Fractional Mirror Symmetry

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Mirror symmetry relates type IIB string theory on a Calabi–Yau 3-fold to type IIA on the mirror CY manifold, whose complex structure and Kähler moduli spaces are exchanged. We show that the mirror map is a particular case of a more general quantum equivalence, fractional mirror symmetry, between Calabi–Yau compactifications and non-CY ones. As was done by Greene and Plesser for mirror symmetry, we obtain these new dualities by considering orbifolds of Gepner models, of asymmetric nature, leading to superconformal field theories isomorphic to the original ones, but with a different target-space interpretation. The associated Landau–Ginzburg models involve both chiral and twisted chiral multiplets hence cannot be lifted to ordinary gauged linear sigma-models.

INTRODUCTION

Mirror symmetry plays a major role in our understanding of Calabi–Yau (CY) manifolds, both in their physical and mathematical aspects \[^1,2\]. It generalizes T-duality to CY sigma-models as one exchanges the axial and vector R-symmetries of the \(\mathcal{N} = (2, 2)\) superconformal algebra \[^3\]. The first concrete realization was obtained by Greene and Plesser \[^4\] using Gepner models \[^5\]; they have shown that an orbifold by the largest subgroup of discrete torus R-symmetries of the minimal models are primaries with quantum numbers, leading to superconformal field theories isomorphic to the original ones, such that the axial and vector R-symmetries for a single minimal model are exchanged. They correspond to ‘hybrid’ Landau–Ginzburg (LG) orbifolds with both chiral and twisted chiral superfields, hence cannot be lifted to Calabi–Yau GLSMs. This is another sign of the non-CY nature of these new dual models; as this map can be applied stepwise to each and every minimal model we give it the name of fractional mirror symmetry.

SIMPLE CURRENTS AND GEPENER MODELS

We first review the simple current formalism \[^11,12\] and its relation with Gepner models. In a CFT a simple current \(J\) is a primary of the chiral algebra whose fusion with a generic primary gives a single primary: \(J \star \phi_\mu = \phi_\nu\). This action defines the monodromy charge of the primary \(Q_\mu(\mu) = \Delta(\phi_\mu) + \Delta(J_1) - \Delta(J_1 \star \phi_\mu) \mod 1\); two-currents are mutually local if \(Q_\mu(J_1) = 0\). A modular-invariant partition function associated with a simple current extension is:

\[
Z = \sum_\mu \prod_{i=1}^M \sum_{b' \in \mathbb{Z}_n} \chi_\mu(q) \chi_{\mu+b'}(q) \delta^{(1)}(Q_\mu(\mu) + X_1 b')
\]

(1)

with \(J_1 \star \phi_\mu = \phi_{\mu+b}\) and \(n_i\) the length of \(J_i\). The symmetric part of the matrix \(X\) is determined by the relative monodromies as \(X_{ij} + X_{ji} = Q_i(J_j) \mod 1\), while the antisymmetric part, discrete torsion, should be such that:

\[
\gcd(n_i, n_j) X_{ij} \in \mathbb{Z}.
\]

(2)

A Gepner model for a CY 3-fold is obtained from a tensor product of \(r\) minimal models whose central charges satisfy \(\sum_{n=1}^r c_n = \sum_{n=1}^r (3-6/k_n) = 9\). One needs to project the theory onto states with odd integer left and right R-charges; this can be rephrased in the simple currents formalism. The simple currents of the minimal models are primaries with quan-
moving space-time chirality at the same time. As super-

of the right R-charge associated with the first minimal

In the right NS sector it is equivalent to change the sign

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torsion, to replace in the original partition function the

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However, twisted sectors associated with the 3-extension

while mutual locality w.r.t. 0 requires \( n_{\mu}/k_n \in \mathbb{Z} \). Extending a Gepner model with

all such simple currents (without discrete torsion) gives

the mirror Gepner model, whose right R-charge has op-

posite sign. This is the basis of the construction of mirror

manifolds by Greene and Plesser [4].

**NON-GEOMETRIC CY ORBIFOLDS**

We consider here simple currents that are not mutually

local w.r.t. the Gepner model currents. A generic cur-

rent \( \mathfrak{J} \) as above is non-local w.r.t. \( J_0 \), hence space-time

supersymmetry is broken. Now comes the key step; there

is a choice of discrete torsion, consistent with eq. (2) for

any \( \{ \rho_n \in \mathbb{Z} \} \), bringing down the \( X \) matrix to a lower-

triangular form, whose only non-zero entries are

\[
X_{33} = \sum_{n=1}^r \frac{\rho_n^2}{k_n}, \quad X_{30} = \sum_{n=1}^r \frac{\rho_n}{k_n}.
\]

Grouping the free-fermion and minimal models character-

\( \chi_{\rho}^\lambda = \theta_{\rho_\eta,2}/\eta \times \prod_{n=1}^r \chi_{m_n,s_n}^\lambda \), the modular-

invariant partition function of the 3-extended Gepner model is

(\( K = \text{lcm}(4,2k_1,\ldots,2k_r) \)) and \( N = \text{lcm} (\text{lcm}(\rho_1,k_1)/\rho_1,\ldots,\text{lcm}(\rho_r,k_r)/\rho_r) \):

\[
Z = \frac{1}{2^r \sqrt{N}} \sum_{\lambda,\mu} \sum_{b_0 \in \mathbb{Z}} (-1)^{b_0} \delta^{(1)} \left( \frac{QR-1}{2} \right) \sum_{B \in \mathbb{Z}} \delta^{(1)} \left( \sum_{n=1}^r \frac{\rho_n(m_n + b_0 + \rho_n B)}{k_n} \right) \times \prod_{n=1}^r \sum_{b_0 \in \mathbb{Z}} \delta^{(1)} \left( \frac{s_0 - s_n}{2} \right) \chi_{\mu}^\lambda(q) \chi_{\mu + \beta_0 b_0 + \beta_1 b_1 + \beta_3 B}^\lambda(q).
\]

Thanks to the discrete torsion the projection onto odd-

integer R-charges is restored in the left-moving sector.

However, twisted sectors associated with the 3-extension

\( (B \neq 0) \) have fractional right R-charge, hence space-time

supersymmetry from the right is generically broken while

supersymmetry from the left is preserved. This construc-

tion provides a whole class of non-geometric quotients of

CY sigma-models at Gepner points.

**FRACTIONAL MIRROR SYMMETRY**

A particular type of \( \mathfrak{J} \)-extensions with discrete torsion

is quite interesting. Extending a Gepner model parti-

tion function with \( \mathfrak{J}_1 = (0|2,0,\ldots,0|0,\ldots,0) \) amounts,

while taking into account the twisted sectors and discrete
torsion, to replace in the original partition function the

anti-holomorphic character for the first minimal model

with the character of opposite \( \mathbb{Z}_{2k_1} \) charge, namely

\[
\lambda_{m_1 + b_0, s_1 + b_0 + 2b_1}^{\mathfrak{J}_1} \xrightarrow{\mathfrak{J}_1 = \mathfrak{J}_1 - m_1 - b_0, s_1 + b_0 + 2b_1} \lambda_{-m_1 - b_0, s_1 + b_0 + 2b_1}^{\mathfrak{J}_1}.
\]

In the right NS sector it is equivalent to change the sign

of the right R-charge associated with the first minimal

model; in the R sector as well if one changes the right-
moving space-time chirality at the same time. As super-

conformal field theories the original model and the new

one are isomorphic, hence the two theories are dual [14].

Starting from a type IIA CY compactification at a Gep-

ner point, we obtain a type IIB theory on a Gepner model

whose right-moving R-charge associated with the first

minimal model has been reversed; with respect to the

original right-moving diagonal R-current the spectrum

of R-charges is not integer-valued hence the model is

not associated with a Calabi–Yau. Put it differently the

quotient does not preserve the holomorphic three-form.

These two models are isomorphic CFTs hence describe

the same physics.

A minimal model is the IR fixed point of a LG model

with superpotential \( W = Z^k \) [15]. Its mirror, obtained

by a \( \mathbb{Z}_k \) quotient, is a LG model for a twisted chiral

superfield \( \tilde{Z} \) with a twisted superpotential \( \tilde{W} = \tilde{Z}^k \).

In the present context we are considering a similar quotient

acting inside a LG orbifold, with a discrete torsion that

disentangles partly the two orbifolds – the diagonal one

ensuring R-charge integrality and the \( \mathbb{Z}_{k_i} \) quotient giving

the fractional mirror. We end up with a model containing

both a twisted chiral superfield \( \tilde{Z}_1 \) and chiral superfields

\( \tilde{Z}_2,\ldots,\tilde{r} \), hence cannot be related to a CY GLSM. This

fractional mirror symmetry can be applied stepwise until

one obtains the mirror Gepner model in the usual sense.

The asymmetric \( K^3 \times T^2 \) Gepner models that we have
constructed in \[9\] can be rephrased in light of this new understanding. One considers two 3-extensions acting respectively in the first and second factors of a K3 Gepner model, and as \(Z_{k_1}\) and \(Z_{k_2}\) shifts along the two-torus. These models are close relatives of T-folds \([10]\), given that the K3 fiber is twisted by non-geometric symmetries as one goes around one-cycles of the base. They interpolate between the K3 sigma–model in the large torus limit, and a 'half-mirror' K3 in the opposite small volume limit.

**DISCUSSION**

We have considered new quantum symmetries associated with two-dimensional superconformal field theories lying the moduli space of Calabi–Yau compactifications. Usual T-dualities are associated with continuous symmetries corresponding to globally defined Killing vectors. Compact Calabi–Yau sigma-models have no continuous symmetries (apart from the axial and vector \(U(1)_R\)) but new dualities exist when they have accidental discrete symmetries. Hence, unlike mirror symmetry, the fractional mirror map is a symmetry only at specific loci where these symmetries are manifest.

To illustrate this point let us consider the quintic, giving the Gepner point \(k_1 = \cdots = k_5 = 5\). Away from the Gepner point, we expect that for every hypersurface

\[
z_1^5 + \sum \alpha_{abc} z_2^{a} z_3^{b} z_5^{c} = 0
\]

a fractional mirror w.r.t. the chiral superfield \(Z_1\) exists. In other words the complex structure deformations that preserve the \(Z_5\) symmetry \(z_1 \rightarrow e^{2\pi i/5} z_1\) are compatible with this duality. Kähler deformations are not allowed, as can be seen explicitly at the Gepner point, excluding the existence of a large-volume limit. When these conditions are met the \(\mathcal{N} = 2\) superconformal algebra can be split into the algebra coming from the LG model \(W = Z_1^5\) and from the LG model for the other multiplets. This allows to dualize \(Z_1\) into a twisted chiral multiplet, giving a more general 'hybrid' LG orbifold with superpotential \(W = \sum \alpha_{abc} Z_2^{a} Z_3^{b} Z_5^{c} Z_1^{5-a-b-c}\) and twisted superpotential \(\tilde{W} = Z_1^{5}\). One expects also that other accidental splittings of the superconformal algebra should give rise to different fractional mirror symmetries.

One may argue that, after all, the quotient is 'almost' geometric as the discrete torsion only plays a role in the twisted sectors. This is not actually correct, as the tensor product of minimal models becomes a CY sigma-model only after the Gepner orbifold has been implemented. The discrete torsion has an effect in the twisted sectors of the \(J_0\)-extension, giving the compactification a non-geometric nature. In particular the quotient has a different action on twisted chiral modules, i.e. on Kähler moduli, compared to the corresponding geometric orbifold.

A more geometrical characterization of the fractional mirrors, that are not expected to be in any Calabi–Yau moduli space, would be very helpful in understanding their properties. Following the approach of Horit and Vafa \([7]\), one may start with a Calabi–Yau GLSM and dualize only part of the chiral superfields.

Finally it would be interesting to find whether these symmetries are related to the Mathieu moonshine, which suggests that K3 compactifications have an underlying \(M_{24}\) symmetry whose origin is not fully understood \([17]\).

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