Arnold-Liouville theorem for integrable PDEs: a case study of the focusing NLS equation

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February 27, 2020

Abstract

We prove an infinite dimensional version of the Arnold-Liouville theorem for integrable non-linear PDEs: In a case study we consider the focusing NLS equation with periodic boundary conditions.

Keywords: Arnold-Liouville theorem, focusing NLS equation, focusing mKdV equation, normal forms, Birkhoff coordinates

2010 MSC: 37K10, 37K20, 35P10, 35P15

1 Introduction

Let us first review the classical Arnold-Liouville theorem ([27, 30, 1]) in the most simple setup: assume that the phase space $M$ is an open subset of $\mathbb{R}^{2n} = \{(x, y) \mid x, y \in \mathbb{R}^n\}$, $n \geq 1$, endowed with the Poisson bracket

$$\{F, G\} = \sum_{j=1}^{n} \left( \partial_{y_j} F \cdot \partial_{x_j} G - \partial_{x_j} F \cdot \partial_{y_j} G \right).$$

The Hamiltonian system with smooth Hamiltonian $H : M \to \mathbb{R}$ then has the form

$$\dot{x} = \partial_y H, \quad \dot{y} = -\partial_x H$$

with corresponding Hamiltonian vector field $X_H = (\partial_y H, -\partial_x H)$ where $\partial_x \equiv (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial_y \equiv (\partial_{y_1}, \ldots, \partial_{y_n})$. Such a vector field is said to be completely integrable if there exist $n$ pairwise Poisson commuting smooth integrals $F_1, \ldots, F_n : M \to \mathbb{R}$ (i.e. $\{H, F_k\} = 0$ and $\{F_k, F_l\} = 0$ for any $1 \leq k, l \leq n$) so that the differentials $d(x, y)F_1, \ldots, d(x, y)F_n \in T^*_{(x, y)} \mathbb{R}^{2n}$ are linearly independent on an open dense subset of points $(x, y)$ in $M$. In this setup the Arnold-Liouville theorem reads as follows (cf. e.g. [1] or [31]):

∗T.K. is partially supported by the Swiss National Science Foundation.
†P.T. is partially supported by the Simons Foundation, Award #526907
**Theorem** (Arnold-Liouville). Assume that \( c \in F(M) \subseteq \mathbb{R}^n \) is a regular value of the momentum map \( F = (F_1, \ldots, F_n) : M \to \mathbb{R}^n \) so that a connected component of \( F^{-1}(c) \), denoted by \( N_c \), is compact. Then \( N_c \) is an \( n \)-dimensional torus that is invariant with respect to the Hamiltonian vector field \( X_H \). Moreover, there exists a \( X_H \)-invariant open neighborhood \( U \) of \( N_c \) in \( M \) and a diffeomorphism 
\[ \Psi : D \times (\mathbb{R}/2\pi)^n \to U, \quad (I, \theta) \mapsto \Psi(I, \theta), \]

where \( D \) is an open disk in \( \mathbb{R}^n \), so that the actions \( I = (I_1, \ldots, I_n) \) and the angles \( \theta = (\theta_1, \ldots, \theta_n) \) are canonical coordinates (i.e., \( \{\theta_j, I_j\} = 1 \) for all \( 1 \leq j \leq n \) whereas all other brackets between the coordinates vanish) and the pull-back \( H = H \circ \Psi^{-1} \) of the Hamiltonian \( H \) depends only on the actions.

An immediate implication of this theorem is that the equation of motion, when expressed in action-angle coordinates, becomes
\[ \dot{\theta} = \partial_I H(I), \quad \dot{I} = -\partial_\theta H(I) = 0. \]

Hence, the actions are conserved and so is the frequency vector \( \omega(I) := \partial_I H(I) \). Therefore, the equation can be solved explicitly,
\[ \theta(t) = \theta(0) + \omega(I) t \left( \text{mod}(2\pi\mathbb{Z})^n \right), \quad I(t) = I(0). \]

In particular, this shows that every solution is quasi-periodic. Furthermore, the invariant tori in the neighborhood \( U \) are of maximal dimension \( n \) and their tangent spaces at any given point \( (x, y) \) are spanned by the Hamiltonian vectors \( X_{F_1}(x, y), \ldots, X_{F_n}(x, y) \). Informally, one can say that the Arnold-Liouville theorem assures that generically there is no room for other types of dynamics besides quasi-periodic motion. In fact, by Sard’s theorem one sees that if \( F \) is e.g. proper then, under the conditions above, the union of invariant tori of maximal dimension \( n \) is open and dense in \( M \). In addition, the set of regular values of the momentum map \( F = (F_1, \ldots, F_n) : M \to \mathbb{R}^n \) is open and dense in \( F(M) \). Typically, action-angle coordinates cannot be extended globally. One reason for this is the existence of singular values of the momentum map \( F : M \to \mathbb{R}^n \). Such a situation appears e.g. in the case of an elliptic fixed point \( \xi \in M \). In that case, it might be possible to extend the coordinates \( X_j = \sqrt{2I_j}\cos \theta_j, \quad Y_j = \sqrt{2I_j}\sin \theta_j, \) to an open neighborhood of \( \xi \) in \( M \). We refer to such coordinates as Birkhoff coordinates and the Hamiltonian \( H \), when expressed in these coordinates, is said to be in Birkhoff normal form. For results in this direction we refer to [35, 7, 37] and references therein. In special cases, such as systems of coupled oscillators, Birkhoff coordinates can be defined on the entire phase space and are referred to as global Birkhoff coordinates. In what follows we will keep this terminology and will call \( (X_j, Y_j) \) defined in terms of the action-angle coordinates as above again Birkhoff coordinates, even if they are non necessarily related to an elliptic fixed point. Other obstructions to globally extend action-angle coordinates are the presence of hyperbolic fixed points as well as focus-focus fixed points (cf. [6, 7, 38]) where one observes topological
obstructions in case of a non-trivial monodromy of the action variables. Finally note that action-angle coordinates are used for obtaining KAM type results for small perturbations of integrable Hamiltonian systems.

Our aim is to present an extension of the above described version of the Arnold-Liouville theorem to integrable PDEs in form of a case study of the focusing nonlinear Schrödinger (fNLS) equation

\[ i \partial_t u = -\partial_x^2 u - 2|u|^2 u, \quad u|_{t=0} = u_0 \]  

with periodic boundary conditions. Our motivation for choosing the fNLS equation stems from the fact that it is known to be an integrable PDE which does not admit local action-angle coordinates in open neighborhoods of specific potentials (see [23]), whereas in contrast, integrable PDEs such as the Korteweg-de Vries equation (KdV) or the defocusing nonlinear Schrödinger equation (dNLS) admit global Birkhoff coordinates ([11, 17]).

To state our results we first need to introduce some notation and review some facts about the fNLS equation. It is well known that (1) can be written as a Hamiltonian PDE. To describe it, let \( L^2 \equiv L^2(T, \mathbb{C}) \) denote the Hilbert space of square-integrable complex valued functions on the unit torus \( T := \mathbb{R}/\mathbb{Z} \) and let \( L^2_c := L^2 \times L^2 \). On \( L^2_c \) introduce the Poisson bracket defined for \( C^1 \)-functions \( F \) and \( G \) on \( L^2_c \) by

\[ \{ F, G \}(\varphi) := -i \int_0^1 \left( \partial_1 F \cdot \partial_2 G - \partial_2 F \cdot \partial_1 G \right) dx \]  

where \( \varphi = (\varphi_1, \varphi_2) \) and \( \partial_j F \equiv \partial_{\varphi_j} F \) for \( j = 1, 2 \) are the two components of the \( L^2 \)-gradient of \( F \) in \( L^2_c \). More generally, we will consider \( C^1 \)-functions \( F \) and \( G \) defined on a dense subspace of \( L^2_c \) having sufficiently regular \( L^2 \)-gradients so that the integral in (2) is well defined when viewed as a dual pairing. The NLS-Hamiltonian, defined on the Sobolev space \( H^1_c = H^1 \times H^1 \), \( H^1 \equiv H^1(T, \mathbb{C}) \), is given by

\[ H_{NLS}(\varphi) := \int_0^1 \left( \partial_x \varphi_1 \cdot \partial_x \varphi_2 + \varphi_1^2 \varphi_2^2 \right) dx. \]

The corresponding Hamiltonian equation then reads

\[ \partial_t (\varphi_1, \varphi_2) = -i \left( \partial_2 H_{NLS}, -\partial_1 H_{NLS} \right). \]  

Equation (1) is obtained by restricting (3) to the real subspace \( iL^2_c := \{ \varphi \in L^2_c | \varphi_2 = -\overline{\varphi_1} \} \) of the complex vector space \( L^2_c \). More precisely, for \( \varphi = i(u, \overline{u}) \), one gets the fNLS equation \( i\partial_t u = -\partial_x^2 u - 2|u|^2 u \). We also remark that when restricting (3) to the real subspace \( L^2_r := \{ \varphi \in L^2_c | \varphi_2 = \overline{\varphi_1} \} \) one obtains the dNLS equation mentioned above. According to [36] equation (3) admits a Lax pair representation (cf. [25])

\[ \partial_t L(\varphi) = [P(\varphi), L(\varphi)] \]

where \( L(\varphi) \) is the Zakharov-Shabat operator (ZS operator)

\[ L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} \]

\[ P(\varphi) := i \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\varphi_2 & -\varphi_1 \end{pmatrix} \]
and $P(\varphi)$ is a certain differential operator of second order. As a consequence, the periodic spectrum of $L(\varphi)$ is invariant with respect to the NLS flow. Actually, we need to consider the periodic and the anti-periodic spectrum of $L(\varphi)$ or, by a slight abuse of notation, of $\varphi$. In order to treat the two spectra at the same time we consider $L(\varphi)$ on the interval $[0, 2]$ and impose periodic boundary conditions. Denote the spectrum of $L(\varphi)$ defined this way by $\text{Spec}_p L(\varphi)$. In what follows we refer to it as the periodic spectrum of $L(\varphi)$ or, by a slight abuse of terminology, of $\varphi$. Note that $\text{Spec}_p L(\varphi)$ is discrete and invariant with respect to the NLS flow on $L^2_c$. We say that the periodic spectrum $\text{Spec}_p L(\varphi)$ is simple if every eigenvalue has algebraic multiplicity one. Furthermore, for any $\psi \in iL^2_r$ introduce the isospectral set, 

$$\text{Iso}(\psi) := \{ \varphi \in iL^2_r \mid \text{Spec}_p L(\varphi) = \text{Spec}_p L(\psi) \}$$

where the equality of the two spectra means that they coincide together with the corresponding algebraic multiplicities of the eigenvalues. Denote by $\text{Iso}_o(\psi)$ the connected component of $\text{Iso}(\psi)$ that contains $\psi$. For any integer $N \geq 0$ introduce the Sobolev spaces $H^N_c := H^N \times H^N$ and the real subspace 

$$iH^N_r := \{ \varphi \in H^N_c \mid \varphi_2 = -\overline{\varphi_1} \}$$

where $H^N \equiv H^N(T,\mathbb{C})$ denotes the Sobolev space of functions $f: T \rightarrow \mathbb{C}$ with distributional derivatives up to order $N$ in $L^2$. In a similar way introduce the following spaces of complex valued sequences 

$$b^N_c := b^N \times b^N,$$

$$b^N_r := \{ (z, w) \in b^N_c \mid w_n = \overline{z}_{-n} \forall n \in \mathbb{Z} \},$$

$$i b^N_r := \{ (z, w) \in b^N_c \mid w_n = -\overline{z}_{-n} \forall n \in \mathbb{Z} \},$$

where 

$$b^N := \{ z = (z_n)_{n \in \mathbb{Z}} \mid \| z \|_N < \infty \}, \quad \| z \|_N := \left( \sum_{j \in \mathbb{Z}} (1 + j^2)^N |z_j|^2 \right)^{1/2}.$$ 

Note that $b^N_c$ and $i b^N_r$ are real subspaces of $b^N_r$. In the case when $N = 0$ we set $\ell^2_c \equiv b^0_c$, $i \ell^2_c \equiv i b^0_r$, and $\ell^2 \equiv b^0$. For simplicity, we will also use the same symbols for the spaces of sequence with indices $|n| > R$ with $R \geq 0$.

Finally, we say that the subset $\mathcal{W} \subseteq iL^2_r$ is saturated if $\text{Iso}_o(\psi) \subseteq \mathcal{W}$ for any $\psi \in \mathcal{W}$. The main result of this paper is the following Theorem.

**Theorem 1.1.** Assume that $\psi \in iL^2_r$ has simple periodic spectrum $\text{Spec}_p L(\psi)$. Then there exist a saturated open neighborhood $\mathcal{W}$ of $\text{Iso}_o(\psi)$ in $iL^2_r$ and a real analytic diffeomorphism 

$$\Psi: \mathcal{W} \rightarrow \Psi(\mathcal{W}) \subseteq i\ell^2_r, \quad \varphi \mapsto \left( (z_n(\varphi))_{n \in \mathbb{Z}}, (w_n(\varphi))_{n \in \mathbb{Z}} \right),$$

onto the open subset $\Psi(\mathcal{W})$ of $i\ell^2_r$ so that the following holds:
(NF1) \( \Psi \) is canonical, i.e., \( \{ z_n, w_{(-n)} \} = -i \) for any \( n \in \mathbb{Z} \) whereas all other brackets between coordinate functions vanish.

(NF2) For any integer \( N \geq 0 \), \( \Psi(W \cap iH^N_r) \subseteq i\hbar^N_r \) and

\[
\Psi : W \cap iH^N_r \to \Psi(W \cap iH^N_r) \subseteq i\hbar^N_r
\]

is a real analytic diffeomorphism onto its image.

(NF3) The pull-back \( \mathcal{H}_{NLS} \circ \Psi^{-1} : \Psi(W \cap iH^N_r) \to \mathbb{R} \) of the fNLS Hamiltonian is a real analytic function that depends only on the actions \( I_n := z_n w_{(-n)} \), \( n \in \mathbb{Z} \).

The open neighborhood \( W \subseteq iL^2_r \) in Theorem 1.1 is chosen in such a way that for any \( \varphi \in W \) the spectrum \( \text{Spec}_p L(\varphi) \) has the property that all multiple eigenvalues are real with algebraic and geometric multiplicity two whereas all simple eigenvalues are non-real and appear in complex conjugate pairs. Hence the periodic eigenvalues of \( L(\varphi) \) have algebraic multiplicity at most two. Recall also that a potential \( \varphi \in iL^2_r \) is called a finite gap potential if the number of simple periodic eigenvalues of \( L(\varphi) \) is finite (cf. e.g. [13, 14]). As an immediate application of Theorem 1.1 one obtains the following

**Corollary 1.1.** Assume that \( \psi \in iH^N_r \) with \( N \in \mathbb{Z}_{\geq 0} \) has simple periodic spectrum \( \text{Spec}_p L(\psi) \). Then for any \( \varphi \in W \) where \( W \) is the open neighborhood of Theorem 1.1 the following holds:

(i) The set \( \Psi(\text{Iso}_o(\varphi)) \) is compact in \( i\hbar^N_r \) and can be represented as a direct product of countably many circles, one for each pair of complex conjugated simple periodic eigenvalues.

(ii) For \( N \geq 1 \), the fNLS equation on \( \Psi(W \cap iH^N_r) \subseteq i\hbar^N_r \) takes the form

\[
\dot{z}_n = -i\omega_n z_n, \quad \dot{w}_n = i\omega_n w_n, \quad n \in \mathbb{Z}
\]

where \( \omega_n \equiv \omega_n(I) := \partial_{I_n}(\mathcal{H}_{NLS} \circ \Psi^{-1}) \) are the NLS frequencies and \( I_n := z_n w_{(-n)} \), \( n \in \mathbb{Z} \), are the actions. Hence, the solutions of the fNLS equation with initial data in \( W \cap iH^N_r \) are globally defined and almost periodic in time.

(iii) The finite gap potentials in \( W \) lie on finite dimensional fNLS invariant tori contained in \( W \cap iH^N_r \) for any \( N \geq 0 \). The set of these potentials is dense in \( W \cap iH^N_r \).

**Remark 1.1.** (i) For any \( \varphi \in W \cap iH^N_r \), \( N \geq 1 \), and \( t \in \mathbb{R} \) denote by \( S_t(\varphi) \) the solution of the fNLS equation obtained in Corollary 1.1(ii) with initial data \( \varphi \). Then for any given \( N \in \mathbb{Z}_{\geq 1} \) and \( t \in \mathbb{R} \), one can prove that the flow map \( S_t : W \cap iH^N_r \to W \cap iH^N_r \) is a homeomorphism (cf. [27]). In addition, the map \( S : \mathbb{R} \times (W \cap iH^N_r) \to W \cap iH^N_r \), \( (t, \varphi) \mapsto S_t(\varphi) \), is continuous.
(ii) It can be shown that the frequencies \( \omega_n, n \in \mathbb{Z} \), extend real analytically to the larger set \( W \cap iL^2_r \) (cf. [21, 16]).

The next result addresses the question of how restrictive the assumption of \( \text{Spec}_p L(\psi) \) being simple is. Let

\[
T := \{ \psi \in iL^2_r \mid \text{Spec}_p L(\psi) \text{ is simple} \}.
\]

Recall that a subset \( A \) of a complete metric space \( X \) is said to be residual if it is the intersection of countably many open dense subsets. By Baire’s theorem, such a set is dense in \( X \). We prove in [24] the following

**Theorem 1.2.** For any integer \( N \geq 0 \), the set \( T \cap iH^N_r \) is residual in \( iH^N_r \).

**Remark 1.2.** It is well known that the fNLS equation is wellposed on \( iH^N_r \) for any integer \( N \geq 0 \) ([3]). It then follows from Theorem 1.1 and Corollary 1.1 that any solution in \( iH^N_r \) with \( N \geq 0 \) can be approximated in \( C([-T,T],iH^N_r) \) by finite gap solutions for any \( T > 0 \).

Finally we mention that the results of Theorem 1.1 apply to any Hamiltonian in the fNLS hierarchy. In particular, these results hold for the focusing modified KdV equation

\[
\partial_t v = -\partial_x^3 v - 6v^2 \partial_x v, \quad v|_{t=0} = v_0
\]

with periodic boundary conditions which can be obtained as the restriction of the Hamiltonian PDE on the Poisson manifold \( L^2_c \) with Hamiltonian

\[
H_{mKdV}(\varphi) := \int_0^1 \left( - (\partial_x \varphi_1)^2 + 3(\varphi_1 \partial_x \varphi_1) \varphi_2^2 \right) dx
\]

to the real subspace of \( iL^2_r \),

\[
\{ \varphi = i(v,v) \in iL^2_r \mid v \text{ real valued} \} \cong L^2(\mathbb{T}, \mathbb{R}).
\]

We remark that Theorem 1.2 also holds in this setup meaning that the subset \( \{ v \in L^2(\mathbb{T}, \mathbb{R}) \mid i(v,v) \in T \cap iH^N_r \} \) is residual in \( H^N(\mathbb{T}, \mathbb{R}) \) for any integer \( N \geq 0 \) – see [24] for more details.

**Method of proof:** The proof of Theorem 1.1 is based on the following key ingredients:

(1) Setup allowing to construct analytic coordinates: One of the principal merits of our analytic setup is that it allows to prove the canonical relations between the action-angle coordinates by a deformation argument using the canonical relations between these coordinates in a neighborhood of the zero potential established in [14] (cf. also [11] for the construction of such coordinates in the defocusing case).

(2) Choice of contours: For an integrable system on a \( 2n \)-dimensional symplectic space \( M \) (as the one discussed at the beginning of the introduction),
action coordinates $I_j$, $1 \leq j \leq n$, on the invariant tori $N_c$ of dimension $n$, smoothly parametrized by regular values $c \in \mathbb{R}^n$ in the image of the momentum map $F : M \to \mathbb{R}^n$, can be defined by Arnold’s formula

$$I_j := \frac{1}{2\pi} \int_{\gamma_j(c)} \alpha, \quad 1 \leq j \leq n,$$

in terms of the canonical 1-form $\alpha$, stemming from the Poisson structure on $M$, and a set of cycles $\gamma_j(c)$, $1 \leq j \leq n$, on $N_c$ that form a basis in the first homology group of $N_c$ and depend smoothly on the parameter $c$. For integrable PDEs such as the KdV or the dNLS equations, Arnold’s procedure for constructing the action coordinates has been successfully implemented in [9] (cf. also [34] and [29]). More precisely, in the case of the dNLS equation, the Dirichlet eigenvalues of $L(\varphi)$ can be used to define cycles $\gamma_j$, $j \in \mathbb{Z}$. Surprisingly, the integrals $\frac{1}{2\pi} \int_{\gamma_j} \alpha$ can be interpreted as contour integrals on the complex plane (cf. e.g. [9, 29, 11]). We emphasize that in the case of the NL equation, treated in the present paper, these contours can not be obtained from the Dirichlet spectrum of $L(\varphi)$. It turns out that the contours in the complex plane which work in the case of the dNLS equation for potentials $\varphi$ near the origin in $H^N_r$ also work in the case of the fNLS for potentials near the origin in $iH^N_r$ ([15]). We then use a deformation argument along an appropriately chosen path, that connects $\psi$ with a small open neighborhood of the zero potential in $iH^N_r$, to obtain contours for potentials in an open neighborhood of the isospectral set $\text{Iso}_\omega(\psi) \cap iH^N_r$.

(3) Normalized differentials: The angle coordinates are defined in terms of a set of normalized differentials on an open Riemann surface (of possibly infinite genus), associated to the periodic spectrum $\text{Spec}_p L(\varphi)$, and the Dirichlet spectrum of $L(\varphi)$ (cf. e.g. [2, 4] for the case of finite gap potentials as well as [11, 28, 29, 8] for the case of more general potentials in $H^N_r$). Such normalized differentials (with properties needed for our purposes) for generic potentials in $iH^N_r$ have been constructed in [22] (cf. also [12]). Note that the case of potentials in $iH^N_r$ is more complicated since the operator $L(\varphi)$ is not selfadjoint. An important ingredient for estimates, needed to construct the angle coordinates, is the rather precise localization of the zeros of these differentials provided in [22]. We emphasize that no assumptions are made on the Dirichlet eigenvalues of $L(\varphi)$. In particular, they might have algebraic multiplicities greater or equal to two. As in the case of the actions we construct the angles using a deformation argument.

(4) Generic spectral properties of non-selfadjoint ZS operators: The deformation argument, briefly discussed in item (1), requires that the path of deformation stays within the part of phase space $iH^N_r$ which admits action-angle coordinates. This part of the phase space contains the set of potentials $\varphi \in iH^N_r$ which have the property that all multiple eigenvalues are real with geometric multiplicity two whereas all simple eigenvalues are non-real and appear in complex conjugated pairs. In [14], it is shown that this set is open and path connected. We remark that in order to prove that the actions and the angles,
first defined in an open neighborhood of the potential $\psi \in iH^N_r$, analytically extend to an open neighborhood of $\text{Iso}_o(\psi)$, we make use of the assumption that all periodic eigenvalues of $L(\psi)$ are simple.

(5) Lyapunov type stability of isospectral sets: To ensure that the Birkhoff map (4) is injective in an open neighborhood $\mathcal{W}$ of $\text{Iso}_o(\psi)$ in $iL^2_r$, we show that for any open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of $\text{Iso}_o(\psi)$ in $iL^2_r$ there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of $\text{Iso}_o(\psi)$ in $iL^2_r$ with the property that $\text{Iso}_o(\varphi) \subseteq \mathcal{U}$ for any $\varphi \in \mathcal{V}$.

Additional comments: Informally, Theorem 1.1 means that $(z_n, w_n), n \in \mathbb{Z},$ can be thought of as nonlinear Fourier coefficients of $\varphi = (\varphi_1, \varphi_2)$. They are referred to as Birkhoff coordinates. The Birkhoff coordinates are constructed in terms of action and angle variables. In the case of a finite gap potential, the angle variables are defined by real valued expressions involving the Abel map of a special curve of finite genus associated to the finite gap potential. The question if these expressions are real valued has been a longstanding issue, raised by experts in the field in connection with special solutions of the fNLS equation, given in terms of theta functions.

In [15], coordinates of the type provided by Theorem 1.1 have been constructed in a open neighborhood of the origin in $iH^N_r$. Note however that $\text{Spec}_o L(0)$ is not simple since it consists of real eigenvalues of algebraic and geometric multiplicity two.

Related work: In the seventies and the eighties, several groups of scientists made pioneering contributions to the development of the theory of integrable PDEs. In the periodic or quasi-periodic setup, deep connections between such equations and complex geometry as well as spectral theory were discovered and much of the efforts were aimed at representing classes of solutions (referred to as finite band solutions) by the means of theta functions, leading to the discovery of finite dimensional invariant tori. See e.g. [25, 5] as well as the books [33, 2, 10] and the references therein. Further developments of these connections allowed to treat more general classes of solutions. In particular, it was established that many integrable PDEs admit invariant tori with infinitely many degrees of freedom. See e.g. [28, 9, 17, 11, 21] and the references therein.

Organization of the paper: In Section 2 we review various results on ZS operators and related topics which are used in the paper. In Section 3 we construct a tubular neighborhood $U_{tn}$ of an appropriate path, connecting a potential $\psi^{(0)} \in \mathcal{T}$ (cf. [5]) near zero with a given potential $\psi \in \mathcal{T}$ and define the actions in $U_{tn}$. In Section 4 we define the angles in $U_{tn}$. In Section 5 we introduce actions and angles in a tubular neighborhood in $iL^2_r$ of the isospectral set $\text{Iso}_o(\psi)$ of $\psi$. In Section 6 we define the pre-Birkhoff map and study its local properties whereas in Section 7 we prove Theorem 1.1.

Acknowledgment: The authors gratefully acknowledge the support and hospitality of the FIM at ETH Zurich and the Mathematics Departments of the Northeastern University and the University of Zurich.
2 Setup

In this Section we review results needed throughout the paper. In particular we recall the spectral properties of ZS operators and the results on Birkhoff coordinates in a neighborhood of the zero potential ([15]).

For \( \varphi = (\varphi_1, \varphi_2) \in L_c^2 \) and \( \lambda \in \mathbb{C} \), let \( M = M(x, \lambda, \varphi) \), \( x \in \mathbb{R} \), be the fundamental solution of \( L(\varphi)M = \lambda M \) satisfying the initial condition \( M(0, \lambda, \varphi) = \text{Id}_{2 \times 2} \), where \( \text{Id}_{2 \times 2} \) is the identity \( 2 \times 2 \) matrix. It is convenient to write

\[
M := \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, \quad M_1 := \begin{pmatrix} m_1 \\ m_3 \end{pmatrix}, \quad M_2 := \begin{pmatrix} m_2 \\ m_4 \end{pmatrix}. \tag{6}
\]

The fundamental solution \( M(x, \lambda, \varphi) \) is a continuous function on \( \mathbb{R} \times \mathbb{C} \times L_c^2 \), for any given \( x \in \mathbb{R} \) it is analytic in \((\lambda, \varphi) \in \mathbb{C} \times L_c^2 \), and for any given \((\lambda, \varphi) \) in \( \mathbb{C} \times L_c^2 \), \( M(\lambda, \varphi) \in H^1([0, 1], \text{Mat}_{2 \times 2}(\mathbb{C})) \) – see [11]. For \( \varphi = 0 \), \( M \) is given by the diagonal \( 2 \times 2 \) matrix \( E_\lambda(x) = \text{diag}(e^{-i\lambda x}, e^{i\lambda x}) \).

**Periodic spectrum:** Recall that a complex number \( \lambda \) is said to be a periodic eigenvalue of \( L(\varphi) \) if there exists a nonzero solution of \( L(\varphi)f = \lambda f \) with \( f(1) = \pm f(0) \). As \( f(1) = M(1, \lambda)f(0) \) it means that 1 or \(-1\) is an eigenvalue of the Floquet matrix \( M(1, \lambda) \). Denote by \( \Delta(\lambda, \varphi) \) the discriminant of \( L(\varphi) \)

\[
\Delta(\lambda, \varphi) := m_1(1, \lambda, \varphi) + m_4(1, \lambda, \varphi)
\]

and note that by the discussion above \( \Delta : \mathbb{C} \times L_c^2 \to \mathbb{C} \) is analytic. It follows easily from the Wronskian identity that \( \lambda \) is a periodic eigenvalue of \( L(\varphi) \) iff \( \Delta(\lambda) \in \{2, -2\} \). Hence the periodic spectrum of \( L(\varphi) \) coincides with the zero set of the entire function

\[
\chi_p(\lambda, \varphi) := \Delta^2(\lambda, \varphi) - 4. \tag{7}
\]

In fact, by [13] Lemma 2.3, the algebraic multiplicity of a periodic eigenvalue coincides with its multiplicity as a root of \( \chi_p(\cdot, \varphi) \). We say that two complex numbers \( a \) and \( b \) are **lexicographically ordered**, \( a \prec b \), if \([\text{Re}(a) < \text{Re}(b)] \) or \([\text{Re}(a) = \text{Re}(b) \text{ and } \text{Im}(a) \leq \text{Im}(b)] \). Furthermore, for any \( n \in \mathbb{Z} \) and \( R \in \mathbb{Z}_{\geq 0} \) introduce the disks

\[
D_n := \{ \lambda \in \mathbb{C} \mid |\lambda - n\pi| < \pi/6 \} \text{ and } B_R := \{ \lambda \in \mathbb{C} \mid |\lambda| < R\pi + \pi/6 \}.
\]

For a proof of the following well known Lemma see e.g. [11].

**Lemma 2.1.** For any \( \psi \in L_c^2 \) there exist an open neighborhood \( V_\psi \) of \( \psi \) in \( L_c^2 \) and \( R_\psi \in \mathbb{Z}_{\geq 0} \) so that for any \( \varphi \in V_\psi \) the following properties hold:

(i) The periodic spectrum \( \text{Spec}_p L(\psi) \) is discrete. The set of eigenvalues counted with their algebraic multiplicities consists of two sequences of complex numbers \( (\lambda^+_n)_{|n| > R_\psi} \) and \( (\lambda^-_n)_{|n| > R_\psi} \) with \( \lambda^+_n, \lambda^-_n \in D_n, \lambda^-_n \leq \lambda^+_n \), and
a set \( \Lambda_{R_p} \equiv \Lambda_{R_p}(\phi) \) of \( 4R_p + 2 \) additional eigenvalues that lie in the disk \( B_{R_p} \). For any \( |n| > R_p \), \( \Delta(\lambda^\pm_n, \phi) = (-1)^n2 \) and
\[
\lambda^\pm_n = n\pi + \ell^2_n
\]
where the remainder \( (\lambda^\pm_n - n\pi)|_{n > R_p} \) is bounded in \( \ell^2 \) locally uniformly in \( \phi \in V_\psi \). (Later we will list the eigenvalues in \( B_{R_p} \) in a way convenient for our purposes.)

(ii) The set of roots of the entire function \( \lambda \mapsto \check{\Delta}(\lambda) \equiv \partial_\lambda \Delta(\lambda, \phi) \), when counted with multiplicities, consists of a sequence \( (\check{\lambda}_n) |_{n > R_p} \), so that \( \check{\lambda}_n \in D_n \), and a set \( \hat{\Lambda}_{R_p} \equiv \hat{\Lambda}_{R_p}(\phi) \) of \( 2R_p + 1 \) additional roots that lie in the disk \( B_{R_p} \). For \( |n| > R_p \), one has
\[
\check{\lambda}_n = \frac{\lambda^+_n + \lambda^-_n}{2} + (\lambda^+_n - \lambda^-_n)^2\ell^2_n
\]
where the remainder \( \ell^2_n \) is bounded in \( \ell^2 \) locally uniformly in \( \phi \in V_\psi \). The roots in \( \hat{\Lambda}_{R_p} \) are listed in lexicographic order and with their multiplicities
\[
\check{\lambda}_{-R_p} \ll \cdots \ll \check{\lambda}_k \ll \cdots \ll \check{\lambda}_{R_p}, \quad -R_p \leq k \leq R_p - 1.
\]

For potentials \( \phi \in iL^2_c \), the periodic spectrum of \( L(\phi) \) has additional properties. By [14, Proposition 2.6] the following holds.

**Lemma 2.2.** For any given \( \phi \in iL^2_c \) any real periodic eigenvalue of \( L(\phi) \) has geometric multiplicity two and even algebraic multiplicity. For any periodic eigenvalue in \( \mathbb{C} \setminus \mathbb{R} \), its complex conjugate \( \bar{\lambda} \) is also a periodic eigenvalue and has the same algebraic and geometric multiplicity as \( \lambda \). The periodic eigenvalues \( \lambda^+_n \) and \( \lambda^-_n \), \( |n| > R_p \), given by Lemma 2.1, satisfy
\[
\text{Im}(\lambda^+_n) \geq 0 \quad \text{and} \quad \lambda^+_n = \overline{\lambda^-_n} \quad \forall |n| > R_p.
\]

**Discriminant:** The following properties of \( \Delta \) and \( \check{\Delta} \) are well known – see e.g. [11, Section 5]. To state them introduce \( \pi_n := n\pi \ \forall n \in \mathbb{Z}\setminus\{0\} \) and \( \pi_0 := 1 \).

**Lemma 2.3.** For \( \phi \in L^2_c \) arbitrary, let \( R_p \in \mathbb{Z}_{\geq 0} \) be as in Lemma 2.1.

(i) The function \( \lambda \mapsto \chi_p(\lambda) = \Delta(\lambda, \phi)^2 - 4 \) is entire and admits the product representation
\[
\chi_p(\lambda) = -4 \left( \prod_{|n| \leq R_p} \frac{1}{\pi^2_n} \right) \cdot \chi_{R_p}(\lambda) \cdot \prod_{|n| > R_p} \frac{(\lambda^+_n - \lambda)(\lambda^-_n - \lambda)}{\pi^2_n}
\]
where \( \chi_{R_p}(\lambda) \equiv \chi_{R_p}(\lambda, \phi) \) denotes the polynomial of degree \( 4R_p + 2 \) given by
\[
\chi_{R_p}(\lambda) := \prod_{\eta \in \Lambda_{R_p}} (\eta - \lambda).
\]
(ii) The function $\lambda \mapsto \Delta(\lambda) \equiv \Delta(\lambda, \varphi)$ is entire and admits the product representation

$$
\Delta(\lambda) = 2 \left( \prod_{|n| \leq R_p} \frac{1}{\pi_n} \right) \prod_{\eta \in \Lambda_{R_p}} (\eta - \lambda) \cdot \prod_{|n| > R_p} \frac{\lambda_n - \lambda}{\pi_n}.
$$

(iii) For any $\varphi \in iL^2$ and $\lambda \in \mathbb{C}$

$$
\Delta(\lambda, \varphi) = \overline{\Delta(\lambda, \varphi)}, \quad \Delta(\lambda, \varphi) = \overline{\Delta(\lambda, \varphi)}.
$$

In particular, the zero set of $\Delta(\cdot, \varphi)$ is invariant under complex conjugation and thus $\lambda_n$ is a simple real root for any $|n| > R_p$.

The spectrum of $L(\varphi)$, $\varphi \in L^2$, when considered as an unbounded operator on $L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C})$ is given by

$$
\text{Spec}_\mathbb{R} L(\varphi) = \{ \lambda \in \mathbb{C} \mid \Delta(\lambda) \in [-2, 2] \}
$$

– see e.g. [13]. Now consider the case $\varphi \in iL^2$. By Lemma 2.3 (iii), $\Delta(\lambda)$ is real for $\lambda \in \mathbb{R}$ and by Lemma 2.2 and the Wronskian identity one concludes (see e.g. [13]) that

$$
\Delta(\lambda, \varphi) \in [-2, 2] \quad \forall \lambda \in \mathbb{R} \quad \forall \varphi \in iL^2.
$$

**Dirichlet spectrum:** Denote by $\text{Spec}_D L(\varphi)$ the Dirichlet spectrum of the operator $L(\varphi)$, i.e. the spectrum of the operator $L(\varphi)$ considered with domain

$$
\{ f = (f_1, f_2) \in H^1([0, 1], \mathbb{C})^2 \mid f_1(0) = f_2(0), \ f_1(1) = f_2(1) \}.
$$

(When the operator $L(\varphi)$ is written as an AKNS operator, the above boundary conditions become the standard Dirichlet boundary conditions – see e.g. [11].) The Dirichlet spectrum is discrete and the eigenvalues satisfy the following Counting Lemma – see e.g. [11].

**Lemma 2.4.** For any $\psi \in L^2$ the Dirichlet spectrum $\text{Spec}_D L(\psi)$ of $L(\psi)$ is discrete. Moreover, there exist $R_D \in \mathbb{Z}_{\geq 0}$ and an open neighborhood $V_\psi$ of $\psi$ in $L^2$ so that for any $\varphi \in V_\psi$, the set of Dirichlet eigenvalues, when counted with their multiplicities, consists of a sequence $(\mu_n)_{|n| > R_D}$ with $\mu_n \in D_\psi$ and a set of $2R_D + 1$ additional Dirichlet eigenvalues that lie in the disk $B_{R_D}$. These additional Dirichlet eigenvalues are listed in lexicographic order and with multiplicities $\mu_{-R_D} \leq \cdots \leq \mu_k \leq \mu_{k+1} \leq \cdots \leq \mu_{R_D}$, $-R_D \leq k \leq R_D$. For $|n| > R_D$, one has

$$
\mu_n = n\pi + \ell_n^2
$$

where the remainder $\ell_n^2$ is bounded in $\ell^2$ locally uniformly in $\varphi \in V_\psi$. If $\lambda$ is a periodic eigenvalue of $L(\varphi)$ of geometric multiplicity two then $\lambda$ is also a Dirichlet eigenvalue.
Note that for any given \( \varphi \in L^2_c \) the Dirichlet spectrum of \( L(\varphi) \) coincides (with multiplicities) with the zeroes of the entire function \( \chi_D(\cdot, \varphi) : \mathbb{C} \to \mathbb{C} \) where
\[
\chi_D : \mathbb{C} \times L^2_c \to \mathbb{C}, \quad (\lambda, \varphi) \mapsto \chi_D(\lambda, \varphi),
\]
is analytic and
\[
2i\chi_D(\lambda, \varphi) := (m_4 + m_3 - m_2 - m_1)\big|_{(\lambda, \varphi)}
\]
(see [11, Theorem 5.1] and the deformation argument in [13, Appendix C]).

**Birkhoff normal form:** We review results from [15] where Birkhoff coordinates near \( \psi = 0 \) were constructed. It follows from Lemma 2.1 and Lemma 2.4 that there exists an open ball \( U_0 \) centered at zero in \( L^2_c \) so that for any \( \varphi \) in \( U_0 \), the periodic eigenvalues of \( L(\varphi) \) are given by two sequences \( \lambda_\pm n, n \in \mathbb{Z} \), so that for any \( n \in \mathbb{Z}, \lambda_\pm n \in D_n \) and \( \Delta(\lambda_\pm n) = 2(-1)^n \). In addition, the Dirichlet eigenvalues and the roots of \( \Delta \) are given by two sequences \( \mu_n, n \in \mathbb{Z}, \) and \( \hat{\lambda}_n, n \in \mathbb{Z}, \) which are all simple and satisfy \( \mu_n, \lambda_n \in D_n \) for any \( n \in \mathbb{Z}. \) By shrinking the ball \( U_0 \) in \( L^2_c \) if necessary so that it is contained in the domain of definition of the Birkhoff map constructed in [15, Theorem 1.1] we obtain the following

**Theorem 2.1.** The claims of Theorem 1.1 hold on the open ball \( W_0 \equiv U_0 \cap iL^2_r. \)
The diffeomorphism \( \Phi : W_0 \to \Phi(W_0) \subseteq i\mathbb{H}_r \) is the restriction to \( U_0 \) of the Birkhoff map constructed in [15, Theorem 1.1].

### 3 Actions on the neighborhood \( U_{tn} \)

By Theorem 2.1 actions have been constructed for potentials in the open ball \( U_0 \) centered at zero in \( L^2_c. \) Our goal is this Section is to show that they analytically extend along a suitably chosen path to an open neighborhood of any given potential \( \psi^{(1)} \) with simple periodic spectrum. First we need to make some preliminary considerations. For a given \( R \geq \mathbb{Z} \geq 0 \) introduce the set \( T^R \subseteq L^2_c, \) defined as follows.

**Definition 3.1.** An element \( \varphi \in L^2_c \) lies in \( T^R \) if the following conditions on the periodic spectrum \( \text{Spec}_p L(\varphi) \) and the Dirichlet spectrum \( \text{Spec}_D L(\varphi) \) (both counted with multiplicities) hold:

- **(R1)** For any \( |n| > R, \) the disk \( D_n \) contains precisely two periodic eigenvalues. The set \( \Lambda_R(\varphi) \) of the remaining periodic eigenvalues consists of \( 4R + 2 \) eigenvalues which are simple and contained in the disk \( B_R, \Lambda_R(\varphi) \subseteq B_R. \)

- **(R2)** For any \( |n| > R, \) the disk \( D_n \) contains precisely one Dirichlet eigenvalue denoted by \( \mu_n \equiv \mu_n(\varphi). \) There are \( 2R + 1 \) remaining Dirichlet eigenvalues which are contained in the disk \( B_R. \) These remaining eigenvalues are listed in lexicographic order with multiplicities \( \mu_{-R} \leq \cdots \leq \mu_k \leq \mu_{k+1} \leq \cdots \leq \mu_R, -R \leq k \leq R - 1. \)
Note that by Lemma \ref{lem:2.1} and Lemma \ref{lem:2.4}, the set $\mathcal{T}^R$ is open in $L^2_c$. For the given potential $\psi^{(1)} \in iL^2_r$ with $\text{Spec}_{p, L}(\psi^{(1)})$ simple we choose $R \in \mathbb{Z}_{>0}$ as follows: Denote by $\ell$ the line segment in $iL^2_r$ connecting $\psi^{(0)}$ with $\psi^{(1)}$. By replacing $\psi^{(1)}$, if necessary, by the first intersection point of $\ell$ with $\text{Iso}_o(\psi^{(1)})$ we can assume without loss of generality that $\ell$ intersects $\text{Iso}_o(\psi)$ only at $\psi^{(1)}$. For any $\varphi \in iL^2_r$ we choose an open ball $U_\varphi$ of $\varphi$ in $L^2_c$ and $R_\varphi \in \mathbb{Z}_{\geq 0}$ so that the statements of Lemma \ref{lem:2.1} and Lemma \ref{lem:2.4} hold with $R_p$ and $R_D$ replaced by $R_\varphi$. In view of the compactness of $\ell$, we can find $\varphi_j$, $1 < j \leq K$, all in $\ell$, so that $\ell \subseteq \bigcup_{1 < j \leq K} U_{\varphi_j}$. Now, define

$$R := \max_{1 \leq j \leq K} R_{\varphi_j}. \quad (10)$$

Further, by using Theorem \ref{thm:1.2} and by arguing as in the proof of \cite[Corollary 3.3]{14}, one can construct a simple (i.e. without self intersections) continuous path

$$\gamma : [0, 1] \mapsto \bigg( \bigcup_{1 < j \leq K} U_{\varphi_j} \bigg) \cap iL^2_r, \quad s \mapsto \psi(s),$$

connecting $\psi^{(0)}$ with $\psi^{(1)}$, so that $\gamma \subseteq \mathcal{T}^R \cap iL^2_r$. In particular, we see that the statements of Lemma \ref{lem:2.1} and Lemma \ref{lem:2.4} still hold uniformly on $\gamma$ with $R_p$ and $R_D$ replaced by $R$. In addition, as $\gamma \subseteq \mathcal{T}^R \cap iL^2_r$, for any $\varphi \in \gamma$ the periodic eigenvalues of $L(\varphi)$ inside $B_{R}$ are simple, non-real (see Lemma \ref{lem:2.2}). This together with the compactness of $\gamma$ and the openness of $\mathcal{T}^R$ in $L^2_c$ implies that there exists a connected open tubular neighborhood $U_{\text{tn}}$ of $\gamma$ in $L^2_c$ so that the following holds:

(T1) The statements of Lemma \ref{lem:2.1} and Lemma \ref{lem:2.4} hold uniformly in $\varphi \in U_{\text{tn}}$ with $R_p$ and $R_D$ replaced by $R$.

(T2) The set $U_{\text{tn}} \cap iL^2_r$ is connected and for any $\varphi \in U_{\text{tn}}$ the periodic eigenvalues of $L(\varphi)$ in the disk $B_R$ are simple, non-real, and have the symmetries of Lemma \ref{lem:2.2}.

It follows from the construction of the neighborhood $\mathcal{U}_0$ of zero in $L^2_c$ (see the discussion ahead of Theorem \ref{thm:2.1}) and the property (T2) that for any $\varphi \in U_{\text{tn}} \cap \mathcal{U}_0$, and for any $n \in \mathbb{Z}$, the disk $D_n$ contains precisely two periodic eigenvalues of $L(\varphi)$. For any $\varphi \in U_{\text{tn}} \cap \mathcal{U}_0$ we list the periodic eigenvalues as follows: for $|n| \leq R$,

$$\lambda_n^+, \lambda_n^- \in D_n \quad \text{with} \quad \pm \text{Im}(\lambda_n^+) > 0,$$

and for $|n| > R$,

$$\lambda_n^+, \lambda_n^- \in D_n \quad \text{with} \quad \lambda_n^- \ll \lambda_n^+.$$

Then for any $|n| \leq R$, in view of their simplicity, the periodic eigenvalues $\lambda_n^+$ and $\lambda_n^-$ of $L(\varphi)$ considered as functions of the potential, $\lambda_n^+, \lambda_n^- : U_{\text{tn}} \cap \mathcal{U}_0 \to \mathbb{C}$, are
analytic. Since by (T2) the simplicity of these eigenvalues holds on the entire tubular neighborhood $U_{tn}$, the analytic maps above extend to analytic maps

$$\lambda_n^+, \lambda_n^- : U_{tn} \to \mathbb{C} \quad \forall |n| \leq R.$$  

(This is in sharp contrast to the eigenvalues outside the disk $B_R$ which, at least for $|n|$ sufficiently large, are not even continuous in view of the lexicographic ordering.) We note that on $U_{tn} \cap iL^n_\tau$,  

$$\lambda_n^- = \overline{\lambda_n^+} \quad \text{where} \quad \text{Im}(\lambda_n^+) > 0 \quad \forall |n| \leq R.$$  

An important ingredient for the construction of the actions on $U_{tn}$ is the choice of pairwise disjoint simple continuous paths $G_n \subseteq \mathbb{C}$, $n \in \mathbb{Z}$, also referred to as cuts, that connect $\lambda_n^-$ with $\lambda_n^+$, and continuous contours $\Gamma_n$ around $G_n$. In a first step we define the cuts $G_n$ along the path $\gamma : [0, 1] \to U_{tn} \cap iL^n_\tau$, $s \to \psi(s)$. To simplify notation, introduce $\lambda_n^\pm(s) := \lambda_n^\pm(\psi(s))$. We note that $\forall s \in [0, 1]$,  

$$\lambda_n^-(s) = \overline{\lambda_n^+(s)} \quad \text{and} \quad \text{Im}(\lambda_n^+(s)) > 0 \quad \forall |n| \leq R$$  

while  

$$\lambda_n^-(s) = \lambda_n^+(s) \quad \text{and} \quad \text{Im}(\lambda_n^+(s)) \geq 0 \quad \forall |n| > R.$$  

For any $s \in [0, 1]$ and $|n| > R$, we define the cuts $G_n(s) := G_n(\psi(s))$ to be the vertical line segments $[\lambda_n^-(s), \lambda_n^+(s)] \subseteq D_n$, parametrized by  

$$G_n : [0, 1] \times [-1, 1] \to \mathbb{C}, \quad (s, t) \mapsto \tau_n(s) + t(\lambda_n^+(s) - \lambda_n^-(s))/2,$$  

where for any $n \in \mathbb{Z}$, $\tau_n(s) := (\lambda_n^-(s) + \lambda_n^+(s))/2$. Note that $G_n(s, -t) = \overline{G_n(s, t)}$ for any $t \in [-1, 1]$. For $|n| \leq R$ and $s$ sufficiently small so that $\psi(s) \in W_0 \equiv U_0 \cap iL^n_\tau$, we define $G_n(s, t)$ in a similar fashion as in the case $|n| > R$ and then show by a somewhat lengthy but straightforward argument that the cuts $G_n(s)$ can be chosen so that they depend continuously on $s \in [0, 1]$. More precisely, the following holds:

**Lemma 3.1.** There exist continuous functions $G_n$, $|n| \leq R$, with values in the disk $B_R$,

$$G_n : [0, 1] \times [-1, 1] \to B_R, \quad (s, t) \mapsto G_n(s, t),$$

so that for any $s \in [0, 1]$ the following properties hold:

(i) $G_n(s) \equiv G_n(s, \cdot)$ is a simple $C^1$-smooth path such that for any $t \in [-1, 1]$,

$$G_n(s, -1) = \lambda_n^-(s), \quad G_n(s, 1) = \lambda_n^+(s), \quad G_n(s, -t) = \overline{G_n(s, t)}.$$  

(ii) The paths $G_n(s)$ and $G_k(s)$ do not intersect for any integer numbers $n$ and $k$, $n \neq k$, such that $|n|, |k| \leq R$. 

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(iii) There exist $0 < \rho_0 < 1$ and $\delta_0 > 0$ such that

$$G_n(s, t) = \tau_n(s) + t(\lambda_n^+(s) - \lambda_n^-(s))/2 \quad \forall t \in [-\rho_0, \rho_0]$$

where $\tau_n(s) := (\lambda_n^+(s) + \lambda_n^-(s))/2$ and

$$\text{Im}(G_n(s,t)) \geq \delta_0 \quad \forall t \in [\rho_0, 1].$$

In particular, the path $G_n(s)$ intersects the real line at the unique point $\tau_n(s) = G_n(s, 0)$.

In the next step, for any $s \in [0, 1]$ and $n \in \mathbb{Z}$ we will choose a counterclockwise oriented simple $C^1$-smooth contour $\Gamma_n(s)$ in $\mathbb{C}$ around the cut $G_n(s)$ so that $\Gamma_n(s)$ is invariant under complex conjugation, $\overline{\Gamma_n(s)} = \Gamma_n(s)$, $\Gamma_n(s) \cap \mathbb{R}$ consists of two points, and $\Gamma_n(s) \cap \Gamma_k(s) = \emptyset$ for $n \neq k$. In the case $|n| > R$ we choose $\Gamma_n(s)$ to be the boundary $\Gamma_n$ of the disk $\{ \lambda \in \mathbb{C} \, | \, |\lambda - n\pi| < \pi/4 \}$ whereas for $|n| \leq R$, the contours $\Gamma_n(s)$ are chosen in the disk $B_R$. In a similar way, for any $s \in [0, 1]$ and $n \in \mathbb{Z}$ we choose a counterclockwise oriented simple $C^1$-smooth contour $\Gamma_n'(s)$ around $G_n(s)$ so that $\Gamma_n'(s)$ lies in the interior domain of the contour $\Gamma_n(s)$, $\overline{\Gamma_n'(s)} = \Gamma_n'(s)$, $\Gamma_n'(s) \cap \mathbb{R}$ consists of two points, and $\Gamma_n'(s) \cap \Gamma_k'(s) = \emptyset$ for $n \neq k$. In the case $|n| > R$ we choose $\Gamma_n'(s)$ to be the boundary of the disk $D_n$ whereas for $|n| \leq R$, the contours $\Gamma_n'(s)$ are chosen so that $\inf_{s \in [0, 1], |n| \leq R} \text{dist}(\Gamma_n'(s), \Gamma_n(s)) > 0$. For any $s \in [0, 1]$ and $n \in \mathbb{Z}$ denote by $D_n(s)$ the interior domain of the contour $\Gamma_n'(s)$ and by $D_n(s)$ the interior domain of $\Gamma_n(s)$. The domains $D_n'(s)$ and $D_n(s)$ are topologically open disks with the property that $G_n(s) \subseteq D_n'(s) \subseteq D_n(s)$. Note that by definition $D_n(s) = D_n$ for any $s \in [0, 1]$ and $|n| > R$ whereas for $|n| \leq R$ this does not necessarily hold.

The contours $\Gamma_n'(s)$, constructed above for any $s \in [0, 1]$ and $n \in \mathbb{Z}$, are now used to choose open neighborhoods $U_s \subseteq U_n$ of $\psi(s)$ in $L^2_\gamma$ and cuts $G_n(s, \varphi) \subseteq D_n(s)$ for any $\varphi \in U_s$ (and not just for $\varphi$ in $\gamma$). In the case $|n| \leq R$ we proceed as follows: for any given $s \in [0, 1]$ consider the potential $\psi(s)$. Then we choose an open ball $U_s \subseteq U_n$ in $L^2_\gamma$ centered at $\psi(s)$ so that for any $\varphi \in U_s$ and $k \in \mathbb{Z}$, the periodic eigenvalues $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$ are in the interior domain $D_n'(s)$ of $\Gamma_n'(s)$. Then for any $|n| \leq R$ we choose a $C^1$-smooth simple path $G_n(s, \varphi)$ in the interior domain $D_n'(s)$ of $\Gamma_n'(s)$, connecting $\lambda_n^-(\varphi)$ with $\lambda_n^+(\varphi)$ so that $G_n(s, \varphi)$ intersects the real axis at a unique point $\tau_n(s, \varphi)$. In the case when $\varphi \in U_s \cap iL^2_\gamma$, in addition to the properties described above, $G_n(s, \varphi)$ is chosen so that $\overline{G_n(s, \varphi)} = G_n(s, \varphi)$. It is convenient to define $G_n(s, \varphi)$ for $\varphi = \psi(s)$ by $G_n(s, \psi(s)) := G_n(s)$. Note that then $\tau_n(s, \psi(s)) = \tau_n(s)$. By construction, the cuts $G_n(s, \varphi)$, $|n| \leq R$, do not necessarily depend continuously on $\varphi \in U_s$. In the case when $|n| > R$ we choose $G_n(s, \varphi)$ to be the line segment $G_n(\varphi) := [\lambda_n^-(\varphi), \lambda_n^+(\varphi)]$ for any $\varphi \in U_n$.

The contours $\Gamma_n(s)$, $0 \leq s \leq 1$, and the cuts $G_n(s, \varphi)$, $\varphi \in U_s$, $n \in \mathbb{Z}$, are a key ingredient not only for the construction of the actions but also for the construction of a family of one forms used in the subsequent section to define
the angles. Let us begin with the one forms. To obtain such a family of one forms, we apply Theorem 1.3 in [22]: shrinking the ball \( U_s \) if necessary, it follows that there exist analytic functions

\[
\zeta^{(s)}_n : \mathbb{C} \times U_s \to \mathbb{C}, \quad n \in \mathbb{Z},
\]

and an integer \( R_s \geq R \), used to describe the location of the zeros of these functions, so that for any \( \varphi \in U_s \) and \( n \in \mathbb{Z} \),

\[
\frac{1}{2\pi} \int_{\Gamma_m(s)} \zeta^{(s)}_n(\lambda, \varphi) \sqrt{\Delta^2(\lambda, \varphi) - 4} d\lambda = \delta_{nm}, \quad m \in \mathbb{Z}.
\] (11)

The canonical root appearing in the denominator of the integrand in (11) is defined by the infinite product

\[
\sqrt[2i]{\Delta^2(\lambda, \varphi) - 4} := 2i \prod_{k \in \mathbb{Z}} \sqrt[2i]{\left(\lambda_k^+(\varphi) - \lambda\right)\left(\lambda_k^{-}(\varphi) - \lambda\right)},
\] (12)

and the standard root \( \sqrt[2i]{\left(\lambda_k^+(\varphi) - \lambda\right)\left(\lambda_k^{-}(\varphi) - \lambda\right)} \) is defined as the unique holomorphic function on \( \mathbb{C} \setminus G_k(s, \varphi) \) satisfying the asymptotic relation

\[
\sqrt[2i]{\left(\lambda_k^+(\varphi) - \lambda\right)\left(\lambda_k^{-}(\varphi) - \lambda\right)} \sim -\lambda \quad \text{as} \quad |\lambda| \to \infty.
\] (13)

Note that by the asymptotic estimate in Lemma 2.1 (i), for any \( \varphi \in U_s \) the canonical root \( \sqrt[2i]{\Delta^2(\lambda, \varphi) - 4} \) is a holomorphic function of \( \lambda \) in the domain \( \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} G_k(s, \varphi) \right) \). In addition, the map

\[
\left( \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} D_k(s) \right) \right) \times U_s \to \mathbb{C}, \quad (\lambda, \varphi) \mapsto \sqrt[2i]{\Delta^2(\lambda, \varphi) - 4},
\] (14)

is analytic and its image does not contain zero.

**Remark 3.1.** The infinite product in (12) is understood as the limit

\[
\lim_{N \to \infty} 2i \prod_{|k| \leq N} \sqrt[2i]{\left(\lambda_k^+(\varphi) - \lambda\right)\left(\lambda_k^{-}(\varphi) - \lambda\right)} \frac{1}{\pi_k}.
\]

In order to see that this limit exists locally uniformly in \( \varphi \in U_s \) and \( \lambda \in \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} D_k(s) \right) \) one combines the terms corresponding to \( k \) and \( -k \) for \( 1 \leq k \leq N \) and notices that in view of Lemma 2.1 (i),

\[
\frac{(\lambda_k^+(\varphi) - \lambda)(\lambda_k^-(\varphi) - \lambda)}{(-\pi_k^2)}, \quad \frac{(\lambda_k^-(\varphi) - \lambda)(\lambda_k^-(\varphi) - \lambda)}{(-\pi_k^2)} = 1 + \frac{a_k}{k},
\]

where the remainder \( (a_k)_{k \in \mathbb{Z}} \) is bounded in \( \ell^2 \) locally uniformly in \( \varphi \in U_s \) and \( \lambda \in \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} D_k(s) \right) \) (cf. [17]).
According to Theorem 1.3 in [22], there exists $R_s > R$ so that for any $\varphi \in U_s$ and $n \in \mathbb{Z}$ the zeros of the entire function $\zeta^{(s)}_n(\cdot, \varphi)$, counted with their multiplicities and listed lexicographically, $\{\sigma^n_k(\varphi) \mid k \in \mathbb{Z} \setminus \{n\}\}$ have the following properties:

(D1) For any $|k| > R_s$, $k \neq n$, $\sigma^n_k(\varphi)$ is the only zero of $\zeta^{(s)}_n(\cdot, \varphi)$ in the disk $D_k$ and the map $\sigma^n_k : U_s \to D_k$ is analytic. Furthermore, for any $|k| \leq R_s$, $k \neq n$, $\sigma^n_k(\varphi) \in \mathbb{B}_{R_s}$.

(D2) For any $|k| > R_s$, $k \neq n$, we have that

$$\sigma^n_k(\varphi) = \tau_k(\varphi) + \gamma_k^2(\varphi) \ell_k^2, \quad \gamma_k(\varphi) := \lambda^+_k(\varphi) - \lambda^-_k(\varphi),$$

uniformly in $n \in \mathbb{Z}$ and locally uniformly in $\varphi \in U_s$.

(D3) The entire function $\zeta^{(s)}_n(\cdot, \varphi)$ admits the product representation

$$\zeta^{(s)}_n(\lambda, \varphi) = -\frac{2}{\pi_n} \prod_{k \neq n} \sigma^n_k(\varphi) - \lambda,$$

Moreover, if $\lambda^+_k(\varphi) = \lambda^-_k(\varphi) = \tau_k(\varphi)$ for some $k \neq n$ then $\tau_k(\varphi)$ is a zero of $\zeta^{(s)}_n(\cdot, \varphi)$.

Finally, we use the compactness of the path $\gamma$ to find finitely many numbers $s_1 < \ldots < s_N$ in the interval $[0, 1]$ so that $\{U_{s_1}, \ldots, U_{s_N}\}$ is an open cover of $\gamma$ in $L^2_k$. We can assume without loss of generality that $s_1 = 0$ and $s_N = 1$. We now shrink $U_{in}$ and set

$$U_{in} := \bigcup_{1 \leq j \leq N} U_{s_j}, \quad R' := \max_{1 \leq j \leq N} R_{s_j}. \quad (15)$$

In the sequel we will always assume that $U_{s_1} \subseteq \mathbb{U}_0$ where $\mathbb{U}_0$ is the open ball in $L^2_k$ centered at zero introduced at the end of Section 2. By the construction above, $U_{in}$ is a connected tubular neighborhood of $\gamma$ so that the properties (T1) and (T2) hold.

**Lemma 3.2.** For any $1 \leq k < l \leq N$, $\varphi \in U_{s_k} \cap U_{s_l}$ and $m \in \mathbb{Z}$ the contours $\Gamma_m(s_k)$ and $\Gamma_m(s_l)$ are homologous within the resolvent set of $L(\varphi)$.

The statement of this Lemma holds since by Lemma 3.1 for any $m \in \mathbb{Z}$ the cut $G_m(s_l)$ is obtained from $G_m(s_k)$ by a continuous deformation $\{G_m(s) \mid s \in [s_k, s_l] \}$ satisfying the properties listed in Lemma 3.1.

Lemma 3.2 implies the following. If $\varphi \in U_{s_k} \cap U_{s_l}$ for some $1 \leq k < l \leq N$, then in view of the normalization condition (11), we conclude that for any $n \in \mathbb{Z}$,

$$\int_{\Gamma_m(s_k)} \frac{\zeta^{(s_k)}_n(\lambda, \varphi)}{\sqrt{\Delta_1^2(\lambda, \varphi) - 4}} d\lambda = \int_{\Gamma_m(s_l)} \frac{\zeta^{(s_l)}_n(\lambda, \varphi)}{\sqrt{\Delta_1^2(\lambda, \varphi) - 4}} d\lambda, \quad m \in \mathbb{Z}.$$
This together with Lemma 3.2 and the definition of the canonical root (12) shows that
\[ \int_{\Gamma_m(n)} \zeta_n^{(s)}(\lambda, \varphi) - \zeta_n^{(s_1)}(\lambda, \varphi) d\lambda = 0, \quad m \in \mathbb{Z}. \]
Now we can apply [22, Proposition 5.2] to conclude that for any \( n \in \mathbb{Z}, \)
\[ \zeta_n^{(s_k)}(\cdot, \varphi) = \zeta_n^{(s_l)}(\cdot, \varphi). \]

**Remark 3.2.** An important point in the above argument is that we apply [22, Proposition 5.2] to the difference of the forms 
\[ \omega_n^{(s_k)} := \zeta_n^{(s_k)} - \zeta_n^{(s_l)}, \]
and \( \omega_n^{(s_l)} := \zeta_n^{(s_l)} - \zeta_n^{(s_k)}. \) More specifically, by construction (see the ansatz (16) in [22]), the two forms have the same “leading” term \( \Omega_n \) which cancels when we take their difference. This allows us to apply [22, Proposition 5.2] to the difference of the forms and then conclude that they coincide.

The above allows us to define \( \zeta_n(\lambda, \varphi) \) for \((\lambda, \varphi) \in \mathbb{C} \times U_{tn}\) by setting
\[ \zeta_n|_{\mathbb{C} \times U_{s_k}} := \zeta_n^{(s_k)}. \]
In this way we proved

**Proposition 3.1.** For any \( n \in \mathbb{Z}, \) the analytic function
\[ \zeta_n : \mathbb{C} \times U_{tn} \to \mathbb{C} \]
satisfies the above properties (D1)–(D3) with \( \zeta_n^{(s)} \) replaced by \( \zeta_n \) and \( R_n \) replaced by \( R' \) uniformly on \( U_{tn} \). In addition, for any \( \varphi \in U_{tn} \) so that \( \varphi \in U_{s_k} \) for some \( 1 \leq k \leq N, \) the analytic function \( \zeta_n \) satisfies the normalization conditions
\[ \frac{1}{2\pi} \int_{\Gamma_n(s_k)} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda = \delta_{nm}, \quad m \in \mathbb{Z}. \]

We now turn to the construction of the actions on the connected tubular neighborhood \( U_{tn} = \bigcup_{1 \leq j \leq N} U_{sj} \) of the path \( \gamma \) defined in (15). Recall that \( s_1 = 0 \) and \( U_{sj} \subseteq U_0 \) where \( U_0 \) is the open ball in \( L^2_c \) centered at zero, introduced at the end of Section 2. For any \( 1 \leq j \leq N \) and \( \varphi \in U_{sj} \) define the (prospective) actions
\[ I_n^{(j)}(\varphi) := \frac{1}{\pi} \int_{\Gamma_n(s_j)} \frac{\lambda \Delta(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda, \quad n \in \mathbb{Z}. \quad (16) \]
This definition is motivated by (15) where actions were defined by formula (16) for potentials \( \varphi \) in the ball \( U_0 \) (cf. Theorem 2.1). Note that the contours of integration \( \Gamma_n(s_j), n \in \mathbb{Z}, \) appearing in (16) are independent of \( \varphi \in U_{sj} \) and are contained in the set \( \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} D_k(s_j) \right) \). In addition, the mapping \( \Delta : \mathbb{C} \times L^2_c \to \mathbb{C} \)

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is analytic and the mapping (14) is analytic and does not have zeros (cf. Section 2). This shows that for any \( n \in \mathbb{Z} \) and \( 1 \leq j \leq N \),
\[
I_n^{(j)} : U_{s_j} \to \mathbb{C}
\]
is analytic. If \( \phi \in U_{s_k} \cap U_{s_l} \) for some \( 1 \leq k < l \leq N \), then for any \( n \in \mathbb{Z} \) the contours \( \Gamma_n(s_k) \) and \( \Gamma_n(s_l) \) are homologous in the resolvent set of \( L(\phi) \) – see Lemma 3.2. This together with the definition of the canonical root (12) shows that for any \( n \in \mathbb{Z} \),
\[
I_n^{(k)}(\phi) = I_n^{(l)}(\phi).
\]
Hence, \( I_n : U_{tn} \to \mathbb{C}, \ I_n|_{U_{s_j}} := I_n^{(j)}, \ 1 \leq j \leq N, \) (17) is well defined. In this way we proved

**Proposition 3.2.** For any \( n \in \mathbb{Z} \), the function \( I_n : U_{tn} \to \mathbb{C} \), defined by (17), is analytic. On \( U_{s_k} \cap U_{s_0} \) the function \( I_n \) coincides with the \( n \)-th action variable constructed in [15].

**Remark 3.3.** Alternatively, one can analytically extend the actions by arguments similar to the ones used in the proof of Proposition 4.1 to analytically extend the angles. Here we use instead a deformation of the cuts (cf. Lemma 3.1) and a subsequent deformation of the contours \( \Gamma_n(s) \) which provide a geometrically simple approach.

In view of Theorem 2.1, for any \( n \in \mathbb{Z} \), the action \( I_n|_{U_{s_1} \cap W_0} \) is real valued and for any \( m, n \in \mathbb{Z} \),
\[
\{ I_m|_{U_{s_1} \cap W_0}, I_n|_{U_{s_1} \cap W_0} \} = 0.
\]
We have the following

**Corollary 3.1.** For any \( m, n \in \mathbb{Z} \), \( \{ I_m, I_n \} = 0 \) on \( U_{tn} \). Moreover, the action \( I_n : U_{tn} \to \mathbb{C} \) is real-valued when restricted to \( U_{tn} \cap iL_2^1 \).

**Proof of Corollary 3.1.** Note that the analyticity of the action \( I_n : U_{tn} \to \mathbb{C} \) implies that for any \( j = 1, 2 \) the \( L^2 \)-gradient \( \partial_j I_n : U_{tn} \to L_2^1 \) is analytic. By the definition (2) of the Poisson bracket we conclude that \( \{ I_n, I_m \} : U_{tn} \to \mathbb{C} \) is analytic for any \( m, n \in \mathbb{Z} \). Since \( U_{tn} \) is connected and \( \{ I_n, I_m \}|_{U_{s_1}} = 0 \) by the considerations above the first statement of the Corollary follows. The second statement follows from Proposition 3.2, the fact that \( I_n : U_{tn} \to \mathbb{C} \) is real-valued when restricted to \( U_{s_1} \cap W_0 \), and Lemma 3.3 below.

In the proof of Corollary 3.1 we used the following result about real analytic functions.

**Lemma 3.3.** Let \( X_r \) be a real subspace inside the complex Banach space \( X_c = X_r \otimes \mathbb{C}, U \) a connected open set in \( X_c \) such that \( U \cap X_r \) is connected, and \( f : U \to \mathbb{C} \) an analytic function. Assume that there exists an open ball \( B(x_0) \) in \( X_c \) centered at \( x_0 \in U \cap X_r \) such that \( B(x_0) \subseteq U \) and \( f|_{B(x_0) \cap X_r} \) is real-valued. Then \( f : U \to \mathbb{C} \) is real-valued on \( U \cap X_r \).
The Lemma follows easily from standard arguments involving Taylor’s series expansions of $f$.

4 Angles on the neighborhood $U_{tn}$

In this Section we analytically extend the angles, constructed in [15] inside $U_{tn}$, along the tubular neighborhood $U_{tn}$ defined by [15]. First we need some preparation. As by construction $R' \geq R$ where $R'$ and $R$ are defined by (10) and, respectively, (11), there is a (possibly empty) set of indices $R < |k| \leq R'$ so that the statements of Lemma 2.1 and Lemma 2.4 hold uniformly in $\varphi \in U_{tn}$ with $R_p$ and $R_D$ replaced by $R'$ but in contrast to (T2), there could be double periodic eigenvalues of $L(\varphi)$ inside the disk $B_{R'}$. More specifically, double periodic eigenvalues in $B_{R'}$ can only appear in the union of disks $\bigcup_{R < |k| \leq R'} D_k$. Since by construction, $\lambda^+_k(\varphi), \lambda^-_k(\varphi) \in D_k \subseteq B_{R'}$ for any $\varphi \in U_{tn}$, $R < |k| \leq R'$.

Next we argue as in the proof of Corollary 3.3 in [14] to construct a continuous path $\tilde{\gamma} : [0, 1] \to U_{tn} \cap iL^2$, so that $\tilde{\gamma}(0) = \psi(0), \tilde{\gamma}(1) = \psi(1)$, and for any potential $\varphi \in \tilde{\gamma}([0, 1])$ the operator $L(\varphi)$ has only simple (and hence non-real) periodic eigenvalues in the disk $B_{R'}$. In view of the compactness of $\tilde{\gamma}$ we can find a connected open neighborhood $V_{tn}$ of $\tilde{\gamma}$ in $L^2$ so that $V_{tn} \subseteq U_{tn}, V_{tn} \cap iL^2$ is connected, and for any $\varphi \in V_{tn}$, the operator $L(\varphi)$ has only simple, non-real periodic eigenvalues in the disk $B_{R'}$. To simplify notation, in the sequel we denote $V_{tn}$ by $U_{tn}$ and $R'$ by $R$. In this way, we obtain

Lemma 4.1. The neighborhood $U_{tn}$ is connected, contains the potentials $\psi(0)$ and $\psi(1)$, and satisfies the properties (T1) and (T2), and Proposition 3.1 with $R'$ replaced by $R$.

Now, we proceed with the construction of the angles. For any given $\varphi \in U_{tn}$ consider the affine curve

$$C_{\varphi} := \{ (\lambda, w) \in \mathbb{C}^2 \mid w^2 = \Delta^2(\lambda, \varphi) - 4 \}$$

and the projection $\pi_1 : C_{\varphi} \to \mathbb{C}, (\lambda, w) \mapsto \lambda$. In fact, in what follows we work only with the following subsets of $C_{\varphi}$:

$$C_{\varphi, R} := \pi_1^{-1}(B_R) \quad (18)$$

and

$$D_{\varphi, k} := \pi_1^{-1}(D_k), \quad |k| > R. \quad (19)$$

By Lemma 4.1 for any $\varphi \in U_{tn}$ there are precisely $4R + 2$ periodic eigenvalues of $L(\varphi)$ inside the disk $B_R$ and they are all simple. This implies that $C_{\varphi, R}$ is an open Riemann surface with $2R + 1$ handles whose boundary in $C_{\varphi}$ is a disjoint union of two circles. The same Lemma also implies that for any $\varphi \in U_{tn}$ and $|k| > R$,

$$\lambda^+_k(\varphi), \lambda^-_k(\varphi), \mu_k(\varphi) \in D_k.$$
and there are no other periodic or Dirichlet eigenvalues of $L(\varphi)$ inside $D_k$. While for any $|k| > R$, the Dirichlet eigenvalue $\mu_k$ is simple and hence depends analytically on $\varphi \in U_{|n|}$, the periodic eigenvalues $\lambda_k^+ \ $ and $\lambda_k^-$ are not necessarily simple. In particular, we see that $D_{\varphi,k}$ is either a Riemann surface diffeomorphic to $(0,1) \times \mathbb{T}$ or, when $\lambda_k^+ = \lambda_k^-$, a transversal intersection in $\mathbb{C}^2$ of two complex disks at their centers. Now, let $\varphi \in U_{|n|}$ and assume that $\varphi \in U_{|n|}$ for some $1 \leq l \leq N$ (cf. (15)). For any $k \in \mathbb{Z}$ consider the contour $\Gamma_k(s_l)$. By Lemma 3.2 the homology class of the cycle $\Gamma_k(s_l)$ within the resolvent set of $L(\varphi)$ is independent of the choice of $1 \leq l \leq N$ with the property that $\varphi \in U_{s_l}$. Denote by $a_k$ the homology class in $C_\varphi$ of the component of $\pi_1^{-1}(\Gamma_k(s_l))$ that lies on the canonical sheet

$$C_\varphi^s := \left\{ (\lambda, w) \mid \lambda \in \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} G_k(s_l, \varphi) \right), \ w = \sqrt{\Delta^2(\lambda, \varphi) - 4} \right\}$$

of the curve $C_\varphi$ where the canonical root $\sqrt{\Delta^2(\lambda, \varphi) - 4}$ is defined by (12). By the discussion above, for any $\varphi \in U_{|n|}$ and $k \in \mathbb{Z}$ the class $a_k$ is independent of the choice of $1 \leq l \leq N$ with the property that $\varphi \in U_{s_l}$. In what follows we will not distinguish between the class $a_k$ and a given $C^1$-smooth representative of $a_k$, which we call an $a_k$-cycle. In a similar way for any $1 \leq |k| \leq R$ we define the $b_k$-cycle. More specifically, given any $1 \leq |k| \leq R$ and $\varphi \in U_{|n|}$ so that $\varphi \in U_{s_l}$ for some $1 \leq l \leq N$, consider the intersection point $\mathcal{K}_k(s_l, \varphi)$ of the cut $G_k(s_l, \varphi)$ with the real axis. Denote by $b_k$ the homology class in $C_\varphi$ of the cycle $\pi_1^{-1}(\mathcal{K}_k(s_l, \varphi), \mathcal{K}_k(s_l, \varphi))$ if $1 \leq k \leq R$ and the cycle $\pi_1^{-1}(\mathcal{K}_k(s_l, \varphi), \mathcal{K}_{k+1}(s_l, \varphi))$ if $-R \leq k \leq -1$ oriented so that the intersection index $a_k \circ b_k$ of $a_k$ with $b_k$ is equal to one. It is not hard to see that for any $1 \leq k \leq R$ the class $b_k$ is independent of the choice of $1 \leq l \leq N$ with the property that $\varphi \in U_{s_l}$. Moreover, the first homology group

$$H_1(C_\varphi, R, \mathbb{Z}) = \text{span}(a_0, a_k, b_k, 1 \leq |k| \leq R)_{\mathbb{Z}} \cong \mathbb{Z}^{2R+1}. \quad (20)$$

In view of Lemma 4.1 and Proposition 5.1 for any $n \in \mathbb{Z}$ the analytic functions $\zeta_n : \mathbb{C} \times U_{|n|} \rightarrow \mathbb{C}$ are well defined and satisfy the normalization conditions

$$\frac{1}{2\pi} \int_{a_k} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\chi_p(\lambda, \varphi)}} d\lambda = \delta_{nk}, \quad k \in \mathbb{Z}, \quad (21)$$

where by (11), $\chi_p(\lambda, \varphi) = \Delta^2(\lambda, \varphi) - 4$. For any $\varphi \in U_{|n|}$, $n \in \mathbb{Z}$, and $1 \leq |k| \leq R$, denote by $p_{nk}$ the $b_k$-period

$$p_{nk} := p_{nk}(\varphi) := \int_{b_k} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\chi_p(\lambda, \varphi)}} d\lambda.$$ 

(Note that $p_{nk}(\varphi)$ is well defined since $b_k$ is independent of the choice of $1 \leq l \leq N$ with the property that $\varphi \in U_{s_l}$.) Recall from Lemma 2.4 that for any $\varphi \in U_{|n|}$ we list the Dirichlet eigenvalues in lexicographic order and with multiplicities $\mu_k = \mu_k(\varphi), \ k \in \mathbb{Z}$. For any $k \in \mathbb{Z}$ we define the Dirichlet divisor

$$\mu_k^*(\varphi) := \left( \mu_k(\varphi), \hat{m_2}(\mu_k(\varphi), \varphi), \hat{m_3}(\mu_k(\varphi), \varphi) \right) \quad (22)$$
where for any $\lambda \in \mathbb{C}$

$$
\left( \hat{m}_1(\lambda, \varphi), \hat{m}_2(\lambda, \varphi) \right) = M(1, \lambda, \varphi)
$$

and $M(x, \lambda, \varphi)$ is the fundamental solution [6]. Note that $\mu_k^*(\varphi)$ lies on the curve $C_\varphi$ since by [11], Lemma 6.6],

$$
(\hat{m}_2(\mu_k) + \hat{m}_3(\mu_k))^2 = (\hat{m}_1(\mu_k) + \hat{m}_4(\mu_k))^2 - 4 = \Delta(\mu_k)^2 - 4.
$$

As the Floquet matrix $M(1, \lambda, \varphi) \in \text{Mat}_{2 \times 2}$ is analytic in $(\lambda, \varphi) \in \mathbb{C} \times L_\varphi^2$, we conclude from (23) and the discussion above that for any $|k| > R$, the mapping $U_{tn} \to \mathbb{C}^2$, $\varphi \mapsto \mu_k^*(\varphi)$, is analytic.

With these preparations we are ready to define for any $\varphi \in U_{tn}$ and $n \in \mathbb{Z}$ the following multivalued functions,

$$
\beta^n(\varphi) := \sum_{|k| \leq R} \frac{\mu_k^*(\varphi)}{\lambda_k(\varphi)} \frac{\chi_p(\lambda_k, \varphi)}{\sqrt{\chi_p(\lambda_k, \varphi)}} d\lambda
$$

and, for any $|k| > R$

$$
\beta^n_k(\varphi) := \int_{\lambda_k(\varphi)} \frac{\mu_k^*(\varphi)}{\lambda_k(\varphi)} \frac{\chi_p(\lambda_k, \varphi)}{\sqrt{\chi_p(\lambda_k, \varphi)}} d\lambda.
$$

Let us discuss the definition of these path integrals in more detail. In (23) the paths of integration are chosen in $C_{\varphi,R}$. The integrals in (24) depend on the choice of the path but only up to integer linear combinations of the periods of the one form $(\zeta_p(\lambda_k, \varphi))$ with respect to the basis of cycles $(a_k)_{|k| \leq R}$ and $(b_k)_{1 \leq |k| \leq R}$ on $C_{\varphi,R}$. More specifically, if $|n| > R$, then since $\int_{a_k} \frac{\zeta_p(\lambda_k, \varphi)}{\sqrt{\chi_p(\lambda_k, \varphi)}} d\lambda = 0$ for any $|k| \leq R$, the quantity $\beta^n(\varphi)$ is defined modulo the lattice

$$
\mathcal{L}_n = \mathcal{L}_n(\varphi) := \left\{ \sum_{1 \leq |k| \leq R} m_k p_{nk}(\varphi) \mid m_k \in \mathbb{Z}, 1 \leq |k| \leq R \right\}
$$

whereas, if $|n| \leq R$, it is defined modulo span$(2\pi, \mathcal{L}_n)\mathbb{Z}$ since $\frac{1}{2\pi} \int_{a_k} \frac{\zeta_p(\lambda_k, \varphi)}{\sqrt{\chi_p(\lambda_k, \varphi)}} d\lambda = \delta_{nk}$. In (24), for $|k| > R$, the path of integration is chosen to be in $D_{\varphi,k}$. If $k \neq n$, the integral $\beta^n_k(\varphi)$ is independent of the path whereas for $k = n$ with $|n| > R$, the integral $\beta^n_n(\varphi)$ is defined modulo $2\pi$ on $U_{tn} \setminus \mathcal{Z}_n$ where

$$
\mathcal{Z}_n := \{ \varphi \in U_{tn} \mid \gamma_n^2 = 0 \}.
$$

Since for any $|n| > R$, $\gamma_n^2$ is analytic on $U_{tn}$ (cf. [11], Lemma 12.4], $\mathcal{Z}_n$ is an analytic subvariety of $U_{tn}$. Furthermore, by the proof of Lemma 7.9 and Proposition 7.10 in [11], $\mathcal{Z}_n \cap iL_\varphi^2$ is a real analytic submanifold of $U_{tn} \cap iL_\varphi^2$ of real codimension two. Defining $\gamma_n := \lambda_n^+ - \lambda_n^-$ for $|n| \leq R$ it is clear that


\[ Z_n := \{ \varphi \in U_{in} \mid \gamma_n^2 = 0 \} = \emptyset \text{ for } |n| \leq R. \]

Note that the integrals in (23) and (24) exist since whenever \( \lambda_k^+ \neq \lambda_k^- \), the integrands have a singularity of the form \( (\lambda - \lambda_k^\pm)^{-1/2} \) for \( \lambda \) near \( \lambda_k^\pm \), and hence are integrable. If \( \lambda_k^+ = \lambda_k^- \) (and hence by the construction of \( U_{in} \) necessarily \(|k| > R\)), the singularity of the integrand in (24) is removable since, by Lemma 4.1 and Lemma 3.1 (see property (D3)), the root \( \sigma_k^n \) of \( \zeta_n(\lambda) \) in \( D_k \) then coincides with \( \tau_k \) which, in view of (12), is a zero of the denominator since \( \sqrt[3]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} = \tau_k - \lambda \). For convenience, we introduce

\[ w_k(\lambda) := \frac{1}{\sqrt[3]{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}. \]  

The aim of this Section is to show that

\[ \theta_n(\varphi) := \tilde{\beta}^n(\varphi) + \sum_{|k| > R} \beta_k^n(\varphi), \quad \varphi \in U_{in} \setminus Z_n, \quad n \in \mathbb{Z}, \]  

are bona fide angle variables, conjugate to the action variables introduced in Section 3. In particular the series in (28) converges. The definition (28) is motivated by [15] where the angle \( \theta_n \) was defined by a formula of the type as in (28) for potentials in \( U_{s1} \setminus Z_n \) (cf. Theorem 2.1). Here \( \tilde{\beta}^n(\varphi) \) denotes an analytic branch of the multivalued function \( \beta^n \), defined by (23), which is well defined modulo \( 2\pi \) and obtained by analytic extension of the corresponding function defined on \( U_{s1} \setminus Z_n \) in [15]. We begin by studying the integrals \( \beta_k^n(\varphi), |k| > R \). First we establish the following estimates.

**Lemma 4.2.** For any \( n \in \mathbb{Z} \) and \( |k| > R, k \neq n \),

\[ \beta_k^n = O\left( \frac{|\gamma_k| + |\mu_k - \tau_k|}{|k - n|} \right) \]

locally uniformly in \( \varphi \in U_{in} \) and uniformly in \( n \in \mathbb{Z} \).

**Proof of Lemma 4.2.** We follow the arguments of the proof of Lemma 5.1 in [11]. It follows from (24), the normalization condition (21), and the discussion above that \( \beta_k^n = 0 \) for any \( \varphi \in U_{in}, n \in \mathbb{Z}, \) and \( |k| > R \) with \( k \neq n \), such that \( \mu_k \in \{ \lambda_k^+, \lambda_k^- \} \). Moreover, in view of the normalization condition (21), the value of \( \beta_k^n \) with \( |k| > R, k \neq n \), will not change if we replace in formula (24) the eigenvalue \( \lambda_k^- \) by \( \lambda_k^+ \). Hence it is sufficient to prove the claimed estimate only for those \( \varphi \in U_{in}, n \in \mathbb{Z}, \) and \( |k| > R \) with \( k \neq n \), for which

\[ \mu_k \neq \{ \lambda_k^+, \lambda_k^- \} \quad \text{and} \quad |\mu_k - \lambda_k^-| \leq |\mu_k - \lambda_k^+|. \]

Using the definition (12) of the canonical root and (D3) we write for \( \lambda \in D_k \),

\[ \frac{\zeta_n(\lambda)}{\sqrt[3]{\lambda_k^\pm(\lambda)}} = \frac{\sigma_k^n - \lambda}{w_k(\lambda)} \zeta_k^n(\lambda) \quad \text{with} \quad \zeta_k^n(\lambda) := \frac{i}{w_n(\lambda)} \prod_{r \neq k, n} \frac{\sigma_k^n - \lambda}{w_r(\lambda)}. \]  

(29)
Arguing as in [11] Corollary 12.7 and in view of the asymptotics for \( (\sigma^n_m)_{m \neq n} \) given by Lemma 4.1 and Proposition 3.1 (property (D2)) we get

\[
\zeta_k^n(\lambda) = O\left(\frac{1}{|n-k|}\right), \quad k \neq n, \quad \lambda \in D_k,
\]

locally uniformly on \( U_{tn} \). To estimate \( \beta^n_k \), we parametrize the interval \([\lambda_k^-, \mu_k] \subseteq \mathbb{C} \) in formula (24), \( t \mapsto \lambda(t) := \lambda_k^- + td_k, \quad t \in [0, 1] \), where \( d_k := \mu_k - \lambda_k^- \) and \( d_k \neq 0 \), and then use (24), (29), and (30), to get

\[
|\beta^n_k| = \left| \int_{\lambda_k^-}^{\mu_k} \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta_k^n(\lambda) d\lambda \right| = O\left(\frac{1}{|n-k|}\right) \int_0^1 \left| \frac{\sigma^n_k - \lambda(t)}{td_k} \right|^{1/2} \left| \frac{\sigma^n_k - \lambda(t)}{\lambda_k^- - \lambda(t)} \right|^{1/2} |d_k| dt.
\]

(31)

Further, we have for any \( 0 \leq t \leq 1, \)

\[
\frac{\sigma^n_k - \lambda(t)}{\lambda_k^- - \lambda(t)} = 1 + \frac{\sigma^n_k - \lambda_k^+}{\lambda_k^- - \lambda(t)}.
\]

Using that \( |\mu_k - \lambda_k^-| \leq |\mu_k - \lambda_k^+| \) one easily sees that \( |\lambda_k^+ - \lambda(t)| \geq |\gamma_k|/2 \) for any \( 0 \leq t \leq 1 \). Then, by Lemma 4.1 and Proposition 3.1 (property (D2)) and the triangle inequality, \( |\sigma^n_k - \lambda_k^-| \leq |\sigma^n_k - \gamma_k| \) and \( |\gamma_k|/2 = O(\gamma_k) \). This implies that

\[
|\sigma^n_k - \lambda(t)| = O(1), \quad t \in [0, 1],
\]

locally uniformly on \( U_{tn} \). On the other hand,

\[
\left| \frac{\sigma^n_k - \lambda(t)}{td_k} \right|^{1/2} = \left( \left| \frac{\sigma^n_k - \lambda_k^-}{\gamma_k} \right| + \left| d_k \right| \right)^{1/2} = O\left( \left( \frac{|\gamma_k| + |d_k|^{1/2}}{\sqrt{t} |d_k|^{1/2}} \right) \right).
\]

(33)

Combining, (31) with (32) and (33) we finally obtain for \( |k| > R, k \neq n, \)

\[
|\beta^n_k| = \left| \int_{\lambda_k^-}^{\mu_k} \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta_k^n(\lambda) d\lambda \right| = O\left( \frac{(|\gamma_k| + |d_k|^{1/2})^{1/2} |d_k|^{1/2}}{|n-k|} \right).
\]

(34)

The claimed estimate then follows by the Cauchy-Schwarz inequality. Going through the arguments of the proof one sees that the estimates for \( \beta^n_k \) hold uniformly in \( n \in \mathbb{Z} \) and locally uniformly on \( U_{tn} \).

The next result claims that \( \beta^n_k, |k| > R, \) are analytic. More precisely, the following holds.

**Lemma 4.3.** Let \( n \in \mathbb{Z} \) be arbitrary.

(i) For any \( |k| > R \) with \( k \neq n \), \( \beta^n_k \) is analytic on \( U_{tn} \).
(ii) For \( |n| > R \), \( \beta^n_k \) is defined modulo \( 2\pi \). It is analytic on \( U_{tn} \setminus \mathcal{Z}_n \) when considered modulo \( \pi \).

**Proof of Lemma 4.3.** We follow the arguments of the proof of Lemma 15.2 in [11]. (i) Fix \( k \neq n \) with \( |k| > R \). In addition to the analytic subvariety \( \mathcal{Z}_k \) introduced in (20) we also consider

\[
\mathcal{E}_k := \{ \varphi \in U_{tn} \mid \mu_k \in \{ \lambda_k^\pm \} \} = \{ \varphi \in U_{tn} \mid \Delta(\mu_k) = 2(-1)^k \}
\]

which clearly is also an analytic subvariety of \( U_{tn} \). We prove that \( \beta^n_k \) is analytic on \( U_{tn} \setminus (\mathcal{Z}_k \cup \mathcal{E}_k) \) when taken modulo \( \pi \), extends continuously to \( U_{tn} \), and has weakly analytic restrictions to \( \mathcal{Z}_k \) and \( \mathcal{E}_k \). It then follows by [11, Theorem A.6] that \( \beta^n_k \) is analytic on \( U_{tn} \). To prove that \( \beta^n_k \) is analytic on \( U_{tn} \setminus (\mathcal{Z}_k \cup \mathcal{E}_k) \) it suffices to prove its differentiability. Note that \( \lambda_k^\pm \) are simple eigenvalues on \( U_{tn} \setminus \mathcal{Z}_k \), but as they are listed in lexicographic order, they are not necessarily continuous. For any given \( \varphi \in U_{tn} \setminus (\mathcal{Z}_k \cup \mathcal{E}_k) \), according to [11, Proposition 7.5], in a neighborhood of \( \varphi \) there exist two analytic functions \( \varphi_k^\pm \) such that \( \{ \lambda_k^\pm \} = \{ \varphi_k^+, \varphi_k^- \} \). Choose \( \varphi_k^\pm \) so that \( \text{dist}(\varphi_k^\pm, \mu_k) \geq \frac{1}{\delta}|\gamma_k| \). In view of the normalization condition in Proposition 3.1 we can write

\[
\beta^n_k = \int_{\varphi_k^-}^{\varphi_k^+} \frac{\zeta_n(\lambda)}{\sqrt{\chi_p(\lambda)}} \, d\lambda
\]

where the integral is taken along any path from \( \varphi_k^- \) to \( \mu_k \) inside \( D_k \) that besides its end point(s) does not contain any point of \( G_k \). The sign of the \( s \)-root along such a path is the one determined by

\[
\sqrt{\chi_p(\mu_k)} = \hat{m}_2(\mu_k) + \hat{m}_3(\mu_k).
\]

As in (20) write

\[
\frac{\zeta_n(\lambda)}{\sqrt{\chi_p(\lambda)}} = \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta^n_k(\lambda)
\]

and let \( d_k := \mu_k - \varphi_k^- \). With the substitution \( \lambda(t) = \varphi_k^- + td_k \) one has \( w_k(\lambda)^2 = td_k(\lambda(t) - \varphi_k^+) \) and as by assumption, \( |\lambda(t) - \varphi_k^+| \geq |\gamma_k/3| \) for \( 0 < t < 1 \), in a neighborhood of \( \varphi \) the argument of \( \lambda(t) - \varphi_k^+ \) is contained in an interval of length strictly smaller than \( \pi \). Hence the square root \( \sqrt{\lambda(t) - \varphi_k^+} \) can be chosen to be continuous in \( t \) and analytic near \( \varphi \). With the appropriate choice of the root \( \sqrt{d_k} \) it then follows that

\[
\beta^n_k = \int_0^1 \frac{1}{\sqrt{t}} \frac{\sigma^n_k - \lambda}{\sqrt{\lambda} - \varphi_k^+} \zeta^n_k(\lambda) \sqrt{d_k} \, dt
\]

is differentiable at \( \varphi \). Next let us show that \( \beta^n_k \) is continuous on \( U_{tn} \). By the previous considerations, \( \beta^n_k \) is continuous in all points of \( U_{tn} \setminus (\mathcal{Z}_k \cup \mathcal{E}_k) \). By (24) and \( \beta^n_k \mid _{E_k} = 0 \), it follows that \( \beta^n_k \) is continuous at points of \( \mathcal{E}_k \). It thus
remains to prove that \( \beta^n_k \) is continuous in the points of \( Z_k \setminus E_k \). First we show that \( \beta^n_k \mid_{Z_k \setminus E_k} \) is continuous. Indeed, on \( Z_k \), \( \lambda_k = \tau_k \) and \( (\sigma^n_k - \lambda)/w_k(\lambda) = 1 \) hence

\[
\beta^n_k \mid_{Z_k \setminus E_k} = \int_{\tau_k}^{\mu_k} \zeta^n_k(\lambda) \, d\lambda \bigg|_{Z_k \setminus E_k}
\]

and it follows that \( \beta^n_k \mid_{Z_k \setminus E_k} \) is continuous. As \( E_k \) is closed in \( U_{\infty} \), it then remains to show that for any sequence \( (\varphi^{(j)})_{j \geq 1} \subseteq U_{\infty} \setminus (Z_k \cup E_k) \) converging to an element \( \varphi \in Z_k \setminus E_k \) one has

\[
\beta^n_k(\varphi^{(j)}) \xrightarrow{j \to \infty} \beta^n_k(\varphi).
\]

Without loss of generality we may assume that \( \inf_{j} \left( (\mu_k - \tau_k)(\varphi^{(j)}) \right) > 0 \), hence

\[
|\lambda_k^+ (\varphi^{(j)}) - \mu_k(\varphi^{(j)})| \geq |\lambda_k^- (\varphi^{(j)}) - \mu_k(\varphi^{(j)})|
\]

(otherwise go to a subsequence of \( \varphi^{(j)} \) and/or, if necessary, switch the roles of \( \lambda_k^+ \) and \( \lambda_k^- \)), and

\[
\sqrt{\chi_p(\mu_k(\varphi^{(j)}))} = \sqrt{\chi_p(\mu_k(\varphi^{(j)}))}.
\]

Let \( 0 < \varepsilon \ll 1 \). As \( \lim_{j \to \infty} \gamma_k(\varphi^{(j)}) = 0 \) as well as \( \lim_{j \to \infty} d_k(\varphi^{(j)}) = \mu_k(\varphi) - \tau_k(\varphi) \neq 0 \) there exists \( j_0 \geq 1 \) so that

\[
\left| \frac{\gamma_k(\varphi^{(j)})}{d_k(\varphi^{(j)})} \right| \leq \varepsilon / 2 \quad \forall j \geq j_0.
\]

(35)

With the substitution \( \lambda(t) = \lambda_k^- + t d_k, \), \( d_k = \mu - \lambda_k^- \), one gets

\[
\beta^n_k(\varphi^{(j)}) = \left( \int_{0}^{\varepsilon} + \int_{\varepsilon}^{1} \right) \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta^n_k(\lambda) \, d_k \, dt.
\]

By using the estimates \( 33-35 \) one sees that for any \( j \geq j_0 \)

\[
\left| \int_{0}^{\varepsilon} \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta^n_k(\lambda) \, d_k \, dt \right| \leq C \sqrt{\varepsilon}
\]

where \( C > 0 \) is a constant independent of \( j \). To estimate the integral

\[
J_{\varepsilon}(\varphi^{(j)}) := \int_{\varepsilon}^{1} \frac{\sigma^n_k - \lambda}{w_k(\lambda)} \zeta^n_k(\lambda) \, d_k \, dt
\]

note that for any \( \varepsilon \leq t \leq 1 \) and \( j \geq j_0 \)

\[
\frac{\gamma_k^2 / 4}{(\tau_k - \lambda)^2} = \left| t \frac{2d_k}{\gamma_k} - 1 \right|^{-2} \leq 3^{-2}
\]

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and thus according to the definition \[ w_k(\lambda) = (\tau_k - \lambda)^{1 - \frac{\gamma_k^2/4}{(\tau_k - \lambda)^2}} \]
for \( \varepsilon \leq t \leq 1 \) and \( \varphi^{(j)} \) with \( j \geq j_0 \). Thus
\[
J_\varepsilon(\varphi^{(j)}) = \int_\varepsilon^1 \left( 1 + \frac{\sigma^n_k - \tau_k}{\tau_k - \lambda} \right) \left( 1 - \frac{\gamma_k^2/4}{(\tau_k - \lambda)^2} \right)^{-1/2} \zeta^n_k(\lambda) \, d_k \, dt.
\]
As \( \sigma^n_k - \tau_k = \gamma_k^2 \varepsilon^n_k^2 \) (property (D2)) one has for \( \varepsilon \leq t \leq 1 \)
\[
\frac{\sigma^n_k - \tau_k}{\tau_k - \lambda} = \frac{\gamma_k^2 \varepsilon^n_k^2}{\tau_k - \lambda_k - t d_k} \to 0 \quad \text{as} \; j \to \infty
\]
as well as
\[
\frac{\gamma_k}{\tau_k - \lambda} = \frac{\gamma_k}{\tau_k - \lambda_k - t d_k} \to 0 \quad \text{as} \; j \to \infty.
\]
By dominated convergence it then follows that
\[
J_\varepsilon(\varphi^{(j)}) \to \int_\varepsilon^1 \zeta^n_k(\lambda, \varphi) \, d_k(\varphi) \, dt \quad \text{as} \; j \to \infty.
\]
But \( \beta^n_k(\varphi) - \int_\varepsilon^1 \zeta^n_k(\lambda, \varphi) \, d_k(\varphi) \, dt = O(\varepsilon) \) by the continuity of \( \zeta^n_k \) in \( \lambda \). Altogether we showed that there exits \( j_1 \geq j_0 \), depending on \( \varepsilon \), so that for any \( j \geq j_1 \),
\[
|\beta^n_k(\varphi^{(j)}) - \beta^n_k(\varphi)| \leq C \sqrt{\varepsilon}
\]
where \( C \) can be chosen independently of \( \varepsilon \). As \( \varepsilon > 0 \) is arbitrarily small this establishes the claimed convergence. It remains to check the weak analyticity on \( \mathcal{E}_k \) and \( \mathcal{Z}_k \). On \( \mathcal{E}_k \) this is trivial since \( \beta^n_k \big|_{\mathcal{E}_k} \equiv 0 \). On \( \mathcal{Z}_k \) we can write \( \beta^n_k = \int_{\tau_k}^{\mu_k} \varepsilon_k \zeta^n_k(\lambda) \, d\lambda \) where \( \varepsilon_k \in \{0, \pm 1\} \) is defined on \( \mathcal{Z}_k \setminus \mathcal{E}_k \) by \( \sqrt{\chi_{\mathcal{F}}(\mu_k)} = \varepsilon_k \sqrt{\chi_{\mathcal{F}}(\mu_k)} \) and is zero on \( \mathcal{Z}_k \cap \mathcal{E}_k \). Now consider a disk \( D \subseteq \mathcal{Z}_k \). As \( \mathcal{E}_k \) is an analytic subvariety one has either \( D \subseteq \mathcal{Z}_k \cap \mathcal{E}_k \) in which case \( \beta^n_k \big|_D \equiv 0 \) or \( D \cap \mathcal{E}_k \) is finite. As \( \int_{\tau_k}^{\mu_k} \zeta^n_k(\lambda) \, d\lambda \big|_D \) is analytic and \( \beta^n_k \) is continuous on \( D \) it then follows that \( \int_{\tau_k}^{\mu_k} \zeta^n_k(\lambda) \, d\lambda \big|_D \equiv 0 \) or \( \varepsilon_k \big|_{D \cap \mathcal{E}_k} \) is constant. In both cases it follows that \( \beta^n_k \big|_D \) is analytic. This establishes the claimed analyticity.

Item (ii) is proved in an analogous way except for the fact that switching from \( \lambda^- \) to \( \lambda^+ \) may change the value of \( \beta^n_k \) by \( \pi \) in view of the normalization condition in Proposition 3.1. Hence we have \( \beta^n_k = \int_{\varepsilon_k}^{\mu_k} \frac{\zeta^n_k(\lambda)}{\sqrt{\chi_{\mathcal{F}}(\lambda)}} \, d\lambda \) modulo \( \pi \). 

As a next step we show that by choosing appropriate integration paths in \[ \mathcal{E}_k \] the quantity \( \beta^n_k(\varphi) \) is well defined modulo \( 2\pi \) on \( U_{\text{tn}} \) and real valued, when restricted to \( U_{\text{tn}} \cap iZ^2 \). In [15] it is proved that such a choice is possible in a small neighborhood of zero in \( L^2 \). More specifically, we have the following
Lemma 4.4. For any $\varphi \in U_{s_1}$, where $U_{s_1} \subseteq U_{\text{tn}} \cap U_0$ and $U_0$ is the open ball in $L^2_\mathbb{C}$ centered at zero, given by Theorem 2.1 and for any $n \in \mathbb{Z}$ and $|k| \leq R$, the path of integration in $\beta^\ast_k = \int_{\Lambda_k} \frac{\zeta_n(\lambda)}{\sqrt{\lambda_n(\lambda)}} \, d\lambda$ can be chosen to lie in the handle $D_{\varphi,k} \equiv \pi_k^{-1}(D_k)$ which is a Riemann surface biholomorphic to $\{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$ since $\lambda_k \neq \lambda_k^\ast$. For $|k| < R$ with $k \neq n$ the quantity $\beta_k^\ast$ is well defined and analytic on $U_{s_1}$ whereas for $k = n$ it is defined modulo $2\pi$ and as such analytic. Furthermore, for any $n \in \mathbb{Z}$ and $|k| \leq R$, $\beta^\ast_k$ is real valued when restricted to $U_{s_1} \cap iL^2_\mathbb{C}$. As a consequence, for any $n \in \mathbb{Z}$, the sum $\beta^n = \sum_{|k| \leq R} \beta^\ast_k$ is real valued when restricted to $U_{s_1} \cap iL^2_\mathbb{C}$.

Let us introduce for any $\varphi \in U_{\text{tn}}$ the set

$$M_R \equiv M_R(\varphi) := \{ \mu_k \mid |k| \leq R \}.$$ 

By the definition (22) for any $|k| \leq R$, the Dirichlet divisor $\mu_k(\varphi)$ on the Riemann surface $C_{\varphi,R}$ is uniquely determined by $\mu_k(\varphi)$. Similarly we introduce

$$\Lambda_k^- \equiv \Lambda_k^{-}\!(\varphi) := \{ \lambda^-_k \mid |k| \leq R \}.$$ 

and recall that $\lambda_k^+(\varphi)$, $|k| \leq R$, are simple periodic eigenvalues of $L(\varphi)$ that satisfy $\text{Im}(\lambda_k^-) < 0$. Then, we have the following important

Proposition 4.1. After shrinking the tubular neighborhood $U_{\text{tn}}$ of the path $\gamma : [0, 1] \to U_{\text{tn}} \cap iL^2_\mathbb{C}$, if necessary, so that $U_{\text{tn}}$ and $U_{\text{tn}} \cap iL^2_\mathbb{C}$ are still connected, there exist for any $\varphi \in U_{\text{tn}}$ a bijective correspondence

$$\Lambda_k^- \!(\varphi) \to M_R(\varphi), \quad z \mapsto \mu_k(z),$$

and for any $z \in \Lambda_k^- \!(\varphi)$ a continuous, piecewise $C^1$-smooth path $P^\ast[z, \mu_k(z)]$ on $C_{\varphi,R}$ from $z^\ast = (z, 0)$ to $\mu_k(\varphi)$ (so that for any $n \in \mathbb{Z}$,

$$\tilde{\beta}^n(\varphi) := \sum_{\Lambda_k^- \!(\varphi)} \int_{P^\ast[z, \mu_k(\varphi) \cap U_{\text{tn}}]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda, \quad (36)$$

defined modulo $2\pi$ if $|n| \leq R$, is analytic on $U_{\text{tn}}$, and real valued when restricted to $U_{\text{tn}} \cap iL^2_\mathbb{C}$.

Remark 4.1. Since the curve $\gamma$ is simple and can be assumed to be piecewise $C^1$-smooth, by shrinking $U_{\text{tn}}$ once more if necessary, we can ensure that $\tilde{\beta}^n$ is single valued on $U_{\text{tn}}$ and hence there is no need to define in the case $|n| \leq R$, $\tilde{\beta}^n$ modulo $2\pi$ as stated in Proposition 4.1. We chose to state Proposition 4.1 as is because we want to use its proof without modification in the subsequent Section when constructing angle coordinates in a tubular neighborhood of the isospectral set $\text{Iso}_n(\psi)$ which is not simply connected.

Proof of Proposition 4.1. For a given $0 \leq \tau \leq 1$, let $\psi := \gamma(\tau)$ and for any $z \in \mathbb{C}$ and $\varepsilon > 0$ denote by $D^\varepsilon(z)$ the open disk of radius $\varepsilon$ in $\mathbb{C}$ centered at $z$. 28
Let $\bar{D}(z)$ be the corresponding closed disk. We refer to the point $Q_z$ on the boundary of $D^\epsilon(z)$ with the smallest real part among the points of boundary of the disk as the base point of $D^\epsilon(z)$. Choose $\varepsilon \equiv \varepsilon_\psi > 0$ so that the following holds: For any $z, z' \in \Lambda_R(\psi) \cup M_R(\psi)$ where $\Lambda_R(\psi) \equiv \{ \lambda^\pm_k \mid |k| \leq R \}$ we have

1. $\bar{D}(z) \cap \bar{D}(z') = \emptyset$ if $z \not= z'$.
2. $\bar{D}(z) \setminus \{z\}$ does not contain any periodic eigenvalue of $L(\psi)$.
3. $\bar{D}(z) \subseteq B_R$ and if $z \in \Lambda_R^-(\psi)$, $\bar{D}(z) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Im}(\lambda) < 0 \}$.

Now we choose an open ball $U_\psi \subseteq U_{\text{in}}$ in $L^2_\psi$ centered at $\psi$ so that for any $\varphi \in U_{\text{in}}$ the following holds: For any Dirichlet eigenvalue $\mu \in M_R(\psi)$ with (algebraic) multiplicity $m_D \geq 1$ there exist exactly $m_D$ Dirichlet eigenvalues of $L(\varphi)$ in the disk $D^\epsilon(\mu)$ and for any periodic eigenvalue $\nu \in \Lambda_R^-(\psi)$ there exists a unique periodic eigenvalue $\nu_\varphi$ of $L(\varphi)$ in $D^\epsilon(\nu)$. Denote by $B^\bullet_{R,\psi}$ the complement in $B_R$ of the union of disks $D^\epsilon(z), z \in \Lambda_R^-(\psi) \cup M_R(\psi)$,

$$B^\bullet_{R,\psi} := B_R \setminus \left( \bigcup_{z \in \Lambda_R^-(\psi) \cup M_R(\psi)} D^\epsilon(z) \right).$$

Furthermore, in what follows, $\pi_1$ will denote the projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, $(\lambda, u) \mapsto \lambda$, onto the first component of $\mathbb{C}^2$. Then the set $(\pi_1|_{C_{\psi,R}})^{-1}(B^\bullet_{R,\psi})$ is a connected Riemann surface obtained from the Riemann surface $C_{\psi,R}$ by removing a certain number of open disks. Choose an arbitrary bijection $\Lambda_R^-(\psi) \rightarrow M_R(\psi), \ \nu \mapsto \mu_\psi(\nu)$.

Then, for any $\nu \in \Lambda_R^-(\psi)$, take a $C^1$-smooth curve in $B^\bullet_{R,\psi}$ that connects the base point of $D^\epsilon(\nu)$ with the base point of $D^\epsilon(\mu_\psi(\nu))$. We denote this curve by $Y_\psi[\nu, \mu_\psi(\nu)]$. In this way we obtain a set of $2R + 1$ curves

$$\{Y_\psi[\nu, \mu_\psi(\nu)] \mid \nu \in \Lambda_R^-(\psi)\}$$

in $B^\bullet_{R,\psi}$. By construction, these curves depend on the choice of $\psi$ and the bijective correspondence. Now take $\varphi \in U_\psi$. For any $z \in \Lambda_R^-(\varphi)$ with the condition that $z \in D^\epsilon(\nu)$ where $\nu \in \Lambda_R^-(\psi)$, denote by $P_{\varphi,\psi}[z, \mu_\varphi(z)]$ the concatenated path

$$[z, Q_\nu] \cup Y_\psi[\nu, \mu_\psi(\nu)] \cup [Q_{\mu_\psi(\nu)}, \mu_\varphi(z)]$$

where by $\mu_\varphi(z)$ we denote one of the Dirichlet eigenvalues in $M_R(\varphi)$ that lies in the disk $D^\epsilon(\mu_\psi(\nu))$ and $[a, b]$ denotes the line segment connecting two complex numbers $a, b \in \mathbb{C}$. Requiring that every Dirichlet eigenvalue in $M_R(\varphi)$ appears as the endpoint of such a concatenated curve precisely once, we construct for the considered potential $\varphi \in U_\psi$ a bijective correspondence

$$\Lambda_R^-(\varphi) \rightarrow M_R(\varphi), \ \ z \mapsto \mu_\varphi(z) \equiv \mu_{\varphi,\psi}(z).$$
Remark 4.2. In the case when \( \mu_\psi(\nu) \) has algebraic multiplicity \( m_D \geq 2 \) there are multiple options for the choice of the third \([Q_{\mu_\psi(\nu)}, \mu_\psi(z)]\) of the path \( P_{\varphi, \psi}[z, \mu_\psi(z)] \). This leads to different bijective correspondences \((38)\). The final result however will not depend on the choice of the correspondence in \((38)\).

We now want to lift the path \( P_{\varphi, \psi}[z, \mu_\psi(z)] \) constructed above to the Riemann surface \( \mathcal{C}_{\varphi, R} \). Let us first treat the case where \( \mu_\psi(\nu) \) is not a ramification point of \( \mathcal{C}_{\varphi, R} \), i.e. \( \mu_\psi(\nu) \notin \Lambda_R(\psi) \). Then the preimage \( (\pi_1|_{\mathcal{C}_{\varphi, R}})^{-1}(D(\mu_\psi(\nu))) \) consists of two disjoint closed disks. Denote by \( Q_{\mu_\psi(\nu), \nu}^* \) the lift of \( Q_{\mu_\psi(\nu)}(z) \) which is in the disk containing the Dirichlet divisor \( \mu_\psi^*(\nu) \). Similarly, for \( \varphi \in U_\psi \) the preimage \( (\pi_1|_{\mathcal{C}_{\varphi, R}})^{-1}(D(\mu_\psi(\nu))) \) consists of two disjoint disks. We denote by \( Q_{\nu, \psi}^* \) the starting point of the lift \( Y_{\nu, \psi}^*[\nu, \mu_\psi(\nu)] \) of \( Y_\psi[\nu, \mu_\psi(\nu)] \) by \( (\pi_1|_{\mathcal{C}_{\varphi, R}})^{-1} \) that is ending at \( Q_{\mu_\psi(\nu), \nu}^* \). By construction,

\[
\pi_1(Q_{\mu_\psi(\nu), \nu}^*) = Q_\nu \quad \text{and} \quad \pi_1(Q_{\nu, \psi}^*) = \mu_\psi(\nu).
\]

This yields a uniquely determined lift \( P_{\varphi, \psi}[\nu, \mu_\psi(z)] \) of \( P_{\varphi, \psi}[z, \mu_\psi(z)] \) to \( \mathcal{C}_{\varphi, R} \), starting at \((z, 0)\) and ending at \( \mu_\psi^*(\nu) \). Now, let us turn to the case where \( \mu_\psi(\nu) \in \Lambda_R(\psi) \) and hence \( \mu_\psi(\nu) \) is a ramification point of \( \pi_1|_{\mathcal{C}_{\varphi, R}} : \mathcal{C}_{\varphi, R} \to \mathbb{C} \). Then the preimage \( (\pi_1|_{\mathcal{C}_{\varphi, R}})^{-1}(D(\mu_\psi(\nu))) \) is a closed disk in \( \mathcal{C}_{\varphi, R} \) and the restriction of the map \( \pi_1 \) to this disk is two-to-one except at the branching point \( \mu_\psi(\nu) \) where it is one-to-one. Denote by \( Q_{\mu_\psi(\nu), \psi}^* \) one of the preimages of \( Q_{\mu_\psi(\nu)}(z) \). For \( \varphi \in U_\psi \) the preimage \( (\pi_1|_{\mathcal{C}_{\varphi, R}})^{-1}(D(\mu_\psi(\nu))) \) is also a closed disk in \( \mathcal{C}_{\varphi, R} \) and contains a unique branched point in its interior. We denote by \( Q_{\nu, \psi}^* \) the preimage of \( Q_{\mu_\psi(\nu)}(z) \) defined uniquely by the condition that the map

\[
U_\psi \to \mathbb{C}^2, \quad \varphi \mapsto Q_{\mu_\psi(\nu), \varphi}^*,
\]

is analytic and \( Q_{\nu, \psi}^*|_{\varphi = \psi} = Q_{\mu_\psi(\nu), \psi}^* \). Then, by proceeding in the same way as in the previous case, we construct the point \( Q_{\nu, \psi}^* \), the lift \( Y_{\nu, \psi}^*[\nu, \mu_\psi(\nu)] \) of \( Y_\psi[\nu, \mu_\psi(\nu)] \), and the uniquely determined lift \( P_{\nu, \psi}[\nu, \mu_\psi(\nu)] \) of \( P_{\varphi, \psi}[\nu, \mu_\psi(\nu)] \) to \( \mathcal{C}_{\varphi, R} \) that starts at \((z, 0)\), passes consecutively through \( Q_{\nu, \psi}^* \) and \( Q_{\mu_\psi(\nu), \psi}^* \), and then ends at the Dirichlet divisor \( \mu_\psi^*(\nu) \). It then follows that for any \( n \in \mathbb{Z} \),

\[
\beta^n_\psi(\varphi) := \sum_{\varphi \in \mathcal{C}_{\varphi, R}} \int_{P_{\nu, \psi}^*[\nu, \mu_\psi(\nu)]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda,
\]

is well defined and analytic on \( U_\psi \). Indeed, if \( \nu \in \Lambda_{\overline{R}}(\psi) \) so that \( \mu_\psi(\nu) \) is a simple Dirichlet eigenvalue of \( L(\varphi) \) the integral \( \int_{P_{\nu, \psi}^*[\nu, \mu_\psi(\nu)]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda \) is clearly analytic on \( U_\psi \) where \( z \in \Lambda_{\overline{R}}(\varphi) \) is the periodic eigenvalue of \( L(\varphi) \) in the disk \( D^*(\nu) \). In case \( \mu_\psi(\nu) \) is a Dirichlet eigenvalue of \( L(\psi) \) of multiplicity \( m_D \geq 2 \), denote by \( \nu_j \), \( 1 \leq j \leq m_D \), the periodic eigenvalues of \( L(\psi) \) such that
\[ \mu_\psi(\nu_j) = \mu_\psi(\nu) \text{ and for any } \varphi \in U_\psi \text{ by } z_j, 1 \leq j \leq m_D, \text{ the periodic eigenvalues of } L(\varphi) \text{ with } z_j \in D^\varphi(\nu_j). \text{ Then the argument principle implies that} \]

\[
\sum_{1 \leq j \leq m_D} \int_{P_{z_j}(\mu_\psi(z_j))} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda,
\]

is analytic in \( U_\psi \) – see Lemma 3.3 in the Appendix for more details. One can easily check that the value \((39)\) does not depend on the choice described in Remark 3.2. Note that the restriction of \( \beta_\psi^n \) to \( U_\psi \cap iL_2^r \) is not necessarily real valued. For latter reference we record that by construction, for any \( \varphi \in U_\psi \), the path \( P_{\varphi, \psi}[z, \mu_\psi(z)] \) is a concatenation of three paths which we write as

\[
P_{\varphi, \psi}[z, \mu_\psi(z)] = [z, Q_\nu]^{*} \cup Y_{\varphi, \psi}^{*}[\nu, \mu_\psi(\nu)] \cup [Q_{\mu_\psi(\nu)}, \mu_\psi(z)]^{*}. \tag{41}
\]

Now consider \( 0 \leq \tau_0 \leq 1 \) with \( \tau_0 \neq \tau \) and let \( \psi_0 := \gamma(\tau_0) \). Assume that following the same construction as for \( \psi = \gamma(\tau) \), one finds an open ball \( U_{\psi_0} \subseteq U_{\psi_0} \) of \( \psi_0 \) in \( L_2^r \) with \( U_\psi \cap U_{\psi_0} \neq \emptyset \) and for any \( \varphi \in U_{\psi_0} \) a system of paths \( P_{\varphi, \psi_0}[z, \mu_\psi, \psi_0(z)] \), \( z \in \Lambda^-(\varphi) \), so that the restriction of \( \beta_{\psi_0}^n \) to \( U_{\psi_0} \cap iL_2^r \) is real valued. We now want to show that one can modify the system of paths \( P_{\varphi, \psi_0}[z, \mu_\psi(z)] \), \( z \in \Lambda^-(\varphi) \), so that for any \( n \in \mathbb{Z} \),

\[
\beta_{\psi_0}^n(\varphi) = \beta_{\psi_0}^n(\varphi), \quad \forall \varphi \in U_\psi \cap U_{\psi_0}, \tag{42}
\]

where \( \beta_{\psi_0}^n(\varphi) \) denotes the value \((39)\) with the system of paths modified. Then by Lemma 3.3 for any \( n \in \mathbb{Z} \), the quantity \( \beta_{\psi_0}^n \) will be real valued when restricted to \( U_\psi \cap iL_2^r \). Indeed, take \( \varphi \in U_\psi \cap U_{\psi_0} \). The first problem arises if the bijection

\[
\mu_{\varphi, \psi_0} : \Lambda^-_R \rightarrow M_R(\varphi)
\]

corresponding to the neighborhood \( U_{\psi_0} \) is different from the bijection

\[
\mu_\varphi \equiv \mu_{\varphi, \psi} : \Lambda^-_R \rightarrow M_R(\varphi)
\]

corresponding to the neighborhood \( U_\psi \). In such a case, denote by \( \sigma_\varphi : \Lambda^-_R(\varphi) \rightarrow \Lambda^-_R(\varphi) \) the permutation with the property that \( \mu_{\varphi, \psi_0} = \mu_{\varphi, \psi} \circ \sigma_\varphi \). Since \( \sigma_\varphi \) can be written as a product of transpositions we can assume without loss of generality that \( \sigma_\varphi \) is a transposition interchanging two periodic eigenvalues \( z \) and \( z' \) in \( \Lambda^-_R(\varphi) \). Then \( z \in D^\varphi(\nu) \) and \( z' \in D^\varphi(\nu') \) for some \( \nu, \nu' \in \Lambda^-_R(\psi) \). In view of \((11)\),

\[
P_{\varphi, \psi}[z, \mu_\psi(z)] = [z, Q_\psi]^{*} \cup Y_{\varphi, \psi}^{*}[\nu, \mu_\psi(\nu)] \cup [Q_{\mu_\psi(\nu)}, \mu_\psi(z)]^{*}
\]

and

\[
P_{\varphi, \psi}[z', \mu_\psi(z')] = [z', Q_\psi]^{*} \cup Y_{\varphi, \psi}^{*}[\nu', \mu_\psi(\nu')] \cup [Q_{\mu_\psi(\nu')}, \mu_\psi(z')]^{*}.
\]

Choose a path \( Y_{\nu, \nu'}^{*} \) on the Riemann surface

\[
C_{\psi, R} := (\pi_1|_{C_{\psi, R}})^{-1}(B_{\psi, R}^{*})
\]

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that connects the point $Q_\nu$ with $Q_{\nu'}$. Let $Y_{\nu,\nu'}$ be its projection into $B_{R,\psi}$ by the map $\pi_1$ and denote by $Y_{\nu,\nu'}^*$ the unique lift of $Y_{\nu,\nu'}$ by $\pi_1|_{C_{\psi,R}}: C_{\psi,R} \to \mathbb{C}$ which connects $Q_{\nu'}$ with $Q_{\nu}$. We then replace $P_{\psi,\nu}[z, \mu_\psi(z)]$ by the concatenated curve

$$[z, Q_\nu] \cup Y_{\nu,\nu'}^* \cup Y_{\nu',\psi}^* [\nu', \mu_\psi(\nu')] \cup [Q_{\mu_\psi(\nu')}, \mu_\psi(z')]^*$$

and $P_{\nu,\psi}^*[z', \mu_\psi(z')]$ by

$$[z', Q_{\nu}] \cup (Y_{\nu,\nu'}^*)^{-1} \cup Y_{\nu,\psi}^* [\nu, \mu_\psi(\nu)] \cup [Q_{\mu_\psi(\nu)}, \mu_\psi(z)]^*$$

where $(Y_{\nu,\nu'}^*)^{-1}$ denotes the path obtained from $Y_{\nu,\nu'}^*$ by traversing it in the opposite direction. Since this change of paths does not affect the value of $\beta^0_\psi(\phi)$ we can assume without loss of generality that $\mu_{\nu,\psi} = \mu_{\nu,\psi_0}$. Therefore, for any $z \in \Lambda_{\psi_0}(\phi)$, the paths $P_{\phi,\nu}^*[z, \mu_\psi(z)]$ and $P_{\nu,\psi}^*[z, \mu_{\psi_0}(\nu)]$ have the same initial points and the same end points. Now, take $\phi_0 \in U_{\psi} \cap U_{\psi_0}$. Then, for any $z \in \Lambda_{\psi_0}(\phi_0)$ choose an arbitrary point on $Y_{\phi_0,\psi}^*[\nu, \mu_\psi(\nu)]$ of the path $P_{\phi,\psi}^*[z, \mu_{\psi_0}(\nu)]$ and modify it by adding a bouquet of a-cycles and b-cycles of the Riemann surface $C_{\phi_0,R}$, contained in $(\pi_1|_{C_{\phi_0,R}})^{-1}(B_{R,\psi_0})$ so that when modified in this way the path $P_{\phi_0,\psi}^*[z, \mu_{\psi_0}(\nu)]$ is homologous to $P_{\phi,\psi}^*[z, \mu_{\psi_0}(\nu)]$ on $C_{\phi_0,R}$. Hence, for any $n \in \mathbb{Z}$, $\beta^n_\psi(\phi_0)$ defined by (59) with these modified paths equals $\beta^n_\psi(\phi_0)$. For any $\nu \in \Lambda_{\psi}^+(\psi)$, let

$$\bar{Y}_\psi^*[\nu, \mu_\psi(\nu)] := \pi_1(\bar{Y}_\psi^*[\nu, \mu_\psi(\nu)])$$

where $\bar{Y}_\psi^*[\nu, \mu_\psi(\nu)]$ denotes the middle part of the path $P_{\phi,\psi}^*[z, \mu_{\psi_0}(\nu)]$ modified as described above. Note that $\bar{Y}_\psi^*[\nu, \mu_\psi(\nu)] \subseteq B_{R}^*$. Now, use $\bar{Y}_\psi^*[\nu, \mu_\psi(\nu)]$ instead of $Y_{\phi}^*[\nu, \mu_\psi(\nu)]$ to construct the paths $P_{\phi,\psi}^*[z, \mu_{\psi_0}(\nu)]$ and their lifts $P_{\phi,\psi}^*[z, \mu_{\psi_0}(\nu)]$ for any $\phi \in U_{\psi}$ and $z \in \Lambda_{\phi}(\phi)$. By this construction and since (60) is independent on the choice described in Remark 3.3, the equality (19) follows. As mentioned above, Lemma 3.3 then implies that $\beta^n_\psi: U_{\psi} \to \mathbb{C}$ is real valued when restricted to $U_{\psi} \cap iL^2_\psi$ for any $n \in \mathbb{Z}$.

Since the set $\gamma([0, 1]) := \{ \gamma(t) \mid t \in [0, 1] \} \subseteq U_{\psi} \cap iL^2$ is compact in $L^2$ there are finitely many potentials $\phi_1, ..., \phi_M \in \gamma([0, 1])$, open balls $U(\phi_1), ..., U(\phi_M)$ in $U_{\psi}$, and for any $1 \leq j \leq M$ and any $\phi \in U(\phi_j)$, a system of paths

$$\{ P_{\phi,\psi}^*[z, \mu_{\phi,\psi}(z)] \mid z \in \Lambda_{\phi}^+(\phi_j) \},$$

constructed as described above so that

$$\gamma([0, 1]) \subseteq \bigcup_{j=1}^{M} U(\phi_j) \subseteq U_{\psi}.$$

Without loss of generality we can assume that $\phi_1 = \psi(0) \in U_0 \cap U_{\psi_0}$ where $U_0$ is the open ball in $L^2$ centered at zero given by Theorem 2.4.1 By choosing for any $\phi \in U(\phi_1)$ the system of paths as above and, at the same time, as described in Lemma 4.4, it follows that $\beta^n_{\phi_1}: U(\phi_1) \to \mathbb{C}$ is real-valued on $U(\phi_1) \cap iL^2_\psi$. 

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By modifying the systems of paths as described above we conclude that for any 
\[1 \leq j < k \leq M\] and \(\varphi \in U(\varphi_j) \cap U(\varphi_k)\) the quantities \(\beta^n_{\varphi_j}(\varphi)\) and \(\beta^n_{\varphi_k}(\varphi)\) are real valued and coincide up to a linear combination of \(a\) and \(b\)-periods with integer coefficients. Taking into account Lemma \[15\] it then follows that modulo \(2\pi\),

\[\beta^n_{\varphi_j}(\varphi) \equiv \beta^n_{\varphi_k}(\varphi), \quad \forall \varphi \in U(\varphi_j) \cap U(\varphi_k).\]

By setting \(\tilde{\beta}^n|_{U(\varphi_j)} := \beta^n_{\varphi_j}\) for \(1 \leq j \leq M\) we then obtain a well defined function modulo \(2\pi\),

\[\tilde{\beta}^n : U'_{tn} \to \mathbb{C}, \quad U'_{tn} := \bigcup_{j=1}^{M} U(\varphi_j),\] (43)

so that \(\tilde{\beta}^n|_{U'_{tn} \cap iL^2}\) is real-valued. This completes the proof of Proposition \[4.1\].

In the proof of Proposition \[4.1\] we used the following

**Lemma 4.5.** For any \(\varphi \in U_{tn} \cap iL^2\) and for any \(n \in \mathbb{Z}\), the lattice of periods \(L_n(\varphi)\), introduced in (25), consists of imaginary numbers, \(L_n(\varphi) \subseteq i\mathbb{R}\).

**Proof of Lemma 4.5.** Take \(\varphi \in U_{tn} \cap iL^2\) and assume that \(\varphi \in U_{sk}\) for some \(1 \leq k \leq N\) (see (15)). Then, by Proposition 3.1 for any \(n \in \mathbb{Z}\),

\[\int_{\Gamma_m(s_k)} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda = 2\pi \delta_{mn}, \quad m \in \mathbb{Z},\] (44)

where by construction,

\[\Gamma_m(s_k) = \Gamma_m(s_k).\] (45)

It follows from Lemma \[2.2\] and the definition of the canonical root (12) that

\[\left(\sqrt{\Delta^2(\lambda, \varphi) - 4}\right) = -\sqrt{\Delta^2(\phantom{\lambda} \varphi) - 4}.\] (46)

By taking the complex conjugate of both sides of (14), then using (15) and (16), and finally by passing to the complex conjugate variable in the integral, we obtain that for any \(n \in \mathbb{Z}\),

\[\int_{\Gamma_m(s_k)} \frac{\zeta_n(\overline{\lambda}, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda = 2\pi \delta_{mn}, \quad m \in \mathbb{Z}.\] (47)

Now, by comparing (14) and (17) we conclude from \[22\] Proposition 5.2] (cf. Remark \[3.2\]) that

\[\zeta_n(\overline{\lambda}, \varphi) = \zeta_n(\lambda, \varphi)\] (48)

for any \(\varphi \in U_{tn} \cap iL^2\), \(\lambda \in \mathbb{C}\), and \(n \in \mathbb{Z}\). In particular, we see that in this case \(\zeta_n(\lambda, \varphi)\) is real-valued for \(\lambda \in \mathbb{R}\). Now consider the \(b\)-period of the one form
Then, by construction \( \pi_1(b_l) = [\kappa_{l-1}(s_k, \varphi), \kappa_l(s_k, \varphi)] \), and therefore

where the integration is performed over the real interval \([\kappa_{l-1}(s_k, \varphi), \kappa_l(s_k, \varphi)]\). By construction \(B_R \cap \mathbb{R}\) does not contain periodic eigenvalues of \(L(\varphi)\). In view of \(18\) and the property that \(\Delta(\lambda, \varphi) \in (-2, 2)\) for \(\lambda \in B_R \cap \mathbb{R}\) (see \(18\)) we then conclude from \((19)\) that the \(b_l\)-period \(p_{nl}\) is an imaginary number.

Arguing as in the proof of Lemma 4.2 one obtains the following estimates for \(\tilde{\beta}^n(\varphi)\).

**Lemma 4.6.** For any \(n \in \mathbb{Z}\),\n
\[
\tilde{\beta}^n = O(1/n) \quad \text{as} \quad |n| \to \infty,
\]

locally uniformly in \(\varphi \in U_{tn}\).

In what follows we assume that the tubular neighborhood \(U_{tn}\) is chosen as in Proposition 4.1. We summarize the results obtained in this section so far as follows.

**Proposition 4.2.** For any \(n \in \mathbb{Z}\), the following statements hold:

(i) \(\sum_{|k| > R} \beta^n_k\) converges locally uniformly on \(U_{tn}\) to an analytic function on \(U_{tn}\) which is of the order \(o(1)\) as \(|n| \to \infty\).

(ii) If \(|n| > R\), then \(\beta^n\) is defined modulo \(2\pi\) on \(U_{tn} \setminus \mathbb{Z}_n\). It is analytic, when taken modulo \(\pi\).

(iii) For any \(\varphi \in U_{tn}\) and \(n \in \mathbb{Z}\), the quantity \(\tilde{\beta}^n(\varphi)\) is defined modulo \(2\pi\) and is analytic on \(U_{tn}\). Furthermore, \(\tilde{\beta}^n = O(1/n)\) locally uniformly on \(U_{tn}\).

**Proof of Proposition 4.2.** (i) By Lemma 4.2 and the Cauchy-Schwarz inequality,

\[
\sum_{|k| > R} |\beta^n_k| = \sum_{0 < |k - n| \leq \frac{1}{2n}, |k| > R} |\beta^n_k| + \sum_{|k - n| > \frac{1}{2n}, |k| > R} |\beta^n_k| \leq C \left( \sum_{|k| \geq |n|/2} |\gamma_k|^2 + |\mu_k - \tau_k|^2 \right)^{1/2} + C \left( ||\gamma||_0 + ||\mu - \tau||_0 \right) \left( \sum_{m \geq \frac{|n|}{2}} \frac{1}{m^2} \right)^{1/2}
\]

where here \(\gamma = (\gamma_m)_{m \in \mathbb{Z}}\) and \(\mu - \tau = (\mu_m - \tau_m)_{m \in \mathbb{Z}}\). Both latter displayed terms converge to zero locally uniformly on \(U_{tn}\) as \(|n|\) tends to infinity whence we have \(\sum_{|k| > R} \beta^n_k = o(1)\). By Lemma 4.3 (i) and \([11\) Theorem A.4], it then follows that \(\sum_{|k| > R} \beta^n_k\) is analytic on \(U_{tn}\). Item (ii) is proved in Lemma 4.3 (ii) and item (iii) follows from Proposition 4.1 and Lemma 4.6. \(\Box\)
As a consequence the angle variables (28),

\[ \theta_n(\varphi) = \tilde{\beta}^n(\varphi) + \sum_{|k| > R} \beta_k^n(\varphi), \quad \varphi \in U_{\text{tn}} \setminus \mathbb{Z}_n, \quad n \in \mathbb{Z}, \quad (50) \]

are well defined.

**Proposition 4.3.** For any \( n \in \mathbb{Z} \), the angle variable \( \theta_n \) are defined on \( U_{\text{tn}} \setminus \mathbb{Z}_n \) modulo \( 2\pi \) by (50). For \( |n| \leq R \), \( \theta_n \) is analytic and for \( |n| > R \), it is analytic on \( U_{\text{tn}} \setminus \mathbb{Z}_n \) when taken modulo \( \pi \). Moreover, (50) is real valued when restricted to \( (U_{\text{tn}} \setminus \mathbb{Z}_n) \cap iL_r^2 \).

**Proof of Proposition 4.3.** In view of Proposition 4.1 and Proposition 4.2 it only remains to prove that for any \( n \in \mathbb{Z} \) the angle variable \( \theta_n \) is real valued when restricted to \( (U_{\text{tn}} \setminus \mathbb{Z}_n) \cap iL_r^2 \). This follows from the analyticity of \( \theta_n \), Lemma 3.3 and the fact that, by Theorem 2.1 \( \theta_n \) is real valued when restricted to the neighborhood of zero \( (U_{s_1} \setminus \mathbb{Z}_n) \cap iL_r^2 \subseteq W_0 \setminus \mathbb{Z}_n \) (cf. 15).

By combining Lemma 4.4, Theorem 2.1 and Proposition 4.3, we obtain by arguing as in the proof of Corollary 3.1 the following theorem, which summarizes the results obtained in Section 3 and Section 4.

**Theorem 4.1.** For any \( k, n \in \mathbb{Z} \) we have \( \{I_n, I_k\} = 0 \) on \( U_{\text{tn}} \), \( \{\theta_n, I_k\} = \delta_{nk} \) on \( U_{\text{tn}} \setminus \mathbb{Z}_n \), and \( \{\theta_n, \theta_k\} = 0 \) on \( U_{\text{tn}} \setminus (\mathbb{Z}_n \cup \mathbb{Z}_k) \). Furthermore, the action \( I_n \) and the angle \( \theta_n \) are real valued when restricted to \( U_{\text{tn}} \cap iL_r^2 \) and, respectively, \( (U_{\text{tn}} \setminus \mathbb{Z}_n) \cap iL_r^2 \).

## 5 Actions and angles in \( U_{\text{iso}} \)

Let \( \psi^{(1)} \in iL_r^2 \) and assume that \( \psi^{(1)} \) has simple periodic spectrum. Following the construction of action-angle coordinates along the path \( \gamma : [0, 1] \to U_{\text{tn}} \cap iL_r^2 \), \( s \to \psi^{(s)} \), in Section 3 and Section 4 we construct in the present Section action-angle coordinates in an open neighborhood of the isospectral set \( \text{Iso}_o(\psi^{(1)}) \).

We prove additional properties of these coordinates that will be used in the subsequent sections.

For any \( n \in \mathbb{Z} \), let \( \Gamma_n := \Gamma_n(s_N), \Gamma'_n := \Gamma'_n(s_N) \) where \( \Gamma_n(s_N) \) and \( \Gamma'_n(s_N) \) are the contours corresponding to the cut \( G_n(s_N) \) constructed in Section 3. Here \( s = s_N = 1 \) is the endpoint of the deformation \( \{G_n(s) \mid n \in \mathbb{Z}\}_{s \in [0, 1]} \) of cuts given by Lemma 3.3. For any \( \psi \in \text{Iso}_o(\psi^{(1)}) \) we choose an open ball \( U_{\psi} \) of \( \psi \) in \( L_r^2 \), an integer \( R_{\psi} \in \mathbb{Z}_{>0} \), and positive constants \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that the following holds:

(1) The statements of Lemma 2.1, Lemma 2.4 and Lemma 8.2 in the Appendix with \( \varepsilon = \varepsilon_0 \) and \( \delta = \delta_0 \), hold uniformly in \( \varphi \in U_{\psi} \) with \( R_p, R_D \), and \( R \) replaced by \( R_{\psi} \). Moreover, for any \( \varphi \in U_{\psi} \) the \( 4R_{\psi} + 2 \) periodic eigenvalues of \( L(\varphi) \) inside the disk \( B_{R_{\psi}} \) are simple.
(I2) For any \( \varphi \in U_\psi \) and for any \( n \in \mathbb{Z} \) the pair of periodic eigenvalues \( \lambda_n^\pm \) is contained in the interior domain \( D_n^r \) encircled by the contour \( \Gamma_n^r \).

(I3) For any \( n \in \mathbb{Z} \) there exists an analytic function \( \zeta_n(\psi) : \mathbb{C} \times U_\psi \to \mathbb{C} \) such that for any \( m \in \mathbb{Z} \) one has

\[
\frac{1}{2\pi} \int_{\Gamma_m} \frac{\zeta_n(\psi)(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda = \delta_{nm}. \tag{51}
\]

Moreover, for any \( \varphi \in U_\psi \) and for any \( n \in \mathbb{Z} \) the zeros \( \{\sigma_k^\psi \mid k \in \mathbb{Z} \setminus \{n\} \} \) of the entire function \( \zeta_n(\psi)(\cdot, \varphi) \), when listed with their multiplicities, satisfy the conditions (D1)-(D3) with \( R_n \) replaced by \( R_\psi \) and \( \zeta_n(\psi) \) replaced by \( \zeta_n(\psi) \). The canonical root in (51) is defined by (12) and the map

\[
\left( \mathbb{C} \setminus \bigcup_{k \in \mathbb{Z}} D_k^\psi \right) \times U_\psi \to \mathbb{C}, \quad (\lambda, \varphi) \mapsto \sqrt{\Delta^2(\lambda, \varphi) - 4},
\]

is analytic.

Since the isospectral set \( \text{Iso}_\psi(\psi^{(1)}) \) is compact (see Proposition 2.2 in [24]), there exist finitely many elements \( \eta^{(j)} \in \text{Iso}_\psi(\psi^{(1)}), 1 \leq j \leq J \), such that \( \text{Iso}_\psi(\psi^{(1)}) \) is contained in the connected open set

\[
U_{\text{iso}} := \bigcup_{1 \leq j \leq J} U_{\eta^{(j)}} \subseteq L_e^2. \tag{52}
\]

Take \( R := \max_{1 \leq j \leq J} R_{\eta^{(j)}} \). Using the compactness of \( \text{Iso}_\psi(\psi^{(1)}) \) and the fact that \( \psi^{(1)} \) has simple periodic spectrum one sees that if necessary, the neighborhood \( U_{\text{iso}} \) can be shrunk so that \( U_{\text{iso}} \) and \( U_{\text{iso}} \cap iL_e^2 \) are connected and for any \( \varphi \in U_{\text{iso}} \) the periodic eigenvalues of \( L(\varphi) \) inside the disk \( B_R \) are simple. Note that if \( \varphi \in U_{\eta^{(k)}} \cap U_{\eta^{(l)}} \) for some \( 1 \leq k < l \leq J \) then in view of the normalization condition \( [51] \) and [22] Proposition 5.2 one concludes (cf. Remark 5.2) that for any \( n \in \mathbb{Z} \) and for any \( \lambda \in \mathbb{C} \) we have that \( \zeta_n(\eta^{(k)})(\lambda, \varphi) = \zeta_n(\eta^{(l)})(\lambda, \varphi) \). This allows us to define for any \( n \in \mathbb{Z} \) the analytic map

\[
\zeta_n : \mathbb{C} \times U_{\text{iso}} \to \mathbb{C}
\]

such that for any \( \varphi \in U_{\text{iso}} \) and for any \( m \in \mathbb{Z} \),

\[
\frac{1}{2\pi} \int_{\Gamma_m} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} \, d\lambda = \delta_{nm}. \tag{53}
\]

Hence the following holds.

**Proposition 5.1.** Let \( \psi^{(1)} \in iL_e^2 \) and assume that \( \text{Spec}_e L(\psi^{(1)}) \) is simple. Then there exist a connected open neighborhood \( U_{\text{iso}} \) of \( \text{Iso}_\psi(\psi^{(1)}) \) in \( L_e^2 \) with \( U_{\text{iso}} \cap iL_e^2 \) connected, analytic maps \( \zeta_n : \mathbb{C} \times U_{\text{iso}} \to \mathbb{C}, n \in \mathbb{Z} \), and an integer \( R \in \mathbb{Z}_{\geq 0} \) such that the conditions (I1)-(I3) above hold with \( U_\psi \) replaced by \( U_{\text{iso}} \), \( \zeta_n(\psi) \) replaced by \( \zeta_n \), and \( R_\psi \) replaced by \( R \). In particular, for any \( \varphi \in U_{\text{iso}} \) and \( m, n \in \mathbb{Z} \) we have the normalization condition (53).
As in Section 3 for any \( n \in \mathbb{Z} \) and any \( \varphi \in U_{\text{iso}} \) define the \( n \)-th action

\[
I_n(\varphi) := \frac{1}{\pi} \int_{I_n} \frac{d\lambda}{\sqrt{\Delta(\lambda, \varphi)}} d\lambda.
\]  

(54)

One easily sees from Proposition 5.1, Corollary 5.1 and Lemma 5.3 that for any \( n \in \mathbb{Z} \), the map \( I_n : U_{\text{iso}} \to \mathbb{C} \) is analytic and real-valued when restricted to \( U_{\text{iso}} \cap iL^2_\varphi \). Next let us define the angle variables. To this end introduce for any \( n \in \mathbb{Z} \) the analytic subvariety of \( U_{\text{iso}} \),

\[
Z_n = \{ \varphi \in U_{\text{iso}} \mid \gamma_n^2(\varphi) = 0 \}.
\]  

(55)

For \( |n| > R \), the set \( Z_n \cap iL^2_\varphi \) is a real analytic submanifold in \( iL^2_\varphi \) of real codimension two. Since for any \( \varphi \in U_{\text{iso}} \) the periodic spectrum of \( L(\varphi) \) is simple inside the disk \( B_R \) we see that \( Z_n = \emptyset \) for any \( |n| \leq R \). Arguing as in the proof of Proposition 4.1 one shows that after shrinking \( U_{\text{iso}} \), if needed, one can analytically extend the angle variable \( \theta_n \), introduced near \( \psi^{(1)} \) in Proposition 4.3 to \( U_{\text{iso}} \setminus Z_n \) so that it is of the form

\[
\theta_n(\varphi) = \tilde{\beta}^n(\varphi) + \sum_{|k| > R} \beta^n_k(\varphi), \quad \beta^n_k(\varphi) = \int_{I_n} \frac{d\lambda}{\sqrt{\Delta(\lambda, \varphi)}} \forall |k| > R,
\]

\[
\tilde{\beta}^n(\varphi) = \sum_{|k| \leq R} \int_{I_n} \frac{d\lambda}{\sqrt{\Delta(\lambda, \varphi)}} \forall |k| > R.
\]  

(56)

defined modulo \( 2\pi \) on \( U_{\text{iso}} \setminus Z_n \), real valued when restricted to \( (U_{\text{iso}} \setminus Z_n) \cap iL^2_\varphi \), and analytic on \( U_{\text{iso}} \setminus Z_n \) when considered modulo \( \pi \) if \( |n| > R \) and modulo \( 2\pi \) if \( |n| \leq R \). By the arguments yielding Theorem 4.1(cf. Corollary 5.1) we then obtain

Theorem 5.1. Let \( \psi \in iL^2_\varphi \) and assume that \( \text{Spec}_p L(\psi) \) is simple and \( U_{\text{iso}} \) is the neighborhood of \( \text{Iso}_1(\psi) \) introduced above. Then for any \( k, n \in \mathbb{Z} \) we have \( \{I_n, I_k\} = 0 \) on \( U_{\text{iso}} \), \( \{\theta_n, I_k\} = \delta_{nk} \) on \( U_{\text{iso}} \setminus Z_n \), and \( \{\theta_n, \theta_k\} = 0 \) on \( U_{\text{iso}} \setminus (Z_n \cup Z_k) \). Furthermore, the action \( I_n \) and the angle \( \theta_n \) are real valued when restricted to \( U_{\text{iso}} \cap iL^2_\varphi \) and, respectively, \( (U_{\text{iso}} \setminus Z_n) \cap iL^2_\varphi \).

Remark 5.1. Note that the statements of Lemma 4.2, Lemma 4.6, and Proposition 4.3 (i) also hold in \( U_{\text{iso}} \).

We finish this Section by proving properties of the actions used in the subsequent Section for constructing Birkhoff coordinates in \( U_{\text{iso}} \).

Lemma 5.1. For any \( \varphi \in U_{\text{iso}} \cap iL^2_\varphi \) and for any \( |n| > R \) we have that \( I_n(\varphi) \leq 0 \). Moreover, \( I_n(\varphi) = 0 \) iff \( \lambda_n^+(\varphi) = \lambda_n^-(\varphi) \).

Proof of Lemma 5.1. We follow the arguments of the proof of Theorem 13.1 in [1]. For any \( \varphi \in U_{\text{iso}} \cap iL^2_\varphi \) and \( |n| > R \) with \( \lambda_n^+ = \lambda_n^- \), one has \( \lambda_n^+ = \lambda_n \) and
any \( \lambda \).

(i) For any Lemma 5.2.

By deforming \( \Gamma_n \) to \( g_n \) and denoting by \( g^+_n \) the arc \( g_n \) with the orientation determined by starting at \( \lambda_n^- \) one gets

\[
I_n = \frac{1}{\pi} \int_{g^+_n} \frac{\lambda \Delta(\lambda)}{i(-1)^{n+1}(4 - \Delta(\lambda))^2} \, d\lambda - \frac{1}{\pi} \int_{g^+_n} \frac{\lambda \Delta(\lambda)}{i(-1)^{n+1}(4 - \Delta(\lambda))^2} \, d\lambda.
\]

Moreover, for any \( \lambda \in g_n \) the quantity \((-1)^n \Delta(\lambda) \pm i \sqrt{4 - \Delta(\lambda)^2} \) is in the domain of the principal branch of the logarithm, \( \log z = \log |z| + i \arg z \) where \(-\pi < \arg(z) < \pi\). Integrating by parts one then gets

\[
I_n = -\frac{1}{\pi} \int_{g^+_n} \log \left((-1)^n \Delta(\lambda) + i \sqrt{4 - \Delta(\lambda)^2}\right) \, d\lambda + \frac{1}{\pi} \int_{g^+_n} \log \left((-1)^n \Delta(\lambda) - i \sqrt{4 - \Delta(\lambda)^2}\right) \, d\lambda.
\]

Let \( f_{\pm}(\lambda) := (-1)^n \Delta(\lambda) \pm i \sqrt{4 - \Delta(\lambda)^2} \) and note that for any \( \lambda \in g_n \), \( |f_{+}(\lambda)| = |f_{-}(\lambda)| = 2 \) whereas \( \arg f_{\pm}(\lambda) = 0 \) for \( \lambda \in \{\lambda_n^+\} \) and \( 0 < \pm \arg f_{\pm}(\lambda) < \pi \) for any \( \lambda \in g_n \setminus \{\lambda_n^\pm\} \). Hence

\[
I_n = -\frac{1}{\pi} \int_{g^+_n} (\arg f_{+}(\lambda) - \arg f_{-}(\lambda)) \, d\lambda < 0. \tag{57}
\]

Lemma 5.1 is applied to study the quotient \( I_n/\gamma_n^2 \). We have the following

**Lemma 5.2.**

(i) For any \( |n| > R \), the quotient \( I_n/\gamma_n^2 : U_{iso} \setminus \mathbb{Z}_n \to \mathbb{C} \) extends analytically to \( U_{iso} \) so that

\[
\frac{4I_n}{\gamma_n^2} = 1 + \ell_n^2, \quad |n| > R
\]

locally uniformly on \( U_{iso} \). Furthermore, \( I_n/\gamma_n^2 \) is real on \( U_{iso} \cap iL^2_r \).

(ii) For any \( |n| > R \) and \( \varphi \in U_{iso} \cap iL^2_r \),

\[
4I_n/\gamma_n^2 > 0.
\]

Furthermore, after shrinking \( U_{iso} \) if necessary, for any \( |n| > R \), the real part of \( 4I_n/\gamma_n^2 \) is bounded away from 0 uniformly in \( |n| > R \) and locally uniformly on \( U_{iso} \). Moreover, the square root \( \xi_n := \sqrt{4I_n/\gamma_n^2} \) satisfies the asymptotics

\[
\xi_n = 1 + \ell_n^2
\]

locally uniformly on \( U_{iso} \).
Proof of Lemma 5.3. We follow the arguments of the proof of Theorem 13.3 in [11]. (i) We show that for any \(|n| > R\) the quantity \(I_n/\gamma_n^2\) continuously extends to all of \(U_{\text{iso}}\) and its restriction to \(Z_n\) is weakly analytic. By [11] Theorem A.6], it then follows that \(I_n/\gamma_n^2\) is analytic on \(U_{\text{iso}}\). Let \(\varphi \in U_{\text{iso}} \setminus Z_n\) for some given \(|n| > R\). By the definition \([12]\) of the canonical root and the product representation of \(\Delta(\lambda)\) (cf. Lemma 2.8) one has for \(\lambda\) near \(\Gamma_n\),

\[
\frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \frac{\dot{\lambda}_n - \lambda}{iw_n(\lambda)} \chi_n(\lambda) \quad \text{and} \quad \chi_n(\lambda) = \prod_{k \neq n} \frac{\dot{\lambda}_k - \lambda}{w_k(\lambda)}
\]

where, by (27), \(w_k(\lambda) = \sqrt{\frac{(\lambda - \dot{\lambda}_k)(\lambda - \dot{\lambda}_k^+)}{\lambda - \lambda_k^+}}\). By the definition (53) of \(I_n\),

\[
I_n = \frac{i}{\pi} \int_{\Gamma_n} \frac{(\lambda - \dot{\lambda}_n)^2}{\sqrt{\Delta(\lambda)^2 - 4}} \chi_n(\lambda) \, d\lambda.
\]

The assumption \(\gamma_n \neq 0\), allows to make the substitution \(\lambda = \tau_n + z\gamma_n/2\), \(\tau_n = (\lambda_n^+ + \lambda_n^-)/2\), which in view of the definition (13) of the standard root leads to

\[
\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)} \big|_{z = 0} = \int_{-1}^1 \frac{\gamma_n}{2} \sqrt{1 - z^2}, \quad -1 \leq z \leq 1.
\]

Hence, with \(z_n = 2(\dot{\lambda}_n - \tau_n)/\gamma_n\),

\[
\frac{4I_n}{\gamma_n^2} = \frac{2}{\pi} \int_{-1}^1 \frac{(z - z_n)^2}{\sqrt{1 - z^2}} \chi_n(\tau_n + z\gamma_n/2) \, dz.
\]

By Lemma 2.1 (ii), \(z_n \to 0\) as \(\gamma_n \to 0\). Thus

\[
\frac{4I_n}{\gamma_n^2} \to \chi_n(\tau_n) \frac{2}{\pi} \int_{-1}^1 \frac{z^2}{\sqrt{1 - z^2}} \, dz = \chi_n(\tau_n).
\]

This shows that \(I_n/\gamma_n^2\) is continuous on all of \(U_{\text{iso}}\). Note that \(\chi_n(\tau_n) \neq 0\). Moreover, by the argument principle, \(\tau_n\) is analytic on \(U_{\text{iso}}\) and arguing as in the proof of [11] Lemma 12.7 one sees that \(\chi_n\) is analytic on \(D_n \times U_{\text{iso}}\). Hence the composition \(\chi_n(\tau_n)\) is analytic on \(U_{\text{iso}}\) and thus in particular weakly analytic on \(Z_n\). By [11] Theorem A.6, \(I_n/\gamma_n^2\) extends analytically to all of \(U_{\text{iso}}\). Arguing as in the proof of [11] Lemma 12.10 one sees that \(\chi_n(\lambda) = 1 + \ell_n^2\) for \(\lambda\) near the interval

\[
[\lambda_n^-, \lambda_n^+] = \{(1 - t)\lambda_n^- + t\lambda_n^+ \mid 0 \leq t \leq 1\}
\]

locally uniformly on \(U_{\text{iso}}\). By the asymptotics \(z_n = \gamma_n\ell_n^2\) (cf. Lemma 2.1 (ii)) it then follows that \(4I_n/\gamma_n^2 = 1 + \ell_n^2\) locally uniformly on \(U_{\text{ iso}}\).

(ii) By Lemma 2.2 one has for any \(\varphi \in U_{\text{iso}} \cap iL_n^2\), \(\gamma_n^2 \leq 0\). By Lemma 5.1 it then follows that for any \(|n| > R\) with \(\gamma_n \neq 0\) we have \(I_n/\gamma_n^2 > 0\) whereas if \(\gamma_n = 0\), one has by the proof of item (i) that \(4I_n/\gamma_n^2 = \chi_n(\tau_n) \neq 0\). By
the continuity of $I_n/\gamma_n^2$ on $U_\text{iso} \cap iL_r^2$ one then concludes that $I_n/\gamma_n^2 > 0$ on $U_\text{iso} \cap iL_r^2$. Furthermore, by the asymptotics established in item (i), $4I_n/\gamma_n^2 \to 1$ locally uniformly on $U_\text{iso}$. By shrinking $U_\text{iso}$ if necessary we can assure that $\text{Re}(4I_n/\gamma_n^2)$ is bounded away from 0 locally uniformly on $U_\text{iso}$ and uniformly in $|n| > R$. Then $\xi_n = \sqrt{4I_n/\gamma_n^2}$ is well defined and real analytic on $U_\text{iso}$. It is positive on $U_\text{iso} \cap iL_r^2$ and satisfies the asymptotics $1 + \ell_n^2$ locally uniformly on $U_\text{iso}$.

6 The pre-Birkhoff map and its Jacobian

In this Section we construct the pre-Birkhoff map $\Phi : U_\text{iso} \to \ell_r^2$ in an open neighborhood $U_\text{iso}$ of the isospectral set $\text{Iso}_n(\psi(1))$ of an arbitrary given potential $\psi(1) \in iL_r^2$ with simple periodic spectrum, using the action and angle variables introduced in Section 5. We then prove that the restriction $\Phi : U_\text{iso} \cap iL_r^2 \to \ell_r^2$ is a local diffeomorphism. Without further reference we will use the notations and results of the previous sections. For any $|n| > R$ and $\varphi \in U_\text{iso} \setminus Z_n$ we define

$$x_n := \frac{\xi_n \gamma_n}{\sqrt{2}} \cos \theta_n, \quad y_n := \frac{\xi_n \gamma_n}{\sqrt{2}} \sin \theta_n$$

(58)

where $\theta_n : U_\text{iso} \setminus Z_n \to \mathbb{C}$ is the $n$-th angle in Section 6 and $\xi_n : U_\text{iso} \to \mathbb{C}$ is the real-analytic non vanishing function introduced in Section 5. Recall from Lemma 5.2 that

$$4I_n = (\xi_n \gamma_n)^2$$

where the $n$-th action $I_n$ is defined by (54) and $\xi_n$ satisfies

$$\xi_n = 1 + \ell_n^2$$

(59)

locally uniformly in $U_\text{iso}$. Note that on $U_\text{iso} \cap iL_r^2$, $\xi_n$ is real valued and $\gamma_n \in i\mathbb{R}$ for any $|n| > R$. Since $\theta_n$, defined modulo $2\pi$, is real valued on $\left(U_\text{iso} \setminus Z_n\right) \cap iL_r^2$, it then follows that $x_n, y_n \in \mathbb{R}$ on $\left(U_\text{iso} \setminus Z_n\right) \cap iL_r^2$, $|n| > R$.

Using the arguments from [11, Section 16] we now show that $(x_n, y_n)_{|n| > R}$ can be analytically extended to $U_\text{iso}$ in $L_r^2$.

First note that due to the lexicographic ordering $\lambda_n^- \leq \lambda_n^+$, the $n$-th gap length $\gamma_n$ is not necessarily continuous on $U_\text{iso}$. On the other hand, by Proposition 4.2 the quantity $\beta_n^+$ is defined modulo $2\pi$ on $U_\text{iso} \setminus Z_n$ and real analytic when taken modulo $\pi$ whereas in view of the definition (70) of the angle $\theta_n$ and Proposition 4.1 the difference $\theta_n - \beta_n^+$ is real analytic on $U_\text{iso}$. We will first focus our attention on the complex functions

$$z_n^+ := \gamma_n e^{i\beta_n^+}, \quad z_n^- := \gamma_n e^{-i\beta_n^-}, \quad |n| > R.$$

Lemma 6.1. The functions $z_n^\pm = \gamma_n e^{\pm i\beta_n^\pm}$ are analytic on $U_\text{iso} \setminus Z_n$ for any $|n| > R$.\footnote{For simplicity of notation we will use the same symbol for the map $\Phi$ and its restriction to $U_\text{iso} \cap iL_r^2$.}
Proof of Lemma 6.1. We follow the arguments of the proof of Lemma 16.1 in [11]. Fix $|n| > R$ arbitrarily. Arguing as in [11, Proposition 7.5] one easily sees that locally around any potential in $U_{iso} \backslash \mathcal{Z}_n$, there exist analytic functions $\varphi^+_n$ and $\varphi^-_n$ such that the set equality $\{\varphi^-_n, \varphi^+_n\} = \{\lambda^-_n, \lambda^+_n\}$ holds. Let
\[
\tilde{\gamma}_n := \varphi^+_n - \varphi^-_n, \quad \tilde{\beta}_n := \int_{\varphi^-_n}^{\varphi^+_n} \frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} \, d\lambda \pmod{2\pi}.
\]
If $\varphi^-_n = \lambda^-_n$ (and then $\varphi^+_n = \lambda^+_n$) one has $\gamma_n = \tilde{\gamma}_n$ and $\beta_n = \tilde{\beta}_n$ whereas if $\varphi^-_n = \lambda^+_n$ (and hence $\varphi^+_n = \lambda^-_n$), in view of the normalization condition
\[
\int_{\lambda^-_n}^{\lambda^+_n} \frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} \, d\lambda \in \pi + 2\pi \mathbb{Z},
\]
on one has
\[
\gamma_n = -\tilde{\gamma}_n, \quad \beta_n = \int_{\lambda^-_n}^{\lambda^+_n} \frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} \, d\lambda = \tilde{\beta}_n + \pi \pmod{2\pi}.
\]
Thus in both cases $\gamma_n e^{\pm i\beta_n} = \tilde{\gamma}_n e^{\pm i\tilde{\beta}_n}$. As the right hand side of the latter identity is analytic the Lemma follows.

Next we study the limiting behavior of $z^+_n, |n| > R,$ as $\varphi$ approaches a potential $\psi \in U_{iso}$ with the $n$-th gap collapsed. This limit is different from zero when $\psi$ is in the set
\[
\mathcal{F}_n := \{\psi \in U_{iso} \mid \mu_n \notin G_n\}
\]
where $G_n = [\lambda^-_n, \lambda^+_n]$. Note that the set $\mathcal{F}_n$ is open. On $\mathcal{F}_n$, the sign function
\[
\varepsilon_n := \frac{\sqrt{\Delta(\mu_n)^2 - 4}}{\sqrt{\Delta(\mu_n)^2 - 4}}
\]
is well defined and locally constant.

Lemma 6.2. Let $|n| > R$ and $\psi \in \mathcal{Z}_n$. If $\varphi \in \mathcal{F}_n \backslash \mathcal{Z}_n$ tends to $\psi$, then
\[
\gamma_n e^{\pm i\beta_n} \to \begin{cases} 
2(\tau_n - \mu_n)(1 \pm \varepsilon_n)e^{\chi_n} & \text{if } \psi \in \mathcal{F}_n \cap \mathcal{Z}_n \\
0 & \text{if } \psi \in \mathcal{Z}_n \backslash \mathcal{F}_n
\end{cases}
\]
where
\[
\chi_n = \frac{Z_n(\tau_n) - Z_n(\lambda)}{\tau_n - \lambda} \, d\lambda, \quad Z_n(\lambda) = -\prod_{m \neq n} \sigma_m - \lambda \wedge m(\lambda),
\]
and $w_m(\lambda) = \sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}$. 

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Proof of Lemma 6.2. We follow the arguments of the proof of Lemma 16.2 in [11]. Without loss of generality we may choose for \( \varphi \in \mathcal{F}_n \setminus Z_n \) a path of integration in the integral of \( \beta^n \) which meets \( G_n \) only at the initial point \( \lambda_n^- \). Using the definition (60) of the sign \( \varepsilon_n \) and the estimate (62) of \( \sqrt{\Delta(\lambda)^2 - 4} \), we then can write

\[
\frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = -i\varepsilon_n Z_n(\lambda) w_n(\lambda)
\]

and get modulo \( 2\pi \),

\[
i\beta^n = i \int_{\lambda_n^->\lambda_n^-} \frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda = \varepsilon_n \int_{\lambda_n^->\lambda_n^-} \frac{Z_n(\lambda)}{w_n(\lambda)} d\lambda,
\]

where \( w_n(\lambda) \) is well defined as we consider a path of integration from \( \lambda_n^- \) to \( \mu_n \) inside \( D_n \) which meets \( G_n \) only at its initial point. We decompose the numerator \( Z_n(\lambda) \) into three terms,

\[
Z_n(\lambda) = (Z_n(\lambda) - Z_n(\tau_n)) + (Z_n(\tau_n) + 1) - 1
\]

and denote the corresponding integrals by \( o_n, v_n, \) and \( \omega_n \), respectively. The limit of the first term is straightforward. Note that \( Z_n \) is analytic in some neighborhood \( D_n \times V_\psi \subseteq \mathbb{C} \times L^2_\psi \) of \( (n\pi, \psi) \). Moreover, if \( \varphi \to \psi \), then \( \lambda_n^- n(\varphi) \to \tau_n(\psi) \), \( w_n(\lambda) \to \tau_n - \lambda \), and \( \mu_n(\varphi) \to \mu_n(\psi) \). Thus by the definition of \( \chi_n(\psi) \),

\[
o_n = \int_{\lambda_n^-}^{\mu_n} \frac{Z_n(\lambda) - Z_n(\tau_n)}{w_n(\lambda)} d\lambda \to \int_{\tau_n^-}^{\mu_n} \frac{Z_n(\lambda) - Z_n(\tau_n)}{\tau_n - \lambda} d\lambda = -\chi_n(\psi).
\]

For the second term we have in view of the identity (62) below and the estimate \( Z_n(\tau_n) + 1 = O(\gamma_n) \) by Lemma 6.3 below

\[
v_n = (Z_n(\tau_n) + 1) \int_{\lambda_n^->\lambda_n^-} \frac{1}{w_n(\lambda)} d\lambda \to 0.
\]

Now consider the third term. Using a limiting argument we can compute it modulo \( 2\pi i \) on \( \mathcal{F}_n \setminus Z_n \) by choosing the line segment \( \lambda = \tau_n + t\tau_n \) with \(-1 \leq t \leq \varrho_n \) as path of integration where \( \varrho_n = 2(\mu_n - \tau_n)/\gamma_n \). In case the interval \( [\lambda_n^-,\mu_n] \) intersects \( G_n \setminus \{\lambda_n^-\} \) it actually contains all of \( G_n \). One easily verifies that in this case the choice of the sign of \( w_n(\lambda) \) along \( G_n \) does not matter. We then get modulo \( 2\pi i \)

\[
\omega_n = \int_{\lambda_n^->\lambda_n^-} \frac{d\lambda}{w_n(\lambda)} = \int_{\lambda_n^->\lambda_n^-} \frac{d\lambda}{\sqrt{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} = \int_{-1}^{\varrho_n} \frac{dt}{\sqrt{(1-t)(-1-t)}}
\]

with \( \sqrt{(1-t)(-1-t)} = -t \sqrt{1-t^2} \) for \( |t| \to \infty \). Note that for \( \varphi \in \mathcal{F}_n \setminus Z_n \) one has \( \varrho_n \in \mathbb{C} \setminus [-1,1] \). We claim that

\[
e^{-\varepsilon_n \omega_n} = -\varrho_n + \varepsilon_n \sqrt{1-\varrho_n}(-1-\varrho_n).
\]
Indeed both sides of (63), viewed as functions of $\varrho_n$, are solutions of the initial value problem

$$\frac{f'(w)}{f(w)} = \frac{-\varepsilon_n}{\sqrt{(1 - w)(1 - w)}}, \quad f(-1) = 1, \quad w \in \mathbb{C} \setminus [-1, 1].$$

Now we compute the limit $\varphi \to \psi$. First consider the case where $\psi \in \mathcal{F}_n \cap Z_n$. Then $\mu_n - \tau_n$ does not converge to zero. This implies $\varrho_n^{-1} \to 0$ and further

$$\gamma_n e^{-\varepsilon_n \omega_n} = -\gamma_n \varrho_n + \varepsilon_n \gamma_n \sqrt{(1 - \varrho_n)(1 - \varrho_n)}$$

$$= -\gamma_n \varrho_n - \varepsilon_n \gamma_n \sqrt{1 - \varrho_n^2}$$

$$= 2(\tau_n - \mu_n)(1 + \varepsilon_n \sqrt{1 - \varrho_n^2})$$

$$\to 2(\tau_n - \mu_n)(1 + \varepsilon_n).$$

(64)

In the case where $\psi \in Z_n \setminus \mathcal{F}_n$, one has $\gamma_n \varrho_n = 2(\mu_n - \tau_n) \to 0$ and thus concludes that

$$\gamma_n e^{-\varepsilon_n \omega_n} = -\gamma_n \varrho_n + \varepsilon_n \gamma_n \sqrt{(1 - \varrho_n)(1 - \varrho_n)} \to 0.$$

(Actually, this case is included in the above result since it does not matter that $\varepsilon_n$ is not well defined outside of $\mathcal{F}_n$.) So in both cases we obtain

$$\gamma_n e^{-\varepsilon_n \omega_n} \to 2(\tau_n - \mu_n)(1 + \varepsilon_n).$$

Combining the results for all three integrals we obtain

$$\gamma_n e^{i\beta_n} = \gamma_n e^{-\varepsilon_n \omega_n} e^{\varepsilon_n (\sigma_n + v_n)} \to 2(\tau_n - \mu_n)(1 + \varepsilon_n) e^{\varepsilon_n \chi_n}. \quad (65)$$

This agrees with the claim for $z^+_{\varepsilon_n}$ for $\varepsilon_n = -1$ where it vanishes and for $\varepsilon_n = 1$, where $e^{\varepsilon_n \chi_n} = e^{\chi_n}$. For $z^-_{\varepsilon_n}$ we just have to switch the sign of $\varepsilon_n$ in (64) to obtain

$$\gamma_n e^{-i\beta_n} = \gamma_n e^{\varepsilon_n \omega_n} e^{-\varepsilon_n (\sigma_n + v_n)} \to 2(\tau_n - \mu_n)(1 - \varepsilon_n) e^{-\varepsilon_n \chi_n}.$$

In particular the limit vanishes for $\varepsilon_n = 1$. \qed

**Lemma 6.3.** For $\lambda \in G_n$, $|n| > R$, one has $Z_n(\lambda) = -1 + O(\gamma_n)$ locally uniformly on $\Gamma_{iso}$.

**Proof of Lemma 6.3.** We follow the arguments of the proof of Lemma 16.3 in [11]. In analogy to [61] we write

$$\frac{\zeta_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \frac{Z_n(\lambda)}{iw_n(\lambda)}, \quad Z_n(\lambda) = -\prod_{m \neq n} \frac{\sigma_m^\gamma - \lambda}{w_m(\lambda)}.$$

Integrating over $\Gamma_n$ and referring to Proposition 5.1 we obtain for $\tau \in G_n$,

$$1 = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{Z_n(\lambda)}{w_n(\lambda)} d\lambda = \frac{1}{2\pi i} Z_n(\tau) \int_{\Gamma_n} \frac{1}{w_n(\lambda)} d\lambda + \frac{1}{2\pi i} \int_{\Gamma_n} \frac{Z_n(\lambda) - Z_n(\tau)}{w_n(\lambda)} d\lambda.$$
Since \( \frac{1}{2\pi i} \int_{\Gamma_n} \frac{Z_n(\lambda) - Z_n(\tau)}{w_n(\lambda)} d\lambda = -1 \) we get

\[
1 = -Z_n(\tau) + \frac{1}{2\pi i} \int_{\Gamma_n} \frac{Z_n(\lambda) - Z_n(\tau)}{w_n(\lambda)} d\lambda = -Z_n(\tau) + O \left( |Z_n - Z_n(\tau)|_{G_n} \right)
\]

where the latter asymptotics follow from [11, Lemma 14.3]. Note that \( Z_n(\lambda) \) is analytic on \( D_n \) by [11, Corollary 12.8]. Since \( Z_n \) is bounded on \( D_n \) locally uniformly in \( \varphi \) and uniformly in \( n \) by [11, Lemma 12.10], it then follows that the same is true for \( \dot{Z}_n(\lambda), \lambda \in G_n \), by Cauchy’s estimate. Therefore

\[
\max_{\lambda \in G_n} |Z_n(\lambda) - Z_n(\tau)| \leq \max_{\lambda \in G_n} |\dot{Z}_n(\lambda)||\gamma_n| = O(\gamma_n)
\]

locally uniformly in \( \varphi \) and uniformly in \( n \). This proves the claim. \( \square \)

For any \( |n| > R \), we now extend \( z_n^\pm \) on all of \( U_{iso} \) as follows

\[
z_n^\pm = \begin{cases} 
2(\tau_n - \mu_n)(1 \pm \varepsilon_n)\chi_n & \text{on } Z_n \cap F_n, \\
0 & \text{on } Z_n \setminus F_n
\end{cases}
\]

where \( \chi_n \) is given by Lemma 6.2. To establish the proper target space of \( (x_n, y_n)_{n \geq 1} \) we need the following asymptotic estimates.

**Lemma 6.4.** For \( |n| > R \) one has \( z_n^\pm = O(|\gamma_n| + |\mu_n - \tau_n|) \) locally uniformly on \( U_{iso} \).

**Proof of Lemma 6.4.** We follow the arguments of the proof of Lemma 16.4 in [11]. From the proof of Lemma 6.2 in particular equations (64) and (65), one sees that for \( \varphi \) in \( F_n \setminus Z_n \),

\[
\gamma_n e^{i\beta_n} = \left( -\gamma_n \varrho_n + \varepsilon_n \gamma_n \sqrt{1 - \varrho_n}(-1 - \varrho_n) \right) e^{\varepsilon_n(v_n + o_n)}.
\]

In the case \( 2|\mu_n - \tau_n| \leq |\gamma_n| \), i.e., \( |\varrho_n| \leq 1 \), one has

\[
\left| -\gamma_n \varrho_n + \varepsilon_n \gamma_n \sqrt{1 - \varrho_n}(-1 - \varrho_n) \right| \leq 3|\gamma_n| \quad (66)
\]

while in the case \( 2|\mu_n - \tau_n| > |\gamma_n| \), i.e., \( |\varrho_n| > 1 \),

\[
\gamma_n e^{i\beta_n} = 2(\tau_n - \mu_n) \left( 1 + \varepsilon_n \sqrt{1 + \varrho_n^{-2}} \right) e^{\varepsilon_n(v_n + o_n)}
\]

yielding the estimate

\[
\left| 2(\tau_n - \mu_n) \left( 1 + \varepsilon_n \sqrt{1 + \varrho_n^{-2}} \right) \right| \leq 6|\mu_n - \tau_n|. \quad (67)
\]

The exponential term \( e^{\varepsilon_n(v_n + o_n)} \) is locally uniformly bounded in view of [11, Lemma 12.10]. So we get on \( F_n \setminus Z_n \),

\[
z_n^+ = O(|\gamma_n| + |\mu_n - \tau_n|). \quad (68)
\]
By Lemma 6.2, (67) continues to hold on $\mathcal{F}_n \cap \mathcal{Z}_n$. Furthermore, one easily verifies that (66) is also valid on $U_{iso} \setminus \mathcal{F}_n$ for any choice of $\varepsilon_n \in \{ \pm 1 \}$. Hence (68) holds in a locally uniform fashion on all of $U_{iso}$. The argument for $\gamma_n e^{-i\beta_n^*}$ is the same.

We are now ready to prove

**Proposition 6.1.** For any $|n| > R$, the functions $z_n^\pm$ are analytic on $U_{iso}$.

**Proof of Proposition 6.1.** We follow the arguments of the proof of Proposition 16.5 in [11]. First, we apply [11, Theorem A.6] to the functions $z_n^\pm$ on the domain $U_{iso}$ with the subvariety $\mathcal{Z}_n$. These functions are analytic on $U_{iso} \setminus \mathcal{Z}_n$ by Lemma 6.1. We claim that they are also continuous at points of $\mathcal{Z}_n$. First note that their restrictions to $\mathcal{Z}_n$ are continuous by inspection. Approaching a point in $\mathcal{Z}_n$ from within $\mathcal{F}_n \cup \mathcal{Z}_n$, the corresponding values $z_n^\pm$ converge to the ones of the limiting potential by Lemma 6.2. On the other hand, approaching a point in $\mathcal{Z}_n$ from outside of $\mathcal{F}_n \cup \mathcal{Z}_n$ one has $|\mu_n - \tau_n| \leq |\gamma_n|$ and thus $z_n^\pm = \gamma_n e^{\pm i\beta_n^*}$ converges to zero by Lemma 6.4. Thus the functions $z_n^\pm$ are continuous on all of $U_{iso}$. To show that their restrictions to $\mathcal{Z}_n$ are weakly analytic, let $D$ be a one-dimensional complex disk contained in $\mathcal{Z}_n$. If the center of $D$ is in $\mathcal{F}_n$, then the entire disk $D$ is in $\mathcal{F}_n$ if chosen sufficiently small. The analyticity of $z_n^\pm = \gamma_n e^{\pm i\beta_n^*}$ on $D$ is then evident from the above formula, the definition of $\chi_n$, and the local constancy of $\varepsilon_n$ on $\mathcal{F}_n$. If the center of $D$ does not belong to $\mathcal{F}_n$, then consider the analytic function $\mu_n - \tau_n$ on $D$. This function either vanishes identically on $D$, in which case $z_n^\pm$ vanishes identically, too. Or it vanishes in only finitely many points. Outside these points, $D$ is in $\mathcal{F}_n$, hence $z_n^\pm$ is analytic there. By continuity and analytic continuation, these functions are analytic on all of $D$.

For $\varphi \in U_{iso}$ we now define $\Phi(\varphi) = (x_n(\varphi), y_n(\varphi))_{n \in \mathbb{Z}}$ where for $|n| > R$,

\[
\begin{cases}
 x_n = \frac{\xi_n}{\sqrt{2}} \left( z_n^+ e^{-i(\theta_n - \beta_n^*)} + z_n^- e^{-i(\theta_n - \beta_n^*)} \right), \\
y_n = \frac{\xi_n}{\sqrt{8i}} \left( z_n^+ e^{-i(\theta_n - \beta_n^*)} - z_n^- e^{-i(\theta_n - \beta_n^*)} \right),
\end{cases}
\]

and for $|n| \leq R$,

\[
x_n = \sqrt{2I_n} \cos \theta_n, \quad y_n = \sqrt{2I_n} \sin \theta_n.
\]

Here the roots $\sqrt{2I_n}$ are defined as follows: First note that by adding constants to the actions and by shrinking the neighborhood $U_{iso}$, if necessary, we can ensure that $\text{Re}(I_n) \leq -1$ on $U_{iso}$ for any $|n| \leq R$. Then, $\sqrt{2I_n}$ is analytic on $U_{iso}$ where $\sqrt{\cdot}$ is the branch of the square root satisfying

\[
\sqrt{2I_n} = i \sqrt{-2I_n}
\]

on $U_{iso} \cap iL_0^2$. From the preceding asymptotic estimates it is then evident that $\Phi$ defines a continuous, locally bounded map into $\ell^2_c = \ell^2 \times \ell^2$ which, when
restricted to $U_{iso} \cap iL^2_r$, takes values in $i\ell^2_c$. Moreover, each component is real analytic. In what follows we identify the real Hilbert space $i\ell^2_c$ with $\ell^2(Z, i\mathbb{R}^2)$ by the $\mathbb{R}$-linear isomorphism

$$i\ell^2(Z, i\mathbb{R}^2) \to i\ell^2_c, \quad (i(u_n, v_n))_{n \in \mathbb{Z}} \mapsto (z_n(\varphi), w_n(\varphi))_{n \in \mathbb{Z}} \quad \text{(70)}$$

where $z_n = (v_n + iu_n)/\sqrt{2}$ and $w_n = -\bar{z}_{(-n)}$ for any $n \in \mathbb{Z}$. We also write $x_n$ for $iu_n$ and $y_n$ for $iv_n$. We have proved the following

**Proposition 6.2.** The map

$$\Phi : U_{iso} \cap iL^2_r \to i\ell^2_c, \quad \varphi \mapsto (x_n(\varphi), y_n(\varphi))_{n \in \mathbb{Z}},$$

is real analytic and extends to an analytic map $U_{iso} \to \ell^2_c$.

Using Theorem 5.1 we now can establish the following

**Proposition 6.3.** For any $m, n \in \mathbb{Z}$ and $\varphi \in U_{iso} \cap iL^2_r$,

$$\{x_m, x_n\} = 0, \quad \{x_m, y_n\} = -\delta_{mn}, \quad \{y_m, y_n\} = 0.$$

**Proof of Proposition 6.3.** Let $n \in \mathbb{Z}$. For any $|n| > R$, the set $\mathbb{Z}_n \cap iL^2_r$ is a real analytic submanifold of $U_{iso} \cap iL^2_r$ of codimension two whereas $\mathbb{Z}_n \cap iL^2_r = \emptyset$ for $|n| \leq R$ (cf. (55)). Since the coordinates $x_n$ and $y_n$ are smooth it suffices to check that for any $m \in \mathbb{Z}$ the claimed commutation relations hold on the subset $(iL^2_r \cap U_{iso}) \setminus (\mathbb{Z}_n \cup \mathbb{Z}_m)$. Then for any $m \in \mathbb{Z}$, on $(U_{iso} \setminus \mathbb{Z}_m) \cap iL^2_r$

$$x_m = i \sqrt{-2}i_m \cos \theta_m, \quad y_m = i \sqrt{-2}i_m \sin \theta_m.$$

Arguing as in the proof of [11] Theorem 18.8] one has by the chain rule

$$\{x_m, y_n\} = \partial_{\theta_m} x_m \partial_{\theta_n} y_n \{\theta_m, \theta_n\} + \partial_{\theta_m} x_m \partial_{I_m} y_n \{\theta_m, I_n\} + \partial_{I_m} x_m \partial_{\theta_n} y_n \{I_m, \theta_n\} + \partial_{I_m} x_m \partial_{I_m} y_n \{I_m, I_n\}.$$

By Theorem 5.1 it then follows that

$$\{x_m, y_n\} = (\partial_{\theta_m} x_m \partial_{I_m} y_n - \partial_{I_m} x_m \partial_{\theta_n} y_n) \delta_{mn}$$

$$= \left(-\sqrt{2}i_m \sin \theta_m \frac{1}{\sqrt{2}i_m} \sin \theta_n - \sqrt{2}i_m \cos \theta_m \frac{1}{\sqrt{2}i_m} \cos \theta_n\right) \delta_{mn}$$

$$= -\delta_{mn}.$$

All other commutation relations between coordinates are verified in a similar fashion. \hfill \Box

In the remaining part of this Section we prove that for any $\varphi \in U_{iso} \cap iL^2_r$, the Jacobian $d_{\varphi} \Phi$ of $\Phi$ is a Fredholm operator of index 0. First we need to introduce some more notation and establish some auxiliary results. Recall that
$\mathbb{Z}_n \cap iL^2_\tau, |n| > R$, is a real analytic submanifold of real codimension two (cf. (55)). Note that for $\varphi \in \mathbb{Z}_n \cap iL^2_\tau$ with $|n| > R$, the double periodic eigenvalue $\lambda_n^+ (\varphi) = \lambda_n^- (\varphi)$ has geometric multiplicity two and hence is also a Dirichlet eigenvalue. It turns out to be convenient to introduce for any $|n| > R$,

$$\mathcal{M}_n := \{ \varphi \in U_{iso} | \mu_n (\varphi) \in \{ \lambda_n^+ (\varphi) \} \}.$$  

Note that $\mathbb{Z}_n \cap iL^2_\tau \subseteq \mathcal{M}_n \cap iL^2_\tau$.

**Lemma 6.5.** For any $|n| > R$, $(U_{iso} \setminus \mathcal{M}_n) \cap iL^2_\tau$ is open and dense in $U_{iso} \cap iL^2_\tau$.

**Proof of Lemma 6.5.** Note that $\mathcal{M}_n = \{ \varphi \in U_{iso} | \chi_p(\mu_n, \varphi) = 0 \}$ where we recall that $\chi_p(\lambda, \varphi) = \Delta(\lambda, \varphi)^2 - 4$. Using that $\chi_p : \mathbb{C} \times L^2 \to \mathbb{C}$, $(\lambda, \varphi) \mapsto \chi_p(\lambda, \varphi)$, and for any $|n| > R$, $\mu_n : U_{iso} \to \mathbb{C}$, $\varphi \mapsto \mu_n(\varphi)$, are analytic it follows that the composition $F_n : U_{iso} \to \mathbb{C}$, $\varphi \mapsto \chi_p(\mu_n(\varphi), \varphi)$ is analytic. The claimed result follows if one can show that $F_n$ does not vanish identically on $U_{iso} \cap iL^2_\tau$.

Assume on the contrary that $F_n$ vanishes identically on $U_{iso} \cap iL^2_\tau$. By analyticity it then vanishes on all of $U_{iso}$. Actually, the $n$-th Dirichlet eigenvalue $\mu_n$ is well defined and analytic on $U_{iso} \cup U_{tn}$ where $U_{tn}$ is the tubular neighborhood introduced in Section 3. Thus $F_n$ defines an analytic function on $U_{iso} \cup U_{tn}$ which vanishes also on $U_{tn}$. Note that $U_{tn}$ contains potentials $\psi$ in $U_{iso} \cap iL^2_\tau$ near zero for which $\lambda_n^- < \lambda_n^+$ and $\{ \varphi \in L^2 | \operatorname{Spec}_p(L(\varphi)) = \operatorname{Spec}_p(L(\psi)) \} \subseteq U_{iso} \cap iL^2_\tau$. The vanishing of $F_n$ on $U_{tn} \cap L^2_\tau$ then contradicts [11, Corollary 9.4].

In a first step we establish asymptotics for the gradients of $z_n^\pm$ as $|n| \to \infty$. Since finite gap potentials are dense in $U_{iso} \cap iL^2_\tau$, it turns out that for our purposes it is sufficient to establish them for such potentials. Recall that for any $n \in \mathbb{Z}$ we have that $e_n^+ = (0, e^{-2\pi inx})$ and $e_n^- = (e^{2\pi inx}, 0)$.

**Lemma 6.6.** At any finite gap potential in $U_{iso} \cap iL^2_\tau$, one has

$$\partial z_n^\pm = -2e_n^\pm + \ell_n^2.$$  

These estimates hold uniformly on $0 \leq x \leq 1$ in the sense that $\| \partial z_n^\pm + 2e_n^\pm \|_{L^\infty} = \ell_n^2$.

**Proof of Lemma 6.6.** We follow the proof of [11, Lemma 17.1]. In view of Lemma 6.5 we may approximate a given potential $\psi \in \mathbb{Z}_n \cap iL^2_\tau, |n| > R$, by potentials in $iL^2_\tau \cap (U_{iso} \setminus \mathcal{M}_n)$. Then it can be approximated by potentials $\varphi$ in $iL^2_\tau \cap U_{iso}$ satisfying either $|\mu_n - \tau_n| \leq |\gamma_n|/2, \mu_n \neq \lambda_n^+, \lambda_n^-$ (case 1) or $|\mu_n - \tau_n| > |\gamma_n|/2$ (case 2). Both cases are treated in a similar fashion so we concentrate on case 1 only. As in (60) we define a sign $\varepsilon_n$ by

$$\varepsilon_n = \sqrt{\frac{\Delta^2(\lambda)}{4} - 4} \bigg|_{\lambda=\mu_n},$$

47
where the root \( \sqrt{\Delta^2(\lambda) - 4} = 2\prod_{m \in \mathbb{Z}} \frac{w_m(\lambda)}{w_m} \) is extended to \( G_n \) by extending \( w_n(\lambda) \) from the left of \( G_n \). Since by definition \( \zeta_n = -\frac{2}{\tau_n} \prod_{m \neq n} \frac{\sigma_m - \lambda}{\pi_m} \) one has

\[
\frac{\zeta_n(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} \bigg|_{\lambda = \mu_n} = -i \varepsilon_n \frac{Z_n(\lambda)}{w_n(\lambda)} \bigg|_{\lambda = \mu_n}
\]

Hence

\[
i \beta_n = \varepsilon_n \int_{\lambda_n}^{\mu_n} \frac{Z_n(\lambda)}{w_n(\lambda)} \, d\lambda
\]

and we are in the same situation as in the proof of Lemma 6.2. With the notation introduced there we conclude again that \( z_n^\pm = \gamma_n e^{i\beta_n} = \gamma_n e^{-\varepsilon_n \omega_n} e^{\varepsilon_n (\theta_n + v_n)} \) and

\[
\gamma_n e^{-\varepsilon_n \omega_n} = \gamma_n \theta_n + \varepsilon_n \gamma_n \sqrt{(1 - \theta_n)(-1 - \theta_n)}
\]

where \( \theta_n = 2(\mu_n - \tau_n)/\gamma_n \). As by assumption, \( 2|\mu_n - \tau_n| \leq |\gamma_n|, \mu_n \notin \{\lambda_n^\pm\} \)

and \( \gamma_n \in i\mathbb{R} > 0 \) one has \( |\theta_n| \leq 1, \theta_n \neq \pm 1 \). By extending the standard root (cf. 13) by continuity from above to the interval \([-1, 1]\), one obtains that

\[
\sqrt{(1 - \theta_n)(-1 - \theta_n)} = \begin{cases} 
-i \sqrt{1 - \theta_n^2} & \text{if } \text{Im}(\theta_n) \geq 0 \\
+i \sqrt{1 - \theta_n^2} & \text{if } \text{Im}(\theta_n) < 0.
\end{cases}
\]

As \(-i\gamma_n > 0 \) it then follows that \( \gamma_n e^{-\varepsilon_n \omega_n} = 2(\tau_n - \mu_n) + 2\varepsilon_n r_n \) where \( r_n = w_n(\mu_n) \). As both \( \gamma_n e^{-\varepsilon_n \omega_n} \) and \( e^{\varepsilon_n (\theta_n + v_n)} \) are analytic and as \( \theta_n + v_n \) vanishes for \( \varphi \to \psi \) (see proof of Lemma 6.2) we get by the product rule

\[
\partial z_n^\pm = \partial (\gamma_n e^{i\beta_n}) \to \partial (\gamma_n e^{-\varepsilon_n \omega_n}) = 2(\partial \tau_n - \partial \mu_n) + 2\varepsilon_n \partial r_n.
\]

To study the gradient of \( r_n \) we use the representation [11] at \( \mu_n \) to write

\[
i \varepsilon_n r_n = \frac{Z_n(\mu_n)}{\zeta_n(\mu_n)} \sqrt{\Delta^2(\mu_n) - 4} = \phi_n \cdot \delta(\mu_n)
\]

where \( \phi_n := \frac{Z_n(\mu_n)}{\zeta_n(\mu_n)} \), \( \delta(\mu_n) := \dot{m}_2(\mu_n) + \dot{m}_3(\mu_n) \) and, by the definition of the \(*\)-root,

\[
\sqrt{\Delta(\mu_n)^2 - 4} = \delta(\mu_n).
\]

By [11] Lemma 4.4 and the asymptotics of [11] Lemma 5.3

\[
i \partial \delta(\mu_n) = (-1)^n (e_n^+ - e_n^-) + \ell_n^2
\]

where these estimates hold uniformly for \( 0 \leq x \leq 1 \). In addition, by [11] Theorem 2.4, Lemma 5.3, \( \delta(\mu_n) = \ell_n^2 \) and \( \delta(\mu_n) = \ell_n^2 \). It then follows from [11] Lemma 7.6] that \( \delta(\mu_n) \theta_n = \ell_n^2 \). Moreover, by Lemma [6.3] \( Z_n(\mu_n) = -1 + O(\gamma_n) \) and by [11] Lemma C.4, \( \zeta_n(\mu_n) = -2(-1)^n + \ell_n^2 \) and thus \( 2\phi_n = (-1)^n + \ell_n^2 \) and with Cauchy’s estimate \( \partial \phi_n = O(1) \), leading to \( \delta(\mu_n) \partial \phi_n = \ell_n^2 \).

On the other hand

\[
2\phi_n \partial (\delta(\mu_n)) = 2\phi_n \partial \delta(\mu_n) + 2\phi_n \delta(\mu_n) \partial \mu_n
\]

\[
= (-1)^n (e_n^+ - e_n^-) \partial \delta(\mu_n) + \ell_n^2 = -i(e_n^+ - e_n^-) + \ell_n^2.
\]
Hence
\[ 2\varepsilon_n \partial r_n = -i\partial(2\phi_n \delta(\mu_n)) = e_n^+ - e_n^- + \ell_n^2. \]

Finally, by [11, Lemma 7.6] and by [11, Lemma 12.3] and its proof as well as Lemma 12.4 in [11]
\[ 2(\partial r_n - \partial \mu_n) = -(e_n^+ + e_n^-) + \ell_n^2. \]

Altogether we thus have proved that for finite gap potentials
\[ \partial z_n^+ = 2(\partial r_n - \partial \mu_n) + 2\varepsilon_n \partial r_n = -2e_n^+ + \ell_n^2. \]

Analogously, one has
\[ \partial z_n^- = 2(\partial r_n - \partial \mu_n) - 2\varepsilon_n \partial r_n = -2e_n^- + \ell_n^2. \]

Lemma 6.7. At any finite gap potential in \( U_{iso} \cap iL_r^2 \),
\[ \partial x_n = -\frac{1}{\sqrt{2}}(e_n^+ + e_n^-) + \ell_n^2, \quad \partial y_n = i\frac{1}{\sqrt{2}}(e_n^+ - e_n^-) + \ell_n^2. \]

Proof of Lemma 6.7. We follow the arguments in the proof of Theorem 17.2 in [11]. By the definition of \( x_n \) and \( y_n \), for any \( |n| > R \),
\begin{align*}
x_n &= \frac{\xi_n}{2\sqrt{2}} \left( z_n^+ e^{i(\theta_n - \beta_n^+)} + z_n^- e^{-i(\theta_n - \beta_n^+)} \right), \quad (71) \\
y_n &= \frac{\xi_n}{2\sqrt{2}i} \left( z_n^+ e^{i(\theta_n - \beta_n^-)} - z_n^- e^{-i(\theta_n - \beta_n^-)} \right). \quad (72)
\end{align*}

At a finite gap potential we have \( z_n^\pm = 0 \) for \( |n| \) sufficiently large, \( \xi_n = 1 + \ell_n^2 \) by (59) and \( \theta_n - \beta_n^\pm = O \left( \frac{1}{n} \right) \) by Remark 5.1 using that \( |\gamma_n| + |\mu_n - \tau_n| = 0 \) for all but finitely many \( m \). Furthermore by Lemma 4.2 and the asymptotics of \( \lambda_n^\pm \) and \( \mu_n \), the \((z_n^\pm)_{n\in\mathbb{Z}}\) are locally bounded in \( \ell^2 \). By applying the product rule and Cauchy’s estimate we thus obtain from the formulas (71) and (72) that
\[ \partial x_n = \frac{1}{2\sqrt{2}}(\partial z_n^+ + \partial z_n^-) + \ell_n^2, \quad \partial y_n = \frac{1}{2\sqrt{2}i}(\partial z_n^+ - \partial z_n^-) + \ell_n^2. \]

From Lemma 6.6 we then get the claimed asymptotics.

Now consider the Jacobian of \( \Phi \). At any \( \varphi \in U_{iso} \cap iL_r^2 \) it is the linear map given by
\[ d\varphi \Phi : iL_r^2 \to i\ell_r^2, \quad h \mapsto \left( (b_n^+, h)_r \right)_{n\in\mathbb{Z}}, \left( (b_n^-, h)_r \right)_{n\in\mathbb{Z}} \]
where
\[ b_n^\pm := \partial x_n, \quad b_n^- := \partial y_n \]
and $\langle \cdot, \cdot \rangle_r$ denotes the bilinear form

$$L^2_c \times L^2_c \to \mathbb{C}, \quad (h, g) \mapsto \int_0^1 (h_1 g_1 + h_2 g_2) dx.$$  

For any $n \in \mathbb{Z}$, introduce

$$d_n^+ := -\frac{1}{\sqrt{2}} (e_n^+ + e_n^-), \quad d_n^- := -\frac{1}{\sqrt{2}} i (e_n^+ - e_n^-).$$

As $d_n^+, d_n^-, n \in \mathbb{Z}$, represent a Fourier basis of $iL^2_r$, the linear map $F : iL^2_r \to i\ell^2_r$, $h \mapsto (\langle d_n^+, h \rangle_r, \langle d_n^-, h \rangle_r)_{n \in \mathbb{Z}}$ is a linear isomorphism. To prove the same for the Jacobian $d\varphi \Phi$ it therefore suffices to show that $B_\varphi := F^{-1} d\varphi \Phi : iL^2_r \to iL^2_r$ is a linear isomorphism for any $\varphi$ in $U_{iso} \cap iL^2_r$. Clearly, $B_\varphi$ is continuous in $\varphi$ by the analyticity of $\Phi$ and is given by

$$B_\varphi h = \sum_{n \in \mathbb{Z}} \langle b_n^+, h \rangle_r d_n^+ + \langle b_n^-, h \rangle_r d_n^-.$$  

Its adjoint $A_\varphi$ with respect to $\langle \cdot, \cdot \rangle_r$ is then a bounded linear operator on $iL^2_r$ which also depends continuously on $\varphi$ and is given by

$$A_\varphi h = \sum_{n \in \mathbb{Z}} \langle d_n^+, h \rangle_r b_n^+ + \langle d_n^-, h \rangle_r b_n^-.$$  

Moreover, $B_\varphi$ is a linear isomorphism if and only if $A_\varphi$ is. We obtain the following

**Lemma 6.8.** For any $\varphi \in U_{iso} \cap iL^2_r$, the differential $d\varphi \Phi$ is a linear isomorphism if and only if the operator $A_\varphi$ is a linear isomorphism. The latter is a compact perturbation of the identity on $iL^2_r$.

**Proof of Lemma 6.8.** We follow the arguments of the proof of Lemma 17.3 in [11]. It remains to prove the compactness claim. At any finite gap potential in $U_{iso} \cap iL^2_r$ we have by Lemma 6.7

$$\sum_{n \in \mathbb{Z}} \| (A_\varphi - Id) d_n^+ \|^2 = \sum_{n \in \mathbb{Z}} \| b_n^+ - d_n^+ \|^2 < \infty.$$  

Thus $A_\varphi - Id$ is Hilbert-Schmidt and therefore compact. Since $A_\varphi$ depends continuously on $\varphi$ and finite gap potentials are dense in $U_{iso} \cap iL^2_r$ we see that $A_\varphi - Id$ is compact for any $\varphi \in U_{iso} \cap iL^2_r$.  

We summarize the main results of this Section as follows.
Theorem 6.1. The map \( \Phi : U_{\text{iso}} \cap iL^2_r \rightarrow i\ell^2_r, \varphi \mapsto (x_n(\varphi), y_n(\varphi))_{n \in \mathbb{Z}} \) is real analytic and for any \( m, n \in \mathbb{Z} \) and \( \varphi \in U_{\text{iso}} \cap iL^2_r \),

\[
\{x_m, x_n\} = 0, \quad \{x_m, y_n\} = -\delta_{mn}, \quad \{y_m, y_n\} = 0.
\]

Furthermore, for any \( \varphi \in U_{\text{iso}} \cap iL^2_r \) we have that \( d_{\varphi} \Phi : iL^2_r \rightarrow i\ell^2_r \) is a linear isomorphism.

Proof of Theorem 6.1. In view of Proposition 6.2 and Proposition 6.3 it remains to prove that for any \( \varphi \in U_{\text{iso}} \cap iL^2_r \) the differential \( d_{\varphi} \Phi : iL^2_r \rightarrow i\ell^2_r \) is a linear isomorphism. By Lemma 6.8 it suffices to show that \( A_{\varphi} : iL^2_r \rightarrow iL^2_r \) is such a map. This follows from Proposition 6.3, Lemma 6.8, and Lemma F.7 in [11]. \( \square \)

7 Proof of Theorem 1.1

In this Section we prove Theorem 1.1 stated in the Introduction. Recall that the map

\( \Phi : U_{\text{iso}} \cap iL^2_r \rightarrow i\ell^2_r \)  

(74)

constructed in Theorem 6.1 is a canonical local diffeomorphism. The map \( \Psi \) of Theorem 1.1 will be obtained from \( \Phi \) by a slight adjustment to ensure that \( \Psi \) is one-to-one. We begin with some preliminary considerations. Let us first study isospectral sets of the ZS operator with potentials in \( iL^2_r \). By Proposition 2.2 in [24], for any \( \psi \in iL^2_r \), the isospectral set \( \text{Iso}(\psi) \) is compact. Furthermore, recall that any connected component in a topological space is closed. If the space is locally path connected, then its connected components are also open (see e.g. [32]). We have

Lemma 7.1. For any \( \psi \in iL^2_r \) with simple periodic spectrum the isospectral set \( \text{Iso}(\psi) \) is locally path connected. In particular, the connected components of \( \text{Iso}(\psi) \) are open and closed. Since \( \text{Iso}(\psi) \) is compact we conclude that \( \text{Iso}(\psi) \) has finitely many connected components.

Proof of Lemma 7.1. It is enough to proof that \( \text{Iso}(\psi) \) is locally path connected. First note that we can assume without loss of generality that \( \psi \in \text{Iso}_0(\psi^{(1)}) \) where \( \psi^{(1)} \in U_{\text{iso}} \cap iL^2_r \) is the potential with simple periodic spectrum used in the construction of the map (74) in Section 6. Since \( \Phi \) is a local diffeomorphism we can find an open neighborhood \( U_\psi \) of \( \psi \) in \( U_{\text{iso}} \cap iL^2_r \) and an open neighborhood \( V_{p^0} \) of \( p^0 := \Phi(\psi) \) in \( i\ell^2_r \) such that

\( \Phi : U_\psi \rightarrow V_{p^0} \)

is a diffeomorphism. Here \( p^0 = (p^0_n)_{n \in \mathbb{Z}} \) with \( p^0_n = i(u^0_n, v^0_n) \) and the neighborhood \( V_{p^0} \) of \( p^0 \) in \( i\ell^2_r \) is chosen of the form

\( V_{p^0} = B_{|n| \leq R'}(p^0) \times B_{|n| > R'}(0) \)  

(75)
where \( R', \delta'_0, \delta''_0, \epsilon_0 > 0 \) are appropriately chosen parameters,

\[
B^{\delta'_0, \delta''_0}_{|n| \leq R'}(p^0) := \prod_{|n| \leq R'} \left\{ p_n = i(u_n, v_n) \in i \mathbb{R}^2 \left| |p_n^0| - |p_n| < \delta'_0, |\theta_n - \theta_0^n| < \delta''_0 \right\},
\]

\[
B^{\epsilon_0}_{|n| > R'}(0) := \left\{ p_n = i(u_n, v_n)_{|n| > R'} \in i\ell_r^2 \left| \left( \sum_{|n| > R'} |p_n|^2 \right)^{1/2} < \epsilon_0 \right\},
\]

and \(|p_n| = \sqrt{u_n^2 + v_n^2}\) and \(\theta_n\) are the polar coordinates of the point \((u_n, v_n)\) in the Euclidean plane \(\mathbb{R}^2\). By construction, for any \(\varphi \in U_{\text{iso}} \cap i\ell_r^2\) and \(p = \Phi(\varphi)\) we have that \(iu_n = x_n(\varphi)\) and \(iv_n = y_n(\varphi)\) and hence

\[
\frac{1}{2} |p_n|^2 = -I_n(\varphi) = -\frac{1}{4} (\xi_n \gamma_n)^2 \geq 0 \quad \text{and} \quad \theta_n = \theta_n(\varphi).
\]

Since \(\psi\) has simple periodic spectrum we have that \(|p_n| > 0\) for any \(n \in \mathbb{Z}\). We will assume that \(0 < \delta'_0 < |p_n^0|\) for \(|n| \leq R'\) and \(0 < \delta''_0 < \pi\).

**Remark 7.1.** Here we used that for any \(p^0 \in i\ell_r^2\) and for any open neighborhood \(W_{\varphi^0}\) of \(p^0\) in \(i\ell_r^2\) there exists an open neighborhood \(V_{\varphi^0}\) of \(p^0\), \(V_{\varphi^0} \subseteq W_{\varphi^0}\) of the form (75). An important property of \(V_{\varphi^0}\) is that its “tail” component \(B^{\epsilon_0}_{|n| > R'}(0)\) is a ball in \(i\ell_r^2\) centered at zero. We will call such a neighborhood a tail neighborhood of \(p^0\) in \(i\ell_r^2\).

Note that the action variables \(I_n, n \in \mathbb{Z}, \) in \(U_{\text{iso}}\) constructed in Section 5 are constant on isospectral potentials. This follows from the fact that the contours \(\Gamma_n, n \in \mathbb{Z},\) used in the definition of the actions on \(U_{\text{iso}}\) are fixed. Hence

\[
\Phi(\text{Iso}(\psi) \cap U_{\psi}) \subseteq \text{Tor}(p^0) \cap V_{\varphi^0}
\]

where for \(q^0 \in i\ell_r^2\) we set

\[
\text{Tor}(q^0) := \left\{ (i(u_n, v_n))_{n \in \mathbb{Z}} \in i\ell_r^2 \left| u_n^2 + v_n^2 = |q_n^0|^2, n \in \mathbb{Z} \right. \right\}.
\]

By a slight abuse of terminology we refer to \(\text{Tor}(q^0)\) as a torus of dimension \(#\{n \in \mathbb{Z}| |q_n^0| > 0\}\). Note that \(\text{Tor}(p^0)\) is a compact set in \(i\ell_r^2\) that is an infinite product of circles whereas the set \(\text{Tor}(p^0) \cap V_{\varphi^0}\) is the product of \(2R' + 1\) arcs

\[
\times \left\{ i(u_n, v_n)_{|n| \leq R'} \left| |p_n| = |p_n^0|, |\theta_n - \theta_0^n| < \delta''_0 \right. \right\}
\]

times the infinite product of circles

\[
\left\{ i(u_n, v_n)_{|n| > R'} \left| |p_n| = |p_n^0|, |n| > R' \right. \right\}.
\]

It follows from Lemma 8.3 in [11] and the fact that the actions are defined only in terms of the discriminant that \(\{I_n, \Delta(\lambda)\} = 0\) on \(U_{\text{iso}} \cap i\ell_r^2\) for any \(n \in \mathbb{Z}\) and \(\lambda \in \mathbb{C}\). This implies that for any \(n \in \mathbb{Z}\) the Hamiltonian vector field \(X_{I_n}\).
corresponding to the action variables $I_n$ in $U_{iso} \cap iL_2^r$ is isospectral, i.e. for any $\varphi \in U_{iso} \cap iL_2^r$ the integral trajectory of $X_{I_n}$ in $U_{iso} \cap iL_2^r$ with initial data at $\varphi$ lies in $Iso(\varphi)$. In view of the commutation relations
\[
\{\theta_n, I_m\} = \delta_{nm}, \quad \{I_n, I_m\} = 0, \quad n, m \in \mathbb{Z},
\]
the fact that $\Phi : U_\psi \to V_{\psi^0}$ is a canonical diffeomorphism and (78), one easily sees from the closedness of $Iso(\psi)$ that the set $Iso(\psi) \cap U_\psi$ is isospectral (via $\Phi$) to the product of the sets (80) and (81). Since this set is locally path connected so is $Iso(\psi)$.

Given a non-empty subset $A$ of $iL_2^r$ and $\delta > 0$ denote by $B_\delta(A)$ the open $\delta$-tubular neighborhood of $A$,
\[
B_\delta(A) := \bigcup_{\varphi \in A} B_\delta(\varphi),
\]
where $B_\delta(\varphi)$ denotes the open ball of radius $\delta$ in $iL_2^r$ centered at $\varphi$. In view of Lemma 7.1, the isospectral set $Iso(\psi)$ of any potential $\psi$ with simple periodic spectrum consists of finitely many connected components. This implies that there exists $\delta_0 \equiv \delta_0(\psi) > 0$ such that for any $0 < \delta < \delta_0$ the $\delta$-tubular neighborhoods of the different connected components of $Iso(\psi)$ do not intersect. We have the following Lyapunov type stability property of $Iso(\psi)$.

**Lemma 7.2.** Assume that $\psi \in iL_2^r$ has simple periodic spectrum. Then, for any $0 < \delta < \delta_0$ there exists $0 < \delta_1 \leq \delta$ such that for any $\varphi \in B_{\delta_1}(Iso_0(\psi))$, one has $Iso_0(\varphi) \subseteq B_\delta(Iso(\psi))$. Since $Iso_0(\varphi)$ is connected and $0 < \delta \leq \delta_0$ we conclude from the choice of $\delta_0 > 0$ that $Iso_0(\varphi) \subseteq B_\delta(Iso_0(\psi))$.

**Proof of Lemma 7.2.** Take $\psi \in iL_2^r$ with simple periodic spectrum and choose $0 < \delta < \delta_0$. We will prove the statement by contradiction. Assume that the statement of Lemma 7.2 does not hold. Then, there exist two sequence $(\psi_k)_{k \geq 1}$ and $(\tilde{\psi}_k)_{k \geq 1}$ in $iL_2^r$ such that
\[
\tilde{\psi}_k \in Iso_0(\psi_k) \quad \text{and} \quad \tilde{\psi}_k \notin B_\delta(Iso(\psi)),
\]
and a sequence $(\tilde{\psi}_k^*)_{k \geq 1}$ in $Iso_0(\psi)$ such that $\|\psi_k^* - \psi_k\| \to 0$ as $k \to \infty$. By using the compactness of $Iso_0(\psi)$ and then passing to subsequences if necessary we obtain that there exists $\psi^* \in Iso_0(\psi)$ such that
\[
\psi_k \to \psi^* \quad \text{as} \quad k \to \infty.
\]
Since by Proposition 2.3 in [24] the $L^2$-norm is a spectral invariant of the ZS operator for potentials in $iL_2^r$, we conclude from (83) that for any $k \geq 1$, $\|\tilde{\psi}_k\| = |\|\psi_k\||$. This together with (84) then implies that
\[
\|\tilde{\psi}_k\| \to |\|\psi^*\|| \quad \text{as} \quad k \to \infty.
\]
The second relation in (83) then implies that \( \Delta(\lambda, \tilde{\psi}_k) \to \Delta(\lambda, \tilde{\psi}) \) as \( k \to \infty \). By (83), for any \( k \geq 1 \) and for any \( \lambda \in \mathbb{C} \),
\[
\Delta(\lambda, \psi_k) \to \Delta(\lambda, \psi^*) \text{ as } k \to \infty.
\]
On the other side, by (83), we conclude that for any \( k \geq 1 \) and for any \( \lambda \in \mathbb{C} \),
\[
\Delta(\lambda, \tilde{\psi}_k) = \Delta(\lambda, \psi_k).
\]
The three displayed formulas above then imply that \( \Delta(\lambda, \tilde{\psi}) = \Delta(\lambda, \psi^*) \) for any \( \lambda \in \mathbb{C} \). Hence,
\[
\tilde{\psi} \in \text{Iso}(\psi^*) = \text{Iso}(\psi).
\]
By Proposition 2.3 in [24], we obtain \( \|\tilde{\psi}\| = \|\psi^*\| \) which, in view of (83), implies that
\[
\|\tilde{\psi}_k\| \to \|\tilde{\psi}\| \text{ as } k \to \infty.
\]
Since \((\tilde{\psi}_k)_{k \geq 1}\) converges weakly to \( \tilde{\psi} \) in \( iL_r^2 \), we conclude that
\[
\tilde{\psi}_k \to \tilde{\psi} \text{ as } k \to \infty.
\]
The second relation in (83) then implies that \( \tilde{\psi} \notin B_\delta(\text{Iso}(\psi)) \) which contradicts (86).

**Remark 7.2.** Assume that \( \psi \in iL_r^2 \) has simple periodic spectrum. It follows from the proof of Lemma [7.4] that there exist an open neighborhood \( U_\psi \) of \( \psi \) in \( U_{i\ell} \cap iL_r^2 \) and a tail neighborhood \( V_{p^0} \) of \( p^0 := \Phi(\psi) \) in \( i\ell_r^2 \) with parameters \( R' > 0, 0 < \delta' < |p^0| \) for \( |n| \leq R' \), \( 0 < \delta'' < \pi \), and \( \epsilon_0 > 0 \) (see (75)) such that \( \Phi : U_\psi \to V_{p^0} \) is a diffeomorphism and for any \( \varphi \in U_\psi \) we have that \( \Phi : U_\psi \to V_{p^0} \) maps the set \( \text{Iso}_\alpha(\varphi) \cap U_\psi \) bijectively onto the set \( \text{Tor}(\Phi(\varphi)) \cap V_{p^0} \).

Now, let \( \psi := \psi^{(1)} \) and \( p^0 := \Phi(\psi) \). In view of the compactness of \( \text{Iso}_\alpha(\psi) \) and Remark [7.2], we can construct a finite set of open neighborhoods \( U_{\psi_j} \) of \( \psi_j \in \text{Iso}_\alpha(\psi) \) in \( U_{i\ell} \cap iL_r^2 \), \( 1 \leq j \leq N \), such that for any \( 1 \leq j \leq N \), \( \Phi : U_{\psi_j} \to V_{p^0} \) is a diffeomorphism onto a tail neighborhood \( V_{p^0} \) of \( \tilde{p}^j := \Phi(\tilde{\psi}_j) \) in \( i\ell_r^2 \). In what follows we set
\[
U_{i\ell} \cap iL_r^2 = \bigcup_{j=1}^N U_{\psi_j}.
\]
Then, by Lemma [7.2] we can choose \( 0 < \delta < \delta_0 \) such that \( B_\delta(\text{Iso}_\alpha(\psi)) \subseteq U_{i\ell} \cap iL_r^2 \) and \( \text{Iso}_\alpha(\varphi) \subseteq U_{i\ell} \cap iL_r^2 \) for any \( \varphi \in B_\delta(\text{Iso}_\alpha(\psi)) \). By Remark [7.2]
Lemma 7.3. For any \( U \) of \( \psi \) in \( U_{iso} \cap iL_2^2 \) and a neighborhood \( V_{p^0} \) of \( p^0 := \Phi(\psi) \) in \( iL_2^2 \) such that \( \Phi : U \rightarrow V_{p^0} \) is a diffeomorphism, \( V_{p^0} \) is a tail neighborhood

\[
V_{p^0} = B_{|n| \leq R'}(p^0) \times B_{|n| > R'}(0)
\]

(88)

with parameters \( R' > 0, 0 < \delta_0 < |p_n^0| \), and \( \epsilon_0 > 0 \), and \( B_{|n| \leq R'}(p^0) \) and \( B_{|n| > R'}(0) \) are given by (76) and (77). By taking the parameters \( \delta_0', \delta_0'' > 0 \) smaller and \( R' > 0 \) larger if necessary we can ensure that \( U_{\psi} \subseteq B_{|n| \leq R'}(0) \) and hence for any \( \varphi \in U_{\psi} \) we have that

\[
Iso_o(\varphi) \subseteq U_{iso} \cap iL_2^2.
\]

(89)

In view of the compactness of the isospectral component \( Iso_o(\varphi) \) for any \( \varphi \in iL_2^2 \) and the fact that the flow of the Hamiltonian vector field \( X_{I_n} \) corresponding to the action variable \( I_n, n \in \mathbb{Z} \), is isospectral (see the proof of Lemma 7.1), we conclude from (89) that the integral trajectory of \( X_{I_n} \) with initial data \( \varphi \in U_{\psi} \) is defined for all \( t \in \mathbb{R} \). Moreover, it follows from the definition (2) of the Poisson bracket on \( iL_2^2 \) that \( X_{I_n} : U_{iso} \cap iL_2^2 \rightarrow iL_2^2, n \in \mathbb{Z} \), is an analytic vector field on \( U_{iso} \cap iL_2^2 \). By (22) the vector fields \( X_{I_n} \) commute for any \( m, n \in \mathbb{Z} \). Further, consider the direct product of \( 2R' + 1 \) open intervals

\[
J_{|n| \leq R'} := \bigtimes_{|n| \leq R'} \left\{ p_n = i(u_n, v_n) \in i\mathbb{R}^2 \mid |p_n^0| - |p_n| < \delta_0', \theta_n = \theta_0^0 \right\}
\]

(90)

and denote by \( T_p \subseteq U_{\psi} \) the preimage of

\[
T_{p^0} := J_{|n| \leq R'} \times B_{|n| > R'}(0) \subseteq V_{p^0}
\]

(91)

under the diffeomorphism \( \Phi : U_{\psi} \rightarrow V_{p^0} \). Since \( \Phi : U_{iso} \rightarrow iL_2^2 \) is canonical we have that (see (22))

\[
\Phi_*(X_{I_n}) = \partial_{\theta_n}, \quad n \in \mathbb{Z}.
\]

(92)

In particular, the vector fields \( X_{I_n}, |n| \leq R' \), are transversal to the submanifold \( T_{\psi} \) in \( U_{\psi} \). Now, take an arbitrary \( \varphi \in T_{\psi} \) and consider the orbit \( G(\varphi) := \{G^\tau(\varphi) \mid \tau \in \mathbb{R}^{2R'+1}\} \) where

\[
G^\tau(\varphi) := G^{(\tau_{-R'})}_{X_{I_{(\tau)}}} \circ \cdots \circ G^{(\tau_{R'})}_{X_{I_{R}}} (\varphi), \quad \tau = (\tau_{-R'}, \ldots, \tau_{R'}) \in \mathbb{R}^{2R'+1},
\]

(93)

and \( G^{(\tau)}_{X_{I_{n}}} \) with \( |n| \leq R' \) is the isospectral flow corresponding to the Hamiltonian vector field \( X_{I_n} \). Since the flows of \( X_{I_n}, n \in \mathbb{Z} \), are isospectral and commute we conclude from (89) that for any \( \varphi \in T_{\psi} \subseteq U_{\psi} \) and for any \( \tau \in \mathbb{R}^{2R'+1} \), \( G^\tau(\varphi) \) in (93) is well defined and \( G(\varphi) \subseteq Iso_o(\varphi) \).

**Lemma 7.3.** For any \( \varphi \in T_{\psi} \) the orbit \( G(\varphi) \) is a compact smooth submanifold of \( U_{iso} \cap iL_2^2 \) of dimension \( 2R' + 1 \). Moreover, for \( \varphi_1, \varphi_2 \in T_{\psi} \) we have that \( G(\varphi_1) \cap G(\varphi_2) = \emptyset \) if \( \varphi_1 \neq \varphi_2 \).
Proof of Lemma 7.3. To see that \( G(\varphi), \varphi \in \mathcal{T}_\psi \), is compact let \((\varphi_k)_{k \geq 1}\) be a sequence in \( G(\varphi) \). Since the flows \( X_{I_n}, n \in \mathbb{Z} \), are isospectral, we have that \( G(\varphi) \subseteq \text{Iso}_n(\varphi) \). We then conclude from the compactness of \( \text{Iso}_n(\varphi) \) that there exist \( \varphi^* \in \text{Iso}_n(\varphi) \) and a subsequence of \((\varphi_k)_{k \geq 1}\), denoted again by \((\varphi_k)_{k \geq 1}\), such that
\[
\varphi_k \to \varphi^*, \quad \text{as} \quad k \to \infty.
\]
(94)

In view of (87) and (89), \( \varphi^* \in \tilde{U}_{\tilde{\psi}} \cap \text{Iso}_n(\varphi) \) where \( \tilde{U}_{\tilde{\psi}} \) is one of the neighborhoods appearing in (87). Since \( \Phi : \tilde{U}_{\tilde{\psi}} \to \tilde{V}_{\tilde{\mu}} \) is a diffeomorphism and since \( \tilde{V}_{\tilde{\mu}} \) is a tail neighborhood, we conclude that \( \Phi(\varphi^*) \in \tilde{V}_{\tilde{\mu}} \cap \text{Tor}(\Phi(\varphi)) \) (see Remark 7.2). It follows from (94) that there exists \( k_0 \geq 1 \) so that \( \Phi(\varphi_k) \in \tilde{V}_{\tilde{\mu}} \cap \text{Tor}(\Phi(\varphi)) \) for any \( k \geq k_0 \). This implies that \( \varphi^* = G^\tau(\varphi_k) \) for some \( \tau \in \mathbb{R}^{2R'+1} \) where \( \varphi_k \in G(\varphi) \). In particular, we see that \( \varphi^* \in G(\varphi) \) and hence \( G(\varphi) \) is compact. The coordinates \((\theta_{-\tau'},...,-\theta_{\tau'})\) on \( \tilde{V}_{\tilde{\mu}} \cap \text{Tor}(\Phi(\varphi)) \), \( 1 \leq j \leq \tilde{N} \), define a smooth (in fact, real analytic) submanifold structure on \( G(\varphi) \) in \( U_{\text{iso}} \cap iL^2_t \). It remains to prove the last statement of the Lemma. Take \( \varphi_1, \varphi_2 \in \mathcal{T}_\psi \) so that \( \varphi_1 \neq \varphi_2 \). Then, in view of the definition of \( \mathcal{T}_\psi \), either there exists \( |n_0| > R' \) such that
\[
(x_{n_0} , y_{n_0})|_{\varphi_1} \neq (x_{n_0} , y_{n_0})|_{\varphi_2}
\]
(95)
or there exists \( |n_0| \leq R' \) such that
\[
I_{n_0}(\varphi_1) \neq I_{n_0}(\varphi_2).
\]
(96)

Since the vector fields \( X_{I_n} \) with \(|n| \leq R' \) preserve the functions \( x_n , y_n \) with \(|n| > R' \) and the functions \( I_n \) with \(|n| \leq R' \) we conclude that for any \( \tau, \mu \in \mathbb{R}^{2R'+1} \) the relations (95) or (96) holds with \( \varphi_1 \) and \( \varphi_2 \) replaced respectively by \( G^\tau(\varphi_1) \) and \( G^\mu(\varphi_2) \). This implies that \( G(\varphi_1) \cap G(\varphi_2) = \emptyset \). \( \square \)

Let us now consider the following set in \( iL^2_t \),
\[
\mathcal{W} := \bigcup_{\varphi \in \mathcal{T}_\psi} G(\varphi).
\]
(97)

Lemma 7.4. \( \mathcal{W} \) is an open neighborhood of \( \psi \) in \( iL^2_t \) that is invariant with respect to the flows of the Hamiltonian vector fields \( X_{I_n}, n \in \mathbb{Z} \).

Proof of Lemma 7.4. It follows from the definition (89) of the tail neighborhood \( V_{\mu} \) and the set \( \mathcal{T}_\psi \subseteq V_{\mu} \) defined in (91) that
\[
U_\psi = \{ G^\tau(\mathcal{T}_\psi) \mid |\tau_n| < \delta_0', |n| \leq R' \}.
\]

This implies that
\[
\mathcal{W} = \bigcup_{\varphi \in \mathcal{T}_\psi} G(\varphi) = \bigcup_{\tau \in \mathbb{R}^{2R'+1}} G^\tau(\mathcal{T}_\psi) = \bigcup_{\tau \in \mathbb{R}^{2R'+1}} G^\tau(U_\psi).
\]
(98)
Since the sets \( G_\tau(U_\psi), \tau \in \mathbb{R}^{2R'+1} \), are open we conclude from (98) that \( W \) is an open set in \( iL^2_r \). The invariance of \( W \) with respect to the flow of \( X_{I_n}, n \in \mathbb{Z} \), follows from the fact that the section \( T_\psi \) in (97) is invariant with respect to the flows of \( X_{I_n}, |n| > R' \), and since \( X_{I_n} \) and \( X_{I_k} \) commute for any \( n, k \in \mathbb{Z} \).

By restricting (93) to \( \mathbb{R}^{2R'+1} \times W \) we obtain a smooth action

\[
G : \mathbb{R}^{2R'+1} \times W \to W.
\]

of \( \mathbb{R}^{2R'+1} \) on \( W \). In view of (92) we have the following commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{G_\tau} & W \\
\downarrow \Phi & & \downarrow \Phi \\
W & \xrightarrow{\rho_\tau} & W
\end{array}
\]

where \( W \) is the tail neighborhood (cf. (88))

\[
W := B_{|n| \leq R'}^{\delta_0,2\pi}(p^0) \times B_{|n| > R'}^{\alpha}(0) \subseteq iL^2_r
\]

(101)

where the parameter \( \delta_0 > 0 \) is replaced by \( 2\pi \) and for any \( \tau = (\tau_{-R'},...,\tau_R) \in \mathbb{R}^{2R'+1} \) and any \( |n| \leq R' \) the map \( \rho_\tau : W \to W \) rotates the component \( p_n = i(u_n,v_n) \in i\mathbb{R}^2 \) of \( p \in W \) by the angle \( \theta_n = \tau_n \) for \( |n| \leq R' \) while keeping the components of \( p \) for \( |n| > R' \) unchanged. The commutative diagram (100) and the invariance of \( W \) (cf. Lemma 7.4) then easily imply that \( \Phi : W \to W \) is onto.

Note also that by (101), the open set \( W \) in \( iL^2_r \) is a direct product of \( 2R'+1 \) two dimensional annuli and an open ball in \( iL^2_r \).

**Corollary 7.1.** For any \( \varphi \in T_\psi \) the orbit \( G(\varphi) \) is diffeomorphic to the \( 2R'+1 \) dimensional torus

\[
G := \mathbb{R}^{2R'+1} / \text{Span}(e_{-R'},...,e_{R'})_\mathbb{Z}
\]

(102)

where \( e_{-R'},...,e_{R'} \in (2\pi\mathbb{Z})^{2R'+1} \) are linearly independent over \( \mathbb{R} \). The vectors \( e_{-R'},...,e_{R'} \) are independent of the choice of \( \varphi \in T_\psi \).

**Proof of Corollary 7.1.** By restricting the action (99) to the orbit \( G(\varphi) \) with \( \varphi \in T_\psi \) we obtain a smooth transitive action

\[
G : \mathbb{R}^{2R'+1} \times G(\varphi) \to G(\varphi)
\]

(103)

of \( \mathbb{R}^{2R'+1} \) on the submanifold \( G(\varphi) \). Since \( G(\varphi) \) is compact (cf. Lemma 7.3) the stabilizer \( \text{St}(\varphi) \) of (103) is of the form

\[
\text{St}(\varphi) = \text{Span}(e_{-R'},...,e_{R'})_\mathbb{Z}
\]

2Note that we do not claim that \( e_{-R'},...,e_{R'} \) is a basis of \( (2\pi\mathbb{Z})^{2R'+1} \) over \( \mathbb{Z} \).
where the vectors $e_{(-R')}, ..., e_{R'} \in \mathbb{R}^{2R'+1}$ form a basis in $\mathbb{R}^{2R'+1}$ and $G(\varphi)$ is diffeomorphic to the factor group $\mathbb{R}^{2R'+1}/\text{St}(\varphi)$, which is a torus of dimension $2R'+1$ (cf. [1, §49]). It then follows from the commutative diagram (100) that

$$e_{(-R')}, ..., e_{R'} \in (2\pi\mathbb{Z})^{2R+1}. \quad (104)$$

Indeed, assume that $\tau \in \text{St}(\varphi)$. Then $G^{\tau}(\varphi) = \varphi$ and, by the commutative diagram (100), we obtain that $\rho^\tau(\Phi(\varphi)) = \Phi(\varphi)$. This implies that $\tau \in (2\pi\mathbb{Z})^{2R'+1}$, and hence, completes the proof of the first statement of the Corollary. Let us now prove the second statement. Take $\varphi^* \in \mathcal{T}_\varphi$ and let $\tau^* \in \text{St}(\varphi^*)$. In view of the continuity of the action (99), there exists an open neighborhood $U_1$ of zero in $\mathbb{R}^{2R'+1}$ and an open neighborhood $U_2$ of $\varphi^*$ in $U_\varphi$ so that for any $(\tau, \varphi) \in U_1 \times U_2$ we have that $G^{\tau^*+\tau}(\varphi) \in U_\varphi$. It then follows from the commutative diagram (100) and the fact that $\tau^* \in (2\pi\mathbb{Z})^{2R'+1}$ that

$$\Phi(G^{\tau^*+\tau}(\varphi)) = \begin{cases} i \|p_n\| (\cos(\theta_n(\varphi) + \tau_n), \sin(\theta_n(\varphi) + \tau_n)), & |n| \leq R', \\ p_n, & |n| > R', \end{cases} \quad (105)$$

where $(p_n)_{n \in \mathbb{Z}} = \Phi(\varphi)$. Since $\Phi : U_\varphi \to V_\varphi$ is a diffeomorphism, we conclude from (105) that $G^{\tau^*}(\varphi) = \varphi$ for any $\varphi \in U_2$. In view of the connectedness of $U_\varphi$ we then obtain that $\tau^* \in \text{St}(\varphi)$ for any $\varphi \in U_\varphi$. Since $\mathcal{T}_\varphi$ is connected this shows that the $2R'+1$ linearly independent vectors in (104) can be chosen independently of $\varphi \in \mathcal{T}_\varphi$.

With these preparations done we are now ready to introduce the slight adjustment of $\Phi$, mentioned at the beginning of the Section. By (97) and Lemma 7.3 the open neighborhood $\mathcal{W}$ of $\psi$ in $iL_2^p$ is foliated by orbits of the action (99) with $\mathcal{T}_\varphi$ as a global section. By Corollary 7.1 any orbit $G(\varphi), \varphi \in \mathcal{T}_\varphi$, is diffeomorphic to the $2R'+1$ dimensional torus (102). Denote by $(t_{-R'}, ..., t_{R'})$ the coordinates corresponding to the frame $(e_{(-R')}, ..., e_{R'})$ in $\mathbb{R}^{2R'+1}$. For any given $\varphi \in \mathcal{T}_\varphi$ denote by $(\theta^*_n, ..., \theta^*_R)$ the coordinates on $G(\varphi)$ that are obtained by taking the pull-back of $(2\pi t_{-R'}, ..., 2\pi t_{R'})$ via the diffeomorphism $G(\varphi) \to G$ given by the action (103). For any $|n| \leq R'$ denote the coordinates of $e_n$ by $\tau_n, |k| \leq R'$,

$$e_n = (\tau_n(-R'), ..., \tau_nR') \in (2\pi\mathbb{Z})^{2R'+1}, \quad |n| \leq R'. \quad (106)$$

For $|n| \leq R'$ consider the (analytic) functions $I^*_n : \mathcal{W} \to \mathbb{R}$,

$$I^*_n := \frac{1}{2\pi} \sum_{|k| \leq R'} \tau_n I_k, \quad (107)$$

where $I_k, |k| \leq R'$, is the $k$-th action on $U_{iso}$. Then, it follows from (107) and the second statement of Corollary 7.1 that $X_{I^*_n} = \sum_{|k| \leq R'} \tau_n X_k$, and by the definition of the action (98) (see also (99)) and the coordinates $(\theta^*_n(-R'), ..., \theta^*_R)$, we conclude from (106) that on any orbit $G(\varphi), \varphi \in \mathcal{T}_\varphi$, we have that

$$\{\theta^*_n, I^*_k\} \equiv (d\theta^*_n)(X_{I^*_n}) = d(2\pi t_n)(e_k/2\pi) = \delta_{nk} \quad (108)$$

58
for any $|n| \leq R'$ and $|k| \leq R'$. This implies that on any orbit $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$,

$$d\theta_n^* = \sum_{|k| \leq R'} \tau_n^* d\theta_k$$

(109)

where $\theta_k : W \to \mathbb{R} / 2\pi$, $|k| \leq R'$, is the $k$-th angle on $U_{iso} \cap iL_2^*$ and $(\tau_n^*)_{|n|,|k| \leq R'}$ are the elements of the non-degenerate $(2R' + 1) \times (2R' + 1)$ matrix $P^* := (P^{-1})^T$ where $P := (\tau_{nk})_{|n|,|k| \leq R'}$ and $(P^{-1})^T$ denotes the transpose of $P^{-1}$. Formula (109) shows that the coordinates $\theta_n^* \mid \ell^2_r$, are real analytic and that their Poisson brackets satisfy the relations stated in the Lemma 7.5. Moreover, by the commutation relations (82), the functions $(\tau_n^* \mid \ell^2_r)$, are real analytic on $W$ and separate the orbits $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$, and for any orbit $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$, and they form a coordinate system on the section $\mathcal{T}_\psi$ in (97). Hence, these functions separate the orbits $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$. Since the functions $(\theta_n^*)_{|n| \leq R'}$ taken modulo $2\pi$ separate the points on any of these orbits, we conclude the proof of the Lemma.

**Lemma 7.5.** The angles $(\theta_n^*)_{|n| \leq R'}$ taken modulo $2\pi$, the actions $(I_n^*)_{|n| \leq R'}$, and $(x_n)_{|n| > R'}$, $(y_n)_{|n| > R'}$, are real analytic on $W$ and separate the points on $W$. Moreover, $\{x_n, y_n\} = 1$, $|n| > R'$, $\{\theta_n^*, I_n^*\} = 1$, $|n| \leq R'$, whereas all other Poisson brackets vanish.

**Proof of Lemma 7.5.** We already proved that the functions in Lemma 7.5 are real analytic and that their Poisson brackets satisfy the relations stated in the Lemma. Moreover, by the commutation relations (82), the functions $(I_n^*)_{|n| \leq R'}$, $(x_n)_{|n| > R'}$, and $(y_n)_{|n| > R'}$, are constant on any of the orbits $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$, and they form a coordinate system on the section $\mathcal{T}_\psi$ in (97). Hence, these functions separate the orbits $G(\varphi)$, $\varphi \in \mathcal{T}_\psi$. Since the functions $(\theta_n^*)_{|n| \leq R'}$ taken modulo $2\pi$ separate the points on any of these orbits, we conclude the proof of the Lemma.

Lemma 7.5 allows us to define the modification $\Psi : W \to i\ell^2_r$ of the map $\Phi : W \to i\ell^2_r$ by setting

$$x_n := \sqrt{2I_r^*} \cos \theta_n^*, \quad y_n := \sqrt{2I_r^*} \sin \theta_n^*, \quad |n| \leq R',$$

(110)

while keeping the other components of $\Phi$ unchanged. By shrinking the open neighborhood $W$ if necessary we can ensure that $\Psi(W)$ is a tail neighborhood of the form (101) with $p' := \Psi(p)$ and the same value of the parameter $t_0 > 0$. For simplicity of notation, we will denote $\Psi(W)$ again by $W$ and for any $|n| \leq R'$, write $\theta_n$ and $I_n$ instead of $\theta_n^*$ and, respectively, $I_n^*$. We have

**Proposition 7.1.** The map

$$\Psi : W \to W$$

(111)

is a canonical real analytic diffeomorphism. Moreover, the open neighborhood $W$ is a saturated neighborhood of $\text{Iso}_r(\psi)$ and $W$ is a tail neighborhood of $\Psi(\psi)$ in $i\ell^2_r$.

**Proof of Proposition 7.1.** The Proposition follows from Lemma 7.5. Indeed, it follows from Lemma 7.5 and (110) that the map (111) is analytic. Furthermore, it follows from (107), (109), (110), and the fact that $\Phi : W \to i\ell^2_r$ is a local diffeomorphism (cf. Theorem 6.1) onto its image, that (111) is a local diffeomorphism onto $W$. Since, by Lemma 7.5 the components of $\Psi$ separate the points on $W,$
we then conclude that $\Psi$ is a diffeomorphism. This map is canonical by the last statement in Lemma 7.5. The statement that $\mathcal{W}$ is a saturated neighborhood of $\text{Iso}_0(\psi)$ follows from the invariance of the neighborhood $\mathcal{W}$ with respect to the complete isospectral flows $X_n$, $n \in \mathbb{Z}$ (see Lemma 7.4), and the arguments in the proof of Lemma 7.1 that allow us, for any $\varphi \in \mathcal{W}$, to identify $\text{Iso}_o(\varphi)$ with $\text{Tor}(\Psi(\varphi))$ (see (72)) in the tail neighborhood $W$.

Finally, we prove

**Proposition 7.2.** For any integer $N \geq 1$ the restriction of the map (111) to $\mathcal{W} \cap iH_r^N$ takes values in $i\mathfrak{h}_r^N$ and $\Psi|_{\mathcal{W} \cap iH_r^N} : \mathcal{W} \cap iH_r^N \to W \cap i\mathfrak{h}_r^N$ is a real analytic diffeomorphism.

**Proof of Proposition 7.2.** Assume that $N \geq 1$. Since the coordinate functions $x_n$ and $y_n$ are real analytic on $\mathcal{W}$ so are their restrictions to $\mathcal{W} \cap iH_r^N$. Recall that by Lemma 6.4 that $z_n^2 = O(|\gamma_n| + |\mu_n - \tau_n|)$ for $|n| > R$, locally uniformly on $U_{iso}$. Taking into account the estimates for $\gamma_n$ on $H_r^N$ in [19, Corollary 1.1] and the ones for $\mu_n - \tau_n$ on $H_r^N$ in [19, Theorem 1.1] and [19, Theorem 1.3] it follows that

$$\sum_{|n| > R} (n)^{2N} |z_n^2|^2 < \infty$$

(112)

locally uniformly on $U_{iso} \cap H_r^N$. Furthermore, recall from Remark 5.1 that $\tilde{\beta}_n$, defined modulo $2\pi$, is of order $O(1/n)$ as $|n| \to \infty$ locally uniformly on $U_{iso} \setminus \mathbb{Z}_n$ and that $\sum_{|k| > R, k \neq n} \beta_n^k = O(1/n)$ as $|n| \to \infty$ locally uniformly on $U_{iso}$. By combining this with (112), we conclude from (99) that the real analytic map $\Psi|_{\mathcal{W} \cap iH_r^N} : \mathcal{W} \cap iH_r^N \to i\mathfrak{h}_r^N$ is locally bounded in a complex neighborhood of $\mathcal{W} \cap iH_r^N$. By (11) Theorem A.5 it then follows that $\Psi|_{\mathcal{W} \cap iH_r^N} : \mathcal{W} \cap iH_r^N \to i\mathfrak{h}_r^N$ is real analytic. Furthermore, by the characterization of potentials $\varphi \in iL_r^2$ to be in $iH_r^N$, provided in [20, Theorem 1.2], it follows that $\Psi(\mathcal{W} \cap iH_r^N) = \mathcal{W} \cap i\mathfrak{h}_r^N$, implying that $\Psi|_{\mathcal{W} \cap iH_r^N} : \mathcal{W} \cap iH_r^N \to W \cap i\mathfrak{h}_r^N$ is bijective. To see that the latter map is a real analytic diffeomorphism it remains to show that for any $\varphi \in \mathcal{W} \cap iH_r^N$, $(d_{\varphi}\Psi)|_{i\mathfrak{h}_r^N} : iH_r^N \to i\mathfrak{h}_r^N$ is an isomorphism.

Since $d_{\varphi}\Psi : L_r^2 \to iL_r^2$ is an isomorphism by Proposition 7.1, we conclude that $\text{Ker} (d_{\varphi}\Psi)|_{i\mathfrak{h}_r^N} = \{0\}$. Hence, we will complete the proof if we show that $(d_{\varphi}\Psi)|_{i\mathfrak{h}_r^N} : iH_r^N \to i\mathfrak{h}_r^N$ is a Fredholm operator of index zero. This will follow once we prove that $(d_{\varphi}\Psi)|_{i\mathfrak{h}_r^N} : iH_r^N \to i\mathfrak{h}_r^N$ is a Fredholm operator of index zero, where $\Phi$ is the map (74). In order to prove this, we show by analytic extension that the formulas for $z_n^2$ in [18, Theorem 2.2], valid for potentials near zero, continue to hold on $U_{iso}$ for any $|n| > R$. These formulas involve the quantities $\tau_n - \mu_n, \delta(\mu_n), \delta_n(\mu_n)$, and $\eta_{n,2}^0$, which can be estimated using [19, Theorem 1.1, Theorem 1.3, and Theorem 1.4] and [11, Lemma 12.7]. In this way we show that for any $N \geq 1$,

$$\Phi - \mathcal{F} : \mathcal{W} \cap iH_r^N \to i\mathfrak{h}_r^{N+1}$$

(113)
Lemma 8.1. The quantity
\[ 1 \leq \text{values of } L = 0 \) of Theorem 1.1 hold (cf. (70)). The statement \( N_L \phi \)
One then concludes by the argument principle that \( \text{of Theorem 20.3 in } [11] \).
from Proposition 7.2. For proving \( (NF_3) \) is analytic in \( U \).
\[ \sum \int_{\mathcal{P}_{\varphi,\psi}[z_j,\mu_{\varphi}(z_j)]} \] \[ \zeta_n(\lambda, \varphi) \] \[ \sqrt{\Delta^2(\lambda, \varphi) - 4} \] \[ d\lambda \] \[ (114) \]
is analytic in \( U_\varphi \).
Proof of Lemma 8.1. In view of the representation \[ (111) \], it is enough to prove that
\[ \sum_{1 \leq j \leq m_D} \int_{\mathcal{P}_{\varphi,\psi}[z_j,\mu_{\varphi}(z_j)]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda \] \[ (115) \]
is analytic in \( U_\varphi \). In the case where \( \mu_{\varphi}(\nu) \notin \Lambda_R(\psi) \) it follows that for any \( \varphi \in U_\varphi \) the initial point of \( [Q_{\mu_{\varphi}(\nu)}, \mu_{\varphi}(z_j)]^* \) does not depend on \( 1 \leq j \leq m_D \).
One then concludes by the argument principle that
\[ \sum_{1 \leq j \leq m_D} \int_{\mathcal{P}_{\varphi,\psi}[z_j,\mu_{\varphi}(z_j)]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda = \frac{1}{2\pi i} \int_{\mathcal{D}_{\mu_{\varphi}(\nu)}} F(\mu, \varphi) \frac{\chi_D(\mu, \varphi)}{\chi_D(\mu, \varphi)} d\mu \]
where \( \chi_D \) is given by \[ (9) \] and
\[ F(\mu, \varphi) := \int_{Q_{\mu_{\varphi}(\nu)}} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda \]
8 Appendix: Auxiliary results
First we provide details of the proof that the sum of the integrals \[ (10) \], introduced in the proof of Proposition 4.1, is analytic. We use the notation introduced there. Assume that for a given \( \nu \in L_N(\psi), \mu_{\varphi}(\nu) \) is a Dirichlet eigenvalue of \( L(\psi) \) of multiplicity \( m_D \geq 2 \). Denote by \( \nu_j, 1 \leq j \leq m_D, \) the periodic eigenvalues of \( L(\psi) \) such that \( \mu_{\varphi}(\nu_j) = \mu_{\varphi}(\nu) \) and for any \( \varphi \in U_\varphi \) denote by \( z_j, 1 \leq j \leq m_D \); the periodic eigenvalues of \( L(\varphi) \) with \( z_j \in D^r(\nu_j) \).
Lemma 8.1. The quantity
\[ \sum_{1 \leq j \leq m_D} \int_{\mathcal{P}_{\varphi,\psi}[z_j,\mu_{\varphi}(z_j)]} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda \] \[ (114) \] is analytic in \( U_\varphi \).
is analytic on the disk $D^\varepsilon(\mu_\psi(\nu))$ and continuous up to its boundary. The case when $\mu_\psi(\nu) \in \Lambda_R(\psi)$ can be treated in a similar way. We only remark that in this case, the initial points $Q^*_{\mu_\psi(\nu),j,\psi}$, $1 \leq j \leq m_D$, of the paths $[Q_{\mu_\psi(\nu),j,\psi}(z_j)]^*$, $1 \leq j \leq m_D$, are not necessarily the same. If $Q^*_{\mu_\psi(\nu),j,\psi} = Q^*_{\mu_\psi(\nu),j,\psi}$ let $P^*_{j,\psi}$ be the constant path $Q^*_{\mu_\psi(\nu),j,\psi}$ and if $Q^*_{\mu_\psi(\nu),j,\psi} \neq Q^*_{\mu_\psi(\nu),j,\psi}$, let $P^*_{j,\psi}$ be the counterclockwise oriented lift of the circle $\partial \overline{D}(\mu_\psi(\nu))$ by $\pi_1(\partial \overline{D}(\mu_\psi(\nu)), \nu) : C_{\nu,R} \to \mathbb{C}$, which connects $Q^*_{\mu_\psi(\nu),j,\psi}$ with $Q^*_{\mu_\psi(\nu),j,\psi}$. We then write the path integral $\int_{Q_{\mu_\psi(\nu),j,\psi}(z_j)}^{Q_{\mu_\psi(\nu),j,\psi}(z_j)} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda$ as a sum of two path integrals with paths $P^*_{j,\psi} \cup [Q_{\mu_\psi(\nu),j,\psi}(z_j)]^*$ and $(P^*_{j,\psi})^{-1}$. By the argument principle,

$$\sum_{1 \leq j \leq m_D} \int_{P^*_{j,\psi} \cup [Q_{\mu_\psi(\nu),j,\psi}(z_j)]^*} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda$$

is analytic on $U_\psi$. Since

$$\sum_{1 \leq j \leq m_D} \int_{(P^*_{j,\psi})^{-1}} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda$$

is also analytic on $U_\psi$, it then follows that $\Delta(\lambda, \varphi)$ analytic on $U_\psi$ in this case.}

The second result characterizes the spectral bands of $\text{Spec}_{\mathbb{R}} L(\varphi)$. It is used in the proof of Lemma 5.1 to show that the actions $I_n$, $|n| > R$, are non positive on $U_{\text{iso}} \cap iL^2$. For any given $\varepsilon > 0$ and $\delta > 0$ sufficiently small, and $z \in \mathbb{C}$, denote by $B^\varepsilon_\delta$ the following box in $\mathbb{C}$,

$$B^\varepsilon_\delta := \{ \lambda \in \mathbb{C} \mid |\text{Re}(\lambda - z)| < \varepsilon, \ |\text{Im}(\lambda - z)| < \delta \}.$$

In the sequel we use results on the discriminant $\Delta(\lambda, \varphi)$ of $L(\varphi)$ reviewed in Section 2.

**Lemma 8.2.** For any $\psi \in iL^2$, choose $R_\psi \in \mathbb{Z}_{>0}$ as in Lemma 8.1. Then there exist an integer $\hat{R} \geq R_\psi$, as well as $\varepsilon > 0$, $\delta > 0$, and a neighborhood $W_\psi$ of $\psi$ in $iL^2$ so that for any $|n| > \hat{R}$ and $\varphi \in W_\psi$, $B^\varepsilon_\delta \cap \text{Spec}_{\mathbb{R}} L(\varphi)$ consists of the interval $(\lambda^+_{\psi}(\varphi) - \varepsilon, \lambda^+_{\psi}(\varphi) + \varepsilon) \subseteq \mathbb{R}$ and a smooth arc $g_n$ connecting $\lambda^+_{\psi}(\varphi)$ with $\lambda^+_{\psi}(\varphi)$ within $B^\varepsilon_\delta$ so that $\Delta(\lambda)$ is real valued on $g_n$ and satisfies $-2 < \Delta(\lambda) < 2$ for any $\lambda \in g_n \setminus \{\lambda^+_{\psi}(\varphi)\}$. In fact for any $\varphi \in W_\psi$, the arc $g_n$, also referred to as spectral band, is the graph $\{a_n(t) + it : |t| \leq \text{Im}(\lambda^+_n)\}$ of a smooth real valued function $a_n : [-\text{Im}(\lambda^+_n), \text{Im}(\lambda^+_n)] \to \mathbb{R}$ with the property that $a_n(0) = \lambda^+_n(\varphi)$, $a_n(-t) = a_n(t)$ for any $0 \leq t \leq \text{Im}(\lambda^+_n)$, and $a_n(\pm \text{Im}(\lambda^+_n)) = \text{Re}(\lambda^+_n)$.

For the convenience of the reader we include the proof of Lemma 8.2 given in [13].
Proof of Lemma 8.2. First let us introduce some more notation. For any $\lambda \in \mathbb{C}$ write $\lambda = u + iv$ with $u, v \in \mathbb{R}$ and $\Delta = \Delta_1 + i\Delta_2$ where for $\phi \in L^2_\ell$ arbitrary, $
abla_j(u, v) \equiv \nabla_j(u, v, \phi)$, $j = 1, 2$, are given by

$$
\Delta_1(u, v) := \text{Re}(\Delta(u + iv, \phi)), \quad \Delta_2(u, v) := \text{Im}(\Delta(u + iv, \phi)).
$$

In a first step we want to study $\Lambda \cap D_n$ for $\phi \in iL^2_\ell$ where

$$
\Lambda = \{u + iv \mid u, v \in \mathbb{R}, \Delta_2(u, v) = 0\}.
$$

By Lemma 2.8 $iii$ for any $\phi \in iL^2_\ell$, $\Delta_2(u, 0, \phi) = 0$. Hence

$$
F : \mathbb{R} \times \mathbb{R} \times iL^2_\ell \to \mathbb{R}, \quad (u, v, \phi) \mapsto \Delta_2(u, v, \phi)/v
$$

is well defined. As $F$ and $\Delta_2$ have the same zero set on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ we investigate $\{(u, v) \in \mathbb{R}^2 : F(u, v, \phi) = 0\}$ further. We note that

$$
F(u, v) = F(u, v, \phi) = \int_0^1 (\partial_s \Delta_2)(u, tv) \, dt
$$

and hence $F(u, 0) = \partial_u \Delta_2(u, 0)$. Since $\Delta_2(u, 0)$ vanishes for any $u \in \mathbb{R}$, so does $\partial_u \Delta_2(u, 0)$. By the Cauchy-Riemann equation it then follows that for $u \in \mathbb{R}$, $\Delta(u, 0) = 0$ if and only if $\partial_u \Delta_2(u, 0) = 0$. The real roots of $F(\cdot, 0) = \partial_u \Delta_2(\cdot, 0)$ are thus given by $\lambda_n$, $n \in \mathbb{Z}$, with $\lambda_n \in \mathbb{R}$. Since $\Delta$ is an analytic function on $\mathbb{C} \times L^2_\ell$ one concludes that $\Delta_2(\cdot, \cdot, \cdot)$ is a real analytic function on $\mathbb{R} \times \mathbb{R} \times iL^2_\ell$, and by formula (116), so is $F$.

For $\psi \in iL^2_\ell$ let $R_\rho \in \mathbb{Z}_{\geq 0}$ and the neighborhood $V_\psi$ of $\psi$ be as in Lemma 2.1. For any $|k| > R_\rho$, let $S_k$ be the map

$$
S_k : B_{\ell^\infty} \times (-1, 1) \times (V_\psi \cap iL^2_\ell) \to \mathbb{R}, \quad (\zeta_n)_{|n| > R_\rho}, v, \phi) \mapsto F(\lambda_k + \zeta_k, v, \phi)
$$

where $\lambda_k = \lambda_k(\phi)$ is in $D_k$ and $B_{\ell^\infty}$ denotes the open unit ball in the space of real valued sequences

$$\ell^\infty = \{\zeta = (\zeta_n)_{|n| > R_\rho} \mid \|\zeta\|_{\infty} < \infty\}, \quad \|\zeta\|_{\infty} := \sup_{|n| > R_\rho} |\zeta_n|.
$$

Note that $S_k$ is the composition of the map $B_{\ell^\infty} \times (-1, 1) \times (V_\psi \cap iL^2_\ell) \to \mathbb{R} \times (-1, 1) \times (V_\psi \cap iL^2_\ell)$,

$$(\zeta, v, \phi) \mapsto (\lambda_k(\phi) + \zeta_k, v, \phi),
$$

with $F$ and hence real analytic. It follows from the asymptotics of $\Delta$ of the Lemma 4.3 and the asymptotics of $\lambda_k$ stated in Lemma 2.1 that for any sequence $\zeta = (\zeta_n)_{|n| > R_\rho} \in B_{\ell^\infty}$, $(S_k)_{|k| > R_\rho}$ is also in $\ell^\infty$. We claim that

$$
S : B_{\ell^\infty} \times (-1, 1) \times (V_\psi \cap iL^2_\ell) \to \ell^\infty, \quad (\zeta, v, \phi) \mapsto (S_k(\zeta, v, \phi))_{|k| > R_\rho}
$$

is smooth. To see it note that by the Cauchy-Riemann equation, $\Delta_2(u, v, \phi)$ and its derivatives, when restricted to $\mathbb{R} \times \mathbb{R} \times iL^2_\ell$, can be estimated in terms
of $\Delta(\lambda, \varphi)$ and its derivatives. Hence the asymptotic estimates of $\Delta(\lambda, \varphi)$ (see [11, Theorem 2.2, Lemma 4.3]) together with Cauchy’s estimate can be used to estimate the derivatives of $S_k$ to conclude that the map $S = (S_k)_{|k| > R_p}$ is smooth.

We now would like to apply the implicit function theorem to $S$. First note that $S|_{\zeta = 0, v = 0, \psi} = 0$ as $\lambda_k(\psi)$, $k \in \mathbb{Z}$, are the roots of $\hat{\Delta}(\cdot, \psi)$. In addition, since $\hat{\lambda}_k(\psi)$, $|k| > R_p$, are real and simple roots of $\hat{\Delta}(\cdot, \psi)$, one has

$$\partial_{\zeta} S_k|_{\zeta = 0, v = 0, \psi} = \partial_{\zeta} \partial_{u} \Delta_2|_{u = \lambda_k(\psi), v = 0, \psi}$$

whereas by the definition of $S_k$, $\partial_{\zeta} S_k$ vanishes identically for any $n \neq k$. By the asymptotics $\lambda_k(\psi) = k\pi + \ell_k^2$ (cf. Lemma 241 and the one of $\Delta$ (cf. [11, Lemma 4.3]) it follows from Cauchy’s estimate that

$$\partial_u \partial_{\zeta} \Delta_2|_{u = \lambda_k(\psi), v = 0} = \partial_u \partial_{\zeta} \Delta(u + iv, 0)|_{u = \lambda_k(\psi), v = 0} + \ell_k^2.$$ 

On the other hand $\Delta(\lambda, 0) = 2 \cos \lambda$ and thus $\text{Im} \left( \Delta(u + iv, 0) \right) = -2 \sin u \sinh v$ implying that

$$\partial_u \partial_{\zeta} \text{Im} \left( \Delta(u + iv, 0) \right)|_{u = \lambda_k(\psi), v = 0} = 2(-1)^{k+1}(1 + \ell_k^2).$$

Altogether we then conclude that

$$\partial_{\zeta} S|_{\zeta = 0, v = 0, \psi} = 2 \text{diag} \left( \left( (-1)^{k+1} \right)_{|k| > R_p} + \ell_k^2 \right).$$

Thus there exists an integer $\tilde{R} \geq R_p$ so that for the restriction of $S$ to $B_{\tilde{R}} \times (-1, 1) \times (V_{\psi} \cap iL_2^2)$,

$$\tilde{S} : \tilde{B}_{\tilde{R}} \times (-1, 1) \times (V_{\psi} \cap iL_2^2) \to \tilde{\mathbb{R}},$$

the differential $\partial_{\zeta} \tilde{S}|_{\zeta = 0, v = 0, \psi}$ is invertible. Here $B_{\tilde{R}}$ denotes the unit ball in $\tilde{\mathbb{R}} = \{ \tilde{\zeta} = (\tilde{\zeta}, |\tilde{\zeta}| > \tilde{R} \subseteq \mathbb{R} | |\tilde{\zeta}| < \infty \}$. By the implicit function theorem there exist $\delta > 0$, a neighborhood $W_\psi \subseteq V_\psi$ of $\psi$ in $L_2^2$, $0 < \varepsilon < 1$, and a smooth map

$$h : (-\delta, \delta) \times W_\psi \to B_{\tilde{R}}(\varepsilon), (v, \varphi) \mapsto h(v, \varphi) = (h_n(v, \varphi))_{|n| > \tilde{R}}$$

so that $h(0, \psi) = 0$ and $\tilde{S}(h(v, \varphi), v, \varphi) = 0$ for any $(v, \varphi) \in (-\delta, \delta) \times W_\psi$. Here

$$B_{\tilde{R}}(\varepsilon) = \{ \tilde{\zeta} \in \tilde{\mathbb{R}} | |\tilde{\zeta}| < \varepsilon \}$$

and $(v, \varphi) \mapsto (h(v, \varphi), v, \varphi)$ parametrizes the zero level set of $\tilde{S}$ in $B_{\tilde{R}}(\varepsilon) \times (-\delta, \delta) \times W_\psi$. In particular, for any $|n| > \tilde{R}$ and $\varphi$ in $W_\psi$, the intersection of $\{ F(u, v, \varphi) = 0 \}$ with the box $B_{\tilde{R}}(\varepsilon)$ is smoothly parametrized by

$$z_n(\cdot, \varphi) := (-\delta, \delta) \to B_{\tilde{R}}(\varepsilon) \times (-\delta, \delta), v \mapsto \hat{\lambda}_n(\varphi) + h_n(v, \varphi) + iv.$$
By choosing \( \hat{R} \) larger and by shrinking \( W_\psi \), if necessary, one can assure that \( \lambda_n^\pm \equiv \lambda_n^\pm(\varphi) \) is in \( B^{\delta,\varepsilon}_{\hat{\lambda}_n}(\varphi) \) for any \( \varphi \in W_\psi \) and \( |n| > \hat{R} \). Since \( \Delta(\lambda_n^\pm) = (-1)^n 2 \) and \( \Delta(\hat{\lambda}_n) \in \mathbb{R} \), the pair \( \lambda_n^+, \lambda_n^- \) as well as \( \hat{\lambda}_n \) are in the range of \( z_n \). In fact, one has

\[
z_n(0, \varphi) = \hat{\lambda}_n \text{ and } z_n \left( \pm \text{Im}(\lambda_n^\pm), \varphi \right) = \lambda_n^\pm, \quad \forall \varphi \in W_\psi.
\]

Furthermore recall that \( \chi_p(\hat{\lambda}_n, \varphi) \leq 0 \) since \( \hat{\lambda}_n \in \mathbb{R} \). As \( \lambda_n^\pm \) are the only roots of \( \chi_p(\cdot, \varphi) \) in \( D_n \), it follows that if \( \hat{\lambda}_n \neq \lambda_n^\pm \), then \( \chi_p(\hat{\lambda}_n, \varphi) < 0 \) and, hence

\[
\chi_p(z_n(v), \varphi) < 0 \quad \forall -\text{Im}(\lambda_n^+) < v < \text{Im}(\lambda_n^+).
\]

Altogether we thus have proved that for any \( |n| > \hat{R} \)

\[
g_n = \{ z_n(v) \mid -\text{Im}(\lambda_n^+) \leq v \leq \text{Im}(\lambda_n^+) \}
\]

is a smooth arc on which \( \Delta \) takes values in \([-2, 2]\). We also have all the properties listed in the statement of the Lemma. \( \square \)

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