The Pricing of Double Trigger Catastrophe Put Option with Default Risk

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Abstract

This study was present a catastrophe put option pricing model that considers default risk. The default of the option issuer can occur at any time before the maturity, and there is a correlation between the total assets of the option issuer, the underlying stock and the zero coupon bond. The explicit solution of option pricing is obtained when the interest rate process follows the Vasicek model and relevant proofs are given. Finally, the value changes under different parameters are discussed through a numerical analysis.

Keywords: Catastrophe option; Vasicek interest rate model; default risk.

1 Introduction

Default risk, also known as credit risk, refers to the risk of one party defaulting in a financial contract before it matures. It has received extensively attention due to the financial crisis during 2007-2008. For example, Cox and Pedersen [1] studied the pricing of catastrophe bonds, and simply discussed the equilibrium pricing principle and its relationship with the standard no arbitrage valuation framework. Dassios and Jang [2] used the Cox process (also known as double stochastic Poisson process) to simulate the arrival process of catastrophe claims, and study the pricing problems of stop loss catastrophe reinsurance contracts and catastrophe insurance derivatives. Liao and Huang 0 constructed the valuation formula of Black-Scholes options affected by interest rate risk and credit risk. Jaimungal and Wang 0 constructed a catastrophe option

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pricing model, in which the loss process is a compound Poisson process and the interest rate is a Vasicek model. The specific pricing formula of options is given and the dynamic hedging problem is studied. Lee and Yu 0 considered a reinsurance contract using the framework of contingent claims and studied how reinsurance companies can reduce the default risk by issuing catastrophe bonds. Jiang et al. 0 proposed a catastrophe option pricing model considering default risk, in which option issuers can only default at maturity. Wang 0 constructed a pricing model of catastrophe put option under default risk and gave the specific pricing formula. The readers are referred to Xu and Wang [8] and the references therein for more catastrophe option pricing models under the credit risk of counterparties.

In this paper, a catastrophe put option pricing model that considers default risk, in which the default of option issuer can happen at any time before the maturity. The correlation between the total asset value of option issuers, the value of underlying stocks and zero coupon bonds are assumed. This study’s primary contribution is to provide an analytical solution for evaluating catastrophe put option with default risk and Vasicek interest model 0 under the forward measure.

2 The Modeling Assumptions

In this section, we propose a simplified form of catastrophe put option pricing model with default risk. Specifically, it is assumed that the loss process is generated by a doubly stochastic Poisson process, the stock price process is characterized by a jump diffusion process related to the loss process and the risk-free short interest rate process and the interest rate process follows the Vasicek model. It is assumed that all processes are carried out under a risk neutral measure \( Q \) on a filtered probability space \( (\Omega, F, (F_t)_{t \in \mathbb{R}^+}, Q) \). At maturity \( T \), consider a special form of double trigger option, which has a payoff

\[
\text{payoff} = 1_{\{r(t) \leq l_i \}} \left( K - S(T) \right) = \begin{cases} 
K - S(T), & S(T) < K \& \ L(T) - L(t) > L, \\
0, & S(T) > K \& \ L(T) - L(t) \leq L,
\end{cases}
\]

Where, \( S(T) \) represents the stock price at time \( T \), and \( L(T) - L(t) \) represents the total loss for the insured in \( [t, T] \). The parameter \( L \) is the trigger level of the loss, and \( K \) represents the strike price of the option.

Suppose \( \{N(t) : t \geq 0\} \) is the number of catastrophe losses, which is a homogeneous Poisson process with intensity \( \lambda \). Meanwhile, \( \{I_i = 1, 2, \cdots\} \) are the size of the \( i \)-th loss with probability density function (hereafter p.d.f.) \( f(i) \), and which are independent and identically distributed. The independence between \( \{I_i = 1, 2, \cdots\} \) and \( \{N(t) : t \geq 0\} \) is also assumed. The cumulative loss at time \( t \) is denoted by \( L(t) = \sum_{i=1}^{N(t)} I_i \).

The stock price process \( \{S(t) : t \geq 0\} \) under \( Q \) measure is assumed to be:

\[
S(t) = X(0) \exp \left( -\alpha (L(t) - \kappa t) + \int_0^t \left( r(u) - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dW_t(u) \right) = \exp \left( -\alpha (L(t) - \kappa t) \right) X(t)
\]

(2.1)

where

\[
X(t) = X(0) \exp \left( \int_0^t \left( r(u) - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dW_t(u) \right)
\]

the constant \( \alpha \) describes the impact of the loss on the stock price, and \( \kappa > 0 \) specifies the compensation for the stock price decline caused by the loss. \( \sigma \) denotes the volatility of the underlying stock, \( \{W_t(t) : t \geq 0\} \) is \( Q \) – standard Brownian motion, and \( \{r(t) : t \geq 0\} \) denotes the Vasicek stochastic interest rate process and the asset dynamic of the option issuer are as follows:
where $\sigma_r$ and $\sigma_v$ denote the volatilities, $\{W_r(t): t \geq 0\}$ and $\{W_v(t): t \geq 0\}$ are $Q$-standard Brownian motions. The constant instantaneous correlation coefficients among the standard Brownian motions are given by $d[W_r(t), W_r(t)] = \rho_{dr} dt$, $d[W_r(t), W_v(t)] = \rho_{dv} dt$ and $d[W_v(t), W_v(t)] = \rho_{vv} dt$.

Assuming that the option issuer’s default may occur at any time before the option’s maturity. If the market value $V(t)$ of the counterparty’s assets is lower than the amount $D$ of all outstanding claims set, a credit loss will occur. The time when the default occurs can be expressed as:

$$\tau := \inf\{t > 0 | V(t) \leq D\},$$

Once credit loss occurs, assuming that the recovery rate is a constant $\beta$, $(1-\beta)V(\tau)$ represents the deadweight cost due to bankruptcy or restructuring, and the remaining value of $\beta V(\tau)$ is paid to holders of options and other liabilities.

### 3 Pricing the Catastrophe Put Option

Based on the model framework described in the previous section, we consider the pricing of catastrophe put option with default risk with maturity $T$ and strike price $K$. Under the risk-neutral measure $\hat{Q}$, the value of the catastrophe put option with default risk can be obtained by discounting the expected return. Let $C'$ be the price of the put option with default risk, then the price of the option is:

$$C' = E \left[ \exp \left\{ - \int_0^\tau r(u) du \right\} I(\tau > T)I(L(T) > L)(K - S(T))^{-} \right]$$

$$+ E \left[ \exp \left\{ - \int_0^\tau r(u) du \right\} I(0 \leq \tau \leq T)I(L(T) > L)(K - S(T))^{-} | F_\tau \right]$$

$$= E \left[ \exp \left\{ - \int_0^\tau r(u) du \right\} I(\tau > T)I(L(T) > L)(K - S(T))^{-} \right]$$

$$+ E \left[ \beta \exp \left\{ - \int_0^\tau r(u) du \right\} I(0 \leq \tau \leq T)I(L(T) > L)(K - S(T))^{-} \right]$$

Because of $I(0 \leq \tau \leq T) = 1 - I(\tau > T)$, then

$$C' = \beta E \left[ \exp \left\{ - \int_0^\tau r(u) du \right\} I(L(T) > L)(K - S(T))^{-} \right]$$

$$+ (1 - \beta) E \left[ \exp \left\{ - \int_0^\tau r(u) du \right\} I(\tau > T)I(L(T) > L)(K - S(T))^{-} \right]$$

If the interest rate is fixed, the discount factor can be extracted from the expectation. In the presence of stochastic interest rate, a forward neutral measure $Q^f$ can be constructed, then the value of the option becomes:

$$C' = \beta P(0, T) E^f \left[ I(L(T) > L)(K - \hat{S}(T))^{-} \right]$$

$$+ (1 - \beta) P(0, T) E^f \left[ I(\tau > T)I(L(T) > L)(K - \hat{S}(T))^{-} \right]$$

$$= E_i + E_f$$
Where,

\[ E_1 = \beta P(0, T) E^\varphi \left[ I(L(T) > L)(K - \tilde{S}(T)) \right], \]  

(3.1)

\[ E_2 = (1 - \beta) P(0, T) E^\varphi \left[ I(\tau > T) I(L(T) > L)(K - \tilde{S}(T)) \right], \]  

(3.2)

\( P(0, T) \) represents the price of a zero-coupon bond with maturity \( T \) at zero time.

Under the Vasicek model, the price of zero-coupon bond \( P(t, T) \) with maturity \( T \) is:

\[ P(t, T) = e^{A(t, T) - r(t) B(t, T)}, \]

Where

\[ A(t, T) = \left( \theta - \frac{\sigma^2}{2k^2} \right) \left[ B(t, T) - (T - t) \right] - \frac{\sigma^2}{4k} B^t(t, T) \]

\[ B(t, T) = \frac{1 - \exp\left(-k(T-t)\right)}{k} \]

And the price of zero-coupon bond satisfies the stochastic differential equation:

\[ \frac{dP(t, T)}{P(t, T)} = r(t) dt - \sigma B(t, T) dW(t) \]

Note that the volatility term is deterministic. The following lemma gives the concrete construction of the forward measure \( Q^T \).

**Lemma 1.** Let \( \eta \) denote the Radon-Nikodym process

\[ \eta = \frac{dQ^T}{dQ} = \frac{P(t, T)}{P(0, T)} D(0, t) = \exp\left\{ -\frac{1}{2} \sigma^2 \int_0^t B(u, T) du - \sigma \int_0^t B(u, T) dW(u) \right\} \]

And for any \( \mathcal{A} \in \mathcal{F} \), there is \( Q^T(\mathcal{A}) = E^Q[1(\mathcal{A}) \eta(T)] \), \( \tilde{W}_r(t) \), \( \tilde{W}_1(t) \) and \( \tilde{W}_2(t) \) are defined as:

\[ \tilde{W}_r(t) = W_r(t) + \int_0^t \sigma B(u, T) du, \]
\[ \tilde{W}_1(t) = W_1(t) + \int_0^t \rho_1 \sigma B(u, T) du, \]
\[ \tilde{W}_2(t) = W_2(t) + \int_0^t \rho_2 \sigma B(u, T) du, \]

Where, \( \tilde{W}_r(t) \), \( \tilde{W}_1(t) \) and \( \tilde{W}_2(t) \) are \( Q^T \) – standard Brownian motion in the forward risk-neutral probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t^c), \mathcal{Q}^T) \).

**Proof.** It can be proved by Ito Lemma and Girsanov theorem.

**Proposition 2.** \( E^T \) is determined by formula (3.1), so we can get

\[ E_i = \beta \sum_{n=1}^{\infty} \frac{(\Delta T)^n}{n!} \exp\left\{ -\Delta T \right\} \int_{-\infty}^{\infty} f^{(i)}_r(y) \left( K P(0, T) \Phi(d_i) - X(0) \exp\left\{ -\alpha (L(T) - \kappa T) \right\} \Phi(d_2) \right\} dy \]
where,

\[
\ln \frac{F_r(0,T)}{D} - \int_0^T \frac{1}{2} \sigma_r^2(u,T) du = \ln \frac{F_r(0,T)}{D} + \int_0^T \frac{1}{2} \sigma_r^2(u,T) du
\]

\[
d_s = \frac{\ln \left( \frac{K}{F_r(0,T)} + \alpha \left( L(T) - \kappa T \right) + \int_0^T \frac{1}{2} \sigma_r^2(u,T) du \right)}{\sqrt{\int_0^T \sigma_r^2(u,T) du}},
\]

\[
d_s = \frac{\ln \left( \frac{K}{F_r(0,T)} + \alpha \left( L(T) - \kappa T \right) + \int_0^T \frac{1}{2} \sigma_r^2(u,T) du \right)}{\sqrt{\int_0^T \sigma_r^2(u,T) du}},
\]

\[
d_s = \frac{\ln \left( \frac{K}{F_r(0,T)} + \alpha \left( L(T) - \kappa T \right) + \int_0^T \frac{1}{2} \sigma_r^2(u,T) du \right)}{\sqrt{\int_0^T \sigma_r^2(u,T) du}}.
\]

Proposition 4. \( E_r \) is determined by formula (3.2), so we can get

\[
E_r = (1 - \beta) \sum_{i=1}^{\infty} \left[ \left( \frac{\sigma_r}{\sigma_y} \right)^2 \Phi \left( d_r, d_r, \rho_{rr} \right) \right] dy
\]

Proof. The proof is shown in Appendix A.

Lemma 3. Define a probability measure \( Q^\alpha \) equivalent to \( Q^\nu \) with the following Radon-Nikodym process

\[
\Lambda(t) = \left( \frac{dQ^\alpha}{dQ^\nu} \right)^{\frac{1}{2}} = E^Q \left[ \exp \left\{ \int_0^t \left( \frac{1}{2} \sigma_r^2(u,T) du + \int_0^t \sigma_r(u,T) d\tilde{W}_r(u) \right) \right\} \right]
\]

where \( \tilde{W}_r(t) \) and \( \tilde{W}_r(t) \) are defined as

\[
\tilde{W}_r(t) = \tilde{W}_r(t) - \int_0^t \sigma_r(u,T) du, \quad \tilde{W}_r(t) = \tilde{W}_r(t) - \int_0^t \sigma_r(r,T) du.
\]

Proof. It can be proved by Girsanov theorem.
\[
\begin{align*}
\ln \frac{K}{F_y(0, T)} + \alpha (L(T) - \kappa T) + \int_0^t \left( \frac{1}{2} \sigma^2_x(t) + \frac{2}{\sigma_x(t)} \right) du + \frac{2}{\sigma_x(t)} \ln \frac{D}{F_x(0, T)} \\
\end{align*}
\]

\[
\begin{align*}
d_s &= \sqrt{\int_0^t \sigma^2_x(u, T) du}, \\
\end{align*}
\]

\[
\begin{align*}
\ln \frac{F_y(0, T)}{D} + \int_0^t \left( -\frac{1}{2} \sigma^2_x(t) + \rho \sigma_x(t) \sigma_y(t) \right) du \\
\end{align*}
\]

\[
\begin{align*}
d_s &= \sqrt{\int_0^t \sigma^2_x(u, T) du}, \\
\end{align*}
\]

\[
\begin{align*}
\ln \frac{D}{F_x(0, T)} + \int_0^t \left( -\frac{1}{2} \sigma^2_x(t) + \rho \sigma_x(t) \sigma_y(t) \right) du \\
\end{align*}
\]

\[
\begin{align*}
d_s &= \sqrt{\int_0^t \sigma^2_x(u, T) du}, \\
\end{align*}
\]

\[
\begin{align*}
\ln \frac{K}{F_x(0, T)} + \alpha (L(T) - \kappa T) - \int_0^t \frac{1}{2} \sigma^2_x(u, T) du + \frac{2}{\sigma_x(t)} \ln \frac{D}{F_y(0, T)} \\
\end{align*}
\]

\[
\begin{align*}
\sigma^2_x(t, T) = \sigma^2_x(t) - 2 \sigma_x(t) \sigma_y(t) B(t, T) \rho_x + \sigma_y^2(t) B^2(t, T), \\
\sigma^2_y(t, T) = \sigma^2_y(t) - 2 \sigma_x(t) \sigma_y(t) B(t, T) \rho_x + \sigma_x^2(t) B^2(t, T), \\
\hat{L} = \max \{ L + L(0) - L(T), 0 \}, \\
f^{(n)}_x(y) \text{ is the n-fold convolution of the loss probability density function } f_x(y) \text{, and } \Phi_x(\cdot, \cdot, \cdot) \text{ is the bivariate normal cumulative distribution function.}
\]

**Proof.** The proof is shown in Appendix B.

From Proposition 2 and Proposition 4, the expression of \( \mathcal{C} \) can be obtained. Assuming that the size of the loss conditional on the loss is fixed to \( l \) and the trigger level is an integer multiple of the size of the loss, that is, \( L = Nl \), we can get the pricing of catastrophe put options with default risk. In this case, the probability density function of the loss size is Dirac density \( f(y) = \delta(y - l) \), and we get the following corollary.
Corollary 5. Suppose that the probability density function of the loss size is Dirac density $f(y) = \delta(y - l)$, then the expression of $C^*$ becomes

$$C^* = \sum_{n=0}^{\infty} \left( \frac{2T}{n!} \exp{-\lambda T} \right) \left[ \beta K P(0,T) \Phi\left( d_1(n) \right) - \lambda X(0) \exp{-anl + \left( 1 - \exp{-al} \right) T} \Phi\left( d_1(n) \right) \right]$$

$$+ \left( 1 - \beta \right) \left[ K P(0,T) \Phi\left( d_1(n) \right) - \lambda X(0) \exp{-anl + \left( 1 - \exp{-al} \right) T} \Phi\left( d_1(n) \right) \right]$$

$$\cdot \left[ \Phi\left( d_1, d_1(n) - \rho_{\ell\ell} \right) - \frac{F \left( 1, 0, T \right) \left( \frac{D}{F \left( 0, T \right)} \right) \int_{0}^{\infty} e^{-u} \exp{-\lambda T} \exp{-anl + \left( 1 - \exp{-al} \right) T} \Phi\left( d_1, d_1(n) - \rho_{\ell\ell} \right)}{\int_{0}^{\infty} e^{-u} \exp{-\lambda T} \exp{-anl + \left( 1 - \exp{-al} \right) T} \Phi\left( d_1, d_1(n) - \rho_{\ell\ell} \right)} \right]$$

Where

$$d_1(n) = \frac{\ln \left( \frac{K}{F \left( 0, T \right)} \right) + anl - \left( 1 - \exp{-al} \right) T + \frac{1}{2} \int_{0}^{T} \sigma^2 \left( u, T \right) du}{\sqrt{\int_{0}^{T} \sigma^2 \left( u, T \right) du}}$$

$$d_1(n) = \frac{\ln \left( \frac{K}{F \left( 0, T \right)} \right) + anl - \left( 1 - \exp{-al} \right) T - \frac{1}{2} \int_{0}^{T} \sigma^2 \left( u, T \right) du}{\sqrt{\int_{0}^{T} \sigma^2 \left( u, T \right) du}}$$

$$d_1(n) = \frac{\ln \left( \frac{K}{F \left( 0, T \right)} \right) + anl - \left( 1 - \exp{-al} \right) T + \frac{1}{2} \int_{0}^{T} \sigma^2 \left( u, T \right) du + \int_{0}^{T} \sigma^2 \left( u, T \right) d\rho \left( u, T \right) \ln \left( \frac{D}{F \left( 0, T \right)} \right)}{\sqrt{\int_{0}^{T} \sigma^2 \left( u, T \right) du}}$$

$$d_1(n) = \frac{\ln \left( \frac{K}{F \left( 0, T \right)} \right) + anl - \left( 1 - \exp{-al} \right) T - \int_{0}^{T} \frac{1}{2} \sigma^2 \left( u, T \right) du}{\sqrt{\int_{0}^{T} \sigma^2 \left( u, T \right) du}}$$

$$d_1(n) = \frac{\ln \left( \frac{K}{F \left( 0, T \right)} \right) + anl - \left( 1 - \exp{-al} \right) T - \int_{0}^{T} \frac{1}{2} \sigma^2 \left( u, T \right) du}{\sqrt{\int_{0}^{T} \sigma^2 \left( u, T \right) du}}$$

Up to now, in the case that the option issuer may default at any time before the maturity of the option, we have proposed the pricing formula of catastrophe put option with default risk. In the next section, we illustrate the changes in the price of catastrophe put options with default risk under different parameter values by assuming that the size of the loss is described by a lognormal random variables.
4 Numerical Example

In this section, we study the changes in the price of catastrophe put options with default risk under different parameter values. In the model proposed above, catastrophe event is modeled by doubly stochastic Poisson process. The probability density function of loss size \( l_i, i = 1, 2, \ldots \) is Dirac density \( f(y) = \delta (y - l) \), and the lognormal distribution is used to describe the size of the loss.

In order to obtain the option price, we now give the following preference parameters. In this case, the option is denominated in currency and issued by a highly leveraged company, and the bankruptcy boundary \( D \) is assumed to be \( 3/4 \) times the initial value of the option issuer's assets \( V(0) \), where \( V(0) = 100 \), \( D = 75 \). In addition, assume that the initial price of the stock \( S(0) = 80 \), the correlation between the forward price of the total assets of the option issuer and the forward price of the underlying stock is \( \rho_{S_t,S} = -0.5 \), the forward initial price of the total assets of the option issuer \( F_t(0,T) = 100 \), the forward initial price of the underlying stock \( F_t(0,T) = 80 \), the strike price \( K = 80 \), the maturity \( T = 3 \), the recovery rate \( \beta = 0.6 \), the impact of the loss \( \alpha = 0.0008 \), the intensity of the loss \( \lambda = 0.5 \), the volatility of the underlying stock \( \sigma_s = 0.2 \), the volatility of the asset of the option issuer \( \sigma_r = 0.2 \), the volatility \( \sigma_r = 0.2 \), the trigger level \( L = 3 \), the speed regression of the Vasicek model \( k = 0.3 \).

The numerical analysis of some basic variables of option prices is shown in Fig. 1 to Fig. 4.

Fig. 1 shows the value of catastrophe put options under different maturities. We take the maturity date \( T = 0.25, 0.5, 0.75, 1.00, 2.00, 3.00, 4.00, 5.00 \) respectively. It can be seen from the figure that when the maturity is one year, the value of catastrophe put option reaches its peak; from zero to one year, the value of catastrophe put option gradually climbs, and increases rapidly; after one year, the value of option begins to decline year by year, but the rate of decline slowed down significantly. Fig. 2 shows the value of catastrophe put options under different strike prices. We take the strike price \( K = 60, 65, 70, 75, 80, 85, 90, 95 \) respectively. From the figure, it can be seen that as the strike price increases, the value of catastrophe put options also gradually increases.

![Fig. 1. Option prices under different maturities](image-url)
Fig. 2. Option prices under different strike prices

Fig. 3. Option prices under different intensities

Fig. 4. Option prices under different correlation coefficients
Fig. 3 shows the value of catastrophe put options under different intensities of loss occurrence. We take the intensity $\lambda = 1.8, 1.9, 2, 2.2, 2.4, 2.6, 2.8, 3$ respectively. From the figure, it can be seen that the option price increases as the loss occurrence intensity increases. Intuitively, the greater the intensity of catastrophe events, the greater the possibility of occurrence. The losses caused by these catastrophe events will lead to the decline of stock prices, making put options more likely to occur. Fig. 4 shows catastrophe put option prices under different correlation coefficients between the forward price of the option issuer’s total assets and the forward price of the underlying stock. We take the correlation coefficient $ho_{F,F} = -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4$ respectively. From the figure, it can be seen that the price of catastrophe put option decreases as the correlation coefficient increases.

5 Conclusion

This paper studies the price of catastrophe stock put option with default risk. Since it is unreasonable to assume that the default of the option issuer occurs only when the option matures, we have considered the continuous defaults, that is, the option issuer’s default can occur at any time before the maturity of the option. It is assumed that the catastrophe loss is generated by a compound doubly stochastic Poisson process, and the catastrophe loss has a certain impact on the dynamics of the underlying stock price. Under the risk neutral measure, the pricing formula in the form of series is obtained. In the numerical part, we use the lognormal distribution to describe the loss severity, and explain the change of catastrophe put option when the parameters values of some basic parameters changes.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Cox SH, Pedersen HW. Catastrophe risk bonds. North American Actuarial Journal. 2000;4(4):56-82.
[2] Dassios A, Jang JW. Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity. Finance & Stochastics. 2003;7(1):73-95.
[3] Liao SL, Huang HH. Pricing Black-Scholes options with correlated interest rate risk and credit risk: An extension. Quantitative Finance. 2005;5(5):443-457.
[4] Jaimungal S, Wang T. Catastrophe options with stochastic interest rates and compound Poisson losses. Insurance Mathematics and Economics. 2005;38(3).
[5] Lee JP, Yu MT. Valuation of catastrophe reinsurance with catastrophe bonds. Insurance: Mathematics and Economics. 2007;41(2):264-278.
[6] Jiang IM, Yang SY, Liu YH, et al. Valuation of double trigger catastrophe options with counterparty risk. North American Journal of Economics & Finance. 2013;25:226-242.
[7] Wang XC. The pricing of catastrophe equity put options with default risk. International Review of Finance. 2016;16(2):181-201.
Appendix A

Proof. Under the forward risk-neutral measure, \( P(t,T) \) is the pricing unit, and the forward prices of the underlying stock and option issuer’s total assets are \( F_S(t,T) = \frac{X(t)}{P(t,T)} \) and \( F_V(t,T) = \frac{V(t)}{P(t,T)} \) respectively, and under the \( Q^T \) measure, it is a martingale. Their dynamics are as follows:

\[
\frac{dF_S(t,T)}{F_S(t,T)} = \sigma_S(t,T) d\tilde{W}_S(t) \tag{A.1}
\]

\[
\frac{dF_V(t,T)}{F_V(t,T)} = \sigma_V(t,T) d\tilde{W}_V(t) \tag{A.2}
\]

Where, \( \tilde{W}_S(t) \) and \( \tilde{W}_V(t) \) are \( Q^T \) – wiener processes, and the correlation coefficients are:

\[
\rho_{S,V} = \frac{\int_T^T \sigma_{S,V}(u,T) du}{\sqrt{\int_T^T \sigma_S^2(u,T) du \sqrt{\int_T^T \sigma_V^2(u,T) du}}}
\]

According to Ito Lemma, the logarithm transformation of formula (A.1) is obtained

\[
\frac{F_S(t,T)}{F_S(0,T)} = \exp \left\{ \int_t^T -\frac{1}{2} \sigma^2_S(u,T) du + \int_t^T \sigma_S(u,T) d\tilde{W}_S(u) \right\}
\]

Therefore, according to formula (2.1), the stock price at the maturity date under the measure \( Q^T \) is

\[
\tilde{S}(T) = \exp \left\{ -\alpha \left\{ L(T) - \kappa T \right\} \right\} F_S(0,T) \exp \left\{ \int_0^T -\frac{1}{2} \sigma^2_S(u,T) du + \int_0^T \sigma_S(u,T) d\tilde{W}_S(u) \right\}
\]

(A.3)

Using the total probability formula, formula (3.1) can be rewritten as

\[
\tilde{E}_S = \beta P(0,T) \sum_{n=1}^{\infty} \frac{(\lambda T)^n}{n!} \exp \left\{ -\lambda T \right\} \int_0^T f_S^{(n)}(y) E^{Q^T} \left\{ \left( \tilde{S}(T) \right)^n \right\} dy
\]

(A.4)
Note that \[ \int_0^\infty -\frac{1}{2} \sigma_{r_t}^2 (u,T)du + \int_0^\infty \sigma_{r_t} (u,T)d\tilde{W}_s(u) \] is a standard normal random variable, which can be proved by the calculation and arrangement of equations (A.3) and (A.4).

**Appendix B**

**Proof.** Similar to the proof of Proposition 2, from the total probability formula and \( \{ \tau > T \} = \{ \min_{t \geq t} F_{t, T} > D \} \), then formula (3.2) can be rewritten as

\[
E_2 = (1 - \beta) P(0, T) \sum_{n=1}^{\infty} \frac{(\Delta T)^n}{n!} \exp \{-\Delta T\} \int_{t}^{\infty} f^{\{\cdot\}}(y) E^{\Phi}(I_{(\tau > T)}(K - \bar{S}(T))) dy
\]

\[
= (1 - \beta) P(0, T) \sum_{n=1}^{\infty} \frac{(\Delta T)^n}{n!} \exp \{-\Delta T\} \int_{t}^{\infty} f^{\{\cdot\}}(y) \left[ KE^{\Phi}(I_{(\tau > T, \bar{S}(T) \leq K)} - E^{\Phi}(\bar{S}(T)I_{(\tau > T, \bar{S}(T) \leq K)}) \right) dy
\]

\[
= (1 - \beta) P(0, T) \sum_{n=1}^{\infty} \frac{(\Delta T)^n}{n!} \exp \{-\Delta T\} \int_{t}^{\infty} f^{\{\cdot\}}(y)(E_{21} - E_{22}) dy
\]

Where,

\[
E_{21} = KE^{\Phi}(I_{\{ \min_{t \geq t} F_{t, T} > D, \bar{S}(T) \leq K \})},
\]

\[
E_{22} = E^{\Phi}(\bar{S}(T)I_{\{ \min_{t \geq t} F_{t, T} > D, \bar{S}(T) \leq K \})}
\]

According to Ito Lemma, the logarithm transformation of formula (A.2) is obtained

\[
\frac{F_r(t, T)}{F_r(0, T)} = \exp \left\{ \int_0^t -\frac{1}{2} \sigma^2_{r_t} (u,T)du + \int_0^t \sigma_{r_t} (u,T)d\tilde{W}_s(u) \right\}
\]

(B.2)

According to the reflection principle of the standard Brownian motion (see Harrison 0 or Musiela and Rutkowski 0), \( E_{21} \) can be rewritten from equations (A.3) and (B.2) to

\[
E_{21} = KQ \left[ \min_{t \geq t} \left\{ \int_0^t -\frac{1}{2} \sigma^2_{r_t} (u,T)du + \int_0^t \sigma_{r_t} (u,T)d\tilde{W}_s(u) \right\} > \ln \frac{D}{F_r(0, T)} \right] 
\]

\[
\left\{ \int_0^t -\frac{1}{2} \sigma^2_{r_t} (u,T)du - \int_0^t \sigma_{r_t} (u,T)d\tilde{W}_s(u) \right\} \geq - \left( \ln \frac{K}{F_r(0, T)} + \alpha \langle L(T) - \kappa T \rangle \right)
\]

(B.3)

\[
= K \left\{ \Phi \left( d_1, d_2, -\rho_{r_t} \right) - \frac{F_r(0, T)}{D} \Phi \left( d_1, d_2, -\rho_{r_t} \right) \right\}
\]
According to Lemma 3 and the reflection principle of standard Brownian motion, and applying equations (A.3) and (B.2), 

\[ E_{\alpha} = F_{\alpha}(0,T) \exp \left\{ -\alpha \left( L(T) - \kappa T \right) \right\} \]

\[ \cdot E^{\nu} \left( \exp \left\{ \int_{t}^{T} -\frac{1}{2} \sigma \nu \left( u, T \right) du + \int_{t}^{T} \sigma \nu \left( u, T \right) d\bar{W} \left( u \right) \right\} I \left( \min_{t \leq u \leq T} F_{\alpha}(t, T) > D, S(T) \leq K \right) \right) \]

\[ = F_{\alpha}(0,T) \exp \left\{ -\alpha \left( L(T) - \kappa T \right) \right\} E^{\nu} \left( I \left( \min_{t \leq u \leq T} F_{\alpha}(t, T) > D, S(T) \leq K \right) \right) \]

\[ = F_{\alpha}(0,T) \exp \left\{ -\alpha \left( L(T) - \kappa T \right) \right\} \]

\[ \cdot Q^{\nu} \left( \min_{t \leq u \leq T} \left\{ -\int_{t}^{T} \frac{1}{2} \sigma \nu \left( u, T \right) du + \int_{t}^{T} \sigma \nu \left( u, T \right) d\bar{W} \left( u \right) > \ln \frac{D}{F_{\alpha}(0,T)}, \ln \frac{K}{F_{\alpha}(0,T)} + \alpha \left( L(T) - \kappa T \right) \right\} \right) \]

\[ = F_{\alpha}(0,T) \exp \left\{ -\alpha \left( L(T) - \kappa T \right) \right\} \]

\[ \cdot \Phi \left( d_{\alpha}, d_{\alpha} - \rho_{\nu} \right) - \frac{F_{\alpha}(0,T)}{D} \left( \frac{D}{F_{\alpha}(0,T)} \right)^{\frac{1}{2}} \frac{\int_{\nu}^{T} \sigma \nu \left( u, T \right) \sigma \nu \left( u, T \right) du}{\int_{\nu}^{T} \sigma \nu \left( u, T \right) \sigma \nu \left( u, T \right) du} \Phi \left( d_{\alpha}, d_{\alpha} - \rho_{\nu} \right) \]

The propositions are proved by the calculation and arrangement of equations (B.1), (B.3) and (B.4).