RIGIDITY RESULTS AND TOPOLOGY AT INFINITY OF TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW

DEBORA IMPERA AND MICHELE RIMOLDI

Abstract. In this paper we obtain rigidity results and obstructions on the topology at infinity of translating solitons of the mean curvature flow in the Euclidean space. Our approach relies on the theory of $f$-minimal hypersurfaces.

1. Introduction

By a translating soliton of the mean curvature flow (translator for short) we mean a connected isometrically immersed complete hypersurface $x : \Sigma^m \to \mathbb{R}^{m+1}$ whose mean curvature vector field $H$ satisfies the equation

\[ H = v^\perp \]

for some fixed unit length vector $v \in \mathbb{R}^{m+1}$, where $(\cdot)^\perp$ denotes the projection on the normal bundle of $\Sigma$. Note that here we are using the convention

\[ H = H\nu = \text{tr}(A), \]

where $\nu$ is the unit normal vector of $\Sigma$, $H$ is the mean curvature function, and the second fundamental form of the immersion is defined as the generalized Hessian

\[ A = Ddx. \]

When possible, we will choose the unit normal $\nu$ to be inward-pointing, and we will agree to say that $\Sigma$ is mean convex if it holds that $H \geq 0$.

Solutions of (1) correspond to translating solutions $\{\Sigma_t = \Sigma + tv\}_{t \in \mathbb{R}}$ of the mean curvature flow, and play a key role in the study of slowly forming singularities; see e.g.
Without loss of generality we can assume that the velocity vector $v$ is given by $v = e_{m+1}$, where $\{e_1, \ldots, e_{m+1}\}$ is the standard orthonormal basis of $\mathbb{R}^{m+1}$. As a first basic example, note that if a translator has zero mean curvature, by (1) we have that $v$ must be tangential to the translator. Consequently, flat hyperplanes tangential to the vector $v$ are the only examples of translators which are also minimal hypersurfaces.

Here are some other typical examples of translators.

Example 1.1. (Grim reaper and grim reaper cylinder) In the case of $\gamma : I \to \mathbb{R}^2$, smooth curve parametrized by its arc–length, (1) becomes

$$\ddot{\gamma} = \langle v, R \dot{\gamma} \rangle,$$

where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of $\frac{\pi}{2}$. By integration of this ODE, one can prove that the only possible translating curve with non–identically zero curvature is given by the graph of $y = -\log(\cos(x))$, where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This curve is called the grim reaper, [15].

Taking the orthogonal product of a grim reaper with $\mathbb{R}^{m-1}$, one easily obtains another (higher dimensional) example of translator in $\mathbb{R}^{m+1}$, the grim reaper cylinder. It can be easily proved that these examples are mean convex. Indeed, they have only one strictly positive principal curvature.

Example 1.2. (Bowl soliton and the translating catenoid) In [1] it is proved that there exist a unique (up to rigid motion) solution of (1) which is rotationally symmetric and strictly convex. When $m = 1$ this is the grim reaper. When $m > 1$, the solution (which is actually an entire graph growing quadratically at infinity) roughly looks like a paraboloid, and is usually called the bowl soliton.

In [42], X.–J. Wang proved that in dimension $m = 2$ any entire convex translator must be rotationally symmetric, and hence the bowl soliton. However, in [42] it is also showed that for $m \geq 3$ there exist entire strictly convex translators that are not rotationally symmetric. See also the very recent [17] by R. Haslhofer, where the uniqueness of the bowl soliton is proved in arbitrary dimension under somewhat different assumptions.

Without assuming convexity, it was proved by J. Clutterbuck, O. Schnürer and F. Schulze, [10], that rotationally symmetric translators coincide (up to rigid motion) either with the bowl soliton or with the translating catenoid, a complete non–convex translator made up of the union of two graphical “winglike” solutions in the complement of a ball that are asymptotic at infinity to a bowl soliton.

Example 1.3. By the desingularization technique, X.–H. Nguyen, [30, 31], exhibited new examples of translators. In particular, note that in these papers are constructed 2–dimensional translators with infinite genus.

As we shall see in a moment, translators of the mean curvature flow turn out to belong to the more general class of $f$–minimal hypersurfaces. This class of hypersurfaces has been extensively studied in recent years due to the fact that, besides minimal hypersurfaces, also self–shrinkers of the mean curvature flow pertain to it; see for instance [9], [12], [13], [18], [22], [27]. On the other hand, the general theory developed for $f$–minimal hypersurfaces hasn’t been exploited yet for the study of translators. The aim of this paper is to highlight how the realm of weighted manifolds and $f$–minimal hypersurfaces can naturally give strong enough characterization and topological results for translators.
Recall that a weighted manifold is a triple $M^{m+1}_f = (M^{m+1}, \langle , \rangle, e^{-f} d\text{vol}_M)$, where $(M^{m+1}, \langle , \rangle)$ is a complete $(m + 1)$-dimensional Riemannian manifold, $f \in C^\infty(M)$, and $d\text{vol}_M$ denotes the canonical Riemannian volume measure on $M$. Following Gromov, [16], if we consider an isometrically immersed orientable hypersurface $\Sigma^m$ in the weighted manifold $M_f$, the weighted mean curvature vector field is defined as

$$H_f = H + (\nabla f) \perp.$$ 

Here we have denoted by $\nabla$ the Levi–Civita connection on $M$. From variational formulae, [4], one can see that $\Sigma$ is $f$–minimal, namely a critical point of the weighted area functional

$$\text{vol}_f(\Sigma) = \int_{\Sigma} e^{-f} d\text{vol}_{\Sigma},$$

if and only if $H_f$ vanishes identically.

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator satisfying (1) for some $v \in \mathbb{R}^{m+1}$. Then, letting $f = -\langle v, x \rangle$, it is easy to see that $\Sigma$ is a $f$–minimal hypersurfaces in $\mathbb{R}^{m+1}$. Moreover, looking at the Bakry–Émery Ricci tensor of $\Sigma$, we have that

$$\text{Ric}_f(X, X) = \text{Ric}(X, X) + \text{Hess}(f)(X, X) = -\langle A^2 X, X \rangle,$$

for any $X \in T\Sigma$; see e.g. Section 7 in [22].

Computing the second variation formula for the weighted area functional, it can be seen that stability properties of $f$–minimal hypersurfaces are taken into account by spectral properties of the weighted Jacobi operator

$$L_f = -\Delta_f - (|A|^2 + \text{Ric}_f(\nu, \nu)),$$

where $\text{Ric}_f$ denotes the Bakry–Émery Ricci tensor of the ambient space and $\Delta_f = \Delta - \langle \nabla f, \cdot \rangle$ is the $f$–Laplacian operator on $\Sigma_f$. In particular, we say that a $f$–minimal hypersurface $\Sigma$ is $f$–stable if for any compactly supported smooth function $u \in C^\infty_0(\Sigma)$, it holds that

$$\int_{\Sigma} uL_f u e^{-f} d\text{vol}_{\Sigma} \geq 0.$$

The $f$–index of $\Sigma$ is the generalized Morse index of $L_f$ on $\Sigma$. Namely, given a bounded domain $\Omega \subset \Sigma$, we define

$$\text{Ind}^{L_f}(\Omega) = \#\{\text{negative eigenvalues of } L_f \text{ on } C^\infty_0 \}.$$

The $f$–index of $\Sigma$ is then defined as

$$\text{Ind}_f(\Sigma) := \text{Ind}^{L_f}(\Sigma) = \sup_{\Omega \subset \subset \Sigma} \text{Ind}^{L_f}(\Omega).$$

In the special case of $x : \Sigma^m \to \mathbb{R}^{m+1}$ translator of the mean curvature flow, the $f$–stability operator take the form

$$L_f = -\Delta_f - |A|^2.$$

It is not difficult to prove that translator hyperplanes are $f$–stable. Moreover, it has been proved by C. Arezzo and J. Sun, [3], but also by L. Ma, using different techniques, [24], that the grim reaper is a $f$–stable translator in $\mathbb{R}^2$. Similar computations show that also the grim reaper cylinders are $f$–stable. More generally, note that L. Shahriyari has proved that translating graphs are $f$–stable, [40].

The first main result of the paper is the following rigidity theorem for $f$–stable translators under a weighted $L^2$–condition on the norm of the second fundamental form.
Theorem A. Let \( x : \Sigma^m \to \mathbb{R}^{m+1} \) be a \( f \)-stable translator of the mean curvature flow. Assume that \( |A| \in L^2(\Sigma_f) \). Then \( \Sigma \) is a translator hyperplane.

An application of the maximum principle and the weighted version of a result in [14], give that translators with mean curvature that does not change sign are \( f \)-stable (generalizing in particular Theorem 1.2.5 in [40]). Hence Theorem A can be applied in this situation.

On the other hand, under this assumption we are able to strengthen the result as follows, improving Theorem 6 in [25].

Theorem B. Let \( x : \Sigma^m \to \mathbb{R}^{m+1} \) be a translator with mean curvature which does not change sign. Suppose that the traceless second fundamental form of the immersion \( \Phi = A - \frac{H}{m}I \) satisfies \( |\Phi| \in L^2(\Sigma_f) \). Then \( \Sigma \) is a translator hyperplane.

In the second part of the paper we focus our attention on topological properties of translators.

Exploiting in an essential way the correspondence, discovered by K. Smoczyk, [41], between translators of the mean curvature flow in \( \mathbb{R}^{m+1} \) and minimal hypersurfaces in the manifold \( \mathbb{R}^{m+1} \times \mathbb{R} \), endowed with a suitable warped product metric, we obtain the validity of a weighted \((m + 1)\)-dimensional \( L^1 \) Sobolev inequality on translators. The validity of this inequality permits to obtain very neat results on the topology at infinity of \( f \)-stable translators and translator with finite \( f \)-index. The main tools here are weighted versions of the Li–Tam theory and of an abstract finiteness result. For more details see [22].

Theorem C. Let \( x : \Sigma^m \to \mathbb{R}^{m+1}, \ m \geq 2, \) be a \( f \)-stable translator. Then \( \Sigma \) has at most one end.

Remark 1.4. As observed above, Theorem C in particular applies to translators with mean curvature which does not change sign. This permit to deduce that, for \( m \geq 2, \) if \( H \) does not change sign then \( |H| > 0 \) everywhere, and the translator has at most one end.

Concerning the topology at infinity of translators with finite \( f \)-index we obtain the following

Theorem D. Let \( x : \Sigma^m \to \mathbb{R}^{m+1}, \ m \geq 2, \) be a translator with finite \( f \)-index. Then \( \Sigma \) has finitely many ends.

Under the geometric assumption that the translator is contained in a halfspace determined by an hyperplane orthogonal to the translating direction \( v \), we are able to guarantee that the standard \((m\)-dimensional) \( L^1 \) Sobolev inequality holds. Hence, as a consequence of Proposition 8 in [22], the same conclusion of Theorem D holds replacing the finiteness of the \( f \)-index by the integrability condition \( |A| \in L^m(\Sigma_f) \). Actually, in this situation we are able to prove the following stronger result.

Theorem E. Let \( x : \Sigma^m \to \mathbb{R}^{m+1}, \ m \geq 3, \) be a translator contained in the halfspace \( \Pi_{v,a} := \{ p \in \mathbb{R}^{m+1} : \langle p, v \rangle \geq a \} \), for some \( a \in \mathbb{R} \). Assume that \( |A| \in L^m(\Sigma_f) \). Then \( \Sigma \) is properly immersed and has finite topological type, i.e., there exists a smooth compact subset \( \Omega \subset \subset \Sigma \) such that \( \Sigma \setminus \Omega \) is diffeomorphic to the half–cylinder \( \partial \Omega \times [0, +\infty) \).

The paper is organized as follows. In Section 2 we collect basic equations that we shall use in the proofs of our results. Section 3 is devoted to the proof of Theorem A and Theorem B. In Section 4 we derive the weighted \( L^1 \) Sobolev inequalities for translators that will be employed in Section 5 to prove Theorems C, D, and E. A final section is devoted to some remarks on boundedness properties of translators.
2. Basic equations

In the following lemma we collect some useful identities for basic geometric quantities on translators, naturally involving the weighted Laplacian $\Delta_f$.

**Lemma 2.1.** Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator of the mean curvature flow, set $f = -\langle x, v \rangle$, and let $\Phi = A - \frac{H}{m} \text{Id}$ be the traceless second fundamental form. Then

\begin{align*}
\Delta f f &= -1, \\
\Delta f |x|^2 &= 2(m + \langle x, v \rangle), \\
\Delta f H &= -|A|^2 H, \\
\Delta f |A|^2 &= 2|\nabla A|^2 - 2|A|^4, \\
\Delta f |\Phi|^2 &= 2|\nabla \Phi|^2 - 2|A|^2 |\Phi|^2.
\end{align*}

**Proof.** To prove (3), letting $Y \in T\Sigma$, we compute

\[ \langle \nabla f, Y \rangle = -Y \langle x, v \rangle = -\langle \nabla_Y x, v \rangle = -\langle Y, v \rangle. \]

Hence, by (1),

\[ \nabla f = -v^T = -v + \langle v, \nu \rangle \nu = -v + H. \]

Letting now $Y, Z \in T\Sigma$, we have that

\[ \text{Hess } f(Y, Z) = \langle \nabla_Y \nabla f, Z \rangle =\langle \nabla_Y \nabla f, Z \rangle = -\langle \nabla_Y v, Z \rangle + \langle v, \nu \rangle \langle \nabla_Y \nu, Z \rangle = -\langle v, \nu \rangle \langle AY, Z \rangle = -H \langle AY, Z \rangle. \]

Taking the trace of this latter and using (1), we obtain that

\[ \Delta f f = \Delta f - |\nabla f|^2 = -H^2 - |v|^2 = -1, \]

that is, (3).

As for (4), with a similar computation, we get that for $Y \in T\Sigma$,

\[ \langle \nabla |x|^2, Y \rangle = Y |x|^2 = 2 \langle \nabla_Y x, x \rangle = 2 \langle Y, x \rangle, \]

i.e.,

\[ \nabla |x|^2 = 2x^T = 2x - 2\langle x, \nu \rangle \nu. \]

Thus, we obtain that for $Y, Z \in T\Sigma$,

\[ \text{Hess } |x|^2(Y, Z) = \langle \nabla_Y \nabla |x|^2, Z \rangle = 2 \langle \nabla_Y (x - \langle x, \nu \rangle \nu), Z \rangle = 2 \langle Y, Z \rangle - 2\langle x, \nu \rangle \langle \nabla_Y \nu, Z \rangle = 2 \langle \langle Y, Z \rangle + \langle x, \nu \rangle \langle AY, Z \rangle \rangle. \]

Taking the trace in the previous equation we get

\[ \Delta |x|^2 = 2 \langle m + H \langle x, \nu \rangle \rangle, \]
and hence,
\[
\Delta |x|^2 = \Delta |x|^2 - \langle \nabla f, \nabla |x|^2 \rangle = 2 \left( m + H \langle x, \nu \rangle + \langle v^T, x^T \rangle \right) \\
= 2 \left( m + H \langle x, \nu \rangle + \langle v, x \rangle - \langle v, \nu \rangle \langle x, \nu \rangle \right) \\
= 2 \left( m + \langle v, x \rangle \right),
\]
that is, (4).

Analogously,
\[
Y(H) = \langle \nabla_Y \nu, v \rangle = - \langle AY, v \rangle = - \langle Av^T, Y \rangle,
\]
for any \( Y \in T\Sigma \), i.e., \( \nabla H = -Av^T \). Hence, for \( Y, Z \in T\Sigma \),
\[
\text{Hess} \, H(Y, Z) = \langle \nabla_Y \nabla H, Z \rangle = - \langle \nabla_Y Av^T, Z \rangle \\
= - \langle (\nabla_Y A) v^T, Z \rangle - \langle A \nabla_Y v, Z \rangle.
\]
Using Codazzi equation, we get
\[
\text{Hess} \, H(Y, Z) = - \langle (\nabla v^T A) Y, Z \rangle - \langle \nabla v \nabla (\text{tr} A), Z \rangle \\
= - \langle (\nabla v^T A) Y, Z \rangle - \langle \nabla (v - H\nu), AZ \rangle \\
= - \langle (\nabla v^T A) Y, Z \rangle + H \langle \nabla \nu, AZ \rangle \\
= - \langle (\nabla v^T A) Y, Z \rangle - H \langle A^2 Y, Z \rangle.
\]
Taking the trace and using the relation
\[
\text{tr} (\nabla v^T A) = \langle v^T, \nabla (\text{tr} A) \rangle,
\]
we obtain (5).

As for the proof of (6), we refer the reader to [8], taking into account that the ambient space is \( \mathbb{R}^{m+1} \) and \( f = -\langle v, x \rangle \).

Finally, in order to obtain (7), note that
\[
|\Phi|^2 = |A|^2 - \frac{H^2}{m}.
\]
Hence, by (6) and (5),
\[
\frac{1}{2} \Delta |\Phi|^2 = \frac{1}{2} \Delta |A|^2 - \frac{1}{2m} \Delta^2 H^2 \\
= |\nabla A|^2 - |A|^4 - \frac{1}{m} \left( H \Delta^2 H + |\nabla H|^2 \right) \\
= |\nabla A|^2 - \frac{1}{m} |\nabla H|^2 - |A|^2 \left( |A|^2 - \frac{H^2}{m} \right) \\
= |\nabla \Phi|^2 - |A|^2 |\Phi|^2.
\]

3. Rigidity results

The proof of Theorem A relies on the following two lemmas. The first is a vanishing result proved in [39], which adapts to the weighted setting a result originally obtained in [33], [34]; see also [37].
Lemma 3.1 (Theorem 8 in [39]). Assume that on a weighted manifold $M_f$ the locally Lipschitz functions $u \geq 0$, $v > 0$ satisfy
\begin{equation}
\Delta_f u + a(x)u \geq 0
\end{equation}
and
\begin{equation}
\Delta_f v + \delta a(x)v \leq 0,
\end{equation}
for some constant $\delta \geq 1$ and $a(x) \in C^0(M)$. If $u \in L^{2\beta}(M_f)$, $1 \leq \beta \leq \delta$, then there exists a constant $C \geq 0$ such that
\begin{equation}
u^\delta = Cv.
\end{equation}
Furthermore,
(i) If $\delta > 1$ then $u$ is constant on $M$ and either $a \equiv 0$ or $u \equiv 0$;
(ii) If $\delta = 1$ and $u \not\equiv 0$, $v$ and therefore $u^\delta$ satisfy (11) with equality sign.

The second preliminary result we need in the proof of Theorem A is the following

Lemma 3.2. Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be a $f$–stable translator and let $\omega \in C^2(\Sigma)$ be a positive solution of the stability equation
\begin{equation}
\Delta_f \omega + |A|^2 \omega = 0.
\end{equation}
If $|A| \in L^2(\Sigma_f)$, then there exists a constant $C \in \mathbb{R}$ such that $H = C\omega$. In particular $H$ does not change sign.

Proof. We follow the argument used in the proof of Lemma 9.25 in [11]. First of all, note that the existence of $\omega$ is guaranteed by the $f$–stability of $\Sigma$, using the weighted version of a result by D. Fischer–Colbrie and R. Schoen, [14], (see e.g. Proposition 3 in [22]). Moreover, recall that, if $u, v \in C^2(\Sigma)$ satisfy
\begin{equation}
\int_\Sigma (|u \nabla v| + |\nabla u||\nabla v| + |u \Delta_f v|) e^{-f} dvol_\Sigma < +\infty,
\end{equation}
then
\begin{equation}
\int_\Sigma u \Delta_f v e^{-f} dvol_\Sigma = -\int_\Sigma \langle \nabla u, \nabla v \rangle e^{-f} dvol_\Sigma.
\end{equation}
Since $|A| \in L^2(\Sigma_f)$, we have that $H \in W^{1,2}(\Sigma_f)$. Furthermore, it follows by (11), that $|H| \leq 1$, and thus
\begin{equation}|H \Delta_f H| = |A|^2 H^2 \leq |A|^2.
\end{equation}
In particular, Equation (11) is satisfied for $u = v = H$, and we get that
\begin{equation}
\int_\Sigma H^2 |A|^2 e^{-f} dvol_\Sigma = -\int_\Sigma H \Delta_f He^{-f} dvol_\Sigma = \int_\Sigma |\nabla H|^2 e^{-f} dvol_\Sigma.
\end{equation}
Let us now prove that $H|\nabla \log \omega| \in L^2(\Sigma_f)$. Given a compactly supported function $\eta$ on $\Sigma$ it holds that
\begin{align*}
\int_\Sigma \langle \nabla \eta^2, \nabla \log \omega \rangle e^{-f} dvol_\Sigma &= -\int_\Sigma \eta^2 \Delta_f \log \omega e^{-f} dvol_\Sigma \\
&= \int_\Sigma \eta^2 \left(|A|^2 + |\nabla \log \omega|^2\right) e^{-f} dvol_\Sigma.
\end{align*}
On the other hand

\[ \int_{\Sigma} \langle \nabla \eta^2, \nabla \log \omega \rangle e^{-f} d\text{vol}_\Sigma \leq \int_{\Sigma} \left[ \frac{\eta^2}{2} \| \nabla \log \omega \|^2 + 2 \| \nabla \eta \|^2 \right] e^{-f} d\text{vol}_\Sigma. \]

Inserting this in the above identity yields

\[ \int_{\Sigma} \eta^2 (|A|^2 + |\nabla \log \omega|^2) e^{-f} d\text{vol}_\Sigma \leq 4 \int_{\Sigma} |\nabla \eta|^2 e^{-f} d\text{vol}_\Sigma. \]

Now let

\[ \eta_j = \begin{cases} 1 & \text{on } B_j(o) \\ j + 1 - r(x) & \text{on } B_{j+1}(o) \setminus B_j(o) \\ 0 & \text{on } \Sigma \setminus B_{j+1}(o), \end{cases} \]

where \( o \in \Sigma \) is a fixed reference point and \( r \) is the distance function from \( o \). Setting \( \eta = \eta_j H \), we have that

\[ \int_{\Sigma} \eta^2 (|A|^2 + |\nabla \log \omega|^2) e^{-f} d\text{vol}_\Sigma \leq 4 \int_{\Sigma} |\nabla (\eta_j H)|^2 e^{-f} d\text{vol}_\Sigma \]

\[ \leq 8 \int_{\Sigma} H^2 |\nabla \eta_j|^2 e^{-f} d\text{vol}_\Sigma + 8 \int_{\Sigma} \eta_j^2 |\nabla H|^2 e^{-f} d\text{vol}_\Sigma \]

\[ \leq 8 \int_{\Sigma} (H^2 + |\nabla H|^2) e^{-f} d\text{vol}_\Sigma < +\infty. \]

Taking the limit as \( j \to +\infty \) and using the dominated convergence theorem, we obtain

\[ \int_{\Sigma} H^2 |\nabla \log \omega|^2 e^{-f} d\text{vol}_\Sigma \leq \int_{\Sigma} H^2 (|A|^2 + |\nabla \log \omega|^2) e^{-f} d\text{vol}_\Sigma < +\infty. \]

Hence, using Hölder inequality, we get

\[ \int_{\Sigma} H^2 |\nabla \log \omega| e^{-f} d\text{vol}_\Sigma \leq \left( \int_{\Sigma} H^2 e^{-f} d\text{vol}_\Sigma \right)^\frac{1}{2} \left( \int_{\Sigma} H^2 |\nabla \log \omega|^2 e^{-f} d\text{vol}_\Sigma \right)^\frac{1}{2} < +\infty, \]

\[ \int_{\Sigma} |\nabla H|^2 |\nabla \log \omega| e^{-f} d\text{vol}_\Sigma = 2 \int_{\Sigma} |H| |\nabla H| |\nabla \log \omega| e^{-f} d\text{vol}_\Sigma \]

\[ \leq \left( \int_{\Sigma} |\nabla H|^2 e^{-f} d\text{vol}_\Sigma \right)^\frac{1}{2} \left( \int_{\Sigma} H^2 |\nabla \log \omega|^2 e^{-f} d\text{vol}_\Sigma \right)^\frac{1}{2} < +\infty. \]

Finally,

\[ \int_{\Sigma} |H^2 \Delta f \log \omega| e^{-f} d\text{vol}_\Sigma \leq \int_{\Sigma} H^2 (|A|^2 + |\nabla \log \omega|^2) e^{-f} d\text{vol}_\Sigma < +\infty. \]

Hence (11) is satisfied with the choices \( u = H^2 \), \( v = \log \omega \) and we deduce that

\[ \int_{\Sigma} \langle \nabla H^2, \nabla \log \omega \rangle e^{-f} d\text{vol}_\Sigma = - \int_{\Sigma} H^2 \Delta f \log \omega e^{-f} d\text{vol}_\Sigma \]

\[ = \int_{\Sigma} (H^2 |A|^2 + H^2 |\nabla \log \omega|^2) e^{-f} d\text{vol}_\Sigma. \]
Substituting (12) in the previous equation we get

\[ 0 = \int_{\Sigma} |\nabla H|^2 + H^2 |\nabla \log \omega|^2 - \langle \nabla H^2, \nabla \log \omega \rangle \, e^{-f} \text{dvol}_\Sigma \]

\[ = \int_{\Sigma} |\nabla H - H \nabla \log \omega|^2 e^{-f} \text{dvol}_\Sigma. \]

Hence \( \nabla H = H \nabla \log \omega \), implying that \( \frac{H}{\omega} \) has to be constant. \( \square \)

**Proof.** (of Theorem A) By (11) we have that

\[ |A|(\Delta_f + |A|^2)|A| = |\nabla A|^2 - |\nabla |A||^2 \geq 0. \tag{13} \]

Moreover, since \( \Sigma \) is \( f \)-stable, there exists a positive \( C^2 \) solution \( \omega \) of

\[ \Delta_f \omega + |A|^2 \omega = 0. \tag{14} \]

We can then apply Lemma 3.1 with the choices \( u = |A|, \ v = \omega, \ a(x) = |A|^2 \) and \( \delta = \beta = 1 \), to deduce that

\[ |A| = C_1 \omega, \tag{15} \]

for some constant \( C_1 \geq 0 \), and that either \( |A| \equiv 0 \) and \( \Sigma \) is necessarily a translator hyperplane, or

\[ |\nabla A|^2 = |\nabla |A||^2. \tag{16} \]

Note now that, combining Lemma 3.2 with (15), we have that

\[ |A| = C_2 H, \tag{17} \]

for some \( C_2 \in \mathbb{R} \). Equations (16) and (17) are the key geometric identities to prove our assertion. Indeed, reasoning as in the proof of Theorem 10.1 in [11], we can proceed as follows.

By Lemma 3.2 we know that \( H \) cannot change sign. Thus, a simple application of the maximum principle to (5) gives that either \( H \equiv 0 \), and \( \Sigma \) is necessarily a translator hyperplane, or \( |H| > 0 \). Therefore, we assume by now to be in the case \( |H| > 0 \).

Let \( p \in \Sigma \), \( \{e_i\}_{i=1}^m \) an orthonormal frame in \( T_p \Sigma \) that diagonalizes \( A_p \), then

\[ A(e_i, e_j) = \lambda_i \delta_{ij}. \]

Reasoning as in [11], we hence have by (16) that

(i) For every \( k \), there exists \( \alpha_k \) such that \( \nabla_{e_k} (A(e_i, e_j)) = \alpha_k \lambda_i, \ i = 1, \ldots, m. \)

(ii) If \( i \neq j \) then \( \nabla_{e_k} (A(e_i, e_j)) = 0. \)

Since \( \nabla A \) is fully symmetric, by Codazzi equations (ii) implies

(iii) \( \nabla_{e_k} A(e_i, e_j) = 0 \) unless \( i = j = k. \)

If \( \lambda_i \neq 0 \) and \( j \neq i \), then \( 0 = \nabla_{e_j} A(e_i, e_i) = \alpha_j \lambda_i, \) so that \( \alpha_j = 0. \)

In particular, if \( \text{rk}(A_p) \geq 2 \), then \( \alpha_j = 0 \) for every \( j \), and thus, by (i), \( (\nabla A)_p = 0. \)

We now consider two cases, depending on the rank of \( A \).

**Case 1.** There exists \( p \in \Sigma \) such that \( \text{rk}(A_p) \geq 2 \). We first show that \( \text{rk}(A) \) is at least two everywhere. For every \( q \in \Sigma \), let \( \lambda_1(q), \lambda_2(q) \) be the two eigenvalues of \( A_q \) that are the largest in absolute value. Define the set

\[ \Sigma_2 = \{ q \in \Sigma \text{ s.t. } \lambda_1(q) = \lambda_1(p), \lambda_2(q) = \lambda_2(p) \}. \]
Then $p \in \Sigma_2$ and, since the $\lambda_i$'s are continuous in $q$, $\Sigma_2$ is a closed non-empty set. Given $q \in \Sigma_2$ it follows that $\text{rk}(A_q) \geq 2$. Since this is an open condition, we get that there exists an open set $\Omega$ containing $q$ where $\text{rk}(A) \geq 2$. However, by the reasoning above, $\nabla A = 0$ on this set. Hence the eigenvalues of $A$ should be constant on $\Omega$, giving that $\Omega \subset \Sigma_2$ and thus that $\Sigma_2$ is an open set. Since $\Sigma$ is connected we obtain that $\Sigma = \Sigma_2$, and thus that $\Sigma$ is an open set. Since $\Sigma$ is connected we obtain that $\Sigma = \Sigma_2$.

According to a theorem by Lawson, we get that $\Sigma$ should be isometric to $S^k(r) \times \mathbb{R}^{m-k}$ for some $r \geq 0$ and $k = 2, \ldots, m$. However, none of these examples is a translator.

**Case 2.** $\text{rk}(A) = 1$. Since $|H| > 0$, the remaining case to consider is when the rank of $A$ is exactly one at every point.

In this case we reason exactly as in Case 2 of the proof of theorem 10.1 in [11] to conclude that $\Sigma$ is invariant under the isometric translations in the $(m - 1)$-dimensional subspace spanned by a global orthonormal frame for $\text{Ker}(A)$.

Therefore $\Sigma$ is a product of a curve $\Gamma \subset \mathbb{R}^2$ and this $(m - 1)$-dimensional subspace. The curve $\Gamma$ has to be a smooth translator curve in $\mathbb{R}^2$ with $|H| > 0$, hence the grim reaper. Thus $\Sigma$ should be isometric to the grim reaper cylinder. On the other hand our integrability assumption $|A| \in L^2(\Sigma_f)$ is not met by this hypersurface, leading to a contradiction.

**Proof.** (of Theorem B) Reasoning as above we have that either $|H| > 0$ or $H \equiv 0$ and $\Sigma$ is a translator hyperplane. Therefore, assume by now that $H > 0$. The case $H < 0$ can be treated exactly in the same way substituting $-H$ to $H$ in the reasoning below. By (7) we have that

\[
|\Phi| [\Delta f|\Phi| + |A|^2|\Phi|] \geq 0.
\]

Hence we can apply Lemma 3.1 as in the proof of the previous theorem, with the choices $u = |\Phi|$, $v = H$, $a(x) = |A|^2$ and $\delta = \beta = 1$. We thus deduce that

\[
|\Phi| = DH,
\]

for some constant $D \geq 0$, and hence, by (5),

\[
|A| = \sqrt{D^2 + \frac{1}{m}} H.
\]

Combining (19) and (5), we deduce that (16) holds. The conclusion follows now reasoning as in the proof of Theorem A and noting that also the assumption $|\Phi| \in L^2(\Sigma_f)$ is not met by a grim reaper cylinder.

**4. Weighted Sobolev inequalities for translators**

The aim of this section is to obtain the validity of a weighted $L^1$ Sobolev inequality on translators of the mean curvature flow. This will be essential in the proof of the topological results in Section 5.

The first result establishes the validity on translators (without any further assumption) of the following $(m + 1)$-dimensional weighted $L^1$ Sobolev inequality. An essential tool in the proof, as we shall see, is the bijective correspondence found out by K. Smoczyk, [41], between translators and minimal hypersurfaces in a suitable warped product.
Lemma 4.1. Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator. Let $h$ be a non-negative compactly supported $C^1$ function on $\Sigma$. Then

\[
\int_{\Sigma} h^{m+1} \frac{m}{m+1} e^{-f} \text{dvol}_\Sigma \leq C \int_{\Sigma} |\nabla h| e^{-f} \text{dvol}_\Sigma.
\]

Proof. Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator and let $f(p) = -\langle p, v \rangle$. We consider the warped product $N = \mathbb{R}^{m+1} \times_{e^{-f}} T$, where $T = T^1 = \mathbb{R}/\mathbb{Z}$, so that $\text{vol}(T) = 1$. Recall that the warped product metric is given by $g_N = \langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}} + e^{-2f} \text{d}t^2$. In the rest of the proof $\{E_i\}_{i=1}^{m+2}$ will be an orthonormal frame at $(p, t) \in M$ such that $\{E_j\}_{j=1}^{m+1}$ is a local orthonormal frame at $p \in \mathbb{R}^{m+1}$ and $E_{m+2} = e^f \frac{\partial}{\partial t} \in T_p T$. Furthermore, we will denote by $(\cdot)$ and by $(\hat{\cdot})$ respectively, the geometric quantities of $\mathbb{R}^{m+1}$ and of $N$. It was proved in [11] (see also [3]) that, letting

\[
\hat{x} : \hat{\Sigma} := \Sigma \times T \to \hat{N},
\]

it holds that

\[
H_{\Sigma}(p, t) = (H_{\Sigma}(p) + \nabla f(p), 0).
\]

In particular $\Sigma$ is a translator if and only if $\hat{\Sigma}$ is a complete minimal hypersurface in $\hat{N}$. Moreover, a direct computation (see e.g. Proposition 2.2. in [11]) shows that the Riemann curvature tensor of $\hat{N}$ satisfies

\[
\hat{R}_{ijkl} = R_{ijkl} \equiv 0,
\]

\[
\hat{R}_{m+2j,m+2j} = \text{Hess} f(E_i, E_i) - \langle \nabla f, E_i \rangle^2,
\]

\[
= - \langle \nabla f, E_i \rangle^2 \leq 0,
\]

\[
\hat{R}_{m+2m+2m+2} = 0,
\]

with $i, j, k, l = 1, \ldots, m + 1$. In particular, $\hat{N}$ is a Cartan–Hadamard manifold and, using a result of D. Hoffmann and J. Spruck, [19], we deduce that the minimal hypersurface $\hat{\Sigma}$ enjoys the $L^1$–Sobolev inequality

\[
\int_{\hat{\Sigma}} (h^{m+1})^{m+1} \text{dvol}_{\hat{\Sigma}} \leq C \int_{\hat{\Sigma}} |\nabla \hat{h}| \text{dvol}_{\hat{\Sigma}} \forall 0 \leq \hat{h} \in C^1_c(\hat{\Sigma}).
\]

Now let $0 \leq h \in C^1_c(\Sigma)$ and set $\hat{h}(x, t) = h(x)$. Then

\[
\int_{\Sigma} h^{m+1} \frac{m}{m+1} e^{-f} \text{dvol}_\Sigma \leq C \int_{\Sigma} |\nabla h| \text{dvol}_\Sigma = C \int_{\Sigma} |\nabla h| e^{-f} \text{dvol}_\Sigma,
\]

proving (20). \hfill \Box

As a consequence of Lemma 4.1, applying (20) to the function $h = u^{2m}/(m-1)$ and using Hölder inequality, we deduce the validity of the following weighted $L^2$ Sobolev inequality

\[
\int_{\Sigma} u^{2(m+1)} \frac{m}{m-1} e^{-f} \text{dvol}_\Sigma \leq \left( \frac{2Cm}{m-1} \right)^2 \int_{\Sigma} |\nabla u|^2 e^{-f} \text{dvol}_\Sigma.
\]
Moreover, adapting to the weighted setting a well–known result due to H. D. Cao, Y. Shen, and S. Zhu, \cite{7} (see also Lemma 7.13 in [34], and [6] for a version of this result in the more general setting of metric measure spaces), we have that if a weighted manifold $M_f$ satisfies for some $0 \leq \alpha < 1$ the inequality
\begin{equation}
\left[ \int_M h^{\frac{2-m}{2}} e^{-f} d\vol_M \right]^{1-\alpha} \leq S(\alpha)^2 \int_M |\nabla h|^2 e^{-f} d\vol_M,
\end{equation}
for some positive constant $S(\alpha)$ and for every $h \in C^\infty_c(M)$, then every end of $M$ either is non–$f$–parabolic or it has finite $f$–volume. Note that, by \cite{21}, this result in particular applies to translators. On the other hand, it is not difficult to prove that if a weighted manifold $M_f$ enjoys \cite{20}, then every end of $M$ has infinite $f$–volume. We hence deduce the following

**Proposition 4.2.** Let $\Sigma^m \to \mathbb{R}^{m+1}$ be a translator. Then every end of $\Sigma$ is non–$f$–parabolic.

However, note that the $(m+1)$–dimensional weighted $L^1$ Sobolev inequality obtained in Lemma \cite{43} does not allow to prove Theorem \cite{5} stated in the introduction. Indeed, in order to obtain an Anderson–type decay estimate we need the usual $(m$–dimensional) inequality. On the other hand, this latter can be obtained with the additional assumption that the translator is contained in a halfspace determined by an hyperplane orthogonal to the translating direction $v$. Namely, we have the validity of the following

**Lemma 4.3.** Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator contained in the halfspace $\Pi_{v,a} = \{ p \in \mathbb{R}^{m+1} : \langle p,v \rangle \geq a \}$, for some $a \in \mathbb{R}$. Let $h$ be a non–negative compactly supported $C^1$ function on $\Sigma$. Then
\begin{equation}
\left[ \int_{\Sigma} h^{\frac{2-m}{2}} e^{-f} d\vol_{\Sigma} \right]^{\frac{m-1}{m}} \leq D \int_{\Sigma} |\nabla h| e^{-f} d\vol_{\Sigma},
\end{equation}

**Proof.** Let $(M^{m+1},g)$ be an $(m+1)$–dimensional Riemannian manifold and $x : \Sigma^m \to M^{m+1}$ be an immersion. Given a smooth function $f \in C^\infty(M)$ we let $\tilde{g} = e^{2f} g$ denote a conformal change of metric on $M$. Then $x$ may induce two isometric immersions of $\Sigma$, that is $(\Sigma,x^*g) \to (M,g)$ and $(\Sigma,x^*\tilde{g}) \to (M,\tilde{g})$ respectively. Moreover $(\Sigma,x^*\tilde{g})$ is minimal in $(M,\tilde{g})$ if and only if $(\Sigma,x^*g)$ is $f$–minimal in $(M,g)$. This simply follows by the variational formula for the area functional, keeping in mind that $d\vol_{\Sigma} = e^{-f} d\vol_{\Sigma}$.

Assume now that $x : \Sigma^m \to \mathbb{R}^{m+1}$ is a translator for the mean curvature flow and let $f(p) = -\langle p,v \rangle$. Denote by $(\cdot,\cdot)_{\mathbb{R}^{m+1}}$ the standard metric on $\mathbb{R}^{m+1}$ and consider on $\mathbb{R}^{m+1}$ the conformal metric $(\cdot,\cdot) : = e^{-\frac{2f}{m}} (\cdot,\cdot)_{\mathbb{R}^{m+1}}$. Then, the previous reasoning implies that $(\Sigma,x^*((\cdot,\cdot)_{\mathbb{R}^{m+1}}))$ is a translator in $(\mathbb{R}^{m+1},(\cdot,\cdot)_{\mathbb{R}^{m+1}})$ if and only if $\tilde{\Sigma} : = (\Sigma,x^*((\cdot,\cdot)))$ is a minimal hypersurface in $(\mathbb{R}^{m+1},(\cdot,\cdot))$.

Using the expression for the curvature tensor of a Riemannian manifold under a conformal change (see e.g. Section 1.1 in [5]) it is not difficult to prove that the curvature tensor $\tilde{R}$ of $(\mathbb{R}^{m+1},(\cdot,\cdot))$ satisfies
\[ \tilde{R}_{ijij} = e^{-\frac{2f}{m}} \left( \langle \nabla f, e_i \rangle^2 + \langle \nabla f, e_j \rangle^2 - |\nabla f|^2 \right) \leq 0, \]
where $\{e_j\}_{j=1}^{m+1}$ denotes the standard orthonormal basis in $\mathbb{R}^{m+1}$. 
In particular \((\mathbb{R}^{m+1}, \langle , \rangle)\) is a Cartan–Hadamard manifold and hence, by [19], the minimal hypersurface \(\tilde{\Sigma}\) enjoys the \(L^1\)–Sobolev inequality
\[
\left( \int_{\tilde{\Sigma}} h^{m-1} \, d\text{vol}_{\tilde{\Sigma}} \right)^{\frac{1}{m-1}} \leq C \int_{\tilde{\Sigma}} |\nabla h| \, d\text{vol}_{\tilde{\Sigma}} \quad \forall \, 0 \leq h \in C^1_0(\Sigma),
\]
where we denoted by \(|\cdot|\) and \(\tilde{\nabla}\) the norm and the Levi–Civita connection on \(\tilde{\Sigma}\) respectively. The conclusion is now straightforward keeping in mind that under a conformal change of the metric, the volume form and the norm of the gradient of a given function satisfy
\[
d\text{vol}_{\tilde{\Sigma}} = e^{-f} d\text{vol}_{\Sigma}
\]
\[
|\tilde{\nabla} h| = e^{\frac{f}{m}} |\nabla h| \leq e^{\frac{a}{m}} |\nabla h|.
\]

5. Topological consequences

Recall that, in the non–weighted setting, there is a well known connection, developed mainly by Li and Tam, [26], between the dimension of the space of \(L^2\)–harmonic forms, the number of non–parabolic ends and the Morse index of the Schrödinger operator \(L = -\Delta - a\), where the function \(a\) is the smallest eigenvalue of the Ricci tensor.

In [22] it is showed that a similar connection remains valid also in the weighted setting. Indeed, adapting Li–Tam theory, it is not difficult to prove that given any relatively compact domain \(D\) on a weighted manifold \(M_f\), the number of non–\(f\)–parabolic ends of \(M_f\) with respect to \(D\) is bounded from above by the dimension of the space of bounded \(f\)–harmonic functions with finite Dirichlet weighted integral on \(M_f\) (see Theorem 4 in [22]). In particular this implies that if \(M_f\) has at least two non–\(f\)–parabolic ends, then it supports a non–constant bounded \(f\)–harmonic function with finite Dirichlet weighted integral. Moreover, combining Theorem 4 in [22] with a weighted version of an abstract finiteness result (see Theorem 3 in [22]) one can also prove the following

**Proposition 5.1** (Corollary 1 in [22]). Let \(M_f\) be a non–compact weighted manifold satisfying
\[
\text{Ric}_f \geq -a(x)
\]
for some \(0 \leq a \in C^0(M)\), and let \(L_f = -\Delta_f - a(x)\). If \(L_f\) has finite Morse index, then \(M_f\) has at most finitely many non–\(f\)–parabolic ends.

We are now in the position to prove Theorem C and Theorem D stated in Section 1.

**Proof.** (of Theorem C). Let \(x : \Sigma^m \rightarrow \mathbb{R}^{m+1}, m \geq 3\), be a \(f\)–stable translator. Note first that, as a consequence of Proposition 4.2 every end of \(\Sigma\) is non–\(f\)–parabolic. We reason by contradiction and assume that \(\Sigma\) has at least two non–\(f\)–parabolic ends. Then, as observed above, there exists a non–constant bounded \(f\)–harmonic function \(u\) such that \(|\nabla u| \in L^2(\Sigma_f)\). Recall the well-known Bochner formula for \(f\)–harmonic functions
\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess}(u)|^2 + \text{Ric}_f(\nabla u, \nabla u).
\]
Using (24), (2) and the Kato inequality, it is not difficult to prove that \(|\nabla u|\) satisfies (9). Moreover, since \(\Sigma\) is \(f\)–stable, there exist a positive function \(v\) satisfying (10) with \(\delta = 1\) and \(a(x) = |A|^2(x)\). Applying Lemma 3.1, we deduce that either \(|\nabla u| \equiv 0\), contradicting
the fact that $u$ is not constant, or $|\nabla u| = Cv$, for some positive constant $C$. Furthermore, in this latter case, inequalities (9) and (10) hold with the equality sign. In particular this implies that

$$\text{Ric}_f(\nabla u, \nabla u) = -|A|^2 |\nabla u|^2.$$ 

Fix an arbitrary point $p \in \Sigma$ and let $\{E_i(p)\}_{i=1}^m$ be an orthonormal basis for $T_p \Sigma$ such that $E_1(p) = \frac{\nabla u}{|\nabla u|}(p)$. Hence at $p$,

$$\text{tr}(\text{Ric}_f) = -|A|^2 - \sum_{i=2}^m |AE_i|^2.$$ 

On the other hand, combining Gauss’ equation with (3), we also have

$$\text{tr}(\text{Ric}_f) = -|A|^2.$$ 

Hence we obtain

$$\sum_{i=2}^m |AE_i|^2 = 0,$$

from which we deduce that $|A|^2(p) = H^2(p)$. Since $p$ is arbitrary this implies that $|A|^2 = H^2$ on $\Sigma$. In particular, as a consequence of Gauss’ equation, the scalar curvature of $\Sigma$ vanishes identically. Using Corollary 2.4 in [28] we conclude that $\Sigma$ has to be either a grim reaper cylinder or a translator hyperplane. However, this contradicts the assumption that $\Sigma$ has at least two (non–f–parabolic) ends.

Proof. (of Theorem D). Recall that, given a translator $x : \Sigma^m \to \mathbb{R}^{m+1}$, the Bakry–Émery Ricci tensor of $\Sigma$ satisfies the curvature condition (2). Moreover, if $\text{Ind}_f(\Sigma) < +\infty$, the assumptions of Proposition 5.1 are met with the choice $a(x) = |A|^2(x)$ and we can conclude that $\Sigma$ has at most finitely many non–f–parabolic ends. The conclusion of the theorem is then straightforward since, according to Proposition 4.2, every end of a translator is non–f–parabolic.

Finally, the third result we aim at proving in this section, namely Theorem E, deals with the connection, for a translator $\Sigma$ contained in a halfspace $\Pi_{v,a}$, between the $L^m$–integrability of its second fundamental form and the property of having finite topological type, i.e., the existence of a smooth compact subset $\Omega \subset \Sigma$ such that $\Sigma \setminus \Omega$ is diffeomorphic to the half–cylinder $\partial \Omega \times [0, +\infty)$. The proof of this theorem relies on the following general result that, in the setting of minimal submanifolds of the Euclidean space, is due to M. Anderson, [2].

**Lemma 5.2.** Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a complete, non-compact hypersurface satisfying

$$\sup_{\Sigma \setminus B_R(o)} |A| = o\left( R^{-1} \right), \text{ as } R \to +\infty.$$ 

Then $x$ is proper and $\Sigma$ has finite topological type.

**Proof.** (of Theorem E). Assume that that $x : \Sigma^m \to \mathbb{R}^{m+1}$, $m \geq 3$, is a complete translator contained in a halfspace $\Pi_{v,a}$, for some $a \in \mathbb{R}$. Then, applying (23) to $h = u^{\frac{2(m-1)}{m}}$, we get that $\Sigma$ enjoys the weighted $L^2$–Sobolev inequality

$$\left[ \int_{\Sigma} u^{\frac{2m}{m-2}} e^{-f} d\text{vol}_\Sigma \right]^{\frac{m-2}{m}} \leq C^2 \int_{\Sigma} |\nabla u|^2 e^{-f} d\text{vol}_\Sigma,$$
for some constant $C > 0$ and for every $u \in C^\infty_c(\Sigma)$. Now we recall that, using (6) and the Kato inequality, the second fundamental form of $\Sigma$ satisfies the Simons–type inequality
\[ \Delta_f |A| + |A|^3 \geq 0. \]
Since $|A| \in L^m(\Sigma_f)$, adapting to the weighted setting the arguments in [30], we are able to obtain the Anderson–type decay estimate (25). The desired conclusion is now a direct consequence of Lemma 5.2.

6. About boundedness properties of translators

A simple application of the maximum principle to (3) gives that $f$ cannot have a local minimum on $\Sigma$. Thus, in particular, there are no compact translators.

In the non–compact setting, reasoning in a similar way, one can make use of a version of the maximum principle at infinity, known as the Omori–Yau maximum principle for $\Delta_f$, to obtain that a suitable class of translators cannot be bounded in $\mathbb{R}^{m+1}$. The validity of the Omori–Yau maximum principle for $\Delta_f$, is known to be equivalent to the $f$–stochastic completeness of the hypersurface, namely the Markovianity of the natural diffusion process associated with the weighted Laplacian, [32], [35].

Proposition 6.1. Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be an $f$–stochastically complete translator. Then $x(\Sigma)$ cannot be bounded.

Proof. Since $\Sigma$ is $f$–stochastically complete, if $u \in C^2(\Sigma)$ is bounded from below we can find a sequence of points $\{p_j\} \subset \Sigma$ such that
\[ (i) \ u(p_j) = \inf_{\Sigma} u + \frac{1}{j} \quad (ii) \Delta_f u(p_j) > -\frac{1}{j}. \]
Assume by contradiction that $x(\Sigma)$ is bounded. Then
\[ \inf_{\Sigma} f \geq -\sup_{\Sigma} |x| > -\infty, \]
and we can find a sequence $\{p_j\} \subset \Sigma$ such that
\[ (i) \ f(p_j) = \inf_{\Sigma} f + \frac{1}{j} \quad (ii) \ 1 = \Delta_f f(p_j) > -\frac{1}{j}. \]
Passing to the limit as $j \to \infty$ in (ii) we reach the desired contradiction. □

Remark 6.2. It is known that $f$–stochastic completeness is guaranteed if the Bakry–Émery Ricci curvature is uniformly bounded from below; see [38], [43], [35]. In particular, by [2], a translator $x : \Sigma^m \to \mathbb{R}^{m+1}$ is $f$–stochastically complete if $|A| \in L^\infty(\Sigma)$.

By the way, note that also properly immersed translators $x : \Sigma^m \to \mathbb{R}^{m+1}$ are $f$–stochastically complete. Indeed, by (4), we get that outside a compact set
\[ \Delta_f |x|^2 \leq 2(m+1)|x|^2. \]
Thus $|x|^2$ is a proper $\lambda$–$f$–superharmonic function, with $\lambda = 2(m+1) > 0$. The $f$–stochastic completeness follows then by the standard Khas’minskii test.

Assuming, instead of an $L^\infty$ condition on $|A|$, the finiteness of the total curvature, we are still able to deduce a non–existence result for bounded translators.

Proposition 6.3. Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator satisfying $|A| \in L^m(\Sigma)$. Then $x(\Sigma)$ cannot be bounded.
Proof. Assume by contradiction that $\Sigma$ is bounded. Then, $f$ is bounded and there exist positive constants $C_1$, $C_2$ such that

$$C_1 \text{dvol}_\Sigma \leq e^{-f} \text{dvol}_\Sigma \leq C_2 \text{dvol}_\Sigma.$$  

Hence, combining (26) with Theorem 2.1 in [29], we obtain that $\Sigma$ enjoys the weighted $L^1$-Sobolev inequality

$$\left[ \int_\Sigma h^\frac{m-1}{m} e^{-f} \text{dvol}_\Sigma \right]^{\frac{m-1}{m}} \leq C \left[ \int_\Sigma (|\nabla h| + |H| h) e^{-f} \text{dvol}_\Sigma \right],$$

for some constant $C > 0$ and for every $0 \leq h \in C^1_0(\Sigma)$. Moreover, the integrability assumption on the second fundamental form implies that $H \in L^m(\Sigma_f)$ and, using the Hölder inequality, it is not difficult to prove that $\Sigma$ enjoys the $L^2$ weighted Sobolev inequality

$$\left[ \int_\Sigma h^\frac{2m}{m-2} e^{-f} \text{dvol}_\Sigma \right]^{\frac{m-2}{m}} \leq D^2 \left[ \int_\Sigma |\nabla h|^2 e^{-f} \text{dvol}_\Sigma \right],$$

for some constant $D > 0$ and for every $h \in C^1(\Sigma)$. Reasoning as in the proof of Theorem 4, we obtain again the Anderson–type decay estimate (25) for $|A|$ and hence the properness of $x$. But then $\Sigma$ has to be $f$–stochastically complete, contradicting Proposition 6.1. □

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