FUNCTORIALITY FOR SYMPLECTIC AND CONTACT CUTTING, AND EQUIVARIANT RADIAL-SQUARED BLOWUPS

YAELE KARSHON

Abstract. We exhibit Lerman’s cutting procedure as a functor from the category of manifolds-with-boundary equipped with free circle actions near the boundary, with so-called equivariant transverse maps, to the category of manifolds and smooth maps. We then apply the cutting procedure to differential forms that are not necessarily symplectic, to distributions that are not necessarily contact, and to submanifolds. We obtain an inverse functor from so-called equivariant radial-squared blowup.

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1. Introduction

The cutting construction.

Lerman’s symplectic cutting procedure [24], and his related contact cutting procedure [25], have had wide applications in symplectic and contact geometry. The symplectic cutting procedure takes a symplectic manifold \( M \) with a Hamiltonian circle action and a momentum map \( \mu: M \to \mathbb{R} \), passes to a regular sub-level set \( \{ \mu \geq \lambda \} \), which is a manifold-with-boundary, and collapses its boundary \( \{ \mu = \lambda \} \) along the circle action. Identifying the resulting space with the symplectic reduction of the diagonal circle action on \( M \times \mathbb{C} \) makes it into a symplectic manifold. The name “cutting” is because taking the two sub-level sets \( \{ \mu \geq \lambda \} \) and \( \{ \mu \leq \lambda \} \) and collapsing their boundaries can be considered as a “cutting” of \( M \) into two pieces. If \( M \) is a symplectic toric manifold and the circle action is a sub-circle of the toric action, then—up to equivariant symplectomorphism—this construction amounts to decomposing the momentum polytope \( \Delta \) of \( M \) into two convex polytopes by slicing it along a hyperplane, and taking the symplectic toric manifolds that correspond to the two convex polytopes.

As noted by Lerman [25, Section 2], one can carry out this construction without a symplectic form, and one can work with one-half of this construction, beginning with a manifold-with-boundary and a
circle action near the boundary\textsuperscript{1}. In this situation, the name “cutting” might be less appropriate; perhaps we can call it “collapsing” (because we collapse the boundary along the circle action) or “closing” (because we create a manifold without boundary), or perhaps “sewing” or “stitching”; though changing the name would hide the connection with classical symplectic cutting. Until this paper appears in print, I would like to hear your opinion on the terminology; drop me an email at karshon@math.toronto.edu and let me know what you think.

This cutting procedure takes a manifold-with-boundary $M$, equipped with a free action of the circle group $S^1$ on a neighbourhood $U_M$ of the boundary,\textsuperscript{2} to the quotient of $M$ by the equivalence relation $\sim$ in which distinct points are equivalent if and only if they are both in the boundary $\partial M$ and are in the same $S^1$ orbit. We denote this quotient by

$$M_{\text{cut}} := M/\sim$$

and the quotient map by

$$c: M \to M_{\text{cut}}.$$

We equip $M_{\text{cut}}$ with the quotient topology.

Lerman [25, Section 2] begins with a circle action on $\partial M$. To obtain a smooth manifold structure on $M_{\text{cut}}$, he uses the collar neighbourhood theorem to identify a neighbourhood of $\partial M$ in $M$ with $\partial M \times [0,1)$ and applies his symplectic cutting procedure (without a two-form and with the momentum map replaced by the projection to the second factor) to $\partial M \times (-1,1)$. Our line of argument is slightly different. First, in order to exhibit the cutting construction as a functor, it is not enough to begin with a circle action on $\partial M$; we need to begin with a circle action near $\partial M$. (The manifold structure on $M_{\text{cut}}$ does depend on the extension of the circle action to a neighbourhood of $\partial M$; see Remark 7.3.) Second, we characterise the manifold structure on $M_{\text{cut}}$ through its space of real valued smooth functions. We do this in Construction 1.3, followed by Lemma 1.4 and Theorem 1.5.

(Warning: the map $c: M \to M_{\text{cut}}$ is not smooth!)

As before, let $M$ be a manifold-with-boundary, with a free circle action on a neighbourhood $U_M$ of the boundary. An invariant boundary defining function\textsuperscript{3} on $M$ is a smooth function $f: M \to \mathbb{R}_{\geq 0}$.

\begin{footnotesize}
\textsuperscript{1}“Near the boundary” means “on some neighbourhood of the boundary”.
\textsuperscript{2}Necessarily, the boundary $\partial M$ is $S^1$-invariant.
\textsuperscript{3}The term “boundary defining function” is promoted in John Lee’s textbook [20, Chapter 5].
\end{footnotesize}
such that $f^{-1}(0) = \partial M$ and $df|_{\partial M}$ never vanishes, and such that $f$ is $S^1$-invariant on some $S^1$-invariant neighbourhood of $\partial M$ in $U_M$.

1.1. Remark. An invariant boundary-defining function always exists: take charts with values in $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ whose domains cover $\partial M$; project to the last coordinate, patch with a partition of unity, restrict to an invariant neighbourhood\footnote{Every neighbourhood of $\partial M$ contains an invariant neighbourhood; see Lemma A.1(4).} of $\partial M$ in $U_M$, and average with respect to the circle action; use a partition of unity to patch with the constant function on $\mathring{M}$ with value 1.

1.2. Remark. If we start from a symplectic manifold with a Hamiltonian circle action, and $M$ is the set $\mu \geq 0$ where $\mu$ is the momentum map and 0 is a regular value for $\mu$, then $f := \mu$ is an invariant boundary defining function.

We now describe the smooth cutting construction, which yields a smooth manifold structure on the topological space $M_{\text{cut}}$. We do this in Construction 1.3, Lemma 1.4, and Lemma 1.5. The technical condition in Construction 1.3(\text{F2}) is motivated by a characterization of smooth functions on $\mathbb{R}^2$ in terms of their expressions in polar coordinates; see Lemma 1.7.

1.3. Construction. Let $f: M \to \mathbb{R}_{\geq 0}$ be an invariant boundary defining function. We then consider the set $\mathcal{F}_M$ of those real valued functions $h: M_{\text{cut}} \to \mathbb{R}$ whose composition $\mathring{h} := h \circ c$ with the quotient map $c: M \to M_{\text{cut}}$ satisfies the following two conditions.

\begin{enumerate}[(\mathcal{F}1)]
\item $\left.\mathring{h}\right|_{M}: \mathring{M} \to \mathbb{R}$ is smooth.
\end{enumerate}
(F2) There exists an $S^1$-invariant open neighbourhood $V$ of $\partial M$ in $U_M$ and a smooth function $H: V \times \mathbb{C} \to \mathbb{R}$ such that
(a) $H(a \cdot x, z) = H(x, az)$ for all $a \in S^1$ and $(x, z) \in V \times \mathbb{C}$; and
(b) $\hat{h}(x) = H(x, \sqrt{f(x)})$ for all $x \in V$.

1.4. Lemma. In the setup of Construction 1.3, the set of functions $\mathcal{F}_M$ is independent of the choice of invariant boundary defining function $f$.

Proof. Let $f_o: M \to \mathbb{R}_{\geq 0}$ be another\footnote{The subscript $o$ stands for “other”} invariant boundary defining function. Then $f$ is the product of $f_o$ with a smooth function on $M$ that is everywhere positive. Let $g: M \to \mathbb{R}_{>0}$ be the square root of this positive function. Then $g$ is smooth and is $S^1$-invariant near the boundary, and $f(x) = g(x)^2 f_o(x)$ for all $x \in M$.

Fix any real valued function $h: M_{\text{cut}} \to \mathbb{R}$. Suppose that $\hat{h} := h \circ c: M \to \mathbb{R}$ satisfies Condition (F2) of Construction 1.3, with an open subset $V$ and a smooth function $H: V \times \mathbb{C} \to \mathbb{R}$. After possibly shrinking $V$, we may assume that $g$ is $S^1$ invariant on $V$. Then $\hat{h}$ satisfies Condition (F2) of Construction 1.3 with $f$ replaced by $f_o$, with the smooth function $H_o: V \times \mathbb{C} \to \mathbb{R}$ defined by $H_o(x, z) := H(x, g(x)z)$.

Because the boundary defining function does not appear in Condition (F1) of Construction 1.3, and because the function $h$ was arbitrary, we conclude that the set of functions $\mathcal{F}_M$ that is obtained from $f$ through Construction 1.3 is contained in the set of functions that is obtained from $f_o$ through Construction 1.3. Flipping the roles of $f$ and $f_o$, we conclude that these two sets of functions coincide. □

1.5. Theorem. In the setup of Construction 1.3, there exists a unique manifold structure on $M_{\text{cut}}$ such that $\mathcal{F}_M$ is the set of real valued smooth functions on $M_{\text{cut}}$.

We prove Theorem 1.5 in Section 7; see the first half of Proposition 7.1.

1.6. Remark. The decomposition $M = \hat{M} \sqcup \partial M$ of $M$ into the disjoint union of its interior and its boundary descends to a decomposition of the cut space,

$$M_{\text{cut}} = \hat{M}_{\text{cut}} \sqcup M_{\text{red}},$$
where $\tilde{M}_{\text{cut}} := c(M)$ and $M_{\text{red}} := c(\partial M) = (\partial M)/S^1$. There exist unique manifold structures on the pieces $\tilde{M}_{\text{cut}}$ and $M_{\text{red}}$ such that the quotient map restricts to a diffeomorphism $c|_{\tilde{M}}: \tilde{M} \to \tilde{M}_{\text{cut}}$ and to a principal $S^1$ bundle $c|_{\partial M}: \partial M \to M_{\text{red}}$. (For $\tilde{M}_{\text{cut}}$, this is because $c|_{\tilde{M}}: x \mapsto \{x\}$ is a bijection onto $\tilde{M}_{\text{cut}}$. For $M_{\text{red}}$, this is because, by Koszul’s slice theorem, the quotient map $\partial M \to (\partial M)/S^1$ is a principal circle bundle; see Lemma A.3(1).)

The topologies for these manifold structures are the quotient topologies induced from $\tilde{M}$ and $\partial M$. These topologies coincide with the subset topologies induced from $M_{\text{cut}}$; this is a consequence of Lemmas 4.1 and A.1(3). The inclusion map of $\tilde{M}_{\text{cut}}$ is a diffeomorphism with an open dense subset of $M_{\text{cut}}$ and the inclusion map of $M_{\text{red}}$ is a diffeomorphism with an embedded submanifold of $M_{\text{cut}}$; see Lemma 7.2 and the second half of Proposition 7.1.

In Remark 7.3 we will see that the manifold structures on $\tilde{M}_{\text{cut}}$ and on $M_{\text{red}}$ (which depend only on the circle action on the boundary $\partial M$ and not on its extension to a neighbourhood of the boundary) do not determine the manifold structure on $M_{\text{cut}}$. ☐

As a prototype for the smooth cutting procedure, and to motivate the technical condition $(\mathcal{F}2)$ in Construction 1.3, in the following lemma we characterize the smooth functions on the plane in terms of the angle and the radius-squared in polar coordinates.

1.7. Lemma. Consider the half-cylinder $M := S^1 \times [0, \infty)$, with the circle acting on the first component, and with the function $f: M \to \mathbb{R}_{\geq 0}$ given by $f(b, s) = s$. Consider the map $M \to \mathbb{C}$ given by $(b, s) \mapsto \sqrt{s} b$. A real-valued function $h: \mathbb{C} \to \mathbb{R}$ is smooth if and only if its pullback to the half-cylinder, $\hat{h}: M \to \mathbb{R}$, satisfies the following condition. There exists a smooth function $H: M \times \mathbb{C} \to \mathbb{R}$ such that

(a) $H(a \cdot x, z) = H(x, az)$ for all $a \in S^1$ and $(x, z) \in M \times \mathbb{C}$; and

(b) $\hat{h}(x) = H(x, \sqrt{f(x)})$ for all $x \in M$.

Proof. Identifying $M \times \mathbb{C}$ with $S^1 \times [0, \infty) \times \mathbb{C}$, the conditions (a) and (b) become

(a’) $H(ab, s, z) = H(b, s, az)$ for all $a \in S^1$ and $(b, s, z) \in S^1 \times [0, \infty) \times \mathbb{C}$; and

(b’) $h(\sqrt{s} b) = H(b, s, \sqrt{s})$ for all $(b, s) \in S^1 \times [0, \infty)$. 

If $h$ is smooth, then $H(b, s, z) := h(bz)$ is smooth and satisfies (a') and (b'). Conversely, if $H$ is smooth and satisfies (a') and (b'), then writing $h(z) = H(1, |z|^2, z)$, we see that $h$ is smooth. \[\square\]

**Figure 3. Cutting a cylinder**

1.8. *Remark* (*Symplectic polar coordinates; cutting*). An equivariant symplectic geometer would recognize Lemma 1.7 as constructing $C$ as the symplectic cut of the cylinder $N := S^1 \times \mathbb{R}_{\geq 0}$. Equip the cylinder with the standard symplectic form $\omega_N := ds \wedge d\theta$, where each point in the cylinder is written as $(e^{i\theta}, s)$. The circle group acts on the first component by left multiplication, with the momentum map $f : N \rightarrow \mathbb{R}$ given by $f(b, s) = s$. Also take $\mathbb{C}$, with the symplectic form $\omega_\mathbb{C} := 2dx \wedge dy$, where the complex coordinate is $z = x + iy$. Then, take the product $N \times \mathbb{C}$, with the circle action $a \cdot (n, z) = (a \cdot n, a^{-1}z)$ and the momentum map $\mu(n, z) = f(n) - |z|^2$. The reduced space $N//S^1 := \mu^{-1}(0)/S^1$ is then a symplectic manifold, which we can identify with the quotient of the super-level-set $\{f \geq 0\}$ by the equivalence relation that collapses the circle orbits in $f^{-1}(0)$.

More generally, in Lerman’s symplectic cutting construction [24], we start with a symplectic manifold $(\tilde{M}, \omega)$ with a circle action and a momentum map $f : \tilde{M} \rightarrow \mathbb{R}$ such that the circle action is free on $f^{-1}(0)$ (hence 0 is a regular level set), and we take $M := \{f \geq 0\}$. Writing an element of $\mathbb{C}$ as $z = x + iy$, we equip $\tilde{M} \times \mathbb{C}$ with the split symplectic form $\omega \oplus (2dx \wedge dy)$, with the circle action $a \cdot (m, z) = (a \cdot m, a^{-1}z)$, and with the momentum map $\mu(m, z) = f(m) - |z|^2$. The symplectic cut of $\tilde{M}$ is the reduced space $(\tilde{M} \times \mathbb{C})//S^1 := \mu^{-1}(0)/S^1$, which is a smooth manifold. A real valued function on the reduced space is smooth iff it lifts to an $S^1$-invariant smooth function on the level set $\mu^{-1}(0)$. This level set is a closed submanifold of $\tilde{M} \times \mathbb{C}$ that is contained in $M \times \mathbb{C}$; its

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6The factor 2 simplifies later formulas
invariant smooth functions are exactly the restrictions to the level set of the invariant smooth functions on $M \times \mathbb{C}$. The map $m \mapsto (m, \sqrt{f(m)})$ from $M$ to the zero level set $\mu^{-1}(0)$ descends to a bijection of $M_{\text{cut}}$ with the reduced space $\mu^{-1}(0)/S^1$; we use this bijection to define the smooth structure on $M_{\text{cut}}$. 

The idea of the proof of Theorem 1.5 is simple: locally near the boundary we can identify $M$ with open subsets of $\mathbb{R}^{n-2} \times S^1 \times \mathbb{R}_{\geq 0}$; the cutting construction makes the $S^1$ and $\mathbb{R}_{\geq 0}$ components into the angle and the radius-squared in polar coordinates on $\mathbb{C}$ as in Lemma 1.7. This implies that a smooth manifold structure with the required properties exists locally on $M_{\text{cut}}$. These smooth local manifold structures are consistent and fit into a global smooth manifold structure.

To provide accurate details, we work with the formalism of differential spaces as axiomatized by Sikorski [28, 29]; see Section 5. The smooth cutting construction makes $M_{\text{cut}}$ into a differential space; see Section 6. The consistency of the local manifold structures follows from functoriality of the smooth cutting construction with respect to inclusion maps of open subsets that are invariant near the boundary. A-priori, the smooth cutting construction defines a functor that takes values in the category of differential spaces; a-posteriori, it takes values in the category of smooth manifolds.

Functoriality first appears in Section 2, where we apply the cutting construction to so-called equivariant transverse maps. Inclusions of open subsets that are invariant near the boundary are examples of equivariant transverse maps.

In Section 3 we give local models for neighbourhoods in $M$ as open subsets of $\mathbb{R}^{n-2} \times S^1 \times [0, \infty)$.

In Section 4 we spell out some point-set-topological properties of the cut space $M_{\text{cut}}$ and of the quotient map $c: M \to M_{\text{cut}}$.

Section 5 contains an introduction to differential structures, and in Section 6 we show that Construction 1.3 makes $M_{\text{cut}}$ into a differential space.

In Section 7 we combine the local models with the functoriality of the cutting construction with respect to inclusions of open subsets that are invariant near the boundary to conclude that the cut space is a manifold.
In Section 8 we show that the cutting construction takes immersions (resp., submersions or embeddings) \( M \to N \) to immersions (resp., submersions or embeddings) \( M_{\text{cut}} \to N_{\text{cut}} \). It follows that “well behaved” submanifolds-with-boundary of \( N \) descend to submanifolds of \( N_{\text{cut}} \); see Corollary 8.2.

Sections 9–12 are about differential forms and distributions (sub-bundles of tangent bundles). For symplectic two-forms, contact one-forms, and contact distributions, we recover Lerman’s earlier results; in particular see [25, Propositions 2.7 and 2.15 and Remark 2.14]. Our current treatment puts these results in a broader context and provides more details.

In Section 9, we show that a differential form \( \beta \) on \( M \) that is basic on \( \partial M \) and invariant near \( \partial M \) descends to a differential form \( \beta_{\text{cut}} \) on \( M_{\text{cut}} \). We show that the map \( \beta \mapsto \beta_{\text{cut}} \) is linear, intertwines exterior derivatives, intertwines wedge products, is one-to-one, and takes non-vanishing forms to non-vanishing forms. We conclude that if \( \beta \) is a symplectic two-form (resp., contact one-form) on \( M \) then \( \beta_{\text{cut}} \) is a symplectic two-form (resp., contact one-form) on \( M_{\text{cut}} \).

In Section 10 we show that the pullback of \( \beta_{\text{cut}} \) under the inclusion map \( M_{\text{red}} \to M_{\text{cut}} \) coincides with the differential form \( \beta_{\text{red}} \) on \( M_{\text{red}} \) whose pullback to \( \partial M \) coincides with the pullback of \( \beta \) to \( \partial M \). We also show that if \( \beta \) is a symplectic two-form (resp., contact one-form) on \( M \) then \( \beta_{\text{red}} \) is a symplectic two-form (resp., contact one-form) on \( M_{\text{red}} \); so \( M_{\text{red}} \) is then a symplectic (resp., contact) submanifold of \( M_{\text{cut}} \).

In Sections 11 and 12, we relate the smooth cutting procedure with differential forms to the classical version of Lerman’s symplectic cutting procedure [24] and contact cutting procedure [25].

In Section 13, we show that a distribution \( E \) on \( M \) that is \( S^1 \)-invariant near \( \partial M \), is transverse to \( \partial M \), and contains the tangents to the \( S^1 \)-orbits along \( \partial M \), descends to a distribution \( E_{\text{cut}} \) on \( M_{\text{cut}} \). A foliation (resp., a contact distribution) on \( M \) with these properties gives a foliation (resp., a contact distribution) on \( M_{\text{cut}} \). In fact, for a contact distribution \( E \), it is enough to assume that \( E \) is \( S^1 \)-invariant near \( \partial M \) and contains the tangents to the \( S^1 \)-orbits along \( \partial M \).

In Appendix A, we recall some facts about actions of compact Lie groups and their quotients.

In Appendix B, we sketch the construction for simultaneous cutting along the facets of a manifold-with-corners that is equipped with
commuting circle actions near its facets. A-posteriori, such a manifold-with-corners is what is sometimes called a manifold-with-faces; in particular, its facets are embedded submanifolds-with-corners. We expect the results of this paper to generalize to this setup of simultaneous cutting. We expect that this generalization would streamline several procedures in the literature, including the unfolding of folded symplectic structures [5, 6], the passage from a non-compact cobordism between compact Hamiltonian $T$-manifolds to a compact cobordism [21], and the equivalence of categories between symplectic toric $T$ manifolds and symplectic toric $T$-bundles [17].

**Equivariant radial-squared blowups.**

Sections 14, 15, and 16 constitute a second part to the paper, in which we describe an inverse to the cutting construction. This inverse is a modification of the radial blowup construction that is inspired by Bredon and which we call the *equivariant radial-squared-blowup* construction. A baby-case is the passage from Cartesian coordinates $(x, y)$ with $x + iy = re^{i\theta}$ to symplectic polar coordinates $r^2$ (instead of $r$) and $\theta$.

We start in Section 14, with a baby-example, describing equivariant diffeomorphisms in symplectic polar coordinates. This section can be read independently of the others. In Section 15 we describe the radial blowup construction of a manifold along a closed manifold. We give more details than we found in the literature. This provides preparation for the equivariant-radial-squared-blowup construction, which we introduce in Section 16.

The radial blowup construction is functorial with respect to diffeomorphisms, so a Lie group action on a manifold $X$ that preserves a closed submanifold $F$ naturally lifts to the radial blowup $X \circ F$ of $X$ along $F$. In contrast, the equivariant radial-squared-blowup, which we denote $X \odot F$, requires a special group action: each isotropy representation must be isomorphic to a product of groups, acting on a product of their representations, where in each factor the action is either trivial or is transitive on the unit sphere. This assumption goes back to so-called *special G-manifolds*, introduced by Jänich [14] and Hsiang-Hsiang [13], and treated by Bredon [4, Chap. VI, Sec. 5 and 6] (though these authors focused on special cases). In the special case of circle actions, the equivariant-radial-squared blowup gives an inverse to the cutting construction.
To put this in context, we digress for a moment to discuss iterated (not simultaneous) radial (not equivariant-radial-squared) blowups. For a proper action of a Lie group $G$ on $X$ such that $X/G$ is connected, iterates of the radial blowup construction along minimal orbit type strata leads to a manifold-with-corners $M$ with a $G$ action with constant orbit type, which is a bundle whose base can be further collapsed to $M/G$. This resolution of a group action using the radial blowup construction goes back to Jänich [15, Section 1.3] and was further elaborated by Michael Davis [9], Duistermaat and Kolk [11, Section 2.9], and Albin and Melrose [3]. Getting back from the manifold-with-corners $M$ to the $G$-manifold $X$ is not straight-forward. Davis [9] does this by keeping track of certain “attaching data” on $M$. This iterated construction is different from the simultaneous construction that we mentioned above.

Finally, we note that, as explained by Lerman [24], his original cutting construction, which starts from a manifold cut along a hypersurface (rather than from a manifold-with-boundary as we do), provides an inverse to Gompf’s symplectic gluing construction [22]. The equivariant case is addressed in [31].

This work is inspired by collaborations with Eugene Lerman, River Chiang, Shintaro Kuroki, Ana Cannas da Silva, and Liat Kessler. With Eugene Lerman, in our classification of not-necessarily-compact symplectic toric manifolds [17], we applied a simultaneous cutting procedure, and we used the functoriality of this procedure with respect to inclusions of invariant open subsets. With River Chiang [8], we use symplectic and contact cutting while keeping track of submanifolds. With Liat Kessler [18], we apply the smooth cutting construction to obtain families of symplectic blowups. With Shintaro Kuroki [16], we use simultaneous cutting to classify smooth manifolds with so-called locally standard torus actions. With Ana Cannas da Silva, we can obtain so-called toric Lagrangians submanifolds of symplectic toric manifolds through a smooth cutting construction, continuing Cannas da Silva’s earlier work with her student Giovanni Ambrosioni.

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2. Functoriality with respect to equivariant transverse maps

Let \( M \) and \( N \) be manifolds-with-boundary equipped with free circle actions on neighbourhoods \( U_M \) and \( U_N \) of the boundaries \( \partial M \) and \( \partial N \). A map \( \psi: M \to N \) from \( M \) to \( N \) is an equivariant transverse map if it has the following properties.

• \( \psi \) is smooth.

• For some—hence every—invariant boundary defining function \( f_N: N \to \mathbb{R}_{\geq 0} \) on \( N \), the composition \( f_N \circ \psi: M \to \mathbb{R}_{\geq 0} \) is an invariant boundary defining function on \( M \).

Hence, \( \psi \) takes \( \mathring{M} \) to \( \mathring{N} \) and \( \partial M \) to \( \partial N \).

• There exists a neighbourhood \( V \) of \( \partial M \) in \( M \), contained in \( U_M \cap \psi^{-1}U_N \) and \( S^1 \)-invariant, such that \( \psi|_V : V \to U_N \) is \( S^1 \)-equivariant.

2.1. Example. Let \( M \) be a manifold-with-boundary, equipped with a free circle action on a neighbourhood \( U_M \) of the boundary. Then the inclusion maps of \( U_M \) and of \( M \) into \( M \) are equivariant transverse maps.

Manifolds-with-boundary equipped with free circle actions on neighbourhoods of the boundary, and their equivariant transverse maps, form a category. The cutting construction gives a functor on this category, which we describe in the following lemmas.

2.2. Lemma. Let \( M \) and \( N \) be manifolds-with-boundary, equipped with free circle actions on neighbourhoods of the boundary. Let \( M_{\text{cut}} \) and \( N_{\text{cut}} \) be the corresponding cut spaces, and let \( c_M: M \to M_{\text{cut}} \) and \( c_N: N \to N_{\text{cut}} \) be the quotient maps. Let \( \psi: M \to N \) be an equivariant transverse map. Then there exists a unique map \( \psi_{\text{cut}}: M_{\text{cut}} \to N_{\text{cut}} \)
such that the following diagram commutes.

\[ \begin{array}{ccc}
M & \xrightarrow{\psi} & N \\
\downarrow{c_M} & & \downarrow{c_N} \\
M_{\text{cut}} & \xrightarrow{\psi_{\text{cut}}} & N_{\text{cut}}
\end{array} \]

Moreover,

- \( \psi_{\text{cut}} \) takes \( M_{\text{cut}} \) to \( N_{\text{cut}} \) and \( M_{\text{red}} \) to \( N_{\text{red}} \).
- If \( \psi \) is one-to-one, so is \( \psi_{\text{cut}} \). If \( \psi \) is onto, so is \( \psi_{\text{cut}} \).
- \( \psi_{\text{cut}} \) is continuous.
- If \( \psi \) is open, so is \( \psi_{\text{cut}} \).
- If \( \psi \) is open as a map to its image, so is \( \psi_{\text{cut}} \).

2.3. Remark. The quotient maps \( c_M \) and \( c_N \) are not open. \( \diamond \)

A subset \( W \) of \( M \) is **saturated** with respect to the equivalence relation \( \sim \) if for every two points \( x \) and \( x' \) in \( M \), if \( x \in W \) and \( x \sim x' \), then \( x' \in W \).

**Proof of Lemma 2.2.** Because \( \psi \) restricts to an \( S^1 \) equivariant map \( \psi|_{\partial M} : \partial M \to \partial N \), it descends to a unique map \( \psi_{\text{cut}} \) such that the diagram commutes. Because \( \psi \) takes \( M \) to \( N \) and \( \partial M \) to \( \partial N \), the map \( \psi_{\text{cut}} \) takes \( M_{\text{cut}} \) to \( N_{\text{cut}} \) and \( M_{\text{red}} \) to \( N_{\text{red}} \). Assuming that \( \psi \) is onto, onto-ness of \( \psi_{\text{cut}} \) follows from that of \( c_N \). Assuming that \( \psi \) is one-to-one, one-to-one-ness of \( \psi_{\text{cut}} \) follows from those of \( \psi|_{\partial M} \) and of \( \psi|_{\partial M} \) and the equivariance of \( \psi|_{\partial M} \). The continuity of \( \psi_{\text{cut}} \) follows by chasing the commuting diagram, noting that \( \psi \) and \( c_N \) are continuous and that the topology of \( M_{\text{cut}} \) is induced from the quotient map \( c_M : M \to M_{\text{cut}} \). Assuming that \( \psi \) is open, the openness of \( \psi_{\text{cut}} \) also follows by chasing the commuting square, noting that \( c_M \) is continuous, that \( \psi \) is open and takes saturated sets to saturated sets, and that the topology of \( N_{\text{cut}} \) is induced from the quotient map \( c_N : N \to N_{\text{cut}} \). A similar argument holds if \( \psi \) is open as a map to its image, noting that the image of \( \psi \) is the preimage of the image of \( \psi_{\text{cut}} \). \( \square \)

2.4. **Lemma.** Let \( M_1, M_2, \) and \( M_3 \) be manifolds with boundary, with free circle actions near the boundary. Let

\[ M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} M_3 \]

be equivariant transverse functions. Then \( (\psi_2 \circ \psi_1)_{\text{cut}} = (\psi_2)_{\text{cut}} \circ (\psi_1)_{\text{cut}} \).
Proof. Since $\psi_1$ takes $\hat{M}_1$ to $\hat{M}_2$ and $\psi_2$ takes $\hat{M}_2$ to $\hat{M}_3$, and by the definition of $(\psi_1)_{\text{cut}}$ and $(\psi_2)_{\text{cut}}$, the equality is true on the open dense subset $(\hat{M}_1)_{\text{cut}}$. By continuity, the equality is true everywhere. □

2.5. Lemma. Let $M$ be a manifold with boundary, with a free circle action near the boundary. The identity map $\text{Id}: M \to M$ is an equivariant transverse map, and $\text{Id}_{\text{cut}}$ is the identity map on $M_{\text{cut}}$.

Proof. By the definition of the map $\text{Id}_{\text{cut}}$, this map restricts to the identity map on $(\hat{M}_1)_{\text{cut}}$. By continuity, $\text{Id}_{\text{cut}}$ is the identity map everywhere. □

2.6. Corollary. Let $M$ and $N$ be manifolds with boundary, with free circle actions near their boundaries. Let $\psi: M \to N$ be an equivariant diffeomorphism; let $\psi^{-1}: N \to M$ be its inverse. Then $\psi_{\text{cut}}: M_{\text{cut}} \to N_{\text{cut}}$ is invertible, and its inverse is $(\psi^{-1})_{\text{cut}}: N_{\text{cut}} \to M_{\text{cut}}$.

Proof. This follows from Lemma 2.4 and Lemma 2.5. □

Thus, we get a functor, from the category whose objects are manifolds-with-boundary equipped with free circle actions near the boundary and whose morphisms are equivariant transverse maps, to the category of topological spaces and their continuous maps.

Until now, we only considered the cut space as a topological space. By Lemmas 2.2, 2.4, and 2.5, the cutting construction defines a functor from the category of manifolds-with-boundary equipped with free circle actions on neighbourhoods of their boundaries, with their equivariant transverse maps, to the category of topological spaces, with their continuous maps. The following lemma shows that the cutting functor also takes smooth maps to smooth maps, where the meaning of “smooth” is in terms of the collections of real valued functions of Construction 1.3.

2.7. Lemma. Let $M$ and $N$ be manifolds with boundary, with free circle actions near the boundary, and let $\mathcal{F}_M$ and $\mathcal{F}_N$ be the sets of real valued functions on $M_{\text{cut}}$ and on $N_{\text{cut}}$ that are obtained from Construction 1.3. Let $\psi: M \to N$ be an equivariant transverse map. Then for each function $h_N$ in $\mathcal{F}_N$, the composition $h_M := h_N \circ \psi_{\text{cut}}$ is in $\mathcal{F}_M$.

Proof. Let $h_N: N_{\text{cut}} \to \mathbb{R}$ be in $\mathcal{F}_N$. We would like to show that $h_M := h_N \circ \psi_{\text{cut}}: M_{\text{cut}} \to \mathbb{R}$ is in $\mathcal{F}_M$. Write

$$(h_M \circ c_M)|_{\hat{M}} = (h_N \circ \psi_{\text{cut}} \circ c_M)|_{\hat{M}} = (h_N \circ c_N) \circ \psi|_{\hat{M}}.$$ 

The right hand side is smooth, because $\psi$ restricts to a smooth function from $\hat{M}$ to $\hat{N}$, and because—since $h_N$ is in $\mathcal{F}_N$—the composition $h_N \circ$
\( h_c \) induces a homeomorphism about the origin in all \( y \) satisfies Condition (\( h \)).

Because \( \psi : M \to N \) is an equivariant transverse map, there exists a neighbourhood \( V_\psi \) of \( \partial M \), contained in \( U_M \cap \psi^{-1}U_N \) and \( S^1 \)-invariant, such that \( \psi|_{V_\psi} : V_\psi \to U_N \) is \( S^1 \) equivariant. Fix such a \( V_\psi \).

Let \( f_N : N \to \mathbb{R}_{\geq 0} \) be an invariant boundary defining function on \( N \). Let \( U_{f_N} \) be a neighbourhood of \( \partial N \), contained in \( U_N \) and \( S^1 \)-invariant, on which \( f_N \) is \( S^1 \)-invariant. Then \( f_M := f_N \circ \psi \) is an invariant boundary defining function on \( M \), and \( U_{f_M} := V_\psi \cap \psi^{-1}U_{f_N} \) is a neighbourhood of \( \partial M \), contained in \( U_M \) and \( S^1 \)-invariant, on which \( f_M \) is \( S^1 \)-invariant.

Because \( h_N \) is in \( \mathcal{F}_N \), there exist an \( S^1 \)-invariant open neighbourhood \( V_N \) of \( \partial N \), contained in \( U_N \) and \( S^1 \)-invariant, and a smooth function \( H_N : V_N \times \mathbb{C} \to \mathbb{R} \), such that \( H_N(a \cdot y, z) = H_N(y, az) \) for all \( a \in S^1 \) and \( (y, z) \in V_N \times \mathbb{C} \), and such that \( h_N(c_N(y)) = H_N(y, \sqrt{f_N(y)}) \) for all \( y \in V_N \). Fix such \( V_N \) and \( H_N \).

\[ V_M := U_{f_M} \cap \psi^{-1}V_N \] is a neighbourhood of \( \partial M \), contained in \( U_{f_M} \) and \( S^1 \)-invariant. Define \( H_M : V_M \times \mathbb{C} \to \mathbb{R} \) by \( H_M(x, z) := H_N(\psi(x), z) \).

Then \( H_M \) is smooth, \( H_M(a \cdot x, z) = H_M(x, az) \) for all \( (x, z) \in V_M \times \mathbb{C} \) and \( a \in S^1 \), and \( h_M(c_M(x)) = H_M(x, \sqrt{f_M(x)}) \) for all \( x \in V_M \). Thus, \( h_M \circ c_M \) satisfies Condition (\( \mathcal{F}2 \))(a,b) of Construction 1.3 with respect to \( f_M \).

\[ \Box \]

3. A local model

Let \( \epsilon > 0 \) be a positive number, let \( D^2 \) be the open disc of radius \( \sqrt{\epsilon} \) about the origin in \( \mathbb{R}^2 \), and let \( D^{n-2} \) be any open disc in \( \mathbb{R}^{n-2} \).

3.1. Lemma. Equip

\[ N := D^{n-2} \times S^1 \times [0, \epsilon) \]

with the circle action \( a \cdot (\xi, b, s) = (\xi, ab, s) \) and with the invariant boundary defining function \( (\xi, b, s) \mapsto s \). Then the map

\[ \tilde{\psi} : N \to D^{n-2} \times D^2 \]

given by \( (\xi, b, s) \mapsto (\xi, b\sqrt{s}) \)

induces a homeomorphism

\[ \psi : N_{\text{cut}} \to D^{n-2} \times D^2 \]
and induces a bijection

\[ f \mapsto f \circ \psi \]

from the set of real valued smooth functions on \( D^{n-2} \times D^2 \) to the set \( \mathcal{F}_N \) of real valued functions on \( N_{\text{cut}} \) that is described in Construction 1.3.

**Proof.** By its definition, the map \( \hat{\psi} \) descends to a bijection \( \psi \), such that we have a commuting diagram

\[ \begin{array}{ccc}
N & \xrightarrow{c} & N_{\text{cut}} \\
\downarrow{\hat{\psi}} & & \downarrow{\psi} \\
\hat{\psi} & & D^{n-2} \times \mathbb{C}.
\end{array} \]

Because \( \hat{\psi} \) is continuous and the topology of \( N_{\text{cut}} \) is induced from the quotient map \( c: N \to N_{\text{cut}} \), the map \( \psi \) is continuous. Because \( \hat{\psi} \) is proper and \( c \) is continuous, \( \psi \) is proper; because the target space \( D^{n-2} \times \mathbb{C} \) is Hausdorff and locally compact, the proper map \( \psi \) is a closed map. Being a continuous bijection and a closed map, \( \psi \) is a homeomorphism.

We need to show that a function \( f: D^{n-2} \times D^2 \to \mathbb{R} \) is smooth iff the function \( \hat{h}(\xi, b, s) := f(\xi, b\sqrt{s}) \) from \( N \to \mathbb{R} \) satisfies Conditions (\( F_1 \)) and (\( F2 \)) of Construction 1.3. Because \( \hat{\psi} \) restricts to a diffeomorphism from \( \hat{\psi} \) to \( D^{n-2} \times (D^2 \setminus \{0\}) \), Condition (\( F1 \)) of Construction 1.3 is equivalent to the restriction of \( f \) to \( D^{n-2} \times (D^2 \setminus \{0\}) \) being smooth.

It remains to show that \( \hat{\psi} \) satisfies Condition (\( F2 \)) of Construction 1.3 if and only if \( f \) is smooth near \( D^{n-2} \times \{0\} \).

We proceed as in Lemma 1.7.

Suppose that \( \hat{h} \) satisfies Condition (\( F2 \)) of Construction 1.3. Let \( V \) be an \( S^1 \)-invariant open neighbourhood of \( \partial N \) and \( H: V \times D^2 \to \mathbb{R} \) a smooth function that satisfies Conditions (\( F2 \))(a,b) of Construction 1.3. Because \( V \) is an \( S^1 \)-invariant open subset of \( N \), its image in \( D^{n-2} \times D^2 \) is open. Writing \( f(\xi, z) = H((\xi, 1, |z|^2), z) \), we see that \( f \) is smooth on this image.

Conversely, suppose that \( f \) is smooth near \( D^{n-2} \times \{0\} \). Let \( V \) be the preimage in \( N \) of an \( S^1 \)-invariant neighbourhood of \( D^{n-2} \times \{0\} \) on which \( f \) is smooth. Then the function \( H: V \times D^2 \to \mathbb{R} \) defined by \( H((\xi, b, s), z) := f(\xi, bz) \) satisfies Conditions (\( F2 \))(a,b) of Construction 1.3. \[ \square \]

**3.2. Proposition.** Let \( M \) be an \( n \) dimensional manifold-with-boundary, equipped with a circle action on a neighbourhood \( U_M \) of the boundary. Let \( f: M \to \mathbb{R} \) be an invariant boundary defining function. Then for
each point \( x \) in \( \partial M \) there exists an \( S^1 \)-invariant open neighbourhood \( W \) of \( x \) in \( U_M \), a positive number \( \epsilon > 0 \), and a diffeomorphism
\[
W \rightarrow D^{n-2} \times S^1 \times [0, \epsilon)
\]
that intertwines the \( S^1 \) action on \( W \) with the rotations of the middle factor and that intertwines the function \( f|_W \) with the projection to the last factor.

Moreover, if \( v \) is a vector field near \( \partial M \) that is transverse to \( \partial M \) and that is \( S^1 \)-invariant near \( \partial M \), then this diffeomorphism can be chosen such that \( v \) is tangent to the fibres of the projection \( W \rightarrow D^{n-2} \times S^1 \).

Proposition 3.2 is a minor variation of the collar neighbourhood theorem. For completeness, we include a proof.

**Proof of Proposition 3.2.** Let \( v \) be a vector field near \( \partial M \) that is transverse to \( \partial M \) and that is \( S^1 \)-invariant near \( \partial M \). (We can obtain such a \( v \) by using local coordinates charts near \( \partial M \) to obtain inward-pointing vector fields, patching these vector fields with a partition of unity, and averaging with respect to the \( S^1 \) action.) After multiplying \( v \) by a non-vanishing smooth function, we may assume that the derivative \( vf \) of \( f \) along \( v \) is equal to one. The forward-flow of the vector field \( v \) determines a diffeomorphism from an open neighbourhood \( U \) of \( \partial M \times \{0\} \) in \( \partial M \times [0, \infty) \) to an open neighbourhood of \( \partial M \) in \( M \) that intertwines the \( S^1 \) action on the \( \partial M \) factor with the \( S^1 \) action on \( M \) and whose composition with \( f \) is the projection to the \( [0, \infty) \) factor.

Let \( x \in \partial M \). Let \( \epsilon > 0 \) be such that \( \{x\} \times [0, \epsilon] \) is contained in \( U \). Using a chart for \( \partial M \) near \( x \), we obtain an open disc \( D^{n-2} \) about the origin in \( \mathbb{R}^{n-2} \) and an embedding of \( D^{n-2} \) into \( \partial M \) that takes the origin to \( x \) and is transverse to the \( S^1 \) orbit through \( x \). “Sweeping” by the circle action, and possibly shrinking \( D^{n-2} \), we obtain an open neighbourhood \( W' \) of \( S^1 \cdot x \) in \( \partial M \) such that \( W' \times [0, \epsilon] \) is still contained in \( U \) and a diffeomorphism from \( W' \) to \( D^{n-2} \times S^1 \) that intertwines the \( S^1 \) action on \( W' \) with the rotations of the \( S^1 \) factor. To finish, take \( W \) to be the image of \( W' \times [0, \epsilon] \) in \( M \), and compose the inverse of the diffeomorphism \( W' \times [0, \epsilon] \rightarrow W \) with the diffeomorphism \( W' \rightarrow D^{n-2} \times S^1 \). \( \square \)

4. The cut space as a topological space

Let \( M \) be an \( n \) dimensional manifold-with-boundary, equipped with a circle action on a neighbourhood \( U_M \) of the boundary. Let \( M_{\text{cut}} = M/\sim \)
be obtained from $M$ by the cutting construction, and let $c : M \to M_{\text{cut}}$ be the quotient map. Equip $M_{\text{cut}}$ with the quotient topology.

Let $W$ be an open subset of $M$; then $W$ is a manifold-with-boundary, and its boundary is given by $\partial W = \partial M \cap W$. We say that such an $W$ is $S^1$-invariant near its boundary if there exists an $S^1$-invariant open neighbourhood of $\partial M$ in $U_M$ whose intersection with $W$ is $S^1$-invariant. For example, open subsets of $M$ and $S^1$-invariant open subsets of $U_M$ are $S^1$-invariant near their boundary (for subsets of $M$, this condition is vacuous).

4.1. Lemma. For any open subset $W$ of $M$ that is $S^1$-invariant near its boundary, $W_{\text{cut}}$ is an open subset of $M_{\text{cut}}$, and the quotient topology of $W_{\text{cut}}$ that is induced from $W$ agrees with the subset topology on $W_{\text{cut}}$ that is induced from $M_{\text{cut}}$.

Proof. Because the open subset $W$ of $M$ is $S^1$-invariant near its boundary, the inclusion map of $W$ into $M$ is an equivariant transverse map. This map is one-to-one, continuous, and open. By Lemma 2.2, the inclusion map of $W_{\text{cut}}$ into $M_{\text{cut}}$ is one-to-one, continuous, and open, so it is a homeomorphism with an open subset of $M_{\text{cut}}$. □

4.2. Lemma. $M_{\text{cut}}$ is Hausdorff.

Proof. Let $y_1$ and $y_2$ be distinct points of $M_{\text{cut}}$.

- Suppose that $y_1 = \{x_1\}$ and $y_2 = \{x_2\}$ for distinct points $x_1$ and $x_2$ of $M$. Because $M$ is Hausdorff, there exist disjoint open subsets $V_1$ and $V_2$ of $M$ such that $x_1 \in V_1$ and $x_2 \in V_2$. Let $V'_1 := M \cap V_1$ and $V'_2 := M \cap V_2$.

- Suppose that $y_1 = S^1 \cdot x_1$ and $y_2 = S^1 \cdot x_2$ are distinct orbits in $\partial M$. Then $y_1$ and $y_2$ are disjoint compact subsets of $M$ (see Lemma A.1(1)). Because $M$ is Hausdorff, there exist disjoint open subsets $V'_1$ and $V'_2$ of $M$ such that $y_1 \subset V'_1$ and $y_2 \subset V'_2$. Let $V'_1 := \bigcap_{a \in S^1} a \cdot (U_M \cap V'_1)$ and $V'_2 := \bigcap_{a \in S^1} a \cdot (U_M \cap V'_2)$; these subsets of $M$ are open (by Lemma A.1(4)).

- Otherwise, after possibly switching $y_1$ and $y_2$, we may assume that $y_1 = S^1 \cdot x_1$ for $x_1 \in \partial M$ and $y_2 = \{x_2\}$ for $x_2 \in M$. As before, $y_1$ and $y_2$ are disjoint compact subsets of $M$, so there exist disjoint open subsets $V'_1$ and $V'_2$ of $M$ such that $y_1 \subset V'_1$ and $y_2 \subset V'_2$. Let $V'_1 := \bigcap_{a \in S^1} a \cdot (U_M \cap V'_1)$ and $V'_2 := M \cap V'_2$; as before, these subsets of $M$ are open.
In each of these cases, $V'_1$ and $V'_2$ are open subsets of $M$ that are $S^1$-invariant near their boundaries, and $y_1 \subset V'_1 \subset V_1$ and $y_2 \subset V'_2 \subset V_2$, where $V_1$ and $V_2$ are disjoint. By Lemma 4.1, $(V'_1)_{\text{cut}}$ and $(V'_2)_{\text{cut}}$ are then disjoint open neighbourhoods of $y_1$ and $y_2$ in $M_{\text{cut}}$. □

4.3. Lemma. $M_{\text{cut}}$ is second countable.

Proof. Let $\mathcal{U}$ be a countable basis for the topology of $M$. We will show that $\mathcal{U}' := \{(M \cap U)_{\text{cut}} \cup \{(S^1 \cdot (U_M \cap U))_{\text{cut}}\}_{U \in \mathcal{U}}$ is a countable basis for the topology of $M_{\text{cut}}$.

The countability of $\mathcal{U}'$ follows from that of $\mathcal{U}$.

For each $U \in \mathcal{U}$, the sets $\hat{M} \cap U$ and $S^1 \cdot (U_M \cap U)$ are open in $M$ (see Lemma A.1(3)) and are $S^1$-invariant near their boundaries. By Lemma 4.1, these sets are open in $M_{\text{cut}}$.

Let $y$ be a point in $M_{\text{cut}}$ and $W$ an open neighbourhood of $y$ in $M_{\text{cut}}$.

- Suppose that $y = \{x\}$ for $x \in \hat{M}$. Let $U \in \mathcal{U}$ be such that $x \in U \subset c^{-1}(W)$. Then $U' := (M \cap U)_{\text{cut}}$ is in $\mathcal{U}'$, and $y \in U' \subset W$.

- Suppose that $y = S^1 \cdot x$ for $x \in \partial M$. By Lemma A.1(4), $V := \bigcap_{a \in S^1} a \cdot (U_M \cap c^{-1}(W))$ is open (in $U_M$, hence) in $M$. Let $U \in \mathcal{U}$ be such that $x \in U \subset V$. Then $U' := (S^1 \cdot (U_M \cap U))_{\text{cut}}$ is in $\mathcal{U}'$, and $y \in U' \subset W$.

In either case, we found an element $U'$ of $\mathcal{U}'$ such that $y \in U' \subset W$. Because $y$ and $W$ were arbitrary, $\mathcal{U}'$ is a basis for the topology. □

4.4. Lemma. Each point in $M_{\text{cut}}$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Proof. Take a point $y$ in $M_{\text{cut}}$.

- Suppose that $y = \{x\}$ for $x \in \hat{M}$. Let $W$ be the intersection of $\hat{M}$ with the domain of a coordinate chart on $M$ that contains $x$. By Lemma 4.1, $W_{\text{cut}}$ is an open neighbourhood of $y$ in $M_{\text{cut}}$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

- Suppose that $y = S^1 \cdot x$ for $x \in \partial M$. Let $W \to D^{n-2} \times S^1 \times [0, \epsilon)$ be a diffeomorphism as in Proposition 3.2. By Lemma 2.2, this diffeomorphism descends to a homeomorphism $W_{\text{cut}} \to (D^{n-2} \times S^1 \times [0, \epsilon))_{\text{cut}}$. 

Lemma 3.1 gives a homeomorphism
\[(D^{n-2} \times S^1 \times [0, \epsilon])_{\text{cut}} \rightarrow D^{n-2} \times D^2(\sqrt{\epsilon}).\]

By Lemma 4.1, \(W_{\text{cut}}\) is an open neighbourhood of \(y\) in \(M_{\text{cut}}\), and its topology as a subset of \(M_{\text{cut}}\) agrees with its quotient topology that is induced from \(W\). Composing the above two homeomorphisms, we conclude that this neighbourhood of \(y\) in \(W_{\text{cut}}\) is homeomorphic to an open subset of \(\mathbb{R}^n\).

\[\square\]

5. Differential spaces

A differential space is a set that is equipped with a collection of real valued functions, of which we think as the smooth functions on the set, that satisfies a couple of axioms. Variants of this structure were used by many authors. In particular, we would like to mention Bredon [4, Chap. VI, §1]. Our definition follows Sikorski [28, 29]. We find it convenient to phrase it in terms of \(C^\infty\) rings, of which we learned from Eugene Lerman. For further developments, see Śniatycki’s book [30].

A non-empty collection \(\mathcal{F}\) of real-valued functions on a set \(X\) is a \(\mathcal{C}^\infty\)-ring (with respect to the usual composition operations) if for any positive integer \(n\) and functions \(h_1, \ldots, h_n\) in \(\mathcal{F}\), and for any smooth real valued function \(g\) on \(\mathbb{R}^n\), the composition \(x \mapsto g(h_1(x), \ldots, h_n(x))\) is in \(\mathcal{F}\).

The initial topology determined by a collection \(\mathcal{F}\) of functions on a set \(X\) is the smallest topology on \(X\) for which the functions in \(\mathcal{F}\) are continuous.

If \(\mathcal{F}\) is a \(\mathcal{C}^\infty\)-ring, then the preimages of open intervals by elements of \(\mathcal{F}\) are a basis (not only a sub-basis) for the initial topology.

5.1. Remark. Let \(X\) be a topological space, and let \(\mathcal{F}\) be a \(\mathcal{C}^\infty\)-ring \(\mathcal{F}\) of real-valued functions on \(X\). Then the initial topology determined by \(\mathcal{F}\) coincides with the given topology on \(X\) iff

- All the functions in \(\mathcal{F}\) are continuous; and
- \(X\) is \(\mathcal{F}\)-regular, in the following sense: for each closed subset \(C\) of \(X\) and point \(x \in X \setminus C\), there exists a function \(f \in \mathcal{F}\), such that \(f(x) \neq 0\) and such that \(f\) vanishes on a neighbourhood of \(C\).
Indeed, the initial topology is contained in the given topology iff the functions in $F$ are continuous, and the given topology is contained in the initial topology iff $X$ is $F$-regular.

Let $F$ be a non-empty collection of real-valued functions on set $X$. Equip $X$ with the initial topology determined by $F$. The collection $F$ is a **differential structure** if it is a $C^\infty$-ring and it satisfies the following **locality condition**. Given any function $h: X \to \mathbb{R}$, if for each point in $X$ there exists a neighbourhood $V$ and a function $g$ in $F$ such that $h|_V = g|_V$, then $h$ is in $F$.

A **differential space** is a set equipped with a differential structure. Given differential spaces $(X, F_X)$ and $(Y, F_Y)$, a map $\psi: X \to Y$ is **smooth** if for any function $h$ in $F_Y$ the composition $h \circ \psi$ is in $F_X$; the map $\psi$ is a **diffeomorphism** if it is a bijection and it and its inverse are both smooth.

On a subset $A$ of a differential space $(X, F)$, the **subset differential structure** consists of those functions $h: A \to \mathbb{R}$ such that, for every point in $A$, there exist a neighbourhood $U$ in $X$ and a function $g$ in $F$ such that $h|_{U \cap A} = g|_{U \cap A}$.

A manifold $M$, equipped with the set of real valued functions that are infinitely differentiable, is a differential space. A map between manifolds is infinitely differentiable if and only if it is smooth in the sense of differential spaces. In this way, we identify manifolds with those differential spaces $X$ that are Hausdorff and second countable and that are locally diffeomorphic to Cartesian spaces in the following sense: for each point in $X$ there exist a neighbourhood $U$ and a diffeomorphism of $U$ with an open subset of a Cartesian space $\mathbb{R}^n$. Manifolds with boundary, and manifolds with corners [20, Chap. 16], are similarly identified with those differential spaces that are Hausdorff and second countable and that are locally diffeomorphic to half-spaces $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ or, respectively, to orthants $\mathbb{R}^n_{\geq 0}$.

### 6. The cut space as a differential space

Let $M$ be a manifold-with-boundary, equipped with a free circle action on a neighbourhood $U_M$ of the boundary. Let $M_{\text{cut}}$ be obtained from $M$ by the cutting construction, and let $c: M \to M_{\text{cut}}$ be the quotient map. Let $F_M$ be the set of real valued functions on $M$ that is described in Construction 1.3 and Lemma 1.4.

6.1. **Lemma.** $F_M$ is a $C^\infty$ ring of real-valued functions on $M_{\text{cut}}$. 


Proof. Let \( n \) be a positive integer, and let \( h_1, \ldots, h_n \) be functions in \( \mathcal{F}_M \). Let \( g \) be a smooth real valued function on \( \mathbb{R}^n \). Define \( h: M_{\text{cut}} \to \mathbb{R} \) by \( h(x) := g(h_1(x), \ldots, h_n(x)) \). We need to prove that the function \( \hat{h} := h \circ c: \hat{M} \to \mathbb{R} \) satisfies Conditions (\( \mathcal{F}1 \)) and (\( \mathcal{F}2 \)) of Construction 1.3.

Let \( \hat{h}_i := h_i \circ c \) for \( i = 1, \ldots, n \).

- The functions \( \hat{h}_1, \ldots, \hat{h}_n \) satisfy Condition (\( \mathcal{F}1 \)) of Construction 1.3, so they are smooth on \( \hat{M} \). Because the function \( \hat{h} \) can be written as the composition \( g(\hat{h}_1(\cdot), \ldots, \hat{h}_n(\cdot)) \) of these functions with the smooth function \( g: \mathbb{R}^n \to \mathbb{R} \), it too is smooth on \( \hat{M} \). So \( \hat{h} \) satisfies Condition (\( \mathcal{F}1 \)) of Construction 1.3.

- The functions \( \hat{h}_1, \ldots, \hat{h}_n \) satisfy Condition (\( \mathcal{F}2 \)) of Construction 1.3, so for each \( i \in \{1, \ldots, n\} \) there exists an \( S^1 \)-invariant neighbourhood \( V_i \) of \( \partial M \) in \( U_M \) and a smooth function \( H_i: V_i \times S^1 \to \mathbb{R} \) such that \( H_i(a \cdot x, z) = H_i(x, az) \) for all \( a \in S^1 \) and \( (x, z) \in V \times S^1 \) and such that \( \hat{h}_i(x) = H_i(x, \sqrt{f(x)}) \) for all \( x \in V_i \). The intersection \( V := V_1 \cap \ldots \cap V_n \) is an \( S^1 \)-invariant open neighbourhood of \( \partial M \) in \( U_M \). Define \( H': V \times S^1 \to \mathbb{R} \) by \( H'(x', z) = g(H_1(x', z), \ldots, H_n(x', z)) \). Then \( H' \) is smooth, \( H'(a \cdot x', z) = H'(x', az) \) for all \( a \in S^1 \) and \( (x', z) \in V \times S^1 \), and \( \hat{h}(x') = H'(x', \sqrt{f(x')}) \) for all \( x' \in V \). So \( \hat{h} \) satisfies Condition (\( \mathcal{F}2 \)) of Construction 1.3.

\( \square \)

6.2. Lemma. The initial topology on \( M_{\text{cut}} \) determined by \( \mathcal{F}_M \) coincides with the given topology of \( M_{\text{cut}} \) (which is the quotient topology induced from \( M \)).

Proof. By Remark 5.1, we need to show two things:

(i) Every function in \( \mathcal{F}_M \) is continuous (with respect to the quotient topology).

(ii) \( M_{\text{cut}} \) (with the quotient topology) is \( \mathcal{F}_M \)-regular.

Proof of (i): Let \( h \) be a function in \( \mathcal{F}_M \), and let \( \hat{h} := h \circ c: M \to \mathbb{R} \). Because \( \hat{h} \) satisfies Condition (\( \mathcal{F}1 \)) of Construction 1.3, it is continuous on \( \hat{M} \). Because \( \hat{h} \) satisfies Condition (\( \mathcal{F}2 \)) of Construction 1.3, it is continuous on an open neighbourhood \( V \) of \( \partial M \) in \( U_M \). Because \( \hat{M} \) and \( V \) are open in \( M \) and their union is \( M \), the function \( \hat{h}: M \to \mathbb{R} \) is
continuous. Chasing the commuting diagram

\[
\begin{array}{ccc}
M & \xrightarrow{c} & M_{\text{cut}} \\
\Downarrow{\hat{h}} & & \Downarrow{h} \\
\hat{h} & \xrightarrow{} & \mathbb{R},
\end{array}
\]

we conclude that \( h: M_{\text{cut}} \to \mathbb{R} \) is continuous with respect to the quotient topology.

**Proof of (ii):** We need to show that for every point \( y \) in \( M_{\text{cut}} \) and neighbourhood \( \mathcal{O} \) of \( y \) there exists a function in \( \mathcal{F}_M \) that is non-zero on \( y \) and whose support in \( M_{\text{cut}} \) is contained in \( \mathcal{O} \).

Let \( y \) be a point in \( M_{\text{cut}} \) and \( \mathcal{O} \) an open neighbourhood of \( y \) in \( M_{\text{cut}} \) (with respect to the quotient topology). Also consider \( y \) as a subset of \( M \) that is contained in \( c^{-1}(\mathcal{O}) \cap M \).

- Suppose that \( y = \{x\} \) for \( x \in M \). Let \( \hat{h}: M \to \mathbb{R} \) be a non-negative smooth function that is positive on \( x \) and that vanishes outside a closed subset \( A \) of \( M \) that is contained in \( c^{-1}(\mathcal{O}) \cap M \).
- Suppose that \( y = S^1 \cdot x \) for \( x \in \partial M \). Then \( U_1 := \bigcap_{a \in S^1} a \cdot (c^{-1}(\mathcal{O}) \cap U_M) \) is an \( S^1 \)-invariant open neighbourhood of \( y \) in \( U_M \) (by Lemma A.1(4)) that is contained in \( c^{-1}(\mathcal{O}) \cap M \). Let \( \hat{h}_1: M \to \mathbb{R} \) be a non-negative smooth function that is positive along \( S^1 \cdot x \) and vanishes outside a closed subset \( A \) of \( M \) that is contained in \( U_1 \). Let \( \hat{h}: M \to \mathbb{R} \) be the function that vanishes outside \( U_M \) and whose restriction to \( U_M \) is the \( S^1 \)-average of the function \( \hat{h}_1 \).

In either case, the function \( \hat{h}: M \to \mathbb{R} \) has the following properties.

(a) \( \hat{h} \) is smooth on \( M \), and it is \( S^1 \)-invariant on some \( S^1 \)-invariant open subset \( V \) of \( U_M \) that contains \( \partial M \). (In the first case we can take \( V = \bigcap_{a \in S^1} a \cdot (U_M \setminus A) \), which is open by Lemma A.1(4). In the second case we can take \( V = U_M \).)

(b) There exists a closed subset \( A_1 \) of \( M \) that contains the carrier \( \{h \neq 0\} \) and is contained in \( c^{-1}(\mathcal{O}) \) and whose complement is \( S^1 \)-invariant along its boundary. (In the first case, we can take \( A_1 = A \). In the second case, we can take \( A_1 = S^1 \cdot A \), which is closed by Lemma A.1(2).)

(c) \( \hat{h} \) is positive along \( y \).

By (a), the function \( \hat{h} \) is constant on the level sets of \( c: M \to M_{\text{cut}} \), so it determines a function \( h: M_{\text{cut}} \to \mathbb{R} \) such that \( \hat{h} = h \circ c \). Also by (a),
the function \( \hat{h} \) satisfies Conditions (F1) and (F2) of Construction 1.3 (with \( H(x, z) := \hat{h}(x) \)), so the function \( h \) is in \( \mathcal{F}_M \). By (b) and (c), the function \( h \) is positive on \( y \) and its support in \( M_{cut} \) is contained in \( \mathcal{O} \).

\[ \square \]

6.3. Lemma. \( \mathcal{F}_M \) satisfies the locality condition of a differential structure.

Proof. By Lemma 6.2, in the formulation of the locality condition, we may take “neighbourhood” to be with respect to the quotient topology on \( M_{cut} \) that is induced from the topology on \( M \).

Take any function \( h: M_{cut} \to \mathbb{R} \). Assume that for each point \( x \in M \) there exists a neighbourhood \( \mathcal{O}_x \) of \( c(x) \) in \( M_{cut} \) and a function \( g_x: M_{cut} \to \mathbb{R} \) that is in \( \mathcal{F}_M \) and such that \( h \) and \( g_x \) coincide on \( \mathcal{O}_x \). We need to prove that the function \( \hat{h} := h \circ c: M \to \mathbb{R} \) satisfies Conditions (F1) and (F2) of Construction 1.3.

For each point \( x \in M \), fix \( \mathcal{O}_x \) and \( g_x \) as above, and let \( \hat{g}_x := g_x \circ c \).

- Let \( x \in \hat{M} \). Because the function \( \hat{g}_x \) satisfies Condition (F1) of Construction 1.3, it is smooth on \( \hat{M} \). Because the function \( \hat{h} \) coincides with \( \hat{g}_x \) on the set \( \hat{M} \cap c^{-1}(\mathcal{O}_x) \) and this set is an open neighbourhood of \( x \) in \( M \), the function \( \hat{h} \) is smooth at \( x \). Because \( x \) was an arbitrary point of \( \hat{M} \), we conclude that \( \hat{h} \) satisfies Condition (F1) of Construction 1.3.

- For each \( x \in \partial M \), because \( \hat{g}_x \) satisfies Condition (F2) of Construction 1.3, there exist an \( S^1 \)-invariant open neighbourhood \( V_x \) of \( \partial M \) in \( U_M \) and a smooth function \( G_x: V_x \times \mathbb{C} \to \mathbb{R} \), such that \( G_x(a \cdot x', z) = G(x', az) \) for all \( a \in S^1 \) and \( (x', z) \in V_x \times \mathbb{C} \), and such that \( \hat{g}_x(x') = G_x(x', \sqrt{j(x')}) \) for all \( x' \in V_x \). Fix such \( V_x \) and \( G_x \). The set \( \mathcal{O}'_x := \bigcap_{a \in S^1} a \cdot (V_x \cap c^{-1}(\mathcal{O}_x)) \) is an open neighbourhood of \( S^1 \cdot x \) in \( U_M \) (see Lemma A.1(4)) on which \( \hat{h} \) coincides with \( \hat{g}_x \).

The sets \( \mathcal{O}'_x \), together with the set \( U_M \cap \hat{M} \), are a covering of \( U_M \) by \( S^1 \)-invariant open sets. Let \( (\rho_i: U_M \to \mathbb{R}_{\geq 0})_{i \geq 0} \) be an \( S^1 \)-invariant partition of unity that is subordinate to this covering (see Lemma A.3(3)), with \( \text{supp} \rho_0 \subset U_M \cap \hat{M} \) and \( \text{supp} \rho_i \subset \mathcal{O}'_{x_i} \) for \( i \geq 1 \).

The set \( V := U_M \setminus \text{supp} \rho_0 \) is an \( S^1 \)-invariant open neighbourhood of \( \partial M \) in \( U_M \), and \( (\rho_i|_V)_{i \geq 1} \) (without \( i = 0 \)) is a partition
of unity on $V$. Because each $\hat{g}_x$ coincides with $\hat{h}$ on the carrier $\{\rho_i \neq 0\}$, we have $\hat{h}|_V = \sum_{i \geq 1} \rho_i \hat{g}_x|_V$.

Define $H: V \times \mathbb{C} \to \mathbb{R}$ by $H := \sum_{i \geq 1} \rho_i G_x|_{V \times \mathbb{C}}$. Then $H$ is smooth, $H(a \cdot x', z) = H(x', az)$ for all $a \in S^1$ and $(x', z) \in V \times \mathbb{C}$, and $\hat{h}(x') = H(x', \sqrt{f(x')})$ for all $x' \in V$. So $\hat{h}$ satisfies Condition $(F2)$ of Construction 1.3.

\[ \square \]

6.4. Corollary. In the setup of Construction 1.3, the set $\mathcal{F}_M$ is a differential structure on $M_{\text{cut}}$.

Proof. By Lemma 6.1, $\mathcal{F}_M$ is a $C^\infty$ ring. By Lemma 6.3, $\mathcal{F}_M$ also satisfies the locality condition of a differential structure. \[ \square \]

6.5. Lemma. Let $M$ be a manifold-with-boundary, equipped with a free circle action on a neighbourhood $U_M$ of the boundary. Let $W$ be an open subset of $M$ that is $S^1$-invariant near its boundary. Then the inclusion of $W$ into $M$ descends to a diffeomorphism (as differential spaces) of $W_{\text{cut}}$ with an open subset of $M_{\text{cut}}$.

Proof. Let $\mathcal{F}_M$ and $\mathcal{F}_W$, respectively, be the differential structures on $M_{\text{cut}}$ and on $W_{\text{cut}}$ that are obtained from Construction 1.3. We need to show that $\mathcal{F}_W$ coincides with the subset differential structure that is induced from $\mathcal{F}_M$.

By Lemma 2.7, the inclusion map of $W_{\text{cut}}$ into $M_{\text{cut}}$ is smooth as a map of differential spaces. So the subset differential structure on $W_{\text{cut}}$ that is induced from $\mathcal{F}_M$ is contained in $\mathcal{F}_W$. For the converse, fix any function $h: W_{\text{cut}} \to \mathbb{R}$ in the differential structure $\mathcal{F}_W$, let $y$ be a point in $W_{\text{cut}}$, and let $\hat{h} := h \circ c: W \to \mathbb{R}$.

- Suppose that $y = \{x\}$ for $x \in \partial M$. Let $\rho: M \to \mathbb{R}_{\geq 0}$ be a smooth function that is equal to 1 near $x$ and whose support is contained in $\bar{M} \cap W$. Let $\hat{g}: M \to \mathbb{R}$ be equal to $\rho \hat{h}$ on $\bar{M} \cap W$ and zero elsewhere. Let $g: M_{\text{cut}} \to \mathbb{R}$ be such that $\hat{g} = g \circ c$.

- Suppose that $y = S^1 \cdot x$ for $x \in \partial M$. Let $\rho: M \to \mathbb{R}_{\geq 0}$ be a smooth function whose support is contained in $U_M \cap W$, that is $S^1$-invariant, and that is equal to 1 near $x$ (see Lemma A.3(2)). Let $\hat{g}: M \to \mathbb{R}$ be equal to $\rho \hat{h}$ on $U_M \cap W$ and zero elsewhere. Let $g: M_{\text{cut}} \to \mathbb{R}$ be such that $\hat{g} = g \circ c$.

In each of these cases, the function $g$ is in $\mathcal{F}_M$, and it is equal to $h$ on some open neighbourhood of $y$ in $M_{\text{cut}}$ (with respect to the given
topology of \( M_{\text{cut}} \) as a quotient of \( M \), hence—by Lemma 6.2—also with respect to the initial topology of \( M_{\text{cut}} \) that is induced from \( \mathcal{F}_M \). Because \( y \) was arbitrary, this implies that \( h : W \to \mathbb{R} \) is in the subset differential structure that is induced from \( \mathcal{F}_M \). Because \( h \) was arbitrary, \( \mathcal{F}_W \) is contained in the subset differential structure on \( W_{\text{cut}} \) that is induced from \( \mathcal{F}_M \). \( \square \)

7. The cut space is a manifold

By Lemmas 2.2, 2.4, and 2.5, the cutting construction gives a functor from the category of manifolds-with-boundary, equipped with free circle actions near the boundary, and their equivariant transverse maps, to the category of topological spaces and their continuous maps. By Corollary 6.4 and Lemma 2.7, when each \( M_{\text{cut}} \) is equipped with the set of real valued functions \( \mathcal{F}_M \) of Construction 1.3, the cutting construction defines a functor to the category of differential spaces and their smooth maps. In Proposition 7.1, which is a reformulation of Theorem 1.5, we use these functoriality properties to give a precise proof that cutting yields smooth manifolds.

7.1. Proposition. Let \( M \) be a manifold-with-boundary, equipped with a free circle action on a neighbourhood \( U_M \) of its boundary. Then the cut space \( M_{\text{cut}} \), equipped with the differential structure \( \mathcal{F}_{M_{\text{cut}}} \) that is obtained from Construction 1.3 (see Corollary 6.4), is a manifold.

Moreover, \( \check{M}_{\text{cut}} := c(\check{M}) \) is an open dense subset of \( M_{\text{cut}} \), and \( M_{\text{red}} := c(\partial M) = (\partial M)/S^1 \) is a codimension two closed submanifold of \( M_{\text{cut}} \).

Proof. By Lemma 6.2, the initial topology on \( M_{\text{cut}} \) determined by \( \mathcal{F}_M \) coincides with the quotient topology on \( M_{\text{cut}} \) induced from \( M \). By Lemmas 4.2 and 4.3, \( M_{\text{cut}} \)—with this topology—is Hausdorff and second countable.

Let \( y \) be a point in \( M_{\text{cut}} \). Let \( n = \dim M \).

- If \( y = \{x\} \) for \( x \in \check{M} \), let \( W \) be the domain of a coordinate chart on \( M \) that satisfies \( x \in W \subset \check{M} \). Then the differential space \( W_{\text{cut}} \) is diffeomorphic to an open subset of \( \mathbb{R}^n \).
- If \( y = S^1 \cdot x \) for \( x \in \partial M \), let \( W \) be an \( S^1 \)-invariant open neighbourhood of \( x \) in \( U_M \) as in Proposition 3.2, and let

\[
W \to \mathbb{D}^{n-2} \times S^1 \times [0, \epsilon)
\]
be a diffeomorphism that intertwines the $S^1$ action on $W$ with the rotations of the middle factor and that intertwines an invariant boundary-defining function with the projection to the last factor. Then the differential space $W_{\text{cut}}$ is diffeomorphic to $\left(D^{n-2} \times S^1 \times [0, \varepsilon]\right)_{\text{cut}}^{-1}$, which, as a consequence of Lemma 3.1, is diffeomorphic to $D^{n-2} \times D^2$.

Thus, in each of these cases, $W_{\text{cut}}$—with the differential structure $F_W$ that is obtained from Construction 1.3—is diffeomorphic to an open subset of $\mathbb{R}^n$. By Lemma 6.5, the inclusion map of $W_{\text{cut}}$ into $M_{\text{cut}}$ is a diffeomorphism with an open neighbourhood of $y$ in $M_{\text{cut}}$ (where $W_{\text{cut}}$ is equipped with the differential structure $F_{\text{cut}}$ and its image is equipped with the subset differential structure induced from $F_{\text{M}}$). So $y$ has a neighbourhood that is diffeomorphic to an open subset of $\mathbb{R}^n$.

Because $\hat{M}$ is open and dense in $M$ and is saturated, $\hat{M}_{\text{cut}}$ is open and dense in $M_{\text{cut}}$. Because each of the above diffeomorphisms $W_{\text{cut}} \to D^{n-2} \times D^2$ takes the intersection $M_{\text{red}} \cap W_{\text{cut}}$ to the subset $D^{n-2} \times \{0\}$ of $D^{n-2} \times D^2$, the subset $M_{\text{red}}$ is a codimension two submanifold of $M_{\text{cut}}$. □

7.2. Lemma. Consider $M_{\text{cut}}$ with its manifold structure that is obtained from $F_M$; see Proposition 7.1. Then the quotient map $c: M \to M_{\text{cut}}$ restricts to a diffeomorphism

$$\hat{M} \to \hat{M}_{\text{cut}}$$

and to a principal circle bundle map

$$\partial M \to M_{\text{red}},$$

where the domains of these maps are equipped with their manifold structures as subsets of $\hat{M}$, and the targets of these maps are equipped with their manifold structures as subsets of $M_{\text{cut}}$.

Proof. By Lemma 6.5, $\hat{M}_{\text{cut}}$ is open in $M_{\text{cut}}$, and its manifold structure as a subset of $M_{\text{cut}}$ coincides with its manifold structure that is obtained from the differential structure $F_{\hat{M}}$ of Construction 1.3. Because $c|_{\hat{M}}: \hat{x} \mapsto \{\hat{x}\}$ is a bijection onto $\hat{M}_{\text{cut}}$, to show that the map $\hat{M} \to \hat{M}_{\text{cut}}$ is a diffeomorphism, we need to show that, for any function $h: M_{\text{cut}} \to \mathbb{R}$, the function $g$ is in $F_{\hat{M}}$ iff its pullback $\hat{h}: \hat{M} \to \mathbb{R}$ is smooth. This, in turn, follows from Condition $(F1)$ of Construction 1.3.

As we already mentioned in Remark 1.6, there exists a unique manifold structure on $M_{\text{red}}$ such that the quotient map $\partial M \to M_{\text{red}}$ is a principal $S^1$ bundle. With this structure, a real valued function on
$M_{\text{red}}$ is smooth iff its pullback to $\partial M$ is smooth. We will show that, for any function $g: M_{\text{red}} \to \mathbb{R}$, the function $g$ is smooth on $M_{\text{red}}$ as a submanifold of $M_{\text{cut}}$ iff its pullback to $\partial M$ is smooth. We will show that, for any function $g: M_{\text{red}} \to \mathbb{R}$, the function $g$ is smooth on $M_{\text{red}}$ as a submanifold of $M_{\text{cut}}$ iff its pullback $\hat{g} := g \circ c: \partial M \to \mathbb{R}$ is smooth.

Suppose that $g: M_{\text{red}} \to \mathbb{R}$ is smooth on $M_{\text{red}}$ as a submanifold of $M_{\text{cut}}$. Because this submanifold is closed, the function $g$ extends to a smooth function $h: M_{\text{cut}} \to \mathbb{R}$. The pullback $\hat{h}: M \to \mathbb{R}$ then satisfies the conditions of Construction 1.3, and its restriction to $\partial M$ coincides with $\hat{g}$. Let $V$ be an invariant open neighbourhood of $\partial M$ in $U_M$ and

$$H: V \times \mathbb{C} \to \mathbb{R}$$

be a smooth function that relates to $\hat{h}$ as in Condition (F2) of Construction 1.3. Then $\hat{g}(x) = \hat{h}(x) = H(x, 0)$ for all $x \in \partial M$. Because $H$ is smooth, $\hat{g}$ is smooth.

Conversely, suppose that $\hat{g}: \partial M \to \mathbb{R}$ is smooth. Let $V$ be an $S^1$-invariant neighbourhood of $\partial M$ in $U_M$ and $\pi: V \to \partial M$ an $S^1$-equivariant collar neighbourhood projection map. Define $\hat{h}: V \to \mathbb{R}$ by $\hat{h}(x) := \hat{g}(\pi(x))$. Then $\hat{h}$ descends to a function $h: V_{\text{cut}} \to \mathbb{R}$ such that $h \circ c = \hat{h}$, and the function $h$ is in the differential structure $\mathcal{F}_V$. (In Condition (F2) of Construction 1.3 applied to $V$, we can take the function $H: V \times \mathbb{C} \to \mathbb{R}$ to be $H(x, z) := \hat{g}(\pi(x))$.) By Lemma 6.5, $V_{\text{cut}}$ is open in $M_{\text{cut}}$, and its manifold structure as an open subset of $M_{\text{cut}}$ coincides with its manifold structure that is obtained from the differential structure $\mathcal{F}_V$. So $h: V_{\text{cut}} \to \mathbb{R}$ is smooth on $V_{\text{cut}}$ as an open subset of $M_{\text{cut}}$, and so $g$, being the restriction of $h$ to the submanifold $M_{\text{red}}$, is smooth on $M_{\text{red}}$ as a submanifold (of $V_{\text{cut}}$, hence) of $M_{\text{cut}}$. □

7.3. Remark. The topology on $M_{\text{cut}}$, and the manifold structures on the pieces $\hat{M}_{\text{cut}}$ and $M_{\text{red}}$, depend only on the circle action on the boundary $\partial M$. But different extensions of this circle action to a neighbourhood the boundary can yield manifold structures on $M_{\text{cut}}$ that are diffeomorphic but not equal. For example, take the half-cylinder $M = S^1 \times [0, \infty)$ with the standard $S^1$-action. Then, define a new $S^1$-action by conjugating the standard $S^1$-action by a diffeomorphism of the form $\psi(b, s) := (b, sg(b)^2)$ where $g: S^1 \to \mathbb{R}_{>0}$ is some smooth function. The function $\hat{h}(e^{i\theta}, s) := g(e^{i\theta})\sqrt{s}\cos \theta$ descends to a real valued function on $M_{\text{cut}}$ that is smooth with respect to the new differential structure but is not smooth with respect to the standard differential structure unless $g$ is constant. (Indeed, the corresponding function on $\mathbb{R}^2$ satisfies

$$h(x, y) = xg(e^{i\theta})$$
whenever \( x = r \cos \theta \) and \( y = r \sin \theta \) with \( r \geq 0 \). So
\[
\lim_{t \to 0} \frac{h(tx, ty) - h(0, 0)}{t} = xg(e^{i\theta})
\]
whenever \( x = \cos \theta \) and \( y = \sin \theta \). If \( h \) is smooth, then, by the chain rule, the limit on the left is equal to \( ax + by \) where \( a = \frac{\partial h}{\partial x}|_{(0, 0)} \) and \( b = \frac{\partial h}{\partial y}|_{(0, 0)} \), so
\[
ax + by = xg(e^{i\theta})
\]
whenever \( x = \cos \theta \) and \( y = \sin \theta \). Substituting \( (x, y) = (1, 0) \) and \( (x, y) = (0, 1) \), we obtain that \( a = g(1) \) and \( b = 0 \), and so
\[
ax = xg(e^{i\theta})
\]
whenever \( x = \cos \theta \). So \( g(e^{i\theta}) = a \) (when \( \cos \theta \neq 0 \), and hence, by continuity,) for all \( e^{i\theta} \in S^1 \). ♦

8. Cuttings with immersions, submersions, embeddings

In this section, we cut submanifolds, and a bit more. Here is a precise statement.

8.1. Lemma. Let \( M \) and \( N \) be manifolds-with-boundary, equipped with free circle actions on neighbourhoods of the boundary. Let \( M_{\text{cut}} \) and \( N_{\text{cut}} \) be obtained from \( M \) and \( N \) by the smooth cutting construction. Let \( \psi: M \to N \) be an equivariant transverse map and \( \psi_{\text{cut}}: M_{\text{cut}} \to N_{\text{cut}} \) the resulting smooth map of the cut spaces.

- If \( \psi \) is an immersion, so is \( \psi_{\text{cut}} \).
- If \( \psi \) is a submersion, so is \( \psi_{\text{cut}} \).
- If \( \psi \) is an embedding, so is \( \psi_{\text{cut}} \).

Proof. Let \( x_M \) be a point in \( M \). Let \( x_N \) be its image in \( N \), and let \( y_M \) and \( y_N \) be the images of \( x_M \) and \( x_N \) in \( M_{\text{cut}} \) and \( N_{\text{cut}} \), so that \( \psi_{\text{cut}}(y_M) = y_N \).

- Suppose that \( x_M \in \tilde{M} \). Then \( x_N \in \tilde{N} \) (see Lemma 2.2). Since the quotient maps \( c_M|_{\tilde{M}}: \tilde{M} \to \tilde{M}_{\text{cut}} \) and \( c_N|_{\tilde{N}}: \tilde{N} \to \tilde{N}_{\text{cut}} \) are diffeomorphisms (see Lemma 7.2), \( d\psi|_{x_M} \) is injective (resp., surjective) iff \( d\psi_{\text{cut}}|_{y_M} \) is injective (resp., surjective).
- Suppose that \( x_M \in \partial M \). Equip \( N \) with any invariant boundary defining function \( f \), and equip \( M \) with the boundary defining function \( f \circ \psi \). Proposition 3.2 implies that we can identify an
open neighbourhood of \( x_M \) in \( M \) with \( U \times S^1 \times [0, \epsilon) \) where \( U \) is open in \( \mathbb{R}^{n-2} \), and an open neighbourhood of \( x_N \) in \( N \) with \( V \times S^1 \times [0, \epsilon) \) where \( V \) is open in \( \mathbb{R}^{k-2} \), such that the map \( \psi \) takes the form

\[
\psi(x, a, s) = (\overline{\psi}(x, s), a b(x, s), s)
\]

for some smooth functions

\[
\overline{\psi}: U \times [0, \epsilon) \to V \quad \text{and} \quad b: U \times [0, \epsilon) \to S^1.
\]

Expressing \( \psi \) as the composition of the diffeomorphism

\[
(x, a, s) \mapsto (x, ab(x, s), s)
\]

with the map

\[
(x, a, s) \mapsto (\overline{\psi}(x, s), a, s),
\]

we see that \( d\psi|_{x_M} \) is injective (resp., surjective) iff \( d\overline{\psi}(\cdot, 0)|_{\overline{x}_M} \) is injective (resp., surjective), where \( \overline{x}_M \) is the corresponding point of \( U \).

By Lemmas 3.1 and 6.5, we can further identify an open neighbourhood of \( y_M \) in \( M_{\text{cut}} \) with \( U \times D^2 \) and an open neighbourhood of \( y_N \) in \( N_{\text{cut}} \) with \( V \times D^2 \), where \( D^2 \) is the open disc of radius \( \sqrt{\epsilon} \) about the origin in \( \mathbb{R}^2 \), and the map \( \psi_{\text{cut}} \) becomes

\[
\psi_{\text{cut}}(x, z) = (\overline{\psi}(x, |z|^2), z b(x, |z|^2)).
\]

Expressing \( \psi_{\text{cut}} \) as the composition of the diffeomorphism

\[
(x, z) \mapsto (x, z b(x, |z|^2))
\]

with the map

\[
(x, z) \mapsto (\overline{\psi}(x, |z|^2), z),
\]

we see that \( d\psi_{\text{cut}}|_{y_M} \) too is injective (resp., surjective) iff \( d\overline{\psi}(\cdot, 0)|_{\overline{x}_M} \) is injective (resp., surjective).

We conclude that \( d\psi|_{x_M} \) is injective (resp., surjective) iff \( d\psi_{\text{cut}}|_{y_M} \) is injective (resp., surjective).

We conclude that if \( \psi \) is an immersion, then so is \( \psi_{\text{cut}}, \) and if \( \psi \) is a submersion, then so is \( \psi_{\text{cut}} \). By the local immersion theorem\(^7\), being an embedding is equivalent to being an immersion and a topological embedding (namely, a homeomorphism with its image). By Lemma 2.2, if \( \psi \) is a topological embedding, then so is \( \psi_{\text{cut}} \). We conclude that if \( \psi \) is an embedding, then so is \( \psi_{\text{cut}} \). \( \square \)

\(^7\)I’ve adopted this name from John Lee [20]
Below, submanifold-with-boundary refers to a subset that, with the subset differential structure, is a manifold-with-boundary.

8.2. **Corollary.** Let $N$ be a manifold-with-boundary, equipped with a free circle action near the boundary, and let $M$ be a submanifold-with-boundary of $N$. Suppose that the boundary of $M$ is contained in the boundary of $N$, that the intersection of $M$ with some invariant neighbourhood of $\partial N$ is invariant, and that the inclusion map of $M$ in $N$ is an equivariant transverse map. Then $M_{\text{cut}}$ is a submanifold of $N_{\text{cut}}$.

**9. Cutting with differential forms**

In this section, we give a criterion for a differential form to descends to a cut space.

Throughout this section, let $M$ be a manifold-with-boundary equipped with a free circle action on a neighbourhood $U_M$ of its boundary, let $M_{\text{cut}}$ be obtained from $M$ by the smooth cutting construction, and let $c: M \to M_{\text{cut}}$ be the quotient map.

9.1. **Lemma.** Let $\beta$ be a differential form on $M$ that is basic on $\partial M$ and invariant near $\partial M$. Then there exists a unique differential form $\beta_{\text{cut}}$ on $M_{\text{cut}}$ whose pullback through $c|_{\hat{M}}: \hat{M} \to \hat{M}_{\text{cut}}$ coincides with $\beta$ on $\hat{M}$.

**Remark.** In Lemma 9.1, the assumption on $\beta$ is that its pullback to $\partial M$ is $S^1$ basic and its restriction to some invariant neighbourhood of $\partial M$ in $U_M$ is $S^1$ invariant.

We summarize Lemma 9.1 in the following diagram, which encodes pullbacks of differential forms. (Note that we do not obtain $\beta$ is a pullback of $\beta_{\text{cut}}$ because the map $M \to M_{\text{cut}}$ is not smooth.)

\[
\begin{array}{ccc}
(M, \beta) & \xleftarrow{\text{inclusion}} & (\hat{M}, \beta|_{\hat{M}}) \\
\downarrow{c|_{\hat{M}}} & & \downarrow{c|_{\hat{M}_{\text{cut}}}} \\
(M_{\text{cut}}, \beta_{\text{cut}}) & \xleftarrow{\text{inclusion}} & (\hat{M}_{\text{cut}}, \beta_{\text{cut}}|_{\hat{M}_{\text{cut}}})
\end{array}
\]

**Proof of Lemma 9.1.** Because $\hat{M}_{\text{cut}}$ is open and dense in $M_{\text{cut}}$ and the map $c|_{\hat{M}}: M \to M_{\text{cut}}$ is a diffeomorphism, it is enough to show that such a form $\beta_{\text{cut}}$ exists locally near each point of $\hat{M}_{\text{cut}}$. Indeed, by continuity, such local forms are unique and patch together into a form on $M_{\text{cut}}$ as required. Because $M_{\text{cut}} = M_{\text{red}} \sqcup \hat{M}_{\text{cut}}$, it is enough to
consider neighbourhoods of points in $M_{\text{red}}$. Thus, we may assume that $eta$ is invariant and (by Proposition 3.2 and Lemma 6.5) that

$$M = D^{n-2} \times S^1 \times [0, \epsilon).$$

We write the components of a point in $M$ as

$$x \in D^{n-2}, a = e^{i\theta} \in S^1, \quad \text{and} \quad s \in \mathbb{R}_{\geq 0}.$$ 

Let $k$ be the degree of $\beta$. The assumption on $\beta$ implies (by Hadamard’s lemma) that we can write

$$\beta = \beta_k + \beta_{k-1} \wedge ds + s\hat{\beta}_{k-1} \wedge d\theta + \beta_{k-2} \wedge ds \wedge d\theta,$$

where for each $\ell$

$$\beta_\ell = \sum_{I = (i_1, \ldots, i_\ell)}^{i_1 < \ldots < i_\ell} b_I(x, s) dx_{i_1} \wedge \ldots \wedge dx_{i_\ell}$$

for some smooth functions $b_I(x, s)$ on $D^{n-2} \times [0, \epsilon)$, and similarly for $\hat{\beta}_\ell$.

If $\ell < 0$, the sum is empty and $\beta_\ell = 0$.

Identify $M_{\text{cut}}$ with $D^{n-2} \times D^2$ as in Lemma 3.1, with coordinates $x$ and $z = u + iv$. The inverse of the diffeomorphism $c|_{M_{\text{cut}}} : M_{\text{cut}} \rightarrow \hat{M}_{\text{cut}}$ is then given by $(x, z) \mapsto (x, a, s)$ with $a = z/|z|$ and $s = |z|^2$. Writing

$$ds = (2udu + 2vdv), \quad s d\theta = udv - vdu, \quad ds \wedge d\theta = 2du \wedge dv,$$

and

$$(\beta_\ell)_{\text{cut}} = \sum_{I = (i_1, \ldots, i_\ell)}^{i_1 < \ldots < i_\ell} b_I(x, |z|^2) dx_{i_1} \wedge \ldots \wedge dx_{i_\ell},$$

and similarly for $\hat{\beta}_\ell$, we can then take

$$\beta_{\text{cut}} := (\beta_k)_{\text{cut}} + (\beta_{k-1})_{\text{cut}} \wedge (2udu + 2vdv)$$

$$+ (\hat{\beta}_{k-1})_{\text{cut}} \wedge (udv - vdu) + (\beta_{k-2})_{\text{cut}} \wedge (2du \wedge dv).$$

\[\square\]

We now give some important properties of the cutting procedure for differential forms.

9.2. Lemma. Let $\beta$ and $\beta'$ be differential forms on $M$ that are basic on $\partial M$ and invariant near $\partial M$. Then

$$(d\beta)_{\text{cut}} = d(\beta_{\text{cut}}) \quad \text{and} \quad (\beta \wedge \beta')_{\text{cut}} = \beta_{\text{cut}} \wedge (\beta')_{\text{cut}}.$$

Proof. Because $c|_{M_{\text{cut}}} : M_{\text{cut}} \rightarrow \hat{M}_{\text{cut}}$ is a diffeomorphism that takes $\beta$ to $\beta_{\text{cut}}$ and $\beta'$ to $(\beta')_{\text{cut}}$, these equalities hold on $M_{\text{cut}}$. By continuity, they hold on all of $M_{\text{cut}}$. \[\square\]
9.3. **Lemma.** Let $\beta$ be a differential form on $M$ that is basic on $\partial M$ and invariant near $\partial M$. Then $\beta$ is closed on $M$ if and only if $\beta_{\text{cut}}$ is closed on $M_{\text{cut}}$.

**Proof.** Because $c|_{\tilde{M}}: \tilde{M} \to \tilde{M}_{\text{cut}}$ is a diffeomorphism that takes $\beta$ to $\beta_{\text{cut}}$, $\beta$ is closed on $\tilde{M}$ if and only if $\beta_{\text{cut}}$ is closed on $\tilde{M}_{\text{cut}}$. By continuity, $\beta$ is closed on $M$ if and only if $\beta_{\text{cut}}$ is closed on $M_{\text{cut}}$. $\square$

9.4. **Lemma.** $\beta \neq 0$ at a point $p$ of $M$ if and only if $\beta_{\text{cut}} \neq 0$ at the point $c(p)$ of $M_{\text{cut}}$.

**Proof.** Because $c|_{\tilde{M}}: \tilde{M} \to \tilde{M}_{\text{cut}}$ is a diffeomorphism that takes $\beta$ to $\beta_{\text{cut}}$, this is true at points of $\tilde{M}$. Near a point in $\partial M$, we may identify $M$ with $D^n - 2 \times S^1 \times [0, \epsilon)$. In the expression in coordinates for $\beta$ and $\beta_{\text{cut}}$ that appear in the proof of Lemma 9.1, we see that the non-vanishing of $\beta$ at any point of $M$ is equivalent to the non-vanishing of at least one of $\beta_k$, $\beta_{k-1}$, or $\beta_{k-2}$ at that point, and that the non-vanishing of $\beta_{\text{cut}}$ at any point of $M_{\text{cut}}$ is equivalent to the non-vanishing of at least one of $(\beta_k)_{\text{cut}}$, $(\beta_{k-1})_{\text{cut}}$, or $(\beta_{k-2})_{\text{cut}}$ at that point. For a point $p$ in $\partial M$ and its image $c(p)$ in $M_{\text{cut}}$, the coordinates of $p$ are $x, \theta, s$ with $s = 0$, and the coordinates of $c(p)$ are $x, z$ with the same $x$ and with $z = 0$. We finish by noting that $\beta|_{s=0}$ and $(\beta_{\ell})_{\text{cut}}|_{z=0}$ are both given by the same expression,  

$$
\sum_{I=\{i_1, \ldots, i_\ell\}} b_I(x, 0) dx_{i_1} \wedge \ldots \wedge dx_{i_\ell} .
$$

$\square$

9.5. **Corollary.** In the setup of Lemma 9.1, the following holds.

- $\beta$ is a symplectic two-form on $M$ iff $\beta_{\text{cut}}$ is a symplectic two-form on $M_{\text{cut}}$.
- $\beta$ is a contact one-form on $M$ iff $\beta_{\text{cut}}$ is a contact one-form on $M_{\text{cut}}$.

**Proof.** First, assume that $M$ has dimension $2n$ and that $\beta$ is a two-form. By Lemmas 9.2 and 9.4, $\beta$ is closed on $M$ iff $\beta_{\text{cut}}$ is closed on $M_{\text{cut}}$, and $\beta^n$ is non-vanishing on $M$ iff $\beta_{\text{cut}}^n$ is non-vanishing on $M_{\text{cut}}$. This gives the first result.

Next, assume that $M$ has dimension $2n + 1$ and that $\beta$ is a one-form. By Lemmas 9.2 and 9.4, $\beta \wedge (d\beta)^n \neq 0$ on $M$ iff $\beta_{\text{cut}} \wedge (d\beta_{\text{cut}})^n \neq 0$ on $M_{\text{cut}}$. This gives the second result. $\square$
10. Relation with reduced forms

Let $M$ be a manifold-with-boundary, equipped with a free circle action near its boundary, let $M_{\text{cut}}$ be obtained from $M$ by the smooth cutting construction, and let $c : M \to M_{\text{cut}}$ be the quotient map. Let $M_{\text{red}} := c(\partial M) = (\partial M)/S^1$ in $M_{\text{cut}}$. So we have a commuting square

\[ \begin{array}{ccc}
\partial M & \xrightarrow{\text{inclusion}} & M \\
\downarrow\text{quotient} & & \downarrow\text{quotient} \\
M_{\text{red}} & \xrightarrow{\text{inclusion}} & M_{\text{cut}},
\end{array} \]

(in which the map $M \to M_{\text{cut}}$ is not smooth).

Here is a quick reminder on basic forms. Let $N$ be a manifold with a free circle action, generated by a vector field $\xi_N$. A differential form $\beta_N$ on $N$ is basic if it is horizontal, which means that $\xi_N \cdot N = 0$, and invariant. This holds iff $\beta_N$ is the pullback to $N$ of a differential form on $N/S^1$. In this situation, the differential form on $N/S^1$ is determined uniquely by $\beta$.

So if $\beta$ is a differential form on $M$ that is basic on $\partial M$ and invariant near $\partial M$, then there exists a unique differential form $\beta_{\text{red}}$ on $M_{\text{red}}$ whose pullback $\beta_{\partial M}$ to $\partial M$ coincides with the pullback of $\beta$ to $\partial M$. We summarize this in the following diagram.

\[ \begin{array}{ccc}
(\partial M, \beta_{\partial M}) & \xrightarrow{\text{inclusion}} & (M, \beta) \\
\downarrow\text{quotient} & & \\
(M_{\text{red}}, \beta_{\text{red}})
\end{array} \]

In Lemma 10.1 we show that $\beta_{\text{red}}$ coincides with the pullback of $\beta_{\text{cut}}$ under the inclusion map $M_{\text{red}} \to M_{\text{cut}}$. This result does not follow immediately from the above commuting square, because the quotient map $M \to M_{\text{cut}}$ is not smooth. We summarize this result in the following diagram.

\[ \begin{array}{ccc}
(\partial M, \beta_{\partial M}) & \xrightarrow{\text{inclusion}} & (M, \beta) \xleftarrow{\text{inclusion}} (\tilde{M}, \beta|_{\tilde{M}}) \\
\downarrow\text{quotient} & & \downarrow c|_{\tilde{M}} \\
(M_{\text{red}}, \beta_{\text{red}}) & \xrightarrow{\text{inclusion}} & (M_{\text{cut}}, \beta_{\text{cut}}) \xleftarrow{\text{inclusion}} (\tilde{M}_{\text{cut}}, \beta_{\text{cut}}|_{\tilde{M}_{\text{cut}}})
\end{array} \]

In Corollary 9.5 we showed that if $\beta$ is a symplectic two-form (resp., a contact one-form) on $M_{\text{cut}}$ then $\beta_{\text{cut}}$ is a symplectic two-form (resp., a
contact one-form) on $M_{\text{cut}}$. In Lemmas 10.2 and 10.3 we will now show that, in these situations, $\beta_{\text{red}}$ is a symplectic two-form (resp., a contact one-form) on $M_{\text{red}}$. Thus, in these situations, $M_{\text{red}}$ is a symplectic (resp., contact) submanifold of $M_{\text{cut}}$.

10.1. Lemma. Let $\beta$ be a differential form on $M$ that is basic on $\partial M$ and invariant near $\partial M$. Let $\beta_{\text{red}}$ and $\beta_{\text{cut}}$ be the induced differential forms on $M_{\text{red}}$ and $M_{\text{cut}}$ as described above. Then $M_{\text{red}}$ coincides with the pullback of $\beta_{\text{cut}}$ under the inclusion map $M_{\text{red}} \to M_{\text{cut}}$.

Proof. As in the proof of Lemma 9.1, we may assume that $M$ is $D^{n-2} \times S^1 \times [0, \epsilon)$, with coordinates $x_i, \theta$, and $s$.

By setting $s = 0$ in the expression in coordinates for $\beta$ in the proof of Lemma 9.1, we obtain that the restriction of $\beta$ to $\partial M$ is

$$\beta|_{\partial M} = \sum_{I = (i_1, \ldots, i_k)} b_I(x, 0) dx_{i_1} \wedge \ldots \wedge dx_{i_k} + \beta_{k-1}|_{s=0} \wedge ds + \beta_{k-2}|_{s=0} \wedge ds \wedge d\theta.$$

Further setting $ds = 0$, we obtain that the pullback of $\beta$ to $\partial M$ is

$$\beta|_{\partial M} = \sum_{I = (i_1, \ldots, i_k)} b_I(x, 0) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

as a differential form on $D^{n-2} \times S^1 \times \{0\}$. Identifying $M_{\text{red}}$ with $D^{n-2}$, we obtain that

$$\beta_{\text{red}} = \sum_{I = (i_1, \ldots, i_k)} b_I(x, 0) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

as a differential form on $D^{n-2}$.

By setting $z = u + iv = 0$ in the expression in coordinates for $\beta_{\text{cut}}$ in the proof of Lemma 9.1, we obtain that the restriction of $\beta_{\text{cut}}$ to $M_{\text{red}}$ is

$$\beta_{\text{cut}}|_{M_{\text{red}}} = \sum_{I = (i_1, \ldots, i_k)} b_I(x, 0) dx_{i_1} \wedge \ldots \wedge dx_{i_k} + (\beta_{k-2})_{\text{cut}} \wedge (2du \wedge dv).$$

Further setting $du = dv = 0$, we obtain that the pullback of $\beta_{\text{cut}}$ to $M_{\text{red}}$ coincides with $\beta_{\text{red}}$, as required. \qed
We now specialize to the context of symplectic cutting and contact cutting. In Corollary 9.5 we showed that, in the setting of Lemma 9.1, if \( \beta \) is a symplectic two-form (resp., contact one-form) on \( M \) then \( \beta_{\text{cut}} \) is a symplectic two-form (resp., contact one-form) on \( M_{\text{cut}} \). In Lemmas 10.2 and 10.3 we will show that, in this setting, if \( \beta \) is a symplectic two-form (resp., contact one-form) on \( M \) then \( \beta_{\text{red}} \) is a symplectic two-form (resp., contact one-form) on \( M_{\text{red}} \).

**10.2. Lemma.** Let \( \beta \) be a symplectic two-form on \( M \) that is basic on \( \partial M \) and invariant near \( \partial M \). Then \( \beta_{\text{red}} \) is a symplectic two-form on \( M_{\text{red}} \).

**Proof.** Because \( \beta \) is closed, its pullback \( \beta_{\partial M} \) to \( \partial M \) is closed, and so \( \beta_{\text{red}} \) is closed. It remains to prove that \( \beta_{\text{red}} \) is non-degenerate.

Let \( 2n = \dim M \).

As in the proofs of Lemmas 9.1 and 10.1, we may assume that \( M = D^{2n-2} \times S^1 \times [0, \varepsilon) \) and \( M_{\text{red}} = D^{2n-2} \), and we obtain the expressions

\[
\beta = \sum_{i<j} b_{i,j}(x, s) dx_i \wedge dx_j
+ \sum_i b_i(x, s) dx_i \wedge ds + s \sum_i \hat{b}_i(x, s) dx_i \wedge d\theta + b_0(x, s) ds \wedge d\theta
\]

and

\[
\beta_{\text{red}} = \sum_{i<j} b_{i,j}(x, 0) dx_i \wedge dx_j,
\]

where \( b_{i,j}, b_i, \hat{b}_i, \) and \( b_0 \) are smooth functions on \( D^{2n-2} \times [0, \varepsilon) \). By setting \( s = 0 \) in the expression for \( \beta \), we obtain that the restriction of \( \beta \) to \( \partial M \) is

\[
\beta|_{\partial M} = \sum_{i<j} b_{i,j}(x, 0) dx_i \wedge dx_j
+ \sum_i b_i(x, 0) dx_i \wedge ds + b_0(x, 0) ds \wedge d\theta.
\]

Taking the \( n \)th wedge and noting that the number of \( x \)-coordinates is \( 2n - 2 \), and then comparing with the expression for \( \beta_{\text{red}} \), we obtain

\[
\beta^n|_{\partial M} = n \left( \sum_{i<j} b_{i,j}(x, 0) dx_i \wedge dx_j \right)^{n-1} \wedge \left( b_0(x, 0) ds \wedge d\theta \right)
= n \left( \pi^* \beta_{\text{red}} \right)^{n-1} \wedge \left( b_0(x, 0) ds \wedge d\theta \right)
\]
where $\pi \colon D^{2n-2} \times S^1 \times [0, \epsilon) \to D^{2n-2}$ is the projection map. The non-vanishing of this form along $\partial M$ implies that $\beta^{n-1}_{\text{red}}$ is non-vanishing and hence that $\beta_{\text{red}}$ is non-degenerate. \hfill \Box

10.3. Lemma. Let $\beta$ be a contact one-form on $M$ that is basic on $\partial M$ and invariant near $\partial M$. Then $\beta_{\text{red}}$ is a contact one-form on $M_{\text{red}}$.

Proof. Let $2n + 1 = \dim M$.

As in the proofs of Lemmas 9.1 and 10.1, we may assume that $M = D^{2n-1} \times S^1 \times [0, \epsilon)$ and $M_{\text{red}} = D^{2n-1}$, and we obtain the expressions

$$\beta = \sum_i b_i(x, s) dx_i + b_0(x, s) ds + s\tilde{b}_0(x, s) d\theta$$

and

$$\beta_{\text{red}} = \sum_i b_i(x, 0) dx_i.$$

From the expression for $\beta$ we get

$$d\beta = \sum_{i,j} \frac{\partial b_i}{\partial x_j}(x, s) dx_j \wedge dx_i + \sum_i \frac{\partial b_i}{\partial s}(x, s) ds \wedge dx_i + \sum_i \frac{\partial b_0}{\partial x_i}(x, s) dx_i \wedge ds + \tilde{b}_0(x, 0) ds \wedge d\theta + s\eta$$

for some two-form $\eta$ on $D^{2n-1} \times S^1 \times [0, \epsilon)$. By setting $s = 0$ in the expressions for $\beta$ and for $d\beta$, we obtain that their restrictions to $\partial M$ are

$$\beta|_{\partial M} = \sum_i b_i(x, 0) dx_i + b_0(x, 0) ds$$

and

$$(d\beta)|_{\partial M} = \sum_{i,j} \frac{\partial b_i}{\partial x_j}(x, 0) dx_j \wedge dx_i + \gamma \wedge ds$$

for some one-form $\gamma$ on $D^{2n-1} \times S^1 \times [0, \epsilon)$ along $\{s = 0\}$. Taking the $n$th wedge and noting that the number of $x$-coordinates is $2n - 1$, we obtain

$$(d\beta)^n|_{\partial M} = n \left( \sum_{i,j} \frac{\partial b_i}{\partial x_j}(x, 0) dx_j \wedge dx_i \right)^{n-1} \wedge \gamma \wedge ds.$$
Taking the wedge of the expression for $\beta|_{\partial M}$ and the expression for $(d\beta)^n|_{\partial M}$, and then comparing with the expression for $\beta_{\text{red}}$, we obtain

$$\beta \wedge (d\beta)^n|_{\partial M} = n \left( \sum_i b_i(x,0)dx_i \right) \wedge \left( \sum_{i,j} \frac{\partial b_i}{\partial x_j}(x,0)dx_j \wedge dx_i \right)^{n-1} \wedge \gamma \wedge ds$$

where $\pi: D^{2n-1} \times S^1 \times [0,\epsilon) \to D^{2n-1}$ is the projection map. The non-vanishing of this form along $\partial M$ implies that $\beta_{\text{red}} \wedge (d\beta_{\text{red}})^{n-1}$ is nonvanishing and hence that $\beta_{\text{red}}$ is a contact one-form.

\[ \square \]

11. Relation with symplectic cutting

We now relate the smooth cutting procedure with differential forms to the classical version of Lerman’s symplectic cutting procedure.

Let $\tilde{M}$ be a manifold equipped with a circle action and with an invariant two-form $\omega$. Let $\mu: M \to \mathbb{R}$ be a corresponding momentum map; this means that

$$\xi_{M/\mu} \omega = -d\mu,$$

where $\xi_M$ is the vector field that generates the circle action. It implies that $\mu$ is invariant. Suppose that the circle action on the zero level set $\mu^{-1}(\{0\})$ is free. The level set $\mu^{-1}(\{0\})$ is regular iff $\xi_M$ is not in the null-space of $\omega$ at any point of $\mu^{-1}(\{0\})$. Assume that this holds; note that it always holds if $\omega$ is non-degenerate. Then

$$M := \mu^{-1}([0, \infty))$$

is a submanifold-with-boundary of $\tilde{M}$, its boundary is $\partial M = \mu^{-1}(\{0\})$, and $\mu|_M$ is an invariant boundary defining function.

Let $M_{\text{cut}}$ be obtained from $M$ by the smooth cutting construction. The equation $\xi_M \omega = -d\mu$ implies that the pullback of $\omega$ to $\partial M$ is basic. By Lemmas 9.1 and 9.3, $\omega$ induces a closed two-form $\omega_{\text{cut}}$ on $M_{\text{cut}}$. By Corollary 9.5, if $\omega$ is symplectic, so is $\omega_{\text{cut}}$.

Suppose that a Lie group $G$ acts smoothly on $M$, commutes with the circle action, and preserves $\omega$. By functoriality, the action descends to a smooth $G$ action on $M_{\text{cut}}$. This action preserves $\omega_{\text{cut}}$ on $M_{\text{cut}}$, hence (by continuity) everywhere.
Suppose now that the $G$ action on $M$ is Hamiltonian with momentum map $\mu_G$. Because the $G$ action preserves $\omega$, commutes with the $S^1$ action, and preserves the zero level set of $\mu$, it preserves $\mu$ near $\partial M$; this implies that the $S^1$ action preserves $\mu_G$ near $\partial M$. By Lemma 9.1, $\mu_G$ descends to a smooth map $(\mu_G)_{cut}$ on $M_{cut}$. The momentum map equation for $G$ holds on $\tilde{M}_{cut}$ and hence (by continuity) everywhere. So $(\mu_G)_{cut}$ is a momentum map for the $G$ action on $(M_{cut}, \omega_{cut})$.

12. Relation with contact cutting

We now relate the smooth cutting procedure with differential forms to the classical version of Lerman’s contact cutting procedure. We give here a version that should be useful for our paper-in-progress [8].

Let $\tilde{M}$ be a manifold equipped with a circle action and with an invariant contact one-form $\beta$. Consider the corresponding momentum map,

$$\mu := \xi_M \hookdot \beta: M \to \mathbb{R}.$$ 

Assume that the circle action is free on the zero level set $\mu^{-1}(\{0\})$. In particular, $\xi_M$ is non-vanishing along $\mu^{-1}(\{0\})$. Because $\xi_M$ is in $\ker \beta$ along $\mu^{-1}(\{0\})$, and because restriction of $d\beta$ to $\ker \beta$ is non-degenerate (because $\beta$ is a contact one-form), it follows that $\xi_M \hookdot d\beta$ is non-vanishing along $\mu^{-1}(\{0\})$. But $\xi_M \hookdot d\beta = -d\xi_M \hookdot \beta = -d\mu$, so the level set $\mu^{-1}(\{0\})$ is regular,

$$M := \mu^{-1}([0, \infty))$$

is a submanifold-with-boundary of $\tilde{M}$, its boundary is $\partial M = \mu^{-1}(\{0\})$, and $\mu|_M$ is an invariant boundary defining function on $M$.

More generally, let $M$ be a manifold with boundary, equipped with a free circle action near the boundary $\partial M$, and let $\beta$ be a contact one-form on $M$ that is invariant near $\partial M$. Assume that the corresponding momentum map

$$\mu := \xi_M \hookdot \beta: M \to \mathbb{R}$$

vanishes along $\partial M$. Then $\mu$ is an invariant boundary defining function near $\partial M$.

Let $M_{cut}$ be obtained from $M$ by the smooth cutting construction. Because $\xi_M \hookdot \beta = \mu$ vanishes along $\partial M$ and is $S^1$ invariant near $\partial M$, the differential form $\beta$ is basic on $\partial M$. By Lemma 9.1 and Corollary 9.5, $\beta$ induces a contact one-form $\beta_{cut}$ on $M_{cut}$. By Lemmas 10.1 and 10.3, the pullback of $\beta_{cut}$ under the inclusion map $M_{red} \to M_{cut}$ is the contact one-form on $M_{red}$, so $M_{red}$ is a contact submanifold of $M_{cut}$. 

13. Cutting with distributions

We begin with a quick reminder on distributions.

A distribution on a manifold $N$ is a sub-bundle $E$ of the tangent bundle $TN$. Lie brackets of vector fields determine a $TN/E$-valued two-form on $E$, which we write as

$$\Omega : E \times E \to TN/E :$$

for any two vector fields $u, v$ with values in $E$, we have the equality $[u, v] + E = \Omega(u, v)$ of sections of $TN/E$. The distribution is involutive if this two-form is zero; it’s contact if $E$ is of codimension one and this two-form is non-degenerate.

A subset $E$ of $TN$ is a codimension-$k$ distribution iff can locally be written as the null-space of a non-vanishing decomposable $k$-form $\beta$. (Namely, $E = \{v \in TM \mid v \cup \beta = 0\}, \beta = \beta_1 \wedge \ldots \wedge \beta_k$ for some one-forms $\beta_j$, and $\beta \neq 0$.) A subset $E$ of $TN$ is a contact distribution iff $\dim N$ is odd, say, $\dim N = 2n + 1$, and $E$ can locally be written as the null-space of a one-form $\beta$ such that $\beta \wedge (d\beta)^n \neq 0$. If this holds, then $\beta \wedge (d\beta)^n \neq 0$ for every local one-form $\beta$ whose null-space is $E$.

13.1. Lemma. Let $M$ be a manifold-with-boundary, with a free circle action on a neighbourhood $U_M$ of the boundary $\partial M$. Let $M_{\text{cut}}$ be obtained from $M$ by the smooth cutting construction, and let $c: M \to M_{\text{cut}}$ be the quotient map. Let $E$ be a distribution on $M$. Assume that $E$ contains the tangents to the $S^1$ orbits along $\partial M$, is $S^1$-invariant near $\partial M$, and is transverse to $\partial M$. Then there exists a unique distribution $E_{\text{cut}}$ on $M_{\text{cut}}$ whose preimage under $c|_{M'} : M \to (M)_{\text{cut}}$ is $E|_{M'}$.

Moreover,

- $E$ is involutive iff $E_{\text{cut}}$ is involutive; and
- $E$ is a contact distribution iff $E_{\text{cut}}$ is a contact distribution.

We resist our temptation to define $E_{\text{cut}}$ to be the preimage of $E$ under the differential of the quotient map $c: M \to M_{\text{cut}}$: this map $c$ is not smooth.

Here is a proof of the lemma.

Proof of Lemma 13.1.

Uniqueness follows from continuity. So does the statement about involutivity.
It is enough to check that a distribution $E_{\text{cut}}$ with the required properties exists locally near each point of $M_{\text{cut}}$: by continuity, such local distributions patch together into a distribution on $M_{\text{cut}}$ as required. Because $c$ restricts to a diffeomorphism from $M$ to $(M)_{\text{cut}}$, it is enough to consider neighbourhoods of points in $M_{\text{red}} := (\partial M)_{\text{cut}} (= (\partial M)/S^1)$.

Assume, first, that $E$ has codimension one. The assumptions on $E$ imply that near each orbit in $\partial M$ we can write $E = \ker \beta$ where $\beta$ is an $S^1$-invariant one-form whose pullback to $\partial M$ is $S^1$-basic and non-vanishing. By Lemmas 9.1 and 9.4, we obtain a differential form $\beta_{\text{cut}}$ on $M_{\text{cut}}$ such that $E_{\text{cut}} := \ker \beta_{\text{cut}}$ is a distribution whose preimage under $c|_{\hat{M}}$ is $E|_{\hat{M}}$.

By Corollary 9.5, $\beta$ is a contact form on $M$ iff $\beta_{\text{cut}}$ is a contact form on $M_{\text{cut}}$. So $E$ is a contact distribution on $M$ iff $E_{\text{cut}}$ is a contact distribution on $M_{\text{cut}}$.

Now, let $E$ be a distribution of any codimension. Let $k$ be the codimension of the distribution $E$.

The assumptions on $E$ imply that $E \cap T\partial M$ is a codimension $k$ distribution on $\partial M$ that is $S^1$-invariant and that contains the tangents to the $S^1$ orbits on $\partial M$. So it determines a distribution $E_{\text{red}}$ on $M_{\text{red}}$ such that $E \cap T\partial M$ is the preimage of $E_{\text{red}}$ under the differential of the quotient map $\partial M \to M_{\text{red}}$.

Fix a point $p \in \partial M$. By Proposition 3.2, we may assume that

$$M = D^{n-2} \times S^1 \times [0, \epsilon),$$
with coordinates $x_i$ on $D^{n-2}$, $\theta \mod 2\pi$ on $S^1$, and $s$ on $[0, \epsilon)$, that $E$ contains the vector field $\partial/\partial s$, and that the point $p$ is given by $x = 0$, $\theta = 0$, and $s = 0$. Identify $M_{\text{red}}$ with the disc $D^{n-2}$.

There exists $S^1$-invariant differential one-forms $\beta_1, \ldots, \beta_k$ on $M$ whose pullbacks to $\partial M$ are basic, whose null-space contains $E$, and such that the $k$-form $\beta := \beta_1 \wedge \ldots \wedge \beta_k$ is non-vanishing (at $p$, hence) near $p$. Shrinking $D^{n-2}$ and $\epsilon$, we may assume that $\beta$ is non-vanishing everywhere.

(Here is how to obtain such one-forms. Equip $M$ with the Riemannian metric that is standard in the coordinates $x, \theta, s$. Let $(\eta_1)_p, \ldots, (\eta_k)_p$ be a basis to the orthocomplement of $E|_p$ in $TM$. Because $E$ contains $\partial/\partial s$, the $(\eta_j)_p$ are tangent to $\partial M$. Extend each $(\eta_j)_p$ to a vector field on $M$ with constant coefficients with respect to our coordinates, and project to the orthocomplement of $E$ in $TM$. We obtain $S^1$-invariant vector fields $\eta_1, \ldots, \eta_k$ that are everywhere orthogonal to $E$ and that at $p$ are a basis to the orthocomplement of $E|_p$. For each $j$, inner product with $\eta_j$ defines a one-form $\beta_j$. These one-forms have the required properties.)

Lemmas 9.1, 9.2, and 9.4 then give one-forms $(\beta_j)_{\text{cut}}$ on $M_{\text{cut}}$ such that $\beta_{\text{cut}} = (\beta_1)_{\text{cut}} \wedge \ldots \wedge (\beta_k)_{\text{cut}}$ is non-vanishing. Let $E_{\text{cut}}$ be the null-space of $\beta_{\text{cut}}$. Then $E_{\text{cut}}$ is a codimension-$k$ distribution on $M_{\text{cut}}$ whose preimage under $c|_M$ is $E|_{\tilde{M}}$.

\[ \square \]

14. DIFFEOMORPHISMS IN SYMPLECTIC POLAR COORDINATES

In this section, which can be read independently of the others, we describe a baby-example that motivates the equivariant-radial-squared-blowup construction of Section 16.

Symplectic geometers find it natural to parametrize a complex number $z = x + iy = re^{i\theta}$ by $s := \frac{1}{2}r^2$ and $\theta$ rather than the standard polar coordinates $r$ and $\theta$. With this parametrization, the standard symplectic form $dx \wedge dy$ becomes $ds \wedge d\theta$. That is, $(s, \theta)$ are symplectic coordinates. This works outside the origin; polar coordinates — symplectic or not — generally fail to be useful at the origin.

In equivariant differential topology, symplectic polar coordinates turn out to be useful even at the origin. We now explain how. To simplify formulas, we now take $s$ to be $r^2$ instead of $\frac{1}{2}r^2$. 

Consider the half-cylinder $S^1 \times \mathbb{R}_{\geq 0}$ with the circle acting on the first component. The map

$$E: S^1 \times \mathbb{R}_{\geq 0} \to \mathbb{C}, \quad E(u, s) := \sqrt{s} u$$

descends to a bijection $\overline{E}: (S^1 \times \mathbb{R}_{\geq 0})/\sim \to \mathbb{C}$, where $\sim$ is the equivalence relation in which distinct points $(u_1, s_1)$ and $(u_2, s_2)$ are equivalent iff $s_1 = s_2 = 0$. Because the map $E$ is continuous and proper, the bijection $\overline{E}$ is a homeomorphism.

The map $E$ is not smooth, but the relation $E \circ \psi = \phi \circ E$, expressed in the commuting diagram below, defines a bijection between $S^1$-equivariant diffeomorphisms of the half-cylinder and $S^1$-equivariant diffeomorphisms of $\mathbb{C}$.

$$\begin{array}{ccc}
S^1 \times \mathbb{R}_{\geq 0} & \xrightarrow{\psi} & S^1 \times \mathbb{R}_{\geq 0} \\
E \downarrow & & \downarrow E \\
\mathbb{C} & \xrightarrow{\phi} & \mathbb{C}.
\end{array}$$

Indeed, every $S^1$-equivariant diffeomorphism $\psi$ of the half-cylinder has the form $\psi(u, s) = (a(s)u, g(s)s)$ for some smooth maps $a: \mathbb{R}_{\geq 0} \to S^1$ and $g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. (This is a consequence of Hadamard’s lemma.) The corresponding map of $\mathbb{C}$ is $\varphi(z) = \sqrt{g(|z|^2)}a(|z|^2)z$, which is smooth. Because we can obtain a smooth inverse for $\varphi$ by applying the same argument to the inverse of $\psi$, we conclude that $\varphi$ is a diffeomorphism.

Conversely, every $S^1$-equivariant diffeomorphism $\varphi$ of $\mathbb{C}$ has the form $\varphi(z) = r(|z|^2)a(|z|^2)z$ where $r: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $a: \mathbb{R}_{\geq 0} \to S^1$ are smooth. (This is a consequence of Hadamard’s lemma and of the fact that any $S^1$-invariant smooth function of $z \in \mathbb{C}$ is smooth as a function of $|z|^2$. This fact, in turn, is a consequence of Whitney’s theorem about smooth even functions [32].) The corresponding map of the half-cylinder is $\psi(u, s) = (a(s)u, r(s)^2s)$, which is smooth. Because we can obtain a smooth inverse for $\psi$ by applying the same argument to the inverse of $\varphi$, we conclude that $\psi$ is a diffeomorphism.

Ordinary polar coordinates do not have this property: with the corresponding map

$$p: S^1 \times \mathbb{R}_{\geq 0} \to \mathbb{C}, \quad p(u, r) = ru,$$
Figure 5. Bijection between equivariant diffeomorphisms

The diagram

\[
\begin{array}{c}
S^1 \times \mathbb{R}_{\geq 0} \xrightarrow{\psi} S^1 \times \mathbb{R}_{\geq 0} \\
p \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
for $z$ near the origin. The function $\psi(u,r) := (\frac{A(ru) \cdot u}{|A(ru) \cdot u|}, r|A(ru) \cdot u|)$ is then a smooth lifting of $\varphi$ near the origin. Because we obtain a smooth inverse of $\psi$ near the origin by applying the same argument to the inverse of $\varphi$, we conclude that $\varphi$ lifts to a diffeomorphism $\psi$ of the half-cylinder. In fact, this argument applies to arbitrary—not necessarily equivariant—diffeomorphisms of $\mathbb{C}$ that fix the origin. This is a special case of the functoriality of the radial blowup construction, which we describe in the next section.

15. Radial blowup

In this section we describe the radial blowup construction, which was described by Klaus Jänich in 1968 [15]. This construction is a generalization of the passage to polar coordinates. This section provides preparation for the next section, where we describe the equivariant-radial-squared blowup construction, which provides an inverse to the cutting construction.

Below, we use the symbol $0$ to denote the zero section in any vector bundle. Whenever we use it, it should be clear from the context what is the ambient vector bundle.

Whereas any vector bundle can be naturally identified with the normal bundle of its zero section, we do not make this identification, so as to not introduce ambiguity on a set-theoretic level.

Radial blowup of a vector bundle along its zero section

Let $E \to F$ be a vector bundle. Consider the sphere bundle

$$S_{E}(0) := (\nu_{E}(0) \smallsetminus 0)/\mathbb{R}_{>0},$$

where $\nu_{E}(0)$ is the normal bundle of the zero section in $E$. As a set, the radial blowup of $E$ along $0$ is the disjoint union

$$E \odot 0 := (E \smallsetminus 0) \sqcup S_{E}(0),$$

equipped with the blow-down map

$$p: E \odot 0 \to E$$

whose restriction to $E \smallsetminus 0$ is the inclusion map into $E$ and whose restriction to $S_{E}(0)$ is induced from the bundle map $\nu_{E}(0) \to 0$.

Let

$$q: E \smallsetminus 0 \to S_{E}(0)$$

be the quotient map to the sphere bundle that is obtained from the natural identification $E \cong \nu_{E}(0)$ by restricting to the complement of
the zero section and composing with the quotient map to $S_E(0)$. Fix a fibrewise inner product on $E$ with fibrewise norm $| \cdot |$. There exists a unique manifold-with-boundary structure on $E \odot 0$ such that the bijection

$$E \odot 0 \to S_E(0) \times \mathbb{R}_{\geq 0}$$

that is given by

$$x \mapsto \begin{cases} (q(x), |x|) & \text{if } x \in E \smallsetminus 0 \\ (x, 0) & \text{if } x \in S_E(0) \end{cases}$$

is a diffeomorphism.

For any two fibrewise inner products on $E$, the ratio of their norms is a smooth function on $E \smallsetminus 0$ that is constant on $\mathbb{R}_{>0}$-orbits, so it has the form $x \mapsto \rho(q(x))$, where

$$\rho: S_E(0) \to \mathbb{R}_{>0}$$

is a smooth function with positive values. The corresponding bijections $E \odot 0 \to S_E(0) \times \mathbb{R}_{\geq 0}$ are then related by the diffeomorphism $(s, r) \mapsto (s, \rho(s)r)$ of $S_E(0) \times \mathbb{R}_{\geq 0}$. It follows that the manifold-with-boundary structure of $E \odot 0$ is independent of the choice of fibrewise inner product.

For any open neighbourhood $\mathcal{O}$ of the zero section $0$ in $E$,

$$\mathcal{O} \odot 0 := (\mathcal{O} \smallsetminus 0) \sqcup S_E(0)$$

is an open subset of the manifold-with-boundary $E \odot 0$, so it too becomes a manifold-with-boundary. Its boundary is

$$\partial(\mathcal{O} \odot 0) = S_E(0),$$

and its interior is $\mathcal{O} \smallsetminus 0$. The manifold structures on $S_E(0)$ and on $\mathcal{O} \smallsetminus 0$ that are induced from $\mathcal{O} \odot 0$ coincide with their manifold structures that are induced from $E$.

![Figure 6. Radial blowup](image-url)
Lifting diffeomorphisms

The following lemma is a crucial step in extending the radial blow-up construction from vector bundles to manifolds.

15.1. Lemma. Let $E \to F$ be a vector bundle and $0$ its zero section. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be open neighbourhoods of the zero section, and let

$$\varphi: \mathcal{O}_1 \to \mathcal{O}_2$$

be a diffeomorphism whose differential along the zero section is the identity map. Then $\varphi$ lifts to a diffeomorphism $\psi: \mathcal{O}_1 \circ 0 \to \mathcal{O}_2 \circ 0$. 

Lemma 15.1 follows from the following more general result.

15.2. Lemma. Let $E_1 \to F_1$ and $E_2 \to F_2$ be vector bundles. Let $\mathcal{O}_1$ be an open neighbourhood of the zero section $0_1$ in $E_1$, let $\mathcal{O}_2$ be an open neighbourhood of the zero section $0_2$ in $E_2$, and let

$$\varphi: \mathcal{O}_1 \to \mathcal{O}_2$$

be a smooth map. Assume that for some—hence every—fibrewise inner product on $E_2$, the function on $\mathcal{O}_1$ that is given by $e \mapsto |\varphi(e)|^2$ vanishes exactly on the zero section, and its Hessian is non-degenerate on the normal bundle to the zero section. Then there exists a unique smooth map $\psi: \mathcal{O}_1 \circ 0_1 \to \mathcal{O}_2 \circ 0_2$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{O}_1 \circ 0_1 & \xrightarrow{\psi} & \mathcal{O}_2 \circ 0_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
\mathcal{O}_1 & \xrightarrow{\varphi} & \mathcal{O}_2
\end{array}
$$

commutes. Moreover, the composition of $\psi$ with any boundary-defining function on $\mathcal{O}_2 \circ 0_2$ is a boundary-defining function on $\mathcal{O}_1 \circ 0_1$. 

Proof. The assumption on $\varphi$ is equivalent to the following property: $\varphi$ takes the zero section $0_1$ to the zero section $0_2$ and the complement $\mathcal{O}_1 \setminus 0_1$ to the complement $\mathcal{O}_2 \setminus 0_2$, and the fibrewise linear map $\nu_{E_1}(0_1) \to \nu_{E_2}(0_2)$ between the normal bundles to the zero sections that is induced from the differential of $\varphi$ is injective at each point of $0_1$.

Because $\varphi$ takes $\mathcal{O}_1 \setminus 0_1$ to $\mathcal{O}_2 \setminus 0_2$, and because the blow-down maps $p_1$ and $p_2$ restrict to the identity maps over the complements of the zero sections, the commuting of the diagram determines the map $\psi$ on the open dense subset $\mathcal{O}_1 \setminus 0_1$. By continuity, if such a lift $\psi$ exists then it is unique. So it is enough to show that a smooth lift $\psi$ of $\varphi$ exists locally near each point of the zero section $0_1$. 
For each $i$, choose a fibrewise inner product on $E_i$, and let $S_i \subset E_i$ be the corresponding unit sphere bundle. The inclusion map of $S_i$ into $E_i$ induces a diffeomorphism $S_i \cong S_{E_i}(0)$, and, furthermore, a diffeomorphism of manifolds-with-boundary

$$S_i \times [0, \infty) \cong E_i \circ 0_i,$$

whose composition with the blow-down maps is given by $(u, r) \mapsto ru$.

For each $i$, let $d_i$ denote the dimension of the base manifold $F_i$, and let $k_i$ denote the rank of the vector bundle $E_i \to F_i$. Because we may work locally, we may assume that each $F_i$ is an open subset of $\mathbb{R}^{d_i}$ and each $E_i$ is the trivial bundle $F_i \times \mathbb{R}^{k_i}$, and we may assume that $O_1 = D \times B$ where $D$ is an open subset of $\mathbb{R}^{d_1}$ and $B$ is a ball, say, of radius $\epsilon$, about the origin in $\mathbb{R}^{k_1}$. So $\varphi$ becomes a smooth map from $D \times B$ to $\mathbb{R}^{d_2} \times \mathbb{R}^{k_2}$, and we seek a smooth map $\psi$ such that the diagram

$$D \times S^{k_1-1} \times [0, \epsilon) \xrightarrow{\psi} \mathbb{R}^{d_2} \times S^{k_2-1} \times [0, \infty) \xrightarrow{(t, u, r) \mapsto (t, ru)} \mathbb{R}^{d_2} \times \mathbb{R}^{k_2}$$

commutes.

Because $\varphi$ maps the zero section to the zero section, Hadamard’s lemma with parameters gives a smooth map

$$A : D \times B \to \mathbb{R}^{k_2 \times k_1},$$

where $\mathbb{R}^{k_2 \times k_1}$ denotes the set of $k_2 \times k_1$ matrices, such that

$$\varphi(t, x) = (\varphi_1(t, x), A(t, x) \cdot x)$$

for all $t \in D$ and $x \in B$, where $\cdot$ denotes the product of a matrix with a column vector and where $\varphi_1$ is the first component of $\varphi$.

Writing $u = x/|x|$ and $r = |x|$, we have

$$\frac{A(t, x) \cdot x}{|A(t, x) \cdot x|} = \frac{A(t, ru) \cdot u}{|A(t, ru) \cdot u|} \quad \text{and} \quad |A(t, x) \cdot x| = r |A(t, ru) \cdot u|$$

whenever the denominator does not vanish. But the matrix $A(t, 0)$ represents the linear map between the normal bundles to the zero sections at the point $(t, 0)$, and the kernel of this linear map is trivial by our assumption on $\varphi$, so $A(t, ru) \cdot u$ is non-vanishing when $r = 0$. Because $\varphi$ takes the complement to the zero section to the complement to the zero section, $A(t, ru) \cdot u$ is non-vanishing also when $r > 0$. So

$$(t, u, r) \mapsto |A(t, ru) \cdot u|$$
defines a smooth map with positive values, and we obtain a smooth lift \( \psi \) of \( \varphi \) with the required properties by setting
\[
\psi(t, u, r) := \left( \varphi_1(t, ru), \frac{A(t, ru) \cdot u}{|A(t, ru) \cdot u|}, r|A(t, ru) \cdot u| \right).
\]

\[
\square
\]

**Radial blowup of a manifold along a submanifold**

Let \( X \) be a manifold, and let \( F \) be a closed submanifold. Consider the sphere bundle
\[
S_X(F) := (\nu_X(F) \setminus 0)/\mathbb{R}_{>0},
\]
where \( \nu_X(F) \) is the normal bundle of \( F \) in \( X \). As a set, the radial blowup of \( X \) along \( F \) is the disjoint union
\[
X \circ F := (X \setminus F) \sqcup S_X(F),
\]
equipped with the blow-down map
\[
p: X \circ F \to X
\]
whose restriction to \( X \setminus F \) is the inclusion map into \( X \) and whose restriction to \( S_X(F) \) is induced from the bundle map \( \nu_X(F) \to F \). This definition is consistent with the earlier definition for the radial blowup of a vector bundle along its zero section.

Let \( O \) be a starshaped open neighbourhood of the zero section \( 0 \) in the normal bundle \( \nu_X(F) \), let
\[
\varphi: O \to X
\]
be a tubular neighbourhood embedding, and let
\[
\psi: O \circ 0 \to X \circ F
\]
be the injection that is induced from \( \varphi \).

(We have \( O \circ 0 = (O \setminus 0) \sqcup S_X(F) \). On the subset \( O \setminus 0 \), the injection \( \psi \) coincides with \( \varphi \). On the subset \( S_X(F) \), the injection \( \psi \) is induced from the identification of \( \nu_{\nu_X(F)}(F) \) with \( \nu_X(F) \).)

Then we have
\[
X \circ F = \psi(O \circ 0) \cup (X \setminus F).
\]
There exists a unique manifold-with-boundary structure on \( X \circ F \) such that the maps
\[
\psi: O \circ 0 \to X \circ F \quad \text{and} \quad X \setminus F \xrightarrow{\text{inclusion}} X \circ F
\]
are diffeomorphisms with open subsets of $X \odot F$, where, in the domains of these maps, $O \odot 0$ is a manifold-with-boundary as an open subset of $E \odot 0$ for the vector bundle $E := \nu_X(F)$, and $X \setminus F$ is a manifold as an open subset of $X$. This is because the preimage in $O \odot 0$ of the intersection $\psi(O \odot 0) \cap (X \setminus F)$ is the open subset $O \setminus 0$, the preimage in $X \setminus F$ of this intersection is the open subset $\varphi(O \setminus 0)$, and $\psi$ restricts to the diffeomorphism $\varphi$ between these preimages. Because each of $O \odot 0$ and $X \setminus F$ is second countable, so is $X \odot F$. To see that $X \odot F$ is Hausdorff, note that any point of $X \setminus F$ has a neighbourhood $W$ in $X$ is disjoint from $F$; let $O_1$ be the preimage in $O$ of the complement $X \setminus W$; then $\psi(O_1 \odot 0)$ is a neighbourhood of $S_X(F)$ in $X \odot F$ that is disjoint from the neighbourhood $W$ of the given point of $X \setminus F$.

The boundary of the manifold-with-boundary $X \odot F$ is

$$\partial(X \odot F) = S_X(F),$$

and the interior of $X \odot F$ is $X \setminus F$. The manifold structures on $S_X(F)$ and on $X \setminus F$ that are induced from $X \odot F$ coincide with their manifold structures that are induced from $X$.

The manifold-with-boundary structure on $X \odot F$ is independent of the choice of tubular neighbourhood embedding. Jänich [15] states this fact without proof, and he writes this: *The proof has nothing to do with “uniqueness of tubular maps”, it is just an exercise in calculus.* He probably had in mind Lemma 15.1. Indeed, for any two tubular neighbourhood embeddings $\varphi_1: O_1 \to X$ and $\varphi_2: O_2 \to X$, their germs along the zero section of $\nu_X(F)$ are related by a diffeomorphism between neighbourhoods of the zero section whose differential along the zero section is the identity map. Lemma 15.1 implies that—after possibly shrinking $O_1$ and $O_2$ so as to have the same image in $X$—the corresponding maps $\psi_1: O_1 \odot 0 \to X \odot F$ and $\psi_2: O_2 \odot 0 \to X \odot F$ are related by a diffeomorphism from $O_1 \odot 0$ to $O_2 \odot 0$. This implies that the identity map on $X \odot F$ is a diffeomorphism from the manifold-with-boundary structure that is induced from the tubular neighbourhood embedding $\varphi_1$ to the manifold-with-boundary structure that is induced from the tubular neighbourhood embedding $\varphi_2$.

16. **Equivariant radial-squared blowup**

In this section we describe the radial-squared blowup, which—for circle actions—provides an inverse to the cutting construction.
Radial-squared blowup of a vector bundle along its zero section

Let $E \to F$ be a vector bundle. As a set, the radial-squared blowup of $E$ along $0$ is the same as the radial blowup:

$$E \odot 0 := (E \setminus 0) \sqcup S_E(0),$$

where $0$ is the zero section and $S_E(0) = (\nu_E(0) \setminus 0)/\mathbb{R}_{>0}$ is the sphere bundle of its normal bundle. We take the same blow-down map

$$p: E \odot 0 \to E$$

and quotient map

$$q: E \setminus 0 \to S_E(0)$$

as before, and fix a fibrewise inner product on $E$ with fibrewise norm $|\cdot|$. There exists a unique manifold-with-boundary structure on $E \odot 0$ such that the bijection

$$E \odot 0 \to S_E(0) \times \mathbb{R}_{\geq 0}$$

that is given by

$$x \mapsto \begin{cases} (q(x), |x|^2) & \text{if } x \in E \setminus 0 \\ (x, 0) & \text{if } x \in S_E(0) \end{cases}$$

(this time with $|x|^2$, not $|x|$) is a diffeomorphism. As before, this manifold-with-boundary structure on $E \odot 0$ is independent of the choice of fibrewise inner product, its boundary is $S_E(0)$, and its interior is $E \setminus 0$.

The identity map

$$E \odot 0 \xrightarrow{\text{identity}} E \odot 0$$

is a homeomorphism and is smooth, but it is not a diffeomorphism. Thus, the radial blowup $E \odot 0$ and the radial-squared blowup $E \odot 0$ yield the same topological manifold, with the same smooth structures on its boundary and on its interior, but with different manifold-with-boundary structures. For any neighbourhood $U$ of the zero section, there exists a diffeomorphism between these manifolds-with-boundary that is supported in $U$ and that restricts to the identity map on the zero section, but we cannot take this diffeomorphism to be the identity map.

Attempt at radial-squared blowup of a manifold along a submanifold
We now discuss the possibility of defining the radial-squared blowup of a manifold $X$ along a closed submanifold $F$. As a set, this will be the same as the radial blowup:

$$X\odot F = (X \setminus F) \sqcup S_X(F),$$

with the same blow-down map $p: X\odot F \to X$. Let $O$ be a starshaped open neighbourhood of the zero section 0 in the normal bundle $\nu_X(F)$, let $O\odot 0 := (O \setminus 0) \sqcup S_E(0)$ be the corresponding open subset of the manifold-with-boundary $E\odot 0$, fix a tubular neighbourhood embedding $O \to X$, and let

$$\psi: O\odot 0 \to X\odot F$$

be the injection that it induced. As before, there exists a unique manifold-with-boundary structure on $X\odot F$ such that the maps

$$\psi: O\odot 0 \to X\odot F \quad \text{and} \quad X \setminus F \xrightarrow{\text{inclusion}} X\odot F$$

are diffeomorphisms with open subsets of $X\odot F$. For this manifold-with-boundary structure to be independent on the choice of the tubular neighbourhood embedding, we need an analogue of Lemma 15.1: for every diffeomorphism $\varphi: O \to O'$ between neighbourhoods of the zero section whose differential along the zero section is the identity map, we would like to have a diffeomorphism $\hat{\psi}$ such that the square

$$
\begin{array}{ccc}
O\odot 0 & \xrightarrow{\hat{\psi}} & O'\odot 0 \\
\downarrow & & \downarrow \\
O & \xrightarrow{\varphi} & O'
\end{array}
$$

commutes.

Unfortunately, such a $\hat{\psi}$ might not exist. To see this, we locally identify $O$ with $D \times B$ where $D$ is an open subset of $\mathbb{R}^d$ and $B$ is a ball, say, of radius-squared $\epsilon$, about the origin in $\mathbb{R}^k$, and we write the diffeomorphism $\varphi$ as

$$\varphi(t, x) = (\varphi_1(t, x), A(t, x) \cdot x),$$

as in the proof of Lemma 15.2, where $\varphi(t, 0) = t$ and $A(t, 0)$ is the identity matrix for all $t$. We then seek a smooth map $\hat{\psi}$ such that the
diagram

\[
\begin{array}{ccc}
D \times S^{k-1} \times [0, \epsilon) & \overset{\hat{\psi}}{\longrightarrow} & \mathbb{R}^d \times S^{k-1} \times [0, \infty) \\
(t,u,s) & \mapsto & (t,\sqrt{s}u) \\
\downarrow & & \downarrow \\
D \times B & \overset{\varphi}{\longrightarrow} & \mathbb{R}^d \times \mathbb{R}^k
\end{array}
\]

commutes. Necessarily,

\[\hat{\psi}(t,u,s) = \left(\varphi_1(t,\sqrt{s}u), \frac{A(t,\sqrt{s}u) \cdot u}{|A(t,\sqrt{s}u) \cdot u|}, s|A(t,\sqrt{s}u) \cdot u|^2\right);\]

this map is continuous, but it might not be smooth. Take, for example, 
\[\varphi(t,x) = (t + L(x), x)\] for some non-trivial linear map \(L: \mathbb{R}^k \to \mathbb{R}^d\). Then \(\hat{\psi}(t,u,s) = (t + \sqrt{s}L(u), u, s)\), which is not smooth along \(\{s = 0\}\).

Fortunately, in the special case that relates to cutting, we can obtain a manifold structure on \(X \ominus F\) through tubular neighbourhood embeddings that are equivariant near \(F\). We will do this in a moment.

In this special case that we need, \(F\) has codimension 2, and the manifold \(X\) is equipped with a circle action near \(F\) that fixes \(F\) and is free outside \(F\). But keeping in mind future applications, we will allow more than circle groups. This is inspired by Jänich’s and Bredon’s work ([14, 15], [4, Chapter VI, Section 6]). Future applications may include simultaneous cutting of toric Lagrangians [2] that are invariant under a product of several copies of \(S^1\) and \(\mathbb{Z}_2\).

**Equivariant radial-squared blowup of a manifold along a submanifold**

Let \(G\) be a compact Lie group, and let \(E \to F\) be a vector bundle, equipped with a fibrewise linear \(G\) action that is transitive on the fibres of the sphere bundle \((E \setminus 0)/\mathbb{R}_{>0}\). Fix a \(G\)-invariant fibrewise inner product on \(E\), and let \(S\) be the unit sphere bundle in \(E\) with respect to the corresponding fibrewise norm \(|\cdot|\). The natural diffeomorphism \(S \cong (E \setminus 0)/\mathbb{R}_{>0}\) gives a diffeomorphism of manifolds-with-boundary

\[S \times [0, \infty) \overset{\sim}{\longrightarrow} E \ominus 0\]

whose composition with the radial-squared-blowdown map is

\[(u, s) \mapsto \sqrt{s}u.\]

Let \(E' \to F'\) be another vector bundle with a \(G\) action with these properties, with zero section \(0'\) and unit sphere bundle \(S'\).

We have the following analogues of Lemmas 15.1 and 15.2.
16.1. Lemma. Let $\mathcal{O}$ and $\mathcal{O}'$ be $G$-invariant open neighbourhoods of the zero section of $E$, and let

$$\varphi: \mathcal{O} \to \mathcal{O}'$$

be a $G$-equivariant diffeomorphism whose differential along the zero section is the identity map. Then $\varphi$ lifts to a diffeomorphism $\psi: \mathcal{O} \cap 0 \to \mathcal{O}' \cap 0$.

Lemma 16.1 follows from the following more general result.

16.2. Lemma. Let $\mathcal{O}$ and $\mathcal{O}'$ be $G$-invariant neighbourhoods of the zero sections in $E$ and $E'$, and let

$$\varphi: \mathcal{O} \to \mathcal{O}'$$

be a $G$-equivariant smooth map. Assume that the function on $\mathcal{O}$ that is given by $e \mapsto |\varphi(e)|^2$ vanishes exactly on the zero section and that its Hessian is non-degenerate on the normal bundle to the zero section. Then there exists a unique smooth map $\psi'$ such that the diagram

$$\begin{array}{ccc}
\mathcal{O} \cap 0 & \xrightarrow{\psi'} & \mathcal{O}' \cap 0' \\
\downarrow & & \downarrow \\
\mathcal{O} & \xrightarrow{\varphi} & \mathcal{O}'
\end{array}$$

commutes. Moreover, $\psi'$ is equivariant, and its composition with any boundary-defining function on $\mathcal{O}' \cap 0'$ is a boundary-defining function on $\mathcal{O} \cap 0$.

Proof. As in the proof of Lemma 15.2, we locally identify $\varphi$ with a map

$$\varphi: D \times B \to \mathbb{R}^d \times \mathbb{R}^{k'}$$

of the form

$$\varphi(t, x) = (\varphi_1(t, x), A(t, x) \cdot x)$$

where $A(t, 0)$ is invertible for all $t$, and we seek a smooth map $\hat{\psi}$ such that the diagram

$$\begin{array}{ccc}
D \times S^{k-1} \times [0, \epsilon) & \xrightarrow{\hat{\psi}} & \mathbb{R}^d \times S^{k'-1} \times [0, \infty) \\
(t, u, s) \mapsto (t, \sqrt{s}u) & & (t, u, s) \mapsto (t, \sqrt{s}u) \\
D \times B & \xrightarrow{\varphi} & \mathbb{R}^d \times \mathbb{R}^{k'}
\end{array}$$

commutes. Because $\varphi$ is $G$ equivariant, $\varphi_1$ and $A(t, x)$ are $G$ invariant. Because the $G$ action is transitive on the fibres of $S$, there exist
functions \( \tilde{\varphi}_1 \) and \( \tilde{A} \) such that
\[
\varphi_1(t, x) = \tilde{\varphi}_1(t, |x|^2) \quad \text{and} \quad A(t, x) = \tilde{A}(t, |x|^2).
\]
Because \( \varphi_1 \) and \( A \) are smooth, the functions \( \tilde{\varphi}_1 \) and \( \tilde{A} \) are also smooth. (This special case of Schwarz’s theorem [27] appears in Bredon’s book [4, Chapter VI, Theorem 5.1]; it can be deduced from Whitney’s theorem about even functions [32].) Then
\[
\hat{\psi}(t, u, s) := \left( \tilde{\varphi}_1(t, s), \frac{\tilde{A}(t, s) \cdot u}{|\tilde{A}(t, s) \cdot u|^2}, s|\tilde{A}(t, s) \cdot u|^2 \right)
\]
is a smooth lift of \( \varphi \) with the required properties. \( \square \)

We can now define the equivariant radial-squared blowup \( X \lozenge F \) of a manifold \( X \) along a submanifold \( F \), when a compact Lie group \( G \) acts on a neighbourhood of \( F \), and when this action fixes \( F \) and is transitive on the fibres of the sphere bundle \( S_X(F) = (\nu_X(F) \setminus 0)/\mathbb{R}_{>0} \). As a set, it is the same as the radial blowup:
\[
X \lozenge F := (X \setminus F) \sqcup S_X(F).
\]

Fix a \( G \)-invariant starshaped open neighbourhood \( O \) of the zero section \( 0 \) in the normal bundle \( \nu_X(F) \) and a \( G \)-equivariant tubular neighbourhood embedding
\[
O \to X,
\]
and let
\[
\psi: O \lozenge 0 \to X \lozenge F
\]
be the injection that is induced from \( \varphi \). There exists a unique manifold-with-boundary structure on \( X \lozenge F \) such that the maps
\[
\psi: O \lozenge 0 \to X \lozenge F \quad \text{and} \quad X \setminus F \xrightarrow{\text{inclusion}} X \lozenge F
\]
are diffeomorphisms with open subsets of \( X \lozenge F \). Lemma 16.1 implies that this manifold-with-boundary structure is independent on the choice of the tubular neighbourhood embedding.

As in the case of a vector bundle, the radial blowup \( X \odot F \) and the radial-squared blowup \( X \lozenge F \) yield the same topological manifold, with the same smooth structures on its boundary and on its interior, but with different manifold-with-boundary structures. For any neighbourhood of \( F \), there exists an equivariant diffeomorphism between these manifolds-with-boundary that is supported in this neighbourhood and that restricts to the identity map on \( F \), but we cannot take this diffeomorphism to be the identity map.

**Functoriality:**
By Lemma 16.2, the radial-squared blowup procedure defines a functor. The domain of this functor is the following category. An object is a pair \((X, F)\), where \(X\) is a manifold and \(F\) is a closed submanifold, equipped with a \(G\)-action on a neighbourhood of \(F\) that fixes \(F\) and acts transitively on the fibres of the sphere bundle of \(\nu_X(F)\). For such a pair \((X, F)\), an **invariant norm-squared-like function** is a function \(X \rightarrow \mathbb{R}_{\geq 0}\) that is \(G\)-invariant near \(F\), that vanishes exactly on \(F\), and whose Hessian is non-degenerate on the normal bundle \(\nu_X(F)\).

A morphism from \((X, F)\) to \((X', F')\) is a smooth map from \(X\) to \(X'\) whose composition with (some, hence every) invariant norm-squared-like function on \(X'\) is an invariant norm-squared-like function on \(X\), and that is \(G\)-equivariant near \(F\).

The functor takes an object \((X, F)\) to \(X \circledast F\) and a morphism \(\varphi: X \rightarrow X'\) to the smooth map \(\psi: X \circledast F \rightarrow X' \circledast F'\) such that the following diagram commutes

\[
\begin{array}{ccc}
X \circledast F & \xrightarrow{\psi} & X' \circledast F' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & X',
\end{array}
\]

where the vertical arrows are the equivariant-radial-squared-blowdown maps. Such a map \(\psi\) exists by Lemma 16.2.

When \(G\) is the circle group and its action is faithful, the transitivity of the action on the fibres of the sphere bundle \(S_E(F)\) implies that \(F\) has codimension two in \(X\) and that the action is free on a punctured neighbourhood of \(F\) in \(X\). The equivariant-radial-squared-blowup functor then takes values in the category whose objects are manifolds-with-boundary, equipped with free circle actions near their boundaries, and whose morphisms are equivariant transverse maps, which we defined in Section 2. We now show that this functor provides an inverse to the cutting functor.

**Radial-squared blowup and cutting:**

We return to the circle group, \(G = S^1\).

Let \(X\) be a manifold and \(F\) a closed codimension two submanifold, and let \(U_X\) be an open neighborhood of \(F\) in \(X\), equipped with a circle action that fixes \(F\) and is free on \(U_X \setminus F\). The radial-squared blowup construction yields a manifold-with-boundary \(X \circledast F\). The circle action on \(U_X\) lifts to a free circle action on the open neighbourhood.
of the boundary. The cutting construction then yields a manifold $(X \circ F)_{\text{cut}}$ and a quotient map $c : X \circ F \to (X \circ F)_{\text{cut}}$. The interior of the manifold-with-boundary $X \circ F$ is $X \setminus F$. We claim that there exists a unique diffeomorphism

$$X \xrightarrow{\cong} (X \circ F)_{\text{cut}}$$

whose restriction to $X \setminus F$ coincides with the quotient map $c$ (which takes each point of $X \setminus F$ to the singleton containing that point).

Conversely, let $M$ be a manifold-with-boundary, equipped with a free circle action on an open neighbourhood $U_M$ of the boundary. The cutting construction yields a manifold $M_{\text{cut}}$ and codimension two submanifold $M_{\text{red}}$, and a quotient map $c : M \to M_{\text{cut}}$. The circle action on $U_M$ descends to a circle action on an open neighbourhood of $M_{\text{red}}$ that fixes $M_{\text{red}}$ and is free outside $M_{\text{red}}$. The radial-squared blowup then yields a manifold-with-boundary $M_{\text{cut}} \circ M_{\text{red}}$ with a free circle action on the open neighbourhood of the boundary. The interior of the manifold-with-boundary $M_{\text{cut}} \circ F$ is $(M)_{\text{cut}}$. We claim that there exists a unique diffeomorphism

$$M \xrightarrow{\cong} M_{\text{cut}} \circ M_{\text{red}}$$

whose restriction to the interior $\hat{M}$ coincides with the quotient map $c$ (which takes each point of $\hat{M}$ to the singleton containing that point).

By functoriality, it is enough to check these claims locally. Locally, these claims follow from the local models for the cutting construction and from the radial–squared blowup construction of a vector bundle.

Namely, if

$$X = D^{n-2} \times D^2 \quad \text{and} \quad F = D^{n-2} \times \{0\}$$

where $D^{n-2}$ is an open subset of $\mathbb{R}^{n-2}$ and $D^2$ is the open disc of radius-squared $\epsilon$ about the origin in $\mathbb{R}^2$, then the definition of the radial-squared blowup gives a diffeomorphism

$$X \circ F \xrightarrow{\cong} D^{n-2} \times S^1 \times [0, \epsilon)$$

that carries the radial-squared-blowdown map $X \circ F \to X$ to

$$(t, u, s) \mapsto (t, \sqrt{s} u),$$

which is exactly the map that induces the diffeomorphism

$$\left( D^{n-2} \times S^1 \times [0, \epsilon) \right)_{\text{cut}} \xrightarrow{\cong} D^{n-2} \times D^2.$$
Conversely, if $M = D^{n-2} \times S^1 \times [0, \epsilon)$, then we have a diffeomorphism $M_{\text{cut}} \cong D^{n-2} \times D^2$ that takes the quotient map $c: M \to M_{\text{cut}}$ to 

$$(t, u, s) \mapsto (t, \sqrt{s}u),$$

which is exactly the map that induces the diffeomorphism $M_{\text{cut}} \times M_{\text{red}} \cong D^{n-2} \times S^1 \times [0, \epsilon)$.

**Appendix A. Actions and Quotients**

In this appendix we collect some well-known facts about actions of compact groups and their quotients.

First, we recall some facts about proper maps. A continuous map $g: A \to B$ between topological spaces is **proper** if the preimage of every compact subset of $B$ is a compact subset of $A$.

**Exercise.** Every closed map with compact level sets is proper. Every proper map has compact level sets. If $B$ is Hausdorff and locally compact, then every proper map to $B$ is a closed map.

Recall that, for a topological space, being $T_1$ means that singletons are closed, and being normal means that for any two disjoint closed sets $C_1$ and $C_2$ there exist open sets $U_1$ and $U_2$ that are disjoint and such that $U_1$ contains $C_1$ and $U_2$ contains $C_2$. Every $T_1$ normal space is Hausdorff.

Manifolds are assumed to be Hausdorff and second countable. Because they are locally compact, these properties imply that they are paracompact (every open cover has a locally finite open refinement) and normal; see, e.g., [19, Theorems 4.77 and 4.81].

**A.1. Lemma.** Let a compact topological group $G$ act continuously on a $T_1$ normal topological space $N$. Consider the action map $\rho: G \times N \to N$, which we write as $(a, x) \mapsto a \cdot x$, and the quotient map $\pi: N \to N/G$. For any subset $A$ of $N$, write $G \cdot A = \{a \cdot x \mid a \in G \text{ and } x \in N\}$. Write the orbit of a point $x \in N$ also as $G \cdot x$ (and not only as $G \times \{x\}$).

1. Orbits in $N$ are compact and closed.
2. The action map $G \times N \to N$ is a closed map.
3. For every open subset $U$ of $N$, the subset $G \cdot U$ of $N$ is open.
4. For every open subset $U$ of $N$, the intersection $\bigcap_{a \in G} a \cdot U$ is open.
(5) The quotient topological space \( N/G \) is \( T_1 \) and normal.

Proof. For any compact subset \( K \) of \( N \), the subset \( G \cdot K \) of \( N \), being the image of the compact set \( G \times K \) under the continuous action map \((a, x) \mapsto a \cdot x\), is also compact.

Applying this to singletons in \( N \), we obtain that \( G \)-orbits in \( N \) are compact. Because \( N \) is Hausdorff, this implies that \( G \)-orbits in \( N \) are closed. This proves (1).

We now show that the action map \( G \times N \to N \) is a proper map. Let \( K \) be a compact subset of \( N \). Because \( N \) is Hausdorff, the compact set \( K \) is closed in \( N \), so its preimage under the continuous action map \((a, x) \mapsto a \cdot x\) is closed. Being a closed subset of the compact set \( G \times (G \cdot K) \), this preimage is compact. Because \( K \) was an arbitrary compact subset of \( N \), the action map is proper.

Because the action map \( G \times N \to N \) is proper and its target space \( N \) is Hausdorff and locally compact, the action map is a closed map. This proves (2).

Let \( U \) be an open subset of \( N \). For each \( a \in G \), because the maps \( x \mapsto a \cdot x \) and \( x \mapsto a^{-1} \cdot x \) are continuous and are inverses of each other, the set \( a \cdot U \) is open. The set \( G \cdot U \), being the union of the open sets \( a \cdot U \) over all \( a \in G \), is then open. This proves (3).

Let \( U \) be an open subset of \( N \). Then the complement of \( U \) in \( N \) is closed in \( N \). By (2), the image of this complement under the action map \( G \times N \to N \) is closed in \( N \). So the complement in \( N \) of this image is open. But this complement is exactly \( \bigcap_{a \in G} a \cdot U \). This proves (4).

Because (by (1)) orbits in \( N \) are closed, the quotient space \( N/G \) is \( T_1 \). It remains to show that the quotient space \( N/G \) is normal. Let \( C_1 \) and \( C_2 \) be disjoint closed subsets of \( N/G \). Then \( \pi^{-1}(C_1) \) and \( \pi^{-1}(C_2) \) are disjoint closed subsets of \( N \). Because \( N \) is normal, there exist open subsets \( \hat{U}_1 \) and \( \hat{U}_2 \) of \( N \) that are disjoint and such that \( \hat{U}_1 \) contains \( \pi^{-1}(C_1) \) and \( \hat{U}_2 \) contains \( \pi^{-1}(C_2) \). Let \( U_1 := \left( \bigcap_{a \in G} a \cdot \hat{U}_1 \right)/G \) and \( U_2 := \left( \bigcap_{a \in G} a \cdot \hat{U}_2 \right)/G \). Then \( U_1 \) and \( U_1 \) are open subsets of \( N/G \) (by (4)), they are disjoint, \( U_1 \) contains \( C_1 \), and \( U_2 \) contains \( C_2 \). Because \( C_1 \) and \( C_2 \) were arbitrary disjoint closed subsets of \( N/G \), we conclude that \( N/G \) is normal. This proves (5). \( \Box \)
A.2. Lemma. Let a compact topological group $G$ act continuously on a topological space $N$. Suppose that $N$ is $T_1$ normal, and second countable. Then $N/G$ is also $T_1$ normal, and second countable.

Proof of Lemma A.2. Let $y$ be a point of $N/G$. By (1), $y$ is closed as a subset of $N$. Because the subset $y$ of $N$ is the preimage under the quotient map $N \to N/G$ of the singleton $\{y\}$, this singleton is closed in $N/G$. Because the point $y$ of $N/G$ was arbitrary, the topological space $N/G$ is $T_1$.

Let $\hat{C}_1$ and $\hat{C}_2$ be disjoint closed subsets of $N/G$. Let $C_1$ and $C_2$ be their preimages in $N$; then $C_1$ and $C_2$ are disjoint closed subsets of $N$. Because $N$ is normal, there exist disjoint open subsets $U_1$ and $U_2$ of $N$, that, respectively, contain $C_1$ and $C_2$. By (4), the $G$-invariant sets $U'_1 := \bigcap_{a \in G} a \cdot U_1$ and $U'_2 := \bigcap_{a \in G} a \cdot U_2$ are open in $N$. The quotients $U'_1/G$ and $U'_2/G$ are disjoint open subsets of $N/G$ that, respectively, contain $\hat{C}_1$ and $\hat{C}_2$. Because the closed subsets $\hat{C}_1$ and $\hat{C}_2$ of $N/G$ were arbitrary, the topological space $N/G$ is normal.

A.3. Lemma. Let a compact Lie group $G$ act freely on a manifold-with-boundary $N$. Then the following holds.

1. There exists a unique manifold-with-boundary structure on $N/G$ such that the quotient map $N \to N/G$ is a submersion. Moreover, this quotient map is a principal $G$-bundle. Finally, the boundary of $N$ maps to the boundary of $N/G$, and the interior of $N$ maps to the interior of $N/G$.

2. For every orbit $G \cdot x$ in $N$ and $G$-invariant open set $U$ that contains the orbit there exists a $G$-invariant smooth function $\rho : N \to \mathbb{R}$ that is equal to 1 on a neighbourhood of the orbit and whose support is contained in the open set $U$.

3. For every cover of $N$ by $G$-invariant open sets there exists a partition of unity by $G$-invariant smooth functions that is subordinate to the cover.

Proof. The following two local facts are special cases of Koszul’s slice theorem and its analogue for manifolds-with-boundary. They can be proved by a straightforward adaptation of the proof of Proposition 3.2.
• Each orbit in the interior of $N$ has a neighbourhood that is equivariantly diffeomorphic to $D^k \times G$, with $G$ acting by left translation on the middle factor, and where $D^k$ is a disc in $\mathbb{R}^k$.

• Similarly, each orbit in the boundary of $N$ has a neighbourhood that is equivariantly diffeomorphic to $D^{k-1} \times G \times [0, \epsilon)$, with $G$ acting by left translation on the middle factor, and where $D^{k-1}$ is a disc in $\mathbb{R}^{k-1}$ and $\epsilon > 0$.

By Lemma A.1(5), the quotient space $N/G$ is $T_1$ and normal. Let $\mathcal{U}$ be a countable basis for the topology of $N$; then $\mathcal{U}' := \{(G \cdot U/G)_{U \in \mathcal{U}}\}$ is a countable basis for the topology of $N/G$. So $N/G$ is second countable.

The first part of the lemma follows from the above two local facts and from $N/G$ being Hausdorff and second countable. The second and third parts of the lemma then follow from the existence of smooth bump functions and smooth partitions of unity on the manifold-with-boundary $N/G$. □

Appendix B. Simultaneous cutting

We expect the results of this paper to generalize to the setups of simultaneous cutting. We include here the relevant definitions, deferring the details to another occasion.

Let $M$ be an $n$ dimensional manifold-with-corners (as introduced by Jean Cerf and by Adrien Douady [7, 10]; see [20, Sect. 16]).

The depth of a point $x \in M$ is the (unique) integer $k$ such that there exists a chart $U \to \Omega$ from a neighbourhood $U$ of $x$ in $M$ to an open subset $\Omega$ of the sector $\mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}$ that takes $x$ to a point in $\{0\}^k \times \mathbb{R}^{n-k}$. The $k$-boundary of $M$, denoted $M^{(k)}$, is the set of points $x \in M$ of depth $k$. The interior of $M$ is its 0-boundary. The strata of $M$ are the connected components of $M^{(k)}$ for $k = 0, \ldots, n$.

Every open subset $U$ of $M$ is also a manifold with corners. The manifold with corners $M$ is a manifold with faces if each point $x$ of depth $k$ is in the closure of $k$ distinct components of $M^{(1)}$.

Now assume that $M$ is an $n$ dimensional manifold with faces. The faces of $M$ are the closures of the strata. The facets of $M$ are the $n-1$ dimensional strata. Let $\mathcal{S}$ denote the set of facets of $M$. Every codimension $k$ face $Y$ of $M$, obtained as the closure of a stratum $\hat{Y}$, is itself a manifold with corners (in fact, with faces), whose interior is the
stratum $\tilde{Y}$. Also, the face $Y$ is the intersection of a unique $k$ element subset $S_Y$ of $\mathcal{S}$.

For each facet $F \in \mathcal{S}$, let $U_F$ be a neighbourhood of $F$ in $M$. Suppose that, for each subset $S$ of $\mathcal{S}$, the neighbourhoods $U_F$, for $F \in S$, have a nonempty intersection only if the facets $F$, for $F \in S$, have a nonempty intersection. This intersection is then the union of those faces $Y$ with $S_Y = S$. Suppose that we are given a free circle action on each neighbourhood $U_F$ and that these actions commute on the intersections of these neighbourhoods.

To each face $Y$ of $M$ we then associate neighbourhood $U_Y$ that is contained in the intersection $\cap\{U_F \mid F \in S_Y\}$ and is invariant with respect to the torus $T_Y := (S^1)^{S_Y}$, which acts on $U_Y$. (If $Y$ has codimension $k$, then $S_Y$ is a set of $k$ elements, so $T_Y$ has dimension $k$.)

Consider the equivalence relation $\sim$ on $M$ such that, for $m \neq m'$, we have that $m \sim m'$ if and only if there exists a codimension $k$ face $Y$ of $M$ and a torus element $a \in T_Y$ such that $m, m' \in Y$ and $m = a \cdot m'$.

Let $M_{\text{cut}} := M/\sim$; equip $M_{\text{cut}}$ with the quotient topology; let $c: M \to M_{\text{cut}}$ denote the quotient map. For each face $Y$ of $M$, denote $X_Y = \tilde{Y}_{\text{cut}}$. There exists a unique manifold structure on $X_Y$ such that a real valued function $h: X_Y \to \mathbb{R}$ is smooth if and only if the function $h \circ c|_{\tilde{Y}}: \tilde{Y} \to \mathbb{R}$ is smooth. Moreover, with this manifold structure, the map $c|_{\tilde{Y}}: \tilde{Y} \to X_Y$ is a principal $T_Y$-bundle. This follows from the slice theorem for the $T_Y$ action on $\tilde{Y}$.

So we have a decomposition of the topological space $M_{\text{cut}}$ into disjoint subsets $X_Y$, and we have a smooth manifold structure on each of the subsets $X_Y$.

For each facet $F \in \mathcal{S}$, fix a smooth function $f^{(F)}: U_F \to \mathbb{R}_{\geq 0}$ whose zero level set is $F$ and such that $df^{(F)}|_F$ never vanishes. Moreover, assume that each $f^{(F)}|_{U_F}$ is invariant, with respect to the circle action on $U_F$, on some invariant neighbourhood of $F$ in $U_F$. Furthermore, assume that for each face $Y$ the functions $f^{(F)}$, for $F \in S_Y$, are $T_Y$-invariant on some neighbourhood of $Y$.

Generalizing Construction 1.3, we define $\mathcal{F}_M$ to be the set of those real valued functions $h: M_{\text{cut}} \to \mathbb{R}$ whose composition $\hat{h} := h \circ c$ with the quotient map $c: M \to M_{\text{cut}}$ satisfies the following two conditions.

($\mathcal{F}_1$) $\hat{h}|_{\tilde{M}}: \tilde{M} \to \mathbb{R}$ is smooth.
(F2) For each face $Y$ of $M$, there exists a $T_Y$-invariant neighbourhood $V$ of $Y$ in $U_Y$ and a smooth function $H : V \times \mathbb{C}^{S_Y} \to \mathbb{R}$ such that

(a) $H(a \cdot x, z) = H(x, az)$ for all $a \in T_Y$ and $(x, z) \in V \times \mathbb{C}^{S_Y}$; and

(b) $\hat{h}(x) = H(x, z)$, where the coordinates of $z$ are given by $z_F = \sqrt{f(F)(x)}$ for all $F \in S_Y$.

As in Lemma 1.4 and Theorem 1.5, we expect the set of functions $\mathcal{F}_M$ to be independent of the choice of invariant boundary defining functions $f(F)$, and we expect there to exist a unique manifold structure on $M_{cut}$ such that $\mathcal{F}_M$ is the set of real valued smooth functions on $M_{cut}$.

We expect the results of this paper (functoriality of $\psi \mapsto \psi_{cut}$, cutting of submanifolds, cutting of differential forms, symplectic and contact cutting) to extend to this setup.

We expect that a simultaneous (not iterated) equivariant-radial-squared-blowup construction would provide an inverse to the simultaneous cutting construction.

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