Geometrodynamics of spacetime “surfing”

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Abstract. The idea of hyper-fast travel in the framework of General Relativity has firstly been introduced by Miguel Alcubierre. It was interesting, as it shows the possibility to locally modified spacetime surrounding an intended vehicle to move at huge speed which is rather different than the method of spacetime topological breaking as in a wormhole theory. Instead of puncturing through a hole in spacetime as in a wormhole, this method of spacetime geometrical modification, manipulates the geometrodynamics contraction and expansion of spacetime curvature so that the intended object’s hypersurface surfs on the surrounding spacetime hyperspace. In this paper we will explore in more detail mathematically, how the concept has been derived in the framework of General Relativity. Starting from developing the metric to the expression of energy density that governs the spacetime “surfing” geometrodynamics in relation with the necessary parameters describing the spacetime warp bubble.

1. Introduction
To be a spacefaring civilization with current rocket propulsion technologies that is utilizing Newtonian orbital mechanics, humankind could only travel within the solar system and yet still acquire years of travel to surpass the outermost regions of the solar system. Thus, not to mention the sensible of humanity’s dreams for interstellar or perhaps intra-galactic travel, if just by using Newtonian mechanics-base propulsion technologies, because even if such endeavour may approach up to 99% the speed of light [1] which is an almost impossible feat by itself, the journey will still take several years, even of tens to hundreds of thousands years [2]. Thus the speed of light, tremendously huge as it may seems, but nevertheless, if the astronomical scale of the universe is to be considered, light speed is still slow. Therefore, the idea of hyper-fast travel that could surpass many folds the speed of light is very much an interesting subject to study [3]

Surpassing the speed of light seems to violate the law of physics. Yes this notion is true if Special Relativity (SR) is taken into consideration where SR tells that any subject of travel that holds any amount of mass could never even achieve the light speed, not to mention exceeding it. This is due to increase in relativistic mass asymptotically toward infinity as the subject’s speed approaching the speed of light, consequently require infinite amount of energy. However, in General Relativity (GR) the concept of spacetime geometrodynamics [4] seems to show that the dynamics of spacetime itself does not constrained toward any limit. These notion has been proven at least in the cosmological scale level when we consider the signatures of the inflationary period of the universe [5] and the currently observable phenomenon of accelerating expansion of the universe [6]. General Relativity has shown that the dynamics of stretching and folding of spacetime may exceed the speed of light by either short cutting the spacetime as in black hole and wormhole [7] or superluminally stretching the spacetime via expansion as during the inflationary period of post big bang or via contraction as during the theoretically possible scenario of the big crunch period [8].
Thus, the idea of superluminal travel is not about violating SR but rather a geometrodynamically possible scenario in various GR frameworks. Essentially there are two concepts of superluminal travel that are of “short cutting” and “surfing” on the spacetime. Short cutting concept is about reducing tremendously the distance between two different coordinates as in the wormhole theory which require topological breaking of the spacetime itself of either naturally (from subatomic quantum foam) [9] or induce (through entanglement) [10]. The surfing concept, that is the subject of this paper is about embedding a bubble of warped spacetime hypersurface onto the surrounding hyperspace and using the transition region of warped spacetime properties between the bubble and the surrounding hyperspace that defines the bubble’s extrinsic curvature to govern its dynamic’s character of expansion and contraction of spacetime. In this paper we investigate the idea of spacetime “surfing” which was firstly introduced by Alqubierre [3] through exploring the mathematical derivation underlying the subject in a rather elaborate details.

2. The metric property of warp bubble
The concept of spacetime “surf” is very much about manipulating spacetime locally, thus we may start by considering a flat spacetime metric that not only describes the hyperspace surrounding the “surfing” spacetime region but also the “interior” of the surfing spacetime region that is by itself a hypersurface [11] with respect to the surrounding region.

\[
ds^2=dr^2+dx^2+dy^2+dz^2
\]

Imagine the “surfing” spacetime region as a warp bubble that moves along the \(x\) axis with a velocity of \(v\),

![Figure 1. Spacetime warp bubble “surf” (moves) along \(x\) axis with velocity of \(v\).](image)

so we may rewrite the metric (1) as:

\[
ds^2=dr^2+(dx-vdt)^2+dy^2+dz^2
\]

(2)

which preserved the essence of equation (1) whereby, if the bubble doesn’t move, the equation (2) will becomes (1) again. For representing the characteristic of the bubble where the center is located at \(r=0\) and the radius of the bubble can be assigned as \(R\) we may consider a function \(f=f(r)\) where \(f=1\) always within the bubble’s radii |\(\rho|=R\).
Thus, the essence of equation (2) is always preserved and we may rewrite equation (2) with the inclusion of the function $f=f(r)$ as:

$$ds^2=dt^2+f(r)\left(dx^2+dy^2+dz^2\right).$$

(3)

If we consider condition at the center of the warp bubble where the spatial coordinate is $(y(r)t,0,0)$ which then obviously shows equation (3) reduce to $ds^2=-dt^2$. Since the different of proper time is defined as minus of the line element thus $d\tau^2=-ds^2$ therefore $d\tau^2=dt^2$ which obviously implies the null-time dilation characteristic of the bubble as proper time is the same as coordinate time $\tau=t$. Inside the bubble, spacetime condition is as normal as in a stationary non relativistic condition.

We may now expand this notion to find the geodesic and free fall characteristic of the warp bubble. By relationship of proper time and line element metric, we may derive the extremum [12] of the proper time as below:

$$\frac{d\tau}{d\xi}=-ds^2=-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi},$$

$$d\tau^2=\left(d\tau\right)^2=\left(\frac{d\tau}{d\xi}\right)^2=-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi},$$

$$\frac{d\xi}{d\tau}=\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}},$$

$$\tau=\int\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}}d\lambda,$$

$$\tau=\int\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}}d\lambda,$$

(4)

which is the extremum expression of the proper time. We may obtain the Lagrangian from Equation (4). But before expanding (4) to obtain the Lagrangian we may have to expand and rearrange metric equation (3) as below

$$ds^2=dt^2+dx^2-2f(r)dxdt+\left(f(r)\right)^2dt^2+dy^2+dz^2,$$

$$=-\left(1-f(r)\right)dt^2-2f(r)dxdt+dx^2+dy^2+dz^2,$$

$$=-\left(1-f(r)\right)dt^2-f(r)dxdt+f(r)dx^2+dy^2+dz^2,$$

(5)

so we identify the metric components which are:
\[ g_{00} = v^2 f(r)^2 - 1, \quad g_{0i} = g_{0i} = -v f(r), \quad g_{ij} = g_{ij} = 1. \]  \hspace{1cm} (6)

By (4):  
\[ \tau = J\left( -g_{00} (x^0)^{2} - 2g_{0i} x^{i} x^{k} - g_{ij} (x^i)^{2} - g_{ij} (x^j)^{2} \right) d\lambda, \]
\[ = J\left( -(1-v^2 f^2)^2 + 2v f x t - x^2 - y^2 - z^2 \right) d\lambda, \]
\[ \Rightarrow \left( \frac{d\tau}{d\lambda} \right)^2 = \frac{dx^2}{d\lambda} = (1-v^2 f^2)^2 + 2v f x t - x^2 - y^2 - z^2 = -g_{ij} x^i x^j. \]  \hspace{1cm} (7)

From the pure definition of Lagrangian \( L = \frac{1}{2}\dot{q}^2 \) where \( q = s(x(t)) \), as a coordinate position of a metric, thus:
\[ \dot{q}^2 = \dot{s}^2 = (-\dot{r})^2 = -\frac{dr^2}{d\lambda} = \frac{dk^2}{d\lambda} = 2L \quad \text{since} \quad -dk^2 = dr^2, \quad ds^2 = -dr^2 \]  \hspace{1cm} (8)

and from (7) and (8)
\[ -g_{ij} x^i x^j = (1-v^2 f^2)^2 + 2v f x t - x^2 - y^2 - z^2 = \frac{d^2 \tau}{d\lambda} \quad \text{therefore:} \]
\[ \left( \frac{dk}{d\lambda} \right)^2 = \frac{dx^2}{d\lambda} = \frac{d\tau}{d\lambda} = \frac{-d^2 \tau}{d\lambda} = -g_{ij} x^i x^j = (1-v^2 f^2)^2 + 2v f x t - x^2 + y^2 + z^2 = 2L, \]

thus the Lagrangian is:
\[ L = \frac{1}{2}\left( -(1-v^2 f^2)^2 + 2v f x t - x^2 + y^2 + z^2 \right). \]  \hspace{1cm} (9)

and rearrange as,
\[ L = \frac{1}{2}\left( -t^2 + y^2 + z^2 + (x-v f)^2 \right). \]  \hspace{1cm} (10)

We may parameterize the geodesic using proper time itself taking \( \lambda = \tau \) thus the dot indicates the derivative with respect to \( \tau \). By Euler Lagrange equation
\[ \frac{\partial}{\partial \dot{\alpha}} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) - \frac{\partial L}{\partial \alpha} = 0 \quad \text{and from (10) we have 4 equations as follow} \]
\[ \frac{\partial}{\partial \dot{\alpha}} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) = \frac{\partial}{\partial \dot{\alpha}} \left( -t^2 + y^2 + z^2 + (x-v f)^2 \right) = 0, \]
\[ \Rightarrow \frac{\partial}{\partial \dot{\alpha}} \left( -t^2 + y^2 + z^2 + (x-v f)^2 \right) = 0. \]  \hspace{1cm} (11)

\[ \frac{\partial}{\partial \dot{\alpha}} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) = \frac{\partial}{\partial \dot{\alpha}} \left( -t^2 + y^2 + z^2 + (x-v f)^2 \right) = 0, \]
\[ \Rightarrow \frac{\partial}{\partial \dot{\alpha}} \left( -t^2 + y^2 + z^2 + (x-v f)^2 \right) = 0. \]  \hspace{1cm} (12)
\[
\frac{\partial}{\partial t}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial z}\right) = 0,
\]
\[
\Rightarrow \frac{\partial f}{\partial t} + (x-yf)\frac{\partial (xf)}{\partial y} = 0,
\]
\[
\frac{\partial}{\partial t}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f^2 + y^2 + z^2 + (x-yf)^2}{\partial z}\right) = 0,
\]
\[
\Rightarrow \frac{\partial f}{\partial t} + (x-yf)\frac{\partial (xf)}{\partial y} = 0,
\]

where obviously the solution shall be

\[
x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (1, yf, 0, 0),
\]

which are 4 velocities for Eulerian observers that follow timelike geodesics. Since \(x = v(t) f(r(t)) x\), as at \(r = 0, f = 1\) and moreover \(\lambda = t\) so that \(t = d\lambda / d\lambda = d\lambda / d\tau = 1\) (since \(t = \tau\) in the warp bubble), thus the spatial velocity of the warp bubble is at \(v\) which does not constrained by speed limit of light speed. This notion of timelike characteristic in the warp bubble can also be proven more by considering the following by referring to the metric equations (6)

\[
\lambda_{\mu} = g_{\mu\nu} x^\nu \quad \text{for} \quad x_0 = g_{00} x^0 + g_{01} x^1 + g_{02} x^2 + g_{03} x^3
\]
\[
= g_{00} x^0 \quad \text{since} \quad g_{02} = g_{03} = 0,
\]
\[
= (v^2 f^2 - 1) \frac{\partial t}{\partial t} + (-vf) \frac{\partial x}{\partial t}.
\]
\[
= (v^2 f^2 - 1) - (vf) v \quad \text{since} \quad \frac{\partial t}{\partial t} = 1, \quad \frac{\partial x}{\partial t} = \frac{\partial t}{\partial t} = v,
\]
\[
= (v^2 - 1) - v v \quad \text{since} \quad f = 1,
\]
\[
x_0 = -1,
\]

for \(x_1 = g_{10} x^0 + g_{11} x^1 + g_{12} x^2 + g_{13} x^3 \),
\[
= g_{10} x^0 \quad \text{since} \quad g_{12} = g_{13} = 0,
\]
\[
= (-vf) \frac{\partial t}{\partial t} + \frac{\partial t}{\partial x},
\]
\[
= (-v) + v, \quad x_1 = 0.
\]

The rest, that is \(x_2 = x_3 = 0\) obviously, which therefore:

\[
x_\mu = (-1, 0, 0, 0)
\]
is defined as covector that is normal to the hypersurface [12] given by $dt = 0$, and by equation (15) $\lambda^i \lambda^i = -1$ confirming the timelike characteristic of the warp bubble.

Defining $\beta = -V(r)$ we may rewrite equation (5) as below:

$$ds^2 = -(1 - \beta^2) dt^2 + 2\beta dx dt + dx^2 + dy^2 + dz^2.$$  

(19)
In tensorial form:

\[
\beta^i = \beta^j, \quad dx^i + dx^j + dx^k = \delta^i_j dx^l dx^l, \\
\beta dx = \beta dx^l.
\]  

(20)

where \( i,j = 1,2,3 \), which by equation (5) obviously shows that \( \beta = -v \) is only in the direction of the bubble’s motion thus otherwise it is zero. Therefore for this case where the bubble is “surfing” along the \( x \) axis only \( \beta_x = \beta = -v \) while others \( \beta_y, \beta_z = \beta_x = 0 \). Now equation (19) in general tensorial form, can be rewritten as

\[
ds^2 = -(1 - \beta) dx^2 + 2\beta dx^i dt + \delta^i_j dx^l dx^l.
\]

(21)

For even more generalized term to suit the metric tensor \( g_{00} \) of equation (21) the term \( \sqrt{-\alpha} = \alpha = 1 \) introduced, thus precisely the equation (21) should be written as

\[
ds^2 = -(\alpha - \beta) dx^2 + 2\beta dx^i dt + \delta^i_j dx^l dx^l
\]

(22)

which is the metric as firstly proposed by Alqubierre [3] describing the spacetime metric of warp bubble hypersurface.

3. Shape function

The function \( f = f(r) \) is considered as the shape function of the bubble where the conditions required are as in Figure 2 where the values of \( f(r) \) at \( r = R \) = 1 in the bubble’s interior and \( f(r) \) = 0 at the exterior of the bubble. Alqubierre [3] has shown that the function may be written as

\[
f(r) = \frac{\tanh(\sigma(r+R)) - \tanh(\sigma(r-R))}{2\tanh(\sigma R)},
\]

(23)

where \( \sigma \) is the bubble’s thickness factor. We may investigate this function in four conditions of warp bubble radii which are as the following when:-

3.1. The radii is at the positive bubble radius (toward the direction of travel) \( r = R \)

At the bubble radius, the function value transition between \( f = 1 \) and \( f = 0 \) is supposed to be abrupt however equation (23) will ensure some delay in the transition but controllable. This transition 1 to 0 is also representing spacetime contraction in front of the warp bubble toward the direction of travel. As \( r = R \) equation (23) can be shown to reduce to:

\[
f(r)|_{r=R} = \frac{\tanh(\sigma(2R))}{2\tanh(\sigma R)} = \frac{\tanh(\sigma R) + \tanh(\sigma R)}{2\tanh(\sigma R)(1 + \tanh^2(\sigma R))},
\]

and thus,

\[
f(r)|_{r=R} = \frac{1}{1 + \tanh^2(\sigma R)}.
\]

(24)
The equation (24) implies that the function’s transition is dependence on the value of the bubble thickness factor. As the bubble thickness factor reduce toward zero the transition will be abrupt from \(f(r)_{\text{fr}}=1\) toward \(f(r)_{\text{fr}}=0\).

3.2. The radii is at the center of the bubble \(r=0\)

The center inside the bubble is where any physical object, test particle or even a vehicle stay stationary. As \(r=0\) equation (23) can be shown deduced to 1 as the following

\[
f(r)_{r=0} = \frac{\tanh(\sigma(R)) - \tanh(\sigma(-R))}{2\tanh(\sigma(R))} \quad \text{and since} \quad \tanh(\sigma(-R))=-\tanh(\sigma(R)),
\]

\[
f(r)_{r=0} = \frac{\tanh(\sigma(R)) + \tanh(\sigma(R))}{2\tanh(\sigma(R))} = 1.
\]

(25)

3.3. The radii is at the negative bubble radius (toward the opposite of the direction of travel) \(r=-R\)

Similar to the case in 3.1 the transition from 0 to 1 at the rear of the bubble radius is supposedly to be abrupt however the equation (23) will ensure controllable delay in the transition. This transition of 0 to 1 is also representing spacetime expansion at the rear of the warp bubble as it moves away toward the frontal spacetime contraction in the direction of travel. As \(r=-R\) equation (23) can be shown to reduce similarly as in the case of 3.1:

\[
f(r)_{r=-R} = \frac{-\tanh(\sigma(-2R))}{2\tanh(\sigma(R))} \quad \text{and since} \quad \tanh(\sigma(-2R))=-\tanh(\sigma(2R)),
\]

\[
f(r)_{r=-R} = \frac{\tanh(\sigma(R)) + \tanh(\sigma(R))}{2\tanh(\sigma(R))(1+\tanh^2(\sigma(R)))},
\]

thus similarly,

\[
f(r)_{r=-R} = \frac{1}{1+\tanh^2(\sigma(R))}.
\]

(26)

3.4 The radii is at the exterior region of the bubble \(|r|>R\)

The exterior region will display flatness of spacetime after the shape function transitions of rear spacetime expansion (\(f=0\rightarrow f=1\)) and of frontal spacetime contraction (\(f=1\rightarrow f=0\)). From equation (23) as \(r<-R, (\eta_{l<R}+R)<0\) and \((\eta_{l<R}-R)<-2R\) thus

\[
f(r)_{r<-R} = \frac{\tanh(\sigma((\eta_{l<R}+R)<0)) - \tanh(\sigma((\eta_{l>R}-R)<-2R))}{2\tanh(\sigma(R))},
\]

since \(\tanh(\sigma((\eta_{l<R}+R)<0)) \approx 1\) and \(\tanh(\sigma((\eta_{l<R}-R)<-2R)) \approx -1\) thus \(f(r)_{r<-R} \approx 0\)

as \(r>R, (\eta_{l<R}+R)>2R\) and \((\eta_{l<R}-R)>0\), thus:

\[
f(r)_{r>R} = \frac{\tanh(\sigma((\eta_{l<R}+R)>2R)) - \tanh(\sigma((\eta_{l<R}-R)>0))}{2\tanh(\sigma(R))},
\]

since \(\tanh(\sigma((\eta_{l>R}+R)>2R)) \approx 1\) and \(\tanh(\sigma((\eta_{l>R}-R)>0) \approx 1\) thus \(f(r)_{r>R} \approx 0\), thus:
By Alqubierre warp bubble shape function equation (23) sharp transition of shape function as in Figure 2 becoming rather gradual as depicted in the following figure

![Figure 3](image)

**Figure 3.** Function \( f = f(r) \) characteristic with bubble radii with gradual transition.

The transition region between the warp bubble interior and the surrounding spacetime is actually the governing factor for the dynamics characteristic of the warp bubble.

### 4. Dynamic of expansion and contraction

The transition region is best described by deriving the extrinsic curvature. The relationship between the extrinsic curvature of the warp bubble and the surrounding spacetime is depicted as below.

![Figure 4](image)

**Figure 4.** The extrinsic curvature between warp bubble and the surrounding spacetime

Beside flat spacetime base metric that represents locality the “surfing” concept is also about embedding a bubble of warped spacetime hypersurface onto the surrounding hyperspace. The transition region of warped spacetime properties between the bubble and the surrounding hyperspace can be elaborated further by the extrinsic curvature. Consider the definition of an extrinsic curvature that is the Lie derivative of the projection tensor \( h_i = h_i^j h_j^k g_{jk} \) which is the rate of change of the projection tensor along the normal vector field \( n_l \) [11]

\[
K_y = \frac{1}{2} h_i^j h_j^k L g_{ik}.
\]  

The normal vector of concern here is actually the vector of warp bubble direction of travel \( n = \sqrt{\eta n} = \sqrt{\beta} = \beta \) thus we may rewrite (28) as below
\[ K_g = \frac{1}{2} \mathbf{h}^6 \mathbf{L}_{\mu g_{\phi}}, \quad (29) \]

which then can be expanded as the covariant derivatives of warp bubble direction of travel vector \( \beta_i \)
and reduce toward the expression derived elaborately as the following

\[
K_g = \frac{1}{2} \mathbf{h}^6 \mathbf{L}_{\mu g_{\phi}} \left( \nabla_i \beta_j + \nabla_j \beta_i \right)
\]

\[
= \frac{1}{2} \mathbf{h}^6 \mathbf{L}_{\mu g_{\phi}} \left( \nabla_i \beta_j + \nabla_j \beta_i - \Gamma^3_{ij} \beta_3 \right)
\]

\[
= \frac{1}{2} \mathbf{h}^6 \mathbf{L}_{\mu g_{\phi}} \left( \nabla_i \beta_j + \nabla_j \beta_i - 2 \Gamma^3_{ij} \nabla^3 \beta_3 \right)
\]

\[
= \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i - \frac{1}{2} \left( \nabla_i g_{\mu j} + \nabla_j g_{\mu i} - \nabla_{\mu} g_{ij} \right) \right) \beta_3
\]

\[
= \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i - \frac{1}{2} \left( \nabla_i g_{\mu j} + \nabla_j g_{\mu i} - \nabla_{\mu} g_{ij} \right) \right) \beta_3
\]

\[
= \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i + \beta_3 \nabla^3 \beta_3 \right)
\]

\[
= \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i + \beta_3 \nabla^3 \beta_3 \right)
\]

\[
= \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i + \beta_3 \nabla^3 \beta_3 \right)
\]

\[
K_g = \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i + \beta_3 \nabla^3 \beta_3 \right) g_{ij} = g_{xx} = g_{yy} = g_{zz} = 1 \quad (30)
\]

thus, reduced to

\[
K_g = \frac{1}{2} \left( \nabla_i \beta_j + \nabla_j \beta_i \right), \quad (31)
\]

which is essentially an equation of motion of the warp bubble. The matrix of the extrinsic curvature is

\[
K = \begin{pmatrix} K_{11} & K_{21} & K_{31} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}, \quad (32)
\]

where its trace describes the spacetime dynamic’s characteristics \( \theta \) indicating the warp bubble
surrounding space of either expansion or contraction

\[
\theta = -d \Gamma^K, \quad (33)
\]

\[
= -\alpha (K_{11} + K_{22} + K_{33}),
\]

\[
= -\alpha \left( \frac{1}{2} (\nabla_i \beta_j + \nabla_j \beta_i) + \frac{1}{2} (\nabla_i \beta_2 + \nabla_j \beta_2) + \frac{1}{2} (\nabla_i \beta_3 + \nabla_j \beta_3) \right),
\]

\[
= -\alpha (\nabla_i \beta_j + \nabla_j \beta_j + \nabla_i \beta_3 + \nabla_j \beta_3).
\]
\[ \theta = -\alpha \left( \partial_x \beta_x + \partial_y \beta_y + \partial_z \beta_z \right), \quad \text{but since} \quad \beta_x = \beta_y = 0, \]

thus reduce to
\[ \theta = -\alpha \partial_x \beta_x. \quad (34) \]

Expanding \( \partial_x \beta_x \):
\[
\partial_x \beta_x = -\frac{\partial f(r)}{\partial x} = -\left( v(t) \frac{\partial f(r)}{\partial x} + f(r) \frac{\partial}{\partial x} \right) = -v(t) \frac{\partial f(r)}{\partial x} \quad (35)
\]

Since,
\[ v = v(t) = \frac{dx(t)}{dt}, \quad \text{thus} \quad \frac{\partial}{\partial x} \frac{dx(t)}{dt} = 0 \quad (36) \]

\[ r = r(x(t)) = \left[ (x-x(t))^2 + y^2 + z^2 \right]^{\frac{1}{2}}, \quad x = x(t) \quad \text{thus} \quad \frac{dx}{dt} = \frac{dx(t)}{dt} \quad (37) \]

\[ \theta = -\alpha \frac{\partial f(r)}{\partial r} \frac{\partial r(x(t))}{\partial x} = -\alpha \frac{\partial f(r)}{\partial r} \left( \frac{1}{2r} \right) \left[ (x-x(t))^2 + y^2 + z^2 \right] \left( x-x(t) \right), \]

\[ = -\alpha \frac{\partial f(r)}{\partial r} \frac{1}{2r} \left[ (x-x(t))^2 + y^2 + z^2 \right] \left( x-x(t) \right), \]

\[ = -\alpha \frac{\partial f(r)}{\partial r} \frac{1}{2r} \left[ (x-x(t))^2 + y^2 + z^2 \right] \left( x-x(t) \right), \]

\[ \partial_y \beta_y = -v \frac{\partial f(r)}{\partial r} \frac{\partial r(x(t))}{\partial y}, \]

\[ \partial_z \beta_z = -v \frac{\partial f(r)}{\partial r} \frac{\partial r(x(t))}{\partial z}, \]

\[ \theta = -\alpha \frac{\partial f(r)}{\partial r} \frac{\partial r(x(t))}{\partial x}, \quad \text{since} \quad \alpha = 1 \quad (38) \]
Interior region of the warp bubble, $f=1$, while at the exterior region $f=0$ thus in these regions:

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=1} = 0 \]

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=-1} = 0 \]

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=0} = 0 \]

**Figure 6.** Spacetime dynamic’s characteristics at the warp bubble interior and exterior regions.

At the transition where $f(r): 0 \leftrightarrow 1$, $\theta \neq 0$, these regions constitute two thickening walls with thickness $\xi = \frac{1}{v} \sigma$ along the direction of the warp bubble travel.

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=1} = 0 \]

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=-1} = 0 \]

**Figure 7.** Spacetime dynamic’s characteristics at the warp bubble’s transition regions.

The transition region will show the characteristic of spacetime expansion and contraction.

At the front of the bubble in the direction of travel:

\[ \theta = \frac{1}{v} \left( x - x_b(t) \right) \frac{\partial f(r)}{\partial r} \bigg|_{(r,R)=1} < 0 \]
Figure 8. Transition region at the front of the bubble.

Since, \(\frac{\partial f(r)}{\partial r} < 0\), thus \(x-x_b(t) > 0\), therefore \(x_b(t) < x\), which represents space contraction. Thus, the transition region at the rear of the bubble, opposite to the direction of travel (as oppose to contraction condition \(\dot{\theta} < 0\)), thus \(\theta > 0\) that is equation (38) is positive in value.

\[
\theta = v \left( x-x_b(t) \right) \left( \frac{\partial f(r)}{\partial r} \right)_{(f=0, \dot{f}=0)} > 0 ,
\]

Figure 9. Transition region at the rear of the bubble.

Since also that \(\frac{\partial f(r)}{\partial r} = \frac{\partial f(-r)}{\partial (-r)} < 0\), which implies that \(\frac{\partial f}{\partial r} < 0\) as always (of either during contraction or expansion), and \(\theta > 0\) (as oppose to contraction condition), thus \(x-x_b(t) < 0\), therefore \(x_b(t) > x\), which represents space expansion.

5. Velocity expression

By the Einstein field equation the relationship of between the warp bubble velocity and the energy density can be acquired. Expanding the Einstein curvature tensor \(G_{\mu\nu}\) in term of Ricci tensor, rearrange in term of the energy momentum tensor and multiply both sides of the equation with timelike vector \(l^\mu\) as below:

\[
\frac{1}{8\pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = T_{\mu\nu} , \quad T_{\mu\nu} l^\mu l^\nu = \frac{1}{8\pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) l^\mu l^\nu .
\]
\(-T_{\mu \nu} = \rho_{\text{warp}}\) can be shown that the energy momentum tensor \(T_{\mu \nu}\) in equation (42) will result to the following as proven by Lobo and Visser [13] but require a little modification since the warp bubble direction of travel is in the \(X\) direction (Fig. 5),

\[
T_{\mu \nu} t^\mu t^\nu = - \frac{v^2}{32\pi} \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2.
\]

(43)

which is the energy density of the warp bubble that violates the energy condition \(T_{\mu \nu} t^\mu t^\nu < 0\) that is

\[
T_{\mu \nu} t^\mu t^\nu = - \rho_{\text{warp}}.
\]

(44)

It is known that the shape function \(f = f(r)\), thus \(\frac{\partial f}{\partial y} = \frac{\partial f(r)}{\partial r} \frac{\partial f}{\partial y} \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial \zeta}\), therefore \(\frac{\partial f}{\partial y} = \frac{\partial f(r)}{\partial r} \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial \zeta}\)

\[
\frac{\partial f}{\partial y} = \frac{\partial f(r)}{\partial r} \frac{\partial (\dot{y} - x_b)}{\partial y} + \frac{y^2 + z^2}{\left(\dot{y} - x_b\right)^2 + y^2 + z^2} \frac{\partial (x - x_b)}{\partial y} = \frac{y \partial f(r)}{r \frac{\partial f}{\partial r}}.
\]

(45)

\[
\frac{\partial f}{\partial z} = \frac{\partial f(r)}{\partial r} \frac{\partial (\dot{z} - x_b)}{\partial z} + \frac{y^2 + z^2}{\left(\dot{z} - x_b\right)^2 + y^2 + z^2} \frac{\partial (x - x_b)}{\partial z} = \frac{z \partial f(r)}{r \frac{\partial f}{\partial r}}.
\]

(46)

By Equations (43) to (46) the expression of the warp bubble energy density is:

\[
\rho_{\text{warp}} = \frac{v^2}{32\pi} \left( \frac{y^2 + z^2}{r^2} \right) \left( \frac{\partial f(r)}{\partial r} \right)^2.
\]

(47)

The derivative \(\frac{\partial f(r)}{\partial r} \approx \sigma\) which is actually the slope of the transition region in the condition where \(f(r) > 0\) and \(\theta \neq 0\) which is also the thickness factor. The thickness of the bubble is \(\xi = \sqrt{\sigma}\) (Fig. 7) thus from (47) the warp bubble velocity expression can be written as:

\[
v \approx \sqrt{\frac{32\pi \rho_{\text{warp}}}{\gamma^2 + \zeta^2}}.
\]

(48)

This shows that the thickness of bubble and the energy density are the two most important factors to determine the warp bubble speed.

6. Conclusion

GR has shown that the superluminal travel is possible by not just through short cutting spacetime but also through the surfing-like dynamics of embedded hypersurface onto a hyperspace. Both have the shortcoming of requiring to violate energy condition, however being optimistic is always justifiable since more than 70% of the universe is still mysteriously encompassed by the dark energy that have the same characteristic of exotic matter for the energy required by both methods of superluminal travel. Superluminal capability of the surfing-like dynamics fundamentally based on the concept of separating inertial reference frames that is between the warp-bubble and its surrounding spacetime. The inertial
reference frame is preserved in the respective region of either interior or exterior of the warp bubble. Spacetime condition in the bubble’s interior is as normal as in a stationary non relativistic condition. Subject of travel could remain stationary inside the bubble whereas the warp bubble hypersurface would carry the subject via unlimited magnitude of velocity unconstrained by any relativistic restriction toward the subject using the hypersurface-hyperspace transition. The transition region of the warp bubble is fundamentally manipulated on how the shape-function characteristic of the warp bubble interact with the surrounding spacetime. This was done by using the extrinsic curvature of warp bubble. The trace of the extrinsic curvature represents the dynamics characteristic of warp bubble interaction with the surrounding spacetime of expansion and contraction. Deriving the dynamics characteristic expression $\theta$ from the trace of the extrinsic curvature provides its relationship with the warp bubble velocity and the slope of the transition region which is also representing the bubble thickness factor. These has shown how the warp bubble propelled through contraction and expansion of hyperspace that is the spacetime surrounding the warp bubble hypersurface. Finally using the Einstein field equation it has been shown that the energy density acquire exotic matter property of negative energy at the slope of the transition region which also indicates that beside the energy density $\rho_{\text{warp}}$, the thickness of the warp bubble boundary $\zeta$ also plays important factor in determining the speed of the warp bubble which by itself does not constrained by any relativistic limitations.

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