Global Minimum Depth In Edwards-Anderson Model

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Abstract. In the literature the most frequently cited data are quite contradictory, and there is no consensus on the global minimum value of 2D Edwards-Anderson (2D EA) Ising model. By means of computer simulations, with the help of exact polynomial Schraudolph-Kamenetsky algorithm, we examined the global minimum depth in 2D EA-type models. We found a dependence of the global minimum depth on the dimension of the problem N and obtained its asymptotic value in the limit $N \rightarrow \infty$. We believe these evaluations can be further used for examining the behavior of 2D Bayesian models often used in machine learning and image processing.

Keywords: spectrum; local minimum; global minimum; spin system; spin glass system; minimization; exact polynomial algorithm; Edwards-Anderson model; planar Ising model

1 Introduction

In many fields of science, it is necessary to know the global energy minimum for different systems. Namely, in informatics we use it when solving problems of quadratic optimization [1–6], developing search algorithms for the global minimum [7–12] and solving max-cut problems [13–17]. In neuroinformatics, we have to know the global minimum when developing associative memory systems [18–21] and constructing neural networks and neural network minimization algorithms [22–24]. In physics, the knowledge of the global energy minimum is most frequently necessary when studying the behavior of spin glass systems [25–35] and even when describing four-photon mixing in nonlinear media [36, 37].

The question of calculation of the global minimum depth has been discussed over the years. However, since it has no decisive answer it remains a highly topical problem up to now. Indeed, in the literature the most frequently cited data are quite contradictory, and there is no consensus on the global minimum value (see references in [27]). To illustrate this statement, we present the values of the global minimum depth obtained by different methods:
\( E_0 = 0 \) \hspace{1cm} \text{TAP (Thouless et al. [25])}

\( E_0 = 1/\sqrt{2\pi} \) \hspace{1cm} \text{mean random field (Klein [26])}

\( E_0 = 0.5 \) \hspace{1cm} \text{partition function (Tanaka and Edwards [27])}

\( E_0 = 2/\pi \) \hspace{1cm} \text{replica (Sherrington and Kirkpatrick [28])}

\( E_0 = 0.76 \sim 0.77 \) \hspace{1cm} \text{Monte Carlo (Sherrington and Kirkpatrick [29])}

Such a spread of values exists because until recently there were no exact calculation algorithms for the determination of \( E_0 \). This was the reason why different authors used different minimization methods, and consequently the obtained estimates were sufficiently far from the true value of \( E_0 \). New algorithms appeared recently. They allow us to calculate \( E_0 \) exactly when examining spin systems on planar graphs with arbitrary boundary conditions [38, 39]. Implementing these algorithms, we were able to refine our results [40] for the Edwards–Anderson model (the EA model).

In the present paper, we present an experimental analysis of the global minimum depth in the EA model, which is a spin system on an \( N = L \times L \) square lattice where only interactions with four nearest neighbors do not equal to zero. Formally, we have in mind a system whose behavior is described by a Hamiltonian

\[
H = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j
\]

defined in the configuration space of states \( S = (s_1, s_2, \ldots, s_N) \) with binary variables \( s_i = \pm 1 \), \( i = 1, N \). Here \( N \) is the number of spins, and \( J_{ij} \) is a symmetric, zero-diagonal matrix (\( J_{ji} = J_{ij} \) and \( J_{ii} = 0 \)).

To describe the spectrum of the system, it is convenient to introduce the depth of the minimum that is defined by equation

\[
E = -\frac{1}{2N\sigma_j} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} s_i s_j
\]

As we show in what follows, the normalization coefficient in Eq. (3) is quite universal since the value of \( E \) is almost independent of the dimension of the problem \( N \) as well as of the normalization of the matrix elements \( J_{ij} \). In these notations, the Hamiltonian of the system has the form \( H = -N\sigma_j E \), and its dependence on the dimension reduces to \( H \sim N \).

As we see from Eq. (1), the results obtained by different authors are so very different that it is hardly possible to use them in the course of calculations. This was the reason why we performed a huge experiment having in mind to determine the basic spectral characteristics such as the mean value of the local minimum depth, the spectrum width, and the depth of the global minimum. Based on the obtained experimental data, we plotted the dependences of these characteristics on the dimension of the problem \( N \) and determined their asymptotic values in the limit \( N \to \infty \).
The structure of the paper is as follows. In Section II, we describe our experiment and analyze the obtained data. In Section III, we discuss the results and the tables showing our experimental data.

2 Experiment

To define the value of $E_0$, we used an algorithm described in [39]. In the course of our experiment, we examined the classical EA-model (with the normal distribution of $J_{ij}$) and the EA*-model (with the uniform distribution of $J_{ij}$). For the chosen model of the given size $N = L \times L$, we generated $M$ matrices $J_{ij}$ and determined $M$ values $E_{m0}$, $m = 1, M$. We used these data to calculate the mean value and the variance of the obtained values:

$$E_0 = M^{-1} \sum_{m=1}^{M} E_{m0}, \quad \sigma_0^2 = M^{-1} \sum_{m=1}^{M} (E_{m0}^2 - E_0^2)$$

(4)

The results of our experiments are collected in Table 1. Based on the obtained data, we derived formulas that described the dependences of $E_0$ and $\sigma_0$ on $N$. We optimized these formulas by means of the least squares method. We minimized the value of the summary relative error and estimated the quality of the approximation formulas by the value of validity defined as

$$R^2 = 1 - \frac{\sum (x_{\text{exp}} - x_{\text{app}})^2}{\sum (x_{\text{exp}} - \bar{x}_{\text{exp}})^2}$$

(5)

where $x_{\text{exp}}$ are the experimental values, $\bar{x}_{\text{exp}}$ are the means of the experimental values, and $x_{\text{app}}$ are the values obtained using the approximation formulas.

2.1 EA-model

This is the Edwards–Anderson model for a two-dimensional lattice where spins interact with their four nearest neighbors only and nonzero matrix elements are normally distributed.

We found that the function of Approximation functions derived as a result of analysis of our experimental data have the form

$$E_0 = 1.3151 - \frac{1.17}{L}, \quad \sigma_0 = 0.74 \frac{0.11}{L}$$

(6)

The validities of these expressions are $R^2 = 0.994$ and $R^2 = 0.993$, respectively.

When comparing the expressions of Eq. (6) with the experiment, we see that these formulas describe them very well. In Fig. 1, we show that the function
$E_0 = E_0(N)$ matches perfectly with data of Table 1. The value of the relative error $Err = 1 - E_0^{(exp)} / E_0^{(approx)}$ is less than $2 \cdot 10^{-3}$.

The function $\sigma_0 = \sigma_0(N)$ (the second expression of Eq. (6)) in Fig. 2 also describes the data of Table 1 very well. The value of the relative error is less than 0.4%.

**Fig. 1.** $E_0$ vs $L$. EA-model: line – eqs.(6), circles – experiment.

**Fig. 2.** $\sigma_0$ vs $L$. EA-model: line – eqs.(6), circles – experiment.
Table 1. Characteristics of global minima

|   | EA-model |   | EA*-model |   |
|---|----------|---|-----------|---|
| L | M        | E₀ | σ₀        | E₀ | σ₀ |
| 10| 200      | 1.20245| 0.07611 | 1.27347| 0.05579 |
| 20| 200      | 1.26305| 0.04049 | 1.32793| 0.02848 |
| 30| 200      | 1.28416| 0.02608 | 1.34307| 0.01934 |
| 40| 200      | 1.28838| 0.01882 | 1.34864| 0.01514 |
| 50| 200      | 1.29569| 0.01493 | 1.35738| 0.01066 |
| 100| 200    | 1.30448| 0.00695 | 1.36681| 0.00517 |
| 200| 200    | 1.30957| 0.00392 | 1.37173| 0.00294 |
| 300| 100    | 1.31136| 0.00254 | 1.37337| 0.00187 |
| 400| 100    | 1.31229| 0.00193 | 1.37467| 0.00144 |
| 500| 30     | 1.31279| 0.00135 | 1.37498| 0.00113 |
| 1000| 30    | 1.31390| 0.00074 | 1.37564| 0.00054 |

2.2 EA*-model

This is the same Edwards-Anderson model for a two-dimensional lattice, but here the uniform distribution is used in place of the normal distribution.

In this case, approximation functions obtained after our analysis of the experimental data have the form

\[ E_0 = 1.3769 - \frac{1.23}{L}, \quad \sigma_0 = \frac{0.54}{L} + \frac{0.2}{L^2} \] \quad (7)

The validities of these expressions are \( R^2 = 0.996 \) and \( R^2 = 0.992 \), respectively.

Comparing the expressions of Eq. (7) with the experiment, we see that they describe it very well. In Fig. 3, we present the dependence \( E_0 = E_0(N) \) (the first expression of Eq. (7)) that matches perfectly with the data from Table 1. When \( L > 50 \), the relative error is less than \( 2 \times 10^{-4} \).

The dependence \( \sigma_0 = \sigma_0(N) \) (the second expression of Eq. (7)) shown in Fig. 4 also describes the data from Table 1 very well. Here the relative error is less than 0.5%.
Fig. 3. $E_0$ vs $L$. EA-model: line – eqs.(6), circles – experiment.

Fig. 4. $\sigma_0$ vs $L$. EA*-model: line – eqs.(7), circles – experiment.

3 Discussion

Our analysis of the two models allowed us to derive empirical relations in Eqs. (6) and (7) for the most important characteristics of the global minima (see Eqs. (6) and
Our goal was to obtain expressions which with a high certainty described the dependences of these characteristics on \( N \) in the whole range of the dimensions of the problem that we were able to examine. Based on these results, we had to determine the asymptotic behavior of these characteristics when \( N \to \infty \). Evidently there are different approaches to approximation of the experimental data in Table 1. Consequently, it is possible to obtain a list of different expressions, and some of them can be even more accurate than the expressions of Eqs. (6) and (7). However, this fact does not change the goal of our study: independent of the form of the obtained approximation functions, they have to describe correctly the behavior of the characteristics inside the test range of \( N \) and provide trustworthy asymptotic values when \( N \to \infty \) (see Table 2).

**Table 2.** Asymptotic values of \( E_0 \) and \( \sigma_0 \) \( (N \to \infty) \).

|          | \( E_0 \)                 | \( \sigma_0 \)       |
|----------|---------------------------|----------------------|
| EA-model | 1.3151 ± 0.002            | 0.74 / \( L \)       |
| EA*-model| 1.3769 ± 0.002            | 0.54 / \( L \)       |

As we see, the data of Table 2 differ significantly from the values presented in Eq. (1). The point is that when minimizing the functional of Eq. (2) with a view to calculating \( E_0 \), different authors used different minimization algorithms. To do that, they defined \( E_0 \) as the energy corresponding to the deepest minimum, which under a reasonable number of tests frequently was far from \( E^\star \). As an example, let us discuss the results of numerical experiments [40] in which they defined the energy of the deepest minimum \( E^\star \). Then, for the relative distance

\[
\delta E = 100\% \cdot \frac{E_0 - E^\star}{E_0} \tag{8}
\]

between \( E_0 \) and \( E^\star \) when \( L \geq 100 \) we obtain:

\[
\delta E = 16.45\% \pm 0.5\%, \quad \text{for EA-model,} \tag{9}
\]

\[
\delta E = 16.61\% \pm 0.4\%, \quad \text{for EA*-model.} \tag{10}
\]

From our point of view, this is a possible reason why the estimates of \( E_0 \) obtained by different authors differ so significantly. Namely, when the size of the system is sufficiently large (\( L \geq 30 \)) such an approach is not applicable since the probability of finding the global minimum in the course of a random search is exponentially small: it is \( \sim \exp(-0.04N) \).
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References

1. Fu, Y., Anderson, P.W.: Application of statistical mechanics to NP-complete problems in combinatorial optimization. J. Phys. A 19, 605–1620 (1986)
2. Hartmann, A.: Calculation of ground states of four-dimensional +/-J Ising spin glasses. Phys. Rev. B 60, 5135–5138 (1999)
3. Hartmann, A.K., Rieger, H.: Optimization Algorithms in Physics. Wiley-VCH, Berlin (2001)
4. Kryzhanovsky, B., Kryzhanovsky, V.: Binary optimization: On the probability of a local minimum detection in random search. In: Artificial Intelligence and Soft Computing — ICAISC 2008, LNCS 5097, pp. 89–100 (2008)
5. Hartmann, A.K., Rieger, H.: New Optimization Algorithms in Physics. Wiley-VCH, Berlin (2004)
6. Litinskii, L.B.: Eigenvalue problem approach to discrete minimization. In: Artificial Neural Networks: Formal Models and Their Applications — ICANN 2005, LNCS 3697, pp. 405–410 (2005)
7. Houdayer, J., Martin, O.C.: Hierarchical approach for computing spin glass ground states. Phys. Rev. E 64, 056704 (2001)
8. Litinskii, L.B., Magomedov, B.M.: Global minimization of a quadratic functional: Neural networks approach. Pattern Recog. Image Anal. 15(1), 80–82 (2005)
9. Karandashev, Y.M., Kryzhanovsky, B.V.: Transformation of energy landscape in the problem of binary minimization. Dokl. Math. 80(3), 927–931 (2009)
10. Liers, F., Junger, M., Reinelt, G., Rinaldi, G.: Computing exact ground states of hard Ising spin glass problems by branch-and-cut. In: New Optimization Algorithms in Physics, pp. 47–68. Wiley (2004).
11. Karandashev, I., Kryzhanovsky, B.: Increasing the attraction area of the global minimum in the binary optimization problem. J. Global Optim. 56(3), 1167–1185 (2013)
12. Karandashev, I.M., Kryzhanovsky, B.V.: Attraction area of minima in quadratic binary optimization. Opt. Memory Neural Networks (Inf. Opt.) 23(2), 84–88, (2014)
13. Goemans, M.X., Williamson, D.P.: .878-approximation algorithms for MAX CUT and MAX 2SAT. In: ACM Symposium on Theory of Computing (STOC), pp. 422–431 (1994).
14. Bellare, M., Goldreich, O., Sudan, M.: Free bits, PCPs and non-approximability. Towards tight on foundations of computer science. In: Proceedings of IEEE 36th Annual Foundations of Computer Science, pp. 422–431 (1995)
15. Marti, R., Duarte, A., Laguna, M.: Advanced scatter search for the max-cut problem. INFORMS J. Comput. 21(2), 26–38 (2009)
16. Burer, S., Monteiro, R.D.C., Zhang, Y.: Rank-two relaxation heuristics for max-cut and other binary quadratic programs. SIAM J. Optim. 12, 503–521 (2000)
17. Karandashev, I., Kryzhanovsky, B.: Mix-matrix transformation method for max-cut problem. In: Artificial Neural Networks and Machine Learning — ICANN 2014, LNCS 8681, pp. 323–330 (2014)
18. Hopfield, J.J.: Neural networks and physical systems with emergent collective computational abilities. Proc. Nat. Acad. Sci. USA. 79, 2554–2558 (1982)
19. Pinkas, G., Dechter, R.: Improving connectionist energy minimization. J. Artif. Intell. Res. 3(195), 23–48 (1995)
20. Smith, K.A.: Neural networks for combinatorial optimization: A review of more than a decade of research. INFORMS J. Comput. 11(1), 15–34 (1999).
21. Kryzhanovsky, V., Malsagov, M., Zhelavskaya, I.: Error probability of search in high-dimensional binary space by scalar neural network tree. In: Proceedings of 8th International Conference on Bioinspired Information and Communications Technologies, pp. 23–27 (2014)
22. Joya, G., Atencia, M., Sandoval, F.: Hopfield neural networks for optimization: Study of the different dynamics. Neurocomputing 43(1–4), 219–237 (2002)
23. Kryzhanovsky, B.V., Litinskii, L.B., Mikaelian, A.L.: Vector-neuron models of associative memory. In: 2004 IEEE International Joint Conference on Neural Networks IJCNN-2004, pp. 909–1004 (2004)
24. Karandashev, I., Kryzhanovsky, B., Litinskii, L.: Weighted patterns as a tool to improve the Hopfield model. Phys. Rev. E 85, 041925 (2012)
25. Thouless, D.J., Anderson, P.W., Palmer, R.G.: Solution of solvable model of a spin glass. Philos. Mag. 35, 593–601 (1977)
26. Klein, M.W.: Comparison of the self-consistent mean-random-field approximation with the n→0 expansion of Sherrington and Kirkpatrick for spin glasses and with experiment. Phys. Rev. B 14, 5008–5017 (1976)
27. Tanaka, F., Edwards, S.F.: Analytic theory of the ground state properties of a spin glass. I. Ising spin glass. J. Phys. F: Metal Phys. 10, 2769–2778 (1980)
28. Sherrington, D., Kirkpatrick, S.: Solvable model of a spin-glass. Phys. Rev. Lett. 35, 1792–1796 (1975)
29. Kirkpatrick, S., Sherrington, D.: Infinite-ranged models of spin-glasses. Phys. Rev. B 17, 4384–4403 (1978).
30. Bolle, D., Dupont, P., Huyghebaert, J.: Thermodynamic properties of the Q-state Potts-glass neural network. Phys. Rev. A 45(6), 4194–4197 (1992)
31. Boettcher, S.: Extremal optimization for Sherrington–Kirkpatrick spin glasses. Eur. Phys. J. B 46, 501 (2005).
32. Horner, H.: Time dependent local field distribution and metastable states in the SK-spin-glass. Eur. Phys. J. B 60(4), 413 (2007)
33. Ananikian, N.S., Ananikyan, L.N., Chakhmakhchyan, L.A., Rojas O.: Thermal entanglement of a spin-1/2 Ising–Heisenberg model on a symmetrical diamond chain. J. Phys.: Condens. Matter 24(25), 256001 (2012).
34. Kryzhanovsky, B., Litinskii, L.: Generalized Bragg–Williams equation for system with an arbitrary long-range interaction. Dokl. Math. 90(3), 784–787 (2014)
35. Kryzhanovsky, B., Litinskii, L.: Approximate method of free energy calculation for spin system with arbitrary connection matrix. In: International Conference on Mathematical Modeling in Physical Sciences IC-MSQUARE, Madrid, Spain, (2014)
36. Kryzhanovsky, B.V., Karapetyan, A.R., Glushko, B.A.: Theory of energy exchange and conversion via four-wave mixing in a nondissipative ⋅(3) material. Phys. Rev. A 44(9), 6036–6042 (1991)
37. Kryzhanovsky, B.V., Melikyan, A.O.: The effects of intensity in resonance fluorescence. Opt. Commun. 29(2), 164 (1979)
38. Karandashev, Y, Malsagov, M.: Polynomial algorithm for exact calculation of partition function for binary spin model on planar graphs. Opt. Memory Neural Networks (Inf. Opt.) 26(2), 87–95 (2017)
39. Schraudolph, N., Kamenetsky, D.: Efficient exact inference in planar Ising models. In: Advances in Neural Information Processing Systems 21 (NIPS 2008) (2008)
40. Kryzhanovsky, B., Magomed Malsagov, M.: The spectra of local minima in spin-glass models. Opt. Memory Neural Networks (Inf. Opt.) 25(1), 1–15 (2016)