Mod $\ell$ representations of arithmetic fundamental groups I

(An analog of Serre’s conjecture for function fields)

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Abstract
There is a well-known conjecture of Serre that any continuous, irreducible representation $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_p)$ which is odd arises from a newform. Here $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$ and $\mathbb{F}_p$ an algebraic closure of the finite field $\mathbb{F}_p$ of $p$ elements. We formulate such a conjecture for $n$-dimensional mod $\ell$ representations of $\pi_1(X)$, for any positive integer $n$, and where $X$ is a geometrically irreducible, smooth curve over a finite field $k$ of characteristic $p$ ($p \neq \ell$), and prove this conjecture in a large number of cases. In fact a proof of all cases of the conjecture for $\ell > 2$ follows from a result announced (conditionally) by Gaitsgory in [Ga]. The methods are completely different.

1 Statement of main result: analog of Serre’s conjecture for function fields

Let $X$ be a geometrically irreducible, smooth curve over a finite field $k$ of characteristic $p$ and cardinality $q$. Denote by $K$ its function field and by

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 its smooth compactification and set $S := \bar{X} \setminus X$. Let $\pi_1(X)$ denote the arithmetic fundamental group of $X$. Thus $\pi_1(X)$ sits in the exact sequence

$$0 \to \pi_1(\bar{X}) \to \pi_1(X) \to G_k \to 0,$$

where $\bar{X}$ is the base change of $X$ to an algebraic closure of $k$, and $G_F$ denotes the absolute Galois group of any field $F$.

We study here mod $\ell$ representations of $\pi_1(X)$, i.e., continuous absolutely irreducible representations $\bar{\rho} : \pi_1(X) \to GL_n(F)$ with $F$ a finite field of characteristic $\ell \neq p$. In this paper we are mainly interested in an analog of (the qualitative part of) Serre’s conjectures in [Se] in the function field situation.

Let us fix once and for all an embedding $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$. Then with respect to this embedding $\iota$, there is a correspondence between $n$-dimensional $\ell$-adic representations of $\pi_1(X \setminus T)$ with finite order determinant and suitably ramified cuspidal eigenforms (or equivalently cuspidal automorphic representations with a newvector fixed by a suitable open compact of $GL_n(\mathbb{A}_K)$) on $GL_n(\mathbb{A}_K)$ with finite central order character, for any finite set $T$ of places of $X$. This correspondence is the global Langlands correspondence for functions due to Drinfeld and Lafforgue (see [Dr] and [Laf]).

We call a residual representation $\bar{\rho}$ automorphic, if it is isomorphic to the residual representation attached to (an integral model of) an $n$-dimensional continuous representation $\pi_1(X \setminus T) \to GL_n(\mathbb{Q}_\ell)$ that is associated to a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$ in [Dr] and [Laf] for some finite set of places $T$. An analog of Serre’s conjecture in the function field setting is therefore that any absolutely irreducible residual representation $\bar{\rho}$ is automorphic. It is worth noting that unlike in the classical setting here there are no “local conditions” that need to be imposed on $\bar{\rho}$ to expect it to be automorphic. In view of [Laf] this conjecture is equivalent to the assertion that any such $\bar{\rho}$ lifts to an $\ell$-adic representation of $\pi_1(X \setminus T)$ of finite order determinant for some finite subset $T$ of $X$.

There is little known about Serre’s original conjecture, while the analog that we study for function fields is more accessible because of the results in [Dr] and [Laf]. Moreover the main result of [dJ] directly implies that the function field analog of Serre’s conjecture holds for $n \leq 2$ (for $n = 1$ it is a simple consequence of class field theory). This is strong evidence in favour of the analog. In fact, the main conjecture made in [dJ] may be regarded as a refinement of the above analog and easily implies it.
In Theorem 2.1 below, we establish the analog in many more cases by producing suitable \( \ell \)-adic liftings of \( \rho \). Our approach uses the Galois cohomological methods of R. Ramakrishna, [Ra], and their further refinements by R. Taylor in [Ta]. The following is an important special case of Theorem 2.1:

**Theorem 1.1** Let \( X \) be a smooth, geometrically irreducible curve defined over a finite field \( k \) of characteristic \( p \), and \( \overline{\rho} : \pi_1(X) \to \text{SL}_n(F) \) be a representation with \( F \) a finite field of characteristic \( \ell \neq p \). Assume that

(i) \( \overline{\rho} \) has full image, \(|F| \geq 4\), \( \ell \nmid n \), and

(ii) at any \( v \in S \) the ramification is either tame or of order prime to \( \ell \).

Then \( \overline{\rho} \) lifts to a representation \( \rho : \pi_1(X \setminus T) \to \text{SL}_n(W(F)) \) with \( T \) a finite set of places of \( X \) and \( W(F) \) the Witt vectors of \( F \). Hence \( \rho \) is automorphic.

What is mainly needed in the proof are that the adjoint representation \( \text{ad}^0(\overline{\rho}) \) of \( \overline{\rho} \) on the traceless matrices of \( M_n(F) \) is irreducible and that \( H^1(\text{im}(\overline{\rho}), \text{ad}^0(\overline{\rho})) \) is (almost) zero. It is this lifting result, which is a consequence of de Jong’s conjecture, and that we prove under the technical assumption above, that seems to be crucial for the applications of de Jong’s conjecture by Drinfeld in [Dr] to some purity conjectures of Kashiwara on perverse sheaves. For \( \ell > 2 \) a proof of all cases of Serre’s conjecture, that depends on some “unpublished mathematics”, follows from the work of Gaitsgory, cf. [Ga]. The methods are completely different and while Gaitsgory’s work should prove the conjecture in totality for \( \ell > 2 \), our methods apply in characteristic 2.

In a continuation (part II) of this work we study the conjecture of A.J. de Jong from [dJ] which is about deformations of representations of the type \( \overline{\rho} \) studied in this paper. For this we will use the lifting result of this paper. In fact proving de Jong’s conjecture was the main motivation for this work. Our results towards de Jong’s conjecture yields that in many cases \( \overline{\rho} \) arises from a cuspidal eigenform form of level the conductor of \( \overline{\rho} \), where by “arises from” we mean is isomorphic to the reduction of the \( n \)-dimensional \( \ell \)-adic representation (which might no longer have coefficients in Witt vectors) associated to the eigenform, thus proving results towards the analog of Serre’s conjecture in its “quantitative aspect”.

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2 Proof of the main result

Our main goal is to prove a general criterion for a residual representation to lift to a characteristic 0 representation which will then give a proof of Theorem 1.1 upon using Lafforgue’s theorem. We start by first making all the necessary definitions to state a result, Theorem 2.1, that is more general than Theorem 1.1 but is more technical to state. After stating the Theorem 2.1, we will first quickly derive Theorem 1.1 from it. Then in the following sections we will, following Ramakrishna, [Ra], and Taylor, [Ta], give the proof of Theorem 2.1. For the general background on Galois cohomology of function fields, the reader is referred to [NSW], Ch. 7 and 8.

Let us fix some notation. For a place \( v \) of \( \tilde{X} \) denote by \( q_v \) the cardinality of the residue field at \( v \). Let \( G_v \supset I_v \supset P_v \) be the absolute Galois group of the completion of \( K \) at \( v \), its inertia and wild inertia subgroup, respectively. We also choose an embedding \( G_v \to G_K \). For any curve \( X \subset \tilde{X} \) this yields morphisms \( G_v/I_v \to \pi_1(X) \), and by \( \text{Frob}_v \in \pi_1(X) \) we denote the corresponding Frobenius-substitution at \( v \in X \).

We fix a residual representation \( \rho: \pi_1(X) \to GL_n(F) \), where \( F \) is some finite field of characteristic \( \ell \neq p \). Denote by \( \text{ad}(\rho) \) the module \( M_n(F) \) considered as a \( \pi_1(X) \)-module via the adjoint action composed with \( \rho \) and by \( \text{ad}^0(\rho) \) the subrepresentation on the traceless matrices \( M^0_n(F) \) of \( M_n(F) \). If \( M \) is an \( F[\pi_1(X)] \)-module, then \( M(i), i \in \mathbb{Z} \), denotes the twist of \( M \) by the \( i \)-th tensor power of the cyclotomic mod \( \ell \) character \( \chi: \pi_1(X) \to F^*_\ell \), and \( M^* \) denotes the representation \( \text{Hom}(M, F) \). Note that \( \text{ad}^0(\rho) \cong \text{ad}^0(\rho)^* \) if \( \ell \not| n \). For \( M \) as above, we also define \( h^i(\pi_1(X), M) := \dim_F H^i(\pi_1(X), M) \) and similarly with \( \pi_1(X) \) replaced by some \( G_v \).

To state the main technical theorem, we need to introduce some further notation. Let \( \zeta_\ell \) be a primitive \( \ell \)-th root of unity of \( K \) and denote by \( E \) the splitting field of \( \overline{\rho} \) over \( K \), i.e., the fixed field of \( \overline{\rho} \) in a fixed separable closure of \( K \). Recall that a matrix \( A \in GL_n(F) \) is called regular, if \( \dim_F M_n(F)^A = n \), where \( A \) operates via the adjoint action, i.e., via conjugation.

We call a conjugacy class \([\sigma]\) of \( \text{Gal}(E(\zeta_\ell)/K) \) an \( R \)-class or Ramakrishna-class for \( \overline{\rho} \) if \( A := \overline{\rho}(\sigma) \) is regular and one of the following two cases holds:

(I) \( \chi(\sigma) \neq 1 \) and \( A \) has distinct simple roots \( \lambda, \lambda' \in F \) with \( \lambda' = \chi(\sigma)\lambda \).

(II) \( \chi(\sigma) = 1 \) and in the Jordan decomposition of \( A \) there occurs at least one \( 2 \times 2 \) block with eigenvalue \( \lambda \in F \).
We call a place $v$ of $X$ an $R$-place for $\overline{\rho}$ if the class of $\text{Frob}_v$ in $\text{Gal}(E(\zeta_\ell)/K)$ is an $R$-class. Note that if an $R$-class exists, then by the Čebotarev density theorem, there exist infinitely many $R$-places.

While the main use of $R$-places is to provide locally some freedom for lifting, they can also be useful to remove global obstructions. To describe this, denote by $V$ the space $F^n$ considered as a representation of $\pi_1(X)$ via $\overline{\rho}$ and let $\sigma$ be an $R$-class of type (II). The indecomposable summands of $V$ are denoted by $V_i$, and $\text{ad}(\overline{\rho})_i$ denotes the corresponding representation on (the trace zero matrices of) $\text{End}(V_i)$, considered as a representation of $\langle \sigma \rangle$. Let $\lambda_i$ be one of the eigenvalues of $V_i$. We define

$$\text{ad}^0(\overline{\rho})_\sigma := \prod_{\lambda_i \in F} \text{ad}^0(\overline{\rho})_i.$$  (1)

Since $V \cong \bigoplus_i V_i$ there is a $\langle \sigma \rangle$-morphism $\text{ad}^0(\overline{\rho}) \to \text{ad}^0(\overline{\rho})_\sigma$. We say that $\overline{\rho}$ admits sufficiently many $R$-classes if there exists at least one $R$-class and if the restriction morphisms (composed with $\text{ad}^0(\overline{\rho}) \to \text{ad}^0(\overline{\rho})_\sigma$ at $R$-places)

$$H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})) \to \prod_{\sigma \text{ an } R\text{-class of type (II)}} H^1(\langle \sigma \rangle, \text{ad}^0(\overline{\rho})_\sigma) \oplus \prod_{v \in S} H^1(\overline{\rho}(I_v), \text{ad}^0(\overline{\rho})),$$

$$H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})(1)) \to \prod_{\sigma \text{ an } R\text{-class of type (II)}} H^1(\langle \sigma \rangle, \text{ad}^0(\overline{\rho})_\sigma) \oplus \prod_{v \in S} H^1(\overline{\rho}(I_v), \text{ad}^0(\overline{\rho})(1))$$

are injective. Note that $\text{ad}^0(\overline{\rho})^*_\sigma(1) = \text{ad}^0(\overline{\rho})^*_\sigma = \text{ad}^0(\overline{\rho})_\sigma$ for $R$-classes of type (II).

Our main result in this chapter is

**Theorem 2.1** Let $X$ be a smooth, geometrically irreducible curve defined over a finite field $k$ of characteristic $p \neq \ell$, and $\overline{\rho} : \pi_1(X) \to \text{GL}_n(F)$ be a continuous representation. Assume that

(a) $\text{ad}^0(\overline{\rho})$ is irreducible over $F_\ell[\text{im}(\overline{\rho})]$,

(b) $\overline{\rho}$ has sufficiently many $R$-classes,

(c) at all $v \in S$ the ramification is either tame or of order prime to $\ell$.

Then $\overline{\rho}$ lifts to a representation $\rho : \pi_1(X\setminus T) \to \text{GL}_n(W(F))$ where
(i) $T$ is a finite set of places of $X$, 
(ii) $\det \rho$ is the Teichmüller lift of $\det \overline{\rho}$, 
(iii) for $v \in S$ the conductors of $\rho$ and $\overline{\rho}$ agree, and 
(iv) if $\overline{\rho}$ is tame at $v$, then $\rho(I_v) \cong \overline{\rho}(I_v)$, i.e., $\rho$ is minimal at $v$.

Note that we do not need that $\text{ad}^0(\overline{\rho})$ is absolutely irreducible. Note also that the condition that $\text{ad}^0(\overline{\rho})$ is irreducible implies that $\ell$ does not divide $n$, since in the case $\ell | n$, the representation $\text{ad}^0(\overline{\rho})$ contains the trivial representation on scalar matrices as a non-trivial submodule.

As an application of Lafforgue’s theorem, we find.

**Corollary 2.2** Any $\overline{\rho}$ as in the previous Theorem is automorphic.

We have the following example for the existence of sufficiently many $R$-classes. Combined with the above theorem it completes the proof of Theorem 1.1.

**Proposition 2.3** Suppose $\overline{\rho}: \pi_1(X) \to \text{SL}_n(F)$ is surjective, $\ell \not| n$, $\ell \neq p$ and $|F| > 4$. Then $\overline{\rho}$ admits sufficiently many $R$-classes.

**Proof:** Let us first show the injectivity of the two restriction morphisms considered above Theorem 2.1. If $|F| > 5$ or $n > 2$, then by [CPS] we have $H^1(\text{SL}_n(F), M^0_n(F)) = 0$. In this case it easily follows, e.g., [Bö1], §5, that

$$H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})) = H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})(1)) = 0.$$

In the remaining case, note first that by loc. cit., if $\chi$ is non-trivial, one has $H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})(1)) = 0$. Finally, in [Ta] it is shown for $n = 2$ and $F = \mathbb{F}_5$ how to find an $R$-class $\sigma$ such that the kernel of

$$F \cong H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})) \to H^1(\langle \sigma \rangle, \text{ad}^0(\overline{\rho}))$$

is trivial (in this particular case one has $\text{ad}^0(\overline{\rho})|_\sigma = \text{ad}^0(\overline{\rho})$). If $\chi$ is trivial, the same class also works for $\text{ad}^0(\overline{\rho})(1) = \text{ad}^0(\overline{\rho})$.

It remains to prove the existence of at least one $R$-class. For this, note that $\text{SL}_n(F)$ has no abelian quotients, and therefore the morphism

$$\overline{\rho} \times \chi: \pi_1(X) \longrightarrow \text{GL}_n(F) \times F$$



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surjects onto $\text{SL}_n(F) \times \text{im}(\chi)$. Since $\text{SL}_n(F)$ contains matrices of type (II), the existence of an $R$-class is obvious. Furthermore, if $\text{im}(\chi)$ is non-trivial, then one may also find matrices of type (I). This completes the proof of the proposition.

2.1 Strategy of the proof of Theorem 2.1

Our method of producing lifts is essentially that of Ramakrishna, cf. [Ra]. However we will follow the more axiomatic treatment as presented in [Ta]. Let us fix from now on a representation $\overline{\rho}: \pi_1(X) \to \text{GL}_n(F)$ which satisfies the conditions of Theorem 2.1 and let $n \geq 2$, since $n = 1$ is trivial by using Teichmüller lifts. Therefore in the following we assume that $\text{ad}^0(\overline{\rho})$ is irreducible over $F_\ell[\text{im}(\overline{\rho})]$ (and hence also $\ell \nmid n$). Also define $\eta: \pi_1(X) \to W(F)$ as the Teichmüller lift of $\det \rho$ and for any place $v$ define restrictions $\eta_v := \eta|_{G_v}$ and $\overline{\rho}_v := \overline{\rho}|_{G_v}$.

The strategy in [Ra] to produce lifts of $\overline{\rho}$ to $W(F)$ is to first consider all deformations of $\overline{\rho}$ which are representations of $\pi_1(X \setminus T)$ for some fixed finite subset $T$ of $R$-places of $X$ and which at the places in $S \cup T$ are allowed to only have ramification of a very specific type. Without loss of generality, we assume that $\overline{\rho}$ is ramified at the places in $S$ and call these residually ramified places or simply $r$-places.

The type of ramification is most conveniently formulated in terms of local deformation problems $C_v$ at places $v \in S \cup T$. In this formulation, locally the crucial requirement is that the resulting versal hull is smooth over the ring $W(F)$ of relative dimension $h^0(G_v, \text{ad}^0(\overline{\rho}))$. In Sections 2.2 and 2.3 we will define such $C_v$ for $R$- and $r$-places, respectively.

The global requirement on $T$ and the types $C_v$ is made in such a way that one can inductively construct lifts of $\overline{\rho}$ to the rings $W_n(F)$ of Witt vectors of length $n$. It can be entirely formulated in terms of Galois cohomology. In this section we will recall the necessary background from [La] and give a proof of the main theorem pending on a key lemma, whose proof will be given in the later Section 2.4.

Let $\mathcal{A}$ denote the category of complete noetherian local $W(F)$-algebras $(R, m_R)$ with residue field $F$ and where morphisms are morphisms of local rings which are the identity on the residue field. By a lift of determinant
η_v of \( \overline{\rho}_v \) we mean a continuous representation \( \rho : G_v \to \text{GL}_n(R) \) for some \((R, m_R) \in A\) such that \( \rho(\text{mod } m_R) = \overline{\rho}_v \) and \( \det \rho = \eta_v \).

We call a pair \((C_v, L_v)\), where \( C_v \) is a collection of lifts of \( \overline{\rho}_v \) of determinant \( \eta_v \), and where \( L_v \) is a subspace of \( H^1(G_v, \text{ad}(\overline{\rho})) \), \((\text{locally}) \text{ admissible and compatible with } \eta_v \) if it satisfies the conditions P1–P7 of [Ta], where in loc. cit. one has to replace \( m \) by \( m_R \) and \( M_2(m) \) by \( M_n(m_R) \) in property P2.

**Remark 2.4** Observe that the conditions of loc. cit. imply in particular that the versal hull of the deformation problem \( C_v \) exists and the corresponding versal deformation ring is smooth over \( W(F) \) of relative dimension \( \dim L_v \). Heuristically one would expect the versal deformation ring of all deformations of \( \rho_v \) with fixed determinant to be a complete intersection, flat over \( W(F) \) and of relative dimension \( h^0(G_v, \text{ad}(\overline{\rho})) \). Therefore one expects \( \dim L_v \leq h^0(G_v, \text{ad}(\overline{\rho})) \).

Suppose one is given a finite set \( T \subset X \) and for each \( v \in S \cup T \) a locally admissible pair \((C_v, L_v)\) compatible with \( \eta_v \). Then a lift of type \( (C_v, \overline{\rho}_v) \) \( v \in S \cup T \), is a continuous representation \( \rho : \pi_1(X \setminus T) \to \text{GL}_n(R) \) for some \((R, m_R) \in A\) such that \( \rho(\text{mod } m_R) = \overline{\rho}_v \), \( \rho|_{G_v} \in C_v \) for all \( v \in S \cup T \) and \( \det \rho = \eta_v \).

To describe tangential conditions on the above lifts, we need to fix some more notation. For \( v \) a place of \( \widetilde{X} \) and \( M \) a \( G_v \)-module, the pairing \( M \times M^* \to F \) defined by evaluation is obviously perfect. Tate localy duality says that the induced pairing

\[
H^1(G_v, M) \times H^1(G_v, M^*(1)) \to H^2(G_v, F(1)) \cong F
\]

is perfect as well, and one denotes for any \( F \)-submodule \( L \subset H^1(G_v, M) \) its annihilator under this pairing by \( L^\perp \subset H^1(G_v, M^*(1)) \). In the particular case of the subspace of unramified cocycles

\[
H^1_{\text{unr}}(G_v, M) := H^1(G_v/I_v, M^I_v) \subset H^1(G_v, M)
\]

one finds \( H^1_{\text{unr}}(G_v, M)^\perp = H^1_{\text{unr}}(G_v, M^*(1)) \).

We now specialize to the situations of interest to us, i.e., to \( M = \text{ad}(\overline{\rho}) \) and \( M = \text{ad}^0(\overline{\rho}) \). Note first that \( \text{ad}(\overline{\rho}) \) is self-dual via the perfect trace pairing \( \text{ad}(\overline{\rho}) \times \text{ad}(\overline{\rho}) \to F : (A, B) \mapsto \text{Trace}(AB) \). Because \( \ell \) does not divide \( n \), this pairing restricts to a perfect pairing

\[
\text{ad}^0(\overline{\rho}) \times \text{ad}^0(\overline{\rho}) \to F,
\]
and local Tate duality induces the perfect pairing
\[ H^1(G_v, \text{ad}^0(\overline{\rho})) \times H^1(G_v, \text{ad}^0(\overline{\rho}))(1) \longrightarrow H^2(G_v, F(1)) \cong F. \] (4)

For a finite subset \( T \) of \( X \) and a collection \( (L_v)_{v \in S \cup T} \) of subspaces of \( H^1(G_v, \text{ad}^0(\overline{\rho})) \) one defines \( H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho})) \) as the kernel of
\[ H^1(\pi_1(X \setminus T), \text{ad}^0(\overline{\rho})) \longrightarrow \bigoplus_{v \in S \cup T} H^1(G_v, \text{ad}^0(\overline{\rho}))/L_v. \]

Note that \( H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho})) \) does depend on \( S \) even though \( S \) does not explicitly appear in the notation.

Ramakrishna’s first observation is the following:

**Lemma 2.5** Suppose one is given locally admissible pairs \((C_v, L_v)_{v \in S \cup T}\) compatible with \( \eta \) such that
\[ H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho}))(1) = 0. \]
Then there exists a lift of \( \overline{\rho} \) to \( W(F) \) of type \((C_v)_{v \in S \cup T}\).

The proof is essentially that of [Ta], Lemma 1.2, and so we omit the details.

**Remark 2.6** Mimicking the proofs of [DDT], Thms. 2.13, 2.14, one obtains for a \( \pi_1(X \setminus T) \)-module \( M \) and subspaces \( L_v \subset H^1(G_v, M) \) for \( v \in S \cup T \) the formula
\[ \frac{|H^1_{\{L_v\}}(T, M)|}{|H^1_{\{L_v\}}(T, M^*(1))|} = \frac{|H^0(\pi_1(X), M)|}{|H^0(\pi_1(X), M^*(1))|} \prod_{v \in S \cup T} \frac{|L_v|}{|H^0(G_v, M)|}. \]

In our situation \( M \cong M^* \cong \text{ad}^0(\overline{\rho}) \), the first quotient on the right is clearly 1. Thus by Remark 2.3, one expects the product on the right to have the value at most 1. Furthermore, this should happen precisely when \( \dim L_v = h^0(G_v, \text{ad}^0(\overline{\rho})) \) for all \( v \in S \cup T \). Therefore if the hypothesis of the above lemma are satisfied, then one expects
\[ \dim H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho})) = 0. \]
In terms of deformation theory, this can be interpreted by saying that the universal deformation ring of type \((C_v)_{v \in S \cup T}\) is smooth over \( W(F) \) of relative dimension zero, i.e., isomorphic to \( W(F) \).
Note that the above formula also holds for $S \cup T = \emptyset$, even though the duality results in [NSW] are not proved in this case. The reason is that in this case the right hand side is 1 and because of $H^0(\pi_1(X), \text{ad}^0(\overline{\rho})) = 0$, the left hand expresses that fact that the Euler-Poincaré characteristic of the unramified $F[\pi_1(\tilde{X})]$-module $\text{ad}^0(\overline{\rho})$ is zero.

We need to slightly generalize the concept of sufficiently many $R$-classes for the following result: Suppose we are given locally admissible $(C_v, L_v)_{v \in S \cup T}$ which are compatible with $\eta$. We say that $\rho$ admits sufficiently many $R$-classes for $(C_v, L_v)_{v \in S \cup T}$ if there exists at least one $R$-class and if the kernels of the restriction morphisms (composed with $\text{ad}^0(\overline{\rho}) \to \text{ad}^0(\overline{\rho})_\sigma$ at $R$-places)

$$H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})) \cap H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho})) \to \prod_{\sigma \text{ an } R\text{-class of type (II)}} H^1(\langle \sigma \rangle, \text{ad}^0(\overline{\rho})_\sigma),$$

$$H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\overline{\rho})(1)) \cap H^1_{\{L_v\}}(T, \text{ad}^0(\overline{\rho})(1)) \to \prod_{\sigma \text{ an } R\text{-class of type (II)}} H^1(\langle \sigma \rangle, \text{ad}^0(\overline{\rho})_\sigma)$$

are zero.

The main observation of Ramakrishna, if adapted to our situation, is the following key lemma:

**Lemma 2.7** Suppose one is given a finite set of places $T' \subset X$ and locally admissible $(C_v, L_v)_{v \in S \cup T'}$ which are compatible with $\eta$ and such that

$$\sum_{v \in S} \dim L_v \geq \sum_{v \in S} h^0(G_v, \text{ad}^0(\overline{\rho})).$$

If $\overline{\rho}$ admits sufficiently many $R$-classes for $(C_v, L_v)_{v \in S \cup T'}$, then one can find a finite set of $R$-places $T \subset X$ and locally admissible $(C_v, L_v)_{v \in T}$ compatible with $\eta$, such that

$$H^1_{\{L_v\}}(T \cup T', \text{ad}^0(\overline{\rho})) = 0.$$

The proof of this lemma will be given in Section 2.4. Let us now explain how this will give a proof of Theorem 2.1.

In the following two sections, we will define good local deformation problems at certain unramified primes and at ramified primes where the ramification is either of order prime to $\ell$ or prime to $p$. We then apply Lemma 2.7 with $T' = \emptyset$ and assume that $\overline{\rho}$ ramifies at all places of $S$. In order to do that we also have to check that if $\overline{\rho}$ has sufficiently many $R$-classes, then this implies that $\overline{\rho}$ has sufficiently many $R$-classes for $(C_v, L_v)_{v \in S}$ where the $(C_v, L_v)$ will be defined below. Once this is shown the theorem follows easily from the above two lemmas. The full proof is given at the end of Section 2.4.
2.2 Local deformations at $R$-places

In this section, we will define locally admissible deformation problems $\mathcal{C}_v$ at $R$-places $v$ compatible with the Teichmüller lift $\eta_v : G_v \to W(F)$ of $\det \overline{\rho}_v$, cf. Proposition 2.10. So for the remainder of this section, we fix an $R$-place $v$, and denote by $\sigma$ the image of $\text{Frob}_v$ in $\text{Gal}(E(\zeta_\ell)/K)$, so that $[\sigma]$ is an $R$-class. We also fix an eigenvalue $\lambda \in F$ of $A := \overline{\rho}(\sigma)$ as required in the definition of an $R$-place.

**The definition of $\mathcal{C}_v$ at an $R$-place**

Using the rational canonical form, we may assume that $A$ is given in the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix},$$

where each $A_i$ is a square matrix of size $n_i$, the matrices $A_i$ for $i > 1$ are in rational canonical form and act indecomposably, and the matrix $A_1$ has the following form depending on our two cases:

$$A_1 = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \quad \text{in case (I)} \quad \text{and} \quad A_1 = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \quad \text{in case (II)}.$$

Note that in case (II) the $A_i$ are in bijection with the irreducible representations $V_i$ used in the defining formula (1). Because the $A_i$ act indecomposably, the eigenvalues form a single Galois orbit and the Jordan canonical form of an $A_i$ consists of identical blocks for each of the eigenvalues. Because $A$ is regular, different $A_i$ have distinct orbits of eigenvalues. Also clearly each $A_i$ is again regular.

For $i = 2, \ldots, r$ we define

$$\rho_{v,i} : G_v \to \text{GL}_n(R_{v,i})$$

as the versal unramified deformation of $\overline{\rho}_{v,i} : G_v \to \text{GL}_n(F)$ defined as the restriction of $\overline{\rho}$ to the $i$-th block.

For the definition in case $i = 1$, let $\hat{Z}$ be the profinite completion of $Z$ and $\hat{Z}'$ the prime-to-$p$ completion of $\hat{Z}$. Let $s, t$ be topological generators of $\hat{Z}$ and $\hat{Z}'$, respectively. For $q$ a power of $p$, and thus prime to $\ell$, define $\overline{\mathcal{G}}_q := \hat{Z}' \rtimes \hat{Z}$, where the semidirect product is given (in multiplicative notation) by the
condition $sts^{-1} = t^q$. Then $G_{q_v}$ can be identified with the tame quotient of $G_v$ in such a way that $t$ is a generator of $I_v/P_v$ and $s$ is a lift of $\text{Frob}_v \in G_v/I_v$.

By $\hat{\mu} \in W(F)$ we denote the Teichmüller lift of any element $\mu$ of $F$, by $\hat{q}_v := q_v/\chi(\sigma)$, and we set $\delta$ to be 0 in case (I) and 1 in case (II). We now define $R_v, 1 := W(F)[x, 0, x, 1]$ and $\varrho_v, 1 : G_v \rightarrow G_{q_v} \rightarrow GL_n(R_{v,i})$ by

$$s \mapsto \left( \begin{pmatrix} \hat{\lambda} & \hat{\beta} \\ 0 & 1 \end{pmatrix} \right)$$

and

$$t \mapsto \left( \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \right).$$

The necessary condition $\rho_v, 1(s) \varrho_v, 1(t) = \rho_v, 1(t)\varrho_v, 1(s)$ can easily be verified.

Combining the above, we define

$$R_v := \bigotimes_{i=1}^r R_v, i / \left( \prod_{i=1}^r \det \rho_v, i(s) - \eta_v(s) \right),$$

with $\bigotimes$ formed over $W(F)$, and the corresponding representation $\varrho_v : G_v \rightarrow GL_n(R_v)$ as $\bigoplus_{i=1}^r \varrho_v, i$ (where the entries are taken modulo the ideal generated by $\prod_{i=1}^r \det \rho_v, i(s) - \eta_v(s)$).

To investigate the resulting representations, we first need a simple result on the individual $\varrho_v, i$. For this we denote by $\text{ad}(\varphi)$ the adjoint representations of the $A_i$ and by $\text{ad}^0(\varphi)$ its subrepresentation on trace zero matrices; i.e., in case (II) they agree with those defined in (1). Then obstruction theory easily shows the following:

**Lemma 2.8** Let $i$ be in $2, \ldots, r$. Then the versal deformation $\varrho_v, i$ is smooth over $W(F)$ of dimension $h^1(G_v, \text{ad}(\varphi)_i) = n_i$. If $\ell \nmid n_i$ and if $\eta_i$ is any lift of $\det \varphi$ to $W(F)$, then the versal deformation of determinant equal to $\eta_i$ is smooth of dimension $h^1(G_v, \text{ad}^0(\varphi)_i) = n_i - 1$.

**Corollary 2.9** Assume that there exists an $i$ such that $\ell$ does not divide $n_i$. Then $R_v$ is smooth over $W(F)$ of relative dimension $n - 1 = h^0(G_v, \text{ad}^0(\varphi))$.

Note that for $\ell \nmid n$ such an $n_i$ always exists.

**Proof:** By the preceeding lemma, the ring $\bigotimes_{i=1}^r R_v, i$ is smooth of dimension $n$. Because the $A_i$ have distinct sets of eigenvalues for different $i$ we have $h^1(G_v, \text{ad}(\varphi)) = \sum_i h^1(G_v, \text{ad}(\varphi)_i) = n$. Moreover if one of the $n_i$ is not
divisible by \( \ell \), then it is easy to see that \( h^1(G_v, \text{ad}^0(\overline{\rho})) = h^1(G_v, \text{ad}(\overline{\rho})) - 1 \). Let \( i_0 \) be the corresponding index.

We now prove the smoothness of \( R_v \). The previous lemma applied to \( i_0 \) say that there is a system of local coordinates of \( R_{v,i} \) such that \( \det \rho_{v,i}(s) = \eta_{i_0}(s)(1 + x) \) where \( x \) is one of these coordinates. If we regroup the defining relation of \( R_v \), it yields therefore the relation
\[
\eta_v(s) \prod_{i \neq i_0} \det \rho_{v,i}^{-1}(s) = \eta_{i_0}(s)(1 + x),
\]
and the variable \( x \) does not occur on the left hand side. Thus the relation eliminates the variable \( x \), which is one of the local coordinates in a suitable set of such for the ring \( \hat{\otimes}_i R_{v,i} \). Because \( \hat{\otimes}_i R_{v,i} \) is smooth over \( W(F) \) of relative dimension \( n \), so is \( R_v \) of relative dimension \( n - 1 \).  

The following defines a pair \((C_v, L_v)\) compatible with \( \eta_v \): The functor \( C_v : A \to \text{Sets} \) is given by
\[
R \mapsto C_v(R) := \{ \rho : G_v \to \text{GL}_n(R) \mid \exists \alpha \in \text{Hom}_A(R_v, R), \exists M \in 1 + M_n(\mathfrak{m}_R) : \rho = M(\alpha \circ \rho_v)M^{-1} \}. 
\]
Moreover, if \( \rho_0 : G_v \to \text{GL}_n(F[\varepsilon]/(\varepsilon^2)) \) denotes the trivial lift of \( \overline{\rho} \), the subspace \( L_v \subset H^1(G_v, \text{ad}^0(\overline{\rho})) \) is the set of 1-cocycles
\[
\left\{ c : G_v \to \text{ad}^0(\overline{\rho}) : g \mapsto \frac{1}{\varepsilon}(\rho(g)\rho_0^{-1}(g) - I) \ \bigg| \ \rho \in C_v(F[\varepsilon]/(\varepsilon^2)) \right\}
\]
and \( L_v,\text{unr} \subset H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho})) \) is the intersection \( L_v \cap H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho})) \).

**Proposition 2.10**  
(i) \( \dim_F L_v = 1 + \dim_F L_v,\text{unr} = n - 1 \).

(ii) The pair \((C_v, L_v)\) satisfies conditions P1–P7 of [Tha].

**Verifying the axioms for \( C_v \)**

Our first aim is to prove the assertions on the dimensions in the above proposition. For this we need a closer analysis of \( H^1(G_v, \text{ad}^0(\overline{\rho})) \). Observe first that by repeatedly applying the Leray-Serre spectral sequence to \( G_v \supset I_v \supset P_v \) and \( \text{ad}^0(\overline{\rho}) \), one obtains the short exact sequence
\[
0 \to \text{ad}^0(\overline{\rho})/(\sigma - 1)\text{ad}^0(\overline{\rho}) \to H^1(G_v, \text{ad}^0(\overline{\rho})) \to (\text{ad}^0(\overline{\rho})(-1))^\sigma \to 0 \quad (5)
\]
and an isomorphism $H^i(G_v, \text{ad}^0(\overline{\mathfrak{p}})) \cong H^i(G_{q_v}, \text{ad}^0(\overline{\mathfrak{p}}))$.

The above short exact sequence can be interpreted in terms of 1-cocycles representing cohomology classes. Namely any 1 cocycle $c$ of $G_{q_v}$ with values in $\text{ad}^0(\overline{\mathfrak{p}})$ is uniquely determined by its values $c(s)$, $c(t)$. These are subject to the conditions $c(s) \in \text{ad}^0(\overline{\mathfrak{p}})$ and $c(t) \in (\text{ad}^0(\overline{\mathfrak{p}})(-1))^s$, i.e., $c(t) \in \text{ad}^0(\overline{\mathfrak{p}})$ satisfies $sc(t) = \frac{1}{s}c(t)$. Furthermore the 1-coboundaries are precisely the 1-cocycles with $c(s) \in (s-1)\text{ad}^0(\overline{\mathfrak{p}})$ and $c(t) = 0$.

To determine $L_v$, we state the following simple lemma without proof:

**Lemma 2.11** Suppose $A \in \text{GL}_n(F)$ is regular. Let $V$ be the $F[A]$-module on $F^n$ defined by $A$ and suppose $V = \bigoplus V_i$ is a direct sum decomposition as a $F[A]$-module. Let the corresponding $\text{ad}(\overline{\mathfrak{p}})_i$ be defined as above. Then $\text{ad}(\overline{\mathfrak{p}})/[\text{ad}(\overline{\mathfrak{p}}),A] = \bigoplus_i \text{ad}(\overline{\mathfrak{p}})_i/[\text{ad}(\overline{\mathfrak{p}})_i,A]$.

**Proof of Proposition 2.10(ii)** Let us fix local coordinates $x_{i,j}$, $j = 1, \ldots, n_i$ of the rings $R_{v,i}$, $i = 2, \ldots, r$. We also enumerate them, so that the variable $t$ in the proof of Corollary 2.9 is given by $x_{i_0,1}$.

Let first $c_0$ be the 1-cocycle that arises from $R_v \to F[\varepsilon]/(\varepsilon^2)$ by mapping the $x_{i,j}$, $j \geq 1$ and $(i,j) \neq (i_0,1)$, to zero and $x_{1,0}$ to $\varepsilon$. (The image of $x_{i_0,1}$ is determined by the $x_{i,j}$ with $j \geq 1$.) The corresponding element in $L_v$ is easily seen to be non-zero and ramified. Moreover all cocycles obtained from an assignment where $x_{1,0}$ maps to zero are unramified. This shows $L_v = L_{v,\text{unr}} + 1$.

Let $L'_v$ be the set of cocycles corresponding to the ring $R'_v := \hat{\otimes}_v R_{v,i}/(x_{1,0})$. The subspace defined from $R_{v,1}/(x_{1,0})$ is 1-dimensional, that from $R_{v,i}$, $i > 1$, has dimension $n_i$ by Lemma 2.8. The previous lemma implies $\dim F L' = 1 + \sum_{i>1} n_i = n - 1$ which in turn is the relative dimension of $R'$ over $W(F)$. Because $R_v/(x_{1,0})$ is a smooth quotient of $R'$ of relative dimension $n - 2$ this yields $\dim F L_{v,\text{unr}} = n - 2$, as asserted.

**Proof of Proposition 2.10(ii)** The only non-trivial condition to verify is P4. For this let us first prove the following:

**Lemma 2.12** Let $\tilde{R}$ be in $A$ and $\alpha, \alpha' \in \text{Hom}_A(R_v, \tilde{R})$ such that there exists $M \in \text{GL}_n(\tilde{R})$ congruent to the identity modulo $\mathfrak{m}_{\tilde{R}}$ with $M(\alpha \circ \rho_v) M^{-1} = \alpha' \circ \rho_v$. Then $\alpha \circ \rho_v(s) = \alpha' \circ \rho_v(s)$, so that in particular $M$ commutes with $\alpha \circ \rho_v(s)$.

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Proof:} We use the same local parameters for $R_v$ as in the proof of Proposition 2.10 (i). The matrix $\rho(s)$ has entries in the power series ring over $W(F)$ in the variables $x_{i,j}$. By $\rho(s)_r$ we denote the part of $\rho(s)$ that is homogeneous of degree $r$, so that $\rho(s) = \sum_{r=0}^{\infty} \rho(s)_r$. The assertion $\dim_F L_{v,unr} = n - 2$ of Proposition 2.10 (i) means precisely that the $n - 2$ matrices $\frac{\partial}{\partial x_{i,j}} \rho(s)_1$ over all $i, j$ with $j \geq 1$ and $(i, j) \neq (i_0, 1)$ form a basis of the vector space $L_{v,unr} \subset \text{ad}^0(\mathfrak{p})/(s-1)\text{ad}^0(\mathfrak{p})$.

Define $\rho := \alpha \circ \rho_v$, $\rho' := \alpha' \circ \rho_v$. Let $\rho_{(m)} := \rho(\text{mod } m_R^{m+1})$ and introduce analogous abbreviations $\rho'_{(m)}$, $\alpha_{(m)}$, $\alpha'_{(m)}$ and $M_{(m)}$. By induction on $m$ we will show that $\alpha_{(m)}(x_{i,j}) = \alpha'_{(m)}(x_{i,j})$ for all $i, j$ with $j \geq 1$. This clearly implies $\rho_{(m)}(s) = \rho'_{(m)}(s)$ for all $i$, and thus the lemma. The case $m = 1$ is clear and so we now carry out the induction step $m \mapsto m + 1$.

By the induction hypothesis $M_{(m)}$ commutes with $\rho_{(m)}(s)$. Because $\mathfrak{p}(s)$ is regular, [Bö2], Lemma 5.6, implies that there exists a lift $M'$ of $M_{(m)}$ to $\text{GL}_n(\tilde{R}/m_R^{m+1})$ which commutes with $\rho_{(m+1)}(s)$. By considering $M'^{-1} M_{(m+1)}$, we may thus assume $M_{(m+1)} = I + \Delta$ for some $\Delta \in M_n(m_R/m_R^{(m+1)})$. We also define elements $\delta_{i,j} := \alpha_{(m+1)}(x_{i,j}) - \alpha'_{(m+1)}(x_{i,j})$ which by the induction hypothesis lie in $m_R^{m+1}$. The expansion of $\rho(s)$ in homogeneous parts shows

$$\rho'_{m+1}(s) = \rho_{m+1}(s) + \rho(s)_1|_{x_{i,j} = \delta_{i,j}} \text{ in } M_n(\tilde{R}/m_R^{m+1}),$$

and so the condition $M_{(m+1)} \rho_{(m+1)}(s) M_{(m+1)}^{-1} = \rho'_{(m+1)}(s)$ yields

$$\sum_{(i,j)} \delta_{i,j} \frac{\partial}{\partial x_{i,j}} \rho(s)_1 = \rho(s)_1|_{x_{i,j} = \delta_{i,j}} = \Delta \rho_{(m+1)}(s) - \rho_{(m+1)}(s) \Delta$$

in $M_n(m_R^{m}/m_R^{(m+1)})$. The right hand side is clearly a linear combination of coboundaries, of $H^1(G_v, \text{ad}^0(\mathfrak{p})) \otimes m_R^{m}/m_R^{(m+1)}$, while the left hand side is a linear combination of a basis of the cohomology. Therefore both sides must vanish. This concludes the induction step. □

To verify P4, suppose that we are given rings $R_1, R_2 \in A$, lifts $\rho_i \in C_v(R_i)$, ideals $I_j \in R_j$, and an identification $R_1/I_1 \cong R_2/I_2$ under which $\rho_1(\text{mod } I_1) \equiv \rho_2 \mod I_2$. We want to glue the $\rho_i$ to an element $\rho$ of $C_v(R)$ for

$$R := \{(r_1, r_2) \in R_1 \oplus R_2 : r_1 \mod I_1 = r_2 \mod I_2\}.$$
So let $\alpha_i \in \text{Hom}_A(R_v, R_i)$ and $M_i \in \text{GL}_n(R_i)$ such that $\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}$, $i = 1, 2$. We claim that there exists $\alpha \in \text{Hom}_A(R_v, R)$ and $M \in \text{GL}_n(R)$ with $M \equiv I \pmod{m_R}$ such that $\rho := M(\alpha \circ \rho_v)M^{-1} = \rho_1 \perp \rho_2$.

By conjugating $\rho_1$ by some lift of $M_2 \pmod{I_1}$ to $R_1$, we may assume that $M_2 = I$. By the lemma, the matrix $M_1 \pmod{I_1}$ commutes with $(\alpha_1 \pmod{I_1}) \circ \rho_v(s) = (\alpha_2 \pmod{I_2}) \circ \rho_v(s)$. Using [Bo2], Lemma 5.6, and the regularity of $\overline{\rho}$, we may choose a lift $M'_1 \in \text{GL}_n(R_1)$ of $M_1 \pmod{I_1}$ which commutes with $\alpha_1 \circ \rho_v(s)$. We now replace $M_1$ by $\tilde{M}_1 := M_1M_1'^{-1}$ and $\alpha_1$ by some $\tilde{\alpha}_1 : R_v \to R_1$, which differs from $\alpha_1$ at most on the variable $x_0$, and such that

$$\tilde{M}_1(\tilde{\alpha}_1 \circ \rho_v)\tilde{M}_1^{-1} = M_1(\alpha_1 \circ \rho_v)M_1^{-1}.$$  

Defining $M := (\tilde{M}_1, I) \in \text{GL}_n(R)$ and $\alpha := (\alpha'_1, \alpha_2) : R_v \to R$, the above claim is satisfied, and the proof of [Ta], P4, completed. ■

**On the local duality pairing**

In analogy to the short exact sequence [4] one also has the short exact sequence

$$0 \to \text{ad}^0(\overline{\rho})(1)/(\sigma - 1)\text{ad}^0(\overline{\rho})(1) \to H^1(G_v, \text{ad}^0(\overline{\rho})(1)) \to \text{ad}^0(\overline{\rho})^\sigma \to 0.$$  

Thus one may view $L_{v, \text{unr}}$ as a subspace of $\text{ad}^0(\overline{\rho})/(\sigma - 1)\text{ad}^0(\overline{\rho})$ where $\sigma$ is the image of Frob in $\text{Gal}(E(\zeta)/K)$ and $L_{v, \text{unr}} := L_v^1 \cap H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho})(1))$ as a subspace of $\text{ad}^0(\overline{\rho})(1)/(\sigma - 1)\text{ad}^0(\overline{\rho})(1)$. While it is clear that the former only depends on $\sigma$ and the choice of $\lambda$, for the latter this is not immediate, since it was defined using the pairing [4] which in turn was defined using Tate local duality. For later use, we now show that in fact also $L_{v, \text{unr}}^1 \subset \text{ad}^0(\overline{\rho})(1)/(\sigma - 1)\text{ad}^0(\overline{\rho})(1)$ only depends on $\sigma$. For this, let us fix $v$ and $\sigma$ as above and also $\lambda$ and, in case (I), $\lambda'$.

Above we observed that cocycles in $H^i(G_v, \text{ad}^0(\overline{\rho}))$ are determined by their values $c(s) \in \text{ad}^0(\overline{\rho})/(\sigma - 1)\text{ad}^0(\overline{\rho})$ and $c(t) \in (\text{ad}^0(\overline{\rho})(-1))^\sigma$. Similar interpretations hold for 1-cocycles in the cohomology group $H^i(G_v, \text{ad}^0(\overline{\rho})(1)) \equiv H^i(G_{q_v}, \text{ad}^0(\overline{\rho})(1))$. We will now make explicit the Tate local duality in terms of cocycle representatives. This requires that we make explicit the pairing [4], as well as the isomorphism $H^2(G_v, F(1)) \cong F$.

It is well-known and, using the Leray-Serre spectral sequence and the fact that $\mathbb{Z}$ and $\mathbb{Z}'$ are of cohomological dimension one, easy to see that
$H^2(G_q, F(1)) \cong H^2(G_{\overline{q}}, F(1)) \cong F$. Recall a) that elements of $H^2(G_{\overline{q}}, F)$ classify extensions of $G_q$ by $F$ and b) that elements of this cohomology group may be represented by normalized 2 cocycles, i.e., maps

$$\langle ., . \rangle: \overline{G_q} \times \overline{G_q} \to F$$

which satisfy $[1, g] = [g, 1] = 0$ for all $g \in G_q$ and

$$f[g, h] - [fg, h] + [f, gh] - [f, g] \quad \forall f, g, h \in \overline{G_q}.$$ 

Note that in our situation $f[g, h] = [g, h]$; but we decided to leave $f$ in the notation, to remind the reader of the condition of a normalized 2-cocycle also in the case of non-trivial coefficients.

Regarding the duality pairing (4), we have the following results:

**Lemma 2.13** In terms of normalized 2-cocycles, the isomorphism

$$H^2(G_{\overline{q}}, F(1)) \cong F$$

is given by

$$\langle ., . \rangle : \overline{G_q} \times \overline{G_q} \to F$$

$$\sum_{i=1}^{q^{\ell-1}-1} [t^i, t] + [t^{q^{\ell-1}}, s^{\ell-1}] - [s^{\ell-1}, t] \in F.$$

**Lemma 2.14** With respect to the isomorphism of the previous lemma, the trace pairing

$$H^1(G_{\overline{v}}, \text{ad}^0(\overline{\rho})) \times H^1(G_{\overline{v}}, \text{ad}^0(\overline{\rho})(1)) \to F,$$

is given explicitly in terms of 1-cocycles as follows: Let $c_1$ and $c_2$ be 1-cocycles of $H^1(G_q, \text{ad}^0(\overline{\rho}))$ and $H^1(G_{\overline{q}}, \text{ad}^0(\overline{\rho})(1))$, respectively. Then the image of $(c_1, c_2)$ under the pairing is given by

$$\text{Trace}(c_1(s)c_2(t) - c_2(s)c_1(t)) \in F,$$

unless $\ell = 2$ and $q \equiv 3 \pmod{4}$. In the latter case it is $\text{Trace}(c_1(s)c_2(t) + c_2(s)c_1(t) + c_1(t)c_2(t))$.

**Corollary 2.15** $L_{\overline{v}, \text{unr}}$ and $L_\lambda^{+\overline{v}, \text{unr}}$ only depend on $\sigma$ and the choice of eigenvalue $\lambda$. 

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Therefore, in later sections we freely write $L_{\sigma,\text{unr}}$ and $L_{\sigma,\text{unr}}^\perp$ for the respective spaces, where $\sigma$ is the image of Frob$_v$ in $\text{Gal}(E(\zeta_\ell)/K)$. If we also want to include the choice of $\lambda$ in the notation, we write $L_{\sigma,\lambda,\text{unr}}$ and $L_{\sigma,\lambda,\text{unr}}^\perp$.

Let us first prove the corollary:

**Proof:** On rings of characteristic $\ell$, the definition of $C_v$ depends only on $\sigma$. This easily implies that $L_v,\text{unr} \subset \text{ad}^0(\overline{\rho})/((\sigma - 1)\text{ad}^0(\rho))$, as well as $M := L_v + H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho})) / H^1_{\text{unr}}(G_v, \text{ad}^0(\rho)) \subset (\text{ad}^0(\rho)(-1))^\sigma$ only depend on $\sigma$.

Let $c_1$ be any cocycle in $L_v$ whose image $\tilde{c}_1$ in $M \subset \text{ad}^0(\overline{\rho})/((\sigma - 1)\text{ad}^0(\rho))$ is a generator. Since $H^1_{\text{unr}}(G_v, \text{ad}^0(\rho))$ and $H^1_{\text{unr}}(G_v, \text{ad}^0(\rho)(1))$ are orthogonal under the trace pairing, Lemma 2.14 shows that $L^\perp_{v,\text{unr}}$ is spanned by the set of all unramified cocycles $c_2: G_v \rightarrow \text{ad}^0(\rho)(1)$ which satisfy $\text{Trace}(\tilde{c}_1 \cdot c_2(s)) = 0$.

This gives a condition on $c_2(s) \in \text{ad}^0(\rho)(1)/(\sigma - 1)\text{ad}^0(\rho)(1)$ which only depends on $M$, i.e. on $\sigma$. The corollary is thus proved.

**Proof of Lemma 2.14:** We assume that we have proved Lemma 2.13. In terms of 1-cocycles, the map

$$H^1(G_v, \text{ad}^0(\overline{\rho})) \times H^1(G_v, \text{ad}^0(\rho)) \rightarrow H^2(G_v, \text{ad}^0(\overline{\rho}) \otimes \text{ad}^0(\rho)(1))$$

is given by mapping a pair $(c_1, c_2)$ to the (normalized) 2-cocycle defined by $[f, g] := c_1(f) \otimes c_2(g)$. If we compose this with the map on cohomology induced from the trace map

$$\text{ad}^0(\overline{\rho}) \otimes \text{ad}^0(\rho)(1) \rightarrow \mathbb{F}(1) : A \otimes B \mapsto \text{Trace}(AB),$$

we obtain the (normalized) 2-cocycle defined by $[f, g] := \text{Trace}(c_1(f)c_2(g)) \in \mathbb{F}(1)$. By Lemma 2.13 it follows that the pair $(c_1, c_2)$ is mapped to

$$\text{Trace}\left(\sum_{i=1}^{q^{\ell-1}-1} c_1(t^i)c_2(t) + c_1(t^{q^{\ell-1}})c_2(s^{\ell-1}) - c_1(s^{\ell-1})c_2(t)\right) \in \mathbb{F}.$$

Because $c_1$ restricted to $\hat{Z}^{\ell}$ is a homomorphism, we have $c_1(t^i) = ic_1(t)$. So the sum simplifies to

$$c_1(t)c_2(t) \sum_{i=1}^{q^{\ell-1}-1} i = c_1(t)c_2(t)q^{\ell-1}(q^{\ell-1} - 1)/2.$$
As \( q^{\ell-1} \equiv 1 \pmod{\ell} \), this sum is zero unless \( \ell = 2 \) and \( q \equiv 3 \pmod{4} \). In the latter case it is \( c_1(t)c_2(t) \). For the same reason, the term \( c_1(t^{q^{\ell-1}}) = q^{\ell-1}c_1(t) = c_1(t) \).

To complete the proof of the lemma, it now suffices to show that we may replace \( c_2(s^{\ell-1}) \) by \( -c_2(s) \) (and similarly \( c_1(s^{\ell-1}) \) by \( -c_1(s) \)). An easy calculation shows that \( \text{Trace}(c_1(t)(\sigma - 1)c_2(s)) = 0 \). Also we have \( c_2(s^{\ell-1}) = (1 + \sigma + \ldots + \sigma^{\ell-2})c_2(s) \). Combining the previous two observations, we find

\[
\text{Trace}(c_1(t)c_2(s^{\ell-1})) = \text{Trace}(c_1(t)(\ell - 1)c_2(s)) = -\text{Trace}(c_1(t)c_2(s)),
\]

as asserted. The argument for \( c_1(s^{\ell-1}) \) is analogous. 

---

**Proof of Lemma 2.13** By the Leray-Serre spectral sequence applied to

\[
\hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}' \rtimes \hat{\mathbb{Z}}
\]

and the module \( \mathbf{F}(1) \), we obtain an isomorphism

\[
H^2(\hat{\mathbb{Z}}' \rtimes \hat{\mathbb{Z}}, \mathbf{F}(1)) \cong (H^2(\hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}}, \mathbf{F})(1))^{\mathbb{Z}/(\ell - 1)}
\]

given by restriction. The point is that the action of \( \hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}} \) on \( \mathbf{F}(1) \) is trivial (the residue field of the corresponding local Galois extension has order \( q^{\ell-1} \), and hence contains a primitive \( \ell - 1 \)-th root of unity). Since both \( H^2(\ldots) \) terms are isomorphic to \( \mathbf{F} \), it is not necessary to take invariants on the right for there to be an isomorphism. So it suffices to show that the identification asserted in the lemma is given by first restricting normalized 2-cocycles and then giving an isomorphism \( H^2(\hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}}, \mathbf{F}) \cong \mathbf{F} \).

For the latter we use the interpretation in terms of extension classes, cf. [WB], §6.6. So let \([\ldots]\) be a normalized 2-cocycle of the latter module. Then the corresponding extension \( G \) can be described as the group whose underlying elements are pairs \((a, x), a \in \mathbf{F}, x \in \hat{\mathbb{Z}}' \rtimes \hat{\mathbb{Z}} \) and whose composition law is given by \((a, x)(b, y) = (a + x \cdot b + [x, y], xy)\). One easily verifies that \( G \) is split if and only if there exist \( a, b \in \mathbf{F} \) such that \( \tilde{s} := (a, s^{\ell-1}) \) and \( \tilde{t} := (b, t) \) satisfy

\[
\tilde{s}\tilde{t} = \tilde{t}^\sigma\tilde{s}.
\]

In terms of 2-cocycles, \( G \) being split is equivalent to \([\ldots]\) being a 2-coboundary. Using the composition law, one can compute both sides. Let us denote
the difference of the $F$-component by $d([.,.])$, so that
\[d([.,.]) = \sum_{i=1}^{q-1} [t^i, t] + [t^{q-1}, s^{q-1}] - [s^{q-1}, t].\]

We have $d([.,.]) = 0$ if and only if $[.,.]$ is a 1-coboundary. Furthermore $d$ is $F$-linear and thus it induces an isomorphism
\[H^2(\hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}}, F) \cong F.\]

Given a 2-cocycle for $\hat{\mathbb{Z}}' \rtimes \hat{\mathbb{Z}}$, restricting it to $\hat{\mathbb{Z}}' \rtimes (\ell - 1)\hat{\mathbb{Z}}$ and applying $d$ yields precisely the formula in the lemma, and so its proof is completed.

### 2.3 Local deformations at $r$-places

Regarding places at which $\overline{\rho}$ is ramified, one has the following results:

**Proposition 2.16** Suppose that $\overline{\rho}(I_v)$ is of order prime to $\ell$. Define the functor $C_v : \mathcal{A} \to \text{Sets}$ by
\[R \mapsto \{\rho : G_v \to \text{GL}_n(R) \mid \rho \text{ (mod } m_R) = \overline{\rho}_v, \rho(I_v) \cong \overline{\rho}(I_v), \det \rho = \eta_v\}\]
and $L_v$ as the corresponding subspace in $H^1(G_v, \text{ad}^0(\overline{\mathfrak{t}}))$. Then $(C_v, L_v)$ satisfies the conditions P1–P7 of [Ta], the conductors of $\overline{\rho}_v$ and of any lift $\rho \in C_v(R)$, $R \in \mathcal{A}$ agree, $L_v = H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\mathfrak{t}}))$ and $\dim L_v = h^0(G_v, \text{ad}^0(\overline{\mathfrak{t}}))$.

**Proof:** Except for the assertion on conductors, this is essentially [Ta], Example E1, and so we only prove the latter part. For any ring $R \in \mathcal{A}$ and $\rho \in C_v(R)$, let $V_\rho(R)$ denote the $R[G_v]$ module defined by $\rho$. The kernel of $\text{GL}_n(R) \to \text{GL}_n(F)$ is a pro-$\ell$ group and thus prime to $p$. Therefore the Swan conductors of $\rho$ and $\overline{\rho}_v$ are the same. The module $V_\rho(R)$ is free over $R$ and thus the difference of the conductors of the two representations is given by
\[
\text{rank}_R V_\rho(R)^{I_v} - \dim_F V_{\overline{\rho}_v}(F)^{I_v}.
\]
Since $G_v$ acts on both representations via the same quotient $\overline{T}_v$ which is prime to $\ell$, there is a natural equivalence between $R[\overline{T}_v]$-representations which are free and finite over $R$ and $F[\overline{T}_v]$-representations given by reduction modulo $m_R$. In particular both categories are semisimple and thus the above expression is well-defined. Furthermore, this implies that the number of trivial components contained in $V_\rho(R)$, as an $\overline{T}_v$-module, is the same as that of $V_{\overline{\rho}_v}(F)$, and hence that the above difference is zero, as asserted. ■
Proposition 2.17 Suppose that $\overline{\rho}_v$ is at most tamely ramified and that $h^0(G_v, \text{ad}^0(\overline{\rho})) < h^0(G_v, \text{ad}(\overline{\rho}))$. Then there exists a pair $(C_v, L_v)$ which satisfies conditions P1–P7 of [Ta] with $L_v = H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho}))$ and $\dim L_v = h^0(G_v, \text{ad}^0(\overline{\rho}))$, is compatible with $\eta_v$ and such that the conductors of $\overline{\rho}_v$ and any lift $\rho \in C_v(R)$, $R \in A$, agree.

Remark 2.18 One can construct examples which show that the condition $\dim_{\mathbb{F}} \text{ad}^0(\overline{\rho})^{G_v} < \dim_{\mathbb{F}} \text{ad}(\overline{\rho})^{G_v}$ is necessary. The latter is automatically satisfied if $\ell \not| n$.

Remark 2.19 If $\overline{\rho}(I_v)$ is of order prime to $\ell p$, we may apply either Proposition 2.16 or Proposition 2.17 to obtain a pair $(C_v, L_v)$. The pairs so obtained do have similar properties and in fact, in Remark 2.25 we will explain why the two are isomorphic.

In the remainder of this section, we shall give the proof of Proposition 2.17. Since all representations that occur in the proof will factor via the tame quotient $G_{qv}$ of $G_v$, we fix the usual generators $s, t$ satisfying the relation $sts^{-1} = t^e_v$.

For $B \in \text{GL}_n(W(\mathbb{F}))$ we denote by $V$ the corresponding $W(\mathbb{F})[X]$-module on $W(\mathbb{F})^n$ by having $X$ act via $B$. Let $Q_{\mathbb{F}}$ denote the fraction field of $W(\mathbb{F})$. We say that $B \in \text{GL}_n(W(\mathbb{F}))$ is a minimal lift of its reduction $\overline{B} \in \text{GL}_n(\mathbb{F})$, if $V = \oplus_i V_{i,s} \otimes_{W(\mathbb{F})} V_{i,u}$ where the $V_{i,?}$ are $W(\mathbb{F})[X]$-modules such that:

(i) on $V_{i,u}$ the matrix representing $X$ is $W(\mathbb{F})$-conjugate to a regular unipotent matrix in Jordan form,

(ii) on $V_{i,s}$ the characteristic polynomial of $X$ is irreducible and its roots are Teichmüller lifts of elements in $\mathbb{F}$.

Lemma 2.20 Any $\overline{B} \in \text{GL}_n(\mathbb{F})$ has a minimal lift to $\text{GL}_n(W(\mathbb{F}))$.

Proof: Let $\overline{V} := F^n$ be the $F[X]$-module $\overline{V} := F^n$ obtained by having $X$ acts as $\overline{B}$. We choose a decomposition $\overline{V} \cong \oplus_i \overline{V}_i$ into indecomposable $\overline{V}_i$. On $V_i$ the action of $X$ decomposes into commuting semisimple and a unipotent parts, defined over $F$. For instance by considering Jordan normal forms over $\overline{F}$, one shows that correspondingly one has $\overline{V}_i \cong \overline{V}_{i,s} \otimes_F \overline{V}_{i,u}$ where $\overline{V}_{i,s}$ is a semisimple representation of $X$ and $\overline{V}_{i,u}$ is a unipotent representation of $X$.

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Because $V_i$ is indecomposable, the characteristic polynomial of $X$ on $V_{i,s}$ is irreducible over $F$. For the same reason, the action of $X$ on $V_{i,u}$ is by a regular unipotent matrix. So we may assume that the operation of $X$ on $V_{i,s}$ is given by a companion matrix whose characteristic polynomial is irreducible over $F$, and on $V_{i,u}$ by a single Jordan block with eigenvalue 1.

We now lift $X$ on $V_{i,u}$ to a single Jordan block with eigenvalue 1 over $W(F)$ and $X$ on $V_{i,u}$ to a companion matrix with eigenvalues the Teichmüller lifts of those of $X$ on $V_{i,u}$. The corresponding representations $V_{i,u}$ and $V_{i,s}$ combine to give a representation of $W(F)[X]$ on $V = \bigoplus_i V_{i,s} \otimes_{W(F)} V_{i,u}$ which has all the required properties. Therefore the matrix representing this $X$ is a minimal lift of $B$. ■

We first prove the following result, of which part (a) in the case where $\rho(I_v)$ is an $\ell$-group is [Bö1], Proposition 3.2.

**Proposition 2.21** Let $B$ be a minimal lift of $\overline{B} := \overline{\rho(t)}$.

(i) $M := M_n(W(F))/\{AB - B^q A \mid A \in W(F)\}$ is flat over $W(F)$.

(ii) There exists a lift $\rho_0 : G_v \longrightarrow G_v \longrightarrow \text{GL}_n(W(F))$ with $\rho_0(t) = B$.

**Proof:** Let $X$ and the $V_{i,s}$ be as in the definition of minimal lift of $B$. It is not difficult to see from condition (ii) that we may assume that $X$ on $V_{i,s}$ is given as a companion matrix $B_{i,s}$. Let now $F'$ be a finite extension of $F$ which contains all eigenvalues of $B_{i,s}$. Then clearly over $W(F')$ the companion matrices $B_{i,s}$ may be diagonalized. Moreover this diagonalization procedure commutes with reduction modulo $\ell$. Since the base change $\otimes_{W(F)} W(F')$ is faithfully flat, we will from now on for the proof of (a) assume that $F$ contains all the eigenvalues of $\overline{B}$.

Reordering matrices in Jordan form and using the relation $A B A^{-1} = B^q$ it is easy to prove the following lemma. We leave details to the reader.

**Lemma 2.22** There exist $\mu_i \in F$ with Teichmüller lifts $\hat{\mu}_i$ and $m_i \in \mathbb{N}$, $i = 1, \ldots, d$, such that

(i) $\mu_i^{m_i} = \mu_i$,

(ii) the elements $\mu_i^{q_j}$, $i = 1, \ldots, d$, $j = 1, \ldots, m_i$ are pairwise disjoint and form a complete list of the eigenvalues of $\overline{B} := \overline{\rho(t)}$, and
(iii) with respect to a suitable basis one has

\[ B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \cdot & \ddots & \cdot & \cdot \\ 0 & \cdots & B_d \end{pmatrix}, \]

where each \( B_i = \begin{pmatrix} B_{i,1} & 0 & \cdots & 0 \\ \cdot & \ddots & \cdot & \cdot \\ 0 & \cdots & 0 & B_{i,m_i} \end{pmatrix} \)
is a square matrix, and for fixed \( i \) the \( B_{i,j} \) can be written as \( B_{i,j} = \mu_i^{q_j-1} U_i \), for some unipotent \( U_i \) in Jordan form, independent of \( j \).

Furthermore if \( B \) is given as above, then \( A := \rho(s) \) takes the form

\[ A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \cdot & \ddots & \cdot & \cdot \\ 0 & \cdots & 0 & A_d \end{pmatrix} \]

with \( A_i = \begin{pmatrix} 0 & \bar{A}_{i,1} & 0 & \cdots & 0 \\ \cdot & 0 & \bar{A}_{i,2} & \cdot & \cdot \\ \cdot & \cdot & \ddots & \ddots & \cdot \\ 0 & \cdots & \cdots & 0 & \bar{A}_{i,m_i-1} \\ \bar{A}_{i,m_i} & 0 & \cdots & 0 \end{pmatrix} \)
such that the \( \bar{A}_{i,j} \) satisfy the relation \( \bar{A}_{i,j} U_i = U_i^{q_j} \bar{A}_{i,j} \).

We apply Proposition 2.21 (b) in the case \( \ell \not| \#(I_v) \), covered by [B61], to obtain matrices \( A_0, i, j \) over \( W(F) \) that satisfy \( A_0, i, j U_i = U_i^{q_j} A_0, i, j \) for all \( i, j \) and whose reduction modulo \( \ell \) agree with \( \bar{A}_{i,j} \). Let \( A_0 \) be composed from the \( A_0, i, j \) in the same way as \( A \) is from the \( A_{i,j} \). Then \( A_0 \) is a lift to \( W(F) \) of \( \bar{A} \) such that \( A_0 B A_0^{-1} = B^{q_v} \). (This proves (b) only under the further hypothesis that \( F \) contains all eigenvalues of \( B \).)

We now consider the exact sequence

\[ 0 \rightarrow K \rightarrow M_n(W(F)) \xrightarrow{A \mapsto AB - B^{q_v} A} M_n(W(F)) \rightarrow \mathcal{M} \rightarrow 0 \]  \hspace{1cm} (7)

where \( K := \{ A \in W(F) : AB = B^{q_v} A \} \). To complete (a), we need to show that the generic rank of \( \mathcal{M} \) is the same as its special rank. To simplify the problem we apply the isomorphism \( M_n(W(F)) \rightarrow M_n(W(F)) : A \mapsto A_0^{-1} A \) to the middle terms in (7). This yields the isomorphic exact sequence

\[ 0 \rightarrow K' \rightarrow M_n(W(F)) \xrightarrow{A' \mapsto A'B - BA'} M_n(W(F)) \rightarrow \mathcal{M'} \rightarrow 0, \]

with kernel \( K' = \{ A' \in M_n(W(F)) : A'B = BA' \} \cong K' \), and cokernel \( \mathcal{M'} = M_n(W(F))/\{ A'B - BA' \mid A' \in W(F) \} \cong \mathcal{M} \). We need to prove that the generic and special ranks of \( \mathcal{M'} \) agree.
Counting dimensions in the above short exact sequence and its reduction modulo \( \ell \), it suffices to show that the dimension of \( K' \otimes_{W(F)} Q_F \) and of \( K := \{ \overline{A} \in M_n(F) \mid \overline{A}B = B\overline{A} \} \) agree. Because the \( B_{i,j} \) have distinct eigenvalues modulo \( \ell \), the matrices \( \overline{A} \in K' \) and \( A' \in K' \), respectively, will have the same block form as \( B \). So we may consider blocks for each pair \( i, j \) separately. Therefore it suffices to prove the assertion in the case where \( B \) is a single Jordan block with eigenvalue 1. This case was treated explicitly in the proof of [6], Proposition 3.2. The proof of (a) is now complete.

It remains to deduce (b) from (a). Because \( M \) is flat, the reduction mod \( \ell \) of the exact sequence remains exact, and so the kernel of the reduction is \( K/\ell K \). The matrix \( \overline{A} = \overline{\rho}(s) \in M_n(F) \) lies in this kernel and is therefore the reduction modulo \( \ell \) of a matrix \( A \in K \). Because \( A \) and \( B \) satisfy the same relations as \( s, t \) the desired lift exists.

As a corollary to the above proof, we record the following technical result, obtained by base change and using flatness.

**Corollary 2.23** Suppose \( B \in \text{GL}_n(W(F)) \) is a minimal lift of \( \overline{B} \in \text{GL}_n(F) \). Then for any \( R \in \mathcal{A} \) the submodules \( K(R) := \{ A \in M_n(R) : AB = B^qA \} \) and \( K'(R) := \{ A \in M_n(R) : AB = BA \} \) of \( M_n(R) \) are free and direct summands of \( R \)-rank independent of \( R \). Moreover \( K'(R) = A_0^{-1}K(R) \) for \( A_0 \) as in the above proof.

**Proof** of Proposition 2.17 Let \( b_1, \ldots, b_m \) be parts of a basis of \( K(W(F)) \). We shall specify more requirements on these elements below. Let \( x_1, \ldots, x_m \) be indeterminates and define

\[
\mathcal{S}_v := A + \sum x_i b_i \in \text{GL}_n(R), \quad \mathcal{T}_v := B,
\]

\[
R_v := W(F)[[x_1, \ldots, x_m]]/(\det \mathcal{S}_v - \eta_v(s)),
\]

\[
\rho_v : G_v \longrightarrow \hat{Z}' \times \hat{Z} \longrightarrow \text{GL}_n(R_v) : s \mapsto \mathcal{S}_v, t \mapsto \mathcal{T}_v,
\]

and the functor \( C_v : \mathcal{A} \rightarrow \text{Sets} \) by

\[
R \mapsto C_v(R) := \{ \rho : G_v \rightarrow \text{GL}_n(R) \mid \exists \alpha \in \text{Hom}_A(R_v, R), \exists M \in 1 + M_n(m_R) : \rho = M(\alpha \circ \rho_v)M^{-1} \}.
\]
Let $L_v$ be the corresponding subspace $L_v \subset H^1(G_v, \text{ad}^0(\overline{\rho}))$. As $\rho_v(t)$ does not deform, the subspace $L_v$ lies inside

$$H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho})) \cong \text{ad}^0(\overline{\rho})^t/(s-1)\text{ad}^0(\overline{\rho})^t.$$  

We denote by $\overline{b}_1, \ldots, \overline{b}_m \in \mathcal{K}(F)$ the reductions of the $b_i$ modulo $\ell$. So the elements $\overline{A}^{-1}\overline{b}_i$ lie in $\mathcal{K}'(F) = \text{ad}^0(\overline{\rho})^t$, and an explicit calculation shows that $L_v$ is spanned by the images in $\text{ad}^0(\overline{\rho})^t/(s-1)\text{ad}^0(\overline{\rho})^t$ of those linear combinations $\sum x_i \overline{A}^{-1}\overline{b}_i$, $x_i \in F$, which are consistent with the determinant condition $\det \mathcal{S}_v = \eta(s)$.

The latter condition modulo $(\ell, m^2_R)$ means

$$\det \overline{A} = \det \overline{A} \cdot \det(I + \overline{A}^{-1}(\sum x_i \overline{b}_i)), \text{ i.e.,}$$

$$1 = 1 + \sum x_i \text{Trace}(\overline{A}^{-1}\overline{b}_i).$$

Thus the above linear combinations satisfy $\sum x_i \text{Trace}(\overline{A}^{-1}\overline{b}_i) = 0$.

Now we fix the choice of the $b_i$. Namely, we take them as a subset of $\mathcal{K}(W(F))$ whose reductions modulo $\ell$ forms a basis of $\mathcal{K}(F)/\{X\overline{A} - \overline{A}X : \overline{X} \in \mathcal{K}(F)\}$. The elements $\overline{A}^{-1}\overline{b}_i$ then form a basis of

$$\text{ad}(\overline{\rho})^t/(s-1)\text{ad}(\overline{\rho})^t = \mathcal{K}'(F)/\{\overline{A}X\overline{A}^{-1} - \overline{X} : \overline{X} \in \mathcal{K}'(F)\}.$$  

To pass from $\text{ad}(\overline{\rho})$ to $\text{ad}^0(\overline{\rho})$ we use our assumption

$$h^0(G_v, \text{ad}(\overline{\rho})) > h^0(G_v, \text{ad}^0(\overline{\rho})).$$

Because $(\text{ad}(\overline{\rho}))^G_v/I_v \cong (\text{ad}(\overline{\rho}))^G_v/I_v$ as an $F$-vector space, and similarly for $\text{ad}^0(\overline{\rho})$, we deduce that $(\text{ad}(\overline{\rho})^t)/(s-1)\text{ad}(\overline{\rho})^t$ properly contains $(\text{ad}^0(\overline{\rho})^t)/(s-1)\text{ad}^0(\overline{\rho})^t$. This in turn shows that any element of the module $(\text{ad}^0(\overline{\rho})^t)/(s-1)\text{ad}^0(\overline{\rho})^t$ can be obtained as a linear combination $\sum x_i \overline{A}^{-1}\overline{b}_i$ which satisfies $\sum x_i \text{Trace}(\overline{A}^{-1}\overline{b}_i) = 0$. This has two consequences: Firstly we have $L_v = H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{\rho}));$ secondly the relation $\det \mathcal{S}_v = \eta(s)$ allows one to eliminate one of the variables $x_i$, since this is possible tangentially, and so $R_v$ is smooth of relative dimension $\dim L_v$ over $W(F)$.

Note also that the determinant of $\rho_v$ is the Teichmüller lift of that of $\overline{\rho}_v$: For $\rho_v(t)$, this follows from the construction of $B$, for $\rho_v(s)$ from the definition of $R_v$. This implies the result in general, since $\rho_v$ is only tamely ramified.
Let us now verify properties P1–P7 of \([Ta]\). As expected the only property that is non-trivial is P4. To verify it, suppose we are given rings \(R_1, R_2 \in \mathcal{A}\), lifts \(\rho_i \in \mathcal{C}_v(R_i)\), ideals \(I_j \in R_j\), and an identification \(R_1/I_1 \cong R_2/I_2\) under which \(\rho_1(\text{mod } I_1) \equiv \rho_2 \text{ (mod } I_2)\). We need to show that \((\rho_1, \rho_2)\) lies in \(\mathcal{C}_v(R)\) for

\[R := \{(r_1, r_2) \in R_1 \oplus R_2 : r_1 \equiv r_2 \text{ (mod } I_1)\} \]

So let \(\alpha_i \in \text{Hom}_A(R_v, R_i)\) and \(M_i \in \text{GL}_n(R_i)\) such that \(\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}, i = 1, 2\). We claim that there exists \(\alpha \in \text{Hom}_A(R_v, R)\) and \(M \in \text{GL}_n(R)\) with \(M \equiv I \text{ (mod } \mathfrak{m}_R)\) such that \((\rho_1, \rho_2) = M(\alpha \circ \rho_v)M^{-1}\). By conjugating \(\rho_1\) by some lift of \(M_2 \text{ (mod } I_1)\) to \(R_1\), we may assume \(M_2 = I\).

By an inductive argument, which is left to the reader, one can show the following auxiliary result:

**Lemma 2.24** Suppose \(\tilde{R} \in \mathcal{A}\), \(\tilde{J}\) is a proper ideal of \(\tilde{R}\), \(A' \in A + \sum \beta_i b_i + M_n(\tilde{J})\) for some \(\beta_i \in \tilde{R}\) and that \(A'B = B^{\alpha'}A'\). Then there exists \(\tilde{\beta}' \in \tilde{R}\) with \(\tilde{\beta}' - \beta_i \in \tilde{J}\) and \(C \in I + M_n(\tilde{J})\) such that

\[A' = C\left(A + \sum \beta_i b_i\right)C^{-1}\]

Continuing with the proof of Proposition 2.17, observe that the condition

\[M_1(\alpha_1 \circ \rho_v)M_1^{-1} \equiv \alpha_2 \circ \rho_v \text{ (mod } I_2)\]  

applied to \(t\) implies that \(M_1(\text{mod } I_1)\) commutes with \(B\). By Corollary 2.23 we can find a lift \(\tilde{M}_1 \in M_n(R_1)\) of \(M_1(\text{mod } I_1)\) which commutes with \(B\).

Because of (8) and the choice of \(\tilde{M}_1\), we can apply the above lemma to \(A' := \tilde{M}_1(\alpha_1 \circ \rho_v(s))\tilde{M}_1^{-1}\). It yields \(\alpha'_1 : R_v \to R_1\) and \(C \in I + M_n(I_1)\) such that

\[C(\alpha'_1 \circ \rho_v(s))C^{-1} = \tilde{M}_1(\alpha_1 \circ \rho_v(s))\tilde{M}_1^{-1}\]

and \(\alpha'_1(\text{mod } I_1) = \alpha_2(\text{mod } I_2)\). Define \(\tilde{C} := M_1M_1^{-1}C \in I + M_n(I_1)\). Then

\[\tilde{C}(\alpha'_1 \circ \rho_v)\tilde{C}^{-1} = M_1(\alpha_1 \circ \rho_v)M_1^{-1} = \rho_1\]

Therefore, if we set \(M := (\tilde{C}, I) \in I + M_n(\mathfrak{m}_R)\) and \(\alpha := (\alpha'_1, \alpha_2) : R_v \to R\), we have

\[(\rho_1, \rho_2) = M(\alpha \circ \rho_v)M^{-1},\]

and so the proof of P4, and hence of all the axioms of Taylor, is completed.
It remains to prove the assertion on the conductors. As in the proof of Proposition 2.16, the difference in conductors is given by

\[ \dim_R V_\rho (R)^{I_v} - \dim_F V_{\overline{\rho}_v} (F)^{I_v}, \]

where the notation is analogous to that in the quoted proof. Since \( I_v \) is topologically generated by the single element \( t \), whose image is the image of the matrix \( B \in \text{GL}_n(W(F)) \), this difference is given by

\[ \dim_R \mathcal{K}(R) - \dim_F \mathcal{K}'(F). \]

By Corollary 2.23, this difference is zero. This shows that the conductors of \( \rho \) and \( \overline{\rho}_v \) agree. 

**Remark 2.25** Suppose now that the image of \( I_v \) under \( \overline{\rho} \) is of order prime to \( \ell p \). Let \((\rho'_v, R'_v)\) be the versal deformation constructed in Proposition 2.16 and \((\rho_v, R_v)\) the one constructed in the previous proof.

The representation \( \rho_v \) was constructed so that \( \rho_v(t) \) was a minimal lift of \( \overline{B} \). Because \( \ell \) does not divide \( \#\overline{\rho}(I_v) \), the matrix \( \overline{B} \) is completely reducible. So the \( V_{i,u} \) in the definition of minimal lift are 1-dimensional. Therefore \( \rho_v(t) \) is completely reducible and of finite order prime to \( \ell \).

The universal property of \((\rho'_v, R'_v)\) shows that there is a morphism \( R' \to R_v \) which induces \( \rho_v \) from \( \rho'_v \). Because it is an isomorphism on mod \( \ell \) tangent spaces, the morphism is surjective. Since both rings are smooth of the same dimension it must be bijective. This shows that the two deformations agree.

### 2.4 Proof of Theorem 2.1 and the key lemma

**Proof of Lemma 2.7** We first prove

**Claim 1:** There exists a finite set \( T'' \) of \( R \)-places of type (II) and for each \( R \)-place \( v \in T'' \) a choice of eigenvalue \( \lambda_v \), as in the definition, such that

\[ H^1_{(L_v)}(T' \cup T'', \text{ad}^0(\overline{\rho})) \cap H^1(\text{Gal}(E(\zeta_\ell)/K), M^0_n(F)) = 0, \quad (9) \]

\[ H^1_{(L_v)}(T' \cup T'', \text{ad}^0(\overline{\rho})(1)) \cap H^1(\text{Gal}(E(\zeta_\ell)/K), M^0_n(F)(1)) = 0. \quad (10) \]

We only give the proof of (9), the one of (10) being analogous.
Let $\sigma_1, \ldots, \sigma_s$ be the different $R$-classes of type (II). For $\sigma_i$, let $\lambda_{i,j}$, $j = 1, \ldots, m_i$, be the list of eigenvalues in $F$ of multiplicity 2 (in the characteristic polynomial). Pick unramified places $v_{i,j}$, $i = 1, \ldots, s$, $j = 1, \ldots, m_i$ such that $\text{Frob}_{v_{i,j}} = \sigma_i$ for all $i, j$. Let $(C_{v_{i,j}}, L_{v_{i,j}})$ be the deformation problem defined in Section 2.2 for the pair $(v_{i,j}, \lambda_{i,j})$ and let $T'' := \{v_{i,j} : i = 1, \ldots, s, j = 1, \ldots, m_i\}$. We consider the following commuting diagram:

\[
H^1(\text{Gal}(E(\zeta_d)/K), \ad^0(\overline{\rho})) \cap H^1_{\{L_v\}}(T', \ad^0(\overline{\rho})) \rightarrow \prod_{v \in T''} H^1_{\{\ sigma_v, \ ad^0(\overline{\rho})_{\sigma_v} \}} \\
\downarrow \\
H^1_{\{L_v\}}(T', \ad^0(\overline{\rho})) \rightarrow \prod_{v \in T''} H^1(G_v, \ad^0(\overline{\rho}))/L_v.
\]

The kernel of the bottom row is $H^1_{\{L_v\}}(T' \cup T'', \ad^0(\overline{\rho}))$. By assumption there are sufficiently many $R$-classes for $(C_v, L_v)_{v \in S}$, and so the top horizontal arrow is injective. For each $i$, the image of the top left term in $\prod_{j=1, \ldots, m_i} H^1_{\{\sigma_{v_{i,j}}, \ ad^0(\overline{\rho})_{\sigma_{v_{i,j}}} \}}$ is diagonal. Therefore we may replace the top right term by $\prod_i H^1(\{\sigma_i, \ ad^0(\overline{\rho})_{\sigma_i} \})$ and still retain the injectivity of the top horizontal map. Below we show that the induced right vertical arrow

\[
\prod_i H^1(\{\sigma_i, \ ad^0(\overline{\rho})_{\sigma_i} \}) \rightarrow \prod_{v \in T''} H^1(G_v, \ad^0(\overline{\rho}))/L_v \tag{11}
\]

is injective. An easy diagram chase then shows that the intersection of $H^1_{\{L_v\}}(T' \cup T'', \ad^0(\overline{\rho}))$ and $H^1\left(\text{Gal}(E(\zeta_d)/K), \ad^0(\overline{\rho})\right)$ is zero, as desired.

The injectivity of (11) may be verified on the morphisms

\[
H^1(\{\sigma_i, \ ad^0(\overline{\rho})_{\sigma_i} \}) \rightarrow \prod_{j=1, \ldots, m_i} H^1(G_{v_{i,j}}, \ad^0(\overline{\rho}))/L_{v_{i,j}},
\]

individually. The representation $\ad^0(\overline{\rho})_{\sigma_i}$ itself is a direct sum of adjoint representations on $2 \times 2$ blocks of matrices of trace zero. Furthermore, the image of $H^1(\{\sigma_i, \ ad^0(\overline{\rho})_{\sigma_i} \})$ lies in $H^1_{\text{unr}}(G_{v_{i,j}}, \ad^0(\overline{\rho}))/L_{v_{i,j}, \lambda_{i,j}, \text{unr}}$, and $H^1_{\text{unr}}(G_{v_{i,j}}, \ad^0(\overline{\rho}))$ itself breaks up into a direct sum over pieces corresponding to the rational canonical form of $\overline{\rho}(\sigma_i)$. So one is reduced to consider a single $2 \times 2$ block, and we may assume $n = 2$ and $m_i = 1$. But then $L_{v, \text{unr}} = 0$ ($v = v_{i,j}$) and the map

\[
H^1(\{\sigma_i, \ ad^0(\overline{\rho})\}) \rightarrow H^1_{\text{unr}}(G_{v_{i,j}}, \ad^0(\overline{\rho}))
\]

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is simply given by inflation and is thus injective. Note finally that (9) is preserved under adding further $R$-primes to $T''$. We have thus proved Claim 1.

By enlarging $T'$ if necessary, we assume from now on that (9) and (10) hold with $T'' = \emptyset$. We now induct on the dimension of $H^1_{\{L_v\}}(T', \text{ad}^0(\overline{p}))$ and assume it contains a non-zero cocycle $\phi$. By the formula in Remark 2.6 and our assumptions, the space $H^1_{\{L_v\}}(T', \text{ad}^0(\overline{p}))$ contains a non-zero cocycle $\psi$, as well.

**Claim 2:** There exists $w \in X \setminus T'$ and an admissible pair $(C_w, L_w)$ compatible with $\eta$ such that the following hold:

1. $n - 1 = \dim L_w = \dim L_{w,\text{unr}} + 1$,
2. $\phi$ does not map to zero in $H^1(G_w, \text{ad}^0(\overline{p}))(1) / L_w^\perp$ and
3. the space $(H^1_{\text{unr}}(G_w, \text{ad}^0(\overline{p}))) + L_w) / L_w$ lies in the image of the morphism $H^1_{\{L_v\}}(T', \text{ad}^0(\overline{p})) \rightarrow H^1(G_w, \text{ad}^0(\overline{p}))/L_w$.

Suppose for a moment that we have proved the above claim and let $T'' := T' \cup \{w\}$. The argument given in [Ta], proof of Lemma 1.2, then shows, by using (i), (iii), that $$H^1_{\{L_v\}}(S \cup T', \text{ad}^0(\overline{p})) = H^1_{\{L_v\}}(S \cup T' \cup \{w\} + H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{p}))) / (T'', \text{ad}^0(\overline{p})),$$
and by using (ii), that $$H^1_{\{L_v\}}(S \cup T'', \text{ad}^0(\overline{p})) \hookrightarrow H^1_{\{L_v\}}(S \cup T' \cup \{w\} + H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{p}))) / (T'', \text{ad}^0(\overline{p}))$$ is a proper containment, so that the proof of the lemma is completed.

The above claim is clearly implied by the following **Claim 3**, which we prove below: There exists $w \in X \setminus T'$ and an admissible pair $(C_w, L_w)$ compatible with $\eta$ such that (i) above holds and furthermore

1. $\phi$ does not map to zero in $H^1_{\text{unr}}(G_w, \text{ad}^0(\overline{p}))(1) / L_w^\perp,_{\text{unr}}$, and
2. $\psi$ does not map to zero in $H^1_{\text{unr}}(G_w, \text{ad}^0(\overline{p}))/L_w,_{\text{unr}} \cong F$.

To prove Claim 3, note that conditions (9) and (10) imply that the cycles $\psi$ and $\phi$ restrict to non-zero homomorphisms $\phi : G_{E(\zeta)} \rightarrow (\text{ad}^0(\overline{p}))(1)$ and $\psi : G_{E(\zeta)} \rightarrow \text{ad}^0(\overline{p})$. 

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Let $E_{\phi}$ and $E_{\psi}$ be the fixed fields of the respective kernels. Depending on whether the cyclotomic character $\chi$ is trivial, they may or may not be equal. The induced morphisms on $\text{Gal}(E_{\phi}/E(\zeta_\ell))$ and $\text{Gal}(E_{\psi}/E(\zeta_\ell))$, respectively, are equivariant for $\text{Gal}(E(\zeta_\ell)/K)$. Because $\text{ad}^0(\overline{\rho})$ is irreducible as a $\mathbb{F}_\ell[\text{im}(\rho)]$-module, the morphisms $\phi$, $\psi$ are bijective. Thereby the group $\text{V} := \text{Gal}(E_{\psi}E_{\phi}/E(\zeta_\ell))$ may be regarded as an $\mathbb{F}[\text{Gal}(E(\zeta_\ell)/K)]$-module which surjects onto $\text{ad}^0(\overline{\rho})$ and $\text{ad}^0(\overline{\rho})(1)$.

Let now $\sigma \in \text{Gal}(E(\zeta_\ell)/K)$ be an $\mathbb{F}$-class. Recall that the subspaces $L_{\text{unr},\sigma}^{\perp}$ of $\text{ad}^0(\overline{\rho})/\text{ad}^0(\overline{\rho})(1)$, and $L_{\text{unr},\sigma}$ of $\text{ad}^0(\overline{\rho})(1)/\text{ad}^0(\overline{\rho})(1)$, defined after Corollary 2.15, are of codimension one with respect to $\mathbb{F}$. Define $\widetilde{L}_{\text{unr},\sigma}$ and $\widetilde{L}_{\text{unr},\sigma}^{\perp}$ as the corresponding 1-codimensional subspaces in $\text{ad}^0(\overline{\rho})$ and $\text{ad}^0(\overline{\rho})(1)$, respectively. Each of the conditions

$$
\psi(\tau) \in -\psi(\overline{\sigma}) + \widetilde{L}_{\text{unr},\sigma} \quad \text{and} \quad \phi(\tau) \in -\phi(\overline{\sigma}) + \widetilde{L}_{\text{unr},\sigma}^{\perp},
$$

$\tau \in \text{V} = \text{Gal}(E_{\psi}E_{\phi}/E(\zeta_\ell))$, determines a hyperplane in the $\mathbb{F}$-vector space $\text{V}$. As we assumed $|\mathbb{F}| > 2$, the join of these two hyperplanes cannot span all of $\text{V}$, and hence there exists $\tau \in \text{Gal}(E_{\psi}E_{\phi}/E(\zeta_\ell))$ which lies on neither. We fix such a $\tau$ and define $\tilde{\tau} := \tau \sigma$.

The extension $E_{\psi}E_{\phi}$ is Galois over $K$, and therefore by the Čebotarev density theorem, we can choose a place $w$ in $X \setminus T'$ such that $\text{Frob}_w = \tilde{\tau}$. Take $C_w$ and $L_w$ as constructed in Section 2.2 so that by Proposition 2.10 condition (i) is satisfied and $(C_w, L_w)$ is compatible with $\eta$. Condition (ii') is satisfied, since the image of $\phi$ in $H^1_{\text{unr}}(G_w, \text{ad}^0(\overline{\rho}))$ is given by $\phi(\text{Frob}_w) = \phi(\overline{\tau}) = \phi(\tau) + \phi(\overline{\sigma}) \in \text{ad}^0(\overline{\rho})/(s - 1)\text{ad}^0(\overline{\rho})$, and hence does not lie in $L_{\sigma,\text{unr}}$. Assertion (iii') is shown in the same way, and whence the proof of the Lemma 2.7 is completed.

**Proof of Theorem 2.1** By enlarging $X$ if necessary, we may assume that $\overline{\mathcal{P}}$ ramifies at all places of $S$. Using Propositions 2.16 and 2.17, there exist locally admissible pairs $(\mathcal{C}_v, L_v)_{v \in S}$ compatible with $\eta$ for which one has $\dim L_v = h^0(G_v, \text{ad}^0(\overline{\rho}))$ and such that the conductor (at $v$) of any lift of type $\mathcal{C}_v$ is the same as that of $\overline{\mathcal{P}}_v$.

We claim that if $\overline{\mathcal{P}}$ admits sufficiently many $R$-classes, then it admits sufficiently many $R$-classes for $(\mathcal{C}_v, L_v)_{v \in S}$. We only verify the first of the two
conditions. For this consider the diagram:

\[
\begin{array}{c}
H^1(\text{Gal}(E(\zeta)/K), \text{ad}^0(\overline{p}))) \rightarrow \prod_{v \in S} H^1(\overline{p}(I_v), \text{ad}^0(\overline{p}))) \quad \rightarrow \quad \prod_{v \in S} H^1(G_v, \text{ad}^0(\overline{p}))/L_v
\end{array}
\]

To prove the claim, it suffices to show that the right vertical arrow is injective. By the propositions quoted above, we have \(L_v = H^1_{\text{unr}}(G_v, \text{ad}^0(\overline{p}))\) for \(v \in S\) and thus by the inflation restriction sequence the morphism \(H^1(G_v, \text{ad}^0(\overline{p}))/L_v \hookrightarrow H^1(I_v, \text{ad}^0(\overline{p}))\) is a monomorphism. Therefore it suffices to show for each \(v\) that

\[
H^1(\overline{p}(I_v), \text{ad}^0(\overline{p})) \hookrightarrow H^1(I_v, \text{ad}^0(\overline{p}))
\]

is injective. This is clear since it is an inflation map.

Applying Lemmas 2.5 and 2.7, the assertions of theorem are now straightforward.

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