The Complexity of Shelflisting

Yongjie Yang and Dinko Dimitrov
Chair of Economic Theory, Saarland University, Germany

Abstract
Optimal shelflisting invites profit maximization to become sensitive to the ways in which purchasing decisions are order-dependent. We study the computational complexity of the corresponding product arrangement problem when consumers are either rational maximizers, use a satisficing procedure, or apply successive choice. The complexity results we report are shown to crucially depend on the size of the top cycle in consumers’ preferences over products and on the direction in which alternatives on the shelf are encountered.

JEL Classification Number: D00; D01; D03; D11; D20
Keywords: bounded rationality, choice from lists, computational complexity, product arrangement, top cycle

1. Introduction

There are at least two main research directions in recent works in economics devoted to the study of order or frame effects on consumers’ behavior. The first one adopts a choice theoretic approach and provides foundations for encompassing non-standard behavior models (cf. Masatlioglu and Ok, 2005; Rubinstein and Salant, 2006; Salant and Rubinstein, 2008; Bernheim and Rangel, 2009), while the second direction incorporates frames as part of players’ strategy spaces and analyzes the structure of the corresponding market equilibrium outcomes (cf. Eliaz and Spiegler, 2011; Spiegler, 2014). A complementary viewpoint is provided by experimental studies in the marketing literature (cf. Valenzuela and Raghurir, 2009; Valenzuela et al., 2013) with the focus being on consumers’ beliefs about the organization of product displays and their impact on position-based consumers’ preferences over products as well as on retailers’ actual shelflistings.

Our starting point in the present paper is that of a single shelf designer who has to arrange a given number of products on a shelf in a way that maximizes his profit. In doing so, he is facing a finite set of consumers who select a single product from the shelf by using a pre-specified choice rule. However, in sharp contrast to the cited works, we analyze the effects of consumers’ behavior on the optimal shelf listing from a computational complexity perspective.

For this, we set the shelf designer’s product arrangement problem (denoted by PA) and study its computational complexity when consumers are either rational maximizers, follow a satisficing choice rule, or apply successive choice (see Section 2 for the corresponding definitions).

We show that this decision problem is computationally easy when consumers are rational (Theorem 1), while turning to be in general more difficult when they use a satisficing choice rule (Theorems 2 and 3) as then to become hard for the case of successive choice (Theorems 4-7). The term “in general” stands as to indicate the sensibility of our results with respect to the following features of the decision problem. First, we allow a consumer to encounter the products on the shelf either from left to right or from right to left (the product arrangement problem when all consumers check the list in the same direction (from left to right) is denoted by SE-PA). Second, in the case of successive choice, allowing for non-transitive consumer preferences makes our results dependent on the size of the corresponding top cycles, that is, on the number of consumers’ top favorite products. This dependence is summarized in Table 1.

Our focus on the three types of consumers’ behavior is partially in line with the corresponding findings in Salant (2011) with respect to their state complexity. Since the problem we study is from the viewpoint of a shelf designer

---

1We will assume familiarity with basic concepts in computational complexity: exponential time, polynomial time, polynomial-time reductions, and NP-hardness.
facing various ways of product selection from lists (and we focus on its computational complexity), it is natural to expect that the corresponding statements go in rather opposite directions. For instance, while the state (or procedural) complexity of rational choice is much higher than the one of satisficing choice (see Propositions 2 and 3 in Salant, 2011), we show that the PA problem in the latter case is NP-hard, while being polynomial-time solvable in the former case. The corresponding statement with respect to the successive choice rule also applies. Notice additionally, that our results do also partially allow for a comparison of the complexity of the SE-PA problem when order effects are taken into account (that is, when consumers are satisficers or use successive choice). More precisely, as \( \log \frac{n}{\alpha} \) is smaller than 1 for large values of \( n \), \( O(n^{2m}) \) is asymptotically greater than \( O(n\log n + m) \). So, our algorithm to solve the SE-PA problem when consumers are satisficers is asymptotically faster than the one to solve this problem when there are at most three top favorite products for each consumer and successive choice is applied.

The rest of the paper is structured as follows. In Section 2 we provide the basic definitions with respect to lists, consumer preferences and the choice functions we consider. Section 3 is then devoted to the problem formulation and the case when consumers are rational maximizers, while Section 4 contains our results for consumers following consumer preferences and the choice functions we consider. Section 3 is then devoted to the problem formulation and the case when consumers are rational maximizers, while Section 4 contains our results for consumers following consumer preferences and the choice functions we consider. Section 5 contains then the ways in which the difficulty of a shelf designer’s task depends on the number of consumers’ top favorite products as well as on the direction in which they encounter the listed alternatives.

### 2. Lists, preferences, and choices

Our setup consists of the following basic ingredients.

#### Products and lists

Let \( P = \{p_1, p_2, ..., p_n\} \) be a set of \( n \) products. A list is a permutation of \( P \). We denote by \( L(P) \) the set of all \( n! \) lists over \( P \). For convenience, for each list \( L = (p_{\beta(1)}, p_{\beta(2)}, ..., p_{\beta(n)}) \in L(P) \) we refer to \( p_{\beta(1)} \) as the leftmost product and to \( p_{\beta(n)} \) as the rightmost product. Moreover, for each \( p_{\beta(i)} \) with \( i \in \{1, ..., n\} \), we refer to \( i \) as the position of the product \( p_{\beta(i)} \) in the list \( L \). Denoting by \( B = \{b_1, ..., b_m\} \) the set of \( m \) buyers (consumers), we assume that each consumer is purchasing the product delivered by her pre-specified choice rule from lists.

#### Preferences

A tournament preference is defined as a complete and asymmetric binary relation \( > \) over \( P \). In particular, \( p_i > p_j \) signifies that \( p_i \) is preferred to \( p_j \). We say that \( p_i \) is weakly preferred over \( p_j \) if either \( p_i \) is preferred to \( p_j \) or \( p_i = p_j \). The top circle of \( > \), denoted by \( TC(\succ) \), is the unique subset \( P' \subseteq P \) of minimum cardinality such that every product in \( P' \) is preferred to every product not in \( P' \). For a buyer with tournament preference \( > \), products in \( TC(\succ) \) are called her top favorite products. The set of all tournament preferences is denoted by \( \mathcal{T}(P) \). If we require a tournament preference to be transitive, then we have a linear preference. Clearly, for each linear preference \( > \), \( |TC(\succ)| = 1 \). The unique element in \( TC(\succ) \) in this case is referred to as the top product according to \( > \). Notice that a linear preference can be seen as a permutation over \( P \). Precisely, the position of a product \( p \) according to \( > \) is \( |p' \in P : p' > p| + 1 \). Hence, \( L(P) \) is the set of all linear preferences.

#### Choice functions

We assume that each consumer purchases the product determined by the outcome of her corresponding choice function. Choice functions from lists were formally introduced in Rubinstein and Salant (2006) (see also Simon, 1955 and Salant, 2003) in order to describe choice behavior potentially affected by the order of the products in a list. We first describe below the three families of such functions we consider as then to provide a more general description.

| Decision problem | Rational choice | Satisficing choice | Successive choice |
|------------------|----------------|-------------------|------------------|
| PA               | \( O(m) \)     | NP-hard           | \( O(nm) \)      |
| SE-PA            | \( O(m) \)     | \( O(n\log n + m) \) | \( O(nm) \)      |

Table 1: A summary of our results. In the entries corresponding to successive choice, \( t \) is the upper bound of the number of top favorite products of each buyer. Here \( n \) is the number of products and \( m \) is the number of buyers.
Rational choice \( f^{RC} \). Each buyer has a linear preference \( \succ \in \mathcal{L}(P) \) and chooses her most preferred product according to \( \succ \), that is, the one in \( TC(\succ) \).

Satisficing choice \( f^{SAT} \). Each buyer \( b \) has a tournament preference \( \succ \in \mathcal{T}(P) \) and a threshold product \( p(b) \in P \). The buyer chooses the first encountered product from a list that is weakly preferred to her threshold product.

Successive choice \( f^{SC} \). Each buyer has a tournament preference \( \succ \in \mathcal{T}(P) \) and chooses her product as follows. The buyer first stores her first encountered product in a register. Then, she goes through the products further and compares the currently encountered product \( p_j \) with the one \( (p_i) \) in the register. If \( p_j > p_i \), then she replaces \( p_i \) by \( p_j \) in the register and goes forward; otherwise, she goes forward without changing the product in the register. After the buyer encounters all products, she purchases the product in the register.

Clearly, a rational choice function is independent of any order effects in a list, while the other two choice functions are sensitive with respect to such order effects. Notice for instance that a consumer  \( b \) using a satisficing procedure is more likely to select \( p(b) \) when one moves \( p(b) \) toward the beginning of a list and thus, primacy effect is displayed. The magnitude of such a primacy effect crucially depends on the fact whether a consumer starts encountering the products in a list from the left side to the right side, or she checks the products from the right side to the left side. In order to pay attention to both ways for encountering the products on a shelf, we will use the following description of choice functions from lists discussed above.

For each \( X \in \{RC, SC\} \), we define \( f^X : \mathcal{L}(P) \times \mathcal{P} \times \{1, 0\} \to P \) to be a function assigning a single element \( f^X(L, Pref, v) \in P \) to every list \( L = (p_1, ..., p_n) \in \mathcal{L}(P) \), every preference \( Pref \in \mathcal{P} \) in the corresponding domain (i.e., \( \mathcal{P} = \mathcal{L}(P) \) for \( X = RC \) and \( \mathcal{P} = \mathcal{T}(P) \) for \( X = SC \)), and every \( v \in \{1, 0\} \). The third component \( v \in \{1, 0\} \) indicates from which side the corresponding buyer begins to go through the products in the list \( L \). In particular, \( "v = 1" \) means that the buyer goes through the products from the left side to the right side, and \( "v = 0" \) means the buyer goes through the products from the right side to the left side. We call buyers in the former case left-biased, and buyers in the latter case right-biased. The choice function \( f^{SAT} : \mathcal{L}(P) \times \mathcal{T}(P) \times \{1, 0\} \times P \to P \) assigns a single element \( f^{SAT}(L, \succ, v, p) \in P \) to every list \( L \), every tournament preference \( \succ \in \mathcal{T}(P) \), every \( v \in \{1, 0\} \) and every \( p \in P \). Here, the fourth component indicates the threshold product of a buyer.

3. Problem formulation and easiest shell listings

As already indicated in the Introduction, the shelf designer’s problem is to determine a product arrangement by taking into account consumers’ choice rules. More precisely, we will consider the complexity of the following problem for each \( f^X \) with \( X \in \{RC, SAT, SC\} \) by letting \( \mathcal{P} = \mathcal{L}(P) \) for \( X = RC \) and \( \mathcal{P} = \mathcal{T}(P) \) for \( X \in \{SAT, SC\} \).

Product Arrangement (PA-f^X)

Input A set \( P = \{p_1, p_2, \ldots, p_n\} \) of \( n \) products each of which has infinite supplies, a profit function \( \mu : P \to \mathbb{R}^+ \), a set \( B = \{b_1, b_2, \ldots, b_m\} \) of \( m \) buyers where each \( b_i \) is associated with a preference \( Pref_{b_i} \in \mathcal{P} \), an entering function \( \omega : B \to \{0, 1\} \), and a real number \( R \). For \( f^{SAT} \), each buyer \( b_i \in B \) is associated with a threshold product \( p(b_i) \).

Question Is there a list \( L \in \mathcal{L}(P) \) such that
\[
\sum_{b_i \in B} \mu \left( f^X(L, Pref_{b_i}, \omega(b_i)) \right) \geq R
\]
for \( X \in \{RC, SC\} \), and
\[
\sum_{b_i \in B} \mu \left( f^X(L, Pref_{b_i}, \omega(b_i), p(b_i)) \right) \geq R
\]
for \( X = SAT \)?

In the above definition, the value of \( \omega(b_i) \) indicates whether \( b_i \) is left-biased (\( \omega(b_i) = 1 \)) or right-biased (\( \omega(b_i) = 0 \)). For each \( p \in P \), \( \mu(p) \) is the profit associated with the product \( p \) when sold.
We will also consider a special case of the above problem where $\omega(b_i) = 1$ for all $b_i \in B$. We denote this problem as \textsc{Single-Enter Product Arrangement} (SE-PA-$f^S$). For simplicity, in this case, we drop $\omega(b_i)$ in the above definition.

Notice that buyers using the rational choice function choose their products regardless of how the products are arranged in a list. This directly leads to the following theorem.

**Theorem 1** SE-PA-$f^{RC}$ and PA-$f^{RC}$ are solvable in $O(m)$-time.

**Proof.** Since each buyer chooses her most preferred product regardless of how the products are arranged in the list and from which direction she encounters the products, to solve the problems stated in the theorem we need only to sum up the profits of the most preferred products of the buyers and compare the sum with $R$. If the sum is greater than or equal to $R$, return “YES”; otherwise, return “NO”.

Since we have in total $m$ buyers and it takes $O(1)$ time to calculate $TC(\succ)$ for each linear preference $\succ$, the algorithm terminates in $O(m)$ time. \qed

4. Biased consumers and satisficing choice

Let us now turn to the situation where consumers use a satisficing procedure for selecting their products from the shelf. As it turns out, the decision problem is polynomial-time solvable, provided that all consumers are encountering the alternatives in the same direction (Theorem 2), while becoming NP-hard when both left-biased and right-biased consumers are allowed (Theorem 3).

**Theorem 2** SE-PA-$f^{SAT}$ is solvable in $O(n\log n + m)$ time.

**Proof.** We prove the theorem by developing a polynomial-time algorithm of corresponding running time for the problem stated in the theorem. In what follows, for each buyer $b \in B$, let $x_\succ_b$ and $p(b)$ be the tournament preference and the threshold product of $b$, respectively.

The algorithm is quite trivial: sort the products according to the profits, from the highest to the lowest, and determine if this results in a solution. Formally, let $L$ be a list $(p_{n(1)}, \ldots, p_{n(0)})$ such that $\mu(p_{n(x)}(b)) \geq \mu(p_{n(0)}(b))$ for every $1 \leq x < y \leq n$. Then, if $\sum_{b \in B} \mu \left( f^{SAT}(L, \succ_b, p(b)) \right) \geq R$, return “YES”; otherwise return “NO”. Such a list $L$ can be constructed in $O(n \log n)$ time by the Merge sort algorithm (cf. Katajainen and Träff, 1997). In addition, it takes $O(nm)$ time for all buyers to go through $L$ and choose their products. Calculating the sum of the profits of the chosen products takes $O(m)$ time. In total, the above algorithm takes $O(m \log n + m)$ time. It remains to prove the correctness of the algorithm. For this, the following claim will be useful.

**Claim** Let $L_1 = (p_{n(1)}, \ldots, p_{n(0)})$ be a list, where $\pi$ is a permutation of $\{1, 2, \ldots, n\}$, and let, for $x \in \{1, 2, \ldots, n-1\}$, $p_{n(x)}$ and $p_{n(x-1)}$ be two consecutive products with $\mu(p_{n(x)}) > \mu(p_{n(x-1)})$. Let $L_2$ be the list obtained from $L_1$ by swapping $p_{n(x)}$ and $p_{n(x-1)}$. Then, it holds that

$$\sum_{b \in B} \mu \left( f^{SAT}(L_2, \succ_b, p(b)) \right) \geq \sum_{b \in B} \mu \left( f^{SAT}(L_1, \succ_b, p(b)) \right).$$

**Proof of the Claim.** Let $b \in B$ be any arbitrary buyer. If there is an $x'$ with $1 \leq x' < x - 1$ such that $p_{n(x')}(b) > p(b)$ or for every $y$ with $1 \leq y \leq x$ it holds that $p(b) > p_{n(y)}(b)$, then $\mu \left( f^{SAT}(L_1, \succ_b, p(b)) \right) = \mu \left( f^{SAT}(L_2, \succ_b, p(b)) \right).$ Otherwise, either $p_{n(x-1)}(b) \neq p_{n(x)}(b)$ or $p_{n(x)}(b)$ is the first encountered product that is preferred to $p(b)$, or equivalently, $f^{SAT}(L_2, b, p(b)) \in \{p_{n(x-1)}(b), p_{n(x)}(b)\}$; If $p_{n(x)}(b) \geq p(b)$, then $f^{SAT}(L_2, b, p(b)) = p_{n(x)}(b)$ and $f^{SAT}(L_1, b, p(b)) \in \{p_{n(x-1)}(b), p_{n(x)}(b)\}$; otherwise, $f^{SAT}(L_2, b, p(b)) = f^{SAT}(L_1, b, p(b)) = p_{n(x-1)}(b)$. Therefore, we have in both cases that $\mu \left( f^{SAT}(L_2, \succ_b, p(b)) \right) \geq \mu \left( f^{SAT}(L_1, \succ_b, p(b)) \right)$. It then directly follows that

$$\sum_{b \in B} \mu \left( f^{SAT}(L_2, \succ_b, p(b)) \right) \geq \sum_{b \in B} \mu \left( f^{SAT}(L_1, \succ_b, p(b)) \right).$$

This completes the proof of the claim.

Due to the above claim, if we have a solution $L$, we can obtain another solution by swapping two consecutive products in $L$ as indicated in the proof of the claim. By exhaustively performing this swap operation, we can arrive
at a list \((p_\pi(1), \ldots, p_\pi(n))\) such that \(\mu(p_\pi(x)) \geq \mu(p_\pi(y))\) for every \(1 \leq x < y \leq n\). The correctness of the algorithm then follows.

Let us now turn to the study of the PA-\(f^{SAT}\) problem for the case where there are both left-biased buyers and right-biased buyers. In a sharp contrast to the results presented so far, we show that PA-\(f^{SAT}\) is NP-hard even when buyers’ preferences are restricted to be linear. Our proof is by reduction from the following problem.

**Restricted Betweenness**

**Input** Two disjoint sets \(U = \{u_1, \ldots, u_k\}\), \(V = \{v_1, \ldots, v_y\}\), an additional element \(w \notin U \cup V\), a collection \(C\) of 3-tuples \((u_i, v_j, u_k)\) such that \(1 \leq i \neq k \leq x\) and \(1 \leq j \leq y\), and a collection \(D\) of 3-tuples \((v_i, w, u_j)\) with \(1 \leq i \neq j \leq x\).

**Question** Is there a linear order over \(U \cup V \cup \{w\}\) such that for every \((a, b, c) \in C \cup D\) the element \(b\) lies between \(a\) and \(c\)?

As the above problem has been shown to be NP-hard (cf. Opatrny, 1979), we cannot expect to have an efficient algorithm to solve the PA-\(f^{SAT}\) problem exactly, unless \(P=NP\) which is commonly believed to be unlikely.

**Theorem 3** PA-\(f^{SAT}\) is NP-hard even when all consumers have linear preferences.

**Proof.** Let \(I = (U, V, w, C, D)\) be an instance of the Restricted Betweenness problem. We create first an instance \(I’\) for the problem stated in the theorem.

The products are as follows. For each \(u \in U\), we create a product \(p(u)\) such that \(\mu(p(u)) = 2\). For each \(v \in V\), we create a product \(p(v)\) such that \(\mu(p(v)) = 1\). In addition, we create for \(w\) a product \(p(w)\) such that \(\mu(p(w)) = 0\). Hence, the set of products created is \(P = \{p(a) : a \in U \cup V\} \cup \{p(w)\}\).

The buyers are as follows. For each 3-tuple \(s = (a, b, c) \in C \cup D\), we have one left-biased consumer \(b’_l\) with threshold product \(p(b’_l)\) and one right-biased consumer \(b’_r\) with threshold product \(p(b’_r)\). Moreover, each of these two buyers prefers each of the products in the set \(\{p(a), p(b), p(c)\}\) over her threshold product, while all remaining products are ordered below it.

According to the above construction, the buyer \(b’_l\) (resp., \(b’_r\)) corresponding to \(s = (a, b, c) \in C \cup D\) chooses her first encountered product in \(\{p(a), p(b), p(c)\}\). More precisely, given a list \(L\), the buyer \(b’_l\) chooses the leftmost product among \(\{p(a), p(b), p(c)\}\) and the buyer \(b’_r\) chooses the rightmost product among \(\{p(a), p(b), p(c)\}\) from the list \(L\). Finally, we set the threshold bound \(R = 4|C| + 3|D|\).

The construction clearly takes polynomial time. It remains to prove the correctness of the reduction.

\((\Rightarrow):\) Let \(L’\) be a linear order over \(U \cup V \cup \{w\}\) such that for every \((a, b, c) \in C \cup D\) the element \(b\) is between \(a\) and \(c\). Let \(L\) be the list obtained from \(L’\) by replacing each element \(a\) of \(L’\) by the corresponding product \(p(a)\). Due to the above discussion, for each \(s = (u_i, v_j, u_k) \in C\), the two corresponding buyers \(b’_l\) and \(b’_r\) choose exactly the products \(p(u_i)\) and \(p(u_k)\), one for each buyer. As \(\mu(p(u)) = 2\) for every \(u \in U\), the total profit of the products chosen by all buyers corresponding to 3-tuples in \(C\) is exactly \(4|C|\). Analogously, for each \(s = (v_i, w, u_j) \in D\), the two corresponding buyers \(b’_l\) and \(b’_r\) choose exactly the products \(p(v_i)\) and \(p(u_j)\) with \(u_j \in U\) and \(v_i \in V\). As \(\mu(p(v)) = 1\) for every \(v \in V\), the total profit of the products chosen by all buyers corresponding to 3-tuples in \(D\) is exactly \(3|D|\).

Hence, the total profit of the products chosen by all buyers is \(R = 4|C| + 3|D|\).

\((\Leftarrow):\) Observe that to achieve the total profit \(R\), every pair of buyers corresponding to a 3-tuple in \(C\) must choose two products whose total profit is at least 4, and every pair of buyers corresponding to a 3-tuple in \(D\) must choose two products whose total profit is at least 3. Due to above discussion, this happens only if there is a list \(L\) such that for every \(s = (a, b, c) \in C \cup D\) the product \(p(b)\) is between the products \(p(a)\) and \(p(c)\). This implies that the linear order obtained from \(L\) by replacing every \(p(a)\) with the corresponding element \(a \in U \cup V \cup \{w\}\) is a solution of \(I\).

**5. Top cycles and successive choice**

Recall that consumers’ preferences in the definition of the successive choice from lists were not necessarily restricted to be transitive, that is, they may contain cycles. Clearly, since consumers go throughout the products in the entire list, the existence (and, as it turns out, the size) of top cycles matters. Lemma 1 stated below relates the consumer’s choice from a list when her corresponding top cycle is of size 3. More precisely, it is always the last
encountered product from the top favorite products that is selected. We use then Lemma 1 and the construction in the proof of Theorem 3 in order to show that the product arrangement problem is NP-hard even when each buyer has 3 most favorite products (Theorem 4).

**Lemma 1** Let \( L = (p_1, p_2, \ldots, p_n) \) be a list over \( P \) and \( b \in B \) a buyer with preference \( > \). If \( TC(>) = \{p_i, p_j, p_k\} \) with \( i < j < k \), then \( f^{SC}(L, >, 1) = p_k \) and \( f^{SC}(L, >, 0) = p_i \).

**Proof.** Consider the first situation corresponding to \( f^{SC}(L, >, 1) = p_k \). Clearly, exactly one product from the set \( \{p_i, p_j\} \) is preferred to \( p_k \). For each \( x \) with \( 1 \leq x \leq n \), let \( \text{register}(x) \) be the product in the register immediately after \( p_k \) has been encountered. Hence, \( \text{register}(1) = p_1 \). We distinguish between the following two cases corresponding to the two possible preference cycles over the product set \( \{p_i, p_j, p_k\} \).

Case A (\( p_i > p_j \)). Clearly, it must be then that \( p_k > p_j \) and \( p_j > p_i \). It is easy to check that \( \text{register}(j - 1) = p_i \), \( \text{register}(j) = \text{register}(k - 1) = p_j \), and \( \text{register}(k) = \text{register}(n) = p_k \). This implies that \( f^{SC}(L, >, 1) = p_k \).

Case B (\( p_k > p_i \)). We have now that \( p_i > p_j \) and \( p_j > p_k \) should hold. It follows that \( \text{register}(j - 1) = \text{register}(j) = \text{register}(k - 1) = p_i \), and \( \text{register}(k) = \text{register}(n) = p_k \). This again implies that \( f^{SC}(L, >, 1) = p_k \).

The proof for the situation corresponding to \( f^{SC}(L, >, 0) = p_i \) can be obtained from the above proof by swapping all occurrences of \( p_i \) and \( p_k \), and replacing all occurrences of \( 1 \) by \( 0 \).

**Theorem 4** PA-\( f^{SC} \) is NP-hard, even when each buyer has 3 top favorite products.

**Proof.** We can use the construction used in the proof of Theorem 3 with the following slight difference when creating the buyers. For each 3-tuple \( s = (a, b, c) \in C \cup D \), we have one left-biased consumer \( b'_i \) (i.e., \( \omega(b'_i) = 1 \)) and one right-biased consumer \( b'_j \) (i.e., \( \omega(b'_j) = 0 \)) whose top favorite products are \( p(a) \), \( p(b) \), and \( p(c) \). Clearly, for each of these buyers, we have either \( p(a) > p(b) > p(c) \), \( p(a) > p(b) > p(c) \), or \( p(b) > p(a) > p(c) \). Without loss of generality, assume that \( TC(\geq) = \{p(a), p(b), p(c)\} \) \( \mu \) with each of these three products being preferred to any of the remaining products. The application of Lemma 1 gives us then the same consumer choices as the ones in the proof of Theorem 3.

Our next lemma states that the outcome of the successive choice rule is always a top favorite product for the corresponding consumer (the proof is straightforward). It then follows via Lemma 2 that if each buyer has only one favorite product, then the PA-\( f^{SC} \) problem becomes polynomial-time solvable (Theorem 5).

**Lemma 2** Let \( L = (p_1, p_2, \ldots, p_n) \) be a list over \( P \) and \( b \in B \) a buyer with preference \( > \). Then, \( f^{SC}(L, >, \omega(b)) \in TC(\geq) \).

**Theorem 5** PA-\( f^{SC} \) is solvable in \( O(nm) \) time if each buyer has only one top favorite product.

**Proof.** Due to Lemma 2, each buyer chooses the product in the top circle of her preference no matter how the products are arranged in the list. Hence, we can directly adopt the algorithm for PA-\( f^{RC} \). However, since it takes \( O(n) \) time to calculate the top circle in this case, the algorithm has running time \( O(nm) \).

Consider now SE-PA-\( f^{SC} \), the special case of PA-\( f^{SC} \) where there are only left-biased consumers. We first show that if each buyer has at most 3 top products, then the problem is polynomial-time solvable.

**Theorem 6** SE-PA-\( f^{SC} \) is solvable in \( O(n^2m) \) time if \( |TC(\geq)| \leq 3 \) for every buyer \( b \in B \), where \( \geq \) is the preference of \( b \).

**Proof.** For \( P' \subseteq P \), let \( \mu_{\max}(P') = \max \{\mu(p) : p \in P'\} \). Consider the following algorithm: if \( \sum_{b \in B} \mu_{\max}(TC(\geq)) \geq R \), return “YES”; otherwise return “NO”. We show that the algorithm correctly solves the SE-PA-\( f^{SC} \) problem.

Let \( A \) be the set of buyers \( b \) such that \( |TC(\geq)| = 1 \), and \( C \) the set of buyers \( b \) such that \( |TC(\geq)| = 3 \). Due to Lemma 2, every buyer in \( A \) chooses the product in \( TC(\geq) \) for every list \( L \) over \( P \). So, the total profits from the products chosen by the buyers in \( A \) is \( \sum_{b \in A} \mu(\max(TC(\geq))) = \sum_{b \in A} \mu_{\max}(TC(\geq)) \). Let \( L = (p_{x(1)}, \ldots, p_{x(n)}) \) be a list such that \( \mu(p_{x(1)}) \leq \mu(p_{x(1)}) \) for every \( x \) and \( y \) such that \( 1 \leq x < y \leq n \). Consider now a buyer \( b \in C \). Without loss of generality, assume that \( TC(\geq) = \{p_{x(1)}, p_{x(2)}, p_{x(3)}\} \) with \( x < y < z \). Due to Lemma 1, \( f^{SC}(L, \geq) = p_{x(1)} \) with \( \mu(f^{SC}(L, \geq)) = \mu_{\max}(TC(\geq)) \) following from the definition of \( L \). In summary, the list \( L \) results in the highest total profit that can be achieved, since with respect to \( L \) every buyer \( b \) chooses a product with a highest profit among all products that are possible for the buyer to choose. The correctness of the algorithm follows.
It remains to show the running time of the algorithm. Calculating $TC(\succ_B)$ for every $b \in B$ can be done in $O(n^2)$ time (cf. Tarjan, 1972 and Sharir, 1981). Since $TC(\succ_B)$ is of size at most 3, we can calculate each $\mu_{\max}(TC(\succ_B))$ in $O(1)$ time. Summing up them takes $O(m)$ time. Since we have $m$ buyers, the whole running time of the algorithm is bounded by $O(n^2m) + O(m) = O(n^2m)$.

We prove now that if we allow each buyer to have one more top favorite product, the problem becomes NP-hard. The proof utilizes Figures 11 and Tables 4, 5 which can be found in the Appendix.

**Theorem 7** SE-PA-f$^SC$ is NP-hard even if each buyer has at most 4 top favorite products.

**Proof.** We prove the theorem by a reduction from the Restricted Betweenness problem. Let $I = (U, V, w, C, D)$ be an instance of the Restricted Betweenness problem. We create first an instance $I'$ for the problem stated in the theorem.

The products are as follows. For each $u \in U$, we create a product $p(u)$ such that $\mu(p(u)) = 31$. For each $v \in V$, we create a product $p(v)$ such that $\mu(p(v)) = 32$. In addition, we create for $w$ a product $p(w)$ such that $\mu(p(w)) = 33$. Finally, we create two dummy products $d_1$ and $d_2$ such that $\mu(d_1) = 1$ and $\mu(d_2) = 35$.

The buyers are as follows. For each 3-tuple $s = (u_i, v_j, u_k) \in C$, we create 4 buyers whose top favorite products and preferences are as shown in Figure 1. Moreover, for each $s = (v_i, w, u_j) \in D$, we create 14 buyers whose top favorite products and preferences are as shown in Figure 2. In total, we have $4|C| + 14|D|$ buyers.

Finally, we set $R = 129|C| + 464|D|$. The construction clearly takes polynomial time. It remains to prove the correctness of the reduction.

$(\Leftarrow)$ Suppose that $I$ is a YES-instance. Let $L'$ be a solution of $I$, and $L'$ be the list obtained from $L'$ by first replacing every $a \in L'$ by $p(a)$ and then installing $d_1$ and $d_2$ as $L'$’s leftmost and rightmost products, respectively. Consider now the profit from the products selected by the four buyers created for a $(u_i, v_j, u_k) \in C$. According to the constructions and Tables 2 and 3, the four buyers choose $p(v_j), p(u_k), p(u_k), d_2$, respectively, if $u_i \in L'$, $v_j \in L'$, $u_k \in L'$. Here $a \in L'$ means that $a$ is on the left side of $b$ in $L'$. In both cases, the total profit from these products is $32 + 31 + 31 + 35 = 32 + 31 + 31 + 35 + 31 = 129$. Therefore, the total profit from the products chosen by buyers corresponding to 3-tuples in $C$ with respect to $L$ is $129|C|$. Consider now a $(v_i, w, u_j) \in D$. According to the construction and Tables 4 and 5, if $v_l \in L' \cap L'$, then the 14 buyers corresponding to $(v_i, w, u_j)$ choose the following products (number of buyers: product):

1: $p(w);$
7: $d_2;$
5 + 1 = 6: $p(u_j).$

On the other side, if $u_j \in L' \cap L'$, then the 14 buyers corresponding to $(v_i, w, u_j)$ choose the following products (number of buyers: product):

1: $p(w);$
1 + 7 = 8: $p(v_i);$
5: $d_2.$

Notice then that the total profit from these products is 464 and thus, the total profit from the choices of the buyers corresponding to 3-tuples in $D$ is 464 $|D|$. Hence, the total profit from the products chosen by all buyers is exactly equal to $R$. We conclude then that $I'$ is a YES-instance.

$(\Rightarrow)$ Suppose that $I'$ is a YES-instance. Let $L$ be a solution of $I'$, and $L'$ be obtained from $L$ by first removing $d_1$ and $d_2$, and then replacing each product $p(a)$ with its corresponding element in $U \cup V \cup \{w\}$. Observe that according to the construction, the maximum total profit from the choices by the four buyers corresponding to a $(u_i, v_j, u_k) \in C$ is 129. Moreover, this happens if and only if the products in the top circles of the corresponding buyers’ preferences fall into one of the following cases (see Tables 2 and 3 for further details):

- $p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$d_1 \land p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$d_1 \land p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$d_1 \land p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;$$d_1 \land p(u_k) \in L \land p(v_j) \in L \land p(u_k) \in L \land d_2;
• $p(u_k) L d_1 L p(v_j) L p(u_k) L d_2$.

The key point here is that $d_2$ has to be chosen at least once, since otherwise the total profit from the products chosen by the four buyers corresponding to a 3-tuple in $C$ can be at most $32 \times 4 = 128$. Moreover, $d_1$ cannot be chosen by anyone, since otherwise the total profit from the products chosen by the four buyers corresponding to a 3-tuple in $C$ can be at most $35 \times 3 + 1 = 106$. So, according to Tables 2 and 3, only the above four cases and $\{(p(u_k), p(v_j)), d_1, p(u_k), d_2\}$ satisfy these conditions. However, given $\{(p(u_k), p(v_j)), d_1, p(u_k), d_2\}$, the total profit from the products chosen by the four buyers corresponding to $(u_i, v_j, u_k)$ is $\mu(u_i) + \mu(u_k) + \mu(d_2) + \mu(u_j) = 31 + 31 + 35 + 31 = 128 < 129$. A further tedious check shows that all of the above cases lead to the same total profit of 129.

Notice further that the maximal total profit from the choices of the 14 buyers corresponding to $\{(v_i, w, u_j) \in D\}$ is 464. Moreover, a tedious check shows that this happens if and only if the products in the top circles of the corresponding buyers’ preferences fall into one of the following cases (see Tables 4 and 5 for further details):

• $p(u_j) L d_1 L p(w) L p(v_j) L d_2$;
• $d_1 L p(u_j) L p(w) L p(v_j) L d_2$;
• $d_1 L p(v_j) L p(w) L p(u_j) L d_2$;
• $p(v_j) L d_1 L p(w) L p(u_j) L d_2$.

As $R = 129|C| + 464|D|$, it follows that for every $(u_i, v_j, u_k) \in C$, either $u_i L' v_j L' u_k$ or $u_k L' v_j L' u_i$ holds. Moreover, for every $(v_j, w, u_j) \in D$, we have either $v_j L' w L' u_j$ or $u_j L' w L' v_j$. This implies that $L'$ is a solution of $I$.

References

[1] Bernheim, B.D. and A. Rangel (2009): Beyond revealed preference: theoretic foundations for behavioral economics, Quarterly Journal of Economics 124, 51-104.
[2] Guttmann, W. and M. Maucher (2006): Variations on an ordering theme with constraints, in: International Federation for Information Processing, Volume 209, Fourth IFIP International Conference on Theoretical Computer Science—TCS 2006, eds. Navarro, G., Bertossi, L., Kohayakwa, Y. (Boston: Springer), pp. 77-90.
[3] Eliaz, K. and R. Spiegler (2011): Consideration sets and competitive marketing, Review of Economic Studies 78, 235-262.
[4] Katajainen, J. and J. L. Träff (1997): A meticulous analysis of mergesort programs, in: Algorithms and Complexity, Lecture Notes in Computer Science 1203, pp. 217-228.
[5] Masatlioglu, Y. and E. Ok (2005): Rational choice with a status-quo bias, Journal of Economic Theory 115, 1-29.
[6] Opatrny, J. (1979): Total ordering problem, SIAM Journal on Computing 8(1), 111-114.
[7] Rubinstein, A. and Y. Salant (2006): A model of choice from lists, Theoretical Economics 1, 3-17.
[8] Salant, Y. (2011): Procedural analysis of choice rules with applications to bounded rationality, American Economic Review 101, 724-748.
[9] Salant, Y. (2003): Limited computational resources favor rationality, Discussion Paper.
[10] Salant, Y. and A. Rubinstein (2008): (A, f): Choice with frames, Review of Economic Studies 75, 1287-1296.
[11] Sharrir, M. (1981): A strong-connectivity algorithm and its applications in data flow analysis, Computers & Mathematics with Applications 7(1), 67-72.
[12] Simon, H. A. (1955): A behavioral model of rational choice, Quarterly Journal of Economics 69, 99-118.
[13] Spiegler, R. (2014): Competitive framing, American Economic Journal: Microeconomics 6(3), 35-58.
[14] Tarjan, R. (1972): Depth-first search and linear graph algorithms, SIAM Journal on Computing 1(2), 446-160.
[15] Valenzuela, A. and P. Raghurib (2009): Position-biased beliefs: the center-stage effect, Journal of Consumer Psychology 19, 185-196.
[16] Valenzuela, A. P. Raghurib, and C. Mitakakis (2013): Shelf space schemes: myth or reality?, Journal of Business Research 66, 881-888.

8
Appendix

The following figures and tables are utilized in the proof of Theorem 7. In all tables shown below, \((a, b, c, d)\) represents the lists \((a, b, c, d)\) and \((b, a, c, d)\).

Figure 1: This figure shows the top favorite products and the preferences of the four buyers created for a \((u_i, v_j, u_k) \in C\) in the NP-hardness of the SE-PA-fSC problem in Theorem 7. An arc from a product to another product means that the former one is preferred to the latter one.

Figure 2: This figure shows the top favorite products and the preferences of the 14 buyers created for an \((v_i, w, u_j) \in \mathcal{D}\) in the NP-hardness of the SE-PA-fSC problem in Theorem 7. The number on the left side of each graph is the number of buyers with the top circle and preferences as shown in the graph on the right side. An arc from a product to another product means that the former one is preferred to the latter one.
Table 3: This table summarizes all lists of products \(p(u), p(v), p(u_i)\) and \(d_i\) chosen by the first two buyers corresponding to a 3-tuple \((u_i, v_j, u_k) \in C\) (see the two graphs above in Figure 1 for the preferences of these two buyers). The results are the products that the two buyers will choose, one for each.

| list                              | results | list                              | results |
|-----------------------------------|---------|-----------------------------------|---------|
| \(((p(u), p(u_j), d_1, p(v_j))\) | \(p(v), d_1\) | \(((p(u), p(d_1), p(v), p(v_j))\) | \(p(v), p(u)\) |
| \(((p(u), p(u_j), p(v), d_1))\)  | \(d_1, d_1\) | \(((p(u), p(u_j), p(v), p(v_j))\) | \(p(v), p(u)\) |
| \(((p(u), p(u_j), p(v), u_k))\) | \(p(v), p(v_j)\) | \(((p(u), p(u_j), p(v), p(v_i))\) | \(p(v), p(u_k)\) |
| \(((p(u), p(v), p(u), d_1))\)   | \(p(v), p(v)\) | \(((p(u), p(u_j), p(v), p(v_i))\) | \(p(v), p(u)\) |
| \(((p(u), p(v), p(u), d_1))\)   | \(p(v), p(v)\) | \(((p(u), p(u_j), p(v), p(v_i))\) | \(p(v), p(u)\) |
| \(((p(u), p(v), p(u_j), d_1))\) | \(d_1, d_1\) | \(((p(u), p(u_j), p(v), p(v_i))\) | \(p(v), p(u)\) |
| \(((p(u), p(v), p(u_j), u_k))\) | \(p(v), p(v)\) | \(((p(u), p(u_j), p(v), p(v_i))\) | \(p(v), p(u)\) |

Table 4: This table summarizes all lists of products \(p(v), p(u_j), p(v), p(w)\) and \(d_i\) chosen by the first two buyers corresponding to a 3-tuple \((v_i, w, v_j) \in D\) (see the two graphs above in Figure 1 for the preferences of these two buyers). The results are the products that the two buyers will choose.

| list                              | results | list                              | results |
|-----------------------------------|---------|-----------------------------------|---------|
| \(((p(v), p(u), p(v), d_1, p(w))\) | \(p(w), d_1\) | \(((p(u), p(d_1), p(v), p(w))\) | \(p(w), p(v)\) |
| \(((p(v), p(u), p(v), p(w), d_1))\) | \(d_1, d_1\) | \(((p(u), p(w), p(v), p(v))\) | \(p(w), p(v)\) |
| \(((p(v), p(u), p(v), p(w), u_k))\) | \(p(w), p(u_k)\) | \(((p(u), p(w), p(v), p(v_i))\) | \(p(w), p(v)\) |
| \(((p(v), p(u), p(v), p(w), d_1))\) | \(d_1, d_1\) | \(((p(u), p(w), p(v), p(v_i))\) | \(p(w), p(v)\) |
| \(((p(v), p(u), p(v), p(w), u_k))\) | \(p(w), p(u_k)\) | \(((p(u), p(w), p(v), p(v_i))\) | \(p(w), p(v)\) |

Table 5: This table summarizes all lists of products \(p(v), p(u), p(w)\) and \(d_i\) chosen by the last 5 + 7 = 12 buyers corresponding to a 3-tuple \((v_i, w, v_j) \in D\) (see the two graphs below in Figure 1 for the preferences of these 12 buyers). The results \(p, p'\) shown in the table means that first 5 buyers choose \(p\) and the last 7 buyers choose \(p'\). So, for a list with results \(p, p'\), the total profits of the products chosen by the 12 buyers is \(5p + 7p'\).

| list                              | results | list                              | results |
|-----------------------------------|---------|-----------------------------------|---------|
| \(((p(u), d_2), p(v_i), p(w))\)  | \(p(w), p(w)\) | \(((p(u), d_2), p(v), p(u)\)\) | \(p(w), p(u)\) |
| \(((p(u), d_2), p(w), p(v))\)   | \(p(w), p(w)\) | \(((d_2, p(v), p(u), p(w))\) | \(p(w), p(v)\) |
| \(((p(u), d_2), p(w), p(v), d_2))\) | \(d_2, p(w)\) | \(((p(u), d_2), p(v), p(u))\) | \(p(u), p(v)\) |
| \(((p(u), d_2), p(w), p(v), u_k))\) | \(p(w), p(u_k)\) | \(((p(u), d_2), p(v), p(u_i))\) | \(p(u), p(v)\) |
| \(((p(u), d_2), p(w), p(v), d_2))\) | \(d_2, p(v)\) | \(((p(u), d_2), p(v), p(u_i))\) | \(p(u), p(v)\) |
| \(((p(u), d_2), p(w), p(v), u_k))\) | \(p(w), p(u_k)\) | \(((p(u), d_2), p(v), p(u_i))\) | \(p(u), p(v)\) |
| \(((p(u), d_2), p(w), p(v), d_2))\) | \(d_2, p(v)\) | \(((p(u), d_2), p(v), p(u_i))\) | \(p(u), p(v)\) |

Table 6: This table summarizes all lists of products \(p(v), p(u), p(w)\) and \(d_i\) chosen by the last 5 + 7 = 12 buyers corresponding to a 3-tuple \((v_i, w, v_j) \in D\) (see the two graphs below in Figure 1 for the preferences of these 12 buyers). The results \(p, p'\) shown in the table means that first 5 buyers choose \(p\) and the last 7 buyers choose \(p'\). So, for a list with results \(p, p'\), the total profits of the products chosen by the 12 buyers is \(5p + 7p'\).