The unique global solvability and optimal time decay rates for a multi-dimensional compressible generic two-fluid model with capillarity effects

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Abstract
The present paper deals with the Cauchy problem of a compressible generic two-fluid model with capillarity effects in any dimension $N \geq 2$. We first study the unique global solvability of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. Due to the presence of the capillary terms, we exploit the parabolic properties of the linearized system for all frequencies which enables us to apply contraction mapping principle to show the unique global solvability of strong solutions close to a stable equilibrium state. Furthermore, under a mild additional decay assumption involving only the low frequencies of the data, we establish the optimal time decay rates for the constructed global solutions.

Keywords: well-posedness, capillary effects, optimal time decay rates, compressible generic two-fluid, critical Besov spaces
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1. Introduction and main results

In the present paper, we consider the following multi-dimensional \((N \geq 2)\) non-conservative viscous compressible two-fluid model with capillarity effects:

\[
\begin{aligned}
\alpha^+ + \alpha^- &= 1, \\
\partial_t (\alpha^+ \rho^+) + \text{div} (\alpha^+ \rho^+ \mathbf{u}^+) &= 0, \\
\partial_t (\alpha^- \rho^-) + \text{div} (\alpha^- \rho^- \mathbf{u}^-) + \alpha^+ \nabla P^+ (\rho^+) - \sigma^+ \alpha^+ \rho^+ \nabla \Delta (\alpha^+ \rho^+) &= \text{div} (\alpha^+ \tau^+), \\
P^+ (\rho^+) &= A^+ (\rho^+)^{\gamma^+} = P^- (\rho^-) = A^- (\rho^-)^{\gamma^-},
\end{aligned}
\]

(1.1)

where the variable \(0 \leq \alpha^+(x, t) \leq 1\) is the volume fraction of fluid + in one of the two gases, and \(0 \leq \alpha^-(x, t) \leq 1\) is the volume fraction of the other fluid -. Moreover, \(\rho^+(x, t) \geq 0\), \(\mathbf{u}^+(x, t)\) and \(P^+(\rho^+) = A^+(\rho^+)^{\gamma^+}\) are, respectively, the densities, the velocities, and the two pressure functions of the fluids. \(\sigma^+ > 0\) are the capillary coefficients. It is assumed that \(\gamma^+ > 1\), \(\gamma^- > 0\) are constants. In what follows, we set \(A^+ = A^- = 1\) without loss of any generality. Also, \(\tau^\pm\) are the viscous stress tensors

\[
\tau^\pm := 2\mu^\pm D(u^\pm) + \lambda^\pm \text{div} u^\pm \text{Id},
\]

(1.2)

where \(D(u^\pm) \overset{\text{def}}{=} \nabla u^\pm \nabla u^\pm \frac{1}{2}\) stand for the deformation tensor, the constants \(\mu^\pm\) and \(\lambda^\pm\) are the (given) shear and bulk viscosity coefficients satisfying \(\mu^\pm > 0\) and \(\lambda^\pm + 2\mu^\pm > 0\). This system is known as a two-fluid flow system with algebraic closure, which is widely used in industrial applications, such as nuclear, power, oil-and-gas, micro-technology, and so on, and we refer readers to references [2–4, 19, 34, 35, 38, 39] for more discussions about this model and related models.

The system (1.1) is a highly nonlinear partial differential system with the mixed hyperbolic–parabolic property. As a matter of fact, there is no diffusion on the mass conservation equations, whereas velocity evolves according to the parabolic equations due to the viscosity phenomena. We should point out that the system (1.1) includes important single phase flow models such as the compressible Navier–Stokes equations (i.e., \(\sigma^\pm = 0\)) and the compressible Navier–Stokes–Korteweg model when one of the two phases volume fraction tends to zero (i.e., \(\alpha^\pm = 0\) or \(\alpha^- = 0\)). As the extremely important models to describe compressible fluids, these two systems have attracted a lot of attention among many analysts and many important theories have been developed. Here, we briefly review some of the most relevant papers about global well-posedness and large time behaviours of the solutions for two systems. For the compressible Navier–Stokes equations, Lions [30] proved the global existence of weak solutions for large initial data. However, the question of uniqueness of weak solutions remains open, even in the two-dimensional case. Nash [33] considered the local well-posedness for smooth data away from a vacuum. Matsumura and Nishida [32] first studied the global existence of smooth solutions close to a stable equilibrium state for the initial data \((\rho_0 - \bar{\rho}, u_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\), where \(\bar{\rho}\) is a positive constant. Matsumura and Nishida [31], provided that the initial perturbation \((\rho_0 - \bar{\rho}, u_0)\) is sufficiently small in \(H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\), obtained the following optimal time decay rates

\[
\| (\rho - \bar{\rho}, u) \|_{L^2} \lesssim C (1 + t)^{-\frac{1}{2}}, \quad t \geq 0.
\]

(1.3)

Later, if the small initial disturbance belongs to \(H^s(\mathbb{R}^N) \cap W^{s,1}(\mathbb{R}^N)\) with the integers \(s \geq \left[\frac{N}{2}\right] + 3\) and the space dimensions \(N = 2, 3\), Ponce [41] established the optimal \(L^p\)-time decay rates
\[
\|\nabla^k (\rho - \bar{\rho}, u) \|_{L^p} \leq C (1 + t)^{-\frac{k}{2} - \frac{1}{p}}, \quad t \geq 0, \quad 0 \leq k \leq 2, \quad 2 \leq p \leq \infty. \quad (1.4)
\]

In 2007, Duan et al [18] proved the optimal time decay rates for the compressible Navier–Stokes equations with potential forces if the small initial disturbance belongs to \(H^1(\mathbb{R}^3)\) and the initial perturbation is bounded in \(L^1(\mathbb{R}^3)\) without the smallness of \(L^1\)-norm of the initial disturbance. Note that \(L^1(\mathbb{R}^3)\) is included in \(B^1_{1,\infty}(\mathbb{R}^3)\) (which will be defined in section 2), Li and Zhang [28] extended the known results from [31, 32] and showed the optimal time decay rates of the density and momentum when initial perturbation is sufficiently small in \(H^1(\mathbb{R}^3) \cap B^1_{1,\infty}(\mathbb{R}^3)\), \(l \geq 4\), and \(s \in [0, 1]\)
\[
\| \partial_t^k (\rho - \bar{\rho}) (t) \|_{L^2} + \| \partial_t^k m(t) \|_{L^2} \lesssim (1 + t)^{-\frac{k}{2} + \frac{1}{2} - s}, \quad \text{for} \ |k| = 0, 1. \quad (1.5)
\]

Recently, the present author and Chi [47] removed the smallness condition of \(B^1_{1,\infty}(\mathbb{R}^3)\) for \(s \in [0, 1]\). Concerning the optimal time decay rates of global solutions to the compressible Navier–Stokes equations (with or without external potential force), we refer the reader to the papers [17, 22, 36, 37, 43] and references therein. In critical framework, Danchin [14] first proved the existence and uniqueness of the global strong solutions for initial data close to a stable equilibrium state. The optimal time decay estimates issue for the compressible Navier–Stokes equations in the critical regularity framework for dimension \(N \geq 3\) has been addressed only very recently by Okita in [40], provided the data are additionally in some super-space of \(L^1(\mathbb{R}^N)\). In the survey paper [15], Danchin proposed another description of the time decay which allows to handle dimension \(N \geq 2\) in the \(L^2\) critical framework. For the compressible Navier–Stokes–Korteweg model, Hattori and Li [25, 26] established the local existence of smooth solutions with large initial data and global existence of smooth solutions around constant states for small initial data \((\rho_0, u_0)\) in Sobolev spaces \(H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)\) with \(s \geq \frac{N}{2} + 4\) and \(N = 2, 3\). Recently, researchers in [46] proved the global existence and obtain the following optimal decay rates of strong solutions for small initial data in some Sobolev spaces which have lower regularity than that of [26] in three dimensional case
\[
\| (\rho - \bar{\rho}, u) \|_{L^2} \leq C (1 + t)^{-\frac{1}{2}}, \quad t \geq 0, \quad (1.6)
\]
\[
\| \nabla (\rho - \bar{\rho}, u) \|_{L^2} \leq C (1 + t)^{-\frac{1}{2}}, \quad t \geq 0, \quad (1.7)
\]
\[
\| (\rho - \bar{\rho}, u) \|_{L^p} \leq C (1 + t)^{-\frac{1}{2} (1 - \frac{1}{p})}, \quad t \geq 0, \quad 2 \leq p \leq \infty. \quad (1.8)
\]

In 2016, Li and Yong [29] investigated the zero Mach number limit for the three-dimensional model in the regime of smooth solutions. In critical Besov spaces, Danchin and Desjardins [16] and Haspot [23, 24] obtained the global well-posedness of strong solutions close to a stable equilibrium state. Charve–Danchin–Xu [5] obtained the global existence, Gevrey analytic and algebraic time-decay estimates of strong solutions when the initial data are close to a stable equilibrium state in \(L^p\)-critical framework. In 2019, Chikami and Kobayashi [11] established the global existence and algebraic time decay estimates of strong solutions when the initial data are close to a stable equilibrium state.

Due to the more complicated coupling between hyperbolic equations and parabolic equations in the two-phase flow model, the mathematical structure of the model is much more complex than that in the case of single phase flow model. Therefore, extending the currently available results for single phase flow models to two-phase models is not an easy task. Recently, more and more researchers pay more attention to the mathematical problems of the generic two-phase model. In [2], Bresch et al first established the existence of global weak solutions to
the 3D generic two-fluid flow model (1.1). Later, Bresch–Huang–Li [3] extended the result in [2] and proved the existence of global weak solutions to the system (1.1) in one space dimension without capillarity terms. In 2016, Evje–Wang–Wen [19] proved the global existence of strong solutions to the generic two-fluid model (1.1) without capillary terms by the standard energy method under the condition that the initial data are close to the constant equilibrium state in $H^2(\mathbb{R}^3)$ and obtained the following optimal time decay rates for strong solutions if the initial data belong to $L^1(\mathbb{R}^3)$ additionally

$$\|((\alpha^+ \rho^+ - 1, u^+, \alpha^- \rho^- - 1, u^-))\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}, \quad t \geq 0, \quad (1.9)$$

$$\|\nabla((\alpha^+ \rho^+ - 1, u^+, \alpha^- \rho^- - 1, u^-))\|_{L^2} \leq C(1 + t)^{-\frac{5}{4}}, \quad t \geq 0. \quad (1.10)$$

Lai–Wen–Yao [27] studied the vanishing capillarity limit of smooth solutions to the system (1.1) with unequal pressure functions if $\|((\alpha^+ \rho^+_0 - 1, \alpha^- \rho^-_0 - 1))\|_{H^2(\mathbb{R}^3)} + \|((u^+_0, u^-_0))\|_{H^2(\mathbb{R}^3)}$ are small enough. Recently, based on the complicated spectral analysis of Green’s function to the linearized system and the elaborate energy estimates to the nonlinear system, for the system (1.1), authors [13] proved global solvability of smooth solutions close to an equilibrium state in $H^s(\mathbb{R}^3)(s \geq 3)$ and further got the following time decay rates when the initial perturbation is bounded in $L^1(\mathbb{R}^3)$

$$\|((\alpha^+ \rho^+ - 1, \alpha^- \rho^- - 1))\|_{L^2} \leq C(1 + t)^{-\frac{1}{4}}, \quad t \geq 0, \quad (1.11)$$

$$\|\nabla((\alpha^+ \rho^+ - 1, \alpha^- \rho^- - 1))\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}, \quad t \geq 0. \quad (1.12)$$

$$\|(u^+, u^-)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}, \quad t \geq 0, \quad (1.13)$$

$$\|\nabla(u^+, u^-)\|_{L^2} \leq C(1 + t)^{-\frac{5}{4}}, \quad t \geq 0. \quad (1.14)$$

Here, it should be pointed out that the functional spaces with high Sobolev regularity are not the lowest index in the sense of the scaling invariant of the associated system (1.1) and the dimension of spaces is only limited to $N = 3$. Moreover, the decay rates of $(\alpha^\pm \rho^\pm - 1)$ in (1.11) and (1.12) are not the optimal.

The purpose of this work is to investigate the mathematical properties of the system (1.1) in critical regularity framework. More specifically, we address the question of whether available mathematical results such as the global well-posedness and the optimal time decay rates in critical Besov spaces to a single fluid governed by the compressible barotropic Navier–Stokes equations may be extended to multi-dimensional non-conservative viscous compressible two-fluid system. At this stage, we are going to use scaling considerations for the system (1.1) to guess which spaces may be critical. As in [8–10, 14, 16], one can check that if $(\alpha^+ \rho^+, u^+, \alpha^- \rho^-, u^-)$ solves the system (1.1), so does $(((\alpha^\pm \rho^\pm)\lambda, u^{\pm}_\lambda, (\alpha^\pm \rho^\pm)\lambda, u^{\pm}_\lambda)$ with

$$(\alpha^+ \rho^+)\lambda(t,x) = (\alpha^+ \rho^+)(\lambda^2 t, \lambda x), \quad u^{\pm}_\lambda(t,x) = \lambda u^{\pm}(\lambda^2 t, \lambda x), \quad (1.15)$$

provided that the pressure laws $P$ have been changed into $\lambda^2 P$. This suggests us to choose initial data $(((\alpha^\pm \rho^\pm)\lambda_0, u^{\pm}_\lambda, (\alpha^\pm \rho^\pm)\lambda, u^{\pm}_\lambda))$ in critical spaces whose norm is invariant for all $\lambda > 0$ by the transformation $(((\alpha^+ \rho^+)\lambda_0, u^+_\lambda, (\alpha^- \rho^-)\lambda_0, u^-_\lambda) \rightarrow (((\alpha^+ \rho^+)\lambda\lambda, u^+_\lambda\lambda, (\alpha^- \rho^-)\lambda\lambda, \lambda u^-_\lambda\lambda))$. For the convenience of the reader,
similar to [2, 13, 27], we also show some derivations for another expression of the pressure gradient in terms of the gradients of $\alpha^+ \rho^+$ and $\alpha^- \rho^-$ by using the pressure equilibrium assumption. Here, we only focus on the case that $\inf \rho^+ > 0, 0 < \alpha^+ < 1$ in our framework. The relation between the pressures of the system (1.1) implies the following differential identities

$$
\frac{dP^+}{\rho^+} = s^2_+ \frac{d\rho^+}{\rho^+}, \quad \frac{dP^-}{\rho^-} = s^2_- \frac{d\rho^-}{\rho^-}, \quad \text{where } s^\pm := \sqrt{\frac{dP^\pm}{d\rho^\pm}(\rho^\pm)} = \sqrt{\frac{\rho^\pm (\rho^\pm)}{\rho^\pm}}, (1.16)
$$

where $s^\pm$ denote the sound speed of each phase respectively.

Let

$$
R^\pm = \alpha^\pm \rho^\pm.
$$

Resorting to (1.1), we have

$$
d\rho^+ = \frac{1}{\alpha^+} (dR^+ - \rho^+ d\alpha^+), \quad d\rho^- = \frac{1}{\alpha^-} (dR^- + \rho^- d\alpha^-). (1.18)
$$

Combining with (1.16) and (1.18), we conclude that

$$
d\alpha^+ = \frac{\alpha^+ s^2_+}{\alpha^+ \rho^+ s^+_+ + \alpha^+ \rho^- s^-_-} dR^+ - \frac{\alpha^+ s^2_-}{\alpha^- \rho^- s^-_- + \alpha^+ \rho^+ s^+_+} dR^-.
$$

Substituting the above equality into (1.18), we obtain

$$
d\rho^+ = \frac{\rho^+ s^2_-}{R^- (\rho^-)^2 s^-_- + R^+ (\rho^+)^2 s^+_+} \left( \rho^- dR^+ + \rho^+ dR^- \right),
$$

and

$$
d\rho^- = \frac{\rho^- s^2_+}{R^- (\rho^-)^2 s^-_- + R^+ (\rho^+)^2 s^+_+} \left( \rho^- dR^+ + \rho^+ dR^- \right),
$$

which give, for the pressure differential $dP^\pm$,

$$
dP^+ = C^2 \left( \rho^- dR^+ + \rho^+ dR^- \right),
$$

and

$$
dP^- = C^2 \left( \rho^- dR^+ + \rho^+ dR^- \right),
$$

where

$$
C^2 \overset{\text{def}}{=} \frac{s^2_+ s^-_-}{\alpha^- \rho^+ s^+_+ + \alpha^+ \rho^- s^-_-}.
$$

Recalling $\alpha^+ + \alpha^- = 1$, we get the following identity:

$$
\frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad \text{and therefore } \rho^- = \frac{R^- \rho^+}{\rho^+ - R^+}. (1.19)
$$

Then it follows from the pressure relation (1.1) that

$$
\varphi(\rho^+) := P^+ (\rho^+) - P^- \left( \frac{R^- \rho^+}{\rho^+ - R^+} \right) = 0. (1.20)
$$
Differentiating $\varphi$ with respect to $\rho^+$, we have

$$\varphi'(\rho^+) = s_+^3 + s_+^2 \frac{R^+ R^+}{(\rho^+ - R^+)^2}.$$  

By the definition of $R^+$, it is natural to look for $\rho^+$ which belongs to $(R^+, +\infty)$. Since $\varphi' > 0$ in $(R^+, +\infty)$ for any given $R^+ > 0$, and $\varphi: (R^+, +\infty) \rightarrow (-\infty, +\infty)$, this determines that $\rho^+ = \rho^+(R^+, R^-) \in (R^+, +\infty)$ is the unique solution of the equation (1.20). Due to (1.18), (1.19) and (1.1)\textsubscript{1}, $\rho^-$ and $\alpha^+$ are defined as follows:

$$\rho^-(R^+, R^-) = \frac{R^- \rho^+(R^+, R^-)}{\rho^+(R^+, R^-) - R^+},$$

$$\alpha^+(R^+, R^-) = \frac{R^+}{\rho^+(R^+, R^-)},$$

$$\alpha^-(R^+, R^-) = 1 - \frac{R^+}{\rho^-(R^+, R^-)} = \frac{R^-}{\rho^-(R^+, R^-)}.$$

Based on the above analysis, the system (1.1) is equivalent to the following form

$$\begin{align*}
\partial_t R^+ + \text{div}(R^+ u^+) &= 0, \\
\partial_t (R^+ u^+) + \text{div}(R^+ u^+ \otimes u^+) + \alpha^+ C^2 [\rho^- \Delta R^+ + \rho^+ \Delta R^-] - \sigma^+ R^+ \nabla \Delta R^+ &= \text{div} (\alpha^+ [\mu^+(\nabla u^+ + \nabla^t u^+) + \lambda^+ \text{div} u^+ \text{Id}]), \\
\partial_t (R^- u^-) + \text{div}(R^- u^- \otimes u^-) + \alpha^- C^2 [\rho^- \Delta R^+ + \rho^+ \Delta R^-] - \sigma^- R^- \nabla \Delta R^- &= \text{div} (\alpha^- [\mu^- (\nabla u^- + \nabla^t u^-) + \lambda^- \text{div} u^- \text{Id}]).
\end{align*}$$

(1.21)

Here, we are concerned with the Cauchy problem of the system (1.21) in $\mathbb{R}_+ \times \mathbb{R}^N$ subject to the initial data

$$(R^+, u^+, R^-, u^-)(x, t)|_{t=0} = (R^+_0, u^+_0, R^-_0, u^-_0)(x), \quad x \in \mathbb{R}^N,$$

(1.22)

and

$$u^+(x, t) \rightarrow 0, \quad u^-(x, t) \rightarrow 0, \quad R^+ \rightarrow R^+_\infty > 0, \quad R^- \rightarrow R^-\infty > 0, \quad \text{as} \ |x| \rightarrow \infty,$$

where $R^\pm_\infty$ denote the background doping profile, and in the present paper $R^\pm_\infty$ are taken as 1 without losing generality.

For simplicity, we take $\sigma^+ = \sigma^- = 1$. Set $c^\pm = R^\pm - 1$. Then, the system (1.21) can be rewritten as

$$\begin{align*}
\partial_t c^+ + \text{div} u^+ &= H_1, \\
\partial_t u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu_1 \Delta u^+ - \nu_2 \nabla \text{div} u^+ - \nabla \Delta c^+ &= H_2, \\
\partial_t c^- + \text{div} u^- &= H_3, \\
\partial_t u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu_1 \Delta u^- - \nu_2 \nabla \text{div} u^- - \nabla \Delta c^- &= H_4,
\end{align*}$$

(1.23)

with initial data

$$(c^+, u^+, c^-, u^-)|_{t=0} = (c^+_0, u^+_0, c^-_0, u^-_0),$$

(1.24)
where $\beta_1 = \frac{c^2(1,1)c^2(1,1)}{\rho^2(1,1)}$, $\beta_2 = \beta_3 = C^2(1, 1)$, $\beta_4 = \frac{\mu^2(1,1)c^2(1,1)}{\rho^2(1,1)}$, $\nu_1^+ = \frac{\mu^2}{\rho^2(1,1)}$, $\nu_2^+ = \frac{\mu^2}{\rho^2(1,1)}$ and the source terms are

\begin{align}
H_1 &= H_1(c^+, u^+) = -\text{div}(c^+ u_i^+) , \\
H_2 &= H_2(c^+, u^+, c^-) = -g_+(c^+, c^-)\partial_te^+ - \tilde{g}_+(c^+, c^-)\partial_t\rho c^- - (u^+ \cdot \nabla)u_i^+ \\
&+ \mu^+ h_+(c^+, c^-)\partial_t c^+\partial_t u_i^+ + \mu^+ k_+(c^+, c^-)\partial_t c^-\partial_t u_i^+ \\
&+ \mu^+ h_+(c^+, c^-)\partial_t c^+\partial_t u_i^+ + \mu^+ k_+(c^+, c^-)\partial_t c^-\partial_t u_i^+ \\
&+ \lambda^+ h_+(c^+, c^-)\partial_t c^+\partial_t u_i^+ + \lambda^+ k_+(c^+, c^-)\partial_t c^-\partial_t u_i^+ \\
&+ \mu^+ l_+(c^+, c^-)\partial_t^2 u_i^+ + (\mu^+ + \lambda^+) l_+(c^+, c^-)\partial_t\partial_t u_i^+ , \quad i, j \in \{ 1, 2, \ldots, N \} ,
\end{align}

\begin{align}
H_3 &= H_3(c^-, u^-) = -\text{div}(c^- u_i^-) , \\
H_4 &= H_4(c^+, u^-, c^-) = -g_-(c^+, c^-)\partial_te^- - \tilde{g}_-(c^+, c^-)\partial_t\rho c^- - (u^- \cdot \nabla)u_i^- \\
&+ \mu^- h_-(c^-, c^-)\partial_t c^+\partial_t u_i^- + \mu^- k_-(c^-, c^-)\partial_t c^-\partial_t u_i^- \\
&+ \mu^- h_-(c^-, c^-)\partial_t c^+\partial_t u_i^- + \mu^- k_-(c^-, c^-)\partial_t c^-\partial_t u_i^- \\
&+ \lambda^- h_-(c^-, c^-)\partial_t c^+\partial_t u_i^- + \lambda^- k_-(c^-, c^-)\partial_t c^-\partial_t u_i^- \\
&+ \mu^- l_-(c^+, c^-)\partial_t^2 u_i^- + (\mu^- + \lambda^-) l_-(c^+, c^-)\partial_t\partial_t u_i^- , \quad i, j \in \{ 1, 2, \ldots, N \} ,
\end{align}

where we define the nonlinear functions of $(c^+, c^-)$ by

\begin{align}
g_+(c^+, c^-) &= \frac{(C^2\rho^-)(c^+ + 1, c^- + 1)}{\rho^+ (c^+ + 1, c^- + 1)} - \frac{(C^2\rho^+)(1, 1)}{\rho^+ (1, 1)} , \\
g_-(c^+, c^-) &= \frac{(C^2\rho^+)(c^+ + 1, c^- + 1)}{\rho^- (c^+ + 1, c^- + 1)} - \frac{(C^2\rho^-)(1, 1)}{\rho^- (1, 1)} , \\
h_+(c^+, c^-) &= \frac{(C^2\alpha^-)(c^+ + 1, c^- + 1)}{(c^+ + 1)s^2(c^+ + 1, c^- + 1)} , \\
h_-(c^+, c^-) &= -\frac{C^2(c^+ + 1, c^- + 1)}{(\rho s^2)(c^+ + 1, c^- + 1)} , \\
k_+(c^+, c^-) &= \frac{C^2(c^+ + 1, c^- + 1)}{(c^+ + 1)s^2(c^+ + 1, c^- + 1)} , \\
k_-(c^+, c^-) &= \frac{(\alpha^- C^2)(c^+ + 1, c^- + 1)}{(c^- + 1)s^2(c^+ + 1, c^- + 1)} , \\
\tilde{g}_+(c^+, c^-) &= \tilde{g}_-(c^+, c^-) = C^2(c^+ + 1, c^- + 1) - C^2(1, 1) , \\
l_+(c^+, c^-) &= \frac{1}{\rho^+(c^+ + 1, c^- + 1)} - \frac{1}{\rho^+(1, 1)} .
\end{align}
Although the non-conservative viscous compressible two-fluid flow model, to some extent, is similar to the compressible Navier–Stokes equations, it is non-trivial to apply directly the ideas used in single-phase models into the two-phase models since the momentum equation is given only for the mixture and that the pressure involves the masses of two phases in a nonlinear way. Now, let us explain some of the main difficulties and techniques involved in the process. First, the presences of the two-phase flows effect results in a different and more intricate coupling relations in the present system, so that the overall analysis is considerably more complicated. Second, in two-fluid model, we find that there have some binary functions such as $g_i(c^+, c^-)$ including the two variables $c^+$ and $c^-$ in (1.26) and (1.28). In order to obtain the estimates of these nonlinear terms in Besov spaces, we need to exploit the continuity for the composition for binary functions (see lemma 2.7 for details), which are distinguished estimates for the complicated two-phase flow model. Third, employing the energy argument of Godunov [21] for partially dissipative first-order symmetric systems (further developed by [20]), the Littlewood–Paley decomposition and Fourier–Plancherel theorem, we can obtain maximal regularity estimates for the linearized system in Besov spaces and show the parabolic properties of $(c^+, u^+, c^-, u^-)$ in all frequencies (see lemma 3.1 for details). Finally, one may wonder how global strong solutions constructed above look like for large time. Under a suitable additional condition involving only the low frequencies of the data and in the $L^2$ critical regularity framework, we exhibit the optimal time decay rates for the constructed global strong solutions. In this part, our main ideas are based on the low–high frequency decomposition and a refined time-weighted energy functional. In low frequencies, making good use of Fourier localization analysis to a linearized parabolic–hyperbolic system in order to obtain smoothing effects of Green’s function and avoid some complex spectral analysis as in [13]. In high frequencies, in order to close the energy estimates, we further exploit some decay estimates with gain of regularity of $(c^+, u^+, c^-, u^-)$. With these analysis tools in hand, we finally establish the global well-posedness and the optimal time decay rates of strong solutions to the Cauchy problem (1.21) and (1.22).

Now we state our main results as follows:

**Theorem 1.1.** Assume that $((\sqrt{\beta_1} + \Lambda)(R_0^+ - 1), u_0^+), (\sqrt{\beta_4} + \Lambda)(R_0^- - 1), u_0^-) \in B^{\frac{N}{2}-1}_{2,1} \times B^{\frac{N-1}{2}}_{2,1} \times B^{\frac{N+1}{2}}_{2,1}$. Then there exists a constant $\eta > 0$ such that if

$$X(0) \overset{\text{def}}{=} \left\| ((\sqrt{\beta_1} + \Lambda)(R_0^+ - 1), u_0^+), (\sqrt{\beta_4} + \Lambda)(R_0^- - 1), u_0^-) \right\|_{B^{\frac{N}{2}}_{2,1}} \leq \eta, \quad (1.34)$$

then the Cauchy problem (1.21) and (1.22) admits a unique global solution $(R^+ - 1, u^+, R^- - 1, u^-)$ satisfying that for all $t \geq 0$,

$$X(t) \lesssim X(0), \quad (1.35)$$

where

$$X(t) \overset{\text{def}}{=} \left\| ((\sqrt{\beta_1} + \Lambda)(R^+ - 1), u^+), (\sqrt{\beta_4} + \Lambda)(R^- - 1), u^-) \right\|_{L^\infty_t \left( B^{\frac{N}{2}}_{2,1} \right)} + \left\| ((\sqrt{\beta_1} + \Lambda)(R^+ - 1), u^+), (\sqrt{\beta_4} + \Lambda)(R^- - 1), u^-) \right\|_{L^1_t \left( B^{\frac{N+1}{2}}_{2,1} \right)}, \quad (1.36)$$

and the fractional derivative operator $\Lambda^{\ell}$ is defined by $\Lambda^{\ell} f \overset{\text{def}}{=} \mathcal{F}^{-1}(|\cdot|^{\ell} \mathcal{F} f)$. 

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Theorem 1.2. Let the data \((R_0^+, u_0^+, R_0^−, u_0^−)\) satisfy the assumptions of theorem 1.1. Denote \((\tau) \overset{\text{def}}{=} \sqrt{1 + \tau^2}\) and \(\alpha \overset{\text{def}}{=} \frac{\sqrt{2}}{2} + \frac{1}{\tau} - \varepsilon > 0\) arbitrarily small. There exists a positive constant \(c\) such that if in addition

\[
D_0 \overset{\text{def}}{=} \left\| \left( (\sqrt{\beta_4} + \lambda)(R_0^+ - 1), u_0^+, (\sqrt{\beta_4} + \lambda)(R_0^- - 1), u_0^- \right) \right\|_{\dot{B}^{\frac{N}{2}}_{2,\infty}} < c, \tag{1.37}
\]

then the global solution \((R^+, u^+, R^-, u^-)\) given by theorem 1.1 satisfies for all \(t \geq 0,

\[
D(t) \leq C \left( D_0 + \| (\Lambda R_0^+, u_0^+, \Lambda R_0^-, u_0^-) \|_{\dot{B}^{\frac{N}{2}}_{2,1}} \right), \tag{1.38}
\]

with

\[
D(t) \overset{\text{def}}{=} \sup_{\tau \in (\varepsilon - \frac{1}{2}, \varepsilon + \frac{1}{2})} \left\| \langle \tau \rangle^{\frac{N}{2} + \frac{1}{2}} \left( (\sqrt{\beta_4} + \lambda)(R^+ - 1), u^+, (\sqrt{\beta_4} + \lambda)(R^- - 1), u^- \right) \right\|_{L^2_t(\dot{B}^{\frac{N}{2}}_{2,1})} + \left\| \tau^{s} \Lambda^2 \left( (\sqrt{\beta_4} + \lambda)(R^+ - 1), u^+, (\sqrt{\beta_4} + \lambda)(R^- - 1), u^- \right) \right\|_{L^2_t(\dot{B}^{\frac{N}{2}}_{2,1})}. \tag{1.39}
\]

Remark 1.3. Compared with [13, 19], in theorems 1.1 and 1.2, we obtain the global well-posedness and the optimal time decay rates for multi-dimensional non-conservative viscous compressible two-fluid system (1.1) in critical regularity framework respectively. Additionally, in theorem 1.2, the regularity index \(s\) can take both negative and nonnegative values, rather than only nonnegative integers, which improves the classical decay results in high Sobolev regularity, such as [13, 19, 45, 46].

Remark 1.4. Due to the embedding \(L^1(\mathbb{R}^3) \hookrightarrow \dot{B}^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^3)\), our results in theorem 1.2 extend the known conclusions in [13, 19, 45, 46]. In particular, our condition involves only the low frequencies of the data and is based on the \(L^2\)-norm framework. Moreover, the decay rates of strong solutions are in the so-called critical Besov spaces in any dimension \(N \geq 2\).

Remark 1.5. It should be pointed out that we only deal with the case for \(L^2\) framework in this paper. The case for \(L^p\) framework will be considered in our future work.

The rest of the paper unfolds as follows. In the next section, we recall some basic facts about Littlewood–Paley decomposition, Besov spaces and some useful lemmas. In section 3, we will exploit maximal regularity estimates for the linearized system in Besov Spaces. Section 4 is devoted to the proof of the global well-posedness for initial data near equilibrium in critical Besov spaces. In section 5, we present the optimal time decay rates of the global strong solutions. At last, in section 6, we show some corollaries and the standard optimal \(L^q-L^r\) time decay rates.

Notations. We assume \(C\) be a positive generic constant throughout this paper that may vary at different places and denote \(A \lesssim CB\) by \(A \lesssim B\). We shall also use the following notations

\[
z^j \overset{\text{def}}{=} \sum_{j \leq k_0} \Delta \bar{\rho} \quad \text{and} \quad z^h \overset{\text{def}}{=} z - z^j, \quad \text{for some} \ k_0, \ \\
\|z\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \overset{\text{def}}{=} \sum_{j < k_0} 2^j \|\Delta \bar{\rho}\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \overset{\text{def}}{=} \sum_{j \geq k_0} 2^j \|\Delta z^j\|_{L^2}, \quad \text{for some} \ k_0.
\]

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Noting the small overlap between low and high frequencies, we have
\[ \|z_\ell\|_{\dot{B}^s_{2,1}} \lesssim \|z\|_{\dot{B}^s_{2,1}} \quad \text{and} \quad \|z_\ell\|_{\dot{B}^s_{2,1}} \lesssim \|z\|_{\dot{B}^s_{2,1}}. \]

2. Littlewood–Paley theory and some useful lemmas

Let us introduce the Littlewood–Paley decomposition. Choose a radial function \( \varphi \in \mathcal{S}(\mathbb{R}^N) \) supported in \( C = \{ \xi \in \mathbb{R}^N, \frac{1}{4} \leq |\xi| \leq 4 \} \) such that
\[ \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all} \quad \xi \neq 0. \]

The homogeneous frequency localization operators \( \dot{\Delta}_q \) and \( \dot{S}_q \) are defined by
\[ \dot{\Delta}_q f = \varphi(2^{-q}D)f, \quad \dot{S}_q f = \sum_{k \leq q-1} \dot{\Delta}_k f \quad \text{for} \quad q \in \mathbb{Z}. \]

With our choice of \( \varphi \), one can easily verify that
\[ \dot{\Delta}_q \dot{\Delta}_k f = 0 \quad \text{if} \quad |q-k| \geq 2 \quad \text{and} \quad \dot{\Delta}_q (\dot{S}_{k-1}f \dot{\Delta}_k f) = 0 \quad \text{if} \quad |q-k| \geq 5. \]

We denote the space \( Z'(\mathbb{R}^N) \) by the dual space of \( Z(\mathbb{R}^N) = \{ f \in \mathcal{S}(\mathbb{R}^N); D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^N \text{ multi–index} \} \). It also can be identified by the quotient space of \( \mathcal{S}'(\mathbb{R}^N)/P \) with the polynomials space \( P \). The formal equality
\[ f = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q f \]
holds true for \( f \in Z'(\mathbb{R}^N) \) and is called the homogeneous Littlewood–Paley decomposition.

The following Bernstein’s inequalities will be frequently used.

**Lemma 2.1.** [6]. Let \( 1 \leq p_1 \leq p_2 \leq +\infty \). Assume that \( f \in L^{p_1}(\mathbb{R}^N) \), then for any \( \gamma \in (\mathbb{N} \cup \{0\})^N \), there exist constants \( C_1, C_2 \) independent of \( f, q \) such that
\[ \sup \{ |\xi| \leq A_1 2^q \} \Rightarrow \|D^{\gamma} f\|_{p_2} \leq C_1 2^{q|\gamma|} \sup \|\partial^{\beta}\xi\|_{p_1}, \]
\[ \sup \{ A_2 2^q \leq |\xi| \leq A_1 2^q \} \Rightarrow \|f\|_{p_1} \leq C_2 2^{-q|\gamma|} \sup \|\partial^{\beta}\xi\|_{p_1}. \]

Let us recall the definition of homogeneous Besov spaces (see [1, 14]).

**Definition 2.2.** Let \( s \in \mathbb{R}, 1 \leq p, r \leq +\infty \). The homogeneous Besov space \( \dot{B}^s_{p,r} \) is defined by
\[ \dot{B}^s_{p,r} = \left\{ f \in Z'(\mathbb{R}^N) : \|f\|_{\dot{B}^s_{p,r}} < +\infty \right\}, \]
where
\[ \|f\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \|2^{qs}\|\dot{\Delta}_q f(0)\|_{p,r}. \]

**Remark 2.3.** Some properties about the Besov spaces are as follows

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We next introduce the Besov–Chemin–Lerner space \( L^\theta_{r1}(B^s_{2,1}) \) where

\[
\|f\|_{L^\theta_{r1}(B^s_{2,1})} \approx \|\nabla f\|_{B^s_{2,1}}; 
\]

- Derivation:

\[
\|f\|_{B^s_{2,1}} \leq \|f\|_{B^s_{2,1}}; 
\]

- Algebraic properties: for \( s > 0 \), \( B^s_{2,1} \cap L^\infty \) is an algebra;

- Interpolation: for \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in [0, 1] \), we have

\[
\|f\|_{B^{s_1+s_2}_{2,1}} \leq \|f\|_{B^{s_1}_{2,1}} \|f\|_{B^{s_2}_{2,1}}; 
\]

\[
\|f\|_{B^{s_1}_{2,1}} \leq \|f\|_{B^{s_1}_{2,1}} \|f\|_{B^{s_2}_{2,1}}; 
\]

\[
\|f\|_{B^{s_1}_{2,1}} \leq \|f\|_{B^{s_1}_{2,1}} \|f\|_{B^{s_2}_{2,1}}. 
\]

**Definition 2.4.** Let \( s \in \mathbb{R} \), \( 1 \leq p, \rho, r \leq +\infty \). The homogeneous space–time Besov space \( L^\rho_{p} (\hat{B}^s_{p,r}) \) is defined by

\[
L^\rho_{p} (\hat{B}^s_{p,r}) = \left\{ f \in \mathbb{R}_+ \times Z'(\mathbb{R}^N) : \|f\|_{L^\rho_{p} (\hat{B}^s_{p,r})} < +\infty \right\}, 
\]

where

\[
\|f\|_{L^\rho_{p} (\hat{B}^s_{p,r})} \overset{\text{def}}{=} \|2^\rho \|\hat{\Delta}_q f\|_{L^\rho_{p} (0,T \mathbb{R}^N)}\|e\|_{L^\rho_{p}}. 
\]

We next introduce the Besov–Chemin–Lerner space \( \hat{L}^\rho_{p} (\hat{B}^s_{p,r}) \) which is initiated in [7].

**Definition 2.5.** Let \( s \in \mathbb{R} \), \( 1 \leq p, q, r \leq +\infty \), \( 0 < T \leq +\infty \). The space \( \hat{L}^\rho_{p} (\hat{B}^s_{p,r}) \) is defined by

\[
\hat{L}^\rho_{p} (\hat{B}^s_{p,r}) = \left\{ f \in \mathbb{R}_+ \times Z'(\mathbb{R}^N) : \|f\|_{\hat{L}^\rho_{p} (\hat{B}^s_{p,r})} < +\infty \right\}, 
\]

where

\[
\|f\|_{\hat{L}^\rho_{p} (\hat{B}^s_{p,r})} \overset{\text{def}}{=} \|2^\rho \|\hat{\Delta}_q f(t)\|_{L^\rho_{p} (0,T \mathbb{R}^N)}\|e\|_{L^\rho_{p}}. 
\]

Obviously, \( \hat{L}^1_{p} (\hat{B}^s_{p,1}) = L^1_{p} (\hat{B}^s_{p,1}) \). By a direct application of Minkowski’s inequality, we have the following relations between these spaces

\[
L^\rho_{p} (\hat{B}^s_{p,1}) \overset{\text{def}}{\hookrightarrow} \hat{L}^\rho_{p} (\hat{B}^s_{p,1}), \quad \text{if } r \geq \rho, 
\]

\[
\hat{L}^\rho_{p} (\hat{B}^s_{p,1}) \overset{\text{def}}{\hookrightarrow} L^\rho_{p} (\hat{B}^s_{p,1}), \quad \text{if } \rho \geq r. 
\]

The usual product is continuous in many Besov spaces. The following proposition will be frequently used and its proof may be found in [42] section 4.4 (see in particular inequality (28) page 174).

**Proposition 2.6.** For all \( 1 \leq r, p, p_1, p_2 \leq +\infty \), there exists a positive universal constant such that

\[
\|fg\|_{\hat{B}^s_{p,r}} \leq \|f\|_{L^\infty} \|g\|_{\hat{B}^s_{p,r}}, \quad \text{if } s > 0; 
\]

\[
\|fg\|_{\hat{B}^{s_1+s_2}_{p,r}} \leq \|f\|_{\hat{B}^{s_1}_{p,r}} \|g\|_{\hat{B}^{s_2}_{p,r}}, \quad \text{if } s_1, s_2 < \frac{N}{p}, \text{ and } s_1 + s_2 > 0; 
\]

\[
\|fg\|_{\hat{B}^s_{p,r}} \leq \|f\|_{\hat{B}^s_{p,r}} \|g\|_{L^\infty}, \quad \text{if } |s| < \frac{N}{p}; 
\]

\[
\|fg\|_{\hat{B}^{s_1}_{p,r}} \leq \|f\|_{\hat{B}^{s_1}_{p,r}} \|g\|_{L^\infty}, \quad \text{if } s \in (-N/2, N/2]. 
\]
Lemma 2.7.

(a) Let $s > 0$, $t \geq 0$, $1 \leq p, q, r \leq \infty$ and $(f, g) \in (\tilde{L}^p(\mathbb{B}_{p, r}) \cap L^\infty_{\text{loc}}(L^\infty)) \times (\tilde{L}^p(\mathbb{B}_{p, r}) \cap L^\infty_{\text{loc}}(L^\infty))$. If $F \in W^{[1]\,\infty}_{\text{loc}}(\mathbb{R}) \times W^{[1]\,\infty}_{\text{loc}}(\mathbb{R})$ with $F(0, 0) = 0$, then $F(f, g) \in \tilde{L}^p(\mathbb{B}_{p, r})$. Moreover, there exists a $C$ depending only on $s, p, N$ and $F$ such that

\[
\|F(f, g)\|_{\tilde{L}^p(\mathbb{B}_{p, r})} \leq C\left(1 + \|f\|_{L^p_{\text{loc}}(L^\infty)}\|g\|_{L^q_{\text{loc}}(L^\infty)}\right)^{[1]+1}\|F(f, g)\|_{\tilde{L}^p(\mathbb{B}_{p, r})}.
\]  

(2.1)

(b) If $(f_1, g_1) \in L^\infty_{\text{loc}}(\mathbb{B}_{p, r}) \times L^\infty_{\text{loc}}(\mathbb{B}_{p, r})$ and $(f_2, g_2) \in L^\infty_{\text{loc}}(\mathbb{B}_{p, r}) \times L^\infty_{\text{loc}}(\mathbb{B}_{p, r})$, and $(f_2 - f_1, g_2 - g_1)$ belongs to $L^\infty_{\text{loc}}(\mathbb{B}_{p, r}) \times L^\infty_{\text{loc}}(\mathbb{B}_{p, r})$ with $s \in (-\infty, 0]$. If $F \in W^{[1]\,\infty}_{\text{loc}}(\mathbb{R}) \times W^{[1]\,\infty}_{\text{loc}}(\mathbb{R})$ with $\partial_1 F(0, 0) = 0$ and $\partial_2 F(0, 0) = 0$, then there exists a $C$ depending only on $s, p, N$ and $F$ such that

\[
\|F(f_2, g_2) - F(f_1, g_1)\|_{\tilde{L}^p(\mathbb{B}_{p, r})} \leq C\left(1 + \|f_1\|_{L^p_{\text{loc}}(L^\infty)}\|g_1\|_{L^q_{\text{loc}}(L^\infty)}\right)^{[2]+1}\times \left(\|f_2 - f_1\|_{\tilde{L}^p(\mathbb{B}_{p, r})} + \|g_2 - g_1\|_{\tilde{L}^p(\mathbb{B}_{p, r})}\right).
\]  

(2.2)

Proof.

(a) From the smoothness of $F$ and $F(0, 0) = 0$, we have the following formal decomposition

\[
F(f, g) = \sum_{j \in \mathbb{Z}} F(\hat{S}_{j+1} f, \hat{S}_{j+1} g) - F(\hat{S}_j f, \hat{S}_j g)
\]

\[
= \sum_{j \in \mathbb{Z}} \int_0^1 \partial_1 F(\hat{S}_j f + \theta \hat{\Delta}_j f, \hat{S}_j g + \theta \hat{\Delta}_j g)d\theta \hat{\Delta}_j f
\]

\[
+ \int_0^1 \partial_2 F(\hat{S}_j f + \theta \hat{\Delta}_j f, \hat{S}_j g + \theta \hat{\Delta}_j g)d\theta \hat{\Delta}_j g
\]

\[
\triangleq \sum_{j} m_j \hat{\Delta}_j f + m_j \hat{\Delta}_j g.
\]

First, we have the following claim:

\[
\|\partial^\alpha \hat{m}_j\|_{L^\infty} \leq C\left(1 + \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}\right)^{[n]2\|\alpha\|}\|\partial F\|_{W^{[1]\,\infty}_{\text{loc}}}, \quad \forall \alpha \in \mathbb{N}^n, \quad i = 1, 2.
\]

(2.3)

Indeed, taking $h^1 = \hat{S}_j f + \theta \hat{\Delta}_j f$, $h^2 = \hat{S}_j g + \theta \hat{\Delta}_j g$, $h_s = (h^1_s, h^2_s) = (\partial^\alpha_s h^1, \partial^\alpha_s h^2)$, from multivariate Faà di Bruno formula [12], we have

\[
\partial^\alpha \hat{m}_j = \int_0^1 \partial^\alpha \partial_1 F(\hat{S}_j f + \theta \partial_1 f, \hat{S}_j g + \theta \hat{\Delta}_j g)d\theta = \int_0^1 \partial^\alpha \partial_1 F(h^1, h^2)d\theta.
\]

\[
= \int_0^1 \sum_{1 \leq |\alpha| \leq |\alpha|} \partial^\alpha \partial_1 F \sum_{j \in \mathbb{Z}} (\alpha_1) \prod_{q=1}^n \left[\frac{h_{q}^{j}_s}{k_q^j}\right]\prod_{q=1}^n \prod_{i=1}^{p_q(j, \lambda)} \left[\frac{h_i}{k_q^j}\right] \prod_{q=1}^n \prod_{i=1}^{p_q(j, \lambda)} \left[\frac{h_i}{k_q^j}\right] d\theta,\]

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where \( k_q = (k_{q1}, k_{q2}), \sum_{q=1}^r k_q = \lambda, \sum_{q=1}^r |k_q| e_q = \alpha, \) \( h_{\ell_q} = [\partial^\ell_h f]_t \). Using Hölder’s inequality and Bernstein’s inequality, we have

\[
||h_{\ell_q}||_{L^\infty} \leq ||[\partial^\ell_h f]||_{L^\infty} ||[\partial^\ell_h g]||_{L^\infty}.
\]

Thus,

\[
2^{j\ell_q} ||h_{\ell_q}||_{L^\infty} \leq C \prod_{q=1}^{[\alpha]} 2^{j\ell_q} ||h_{\ell_q}||_{L^\infty} ||[\partial^\ell_h f]||_{W^{[\alpha], [\infty]}}.
\]

where we used

\[
\sum_{q=1}^{[\alpha]} |\ell_q| k_q = |\alpha|.
\]

Next, we prove that \( F(f, g) \) belongs to \( \tilde{L}^2(\tilde{B}_{[\nu]}^\alpha) \). For all \( j' \in \mathbb{Z} \), we have

\[
\hat{J}_{j'} F = \sum_{j \neq j'} \hat{J}_{j'}(m^j_{j'} \hat{J}_{j'} f) + \sum_{j \neq j'} \hat{J}_{j'}(m^j_{j'} \hat{J}_{j'} g) + \sum_{j \neq j'} \hat{J}_{j'}(m^j_{j'} \hat{J}_{j'} g)
\]

\[
\Delta = \Delta_{1,j'} + \Delta_{2,j'} + \Delta_{3,j'}.
\]

For the first term \( \Delta_{1,j'} \), taking advantage of Bernstein’s inequality, Leibniz’s formula and (2.3), we have

\[
2^{j_1} ||\Delta_{1,j'}||_{L^2_j(L^p)} \leq \sum_{j \neq j'} 2^{j_1} ||\Delta_{j'}(m^j_{j'} \hat{J}_{j'} f)||_{L^2_j(L^p)}
\]

\[
\leq C \sum_{j \neq j'} 2^{j_1} ||\Delta_{j'}(m^j_{j'} \hat{J}_{j'} f)||_{L^2_j(L^p)}
\]

\[
\leq C \sum_{j \neq j'} 2^{j_1} \bigg(1 + ||f||_{L^2_j(L^p)} ||g||_{L^2_j(L^p)}\bigg)^{[\alpha]} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}}
\]

\[
\leq C \sum_{j \neq j'} 2^{j_1} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}} \bigg(1 + ||f||_{L^2_j(L^p)} ||g||_{L^2_j(L^p)}\bigg)^{[\alpha]} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}}
\]

\[
\leq C \sum_{j \neq j'} 2^{j_1} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}} \bigg(1 + ||f||_{L^2_j(L^p)} ||g||_{L^2_j(L^p)}\bigg)^{[\alpha]} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}}
\]

\[
\leq C \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}} \bigg(1 + ||f||_{L^2_j(L^p)} ||g||_{L^2_j(L^p)}\bigg)^{[\alpha]} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}}
\]

\[
\leq C \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}} \bigg(1 + ||f||_{L^2_j(L^p)} ||g||_{L^2_j(L^p)}\bigg)^{[\alpha]} \|\partial_{j'} f\|_{W^{[\alpha], [\infty]}}
\]
Taking \(|\sigma| = |s| + 1\), convolution inequality enables us to conclude that

\[
\left( \sum \frac{2^{j''}}{r} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)}^r \right)^{\frac{1}{r}} \leq C\| \partial_1 F \|_{W^{m-n}} \left( 1 + \| f \|_{L^\infty_t(L^\infty)} \| g \|_{L^\infty_t(L^\infty)} \right)^{[s]+1} \| f \|_{L^2_t(L^p)},
\]

Similarly,

\[
\left( \sum \frac{2^{j''}}{r} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)}^r \right)^{\frac{1}{r}} \leq C\| \partial_2 F \|_{W^{m-n}} \left( 1 + \| f \|_{L^\infty_t(L^\infty)} \| g \|_{L^\infty_t(L^\infty)} \right)^{[s]+1} \| g \|_{L^2_t(L^p)}.
\]

Bounding the term pertaining to \( \hat{\Delta}^b_j \) is easy. Indeed, we have according to Hölder’s inequality and (2.3),

\[
2^{-j'} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)} \leq \sum_{j \leq i} 2^{-j'} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)} \leq C \sum_{j \leq i} 2^{j'} |m|^j \| \hat{\Delta}^b_j \|_{L^r_t(L^p)} \leq C \| \partial_1 F \|_{L^\infty} \sum_{j \leq i} 2^{j' - j''} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)}.
\]

Due to \( s > 0 \), we have

\[
\left( \sum \frac{2^{j''}}{r} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)}^r \right)^{\frac{1}{r}} \leq C\| \partial_1 F \|_{L^\infty} \| f \|_{L^2_t(L^p)}.
\]

Similarity,

\[
\left( \sum \frac{2^{j''}}{r} \| \hat{\Delta}^b_j \|_{L^r_t(L^p)}^r \right)^{\frac{1}{r}} \leq C\| \partial_2 F \|_{L^\infty} \| g \|_{L^2_t(L^p)}.
\]

Thus, we deduce (2.1).

(b) We use the following identity

\[
F(f_2, g_2) - F(f_1, g_1) = \int_0^1 \partial_1 F(f_1 + \theta(f_2 - f_1), g_1 + \theta(g_2 - g_1))d\theta(f_2 - f_1)
\]

\[
+ \int_0^1 \partial_2 F(f_1 + \theta(f_2 - f_1), g_1 + \theta(g_2 - g_1))d\theta(g_2 - g_1)
\]

\[
= M_1(f_2 - f_1) + M_2(g_2 - g_1).
\]

From proposition 2.6 we obtain

\[
\| F(f_2, g_2) - F(f_1, g_1) \|_{L^2_t(L^p)} \leq \| M_1 \|_{L^\infty_t(L^p)} \| f_2 - f_1 \|_{L^2_t(L^p)}
\]

\[
+ \| M_2 \|_{L^\infty_t(L^p)} \| g_2 - g_1 \|_{L^2_t(L^p)}.
\]
Furthermore, for $\|M_i\|_{L_t^\infty(B_{\mu_i}^\infty)}$ $(i = 1, 2)$, by (a) we have

$$
\|M_i\|_{L_t^\infty(B_{\mu_i}^\infty)} \leq C \left(1 + \|(f_1, f_2)\|_{L_t^\infty(L^\infty)}\|(g_1, g_2)\|_{L_t^\infty(L^\infty)}\right)^{\frac{1}{2}} \times \left(\|(f_1, g_1)\|_{L_t^\infty(B_{\mu_i}^\infty)} + \|(f_2, g_2)\|_{L_t^\infty(B_{\mu_i}^\infty)}\right).
$$

Then, we conclude (2.2).

We finish this subsection by listing an elementary but useful inequality.

**Lemma 2.8.** [32]. Let $r_1, r_2 > 0$ satisfy $\max\{r_1, r_2\} > 1$. Then

$$
\int_0^t (1 + t - \tau)^{-r_1}(1 + \tau)^{-r_2}d\tau \leq C(r_1, r_2)(1 + t)^{-\min(r_1, r_2)}.
$$

### 3. Maximal regularity estimates for the linearized system in Besov spaces

In this section, we will exhibit the parabolic properties of the linearized system for all frequencies of the Cauchy problem (1.23) and (1.24). The key to these remarkable properties is given by the following lemma.

**Lemma 3.1.** Let $T > 0$, $s \in \mathbb{R}$, $1 \leq r \leq q \leq \infty$. Assume that $(c^+, u^+, c^-, u^-)$ is a solution to system (1.23) and (1.24) on $[0, T] \times \mathbb{R}^N$, then,

$$
\left\| \left(\sqrt{\beta_1} + \Lambda\right)c^+, u^+; \left(\sqrt{\beta_4} + \Lambda\right)c^-, u^- \right\|_{L_t^2(B_{s+\frac{4}{3}}^2)} \leq C \left\| \left(\sqrt{\beta_1} + \Lambda\right)c_0^+, u_0^+; \left(\sqrt{\beta_4} + \Lambda\right)c_0^-, u_0^- \right\|_{B_{s,1}^1}
$$

$$
+ \left\| \left(\sqrt{\beta_1} + \Lambda\right)H_1, H_2; \left(\sqrt{\beta_4} + \Lambda\right)H_3, H_4 \right\|_{L_t^2(B_{s-\frac{2}{3}}^2)} \text{ for } \forall t \in [0, T].
$$

**Proof.** Applying the orthogonal projectors $\mathcal{P}$ and $\mathcal{Q}$ over divergence-free and potential vector-fields, respectively, to the second equation and the fourth equation, and setting $\nu^+ \defeq \nu^+_1 + \nu^+_2$, system (1.23) translates into

$$
\left\{ \begin{array}{l}
\partial_t \mathcal{P}u^+ - \nu^+_1 \Delta \mathcal{P}u^+ = \mathcal{P}H_2, \\
\partial_t \mathcal{P}u^- - \nu^-_1 \Delta \mathcal{P}u^- = \mathcal{P}H_4,
\end{array} \right. \quad (3.2)
$$

and

$$
\left\{ \begin{array}{l}
\partial_t c^+ + \text{div } \mathcal{Q}u^+ = H_1, \\
\partial_t \mathcal{Q}u^+ + \beta_1 \nabla c^+ + \beta_2 \nabla c^- - \nu^+ \Delta \mathcal{Q}u^+ - \nabla \Delta c^+ = \mathcal{Q}H_2, \\
\partial_t c^- + \text{div } \mathcal{Q}u^- = H_3, \\
\partial_t \mathcal{Q}u^- + \beta_3 \nabla c^+ + \beta_4 \nabla c^- - \nu^- \Delta \mathcal{Q}u^- - \nabla \Delta c^- = \mathcal{Q}H_4.
\end{array} \right. \quad (3.3)
$$

Note that taking advantage of Duhamel’s formula reduces the proof to the case where $H_1 \equiv 0$, $H_2 \equiv 0$, $H_3 \equiv 0$ and $H_4 \equiv 0$. Thus, from (3.2) we readily get, after taking the (space) Fourier transform,
\[
\frac{1}{2} \frac{d}{dt} \left( |\mathcal{P}u^+|^2 + |\mathcal{P}u^-|^2 \right) + \nu_1 |\xi| |\mathcal{P}u^+|^2 + \nu_1 |\xi| |\mathcal{P}u^-|^2 = 0. \tag{3.4}
\]

Here, it is convenient to set \(d^\pm := \Lambda^{-1} \text{div} u^\pm\), keeping in mind that bounding \(d^\pm\) or \(Qu^\pm\) is equivalent, as one can go from \(d^\pm\) to \(Qu^\pm\) or from \(Qu^\pm\) to \(d^\pm\) by means of a 0 order homogeneous Fourier multiplier. Hence, from system (3.3), we discover that \((e^+, d^+)\) satisfies

\[
\begin{align*}
\frac{\partial e^+}{\partial t} + \Lambda d^+ &= 0, \\
\frac{\partial d^+}{\partial t} - \beta_1 \Lambda e^+ - \beta_2 \Lambda e^- - \nu^+ \Lambda d^+ - \Lambda^3 e^+ &= 0, \\
\frac{\partial e^-}{\partial t} + \Lambda d^- &= 0, \\
\frac{\partial d^-}{\partial t} - \beta_3 \Lambda e^+ - \beta_4 \Lambda e^- - \nu^- \Lambda d^- - \Lambda^3 e^- &= 0.
\end{align*} \tag{3.5}
\]

By using the Fourier transform, from (3.5), we have

\[
\begin{align*}
\frac{\partial \mathcal{F} e^+}{\partial t} + |\xi| \mathcal{F} d^+ &= 0, \\
\frac{\partial \mathcal{F} d^+}{\partial t} - \beta_1 |\xi| \mathcal{F} e^+ - \beta_2 |\xi| \mathcal{F} e^- + \nu^+ |\xi|^2 \mathcal{F} d^+ - |\xi|^3 \mathcal{F} e^+ &= 0, \\
\frac{\partial \mathcal{F} e^-}{\partial t} + |\xi| \mathcal{F} d^- &= 0, \\
\frac{\partial \mathcal{F} d^-}{\partial t} - \beta_3 |\xi| \mathcal{F} e^+ - \beta_4 |\xi| \mathcal{F} e^- + \nu^- |\xi|^2 \mathcal{F} d^- - |\xi|^3 \mathcal{F} e^- &= 0.
\end{align*} \tag{3.6}
\]

Applying the energy argument of Godunov [21] for partially dissipative first-order symmetric systems (and developed further in [20]) to the system (3.6). Multiplying the first equation in (3.6) by the conjugate \(\mathcal{F} e^+\) of \(\mathcal{F} e^+\), the second one by \(\mathcal{F} d^+\), the third one by \(\mathcal{F} e^-\) and the fourth one by \(\mathcal{F} d^-\), we get

\[
\frac{1}{2} \frac{d}{dt} |\mathcal{F} e^+|^2 + |\xi| |\text{Re}(\mathcal{F} e^+ d^+)| = 0 \tag{3.7}
\]

and, because \(\text{Re}(\mathcal{F} e^+ d^+) = \text{Re}(\mathcal{F} d^+ e^+)\),

\[
\frac{1}{2} \frac{d}{dt} |\mathcal{F} d^+|^2 + \nu^+ |\xi|^2 |\mathcal{F} d^+|^2 - |\xi| (\beta_1 + |\xi|^2) |\text{Re}(\mathcal{F} e^+ d^+) - \beta_2 |\xi| |\text{Re}(\mathcal{F} e^- d^+)| = 0, \tag{3.8}
\]

\[
\frac{1}{2} \frac{d}{dt} |\mathcal{F} e^-|^2 + |\xi| |\text{Re}(\mathcal{F} e^- d^-)| = 0, \tag{3.9}
\]

and

\[
\frac{1}{2} \frac{d}{dt} |\mathcal{F} d^-|^2 + \nu^- |\xi|^2 |\mathcal{F} d^-|^2 - |\xi| (\beta_4 + |\xi|^2) |\text{Re}(\mathcal{F} e^- d^-) - \beta_3 |\xi| |\text{Re}(\mathcal{F} e^+ d^-)| = 0. \tag{3.10}
\]

Further, multiplying the first and third equations of (3.6) by \(\mathcal{F} e^-\) and \(\mathcal{F} e^+\) respectively, we have

\[
\frac{d}{dt} |\text{Re}(\mathcal{F} e^+ e^-)| + |\xi| |\text{Re}(\mathcal{F} e^+ d^-)| + |\xi| |\text{Re}(\mathcal{F} e^- d^-)| = 0. \tag{3.11}
\]
In order to track dissipations arising for $c^+$ and $c^-$, let us multiply the first and second equations of (3.6) by $-|\xi|d^+$ and $-|\xi|d^-$, the third one by $-|\xi|\overline{d^+}$ and the fourth one by $-|\xi|\overline{d^-}$, respectively. Adding them, we get

$$
\frac{d}{dt} \left( -|\xi|\text{Re}(\overline{c^+}d^+) - |\xi|\text{Re}(\overline{c^-}d^-) \right) + |\xi|^2 (\beta_1 + |\xi|^2) |c^+|^2 + |\xi|^2 (\beta_4 + |\xi|^2) |c^-|^2 \\
+ \beta_2 |\xi|^2 \text{Re}(\overline{c^+}\overline{c^-}) + \beta_3 |\xi|^2 \text{Re}(\overline{c^-}\overline{c^+}) - \nu^+ |\xi|^3 \text{Re}(c^+ \overline{d^+}) - \nu^- |\xi|^3 \text{Re}(c^- \overline{d^-}) \\
- |\xi|^2 |\overline{d^+}|^2 - |\xi|^2 |\overline{d^-}|^2 = 0.
$$

Adding (3.7) $\times \nu^+ |\xi|^2$ and (3.9) $\times \nu^- |\xi|^2$ to (3.13), we get

$$
\frac{1}{2} \frac{d}{dt} \left( \nu^+ |\xi|^2 |\overline{d^+}|^2 + \nu^- |\xi|^2 |\overline{d^-}|^2 - 2|\xi|\text{Re}(\overline{c^+}d^+) - 2|\xi|\text{Re}(\overline{c^-}d^-) \right) \\
+ |\xi|^2 (\beta_1 + |\xi|^2) |c^+|^2 + |\xi|^2 (\beta_4 + |\xi|^2) |c^-|^2 + 2\beta_2 \text{Re}(\overline{c^+}\overline{c^-}) + 2\beta_3 \text{Re}(\overline{c^-}\overline{c^+}) \\
- |\xi|^2 |\overline{d^+}|^2 - |\xi|^2 |\overline{d^-}|^2 = 0.
$$

Therefore, by multiplying (3.14) by a small enough constant $\delta > 0$ (to be determined later) and adding it to (3.12), we get

$$
\frac{1}{2} \frac{d}{dt} \left( (\beta_1 + |\xi|^2) |c^+|^2 + (\beta_4 + |\xi|^2) |c^-|^2 + 2\beta_2 \text{Re}(\overline{c^+}\overline{c^-}) + |\overline{d^+}|^2 + |\overline{d^-}|^2 \\
+ \delta \nu^+ |\xi|^2 |c^+|^2 + \delta \nu^- |\xi|^2 |c^-|^2 - 2\delta |\xi|\text{Re}(\overline{c^+}d^+) - 2\delta |\xi|\text{Re}(\overline{c^-}d^-) \right) \\
+ \delta |\xi|^2 (\beta_1 + |\xi|^2) |c^+|^2 + \delta |\xi|^2 (\beta_4 + |\xi|^2) |c^-|^2 + \delta \beta_2 |\xi|^2 \text{Re}(\overline{c^+}\overline{c^-}) \\
+ \delta \beta_3 |\xi|^2 \text{Re}(\overline{c^-}\overline{c^+}) + (\nu^+ - \delta) |\xi|^2 |\overline{d^+}|^2 + (\nu^- - \delta) |\xi|^2 |\overline{d^-}|^2 = 0.
$$

By Young’s inequality, for $\delta < \frac{1}{4}$, we have

$$
|2\delta |\xi|\text{Re}(\overline{c^+}d^+)| \leq 2\delta \left( \frac{|\xi|^2 |c^+|^2}{2} + \frac{1}{2} |\overline{d^+}|^2 \right) \\
\leq \frac{1}{3} |\xi|^2 |c^+|^2 + \frac{1}{3} |\overline{d^+}|^2,
$$

$$
|2\delta |\xi|\text{Re}(\overline{c^-}d^-)| \leq 2\delta \left( \frac{|\xi|^2 |c^-|^2}{2} + \frac{1}{2} |\overline{d^-}|^2 \right) \\
\leq \frac{1}{3} |\xi|^2 |c^-|^2 + \frac{1}{3} |\overline{d^-}|^2.
$$
Using further $\beta^2 = \beta_1^2 = \beta_1\beta_4$ and Young’s inequality, and choosing $M = \frac{\beta_1}{2}$, we get
\[
|2\beta_2 \text{Re}(\hat{c}^+ \overline{\hat{c}^+})| \leq M\beta_2|\hat{c}^+|^2 + \frac{\beta_2}{M}|\hat{c}^-|^2
\leq \beta_1|\hat{c}^+|^2 + \beta_4|\hat{c}^-|^2.
\]
Thus,
\[
\mathcal{L}^2 \approx (\beta_1 + |\xi|^2)|\hat{c}^+|^2 + (\beta_4 + |\xi|^2)|\hat{c}^-|^2 + |\hat{d}^+|^2 + |\hat{d}^-|^2.
\]
where $\mathcal{L} := \mathcal{L}^2(\xi, \nu)$ is a Lyapunov functional defined by
\[
\mathcal{L}^2 = (\beta_1 + |\xi|^2)|\hat{c}^+|^2 + (\beta_4 + |\xi|^2)|\hat{c}^-|^2 + 2\beta_2 \text{Re}(\hat{c}^+ \overline{\hat{c}^-}) + |\hat{d}^+|^2 + |\hat{d}^-|^2 + \delta|\nu^+||\hat{c}^+|^2 + \delta|\nu^-||\hat{c}^-|^2 - 2\delta|\xi|\text{Re}(\hat{c}^+ \overline{\hat{d}^+}) - 2\delta|\xi|\text{Re}(\hat{c}^- \overline{\hat{d}^-}).
\]
On the other hand, taking $\delta = \min\{\frac{1}{2}, \nu^+, \nu^-\}$, there exists a positive constant $C$ such that
\[
\begin{align*}
\delta|\xi|^2(\beta_1 + |\xi|^2)|\hat{c}^+|^2 + \delta|\xi|^2(\beta_4 + |\xi|^2)|\hat{c}^-|^2 + \delta\beta_2|\xi|^2\text{Re}(\hat{c}^+ \overline{\hat{c}^-}) \\
+ \delta\beta_3|\xi|^2\text{Re}(\hat{c}+ \overline{\hat{c}^-}) + (\nu^+ - \delta)|\xi|^2|\hat{d}^+|^2 + (\nu^- - \delta)|\xi|^2|\hat{d}^-|^2
\geq C|\xi|^2 \left((\beta_1 + |\xi|^2)|\hat{c}^+|^2 + (\beta_4 + |\xi|^2)|\hat{c}^-|^2 + |\hat{d}^+|^2 + |\hat{d}^-|^2\right).
\end{align*}
\]
Thus,
\[
\frac{d}{dt}\mathcal{L}^2 + C|\xi|^2\mathcal{L}^2 \leq 0. \tag{3.16}
\]
Combining (3.4) and (3.16), we get
\[
\left| \left( \sqrt{\beta_1}e^+, |\xi|e^+, \sqrt{\beta_2}e^-, |\xi|e^-, u^+ \right) \right| \\
\leq Ce^{-\omega|\xi|^2} \left| \left( \sqrt{\beta_1}e^+, |\xi|e^+, \sqrt{\beta_2}e^-, |\xi|e^-, u^- \right) \right| (0). \tag{3.17}
\]
Granted with the above above estimates, by means of the definition of Littlewood–Paley decomposition and Fourier–Plancherel theorem, we get
\[
\left\| \left( \sqrt{\beta_1} + \Lambda \right)e^+, u^+, \sqrt{\beta_4}e^-, u^- \right\|_{L_t^2(L_x^2)^2}
\leq C \left\| \left( \sqrt{\beta_1} + \Lambda \right)e^0_+, u^0_+, \sqrt{\beta_4}e^0_-, u^-_0 \right\|_{L_t^2} \tag{3.18}
\]
Then, for general source terms, from Duhamel’s formula we finally obtain (3.1). This completes the proof of lemma 3.1.
4. The unique global solvability

4.1. Global a priori estimates

The subsection is devoted to exploiting an important global a priori estimates for the Cauchy problems (1.23) and (1.24).

**Proposition 4.1.** Let $T \geq 0$, $N \geq 2$, and $(e^+, u^+, c^-, \dot{u}^-)$ be a solution to the Cauchy problems (1.23) and (1.24) on $[0, T] \times \mathbb{R}^N$, we have

$$X(t) \leq C \left( X(0) + (1 + X^2(t))^{\frac{B}{2}} (X^3(t) + X^5(t)) \right), \text{ for } t \in [0, T], \quad (4.1)$$

where

$$X(t) \overset{\text{def}}{=} \left\| \left( \sqrt{\beta_1} + \Lambda \right)c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right)c^-, \dot{u}^- \right\|_{L^\infty([0,b_{2,1}])}$$

$$+ \left\| \left( \sqrt{\beta_1} + \Lambda \right)c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right)c^-, \dot{u}^- \right\|_{L^1([0,b_{2,1}])}.$$ 

**Proof.** In lemma 3.1, taking $q = 1, \infty, r = 1$ and $s = \frac{N}{2} - 1$, for $t \in [0, T]$, we have

$$X(t) \leq C \left\| \left( \sqrt{\beta_1} + \Lambda \right)c_0^+, u_0^+, \left( \sqrt{\beta_4} + \Lambda \right)c_0^-, \dot{u}_0^- \right\|_{B_{2,1}^{\frac{N}{2}}}$$

$$+ \left\| \left( \sqrt{\beta_1} + \Lambda \right)H_1, H_2, \left( \sqrt{\beta_4} + \Lambda \right)H_3, H_4 \right\|_{L^1([0,b_{2,1}])}, \quad (4.2)$$

In what follows, we derive some nonlinear estimates from the terms $\left\| \left( \sqrt{\beta_1} + \Lambda \right)H_1, H_2, \left( \sqrt{\beta_4} + \Lambda \right)H_3, H_4 \right\|_{L^1([0,b_{2,1}])}$. First, by proposition 2.6 and the embedding $B_{2,1}^{\frac{N}{2}} \hookrightarrow L^\infty$, we have

$$\left\| \left( \sqrt{\beta_1} + \Lambda \right)H_1 \right\|_{L^1\left( B_{2,1}^{\frac{N}{2}} \right)}$$

$$\leq C \left\| \left( \sqrt{\beta_1} + \Lambda \right) (c^+ u^+) \right\|_{L^1\left( B_{2,1}^{\frac{N}{2}} \right)}$$

$$\leq C \left\| c^+ u^+ \right\|_{L^1\left( B_{2,1}^{\frac{N}{2}} \right)} + C \left\| c^+ u^+ \right\|_{L^1\left( B_{2,1}^{\frac{N}{2} + 1} \right)}$$

$$\leq C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} + C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2} + 1} \right)}$$

$$+ C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2} + 1} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)}$$

$$\leq C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} + C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2} + 1} \right)}$$

$$+ C \left\| c^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2} + 1} \right)} \left\| u^+ \right\|_{L^2\left( B_{2,1}^{\frac{N}{2}} \right)}$$
and

\[ \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \leq \left( \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}} \left( \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}}, \]

where we have used the following interpolation inequalities,

\[ \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \leq \left( \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}} \left( \|c\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}}, \]

and

\[ \|u\|_{L^2(\frac{\theta}{\lambda + 1})} \leq \left( \|u\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}} \left( \|u\|_{L^2(\frac{\theta}{\lambda + 1})} \right)^{\frac{1}{2}}. \]

Similarly,

\[ \|(\sqrt{\theta} + \Lambda)H_3\|_{L^1(\frac{\theta}{\lambda + 1})} \leq CX^2(t). \] (4.4)

Next, we bound the term \( H_2 \). By lemma 2.7(a) we get

\[ \|g^+ (c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})} \leq C \left( 1 + \|c^+\|_{L^2(\lambda + 1)} \|c^-\|_{L^2(\lambda + 1)} \right) \left( \frac{\theta}{\lambda + 1} \right)^{\frac{1}{2}} \|(c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})}, \]

\[ \|g^+ (c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})} \leq C \left( 1 + \|c^+\|_{L^2(\lambda + 1)} \|c^-\|_{L^2(\lambda + 1)} \right) \left( \frac{\theta}{\lambda + 1} \right)^{\frac{1}{2}} \|(c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})}, \]

\[ \|I^+ (c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})} \leq C \left( 1 + \|c^+\|_{L^2(\lambda + 1)} \|c^-\|_{L^2(\lambda + 1)} \right) \left( \frac{\theta}{\lambda + 1} \right)^{\frac{1}{2}} \|(c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})}, \]

\[ \|h^+ (c^+, c^-) - \frac{C^2}{\alpha^2 (1, 1)} \|_{L^2(\frac{\theta}{\lambda + 1})} \leq C \left( 1 + \|c^+\|_{L^2(\lambda + 1)} \|c^-\|_{L^2(\lambda + 1)} \right) \left( \frac{\theta}{\lambda + 1} \right)^{\frac{1}{2}} \|(c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})}, \]

and

\[ \|k^+ (c^+, c^-) - \frac{C^2}{\alpha^2 (1, 1)} \|_{L^2(\frac{\theta}{\lambda + 1})} \leq C \left( 1 + \|c^+\|_{L^2(\lambda + 1)} \|c^-\|_{L^2(\lambda + 1)} \right) \left( \frac{\theta}{\lambda + 1} \right)^{\frac{1}{2}} \|(c^+, c^-)\|_{L^2(\frac{\theta}{\lambda + 1})}. \]

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Based on these bounds, thanks to proposition 2.6, lemma 2.7(a) and the embedding \( \dot{B}^{\frac{N}{2}}_{2,1} \hookrightarrow L^\infty \), we easily infer the following estimates

\[
\| g_+ (c^+, c^-) \partial_t c^+ - \tilde{g}_+ (c^+, c^-) \partial_t c^- \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \| g_+ (c^+, c^-) \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \partial_t c^+ \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} + \| \tilde{g}_+ (c^+, c^-) \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \partial_t c^- \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \| g_+ (c^+, c^-) \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \partial_t c^+ \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} + \| \tilde{g}_+ (c^+, c^-) \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| c^- \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \left( 1 + \| c^+ \|_{L^2_t (L^\infty)} \| c^- \|_{L^2_t (L^\infty)} \right) \| \tilde{g}_+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| (c^+, c^-) \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}^2
\leq C \left( 1 + X^2 (t) \right)^{\frac{N}{2} + 1} X^2 (t),
\]

\[
\| u^+ \cdot \nabla u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \leq C \| u^+ \|_{L^\infty_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \nabla u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \| u^+ \|_{L^\infty_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| u^+ \|_{L^1_t \left( \dot{B}^{N+1}_{2,1} \right)}
\leq CX^2 (t),
\]

\[
\| h_+ (c^+, c^-) \partial_t^2 u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \| h_+ (c^+, c^-) \|_{L^\infty_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \partial_t^2 u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \left( 1 + \| c^+ \|_{L^2_t (L^\infty)} \| c^- \|_{L^2_t (L^\infty)} \right) \| \tilde{g}_+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| (c^+, c^-) \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| u^+ \|_{L^1_t \left( \dot{B}^{N+1}_{2,1} \right)}
\leq C \left( 1 + X^2 (t) \right)^{\frac{N}{2} + 1} \| X^2 (t),
\]

\[
\| h_+ (c^+, c^-) \partial_t c^+ \partial_t u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \| h_+ (c^+, c^-) \|_{L^\infty_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \partial_t c^+ \partial_t u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \left( 1 + \| c^+ \|_{L^2_t (L^\infty)} \| c^- \|_{L^2_t (L^\infty)} \right) \| \tilde{g}_+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| (c^+, c^-) \|_{L^2_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| u^+ \|_{L^1_t \left( \dot{B}^{N+1}_{2,1} \right)}
\times \| \nabla c^+ \|_{L^\infty_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)} \| \nabla u^+ \|_{L^1_t \left( \dot{B}^{\frac{N}{2}}_{2,1} \right)}
\leq C \left( 1 + X^2 (t) \right)^{\frac{N}{2} + 1} X^3 (t),
\]
and

\[ \| k_\pm (c^+, c^-) \partial_x c^- \partial^2 \|_{L^1_t(B_{2,1}^N)} \]
\[ \leq C \left( 1 + \| k_+ (c^+, c^-) - \frac{C^2(1,1)}{s^2} \rho^+(1,1) \|_{L^\infty_t(B_{2,1}^N)} \right) \| \partial_x c^- \partial^2 \|_{L^1_t(B_{2,1}^N)} \]
\[ \leq C \left( 1 + \| c^+ \|_{L^2_t(L^\infty_x)} \| c^- \|_{L^2_t(L^\infty_x)} \right) \| \nabla c^- \|_{L^\infty_t(B_{2,1}^N)} \| \nabla u^+ \|_{L^1_t(B_{2,1}^N)} \]
\[ \leq C \left( 1 + X^2(t) \right)^{|\frac{N}{2}+1} X^3(t). \]

Hence, we gather that

\[ \| H_2 \|_{L^1_t(B_{2,1}^N)} \leq C \left( 1 + X^2(t) \right)^{|\frac{N}{2}+1} \left( X^2(t) + X^3(t) \right). \] (4.5)

Similarly, we also have

\[ \| H_4 \|_{L^1_t(B_{2,1}^N)} \leq C \left( 1 + X^2(t) \right)^{|\frac{N}{2}+1} \left( X^2(t) + X^3(t) \right). \] (4.6)

Substituting (4.3) and (4.6) into (4.2), we finally get (4.1). This completes the proof of proposition 4.1.

### 4.2. Global existence and uniqueness

In order to solve the system (1.23) and (1.24) by fixed point theorem, we define the following map

\[ \Phi : (c^+, u^+, c^-, u^-) \rightarrow (b^+, v^+, b^-, v^-) \] (4.7)

with \((b^+, v^+, b^-, v^-)\) the solution to

\[
\begin{align*}
\partial_t b^+ + \text{div} \, v^+ &= H_1(c^+, u^+), \\
\partial_t v^+ + \beta_1 \nabla b^+ + \beta_2 \nabla b^- - \nu_1^+ \Delta v^+ - \nu_2^+ \text{div} \, v^+ - \nabla \Delta b^+ &= H_2(c^+, u^+, c^-), \\
\partial_t b^- + \text{div} \, v^- &= H_3(c^-, u^-), \\
\partial_t v^- + \beta_3 \nabla b^+ + \beta_4 \nabla b^- - \nu_1^- \Delta v^- - \nu_2^- \text{div} \, v^- - \nabla \Delta b^- &= H_4(c^+, u^-, c^-).
\end{align*}
\] (4.8)

Obviously, to prove the existence part of the theorem, we just have to show that \(\Phi\) is a contraction map in a ball of \(X(t)\), where \(X(t) \equiv L^\infty_t(B_{2,1}^N) \cap L^1_t(B_{2,1}^{\frac{N}{2}+1})\) equipped with a norm

\[ \|(c^+, u^+, c^-, u^-)\|_{X(t)} = X(t). \]
We define a ball $B(0, R)$ centred at the origin by

$$B(0, R) = \{ (c^+, u^+, c^-, u^-) \in \mathbb{X}(t) : \| (c^+, u^+, c^-, u^-) \|_{\mathbb{X}(t)} \leq R \}.$$  \hfill (4.9)

Assuming $R \leq 1$, from proposition 4.1 we have

$$\| \Phi(c^+, u^+, c^-, u^-) \|_{\mathbb{X}(t)} \leq C \left( X(0) + \frac{(c^+, u^+, c^-, u^-)^2}{\mathbb{X}(t)} \right)^{\frac{1}{2}} + \left( \frac{(c^+, u^+, c^-, u^-)^3}{\mathbb{X}(t)} \right)^{\frac{1}{3}} \leq C \left( \eta + (1 + R^2)^{\frac{1}{2} + 1} (R^2 + R^3) \right) \leq C \left( \eta + 2R^2 \right).$$  \hfill (4.10)

Choosing $(R, \eta)$ such that

$$R = \min \{ 1, (4C)^{-1} \} \quad \text{and} \quad \eta \leq 2R^2.$$  \hfill (4.11)

Thus, from (4.10), we finally deduce that

$$\Phi(B(0, R)) \subseteq B(0, R).$$

In order to show $\Phi$ is a contraction map, one chooses two elements $(c_1^+, u_1^+, c_1^-, u_1^-)$ and $(c_2^+, u_2^+, c_2^-, u_2^-)$ in $B(0, R)$. According to (4.2), (4.7) and (4.8), we have

$$\| \Phi(c_1^+, u_1^+, c_1^-, u_1^-) - \Phi(c_2^+, u_2^+, c_2^-, u_2^-) \|_{\mathbb{X}(t)} \leq C \left( \| (\sqrt{\beta_{13}} + \Lambda)(H_1(c_1^+, u_1^+) - H_1(c_2^+, u_2^+)) \|_{L^1(\mathbb{X}_{c_1, c_2})} \right) + \| H_2(c_1^+, u_1^+, c_1^-) - H_2(c_2^+, u_2^+, c_2^-) \|_{L^1(\mathbb{X}_{c_1, c_2})} + \| H_3(c_1^+, u_1^+, c_1^-) - H_3(c_2^+, u_2^+, c_2^-) \|_{L^1(\mathbb{X}_{c_1, c_2})}.$$  \hfill (4.12)

Further, using lemma 2.7(b) similar to the estimate (4.1), we have

$$\| \Phi(c_1^+, u_1^+, c_1^-, u_1^-) - \Phi(c_2^+, u_2^+, c_2^-, u_2^-) \|_{\mathbb{X}(t)} \leq C \left( \left( \| (c_1^+, u_1^+, c_1^-, u_1^-) \|_{\mathbb{X}(t)} + \| (c_2^+, u_2^+, c_2^-, u_2^-) \|_{\mathbb{X}(t)} \right) \times \| (c_1^+ - c_2^+, u_1^+ - u_2^+, c_1^- - c_2^-) u_1^+ - u_2^+ \|_{\mathbb{X}(t)} \right).$$  \hfill (4.13)

From (4.11) we finally deduce that

$$\| \Phi(c_1^+, u_1^+, c_1^-, u_1^-) - \Phi(c_2^+, u_2^+, c_2^-, u_2^-) \|_{\mathbb{X}(t)} \leq \frac{1}{2} \| (c_1^+ - c_2^+, u_1^+ - u_2^+, c_1^- - c_2^-) u_1^+ - u_2^+ \|_{\mathbb{X}(t)}.$$  \hfill (4.14)

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and the proof of the existence part of theorem 1.1 is achieved. Moreover, the uniqueness part of theorem 1.1 in $B(0, R)$ naturally follows.

5. Time decay estimates

In this section, we will establish the optimal time decay rates of the global strong solutions constructed in theorem 1.1. We divide the proof into two steps.

**Step 1. Low frequencies**

Denoting by $A(D)$ the semi-group associated to the system (1.23) with $H_1 \equiv H_2 \equiv H_3 \equiv H_4 \equiv 0$, for $U = \langle (\sqrt{\beta_1} + \Lambda)e^+, u^+, (\sqrt{\beta_4} + \Lambda)e^-, u^- \rangle$, from (3.17) and using Parseval’s equality and the definition of $\Delta_q$, we get

$$\|e^{tA(D)}\Delta_q U\|_{L^2} \lesssim e^{-c_0 t^{3/2}} \|\Delta_q U\|_{L^2}.$$  

Hence, multiplying by $r^{5/2} 2^q r^s$ and summing up on $q \lesssim q_0$, we readily have

$$r^{5/2} \sum_{q \lesssim q_0} 2^{q r} \|e^{tA(D)}\Delta_q U\|_{L^2} \lesssim \sum_{q \lesssim q_0} 2^{q r} e^{-c_0 t^{3/2}} \|\Delta_q U\|_{L^2} r^{5/2} 2^q r^s \lesssim \sum_{q \lesssim q_0} 2^{(q+5/2) r} e^{-c_0 t^{3/2}} \|\Delta_q U\|_{L^2} r^{5/2} 2^q r^s \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, \infty}} \sum_{q \lesssim q_0} 2^{(q+5/2) r} e^{-c_0 t^{3/2}} r^{5/2} 2^q r^s \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, \infty}} \sum_{q \lesssim q_0} 2^{(q+5/2) r} e^{-c_0 t^{3/2}} r^{5/2} 2^q r^s. \quad (5.1)$$

As for any $\sigma > 0$ there exists a constant $C_\sigma$ so that

$$\sup_{t \geq 0} \sum_{q \lesssim 2^r} 2^{q r} e^{-c_0 t^{3/2}} \lesssim C_\sigma. \quad (5.2)$$

We get from (5.1) and (5.2) that for $s > -N/2$,

$$\sup_{t \geq 0} r^{5/2} \|e^{tA(D)}U\|_{\tilde{B}^{\frac{5}{2}, \infty}} \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, \infty}}.$$  

Furthermore, it is obvious that for $s > -N/2$,

$$\|e^{tA(D)}U\|_{\tilde{B}^{\frac{5}{2}, 1}} \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, 1}} \sum_{q \lesssim q_0} 2^{(q+5/2) r} \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, 1}}.$$  

Hence, setting $\langle t \rangle \equiv \sqrt{1 + t^2}$, we get

$$\sup_{t \geq 0} \langle t \rangle^{5/2} \|e^{tA(D)}U\|_{\tilde{B}^{\frac{5}{2}, 1}} \lesssim \|U\|_{\tilde{B}^{\frac{5}{2}, 1}}. \quad (5.3)$$

Thus, from (5.3) and Duhamel’s formula, we have

$$\|\left( (\sqrt{\beta_1} + \Lambda)e^+, u^+, (\sqrt{\beta_4} + \Lambda)e^-, u^- \right)\|_{\tilde{B}^{\frac{5}{2}, 1}}.$$
Further, employing a low-high decomposition and \((s)\) instead of \(H\)

\[\lesssim \sup_{t \geq 0} \langle t \rangle^{-\frac{\alpha}{2} + \frac{1}{2}} \left\| \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right\|_{L^2_{t,x}}^h \]

\[+ \int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2} + \frac{1}{2}} \left\| \left( \sqrt{\beta_1} + \Lambda \right) H_1, H_2, \left( \sqrt{\beta_4} + \Lambda \right) H_3, H_4 \right\|_{L^2_{t,x}}^h \, d\tau. \tag{5.4}\]

We claim that for all \(s \in (-N/2, 2]\) and \(t \geq 0\), then

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha}{2} + \frac{1}{2}} \left\| \left( \sqrt{\beta_1} + \Lambda \right) H_1, H_2, \left( \sqrt{\beta_4} + \Lambda \right) H_3, H_4 \right\|_{L^2_{t,x}}^h \, d\tau
\leq \langle t \rangle^{-\frac{\alpha}{2} + \frac{1}{2}} \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right), \tag{5.5}\]

where \(X(t) \) and \(D(t) \) have been defined in (1.36) and (1.39), respectively.

Owing to the embedding \(L^1 \hookrightarrow \dot{B}^{\frac{N}{2}}_{\infty, \infty}\), it suffices to prove (5.5) with \(\| (H_1, H_2, H_3, H_4)(\tau) \|_{L^2_{t,x}}^h \) instead of \(\| (H_1, H_2, H_3, H_4)(\tau) \|_{L^2_{t,x}}^h \).

In order to prove our claim, we first present the following two important inequalities which will be frequently used in our process later. By the smoothing effects of \((c^+, u^+, c^-, u^-)\) in all frequencies, we have

\[
\| \langle \tau \rangle^\alpha \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \|_{L^2(x)}^{\frac{\alpha}{2} - 1} \]

\[\lesssim \| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \|_{L^2(x)}^{\frac{\alpha}{2}} \]

\[+ \| \tau^\alpha \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \|_{L^2(x)}^{\frac{\alpha}{2} - 1}\]

\[\lesssim X(t) + D(t). \tag{5.6}\]

Further, employing a low-high decomposition and (5.6), for \(N \geq 2\) and \(\alpha \geq \frac{N}{2}\), we obtain

\[
\sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{\alpha}{2} \left\| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \right\|_{L^2_{t,x}}^h
\lesssim \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{\alpha}{2} \left\| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \right\|_{L^2_{t,x}}^h

\[+ \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{\alpha}{2} \left\| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \right\|_{L^2_{t,x}}^h
\lesssim \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\frac{\alpha}{2} \left\| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \right\|_{L^2_{t,x}}^h

\[+ \sup_{0 \leq \tau \leq t} \langle \tau \rangle^\alpha \left\| \left( \left( \sqrt{\beta_1} + \Lambda \right) c^+, u^+, \left( \sqrt{\beta_4} + \Lambda \right) c^-, u^- \right) \right\|_{L^2_{t,x}}^{\frac{\alpha}{2} - 1}
\lesssim X(t) + D(t). \tag{5.7}\]

To bound the term with \(H_1\), we use the following decomposition:

\[H_1 = u^+ \cdot \nabla c^+ + c^+ \text{div}(u^+) + c^+ \text{div}(u^+)^h.\]
Now, from Hölder’s inequality, the embedding $\dot{B}^{2,1}_{2,1} \hookrightarrow L^2$, the definitions of $D(t)$, $\alpha$, lemma 2.8, (5.6) and (5.7), one may write for all $s \in (\varepsilon - \frac{1}{2}, 2]$,\,
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| \nabla c^+ \|_{L^1} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| u^+ \|_{L^2} \| \nabla c^+ \|_{L^2} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| u^+ \|_{\dot{B}^0_{2,1}} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| u^+ \|_{\dot{B}^0_{2,1}} \left( \| \nabla c^+ \|_{\dot{B}^0_{2,1}}^\alpha + \| \nabla c^+ \|_{\dot{B}^0_{2,1}}^\beta \right) d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| u^+ \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \\
\times \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau \\
\times \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| u^+ \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \\
\times \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| u^+ \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \\
\times \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| \nabla c^+ \|_{\dot{B}^0_{2,1}} \right) \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \left( D^2(t) + X^2(t) \right). \quad (5.8)
\end{align*}

For the term $c^+ \text{ div}(u^+)'$, using the definitions of $D(t)$, (5.7) and lemma 2.8, we have
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| c^+ \text{ div}(u^+)' \|_{L^1} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| c^+ \|_{L^2} \| \text{div}(u^+)' \|_{L^2} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \| c^+ \|_{\dot{B}^0_{2,1}} \| \nabla u^+ \|_{\dot{B}^0_{2,1}} d\tau \\
\lesssim \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| c^+ \|_{\dot{B}^0_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{\alpha}{2}} \| u^+ \|_{\dot{B}^0_{2,1}} \right) \\
\times \int_0^t \langle t - \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{1}{2}\right)} \left( D^2(t) + X^2(t) \right). \quad (5.9)
\end{align*}
Regarding the term with $c^+ \text{div}(u^+)^h$, we get for all $t \geq 2$ that
\[
\int_0^t (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ \text{div}(u^+)^h(\tau)||_{L^1} d\tau \\
\lesssim \int_0^1 (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ (\tau)||_{|t^0|} ||\text{div} u^+(\tau)||^h_{|t^0|} d\tau \\
\lesssim \int_0^1 (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ (\tau)||_{|t^0|} ||\text{div} u^+(\tau)||^h_{|t^0|} d\tau \\
+ \int_1^t (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ (\tau)||_{|t^0|} ||\text{div} u^+(\tau)||^h_{|t^0|} d\tau \\
\overset{\text{def}}{=} I_1 + I_2.
\]

From the definitions of $X(t)$ and $D(t)$, we obtain
\[
I_1 \lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \sup_{0 \leq \tau \leq 1} ||c^+ (\tau)||_{|t^0|} \int_0^1 ||\text{div} u^+(\tau)||^h_{|t^0|} d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \sup_{0 \leq \tau \leq 1} ||c^+ (\tau)||_{|t^0|} \int_0^1 ||u^+(\tau)||^h_{|t^0|} d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} D(1) X(1),
\]

and, using (5.6) and (5.7) and the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$, we get
\[
I_2 \lesssim \int_1^t (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ (\tau)||_{|t^0|} ||\text{div} u^+(\tau)||^h_{|t^0|} d\tau \\
\lesssim \left(\sup_{1 \leq \tau \leq t} ||c^+ (\tau)||_{|t^0|}\right) \left(\sup_{1 \leq \tau \leq t} ||\tau \nabla u^+(\tau)||^h_{|t^0|}\right) \int_1^t (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} (\tau)^{-\left(\frac{\alpha}{2} + 1\right)} d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left(D^2(t) + X^2(t)\right).
\]

Thus, for $t \geq 2$, we conclude that
\[
\int_0^t (t - \tau)^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} ||c^+ \text{div}(u^+)^h(\tau)||_{L^1} d\tau \lesssim \langle t \rangle^{-\left(\frac{\alpha}{2} + \frac{3}{2}\right)} \left(D^2(t) + X^2(t)\right). \tag{5.10}
\]

The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and
\[
\int_0^t ||c^+ \text{div}(u^+)^h||_{L^1} d\tau \lesssim ||c^+||_{L^2(\Omega^2)} ||\text{div} u^+||^h_{L^1(\Omega^2)} \\
\lesssim ||c^+||_{L^2(\Omega^2)} ||\text{div} u^+||^h_{L^1(\Omega^2)} \\
\lesssim ||c^+||_{L^2(\Omega^2)} ||u^+||^h_{L^1(\Omega^2)} \\
\lesssim X(t) D(t). \tag{5.11}
\]
From (5.8)–(5.11), we get
\[
\int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| H_1(\tau) \|^2_{H_{-\frac{d}{2},\infty}} \, d\tau \lesssim (t)^{-\left(\frac{d}{2} + \theta \right)} (X^2(t) + D^2(t)) .
\]
The term $H_3$ may be treated along the same lines, and we obtain
\[
\int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| H_3(\tau) \|^2_{H_{-\frac{d}{2},\infty}} \, d\tau \lesssim (t)^{-\left(\frac{d}{2} + \theta \right)} (X^2(t) + D^2(t)) .
\]
Next, we bound $H^2$ as follows. For the first part of $H^2$, employing (5.7) and lemma 2.7(a) we write that
\[
\int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| g_+(c^+, c^-) \|_{L^1} \, d\tau
\lesssim \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| g_+(c^+, c^-) \|_{L^2} \| \nabla c^+ \|_{L^2} \, d\tau
\lesssim \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| g_+(c^+, c^-) \|_{H^{\theta}_{2,1}} \| \nabla c^+ \|_{H^{\theta}_{2,1}} \, d\tau
\lesssim (1 + X^2(t))^{\frac{3}{2} + 1} \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| (c^+, c^-) \|_{H^{\theta}_{2,1}} \left( \| \nabla c^+ \|_{H^{\theta}_{2,1}} + \| \nabla c^+ \|_{H^{\theta}_{2,1}} \right) \, d\tau
\lesssim (1 + X^2(t))^{\frac{3}{2} + 1} \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{2}} \| (c^+, c^-)(\tau) \|_{H^{\theta}_{2,1}} \right) \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{2}} \| \nabla c^+ (\tau) \|_{H^{\theta}_{2,1}} \right)
\times \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \langle \tau \rangle^{-\left(\frac{d}{2} + \theta \right)} \, d\tau + (1 + X^2(t))^{\frac{3}{2} + 1} \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\frac{d}{2}} \| (c^+, c^-)(\tau) \|_{H^{\theta}_{2,1}} \right)
\times \left( \sup_{0 \leq \tau \leq t} \langle \tau \rangle^{\alpha} \| \nabla c^+ (\tau) \|_{H^{\theta}_{2,1}} \right) \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \langle \tau \rangle^{-\left(\frac{d}{2} + \theta \right)} \, d\tau
\lesssim (1 + X^2(t))^{\frac{3}{2} + 1} (D^2(t) + X^2(t)) \int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \langle \tau \rangle^{-\min \left(\frac{d}{2} + \theta, \alpha + \frac{d}{2} \right)} \, d\tau
\lesssim (t)^{-\left(\frac{d}{2} + \theta \right)} (1 + X^2(t))^{\frac{3}{2} + 1} (D^2(t) + X^2(t)) ,
\]
where $g_+$ stands for some smooth function vanishing at 0.
Similarly,
\[
\int_0^t (t - \tau)^{-\left(\frac{d}{2} + \theta \right)} \| \tilde{g}_+(c^+, c^-) \|_{L^2} \, d\tau \lesssim (t)^{-\left(\frac{d}{2} + \theta \right)} (1 + X^2(t))^{\frac{3}{2} + 1} (D^2(t) + X^2(t)) .
\]
To bound the term with $(a^+ \cdot \nabla) a^+_i$, we employ the following decomposition:
\[
(a^+ \cdot \nabla) a^+_i = (a^+ \cdot \nabla)(a^+_i) + (a^+ \cdot \nabla)(a^+_i)^S .
\]
By the same estimate of the term $c^+ \text{div}(u^+)$, we have
\[ \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| (u^+ \cdot \nabla)(u^+) \tau \|_{L^2} \, d\tau \lesssim \langle t \rangle^{-\frac{N}{2}+\frac{1}{2}} (D^2(t) + X^2(t)) \, . \quad (5.14) \]

To deal with the term $\mu^+ h_+ (c^+, c^-) \partial c^+ \partial u^+$, we take the following decomposition:
\[ \mu^+ h_+ (c^+, c^-) \partial c^+ \partial u^+ = \mu^+ h_+ (c^+, c^-) \partial c^+ \partial (u^+) + \mu^+ h_+ (c^+, c^-) \partial c^+ \partial (u_i^+) g^i \, . \]

For the term $\mu^+ h_+ (c^+, c^-) \partial c^+ \partial (u_i^+) g^i$, we have
\[ \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| \mu^+ h_+ (c^+, c^-) \partial c^+ \partial (u_i^+) g^i \|_{L^2} \, d\tau \]
\[ \lesssim \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| h_+ (c^+, c^-) \|_{L^2} \| \nabla c^+ \nabla (u^+) g^i \|_{L^2} \, d\tau \]
\[ \lesssim \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| \nabla c^+ \|_{B_{2,1}^0} \| \nabla (u^+) g^i \|_{B_{2,1}^0} \, d\tau \]
\[ \quad + \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| (1 + X^2(t))^\frac{1}{2} \|_{B_{2,1}^0} \| \nabla c^+ \|_{B_{2,1}^0} \| \nabla (u^+) g^i \|_{B_{2,1}^0} \, d\tau \]
\[ \overset{\text{def}}{=} L_1 + L_2 \, . \]

We bound the two terms $L_1$ and $L_2$ as follows respectively, from (5.7), we have
\[ L_1 \lesssim \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| \nabla c^+ \|_{B_{2,1}^0} \| \nabla u^+ \|_{B_{2,1}^0} \, d\tau \]
\[ \lesssim \left( \sup_{0 \leq t \leq T} \langle \tau \rangle^\frac{N}{2} \| \nabla c^+ (\tau) \|_{B_{2,1}^0} \right) \left( \sup_{0 \leq t \leq T} \langle \tau \rangle^\frac{N}{2} \| \nabla u^+ (\tau) \|_{B_{2,1}^0} \right) \]
\[ \times \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \langle \tau \rangle^{-\frac{N}{2}+\frac{1}{2}} \, d\tau \]
\[ \lesssim \langle t \rangle^{-\frac{N}{2}+\frac{1}{2}} (D^2(t) + X^2(t)) \, , \]

and
\[ L_2 \lesssim \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| (1 + X^2(t))^\frac{1}{2} \|_{B_{2,1}^0} \| \nabla c^+ \|_{B_{2,1}^0} \| \nabla u^+ \|_{B_{2,1}^0} \, d\tau \]
\[ \lesssim (1 + X^2(t))^\frac{1}{2} \left( \sup_{0 \leq t \leq T} \langle \tau \rangle^\frac{N}{2} \| (c^+, c^-)(\tau) \|_{B_{2,1}^0} \right) \]
\[ \times \langle \tau \rangle^\frac{N}{2} \| \nabla c^+ (\tau) \|_{B_{2,1}^0} \| \nabla u^+ (\tau) \|_{B_{2,1}^0} \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \langle \tau \rangle ^{-\frac{N}{2}+\frac{1}{2}} \, d\tau \]
\[ \lesssim \langle t \rangle^{-\frac{N}{2}+\frac{1}{2}} (1 + X^2(t))^\frac{1}{2} (D^2(t) + D^3(t) + X^4(t)) \, . \]

Thus,
\[ \int_0^t (t-\tau)^{-\frac{N}{2}+\frac{1}{2}} \| \mu^+ h_+ (c^+, c^-) \partial c^+ \partial (u_i^+) g^i \|_{L^2} \, d\tau \]
\[ \lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} (1 + X^2(t))^{\left(\frac{N}{4} + 1\right)} \left(2D^2(t) + X^2(t) + D^3(t) + X^4(t)\right). \] (5.15)

Regarding the term \( M \defeq M_1 + M_2 \), we also get,

\[
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \| h_+^\pm (c^\pm) \partial_x^\pm \partial_j (u^+_j) \|_{L^1} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \| h_+^\pm (c^\pm) \partial_x^\pm \partial_j (u^+_j) \|_{L^2} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \left(1 + (1 + X^2(t))^{\left(\frac{N}{4} + 1\right)} \| (c^+, c^-)(\tau) \|_{B_{2,1}^N} \right) \\
\times \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \| \nabla u^+ (\tau) \|_{B_{1,1}^N} \ d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \| \nabla u^+ (\tau) \|_{B_{2,1}^N} \ d\tau \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} (1 + X^2(t))^{\left(\frac{N}{4} + 1\right)} \\
\times \| (c^+, c^-)(\tau) \|_{B_{2,1}^N} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \| \nabla u^+ (\tau) \|_{B_{1,1}^N} \ d\tau \\
\defeq M_1 + M_2.
\end{align*}
\]

We deal with the two terms \( M_1 \) and \( M_2 \) in the following, if \( t \geq 2 \),

\[
M_1 \lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \| \nabla u^+ (\tau) \|_{B_{2,1}^N} \ d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \| \nabla u^+ (\tau) \|_{B_{2,1}^N} \ d\tau \\
\defeq M_{11} + M_{12}.
\]

Using the definitions of \( X(t) \) and \( D(t) \), we obtain

\[
M_{11} \lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \sup_{0 \leq \tau \leq 1} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \int_0^1 \| \nabla u^+ (\tau) \|_{B_{2,1}^N} \ d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} D(1) X(1),
\]

and, employing (5.7) and the fact that \( \langle \tau \rangle \approx \tau \) when \( \tau \geq 1 \), we have

\[
M_{12} \lesssim \left( \sup_{1 \leq \tau \leq t} \| \nabla c^+ (\tau) \|_{B_{2,1}^N} \right) \left( \sup_{1 \leq \tau \leq t} \| \tau \nabla u^+ (\tau) \|_{B_{2,1}^N} \right) \int_1^t \langle t - \tau \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} \langle \tau \rangle^{-\left(\frac{N}{4} + 1\right)} \ d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{N}{4} + \frac{1}{2}\right)} D^2(t).
\]

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For the term $M_2$, we have
\[
M_2 \lesssim \int_0^1 \langle t - \tau \rangle^{-\frac{3}{2} + \frac{5}{2}} \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( c^+, c^- \right) (\tau) \| \nabla c^+ (\tau) \|_{\tilde{\mathcal{H}}_{2,1}} \| \nabla u^+ (\tau) \|^h_{\tilde{\mathcal{H}}_{2,1}} d\tau \\
+ \int_1^t \langle t - \tau \rangle^{-\frac{3}{2} + \frac{5}{2}} \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( c^+, c^- \right) (\tau) \| \nabla c^+ (\tau) \|_{\tilde{\mathcal{H}}_{2,1}} \| \nabla u^+ (\tau) \|^h_{\tilde{\mathcal{H}}_{2,1}} d\tau \\
def= M_{21} + M_{22}.
\]

Remembering the definitions of $X(t)$ and $D(t)$, we obtain
\[
M_{21} \lesssim \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( X^2(1) + D^2(1) \right),
\]
and
\[
M_{22} \lesssim \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( \sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha || \nabla u^+ (\tau) \|_{\tilde{\mathcal{H}}_{2,1}} \right) \int_1^t \langle t - \tau \rangle^{-\frac{3}{2} + \frac{5}{2}} \langle \tau \rangle^{-\frac{3}{2} + \frac{5}{2}} d\tau \\
\lesssim \langle t \rangle^{-\frac{3}{2} + \frac{5}{2}} \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right) .
\]

Therefore, for $t \geq 2$, we obtain
\[
\int_0^t \langle t - \tau \rangle^{-\frac{3}{2} + \frac{5}{2}} || \mu^+ h_\tau (c^+, c^-) \partial e^+ \partial_l (a^+)^\delta (\tau) ||_{L^1} d\tau \\
\lesssim \langle t \rangle^{-\frac{3}{2} + \frac{5}{2}} \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right) . \tag{5.16}
\]

The case $t \leq 2$ is obvious as $\langle t \rangle \approx 1$ and $\langle t - \tau \rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and
\[
\int_0^t || \mu^+ h_\tau (c^+, c^-) \partial e^+ \partial_l (a^+)^\delta (\tau) ||_{L^1} d\tau \\
\lesssim \| h_\tau (c^+, c^-) \|_{L^\infty (U_2)} \| \nabla c^+ \nabla (u^+)^h \|_{L^1(U_2)} \\
\lesssim \left[ \| h_\tau (c^+, c^-) \|_{L^\infty (U_2)} \right] \frac{\langle (C^2 - \alpha^-)(1, 1) \rangle}{\langle \tau \rangle^\alpha (1, 1) \| \nabla c^+ \|_{L^\infty (d \gamma_{2,1})} \| \nabla u^+ \|^h_{L^1(d \mathcal{H}^h_{2,1})}} \\
\lesssim \left[ \left( 1 + (1 + X^2(t)) \left[ \frac{1}{2} + 1 \right] || \nabla c^+ \|_{L^\infty (d \gamma_{2,1})} \| \nabla u^+ \|^h_{L^1(d \mathcal{H}^h_{2,1})} \right) \frac{\langle (C^2 - \alpha^-)(1, 1) \rangle}{\langle \tau \rangle^\alpha (1, 1) \| \nabla c^+ \|_{L^\infty (d \gamma_{2,1})} \| \nabla u^+ \|^h_{L^1(d \mathcal{H}^h_{2,1})}} \right] \\
\lesssim \left( 1 + X^2(t) \right) \left[ \frac{1}{2} + 1 \right] \left( X^2(t) + D^2(t) + D^3(t) \right) . \tag{5.17}
\]
From (5.15)–(5.17), we finally conclude that
\[
\begin{align*}
\int_0^t \| \mu^+ h_+ (c^+, c^-) & \partial_x c^+ \partial_x u^+ (\tau) \|_{L^1} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} & \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right). 
\end{align*}
\] (5.18)

Similarly, we also obtain the corresponding estimates of other terms \( \mu^+ k_+ (c^+, c^-) \partial_x c^+ \partial_x u^+ \), \( \mu^+ h_+ (c^+, c^-) \partial_x c^+ \partial_x u^+ \), \( \mu^+ k_+ (c^+, c^-) \partial_x c^+ \partial_x u^+ \), \( \mu^+ h_+ (c^+, c^-) \partial_x c^+ \partial_x u^+ \), and \( \lambda^+ k_+ (c^+, c^-) \partial_x c^+ \partial_x u^+ \). Here, we omit the details.

From the low–high frequency decomposition for \( \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \), we have
\[
\mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ = \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \mu^+ + \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \mu^+ ,
\]
where \( l_+ \) stands for some smooth function vanishing at 0. Thus,
\[
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} & \| \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \mu^+ \|_{L^1} \, d\tau \\
\lesssim \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} & \left( \sup \langle \tau \rangle^{\frac{\mu \beta}{2}} \| (c^+, c^-)(\tau) \|_{B_{2,1}^0} \right) \left( \sup \langle \tau \rangle^{\frac{k}{2} + 1} \| \nabla^2 u^+ \|_{B_{2,1}^0} \right) \\
\times \int_0^t \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right) + \left(\frac{k}{2} + 1\right)} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} & \left( X^2(t) + D^2(t) \right). 
\end{align*}
\] (5.19)

To handle the term \( \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \mu^+ \), we consider the cases \( t \geq 2 \) and \( t \leq 2 \) respectively. When \( t \geq 2 \), then we have
\[
\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} & \| \mu^+ l_+ (c^+, c^-) \partial_x^2 u^+ \mu^+ \|_{L^1} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \| l_+ (c^+, c^-) \|_{B_{2,1}^0} \| \nabla^2 u^+ \|_{B_{2,1}^0} \, d\tau \\
\lesssim \int_0^1 \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} & \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} \| (c^+, c^-) \|_{B_{2,1}^0} \| \nabla^2 u^+ \|_{B_{2,1}^0} \, d\tau \\
+ \int_0^1 \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} & \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} \| (c^+, c^-) \|_{B_{2,1}^0} \| \nabla^2 u^+ \|_{B_{2,1}^0} \, d\tau \\
\overset{\text{def}}{=} N_1 + N_2,
\end{align*}
\]

From the definitions of \( X(t) \) and \( D(t) \), we obtain
\[
N_1 = \int_0^1 \langle t - \tau \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} \| (c^+, c^-) \|_{B_{2,1}^0} \| \nabla^2 u^+ \|_{B_{2,1}^0} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} \left( \sup \| (c^+, c^-) \|_{B_{2,1}^0} \right) \int_0^1 \| u^+ \|_{B_{2,1}^0} \, d\tau \\
\lesssim \langle t \rangle^{-\left(\frac{\mu \beta}{2} + \frac{1}{2}\right)} \left(1 + X^2(t)\right)^{\frac{k}{2} + 1} D(1) X(1),
\]

and, using the fact that \((\tau) \approx \tau\) when \(\tau \geq 1\), from (5.7) we have

\[
N_2 = \int_1^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left[ \left( c^+, c^- \right) \|u_\mu \|_{B_{L^1}^2} \right] \|\nabla u^+\|_{B_{L^1}^2} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( \sup_{0 \leq \tau \leq t} \|u_\mu \|_{B_{L^1}^2} \right) \left( \sup_{0 \leq \tau \leq t} \|\nabla u(\tau)\|_{B_{L^1}^2} \right) \int_1^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \left( \tau \right)^{-\frac{N}{2} + \frac{3}{2}} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( (D^2(t) + X^2(t)) \right).
\]

Thus, for \(t \geq 2\), we arrive at

\[
\int_0^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \left\| \mu^+ L_+(c^+, c^-) \partial_j \mu^+ \right\|_{L^1} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( \sup_{\tau \in [0,1]} \|\nabla u(\tau)\|_{B_{L^1}^2} \right) \int_0^T \left\| u^+ \right\|_{B_{L^1}^2} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} (D(t)X(t)).
\]

From (5.19)–(5.21), we get

\[
\int_0^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \left\| \mu^+ L_+(c^+, c^-) \partial_j \mu^+ \right\|_{L^1} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( (D^2(t) + X^2(t)) \right).
\]

Similarly,

\[
\int_0^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \left\| (\mu^+ + \lambda^+) L_+(c^+, c^-) \partial_j \mu^+ \right\|_{L^1} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( (D^2(t) + X^2(t)) \right).
\]

Thus,

\[
\int_0^T (t - \tau)^{-\frac{N}{2} + \frac{3}{2}} \| H_2(\tau)\|_{B_{L^1}^2} \, d\tau
\]

\[
\lesssim \left( 1 + X^2(t) \right)^{\frac{N}{2} + 1} \left( (X^2(t) + D^2(t) + D^3(t) + D^4(t)) \right).
\]
The term $H_4$ may be treated along the same lines, and we have
\[
\int_0^t (t - \tau)^{-\frac{1}{2}} \left\| H_4(\tau) \right\|_{\dot{B}_{\infty, \infty}^{\frac{\alpha}{2}}} \, d\tau \\
\lesssim (t)^{-\frac{1}{2}} \left( 1 + X^2(t) \right)^{\frac{3}{8}} \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right). 
\]

Thus, we complete the proof of (5.5).

Combining with (5.4) and (5.5), we conclude that for all $t \geq 0$ and $s \in (\varepsilon - \frac{N}{2}, 2]$,
\[
(t)^{\frac{N}{2} + \frac{5}{2}} \left\| (c^+, u^+, c^-, u^-) \right\|_{\dot{B}_{2,1}^\infty} \lesssim D_0 + \left( 1 + X^2(t) \right)^{\frac{3}{8}} \left( X^2(t) + D^2(t) + D^3(t) + D^4(t) \right). 
\]

(5.24)

**Step 2. High frequencies**

This step is devoted to bounding the last term of $D(t)$. We first introduce the following system in terms of the weighted unknowns the term $(t^\alpha c^+, t^\alpha u^+, t^\alpha c^-, t^\alpha u^-)$

\[
\begin{aligned}
\partial_t (t^\alpha c^+) + \text{div}(t^\alpha u^+) &= \alpha t^{\alpha-1} c^+ + t^\alpha H_1, \\
\partial_t (t^\alpha u^+) + \beta_1 \nabla (t^\alpha c^+) + \beta_2 \nabla (t^\alpha c^-) - \nu_1 t^\alpha \Delta (t^\alpha u^+) - \nu_2 \text{div}(t^\alpha u^+) &= \alpha t^{\alpha-1} u^+ + t^\alpha H_2, \\
\partial_t (t^\alpha c^-) + \text{div}(t^\alpha u^-) &= \alpha t^{\alpha-1} c^- + t^\alpha H_3, \\
\partial_t (t^\alpha u^-) + \beta_1 \nabla (t^\alpha c^-) + \beta_4 \nabla (t^\alpha c^-) - \nu_1 t^\alpha \Delta (t^\alpha u^-) - \nu_2 \text{div}(t^\alpha u^-) &= \alpha t^{\alpha-1} u^- + t^\alpha H_4.
\end{aligned}
\]

(5.25)

From lemma 3.1, we have
\[
\begin{aligned}
\left\| t^\alpha \left( \sqrt{\beta_1 + \Lambda} c^+, u^+, \sqrt{\beta_4 + \Lambda} c^-, u^- \right) \right\|_{L^p_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \\
\lesssim \left\| (\sqrt{\beta_1 + \Lambda} (\alpha t^{\alpha-1} c^+) + t^\alpha H_1) \right\|_{L^2_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} + \left\| \alpha t^{\alpha-1} u^+ + t^\alpha H_2 \right\|_{L^2_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \\
+ \left\| (\sqrt{\beta_4 + \Lambda} (\alpha t^{\alpha-1} c^- + t^\alpha H_3) \right\|_{L^2_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} + \left\| \alpha t^{\alpha-1} u^- + t^\alpha H_4 \right\|_{L^2_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}}).}
\end{aligned}
\]

(5.26)

We now handle the lower order linear terms on the right-hand side of the above, for $v \in \{ (\sqrt{\beta_1 + \Lambda} c^+, u^+, (\sqrt{\beta_4 + \Lambda} c^-, u^-) \}$, we have
\[
\left\| \alpha t^{\alpha-1} v \right\|_{L^p_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \lesssim \left\| v \right\|_{L^p_t([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \lesssim X(t), \quad \text{for } 0 \leq \tau \leq t \leq 2.
\]

When $t \geq 2$, for $0 \leq \tau \leq 1$, we get
\[
\left\| \alpha t^{\alpha-1} v \right\|_{L^p([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \lesssim \left\| v \right\|_{L^p([0, 1]; \dot{B}_{2,1}^{\frac{\alpha}{2}})} \lesssim X(t).
\]

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When $t \geq 2$, for $1 \leq \tau \leq t$, we have
\[
\|\tau^{\alpha-1}v\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} = \alpha \sum_{j_0} 2^{j_0+1} \|\tau^{\alpha-1} \tilde{\Delta}_j v\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^2)} \\
\lesssim \alpha 2^{-2j_0} \|\tau^{\alpha} \tilde{\Delta}_j v\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^2)} \\
\lesssim \alpha 2^{-2j_0} \|	au^\alpha v\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})}. \tag{5.27}
\]
Choosing $j_0$ large enough such that
\[
C\alpha 2^{-2j_0} \lesssim \frac{1}{4},
\]
which implies that (5.27) may be absorbed by the left-hand side of (5.26). Thus
\[
\|\tau^\alpha \left( (\sqrt{\beta_1} + \Lambda)c^+, u^+, (\sqrt{\beta_2} + \Lambda)c^-, u^- \right) \|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim X(t) + \|\tau^\alpha (H_1, H_2, H_3, H_4)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})}. \tag{5.28}
\]
It now comes down to estimating the above nonlinear terms. We first show the following inequalities which are repeatedly used later. For $\alpha = \frac{1}{4}(N + 1 - \varepsilon)$, we have
\[
\|\tau^\alpha \nabla (c^+, c^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \lesssim \|\tau^\alpha \nabla (c^+, c^-)\|^f_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha \nabla (c^+, c^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim \|\tau^\alpha (c^+, c^-)\|^f_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha \nabla (c^+, c^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim D(t), \tag{5.29}
\]
\[
\|\tau^\alpha (u^+, u^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \lesssim \|\tau^\alpha (u^+, u^-)\|^f_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha (u^+, u^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim \|\tau^\alpha (u^+, u^-)\|^f_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha (u^+, u^-)\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim D(t). \tag{5.30}
\]
For $\|\tau^\alpha H_1\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})}$, from (5.29), (5.30) and proposition 2.6, we have
\[
\|\tau^\alpha H_1\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \lesssim \|\tau^\alpha \text{div } u^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha u^+ \cdot \nabla c^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim \|\tau^\alpha \text{div } u^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|\tau^\alpha u^+ \cdot \nabla c^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim \|c^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \|\tau^\alpha u^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} + \|u^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \|\tau^\alpha \nabla c^+\|^h_{L^\infty((1,t)B_{\mathcal{X}_1}^{\frac{N}{2}+1})} \\
\lesssim X(t)D(t). \tag{5.31}
\]
Similarly,
\[
\|\tau^\alpha H_3\|_{L^\infty_t(H^{k-1}_{2,1})} \lesssim X(t)D(t). \tag{5.32}
\]

Next, we bound the term \(\|\tau^\alpha H_3\|_{L^\infty_t(H^{k-1}_{2,1})}\) as follows. To bound the first part of \(H_2\), employing (5.29), proposition 2.6 and lemma 2.7(a) we write that
\[
\|\tau^\alpha g_+(c^+,c^-)\partial c^+\|_{L^\infty_t(H^{k-1}_{2,1})} \lesssim \|g_+(c^+,c^-)\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \partial c^+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} \|\langle c^+,c^-\rangle\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla c^+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} (D^2(t) + X^2(t)). \tag{5.33}
\]

To bound the term with \((u^+ \cdot \nabla)u^+_+\), from proposition 2.6 and (5.30), we get
\[
\|\tau^\alpha (u^+ \cdot \nabla)u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})} \lesssim \|u^+\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim D^2(t) + X^2(t). \tag{5.34}
\]

Using (5.30), proposition 2.6 and lemma 2.7(a) we deduce that
\[
\|\tau^\alpha \mu^+ h_+(c^+,c^-)\partial c^+ \partial \mu^+\|_{L^\infty_t(H^{k-1}_{2,1})} \lesssim (1 + X^2(t))^{\frac{k}{2} + 1} \|\langle c^+,c^-\rangle\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} \|\langle c^+,c^-\rangle\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} \|\langle c^+,c^-\rangle\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} (D^2(t) + X^2(t)). \tag{5.36}
\]

Similarly, we also obtain the corresponding estimates of other terms \(\mu^+ k_+(c^+,c^-)\partial c^+ \partial \mu^+, \mu^+ h_+(c^+,c^-)\partial c^+ \partial \mu^+, \mu^+ k_+(c^+,c^-)\partial c^+ \partial \mu^+, \lambda^+ k_+(c^+,c^-)\partial c^+ \partial \mu^+\). Here, we omit the details.

From (5.30), proposition 2.6 and lemma 2.7(a) we have
\[
\|\tau^\alpha \mu^+ I_+(c^+,c^-)\partial^2 u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})} \lesssim (1 + X^2(t))^{\frac{k}{2} + 1} \|\langle c^+,c^-\rangle\|_{L^\infty_t(H^{k-1}_{2,1})} \|\tau^\alpha \nabla^2 u^+_+\|_{L^\infty_t(H^{k-1}_{2,1})}
\]
\[
\lesssim (1 + X^2(t))^{\frac{k}{2} + 1} (D^2(t) + X^2(t)). \tag{5.37}
\]
Similarly,
\[
\|\tau^\alpha (\mu^+ + \lambda^+)^j (c^+, c^-) \partial_t \mu_j^+ \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} 
\lesssim (1 + X^2(t))^{\frac{h}{\ell} + 1} \| (c^+, c^-) \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} \| \tau^\alpha \nabla^2 u^+ \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} 
\lesssim (1 + X^2(t))^{\frac{h}{\ell} + 1} (D^2(t) + X^2(t)). \tag{5.38}
\]

Combining with (5.33)–(5.38), we have
\[
\| \tau^\alpha H_4 \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} \lesssim (1 + X^2(t))^{\frac{h}{\ell} + 1} (X^2(t) + D^2(t) + X^3(t) + 4^4(t)). \tag{5.39}
\]

The term \( H_4 \) may be treated along the same lines, and we have
\[
\| \tau^\alpha H_4 \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} \lesssim (1 + X^2(t))^{\frac{h}{\ell} + 1} (X^2(t) + D^2(t) + X^3(t) + 4^4(t)). \tag{5.40}
\]

Adding up (5.31), (5.32), (5.39) and (5.40) to (5.28) yields
\[
\| \tau^\alpha \left( (\sqrt{\beta_1} + \Lambda) c^+, u^+ , (\sqrt{\beta_2} + \Lambda) c^- , u^- \right) \|^{\frac{h}{\ell}}_{L^\infty (B_{2,1}^1)} \lesssim X(t) + (1 + X^2(t))^{\frac{h}{\ell} + 1} (X^2(t) + D^2(t) + X^3(t) + 4^4(t)). \tag{5.41}
\]

Finally, combining with (5.24) and (5.41), for all \( t \geq 0 \), we have
\[
D(t) \leq D_0 + X(t) + (1 + X^2(t))^{\frac{h}{\ell} + 1} (X^2(t) + D^2(t) + X^3(t) + 4^4(t)).
\]

As theorem 1.1 ensures that \( X(t) \leq X(0) \) with \( X(0) \) being small, and \( X(0)' = \| (c_0^+, u_0^+, c_0^-, u_0^-) \|^{\frac{h}{\ell}}_{B_{2,1}^1} \leq \| (c_0^+, u_0^+, c_0^-, u_0^-) \|^{\frac{h}{\ell}}_{B_{2,1}^1} \), one can conclude that (1.38) is fulfilled for all time if \( \| (\Lambda R_0^+, u_0^+ , \Lambda R_0^-, u_0^-) \|^{\frac{h}{\ell}}_{B_{2,1}^1} \) and \( D_0 \) are small enough. This completes the proof of theorem 1.2.

6. More decay estimates

In this section, we present some corollaries of theorem 1.2, which implies that the standard optimal \( L^s-L^t \) time decay rates of \( (R^+ - 1, u^+, R^- - 1, u^-) \).

**Corollary 6.1.** The solution \( (R^+ - 1, u^+, R^- - 1, u^-) \) constructed in theorem 1.1 satisfies
\[
\| \Lambda^s (R^+ - 1, R^- - 1) \|_{L^2} \lesssim \left( D_0 + \| (\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-) \|^{\frac{h}{\ell}}_{B_{2,1}^1} \right) \langle t \rangle^{-\frac{N}{2} - \frac{N}{4}}
\]
if \( -N/2 < s \leq \min\{2, N/2\} \),
\[
\| \Lambda^s (u^+, u^-) \|_{L^2} \lesssim \left( D_0 + \| (\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-) \|^{\frac{h}{\ell}}_{B_{2,1}^1} \right) \langle t \rangle^{-\frac{N}{2} - \frac{N}{4}}
\]
if \( -N/2 < s \leq \min\{2, N/2 - 1\} \).
Proof. For the solution \((R^+ - 1, u^+, R^- - 1, u^-)\) constructed in theorem 1.1, applying to homogeneous Littlewood–Paley decomposition for \(R^+ - 1\), we have
\[
\|\Lambda'(R^+ - 1)\|_{L^2} \lesssim \sum_{q \in \mathbb{Z}} \|\hat{\Lambda}_q \Lambda'(R^+ - 1)\|_{L^2} = \|\Lambda'(R^+ - 1)\|_{B^1_{2,1}}.
\]

Based on Bernstein’s inequalities and the low–high frequencies decomposition, we may write
\[
\sup_{t \in [0, T]} \|t^{\frac{N}{2} + \frac{2}{s}} \Lambda'(R^+ - 1)\|_{B^1_{2,1}} \lesssim \|t^{\frac{N}{2} + \frac{2}{s}} \Lambda'(R^+ - 1)\|_{L_t^T(B^1_{2,1})} + \|\Lambda'(R^+ - 1)\|_{L_t^T(B^1_{2,1})},
\]

It follows from inequality (1.38) and the definitions of \(D(t)\) and \(\alpha\) that
\[
\|t^{\frac{N}{2} + \frac{2}{s}} \Lambda'(R^+ - 1)\|_{L_t^T(B^1_{2,1})} \lesssim D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}} \quad \text{if } -N/2 < s \leq 2.
\]

On the other hand, for \(\alpha \geq \frac{N}{2} + \frac{2}{s}, s \leq \min\{2, N/2\}\), by (5.6) we have
\[
\|t^{\frac{N}{2} + \frac{2}{s}} \Lambda'(R^+ - 1)\|_{B^1_{2,1}} \lesssim X(t) + D(t) \quad \text{if } s \leq \min\{2, N/2\}.
\]

Furthermore, from theorem 1.1 we have \(X(t) \lesssim X(0)\), and \(X(0) = \|\left(c_0^+, u_0^+, c_0^-, u_0^-\right)\|_{B^1_{2,1}} \lesssim \|\left(c_0^+, u_0^+, c_0^-, u_0^-\right)\|_{B^1_{2,1}}\). Combining with (1.38), we get
\[
\|t^{\frac{N}{2} + \frac{2}{s}} \Lambda'(R^+ - 1)\|_{L_t^T(B^1_{2,1})} \lesssim D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}} \quad \text{if } s \leq \min\{2, N/2\}.
\]

Thus, we obtain the following desired result for \(R^+ - 1\)
\[
\|\Lambda'(R^+ - 1)\|_{L^2} \lesssim \left(D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}}\right)\langle t \rangle^{-\frac{N}{2} - \frac{2}{s}}
\]

if \(-N/2 < s \leq \min\{2, N/2\}\).

Similarly,
\[
\|\Lambda^+ u^+\|_{L^2} \lesssim \left(D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}}\right)\langle t \rangle^{-\frac{N}{2} - \frac{2}{s}}
\]

if \(-N/2 < s \leq \min\{2, N/2 - 1\}\),

\[
\|\Lambda'(R^- - 1)\|_{L^2} \lesssim \left(D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}}\right)\langle t \rangle^{-\frac{N}{2} - \frac{2}{s}}
\]

if \(-N/2 < s \leq \min\{2, N/2\}\),

\[
\|\Lambda^- u^-\|_{L^2} \lesssim \left(D_0 + \|\left(\nabla R_0^+, u_0^+, \nabla R_0^-, u_0^-\right)\|_{B^1_{2,1}}\right)\langle t \rangle^{-\frac{N}{2} - \frac{2}{s}}
\]

if \(-N/2 < s \leq \min\{2, N/2 - 1\}\).
This completes the proof of the corollary.

**Remark 6.2.** In corollary 6.1, taking $s = 0, 1$ leads back to the standard optimal $L^1 - L^2$ time decay rates (1.9) and (1.10) in [19] and (1.13) and (1.14) in [13] when $N = 3$, which is also consistent with the optimal time decay rates (1.6) and (1.7) for a single phase flow model in [45, 46]. Note however that our estimates hold in the $L^2$ critical framework. Meanwhile, we also obtain the optimal time decay rates for $(R^+ - 1, R^- - 1)$ in (1.3), which improves the decay results (1.11) and (1.12) in [13].

As a consequence of the Gagliardo–Nirenberg type inequalities and corollary 6.1, similar to the optimal time decay rates (1.4) and (1.8) for single phase flow models, one can also get the following more $L^q - L^r$ decay estimates:

**Corollary 6.3.** Let the assumptions of theorem 1.2 be fulfilled. Then the corresponding solution $(R^+ - 1, u^+, R^- - 1, u^-)$ constructed in theorem 1.1 satisfies

$$
\| \Lambda^k (R^+ - 1, u^+, R^- - 1, u^-) \|_{L^p} \lesssim \left( D_0 + \| (\nabla R^+_0, u^+_0, \nabla R^-_0, u^-_0) \|_{B^{s+1}_{2,1}} \right) \left( t \right)^{-\frac{N}{2} \frac{1}{p} - \frac{1}{2}},
$$

(6.1)

for all $2 \leq p \leq \infty$ and $k \in \mathbb{R}$ satisfying $-\frac{N}{2} < k + N \left( \frac{1}{2} - \frac{1}{p} \right) < \min \left( 2, \frac{N}{2} - 1 \right)$.

**Proof.** For the following Gagliardo–Nirenberg type inequalities [44]

$$
\| \Lambda^k f \|_{L^p} \lesssim \| \Lambda^m f \|_{L^q} \| \Lambda^n f \|_{L^r},
$$

whenever $0 \leq \theta \leq 1, 1 \leq q \leq p \leq \infty$ and

$$
k + N \left( \frac{1}{q} - \frac{1}{p} \right) = m(1 - \theta) + n\theta,
$$

we take $q = 2, m = \min(2, \frac{N}{2} - 1), n = -\frac{N}{2} + \epsilon$ with $\epsilon$ small enough and define $\theta$ by the relation

$$
n\theta + m(1 - \theta) = k + N \left( \frac{1}{2} - \frac{1}{p} \right).
$$

Thus, from corollary 6.1 we have

$$
\| \Lambda^k (R^+ - 1, u^+, R^- - 1, u^-) \|_{L^p} \lesssim \| \Lambda^m (R^+ - 1, u^+, R^- - 1, u^-) \|_{L^2}^{1-\theta} \| \Lambda^n (R^+ - 1, u^+, R^- - 1, u^-) \|_{L^2}^\theta
$$

$$
\lesssim \left( D_0 + \| (\nabla R^+_0, u^+_0, \nabla R^-_0, u^-_0) \|_{B^{s+1}_{2,1}} \right) \left( t \right)^{-\frac{N}{2} \frac{1}{p} - \frac{1}{2}} \left( t \right)^{-\frac{N}{2} + \epsilon}
$$

$$
= \left( D_0 + \| (\nabla R^+_0, u^+_0, \nabla R^-_0, u^-_0) \|_{B^{s+1}_{2,1}} \right) \left( t \right)^{-\frac{N}{2} \frac{1}{p} - \frac{1}{2} + \epsilon},
$$

which completes the proof of the corollary.
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