Free vibrations of bevelled cone thin-walled constructions of variable thickness

Vladimir A Kozlov¹, Darya A Kashirina²

¹ Department of theoretical and applied mechanics. Voronezh state technical university, 20 let Oktyabrya st., 84, Voronezh, Russian Federation

² Department of theoretical and applied mechanics. Voronezh state technical university, 20 let Oktyabrya st., 84, Voronezh, Russian Federation

E-mail: v.a.kozlov1@yandex.ru

Abstract. One of the possible approaches to dynamic calculation of arbitrary profile supported by conic shell restrained on bevelled end is presented in the paper. Resolving system of ordinary differential equations is obtained in the frame of technical theory of shells with the help of Lagrange variation principle. Boundary value problem on definition of natural frequencies and oscillations forms of thin-walled construction of swept-back box beam of wing with shell variable thickness is solved by the finite-difference method of increased accuracy. The results of calculation of oscillation frequencies in dependence of shell geometrical parameters are tabulated; the forms of longitudinal oscillations are presented graphically. Unlike the well-known works the variable thickness of a construction in length is under consideration. The change of thin elastic construction thickness influences greatly on characteristic values of frequencies and forms of free oscillations which should be account while designing the construction elements of public and transport mechanical engineering, building constructions.

Thin cone shell with rectilinear generatrix supported by transversal (ribs) and longitudinal (stringers) ensemble of regular structure is the object of investigation in the suggested paper. Having skewness simply - or multiply connected support profile of free configuration, variable thickness closing and cross-sectional area of longitudinal reinforcement set such a shell is a universal design model of thin-wall constructions different types including swept-back box beam of the aircraft wing.

The solutions are made in the frame of technical theory of thin shells based on Lagrange variation principle [1]. The suggested approach while determining deflected mode of thin-walled system under analysis in some cases allowed obtaining both closed analytical [2-6] and numerical [7-9] solutions of corresponding boundary problems.

Imagine the vector of elastic displacement $\vec{U}$ as it is in the work [1] as the sum of two vector-functions

$$\vec{U}(\vec{Z},S) = \vec{U}^0(\vec{Z},S) + \vec{U}^*(\vec{Z},S),$$

(1)
dependent on shell curvilinear coordinates: \( \bar{Z} \) is a longitudinal coordinate, presenting the distance from the guide plane and calculated along \( l \); \( S \) – parameter calculated shell contour.

**Figure 1.** Shell design module.

In decomposition (1) the first function \( \bar{U}^0 \) corresponds with derived special displacement of the profile \( \bar{Z} = \text{const} \) as a solid and answers to the rather a rough but physically obvious conception about the construction operation character determining the “generalized” law of plane section. The second vector-function \( \bar{U}^* \) is an amendment to \( \bar{U}^0 \) and determines the profile warping. The expression for \( \bar{U}^0 \) can be written down as the final sum

\[
\bar{U}^0(\bar{Z}, S) = \sum_{i=1}^{6} V_i(\bar{Z}) \cdot \bar{\phi}_i(S). \tag{2}
\]

Here the first three unknown scalar functions \( V_i(\bar{Z}) \) are progressive shift of the counter \( \bar{Z} = \text{const} \) along the corresponding axes \( i = 1,2,3 \), three others \( i = 4,5,6 \) are its turns around these axes, \( \bar{\phi}_i(S) \) are the unknown vector-functions.

Concerning the structure of the vector-functions \( \bar{U}^*(\bar{Z}, S) \) do some hypotheses. Let the shell is supported by a transverse set of ribs located with a fairly small step. This set plays a role of additional bracings preventing the distortion of the shell profile in the plane of the diaphragms as the last ones are suggested to be hard in their plane and relocatable from it. The more so as it follows from [1], the deflected mode of the shell less depends on ribs orientation: downstream or normal to generatrix. That is why let the direction \( \bar{U}^* \) be along the forming shell. Thus by analogy with (2) imagine the depletion vector-function \( \bar{U}^*(\bar{Z}, S) \) as the sum of the products of unknown scalar functions \( V_i(\bar{Z}) \) by the system of designed coordinate vector-functions \( \bar{\phi}_i = \bar{\phi}_i(S) \)

\[
\bar{U}^*(\bar{Z}, S) = \sum_{i=1}^{6} V_i(\bar{Z}) \phi_i(S) \bar{u}_i, \tag{3}
\]

where \( n \) – the number of t contour freedom \( \bar{Z} = \text{const} \) at its warping;
\( \phi_{i1} = \bar{\phi}_i \cdot \bar{t} \) \( \phi_{i2} = \bar{\phi}_i \cdot \bar{r} \) \( \phi_{im} = \bar{\phi}_i \cdot \bar{\eta} \);
\( \mathbf{t}_1, \mathbf{t}_2, \mathbf{n} \) – unit vector corresponding to curvilinear coordinates \((Z, S)\).

In the case of shell vibration the unknown colligated displacements \( V_i \), except the coordinate \( Z \) also depend on time \( t \). Finally with account of decomposition (2), (3) the vector of elastic displacement present as

\[
\bar{U}(Z, S, t) = \sum_{i=1}^{6+n} V_i(Z, t) \cdot \check{\phi}_i(S),
\]

where \( V_i \) is called colligated displacements and corresponding to them vector-functions \( \check{\phi}_i \) – colligated coordinates.

There will be received the decisive system of differential equations based on the Lagrangian principle of virtual displacement

\[
\delta A - \delta \Pi = 0,
\]

where \( \delta A \) – virtual work of external forces, \( \delta \Pi \) – variation of potential energy.

Let the shell be under the influence of external load \( \bar{p} \) including inertial forces:

\[
\bar{p} = p^0 + p^U \bar{U},
\]

where \( p^0 \) – vector of designed external load; \( p^U \) – operator of inertial force of a shell,

\[
p^U = -\rho \frac{\partial^2}{\partial t^2}; \quad \rho \quad \text{– material mass density.}
\]

With account of (4) there is received

\[
p^U \bar{U} = -\rho h \sum_{i=1}^{6+n} \frac{\partial^2 V_i}{\partial t^2} \cdot \check{\phi}_i.
\]

Virtual work of external force

\[
\delta A = \oint_0 \bar{p}^0 \cdot \delta \bar{U} \sin \chi dS \bar{Z} + \oint_0 (p^U \bar{U}) \cdot \delta \bar{U} \sin \chi dS \bar{Z} + \\
+ \oint \bar{q}_0 \cdot \delta \bar{U} dS \bigg|_{Z=0} + \oint \bar{q}_1 \cdot \delta \bar{U} dS \bigg|_{Z=Z_1},
\]

where \( \bar{q}_0(S), \bar{q}_1(S) \) – vectors of external load per unit of a profile length applied to the ends \( Z = 0 \) and \( Z = Z_1 \),

Accordingly to (4) \( \delta \bar{U} = \sum_{i=1}^{6+n} \partial V_i \cdot \check{\phi}_i \). Then

\[
\delta A = \int_0^{Z} \sum_{i=1}^{6+n} R_i^0 \partial V_i d\bar{Z} + \int_0^{Z} \sum_{i=1}^{6+n} R_i^U \partial V_i d\bar{Z} + \sum_{i=1}^{6+n} P_i^0 \partial V_i \bigg|_{Z=0}^{Z=Z_1},
\]

where with account of expressions \( A = l, B = \bar{Z} = 1 - \bar{Z} \)

\[
R_i^0 = \bar{Z} \oint \bar{p}^0 \cdot \check{\phi}_i l \sin \chi dS, \quad R_i^U = \bar{Z} \oint \left(p^U \bar{U}\right) \cdot \check{\phi}_i l \sin \chi dS; \quad P_i^0(0) = \oint \bar{q}_0 \cdot \check{\phi}_i dS, \quad P_i^0(Z_1) = \bar{Z} \oint \bar{q}_1 \cdot \check{\phi}_i dS.
\]
In the accepted geometrical model tangential force factors play a decisive role [1]. Thus a variation of potential energy with an account of tangential forces is determined by the expression

\[
\delta \Pi = \int_0^Z \left( T_1 \partial \varepsilon_1 + T_2 \partial \varepsilon_2 + T^2 \partial \varepsilon^2 \right) A B \sin \chi dS dZ = \int \sum_{i=1}^{6+n} (P_i \partial V_i' + Q_i \partial V_i) dZ,
\]

(10)

where \( P_i(\bar{Z}) = G \left[ \bar{Z} \sum_{j=1}^{6+n} a_{ij} V_j' + \sum_{j=1}^{6+n} b_{ij} V_j \right] \), \( Q_i(\bar{Z}) = G \left[ \sum_{j=1}^{6+n} b_{ij} V_j' + \frac{1}{\bar{Z}} \sum_{j=1}^{6+n} c_{ij} V_j \right] \);

\[
G = E / 2(1 + \nu) - \text{shear modulus};
\]

\[
a_{ij} = \int \left\{ \frac{2}{1 - \nu} \left[ \phi_{i1} \phi_{j1} - \text{ctg} \chi \left( \phi_{i1} \phi_{j1}^2 + \phi_{j1}^2 \phi_{j1} \right) + \text{ctg}^2 \chi \phi_{i1}^2 \phi_{j1}^2 \right] \right\} \sin \chi \ h dS;
\]

\[
b_{ij} = \int \left[ \frac{2}{1 - \nu} \left( \nu \phi_{i1} - \text{ctg} \chi \phi_{i1}^2 \right) \right] \left\{ \phi_{j1}^2 - \frac{\phi_{j1}}{l} - \frac{\phi_{j1}}{R_0} \right\} \sin \chi h dS;
\]

\[
c_{ij} = \int \left[ \frac{2}{1 - \nu} \left( \frac{\phi_{j1}^2}{\sin \chi} - \frac{\phi_{j1}}{l} - \frac{\phi_{j1}}{R_0} \right) \phi_{j1}^2 - \frac{\phi_{j1}}{l} - \frac{\phi_{j1}}{R_0} \right] \sin \chi h dS.
\]

Integrating in (10) the first term under the sign of the sum in parts we get

\[
\delta \Pi = \int_{Z=0}^{Z_1} \sum_{i=1}^{6+n} (P_i' + Q_i) \partial V_i dZ + \sum_{i=1}^{6+n} P_i \partial V_i \left[ Z - Z_i \right].
\]

(11)

Revealing the variation equation (5) with account of (8) and (11) there is received

\[
\int_{Z=0}^{Z_1} \sum_{i=1}^{6+n} (P_i' - Q_i + R'_i + R_i^0) \partial V_i dZ + \sum_{i=1}^{6+n} (P_i' - P_i) \partial V_i \left|_{Z=0}^{Z_1} \right. = 0.
\]

(12)

As the variations of colligated displacements \( \partial V_i \) on an interval of integration are optional and independent the first expressions should be equal to zero. That is why the system of decisive differential equations and natural boundary conditions come from (12)

\[
P_i' - Q_i + R'_i + R_i^0 = 0,
\]

(13)

and natural boundary conditions

\[
(P_i' - P_i) \partial V_i \left|_{Z=0}^{Z_1} = 0 \quad i = 1,2,\ldots,6+n. \right.
\]

(14)

Revealing the equation (13) there is received
where \( \tilde{Z} = 1 - \frac{Z}{Z} \);

coefficients \( d_{ij} \) are found from the expression

\[
d_{ij} = \frac{1}{G} \int (\varphi_i \varphi_j + \varphi_i^2 \varphi_j^2 + \varphi_i \varphi_j \varphi_{ij} \varphi_{ij}) \rho \sin \chi \mathrm{d}S.
\]

General decision of the system (15) contains 2(6+n) of arbitrary constants found from the natural boundary conditions (14) of the variation problem. If the displacements \( V_i^* \) are given on one of the ends then at a corresponding value \( \tilde{Z} \) we have

\[
V_i = V_i^*, \quad i = 1,2,...,6+n.
\]  

If there are force factors \( P_i^* \) on one of the ends the boundary conditions (14) are formed as the equality of unknown colligated forces \( P_i \) with specified value \( P_i^* \)

\[
G \left( \tilde{Z} \sum_{j=1}^{6+n} a_{ij} V_j^* + \sum_{j=1}^{6+n} b_{ij} V_j^* \right) \bigg|_{Z=\tilde{Z}_i} = P_i^*(\tilde{Z}_i), \quad i = 1,2,...,6+n.
\]  

So the system of equations (15) and boundary conditions (16), (17) allows solving the problems of self-and forced oscillations of the shells under analysis. In contrast to [1] for the shells of variable rigidity the system of equations (15) contains variable coefficients and does \( h = h(\tilde{Z}) \) not limited to the types of Euler equations.

Base on the presented theory there is solved a boundary value problem for determination of the frequencies and forms of oscillations of conic shell of variable thickness rigidly pinched at the beveled edge. A decisive system of differential equations of free motions for a conic shell with a profile section symmetrical about the axes \( Ox \) and \( Oy \) is written down as

\[
\tilde{Z} a_{ij} U_i^* + \tilde{Z}^2 a_{ij} U_i^* + \tilde{Z} a_{ij} U_i^* + (\tilde{Z}^2 a_{ij}) Y^* + (\tilde{Z}^2 a_{ij}) Y^* + \tilde{Z} d_{ij} \frac{\partial^2 U_i^*}{\partial t^2} = - R^0_i \quad i = 1,2,\ldots,6+n,
\]  

where \( \tilde{Z} = 1 - \frac{Z}{Z} \);
\[ +[(\xi^2_2 a_{48})' - \xi(b_{48} + a_{48}/l_0)]U_6' + [(\xi^2_2 a_{68})' - \xi(b_{68} + a_{68}/l_0)]U_6' + \\
+ (b_{07} - b_{08})U_7' + (\xi a_{08})U_8 - [b_{48}/l_0 - (a_{48} \xi/l_0)'] + c_{48}]U_4 + \\
+[b_{68}/l_0 - (a_{68} \xi/l_0)']U_6 + b_{87}U_7 + (b_{87}' - c_{88}/\xi)U_8 + \omega^2 ZU_8 = 0. \]  

(18)

Boundary conditions for the shell rigidly embedded in the sloping section have the form at \( z = 0 \):

\[ U_2|_{z=0} = 0, \quad U_4|_{z=0} = 0, \quad U_6|_{z=0} = 0, \quad U_7|_{z=0} = 0, \quad U_8|_{z=0} = 0; \]

(19)

at \( z = l_1 \):

\[ \xi_1 a_{22}U_2' + \xi_1^2 a_{24}U_4' + \xi_1^2 a_{26}U_6' + \xi_1 a_{28}U_8' + \xi_1 (b_{24} - a_{24}/l_0)U_4 - \\
- (\xi_1^2 a_{26}/l_0)U_6 + b_{22}U_7 + b_{22}U_8 = 0, \\
\xi_1 a_{24}U_2' + \xi_1^2 a_{44}U_4' + \xi_1^2 a_{46}U_6' + \xi_1 a_{48}U_8' - \xi_1 (b_{44} - a_{44}/l_0)U_4 - \\
- (\xi_1^2 a_{46}/l_0)U_6 + b_{24}U_7 + b_{24}U_8 = 0, \\
\xi_1 a_{26}U_2' + \xi_1^2 a_{64}U_4' + \xi_1^2 a_{66}U_6' + \xi_1 a_{68}U_8' - (\xi_1 a_{66}/l_0)U_4 - \\
- (\xi_1 a_{66}/l_0)U_6 + b_{26}U_7 + b_{26}U_8 = 0, \\
\xi_1 a_{28}U_2' + \xi_1^2 a_{84}U_4' + \xi_1^2 a_{86}U_6' + \xi_1 a_{88}U_8' - (\xi_1 a_{88}/l_0)U_4 - \\
- (\xi_1 a_{88}/l_0)U_6 + b_{28}U_7 + b_{28}U_8 = 0, \]

(20)

where coefficients \( a_l, b_l, c_l, d_l \) meet shell geometry but unknowns \( U_2, U_4, U_6, U_7, U_8 \) are connected with colligated displacements \( V_2, V_4, V_6, V_7, V_8 \) by equations \( V_2 = U_2, V_4 = \tilde{Z}U_4, V_6 = \tilde{Z}U_6, V_7 = U_7, V_8 = U_8, \) \( \xi_1 = 1 - l_1/l_0 \).

Differential equations (18) form in the contrast to the corresponding equations of a straight shell [7] connected system. The target of the finding characteristic values and forms of final oscillations from the obtained system (18) in boundary conditions (19), (20) by the finite difference method of an increased accuracy comes to the finding of characteristic values and vectors of a block-diagonal matrix of \( 5N \times 5N \) dimension, where \( N \) – is a number of intervals in which the interval of integration is divided. Calculation stability is received at \( N = 34 \). While solving the problem with the help of a computer there was applied the method of double QR-iteration above the matrix coming to the upper practically triangular shape (Hessenberg matrix) by Householder's method [10].

In table 1 there are presented the values \( \omega \) at different taper of beveled slightly conical shell with right-angled profile of cross section \( (d_1, d_2) \) – its height and width relatively) at the following geometrical characteristics: \( l_1 = 0.9 \) m, \( d_1 = 8 \cdot 10^{-2} \) m, \( d_2 = 0.3 \) m, \( \chi_0 = \pi/3 \), \( h_1 = 2 \cdot 10^{-3} \) m, \( h_2 = 5 \cdot 10^{-4} \) m, \( l_0 = 2 \) m, 3 m, 5 m. As it is shown on the data from table1, with increase of \( l_0 \), and consequently with decrease of taper the characteristic values also decrease. At great \( l_0 \) the solution are conform with the solutions for beveled cylindrical shell having analogous geometrical characteristics.

**Table 1.** Natural frequencies in dependence of conicity.

| \( l_0 \times 10^4 \) | 2 | 3 | 5 |
|-------------------|---|---|---|
| 0.89766           | 0.65797 | 0.50846 |
| 2.76733           | 2.25106 | 1.85003 |
Table 2. Natural frequencies in dependence of thickness difference.

| $\vec{h} = 1$ | $\vec{h} = 4/3$ | $\vec{h} = 2$ | $\vec{h} = 4$ |
|---------------|-----------------|--------------|--------------|
| 5.38903       | 4.52698         | 3.63401      | 6.89755      |
| 6.89755       | 6.08484         | 5.03433      | 5.68818      |
| 8.32780       | 7.01875         | 5.86184      | 6.00000      |

In Table 2 there are given the values of natural frequencies $\omega$ at different values $h_2$. Geometrical parameters are the same, which while receiving the data for Table 1, in exception of angle of bevel: $\chi_0 = \pi / 4$, herewith $l_0 = 2$ m, $h_1 = 2 \cdot 10^{-3}$ m at linear changing of thickness of the span. The first column in Table 2 is given for the shell of constant thickness $h = h_2 = 2 \cdot 10^{-3}$ m. The analysis of presented in the table data allows making a conclusion that the change of a shell thickness influences on some first natural frequencies $\omega$ most of all. With a growth of ordinal number $\omega$ less dependents on dimensionless thickness $\vec{h} = h_1 / h_2$.

Figure 2. Forms of shell natural frequencies.
In figure 2 there are given first forms (1, 2, 3) of longitudinal oscillations corresponding to colligated displacements $U_1, U_4, U_7, U_8$ for the same shell of a linear-variable thickness at $\chi_0 = \pi / 3$ and $h_s = 1.5 \cdot 10^{-3} \text{m}$; the rest parameters are the same as in table 2. The oscillations corresponding to shell displacements as a solid $(U_2, U_4)$, with the growth of the sequence number of the eigenvalues, as it should be, have a subside character. Maximal values of oscillations amplitudes are moved to the free end of the shell (figure 2, a, b). Oscillations deplanation forms corresponding to $U_7, U_8$ have the other behavior (figure 2, c, d). Oscillations corresponding to $U_7$ subside onto the free end but with the growth of a relevant proper number increases the amplitude in direct nearness from the section $z = l_1$. At that shell oscillation forms conditioned by profile deplanation with the growth of ordinal figure of proper numbers do not subside as it is at displacement of the profile $Z = \text{const}$ as a solid body.

The results analogical to the presented ones in the very work for the straight cylindrical shell of variable rigidity are given in the article [7]. Some researchers conducted in the under observation field by the foreign authors are given in the works [11-13].

References

[1] Obrastsoy I F and Onanov G G 1973 Structural Mechanics of beveled thin-walled systems (Moscow: Mechanical engineering) pp 319-36
[2] Bulatov S N and Kozlov V A 1987 To the calculation of many closed structure of a variable rigidity Mechanical Engineering 2 68-73
[3] Bulatov S N and Kozlov V A 1988 To the design of swept-back box beam of the multispar wing with account of variable force elements News of universities. Avition technology 3 75-7
[4] Bulatov S N and Kozlov V A 2000 Solution of some applied problems of the theory of conic shells of compound geometry Problems of mechanical engineering and vehicles durability 5 102-8
[5] Kozlov V A 2014 Stress state of beveled non circular conic thin-walled of variable thickness Materials of VIII All-Russian conference on Mechanics of deformed solid (Cheboksari, 16-21 June 2014) in 2 parts ed Morozov N F, Mironov B G and Manzhirov A V (Chuvash state pedagogical university) P.1 pp 219-22
[6] Kozlov V A 2016 The deflected mode of multi coherent prismatic constructive elements of bridge constructions Scientific Herald of the Voronezh State University of architecture and civil engineering 31 41-50
[7] Kozlov V A 2013 Free oscillations of cantilever pinched prism thick-walled constructions Scientific bulletin of VGASU. Construction and architecture 30 9-17
[8] Kozlov V A 2016 Stress and strain of multiply connected prismatic structures, mounted on a skewed cross-section Scientific Herald of the Voronezh State University of architecture and civil engineering. Construction and architecture 30 17-23
[9] Kozlov V A 2018 Stress-strain of elements of bridge structures with a varying thickness of walls along the length Russian journal of building construction and architecture 37 67-80
[10] Wilkinson G Kh 1970 Algebra problem of characteristic constants (Moscow: Science) pp 365-78
[11] Liu L, Liu L, Liu G R and Tan V B C 2002 Element free method for static and free vibration analysis of spatial thin shell structures Computer Methods in applied mechanics and engineering vol 191 5923-42
[12] Piovan M T and Cortinez V H 2007 Mechanics of thin-walled curved beams made of composite materials, allowing for shear deformability Thin-walled Structures vol 45 759-89
[13] Wang Y, Xu Y, Liu Z, Wu H, Yan L and Wang Y 2016 Spatial finite element analysis for dynamic response of curved thin-walled box girder bridges Mathematical problems in engineering ID 8460130.