Finite-Sum Compositional Stochastic Optimization: Theory and Applications

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Abstract

This paper studies stochastic optimization for a sum of compositional functions, where the inner-level function of each summand is coupled with the corresponding summation index. We refer to this family of problems as finite-sum coupled compositional optimization (FCCO). It has broad applications in machine learning for optimizing non-convex or convex compositional measures/objectives such as average precision (AP), $p$-norm push, listwise ranking losses, neighborhood component analysis (NCA), deep survival analysis, deep latent variable models, softmax functions, and model agnostic meta-learning, which deserves finer analysis. Yet, existing algorithms and analysis are restricted in one or other aspects. The contribution of this paper is to provide a comprehensive analysis of a simple stochastic algorithm for both non-convex and convex objectives. The key results are improved oracle complexities with the parallel speed-up by the moving-average based stochastic estimator with mini-batching. Our theoretical analysis also exhibits new insights for improving the practical implementation by sampling the batches of equal size for the outer and inner levels. Numerical experiments on AP maximization and $p$-norm push optimization corroborate some aspects of the theory.

1 Introduction

A fundamental problem in machine learning (ML) that has been studied extensively is the empirical risk minimization (ERM), whose objective is a sum of individual losses on training examples, i.e.,

$$
\min_{w \in \Omega} F(w), \quad F(w) := \frac{1}{n} \sum_{z_i \in D} \ell(w; z_i),
$$

where $w$ and $\Omega \subseteq \mathbb{R}^d$ denotes the model parameter and its domain, $D$ denotes the training set of $n$ examples, and $z_i$ denotes an individual data. However, ERM cannot cover many interesting measures/objectives and hide the complexity of problems whose individual loss and its gradient cannot be easily computed. Instead, in this paper we study a new family of problems that aims to optimize the following compositional objective:

$$
\min_{w \in \Omega} F(w), \quad F(w) := \frac{1}{n} \sum_{z_i \in D} f_i(g(w; z_i; S_i)),
$$

where $g : \Omega \mapsto \mathcal{R}^d$, $f_i : \mathcal{R} \mapsto \mathcal{R}$, and $S_i$ denotes another (finite or infinite) set of examples that could be either dependent or independent of $z_i$. We give an example for each case: 1) In the bipartite ranking, $D$ represents the positive data while $S = S_i$ represents the negative data; 2) In the robust learning (e.g. the invariant logistic regression in [9]), $D$ represents the training data set while $S_i$ denotes the set of perturbed observations for data $z_i \in D$, where $S_i$ depends on $z_i$. We are particularly interested in the case that set $S_i$
is infinite or contains a large number of items, and assume that an unbiased stochastic estimators of $g$ and $\nabla g$ can be computed via sampling from $S_i$. We refer to (1) as finite-sum coupled compositional optimization (FCCO) and its objective as finite-sum coupled compositional risk (FCCR), where for each data $z_i$, the risk $f_i(g(w; z_i, S_i))$ is of a compositional form such that $g$ couples each $z_i$ and items in $S_i$. It is notable that $f_i$ could be stochastic or has a finite-sum structure that depends on a large set of items. For simplicity of presentation and discussion, we focus on the case that $f_i$ is a simple deterministic function whose value and gradient can be easily computed, which covers many interesting objectives of interest. The algorithms and analysis can be extended to the case that $(f_i, \nabla f_i)$ are estimated by their unbiased stochastic versions using random samples (discussed in Appendix D). For simplicity, we denote $g(w; z_i, S_i)$ by $g_i(w)$ in the sequel.

Applications of FCCO. The Average Precision (AP) maximization problem studied in Qi et al. [15] is an example of FCCO. Nevertheless, we notice that the application of FCCO is much broader beyond the AP maximization, including but are not limited to p-norm push optimization [16], listwise ranking objectives (e.g., ListNet [2], ListMLE [21], NDCG), neighborhood component analysis (NCA) [6], deep survival analysis, deep latent variable models, softmax functions, model-agnostic meta learning (MAML) [1]. We postpone the details of some of these problems to Sections 3 and 4. We would like to emphasize that efficient stochastic algorithms for solving these problems are lacking or underdeveloped when the involved set $S_i$ is big and/or the predictive model is nonconvex.

1.1 Related Work

The most straightforward approach to solve the FCCO problem in (1) is to compute the gradient $\nabla F(w) = \frac{1}{n} \sum_{z_i \in D} \nabla f_i(g(w; z_i, S_i)) \nabla g_i(w; z_i, S_i)$ and then use the gradient descent method. However, it can be seen that computing the gradient $\nabla F(w)$ is very expensive (if not infeasible) when $|D|$ or $|S_i|$ is large. Thus, a natural idea is to sample mini-batches $B_1 \subset D$ and $B_{2,i} \subset S_i$ and compute the stochastic gradient $v = \frac{1}{|B_1|} \sum_{z_i \in B_1} \nabla f_i(g(w; z_i, B_{2,i})) \nabla g_i(w; z_i, B_{2,i})$ and update the parameter as $w \leftarrow w - \eta v$, where $g(w; z_i, B_{2,i}) := \frac{1}{|B_{2,i}|} \sum_{\xi_{ij} \in B_{2,i}} g(w; z_i, \xi_{ij})$. The resulting algorithm is named as biased stochastic gradient descent (BSGD) in [9] and the convergence guarantees of BSGD under different assumptions are established. For the smooth nonconvex problem, they further improve the oracle complexity of BSGD by incorporating the recently proposed SpiderBoost [19] method for variance reduction. Actually, Hu et al. [9] study the more general problem $E_{\xi} f_\xi(E_{\xi}(g(w; \xi)))$, which is referred to as conditional stochastic optimization (CSO). The FCCO problem can be mapped to the CSO problem by $z_i = \xi, \zeta \in S_i$. The only difference is that FCCO specifies the finite-sum structure of the outer-level function. Unfortunately, both BSGD and its SpiderBoost variant require unrealistically large batch sizes to ensure the convergence from the theoretical perspective (Please refer to columns 5 and 6 of Table 1). As a comparison, our approach explicitly exploits the finite support of the outer level and can ensure the convergence even with mini-batch sizes $|B_1| = O(1)$, $|B_{2,i}| = O(1)$.

A closely related class of problems: stochastic compositional optimization (SCO) has been extensively studied in the literature. In particular, the SCO problem with the finite support in the outer level is in the form of $F(w) = \frac{1}{n} \sum_{i=0}^{n} f_i(g_i(w; S_i))$, where $S$ might be finite or not. The SCGD algorithm [18] is a seminal work in this field, which track the unknown $g_i(w; S_i)$ with an auxiliary variable $u_i$ that is updated by the exponential moving average $u_i \leftarrow (1 - \gamma)u_{i-1} + \gamma g_i(w; S_i)$, which circumvents the unrealistically large batch size required by the sample average approximation approach. For example, we can sample a mini-batch $B_2 \subset S$ and compute $g = g(w; B_2) := \frac{1}{|B_2|} \sum_{\xi_{ij} \in B_2} g(w; \xi_{ij})$. Then, the stochastic estimator of $\nabla F(w)$ can be computed as $v_t = \frac{1}{|B_2|} \sum_{z_i \in B_2} \nabla f_i(u_i) \nabla g(w; B_2)$. More recently, the NASA algorithm [5] modifies the SCGD algorithm by adding the exponential moving average (i.e., the momentum) to the gradient estimator, i.e., $v_t \leftarrow (1 - \beta) v_{t-1} + \beta \frac{1}{|B_2|} \sum_{z_i \in B_2} \nabla f_i(u_i) \nabla g(w; B_2)$, $\beta \in (0, 1)$, which improves upon the convergence rates of SCGD. When $f_i$ is convex and monotone ($d' = 1$) and $g$ is convex, Zhang and Lan [24] provide a more involved analysis for the two-batch SCGD in its

\[3\]In the original SCGD algorithm [18], they use the same batch $B_2$ to update $u_i$ by $g_i(w; B_2)$ and to compute the gradient estimator by $\nabla g(w; B_2)$. In the work of Zhang and Lan [24], they analyze the two-batch version SCGD which uses independent batches $B_2$ and $B_3'$ for $g_i(w; B_2)$ and $\nabla g(w; B_3')$. The two-batch version with independent $B_2, B_3'$ is definitely less efficient, but it considerably simplifies the analysis.
primal-dual form and derive the optimal rate for this kind of problems.

The difference between FCCO and SCO is that the inner function \( g(w; S) \) in SCO does not depend on \( z_i \) of the outer summation. If we define \( g(w) = [g_1(w)^T, \ldots, g_n(w)^T]^T \in \mathbb{R}^{nd} \) and \( \tilde{f}_i(\cdot) := f_i(I_i), \ I_i := [0_{d \times d}, \ldots, I_{d \times d}, \ldots, 0_{d \times d}] \in \mathbb{R}^{d \times nd} \) (the \( i \)-th block in \( I_i \) is an identity matrix while the others are zeros), the FCCO problem \( F(w) = \frac{1}{n} \sum_{i=1}^{n} f_i(g(w)) \) can be reformulated as an SCO problem \( F(w) = \frac{1}{n} \sum_{i=1}^{n} \tilde{f}_i(g(w)) \) such that the existing algorithms for the SCO problem can be applied to our FCCO problem. Unfortunately, naïve deployments of SCGD and NASA on the FCCO problem via its SCO reformulation need many oracles for the inner function \( g(w) \) (one oracle for each component \( g_i(w) \)) and update all \( n \) components of \( u \) at any time step \( t \) due to the reformulation above even if we only sample one data point \( z_{ti} \in D \) in the outer level, which could be expensive or even infeasible.

Thus, the SCGD and NASA algorithms for SCO can be tailored for the FCCO problem if it selectively samples \( B_{2,i} \) and selectively updates those coordinates \( u_i \) for those sampled \( z_i \in B_1 \) at each time step, instead of sampling \( B_{2,i} \) for all \( z_i \in D \) and update all \( n \) coordinates of \( u = [u_1^T, \ldots, u_n^T]^T \). Formally, the update rule of \( u = [u_1^T, \ldots, u_n^T]^T \) can be expressed as

\[
\begin{aligned}
    u_i &\leftarrow \begin{cases} 
    (1 - \gamma)u_i + \gamma g(w; z_i, B_{2,i}) & i \in B_1 \\
    u_i & i \notin B_1 
    \end{cases}, 
\end{aligned}
\]

which is exactly the SOAP algorithm [15] (together with \( v \leftarrow \frac{1}{|B_1|} \sum_{z_i \in B_1} \nabla f_i(u_i) \nabla g(w; z_i, B_{2,i}) \) and \( w \leftarrow w - \eta v \)). Although SOAP is proposed for the average precision (AP) maximization problem in the first place, it can be treated as the SCGD variant for the general FCCO problem. Wang et al. [17] attempted to do the same adaptation for NASA by an algorithm called MOAP-v1. Unfortunately, MOAP-v1 fails to improve upon the rate of SOAP like that SCGD improves upon NASA. As a compromise, they propose MOAP-v2 that applies the uniform random sparsification [20] (or the uniform randomized block coordinate sampling [14]) to the whole \( g(w) \) and derive the improved rate compared to SOAP. To be specific, the update rule of \( u \) in MOAP-v2 is

\[
\begin{aligned}
    u_i &\leftarrow \begin{cases} 
    (1 - \gamma)u_i + \gamma \frac{n}{|B_1|} g(w; z_i, B_{2,i}) & i \in B_1 \\
    (1 - \gamma)u_i & i \notin B_1 
    \end{cases}. 
\end{aligned}
\]

The update rule [3] has several drawbacks: a) It requires extra costs to update all \( u_i \) at each iteration, while [2] only needs to update \( u_i \) for the sampled \( z_i \in B_1 \); b) For the large-scale problems (i.e., \( n \) is large), multiplying \( g(w; z_i, B_{2,i}) \) by \( \frac{n}{|B_1|} \) might lead to numerical issue; c) Due to the property of random sparsification/block coordinate sampling (see Proposition 3.5 in Khirirat et al. [12]), it does not enjoy any benefit in terms of iteration complexity by increasing \( |B_{2,i}| \). It is worth mentioning that the convergence guarantees for SOAP and MOAP-v1 are only established for the smooth nonconvex problems.

From the algorithmic perspective, our SOX algorithm makes a small modification on the MOAP-v1 algorithm (Please refer to Section 2.1). However, this change enable us to establish a more involved analysis on the convex and nonconvex problems, which makes SOX have several theoretical advantages compared to existing algorithms (See Table 1 for a head-to-head comparison).

- On the convex and nonconvex problems, our SOX algorithm can guarantee the convergence but does not suffer from some limitations in previous methods such as the unrealistically large batch in BSGD [9], the
Table 1: Summary of iteration complexities of different methods for different problems. “NC” means non-convexity of $F$, “C” means convexity of $F$, “SC” means strongly convex and “PL” means PL condition for a non-convex function. “N/A” means not applicable or not available. For the complexities of our method, we omit other constants that are independent of $n$ or the batch sizes. * denotes additional conditions that $\Omega$ is bounded, $f$ is convex and monotone while $g_i$ is convex (but the smoothness of $g_i$ is not required). † assumes a stronger SC condition, which is imposed on a stochastic function instead of the original objective. We suppose $B_1 = |B'_1|$ and $B_2 = |B'_2|$ for simplicity.

| Method                  | NC     | C      | SC (PL) | Outer Batch Size $|B_1|$ | Inner Batch Size $|B_2|$ | Parallel Speed-up |
|-------------------------|--------|--------|---------|--------------------------|--------------------------|-------------------|
| BSGD [9]                | $O(\epsilon^{-4})$ | $O(\epsilon^{-2})$ | $O(\mu^{-1}\epsilon^{-1})$ † | 1                         | $O(\epsilon^{-2})$ (NC) | N/A               |
| BSpiderBoost [9]        | $O(\epsilon^{-2})$ | -      | -       | $n + 1$                  | $O(\epsilon^{-2})$ (C/SC) | N/A               |
| MOAP-v1 [17] (Theorem 2)| $O(n\epsilon^{-5})$ | -      | -       | 1                        | 1                        | N/A               |
| MOAP-v2 [17] (Theorem 3)| $O(n\epsilon^{-3})$ | -      | -       | $B_1$                    | $B_2$                    | Partial           |
| SOX/SOX-boost (this work) | $O(n\epsilon^{-4})$ | $O(n\epsilon^{-2})$ | $O(n^{-2}B_{1}B_{2})$B_{1} | $B_1$                  | $B_2$                  | Yes               |
| SOX ($\beta = 1$) (this work) | - | $O(n\epsilon^{-2})$ | $O(n^{-2}B_{1}B_{2})$B_{1} | $B_1$                  | $B_2$                  | Partial           |

two independent batches for oracles of the inner level in SCGD [24], and the possibly inefficient/unstable update rule in MOAP-v2.

- On the smooth nonconvex problem, SOX has an improved rate compared to SOAP and MOAP-v1 and enjoys a better dependence on $|B_{2,i}|$ compared to MOAP-v2.
- Beyond the smooth nonconvex problem, we also establish the convergence guarantees of SOX for problems that $F$ is convex/strongly convex/$\mu$-PL, which are better than BSGD in terms of oracle complexities.
- Moreover, we carefully analyze how mini-batching in the inner and outer levels improve the worst-case convergence guarantees of SOX in terms of iteration complexity, i.e., the parallel speed-up effect. The theoretical insights are numerically verified in our experiments.

2 Stochastic Algorithm and Convergence

Notations. For machine learning applications, we let $\mathcal{D} = \{z_1, \ldots, z_n\}$ denote a set of training examples for general purpose, let $w \in \Omega$ denote the model parameter (e.g., the weights of a deep neural network). Denote by $h_w(z)$ a prediction score of the model on the data $z$. A function $f$ is Lipchitz continuous on the domain $\Omega$ if there exists $C > 0$ such that $\|f(w) - f(w')\| \leq C \|w - w'\|$ for any $w, w' \in \Omega$, and is smooth if its gradient is Lipchitz continuous. A function $F$ is convex if it satisfies $F(w) \geq F(w') + \nabla F(w')^\top (w - w')$ for all $w, w' \in \Omega$, is $\mu$-strongly convex if there exists $\mu > 0$ such that $F(w) \geq F(w') + \nabla F(w')^\top (w - w') + \frac{\mu}{2} \|w - w'\|^2$ for all $w, w' \in \Omega$. A smooth function $F$ is said to satisfy $\mu$-PL condition if there exists $\mu > 0$ such that $\|\nabla F(w)\|^2 \geq \mu(F(w) - \min_w F(w))$, $w \in \Omega$.

We make the following assumptions throughout the paper:

Assumption 1. We assume that (i) $f_i$ is $L_f$-smooth and $C_f$-Lipchitz continuous; (ii) $g_i(w)$ is differentiable, $L_g$-smooth and $C_g$-Lipchitz continuous; (iii) $F$ is lower bouned by $F^*$.

4 “PL” refers to the Polyak-Lojasiewicz condition.
5 The result in Theorem [3] does not need $g_i$ to be smooth.
Algorithm 1 SOX($w_0, u_0, v_0, \eta, \beta, \gamma, T$)

1: for $t = 1, \ldots, T$ do
2: Draw a batch of samples $B^t_1 \subseteq D$
3: if $z_i \in B^t_1$ then
4: Update the estimator of function value $g_i(w^t)$
   \begin{equation*}
   u^t_i = (1 - \gamma)u^{t-1}_i + \gamma g(w^t; z_i, B^t_{2,i})
   \end{equation*}
5: end if
6: Update the estimator of gradient $\nabla F(w^t)$
   \begin{equation*}
   v^t = (1 - \beta)v^{t-1} + \beta \sum_{z_i \in B^t_1} \nabla f_i(u^{t-1}_i) \nabla g(w^t; z_i, B^t_{2,i})
   \end{equation*}
7: Update the model parameter $w^{t+1} = w^t - \eta_t v^t$
8: end for

Remark: If the assumption above is satisfied, it is easy to verify that $F(w)$ is $L_F$-smooth, where $L_F := C_f L_g + C_g^L L_f$ (see Lemma 4.2 in Zhang and Xiao [23]). The assumption that $f_i$ is smooth and Lipchitz continuous seems to be strong. However, the image of $g_i$ is bounded on domain $\Omega$ in many applications, hence $f_i$ is smooth and Lipchitz continuous in a bounded domain is enough for proving the theoretical results.

2.1 Algorithm Outline of SOX

For simplicity of algorithm and analysis, we assume $\Omega = \mathbb{R}^d$ unless it is explicitly specified. It is notable that the following convergence results of SOX can be extended to the case that $\Omega \neq \mathbb{R}^d$ by using the projection operator for the update of $w_t$ following [22]. The detailed steps of the proposed algorithm are presented in Algorithm 1, which is referred to as SOX meaning stochastic optimization of the X measures/objectives listed in Sections 3 and 4. It is worth noting that SOX only replaces the $u^t_i$ in MOAP-v1 [17] by $u^{t-1}_i$ (highlighted in cyan). As a result, $\nabla f_i(u^{t-1}_i) \nabla g(w^t; z_i, B^t_{2,i})$ does not depend on $B^t_1$, which simplifies the dependence relationship in the analysis.

Model parameter is updated in step 7 based on $v_t$, which is known as the momentum update if the step size $\eta$ is non-adaptive. We only consider the non-adaptive step size in our theoretical analysis. However, it can be easily extended to adaptive Adam-style step sizes by using the techniques in [7].

2.2 Convergence Analysis for the Smooth Nonconvex Problem

In this subsection, we present the convergence analysis. We will highlight the key differences from the previous analysis. We use the following assumption, which is also used in previous works [15, 17].

Assumption 2. We assume that $E[\|g_i(w^t; B^t_{2,i}) - g_i(w^t)\|^2] \leq \frac{\sigma^2}{B^t_2}$ and $E[\|\nabla g_i(w^t; B^t_{2,i}) - g_i(w^t)\|^2] \leq \frac{\zeta^2}{B^t_2}$, where $g_i(w; B_{2,i}) := \frac{1}{B^t_{2,i}} \sum_{\xi_{ij} \in S_i} g_i(w; \xi_{ij})$.

The algorithms aim to find the approximate stationary points.

Definition 1. Any $w \in \Omega$ is called an $\epsilon$-stationary point if $\|\nabla F(w)\| \leq \epsilon$.

Next we build a recursion for the gradient variance $\Delta^t := \|v^t - \nabla F(w^t)\|^2$ by proving the following lemma.
Lemma 1. If $\beta \leq \frac{2}{n}$, we have

$$
\mathbb{E} [\Delta^{t+1}] \leq (1 - \beta) \mathbb{E} [\Delta^t] + \frac{2L^2\eta^2}{\beta} \mathbb{E} \left[ \left\| \mathbf{v}^t \right\|^2 \right] + \frac{3L^2 \beta^2}{n} \mathbb{E} \left[ \sum_{z_i \in \mathcal{B}_1^t} \left\| u_i^{t+1} - u_i^t \right\|^2 \right] + \frac{2\beta^2 C^2 (c^2 + C^2)}{\min\{B_1, B_2\}} + 5\beta L^2 C^2 \mathbb{E} [\Xi_{t+1}],
$$

where $\Xi_t = \left\| u_i^t - g(w^t) \right\|^2$, $u_i^t = [u_i^1, \ldots, u_i^n]^\top$, $g(w^t) = [g_1(w^t), \ldots, g_n(w^t)]^\top$.

The cyan term above is generated by replacing $u_i^t$ by $u_i^{t-1}$ in our algorithm design. However, it can be canceled with the last term in Lemma 2. The next step is to build a recursion for the variance of inner function value estimation $\Xi_t := \left\| u_i^t - g(w^t) \right\|^2$ with the following lemma.

Lemma 2. If $\gamma \leq 1/5$, the following equation holds

$$
\mathbb{E} [\Xi_{t+1}] \leq \left( 1 - \frac{\gamma B_1}{4n} \right) \mathbb{E} [\Xi_t] + \frac{5\eta \gamma^2 C^2 \mathbb{E} \left[ \left\| v^t \right\|^2 \right]}{\gamma B_1} + \frac{2\gamma^2 \sigma^2 B_1}{n B_2} \frac{1}{4n} \mathbb{E} \left[ \sum_{z_i \in \mathcal{B}_1^t} \left\| u_i^{t+1} - u_i^t \right\|^2 \right].
$$

This is a key lemma that is different from [13]. Qi et al. [13] only bound $\sum_i \left\| u_i^t - g_i(w^t) \right\|^2$ for a sampled $z_i \in \mathcal{D}$ at each iteration. In contrast, Lemma 3 will be used to bound $\sum_i \left\| u_i^t - g^t(w^t) \right\|^2$ that includes all coordinates of $u_i^t$ at each iteration. In order to build the recursion above, we consider a strongly convex minimization problem $\min_u \frac{1}{2} \left\| u - g(w^t) \right\|^2$ that is equivalent to

$$
\min_u \frac{1}{2} \sum_{z_i \in \mathcal{D}} \left\| u_i - g_i(w^t) \right\|^2.
$$

Then, the step 4 in SOX can be viewed as stochastic block coordinate descent algorithm applied to [6], i.e.,

$$
u_i^t = \begin{cases} u_i^{t-1} - \gamma \left( u_i^{t-1} - g(w^t; z_i, B_{i,1}^t) \right) & z_i \in B_1^t \\ u_i^{t-1} & z_i \notin B_1^t \end{cases}
$$

where $u_i^{t-1} - g_i(w^t; B_{i,1}^t)$ is a stochastic gradient $\frac{1}{2} \left\| u_i - g_i(w^t) \right\|^2$ of the $i$-th coordinate in the objective [6]. This enables us to use the proof technique of stochastic block coordinate descent methods to build the recursion of $\Xi_t$ and derive the improved rates.

By combining the two lemmas above with Lemma 3 in the supplement, we prove the convergence for finding an $\epsilon$-stationary point, as stated in the following theorem.

**Theorem 1.** Under Assumptions 1 and 2 SOX (Algorithm 1) with $\beta = O(\min\{B_1, B_2\}^2), \gamma = O(\sigma^2)$, and $\eta = \min \left\{ \frac{\beta}{4L^2}, \frac{\gamma B_1}{4BLnC^2} \right\}$ can find an $\epsilon$-stationary point in $T = O \left( \max \left\{ \frac{n}{B_1B_2\sigma^2}, \frac{1}{\min\{B_1, B_2\}^2} \right\} \right)$ iterations.

**Remark:** The above theory suggests that given a budget on the mini-batch size $B_1 + B_2 = B$, the best value of $B_1$ is $B_1 = B/2$. We will verify this result in experiments.

### 2.3 Improved Rates for (Strongly) Convex Objectives

In this subsection, we prove improved rates for SOX for convex and strongly convex objectives. One might directly analyze SOX with different decreasing step sizes for convex and strongly convex objectives separately as in [9]. However, to our knowledge, this strategy does not yield an optimal rate for strongly convex functions. To address this challenge, we provide a unified algorithmic framework for both convex and strongly convex functions and derive the improved rates. The idea is to use the stagewise framework given
in Algorithm 2 to boost the convergence. Our strategy is to prove an improved rate for an objective that satisfies a $\mu$-PL condition, i.e., satisfying:

$$\|\nabla F(w)\|^2 \geq \mu (F(w) - F(w^*)),$$

where $w^*$ is a global minimum. Then, we use this result to derive the improved rates for strongly convex objectives and convex objectives.

**Theorem 2.** Assume $F$ satisfying the PL condition (8), by setting $\epsilon_k = O(1/2^{k-1})$, and $\beta_k = O(\eta_k), \gamma_k = O(\eta_k B_k), T_k = O(\frac{1}{\mu B_k}), \eta_k = O(\min(\mu, \min(B_1, B_2) \epsilon_k, \frac{1}{\min(B_1, B_2) \beta_k})), and K = \log(1/\epsilon)$, SOX-boost ensures that $E[F(w_K) - F(w^*)] \leq \epsilon$, which implies a total iteration complexity of $T = O(\max(\frac{n}{\mu^2 B_1 B_2 \epsilon}, \frac{1}{\min(B_1, B_2) \epsilon})).$

Specific values of the parameters in Theorem 2 can be found in Theorem 3 in the appendix. The result above directly implies the improved complexity for $\mu$-strongly convex function, as it automatically satisfies the PL condition. For a convex function, we use a common trick to make it strongly convex by contracting $\hat{F}(w) = F(w) + \frac{\delta}{2}\|w\|^2$, then we use SOX-boost to optimize $\hat{F}(w)$ with a small $\mu$. Its convergence is summarized by the following corollary.

**Corollary 1.** Assume $F$ is convex, by setting $\mu = O(\epsilon), \eta_k, \gamma_k, \beta_k, T_k$ according to Theorem 2 then after $K = \log(1/\epsilon)$-stages SOX-boost for optimizing $\hat{F}$ ensures that $E[F(w_K) - F(w^*)] \leq \epsilon$, which implies a total iteration complexity of $T = O(\max(\frac{n}{\mu^2 B_1 B_2 \epsilon}, \frac{1}{\min(B_1, B_2) \epsilon})))$ for ensuring $E[F(w_K) - F(w^*)] \leq \epsilon$.

**Remark:** The above result implies an complexity of $T = O(\max(\frac{n}{\mu^2 B_1 B_2 \epsilon}, \frac{1}{\min(B_1, B_2) \epsilon})))$ for optimizing a convex function.

**Further improvements for convex objectives.** Under additional conditions that $f_i$ is monotone and convex, and $g_i$ is convex, we can establish an improved rate for SOX in the order of $O(1/\epsilon^2)$, which is optimal in terms of $\epsilon$. The analysis is inspired by Zhang and Lan [24], which proves the improved complexities for the SCO with convex and strongly convex objectives. We extend their analysis to handle the FCCO problem the involves the selectively sample and update in the inner level. Note that our analysis gets rid of one drawback in Zhang and Lan [24] that needs two independent batches for estimating $g_i$ and $\nabla g_i$.

**Theorem 3.** Assume $d' = 1$, $\Omega$ is a convex and bounded set, $f_i$ is monotone, convex, smooth and Lipschitz-continuous while $g_i$ is convex and smooth. If $\eta = O(\min(\min\{B_1, B_2\} \epsilon, \frac{B_1 \epsilon}{\mu}), \gamma = O(\beta_2 \epsilon), \beta = 1$, and $w_T = \sum_t w_t/T$, SOX ensures that $E[F(w_T) - F(w^*)] \leq \epsilon$ after $T = O(\max(\frac{n}{\mu^2 B_1 B_2 \epsilon}, \frac{1}{\min(B_1, B_2) \epsilon})))$ iterations.

### 3 Experiments

In this section, we provide some experimental results to verify some aspects of our theory and compare SOX with other baselines for two applications, namely deep AP maximization and p-norm push optimization.

#### 3.1 Deep AP Maximization

AP maximization in the form of FCCO has been considered in [15][17]. For a binary classification problem, let $S_+, S_-$ denote the set of positive and negative examples, respectively, $S = S_+ \cup S_-$ denote the set of all
A smooth surrogate objective for maximizing AP can be formulated as:

\[
F(w) = \frac{1}{|S_+|} \sum_{x_i \in S_+} \frac{\sum_{x \in S} \ell(\ell(h_w(x) - h_w(x_i)))}{\sum_{x \in S} \ell(h_w(x) - h_w(x_i))},
\]

where \(\ell(\cdot)\) is a surrogate function that penalizes large input. It is a special case of FCCR by defining

\[
g_i(w) = [\sum_{x \in S} \ell(h_w(x) - h_w(x_j)), \sum_{x \in S} \ell(h_w(x) - h_w(x_i))] \quad \text{and} \quad f(g_i(w)) = -\frac{g_i(w)y_i}{g_i(w)\|i\|^2_2}.
\]

**Setting.** We conduct experiments on two image datasets, namely CIFAR-10, CIFAR-100. We use the dataloader provided in the released code of [15], which constructs the imbalanced versions of binarized CIFAR-10 and CIFAR-100. We consider two tasks: training ResNet18 on the CIFAR-10 data set and training ResNet34 on the CIFAR-100 data set. We follow the same procedure as in [17] that first pre-trains the network by optimizing a cross-entropy loss and then fine-tunes all layers with the randomly initialized classification layer. We also use the same squared hinge loss as in [15]. We aim to answer the following four questions related to our theory:

**Q1:** Given a batch size \(B\), what is the best value for \(B_1, B_2\), i.e., the sizes of \(B'_1\) and \(B'_2\)?

**Q2:** Is there parallel speed-up by increasing the total batch size \(B = B_1 + B_2\)?

**Q3:** What is the best value of \(\gamma\)?

**Q4:** Does SOX converge faster than SOAP (SGD-style) and MOAP-v2? For all of these questions, we focus on the comparison of convergence on the training data. All algorithms are run for 200 epochs. In all experiments, we tune the initial learning rate in a range \(10^{-4}:1:-1\) to achieve the best validation error, and decrease the learning rate at 50% and 75% of total epochs. The experiments are performed on a node of a cluster with single GeForce RTX 2080 TI GPU.

To answer Q1, we fix the total batch size \(B\) as 64 and vary \(B_1\) in the range \(\{4, 8, 16, 32\}\). For each value of \(B_1\), we tune the value of \(\gamma\) and fix \(\beta = 0.1\). The curves of training AP are shown in Figures 2(a) and (e) on the two datasets. We can see that when \(B_1 = 32 = B/2\) SOX has the fastest convergence in terms of number of iterations. This is consistent with our convergence theory.

To answer Q2, we fix \(B_1 = B_2\) and vary \(B\) in the range \(\{32, 64, 128, 256\}\). For each value of \(B\), we tune the value of \(\gamma\) and fix \(\beta = 0.1\). The curves of training AP are shown in Figures 2(b) and (f) on the two datasets. We can see that the iteration complexity of SOX decreases as \(B\) increases, which is also consistent with our convergence theory.

To answer Q3, we fix \(B_1 = B_2 = B/2 = 32\) and \(\beta = 0.1\), and run SOX with different values of \(\gamma\). We can see that \(\gamma = 1\) does not give the best result, which means the naive mini-batch estimation of \(g_i(w)\) is worse than the moving average estimator with a proper value of \(\gamma\). Moreover, we also observe that the best value of \(\gamma\) depends on the task. For example, \(\gamma = 0.1\) gives the fastest convergence on the task of training ResNet18.
with CIFAR-10 while $\gamma = 0.5$ is the best for training ResNet34 with CIFAR-100.

The Figures 2 (d) and (h) answer Q4, which indicates that SOX converges faster than MOAP-v2, which is faster than SOAP (SGD-style) and BSGD.

### 3.2 p-norm Push Optimization

For a binary classification problem, a $p$-norm push objective can be defined as \cite{16}:

$$
F(w) = \frac{1}{|S_+|} \sum_{z_i \in S_+} \left( \frac{1}{|S_-|} \sum_{z_j \in S_-} \ell(h_w(z_j) - h_w(z_i)) \right)^p
$$

where $p > 1$ and $\ell(\cdot)$ is similar as above. We can cast this function into FCCR by defining $D = S_+, S_i = S_-$, $g_i(w) = \frac{1}{|S_-|} \sum_{z_j \in S_-} \ell(h_w(z_j) - h_w(z_i))$ that couples each positive example $z_i$ with all negative samples, $f(g) = g^p$. We can also switch $S_+$ and $S_-$ to define another version of $p$-norm push objective as in the original paper, in which the author only provides a boosting-style algorithm that is not scalable to big data as it has to either process all $|S_+|$ positive and $|S_-|$ negative instances at each iteration.

| Dataset: german | BS-PnP | BSGD | SOX |
|----------------|-------|------|-----|
| Test Loss ($\downarrow$) | 0.849 | 0.660 ± 0.025 | **0.632 ± 0.031** |
| Test AUC ($\uparrow$) | 0.676 | 0.777 ± 0.008 | **0.791 ± 0.005** |
| Time (s) ($\downarrow$) | 1.118 | 0.089 ± 0.002 | 0.090 ± 0.003 |

| Dataset: ijcnn1 | BS-PnP | BSGD | SOX |
|----------------|-------|------|-----|
| Test Loss ($\downarrow$) | 0.908 | 0.671 ± 0.013 | **0.618 ± 0.011** |
| Test AUC ($\uparrow$) | 0.653 | 0.836 ± 0.004 | **0.840 ± 0.005** |
| Time (s) ($\downarrow$) | 1384 | 0.157 ± 0.041 | 0.177 ± 0.049 |

We compare with the boosting-style algorithm proposed by \cite{16} named BS-PnP, and the baseline BSGD \cite{9}. Following \cite{16}, we choose $\ell(\cdot)$ to be the exponential function. We conduct our experiment on two LibSVM datasets: german and ijcnn1. For the german dataset, we randomly choose 90% of the data for training and the rest of data is for testing. For the ijcnn1, we use the original train-test split. For this experiment, we learn a linear ranking function $h_w(x) = \langle w, x \rangle$ and $p = 4$. For each algorithm, we run it with 3 different random seeds and report the average test loss with standard deviation. Besides, we also report AUC on the testing data and the running time. For the stochastic algorithms SOX and BSGD, we choose $B = 64$ and $B_1 = B_2$. The algorithms are implemented with Python and run on a server with 12 cores and Intel(R) Xeon(R) CPU E5-2697 v2 @ 2.70GHz.

As shown in Table 2, the BS-PnP algorithm is indeed not scalable and takes much longer time than BSGD and SOX on the relatively larger ijcnn1 dataset ($n = 35000$). Besides, SOX consistently outperforms BS-PnP and BSGD in terms of both $p$-norm push loss and the AUC score on the test data. We also verify the parallel speed-up of SOX for learning a linear model. The results are shown in Figure 3 on the two datasets.
4 More Applications of SOX

In this section, we present more applications of the proposed algorithm in ML/AI, and highlight the potential of the proposed algorithm in addressing their computational challenges. Providing experimental results of these applications is beyond the scope of this paper. We consider the objective $F(w)$ to be minimized.

**Neighborhood Component Analysis (NCA).** NCA was proposed in [6] for learning a Mahalanobis distance measure. Given a set of data points $D = \{x_1, \ldots, x_n\}$, where each data point $x_i$ has a class label $y_i$. The objective of NCA is defined as

$$F(A) = -\sum_{i=1}^{n} \sum_{x_i \in C_i} \exp(-\|Ax_i - Ax_i\|^2) - \sum_{x_i \in S_i} \exp(-\|Ax_i - Ax_i\|^2),$$

where $C_i = \{x_j \in D : y_j = y_i\}$ and $S_i = D \setminus \{x_i\}$. We can map the above objective as a FCCR by defining $g_i(A) = \sum_{x_i \in C_i} \exp(-\|Ax_i - Ax_i\|^2) + \sum_{x_i \in S_i} \exp(-\|Ax_i - Ax_i\|^2)$ and $f(g_i(A)) = -|\{x \in A : i \in C_i\}| / \sqrt{\sum_{x \in A} (i \in C_i)}$. One might also consider learning a non-linear encoder network by replacing $Ax$ as an output of the encoder network $h_w(x) \in \mathbb{R}^d$.

**Listwise ranking objectives/measures.** In learning to rank (LTR), we are given a set of queries $Q = \{q_1, \ldots, q_n\}$. For each query, a set of items with relevance scores are provided $S_q = \{(x_1^q, y_1^q), \ldots, (x_n^q, y_n^q)\}$, where $x_i^q$ denotes the input data, and $y_i^q \in \mathbb{R}^+$ denotes its relevance score with $y_i^q = 0$ meaning irrelevant. For LTR, there are many listwise objectives and measures that can be formulated as FCCR, e.g., ListNet, ListMLE, NDCG. Due to limit of space, we only consider that of ListNet and ListMLE.

**ListNet & ListMLE.** The objective function of ListNet [2] can be defined by a cross-entropy loss between two probabilities of list of scores:

$$F(w) = -\sum_q \sum_{x_i^q \in S_q} P(y_i^q) \log \frac{\exp(h_w(x_i^q; q))}{\sum_{x \in S_q} \exp(h_w(x; q))} = \sum_q \sum_{x_i^q \in S_q} P(y_i^q) \log(\sum_{x \in S_q} \exp(h_w(x; q) - h_w(x_i^q; q))),$$

where $h_w(x_i^q; q)$ denotes the prediction score of the item $x_i^q$ with respect to the query $q$, $P(y_i^q)$ denotes a probability for a relevance score $y_i^q$ (e.g., $P(y_i^q = x_i^q)$). We can map the above function into FCCR, where $g(w; x_i^q, S_q) = \sum_{x \in S_q} \exp(h_w(x; q) - h_w(x_i^q; q))$ and $f(g) = \log(g)$, $D = \{(q, x_i^q) : P(y_i^q = x_i^q) > 0\}$. The original paper of ListNet uses a gradient method for optimizing the above objective, which has a complexity of $O(|\{q, x_i^q\}|)$, and is inefficient when $S_q$ contains a large number of items. ListMLE [21] has a similar form with its objective replaced by $\sum_q \sum_{x_i^q \in S_q} \log(\sum_{x \in S_q} \exp(h_w(x; q) - h_w(x_i^q; q)))$, where $S_q^q = \{x_i^q \in S_q : y_i^q \leq y_i^q\}$. We notice that in the previous works, $S_q$ is restricted to only a small set (e.g., by sampling 100~1000 items out of a large set). The proposed algorithm makes it possible to handle hundreds of thousands of items in $S_q$.

**Deep Survival Analysis (DSA).** The survival analysis in medicine is to explore and understand the relationships between patients’ covariates (e.g., clinical and genetic features) and the effectiveness of various [Figure 3: The parallel speed-up effect on the task of optimizing p-norm push with linear model.]
treatment options. Using a cox model for modeling the hazard function, the negative log-likelihood can be written as \[1\]:

\[
F(w) = \frac{1}{n} \sum_{i:E_i=1} \log \left( \sum_{j \in S(T_i)} \exp(h_w(x_j) - h_w(x_i)) \right),
\]

where \(x_i\) denote the input feature of a patient, \(h_w(x_i)\) denotes the risk value predicted by the network, \(E_i = 1\) denotes an observable event of interest (e.g., death), \(T_i\) denotes the time interval between the time in which the baseline data was collected and the time of the event occurring, and \(S(t) = \{i : T_i \geq t\}\) denotes the set of patients still at risk of failure at time \(t\). This is similar to the objective of ListMLE. The proposed algorithm is appropriate when both \(\{i : E_i = 1\}\) and \(S(T_i)\) are large.

**Deep Latent variable models (DLVM).** Latent variable models refer to a family of generative models that use latent variables to model the observed data, which can be used in both unsupervised learning and supervised learning. Below, we consider supervised learning. In particular, given a set of observed data \(D = \{(x_1, y_1), \ldots, (x_n, y_n)\}\), we model the probability of \(\text{Pr}(y|x)\) by introducing a discrete latent variable \(z\), i.e., \(\text{Pr}(y|x) = \sum_{z \in Z} \text{Pr}(y|x, z) \text{Pr}(z|x)\), where \(Z\) denotes the support set of the latent variable \(z\) and both \(\text{Pr}(y|x, z)\) and \(\text{Pr}(z|x)\) could be parameterized by a deep neural network. Then by minimizing negative loglikelihood of observed data, we have the following objective function:

\[
F(w) = -\sum_{(x_i, y_i) \in D} \log \sum_{z \in Z} \text{Pr}(y_i|x_i, z) \text{Pr}(z|x_i).
\]

When \(Z\) is a large set, then evaluating the inner sum is expensive. While the above problem is traditionally solved by EM-type algorithms, however, a stochastic algorithm based on back-propogation is used more often in modern deep learning. We consider an application in NLP for retrieve-and-predict language model pre-training \[8\]. In particular, \(x_i\) denotes an masked input sentence, \(y\) denotes masked tokens, \(z\) denotes a document from a large corpus \(Z\) (e.g., wikipedia). In \[8\], \(\text{Pr}(z|x_i) = \frac{\exp(E(x)^\top E(z))}{\sum_{z' \in Z} \exp(E(x)^\top E(z'))}\), where \(E(\cdot)\) is a document embedding network, and \(\text{Pr}(y|x, z)\) is computed based on a masked language model that a joint-embedding \(x, z\) is used to make the prediction. Hence, we can write \(F(w)\) as

\[
F(w) = -\sum_{i=1}^{n} \log \sum_{z \in Z} \text{Pr}(y_i|x_i, z) \exp(E(x_i)^\top E(z))
+ \sum_{i=1}^{n} \log \left( \sum_{z \in Z} \exp(E(x_i)^\top E(z')) \right).
\]

Note that both terms in the above is a special case of FCCR. The proposed algorithm gives an efficient way to solve this problem when \(Z\) is very large. Guo et al. \[8\] address the challenge by approximating the inner summation by summing over the top \(k\) documents with highest probability under \(\text{Pr}(z|x)\), which is retrieved by using maximum inner product search with a running time and storage space that scale sub-linearly with the number of documents. In contrast, SOX has a complexity independent of the number of documents per-iteration, which depends on the batch size.

**Softmax Functions.** One might notice that in the considered problems ListNet, ListMLE, NCA, DSA, DLVM, a common function that causes the difficulty in optimization is the softmax function in the form \(\frac{\exp(h(x_i))}{\sum_{x' \in X} \exp(h(x'))}\) for a target item \(x_i\) out of a large number items in \(X\). This also occurs in many other ML/AI methods, e.g., NLP pre-training methods that predicts masked tokens out of billions/trillions of tokens \[1\]. Taking the logarithmic of the softmax function gives the coupled compositional form \(\log \sum_{x' \in X} \exp(h(x) - h(x_i))\), and summing over all items gives the considered FCCR.

Finally, we would like to emphasize it might require additional and individual efforts to address the above problems by using SOX, which we will pursue in the future work.

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A Convergence Analysis for the Nonconvex Smooth Objectives

We first present a basic lemma below. Then, we provide the proof of Lemma 1 and Lemma 2. Finally, we prove Theorem 1.

Lemma 3 (Lemma 2 in Li et al. [13]). Consider a sequence \( w^{t+1} = w^t - \eta \nu^t \) and the \( L_F \)-smooth function \( F \) and the step size \( \eta L_F \leq 1/2 \).

\[
\begin{align*}
F(w^{t+1}) &\leq F(w^t) + \frac{\eta}{2} \Delta - \frac{\eta}{2} \| \nabla F(w^t) \|^2 - \frac{\eta}{4} \| \nu^t \|^2, \\
\end{align*}
\]

where \( \Delta := \| \nu^t - \nabla F(w^t) \|^2 \).

A.1 Proof of Lemma 1

We define that \( \Delta := \| \nu^t - \nabla F(w^t) \|^2 \) and \( G(w^{t+1}) = \frac{1}{B_1} \sum_{z_i \in B'_i} \nabla f(u^t_i) \nabla g_i(w^{t+1}; B^t_{2,i}) \). Based on the rule of update \( v^{t+1} = (1 - \beta) v^t + \beta G(w^{t+1}) \), we have

\[
\begin{align*}
\Delta^{t+1} &= \left\| (1 - \beta) v^t + \beta \frac{1}{B_1} \sum_{z_i \in B'_i} \nabla f(u^t_i) \nabla g_i(w^{t+1}; B^t_{2,i}) - \nabla F(w^{t+1}) \right\|^2 \\
&= \left\| (1 - \beta) v^t + \beta \frac{1}{B_1} \sum_{z_i \in B'_i} \nabla f(u^t_i) \nabla g_i(w^{t+1}; B^t_{2,i}) - \nabla F(w^{t+1}) \right\|^2 \\
&= \left\| (1) + (2) + (3) + (4) \right\|^2,
\end{align*}
\]

where (1), (2), (3), (4) are defined as

\[
\begin{align*}
1 &= (1 - \beta)(v^t - \nabla F(w^t)), \\
2 &= (1 - \beta)(\nabla F(w^t) - \nabla F(w^{t+1})), \\
3 &= \beta \frac{1}{B_1} \sum_{z_i \in B'_i} \left( \nabla f(u^t_i) \nabla g_i(w^{t+1}; B^t_{2,i}) - \nabla f(g_i(w^{t+1}; B^t_{2,i})) \right), \\
4 &= \beta \left( \frac{1}{B_1} \sum_{z_i \in B'_i} \nabla f(g_i(w^{t+1}; B^t_{2,i})) \nabla g_i(w^{t+1}; B^t_{2,i}) - \nabla F(w^{t+1}) \right).
\end{align*}
\]

Note that \( E_t \left[ \| (1), (4) \| \right] = E_t \left[ \| (2), (4) \| \right] = 0 \). Then, the Young’s inequality for products implies that

\[
\begin{align*}
E_t \left[ \| (1) + (2) + (3) + (4) \|^2 \right] &\leq (1 + \beta) \| (1) \|^2 + 2 \left( 1 + \frac{1}{\beta} \right) \| (2) \|^2 + \frac{2 + 3\beta}{\beta} E_t \left[ \| (3) \|^2 \right] + 2E_t \left[ \| (4) \|^2 \right].
\end{align*}
\]
Besides, we have
\[
(1 + \beta) \| \bar{v} \|^2 = (1 + \beta)(1 - \beta)^2 \| v' - \nabla F(w') \|^2 \leq (1 - \beta) \| v' - \nabla F(w') \|^2,
\]
\[
2 \left(1 + \frac{1}{\beta}\right) \| \bar{w} \|^2 = 2 \left(1 + \frac{1}{\beta}\right)(1 - \beta)^2 \| \nabla F(w') - \nabla F(w^{t+1}) \|^2 \leq \frac{2L_\eta^2}{\beta} \| v' \|^2,
\]
\[
\frac{2 + 3\beta}{\beta} \| \bar{q} \|^2 = \frac{2 + 3\beta}{\beta} \sum_{\bar{z}_i \in B'_1} \| \nabla g_i(w^{t+1}; B'_{2,i}) \|^2 \| \nabla f(u^t_i) - \nabla f(g_i(w^{t+1})) \|^2
\]
\[
\leq \frac{(2 + 3\beta)L_\eta^2}{B_1} \sum_{\bar{z}_i \in B'_1} \| \nabla g_i(w^{t+1}; B'_{2,i}) \|^2 \| u^t_i - g_i(w^{t+1}) \|^2.
\]

Consider that \( w^{t+1} \) and \( u_t \) do not depend on either \( B'_1 \) or \( B'_{2,i} \).
\[
(2 + 3\beta)\beta L_\eta^2 E_t \left[ \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \| \nabla g_i(w^{t+1}; B'_{2,i}) \|^2 \| u^t_i - g_i(w^{t+1}) \|^2 \right]
\]
\[
= (2 + 3\beta)\beta L_\eta^2 E_t \left[ \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} E_t \left[ \| \nabla g_i(w^{t+1}; B'_{2,i}) \|^2 | z_i \in B'_1 \right] \| u^t_i - g_i(w^{t+1}) \|^2 \right]
\]
\[
\leq (2 + 3\beta)\beta L_\eta^2 C_1^2 E_t \left[ \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \| u^t_i - g_i(w^{t+1}) \|^2 \right]
\]
\[
\leq \frac{(2 + 3\beta)\beta (1 + \delta) L_\eta^2 C_1^2}{n} \sum_{\bar{z}_i \in \mathcal{D}} E_t \left[ \| u^t_i + 1 - g_i(w^{t+1}) \|^2 \right] + \frac{(2 + 3\beta)\beta (1 + 1/\delta) L_\eta^2 C_1^2}{n} E_t \left[ \sum_{\bar{z}_i \in \mathcal{D}} \| u^t_i + 1 - u_i \|^2 \right],
\]
where \( C_1^2 := C_g^2 + c^2 / B \) and \( \delta > 0 \) is a constant to be determined later. Note that we have \( u^t_i + 1 = u_i^t \) for all \( i \notin B'_1 \).
\[
(2 + 3\beta)\beta L_\eta^2 E \left[ \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \| \nabla g_i(w^{t+1}; B'_{2,i}) \|^2 \| u^t_i - g_i(w^{t+1}) \|^2 \right]
\]
\[
\leq (2 + 3\beta)\beta (1 + \delta) L_\eta^2 C_1^2 E \left[ \frac{1}{n} \sum_{\bar{z}_i \in \mathcal{D}} \| u^t_i + 1 - g_i(w^{t+1}) \|^2 \right] + (2 + 3\beta)\beta (1 + 1/\delta) L_\eta^2 C_1^2 E \left[ \frac{1}{n} \sum_{\bar{z}_i \in \mathcal{D}} \| u^t_i + 1 - u_i \|^2 \right].
\]
If \( \beta \leq \frac{2}{\eta} \) and \( \delta = \frac{3\beta}{2} \), we have \( (2 + 3\beta)\beta (1 + \delta) \leq 5\beta \) and \( (2 + 3\beta)\beta (1 + 1/\delta) \leq 3. \)
\[
E \left[ \frac{2 + 3\beta}{\beta} \| \bar{q} \|^2 \right] \leq 5\beta L_\eta^2 C_1^2 E [\bar{X}_{t+1}] + \frac{3L_\eta^2 C_1^2}{n} E \left[ \sum_{\bar{z}_i \in B'_1} \| u^t_i + 1 - u_i \|^2 \right].
\]

Next, we upper bound the term \( E_t \left[ \| \bar{q} \|^2 \right] \).
\[
E_t \left[ \| \bar{q} \|^2 \right] = \beta^2 E_t \left[ \left| \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}; B'_{2,i}) - \frac{1}{n} \sum_{\bar{z}_i \in \mathcal{D}} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}) \right|^2 \right]
\]
\[
= \beta^2 E_t \left[ \left| \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}; B'_{2,i}) - \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}) \right|^2 \right]
\]
\[
+ \beta^2 E_t \left[ \left| \frac{1}{B_1} \sum_{\bar{z}_i \in B'_1} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}) - \frac{1}{n} \sum_{\bar{z}_i \in \mathcal{D}} \nabla f(g_i(w^{t+1})) \nabla g_i(w^{t+1}) \right|^2 \right] \leq \beta^2 C_1^2 (c^2 + C_g^2) \min \{B_1, B_2\}.
\]
A.2 Proof of Lemma 2

Based on Algorithm 1, the update rule of $u_i$ is

$$u_i^{t+1} = \begin{cases} (1 - \gamma)u_i^t + \gamma g_i(w_{t+1}; B_{2,i}^t) & \text{if } z_i \in B_1^t \\ u_i^t & \text{if } z_i \notin B_1^t \end{cases}.$$

We can re-write it into the equivalent expression below.

$$u_i^{t+1} = \begin{cases} u_i^t - \gamma \left( g_i(w_{t+1}; B_{2,i}^t) \right) & \text{if } z_i \in B_1^t \\ u_i^t & \text{if } z_i \notin B_1^t \end{cases}.$$

Let us define $\phi_t(u) = \frac{1}{2} \|u - g(w^t)\|^2 = \frac{1}{2} \sum_{z_i \in D} \|u_i - g_i(w^t)\|^2$, which is a 1-strongly convex function. Then, the update rule (7) can be viewed as one step of the stochastic block coordinate descent algorithm (Algorithm 2 in Dang and Lan [3]) for minimizing $\phi_{t+1}(u)$, where the Bregman divergence is associated with the quadratic function. We follow the analysis of Dang and Lan [3].

$$\phi_{t+1}(u^{t+1}) = \frac{1}{2} \|u^{t+1} - g(w^{t+1})\|^2$$

$$= \frac{1}{2} \|u^t - g(w^{t+1})\|^2 + \langle u^t - g(w^{t+1}), u^{t+1} - u^t \rangle + \frac{1}{2} \|u^{t+1} - u^t\|^2$$

$$= \frac{1}{2} \|u^t - g(w^{t+1})\|^2 + \sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), u_i^{t+1} - u_i^t \rangle + \frac{1}{2} \sum_{z_i \in B_1^t} \|u_i^{t+1} - u_i^t\|^2$$

$$+ \sum_{z_i \in B_1^t} \langle g_i(w_{t+1}; B_{2,i}^t), g_i(w_{t+1}), u_i^{t+1} - u_i^t \rangle.$$

Note that $u_i^t - g_i(w^{t+1}; B_{2,i}^t) = (u_i^t - u_i^{t+1})/\gamma$ and $2(b - a, a - c) \leq \|b - c\|^2 - \|a - b\|^2 - \|a - c\|^2$.

$$\sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), u_i^{t+1} - u_i^t \rangle$$

$$= \sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), g_i(w^{t+1}) - u_i^t \rangle + \sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), u_i^{t+1} - g_i(w^{t+1}) \rangle$$

$$= \sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), g_i(w^{t+1}) - u_i^t \rangle + \frac{1}{\gamma} \sum_{z_i \in B_1^t} \langle u_i^t - u_i^{t+1}, u_i^{t+1} - g_i(w^{t+1}) \rangle$$

$$\leq \sum_{z_i \in B_1^t} \langle u_i^t - g_i(w^{t+1}; B_{2,i}^t), g_i(w^{t+1}) - u_i^t \rangle$$

$$+ \frac{1}{2\gamma} \sum_{z_i \in B_1^t} \left( \|u_i^t - g_i(w_{t+1})\|^2 - \|u_i^{t+1} - u_i^t\|^2 - \|u_i^{t+1} - g_i(w^{t+1})\|^2 \right).$$

If $\gamma < \frac{1}{5}$, we have

$$- \frac{1}{2} \left( \frac{1}{\gamma} - 1 - \frac{1}{\gamma} + \frac{1}{4\gamma} \right) \sum_{z_i \in B_1^t} \|u_i^{t+1} - u_i^t\|^2 + \sum_{z_i \in B_1^t} \langle g_i(w^{t+1}; B_{2,i}^t) - g_i(w^{t+1}), u_i^{t+1} - u_i^t \rangle$$

$$\leq - \frac{1}{4\gamma} \sum_{z_i \in B_1^t} \|u_i^{t+1} - u_i^t\|^2 + \gamma \sum_{z_i \in B_1^t} \|g_i(w^{t+1}; B_{2,i}^t) - g_i(w^{t+1})\|^2 + \frac{1}{4\gamma} \sum_{z_i \in B_1^t} \|u_i^{t+1} - u_i^t\|^2$$

$$= \gamma \sum_{z_i \in B_1^t} \|g_i(w^{t+1}; B_{2,i}^t) - g_i(w^{t+1})\|^2.$$
Then, we have
\[
\begin{align*}
\frac{1}{2} \left\| u^{t+1} - g(w^{t+1}) \right\|^2 & \leq \frac{1}{2} \left\| u^t - g(w^{t+1}) \right\|^2 + \frac{1}{2\gamma} \sum_{z \in B_t} \left\| u^t - g_i(w^{t+1}) \right\|^2 - \frac{1}{2\gamma} \sum_{z \in B_t} \left\| u^{t+1} - g_i(w^{t+1}) \right\|^2 - \frac{(\gamma + 1)}{8\gamma} \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2 \\
& + \gamma \sum_{z \in B_t} \left\| g_i(w^{t+1}; B_{2,i}^t) - g_i(w^{t+1}) \right\|^2 + \sum_{z \in B_t} \left\langle u^t - g_i(w^{t+1}; B_{2,i}^t), g_i(w^{t+1}) - u^t \right\rangle.
\end{align*}
\]

Note that \( \frac{1}{2\gamma} \sum_{z \in B_t} \left\| u^t - g_i(w^{t+1}) \right\|^2 = \frac{1}{2\gamma} \sum_{z \in B_t} \left\| u^{t+1} - g_i(w^{t+1}) \right\|^2 \) based on Algorithm 1 which implies that
\[
\frac{1}{2\gamma} \sum_{z \in B_t} \left( \left\| u^t - g_i(w^{t+1}) \right\|^2 - \left\| u^{t+1} - g_i(w^{t+1}) \right\|^2 \right) = \frac{1}{2\gamma} \left( \left\| u^t - g(w^t) \right\|^2 - \left\| u^{t+1} - g(w^t) \right\|^2 \right).
\]

Besides, we also have \( \mathbb{E} \left[ \sum_{z \in B_t} \left\| g_i(w^{t+1}; B_{2,i}^t) - g_i(w^{t+1}) \right\|^2 \right] \leq \frac{B_i \sigma^2}{B_2^t} \) and
\[
\mathbb{E}_t \left[ \sum_{z \in B_t} \left\langle u^t - g_i(w^{t+1}; B_{2,i}^t), g_i(w^{t+1}) - u^t \right\rangle \right] = \frac{B_1}{n} \sum_{z \in D} \left\langle u^t - g_i(w^{t+1}), g_i(w^{t+1}) - u^t \right\rangle = -\frac{B_1}{n} \left\| u^t - g_i(w^{t+1}) \right\|^2.
\]

Then, we can obtain
\[
\frac{\gamma + 1}{2} \mathbb{E} \left[ \left\| u^{t+1} - g(w^{t+1}) \right\|^2 \right] \leq \frac{\gamma (1 - \frac{B_i}{n}) + 1}{\gamma + 1} \mathbb{E} \left[ \left\| u^t - g(w^t) \right\|^2 \right] + \frac{\gamma^2 B_1 \sigma^2}{B_2^t} - \frac{(\gamma + 1)}{8} \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2.
\]

Further divide \( \frac{\gamma + 1}{2} \) and take full expectation on both sides
\[
\mathbb{E} \left[ \left\| u^{t+1} - g(w^{t+1}) \right\|^2 \right] \leq \frac{\gamma (1 - \frac{B_i}{n}) + 1}{\gamma + 1} \mathbb{E} \left[ \left\| u^t - g(w^t) \right\|^2 \right] + \frac{2\gamma^2 \sigma^2 B_1}{B_2^t} - \frac{1}{4} \mathbb{E} \left[ \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2 \right].
\]

Note that \( \frac{(1 - \frac{B_i}{n}) + 1}{\gamma + 1} \leq 1 - \frac{2B_i}{\gamma n} \) and \( \frac{1}{\gamma + 1} \leq 1 \) for \( \gamma \in (0, 1] \). Besides, we have \( \left\| u^t - g(w^{t+1}) \right\|^2 \leq \left( 1 + \frac{\gamma B_1}{4\gamma} \right) \left\| u^t - g(w^t) \right\|^2 + \left( 1 + \frac{4\gamma}{\gamma B_2^t} \right) \left\| g(w^{t+1}) - g(w^t) \right\|^2 \) due to Young’s inequality and \( \left\| g(w^t) - g(w^{t-1}) \right\|^2 \leq nC_2^t \left\| w^{t+1} - w^t \right\|^2 = n\eta^2 C_2^t \left\| w^t \right\|^2.\]

\[
\mathbb{E} \left[ \left\| u^{t+1} - g(w^{t+1}) \right\|^2 \right] \leq \left( 1 - \frac{\gamma B_1}{2n} \right) \mathbb{E} \left[ \left\| u^t - g(w^{t+1}) \right\|^2 \right] + \frac{2\gamma^2 \sigma^2 B_1}{nB_2^t} - \frac{1}{4n} \mathbb{E} \left[ \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2 \right].
\]

\[
\mathbb{E} \left[ \left\| \Xi_{t+1} \right\|^2 \right] \leq \left( 1 - \frac{\gamma B_1}{2n} \right) \mathbb{E} \left[ \left\| u^t - g(w^{t+1}) \right\|^2 \right] + \frac{5nC_2^t}{\gamma B_1} \mathbb{E} \left[ \left\| w^{t+1} - w^t \right\|^2 \right] + \frac{2\gamma^2 \sigma^2 B_1}{nB_2^t} - \frac{1}{4n} \mathbb{E} \left[ \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2 \right].
\]

\[
\leq \left( 1 - \frac{\gamma B_1}{4\gamma} \right) \mathbb{E} \left[ \left\| u^t - g(w^t) \right\|^2 \right] + \frac{2\gamma^2 \sigma^2 B_1}{nB_2^t} - \frac{1}{4n} \mathbb{E} \left[ \sum_{z \in B_t} \left\| u^{t+1} - u^t \right\|^2 \right].
\]
A.3 Proof of Theorem 1

Based on (9), (10), and (11), we have

\[
\mathbb{E} \left[ F(w^{t+1}) - F^* \right] \leq \mathbb{E} \left[ F(w^t) - F^* \right] + \frac{\beta}{2} \mathbb{E} \left[ \Delta_t \right] - \frac{\gamma}{2} \mathbb{E} \left[ \left\| \nabla F(w^t) \right\|^2 \right] - \frac{\gamma}{4} \mathbb{E} \left[ \left\| v^t \right\|^2 \right]
\]

(10)

\[
\mathbb{E} \left[ \Delta_{t+1} \right] \leq (1 - \beta) \mathbb{E} \left[ \Delta_t \right] + \frac{2L_k^2 \eta^2}{\beta} \mathbb{E} \left[ \left\| v^t \right\|^2 \right] + 5\beta L_k^2 C^2 \mathbb{E} \left[ \Xi_{t+1} \right] + \frac{2\beta^2 C^2 \beta + C^2}{\min \{ B_1, B_2 \}} + \frac{3L_k^2 C^2}{n} \mathbb{E} \left[ \sum_{z_i \in B^1_t} \left\| u_i^{t+1} - u_i^t \right\|^2 \right]
\]

(11)

\[
\mathbb{E} \left[ \Xi_{t+1} \right] \leq \left( 1 - \gamma \frac{B_1}{4n} \right) \mathbb{E} \left[ \Xi_t \right] + \frac{5n_\eta^2 C^2_\sigma \mathbb{E} \left[ \left\| v^t \right\|^2 \right]}{\gamma B_1} + \frac{2\gamma \sigma^2 B_1}{n B_2} - \frac{1}{4n} \mathbb{E} \left[ \sum_{z_i \in B^1_t} \left\| u_i^{t+1} - u_i^t \right\|^2 \right]
\]

(12)

Summing (10), \( \frac{\beta}{4} \times (11) \), and \( \frac{20L_k^2 C^2 \eta \sqrt{n}}{\gamma B_1} \times (12) \) leads to

\[
\mathbb{E} \left[ F(w^{t+1}) - F^* + \frac{\eta}{\beta} \Delta_{t+1} + \frac{20L_k^2 C^2 \eta n}{\gamma B_1} \left( 1 - \frac{\gamma B_1}{4n} \right) \Xi_{t+1} \right]
\]

\[
\leq \mathbb{E} \left[ F(w^t) - F^* + \frac{\eta}{\beta} \left( 1 - \frac{\beta}{2} \right) \Delta_t + \frac{20L_k^2 C^2 \eta n}{\gamma B_1} \left( 1 - \frac{\gamma B_1}{4n} \right) \Xi_t \right] - \frac{L_k^2 C^2 \eta}{\gamma B_1} \left( 1 - \frac{\beta}{2} \right) \mathbb{E} \left[ \left\| v^t \right\|^2 \right]
\]

\[
- \frac{\beta}{4} \mathbb{E} \left[ \left\| \nabla F(w^t) \right\|^2 \right] - \eta \left( \frac{1}{4} - \frac{2L_k^2 \eta^2}{\beta^2} - \frac{100L_k^2 n C^2 \eta^2 C^2_{\sigma^2}}{\gamma^2 B_1^2} \right) \mathbb{E} \left[ \left\| v^t \right\|^2 \right] + \frac{2\beta^2 C^2 \beta + C^2}{\min \{ B_1, B_2 \}} + \frac{40\eta \beta C^2 \sigma^2}{B_2}.
\]

If \( \gamma \leq \frac{5n}{2B_1^2 \beta} \), we have \( \frac{5n}{\gamma B_1} - \frac{3}{\beta} \geq 0 \). Set \( \beta = \min \left\{ \frac{\eta B_1}{12C^2 \frac{C^2 + C^2_{\sigma^2}}{\gamma B_1^2}}, \frac{1}{2} \right\} \), and \( \eta = \min \left\{ \frac{\beta}{\beta B_1}, \frac{5n_\eta C^2_{\sigma^2}}{3C^2 \frac{C^2 + C^2_{\sigma^2}}{B_1^2}} \right\} \). Define the Lyapunov function as \( \Phi_t := F(w^t) - F^* + \frac{\eta}{\beta} \Delta_t + \frac{20L_k^2 C^2 \eta n}{\gamma B_1} \left( 1 - \frac{\gamma B_1}{4n} \right) \Xi_t \). If we initialize \( u^1 \) and \( v^1 \) as \( u_1^1 = g_1(w^1; B_{z_1}^1) \) for \( z_i \in B_{z_1}^1 \) and \( v^1 = 0 \), we have \( \mathbb{E} \left[ \Delta_1 \right] \leq C_1^2 C_2^2 \) and \( \mathbb{E} \left[ \Xi_1 \right] \leq \frac{\sigma^2}{B_2} \). Then,

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \left\| \nabla F(w^t) \right\|^2 \right] \leq \frac{2A^2}{\eta T} + \frac{4\beta^2 C^2 \frac{C^2 + C^2_{\sigma^2}}{\min \{ B_1, B_2 \}}} + \frac{80\gamma \beta C^2 \sigma^2}{B_2}, \tag{13}
\]

where we define \( A^2 := \Delta F + \frac{1}{\beta T} C^2 C^2_{\sigma^2} + \frac{2L_k C_{\sigma^2}}{\min \{ B_1, B_2 \}} \geq \mathbb{E} \left[ \Phi_1 \right] \). After

\[
T = \frac{6A^2}{\epsilon^2} \max \left\{ \frac{48C^2 \left( C^2 + C^2_{\sigma^2} \right) L_F}{\min \{ B_1, B_2 \} \epsilon^2}, 14L_F, \frac{7200nL_F C^2 C_{\sigma^2}}{B_1 B_2 \epsilon^2}, \frac{150L_n C_1 C_2}{B_1}, \frac{216L_F C_1 C_2 C_{\sigma^2}}{B_2}, \frac{216L_F C_1 C_2 C_{\sigma^2}}{B_2}, \frac{216L_F C_1 C_2 C_{\sigma^2}}{B_2} \right\}
\]

iterations, we have \( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \left\| \nabla F(w^t) \right\|^2 \right] \leq \epsilon^2 \).

B Convergence Analysis of SOX-boost for the (Strongly) Convex Objectives

Lemma 4. Under assumptions 7, 3, and the \( \mu \)-PL of \( F \), the \( k \)-th epoch of applying Algorithm 2 leads to

\[
\mathbb{E} \left[ \Gamma_{k+1} \right] \leq \frac{\mathbb{E} \left[ \Gamma_k \right]}{\mu n_k T_k} + \frac{\mathbb{E} \left[ \Delta_k \right]}{\mu \beta_k T_k} + \frac{20nL_k^2 C^2 \mathbb{E} \left[ \Xi_k \right]}{B_1 n \gamma_k T_k} + \frac{\beta_k C^2}{\mu \min \{ B_1, B_2 \}} + \frac{\gamma C^2_{\sigma^2}}{\mu B_2}, \tag{14}
\]

where \( \Gamma_k := F(w^k) - F(w^*) \), \( C^2_1 := C^2_{\sigma^2} + \frac{C^2}{B_2} \), \( C^2_2 := 2\beta C^2 \frac{C^2 + C^2_{\sigma^2}}{\gamma B_1^2} \), \( C^2_3 := 40L_k^2 C^2_{\sigma^2} \).
Proof. Since $F$ is $\mu$-PL, we have $F(w) - F(w^*) \leq \frac{1}{2\mu} \|\nabla F(w)\|^2$. We define $\Gamma_k := F(w^k) - F(w^*)$, $\Delta_k := \|v^k - \nabla F(w^k)\|^2$ and $\Xi_k := \frac{1}{n} \|u^k - g(w^k)\|^2$. Applying Theorem 1 to one epoch of SOX-boost leads to

$$
E[\Gamma_{k+1}] \leq \frac{1}{2\mu} E\left[\|\nabla F(w_{k+1})\|^2\right] \leq \frac{1}{2\mu T_k} \sum_{i=1}^{T_k} E\left[\|\nabla F(w^i)\|^2\right]
$$

$$
\leq \frac{E[\Gamma_k]}{\mu T_k} + \frac{E[\Delta_k]}{\mu T_k} + 20nL_f^2C_2^2E[\Xi_k] + \frac{\beta_k C_4^2}{B_1 \mu \gamma_k T_k} + \frac{\beta_k C_4}{2 \mu \min\{B_1, B_2\}} + \frac{\gamma_k C_4^2}{\mu B_2},
$$

where we define $C_2^2 := 2C_f^2(\zeta^2 + C_g^2)$, $C_4^2 := 40L_f^2C_f^2\sigma^2$.

Lemma 5. Under assumptions [1] the $k$-th epoch of Algorithm 2 leads to

$$
E[\Delta_{k+1} + C_5 \Xi_{k+1}] \leq \frac{6\Gamma_k + 10C_4 \Delta_k + 7C_4 C_5 \Xi_k}{\eta_k T_k} + \frac{10\eta_k C_4^2}{C_4 \min\{B_1, B_2\}} + \frac{80nC_4^2 \eta_k}{3B_1 B_2^2},
$$

where $\beta_k \leq \min\{\frac{3B_1}{\min\{B_1, B_2\}}, \frac{2}{\eta_k}\}$, $\gamma_k = \frac{10n}{3B_1} \beta_k$, $\eta_k = \beta_k C_4$, $C_4 := \min\{1/4L_f, 1/9L_f C_1 C_g\}$, $C_5 := 12L_f^3 C_1^2$.

Proof. Applying Lemma 3 to single iteration in any epoch of SOX-boost with $\beta_k \leq \frac{2}{\eta_k}$ leads to

$$
E[\Delta_{i+1}] \leq (1 - \beta_k)E[\Delta_i] + \frac{4L_f^2 \eta_k^2}{\beta_k} E[\Delta_i] + \frac{4L_f^2 \eta_k^2}{\beta_k} E\left[\|\nabla F(w^i)\|^2\right]
$$

$$
+ 5\beta_k L_f^2 C_f^2 E[\Xi_{i+1}] + \frac{2\beta_k^2 C_f^2(\zeta^2 + C_g^2)}{\min\{B_1, B_2\}} + \frac{3L_f^2 C_f^2}{n} E\left[\sum_{i \in B_1} \|u^{i+1}_i - u^i_i\|^2\right].
$$

Applying Lemma 2 to one iteration in any epoch of SOX-boost with $\gamma_k \leq \frac{1}{\eta_k}$ leads to

$$
E[\Xi_{i+1}] \leq \left(1 - \frac{\gamma_k B_1}{4n}\right) E[\Xi_i] + \frac{10m \eta_k^2 C_g^2}{\gamma_k B_1} E[\Delta_i] + \frac{10m \eta_k C_g^2}{\gamma_k B_1} E\left[\|\nabla F(w^i)\|^2\right]
$$

$$
+ \frac{2\gamma_k \sigma^2 D}{n B_2} - \frac{1}{4n} E\left[\sum_{i \in B_1} \|u^{i+1}_i - u^i_i\|^2\right].
$$

The following holds by summing up $E[\Delta_{k+1}]$ and $\frac{40L_f^3 C_f^2 n \beta_k}{\gamma_k B_1} \times E[\Xi_{k+1}]$ and noticing $(1 - \frac{\gamma_k B_1}{4n}) \leq (1 - \frac{2 \gamma_k B_1}{8n})^2$.

$$
E\left[\Delta_{k+1} + \frac{40L_f^3 C_f^2 n \beta_k}{\gamma_k B_1} \Xi_{k+1}\right] \leq \left(1 - \beta_k + \frac{4L_f^2 \eta_k^2}{\beta_k} + \frac{400n^2 \eta_k^2 \beta_k C_4^2 L_f^2 C_1^2}{\gamma_k^2 B_1^2}\right) \Delta_i + \frac{4L_f^2 C_f^2 n \beta_k}{\gamma_k B_1} \left(1 - \frac{\gamma_k B_1}{8n}\right) \Xi_i
$$

$$
+ \left(\frac{4L_f^2 \eta_k^2}{\beta_k} + \frac{400n^2 \eta_k^2 \beta_k C_2^2}{\gamma_k^2 B_1^2}\right) E\left[\|\nabla F(w^i)\|^2\right] + \frac{2\beta_k^2 C_f^2(\zeta^2 + C_g^2)}{\min\{B_1, B_2\}} + \frac{80 \gamma_k \beta_k L_f^3 C_f^2 \sigma^2}{B_2}
$$

$$
- \frac{L_f^2 C_1^2}{\gamma_k B_1^2} \left(10n \beta_k \right) - \frac{3}{n} \sum_{i \in B_1} \|u^{i+1}_i - u^i_i\|^2
$$

If we set $\gamma_k \leq \frac{10n}{3B_1} \beta_k$, $\eta_k \leq \frac{\beta_k}{4L_f}$ and $\eta_k \leq \frac{\gamma_k B_1}{30L_f C_1 C_g}$,

$$
1 - \beta_k + \frac{4L_f^2 \eta_k^2}{\beta_k} + \frac{400n^2 \eta_k^2 \beta_k C_2^2 L_f^2 C_1^2}{\gamma_k^2 B_1^2} \leq 1 - \beta_k + \frac{\eta_k}{4} \leq 1 - \frac{\beta_k}{4}.
$$

If $\beta_k \leq \min\{\frac{3B_1}{\gamma_k B_1}, \frac{2}{\eta_k}\}$, we can set $\gamma_k = \frac{10n}{3B_1} \beta_k$ and $\eta_k = \beta_k C_4$, where $C_4 := \min\{1/4L_f, 1/9L_f C_1 C_g\}$. Then, $1 - \frac{\gamma_k B_1}{8n} = 1 - \frac{5 \beta_k}{12} \leq 1 - \frac{\beta_k}{4}$. Besides, we define $C_5 := \frac{40L_f^3 C_f^2 n \beta_k}{\gamma_k B_1} = 12L_f^3 C_1^2$.

$$
E[\Delta_{k+1} + C_5 \Xi_{k+1}] \leq \left(1 - \frac{\beta_k}{4}\right) E[\Delta_i + C_5 \Xi_i] + \frac{3 \beta_k}{4} E\left[\|\nabla F(w^i)\|^2\right] + \frac{2\beta_k^2 C_f^2(\zeta^2 + C_g^2)}{\min\{B_1, B_2\}} + \frac{2 \gamma_k \beta_k C_4^2}{B_2}.
$$
Telescoping the equation above from 1 to $T_k$ iterations in epoch $k$ leads to

$$E[\Delta_{k+1} + C_5 \Xi_{k+1}] = E \left[ \frac{1}{T_k} \sum_{t=0}^{T_k} \Delta_{t+1} + C_5 \frac{1}{T_k} \sum_{t=0}^{T_k} \Xi_{t+1} \right]$$

$$\leq \frac{4C_4 E[\Delta_k + C_5 \Xi_k]}{\eta_k T_k} + 3 \frac{1}{T_k} \sum_{t=1}^{T_k} E \left[ \| \nabla F(w^t) \|^2 \right] + \frac{4\beta_k C_2^2}{\min\{B_1, B_2\}} + \frac{20nC_3^2 \eta_k}{3B_1 B_2 C_4}.$$  

Applying (13), we can further derive that

$$E[\Delta_{k+1} + C_5 \Xi_{k+1}] \leq \frac{E[6\Gamma_k + 10C_4 \Delta_k + 7C_4 C_5 \Xi_k]}{\eta_k T_k} + \frac{10n\eta_k C_2^2}{C_4 \min\{B_1, B_2\}} + \frac{80nC_3^2 \eta_k}{3B_1 B_2 C_4}.$$

**Lemma 6.** If we set $\eta_k = \min \left\{ \frac{\mu C_4 \min\{B_1, B_2\} \epsilon_k}{6C_2^2}, \frac{\min\{B_1, B_2\} \epsilon_k}{60C_2^2}, \frac{\mu C_4 B_1 B_2 \epsilon_k}{20nC_3^2}, \frac{B_1 B_2 C_4 \epsilon_k}{200nC_3^2}, \frac{3B_1 C_4}{50n}, \frac{2C_4}{7} \right\}$ and $T_k = \max \left\{ \frac{12 \eta_k}{C_3}, \frac{10n\eta_k C_2^2}{\min\{B_1, B_2\}} \right\}$, we can conclude that $E[\Gamma_{k+1}] \leq \epsilon_k$ and $E[\Delta_k + C_5 \Xi_k] \leq \frac{\epsilon_k}{C_4}$, where $\epsilon_k = \max \left\{ \Delta_F, C_4 \frac{C_2^2 C_5^2 + C_5 \sigma^2}{B_2} \right\}$ and $\epsilon_1 = \epsilon_1/2k^{-1}$ for $k \geq 1$.

**Proof.** We prove this lemma by induction. First, we define $\Delta_F := F(w^1) - F(w^*)$, $\Gamma_k := F(w^k) - F(w^*)$, and $\epsilon_k = \max \left\{ \Delta_F, C_4 \frac{C_2^2 C_5^2 + C_5 \sigma^2}{B_2} \right\}$, where $C_4 := \min \{1/4L, 1/9L_C C_1, C_5 := 12L_2^2 C_2^2\}$.

If we initialize $u_1^i$ and $v_1^i$ as $u_1^i = g_i(w^1; B_1^1)$ for $i \in D_1$ and $v_1^i = 0$, we have $E[\Gamma_1] \leq \epsilon_1$ and $E[\Delta_1 + C_5 \Xi_1] \leq \frac{\epsilon_1}{C_4}$. Next, we consider the $k \geq 2$ case. Assume $E[\Gamma_{k+1}] \leq \epsilon_k$ and $E[\Delta_k + C_5 \Xi_k] \leq \frac{\epsilon_k}{C_4}$. We define $\epsilon_k = \epsilon_1/2k^{-1}$ for $k \geq 2$. We choose $\beta_k \leq \min \left\{ \frac{3B_1}{4}, \frac{2C_4}{7} \right\}$, $\gamma_k = \frac{10n\beta_k}{\min\{B_1, B_2\}}$, $\eta_k = \beta_k C_4$. Based on Lemma 3, we have

$$E[\Gamma_{k+1}] \leq \frac{E[\Gamma_k + C_4 (\Delta_k + C_5 \Xi_k)]}{\eta_k T_k} + \frac{\eta_k C_2^2}{\mu C_4 \min\{B_1, B_2\}} + \frac{10n\eta_k C_2^2}{3B_1 B_2 C_4}.$$  

Besides, Lemma 3 implies that

$$E[\Delta_{k+1} + C_5 \Xi_{k+1}] \leq \frac{E[6\Gamma_k + 10C_4 (\Delta_k + C_5 \Xi_k)]}{\eta_k T_k} + \frac{10n\eta_k C_2^2}{C_4 \min\{B_1, B_2\}} + \frac{80n\eta_k C_3^2}{3B_1 B_2 C_4}.$$  

The following choices of $\eta_k$ and $T_k$ makes $E[\Gamma_{k+1}] \leq \epsilon_{k+1} = \frac{\epsilon_k}{2k}$ and $E[\Delta_{k+1} + C_5 \Xi_{k+1}] \leq \frac{\epsilon_{k+1}}{C_4} = \frac{\epsilon_k}{2C_4}$. We define $C_k := \min\{C_4, \frac{1}{200n} \}$.

$$\eta_k = \min \left\{ \frac{\mu C_4 \min\{B_1, B_2\} \epsilon_k}{6C_2^2}, \frac{\min\{B_1, B_2\} \epsilon_k}{60C_2^2}, \frac{\mu C_4 B_1 B_2 \epsilon_k}{20nC_3^2}, \frac{B_1 B_2 C_4 \epsilon_k}{200nC_3^2}, \frac{3B_1 C_4}{50n}, \frac{2C_4}{7} \right\}, \quad T = \frac{\max \left\{ \frac{12}{n}, \frac{96C_4}{\eta_k} \right\}}{\eta_k}. $$

**Theorem 4** (Detailed Version of Theorem 2). Under assumptions $\|\|$ and the $\mu$-PL of $F$, SOX-boost (Algorithm 2) can find an $w$ satisfying that $E[f(w) - f(w^*)] \leq 2\epsilon$ after

$$T = C_1/\mu \max \left\{ \frac{6C_2^2}{\mu C_4 \min\{B_1, B_2\} \epsilon}, \frac{60C_2^2}{\min\{B_1, B_2\} \epsilon}, \frac{20nC_3^2}{\mu C_4 B_1 B_2 \epsilon}, \frac{200nC_3^2}{B_1 B_2 C_4 \epsilon}, \frac{50n \log(\epsilon_1/\epsilon)}{3B_1 C_4}, \frac{7 \log(\epsilon_1/\epsilon)}{2C_4} \right\}$$

iterations, where $C_1/\mu = \max \left\{ \frac{12}{\mu}, 96C_4 \right\}$. 
Proof. According to Lemma 6, the total number of iterations to achieve target accuracy $\mathbb{E}[\Gamma_k] \leq \epsilon$ can be represented as:

$$T = \sum_{k=1}^{\log(\epsilon_1/\epsilon)} T_k$$

$$= C_{1/\mu} \max \left\{ \frac{6C_2^2}{\mu C_4 \min\{B_1, B_2\} \epsilon}, \frac{60C_2^2}{\mu C_4 B_1 B_2 \epsilon}, \frac{80nC_2^2}{B_1 B_2 C_4 \epsilon}, \frac{200nC_2^2}{3B_1 C_4}, \frac{50n \log(\epsilon_1/\epsilon)}{2C_4} \right\},$$

where $C_{1/\mu} = \max \left\{ \frac{12}{\mu}, 96C_4 \right\}$. \hfill \Box

Proof of Corollary 7. Suppose that $w^*$ is a minimum of $F$ and $\hat{w}^*$ is the minimum of the strongly convexified $F$. If $\mathbb{E}\left[\hat{F}(w) - \hat{F}(w^*)\right] \leq \epsilon$, we have

$$\mathbb{E}[F(w)] \leq \mathbb{E}\left[\hat{F}(w)\right] \leq \hat{F}(w^*) + \epsilon \leq F(w^*) + \epsilon = F(w^*) + \lambda \frac{1}{2} \|w^*\|^2 + \epsilon.$$

Thus, if the minimum $w^*$ of $F$ is in a bounded domain $\|w^*\| \leq C_\ast$ and we choose $\lambda = \frac{2\mu}{C_\ast}$, we also have $\mathbb{E}[F(w) - F(w^*)] \leq 2\epsilon$. \hfill \Box

C Proof of Theorem 3

We analyze the SOX algorithm shown in Algorithm 3 that is tailored for the convex FCCO problem on the domain $\Omega$.

$$\min_{w \in \Omega} F(w), \quad F(w) = \frac{1}{n} \sum_{i \in D} f_i(w).$$

Note that we specifically set $\gamma = \frac{1}{1+\tau}$ with a $\tau > 0$. 

Algorithm 3 SOX($w_0, u_0, \eta, \tau, T$)

1: for $t = 0, \ldots, T-1$ do
2: Randomly sample a batch $B_t^i \subset D$
3: if $z_i \in B_t^i$ then
4: Update the estimator of function value $g_i(w^{t+1})$ by $u_i^{t+1} = \frac{\tau}{1+\tau} u_i^t + \frac{1}{1+\tau} g_i(w^t; B_t^i)$
5: end if
6: Update the model parameter $w^{t+1} = \text{Proj}_\Omega \left[ w^t - \eta \frac{1}{B_t^i} \sum_{z_i \in B_t^i} \nabla f_i(u_i^t) \nabla g_i(w^t; B_t^i) \right]$
7: end for

We can reformulate the convex FCCO problem as the saddle point problem below:

$$\min_{w \in \Omega} \max_{\pi_1 \in \Pi_1, \pi_2 \in \Pi_2} \mathcal{L}(w, \pi_1, \pi_2), \quad \mathcal{L}(w, \pi_1, \pi_2) = \frac{1}{n} \sum_{z_i \in D} \mathcal{L}_i(w, \pi_{i,1}, \pi_{i,2}),$$

where $\pi_1 = [\pi_{1,1}, \ldots, \pi_{n,1}]^\top$, $\pi_2 = [\pi_{1,2}, \ldots, \pi_{n,2}]^\top$, $\mathcal{L}_i(w, \pi_{i,1}, \pi_{i,2}) = \pi_{i,1} \mathcal{L}_{i,1}(w, \pi_{i,1}, \pi_{i,2}) - f^\ast(\pi_{i,1})$ and $\mathcal{L}_{i,2}(w, \pi_{i,2}) = \langle \pi_{i,2}, w \rangle - g_i^\ast(\pi_{i,2})$. Here $f^\ast(\cdot)$ and $g_i^\ast(\cdot)$ are the convex conjugates of $f$ and $g_i$, respectively. Then, the SOX algorithm can be viewed as the Stochastic Sequential Dual (SSD) algorithm (Algorithm in Zhang and Lan [21]) for the FCCO problem. The analysis of SSD for solving the traditional compositional optimization problem has been established in Zhang and Lan [21]. However, their analysis is not directly applicable to our FCCO problem since the updates of the stochastic estimators $u_1^t, \ldots, u_n^t$ for the inner function values $g_1(w^t), \ldots, g_n(w^t)$ depend on the sampled batch $B_t^i$ in the outer level. In this paper, we fill in this gap.

For any iteration $t$, we can define $z^t = (w^t, \pi_{1,t}^t, \pi_{2,t}^t)$ and $\pi_{1,t}^t = [\pi_{1,1}^{t+1}, \ldots, \pi_{n,1}^{t+1}]^\top$, $\pi_{2,t}^t = [\pi_{1,2}^{t+1}, \ldots, \pi_{n,2}^{t+1}]^\top$, where $\pi_{1,1}^{t+1} = \nabla f_i(u_i^{t+1})$ and $\pi_{i,2}^{t+1} = \nabla g_i(w_i)$. We define that $z^* := (w^*, \pi_{1}^*, \pi_{2}^*)$ where $\pi_{1}^*$, $\pi_{2}^*$ are defined.
as \( \bar{\pi}_{i, 1} = \arg \max_{\pi_i} \mathcal{L}_{i, 2} (\tilde{w}^T, \bar{\pi}_{i, 2}) - f^* (\pi_i, 1) \) and \( \bar{\pi}_{i, 2} = \arg \max_{\pi_i} \langle \pi_i, 2, \bar{w}^T \rangle - g_i^* (\pi_i, 2) \), we can define the gap \( Q(z', z^*) \) as

\[
Q(z', z^*) = \mathcal{L}(w', \bar{\pi}_{i, 1}^+, \bar{\pi}_{i, 2}^+) - \mathcal{L}(w^*, \pi_{i, 1}^{t+1}, \pi_{i, 2}^{t+1}) = \frac{1}{n} \sum_{z_i \in \Omega} \left( \frac{\mathcal{L}_{i, 1}(w', \bar{\pi}_{i, 1}^+, \bar{\pi}_{i, 2}^+) - \mathcal{L}_{i, 1}(w^*, \pi_{i, 1}^{t+1}, \pi_{i, 2}^{t+1})}{\bar{Q}(z', z^*)} \right).
\]

We can decompose \( Q_i(z', z^*) \) into three terms

\[
Q_{i, 1}(z', z^*) = \mathcal{L}_{i, 1}(w', \bar{\pi}_{i, 1}^+, \bar{\pi}_{i, 2}^+) - \mathcal{L}_{i, 1}(w^*, \pi_{i, 1}^{t+1}, \pi_{i, 2}^{t+1}) = \bar{\pi}_{i, 1}^+ \left( (\bar{\pi}_{i, 2}^+, w^t) - g_i^* (\pi_{i, 2}^t) - (\pi_{i, 2}^{t+1}, w^t) + g_i^* (\pi_{i, 2}^{t+1}) \right),
\]

\[
Q_{i, 2}(z', z^*) = \mathcal{L}_{i, 1}(w^*, \bar{\pi}_{i, 1}^+, \pi_{i, 2}^{t+1}) - \mathcal{L}_{i, 2}(w^*, \pi_{i, 1}^{t+1}, \pi_{i, 2}^{t+1}) = \bar{\pi}_{i, 1}^+ \mathcal{L}_{i, 2}(w^t, \pi_{i, 2}^{t+1}) - f^* (\pi_{i, 2}^t) - \pi_{i, 1}^{t+1} \mathcal{L}_{i, 2}(w^t, \pi_{i, 2}^{t+1}) + f^* (\pi_{i, 2}^{t+1}),
\]

\[
Q_{i, 0}(z', z^*) = \mathcal{L}_{i, 1}(w^*, \pi_{i, 1}^{t+1}, \pi_{i, 2}^{t+1}) - \mathcal{L}_{i, 1}(w^*; \pi_{i, 1}^+, \pi_{i, 2}^{t+1}) = \pi_{i, 1}^{t+1} \langle \pi_{i, 2}^{t+1}, w^t - w^* \rangle.
\]

We make the following additional assumption for our analysis.

**Assumption 3.** The domain \( \Omega \) is bounded such that \( \max_{w \in \Omega} \| w - w^* \| \leq C_\Omega \) and \( \max_{w \in \Omega} \| g_i (w; \xi) \| \leq D_\eta \) for any \( i \) and random variable \( \xi \).

**Lemma 7.** For any \( \pi_{i, 1} \) such that \( \pi_{i, 1} = \nabla f(u) \) for some bounded \( u_i \), then there exists \( C_{f_*} \) such that \( |f^*(\pi_{i, 1})| \leq C_{f_*} \).

**Proof.** Due to the definition of convex conjugate, we have \( f(u_i) + f^*(\pi_{i, 1}) = \pi_{i, 1}^+ u_i \). Due to that \( \pi_{i, 1} \) and \( u_i \) are bounded and \( f(u_i) \) is bounded due to its Lipschitz continuity. As a result, \( f^*(\pi_{i, 1}) \) is bounded by some constant \( C_{f_*} \).

**Lemma 8.** For Algorithm 3, we have \( Q_{i, 2}(z_i, z^*) \leq 0 \) for any \( z_i \in \mathcal{D} \) and any \( t = 0, \ldots, T - 1 \).

**Proof.** Since \( f \) is Lipschitz-continuous, convex and monotonically increasing, we have that \( \pi_{i, 1}^t \leq C_f \). Besides, \( \langle \bar{\pi}_{i, 2}^t, w_t \rangle - g_i^* (\bar{\pi}_{i, 2}^t) \leq \langle \pi_{i, 2}^{t+1}, w_t \rangle - g_i^* (\pi_{i, 2}^{t+1}) \) due to \( \pi_{i, 2}^{t+1} = \arg \max_{\pi_{i, 2}} \langle \pi_{i, 2}, w_t \rangle - g_i^* (\pi_{i, 2}) \). We can conclude that \( Q_{i, 2}(z_i, z^*) \leq 0 \).

**Lemma 9.** For any \( \pi_1 \in \Pi \) and the sequences \( \{ \bar{\pi}_1^t \} \) and \( \bar{\pi}_1^t \) as defined in \( \bar{\pi}_1^t = \arg \min_{\pi_1} \langle \tilde{g}(w; B_2^t) - g(w^t, \pi_1) \rangle + \tau' D_{f^*}(\bar{\pi}_1^t, \pi_1)^t \bar{\pi}_1^t + (\tilde{g}(w^t; B_2^t) - g(w^t, \pi_1)) \), we have

\[
\langle \tilde{g}(w^t; B_2^t) - g(w^t), \pi_1 - \bar{\pi}_1^t \rangle \geq \tau^t (D_{f^*}(\bar{\pi}_1^{t+1}, \pi_1) - D_{f^*}(\bar{\pi}_1^t, \pi_1)) + \frac{1}{2 \tau^t} \| \tilde{g}(w^t; B_2^t) - g(w^t; B_2^t) \|^2.
\]

**Proof.** The proof of this lemma is almost the same to that of Lemma 4 in Juditsky et al. [10]. Due to the three-point inequality (Lemma 1 in Zhang and Lan [24]), we have:

\[
D_{f^*}(\bar{\pi}_1^{t+1}, \pi_1) \geq D_{f^*}(\bar{\pi}_1^t, \pi_1) - \frac{1}{\tau^t} \langle \tilde{g}(w^t; B_2^t) - g(w^t), \bar{\pi}_1^{t+1} - \bar{\pi}_1^t \rangle + D_{f^*}(\bar{\pi}_1^t, \bar{\pi}_1^t),
\]

\[
D_{f^*}(\bar{\pi}_1^{t+1}, \pi_1) \leq D_{f^*}(\bar{\pi}_1^t, \pi_1) + \frac{1}{\tau^t} \langle \tilde{g}(w^t; B_2^t) - g(w^t), \pi_1 - \bar{\pi}_1^{t+1} \rangle - D_{f^*}(\bar{\pi}_1^t, \bar{\pi}_1^t),
\]

where \( \pi_1 \) could be any \( \pi_1 \in \Pi_1 \). The last term on the R.H.S. of [16] can be upper bounded by [15].

\[
D_{f^*}(\bar{\pi}_1^{t+1}, \pi_1) \leq D_{f^*}(\bar{\pi}_1^1, \pi_1) + \frac{1}{\tau^t} \langle \tilde{g}(w^t; B_2^t) - g(w^t), \pi_1 - \bar{\pi}_1^t \rangle + \frac{1}{\tau^t} \langle \tilde{g}(w^t; B_2^t) - \tilde{g}(w^t; B_2^t), \pi_1 - \bar{\pi}_1^{t+1} \rangle - D_{f^*}(\bar{\pi}_1^t, \bar{\pi}_1^t) - D_{f^*}(\bar{\pi}_1^t, \bar{\pi}_1^{t+1})
\]
Considering the strong convexity, we have
\[
D_{f^*}(\bar{v}_1^{t+1}, \pi_1) \leq D_{f^*}(\bar{v}_1^t, \pi_1) + \frac{1}{\tau} \langle \hat{g}(w^t; B_2), \bar{v}_1^t - \bar{v}_1^{t+1} \rangle \\
+ \frac{1}{\tau} \langle \hat{g}(w^t; B_2^\prime) - \hat{g}(w^t; B_2^\prime_2), \bar{v}_1^t - \bar{v}_1^{t+1} \rangle - \frac{1}{2L_f} \| \bar{v}_1^t - \bar{v}_1^t \|^2 - \frac{1}{2L_f} \| \bar{v}_1^t - \bar{v}_1^{t+1} \|^2
\]

Based on the Young’s inequality, we further have
\[
\frac{1}{\tau} \langle \hat{g}(w^t; B_2^\prime) - \hat{g}(w^t; B_2^\prime_2), \bar{v}_1^t - \bar{v}_1^{t+1} \rangle \leq \frac{1}{2(\tau^2)^2} \| \hat{g}(w^t; B_2^\prime) - \hat{g}(w^t; B_2^\prime_2) \|^2 + \frac{1}{2L_f} \| \bar{v}_1^t - \bar{v}_1^{t+1} \|^2.
\]

Re-arranging the terms leads to
\[
\langle \hat{g}(w^t; B_2^\prime_2) - g(w^t), \pi_1 - \bar{v}_1^t \rangle \geq \tau' \langle D_{f^*}(\bar{v}_1^{t+1}, \pi_1) - D_{f^*}(\bar{v}_1^t, \pi_1) \rangle + \frac{1}{2(\tau^2)^2} \| \hat{g}(w^t; B_2^\prime) - \hat{g}(w^t; B_2^\prime_2) \|^2.
\]

**Lemma 10.** For Algorithm  with initialization \( u_0^t = g_i(w^0) \), the term \( \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i \in D} Q_{i,1}(z^t, z^*) \right] \) can be upper bounded as
\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i \in D} Q_{i,1}(z^t, z^*) \right] \\
\leq \frac{2n(C_f C_C C_{\Omega} + C_{f^*})}{B_1} + \frac{mnCT \tau_{B_1}^2}{B_1} + \frac{L_f T \sigma^2}{\tau B_2} + \frac{B_1 T}{n \tau B_2} - \frac{\tau}{2B_1} \sum_{t=0}^{T-1} D_{f^*}(\pi_{i,1}^{t+1}, \pi_{i,1}^t)
\]

Proof. We define the following notations \( D_{f^*}(\pi_{i,1}, \pi_{i,1}^t) = \sum_{z_i \in D} D_{f^*}(\pi_{i,1}, \pi_{i,1}^t) \) for any \( \pi_i, \pi_i^t \in \Pi_i \), \( f^*(\pi_i) = -\sum_{z_i \in D} f_i^*(\pi_{i,1}) = -\sum_{z_i \in D} f_i^*(\pi_{i,1}) + \sum_{z_i \in D} f_i^*(\pi_{i,1}) \). Due to the update rule of \( \pi_{i,1}^t = \pi_{i,1}^t \) for \( i \notin B_1^t \) and the convexity, we have
\[
\begin{align*}
\hat{g}(w^t, \pi_{i,1}^{t+1}) &= \phi(w^t, \pi_{i,1}^t) + (\phi(w^t, \pi_{i,1}^{t+1}) - \phi(w^t, \pi_{i,1}^t)) + f^*(\pi_{i,1}^{t+1}) \\
&\leq \phi(w^t, \pi_{i,1}^t) - \sum_{z_i \in B_1^t} g_i(w^t; B_{i,2}^t)(\pi_{i,1}^{t+1} - \pi_{i,1}^t) + \sum_{z_i \in B_1^t} f^*(\pi_{i,1}^{t+1}) + \sum_{z_i \notin B_1^t} f_i^*(\pi_{i,1}^{t+1}) \\
&\quad + \sum_{z_i \notin B_1^t} (g_i(w^t; B_{i,2}^t) - g_i(w^t))(\pi_{i,1}^{t+1} - \pi_{i,1}^t) \\
&\hspace{2cm} \vdots \ddots
\end{align*}
\]

Applying the three-point inequality (e.g. Lemma 1 of Zhang and Lan [24]) leads to
\[
-\pi_{i,1}^{t+1} g_i(w^t; B_{i,2}^t) + f^*(\pi_{i,1}^{t+1}) + \tau D_{f^*}(\pi_{i,1}^{t+1}, \pi_{i,1}^t) + \tau D_{f^*}(\pi_{i,1}^{t+1}, \pi_{i,1}^t)
\leq -\pi_{i,1}^{t+1} g_i(w^t; B_{i,2}^t) + f^*(\pi_{i,1}^t) + \tau D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t), \quad z_i \in B_1^t.
\]

Add \( \pi_{i,1}^t g_i(w^t; B_{i,2}^t) \) on both sides and re-arrange the terms. For \( z_i \in B_1^t \), we have
\[
-\pi_{i,1}^{t+1} g_i(w^t; B_{i,2}^t)(\pi_{i,1}^{t+1} - \pi_{i,1}^t) + f^*(\pi_{i,1}^{t+1})
\leq -\pi_{i,1}^t g_i(w^t; B_{i,2}^t)(\pi_{i,1}^t - \pi_{i,1}^t) + f^*(\pi_{i,1}^t) + \tau D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t) - \tau D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t), \quad z_i \in B_1^t.
\]

The \( \cdots \) term can be upper bounded by summing the inequality above over all \( z_i \in B_1^t \). Besides, note that \( \pi_{i,1}^{t+1} = \pi_{i,1}^t \) for \( z_i \notin B_1^t \) such that \( \sum_{z_i \notin B_1^t} f^*(\pi_{i,1}^t) = \sum_{z_i \notin B_1^t} f^*(\pi_{i,1}^t) \).
\[
h(w^t, \pi_{i,1}^{t+1})
\leq \phi(w^t, \pi_{i,1}^t) - \sum_{z_i \in B_1^t} g_i(w^t; B_{i,2}^t)(\pi_{i,1}^{t+1} - \pi_{i,1}^t) + \sum_{z_i \notin B_1^t} f^*(\pi_{i,1}^t) + \sum_{z_i \in B_1^t} f_i^*(\pi_{i,1}^t) + \tau \sum_{z_i \in B_1^t} D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t)
\]

\[
- \tau \sum_{z_i \in B_1^t} D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t) - \tau \sum_{z_i \in B_1^t} D_{f^*}(\pi_{i,1}^t, \pi_{i,1}^t) + \sum_{z_i \notin B_1^t} (g_i(w^t; B_{i,2}^t) - g_i(w^t))(\pi_{i,1}^{t+1} - \pi_{i,1}^t)
\]
Based on the $\frac{1}{L_f}$-strong convexity of $f^*(\pi_{i,1})$, the $\otimes$ term for $z_i \in B_1^i$ can be bounded as

$$(g_i(w^t; B_1^i) - g_i(w^{t+1})) (\pi_{i,1}^{t+1} - \pi_{i,1}^t) \leq \frac{L_f}{\tau} \|g_i(w^t; B_1^i) - g_i(w^t)\|^2 + \frac{\tau}{4L_f} \|\pi_{i,1}^{t+1} - \pi_{i,1}^t\|^2$$

Taking the upper bound of $\otimes$ into consideration leads to

$$h(w^t, \pi_{i,1}^{t+1}) \leq \phi(w^t, \pi_{i,1}^t) - \sum_{z_i \in B_1^i} g_i(w^t; B_1^i) (\pi_{i,1}^{*,i} - \pi_{i,1}^t) + \sum_{z_i \in B_1^i} f^*(\pi_{i,1}^{*,i}) + \sum_{z_i \notin B_1^i} f^*(\pi_{i,1}^t) - \frac{\tau}{2} \sum_{z_i \in B_1^i} D_f^*(\pi_{i,1}^{t+1}, \pi_{i,1}^t)$$

Define that $\hat{g}(w^t; B_2^i) = \sum_{z_i \in B_2^i} \frac{n}{B_i^i} g_i(w^t; B_1^i) e_i$ and $g(w^t) = \sum_{z_i \in D} g_i(w^t) e_i$, where $e_i \in \mathbb{R}^n$ is the indicator vector that only the $i$-th element is 1 while the others are 0. Note that $E[\hat{g}(w^t; B_2^i)] = g(w^t)$. Then, $\mathbf{\hat{\otimes}}$ can be decomposed as

$$\mathbf{\hat{\otimes}} = -\frac{B_1}{n} \sum_{z_i \in B_1^i} \frac{n}{B_i^i} g_i(w^t; B_1^i) (\pi_{i,1}^{*,i} - \pi_{i,1}^t)$$

where $\pi_{i}^{t+1}$ is defined as

$$\pi_{i}^{t+1} = \arg\min_{\pi_1} \langle \hat{g}(w^t; B_2^i) - g(w^t), \pi_1 \rangle + \tau' D_f^*(\pi_{i,1}^{t+1}, \pi_{i,1}^t),$$

where $\tau' > 0$, $B_1^i$ is a “virtual batch” (never sampled in the algorithm) that is independent of but has the same size as $B_2^i$. Based on Lemma [9], $\hat{\otimes}$ can be lower bounded as

$$\langle \hat{g}(w^t; B_2^i) - g(w^t), \pi_{i}^{t+1} - \pi_{i}^t \rangle \geq -\tau' D_f^*(\pi_{i,1}^{t+1}, \pi_{i,1}^t) + \tau' D_f^*(\pi_{i}^{t+1}, \pi_{i}^t) - \frac{1}{2(\tau')^2} \|\hat{g}(w^t; B_2^i) - g(w^t; B_2^i)\|^2.$$

Thus, taking the expectation of the equation above w.r.t. the randomness in iteration $t$ leads to

$$E_t \left[ \frac{B_1}{n} \langle \hat{g}(w^t; B_2^i) - g(w^t), \pi_{i}^{t+1} - \pi_{i}^t \rangle \right] \geq \frac{\tau' B_1}{n} \left( -E_t [D_f^*(\pi_{i,1}^{t+1}, \pi_{i}^t)] + E_t [D_f^*(\pi_{i,1}^{t+1}, \pi_{i}^{t+1})] \right) - \frac{n}{(\tau')^2} B_2.$$
\[\pi_{i+1} = \pi_i \text{ for } i \notin B_1.\]

\[\mathbb{E}_t [h(w^t, \pi_{i+1}^t)] \leq \phi(w^t, \pi_i^0) + \left(1 - \frac{B_1}{n}\right) \Phi^*(\pi_i^0) + \frac{B_1}{n} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] + \frac{B_1}{n} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] - \frac{\tau}{2} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] + \frac{L_f B_1 \sigma^2}{\tau B_2}
\]

Subtract \(\mathbb{E}_t [h(w^t, \pi_i^0)]\) from both sides and use the tower property of conditional expectation.

\[\mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] = \left(1 - \frac{B_1}{n}\right) \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] + \frac{B_1}{n} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] - \frac{\tau}{2} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] + \frac{L_f B_1 \sigma^2}{\tau B_2}
\]

Let \(\Delta^t = h(w^t, \pi_{i+1}^t) - h(w^t, \pi_i^0)\). Thus,

\[\mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] = \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] - \frac{\tau}{2} \mathbb{E}_t \left[D_f(\pi_i^0, \pi_i^1)\right] + \frac{L_f B_1 \sigma^2}{\tau B_2}
\]

We define \(C = C_f C_g \sqrt{C_f^2 (C_g^2 + \zeta^2/B_2)}.\) Do the telescoping sum for \(t = 1, \ldots, T\).

\[\mathbb{E}_t \left[\sum_{t=0}^{T-1} \frac{B_1}{n} \Delta^t\right] \leq (\Delta^0 - \Delta^T) + \mathbb{E}_t \left[\tau D_f(\pi_i^0, \pi_i^1)\right] + \frac{L_f B_1 \sigma^2 T}{\tau B_2} + \frac{nT}{\tau B_2} + \frac{\tau B_1}{n} D_f(\pi_i^0, \pi_i^1).
\]

Consider that \(\frac{1}{n} \sum_{z_i \in D} Q_i(1)(z^i, z^*) = \frac{1}{n} \left(h(w^t, \pi_{i+1}^t) - h(w^t, \pi_i^0)\right) = \Delta^t.\)

\[\sum_{t=0}^{T-1} \mathbb{E}_t \left[\frac{1}{n} \sum_{z_i \in D} Q_i(1)(z^i, z^*)\right] \leq \Delta^0 - \Delta^T + \mathbb{E}_t \left[\tau D_f(\pi_i^0, \pi_i^1)\right] + \frac{L_f B_1 \sigma^2 T}{\tau B_2} + \frac{nT}{\tau B_2} + \frac{\tau D_f(\pi_i^0, \pi_i^1)}{n}
\]

The numerator in the first term on the right hand side can be upper bounded as follows.

\[\Delta^0 - \Delta^T = h(w^0, \pi_i^0) - h(w^0, \pi_i^1) - h(w^T, \pi_i^1) + h(w^T, \pi_i^0)\]

\[= - \sum_{z_i \in D} (\pi_i^1 - \pi_i^0) g_i(w^0) - \sum_{z_i \in D} (\pi_i^1 - \pi_i^0) g_i(w^T) + \sum_{z_i \in D} f^*(\pi_i^0) - \sum_{z_i \in D} f^*(\pi_i^1)
\]

\[\leq 2nC_f C_g C_2 + 2nC_f.
\]

On the other hand, if we set \(\pi_i^0 = \nabla f(u_i^0)\) and \(u_i^0 = g_i(w^0)\), we have \(D_f(\pi_i^0, \pi_i^1) = D_f(\nabla f(u_i^0), \nabla f(g_i(w)))\).

The proof concludes by setting \(\tau = \frac{n^2}{B_1^2}\).
Lemma 11. For Algorithm 3 we have

\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} Q_{i,0}(z^t, z^*) \right] \leq \eta TC^2/c^2_g + \eta TC^2/(c^2 + C_g^2) \min\{B_1, B_2\} + \frac{C_g^2}{2T} + \frac{\tau}{2B_1} \mathbb{E} \left[ \sum_{t=0}^{T-1} D_f(z^t, z^{t+1}) \right] + \frac{B_1 L_f C_g^2 C_d^2 T}{nT}.
\]

Proof. Based on the definition of \(Q_{i,0}(z^t, z^*)\), we can derive that

\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} Q_{i,0}(z^t, z^*) \right] = \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} \langle \pi^t_{i,1} \pi^t_{i,2} \rangle, w^t - w^* \rangle \right] = \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} \langle \pi^t_{i,1} \pi^t_{i,2}, w^t - w^* \rangle \right] + \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} \langle \pi^t_{i,1} \pi^t_{i,2}, w^t - w^t+1 \rangle \right]
\]

The second term on the right hand side of (17) can be upper bounded by

\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} \langle \pi^t_{i,1} \pi^t_{i,2}, w^t - w^t+1 \rangle \right] \leq \mathbb{E} \left[ \sum_{t=0}^{T-1} \left( \eta C^2_g + \frac{1}{4\eta} \|w^t+1 - w^t\|^2 \right) \right].
\]

The first term on the right hand side of (17) can be upper bounded by

\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{n} \sum_{z_i \in D} \langle \pi^t_{i,1} \pi^t_{i,2}, w^t + w^t+1 \rangle \right] = \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{z_i \in D} \pi^t_{i,1} \pi^t_{i,2}, w^t+1 - w^* \right\rangle \right] + \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\langle \frac{1}{B_1} \sum_{i \in B'_1} \pi^t_{i,1} \pi^t_{i,2}(B'_i, 2), w^t+1 - w^* \right\rangle \right]
\]

The last equality above uses \(\mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{z_i \in D} \pi^t_{i,1} \pi^t_{i,2}, w^t - w^* \right\rangle \right] = 0\). Moreover,

\[
\mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{z_i \in D} \pi^t_{i,1} \pi^t_{i,2} - \frac{1}{B_1} \sum_{i \in B'_1} \pi^t_{i,1} \pi^t_{i,2}(B'_i), w^t - w^* \right\rangle \right] \leq \eta C^2_g/(c^2 + C_g^2) \min\{B_1, B_2\} + \frac{1}{4\eta} \|w^t+1 - w^t\|^2. \hspace{1cm} (18)
\]

According to the three-point inequality (Lemma 1 in Zhang and Lan [24]), we have

\[
\sum_{t=0}^{T-1} \mathbb{E} \left[ \left\langle \frac{1}{B_1} \sum_{i \in B'_1} \pi^t_{i,1} \pi^t_{i,2}(B'_i, 2), w^t+1 - w^* \right\rangle \right] \leq \frac{1}{2\eta} \|w^t - w^*\|^2 - \frac{1}{2\eta} \|w^t+1 - w^*\|^2 - \frac{1}{2\eta} \|w^t - w^t+1\|^2. \hspace{1cm} (19)
\]
Besides, the third term on the R.H.S. of (17) can be upper bounded as follows based on the Young’s inequality with a constant \( \rho > 0 \).

\[
\sum_{t=0}^{T-1} E \left[ \frac{1}{n} \sum_{z_i \in D} \langle (\pi_{i,t}^{t+1} - \pi_{i,t-1}^{t+1}, w_{t+1} - w^\star) \rangle \right] \leq \sum_{t=0}^{T-1} E \left[ \frac{C^2 \rho}{2n} \sum_{z_i \in D} \|\pi_{i,t}^{t+1} - \pi_{i,t}^{t-1}\|^2 + \|w_{t+1} - w^\star\|^2 / 2 \rho \right]
\]

\[
\leq \sum_{t=0}^{T-1} E \left[ \frac{L_f C^2 \rho}{n} D_f \cdot \langle \pi_{i,t}^{t+1}, \pi_{i,t}^{t+1} \rangle + \|w_{t+1} - w^\star\|^2 / 2 \rho \right],
\]

where the last inequality is due to \( L_f \)-smoothness of \( f \). Choose \( \rho = \frac{n^2}{2B L_f C^2} \). Besides, we also have

\[
\|w_{t+1} - w^\star\|^2 \leq C^2 \Omega \text{ according to Assumption 3.}
\]

\[
\sum_{t=0}^{T-1} E \left[ \frac{1}{n} \sum_{z_i \in D} \langle (\pi_{i,t}^{t+1} - \pi_{i,t-1}^{t+1}, w_{t+1} - w^\star) \rangle \right] \leq \frac{\tau}{2B_1} E \left[ \sum_{t=0}^{T-1} D_f \cdot \langle \pi_{i,t}^{t+1}, \pi_{i,t}^{t+1} \rangle \right] + \frac{B_1 L_f C^2 \Omega^2 T}{n \tau}.
\]

Plugging (18), (19), and (20) into (17) leads to

\[
\sum_{t=0}^{T-1} E \left[ \frac{1}{n} \sum_{z_i \in D} Q_{i,0}(z_i^t, z_i^\star) \right] \leq \eta T C^2 \Omega^2 + \frac{\eta C^2 (\Omega^2 + C^2) \Omega}{2 \eta} \text{min} \{B_1, B_2\} + \frac{\tau}{2B_1} E \left[ \sum_{t=0}^{T-1} D_f \cdot \langle \pi_{i,t}^{t+1}, \pi_{i,t}^{t+1} \rangle \right] + \frac{B_1 L_f C^2 \Omega^2 T}{n \tau}.
\]

**Theorem 5.** Under assumptions 2, 3 and that \( f \) is convex, monotone, Lipschitz-continuous, and smooth while \( g_i \) is convex and Lipschitz-continuous, SOX (Algorithm 3) can find \( w \) that \( E[F(w) - F(w^\star)] \leq \epsilon \) after

\[
T = \max \left\{ \frac{18n(C_f C_g C_\Omega + C_f \cdot)}{B_1 \epsilon}, \frac{81nL_f^2 C^2_g C_\Omega^2 \sigma^2}{B_1 B_2 \epsilon^2}, \frac{81L_f^2 C^2_g C_\Omega^2}{B_2 \epsilon^2}, \frac{81L_f^2 C^4_g C_\Omega^4}{2B_1 \epsilon^2}, \frac{81nCC_2 C_\Omega^2}{2 \epsilon^2}, \frac{81C^2_f C^2_\Omega}{\epsilon^2}, \frac{81C^2_f C^2_\Omega}{2 \epsilon^2}, \frac{81C^2_f C^2_\Omega (\Omega^2 + C^2)}{2 \epsilon^2}, \frac{81nC_\Omega^2}{2 \epsilon^2} \right\}
\]

iterations by setting \( \eta = \min \left\{ \frac{B_1 \sigma^2}{B_2 \epsilon}, \frac{B_1}{n \epsilon \sigma^2}, \frac{9B_1 L_f C^2 \Omega^2}{n \epsilon} \right\} \) and \( \tau = \max \left\{ \frac{9L_f \sigma^2}{B_2 \epsilon}, \frac{9B_1}{n \epsilon \sigma^2}, \frac{9B_1 L_f C^2 \Omega^2}{n \epsilon} \right\} \).

**Proof.** Based on Lemma 8, Lemma 10 and Lemma 11 we have

\[
E \left[ \frac{1}{T} \sum_{t=0}^{T-1} Q(z_i^t, z_i^\star) \right] \leq \frac{2n(C_f C_g C_\Omega + C_f \cdot)}{B_1 T} + \frac{\eta nC}{B_1 T} + \frac{\tau L_f^2 \sigma^2}{\eta T \epsilon^2} + \frac{\eta L_f^2 \sigma^2}{\eta T \epsilon^2} + \frac{L_f^2 \sigma^2}{\eta T \epsilon^2} + \frac{\eta C^2 (\Omega^2 + C^2) \Omega}{2 \eta} \text{min} \{B_1, B_2\} + \frac{\tau}{2B_1} E \left[ \sum_{t=0}^{T-1} D_f \cdot \langle \pi_{i,t}^{t+1}, \pi_{i,t}^{t+1} \rangle \right] + \frac{B_1 L_f C^2 \Omega^2 T}{n \tau},
\]

where \( F(\bar{w}^T) = L(\bar{w}, \bar{\pi}_{1,1}^T, \bar{\pi}_{2,1}^T), \bar{w} = \frac{1}{n} \sum_{t=0}^{T-1} w_i^t \) and \( L_i,1(w^\star, \pi_{i,1}^{t+1}, \pi_{i,2}^{t+1}) \leq f(g_i(w^\star)) \) such that we have

\[
E \left[ \frac{1}{T} \sum_{t=0}^{T-1} Q(z_i^t, z_i^\star) \right] = E \left[ \frac{1}{T} \sum_{t=0}^{T-1} (L(w^t, \pi_{1,1}^t, \pi_{2,1}^t) - L(w^\star, \pi_{1,1}^{t+1}, \pi_{2,1}^{t+1})) \right] \geq E \left[ F(w) - F(w^\star) \right]
\]

Thus, to ensure that \( E[F(w) - F(w^\star)] \leq \epsilon \), we can make

\[
\eta = \min \left\{ \frac{B_1 \epsilon}{9nC}, \frac{\epsilon}{9C^2 \Omega^2}, \frac{\epsilon}{9C^2 (\Omega^2 + C^2)}, \frac{\epsilon}{9C^2 (\Omega^2 + C^2)}, \frac{\epsilon}{9C^2 (\Omega^2 + C^2)}, \frac{\epsilon}{9C^2 (\Omega^2 + C^2)} \right\}, \quad \tau = \max \left\{ \frac{9L_f \sigma^2}{B_2 \epsilon}, \frac{9B_1}{n \epsilon \sigma^2}, \frac{9B_1 L_f C^2 \Omega^2}{n \epsilon} \right\},
\]

\[
T = \max \left\{ \frac{18n(C_f C_g C_\Omega + C_f \cdot)}{B_1 \epsilon}, \frac{81nL_f^2 C^2_g C_\Omega^2 \sigma^2}{B_1 B_2 \epsilon^2}, \frac{81L_f^2 C^2_g C_\Omega^2}{B_2 \epsilon^2}, \frac{81L_f^2 C^4_g C_\Omega^4}{2B_1 \epsilon^2}, \frac{81nC_\Omega^2}{2 \epsilon^2}, \frac{81C^2_f C^2_\Omega}{\epsilon^2}, \frac{81C^2_f C^2_\Omega}{2 \epsilon^2}, \frac{81C^2_f C^2_\Omega (\Omega^2 + C^2)}{2 \epsilon^2}, \frac{81nC_\Omega^2}{2 \epsilon^2} \right\}.
\]
D Extensions for a More general Class of Problems

In this section, we briefly discuss the extension when \( f \) is also a stochastic function such that we can only get an unbiased estimate of its gradient, which has an application in MAML. To this end, we assume a stochastic oracle of \( f_i \) that given any \( g \) returns \( \nabla f_i(g; B_{3,i}) \) such that \( \mathbb{E}[\nabla f_i(g; B_{3,i})] = \nabla f_i(g) \), \( \mathbb{E}[\|\nabla f_i(g; B_{3,i}) - \nabla f_i(g)\|^2] \leq \frac{\chi^2}{B_3^2} \). We can extend our results for the smooth nonconvex problems by the modifications as follows: First, there will an additional term \( \frac{10 \beta L^2 C_i^2 \xi^2}{B_1 B_3} \) on the right hand side of Lemma 1 and the \( \frac{2 \beta^2 C_i^2 (C^2 + C_g^2)}{\min(B_1, B_2)} \) should be replaced by \( \frac{2 \beta^2 (\chi^2 C_i^2 + (C_f^2 + \chi^2/B_3)(C^2 + C_g^2))}{\min(B_1, B_2, B_3)} \). Note that Lemma 2 remains the same and Theorem 1 does not change (up to a constant factor).