A note on logarithmic growth Newton polygons of $p$-adic differential equations

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Abstract

In this paper, we answer a question due to Y. Andrè related to B. Dwork’s conjecture on a specialization of the logarithmic growth of solutions of $p$-adic linear differential equations. Precisely speaking, we explicitly construct a $\nabla$-module $M$ over $\mathbb{Q}_p[[X]]_0$ of rank 2 such that the left endpoint of the special log-growth Newton polygon of $M$ is strictly above the left endpoint of the generic log-growth Newton polygon of $M$.

1 Introduction

We consider an ordinary linear $p$-adic differential equation

$$Dy = y^{(\mu)} + f_{\mu-1}y^{(\mu-1)} + \cdots + f_0y = 0,$$

where the $f_i$’s are bounded analytic functions in the unit disc $|x| < 1$, with coefficients in a $p$-adic field. We assume that the differential equation has a full set of solutions $y$ that are analytic in the unit disc. For example, this assumption is satisfied for Picard-Fuchs equations.

In [Dwo73], B. Dwork studied asymptotic behavior of the solutions around the boundary $|x| = 1$ and he proved that $y$ has at most logarithmic growth (log-growth) of order $\mu - 1$, that is,

$$|y|_0(r) = O((\log (1/r))^{1-\mu})$$

as $r \uparrow 1$,

where $| \cdot |_0(r)$ means the $r$-Gaussian norm with center 0. To obtain more precise information about the log-growth of the solutions of $Dy = 0$, he defined the log-growth Newton polygon $NP_{\log, 0}(D)$.

Then, Dwork made the following observations and stated two fundamental conjectures on log-growth Newton polygons (see [Dwo73, Concluding Remark 3] for details): He first defined the notion of a Frobenius structure for a $p$-adic differential equation. If $Dy = 0$ admits a Frobenius structure, then the solution space of $Dy = 0$ is endowed with a canonical Frobenius structure. Then, the associated Frobenius Newton polygon is called the special Frobenius Newton polygon $NP_{\varphi, 0}(D)$. If we pull back the unit disc to the unit disc around a generic point $t$ of the unit disc, then we can obtain a $p$-adic differential equation $D_t y = 0$, which is defined on the disc $|X - t| < 1$. Then, we can compute the log-growth Newton polygon associated to $D_t y = 0$, which is called the generic log-growth Newton polygon $NP_{\log, t}(D_t)$. Moreover, $D_t y = 0$ is also endowed with a Frobenius structure, and the corresponding Frobenius Newton polygon is called the generic Frobenius Newton polygon $NP_{\varphi, t}(D_t)$. Then, based on a calculation for the hypergeometric differential equation with parameters $(1/2, 1/2; 1)$, Dwork conjectured that

**Conjecture 1.1.** $NP_{\log, 0}(D) = NP_{\varphi, 0}(D)$ and $NP_{\log, t}(D_t) = NP_{\varphi, t}(D_t)$.

Note that if Conjecture [1] is true, then the special log-growth Newton polygon is above the generic log-growth Newton polygon by Grothendieck’s specialization theorem for $F$-isocrystals. Thus, he also conjectured that

**Conjecture 1.2.** $NP_{\log, 0}(D)$ is above $NP_{\log, t}(D_t)$.

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Dwork stated these conjectures vaguely and even the precise formulations of the conjectures were given only recently: Conjecture 1.1 was formulated accurately by B. Chiarellotto and N. Tsuzuki ([CT09, Conjectures 6.8, 6.9]) and they proved their conjecture in the special cases where \( \mu \) is \( \leq 2 \) (Theorem 7.1 (2) loc. cit.), or where \( D \) satisfies a certain technical condition called HPBQ ([CT11, Theorem 6.5]). The conjecture in the generic case was proved without any assumption ([CT11, Theorem 7.1]). Conjecture 1.2 was proved by Y. André in the following form: Since Dwork did not fix the endpoints of the log-growth Newton polygons, he first fixed the right endpoints of the log-growth Newton polygons at \((\mu, 0)\). Under this convention, he proved that \( \NP_{\log,0}(D) \) is above \( \NP_{\log,i}(D_i) \) without assuming the existence of a Frobenius structure on \( D\bar{y} = 0 \) ([And08, Theorem 4.1.1]). Then, André asked whether or not the left endpoint of the log-growth Newton polygon is stable under specialization. In any known example of \( p \)-adic differential equations at that point, the left endpoints of the special and generic Newton polygons coincide with each other. We also note that if \( D\bar{y} = 0 \) admits a Frobenius structure and Chiarellotto-Tsuzuki’s conjecture (in the special case) is true for \( D\bar{y} = 0 \), then the left endpoints of the special and generic Newton polygons coincide with each other ([CT11, Theorem 8.1]).

The aim of this paper is to answer André’s question “negatively”: We will explicitly construct a \( \nabla \)-module \( M \) of rank 2 such that the left endpoint of the special log-growth Newton polygon of \( M \) is strictly above the left endpoint of the generic log-growth Newton polygon of \( M \). Also, we prove that our example does not admit a Frobenius structure. Hence, the existence of our example means that equality of the endpoints is apparently a special feature of log-growth in the presence of a Frobenius structure.

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2 Construction of a \( p \)-adic differential equation

We first recall some notation in [And08].

Notation 2.1. Let \( p \) be a prime number. Let \( v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\} \) be a discrete valuation such that \( v_p(p) = 1 \). We define a norm \( | \cdot |_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0} \) by \( |x|_p = p^{-v_p(x)} \). For a complete valuation field \( K \) with integer ring \( \mathcal{O}_K \), let \( K[[X]]_0 := \mathcal{O}_K[[X]][p^{-1}] \). We denote by \( t \) a formal variable, i.e., a Dwork generic point and let \( \mathbb{Q}_p\{\{t\}\} \) be the fraction field of the \( p \)-adic completion of \( \mathbb{Z}_p[[t]][t^{-1}] \). For \( x \in \mathbb{R} \), let \( |x| := \max\{n \in \mathbb{Z} : n \leq x\} \). Let \( M \) be a \( \nabla \)-module over \( \mathbb{Q}_p[[X]]_0 \) ([CT09, § 0.3]). We denote \( \bar{M}_t := M \otimes_{\mathbb{Q}_p\{\{t\}\}} \mathbb{Q}_p\{\{t\}\}[[X-t]]_0 \). We denote the special log-growth Newton polygon of \( M \) by \( \NP_{\log,0}(M) \) ([And08, § 3.3]). We also denote the generic log-growth Newton polygon of \( M \) by \( \NP_{\log,i}(M_t) \) ([And08, § 3.4]).

Definition 2.2. Let \( \sigma \in \mathbb{R}_{\geq 0} \). We define \( P_{\sigma} \subset \mathbb{R}^2 \) as the lower convex polygon defined by the vertices \((0, -\sigma), (1, -\sigma), \) and \((2, 0)\). Obviously, the slope set of \( P_{\sigma} \) is \( \{0, \sigma\} \).

Theorem 2.3. Let \( \sigma, \sigma' \) be real numbers satisfying \( 0 \leq \sigma' < \sigma < 1 \). Then, there exists a \( \nabla \)-module \( M = M_{\sigma,\sigma'} \) over \( \mathbb{Q}_p[[X]]_0 \) of rank 2 such that

\[
\NP_{\log,0}(M) = P_{1-\sigma}, \quad \NP_{\log,i}(M_t) = P_{1-\sigma'}.
\]

In particular, the left endpoint of \( \NP_{\log,0}(M) \) is strictly above the left endpoint of \( \NP_{\log,i}(M_t) \).

In the following, we will construct \( M = M_{\sigma,\sigma'} \). Denote \( \delta := (\sigma - \sigma')/(1 - \sigma) \in \mathbb{R}_{\geq 0} \). For \( n \in \mathbb{N} \), we put

\[
a_n := \begin{cases} p^{\delta r} & \text{if } n = p^r(p^{\delta r} + 1) - 1 \text{ for some } r \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
f := \sum_{n \in \mathbb{N}} a_n X^n = X^p + \cdots \in \mathbb{Z}_p[[X]].
\]
We define $M := \mathbb{Q}_p[[X]] e_1 \oplus \mathbb{Q}_p[[X]] e_2$ endowed with an action of $d/dX$ given by
\[
\nabla \left( \frac{d}{dX} \right) (e_1 e_2) = (e_1 e_2) \begin{pmatrix} 0 & -f \\ 0 & 0 \end{pmatrix}.
\]
We put
\[
y_s := \sum_{n \in \mathbb{N}} \frac{1}{n+1} a_{n+1} X^{n+1} \in \mathbb{Q}_p[[X]],
\]
\[
y_g := \sum_{n \in \mathbb{N}} (X-t)^{n+1} \sum_{k \geq n} a_k t^{k-n} \binom{k}{n} \in \mathbb{Q}_p \{ \{ t \} \} [X-t].
\]
Then, the space of the horizontal sections of $M_{\mathbb{Q}_p[[X]]}$ and $M_{\mathbb{Q}_p \{ \{ t \} \} [[X-t]]}$ admit basis $\{ e_1, y_s e_1 + e_2 \}$ and $\{ e_1, y_g e_1 + e_2 \}$. Therefore, to prove Theorem 2.3, we have only to prove that
\[
y_s \text{ is exactly of log-growth } 1 - \sigma,
\]
\[
y_g \text{ is exactly of log-growth } 1 - \sigma'.
\]

Remark 2.4. Our example $M_{\sigma, \sigma'}$ with $\sigma = \sigma'$ coincides with [CT09, Example 5.3]. Also, $M_{\sigma, \sigma'}$ with $\sigma \neq \sigma'$ does not admit a Frobenius structure. In fact, if $M_{\sigma, \sigma'}$ admits a Frobenius structure, then the left endpoints of the special and generic log-growth Newton polygons coincide with each other by [CT09, Theorem 7.1 (2)] and [CT11, Theorem 8.1], which contradicts to Theorem 2.3.

2.1 Calculation of the log-growth of $y_s$
We estimate $a_n/(n+1)$, which is the coefficient of $y_s$ at $X^{n+1}$, as follows. Let $\lambda \in \mathbb{R}_{\geq 0}$ and $n = p^r (p^{|\delta r|+1} + 1) - 1$ for some $r \in \mathbb{N}$. Then, we have
\[
\left| \frac{a_n}{n+1} \right|_p / (n+1)^\lambda = p^{-|\sigma'| - r + |\delta r| + 1} 
\]
\[
\frac{1}{(1+p^{-|\delta r|-1})^\lambda}.
\]
When $(1-\sigma')/(1+\delta)(= 1 - \sigma) \leq \lambda$, we have
\[
- |\sigma'| r + r - (r + |\delta r| + 1) \lambda 
\]
\[
\leq - \sigma' r + 1 + r - (r + |\delta r|) \lambda
\]
\[
\leq - \sigma' r + 1 + r - (r + |\delta r|) \left( \frac{1 - \sigma'}{1+\delta} \right) = 1.
\]
When $(1-\sigma')/(1+\delta) > \lambda$, we have
\[
- |\sigma'| r + r - (r + |\delta r| + 1) \lambda
\]
\[
\geq - \sigma' r + r - (r + |\delta r| + 1) \lambda
\]
\[
=r(-\sigma' + 1 - (1+\delta) \lambda) - \lambda
\]
\[
=r(1+\delta) \left( \frac{1 - \sigma'}{1+\delta} - \lambda \right) - \lambda,
\]
where the last term tends to $\infty$ as $n \to \infty$. Hence, we have
\[
\left| \frac{a_n}{n+1} \right|_p / (n+1)^\lambda \begin{cases} O(1) \text{ as } n \to \infty & \text{if } 1 - \sigma \leq \lambda \\ \to \infty \text{ as } n \to \infty & \text{if } 1 - \sigma > \lambda,
\end{cases}
\]
which implies that $y_s$ is exactly of log-growth $1 - \sigma$. 3
2.2 Calculation of the log-growth of $y_g$

First, we prove that $y_g$ is not of log-growth $\lambda$ for any $\lambda \in [0, 1 - \sigma')$. We will use the following lemma:

**Lemma 2.5.** Let $u \in \mathbb{Z}_p$ and $0 \leq r \leq s \in \mathbb{N}$. Then, we have

\[
\left( \frac{p^ru - 1}{p^r - 1} \right) \in \mathbb{Z}_p^*.
\]

**Proof.** In $\mathbb{F}_p[[X]]/X^p \mathbb{F}_p[[X]]$, we have

\[
(1 + X)^{p^ru - 1} = (1 + X)^{p^ru}(1 + X)^{-1} = (1 + X)^{p^r u}(1 + X)^{-1} = (1 + X)^{-1} = \sum_{i \in \mathbb{N}} (-X)^i.
\]

Hence, we have $(\frac{p^ru - 1}{p^r - 1}) = (\frac{1}{p^r - 1}) \neq 0$ in $\mathbb{F}_p$, which implies the assertion. \hfill \Box

When $n = p^r - 1$ for some $r \in \mathbb{N}$, the coefficient of $y_g$ at $(X - t)^{n+1}p^{r+1}$ is equal to

\[
\frac{1}{n + 1} a_{p^{r+1}} \left( \frac{p^{r+1}}{n+1} - 1 \right),
\]

By Lemma 2.5, we have $(\frac{p^{r+1}}{n+1} - 1) \in \mathbb{Z}_p^*$. We also have

\[
\left( \frac{a_{p^{r+1}}}{n + 1} \right)_p / (n + 1) = p^{-(\sigma' r - r - \lambda)} \geq p^{-(\sigma' r + r - \lambda)} = p^{-(\sigma' + 1 - \lambda)}.
\]

Since $-\sigma' + 1 - \lambda > 0$ by assumption, \(O\) tends to $\infty$ as $n \to \infty$, which implies the assertion.

Finally, we prove that $y_g$ is of log-growth $1 - \sigma'$. Put $\lambda := 1 - \sigma'$. For $n \in \mathbb{N}$, we define

\[
S^+(n) := \{ k \geq n; k = p^r(p^{[\delta r]} + 1) - 1 \text{ for some } r \in \mathbb{N}_{\geq v_p(n+1)} \},
\]

\[
S^-(n) := \{ k \geq n; k = p^r(p^{[\delta r]} + 1) - 1 \text{ for some } r \in \mathbb{N}_{v_p(n+1)} \}.
\]

In the following, we fix $n \in \mathbb{N}$ and estimate the Gaussian norm of

\[
\sum_{k \in S^+(n) \cup S^-(n)} \frac{a_k}{n+1} \binom{k}{n} t^{k-n} \in \mathbb{Q}_p \{ \{ t \} \},
\]

which is the coefficient of $y_g$ at $X^{n+1}$.

Case 1: $k = p^r(p^{[\delta r]} + 1) - 1 \in S^+(n)$.

We have

\[
\left| \frac{a_k}{n+1} \binom{k}{n} \right|_p / (n+1)^{\lambda} \leq \left| \frac{a_k}{n+1} \binom{k}{n} \right|_p / (n+1)^{\lambda} \leq p^{-(\sigma' r + v_p(n+1) - v_p(n+1)\lambda)} \leq p^{-(\sigma' r + v_p(n+1)\lambda)} \leq p.
\]

Case 2: $k = p^r(p^{[\delta r]} + 1) - 1 \in S^-(n)$.

Since $(n+1)^{-1}(\frac{k}{n+1}) = (k+1)^{-1}(\frac{k}{n+1})$, we have

\[
\left| \frac{a_k}{n+1} \binom{k}{n} \right|_p / (n+1)^{\lambda} \leq \left| \frac{a_k}{k+1} \binom{k}{n} \right|_p / (n+1)^{\lambda} \leq p^{-(\sigma' r + r - v_p(n+1)\lambda)} < p^{-(\sigma' + r - r - \lambda)} = p.
\]

Hence, \(O\) is equal to $O((n+1)^{1-\sigma'})$ as $n \to \infty$, which implies that $y_g$ is of log-growth $1 - \sigma'$.

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