Key distillation from Gaussian states by Gaussian operations

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We study the secrecy properties of Gaussian states under Gaussian operations. Although such operations are useless for quantum distillation, we prove that it is possible to distill a secret key secure against any attack from sufficiently entangled Gaussian states with non-positive partial transposition. Moreover, all such states allow for key distillation, when Eve is assumed to perform finite-size coherent attacks before the reconciliation process.

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Quantum Information Theory (QIT) analyzes the possibilities offered by quantum states for information encoding and transmission. New processes such as teleportation 1, more powerful algorithms 2 or completely secure data transmission 3 become possible by exploiting quantum effects. Most of the QIT applications use entanglement as resource. It is therefore one of the most important tasks of QIT to determine if a given state is entangled, and if yes, whether its entanglement is “useful” for information processing. Distillable states 4 5 possess such useful entanglement: using several copies of them and local operations and classical communication (LOCC), one can create a smaller number of maximally entangled states, which can be used for variety of quantum information tasks. Given a state, its distillable entanglement, $E_D$, measures the amount of pure-state entanglement that can be extracted from it. Particularly important for cryptography is another measure of entanglement $K_D$: it specifies the number of secret bits that can be extracted from a quantum state using LOCC. Obviously, $K_D \geq E_D$, since a secret key can always be extracted from distilled maximally entangled states, using, for instance, the Ekert protocol 6. Entangled states which cannot be distilled, i.e. for which $E_D = 0$, exist and are termed “bound entangled” 8. Although it was unclear whether these states could contain useful entanglement, it has been recently shown that there exist bound entangled states for which $K_D > E_D = 0$ 2.

All of the above results were originally considered for finite dimensional systems. More recently, many of these concepts have been translated to the infinite dimensional case 7 8. In these systems, a key role is played by the set of Gaussian states and Gaussian operations. First of all, it naturally appears in experiments. In fact, non-Gaussian operations turn out to be very challenging from an experimental point of view. Moreover, the theoretical analysis of Gaussian states and operations is also simplified since all their properties can be expressed in terms of finite-dimensional matrices.

To analyze the limitations and possibilities offered by the Gaussian scenario is a relevant issue. The teleportation of coherent states of light have been experimentally demonstrated 10. Quantum cryptography has also been successfully translated into the Gaussian regime. Gottesman and Preskill proposed to use squeezed states and homodyne measurements in a “prepare & measure” scheme 11. Actually, no squeezing is required, since coherent states are already sufficient for a secure key distribution with Gaussian operations 12 13. The experimental implementation of a coherent-state protocol has been recently realized in 14. From a more fundamental point of view, it is interesting to know which information tasks can be achieved in the Gaussian regime. It is known that all Gaussian states have positive Wigner function, i.e. there is a local variable model reproducing all the (symmetrically ordered) correlations for Gaussian states. Despite the existence of this local description, these states and operations are useful for teleporting quantum states, or for secure information transmission, applications that are impossible in a classical scenario.

An important negative result in this context was obtained in Refs. 15 16: quantum distillation by Gaussian operations is impossible. That is, although it is known that all Gaussian states with non-positive partial transposition (NPPT) are distillable 17, any distillation protocol must include a non-Gaussian operation. This can be rephrased saying that all entangled mixed states are bound entangled in the Gaussian scenario. In this scenario, it is natural to define the corresponding Gaussian versions of entanglement measures, that specify the entanglement properties of these states under Gaussian Local Operations and Classical Communication (GLOCC).

For example, $GE_D$ defines the Gaussian Distillable Entanglement. The results of 15 16 imply that $GE_D = 0$. However, these states may still be useful, since perhaps secret bits can be extracted from them using GLOCC.

In a similar way as for distillable entanglement, for any entangled state one can denote by $GK_D$ the amount of secret bits that can extracted by GLOCC. This quantity refers to the distribution of a secret key, and in principle is independent of entanglement distillability. Thus, secret bits $(GK_D)$ appear as a resource (measure) es-
pecially suitable for the analysis of entangled Gaussian states under Gaussian operations: while \(GE_D = 0\) for all states, there are states with positive \(FK_D\).

In this work we analyze the secrecy properties of Gaussian states under Gaussian operations. We consider a protocol where Alice and Bob measure their shared entangled state at the single-copy level and process the obtained classical results. This type of protocols can easily be translated to prepare & measure schemes. First, we study the security of our protocol when Eve applies an incoherent attack, that is she individually measures her state before any key extraction process is started. Under these conditions, all NPPT states turn out to be secure. The same conclusion holds when Eve applies a coherent attack on a finite number of symbols. All these cases would correspond to a situation where Eve’s quantum memory has a finite coherence time. Then, we consider more powerful attacks and show that our security proof ceases to work for some NPPT states. Finally, using the recent results of Ref. [20], we demonstrate the security of our protocol when Eve applies an attack on a finite number of symbols. All these cases provide evidence that our security proof is robust against various attacks.

We consider quantum systems of \(n\) canonical coordinates of freedom, often called modes, \(H = L^2([0, \infty[^n)\). The commutation relations for the canonical coordinates \(R = (X_1, P_1, \ldots, X_m, P_n) = (R_1, R_2n) = i(J_n)_{ab}\), where \(a, b = 1, \ldots, 2n\) and

\[
J_n = \oplus_{i=1}^n J
\]

The characteristic function, \(\chi\), of a state \(\rho\) is defined as \(\chi(x) = \text{tr}(\rho W(x))\), where \(W(x) = \exp(-ix^T R)\) are the so-called Weyl operators. Gaussian states are those states such that \(\chi\) is a Gaussian function,

\[
\chi(x) = \exp(i x^T \gamma - \frac{1}{4} x^T \gamma x).
\]

where \(\gamma\) is a \(2n\) real vector, called displacement vector (DV), and \(\gamma\) is a \(2n \times 2n\) symmetric real matrix, known as covariance matrix (CM). The positivity condition of \(\rho\) implies that \(\gamma - i J_n \geq 0\). All the information about \(d\) and \(\gamma\) is contained in the first and second moments \(\text{tr}(\rho R_i)\) and \(\text{tr}(\rho R_i R_j)\).

In what follows we consider two parties, Alice and Bob, that share a state \(\rho\) in a composite systems of \(n + m\) modes. The global CM is

\[
\gamma_{AB} = \begin{pmatrix} \gamma_A & C \\ C^T & \gamma_B \end{pmatrix} \geq iJ_{n+m},
\]

where \(\gamma_A (\gamma_B)\) is the CM for the \(n\) mode (\(m\) mode) Gaussian state of system \(A (B)\). The entanglement properties of \(\rho\) are completely specified by its CM.

The effect of partial transposition at the level of CMs can be understood from the fact that this map is equivalent to time-reversal. After partial transposition on, say, \(A\), the sign of Alice’s momenta is changed while the rest of canonical coordinates is kept unchanged. Denote by \(\theta\) the matrix equal to the identity for the position coordinates and minus the identity for the momenta. Partial transposition means that \(\gamma_{AB} \rightarrow \gamma_{AB}' = \theta A \gamma_{AB} \theta_A\). Therefore, the state \(\rho\) has positive partial transposition (PPT) when \(\gamma_{AB}'\) defines a positive operator, that is \(\gamma_{AB}' \geq iJ_{n+m}\). The PPT criterion provides a necessary and sufficient condition for separability for \(1 \times 1\) Gaussian states. As said above, the non-positivity of the partial transposition is a necessary and sufficient condition for distillability.

Two known results will play an important role in what follows. First, any NPPT Gaussian state of \(n + m\) modes can be mapped into GLOCC into an NPPT \(1 \times 1\) Gaussian and symmetric state [17], whose CM, see Eq. (3), is

\[
\gamma_{A} = \gamma_{B} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad C = \begin{pmatrix} c_x & 0 \\ 0 & -c_p \end{pmatrix}
\]

where \(\lambda \geq 0\) and \(c_x, c_p \geq 0\). The positivity condition reads \(\lambda^2 + c_x c_p - 1 \geq \lambda(c_x + c_p)\) while the entanglement (NPPT) condition gives

\[
\lambda^2 + c_x c_p - 1 < \lambda(c_x + c_p).
\]

And second, given an \(n\)-mode Gaussian state \(\rho_1\) with CM \(\gamma_1\), it is always possible to construct a \(2n\) mode pure Gaussian state \(|\Psi_{12}\rangle\) such that \(\text{tr}_2(\rho_{12}^1) = \rho_1\). The global CM \(\gamma_{12}\), see Eq. (4), has \(\gamma_A = \gamma_1\) and

\[
\gamma_{B} = \theta \gamma_1 \theta
\]

\[
C = J_n S \left( \oplus_{i=1}^n \lambda^2_k \mathbb{I}_2 \right) S^{-1}\theta,
\]

where \(\{\lambda_k\}\) defines the symplectic spectrum of \(\gamma_1\) and \(S\) is the symplectic matrix such that \(S^T \gamma_1 S\) is diagonal.

Having collected all these facts, let us describe how Alice and Bob can distill a key from a distillable \(n \times n\) Gaussian state using only Gaussian operations. Since all the NPPT Gaussian states can be mapped into symmetric and entangled states of two modes by GLOCC, we restrict our analysis to this type of states. In equivalent terms, one can think that the first step in the key distillation protocol is the GLOCC transformation [17] that transforms any NPPT state into an entangled state of this family. Then, both parties measure the \(X\) quadrature, where \(X_A\) and \(X_B\) denote the measured operator and \(x_A\) and \(x_B\) the obtained outcome. After communication, they only accept those cases where \(|x_A| = |x_B| = x_0\). Each party associates the logical bit \(0 (1\) to a positive (negative) result with the probability \(p(i, j)\), with \(i, j = 0, 1\). This process transforms the quantum state into a list of correlated classical bits between Alice and Bob. Their error probability, that is the probability that their symbols do not coincide, is given by

\[
\epsilon_{AB} = \frac{\sum_{i \neq j} p(i, j)}{\sum_{i, j} p(i, j)} = \frac{1}{1 + \exp\left(\frac{4\epsilon x^2}{\lambda^2 - c_p}\right)}.
\]
In order to establish a key, Alice and Bob will now apply the advantage distillation protocol introduced by Maurer [27]. Alice generates the random bit $b$. Then, she chooses $N$ items from her list of symbols, $b_A = (b_{A1}, b_{A2}, \ldots, b_{AN})$, and sends to Bob the vector $b$ such that $b_{Ai} + b_i = b \mod 2, \forall i = 1, \ldots, N$, together with the list of chosen symbols. Bob computes $b_{Bi} + b_i$ for his corresponding symbols, and if all the results are equal, $b_{Bi} + b_i = b'$, $\forall i$, the bit is accepted. If not, the symbols are discarded and the process is repeated for another vector. The new error probability is (see also [26, 27])

$$\epsilon_{AB,N} = \frac{(\epsilon_{AB})^N}{(1 - \epsilon_{AB})^N + (\epsilon_{AB})^N} \leq \left( \frac{\epsilon_{AB}}{1 - \epsilon_{AB}} \right)^N,$$

that tends to an equality for $N \rightarrow \infty$.

What is the information that Eve can obtain? As usual, all the environment, all the degrees of freedom outside Alice and Bob’s systems should be accessible to her. This means that the global state including Eve is pure, $|\Psi_{ABE}\rangle$, and such that $\text{tr}_E(|\Psi_{ABE}\rangle \langle \Psi_{ABE}|) = \rho_{AB}$ [28]. Denote by $|e_{\pm}\rangle$ Eve’s states when Alice and Bob have projected onto $|\pm x_0\rangle$. For the case of individual attacks, it was shown in [26] that Eve’s error in the estimation of the final bit $b$ is bounded from below by a term proportional to $(|e_+|e_-|)^N$. Therefore, Alice and Bob can establish a key if (see [26] for more details)

$$\frac{\epsilon_{AB}}{1 - \epsilon_{AB}} < |\langle e_+|e_-\rangle|.$$

More precisely, if this condition is fulfilled, there is always a finite $N$ such that the new list of symbols can be distilled into a secret key using one-way protocols [29].

From Eq. (9), one can compute the global pure state including Eve. Note that taking the Gaussian purification does not imply any loss of generality on Eve’s individual attack [28]. After projecting on $|\pm x_0\rangle$, Eve has a Gaussian state of two modes, with the CM and DV for the states $|e_{\pm}\rangle$ given by

$$d_{++} = -\frac{\sqrt{\lambda^2 + \lambda(c_x - c_p) - c_xc_p - 1}}{\lambda + c_x} (0, 0, x_0, x_0),$$

$$\gamma_{++} = \begin{pmatrix} \gamma_x & 0 \\ 0 & \gamma_x^{-1} \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} \lambda & c_x \\ c_x & \lambda \end{pmatrix},$$

while $\gamma_{--} = \gamma_{++}$ and $d_{--} = -d_{++}$. Now, the overlap between these two states is given by

$$|\langle e_+|e_-\rangle|^2 = \exp \left(-\frac{4(\lambda^2 + \lambda(c_x - c_p) - c_xc_p - 1)x_0^2}{\lambda + c_x} \right).$$

Substituting Eqs. (8) and (11) in (9) one can check, after some algebra, that this condition is equivalent to the entanglement condition of [4]. That is, all the distillable (NPPT) Gaussian states allow a secure key distribution under individual attacks using Gaussian operations.

Moreover, the limits for NPPT entanglement and key distillation also coincide if Eve measures in a coherent way a finite number $N_E \ll N$ of states before the reconciliation process [30]. Interestingly, these limits hold for any $x_0$, and measurements of arbitrary resolution [24].

One may wonder what happens when Eve is no longer restricted to individual attacks. Indeed, a powerful Eve could wait until the end of the advantage distillation protocol and measure in a coherent way all her $N$ symbols [31]. One can see that the corresponding security condition is similar to Eq. (9), but replacing Eve’s states overlap by its square. This new inequality is violated by some NPPT states (see figure 1). Note that this only implies that the analyzed protocol is not good for these states in this more general scenario.

Nevertheless, using the recent techniques of Ref. [20], we can find states for which the presented protocol allows to extract common bits secure against any attack. After a successful $X$ measurement on the state $\Psi$, Alice, Bob and Eve share an effective state

$$|\Psi_{ABE}^{2x2}\rangle = \sqrt{\frac{1 - \epsilon_{AB}}{2}}(|++\rangle|e_+\rangle + |--\rangle|e_-\rangle) + \sqrt{\frac{\epsilon_{AB}}{2}}(|+-\rangle|e_+\rangle + |+\rangle|e_-\rangle).$$

Then, it has been shown in Ref. [20] that the amount of secret bits, $R$, that Alice and Bob can extract from their known quantum state $\rho_{AB}$ by means of protocols using one-way communication is

$$R \geq I(x_A : x_B) - S(\rho_{AB}^{2x2}),$$

where $I$ is the mutual information between their measurement outputs and $S(\rho_{AB}^{2x2})$ the von Neumann entropy of their reduced state. In this case, this condition turns out to be dependent on $x_0$. Thus, for any state, one has to
look for a value of $x_0$ such that $R > 0$. Although we were not able to solve this problem analytically, it can be attacked using numerical methods. For example, the security curve for those states such that $c_x = c_p$ (see Eq. \[3\]) is shown in figure 1.

One can envisage different ways of improving the previous security analysis, e.g. finding better measurements for Alice and Bob or new ways of processing their measurement outcomes. A more interesting possibility consists of allowing the honest parties to manipulate in a coherent way several copies of their local states. Actually, in the study of $GK_D$ with full generality, one should deal with joint (although local and Gaussian) operations by Alice and Bob. This defines a new type of Gaussian quantum privacy amplification protocols different from entanglement distillability where Alice and Bob’s goal is simply to factor Eve out [13, 14].

A related open question that deserves further investigation is whether secret bits can be extracted from PPT Gaussian states, i.e. strict bound entangled states (cf. [7]). At present, we know that our scheme does not work for any PPT state. The details of the proof are quite involved and will be given elsewhere.

Quantum and classical distillation (QD and CD) protocols are two techniques that allow to extract secret bits from entangled states. A schematic comparison between them is shown in figure 2. In finite systems, there are examples of non-distillable quantum states for which the CD branch is possible [7]. Moving to continuous variables systems and the Gaussian scenario, QD techniques are useless for key-agreement [15, 16]. Our analysis proves that CD is still useful for (i) all NPPT states under finite coherent attacks and (ii) sufficiently entangled NPPT states under general attacks. Thus, our results close the gap between NPPT entanglement and security for the case of finite-size coherent attacks.

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![Diagram](image_url)