NEUTRINO OSCILLATIONS and ENERGY-MOMENTUM CONSERVATION

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ABSTRACT

A description of neutrino oscillation phenomena is presented which is based on relativistic quantum mechanics with 4-momentum conservation. This is different from both conventional approaches which arbitrarily use either equal energies or equal momenta for the different neutrino mass eigenstates. Both entangled state and source dependence aspects are also included. The time dependence of the wavefunction is found to be crucial to recovering the conventional result to second order in the neutrino masses. An ambiguity appears at fourth order which generally leads to source dependence, but the standard formula can be promoted to this order by a plausible convention.

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I. Introduction

Lipkin[1] has raised (and continues to raise) questions regarding the standard derivation of neutrino oscillation phenomena[2, 3], which has some unsatisfactory assumptions. While it is straightforward to verify that assuming equal energies or equal momenta both give the same result for the oscillation length[3, 4], it is unclear why either starting point is valid. However, it seems that the view taken in Ref.[1] that oscillations are a momentum/spatial phenomenon and do not involve energy/time is at least incomplete, if not wrong[5]. Lipkin’s very first example claims that the energy of a $K^0$ produced in a two body reaction is fixed by energy conservation, so the momenta of the $K_L$ and $K_S$ components must be different. However, the momentum is also fixed by a conservation law. How then does one get around this conundrum, since the states observed in experiments certainly propagate long enough distances to be considered on-shell? I present here an analysis which includes elements of covariance and entangled states which do not seem to have been discussed in the literature for the particular case of neutrino oscillations. Questions of wavepackets or coherence, which have been discussed in detail in Refs.(2, 3), are applied ex post facto.

II. Calculation

We commence with a decay source of mass $M$ which emits neutrino flavor eigenstates and consider it in its rest frame, with no momentum dispersion. Such a state must be infinitely spread out in space, but we will return to the question of packeting the wavefunction later. The ket transformation is then

$$|M, 0, M^2\rangle \rightarrow c|\sqrt{k^2 + m_1^2}, \bar{k}, m_1^2\rangle \times |E_1, -\bar{k}, M_f^2\rangle + s|\sqrt{q^2 + m_2^2}, \bar{q}, m_2^2\rangle \times |E_2, -\bar{q}, M_f^2\rangle$$

(1)

where $m_{1,2}$ are the masses of the neutrino mass eigenstates, $c, s$ are the cosine and sine of the mixing angle, $\theta$, between the mass and flavor eigenstates, $M_f$ is the mass of the recoiling final state of the source, $\bar{k}, \bar{q}$ are the three-momenta of the two neutrino mass eigenstates, and $E_{1,2}$ are the energies of the recoiling source (daughter) final states for each neutrino mass eigenstate.

It is important to recognize that the neutrino oscillation here is determined by a state preparation, which is in turn defined by the recoil mass of the entangled component. A fixed invariant mass of that component does not define which neutrino mass eigenstate component has been produced, but allows both mass eigenstates to appear. It is only if we measure the energy or momentum of the recoil entangled state separately to sufficient accuracy that projection of one particular neutrino mass eigenstate occurs. In the normal quantum mechanical way, usually discussed in terms of two slit gedanken experiments, when the interference occurs, it is that of this single neutrino wave (specified by $M_f^2$) with itself.

To keep a concrete picture in mind, one may think of the example of a source initial state consisting of a pion and the corresponding recoiling final state of a muon. However, the formulation is quite general, and simply reflects the fact that sufficiently accurate measurement of the recoiling component of the entangled states can in principle allow a particular mass eigenstate to be determined, eliminating any oscillations in the usual quantum mechanical manner. If the source is more complicated, such as a $\beta$-decaying nucleus, or a muon, we need only integrate over the allowed range of $M_f$, which is the invariant mass constructed from the sum of the four-momenta of the recoiling particles: (possibly excited) final state nucleus plus electron or ‘other’ neutrino plus electron, in the two example cases. The argument here is focused on interference of neutrinos for the case with a given, fixed recoiling invariant mass because that is clearly a definable, (radiative corrections to the charged particle recoil notwithstanding,) and
experimentally common case, as in two body weak decay of charged pions. In the normal quantum mechanical fashion, we now implicitly integrate over the recoiling states (over all energies and momenta but with fixed invariant total mass), and find the evolved neutrino ket after time $t$ is

$$|\nu, t\rangle = c|k, m_1\rangle e^{-i(\sqrt{k^2+m_1^2} t-k \cdot \vec{x})} + s|q, m_2\rangle e^{-i(\sqrt{q^2+m_2^2} t-q \cdot \vec{x})}$$

(2)

where we have assumed the momentum and position vectors are parallel to simplify the notation. Using the $t = 0$ definition of the flavor eigenstate to describe the amplitude for detecting it at the event $(t, \vec{x})$, we have

$$\langle \nu, 0 | \nu, t \rangle = c^2 e^{-i(E_k t-k \cdot \vec{x})} + s^2 e^{-i(E_q t-q \cdot \vec{x})}$$

(3)

with the obvious simplifications in notation. Note that the event describes the registration of the neutrino in the experimental detection apparatus, and has, in principle, nothing to do (for quantum waves) with the transit time from the source to the location of the detection in the apparatus, for sufficiently broad wavepackets. Indeed, if the latter relation could be made precise, the interference would be destroyed just as by the sufficiently accurate detection of energy or momentum of the recoiling source final state.

There may be concern here that the delta-function momenta of the recoiling final state components eliminates the coherence of the two terms in Eq.(2) and so obviates Eq.(3). Indeed, a density-matrix view would suggest as much. However, there would be no question about this issue if the source were in a packet of finite energy-momentum width, such as is necessary to localize the source in the first place. What is being done here is to analyze the contributions of such a packet by each (source) spectral line within it, so as to make energy-momentum conservation manifest. As a result, even if the recoiling state were measured with sufficient accuracy to distinguish the two components, still no determination could be made since it would remain unknown from which spectral component the momenta originated. Very recently, questions have been raised regarding possible interferences from the different spectral components of the initial state \[7, 8\]. That question is not addressed here.

A few kinematical relations are now useful. Representing either momentum case by $\vec{p}$ and either neutrino mass by $m$, we have

$$\sqrt{\vec{p}^2 + M_f^2} + \sqrt{\vec{p}^2 + m^2} = M$$

(4)

which can be solved for the neutrino energy to show that

$$\sqrt{\vec{p}^2 + m^2} = \frac{M^2 - M_f^2 + m^2}{2M}$$

(5)

Thus, we have explicitly for the neutrino energies:

$$E_k = \sqrt{k^2 + m_1^2} = \frac{M}{2} - \frac{M_f^2}{2M} + \frac{m_1^2}{2M}$$

(6)

and

$$E_q = \sqrt{q^2 + m_2^2} = \frac{M}{2} - \frac{M_f^2}{2M} + \frac{m_2^2}{2M}$$

(7)
We now want the differences between these energies and momenta, and it is convenient to expand all cases about the averages of each pair, which we do to fourth order in the neutrino masses. For the energy we have

\[ E_{\nu}^{av} = \frac{M^2 - M_f^2}{2M} + \frac{m_1^2 + m_2^2}{4M} \]

and for the momentum, we solve Eqs. (6,7), for \( k \) and \( q \) respectively, and perform a Taylor expansion to fourth order in the neutrino masses

\[ k \approx \frac{M^2 - M_f^2}{2M} - \frac{m_1^2 (M^2 + M_f^2)}{2M (M^2 - M_f^2)} - \frac{m_1^4 M^2 M_f^2}{M (M^2 - M_f^2)^3} \]

and

\[ q \approx \frac{M^2 - M_f^2}{2M} - \frac{m_2^2 (M^2 + M_f^2)}{2M (M^2 - M_f^2)} - \frac{m_2^4 M^2 M_f^2}{M (M^2 - M_f^2)^3}, \]

then average, to obtain

\[ p_{\nu}^{av} = \frac{M^2 - M_f^2}{2M} - \frac{(m_1^2 + m_2^2) (M^2 + M_f^2)}{4M (M^2 - M_f^2)} - \frac{(m_1^2 + m_2^2) M^2 M_f^2}{2M (M^2 - M_f^2)^3}. \]

We next define

\[ \Delta E = \frac{\Delta m^2}{2M} \]

and

\[ \Delta p = -\frac{\Delta m^2 (M^2 + M_f^2)}{2M (M^2 - M_f^2)} - \frac{\Delta m^2 (m_1^2 + m_2^2) MM_f^2}{(M^2 - M_f^2)^3} \]

where \( \Delta m^2 = m_2^2 - m_1^2 \), so that

\[ E_{q,k} = E_{\nu}^{av} \pm \frac{\Delta E}{2} \]

and

\[ p_{q,k} = p_{\nu}^{av} \pm \frac{\Delta p}{2}. \]

We now factor out the average phase quantities, to rewrite Eq. (3) as

\[ \langle \nu, 0|\nu, t \rangle = e^{-i \phi} \left( e^{2s(\Delta E - \Delta p)} + s^2 e^{-i(\Delta E + \Delta p)} \right) \]

where

\[ \phi = (E_{\nu}^{av} t - p_{\nu}^{av} x) = \left[ \frac{M^2 - M_f^2}{2M} - \frac{m_1^2 + m_2^2}{4M} \right] t - \left[ \frac{M^2 - M_f^2}{2M} - \frac{m_1^2 + m_2^2 M^2 + M_f^2}{4M (M^2 - M_f^2)} - \frac{m_1^2 + m_2^2 M^2 M_f^2}{2M (M^2 - M_f^2)^3} \right] x. \]
Squaring Eq. (16) and substituting from Eqs. (12, 13) now gives the persistence probability, \( P(t, x) \), of the initial neutrino flavor to fourth order in the neutrino masses as

\[
P(t, x) \equiv |\langle \nu, 0 | \nu, t \rangle|^2 = c^4 + s^4 + 2c^2s^2 \cos \left( \frac{\Delta m^2 t}{2M} + \frac{\Delta m^2 x}{2M} \left( \frac{M^2 + M_f^2}{M^2 - M_f^2} + \frac{2(m_1^2 + m_2^2)M^2 M_f^2}{(M^2 - M_f^2)^3} \right) \right).
\]

(18)

Experimentally, one usually measures \( x \) and implicitly infers the value of \( t \) to compare with this formula.\[1\] A question has been raised in other contexts\[9\] of what relation is the correct one to use in this regard. Here we evaluate the relation by averaging the classical velocities of the two components in a conventional manner, assuming, as usual, that the \( m_i^2 \) are tiny compared to \( M^2, M_f^2 \), and \( M^2 - M_f^2 \), and that this reflects the motion of centroids of wavepackets. To zeroth order in the neutrino masses, this sets \( t = x \) (in units where \( c = 1 \)). We will need the result to second order in the neutrino masses, and find it by defining the average velocity

\[
v_{av} = \frac{1}{2} \left( \frac{k}{E_k} + \frac{q}{E_q} \right)
\]

\[= 1 - \frac{(m_1^2 + m_2^2)M^2}{(M^2 - M_f^2)^2},\]

(19)

where we have used Eqs. (6, 7, 9, 10) and then set \( t = t_{av}(x) \) where

\[
t_{av}(x) = x/v_{av}
\]

\[= x \left[ 1 + \frac{(m_1^2 + m_2^2)M^2}{(M^2 - M_f^2)^2} \right].\]

(20)

We emphasize that this is only necessary to go beyond leading order, and note that the same result may be obtained to this order from \( p_{av}^\nu / E_{av}^\nu \).

Using the usual trigonometric identities for oscillations in Eq. (18) and substituting for \( t \) from Eq. (20), we finally obtain

\[
P(t_{av}(x), x) = 1 - \sin^2(2\theta) \sin^2 \left( \frac{\Delta m^2 x}{4M} \left[ \frac{2M^2}{M^2 - M_f^2} \times \left( 1 + \frac{(m_1^2 + m_2^2)(M^2 + M_f^2)}{2(M^2 - M_f^2)^2} \right) \right] \right).
\]

(21)

III. Comparison with Conventional Results

To compare this formula with the conventional result, obtained by assuming that either the energies or momenta of the two neutrino components are the same, recall that\[2-4\]

\[P_{conv.}(x) = 1 - \sin^2(2\theta) \sin^2 \left[ \frac{\Delta m^2 x}{4k} \right].\]

(22)

where the usual relativistic approximations are used that \( k \approx E_\nu \) and the signal in the detector occurs at \( t = x \), and which can be derived assuming either that the two neutrino components have the same energy or that they have the same momentum.
To complete the comparison, we simply need to recognize that, in Eq. (21), \( 2M/(M^2 - M_f^2) = 1/p_{\nu}^\ast = 1/E_{\nu}^\ast \) to leading order. Hence the result to second order in the neutrino masses is identical to that produced by the conventional analyses. That is,

\[
P(x, x)_{2nd \, order} = P_{\text{conv.}}(x).
\]

(23)

It is also interesting to extend the interpretation of the result, Eq. (21), to fourth order in the neutrino masses. To do this, we need to recognize that the correction factor multiplying the second order result,

\[
1 + \frac{(m_1^2 + m_2^2)(M^2 + M_f^2)}{2(M^2 - M_f^2)^2},
\]

is precisely that needed to promote the factor of \( 2M/(M^2 - M_f^2) \), interpreted as \( 1/p_{\nu}^\ast \), from its zeroth order value in the neutrino masses to second order, as can be seen from Eq.(11). This would not be true if we were to interpret the factor \( 2M/(M^2 - M_f^2) \) as \( 1/E_{\nu}^\ast \). Thus, there are no fourth order effects at all if we define the neutrino oscillation formula as

\[
P(t_{av}(x), x) = 1 - \sin^2(2\theta) \sin^2 \left\{ \frac{\Delta m^2 x}{4p_{\nu}^\ast} \right\}.
\]

(24)

One could also argue that the procedure used here of averaging the velocities of the two neutrinos to infer a transit time is incorrect. (Perhaps this definitional ambiguity of the detection time is related to what Lipkin\[1\] has in mind.) Certainly the quantum mechanical interference will disappear if the value of the transit time were, in fact, measured to sufficiently high levels of accuracy, as one may then separate (in principle) the two velocity components. More simply, one could ask if the averaging should not be weighted by the relative amplitudes (or probabilities) of the two neutrino mass eigenstates.\[10\] To illustrate the level of sensitivity, we can look at an extreme version of this question: What if we were to take \( t = x \) and correspondingly identify the energy/momentum factor \( (E_0 = (M^2 - M_f^2)/(2M)) \) that both neutrinos would have if they were massless? This is reasonable since the neutrino energy is generally not known accurately in experiments. We would then have

\[
P(x, x) = 1 - \sin^2(2\theta) \sin^2 \left\{ \frac{\Delta m^2 x}{4E_0} \left[ 1 + \frac{(m_1^2 + m_2^2)M_f^2}{(M^2 - M_f^2)^2} \right] \right\}
\]

(25)

where \( E_0 \) is the energy given by Eq.(10) or Eq.(14) at zero neutrino mass.

The result Eq. (25) is source dependent: The coefficient of the sum of neutrino squared masses varies by almost three orders of magnitudes from 170 GeV\(^{-2}\) for muon-neutrinos from \( \pi \rightarrow \mu \) to 0.21 GeV\(^{-2}\) for those from \( K \rightarrow \mu \) two body decay. In view of the experimental bounds on the neutrino masses themselves, however, it seems unlikely that any such higher order source term dependence effect would be experimentally measurable in the foreseeable future.

The discussion has been limited here to the two neutrino mixing case for simplicity. The extension to the case of three neutrino mixing is immediate. See for example Ref.(11).

IV. Conclusion

Starting from a detailed representation of the neutrino source, and fixing only the invariant mass of the recoiling component of resulting entangled state, the evolution of the neutrino amplitude consisting of two mass eigenstates has been shown to produce, to second order in the neutrino eigenmasses, the same oscillation relation as obtained by the usual (but unjustified)
assumption of either equal momenta or energy for the two neutrino components. Note that the variations due to the time between the production of the neutrino by the source and its detection, and the distance from the source to the point of detection must both be included to reproduce the usual result, contrary to the assertions in Ref. [1].

Dependence on the source, through the invariant mass of the final state, does not develop even at the level of fourth order terms in the neutrino masses, provided the standard neutrino oscillation formula is defined in terms of the average momentum of the two (or more) neutrino mass eigenstates. Use of the momentum that would be carried by a zero mass neutrino will induce apparent fourth order corrections which are certainly negligibly small for present day experiments. Nonetheless, we note that these effects can be determined precisely when the source final state recoil invariant mass is itself defined precisely, as in the case of two body decay sources (such as pions).

The calculation discussed here has been carried out for a neutrino source at rest in the same frame as that of the neutrino detector. However, the boost of that source relative to the detector rest frame can only affect the energy and momentum of the observed neutrino by the standard relativistic transformation. Therefore the results will also be applicable to all currently achievable experimental conditions. The extension to cases where the neutrino energies deriving from the source are nonrelativistic, (that is, for $M^2 - M^2_f \approx m^2$,) does not seem likely to be of use at present [12].

Related discussions, in the more accessible case of neutral kaon oscillations, may be found in Refs. [13, 14].

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Note added. This paper first appeared as "Source Dependence of Neutrino Oscillations", hep-ph/9604357v1 but is only slightly modified here. Shortly after this work was completed, Refs. [16, 17] were brought to my attention. In the first, the approach is to treat the states involved as off-shell. From my analysis, this does not seem to be necessary. It should be noted, however, that the paper carefully justifies the assumption, used here, that the spatial position difference vector from source to detection point is parallel to the momentum vector for the interfering neutrino amplitudes. In the second, pion decay with differing neutrino energies and momenta is considered. However, there the emphasis is placed on the time variation of the phase, [completely opposite to the view in Ref. [1]], with yet another averaging procedure (justified by an aside on the question of coherence) used to confirm the conventional leading order result. There have been too many papers since to include as references. The only possible problem raised here appears independently of these issues and only at fourth order in the neutrino masses.
a muon from a decay with a recoiling electron which only occurs with a branching ratio of $\approx 10^{-4}$ anyway.

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