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VERTICALLY \(N\)-CONTRACTIBLE ELEMENTS IN 3-CONNECTED MATROIDS

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ABSTRACT. In this paper we establish a variation of the Splitter Theorem. Let \(M\) and \(N\) be simple 3-connected matroids. We say that \(x \in E(M)\) is vertically \(N\)-contractible if \(s_i(M/x)\) is a 3-connected matroid with an \(N\)-minor. Whittle (for \(k = 1, 2\)) and Costalonga (for \(k = 3\)) proved that, if \(r(M) - r(N) \geq k\), then \(M\) has a \(k\)-independent set of vertically \(N\)-contractible elements. Costalonga also characterized the obstruction for the existence of such a 4-independent set \(I\) in the binary case, provided \(r(M) - r(N) \geq 5\), and improved this result when \(r(M) - r(N) \geq 6\), and in the graphic case. In this paper we generalize such results to the non-binary case.

Moreover, we apply our results to the study of properties similar to 3-roundedness in classes of matroids.

1. INTRODUCTION

We follow the definitions and notations set by Oxley [9], with the following addendum: if \(I\) is an independent set of \(M\) spanning \(x\), we denote \(C_M(x, I)\) as the circuit of \(M\) contained in \(I \cup x\).

Let \(M\) be a 3-connected matroid, \(x \in E(M)\) or \(X \subseteq E(M)\) is vertically contractible in \(M\) if \(s_i(M/x)\) or \(s_i(M/X)\) is 3-connected, respectively. If \(M/x\) (or \(M/X\)) is 3-connected, we say that \(x\) (or \(X\)) is contractible in \(M\). If \(N\) is a matroid, we say that \(x\) (or \(X\)) is vertically \(N\)-contractible in \(M\) if \(s_i(M/x)\) (or \(s_i(M/X)\)) is a 3-connected matroid with an \(N\)-minor. Analogously, \(x\) (or \(X\)) is \(N\)-contractible if \(M/x\) (or \(M/X\)) is 3-connected with an \(N\)-minor.

Contractible and vertically contractible elements are widely used for inductive proofs of results about 3-connected matroids. When working in a class of matroids with an specific \(N\)-minor, this role is better played by \(N\)-contractible and vertically \(N\)-contractible elements.

The most known result about \(N\)-contractible elements is Seymour’s Splitter Theorem [12]. Several variations of this Theorem have been published, as in [2], [3], [6], [4], [5], [7] and [15]. The reader can see Chapter 12, especially Section 12.3 of [9] for a better contextualization.

Whittle [15] (for \(k = 1, 2\)) and Costalonga [5], (for \(k = 3\)) proved:

**Theorem 1.1.** Let \(k \in \{1, 2, 3\}\). Let \(N\) be a 3-connected minor of a 3-connected matroid \(M\), satisfying \(r(M) - r(N) \geq k\). Moreover, suppose that \(N\) is simple or that \(r(M) \neq 2\). Then \(M\) has an \(k\)-independent set, whose elements are vertically \(N\)-contractible.

We remark that the hypothesis of \(N\) being simple when \(r(M) = 2\) was not observed in [15] and [5]. Their proofs use implicitly the fact that \(N\) is simple, being correct under this assumption. But we have a counter-example when \(M \cong U_{2,n}\) and \(N \in \{U_{1,2}, U_{1,3}\}\). In other hand, it is straightforward to check that the simplicity of \(N\) is not required if \(r(M) \neq 2\). With the lemmas we establish here we give a short proof for Theorem [1.1]in the end of Section 2.

The previous proofs of Theorem [1.1] were based on the reduction to a matroid in the form \(M \setminus S\), where \(S\) is a maximal subset of \(E(M)\) such that \(M \setminus S\) is 3-connected and \(r(M) = r(M \setminus S)\). The problem in using this technique to identify the obstructions for Theorem [1.1] to hold for greater values of \(k\) is that we will find then in \(M \setminus S\) and it is hard to know how they extend to \(M\).
The main result established here is the next theorem, but, before we state it, we will need a few definitions. We say that a list \( x_1, x_2, p_1, p_2, p_3 \) of elements of \( M \) is an \((M, N)\text{-biweb}\), with edges \( x_1 \) and \( x_2 \) if:

- (BW1) \( \{p_1, p_2, p_3\} \) is a vertically \( N \)-contractible triangle of \( M \),
- (BW2) \( x_1 \) and \( x_2 \) are vertically \( N \)-contractible,
- (BW3) for \( i, j \in \{1, 2\} \), \( \{x_i, p_j, p_3\} \) is a triad of \( M \) and \( \{x_i, p_j\} \) is vertically \( N \)-contractible in \( M \).

We say that a list \( x_1, x_2, x_3, p_1, p_2, p_3 \) is an \((M, N)\text{-triweb}\), with edges \( x_1 \) and \( x_3 \) if

- (TW1) \( x_i, x_j, p_i, p_j, p_k \) is an \((M, N)\)-biweb for \( i, j, k \in \{1, 2\} \) and
- (TW2) \( x_1, x_2, x_3 \), and \( \{p_1, p_2, p_3\} \) are \( N \)-contractible in \( M \).

In Lemma 2.13 we prove that (TW1) implies (TW2) and that (BW1), (BW2) and (BW3) follows from the apparently weaker condition:

- (BW) In \( M, T := \{p_1, p_2, p_3\} \) is a triangle, \( \{x_1, p_2, p_3\} \) are triads and \( \{x_1, p_1\} \) is vertically \( N \)-contractible.

We define \( x_1, x_2, x_3, p_1, p_2, p_3, q_1, q_2, q_3 \) to be an \((M, N)\text{-prism}\) with edges \( x_1 \) and \( x_3 \) if both \( x_1, x_2, x_3, p_1, p_2, p_3 \) and \( x_1, x_2, x_3, q_1, q_2, q_3 \) are \((M, N)\)-triwebs. In Figure 1 there is a graphic representation of these concepts. The circled vertices represent triads.

![Figure 1](image)

We denote the set of the vertically \( N \)-contractible elements of \( M \) by \( V_N(M) \).

**Theorem 1.2.** Let \( M \) and \( N \) be a 3-connected matroids such that \( M \) has an \( N \)-minor, \( r(M) \geq 5 \), \( r(M) \geq r(N) \geq 4 \) and \( r_M(V_N(M)) \leq 3 \). Then

(a) \( M \) has \( N \)-triweb \( x_1, x_2, x_3, p_1, p_2, p_3 \) and \( V_N(M) = \{x_1, x_2, x_3\} \), or

(b) \( M \) has \( N \)-biweb \( x_1, x_2, p_1, p_2, p_3 \), \( V_N(M) = \{x_1, x_2, p_3\} \).

Moreover, (a) holds if \( r(M) \geq 5 \) and \( M \) has an \( N \)-prism if \( r(M) \geq 6 \).

If we combine this Theorem with Lemma 3.6 we conclude:

**Corollary 1.3.** If \( M \) and \( N \) satisfy the hypothesis of Theorem 1.2 and \( M \) has no \( N \)-prism, then \( V_N(M) = \{x_1, x_2, x_3\} \) if (a) holds and \( V_N(si(M/x)) \subseteq \{x_1, x_2, p_1, p_2, p_3\} \) for each \( x \in V_N(M) \) if (b) holds.

In the graphic case, Costalonga [5] proved the following:

**Theorem 1.4.** Let \( G \) be a simple 3-connected graph with a simple 3-connected minor \( H \) such that \( |V(G)| - |V(H)| \geq 6 \). Then there is a 4-independent set \( I \) of \( M(G) \) such that, for each \( x \in I \), \( G/x \) is a 3-connected graph with an \( H \)-minor.

If we combine Theorem 1.1 and Theorem 3.2 of [15], we have:
Corollary 1.5. If \(N\) is a 3-connected minor of a 3-connected matroid \(M\), satisfying \(r(M) - r(N) \geq 3\), then for every vertically \(N\)-contractible element \(x\) of \(M\), there are elements \(y, z \in E(M) - x\) such that \(y, z\) and \(|x, y|\) are vertically \(N\)-contractible and \(|x, y, z|\) is independent in \(M\).

Next we present an application of the main result. Let \(F\) be a class of matroids. We say that a matroid \(M\) has an \(F\)-minor if it has a minor isomorphic to a matroid in \(F\). We define \(F\) to be \(k\)-rounded if:

(RD1) each member of \(F\) is \((k + 1)\)-connected, and

(RD2) if \(M\) is a \((k + 1)\)-connected matroid with an \(F\)-minor and \(X\) is a \(k\)-subset of \(E(M)\), then \(M\) has an \(F\)-minor with \(X\) contained in its ground set.

Seymour [14] proved, for \(k = 1, 2\) that the task of verifying if a finite class of matroids is \(k\)-rounded is finite, in particular, the verifications of (RD2) are reduced to the 3-connected single-element extensions and coextensions of the elements of \(M\). The most known 2-rounded class of matroids is \(\{U_{2,4}\}\) (Seymour [13]). Several other examples may be found in [9], page 481.

Let \(\mathcal{N}\) be a class of matroids. We say that a class \(F\) of 3-connected matroids is \((3, k)\)-rounded in \(\mathcal{N}\) provided

\[(3KR)\] if \(M\) is a 3-connected matroid in \(\mathcal{N}\) with an \(F\)-minor and \(X \subseteq E(M)\) such that \(|X| = k\), then \(M\) has an \(F\)-minor with \(X\) contained in its ground set.

Although such terminology was introduced here, Oxley [10] already proved that \(\{U_{2,4}, \mathcal{W}^3\}\) and \(\{M(\mathcal{W}_3), \mathcal{W}^3, P_6, Q_6, U_{3,6}\}\) are \((3, 3)\)-rounded in the class of all matroids. Note that \((3, 2)\)-rounded corresponds to 2-rounded.

We say that a matroid \(M\) is minimal in a class if such class has no matroid isomorphic to a proper minor of \(M\). We denote \(F^* : \{M^* : M \in F\}\), \(\bar{r}(F) := \max\{r(M) : M\) is minimal in \(F\}\) and \(\bar{r}^*(F) := \bar{r}(F^*)\). We also define, for \(k = \{1, 2, 3\}\), \(F[\mathcal{N}, 3, k]\) to be the class of the 3-connected matroids of \(\mathcal{N}\) with an \(F\)-minor and with rank at most \(\bar{r}(F) + k + \lceil \frac{k-1}{2}\rceil\)

Theorem 1.6. Let \(F\) be a finite class of 3-connected matroids and \(\mathcal{N}\) be a class of matroids closed under minors, duals and isomorphisms. For \(k = \{1, 2, 3\}\), if condition \((3KR)\) holds each matroid in \(F[\mathcal{N}, 3, k] \cap F^*[\mathcal{N}, 3, k]\), then \(F\) is \((3, k)\)-rounded in \(\mathcal{N}\).

We say that a class of matroids \(R\) is a class of representatives of \(F\) if each matroid in \(F\) is isomorphic to a matroid in \(R\). Since \(F[\mathcal{N}, 3, k] \cap F^*[\mathcal{N}, 3, k]\) has a finite class of representatives when \(F\) is finite, Theorem 1.6 proves that, in finitely many steps, we can check if \(F\) is \((3, k)\)-rounded. The next corollary gives a strategy to find interesting \((3, k)\)-rounded classes of matroids.

Corollary 1.7. Let \(\mathcal{N}\) be a class of matroids closed under minors, duals and isomorphisms and let \(k \in \{1, 2, 3\}\). If \(F\) is a finite class of 3-connected matroids, then there is a finite \((3, k)\)-rounded class of matroids containing \(F\) and whose minimal elements are in \(F\). In particular one of these classes is a minimal class of representatives of \(F[\mathcal{N}, 3, k] \cap F^*[\mathcal{N}, 3, k]\).

We define a class \(F\) of 3-connected matroids to be \((3, k, l)\)-rounded in \(\mathcal{N}\) provided

\[(3KLR)\] if \(M\) is a 3-connected matroid in \(\mathcal{N}\) with an \(F\)-minor and \(X \subseteq E(M)\) such that \(|X| = k\) and \(r(X) \leq l\), then \(M\) has an \(F\)-minor with \(X\) contained in its ground set.

In [1] it is proved that the following classes are \((3, 3, 2)\)-rounded: \(\{U_{2,4}\}\) and \(\{U_{2,4}, M^*(K_{3,3}), F_7\}\) in the class of all matroids, \(\{M^*(K_{3,3})\}\) in the class of cographic matroids and \(\{F_7\}\) in the class of binary matroids. Another result we present here is:
Theorem 1.8. Let $\mathcal{F}$ be a finite class of 3-connected matroids and $\mathcal{N}$ be a class of matroids closed under minors, duals and isomorphisms. For $k \in \{1,2,3\}$, if condition (3KLR) holds for each matroid in $\mathcal{F}[\mathcal{N},3,k]$ then $\mathcal{F}$ is $(3,k,l)$-rounded in $\mathcal{N}$.

Naturally, we have an analog of Corollary 1.7 for this case. Moreover, if we switch the condition “$|X| = k$” in (3KLR) by “$|X| \leq k$” or “$|X| \geq k$” or other suitable predicates, analogous theorems will hold. Similarly we may change “$r(X) \leq l$” by “$r(X) = l$”.

The criteria given in these theorems have a rank and corank gap that make its verification computationally hard. But in some cases, as when $\mathcal{N}$ is the class the matroids representable over a certain field, computer-based approach is feasible. In other cases, some particularities of the class $\mathcal{F}$ may be used. Theorem 1.8 shows that if the subclass of $\mathcal{N}$ formed by the matroids up to a fixed rank has a finite set of representatives under isomorphisms, then the task of deciding if $\mathcal{F}$ is vertically $(3,k,l)$-rounded in $\mathcal{N}$ is finite for $l = 2,3$, again the class of representable matroids over a field is a feasible example. Moreover if $k = 2,3$, then this last hypothesis over $\mathcal{N}$ can be dropped, in this case proceed as in the proof of Theorem 1.6.

2. Configurations

We define, for 3-connected matroids $M$ and $N$, $(C^*,p)$ to be an $(M,N)$-configuration if $C^*$ is a rank-3 cocircuit of $M$, $p \in cl_M(C^*) - C^*$ and $(x,p)$ is vertically $N$-contractible for some $x \in C^*$ (and therefore, for every $x \in C^*$, according to Lemma 2.2). We say that $(C^*,p)$ is an $(M,N)$-configuration containing $x$ if $x \in C^* \cup p$.

Whittle [15] established the following three lemmas:

Lemma 2.1. Let $M$ be a 3-connected matroid with a 3-connected minor $N$. Suppose that $x$ and $p$ are elements of $M$ such that $x$ and $(x,p)$ are vertically $N$-contractible but $p$ is not vertically $N$-contractible in $M$. Then there is an $(M,N)$-configuration $(C^*,p)$ containing $x$.

Lemma 2.2. Let $M$ be a 3-connected matroid with a 3-connected minor $N$. Suppose that $(C^*,p)$ is an $(M,N)$-configuration.

(a) If $x, y \in C^*$, then $si(M/p,x) \equiv si(M/p,y)$.

(b) For each $x \in C^*$, $(x,p)$ is vertically $N$-contractible.

Lemma 2.3. Let $C^*$ be a rank-3 cocircuit of a 3-connected matroid $M$. If $x \in C^*$ has the property that $cl_M(C^*) - x$ contains a triangle of $M/x$, then $si(M/x)$ is 3-connected.

Corollary 2.4. Suppose that, in a 3-connected matroid $M$, there is a triangle $T$ intersecting a triad $T^*$. Let $T^* - T = \{x\}$. Then $si(M/x)$ is 3-connected.

An $(M,N)$-configuration $(C^*,p)$ is said to be connected or disconnected according whether $M|(C^* \cup p)$ is a connected or disconnected matroid respectively. For a rank-3 simple matroid $H$ it is easy to verify that $H$ is connected having a 4-circuit or $H$ has a coloop. So, if we define $H := M|(C^* \cup p)$ we may check that, in the case that $H$ is disconnected, there is a line $L$ of $M$ that contains $p$ and an element $x \in C^*$ such that $E(H) = L \cup x$, where $x$ is a coloop of $H$. In such situation we simply define $x$ as the coloop and $L$ as the line of such disconnected $(M,N)$-configuration.

It follows from Lemma 2.3 and from the observations above that:

Lemma 2.5. Let $M$ and $N$ be a 3-connected matroids. Suppose that $(C^*,p)$ is an $(M,N)$-configuration, then every element of $C^*$ in a 4-circuit of $M|(C^* \cup p)$ is vertically $N$-removable in $M$. Moreover, if $(C^*,p)$ is a connected $(M,N)$-configuration, then $r_M(V_N(M) \cap C^*) = 3$. 

It is also straightforward from Lemma 2.3 that:

**Lemma 2.6.** For 3-connected matroids $M$ and $N$, the coloop of every disconnected $(M, N)$-configuration is vertically $N$-contractible in $M$.

The next two lemmas also have straightforward proofs.

**Lemma 2.7.** In a 3-connected matroid with rank at least 4, there is no pair of distinct rank-3 cocircuits with the same closure.

**Lemma 2.8.** For 3-connected matroids $M$ and $N$, with $r(M) \geq 4$, there is no distinct disconnected $(M, N)$-configurations with same coloop and line.

**Lemma 2.9.** Let $M$ and $N$ be 3-connected matroids and let $(C^*, p)$ be a disconnected $(M, N)$-configuration with line $L$ and coloop $x$. Suppose that $y \in L - V_N(M)$ and $|L - (V_N(M) \cup p)| \geq 2$. Then there exists a cocircuit $D^*$ of $M$ such that $(D^*, y)$ is an $(M, N)$-configuration, $x \notin cl_M(D^*)$, and $L - y \subseteq D^*$.

**Proof.** First we show:

**2.8.1.** $x$ is in no triangle of $M$ and all triangles of $M$ that meet $L - p$ are contained in $L$.

Let $T$ be a triangle of $M$ not contained in $L$. By orthogonality with $C^*$, $x \in T$ if and only if $T$ meets $L - p$. So, if $x \in T$, then, since $|L - (V_N(M) \cup p)| \geq 2$, there is an element $y' \in L - (V_N(M) \cup T \cup p)$. So, $T$ is a triangle of $M/y'$ contained in $cl_M(C^*)$. Thus, by Lemma 2.3 $y' \in V_N(M)$, a contradiction. This proves 2.8.1.

By 2.8.1 $si(M/x) = M/x$, and, by Lemma 2.6 $x$ is $N$-contractible. Let $K := L - \{y, p\}$. By 2.8.1 $si(M/y) \subseteq M \setminus K / y$. But $x$ and $y$ are in series in $M \setminus K$. So $M \setminus K / x \subseteq M \setminus K / y \subseteq si(M/y)$, which is not 3-connected. Let $A_1, A_2$ be a 2-separation for $M \setminus K / x$, with $y \in A_1$.

Next we show:

**2.8.2.** $r_{M \setminus x}(A_i \cup K) = r_{M \setminus x}(A_i) + 1$ for $i = 1, 2$. Moreover $p \in A_2$.

Since $K \subseteq L$ and $y \in L \cap A_1$, it follows that $r_{M\setminus x}(A_1 \cup K) \leq r_{M\setminus x}(A_1) + 1$. Now, if $r_{M\setminus x}(A_1 \cup K) = r_{M\setminus x}(A_1)$, we have:

$$r_{M\setminus x}(A_1 \cup K) + r_{M\setminus x}(A_2) = r_{M\setminus x}(A_1) + r_{M\setminus x}(A_2) = r(M/x \setminus K) + 1 = r(M/x) + 1,$$

a contradiction to the 3-connectivity of $M/x$. So the first part of 2.8.2 holds for $i = 1$. In particular, note that the first part of 2.8.2 for $i = 1$ implies that $p \in A_2$. Then, analogously, 2.8.2 holds also for $i = 2$.

From the fact that $p \in cl_{M\setminus x}(A_1 \cup K)$ and from 2.8.2 for $i = 1$, it follows that:

$$r_{M\setminus x,p}(A_1 \cup K) = r_{M\setminus x}(A_1 \cup K) - 1 = r_{M\setminus x}(A_1),$$

and as $\{A_1, A_2\}$ is a 2-separation for $M/x \setminus K$, then:

$$r_{M\setminus x,p}(A_1 \cup K) + r_{M\setminus x,p}(A_2 - p) = r_{M\setminus x}(A_1) + r_{M\setminus x}(A_2) - 1 \leq r(M/x \setminus K) = r(M/x, p) + 1.$$

As $si(M/x, p)$ is 3-connected, it follows that $r_{M\setminus x,p}(A_1 \cup K) \leq 1$ or $r_{M\setminus x,p}(A_2 - p) \leq 1$. We will verify now that the latter of these inequalities holds. Suppose for a contradiction that the former one holds. So $r_M(A_1 \cup \{x, p\}) \leq r_{M\setminus x,p}(A_1) + 2 \leq 3$. Let $w \in A_1 - y$. Since $x$ is in no triangle of $M$ and $w \notin L$, so $\{x, y, w, p\}$ is a circuit of $M$. Thus $\{x, w, p\}$ is a triangle of $M/y$ contained in $cl_M(C^*)$, and, by Lemma 2.3 $y \in V_N(M)$, a contradiction. Hence $r_{M\setminus x,p}(A_2 - p) \leq 1$. Since swapping the labels of $x$ and $y$ induces an isomorphism between $M \setminus K / x$ and $M \setminus K / y$, then:

$$r_{M\setminus K / y, p}(A_2 - p) = r_{M\setminus v_N(M), p}(A_2 - p) = r_{M\setminus x,p}(A_2 - p) \leq 1.$$
and, therefore, \( r_M(A_2 \cup K) = r_M(A_2 \cup y) \leq 3 \). Now we prove:

**2.8.3.** There is a rank-3 cocircuit \( D^* \) of \( M/x \) meeting \( K \) and contained in \( A_2 \cup K \).

Suppose that 2.8.3 does not hold. Hence \( r^*_{M/x}(A_2 \cup K) = r^*(A_2) + |K| \). Therefore:

\[
\begin{align*}
  r^*_{M/x}(A_1 \cup K) + r^*_{M/x}(A_2) &= r^*_{M/x}(A_1 \cup K) + r^*_{M/x}(A_2 \cup K) - |K| \\
  &= r^*_{M/x}(A_1) + r^*_{M/x}(A_2) - |K| \\
  &= r^*(M/x) + |K| \\
  &= r^*(M) + 1,
\end{align*}
\]

a contradiction to the fact that \( M/x \) is 3-connected. So 2.8.3 holds.

By orthogonality with \( L \), we have \( D^* \cup = \{y\} \). Since \( M/x \) has no cocircuits with rank less than 3, it follows that \( 3 \leq r_M(D^*) \leq r_M(D^*) \leq r_M(A_2 \cup K) \leq 3 \). This proves the lemma. □

**Lemma 2.10.** Let \( M \) and \( N \) be 3-connected matroids and \( (C^*, p) \) a disconnected \((M, N)\)-configuration with line \( L \) and coloop \( x \). Then, for each \( y \in L - V_N(M) \), there is, in \( M \), a cocircuit of \( D^* \) containing \( L - y \), such that \( (D^*, y) \) is an \((M, N)\)-configuration.

**Proof.** Let \( y \in L - V_N(M) \). If \( L \cap V_N(M) = \emptyset \), then the result follows from Lemma 2.9. So, we may suppose that there is \( z \in L \cap V_N(M) \). So \( s(M/z, p) \cong s(M/z, y) \) is 3-connected with an \( N \)-minor, but \( s(M/y) \) is not 3-connected. Therefore, by Lemma 2.1, there is an \((M, N)\)-configuration \((D^*, y)\) containing \( z \). By orthogonality \( L - y \subseteq D^* \).

**Corollary 2.11.** Let \( M \) and \( N \) be 3-connected matroids such that all \((M, N)\)-configurations are disconnected and minimum. Suppose that \((T^*_y, p)\) is an \((M, N)\)-configuration with line \( \{x_1, x_2, p\} \) and coloop \( x \). For \( \{i, j\} = 1, 2 \), if \( x_i \notin V_N(M) \), then there is an element \( y_i \in V_N(M) - x \) such that \( \{x_j, y_i, p\} \) is a triad of \( M \). Moreover \( y_1 \neq y_2 \) provided both elements exist.

**Lemma 2.12.** Let \( M \) and \( N \) be 3-connected matroids and suppose that there is no connected \((M, N)\)-configurations. If \((C^*_y, y)\) is an \((M, N)\)-configuration with line \( L \), then \( r_M(V_N(M)) \geq |L - V_N(M)| \).

**Proof.** By Lemma 2.10 and by hypothesis, for each \( y \in L - V_N(M) \), there is a disconnected \((M, N)\)-configuration \((C^*_y, y)\) with line \( L \), whose coloop we denote \( x_y \). Write \( X := \{x_y : y \in L - V_N(M)\} \). By Lemma 2.8, \( |X| = |L - V_N(M)| \). Since, for each \( y \), \( C^*_y \cap X = \{x_y\} \), then by orthogonality, \( X \) is independent in \( M \). The lemma is proved. □

**Lemma 2.13.** Let \( M \) be a 3-connected matroid not isomorphic to \( M(\mathcal{W}_3) \). Suppose that \( E(M) \) has a 5-subset \( X := \{x_1, x_2, p_1, p_2, p_3\} \) such that \( T^*_1 := \{x_1, p_2, p_3\} \) and \( T^*_2 := \{p_1, x_2, p_3\} \) are triads and \( T := \{p_1, p_2, p_3\} \) is a triangle of \( M \). Then

(a) if \( r_M(X) = 3 \), then \( M \cong \mathcal{W}_3 \),

(b) \( p_3 \) is in no triangle of \( M \) but \( T \),

(c) if there is an element \( x_3 \) such that \( T^*_3 := \{p_1, p_2, x_3\} \) is a triad of \( M \), then \( x_1, x_2, x_3 \) are contractible in \( M \) and \( M/T \) is simple.

(d) if, for a matroid \( N \), \( \{x_1, p_1\} \) is \( N \)-contractible then, \( x_1, x_2, p_1, p_2, p_3 \) is an \( N \)-biweb.

Moreover, if such element \( x_3 \) as in item (c) exists, then \( x_1, x_2, x_3, p_1, p_2, p_3 \) is an \( N \)-triewb.

**Proof.** Suppose that \( r_M(X) = 3 \). Then \( r_M(X) + r^*_M(X) - |X| \leq 1 \). So, \( r(M) = 3 \) and \( E(M) = 6 \). Let \( x_3 \in E(M) - X \). Let \( T^* := C_M(x_3, T^*_2) \), by the simplicity of \( M \) and by orthogonality with \( T^*_2 \), it follows that \( T^* := \{x_2, y_3, x_3\} \). Now (a) follows from Proposition 8.8.5 (iii). Item (b) follows from [111], Lemma 3.4. Let us prove (c) now. By item (b), \( p_3 \) is in no triangle of \( M \) but \( T \).
Analogously, this holds also for $p_1$ and $p_2$. By orthogonality with the $T_i^*$’s the $x_i$’s are in no triangles. Moreover, by Corollary \[2.4\] $x_1$, $x_2$ and $x_3$ are contractible. Let us prove that $M/T$ is simple now. Let $C$ be a circuit of $M$ satisfying $|C - T| \leq 2$. We shall prove that $C = T$. Suppose that contrary. It is easy to see that $T$ meets no other triangle of $M$. Thus $|C| = 4$. We may assume that $C \cap T = \{p_1, p_2\}$. By orthogonality with $T_2^*$ and $T_3^*$, $C - T = \{x_1, x_2\}$. Thus $r_M(X) = 3$. A contradiction to item (a). Therefore $M/T$ is simple. This proves item (c). To prove item (d), suppose that $\{x_1, p_1\}$ is vertically $N$-contractible. By Lemma \[2.2\] applied twice:

$$si(M/x_1, p_1) \equiv si(M/p_2, p_1) \equiv si(M/x_2, p_2).$$

By Corollary \[2.4\] and since $si(M/p_1, p_2) \equiv si(M/T)$, then $x_1$, $x_2$ and $T$ are vertically $N$-contractible. This proves the first part of item (d). The second part follows from item (c).

**Proof of Theorem 2.7** Since every 3-connected matroid with rank at least 3 has an $U_{1,3}$-minor, we may suppose that $N$ is simple.

For $k = 1$, the result is straightforward from the Splitter Theorem (see \[9\], Lemma 12.3.11 for details). Let $M$ and $N$ be a counter-example to the theorem minimizing $k$. Then $k \geq 2$. By the minimality of $k$ applied twice, there is an element $x \in V_N(M)$ and an element $y \in V_N(si(M/x_1)) - V_N(M)$. So, by Lemma 2.1 $r(M) \geq 4$ and there is an $(M, N)$-configuration $(C^*_1, y_1)$. If such $(M, N)$-configuration is connected, the result follows from Lemma 2.5. So we may assume that $M$ has no connected $(M, N)$-configurations. Let $L := \{y_1, \ldots, y_n\}$ be the line and $x_1$ be the coclosure of $(C^*_1, y_1)$. Say that $y_{m+1}, \ldots, y_n \in V_N(M)$. By Lemma 2.12 $m \leq 2$.

If $n - m \geq 2$, then $\{x_1, y_m, y_{n-1}\}$ is a rank-3 subset of $V_N(M)$. Thus $m = 2$ and $n = 3$. So, we may assume that all $(M, N)$-configurations are minimum. By Lemma 2.11 there is an element $x_2$ in a triad with $y_3$ and $y_1$. Thus $\{x_1, x_2, y_3\} \subseteq V_N(M)$. Hence $r_M([x_1, x_2, y_3]) = 3$. So, $r_M([x_1, x_2, y_1, y_2, y_3]) = 3$. By Lemma 2.13 $M \cong \mathcal{N}$ and the theorem holds.

### 3. Critical scenes

For the purposes of our proof, we define a pair of 3-connected matroids $(M, N)$ to be a **critical scene** if $M$ has an $N$-minor, $r(M) - r(N) \geq 4$ and $r_M(V_N(M)) = 3$. From Lemma 2.5 we conclude:

**Corollary 3.1.** Let $(M, N)$ be a critical scene and $(C^*, p)$ be a connected $(M, N)$-configuration. Then $V_N(M) \subseteq cl_M(C^*)$.

**Lemma 3.2.** If $(M, N)$ is a critical scene and $(C^*, p)$ is an $(M, N)$-configuration, then there is no circuit contained in $C^*$ that meets $V_N(M)$.

**Proof.** Suppose for a contradiction that there is a circuit $C \subseteq C^*$ and an element $x \in C \cap V_N(M)$. Let $M_x = si(M/x)$. By Theorem 1.1, $r_M(V_N(M_x)) \geq 3$. So there is a $q \in V_N(M_x) - cl_M(C^*)$. By Lemma 2.1 there is an $(M, N)$-configuration $(D^*, q)$ containing $x$. By orthogonality, there is a $y \in (D^* \cap C) - x$. As $q \in cl_M(D^*) - cl_M(C^*)$, there is a $z \in D^* - cl_M(C^*)$. Let $D := C_M(q, \{x, y, z\})$. Note that $z \in D$, because $q \notin cl_M(C^*)$. By orthogonality with $C^*$, $D = \{x, y, z, q\}$. By Lemma 2.5 applied to $(D^*, q)$, $D \subseteq V_N(M)$. Since $(C^*, p)$ is a connected $(M, N)$-configuration, $M|(C^* \cup p)$ has 4-circuit $C'$. By Lemma 2.5 $C' \subseteq V_N(M)$. So $r_M(D \cup C') \geq 4$ and $D \cup C' \subseteq V_N(M)$, a contradiction.

We say that an $(M, N)$-configuration $(C^*, p)$ is **minimum** if $|C^*| = 3$. Next we show:

**Lemma 3.3.** Suppose that $(M, N)$ is a critical scene, then all connected $(M, N)$-configurations are minimum.
\textbf{Proof.} Let \((C^*, p)\) be a connected \((M, N)\)-configuration. By Corollary 3.1, there is a basis \(B\) for \(M|V_N(M)\) contained in \(C^*\). If \(|C^*| \geq 4\), there is \(z \in C^* - B\) and \(C_M(z, B)\) is a circuit that contradicts Lemma 3.2. Thus \(|C^*| = 3\) and the lemma holds. \hfill \square

\textbf{Lemma 3.4.} If \((M, N)\) is a critical scene such that \(r(M) \geq 5\), then all \((M, N)\)-configurations are disconnected.

\textbf{Proof.} Suppose that \((C^*, p)\) is a connected \((M, N)\)-configuration. By Lemma 3.3 \((C^*, p)\) is minimum. Write \(C^* = \{x_1, x_2, x_3\}\). Since \(M|(C^* \cup p)\) is connected, \(C = \{x_1, x_2, x_3, p\}\) is a circuit of \(M\). For \(i \in \{1, 2, 3\}\), let \(M_i := s_i(M/x_i)\). As \(r(M_i \cap E(M_i)) = 2\), there is \(q_i \in V_N(M_i) - cl_M(C^*) \subseteq V_N(M_i) - V_N(M)\). By Lemma 2.1, there is an \((M, N)\)-configuration \((C^*_i, q_i)\) containing \(x_i\). Since \(q_i \in cl_M(C^*_i) - cl_M(C^*)\), there is \(y_i \in C^*_i - cl_M(C^*)\). Now we show:

3.4.1. For \(i = 1, 2, 3\), \(C^*_i \cap C = \{p, x_i\}\).

By orthogonality with \(C\) there is \(x \in (C \cap C^*_j) - x_j\). Suppose for a contradiction that \(x \neq p\). By orthogonality between \(C\) and \(C^*, x \in C^* - x_j\). Let \(D := C_M(q_i, \{x, x_i, y_i\})\). As \(q_i \notin cl(C^*)\), then \(y_i \in D\). By orthogonality with \(C^*, (x, x_i) \subseteq D\). Thus \(D = \{x, x_i, y_i, q_i\}\). By Lemma 2.5 \(y_i \in V_N(M)\), contradicting Corollary 3.1. Thus 3.4.1 holds.

Let \(C_2 = C_M(q_2, \{p, x_2, y_2\})\). As argued before, \(y_2 \in C_2\). By orthogonality with \(C^*, C_2 = \{p, y_2, q_2\}\). Since \(p \in C^*_i\), by orthogonality with \(C^*_i, y_2 \in C^*_i\) or \(q_2 \in C^*_i\). Hence \(C^*_2 \subseteq cl_M(C^* \cup C^*_i)\). Similarly, \(C^*_2 \subseteq cl_M(C^* \cup C^*_i)\). Let \(X = C^*_1 \cup C^*_2 \cup C^*_3 \cup C^*\). Then \(r_M(X) = 4\). As each \(x_i\) is in \(C^*_i\) but not in \(C^*_j\) for \(j \neq i\), thus \(r_M(X) = |X|\). Since \(M\) is 3-connected with more than three elements, this implies that \(r_M = r_M(X) = 4\), a contradiction. \hfill \square

\textbf{Lemma 3.5.} Let \((M, N)\) be a critical scene such that \(r(M) \geq 5\). Then all \((M, N)\)-configurations are disconnected and minimum.

\textbf{Proof.} We have proved on Lemma 3.4 that such \((M, N)\)-configurations must be disconnected. Suppose that \((C^*_i, y_i)\) is an \((M, N)\)-configuration with line \(L\). We shall prove that \(|L| \leq 3\). Suppose the contrary. By Lemma 3.2 \((L - y_1) \cap V_N(M) = \emptyset\). Thus, by Lemma 2.12 \(r_M(V_N(M)) \geq 4\), a contradiction. \hfill \square

\textbf{Lemma 3.6.} Let \((M, N)\) be a critical scene. Suppose that \((T^*, p_1)\) is a disconnected \((M, N)\)-configuration with line \(T := \{p_1, p_2, p_3\}\) and co-loop \(x_1\). Choose the ground set of \(M_i := s_i(M/x_1)\) maximizing \(|E(M_i) \cap T|\). If there exists \(q_1 \in V_N(M_i) - (V_N(M) \cup T)\), then there are elements \(x_2, x_3, q_1, q_2, q_3\), such that \(x_1, x_2, x_3, p_1, p_2, p_3, q_1, q_2, q_3\) is an \(N\)-prism.

\textbf{Proof.} By hypothesis, \(x_1\) and \(\{x_1, q_1\}\) are vertically \(N\)-contractible in \(M\), but \(q_1\) is not. By Lemma 2.1 there is an \((M, N)\)-configuration \((U^*, q_1)\) containing \(x_1\). By Lemma 3.5 \(U^*\) is a triad of \(M\) and there is a triangle \(U\) of \(M\), such that \(q_1 \in U \subseteq U^* \cup q_1\).

Let us verify that \(x_1 \notin U\). If \(x_1 \in U\), then, by orthogonality with \(T^*, U\) intersects \(T\). So, in \(M/x_1, q_1\) is in parallel with an element of \(T\). This contradicts the maximality of \(|E(M_i) \cap T|\). Thus \(x_1 \notin U\).

Next we show that \(U \cap T = \emptyset\). Assume the contrary. As \(q_1 \notin T\) and \(U - q_1 = U^* - x_1\), then \(U^*\) intersects \(T\). But \(x_1 \in U^* - (U \cup T)\). Thus \(U^* \cap T = U^* - x_1 = U - q_1\). So \(M | U \cup T \cong U_{2,4}\), a contradiction. Therefore \(U \cap T = \emptyset\).

Hence the sets \(U \cap cl^*_M(T)\) and \(T \cap cl^*_M(U)\) are empty. By Corollary 2.11 applied on \((U^*, q)\) and \((T^*, p)\), \(V_N(M) \subseteq cl^*_M(T) \cap cl^*_M(U)\), so \(V_N(M) \cap (T \cup U) = \emptyset\). In particular, Corollary 2.11 implies that each element of \(V_N(M)\) is in two triads of \(M\), one with a pair of elements
of \( T \), other with a pair of elements of \( U \). It implies that \(|V_N(M)| = 3\). Say that \( V_N(M) = \{x_1, x_2, x_3\} \) and \( U = \{q_1, q_2, q_3\} \). We may choose the labels of the elements in such a way that for \( \{i, j, k\} = \{1, 2, 3\} \), \( \{x_i, p_j, p_k\} \) and \( \{x_i, q_j, q_k\} \) are triads of \( M \).

The lemma follows from the application of Lemma 2.13 (d) on the \( x_i \)'s and \( p_i \)'s and on the \( x_i \)'s and \( q_i \)'s.

\[ \square \]

4. PROOFS FOR THE THEOREMS

Proof of Theorem 1.2: Suppose that \((M, N)\) is a critical scene. By Theorem 1.1, \( r_M(V_N(M)) = 3\). Let \( x \in V_N(M) \). By Theorem 1.1, again, there is an element \( p_1 \in V_N(si(M/x)) - cl_M(V_N(M)) \).

By Lemma 2.1, there is an \((M, N)\)-configuration \((T^*, p_1)\). By Lemma 3.5 and \( T^* \) is a triad and there is a triangle \( T \) such that \( p_1 \in T \subseteq T^* \cup p_1 \) and, moreover, by Corollary 2.11,

4.0.1. Each element of \( V_N(M) - T \) is in a triad with two elements of \( T \).

Write \( T^* - T = \{x_1\} \), \( T \cap T^* = \{p_1, p_2\} \) and \( M := si(M/x_1) \). Moreover, choose \( E(M_1) \) maximizing \( |E(M_1) \cap T| \).

We may suppose that \( M \) has no \( N \)-prisms. Then, by Lemma 3.6, \( V_N(M_1) \subseteq T \cup V_N(M) \).

By Theorem 1.1, \( V_N(M_1) \nsubseteq T \), so there is an element \( x_2 \in V_N(M_1) - (T \cup x_1) \). Since \( V_N(M_1) \subseteq T \cup V_N(M) \), by 4.0.1, \( x_2 \) is in a triad \( T^*_2 \) with two elements of \( T \). We may suppose that \( T^*_2 \) = \( \{p_1, x_2, x_3\} \) because there is no coline with four elements meeting \( T \). The first part of the theorem follows from Lemma 2.13 (d).

For the second part, first suppose that \( r(M) - r(N) \geq 5 \). By the first part of the Theorem applied to \( M_1 \), there is an element \( x_3 \in V_N(M_1) - (T \cup x_2) \). Then \( x_3 \in V_N(M) - T \). Thus, by 4.0.1, there is a triad \( T^*_3 \) contained in \( T \cup x_3 \). Since there is no coline with four elements meeting \( T \), \( T^*_3 = \{p_1, p_3, x_3\} \), and (a) holds form Lemma 2.13 (d) again.

Now, suppose that \( r(M) - r(N) \geq 6 \). Applying what we proved to \( M_1 \), it follows that \( M_1 \) has an \( N \)-trielweb and, therefore, \( V_N(M_1) \nsubseteq T \cup \{x_1, x_2\} = (T \cup V_N(M)) \cap E(M_1) \). Now the result follows from Lemma 4.6.

Proof of Theorem 1.8: Let us make the proof for \( k = 3 \), the other cases are more simple and analogous. By hypothesis, property (3KR) holds for matroids in \( \mathcal{F}' := \mathcal{F}[\mathcal{N}, 3, k] \cap \mathcal{F}^*[\mathcal{N}, 3, k] \). Suppose that \( M \) is a matroid in \( \mathcal{N} - \mathcal{F}' \) for which (3KR) fails, minimizing \( |E(M)| \). Let \( X \) be a 3-subset of \( E(M) \). Reducing the proof to the dual case if necessary, we may assume that \( r(M) \geq r(\mathcal{F}) + 5 \). Let \( N \) be a matroid of \( \mathcal{F} \) such that \( r(N) \leq r(\mathcal{F}) \) and \( M \) has an \( N \)-minor. If there is \( w \in V_N(M - cl_M(X)) \), then \( si(M/w) \) contradicts the minimality of \( M \). Thus \( V_N(M) \subseteq X \) and, therefore \( r_M(V_N(M)) \leq 3 \). By Theorem 1.2 (b), \( M \) has an \( N \)-trielweb \( x_1, x_2, x_3, p_1, p_2, p_3 \), and \( V_N(M) = \{x_1, x_2, x_3\} = X \). Now \( M/p_1, p_2, p_3 \) contradicts the minimality of \( M \) and the theorem is proved.

The proof of Theorem 1.8 is similar to the preceding one and is left to the reader.

Acknowledgments

The author thanks the staff and professors in the Mathematics Department at UEM for the efforts in provide the adequate conditions for his work although the circumstances. Special thanks to Professors Emerson Carmello, Irene Nakaoaka and Ednei Santulo.

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