Casimir energy between a plane and a sphere in electromagnetic vacuum

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Collaborators

- Plane-sphere geometry
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Outline

- Geometry and the Casimir effect
- Scattering approach
- Plane-sphere geometry
Geometry and the Casimir effect

Motivation:

- Many modern experiments use the plane-sphere geometry: Lamoreaux, Mohideen, Capasso, Decca, ...

Proximity Force (Derjaguin) Approximation (PFA): take the plane-plane result for the local distances

How accurate is the PFA for a given $L/R$?

- See Krause+Decca+Lopez+Fischbach for experimental approach (2007)

- Proposed experiment with plane and cylinder (Onofrio et al)
Geometry and the Casimir effect (motivation)

Another relevant issue for mastering the quality of theory/experiment comparisons…

- Roughness correction to the Casimir attraction

\[ \langle h_j(r) \rangle = 0, \ j = 1, 2 \]
Such that

\[ \sqrt{\langle h^2 \rangle} \sim \text{nm} \]

More important at short distances
Geometry and the Casimir effect

Theory/experiment comparison: we need to consider...

- the electromagnetic field
- real metals with finite conductivity
Geometry and the Casimir effect - theory

Scalar vs electromagnetic: not a simple factor 2 from polarization!

Example with **plane symmetry**: dissipative force on a single moving mirror (velocity $v(t)$, area $A$)

3+1 (nonrelativistic limit: $v/c << 1$)

| Scalar vacuum field | Electromagnetic vacuum field |
|---------------------|------------------------------|
| Ford+Vilenkin (1982) | PAMN (1994)                  |

Scalar vacuum field:

$$F(t) = -\frac{1}{360\pi^2} \frac{\hbar A}{c^4} \frac{d^4v(t)}{dt^4}$$

Electromagnetic vacuum field:

$$F(t) = -\frac{1}{30\pi^2} \frac{\hbar A}{c^4} \frac{d^4v(t)}{dt^4}$$

12 x larger!!
...transition to electromagnetic (EM) case is often simpler when geometry contains a direction of translational symmetry..

.....not trivial for spherical geometries...

**Different physics** involved: no s-wave scattering in the EM case!
Geometry and the Casimir effect - theory

Why real materials?

Casimir energy (as a function of separation distance) with plasma model for metallic media

\[ \frac{c}{L} \gg \omega_P \rightarrow \text{plasma frequency} \]

Surface plasmons (Van Kampen et al. 1968)

\[ E_{PP}(L) = -0.0245 \, \omega_P \, \frac{\hbar c A}{2\pi L^2} \]

\[ \frac{c}{L} \ll \omega_P \]

Perfect reflectors - Casimir 1948

\[ E_{PP}(L) = -\frac{\pi^2}{720} \frac{\hbar c A}{L^3} \]

Power law modification as in Van der Waals – Casimir-Polder interatomic potential

\[ \lambda_P = 2\pi c / \omega_P \]

\[ \lambda_P = 137 \text{ nm (gold)} \]

-\[ E_C/(\hbar c A) \text{ [nm]} \]

\[ 1 \times 10^{-13} \]

\[ 1 \times 10^{-11} \]

\[ 1 \times 10^{-9} \]

\[ 1 \times 10^{-7} \]

\[ 1 \times 10^{-5} \]

\[ 1 \times 10^{-3} \]

\[ L \text{ (nm)} \]

exact - from Lifshitz theory (1956)

L \ll \lambda_P: 0.0245/[(\lambda_P L^2)] (plasmon)

L \gg \lambda_P: \pi^2/(720 \, L^3) (Casimir)
Why real materials?

Casimir energy proportional to $\omega_P$ for small $\omega_P$ ...

and satures at value independent of $\omega_P$ for large $\omega_P$

Beyond the plasma model:
A. Lambrecht and S. Reynaud, Eur. Phys. J. D8 309 (2000)
Geometry and the Casimir effect - theory

Approximation methods

- **Proximity Force Approximation (Derjagin)** - take local distances

- **Pairwise summation of vdWaals/Casimir-Polder interatomic potentials (Hamaker approach)**

Problem if medium is not rarefied: van der Waals interaction is not additive!!

(Dipole moments are not prescribed: they are induced by the other particles’ dipole moments)

Plane-plane case must be corrected by comparison with exact result ...

... and then the same correction factor is employed for different geometries (ok for nearly plane surfaces – PFA limit)
Geometry and the Casimir effect - theory

Some theoretical tools

**Numerical approaches**

World-line Monte-Carlo – Gies, Langfeld (2001) - no EM implementation so far

Finite-difference numerical evaluation of Green function – Rodriguez et al (2007) – EM, real materials.

**Scattering approach and non-trivial geometries**

Balian+Duplantier - Multiple scattering

Lambrecht+PAMN+Reynaud - Lifshitz formula generalized for non-planar scatterers

Kenneth+Klich - ‘TGTG’ formula, Bordag, Bulgac+Magierski+Wirzba, Milton +Wagner, Emig+Graham+Jaffe+Kardar - displacement+T
Scattering approach

Lifshitz formula

\[ E_{PP} = \frac{\hbar A}{2\pi} \int_0^\infty \frac{d\xi}{2\pi} \int \frac{d^2k}{(2\pi)^2} \sum_p \ln \left( 1 - r_{1;p}(k)r_{2;p}(k) e^{-2\kappa L} \right) \]

\[ \kappa = \sqrt{(\xi/c)^2 + k^2} \]

Lifshitz (1956) in a particular case (3 media with two plane interfaces)

Kats (1977): expression in terms of reflection coefficients

normal modes: \[ r_1 r_2 e^{-2\kappa L} E = E \]

More general derivations along the time...

Lossy plates: Genet+Lambrecht+Reynaud Phys Rev A (2003)

Also applies for magnetic media, generalizations for anisotropic materials
(see Felipe da Rosa talk for applications to metamaterials)
Scattering approach

\[ E_{FP} = \hbar A \int_0^\infty \frac{d\xi}{2\pi} \int \frac{d^2 k}{(2\pi)^2} \sum_p \ln \left( 1 - r_{1;p}(k)r_{2;p}(k) e^{-2\kappa L} \right) \]

Generalizing for non-planar surfaces

\[ E = \hbar \int_0^\infty \frac{d\xi}{2\pi} \text{Tr} \ln \left( 1 - R_1(i\xi)e^{-\kappa L}R_2(i\xi)e^{-\kappa L} \right) \]

\[ \text{Tr}(...) \equiv \int \frac{d^2 k}{(2\pi)^2} \sum_p \langle k, p | (...) | k, p \rangle \]

Non-specular reflection operators

\( R_1 \) and \( R_2 \): change \( k \) and polarization \( p \)

(Diego’s talk)

Lifshitz formula as a limiting case: \( R_1 \) and \( R_2 \) diagonal (specular reflection)
Second-order roughness correction –
PAMN, Lambrecht, Reynaud 2005

Second-order contribution of first-order reflection coefficients (at the same mirror) to closed loops

\[ \delta g_p^{(2)}(k, \omega) = \sum_p \int \frac{d^2 \vec{k}}{(2\pi)^2} L \left( e^{ik_z L r_{1;p}}(k) e^{ik_z L R_{2;p\bar{p}}(k, \bar{k}; \omega)} e^{i\vec{k}_z L r_{1;\bar{p}}(\bar{k})} e^{i\vec{k}_z L R_{2;\bar{p}\bar{p}}(\bar{k}, k; \omega)} + \ldots \right) \left| H_2(\bar{k} - k) \right|^2 + \ldots \left| H_1(\bar{k} - k) \right|^2 \]

Results from scattering formula coincide with those from more ‘standard’ approaches!

\[ \mathcal{E} = \hbar \int_0^\infty \frac{d\xi}{2\pi} \text{Tr} \ln \left( 1 - \mathcal{R}_1(i\xi) e^{-\kappa L} \mathcal{R}_1(i\xi) e^{-\kappa L} \right) \]
Scattering approach

another application:

Casimir Torque between corrugated surfaces: Rodrigues+PAMN+Lambrecht +Reynaud (2006)
Scattering approach

Plane-sphere Casimir energy within the scattering approach

Bordag, Bulgac+Magierski+Wirzba: scalar field models, analytical results for first-order correction to PFA

Emig 2007: formula for a body in front of a perfectly reflecting plane, based on the method of images.

PAMN+Lambrecht+Reynaud (march 2008): formalism for real materials (ex: metals with finite conductivity)
Plane-sphere geometry

Casimir energy

\[ \mathcal{E}_{PS} = \hbar \int_0^\infty \frac{d\xi}{2\pi} \log \det \mathcal{D} \]

\[ \mathcal{D} = 1 - R_S e^{-\kappa L} \cdot R_P e^{-\kappa L} \]

Plane wave basis vs Multipole basis

better adapted to...

- Scattering by plane (P) and free propagation
- Scattering by sphere (S) and determinant evaluation
Plane waves for a given frequency $\omega$

$$\{|k, \phi, p\}, k \in \mathbb{R}^2, \phi = \pm 1, p = \text{TE, TM}\$$

$$k_z = \phi \sqrt{\omega^2/c^2 - k^2}$$

Multipoles for a given frequency $\omega$

$$\{|\ell, m, P\}, \ell = 1, 2, 3..., m = -\ell, ..., \ell, P = \text{E, M}\$$

$E = \text{electric multipoles, } M = \text{magnetic multipoles}$
Plane-sphere geometry

\[ \mathcal{E}_{PS} = \hbar \int_0^\infty \frac{d\xi}{2\pi} \log \det \mathcal{D} \]

\[ \mathcal{D} = 1 - \mathcal{R}_S e^{-\kappa \mathcal{L}} \mathcal{R}_P e^{-\kappa \mathcal{L}} \]

Axial symmetry:

\[ \langle \ell_1, m_1, P_1 | \mathcal{D} | \ell_2, m_2, P_2 \rangle \propto \delta_{m_1,m_2} \]

\[ \langle \ell_1, m, P_1 | \mathcal{D} | \ell_2, m, P_2 \rangle \equiv \mathcal{D}_{1,2}^{(m)} \]

\[ \mathcal{D}_{1,2}^{(m)} = \delta_{1,2} - \int \frac{d^2k}{(2\pi)^2} \sum_{p=TE,TM} \langle \ell_1 m P_1 | \mathcal{R}_S | k, +, p \rangle r_p(k) e^{-\kappa \mathcal{L}} \langle k, -, p | \ell_2 m P_2 \rangle \]

Usual Fresnel reflection coefficients (as in Lifshitz formula!)

Diagram:
- Plane-sphere geometry
- Axial symmetry
- Usual Fresnel reflection coefficients (as in Lifshitz formula!)

Mathematical expressions and diagrams represent the analysis of plane-sphere geometry with axial symmetry. The equations describe the relationship between the wave functions and reflection coefficients in a plane-sphere system.
**Plane-sphere geometry**

**Scattering by the sphere:** in classical optics, one usually needs just the matrix elements (propagation along z-axis)

| Electric multipoles: | Magnetic multipoles: |
|----------------------|----------------------|
| \( P = E \)          | \( P = M \)          |

scattering amplitudes:

\[
\langle \ell m P | \mathcal{R}_S | 0, +, p \rangle
\]

Mie coefficients

\[
a_\ell (i\xi) \quad b_\ell (i\xi)
\]

Rotating the incident plane wave with the help of finite rotation matrix elements

\[
d_{m,m'}^\ell (\theta) \equiv \langle \ell m | e^{-iJ_y \theta / \hbar} | \ell m' \rangle
\]

\[
\langle \ell m E | \mathcal{R}_S | k, +, TE \rangle \sim a_\ell (i\xi) \left( d_{m,1}^\ell (\theta_k) + d_{m,-1}^\ell (\theta_k) \right)
\]

\[
(\ell, m, P) \quad (k, +, p) \quad (0, +, p)
\]

\[
R
\]

\[
Z
\]
Mie scattering with very small spheres: **Rayleigh limit**

- If \( R << \lambda_{\text{vac}}/n, \lambda_{\text{vac}} \), then \( a_1 \) (electric dipole) dominates over higher multipoles (including magnetic dipole \( b_1 \))

- Translating to Casimir theory: \( \lambda_{\text{vac}} \sim L \), then condition reads

\[
R << L, L/n
\]

- We find from the previous results (\( \alpha \) is the sphere polarizability)

\[
a_1(i\xi) \approx -\frac{2}{3} \left( \frac{\xi}{c} \right)^3 \frac{\alpha}{4\pi\epsilon_0}
\]

\[
\alpha = 4\pi\epsilon_0 \frac{\epsilon - 1}{\epsilon + 2}
\]

\[
\mathcal{E}_C = \frac{\hbar}{2\epsilon_0} \int_0^\infty d\xi \frac{d^2k}{2\pi (2\pi)^2} e^{-2\kappa L \hat{\xi}^2 \kappa} \alpha(i\xi) \left[ r_{TE}(k, \xi) - (1 + 2k^2/\hat{\xi}^2)r_{TM}(k, \xi) \right]
\]

vdWaals/Casimir-Polder interaction !!

Diego’s talk
Plane-sphere geometry

\[ D_{1,2}^{(m)} = \delta_{1,2} - \int \frac{d^2k}{(2\pi)^2} \sum_{p=\text{TE,TM}} \langle \ell_1 m P_1 | R_S | k, +, p \rangle r_p(k) e^{-2kL} \langle k, -, p | \ell_2 m P_2 \rangle \]

Simpler expressions in the case of perfect reflectors, taken as the limiting case of the plasma model for very short plasma wavelengths \( \lambda_p \)

- For the plane: \( r_{\text{TE}} = -1, \quad r_{\text{TM}} = 1 \) \( \rightarrow \) allows us to add analytically over \( p \)

- For the sphere, it is not sufficient to consider the limit \( n >> 1 \) if the sphere is small...one also needs \( R >> \lambda_{\text{vac}}/n \) ...

..translating to Casimir theory: \( \lambda_p << L \) and \( \lambda_p << R \)

No intersection with Rayleigh limit!
Perfectly-reflecting limit:

\[ D_{\ell_1E,\ell_2E}^{(m)} = \delta_{\ell_1 \ell_2} + \frac{1}{2} \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)} a_{\ell_1} F_{\ell_1,\ell_2,m}^{(+)a} \]
\[ D_{\ell_1M,\ell_2M}^{(m)} = \delta_{\ell_1 \ell_2} - \frac{1}{2} \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)} b_{\ell_1} F_{\ell_1,\ell_2,m}^{(+b)} \]
\[ D_{\ell_1E,\ell_2M}^{(m)} = \frac{i}{2} \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)} a_{\ell_1} F_{\ell_1,\ell_2,m}^{(-a)} \]
\[ D_{\ell_1M,\ell_2E}^{(m)} = \frac{i}{2} \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)} b_{\ell_1} F_{\ell_1,\ell_2,m}^{(-b)} \]

\[ F_{\ell_1,\ell_2,m}^{(\pm)} = (-)^{\ell_2 + m} \int_1^{\infty} d\cos \theta \ e^{-2\xi L \cos \theta / c} \left[ d_{m,1}^{\ell_1}(\theta)d_{m,1}^{\ell_2}(\theta) \pm (-)^{\ell_1 - \ell_2} d_{m,1}^{\ell_1}(\pi - \theta)d_{m,1}^{\ell_2}(\pi - \theta) \right] \]

PFA limit:

When \( L << R \), we have \( \xi R/c >> 1 \) (for typical values of \( \xi \)) and then \( a_{\ell}(i\xi), b_{\ell}(i\xi) \sim \exp(2\xi R/c) \)

\[ a_{\ell} F^{\pm}, b_{\ell} F^{\pm} \sim \exp \left( -\frac{2\xi L}{c} \right) \]
Plane-sphere geometry

Small, perfectly-reflecting sphere: $\lambda_p \ll R \ll L$

Electric and magnetic dipoles are of the same order!

$$a_1(i\xi) \approx -2b_1(i\xi) \approx -\frac{2}{3} \left( \frac{\xi R}{c} \right)^3$$

Neglect higher multipoles

‘Casimir-Polder’ with magnetic dipole contribution

$$\varepsilon_{PS} = -\frac{9 \hbar c R^3}{16\pi L^4}$$

T. Emig (2008) from model of perfect reflectivity

no intersection with Rayleigh limit

$$(R \ll \lambda_p \ll L)$$

$$\varepsilon_{PS} = -\frac{3\hbar c R^3}{8\pi L^4}$$
What are the typical values of angular momentum $\ell$ when we approach the PFA limit?

**Localization principle**

When $\lambda = 2\pi/k \ll R$, $\ell$ corresponds to rays with impact parameter $B$ given by

$$\ell = k B$$

Rays with $B > R$ provide negligible contributions

$$\ell \lesssim kR \sim \frac{R}{L} \gg 1$$

H. M. Nussenzveig, *Diffraction Effects in Semiclassical Scattering*, Cambridge 1992
A typical Mie scattering numerical calculation from R S Dutra, N B Viana, PAMN and H M Nussenzveig, J. Opt. A: Pure Appl. Opt. 9 (2007) S221.

Plane-sphere geometry

- $R = 4.5 \, \mu m$
- $2\pi R/(\lambda/1.33) = 47.7$

Mie resonances:
- optical whispering gallery modes
- contribution of (slightly) above-edge rays: $\ell > \omega R/c$
- lie close to the real axis ....ouufff...!

localization principle $\ell = \omega B/c$
- converges at $\ell \sim \omega R/c$
- partial sum oscillates with $\ell$
- strong variation with frequency
Plane-sphere geometry

Numerical calculation

Comparing with the Proximity Force Approx (PFA) result

\[ \rho \equiv \frac{\mathcal{E}}{\mathcal{E}_{\text{PFA}}} , \quad \mathcal{E}_{\text{PFA}} = -\frac{\hbar c \pi^3 R}{720 L^2} \]

\[ 0.16 \]
Scattering approach was employed to compute the Casimir interaction energy between a plane and a sphere.

Our approach allows for the computation in the case of real metals with finite conductivity.

Numerical calculation in the case of perfectly-reflecting surfaces. PFA is less accurate in the electromagnetic case (than in the scalar model) – correction is $\sim 8 \times$ larger in the EM case!

Numerical calculation in the more general case will provide the correction to the Proximity Force Approx. result for a given $L/R$ under realistic conditions.

For details see PAMN, A Lambrecht and S Reynaud, Phys. Rev. A 78, 012115 (2008)