Forced vibrations of beveled non-circular conical thin-walled structures of variable thickness

Vladimir A Kozlov and Andrey V Chernikov
Department of Structural Mechanics. Voronezh State Technical University, 20 Let Oktyabrya st., 84, Voronezh, Russian Federation
E-mail: v.a.kozlov1@yandex.ru

Abstract. The stress distribution in shells of variable thickness must be taken into account when designing various elements of general and transport engineering, aviation, rocket and space technology, structures. The paper presents one of the possible approaches to the dynamic calculation of a thin-walled non-circular conical structure rigidly fixed along a beveled end section. The resolving system of ordinary differential equations was obtained in the framework of the technical theory of shells using the variational Lagrange principle. The boundary-value problem of determining dynamic stresses in the structure under consideration under the influence of a harmonic load is solved numerically by the method of orthogonal sweeping of a system of linear ordinary differential equations of the first order. In contrast to the known works, a variable thickness of the structure along the length is considered.

1. Introduction
Shell structures, due to their high operational, strength and aesthetic qualities, are widely used in various industries: civil and industrial construction, aviation, rocket and space technology, transport engineering, construction of bridges, etc. In order to economize consumption of structural material, increase the reliability and durability of structures, the lining and its reinforcing elements are made of variable stiffness, as, for example, in modern aircraft wings. This article studies the forced vibrations of a tapered conical thin-walled structure having reinforcements in the form of longitudinal (stringers) and transverse (ribs) sets of regular structure. In this case, the contour of the cross section can be mono- or multiply connected and, in the general case, of arbitrary shape. In addition, the shell thickness, as is customary in the wings of modern aircraft, is assumed to be variable both in scope and in chord. The area of the transverse reinforcing set may also vary along the length of the structure. The calculation model of the structure under consideration is presented in Figure 1.

Decisions are constructed within the framework of the technical theory of thin shells based on the Lagrange variational principle [1, 2]. The proposed approach in determining the stress-strain state of the thin-walled structures under consideration allows one to obtain both numerical [2, 3] and closed analytical [4, 5] solutions of the corresponding boundary value problems. In the field under consideration there are modern experimental [8], theoretical [9] and dissertation [10] studies.
2. Materials and methods

We decompose elastic displacement vector $\vec{U}$, as in [1, 2], into two components

$$\vec{U}(Z,S) = \vec{U}^0(Z,S) + \vec{U}^*(Z,S),$$

which depend on the curvilinear coordinates of the shell: $Z$ - the coordinate along the length of the shell, which determines the distance from the plane of the contour of the cross section with the origin and is measured in fractions of the generator / total length; $S$ - coordinate measured along the contour of the shell cross section (Figure 1).

The first term $\vec{U}^0$ determines the arbitrary spatial displacement of the contour $Z = const$ as a solid and corresponds to the approximate nature of the construction with a physically clear interpretation of the processes. When considering prismatic and direct shells, the first term $\vec{U}^0$ corresponds to the distribution of normal stresses in the cross section according to a plane law and determines the elementary solutions that are considered in the resistance of materials. In the case of conical shells with an arbitrary contour of the cross section, the displacement of the cross section $Z = const$ as a solid in the general case does not determine the stress distribution in it in accordance with the plane law. We can talk about that $\vec{U}^0$ determines the law of plane sections in a generalized form. The second term $\vec{U}^*$ in the above expansion is a refinement of the approximation $\vec{U}^0$; it determines the deplanation both from the contour of the cross section and in its plane.

In the course of theoretical mechanics, the movement of a solid body in space is determined by the sum of the translational motion of an arbitrary point of this body selected for the pole, and the rotational motion of the body relative to this pole. Having defined the pole, the displacement $\vec{U}^0$ for the contour $Z = const$ can be represented as

$$\vec{U}^0(Z,S) = \vec{H}(Z) + \vec{\Omega}(Z) \times [\vec{r}(Z,S) - \vec{r}_0(Z)],$$

where $\vec{H} = \vec{H}(Z)$ – pole displacement vector; $\vec{\Omega} = \vec{\Omega}(Z)$ – contour angle vector $Z = const$ around the instantaneous axis; $\vec{r}, \vec{r}_0$ – radius vectors of the pole and an arbitrary point on the middle surface, respectively.
Taking as the pole the point of intersection of the contour \( \vec{Z} = \text{const} \) with the \( Oz \) axis, we represent the vector function \( \vec{U}^0 \) in the form

\[
\vec{U}^0(\vec{Z}, S) = H_x(\vec{Z}) \cdot \vec{i} + H_y(\vec{Z}) \cdot \vec{j} + H_z(\vec{Z}) \cdot \vec{k} + \Omega_x(\vec{Z}) \vec{z} \times \left[ y_0(S) \cdot \vec{k} - x_0(S) \cdot \vec{j} + \vec{\Omega}(\vec{Z}) \vec{Z} \right]
\]

where \( H_x, H_y, H_z \) – contour offsets \( \vec{Z} = \text{const} \) along the axes \( Ox, Oy, Oz \) respectively; \( \Omega_x, \Omega_y, \Omega_z \) – turning angles \( \vec{Z} = \text{const} \) around the axes \( Ox, Oy, Oz \); \( \vec{Z} = 1 - \vec{Z} \); \( x_0(S), y_0(S) \) determine the outline of the guide.

Expression for \( \vec{U}^0 \) can be written as the final sum

\[
\vec{U}^0(\vec{Z}, S) = \sum_{j=1}^{6} \vec{V}_j(\vec{Z}) \cdot \vec{\phi}_j(S).
\]

Here \( \vec{V}_j(\vec{Z}) \) – unknown scalar functions, the first three \( (i = 1, 2, 3) \) represent translational displacements of a contour \( \vec{Z} = \text{const} \) along the axes \( Ox, Oy, Oz \), rest \( (i = 4, 5, 6) \) – rotations of this contour around the axes \( Ox, Oy, Oz \):

\[
\begin{align*}
\vec{V}_1(\vec{Z}) &= H_x(\vec{Z}), \\
\vec{V}_2(\vec{Z}) &= H_y(\vec{Z}), \\
\vec{V}_3(\vec{Z}) &= H_z(\vec{Z}), \\
\vec{V}_4(\vec{Z}) &= \vec{Z} \vec{\Omega}_x(\vec{Z}), \\
\vec{V}_5(\vec{Z}) &= \vec{Z} \vec{\Omega}_y(\vec{Z}), \\
\vec{V}_6(\vec{Z}) &= \vec{Z} \vec{\Omega}_z(\vec{Z});
\end{align*}
\]

\( \vec{\phi}_j(S) \) – known vector functions:

\[
\begin{align*}
\vec{\phi}_1(S) &= \vec{i}, \\
\vec{\phi}_2(S) &= \vec{j}, \\
\vec{\phi}_3(S) &= \vec{k}, \\
\vec{\phi}_4(S) &= -x_0(S) \cdot \vec{j} + y_0(S) \cdot \vec{k}; \\
\vec{\phi}_5(S) &= x_0(S) \cdot \vec{k} - x_0(S) \cdot \vec{j}, \\
\vec{\phi}_6(S) &= -y_0(S) \cdot \vec{i} + x_0(S) \cdot \vec{j}.
\end{align*}
\]

The structure of vector functions \( \vec{U}^0(\vec{Z}, S) \) is completely defined by the equalities presented above, and with respect to the structure of vector functions \( \vec{U}^* (\vec{Z}, S) \) we consider assumptions similar to [1, 2] about the direction of this vector only along the generator. As in [2], to obtain a system of resolving differential equations (1) and natural boundary conditions (2) we use the Lagrange variational principle:

\[
\sum_{j=1}^{6+n} \left( Z_{aij} V_j' + b_{ij} V_j \right)' - c_{ij} V_j - \frac{c_{ij} V_j}{Z} - \tilde{Z} d_{ij} \frac{\partial^2 V_i}{\partial t^2} - \tilde{Z} R^A_i(V_1, V_2, ..., V_{6+n}) \right] = -\frac{R_i^0}{G};
\]

\[
\left( p_i^* - p_i \right) \frac{\partial V_i}{\partial Z} \bigg|_{Z=0} = 0, \quad i = 1, 2, ..., 6 + n.
\]

Here \( a_{ij}, b_{ij}, c_{ij}, d_{ij} \) – coefficients, which, unlike [1], are variable and depend on the law of variation of the shell thickness and the cross-sectional area of the stringer set in scope; \( \vec{Z} = Z / l \) - the relative coordinate on the middle surface of the shell, measured along the generatrix in fractions of its full length \( l \); \( Z \) - dimensional coordinate corresponding to \( \vec{Z} \) and forming an angle with the Cartesian axis \( \pi / 2 - \chi_0 \); \( R_i, R^A_i \) - operators corresponding to external load and non-conservative forces; \( \forall V_j = U_j(\vec{Z}) \vec{\alpha}_j(\vec{Z}) \), \( V_j \) - the desired displacement functions and their derivatives with respect to the coordinate \( Z \); \( U_j \) - generalized displacements, moreover, \( U_1, U_2, U_3 \) correspond to the displacements
of the contour $\bar{Z} = \text{const}$ as a solid along the axes $Ox, Oy, Oz$, and $U_4, U_5, U_6$ determine the angles of rotation of this contour around the axes, the rest $n$ correspond to the deplanation displacements of the contour of the cross section; $\lambda_j = \bar{Z}$ for $j = 4 – 6$; for all other values $j \lambda_j = 1$; $G$ – shear modulus; $P_i^*, P_i$ – respectively, given and unknown generalized forces in the considered cross sections of the shell.

In accordance with the order of the differential equations included in system (1), during the solution using the natural boundary conditions (2) of the variational problem under consideration, it is necessary to determine the integration constants with a total number of $2(6 + n)$. If displacements are specified on one of the end sections, then with the corresponding value $\bar{Z}$ we will have the equality of unknown generalized displacements of the contour $\bar{Z} = \text{const}$ given $V_i^*$:

$$V_i = V_i^* \quad i = 1, 2, \ldots, 6 + n.$$  \tag{3}

Equality (3) can be written as $\partial V_i = 0$, and condition (2) for one of the given ones $\bar{Z}$ is satisfied. In case of setting force factors at one of the end sections $P_i^*$, to fulfill condition (2) with the corresponding value $\bar{Z}$, it is necessary to put

$$P_i = P_i^* \quad i = 1, 2, \ldots, 6 + n.$$  

If we expand the last condition for the equality of unknown generalized forces given, we get the expression:

$$G\lambda_i \left( \bar{Z} \sum_{j=1}^{6+n} a_{ij} V_j^* \right)_{\bar{Z} = Z_1} = P_i^* \left( Z_1 \right).$$  \tag{4}

At the same time, generalized forces $P_j$ are the sum of the work of internal efforts in the considered section $\bar{Z} = \text{const}$ of shell on possible movements allowed by superimposed bonds and determined by conditions

$$U_j = \begin{cases} 1, & j = i \\ 0, & j \neq i. \end{cases}$$

The system of differential equations (1) obtained in general form under the fulfillment of boundary conditions (3) and (4) allows one to consider the natural and forced vibrations of shells of noncanonical forms and solve the problems of aeroelasticity (flutter, divergence of the wing box, etc.). Due to the presence of variable coefficients $h = h(\bar{Z})$ for shells of variable stiffness, system (1), unlike [1], cannot be reduced to equations of Euler type.

Under the influence of any load applied to the thin-walled structure embedded along the sloping edge, in cross sections parallel to the embedment plane, a torque arises along with the bending moment. Therefore, when bending, torsion always appears and vice versa, the concept of “axis of rigidity” for a beveled shell does not make sense in a strict understanding of this term. The consequence of this is that when performing calculations from six generalized displacements corresponding to the displacement of the contour $\bar{Z} = \text{const}$ as a solid, only three $U_2, U_4, U_6$ can be held. The need to take into account the remaining three $(U_1, U_3, U_5)$ arises if the contour of the cross section is asymmetric, or if the components of the external load directed along these generalized displacements are applied to the shell: $Q_x, Q_z$ are the bending
and longitudinal forces acting along the axes $Ox$, $Oz$, respectively; $M_i$ - torque about the axis $Oy$. The equilibrium equations $\Sigma X = 0$, $\Sigma Z = 0$, $\Sigma M_y = 0$ in the general case are not satisfied. But when considering the caissons of the wings of aircraft for typical cross sections and the characteristic loads for the wings, this error is not so large and has a negligible effect on the results [1].

So, system (1) is a relative system. Therefore, generalized displacements $U_j$ that correspond to bending and torsion can only be determined jointly. In some special cases, the system of differential equations (1) is simplified. For example, if there is an axis of symmetry of the contour of the cross section relative to the axis $Ox$, out of six generalized displacements corresponding to displacements of the contour $\vec{Z} = \text{const}$ as a solid, only three can be held: displacement of the contour $\vec{Z} = \text{const}$ along the axis of $Oy$ ($U_2$) and rotations of this section relative to the axes $Ox$, $Oz$ ($U_4, U_6$). In this case, the equilibrium equations $\Sigma X = 0$, $\Sigma Z = 0$, $\Sigma M_y = 0$, corresponding to the displacements of the contour $\vec{Z} = \text{const}$ along the axes $Ox$, $Oz$ ($U_1, U_3$) and rotation about the axis $Oy$ ($U_5$) are carried out identically. If the cross section is symmetrical both with respect to the $Ox$ and $Oy$ axis, then a number of coefficients of the system included in system (1) vanish.

As a contour pole $\vec{Z} = \text{const}$, we choose the point of intersection of the plane of this contour with the axis $Oz$. In this case, the elastic displacement vector included in the expansion

$$\vec{U}(\vec{Z}, S) = \sum_{i=1}^{6+n} U_i(Z) \phi_i(S)$$

vector functions $\phi_i(S)$, which correspond to the displacement of the contour $\vec{Z} = \text{const}$ as a solid, are defined by the following expressions

$$\phi_1 = i, \phi_2 = j, \phi_3 = k, \phi_4 = y_0 k - x_0 ctg Z_0 j, \phi_5 = x_0 (ctg Z_0 i - k), \phi_6 = x_0 j - y_0 i.$$

In this case, the corresponding system of scalar coordinate functions is written in the form:

$$\phi_{11} = x_0 / l_0, \phi_{21} = -y_0 / l_0, \phi_{31} = 1 - x_0 ctg Z_0 / l_0, \phi_{41} = \phi_{51} = \phi_{61} = 0,$$

$$\phi_{12} = x_0 / \sin \chi + x_0 ctg \chi / l_0, \phi_{22} = y_0 / \sin \chi + y_0 ctg \chi / l_0,$$

$$\phi_{32} = x_0 ctg Z_0 ctg \chi / l_0, \phi_{42} = [l_0 y_0' - (x_0' y_0 - x_0 y_0') ctg Z_0] / \sin \chi,$$

$$\phi_{52} = -l_0 x_0 / \sin \chi, \phi_{62} = (x_0 y_0 - x_0 y_0') \sin \chi,$$

$$\phi_{13} = -y_0 / \sin \chi, \phi_{23} = x_0 / \sin \chi, \phi_{33} = (x_0 y_0 - x_0 y_0') / l_0 \sin \chi,$$

$$\phi_{43} = l_0 x_0 / \sin \chi, \phi_{53} = l_0 y_0 / \sin \chi, \phi_{63} = (x_0 y_0 - y_0 y_0') / \sin \chi.$$

In [1], it was pointed out that in calculating shells of constant thickness, taking into account the cross section deplanation, two coordinate vector functions $\phi_7$ are decisive: corresponding to bending, and $\phi_8$ corresponding to torsion. The corresponding components along the longitudinal coordinate have the form

$$\phi_{71} = x_0 y_0, \phi_{81} = x_0^2 y_0 + k y_0,$$

where the orthogonalization coefficient $k$ is determined by the equality

$$k = \int x_0^2 y_0 dS / \int y_0^2 dS.$$

Accepting the hypothesis about the direction of deployment of contour points $\vec{Z} = \text{const}$ from its plane [1], we obtain that the components $\phi_{72} = \phi_{7n} = 0, \phi_{82} = \phi_{8n} = 0$. Then, in the resolving system
of differential equations (1) for a beveled shell with a rectangular contour of the cross section (Figure 1), the following coefficients will be different from zero, depending on the variable thickness $h(Z)$:

$$a_{22} = d_1 \left[ \frac{d_1^2}{(1 - \nu)l_0} \left( \frac{d_2}{\sin^2 \chi_0} + \frac{d_1}{3} \right) + 2 \right] h, \quad a_{24} = a_{42} = d_1 \left[ \frac{d_1 d_2}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^2 \chi_0} + 2l_0 \right] h,$$

$$a_{26} = a_{62} = -\frac{d_1^2 d_2}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} h, \quad a_{28} = a_{82} = -\frac{d_1^2 d_2}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} \left( \frac{d_2^2}{12} + K \right) h,$$

$$a_{44} = d_1 \left[ \frac{d_1 d_2}{(1 - \nu)l_0} \left( \frac{d_2}{\sin^2 \chi_0} + 1 \right) + \frac{d_2^2}{2} \right] h,$$

$$a_{46} = a_{64} = d_1 \left[ \frac{d_1 d_2}{(1 - \nu)l_0} \left( \frac{d_2}{\sin^2 \chi_0} + 1 \right) + d_2 \right] h,$$

$$a_{48} = a_{84} = \frac{1 + \nu}{1 - \nu} \frac{d_1^2 d_2}{2} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} \left( \frac{d_2^2}{12} + K \right) h, \quad a_{66} = d_1 d_2 \left( \frac{d_1 d_2}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} + \frac{d_1 + d_2}{2} \right) h,$$

$$a_{68} = a_{86} = \frac{1 + \nu}{1 - \nu} \frac{d_1^2 d_2}{2} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} \left( \frac{d_2^2}{12} + K \right) h, \quad a_{77} = \frac{d_1^2 d_2}{24} \left[ \frac{2}{1 - \nu} (d_1 + d_2) + d_2 \frac{\cos^2 \chi_0}{\sin^3 \chi_0} \right] h,$$

$$b_{24} = 2d_1 h, \quad b_{44} = l_0 b_{24}, \quad b_{27} = \frac{d_1^2 d_2 \frac{\cos^2 \chi_0}{l_0}}{l_0} h, \quad b_{28} = d_1 \left[ d_2 \left( \frac{d_1}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} + \frac{d_2}{2} \right) \right] h,$$

$$b_{47} = \frac{d_1 d_2}{2} \frac{\cos^2 \chi_0}{l_0} (d_1 - d_2) h, \quad b_{48} = l_0 b_{28}, \quad b_{67} = \frac{d_1 d_2}{2} (d_2 - d_1) h, \quad b_{68} = -\frac{d_1^2 d_2}{(1 - \nu)l_0} \frac{\cos^2 \chi_0}{\sin^3 \chi_0} \left( \frac{d_2^2}{12} + K \right) h,$$

$$b_{77} = -\frac{\nu}{1 - \nu} \frac{d_1^2 d_2}{12l_0} (d_1 + d_2) h, \quad b_{78} = -\frac{d_1^2 d_2}{12} \frac{\cos^2 \chi_0}{l_0} h, \quad b_{87} = -\frac{d_1^2 d_2}{2} \frac{\cos^2 \chi_0}{(1 - \nu)l_0} \left( \frac{d_2^2}{12} + K \right) h,$$

$$b_{88} = -\frac{\nu}{1 - \nu} \frac{1}{l_0} \left[ d_1^2 d_2 \left( \frac{d_2^4}{80} + K \frac{d_2^2}{6} + K^2 \right) + \frac{d_1^3}{3} \left( \frac{d_2^4}{16} + K \frac{d_2^2}{2} + K^2 \right) \right] h;$$

$$c_{44} = b_{24}, \quad c_{48} = c_{84} = d_1 \left( \frac{d_2^2}{2} + 2K \right) h, \quad c_{77} = \frac{d_1 d_2}{2} (d_1 + d_2) \left( 1 + \frac{d_1 d_2}{6(1 - \nu)l_0^2} \right) h,$$

$$c_{88} = \left[ \frac{1}{(1 - \nu)l_0} \right] \left[ d_1^2 d_2 \left( \frac{d_2^4}{80} + K \frac{d_2^2}{6} + K^2 \right) + \frac{d_1^3}{3} \left( \frac{d_2^4}{16} + K \frac{d_2^2}{2} + K^2 \right) \right] + \frac{d_1^2 d_2}{6} \left( \frac{d_2^4}{8} + K^2 \right) \right] h, \quad K = -\frac{d_1^2 (d_1 + d_2)}{12(d_1/3 + d_2)};$$

$$d_{22} = \frac{\rho}{G} \left[ \frac{d_2^2}{2l_0} \left( d_2 \frac{\cos^2 \chi_0}{(1 - \nu)l_0} + \frac{d_1}{3} \right) + 2l_0 (d_1 + d_2) \right] h.$$
\[ d_{24} = \frac{\rho}{G} \left[ \frac{d_1^2 d_2}{2} \ctg \chi_0 + 2l_0^2 (d_1 + d_2) \right] h, \quad d_{26} = -\frac{\rho}{G} \frac{d_1^2 d_2}{2} \ctg \chi_0 h, \]
\[ d_{44} = \frac{\rho}{G} \frac{l_0}{2} \left[ \frac{d_2}{2} \ctg \chi_0 \left( d_1 (d_1 + d_2) + \frac{d_2^2}{3} \right) + 2l_0^2 (d_1 + d_2) \right] h, \]
\[ d_{46} = \frac{\rho}{G} \frac{d_2}{2} \frac{l_0 \ctg \chi_0}{2} \left[ \frac{d_2^2}{3} \ctg \chi_0 - d_1 (d_1 + d_2) \right] h, \]
\[ d_{66} = \frac{\rho}{G} \frac{d_2}{2} \frac{l_0}{2} \left[ \frac{d_2}{3} \ctg \chi_0 + d_1 (d_1 + d_2) \right] h, \quad d_{77} = \frac{\rho}{G} \frac{d_1 d_2^2}{24} (d_1 + d_2) l_0 h, \]
\[ d_{88} = \frac{\rho}{G} \frac{l_0}{2} \left[ \frac{d_1}{2} \frac{d_2}{2} \left( \frac{d_1^3}{80} + K \frac{d_2^2}{6} + K^2 \right) + \frac{d_1}{3} \left( \frac{d_1^2}{16} + K \frac{d_2^2}{2} + K^2 \right) \right] h. \]

Let us consider the forced vibrations of a thin-walled prismatic structure rigidly clamped in the section \( Z = 0 \). In the end section \( Z = Z_1 \), we apply an external load \( \bar{p} \) and generalized forces \( P_i^* \) that vary in harmonic law

\[ \bar{p}(Z,t) = \bar{p}(Z) \cos pt, \quad P_i^*(t) = P_i^* \cos pt, \quad (5) \]

where \( p \) – disturbance frequency.

Owing to the presence of resistance to motion, the natural oscillations damp over time, and the generalized displacements will change according to a harmonic law corresponding to external perturbations

\[ U_j(Z,t) = U_j(Z) \cos pt. \quad (6) \]

We assume that the shell under consideration has a variable thickness along the length corresponding to a power law

\[ h(Z) = (b - \beta Z)^k, \quad (7) \]

where \( b = h_1^{1/k}, \quad \beta = \left( h_1^{1/k} - h_2^{1/k} \right) / l_1. \)

From the external disturbing factors, we take the transverse force applied \( Q_y = Q_0 \cos pt \) in the end section, bending moment \( M_x = M_{0x} \cos pt \) and torque \( M_z = M_{0z} \cos pt \) (fig. 1), which vary according to the law (5), and \( \bar{p} = 0 \). Expanding (1) taking into account (6), we obtain a system of ordinary differential equations with variable coefficients:

\[ \xi a_{22} U_x^2 + \xi^2 a_{24} U_x^4 + \xi^2 a_{26} U_6^2 + \xi a_{28} U_8^2 + (\xi a_{22})' U_2 + [(\xi^2 a_{24})'] U_2^2 + \xi (b_{24} - a_{24} / l_0) U_4 + [(\xi^2 a_{26})' - a_{26} \xi / l_0] U_6^2 + b_{23} U_7 + [\xi a_{28}'] U_8^2 + b_{28} U_8^2 + [\xi (b_{24} - a_{24} / l_0)] U_4 - (a_{26} \xi / l_0) U_6^2 + b_{27} U_7 + b_{28} U_8^2 + p^2 \xi d_{22} U_2^2 + \xi d_{24} U_4^2 + \xi d_{26} U_6^2) = 0, \]
\[ \xi a_{24} U_x^2 + \xi^2 a_{44} U_x^4 + \xi^2 a_{46} U_6^2 + \xi a_{48} U_8^2 + [(\xi^2 a_{24})' - b_{24} U_2^2 + [(\xi^2 a_{44})' - a_{44} \xi / l_0] U_4 + [(\xi^2 a_{46})' - a_{46} \xi / l_0] U_6^2 + b_{47} U_7 + [\xi a_{48}'] U_8^2 + b_{48} U_8^2 + [\xi (a_{44} / l_0 - b_{44})] U_4 - (a_{46} \xi / l_0) U_6^2 + b_{47} U_7 \]

7.
\[
+(b'_{48} - c_{48}/\xi)U_8 + p^2 \xi (d_{24}U_2 + \xi d_{44}U_4 + \xi d_{46}U_6) = 0,
\]
\[
\xi a_{26}U'_2 + \xi^2 a_{48}U'_4 + \xi^2 a_{66}U'_6 + \xi a_{68}U'_8 + (\xi a_{26})'U'_2 + [(\xi^2 a_{46})'U' - a_{46}Z/l_0)U' + [(\xi^2 a_{66})'U_6 + b_{67}U'_7 + [(\xi a_{68})'U_8 + b_{68}U'_8
\]
\[
- (a_{46}Z/l_0)'U_4 + (a_{66}Z/l_0)'U_6 + b_{67}U'_7 + b_{68}U'_8
\]
\[+ p^2 \xi (d_{26}U_2 + \xi d_{46}U_4 + \xi d_{66}U_6) = 0,
\]
\[
\xi a_{77}U'_7 - b_{27}U'_2 - \xi a_{47}U'_4 - \xi a_{87}U'_8 + (\xi a_{77})'U'_2 + (b_{78} - b_{87})U'_8
\]
\[+ (b_{47}Z/l_0)U_4 + (b_{67}Z/l_0)U_6 + (b_{77} - c_{77}/\xi)U_7 + b_{78}U_8 + p^2 \xi d_{77}U_7 = 0,
\]
\[
\xi a_{22}U'_2 + \xi^2 a_{48}U'_4 + \xi^2 a_{66}U'_6 + \xi a_{68}U'_8 + [(\xi a_{28})'U'_2 - (b_{48} + a_{48}/l_0)U'_4 + [(\xi a_{48})'U'_6 - (b_{68} + a_{68}/l_0)U'_6
\]
\[+ (b_{87} - b_{87})U'_7 + [(\xi a_{88})'U'_8 - b_{48}/l_0 - (a_{48}Z/l_0)'U'_4 + c_{48}U'_4
\]
\[+ [b_{68}/l_0 - (a_{68}Z/l_0)'U_6 + b_{67}U_7 + (b_{68} - c_{68}/\xi)U_8 + p^2 \xi d_{88}U_8 = 0, \tag{8}
\]

where \( \xi = 1 - Z/l_0. \)

Limit conditions (3), (4) at \( z = 0 \)
\[
U_2 = 0, \quad U_4 = 0, \quad U_6 = 0, \quad U_7 = 0, \quad U_8 = 0. \tag{9}
\]

In the end section of the shell when \( z = l_1, \) we obtain a system of inhomogeneous differential equations:

\[
\xi_1 a_{22}U'_2 + \xi_1^2 a_{44}U'_4 + \xi_1^2 a_{66}U'_6 + \xi_1 a_{88}U'_8 + \xi_1 (b_{24} - a_{24}/l_0)U_4
\]
\[+ (\xi_1 a_{46}/l_0)U_6 + b_{27}U_7 + b_{28}U_8 = Q_0 / G,
\]
\[
\xi_1 a_{22}U'_2 + \xi_1^2 a_{44}U'_4 + \xi_1^2 a_{66}U'_6 + \xi_1 a_{88}U'_8 - \xi_1 (b_{44} - a_{44}/l_0)U_4
\]
\[+ (\xi_1 a_{46}/l_0)U_6 + b_{47}U_7 + b_{48}U_8 = M_{0x} + Q_0 (l_0 - l_1) / G \xi_1,
\]
\[
\xi_1 a_{22}U'_2 + \xi_1^2 a_{44}U'_4 + \xi_1^2 a_{66}U'_6 + \xi_1 a_{88}U'_8 - (\xi_1 a_{46}/l_0)U_4
\]
\[+ (\xi_1 a_{46}/l_0)U_6 + b_{67}U_7 + b_{68}U_8 = M_{0z} / G \xi_1,
\]
\[
\xi_1 a_{77}U'_7 + b_{77}U_7 + b_{78}U_8 = 0,
\]
\[
\xi_1 a_{28}U'_2 + \xi_1^2 a_{48}U'_4 + \xi_1^2 a_{68}U'_6 + \xi_1 a_{88}U'_8 - (\xi_1 a_{88}/l_0)U_4
\]
\[+ (\xi_1 a_{88}/l_0)U_6 + b_{87}U_7 + b_{88}U_8 = 0. \tag{10}
\]

where \( \xi_1 = 1 - l_1/l_0. \)

It is not possible to obtain an analytical solution to the obtained quasistatic problem (8) - (10). Therefore, a numerical approach was applied, based on the algorithm of orthogonal sweeping of the equations of the system of linear ordinary differential equations of the first order [6] when solving the boundary value problem under consideration.

When lowering the order of equations of system (8) from the second to the first of the first, second, third and fifth equations, they were expressed as \( U'_2, U'_4, U'_6, U'_8, \) and the system of 5 differential equations of the 2nd order was reduced to a system of 10 equations of the 1st order. The stability of the numerical calculation was controlled by gradually decreasing the integration step until the difference between the two solutions with different steps amounted to about \( 10^{-7}. \)

For a weakly conical shell with a rectangular contour of the cross section fixed along the beveled end, some results of numerical calculations are presented in Figure 2 and 3.
Figure 2. Dynamic stresses under the action of harmonic torque.

Figure 3. Dynamic stresses under the action of harmonic bending force.
3. Results and discussion
Let us analyze the obtained graphic relationships. From Figure 2, it follows that in the case of torsion of a tapered thin-walled conical structure in the embedment area and at the free end face, there is a boundary effect, which is due to the constraint of the cross section deplanation in the embedment area, as well as the presence of internal constraint due to the conicity of the shell. In [1], for shells of constant thickness, this effect was also noted. If the design has a variable thickness in scope, then this effect is enhanced, especially at the free end. When designing full-scale objects, the noted feature of tapered conical thin-walled structures must be taken into account. With an increase in the value of the frequency $p$ of forced oscillations under the action of the moment $M_i (N\cdot m)$, the voltages $\sigma_1 (Pa)$ first decrease (at $p^2 = 5\cdot 10^4$), then they go into the negative region and subsequently change the nature of the distribution over the span. The stress distribution $\sigma_1$ along the short and long ribs is also different.

The distribution of normal stresses $\sigma_1 (Pa)$ over the span under the influence of an alternating transverse perturbing force $Q_i (N)$ is presented in Figure 3. As the frequency $p$ of the perturbing factor increases, the normal stresses $\sigma_1$ also increase. But the edge effect takes place only in the embedment region, since in the end section, as one would expect, $\sigma_1 \rightarrow 0$ due to free deplanation at $z = l$. The distribution of the curves in fig. 3 shows that along the short rib $A - A$, the magnitude of the normal stresses $\sigma_1$ is higher than along the long rib $B - B$, ceteris paribus. The span distribution of stresses under the influence of a static load is characterized by curves at $p = 0$. It should be noted that the nature of the distribution of static stresses along $z$ is in qualitative agreement with the experimental data performed on the large-sized model of the swept wing and presented in [7].

4. Conclusion
From the presented general solution of the boundary value problem, a number of special cases can be obtained. So, setting $h = \text{const}$, we can obtain solutions for shells of constant thickness. When $l_0 \rightarrow \infty$ we come to the beveled cylindrical shells, and taking $\varphi_0 = \pi/2$, we get conical and cylindrical shells. Therefore, the presented solution generalizes the known results for conical shells with a non-circular contour of the cross section and a constant thickness over the span [1].

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