A replica-symmetric Hamiltonian describes the correlations of the critical state of spin glasses in a field (and might be relevant for other glass formers too)

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(Dated: February 2, 2022)

A growing body of evidence indicates that the sluggish low-temperature dynamics of glass formers (e.g. supercooled liquids, colloids or spin glasses) is due to a growing correlation length. Which is the effective field theory that describes these correlations? The natural field theory was drastically simplified by Bray and Roberts in 1980. More than forty years later, we confirm the tenements of Bray and Roberts theory by studying the Ising spin glass in an externally applied magnetic field, both in four spatial dimensions (data obtained from the Janus collaboration) and on the Bethe lattice.

Spin glasses [1,8] in a magnetic field (but above the de Almeida-Touless (dAT) line [4]), structural glasses close to (but above) their mode coupling temperature [5] or hard spheres above the Gardner transition [6], all display large correlation lengths and slow relaxations that are typical of a second-order phase transition. These features are predicted by mean field (MF) theory [1], and have been identified both in experiments and in numerical simulations [7,12]. However, the very existence of the phase transition has been long debated [33-36]. In other words, the divergence at $T_c$ is more violent for two of the three susceptibilities (there are eight different coupling constants) [45]. Indeed, it has been frequently suggested that these critical features might be connected to a crossover, rather than to a true phase-transition [57,43]: the corrections to MF theory would destroy the transition, or (in some cases) move it to zero temperature. In this paper we do not claim against, nor in favor, of the presence of a transition. Instead, our aim is understanding in detail the properties of the correlations in the region where the susceptibilities are large (e.g. $10^4$ times their natural value).

Let us consider the framework of spin glasses in a magnetic field. The theory is complex [44]. Three different two-point correlators (and their associated susceptibilities) become critical. We also have eight non-linear susceptibilities associated to the the eight three-point correlators (there are eight different coupling constants) [45]. However, in an expansion around MF, one finds a linear transformation such that only one of the three susceptibilities is divergent at the critical temperature $T_c$. Similarly, the divergence at $T_c$ is more violent for two of the non-linear susceptibilities: at first order in perturbation theory, they scale as $1/(T - T_c)^3$, while two non-linear susceptibilities diverge as $1/(T - T_c)^2$, another one as $1/(T - T_c)$ and the remaining three are finite at $T_c$. As expected, only the couplings that correspond to the most divergent non-linear susceptibilities are relevant near the transition. The linear transformations that diagonalize the singularity structure are well known, and they have a physical meaning. Corrections to MF could completely destroy this divergences structure (or they may just modify the values of the critical exponents). A systematic investigation of the correctness of the above picture has never been attempted using numerical simulations. This paper fills the lacuna in the particular case of spin glasses. We show that these qualitative predictions are satisfied in the region of large susceptibilities. It is quite possible that the same situation is present in other contexts, beyond spin glasses.

The standard tool to understand the fate of a transition in finite spatial dimension $D$ is the Wilsonian Renormalization Group (RG) [46]. Unfortunately, the standard perturbative construction fails in these models: the most relevant corrections to MF theory are due to the presence of cubic terms in the effective Landau-Ginsburg theory (LGT), see Eq. (1) in [47], and two coupling ($\tilde{w}$ and $\tilde{w}_2$) are known to be relevant for $D \lesssim 6$. In fact, in spin glasses and also in models with the same LGT, the construction of the $D = 6 - \epsilon$ expansion fails because no fixed point is present in the weak-coupling region $\beta_0$. The action of the RG brings the corrections to the Gaussian behavior in the region where the effective couplings are large. The fate of the parameter $\lambda_i = \tilde{w}_{i,r}/\tilde{w}_{1,r}$ ($\tilde{w}_{i,r}$, $i = 1, 2$, are the renormalized couplings, see e.g. [44]) is of particular interest. Indeed, $\lambda_i$ plays a crucial role in the mode coupling theory where it must be $0 \leq \lambda_i \leq 1$. Moreover, as discovered by Gross et al. [48], and recently stressed by Höller and Read [49], having $\lambda_i > 1$ would...
imply a peculiar first-order like transition, like the calorimetric transition of glasses (see e.g. [3]).

Unfortunately, in spite of the relevance of the renormalized parameters [50], they have not been obtained in simulations, partly because of the complexity of the computation. Here we show that such a computation is feasible: we present results for spin glasses in a magnetic field, both in the Bethe lattice and in the $D = 4$ hypercubic lattice. Our model choice is based on its relative simplicity, but our techniques can be straightforwardly extended to more complex models. The Bethe lattice computation is a test of the viability of the approach and of the formulae used. Indeed, corrections to MF disappear in an infinite Bethe lattice and the value of $\lambda_r$, which is unaffected by fluctuations (i.e. loop corrections), is analytically known. On the other hand, the $D = 4$ Edwards-Anderson (EA) mode may be well thermalized in the region of very large susceptibilities and we have some estimates of the position of the extrapolated dAT transition [24]. Our results are suggestive of the presence of a fixed-point value $\lambda_r \approx 0.5$, and clearly exclude a value of $\lambda$ greater than 1.

Let us summarize the theoretical understanding for spin glasses in a magnetic field $h$. The effective action can be written using the replica formalism (we recall in [47] the main results, that are well described in the literature). We aim to express all our results in terms of correlation functions than can be computed in a numerical simulation. Let us start from the two points correlation functions. As usual in disordered systems, we need to distinguish between the thermal average, $\langle \cdots \rangle$, and the average over disorder, $\langle \cdots \rangle_r$. For a system of linear size $L$, with $N = LD$ spins $S_i = \pm 1$, we have three relevant susceptibilities:

$$\chi_1 \equiv \frac{1}{N} \sum_{ij} (S_i S_j)^2 - q^2,$$

$$\chi_2 \equiv \frac{1}{N} \sum_{ij} (S_i S_j) \langle S_i \rangle \langle S_j \rangle - q^2,$$

$$\chi_3 \equiv \frac{1}{N} \sum_{ij} (S_i^2 S_j^2) - q^2,$$

where $q \equiv \langle S_i \rangle^2$ is the average overlap. If we expand around the MF solution we find at all orders of the perturbation theory that the so-called replica susceptibility is divergent near the transition:

$$\chi_R \equiv \chi_{SG} \equiv \frac{1}{N} \sum_{ij} (S_i S_j)^2_r = \chi_1 - 2\chi_2 + 3\chi_3,$$

where by $\langle \cdots \rangle_r$ we denote the connected correlation function (e.g. $\langle S_i S_j \rangle_r = \langle S_i \rangle \langle S_j \rangle - \langle S_i S_j \rangle$, see for instance [51]). For later use we introduce the longitudinal and anomalous susceptibilities, $\chi_L$ and $\chi_A$. The two are degenerated in presence of a magnetic field,

$$\chi_L = \chi_A = \chi_1 - 4\chi_2 + 3\chi_3.$$

If we consider Gaussian-distributed random magnetic fields, $\chi_L$ is proportional to the staggered magnetic susceptibility [47]. Then, the physically motivated assumption that the magnetic susceptibility is not critical implies that $\chi_L$ is not critical either. Only the average of the (squared) connected-correlator becomes critical. This is in sharp contrast, with the $h = 0$ case where $\chi_2 = \chi_3 = 0$ and $\chi_A = \chi_L = \chi_R$. We expect a crossover region for small $L$ and $h$, where $\chi_L$ and $\chi_A$ seem critical (because $\chi_L$ and $\chi_A$ are critical at the $h = 0$ transition).

It is not a surprise that the two cubic renormalized-couplings $\tilde{w}_{1,r}$ and $\tilde{w}_{2,r}$ are proportional to connected-correlations at zero external momentum,

$$\omega_1 \equiv \frac{1}{N} \sum_{ijk} \langle S_i S_j \rangle_c \langle S_j S_k \rangle_c \langle S_k S_i \rangle_c,$$

$$\omega_2 \equiv \frac{1}{2N} \sum_{ijk} \langle S_i S_j S_k \rangle^2,$$

given that (see sec. IV of Ref. [44])

$$\tilde{w}_{1,r} = \frac{\omega_1}{\chi_R \xi^2}, \quad \tilde{w}_{2,r} = \frac{\omega_2}{\chi_R \xi^2},$$

where $\xi$ is the second-moment correlation length. It follows that

$$\lambda_r = \frac{\tilde{w}_{1,r}}{\tilde{w}_{2,r}} = \frac{\omega_1}{\omega_2} \chi_R \xi^2.$$

In addition, at very large volumes [44] $\tilde{w}_i = \omega_i/\chi_R^3$ ($i = 1, 2$) and then $\lambda_r = \tilde{w}_1/\tilde{w}_2$: hence, $\lambda$ does not renormalize and we will drop thereafter the sub-index $r$.

Finite-volume corrections are very strong so we do not consider here the computation of the renormalized couplings $w_{i,r}$. However, we can introduce the dimensionless quantities

$$\Lambda_1 = \frac{\omega_1}{\chi_R^{3/2} L^{D/2}}, \quad \Lambda_2 = \frac{\omega_2}{\chi_R^{3/2} L^{D/2}},$$

that should scale with $L$ as Binder’s cumulant [52]. Notice that, at the critical point, $\Lambda_1 \propto \tilde{w}_1$, $r$.

Before discussing our numerical findings for $\lambda$, it is important to stress that there are nonequivalent ways of taking the relevant limit for $\Lambda_1(L, T)$ in the onset of a second order phase transition at $T_c$:

$$\lambda^* = \lim_{L \to \infty} \lim_{T \to T_c^+} \lambda(L, T), \quad \lambda(T_c^+) = \lim_{T \to T_c^+} \lim_{L \to \infty} \lambda(L, T).$$

The fact that $\lambda^* \neq \lambda(T_c^+)$ is hardly surprising [53]. Similarly, the corresponding limits for renormalized coupling $\tilde{w}_{1,r}$ and $\tilde{w}_{2,r}$ do not commute. $\lambda(T_c^+)$ is in general more difficult to estimate than $\lambda^*$, but the former could be more desirable given that the RG $\beta$-functions(see, e.g., [51] [54] [55]) are typically expressed in terms of the thermodynamic quantities in analytical computations.
In a simulation, these quantities are computed from real replicas (i.e., systems that evolve independently under the same coupling constants). It is well known (see [47] for a quick reminder) that one needs two real replicas to compute \( q \), four replicas for the three susceptibilities, and six replicas for the \( \omega_i \) in Eqs. (6,7). In spite of this and only at the critical point, it is possible to compute both \( \omega_i \) using only three and four replicas [47]. We shall denote the estimate obtained with \( R \) replicas by \( \omega_i^{(R=3,4)} \). Away from the critical point, one has for the differences \( \omega_i^{(3)} - \omega_i^{(4)} \), \( \omega_i^{(3)} \), \( \omega_i^{(4)} \) and \( \lambda(T_c) \). 

**Numerical results in the Bethe-Lattice.** To study the behavior of the three- and four-replicas estimators in a controlled setting, we have simulated an Ising spin glass in a magnetic field on a Bethe lattice (random regular graph with fixed-degree 4). In this case, there is little doubt that a true dAT transition is present. Furthermore, the divergence of the susceptibilities (both linear and non-linear) closely matches our description above.

In Fig. 1 we plot the parameter \( \lambda \) for the Bethe lattice, as obtained from the exact expression together with the three- and four-replica estimators \( \lambda^{(3)} = \omega_2^{(3)}/\omega_1^{(3)} \) and \( \lambda^{(4)} = \omega_2^{(4)}/\omega_1^{(4)} \). In this case \( T_c \) and \( \lambda(T_c^+) \) are known analytically [56] and we see that the estimators extrapolate to the correct value at the critical temperature, although close to the critical point there are finite size corrections. Note as well that the finite-size corrections of the true \( \lambda \) (i.e., the six-replica estimator) and of the four-replica estimator coincide in the critical region. The same effect is expected for the three-replica estimator but it is masked by pre-asymptotic effects at the sizes considered. At any rate, we find that the deviations are consistent with the predicted MF values \( \omega_i^{(3)} = O(|T - T_c|) \) and \( \omega_i^{(4)} = O(|T - T_c|^2) \) [57].

**Numerical results in four dimensions.** The discussion of the three- and four-replica estimators is of great practical and theoretical importance in this case.

The theoretical importance relies on the fact that, at variance with the Bethe lattice case, one cannot take for granted that the transition is described by the theory outlined above. For instance, we could have a continuous transition described by a different theory and therefore the three- and four-replica estimators would yield conflicting results, thus indicating a wrong choice for the starting field-theory. Furthermore, due to the lack of a perturbative RG fixed-point below six dimensions, one could even question the very existence of such a theory for \( D < 6 \). Thus the fact that the three- and four-replica expressions yield consistent estimates provides a non trivial indication that the region of large susceptibilities is actually described by the Replica-Symmetric field theory of Bray and Roberts [58].

The practical importance of the three- and four-replica estimators lies in that, in the present study, we have re-analyzed equilibrium configurations obtained by the Janus Collaboration [24] using the Janus-I supercomputer [58]. Those equilibrium configurations were obtained only for four real replicas. Therefore, \( \lambda \) can be computed only through the three- and four-replica estimators (although the computation will not be exact away from the dAT line).
In the critical temperatures and the critical exponents were estimated for three different magnetic fields \((h = 0.075, 0.15\) and \(0.3\)), by looking only to one of the two-point correlators, namely the replicon. We study the same magnetic fields considered in \([21]\), for temperatures near (but above) their estimated critical temperatures.

We start by studying in Fig. 2 the replicon and longitudinal susceptibilities, recall Eqs. 4 and 5. We clearly see that \(\chi_R\) increases and becomes very large as the temperature is lowered, while \(\chi_L\) saturates at a much smaller plateau value \([59]\). We conclude, in agreement with our MF-based expectations and with previous dynamic investigations in \(D = 3\) \([22]\), that correlations extend to much larger distances for the replicon mode than for the longitudinal one, thus excluding the possibility that the critical behavior in \(\chi_R\) is due to the \(h = 0\) fixed point.

We have considered also the non-linear susceptibilities, the most divergent ones being \(\omega_1\) and \(\omega_2\), see Eqs. 6 and 7. We find that \(\omega_1\) grows significantly upon decreasing \(T\) and (at a fixed, low \(T\)) upon increasing \(L\), see Fig. 3 (left). The suggested divergence in \(\omega_1\) makes it advisable to consider the dimensionless \(\Lambda_1(L, T)\), recall Eq. 10. At a critical point, the curves of \(\Lambda_1\) as function of \(T/c\), computed for different sizes \(L\), should cross or merge at \(T_c\). Our data for \(L = 10, 12\) and \(16\) in Fig. 3 (right) do not clearly cross nor merge, making it difficult to compute \(T_c\) from these data (indeed, the authors of Ref. 21 could locate \(T_c\) only by considering quantities at non-zero external momentum). The crucial point, however, is the absence of any evidence in Fig. 3 (right) for a runaway trajectory where \(\Lambda_1\) becomes bigger and bigger upon increasing \(L\). This observation makes unlikely the scenario with a first order transition \([39]\).

Once we know that \(\omega_{1,2}\) behave as expected, we can consider their ratio \(\lambda\), which is the main quantity of interest. Fig. 4 shows the three- and four-replica estimators for magnetic fields \(h = 0.075, 0.15\) and \(0.30\). At variance with our findings for the Bethe lattice (where the difference between \(\lambda(T_c^+\)) and \(\lambda^*\) is very clear, recall Eq. 11 and Fig. 1), our data for the 4D case shown in Fig. 4 do not manifest large size effects approaching the critical point: data barely depend on temperature for \(T < T_c(h = 0)\), thus suggesting \(\lambda^*\) and \(\lambda(T_c^+)\) should be very close. The only visible finite size effect in 4D data is a monotonic in \(L\) decrease for \(R = 3\) and increase for \(R = 4\), that actually helps in bracketing \(\lambda^*\) between the values measured on the largest lattice \(L = 16\). Indeed, our data are consistent with a universal value \(\lambda^* \approx 0.55\) at the critical temperature. We remark as well that both the \(R = 3\) and the \(R = 4\) estimates verify \(\lambda(L, T) < 1\). Hence, we conclude \(\lambda(T_c^+) < 1\) in 4D spin glasses in a field, which is the main result of this paper.

Discussion. Irrespective of the on-going debate about whether the glass transition is a true phase transition or a crossover, it is undeniable that glass formers display slow dynamics and large correlations. When the lengthscales for fluctuations becomes large, the natural tool to study the problem is a Field Theory. Unfortunately, symmetry considerations do not constrain much the Hamiltonian: in the particular case of spin glasses in a magnetic field, we end with a extremely complex theory containing eight different coupling-constants. Bray and Roberts \([33]\) drastically simplified the theory. Their so-called replica symmetric Hamiltonian has been the basis for many analysis. In spite of this, up to now it was not possible to test in a non-trivial problem the basic hypothesis underlying the theory. We have overcome this challenge thanks to two crucial ingredients: (i) a detailed scaling description for the many linear and non-linear susceptibilities in the problem \([44]\), and (ii) a reanalysis of the equilibrated configurations obtained with the Janus I supercomputer \([21]\). We have found that the crucial scaling relations are filled beyond Mean Field approximation, close (but above) the de Almeida-Thouless line. Furthermore, it is quite probable that our approach will be relevant for the study of other physical systems as well (e.g. glass-forming liquids). Besides, our results for the renormalized coupling \(\lambda\) seem to exclude the suggested scenario of a first-order transition \([39]\).

The authors wish to thank the Janus Collaboration for allowing us to analyze their data. We would like also to thank E. Marinari for interesting discussions. The analysis of the Janus configurations was performed at ICCAEx supercomputer center in Badajoz, we thank its staff for their assistance.

This work was supported by the European Research Council under the European Unions Horizon 2020 research and innovation programme (grant No. 694925, G. Parisi), by Ministerio de Economía y Competitivi-
Figure 4. Three- and four-replicas estimators for $\lambda$ as a function of the temperature in the $D = 4$ Ising spin glass (the value of the magnetic field is indicated above each panel). Vertical lines report the three critical temperatures taken from [24]. The band around $\lambda^* \approx 0.55$ is our best $L \to \infty$ extrapolation, assuming three- and four-replicas estimators converge to a common value for all the three simulated values of the magnetic field (the width of the band represents the uncertainty in our extrapolation for $h = 0.075$).
As usual the renormalized couplings are defined in terms of the renormalized correlation functions at zero momentum.

G. Parisi, *Statistical Field Theory* (Addison-Wesley, 1988).

K. Binder, *Z. Phys. B — Condensed Matter* 43, 119 (1981).

J. Salas and A. D. Sokal, *Journal of Statistical Physics* 98, 551 (2000).

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Clarendon Press, Oxford, 2005).

D. J. Amit and V. Martin-Mayor, *Field Theory, the Renormalization Group and Critical Phenomena*, 3rd ed. (World Scientific, Singapore, 2005).

G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, *Journal of Statistical Mechanics: Theory and Experiment* 2014, P04013 (2014).

M. Veca, *Numerical Estimation of the exponent parameter of a spin glass model defined on random regular graphs in field*, Master’s thesis, Università La Sapienza-Roma (2021).

F. Belletti, M. Cotallo, A. Cruz, L. A. Fernandez, A. Gordillo, A. Maiorano, F. Mantovani, E. Mariani, V. Martin-Mayor, A. Muñoz Sidiupe, D. Navarro, S. Perez-Gavirio, J. J. Ruiz-Lorenzo, S. F. Schifano, D. Sciretti, A. Tarancón, R. Tripiccione, and J. L. Velasco (Janus Collaboration) *Comp. Phys. Comm*. 178, 208 (2008), arXiv:0704.3573.

The plateau value of $\chi_L$ in Fig. 2 approximately scale as $h^{-\omega}$, with $\omega$ between 2 and 3. In MF, $\chi_L$ near $T_c$ is proportional to $h^{-2/3}$ [60].

T. Temesvári, C. De Dominicis, and I. R. Pimentel, *Eur. Phys. J. B* 25, 361 (2002).

T. Rizzo, *Phys. Rev. E* 88, 032135 (2013).

F. Caltagirone, U. Ferrari, L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, *Phys. Rev. Lett.* 108, 085702 (2012), arXiv:1111.6420.

F. Caltagirone, G. Parisi, and T. Rizzo, *Phys. Rev. E* 87, 032134 (2013).

M. A. Moore and N. Read, *Phys. Rev. Lett.* 120, 130602 (2018).

I. R. Pimentel, T. Temesvári, and C. De Dominicis, *Phys. Rev. B* 65, 224420 (2002).

**Supplemental Information**

**The replica-symmetric field theory**

Standard arguments [33, 34] tell us that the $D$-dimensional Ising spin glass in presence of a magnetic field is described at criticality by the following RS Hamiltonian for the replicated overlap $\phi_{ab}(x) (\phi_{ab}(x) = 0)$:

$$
\mathcal{H} = \frac{1}{2} \int d^Dx \left[ m_1 \sum_{ab} \phi_{ab}^2 + \frac{1}{2} \sum_{ab} (\nabla \phi_{ab})^2 + m_2 \sum_{abc} \phi_{ab} \phi_{ac} + m_3 \sum_{abcd} \phi_{ab} \phi_{cd} + \frac{1}{6} \sum_{abc} \phi_{ab} \phi_{bc} \phi_{ca} - \frac{1}{6} \sum_{ab} \phi_{ab}^3 \right].
$$

(12)
Notice that we have changed slightly the notation of Ref. [44] by adding a tilde to the couplings $\tilde{\omega}_i$ in order to improve the readability of this manuscript. We also note that the non-linear susceptibilities $\omega_i$ are the coefficients of the cubic terms in the associated Gibbs free energy (obtained by taking the Legendre transform of the previous Hamiltonian) [44].

At the MF level, $m_1$ vanishes linearly on the dAT line and, in the SG phase, the solution displays Replica-Symmetry-Breaking (RSB) with a breaking point at a value equal to $w_2/\tilde{w}_1$ [53 61]: it follows that $\lambda \equiv w_2/\tilde{w}_1$ must be smaller than one for consistency. It should be also noted that the parameter $\lambda$ controls the MF values of equilibrium and off-equilibrium dynamical exponents in a variety of contexts [44 62 63].

The idea of Höller and Read [49] (that started from [61]), is to apply the RG to the above replicated Hamiltonian until the mass term $m_1$ (which is initially small because we start close to the dAT line) becomes equal to one, then the RG flow is stopped and the new Hamiltonian is analyzed at the MF level. Note that they actually follow Bray and Roberts [33] and project on the replica subspace effectively sending the longitudinal and anomalous masses to infinity. To obtain subcritical behavior one must keep the massive modes finite, see ref. [60 65] for a thorough comparative discussion of the two approaches. Höller and Read suggest that below the upper critical dimension, $\lambda$ becomes larger than one under the RG flow on the whole dAT line and therefore the transition becomes first-order. One should note that treating a Wilson Hamiltonian at the MF level is always an approximation, although it may be accurate close to the upper critical dimension. Essentially, one is approximating the true Gibbs free energy with the Wilson’s Hamiltonian, i.e. fluctuations are neglected. While the coefficients of the Wilson’s Hamiltonian are bare parameters that cannot be measured, the coefficients of the Gibbs free energy (proportional of the renormalized couplings) can be expressed in terms of physical observables and thus are directly accessible to measurements [51 54 55].

The renormalized couplings $\tilde{\omega}_{1,r}$ and $\tilde{\omega}_{2,r}$ have finite and model-dependent values except at the critical temperature where, if scaling holds, they have finite universal values $\tilde{\omega}_{1,r}^*$ and $\tilde{\omega}_{2,r}^*$. The spin-glass susceptibility and correlation length diverge as $\chi_R \propto |T - T_c|^{-\gamma_3}$ and $\xi_2 \propto |T - T_c|^{-\nu}$ respectively, and consistently $\omega_1$ and $\omega_2$ diverge as:

$$\omega_{1,2} \propto |T - T_c|^{-\gamma_3}, \quad \gamma_3 = 3\nu - \frac{3}{2} \nu D + \frac{\nu D}{2}. \quad (13)$$

Notice that renormalized couplings constants $\tilde{\omega}_{1,r}$ and $\tilde{\omega}_{2,r}$ are universal quantities at criticality and play a key role in computations of critical exponents [51 53 55], being the zeroes of the $\beta$-functions.

Note that Eqs. (8) in the main text follow from Eqs. (91) in Ref. [44] noticing that when the RG flow is stopped the overlaps are effectively rescaled by a factor $\chi_R^{1/2}$ and the length is rescaled by a factor $\xi_2$ since the coefficient of the term $(\nabla \phi_{ab})^2$ is fixed to one in the RG flow.

### Computing $\omega_1$ and $\omega_2$ using three, four and six replicas

In order to compute $\omega_1$ and $\omega_2$ we need to evaluate numerical quantities like

$$m_2^2 = \langle \sigma_i^2 \rangle, \quad m_4^1 = \langle \sigma_i^4 \rangle, \quad m_6^0 = \langle \sigma_i^6 \rangle. \quad (14)$$

The standard approach consists in introducing $K$ independent replicas of the system sharing the same disorder $(\sigma^{(i)}, i = 1, \ldots, K)$ obtaining

$$m_2^i = \langle \sigma_i^{(1)} \sigma_i^{(2)} \rangle, \quad m_4^i = \langle \sigma_i^{(1)} \sigma_i^{(2)} \sigma_i^{(3)} \sigma_i^{(4)} \rangle, \quad m_6^i = \langle \sigma_i^{(1)} \sigma_i^{(2)} \sigma_i^{(3)} \sigma_i^{(4)} \sigma_i^{(5)} \sigma_i^{(6)} \rangle. \quad (15)$$

Both non-linear susceptibilities, $\omega_1$ and $\omega_2$, are suitable for numerical evaluation once expressed as [44]

$$\omega_1 = W_1 - 3W_5 + 3W_7 - W_8, \quad \omega_2 = \frac{1}{2} W_2 - 3W_3 + \frac{3}{2} W_4 + 3W_6 + 2W_7 - 6W_8 + 2W_9,$$

and

$$W_1 = N^2 \langle \delta Q_{12}\delta Q_{23}\delta Q_{31} \rangle, \quad W_2 = N^2 \langle \delta Q_{12} \rangle^2, \quad W_3 = N^2 \langle \delta Q_{12}\delta Q_{13} \rangle, \quad W_4 = N^2 \langle \delta Q_{12}\delta Q_{34} \rangle, \quad W_5 = N^2 \langle \delta Q_{12}\delta Q_{13}\delta Q_{24} \rangle,$$  
$$W_6 = N^2 \langle \delta Q_{12}\delta Q_{13}\delta Q_{14} \rangle, \quad W_7 = N^2 \langle \delta Q_{12}\delta Q_{13}\delta Q_{45} \rangle, \quad W_8 = N^2 \langle \delta Q_{12}\delta Q_{34}\delta Q_{56} \rangle,$$

where overlap fluctuations can be written in terms of independent real replicas with the same quenched disorder

$$\delta Q_{ab} = \frac{1}{N} \sum_i s_i^a s_i^b - \frac{1}{N} \sum_i \langle s_i \rangle^2. \quad (16)$$

Each correlator $W_i$ requires a number of different real replicas equal to the largest index in its expression (right hand side).

Hence, we recall, we need two replicas to compute the overlap, four for the susceptibilities and six for $\omega_1$ and $\omega_2$.

Can we use use a smaller number of replicas? The theory predicts that there are six linear combination of the $W_i$’s that diverge less than the $W_i$ separately. Using
these linear relationships one can express the eight coefficients in terms of only the three-replicas estimators \[44\]:

\[
\omega^{(3)}_1 = \frac{11}{30} W_1 - \frac{2}{15} W_2, \\
\omega^{(3)}_2 = \frac{4}{15} W_1 - \frac{1}{15} W_2.
\]

(17)

(18)

Alternatively the theory predicts that there are three linear combinations of the \( W \)'s that remain finite at the critical temperature. Therefore one can express \( W_7 \) and \( W_8 \) as a function of the remaining cumulants obtaining the four-replicas estimators \[57\]:

\[
\omega^{(4)}_1 = \frac{23 W_1}{30} + \frac{W_2}{20} - \frac{3 W_3}{5} + \frac{9 W_4}{20} - \frac{6 W_5}{5} + \frac{W_6}{2}, \\
\omega^{(4)}_2 = \frac{7 W_1}{15} - \frac{2 W_2}{5} + \frac{9 W_3}{5} - \frac{3 W_4}{5} + \frac{3 W_5}{5} + W_6.
\]

(19)

(20)

Within the RS theory, the three- and four-replicas estimators are different from the true \( \omega_1 \) and \( \omega_2 \) at any given temperature but coincide with them at the critical temperature. At a generic temperature \( \bar{w}_{1,r}, \bar{w}_{2,r} \) and \( \lambda \) have model-dependent values and we are only interested in the universal values they take at the critical temperature. More precisely one can show that close to the critical point

\[
\omega_i - \omega^{(3)}_i = O(|T - T_c|^{\gamma_\Delta}), \quad \omega_i - \omega^{(4)}_i = O(|T - T_c|^{\gamma_3}),
\]

where the exponent \( \gamma_\Delta \) is expected to be smaller than \( \gamma_3 \) (e.g. in MF one finds \( \gamma_\Delta = 1 \) and \( \gamma_3 = 3 \)).

Finiteness of the longitudinal susceptibility

Let us consider the model in presence of a Gaussian magnetic field which generates a new term in the Hamiltonian: \(+ h_o \sum_i h_i S_i\), where \( h_i \) are independent Gaussian variables with zero mean and unit variance. The staggered magnetization is defined as

\[
m_{st} \equiv \langle h_i \sigma_i \rangle
\]

(19)

where \( \langle \cdots \rangle \) is the joint average over the couplings and the Gaussian magnetic field. Its susceptibility is

\[
\chi_{st} = \frac{\partial m_{st}}{\partial h_0} = -\beta \sum_i \left( \langle h_i S_i h_i S_i \rangle - \langle h_i S_i \rangle \langle h_i S_i \rangle \right).
\]

(20)

Integrating by parts Eq. (20) one can finally obtain that \( \chi_{st} = 2\beta \chi_L \). Therefore, if the magnetic susceptibility does not diverge, neither does the longitudinal susceptibility.

The models

We study the 4D-dimensional EA model in a field \( h \) where \( N = L^4 \) Ising spins interact via

\[
\mathcal{H} = -\sum_{(xy)} J_{xy} S_x S_y + h \sum_x S_x,
\]

(21)

where the first sum is over nearest-neighbor pairs and \( J_{xy} = \pm 1 \) with 50% probability. In our 4D computation, the spins are located in the nodes of a hypercubic lattice with periodic boundary conditions.

We have also simulated the model on a Bethe lattice where the spins occupy the vertices of a random-regular graph with connectivity 4.