Efficient entanglement concentration for three-photon W states with parity check measurement

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We present an optimal entanglement concentration ECP for an arbitrary less-entangled W state. By two of the parties say Alice and Charlie performing one parity check measurements, they can obtain the maximally entangled W state with a certain probability. Otherwise, they can obtain another lesser-entangled W state with another probability, which can be used to reconverted into a maximally entangled W state. By iterating this ECP several times, it has the maximal success probability. This ECP maybe an optimal one and is useful in current quantum information processing.

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I. INTRODUCTION

Entanglement is an essential role in quantum information processing. Quantum teleportation, quantum dense coding, quantum key distribution protocols(QKD), quantum secure direct communication, quantum secret sharing, quantum information processing. Quantum teleportation[4,5], quantum key distribution protocol(ECP), called the Schmidt projection method[33]. An ECP based on the quantum swapping was proposed and it was developed by Shi et al.[34,35]. In 2001, Yamamoto et al. and Zhao et al. proposed two ECPs based on the linear optics independently[37,40]. Both ECPs were realized experimentally[38,40]. ECPs based on the cross-Kerr nonlinearity were proposed[44,47]. In 2011, Wang et al. proposed an efficient ECP based on the quantum dot in an optical cavity[20]. In 2012, we proposed an efficient ECP with the help of single photons[13]. The ECPs described above in optical system are all focused on the two-particle Bell-state. They are all used to convert a less-entangled state \(\alpha|H\rangle|H\rangle + \beta|V\rangle|V\rangle\) into a maximally entangled state \(\frac{1}{\sqrt{2}}(|H\rangle|H\rangle + |V\rangle|V\rangle)\). Here \(|H\rangle\) and \(|V\rangle\) represent the horizontal and the vertical polarizations of photons. Unusually, this kind of ECPs can be extended to concentrate the multipartite less-entangled GHZ state \(\alpha|H\rangle|H\rangle|H\rangle + \beta|V\rangle|V\rangle|V\rangle\) into a maximally entangled GHZ state \(\frac{1}{\sqrt{2}}(|H\rangle|H\rangle|H\rangle + |V\rangle|V\rangle|V\rangle)\). Unfortunately, they cannot concentrate the less-entangled W state \(\alpha|V\rangle|H\rangle|H\rangle + \beta|H\rangle|V\rangle|H\rangle + \gamma|H\rangle|H\rangle|V\rangle\) into a maximally entangled W state \(\frac{1}{\sqrt{3}}(|V\rangle|H\rangle|H\rangle + |H\rangle|V\rangle|H\rangle + |H\rangle|H\rangle|V\rangle)\) for the W state cannot be converted into a GHZ state in LOCC[50]. In 2003, Cao and Yang proposed an ECP for W class state with the help of joint unitary transformation[51]. In 2007, Zhang et al. proposed an ECP based on the Bell-state measurement[52]. In 2010, an ECP for a special W state \(\alpha|H\rangle|H\rangle|V\rangle + \beta(|H\rangle|V\rangle|H\rangle + |V\rangle|H\rangle|H\rangle)\) was proposed.

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They also proposed an ECP for three-atom W state in cavity QED system with the same idea \cite{54}. Yildiz proposed an optimal distillation of three-qubit asymmetric W states. In 2011, we also proposed an ECP for W state resorting to the cross-Kerr nonlinearity and some special polarized single photons.

In this paper, we describe an efficient ECP which is focused on an arbitrary three-photon W state. We concentrate an arbitrary less-entangled W state \( \alpha|V\rangle|H\rangle|H\rangle + \beta|H\rangle|V\rangle|H\rangle + \gamma|H\rangle|H\rangle|V\rangle \) into a maximally entangled W state \( \frac{1}{\sqrt{2}}(|V\rangle|H\rangle + |H\rangle|V\rangle + |H\rangle|H\rangle|V\rangle) \), with the help of two same conventional single photons of the form \( \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \). This ECP resorts the cross-Kerr nonlinearity to perform the parity check measurement. By two of the parities performing this ECP one time, they can obtain a maximally entangled W state with a certain success probability. Compared with the other ECPs for W state, it has several advantages. First, it can concentrate an arbitrary less-entangled W state and conventional ECPs unusually focus on some special W states. Second, it only requires the conventional single photons, and other ECPs all need two copies of less-entangled pairs or some special polarized single photons. Third, the cross-Kerr nonlinearity can work on the weak area, and we do not require the large phase shift. This advantage makes this protocol more feasible in current technology. Fourth, this ECP can be repeated to get a higher success probability.

This paper is organized as follows: in Sec. II, we first describe the basic element for this ECP, say, the parity check measurement (PCM) constructed by cross-Kerr nonlinearity. In Sec. III, we explain this ECP. In Sec. IV, we calculate the entanglement transformation efficiency for our ECP. Finally, in Sec. V, we present a discussion and summary.

II. PARITY CHECK MEASUREMENT

Before we start this ECP, we first introduce the key element PCM constructed by cross-Kerr nonlinearity. The cross-Kerr nonlinearity has been widely studied in current quantum information processing, such as construction of CNOT gate \cite{57}, making a Bell-state measurement \cite{57, 58}, performing the entanglement purification and concentration \cite{59, 44, 44, 47, 48, 49}, and so on \cite{59, 44, 44, 47, 48, 49}. From Fig. 1, the Hamiltonian of a cross-Kerr nonlinearity is \cite{56, 57}

\[
H = h\chi a_p^\dagger a_s^\dagger a_s a_p.
\]

Here \( \chi \) is the coupling strength of the nonlinearity. \( a_s^\dagger \) and \( a_p^\dagger \) are the creation operations, and \( a_s \) and \( a_p \) are the destruction operations. We consider two single photon states \( |\psi_1\rangle = c_1|H\rangle + d_1|V\rangle \) and \( |\psi_2\rangle = c_2|H\rangle + d_2|V\rangle \) coupled with a coherent state \( |\alpha\rangle \) shown in Fig. 1. Here \( |c_1|^2 + |d_1|^2 = 1 \), and \( |c_2|^2 + |d_2|^2 = 1 \). The state \( |\psi_1\rangle \) is in the spatial mode \( a_1 \) and \( |\psi_2\rangle \) is in the spatial mode \( a_2 \). The whole system can be written as:

\[
|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\alpha\rangle = (c_1|H\rangle + d_1|V\rangle) \otimes (c_2|H\rangle + d_2|V\rangle) |\alpha\rangle \nonumber
\]

\[
= (c_1c_2|H\rangle |H\rangle + c_1d_2|H\rangle |V\rangle + c_2d_1|V\rangle |H\rangle + d_1d_2|V\rangle |V\rangle) |\alpha\rangle \nonumber
\]

\[
+ (d_1c_2|V\rangle |H\rangle + d_1d_2|V\rangle |V\rangle) |\alpha\rangle \nonumber
\]

\[
+ (c_1d_2|H\rangle |V\rangle |\alpha e^{-2i\theta}\rangle + d_1c_2|V\rangle |H\rangle |\alpha e^{2i\theta}\rangle). \quad (2)
\]

From Eq. (2), it is obvious that the even parity states \( |H\rangle |H\rangle \) and \( |V\rangle |V\rangle \) lead to the coherent state pick up no phase shift. The odd parity state \( |H\rangle |V\rangle \) leads to the coherent state pick up \(-2\theta \) phase shift and \( |V\rangle |H\rangle \) picks up the \( 2\theta \) phase shift. With an X quadrature measurement in which the states \( |\alpha e^{\pm 2\theta}\rangle \) cannot be distinguished, one can distinguish the even parity state from the odd parity state according to the different phase shift in the coherent state \cite{56}. Therefor, the setup shown in Fig. 2 can achieve the function of a PCM.

III. ECP WITH THE PCM

With the PCM shown in Fig. 1, the principle of our ECP for less-entangled W state is shown in Fig. 2. Three photons emitted from the source \( S_1 \) are sent to Alice, Bob and Charlie from the spatial modes \( a_1, b_1 \) and \( c_1 \). The less-entangled photon pair is described as

\[
|\Psi\rangle = |\Phi\rangle_{a_1b_1c_1} \otimes |\Phi\rangle_{a_2} \otimes |\Phi\rangle_{a_3} = (|\alpha\rangle |V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{c_1} \nonumber
\]

\[
+ \beta|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{c_1} + \gamma|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{c_1}) \nonumber
\]

\[
\otimes \left( \frac{1}{\sqrt{2}}(|H\rangle_{a_2} + |V\rangle_{a_2}) \right) \otimes \left( \frac{1}{\sqrt{2}}(|H\rangle_{c_2} + |V\rangle_{c_2}) \right) \nonumber
\]

\[
= |\alpha\rangle |V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{c_1} \otimes (|H\rangle_{a_2} |H\rangle_{c_2} + |H\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ |V\rangle_{a_2} |H\rangle_{c_2} + |V\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ \beta |H\rangle_{a_1} |V\rangle_{b_1} |H\rangle_{c_1} \otimes (|H\rangle_{a_2} |H\rangle_{c_2} + |H\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ |V\rangle_{a_2} |H\rangle_{c_2} + |V\rangle_{a_2} |V\rangle_{c_2} \right). \quad (3)
\]

in the spatial mode \( a_2 \) and \( c_2 \), respectively. The five-photon system can be written as

\[
|\Psi\rangle = |\Phi\rangle_{a_1b_1c_1} \otimes |\Phi\rangle_{a_2} \otimes |\Phi\rangle_{a_3} = (|\alpha\rangle |V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{c_1} \nonumber
\]

\[
+ \beta|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{c_1} + \gamma|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{c_1}) \nonumber
\]

\[
\otimes \left( \frac{1}{\sqrt{2}}(|H\rangle_{a_2} + |V\rangle_{a_2}) \right) \otimes \left( \frac{1}{\sqrt{2}}(|H\rangle_{c_2} + |V\rangle_{c_2}) \right) \nonumber
\]

\[
= |\alpha\rangle |V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{c_1} \otimes (|H\rangle_{a_2} |H\rangle_{c_2} + |H\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ |V\rangle_{a_2} |H\rangle_{c_2} + |V\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ \beta |H\rangle_{a_1} |V\rangle_{b_1} |H\rangle_{c_1} \otimes (|H\rangle_{a_2} |H\rangle_{c_2} + |H\rangle_{a_2} |V\rangle_{c_2} \nonumber
\]

\[
+ |V\rangle_{a_2} |H\rangle_{c_2} + |V\rangle_{a_2} |V\rangle_{c_2} \right). \quad (5)
\]

Then Alice and Charlie both let his two photons pass through their PCM gates respectively to make a parity
check measurement. If they both pick up the even parity states, the state of Eq. (6) collapses to

\[
|\Psi\rangle \to |\Psi\rangle' = \alpha|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}|V\rangle_{a2}|H\rangle_{c2} + \beta|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}|V\rangle_{a2}|H\rangle_{c2} + \gamma|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|V\rangle_{a2}|V\rangle_{c2},
\]

with a probability of 1/4. Certainly, there are other cases with the same probability that Alice picks up the even parity states but Charlie picks up the odd parity states, or Alice picks up the odd parity states but Charlie picks up the even parity state. Also they will both pick up the odd parity states

\[
\to \alpha|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}|V\rangle_{a2}|V\rangle_{c2} + \beta|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}|V\rangle_{a2}|V\rangle_{c2} + \gamma|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|V\rangle_{a2}|V\rangle_{c2}.
\]

If Alice or Charlie picks up the odd parity states, he or she performs a bit-flip operation \(\sigma_x = |H\rangle\langle V| + |V\rangle\langle H|\) on one of his or her photon to change it into the both even parity state in Eq. (6). Then we can rewrite the state \(|\Psi\rangle'\) under the two orthogonal basis

\[
|\varphi_1\rangle_a = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}|H\rangle + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}|V\rangle,
\]

\[
|\varphi_1^+\rangle_a = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}|H\rangle - \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}|V\rangle,
\]

and

\[
|\varphi_2\rangle_c = \frac{\gamma}{\sqrt{\gamma^2 + \beta^2}}|H\rangle + \frac{\beta}{\sqrt{\gamma^2 + \beta^2}}|V\rangle,
\]

\[
|\varphi_2^+\rangle_c = \frac{\beta}{\sqrt{\gamma^2 + \beta^2}}|H\rangle - \frac{\gamma}{\sqrt{\gamma^2 + \beta^2}}|V\rangle,
\]

i.e.,

\[
|\Psi\rangle' = \frac{\alpha \beta \gamma}{\sqrt{\alpha^2 + \beta^2} \sqrt{\gamma^2 + \beta^2}}(|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1} + |H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} + |H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1} + |V\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}).
\]
Eq. (10), if Alice and Charlie obtain the states $|\varphi_1\rangle_a$ and $|\varphi_2\rangle_c$, respectively when they measure their photons in the spatial modes $d2$ and $e2$, the remaining photon pair is essentially the maximally entangled W state $|\Phi\rangle_{a1b1c1} = \frac{1}{\sqrt{3}}(|V\rangle|H\rangle|H\rangle + |H\rangle|V\rangle|H\rangle + |H\rangle|H\rangle|V\rangle)$. The success probability is

$$P_0 = \frac{3\alpha^2\beta^2\gamma^2}{(\alpha^2 + \beta^2)(\beta^2 + \gamma^2)}. \quad (11)$$

Otherwise, if Alice and Charlie obtain the other states, they cannot get the maximally entangled state. For example, if Alice obtains $|\varphi_1\rangle_a$ and Charlie obtains $|\varphi_2^\perp\rangle_c$, they will get another lesser-entangled state

$$|\Phi_1\rangle_{a1b1c1} = \left(\frac{\alpha\beta^2}{\sqrt{\alpha^2 + \beta^2}}|V\rangle|H\rangle_{b1}|H\rangle_{c1} + \frac{\alpha\beta^2}{\sqrt{\alpha^2 + \beta^2}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} - \frac{\alpha\gamma^2}{\sqrt{\alpha^2 + \beta^2}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}\right), \quad (12)$$

with the probability of

$$P_1 = \frac{2\alpha^2\beta^4 + \alpha^2\gamma^4}{(\alpha^2 + \beta^2)(\beta^2 + \gamma^2)}. \quad (13)$$

It can be written as

$$|\Phi_1\rangle_{a1b1c1} = \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1} + \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} - \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}. \quad (14)$$

If Alice obtains the state $|\varphi_1^\perp\rangle_a$ and Charlie obtains the state $|\varphi_2\rangle_c$, they will get

$$|\Phi_2\rangle_{a1b1c1} = -\frac{\alpha^2}{\sqrt{\alpha^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1} + \frac{\beta^2}{\sqrt{\alpha^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} + \frac{\beta^2}{\sqrt{\alpha^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}, \quad (15)$$

with the probability of

$$P_2 = \frac{\alpha^2\gamma^2 + 2\beta^4\gamma^2}{(\alpha^2 + \beta^2)(\beta^2 + \gamma^2)}. \quad (16)$$

If Alice obtain the state $|\varphi_1^\perp\rangle_a$ and Charlie obtains the state $|\varphi_2^\perp\rangle_c$, they will get

$$|\Phi_3\rangle_{a1b1c1} = -\frac{\alpha^2}{\sqrt{\alpha^4 + \beta^4 + \gamma^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1} + \frac{\beta^2}{\sqrt{\alpha^4 + \beta^4 + \gamma^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} + \frac{\gamma^2}{\sqrt{\alpha^4 + \beta^4 + \gamma^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}. \quad (17)$$

with the probability of

$$P_3 = \frac{\alpha^4\beta^2 + \beta^6 + \beta^2\gamma^4}{(\alpha^2 + \beta^2)(\beta^2 + \gamma^2)}. \quad (18)$$

FIG. 3: The relationship between the entanglement transformation efficiency $\eta_{AB}$ of the present ECP and the initial coefficient of the less-entangled state $\alpha^\prime$, when $\gamma = \frac{1}{\sqrt{3}}$. $\eta_{AB}$ can reach 1 when $\alpha = \beta = \gamma = \frac{1}{\sqrt{3}}$.

Interestingly, all states of Eqs. (14), (15) and (17) are the lesser-entangled W states. We first compare the state of Eq. (17) with the original less-entangled state of Eq. (3). They have essentially the same form. That is, we choose $\alpha^\prime = \frac{\alpha^2}{\sqrt{\alpha^4 + 2\beta^4}}$, $\beta^\prime = \frac{\beta^2}{\sqrt{\alpha^4 + 2\beta^4}}$ and $\gamma^\prime = \frac{\gamma^2}{\sqrt{\alpha^4 + 2\beta^4}}$. In order to obtain the maximally entangled W state from Eq. (17), they only need to perform the ECP in a second round, following the same principle described above.

Now let us discuss another states of Eqs. (14) and (15). Both $|\Phi_1\rangle_{a1b1c1}$ and $|\Phi_2\rangle_{a1b1c1}$ are the lesser-entangled states, but they are different from the $|\Phi_3\rangle_{a1b1c1}$, because they essentially only have two different coefficients. They also can be concentrated into a maximally entangled state. Interestingly, this kind of states are much easier to be concentrated than the original state $|\Phi\rangle_{a1b1c1}$, because one of them only need one single-photon to perform this ECP. We take state of Eq. (14) as an example. If they obtain Eq. (14), they first perform a phase-flip operation and make it become

$$|\Phi_1\rangle_{a1b1c1} = \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1} + \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1} + \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}. \quad (19)$$
Then source of $S_2$ emits a single photon to Charlie, and the four-photon state can be written as

$$|\Phi'_c\rangle_{a1b1c1} \otimes \frac{1}{\sqrt{2}}(|H\rangle_{c2} + |V\rangle_{c2})$$

$$= \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}$$

$$+ \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}$$

$$+ \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1} \otimes \frac{1}{\sqrt{2}}(|H\rangle_{c2} + |V\rangle_{c2})$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}|H\rangle_{c2}ight.$$

$$+ \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}|H\rangle_{c2}

$$+ \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|V\rangle_{c2}

$$+ \frac{1}{\sqrt{2}}\left(\frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|H\rangle_{c2}ight.$$

$$+ \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|V\rangle_{c1}|H\rangle_{c2}

$$+ \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|V\rangle_{c2}\right). \quad (20)$$

After passing through the PCM, if Charlie picks up the even parity state, Eq. (20) collapses to

$$\rightarrow \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}|H\rangle_{c2}$$

$$+ \frac{\beta^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}|H\rangle_{c2}

$$+ \frac{\gamma^2}{\sqrt{\gamma^4 + 2\beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}|V\rangle_{c2}. \quad (21)$$

We can rewrite above state under the orthogonal basis

$$|\varphi_2\rangle'_{c} = \frac{\gamma^2}{\sqrt{\gamma^4 + \beta^4}}|H\rangle + \frac{\beta^2}{\sqrt{\gamma^4 + \beta^4}}|V\rangle,$$

$$|\varphi_2\rangle'_{c} = \frac{\beta^2}{\sqrt{\gamma^4 + \beta^4}}|H\rangle - \frac{\gamma^2}{\sqrt{\gamma^4 + \beta^4}}|V\rangle. \quad (22)$$

Then Charlie lets his photon in mode $c2$ pass through the PBS$_2$, whose optical axis is placed at the angle $\varphi'_2$, and two detectors to complete the measurement on the photon in the mode $c2$. Here $\cos\varphi'_2 = \frac{\gamma^2}{\sqrt{\gamma^4 + \beta^4}}$ and $\sin\varphi'_2 = -\frac{\beta^2}{\sqrt{\gamma^4 + \beta^4}}$. If Charlie obtains the state $|\varphi_2\rangle'_{c}$, they will obtain the maximally entangled state. If Charlie obtains $|\varphi_2\rangle'_{c}$, they will obtain another lesser-entangled state of the form

$$\rightarrow \frac{\beta^4}{\sqrt{\gamma^4 + 2\beta^4}\sqrt{\gamma^4 + \beta^4}}|V\rangle_{a1}|H\rangle_{b1}|H\rangle_{c1}$$

$$+ \frac{\beta^4}{\sqrt{\gamma^4 + 2\beta^4}\sqrt{\gamma^4 + \beta^4}}|H\rangle_{a1}|V\rangle_{b1}|H\rangle_{c1}$$

$$- \frac{\gamma^4}{\sqrt{\gamma^4 + 2\beta^4}\sqrt{\gamma^4 + \beta^4}}|H\rangle_{a1}|H\rangle_{b1}|V\rangle_{c1}. \quad (23)$$

Eq. (23) is also a lesser-entangled W state which can be reconcentrated into a maximally entangled W state in a third round. Certainly, from Eq. (20), if Charlie obtains an odd parity state, he needs to perform a bit-flip operation. Then the following steps are the same as described above. In this way, they can ultimately obtain a maximally entangled W state by repeating this ECP. On the other hand, if the measurements results of the first time performed by Alice and Charlie are $|\varphi_1\rangle_{a}$ and $|\varphi_2\rangle_{c}$, and lead to the collapsed state of Eq. (15), it can also be reconcentrated in a second round and only Alice needs to perform the ECP similar to Charlie’s operation described above. That is, all the remaining lesser-entangled states can be reconcentrated. If the remaining state is Eq. (14), only Charlie needs to repeat this ECP. If the remaining state is Eq. (15), only Alice needs to repeat this ECP, and if the remaining state is Eq. (17), both Alice and Charlie should repeat the whole ECP, following the same principle described above.

### IV. ENTANGLEMENT TRANSFORMATION EFFICIENCY

Thus far, we have fully described this ECP. It is well known that LOCC cannot increase entanglement. Therefore, this ECP under LOCC must be the entanglement transformation. Similar to Ref. [43], we can calculate the entanglement transformation efficiency after performing this ECP. In Ref. [43], we calculated the entanglement transformation efficiency for GHZ state concentration. However, different from the GHZ state, the W state has the different entanglement structure from the GHZ state. The entanglement of GHZ state is a global entanglement.
That is, if we trace over any one of the particle, the remaining particles do not entangle anymore. But the W state is the entanglement between each particles. If we trace over any one of the particle, the remaining two particles still entangle. Therefore, for a W state described in Eq. (3), the entanglement of three-tangle is equal to 0. In this paper, we use the concurrence described in Refs. \[68, 72\] to calculate the entanglement between each particles.

Before performing the ECP, the concurrence between particles A and B in Eq. (3) is \(C_{AB} = 2|\alpha\beta|\). The concurrence between particles A and C in Eq. (3) is \(C_{AC} = 2|\alpha\gamma|\), and the concurrence of the two subsystem A and BC is \(C_{A(BC)} = 2|\alpha|\sqrt{|\beta|^2 + |\gamma|^2}\). We denote the entanglement transformation efficiency as

\[
\eta_{AB} = \frac{C'_{AB}}{C_{AB}},
\]

where

\[
C'_{AB} = \frac{2}{3}P_0 + \frac{2\beta^4}{\gamma^4 + 2\beta^4}P_1 + \frac{2\alpha^2\beta^2}{\alpha^4 + 2\beta^4}P_2 + \frac{2\alpha^2\beta^2}{\alpha^4 + \beta^4 + \gamma^4}P_3.
\]

(25)

The \(C'_{AB}\) is the concurrence between particle A and B after performing this ECP. From Eq. (25), it comprises four terms. The first term \(\frac{2}{3}P_0\) means that after concentration, they obtain the maximally entangled W state with the probability of \(P_0\) and it describes the concurrence between A and B for the maximally entangled W state. The second term and other terms are analogy with the first one. For instance, the second term means that...
with the probability of $P_1$, they obtain the state of Eq. (13), and item $\gamma^2 + 2\beta^2$ describes the concurrence between A and B if they obtain such state. $C'_{AC}$ and $C'_{A(BC)}$ are analogy with $C'_{AB}$. $C'_{AC}$ means the concurrence between particles A and C after concentration and $C'_{A(BC)}$ means the concurrence between subsystem A and BC after performing the ECP.

$$C'_{AC} = \frac{2}{3}P_0 + \frac{2\beta^2 \gamma^2}{\gamma^2 + 2\beta^2}P_1 + \frac{2\alpha^2 \beta^2}{\alpha^2 + 2\beta^2}P_2 + \frac{2\alpha^2 \gamma^2}{\alpha^2 + \beta^2 + \gamma^2}P_3.$$  

(26)

$$C'_{A(BC)} = \frac{2\sqrt{2}}{3}P_0 + \frac{2\beta^2}{\sqrt{\gamma^2 + 2\beta^2}}\sqrt{\frac{\beta^4}{\gamma^2 + 2\beta^2} + \frac{\beta^4}{\gamma^2 + 2\beta^2}}P_1 + \frac{2\alpha^2}{\sqrt{\alpha^2 + 2\beta^2}}\sqrt{\frac{\beta^4}{\alpha^2 + 2\beta^2} + \frac{\beta^4}{\alpha^2 + 2\beta^2}}P_2 + \frac{2\alpha^2}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}\sqrt{\frac{\beta^4}{\alpha^2 + \beta^2 + \gamma^2} + \frac{\beta^4}{\alpha^2 + \beta^2 + \gamma^2} + \frac{\gamma^4}{\alpha^2 + \beta^2 + \gamma^2}P_3}.$$  

(27)

We calculate the entanglement transformation efficiency $\eta_{AB}$, $\eta_{AC}$ and $\eta_{A(BC)}$ altered by the coefficient $\alpha$. Here, we choose $\gamma = \frac{1}{\sqrt{3}}$, change $\alpha \in (0, \sqrt{\frac{1}{3}})$. The relationship between the coefficient $\alpha^2$ and entanglement transformation efficiency $\eta_{AB}$ is shown in Fig. 3. Fig. 4 and Fig. 5 show the relationship between $\alpha^2$ and $\eta_{AC}$ and $\eta_{A(BC)}$, respectively. It is interesting to see that if $\alpha = \beta = \gamma = \frac{1}{\sqrt{3}}$, the $\eta_{AB} = \eta_{AC} = \eta_{A(BC)} = 1$. The $\eta_{AB}$, $\eta_{AC}$, and $\eta_{A(BC)}$ are not fixed values but all increases with the initial entanglement. They can reach the maximally value 1 when $\alpha = \frac{1}{\sqrt{3}}$. These result are consistent with the result shown in Ref. [45]. Interestingly, we also calculate the entanglement transformation efficiency $\eta_{AB}$, $\eta_{AC}$ and $\eta_{A(BC)}$ altered by the coefficient $\alpha$ when $\gamma = \frac{1}{3}$. In this case, the original state cannot reach the maximally entangled state. From Fig. 6, $\eta_{AB}$ can also reach the max value 1 when $\alpha^2 = \beta^2 = \frac{1}{2} = 1$. But $\eta_{AC}$ cannot reach 1 when we change $\alpha$, shown in Fig. 7. Our numerical simulation shows that the max value $\eta_{AC} \approx 0.7078$ when $\alpha \approx 0.6757$. In Fig. 8, we calculate the $\eta_{A(BC)}$ alters with $\alpha^2$. It is shown that $\eta_{A(BC)} = 1$ when $\alpha^2 = \frac{15}{37}$, which is similar to Fig. 6.

V. DISCUSSION AND SUMMARY

Thus far, we have fully explained this ECP. In this paper, the most important element for us to complete this ECP is the PCM shown in Fig. 1, constructed by cross-Kerr nonlinearity. At present, in the optical single-photon regime, it is still a quite-controversial assumption to have a clean cross-Kerr nonlinearity. Natural cross-Kerr nonlinearity is only $\tau \approx 10^{-18}$ in the optical single-photon regime [72, 74]. On the other hand, Hofmann et al. pointed out that with a single two-level atom in a one-side cavity, a $\pi$ phase shift can be reached [73]. Fortunately, the PCM in our ECP does not require a strong nonlinearity and it works for weak vales of the cross-Kerr coupling. Gea-Banacloche argued that it is impossible to obtain a giant Kerr effect with a single-photon wave packet [76], which is consistent with the Shapiro and Razavi [77, 78]. He et al. showed that the high fidelities, nonzero conditional phases and high photon numbers are compatible if the transverse-mode effects can be suppressed [64]. Feizpour et al. also showed that it is possible to amplify a cross-Kerr phase shift to an observable value [79]. Recently, Zhu and Zhang discussed that coupled with a four-level, double A-type configuration, giant cross-Kerr nonlinearities was also obtained with nearly vanishing optical absorption [80].

In summary, we present an efficient ECP for an arbitrary three-photon W state. Alice and Charlie exploit the same PCM based on the cross-Kerr nonlinearity to perform this ECP. By iterating this protocol several times, it can reach a higher success probability. This ECP is also quite different from the conventional ECPs for W state. First, it is focused on an arbitrary W state and does not resort to the unitary evolution. Second, it only requires single photons of the same form $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. Third, this PCM can works on the weak regime of cross-Kerr nonlinearity, which is suitable for current technology. We hope this ECP maybe useful in a practical quantum information processing.

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[59] H. Jeong, and N. B. An, Phys. Rev. A 74, 022104 (2006).
[60] G. S. Jin, Y. Lin, B. Wu, Phys. Rev. A 75, 054302 (2007).
[61] Y. M. Li, K. S. Zhang, and K. C. Peng, Phys. Rev. A 77, 015802 (2008).
[62] B. He, J. A. Bergou, and Y.-H. Ren, Phys. Rev. A 76, 032301 (2007); B. He, M. Nadeem, and J. A. Bergou, ibid. 79, 035802 (2009); B. He, Y.-H. Ren, and J. A. Bergou, ibid. 79, 052323 (2009); J. Phys. B 43, 025502 (2010).
[63] B. He, Q. Lin, and C. Simon, Phys. Rev. A 83, 053826 (2011).
[64] B. He and J. A. Bergou, Phys. Rev. A 78, 062328 (2008).
[65] Q. Lin and J. Li, Phys. Rev. A 79, 022301 (2009).
[66] Q. Lin and B. He, Phys. Rev. A 80, 042310 (2009); Q. Lin, B. He, J. A. Bergou, and Y.-H. Ren, ibid. 80, 042311 (2009); Q. Lin and B. He, ibid. 80, 062312 (2009); Q. Lin and B. He, ibid. 82, 022331 (2010); Q. Lin and B. He, ibid. 82, 064303 (2010).
[67] Q. Guo, J. Bai, L. Y. Cheng, X. Q. Shao, H. F. Wang, and S. Zhang, Phys. Rev. A 83, 054303 (2011).
[68] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[69] M. Plesch, and V. Bužek, Phys. Rev. A 68, 012313 (2003).
[70] B. K. Zhao, F. G. Deng, F. S. Zhang, and H. Y. Zhou, Phys. Rev. A 80, 052106 (2009).
[71] B. K. Zhao, and F. G. Deng, Phys. Rev. A 82, 014301 (2010).
[72] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[73] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowing, and G. J. Milburn, Rev. Mod. Phys. 79, 135 (2007).
[74] P. Kok, H. Lee, and J. P. Dowling, Phys. Rev. A 66, 063814 (2002).
[75] H. F. Hofmann, K. Kojima, S. Takeuchi, and K. Sasaki, J. Opt. B 5, 218 (2003).
[76] J.Gea-Banacloche, Phys. Rev. A 81, 043823 (2010).
[77] J. H. Shapiro, Phys. Rev. A 73, 062305 (2006).
[78] J. H. Shapiro and M. Razavi, New J. Phys. 9, 16 (2007).
[79] A. Feizpour, X. Xing, and A. M. Steinberg, Phys. Rev. Lett. 107, 133603 (2011).
[80] C. Zhu and G. Huang, Optics Express, 19, 23364 (2011).