Power-Law Size Distribution of Supercritical Random Trees

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Abstract. – The probability distribution $P(k)$ of the sizes $k$ of critical trees (branching ratio $m = 1$) is well known to show a power-law behavior $k^{-3/2}$. Such behavior corresponds to the mean-field approximation for many critical and self-organized critical phenomena. Here we show numerically and analytically that also supercritical trees (branching ratio $m > 1$) are "critical" in that their size distribution obeys a power-law $k^{-2}$. We mention some possible applications of these results.

Introduction. – Ideal trees, either regular or random, play a great role in the description of natural phenomena: many systems, from rivers to blood vessels and lungs, to real trees can be described by geometrical branched structures. Other phenomena can be described by branched structures in time (i.e. branching processes), with cascades of events generating new events in a multiplicative fashion: examples span from physics (e.g. nuclear chain reactions and directed percolation) to biology (speciation) and many others disciplines. More recently, trees (and branching processes) have been used to model the mean-field approximation of many non-equilibrium, self-organized critical systems such as the Bak-Tang-Wiesenfeld sandpile and the Bak-Sneppen model. The dynamics of these models has a branching structure, where some local events can trigger more events of the same kind, with some rules given by the branching probability distribution (i.e. the probability $p_n$ to trigger $n$ new events) and by the geometric constraints induced by the finite spatial dimension $d$ (that is, different branches can interact when coming to the same region). To model branching processes in infinite dimension (mean-field) the usual assumption is that there are no interactions between different branches: the statistical properties of branching, $p_n$, are preserved, but any spatial constraints are lost. In particular, the mean-field limit of the above mentioned critical models is recovered considering critical branching processes. A critical branching process is characterized by an average branching ratio $m = \sum_n n p_n = 1$, so that every generation of branching is (on the average) identical to the preceding ones. What people are usually interested in is the probability distribution $P(k)$ of the tree sizes $k$, that is, the number of sites (or of branching events, in a branching process jargon) making up the trees. It is a well-known result of branching process theory that $P(k) \sim k^{-3/2}$ for critical trees. This power-law behavior is consistent with the assumption of criticality, as common wisdom suggests.

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Interestingly enough, much less is known about the size distribution of supercritical trees. In this paper we show, through simulations and analytical arguments, that also the supercritical case \((m > 1)\) exhibits a power-law distribution of tree sizes \(P(k) \sim k^{-2}\). Some connections of this result to the structure of the Internet and to taxonomic systems are proposed.

Critical and Supercritical Trees: Numerical and Analytical Results. – Starting from a root site \((\text{generation } 0)\), a random tree is grown letting every site at generation \(t\) \((0 \leq t < t_{\text{max}})\) branch into \(n\) new sites \((\text{generation } t+1)\) with probability \(p_n\). An example of a random tree is shown in Fig.1. We are interested in the size distribution of the subtrees: picking a site at random on the tree, what is the probability \(P(k)\) that the subtree that is rooted on it has size \(k\)? In Fig.1 such sizes are also marked for every site. Sites at generation \(t_{\text{max}}\) are assigned a size 1 (just themselves).

We have performed simulations for different choices of \(p_n\), both with \(m = 1\), finding the known result \(P(k) \sim k^{-3/2}\) and with \(m > 1\), finding \(P(k) \sim k^{-2}\), as announced above (see Fig.2).

To get an analytical insight in the origin of this behavior, we must decompose \(P(k)\) in generation dependent probabilities \(P_t(k)\) \((P_t(k) = 0\) if \(t > t_{\text{max}}\) since trees are grown up to \(t_{\text{max}}\)). We write explicitly the way in which generation dependent quantities sum up to give the tree distribution \(P(k)\):

\[
P(k) = \left\langle \sum_{t=0}^{t_{\text{max}}} \frac{N_t(k)}{N_{\text{tot}}} \right\rangle = \left\langle \sum_{t=0}^{t_{\text{max}}} \frac{N_t}{N_{\text{tot}}} P_t(k) \right\rangle .
\]  

In (1) \(N_t(k)\) is the number of sites at generation \(t\) that are roots of a subtree of size \(k\), \(N_t\) is the total number of sites at generation \(t\), \(N_{\text{tot}}\) is the total number of sites on the tree. \(< \cdot >\) indicates the average over many realization of the system. Indeed, \(N_t, N_{\text{tot}}\) and \(N_t(k)\) change from realization to realization. As a first approximation we assume that, over many realizations, the generation dependent quantities converge to their average values: \(N_t = m^t, N_{\text{tot}} = (m^{t_{\text{max}}+1} - 1)/(m - 1)\). Then we can write

\[
P(k) = (m - 1) \sum_{t=0}^{t_{\text{max}}} m^t \left( \frac{m^{t_{\text{max}}+1} - 1}{m^{t_{\text{max}}+1} - 1} \right) P_t(k) \simeq (m - 1) \sum_{t=0}^{t_{\text{max}}} m^{-(t_{\text{max}} - t + 1)} P_t(k) \tag{2}
\]

where the last equality holds in the limit of large \(t_{\text{max}}\). We checked the reliability of such approximation, simulating the same process on regularly growing lattices. Starting at generation 0 with \(N_0\) sites, then \(N_t = m^t N_0\) (approximating it to the closest integer). Then each site at generation \(N_{t+1}\) is assigned to an ancestor in generation \(t\). The branching distribution is then the binomial

\[
p_n(t) = \binom{N_t+1}{n} \left( \frac{1}{N_t} \right)^n \left( 1 - \frac{1}{N_t} \right)^{N_t+1-n} \tag{3}
\]

that in the limit \(N_t \gg 1\) becomes the Poisson distribution \(p_n = e^{-m} m^n / n!\), independent from \(t\). This process gives the same results as the simulation of the genuine random trees, that is \(P(k) \sim k^{-3/2}\) if \(m = 1\), \(P(k) \sim k^{-2}\) if \(m > 1\), as shown in Fig.3.

Problem (2) can be formally solved using the recursion relation for the generating functions of \(P_t(k)\). Indeed we can write a relation between the probabilities \(P_t(k)\) and \(P_{t+1}(k)\):

\[
P_t(k) = \sum_n p_n \sum_{k_1=1}^{k} \cdots \sum_{k_n=1}^{k} P_{t+1}(k_1) \cdots P_{t+1}(k_n) \delta_{k_1+\cdots+k_n+1,k} \tag{4}
\]
We define the generating functions $G_t(z) = \sum_k P_t(k)z^k$ and $f(z) = \sum_n p_n z^n$. Then, multiplying both sides of (6) by $z^k$ and summing over $k$, we obtain a recursion relation for the generating functions at successive generations
\[ G_t(z) = zf[G_{t+1}(z)] \tag{5} \]
that is a very well known result from branching process theory [12]. Eq. (5) can be formally solved using the "initial condition" that is a very well known result from branching process theory [12].

With the "initial" condition $zf[G(z)]$ of $G_0(z) = z$, we can iterate Eq. (5) back to generation 0 to have an explicit expression of every $G_t(z)$, and, from (5) obtain the generating function $G(z) = \sum_k P(k)z^k$. Knowing $G(z)$ we could, in principle, get the asymptotic behavior of $P(k)$: if $P(k) \sim k^{-\tau}$ for large $k$, then $G(z) \sim 1 - c(1-z)^{\tau-1}$ for $z \to 1$. Unfortunately, if $P(k) \sim k^{-2}$, then we could expect a non-analytic behavior $G(z) \sim 1 - c(1-z) \ln(1-z)$, that cannot be obtained from an expansion of $G(z)$ for $z \to 1$. Instead, we should expand around $z = 0$, and resum the terms power-by-power of $z$, which entails to computing explicitly $P(k)$ directly from (6). Such an observation is of some relevance since in the critical case $m = 1$, invoking the time-translational invariance of the process (that is, setting $P_t(k) = P_{t+1}(k) = P(k)$ so that $G_t(z) = G_{t+1}(z) = G(z)$), Eq. (5) becomes $G(z) = zf[G(z)]$. Expanding both hands for $z \to 1$ and using the corresponding form $G(z) \sim 1 - c(1-z)^{\tau-1}$, one can find that to match all the powers of $(1-z)$ on both sides the only choice is $\tau = 3/2$. Some simple cases for which $G(z) = zf[G(z)]$ can be solved exactly are given by the branching probabilities $p_n = 0$ if $n > 2$, with $p_1 + 2p_2 = 1$. In these cases the recursion equation reduces to a second order equation in $G(z)$, that can be readily solved to explicitly show that the leading non-analytic behavior for $z \to 1$ is given by $(1-z)^{1/2}$, hence $P(k) \sim k^{-3/2}$.

Still, even when the system is supercritical, the generating function formulation is not fruitless. Indeed it is possible to use it to compute the average subtree size $\langle k \rangle = \sum_k kP(k)$. Given the generating function $G(z)$ it is easy to see that the average size is
\[ \langle k \rangle = \left[ \frac{d}{dz} G(z) \right]_{z=1} = (m - 1) \sum_{t=0}^{t_{\text{max}}} m^{-(t_{\text{max}}-t+1)} G'_t(1) \tag{6} \]
Taking the derivative of (6) we obtain a recursion relation for $G'_t(1)$,
\[ G'_t(1) = 1 + mg'_{t+1}(1) \tag{7} \]
With the "initial" condition $G'_{t_{\text{max}}} = 1$, we obtain
\[ G'_t(1) = \frac{m^{t_{\text{max}}-t+1} - 1}{m - 1} \tag{8} \]
from which we finally get
\[ \langle k \rangle = \sum_{t=0}^{t_{\text{max}}} m^{-(t_{\text{max}}-t+1)}(m^{t_{\text{max}}-t+1} - 1) \sim t_{\text{max}} + 1 \tag{9} \]

We find therefore that the average value $\langle k \rangle$ diverges as $t_{\text{max}} \to \infty$, a clear indication of a power-law behavior. Yet, it diverges as the logarithm of the maximal allowed size, that is $k_{\text{max}} \sim m^{t_{\text{max}}+1}$, which is the average total number of sites on the tree. This is already a strong indication that asymptotically $P(k) \sim k^{-2}$.

We then look at the behavior of $P_t(k)$ from simulations. We plot in Fig. 3 the generation probabilities $P_t(k)$ at generations $t = 8, 10, 12$ (trees are grown for 40 generations, with $m =$
1.2). They clearly decay exponentially for large values of \( k \). Such an exponential decay is suggestive of the presence of a size \( k_t \) characteristic of each generation. There is a simple and intuitive relation between \( P_t(k) \) and \( P_{t+1}(k) \), namely,

\[
P_{t+1}(k) = m P_t(mk)
\]

The relation between the arguments is readily understood thinking that, on the average, a site at generation \( t \) is the root of a subtree of size \( mk \) only if each of its \( m \) descendents at generation \( t+1 \) is the root of a subtree of size \( k \) (and formally it is expressed by Eq. 5). The prefactor \( m \) is due to normalization. Interestingly, (10) is the generalization to the supercritical case of the time-translational invariance assumed to hold in the critical case \( m = 1 \). Relation (10) is numerically verified, for \( t ≪ t_{\text{max}} \), where the asymptotic behavior has been reached, as shown in the inset of Fig. 3.

Using (10) it is at last possible to have further evidence that \( P(k) \sim k^{-2} \) asymptotically.

We write

\[
P(mk) = (m - 1) \sum_{t=0}^{t_{\text{max}}} m^{-(t_{\text{max}} - t + 1)} P_t(mk)
\]

\[
= (m - 1) \sum_{t=0}^{t_{\text{max}}} m^{-(t_{\text{max}} - t + 2)} P_{t+1}(k)
\]

\[
= (m - 1) \sum_{t=1}^{t_{\text{max}}} m^{-(t_{\text{max}} - t + 3)} P_t(k) \sim m^{-2} P(k)
\]

where we have used \( P_t(k) = 0 \) if \( t > t_{\text{max}} \). The final equality is consistent with a power-law decay with exponent \(-2\) for large \( k \).

To give further support to the idea that a functional form of \( P_t(k) \) that satisfies (10) generically implies inverse square decay, we assume the form \( P_t(k) = (1 - a_t) a_t^{k-1} \) with \( a_t = \exp(-m^{-(t_{\text{max}} - t)}) \), that obeys (10). Then

\[
P(k) = (m - 1) \left\{ \sum_{t=0}^{t_{\text{max}}} a_t^{k-1} m^{-(t_{\text{max}} - t + 1)} - \sum_{t=0}^{t_{\text{max}}} a_t^{k} m^{-(t_{\text{max}} - t + 1)} \right\}
\]

(12)

The second sum on the \( r.h.s \) of (12) is

\[
\sum_{t=0}^{t_{\text{max}}} a_t^{k} m^{-(t_{\text{max}} - t + 1)} = \frac{1}{m} \sum_{t=0}^{t_{\text{max}}} m^{-k} m^{-(t_{\text{max}} - t)} = \frac{1}{m} \sum_{t=0}^{t_{\text{max}}} e^{-km} m^{-t}
\]

(13)

Going from sums to integrals we write

\[
\sum_{t=0}^{t_{\text{max}}} e^{-km} m^{-t} \sim \frac{1}{\ln m k} \int_{e^{-k}}^{1} dy
\]

(14)

where we posed \( y = e^{-km} \), such that, in the limit of large \( t \), \( \Delta y \sim m^{-t} \Delta t \to 0 \) and the switch from sums to integrals is justified. Moreover, the upper limit of integration, 1, comes after taking the limit \( t_{\text{max}} \to \infty \). After applying the same approximation on the first sum on the \( r.h.s \) of (12), eventually we have

\[
P(k) \sim \frac{1}{k - 1} \int_{e^{-k}}^{1} dy - \frac{1}{k} \int_{e^{-k}}^{1} dy \sim \frac{1}{k^2}
\]

(15)
for large values of $k$.

A derivation of analogous results has been given in [13] for trees with a fixed branching ratio. Such a derivation based on the Mellin transform, although extremely instructive, cannot be extended directly to variable branching ratios and in particular for a non zero death probability.

Conclusions. – We have studied the size distribution $P(k)$ of random trees and subtrees. Alongside with the well known result $P(k) \sim k^{-3/2}$ for the case of critical trees (unitary branching ratio), we have found that also supercritical trees show a power-law behavior $P(k) \sim k^{-2}$. Some occurrences of this behavior have already been found.

In taxonomy [6,7], it has been found that the distribution of taxa according to the number of their subtaxa has a power-law behavior, with exponents scattered around 2. Since taxonomy is intrinsically related to a branch tree organization of genera, families, species etc., it is suggestive to think that such exponents could be related to the tree-like nature of the system itself, rather than to some underlying dynamical critical process.

Even more recently it has been pointed out that such a power-law behavior is present also for trees generated by connections on the Internet [14]: there each node of the tree is a site on the Internet, and the branches correspond to the sites it is linked to (actually the structure of the Internet is that of a network, but search and trace programs superimpose to it a corresponding tree structure). The number of sites on the Internet that collect an area (number of sites) $k$ scales as $k^{-z}$ with $z$ close to 2. Using this result we can infer that the Internet, when looked at as a tree, is exponentially growing.

Recently, the inverse-square law has emerged, for similar reasons, in marine ecology [15].

From a more general viewpoint, the emergence of ”criticality” in such a simple and well known framework as supercritical trees is suggestive of a whole new family of systems (webs and supercritical webs) still hiding interesting properties. Some preliminary simulations for directed webs show that this seems to be the case.

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Fig. 1 – A random Cayley tree grown for five generations (from 0 to 4). The last generation sizes are set to 1, and all the other subtree sizes are also explicitly written.

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Fig. 2 – $P(k)$ for $m = 1$, $m = 1.2$ and $m = 1.5$ (for this last case we show the results both on random trees and on the regular exponentially growing lattice: both exhibit the same behavior); the trees are grown for 100 ($m = 1$), 50 ($m = 1.2$) and 30 ($m = 1.5$) generations, and averages are taken over $10^5$ ($m = 1, 1.2$) and $10^3$ ($m = 1.5$) realizations. The data are binned on intervals growing as powers of 2 (the $m = 1.2$ data have also been shifted to fit together into the same graph).
Fig. 3 – Generation distributions $P_t(k)$ for $t = 8, 10, 12$ from upper to lower, respectively, in correspondence with the arrow. The average branching ratio is $m = 1.2$, and trees are grown for 40 generations. Averages are taken over $10^6$ trees. In the inset the distributions rescaled according to (10) are shown to nicely collapse.