Gravitational Radiation from Nonaxisymmetric Instability in a Rotating Star

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Abstract

We present the first calculations of the gravitational radiation produced by nonaxisymmetric dynamical instability in a rapidly rotating compact star. The star deforms into a bar shape, shedding $\sim 4\%$ of its mass and $\sim 17\%$ of its angular momentum. The gravitational radiation is calculated in the quadrupole approximation. For a mass $M \sim 1.4 \, M_\odot$ and radius $R \sim 10 \, \text{km}$, the gravitational waves have frequency $\sim 4 \, \text{kHz}$ and amplitude $h \sim 2 \times 10^{-22}$ at the distance of the Virgo Cluster. They carry off energy $\Delta E/M \sim 0.1\%$ and radiate angular momentum $\Delta J/J \sim 0.7\%$.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.–s
The detection of gravitational waves from astrophysical sources is one of the key frontiers for research in general relativity [1]. Among the sources that might be observed by detectors currently under development are those due to global rotational instabilities in collapsing or compact stars [2,3]. For example, a rapidly rotating stellar core that has exhausted its nuclear fuel and is prevented from collapsing to neutron star size by centrifugal forces could become unstable, potentially shedding enough angular momentum to allow it to collapse and form a supernova [4]. In addition, neutron stars spun up by accretion of mass from a binary companion could possibly reach fast enough rotation rates to go unstable [5,6]. The prospect of several gravitational wave detectors becoming operational within a decade means that detailed modeling of such sources and the radiation they produce has a high priority.

Global rotational instabilities arise in fluids from growing modes $e^{\pm im\phi}$, where $m = 2$ is the so-called “bar mode”. They are conveniently parametrized by $\beta \equiv T/|W|$, where $T$ is the rotational kinetic energy and $W$ is the gravitational potential energy; see [3,7,8] for reviews. We focus on the bar instability since it is expected to be the fastest growing mode. There are two different physical mechanisms by which this instability can develop. The dynamical bar instability is driven by Newtonian hydrodynamics and gravity. It operates for fairly large values $\beta > \beta_d$ and develops on the timescale of about a rotation period. The secular instability is due to dissipative processes such as gravitational radiation reaction and occurs for typically lower values $\beta_s < \beta < \beta_d$. It develops on a timescale of several rotation periods or longer [3]. For the uniformly rotating, incompressible, constant density Maclaurin spheroids $\beta_s \approx 0.14$ and $\beta_d \approx 0.27$. Differentially rotating fluids with a polytropic equation of state $P \propto \rho^{1+1/n}$, where $n$ is the polytropic index, are believed to undergo secular and dynamical bar instabilities at about these same values of $\beta$ [7,8].

Astrophysical sources driven by rotational instabilities are nonlinear, time-dependent, and fully 3-dimensional systems; calculation of the gravitational radiation they produce requires numerical modeling. We have carried out computer simulations of a differentially rotating compact star with a polytropic equation of state undergoing the dynamical bar...
instability. This instability has previously been modeled numerically by Tohline and collaborators in the context of star formation [7,10–12]. Our work is the first to calculate the gravitational radiation produced by this instability, including waveforms and luminosities. It is also a significant advance over the earlier studies because, in addition to using better numerical techniques, we model the fluid correctly using an energy equation. This is essential due to the generation of entropy by shocks during the later stages of the evolution.

The gravitational radiation is calculated using Newtonian gravity in the quadrupole approximation [13]. The gravitational waveforms are given by the transverse-traceless (TT) components of the metric perturbation \( r h_{ij}^{\text{TT}} = 2 \ddot{f}_{ij}^{\text{TT}} \), where \( f_{ij} = \int \rho (x_i x_j - \frac{1}{3} \delta_{ij} r^2) \, d^3r \) is the trace-reduced quadrupole moment of the source and we use units in which \( c = G = 1 \). Here, \( r^2 = x^2 + y^2 + z^2 \), spatial indices \( i, j = 1, 2, 3 \), and a dot indicates a time derivative \( \frac{d}{dt} \). For an observer located on the axis at \( \theta = 0, \phi = 0 \) in spherical coordinates centered on the source, the waveforms take the simple form \( r h_{ij}^{\text{TT}} = (\ddot{f}_{xx} - \ddot{f}_{yy}) e_+ + 2 \ddot{f}_{xy} e_\times \), where \( e_+ \) and \( e_\times \) are basis tensors for the two polarization states [14]. The gravitational wave luminosity is \( L = \frac{1}{5} \langle f_{ij}^{(3)} f_{ij}^{(3)} \rangle \) and the angular momentum lost is \( \frac{dJ_i}{dt} = \frac{2}{5} \epsilon_{ijk} \langle f_{jm}^{(2)} f_{km}^{(3)} \rangle \), where the superscript (3) indicates 3 time derivatives, there is an implied sum on repeated indices, and the angle-brackets indicate an average over several wave periods. For these burst sources such averaging is not well-defined; therefore we display the unaveraged quantities \( \frac{1}{5} f_{ij}^{(3)} f_{ij}^{(3)} \) and \( \frac{2}{5} \epsilon_{ijk} f_{jm}^{(2)} f_{km}^{(3)} \) below. The energy emitted as gravitational radiation is \( \Delta E = \int L \, dt \) and the angular momentum carried away by the waves is \( \Delta J_i = \int (dJ_i / dt) \, dt \).

The equations of hydrodynamics are integrated using the techniques of smoothed particle hydrodynamics (SPH) [15], in which the fluid is modeled as a collection of particles having non-zero extent given by a smoothing length \( h \). The value of any physical field is then obtained by averaging over all particles within \( 2h \) of a given point using kernel estimation. We have used the implementation of SPH by Hernquist and Katz known as TREESPH [16], which has variable smoothing lengths and individual particle timesteps. This code has been well-tested by its developers; see [16] for details. To reduce the numerical noise that can arise from taking derivatives of \( f_{ij} \) numerically, we calculate \( \ddot{f}_{ij} \) analytically from the
SPH equations of motion and obtain the gravitational waveforms directly from quantities already available in the code (cf. [17]); this produces very smooth profiles. The gravitational radiation calculation and the use of artificial viscosity to handle shocks have been extensively tested and results are available in [16] and [18].

Our calculations begin with an initially axisymmetric differentially rotating star with a polytropic equation of state. This equilibrium model is produced using the self-consistent field method of Smith and Centrella [19], which is based on earlier work of Ostriker and Mark [20] and Hachisu [21]. This method is an iterative technique carried out in cylindrical coordinates \((\varpi, z, \phi)\). A rotation law is specified by giving the specific angular momentum as a function of mass interior to a cylinder of radius \(\varpi\). Following the convention of earlier work [10–12,22], we use the rotation law for the rigidly rotating, uniform density Maclaurin spheroids. Since polytropes do not have uniform density, this produces differentially rotating models. The resulting density \(\rho(\varpi, z)\) and angular velocity \(\Omega(\varpi)\) are then converted into a particle model to be evolved with TREESPH.

The star has polytropic index \(n = 3/2\), total mass \(M\), equatorial radius \(R\), and \(\beta = 0.30\). This rapidly rotating model is highly flattened, with polar radius \(\sim 0.21 R\). Time is measured in units of the dynamical timescale for a sphere of radius \(R\), \(t_D \equiv (R^3/M)^{1/2}\). No perturbations are imposed upon the initial axisymmetric rotating equilibrium model; the dynamical bar instability grows spontaneously from small deviations from axisymmetry in the particle model. We carried out runs with \(N = 2000, 4000, 8000, \) and \(16,000\) particles. All models were run for \(20t_D\), conserved total energy to \(\sim 1\%\) and angular momentum to \(\sim 0.1\%\), and produced generally similar results. Table I shows how some of the bulk properties of the star vary with particle number. We ran a set of tests to investigate the effect of artificial viscosity on the growth of the bar instability, comparing our results with the linearized tensor virial analysis [10]. Since the artificial viscosity used in ref. [18] produced results that most closely matched the analytic ones, we used this form in the runs discussed here. Both the growth rate and rotation speed of the \(m = 2\) bar mode in all our runs were closer to the analytic results than the previous numerical studies [10].
Fig. 1 shows the particle positions for the case \( N = 16,000 \) projected onto the \( x-y \) plane at 6 different times during the calculation. The star is rotating counterclockwise about the \( z \) axis. The simulation starts at \( t = 0 \), Fig. 1(a). By \( t = 6.6t_D \), Fig. 1(b), nonaxisymmetric structure is apparent. As this bar shaped structure develops, the amplitude of the \( m = 2 \) mode grows exponentially until \( t \sim 10t_D \). During this time a spiral arm pattern develops, with mass being shed from the ends of the bar, as seen in Fig. 1(c) and 1(d). The bar and spiral arms exert gravitational torques that cause angular momentum to be transported outward [7,10–12]. The spiral arms expand supersonically and merge together, causing shock heating and dissipation; see Fig. 1(e). The system evolves toward a nearly axisymmetric state with a core of equatorial radius \( \sim R \) and \( \beta \sim 0.26 \), as shown in Fig. 1(f); c.f. [23]. The halo contains \( \sim 4\% \) of the total mass and \( \sim 17\% \) of the angular momentum, and is bound. Throughout its evolution the entire system remains flattened, with final polar radius \( \sim 0.25R \).

The gravitational waveforms \( r h_+ \) (solid line) and \( r h_\times \) (dashed line) for this run are shown in Fig. 2 for an observer on the axis at \( \theta = 0, \phi = 0 \). Fig. 3 shows the gravitational wave luminosity \( L \) and Fig. 4 shows the energy \( \Delta E/M \) emitted as gravitational waves. The rate \( dJ_z/dt \) at which angular momentum is carried away by the gravitational waves is given in Fig. 5 and the angular momentum lost to gravitational radiation \( \Delta J_z/J \) is shown in Fig. 6. Scaling these results to a mass \( M \sim 1.4 M_\odot \) and radius \( R \sim 10 \) km, we find that the gravitational radiation has frequency \( f \sim 4 \) kHz. The dimensionless amplitude of the waves is \( h \sim 4 \times 10^{-19} \) at a distance \( r \sim 10 \) kpc, typical of sources within the Milky Way, and \( h \sim 2 \times 10^{-22} \) at a distance of \( r \sim 20 \) Mpc, typical of sources in the Virgo Cluster. The energy lost to gravitational radiation is \( \Delta E/M \sim 0.1\% \) and the angular momentum radiated away is \( \Delta J/J \sim 0.7\% \).

In this Letter we present the first calculations of the gravitational radiation produced by the dynamical bar instability. This is an important step in understanding astrophysical sources driven by global rotational instabilities. The dynamical instability may be the mechanism by which a rapidly rotating, contracting stellar core sheds enough angular mo-
mentum to allow collapse to neutron star densities [24], and methods developed in this work can be applied to study such collapses. Neutron stars spun up by accretion are expected to be subject to the secular instability [5]. Modeling this process will require including the effects of gravitational radiation reaction [25] and viscous fluids [26] in the hydrodynamical equations.

We thank L. Hernquist for supplying a copy of TREESPH and S. McMillan, K. Thorne, and C. Cutler for interesting and helpful discussions. This work was supported by NSF grant PHY-9208914 and the calculations were performed at the Pittsburgh Supercomputing Center.
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FIGURES

FIG. 1. Particle positions are shown projected onto the $x - y$ plane for various times in the evolution of the model with $N = 16,000$ particles.

FIG. 2. Gravitational waveforms $r h_+$ (solid line) and $r h_\times$ (dashed line) are shown for the run with $N = 16,000$.

FIG. 3. The gravitational wave luminosity $L (M/R)^{-5}$ is shown. This profile has been smoothed using simple averaging over a fixed interval of $0.1 t_D$ centered on each point.

FIG. 4. The energy $[\Delta E/M] (M/R)^{-7/2}$ emitted in the form of gravitational waves is given.

FIG. 5. The angular momentum $[d J_z/dt] M^{-1}(M/R)^{-7/2}$ carried away by gravitational waves is shown.

FIG. 6. The total angular momentum $[\Delta J_z/J] (M/R)^{-5/2}$ lost to gravitational radiation is shown.
TABLES

TABLE I. The mass and angular momentum shed to the halo, and the energy and angular momentum carried by the waves (for $M = 1.4 \, M_\odot$ and $R = 10 \, \text{km}$), for runs with different particle number.

| $N$    | 2000  | 4000  | 8000  | 16,000 |
|--------|-------|-------|-------|--------|
| mass in halo | 6.0%  | 6.1%  | 5.7%  | 4.4%   |
| $J$ in halo    | 19%   | 19%   | 18%   | 17%    |
| $\Delta E/M$  | .057% | .075% | .075% | .10%   |
| $\Delta J_z/J$| .36%  | .52%  | .52%  | .72%   |