or dipoles (cf. Table. II). The gauge field operators respect the charge conservation globally, but not locally. Mathematically, that is to say the gauge field themselves are not gauge invariant.

To respect the Gauss’ law in any infinitesimal, local subregion, the dipoles operators have to be connected head-to-tail together to form a loop. The minimal loop is the magnetic field $B = \epsilon^{ij} \nabla_i A_j$ (cf. Table. II), which is now gauge invariant.

### TABLE I. Comparison of the holographic entanglement properties between genuine AdS/CFT, bit-thread type holographic toy models, and random tensor-networks

The Hamiltonian of Eq. (9) is also a case of Lifshitz gravity [27]. Treating $A^{ij}$ as the perturbation of the metric, the magnetic field squared term $B^2$ is equivalent to $R^2$, $R$ being the Ricci scalar. Here, the conventional linear term $R$ and cosmological constant $\Lambda$ in general relativity, as well as the self-interacting, non-linear terms are forbidden due to the time-reversal, lattice translation, and spatial reflection symmetries. This was carefully analyzed in Ref. [27]. So the theory of Eq. 9 can be viewed as a special version of linearized gravity.
To be an eigenstate of the magnetic field term $B^2$, the vacuum of the system is the fluctuation of the electric-field-line loops, or a superposition of all loop configurations [41]. This enables the calculation of topological entanglement entropy.

Here we can identify the crucial chain of logic: the gauge symmetry chosen determines the Gauss’ laws; the gauge operators are those objects (dipoles) obeying Gauss’ law globally but not locally; they can be used to construct the magnetic field that respect Gauss’ law in any local region (minimal loops); the magnetic field determine the configuration of electric field lines at low energy (all loop configurations), which then determine the entanglement structure of the system.

In Table II, the above logical chain is shown on the second row.

**ENTANGLEMENT STRUCTURE OF R2-U1 IN ADS SPACE: GEODESIC STRING CONDENSATION**

Now let us examine the entanglement structure of R2-U1 in the 2-dimensional AdS space following the same mechanism. We will see that instead of string-net condensation, the picture will be “geodesic string condensation.” That is, the strings of electric fields travel along geodesics only, and their superposition as the vacuum determines the entanglement structure. The facts that the charge is a vector, and space is curved, play crucial roles in determining the entanglement structure.

Like in the previous section, the gauge symmetry and Gauss’ laws determine the effects of gauge operators in terms of creating vector charge multipoles. They are listed in Table II. The diagonal terms $A^{xx}$, $A^{yy}$, or in general $A^{ij}$, for direction $\hat{s}$ is to move a vector charge along the direction it points. The off-diagonal term $A^{xy}$ creates a vector charge multipole with vanishing $\int \rho$ and $\int \rho \times x$.

The dynamics, or magnetic fields, however, are very different in the curved space. It has been carefully studied by Slagle et al. in Ref. [40]. When a vector charge is parallel transported around a finite region back to its starting point, it will in general be different from the original vector due to the spatial curvature. So such parallel transport over a closed loop is energy-costly.

Consequently, the local dynamics of $B$ (Eq. (8)) is forbidden. The pictorial intuition is that the dynamics of $B$ as illustrated by Fig. 2 always happen over a finite-sized plaquette in the system. In flat lattice, such combination of $A_{i}^{j}$ operators does not violate the Gauss’ laws (Eqs. (1,4)) in any microscopic region. But in curved space it is not true anymore.

To convince ourselves, we find that $B$ (Eq. (8)) is not gauge invariant in the presence of curvature. Promoting $\nabla$ to the covariant derivative, we have

$$B \rightarrow B - R g^{ij} \nabla_i \lambda_j,$$  \hspace{1cm} (12)

under gauge transformation, where $R$ is the Ricci scalar [40]. In fact, higher order local terms up to $\nabla_a \nabla_b \nabla_c \nabla_d \lambda^e$ was systematically explored but no gauge invariant $B$ term was found in Ref. [40].

So what are the dynamics allowed in the AdS space? We note that, the difficulty is rooted in parallel transporting a vector charge around a loop. To avoid this, we have to consider instead parallel transporting the charge on a geodesic, extending from one infinity to the other.

In the lattice model, it has the following picture: A vector charge, for example, $\rho = (\rho^x, 0)$, can be moved along $x$—direction by acting $A^{xx}$ operators on the path. To make sure that any local region respects Gauss’ laws, however, such line-operation has to extend to infinity in both directions.

In the field theory, for a given geodesic $g$ with unit vector $\hat{s}$ along it, this operation is the dynamics

$$B_g = \int_g ds A^{ij} \hat{s}_i \hat{s}_j.$$  \hspace{1cm} (13)

The fact that locally no Gauss’ laws are broken is reflected by its invariance under gauge transformation

$$B_g \rightarrow B_g + \int_g ds (\nabla_i \lambda^j + \nabla^i \lambda_j) \hat{s}_i \hat{s}_j$$

$$= B_g + 2(\lambda \cdot \hat{s}) \bigg|_{-\inf}^{\inf},$$  \hspace{1cm} (14)

where the second term vanishes assuming vanishing gauge transformation at infinity.

We can thus write down the theory as

$$H_{R2-U1,AdS} = \int dv U E^{ij} E_{ij} + \sum_{g \in \text{all geodesics}} t_g B_g^2$$  \hspace{1cm} (15)

Such non-local dynamics are normally unfavored in many disciplines of physics. However, they are the ones stable in the presence of spatial curvature. Let us bear with them, and examine the corresponding of entanglement structure.

**FIG. 2.** The operator $B$ of rank-2 U(1) theory in the flat space (Eq. (8)). It involves multiple $A^{xx}$, $A^{yy}$ and $A^{xy}$ operators. It acts on a finite-area plaquette in the system, and does not survive the spatial curvature.
TABLE II. From gauge symmetry to the entanglement structure. This table demonstrates the logical chain leading from the gauge symmetry to the configurations of electric field lines as the dynamical variables. The second row is for conventional U(1), where the electric field lines can be arbitrary loops. The third row is for rank-2 U(1) in AdS space, where the electric field lines are on the geodesics extending to infinity.

With such dynamics on geodesics, a drastic change happens for the electric field lines. In AdS space, instead of forming loops, they travel along geodesics from one boundary point to another. The vacuum is then a superposition of all possible geodesic electric field line configurations. We name this the “geodesic string condensation.” As a result, the entanglement structure for each geodesic string is that the two boundary points are entangled by the corresponding geodesic dynamics. As the $B_g$ distribute in AdS space homogeneously and isotropically, we have exactly the continuous bit-thread picture we speculated at the beginning of this work. Upon lattice discretization, and also assigning $E$ discrete/continuous values, one can obtain toy models of the same universal picture but different in details, including the perfect tensor-networks and the hyperbolic fracton models.

DISCUSSION

In this work we obtained a very pictorial, intuitive understanding of the “leading order” entanglement structure of holography, and the mechanism generating it. The “leading order” entanglement structure is a web of evenly distributed bit-threads, and we noted that several holographic toy models belong to this picture. We reason that, taking the linearized diffeomorphism as the gauge symmetry, the corresponding symmetric tensor gauge theory gives rise to this picture by geodesic string condensation. Retrospectively, it is sensible that a theory mimicking gravity at first order has the entanglement structure of gravity also at first order.

Many questions follow. One exciting question to ask is that, can we understand the finer entanglement structure (some are listed in Table. I) in a similar way? For example, the random tensor-networks proposed by Yang et al. [11–13] satisfy the RT-formula for arbitrary boundary subregion. What modification of the geodesic string condensation picture is needed to capture such properties? Another question is how to introduce non-flat entanglement spectrum to match the $n$–th Rényi entropy. These projects will be very useful for us to gain improved intuition of the entanglement structure in holography.

The argument presented here is based on a chain of reasoning at a qualitative level. It would deepen our understanding to re-derive these results through more explicit calculations of the entanglement entropy for the R2-U1, or its different variations, on flat and AdS space. It is also intriguing to know if the physics demonstrated in the work is connected to other aspects of gravity, including its ground state degeneracy [45], and its relation to topological order [46].

We hope our work will provide new, useful insight in both understanding fracton states of matter and quantum gravity.

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