On the dressing method for Dunajski anti-self-duality equation

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Abstract

A dressing scheme applicable to Dunajski equation is developed. Simple example of constructing solutions in terms of implicit functions is considered. Dunajski equation hierarchy is described, its Lax-Sato form is presented. Dunajsky equation hierarchy is characterised by conservation of three-dimensional volume form, in which a spectral variable is taken into account. Some reductions of Dunajsky equation hierarchy, including waterbag-type reduction, are studied.

1 Dunajski equation

Dunajski equation [1] is a representative of the class integrable systems arising in the context of complex relativity [2]-[6]. It is closely connected to the celebrated Plebański heavenly equations [2] and in some sense generalizes them. It describes anti-self-dual null-Kähler structures. In [1] it was shown that all null-Kähler metrics (signature (2,2)) locally admit a canonical Plebański form

\[ g = dwdx + dzdy - \Theta_{xx}dz^2 - \Theta_{yy}dw^2 + 2\Theta_{xy}dw dz. \]  

The conformal anti-self-duality (ASD) condition leads to Dunajski equation

\[ \Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \]  

\[ \Box f = f_{wx} + f_{yz} + \Theta_{yy}f_{xx} + \Theta_{xx}f_{yy} - 2\Theta_{xy}f_{xy} = 0. \]
Equations (2,3) represent a compatibility condition for the linear system
\[ L_0 \Psi = L_1 \Psi = 0, \]
where \( \Psi = \Psi(w, z, x, y, \lambda) \) and
\[
L_0 = (\partial w - \Theta_{xy} \partial y + \Theta_{yy} \partial x) - \lambda \partial y + f_y \partial \lambda,
\]
\[
L_1 = (\partial z + \Theta_{xx} \partial y - \Theta_{xy} \partial x) + \lambda \partial x - f_x \partial \lambda.
\]

The case \( f = 0 \) corresponds to metrics of the form (1) satisfying Einstein equations, and Dunajski equation (2), (3) reduces to Plebański second heavenly equation [2].

In the case
\[
\Theta = -1/6 a_1(w, z)x^3 + 1/2 a_2(w, z)x^2y - 1/2 a_3(w, z)xy^2 + 1/6 a_4(w, z)y^3
\]
the geodesics of the metrics (1) are determined by solutions of projectively flat second order ODE
\[
\frac{d^2}{dz^2} w(z) + a_1(w, z) \left( \frac{d}{dz} w(z) \right)^3 + 3 a_2(w, z) \left( \frac{d}{dz} w(z) \right)^2 + 3 a_3(w, z) \frac{d}{dz} w(z) + a_4(w, z) = 0.
\]
As an example [7], it is possible to consider equation with the coefficients
\[
a_1 = 2 \exp(\phi(w, z)), \quad a_2 = -\phi_z, \quad a_3 = \phi_w, \quad a_4 = -2 \exp(\phi(w, z)),
\]
where \( \phi(w, z) \) is solution of the Wylczynski-Tzitzeika equation
\[
\phi_{zz} = 4 \exp(2\phi) - \exp(-\phi).
\]

In this paper we develop methods of integrable systems theory for Dunajski equation, using the ideas of the works [8]-[12] and also [13, 14]. We present the results without many technical details, which will be described in a more extensive text.

2 Dressing scheme

Let us consider nonlinear vector Riemann problem of the form
\[ S_+ = F(S_-), \]
where $S_+, S_-$ denote the boundary values of the N-component vector function on the sides of some oriented curve $\gamma$ in the complex plane of the variable $\lambda$. The problem is to find the function analytic outside the curve with some fixed singularities at infinity which satisfies (5). This problem is connected with a class of integrable equations, which can be represented as a commutation relation for vector fields containing a derivative on the spectral variable.

The more specific setting relevant for Dunajski equation is the following. We consider three-component problem (5) for the functions

\[ S^0 \to \lambda + O(\frac{1}{\lambda}), \]
\[ S^1 \to -\lambda z + x + O(\frac{1}{\lambda}), \]
\[ S^2 \to \lambda w + y + O(\frac{1}{\lambda}), \quad \lambda \to \infty, \]

where $x, y, w, z$ are the variables of Dunajski equation (‘times’). We suggest that for given $F$ solution of the problem (5) exists and is unique (at least locally in $x, y, w, z$).

Let us consider a linearized problem

\[ \delta S^i_+ = \sum_j F^i_{j} \delta S^j_. \]

The linear space of solutions of this problem is spanned by the functions $S_x, S_y, S_\lambda$, which can be multiplied by arbitrary function of spectral variable and times. The presence of $S_\lambda$ in the basis is the main difference between the dressing schemes for the heavenly equation [8, 11] and Dunajski equation.

Expanding the functions $S_z, S_w$ into the basis, we obtain linear equations

\[ ((\partial_w + u_y \partial_y + v_y \partial_x) - \lambda \partial_y + f_y \partial_\lambda) S = 0, \]
\[ ((\partial_z - u_x \partial_y - v_x \partial_x) + \lambda \partial_x - f_y \partial_\lambda) S = 0, \] (6)

where $u, v, f$ can be expressed through the coefficients of expansion of $S^0, S^1, S^2$ at $\lambda = \infty$.

\[ u = S^2_1 - w S^0_1, \quad v = S^1_1 + z S^0_1, \quad f = S^0_1, \] (7)
\[ S^0 = \lambda + \sum_{n=1}^{\infty} \frac{S^0_n}{\lambda^n}, \quad S^1 = -z \lambda + x + \sum_{n=1}^{\infty} \frac{S^1_n}{\lambda^n}, \]
\[ S^2 = w \lambda + y + \sum_{n=1}^{\infty} \frac{S^2_n}{\lambda^n}, \quad \lambda = \infty. \]
To get a Lax pair for Dunajski equation, we should consider the reduction \( v_y = -u_x \), then we can introduce a potential \( \Theta \),

\[
v = \Theta_x, \ u = -\Theta_y.
\] (8)

**Proposition 1** Sufficient condition to provide the reduction \( v_y = -u_x \) in terms of the Riemann problem (5) is

\[
\det F^i_j = 1.
\] (9)

2.1 An example

Now will consider a simple example of constructing solution to Dunajski equation using the problem (5). We use the problem of the form

\[
S^1_+ = S^1_-, \quad S^2_+ = S^2 F^{-1}(S^2_0 \cdot S^0_0, S^1_0), \quad S^0_+ = S^0 F(S^2_0 \cdot S^0_0, S^1_0),
\] (10) (11) (12)

where \( F \) is an arbitrary function of two variables. It is easy to check that the reduction condition (9) is indeed satisfied in this case. Equation (10) implies that \( S^1 = -\lambda z + x \). Substituting this solution to linear equations (6) (or using (7)), we obtain \( v = zf \).

The second important property of the problem we use is that the function \( S^2 \cdot S^0 \) is analytic. Then we get an expression

\[
\phi = S^2 \cdot S^0 = \lambda^2 w + \lambda y + 2fw + u.
\] (13)

Equation (12) now reads

\[
S^0_+ = S^0 F(\phi, -\lambda z + x).
\]

The solution to this equation looks like

\[
S^0 = \lambda \exp \left( \frac{1}{2\pi i} \int_\gamma \frac{d\lambda'}{\lambda - \lambda'} \ln F(\phi(\lambda'), -\lambda' z + x) \right)
\]

Considering the expansion of this expression in \( \lambda \), we obtain the equations

\[
\frac{1}{2\pi i} \int_\gamma d\lambda \ln F(\phi(\lambda), -\lambda z + x) = 0,
\] (14)

\[
\frac{1}{2\pi i} \int_\gamma \frac{d\lambda}{\lambda} \ln F(\phi(\lambda), -\lambda z + x) = f.
\] (15)
Taking into account expression (13), we come to the conclusion that these equations define the functions \( u, f \) as implicit functions. Solution to Dunajski equation is then defined by the relation

\[ u = -\Theta_y. \]

Thus we have obtained a solution to Dunajski equation, depending on arbitrary function of two variables, in terms of implicit functions.

Functional dependence on the function of two variables indicates that the solution we have constructed corresponds to some (2+1)-dimensional reduction of Dunajski equation. It is possible to find the reduced equations explicitly, using the fact that linear equations (4) have analytic solutions \( \phi \) and \(-\lambda z + x\). Substituting these solutions to (4) and using (8), we obtain

\[
\begin{align*}
(\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x)(2wf - \Theta_y) + yf_y &= 0, \\
(\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x)(2wf - \Theta_y) - yf_x &= 0,
\end{align*}
\]

\[ z f = \Theta_x. \]

### 3 Dunajski equation hierarchy

The framework developed here is closely connected with the framework of hyper-Kähler hierarchy developed by Takasaki [13, 14], see also [11, 12]. Though there are some essential differences (the volume form is used instead of symplectic form, the spectral variable is included to the form), the technique and ideas of the proofs are very similar. Here we omit the details, planning to present them later.

To define Dunajski equation hierarchy, we consider three formal Laurent series in \( \lambda \), depending on two infinite sets of additional variables (‘times’)

\[
S^0 = \lambda + \sum_{n=1}^{\infty} S^0_n(t^1, t^2)\lambda^{-n},
\]

\[
S^1 = \sum_{n=0}^{\infty} t^1_n(S^0)^n + \sum_{n=1}^{\infty} S^1_n(t^1, t^2)\lambda^{-n},
\]

\[
S^2 = \sum_{n=0}^{\infty} t^2_n(S^0)^n + \sum_{n=1}^{\infty} S^2_n(t^1, t^2)\lambda^{-n},
\]

We denote \( x = t^1_0, y = t^2_0, \mathbf{S} = \begin{pmatrix} S^1 \\ S^2 \end{pmatrix}, \partial_n^1 = \frac{\partial}{\partial t^1_n}, \partial_n^2 = \frac{\partial}{\partial t^2_n} \) and introduce the projectors \((\sum_{-\infty}^{\infty} u_n z^n)^+ = \sum_{n=0}^{\infty} u_n z^n, (\sum_{-\infty}^{\infty} u_n z^n)^- = \sum_{n=-\infty}^{n=-1} u_n z^n.\)
Dunajski equation hierarchy is defined by the relation

\[(dS^0 \wedge dS^1 \wedge dS^2)_- = 0, \quad (17)\]

where the differential includes both times and a spectral variable,

\[df = \sum_{n=0}^{\infty} \partial_t^n f dt^n + \sum_{n=0}^{\infty} \partial_t^n f dt^n + \partial_\lambda f d\lambda.\]

This is a crucial difference with the heavenly equation hierarchy, where only the times are taken into account. Relation (17) plays a role similar to the role of the famous Hirota bilinear identity for KP hierarchy. This relation is equivalent to the Lax-Sato form of Dunajski equation hierarchy.

**Proposition 2** The relation (17) is equivalent to the set of equations

\[\partial_1^n S = \sum_{i=0,1,2} (J^{-1}_{1i}(S^0)^n) \partial_i S, \quad (18)\]

\[\partial_2^n S = \sum_{i=0,1,2} (J^{-1}_{2i}(S^0)^n) \partial_i S, \quad (19)\]

\[\det J = 1, \quad (20)\]

where

\[J = \begin{pmatrix} S^0_x & S^1_x & S^2_x \\ S^0_y & S^1_y & S^2_y \end{pmatrix},\]

\[\partial_0 = \partial_\lambda, \quad \partial_1 = \partial_x, \quad \partial_2 = \partial_y.\]

**Remark** Formula (20) defines a reduction for equations (18, 19). The general hierarchy in the unreduced case is given by equations (18, 19), and the analogue of relation (17) is

\[((\det J)^{-1} dS^0 \wedge dS^1 \wedge dS^2)_- = 0. \quad (22)\]

In a more explicit form, Dunajski equation hierarchy (18, 19) can be written as

\[\partial_1^n S = \begin{pmatrix} (S^0)^n \bigg| S^0_x \\ S^0_y \bigg| S^0_y \\ S^0_y \bigg| S^0_y \end{pmatrix} \partial_x S - \begin{pmatrix} (S^0)^n \bigg| S^0_x \\ S^0_y \bigg| S^0_x \\ S^0_y \bigg| S^0_x \end{pmatrix} \partial_y S + \begin{pmatrix} (S^0)^n \bigg| S^0_x \\ S^0_y \bigg| S^0_x \\ S^0_y \bigg| S^0_x \end{pmatrix} \partial_\lambda S, \quad (23)\]
\[
\partial_n^2 S = - \begin{pmatrix} (S_0^0)^n & S_0^1 \\ S_0^1 & S_0^2 \end{pmatrix} + \partial_x S + \begin{pmatrix} (S_0^0)^n & S_0^1 \\ S_0^1 & S_0^2 \end{pmatrix} + \partial_y S - \begin{pmatrix} (S_0^0)^n & S_0^1 \\ S_0^1 & S_0^2 \end{pmatrix} + \partial_\lambda S. \tag{24}
\]

It is easy to check that for \( S_0^0 = \lambda \) Dunajski equation hierarchy reduces to heavenly equation hierarchy [13, 14], while for \( S_2^2 = y \) it reduces to dispersionless KP hierarchy.

### 3.1 Waterbag-type reduction

Discussing a special solution of Dunajski equation, we have considered a reduction (16) characterized by the existence of two analytic solutions of linear equations (4). Now we will introduce waterbag-type reduction

\[
S_0^0 = \lambda + \sum_{n=1}^{N} \ln \left( \frac{\lambda - u_n^0}{\lambda - v_n^0} \right),
\]

\[
S_1^1 = \sum_{n=0}^{\infty} t_n^1 (S_0^0)^n + \sum_{n=1}^{N} \ln \left( \frac{\lambda - u_n^1}{\lambda - v_n^1} \right),
\]

\[
S_2^2 = \sum_{n=0}^{\infty} t_n^2 (S_0^0)^n + \sum_{n=1}^{N} \ln \left( \frac{\lambda - u_n^2}{\lambda - v_n^2} \right),
\]

where the functions \( u, v \) depend only on times of the hierarchy. This anzats is consistent with equations (18,19,20) and defines (1+1)-dimensional reduction of Dunajski equation hierarchy. The first equation of this hierarchy, containing only the variables \( x, y \), is obtained by the substitution of the anzatz to relation (20). The expression for the determinant will have 6\( N \) simple poles, and relation (20) will give a closed system of 6\( N \) equations for 6\( N \) functions \( u, v \).

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