On the Extension Complexity of Polytopes Separating Subsets of the Boolean Cube

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Abstract
We show that for every $A \subseteq \{0, 1\}^n$, there exists a polytope $P \subseteq \mathbb{R}^n$ with $P \cap \{0, 1\}^n = A$ and extension complexity $O(2^{n/2})$, and that there exists an $A \subseteq \{0, 1\}^n$ such that the extension complexity of any $P$ with $P \cap \{0, 1\}^n = A$ must be at least $2^{n(1-o(1))/3}$. We also remark that the extension complexity of any 0/1-polytope in $\mathbb{R}^n$ is at most $O(2^n/n)$ and pose the problem whether the upper bound can be improved to $O(2^{cn})$, for $c < 1$.

Keywords Polytopes · Extension complexity · Boolean cube · Sign patterns

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1 Introduction

A polytope $P \subseteq \mathbb{R}^n$ with many facets can often be expressed as a projection of a higher-dimensional polytope $Q \subseteq \mathbb{R}^m$ with much fewer facets. This is especially significant in the context of linear programming: instead of optimizing a linear function over $P$, it is more efficient to optimize over $Q$. Extension complexity of $P$ is defined as the smallest $k$ so that $P$ is an affine image of a polytope with $k$ facets. Extension complexity has been studied in the seminal paper of Yannakakis [25], Fiorini et al. [10], Rothvoß [22], and others. In [22], Rothvoß has shown that there exist 0/1-polytopes in $\mathbb{R}^n$ with...
extension complexity $2^{n(1-o(1))/2}$; in fact, a random polytope has this property. Our paper originated with the question whether the bound of Rothvoß is tight.

**Problem 1.1** Can every $0/1$-polytope $P$ be expressed as a projection of a polytope with $O(2^n)$ facets, for a constant $c < 1$?

Note that $P$ itself can have many more than $2^n$ facets [5]. Extension complexity, however, can be bounded by the number of vertices and hence the trivial upper bound is $2^n$. In Problem 1.1, we want to know which of the bounds, $2^{n/2}$ vs. $2^n$, is closer to the truth. This is reminiscent of a similar problem in $\mathbb{R}^2$. In [10], Fiorini et al. have shown that there exist polygons in $\mathbb{R}^2$ with $k$ vertices and extension complexity $\Omega(\sqrt{k})$. Quite surprisingly, Shitov [23] has shown that every $k$-vertex polygon has extension complexity $O(k^{2/3})$. Furthermore, there is an $O(k^{1/2})$ upper bound for several natural classes of polygons [19].

Problem 1.1 is related to a similar question about graphs. Given an $n$-vertex graph, let $P_G \subseteq \mathbb{R}^n$ be the convex hull of characteristic vectors of its edges. An explicit description of $P_G$ in terms of inequalities is known [17, 20], and it is especially simple in the case of bipartite graphs. The trivial upper bound on $\text{xc}(P_G)$ is $O(n^2)$. We point out that any improvement on this trivial bound gives an improvement on extension complexity of $0/1$-polytopes. This reduction is similar to the so-called graph complexity (see [16, 21]) where an $n$ variate Boolean function is interpreted as defining a graph on exponentially many vertices. Extension complexity of $P_G$ has been investigated by Fiorini et al. in [8], where a non-trivial upper bound $O(n^2/\log n)$ was presented (cf. [3]). This translates to a modest contribution to Problem 1.1: $P$ is a projection of a polytope with $O(2^n/n)$ facets. The latter bound has also been obtained by Averkov et al. in [4].

We mainly focus on a relaxation of Problem 1.1. Given $A \subseteq \{0, 1\}^n$, a polytope $P \subseteq \mathbb{R}^n$ will be called *separating* for $A$ if $P \cap \{0, 1\}^n = A$. In other words, $P$ separates Boolean points in $A$ from those outside of $A$. The smallest separating polytope is $\text{conv}(A)$ itself. Extension complexity of separating polytopes has several connections with computational complexity, as extensively discussed in [12]. Here we show that every $A \subseteq \{0, 1\}^n$ has a separating polytope with extension complexity $O(2^{n/2})$. This is achieved by the aforementioned reduction to graphs, and by showing that the set of edges of $G$ has a separating polytope of linear size. This quantitatively matches the lower bound of Rothvoß—except that the assumptions are different. There are infinitely many separating polytopes other than $\text{conv}(A)$ itself and so the lower bound is not applicable. In [13], a lower bound of $2^{\Omega(n)}$ on extension complexity of separating polytopes has been given. The constant in the exponent hinges on known bounds on quantifier elimination and it is not hard to see that the proof from [13] gives $2^{n(1-o(1))}/5$.

We will improve this bound to $2^{n(1-o(1))/3}$ using a more geometrical argument.

**Organization of the paper.** In Sect. 2 we give basic definitions and state the main result (Theorem 2.1). In Sect. 3 we give examples of bounds on the number of facets of separating polytopes. In Sect. 4 we prove the upper bound (Theorem 4.2) and discuss the connection with graph complexity. Finally, in Sect. 5 we prove the lower bound (Theorem 5.7).
2 The Main Result

A polytope $P \subseteq \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$. It can also be viewed as a bounded set defined by a finite number of linear constraints. The extension complexity of a polytope $P$, $\text{xc}(P)$, is the smallest $k$ so that there exists a polytope $Q \subseteq \mathbb{R}^m$ with $k$ facets and an affine map $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $P = \pi(Q)$. Given $A \subseteq \{0, 1\}^n$, its separation complexity, $\text{sep}(A)$, is the minimum $\text{xc}(P)$ over all polytopes $P \subseteq \mathbb{R}^n$ with

$$P \cap \{0, 1\}^n = A;$$

such a $P$ is called a separating polytope for $A$. We provide non-trivial upper and lower bounds on $\text{sep}(A)$:

**Theorem 2.1** (i) For every $A \subseteq \{0, 1\}^n$, $\text{sep}(A) \leq O(2^{n/2})$.
(ii) There exists $A \subseteq \{0, 1\}^n$ with $\text{sep}(A) \geq 2^{n(1-o(1))/3}$.

**Remark 2.2** In [12, 13], separation complexity is defined slightly differently with $P$ allowed to be an unbounded polyhedron. This is just a cosmetic detail—we can intersect $P$ with $[0, 1]^n$ (or a simplex containing it) which increases its complexity by an additive term of $O(n)$.

As observed in [12, 25], a Boolean circuit of size $s$ which accepts precisely the inputs from $A$ gives a separating polytope for $A$ with extension complexity $O(s + n)$. Hence an upper bound of $O(2^n/n)$ on $\text{sep}(A)$ can be obtained from known upper bounds on Boolean circuit size due to Lupanov (see [15, Sect. 1.4.1]). The upper bound from Theorem 2.1 is more intimately related to a result of Dančík [7] who has obtained an $O(2^{n/2})$ upper bound for depth three circuits with unbounded fan-in $\land, \lor$-gates. Our proof is virtually the same though it does not seem to follow from Dančík’s result in a black box fashion. A lower bound of $2^{\Omega(n)}$ on separation complexity has been obtained in [13]. The estimate presented in Theorem 2.1 is quantitatively stronger.

3 Simple Bounds on the Number of Facets

We first give some elementary bounds on the number of facets of separating polytopes. This is mainly to contrast it both with extension complexity and the number of facets of $0/1$-polytopes. These results\(^1\) have been previously obtained by Jeroslow [14], and we include them for completeness.

**Proposition 3.1** Every $A \subseteq \{0, 1\}^n$ has a separating polytope with at most $2^n$ facets.

**Proof** For $x \in \mathbb{R}^n$ and $\sigma \in \{0, 1\}^n$, define

$$h_\sigma(x) := \sum_{i=1}^n x_i (1 - \sigma_i) + (1 - x_i)\sigma_i.$$

\(^1\) In fact, Jeroslow obtains the tight estimate $2^n - 1$ in Proposition 3.1 while allowing the separating polyhedron to be unbounded.
If $x$ is Boolean, $h_{\sigma}(x)$ is the Hamming distance between $x$ and $\sigma$. Define $P \subseteq \mathbb{R}^n$ by the constraints $h_{\sigma}(x) \geq 1$ for every $\sigma \in \{0, 1\}^n \setminus A$. Then indeed $P \cap \{0, 1\}^n = A$. $P$ may be possibly unbounded. This can be remedied by adding the constraints $h_{\sigma}(x) \geq 0$, $\sigma \in A$. 

Let $ODD_n \subseteq \{0, 1\}^n$ be the set of Boolean strings with odd number of ones.

**Proposition 3.2** If $n \geq 2$, every separating polytope for $ODD_n$ has at least $2^{n-1}$ facets.

**Proof** Let $H$ be a closed half-space $\{x \in \mathbb{R}^n : \sum_i a_i x_i \geq b\}$. We claim the following: if $ODD_n \subseteq H$ then $\tilde{H} := \mathbb{R}^n \setminus H$ contains at most one even string $\sigma \in \{0, 1\}^n \setminus ODD_n$. To see this, assume an even $\sigma$ is contained in $\tilde{H}$. Without loss of generality, we can assume that $\sigma$ is the zero vector; otherwise apply an affine map that flips 0 and 1 for the 1-coordinates of $\sigma$. Since $\sigma \notin H$, we must have $b > 0$. Since every unit vector is in $ODD_n \subseteq H$, we have $a_i \geq 0$ for every $i$. This means that $\{0, 1\}^n \setminus \{\sigma\} \subseteq H$ and no other even string can be in $\tilde{H}$.

If $n = 2$, the statement of the proposition is clear. Let $n \geq 3$ and assume that $P$ is a separating polytope for $ODD_n$ with $r$ facets. Then $P = \bigcap_{i=1}^r H_i$ where $H_i$ are closed half-spaces. (This is because $P$ is full-dimensional for $n \geq 3$). We have $ODD_n \subseteq H_i$ for every $i$, and every even $\sigma$ is contained in at least one $H_i$. Since $|\{0, 1\}^n \setminus ODD_n| = 2^{n-1}$, this gives $r \geq 2^{n-1}$. 

By the result of Bárány [5], $A$ can have $2^{\Omega(n \log n)}$ facets—hence Proposition 3.1 shows that a separating polytope for $A$ can have much fewer facets than $\text{conv}(A)$. The convex hull of $ODD_n$, also known as the parity polytope, has extension complexity $O(n)$ (see [6]), and it is trivially a separating polytope for $ODD_n$—hence Proposition 3.2 shows that taking extensions into a higher dimension can be exponentially powerful. It also shows that the $O(2^{n/2})$ upper bound from Theorem 2.1 cannot be achieved simply by counting the facets of the separating polytope.

### 4 The Upper Bound

We now prove the upper bound from Theorem 2.1. Let $B_2^n \subseteq \{0, 1\}^n$ be the set of Boolean vectors of Hamming weight two (i.e., with exactly two ones). For a natural number $n$, $[n]$ will denote the set $\{1, \ldots, n\}$.

**Lemma 4.1** Let $H \subseteq B_2^n$. Then there exists a polytope $R_H \subseteq [0, 1]^n \subseteq \mathbb{R}^n$ with at most $2n$ facets such that $R_H \cap [0, 1]^n = H$.

**Proof** It is convenient to view $H$ as representing edges of a graph with vertex set $[n]$. Namely, $i \neq j$ are adjacent iff $e_i + e_j \in H$, where $e_i$ is the $i$-th unit vector. Given $i \in [n]$, let $N(i)$ be the set of vertices adjacent with $i$. Let $R_H$ be defined by the following constraints: $0 \leq x_i$ for $i \in [n]$ and

\[
\sum_{i \in [n]} x_i = 2, \quad (1)
\]
\[
\sum_{j \in N(i)} x_j \geq x_i, \quad i \in [n]. \quad (2)
\]
There are $2n$ inequalities. It is easy to see they imply $x_i \leq 1$ for every $i \in [n]$ and so $R_H \subseteq [0, 1]^n$. Given $e_i + e_j \in H$, the constraints defining $R_H$ are satisfied and so $H \subseteq R_H$. If $\sigma \in \{0, 1\}^n \setminus R_H$ then either $\sigma \notin B^n_2$, and then $\sigma$ falsifies (1), or $\sigma = e_i + e_j$ with $j \notin N(i)$, and then $\sigma$ falsifies $x_i \leq \sum_{k \in N(i)} x_k$. Hence $R_H \cap \{0, 1\}^n = H$. \hfill $\Box$

**Theorem 4.2** Let $A \subseteq \{0, 1\}^n$. Then there exists a polytope $P \subseteq [0, 1]^n$ with $xc(P) = O(2^{n/2})$ and $P \cap \{0, 1\}^n = A$.

**Proof** Without loss of generality, assume that $n$ is even and $N := 2^{n/2}$. Assume $A \subseteq \{0, 1\}^n$ and partition $[n]$ into two equal parts $X_1$ and $X_2$. Let $F_1 := \{0, 1\}^{X_1}$ and $F_2 := \{0, 1\}^{X_2}$. Hence every $\sigma \in \{0, 1\}^n$ can be uniquely written as $\sigma = \sigma_1 \cup \sigma_2$ with $\sigma_1 \in F_1$, $\sigma_2 \in F_2$. We identify $\mathbb{R}^{2N}$ with $\mathbb{R}^{F_1 \cup F_2}$, so that the coordinates are indexed by elements of $F_1 \cup F_2$. The standard unit vectors are $e_{\sigma_1}, e_{\sigma_2}, \sigma_1 \in F_1, \sigma_2 \in F_2$. Let $H \subseteq B_2^{2N}$ be defined as

$$H := \{e_{\sigma_1} + e_{\sigma_2} : \sigma_1 \cup \sigma_2 \in A\}.$$  

Let $R_H$ be the polytope from the previous lemma. We want to express $P$ in terms of $R_H$. Let $T \subseteq \mathbb{R}^{2N}$ be the intersection of $[0, 1]^{2N}$ with the hyperplanes

$$\sum_{\sigma_1 \in F_1} x_{\sigma_1} = 1, \quad \sum_{\sigma_2 \in F_2} x_{\sigma_2} = 1. \quad \text{(3)}$$

Let $\pi : \mathbb{R}^{2N} \to \mathbb{R}^n$ be the linear map so that for every $\sigma_1 \in F_1, \sigma_2 \in F_2, \pi(e_{\sigma_1}) = \sigma_1 \cup 0$ and $\pi(e_{\sigma_2}) = 0 \cup \sigma_2$ (where 0 is the zero vector in $F_2$ and $F_1$, respectively). This guarantees

$$\pi(e_{\sigma_1} + e_{\sigma_2}) = \sigma_1 \cup \sigma_2.$$

Moreover, for every $\sigma \in \{0, 1\}^n$ with $\sigma = \sigma_1 \cup \sigma_2$, $e_{\sigma_1} + e_{\sigma_2}$ is the unique vector $x \in T$ with $\pi(x) = \sigma$. For if $\pi(x) = \sigma$, we have $\sum_{\beta \in F_1} x_{\beta} = 1$ and $\sum_{\beta \in F_1} x_{\beta} = 1$, $0 \leq x_\beta \leq 1$, by (3). In other words, $x$ gives a convex combination of $\sigma_1$ in terms of the Boolean vectors in $F_1$ which is easily seen to be unique. (Similarly for $\sigma_2$). Finally, let $P := \pi(R_H \cap T)$. Then $xc(P) \leq 2N$. Given $\sigma \in \{0, 1\}^n$, we have $\sigma \in P$ if and only if $e_{\sigma_1} + e_{\sigma_2} \in R_H$. By the definition of $H$, this is equivalent to $\sigma \in A$. Hence indeed $P \cap \{0, 1\}^n = A$. \hfill $\Box$

### 4.1 Graphs and Problem 1.1

Given a (simple undirected) graph $G$ with vertex set $[n]$, let $P_G \subseteq \mathbb{R}^n$ be the convex hull of characteristic vectors of edges in $G$:

$$P_G = \text{conv} \{(e_i + e_j : i \neq j \text{ are adjacent in } G)\}.$$  

Note that $P_G$ has at most $\binom{n}{2}$ vertices and so $xc(P_G)$ is at most quadratic. Fiorini et al. in [8] have given an improved bound $xc(P_G) \leq O(n^2/\log n)$ for any graph. It is however

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not known whether $\text{xc}(P_G) \leq O(n^c)$ for some constant $c < 2$. We summarize the connection between this problem and Problem 1.1 as follows:\footnote{The first part of the statement can also be found in [4].}:

**Proposition 4.3** Let $A \subseteq \{0, 1\}^n$. Then $\text{xc}(\text{conv}(A)) \leq O(2^n/n)$. Moreover, assume that for every bipartite graph $G$ on $2m$ vertices (with the parts of equal size), $\text{xc}(P_G) \leq O(m^c)$, where $c \leq 2$ is an absolute constant. Then $\text{xc}(A) \leq O(2^{cn/2})$

**Proof** This is analogous to the proof of Theorem 4.2. The set $H$ corresponds to a bipartite graph $G$ on $2N$ vertices with $N = 2^n/2$. The projection $\pi$ maps vertices of $P_G \subseteq T$ to Boolean vectors in $\mathbb{R}^n$. Hence $\pi(P_G) = \text{conv}(A)$ and $\text{xc}(\text{conv}(A)) \leq \text{xc}(P_G)$. From [8, Lem. 3.4] we know $\text{xc}(P_G) \leq O(N^2/\log N)$ which gives $\text{xc}(\text{conv}(A)) \leq O(2^n/n)$; the “moreover” part is similar. \hfill $\square$

An explicit description of $P_G$ in terms of linear inequalities can be found in [17, 20]. Apart from the general constraints $\sum x_i = 2$, $0 \leq x_i$, every inequality $\sum_{i \in S} x_i \leq 1$ is valid whenever $S$ is an independent set. In the case of bipartite $G$, this indeed gives a complete description of $P_G$. This also means that $P_G$ can have exponentially many facets—in particular, the polytope from Lemma 4.1 must be strictly larger than $P_G$ for some $G$.

The lemma can be somewhat strengthened when considering independent sets of size 2. Let $Q_G$ be the polyhedron defined by the constraints $x_i + x_j \leq 1$ for every $i \neq j$ not adjacent in $G$. Clearly, $P_G \subseteq Q_G$ and they contain the same set of Boolean vectors of Hamming weight two (i.e., the edges of $G$).

**Remark 4.4** Let $G$ be a bipartite $n$-vertex graph. Then there exists a polytope $R'_G$ with $O(n)$ facets with $P_G \subseteq R'_G \subseteq Q_G$.

**Proof** Let $L$ and $R$ be the parts of $G$ with $L \cup R = [n]$. Then $R'_G$ defined by the following constraints has the desired properties: $0 \leq x_i$ for $i \in [n]$,

$$\sum_{i \in L} x_i = 1, \quad \sum_{i \in R} x_i = 1, \quad \text{and} \quad x_i + \sum_{j \in R \setminus N(i)} x_j \leq 1, \quad i \in L. \quad \square$$

Using the machinery of non-negative rank factorizations of slack matrices (see, e.g., [9, 22, 25]), the quantity $\text{xc}(P_G)$ can be captured by the nonnegative rank of an explicit matrix $\text{EIS}_G$: its rows are indexed by edges $e$ of $G$, columns by independent sets $S$. The entry corresponding to $e$ and $S$ equals 1, if $e$ and $S$ are disjoint, and 0 otherwise. This matrix is intimately related to the famous Clique vs Independent Set problem of Yannakakis [25]; see also [11]. If $G$ is bipartite, $\text{xc}(P_G)$ corresponds to the non-negative rank of $\text{EIS}_G$. An interesting submatrix of $\text{EIS}_G$ is the $\text{ENE}_G$ matrix obtained by restricting the columns to independent sets of size two (i.e., non-edges). A similar matrix has been considered in [18] from the point of view of communication complexity. Since $\text{ENE}_G$ can have size $O(n^2) \times O(n^2)$, one may perhaps hope to obtain quadratic lower bounds on $\text{xc}(P_G)$ using the non-negative rank of $\text{ENE}_G$. We note that this is impossible.\footnote{This could also be concluded from Remark 4.4.}
Remark 4.5 If \( G \) is a bipartite \( n \)-vertex graph, \( \text{ENE}_G \) can be written as a sum of \( O(n) \) 0/1-matrices. For a non-bipartite \( G \), the bound is \( O(n \log n) \).

**Proof** Let \( G \) be a bipartite graph on vertices \( L \cup R \). Let \( E \) be the set of edges of \( G \) and \( \bar{E} \) the set of non-edges. Given \( \ell \in L, r \in R \), define the following sets \( A_\ell, B_r, C_\ell \subseteq E \times \bar{E} \) of edge/non-edge pairs.

- \( A_\ell \) consists of pairs with \( e = (\ell_1, r_1), \bar{e} = (\ell, r_2) \) with \( \ell \neq \ell_1 \in L, r_1, r_2 \in R \), and \( r_1 \in N(\ell) \).
- \( B_r \) consists of pairs \( e = (\ell_1, r), \bar{e} = (v_1, v_2) \), where either \( v_1 \neq v_2 \in R \setminus \{r\} \), or \( v_1 \in L, v_2 \in R \setminus \{r\} \), and \( v_1 \notin N(r) \).
- \( C_\ell \) are the pairs \( (\ell, r_1), (\ell_1, \ell_2) \) with \( \ell_1 \neq \ell_2 \in L \setminus \{\ell\} \).

It is easy to see that the sets form a partition of the set of disjoint edge/non-edge pairs. Moreover, each of the sets is a product set (of the form \( C \times C' \) with \( C \subseteq E, C' \subseteq \bar{E} \)). Identifying a subset of \( E \times \bar{E} \) with the 0/1-matrix representing its characteristic function, we can thus write

\[
\text{ENE}_G = \sum_\ell A_\ell + \sum_r B_r + \sum_\ell C_\ell,
\]

where the summands are 0/1-matrices of rank one. A general \( n \)-vertex graph can be expressed as an edge-disjoint union of a bipartite graph and two graphs with \( \lceil n/2 \rceil \) vertices, and we can proceed by induction. \( \square \)

## 5 The Lower Bound

Our proof of the lower bound from Theorem 2.1 uses Warren’s estimate on the number of sign patterns of a polynomial map, and Alon’s bound on the number of combinatorial types of polytopes. We overview these results first.

For \( b \in \mathbb{R} \) define

\[
\text{sgn}(b) = \begin{cases} 
1, & b > 0, \\
0, & b = 0, \\
-1, & b < 0.
\end{cases}
\]

For a sequence \( f = (f_1(y_1, \ldots, y_p), \ldots, f_s(y_1, \ldots, y_p)) \) of real functions, \( b \in \mathbb{R}^p \), let \( \text{sgn}(f(b)) := (\text{sgn}(f_1(b)), \ldots, \text{sgn}(f_s(b))) \in \{-1, 0, +1\}^s \), which we call the sign-pattern of \( f \) at \( b \). A result of Warren and its extension by Alon gives a bound on the number of sign patterns when \( f_i \) are polynomials of degree at most \( d \).

**Theorem 5.1** (Warren [24], Alon [2]) Let \( f \) be a sequence of \( s \) polynomials of degree at most \( d \geq 1 \) in the same set of \( p \) variables with \( 2s \geq p \). Then \( |\{\text{sgn}(f(b)) : b \in \mathbb{R}^p\}| \leq (8eds/p)^p \).

Given a polytope \( P \), the face lattice of \( P \), \( L(P) \), is the poset of the faces of \( P \) ordered by inclusion (including \( \emptyset \) and \( P \) itself). It is naturally equipped with join and meet operations.
operations, hence it is a lattice. See, e.g., [26] for details. The lattice-isomorphism
equivalence class of $L(P)$ captures the combinatorial type of $P$.

**Theorem 5.2** (Alon [1]) The number of non-isomorphic face lattices arising from
polytopes with $r$ vertices is at most $2^{r^3(1+o(1))}$.

By duality, this implies:

**Corollary 5.3** The number of non-isomorphic face lattices arising from polytopes with
$r$ facets is at most $2^{r^3(1+o(1))}$.

We now proceed to prove the lower bound from Theorem 2.1. We call a set $S \subseteq \mathbb{R}^n$
full-dimensional if no hyperplane in $\mathbb{R}^n$ contains $S$. Note that if $A$ is full-dimensional
then so is any separating $P$ for $A$.

**Lemma 5.4** There are at least $2^{2n(1-o(1))}$ full-dimensional subsets of $\{0, 1\}^n$.

**Proof** If $A$ contains 0 and the $n$ unit vectors, it is full-dimensional. There are $2^{2n} - n - 1$
such $A$’s.

**Lemma 5.5** For every $m \geq n$ there are polynomials $f_1, \ldots, f_s$ in $m n$ variables such
that

- $s = O(m^{n+1})$, each $f_i$ has degree at most $n$ and has at most $2^{O(n \log n)}$ non-zero
  coefficients,
- for every $V \in \mathbb{R}^{m \times n}$ viewed as $m$ points in $\mathbb{R}^n$, if $\text{conv}(V)$ is full-dimensional then
  the set $\text{conv}(V) \cap \{0, 1\}^n$ is uniquely determined by $(\text{sgn}(f_1(V)), \ldots, \text{sgn}(f_s(V)))$.

**Proof** We will construct a set of polynomials such that for any $V = \{v_1, \ldots, v_m\}$,
we can determine $\text{conv}(V) \cap \{0, 1\}^n$ by evaluating the signs of these polynomials
on $V$. The idea is as follows. For every set $V'$ of $n$ points from $V$, we can compute the
unique hyperplane $H$ passing through $V'$ (if such a unique hyperplane exists). If all
points in $V$ lie in the same closed half-space determined by $H$, then $\text{conv}(V) \cap H$ is
a facet of $\text{conv}(V)$. Let us call such closed half-space good. Then, given $\sigma \in \{0, 1\}^n$,
we can determine whether $\sigma \in \text{conv}(V)$ by checking whether it appears in all good
half-spaces.

We now formally define our set of polynomials. Given $S \in \binom{\mathbb{R}^{m \times n}}{n}$ and $V \in \mathbb{R}^{m \times n}$,
let $V_S$ be the set of vectors $\{v_i : i \in S\}$. We start by constructing the following
polynomials/sets of polynomials. They take $V_S$ as input, but we hide the dependence.

(i) $a_{S,1}, \ldots, a_{S,n}$ are polynomials of degree $n - 1$ such that $V_S$ is affinely independent
iff some $a_{S,i}$ is non-zero,

(ii) $b_S$ is a polynomial of degree $n$ such that whenever $V_S$ is affinely independent then
$H_S(V) := \{x \in \mathbb{R}^n : \sum_i a_{S,i} x_i = b_S\}$ is the unique hyperplane passing
through $V_S$,

(iii) $F_S$ is a set of $m - n$ polynomials of degree $n$ such that if $V_S$ is affinely independent,
then $\text{conv}(V) \cap H_S(V)$ is a facet of $\text{conv}(V)$ iff all polynomials in $F_S$ are all non-positive or all non-negative.

Parts (i) and (ii) are an exercise in linear algebra. $F_S$ is obtained by evaluating the
hyperplane equation from (ii) on all points from $V \setminus V_S$—the hyperplane defines a
 facet if all points in \( V \) lie on the same side. Let \( F \) be the set of polynomials containing \( F_S, a_{S,1}, \ldots, a_{S,n} \) and
\[
\sum_i a_{S,i} \sigma_i - b_S,
\]
for every \( S \in \binom{[m]}{n} \) and \( \sigma \in \{0, 1\}^n \). Then \( \text{conv}(V) \cap \{0, 1\}^n \) is uniquely determined by the signs of polynomials in \( F \). The number of polynomials is \((2^n + n + 1 + (m - n)) \binom{m}{n} \leq (2^n + m + 1)m^n/n! \leq O(m^{n+1})\) and their degrees are at most \( n \). The bound on the number of non-zero coefficients follows by noting that each polynomial depends on \( O(n^2) \) variables.

Before proceeding to the next lemma, let us make some comments about rational functions. Given a \( p \)-variate rational function \( f = g/h \) with \( g, h \) coprime polynomials and \( h \neq 0 \), define its degree as the maximum of the degrees of \( g \) and \( h \). Thus \( f \) defines a partial function \( \mathbb{R}^p \to \mathbb{R} \). Warren’s estimate can be extended to rational functions as follows. Given a rational map \( f = \langle f_1, \ldots, f_s \rangle \) from \( \mathbb{R}^p \) to \( \mathbb{R}^n \) with each \( f_i \) of degree at most \( d \geq 1 \), we have
\[
|\{ \text{sgn}(f(b)) : b \in \mathbb{R}^p, f(b) \text{ is defined} \}| \leq (cd^s)^p,
\]
where \( c > 0 \) is an absolute constant. This follows from Theorem 5.1 by considering signs of numerators and denominators separately.\(^4\)

Furthermore, we need the following estimate on the degree of composition. Suppose that \( f(x_1, \ldots, x_m) \) is a polynomial of degree \( d_1 \) with \( k \) non-zero coefficients and \( g_1, \ldots, g_m \) are rational functions of degree at most \( d_2 \). Then it is easy to see that the degree of \( f(g_1, \ldots, g_m) \) is at most \( kd_1d_2 \).

**Lemma 5.6** Let \( L \) be a face lattice of a \( d \)-dimensional polytope with \( r \geq d \) facets. Assume \( d \geq n \) and let \( S_L \) be the set of \( A \subseteq \{0, 1\}^n \) such that \( A \) is full-dimensional and there exists a polytope \( Q \) in \( \mathbb{R}^d \) with combinatorial type \( L \) such that the projection of \( Q \) on the first \( n \) coordinates is separating for \( A \). Then \( |S_L| \leq 2^{O(nr^3)} \).

**Proof** Consider a polytope \( Q \) in \( \mathbb{R}^d \) with face lattice \( L \). Since \( Q \) is full-dimensional, we can write it as \( \{ y \in \mathbb{R}^d : By \leq b \} \) where \( b \in \mathbb{R}^r \), \( B \in \mathbb{R}^{r \times d} \). Hence \( Q \) can be described using \( p := (r + 1)d \leq O(r^2) \) constants \( z = \langle B, b \rangle \). Let \( U \) be the vertices of \( Q \). Then \( |U| \leq 2^r \). Every vertex is the unique intersection of \( d \) hyperplanes defining facets of the polytope. Furthermore, the lattice \( L \) specifies for each vertex, which facets it is contained in and, moreover, which \( d \) of them have the desired unique intersection (see, e.g., [26]). For each vertex \( u \in U \), canonically pick \( d \) such facets. Then \( u \) is the unique solution to a system of \( d \) linear equations, and its coordinates can be seen as rational functions of \( z \). More exactly, using Cramer’s rule, we can write \( u(z) = u_0(z)^{-1}(u_1(z), \ldots, u_d(z)) \), where \( u_0, \ldots, u_d \) have degree \( d \). Note that \( u_0(z) \) is non-zero whenever the polytope described by \( z \) is indeed of type \( L \).

---

\(^4\) We also have no assumption on \( p \) since the number of sign patterns can be trivially bounded by \( 3^p \).
During the current study.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed at the Simons Institute for the Theory of Computing.

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Data Availability

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Theorem 5.7 There exists \( A \in \{0, 1\}^n \) such that \( \text{sep}(A) \geq 2^{n(1-o(1))/3} \).

Proof Let \( \mathcal{A} \) be the set of full-dimensional subsets \( A \subseteq \{0, 1\}^n \). Let \( r \geq n \) be such that every \( A \in \mathcal{A} \) has separation complexity at most \( r \). Without loss of generality, assume that this is exhibited by a full-dimensional polytope \( Q \subseteq \mathbb{R}^d \) with \( r \) facets such that the projection of \( Q \) on the first \( n \) coordinates is a separating polytope for \( A \), and \( n \leq d \leq r \). We then have

\[
|\mathcal{A}| \leq |\mathcal{L}| \cdot \max_{L \in \mathcal{L}} |\mathcal{S}_L|,
\]

where \( \mathcal{L} \) is the set of combinatorial types of polytopes with \( r \) facets. By Lemma 5.6, \( |\mathcal{S}_L| \leq 2^O(nr^3) \). By Corollary 5.3, we have \( |\mathcal{L}| \leq 2^{r^3(1+o(1))} \). Therefore \( |\mathcal{A}| \leq 2^{cnr^3} \) (for some constant \( c \) and \( n \) sufficiently large). By Lemma 5.4, we must have \( 2^{cnr^3} \geq 2^{2n(1-o(1))} \) and thus \( r \geq 2^{n(1-o(1))/3} \).

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Data Availability

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