Abstract. We study wall-crossing phenomena in the McKay correspondence. Craw–Ishii show that every projective crepant resolution of a Gorenstein abelian quotient singularity arises as a moduli space of $\theta$-stable representations of the McKay quiver. The stability condition $\theta$ moves in a vector space with a chamber decomposition in which (some) wall-crossings capture flops between different crepant resolutions. We investigate where chambers for certain resolutions with Hilbert scheme-like moduli interpretations – iterated Hilbert schemes, or ‘Hilb of Hilb’ – sit relative to the principal chamber defining the usual $G$-Hilbert scheme. We survey relevant aspects of wall-crossing, pose our main conjecture, prove it for some examples and special cases, and discuss connections to other parts of the McKay correspondence.

1. Introduction

1.1. Overview. The McKay correspondence studies minimal or crepant resolutions of Gorenstein quotient singularities $\mathbb{C}^n/G$ for $G \subseteq \text{SL}_n(\mathbb{C})$ a finite subgroup. In dimensions 2 and 3, where there is the guarantee of at least one crepant resolution, there are established connections between the geometry of such resolutions and representation theory (either directly of the group $G$, or of related objects) [1,2,21,24,28]. We will subsequently focus on dimension 3.

The first crepant resolution of $\mathbb{C}^3/G$ that was studied in detail is the $G$-Hilbert scheme $G\text{-Hilb} \mathbb{C}^3$. This space is the moduli space of $G$-clusters – 0-dimensional, $G$-invariant subschemes $Z \subseteq \mathbb{C}^3$ with $H^0(O_Z) \cong \mathbb{C}[G]$ as $G$-modules – and was shown to be smooth for all $G \subseteq \text{SL}_3(\mathbb{C})$ in [6] after the abelian case was studied explicitly in [25].

Quivers and their representation theory have been known to enter the picture for some time [3,8,13]. One notable instance is [11, Thm. 1.1] where it is shown that for abelian $G$ every crepant resolution of $\mathbb{C}^3/G$ can be realised as a moduli space of quiver representations. The quiver in this situation is the McKay quiver $Q_G$, which is built out of the representation theory of $G$. Namely, $Q_G$ has a vertex for each irreducible representation of $G$ and the number of arrows between two vertices $\rho$ and $\rho'$ is given by

$$\dim \text{Hom}_G(\rho', \rho \otimes \mathbb{C}^3)$$

which is the multiplicity of $\rho'$ in the decomposition of $\rho \otimes \mathbb{C}^3$ into irreducible representations.

The focus of this paper is in developing an understanding of how different crepant resolutions of $\mathbb{C}^3/G$ are related, both geometrically by flops and more delicately by GIT wall-crossing as in [11], with especial focus on the iterated Hilbert schemes (or ‘Hilb of Hilb’) studied in [18]. We will describe our approach in §2-3, state our main conjecture that epitomises it in §4.1, and work out this conjecture in some examples and special cases in §4.2.

1.2. Toric geometry. When $G$ is abelian the singularity $\mathbb{C}^3/G$ and its crepant resolutions are toric 3-folds, enabling one to use combinatorial methods to examine them. We briefly recall the setup for toric geometry and fix notation that we will use throughout the paper. We will write $G = \frac{1}{r}(a,b,c)$ to mean that $G \cong \mathbb{Z}/r$ is generated by

$$\left(\begin{array}{c} e^a \\ e^b \\ e^c \end{array}\right)$$

MSC 2020: 14E16 (primary), 14M25, 14J17 (secondary)
where \( \epsilon \) is a primitive \( r \)th root of unity. We will assume \( a + b + c \equiv 0 \mod r \); in other words, that \( G \subseteq \text{SL}_3(\mathbb{C}) \).

Let \( N = \mathbb{Z}^3 \) and \( N' = \mathbb{Z}^3 + \mathbb{Z} \cdot (\frac{a}{r}, \frac{b}{r}, \frac{c}{r}) \). Let \( \sigma \) (resp. \( \sigma' \)) denote the cone in \( N_\mathbb{R} \) (resp. in \( N'_\mathbb{R} \)) generated by the standard basis vectors \( e_1 = (1,0,0) \), \( e_2 = (0,1,0) \), \( e_3 = (0,0,1) \). The toric variety \( \Delta_t \) associated to \( \sigma \) is \( \mathbb{C}^3 \), and to \( \sigma' \) is \( \mathbb{C}^3/G \) with the inclusion of lattices \( N \subseteq N' \) inducing the quotient map \( \mathbb{C}^3 \to \mathbb{C}^3/G \). We refer to [7] for general information on toric varieties. Denote by \( \Delta_1 \) the slice \( \sigma' \cap (e^1 + e^2 + e^3 = 1) \), where \( e^i \) are the dual basis of \( N'_{\mathbb{R}} \) to \( e_i \). We call \( \Delta_1 \) the junior simplex. With this setup a crepant resolution of \( \mathbb{C}^3/G \) corresponds to a triangulation of \( \Delta_1 \) with vertices in \( N' \) such that each triangle is unimodal, meaning that its vertices form a \( \mathbb{Z} \)-basis of \( N' \).

Let \( \pi: Y \to \mathbb{C}^3/G \) be a crepant resolution and let \( T \) be the corresponding triangulation. We have the following correspondences between the geometry of \( Y \) and the combinatorics of \( T \):

- torus-invariant exceptional curves in \( Y \) \( \leftrightarrow \) edges in \( T \)
- torus-invariant exceptional divisors in \( Y \) \( \leftrightarrow \) vertices in \( T \)
- torus-invariant compact exceptional divisors in \( Y \) \( \leftrightarrow \) interior vertices in \( T \)
- torus-fixed points in \( Y \) \( \leftrightarrow \) triangles in \( T \)

The combinatorial avatar of the flop in a torus-invariant exceptional curve \( C \) corresponds to flipping the edge corresponding to \( C \) in \( T \) as shown in Fig. 1.

![Figure 1. Flipping an edge](image)

Craw–Reid [12] describe an algorithm for computing the triangulation for \( G \)-\text{Hilb} \( \mathbb{C}^3 \), which works by dividing the junior simplex into a collection of ‘regular triangles’ [12, §1.2] and then applying a standard subdivision to each of these pieces.

1.3. Quiver representations. Let \( Q = (Q_0, Q_1) \) be a quiver, with vertex set \( Q_0 \) and arrow set \( Q_1 \). For an arrow \( \alpha \in Q_1 \) we denote by \( h(\alpha) \) and \( t(\alpha) \) its head and its tail respectively. A representation of \( Q \) with dimension vector \( d = (d_i)_{i \in Q_0} \) is an assignment of a \( d_i \)-dimensional complex vector space \( V_i \) to each vertex \( i \), and a linear map

\[
f_\alpha: V_{t(\alpha)} \to V_{h(\alpha)}
\]

to each arrow \( \alpha \in Q_1 \). If \( V \) is a representation of \( Q \), we write \( \text{dim}(V) = (\dim V_i)_{i \in Q_0} \). Choose a ‘stability condition’

\[
\theta \in \Theta(Q, \underline{d}) := \{ \vartheta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Q}) : \vartheta(\underline{d}) = 0 \}
\]

We say that a representation \( V \) of \( Q \) is \( \theta \)-stable if every nontrivial proper subrepresentation \( U \subseteq V \) has \( \theta(\text{dim}(U)) > 0 \). It is \( \theta \)-semistable if the strict inequality is weakened to \( \geq \). This data produces a moduli space \( \mathcal{M}_\theta(Q, \underline{d}) \) parameterising \( \theta \)-(semi)stable representations \( V \) of \( Q \) with \( \dim(V) = \underline{d} \). We will in fact usually work with representations in which the linear maps \( f_\alpha \) satisfy certain relations coming from a superpotential [5]. The setup above is unmodified for this situation, and so we will not spell it out here.

It turns out that for generically chosen \( \theta \) the moduli space does not depend on small perturbations of \( \theta \) in \( \Theta(Q, \underline{d}) \). Thus there is a chamber decomposition

\[
\Theta(Q, \underline{d}) = \bigcup \Theta
\]
where each open chamber $\mathbb{C}$ has the property that $\theta, \delta \in \mathbb{C}$ implies $M_{\theta}(Q, d) \cong M_{\delta}(Q, d)$. We denote this space by $M_{\theta}$. Note that there may be many chambers describing isomorphic moduli spaces. In the particular situation of the McKay quiver $Q_G$ equipped with a natural dimension vector and certain ‘preprojective’ relations, the moduli spaces $M_{\theta}(Q_G, d)$ are crepant resolutions of $\mathbb{C}^3/G$.

**Theorem 1.1** ([6, Thm. 1.2] + [11, Thm. 1.1]). Let $G \subseteq \text{SL}_3(\mathbb{C})$ be a finite subgroup. Set $d = (\dim \rho)_{\rho \in \text{Irr}(G)}$. For each chamber $\mathbb{C} \subseteq \Theta(Q_G, d)$ the moduli space $M_{\theta}$ is a crepant resolution of $\mathbb{C}^3/G$. When $G$ is abelian, every projective crepant resolution of $\mathbb{C}^3/G$ arises in this way.

Since the vertex set of $Q_G$ is identified with the set $\text{Irr}(G)$ of irreducible representations of $G$, we can regard stability conditions $\theta$ as maps from the representation ring of $G$ to $\mathbb{Q}$. In this version the condition $\theta(d) = 0$ translates to $\theta$ evaluating to zero on the regular representation. Stability conditions $\theta$ that evaluate positively on nontrivial representations give rise to moduli spaces isomorphic to $G$-Hilb $\mathbb{C}^3$. We denote the set of such stability conditions by $\Theta^+(Q_G, d)$ and the chamber containing them by $\mathbb{C}_0$.

2. Wall-crossing

As one passes from a chamber $\mathbb{C} \subseteq \Theta(Q_G, d)$ to another chamber $\mathbb{C'}$ through the ‘wall’ $\overline{\mathbb{C}} \cap \overline{\mathbb{C}'}$ there is a birational map $M_{\mathbb{C}} \to M_{\mathbb{C}'}$ obtained by factoring through the moduli space of semistable representations corresponding to a generic stability condition on the wall.

Let $\pi_{\mathbb{C}}$ denote the morphism $M_{\mathbb{C}} \to \mathbb{C}^3/G$. Often some exceptional locus $E \subseteq \pi_{\mathbb{C}}^{-1}(0)$ is contracted in passing to the moduli of semistable representations. This phenomenon has been studied in [11] and in depth for the chamber $\mathbb{C}_0$ giving $G$-Hilb $\mathbb{C}^3$ in [31] when $G$ is abelian. One of the key tools in the latter is Reid’s recipe [9, 27], which labels strata of $\pi_{\mathbb{C}_0}^{-1}(0)$ with irreducible representations of $G$.

**Example 2.1.** Consider $G = \frac{1}{3}(1, 2, 3)$. Reid’s recipe for this group is shown in Fig. 2. An integer $a$ denotes the representation $\rho_a : G \to \mathbb{C}^\times, g \mapsto e^a$.

![Figure 2. Reid’s recipe for $G = \frac{1}{3}(1, 2, 3)$](image_url)

The main result of [31] classifies the walls of $\mathbb{C}_0$ when $G$ is abelian and finds the unstable and contracted loci for each wall. We need some more language in order to state it.

Let $\rho$ be a character marking a curve in $G$-Hilb. We say that the collection or ‘chain’ of curves marked with $\rho$ is a generalised long side [31, Def. 4.12] if it starts and ends on the boundary of the junior simplex, and if all the corresponding edges along the $\chi$-chain are boundary edges of regular triangles. We exclude the fundamentally different case of three lines meeting at a trivalent vertex if there is a meeting of champions [12, §2.8.2] of side length 0. We call a curve in a generalised long side final [31, Def. 4.14] if it is the furthest curve along away from a vertex along a straight line segment.

On the left of Figure 3 we show the two generalised long sides for $G = \frac{1}{35}(1, 3, 31)$ with dashed lines, and on the right we show the corresponding final curves bolded.
Theorem 2.2 ([31, Thm. 4.17]). Let $G \subseteq \text{SL}_3(\mathbb{C})$ be abelian. The walls of $\mathcal{C}_0$ are as follows:

0. one wall for each irreducible exceptional divisor,
I. one wall for each exceptional $(-1, -1)$-curve,
III. one wall for each generalised long side,
0'. the remaining walls come from potentially reducible exceptional divisors.

The unstable locus in each case is:

0. the irreducible divisor (not contracted),
I. the $(-1, -1)$-curve (contracted to a point),
III. the ruled surface swept out by a final curve of the generalised long side (contracted to a curve),
0'. the potentially reducible divisor (not contracted).

We use the enumeration 0-III of [11, 29] with the slight modification of distinguishing between the two possibilities for type 0 walls. In this way we can associate at most two representations to each wall of $\mathcal{C}_0$.

Definition 2.3. Let $G \subseteq \text{SL}_3(\mathbb{C})$ be abelian. To a wall $\mathcal{W}$ of $\mathcal{C}_0$ we associate a (possibly empty) set $\chi(\mathcal{W})$ of representations of $G$ as follows:

0. If $\mathcal{W}$ corresponds to an irreducible divisor $D$, let $\chi(\mathcal{W})$ be the set of (at most two) representations labelling $D$,
I. If $\mathcal{W}$ corresponds to a $(-1, -1)$-curve $C$, let $\chi(\mathcal{W})$ be the singleton consisting of the representation labelling $C$,
III. If $\mathcal{W}$ corresponds to a generalised long side with final curves $C_1$ and $C_2$, let $\chi(\mathcal{W})$ be the singleton consisting of the representation labelling $C_1$ and $C_2$,
0'. Set $\chi(\mathcal{W}) = \emptyset$ for all other walls.

Let $\gamma: [0, 1] \to \Theta(QG, d)$ be a path starting in $\mathcal{C}_0$. Suppose $\gamma$ only meets walls generically. Let $\gamma$ pass through walls $\mathcal{W}_1, \ldots, \mathcal{W}_m$ at times $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$. Write $\mathcal{C}_i$ for the chamber $\gamma(t)$ lives in for $t_i < t < t_{i+1}$. Notice that this is consistent with our labelling of $\mathcal{C}_0$. Write $\mathcal{M}_i = \mathcal{M}_{\mathcal{C}_i}$. We suppose further that the $\mathcal{C}_i$ are distinct. By [11, §8] we may restrict attention to the case that each wall $\mathcal{W}_i$ is not of type III from Thm. 2.2. From [11, Prop. 6.1] we see that the unstable locus for each such $\mathcal{W}_i$ is either a single exceptional $(-1, -1)$-curve or a divisor as in Thm. 2.2.

We can associate representations to the walls $\mathcal{W}_i$ inductively by essentially propagating Reid’s recipe to different crepant resolutions in such a way that is ‘oriented’ by $\gamma$. Suppose $\mathcal{W}_2$ has unstable locus $E \subseteq \mathcal{M}_1$. This is either:

0. an irreducible exceptional divisor $D$,
I. an exceptional curve $C$, 
0’. a reducible exceptional divisor.

Note that in type 0 the divisor $D$ corresponds to a vertex in the interior of the junior simplex. We mark $w_i$ with the set of representations marking the divisor in $G$-$\text{Hilb} \mathbb{C}^3$ corresponding to that vertex. In type I there are two possibilities: either $w_1$ was type I and $C$ is the curve resulting from the flop, or $C$ is a different curve. If $C$ came from the flop, we let $\chi(w_1) = \chi(w_1)$; if $C$ is a different curve we let $\chi(w_1)$ be the singleton consisting of the representation marking the transform of that $C$ in $G$-$\text{Hilb} \mathbb{C}^3$. We continue inductively, thus associating a set of representations $\chi(w_i)$ to each $w_i$. We write

$$\chi(\gamma) := \bigcup_{i=1}^{m} \chi(w_i)$$

**Example 2.4.** We describe this procedure for a path in $\Theta(Q_G, \underline{d})$ for $G = \frac{1}{6}(1,2,3)$. On the left is the triangulation for $G$-$\text{Hilb} \mathbb{C}^3$ with the labels from Reid’s recipe for two exceptional curves shown. There is a path $\gamma$ starting in $\mathbb{C}_0$ and inducing in turn the two flops shown in the centre and rightmost triangulations: first flopping the curve marked with 4, and then flopping the curve marked with 2 that is not initially floppable but becomes so after the first flop. In this situation, the path $\gamma$ need only pass through two walls $w_1$ and $w_2$ corresponding to these flops. As discussed – and as seen explicitly for a group of order 12 in [11, Ex. 9.13] – in general $\gamma$ may need to cross several walls of type 0 or 0’ prior to realising flops not directly from $G$-$\text{Hilb} \mathbb{C}^3$. The first wall has $\chi(w_1) = \{4\}$, and the second has $\chi(w_2) = \{2\}$ since the curve flopped by crossing $w_2$ was marked with 2 in $G$-$\text{Hilb} \mathbb{C}^3$.

![Diagram showing wall-crossings](image)

**Figure 4.** Wall-crossings for $G = \frac{1}{6}(1,2,3)$

### 3. Iterated Hilbert schemes

Suppose $G \subseteq \text{SL}_3(\mathbb{C})$ is a finite subgroup with a normal subgroup $A \trianglelefteq G$ and quotient $T = G/A$. $T$ acts on $A$-$\text{Hilb} \mathbb{C}^3$ producing a crepant resolution

$$\text{T-Hilb} A$-$\text{Hilb} \mathbb{C}^3 \to \mathbb{C}^3/G$$

One can easily extend this construction to longer chains of normal subgroups $A_1 \trianglelefteq \cdots \trianglelefteq A_s \trianglelefteq G$. We call the crepant resolutions constructed by this procedure *iterated Hilbert schemes* (referred to as ‘Hilb of Hilb’ in [18]). An explicit stability condition $\delta$ was constructed in *ibid.* to express $T$-$\text{Hilb} A$-$\text{Hilb} \mathbb{C}^3$ as a moduli space of quiver representations.

Suppose $\theta_A$ and $\theta_T$ are stability conditions for $Q_A$ and $Q_T$ respectively. As above it will be convenient to view $\theta_A$ (resp. $\theta_T$) as a map from $\text{Irr}(A)$ (resp. $\text{Irr}(T)$) to $Q$, or more generally from the representation ring of $A$ (resp. $T$) to $Q$. Suppose further that $\theta_A$ and $\theta_T$ are positive on all nontrivial irreducible representations of $A$ and $T$ respectively – we call such stability conditions *zero-generated*. A stability condition $\delta \in \Theta(Q_G, \underline{d})$ producing $T$-$\text{Hilb} A$-$\text{Hilb} \mathbb{C}^3$ is defined by

$$\delta(\rho) = \begin{cases} 
\theta_A(\rho|_A) & \rho \notin \text{Irr}(T) \\
\theta_A(\rho|_A) + \epsilon \cdot \theta_T(\rho|_T) & \rho \in \text{Irr}(T)
\end{cases}$$
for sufficiently small $\varepsilon > 0$. By $\rho \in \text{Irr}(T)$ we mean that $\rho$ is an irreducible representation of $G$ that is lifted from $T$ via the map $G \to T$; that is, $\rho|_A$ is trivial.

In the case that $G$ is abelian, Ishii–Ito–Nolla [18, §4.1] constructed a triangulation of the junior simplex giving $G$-Hilb $\mathbb{C}^3$ as a toric variety.

**Example 3.1.** Consider $G = \frac{1}{6}(1, 2, 3)$. We consider the normal subgroup $A \cong \mathbb{Z}/2$ inside $G$ with quotient $T \cong \mathbb{Z}/3$. We show the triangulation for $T$-Hilb $A$-Hilb $\mathbb{C}^3$ on the left of Fig. 5. Note that this is the second flop of $G$-Hilb $\mathbb{C}^3$ considered in Ex. 2.4. One can similarly start with the normal subgroup $A' \cong \mathbb{Z}/3$ with quotient $T' \cong \mathbb{Z}/2$. The iterated Hilbert scheme $T'$-Hilb $A'$-Hilb $\mathbb{C}^3$ is shown on the right of Fig. 5. This is obtained by flopping the curve marked by 3 in $G$-Hilb $\mathbb{C}^3$.

![Figure 5. Iterated Hilbert schemes for $G = \frac{1}{6}(1, 2, 3)$](image)

4. Connecting $G$-Hilb and iterated Hilbert schemes

4.1. **Main conjecture.** We start with the following lemma.

**Lemma 4.1.** With $G, A, T, \delta$ as above we have $\delta(\rho) < 0$ if and only if $\rho \in \text{Irr}(T)$.

For convenience we denote the set of nontrivial irreducible representations of a group $H$ by $\text{Irr}^*(H)$. We also write $\rho^A$ for the $A$-invariant part of a $G$-representation $\rho$, and $\rho_0$ for the trivial representation.

**Proof.** It is irrelevant which zero-generated stability condition $\theta_A$ we choose for $A$ and $T$ and so we use

$$\theta_A(\rho) = \dim \rho \text{ for } \rho \in \text{Irr}^*(G)$$

We have

$$0 = \theta_A(C[A]) = \theta_A(\rho_0) + \theta_A(\bigoplus_{\rho \in \text{Irr}^*(G)} \dim \rho \cdot \rho) = \theta_A(\rho_0) + \sum_{\rho \in \text{Irr}^*(G)} (\dim \rho)^2 = \theta_A(\rho_0) + |A| - 1$$

and so $\theta_A(\rho_0) = 1 - |A|$. This stability condition has the benefit that $\theta_A(\psi) = \dim \psi$ for any representation $\psi$ of $A$ with no trivial summand. It is clear that choosing $\varepsilon$ sufficiently small causes $\delta$ to be negative on characters lifted from $T$ since $\theta_A(\rho|_A) < 0$ for such $\rho$. Thus, suppose $\rho \in \text{Irr}^* G$ is not lifted from $T$ and yet $\delta(\rho) < 0$. Then

$$0 > \theta_A(\rho|_A) = \theta_A(\rho^A) + \theta_A(\rho|_A/\rho^A) = \dim \rho^A \cdot (1 - |A|) + \dim \rho/\rho^A = \dim \rho - \dim \rho^A \cdot |A|$$

since $\rho^A$ is a trivial $A$-module. Note that $\rho^A$ is also a $G$-module hence, since $\rho$ is irreducible, either $\rho^A = \rho$ in which case $\rho \in \text{Irr}(T)$, or $\rho^A = 0$ in which case

$$\dim \rho > 0 = |A| \cdot \dim \rho^A$$

and so the inequality ($\ast$) cannot be satisfied. $\square$

This lemma states that the representations where the sign of $\delta$ differs from a stability condition in $\Theta^+(Q_G, d)$ defining $G$-Hilb $\mathbb{C}^3$ are exactly those lifted from $T$. We take this much further in the following conjecture.
Conjecture 4.2. Let $G \subseteq \text{SL}_3(\mathbb{C})$ be abelian, and let $A \triangleleft G$ be a normal subgroup with quotient $T = G/A$. There is a path $\gamma : [0, 1] \rightarrow \Theta(Q_c, \mathcal{D})$ passing through walls $w_1, \ldots, w_m$ that satisfies the conditions in §2 such that:

(i) $\gamma(0) \in \mathcal{C}_0$

(ii) $\mathcal{M}_{\gamma(1)}(Q_c, \mathcal{D}) \cong T\text{-Hilb } A\text{-Hilb } C^3$

(iii) All nontrivial irreducible representations lifted from $T$ are contained in $\bigcup_{i=1}^m \chi(w_i)$.

We are curious if anything stronger is true especially when $T$ is small, where it may be possible that there is a path from $G\text{-Hilb } C^3$ to $T\text{-Hilb } A\text{-Hilb } C^3$ for which ‘most’ of the walls it crosses are labelled with a lifted character from $T$. One can formulate a version of this conjecture for longer chains of normal subgroups and the corresponding iterated Hilbert schemes.

4.2. Partial results and remarks. We conclude this section with some glimpses of how one might approach Conjecture 4.2.

Proposition 4.3. Let $G/A = T$ as above. Suppose edges $l_1, \ldots, l_p$ labelled with representations $\rho_1, \ldots, \rho_p$ in the triangulation for $G\text{-Hilb } C^3$ are not in the triangulation for $T\text{-Hilb } A\text{-Hilb } C^3$. Then there is a path $\gamma$ satisfying the conditions of Conjecture 4.2 with $\{\rho_1, \ldots, \rho_p\} \subseteq \chi(\gamma)$.

Proof. We note that the curves corresponding to the edges $l_j$ must be flopped (possibly multiple times) in order to obtain the triangulation for $T\text{-Hilb } A\text{-Hilb } C^3$ from the triangulation for $G\text{-Hilb } C^3$. We thus apply the methods of [11, §8] to find a path $\gamma$ passing through walls $w_1, \ldots, w_q$ such that:

- for each $j = 1, \ldots, p$ there is $i_j \in \{1, \ldots, q\}$ such that crossing $w_{i_j}$ realises a flop in the curve corresponding to the edge $l_j$,
- every other wall that $\gamma$ passes through is type $0$, $0'$, or $I$.

By construction $\chi(w_{i_j}) = \{\rho_j\}$, which gives the result. ☐

Corollary 4.4. Continuing the notation of Prop. 4.3, if all the representations lifted from $T$ label edges in the triangulation for $G\text{-Hilb } C^3$ that are not in the triangulation for $T\text{-Hilb } A\text{-Hilb } C^3$, then Conjecture 4.2 is true for $G$.

From Ex. 2.4 and Ex. 3.1 this applies to both decompositions of $\frac{1}{3}(1, 2, 3)$ and to several other examples, such as when $A$ is any subgroup of $G = \frac{1}{30}(2, 3, 25)$ except $\frac{1}{2}(0, 1, 1)$. We note however that there are many examples where representations lifted from a quotient label divisors, such as for $G = \frac{1}{30}(1, 3, 21)$ and $A = \frac{1}{2}(1, 3, 1)$ as can be seen from [31, Fig. 29].

We next interpret and verify Conjecture 4.2 for several polyhedral subgroups. A finite subgroup of $\text{SL}_3(\mathbb{C})$ is polyhedral if it is conjugate to a subgroup of $\text{SO}(3)$. The conjugacy classes of these subgroups are classified by the ADE Dynkin diagrams; the groups themselves are discussed in [26, §2].

Example 4.5. We first consider the subgroup $G$ of type $D_{2n}$ with its index 2 subgroup $A \cong \mathbb{Z}/n$. As usual, denote the quotient $G/A = T \cong \mathbb{Z}/2$. We can compute the stability condition $\delta$ from [18, Def. 2.4] giving $T\text{-Hilb } A\text{-Hilb } C^3$. We see, as expected from Lemma 4.1 that $\delta$ is only negative on the two one dimensional representations of $G$ that are lifted from $T$. Write $L$ for the nontrivial such representation. We use the explicit calculation of chambers in [26, Thm. 6.4(i)-(ii)].

When $n$ is odd we see that $\delta$ lives in the chamber adjacent to $\mathcal{C}_0$ found by crossing the wall $w$ with unstable locus the curve $C$ corresponding to $L$. By the heuristic of §2 it is very reasonable to label the wall with the representation $L$, and hence immediately conclude that this version of Conjecture 4.2 holds for such subgroups.

In the case when $n$ is even a similar calculation can be implemented to show that $\delta$ resides in a chamber adjacent to $\mathcal{C}_0$ found by crossing a wall that realises the flop in the curve corresponding
to the nontrivial irreducible representation lifted from $T$, hence giving an appropriately modified version of Conjecture 4.2 as for odd $p$.

The last case we treat, following [26], is the tetrahedral group $G \subseteq \text{SO}(3)$ of order 12. This has a normal subgroup $A = \frac{1}{2}(1,1,0) \times \frac{1}{2}(1,0,1) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ with quotient $T \cong \mathbb{Z}/3$. The three 1-dimensional representations of $G$ are those lifted from $T$ and so the stability condition $\mathfrak{s}$ describing $\text{-Hilb} \ A - \text{Hilb} \ C^3$ is negative on them. [26, Thm. 6.4(iii)] describes the chamber structure in this case. We see from this that the resolution $X_{12}$ (in Nolla–Sekiya’s notation) obtained by flopping the curves corresponding to the two nontrivial representations lifted from $T$ in any order is isomorphic to $\text{-Hilb} \ A - \text{Hilb} \ C^3$. Thus again we have a suitable version of Conjecture 4.2.

**Remark 4.6.** A large family of examples that would be interesting to study are the trihedral groups [20,23,30] of which the tetrahedral group is the smallest. These are groups of the form $A \rtimes T$ where $T \cong \mathbb{Z}/3$ is generated by

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Recent, as-yet-unpublished work of Nolla constructs a version of Reid’s recipe for the next smallest trihedral group $\frac{1}{2}(1,2,4) \times T$, and Ito–Takahashi have built $T$-$\text{Hilb} \ A$-$\text{Hilb} \ C^3$ for this example. A natural next step is to compare these through the lens of Conjecture 4.2.

**Remark 4.7.** One of the tools that supports extra insight in the polyhedral case is a bijection between crepant resolutions of $C^3/G$ and mutations of a ‘quiver with potential’ $(Q,W)$ – see [14,15] for much of the background – such that mutation in a vertex corresponds to flopping a corresponding curve. This quiver with potential is constructed from the representation theory of $G$; indeed, $Q = Q_G$ is just the McKay quiver.

It is natural to conjecture that there is a quiver with potential $(Q,W)$ whose mutations similarly biject with the crepant resolutions of $C^3/G$ for abelian (or even all) subgroups $G \subseteq \text{SL}_3(\mathbb{C})$. In the abelian case it is equivalent to find a quiver with potential whose mutations are in bijection with unimodular triangulations of the junior simplex, and for which mutation in a vertex corresponds to flipping a corresponding edge. One might compare the various works in the mainline theory of cluster algebras constructing quivers (sometimes with potential) to capture triangulations of marked surfaces, for instance [16,17,22].

**Remark 4.8.** Both Conjecture 4.2 and the conjecture of Rem. 4.7 have analogues for more general Gorenstein toric singularities via the theory of *dimer models* [4,19], which may even be a more amenable setting in which to study them; especially with the recent developments in Reid’s recipe for dimer models [10].

**Acknowledgements.** The author is grateful for helpful conversations with Alastair Craw, Tom Ducat, Álvaro Nolla de Celis, Yukari Ito, Jonathan Lai, and Tim Magee, and to the organisers of the conference ‘The McKay Correspondence, Mutations, and Related Topics’ hosted remotely by Kavli IPMU in July–August 2020 for providing an excellent opportunity – especially given the backdrop of the covid-19 pandemic – to discuss and share some of these ideas.

**References**

[1] Artin, M., and Verdier, J.-L. Reflexive modules over rational double points. *Mathematische Annalen* 270, 1 (1985), 79–82.

[2] Batyrev, V. V., and Dais, D. I. Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry. *Topology* 35, 4 (1996), 901–929.

[3] Bezrukavnikov, R. V., and Kaledin, D. B. McKay equivalence for symplectic resolutions of quotient singularities. *Tr. Mat. Inst. Steklova* 246 (2004), 20–42.
[4] Bocklandt, R., Craw, A., and Vélez, A. Q. Geometric Reid’s recipe for dimer models. *Mathematische Annalen* 361, 3-4 (2015), 689–723.

[5] Bocklandt, R., Schedler, T., and Wemyss, M. Superpotentials and higher order derivations. *Journal of pure and applied algebra* 214, 9 (2010), 1501–1522.

[6] Bridgeland, T., King, A., and Reid, M. The McKay correspondence as an equivalence of derived categories. *Journal of the American Mathematical Society* 14, 3 (2001), 535–554.

[7] Cox, D. A., Little, J. B., and Schenck, H. K. *Toric varieties*. American Mathematical Soc., 2011.

[8] Craw, A. *The McKay correspondence and representations of the McKay quiver*. PhD thesis, University of Warwick, 2001.

[9] Craw, A. An explicit construction of the McKay correspondence for $A$-Hilb $C^3$. *Journal of Algebra* 285, 2 (2005), 682–705.

[10] Craw, A., Heuberger, L., and Amador, J. T. Combinatorial Reid’s recipe for consistent dimer models. *arXiv preprint arXiv:2001.07506* (2020).

[11] Craw, A., and Ishii, A. Flops of $G$-Hilb and equivalences of derived categories by variation of GIT quotient. *Duke Mathematical Journal* 124, 2 (2004), 259–307.

[12] Craw, A., and Reid, M. How to calculate $A$-Hilb $C^3$. In *Séminaires & Congrès* (2002), Citeseer.

[13] de Cecc, A. N. Dihedral G-Hilb via representations of the McKay quiver. *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 88, 5 (2012), 78–83.

[14] Derksen, H., Weyman, J., and Zelevinsky, A. Quivers with potentials and their representations I: Mutations. *Selecta Mathematica* 14, 1 (2008), 59–119.

[15] Derksen, H., Weyman, J., and Zelevinsky, A. Quivers with potentials and their representations II: applications to cluster algebras. *Journal of the American Mathematical Society* 23, 3 (2010), 749–790.

[16] Fomin, S., Shapiro, M., and Thurston, D. Cluster algebras and triangulated surfaces. part i: Cluster complexes. *Acta Mathematica* 201, 1 (2008), 83–146.

[17] Fomin, S., and Thurston, D. Cluster algebras and triangulated surfaces. part ii: Lambda lengths. *arXiv preprint arXiv:1210.5569* (2008).

[18] Ishii, A., Ito, Y., and de Cecc, A. N. On G/N-Hilb of N-Hilb. *Kyoto Journal of Mathematics* 53, 1 (2013), 91–130.

[19] Ishii, A., and Ueda, K. Dimer models and the special McKay correspondence. *Geometry & Topology* 19, 6 (2016), 3405–3466.

[20] Ito, Y. Crepant resolution of trihedral singularities and the orbifold Euler characteristics. *International Journal of Mathematics* 6, 1 (1995), 33–44.

[21] Ito, Y., and Nakajima, H. McKay correspondence and Hilbert schemes in dimension three. *Topology* 6, 39 (2000), 1155–1191.

[22] Labardini-Fragoso, D. On triangulations, quivers with potentials and mutations. *Mexican Mathematicians Abroad* 657 (2016), 103.

[23] Leng, R. C. *The McKay correspondence and orbifold Riemann-Roch*. PhD thesis, University of Warwick, 2002.

[24] McKay, J. Cartan matrices, finite groups of quaternions, and Kleinian singularities. *Proceedings of the American Mathematical Society* 81, 1 (1981), 153–154.

[25] Nakamura, I. Hilbert schemes of abelian group orbits. *Journal of Algebraic Geometry* 10, 4 (2001), 757–780.

[26] Nolla de Cecc, Á., and Sekiya, Y. Flops and mutations for crepant resolutions of polyhedral singularities. *Asian Journal of Mathematics* 21, 1 (2017), 1–46.

[27] Reid, M. McKay correspondence. In *Proceedings of Algebraic Geometry symposium* (Kinosaki, Nov 1996) (1997), pp. 14–41.

[28] Reid, M. La correspondance de McKay. *Astérisque-Societe Mathematique de France* 276 (2002), 53–72.

[29] Wilson, P. M. H. The Kähler cone on Calabi–Yau threefolds. *Inventiones mathematicae* 107, 1 (1992), 561–583.

[30] Wormleighton, B. On the nonabelian McKay correspondence. Master’s thesis, University of Warwick, 2015.

[31] Wormleighton, B. Walls for G-Hilb via Reid’s recipe. *Symmetry, Integrability and Geometry: Methods and Applications* 16 (2020), 106–144.

Department of Mathematics & Statistics, Washington University in St. Louis, St. Louis, MO, 63130, USA
Email address: bern@wustl.edu