Optimizing the Forecast Interval for Determine the Parameters of the Earth Pole Motion Model

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Abstract. The paper provides a solution to the problem of determining the parameters of the model of the Earth pole motion that are optimal from the point of view of the standard deviation. The choice among the considered models is made according to the most accurate forecast on the different time intervals, and the estimate of the approximation interval duration is given that should be used to determine the unknown model parameters from the observational data.

1. Introduction
Mathematical models of the rotational-oscillatory motion of the deformable Earth, which with high accuracy identify its orientation parameters based on the observational data from the International Earth Rotation and Reference Systems Service (IERS), and also give a reliable forecast of these parameters, are fundamental when studying a number of astrometric, geodynamic and navigation problems [1-5].

In particular, because of the development of navigation systems, it has become necessary to achieve high accuracy of coordinate-time and navigation support for spacecraft and satellite Earth stations. In this problem forecasting the Earth pole coordinates and the irregularity of the Earth axial rotation plays a significant role.

Currently, one of the ways to improve the accuracy of navigation systems is to adapt the existing models of the pole oscillations for their autonomous use directly on board spacecraft with minimal correction from the Earth. In this paper results of studying the accuracy characteristics of the existing few-parameter models of the Earth pole models [2-5] by estimating the root mean square error (RMSE) of the forecast calculated for different time scales and analyzing its dependence of the approximation interval required to determine the unknown parameters of the model. The forecast is verified by the IERS observational data.

2. Models of the Earth pole motion
In order to describe the Earth pole oscillations and construct a numerical-analytical model that can give high-precision forecast the celestial-mechanical approach "deformable Earth - Moon in the Sun's gravitational field" was developed by Yu.G. Markov, V.V. Perepyolkin and others [2-5]. This approach allows one to develop few-parameter models with undefined coefficients that can be obtained from observational data. These models, on the one hand, give a sufficiently accurate forecast, adequate to the IERS observations, and on the other hand, they have a small number of parameters and
do not have great computational complexity. Therefore, it is advisable to use these models in the coordinate-time support of spacecraft is appropriate.

The models of the Earth pole motion developed in [2-5] can be represented as:

\[ x_p = c_x(\tau) + \sum_{i=1}^{n}(a_i^x \cos 2\pi f_i \tau + a_i^y \sin 2\pi f_i \tau), \]

\[ y_p = c_y(\tau) + \sum_{i=1}^{n}(b_i^x \cos 2\pi f_i \tau + b_i^y \sin 2\pi f_i \tau), \]

(1)

where \( c_{x,y}(\tau) \) is the pole trend of the \( x \) and \( y \) coordinate respectively, \( a_i^{x,y}, b_i^{x,y} \) are the unknown parameters that are to be determined from observations, and the number of frequencies \( n \) is obtained from the spectral analysis of the IERS observational data.

In this paper, a comparison is made between a two-frequency model with linear trend:

\[ x_p = c_{0x} + c_{1x} \tau + \tilde{d}_x^1 \cos 2\pi N \tau + \tilde{d}_y^2 \cos 2\pi \tilde{d}_x^1 \sin 2\pi \tilde{d}_x^1, \]

\[ y_p = c_{0y} + c_{1y} \tau + \tilde{d}_y^2 \cos 2\pi N \tau + \tilde{d}_y^2 \cos 2\pi \tilde{d}_y^2 \sin 2\pi \tilde{d}_y^2; \]

(2)

and quadratic trend:

\[ x_p = c_{0x} + c_{1x} \tau + c_{2x} \tau^2 + \tilde{d}_x^1 \cos 2\pi N \tau + \tilde{d}_y^2 \cos 2\pi \tilde{d}_x^1 \sin 2\pi \tilde{d}_x^1, \]

\[ y_p = c_{0y} + c_{1y} \tau + c_{2y} \tau^2 + \tilde{d}_y^2 \cos 2\pi N \tau + \tilde{d}_y^2 \cos 2\pi \tilde{d}_y^2 \sin 2\pi \tilde{d}_y^2; \]

(3)

and a seven-frequency model with a quadratic trend

\[ x_p = c_{0x} + c_{1x} \tau + c_{2x} \tau^2 + \sum_{i=1}^{7}(a_i^x \cos 2\pi f_i \tau + a_i^y \sin 2\pi f_i \tau), \]

\[ y_p = c_{0y} + c_{1y} \tau + c_{2y} \tau^2 + \sum_{i=1}^{7}(b_i^x \cos 2\pi f_i \tau + b_i^y \sin 2\pi f_i \tau). \]

(4)

In models (2) - (3) there are only two basic frequencies - Chandler frequency \( N \), equal to 0.845 and annual, equal to 1. In seven-frequency model (4) the following frequencies are added based on the spectral analysis: semiannual, equal to 2; monthly, equal to 13.28; bi-weekly, equal to 26.28; nine-day, equal to 40.58; weekly equal to 52.18.

To improve the accuracy of the coordinate-time and navigation support of spacecraft, it is necessary to analyze the available few-parameter models of the Earth pole oscillations [2-5], and, depending on the length of the forecasting interval and the forecast accuracy, to choose the most optimal model to use on board the spacecraft. Models (2) - (4) have a different number of unknown coefficients that should be determined by the least squares method based on the IERS observational data. Therefore, it is required to determine which model and with what number of parameters will give the best forecast for different time intervals.

Formally, this can be written as follows. Let \( RMSE(f_i(c_1, c_2, ..., c_m), \zeta_j, d_k) \) be the RMSE of the Earth pole coordinates forecast, built according to a model \( f_i(c_1, c_2, ..., c_m) \) containing the number of parameters \( m_i \) determined from the IERS observational data on an interval \( \zeta_j \) for a certain time interval \( d_k \). Let \( F = \{f_i(c_1, c_2, ..., c_m)\}_i^{n} \) be a set of models of the Earth pole coordinates with a different number of unknown parameters, \( m_i \), and \( n \) is the number of studied models.

It is required to find the model that will have the minimum forecast deviation from the IERS observational data for a given number of days:

\[ RMSE(F, \zeta_j, d_k) \rightarrow \min. \]
Forecasting intervals are divided into ultra-short (intraday), short-term (1-30 days), medium-term (several months) and long-term (several years). In this work, forecasts for short-term, medium-term and long-term time intervals have been investigated. As an example, the forecasts calculations for 7 and 45 days and for 2 years are given.

3. **Algorithm to determine the length of the approximation interval**

To minimize the root mean square error of the forecast when compared to the IERS observational data, an algorithm was developed that calculates the forecast RMSE when approximating the unknown model coefficients using the observation intervals of different lengths by the least squares method, and analyze the dependence of the RMSE on the length of the approximation interval and on the number of forecast days. The algorithm was implemented in the Maple.

One of the models (2) - (4) of Earth pole motion was selected, the required number of forecast days was set, and then the selected model was approximated, increasing the approximation interval with each step. The maximum and minimum values for the approximation interval were set depending on the forecast duration, the period of frequencies modulations included in the model and the general approximation principles.

At each step, the model was verified by constructing a forecast and comparing it with the IERS observational data by calculating the forecast RMSE, followed by comparison with the best RMSE value based on the results of previous calculations. If for both \( x_p \) and \( y_p \) the obtained value of RMSE was less than the previous one, then its value and the corresponding approximation interval were recorded as the best approximation. In order to avoid the data loss because of the calculation error, which inevitably appears during the approximation, the last three pairs of \{RMSE - \( \zeta \)\} are recorded.

Also, for the accuracy estimation the matrix of the RMSE evolution was formed, in which all pairs \{RMSE - \( \zeta \)\} were written.

Then, after all the pairs within the specified limits were calculated, the first day of the forecast was shifted by a step, and the procedure was repeated. The maximum approximation interval was chosen to be 6 years, which approximately corresponds to the main modulation period of the Earth pole observed oscillations.

4. **Analysis of the computational results**

The following results were obtained. For a 7-day forecast on Fig. 1 a comparison between the best value of the approximation interval \( \zeta \) according to the model (2) (red line) and (4) (black line) is shown as well as their corresponding root mean square error \( RMSE_{x,y} \) of the best forecast for coordinates \( x_p \) and \( y_p \). Here and further on the abscissa is the date in MJD format, representing the first day of the forecast – the date when the forecast was calculated.

In Fig. 1 it can be seen that the RMSE of the forecasts is higher for the seven-frequency model (4) than for the two-frequency model (2). Therefore, to determine the dependence of the forecast RMSE on the length of the approximation interval the model (2) was analyzed.

For the two-frequency model (2) the comparison between the first three best approximation intervals \( \zeta \) that correspond to the best three values for the forecast accuracy and the forecast RMSE for \( x_p \) and \( y_p \) is presented on the Fig. 2. The most accurate forecast was obtained with the approximation shown by the red line, the second most accurate forecast - by the green line, and the third most accurate forecast - by the blue line (Fig. 2). Since the first two best results have the equal RMSE up to the 4\(^{th}\) or 5\(^{th}\) decimal place, the red line is not visible on the figure and is completely under the blue line. That means that though the first best forecast has slightly better accuracy, it has much larger approximation interval than the second one, so it’s sufficient to take the smaller interval.
**Fig. 1.** Comparison between the best approximation intervals $\zeta$ for models (2) (red line) and (4) (black line) and their corresponding RMSE for $x_p$ and $y_p$.

**Fig. 2.** Comparison of the approximation interval $\zeta$ for model (2) and the corresponding RMSE for $x_p$ and $y_p$ (red line best forecast, green line - for the second best forecast, blue line - for the third best forecast)
The evolution of RMSE depending on the forecast date and approximation interval duration is also obtained (Fig. 3) for model (2). It is used to estimate the length of the approximation interval $\zeta$.

![Fig. 3. Evolution of RMSE for $x_p$ (left) and $y_p$ (right) according to model (2).](image)

Using the RMSE evolution its estimation was calculated with the dependence on the length of the approximation interval (Fig. 4).

![Fig. 4. RMSE estimation of the $x_p$ forecast depending on the of the approximation interval length](image)

When setting the required forecast accuracy, it is possible to obtain a range of values in which the length of the approximation interval can be varied. For example, if one set the deviation of the forecast accuracy from the minimum by no more than 1 mas, then the approximation interval can be taken from 15 to 37 days (Fig. 5).

![Fig. 5. RMSE estimation of the $x_p$ forecast depending on the of the approximation interval length for a given forecast accuracy](image)
In a similar way, it was found that for medium-term forecasts, for example, 45 days, the most accurate forecast is given by a two-frequency model with a linear trend (2). If one set the forecast accuracy, for example, the difference from the minimum value is no more than 1 mas, then the approximation interval can be taken from 119 to 198 days.

For long-term forecasting of the Earth pole coordinates the best of the three models in terms of forecast accuracy was a two-frequency model with a quadratic trend (3). If one set as the forecast accuracy, for example, the difference from the minimum value is no more than 1 mas, then the approximation interval can be taken from 1560 to 2030 days.

The accuracy of the expectation and variance of the length of the approximation interval and the corresponding RMSE are given in Tables 1-3.

**Table 1.** Expectation and variance of the approximation interval length $\zeta$.

| Forecast interval | Expectation | Variability |
|-------------------|-------------|-------------|
|                   | Model (2)   | Model (3)   | Model (4) | Model (2) | Model (3) | Model (4) |
| 7 days            | 161.230     | 61.638      | 77.74     | 146.47    | 40.299    | 35.384    |
| 45 days           | 356.936     | 85.114      | 84.217    | 235.7     | 44.339    | 35.975    |
| 2 years           | 1028.905    | 1178.055    | 1195.832  | 17.945    | 304.937   | 294.13    |

**Table 2.** Expectation and variance of the RMSE of $x_\rho$ coordinate.

| Forecast interval | Expectation | Variability |
|-------------------|-------------|-------------|
|                   | Model (2)   | Model (3)   | Model (4) | Model (2) | Model (3) | Model (4) |
| 7 days            | 1.075       | 1.093       | 1.306     | 0.668     | 0.709     | 0.793     |
| 45 days           | 5.39        | 9.095       | 15.595    | 3.197     | 6.737     | 11.569    |
| 2 years           | 28.733      | 26.33       | 26.473    | 8.618     | 8.883     | 8.656     |

**Table 3.** Expectation and variance of the RMSE of $y_\rho$ coordinate.

| Forecast interval | Expectation | Variability |
|-------------------|-------------|-------------|
|                   | Model (2)   | Model (3)   | Model (4) | Model (2) | Model (3) | Model (4) |
| 7 days            | 1.05        | 1.008       | 1.213     | 0.733     | 0.739     | 0.864     |
| 45 days           | 4.658       | 9.1         | 15.734    | 3.017     | 7.427     | 13.203    |
| 2 years           | 29.333      | 27.027      | 27.268    | 8.691     | 5.923     | 5.819     |
5. Conclusions
Thus, from the considered models (2) - (34), it was determined that for short and medium forecast the best is a two-frequency model with a linear trend (2), and for a long-term forecast a two-frequency model with a quadratic trend (3) is the best of the three. Considering the results taking into account additional frequencies in the models worsens the forecast accuracy.

The results obtained make it possible, by setting the required forecast accuracy, to select an area in which the length of the approximation interval can be varied to determine the unknown coefficients of the model from the IERS observational data.

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