Abstract. Black hole entropy has been shown by ’t Hooft to diverge at the horizon, whereas entanglement entropy in general does not. We show that because the region near the horizon is a thermal state, entropy is linear to energy, and energy at a barrier is inversely proportional to barrier slope, and diverges at an infinitely sharp barrier as a result of position/momentum uncertainty. We show that ’t Hooft’s divergence at the black hole is also an example of momentum/position uncertainty, as seen by the fact that the “brick wall” which corrects it in fact smooths the sharp boundary into a more gradual slope. This removes a major obstacle to identification of black hole entropy with entanglement entropy.

1. Introduction

Two of the possible characterizations of black hole entropy are both proportional to area but they apparently do not coincide. One is entanglement entropy, arising from correlations between that within the black hole and that without [1][2][3], and the other is the statistical mechanics definition related to the number of states of a system in the vicinity of a black hole [4][5]. Both types of entropy are expressed by calculating $-\operatorname{Tr}(\rho \ln \rho)$ of fields exterior to the black hole and the question naturally arises: do they both refer to the same thing? One problem in uniting these descriptions is the question of divergence at the black hole horizon. Entanglement entropy may have UV divergence which will need to be renormalized, but it does not diverge at a particular location. However, in the statistical mechanics calculation first carried out by ’t Hooft, the integrand diverges at the black hole boundary. We will show that the problem of divergence at the boundary can be resolved. It is not a unique black hole characteristic but rather a result of quantum uncertainty, and the correct expression must involve smearing out the boundary. In this way the two expressions may coincide, and we may consider whether black hole entropy is simply entanglement entropy. This talk is based on [6].

For a massive black hole in the region just outside the horizon, it has been shown that entanglement entropy coincides with the thermodynamic definition of entropy [7][8]. Thermodynamic entropy is linear to energy. The expectation value for energy is also linear to momentum uncertainty, $\langle E \rangle \sim \langle p^2 \rangle = \Delta p^2$. Therefore whenever energy is calculated at a precisely defined location we expect it to diverge due to the uncertainty relationship between $x$ and $p$. We show that when any system is cut off by a barrier, the increase in energy is proportional to the narrowness of the barrier, and diverges as the barrier becomes infinitely sharp. Thus black hole entropy, which is proportional to energy, will diverge on the horizon because the horizon is a sharply defined barrier. We also show that this is consistent with ’t Hooft’s “brick wall” cutoff.
First we review these two characterizations of black hole entropy, entanglement and statistical, focusing on behavior at the horizon. Then we show in detail the relationship between energy and entropy in a massive black hole. We examine behavior of energy at a boundary between two subsystems, first for the non-relativistic and then for the relativistic case, and show that in both cases energy diverges as the boundary becomes sharp. Finally, we examine 't Hooft’s calculation leading to divergence at the boundary, and find that his relocation of the boundary to avoid divergence is equivalent to smearing out the boundary. Therefore here too the divergence is related to sharpness of the boundary, and thus it reflects behavior of any thermal system at a boundary, and is not unique to a black hole. This conclusion lends weight to the possibility that black hole entropy is in fact entanglement entropy, because the peculiar divergence which seemed to single it out from other entangled systems is just a result of quantum x/p uncertainty.

2. Entropy at the horizon

The area law was first formulated by Bekenstein, beginning from the observation that black hole area must increase as does entropy [9], and strengthened by thermodynamic calculations [4]. Bekenstein did not call this entanglement entropy but he did describe it as a measure of the inaccessibility of information to an exterior observer as to which internal configuration of the black hole is realized in a given case. In fact black holes have entanglement entropy by definition. Entanglement entropy quantifies the extent to which a state is mixed. If the universe is described as a pure state, the black hole horizon divides it into that within and that without the hole, each of which is a mixed state. Therefore black holes have entanglement entropy, and the question is whether black hole entropy is anything more, or whether entanglement entropy saturates the definition.

't Hooft calculated thermodynamic characteristics of a black hole, among them entropy, and in doing so found a divergence of the integrand at the horizon [5]. He overcame the problem by adjusting the limits of integration to a “brick wall” a finite infinitesimal distance from the horizon, and various other solutions have been adopted since [10]. Entanglement entropy, on the other hand, has ultraviolet divergence, and a UV cutoff must be employed, but it does not diverge at any particular location. Srednicki in his landmark paper [1] calculated entropy by tracing over the degrees of freedom of part of the system, and found it proportional to area as well. He numerically obtained an expression for entropy $S = 0.30M^2 R^2$, where $R = (n + \frac{1}{2})a$, $a$ is lattice spacing and $n$ the number of discrete oscillators. He also defines $M$ as the inverse lattice spacing $a^{-1}$ so that the actual expression is $S = 0.30(n + \frac{1}{2})^2$ which diverges for an infinite number of oscillators, but not at a particular location. Other treatments of entanglement entropy in general [11][12] also find UV divergence but not at any particular location. Therefore horizon divergence would seem be a special characteristic of black hole entropy, and would preclude its being a manifestation of entanglement entropy as such. We show below that this is not the case.

For a massive BH in equilibrium the space just outside the hole near the horizon can be treated as a thermal state in Rindler space [7][8]. In this case entanglement entropy coincides with thermal entropy, as follows. To find entanglement entropy we take the trace of part of the system. If that part of the system is a thermal state, the partial trace is a thermal density
matrix.

\[ S = - \text{Tr} (\rho_{\text{part}} \ln \rho_{\text{part}}) \]

\[ \rho_{\text{part}} = \frac{1}{Z} \sum_i e^{-\beta E_i} |E_i\rangle \langle E_i| \]

\[ \langle E \rangle = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} \]

\[ S = \left( -\frac{1}{Z} \sum_i e^{-\beta E_i} \right) \times \left( -\beta \sum_i E_i - \ln Z \right) \]

\[ = \beta < E > + \ln Z. \quad (1) \]

For a scalar field with finite temperature ln Z is a constant, so the entropy is linear to the expectation value of the energy. Therefore in the case of a massive black hole the entanglement entropy behaves as does the energy. So instead of calculating entropy, which is a complicated non local quantity, we can calculate the reduced density matrix of a subsystem and look at the behavior of its energy, which is a simpler local quantity.

3. Nonrelativistic treatment

We first write down the expectation value for energy in a nonrelativistic many body system. In order to look at behavior at a boundary, we insert a window function into the system. The window function mimics the horizon by “truncating” part of space for the particles. However it does not affect boundary values of the system. An example of such a window function would be the Heaviside step function. Here we provide our window function with a varying slope to see how sharp localization affects the energy divergence.

In a nonrelativistic many body system, taking spinless particles for simplicity, the field state and the kinetic energy operator for a free particle are as follows:

\[ \left| \Psi^\dagger (\vec{r}) \right> = \Psi^\dagger (\vec{r}) |0\rangle = \sum_{\vec{p}} \frac{e^{-i\vec{p} \cdot \vec{r}}}{\sqrt{V}} a^\dagger_{\vec{p}} |0\rangle \]

The energy operator is \( T = \sum_{\vec{p}} \frac{p^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}} \) and its expectation value is \( \langle E \rangle = \langle 0 | \Psi T \Psi^\dagger |0 \rangle \). In configuration space

\[ \Psi T \Psi^\dagger = \int_{-\infty}^{\infty} d^3r \frac{1}{2m} \nabla_r \Psi (\vec{r}) \nabla_r \Psi^\dagger (\vec{r}) \quad (2) \]

With a window function \( f(\vec{r}) \), the field operator becomes

\[ \Psi_W = \int d^3r f (\vec{r}) \Psi^\dagger (\vec{r}) = \int d^3r f (\vec{r}) \sum_{\vec{p}} \frac{e^{-i\vec{p} \cdot \vec{r}}}{\sqrt{V}} a^\dagger_{\vec{p}} \]

The ladder operators on the vacuum give delta functions, resulting in

\[ \langle E \rangle = \frac{1}{2m} \int_{-\infty}^{\infty} d^3r \nabla_r f^* (\vec{r}) \nabla_r f (\vec{r}) \quad (3) \]

For details see [6]. This represents the contribution of the window to the expectation value of the energy.\(^1\)

\(^1\) In addition the state would have a wave function, e.g. \( g(\vec{r}) \), which would give us the energy of the particle regardless of window, but here we have taken \( g(\vec{r}) = 1 \) for simplicity.
We take the window function to represent a boundary with a varying slope. Technically we achieve this by combining a Gaussian with the Heaviside step function, and bound the step for purposes of normalization (see figure 1). In the figure, the amplitude to see the particle through the window falls off gradually with the slope, so that on the left side (from $-1$ to $0$ in the graph) it is constant, and then it slopes down to zero with width $L$. The sharp drop on the left (at $-1$ in the graph) is for purposes of normalization, but does not affect the result; taking the cutoff at other values gives essentially the same result: energy which is inversely proportional to $L$. This is graphed in one dimension for clarity of graphing, but three dimensions give a symmetrical version of the same results.

![Figure 1](image1.png)

**Figure 1.** Half-Gaussian window, bounding the wave function, where we look at the part on the left of the slope. The horizon varies from a sharp step at $x=0$ to a gradual slope as in the drawing. The step at $x = -1$ is just for normalization; in fact space on the left continues to $-\infty$.

We calculate the contribution to the energy expectation value for a system with this window function and obtain

$$\langle E \rangle = \frac{1}{2m \left(L^2 + 2L \sqrt{\frac{2}{\pi}}\right)}$$  \hspace{1cm} (4)

where $L$ refers to width of the slope and $m$ is particle mass. Obviously this diverges when the slope becomes perfectly sharp, $L=0$, as seen in figure 2 where we have taken $m = 1$.

![Figure 2](image2.png)

**Figure 2.** $\langle E \rangle$ for sloping window, as a function of slope width.

We have obtained the same result using completely different functions for the slope. For example, taking the window function as arctangent gives a slope which can be varied by using $\text{Arctan}[nx]$; larger $n$ gives steeper slope. We took $\sqrt{\text{Arctan}[nx]} + \frac{\pi}{2}$ so that all values would
be positive and to simplify the numerical calculations, and again the energy was seen to be inversely proportional to width of slope. Thus we have seen that the energy increases as the barrier grows sharper. That is, the more sharply the position of the dividing barrier is specified, the closer the energy gets to divergence. We now note that $E = \frac{1}{2m} \left< \psi \left| p^2 \right| \psi \right>$. Since $\left< \psi \right| p \left| \psi \right> = 0$, $\left< \psi \right| p^2 \left| \psi \right> = (\Delta p)^2$, so that as energy diverges so do fluctuations in momentum. This divergence occurs as the width is taken to zero, and may be seen as an example of position/momentum uncertainty.

### 4. Relativistic treatment

In a relativistic system the energy operator is taken from the energy momentum tensor:

$$\hat{H} = T^{00} = \int \frac{dk_0}{(2\pi)^3} k_0 a_k^\dagger a_k$$

taking $c = \hbar = 1$. A state with window function, as before, is

$$\int d^3r f(\vec{r}) \Psi(\vec{r}) |0\rangle$$

where the field operator here is the relativistic one.

In order to look for the various expectation values we need the relativistic scalar product:

$$\left< \phi | \phi \right> = \frac{1}{2} \int d^3x (\phi i \partial_0 \phi - (i \partial_0 \phi) \phi)$$

where the expression is sandwiched between the vacuum.\(^2\) Thus for energy we write

$$\left< \varphi \left| \hat{H} \right| \varphi \right> = \frac{1}{2} \int d^3x \left( \varphi i \partial_0 \left( \hat{H} \varphi \right) - (i \partial_0 \varphi) \hat{H} \varphi \right).$$

Next we plug in the state with the window function. Since in the relativistic case $E^2 \sim p^2$ and we are interested in momentum fluctuations, we calculate $\left< \varphi \left| \hat{H}^2 \right| \varphi \right>$ and obtain

$$\frac{1}{2} \int d^3r \nabla_r f^*(\vec{r}) \nabla_r f(\vec{r}).$$

For details see [6]. This clearly has the same behavior as the non relativistic case.

We have seen that in just as in the nonrelativistic treatment, energy is seen to diverge the more sharply position is specified. For the non relativistic case we had $E \sim p^2$ whereas in the relativistic case $E^2 \sim p^2$ but in both cases the divergence may be seen as an example of position/momentum uncertainty.

### 5. Statistical mechanics calculation

't Hooft treats the black hole as a quantum system, and solves the wave equation in the Schwartzschild metric:

$$\left( 1 - \frac{2M}{r} \right)^{-1} E^2 \varphi + \frac{1}{r^2} \partial_r \left( r (r - 2M) \partial_r \right) \varphi - \left( \frac{l(l+1)}{r^2} + m^2 \right) \varphi = 0.$$  

(8)

He identifies $k$, the wave number, by isolating the radial derivative on one side of the equation and the rest of the expression on the other. Thus he obtains an expression for $k^2(r) \varphi$, where $k^2(r)$ is the eigenvalue of the second radial derivative. Labelling the metric coordinates by indices for short, he obtains

$$k^2 = g_{rr} \left( g^{tt} E^2 - \left( \frac{l(l+1)}{r^2} + m^2 \right) \right)$$

\(^2\) If this use of scalar product is unfamiliar, note that $\left< \hat{H} \right> = \left< \varphi \left| i \partial_0 \right| \varphi \right>$ with this formula gives the same expression as $\left< \int d^3r T^{00} \right>$. 

5
Using a WKB approximation he obtains the number of modes by integrating the wave number over the region outside the black hole horizon:

\[ \pi N = \int_{2M}^{L} dr k(r, l, m) \]

\[ = \int_{2M}^{L} dr \left(1 - \frac{2M}{r}\right)^{-1} \int (2l + 1)dl \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + m^2\right)} \]

where \(L\) is an infrared cutoff. However the redshift leads this to diverge at the lower limit, that is, the horizon itself. To avoid this 't Hooft takes the lower limit a slight distance away from the horizon, his well known “brick wall,” so that the lower limit becomes \(2M + h\). From this expression he obtains the energy and entropy, which diverge as \(h \to 0\).

't Hooft’s adjustment of the lower bound of the integral from \(2M\) to \(2M + h\) is equivalent to a change of variable which leaves the bound at \(2M\) but adds a factor into the redshift:

\[ \int_{2M+h}^{L} dr \left(1 - \frac{2M}{r}\right)^{-1} = \int_{2M}^{L} \tilde{r} \left(1 - \frac{2M}{\tilde{r} + h}\right)^{-1} \]

We examine the change in the redshift. In the \(\tilde{r}\) system the redshift is

\[ \left(1 - \frac{2M}{\tilde{r}}\right)^{-1}. \]

So the altered redshift is the same as taking the redshift and multiplying it by another function, thus:

\[ \left(1 - \frac{2M}{\tilde{r} + h}\right)^{-1} = \left(1 - \frac{2M}{\tilde{r}}\right)^{-1} f(r, h) \]

\[ f(r, h) = \frac{(\tilde{r} + h)(\tilde{r} - 2M)}{\tilde{r}(\tilde{r} - 2M + h)} \]

\[ \int_{0}^{L} d\tilde{r} g_{rr}(r) f(r, h) \Theta(r - 2M) = \int_{0}^{L} d\tilde{r} g_{rr}(r) \tilde{\Theta}(r - 2M) \]

Thus we see that 't Hooft’s changed lower limit is exactly equivalent to modification of the step function. The new step function is no longer a sharp step, but rather has a width of \(h\) (for \(r = 2M + h, \tilde{\Theta} = 1/2\)). Figure 3 shows a graph of this softened step function \(\tilde{\Theta}(r - 2M)\) for various values of \(h\), ranging from a sharp step at \(h = 0\) to a wider and wider slope. For \(h = 0\) 't Hooft had a divergence and when he changed the lower limit, equivalent to widening the horizon slope, the divergence vanished.

This is in fact position/momentum uncertainty. A closer look at eq.9 and at the effect of the modifying function \(f(r, h)\) makes this clear. The integrand of eq.9 is \(k(r)\). Just as in the previous sections \(\langle k^2 \rangle = \Delta k^2\), and its root is just the integrand of eq. 9. Multiplying this integrand by the modifying function \(f(r, h)\) is equivalent to multiplying it by \(\Delta x\) since as we have shown, it widens the slope. Eq. 9 with the “brick wall” expressed as a softening function

\[ \int_{2M}^{L} \]
becomes ∫_{2M}^{\infty} dr \Delta k \Delta x. Without the brick wall, \Delta x = 0 and we had \Delta k \rightarrow \infty. Adding the brick wall is directly equivalent to widening \Delta x and as a result the expression becomes finite.

Another way of looking at it is that the “brick wall” is equivalent to a modification of the redshift: g_{rr} \rightarrow \tilde{g}_{rr} = g_{rr} f(r, h). This may become more intuitive from the following illustration: I hover outside a black hole and send an astronaut in towards it radially, and require of the astronaut to send me back progress reports every kilometer; we all know the progress reports will come more and more slowly, so that it seems to me that he has slowed down, and come to a full stop at the horizon. I send another astronaut after him and the same thing happens. So I send a whole fleet of astronauts, tied together by a rope at regular intervals, as in dangerous mountain climbing. It will seem to me that as they near the horizon they all slow down, the rope slackens so they grow closer and closer, and at the horizon eventually they will all crowd together and I’ll have an infinite density of astronauts there. This is the effect of the redshift. However, if I smooth out the horizon - stretch out the redshift, so to speak - they will not crowd up at one point and the density won’t be infinite. This is, so to speak, quantum uncertainty \delta(x) \delta(\text{astronaut}).

’t Hooft’s displacement of the boundary prevents divergence at the black hole horizon by distorting the horizon so that it is no longer a sharply located step function but rather a more gradual transition. By doing so he actually shrinks the momentum uncertainty. Therefore we see that this too is an example of x/p quantum uncertainty, and not a unique characteristic of a black hole.

6. Discussion

Energy has been shown to diverge as the boundary between two subsystems becomes sharp. The divergence is due to the fact that the energy is a simple function of momentum. For the nonrelativistic case it is easy to see that \langle E \rangle = \frac{1}{2m} \langle p^2 \rangle = \frac{1}{2m} \Delta p^2 and so divergence at a sharp boundary is just due to the quantum \Delta p \Delta x uncertainty. The relativistic expression for \langle E^2 \rangle has exactly the same form as the nonrelativistic result and so in both cases energy divergence at an infinitely sharp boundary is a consequence of x/p uncertainty.

The region near the boundary of a black hole is a thermal state, where the entropy is linear to energy. Therefore black hole entropy will diverge at the boundary as well. We have not proven that there is no other cause of the divergence, unique to a black hole. But we have shown that regardless of any other cause, there would be divergence at the boundary as a result of the uncertainty principle. We have also shown that ’t Hooft’s divergence at the black hole is also an example of momentum/position uncertainty, as seen by the fact that the “brick wall” which
corrects it in fact smooths the sharp boundary into a more gradual slope.

We may now consider whether the entanglement and statistical mechanics definitions of black hole entropy might refer to the same thing. Both are proportional to area. The UV divergence may be renormalized with a cutoff, and the boundary divergence by smearing out the boundary, so that these no longer preclude unification of the two expressions. The argument can then be made that black hole entropy is due to entanglement, that is, to quantum correlations between the two parts of the system inside and outside the black hole, and that counting the number of states is tantamount to counting the correlations.

However there is a third definition with which these two must now be reconciled. Black hole entropy has also been shown - from thermodynamic considerations [9] as well as explicit calculations in string theory [13] to equal one fourth of the horizon area. An open problem is to obtain the factor of 1/4 in these two cases as well.

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