A quantum linguistic characterization of the reverse relation between confidence interval and hypothesis testing

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Abstract

Although there are many ideas for the formulations of statistical hypothesis testing, we consider that the likelihood ratio test is the most reasonable and orthodox. However, it is not handy, and thus, it is not usual in elementary books. That is, the statistical hypothesis testing written in elementary books is different from the likelihood ratio test. Thus, from the theoretical point of view, we have the following question:

• What is the statistical hypothesis testing written in elementary books?

For example, we consider that even the difference between "one sided test" and "two sided test" is not clear yet. In this paper, we give an answer to this question. That is, we propose a new formulation of statistical hypothesis testing, which is contrary to the confidence interval methods. In other words, they are two sides of the same coin. This will be done in quantum language (or, measurement theory), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics, and also, a kind of system theory such that it is applicable to both classical and quantum systems. Since quantum language is suited for theoretical arguments, we believe that our results are essentially final as a general theory.

Key words: Quantum language, Statistical hypothesis testing, Confidence interval, Chi-squared distribution, Student’s t-distribution

1 Introduction

1.1 Quantum language (Axioms and Interpretation)

As mentioned in the above abstract, our purpose is to answer the following question:

(A) What is the statistical hypothesis testing written in elementary books?

This will be answered in terms of quantum language.

According to ref. [8], we shall mention the overview of quantum language (or, measurement theory, in short, MT).

Quantum language is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics(cf. ref. [5], [10]). Quantum language (or, measurement theory ) has two simple rules (i.e. Axiom 1(concerning measurement) and Axiom 2(concerning causal relation)) and the linguistic interpretation (= how to use the Axioms 1 and 2). That is,

\[
\text{Quantum language (MT(measurement theory))} = \text{Axiom 1(measurement)} + \text{Axiom 2(causality)} + \text{linguistic interpretation (how to use Axioms)}
\]

(cf. refs. [2]-[9]).

This theory is formulated in a certain $C^*$-algebra $\mathcal{A}$ (cf. ref. [11]), and is classified as follows:
(B) MT \[ \text{quantum MT} \quad \text{when } \mathcal{A} \text{ is non-commutative} \]
\[ \text{classical MT} \quad \text{when } \mathcal{A} \text{ is commutative, i.e., } \mathcal{A} = C_0(\Omega) \]

where $C_0(\Omega)$ is the $C^*$-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space $\Omega$.

Since our concern in this paper is concentrated to the usual statistical hypothesis test methods in statistics, we devote ourselves to the commutative $C^*$-algebra $C_0(\Omega)$, which is quite elementary. Therefore, we believe that all statisticians can understand our assertion (i.e., a new viewpoint of the confidence interval methods).

Let $\Omega$ be a locally compact Hausdorff space, which is also called a state space. An element $\omega (\in \Omega)$ is said to be a state. Let $C(\Omega)$ be the $C^*$-algebra composed of all bounded continuous complex-valued functions on a locally compact Hausdorff space $\Omega$. The norm $\| \cdot \|_{C(\Omega)}$ is usual, i.e., $\| f \|_{C(\Omega)} = \sup_{\omega \in \Omega} |f(\omega)| (\forall f \in C(\Omega))$.

Motivated by Davies' idea (cf. ref. [1]) in quantum mechanics, an observable $O = (X, F, F)$ in $C_0(\Omega)$ (or, precisely, in $C(\Omega)$) is defined as follows:

$(C_1)$ $X$ is a topological space, $F \subseteq 2^X$ (i.e., the power set of $X$) is a field, that is, it satisfies the following conditions (i)–(iii): (i): $\emptyset \in F$, (ii): $\exists \Xi \in F \implies X \setminus \Xi \in F$, (iii): $\Xi_1, \Xi_2, \ldots, \Xi_n \in F \implies \cup_{k=1}^n \Xi_k \in F$.

$(C_2)$ The map $F : F \to C(\Omega)$ satisfies that

$0 \leq [F(\Xi)](\omega) \leq 1, \quad [F(X)](\omega) = 1 \quad (\forall \omega \in \Omega)$

and moreover, if

$\Xi_1, \Xi_2, \ldots, \Xi_n, \ldots \in F, \quad \Xi_m \cap \Xi_n = \emptyset \quad (m \neq n), \quad \Xi = \cup_{k=1}^\infty \Xi_k \in F,$

then, it holds

$[F(\Xi)](\omega) = \lim_{n \to \infty} \sum_{k=1}^n [F(\Xi_k)](\omega) \quad (\forall \omega \in \Omega)$

Note that Hopf extension theorem (cf. ref. [12]) guarantees that $(X, F, [F(\cdot)](\omega))$ is regarded as the mathematical probability space.

**Example 1** [Normal observable]. Let $\mathbb{R}$ be the set of the real numbers. Consider the state space $\Omega = \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ = \{ \sigma \in \mathbb{R} | \sigma > 0 \}$. Define the normal observable $O_N = (\mathbb{R}, B_{\mathbb{R}}, N)$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$ such that

$[N(\Xi)](\omega) = -\frac{1}{\sqrt{2\pi} \sigma} \int_{\Xi} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} dx \quad (\forall \Xi \in B_{\mathbb{R}} (=\text{Borel field in } \mathbb{R}), \quad \forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+).$

(2)

In this paper, we devote ourselves to the normal observable.

Now we shall briefly explain "quantum language (1)" in classical systems as follows: A measurement of an observable $O = (X, F, F)$ for a system with a state $\omega (\in \Omega)$ is denoted by $M_{C_0(\Omega)}(O, S[\omega])$. By the measurement, a measured value $x(\in X)$ is obtained as follows:

**Axiom 1** (Measurement)

- The probability that a measured value $x (\in X)$ obtained by the measurement $M_{C_0(\Omega)}(O \equiv (X, F, F), S[\omega_0])$ belongs to a set $\Xi (\in F)$ is given by $[F(\Xi)](\omega_0)$.

**Axiom 2** (Causality)

- The causality is represented by a Markov operator $\Phi_{21} : C_0(\Omega_2) \to C_0(\Omega_1)$. Particularly, the deterministic causality is represented by a continuous map $\pi_{12} : \Omega_1 \to \Omega_2$. 

Interpretation (Linguistic interpretation). Although there are several linguistic rules in quantum language, the following is the most important:

- Only one measurement is permitted.

In order to read this paper, it suffices to understand the above three. For the further arguments, see refs. [2]-[9].

Consider measurements \( M_{C_0(\Omega)}(O_k \equiv (X_k, F_k, F_k), S_{[\omega]}), (k = 1, 2, \ldots, n) \). However, the linguistic interpretation says that only one measurement is permitted. Thus we must consider a simultaneous measurement or a parallel measurement.

**Definition 1** (i): Simultaneous observable. Let \( O_k \equiv (X_k, F_k, F_k) \) be an observable in \( \Omega \). The simultaneous observable \( \times_{k=1}^n O_k \equiv (\times_{k=1}^n X_k, \times_{k=1}^n F_k, \times_{k=1}^n F_k) \) in \( \Omega \) is defined by

\[
[\hat{F}(\Xi_1 \times \cdots \times \Xi_n)](\omega)(\equiv [(\times_{k=1}^n F_k)(\Xi_1 \times \cdots \times \Xi_n)](\omega)) = \times_{k=1}^n [F_k(\Xi_k)](\omega)
\]

\[(\forall \Xi_k \in F_k, (k = 1, \ldots, n), \forall \omega \in \Omega)\]

Here, \( \times_{k=1}^n F_k \) is the smallest field including the family \( \{\times_{k=1}^n \Xi_k : \Xi_k \in F_k \} \). If \( O \equiv (X,F,F) \) is equal to \( O_k \equiv (X_k, F_k, F_k) \) \((k = 1, 2, \ldots, n)\), then the simultaneous observable \( \times_{k=1}^n O_k \equiv (\times_{k=1}^n X_k, \times_{k=1}^n F_k, \times_{k=1}^n F_k) \) is denoted by \( O^n \equiv (X^n, F^n, F^n) \).

(ii): Parallel observable. Let \( O_k \equiv (X_k, F_k, F_k) \) be an observable in \( \Omega(n) \), \((k = 1, 2, \ldots, n)\). The parallel observable \( \times_{k=1}^n O_k \equiv (\times_{k=1}^n X_k, \times_{k=1}^n F_k, \times_{k=1}^n F_k) \) in \( \Omega(n) \) is defined by

\[
[\hat{F}(\Xi_1 \times \cdots \times \Xi_n)](\omega_1, \omega_2, \ldots, \omega_n)(\equiv [(\times_{k=1}^n F_k)(\Xi_1 \times \cdots \times \Xi_n)](\omega_1, \omega_2, \ldots, \omega_n)) = \times_{k=1}^n [F_k(\Xi_k)](\omega_k)
\]

\[(\forall \Xi_k \in F_k, (k = 1, \ldots, n))\]

**Definition 2** [Image observable]. Let \( O \equiv (X,F,F) \) be observables in \( \Omega(n) \). The observable \( f(O) \equiv (Y,G,G(\equiv F \circ f^{-1})) \) in \( \Omega(n) \) is called the image observable of \( O \) by a map \( f : X \rightarrow Y \), if it holds that

\[
G(\Gamma) = F(f^{-1}(\Gamma)) \quad (\forall \Gamma \in G)
\]

**Example 2** [Simultaneous normal observable]. Let \( O_n = (\mathbb{R}, B_{\mathbb{R}}, N) \) be the normal observable in \( \Omega(n) \) in Example 1. Let \( n \) be a natural number. Then, we get the simultaneous normal observable \( O^n_N = (\mathbb{R}^n, B^n_{\mathbb{R}}, N^n) \) in \( \Omega(n) \). That is,

\[
[N^n(\times_{k=1}^n \Xi_k)](\omega) = \times_{k=1}^n [N(\Xi_k)](\omega)
\]

\[
= \frac{1}{(\sqrt{2\pi}^n)^n} \int \cdots \int \exp\left[ -\sum_{k=1}^n (x_k - \mu)^2 / 2\sigma^2 \right] dx_1 dx_2 \cdots dx_n
\]

\[(\forall \Xi_k \in B_{\mathbb{R}}(k = 1, \ldots, n), \forall \omega \in \Omega \in \mathbb{R} \times \mathbb{R}_+).\]

Consider the maps \( \overline{f} : \mathbb{R}^n \rightarrow \mathbb{R}, \overline{SS} : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( \overline{\sigma} : \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[
\overline{f}(x) = \overline{f}(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad (\forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n)
\]

\[
\overline{SS}(x) = \overline{SS}(x_1, x_2, \ldots, x_n) = \sum_{k=1}^n (x_k - \overline{f}(x))^2 \quad (\forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n)
\]

\[
\overline{\sigma}(x) = \overline{\sigma}(x_1, x_2, \ldots, x_n) = \sqrt{\frac{\sum_{k=1}^n (x_k - \overline{f}(x))^2}{n}} \quad (\forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n)
\]
Thus, we have two image observables $\mathfrak{P}(O^0_N) = (\mathbb{R}, B_\mathbb{R}, N^0 \circ \mathfrak{P}^{-1})$ and $\mathfrak{SS}(O^0_N) = (\mathbb{R}_+, B_{\mathbb{R}_+}, N^0 \circ \mathfrak{SS}^{-1})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. 

It is easy to see that

$$[(N^0 \circ \mathfrak{P}^{-1})(\Xi_1)](\omega) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\Pi(x) \in \Xi_1} \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \ldots dx_n$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\Xi_1} \exp\left[-\frac{n(x - \mu)^2}{2\sigma^2}\right] dx \quad (9)$$

and

$$[(N^0 \circ \mathfrak{SS}^{-1})(\Xi_2)](\omega) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\Pi(x) \in \Xi_2} \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \ldots dx_n$$

$$= \int_{\Xi_2/\sigma^2} p_{n-1}^2(x) dx \quad \forall \Xi_1 \in B_\mathbb{R}, \forall \Xi_2 \in B_{\mathbb{R}_+}, \forall \omega = (\mu, \sigma) \in \Omega \equiv \mathbb{R} \times \mathbb{R}_+. \quad (10)$$

Here, $p_{n-1}^2(x)$ is the chi-squared distribution with $n - 1$ degrees of freedom. That is,

$$p_{n-1}^2(x) = \frac{x^{(n-1)/2-1}e^{-x/2}}{2^{(n-1)/2}\Gamma((n-1)/2)} \quad (x > 0) \quad (11)$$

where $\Gamma$ is the gamma function.

### 1.2 Fisher’s maximum likelihood method

It is usual to consider that we do not know the pure state $\omega_0 (\in \Omega)$ when we take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$. That is because we usually take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ in order to know the state $\omega_0$. Thus, when we want to emphasize that we do not know the state $\omega_0$, $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ is denoted by $M_{C_0(\Omega)}(O, S_{[\omega_0]})$. Also, if we know (or, postulate) that a state $\omega_0$ belongs to a certain suitable set $K (\subseteq \Omega)$, the $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ is denoted by $M_{C_0(\Omega)}(O, S_{[\omega_0]}(K))$. 

**Theorem 1** [Fisher’s maximum likelihood method (cf. refs. [3], [4])]. Consider a measurement $M_{C_0(\Omega)}(O = (X, F, F), S_{[\omega]}(K))$. Assume that we know that the measured value $x (\in X)$ obtained by a measurement $M_{C_0(\Omega)}(O = (X, F, F), S_{[\omega]}(K))$ belongs to $\Xi (\in F)$. Then, there is a reason to infer that the unknown state $[\omega]$ is equal to $\omega_0 (\in K)$ such that

$$\min_{\omega_1 \in K} \frac{|F(\Xi)(\omega_0)|}{|F(\Xi)(\omega_1)|} \left(= \max_{\omega_1 \in K} \frac{|F(\Xi)(\omega_0)|}{|F(\Xi)(\omega_1)|} \right) = 1 \quad (13)$$

if the righthand side of this formula exists. Also, if $\Xi = \{x\}$, it suffices to calculate the $\omega_0 (\in K)$ such that

$$L(x, \omega_0) = 1$$

where the likelihood function $L(x, \omega) (\equiv L_x(\omega))$ is defined by

$$L(x, \omega) = \inf_{\omega_1 \in K} \left[ \lim_{\Xi \ni \{x\}, |F(\Xi)(\omega_1)| \neq 0} \frac{|F(\Xi)(\omega)|}{|F(\Xi)(\omega_1)|} \right] \quad (14)$$
Example 3 [Fisher’s maximum likelihood method]. Consider the simultaneous normal observable \( O^N = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, N^n) \) in \( C_0(\mathbb{R} \times \mathbb{R}_+) \) in the formula (6). Thus, we have the simultaneous measurement \( M_{C_0(\mathbb{R} \times \mathbb{R}_+)}(O^N) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, N^n, S|_{\{\theta\}}(K)) \) in \( C_0(\mathbb{R} \times \mathbb{R}_+) \). Assume that a measured value \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) is obtained by the measurement. Since the likelihood function \( L_x(\mu, \sigma) = L(x, (\mu, \sigma)) \) is defined by

\[
L_x(\mu, \sigma) = \frac{1}{(2\pi\sigma)^n} \exp\left[ -\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{2\sigma^2} \right]
\]

or, in the sense of (14),

\[
L_x(\mu, \sigma) = \frac{1}{(2\pi\sigma)^n} \exp\left[ -\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{2\sigma^2} \right] \frac{1}{(2\pi\tilde{\sigma}(x))^n} \exp\left[ -\frac{\sum_{k=1}^{n}(x_k - \tilde{\sigma}(x))^2}{2\tilde{\sigma}(x)^2} \right]
\]

(\( \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \ \forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+ \)).

it suffices to calculate the following equations:

\[
\frac{\partial L_x(\mu, \sigma)}{\partial \mu} = 0, \quad \frac{\partial L_x(\mu, \sigma)}{\partial \sigma} = 0
\]  

(16)

For example, assume that \( K = \mathbb{R} \times \mathbb{R}_+ \). Solving the equation (16), we can infer, by Theorem 1 (Fisher’s maximum likelihood method), that \([\ast] = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+\) such that

\[
\mu = \overline{x}(x) = \frac{x_1 + x_2 + \ldots + x_n}{n}, \quad \sigma = \tilde{\sigma}(x) = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \overline{x}(x))^2}{n}} = \sqrt{\frac{n-1}{n} \tilde{\sigma}(x)}
\]  

(17)

1.3 The orthodox characterization of statistical hypothesis testing (the likelihood ratio test)

Our purpose of this paper is to propose a kind of statistical hypothesis test which is characterized as "the reverse confidence reverse" in the following Section 2. However, before it, we mention the standard statistical hypothesis test (i.e., the likelihood ratio test) as follows.

Consider a measurement \( M_{C_0(\Omega)}(O \equiv (X, \mathcal{F}, F), S|_{\{\theta\}}) \) formulated in \( C_0(\Omega) \). Here, we assume that \((X, \tau_X)\) is a topological space, where \( \tau_X \) is the set of all open sets. And assume that \( \mathcal{F} = \mathcal{B}_X \); the Borel field, i.e., the smallest \( \sigma \)-field that contains all open sets in \( X \). Note that we can assume, without loss of generality, that \( F(\Xi) \neq 0 \) for any open set \( \Xi \in \tau_X \) such that \( \Xi \neq \emptyset \). That is because, if \( F(\Xi) = 0 \), it suffices to redefine \( X \) by \( X \setminus \Xi \). Let \( \Theta \) be a locally compact space with the Borel field \( \mathcal{B}_\Theta \). Let \( \pi : \Omega \to \Theta \) be a continuous map, which is a kind of causal relation (in Axiom 2), and called “quantity”, and let \( E : X \to \Theta \) be a continuous (or more generally, measurable) map, which is called “estimator”.

Assume the following hypothesis called “null hypothesis”:

(D) \( \pi(\ast) \) (where \([\ast] \) is the unknown state in \( M_{C_0(\Omega)}(O, S|_{\{\theta\}}) \) belongs to a set \( H_N \subseteq \Theta \).

In short, the set \( H_N \) is also called “null hypothesis”.

In order to deny this hypothesis (D), we define the rejection region \( \hat{R}^\alpha_{H_N} \in \mathcal{B}_\Theta \) as follows.

(E) For sufficiently small significance level \( \alpha \ (0 < \alpha \ll 1 \text{, e.g., } \alpha = 0.05) \), define the rejection region \( \hat{R}^\alpha_{H_N} \in \mathcal{B}_\Theta \) such that

\[
(E_1) \sup_{\omega \in \pi^{-1}(\{\theta\})} [F(E^{-1}(\hat{R}^{\alpha}_{H_N}))](\omega) \leq \alpha \quad (\forall \theta \in H_N(\subseteq \Theta))
\]

\[
(E_2) \text{If } \hat{R}^{\alpha,1}_{H_N}(\in \mathcal{B}_\Theta) \text{ and } \hat{R}^{\alpha,2}_{H_N}(\in \mathcal{B}_\Theta) \text{ satisfy } (E_1) \text{ and } \hat{R}^{\alpha,1}_{H_N} \subseteq \hat{R}^{\alpha,2}_{H_N}, \text{ then, choose } \hat{R}^{\alpha,2}_{H_N}.
\]
Figure 1. Null Hypothesis $H_N$

Then, Axiom 1 says that

(F) if $\pi(*) \in H_N$, the following (F₁) (or, equivalently, (F₂)) holds:

(F₁) the probability that a measured value obtained by $M_{C_0(\Omega)}(O \equiv (X, F, S_\omega))$ belong to $E^{-1}(\hat{R}_H^\alpha)$ is less than or equal to $\alpha$.

(F₂) the probability that a measured value obtained by $M_{C_0(\Omega)}(EO \equiv (\Theta, B_\Theta, F \circ E^{-1}), S_\omega)$ belong to $\hat{R}_H^\alpha$ is less than or equal to $\alpha$.

Therefore, if $\pi(*) \in H_N$, and if $\alpha$ is sufficiently small, then there is a reason to deny the hypothesis (D).

It is clear that the rejection region $\hat{R}_H^\alpha$ is not uniquely determined in general. Thus, we have the following problem:

(G) Find the most proper rejection region $\hat{R}_H^\alpha$.

In what follows, we shall answer this (G) as "the likelihood ratio test".

Let $E(O) \equiv (\Theta, B_\Theta, F \circ E^{-1})$ be the image observable of the $O \equiv (X, F, F)$ in a commutative $C^*$-algebra $C_0(\Omega)$. Define the likelihood function $L : \Theta \times \Omega \rightarrow [0, 1]$ of the image observable $E(O)$ by (14). Let $H_N$ be as in (D). Here define the function $\Lambda_{H_N} : \Theta \rightarrow [0, 1]$ such that:

$$\Lambda_{H_N}(\theta) = \sup_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} L(\theta, \omega) \quad (\forall \theta \in \Theta). \quad (18)$$

Also, for any $\epsilon (0 < \epsilon \leq \frac{1}{2})$, define $R_{\epsilon}^H (\in B_\Theta)$ such that

$$R_{\epsilon}^H = \{\theta \in \Theta \mid \Lambda_{H_N}(\theta) \leq \epsilon\}. \quad (19)$$

Figure 2. $R_{\epsilon}^H$

Consider a positive number $\alpha$ (called a significance level) such that $0 < \alpha \ll 1$ (e.g. $\alpha = 0.05$). Thus we can define $\epsilon(\alpha)$ such that:

$$\epsilon(\alpha) = \sup\{\epsilon \mid \sup_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} [F(E^{-1}(R_{\epsilon}^H))](\omega) \leq \alpha\}. \quad (20)$$

Thus, as our answer to the problem (G), we can assert the following theorem, which is a slight generalization of our result in refs. [4], [8].

**Theorem 2 [Likelihood ratio test].** Assume the above notations. Then, the $R_{\epsilon(\alpha)}^H$ satisfies the condition (F). And thus, the rejection region $\hat{R}_H^\alpha$ is given by $R_{\epsilon(\alpha)}^H$.

We believe that this theorem is the most orthodox answer to Problem (G). However, in Section 2.2, we will propose another answer to Problem (G).
2 The reverse relation between confidence interval method and statistical hypothesis testing

In this main section, we propose a new formulation of the confidence interval methods and statistical hypothesis testing, and show that they can be understood as two sides of the same coin.

2.1 Confidence interval method

Let \( \mathcal{O} = (X, \mathcal{F}, F) \) be an observable formulated in a commutative \( C^* \)-algebra \( C_0(\Omega) \). Let \( \Theta \) be a locally compact space with the semi-distance \( d_x^\Theta (\forall x \in X) \), that is, for each \( x \in X \), the map \( d_x^\Theta : \Theta^2 \to [0, \infty) \) satisfies that (i): \( d_x^\Theta (\theta, \theta) = 0 \), (ii): \( d_x^\Theta (\theta_1, \theta_2) = d_x^\Theta (\theta_2, \theta_1) \), (iii): \( d_x^\Theta (\theta_1, \theta_3) \leq d_x^\Theta (\theta_1, \theta_2) + d_x^\Theta (\theta_2, \theta_3) \).

Let \( \pi : \Omega \to \Theta \) be a continuous map, which is a kind of causal relation (in Axiom 2), and called “quantity”. Let \( E : X \to \Theta \) be a continuous (or more generally, measurable) map, which is called “estimator”.

Theorem 3 [Confidence interval method(cf. ref. [9])]. Let \( \gamma \) be a real number such that \( 0 \ll \gamma < 1 \), for example, \( \gamma = 0.95 \). For any state \( \omega (\in \Omega) \), define the positive number \( \eta_\gamma^\omega \) such that:

\[
\eta_\gamma^\omega = \inf \{ \eta > 0 : \{ x \in X : d_x^\Theta (E(x), \pi(\omega)) < \eta \} \} \geq \gamma
\]

Then we say that:

\( (H_1) \) the probability, that the measured value \( x \) obtained by the measurement \( M_{C_0(\Omega)}(\mathcal{O} := (X, \mathcal{F}, F), S[\omega_0]) \) satisfies the following condition (22), is more than or equal to \( \gamma \) (e.g., \( \gamma = 0.95 \)).

\[
d_x^\Theta (E(x), \pi(\omega_0)) < \eta_\gamma^\omega
\]

And further, put

\[
D_x^\gamma = \{ \pi(\omega) : d_x^\Theta (E(x), \pi(\omega)) < \eta_\gamma^\omega \}
\]

which is called the \((\gamma)\)-confidence interval. Here, we see the following equivalence:

\[
(22) \iff \bigcup_{\omega \in \Omega} \{ x \in X : d_x^\Theta (E(x), \pi(\omega)) < \eta_\gamma^\omega \} \ni \pi(\omega_0).
\]

Figure 3. Confidence interval \( D_x^\gamma \)

The following corollary 1 may not be useful. However, it should be compared with Theorem 4.

Corollary 1 Further, consider a subset \( H_S \) of \( \Theta \), which is called a "sure hypothesis". Put

\[
\hat{D}_H^\gamma = \bigcup_{\omega \in \Omega} \{ x \in X : d_x^\Theta (E(x), \pi(\omega)) < \eta_\gamma^\omega \}
\]

Then we say that:
(H2) the probability, that the measured value \( x \) obtained by the measurement \( M_{C_0(\Omega)}(O := (X, \mathcal{F}, F), S_{[\pi]}(\pi^{-1}(H_S))) \) (cf. (12)) satisfies the following condition (26), is more than or equal to \( \gamma \) (e.g., \( \gamma = 0.95 \)).

\[ \hat{D}_{H_S}^\gamma \ni E(x). \]  

(26)

2.2 Statistical hypothesis testing

The following theorem is our main theorem in this paper, which says that it is contrary to Theorem 3 (the confidence interval method). In other words, they are two sides of the same coin.

**Theorem 4** [Statistical hypothesis testing]. Let \( \alpha \) be a real number such that \( 0 < \alpha \ll 1 \), for example, \( \alpha = 0.05 \). For any state \( \omega (\in \Omega) \), define the positive number \( \eta^\alpha_\omega \) (\( > 0 \)) such that:

\[ \eta^\alpha_\omega = \inf \{ \eta > 0 : [F(\{ x \in X : d_\pi^\alpha(E(x), \pi(\omega)) \geq \eta \})](\omega) \leq \alpha \} \]

\[ = \inf \{ \eta > 0 : [F(\{ x \in X : d_\pi^\alpha(E(x), \pi(\omega)) < \eta \})](\omega) \geq 1 - \alpha \} = \eta^1_{\alpha} \text{ in (21)} \]

(27)

Then we say that:

(I1) the probability, that the measured value \( x \) obtained by the measurement \( M_{C_0(\Omega)}(O := (X, \mathcal{F}, F), S_{[\pi]} \omega) \) satisfies the following condition (28), is less than or equal to \( \alpha \) (e.g., \( \alpha = 0.05 \)).

\[ d_\pi^\alpha(E(x), \pi(\omega_0)) \geq \eta^\alpha_\omega \]  

(28)

Further, consider a subset \( H_N \) of \( \Theta \), which is called a "null hypothesis". Put

\[ \hat{R}_{H_N}^\alpha = \bigcap_{\omega \in \Omega} \{ E(x) \in \Theta : d_\pi^\alpha(E(x), \pi(\omega)) \geq \eta^\alpha_\omega \}, \]

(29)

which is called the \( (\alpha) \)-rejection region of the null hypothesis \( H_N \). Then we say that:

(I2) the probability, that the measured value \( x \) obtained by the measurement \( M_{C_0(\Omega)}(O := (X, \mathcal{F}, F), S_{[\pi]}(\pi^{-1}(H_N))) \) (cf. (12)) satisfies the following condition (30), is less than or equal to \( \alpha \) (e.g., \( \alpha = 0.05 \)).

\[ \hat{R}_{H_N}^\alpha \ni E(x). \]  

(30)

FIGURE 4. Rejection region \( \hat{R}_{H_N}^\alpha \) (when \( H_N = \{ \pi(\omega_0) \} \))

**Remark 1** [The statistical meaning of Theorems 3 and 4]. (i): The \( \hat{D}_{H_S}^\gamma \) in (25) is the compliment of \( \hat{R}_{H_S}^\gamma \), however, Corollary 1 may not be useful.
(ii): Consider the simultaneous measurement \( M_{\mathcal{C}_0(\Omega)}(\mathcal{O}^J := (X^J, \mathcal{F}^J, F^J), S_{\omega}), \) and assume that a measured value \( x = (x_1, x_2, \ldots, x_J) \) is obtained by the simultaneous measurement. Recall the formula (24). Then, it surely holds that
\[
\lim_{J \to \infty} \frac{\text{Num}[\{ j \mid D^\gamma_{\omega} \ni \pi(\omega) \}]}{J} \geq \gamma (= 0.95)
\] (31)
where \( \text{Num}[A] \) is the number of the elements of the set \( A \). Hence Theorem 3 can be tested by numerical analysis (with random number). Similarly, Theorem 4 can be tested.

3 Examples

The arguments in this section are continued from Example 2. Let \( \alpha \) be a real number such that \( 0 < \alpha \ll 1 \), for example, \( \alpha = 0.05 \). From the reverse relation between Theorem 3 (the confidence interval method) and Theorem 4 (statistical hypothesis testing), Examples 4-10 in this section may be essentially the same as the examples of ref. [9].

3.1 Population mean

Example 4 [Rejection region of \( H_N = \{ \mu_0 \} \subseteq \Theta = \mathbb{R} \)]. Consider the simultaneous measurement \( M_{\mathcal{C}_0(\mathbb{R} \times \mathbb{R}^+)}(\mathcal{O}_N = (\mathbb{R}^n, \mathbb{R}_+, N^n), S_{[(\mu, \sigma)]}) \) in \( \mathcal{C}_0(\mathbb{R} \times \mathbb{R}^+) \). Thus, we consider that \( \Theta = \mathbb{R} \times \mathbb{R}^+, X = \mathbb{R}^n \). Assume that the real \( \sigma \) in a state \( \omega = (\mu, \sigma) \in \Omega \) is fixed and known. Put
\[
\Theta = \mathbb{R}
\]
The formula (17) urges us to define the estimator \( E : \mathbb{R}^n \to \Theta(= \mathbb{R}) \) such that
\[
E(x) = E(x_1, x_2, \ldots, x_n) = \overline{x}(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}
\] (32)
And consider the quantity \( \pi : \Omega \to \Theta \) such that
\[
\Omega = \mathbb{R} \times \mathbb{R}^+ \ni \omega = (\mu, \sigma) \mapsto \pi(\omega) = \mu \in \Theta = \mathbb{R}
\]
Consider the following semi-distance \( d^{(1)}_{\Theta} \) in \( \Theta(= \mathbb{R}) \):
\[
d^{(1)}_{\Theta}(\theta_1, \theta_2) = |\theta_1 - \theta_2|
\] (33)
Define the null hypothesis \( H_N \) such that
\[
H_N = \{ \mu_0 \}(\subseteq \Theta(= \mathbb{R}))
\]
For any \( \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}^+ \), define the positive number \( \eta_0^\omega \) \( (> 0) \) such that:
\[
\eta_0^\omega = \sup \{ \eta > 0 : [F,E^{-1}(\text{Ball}^C_{d^{(1)}_{\Theta}}(\pi(\omega); \eta))](\omega) \leq \alpha \}
\]
where \( \text{Ball}^C_{d^{(1)}_{\Theta}}(\pi(\omega); \eta) = \{ \theta(\in \Theta) : d^{(1)}_{\Theta}(\mu, \theta) \geq \eta \} = \left( (0, -\mu + \eta, -\mu + \infty) \cup (0, \mu + \eta, \infty) \right) \)
Hence we see that
\[
E^{-1}(\text{Ball}^C_{d^{(1)}_{\Theta}}(\pi(\omega); \eta)) = E^{-1}\left( (0, -\mu + \eta, -\mu + \infty) \cup (0, \mu + \eta, \infty) \right)
\]
\[
= \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{x_1 + \cdots + x_n}{n} \leq \mu - \eta \text{ or } \mu + \eta \leq \frac{x_1 + \cdots + x_n}{n} \}
\]
\[
= \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{(x_1 - \mu) + \cdots + (x_n - \mu)}{n} \geq \eta \}
\] (34)
Thus,

\[ [N^n(E^{-1}(\text{Ball}_{\Theta_n}^C(\pi(\omega); \eta)))](\omega) \]

\[ = \frac{1}{(\sqrt{2\pi} \sigma)^n} \int \cdots \int_{|x_1+\ldots+x_n-\mu| \geq \eta} \exp\left[-\frac{\sum_{k=1}^n (x_k-\mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \]

\[ = \frac{1}{(\sqrt{2\pi} \sigma)^n} \int \cdots \int_{|x_1+\ldots+x_n| \geq \eta} \exp\left[-\frac{\sum_{k=1}^n (x_k)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \]

\[ = \frac{\sqrt{n}}{(2\pi \sigma)^n} \int_{x \geq \eta} \exp\left[-\frac{n x^2}{2\sigma^2}\right] dx = \frac{1}{\sqrt{2\pi}} \int_{x \geq \sqrt{n}\eta/\sigma} \exp\left[-\frac{x^2}{2}\right] dx \]  

(35)

Solving the following equation:

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z(\alpha/2)} \exp\left[-\frac{x^2}{2}\right] dx = \frac{1}{\sqrt{2\pi}} \int_{z(\alpha/2)}^{\infty} \exp\left[-\frac{x^2}{2}\right] dx = \frac{\alpha}{2} \]  

(36)

we define that

\[ \eta_\alpha^\sigma = \frac{\sigma}{\sqrt{n}} z(\frac{\alpha}{2}) \]  

(37)

Therefore, we get \( \hat{R}_{H_N}^\alpha \) (the (\( \alpha \))-rejection region of \( H_N (= \{ \mu_0 \} \subseteq \Theta (= \mathbb{R}) ) \) as follows:

\[ \hat{R}_{\{\mu_0\}}^\alpha = \bigcap_{\pi(\omega) = \mu_0} \{ E(x) (\in \Theta = \mathbb{R}) : d_{\Theta}^{(1)}(E(x), \pi(\omega)) \geq \eta_\alpha^\sigma \} \]

\[ = \{ E(x) = \frac{x_1 + \ldots + x_n}{n} \in \mathbb{R} : \overline{\mu}(x) - \mu_0 = \frac{x_1 + \ldots + x_n}{n} - \mu_0 \geq \frac{\sigma}{\sqrt{n}} z(\frac{\alpha}{2}) \} \]  

(38)

Remark 2 Note that the \( \hat{R}_{\{\mu_0\}}^\alpha \) (the (\( \alpha \))-rejection region of \( \{ \mu_0 \} ) \) depends on \( \sigma \). Thus, putting

\[ \hat{R}_{\{\mu_0\} \times \mathbb{R}_+}^\alpha = \{ (\overline{\mu}(x), \sigma) \in \mathbb{R} \times \mathbb{R}_+ : |\mu_0 - \overline{\mu}(x)| = |\mu_0 - \frac{x_1 + \ldots + x_n}{n}| \geq \frac{\sigma}{\sqrt{n}} z(\frac{\alpha}{2}) \} \]  

(39)

we see that \( \hat{R}_{\{\mu_0\} \times \mathbb{R}_+}^\alpha = " \)the slash part in Figure 5".

\[ \sigma \]

\[ \hat{R}_{\{\mu_0\} \times \mathbb{R}_+}^\alpha \]

\[ \mu_0 \]  

\[ \mathbb{R} \]

Figure 5. Rejection region \( \hat{R}_{\{\mu_0\}}^\alpha \) (which depends on \( \sigma \)
Example 5 [Rejection region of $H_N = (-\infty, \mu_0] \subseteq \Theta(= \mathbb{R})$. Consider the simultaneous measurement $M_{C_0(\mathbb{R} \times \mathbb{R}_+)}(O_N^\theta = (\mathbb{R}^n, B_{\mathbb{R}^n}, N^n), S_{\{\mu, \sigma\}})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. Thus, we consider that $\Omega = \mathbb{R} \times \mathbb{R}$, $X = \mathbb{R}^n$. Assume that the real $\sigma$ in a state $\omega = (\mu, \sigma) \in \Omega$ is fixed and known. Put 

$$\Theta = \mathbb{R}$$

The formula (17) urges us to define the estimator $E : \mathbb{R}^n \rightarrow \Theta(= \mathbb{R})$ such that

$$E(x) = \overline{\mu}(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$  \hspace{1cm} (40)

And consider the quantity $\pi : \Omega \rightarrow \Theta$ such that

$$\Omega = \mathbb{R} \times \mathbb{R}_+ \ni \omega = (\mu, \sigma) \mapsto \pi(\omega) = \mu \in \Theta = \mathbb{R}$$

Consider the following semi-distance $d_{d_{\Theta}}^{(2)}$ in $\Theta(= \mathbb{R})$:

$$d_{d_{\Theta}}^{(2)}((\theta_1, \theta_2)) = \begin{cases} |	heta_1 - \theta_2| & \theta_0 \leq \theta_1, \theta_2 \\ |	heta_2 - \theta_0| & \theta_1 \leq \theta_0 \leq \theta_2 \\ |	heta_1 - \theta_0| & \theta_2 \leq \theta_0 \leq \theta_1 \\ 0 & \theta_1, \theta_2 \leq \theta_0 \end{cases}$$  \hspace{1cm} (41)

Define the null hypothesis $H_N$ such that

$$H_N = (-\infty, \mu_0](\subseteq \Theta(= \mathbb{R}))$$

For any $\omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+$, define the positive number $\eta_\omega^\alpha$ ($> 0$) such that:

$$\eta_\omega^\alpha = \sup \{ \eta > 0 : |F(E^{-1}(\text{Ball}_{d_{d_{\Theta}}}^{C_0}(\pi(\omega); \eta)))| \leq \alpha \}$$

where $\text{Ball}_{d_{d_{\Theta}}}^{C_0}(\pi(\omega); \eta) = \{ \theta(\in \Theta) : d_{d_{\Theta}}^{(2)}(\mu, \theta) \geq \eta \} = \left( -\infty, \mu - \eta \right] \cup \left[ \mu + \eta, \infty \right) )$

Hence we see that

$$E^{-1}(\text{Ball}_{d_{d_{\Theta}}}^{C_0}(\pi(\omega); \eta)) = E^{-1}\left( [\mu + \eta, \infty) \right)$$

=\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \mu + \eta \leq \frac{x_1 + \cdots + x_n}{n} \}

=\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{(x_1 - \mu) + \cdots + (x_n - \mu)}{n} \geq \eta \}$$  \hspace{1cm} (42)

Thus,

$$[N^n(E^{-1}(\text{Ball}_{d_{d_{\Theta}}}^{C_0}(\pi(\omega); \eta))))(\omega)]$$

$$= \frac{1}{(\sqrt{2\pi})^n} \int_{\frac{x_1 - \mu + \cdots + x_n - \mu}{\sqrt{n}} \geq \eta} \cdots \int_{\frac{x_n - \mu}{\sqrt{n}} \geq \eta} \exp[-\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{2\sigma^2}] dx_1 dx_2 \cdots dx_n$$

$$= \frac{1}{(\sqrt{2\pi})^n} \int_{\frac{x_1 - \mu + \cdots + x_n - \mu}{\sqrt{n}} \geq \eta} \cdots \int_{\frac{x_n - \mu}{\sqrt{n}} \geq \eta} \exp[-\frac{\sum_{k=1}^{n}(x_k)^2}{2\sigma^2}] dx_1 dx_2 \cdots dx_n$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi} \sigma} \int_{|x| \geq \eta} \exp[-\frac{nx^2}{2\sigma^2}] dx = \frac{1}{\sqrt{2\pi}} \int_{|x| \geq \sqrt{n}/\sigma} \exp[-\frac{x^2}{2}] dx$$  \hspace{1cm} (43)

Solving the following equation:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z(\alpha/2)} \exp[-\frac{x^2}{2}] dx + \frac{1}{\sqrt{2\pi}} \int_{z(\alpha/2)}^{\infty} \exp[-\frac{x^2}{2}] dx = \alpha$$  \hspace{1cm} (44)
we define that
\[ \eta_\alpha^0 = \frac{\sigma}{\sqrt{n}} z(\alpha) \] (45)

Therefore, we get \( \hat{R}_{H_N}^\alpha \) (the \((\alpha)\)-rejection region of \( H_N \((= (-\infty, \mu_0] \subseteq \Theta(= \mathbb{R}) \)) \) as follows:
\[
\hat{R}_{\alpha} = \bigcap_{\pi(\omega) \in (-\infty, \mu_0]} \{ E(x) \in \Theta = \mathbb{R} : d_\Theta^{(2)}(E(x), \pi(\omega)) \geq \eta_\alpha^0 \}
\]
\[
\hat{R}_{\alpha} = \{ E(x) = \frac{x_1 + \ldots + x_n}{n} \in \mathbb{R} : \frac{x_1 + \ldots + x_n}{n} - \mu_0 \geq \frac{\sigma}{\sqrt{n}} z(\alpha) \} \] (46)

Thus, in a similar way of Remark 2, we see that \( \hat{R}_{(-\infty, \mu_0] \times \mathbb{R}_+} \) = “the slash part in Figure 6”, where
\[
\hat{R}_{(-\infty, \mu_0] \times \mathbb{R}_+} = \{ (E(x) = \frac{x_1 + \ldots + x_n}{n}, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \frac{x_1 + \ldots + x_n}{n} - \mu_0 \geq \frac{\sigma}{\sqrt{n}} z(\alpha) \} \] (47)

![Figure 6. Rejection region \( \hat{R}_{(-\infty, \mu_0]} \) (which depends on \( \sigma \)](image)

### 3.2 Population variance

**Example 6** [Rejection region of \( H_N = \{ \sigma_0 \} \subseteq \Theta(= \mathbb{R}_+) \). Consider the simultaneous measurement \( M_{C_0(\mathbb{R} \times \mathbb{R}_+) \mathbb{R}_, N^\alpha} \) in \( C_0(\mathbb{R} \times \mathbb{R}_+) \). Thus, we consider that \( \Omega = \mathbb{R} \times \mathbb{R}_+ \), \( X = \mathbb{R}_n \). Assume that the real \( \mu \) in a state \( \omega = (\mu, \sigma) \in \Omega \) is fixed and known. Put
\[
\Theta = \mathbb{R}_+ \]

The formula (17) may urge us to define the estimator \( E : \mathbb{R}_n \rightarrow \Theta(= \mathbb{R}_+) \) such that
\[
E(x) = E(x_1, x_2, \ldots, x_n) = \sigma(x) = \sqrt{\frac{\sum_{k=1}^{n}(x_k - x))^2}{n}} \] (48)

And consider the quantity \( \pi : \Omega \rightarrow \Theta \) such that
\[
\Omega = \mathbb{R} \times \mathbb{R}_+ \ni \omega = (\mu, \sigma) \mapsto \pi(\omega) = \sigma \in \Theta = \mathbb{R}_+ \]

Define the null hypothesis \( H_N \) such that
\[
H_N = \{ \sigma_0 \} \subseteq \Theta(= \mathbb{R}_+) \]
Consider the following semi-distance $d^{(1)}_{\Theta}$ in $\Theta(=\mathbb{R}_+)$:

$$d^{(1)}_{\Theta}(\theta_1, \theta_2) = |\int_{\theta_1}^{\theta_2} \frac{1}{\sigma} d\sigma| = |\log \sigma_1 - \log \sigma_2|$$  \hspace{1cm} (49)

For any $\omega = (\mu, \sigma)(\in \Omega = \mathbb{R} \times \mathbb{R}_+)$, define the positive number $\eta_0^\alpha$ ($> 0$) such that:

$$\eta_0^\alpha = \sup \{\eta > 0 : [F(E^{-1}(\text{Ball}^{C}_{d^{(1)}_{\Theta}}(\omega; \eta)))|/(\omega) \leq \alpha\}$$  \hspace{1cm} (50)

where

$$\text{Ball}^{C}_{d^{(1)}_{\Theta}}(\omega; \eta) = \text{Ball}^{C}_{d^{(1)}_{\Theta}}((\mu; \sigma), \eta) = \mathbb{R} \times \{\sigma' : |\log(\sigma'/\sigma)| \geq \eta\} = \mathbb{R} \times ((0, \sigma e^{-\eta}] \cup [\sigma e^\eta, \infty))$$  \hspace{1cm} (51)

Then,

$$E^{-1}(\text{Ball}^{C}_{d^{(1)}_{\Theta}}(\omega; \eta)) = E^{-1}\left(\mathbb{R} \times ((0, \sigma e^{-\eta}] \cup [\sigma e^\eta, \infty))\right)$$

$$= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \left(\sum_{k=1}^{n}(x_k - \mu)^2/n\right)^{1/2} \leq \sigma e^{-\eta} \text{ or } \sigma e^\eta \leq \left(\sum_{k=1}^{n}(x_k - \mu)^2/n\right)^{1/2}\}$$  \hspace{1cm} (52)

Hence we see, by (10), that

$$\left[N^\alpha(E^{-1}(\text{Ball}^{C}_{d^{(1)}_{\Theta}}(\omega; \eta)))|(\omega)\right]$$

$$= \frac{1}{(2\pi\sigma)^n} \int \cdots \int \exp[-\sum_{k=1}^{n}(x_k - \mu)^2/(2\sigma^2)] dx_1 dx_2 \cdots dx_n$$

$$E^{-1}(\mathbb{R} \times ((0, \sigma e^{-\eta}] \cup [\sigma e^\eta, \infty)))$$

$$= \int_{0}^{\sigma e^{-\eta}} p_{n-1}(x)dx + \int_{\sigma e^\eta}^{\infty} p_{n-1}(x)dx = 1 - \int_{\sigma e^{-\eta}}^{\sigma e^\eta} p_{n-1}(x)dx$$  \hspace{1cm} (53)

Using the chi-squared distribution $p_{n-1}(x)$ (with $n - 1$ degrees of freedom) in (11), define the $\eta_0^\alpha$ such that

$$1 - \alpha = \int_{\sigma e^{-\eta_0^\alpha}}^{\sigma e^\eta_0^\alpha} p_{n-1}(x)dx$$  \hspace{1cm} (54)

where it should be noted that the $\eta_0^\alpha$ depends on only $\alpha$ and $n$. Thus, put

$$\eta_0^\alpha = \eta_n^\alpha$$  \hspace{1cm} (55)

Hence we get the $\hat{R}^\alpha_{H_N}$ (the $(\alpha)$-rejection region of $H_N = \{\sigma_0\} \subseteq \Theta = \mathbb{R}_+$) as follows:

$$\hat{R}^\alpha_{H_N} = \hat{R}^\alpha_{(\sigma_0)} = \cap_{\pi(\omega) = \sigma_0} \{E(x)(\in \Theta) : d^{(2)}(E(x), \omega) \geq \eta_\omega^\alpha\}$$

$$= \{E(x)(\in \Theta = \mathbb{R}_+) : d^{(2)}(E(x), (\mu, \sigma_0)) \geq \eta_\sigma^\alpha\}$$

$$= \{\sigma(x)(\in \Theta = \mathbb{R}_+) : \sigma(x) \leq \sigma_0 e^{-\eta_0^\alpha} \text{ or } \sigma_0 e^{\eta_0^\alpha} \leq \sigma(x)\}$$  \hspace{1cm} (56)

where $\sigma(x) = \left(\sum_{k=1}^{n}(x_k - \mu)^2/n\right)^{1/2}$.

Thus, in a similar way of Remark 2, we see that $\hat{R}^\alpha_{\mathbb{R} \times \{\sigma_0\}} =$”the slash part in Figure 7”, where

$$\hat{R}^\alpha_{\mathbb{R} \times \{\sigma_0\}} = \{(\mu, \sigma(x)) \in \mathbb{R} \times \mathbb{R}_+ : \sigma(x) \leq \sigma_0 e^{-\eta_0^\alpha} \text{ or } \sigma_0 e^{\eta_0^\alpha} \leq \sigma(x)\}$$  \hspace{1cm} (57)
Example 7 [Rejection region of $H_N = (-\infty, \sigma_0] \subseteq \Theta(= \mathbb{R}_+)$. Consider the simultaneous measurement $M_{C_0(\mathbb{R} \times \mathbb{R}_+)} (\mathcal{O}_N^0 = (\mathbb{R}^n, B_0^\mathbb{R}, N^k, S_{[\mu, \sigma]}))$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. Thus, we consider that $\Omega = \mathbb{R} \times \mathbb{R}_+$, $X = \mathbb{R}^n$. Assume that the real $\mu$ in a state $\omega = (\mu, \sigma) \in \Omega$ is fixed and known. Put $\Theta = \mathbb{R}_+$

The formula (17) may urge us to define the estimator $E : \mathbb{R}^n \to \Theta(= \mathbb{R}_+)$ such that

$$E(x) = E(x_1, x_2, \ldots, x_n) = \sigma(x) = \sqrt{\frac{1}{n} \sum_{k=1}^n (x_k - \mu(x))^2} \quad (58)$$

And consider the quantity $\pi : \Omega \to \Theta$ such that

$$\Omega = \mathbb{R} \times \mathbb{R}_+ \ni \omega = (\mu, \sigma) \mapsto \pi(\omega) = \sigma \in \Theta = \mathbb{R}_+$$

Define the null hypothesis $H_N$ such that

$$H_N = (-\infty, \sigma_0] \subseteq \Theta(= \mathbb{R}_+)$$

Consider the following semi-distance $d_{\Theta}^{(2)}$ in $\mathbb{R} \times \mathbb{R}_+$:

$$d_{\Theta}^{(2)}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \begin{cases} |\int_{\sigma_2}^{\sigma_1} \frac{1}{\sigma} d\sigma| = |\log \sigma_1 - \log \sigma_2| & (\sigma_0 \leq \sigma_1, \sigma_2) \\ |\int_{\sigma_1}^{\sigma_2} \frac{1}{\sigma} d\sigma| = |\log \sigma_0 - \log \sigma_2| & (\sigma_1 \leq \sigma_0 \leq \sigma_2) \\ |\int_{\sigma_0}^{\sigma_1} \frac{1}{\sigma} d\sigma| = |\log \sigma_0 - \log \sigma_1| & (\sigma_2 \leq \sigma_0 \leq \sigma_1) \\ 0 & (\sigma_1, \sigma_2 \leq \sigma_0) \end{cases} \quad (59)$$

For any $\omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+$, define the positive number $\eta^0_\omega$ ($> 0$) such that:

$$\eta^0_\omega = \sup\{\eta > 0 : [F(E^{-1}(\text{Ball}^{C}_{d_{d^2}^{(2)}}(\omega; \eta)))](\omega) \leq \alpha\} \quad (60)$$

where

$$\text{Ball}^{C}_{d_{d^2}^{(2)}}(\omega; \eta) = \text{Ball}^{C}_{d_{d^2}^{(2)}}((\mu; \sigma), \eta) = \mathbb{R} \times [\sigma e^\eta, \infty) \quad (61)$$

Then,

$$E^{-1}(\text{Ball}^{C}_{d_{d^2}^{(2)}}(\omega; \eta)) = E^{-1}(\{\sigma e^\eta, \infty\}) = \{x_1, \ldots, x_n \in \mathbb{R}^n : \sigma e^\eta \leq \sigma(x) = \left(\frac{\sum_{k=1}^n (x_k - \mu(x))^2}{n}\right)^{1/2} \} \quad (62)$$
Hence we see, by (10), that
\[ [N^n(E^{-1}(\text{Ball}_{d_{\mathcal{D}}^{(2)}}(\omega; \eta)))](\omega) \]
\[ = \frac{1}{(\sqrt{2\pi\sigma})^n} \int \cdots \int \exp\left[-\sum_{k=1}^{n} \frac{(x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \]
\[ = \int_{\sigma_0}^{\infty} p_{n-1}^2(x)dx \]
\[ \leq \int_{\sigma_0}^{\infty} p_{n-1}^2(x)dx \]
(63)

Solving the following equation, define the \((\eta_n^a)'(>0)\) such that
\[ \alpha = \int_{\sigma_0}^{\infty} p_{n-1}^2(x)dx \]
(64)

Hence we get the \(\hat{R}_H^\alpha\) (the \((\alpha)\)-rejection region of \(H_N = \mathbb{R} \times (0, \sigma_0)\) ) as follows:
\[ \hat{R}_H^\alpha = \hat{R}_{\mathbb{R} \times (0, \sigma_0)}^\alpha = \bigcap_{\pi(\omega) = \omega \in \mathbb{R} \times (0, \sigma_0)} \{ E(x)(\in \Omega) : d_{\mathcal{D}}^{(2)}(E(x), \omega) \geq \eta_n^a \} \]
\[ = \{ E(x)(\in \Omega) : d_{\mathcal{D}}^{(2)}(E(x), \omega) \geq (\eta_n^a)’ \} \]
\[ = \{ (\mu, \sigma(=\sigma(x))) \in \mathbb{R} \times \mathbb{R}^+ : \sigma_0 e^{(\eta_n^a)’} \leq \sigma(x) \} \]
(65)

where \(\sigma(x) = \left(\sum_{k=1}^{n} \frac{(x_k - \bar{x})^2}{n}\right)^{1/2}\).

Thus, in a similar way of Remark 2, we see that \(\hat{R}_{\mathbb{R} \times (0, \sigma_0)}^\alpha = \text{“the slash part in Figure 8”}\), where
\[ \hat{R}_{\mathbb{R} \times (0, \sigma_0)}^\alpha = \{ (\mu, \sigma(x)) \in \mathbb{R} \times \mathbb{R}^+ : \sigma_0 e^{(\eta_n^a)’} \leq \sigma(x) \} \]
(66)

\[ \text{Figure 8. Rejection region} \hat{R}_{(0,\sigma_0)}^\alpha \]

3.3 The difference of the population means

The arguments in this section are continued from Example 2.

Example 8 [Rejection region in the case that \(\pi(\mu_1, \mu_2) = \mu_1 - \mu_2^-\)]. Consider the parallel measurement \(M_{\pi_0}((\mathbb{R} \times \mathbb{R}^+)(\mathbb{R} \times \mathbb{R}^+)) = (\mathbb{R} \times \mathbb{R}^+, \mathcal{B}^{(2)}_{\mathbb{R} \times \mathbb{R}^+}, N^n \otimes N^m), S_{\pi_0}(\mu_1, \sigma_1, \mu_2, \sigma_2)\) in \(C_0((\mathbb{R} \times \mathbb{R}^+) \times (\mathbb{R} \times \mathbb{R}^+))\).
Assume that σ₁ and σ₂ are fixed and known. Thus, this parallel measurement is represented by \( M_{(\mathbb{R} \times \mathbb{R})} \cdot (O_{N_1} \otimes O_{N_2}) = (\mathbb{R}^n \times \mathbb{R}^m, \mathcal{B}_\mathbb{R}^n \otimes \mathcal{B}_\mathbb{R}^m, N_{(\mu_1, \mu_2)}) \) in \( C_0(\mathbb{R} \times \mathbb{R}) \). Here, recall the (2), i.e.,

\[
[N_\sigma(\Xi)](\mu) = \frac{1}{\sqrt{2\pi\sigma}} \int_\Xi \exp[- \frac{(x - \mu)^2}{2\sigma^2}] dx \quad (\forall \Xi \in \mathcal{B}_\mathbb{R}(\text{=Borel field in } \mathbb{R})), \quad \forall \mu \in \mathbb{R}).
\]

Therefore, we have the state space \( \Omega = \mathbb{R}^2 \). Let \( \omega = (\mu_1, \mu_2) : \mu_1, \mu_2 \in \mathbb{R} \). Put \( \Theta = \mathbb{R} \) with the distance \( d_{\Theta}^{(1)}(\theta_1, \theta_2) = |\theta_1 - \theta_2| \) and consider the quantity \( \pi : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
\pi(\mu_1, \mu_2) = \mu_1 - \mu_2
\]

The estimator \( E : \tilde{X}(=X \times Y = \mathbb{R}^n \times \mathbb{R}^m) \rightarrow \Theta(= \mathbb{R}) \) is defined by

\[
E(x_1, \ldots, x_n, y_1, \ldots, y_m) = \frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^m y_k}{m}
\]

For any \( \omega = (\mu_1, \mu_2)(\in \Omega = \mathbb{R} \times \mathbb{R}) \), define the positive number \( \eta^\alpha \) (\( > 0 \)) such that:

\[
\eta^\alpha = \inf\{ \eta > 0 : |F(E^{-1}(\text{Ball}_{d_{\Theta}^{(1)}}^C(\pi(\omega); \eta))))(\omega) | \geq \alpha \}
\]

where \( \text{Ball}_{d_{\Theta}^{(1)}}^C(\pi(\omega); \eta) = (-\infty, \mu_1 - \mu_2 - \eta] \cup [\mu_1 - \mu_2 + \eta, \infty) \). Define the null hypothesis \( H_N (\subseteq \Theta = \mathbb{R}) \) such that

\[
H_N = \{ \theta_0 \}
\]

Now let us calculate the \( \eta^\alpha \) as follows:

\[
E^{-1}(\text{Ball}_{d_{\Theta}^{(1)}}^C(\pi(\omega); \eta)) = \frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^m y_k}{m} \geq \mu_1 - \mu_2 - \eta
\]

Thus,

\[
\frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^m y_k}{m} \geq \mu_1 - \mu_2 - \eta
\]

Using the \( z(\alpha/2) \) in (36), we get that

\[
\eta^\alpha = \left( \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \right)^{1/2} z(\alpha/2)
\]
Therefore, we get \( \hat{R}_N^0 \) (the \((\alpha)\)-rejection region of \( H_N = \{\theta_0^1(\subseteq \Theta)\) as follows:

\[
\hat{R}_N^0 = \bigcap_{\omega = (\mu_1, \mu_2) \in \Omega(= \mathbb{R}^2)} \{ E(\tilde{x})(\in \Theta) : d_0(1)(E(\tilde{x}), \pi(\omega)) \geq \eta_0^0 \}
\]

\[
= \{ \overline{p}(x) - \overline{p}(y) \in \Theta(= \mathbb{R}) : |\overline{p}(x) - \overline{p}(y) - \theta_0| \geq \left( \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \right)^{1/2} z(\alpha) \} \quad (73)
\]

where

\[
\overline{p}(x) = \frac{\sum_{k=1}^n x_k}{n}, \quad \overline{p}(y) = \frac{\sum_{k=1}^m y_k}{m}.
\]

**Remark 3** [The case that \( H_N = (-\infty, \theta_0) \).] If the null hypothesis \( H_N \) is assumed as follows:

\[ H_N = (-\infty, \theta_0), \]

it suffices to define the semi-distance

\[
d_0(1; \theta_1; \theta_2) = \begin{cases} \max\{\theta_1, \theta_2\} - \theta_0 & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \theta_0 \leq \theta_1, \theta_2) \\
0 & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \min\{\theta_1, \theta_2\} \leq \theta_0 \leq \max\{\theta_1, \theta_2\}) \end{cases} \quad (74)
\]

Then, we can easily see that

\[
\hat{R}_N^0 = \bigcap_{\omega = (\mu_1, \mu_2) \in \Omega(= \mathbb{R}^2)} \{ E(\tilde{x})(\in \Theta) : d_0(1)(E(\tilde{x}), \pi(\omega)) \geq \eta_0^0 \}
\]

\[
= \{ \overline{p}(x) - \overline{p}(y) \in \mathbb{R} : |\overline{p}(x) - \overline{p}(y) - \theta_0| \geq \left( \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m} \right)^{1/2} z(\alpha) \} \quad (75)
\]

### 3.4 The ratio of the population variances

**Example 9** [Rejection region in the case that "\( \pi(\sigma_1, \sigma_2) = \sigma_1/\mu_2 \)." Consider the parallel measurement \( M_{C_0(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})} (\mathbb{O}_n \otimes \mathbb{O}_m = (\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^n \otimes \mathbb{R}^m, N^m, N^m, S_{(\mu_1, \mu_2, \sigma_2)})) \text{ in } C_0(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}).

Put \( \Theta = \mathbb{R}_+ \) with the distance \( d_0(2; \theta_1; \theta_2) = |\log \theta_1 - \log \theta_2| = |\log \theta_1/\theta_2| \) and consider the quantity

\[
\pi : \Theta = (\mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \Theta(= \mathbb{R}_+) \text{ by }
\]

\[
\pi((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \sigma_1/\sigma_2 \quad (76)
\]

The estimator \( E : \tilde{X} (= X \times Y = \mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \Theta(= \mathbb{R}_+) \) is defined by

\[
E(x_1, \ldots, x_n, y_1, \ldots, y_m) = \frac{\overline{x}(x)}{\overline{x}(y)} \quad (\text{Recall (17)}).
\]

For any \( \omega = ((\mu_1, \sigma_1), (\mu_2, \sigma_2)) \in \Omega = (\mathbb{R}_+ \times \mathbb{R}_+) \times (\mathbb{R}_+ \times \mathbb{R}_+) \), define the positive number \( \eta_0^0 \) (\( > 0 \)) such that:

\[
\eta_0^0 = \inf\{ \eta > 0 : |F(E^{-1}(\text{Ball}_{d_0(2)}(\pi(\omega); \eta))))(\omega) \geq \alpha \}
\]

where \( \text{Ball}_{d_0(2)}(\pi(\omega); \eta) = (0, (\sigma_1/\sigma_2) e^{-\eta}] \cup [(\sigma_1/\sigma_2) e^\eta, \infty) \). Define the null hypothesis \( H_N \subseteq \Theta(= \mathbb{R}_+) \) such that

\[
H_N = \{ r_0 \}
\]

Now let us calculate the \( \eta_0^0 \) as follows:

\[
E^{-1}(\text{Ball}_{d_0(2)}(\pi(\omega); \eta)) = E^{-1}((0, (\sigma_1/\sigma_2) e^{-\eta}] \cup [(\sigma_1/\sigma_2) e^\eta, \infty))
\]

\[
= \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m : \frac{\overline{x}(x)/\sigma_1}{\overline{y}(y)/\sigma_2} \leq e^{-\eta} \text{ or } \frac{\overline{x}(x)/\sigma_1}{\overline{y}(y)/\sigma_2} \geq e^\eta \} \quad (77)
\]


Thus,

\[
1 - [(N_{\sigma_1^n} \times N_{\sigma_2^m})(E^{-1}(\text{Ball}_{d_{n0}}^{(2)}(\pi(\omega); \eta)))](\omega)
= \frac{1}{(\sqrt{2\pi} \sigma_1^n)(\sqrt{2\pi} \sigma_2^m)} \times \int \cdots \int_{e^{-n} \leq \frac{\sigma}{\sigma_1^n} \leq e^n} \exp\left[-\sum_{k=1}^{n} \frac{(x_k - \mu_1)^2}{2\sigma_1^2} - \sum_{k=1}^{m} \frac{(y_k - \mu_2)^2}{2\sigma_2^2}\right] dx_1 dx_2 \cdots dx_n dy_1 dy_2 \cdots dy_m
= \frac{1}{(\sqrt{2\pi} \sigma_1^n)(\sqrt{2\pi} \sigma_2^m)} \int_{e^{-n} \leq \frac{\sigma}{\sigma_1^n} \leq e^n} \exp\left[-\sum_{k=1}^{n} \frac{x_k^2}{2} - \sum_{k=1}^{m} \frac{y_k^2}{2}\right] dx_1 dx_2 \cdots dx_n dy_1 dy_2 \cdots dy_m
= \int_{e^{-n}}^{e^{2n}} p_{n-1,m-1}^F(x) dx
\]  

where \(p_{n-1,m-1}^F(x)\) is the \(F\)-distribution with \((n-1, m-1)\) degrees of freedom. Define the positive \(\eta_n^\alpha\) such that

\[
1 - \alpha = \int_{e^{-n}}^{e^{2n}} p_{n-1,m-1}^F(x) dx
\]

Since \(\eta_n^\alpha\) does not depend on \(\omega\), we can put \(\eta_n^\alpha = \eta_n^\alpha\). Therefore, we get \(\hat{R}_{H_N}^\alpha\) (the \((\alpha)\)-rejection region of \(H_N(=\{r\})\)) as follows:

\[
\hat{R}_{H_N}^\alpha = \bigcap_{\omega=(\mu_1, \sigma_1, \mu_2, \sigma_2) \in \Omega} \{E(\bar{x})(\in \Theta) : d_{\Theta}^{(2)}(E(\bar{x}), \pi(\omega)) \geq \eta_n^\alpha\}
= \bigcap_{\sigma_1^2 = r_0} \{\frac{\nu_1}{\sigma_1^2}(x) \in \Theta = \mathbb{R}_+ : \frac{\nu_1}{\sigma_1^2}(x) \sigma_1^2 / \sigma_2^2 \leq e^{-\eta_n^\alpha} \text{ or } \frac{\nu_1}{\sigma_1^2}(x) \sigma_1^2 / \sigma_2^2 \geq e^{\eta_n^\alpha}\}
= \{\frac{\nu_1}{\sigma_2^2}(y) \in \Theta = \mathbb{R}_+ : \frac{\nu_1}{\sigma_2^2}(y) \leq r_0 e^{-\eta_n^\alpha} \text{ or } \frac{\nu_1}{\sigma_2^2}(y) \geq r_0 e^{\eta_n^\alpha}\}
\]

**Remark 4** [The case that \(H_N = (0, r_0) \subseteq \Theta = \mathbb{R}_+\]. If the null hypothesis \(H_N\) is assumed as follows:

\[
H_N = (0, r_0],
\]

it suffices to define the semi-distance

\[
d_{\Theta}^{(2)}(\nu_1, \nu_2) = \begin{cases} 
\log(\theta_1 \theta_2) & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } r_0 \leq \theta_1, \theta_2) \\
\log(\max\{\theta_1, \theta_2\}/r_0) & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \min\{\theta_1, \theta_2\} \leq r_0 \leq \max\{\theta_1, \theta_2\}) \\
0 & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \theta_1, \theta_2 \leq r_0)
\end{cases}
\]

Then, we can easily see that

\[
\hat{R}_{H_N}^\alpha = \bigcap_{\omega=(\mu_1, \mu_2) \in \Omega(=\mathbb{R}^2)} \{E(\bar{x})(\in \Theta) : d_{\Theta}^{(2)}(E(\bar{x}), \pi(\omega)) \geq \eta_n^\alpha\}
= \{\frac{\nu_1}{\sigma_2^2}(y) \in \Theta = \mathbb{R}_+ : \frac{\nu_1}{\sigma_2^2}(y) \geq r_0 e^{(\eta_n^\alpha)}\}
= \{r_0 e^{(\eta_n^\alpha)}, \infty\}
\]

where the positive \((\eta_n^\alpha)\) such that

\[
\alpha = \int_{e^{2(\eta_n^\alpha)}}^{\infty} p_{n-1,m-1}^F(x) dx
\]
3.5 The case that \( d^α_Θ \) depends on \( x \); Student’s t-distribution

The arguments in this section are continued from Example 2.

**Example 10** [Student’s t-distribution]. Consider the simultaneous measurement \( M_{C_0(\mathbb{R} \times \mathbb{R}_+)} (O^n = (\mathbb{R}^n, B^n, N^n), S_{(\mu, \sigma)}) \) in \( C_0(\mathbb{R} \times \mathbb{R}_+) \). Thus, we consider that \( \Omega = \mathbb{R} \times \mathbb{R}_+ \), \( X = \mathbb{R}^n \). Put \( \Theta = \mathbb{R} \) with the semi-distance \( d^α_Θ(∀x ∈ X) \) such that

\[
d^α_Θ(θ₁, θ₂) = \frac{|θ₁ - θ₂|}{σ'_{(x)}/\sqrt{n}} \quad (∀x ∈ X = \mathbb{R}^n, ∀θ₁, θ₂ ∈ Θ = \mathbb{R}) \tag{83}
\]

where \( σ'(x) = \sqrt{\frac{1}{n-1}σ(x)} \). The quantity \( π : \Omega(= \mathbb{R} \times \mathbb{R}_+) → \Theta(= \mathbb{R}) \) is defined by

\[
Ω(= \mathbb{R} \times \mathbb{R}_+) \ni Ω(= \mathbb{R}) \ni π(\mu, σ) = μ ∈ Θ(= \mathbb{R}) \tag{84}
\]

Also, define the estimator \( E : X(= \mathbb{R}^n) → Θ(= \mathbb{R}) \) such that

\[
E(x) = E(x₁, x₂, ..., xₙ) = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{x₁ + x₂ + ... + xₙ}{n} \tag{85}
\]

Define the null hypothesis \( H_N (⊆ Θ = \mathbb{R}) \) such that

\[
H_N = \{μ₀\} \tag{86}
\]

Thus, for any \( ω = (μ₀, σ)(∈ Ω = \mathbb{R} \times \mathbb{R}_+) \), we see that

\[
\begin{align*}
&\mathbb{P}(x ∈ X : d^α_Θ(E(x), π(ω)) ≥ η) |
&\mathbb{P}(x ∈ X : \frac{|\bar{μ}(x) - μ₀|}{σ'_{(x)}/\sqrt{n}} ≥ t(α/2)) |
&\mathbb{P}(x ∈ X : \frac{|\bar{μ}(x) - μ₀|}{σ'_{(x)}/\sqrt{n}} ≥ \frac{1}{(\sqrt{2π})^n} \int_{x ≤ \bar{μ}(x) - μ₀} \exp\left(-\frac{1}{2σ²} \sum_{k=1}^{n} (x_k - μ₀)^2\right) dx₁dx₂...dxₙ |
&\mathbb{P}(x ∈ X : \frac{|\bar{μ}(x) - μ₀|}{σ'_{(x)}/\sqrt{n}} ≥ \frac{1}{(\sqrt{2π})^n} \int_{x ≤ \bar{μ}(x) - μ₀} \exp\left(-\frac{1}{2σ²} \sum_{k=1}^{n} (x_k - μ₀)^2\right) dx₁dx₂...dxₙ |
&\mathbb{P}(x ∈ X : \frac{|\bar{μ}(x) - μ₀|}{σ'_{(x)}/\sqrt{n}} ≥ \int_{-∞}^{η} t_{n-1}^α(x)dx)
\end{align*}
\]

where \( t^α_{n-1} \) is the t-distribution with \( n-1 \) degrees of freedom. Solving the equation \( 1 - α = \int_{-∞}^{η_{n-1}^α} t_{n-1}^α(x)dx \), we get \( η_{n}^α = t(α/2) \).

Therefore, we get \( \hat{R}^α_{H_N} (the (α)-rejection region of H_N(= \{μ₀\}) ) \) as follows:

\[
\hat{R}^α_{H_N} = \bigcap_{ω=(μ, σ)∈Ω(= \mathbb{R} \times \mathbb{R}_+)} \{E(x)(∈ Θ) : d^α_Θ(E(x), π(ω)) ≥ η_{n}^α\}
\]

\[
= \{\bar{μ}(x) ∈ Θ(= \mathbb{R}) : \frac{|\bar{μ}(x) - μ₀|}{σ'_{(x)}/\sqrt{n}} ≥ t(α/2)\}
\]

\[\begin{align*}
&\{\bar{μ}(x) ∈ Θ(= \mathbb{R}) : μ₀ ≤ \bar{μ}(x) - \frac{σ'_{(x)}}{\sqrt{n}}t(α/2) \text{ or } \bar{μ}(x) + \frac{σ'_{(x)}}{\sqrt{n}}t(α/2) ≤ μ₀\}
&\{\bar{μ}(x) ∈ Θ(= \mathbb{R}) : μ₀ ≤ \bar{μ}(x) - \frac{σ'(x)}{\sqrt{n}}t(α/2) \text{ or } \bar{μ}(x) + \frac{σ'(x)}{\sqrt{n}}t(α/2) ≤ μ₀\}
\end{align*}\tag{88}
\]

**Remark 5** [The case that \( H_N = (−∞, μ₀) \)]. If the null hypothesis \( H_N \) is assumed as follows:

\[
H_N = (−∞, μ₀),
\]


it suffices to define the semi-distance

\[ d_{\Theta}(\theta_1, \theta_2) = \begin{cases} \frac{|\theta_1 - \theta_2|}{\sigma_x(x) \sqrt{n}} & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \mu_0 \leq \theta_1, \theta_2) \\ \frac{\max{|\theta_1, \theta_2|}}{\sigma_x(x) \sqrt{n}} - \mu_0 & (\forall \theta_1, \theta_2 \in \Theta = \mathbb{R} \text{ such that } \min{\theta_1, \theta_2} \leq \mu_0 \leq \max{\theta_1, \theta_2}) \end{cases} \]

(89)

for any \( x \in X = \mathbb{R}^n \). Then, we can easily see that

\[ \tilde{R}_N^\alpha = \bigcap_{\omega=(\mu, \sigma) \in \Omega(=\mathbb{R} \times \mathbb{R}_+)} \{ E(x)(\in \Theta) : d_{\Theta}(E(x), \pi(\omega)) \geq \eta^\alpha \} \]

\[ = \{ \bar{\mu}(x) \in \Theta(= \mathbb{R}) : \mu_0 \leq \bar{\mu}(x) - \frac{\sigma_x(x)}{\sqrt{n}} t(\alpha) \} \]

(90)

4 Conclusions

It is sure that statistics and (classical) quantum language are similar, however, quantum language has the firm structure (1), i.e.,

\[
\text{Quantum language} = \begin{array}{l}
\text{Axiom 1 (measurement)} + \\
\text{Axiom 2 (causality)} + \\
\text{linguistic interpretation (how to use Axioms)}
\end{array}
\]

(91)

Hence, as seen in Theorems 1-4 of this paper, every argument cannot but become clear in quantum language. Particularly, the following two statistical hypothesis tests (J1) and (J2), that is,

(J1) Theorem 2 (Likelihood ratio test)

key-words: Estimator \( E : X \to \Omega \), Quantity \( \pi : \Omega \to \Theta \), Likelihood function \( L_\theta(\omega) \) in (18)

(J2) Theorem 4 (Reverse confidence interval method)

key-words: Estimator \( E : X \to \Theta \), Quantity \( \pi : \Omega \to \Theta \), Semi-distance \( d_{\Theta}^\alpha \) on \( \Theta \).

should be compared and examined.

For example, we remark that the difference between "one sided test" and "two sided test" is due to the difference of the semi-distances. And further, we see the peculiarity of the student's \( t \)-distribution in Example 10, however, we have no firm answer to the following question:

(K) Can Example 10 (Student’s \( t \)-distribution) be naturally understood in Theorem 2 (Likelihood ratio test)?

Although Theorem 2 (Likelihood ratio test) is orthodox, it is not handy. On the other hand, we believe that Theorems 4 (Reverse confidence interval method) may be usual, though it is not presented as a general theorem in the elementary books of statistics.

Since quantum language is suited for theoretical arguments, we believe, from the theoretical point of view, that our results (i.e., Theorems 1-4) are final in classical systems. We hope that our assertions will be examined from various points of view.

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