Entropy of extreme three-dimensional charged black holes

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Abstract

It is shown that three-dimensional charged black holes can approach the extreme state at nonzero temperature. Unlike even dimensional cases, the entropy for the extreme three-dimensional charged black hole is uniquely described by the Bekenstein-Hawking formula, regardless of different treatments of preparing the extreme black hole, namely, Hawking’s treatment and Zaslavskii’s treatment.

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There has been recently a great deal of interest in the study of the extreme black hole (EBH) entropy. The interest was first heated up by the findings that the four-dimensional (4D) Reissner-Nordström (RN) EBH and the non-extreme black hole (NEBH) are different objects due to their drastically different topological properties and RN EBH has zero entropy regardless of its nonzero horizon area [1,2]. However, using the grand canonical ensemble, Zaslavskii argued that a 4D RN black hole in a finite size cavity can approach the extreme state as closely as one likes and the Bekenstein-Hawking formula is still expected to hold for RN EBH [3]. The geometrical and topological properties were also claimed of nonextreme sectors [4,5]. Support for this view is also provided by state-counting calculations of certain extreme and near-extreme black holes in string theory, see [6] for a review. These different results indicate that EBHs have a special and controversial role in black hole thermodynamics and topologies.

Comparing [1,2] and [3-5], it seems that the clash comes from two different treatments: one refers to Hawking’s treatment by starting with the original EBH [1,2] and the other Zaslavskii’s treatment by first taking the boundary limit and then the extreme limit to get the EBH from its nonextreme counterpart [3-5]. Recently by using these two treatments, the geometry and intrinsic thermodynamics have been investigated in detail for a wide class of EBHs including 4D and two-dimensional (2D) cases [7-10]. It was found that these different treatments lead to two different topological objects represented by different Euler characteristics and show drastically different intrinsic thermodynamical properties both classically and quantum-mechanically. Based upon these results it was suggested that there maybe two kinds of EBHs in the nature: the first kind suggested by Hawking et al with the extreme topology and zero entropy, which can only be formed by pair creation in the early universe; on the other hand, the second kind, suggested by Zaslavskii, has the topology of the nonextreme sector and the entropy is still described by the Bekenstein-Hawking formula, which can be developed from its nonextreme counterpart through second order phase transition [11-13]. This speculation has been further confirmed recently in a Hamiltonian framework [14] and the grand canonical ensemble [15] as well as canonical ensemble [16] formulation for RN anti-de Sitter black hole.

All these results available for EBHs’ entropy are limited to even dimensions. Whether these results can be extended to odd dimensions is unclear. This paper evolves from an attempt to study this problem by using (2+1)-dimensional (3D) charged black hole as an
The metric of the 3D charged black hole reads [17]

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2$$ (1)

where

$$N^2 = -M + r^2 \frac{\epsilon^2}{l^2} - \frac{\epsilon^2}{2} \ln \frac{r}{r_0}$$ (2)

with $-\infty < t < +\infty, 0 < r < \infty$ and $0 \leq \phi \leq 2\pi$, $M$ and $\epsilon$ in the above metric associated respectively with the mass and the charge of the black hole, $-l^{-2}$ is the negative cosmological constant and $r_0$ is a constant. When $r > r_0$, the 3D charged black hole is described by the Penrose diagram as usual[18]. The electric potential of the charge is

$$A_0(r) = -\epsilon \ln \frac{r}{r_0}$$ (3)

This black hole has two, one, or no horizons, depending on whether [19]

$$M - \left(\frac{\epsilon^2}{4} - \frac{\epsilon^2}{4} \ln \frac{\epsilon^2 l^2}{4r_0^2}\right)$$ (4)

is greater than, equal to or less than zero, respectively.

Now we can directly make use of the approach of [20] to study the black hole thermodynamics in a grand canonical ensemble where we consider the black hole in a cavity with radius $r_B$. The temperature on the boundary of the cavity is $T_W = T_H/N(r_B)$, where $T_H = k/2\pi$ is the Hawking temperature and $k$ is the surface gravity.

For our metric (1), the local temperature has the form

$$T_W = \frac{T_H}{\sqrt{-M + r_B^2/l^2 - \epsilon^2/2 \ln(r_B/r_0)}}$$ (5)

$$T_H = \frac{2r_+/l^2 - \epsilon^2/2r_+}{4\pi}$$ (6)

When a black hole approaches the extreme state ($M = \frac{\epsilon^2}{4} - \frac{\epsilon^2}{4} \ln \frac{\epsilon^2 l^2}{4r_0^2}, \epsilon^2 = \frac{4r_+^2}{l^2}$), according to (6) $T_H \to 0$. The simplest choice is to take the limit $T_W \to 0$. One might refer to the third law of thermodynamics to argue that the EBH cannot be achieved because the absolute zero temperature is unachievable.

However, it is interesting to point out that although $T_H \to 0$, the square root in (5) tends to zero as well if we take $r_+ \to r_B$, thus the extreme state with nonzero local temperature
does exist. Indeed, taking $r_+$ and $r_-$ as corresponding to event horizon and Cauchy horizons, we have

$$
\epsilon^2 = \frac{2(r_+^2 - r_-^2)}{l^2 \ln(r_+/r_-)},
$$

(7)

and we can readily see that although $T_H$ has a simple zero in $r_+ \rightarrow r_-$, the expression in the square-root in the denominator of $T_W$ has a double-zero, i.e.,

$$
-M + \frac{r_B^2}{l^2} - \frac{\epsilon^2}{2} \ln \frac{r_B}{r_0} = \frac{r_+^2 - r_-^2}{l^2} \left(1 - \frac{\ln r_+ - \ln r_-}{\ln r_+ - \ln r_-}\right),
$$

(8)

therefore $T_W$ tends to a constant value in the EBH case. Recall that in the grand canonical ensemble, only the temperature on the boundary has physical meaning, whereas $T_H$ can always be rescaled without changing observable quantities [21]. Therefore analogous to the 4D RN case [3], there exists a well defined extreme state of the 3D charged black hole in the grand canonical ensemble and no contradiction with the third law arise.

Now it is of interest to investigate whether two different treatments applied in even dimensions will lead to similar different entropy results for 3D charged EBH. The action for the Euclidean version of the 3D charged black hole on a 3D manifold $M$ with a boundary is given by

$$
I = -\frac{1}{2\pi} \int_M d^3x \sqrt{-g} \left(R + \frac{2}{l^2} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) + \frac{1}{\pi} \int_{\partial M} d^2x \sqrt{-\gamma} \left(K - K_0\right)
$$

(9)

Here $\gamma$ is the induced metric on the boundary $\partial M$ and $K$ is the extrinsic curvature of the boundary. $K_0$ is a constant independent on the metric of 3D spacetime and we choose it to be zero to normalize the thermodynamic energy in a flat spacetime.

Introducing the Gaussian normal coordinates near every point on the surface of the cavity, the timelike coordinate of this system is the proper time $\tau$ for an observer on the surface and the coordinates on the surface are $(\tau, \phi)$. Defining $\vec{N}$ as the unit spacelike vector orthogonal to the surface and $\vec{U}$ the velocity of a mass element of this surface, the orthogonal condition becomes

$$
\vec{N} \cdot \vec{U} = 0
$$

(10)

The velocity is $\vec{U} = i\partial_\tau + i\partial_\phi$ where the overdot denotes differentiation with respect to $\tau$. We obtain $\vec{N} = (|g_{tt}|)^{-1}i\partial_\tau + |g_{tt}| i\partial_\phi$ from Eq(10). The normalization conditions are $\vec{N} \cdot \vec{N} = 1, \vec{U} \cdot \vec{U} = -1$. The extrinsic curvatures relative to the Gaussian normal coordinates

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are
\begin{align}
K_{\tau\tau} &= N_{\tau\tau} = U^\mu U^\nu N_{\mu\nu} \\
K_{\phi\phi} &= N_{\phi\phi}.
\end{align}

The action for the black hole with the cavity at \( r = r_B \) is
\begin{equation}
\frac{\beta}{N_{\tau\tau}} \left[ \frac{4}{l^2}(r_B^2 - r_+^2) - \frac{e^2}{4} \ln \frac{r_B}{r_+} \right] + 2\beta \left[ \frac{r_B}{2N_{\tau\tau}} \frac{dN^2}{dr} \right] r_B + N_{\tau\tau},
\end{equation}
where the relation \( \beta = T_W^{-1} = \int_0^{2\pi} N(r_B) d\tau \) has been used.

The free energy is given by the expression
\begin{equation}
F = \frac{I}{\beta} = \frac{1}{N_{\tau\tau}} \left[ \frac{4}{l^2}(r_B^2 - r_+^2) - \frac{e^2}{4} \ln \frac{r_B}{r_+} \right] + 2\frac{r_B}{2N_{\tau\tau}} \left( \frac{dN^2}{dr} \right) r_B + 2N_{\tau\tau},
\end{equation}
while the entropy can be calculated by means of the formula
\begin{equation}
S = -\left( \frac{\partial F}{\partial T_W} \right)_D = -\left( \frac{\partial F}{\partial r_+} \right)_D \left( \frac{dT_W}{dr_+} \right)_D^{-1}.
\end{equation}

We have,
\begin{equation}
S = -4\pi \frac{dN_{\tau\tau}}{dr_+} \left[ \frac{dN^2}{dr} \right]_{r_+} - 8\pi N_{\tau\tau} \frac{d}{dr_+} \left( \frac{dN^2}{dr} \right)_{r_+}
- \frac{r_B}{2N_{\tau\tau}} \frac{dN_{\tau\tau}}{dr_+} \left( \frac{dN^2}{dr} \right)_{r_B} + \frac{r_B}{2N_{\tau\tau}} \frac{d}{dr_+} \left( \frac{dN^2}{dr} \right)_{r_B} + \frac{dN_{\tau\tau}}{dr_+} \left( \frac{dN^2}{dr} \right)_{r_B}.
\end{equation}

Taking the boundary limit \( r_+ \to r_B, (N_{\tau\tau} \to 0) \), we find
\begin{equation}
S = 4\pi r_+
\end{equation}
This is just the entropy for the 3D charged NEBH [17]. We note that the first term in (16) does not contribute to the entropy, which is similar to the even dimensional cases, where the entropy result is only attributed to the surface term of the Euclidean action.

We are now in position to extend the above calculations to EBH. We are facing two limits, namely, the boundary limit \( r_+ \to r_B \) and the extreme limit \( M = e^2/4 \ln e^2l^2/4r_0^2, e^2 = 4r_+^2/l^2 \). We follow two different treatments while taking these two limits: (A) first take the boundary limit and then the extreme limit, which corresponds to the treatment adopted in [3-5]; and
(B) first take the extreme limit and then the boundary limit, which corresponds to starting with the original EBH in [1,2]. From Eq.(14), it is easy to find that both the first term and the third term in the free energy will vanish either in treatment (A) or (B) due to the limit $r_+ \to r_B$. Therefore only the second term of the free energy has the contribution to the entropy in these two treatments. Using (15), we have

$$S(A) = \left[ \frac{4\pi r_B}{dr_+} \left( \frac{dN^2_B}{dr_+} \right) r_B \right]_{r_+ \to r_B} = 4\pi r_+ \left( 2 \ln \frac{r_B^2}{l^2} - 1 \right)$$

$$S(B) = \left[ \frac{4\pi r_B}{dr_+} \left( \frac{dN^2_B}{dr_+} \right) r_B \right]_{r_+ \to r_B} = \lim_{r_+ \to r_B} 4\pi r_B \frac{r_+}{r_B} \ln \frac{r_B^2}{l^2} - 1 \right)$$

These two different ways of taking the limits lead to the same entropy for 3D charged EBH, and entropy never vanishes. This result can also be extended to 3D rotating black hole.

Thus we have shown that in the grand canonical ensemble, the 3D charged black hole can approach the extreme state at nonzero temperature. Unlike even dimensional cases, the entropy of the 3D charged EBH is uniquely described by the Bekenstein-Hawking formula regardless of the different ways of taking the limits.

As a matter of fact, in even dimensions it is usual to classify the topology of the manifold in terms of the Chern class, in dimensions multiple of four also by the Pontryagin number. The problem of the black hole entropy has been generally related to the Euler characteristic of the manifold, rather useful in the context of general relativity due to the Gauss-Bonnet theorem. Thus, for even dimensional NEBHs and EBHs, there are direct relations between the black hole entropy and the topological properties represented by Euler characteristics obtained from Gauss-Bonnet theorem [22,23,24,25,2,8-11]. In odd dimensional space-time the situation is much more difficult, since most of the tradicional topological invariants do not exist, in spite of the fact that the topology may be far from trivial. In three dimensions, in particular, the Gauss-Bonnet theorem does not exist, and the relations between the entropies and topologies either in NEBH or EBH valid in even dimensions are not appropriate. As far
as the traditional invariants are concerned, our result is compatible with having the same
topology for both, the extreme and non-extreme black holes in three dimensions, thus we find
no contradiction when the extreme limit is taken. However, for more general configurations
we certainly need a finer analysis. We can say that the relation between the entropy result
and their topological properties in odd dimensions is still unclear and needs further study.

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