Discrete convolution operators and Riesz systems generated by actions of abelian groups

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Abstract
We study the bounded endomorphisms of $\ell^2(G) \times \cdots \times \ell^2(G) = \ell^2_N(G)$ that commute with translations, where $G$ is a discrete abelian group. It is shown that they form a C*-algebra isomorphic to the C*-algebra of $N \times N$ matrices with entries in $L^\infty(\hat{G})$, where $\hat{G}$ is the dual space of $G$. Characterizations of when these endomorphisms are invertible, and expressions for their norms and for the norms of their inverses, are given. These results allow us to study Riesz systems that arise from the action of $G$ on a finite set of elements of a Hilbert space.

Keywords Discrete convolution · C*-algebra · Multiplier · Shift-invariant space · Discrete abelian group and Riesz basis

Mathematics Subject Classification 47L25 · 43A99 · 46L99

1 Introduction

Let $G$ be a discrete abelian group. The first aim of this work is to study operators of the type

$$A : \ell^2_N(G) \to \ell^2_M(G), \quad A(x) = A \ast x = \sum_{g \in G} A(g) \cdot (x \cdot -g),$$

where $A \in \mathcal{M}_{M \times N}(\ell^2(G))$ is a $M \times N$ matrix whose entries are elements of $\ell^2(G)$ and $A \ast x$ is the convolution of the matrix $A$ with the vector $x \in \ell^2_N(G) = \ell^2(G) \times \cdots \times \ell^2(G)$ ($N$ times).
The bounded convolution operators of this type can also be described as those bounded linear operators that commute with translations (see Theorem 3.4). When $G = \mathbb{Z}^d$ they are called in discrete signal processing, where they are widely used, Linear Time Invariant (LTI) Multi-Input Multi-Output (MIMO) transformations. See for example [15].

In many situations, operators between some spaces of functions or measures, on a locally compact abelian group $G$, that commute with translations coincide with those that can be expressed as a multiplication in the Fourier domain (see Theorem 3.4 for the discrete vectorial case here considered). They are called multipliers and have been widely studied for scalar functions and measures. See for example [16], in particular the closest result to this work [16, Theorem 4.3.1], where characterizations of the multipliers in $\mathcal{B}(L^2(G))$ are given for a general locally compact abelian group $G$.

We give, in Sect. 3, characterizations of the multipliers from $\ell^2_N(G)$ into $\ell^2_M(G)$, where $G$ is a discrete abelian group. We study some of the characteristics of these discrete vectorial convolution operators. Special attention is devoted to the convolution operators in $\mathcal{B}(\ell^2_N(G))$, the space of bounded endomorphisms of $\ell^2_N(G)$. It is proved that they form a C*-subalgebra of $\mathcal{B}(\ell^2_N(G))$ which is isomorphic to $\mathcal{M}_N(L^\infty(\hat{G}))$, the C*-algebra of the $N \times N$ matrices with entries in $L^\infty(\hat{G})$, where $\hat{G}$ is the dual space of $G$. For instance, the set of convolution operators of $\mathcal{B}(\ell^2_N(\mathbb{Z}))$ form a C*-algebra isomorphic to $\mathcal{M}_N(L^\infty(\mathbb{T}))$ where $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$, or those in $\mathcal{B}(\ell^2_N(\mathbb{Z}^2))$ form a C*-algebra isomorphic to $\mathcal{M}_N(L^\infty(\mathbb{T}^2))$.

By means of this C*-isomorphism, we characterize the invertible convolution operators, and we give a suitable expression for the norm of the inverse.

These results about discrete convolution operators in this general setting could be useful in future applications, specially in discrete signal processing where not only the discrete group $\mathbb{Z}^d$, but also finite groups such as $\mathbb{Z}_s^d$ or direct products as $\mathbb{Z}_s \times \mathbb{Z}_r$ or $\mathbb{Z}_s \times \mathbb{Z}^d$, are often used. The second part of this article, shows that by means of these convolution operators, an interesting generalization of some relevant results about shift-invariant systems can be obtained. In reference [11], it is shown that they are also useful in order to obtain a regular sampling theory in a very general context.

The second aim of this work is to study Bessel and Riesz systems generated by actions of abelian groups. The development of wavelet and approximation theories in different directions has led to the consideration and analysis of various generalizations of the classical shift-invariant spaces in $\mathbb{R}$,

$$V_{\Phi} = \left\{ \sum_{n=1}^{N} \sum_{\alpha \in \mathbb{Z}} x_n(\alpha) \varphi_n(\cdot - \alpha) : (x_1, \ldots, x_N) \in \ell^2_N(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

where $\Phi = \{ \varphi_1, \ldots, \varphi_N \}$ denotes the set of generators. Here, we consider the spaces

$$V_{\Phi} = \left\{ \sum_{n=1}^{N} \sum_{g \in G} x_n(g) \pi_g \varphi_n : (x_1, \ldots, x_N) \in \ell^2_N(G) \right\} \subset \mathcal{H}.$$
where $\mathcal{H}$ is a Hilbert space, $G$ a discrete abelian group, and $\pi$ a unitary representation of $G$ on $\mathcal{H}$. This generalization of shift-invariant spaces is related with many of the generalizations previously considered, see [1–3,7,13] and Sect. 4 for more details.

Characterizations of when the system \( \{ \pi_g \varphi_n \}_{n=1,\ldots,N,g \in G} \) is a Riesz basis for $V_\Phi$, or when it is a Bessel sequence of $\mathcal{H}$ and suitable expressions, in the Fourier domain, for the optimal Riesz bounds are provided in Sect. 4 using the results on convolution operators of the Sect. 3.

2 Notation and preliminaries

Throughout the paper we assume that $G$ is a discrete abelian group (with additive notation), that $\mathcal{H}$ is a separable complex Hilbert space, and we use the following notation:

1. \( \ell^2(G) = \{ x : G \to \mathbb{C} : \| x \|_{\ell^2(G)}^2 = \sum_{g \in G} | x(g) |^2 < \infty \} \), \( \ell^2_N(G) = \ell^2(G) \times \cdots \times \ell^2(G) \) (N times) and \( [x]_m = x_m \) denotes the m-th entry of \( x \in \ell^2_N(G) \).

2. \( T_g x = x(\cdot - g) \) denotes the translation operator in \( \ell^2_N(G) \).

3. \( \mathcal{M}_{M \times N}(\ell^2(G)) \) is the set of \( M \times N \) matrices with entries in \( \ell^2(G) \) and \( \mathcal{M}_{N}(\ell^2(G)) = \mathcal{M}_{N \times N}(\ell^2(G)) \).

4. \( B(\ell^2_N(G)) \) is the algebra of the bounded endomorphisms of \( \ell^2_N(G) \).

5. The symbol \( \ast \) denotes convolution, namely; \( x \ast y = \sum_{g \in G} x(g)y(\cdot - g) \), for \( x, y \in \ell^2(G) \); \( A \ast B = \sum_{g \in G} A(g)B(\cdot - g) \), for \( A, B \in \mathcal{M}_{N}(\ell^2(G)) \); \( A \ast x = \sum_{g \in G} A(g)x(\cdot - g) \), for \( A = [a_{m,n}] \in \mathcal{M}_{M \times N}(\ell^2(G)) \) and \( x \in \ell^2_N(G) \), or equivalently \( A \ast x \) is the vector whose \( m \)-th entry is \( [A \ast x]_m = \sum_{n=1}^{N} a_{m,n} \ast x_n \).

6. \( \| D \|_2 \) is the spectral norm of a matrix \( D \). The symbols \( \lambda_{\text{min}}[C] \) and \( \lambda_{\text{max}}[C] \) denote the minimum and the maximum eigenvalue of a positive semidefinite matrix \( C \).

7. \( \pi \) is a unitary representation of the group \( G \) on \( \mathcal{H} \) i.e. a homomorphism \( \pi : G \to U(\mathcal{H}) \), where \( U(\mathcal{H}) \) is the group of unitary operators of \( \mathcal{H} \), that satisfies \( \pi_{g+g'} = \pi_g \pi_g' \) and \( \pi_{-g} = (\pi_g)^{-1} \) for all \( g, g' \in G \).

8. Since \( G \) is discrete, its dual space $\widehat{G}$ is compact. We normalize the Haar measure of $\widehat{G}$ so that $\mu(\widehat{G}) = 1$ and we define $L^\infty(\widehat{G})$ and $L^2(\widehat{G})$ as usual. Let $\mathcal{F}(x) = \widehat{x}$ denote the Fourier transform of $x$ which is defined by $\widehat{x}(\xi) = \sum_{g \in G} x(g)(-g, \xi)$ for $x$ in $\ell^1(G)$ and it is extended by density to a bijective isometry between $\ell^2(G)$ and $L^2(\widehat{G})$. See [9, 4.5 and 4.6] or [18, 1.2.7 and 2.2.2] for the most common cases, $G = \mathbb{Z}, \mathbb{Z}^d, \mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z}, \mathbb{Z}_d^s \times \mathbb{Z}_s, \mathbb{Z}^d \times \mathbb{Z}_r, \mathbb{Z} \times \mathbb{Z}_r \times \mathbb{Z}_s$. For instance:

\[
\mathbb{Z} \cong \mathbb{T} \quad \text{and} \quad \widehat{\mathbb{Z}}(z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n}, \quad z \in \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \},
\]

\[
\mathbb{Z}_s \cong \mathbb{Z} \quad \text{and} \quad \widehat{\mathbb{Z}_s}(z) = \sum_{n \in \mathbb{Z}_s} x(n)z^{-n}, \quad z = (z_1, \ldots, z_d) \in \mathbb{T}^d, \text{ where } z^n = z_1^{n_1} \cdots z_d^{n_d},
\]

\[
\mathbb{Z}_s \cong \mathbb{Z} \quad \text{and} \quad \widehat{\mathbb{Z}_s}(m) = \sum_{n \in \mathbb{Z}_s} x(n)W_s^{-nm}, \quad m \in \mathbb{Z}_s, \text{ where } W_s = e^{2\pi i/s},
\]
\[ \mathbb{Z}^d \times \mathbb{Z}_s = \mathbb{T}^d \times \mathbb{Z}_s \text{ and } \hat{\mathcal{F}}(\mathbf{z}, m) = \sum_{\mathbf{z} \in \mathbb{T}^d, m \in \mathbb{Z}_s} x(\mathbf{z}, n) \mathbf{z}^{-n} W_{s}^{-nm}, \mathbf{z} \in \mathbb{T}^d, m \in \mathbb{Z}_s. \]

(9) \( \mathcal{M}_{M \times N}(L^\infty(\hat{G})) \) is the set of \( M \times N \) matrices with entries in \( L^\infty(\hat{G}) \).

(10) For two functions \( X, Y : \hat{G} \to \mathbb{C}^M \) the notation \( X = Y \) means that \( X(\xi) = Y(\xi) \) a.e. \( \xi \in \hat{G} \).

### 3 Discrete convolution operators

We said that \( A : \ell^2_N(G) \to \ell^2_M(G) \) is a LTI operator if it commutes with translations, i.e. \( A T_g = T_g A \), for all \( g \in G \). A bounded LTI operator \( A \) can be expressed in the form

\[
[A(\mathbf{x})]_1 = a_{1,1} \ast x_1 + \cdots + a_{1,N} \ast x_N \\
\vdots \quad \vdots \quad \vdots \\
[A(\mathbf{x})]_M = a_{M,1} \ast x_1 + \cdots + a_{M,N} \ast x_N
\]

or in matrix notation

\[
A(\mathbf{x}) = A \ast \mathbf{x}, \quad \mathbf{x} \in \ell^2_N(G), \quad A = [a_{m,n}] \in \mathcal{M}_{M \times N}(\ell^2(G)).
\]

Indeed, if \( A \) is LTI and bounded,

\[
[A(\mathbf{x})]_m = \left[ A \left( \sum_{n=1}^{N} \sum_{g \in G} x_n(g) T_g(\delta e_n) \right) \right]_m = \sum_{n=1}^{N} \sum_{g \in G} x_n(g) T_g[A(\delta e_n)]_m
\]

\[
= \sum_{n=1}^{N} x_n \ast a_{m,n},
\]

for all \( \mathbf{x} \in \ell^2_N(G) \), where \( a_{m,n} = [A(\delta e_n)]_m \), \( \delta \) is the dirac delta and \( e_n \) the nth column of the identity matrix \( I_N \). Reciprocally, a convolution operator \( A : \ell^2_N(G) \to \ell^2_M(G) \), given by \( A(\mathbf{x}) = A \ast \mathbf{x} \), with \( A \in \mathcal{M}_{M \times N}(\ell^2(G)) \) commutes with translations. In the following theorem we characterize when it is bounded in terms of the Fourier transform of the matrix \( A \), which we call, as usual, the transfer matrix of \( A \),

\[
\hat{A} = [\hat{a}_{m,n}] \in \mathcal{M}_{M \times N}(L^\infty(\hat{G})).
\]

To prove the theorem we need the Lemma 2.2 of ref. [10]:

**Lemma 3.1** If \( x, y \in \ell^2(G) \) and \( \hat{x} \cdot \hat{y} \in L^2(\hat{G}) \) then \( x \ast y \in \ell^2(G) \) and \( \hat{x} \ast \hat{y} = \hat{x} \cdot \hat{y} \).

**Theorem 3.2** Let \( A \in \mathcal{M}_{M \times N}(\ell^2(G)) \). Then, \( A : \mathbf{x} \mapsto A \ast \mathbf{x} \) is a well defined bounded operator from \( \ell^2_N(G) \) into \( \ell^2_M(G) \) if and only if \( \hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{G})) \).
Proof Assume first that $\mathcal{A}$ is a well defined bounded operator from $\ell^2_n(G)$ into $\ell^2_m(G)$. Denoting by $a_{m,n}$ the $m$, $n$ entry of $A$ and $e_n$ the $n$th column of $\mathbb{I}_N$, we have that, for any $x \in \ell^2(G)$,

$$\|a_{m,n} \ast x\|_{\ell^2(G)} = \| [A \ast x e_n]_m \|_{\ell^2(G)} \leq \|A\| \|x e_n\|_{\ell^2_N(G)} = \|A\| \|x\|_{\ell^2(G)} \quad (3.2)$$

On the other hand, let us assume that $C$ is a number such that $|\hat{a}_{m,n}(\xi)| > C$ for all $\xi$ in a set $\Omega$ of positive measure. Since $\hat{G}$ is compact, $\chi_\Omega$, the characteristic function of $\Omega$, belongs to $L^2(\hat{G})$ and then $x = \mathcal{F}^{-1}(\chi_\Omega) \in \ell^2(G)$. By applying Lemma 3.1, we obtain, that for such $C$ and $x$ it is satisfied

$$\|a_{m,n} \ast x\|_{\ell^2(G)} = \|\hat{a}_{m,n} \ast \chi_\Omega\|_{L^2(\hat{G})} \leq C \|\chi_\Omega\|_{L^2(\hat{G})} = C \|x\|_{\ell^2(G)},$$

and then, from (3.2), $C \leq \|A\|$. Hence, we deduce $|\hat{a}_{m,n}(\xi)| \leq \|A\|$ a.e. and then $\hat{a}_{m,n} \in L^\infty(\hat{G})$.

Reciprocally, we assume that $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{G}))$. By applying Lemma 3.1, we obtain that $a_{m,n} \ast x = \hat{a}_{m,n} \ast \hat{x}$, for all $x \in \ell^2(G)$. Then, for any $x \in \ell^2(G)$,

$$\|a_{m,n} \ast x\|_{\ell^2(G)} = \|\hat{a}_{m,n} \ast \hat{x}\|_{L^2(\hat{G})} \leq \|\hat{a}_{m,n}\|_{L^\infty(\hat{G})} \|x\|_{\ell^2(G)}.$$ 

Using this inequality and (3.1), it follows easily that $\mathcal{A}$ is bounded. \hfill $\square$

Remark 3.3 If $A \in \mathcal{M}_{M \times N}(\ell^1(G))$ then $\hat{A} \in \mathcal{M}_{M \times N}(C(\hat{G})) \subseteq \mathcal{M}_{M \times N}(L^\infty(\hat{G}))$ and then $\mathcal{A}$ is bounded. The same is true for $A \in \mathcal{M}_{M \times N}(C^*(G))$, where $C^*(G) = \mathcal{F}^{-1}(C(\hat{G}))$, the group C*-algebra of $G$ [9, Section 7.1].

Theorem 3.4 Let $\mathcal{A} : \ell^2_N(G) \rightarrow \ell^2_M(G)$ a linear operator. Then the following statements are equivalent:

(a) $\mathcal{A}$ commutes with translations and is bounded,

(b) There exists $A \in \mathcal{M}_{M \times N}(\ell^2(G))$, such that $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{G}))$ and $A(x) = A \ast x$ for all $x \in \ell^2_N(G)$.

(c) There exists $\Lambda \in \mathcal{M}_{M \times N}(L^\infty(\hat{G}))$ such that $\hat{A}(x) = \Lambda \cdot \hat{x}$ for all $x \in \ell^2_N(G)$.

In this case, the matrices $A$ and $\Lambda$ in (b) and (c) are unique and $\Lambda = \hat{A}$.

Proof (a) $\iff$ (b) If (a) holds, from (3.1), $\mathcal{A}(x) = A \ast x, x \in \ell^2_N(G)$, where $A := [a_{m,n}]$ with $a_{m,n} = [\mathcal{A}(\delta e_n)]_m$. From Theorem 3.2, $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{G}))$. Reciprocally, if (b) holds, $\mathcal{A}$ is bounded from Theorem 3.2 and it commutes with translations since convolution operators do.

(b) $\iff$ (c) If (b) holds, by applying Lemma 3.1, we obtain that $\hat{A}(x) = \hat{A} \cdot \hat{x}$, for $x \in \ell^2_N(G)$, which proves (c). Reciprocally, we assume that (c) holds. Since $\hat{G}$ is compact, $L^\infty(\hat{G}) \subseteq L^2(\hat{G})$. Thus, $\Lambda \in \mathcal{M}_{M \times N}(L^2(\hat{G}))$ and then $A = \mathcal{F}^{-1}(\Lambda) \in \mathcal{M}_{M \times N}(\ell^2(G))$. Since the entries $a_{m,n}$ of $A = \Lambda$, belong to $L^\infty(\hat{G})$ by using Lemma 3.1 we obtain that, for any $x \in \ell^2_N(G)$,
\[ \widehat{A}(x) = \Lambda \cdot \hat{x} = \hat{A} \cdot \hat{x} = \left[ \sum_{n=1}^{N} a_{m,n} \cdot \hat{x}_n \right]_{m=1,\ldots,M} = \left[ \sum_{n=1}^{N} a_{m,n} \cdot \hat{x}_n \right]_{m=1,\ldots,M} \]

and then \( A(x) = \left[ \sum_{n=1}^{N} a_{m,n} \cdot x_n \right]_{m=1,\ldots,M} = A \cdot x \), which proves (b).

The matrix \( A \) satisfying (b) is unique since, from (3.1), the entry \( m, n \) of such matrix is necessarily given by \( a_{m,n} = \left[ A \ast (\delta e_n) \right]_m \). If \( \Lambda \) satisfies (c) we have proved that \( A(x) = \mathcal{F}^{-1}(\Lambda) \ast x \) for all \( x \in \ell^2_N(G) \). Therefore, the matrix \( \Lambda \) is necessarily \( \mathcal{F}(A) \), where \( A \) is the unique matrix satisfying (b).

For the simplest scalar case \( N = M = 1 \), Theorem 3.4 can be obtained as a corollary of [16, Theorems 4.1.1 and 4.3.1].

### 3.1 The algebra \( B_{\text{lt1}}(\ell^2_N(G)) \)

Let us denote by \( B_{\text{lt1}}(\ell^2_N(G)) \) the set of bounded LTI endomorphism of \( \ell^2_N(G) \),

\[ B_{\text{lt1}}(\ell^2_N(G)) := \left\{ A \in B(\ell^2_N(G)) : AT_g = T_g A \text{ for all } g \in G \right\}. \]

The following lemma proves that \( B_{\text{lt1}}(\ell^2_N(G)) \) is an algebra isomorphic to \( \mathcal{M}_N(L^\infty(\widehat{G})) \). Recall that \( \mathcal{M}_N(L^\infty(\widehat{G})) \) is an involution algebra. The product is the pointwise multiplication and the involution of a matrix \( \Lambda \) is the adjoint matrix \( \Lambda^* \).

**Lemma 3.5** \( B_{\text{lt1}}(\ell^2_N(G)) \) is a \(*\)-subalgebra of \( B(\ell^2_N(G)) \) and it is \(*\)-isomorphic to the algebra \( \mathcal{M}_N(L^\infty(\widehat{G})) \). Namely,

\[ B_{\text{lt1}}(\ell^2_N(G)) \ni A \mapsto \widehat{A} \in \mathcal{M}_N(L^\infty(\widehat{G})) \]

is a \(*\)-isomorphism.

**Proof** The transform \( A \rightarrow \widehat{A} \) is obviously linear and Theorem 3.4 proves that it is bijective from \( B_{\text{lt1}}(\ell^2_N(G)) \) onto \( \mathcal{M}_N(L^\infty(\widehat{G})) \). Let \( A, B \in B_{\text{lt1}}(\ell^2_N(G)) \). By applying Theorem 3.4 twice we obtain that

\[ \widehat{BA}(x) = \widehat{B} \cdot \widehat{A}(x) = \widehat{B} \cdot \widehat{A} \cdot \hat{x} \quad x \in \ell^2_N(G). \]

where \( \widehat{A}, \widehat{B} \in \mathcal{M}_N(L^\infty(\widehat{G})) \). Since \( \widehat{B} \cdot \widehat{A} \in \mathcal{M}_N(L^\infty(\widehat{G})) \) and Theorem 3.4, \( BA \in B_{\text{lt1}}(\ell^2_N(G)) \) and \( BA \rightarrow \widehat{B} \cdot \widehat{A} \). Using Theorem 3.4 we obtain, that for all \( x, y \in \ell^2_N(G) \),

\[ \langle \hat{x}, A^* \hat{y} \rangle_{L^2_N(\widehat{G})} = \langle x, A^* y \rangle_{\ell^2_N(G)} = \langle Ax, y \rangle_{\ell^2_N(G)} = \langle \widehat{A} \cdot \hat{x}, \hat{y} \rangle_{L^2_N(\widehat{G})} = \langle \widehat{A} \cdot \hat{x}, \hat{y} \rangle_{L^2_N(\widehat{G})} = \langle \hat{x}, (A^*) \hat{y} \rangle_{L^2_N(\widehat{G})}. \]
Hence \( \mathcal{A}^* \hat{\mathbf{y}} = (\hat{A})^* \cdot \hat{\mathbf{y}} \), for all \( \mathbf{y} \in \ell^2_N(G) \). Hence, since \((\hat{A})^* \in \mathcal{M}_N(L^\infty(\hat{G}))\) and Theorem 3.4 we obtain that \( \mathcal{A}^* \in \mathcal{B}_{LT1}(\ell^2_N(G))\) and \( \mathcal{A}^* \rightarrow (\hat{A})^* \). \( \square \)

**Remark 3.6** Denote by

\[
P(G) := \mathcal{F}^{-1}[L^\infty(\hat{G})] = \{ x \in \ell^2(G) : \hat{x} \in L^\infty(\hat{G}) \},
\]

the space of pseudomeasures on \( G \), see [16, Section 4.2] and [4, Section 3.1.8]. From Lemma 3.5, it could be easily proved that the set of matrices \( \mathcal{M}_N(P(G)) \) is an involution algebra where the product is the convolution and the involution \( A^* \) of \( A \), is not the adjoint of \( A \), but

\[
A^* = [a^*_{m,n}]^T \in \mathcal{M}_N(P(G)), \quad \text{where} \quad a^*_{m,n}(g) := a_{m,n}(-g), \quad g \in G. \tag{3.3}
\]

Besides \( \mathcal{B}_{LT1}(\ell^2_N(G)) \ni A \mapsto A \in \mathcal{M}_N(P(G)) \) is a \(*\)-isomorphism.

**3.2 The norm in \( \mathcal{B}_{LT1}(\ell^2_N(G)) \)**

The algebra \( \mathcal{M}_N(L^\infty(\hat{G})) \) is a \( C^* \)-algebra, namely it is the matrix algebra on the \( C^* \)-algebra \( L^\infty(\hat{G}) \) [6, Section II.6.6]. The faithful representation of \( L^\infty(\hat{G}) \), \( \sigma : L^\infty(\hat{G}) \rightarrow \mathcal{B}(L^2_N(\hat{G})) \), defined by \( \sigma(y)x = y \cdot x \) gives the faithful representation of \( \mathcal{M}_N(L^\infty(\hat{G})) \),

\[
\gamma : \mathcal{M}_N(L^\infty(\hat{G})) \rightarrow \mathcal{B}(L^2_N(\hat{G})), \quad \gamma(\Lambda)X = \Lambda \cdot X,
\]

which provides an expression for the \( C^* \)-norm of \( \mathcal{M}_N(L^\infty(\hat{G})) \),

\[
\| \Lambda \|_{\mathcal{M}_N(L^\infty(\hat{G}))} = \| \gamma(\Lambda) \|_{\mathcal{B}(L^2_N(\hat{G}))} = \sup_{\| X \|_{L^2_N(\hat{G})} = 1} \| \Lambda \cdot X \|_{L^2_N(\hat{G})}. \tag{3.4}
\]

The algebra \( \mathcal{M}_N(L^\infty(\hat{G})) \) can also be seen as the algebra \( L^\infty(\hat{G}, \mathcal{M}_N(\mathbb{C})) \), that is, the algebra of the essentially bounded functions from \( \hat{G} \) into the \( C^* \)-algebra \( \mathcal{M}_N(\mathbb{C}) \). This provides a simpler expression for the \( C^* \)-norm of \( \mathcal{M}_N(L^\infty(\hat{G})) \),

\[
\| \Lambda \|_{\mathcal{M}_N(L^\infty(\hat{G}))} = \text{ess sup}_{\xi \in \hat{G}} \| \Lambda(\xi) \|_2. \tag{3.5}
\]

**Theorem 3.7** \( \mathcal{B}_{LT1}(\ell^2_N(G)) \) is a \( C^* \)-subalgebra of \( \mathcal{B}(\ell^2_N(G)) \) which is \(*\)-isometric to the \( C^* \)-algebra \( \mathcal{M}_N(L^\infty(\hat{G})) \). Namely

\[
\mathcal{B}_{LT1}(\ell^2_N(G)) \ni A \mapsto \hat{A} \in \mathcal{M}_N(L^\infty(\hat{G}))
\]

is an isometry \(*\)-isomorphism (a \( C^* \)-isomorphism).
Proof By using Theorem 3.4, that the Fourier transform is a isometric isomorphism from $\ell^2_N(G)$ onto $L^2_N(\widehat{G})$, and (3.4), we obtain that, for any $A \in B_{LT}(\ell^2_N(G))$, 
\[
\|A\| = \sup_{\|x\|^2_{\ell^2_N(G)}=1} \|A(x)\|_{\ell^2_N(G)} = \sup_{\|x\|^2_{\ell^2_N(G)}=1} \|\widehat{A}(x)\|_{L^2_N(\widehat{G})} = \sup_{\|x\|^2_{\ell^2_N(G)}=1} \|\widehat{A} \cdot \hat{x}\|_{L^2_N(\widehat{G})} = \sup_{\|X\|^2_{L^2_N(\widehat{G})}=1} \|\widehat{A} \cdot X\|_{L^2_N(\widehat{G})} = \|A\|_{\mathcal{M}_N(L^\infty(\widehat{G}))}.
\]

Now the theorem follows from Lemma 3.5. \hfill \Box

From Theorem 3.7, having in mind the expression for the norm (3.5), we obtain the following consequence.

Corollary 3.8 For any $A \in B_{LT}(\ell^2_N(G))$ with transfer matrix $\widehat{A}$, we have 
\[
\|A\| = \operatorname{ess sup}_\xi \|\widehat{A}(\xi)\|_2.
\]

For the case $G = \mathbb{Z}$ this expression for the norm was given in [1, Theorem 2.2].

Remark 3.9 The norm given by this corollary is difficult to compute. Reference [5, Theorem 3] provides the estimation $\|A\| = \|\widehat{A}\|_{\mathcal{M}_N(L^\infty(\widehat{G}))} \leq \|A\|_2$ where $A = [\|\widehat{a}_{m,n}\|_{L^\infty(\widehat{G})}] \in \mathcal{M}_N(\mathbb{R})$.

Remark 3.10 The involution algebra $\mathcal{M}_N(P(G))$ defined in Remark 3.6 with the norm $\|A\| = \operatorname{ess sup}_\xi \|\widehat{A}(\xi)\|_2$ is a C*-algebra which is C*-isometric to $B_{LT}(\ell^2_N(G))$ and to $\mathcal{M}_N(L^\infty(\widehat{G}))$.

### 3.3 The invertible elements in $B_{LT}(\ell^2_N(G))$

The following theorem characterizes the units of the algebra $B_{LT}(\ell^2_N(G))$ and provides an expression for the norm of the inverse.

Theorem 3.11 Let $A \in B_{LT}(\ell^2_N(G))$ with transfer matrix $\widehat{A}$. Then $A$ is invertible in the C*-algebra $B_{LT}(\ell^2_N(G))$ if and only if $\operatorname{ess inf}_\xi \det(\widehat{A}(\xi)) > 0$. In this case 
\[
\|A^{-1}\| = \left(\operatorname{ess inf}_\xi \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]\right)^{-1/2}.
\]

Proof We will prove the equivalent assertion (see Theorem 3.7): $\Lambda \in \mathcal{M}_N(L^\infty(\widehat{G}))$ is invertible in the C*-algebra $\mathcal{M}_N(L^\infty(\widehat{G}))$ if and only if $\operatorname{ess inf}_\xi \det(\Lambda(\xi)) > 0$, and in this case 
\[
\|\Lambda^{-1}\|_{\mathcal{M}_N(L^\infty(\widehat{G}))} = \left(\operatorname{ess inf}_\xi \lambda_{\min}[\Lambda(\xi)^*\Lambda(\xi)]\right)^{-1/2}.
\]
If \( \text{ess inf}_{\xi \in G} |\det \Lambda(\xi)| > 0 \) then there exists the inverse matrix \( [\Lambda(\xi)]^{-1} \) a.e. \( \xi \in \hat{G} \). Besides, since \( \Lambda \in \mathcal{M}_N(L^\infty(\hat{G})) \) and \( \text{ess inf}_{\xi \in G} |\det \Lambda(\xi)| > 0 \), we deduce that \( \Lambda^{-1} \in \mathcal{M}_N(L^\infty(\hat{G})) \). Reciprocally, if \( \Lambda^{-1} \in \mathcal{M}_N(L^\infty(\hat{G})) \) then \( \det \Lambda^{-1} \in L^\infty(\hat{G}) \), and then, having in mind that \( \det \Lambda \det \Lambda^{-1} = 1 \), we deduce that \( \text{ess inf}_{\xi \in G} |\det \Lambda(\xi)| > 0 \).

In order to obtain the expression for the norm, note that since \( |\det \Lambda(\xi)| > 0 \) a.e. we have \( \lambda_{\min}[\Lambda(\xi)^*\Lambda(\xi)] > 0 \) a.e. Having in mind this fact and (3.5), we obtain that

\[
\|\Lambda^{-1}\|_{\mathcal{M}_N(L^\infty(\hat{G}))}^{-2} = \left( \text{ess sup}_{\xi \in G} \|\Lambda^{-1}(\xi)\|_2 \right)^{-2} = \left( \text{ess sup}_{\xi \in G} \lambda_{\min}^{-1}[\Lambda(\xi)^*\Lambda(\xi)] \right)^{-1}
\]

\[
= \text{ess inf}_{\xi \in G} \lambda_{\min}[\Lambda(\xi)^*\Lambda(\xi)].
\]

Whenever \( \hat{A} \in \mathcal{M}_{m \times n}(C(\hat{G})) \), the characterization in Theorem 3.11 can be obtained as a corollary of [5, Theorem 4] and Theorem 3.7.

\[\Box\]

4 Riesz systems generated by actions of abelian groups

Let \( \mathcal{H} \) be a separable complex Hilbert space and \( \pi \) a unitary representation of the group \( G \) on \( \mathcal{H} \), i.e. a homomorphism \( \pi : G \to U(\mathcal{H}) \), where \( U(\mathcal{H}) \) denotes the group of unitary operators of \( \mathcal{H} \), that satisfies \( \pi_g g' = \pi_g \pi_g' \), \( \pi_{-g} = \pi^{-1}_g = \pi^*_g \), for all \( g, g' \in G \).

For a set \( \Phi = \{\varphi_1, \ldots, \varphi_N\} \) of \( N \) elements of \( \mathcal{H} \), we consider the system

\[
\{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G},
\]

where \( \mathcal{N} = \{1, 2, \ldots, N\} \). It is said that \( \{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G} \) is a Riesz sequence of \( \mathcal{H} \) when there exist constants \( 0 < \alpha \leq \beta < \infty \) such that

\[
\alpha \|x\|_{\ell^2_N(G)}^2 \leq \sum_{n \in \mathcal{N}, g \in G} x_n(g) \pi_g \varphi_n \|_{\mathcal{H}}^2 \leq \beta \|x\|_{\ell^2_N(G)}^2,
\]

for all \( x = (x_1, \ldots, x_N) \in \ell^2_N(G) \). In this case, the system \( \{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G} \) is a Riesz basis for the space [8, Section 3.6]

\[
V_\Phi = \text{span}\{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G} = \left\{ \sum_{n \in \mathcal{N}, g \in G} x_n(g) \pi_g \varphi_n : (x_1, \ldots, x_N) \in \ell^2_N(G) \right\}.
\]

Note that \( V_\Phi \) is invariant by actions of the group \( G \). When \( \mathcal{H} = L^2(\mathbb{R}^d) \), \( G = \mathbb{Z}^d \) and \( \pi_g f = T_g f = f(\cdot - g) \), these spaces, called shift-invariant spaces, have been widely studied given its importance in wavelets and approximation theory.
The largest $\alpha$ and the smallest $\beta$ satisfying (4.1) are called the optimal Riesz bounds. When the right inequality in (4.1) holds, it is said that $\{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G}$ is a Bessel sequence of $\mathcal{H}$ with Bessel bound $\beta$.

Let

$$a_{m,n}(g) := \langle \varphi_n, \pi_g \varphi_m \rangle \mathcal{H}, \quad A(g) := [a_{m,n}(g)] \in \mathcal{M}_N(\mathbb{C}), \quad g \in G.$$  \hspace{1cm} (4.2)

For any finite sequences $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$, we have

$$\left\langle \sum_{n \in \mathcal{N}, g' \in G} x_n(g') \pi_g \varphi_n, \sum_{m \in \mathcal{N}} y_m(g) \pi_g \varphi_m \right\rangle = A(g) \left\langle \sum_{n \in \mathcal{N}, g \in G} x_n(g') \pi_g \varphi_n, \sum_{m \in \mathcal{N}} y_m(g) \pi_g \varphi_m \right\rangle \mathcal{H} y_m(g)$$

$$= \sum_{n,m \in \mathcal{N}, g \in G} (x_n \ast a_{m,n})(g) y_m(g) = \sum_{m \in \mathcal{N}} [A \ast x(g)]_m y_m(g) = \left\langle A \ast x, y \right\rangle_{\ell^2(G)}.$$  \hspace{1cm} (4.3)

Thus, the properties of the system $\{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G}$ are related to the properties of the convolution operator $A(x) = A \ast x$.

Note that, whenever equalities (4.3) also hold for any $x, y \in \ell^2_N(G)$, the operator $A$ is positive since in this case $\left\langle A(x), x \right\rangle_{\ell^2_N(G)} = \left\| \sum_{n \in \mathcal{N}, q \in G} x_n(q) \pi_q \varphi_n \right\|_\mathcal{H}^2 \geq 0$ for all $x \in \ell^2_N(G)$.

In order to use the usual notation in this context, we state the results in terms of the transpose of the transfer matrix (the Gram matrix),

$$\mathcal{G} := \hat{A}^\top,$$  \hspace{1cm} (4.4)

which is well defined and belongs to $\mathcal{M}_N(L^2(\hat{G}))$ provided that $A$ belongs to $\mathcal{M}_N(\ell^2(G))$.

**Lemma 4.1** Let $A \in \mathcal{B}(\ell^2_N(G))$ be a positive operator. Then, the operator $A$ is invertible if and only if $\inf_{\|x\|=1} \left\langle A(x), x \right\rangle_{\ell^2_N(G)} > 0$. In this case $\|A^{-1}\| = \left(\inf_{\|x\|=1} \left\langle A(x), x \right\rangle_{\ell^2_N(G)}\right)^{-1}$.

**Proof** Since $\left\langle A(x), x \right\rangle_{\ell^2_N(G)} = \|A^{1/2}(x)\|^2_{\ell^2_N(G)}$, if $\inf_{\|x\|=1} \left\langle A(x), x \right\rangle_{\ell^2_N(G)} > 0$, $A^{1/2}$ is invertible and then $A$ is invertible. Reciprocally, if $A$ is invertible then $A^{1/2}$ is invertible and

$$\inf_{\|x\|=1} \left\langle A(x), x \right\rangle_{\ell^2_N(G)} = \inf_{\|x\|=1} \|A^{1/2}(x)\|^2_{\ell^2_N(G)} = \|A^{-1/2}\|^{-2} = \|A^{-1}\|^{-1} > 0.$$

$\square$

**Theorem 4.2** Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{H}$ and $A, \mathcal{G}$ the matrices defined in (4.2) and (4.4). We assume that $A$ belongs to $\mathcal{M}_N(\ell^2(G))$. Then

(a) The system $\{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G}$ is a Bessel sequence of $\mathcal{H}$ if and only if $\mathcal{G} \in \mathcal{M}_N(L^\infty(\hat{G}))$. In this case, the matrix $\mathcal{G}(\xi)$ is semidefinite positive a.e. $\xi \in \hat{G}$, and the optimal Bessel bound is $\text{ess sup}_{\xi \in \hat{G}} \lambda_{\text{max}} [\mathcal{G}(\xi)]$.
(b) The system \( \{ \pi_g \varphi_n \}_{n \in \mathcal{N}, g \in G} \) is a Riesz sequence of \( \mathcal{H} \) if and only if

\[
\mathcal{G} \in \mathcal{M}_\mathcal{N}(L^\infty(\hat{G})) \quad \text{and} \quad \text{ess inf} \det \mathcal{G}(\xi) > 0.
\]

In this case, the optimal Riesz bounds are

\[
\text{ess inf} \lambda_{\min}[\mathcal{G}(\xi)] \quad \text{and} \quad \text{ess sup} \lambda_{\max}[\mathcal{G}(\xi)].
\]

Proof Let \( \mathcal{A} \) denote the operator defined by \( \mathcal{A}(x) = A * x, x \in \ell^2_N(G) \).

(a) We assume first that \( \{ \pi_g \varphi_n \}_{n \in \mathcal{N}, g \in G} \) is a Bessel sequence of \( \mathcal{H} \) with Bessel bound \( \beta \). Then, for any \( x \in \ell^2_N(G) \), the series \( \sum_{n \in \mathcal{N}, g \in G} x_n(q) \pi_g \varphi_n \) converges in \( \mathcal{H} \). Hence, we deduce that the equalities (4.3) hold for any sequences \( x, y \in \ell^2_N(G) \). For \( x = (x_1, \ldots, x_N) \) let us denote

\[
f_x = \sum_{n \in \mathcal{N}, g \in G} x_n(g) \pi_g \varphi_n.
\]

Since \( \{ \pi_g \varphi_n \}_{n \in \mathcal{N}, g \in G} \) is a Bessel sequence with bound \( \beta \), we have \( \| f_x \|^2_\mathcal{H} \leq \beta \| x \|^2_{\ell^2_N(G)} \) for all \( x \in \ell^2_N(G) \). Then, using (4.3), we obtain that, for any \( x, y \in \ell^2_N(G) \),

\[
|\langle \mathcal{A}(x), y \rangle_{\ell^2_N(G)}| = |\langle f_x, f_y \rangle_{\mathcal{H}}| \leq \| f_x \|_{\mathcal{H}} \| f_y \|_{\mathcal{H}} \leq \beta \| x \|_{\ell^2_N(G)} \| y \|_{\ell^2_N(G)}.
\]

Hence, the sesquilinear functional \( (x, y) \mapsto \langle \mathcal{A}(x), y \rangle_{\ell^2_N(G)} \) is bounded. Then, the convolution operator \( \mathcal{A} \) belongs to \( \mathcal{B}(\ell^2_N(G)) \). Then, from Theorem 3.2, the matrix \( \mathcal{G} = \hat{A}^\top \) belongs to \( \mathcal{M}_\mathcal{N}(L^\infty(\hat{G})) \).

Reciprocally, assume now that \( \mathcal{G} \in \mathcal{M}_\mathcal{N}(L^\infty(\hat{G})) \). Then, from Theorem 3.2, the operator \( \mathcal{A} \) belongs to \( \mathcal{B}(\ell^2_N(G)) \). By using (4.3), we obtain that for any finite sequence \( x \),

\[
\left\| \sum_{n \in \mathcal{N}, g \in G} x_n(g) \pi_g \varphi_n \right\|^2_\mathcal{H} = \langle \mathcal{A}(x), x \rangle_{\ell^2_N(G)} \leq \| \mathcal{A} \| \| x \|^2_{\ell^2_N(G)}.
\]

Then, \( \{ \pi_g \varphi_n \}_{n \in \mathcal{N}, g \in G} \) is a Bessel sequence of \( \mathcal{H} \) [8, Theorem 3.6.6].

Assume now that the equivalent conditions of (a) hold. Hence, (4.3) holds for any sequences \( x, y \in \ell^2_N(G) \). Then \( \mathcal{A} \) is a positive operator, and thus a positive element of the \( \mathcal{C}^* \)-algebra \( \mathcal{B}(\ell^2_N(G)) \). Hence, it is a positive element of the \( \mathcal{C}^* \)-subalgebra \( \mathcal{B}_{LTI}(\ell^2_N(G)) \) [6, II.3.1]. Using Theorem 3.7, we obtain that there exists \( \Lambda \in \mathcal{M}_\mathcal{N}(L^\infty(\hat{G})) \) such that \( \hat{A} = \Lambda^* \Lambda \), and thus the matrix \( \hat{A}(\xi) = \mathcal{G}(\xi)^\top \) is semidefinite positive a.e. \( \xi \in \hat{G} \). Then, from Corollary 3.8 and (4.3),

\[\text{ess inf} \lambda_{\min}[\mathcal{G}(\xi)] \quad \text{and} \quad \text{ess sup} \lambda_{\max}[\mathcal{G}(\xi)].\]
\[
\text{ess sup } \lambda_{\text{max}} [G(\xi)] = \text{ess sup } \|A(\xi)\|_2 = \|A\| = \sup_{\|x\|_{G(G)} = 1} \left\{ A(x), x \right\}_{\ell^2_N(G)} \]

\[
= \sup_{\|x\|_{G(G)} = 1} \left\| \sum_{n\in\mathbb{N}, q\in\mathcal{G}} x_n(q) \pi_g \phi_n \right\|^2_{\mathcal{H}},
\]

which proves (a).

(b) Since (a), we just have to prove that \(\text{ess inf } \xi \in \hat{G} \det G(\xi) > 0\) if and only if the left inequality in (4.1) holds, and that the lower optimal Riesz bound is \(\text{ess inf } \xi \in \hat{G} \lambda_{\text{min}}(G(\xi))\). Besides, we have proved that in any of the hypotheses in (b), the equalities (4.3) hold for any \(x, y \in \ell^2_N(G)\), \(A\) is positive operator, and \(\hat{A}(\xi)\) and \(G(\xi)\) are semidefinite positive a.e. \(\xi \in \hat{G}\).

Assume first that \(\text{ess inf } \xi \in \hat{G} \det G(\xi) > 0\). From Theorem 3.11, the operator \(A\) is invertible, and

\[
0 < \|A^{-1}\|^{-2} = \text{ess inf } \lambda_{\text{min}} [\hat{A}(\xi)^* \hat{A}(\xi)] = \text{ess inf } \lambda_{\text{min}}^2 [\hat{A}(\xi)] \]

\[
= \text{ess inf } \lambda_{\text{min}}^2 [G(\xi)].
\]

Hence, using (4.3) and Lemma 4.1, we obtain

\[
\inf_{\|x\|=1} \left\| \sum_{n\in\mathbb{N}, q\in\mathcal{G}} x_n(q) \pi_g \phi_n \right\|^2_{\mathcal{H}} = \inf_{\|x\|=1} \left\{ A(x), x \right\}_{\ell^2_N(G)} = \|A^{-1}\|^{-1}
\]

\[
= \text{ess inf } \lambda_{\text{min}} [G(\xi)] > 0.
\]

Therefore \(\{\pi_g(\phi_m)\}_{n\in\mathbb{N}, g\in\mathcal{G}}\) is a Riesz sequence of \(\mathcal{H}\) and the optimal lower Riesz bound is \(\text{ess inf } \xi \in \hat{G} \lambda_{\text{min}}[G(\xi)]\).

To prove the reciprocal, assume now that \(\{\pi_g \phi_n\}_{n\in\mathbb{N}, g\in\mathcal{G}}\) is a Riesz sequence of \(\mathcal{H}\). Then, from (4.3), we have that

\[
\inf_{\|x\|=1} \left\{ A(x), x \right\}_{\ell^2_N(G)} = \inf_{\|x\|=1} \left\| \sum_{n\in\mathbb{N}, q\in\mathcal{G}} x_n(q) \pi_g \phi_n \right\|^2_{\mathcal{H}} > 0.
\]

Hence, from Lemma 4.1, we obtain that the operator \(A\) is invertible. Hence, from Theorem 3.11, \(\text{ess inf } \xi \in \hat{G} \det G(\xi) = \text{ess inf } \xi \in \hat{G} \det \hat{A}(\xi) > 0\).

For the classical shift-invariant systems, \(\{\pi_g(\phi_m)\}_{n\in\mathbb{N}, g\in\mathcal{G}} = \{T_g \phi_n\}_{n\in\mathbb{N}, g\in\mathbb{Z}^d} \subset L^2(\mathbb{R}^d)\), the result of this theorem is very well known (see for example [14,17]), and it has many applications in wavelet theory and approximation theory. It is given usually in terms of

\[
\sum_{\alpha\in\mathbb{Z}^d} \hat{\phi}_n(\xi + 2\pi \alpha) \hat{\phi}_m(\xi + 2\pi \alpha)
\]
which is equal to \( \hat{a}_{m,n}(e^{i\xi}) \) under appropriate conditions, see [13, eq. 4.1] and [14, Thm. 3.2].

The result given in Theorem 4.2 is related to many of the generalizations previously considered: For the systems \( \{\pi_g \varphi_n\}_{n \in \mathcal{N}, g \in G} = \{U_1^{g_1} \cdots U_d^{g_d} \varphi_n\}_{n \in \mathcal{N}, (g_1, \ldots, g_d) \in \mathbb{Z}^d} \), where \( U_1, \ldots, U_d \) are unitary operators of \( \mathcal{H} \), the result was given in [1,12]; For the systems \( \{T_g \varphi_n\}_{n \in \mathcal{N}, g \in G} \), where the set of generators \( \{\varphi_n\}_{n \in \mathcal{N} \subset L^2(S)} \) can be countable, \( S \) is a LCA group, and \( G \) is a discrete subgroup of \( S \) such that \( S/G \) is compact, the corresponding result was given in [7]; For the systems \( \{\pi_g \varphi\}_{g \in G} \), where the representation \( \pi \) satisfies the, so called, dual integrability condition, the result was given in [13], see [2] for the non abelian case with a countable set of generators.

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