THURSTON'S H-PRINCIPLE AND FLEXIBILITY OF POISSON STRUCTURES

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Abstract. We prove an analogue of Thurston's h-principle for 2-dimensional foliations on manifolds of dimension bigger or equal to 4, in the presence of a fiber-wise non-degenerate 2-form. This helps us understand the flexibility of rank 2 regular Poisson structures on open manifolds with dimension bigger or equal to 4 and it also helps us understand the flexibility of Poisson structures (not regular) on closed 4-manifolds.

1. INTRODUCTION

An h-principle for Poisson structures on open manifolds has been proved by Fernandes and Frejlich in [6]. We state their result below.

Let $M^{2n+q}$ be a $C^\infty$-manifold equipped with a co-dimension-$q$ foliation $\mathcal{F}_0$ and a 2-form $\omega_0$ such that $(\omega_0^n|_{\mathcal{T}\mathcal{F}_0}) \neq 0$. Denote by $\text{Fol}_q(M)$ and $\text{Dist}_q(M)$ the spaces of co-dimension-$q$ foliations and distributions on $M$ respectively identified as a subspace (the entire space in case of distributions) of $\Gamma(Gr_{2n}(M))$, where $Gr_{2n}(M) \xrightarrow{pr} M$ be the grassmann bundle, i.e, $pr^{-1}(x) = Gr_{2n}(T_xM)$ and $\Gamma(Gr_{2n}(M))$ is the space of sections of $Gr_{2n}(M) \xrightarrow{pr} M$ with compact open topology. Define

$$\Delta_q(M), \tilde{\Delta}_q(M) \subset \text{Dist}_q(M) \times \Omega^2(M)$$

$$\Delta_q(M) := \{(\mathcal{F}, \omega) : \omega^n|_{\mathcal{T}\mathcal{F}} \neq 0\}$$

$$\tilde{\Delta}_q(M) := \{(\mathcal{D}, \omega) : \omega^n|_{\mathcal{T}\mathcal{D}} \neq 0\}$$

Obviously $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$. In this setting Fernandes and Frejlich has proved the following

**Theorem 1.1.** ([6]) Let $M^{2n+q}$ be an open manifold with $(\mathcal{F}_0, \omega_0) \in \Delta_q(M)$ be given. Then there exists a homotopy $(\mathcal{F}_t, \omega_t) \in \Delta_q(M)$ such that $\omega_1$ is $d_{\mathcal{F}_1}$-closed (actually exact).

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In the language of poisson geometry the above result \([\ref{1}].\) takes the following form. Let \(\pi \in \Gamma(\wedge^2 T M)\) be a bi-vectorfield on \(M\), define \(\# \pi : T^* M \to TM\) as \(\# \pi(\eta) = \pi(\eta, -)\). If \(\text{Im}(\# \pi)\) is a regular distribution then \(\pi\) is called a regular bi-vectorfield.

**Theorem 1.2.** Let \(M^{2n+q}\) be an open manifold with a regular bi-vectorfield \(\pi_0\) on it such that \(\text{Im}(\# \pi_0)\) is an integrable distribution then \(\pi_0\) can be homotoped through such bi-vectorfields to a poisson bi-vectorfield \(\pi_1\).

**Remark 1.3.** Fernandes and Frejlich has also shown by example in \([6]\) that if the condition that \(\text{Im}(\# \pi_0)\) being integrable is removed then the result fails. This is because in general a distribution need not have a foliation in its homotopy class. On open manifolds the obstruction is known by Haefliger in \([7]\).

In \([1]\) above \(d\mathcal{F}\) is the tangential exterior derivative, i.e, for \(\eta \in \Gamma(\wedge^k T^* \mathcal{F})\), \(d\mathcal{F}\eta\) is defined by the following formula

\[
d\mathcal{F}\eta(X_0, X_1, ..., X_k) = \sum_i (-1)^i X_i(\eta(X_0, ..., \hat{X}_i, ..., X_k)) + \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k)
\]

where \(X_i \in \Gamma(T \mathcal{F})\). So if we extend a \(\mathcal{F}\)-leafwise closed \(k\)-form \(\eta\), i.e, \(d\mathcal{F}\eta = 0\), to a form \(\eta'\) by the requirement that \(\ker(\eta') = \nu \mathcal{F}\), where \(\nu \mathcal{F}\) is the normal bundle to \(\mathcal{F}\), then \(d\eta' = 0\).

In order to fix the foliation in \([1]\) one needs to impose an openness condition on the foliation, we refer the readers to \([1]\) for precise definition of this openness condition. Under this hypothesis Bertelson proved the following

**Theorem 1.4.** (\([1]\)) If \((M, \mathcal{F})\) be an open foliated manifold with \(\mathcal{F}\) satisfies some openness condition and let \(\omega_0\) be a \(\mathcal{F}\)-leaf wise 2-form then \(\omega_0\) can be homotoped through \(\mathcal{F}\)-leaf wise 2-forms to a \(\mathcal{F}\)-leaf wise symplectic form.

She also constructed counter examples in \([2]\) that without this openness condition the above theorem fails. A contact analogue of Bertelson’s result on any manifold (open or closed) has recently been proved in \([3]\) by Borman, Eliashberg and Murphy.

\([1.1]\) and \([1.2]\) has been generalized to closed manifolds in \([10]\).

**Theorem 1.5.** Let \(M^{2n+q}\) be a closed manifold with \(q = 2\) and \((\mathcal{F}_0, \omega_0) \in \Delta_q(M)\) be given. Then there exists a homotopy \(\mathcal{F}_t\) of singular foliations on \(M\) with singular locus \(\Sigma_t\) and a
homotopy of two forms $\omega_t$ such that the restriction of $(\omega_t)$ to $TF_t$ is non-degenerate and $\omega_1$ is $d_{F_t}$-closed.

In terms of poisson geometry [1.5] states

**Theorem 1.6.** Let $M^{2n+q}$ be a closed manifold with $q = 2$ and $\pi_0$ be a regular bi-vectorfield of rank $2n$ on it such $\text{Im}(\#\pi_0)$ is integrable distribution. Then there exists a homotopy of bi-vectorfields $\pi_t$, $t \in I$ (not regular) such that $\text{Im}(\#\pi_t)$ integrable and $\pi_1$ is a poisson bi-vectorfield.

Now we come to the main topic of this paper. In [12] Thurston has proved the following

**Theorem 1.7.** Any $C^\infty$ 2-plane field on a manifold $M$ of dimension at least 4 is homotopic to an integrable one. Relative version of this result is also true.

In this paper we prove an analogue of this result in the presence of a fiber-wise non-degenerate 2-form. The main theorem of this paper is the following. For a 2-distribution $D$ and for a section $\tau$ of $D$, we set $\omega|_D = \omega|_{\tau}$.

**Theorem 1.8.** Let $M^n$ be a manifold of dimension at least 4, i.e, $n \geq 4$ and $(\tau_0, \omega_0) \in \bar{\Delta}_{n-2}(M)$. Then $(\tau_0, \omega_0)$ is homotopic to $(\tau_1, \omega_1)$ through a homotopy $(\tau_t, \omega_t) \in \bar{\Delta}_{n-2}(M)$, $t \in [0,1]$ such that $\tau_1$ is integrable.

Using [1.8] in [1.1] and [1.5] (or equivalently in [1.2] and [1.6]) we get the following.

**Corollary 1.9.** If there exists a regular rank 2 bivector field $\pi_0$ on an open manifold $M$ of dimension bigger or equal to 4 then there exists a homotopy $\pi_t$, $t \in [0,1]$ of regular bivector fields of rank 2 such that $\pi_1$ is regular Poisson.

If on the other hand $M$ is closed 4-manifold and $\pi_0$ a regular rank 2 bivector field on it then there would exists a homotopy of bivector fields $\pi_t$, $t \in [0,1]$ (not regular) such that $\pi_1$ is Poisson (not regular).

2. Plan of the proof of [1.8]

We closely follow [9] in order to prove [1.8] We state the following proposition.

**Proposition 2.1.** Let $(\tau_0, \omega_0) \in \bar{\Delta}_{n-2}(\mathbb{R}^n)$ then $(\tau_0, \omega_0)$ is homotopic through $(\tau_t, \omega_t) \in \bar{\Delta}_{n-2}(\mathbb{R}^n)$, $t \in [0,1]$ such that $\tau_1$ is integrable in a neighborhood of $[-1,1]^n$ and $(\tau_t, \omega_t)$ is constant in a neighborhood of $\mathbb{R}^n - (-2,2)^n$. 
Proof. (Proof of 1.8) Let \( \{ h_i : U_i \to \mathbb{R}^n \}_{i\in\mathbb{N}} \) be a countable atlas with the property that \( \{ h_i^{-1}((-1,1)^n) \}_{i\in\mathbb{N}} \) covers \( M \). Define
\[
(\tau'_0, \omega'_0) = ((h_1)_*(\tau_0), (h_1)_*(\omega_0))
\]
Using 2.1 for \((\tau'_0, \omega'_0)\) we get \((\tau'_t, \omega'_t) \in \bar{\Delta}_{n-2}(\mathbb{R}^n)\) such that \(\tau'_1\) is integrable in a neighborhood of \([-1,1]^n\) and \((\tau'_t, \omega'_t)\) is constant on a neighborhood of \(\mathbb{R}^n - (-2,2)^n\).

\(h_1^*\tau'_1\) can be extended by \(\tau\) to a plane field \(\tau_1\) and do the same for \(h_1^*\omega'_1\) and let us call it \(\omega_1\).

Continue by replacing \(\tau_0, \omega_0\) and \(h_1\) by \(\tau_1, \omega_1\) and \(h_2\). If \(x \in \bigcup_{i=1}^N h_i^{-1}((-1,1)^n)\), then there is a neighborhood \(V\) of \(x\) such that all homotopies after \(N\) steps are constant and if \(V\) is small enough, all homotopies from step 1 to \(N\) are constant on \(V\) and hence the sequence \((\tau_0, \omega_0), (\tau_1, \omega_1), ...\) converges. \(\square\)

So the proof of 2.1 is the main task. It shall be done in the following three steps namely

1. Triangulate \(\mathbb{R}^n\) so that \(\tau_0\) is in general position on a neighborhood of \([-1,1]^n\).

2. Deform \((\tau_0, \omega_0)\) into one which is civilized in near \([-1,1]^n\) with respect to the triangulation in the first step.

3. Filling the holes.

These steps will be covered in the following sections bellow.

3. Triangulation in general position

This part is exactly same as [9], we just outline the main definitions and results for completion.

Definition 3.1. For a \(k\)-plane field \(\tau\) on \(\mathbb{R}^n\), an \(n\)-simplex \(\sigma\) is in general position with respect to \(\tau\) if for all \(x \in \sigma\) the orthogonal projection along \(\tau(x)\) from the tangent plane of every \((n-k)\)-face to \(\tau(x)\) is injective. A triangulation is said to be in general position with respect to \(\tau\) in a neighborhood of a closed set if every \(n\)-simplex of this triangulation intersecting the closed set is in general position.
Definition 3.2. An affine triangulation $T'$ is said to be an $\varepsilon$-jiggling of a given affine triangulation $T$ for a given $\varepsilon > 0$ if there is a simplicial isomorphism $T \xrightarrow{\phi} T'$ such that $|\phi(v) - v| < \varepsilon$ for every vertex $v \in T$

Proposition 3.3. For a given $C^1$-plane field $\tau$ on $\mathbb{R}^n$ with $1 \leq k \leq n - 1$, a compact set $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists an $L \in \mathbb{N}$ such that whenever $l \geq L$ there exists an $\varepsilon$-jiggling of the triangulation associated to the cubical lattice $(\frac{1}{l}\mathbb{Z})^n$ which is in general position with respective to $\tau$ near $K$.

4. Civilization

Let $T$ be an $\varepsilon$-jiggling of the standard triangulation associated to the lattice $(\frac{1}{l}\mathbb{Z})^n$ which is in general position with respect to $\tau$ in a neighborhood of $[-2, 2]^n$. Take $l$ large enough so that the following holds.

(A) If $x$ is a vertex of $T$ with $\bar{s}I(x, T) \cap [-1, 1]^n$ being non-empty then $\bar{s}I(x, T) \subset [-\frac{3}{2}, \frac{3}{2}]^n$.

(B) If $\sigma$ is a $n$-simplex of $T$ which intersects $[-2, 2]^n$ then for any $x, y \in \sigma$ the plane field $\tau(y)$ is the graph of a linear map $L_{xy}: \tau(x) \to \tau(x)^\perp$ of norm less than one.

Let $T_1$ be the union of all simplices which are faces of $n$-simplices of $T$ intersecting $[-1, 1]^n$.

We shall deform $(\tau, \omega)$ into one whose plane field is integrable in a neighborhood of the $(n - 1)$-skeleton of $T_1$. The deformation will be done through $(\tau_t, \omega_t) \in \bar{\Delta}_{n-2}(\mathbb{R}^n)$ such that $T$ is in general position with respect to $\tau_t$ near $[-2, 2]^n$ and which satisfies (B).

We first give the definition of civilized pair $(\tau, \omega)$.

Definition 4.1. Let $0 \leq j \leq n - 2$. We say $(\tau_j, \omega_j)$ civilized on the $j$-skeleton of $T_1$ if $T_1$ is in general position with respect to $\tau_j$ near $[-2, 2]^n$ and there are real numbers $\delta_0 > ... > \delta_j > 0$ and $\eta_0 > ... > \eta_j > 0$ satisfying

(C) Let $\sigma$ be an $i$-simplex of $T_1$ with $0 \leq i \leq j$. For $x \in \sigma$ consider the planes $x + \tau_j(x)$ and $x + E_x$, where $E_x = (\tau_j(x) + T\sigma)^\perp$. Let $B_x(\delta)$ and $E_x(\eta)$ be the closed $\delta$ and $\eta$ neighborhoods of $x$ in the planes $x + \tau_j(x)$ and $x + E_x$ respectively. From [3.1] it follows that the disk $B_x(\delta) \times E_x(\eta)$ has dimension $(n - i)$. Then the disk $B_x(\delta_i) \times E_x(\eta_i)$ is the fiber of a tubular neighborhood $N(\sigma)$ of $\sigma$ in $\mathbb{R}^n$ moreover any $(n - 2)$-simplex
of $T_1$ which has $\sigma$ as a face intersects the boundary of the disk $B_x(\delta_1) \times E_x(\eta_1)$ in a subset of $\text{int}(B_x(\delta_1)) \times \partial(E_x(\eta_1))$.

(D) On the fibers $B_x(\delta_1) \times E_x(\eta_1)$, $(\tau_j, \omega_j) = (\tau_j(x), \omega_j(x))$.

(E) For two simplices $\sigma, \sigma'$ of $T_1$ of dimension less or equal to $j$, we must have $N(\sigma) \cap N(\sigma') \subset N(\sigma \cap \sigma')$. If $\sigma'$ is a proper face of $\sigma$ and $N(\sigma) \cap N(\sigma')$ non-empty so that $(y_1, y_2) \in B_x(\delta_1) \times E_x(\eta_1)$ lies in $B_v(\delta_{v'}) \times \{y'_2\}$ with $v \in \sigma'$ and $y'_2 \in E_v(\eta_{v'})$ with $i' = \dim \sigma'$, then we must have $B_x(\delta_1) \times \{y_2\} \subset \text{int}(B_v(\delta_{v'})) \times \{y'_2\}$. We must also have $N(\sigma) \cap \sigma''$ be empty, where $\sigma''$ is of dimension at least $j + 1$ for which $\sigma$ is not a face.

For the case $j = n - 1$ the conditions (C), (D) and (E) will have to be replaced by (C'), (D') and (E') respectively.

(C') For a $(n - 1)$-simplex $\sigma$ of $T_1$ let $F_x(\delta)$ be the closed $\delta$-neighborhood of $x$ in the line $x + F_x$, where $F_x$ is the the orthogonal complement of $\tau_{n-1}(x) \cap T\sigma$ in $\tau_{n-1}(x)$. Then $F_x(\delta_{n-1})$ is the fiber of a tubular neighborhood $N(\sigma)$ of $\sigma$ in $\mathbb{R}^n$.

(D') If $x \in \sigma$, where $\sigma$ is a $(n - 1)$-simplex of $T_1$. Then on $F_x(\delta_{n-1})$ the pair $(\tau_{n-1}, \omega_{n-1})$ is equal to $(\tau_{n-1}(x), \omega_{n-1}(x))$.

(E') (E) holds and if $\sigma, \sigma'$ be $(n-1)$-simplices then we must have $N(\sigma) \cap N(\sigma') \subset N(\sigma \cap \sigma')$. If $\sigma''$ is a proper face of $\sigma$ so that $F_x(\delta_{n-1})$ intersects $N(\sigma')$, say $y \in F_x(\delta_{n-1})$ also belongs to $B_v(\delta_{v'}) \times \{y'_2\}$ with $v \in \sigma'$ and $y' \in E_v(\eta_{v'})$, $i' = \dim \sigma'$, then $F_x(\delta_{n-1}) \subset \text{int}(B_v(\delta_{v'})) \times \{y'_2\}$. Moreover $N(\sigma) \cap \sigma'''$ is empty for any $n$-simplex $\sigma'''$ for which $\sigma$ is not a face.

Let us set that for any 2-plane field for which $T$ is in general position near $[-2, 2]^n$ to be civilized on the $-1$-skeleton of $T_1$ and $\delta_{-1}, \eta_{-1} = \infty$.

**Proposition 4.2.** Let $-1 \leq p \leq n - 2$ and let $(\tau_{p-1}, \omega_{p-1}) \in \tilde{\Delta}_{n-2}(\mathbb{R}^n)$ which is civilized on the $(p-1)$-skeleton of $T_1$ and let $\delta_0 > ... > \delta_{p-1} > 0$ and $\eta_0 > ... > \eta_{p-1} > 0$ be the associated real numbers. Then there exists

$$(\tau_p, \omega_p) \in \tilde{\Delta}_{n-2}(\mathbb{R}^n), \ 0 < \delta_p < \delta_{p-1} \text{ and } 0 < \eta_p < \eta_{p-1}$$
such that properties (C) to (E) holds for \( j = p \) (for \( j = p = n−1 \) (C) to (E) has to be replaced by (C') to (E')) and \( \delta_i, \eta_i, i = 0, ..., p \). Moreover \( (\tau_p, \omega_p) \) is homotopic to \( (\tau_{p−1}, \omega_{p−1}) \) in \( \Delta_{n−2}(\mathbb{R}^n) \) through a homotopy for which \( T \) remains in general position near \([-2, 2]^n\) and which satisfies (B) through out the homotopy.

**Proof.** We shall only deal with the case \( 0 \leq j \leq n−2 \). The \( j = n−1 \) case is similar.

Consider all \( p \)-simplices \( \sigma \) of \( T_1 \). Let \( \sigma' \) be a proper face of \( \sigma \). Then \( (\tau_{p−1}, \omega_{p−1}) \) is constant on \( B_y(\delta_i) \times E_y(\eta_i), i = \text{dim} \sigma' \) and \( \sigma \) intersects \( B_y(\delta_i) \times E_y(\eta_i) \) in \( \text{int}(B_y(\delta_i)) \times E_y(\eta_i) \) by (C). So if \( x \in \sigma \cap B_y(\delta_i) \times E_y(\eta_i) \) then \( \tau_{p−1}(x) = \tau_{p−1}(y) \) and \( E_x \leq E_y \) where \( E_x = (T \sigma + \tau_{p−1}(x))' \). So we can find \( \delta_p \) and \( \eta_p \) so that (C) and (E) holds and (D) holds for those

\[
x \in \sigma \cap N(\sigma')
\]

where \( \sigma' \) a proper face of \( \sigma \). The last condition of (C) holds if \( \frac{\partial x}{\partial \delta_p} \) is small.

So we need to deform \( (\tau_{p−1}, \omega_{p−1}) \) so that (D) holds for all \( x \in \sigma \) and the deformation will be supported in the complement of \( N(\sigma') \).

By (B) and (E), for all \( x \in \sigma \) and \( z \in B_x(\delta_p) \times E_x(\eta_p) \) the plane \( \tau_{p−1}(z) \) is the graph of a linear map \( L_{xz} : \tau_{p−1}(x) \to \tau_{p−1}(x)' \).

Choose \( \bar{\delta}_p > \delta_p \) and \( \bar{\eta}_p > \eta_p \) such that \( B_x(\bar{\delta}_p) \times E_x(\bar{\eta}_p) \) are still fibers of a tubular neighborhood \( \bar{N}(\delta) \).

Now on \( B_x(\delta_p) \times E_x(\eta_p) \) we set

\[
(\tau_p, \omega_p) = (\tau_{p−1}(x), \omega_{p−1}(x))
\]

Let the discs \( B_x(\delta_p) \times E_x(\eta_p) \) and \( B_x(\bar{\delta}_p) \times E_x(\bar{\eta}_p) \) have radii \( r, \bar{r} \) respectively. Define a continuous map \( f : [r, \bar{r}] \to [0, \bar{r}] \) such that \( f(r) = 0 \) and \( f(\bar{r}) = \bar{r} \). Now for \( y \in B_x(\bar{\delta}_p) \times E_x(\bar{\eta}_p) \cap B_x(\delta_p) \times E_x(\eta_p) \) set \( (\tau_p(y), \omega_p(y)) = (\tau_{p−1}(f(|y|)y), \omega_{p−1}(f(|y|)y)) \).

5. **Filling The Hole**

Now we can assume that \( (\tau_0, \omega_0) \) satisfies the desired properties of (2.1) on \( N(\partial \sigma) \), where \( \sigma \) is an \( n \)-simplex of \( T_1 \). A subset of \( \sigma \) diffeomorphic to \( B^2 \times B^{n−2} \) containing the complement of \( N(\partial \sigma) \) is called a hole.
Proposition 5.1. For each n-simplex $\sigma$ of $T_1$ there exists an embedding $\phi: D^2 \times D^{n-2} \to \text{int}(\sigma)$ such that $(\tau', \omega') = (\phi^*\tau, \phi^*\omega) \in \tilde{\Delta}_{n-2}(D^n)$ satisfying

1. Near $D^2 \times \partial D^{n-2}$, $\tau'$ is the kernel of the projection to $D^{n-2}$ factor.
2. $\tau'$ is $\cap$ to $\partial D^2 \times D^{n-2}$ and in a neighborhood of $\partial D^2 \times D^{n-2}$ is the pull back of the line field $\tau' \cap T(\partial D^2 \times D^{n-2})$ by the projection $(D^2 - \{0\}) \times D^{n-2} \to \partial D^2 \times D^{n-2}$. Furthermore the line field $\tau' \cap T(\partial D^2 \times D^{n-2})$ is $\cap$ to $\{x\} \times D^{n-2}$ for all $x \in D^2$.

Proof. The proof is same as (4.1)-(4.3) in [9]. Only one needs to observe that the non-degeneracy condition is preserved by pull back under diffeomorphisms. □

5.1. Another Pair. For a $k$-manifold $M$ a foliated $M$-product over the circle $S^1$ is a co-dimension $k$ foliation on $S^1 \times M$ which is $\cap$ to the second factor. A foliated $M$-product over $S^1$ is said to have compact support if there exists a compact set $C \subset M$ such that on $S^1 \times (M - C)$ the foliation is given by the projection onto $M - C$.

Let $\xi$ be a foliated $M$-product with compact support and let $\alpha \in \Omega^1(S^1 \times M)$. Consider

$$(\xi, \alpha) \in \text{Fol}_k(S^1 \times M) \times \Omega^1(S^1 \times M)$$

satisfying $\alpha|_{T\xi} \neq 0$. By 5.1 $\tau'$ defines a foliated $\mathbb{R}^{n-2}$-product on $S^1 \times \mathbb{R}^{n-2}$. Here we identify $\text{int}(D)_{n-2}$-product over $S^1$ with $\mathbb{R}^{n-2}$-product over $S^1$.

Define the vector field $X := \frac{\tau'|_{S^1 \times D^{n-2}}}{T\xi}$, where $\xi'$ is the foliated $\mathbb{R}^{n-2}$ structure as identified with $\tau'$ by 5.1. Then define $\alpha' = \omega'|_{S^1 \times D^{n-2}}(X, -)$. Observe that the pair $(\xi', \alpha')$ satisfies the non-degeneracy condition $\alpha'|_{T\xi} \neq 0$. From now on whenever we write the pair $(\xi, \alpha)$ we mean that it satisfies this non-degeneracy condition.

Thus we need to find a pair $(\tau'', \omega'') \in \tilde{\Delta}_{n-2}(D^2 \times \mathbb{R}^{n-2})$ with $\tau'' = TF''$ integrable satisfying

(*) Outside some compact set $D^2 \times C$, $\tau''$ is given by the projection onto $\mathbb{R}^{n-2}$.

(**) $\tau'' \cap rS^1 \times \mathbb{R}^{n-2}$ for $r$ close to 1 and induces there the given pair $(\xi', \alpha')$. 
\[ (***) \ (\tau'', \omega'') \text{ is homotopic in } \Delta_{n-2}(S^1 \times \mathbb{R}^{n-2}) \text{ to an one for which the 2-plane field is } \phi \text{ to the } \mathbb{R}^{n-2}\text{-factor by a homotopy constant near } S^1 \times \mathbb{R}^{n-2}. \]

Now as in \[9\] we identify \( \xi' \) with a periodic path \( \gamma' : \mathbb{R} \to Diff_c\mathbb{R}^{n-2} \).

**Definition 5.2.** We call a pair \((\xi, \alpha)\) or equivalently \((\gamma, \alpha)\) fillable if there exists a pair \((\tau = TF, \omega)\) inducing \((\xi, \alpha)\) on \(S^1 \times \mathbb{R}^{n-2}\) and which satisfies \((*)\) and \((***)\).

**Definition 5.3.** A path \(\gamma : [0, 1] \to Diff_c\mathbb{R}^{n-2}\) starting at identity \(id\) is called horizontal on an interval \(J \subset [0, 1]\) if it is constant on \(J\). A path which is horizontal near the starting and end is called adjusted.

**Concatenation:** Let \((\gamma_i, \alpha_i), \ i = 1, 2\) be two adjusted pairs. Define the immersion

\[
\{(exp(2\pi it), \gamma_1(2t, y)) : y \in \mathbb{R}^{n-2}, \ t \in [0, 1/2]\} \xrightarrow{\phi_1} \{(exp(2\pi it), \gamma_1(t, y)) : y \in \mathbb{R}^{n-2}, t \in [0, 1]\}
\]

by \(\phi_1(exp(2\pi it), \gamma_1(2t, y)) = (exp(2\pi i.2t), \gamma_1(2t, y))\). Set \(\bar{\alpha}_1 = \phi^*\alpha_1\).

Similarly define

\[
\{(exp(2\pi it), \gamma_2(2t - 1, \gamma_1(1, y))) : y \in \mathbb{R}^{n-2}, \ t \in [1/2, 1]\}
\]

\[ \xrightarrow{\phi_2} \{(exp(2\pi it), \gamma_2(t, \gamma_1(1, y))) : y \in \mathbb{R}^{n-2}, t \in [0, 1]\} \]

by \(\phi_2(exp(2\pi it), \gamma_2(2t - 1, \gamma_1(1, y))) = (exp(2\pi i(2t - 1)), \gamma_2(2t - 1, \gamma_1(1, y)))\). Set \(\bar{\alpha}_2 = \phi^*\alpha_2\).

The new pairs still satisfies the non-degeneracy condition on the respective domains. Now define the concatenation \((\gamma_1, \alpha_1) \ast (\gamma_2, \alpha_2) = (\gamma, \alpha)\) as follows. The concatenation of the paths namely \(\gamma\) is given by \(\gamma(t) = \gamma_1(2t), \ t \in [0, 1/2]\) and \(\gamma(t) = \gamma_2(2t - 1) \circ \gamma_1(1), \ t \in [1/2, 1]\).

From the fact that \(\gamma_i\)’s being adjusted it follows that \(\bar{\alpha}_i\)’s match nicely and we get \(\alpha\).

**Lemma 5.4.** The concatenation of two fillable adjusted pair \((\gamma_1, \alpha_1)\) and \((\gamma_2, \alpha_2)\) is fillable.

**Proof.** Let us consider \((\gamma_1, \alpha_1)\). Define

\[
\{(r exp(2\pi it), \gamma_1(2t, y)) : y \in \mathbb{R}^{n-2}, \ t \in [0, 1/2], \ r \in [0, 1]\}
\]

\[ \xrightarrow{\phi_1} \{(r exp(2\pi it), \gamma_1(t, y)) : y \in \mathbb{R}^{n-2}, t \in [0, 1], \ r \in [0, 1]\} \]
by $\Phi_1(rexp(2\pi i t), \gamma_1(2t, y)) = (rexp(2\pi i, 2t), \gamma_1(2t, y))$. Set $(\bar{\mathcal{F}}_1, \bar{\omega}_1) = (\Phi^*\mathcal{F}_1, \Phi^*\omega_1)$, where $(T\mathcal{F}_1, \omega_1) \in \bar{\Delta}_{n-2}(D^2 \times \mathbb{R}^{n-2})$ be the pair given by the fillability of $(\gamma_1, \alpha_1)$. Again the new pair satisfies the non-degeneracy condition.

Do the similar arrangement in view of the concatenation arrangement. Now the rest of the proof is same as Lemma 4.10 in [9] and we skip it.

\[ \square \]

Remark 5.5. Observe that in the concatenation process $\gamma(1) = \gamma_2(1) \circ \gamma_1(1)$ and we shall treat the product $\gamma_2(1) \circ \gamma_1(1)$ and concatenation in place of each other without mention.

### 5.2. An Example.

Consider the foliation on the torus $S^1 \times S^1$ by lines of constant slope $a$, i.e., the foliation is given by the kernel of the one form $d\theta - ad\phi$ where $(\theta, \phi)$ are coordinates on $S^1 \times S^1$. Let $D^2$ have the polar coordinate $(r, \phi)$ and let $\lambda_0, \lambda_1/2, \lambda_1$ be cutoff functions on $[0, 1]$ such that $\lambda_i = 1$ on a neighborhood of $i \in [0, 1]$ and $\text{supp}(\lambda_0) \cap \text{supp}(\lambda_1)$ is empty. Then the kernel of the one form

$$\lambda_1(r)(d\theta - ad\phi) + \lambda_{1/2}(r)dr + \lambda_0(r)d\theta$$

on $D^2 \times S^1$ with coordinates $(r, \phi, \theta)$ defines a co-dimension-1 foliation $\mathcal{F}$ on the solid torus $D^2 \times S^1$ whose restriction on the boundary torus $S^1 \times S^1$ is the given foliation, i.e., $\text{Ker}(\theta - ad\phi)$.

On the inner solid torus $D^2_{1/2} \times S^1$ the foliation is given by the taut foliation.

On the torus consider the one form $\alpha = ad\theta + d\phi$. Observe that

$$\text{Ker}(d\theta - ad\phi) = \langle (a\partial_\theta + \partial_\phi) \rangle$$

So $\alpha|_{\text{Ker}(d\theta - ad\phi)} \neq 0$.

Now consider the two form $\omega$ whose restriction on $\{(r, \phi) : 0 \leq r \leq 1/2\} \times S^1$ is given by $(\lambda_0(r)rdr + \lambda_{1/2}(r)d\theta) \wedge d\phi$ and whose restriction on $\{(r, \phi) : 1/2 \leq r \leq 1\} \times S^1$ is given by

$$\frac{1}{2}(2\lambda_1(r)dr + \lambda_{1/2}(r)d\theta) \wedge d\phi + \frac{1}{2}d\theta \wedge (\lambda_{1/2}(r)d\phi - 2a\lambda_1(r)dr)$$

Observe that on the boundary torus $S^1 \times S^1$, $\omega(\partial_r, -) = \alpha$ and $\omega|_{\mathcal{T}, \mathcal{F}} \neq 0$.

A tubular neighborhood of $S^1 \subset \mathbb{R}^{n-2}$ is given by $S^1 \times D^{n-3}$, $[n \geq 4]$. Consider a periodic curve $[0, 1] \to Diff_c \mathbb{R}^{n-2}$ which is constant outside $S^1 \times D^{n-3}$ and whose restriction
to $S^1 \times D^{n-3}$ induces a foliated $S^1 \times D^{n-3}$-product structure $\xi$ over $S^1$, i.e., a co-dimension-$(n-2)$ foliation on $S^1 \times S^1 \times D^{n-3}$ whose restriction on $S^1 \times S^1 \times \{x\}$, $x \in D^{n-3}$ is given by $\text{Ker}(d\theta - f(x)d\phi)$, where $f : D^{n-3} \to [0, 1]$ is a non-vanishing map which is zero near $\partial D^{n-3}$.

Observe that if we set $f(x) = a$ we get the foliation mentioned above. Let $\omega(x)$ be the two form achieved by replacing $a$ by $f(x)$ in $\omega$. Now one just need to interpolate by introducing the function $g : D^{n-3} \to [0, 1]$ such that $g \equiv 0$ in a neighborhood of $\partial D^{n-3}$ and $g \equiv 1$ on $\text{supp}(f)$. The resulting one form whose kernel defines the foliation is given by

$$(1 - g(x))d\theta + g(x)(\lambda_1(r)(d\theta - f(x)d\phi) + \lambda_{1/2}(r)dr + \lambda_0(r)d\theta)$$

and the two form is given by

$$(1 - g(x))rdr \wedge d\theta + g(x)\omega(x)$$

5.3. The group $\text{Diff}_c\mathbb{R}^{n-2}$. In this subsection we recall some results regarding the group $\text{Diff}_c\mathbb{R}^{n-2}$. Let us set $k = n - 2$. Let $C^\infty_c(\mathbb{R}^k, \mathbb{R}^k)$ be the vector space of smooth maps from $\mathbb{R}^k$ to itself with support a compact set $K \subset \mathbb{R}^k$ and equipped with $C^\infty$-topology. The inductive limit of $C^\infty_c(\mathbb{R}^k, \mathbb{R}^k)$ on $K$ will be denoted by $C^\infty_c(\mathbb{R}^k, \mathbb{R}^k)$. Now $\text{Diff}_c\mathbb{R}^{n-2}$ is a manifold modeled on $C^\infty_c(\mathbb{R}^k, \mathbb{R}^k)$. An atlas is given by different translates by elements $g \in \text{Diff}_c\mathbb{R}^k$ of small enough neighborhoods $U_g$ depending on $g$ such that $U_g$ is so small that $g + U_g \subset \text{Diff}_c\mathbb{R}^k$. For any map $f : J \to C^\infty_c(\mathbb{R}^k, \mathbb{R}^k) \subset \text{Diff}_c\mathbb{R}^k$ is contained in some $C^\infty_c(\mathbb{R}^k, \mathbb{R}^k)$ for some compact set $K$. For more detailed information we refer to [9].

**Proposition 5.6.** ([8]) For $k \geq 2$ and $B \subset \mathbb{R}^k$ be open and bounded. Then there exists compactly supported smooth vector fields $X_1, \ldots, X_6$ on $\mathbb{R}^k$, a $C^\infty$-open neighborhood $\mathcal{W}$ of the identity in $\text{Diff}_c^\infty(\mathbb{R}^k)$, and a smooth mappings $\sigma_1, \ldots, \sigma_6 : \mathcal{W} \to \text{Diff}_c^\infty(\mathbb{R}^k)$ such that for all $g \in \mathcal{W}$,

$$g = [\sigma_1(g), \exp X_1] \circ \ldots \circ [\sigma_6(g), \exp X_6]$$

where the commutator $[a, b] = aba^{-1}b^{-1}$. Moreover the vector fields are close to zero.

**Proposition 5.7.** ([11]) Let $U$ be an open subset of a manifold $M$ and let $h$ be a diffeomorphism of $M$ such that $U \cap f(U)$ is empty also assume that $a, b \in \text{Diff}_c^\infty(M)$ are supported in $U$, then the commutator is the product of four conjugates of $h$ and $h^{-1}$, more precisely

$$[a, b] = h(ch^{-1}c^{-1})(bch^{-1}b^{-1})(bh^{-1}b^{-1})$$

where $c = h^{-1}ah$. 
Corollary 5.8. ([9]) Let $M^k$ be a smooth connected manifold and let $h \in \text{Diff}_c^\infty(M^k)$ any element other than identity. Let $a_i, b_i, i = 1, ..., r$ be elements of $\text{Diff}_c^\infty(M^k)$ such that for each $i$ the diffeomorphisms $a_i$, $b_i$ have support in $\text{int}(U_i)$ where $U_i$ is a closed $k$-ball in $M^k$. Then

$$f := \prod_{i=1}^{r}[a_i, b_i]$$

is a product of $4r$ conjugates of $h^{-1}$

5.4. Concluding The Proof of 2.1. We consider a pair $(\gamma, \alpha)$ and show that it is fillable. The proof is similar to [9].

(a) Define $V_\varepsilon := \{\text{id} + e : e \in C_c^\infty(\mathbb{R}^k, \mathbb{R}^k) \text{ with } \max_x |de_x| < \varepsilon\}$. $V_1$ is an open contractible neighborhood of $\text{id}$ in $\text{Diff}_c^\infty(\mathbb{R}^{n-2})$. Then as in [9] there exists $\varepsilon > 0$ such that any composition of 72 elements of $V_\varepsilon$ is in $V_1$.

(b) Recall from 5.3 that the periodic curve of the pair that has been filled is of the form $h_f(t, z) = z$, $z \notin S^1 \times D^{n-3}$ and $h_f(t, z) = (\theta + t.f(x), x)$, $z = (\theta, x) \in S^1 \times D^{n-3}$. Here $f : D^{n-3} \to [0, 1]$ is a smooth map vanishing near the boundary. Observe that $h_f(t)^{-1} = h_{-f}(t)$. So for $f$ small enough $h_f(t)^{-1}, h_{-f}(t) \in V_\varepsilon$. Set $h = h_f(1)$.

(c) $U \subset S^1 \times D^{n-3}$ be an open ball such that $U \cap h(U)$ is empty. Let $B \subset \mathbb{R}^{n-2}$ be an open ball which contains $S^1 \times D^{n-3}$ and $A$ be an open ball in $\mathbb{R}^{n-2}$ containing $\bar{B}$.

(d) $g_t : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$, $t \in [0, 1]$ be a compactly supported isotopy with $g_0 = \text{id}$ and $g_1(\bar{A}) \subset U$. Set $g = g_1$.

(e) By [5,6] there exists $W \subset V_1$ a $C^\infty$-open neighborhood of $\text{id}$ in $\text{Diff}_c^\infty(B)$, $\sigma_i : W \to \text{Diff}_c^\infty(A)$, $i = 1, ..., 6$ and compactly supported vector fields $X_i, i = 1, ..., 6$. We can assume that $\text{supp}(X_i) \subset A$, $i = 1, ..., 6$. We shall later refer how small $W$ needs to be.

Remark 5.9. Now observe that $(\gamma, \alpha)$ is fillable if and only if $(g \circ \gamma \circ g^{-1}, (g^{-1})^*\alpha)$ is fillable.

As in [9] $g \circ \gamma \circ g^{-1}$ is the concatenation of of $g \circ \gamma_i \circ g^{-1}$, $i = 1, ..., q$, where $\gamma_i(t) = \gamma(\frac{t}{q}) \circ \gamma(\frac{1}{q})^{-1}$. All of then have images in $W$. 
By 5.6 each $\gamma_i(1)$ is a product of the commutators $[\sigma_j(\gamma_i(1)), \exp X_j]$, $j = 1, \ldots, 6$. As we know that the conjugate of the commutator is equal to the commutator of the conjugates, we get

$$g[\sigma_j(\gamma_i(1)), \exp X_j]g^{-1} = h \circ ((h^{-1}g\sigma_j(\gamma_i(1))g^{-1}h) \circ h^{-1} \circ (h^{-1}g\sigma_j(\gamma_i(1))^{-1}g^{-1}h)) \circ$$

$$((g \exp X_j g^{-1}) \circ (h^{-1}g\sigma_j(\gamma_i(1))g^{-1}h) \circ h \circ (h^{-1}g\sigma_j(\gamma_i(1))^{-1}g^{-1}h) \circ (g \exp X_j^{-1}g^{-1})) \circ$$

$$((g \exp X_j g^{-1}) \circ h^{-1} \circ (g \exp X_j^{-1}g^{-1}))$$

Observe that it is a product of 12 diffeomorphisms and $j$ runs from 1 to 6. So in total it is a product of 72 diffeomorphisms. So we choose $W$ from item (e) above accordingly.

Also observe that in the above expression for $g[\sigma_j(\gamma_i(1)), \exp X_j]g^{-1}$ is a product of different conjugates of $h$ and $h^{-1}$. So from 5.9 and 5.5 it is fillable.

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