THE COMMENTS ON QED CONTRIBUTIONS TO \((g - 2)_\mu\)

A.L. Kataev\textsuperscript{a, b}

\textit{Institute for Nuclear Research, 117312 Moscow, Russia}

Abstract. The comparison of definite numerical values of analytically evaluated asymptotic expressions for order \(\alpha^4\) and \(\alpha^5\) QED contributions to the muon anomalous magnetic moment with the results of numerical calculations are presented. It is stressed that observed agreement can be considered as the additional argument in favour of correctness of the recent direct numerical calculations.

1 Introduction

The calculations of perturbative QED contributions to the muon anomalous magnetic moments \(a_\mu\) usually attract special interest. It is supported by the fact that for this low-energy quantity the interplay between pure QED contributions, electroweak (EW) and strong interactions effects plays important role. Moreover, the importance of taking into account strong interaction contributions to \(a_\mu\) was clearly demonstrated by comparison of the most recent experimental data for \(a_\mu\) [1] with available theoretical predictions (for a review see [2]). It reveals the appearance of definite, though not so clean, deviations of theoretical and experimental results. The existence of this deviation pushed ahead the desire to improve the knowledge about high order QED corrections to \(a_\mu\). As a result, the numerical calculations of the sum of 2958 most important five-loop diagrams, which depends on \(m_\mu/m_e\)-ratio, were recently performed [3].

Here we concentrate ourselves on the comparison of the numerical results for order \(\alpha^4\) and \(\alpha^5\) contributions to \(a_\mu\), obtained in Refs. [4]-[9] from analytical high-loop contributions to the photon vacuum polarization function, with the existing results of direct numerical calculations.

2 The four-loop contributions

The renormalization group method plays an important role in the high-order perturbative calculations. As was shown in Ref. [10], the coefficients of the \(\ln(m_\mu/m_e)\)-terms in the expression for the vacuum polarization insertions into \(a_\mu\) can be determined with the help of the renormalization group equations in the on-shell scheme. The general expression for the contributions to \(a_\mu\) of the diagrams with dressed by electron loops internal photon line in the muon vertex reads

\[
a_\mu = \frac{\alpha}{\pi} \int_0^1 dy (1 - y)[d_R^\infty \left(\frac{-y^2 m_\mu^2}{1 - y m_e^2}, \alpha\right) - 1]
\]

\textsuperscript{a}e-mail: kataev@ms2.inr.ac.ru

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where \( d_{\infty}^R(\alpha, x) = [1 + (\alpha/\pi)\Pi(\alpha, x)]^{-1} \) is the asymptotic photon propagator. Taking into account analytical expressions for the constant and logarithmic contributions to the photon vacuum polarization function in the on-shell scheme, one can obtain the number of asymptotic expressions for the diagrams, contributing to \( a_\mu \). The coefficients for the photon vacuum polarization function can be obtained using the renormalization group relations between QED \( \beta \)-function in the on-shell scheme and QED \( \beta \)-function in the momentum subtractions scheme which is known in the literature as the \( \Psi \)-function (see e.g. [11,12]).

The example of application of this formalism is the analytical calculation of the asymptotic limit of the subset of four-loop diagrams, which contain in the photon line of the external muon vertex three-loop photon vacuum polarization contribution with two electron loops (Subset 1). The preliminary consideration of Ref. [4] (see [13]) was based on the application of analytical results for the four-loop contributions to the \( \Psi \)-function from the photon vacuum polarization diagram with three electron loops [14] (confirmed later on in [15]) and of the analogous four-loop corrections to the \( \beta \)-function in the on-shell scheme, taken from Ref. [16]. These considerations gave the following numerical contribution to \( a_\mu \) [13] (see also [4])

\[
a_\mu(\text{Subset 1}) = [0.923374\ldots + O(m_e/m_\mu)] \left( \frac{\alpha}{\pi} \right)^4
\] (2)

which was in disagreement with the result of numerical calculations of Ref. [17], namely

\[
a_\mu(\text{Subset 1}) = 1.4416(18) \left( \frac{\alpha}{\pi} \right)^4.
\] (3)

As was found in Ref. [5], this disagreement came from a theoretical error, which entered on-shell studies of Ref. [16]. It was corrected in Ref. [5] and the new asymptotic expression was obtained [4,5]:

\[
a_\mu(\text{Subset 1}) = [1.452570\ldots + O(m_e/m_\mu)] \left( \frac{\alpha}{\pi} \right)^4.
\] (4)

It differs now from Eq.(3) in the third significant digit only. This discrepancy can be explained by the existence in Eq.(4) of the \( O(m_\mu/m_e) \)-corrections. The problem of comparing four-loop asymptotic and numerical contributions to \( a_\mu \) appeared again in the process of consideration of the subset of diagrams, generated by the insertion into the internal photon line of the muon vertex three-loop vacuum polarization graphs in the “quenched” approximation, which do not contain internal loops. (Subset 2). Detailed calculational analysis of Ref. [8] gave the following numerical expression:

\[
a_\mu(\text{Subset 2}) = [-0.290987\ldots + O(m_e/m_\mu)] \left( \frac{\alpha}{\pi} \right)^4.
\] (5)
while the numerical results of Ref. [17] were

$$a_\mu(\text{Subset 2}) = -0.7945(202) \left( \frac{\alpha}{\pi} \right)^4.$$  \hspace{1cm} (6)

The disagreement was mostly removed in Ref. [18] after re-evaluating these diagrams using the integration VEGAS routine (the result of Eq.(6) was obtained using the integration routine RIWiAD). It was explained in Ref. [18] that the new number

$$a_\mu(\text{Subset 2}) = -0.2415(19) \left( \frac{\alpha}{\pi} \right)^4$$  \hspace{1cm} (7)

differed from the one of Eq.(6) due to severe underestimation of errors, which appeared in the process of the calculations of Ref [17]. Moreover, it is closer to the asymptotic expression of Eq.(5), though definite difference of order $O(m_e/m_\mu)$ still remained. This difference stimulated the authors of Ref. [9] to improve the precision of Eq.(5), combining the asymptotic [19, 20] and threshold [21] results and applying the developed in Ref. [22] variant of the Padé resummation technique. After these improvements Eq.(5) moved in the direction of numerical value of Eq.(7). Indeed, the delicate calculation of Ref. [9] gave

$$a_\mu(\text{Subset 2}) = -0.230362(5) \left( \frac{\alpha}{\pi} \right)^4.$$  \hspace{1cm} (8)

Finally, in the work of Ref. [23] the following new result was obtained:

$$a_\mu(\text{Subset 2}) = -0.230596(416) \left( \frac{\alpha}{\pi} \right)^4.$$  \hspace{1cm} (9)

It is in perfect agreement with Eq.(8). The remaining difference between the results of Eq.(9) and Eq.(7) was traced to a problem discovered in Ref. [23] of rounding-off errors caused by keeping an insufficient number of effective digits in carrying out renormalization by numerical means. Thus the discovered in Ref. [8] discrepancy between the results of analytically-oriented and numerical calculations was finally eliminated.

3 The five-loop contributions

The calculations of the asymptotic contributions to $a_\mu$ from the diagrams with single dressed internal photon line was continued at the five-loop level in Refs. [6]- [8]. In particular, Ref. [6] shows that the coefficient of the subset of diagrams with internal four-loop light-by-light scattering graphs, composed from diagrams with two electron loops (Subset 3), is small

$$a_\mu(\text{Subset 3}) = \left[ -\frac{a_0^{[2.1-1]}}{2} - 2.0237 + O\left( \frac{m_e}{m_\mu} \right)\right] \left( \frac{\alpha}{\pi} \right)^5.$$  \hspace{1cm} (10)
where \( a^{[2,l-l]}_4 \) is the constant term of the logarithmically divergent sum of the four-loop light-by-light scattering contributions to the photon vacuum polarization function. The subset of the five-loop contributions to \( a_\mu \), formed by insertion into photon line of the sum of the four-loop “quenched” vacuum polarization graphs (Subset 4), gives the similar five-loop correction [7]

\[
a_\mu(\text{Subset } 4) = \left[ -\frac{a^{[1]}_4}{2} - 0.7334 + O\left(\frac{m_e}{m_\mu}\right)\right] \left(\frac{\alpha}{\pi}\right)^5
\]  

where \( a^{[1]}_4 \) is the constant term of the sum of the four-loop “quenched” vacuum polarization diagrams. The contribution to \( a_\mu \) from the five-loop diagrams of Subset 5, which is generated by inserting into photon line of the four-loop vacuum polarization graphs with two electron loops, also contain still unknown constant terms [7]:

\[
a_\mu(\text{Subset } 5) = \left[ -\frac{a^{[2]}_4}{2} + 0.6642 + O\left(\frac{m_e}{m_\mu}\right)\right] \left(\frac{\alpha}{\pi}\right)^5
\]  

The numerical calculations of the diagrams contributing to Eq.(10)-Eq.(12) are still unavailable. However, there are subsets of five-loop diagrams contributing to \( a_\mu \), which can be approximated by their asymptotic expressions and were numerically calculated recently in the process of the work of Ref. [3]. These are the diagrams of six concrete subsets. Subset 6 is formed by insertion into photon line of the muon vertex four-loop vacuum polarization graphs with three electron loops. Subset 7 consists of diagrams, generated by dressing this photon line by two two-loop vacuum polarization insertions with electron loops. The photon line of the diagrams of Subset 8 contains three-loop photon vacuum polarization insertions with two electron loops and the additional one-loop bubble. Subset 9 represents the sum of the diagrams, which have “quenched” three-loop and one-loop subsequent vacuum polarization insertions into photon line. Thus, Subset 8 and Subset 9 can be generated from the four-loop Subset 1 and Subset 2 by inserting into their dressed photon line the additional one-loop electron bubble. Subset 10 consists from the five-loop diagrams with the photon line, dressed by two-photon photon vacuum polarization contribution and two electron loops. Subset 11 is formed by inserting into photon line of four subsequent one-loop vacuum polarization contributions. It is necessary to emphasize, that all vacuum polarization subgraphs of the diagrams, considered in this report, are containing electron loops only. Let us compare the numerical analogs of analytical asymptotic five-loop contributions to \( a_\mu \), calculated in Ref. [7], with the concrete results [24] of direct numerical calculations of Ref. [3]. The value of the constant, related to Subset 9, was derived in the process of this work by more careful numerical studies of the analytical result of Ref. [7]. The asymptotic analytical expression of the Subset 11 was confirmed in [25].
Table 1. The coefficients for several order $(\alpha/\pi)^5$ contributions to $a_\mu$.

| Subset of diagrams | asymptotic result | numerical result |
|--------------------|-------------------|-----------------|
| $a_\mu$ (Subset 6) | $-\frac{\alpha^5}{2} + 2.8523 + O(\frac{m_e}{m_\mu})$ [7] | 2.88598(9) [24] |
| $a_\mu$ (Subset 7) | $4.8176... + O(\frac{m_e}{m_\mu})$ [7] | 4.74212(14) [24] |
| $a_\mu$ (Subset 8) | $7.4491... + O(\frac{m_e}{m_\mu})$ | 7.5270(88) [24] |
| $a_\mu$ (Subset 9) | $-1.3314... + O(\frac{m_e}{m_\mu})$ [8] | -1.20841(70) [24] |
| $a_\mu$ (Subset 10) | $27.7188... + O(\frac{m_e}{m_\mu})$ [7] | 27.69038(30) [24] |
| $a_\mu$ (Subset 11) | $20.1832... + O(\frac{m_e}{m_\mu})$ [7] | 20.14293(23) [24] |

We observe satisfactory agreement between the entries to the second and third column of Table 1. The existing difference can be explained by the effects of the $O(m_e/m_\mu)$-corrections. Indeed, for the several subsets the five-loop numerical calculations Ref. [3] are in agreement with the related analytical expressions obtained in Ref. [26] both for the asymptotic and $O(m_e/m_\mu)$-contributions. Among these subsets are Subsets 7,10,11. Thus, their asymptotic analytical results, calculated in Ref. [7], and confirmed in Ref. [26], really differ from the numerical ones by the leading $O(m_e/m_\mu)$-terms. In the case of the five-loop diagrams from Subset 9, the $O(m_e/m_\mu)$-terms should also be responsible for elimination of minor differences between asymptotic and numerical results. Indeed, since this statement is correct for the four-loop diagrams of Subset 2 (compare Eq.(6) and Eq.(9) with Eq.(10)), it should also be correct in the case of five-loop diagrams from Subset 9, which have similar topological structure. A similar argument may also hold for the five-loop diagrams from Subset 8.

The discussions presented in this summary provide additional support for the results of the numerical calculations [3] of the definite five-loop contributions to $a_\mu$. However, the dominant contribution of evaluated 2958 diagrams is coming from the subset with light-by-light scattering electron loop internal insertion, which at present is difficult to check. Moreover, other 6122 diagrams still remained uncalculated. In spite of this we think that that the agreement of the new tenth-order QED contribution to $a_\mu$, namely $663(20)(\alpha/\pi)^5$ [3], with the estimate, $\approx 658(\alpha/\pi)^5$ [27], obtained by taking into account the results of the 10th-order calculations in the improved renormalization-group inspired estimates of the high-order corrections to is of theoretical interest and deserve further attention. The current difference between phenomenological and experimental results for $a_\mu$ can not be described by the five-loop corrections (for a review see [2]). They may become really important for planning new $a_\mu$ experiments.

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