Nonconvex Factorization and Manifold Formulations are Almost Equivalent in Low-rank Matrix Optimization

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Abstract

In this paper, we consider the geometric landscape connection of the widely studied manifold and factorization formulations in low-rank positive semidefinite (PSD) and general matrix optimization. We establish a sandwich relation on the spectrum of Riemannian and Euclidean Hessians at first-order stationary points (FOSPs). As a result of that, we obtain an equivalence on the set of FOSPs, second-order stationary points (SOSPs), and strict saddles between the manifold and the factorization formulations. In addition, we show the sandwich relation can be used to transfer more quantitative geometric properties from one formulation to another. Similarities and differences in the landscape connection under the PSD case and the general case are discussed. To the best of our knowledge, this is the first geometric landscape connection between the manifold and the factorization formulations for handling rank constraints, and it provides a geometric explanation for the similar empirical performance of factorization and manifold approaches in low-rank matrix optimization observed in the literature. In the general low-rank matrix optimization, the landscape connection of two factorization formulations (unregularized and regularized ones) is also provided. By applying these geometric landscape connections, in particular, the sandwich relation, we are able to solve unanswered questions in literature and establish stronger results in the applications on geometric analysis of phase retrieval, well-conditioned low-rank matrix optimization, and the role of regularization in factorization arising from machine learning and signal processing.

1 Introduction

Low-rank optimization problems are ubiquitous in a variety of fields, such as optimization, machine learning, signal processing, scientific computation, and statistics. One popular formulation is the following rank constrained optimization:

\textbf{PSD case :} \[ \min_{X \in \mathbb{S}^{p \times p} \succeq 0, \text{rank}(X) = r} f(X), \quad 0 < r \leq p, \] \hspace{1cm} (1)

\textbf{general case :} \[ \min_{X \in \mathbb{R}^{p_1 \times p_2}, \text{rank}(X) = r} f(X), \quad 0 < r \leq \min\{p_1, p_2\}. \] \hspace{1cm} (2)

In the positive semi-definite (PSD) case, without loss of generality, we assume \( f \) is symmetric in \( X \), i.e., \( f(X) = f(X^\top) \); otherwise, we can set \( \tilde{f}(X) = \frac{1}{2}(f(X) + f(X^\top)) \) and have \( \tilde{f}(X) = f(X) \) for all \( X \succeq 0 \) [Bhojanapalli et al, 2016a]. In both cases, we assume \( f \) is twice continuously differentiable.
differentiable with respect to $X$ and the Euclidean metric. Viewed as optimization problems over low-rank matrix manifolds under the embedded geometry (Absil et al., 2009; Boumal, 2020), (1) and (2) can be solved via various manifold optimization methods. On the other hand, to accelerate the computation and to better cope with the rank constraint, a line of research studied the following nonconvex factorization formulation (Burer and Monteiro, 2005):

$$\text{PSD case : } \min_{Y \in \mathbb{R}^{p \times r}} g(Y) := f(YY^\top),$$

$$\text{general case : } \min_{L \in \mathbb{R}^{p_1 \times r}, R \in \mathbb{R}^{p_2 \times r}} g(L, R) := f(LR^\top).$$

In the general asymmetric case, to promote balance between two factors $L$ and $R$ in (4), the following regularized optimization problem has also been widely studied (Tu et al., 2016):

$$\min_{L \in \mathbb{R}^{p_1 \times r}, R \in \mathbb{R}^{p_2 \times r}} g_{\text{reg}}(L, R) := f(LR^\top) + \frac{\mu}{2} \| L^\top L - R^\top R \|^2_F,$$

where $\mu > 0$ is some properly chosen regularization parameter. Note that (3), (4), and (5) are unconstrained, and thus can be tackled by running unconstrained optimization algorithms. Indeed, under proper assumptions, a number of algorithms with theoretical guarantees have been proposed for both the manifold and the factorization formulations (Chi et al., 2019; Cai and Wei, 2018a). See Section 1.2 for a review of existing results.

On the other hand, the manifold and the factorization formulations are more or less treated as two different approaches for low-rank matrix optimization in the literature and they are not obviously related. Similar algorithmic guarantees under these two formulations, including convergence rate and sample complexity for successful recovery, were observed in a number of matrix inverse problems (Wei et al., 2016; Luo et al., 2020; Cai and Wei, 2018b; Zhang and Yang, 2018; Keshavan et al., 2009; Ma et al., 2019; Cai and Zhang, 2015; Chen and Wainwright, 2015; Hardt, 2014; Zhao et al., 2015; Zheng and Lafferty, 2015; Wang et al., 2017b; Tong et al., 2020a), while there are little studies on the reason behind. Moreover, most of the existing geometric analyses in low-rank matrix optimization are performed under the factorization formulation (Bhojanapalli et al., 2016b; Ge et al., 2017; Zhang et al., 2019; Zhu et al., 2018; 2021; Li et al., 2019a; Park et al., 2017). It has been asked by Cai and Wei (2018b); Li et al. (2019d) whether it is possible to investigate the geometric landscape directly on the low-rank matrix manifolds as the manifold formulation avoids unidentifiable parameterizations of low-rank matrices and explicit regularizations to cope with the unbalanced factorization in (5). In this work, we make the first attempt to answer these questions by investigating the geometric landscape connections between the manifold and the factorization formulations in low-rank matrix optimization.

1.1 Our Contributions

First, we establish a sandwich relation on the spectrum between Riemannian and Euclidean Hessians at FOSPs under the manifold and the factorization formulations. In particular, sandwich inequalities between Riemannian and Euclidean Hessians are established for (1), (3) and (2), (4); a partial sandwich inequality is built between (2) and (5). As an immediate corollary, we obtain an equivalence on the set of first-order stationary points (FOSPs), second-order stationary points (SOSPs) and strict saddles between the manifold formulation under the embedded geometry and the factorization formulation in both the PSD and the general low-rank matrix optimization. In addition, we demonstrate the sandwich relation is useful in transferring more geometric landscape properties, such as the strict saddle property, from one formulation to another. To the best of
our knowledge, this is the first equivalence geometric landscape connection between the manifold and the factorization formulations for low-rank matrix optimization. Key technical ingredients to establish these results include a characterization of the zero eigenspace of the Hessian of the factorization objective and a bijection between its orthogonal complement and the tangent space of the fixed-rank $r$ manifold at the reference point. In addition, a few similarities and key differences in the landscape connection under the PSD case and the general case are identified.

We also provide a geometric landscape connection between the unregularized and the regularized factorization formulations ((4) and (5)) and give a sandwich inequality on the spectrum of Euclidean Hessians of two factorization formulations at rank $r$ FOSPs.

Furthermore, we apply our main results in three applications from machine learning and signal processing. By our geometric landscape connections between the manifold and the factorization formulations, we provide the first global optimality result for phase retrieval with a rate-optimal sample complexity under the manifold formulation and specifically show there is a unique Riemannian SOSP that is the global optima and all other Riemannian FOSPs are strict saddles with an explicit upper bound on the negative eigenvalue. We also prove the global optimality result for generic well-conditioned low-rank matrix optimization under the manifold formulation in both exact-parameterization and over-parameterization settings. Finally, we provide a geometric analysis on the role of regularization in the factorization formulation for a general $f$; when $f$ is further well-conditioned, we give a global optimality result under the formulation (4). All of these results rely critically on the sandwich inequalities we establish between the Riemannian and Euclidean Hessians under the manifold and the factorization formulations.

In a broad sense, manifold and factorization can be treated as two different approaches in handling the rank constraint in optimization problems. This paper bridges them from a geometric point of view and demonstrates that the manifold and the factorization approaches are indeed strongly connected in solving low-rank matrix optimization problems.

1.2 Related Literature

This work is related to a range of literature on low-rank matrix optimization, manifold/nonconvex optimization, and geometric landscape analysis arising from a number of communities, such as optimization, machine learning and signal processing.

First, from an algorithmic perspective, a number of algorithms, including the penalty approaches, gradient descent, alternating minimization, and Gauss-Newton, have been developed either for solving the manifold formulation (Bi et al., 2020; Gao and Sun, 2010; Boumal and Absil, 2011; Mishra et al., 2014; Meyer et al., 2011; Mishra et al., 2014; Vandereycken, 2013; Huang and Hand, 2018; Luo et al., 2020) or the factorization formulation (Candès et al., 2015; Jain et al., 2013; Sun and Luo, 2015; Tran-Dinh, 2021; Tu et al., 2016; Wen et al., 2012; Bauch et al., 2021). We refer readers to Chi et al. (2019); Cai and Wei (2018a) for the recent algorithmic development under two formulations. Many algorithms developed under the manifold formulation involve Riemannian optimization techniques and can be more complex than the ones developed under the factorization formulation. On the other hand, similar guarantees were observed for both lines of algorithms under two formulations in various matrix inverse problems (Miao et al., 2016; Wei et al., 2016; Luo et al., 2020; Hou et al., 2020; Cai and Wei, 2018b; Zhang and Yang, 2018; Keshavan et al., 2009; Bhojanapalli et al., 2016a; Park et al., 2018; Li et al., 2019b; Ma et al., 2019; Sanghavi et al., 2017; Chen and Wainwright, 2015; Hardt, 2014; Zhao et al., 2015; Zheng and Lafferty, 2015; Wang et al., 2017b; Tong et al., 2020a). Our results on the geometric connection between two formulations shed light on this phenomenon by showing that these two approaches are in essence closely related.

Second, from a geometric landscape perspective, a body of work showed that factorization
will not introduce spurious local minima compared to the original rank constrained optimization problem when the objective \( f \) is well-conditioned (Bhojanapalli et al., 2016b; Ge et al., 2017; Zhang et al., 2019; Zhu et al., 2018; Chen and Li, 2019; Park et al., 2017; Zhang et al., 2018, 2021). Similar benign landscape results were proved for factorization in solving semidefinite programs and convex programs on PSD matrices or with a nuclear norm regularization (Boumal et al., 2020; Journée et al., 2010; Yamakawa et al., 2021; Li et al., 2019a). On the other hand, it is much less explored for the geometric analysis under the manifold formulation. Maunu et al. (2019); Ahn and Suarez (2021) provided landscape analyses for robust subspace recovery and matrix factorization over the Grassmannian manifold. Under the embedded manifold, Uschmajew and Vandereycken (2020) showed the benign landscape of (2) when \( f \) is quadratic and satisfies certain restricted spectral bounds properties. Different from both lines of work focusing on the landscape under either the factorization or the manifold formulation when \( f \) is well-conditioned, i.e., \( f \) satisfies the restricted strong convexity and smoothness or restricted spectral bounds properties, here we study the geometric landscape connection between the factorization and the manifold formulations in low-rank matrix optimization for a general \( f \).

The closest work in the literature related to ours is Ha et al. (2020), where they study the relationship between Euclidean FOSPs and SOSPs under the factorization formulation and fixed points of the projected gradient descent (PGD) in the general low-rank matrix optimization. They show while the sets of FOSPs of (4) and (5) can be larger, the sets of SOSPs of these two factorization formulations are contained in the set of fixed points of the PGD with a small stepsize. Complementary to their results, here we consider the geometric landscape connection between the manifold and the factorization formulations under both the PSD and the general low-rank matrix optimization and establish a stronger equivalence on sets of FOSPs as well as SOSPs of manifold and factorization formulations.

### 1.3 Organization of the Paper

The rest of this article is organized as follows. After a brief introduction of notation, we introduce Riemannian optimization and some preliminary results on the Riemannian geometry of low-rank matrices in Section 2.1. Our main results on the geometric landscape connection between the manifold and the factorization formulations in low-rank PSD and general matrix optimization are presented in Sections 3 and 4, respectively. In Section 5, we present three applications of our main results in machine learning and signal processing. Conclusion and future work are given in Section 6. We present the proofs of the main results in the main text and additional proofs and lemmas are presented in Appendices A and B, respectively.

### 2 Notation and Preliminaries

The following notation will be used throughout this article. \( \mathbb{R}^{p_1 \times p_2} \) and \( \mathbb{S}^{p \times p} \) denote the spaces of \( p_1 \)-by-\( p_2 \) real matrices and \( p \)-by-\( p \) real symmetric matrices, respectively. Uppercase and lowercase letters (e.g., \( A, B, a, b \)), lowercase boldface letters (e.g., \( \mathbf{u}, \mathbf{v} \)), uppercase boldface letters (e.g., \( \mathbf{U}, \mathbf{V} \)) are used to denote scalars, vectors, matrices, respectively. We denote \([p_k] \) as the set \( \{1, \ldots, p_k\} \). For any \( a, b \in \mathbb{R} \), let \( a \wedge b := \min\{a, b\} \), \( a \vee b := \max\{a, b\} \). For any vector \( \mathbf{v} \), denote its \( \ell_1 \) and \( \ell_2 \) norms as \( \|\mathbf{v}\|_1 \) and \( \|\mathbf{v}\|_2 \), respectively. For any matrix \( \mathbf{X} \in \mathbb{R}^{p_1 \times p_2} \) with singular value decomposition (SVD) \( \sum_{i=1}^{\min\{p_1, p_2\}} \sigma_i(\mathbf{X}) \mathbf{u}_i \mathbf{v}_i^\top \), where \( \sigma_1(\mathbf{X}) \geq \sigma_2(\mathbf{X}) \geq \cdots \geq \sigma_{\min\{p_1, p_2\}}(\mathbf{X}) \), denote \( \|\mathbf{X}\|_F = \sqrt{\sum \sigma_i^2(\mathbf{X})} \) and \( \|\mathbf{X}\| = \sigma_1(\mathbf{X}) \) as its Frobenius norm and spectral norm, respectively. Also, we use \( \mathbf{X}^{-1}, \mathbf{X}^{-\top} \) and \( \mathbf{X}^\dagger \) to denote the inverse, transpose inverse, and Moore-Penrose inverse of \( \mathbf{X} \), respectively.
For any real symmetric matrix $X \in \mathbb{S}^{p \times p}$ having eigendecomposition $U \Sigma U^T$ with non-increasing eigenvalues on the diagonal of $\Sigma$, let $\lambda_i(X)$ be the $i$th largest eigenvalue of $X$, $\lambda_{\min}(X)$ be the least eigenvalue of $X$ and $X^{1/2} = U \Sigma^{1/2} U^T$. We say a symmetric matrix $X$ is positive semidefinite (PSD) and denote $X \succeq 0$ if and only if (iff) for any vector $y \in \mathbb{R}^p$, $y^T X y \geq 0$. For two symmetric matrices $X, Y$, we say $X \succeq Y$ iff $X - Y \succeq 0$. Throughout the paper, the SVD or eigendecomposition of a rank $r$ matrix $X$ refers to its economic version. We use bracket subscripts to denote sub-matrices. For example, $X_{[i_1:i_2,j_1:j_2]}$ is the entry of $X$ on the $i_1$-th row and $i_2$-th column; $X_{[(r+1):p_1],:}$ contains the $(r+1)$-th to the $p_1$-th rows of $X$. In addition, $I_r$ is the $r$-by-$r$ identity matrix and $I$ denotes an identity operator. Let $\mathcal{O}_{p,r} = \{ U \in \mathbb{R}^{p \times r} : U^T U = I_r \}$ be the set of all $p$-by-$r$ matrices with orthonormal columns and $\mathcal{O}_r := \mathcal{O}_{r,r}$. For any $U \in \mathcal{O}_{p,r}$, $P_U = U U^T$ represents the orthogonal projector onto the column space of $U$; we also note $U_\perp \in \mathcal{O}_{p,p-r}$ as the orthonormal complement of $U$. For any linear operator $L$, we denote $L^*$ as its adjoint operator. Finally, for a linear space $V$, we denote its dimension as $\dim(V)$. For two linear spaces $V_1, V_2$, the sum of $V_1$ and $V_2$ is denoted by $V_1 + V_2 := \{ v_1 + v_2 | v_1 \in V_1, v_2 \in V_2 \}$. If every vector in $V_1 + V_2$ can be uniquely decomposed into $v_1 + v_2$, where $v_1 \in V_1, v_2 \in V_2$, then we call the sum of $V_1$ and $V_2$ as the direct sum, denoted by $V_1 \oplus V_2$, and have $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$. For two Euclidean spaces $V_1$ and $V_2$, we say $V_1$ is orthogonal to $V_2$ and denote by $V_1 \perp V_2$ iff $\langle v_1, v_2 \rangle = 0$ for any $v_1 \in V_1, v_2 \in V_2$.

Given differentiable scalar and matrix-valued functions $f : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R}$ and $\phi : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R}^{q_1 \times q_2}$. The Euclidean gradient of $f$ at $X$ is denoted as $\nabla f(X)$ and $(\nabla f(X))_{[i,j]} = \frac{\partial f(X)}{\partial X_{[i,j]}}$ for $i \in [p_1], j \in [p_2]$. The Euclidean gradient of $\phi$ is a linear operator from $\mathbb{R}^{p_1 \times p_2}$ to $\mathbb{R}^{q_1 \times q_2}$ defined as $(\nabla \phi(X)[Z])_{[i,j]} = \sum_{k \in [p_1], l \in [p_2]} \frac{\partial \phi(X)}{\partial X_{[k,l]}} Z_{[k,l]}$ for any $Z \in \mathbb{R}^{p_1 \times p_2}, i \in [q_1], j \in [q_2]$. Using this notation, given a twice continuously differentiable scalar function $f : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R}$, we denote its Euclidean Hessian by $\nabla^2 f(X)[,]$, which is the gradient of $\nabla f(X)$ and can be viewed as a linear operator from $\mathbb{R}^{p_1 \times p_2}$ to $\mathbb{R}^{p_1 \times p_2}$ satisfying

$$
(\nabla^2 f(X)[Z])_{[i,j]} = \sum_{k \in [p_1], l \in [p_2]} \frac{\partial (\nabla f(X))_{[i,j]}}{\partial X_{[k,l]}} Z_{[k,l]} = \sum_{k \in [p_1], l \in [p_2]} \frac{\partial^2 f(X)}{\partial X_{[k,l]} \partial X_{[i,j]}} Z_{[k,l]}.
$$

We also define the bilinear form for the Hessian of $f$ as $\nabla^2 f(X)[Z_1, Z_2] := \langle \nabla^2 f(X)[Z_1], Z_2 \rangle$ for any $Z_1, Z_2 \in \mathbb{R}^{p_1 \times p_2}$. Apart from the matrix representation of $\nabla^2 f(X)$ above, we can also view $\nabla^2 f(X)$ as a $(p_1p_2)$-by-$(p_1p_2)$ symmetric matrix and define its spectrum in the classic way. We say $X$ is a Euclidean first-order stationary point (FOSP) of $f$ iff $\nabla f(X) = 0$ and a Euclidean second-order stationary point (SOSP) of $f$ iff $\nabla^2 f(X) \succeq 0$ and $\nabla^2 f(X) \succeq 0$. Finally, we say a pair of matrices $(L, R) \in \mathbb{R}^{p_1 \times r} \times \mathbb{R}^{p_2 \times r}$ is of rank $r$ if $LR^T$ has rank $r$.

### 2.1 Riemannian Optimization and Riemannian Geometry of Low-rank PSD and General Matrices

In this section, we first give a brief introduction to Riemannian optimization and then present the necessary preliminaries to perform Riemannian optimization on $\mathbb{S}^n$ and $\mathbb{R}^{n \times n}$. Finally, we provide the Euclidean/Riemannian gradient and Hessian expressions for the optimization problems (1)-(5) considered in this paper.

Riemannian optimization concerns optimizing a real-valued function $f$ defined on a Riemannian manifold $\mathcal{M}$, for which the readers are referred to Absil et al. (2009); Boumal (2020); Hu et al. (2020) for more details. Algorithms for continuous optimization over the Riemannian manifold often require calculations of Riemannian gradients and Riemannian Hessians. Suppose $X \in \mathcal{M}$ and the Riemannian metric and tangent space of $\mathcal{M}$ at $X$ are $\langle \cdot, \cdot \rangle_X$ and $T_X \mathcal{M}$, respectively. Then the Riemannian gradient of a smooth function $f : \mathcal{M} \rightarrow \mathbb{R}$ at $X$ is defined as the unique tangent vector
grad $f(X) \in T_X \mathcal{M}$ such that $\langle \nabla f(X), Z \rangle_X = Df(X)[Z]$, $\forall Z \in T_X \mathcal{M}$, where $Df(X)[Z]$ denotes the directional derivative of $f$ at point $X$ along the direction $Z$. The Riemannian Hessian of $f$ at $X \in \mathcal{M}$ is the linear map $\text{Hess}_f(X)$ of $T_X \mathcal{M}$ onto itself defined as

$$\text{Hess}_f(X)[Z] = \nabla_Z \nabla f, \quad \forall Z \in T_X \mathcal{M},$$

(6)

where $\nabla$ is the Riemannian connection on $\mathcal{M}$ and can be viewed as the generalization of the directional derivative on the manifold (Abisil et al. 2009 Section 5.3). The bilinear form of Riemannian Hessian is defined as $\text{Hess}_f(X)[Z_1, Z_2] := \langle \text{Hess}_f(X)[Z_1], Z_2 \rangle_X$ for any $Z_1, Z_2 \in T_X \mathcal{M}$. We say $X \in \mathcal{M}$ is a Riemannian FOSP of $f$ iff $\text{grad}_f(X) = 0$ and a Riemannian SOSP of $f$ iff $\text{grad}_f(X) = 0$ and $\text{Hess}_f(X) \succeq 0$. Moreover, we call a Riemannian or Euclidean FOSP a strict saddle if and only if the Riemannian or Euclidean Hessian evaluated at this point has a strict negative eigenvalue.

For optimization problems in [1] and [2], two manifolds of particular interest are the set of rank $r$ PSD matrices $\mathcal{M}_{r+} := \{ X \in \mathbb{S}^{p \times p} | \text{rank}(X) = r, X \succeq 0 \}$ and the set of rank $r$ matrices $\mathcal{M}_r := \{ X \in \mathbb{R}^{p_1 \times p_2} | \text{rank}(X) = r \}$. Lee (2013), Helmke and Moore (2012), Vandereycken and Vandewalle (2010) showed that $\mathcal{M}_{r+}$ and $\mathcal{M}_r$ are smooth embedded submanifolds of $\mathbb{R}^{p \times p}$ and $\mathbb{R}^{p_1 \times p_2}$, respectively and established their Riemannian geometry as follows.

**Lemma 1.** ([Helmke and Moore] 2012 Chapter 5), ([Vandereycken and Vandewalle] 2010 Proposition 5.2), ([Lee] 2013 Example 8.14)) $\mathcal{M}_{r+}, \mathcal{M}_r$ are smooth embedded submanifolds of $\mathbb{R}^{p \times p}$ and $\mathbb{R}^{p_1 \times p_2}$ with dimensions $(pr - r(r - 1)/2)$ and $(p_1 r + p_2 - r)$, respectively. The tangent space $T_X \mathcal{M}_{r+}$ at $X \in \mathcal{M}_{r+}$ with the eigendecomposition $X = U \Sigma U^\top$ is given by

$$T_X \mathcal{M}_{r+} = \left\{ [U \quad U_{\perp}] \begin{bmatrix} S & D \end{bmatrix}^\top : S \in \mathbb{S}^{r \times r}, D \in \mathbb{R}^{(p-r) \times r} \right\}.$$  

(7)

The tangent space $T_X \mathcal{M}_r$ at $X \in \mathcal{M}_r$ with SVD $X = U \Sigma V^\top$ is given by

$$T_X \mathcal{M}_r = \left\{ [U \quad U_{\perp}] \begin{bmatrix} S & D_1 \end{bmatrix}^\top : S \in \mathbb{R}^{r \times r}, D_1 \in \mathbb{R}^{(p_1 - r) \times r}, D_2 \in \mathbb{R}^{(p_2 - r) \times r} \right\}.$$  

(8)

Throughout the paper, we equip the Riemannian manifolds $\mathcal{M}_{r+}$ and $\mathcal{M}_r$ with the metric induced by the Euclidean inner product, i.e., $\langle U, V \rangle = \text{trace}(U^\top V)$. Given $X \in \mathcal{M}_{r+}$ with eigendecomposition $U \Sigma U^\top$ or $X \in \mathcal{M}_r$ with SVD $U \Sigma V^\top$, the orthogonal projectors $P_{T_X \mathcal{M}_{r+}}(\cdot)$ and $P_{T_X \mathcal{M}_r}(\cdot)$, which project any matrix onto $T_X \mathcal{M}_{r+}$ and $T_X \mathcal{M}_r$, are given as follows

$$P_{T_X \mathcal{M}_{r+}}(Z) = P_U Z P_U + P_{U_{\perp}} Z P_{U_{\perp}} + P_{U} Z P_{U_{\perp}}, \quad \forall Z \in \mathbb{S}^{p \times p},$$

$$P_{T_X \mathcal{M}_r}(Z) = P_U Z P_V + P_{U_{\perp}} Z P_{V_{\perp}} + P_{U} Z P_{V_{\perp}}, \quad \forall Z \in \mathbb{R}^{p_1 \times p_2}.\quad (9)$$

Next, we collectively give the expressions for gradients and Hessians under both manifold formulations [1] and [2] and factorization formulations [3], [4] and [5] in Propositions 1 and 2 respectively. The proof is postponed to Appendix.

**Proposition 1** (Riemannian and Euclidean Gradients). The Riemannian and Euclidean gradients under the manifold and the factorization formulations are:

- **PSD case:**

$$\nabla g(Y) = 2 \nabla f(Y Y^\top) Y.$$

Here $U \in \mathbb{O}_{p,r}$ is formed by the top $r$ eigenvectors of $X$. 

$$\nabla g(Y) = 2 \nabla f(Y Y^\top) Y.$$
that throughout the paper. It is relatively easy to obtain the general bilinear expressions by noting

the results in PSD case are easier to follow and also convey most of the essential messages in the
covariance sensing (Chen et al., 2015; Cai and Zhang, 2015) and is interesting on its own; third,
low-rank PSD optimization appears in many real applications such as phase retrieval (Fienup, 1982), rank-1
the PSD case from the general case and give it a detailed discussion for a few reasons: first, the

In this section, we present the geometric landscape connections between the manifold formulation

Proposition 2 (Riemannian and Euclidean Hessians). The Riemannian and Euclidean Hessians
under the manifold and the factorization formulations are:

• PSD case: Suppose $X \in M_{r+}$ has eigendecomposition $U \Sigma U^T$, $\xi = [U \ U_\bot] \begin{bmatrix} S & D \end{bmatrix}^T [U \ U_\bot]^T \in T_X M_{r+}$ and $A \in \mathbb{R}^{p \times r}$. Then

\[
\text{Hess} f(X)[\xi, \xi] = \nabla^2 f(X)[\xi, \xi] + 2\langle \nabla f(X), U_\bot D \Sigma^{-1} D^T U_\bot \rangle,
\]

\[
\nabla^2 g(Y)[A, A] = \nabla^2 f(Y Y^T)[YA^T + AY^T, YA^T + AY^T] + 2\langle \nabla f(Y Y^T), AA^T \rangle.
\]

• General case: Suppose $X \in M_r$ has SVD $U \Sigma V^T$, $\xi = [U \ U_\bot] \begin{bmatrix} S & D \end{bmatrix}^T [V \ V_\bot]^T \in T_X M_r$
and $A = [A_L^T \ A_R^T]^T$ with $A_L \in \mathbb{R}^{p_1 \times r}$, $A_R \in \mathbb{R}^{p_2 \times r}$. Then

\[
\text{Hess} f(X)[\xi, \xi] = \nabla^2 f(X)[\xi, \xi] + 2\langle \nabla f(X), U_\bot D_1 \Sigma^{-1} D_2^T V_\bot \rangle,
\]

\[
\nabla^2 g(L, R)[A, A] = \nabla^2 f(L R^T)[LA_R + A_L R^T, LA_R + A_L R^T] + 2\langle \nabla f(L R^T), A_L A_R \rangle,
\]

\[
\nabla^2 g_{\text{reg}}(L, R)[A, A] = \nabla^2 g(L, R)[A, A] + \mu \|LA_L + A_L L - R^T A_R - A_R^T R\|_F^2
\]

\[
+ 2\mu \langle L^T L - R^T R, A_L^T A_L - A_R^T A_R \rangle.
\]

In Proposition 2 we give the quadratic expressions of the Hessians as we use them exclusively throughout the paper. It is relatively easy to obtain the general bilinear expressions by noting that \( \nabla^2 g(Y)[A, B] = (\nabla^2 g(Y)[A + B, A + B] - \nabla^2 g(Y)[A - B, A - B]) / 4 \) and similarly for the Riemannian Hessian.

3 Geometric Connection of Manifold and Factorization Formulations: PSD Case

In this section, we present the geometric landscape connections between the manifold formulation
[1] and the factorization formulation [2] in low-rank PSD matrix optimization. Here we single out the
PSD case from the general case and give it a detailed discussion for a few reasons: first, the
low-rank PSD matrix manifold is different from the low-rank matrix manifold; second, low-rank
PSD optimization appears in many real applications such as phase retrieval (Fienup, 1982), rank-1
covariance sensing (Chen et al., 2015; Cai and Zhang, 2015) and is interesting on its own; third,
the results in PSD case are easier to follow and also convey most of the essential messages in the
general setting.
We begin with a few more definitions. Suppose $Y \in \mathbb{R}^{p \times r}$ is of rank $r$, $X = YY^T$ has eigendecomposition $U \Sigma U^T$, and $P = U^T Y$. For any $A \in \mathbb{R}^{p \times r}$, define

$$\xi_A^Y := YA^T + AY^T = [U \ U_\perp] \begin{bmatrix} PA^T U + U^T A P^T & PA^T U_\perp \\ U^T A P^T & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^T \in T_X M_{r+}. \quad (12)$$

For any $\xi = [U \ U_\perp] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^T \in T_X M_{r+}$ with $S \in \mathbb{S}^{r \times r}$, define

$$\mathcal{A}_Y^\xi := \begin{bmatrix} A : YA^T + AY^T = \xi \end{bmatrix}. \quad (13)$$

In the following Lemma 2, we quickly check $\mathcal{A}_Y^\xi$ is nonempty and forms an $(r^2 - r)/2$ dimensional subspace.

**Lemma 2.** Suppose $Y \in \mathbb{R}^{p \times r}$ is of rank $r$, $X = YY^T$ has eigendecomposition $U \Sigma U^T$, and $P = U^T Y$. Given any $\xi \in T_X M_{r+}$, it holds $\mathcal{A}_Y^\xi := \{ A : A = (US_1 + U_\perp D)P^{-T} \in \mathbb{R}^{p \times r}, S_1 + S_1^T = S \}$. The motivation behind the constructions of $\xi_A^Y$ and $\mathcal{A}_Y^\xi$ is to find a correspondence between $\mathbb{R}^{p \times r}$ and $T_X M_{r+}$. Later, this correspondence will be used to establish the connections of Riemannian and Euclidean Hessians in Theorem 1.

From Lemma 2, we can see there is no one-to-one correspondence between $\mathbb{R}^{p \times r}$ and $T_X M_{r+}$ due to their mismatched dimensions, and in particular given $Y$, $A$ and $\xi$, $\xi_A^Y$ is a single matrix while there exists an $(r^2 - r)/2$ dimensional subspace $\mathcal{A}_Y^\xi$ such that $YA^T + A' Y^T = \xi$ for any $A' \in \mathcal{A}_Y^\xi$. To handle the mismatch, we introduce the following decomposition of $\mathbb{R}^{p \times r}$:

**Lemma 3.** Suppose the conditions in Lemma 2 hold. Then $\mathbb{R}^{p \times r} = \mathcal{A}_Y^\xi \oplus \mathcal{A}_Y^{\xi \text{null}}$ for

$$\mathcal{A}_Y^{\xi \text{null}} = \{ A : A = USP^{-T}, S + S^T = 0 \in \mathbb{R}^{p \times r} \},$$

$$\mathcal{A}_Y^\xi = \{ A : A = (US + U_\perp D)P^{-T}, D \in \mathbb{R}^{(p-r) \times r}, S \Sigma^{-1} \in \mathbb{S}^{r \times r} \}. \quad (14)$$

Moreover, $\dim(\mathcal{A}_Y^{\xi \text{null}}) = (r^2 - r)/2$, $\dim(\mathcal{A}_Y^\xi) = pr - (r^2 - r)/2$, and $\mathcal{A}_Y^{\xi \text{null}}$ is orthogonal to $\mathcal{A}_Y^\xi$, i.e., $\mathcal{A}_Y^{\xi \text{null}} \perp \mathcal{A}_Y^\xi$.

By decomposing $\mathbb{R}^{p \times r}$ into $\mathcal{A}_Y^{\xi \text{null}}$ and $\mathcal{A}_Y^\xi$, it is easy to check $\xi_A^Y = 0$ if and only if $A \in \mathcal{A}_Y^{\xi \text{null}}$. In the following Proposition 3, we show $\mathcal{A}_Y^\xi$ can be decomposed as the direct sum of $\mathcal{A}_Y^{\xi \text{null}}$ and a singleton from $\mathcal{A}_Y^{\xi \text{null}}$. In addition, there is a bijective linear map, $L_Y$, between $\mathcal{A}_Y^{\xi \text{null}}$ and $T_X M_{r+}$ and $\mathcal{A}_Y^{\xi \text{null}}$ is the null space of $L_Y$. A pictorial illustration of the relationship of these subspaces is given in Figure 1.

**Proposition 3 (Decomposition of $\mathcal{A}_Y^\xi$ and Bijection Between $\mathcal{A}_Y^{\xi \text{null}}$ and $T_X M_{r+}$).** Suppose the conditions in Lemma 2 hold and $\xi = [U \ U_\perp] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^T \in T_X M_{r+}$. Then $\mathcal{A}_Y^\xi = A_Y^\xi \oplus \mathcal{A}_Y^{\xi \text{null}}$, where $A_Y^\xi = (US + U_\perp D)P^{-T} \in \mathcal{A}_Y^{\xi \text{null}}$ and $S$ is the unique solution of the linear equation system $S \Sigma^{-1} = \Sigma^{-1} S^T$ and $\bar{S} + \bar{S}^T = S$.

Moreover, there is a bijective linear map $L_Y$ between $\mathcal{A}_Y^{\xi \text{null}}$ and $T_X M_{r+}$ given as follows

$$L_Y : A \in \mathcal{A}_Y^{\xi \text{null}} \rightarrow \xi_A^Y \in T_X M_{r+} \quad \text{and} \quad L_Y^{-1} : \xi \in T_X M_{r+} \rightarrow A_Y^\xi \in \mathcal{A}_Y^{\xi \text{null}}.$$  

Finally, recall $X = YY^T$, we have the following spectrum bounds for $L_Y$:

$$2\sigma_r(X)\|A\|_F^2 \leq \|L_Y(A)\|_F^2 \leq 4\sigma_1(X)\|A\|_F^2, \quad \forall A \in \mathcal{A}_Y^{\xi \text{null}}.$$  

(14)
Proof of Proposition 3. We divide the proof into two steps: in Step 1, we prove the decomposition for \( \mathcal{A}_Y^\xi \); in Step 2, we show \( \mathcal{L}_Y \) is a bijection and prove the spectrum bounds.

**Step 1.** Noting that \( \Sigma = \{ \{ \Sigma_{[i,j]} \}_{j=1}^{r} \} \) is a diagonal matrix, the linear equation system \( \Sigma \Sigma^{-1} = \Sigma^{-1} \tilde{S}^T \), \( \tilde{S} + \tilde{S}^T = S \) is equivalent to
\[
\tilde{S}_{[i,j]} \Sigma_{[i,j]} = \Sigma_{[i,i]} \tilde{S}_{[i,j]}, \quad \tilde{S}_{[i,j]} + \tilde{S}_{[j,i]} = S_{[i,j]}, \quad 1 \leq i, j \leq r.
\]
Here, we use the fact that \( S \) is symmetric, i.e., \( S_{[i,j]} = S_{[j,i]} \). By calculations, we know the equation system above is further equivalent to
\[
S_{[i,j]} = S_{[i,j]} \Sigma_{[i,j]} / (\Sigma_{[i,i]} + \Sigma_{[j,j]}), \quad \tilde{S}_{[i,j]} = S_{[i,j]} \Sigma_{[i,j]} / (\Sigma_{[i,i]} + \Sigma_{[j,j]}), \quad 1 \leq i, j \leq r.
\]
Namely, \( \Sigma \Sigma^{-1} = \Sigma^{-1} \tilde{S}^T \), \( \tilde{S} + \tilde{S}^T = S \) has the unique solution as presented in (15). Therefore, \( A_Y^\xi \) is well-defined for any \( \xi \in T_X \mathcal{M}_{r+} \).

At the same time, given \( A = (US + U \perp D)P^{-T} \in \mathcal{A}_Y^\xi \), we can check \( A - A_Y^\xi \in \mathcal{A}_Y \). In addition, \( A_Y^\xi + A \in \mathcal{A}_Y \) for any \( A \in \mathcal{A}_Y \). This shows \( \mathcal{A}_Y^\xi = \mathcal{A}_Y^\xi + \mathcal{A}_Y \).

**Step 2.** Notice both \( \mathcal{A}_Y \) and \( T_X \mathcal{M}_{r+} \) are of dimension \( (p r - (r^2 - r)/2) \). Suppose \( \mathcal{L}_Y : \xi \in T_X \mathcal{M}_{r+} \rightarrow A_Y^\xi \in \mathcal{A}_Y \). For any \( \xi = [U \ U \perp] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} [U \ U \perp]^T \in T_X \mathcal{M}_{r+} \), we have
\[
\mathcal{L}_Y(\mathcal{L}_Y(\xi)) = \mathcal{L}_Y(A_Y^\xi) = [U \ U \perp] \begin{bmatrix} PA_Y^T U + U^T A_Y^\xi P^T & PA_Y^T U \ U \perp A_Y^\xi P^T \end{bmatrix} [U \ U \perp]^T = \xi.
\]

Since \( \mathcal{L}_Y \) and \( \mathcal{L}_Y' \) are linear maps, (16) implies \( \mathcal{L}_Y \) is a bijection and \( \mathcal{L}_Y' = \mathcal{L}_Y^{-1} \).

Next, we provide the spectrum bounds for \( \mathcal{L}_Y \). Suppose \( A = (US + U \perp D)P^{-T} \in \mathcal{A}_Y \). Then,
\[
\|A\|_F^2 \leq (\|S\|_F^2 + \|D\|_F^2)\sigma_1^2(\Sigma^{-1} P^{-T})^{PP^T = \Sigma} (\|S\|_F^2 + \|D\|_F^2) / \sigma_r(X); \tag{17}
\]
\[
\langle S^T, S \rangle \overset{a}{=} \langle SS \Sigma^{-1}, S \rangle = \langle \Sigma^{1/2} S \Sigma^{-1/2}, \Sigma^{1/2} S \Sigma^{-1/2} \rangle \geq 0.
\]
Here (a) is because \( SS \Sigma^{-1} = \Sigma^{-1} S^T \) by the definition of \( \mathcal{A}_Y \). So
\[
\|\mathcal{L}_Y(A)\|_F^2 = \|A_Y^\xi\|_F^2 \overset{(12)}{=} \|PA^T U + U^T A \Sigma P^T\|_F^2 + 2\|U^T A \Sigma P^T\|_F^2 \geq 2\|S^T + S\|_F^2 + 2\|D\|_F^2 \overset{(17)}{=} 2(\|S\|_F^2 + \|D\|_F^2) \geq 2\sigma_r(X)\|A\|_F^2,
\]

Figure 1: Relationship of \( \mathbb{R}^{p \times r} \), \( T_X \mathcal{M}_{r+} \), \( \mathcal{A}_Y^\xi \), \( \mathcal{A}_Y \), and \( \mathcal{A}_Y^\xi \).
This finishes the proof of this proposition. ■

Next, we present our first main result on the geometric landscape connection between formulations (1) and (3).

Theorem 1. (Geometric Landscape Connection Between Manifold and Factorization Formulations (PSD Case)) Suppose $Y \in \mathbb{R}^{p \times r}$ is of rank $r$ and $X = YY^\top$. Then

$$\nabla f(X) = (\nabla g(Y)Y^\top + (\nabla g(Y)Y^\top)^\top(I_p - YY^\top))/2 \quad \text{and} \quad \nabla g(Y) = 2\nabla f(X)Y. \quad (18)$$

Furthermore, if $Y$ is a Euclidean FOSP of (3), we have:

$$\nabla^2 g(Y)[A, A] = \text{Hess} f(X)[\xi, \xi], \quad \forall A \in \mathbb{R}^{p \times r};$$

$$\text{Hess} f(X)[\xi, \xi] = \nabla^2 g(Y)[\mathcal{L}^{-1}(Y), \mathcal{L}^{-1}(Y)], \quad \forall \xi \in T_X M_r. \quad (19)$$

More precisely,

$$\nabla^2 g(Y)[A] = 0, \quad \forall A \in \mathcal{A}^Y;$$

$$\nabla^2 g(Y)[A, A] = \text{Hess} f(X)[\mathcal{L}^{-1}(Y), \mathcal{L}^{-1}(Y)], \quad \forall A \in \mathcal{A}^Y. \quad (20)$$

Finally, $\text{Hess} f(X)$ has $(pr - (r^2 - r)/2)$ eigenvalues and $\nabla^2 g(Y)$ has $pr$ eigenvalues. $\nabla^2 g(Y)$ has at least $(r^2 - r)/2$ zero eigenvalues which correspond to the zero eigenspace $\mathcal{A}^Y$. Denote the rest of the $(pr - (r^2 - r)/2)$ possibly non-zero eigenvalues of $\nabla^2 g(Y)$ from the largest to the smallest as $\lambda_1(\nabla^2 g(Y)), \ldots, \lambda_{pr-(r^2-r)/2}(\nabla^2 g(Y))$. Then for $i = 1, \ldots, pr - (r^2 - r)/2$:

$$\lambda_i(\nabla^2 g(Y)) \text{ is sandwiched between } 2\sigma_r(X)\lambda_i(\text{Hess} f(X)) \text{ and } 4\sigma_1(X)\lambda_i(\text{Hess} f(X)).$$

Proof of Theorem 1. First, suppose $YY^\top$ has eigendecomposition $U\Sigma U^\top$. Then $Y$ lies in the column space spanned by $U$ and $YY^\top = P_U$. So (18) is by direct calculation from the gradient expressions in Proposition 1. The rest of the proof is divided into two steps: in Step 1, we prove (19) and (20); in Step 2, we prove the individual eigenvalue connection between $\text{Hess} f(X)$ and $\nabla^2 g(Y)$.

Step 1. We begin by proving the first equality in (19). Since $Y$ is a Euclidean FOSP of (3), by (18) we have $X = YY^\top$ is a Riemannian FOSP of (1) and $\nabla f(X) = P_{U\perp}\nabla f(X)P_{U\perp}$. Let $P = U^\top Y$. Given $A \in \mathbb{R}^{p \times r}$, we have

$$\langle \nabla f(X), P_{U\perp}AP^\top\Sigma^{-1}PA^\top P_{U\perp} \rangle = \langle \nabla f(X), AA^\top \rangle, \quad (21)$$

where the equality is because $P$ is nonsingular, $PP^\top = \Sigma$ and $\nabla f(X) = P_{U\perp}\nabla f(X)P_{U\perp}$.

Then by Proposition 2

$$\nabla^2 g(Y)[A, A] = \nabla^2 f(YY^\top)[YA^\top + AY^\top, YA^\top + AY^\top] + 2\langle \nabla f(YY^\top), AA^\top \rangle$$

$$\text{Lemma 2 (21)} \quad \nabla^2 f(X)[\xi_Y, \xi_Y] + 2\langle \nabla f(X), P_{U\perp}AP^\top\Sigma^{-1}PA^\top P_{U\perp} \rangle = \text{Hess} f(X)[\xi_Y, \xi_Y].$$
where (a) is because \( \xi^A \) is because \( \xi^L \) is an eigenvalue of Hess \( f \) respectively. This finishes the proof of the first equality in (19).

Moreover, for any \( A \in \mathcal{X}_{\text{null}}^\mathcal{Y}, B \in \mathbb{R}^{p \times r} \):

\[
\nabla^2 g(Y)[A, B] = \left( \nabla^2 g(Y)[A + B, A + B] - \nabla^2 g(Y)[A - B, A - B] \right) / 4
\]

\[
= \left( \text{Hess} f(X)[\xi^A + \xi^B, \xi^A + \xi^B] - \text{Hess} f(X)[\xi^A - \xi^B, \xi^A - \xi^B] \right) / 4
\]

\[
= \text{Hess} f(X)[\xi^A, \xi^B] \overset{(a)}{=} 0,
\]

where (a) is because \( \xi^A = 0 \) for \( A \in \mathcal{X}_{\text{null}}^\mathcal{Y} \). This implies the first equality in (20). The second equality in (20) follows directly from the first equality in (19) and the definition of \( \mathcal{L}_Y \). Finally, since \( \mathcal{L}_Y \) is a bijection, the second equality in (19) follows from the second equality in (20).

**Step 2.** Hess \( f(X) \) and \( \nabla^2 g(Y) \) are by definition linear maps from \( T_X \mathcal{M}_{r+} \) and \( \mathbb{R}^{p \times r} \) to \( T_X \mathcal{M}_{r+} \) and \( \mathbb{R}^{p \times r} \), respectively. Because \( T_X \mathcal{M}_{r+} \) is of dimension \( (pr - (r^2 - r)/2) \), the number of eigenvalues of Hess \( f(X) \) and \( \nabla^2 g(Y) \) is \( (pr - (r^2 - r)/2) \) and \( pr \), respectively. By the first equality in (20), we have \( \mathcal{X}_{\text{null}}^\mathcal{Y} \) is the eigenspace of \( (r^2 - r)/2 \) zero eigenvalues of \( \nabla^2 g(Y) \) and the rest of the \( (pr - (r^2 - r)/2) \) possibly non-zero eigenvalues of \( \nabla^2 g(Y) \) span the eigenspace \( \mathcal{X}_{\text{null}}^\mathcal{Y} \). Restricting to \( \mathcal{X}_{\text{null}}^\mathcal{Y} \) and \( T_X \mathcal{M}_{r+} \) and using (20), (14) and Lemma 7 in the Appendix, we have \( \lambda_i(\nabla^2 g(Y)) \) is sandwiched between \( 2\sigma_r(X)\lambda_i(\text{Hess} f(X)) \) and \( 4\sigma_1(X)\lambda_i(\text{Hess} f(X)) \). 

**Remark 1. (Necessity of First-order Property in Connecting Riemannian and Euclidean Hessians)** The following example shows that the assumption on the first-order stationary property is necessary for establishing the connection of the Riemannian and the Euclidean Hessians in Theorem 2. Consider a special case that \( p = r = 1 \), the objective functions are \( f(x) (x > 0) \) and \( g(y) = f(y^2) \), where both parameters \( x, y \) are scalars. In such a scenario, when \( x = y^2 \) we have

\[
g'(y) = 2y f'(y^2); \quad g''(y) = \frac{\partial^2}{\partial y^2} f(y^2) = 2 f'(y^2) + 4y^2 f''(y^2); \quad \text{Hess} f(x) = f''(y) = f''(y^2),
\]

where \( f' \) and \( f'' \) denote the first and second derivatives of \( f \). If \( f \) is not a first-order stationary point of \( g \) (namely \( f'(y) \) can be non-zero without any constraint), \( 2 f'(y^2) + 4y^2 f''(y^2) \) and \( f''(y) \) do not necessarily share the same sign or hold any sandwich inequality.

**Remark 2 (Connection of \( \mathcal{X}_{\text{null}}^\mathcal{Y} \) and Rotational Invariance of \( g(Y) \)).** We note \( \mathcal{X}_{\text{null}}^\mathcal{Y} \) is also called the vertical space in studying the Riemannian quotient geometry of \( \mathbb{R}^{p \times r}_* \), where \( \mathbb{R}^{p \times r}_* \) is the set of \( p \)-by-\( r \) full column rank matrices (Massart and Absil 2020). It has also appeared in (Li et al. 2019c, Theorem 2, Example 4) in analyzing the landscape of low-rank PSD matrix factorization. By assuming \( f \) is convex, (Li et al. 2019c) showed via invariance theory that \( \mathcal{X}_{\text{null}}^\mathcal{Y} \) has the property \( \nabla^2 g(Y)[A] = 0, \forall A \in \mathcal{X}_{\text{null}}^\mathcal{Y} \) at FOSP \( Y \). Here, by establishing the connection between the Riemannian and the Euclidean Hessians, we can establish the same result without assuming \( f \) is convex. Moreover, in the later Theorems 3 and 4 we extend our result to the general case and provide explicit expressions for the eigenspace corresponding to the zero eigenvalues there. Interested readers are also referred to the recent survey (Zhang et al. 2020) for the discussion on the effect of invariance in geometry of nonconvex problems.

Theorem 1 immediately shows the following equivalence of FOSPs and SOSPs between the manifold and the factorization formulations in low-rank PSD matrix optimization.
Corollary 1. (Equivalence on FOSPs, SOSP and Strict Saddles of Manifold and Factorization Formulations (PSD Case)) (a) If $Y$ is a rank $r$ Euclidean FOSP or SOSP or strict saddle of (3), then $X = YY^T$ is a Riemannian FOSP or SOSP or strict saddle of (1); (b) if $X$ is a Riemannian FOSP or SOSP or strict saddle of (1), then any $Y$ such that $YY^T = X$ is an Euclidean FOSP or SOSP or strict saddle of (3).

Remark 3. (Geometric Landscape Connection Between Manifold and Factorization Formulations on FOSPs, SOSP and Strict Saddles) We show in Corollary 1 that when constraining the Euclidean FOSPs/SOSPs/strict saddles of $g_Y$ to be rank $r$, the sets of matrices $YY^T$ are exactly the same as the sets of Riemannian FOSPs/SOSPs/strict saddles under the manifold formulation. On the other hand, we note the factorization formulation (3) can have many rank degenerate FOSPs: one canonical example is $Y = 0$.

In addition, we would like to mention that Corollary 1 can also be obtained via (Boumal, 2020, Proposition 9.6). But to our knowledge, that result is included recently after the first preprint of our manuscript. Having said that, our sandwich inequalities established in Theorem 1 are novel and not covered by theirs. Our sandwich inequalities reveal a finer connection on the spectrum of the Riemannian and the Euclidean Hessians at FOSPs: (1) $\nabla g^2_Y(Y)$ has $(r^2 - r)/2$ zero eigenvalues with zero eigenspace $\mathcal{X}_\text{null}$; (2) each of the other eigenvalues of $\nabla g^2_Y(Y)$ is sandwiched by the corresponding eigenvalues of $\text{Hess}f(YY^T)$ with explicit sandwich constants. In Section 5, we will illustrate the power of these sandwich inequalities in transferring the strict saddle property (Ge et al., 2015; Lee et al., 2019) quantitatively from the factorization formulation to the manifold formulation.

Remark 4. (Implication on Connection of Different Approaches for Rank Constrained Optimization) Broadly speaking, manifold and factorization are two different ways to handle the rank constraint in matrix optimization problems (see also discussion in the Introduction of paper Absil et al. (2007) on the relationship between manifold optimization and constrained optimization in the Euclidean space). Manifold formulation deals with the rank constraint explicitly via running Riemannian optimization algorithms on the manifold, while the factorization formulation treats the constraint implicitly via factorizing $X$ into $YY^T$ and running the unconstrained optimization algorithms in the Euclidean space. Theorem 1 establishes a strong geometric landscape connection between two formulations and this provides an example under which the two different approaches are indeed connected in treating the rank constraint.

Remark 5. Currently, the problem (1) we considered only has the PSD and rank constraints. Standard SDP problems may have additional linear constraints such as $\langle A_i, X \rangle = b_i$ for $i = 1, \ldots, m$. An interesting research direction is to extend current results to such settings. One strategy to handle these linear constraints is to add a quadratic penalty to the objective and considering solving

$$\min_{X \in \mathbb{S}^p \times \mathbb{S}^p \neq 0, \text{rank}(X) = r} f(X) + \mu \sum_{i=1}^{m} (\langle A_i, X \rangle - b_i)^2,$$

where $\mu > 0$ is a penalty parameter. The landscape of (24) under the factorization formulation has been considered in Bhojanapalli et al. (2018). Using our results, one can also transfer the landscape characterization to the corresponding manifold formulation.

4 Geometric Connection of Manifold and Factorization Formulations: General Case

In this section, we present the geometric landscape connection of the manifold formulation (2) and factorization formulations without regularization (1) or with regularization (5). Given $L \in \mathbb{R}^{p \times r}$,
Let $R \in \mathbb{R}^{p_2 \times r}$, suppose that $X = LR^\top$ is of rank $r$ and has SVD $X = USV^\top$. Let $P_1 = U^\top L$ and $P_2 = V^\top R$. Given any $A = [A_L \quad A_R]^\top$ with $A_L \in \mathbb{R}^{p_1 \times r}$, $A_R \in \mathbb{R}^{p_2 \times r}$, define
\[
\xi_{LR}^A := LA_R^\top + A_L R^\top = [U \quad U_{\perp}][P_1 A_R^\top V + U_{\perp} A_L P_2^\top \quad P_1 A_R^\top V_{\perp} \quad 0][V \quad V_{\perp}]^\top \in T_X M_r; \tag{25}
\]
and at the same time, given any $\xi = [U \quad U_{\perp}][S \quad D_1 \quad 0][V \quad V_{\perp}]^\top \in T_X M_r$, define
\[
\tilde{\xi}_{LR}^A = \{ A = [A_L \quad A_R]^\top : LA_R^\top + A_L R^\top = \xi \} \tag{26}
\]
Note that (25) and (26) are generalizations of [12] and [13], respectively and are used to connect the landscape geometry of the manifold formulation [2] and the unregularized factorization formulation [3]. To further incorporate the geometry of the regularized formulation [5], we introduce:
\[
\tilde{\bar{\xi}}_{LR}^A = \{ A = [A_L \quad A_R]^\top : LA_R^\top + A_L R^\top = \xi \quad \text{and} \quad L^\top A_L + A_L^\top L - R^\top A_R - A_R^\top R = 0 \}.
\]
Compared to $\xi_{LR}^A$, there is one additional constraint in the definition of $\tilde{\bar{\xi}}_{LR}^A$ corresponding to $\nabla^2 \tilde{g}_{\text{reg}}(L, R)$ and it is useful in connecting $\nabla^2 \tilde{g}_{\text{reg}}(L, R)$ with $\nabla^2 g(L, R)$ and Hess$f(X)$ as we will see in Theorem 3. The following lemma shows the affine space $\tilde{\bar{\xi}}_{LR}^A$ is nonempty and establishes the dimension and some properties of $\xi_{LR}^A$ and $\tilde{\bar{\xi}}_{LR}^A$.

**Lemma 4.** Given $L \in \mathbb{R}^{p_1 \times r}$, $R \in \mathbb{R}^{p_2 \times r}$, suppose that $X = LR^\top$ is of rank $r$ and has SVD $U \Sigma V^\top$. Let $P_1 = U^\top L$ and $P_2 = V^\top R$. Given any $\xi \in T_X M_r$, we have $\dim(\xi_{LR}^A) = r^2$, $\dim(\tilde{\bar{\xi}}_{LR}^A) = (r^2 - r)/2$, and
\[
\xi_{LR}^A := \{ A = [A_L \quad A_R]^\top : A_L = (US_1 + U_{\perp} D_1) P_2^{-\top} \in \mathbb{R}^{p_1 \times r} \quad \text{and} \quad S_1 + S_2 = S \}, \tag{27}
\]
and
\[
\tilde{\bar{\xi}}_{LR}^A := \{ A = [A_L \quad A_R]^\top : A_L = (US_1 + U_{\perp} D_1) P_2^{-\top} \in \mathbb{R}^{p_1 \times r}, \quad A_R = (VS_2^\top + V_{\perp} D_2) P_1^{-\top} \in \mathbb{R}^{p_2 \times r}, \quad S_1 + S_2 = S, \quad P_1 S_1 P_2^{-\top} + P_2^{-1} S_1 P_1 - P_2 S_2 P_1^{-\top} - P_1^{-1} S_2 P_2 = 0 \}. \tag{28}
\]
Similar to the PSD case discussed in Section 3, we construct $\xi_{LR}^A$, $\xi_{LR}^A$, and $\tilde{\bar{\xi}}_{LR}^A$ to find a correspondence between $\mathbb{R}^{(p_1+p_2) \times r}$ and $T_X M_r$. On the other hand, we note given $(L, R), A \in \mathbb{R}^{(p_1+p_2) \times r}$ and $\xi \in T_X M_r$, $\xi_{LR}^A$ is a single matrix while $\xi_{LR}^A$ forms a subspace of $\mathbb{R}^{(p_1+p_2) \times r}$ with dimension $r^2$ and $\tilde{\bar{\xi}}_{LR}^A \subset \xi_{LR}^A$ forms a subspace of $\mathbb{R}^{(p_1+p_2) \times r}$ with dimension $(r^2 - r)/2$. To deal with this ambiguity, we introduce the following two decompositions for $\mathbb{R}^{(p_1+p_2) \times r}$ tailored to $\xi_{LR}^A$ and $\tilde{\bar{\xi}}_{LR}^A$, respectively.

**Lemma 5.** Under the conditions in Lemma 4, it holds that:
- $\mathbb{R}^{(p_1+p_2) \times r} = \xi_{LR}^A \oplus \xi_{LR}^\null$ with $\dim(\xi_{LR}^A) = r^2$, $\dim(\xi_{LR}^\null) = (p_1+p_2-r)r$ and $\xi_{LR}^A \perp \xi_{LR}^\null$, where
  \[
  \xi_{LR}^\null = \{ A : A = \begin{bmatrix} US P_2^{-\top} \\ -VS P_1^{-\top} \end{bmatrix}, S \in \mathbb{R}^{r \times r} \};
  \]
  \[
  \xi_{LR}^\null = \{ A : A = \begin{bmatrix} US P_2^{-\top} + U D_1 P_2^{-\top} \\ (VS P_1 P_1^{-\top} + V D_2 P_1^{-\top}) \end{bmatrix}, D_1 \in \mathbb{R}^{(p_1-r) \times r}, D_2 \in \mathbb{R}^{(p_2-r) \times r}, S \in \mathbb{R}^{r \times r} \}.
  \]
Lemma 4, let \( \mathcal{A}_{L,R}^{\xi} \) be decomposed as the direct sum of eigenvalues of \( A \). Derive the following three results in Proposition 4: first, we show \( \mathcal{A}_{L,R}^{\xi} \) and \( \mathcal{A}_{L,R}^{\xi} \); second, \( \mathcal{A}_{L,R}^{\xi} \) and \( \mathcal{A}_{L,R}^{\xi} \); third, we construct a “pseudobijection” between \( \mathcal{A}_{L,R}^{\xi} \) and \( T_{X}M_{r} \).

As we will see in Theorems 2 and 3, \( \mathcal{A}_{L,R}^{\xi} \) and \( \mathcal{A}_{L,R}^{\xi} \) correspond to the eigen space of zero eigenvalues of \( \nabla^{2}g(L,R) \) and \( \nabla^{2}g_{reg}(L,R) \), respectively. By the decompositions in Lemma 5, we derive the following three results in Proposition 4: first, we show \( \mathcal{A}_{L,R}^{\xi} \) and \( \mathcal{A}_{L,R}^{\xi} \) can be further decomposed as the direct sum of \( \mathcal{A}_{null}^{\xi} \) and \( \mathcal{A}_{null}^{\xi} \) with the same single matrix from \( \mathcal{A}_{null}^{\xi} \); second, we find there exists a bijection between \( \mathcal{A}_{null}^{\xi} \) and \( T_{X}M_{r} \); third, we construct a “pseudobijection” between \( \mathcal{A}_{null}^{\xi} \) and \( T_{X}M_{r} \) since a bijection between them is impossible due to the mismatch of their dimensions. A pictorial illustration of the relationship of subspaces in Lemma 5 is given in Figure 2.

Proposition 4 (Decompositions of \( \mathcal{A}_{L,R}^{\xi} \) and \( \mathcal{A}_{L,R}^{\xi} \) and Maps Between \( \mathcal{A}_{null}^{\xi} \) and \( T_{X}M_{r} \)).

Under the conditions in Lemma 4, let \( \xi = [U \ U_{\perp}] \left[ \begin{array}{c} \mathbf{S} \\ \mathbf{D}_{1} \\ 0 \end{array} \right] \left[ \begin{array}{c} \mathbf{V} \\ \mathbf{V}_{\perp} \end{array} \right]^{T} \in T_{X}M_{r} \). Then

\[
\mathcal{A}_{L,R}^{\xi} = A_{L,R}^{\xi} \oplus \mathcal{A}_{null}^{\xi} \\
\mathcal{A}_{L,R}^{\xi} = A_{L,R}^{\xi} \oplus \mathcal{A}_{null}^{\xi} \\
\mathcal{A}_{L,R}^{\xi} = (US_{1}P_{2}^\top + U_{\perp}D_{1})P_{2}^\top + (VS_{1}P_{1}^\top + V_{\perp}D_{2})P_{1}^\top \in \mathcal{A}_{null}^{\xi} \subseteq \mathcal{A}_{null}^{\xi}.
\]

Figure 2: Relationship of subspaces involved in two decompositions in Lemma 5. Left hand side: first decomposition in Lemma 5 on relationship between \( \mathbb{R}^{(p_{1}+p_{2})\times r} \), \( T_{X}M_{r} \), \( \mathcal{A}_{null}^{L,R} \), \( \mathcal{A}_{null}^{\xi} \), \( A_{L,R}^{\xi} \), \( \mathcal{A}_{null}^{\xi} \), and \( \mathcal{A}_{null}^{\xi} \); Right hand side: second decomposition in Lemma 5 on relationship between \( \mathbb{R}^{(p_{1}+p_{2})\times r} \), \( T_{X}M_{r} \), \( \mathcal{A}_{null}^{L,R} \), \( \mathcal{A}_{null}^{\xi} \), \( A_{L,R}^{\xi} \), and \( \mathcal{A}_{null}^{\xi} \).
where $\bar{S}$ is the unique solution of the following Sylvester equation $P_2P_2^T\bar{S}^T + \bar{S}^TP_1P_1^T = S^T$.

Moreover, there is a bijective linear map $\mathcal{L}_{LR}$ between $\mathcal{A}_{LR}^{\text{null}}$ and $T_{X_M}r$ given as follows

$$\mathcal{L}_{LR} : A \in \mathcal{A}_{LR}^{\text{null}} \rightarrow \xi_{LR}^A \in T_{X_M}r \quad \text{and} \quad \mathcal{L}_{LR}^{-1} : \xi \in T_{X_M}r \rightarrow A^\xi_{LR} \in \mathcal{A}_{LR}^{\text{null}}.$$  \hfill (31)

In addition, there is a surjective linear map $\tilde{\mathcal{L}}_{LR}$ between $\tilde{\mathcal{A}}_{LR}^{\text{null}}$ and $T_{X_M}r$ given as follows

$$\tilde{\mathcal{L}}_{LR} : A \in \tilde{\mathcal{A}}_{LR}^{\text{null}} \rightarrow \xi_{LR}^A \in T_{X_M}r,$$  \hfill (32)

and it satisfies $\tilde{\mathcal{L}}_{LR}(\mathcal{L}_{LR}^{-1}(\xi)) = \xi$ for any $\xi \in T_{X_M}r$.

Finally, we have the following spectrum bounds for $\mathcal{L}_{LR}$, $\mathcal{L}_{LR}^{-1}$ and $\tilde{\mathcal{L}}_{LR}$, respectively:

$$(\sigma_r(L) \land \sigma_r(R))^2 \|A\|_F^2 \leq \|\mathcal{L}_{LR}(A)\|_F^2 \leq 2(\sigma_r(L) \lor \sigma_r(R))^2 \|A\|_F^2, \quad \forall A \in \mathcal{A}_{LR}^{\text{null}},$$

$$(\sigma_r(L) \lor \sigma_r(R))^{-2} \|\xi\|_F^2/2 \leq \|\mathcal{L}_{LR}^{-1}(\xi)\|_F^2 \leq (\sigma_r(L) \land \sigma_r(R))^{-2} \|\xi\|_F^2, \quad \forall \xi \in T_{X_M}r,$$  \hfill (33)

$$\|\tilde{\mathcal{L}}_{LR}(A)\|_F^2 \leq 2(\sigma_2(L) \lor \sigma_2(R))^2 \|A\|_F^2, \quad \forall A \in \tilde{\mathcal{A}}_{LR}^{\text{null}}.$$  \hfill (34)

**Proof of Proposition**. We divide the proof into two steps: in Step 1, we prove the decomposition results for $\mathcal{A}_{LR}^\xi$ and $\mathcal{A}_{LR}^{\text{null}}$; in Step 2, we show $\mathcal{L}_{LR}$ is a bijection, $\tilde{\mathcal{L}}_{LR}(\mathcal{L}_{LR}^{-1}(\xi)) = \xi$, and prove their spectrum bounds.

**Step 1.** First, the uniqueness of $\bar{S}$ is guaranteed by the fact $P_1P_1^T$ and $-P_2P_2^T$ have disjoint spectra and (Bhatia, 2013, Theorem VII.2.1). Next, we prove $\mathcal{A}_{LR}^\xi = \mathcal{A}_{LR}^{\text{null}} \cup \mathcal{A}_{LR}^{\text{null}}$. Recall

$$A^\xi_{LR} = \left[ \begin{array}{c} (US_2P_2^T + U_1D_1)P_2^{-T} \\ (VS_2P_1^T + V_1D_2)P_1^{-T} \end{array} \right] \in \mathcal{A}_{LR}^{\text{null}}, \quad \text{given} \quad A = \left[ \begin{array}{c} (US_1 + U_1D_1)P_2^{-T} \\ (VS_1^T + V_1D_2)P_1^{-T} \end{array} \right] \in \mathcal{A}_{LR}^{\text{null}}.$$

$$A - A^\xi_{LR} = \left[ \begin{array}{c} U(S_1 - \bar{S}P_2^TP_2^{-T})P_2^{-T} \\ V(S_1^T - \bar{S}^TP_1P_1^T)P_1^{-T} \end{array} \right] \in \mathcal{A}_{LR}^{\text{null}},$$  \hfill (34)

where (a) is because $SP_2^TP_2^T + P_1P_1^T\bar{S} = S$ and $S_1 + S_2 = S$. Moreover, $A^\xi_{LR} + A \in \mathcal{A}_{LR}^{\text{null}}$ for any $A \in \mathcal{A}_{LR}^{\text{null}}$. This proves $\mathcal{A}_{LR}^\xi = \mathcal{A}_{LR}^{\text{null}} \cup \mathcal{A}_{LR}^{\text{null}}$.

Next, we prove the second decomposition result $\tilde{\mathcal{A}}_{LR}^{\text{null}} = \mathcal{A}_{LR}^{\text{null}} \cup \mathcal{A}_{LR}^{\text{null}}$. Given $A_{LR}^\xi = \left[ \begin{array}{c} (US_2P_2^T + U_1D_1)P_2^{-T} \\ (VS_2P_1^T + V_1D_2)P_1^{-T} \end{array} \right]$ and $A = \left[ \begin{array}{c} (US_1 + U_1D_1)P_2^{-T} \\ (VS_1^T + V_1D_2)P_1^{-T} \end{array} \right] \in \mathcal{A}_{LR}^{\text{null}}$, we have

$${P_1^T}(P_1^TP_1^T\bar{S} - S_2)P_2^{-T} + (P_1^T(P_1^T\bar{S} - S_2)P_2^{-T})^T + P_1^{-1}(P_1^TP_1^T\bar{S} - S_2)P_2 + (P_1^{-1}(P_1^TP_1^T\bar{S} - S_2)P_2)^T$$

$$(a) \left[ \begin{array}{c} P_1^T(S - \bar{S}P_2^TP_2^{-T})P_2^{-T} + P_2^{-1}(S - P_2P_2^T\bar{S})P_1 + P_1^TS_2P_2 + P_2^TS_1P_1 \\ -P_1^TS_2P_2^{-T} - (P_1^TS_2P_2^{-T})^T - P_1^{-1}S_2P_2 - (P_1^{-1}S_2P_2)^T \end{array} \right]$$

$$(b) \left[ \begin{array}{c} P_1^TSP_2^T + P_2^{-1}S^TP_1 - P_1^TSP_2^T - (P_1^TSP_2^T)^T - P_1^{-1}S_2P_2 - (P_1^{-1}S_2P_2)^T \end{array} \right]$$

$$(c) \left[ \begin{array}{c} P_1^TS_2P_2^{-T} + (P_1^TSP_2^T)^T - P_1^{-1}S_2P_2 - (P_1^{-1}S_2P_2)^T \end{array} \right] 0.$$  \hfill (35)
Here (a) is because \( \bar{SP}_2P_2^T + P_1P_1^T \bar{S} = S \), (b) is because \( S_1 + S_2 = S \) and (c) is by the constraints of \( S_1, S_2 \) given in \( \mathcal{A}_{\text{LR}}^\xi \). So (35) shows \( P_1P_1^T \bar{S} - S_2 \in \mathcal{R}_{\text{LR}} \) and we have

\[
A - A_{\text{LR}}^\xi \in \mathcal{R}_{\text{null}} \implies \left( U(-S_2 + P_1P_1^T \bar{S})P_2^T \right) \in \mathcal{R}_{\text{null}}.
\]

Moreover, for \( A_{\text{LR}}^\xi \) and any \( A = \begin{bmatrix} USP_2^T & -VS^TP_1^T \end{bmatrix} \in \mathcal{R}_{\text{null}} \) with \( S \in \mathcal{R}_{\text{LR}} \), we have

\[
(\bar{SP}_2P_2^T + S) + (S^TP_1P_1^T - S^T)^T \overset{(a)}{=} S;
\]

\[
P_1^T(\bar{SP}_2P_2^T + S)P_2^T + P_2^T(P_2P_2^TS^T + S^TP_1P_1^T - S^T)P_1^T - P_2^T(S^TP_1P_1^T - S^T)P_1^T - P_1^T(\bar{SP}_2P_2^T + S)P_2^T
\]

\[
= P_1^TSP_2^T + (P_1^TSP_2^T)^T + P_1^TS^TP_2 + (P_1^TSP_2^T)^T = 0.
\]

(36)

Here (a) is because \( \bar{SP}_2P_2^T + P_1P_1^T \bar{S} = S \) and (b) is because \( S \in \mathcal{R}_{\text{LR}} \). Thus,

\[
A_{\text{LR}}^\xi + A = \begin{bmatrix} USP_2^T + (S^TP_1P_1^T - S^T)^T + (V^THP_1P_1^T - S^T)P_1^T - (V^THP_1P_1^T - S^T)P_1^T + (V^THP_1P_1^T - S^T)P_1^T + (V^THP_1P_1^T - S^T)P_1^T \end{bmatrix} \in \mathcal{R}_{\text{LR}}^\xi.
\]

This finishes the proof for \( \mathcal{A}_{\text{LR}}^\xi = A_{\text{LR}}^\xi \oplus \mathcal{R}_{\text{LR}}^\xi \).

**Step 2.** We begin by proving \( \mathcal{L}_{\text{LR}}^\xi \) is a bijection. Note that both \( \mathcal{A}_{\text{LR}}^\xi \) and \( T_XM_r \) have dimension \( p_1 + p_2 - r \). Suppose \( L_{\text{LR}}^\xi : \xi \in T_XM_r \rightarrow A_{\text{LR}}^\xi \in \mathcal{R}_{\text{LR}}^\xi \). Then for any \( \xi = [U~ U_\perp] \begin{bmatrix}S & D_2^T \end{bmatrix} [V~ V_\perp]^T \in T_XM_r \), we have

\[
L_{\text{LR}}(L_{\text{LR}}^\xi(\xi)) = L_{\text{LR}}(A_{\text{LR}}^\xi) = [U~ U_\perp] \begin{bmatrix} \bar{SP}_2P_2^T + P_1P_1^T \bar{S} & D_2^T \end{bmatrix} [V~ V_\perp]^T = \xi.
\]

(37)

Since \( \mathcal{L}_{\text{LR}}^\xi \) and \( L_{\text{LR}}^\xi \) are linear maps, (37) implies \( \mathcal{L}_{\text{LR}}^\xi \) is bijection and \( L_{\text{LR}}^\xi = L_{\text{LR}}^{-1} \). Following a similar proof of (37), we can also show \( \mathcal{L}_{\text{LR}}(L_{\text{LR}}^{-1}(\xi)) = \xi \) holds for any \( \xi \in T_XM_r \) and this implies \( \mathcal{L}_{\text{LR}} \) is surjective.

Next, we provide the spectrum bounds for \( \mathcal{L}_{\text{LR}}^\xi \). Suppose \( A = [A_L^T ~ A_R^T] \in \mathcal{R}_{\text{null}}^\xi \), where

\[
A_L = (USP_2P_2^T + U_1D_1)P_2^T, \quad A_R = (VS^TP_1P_1^T + V_1D_2)P_1^T.
\]

Then

\[
\|A\|_F^2 \leq \left( \|SP_2P_2^T\|_F^2 + \|D_1\|_F^2 \right) \sigma_1^2(P_2^T) + \left( \|P_1P_1^T\|_F^2 + \|D_2\|_F^2 \right) \sigma_1^2(P_1^T)
\]

\[
\overset{(a)}{\leq} \left( \|SP_2P_2^T\|_F^2 + \|P_1P_1^T\|_F^2 \right) + \|D_1\|_F^2 + \|D_2\|_F^2)/(\sigma_r(L) \wedge \sigma_r(R))^2;
\]

where in (a), we use the fact \( L = UP_1, R = VP_2 \) and \( L, R \) share the same spectrum as \( P_1, P_2 \). In addition,

\[
\langle P_1P_1^T\bar{S}, SP_2P_2^T \rangle = \langle (P_1P_1^T)^{1/2}S(P_2P_2^T)^{1/2}, (P_1P_1^T)^{1/2}S(P_2P_2^T)^{1/2} \rangle \geq 0.
\]

(39)

So

\[
\|\mathcal{L}_{\text{LR}}(A)\|_F^2 = \|A\|_F^2 \overset{25}{=} \|P_1A_R^TV + U^TA_LP_2^T\|_F^2 + \|U_1A_LP_2^T\|_F^2 + \|P_1A_R^TV\|_F^2
\]

\[
= \|P_1P_1^T\bar{S} + SP_2P_2^T\|_F^2 + \|D_1\|_F^2 + \|D_2\|_F^2
\]

\[
\overset{39}{=} \|SP_2P_2^T\|_F^2 + \|P_1P_1^T\bar{S}\|_F^2 + \|D_1\|_F^2 + \|D_2\|_F^2 \overset{38}{=} (\sigma_r(L) \wedge \sigma_r(R))^2||A||_F^2,
\]

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and

\[ \| L_{LR}(A) \|_F^2 = \| \xi_{LR}^A \|_F^2 + \| P_1 A_R^T V + U^T A_L P_2 \|_F^2 + \| U^T A_L P_2 \|_F^2 + \| P_1 A_R^T V \|_F^2 \]

\[ \leq 2(\| P_1 A_R^T V \|_F^2 + \| U^T A_L P_2 \|_F^2 + \| P_1 A_R^T V \|_F^2)
\]

\[ = \| P_1 A_R^T V \|_F^2 + \| U^T A_L P_2 \|_F^2 + \| A_L P_2 \|_F^2 + \| P_1 A_R^T \|_F^2 \]

\[ \leq 2(\sigma_1(L) + \sigma_1(R))^2 \| A \|_F^2. \]

By the relationship of the spectrum of an operator and its inverse, the spectrum bounds for \( L_{LR}^{-1} \) follow from the ones of \( L_{LR} \). Finally, since \( \tilde{L}_{LR} \) is surjective and the “pseudoinverse” of \( L_{LR}^{-1} \), its spectrum upper bound follows from the spectrum lower bound of \( L_{LR}^{-1} \). This finishes the proof of this proposition.  

Now, we are ready to present our main results on the geometric landscape connection of the manifold and the factorization formulations in the general low-rank matrix optimization.

**Theorem 2** (Geometric Landscape Connection of Formulations [2] and [4]). Suppose \( L \in \mathbb{R}^{p1 \times r} \), \( R \in \mathbb{R}^{p2 \times r} \) and \( X = LR^T \) are of rank \( r \). Then

\[ \nabla f(X) = \nabla_L g(L, R) R^\dagger + (\nabla_R g(L, R) L^\dagger) (I_{p2} - RR^\dagger) \quad \text{and} \quad \nabla g(L, R) = \begin{bmatrix} \text{grad} f(X) R \\ (\text{grad} f(X))^\dagger L \end{bmatrix}. \]

Furthermore, if \((L, R)\) is a Euclidean FOSP [4], then we have

\[ \nabla^2 g(L, R)[A, A] = \text{Hess} f(X)[\xi_{LR}^A, \xi_{LR}^A], \quad \forall A \in \mathbb{R}^{(p1 + p2) \times r}; \]

\[ \text{Hess} f(X)[\xi, \xi] = \nabla^2 g(L, R)[L_{LR}^{-1}(\xi), L_{LR}^{-1}(\xi)], \quad \forall \xi \in T_X M_r. \]

More precisely,

\[ \nabla^2 g(L, R)[A] = 0, \quad \forall A \in \mathcal{A}_{null}^LR; \]

\[ \nabla^2 g(L, R)[A] = \text{Hess} f(X)[L_{LR}(A), L_{LR}(A)], \quad \forall A \in \mathcal{A}_{null}^{LR}. \]

Finally, \( \text{Hess} f(X) \) has \((p1 + p2 - r) r\) eigenvalues and \( \nabla^2 g(L, R) \) has \((p1 + p2 - r) r\) eigenvalues. \( \nabla^2 g(L, R) \) has at least \( r^2 \) zero eigenvalues with the corresponding zero eigenspace \( \mathcal{A}_{null}^LR \). Denote the rest of the \((p1 + p2 - r) r\) possibly non-zero eigenvalues of \( \nabla^2 g(L, R) \) from the largest to the smallest as \( \lambda_1(\nabla^2 g(L, R)), \ldots, \lambda_{(p1 + p2 - r)}(\nabla^2 g(L, R)) \). Then for \( i = 1, \ldots, (p1 + p2 - r) r \): \( \tilde{\lambda}_i(\nabla^2 g(L, R)) \) is sandwiched between \((\sigma_1(L) \wedge \sigma_r(R))^2 \lambda_i(\text{Hess} f(X)) \) and \( 2(\sigma_1(L) \vee \sigma_r(R))^2 \lambda_i(\text{Hess} f(X)). \)

**Proof of Theorem 2** [2]. First, suppose \( X = LR^T \) has SVD \( U \Sigma V^T \). Then \( LL^\dagger = P_U, RR^\dagger = P_V \) and \( L \) and \( R \) lie in the column spaces of \( U \) and \( V \), respectively. So [40] is by direct calculation from the expressions of Riemannian and Euclidean gradients given in Proposition [1]. The rest of the proof is divided into two steps: in Step 1, we prove [41] and [42]; in Step 2, we prove the individual eigenvalue connection between \( \text{Hess} f(X) \) and \( \nabla^2 g(L, R) \).

**Step 1.** We begin by showing the first equality in [41]. Let \( P_1 = U^T L, P_2 = V^T R \), it is easy to verify \( P_1 P_2^\dagger = U^T L R^T V = \Sigma \). Since \((L, R)\) is a Euclidean FOSP of [4], by [40] we have \( X \) is a Riemannian FOSP of [2]. So \( \nabla f(X) = P_U \nabla f(X) P_V \). Given \( A = [A_L, A_R] \in \mathbb{R}^{(p1 + p2) \times r} \), we have

\[ \langle \nabla f(X), U \Sigma L^\dagger A_L P_2 \Sigma^{-1} P_1 A_R^T V \rangle = \langle \nabla f(X), A_L A_R^T \rangle. \]
Here (a) is because $\nabla f(X) = P_U \nabla f(X) P_{\perp}$ and $P_1 P_2^T = \Sigma$.

Then by Proposition 2,

$$\nabla^2 g(L,R)[A,B] = 2\langle \nabla f(LR^T), A_L A_R^T \rangle + \nabla^2 f(LR^T)[LA_R^T + A_L R^T, LA_R^T + A_L R^T]$$

$$= 2\langle \nabla f(X), U \perp U^T A_L P_2 \Sigma^{-1} P_1 A_R \nabla_\perp \nabla^2 f(X) |\xi_L^R, \xi_L^R \rangle$$

$$= \text{Hess}_f(X)[\xi_L^R, \xi_L^R],$$

(44)

where the last equality follows from the expressions of $\text{Hess}_f(X)$ and $\xi_L^R$ in (11) and (25), respectively. This finishes the proof for the first equality in (41). Meanwhile, by a similar argument as (23), we have

$$\nabla^2 g(L,R)[A,B] = \text{Hess}_f(X)[\xi_L^R, \xi_L^R] = 0, \quad \forall A \in \mathcal{A}_{\text{null}}^L, \forall B \in \mathbb{R}^{(p_1 + p_2) \times r}.$$ 

This implies the first equality in (42).

The second equality in (42) follows directly from the first equality in (41) and the definition of $\mathcal{L}_{L,R}$. Finally, by the bijectivity of $\mathcal{L}_{L,R}$, the second equality in (41) follows from the second equality in (42).

Step 2. $\text{Hess}_f(X)$ and $\nabla^2 g(L,R)$ are by definition linear maps from $T_XM_r$ and $\mathbb{R}^{(p_1 + p_2) \times r}$ to $T_XM_r$ and $\mathbb{R}^{(p_1 + p_2) \times r}$, respectively. Because $T_XM_r$ is of dimension $(p_1 + p_2 - r)r$, the number of eigenvalues of $\text{Hess}_f(X)$ and $\nabla^2 g(L,R)$ are $(p_1 + p_2 - r)r$ and $(p_1 + p_2)r$, respectively. By the first equality in (42), we have $\mathcal{A}_{\text{null}}^L$ is the eigenspace of $r^2$ zero eigenvalues and the rest of the $(p_1 + p_2 - r)r$ possibly non-zero eigenvalues of $\nabla^2 g(L,R)$ span the eigenspace $\mathcal{A}_{\text{null}}^L$. Restricting to $\mathcal{A}_{\text{null}}^L$ and $T_XM_r$ and using (42), (33) and Lemma 7 in the Appendix, we have $\lambda_i(\nabla^2 g(L,R))$ is sandwiched between $(\sigma_r(L) \wedge \sigma_r(L))^2 \lambda_i(\text{Hess}_f(X))$ and $2(\sigma_1(L) \vee \sigma_1(L))^2 \lambda_i(\text{Hess}_f(X))$. This finishes the proof.

\begin{theorem}[Geometric Landscape Connection of Formulations (2) and (5)] Suppose $L \in \mathbb{R}^{p_1 \times r}, R \in \mathbb{R}^{p_2 \times r}$ and $X = LR^T$ are of rank $r$ and $(L,R)$ is a Euclidean FOSP of (5). First, we have

$$L^T L = R^T R \quad \text{and} \quad \nabla_{\text{reg}} g(L,R) = \nabla g(L,R),$$

(45)

and for any $A = [A_L^T, A_R^T]^T \in \mathbb{R}^{(p_1 + p_2) \times r}$,

$$\nabla^2 g_{\text{reg}}(L,R)[A,A] = \nabla^2 g(L,R)[A,A] + \mu \|L^T A_L + A_L L - R^T A_R - A_R^T R\|_{F}^2.$$ 

(46)

Second,

$$\nabla^2 g_{\text{reg}}(L,R)[A,A] = \text{Hess}_f(X)[\xi_L^R, \xi_L^R] + \mu \|L^T A_L + A_L L - R^T A_R - A_R^T R\|_{F}^2, \quad \forall A \in \mathbb{R}^{(p_1 + p_2) \times r};$$

(47)

$$\text{Hess}_f(X)[\xi, \xi] = \nabla^2 g_{\text{reg}}(L,R)[\mathcal{L}_{L,R}^{-1}(\xi), \mathcal{L}_{L,R}^{-1}(\xi)], \quad \forall \xi \in T_XM_r,$$

where $\mathcal{L}_{L,R}^{-1}$ is the bijective map given in (31). More precisely,

$$\nabla^2 g_{\text{reg}}(L,R)[A] = 0, \quad \forall A \in \mathcal{A}_{\text{null}}^L,$$

$$\nabla^2 g_{\text{reg}}(L,R)[A] = \text{Hess}_f(X)[\hat{\mathcal{L}}_{L,R}(A), \hat{\mathcal{L}}_{L,R}(A)] + \mu \|L^T A_L + A_L L - R^T A_R - A_R^T R\|_{F}^2, \quad \forall A \in \mathcal{A}_{\text{null}}^L.$$ 

(48)

\end{theorem}
Finally, $\nabla^2_{\text{reg}}(L, R)$ has $(p_1 + p_2)r$ eigenvalues and at least $(r^2 - r)/2$ of them are zero spanning the eigenspace $\mathcal{A}^{\text{null}}_{L,R}$. Denote the rest of the $((p_1 + p_2)r - (r^2 - r)/2)$ possibly non-zero eigenvalues of $\nabla^2_{\text{reg}}(L, R)$ from the largest to the smallest as $\lambda_1(\nabla^2_{\text{reg}}(L, R)), \ldots, \lambda_{(p_1 + p_2)r - (r^2 - r)/2}(\nabla^2_{\text{reg}}(L, R))$. Then

- the following lower bounds for $\lambda_i(\nabla^2_{\text{reg}}(L, R))$ hold:

$$
\lambda_i(\nabla^2_{\text{reg}}(L, R)) \geq (\sigma_r(X)\lambda_i(\text{Hess} f(X))) \wedge (2\sigma_1(X)\lambda_i(\text{Hess} f(X))), \text{ for } 1 \leq i \leq (p_1 + p_2 - r)r,
$$

$$
\lambda_i(\nabla^2_{\text{reg}}(L, R)) \geq 2\sigma_1(X)\lambda_{\min}(\text{Hess} f(X)) \wedge 0, \text{ for } (p_1 + p_2 - r)r + 1 \leq i \leq (p_1 + p_2)r - (r^2 - r)/2; \quad (49)
$$

- the following upper bounds for $\lambda_i(\nabla^2_{\text{reg}}(L, R))$ hold:

$$
\lambda_i(\nabla^2_{\text{reg}}(L, R)) \leq 2\sigma_1(X)((\lambda_i(\text{Hess} f(X)) \vee 0) + 4\mu), \text{ for } 1 \leq i \leq (r^2 + r)/2,
$$

$$
\lambda_i(\nabla^2_{\text{reg}}(L, R)) \leq (2\sigma_1(X)\lambda_{i-(r^2+2)/2}(\text{Hess} f(X))) \vee (\sigma_r(X)\lambda_{i-(r^2+2)/2}(\text{Hess} f(X))), \quad (50)
$$

for $(r^2 + r)/2 + 1 \leq i \leq (p_1 + p_2)r - (r^2 - r)/2$.

**Proof of Theorem 3**. Since $(L, R)$ is a FOSP of (5), the first result in (45) is by Theorem 3 of Zhu et al. [2018]. The second result in (45) is by $L^\top L = R^\top R$ and Proposition 1. In addition, for any $A = [A_L \ A_R]^T \in \mathbb{R}^{(p_1 + p_2) \times r}$, (46) follows from (45) and Proposition 2.

The rest of the proof is divided into two steps. In Step 1, we prove the second part of Theorem 3 i.e., (47) and (48); in Step 2, we prove the final part of the theorem, i.e., the spectrum bounds in (49) and (50).

**Step 1.** First, by the first equality in (41) and (46), we obtain the first equality in (47). Since $(L, R)$ is a FOSP of (5), from (45), we see that $(L, R)$ is also a Euclidean FOSP of (4). Recalling the definition of $\mathcal{L}^{-1}_{L,R}(X)$ in (31), given $X \in T_XM_r$, we see that $\mathcal{L}^{-1}_{L,R}(X) = A^\xi_{L,R} = [A^\xi_{L} \ A^\xi_{R}]^T$ satisfies

$$
L^\top A^\xi_L + A^\xi_L L - R^\top A^\xi_R - A^\xi_R R = P^\top S^P + (P^\top S^P)^\top - P^\top S^P - (P^\top S^P)^\top = 0.
$$

So the second equality in (47) follows from (46) and the second equality in (41).

The second equality in (48) directly follows from the first equality in (47) and the definition of $\mathcal{L}^{-1}_{L,R}$. Next, we prove the first equality in (48). For any $A = [A_L \ A_R]^T \in \mathcal{A}^{\text{null}}_{L,R}$, $B = [B_L \ B_R]^T \in \mathbb{R}^{(p_1 + p_2) \times r}$,

$$
\nabla^2_{\text{reg}}(L, R)[A, B] = (\nabla^2_{\text{reg}}(L, R)[A + B, A + B] - \nabla^2_{\text{reg}}(L, R)[A - B, A - B]) / 4
$$

$$
\begin{align*}
&= (\text{Hess} f(X)[\xi^A_{L,R} + \nu_{L,R}^A] + \mu_{L,R}^A + B^L_L L - R^\top B_R - B^R_R R) / 4 \\
&- \text{Hess} f(X)[\xi^A_{L,R} - \nu_{L,R}^A - \mu_{L,R}^A + B^L_L L - R^\top B_R - B^R_R R] / 4 \\
&= \text{Hess} f(X)[\xi^A_{L,R} + \xi^B_{L,R} - \xi^A_{L,R}] / 4 \\
&= \text{Hess} f(X)[\xi^A_{L,R}, \xi^B_{L,R}] / 4,
\end{align*}
$$

where (a) is because of (47) and $L^\top A_L + A^\top_L L - R^\top A_R - A^\top_R R = 0$ for any $A \in \mathcal{A}^{\text{null}}_{L,R}$ and (b) is because $\xi^A_{L,R} = 0$ for any $A \in \mathcal{A}^{\text{null}}_{L,R}$. This implies the first equality in (48) and finishes the proof of this part.

**Step 2.** It is easy to check the number of eigenvalues of $\text{Hess} f(X)$ and $\nabla^2_{\text{reg}}(L, R)$ are $(p_1 + p_2 - r)r$ and $(p_1 + p_2)r$, respectively. By the first equality in (48), we have $\mathcal{A}^{\text{null}}_{L,R}$ is the
eigenspace of \((r^2 - r)/2\) zero eigenvalues and the rest of the \(((p_1 + p_2)r - (r^2 - r)/2)\) possibly non-zero eigenvalues of \(\nabla^2g_{reg}(L, R)\) span the eigenspace \(\tilde{\mathcal{L}}_{LR}^{1}\). Restricting \(\text{Hess} f(X)\) and \(\nabla^2g_{reg}(L, R)\) to \(T_XM_r\) and \(\tilde{\mathcal{L}}_{LR}^{1}\), respectively, next we prove the inequalities in (49) and (50) sequentially.

**Proof of the first inequality in (49).** Define the linear map \(\mathcal{P} : T_XM_r \rightarrow \tilde{\mathcal{L}}_{LR}^{1}\) as \(\mathcal{P}(\xi) = A_L^T\). By the definition of \(\mathcal{L}_{LR}^{1}\) and second equality in (47), we have

\[
\text{Hess} f(X)[\xi, \xi] = \nabla^2g_{reg}(L, R)[\mathcal{L}_{LR}^{1}(\xi), \mathcal{L}_{LR}^{1}(\xi)] = \nabla^2g_{reg}(L, R)[\mathcal{P}(\xi), \mathcal{P}(\xi)], \quad \forall \xi \in T_XM_r,
\]

i.e.,

\[
\text{Hess} f(X) = \mathcal{P}^*\nabla^2g_{reg}(L, R)\mathcal{P}.
\]

Moreover, by the construction of \(\mathcal{P}\) and \(33\), we have for any \(\xi \in T_XM_r\),

\[
(2\sigma_1(X))^{-1}\|\xi\|_F^2 \overset{(a)}{=} (\sigma_1(L) \vee \sigma_1(R))^{-2}\|\xi\|_F^2/2 \leq \|\mathcal{P}(\xi)\|_F^2 \leq (\sigma_r(L) \wedge \sigma_r(R))^{-2}\|\xi\|_F^2 \overset{(b)}{=} \sigma_r(X)^{-1}\|\xi\|_F^2.
\]

In (a) and (b), we use the fact that when \(45\) holds, we have

\[
\sigma_1(L) = \sigma_1(R) = \sigma_1^{1/2}(X), \quad \sigma_r(L) = \sigma_r(R) = \sigma_r^{1/2}(X).
\]

Finally, by \(51\), \(52\) and Lemma 8(i) in the Appendix, we have obtained the first inequality in (49).

**Proof of the second inequality in (49).** By the second equality in (48), we have \(\nabla^2g_{reg}(L, R) \succ \tilde{\mathcal{L}}_{LR}^\ast \text{Hess} f(X)\tilde{\mathcal{L}}_{LR}^\ast\). Then by \(33\), \(53\) and Lemma 8(iii) in the Appendix, we have for \((p_1 + p_2 - r)r + 1 \leq i \leq (p_1 + p_2)r - (r^2 - r)/2\),

\[
\tilde{\lambda}_i(\nabla^2g_{reg}(L, R)) \geq \tilde{\lambda}_{(p_1 + p_2)r - (r^2 - r)/2}(\nabla^2g_{reg}(L, R)) \geq 2\sigma_1(X)\lambda_{\min}(\text{Hess} f(X)) \wedge 0.
\]

**Proof of the first inequality in (50).** By the second equality in (48), we have for any \(A \in \tilde{\mathcal{L}}_{LR}^{1}\),

\[
\nabla^2g_{reg}(L, R)[A, A] = \text{Hess} f(X)[\tilde{\mathcal{L}}_{LR}(A), \tilde{\mathcal{L}}_{LR}(A)] + \mu\|L^\top A_L + A_L^\top L - R^\top A_R - A_R^\top R\|_F^2 \overset{\text{Lemma 10, 53}}{\leq} \text{Hess} f(X)[\tilde{\mathcal{L}}_{LR}(A), \tilde{\mathcal{L}}_{LR}(A)] + 8\mu\sigma_1(X)\|A\|_F^2.
\]

So we have \((\nabla^2g_{reg}(L, R) - 8\mu\sigma_1(X)I) \preceq \tilde{\mathcal{L}}_{LR}^\ast \text{Hess} f(X)\tilde{\mathcal{L}}_{LR}^\ast\) where \(I\) denotes an identity operator. Then by \(33\), \(53\) and Lemma 8(iv) in the Appendix, we have for \(1 \leq i \leq (r^2 + r)/2\):

\[
\tilde{\lambda}_i(\nabla g_{reg}(L, R)) - 8\mu\sigma_1(X) \leq \tilde{\lambda}_i(\nabla g_{reg}(L, R)) - 8\mu\sigma_1(X) \leq 2\sigma_1(X)\lambda_{\min}(\text{Hess} f(X)) \wedge 0.
\]

**Proof of the second inequality in (50).** The desired inequality can be obtained by \(51\), \(52\) and Lemma 8(ii) in the Appendix. This finishes the proof.

**Remark 6** (Comparison of Regularized and Unregularized Factorization Formulations). Compared to Theorem \(3\) the gap of the sandwich inequality in Theorem \(3\) depends explicitly on the spectrum of \(L\) and \(R\) and can be arbitrarily large for ill-conditioned \((L, R)\) pairs. Such an issue makes the geometry analysis for the unregularized factorization \(4\) hard \(\{Zhang et al., 2020\}\). On the other hand, any Euclidean FOSL of the regularized formulation \(5\) is always balanced \(\{Zhu et al., 2018\}\), i.e., satisfying \(L^\top L = R^\top R\), and the gap of the sandwich inequality in Theorem \(4\) only depends on \(X = LR^\top\), not individual \(L\) or \(R\).
In addition, comparing two factorization formulations \((4)\) and \((5)\), \(\nabla^2 g_{\text{reg}}(L, R)\) has \((r^2 + r)/2\) less zero eigenvalues than \(\nabla^2 g(L, R)\) as the regularization reduces the ambiguity set from invertible transforms to rotational transforms. On the other hand, it is difficult to control these potentially non-zero \((r^2 + r)/2\) eigenvalues in \(\nabla^2 g_{\text{reg}}(L, R)\) due to the complex interaction between the regularization and the original objective function. So that is why in Theorem 3 we can only get a partial sandwich inequality between \(\lambda_i(\nabla^2 g_{\text{reg}}(L, R))\) and \(\lambda_i(\text{Hess} f(X))\) while in Theorems 1 and 2 we have full sandwich inequalities.

**Remark 7 (Comparison of PSD and General Cases).** There are a few similarities and key differences in the landscape connection under the PSD case and the general case. First, in both cases, we tackle the problem via finding a connection between Riemannian and Euclidean Hessians on some carefully constructed points. However, exact Riemannian and Euclidean Hessian connections between \((1)\) and \((3)\) as well as \((2)\) and \((4)\) are available, while the Hessian connection between \((2)\) and \((5)\) is weaker. Second, although sandwich inequalities between the spectrum of Riemannian and Euclidean Hessians can be established in both the PSD case \((1)\) and \((3)\) and the general case \((2)\) and \((4)\), the gap of the sandwich inequality in the general case depends on the balancing of two factors \(L, R\) as we mentioned in Remark 6 while there is no such an issue in the PSD case. Finally, compared to the PSD one, there are two factorization formulations in the general case (unregularized and regularized ones) and it is nontrivial to extend the results from the PSD case to the general case. In particular, the regularized factorization formulation can potentially have a distinct landscape geometry from the unregularized one and establishing the landscape connection between \((2)\) and \((5)\) is much harder than \((1)\) and \((3)\) or \((2)\) and \((4)\).

By Theorems 2 and 3 we have the following Corollary 2 on the equivalence of FOSPs, SOSPs and strict saddles between the manifold and the factorization formulations in the general low-rank matrix optimization.

**Corollary 2. (Equivalence on FOSPs, SOSPs and Strict Saddles of Manifold and Factorization Formulations (General Case))** (a) If \((L, R)\) is a rank \(r\) Euclidean FOSP or SOSP or strict saddle of \((4)\) or \((5)\), then \(X = LR^\top\) is a Riemannian FOSP or SOSP or strict saddle of \((2)\); (b) if \(X\) is a Riemannian FOSP or SOSP or strict saddle of \((2)\), then any \((L, R)\) such that \(L R^\top = X\) is a Euclidean FOSP or SOSP or strict saddle of \((4)\) and any \((L, R)\) such that \(L R^\top = X, L^\top L = R^\top R\) is a Euclidean FOSP or SOSP or strict saddle of \((5)\).

In Theorems 2 and 3 we present the geometric landscape connection between the manifold and the two factorization formulations in the general low-rank matrix optimization. There is also a simple landscape connection between the two factorization formulations \((4)\) and \((5)\). This connection will be used to analyze the role of regularization in Section 5.3.

**Theorem 4. (Geometric Landscape Connection of Unregularized Formulation \((4)\) and Regularized Formulation \((5)\))** Suppose \((L, R)\) and \((L_{\text{reg}}, R_{\text{reg}})\) are rank \(r\) Euclidean FOSPs of \(g(L, R)\) and \(g_{\text{reg}}(L, R)\), respectively and \(L R^\top = L_{\text{reg}} R_{\text{reg}}^\top\). Let \(\Delta = L^\top L_{\text{reg}}\). Then \(\Delta\) is nonsingular and we can find a linear bijection \(\mathcal{J}\) on \(\mathbb{R}^{(p_1 + p_2) \times r}\) such that

\[
\mathcal{J}: A = \begin{bmatrix} A_L \\ A_R \end{bmatrix} \in \mathbb{R}^{(p_1 + p_2) \times r} \mapsto A' = \begin{bmatrix} A_L \Delta^{-1} \\ A_R \Delta^\top \end{bmatrix} \in \mathbb{R}^{(p_1 + p_2) \times r},
\]

such that

\[
\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}})[A, A] = \mu \| L_{\text{reg}}^\top A_L + A_L^\top L_{\text{reg}} - R_{\text{reg}}^\top A_R - A_R^\top R_{\text{reg}} \|^2_F = \nabla^2 g(L, R)[\mathcal{J}(A), \mathcal{J}(A)]
\]

(54)
holds for any $A = [A_L^T A_N^T]^T \in \mathbb{R}^{(p_1 + p_2) \times r}$.
Moreover, we have the following spectrum bounds for $J$:

$$
\theta^2_\Delta \|A\|_F^2 \leq \|J(A)\|_F^2 \leq \theta^2_{\Delta^a} \|A\|_F^2, \quad \forall A \in \mathbb{R}^{(p_1 + p_2) \times r},
$$

(55)

where $\Theta_\Delta := \sigma_1(\Delta) \vee (1/\sigma_r(\Delta))$ and $\theta_\Delta := 1/\Theta_\Delta = (1/\sigma_1(\Delta)) \wedge \sigma_r(\Delta)$.

Finally, for $1 \leq i \leq (p_1 + p_2)r$, the following connections on individual eigenvalues between $\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}})$ and $\nabla^2 g(L, R)$ hold:

$$
\lambda_i(\nabla^2 g(L, R)) \leq (\Theta^2_\Delta \lambda_i(\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}}))) \vee (\theta^2_\Delta \lambda_i(\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}})))
$$

and

$$
\lambda_i(\nabla^2 g(L, R)) \geq (\Theta^2_\Delta (\lambda_i(\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}})) - 8\mu \sigma_1(L_{\text{reg}} R_{\text{reg}})))
$$

(56)

5 Applications

In this section, we apply our main results to three specific problems from machine learning and signal processing.

5.1 Global Optimality for Phase Retrieval Under Manifold Formulation

We first consider the following real-valued quadratic equation system

$$
y_i = \langle a_i, x^* \rangle^2 \quad \text{for} \quad 1 \leq i \leq n,
$$

(57)

where $y \in \mathbb{R}^n$ and covariates $\{a_i\}_{i=1}^n \in \mathbb{R}^p$ are known whereas $x^* \in \mathbb{R}^p$ is unknown. The goal is to recover $x^*$ based on $\{y_i, a_i\}_{i=1}^n$. One important application is known as phase retrieval arising from physical science due to the nature of optical sensors (Fienup, 1982). A common formulation to solve (57) is the following least squares formulation:

$$
\tilde{g}(x) = \frac{1}{2n} \sum_{i=1}^n (y_i - \langle a_i, x \rangle^2)^2.
$$

(58)

In the literature, both convex relaxation (Candès et al., 2013; Waldspurger et al., 2015) and non-convex approaches (Candès et al., 2015; Chen and Candès, 2017; Ma et al., 2019; Netrapalli et al., 2013; Sanghavi et al., 2017; Wang et al., 2017a; Cai and Wei, 2018b) have been proposed to solve (58) with provable recovery guarantees. In terms of the geometric landscape analysis, Sun et al. (2018) showed that under the Gaussian design, i.e., $a_i$s are drawn from i.i.d. Gaussian distribution, $\tilde{g}(x)$ does not have any spurious local minima if $n \geq C p \log^3 p$ for some positive constant $C$. Later, the sample complexity requirement for the global optimality in phase retrieval under Gaussian design was improved to $n \geq C p$ for a slightly modified loss function (Li et al., 2019d):

$$
g(x) = \frac{1}{2n} \sum_{i=1}^n (y_i - \langle a_i, x \rangle^2)^2 h \left( \frac{\langle a_i, x \rangle^2}{\|x\|_2^2} \right) h \left( \frac{ny_i}{\|y\|_1} \right),
$$

(59)

where for two predetermined universal parameters $1 < \beta < \gamma$, the twice continuously differential activation function $h(a)$ satisfies:

$$
\begin{align*}
  h(a) &= 1 \quad \text{if} \quad 0 \leq a \leq \beta, \\
  h(a) &= [0, 1] \quad \text{if} \quad a \in (\beta, \gamma), \\
  h(a) &= 0 \quad \text{if} \quad a \geq \gamma
\end{align*}
$$

(60)
Comparing objectives in (58) and (59), \( g(x) \) incorporates a smooth activation function \( h \) to handle the heavy-tailedness of the fourth moment of Gaussian random variables in \( \tilde{g}(x) \). On the other hand, for both (58) and (59), the geometric landscape analyses performed in \( \text{Sun et al.} \, (2018) \); \( \text{Li et al.} \, (2019d) \) are carried out in terms of \( x \) in the vector space. However, it is known that by lifting \( x \) to \( X = xx^\top \), both (58) and (59) can be recast as a rank-1 PSD matrix recovery problem, e.g., \( \min_{x \in \mathbb{R}^d} g(x) \) has the following equivalent PSD manifold formulation:

\[
\min_{X \in S^{d \times d}, \text{rank}(X) = 1, X > 0} f(X) := \frac{1}{2n} \sum_{i=1}^{n} (y_i - \langle A_i, X \rangle)^2 h \left( \frac{\langle A_i, X \rangle}{\|X\|_F} \right) h \left( \frac{n y_i}{\|y\|_1} \right),
\]

where \( A_i = a_i a_i^\top \) for \( i = 1, \ldots, n \). Since \( h \) is twice continuously differentiable, the objective function in (61) is also twice continuously differentiable over rank-1 PSD matrices \( \text{Cai and Wei} \, (2018b) \); \( \text{Li et al.} \, (2019d) \) asked whether it is possible to investigate the geometric landscape of the phase retrieval problem directly on the rank-1 PSD matrix manifold. By our Theorem 1 and Corollary 1 we provide an affirmative answer to their question and give the first global optimality result for phase retrieval under the manifold formulation with a rate-optimal sample complexity.

**Theorem 5** (Global Optimality for Phase Retrieval under Manifold Formulation). In (61), suppose \( a_i \overset{i.i.d.}{\sim} N(0, I_p) \), \( \gamma > \beta > 1 \) are sufficiently large in the smooth activation function \( h \), and \( n \geq C p \) for large enough positive constant \( C \). Then with probability at least \( 1 - \exp(-C'n) \) for \( C' > 0 \), \( X^* = x^* x^{*\top} \) is the unique Riemannian SOSP of (61) and any other Riemannian FOSP \( X \) is a strict saddle with \( \lambda_{\text{min}}(\text{Hess} f(X)) \leq -\frac{3 \delta_1(X^*)}{4 \delta_1(X)} \).

**Remark 8** (Transferring the Strict Saddle Property). The key reason we can establish the global optimality and strict saddle results for phase retrieval under the manifold formulation is attributed to the spectrum connection of the Riemannian and the Euclidean Hessians given in Theorem 7. As a result of that, we can transfer the strict saddle property \( \text{Ge et al.} \, 2015 \); \( \text{Lee et al.} \, 2019 \), which states that the function has a strict negative curvature at all stationary points but local minima, from the factorization formulation of phase retrieval to the manifold one. This is fundamentally different from the results in \( \text{Ha et al.} \, 2020 \) where only the connection between Euclidean SOSP and fixed points of PGD was established without giving the estimation on the curvature of the Hessian. With this strict saddle property, various gradient descent and trust region methods are guaranteed to escape all strict saddles and converge to a SOSP \( \text{O'Neil and Wright} \, 2020 \); \( \text{Ge et al.} \, 2015 \); \( \text{Lee et al.} \, 2019 \); \( \text{Jin et al.} \, 2017 \); \( \text{Paternain et al.} \, 2019 \); \( \text{Sun et al.} \, 2018, 2019 \); \( \text{Criscitiello and Boumal} \, 2019 \); \( \text{Boumal et al.} \, 2019 \); \( \text{Han and Gao} \, 2020 \). Finally, we note here an explicit upper bound on the negative eigenvalue of the strict saddle in Theorem 5 can be helpful in determining the convergence rate of perturbed GD to the global minima \( \text{Jin et al.} \, 2017 \); \( \text{Sun et al.} \, 2019 \); \( \text{Criscitiello and Boumal} \, 2019 \).

### 5.2 Global Optimality of General Well-Conditioned Low-rank Matrix Optimization Under Manifold Formulation

In the existing literature on low-rank matrix optimization, most of the geometric landscape analyses focused on the factorization formulation. They showed that doing factorization for a rank constrained objective will not introduce spurious local minima when the objective \( f \) satisfies the restricted strong convexity and smoothness property (see the upcoming Definition 1) \( \text{Bhojanapalli et al.} \, 2016b \); \( \text{Ge et al.} \, 2017 \); \( \text{Zhang et al.} \, 2019 \); \( \text{Zhu et al.} \, 2018, 2021 \); \( \text{Park et al.} \, 2017 \). On the
other hand, the geometric analysis performed directly under the rank constrained manifold formulation is scarce. [Uschmajew and Vandereycken (2020)] showed the benign landscape of (2) under the embedded manifold geometry and exact-parameterization setting, i.e., \( r = \text{rank}(X^*) \), where \( X^* \) is a low-rank parameter matrix of interest, when \( f \) is quadratic and satisfies certain restricted spectral bounds property.

**Definition 1.** We say \( f : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R} \) satisfies the \((2r, 4r)\)-restricted strong convexity and smoothness property if for any \( X, G \in \mathbb{R}^{p_1 \times p_2} \) with \( \text{rank}(X) \leq 2r \) and \( \text{rank}(G) \leq 4r \), the Euclidean Hessian of \( f \) satisfies

\[
\alpha_1 \|G\|_F^2 \leq \nabla^2 f(X)[G, G] \leq \alpha_2 \|G\|_F^2
\]

for some \( \alpha_2 \geq \alpha_1 > 0 \).

By Theorem 3 and Corollary 2, we can leverage the existing results in [Zhu et al. (2018)] to provide the first global optimality result of (2) under the manifold formulation for a generic objective \( f \) satisfying the restricted strong convexity and smoothness property. Moreover, our results cover both over-parameterization \((r > \text{rank}(X^*))\) and exact-parameterization \((r = \text{rank}(X^*))\) settings compared with [Uschmajew and Vandereycken (2020)].

**Theorem 6.** (Global Optimality of Well-Conditioned Low-rank Matrix Optimization under Manifold Formulation) Consider the optimization problem (2). Suppose there exists a rank \( r^* (r^* \leq r) \) matrix \( X^* \) s.t. \( \nabla f(X^*) = 0 \) and \( f \) satisfies the \((2r, 4r)\)-restricted strong convexity and smoothness property (62) with positive constants \( \alpha_1 \) and \( \alpha_2 \) satisfying \( \alpha_2 / \alpha_1 \leq 1.5 \). Then,

- if \( r = r^* \), \( X^* \) is the unique Riemannian SOSP of (2) and any other Riemannian FOSP \( X \) is a strict saddle with \( \lambda_{\min}(\text{Hess}f(X)) \leq -0.04\alpha_1 \sigma_r(X^*)/\sigma_1(X) \);
- if \( r > r^* \), there is no Riemannian SOSP of (2) and any Riemannian FOSP \( X \) is a strict saddle with \( \lambda_{\min}(\text{Hess}f(X)) \leq -0.05\alpha_1 (\sigma_{r}(X) \wedge \sigma_{r^*}(X^*)) / \sigma_1(X) \).

**Remark 9.** As guaranteed by Proposition 1 of [Zhu et al. (2018)], the \((2r, 4r)\)-restricted strong convexity and smoothness property of \( f \) ensures \( X^* \) in Theorem 4 is the unique global minimizer of \( \min_{X: \text{rank}(X) \leq r} f(X) \). So Theorem 6 shows that if the input rank \( r \) is equal to the true rank \( r^* \), i.e., under exact-parameterization, then (2) has no spurious local minimizer other than the global minima \( X^* \) and any other Riemannian FOSP is a strict saddle. These two facts together ensure the recovery of \( X^* \) by many iterative algorithms [Lee et al. 2017, Sun et al. 2018, 2019, Criscitiello and Boumal 2017].

On the other hand, when the input rank \( r \) is greater than the true rank \( r^* \), i.e., under over-parameterization, Theorem 6 shows that there is no Riemannian SOSP for (2) and all Riemannian FOSP are strictly saddles. In addition, the upper bound on the negative curvature of the strict saddle implies that when running algorithms with guaranteed strict-saddle escaping property, the least singular value, i.e., \( \sigma_r(X) \), of the iterates will converge to zero. This suggests that the iterates tend to enter a lower rank matrix manifold and we can adopt some rank-adaptive Riemannian optimization methods to accommodate this [Zhou et al. 2016, Gao and Absil 2021]. We note this observation is only possible due to an explicit upper bound on the negative eigenvalue at strict saddles powered by the sandwich inequalities Theorem 3.

### 5.3 Role of Regularization in Nonconvex Factorization for Low-rank Matrix Optimization

As we have discussed in the introduction, for nonconvex factorization of the general low-rank matrix optimization, the regularized formulation (5) is often considered. The regularization is introduced
to balance the scale of two factors $L, R$ and it facilitates both algorithmic and geometric analyses in the nonconvex factorization formulation (Tu et al., 2016; Zheng and Lafferty, 2015; Ma et al., 2019; Wang et al., 2017b; Park et al., 2018; Zhu et al., 2018; Ge et al., 2017). On the other hand, it has been first observed empirically (Zhu et al., 2018), and then recently proved that the regularization is not necessary for iterative algorithms to converge in a number of smooth and non-smooth formulated matrix inverse problems (Du et al., 2018; Charisopoulos et al., 2021; Tong et al., 2019b; Ma et al., 2021; Ye and Du, 2021). Moreover, Li et al. (2020) showed from a geometric point of view that without regularization, the landscape of the factorization formulation (4) is still benign when $f$ satisfies the restricted strong convexity and smoothness property (62).

In this paper, we provide more geometric landscape connections between two factorization formulations (4) and (5) under a general $f$. Specifically, by connecting them with the manifold formulation, we show in Corollary 2 that the sets of $LR^\top$’s formed by rank $r$ Euclidean FOSPs and SOSP of two factorization formulations are exactly the same. By Theorem 4, we also have a connection on the spectrum of Hessians at Euclidean FOSPs under two factorization formulations. If we further assume $f$ is well-conditioned as in Section 5.2, then we can have the following global optimality result under the unregularized formulation (4).

\textbf{Theorem 7.} Consider the optimization problem (4). Suppose there exists a rank $r$ matrix $X^*$ s.t. $\nabla f(X^*) = 0$ and $f$ satisfies the $(2r, 4r)$-restricted strong convexity and smoothness property (62) with positive constants $\alpha_1$ and $\alpha_2$ satisfying $\alpha_2/\alpha_1 \leq 1.5$. Then for any rank $r$ Euclidean FOSP $(L, R)$ of $g(L, R)$, it is either a Euclidean SOSP and satisfies $LR^\top = X^*$, or a strict saddle with $\lambda_{\min}(\nabla^2 g(L, R)) \leq -0.08 \left( (\sigma_1^2(L)/\sigma_1(LR^\top)) \wedge (\sigma_1(LR^\top)/\sigma_1^2(L)) \right) \cdot \alpha_1 \sigma_r(X^*)$.

Part of the results in Theorem 7 have appeared in the recent work Li et al. (2020), but here we provide a precise upper bound on the negative curvature of the strict saddle that is absent in Li et al. (2020). Again, the precise upper bound on the negative curvature of the strict saddle is helpful in determining the convergence rate of perturbed GD to the global minima as we mentioned in Remark 8.

6 Conclusion and Discussions

In this paper, we consider the geometric landscape connection of the manifold and the factorization formulations in low-rank matrix optimization. We establish sandwich inequalities on the corresponding eigenvalues of the Riemannian and Euclidean Hessians and show an equivalence on the sets of FOSPs, SOSP and strict saddles between two formulations. These results provide partial reasons for the similar empirical performance of manifold and factorization approaches in low-rank matrix optimization. Finally, we apply our main results to three applications in machine learning and signal processing.

There are many interesting extensions to the results in this paper to be explored in the future. First, as we have mentioned in Remark 11 our results on the connection of Riemannian and Euclidean Hessians are established at FOSPs. It is interesting to explore whether it is possible to connect the geometry of the manifold and the factorization formulations of low-rank matrix optimization at non-stationary points. By achieving this we can (1) connect approximate SOSP\footnote{An approximate SOSP means the gradient norm at the point is small and the least eigenvalue of the Hessian at the point is lower bounded by a small negative constant (Jin et al., 2017).} between two formulations, which is useful in practice as standard optimization methods such as stochastic or perturbed gradient descent can only find approximate SOSP\footnote{An approximate SOSP means the gradient norm at the point is small and the least eigenvalue of the Hessian at the point is lower bounded by a small negative constant (Jin et al., 2017).}; (2) transfer the global geometry
properties (the landscape property of the objective in the whole space rather than at stationary points) between two formulations (Zhu et al., 2021; Li et al., 2019c). Second, in this work, we consider the natural embedded geometry of low-rank matrices in the manifold formulation. Another choice for handling low-rank matrices is the quotient manifold (Mishra et al., 2014). The follow-up work investigates the landscape connection of an optimization problem under the embedded and quotient geometries. Third, it is interesting to explore how will the landscape connect under two formulations when the objective function is nonsmooth. The connection of FOSPs might still be possible based on the notion of Clarke subdifferential (?), but some regularity condition on f might be needed. Finally, the manifold approach is a general way to deal with geometric constraints in optimization problems and here we show a strong geometric connection of it to the factorization approach in dealing with the rank constraint in matrix optimization. From an algorithmic perspective, connections of manifold methods with the sequential quadratic programming (SQP) method for solving equality constrained optimization problems and common nonlinear programming methods for handling orthogonal constraints were revealed in Edelman et al. (1998); Mishra and Sepulchre (2010) and Edelman et al. (1998), respectively. It is interesting future work to find more instances under which the manifold approach is geometrically or algorithmically connected with other well-known approaches in general nonlinear optimization.

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A Additional Proofs

### A.1 Additional Proofs in Section 2.1

**Proof of Proposition 1.** The expressions for Euclidean gradients are obtained via direct computation. For the Riemannian gradient, since $M_{r+}$ and $M_r$ are embedded submanifolds of $\mathbb{R}^{p \times p}$ and $\mathbb{R}^{p_1 \times p_2}$, respectively and the Euclidean metric is considered, from (Absil et al., 2009, (3.37)), we know the Riemannian gradients are the projections of the Euclidean gradients onto the corresponding tangent spaces. The results follow by observing the projection operator onto $T_{\mathcal{X}} M_{r+}$ and $T_{\mathcal{X}} M_r$ given in (9). ■

**Proof of Proposition 2.** First the expressions for $\nabla^2 g(L, R)[A, A]$ and $\nabla^2 g_{reg}(L, R)[A, A]$ are given in (Ha et al., 2020, Eq. (2.8)) and (Zhu et al., 2018, Section IV-A and Remark 8), respectively. The expressions for $\nabla^2 g(Y)[A', A']$ can be obtained by letting $A = [A^\top \quad A'^\top]^\top$ and $L = R = Y$ in $\nabla^2 g(L, R)[A, A]$.

Next, we derive the Riemannian Hessian of $f$. The Riemannian Hessian of an objective function $f$ is usually defined in terms of the Riemannian connection as in (6). But in the case of embedded submanifolds, it can also be defined by means of the so-called second-order retractions.

Given a general smooth manifold $\mathcal{M}$, a retraction $R$ is a smooth map from $T \mathcal{M}$ to $\mathcal{M}$ satisfying i) $R(\mathcal{X}, 0) = \mathcal{X}$ and ii) $\frac{d}{dt} R(\mathcal{X}, t \eta)|_{t=0} = \eta$ for all $\mathcal{X} \in \mathcal{M}$ and $\eta \in T_{\mathcal{M}} \mathcal{M}$, where $T \mathcal{M} = \{(\mathcal{X}, T_{\mathcal{X}} \mathcal{M}) : \mathcal{X} \in \mathcal{M}\}$, is the tangent bundle of $\mathcal{M}$ (Absil et al., 2009, Chapter 4). We also let $R_{\mathcal{X}}$ to be the restriction of $R$ to $T_{\mathcal{X}} \mathcal{M}$ and it satisfies $R_{\mathcal{X}} : T_{\mathcal{X}} \mathcal{M} \rightarrow \mathcal{M}, \xi \mapsto R(\mathcal{X}, \xi)$. Retraction is in general a first-order approximation of the exponential map (Absil et al., 2009, Chapter 4). A second-order retraction is the retraction defined as a second-order approximation of the exponential map (Absil and Malick, 2012). As far as convergence of Riemannian optimization methods goes, first-order retraction is sufficient (Absil et al., 2009, Chapter 3), but second-order retraction enjoys the following nice property: the Riemannian Hessian of an objective function $f$ coincides with the Euclidean Hessian of the lifted objective $f_{\mathcal{X}} := f \circ R_{\mathcal{X}}$.

**Lemma 6** (Proposition 5.5.5 of Absil et al. (2009)). Let $R_{\mathcal{X}}$ be a second-order retraction on $\mathcal{M}$. Then $\text{Hess} f(\mathcal{X}) = \nabla^2 (f \circ R_{\mathcal{X}})(0)$ for all $\mathcal{X} \in \mathcal{M}$.

We present the second-order retractions under both PSD and general low-rank matrix settings in the following Proposition 5.
Proposition 5 (Second-order Retractions in PSD and General Low-rank Matrix Manifolds).

- **PSD case:** Suppose \( X \in \mathcal{M}_{r+} \) has eigendecomposition \( U \Sigma U^\top \). Then the mapping \( R_X^{(2)} : T_X \mathcal{M}_{r+} \to \mathcal{M}_{r+} \) given by

\[
R_X^{(2)} : \xi = [U U_\perp] \begin{bmatrix} S & D^\top \\ D & 0 \end{bmatrix} [U U_\perp]^\top \to WX^\top W^\top
\]

is a second-order retraction on \( \mathcal{M}_{r+} \), where \( W = X + \frac{1}{2} \xi^s + \xi^p - \frac{1}{8} \xi^s X^\top \xi^s - \frac{1}{2} \xi^p X^\top \xi^s \), \( \xi^s = P_\perp \xi P_\perp \) and \( \xi^p = P_\perp \xi P_\perp \). Furthermore, we have

\[
R_X^{(2)}(\xi) = X + \xi + U_\perp D\Sigma^{-1}D^\top U_\perp + O(\|\xi\|_F^3), \quad \text{as } \|\xi\|_F \to 0.
\]

- **General case:** Suppose \( X \in \mathcal{M}_r \) has SVD \( U \Sigma V^\top \). Then the mapping \( R_X^{(2)} : T_X \mathcal{M}_r \to \mathcal{M}_r \) given by

\[
R_X^{(2)} : \xi = [U U_\perp] \begin{bmatrix} S \quad \Sigma & D_2^\top \\ D_1 & 0 \end{bmatrix} [V V_\perp]^\top \to WX^\top W
\]

is a second-order retraction on \( \mathcal{M}_r \), where \( W = X + \frac{1}{2} \xi^s + \xi^p - \frac{1}{8} \xi^s X^\top \xi^s - \frac{1}{2} \xi^p X^\top \xi^s - \frac{1}{2} \xi^s X^\top \xi^p \), \( \xi^s = P_\perp \xi P_\perp \) and \( \xi^p = P_\perp \xi P_\perp \). Furthermore, we have

\[
R_X^{(2)}(\xi) = X + \xi + U_\perp D_1\Sigma^{-1}D_2^\top V_\perp + O(\|\xi\|_F^3), \quad \text{as } \|\xi\|_F \to 0.
\]

**Proof of Proposition 5**

The results for the PSD case can be found in [Vandereycken and Vandewalle 2010, Proposition 5.10] and the results under the general case can be found in [Vandereycken 2013, Proposition A.1] and [Shalit et al. 2012, Theorem 3].

By Lemma 6 and the property of second-order retraction, the sum of the first three dominating terms in the Taylor expansion of \( f \circ R_X^{(2)}(\xi) \) w.r.t. \( \xi \) are \( f(X) + \langle \nabla f(X), \xi \rangle + \frac{1}{2} \text{Hess}_f(X)[\xi, \xi] \). By matching the corresponding terms and the expressions of \( R_X^{(2)} \) in Proposition 5, we can get the quadratic expression for \( \text{Hess}_f(X)[\xi, \xi] \).

Next, we discuss how to obtain \( \text{Hess}_f(X)[\xi, \xi] \) in PSD and general low-rank matrix manifolds, respectively.

**PSD case:** Given small enough \( \xi = [U U_\perp] \begin{bmatrix} S \quad D^\top \\ D & 0 \end{bmatrix} [U U_\perp]^\top \), define \( U_p = U_\perp D \). By Proposition 5 and Taylor expansion, we have

\[
f \circ R_X^{(2)}(\xi) = f(X) + \xi + U_p \Sigma^{-1}U_p^\top + O(\|\xi\|_F^3)
\]

\[
= f(X) + \xi + U_p \Sigma^{-1}U_p^\top + O(\|\xi\|_F^3)
\]

\[
= f(X) + \langle \nabla f(X) + \xi, U_p \Sigma^{-1}U_p^\top \rangle + O(\|\xi\|_F^3)
\]

\[
= f(X) + \langle \nabla^2 f(X) \xi, \xi \rangle + \frac{1}{2} \text{Hess}_f(X)[\xi, \xi] + \langle \nabla f(X), U_p \Sigma^{-1}U_p^\top \rangle + O(\|\xi\|_F^3). \tag{63}
\]

Since \( \xi^p X^\top \xi^p = U_p \Sigma^{-1}U_p^\top \), where \( \xi^p = P_\perp \xi P_\perp + P_\perp \xi P_\perp \), the second order term in (63) is \( \frac{1}{2} \text{Hess}_f(X)[\xi, \xi] + \langle \nabla f(X), U_p \Sigma^{-1}U_p^\top \rangle \) and it equals to \( \frac{1}{2} \text{Hess}_f(X)[\xi, \xi] \).
General case: Given small enough \( \xi = \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} S & D_1^T & D_2^T \\ D_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T \), define \( U_p = U_\perp D_1 \) and \( V_p = V_\perp D_2 \). By Proposition 5 and Taylor expansion, we have

\[
f \circ R_X^{(2)}(\xi) = f(X + \xi + U_p \Sigma^{-1} V_p^T + O(\|\xi\|_F^2))
= f(X + \xi + U_p \Sigma^{-1} V_p^T) + O(\|\xi\|_F^3)
= f(X + \xi) + \langle \nabla f(X), U_p \Sigma^{-1} V_p^T \rangle + O(\|\xi\|_F^3)
= f(X) + \langle \nabla f(X), \xi \rangle + \frac{1}{2} \nabla^2 f(X)[\xi, \xi] + \langle \nabla f(X), U_p \Sigma^{-1} V_p^T \rangle + O(\|\xi\|_F^3).
\]

(64)

Since \( \xi P X \xi^p = U_p \Sigma^{-1} V_p^T \), where \( P U_\perp \xi P V + P U \xi P V_\perp \), the second order term in (64) is \( \frac{1}{2} \nabla^2 f(X)[\xi, \xi] + \langle \nabla f(X), U_p \Sigma^{-1} V_p^T \rangle \) and it equals to \( \frac{1}{2} \text{Hess} f(X)[\xi, \xi] \). This finishes the proof of this proposition.

We note the proof technique for deriving the Riemannian Hessian is analogous to the proof of (Vandereycken 2013, Proposition 2.3). Here we extend it to the setting for a general twice differentiable function \( f \).

A.2 Additional Proofs in Section 3

Proof of Lemma 2 Suppose \( X \) has the eigendecomposition \( U \Sigma U^T \) and \( P = U^T Y \). Given \( \xi = \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} S & D_1^T \\ D_1 & 0 \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^T \). For any \( A \in \mathcal{A}_Y^\xi \), it is easy to check \( YA^T + AY^T = \xi \), so \( \mathcal{A}_Y^\xi \subseteq \{ A : YA^T + AY^T = \xi \} \). For any \( A \) such that \( YA^T + AY^T = \xi \), we have

\[
\begin{bmatrix} S & D_1^T \\ D_1 & 0 \end{bmatrix} \xi [U & U_\perp] = [U^T & U_\perp] (YA^T + AY^T)[U & U_\perp] = \begin{bmatrix} PA^T U + U^T A P^T & PA^T U_\perp \\ U_\perp A P^T & 0 \end{bmatrix}
\]

by observing \( Y = UP \). This implies \( U_\perp U_\perp^T A = U_\perp D P^T \) and \( PA^T U + U^T A P^T = S \). By denoting \( S_1 = U^T A P^T \), we have \( S_1 + S_1^T = S \) and \( UU^T A = US_1 P^T \). Finally, \( A = UU^T A + U_\perp U_\perp^T A = (UU^T + U_\perp U_\perp^T) P^T \in \mathcal{A}_Y^\xi \). This proves \( \mathcal{A}_Y^\xi \subseteq \{ A : YA^T + AY^T = \xi \} \) and finishes the proof.

Proof of Lemma 3 First, it is easy to check the dimensions of \( \mathcal{A}_\text{null}^Y \) and \( \mathcal{A}_\text{null}^\xi \) are \((r^2 - r)/2\) and \( pr - (r^2 - r)/2 \), respectively. Since \((r^2 - r)/2 + pr - (r^2 - r)/2 = pr \), to prove \( \mathbb{R}^{p \times r} = \mathcal{A}_\text{null}^Y \oplus \mathcal{A}_\text{null}^\xi \), we only need to show \( \mathcal{A}_\text{null}^Y \) is orthogonal to \( \mathcal{A}_\text{null}^\xi \). Suppose \( A = USP^T \in \mathcal{A}_\text{null}^Y \) and \( A' = (US' + U_\perp D)' P^T \in \mathcal{A}_\text{null}^\xi \). Then

\[
\langle A, A' \rangle = \langle SP^{-T}, S'P^{-T} \rangle = \langle S, S'P^{-T}P^{-1} \rangle \overset{(a)}{=} \langle S, S'S^{-1} \rangle \overset{(b)}{=} -\langle S^T, (S'S^{-1})^T \rangle = -\langle A, A' \rangle,
\]

where (a) is because \( PP^T = \Sigma \), (b) is because \( S + S^T = 0 \), and \( S'S^{-1} \) is symmetric by the construction of \( \mathcal{A}_\text{null}^Y \) and \( \mathcal{A}_\text{null}^\xi \), respectively. So we have \( \langle A, A' \rangle = 0 \) and this finishes the proof of this lemma.

Proof of Corollary 1 First, by the connection of Riemannian and Euclidean gradients in (18), the connection of FOSPs under two formulations clearly holds.

Suppose \( Y \) is a rank \( r \) Euclidean SOSP of (3) and let \( X = YY^T \). Given any \( \xi \in T_X M_{r^+} \), we have

\[
\text{Hess} f(X)[\xi, \xi] \overset{19}{=} \nabla^2 g(Y)[\mathcal{L}^{-1}_Y(\xi), \mathcal{L}^{-1}_Y(\xi)] \geq 0,
\]

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where the inequality is by the SOSP assumption on $Y$. Combining the fact $X$ is a Riemannian FOSP of (1), this shows $X = YY^T$ is a Riemannian SOSP of (1).

Next, let us show the other direction: suppose $X$ is a Riemannian SOSP of (1), then for any $Y$ such that $YY^T = X$, it is a Euclidean SOSP of (3). To see this, first $Y$ is of rank $r$ and we have shown $Y$ is a Euclidean FOSP of (3). Then by (19), we have for any $A \in \mathbb{R}^{p \times r}$:

$$\nabla^2 g(Y)[A, A] = \text{Hess} f(X)[\xi^\alpha, \xi^\beta] \geq 0.$$ 

Suppose $Y$ is a rank $r$ Euclidean strict saddle of (3) and let $X = YY^T$. It implies that there exists $A \in \mathcal{A}_{\text{null}}^Y$ such that $\nabla^2 g(Y)[A, A] < 0$. Then by (19) $\nabla^2 g(Y)[A, A] = \text{Hess} f(X)[\mathcal{L}(A), \mathcal{L}(A)] < 0$, and this implies that $\text{Hess} f(X)$ also has at least one eigenvalue. Thus, $X$ is a Riemannian strict saddle. The proof for the other direction is similar and for simplicity, we omit it here. ■

A.3 Additional Proofs in Section 4

Proof of Lemma 4  Given any tangent vector $\xi = [\mathbf{U} \quad \mathbf{U}_\perp]^T \begin{bmatrix} S & D_2^T \\ D_1 & 0 \end{bmatrix} \begin{bmatrix} V \\ V_\perp \end{bmatrix}^T$ in $T \mathcal{X}M_r^r$, denote $\mathcal{A}_1 = \{A = [A_L^T \quad A_R^T]^T : \mathbf{L}A_R^T + A_L^T \mathbf{R}^T = \xi\}$ and $\mathcal{A}_2 = \{A = [A_L^T \quad A_R^T]^T : \mathbf{L}A_R^T + A_L^T \mathbf{R}^T = \xi\}$. The rest of the proof is divided into two steps: in Step 1 we show the results on $\mathcal{A}_L^\xi$, in Step 2 we show the results on $\mathcal{A}_R^\xi$.

**Step 1.** It is clear $\dim(\mathcal{A}_L^\xi) = r^2$. For any $A = [A_L^T \quad A_R^T]^T \in \mathcal{A}_L^\xi$, it is straightforward to check $\mathbf{L}A_R^T + A_L^T \mathbf{R}^T = \xi$, so $\mathcal{A}_L^\xi \subseteq \mathcal{A}_1$. For any $A$ such that $\mathbf{L}A_R^T + A_L^T \mathbf{R}^T = \xi$, we have

$$\begin{bmatrix} S & D_2^T \\ D_1 & 0 \end{bmatrix} \xi = \begin{bmatrix} U^T \\ U_\perp \end{bmatrix} \begin{bmatrix} U^T & (\mathbf{L}A_R^T + A_L^T \mathbf{R}^T) \end{bmatrix} \begin{bmatrix} V \\ V_\perp \end{bmatrix} = \begin{bmatrix} P_1 A_R^T V + U^T A_L P_2^T \\ U_\perp A_L P_2^T \end{bmatrix} \begin{bmatrix} P_1 A_R^T V \\ 0 \end{bmatrix}$$

by observing $\mathbf{L} = \mathbf{U} \mathbf{P}_1, \mathbf{R} = \mathbf{V} \mathbf{P}_2$. This implies $P_{U_\perp} A_L = U_\perp D_1 P_2^T, P_{U^T} A_R = V_\perp D_2 P_1^T$ and $P_1 A_{R}^T V + U^T A_L P_2^T = \mathbf{S}$. By denoting $S_1 = U^T A_L P_2^T$ and $S_2 = V^T A_R P_1^T$, we have $S_1 + S_2 = \mathbf{S}, P_{U_\perp} A_L = \mathbf{U} \mathbf{S}_1 P_2^T$ and $P_{U^T} A_R = \mathbf{V} \mathbf{S}_1 P_1^T$. Finally, $A_L = P_{U_\perp} A_L + P_{U^T} A_L = (\mathbf{U} \mathbf{S}_1 + \mathbf{U} \mathbf{D}_1) P_2^T$, $A_R = P_{V} A_R + P_{V^T} A_R = (\mathbf{V} \mathbf{S}_2 + \mathbf{V} \mathbf{D}_2) P_1^T$. So $A = [A_L^T \quad A_R^T]^T \in \mathcal{A}_L^\xi$, and $\mathcal{A}_L^\xi \supseteq \mathcal{A}_1$. This proves the first result.

**Step 2.** Let us begin by proving $\dim(\mathcal{A}_R^\xi) = (r^2 - r)/2$. First, by simple computation, we have $\dim(\mathcal{A}_R^\xi) = \dim(\mathcal{J})$ where

$$\mathcal{J} := \left\{ S_1 \in \mathbb{R}^{r \times r} : \mathbf{P}_1^T S_1 \mathbf{P}_2^T - \mathbf{P}_1^T S_2 \mathbf{P}_1^T + \mathbf{P}_1^{-1} S_1 \mathbf{P}_2 + (\mathbf{P}_1^{-1} S_2 \mathbf{P}_2)^T = 0 \right\}.$$ 

Next, we show $\mathcal{J}$ is of dimension $(r^2 - r)/2$. Construct the following linear map $\varphi_{L,R} : S' \to \mathbf{P}_1^T S' \mathbf{P}_2^T - \mathbf{P}_1^{-1} S' \mathbf{P}_2$. We claim $\varphi_{L,R}$ is a bijective linear map over $\mathbb{R}^{r \times r}$:

- **injective part**: suppose there exists $S_1', S_2' \in \mathbb{R}^{r \times r}$ such that $S_1' \neq S_2'$ and $\varphi_{L,R}(S_1') = \varphi_{L,R}(S_2')$. Then by definition of $\varphi_{L,R}$, we have $\mathbf{P}_1^T (S_1' - S_2') \mathbf{P}_2^T + \mathbf{P}_1^{-1} (S_1' - S_2') \mathbf{P}_2 = 0$. It further implies $\mathbf{P}_1^T (S_1' - S_2') + (S_1' - S_2') \mathbf{P}_2 \mathbf{P}_2^T = 0$. This is a Sylvester equation with respect to $(S_1' - S_2')$ and we know from [Bhatia 2013 Theory VII.2.1] that it has a unique solution $0$ due to the fact $\mathbf{P}_1^T$ and $-\mathbf{P}_2 \mathbf{P}_2$ have disjoint spectra. So we get $S_1' = S_2'$, a contradiction.
• **surjective part:** for any \( \tilde{S} \in \mathbb{R}^{r \times r} \), we can find a unique \( \tilde{S}' \) such that \( \varphi_{L,R}(\tilde{S}') = \tilde{S} \). This follows from the facts: (1) \( \{S': P_1^T S' P_2^{-1} + P_1^{-1} S' P_2 = \tilde{S}\} = \{S': P_1^T S' + S' P_2 P_2^{-1} = \tilde{S} P_2^T\} \); (2) \( P_1 P_1^* S' + S' P_2 P_2^{-1} = \tilde{S} P_2^T \) is a Sylvester equation with respect to \( S' \) which has a unique solution again by [Bhatia 2013 Theorem VII.2.1](#). Then we have \( \mathcal{J} = \{\varphi_{L,R}^{-1}(S') : S' + S'^* = P_1^{-1} S P_2 + (P_1^{-1} S P_2)^*\} \) and

\[
\dim(\mathcal{J}) = \dim(\{\varphi_{L,R}^{-1}(S') : S' + S'^* = P_1^{-1} S P_2 + (P_1^{-1} S P_2)^*\}) = \dim(\{S' : S' + S'^* = P_1^{-1} S P_2 + (P_1^{-1} S P_2)^*\}) = (r^2 - r)/2.
\]

Finally, we show the second result. For any \( A = [A_L^T \quad A_R^T]^T \in \mathcal{A}_L \), it is straightforward to check \( L A_R^T + A_L R^T = \xi \) and \( L^T A_L + A_L^T L - R^T A_R - A_R^T R = 0 \). So \( \mathcal{A}_L \subseteq \mathcal{A}_L^L \). For any \( A \in \mathcal{A}_L^L \), following the same proof of [65], we have \( A_L = (U S_1 + U \perp D_1) P_2^{-1} \), \( A_R = (V S_2 + V \perp D_2) P_1^{-1} \) where \( S_1 = U^T A_L P_2^T \), \( S_2 = V^T A_R P_1^T \) and they satisfy \( S_1 + S_2 = S \). \( L^T A_L + A_L^T L - R^T A_R - A_R^T R = 0 \) further requires \( S_1, S_2 \) to satisfy \( P_1^T S_1 P_2^{-1} + P_2^{-1} S_2^T P_1 - P_1^T S_2^T P_2^{-1} - P_1^T S_2 P_2 = 0 \). So \( A = [A_L^T \quad A_R^T]^T \in \mathcal{A}_L^L \) and \( \mathcal{A}_L^L \cong \mathcal{A}_L \). This finishes the proof of this lemma. 

**Proof of Lemma 5** We first consider the result of \( \mathcal{A}_L^L \) and \( \mathcal{A}_L^L R^{null} \). It is easy to check \( \mathcal{A}_L^L \) and \( \mathcal{A}_L^L R^{null} \) are of dimensions \( r^2 \) and \( (p_1 + p_2 - r)r \), respectively. Since \( r^2 + (p_1 + p_2 - r)r = (p_1 + p_2)r \), to prove \( R^{(p_1+p_2) \times r} = \mathcal{A}_L^{null} \oplus \mathcal{A}_L^{null} \), we only need to show \( \mathcal{A}_L^{null} \) is orthogonal to \( \mathcal{A}_L^{null} \). Indeed, for any \( A = \begin{bmatrix} U S P_2^{-1} \\ -V S P_1^{-1} \end{bmatrix} \in \mathcal{A}_L^{null} \), and \( A' = \begin{bmatrix} (U S P_2 P_2^{-1} + U \perp D_1) P_2^{-1} \\ (V S P_1 P_1^{-1} + V \perp D_2) P_1^{-1} \end{bmatrix} \in \mathcal{A}_L^{null} \), by simple calculations, we have \( \langle A, A' \rangle = \langle S, S' \rangle = \langle S, S' \rangle = 0 \).

Next, we prove the result of \( \mathcal{A}_L^{null} \) and \( \mathcal{A}_L^{null} \). From the dimension of \( \mathcal{J} \) in Step 2 of the proof of Lemma 4, we have \( \dim(\mathcal{J}^L) = (r^2 - r)/2 \). As a result of this, we have \( \mathcal{J}^L \) is of dimension \( (r^2 - r)/2 \). Thus, \( (S_1 - S_2) \perp \mathcal{J}^L \) in the definition of \( \mathcal{J}^{null} \) adds \( (r^2 - r)/2 \) constraints and \( \dim(\mathcal{J}^{null}) = (p_1 + p_2)r - (r^2 - r)/2 \). Now, to prove \( R^{(p_1+p_2) \times r} = \mathcal{A}_L^{null} \oplus \mathcal{A}_L^{null} \), we only need to show \( \mathcal{J}^{null} \) is orthogonal to \( \mathcal{J}^{null} \). In fact, for any \( A = \begin{bmatrix} U S P_2^{-1} \\ -V S P_1^{-1} \end{bmatrix} \in \mathcal{J}^{null} \) and \( A' = \begin{bmatrix} (U S P_2 P_2^{-1} + U \perp D_1) P_2^{-1} \\ (V S P_1 P_1^{-1} + V \perp D_2) P_1^{-1} \end{bmatrix} \in \mathcal{J}^{null} \), we have \( \langle A, A' \rangle = \langle S, S' \rangle = 0 \), where the second equality is because \( S \in \mathcal{J}_L^L \) and \( (S'_1 - S'_2) \perp \mathcal{J}_L^L \) by the construction of \( \mathcal{J}_L^L \) and \( \mathcal{J}^{null}_L \), respectively. This finishes the proof of this lemma.

**Proof of Corollary 2** First, for any Euclidean FOSP \( (L, R) \) of [65] or \( (L, R) \) such that \( L^T L = R^T R \), we have \( \nabla g_{reg}(L, R) = \nabla g(L, R) \) by [65] and Proposition 1, respectively. The connection on FOSPs of different formulations can be easily obtained by the connection of Riemannian and Euclidean gradients given in [10]. Next, we show the equivalence on SOSP of different formulations.

Suppose \( X \) is a Riemannian SOSP of [2], we claim any \( (L, R) \) such that \( L^T = X \) is a Euclidean SOSP of [4] and any \( (L, R) \) such that \( L^T = X \) and \( L^T L = R^T R \) is a Euclidean SOSP of [5]. To see it, first \( (L, R) \) in both cases are Euclidean FOSP of [4] and [5] as mentioned before. For
any \( A = [A_L^T \ A_R^T]^T \in \mathbb{R}^{(p_1+p_2) \times r} \), by Theorems 2 and 3 we have

\[
\nabla^2 g(L, R)[A, A] \overset{11}{=} \nabla^2 g(L, R)[\xi_{L,R}^A, \xi_{L,R}^A] \geq 0;
\]

\[
\nabla^2 g_{\text{reg}}(L, R)[A, A] \overset{17}{=} \nabla^2 g_{\text{reg}}(L, R)[\xi_{L,R}^A, \xi_{L,R}^A] \geq 0.
\]

Next we show the reverse direction: suppose \((L, R)\) is a rank \( r \) Euclidean SOSP of \((4)\) or \((5)\), then \( X = LR^T \) is a Riemannian SOSP of \((2)\). To see this, for any \( \xi \in T_X M_r \),

\[
Hess f(LR^T)[\xi, \xi] \overset{11}{=} \nabla^2 g(L, R)[\xi, \xi] \geq 0,
\]

\[
Hess f(LR^T)[\xi, \xi] \overset{17}{=} \nabla^2 g_{\text{reg}}(L, R)[\xi, \xi] \geq 0.
\]

This shows \( X \) is a Riemannian SOSP of \((2)\).

Suppose \((L, R)\) is a rank \( r \) Euclidean strict saddle of \((4)\) or \((5)\), and let \( X = LR^T \). Then by definition there exists \( A_1, A_2 \in \mathbb{R}^{(p_1+p_2) \times r} \) such that \( \nabla^2 g(L, R)[A_1, A_1] < 0 \) and \( \nabla^2 g_{\text{reg}}(L, R)[A_2, A_2] < 0 \). Then

\[
Hess f(X)[\xi, \xi] \overset{11}{=} \nabla^2 g(L, R)[A_1, A_1] < 0;
\]

\[
Hess f(X)[\xi, \xi] \overset{17}{=} \nabla^2 g_{\text{reg}}(L, R)[A_2, A_2] < 0.
\]

This implies that \( Hess f(X) \) has negative eigenvalues in both cases, i.e., \( X \) is a Riemannian strict saddle. The proof for the reverse direction is similar and for simplicity, we omit it here. ■

**Proof of Theorem 4** This proof is divided into two steps. In Step 1, we show \((64)\); in Step 2, we give the spectrum bounds for the bijection map \( J \) and the spectrum connection between \( \nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}}) \) and \( \nabla^2 g(L, R) \).

**Step 1.** First, since \( L_{\text{reg}} R_{\text{reg}}^T = LR^T \), \( L_{\text{reg}} \) and \( L \) share the same leftsingular subspace. Thus \( L_\Delta = LL_{\text{reg}} \) and \( \Delta \) is of rank \( r \). Meanwhile, by \( LR^T = L_{\text{reg}} R_{\text{reg}}^T \), we have

\[
\Delta R_{\text{reg}}^T = L_{\text{reg}} R_{\text{reg}}^T = L_{\text{reg}} LR^T = L^T. \quad \text{Moreover, as } (L_{\text{reg}}, R_{\text{reg}}) \text{ is a Euclidean FOSP of } (5), \text{ by } \overset{46}{\text{we have for any } A = [A_L^T \ A_R^T]^T \in \mathbb{R}^{(p_1+p_2) \times r};}
\]

\[
\nabla^2 g_{\text{reg}}(L_{\text{reg}}, R_{\text{reg}})[A, A] = \nabla^2 g(L, R)[J(A), J(A)]. \quad \text{By Proposition } \overset{2}{\text{we have}}
\]

\[
\nabla^2 g(L_{\text{reg}}, R_{\text{reg}})[A, A] = \nabla^2 g(L, R)[J(A), J(A)].
\]

This finishes the proof for the first part.

**Step 2.** Next, we provide the spectrum bounds for the bijection operator. Suppose \( A = [A_L^T \ A_R^T]^T \) and \( J(A) = [A_L^T \ A_R^T]^T \). Then

\[
\|J(A)\|^2_F = \|A_L'\|^2_F + \|A_R'\|^2_F = \|A_L\Delta^{-1}\|^2_F + \|A_R\Delta^{-1}\|^2_F \leq (\sigma_1(\Delta) \vee (1/\sigma_r(\Delta)))^2 \|A\|^2_F,
\]

\[
\|A\|^2_F = \|A_L\|^2_F + \|A_R\|^2_F = \|A_L'\|^2_F + \|A_R'\|^2_F \leq (\sigma_1(\Delta) \vee (1/\sigma_r(\Delta)))^2 \|J(A)\|^2_F.
\]

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Finally, we provide a spectrum connection of two Euclidean Hessians at FOSPs. By (54), we have $\nabla^2_{g_{\text{reg}}}(L_{\text{reg}}, R_{\text{reg}}) \succeq J^* \nabla^2 g(L, R) J$. So the first inequality of (56) follows from Lemma 8(ii) in the Appendix and (55). Also by (45), (54) and Lemma 10, we have $\nabla^2_{g_{\text{reg}}}(L_{\text{reg}}, R_{\text{reg}}) - 8\mu_1 (L_{\text{reg}} R_{\text{reg}}^T) I \succeq J^* \nabla^2 g(L, R) J$ and the second inequality in (56) follows from Lemma 8(i) and (55). This finishes the proof. ■

A.4 Additional Proofs in Section 5

Proof of Theorem 5. By Theorem I.1 and Theorem II.2 of [Li et al. (2019d)], we have with probability at least $1 - \exp(-C'n)$, the factorization formulation $g(x)$ in (59) has the following geometric landscape properties: (1) $x^*$ is the unique Euclidean SOSP of $g(x)$; (2) for any other non-zero Euclidean FOSP $x$ of $g(x)$, it satisfies $\lambda_{\min}(\nabla^2 g(x)) \leq -3\|x^*\|^2 = -3\sigma_1(X^*)$ under the assumptions of Theorem 5.

By Corollary 1 we have $X^* = x^* x^T$ is the unique Riemannian SOSP of (61). In addition, by Theorem 1 for any other Riemannian FOSP $X$ of (61), we have

$$\lambda_{\min}(\text{Hess} f(X)) \leq \frac{1}{4\sigma_1(X)} \lambda_{\min}(\nabla^2 g(x)) \leq -\frac{3\sigma_1(X^*)}{4\sigma_1(X)},$$

where $x$ is any Euclidean FOSP satisfying $xx^T = X$. ■

Proof of Theorem 6. First, [Zhu et al. (2018)] considered the geometric landscape of (5) when $f$ satisfies the $(2r, 4r)$-restricted strong convexity and smoothness property. Under the assumptions of Theorem 6, Theorem 3 of [Zhu et al. (2018)] shows any Euclidean SOSP $(L, R)$ of the regularized factorization formulation satisfies $LR^T = X^*$. By Corollary 2 of this paper, we further conclude if the input rank $r = r^*$ in (2), then $X^*$ is the unique Riemannian SOSP of (2) and if $r > r^*$, there is no Riemannian SOSP of (2).

At the same time, by Theorem 3 of [Zhu et al. (2018)], any Euclidean FOSP $(L, R)$ of (5) that is not a SOSP must be a strict saddle and satisfy

$$\lambda_{\min}(\nabla^2_{g_{\text{reg}}}(L, R)) \leq \begin{cases} -0.08\alpha_1 \sigma_r(X^*), & \text{if } r = r^*; \\
-0.05\alpha_1 \cdot (\sigma^2_{\ell}(W) \wedge 2\sigma_{r^*}(X^*)), & \text{if } r > r^*, \end{cases}$$

where $W = [L^T \quad R^T]^T$ and $r^c$ is the rank of $W$. Under the manifold formulation (2), by Theorem 3 any Riemannian FOSP $X$ that is not a Riemannian SOSP must satisfy

$$\lambda_{\min}(\text{Hess} f(X)) \leq \lambda_{\min}(\nabla^2_{g_{\text{reg}}}(L', R'))/2\sigma_1(X) \leq \begin{cases} -0.08\alpha_1 \sigma_r(X^*)/(2\sigma_1(X)), & \text{if } r = r^*; \\
-0.05\alpha_1 \cdot (\sigma^2_{\ell}(W') \wedge 2\sigma_{r^*}(X^*))/(2\sigma_1(X)), & \text{if } r > r^*, \end{cases}$$

where $W' = [L'^T \quad R'^T]^T$ and $(L', R')$ is a rank $r$ Euclidean FOSP of (5) satisfying $L'R'^T = X$. Finally, we only need to compute $\sigma^2_{\ell}(W')$. By Lemma 11 we have $L' = UP$ and $R' = VP$ for some invertible $P \in \mathbb{R}^{r \times r}$, where $U, V$ are the left and right singular subspaces of $X$. So $\sigma_{r}(W') = \sigma_{r}([L'^T \quad R'^T]^T) = \sqrt{2}\sigma_{r}(P) = \sqrt{2}\sigma_{r}(X)$. This finishes the proof of this theorem. ■

Proof of Theorem 7. Under the assumptions of Theorem 3, by Theorem 3 of [Zhu et al. (2018)] we have for a rank $r$ Euclidean FOSP $(L_{\text{reg}}, R_{\text{reg}})$ of the regularized formulation (5), it is either a Euclidean SOSP satisfying $L_{\text{reg}} R_{\text{reg}}^T = X^*$ or a strict saddle with $\lambda_{\min}(\nabla^2_{g_{\text{reg}}}(L_{\text{reg}}, R_{\text{reg}})) \leq -0.08\alpha_1 \sigma_r(X^*)$. 38
By Corollary 2 and Theorem 4, we have for any rank r Euclidean FOSP \((L,R)\) of (4), it is either a Euclidean SOSP satisfying \(LR^T = X^*\) or a strict saddle with
\[
\lambda_{\min}(\nabla^2 g(L,R)) \leq \theta^2 \sigma_{\min}(\nabla^2_{reg} (L_{reg}', R_{reg}')) \leq -0.08 \theta^2 \alpha_1 \sigma_r(X^*),
\]
where \(\theta_\Delta := (1/\sigma_1(\Delta)) \wedge \sigma_r(\Delta), \quad \Delta = L' L_{reg} \) and \((L_{reg}', R_{reg}')\) is a rank \(r\) Euclidean FOSP of (5) satisfying \(L_{reg}' R_{reg}' = LR^T =: X\).

Finally, we give a lower bound for \(\theta_\Delta\). Notice \(L \Delta = L_{reg}'\), and
\[
\sigma_1(\Delta) = \sigma_1(L'L_{reg}') \leq \sigma_1(L') \sigma_{1}(L_{reg}') \leq \sigma_1(L') \sigma_1(L_{reg}') = \sigma_1(L'L_{reg}) = \frac{\sigma_1(L')}{\sigma_r(L)},
\]
\[
\sigma_r^{1/2}(X) \text{ Lemma 15} \sigma_r(L_{reg}') = \sigma_r(L) = \inf_{x: ||x||_2 = 1} ||L \Delta x||_2 \leq \sigma_1(L) \inf_{x: ||x||_2 = 1} ||\Delta x||_2 = \sigma_1(L) \sigma_r(\Delta).
\]
We have \(\theta_\Delta := (1/\sigma_1(\Delta)) \wedge \sigma_r(\Delta) \geq (\sigma_r(L)/\sigma_1^{1/2}(X)) \wedge (\sigma_r^{1/2}(X)/\sigma_1(L))\). This finishes the proof of this theorem. 

### B Additional Lemmas

Recall \(\lambda_k(\cdot)\) and \(\sigma_k(\cdot)\) are the \(k\)th largest eigenvalue and \(k\)th largest singular value of matrix \((\cdot)\). Also \(\lambda_{\max}(\cdot)\), \(\lambda_{\min}(\cdot)\) denote the largest and least eigenvalue of matrix \((\cdot)\).

**Lemma 7.** Suppose \(A \in \mathbb{S}^{p \times p}\) is symmetric and \(P \in \mathbb{R}^{p \times p}\) is invertible. Then \(\lambda_k(P^T AP)\) is sandwiched between \(\sigma_1^2(P) \lambda_k(A)\) and \(\sigma_r^2(P) \lambda_k(A)\) for \(k = 1, \ldots, p\).

**Proof.** Suppose \(u_1, \ldots, u_p\) are eigenvectors corresponding to \(\lambda_1(A), \ldots, \lambda_p(A)\) and \(v_1, \ldots, v_p\) are eigenvectors corresponding to \(\lambda_1(P^T AP), \ldots, \lambda_p(P^T AP)\). For \(k = 1, \ldots, p\), define
\[
\mathcal{U}_k = \text{span}\{u_1, \ldots, u_k\}, \quad \mathcal{U}_k' = \text{span}\{P^{-1} u_1, \ldots, P^{-1} u_k\},
\]
\[
\mathcal{V}_k = \text{span}\{v_1, \ldots, v_k\}, \quad \mathcal{V}_k' = \text{span}\{P v_1, \ldots, P v_k\}.
\]

Let us first consider the case that \(\lambda_k(A) \geq 0\). By Lemma 9, we have
\[
\lambda_k(P^T AP) \geq \min_{u \in \mathcal{U}_k, u \neq 0} \frac{u^T P^T AP u}{||u||^2} = \min_{u \in \mathcal{U}_k, u \neq 0} \frac{u^T A u}{||P^{-1} u||^2} \geq \min_{u \in \mathcal{U}_k, u \neq 0} \frac{\lambda_k(A) ||u||^2}{||P^{-1} u||^2} \geq \lambda_k(A) \sigma_1^2(P) \geq 0.
\]

On the other hand, we have
\[
\lambda_k(A) \geq \min_{u \in \mathcal{V}_k, u \neq 0} \frac{u^T P^T A P^{-1} u}{||u||^2} = \min_{v \in \mathcal{V}_k, v \neq 0} \frac{v^T A P v}{||P v||^2} \geq \min_{v \in \mathcal{V}_k, v \neq 0} \frac{\lambda_k(P^T AP) ||v||^2}{||P v||^2} \geq \frac{\lambda_k(P^T AP)}{\sigma_1^2(P)}.
\]

So we have proved the result for the case that \(\lambda_k(A) \geq 0\). When \(\lambda_k(A) < 0\), we have \(\lambda_{p+1-k}(-A) = -\lambda_k(A) > 0\). Following the same proof of (66) and (67), we have
\[
-\lambda_k(P^T AP) = \lambda_{p+1-k}(-P^T AP) \geq \sigma_2^2(P) \lambda_{p+1-k}(-A) = -\sigma_2^2(P) \lambda_k(A) > 0,
\]
\[
-\lambda_k(A) = \lambda_{p+1-k}(-A) \geq \lambda_{p+1-k}(-P^T AP)/\sigma_1^2(P) = -\lambda_k(P^T AP)/\sigma_1^2(P).
\]
This finishes the proof of this lemma. 

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Lemma 8. Suppose $A \in \mathbb{S}^{p \times p}, B \in \mathbb{S}^{q \times q}$ are symmetric matrices with $q \geq p$ and $P \in \mathbb{R}^{q \times p}, Q \in \mathbb{R}^{p \times q}$.

(i) If $P^TBP \succeq A$, then $\lambda_k(B)\sigma_1^2(P) \vee \lambda_k(B)\sigma_2^2(P) \geq \lambda_k(A)$ holds for $k = 1, \ldots, p$.

(ii) If $P^TBP \preceq A$, then $\lambda_{k+q-p}(B)\sigma_1^2(P) \land \lambda_{k+q-p}(B)\sigma_2^2(P) \leq \lambda_k(A)$ holds for $k = 1, \ldots, p$.

(iii) If $Q^TAQ \succeq B$, then $\lambda_{\min}(B) \geq \sigma_1^2(Q)\lambda_{\min}(A)$ \& 0.

(iv) If $Q^TAQ \preceq B$, then $\lambda_1(B) \leq \sigma_1^2(Q)\lambda_{\max}(A) \lor 0$.

Proof. We first prove the first and the second claims under the assumption that $\sigma_p(P) > 0$, i.e., all $p$ columns of $P$ are linearly independent.

Suppose $u_1, \ldots, u_p$ are eigenvectors corresponding to $\lambda_1(A), \ldots, \lambda_p(A)$, respectively and let $U_k = \text{span}\{u_1, \ldots, u_k\}$. Then

$$
\lambda_k(B) \geq \inf_{u \in U_k} \frac{\|u^T P^T B P u\|_2}{\|u\|_2^2} \geq \inf_{u \in U_k} \frac{\|u^T A u\|_2}{\|u\|_2^2} \geq \inf_{u \in U_k} \lambda_k(A)\|u\|_2^2 \geq \left\{ \begin{array}{ll}
\frac{\rho_2(B)}{\rho_p(B)} \lambda_{k+q-p}(B), & \text{if } \lambda_{k+q-p}(B) \geq 0 \\
\frac{\rho_1(B)}{\rho_p(B)} \lambda_{k+q-p}(B), & \text{if } \lambda_{k+q-p}(B) < 0
\end{array} \right.
$$

Here (a) is because \{P_{u_1}, \ldots, P_{u_k}\} forms a $k$ dimensional subspace in $\mathbb{R}^q$ and Lemma 9.

To prove the second claim under $\sigma_p(P) > 0$, suppose $v_1, \ldots, v_q$ are eigenvectors corresponding to $\lambda_1(B), \ldots, \lambda_q(B)$ and let $V_{k+q-p} = \text{span}\{v_1, \ldots, v_{k+q-p}\}$.

$$
\lambda_k(A) \geq \inf_{v : P v \in \mathbb{C}^{k+q-p}} \frac{v^T A v}{\|v\|_2^2} \geq \inf_{v : P v \in \mathbb{C}^{k+q-p}} \frac{v^T P^T B P v}{\|v\|_2^2} \geq \inf_{v : P v \in \mathbb{C}^{k+q-p}} \frac{\lambda_{k+q-p}(B)\|v\|_2^2}{\|v\|_2^2}
$$

Here (a) is because of Lemma 9 and the fact \{v : P v \in \mathbb{C}^{k+q-p}\} has dimension at least $k$.

When $\sigma_p(P) = 0$, we construct a series of matrices $P_l$ such that $\lim_{l \to \infty} P_l = P$ and $\sigma_p(P_l) > 0$. According to the previous proofs,

$$
\lambda_k(B)\sigma_1^2(P_l) \lor \lambda_k(B)\sigma_2^2(P_l) \geq \lambda_k(P_l^T B P_l), \quad \lambda_{k+q-p}(B)\sigma_1^2(P_l) \land \lambda_{k+q-p}(B)\sigma_2^2(P_l) \leq \lambda_k(P_l^T B P_l).
$$

Since $\sigma_k(\cdot)$ and $\lambda_k(\cdot)$ are continuous functions of the input matrix, by taking $l \to \infty$, we have

$$
\lambda_k(B)\sigma_1^2(P) \lor \lambda_k(B)\sigma_2^2(P) \geq \lambda_k(P^T B P) \underset{(a)}{\geq} \lambda_k(A), \quad \text{under the assumption of Claim 1;}
$$

$$
\lambda_{k+q-p}(B)\sigma_1^2(P) \land \lambda_{k+q-p}(B)\sigma_2^2(P) \leq \lambda_k(P^T B P) \underset{(a)}{\leq} \lambda_k(A), \quad \text{under the assumption of Claim 2.}
$$

Here in (a) we use the fact for any two $p_l$-by-$p_l$ symmetric matrices $W_1, W_2, W_1 \succeq W_2$ implies $\lambda_k(W_1) \geq \lambda_k(W_2)$ for any $k \in [p_l]$. This finishes the proof for the first two claims.

To prove the third claim, suppose $v_{\min}$ is the eigenvector corresponding to the smallest eigenvalue of $B$, then

$$
\lambda_{\min}(B) = v_{\min}^T B v_{\min} \geq v_{\min}^T Q^T A Q v_{\min} \geq \lambda_{\min}(A)\|Q v_{\min}\|_2^2 \geq \left\{ \begin{array}{ll}
0, & \text{if } \lambda_{\min}(A) \geq 0; \\
\sigma_1^2(Q)\lambda_{\min}(A), & \text{if } \lambda_{\min}(A) < 0.
\end{array} \right.
$$

To prove the last claim, suppose $v_{\max}$ is the eigenvector corresponding to the largest eigenvalue of $B$, then

$$
\lambda_1(B) = v_{\max}^T B v_{\max} \leq v_{\max}^T Q^T A Q v_{\max} \leq \lambda_{\max}(A)\|Q v_{\max}\|_2^2 \leq \left\{ \begin{array}{ll}
0, & \text{if } \lambda_{\max}(A) < 0; \\
\sigma_1^2(Q)\lambda_{\max}(A), & \text{if } \lambda_{\max}(A) \geq 0.
\end{array} \right.
$$

This finishes the proof of this lemma. ■
Lemma 9. (Max-min Theorem for Eigenvalues [Bhatia 2013, Corollary III.1.2]) For any $p$-by-$p$ real symmetric matrix $A$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. If $C_k$ denotes the set of subspaces of $\mathbb{R}^p$ of dimension $k$, then $\lambda_k = \max_{C \in C_k} \min_{u \in C, u \neq 0} u^T A u / \|u\|^2$.

Lemma 10. Suppose $L \in \mathbb{R}^{p_1 \times r}$ and $R \in \mathbb{R}^{p_2 \times r}$. Then for any $[A_L^T \ A_R^T]^T \in \mathbb{R}^{(p_1+p_2) \times r}$,

$$\|L^T A_L + A_L^T L - R^T A_R - A_R^T L\|_F^2 \leq 8(\sigma_1(L) \lor \sigma_1(R))^2 (\|A_R\|_F^2 + \|A_L\|_F^2).$$

Proof.

$$\|L^T A_L + A_L^T L - R^T A_R - A_R^T L\|_F^2 \leq 2(||L^T A_L + A_L^T L\|_F^2 + ||R^T A_R + A_R^T L\|_F^2) \leq 2(4\|L^T A_L\|_F^2 + 4\|R^T A_R\|_F^2) \leq 8(\sigma_1(L) \lor \sigma_1(R))^2 (\|A_R\|_F^2 + \|A_L\|_F^2).$$

This finishes the proof. \[\Box\]

Lemma 11. Suppose $L \in \mathbb{R}^{p_1 \times r}$, $R \in \mathbb{R}^{p_2 \times r}$ are two rank $r$ matrices and $L^T L = R^T R$. Let $U \Sigma V^T$ be a SVD of $LR^T$. Then we have $L = UP$, $R = VP$ for some $r$-by-$r$ full rank matrix $P$ satisfying $PP^T = \Sigma$.

Proof. First since $LR^T$ has SVD $U \Sigma V^T$, we have $L = UP_1$ and $R = VP_2$. Next we show $P_1 = P_2$. Since $P_1 P_2^T = \Sigma$, we have

$$\Sigma^2 = P_1 P_2^T P_2 P_1 = (a) P_1 P_1^T P_1^T (b) \Rightarrow \Sigma = P_1 P_1^T.$$

Here (a) is because $L^T L = R^T R$ implies $P_1^T P_1 = P_2^T P_2$; and (b) is because a PSD matrix has a unique principal square root [Johnson et al. 2001]. This finishes the proof of this lemma. \[\Box\]