Behavior of Gabled Hyperbolic Paraboloid Shells

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Abstract
Previous studies on gabled hyperbolic paraboloid shells (gabled hypar) revealed that shell loads are transferred to the supports mainly through diagonal arch action, and the contribution of edge beams, which have been traditionally included based on the assumptions of membrane theory, is actually very limited. This finding introduced a new shape of gabled hypars in which the edge beams are removed. This paper investigated the behaviors of gabled hypars with and without edge beams for various cases considering the effects of rise-to-span ratio (RSR) and lateral support movement. The FE analyses results indicated that, when the RSR was low, distribution of shell stress showed large variations. Lateral support movement caused an increase of tensile stresses, a decrease of compressive stresses, and intensified stress variation. When edge beams were not used, deflections were increased substantially, and local fluctuation of stress in the vicinity of the supports was intensified. Such behaviors were aggravated when RSR was low and proper constraints against the lateral support movement were not provided, thus resulting in inefficient systems. As such, gabled hypars without edge beams should be designed with caution.

Keywords: gabled hypar; hyperbolic paraboloid shell; finite element analysis; membrane theory

1. Introduction
Shell structures have been used historically in large spatial structures (Park, 2005; Lainez et al. 2009) and the concept of the form-resistant-structure is continuously adopted in current architectural forms (Kim and Shin, 2011). The gabled hyperbolic paraboloid shell (referred to as gabled hypar) is one of the most common types of hyperbolic paraboloid shell structures. The basic unit of the gabled hypar consists of four shell panels, which have straight edges and four columns as supports. Sometimes multiples of the basic units are combined together to form a shell cluster. The behavior of a single gabled hypar is largely influenced by the support conditions. Usually four corners of a single gabled hypar are confined using tie rods or tie beams in order to restrain excessive lateral movements.

Shell panels have zero curvature along the direction of the perimeter but have convex or concave surfaces along the diagonal direction (Fig.1.a). In general, beams are located at the boundaries of shell panels. The beams located along the sloping lines of shell panels are referred to as edge beams, and beams located along the straight lines where the shell panels meet are called ridge or crown beams (Fig.1.b).

The rationale for locating beams at the edges of the shell panels stems from the membrane theory. In the membrane theory, it is assumed that the gravity load acting on the shell panels is resisted by the in-plane shear stress of the shell, and the shearing forces are transformed into tensile and compressive forces by concave or convex arch action and then transferred to the perimeter (i.e., edge beams and ridge beams) as axial forces. Since the membrane theory is based on a simplified assumption that the external force is resisted only by the in-plane shearing forces, the effects of the flexural stress and out-of-plane shearing force of the shell panels cannot be considered (Billington, 1982).

Numerical analysis using the finite element method is capable of considering the flexural moment, out-of-plane shearing forces, and their interactions with the in-plane shearing forces. The results from the refined finite element analysis show that shell loads are transferred to the supports through a different mechanism, which cannot be explained using the membrane theory. That is, the shell loads are directly transferred to corner supports by the convex arch action along the diagonal direction of the shell panel (from the crown to the corner supports), and the tensile forces created by concave arch action in the
perpendicular direction are comparatively insignificant (Shaaban and Ketchum, 1976; Simmonds, 1989). As such, stresses on perimeter beams are reduced; however, shell panels, especially those in the vicinity of the supports, are subjected to very high compressive stress. Reduced significance of the perimeter beams resulted in the proposal of a new and simple gabled hypar shape, in which the edge beams along the slant edge line were removed (Jadik and Billington, 1995).

In this paper, behaviors of gabled hypars without edge beams were investigated extensively and compared with those of traditional gabled hypars. Specifically, the influences of the rise ratio (the ratio of the height to width of a gabled hypar) on the global behavior of the structure, including buckling behavior, were investigated. In addition, the effects of lateral movement between the supports were also investigated.

2. Analytical Model

Fig. 1 shows a typical configuration of the traditional gabled hypar, which includes ridge beams and edge beams. Shaaban and Ketchum (1976) analyzed the structural behavior of the traditional gabled hypar subjected to gravity load when support movements were restrained. They revealed that the shell loads were transferred mainly by the compressive force due to the arch action from the crown to corner supports. Tensile forces due to the concave arch action in the perpendicular direction diminished due to the inward displacement of the perimeter beams. Jadik and Billington (1995) removed the edge beams since the tensile action between end points of the ridge became insignificant. Based on the FE analysis results, they asserted that a gabled hypar can be designed without edge beams if there is sufficient shell thickness in the vicinity of the supports in order to resist high compressive stress. However, the gabled hypar model that Jadik and Billington (1995) used for their study was very limited. The model used in their study was a 24 m x 24 m shell structure composed of four 12 m x 12 m shell panels. The rise-to-span ratio was fixed at 0.1 (i.e., the height of the ridge was 2.4 m), and all supports were assumed to be completely restrained. However, the geometry, especially the rise ratio, may affect the global behavior of the gabled hypar, and the effects of lateral movement of supports should be considered. Thus, in this study, the effects of such factors were examined using FE analyses for the cases when the edge beams were and were not used, respectively.

Fig. 2 and Table 1 show the geometry of the gabled hypars and dimensions of members used in this study, which are basically identical to previous studies except that various rise ratios ranging from 0.05 to 0.25 were used. Each shell panel, which consists of one-quarter of a roof shell, has a dimension of 12 m x 12 m and was divided into 40 elements with a grid of 300 mm spacing in each direction. The dimensions of ridge beams are 600 mm wide and 200 mm deep, including the shell thickness.

![Fig. 1. Gabled Hyperbolic Paraboloid Shell](image1)

![Fig. 2. Geometry of the Gabled Hypar Model](image2)
The size of edge beams, if included, is 300 mm x 400 mm. In the case of hypars without edge beams, the thickness of the shell around the corners was increased as shown in Fig.2.c. The numbers inside the elements in Fig.2.c denote the shell thickness in millimeters. The dots in Fig.2.c indicate the supported nodes by the columns, which were distributed to avoid stress concentration. The shell and beams were modeled as homogenous concrete with a unit weight of 24 kN/m³, a modulus of elasticity of 21 GPa, and a Poisson’s ratio of 0.15.

Table 1. Member Size of the Gabled HP Shell Model

| Component | Sizes          |
|-----------|----------------|
| Shell     | 24 m x 24 m    |
| Thickness | 75 mm          |
| Ridge Beam| 600 mm x 200 mm|
| Edge Beam | 300 mm x 400 mm|
| Height    |                |
|           | 1.2 m (0.05)   |
|           | 2.4 m (0.1)    |
|           | 3.6 m (0.15)   |
|           | 4.8 m (0.2)    |
|           | 6.0 m (0.25)   |

Modeling and analysis were carried out using the finite element analysis software package, SAP2000 (CSI, 2005). Shell panels were modeled using four-node quadrilateral shell elements with conventional Kirchhoff's thin-plate theory, which neglects the out-of-plane shear effects. Edge beams were modeled as a typical frame element. Ridge beams combined with a shell, as shown in Fig.2.b, were modeled using a layered shell element, which is a built-in element in the SAP2000 element library. This element is useful when the centerline of the shell is not aligned with the centerline of the beams and yields reasonable results (Kim et al., 2012).

3. Analysis Results

3.1 Effects of Rise-to-Span Ratio

The effects of rise-to-span ratio (RSR) on the behavior of the hypars were investigated in this section. All corners of the hypars were supported as shown in Fig.2.c, and no lateral movements were considered.

Fig.3.a shows typical distributions of principal stress in a concave direction measured along the convex direction as indicated in Fig.4.b. Unlike the assumptions in the membrane theory, the tensile stress, which is the indication of concave arch action, is observed only in less than half of the diagonal. When the edge beams were not used, the tensile stress region decreased, which is believed to be the result of weak lateral restraints.

Fig.3.b, which is a plot of the maximum tensile stress magnitudes with respect to the RSR, indicates that the peak stress magnitudes were similar to those obtained by the membrane theory when the RSR was relatively high. However, when the RSR decreased, the magnitude of peak tensile stress was much smaller than that expected by the membrane theory. In membrane theory, the magnitudes of principal tensile and compressive stress are identical and are given by Eq. (1).

\[ \sigma = \frac{p}{2kt} \]  

where \( p \) is the uniform vertical pressure, \( k = \frac{c}{ab} \) is the twist of the surface, \( a \) and \( b \) are projected dimensions in each direction, and \( c \) and \( t \) are the rise and thickness of the shell, respectively. Thus, the magnitude of principal stress is inversely proportional to the RSR.

In the case of principal compressive stress, when the RSR is high and edge beams are used, an almost uniform compressive stress is observed. On the contrary, when the RSR is low and edge beams are not used, compressive stress is concentrated at the middle (i.e., diagonal line from the crown to edge), as can be seen in Fig.3.c. Note that Fig.3.c was plotted along the diagonal direction shown in Fig.4.c, which is perpendicular to the direction shown in Fig.4.b. The magnitude of principal compressive stresses was higher than predicted by the membrane theory regardless of RSR (Fig.3.d), which implies that the load is transferred to the edge through convex arch action (i.e., compressive arch action).

From the results above, authors can conclude that the load resisting capacity of hypar becomes superior when the RSR is high and edge beams are used, since lower but evenly distributed stress was developed, which is an indication of an efficient structural system.

The behavior of each hypar can be presumed from the deflection of shell panels. The graphs in Fig.5. indicate the overall deflected shape of a panel (Fig.5.a), the maximum deflection of shell panels in a parallel direction (Fig.5.b), in a perpendicular direction (Fig.5.c) of the edge line, and in a vertical direction (Fig.5.d).

It can be seen that the displacement of shell panels is larger in all directions when edge beams are not used. Fig.5.c suggests a plausible indication of tensile stress distributions in Fig.3.a (note the smaller tensile regions when edge beams were not used) since the larger lateral (inward) deflection of edge beam or edge line caused a decrease of tensile arch action.

Fig.6. shows shear force distribution along the edge beams and maximum shear force with respect to the RSR. In membrane theory, the edge beams are required to resist shear force, which is developed due to shell stress via tensile and compressive stresses given by Eq. (1). However, FE analysis indicated that the magnitudes of shear force at the edge beams are very small regardless of RSR. The FE analyses also showed that fluctuations in the vicinity of support, which is believed to be a consequence of stress concentration, increase with a decrease of RSR (Fig.6.b). Thus, when the RSR is low, edge beams may be required to resist high peak shear force. Local reinforcement of the shell near the supports also may be required as the shell stress shows large variation (Fig.3.a).

Linear buckling analyses were performed to investigate the buckling loads which cause instability.
of the structure. The obtained buckling loads are expressed in Table 2. in terms of buckling factor, which is the ratio of buckling load to the self-weight of the structure. Generally, lower buckling factors were obtained when the RSR decreased and when edge beams were not used. It should be noted that, in the case when RSR was 0.05 and edge beams were not used, a buckling factor of less than 1.0 was obtained, which indicates that the structure cannot support even its own weight.

| R/S Ratio (RSR) | w/edge beams | w/o edge beams |
|-----------------|--------------|---------------|
| 0.05            | 2.15         | 0.92          |
| 0.10            | 6.28         | 2.29          |
| 0.15            | 12.22        | 3.96          |
| 0.20            | 19.08        | 5.70          |
| 0.25            | 26.09        | 7.26          |

Fig.3. Principal Shell Stresses; (a) Distribution of Shell Stress in the Concave Direction (Along Convex Diagonal), (b) Maximum Tensile Stress, (c) Distribution of Shell Stress in the Convex Direction (Along Concave Diagonal), and (d) Maximum Compressive Stress

Fig.4. Locations and Directions of Presented Stresses; (a) Observed Shell Panel, (b) Direction of Diagonal and Shell Stress Associated with Concave Arch Action, and (c) Direction of Diagonal and Shell Stress Associated with Convex Arch Action
3.2 Effects of Lateral Support Movement

Lateral support movements have detrimental effects on the load-resisting capacity of hypars since they decrease the arching action. Thus, usually four corners of single gabled hypars are supported with very stiff columns or confined using tie rods or tie beams in order to restrain excessive lateral movements. In this section, the effect of lateral support movement was investigated by analyzing hypars in which supports were restrained by tie rods. Two different tie rod sizes (53 mm and 76 mm) were used to change the degree of restraint. Tie rods were modeled using the truss element with 200 GPa of elastic modulus. Supports were modeled with a two-way roller and a pin at opposite supports of a diagonal, while the other corners were modeled with one-way rollers. Columns were not considered in the models.

The graphs in Fig. 7. show behaviors of hypars with tie rods compared with those of fully restrained hypars. As expected, displacements of hypars caused by roof load increased with the decrease of the RSR and size of tie rods. When the RSR exceeded 0.15,
existence of edge beams did not affect the magnitude of displacements. The largest lateral support movement occurred in the case when both RSR and the diameter of tie rods were smallest and the edge beams were not used. (Fig.7.b) However, the magnitude of lateral support movement in such a case was about 70 mm, as small as 3% of the distance between the supports.

Figs.7.c and 7.d indicate that, when the supports are not completely restrained, inward displacement and downward deflection of the edge beam or edge line drastically increased. For example, when lateral movements were completely restrained, the inward displacement of the edge line was about 1.5 mm, as can be seen in Fig.5.c (RSR=0.15 and edge beams were not used). However, when the supports were partially restrained, the displacement increased almost 20-fold (Fig.7.c). Downward deflection of the edge showed a similar result. It should be noted that, when the RSR is low (less than 0.10), the edge beams have some contribution in decreasing the displacement, as mentioned above.

Figs.8.a and 8.c show distributions of principal tensile and compressive shell stress, respectively, depending on the degree of lateral restraints. The tensile region increased when lateral supports were allowed to move. Hence, it seems that the edge beam does not alter the principal tensile stress distribution. On the contrary, the distribution of stress in the compressive diagonal differed greatly when the lateral movement of supports was allowed. When the lateral movement was restrained, the diagonals from the crown to edge were subjected to the largest compressive stress. However, as the degree of restraint decreased, compressive stress along the diagonal line also decreased, which indicates that the mechanism of compression arch was diminished. As a result, variations in magnitude of compressive stress were increased.

Peak tensile and compressive principal stresses were plotted with the RSR shown in Figs.5.b and 5.d, respectively. When lateral movements were restrained, the magnitudes of principal tensile stress were similar to or less than those predicted by membrane theory, while principal compressive stresses were about twice as large as those suggested by membrane theory (Fig.5). When lateral support movements were allowed, the magnitude of peak tensile stresses increased, and the magnitude of peak compressive stresses decreased. This tendency is clear when the RSR is reduced. Specifically, for the hypars with RSR=0.05 not using edge beams, the shell is subjected to very large tensile stress and relatively large compressive stress, which is believed to be the consequence of a large fluctuation of stress distribution. However, this tendency was considerably reduced when edge beams were used.

Fig.9. is the contour plot of the principal shell stress for RSR=0.1. It is clearly shown that, when lateral movements were allowed, the area of tensile stress in
the concave direction increased (Figs. 9.a and 9.b), and the area of compressive stress in the convex direction decreased (Figs.9.c and 9.d).

In summary, when the lateral movements of supports were not completely restrained, hypar displacement increased drastically. In addition, greater tensile stress and a large stress fluctuation were observed in the shell stress, which indicates that the contribution of diagonal compressive arching action was reduced. These tendencies were apparent when the RSR decreased. When the RSR was greater than 0.15, the use of edge beams did not make significant differences. However, when the RSR was low, edge beams were effective in decreasing the displacement and peak shell stress.

4. Conclusion

This paper investigated the behaviors of gabled hypars with and without edge beams for various cases considering the effects of rise-to-span ratio (RSR) and lateral support movement. The FE analyses performed in this study indicated that, when the RSR is low, distribution of shell stress showed large variations and the critical buckling load was reduced. Lateral support movement tends to affect the stability of the structure. Such movement caused a considerable increase of shell deflection, an increase of the tensile stresses, a decrease of compressive stresses, and an intensified stress variation at the shell, which seemed to be reflective of a weakened compressive arch action.

Removal of the edge beams decreased the area of tensile stress at the shell due to the inward deflection of the edge line. However, the peak values of the shell stresses were almost the same as those when the edge beams were used. In addition, the magnitudes of shear stress carried by the edge beams were always small irrespective of the RSR, which was the reason for removing the edge beams. However, when the edge beams were not used, deflections increased substantially and local fluctuation of stress at the shell panel and edge beam in the vicinity of the supports was intensified. These behaviors were aggravated when the RSR was low, and strong constraints against the lateral support movement were not provided. Thus, although removing the edge beam can provide flexibility in design, gabled hypars without edge beams in such cases may result in inefficient structural systems from the viewpoint of structural design.

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Fig. 9. Principal Shell Stress Contours for RSR=0.10; (a) Stresses in the Concave Direction when Fully Constrained, (b) Stresses in the Concave Direction when Restrained with 53 mm Tie Rods, (c) Stresses in the Convex Direction when Fully Constrained, and (d) Stresses in the Convex Direction when Restrained with 53 mm Tie Rods.