Absolute Stability via Lifting and Interpolation

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Absolute stability

Find conditions on the LTI system $G$ with state $x$ that ensure stability for all initial states and all nonlinearities $\phi$ in a function class.

Contributions

• directly construct a Lyapunov function
• simple dissipation proof that naturally generalizes to other settings
• relate the set of all valid multipliers to interpolation
Absolute stability

Find conditions on the LTI system $G$ with state $x$ that ensure stability for all initial states and all nonlinearities $\phi$ in a function class.

**Assumptions** (non-restrictive)

- $G$ is SISO
- $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex with $f(0) = 0$ and $\nabla f(0) = 0$
Quadratic constraints

Quadratic inequalities that hold between $u$ and $y$ when $u_t = \nabla f(y_t)$ for some convex $f$.

- **Willems, Brockett**

$$\begin{bmatrix} u_\ell \\ \vdots \\ u_0 \end{bmatrix}^T M \begin{bmatrix} y_\ell \\ \vdots \\ y_0 \end{bmatrix} \geq 0 \quad \text{where} \quad M \text{ doubly hyperdominant}$$

- **Zames, Falb, O’Shea**

$$\sum_{t=-\infty}^{\infty} u_t (\Pi y)_t \geq 0 \quad \text{where} \quad \sum_{t=-\infty}^{\infty} \pi_t \geq 0 \text{ and } \pi_t \leq 0 \text{ for } t \neq 0$$
The system is absolutely stable if there exists a multiplier $\Pi$ such that

$$\text{Re}\{\Pi(z)G(z)\} < 0$$

for all $z$ on the unit circle.

- follows from the main IQC result (Megretski & Rantzer, 1996)
- frequency domain inequality must hold at an infinite number of points
- tractable search in the time domain over a subset of multipliers
Multiplier factorization

If $\pi$ has finite duration $\ell$, then we can factor the multiplier as

$$\Pi(z) = \Psi(z)^* \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} \Psi(z)$$

where

$$\Psi(z) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ z^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ z^{-\ell} & 0 & \cdots & 0 \end{bmatrix}$$

and

$$M = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_\ell \\ \pi_{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \cdots & 0 \end{bmatrix}$$

Use the factorization to define the augmented system with state $x_t$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Psi(z) \begin{bmatrix} G(z) \\ 1 \end{bmatrix}$$
Time domain

Apply the positive real lemma to the FDI to obtain the equivalent LMI

\[ P \succ 0 \]

\[ 0 \succ \begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} + \frac{1}{2} [C \quad D]^T \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} [C \quad D] \]

| LMI feasible | \( \Rightarrow \) | FDI feasible | \( \Rightarrow \) | IQC | absolute stability |

Issues

- does not produce a Lyapunov function
- how to construct the multipliers for other function classes?
A simple time-domain proof

Multiply the LMI by \((x_t, u_t)\) to obtain the dissipation inequality

\[
0 > x_{t+1}^T P x_{t+1} - x_t^T P x_t + \left[ \begin{array}{c} u_t \\ u_{t-1} \\ \vdots \\ u_{t-\ell} \end{array} \right]^T \left[ \begin{array}{cccc} \pi_0 & \pi_1 & \cdots & \pi_{\ell} \\ \pi_{\ell} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \cdots & 0 \end{array} \right] \left[ \begin{array}{c} y_t \\ y_{t-1} \\ \vdots \\ y_{t-\ell} \end{array} \right]
\]

Then sum over \(t\) from 0 to \(T\)

\[
0 > x_{T+1}^T P x_{T+1} - x_0^T P x_0 + \left[ \begin{array}{c} u_T \\ u_{T-1} \\ \vdots \\ u_0 \end{array} \right]^T \left[ \begin{array}{cccc} \pi_0 & \pi_1 & \cdots & \pi_T \\ \pi_1 & \pi_0 & \cdots & \pi_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-T} & \pi_{1-T} & \cdots & \pi_0 \end{array} \right] \left[ \begin{array}{c} y_T \\ y_{T-1} \\ \vdots \\ y_0 \end{array} \right]
\]

From the conditions on the multiplier, the matrix involving \(\pi\) is doubly hyperdominant, so the quadratic form is nonnegative.
A conservative approach

Feasibility of the LMI implies $V(x) = x^TPx$ is a Lyapunov function.

$$P > 0$$

$$0 > \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + \frac{1}{2} \begin{bmatrix} C & D \end{bmatrix}^T \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$M$ doubly hyperdominant

But this approach is very conservative!
Main idea

• lift the iterates to a higher-dimensional space

\[
\begin{align*}
\mathbf{y}_t &= \begin{bmatrix} y_t \\ \vdots \\ y_{t-\ell} \end{bmatrix}, & \mathbf{u}_t &= \begin{bmatrix} u_t \\ \vdots \\ u_{t-\ell} \end{bmatrix}, & \mathbf{f}_t &= \begin{bmatrix} f_t \\ \vdots \\ f_{t-\ell} \end{bmatrix}
\end{align*}
\]

• use interpolation to find all quadratic-plus-linear inequalities

\[\mathbf{y}_t^T \mathbf{M} \mathbf{u}_t + \mathbf{m}^T \mathbf{f}_t \geq 0\]

• use the inequalities to search for a common quadratic Lyapunov function in the lifted space

\[V(\mathbf{x}_t, \mathbf{f}_t) = \mathbf{x}_t^T \mathbf{P} \mathbf{x}_t + \mathbf{p}^T \mathbf{f}_t\]

This approach constructs a Lyapunov function and recovers the best known results.
Lifted system

\[\begin{bmatrix} x_{t+1} \\ y_t \\ u_t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} \quad \text{and} \quad F f_{t+1} = F + f_t \]

where

\[\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \Psi(z) \begin{bmatrix} G(z) \\ 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0_{\ell \times 1} & I_\ell \end{bmatrix} \quad F_+ = \begin{bmatrix} I_\ell & 0_{\ell \times 1} \end{bmatrix}\]
Convex interpolation

When does there exist a convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $\nabla f(0) = 0$ such that $u_i = \nabla f(y_i)$ and $f_i = f(y_i)$?

Necessary and sufficient conditions (Taylor, Hendrickx, Glineur, 2017)

$$f_i \geq f_j + u_j^T (y_i - y_j), \quad u_i^T y_i \geq f_i, \quad f_i \geq 0 \quad \text{for all } i, j$$
Interpolation cone

\[ \mathcal{K} = \left\{ \left( \begin{bmatrix} y_\ell^T u_\ell & \ldots & y_\ell^T u_1 \\ \vdots & \ddots & \vdots \\ y_1^T u_\ell & \ldots & y_1^T u_1 \end{bmatrix}, \begin{bmatrix} f_\ell \\ \vdots \\ f_1 \end{bmatrix} \right) \mid (y_i, u_i, f_i) \text{ is interpolable} \right\} \]

For convex functions,

\[ \mathcal{K} = \{(G, f) \mid f_i \geq f_j + G_{ij} - G_{jj}, \ G_{ii} \geq f_i, \text{ and } f_i \geq 0 \text{ for all } i, j \text{ and } \text{rank}(G) = 1 \} \]

The interpolation cone \( \mathcal{K} \) characterizes the set of Gramians \( G \) and function values \( f \) that are interpolable.
Quadratic-plus-linear inequalities

\[ y_t^T M u_t + m^T f_t \geq 0 \]

This holds for all multipliers \((M, m)\) in the dual of the interpolation cone.

\[ \mathcal{K}^* = \{ (M, m) \mid \text{tr}(M^T G) + m^T f \geq 0 \text{ for all } (G, f) \in \mathcal{K} \} \]

For convex functions,

\[ \mathcal{K}^* = \{ (M, m) \mid M^T 1 \geq 0, \; M 1 + m \geq 0, \; \text{and } M_{ij} \leq 0 \text{ for all } i \neq j \} \]

The set of all multipliers is the dual of the interpolation cone.
Lyapunov function

\[ V(x, f) = x^T P x + p^T f \]

- **Dissipation inequality**
  \[ V(x_{t+1}, f_{t+1}) - V(x_t, f_t) + \sigma_1(y_t, u_t, f_t) \leq 0 \]

- **Positivity**
  \[ \|x_t\|^2 - V(x_t, f_t) + \sigma_2(y_t, u_t, f_t) \leq 0 \]

- **Multipliers**
  \[ \sigma_i(y_t, u_t, f_t) = y_t^T M_i u_t + m_i^T f_t \quad (M_i, m_i) \in K^* \]

Proof: \[ \|x_t\|^2 \leq V(x_t, f_t) \leq V(x_{t-1}, f_{t-1}) \leq \ldots \leq V(x_0, f_0) \]
Main result

\[
\begin{bmatrix}
A^T P A - P & A^T P B \\
B^T P A & B^T P B
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
C & D
\end{bmatrix}^T \begin{bmatrix}
0 & M_1^T \\
M_1 & 0
\end{bmatrix} \begin{bmatrix}
C & D
\end{bmatrix} \preceq 0
\]

\[
(I - P \ 0) + \frac{1}{2} \begin{bmatrix}
C & D
\end{bmatrix}^T \begin{bmatrix}
0 & M_2^T \\
M_2 & 0
\end{bmatrix} \begin{bmatrix}
C & D
\end{bmatrix} \preceq 0
\]

\[
(F_+ - F)^T p + m_1 \leq 0
\]

\[
-(F^T p + m_2) \leq 0
\]

- symmetric matrix \( P \)
- vector \( p \)
- multipliers \((M_1, m_1)\) and \((M_2, m_2)\) in the dual cone \( K^* \)

Feasibility of the LMI implies that \( V(x, f) \) is a Lyapunov function.
Numerical examples

- $G(z)$ in negative feedback with slope-restricted nonlinearity in $(0, \alpha)$
- find the largest $\alpha$ for which the system is absolutely stable

| Ex. | Plant $G(z)$ | $\alpha$ | $(n_b, n_f)$ | $\ell$ |
|-----|--------------|----------|--------------|--------|
| 1   | $\frac{0.1z}{z^2-1.8z+0.81}$ | 12.9960  | (1, 0)       | 1      |
| 2   | $\frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$ | 0.8027  | (1, 4)       | 4      |
| 3   | $\frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$ | 0.3054  | (0, 1)       | 1      |
| 4   | $\frac{4.4z^5-8.957z^4+9.893z^3-5.671z^2+2.207z-0.5}{-0.5z+0.1}$ | 3.8240  | (0, 4)       | 4      |
| 5   | $\frac{z^3-0.9z^2+0.79z+0.089}{2z+0.92}$ | 2.4475  | (0, 1)       | 1      |
| 6   | $\frac{z^2+0.5z}{z^2-0.5z}$ | 0.9114  | (1, 2)       | 2      |
| 7   | $\frac{1.341z^4-1.221z^3+0.6285z^2-0.5618z+0.1993}{z^5-0.935z^4+0.7697z^3-1.118z^2+0.6917z-0.1352}$ | 0.4347  | (3, 3)       | 3      |

Equivalent to $-(1 + \alpha G)$ in positive feedback with the gradient of a convex function.
Lyapunov function for Example 6

\[ V(x_t, f_t) = \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix}^T P \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix} + p^T \begin{bmatrix} f_{t-1} \\ f_{t-2} \end{bmatrix} \]

\[ P = \begin{bmatrix} 1.4483 & -0.2173 & -2.4073 & -2.4262 \\ -0.2173 & 0.8523 & -2.6369 & 0.1214 \\ -2.4073 & -2.6369 & 2.4142 & -1.5938 \\ -2.4262 & 0.1214 & -1.5938 & 0.4756 \end{bmatrix} \quad p = \begin{bmatrix} -6.1534 \\ -3.2837 \end{bmatrix} \]

\[ M_1 = \begin{bmatrix} 8.6813 & -8.6813 & -0.0000 \\ -0.0000 & 5.8115 & -5.8115 \\ -2.5025 & -0.0000 & 2.5278 \end{bmatrix} \quad m_1 = \begin{bmatrix} -6.1788 \\ 2.8698 \\ 3.2837 \end{bmatrix} \]

\[ M_2 = \begin{bmatrix} 11.2412 & -3.3521 & -1.6564 \\ -1.6595 & 10.6892 & -2.7451 \\ -1.3351 & -1.5047 & 5.9290 \end{bmatrix} \quad m_2 = \begin{bmatrix} -8.2467 \\ -5.8325 \\ -0.0000 \end{bmatrix} \]
Extensions

• continuous time

\[ 0 \geq \frac{d}{dt} V(x(t)) + \sigma(y(t), u(t), f(t)) \]

• exponential stability

\[ 0 \geq V(x_{t+1}, f_{t+1}) - \rho^2 V(x_t, f_t) + \sigma(y_t, u_t, f_t) \]
For all exogenous input signals $w$,\[ V(x_{t+1}, f_{t+1}) - V(x_t, f_t) + \sigma_1(y_t, u_t, f_t) \leq \sigma_p(w_t, z_t) \]
\[ \|x_t\|^2 - V(x_t, f_t) + \sigma_2(y_t, u_t, f_t) \leq 0 \]
\[
\sum_{t=0}^{T} \sigma_p(w_t, z_t) \geq 0
\]

For example, if $\sigma_p := \gamma^2\|w_t\|^2 - \|z_t\|^2$, then $G$ has a robust $\ell_2$ gain from $w \to z$ of $\gamma$. 
Robust stochastic performance

Suppose $w_t$ is i.i.d. zero-mean random noise with covariance $\Sigma$.

\[
V(x_{t+1}, f_{t+1}) - V(x_t, f_t) + \sigma_1(y_t, u_t, f_t) + \|z_t\|^2 \leq 0 \\
-V(x_t, f_t) + \sigma_2(y_t, u_t, f_t) \leq 0 \\
\text{tr} \left( PB_w \Sigma B_w^T \right) \leq \gamma^2
\]

\[
\limsup_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \|z_t\|^2 \right] \leq \gamma^2
\]
Approach

• lift the iterates to a higher-dimensional space
• use interpolation to find all quadratic-plus-linear inequalities
• search for a common quadratic Lyapunov function in the lifted space

Benefits

• directly construct a Lyapunov function
• simple dissipation proof that naturally generalizes to other settings
• relate the set of all valid multipliers to interpolation