On $Mp^c$-structures and Symplectic Dirac Operators

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Abstract

We prove that the kernels of the restrictions of symplectic Dirac or symplectic Dirac-Dolbeault operators on natural subspaces of polynomial valued spinor fields are finite dimensional on a compact symplectic manifold. We compute those kernels for the complex projective spaces. We construct injections of subgroups of the symplectic group (the pseudo-unitary group and the stabilizer of a Lagrangian subspace) in the group $Mp^c$ and classify $G$-invariant $Mp^c$-structures on symplectic spaces with a $G$-action. We prove a variant of Parthasarathy’s formula for the commutator of two symplectic Dirac-type operators on a symmetric symplectic space.
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1 Introduction

This paper is a contribution to the growing literature on Dirac operators in a symplectic context. In order for it to be self-contained, we have included all relevant definitions and constructions. The first introduction of symplectic spinors on a symplectic manifold $(M, \omega)$ was made by Kostant [9] using an auxiliary metaplectic structure. Metaplectic structures do not always exist, there is a topological obstruction of being a spin manifold so that important examples such as $\mathbb{C}P^{2n}$ are not metaplectic. In the metaplectic setting, using a symplectic connection, K. Habermann was the first to introduce [5] the notion of a symplectic Dirac operator $D$. A few years later, K. and L. Habermann used a compatible positive almost complex structure $J$ on $(M, \omega)$ and a linear connection preserving $J$ and $\omega$ (which may have torsion) to define a second Dirac operator $\tilde{D}$, [6]. They showed further that the commutator $[D, \tilde{D}]$ is elliptic. In [3], three of the present authors modified Habermann’s construction using an $Mp^c$-structure on $(M, \omega)$ which always exists and an $Mp^c$-connection to define symplectic Dirac operators on any symplectic manifold. The introduction of a compatible positive almost complex structure $J$ on $(M, \omega)$ and an $MU^c$-connection allows again the construction of a second operator $D_J$. The commutator $[D, D_J]$ is again elliptic. In [2], we observed that similar constructions could be performed on symplectic manifolds which admit a field $A$ of infinitesimally symplectic endomorphisms of the tangent bundle, yielding an operator $D_A$. It can be similarly extended in a Riemannian or pseudo-Riemannian context.

A compatible positive almost complex structure $J$ gives a splitting of the complexified tangent bundle, which led in [3] to the definition of two partial Dirac operators $D'$ and $D''$ such that

$$D = D' + D'' \quad \quad D_J = i(-D' + D'').$$

These operators were rediscovered in [8] in the framework of Kähler manifolds and named symplectic Dolbeault operators. We shall call them here symplectic Dirac–Dolbeault operators. When the connection is chosen so that the torsion vector of the induced linear connection vanishes, the operator $D''$ is the formal adjoint of $D'$. One can always choose such a connection (but not uniquely).

The choice of $J$ defines, in the space of spinor fields, a dense subspace consisting of polynomial-valued spinor fields. The operator $D'$ raises the degree of a polynomial valued spinor field by 1 whereas $D''$ lowers this degree by 1. The commutator is an elliptic operator which preserves the degree. Theorem [12] proves that the kernel of the operator $D$ restricted to spinor fields with values in polynomials of degree at most $k$ is finite dimensional for all $k$ on any compact symplectic manifold, for any choices made (i.e. any compatible positive almost complex structure $J$, any $Mp^c$-structure and any $MU^c$-connection with vanishing...
One way to limit the degree of arbitrariness in the construction is to restrict oneself to an invariant situation. We consider symplectic manifolds endowed with a symplectic action of a Lie group $G$ and we study in Section 3.2 the construction and classification of $G$-invariant $Mp^c$-structures. Although $Mp^c$-structures exist on any homogeneous manifolds, it is not guaranteed that homogeneous $Mp^c$-structures exist on any homogeneous manifolds. As could be guessed, the existence of a $G$-invariant $Mp^c$-structure on a $G$-homogeneous symplectic manifold is equivalent to the existence of a lift of the isotropy representation to the group $Mp^c$. In Section 2.7, we give necessary and sufficient conditions for the existence of a lift of a subgroup $H$ of the symplectic group into the group $Mp^c$. Proposition 2.18 gives such a lift for the pseudo-unitary group, and Proposition 2.22 for the subgroup preserving a complex Lagrangian subspace.

The last part of the paper deals with Dirac operators on symplectic symmetric spaces. Proposition 4.4 gives a Parthasarathy type formula for the operators of the type $[D', D'']$ and exhibit the dependence in the character of the isotropy group which characterizes the choice of the invariant $Mp^c$-structure. Section 4.3 gives the spectrum of the operator $[D', D'']$ and the kernels of the operators $D$, $D'$ and $D''$ restricted to polynomial valued spinor fields on complex projective spaces.

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2 Subgroups of $Sp(V, \Omega)$ and lifts to $Mp^c(V, \Omega, j)$

2.1 The symplectic Clifford algebra

Let $(V, \Omega)$ be a finite-dimensional real symplectic vector space of dimension $2n$. The symplectic Clifford Algebra $Cl(V, \Omega)$ is the associative unital complex algebra generated by $V$ with the relation

$$u \cdot v - v \cdot u = \frac{i}{\hbar} \Omega(u, v) 1$$

where $\hbar$ is a positive real number and $\hbar = \frac{\hbar}{2\pi}$.

A symplectic spinor space is a vector space carrying a representation of the symplectic Clifford algebra. This representation, called Clifford multiplication and denoted by $cl$, is derived from an irreducible unitary representation of the Heisenberg group. There are many
ways to construct such a representation; in the next subsection we describe a particular form of the Fock representation which is adapted to our applications.

2.2 The Fock representation of the Heisenberg group

Let \((V, \Omega)\) be a finite-dimensional real symplectic vector space of dimension \(2n\). Consider the Heisenberg Lie group \(H(V, \Omega)\) whose underlying manifold is \(V \times \mathbb{R}\) with multiplication

\[
(v_1, t_1)(v_2, t_2) = (v_1 + v_2, t_1 + t_2 - \frac{1}{2}\Omega(v_1, v_2)).
\]

Its Lie algebra \(\mathfrak{h}(V, \Omega)\) has underlying vector space \(V \oplus \mathbb{R}\) with bracket

\[
[(v, s), (w, s')] = (0, -\Omega(v, w))
\]

and is two-step nilpotent. The exponential map is \(\exp(v, s) = (v, s)\).

In any (continuous) unitary irreducible representation \(U\) of the Heisenberg group on a separable Hilbert space \(\mathcal{H}\), the centre \(\{0\} \times \mathbb{R}\) acts by multiples of the identity: \((0, t) \mapsto e^{i\lambda t}I_{\mathcal{H}}\) for some real number \(\lambda\) and it is known that any two irreducible unitary representations with the same non-zero central parameter are unitarily equivalent (Stone–von Neumann Uniqueness Theorem). Up to scaling, complex conjugation and equivalence there is just one infinite dimensional unitary irreducible representation fixed by specifying its parameter \(\lambda\) which we take as \(\lambda = -1/\hbar\). Note that a representation of the Clifford algebra corresponds to a representation of the Lie algebra of the Heisenberg group with prescribed central character equal to \(-i/\hbar\).

The infinite dimensional unitary representation of the Heisenberg group with central parameter \(-1/\hbar\) can be constructed in a number of ways, for example on \(L^2(V/W)\) with \(W\) a Lagrangian subspace of \((V, \Omega)\) (Schrödinger picture). For our purposes it is most useful to realise it on a Hilbert space of holomorphic functions (Fock picture). For this we fix a positive compatible complex structure (PCCS) \(j\) on \((V, \Omega)\).

**Definition 2.1** A compatible complex structure \(\tilde{j}\) on \((V, \Omega)\) is a (real) linear map of \(V\) which is symplectic, \(\Omega(\tilde{j}v, \tilde{j}w) = \Omega(v, w)\), and satisfies \(\tilde{j}^2 = -\text{Id}_V\). We denote by \(j(V, \Omega)\) the set of compatible complex structures. Given such a \(\tilde{j} \in j(V, \Omega)\), the map \((v, w) \mapsto G_{\tilde{j}}(v, w) := \Omega(v, \tilde{j}w)\) is a non-degenerate symmetric bilinear form. The symplectic group acts by conjugation on the space of compatible complex structures and the orbits are characterised by the signature of the form \(G_{\tilde{j}}\). We say \(\tilde{j}\) is positive if this form is positive definite. Let \(j_+(V, \Omega)\) denote the set of PCCS.

Picking \(j \in j_+(V, \Omega)\) gives a complex Hilbert space \((V, \Omega, j)\) of complex dimension \(n = \frac{1}{2}\dim_{\mathbb{R}} V\) with the Hermitean structure

\[
\langle v, w \rangle_j = \Omega(v, jw) - i\Omega(v, w) = G_j(v, w) - i\Omega(v, w), \quad |v|^2_j = \langle v, v \rangle_j
\] (2)
(which is complex linear in the first argument, anti-linear in the second).

We may consider the Hilbert space $\mathcal{H}(V, \Omega, j)$ of holomorphic functions $f(z)$ on $(V, \Omega, j)$ which are $L^2$ in the sense that the norm $\|f\|_j$ given by

$$
\|f\|_j^2 = h^{-n} \int_V |f(z)|^2 e^{-|z|^2/2} dz
$$

is finite, where $dz$ denotes the normalised Lebesgue volume on $V$ for the norm $|\cdot|_j$. The Heisenberg group $H(V, \Omega)$ acts unitarily and irreducibly on $\mathcal{H}(V, \Omega, j)$ by the representation $U_j$ where

$$(U_j(v, t)f)(z) = e^{-it/h + \langle z, v \rangle_j/2h - |v|^2/4h} f(z - v).$$

The Heisenberg Lie algebra $h(V, \Omega)$ then has a skew-Hermitean representation on the dense subspace of smooth vectors $\mathcal{H}(V, \Omega, j)^\infty$ of this representation. For results and references concerning the space of smooth vectors of a unitary representation of a Lie group see [4]. If $f \in \mathcal{H}(V, \Omega, j)^\infty$ we have

$$(\dot{U}_j(v, s)f)(z) = -is/h f(z) + \frac{1}{2h} \langle z, v \rangle_j f(z) - (\partial_z f)(v)$$

where $(\partial_z f)(v) = \sum_{k=1}^n v^k \partial_{z^k} f$ denotes the holomorphic derivative of $f$ in the direction of $v$. If we extend the representation of $h(V, \Omega)$ to its enveloping algebra then $\mathcal{H}(V, \Omega, j)^\infty$ becomes a Fréchet space with seminorms $f \mapsto \|u \cdot f\|_j$ for $u$ in the enveloping algebra, and its dual $\mathcal{H}(V, \Omega, j)^{-\infty}$ can be viewed as containing $\mathcal{H}(V, \Omega, j)$ so we have a Gelfand triple $\mathcal{H}(V, \Omega, j)^\infty \subset \mathcal{H}(V, \Omega, j) \subset \mathcal{H}(V, \Omega, j)^{-\infty}$ on which the enveloping algebra acts compatibly. Any of those spaces can be considered as symplectic spinor space with the corresponding representation $cl$ of the symplectic Clifford algebra $cl(V, \Omega)$ defined by extending the Clifford multiplication $cl(v) := \dot{U}_j(v, 0)$:

$$(cl(v)f)(z) = \frac{1}{2h} \langle z, v \rangle_j f(z) - (\partial_z f)(v)$$

which satisfies

$$cl(v)(cl(w)f) - cl(w)(cl(v)f) = \frac{i}{h} \Omega(v, w)f.$$  

Note, in this definition $f$ is initially taken in the smooth vectors $\mathcal{H}(V, \Omega, j)^\infty$ but $cl(V, \Omega)$ can be viewed as also acting on $\mathcal{H}(V, \Omega, j)$ or $\mathcal{H}(V, \Omega, j)^{-\infty}$ in the distributional sense with the same formulas.

### 2.3 Definition of the $Mp^c$ group

We denote by $Sp(V, \Omega)$ the Lie group of invertible linear maps $g : V \to V$ such that $\Omega(gv, gw) = \Omega(v, w)$ for all $v, w \in V$. Its Lie algebra $sp(V, \Omega)$ consists of linear maps
\(\xi : V \to V\) with \(\Omega(\xi v, w) + \Omega(v, \xi w) = 0\) for all \(v, w \in V\) or equivalently \((u, v) \mapsto \Omega(u, \xi v)\) is a symmetric bilinear form.

\(Sp(V, \Omega)\) acts as a group of automorphisms of the Heisenberg group \(H(V, \Omega)\) by

\[g \cdot (v, t) = (g(v), t)\]

By composing the representation \(U_j\) of \(H(V, \Omega)\) on \(H(V, \Omega, j)\) with an automorphism \(g \in Sp(V, \Omega)\) we get a second representation of \(H(V, \Omega)\) also on \(H(V, \Omega, j)\):

\[U^g_j(v, t) = U_j(g \cdot (v, t)) = U_j(g(v), t)\]

which is still irreducible and has the same central parameter \(-\frac{1}{\hbar}\). By the Stone–von Neumann Uniqueness Theorem there is a unitary transformation \(U\) of \(H(V, \Omega, j)\) such that

\[U^g_j = U U_j U^{-1}.\] (4)

Since \(U_j\) is irreducible, the operator \(U\) is determined up to a scalar multiple by the corresponding elements \(g\) of \(Sp(V, \Omega)\), and it is known to be impossible to make a continuous choice \(U_g\) which respects the group multiplication.

**Definition 2.2** The group \(Mp^c(V, \Omega, j)\) consists of the pairs \((U, g)\) of unitary transformations \(U\) of \(H(V, \Omega, j)\) and elements \(g\) of \(Sp(V, \Omega)\) satisfying

\[U_j(g(v), t) = U U_j(v, t) U^{-1} \quad \forall (v, t) \in H(V, \Omega),\] (II)

with diagonal multiplication law.

The map

\[\sigma : Mp^c(V, \Omega, j) \to Sp(V, \Omega) : (U, g) \mapsto \sigma(U, g) := g\] (5)

is a surjective homomorphism with kernel consisting of all unitary multiples of the identity. So we have a central extension

\[1 \to U(1) \to Mp^c(V, \Omega, j) \xrightarrow{\sigma} Sp(V, \Omega) \to 1\] (6)

which does not split.

### 2.4 Parametrising the symplectic group

Choose and fix \(j \in j_+ (V, \Omega)\). A parametrisation of the real symplectic group, which will be useful to describe a parametrisation of the \(Mp^c\) group, and which depends on the triple \((V, \Omega, j)\) is given as follows. Consider \(GL(V, j) = \{g \in GL(V) \mid gj = jg\}\) and observe that \(U(V, \Omega, j) = Sp(V, \Omega) \cap GL(V, j)\) is the unitary group of the Hilbert space \((V, \Omega, j)\).
Any \( g \in Sp(V, \Omega) \) decomposes uniquely as a sum \( C_g + D_g \) of a \( j \)-linear and \( j \)-antilinear part,

\[
g = C_g + D_g, \quad C_g = \frac{1}{2}(g - jgj), \quad D_g = \frac{1}{2}(g + jgj).
\]

For any \( 0 \neq v \in V \), we have \( 4\Omega(C_gv, jC_gv) = 2\Omega(v, jv) + \Omega(gv, jgv) + \Omega(gjv, jgjv) > 0 \) so that \( C_g \) is invertible. Define

\[
Z_g = C_g^{-1}D_g.
\]

Clearly \( g = C_g(1 + Z_g) \) and \( Z_g \) is \( \mathbb{C} \)-antilinear.

Equating \( \mathbb{C} \)-linear and \( \mathbb{C} \)-antilinear parts in \( 1 = g^{-1}g \) gives

\[
1 - Z_g^{-1} = -C_gZ_gC_g^{-1} \quad \text{and} \quad 1 = C_{g^{-1}}C_g(1 - Z_g^2).
\]

Thus \( 1 - Z_g^2 \) is invertible with \( (1 - Z_g^2)^{-1} = C_{g^{-1}}C_g \). Decomposing a product yields

\[
C_{g_1g_2} = C_{g_1}(C_{g_2} + Z_{g_1}C_{g_2}Z_{g_2}) = C_{g_1}(1 - Z_{g_1}Z_{g_2}^{-1})C_{g_2}
\]

and

\[
Z_{g_1g_2} = C_{g_2}^{-1}(1 - Z_{g_1}Z_{g_2}^{-1})^{-1}(Z_{g_1} - Z_{g_2}^{-1})C_{g_2}.
\]

Adding and subtracting \( \langle gu, v \rangle_j \) and \( \langle jgju, v \rangle_j \) yields

\[
C_g^* = C_{g^{-1}}, \quad 1 - Z_g^2 = (C_{g^{-1}}C_g)^{-1} = (C_g^*C_g)^{-1},
\]

which is positive definite, and

\[
\langle Z_gu, v \rangle_j = \langle Z_gv, u \rangle_j.
\]

Hence any \( Z_g \) has the three following properties: it is \( \mathbb{C} \)-antilinear, the map \( (v, w) \mapsto \langle v, Zgw \rangle_j \) is complex bilinear symmetric, and \( 1 - Z_g^2 \) is self adjoint and positive definite.

Let \( \mathbb{B}(V, \Omega, j) \) be the Siegel domain consisting of \( Z \in \text{End}(V) \) such that

\[
Zj = -jZ, \quad \langle v, Zw \rangle_j = \langle w, Zv \rangle_j, \quad \text{and} \quad 1 - Z^2 \text{ is positive definite.}
\]

**Theorem 2.3** [10, 3] There is an injective map

\[
Sp(V, \Omega) \rightarrow GL(V, j) \times \mathbb{B}(V, \Omega, j) : g \mapsto (C_g, Z_g),
\]

whose image is the set \( \{(C, Z) \mid 1 - Z^2 = (C^*C)^{-1}\} \).

Indeed, for any such \( (C, Z) \), defining \( g = C(1 + Z) \), we have

\[
\Omega(gu, gv) = -\text{Im}\langle C(1 + Z)u, C(1 + Z)v \rangle_j = -\text{Im}\langle (1 + Z)u, C^*C(1 + Z)v \rangle_j
\]

\[
= \text{Im}\langle (1 - Z)C^*C(1 + Z)v, u \rangle_j = \text{Im}\langle v, u \rangle_j = \Omega(u, v).
\]
Thus $C$ and $Z$ are parameters (but not independent) for an element of $Sp(V,\Omega)$. In order to parametrise $Mp^c(V,\Omega,j)$ in a similar fashion we observe that if $Z_1, Z_2 \in \mathbb{B}(V,\Omega,j)$, then $1-Z_1Z_2 \in GL(V,j)$ and its real part, $\frac{1}{2}((1-Z_1Z_2)+(1-Z_1Z_2)^*) = \frac{1}{2}((1-Z_1Z_2)+(1-Z_2Z_1))$, is positive definite.

Any $g \in GL(V,j)$ can be written uniquely in the form $X + iY$ with $X$ and $Y$ self-adjoint, and $g \in GL(V,j)_+ = \{ g \in GL(V,j) \mid g + g^* \text{ is positive definite} \}$ when $X$ is positive definite. Positive definite self-adjoint operators $X$ are of the form $X = e^Z$ with $Z$ self-adjoint and $Z \mapsto e^Z$ is a diffeomorphism of all self-adjoint operators with those which are positive definite. Given self-adjoint operators $X$ and $Y$ with $X$ positive definite then $X + iY$ has no kernel, so is in $GL(V,j)$. Thus $GL(V,j)_+$ is an open set in $GL(V,j)$ diffeomorphic to the product of two copies of the real vector space of Hermitean linear maps of $(V,\Omega,j)$. In particular $GL(V,j)_+$ is contractible so simply-connected. Thus there is a unique smooth function $a:GL(V,j)_+ \rightarrow \mathbb{C}$ such that

$$\text{Det}_j g = e^{a(g)}, \quad g \in GL(V,j)_+$$

and normalised by $a(I) = 0$. Here $\text{Det}_j g$ is the determinant of $g$ considered as a complex transformation of $V$ viewed as a complex space using $j$. Further, since $\text{Det}_j$ is a holomorphic function on $GL(V,j)$, $a$ will be holomorphic on $GL(V,j)_+$. In particular, for any $Z_1, Z_2 \in \mathbb{B}(V,\Omega,j)$, we have

$$\text{Det}_j(1 - Z_1Z_2) = e^{a(1-Z_1Z_2)} \quad (9)$$

and $a(1-Z_1Z_2)$ is a holomorphic function of $Z_1$ and anti-holomorphic function of $Z_2$.

### 2.5 Parametrising the $Mp^c$ group

The group $Mp^c(V,\Omega,j)$ is defined as pairs $(U,g)$ with $U$ a unitary operator on $\mathcal{H}(V,\Omega,j)$ and $g \in Sp(V,\Omega)$ satisfying (4). One can determine the form of the operator $U$ in terms of the parameters $C_g, Z_g$ of $g$ introduced in the previous paragraph. Fixing $j \in j_+(V,\Omega)$, any bounded operator $A$ on $\mathcal{H}(V,\Omega,j)$ is determined by its Berezin kernel

$$A(z,v) = (Ae_v, e_z)_j = (Ae_v)(z)$$

where the $e_v$ are the coherent states of $\mathcal{H}(V,\Omega,j)$ defined by $(e_v)(z) = e^{\frac{1}{2}j(z,v)}$ and one gets

\[ U(z,v) = \lambda \exp \frac{1}{4\hbar} \{ 2\langle C_g^{-1}z,v \rangle_j - \langle z, Z_g^{-1}z \rangle_j - \langle Z_g v, v \rangle_j \} \quad (10) \]

for some $\lambda \in \mathbb{C}$ with $|\lambda^2 \text{Det}_j C_g| = 1$. Moreover $\lambda = (Ue_0)(0) = (Ue_0, e_0)_j$. 

**Theorem 2.4** [10, 11] If $(U,g) \in Mp^c(V,\Omega,j)$ then the Berezin kernel $U(z,v)$ of $U$ has the form

$$U(z,v) = \lambda \exp \frac{1}{4\hbar} \{ 2\langle C_g^{-1}z,v \rangle_j - \langle z, Z_g^{-1}z \rangle_j - \langle Z_g v, v \rangle_j \}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda^2 \text{Det}_j C_g| = 1$. Moreover $\lambda = (Ue_0)(0) = (Ue_0, e_0)_j$. 


We call \( g, \lambda \) given by Theorem 2.4 the parameters of the element \((U, g) \in Mp^c(V, \Omega, j)\). If \( g \) is symplectic and \( \lambda \in \mathbb{C} \) satisfies \(|\lambda^2 \det_j C_g| = 1\) we denote by \( U_{g, \lambda} \) the unitary transformation whose kernel is given by (10). Writing the multiplication in \( Mp^c(V, \Omega, j) \) in terms of the parameters, we get

**Theorem 2.5** [10, 3] The product in \( Mp^c(V, \Omega, j) \) of \((U_i, g_i)\) with parameters \( g_i, \lambda_i, i = 1, 2 \) has parameters \( g_1 g_2, \lambda_{12} \) with

\[
\lambda_{12} = \frac{1}{4} a \left( 1 - z_1 z_2^{-1} \right)
\]

(11)

where \( a \) is defined as in (9).

**Corollary 2.6** The group \( Mp^c(V, \Omega, j) \) is a Lie group. It admits a smooth character \( \eta \) given by

\[
\eta(U, g) = \lambda^2 \det_j C_g \in U(1)
\]

(12)

if \( g, \lambda \) are the parameters of \((U, g)\). The restriction of \( \eta \) to the central \( U(1) \) is the squaring map. The inclusion \( U(1) \hookrightarrow Mp^c(V, \Omega, j) \) sends \( \lambda \in U(1) \) to \((\lambda I_V, I_V)\) which has parameters \( I_V, \lambda \). The exact sequence (6) is a central extension of Lie groups.

**Definition 2.7** The metaplectic group is the kernel of \( \eta \); it is given by

\[
Mp(V, \Omega, j) = \{(U, g) \in Mp^c(V, \Omega, j) \mid \lambda^2 \det_j C_g = 1\}
\]

with the multiplication rule given by Theorem 2.5.

For \((U, g) \in Mp(V, \Omega, j)\) it is clear from the definition that \( g \) determines \( \lambda \) up to sign and so \( Mp(V, \Omega, j) \) is a double covering of \( Sp(V, \Omega) \).

Let \( mp^c(V, \Omega, j) \) be the Lie algebra of \( Mp^c(V, \Omega, j) \). Differentiating (6) gives an exact sequence of Lie algebras

\[
0 \longrightarrow u(1) \longrightarrow mp^c(V, \Omega, j) \xrightarrow{\sigma} sp(V, \Omega) \longrightarrow 0.
\]

(13)

We denote by \( \eta_* : mp^c(V, \Omega, j) \longrightarrow u(1) \) the differential of the group homomorphism \( \eta \), and observe that \( \frac{1}{2} \eta_* \) is a map to \( u(1) \) which is the identity on the central \( u(1) \) of \( mp^c(V, \Omega, j) \). Hence (13) splits as a sequence of Lie algebras.

We let \( MU^c(V, \Omega, j) \) be the inverse image of \( U(V, \Omega, j) \) under \( \sigma \) so that (13) induces a corresponding short exact sequence

\[
1 \longrightarrow U(1) \longrightarrow MU^c(V, \Omega, j) \xrightarrow{\sigma} U(V, \Omega, j) \longrightarrow 1.
\]

(14)

\( MU^c(V, \Omega, j) \) is a maximal compact subgroup of \( Mp^c(V, \Omega) \).
Proposition 2.8 If \((U, k) \in MU^c(V, \Omega, j)\) has parameters \(k\) and \(\lambda\) then \(\lambda(U, k) = \lambda\) is a character of \(MU^c(V, \Omega, j)\). If \(f \in \mathcal{H}(V, \Omega, j)\) then \((Uf)(z) = \lambda f(k^{-1}z)\), so unlike the exact sequence (6), (14) does split canonically by means of the homomorphism 
\[\lambda: MU^c(V, \Omega, j) \to U(1).\]

This gives an isomorphism
\[MU^c(V, \Omega, j) \xrightarrow{\sigma \lambda} U(V, \Omega, j) \times U(1). \tag{15}\]

In addition we have the determinant character \(\text{Det}_j: U(V, \Omega, j) \to U(1)\) which can be composed with \(\sigma\) to give a character \(\text{Det}_j \circ \sigma\) of \(MU^c(V, \Omega, j)\). The three characters \(\eta, \lambda\) and \(\text{Det}_j \circ \sigma\) are related by
\[\eta = \lambda^2 \text{Det}_j \circ \sigma. \tag{16}\]

Remark 2.9 The map \(F_j: U(V, \Omega, j) \to Mp^c(V, \Omega, j) : g \mapsto (g, 1)\) is an isomorphism by Proposition 2.8. We say \(U(V, \Omega, j)\) is liftable to \(Mp^c(V, \Omega, j)\). In section 2.7, we look at other subgroups of \(Sp(V, \Omega)\) which are liftable.

### 2.6 Embedding of \(mp^c(V, \Omega, j)\) into the Clifford Algebra

For any \(v \in V\) we denote by \(v\) the linear form on \(V\) defined by
\[v(x) = \Omega(v, x)\]
so that \(\mathfrak{sp}(V, \Omega)\) is spanned by the elements
\[v \otimes w + w \otimes v \quad \forall v, w \in V.\]

We endow the symplectic Clifford algebra with the Lie algebra structure \([\ , \]_\text{Cl}\) defined by skewsymmetrising the associative product \(\cdot\), \([a, b]_\text{Cl} := a \cdot b - b \cdot a\).

**Lemma 2.10** The map
\[\nu: \mathfrak{sp}(V, \Omega) \to Cl(V, \Omega): v \otimes w + w \otimes v \mapsto -\frac{i\hbar}{2}(v \cdot w + w \cdot v) \tag{17}\]
is a homomorphism of Lie algebras. Furthermore
\[\left[\nu(A), v\right]_\text{Cl} = Av \quad \text{for any } A \in \mathfrak{sp}(V, \Omega) \text{ and } v \in V. \tag{18}\]

**Proof** It is enough to observe that \([v \otimes v, w \otimes w] = \Omega(v, w)(v \otimes w + w \otimes v)\) and
\[v \cdot [v, w]_\text{Cl} = v \cdot [v, w]_\text{Cl} \cdot w + [v, w]_\text{Cl} \cdot v \cdot w + w \cdot v \cdot [v, w]_\text{Cl} + w \cdot [v, w]_\text{Cl} \cdot v = 2i\frac{\hbar}{2}\Omega(v, w)(v \cdot w + w \cdot v).\]
Also \((v \otimes v)(x) = \Omega(v, x)v\) and \([v \cdot v, x]_{Cl} = v \cdot [v, x]_{Cl} + [v, x]_{Cl} \cdot v = 2\frac{1}{\hbar} \Omega(v, x)v\).

The group \(Mp^c(V, \Omega, j)\) is the set of pairs of elements \((U, g)\) with \(U\) a unitary operator on \(H(V, \Omega, j)\) and \(g \in Sp(V, \Omega)\) satisfying (4) i.e. \(U_j(g(v), t) = U U_j(v, t) U^{-1}\) \(\forall v, t\). We have a natural representation, denoted \(U\), of \(Mp^c(V, \Omega, j)\) on \(H(V, \Omega, j)\) defined by

\[
U(U, g) := U.
\]

The smooth vectors for this representation coincide with the smooth vectors for the representation of \(H(V, \Omega, j)\). Differentiating this relation we obtain a representation \(U^* : mp^c(V, \Omega, j) \rightarrow \text{End}(H(V, \Omega, j)^\infty)\) on the smooth vectors and for any \(X \in mp^c(V, \Omega, j)\) and \(f \in H(V, \Omega, j)^\infty\) we have

\[
\dot{U}_j(\sigma_j(X)v, s)f = \left[ U_*(X), \dot{U}_j(v, s) \right] f. \tag{19}
\]

**Lemma 2.11** For any \(X \in mp^c(V, \Omega, j)\) we have

\[
U_*(X) = \text{cl} (\nu(\sigma_j(X))) + \frac{1}{2} \eta_*(X) \text{Id.} \tag{20}
\]

**Proof** By equation (19) and the definition of the Clifford multiplication, we have

\[
\left[ U_*(X), \dot{U}_j(v, 0) \right] f = [U_*(X), \text{cl}(v)] f = \text{cl}(\sigma_j(X)v)f;
\]

and by Lemma 2.10 \(\sigma_j(X)v = [\nu(\sigma_j(X)), v]_{Cl}\) so that

\[
[U_*(X), \text{cl}(v)] f = \text{cl} ([\nu(\sigma_j(X)), v]_{Cl}) f = [\text{cl}(\nu(\sigma_j(X)), \text{cl}(v)] f
\]

which shows that \(U_*(X) - \text{cl} (\nu(\sigma_j(X)))\) commutes with the action of the Clifford multiplication, hence commutes with the representation of the Heisenberg Lie algebra and so is a multiple of the identity (see [4]). Since \(mp^c(V, \Omega, j) \xrightarrow{\sigma_j \times \eta_*} \text{sp}(V, \Omega) \times u(1)\) is an isomorphism of Lie algebras and \(\text{sp}(V, \Omega)\) is semisimple, any character of \(mp^c(V, \Omega, j)\) is a multiple of \(\eta_*\) and we have

\[
U_*(X) = \text{cl}(\nu(\sigma_j(X))) + a \eta_*(X) \text{Id}
\]

for an \(a \in \mathbb{C}\). Evaluating the above for an element \(X \in mu^c(V, \Omega, j) \xrightarrow{\sigma_j \times \lambda_*} u(V, \Omega, j) \times u(1)\) such that \(\sigma_j X = 0\) and \(\lambda_*(X) = c\) we have, since \(U_*(X)f = cf\) and \(\eta_*(X) = 2\lambda_*(X)\), that \(a = \frac{1}{2}\).
2.7 Subgroups of $Sp(V, \Omega)$ lifting to $Mp^c(V, \Omega, j)$

**Definition 2.12** If $H$ is a Lie subgroup of $Sp(V, \Omega)$ we say that $H$ lifts or is liftable to $Mp^c(V, \Omega, j)$ if there is a smooth homomorphism $F : H \to Mp^c(V, \Omega, j)$ with $\sigma \circ F = \text{Id}_H$.

**Example 2.13** We have seen (Remark 2.13) that given any positive compatible complex structure $j \in j_+(V, \Omega)$, the inclusion of the unitary group $U(V, \Omega, j)$ in the symplectic group $Sp(V, \Omega)$ lifts to an embedding $F_j : U(V, \Omega, j) \to Mp^c(V, \Omega, j)$. An element $g \in U(V, \Omega, j)$ is mapped to the element $F_j(g)$ in $Mp^c(V, \Omega, j)$ parametrised by $g, 1$, so $F_j(g) = (U, g)$ where the kernel $U(z, v)$ of $U$ is given by $U(z, v) = \exp \frac{1}{2\hbar}\langle g^{-1}z, v \rangle_j = e_v(g^{-1}z)$. Hence

$$F_j : U(V, \Omega, j) \to Mp^c(V, \Omega, j) : g \mapsto F_j(g) = (U, g) \text{ with } (Uf)(z) = f(g^{-1}z). \quad (21)$$

We look at some other subgroups of $Sp(V, \Omega)$ which are liftable. A Lie subgroup $H$ of $Sp(V, \Omega)$ is liftable if there is a homomorphism $F : H \to Mp^c(V, \Omega, j)$ with $F(g) = (U, g)$ where the unitary operator $U$ has parameters $(g, f(g))$. By Theorem 2.14 what is needed to obtain such a homomorphism is a smooth $\mathbb{C}^*$-valued function $f$ on $H$ such that

$$f(g_1g_2) = f(g_1)f(g_2)e^{-\hbar \langle 1 - Z_{g_1}Z_{g_2}^{-1} \rangle} \quad (22)$$

and

$$|f(g)|^2 \text{Det}_j C_g = 1, \quad \forall g \in H. \quad (23)$$

Note that from equations (22,23) two lifts of $H$ differ by a homomorphism $H \to U(1)$, so in general we may not have uniqueness of the lift. In summary

**Theorem 2.14** A Lie subgroup $H$ of $Sp(V, \Omega)$ is liftable to $Mp^c(V, \Omega, j)$ if and only if there is a smooth complex-valued function $f$ on $H$ satisfying equations (22,23). The group $\text{Hom}(H, U(1))$ acts simply transitively on the set of lifts of $H$.

**Remark 2.15** In the case of the unitary group $U(V, \Omega, j)$ we can simply take $f \equiv 1$.

If we square equation (22) we have

$$f(g_1g_2)^2 = f(g_1)^2f(g_2)^2 \text{Det}_j \left(1 - Z_{g_1}Z_{g_2}^{-1}\right)^{-1} = f(g_1)^2f(g_2)^2 \text{Det}_j \left(C_{g_1}C_{g_2}^{-1}C_{g_2}^{-1}\right)^{-1}$$

and hence the function $f$ must satisfy

$$g \mapsto f(g)^2 \text{Det}_j(C_g)$$

is a homomorphism from $H$ to $U(1)$. Thus in order to have a lift $H$ must consist of a submanifold of elements $g$ for which $\text{Det}_j(C_g)$ differs from a smooth square by a homomorphism. In fact this property is almost all that is required apart from a means of picking smooth square roots as the following Lemma shows:
Lemma 2.16 Let $j \in j_+(V, \Omega)$ and $H \subset Sp(V, \Omega)$ be a Lie subgroup such that $H_1 = H \cap U(V, \Omega, j)$ meets every connected component of $H$. Suppose there is a smooth map $f : H \to \mathbb{C}^*$ such that

(i) $f|_{H_1}$ is a homomorphism of $H_1$ into $\mathbb{C}^*$;

(ii) $g \mapsto f(g)^2 \text{Det}_j C_g$ is a homomorphism of $H$ into $U(1)$.

Then $(g, f(g))$ are the parameters of an element $F(g) = (U, g)$ of $Mp^c(V, \Omega, j)$ and $F : H \to Mp^c(V, \Omega, j)$ is a homomorphism with $\sigma \circ F = \text{Id}_H$.

**Proof** Let $\psi(g) = f(g)^2 \text{Det}_j C_g$ then by (ii) $\psi$ is a homomorphism into $U(1)$ so in particular $|f(g)^2 \text{Det}_j C_g| = 1$ hence $(g, f(g))$ are the parameters of an element $F(g) = (U, g)$ of $Mp^c(V, \Omega, j)$. We can then write $f(g)^2 = \psi(g) \text{Det}_j C_g^{-1}$ for a homomorphism $\psi$ so

$$f(g_1g_2)^2 f(g_1)^{-2} f(g_2)^{-2} = \text{Det}_j \left( C_{g_2}^{-1} C_{g_1} C_{g_2} \right) = \text{Det}_j \left( 1 - Z_{g_1} Z_{g_2}^{-1} \right)^{-1} = e^{-\alpha(1 - Z_{g_1} Z_{g_2}^{-1})}.$$ 

Thus

$$\varepsilon(g_1, g_2) = f(g_1g_2)f(g_1)^{-1}f(g_2)^{-1}e^{\frac{1}{2} \alpha(1 - Z_{g_1} Z_{g_2}^{-1})}$$

is a continuous function on $H \times H$ taking values in $\{1, -1\}$. On $H_1$ $f$ is a homomorphism and $Z_g = 0$. Thus $\varepsilon = 1$ on $H_1 \times H_1$. Since $H_1$ meets every component of $H$ it follows that $\varepsilon = 1$ on $H \times H$. We thus have

$$f(g_1g_2) = f(g_1)f(g_2)e^{-\frac{1}{2} \alpha(1 - Z_{g_1} Z_{g_2}^{-1})}$$

which makes $F$ a homomorphism by Theorem 2.14. \hfill \blacksquare

Remark 2.17 This Lemma also determines which subgroups $H$ are liftable to the metaplectic group $Mp(V, \Omega, j)$ since, by Definition 2.7, the character in (ii) has to be trivial. Thus we need a smooth square root $f$ on $H$ of the function $\text{Det}_j(C_g)^{-1}$ which is a character on $H \cap U(V, \Omega, j)$.

2.7.1 The pseudo-unitary subgroup

Suppose we have a compatible complex structure $\tilde{j} \in j(V, \Omega)$ which is not (necessarily) positive definite. It is always possible to find a positive compatible complex structure $j \in j_+(V, \Omega)$ which commutes with $\tilde{j}$. One has:
Proposition 2.18 The pseudo-unitary group $U(V, \Omega, \tilde{j}) := \{ g \in Sp(V, \Omega) \mid gj = \tilde{g}j \}$ embeds in $Mp^c(V, \Omega, j)$ for a $j \in j_+(V, \Omega)$ commuting with $\tilde{j}$. The map

$$F_{\tilde{j}, j} : U(V, \Omega, \tilde{j}) \to Mp^c(V, \Omega, j) : g \mapsto F_{\tilde{j}, j}(g) = (U_{g, \lambda}, g)$$

(24)

with $\lambda = (\text{Det}_j C_g^-)^{-1}$ where $C_g^-$ is the restriction of $C_g = \frac{1}{2}(g - jgj)$ to the invariant subspace $V_- = \{ v \in V \mid jv = -jv \}$, is an injective homomorphism lifting the inclusion of $U(V, \Omega, \tilde{j})$ in $Sp(V, \Omega)$.

**Proof** Since $\tilde{jj} = j\tilde{j}$, we have $(\tilde{jj})^2 = \text{Id}$ so that

$$V = V_+ \oplus V_-$$

where

$$V_\pm = \{v \in V \mid \tilde{jj}v = \mp v\} = \{v \in V \mid jv = \pm jv\}$$

are symplectic subspaces of $V$ which are $\tilde{j}$- and $j$-stable, and orthogonal for $\Omega$, $G_j$ and $G_{\tilde{j}}$ (as in Definition 2.1).

Picking $g \in U(V, \Omega, \tilde{j})$, we write $g = C_g(\text{Id} + Z_g)$ with $C_g = \frac{1}{2}(g - jgj)$ in $GL(V, j)$ and $C_g Z_g = \frac{1}{2}(g + jgj)$. Since $g$ and $j$ commute with $\tilde{j}$, $C_g$ commutes with $\tilde{j}$ (and with $j$) hence $C_g$ preserves the splitting $V = V_+ \oplus V_-$, so gives elements $C_g^\pm$ of $GL(V_+, j)$ and $C_g^-$ of $GL(V_-, j)$. On the other hand $Z_g$ is $\tilde{j}$-linear but $j$-antilinear, so $Z_g(V_\pm) \subset V_\mp$; let $Z_g' := Z_g|_{V_+} : V_+ \to V_-$ and $Z_g'' := Z_g|_{V_-} : V_- \to V_+$.

Since $g$ is $\tilde{j}$-linear, we want to express its complex determinant $\text{Det}_j g$ in terms of those parameters:

$$\text{Det}_j g = \text{Det}_j(C_g) \text{Det}_j(\text{Id} + Z_g) = \text{Det}_j(C_g^+) \text{Det}_j(C_g^-) \text{Det}_j(\text{Id} + Z_g) = \text{Det}_j(C_g^+) \text{Det}_j(C_g^-) \text{Det}_j(\text{Id} + Z_g).$$

In a basis adapted to the decomposition $V = V_+ \oplus V_-$ the matrix of $Z_g$ has the form

$$\left( \begin{array}{cc} 0 & Z_g'' \\ Z_g' & 0 \end{array} \right)$$

so that

$$\text{Id} + Z_g = \left( \begin{array}{cc} \text{Id} & Z_g'' \\ Z_g' & \text{Id} \end{array} \right) = \left( \begin{array}{cc} \text{Id} & 0 \\ Z_g' & \text{Id} \end{array} \right) \cdot \left( \begin{array}{cc} \text{Id} & Z_g'' \\ 0 & \text{Id} - Z_g'Z_g'' \end{array} \right).$$

Thus $\text{Det}_j(\text{Id} + Z_g) = \text{Det}_j^{V_+}(\text{Id} - Z_g'Z_g'') = \text{Det}_j^{V_-}(\text{Id} - Z_g''Z_g''')$.

Now $\text{Id} - Z_g^2 = (C_g^+ C_g^-)^{-1}$; hence

$$\text{Id} - Z_g^2 = \left( \begin{array}{cc} \text{Id} - Z_g''Z_g' & 0 \\ 0 & \text{Id} - Z_g'Z_g'' \end{array} \right) = \left( \begin{array}{cc} (C_g^+ C_g^-)^{-1} & 0 \\ 0 & (C_g^+ C_g^-)^{-1} \end{array} \right).$$

so that $\text{Det}_j^{V_-}(\text{Id} - Z_g'Z_g'') = \text{Det}_j \left( C_g^+ C_g^- \right)^{-1} = \text{Det}_j(C_g^-)^{-1} \text{Det}_j(C_g^-)^{-1}$. Hence we have

$$\text{Det}_j g = \text{Det}_j(C_g^+) \text{Det}_j(C_g^-)^{-1} \text{Det}_j(C_g^-)^{-1} \text{Det}_j(C_g^-)^{-1} = \text{Det}_j(C_g^+) \text{Det}_j(C_g^-)^{-1}$$
so that
\[ \text{Det}_j C_g = \text{Det}_j (C_g^+ \text{Det}_j (C_g^-)) = \text{Det}_j (\text{Det}_j (C_g^-))^2. \] (25)

Equation (25) suggests defining, for \( g \in U(V, \Omega, \overline{j}) \),
\[ f_{\overline{j},j}(g) = \text{Det}_j (C_g^-)^{-1} \]
so that \( f_{\overline{j},j}(g)^2 \text{Det}_j C_g = \text{Det}_j g \) is a homomorphism of \( U(V, \Omega, \overline{j}) \to U(1) \). On \( U(V, \Omega, \overline{j}) \cap U(V, \Omega, j) \) we have \( C_g^- = g|_{V_-} \) and so \( f_{\overline{j},j} \) is a homomorphism. Further \( U(V, \Omega, \overline{j}) \) is connected so that all the conditions of Lemma 2.16 are satisfied. Hence we have a homomorphism
\[ F_{\overline{j},j} : U(V, \Omega, \overline{j}) \to Mp_c(V, \Omega, j), \quad F_{\overline{j},j}(g) = (Ug, \lambda, g) \]
with \( \lambda = f_{\overline{j},j}(g) = \text{Det}_j (C_g^-)^{-1} \) as claimed.

2.7.2 The stabiliser of a real Lagrangian subspace

If \((V, \Omega)\) is a real symplectic vector space of dimension \(2n\) and \(D \subset V\) is a subspace then we denote by \(D^\perp\) its \(\Omega\)-orthogonal
\[ D^\perp = \left\{ v \in V \mid \Omega(v, w) = 0, \forall w \in D \right\}. \]
\(D\) is called isotropic if \(D \subset D^\perp\) and (real) Lagrangian if it is maximal isotropic which is the case when \(D = D^\perp\) so its dimension is \(n\), half the dimension of \(V\). We denote by \(\Lambda(V, \Omega)\) the set of real Lagrangian subspaces of \((V, \Omega)\). For \(F \in \Lambda(V, \Omega)\) let
\[ Sp(V, \Omega, F) = \left\{ g \in Sp(V, \Omega) \mid g(F) \subset F \right\}. \]

If we take a PCCS \(j \in j_{+}(V, \Omega)\) then \(jF \in \Lambda(V, \Omega)\) and \(F\) and \(jF\) are \(G_j\)-orthogonal so \(V = F + jF\) is a \(G_j\)-orthogonal decomposition. Relative to the direct sum \(F \oplus F\), an element \(g \in Sp(V, \Omega, F)\) will be triangular
\[ g \leftrightarrow \begin{pmatrix} A_g & B_g \\ 0 & E_g \end{pmatrix} \]
with \(A_g, E_g \in GL(F), B_g \in \text{End}(F, F)\), and \(g \mapsto A_g\) is a homomorphism from \(Sp(V, \Omega, F)\) to \(GL(F)\). Here \(g\) acts on \(v + jw\) whilst the RHS acts on \(\begin{pmatrix} v \\ w \end{pmatrix}\). For such a \(g\) to be symplectic we have
\[
\Omega(v_1 + jw_1, v_2 + jw_2) = \Omega(g(v_1 + jv_1), g(v_2 + jw_2)) = \Omega(A_g v_1 + B_g w_1 + jE_g w_1, A_g v_2 + B_g w_2 + jE_g w_2) = G_j(A_g v_1 + B_g w_1, E_g w_2) - G_j(E_g w_1, A_g v_2 + B_g w_2)
\]
whilst
\[ \Omega(v_1 + jw_1, v_2 + jw_2) = G_j(v_1, w_2) - G_j(w_1, v_2). \]

Hence
\[ G_j(A_g v_1, E_g w_2) = G_j(v_1, w_2) \quad \text{and} \quad G_j(B_g w_1, E_g w_2) - G_j(E_g w_1, B_g w_2) = 0. \]

Thus
\[ E_g = (A_g^T)^{-1} \quad \text{and} \quad E_g^T B_g = B_g^T E_g \]

where the transpose is taken relative to \( G_j \). Then \( B_g = A_g S_g \) for a symmetric endomorphism \( S_g \) of \( F \). Thus
\[ g \leftrightarrow \begin{pmatrix} A_g & A_g S_g \\ 0 & (A_g^T)^{-1} \end{pmatrix} \]

In these terms,
\[ j \leftrightarrow \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

and so
\[ 2C_g \leftrightarrow \begin{pmatrix} A_g & A_g S_g \\ 0 & (A_g^T)^{-1} \end{pmatrix} - \begin{pmatrix} -(A_g^T)^{-1} & 0 \\ A_g S_g & -A_g \end{pmatrix} = \begin{pmatrix} A_g + (A_g^T)^{-1} & A_g S_g \\ -A_g S_g & A_g + (A_g^T)^{-1} \end{pmatrix}. \]

Note that the complex vector space \((V,j)\) can be identified with \( F^C \) and under this identification \( C_g \) becomes a complex linear endomorphism of \( F^C \)
\[ C_g \leftrightarrow \frac{1}{2} \left( A_g + (A_g^T)^{-1} - i A_g S_g \right) = A_g \left( \frac{1}{2} (I_F + (A_g^T A_g)^{-1} - i S_g) \right). \]

In particular
\[ \text{Det}_j C_g = \text{Det}_F A_g \text{Det}_{F^C} \left( \frac{1}{2} (I_F + (A_g^T A_g)^{-1} - i S_g) \right). \]

We now proceed as in the pseudo-Hermitean case to construct a lift using a smooth square root of the last determinant which exists since the term \( \frac{1}{2} (I_F + (A_g^T A_g)^{-1} - i S_g) \) has strictly positive real part in the complex general linear group. We set
\[ f_F(g) = \left| \text{Det}_F(A_g) \right|^{-\frac{1}{2}} e^{-\frac{1}{2} a_F \left( \frac{1}{2} (I_F + (A_g^T A_g)^{-1} - i S_g) \right)}, \]

where \( a_F \) is the smooth complex valued function on the open set in \( GL(F^C)_+ \) of elements with positive definite real part relative to the Hermitean extension of \( G_j \) to \( F^C \) such that
\[ \text{Det}_{F^C}(g) = e^{a_F(g)}, \quad a_F(I) = 0 \]

analogously to \( a \) in (9). Then
\[ f_F(g)^2 \text{Det}_j C_g = \text{Det}_F(A_g)/|\text{Det}_F(A_g)| \in U(1) \]
and the RHS is a homomorphism so \((ii)\) of Lemma \(2.16\) holds whilst \(g \in H_1 = Sp(V, \Omega, F) \cap U(V, \Omega, j)\) implies \(gjF = jgF \subset jF\) so \(S_g = 0\) and \(A_g\) is in the orthogonal group of \((F, G_j)\). Thus \(H_1\) meets all components of \(Sp(V, \Omega, F)\) and \(f_F|H_1 \equiv 1\) so \((i)\) of Lemma \(2.16\) also holds. Thus all the conditions of Lemma \(2.16\) are satisfied proving

**Proposition 2.19** If \(F \subset V\) is a real Lagrangian subspace of \((V, \Omega)\) and \(j \in j_+(V, \Omega)\) then \(Sp(V, \Omega, F)\) has a homomorphism into \(Mp^c(V, \Omega, j)\) lifting the inclusion \(Sp(V, \Omega, F) \subset Sp(V, \Omega)\). This lifting maps \(g \in Sp(V, \Omega, F)\) to the element of \(Mp^c(V, \Omega, j)\) with parameters \((g, f_F(g))\) where

\[
f_F(g) = |\text{Det}_F(A_g)|^{-\frac{1}{2}} e^{-\frac{1}{2}a_F \left(\frac{1}{2}(I_F + (A_g^TA_g)^{-1})iS_g\right)}
\]

where \(A_g = g|_F\) and \(S_g\) is the endomorphism of \(F\) determined by: \(A_gS_gv\) is the \(F\) component of \(gjv\) relative to the decomposition \(V = F + jF\).

### 2.7.3 The stabiliser of a complex Lagrangian subspace

A subspace \(F\) of \(V^C\) which is Lagrangian for the complex bilinear extension of \(\Omega\), in an abuse of terminology, is called a complex Lagrangian subspace of \((V, \Omega)\). We denote the set of complex Lagrangian subspaces by \(\Lambda(V^C, \Omega)\). We consider the group

\[
Sp(V, \Omega, F) = \{g \in Sp(V, \Omega) \mid g(F) \subset F\}
\]

where \(g\) has been extended complex-linearly to act on \(V^C\). We shall prove that this subgroup is liftable to \(Mp^c\).

**Remark 2.20** If \(\overline{F} = F\), then \(F = L^C\) where \(L\) is a Lagrangian subspace of \(V\). Then \(Sp(V, \Omega, F) = Sp(V, \Omega, L)\) is liftable by Proposition \(2.19\).

If \(F \cap \overline{F} = \{0\}\), then \(V^C = F \oplus \overline{F}\) and there is a complex linear map \(\overline{j} : V^C \to V^C\) so that \(\overline{j}|_F = i \text{Id}_F\) and \(\overline{j}|_{\overline{F}} = -i \text{Id}_{\overline{F}}\). The map \(\overline{j}\) is the complex extension of a complex structure \(\overline{j}\) on \(V\) which is compatible with \(\Omega\). The complex extension of \(g \in Sp(V, \Omega)\) maps \(F\) into \(F\) if and only if \(g \circ \overline{j} = \overline{j} \circ g\). Hence \(Sp(V, \Omega, F) = U(V, \Omega, \overline{j})\) and this subgroup is liftable by proposition \(2.18\).

For a general complex Lagrangian \(F\), \(F \cap \overline{F} = D^C\) where \(D = F \cap V\) is an isotropic subspace of \(V\). The symplectic orthogonal \(D^\perp\) of \(D\) is given by \((D^\perp)^C = F + \overline{F}\). The quotient space \(V' : = D^\perp/D\) is naturally endowed with a symplectic structure \(\Omega'\) induced by \(\Omega\) and \(F' : = F/D^C\) is a complex Lagrangian of \((V', \Omega')\). We have \(V'^C = F' \oplus \overline{F'}\) so there is a complex structure \(\overline{j}'\) on \(V'\) whose complex linear extension is the multiplication by \(i\) on \(F'\).

Conversely given a real isotropic subspace \(D\) of \(V\) and a complex structure \(\overline{j}'\) on \(D^\perp/D\) which is compatible with the symplectic structure induced by \(\Omega\), define \(F : = p^{-1}F'\) where
$F' \subset (D^\perp/D)^C$ is the $+i$ eigenspace of the complex extension of $\tilde{\gamma}$ and where $p : (D^\perp)^C \to (D^\perp/D)^C$ is the canonical projection. Then $F$ is a complex Lagrangian of $(V, \Omega)$. Hence

**Lemma 2.21** A complex Lagrangian $F$ of $(V, \Omega)$ determines a pair $(D, \tilde{\gamma})$ consisting of a real isotropic subspace $D$ of $V$ and of a complex structure $\tilde{\gamma}$ on $D^\perp/D$ which is compatible with the symplectic structure induced by $\Omega$, and vice versa.

In particular, an element $g \in Sp(V, \Omega)$ is in $g \in Sp(V, \Omega, F)$ if and only if $g(D) \subset D$ and $g' \circ \tilde{\gamma} = \tilde{\gamma} \circ g'$ where $g'$ is the linear endomorphism of $D^\perp/D$ induced by $g$.

We first examine the stabiliser of an isotropic subspace. Let $D \subset V$ be isotropic then, as above, $D$ is the kernel of the restriction of $\Omega$ to $D^\perp$ and so $\Omega$ induces a non-degenerate bilinear form $\Omega'$ on the quotient space $D^\perp/D$ making $(D^\perp/D, \Omega')$ a symplectic vector space called the *symplectic quotient* of $(V, \Omega)$ by the isotropic subspace $D$.

Let $j \in j_+(V, \Omega)$ and $D$ be isotropic then $jD$ is $G_j$-orthogonal to $D^\perp$ and so $V = D^\perp + jD$ is a direct sum. $jD$ is also isotropic and $D^\perp \cap (jD)^\perp = (D + jD)^\perp$ is a $j$-stable symplectic subspace with $((D + jD)^\perp, \Omega)$ symplectically isomorphic to $(D^\perp/D, \Omega')$. Thus $j$ on $(D + jD)^\perp$ induces $j' \in j_+(D^\perp/D, \Omega')$.

Denote by $Sp(V, \Omega, D)$ the subgroup of $Sp(V, \Omega)$ of elements which stabilise $D$. If $g \in Sp(V, \Omega, D)$ then $g$ preserves $D$ and $D^\perp$ and so induces a transformation $g'$ of $D^\perp/D$ which clearly lies in $Sp(D^\perp/D, \Omega')$. We thus have two homomorphisms

$$a : Sp(V, \Omega, D) \to GL(D) \quad \text{and} \quad b : Sp(V, \Omega, D) \to Sp(D^\perp/D, \Omega').$$

Fix $j \in j_+(V, \Omega)$ and let $Q = (D + jD)^\perp$ then $V = D + jD + Q$ is a $G_j$-orthogonal direct sum and $(Q, \Omega)$ is symplectically isomorphic to $(D^\perp/D, \Omega')$. We write $g$ in terms of the corresponding transformation $\gamma$ of $V = D \oplus D \oplus Q$ and the map

$$\gamma \to V, \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto u + jv + w.$$  

Then using the fact that $g$ is symplectic as well as preserving $D$ and $D^\perp$ but not necessarily $jD$, we have

$$\gamma = \begin{pmatrix} a(g) & a(g) \left( s(g) - \frac{1}{2} e(g)^* e(g) \right) & -a(g) e(g)^* \\ 0 & (a(g)^T)^{-1} & 0 \\ 0 & b(g) e(g) & b(g) \end{pmatrix}.$$

Here $x^T$ is the transpose of $x \in \text{End}(D)$ relative to the inner product $G_j$, $s(g)$ is a symmetric endomorphism of $D$, $e(g)$ is a linear map from $D$ to $Q$ and $e(g)^*$ is the transposed linear map relative to $G_j$ and $\Omega|_{Q \times Q}$ from $Q$ to $D$ such that

$$G_j(e(g)^* w, v) = \Omega(w, e(g)v).$$
If we transfer \( j \) from \( V \) to \( \overline{V} \) as \( \overline{j} \) then it has block matrix form

\[
\overline{j} = \begin{pmatrix}
0 & -I_D & 0 \\
I_D & 0 & 0 \\
0 & 0 & j'
\end{pmatrix}.
\]

Finally, if \( \overline{C_g} = \frac{1}{2}(\overline{g} - \overline{g}j) \) be \( C_g \) transported to \( \overline{V} \) then

\[
\overline{C_g} = \begin{pmatrix}
\frac{1}{2} (a(g) + (a(g))^T)^{-1} & \frac{1}{2} a(g) \left(s(g) - \frac{i}{2} e(g)\ast e(g)\right) & -\frac{1}{2} a(g)\ast e(g) \\
-\frac{1}{4} a(g) \left(s(g) - \frac{i}{2} e(g)\ast e(g)\right) & \frac{1}{4} (a(g) + (a(g))^T)^{-1} & \frac{1}{4} a(g)\ast e(g) \ast j' \\
-\frac{1}{4} j' b(g) e(g) & \frac{1}{4} j' b(g) e(g) & C_{b(g)}^r
\end{pmatrix}.
\]

Here \( C_{b(g)}^r \) arises from \( j' \) on \( D^1 / D \).

Clearly, \( \text{Det}_j(C_g) = \text{Det}_{\overline{j}}(\overline{C_g}) \) and to compute the latter we write \( D \oplus D \) as \( D^C \) and replace \( Q \) by \( Q^+ \) the +i eigenspace of \( j' \). On \( D^C \oplus Q^+ \), \( C_g \) becomes a 2 \times 2 complex block matrix

\[
\begin{pmatrix}
\frac{1}{2} (a(g) + (a(g))^T)^{-1} - ia(g) \left(s(g) - \frac{i}{2} e(g)\ast e(g)\right) & -a(g)\ast e(g) \\
-\frac{1}{4} (1 - ij') b(g) e(g) & C_{b(g)}^r
\end{pmatrix}
\]

and hence

\[
\text{Det}_j(C_g) = \text{Det}_D(a(g)) \text{Det}_{j'}(C_{b(g)}^r) \times \\
\text{Det}_D \left( \frac{1}{2} (I_D + (a(g))^T a(g))^{-1} - is(g) + \frac{i}{2} e(g)\ast e(g) - \frac{i}{4} e(g)\ast (1 - ij') C_{b(g)}^r b(g) e(g) \right).
\]

Now \( C_{b(g)}^r b(g) = I + Z_{b(g)}' \) so that the argument of the last determinant becomes

\[
\frac{1}{2} (I_D + (a(g))^T a(g))^{-1} - is(g) + \frac{i}{4} e(g)\ast e(g) - \frac{i}{4} e(g)\ast (1 - ij') C_{b(g)}^r b(g) e(g) = \\
\frac{1}{2} (I_D + (a(g))^T a(g))^{-1} - is(g) - \frac{i}{4} e(g)\ast j' e(g) - \frac{i}{4} e(g)\ast j' Z_{b(g)}^r e(g) - \frac{1}{4} e(g)\ast j' Z_{b(g)}^r e(g).
\]

One can easily check that each of the last three terms is symmetric so when one takes real parts as linear operators on a complex Hermitian vector space what remains is

\[
\frac{1}{2} (I_D + (a(g))^T a(g))^{-1} - \frac{1}{4} e(g)\ast j' e(g) - \frac{1}{4} e(g)\ast j' Z_{b(g)}^r e(g).
\]

Now

\[
-G_j'(e(g)\ast j' e(g) u, u) = -\Omega'(j' e(g) u, e(g) u) = G_j'(e(g) u, e(g) u)
\]

and

\[
-G_j(e(g)\ast j' Z_{b(g)}^r e(g) u, u) = -\Omega'(j' Z_{b(g)}^r e(g) u, e(g) u) = G_j'(e(g) u, Z_{b(g)}^r e(g) u).
\]
Then
\[
G_j'(e(g)u, e(g)u) + G_j'(e(g)u, Z_{b(g)}' e(g)u) =
\frac{1}{2} G_j'(e(g)u, (1 - (Z_{b(g)}')^2) e(g)u) + \frac{1}{2} G_j'((I + Z_{b(g)})e(g)u, (I + Z_{b(g)})e(g)u)
\]
which is non-negative, and hence the last two terms in (26) define a positive operator. Since the first term is positive definite the argument of the determinant is in $GL(D^C)$, so there is a smooth square root of the determinant, say $\delta_D(g)$ with $\delta_D(I_V) = 1$:
\[
\text{Det}_j(C_g) = \text{Det}_D(a(g)) \text{Det}_j(C_{b(g)}') \delta_D(g)^2.
\tag{27}
\]
This shows that a subgroup of $Sp(V, \Omega, D)$ will have a lift to $Mp^f$ precisely when its image in the symplectic group, $Sp(D^\perp/D, \Omega')$, of the reduced space has a lift.

Let $F \in \Lambda(V^C, \Omega)$ be complex Lagrangian with $D = F \cap V$ as above. We shall assume that the dimension of $D$ is neither 0 nor $n$ so $F$ is neither real nor pseudo-Hermitean (since these are cases we already dealt with in Propositions 2.18, 2.19), but a mixture of both.

Then $F \subset (D^\perp)^C$ so projects to a subspace $F/D^C$ of $(D^\perp/D)^C$ which is clearly a complex Lagrangian subspace of the symplectic quotient. Now, however, $F/D^C$ has no real part, so there is a real endomorphism $\tilde{j}$ of $D^\perp/D$ whose complexification has $+i$ eigenspace given by $F/D^C$. The image of $Sp(V, \Omega, F)$ in $Sp(D^\perp/D, \Omega')$ will then be a pseudo-unitary group $U(D^\perp/D, \Omega', \tilde{j})$.

We fix $j \in J_+(V, \Omega)$ so that $j'$ and $\tilde{j}$ commute. This can always be done since we can choose a positive $j'$ on $D^\perp/D$ commuting with $\tilde{j}$, pick any $j_1 \in J_+(V, \Omega)$ lift $j'$ to $D^\perp\cap(j_1 D)^\perp$ by the isomorphism and extend by $j_1$ on $D + j_1 D$.

If $g \in Sp(V, \Omega, F)$ then $g \in Sp(V, \Omega, D)$ and so we have $b(g) \in U(D^\perp/D, \Omega', \tilde{j})$ and
\[
\text{Det}_j(C_{b(g)}) = \text{Det}_j(b(g)) \text{Det}_j(C_{b(g)}')^2.
\]
Combining this with equation (27) we have
\[
\text{Det}_j(C_g) = \text{Det}_D(a(g)) \text{Det}_j(b(g)) \text{Det}_j(C_{b(g)}')^2 \delta_D(g)^2
= \left| \frac{\text{Det}_D(a(g))}{\text{Det}_D(a(g))} \right| \text{Det}_j(b(g)) \left| \frac{\text{Det}_D(a(g))}{\text{Det}_D(a(g))} \right| \text{Det}_j(C_{b(g)}') \delta_D(g) \right|^2
\]
If we set
\[
f_F(g) = \left| \frac{\text{Det}_D(a(g))}{\text{Det}_D(a(g))} \right| \text{Det}_j(C_{b(g)}') \delta_D(g) \right|^{-1}
\tag{28}
\]
we claim all the conditions of Lemma 2.10 are satisfied with $H = Sp(V, \Omega, F)$.

Firstly, when $g \in H_1$, $e(g) = 0$, $s(g) = 0$, $a(g) \in O(D)$, and $b(g) \in U(D^\perp/D, \Omega', j')$ so $H_1$ meets all components of $H$ and $\delta_D(g) = 1$, and so $f_F$ is a homomorphism.
Secondly, for \( g \in H \)

\[
f_F(g)^2 \text{Det}_j(C_g) = \frac{\text{Det}_D(a(g))}{|\text{Det}_D(a(g))|} \text{Det}_j'(b(g))
\]

is a homomorphism into \( U(1) \) since \( b(g) \) is pseudo-unitary. Thus we have proved:

**Proposition 2.22** Let \( F \in \Lambda(V^C, \Omega) \) be a complex Lagrangian subspace, put \( D = F \cap V, \)
\( \tilde{j}' \) the compatible complex structure induced on \( (D^\perp / D, \Omega) \) and \( j \in j_+(V, \Omega) \) be such that the induced \( j' \) on \( (D^\perp / D, \Omega) \) commutes with \( \tilde{j}' \). Let \( f_F \) be defined as in \( (28) \) then for \( g \in \text{Sp}(V, \Omega, F) \), \( g, f_F(g) \) are the parameters of an element \( F_F(g) \) of \( Mp^c(V, \Omega, j) \) and \( F_F \) is a lift of \( \text{Sp}(V, \Omega, F) \) into \( Mp^c(V, \Omega, j) \).

3 Invariant \( Mp^c \)-structures on homogeneous spaces

3.1 \( Mp^c \)-structures

Let \((M, \omega)\) be a symplectic manifold. Consider its symplectic frame bundle \( \text{Sp}(M, \omega) \) whose fibre at \( x \in M \) consists of all symplectic isomorphisms \( b : (V, \Omega) \to (T_xM, \omega_x) \).

**Definition 3.1** An \( Mp^c \)-structure on a symplectic manifold \((M, \omega)\) is a pair \((P, \phi)\) of a principal \( Mp^c(V, \Omega, j) \) bundle \( P \xrightarrow{\pi} M \) with a fibre-preserving map \( \phi : P \to \text{Sp}(M, \omega) \) such that for all \( U \in Mp^c(V, \Omega, j) \) and for all \( p \in P \):

\[
\phi(p, U) = \phi(p).\sigma(U).
\]

**Notation 3.2** If \( M \) and \( N \) are two manifolds and if \( G \) is a Lie group acting on the right on \( M \) and acting on the left on \( N \), then we denote by

\[
M \times_G N
\]

the manifold whose points are equivalence classes in \( M \times N \) for the equivalence defined by the actions of \( G \):

\[
(x, y) \sim (x \cdot g, g^{-1} \cdot y), \quad \forall x \in M, y \in N, g \in G.
\]

In particular, when \( N = H \) is a Lie group and when \( \mu : G \to H \) is a Lie group homomorphism, we consider the left action of \( G \) on \( H \) defined by \( g \cdot h := \mu(g)h \) and the corresponding \( M \times_G H \) is denoted

\[
M \times_{G, \mu} H;
\]

the equivalence class of \((x, h)\) is denoted by \([x, h]\) with \( x \in M, h \in H \) so that \([x, h] = [x \cdot g, \mu(g^{-1})h]\).
Since there is a character $\eta$ defined on the group $Mp^c(V,\Omega, j)$, one can define a complex line bundle (with a natural Hermitean structure) associated to a $Mp^c$-structure

$$P(\eta) := P \times_{Mp^c(V,\Omega, j), \eta} \mathbb{C}. \quad (29)$$

**Remark 3.3** Every symplectic manifold $(M, \omega)$ admits an $Mp^c$-structure, and the isomorphism classes of $Mp^c$-structures are parametrised by equivalence classes of complex line bundles with Hermitean structure over $M$. We briefly recall how to establish these facts. One chooses a positive compatible almost complex structure $J$ on $(M, \omega)$; this is always possible as the bundle of fibrewise positive $\omega$-compatible complex structures has contractible fibres. Choosing such a positive $J$, those symplectic frames which are also complex linear form a principal $U(V, \Omega, j)$-bundle called the unitary frame bundle which we denote by $U(M,\omega,J)$.

Let $L$ be any complex line bundle over $M$ endowed with a Hermitean structure $h$; let $L^{(1)} = \{ u \in L \mid h(u,u) = 1 \}$ be the associated $U(1)$-bundle. Define

$$P(L,J) := (U(M,\omega,J) \times_M L^{(1)}) \times_{MU^c(V,\Omega, j)} Mp^c(V,\Omega, j)$$

with the right action of $MU^c(V,\Omega, j)$ on the right-hand side given via $\sigma \times \lambda$ by the right action of the group $U(V,\Omega, j)$ on $U(M,\omega,J)$ and of $U(1)$ on $L^{(1)}$. Define

$$\phi(L,J) : P(L,J) \to Sp(M,\omega) : [(b,s), A] = b \cdot \sigma(A).$$

Then $(P(L,J),\phi(L,J))$ is a $Mp^c$-structure on $(M,\omega)$.

Conversely, if $(P,\phi)$ is any $Mp^c$-structure on $(M,\omega)$, we define the subset $P_J$ of $P$ lying over the unitary frames

$$P_J := \phi^{-1}(U(M,\omega,J)).$$

This will be a principal $MU^c(V,\Omega, j) \simeq_{\sigma \times \lambda} U(V,\Omega, j) \times U(1)$ bundle.

The complex line bundle associated to $P_J$ by the character $\lambda$ is denoted by $P_J(\lambda)$

$$P_J(\lambda) := P_J \times_{MU^c(V,\Omega, j), \lambda} \mathbb{C}; \quad (30)$$

it carries a natural Hermitean structure. Remark that the line bundle $P_J(\lambda)$ associated to $(P(L,J),\phi(L,J))$ is $L$.

Now $(P,\phi)$ is completely determined by $(P_J,\phi|_{P_J})$ via $P \simeq P_J \times_{MU^c(V,\Omega, j)} Mp^c(V,\Omega, j)$ and $\phi[p,A] = \phi(p) \cdot \sigma(A)$. The map $\check{\lambda} : P_J \to P_J^{(1)}(\lambda) : \xi \mapsto [\xi,1]$ allows to write an isomorphism

$$\phi \times \check{\lambda} : P_J \to U(M,\omega,J) \times_M P_J^{(1)}(\lambda) : \xi \mapsto (\phi(\xi),[\xi,1]).$$

Hence $(P,\phi)$ is isomorphic to $(P(L,J),\phi(L,J))$ for $L = P_J(\lambda)$. 
The isomorphism class of the line bundle $P_J(\lambda)$ is independent of the choice of $J$. This class is called the class of the $Mp^c$-structure $(P, \phi)$.

Remark that the relation (16) between the characters gives

$$P(\eta) = (P_J(\lambda))^{\otimes 2} \otimes \Lambda^n(T^1,0_J M)$$

where $\Lambda^n(T^1,0_J M) = U(M, \omega, J) \times_{U(V, \Omega, J)} \mathbb{C}$ is the line bundle whose class is the first Chern class of the symplectic structure.

The above shows that we have an $Mp^c$-structure $(P, \phi)$ for which the class is zero, i.e. the line bundle $P_J(\lambda)$ is trivial, and this $Mp^c$-structure is unique up to isomorphism. That is, unlike metaplectic structures where there is no base-point, we have an $Mp^c$-structure from which the others can be obtained by twisting.

Definition 3.4 A basic $Mp^c$-structure on a symplectic manifold is an $Mp^c$-structure $(P, \phi)$ whose class is zero. Up to isomorphism, it is unique and can be constructed, using a positive compatible almost complex structure $J$ on $(M, \omega)$, as

$$P_{\text{basic}, J} := U(M, \omega, J) \times_{U(V, \Omega, J)} Mp^c(V, \Omega, j)$$

where $F_j$ is the embedding of $U(V, \Omega, j)$ into $Mp^c(V, \Omega, j)$ as in formula (21) and $\phi$ is the natural projection

$$\phi : P_{\text{basic}, J} \rightarrow Sp(M, \omega) : [b, A] \rightarrow b \cdot \sigma(A).$$

Definition 3.5 A metaplectic structure on a symplectic manifold $(M, \omega)$ is a pair $(B, \psi)$ of a principal $Mp(V, \Omega, j)$-bundle $B \rightarrow M$ with a fibre-preserving map $\psi : B \rightarrow Sp(M, \omega)$ such that for all $U \in Mp(V, \Omega, j)$ and for all $p \in B$:

$$\psi(p, U) = \psi(p) \cdot \sigma(U).$$

Lemma 3.6 There exists a metaplectic structure on $(M, \omega)$ if and only if the canonical line bundle $\Lambda^n T^1,0_J M$ admits a square root.

Proof Since $Mp(V, \Omega, j) = \ker \eta = \{(U, g) \in Mp^c(V, \Omega, j) \mid \lambda^2 \text{Det}_j C_g = 1\}$, any metaplectic structure $(B, \psi)$ yields an $Mp^c$-structure defined by

$$P = B \times_{Mp(V, \Omega, j)} Mp^c(V, \Omega, j) \quad \phi : P \rightarrow Sp(M, \omega) : [p, A] \rightarrow \psi(p) \cdot \sigma(A)$$

and the line bundle associated to $P$ and the character $\eta$, $P(\eta)$, is trivial; equivalently, by (31), $(P_J(\lambda))^{\otimes 2} \otimes \Lambda^n T^1,0_J M$ is trivial so $P_J(\lambda^{-1})$ is a square root of the canonical bundle.
Reciprocally, if there is an $Mp^{c}$-structure $(P, \phi)$ such that $P(\eta)$ is trivial, i.e. if there is a square root of the canonical bundle, then there is an associated metaplectic structure defined as follows. If $T : P(\eta) \to M \times \mathbb{C}$ is a trivialisation, define

$$B := T^{-1}(M \times \{1\}), \quad \psi := \phi|_B.$$  

\[ 3.1.1 \quad Mp^{c}$-structure associated to a compatible almost complex structure \]

Given any compatible almost complex structure (not necessarily positive!) $\tilde{J}$ on $(M, \omega)$, we construct $Mp^{c}$-structures in a similar way. Symplectic frames which are also complex linear for $\tilde{J}$ form a principal $U(V, \Omega, \tilde{J})$-bundle called the pseudo-unitary frame bundle which we denote by $U(M, \omega, \tilde{J})$ where $\tilde{J}$ is chosen so that $G_{\tilde{J}}$ has the same signature as $G_{\tilde{J}}$. We define a pseudo-basic $Mp^{c}$-structure associated to $\tilde{J}$ by:

$$P_{psbasic,\tilde{J}} := U(M, \omega, \tilde{J}) \times_{U(V,\Omega,\tilde{J}),F_{\tilde{J},\tilde{J}}} Mp^{c}(V, \Omega, j),$$  

with $F_{\tilde{J},\tilde{J}} : U(V, \Omega, \tilde{J}) \to Mp^{c}(V, \Omega, j)$ defined as in (24). Remark that as before, any other $Mp^{c}$-structure is given up to isomorphism by tensoring the above one with a circle bundle.

Observe that this pseudo-basic bundle is not basic in general. Indeed, let us choose a $Mp^{c}$-structure associated to a compatible almost complex structure $J$ on $(M, \omega)$ which commutes with $\tilde{J}$. [This is always possible; indeed, choosing any Riemannian metric $g_0$ on $M$, setting $g_1(X, Y) = g_0(X, Y) + g_0(\tilde{J}X, \tilde{J}Y)$ and defining the field $A$ of linear endomorphisms by $\omega(X, Y) = g_1(AX, Y)$, then $A$ commutes with $\tilde{J}$, the transpose $A^*$ of $A$ relative to the metric $g_1$ is equal to $-A$, $AA^* = -A^2$ is symmetric and positive definite and we can define $J$ using the polar decomposition of $A$ as $J = (-A^2)^{-\frac{1}{2}} A$.]

At each point $x$ the tangent space $T_x M$ splits as a direct sum $T_x M^+ \oplus T_x M^-$ of $-1$ and $+1$ eigenspaces of $J_x \tilde{J}_x$, and this is a $\omega_x$-orthogonal splitting, with each subspace stable by $J_x$ (and $\tilde{J}_x$ since $J_x = \pm \tilde{J}_x = J_x^{\pm}$ on $T_x M^{\pm}$). Symplectic frames which are complex linear both for $J$ and for $\tilde{J}$ consist of pairs of symplectic unitary frames of $T_x M^+$ and $T_x M^-$. Those form a principal $U(V_+) \oplus U(V_-)$-bundle which we denote $U(M, \omega, J, \tilde{J})$. The pseudo-basic $Mp^{c}$-structure associated to $\tilde{J}$ has the form:

$$P_{psbasic,\tilde{J}} = U(M, \omega, J, \tilde{J}) \times_{U(V_+)\oplus U(V_-),F_{\tilde{J},\tilde{J}}} Mp^{c}(V, \Omega, j),$$  

where $i$ is the natural injection of $U(V_+) \oplus U(V_-)$ into $U(V = V_+ \oplus V_-, \Omega, j)$. If $g \in U(V_+) \oplus U(V_-)$ then $i(g) = \begin{pmatrix} U^+ & 0 \\ 0 & U^- \end{pmatrix}$ so that $\text{Det}_j(C_g) = \text{Det}_j(U_-)$. Hence the line bundle associated to the $Mp^{c}$-structure $P_{psbasic,\tilde{J}}$ is

$$P^{\tilde{J}}_{psbasic,\tilde{J}}(\lambda) = U(M, \omega, J, \tilde{J}) \times_{U(V_+)\oplus U(V_-),\lambda} \mathbb{C}$$  

(34)
where \( \chi = \lambda \circ F_j \circ i \) so that \( \chi \left( \begin{array}{cc} U^+ & 0 \\ 0 & U^- \end{array} \right) = \text{Det}_j(U_-)^{-1} \). Since \( \chi^2(g) = \text{Det}_j g \text{Det}_j^{-1} g \)

\[
P_{psbasic, j}(\eta) = U(M, \omega, \tilde{J}) \times_{U(V, \Omega), \text{Det}_j} \mathbb{C}. \tag{35}
\]

### 3.1.2 \( Mp^c \)-structure associated to bi-Lagrangian or liftable \( H \)-structures.

Given a real Lagrangian distribution on a symplectic manifold \((M, \omega)\) (i.e. a smooth distribution \( L \), with \( L_x \subset T_x M \) real Lagrangian subspace for each \( x \in M \)), we construct \( Mp^c \)-structures adapted to this situation in a similar way. Symplectic frames whose first elements yield a basis of the distribution form a principal \( Sp(V, \Omega, F) \)-bundle which we denote by \( Sp(M, \omega, L) \). We define a \( L \)-basic \( Mp^c \)-structure as

\[
P_{L-basic} := Sp(M, \omega, L) \times_{Sp(V, \Omega, F), LF_j} Mp^c(V, \Omega, j), \tag{36}
\]

where \( LF_j : Sp(V, \Omega, F) \to Mp^c(V, \Omega, j) \) is the lift defined in Proposition \[2.19\] Remark that any other \( Mp^c \)-structure is given up to isomorphism by tensoring the above one with a circle bundle.

More generally we get

**Definition 3.7** An \( H \)-structure on the symplectic manifold \((M, \omega)\) is the data of

- a principal \( H \)-bundle \( B \xrightarrow{\pi_B} M \), and
- a homomorphism \( \tau : H \to Sp(V, \Omega) \) so that

\[
Sp(M, \omega) \simeq B \times_{H, \tau} Sp(V, \Omega).
\]

It is said to be liftable if there exists a lift, that is

- a group homomorphism \( \tilde{\tau} : H \to Mp^c(V, \Omega, j) \), for a choice of PCCS \( j \), so that \( \sigma \circ \tilde{\tau} = \tau \).

Given a liftable \( H \)-structure on \((M, \omega)\) we define the \( H \)-basic \( Mp^c \)-structure as

\[
P_{Hbasic} := B \times_{H, \tilde{\tau}} Mp^c(V, \Omega, j), \quad \phi_{Hbasic} : P_{Hbasic} \to Sp(M, \omega) : [b, A] \mapsto i(b)\sigma(A), \tag{37}
\]

where

\[
i : B \to Sp(M, \omega) = B \times_{H, \tau} Sp(V, \Omega) : b \mapsto [b, 1]. \tag{38}
\]

Up to isomorphism, any other \( Mp^c \)-structure over \((M, \omega)\) is given by tensoring the above with a circle bundle.
An example of liftable $H$-structure is given by the choice on $(M, \omega)$ of a bi-Lagrangian structure, i.e. a field $A$ of endomorphisms of the tangent bundle so that $A_x \in \mathfrak{sp}(T_xM, \omega_x)$ and $A_x^2 = \text{Id}_{T_xM} \forall x \in M$. The $\pm 1$ eigenspaces $A_+$ and $A_-$ of $A$ define supplementary Lagrangian distributions. Given any basis $\{e_1, \ldots, e_n\}$ of $A_+\times A_-$, there is a unique basis $\{f_1, \ldots, f_n\}$ of $A_-\times A_+$ so that $\omega_x(e_j, f_k) = \delta_{jk}$. The bundle of such adapted frames is a $GL(F)$ principal bundle $B \to M$ (where $F$ is a Lagrangian subspace of $V$). This defines a liftable $GL(F)$-structure with

$$\tau_F : GL(F) \to Sp(V = F \oplus jF, \Omega) : C \mapsto \tau_F(C) := \begin{pmatrix} C & 0 \\ 0 & (C^T)^{-1} \end{pmatrix}. \quad (39)$$

The lift $\tilde{\tau}_F : GL(F) \to Mp^\varepsilon(V = F \oplus jF, \Omega, j)$ is given as in Proposition 2.19 by

$$\tilde{\tau}_F(C) = (U_{\tau_F(C)}, f_F(C), \tau(C)) \quad \text{with} \quad f_F(C) = \left(\text{Det}_F C\right)^{-\frac{1}{2}} e^{-\frac{1}{2} \varphi_F(\frac{1}{2}(\text{Id}_F + (C^T)C)^{-1})}. \quad (40)$$

The case of a field $A$ of endomorphisms of the tangent bundle so that $A_x \in \mathfrak{sp}(T_xM, \omega_x)$ and $A_x^2 = -\text{Id}_{T_xM} \forall x \in M$, is the case of a compatible almost complex structure (not necessarily positive!) $\tilde{J}$ on $(M, \omega)$. It corresponds to the liftable $U(V, \Omega, \tilde{J})$-structure defined by the pseudo-unitary frame bundle $U(M, \omega, \tilde{J})$ and the lift $F_{\tilde{J}j} : U(V, \Omega, \tilde{J}) \to Mp^\varepsilon(V, \Omega, j)$ of the inclusion map, as described in section 3.1.1.

### 3.1.3 Action of $U(1)$-principal bundles on $Mp^\varepsilon$-structures

The action of $U(1)$-principal bundles on $Mp^\varepsilon$-structures is made explicit and canonical (not depending on the choice of an almost complex structure) in the following two lemmas:

**Lemma 3.8** Given an $Mp^\varepsilon$-structure on $M$, $(P, \tilde{\pi} : P \to \pi : P \to Sp(M, \omega))$, and given a principal $U(1)$-bundle over $M$, $L^{(1)} \xrightarrow{\tilde{\pi}} M$, one can define a new $Mp^\varepsilon$-structure on $M$, $(P', \pi' : P' \to Sp(M, \omega))$ denoted $L^{(1)} \cdot P$ as follows. One first considers the fibrewise product of $L^{(1)}$ and $P$ over $M$

$$L^{(1)} \times_M P := \{ (s, p) \in L^{(1)} \times P \mid \tilde{\pi}(s) = \pi(p) \}.$$ 

It is a principal $U(1) \times Mp^\varepsilon(V, \Omega, j)$ bundle over $M$. One defines

$$P' = \left( L^{(1)} \times_M P \right) \times_{(U(1) \times Mp^\varepsilon(V, \Omega, j), \tilde{\rho})} Mp^\varepsilon(V, \Omega, j) \quad (41)$$

for the homomorphism given by the embedding of $U(1)$ in $Mp^\varepsilon(V, \Omega, j)$ and multiplication

$$\tilde{\rho} : U(1) \times Mp^\varepsilon(V, \Omega, j) \to Mp^\varepsilon(V, \Omega, j) : (e^{i\theta}, A) \mapsto e^{i\theta} A.$$

The projection $\pi' : P' \to M$ is defined by

$$\pi'([(s, p), A]) := \pi(p)(= \tilde{\pi}(s)), \quad (s, p) \in L^{(1)} \times_M P, A \in Mp^\varepsilon(V, \Omega, j).$$
and the map $\phi' : P' \to Sp(M, \omega)$ is defined by

$$\phi'([s, p], A) = \phi(p \cdot A) = \phi(p) \cdot \sigma(A).$$

At the level of isomorphism classes this Lemma gives an action of $H^2(M, \mathbb{Z})$ on the set of isomorphism classes of $Mp^c$-structures for a fixed symplectic structure.

**Lemma 3.9** Given two $Mp^c$-structures $(P, \phi)$ and $(P', \phi')$ over the same symplectic manifold $(M, \omega)$, there is a canonical principal $U(1)$-bundle $L^{(1)}$ over $M$ constructed in the following way. One first defines the fibrewise product of $P$ and $P'$ viewed as principal $U(1)$-bundles over $Sp(M, \omega)$:

$$P \times_{\phi \times \phi'} P' := \{ (p, p') \in P \times P' \mid \phi(p) = \phi'(p') \}.$$

One defines the group

$$Mp^c(V, \Omega, j) \times_\sigma Mp^c(V, \Omega, j) := \{(A, B) \mid A, B \in Mp^c(V, \Omega, j), \sigma(A) = \sigma(B)\}$$

which acts on the right on $P \times_{\phi \times \phi'} P'$ in the obvious way $(p, p') \cdot (A, B) := (p \cdot A, p' \cdot B)$ and makes $P \times_{\phi \times \phi'} P'$ into a principal $Mp^c(V, \Omega, j) \times_\sigma Mp^c(V, \Omega, j)$ bundle over $M$. Then

$$L^{(1)} := (P \times_{\phi \times \phi'} P') \times_{(Mp^c(V, \Omega, j) \times_\sigma Mp^c(V, \Omega, j), \rho)} U(1)$$

for the homomorphism

$$\rho : Mp^c(V, \Omega, j) \times_\sigma Mp^c(V, \Omega, j) \to U(1) : (A, B) \mapsto AB^{-1}.$$  

Clearly

$$P = L^{(1)} \cdot P'.$$

This Lemma shows in particular that the action is simply transitive.

### 3.2 Invariant $Mp^c$-structures

**Definition 3.10** If there is a symplectic action $\rho^M$ of the Lie group $G$ on the symplectic manifold $(M, \omega)$, a $G$-invariant $Mp^c$-structure on $(M, \omega, \rho^M)$ is an $Mp^c$-structure $(P \xrightarrow{\phi} M, \phi : P \to Sp(M, \omega))$ and an action $\rho^P$ of $G$ on $P$, commuting with the right action of the group $Mp^c$, and such that

$$\phi \circ \rho^P(g) = \tilde{\rho}(g) \circ \phi \quad \forall g \in G$$

where $\tilde{\rho}$ is the action induced by $\rho^M$ on $Sp(M, \omega)$ i.e.

$$\tilde{\rho}(g)b = \rho^M(g)_{*x} \circ b \text{ for } b : V \to T_xM.$$
Proof The group $\mathcal{G}$, the first one by a $\mathcal{G}$-invariant $\mathcal{G}$-structure on a symplectic manifold $\mathcal{M}$, is defined in Lemma 3.8, is $\mathcal{G}$-invariant in a canonical way. Given an action $\rho^\mathcal{M}$ of the Lie group $\mathcal{G}$ on the symplectic manifold $(\mathcal{M},\omega)$ and given two $\mathcal{G}$-invariant $\mathcal{G}$-structures $(\mathcal{P},\phi,\rho^\mathcal{P})$ and $(\mathcal{P}',\phi',\rho'^\mathcal{P})$ over $(\mathcal{M},\omega,\rho^\mathcal{M})$, the canonical principal $U(1)$-bundle $L^{(1)}$ over $\mathcal{M}$ constructed in Lemma 3.3 is $\mathcal{G}$-invariant in a canonical way.

**Proof** One first defines the action of $\mathcal{G}$ on the fibrewise product $\mathcal{P} \times_{\phi,\phi'} \mathcal{P}'$ of $\mathcal{P}$ and $\mathcal{P}'$ over $Sp(\mathcal{M},\omega)$:

$$g \cdot (p,p') := (\rho^\mathcal{P}(g)p,\rho'^\mathcal{P}(g)p')$$

this is indeed an action on $\mathcal{P} \times_{\phi,\phi'} \mathcal{P}'$ since $\phi(p,g\cdot p) = \delta(p)\phi(p) = \delta(p)\phi(p') = \phi'(\rho'^\mathcal{P}(g)p')$ when $\phi(p) = \phi'(p')$. Then

$$g \cdot ((p,p') \cdot (A,B)) = (\rho^\mathcal{P}(g)(p \cdot A),\rho'^\mathcal{P}(g)(p' \cdot B)) = (g \cdot (p,p')) \cdot (A,B)$$

for all $(A,B) \in (\mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho) \times_{\sigma} \mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho))$ so that $\mathcal{G}$ acts on the circle bundle

$$L^{(1)} := (\mathcal{P} \times_{\phi,\phi'} \mathcal{P}') \times_{\mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho)} \mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho) U(1)$$

via $g \cdot [(p,p'),A] := [g \cdot (p,p'),A]$. □

**Lemma 3.13** Consider an action $\rho^\mathcal{M}$ of the Lie group $\mathcal{G}$ on the symplectic manifold $(\mathcal{M},\omega)$. Given a $\mathcal{G}$-invariant $\mathcal{G}$-structure $(\mathcal{P},\phi,\rho^\mathcal{P})$ and a $\mathcal{G}$-invariant principal $U(1)$-bundle over $\mathcal{M}$, i.e. a $U(1)$-bundle $L^{(1)} \xrightarrow{\tilde{\pi}} \mathcal{M}$ with a left action $\rho^L$ of $\mathcal{G}$ on $L^{(1)}$ commuting with the right action of $U(1)$ and so that $\tilde{\pi} \circ \rho^L(g) = \rho^\mathcal{M}(g) \circ \tilde{\pi}$, the new $\mathcal{G}$-structure $L^{(1)} \cdot \mathcal{P}$ on $\mathcal{M}$ defined in Lemma 3.8 is $\mathcal{G}$-invariant in a canonical way.

**Proof** The group $\mathcal{G}$ acts on the fibrewise product of $L^{(1)}$ and $\mathcal{P}$ over $\mathcal{M}$, $L^{(1)} \times_{\mathcal{M}} \mathcal{P}$ via

$$g \cdot (s,p) := (\rho^L(g)s,\rho^P(g)p)$$

and this action commutes with the right action of $U(1) \times \mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho)$ so that $\mathcal{G}$ acts on $L^{(1)} \cdot \mathcal{P} = (L^{(1)} \times_{\mathcal{M}} \mathcal{P}) \times_{(U(1) \times \mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho))} \mathcal{M}^\mathcal{P}(\mathcal{V},\Omega,\rho)$ via

$$\rho^{L^{(1)} \cdot \mathcal{P}}(g)[(s,p),A] := [g \cdot (s,p),A] = [(\rho^L(g)s,\rho^P(g)p),A].$$

□

The two lemmas above show that if there exists one $\mathcal{G}$-invariant $\mathcal{G}$-structure on a symplectic manifold with a symplectic action of $\mathcal{G}$, then any other one is obtained by acting on the first one by a $\mathcal{G}$-invariant $U(1)$-bundle over $\mathcal{M}$. 
Let us remark that in general there is no basic $G$-invariant $Mp^c$-structure; this will happen in particular on some pseudo-Hermitean symmetric spaces; on those, there is a pseudo-basic invariant $Mp^c$-structure. This stresses again the point that one should not restrict the study to basic $Mp^c$-structures.

**Remark 3.14** Let us observe that if there is a symplectic action $\rho^M$ of the Lie group $G$ on the symplectic manifold $(M,\omega)$, and if the symplectic manifold is endowed with a liftable $H$-structure $(B,\tau,\tilde{\tau})$, which is $G$-invariant (i.e. there is an action $\rho^B$ of $G$ on $B$, commuting with the right action of the group $H$, and such that $i(\rho^B(g)b) = \rho^M(g)_{\pi_B}(\tilde{\tau}(b)) \circ i(b)$ with $i$ the natural bundle map $i : B \to Sp(M,\omega)$ defined in (38)), then $P_{H\text{basic}}$ is $G$-invariant.

### 3.2.1 Invariant $Mp^c$-structures on homogeneous spaces

Consider a transitive symplectic action $\rho^M$ of the Lie group $G$ on the symplectic manifold $(M,\omega)$. Choose a base point $p_0 \in M$ and let $K$ be the stabilizer in $G$ of this point. The canonical projection

$$\pi : G \to M : g \mapsto gp_0$$

gives an identification $M \simeq G/K$ and the differential $\pi_{se}$ at the neutral element $e \in G$ identifies the tangent space $T_{p_0}M$ with the quotient $g/k$, which becomes a symplectic vector space. The differential at $p_0$ of the action $\rho^M$ restricted to $K$ yields a homomorphism

$$K \to Sp(T_{p_0}M,\omega_{p_0}) : k \mapsto (\rho^M(k))_{\pi_{p_0}}$$

with the identification of $T_{p_0}M$ with $g/k$. Having chosen a symplectic frame $f_0 : V \to T_{p_0}M$ at $p_0$, any symplectic frame at $\rho^M(g)p_0$ is of the form

$$(\rho^M(g))_{\pi_{p_0}} \circ f_0 \circ A \text{ for an } A \in Sp(V,\Omega)$$

so the bundle of symplectic frames is given by

$$Sp(M,\omega) = G \times_{K,\tau} Sp(V,\Omega)$$

for

$$\tau : K \to Sp(V,\Omega) : k \mapsto f_0^{-1} \circ (\rho^M(k))_{\pi_{p_0}} \circ f_0.$$
left multiplication on the first factor \( \rho^P(g)[g', B] := [gg', B] \).

Reciprocally, if \((P, \phi)\) is any \(G\)-invariant \(Mp^c\)-structure on \(M\) and if \(\xi_0\) belongs to \(\phi^{-1}(f_0)\), then any element of \(P\) above \(\rho^M(g)p_0\) is of the form
\[
\rho^P(g) \circ \xi_0 \circ B
\]
for a \(B \in Mp^c(V, \Omega, j)\). so that
\[
P = G \times_{K, \tilde{\tau}} Mp^c(V, \Omega, j)
\]

for
\[
\tilde{\tau} : K \to Mp^c(V, \Omega, j) \text{ defined by } k \cdot \xi_0 = \xi_0 \circ \tilde{\tau}(k)
\]
which is clearly a lift of \(\tau\). Hence we have

**Proposition 3.15** Given a transitive symplectic action \(\rho^M\) of a Lie group \(G\) on a symplectic manifold \((M, \omega)\), there exists a \(G\)-invariant \(Mp^c\)-structure on \(M\) if and only if there exists a lift \(\tilde{\tau} : K \to Mp^c(V, \Omega, j)\) of the isotropy representation
\[
\tau : K \to Sp(V, \Omega) : k \mapsto f_0^{-1} \circ (\rho^M(k))_{\ast p_0} \circ f_0
\]
where \(p_0\) is a chosen point in \(M\), where \(K\) is the stabilizer in \(G\) of this point and where \(f_0\) is a chosen symplectic frame at \(p_0\). Furthermore, any \(G\)-invariant \(Mp^c\)-structure on \(M\) is of the form
\[
P = G \times_{K, \tilde{\tau}} Mp^c(V, \Omega, j)
\]
with \(\tilde{\tau}\) such a lift.

We have seen that such lifts exist when \(\tau(K)\) is in the pseudo-unitary group or in the group of symplectic endomorphisms which stabilize a real or complex Lagrangian subspace.

### 3.2.2 Pseudo-Hermitean or bi-Lagrangian homogeneous spaces

A \(G\)-homogeneous space endowed with a \(G\)-invariant symplectic structure and a \(G\)-invariant compatible almost complex structure \(\tilde{J}\) is called a \textit{pseudo-Hermitean homogeneous space}. The stabilizer \(K\) of a point \(p_0\) acts on \(T_{p_0}M\) (endowed with \(\omega_{p_0}\) and \(\tilde{J}_{p_0}\)) in a pseudo-unitary way. Hence if \((M, \omega, \tilde{J})\) is a pseudo-Hermitean homogeneous manifold, it is endowed with a \(G\)-invariant liftable \(K\)-structure, where the \(K\)-principal bundle is \(G \to G/K\), where
\[
\tau = K \to U(V, \Omega, \tilde{J}) : k \mapsto \tau(k) := f_0^{-1} \circ \rho^M(k)_{\ast p_0} \circ f_0
\]
for a pseudo-unitary frame at \(p_0\), \(f_0 : (V, \Omega, \tilde{J}) \to (T_{p_0}M, \omega_{p_0}, \tilde{J}_{p_0})\). Any homogeneous Hermitean line bundle \(L\) over \(M\) is defined by a character \(\chi : K \to U(1)\) via \(L = G \times_{K, \chi} \mathbb{C}\). Hence:
Proposition 3.16 Any $G$-invariant $M^\varphi$-structure over a pseudo-Hermitean homogeneous manifold $(M, \omega, \tilde{J})$ is of the form

$$P(G, K, \omega, \tilde{J}, \chi) := G \times_{K, \chi \times (F_{\tilde{J}, \tilde{J}} \circ \tau)} M^\varphi(p, \Omega, j). \quad (44)$$

with $K$ the stabilizer of a point $p_0 \in M$, with $\chi$ a unitary character of $K$, $\chi : K \to U(1)$, with $\tau(k) := f_0^{-1} \circ \rho^M(k)|_{p_0} \circ f_0$ where $f_0$ is a pseudo-unitary frame at $p_0$, and with $F_{\tilde{J}, j} : U(V, \Omega, \tilde{J}) \to M^\varphi(p, \Omega, j)$ defined as in $[24]$. A $G$-homogeneous space endowed with a $G$-invariant symplectic structure and a $G$-invariant bi-Lagrangian structure (i.e. a $G$-invariant field $A$ of endomorphisms of the tangent bundle so that $A_x \in \mathfrak{sp}(T_x M, \omega_x)$ and $A_x^2 = \text{Id}_{T_x M}$ $\forall x \in M$) is called a bi-Lagrangian homogeneous space. For a bi-Lagrangian homogeneous space the stabilizer $K$ of a point $p_0$ acts on $T_{p_0}M$ by elements in the symplectic group which commute with $A_{p_0}$, hence of the form

$$\left( \begin{array}{cc} (\rho^M(k)|_{p_0})|_{T_+} & 0 \\ 0 & (\rho^M(k)|_{p_0})|_{T_-} \end{array} \right) \in \tau_{T_+} GL(T_+)$$

under the decomposition $T_{p_0}M = T_+ \oplus T_-$ into $\pm 1$ eigenspaces for $A_{p_0}$, with $\tau_{T_+}$ defined as in $[39]$. Hence if $(M, \omega, A)$ is a bi-Lagrangian homogeneous manifold, it is endowed with a liftable $K$-structure. The $K$-principal bundle is $G \to G/K$ with

$$\tau : K \to \tau F GL(F) \subset Sp(V, \Omega) : k \mapsto \tau_F \left( f_{+0}^{-1} \circ (\rho^M(k)|_{p_0})|_{T_+} \circ f_{+0} \right) \quad (45)$$

for any frame $f_{+0}$ of $T_+$, $f_{+0} : F \to T_+$ with $F$ a Lagrangian subspace of $(V, \Omega)$. Any homogeneous hermitian line bundle $L$ over $M$ is defined by a character $\chi : K \to U(1)$ via $L = G \times_{K, \chi} \mathbb{C}$. Hence:

Proposition 3.17 Any $G$-invariant $M^\varphi$-structure over a bi-Lagrangian homogeneous manifold $(M, \omega, A)$ is of the form

$$P(G, K, \omega, A, \chi) := G \times_{K, \chi \times (\tau F \circ \rho_+)} M^\varphi(p, \Omega, j). \quad (46)$$

with $K$ the stabilizer of a point $p_0 \in M$, with $\chi$ a unitary character of $K$, $\chi : K \to U(1)$, with $\rho_+(k) := f_{+0}^{-1} \circ (\rho^M(k)|_{p_0})|_{T_+} \circ f_{+0}$ where $T_+$ is the $+1$ eigenspace for $A_{p_0}$ and where $f_{+0} : F \to T_+$ is a frame of $T_+$ at $p_0$, and with $\tau F : GL(F) \to M^\varphi(p, \Omega, j)$ defined as in $[40]$.  

3.3 $G$-invariant $M^\varphi$-connections

Definition 3.18 An $M^\varphi$-connection on an $M^\varphi$-structure $(P, \phi)$ is a principal connection $\alpha$ on $P$; in particular, it is a 1-form on $P$ with values in $\mathfrak{m}^\varphi \simeq_{\sigma_x \times \eta_\Phi} \mathfrak{sp}(V, \Omega) \oplus \mathfrak{u}(1)$. We decompose it accordingly as

$$\alpha = \alpha_1 + \alpha_0.$$
The character $\eta$ yields the construction of a $U(1)$ principal bundle

$$P^1(\eta) := P \times_{Mp^\ell(V,\Omega,j)} U(1)$$

and there exists a map

$$\tilde{\eta} : P \to P^1(\eta) : p \mapsto [p, 1].$$

Then $\alpha_0$ is the pull-back of a $u(1)$-valued 1-form on $P^1(\eta)$ under the differential of $\tilde{\eta}$ and

$$\alpha_0 = 2\tilde{\eta}^* \beta_0$$

where $\beta_0$ is a principal $U(1)$ connection on $P^1(\eta)$, because $\eta$ is the squaring map on the central $U(1)$.

Similarly $\alpha_1$ is the pull-back under the differential of $\phi : P \to Sp(M,\omega)$ of a $sp(V,\Omega)$-valued 1-form $\beta_1$ on $Sp(M,\omega)$ and

$$\alpha_1 = \phi^* \beta_1$$

where $\beta_1$ is a principal $Sp(V,\Omega)$ connection on $Sp(M,\omega)$, hence corresponding to a linear connection $\nabla$ on $M$ so that $\nabla \omega = 0$.

Thus a $Mp^\ell$-connection on $P$ induces connections in $TM$ preserving $\omega$ and in $P^1(\eta)$. The converse is true – we pull back and add connection 1-forms in $P^1(\eta)$ (with a factor 2) and in $Sp(M,\omega)$ to get a connection 1-form on $P$.

**Remark 3.19** Let us observe that if the symplectic manifold is endowed with a liftable $H$-structure, then any principal connection $\gamma$ on the principal $H$-bundle $B$ induces a unique $Mp^\ell$-connection on each $Mp^\ell$-structure of the form $P = B \times_{H,\chi,\tilde{\tau}} Mp^\ell(V,\Omega,j)$, where $\chi$ is a character of $H$ via

$$\alpha_{(b, U)}(g_*(X_b + L_U A)) = \text{Ad} U^{-1}(\chi \times \tilde{\tau}_*)(\gamma_b(X_b)) + A$$

where $q : B \times Mp^\ell(V,\Omega,j) \to P$ is the canonical projection.

Such a connection is said to be **compatible with the liftable $H$-structure**.

**Definition 3.20** A $G$-invariant $Mp^\ell$-connection on a $G$-invariant $Mp^\ell$-structure $P$ is a connection 1-form $\alpha$ on the principal bundle $P \to M$ such that $\rho^p(g)^* \alpha = \alpha$ for all $g \in G$. Observe that this is the case if and only if $\alpha = \alpha_1 + \alpha_0$ with $\alpha_0 = 2\tilde{\eta}^* \beta_0$ where $\beta_0$ is a $G$-invariant principal $U(1)$ connection on $P^1(\eta)$ and with $\alpha_1 = \phi^* \beta_1$ where $\beta_1$ is a $G$-invariant principal $Sp(V,\Omega)$ connection on $Sp(M,\omega)$.

Let us observe that if the symplectic manifold is endowed with a $G$-invariant liftable $H$-structure $(B, \tau, \tilde{\tau})$, then any $G$-invariant $Mp^\ell$-connection which is compatible with the liftable $H$-structure is defined by a $G$-invariant principal connection $\gamma$ on the principal
$H$-bundle $B$. In the case where the manifold is homogeneous for $G$, $M = G/H$, such a connection is of the form

$$\alpha_g(L_g*X) = \nu(X) \text{ for any } X \in \mathfrak{g}$$

where $\nu : \mathfrak{g} \to \mathfrak{h}$ is linear, vanishes on $\mathfrak{h}$ and is $\text{Ad} H$ equivariant.

**Definition 3.21** If $M = G/K$ is a reductive homogeneous manifold, i.e. when one can write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p}$ an $\text{Ad} K$ invariant subspace of $\mathfrak{g}$ supplementary to $\mathfrak{k}$, the reductive connection one form $\alpha$ is defined on the $K$-principal bundle $G p \to G/K$ by

$$\alpha_g(L_g*X) = X_\mathfrak{k} \text{ for any } X \in \mathfrak{g}$$

where $U_\mathfrak{k}$ denotes the projection of $U \in \mathfrak{g}$ on $\mathfrak{k}$ relatively to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The corresponding horizontal subspace of the tangent space $T_gG = L_g*\mathfrak{g}$ is given by $L_g*\mathfrak{p}$ and is supplementary to the vertical subspace $\ker p_{*g} = L_g*\mathfrak{k}$. This connection 1-form on $G$ induces a $G$-invariant covariant derivative on $TM$ and on any vector bundle associated to $G$. It also induces a $G$-invariant connection on any principal bundle of the form $G \times_{K,\nu} H$ for any group homomorphism $\nu : K \to H$, asking the horizontal subspace at $\iota(g) := [g,1]$ to be $\iota_\mathfrak{k}L_g*\mathfrak{p}$.

If $M = G/K$ is a symmetric space, it yields the unique linear connection – called the symmetric connection – which is invariant under all symmetries. In the non-symmetric case, any other $G$-invariant connection on the $K$-principal bundle $G p \to G/K$ is of the form

$$\alpha_g(L_g*X) = X_\mathfrak{k} + \lambda(X) \text{ for any } X \in \mathfrak{g}$$

where $\lambda : \mathfrak{g} \to \mathfrak{k}$ is a linear map, vanishing on $\mathfrak{k}$ and equivariant for the adjoint action of $K$.

If $(M = G/K, \omega, A)$ is a pseudo-Hermitean reductive homogeneous manifold ($A = \bar{J}$ i.e. $A^2 = -\text{Id}$), or a bi-Lagrangian reductive homogeneous space ( i.e. $A^2 = \text{Id}$), the linear connection induced by the reductive connection preserves the symplectic form and preserves $A$ but may have torsion. It is without torsion if $G/K$ is symmetric.

## 4 Symplectic Dirac operators

### 4.1 Definition and some spectral properties of symplectic Dirac operators

Consider a symplectic manifold $(M, \omega)$ with an $Mp^c$-structure $(P, \phi)$ and an $Mp^c$-connection $\alpha = 2\eta^*\beta_0 + \phi^*\beta_1$ where $\beta_0$ is a principal $U(1)$ connection on $P^1(\eta)$ and $\beta_1$ is a principal $Sp(V, \Omega)$ connection on $Sp(M, \omega)$ corresponding to a linear connection $\nabla$ on $M$ so that $\nabla \omega = 0$. Recall that $TM = P \times_{Mp^c,\sigma} V$ with $[p, v] = \phi(p)(v)$. 
The *symplectic spinor bundle* associated to $P$ is defined as

$$ \mathcal{S} = P \times_{M^p, U} \mathcal{H}^{\pm \infty} \quad (47) $$

where $U$ denotes the $M^p$-representation on the space $\mathcal{H}^{\infty}$ of smooth vectors in $\mathcal{H}$ or on its dual $\mathcal{H}^{-\infty}$; its sections are the *symplectic spinor fields*.

The *symplectic Clifford multiplication* is the map

$$ TM \otimes \mathcal{S} \rightarrow \mathcal{S} : [p(v), f] \mapsto [p, Cl(v) f]. \quad (48) $$

This Clifford multiplication is parallel i.e.

$$ \nabla_U^{(\alpha)} (Cl(V) \varphi) = Cl(\nabla_U V) \varphi + Cl(V) \nabla_U^{(\alpha)} \varphi $$

for any spinor field $\varphi$ and any smooth vector fields $U, V$ on $M$, $\nabla^{(\alpha)}$ denoting the covariant derivative associated to $\alpha$ in the spinor bundles.

The *symplectic Dirac operator* (associated to $P$ and $\alpha$) is the operator acting on symplectic spinor fields, defined by

$$ D^{(\alpha)} \varphi = \sum_a Cl(e_a) \nabla^{(\alpha)}_{e_a} \varphi = - \sum_{a,b} \omega^{ab} Cl(e_a) \nabla^{(\alpha)}_{e_b} \varphi \quad (49) $$

where $e_a$ is any local frame of the tangent bundle, $\omega^{ab}(x)$ denotes the coefficients of the inverse of the matrix $\omega_{ab}(x) := \omega_x(e_a(x), e_b(x))$, and $e^a$ is the dual frame defined by $\omega(e_a, e^b) = \delta^b_a$.

Given a compatible positive almost complex structure $J$ on $(M, \omega)$, one defines (following K. Habermann) a second first order operator, the *$J$-twisted symplectic Dirac operator*

$$ D_J^{(\alpha)} \varphi = \sum_a Cl(J e_a) \nabla^{(\alpha)}_{e_a^c} \varphi = \sum_{a,b} g^{ab} Cl(e_a) \nabla^{(\alpha)}_{e_b} \varphi \quad (50) $$

with $g^{ab}(x)$ the inverse of the matrix $g_{ab}(x) := g_{Jx}(e_a(x), e_b(x)) = \omega_x(e_a(x), J e_b(x))$, having chosen the $M^p$-connection $\alpha$ so that the induced linear connection preserves $J$ ($\nabla J = 0$).

This is the case if and only if the $M^p$-connection is induced by a $MU^c$-connection on the principal $MU^c(V, \Omega, j)$-bundle $P_J := \phi^{-1}(U(M, \omega, J))$ lying over the unitary frames. There always exists a linear connection preserving $\omega$ and $J$; it may have torsion, but one can always assume that the *torsion vector* ($= \sum_{a,b} \omega^{ab} T^{\nabla}(e_a, e_b)$) vanishes.

Given $J$, the symplectic spinor bundle can also be written

$$ \mathcal{S} = P_J \times_{MU^c, U} \mathcal{H}^{\pm \infty}. $$

Since the action of $MU^c$ on the subspace of $Pol(V, j) \subset \mathcal{H}^{\pm \infty}$ consisting of polynomials (in the Fock realization) preserves the degree, we can consider the dense subspace of polynomial-valued spinor fields which are sections of

$$ \mathcal{S}_J = P_J \times_{MU^c, U} Pol(V, j). $$
In order to have such a decomposition, we have assumed that the chosen compatible almost complex structure $J$ is positive. Now, in the presence of an almost complex structure $J$ it is convenient to write derivatives in terms of their $(1, 0)$ and $(0, 1)$ parts. That is we complexify $TM$ and then decompose $TM^C$ into the $\pm i$ eigenbundles of $J$ which are denoted by $T'M$ and $T''M$. If $X$ is a tangent vector then it decomposes into two pieces $X = X' + X''$ lying in these two subbundles so $JX' = iX'$ and $JX'' = -iX''$. We can then define

$$\nabla_X^{(a)} := \nabla_{X'}^{(a)}, \quad \nabla_X^{(a)_\prime} := \nabla_{X''}^{(a)}$$

after extending $\nabla^{(a)}$ by complex linearity to act on complex vector fields. We have defined in [3] two partial Dirac operators, the Dirac–Dolbeault symplectic operators $D^{(a,J)}$ and $D^{(a,J)_\prime}$ by

$$D^{(a,J)} \varphi = \sum_a \text{Cl}(e_a) \nabla^{(a)}_{e^a} \varphi, \quad D^{(a,J)_\prime} \varphi = \sum_a \text{Cl}(e_a) \nabla^{(a)}_{e^a} \varphi.$$ 

Then

$$D^{(a)} = D^{(a,J)} + D^{(a,J)_\prime}, \quad D^{(a)_\prime} = -iD^{(a,J)} + iD^{(a,J)_\prime}.$$

We have proven in [3] that $D^{(a,J)_\prime}$ is the adjoint of $D^{(a,J)}$ when the torsion vector of the linear connection induced by the $MU^c$ connection vanishes; we shall assume this is the case from now on. (Recall that a connection preserving $\omega$ and $J$ and with vanishing torsion vector always exists).

The choice of $J$ on $(M, \omega)$ allows us to split Clifford multiplication into creation and annihilation parts. Remark that the Clifford multiplication on $H^{\pm\infty}(V, \Omega, j)$ splits as

$$\text{cl}(v) = c(v) - a(v) \text{ where } (c(v)f)(z) := \frac{1}{2\hbar} (z, v)_j f(z) \text{ and } (a(v)f)(z) = (\partial_z f)(v). \quad (51)$$

Consider $TM = P_J \times_{MU^c(V, \Omega, j)} S$, $S = P_J \times_{MU^c(V, \Omega, j)} H^{\pm\infty}(V, \Omega, j)$ and define the operators:

$$C_J : TM \otimes S \to S : (X = [p', v]) \otimes (\psi = [p', f]) \mapsto C_J(X)\psi := [p', c(v)f];$$

$$A_J : TM \otimes S \to S : (X = [p', v]) \otimes (\psi = [p', f]) \mapsto A_J(X)\psi := [p', a(v)f];$$

this is well defined because $a(gv)Uf = Ua(v)f$ for any $(U, g)$ in $MU^c(V, \Omega, j)$. We have $A_J(JX) = iA_J(X)$, $C_J(JX) = -iC_J(X)$, $h(A_J(X)\psi, \psi') = h(\psi, C_J(X)\psi')$. Hence, having chosen a $MU^c$ connection $\alpha$ on $P_J$, we have

$$D^{(a,J)} \varphi = \sum_a C_J(e_a) \nabla^{(a)}_{e^a} \varphi = -\sum_{ab} \omega^{ab} C_J(e_a) \nabla^{(a)}_{e_b} \varphi = \sum_a C_J(e_a) \nabla^{(a)}_{e^a} \varphi, \quad (52)$$

$$D^{(a,J)_\prime} \varphi = -\sum_a A_J(e_a) \nabla^{(a)}_{e^a} \varphi = \sum_{ab} \omega^{ab} A_J(e_a) \nabla^{(a)}_{e^b} \varphi = -\sum_a A_J(e_a) \nabla^{(a)}_{e^a} \varphi. \quad (53)$$
The space of polynomial spinor fields of degree $\leq q$ is the space of sections of the bundle

$$S^q_j := P_J \times_{MU^c} Pol^q(V, j) = \oplus_{k=1}^q L \otimes S^k(T^*(1,0)M)$$

where $Pol^q(V, j)$ is the space of holomorphic polynomials of degree $\leq q$ on $(V, j)$, and where $L = P_J(\lambda)$ is the line bundle whose class characterises the isomorphism class of the $Mp^c$ structure. Observe that $D^{(\alpha, J)}$ raises the degree by 1 whilst $D''^{(\alpha, J)}$ lowers it by 1.

Given a differential operator $Op$ of order $k$ acting from the space of sections of a vector bundle $E \rightarrow M$ to the space of sections of $E' \rightarrow M$, its principal symbol $ps(Op)$ associates to any point $x \in M$ and any non-zero element $\xi \in T^*_x M$ the linear endomorphism of the fibres

$$ps(Op)(x, \xi) : E_x \rightarrow E'_x : ps(Op)(x, \xi)f := \frac{d^k}{dt^k}e^{-ith}Op(e^{ith}\psi)(x)|_{t=0}$$

where $h$ is any function on $M$ so that $dh_x = \xi$ and $\psi$ is any local section of $L$ with $\psi(x) = f$.

**Proposition 4.1** The principal symbols of the operators $D^{(\alpha)}, D^{(\alpha, J)}, D'^{(\alpha, J)}$ and $D''^{(\alpha, J)}$ are given by

$$ps(D^{(\alpha)})(x, \xi) = iCl(\xi^\sharp)$$
$$ps(D^{(\alpha, J)})(x, \xi) = iC_J(\xi^\sharp)$$

$$ps(D'^{(\alpha, J)})(x, \xi) = iC_J(\xi^\sharp)$$
$$ps(D''^{(\alpha, J)})(x, \xi) = -iA_J(\xi^\sharp)$$

with $\xi^\sharp \in T^*_x M$ defined so that $\omega_x(\xi^\sharp, \cdot) = \xi$.

In particular, the symbol of the operator $D^{(\alpha, J)}_q$, which is the restriction of $D^{(\alpha, J)}$ to $S^q_j$ with values in $S^{q+1}_j$, is injective for all $q$. The operator $D''^{(\alpha, J)}D''^{(\alpha, J)}$, which is the restriction of $D''^{(\alpha, J)}D^{(\alpha, J)}$ to $S^q_j$, is self-adjoint, elliptic, and preserves the degree.

**Theorem 4.2** Consider a compact symplectic manifold $(M, \omega)$, with

* an $Mp^c$-structure $(P, \phi)$;
* a compatible positive almost complex structure $J$;
* an $MU^c$ connection $\alpha$ on the $MU^c$-structure $P_J = \phi^{-1}U(M, \omega, J)$ chosen so that the induced linear connection has vanishing torsion vector.

Then, for each integer $q \geq 0$, the operator $D^{(\alpha)}_q$, which is the restriction of $D^{(\alpha)}$ to $S^q_j$ with values in $S^{q+1}_j$, has a finite dimensional kernel.

**Proof** Given any polynomial spinor field $\psi \in S^q_j$, we decompose it by degrees as

$$\psi = \psi_q + \psi_{q-1} + \ldots + \psi_1 + \psi_0.$$
Then $D^{(\alpha)}\psi = 0$ is equivalent to $D^{(\alpha,J)}\psi_q = 0$, $D^{(\alpha,J)}\psi_{q-1} = 0$, $D^{(\alpha,J)}\psi_{k-2} = -D^{(\alpha,J)}\psi_k$ for $2 \leq k \leq q$ and $D^{(\alpha,J)}\psi_1 = 0$. Now an element is in the kernel of $D^{(\alpha,J)}$ if and only if it is in the kernel of $D^{(\alpha,J)}D^{(\alpha,J)}$ and this kernel is finite dimensional since $D^{(\alpha,J)}D^{(\alpha,J)}$ is self-adjoint, elliptic, and acts on sections of a finite dimensional bundle over a compact manifold. So $\psi_q$ and $\psi_{q-1}$ belong to a finite dimensional subspace and inductively for decreasing $k$‘s, each $\psi_k$ belongs to the finite dimensional subspace whose image under $-D^{(\alpha,J)}$ is the finite dimensional subspace which is the image under $D^{(\alpha,J)}$ of possible $\psi_{k+2}$‘s.

The commutator of $D^{(\alpha)}$ and $D^{(\alpha,J)}$ yields a second order operator introduced by Habermann

$P^{(\alpha,J)} = i[D^{(\alpha)}, D^{(\alpha,J)}]; \quad (56)$

it is now given by $P^{(\alpha,J)} = 2[D^{(\alpha,J)}, D^{(\alpha,J)}]$; it is elliptic [6] and preserves the degree. Thus, on the dense subspace of polynomial spinor fields, the operator $P^{(\alpha,J)}$ is a direct sum of operators acting on sections of finite rank vector bundles.

More generally, we consider a field $A$ of endomorphisms of the tangent bundle $TM$ of a symplectic manifold $(M, \omega)$, such that

$\omega(AX, Y) = -\epsilon(A)\omega(X, AY) \quad \forall X, Y \in \Gamma(M, TM),$

with $\epsilon(A) = \pm 1$ and such that there is a linear connection $\nabla$ preserving $\omega$ and $A$, i.e.

$\nabla \omega = 0 \quad \text{and} \quad \nabla A = 0.$

We consider a $Mp^c$-connection $\alpha = \alpha_1 + \alpha_0$ so that $\alpha_1 = \phi^*\beta_1$ where $\beta_1$ is the connection 1-form on $Sp(M, \omega)$ defined by the linear connection $\nabla$. We define a new $A$-twisted Dirac operator which is a first order differential operator acting on symplectic spinor fields:

$D^{(\alpha)}_A \varphi := \sum_a Cl(Ae_a)\nabla^{(\alpha)}_a \varphi = -\sum_{a,b,d} A^d \omega^{ab} Cl(e_d)\nabla^{(\alpha)}_{e_b} \varphi \quad (57)$

where $e_a$ is a local frame field for $TM$ and $e^a$ is defined by $\omega(e^a, e^b) = \delta^b_a$.

The $Mp^c$ symplectic Dirac operator $D^{(\alpha)}$ corresponds to $A = \text{Id}$, $\epsilon(A) = 1$. The operator $D^{(\alpha)}_J$ corresponds to $A = J$, $\epsilon(A) = -1$ for $J$ a positive compatible almost complex structure on $(M, \omega)$.

### 4.2 Symplectic Dirac operators on symmetric spaces

Consider $(M = G/K, \omega, A)$, a pseudo-Hermitean or bi-Lagrangian symmetric space, or more generally a symplectic symmetric space with a $G$-invariant parallel field $A$ of invertible endomorphisms so that $A_x \in \mathfrak{sp}(T_x M, \omega_x)$ ($A^2 = -\text{Id}$ in the pseudo-Hermitean case and
$A^2 = \text{Id}$ in the bi-Lagrangian case. Observe that a pseudo-Hermitean symmetric space is pseudo-Kählerian since the torsion vanishes and $\nabla J = 0$.

We write as before $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and consider the non-degenerate skewsymmetric $\text{Ad} K$ invariant 2-form $\Omega$ on $\mathfrak{p}$ and the $\text{Ad} K$ invariant endomorphism $\tilde{\mathcal{A}}$ of $\mathfrak{p}$ (which is a complex structure $\tilde{\jmath}$ on $\mathfrak{p}$ if $A^2 = -\text{Id}$ or gives a splitting into $\pm 1$-eigenvalues $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ if $A^2 = \text{Id}$).

**Definition 4.3** We define the symmetric non-degenerate 2-form $B_{\tilde{\mathcal{A}}}$ on $\mathfrak{p}$

$$B_{\tilde{\mathcal{A}}}(X, Y) = -\Omega(X, \tilde{\mathcal{A}}^{-1}Y);$$

it is $\text{Ad} K$ invariant; we extend $B_{\tilde{\mathcal{A}}}$ to a $\text{an Ad} K$ and $\text{ad} \mathfrak{g}$ invariant symmetric bilinear form $\tilde{\mathcal{B}}_{\tilde{\mathcal{A}}}$ on $\mathfrak{g}$, using the fact that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ through

$$\tilde{\mathcal{B}}_{\tilde{\mathcal{A}}}(X, [Y, Z]) := 0 \quad \text{and} \quad \tilde{\mathcal{B}}_{\tilde{\mathcal{A}}}([X, T], [Y, Z]) := B_{\tilde{\mathcal{A}}}(X, [T, [Y, Z]])$$

for any $X, Y, Z, T \in \mathfrak{p}$. This is well defined since $B_{\tilde{\mathcal{A}}}(X, [T, [Y, Z]]) = B_{\tilde{\mathcal{A}}}(Y, [Z, [X, T]])$ by the usual symmetry properties of a pseudo-Riemannian curvature tensor.

**We assume that there is a liftable $G$-invariant $K$-structure** on $(M, \omega)$ defined by the principal bundle $G \to M$, the homomorphism

$$\tau : K \to GL(\mathfrak{p}, \Omega, \tilde{\mathcal{A}}) \subset Sp(\mathfrak{p}, \Omega) : k \mapsto \tau(k) := \text{Ad}(k)_{|\mathfrak{p}}$$

with $GL(\mathfrak{p}, \Omega, \tilde{\mathcal{A}}) = \{ g \in Sp(V, \Omega) \mid g\tilde{\mathcal{A}} = \tilde{\mathcal{A}}g \}$ and a lift

$$\tilde{\tau} : K \to Mp^c(\mathfrak{p}, \Omega, j) : k \mapsto \tilde{\tau}(k).$$

This is always true in the pseudo-Hermitean case: when $A = \tilde{\jmath}$, $GL(\mathfrak{p}, \Omega, \tilde{\mathcal{A}}) = U(\mathfrak{p}, \Omega, \tilde{\jmath})$ and we consider the lift $\tilde{\tau} = F_{\tilde{\jmath}, \jmath} \circ \tau$ with $F_{\tilde{\jmath}, \jmath} : U(V, \Omega, \tilde{\jmath}) \to Mp^c(V, \Omega, j)$ as in (24).

It is also always true in the bi-Lagrangian case: when $A^2 = \text{Id}$, $GL(\mathfrak{p}, \Omega, \tilde{\mathcal{A}}) = \tau_{\mathfrak{p}+}GL(\mathfrak{p}_+)$ and we have the lift $\tilde{\tau} = L \circ \text{Ad}|_{\mathfrak{p}_+}$ with $L : GL(\mathfrak{p}_+) \to Mp^c(\mathfrak{p}, \Omega, j)$ defined as in (10).

Choose any $G$-invariant $Mp^c$-structure on $(M = G/K, \omega)$, then

$$P_{\chi} := G \times_{K, \chi \times \tilde{\tau}} Mp^c(V, \Omega, j)$$

corresponding to the choice of a character $\chi : K \to U(1)$. Observe that the canonical line bundle $P_{\chi}(\eta)$ is given by

$$P_{\chi}(\eta) = G \times_{K, \chi'} C \quad \text{with} \quad \chi'(k) = \eta(\chi(k)\tilde{\tau}(k)) = \chi(k)^2 \eta \circ \tilde{\tau}(k). \quad (58)$$

The spinor bundle is given by

$$S = G \times_{K, U(\chi \times \tilde{\tau})} \mathcal{H}^{\pm \infty} \quad (59)$$
and any spinor field $\varphi$ is viewed as a $K$-equivariant function $\tilde{\varphi}$

$$\tilde{\varphi} : G \rightarrow \mathcal{H}^\infty \quad \text{with} \quad \varphi(\pi(g)) = [g, \tilde{\varphi}(g)]$$

so that $\tilde{\varphi}(gk) = U_k \tilde{\varphi} (g)$.

In this context of symmetric spaces, we **choose the symmetric connection** $\alpha$ defined on the $K$-principal bundle $G \rightarrow G/K$:

$$\alpha_g(L_g X) = X_k \quad \forall X \in \mathfrak{g}$$

where $U_k$ denotes the projection of $U$ on $\mathfrak{k}$ relatively to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The vector $L_g X$ for $X \in \mathfrak{p}$ is the horizontal lift of the vector $\rho^M(g, \mathfrak{k})$ with $\mathfrak{p}$ identified with $T_{eK}G/K$.

Hence

$$\left( \tilde{\nabla}_{[g,X]}^{\alpha} \right) (g) = \frac{d}{dt} \bigg|_0 \tilde{\varphi}(g \exp(tX)) = \tilde{X}_g \tilde{\varphi} \quad \forall X \in \mathfrak{p}$$

where $\tilde{X}$ is the left invariant vector field on $G$ corresponding to $X \in \mathfrak{p} \subset \mathfrak{g}$. Since this choice of connection is natural in the symmetric context, we shall from now on often drop the $(\alpha)$ in all notations concerning Dirac operators and covariant derivatives.

The choice of a basis $X_a$ of $\mathfrak{p}$ and the choice of an element $g \in G$ induce a frame at the point $x = gK$ given by

$$e_a(x) := [g, X_a] = \rho^M(g, \mathfrak{k}) X_a$$

so that the Dirac operator has the global form

$$D = -\sum_{a,b} \Omega^{ab} \partial c l(X_a) \otimes \tilde{X}_b$$

where $\Omega^{ab}$ denotes the coefficients of the inverse of the matrix $\Omega_{ab} := \Omega(X_a, X_b)$. The twisted Dirac operator is given by

$$D_A = -\sum_{a,b} \Omega^{ab} \partial c l(\tilde{A}X_a) \otimes \tilde{X}_b = -\sum_{acd} \Omega^{ad} \tilde{A}_a \partial c l(X_c) \otimes \tilde{X}_d = \sum_{cd} B^{cd} \partial c l(X_c) \otimes \tilde{X}_d$$

where $B^{cd}$ denotes the components of the inverse of the symmetric matrix defined by $B_{ab} = B_{\tilde{A}}(X_a, X_b) = \Omega(X_a, \tilde{A}^{-1}X_b)$ as in Definition 4.3.

In the Kählerian symmetric case the Dirac–Dolbeault symplectic operators can be written

$$D^{(J)} = -\sum_{ab} \Omega^{ab} c(X_a) \otimes \tilde{X}_b = -\frac{i}{2} \sum_{ab} \Omega^{ab} c(X_a) \otimes (\tilde{X}_b - ij\tilde{X}_b)$$

$$D''^{(J)} = \sum_{ab} \Omega_{ab} a(X_a) \otimes \tilde{X}_b = \frac{i}{2} \sum_{ab} \Omega^{ab} a(X_a) \otimes (\tilde{X}_b + ij\tilde{X}_b)$$

with creation and annihilation operators $c$ and $a$ defined by (51).
4.2.1 Parthasarathy formula for the commutator \([D, D_A]\)

The commutator of \(D\) and \(D_A\) is given by

\[
(DD_A - D_A D)\varphi = - \sum_{abcd=1}^{2n} \Omega^{ab} B^{cd} \left( cl(X_a X_c) X_b X_d \varphi - cl(X_c X_a) X_b X_d \varphi \right)
\]

\[
= - \sum_{abcd=1}^{2n} \Omega^{ab} B^{cd} \left( (cl(X_a X_c) - cl(X_c X_a)) X_b X_d \varphi \right)
\]

\[
+ cl(X_c X_a) \left( X_b X_d - X_d X_b \right) \varphi
\]

\[
= \frac{i}{\hbar} \sum_{bd=1}^{2n} B^{bd} \tilde{B}_b X_d \varphi - \sum_{abcd=1}^{2n} \Omega^{ab} B^{cd} cl(X_c X_a) [X_b, X_d] \varphi.
\]

Introducing a basis \(\{W_1, \ldots, W_k\}\) of \(\mathfrak{g}\) we can write

\[
[X_b, X_d] = \sum_{rs=1}^{k} \tilde{B}_{\tilde{A}}([X_b, X_d], W_r) \tilde{B}^{rs} W_s = - \sum_{rs=1}^{k} \tilde{B}_{\tilde{A}}(X_c, [X_b, W_r]) \tilde{B}^{rs} W_s
\]

\[
= \sum_{rs=1}^{k} B_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) \tilde{B}^{rs} W_s.
\]

where \(\tilde{B}^{rs}\) are the coefficients of the matrix which is the inverse of the matrix \(\tilde{B}_{pq} := \tilde{B}_{\tilde{A}}(W_p, W_q)\). On the other hand we have \(\mathrm{ad} W_r(X_b) = \sum_{d=1}^{2n} B_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) B^{dc} X_c\) so that

\[
\mathrm{ad} W_r|_p = \sum_{abcd=1}^{2n} \tilde{B}_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) B^{dc} \Omega^{ka} X_a \otimes X_c.
\]

Remark that \(\sum_{d=1}^{2n} B_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) B^{dc} \Omega^{ka} = \sum_{d=1}^{2n} (\mathrm{ad} W_r)_b \Omega^{ka}\) is symmetric in \(ac\) since \(\mathrm{ad} W_r\) is in \(\mathfrak{sp}(p, \Omega)\). Hence, using formula (17), we have

\[
(cl \circ \nu)(\mathrm{ad} W_r|_p) = \frac{i}{2 \hbar} \sum_{abcd=1}^{2n} B_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) B^{dc} \Omega^{ka} cl(X_c X_a).
\]

The last term in the formula for the commutator can be rewritten as

\[
\sum_{abcd=1}^{2n} \Omega^{ab} B^{cd} cl(X_c X_a) [X_b, X_d] \varphi = \sum_{abcd=1}^{2n} \Omega^{ab} B^{cd} \sum_{rs=1}^{k} B_{\tilde{A}}(\mathrm{ad} W_r(X_b), X_d) \tilde{B}^{rs} cl(X_c X_a) \tilde{W}_s \varphi
\]

\[
= - \frac{2i}{\hbar} \sum_{rs=1}^{k} \tilde{B}^{rs} (cl \circ \nu)(\mathrm{ad} W_r|_p) \tilde{W}_s \varphi
\]

so that

\[
(DD_A - D_A D)\varphi = \frac{i}{\hbar} \sum_{bd=1}^{2n} B^{bd} \tilde{B}_b X_d \varphi + 2 \frac{i}{\hbar} \sum_{rs=1}^{k} \tilde{B}^{rs} (cl \circ \nu)(\mathrm{ad} W_r|_p) \tilde{W}_s \varphi. \tag{66}
\]
We can introduce the Casimir element $\Omega^\tilde{B}_g$ in the center of the universal enveloping algebra of $\mathfrak{g}$ defined by the invariant symmetric 2-form $\tilde{B}_A$; since $\{W_1, \ldots, W_k, X_1, \ldots X_{2n}\}$ is a basis of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and since $\mathfrak{k}$ and $\mathfrak{p}$ are $\tilde{B}_A$-orthogonal, we have

$$\Omega^\tilde{B}_g = \sum_{bd=1}^{2n} B_{bd} X_b \cdot X_d + \sum_{rs=1}^k \tilde{B}^{rs} W_r \cdot W_s. \quad (67)$$

Similarly the Casimir element $\Omega^\tilde{B}_t$ in the center of the universal enveloping algebra of $\mathfrak{k}$ is defined by the invariant symmetric 2-form $\tilde{B}_A$ restricted to $\mathfrak{k}$; so that

$$\Omega^\tilde{B}_t = \sum_{rs=1}^k \tilde{B}^{rs} W_r \cdot W_s. \quad (68)$$

Those act as $G$-invariant differential operators $\tilde{\Omega}^\tilde{B}_g$ and $\tilde{\Omega}^\tilde{B}_t$ on the space of spinors (viewed as functions on $G$). Spinors are equivariant functions so that for any $W \in \mathfrak{k}$ we have

$$\left( \tilde{W} \tilde{\varphi} \right)(g) = -U_s(\chi_s(W) + \tilde{r}_s(W)) \left( \tilde{\varphi}(g) \right).$$

Since $U_s(X) = \text{cl}(\nu(\sigma_s(X))) + \frac{1}{2} \eta_s(X) \text{Id}$ by equation (20) and $\sigma_s \tilde{r}_s(W) = \text{ad} W|_{\mathfrak{p}}$ we have

$$\left( \tilde{W} \tilde{\varphi} \right)(g) = -\left( \chi_s(W) + \frac{1}{2} \eta_s \tilde{r}_s(W) + \left( \text{cl} \circ \nu \right)(\text{ad} W|_{\mathfrak{p}}) \right) \left( \tilde{\varphi}(g) \right) = -\left( \chi'_s(W) + \left( \text{cl} \circ \nu \right)(\text{ad} W|_{\mathfrak{p}}) \right) \left( \tilde{\varphi}(g) \right),$$

where $\chi'$ is the character defining the canonical bundle $P_\chi(\eta)$ associated to the $Mp^\mathfrak{f}$-structure $P_\chi$. Hence we have

$$(DD_A - D_A D) \tilde{\varphi}$$

$$= \frac{i}{\hbar} (\tilde{\Omega}^\tilde{B}_t - \tilde{\Omega}^\tilde{B}_g) \tilde{\varphi} - 2\frac{i}{\hbar} \sum_{rs=1}^k \tilde{B}^{rs} (\text{cl} \circ \nu)(\text{ad} W_r|_{\mathfrak{p}}) \left( \frac{1}{2} \chi'_s(W_s) + \left( \text{cl} \circ \nu \right)(\text{ad} W_s|_{\mathfrak{p}}) \right) \tilde{\varphi}$$

$$= \frac{i}{\hbar} \tilde{\Omega}^\tilde{B}_g \tilde{\varphi} - \frac{i}{\hbar} \sum_{rs=1}^k \tilde{B}^{rs} \left( \frac{1}{2} \chi'_s(W_s) + \left( \text{cl} \circ \nu \right)(\text{ad} W_s|_{\mathfrak{p}}) \right) \left( \frac{1}{2} \chi'_s(W_s) + \left( \text{cl} \circ \nu \right)(\text{ad} W_s|_{\mathfrak{p}}) \right) \tilde{\varphi}$$

$$- 2\frac{i}{\hbar} \sum_{rs=1}^k \tilde{B}^{rs}(\text{cl} \circ \nu)(\text{ad} W_r|_{\mathfrak{p}}) \left( \frac{1}{2} \chi'_s(W_s) + \left( \text{cl} \circ \nu \right)(\text{ad} W_s|_{\mathfrak{p}}) \right) \tilde{\varphi}.$$

**Proposition 4.4** Consider a symplectic symmetric space $(M = G/K, \omega)$ endowed with a $G$-invariant field $A$ of invertible endomorphisms so that $A_x \in \mathfrak{sp}(T_x M, \omega_x)$ and assume that there is a liftable $G$-invariant $K$-structure on $(M, \omega)$. Consider the $G$-invariant $Mp^\mathfrak{f}$-structure associated to the character $\chi$ of $K$ and the symmetric connection. With $\tilde{B}_A$ defined
as in 4.3 with $\chi'$ the character corresponding to the canonical line bundle as in (58), and with the notations above, the commutator of the Dirac operator $D$ and the $A$ twisted Dirac operator $D_A$ has the form:

$$
(DD_A - D_A D)\tilde{\phi} = \left( \frac{i}{\hbar}\Omega^B_0 - \frac{3i}{\hbar}(cl \circ \nu \circ \text{ad}|_p)(\Omega^A_0) - \frac{i}{4\hbar} \sum_{rs=1}^k \tilde{B}^{rs} \chi'_s(W_r)\chi'_s(W_s)
\right)
\tilde{\phi}
$$

$$
- \frac{2i}{\hbar} \sum_{rs=1}^k \tilde{B}^{rs} \chi'_r(W_r) (cl \circ \nu \circ \text{ad}|_p)(W_s) \tilde{\phi}
$$

This clearly simplifies drastically when the $Mp^c$-structure comes from a metaplectic structure (i.e. $\chi' = 0$).

4.3 The example of $\mathbb{C}P^n$

In this section, we shall illustrate the construction of the homogeneous $Mp^c$-structures, the invariant symplectic Dirac operators $D, D_J, D'^{(J)}, D''^{(J)}$ and the elliptic operator $\mathcal{P}^{(J)}$ on the complex projective spaces. We shall use the standard invariant complex structure on $\mathbb{C}P^n$ and thus we shall drop the superscript $(J)$. We shall compute the eigenvalues of $\mathcal{P}$ and the kernel of $D'$. The case of $\mathbb{C}P^1$ was treated by Brasch, Habermann and Habermann in [1], using metaplectic structures. It was extended in higher odd dimensions using metaplectic structure by Christian Wyss in his Diplomarbeit at the Universität Bremen in 2003. The operators $D'$ and $D''$ were also explored on $\mathbb{C}P^1$ by Korman [8], where they are called symplectic Dolbeault operators.

4.3.1 Homogeneous $Mp^c$-structures and spinor fields on $\mathbb{C}P^n$

The complex projective space $\mathbb{C}P^n$ has a natural structure of symmetric space. Viewing $\mathbb{C}P^n$ as the set of complex lines in $\mathbb{C}^{n+1}$, we have $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where the equivalence relation is defined by $z = (z^0, \ldots, z^n) \sim z' = (z'^0, \ldots, z'^n)$ if and only if $z' = \lambda z$ for a non-vanishing $\lambda \in \mathbb{C}$. We denote by $[z]$ the equivalence class of $z \in \mathbb{C}^{n+1} \setminus \{0\}$. The group $SU(n+1)$ acts on $\mathbb{C}P^n$ through $A \cdot [z] := [Az]$. This action is transitive. The stabiliser $K$ of the point $[1,0,\ldots,0]$ is isomorphic to $U(n)$:

$$K := \left\{ \tilde{g}(A) := \begin{pmatrix} \det A^{-1} & 0 \\ 0 & A \end{pmatrix} \bigg| A \in U(n) \right\} \subset SU(n+1).$$
Observe that $K$ is the subgroup of $SU(n+1)$ of the elements which are stable under the involutive automorphism $\sigma$ of $SU(n+1)$ defined by

$$\sigma(g) = S g S^{-1} \quad \text{with} \quad S = \begin{pmatrix} -1 & 0 \\ 0 & \text{Id}_n \end{pmatrix}.$$ 

This gives

$$\mathbb{C}P^n = SU(n+1)/K$$

the structure of a symmetric space. The Lie algebra $\mathfrak{su}(n+1)$ decomposes into $\pm 1$ eigenspaces for the differential $\tilde{\sigma} = \sigma^*$ as

$$\mathfrak{su}(n+1) = \mathfrak{k} \oplus \mathfrak{p} \quad \text{where} \quad \mathfrak{k} := \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \bigg| a \in i\mathbb{R}, A \in \mathfrak{u}(n), a + \text{Tr}(A) = 0 \right\}$$

is the Lie algebra of $K$ and

$$\mathfrak{p} := \left\{ q(\alpha) := \begin{pmatrix} 0 & -\bar{\alpha}^T \\ \alpha & 0 \end{pmatrix} \bigg| \alpha \in \mathbb{C}^n \right\} \cong \mathbb{C}^n.$$ 

Let $\tilde{B}$ be the invariant positive definite symmetric two form on $\mathfrak{su}(n+1)$ defined by $\tilde{B}(x, y) := -\frac{1}{2} \text{Tr}(xy)$; its restriction $B$ to $\mathfrak{p}$ defines an $\text{Ad}(K)$-invariant metric

$$B(q(\alpha), q(\beta)) = \text{Re}(\alpha^T \bar{\beta})$$

which coincides with the real part of the standard Hermitean form on $\mathbb{C}^n$. The $\text{Ad}(K)$-invariant complex structure $j$ on $\mathfrak{p}$ is given by

$$j := \text{Ad} w \quad \text{with} \quad w = \begin{pmatrix} e^{-\frac{\pi i}{n+1}} & 0 \\ 0 & e^{\frac{\pi i}{n+1}} \text{Id}_n \end{pmatrix} \quad \text{so that} \quad j(q(\alpha)) = q(i\alpha)$$

coincides with the multiplication by $i$ on $\mathbb{C}^n$. The $\text{Ad}(K)$-invariant symplectic form on $\mathfrak{p}$ is given by $\Omega(\cdot, \cdot) = B(j \cdot, \cdot)$ so that

$$\Omega(q(\alpha), q(\beta)) = -\text{Im}(\alpha^T \bar{\beta})$$

coincides with minus the imaginary part of the standard Hermitean form on $\mathbb{C}^n$.

The tangent bundle to $\mathbb{C}P^n$ is canonically identified with

$$T\mathbb{C}P^n \cong SU(n+1) \times_{K, \text{Ad}|_p} \mathfrak{p}$$

and $B, j$ induce an $SU(n+1)$-invariant Kähler structure $(g, J)$ on $\mathbb{C}P^n$ as before:

$$g_{g'K}(\{g', X\}, \{g', Y\}) = B(X, Y), \quad J_{g'K}(\{g', X\}) := \{g', j(X)\},$$
for $g' \in SU(n + 1), X, Y \in \mathfrak{p}$. The corresponding symplectic form is the Kähler form

$$\omega_{g',K}([g', X], [g', Y]) := \Omega(X, Y).$$

The unitary frame bundle takes the form

$$U(M, \omega, J) = SU(n + 1) \times_{K, \tau} U(\mathfrak{p}, \Omega, j) \quad \text{with} \quad \tau(B)(U) := \text{Ad}(B)|_{\mathfrak{p}} \circ U. \quad (71)$$

For $\mathbb{CP}^n$ we have $\text{Ad}(\tilde{q}(A))|_{\mathfrak{p}} = \det(A)A \in U(\mathfrak{p}, \Omega, j)$ for $A \in U(n)$.

$Mp^c$-structures depend on the choice of a character of $K \simeq U(n)$. Such a character is of the form $\chi_k(A) := \det^k(A)$ for $A \in U(n)$ and the corresponding homogeneous $Mp^c$-structure is

$$P_{\chi_k} := SU(n + 1) \times_{U(n), \det^k} SU(\mathfrak{p}, \Omega, j), \quad (72)$$

for the map $\det^k : U(n) \to MU^c(\mathfrak{p}, \Omega, j) : B \mapsto (\det^k(B), \det(B))$.

The spinor bundle $S_{\chi_k}$ associated to the $Mp^c$-structure $P_{\chi_k}$ is given by

$$S_{\chi_k} = SU(n + 1) \times_{U(n), \text{Ad}(\tilde{\chi_k})} H^\infty. \quad (73)$$

A spinor field is a section of this bundle; equivalently it is a $U(n)$-equivariant map

$$\varphi : SU(n + 1) \to H^\infty \quad \text{such that} \quad (\varphi(A \tilde{q}(B)))(z) = (U(\chi_k(B^{-1}), \tau(B^{-1})), \varphi(A))(z) = \det^k(B^{-1})\varphi(A)(\det(B)Bz),$$

for all $A \in SU(n + 1)$ and $B \in U(n)$ and $z \in \mathbb{C}^n$.

The space $H^\infty$ contains the dense subspace $E := \bigoplus_{l \in \mathbb{N}} S^l(\mathbb{C}^n)$ where $S^l(\mathbb{C}^n)$ is the space of homogeneous polynomials of degree $l$ in $n$ complex variables and each subspace $S^l(\mathbb{C}^n)$ is stable under the action of the group $MU^c(\mathfrak{p}, \Omega, j)$. Hence, the space of sections of the spinor bundle decomposes as the sum of spaces of $K$-equivariant functions $\varphi : SU(n + 1) \to S^l(\mathbb{C}^n)$. We will denote this space by $E_{n}^l$. Remark that the equivariance of the map $\varphi$ takes the form

$$(\varphi(A \tilde{q}(B)))(z) = \det^{l-k}(B)\varphi(A)(Bz).$$

The commutator $P^{(j)} = 2[D^{(j)}, D''^{(j)}] = i[D_J, D]$ preserves this decomposition on spinors.

We now decompose the space of spinor fields in $E_{n}^l$ under the action of $SU(n + 1)$, using the Peter–Weyl Theorem and we get

**Lemma 4.5** Let $I$ be the set of highest weights parametrizing the irreducible representations of $SU(n + 1)$. The space $E_{n}^l$ decomposes as the sum

$$\sum_{\lambda \in I} V_{\lambda} \otimes \text{Hom}_{K,k}(V_{\lambda}, S^l(\mathbb{C}^n)), \quad (75)$$
where \((V_\lambda, \pi_\lambda)\) is an irreducible representation of \(SU(n+1)\) with highest weight \(\lambda\) and the space \(\text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\) denotes the space of \(K\)-intertwining homomorphisms from \((V_\lambda, \pi_\lambda|_K)\) to \((S^l(\mathbb{C}^n), \rho')\) with

\[
(\rho'(q^v(B)))f(z) := \det^{k-l}(B)f(B^{-1}z) \quad (76)
\]

A tensor \(v \otimes L \in V_\lambda \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\) corresponds to a function \(\varphi : SU(n+1) \rightarrow S^l(\mathbb{C}^n) : g \mapsto L(\pi_\lambda(g^{-1})v)\).

**Theorem 4.6** For each \(\lambda \in I\) the Dirac operators \(D, D_j, D'\) and \(D''\) preserve the spaces

\[\oplus_l V_\lambda \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\];

they induce operators on \(\oplus_l \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\) : \(D^{(\lambda, k)}, D_j^{(\lambda, k)}, D'^{(\lambda, k)}\) and \(D''^{(\lambda, k)}\). For \(L \in \oplus_l \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\) we have

\[
D^{(\lambda, k)}L := \sum_{ij} \Omega^{ij} cl(X_i) \circ L \circ \pi_\lambda (X_j), \quad D_j^{(\lambda, k)}L := -\sum_{ij} B^{ij} cl(X_i) \circ L \circ \pi_\lambda (X_j). \quad (77)
\]

\[
D'\lambda, k) L := \sum_{rs} \Omega^{rs} c(X_r) \circ L \circ \pi_\lambda (X_s) = \frac{1}{2} \sum_{rs} \Omega^{rs} c(X_r) \circ L \circ \pi_\lambda (X_s - ijX_s) \quad (78)
\]

\[
D''\lambda, k) L := -\sum_{rs} \Omega^{rs} a(X_r) \circ L \circ \pi_\lambda (X_s) = -\frac{1}{2} \sum_{rs} \Omega^{rs} c(X_r) \circ L \circ \pi_\lambda (X_s + ijX_s). \quad (79)
\]

where \(X_1, \ldots, X_{2n}\) is a basis of \(\mathfrak{p}\).

**Proof** For \(v \in V_\lambda, L \in \oplus_l \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\) and \(X \in \mathfrak{p}\) we have

\[
\widetilde{X}_j(v \otimes L) = -v \otimes (L \circ \pi_\lambda (X_j)),
\]

hence, using the expression of the symplectic Dirac operator on a symmetric space [62].

\[
D(v \otimes L)(g) = -\sum_{i,j} \Omega^{ij} cl(X_i) \widetilde{X}_j(v \otimes L) = \sum_{i,j} \Omega^{ij} cl(X_i) (v \otimes (L \circ \pi_\lambda (X_j))) (g).
\]

The other operators work similarly using [63] [64] [65].

### 4.3.2 The spinor fields in \(V_\lambda \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\)

We take the maximal torus \(T = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{i\theta_{n+1}} \end{pmatrix} \bigg| e^{i(\theta_1 \ldots + \theta_{n+1})} = 1 \right\} \) in \(SU(n+1)\);

it is also a maximal torus in \(K\). A weight of the Lie algebra \(\mathfrak{t}\) of \(T\) is an imaginary valued linear form on \(\mathfrak{t}\); it will be written as \(b_1 \epsilon_1 + \ldots + b_n \epsilon_n\) with

\[
(b_1 \epsilon_1 + \ldots + b_n \epsilon_n) \begin{pmatrix} i\theta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & i\theta_{n+1} \end{pmatrix} = i(b_1 \theta_1 + \ldots + b_n \theta_n).
\]
It is a weight of the group $T$ if it lifts to a group homomorphism from $T$ to $U(1)$, i.e. if the $b_i$ are integers; then (with a slight abuse of notation), the homomorphism is given by

$$(b_1\varepsilon_1 + \ldots + b_n\varepsilon_n) \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{i\theta_{n+1}} \end{pmatrix} = e^{i(b_1\theta_1 + \ldots + b_n\theta_n)}.$$

The set of roots of $\mathfrak{su}(n+1)$ is $\{\pm(\varepsilon_i - \varepsilon_j); 1 \leq i < j \leq n+1\}$ with $\varepsilon_{n+1} := -(\varepsilon_1 + \ldots + \varepsilon_n)$. The set of roots of $\mathfrak{t}$ is $\{\pm(\varepsilon_i - \varepsilon_j); 2 \leq i < j \leq n+1\}$. We choose as positive roots the $\varepsilon_i - \varepsilon_j$ for $i < j$. Any irreducible representation of a compact Lie group is characterised by a highest weight. The highest weight for an irreducible representation of $SU(n+1)$ is an $n$-tuple of non-increasing non-negative integers:

$$\lambda = m_1\varepsilon_1 + \ldots m_n\varepsilon_n \text{ with } m_1 \geq m_2 \geq \ldots \geq m_{n-1} \geq m_n \geq 0.$$

The highest weight for an irreducible representation of $K \cong U(n)$ is a $n$-tuple of integers, the last $n - 1$ being non-increasing and non-negative:

$$\mu = k_1\varepsilon_1 + \ldots k_n\varepsilon_n \text{ with } k_2 \geq k_3 \geq \ldots \geq k_{n-1} \geq k_n \geq 0.$$

**Theorem 4.7 (A. Ikeda and Y. Tanigushi [7])** An irreducible representation $(V, \rho)$ of $SU(n+1)$ of highest weight $\lambda = m_1\varepsilon_1 + \ldots m_n\varepsilon_n$ decomposes, as a $K$-module, into irreducible $K$-modules as

$$V = \bigoplus V_{k_1\varepsilon_1 + \ldots k_n\varepsilon_n}^K,$$

where $V_{k_1\varepsilon_1 + \ldots k_n\varepsilon_n}^K$ is the irreducible $K$-module of highest weight $k_1\varepsilon_1 + \ldots k_n\varepsilon_n$, and where the summation runs over all integers $k_1, \ldots, k_n$ for which there exists an integer $\hat{k}$ satisfying

$$m_1 \geq k_2 + \hat{k} \geq m_2 \geq k_3 + \hat{k} \geq m_3 \geq \ldots \geq m_{n-1} \geq k_n + \hat{k} \geq m_n \geq \hat{k} \geq 0,$$

and

$$k_1 = \sum_{i=1}^n m_i - \sum_{j=2}^n k_j - (n+1)\hat{k}.$$

The representation $\rho'$ of $K$ on $S^l(\mathbb{C}^n)$ defined by (76) has a unique highest weight vector the polynomial $f(z = (z_1, \ldots, z_n)) = z_1^l$ (up to multiples). Hence the representation is irreducible of highest weight

$$(2l-k)\varepsilon_1 + l\varepsilon_2 + \ldots + l\varepsilon_n.$$

**Theorem 4.8** If $(V_\lambda, \pi_\lambda)$ is an irreducible representation of $SU(n+1)$ (with $n \geq 2$) of highest weight $\lambda = m_1\varepsilon_1 + \ldots m_n\varepsilon_n$ ($m_1 \geq m_2 \geq \ldots \geq m_{n-1} \geq m_n \geq 0$), the space
\[ \text{Hom}_{K,(k)}(V_\lambda, S^l(C^n)) \text{ vanishes unless} \]
\[ m_1 + m_n + k = 3r \quad \text{for an integer } r \text{ satisfying } m_n \leq r \leq m_1 \]
\[ r - m_n \leq l \leq r \]
\[ m_2 = m_3 = \ldots = m_{n-1} = r \quad \text{when } n > 2. \]

and under those conditions, \( \text{Hom}_{K,(k)}(V_\lambda, S^l(C^n)) \) is one-dimensional.

In particular, in the case of \( \mathbb{CP}^2 \), the space of spinor fields for the \( Mp^c \)-structure corresponding to \( k = 0 \) is given by
\[ \bigoplus_{a \geq 0, r \geq a} V_{(2r+a)e_1 + (r-a)e_2} \otimes \bigoplus_{a \leq l \leq r} \text{Hom}_{U(2)}(V_{(2r+a)e_1 + (r-a)e_2}, S^l(C^2)) \]
and each of those \( \text{Hom}_{U(2)}(V_{(2r+a)e_1 + (r-a)e_2}, S^l(C^2)) \) is one-dimensional.

### 4.3.3 The spectrum of \( \mathcal{P} \) on \( \mathbb{CP}^n \)

To compute the spectrum of the commutator \( \mathcal{P} \), we use Parthasarathy formula as given in subsection 4.2.1:

\[ \mathcal{P} \tilde{\varphi} = -i(DD_J - D_JD)\tilde{\varphi} \]
\[ = \frac{1}{\hbar} \left( \tilde{\Omega}^\mathcal{B}_{\text{su}(n+1)} + (cl \circ \nu \circ \text{ad}|_p)(\Omega^\mathcal{B}) - 4(cl \circ \nu \circ \text{ad}|_p + \frac{1}{4} \check{\chi}_\lambda^\prime)(\Omega^\mathcal{B}) \right) \tilde{\varphi} \quad (80) \]

where \( \tilde{\Omega}^\mathcal{B}(X,Y) = -\frac{i}{2} \text{Tr } XY \).

**Lemma 4.9** Let \( \mathfrak{t} \) be the Lie algebra of a maximal torus \( T \) in the Lie algebra \( \mathfrak{g} \) of a compact Lie group \( G \). Let \( \Phi \) be the set of roots and \( \Phi^+ \) be the chosen set of positive roots. Let \( \mathcal{B} \) be an invariant symmetric non-degenerate real bilinear form on \( \mathfrak{g} \); it induces an isomorphism between \( \mathfrak{t}^* \) and \( \mathfrak{t} \), and a scalar product on \( \mathfrak{t}^* \) which we denote by \( <, >^\mathcal{B} \). The Casimir operator \( \tilde{\Omega}^\mathcal{B}_\mathfrak{g} = \sum_{r,s} \mathcal{B}^{rs} X_r \otimes X_s \) (where the \( X_r \) form a basis of \( \mathfrak{g} \), \( \mathcal{B}^{rs} = \mathcal{B}(X_r, X_s) \) and \( \mathcal{B}^{rs} \) is the inverse matrix) acts on an irreducible representation \( (V, \pi) \) of the Lie algebra \( \mathfrak{g} \) of highest weight \( \lambda \in \mathfrak{t}^* \) by a multiple of the identity given by

\[ \pi \left( \Omega^\mathcal{B}_\mathfrak{g} \right) |_{V_\lambda} = c^\mathcal{B}_\lambda \text{Id} |_{V_\lambda} \quad \text{with} \quad c^\mathcal{B}_\lambda := < \lambda, 2\rho + \lambda >^\mathcal{B} \]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) is half the sum of the positive roots.

**Proof** Since the Casimir is central in the universal enveloping algebra of \( \mathfrak{g} \) by invariance of \( \mathcal{B} \), it acts on any irreducible representation of the Lie algebra \( \mathfrak{g} \) by a multiple of the identity. To compute this multiple on an irreducible representation of highest \( \lambda \) we evaluate its action on a highest vector \( v_\lambda \).
The complexified Lie algebra \( \mathfrak{g}^\mathbb{C} \) decomposes as \( \mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \). Extending \( \widetilde{B} \) \( \mathbb{C} \)-linearly to \( \mathfrak{g}^\mathbb{C} \), invariance implies \( \widetilde{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \) unless \( \alpha + \beta = 0 \), and the restriction of \( \widetilde{B} \) to \( \mathfrak{t}^\mathbb{C} \) is non-degenerate. For any root \( \alpha \in \Phi \) we define \( u_\alpha \in \mathfrak{t}^* \) so that \( \alpha(h) = \widetilde{B}(u_\alpha, h) \forall h \in \mathfrak{t} \). Choosing \( E_\alpha \in \mathfrak{g}_\alpha \) for any positive root \( \alpha \in \Phi^+ \) and choosing a basis \( \{ T_1, \ldots, T_k \} \) of \( \mathfrak{t} \), a basis for \( \mathfrak{g} \) is given by \( \{ X_\alpha := \frac{1}{2}(E_\alpha + \overline{E}_\alpha), Y_\alpha := \frac{1}{2i}(E_\alpha - \overline{E}_\alpha) \text{ for all } \alpha \in \Phi^+, T_1, \ldots, T_k \} \). Observe that the conjugate \( \overline{E}_\alpha \) is in \( \mathfrak{g}_{-\alpha} \). By invariance, we have \( [E_\alpha, E_\alpha] = \widetilde{B}(E_\alpha, \overline{E}_\alpha) u_\alpha \) so that the \( X_\alpha, Y_\alpha \)'s are all \( \widetilde{B} \) orthogonal and \( \widetilde{B}(X_\alpha, X_\alpha) = \frac{1}{2} \widetilde{B}(E_\alpha, \overline{E}_\alpha) = -\widetilde{B}(Y_\alpha, Y_\alpha) \). Since \( \pi(\mathfrak{g}_\alpha)v_\lambda = 0 \) for any \( \alpha \in \Phi^+ \) the action of the Casimir operator on the highest vector \( v_\lambda \) can be written

\[
\pi \left( \Omega^\widetilde{B}_{\mathfrak{g}} \right) v_\lambda = \left( \sum_{\alpha \in \Phi^+} \frac{2}{B(E_\alpha, E_\alpha)} \left( \pi(X_\alpha) \circ \pi(X_\alpha) + \pi(Y_\alpha) \circ \pi(Y_\alpha) \right) + \sum_{i,j=1}^{k} \widetilde{B}^{ij} T_i \circ T_j \right) v_\lambda
\]

\[
= \left( \sum_{\alpha \in \Phi^+} \frac{1}{B(E_\alpha, E_\alpha)} \left( \pi(E_\alpha) \circ \pi(E_\alpha) + \pi(\overline{E}_\alpha) \circ \pi(\overline{E}_\alpha) \right) + \sum_{i,j=1}^{k} \widetilde{B}^{ij} T_i \circ T_j \right) v_\lambda
\]

\[
= \sum_{\alpha \in \Phi^+} \frac{1}{B(E_\alpha, E_\alpha)} \left[ \pi(E_\alpha, \pi(\overline{E}_\alpha)) \right] v_\lambda + \sum_{i,j=1}^{k} \widetilde{B}^{ij} \lambda(T_i) \lambda(T_j) v_\lambda
\]

\[
= \sum_{\alpha \in \Phi^+} \pi(u_\alpha) v_\lambda + \sum_{i,j=1}^{k} \widetilde{B}^{ij} \widetilde{B}(u_\lambda, T_i) \widetilde{B}(u_\lambda, T_j) v_\lambda
\]

\[
= \left( \sum_{\alpha \in \Phi^+} \lambda(u_\alpha) + \widetilde{B}(u_\lambda, u_\lambda) \right) v_\lambda
\]

\[
= (\langle \lambda, 2\rho \rangle_\widetilde{B} + \langle \lambda, \lambda \rangle_\widetilde{B} ) v_\lambda.
\]

We consider a spinor field \( \widetilde{\varphi} = v \otimes L \) belonging to \( V_\lambda \otimes \text{Hom}_K(V_\lambda, S^1(\mathbb{C}^n)) \) for an irreducible representation \( (V_\lambda, \pi_\lambda) \) of \( SU(n + 1) \) of highest weight

\[
\lambda = m_1 \epsilon_1 + \cdots + m_n \epsilon_n,
\]

with \( m_1 \geq m_2 \geq \cdots \geq m_n \) satisfying the conditions of Theorem 4.7. Since \( \widetilde{X}(v \otimes L) = -L \circ \pi_\lambda(X) \) for all \( X \in \mathfrak{su}(n + 1) \) we have

\[
(\Omega^\widetilde{B}_{\mathfrak{su}(n+1)}(v \otimes L))(g) = \left( \pi_{\lambda_2}(\Omega^\widetilde{B}_{\mathfrak{su}(n+1)})v \otimes L \right)(g)
\]

\[
= \pi_{\lambda_2}(\mathfrak{su}(n+1), \widetilde{B})(v \otimes L)(g).
\]

Also \( \left( (cl \circ \nu \circ \text{ad}|_\mathfrak{p})(\Omega^\widetilde{F}_r \right)(v \otimes L))(g) = \left( (cl \circ \nu \circ \text{ad}|_\mathfrak{p})(\Omega^\widetilde{F}_r \right)((v \otimes L)(g)) \). By equation (20) and the definition (21) of \( F_j \), we have \( cl \circ \nu(Y) = U_*((F_j(Y)) - \frac{1}{2} \text{Tr}_j(Y) \text{Id} \) for all
Y ∈ u(n). Since Ad|p ˜q(A) = (det$_j$ A)A and (U(£j((det$_j$ A)A))f)(z) = f((det$_j$ A)$^{-1}$ A$^{-1}$ z) for all A ∈ U(n), the representation (cl$\circ$ ν$\circ$ ad|p) of $\mathfrak{k}$ on $S^l$($\mathbb{C}^n$) is given by

$$((cl$\circ$ ν$\circ$ ad|p)(q,b)f)(z) = \left(-l - \frac{n+1}{2}\right) Tr_j b f(z) + \frac{d}{dt} f(exp - tb z) \quad \forall b \in u(n).$$

For only highest weight vector we get the polynomial $f(z = (z_1, \ldots, z_n)) = z_n^l$ (up to a multiple). Hence the representation is irreducible of highest weight

$$\beta = \left(2l + \frac{n+1}{2}\right) \epsilon_1 + l\epsilon_2 + \ldots + l\epsilon_n. \quad (82)$$

We get

$$\left((cl$\circ$ ν$\circ$ ad|p)(\Omega^{\tilde{B}}_t) (v \otimes L)\right) (g) = c^{\tilde{B}}_{\beta}(v \otimes L)(g) \quad \text{and}$$

$$\left((cl$\circ$ ν$\circ$ ad|p + \frac{1}{4} c\tilde{B}) (v \otimes L)\right) (g) = c^{\tilde{B}}_{\gamma}(v \otimes L)(g)$$

because cl$\circ$ ν$\circ$ ad|p + $\frac{1}{4} c\tilde{B}$ is again an irreducible representation of $\mathfrak{k}$ on $S^l$($\mathbb{C}^n$). Since $\chi'(\tilde{q}(A)) = \eta(\chi \otimes F_j \circ ad|p)(\tilde{q}(A)) = \eta((det$_j$ A)$^k$, (det$_j$ A)A) = (det$_j$ A)$^{2k}$ det$_j$((det$_j$ A)A) = (det$_j$ A)$^{2k+n+1}$, it has highest weight

$$\gamma = \left(2l + \frac{n+1}{2} - \frac{2k+n+1}{4}\right) \epsilon_1 + l\epsilon_2 + \ldots + l\epsilon_n. \quad (83)$$

Hence

**Lemma 4.10** For the $Mp^c$-structure on $\mathbb{C} P^n$ defined by $\chi_k$ (as in (73)), the elliptic operator $\mathcal{P} = -i(DDJ - DJD)$ acts on the subspace of spinor fields $V_\lambda \otimes \text{Hom}_{K,(l)}(V_\lambda, S^l$($\mathbb{C}^n$)) (when it does not vanish) as a multiple of the identity given by

$$\frac{1}{\hbar} \left(c^{su(n+1),\tilde{B}}_{\chi} + c^{B}_{\beta} - 4c^{B}_{\gamma}\right) = \frac{1}{\hbar} \left(<\lambda, 2\rho_{su(n+1)} + \lambda >_B + <\beta, 2\rho_{l} + \beta >_B\right)$$

$$-4 < \gamma, 2\rho_{l} + \gamma >_B$$

for $\beta = (2l + \frac{n+1}{2})\epsilon_1 + l\epsilon_2 + \ldots + l\epsilon_n$ and $\gamma = \beta - \frac{2k+n+1}{4}\epsilon_1$.

In our situation, for $\mathfrak{g} = su(n + 1)$ or $\mathfrak{g} = \mathfrak{k}$, with $\tilde{B}(X, Y) = -\frac{1}{2} \text{Tr} XY$, an element $\kappa = k_1\epsilon_1 + \ldots k_n\epsilon_n$ in $i^* \mathfrak{k}$ corresponds to the element

$$u_\kappa = 2 \left(\begin{array}{cccccc}
-k_1 + \frac{\sum_{i=1}^{n} k_i}{n+1} & 0 & \ldots & \ldots & 0 \\
0 & -k_2 + \frac{\sum_{i=1}^{n} k_i}{n+1} & 0 & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ldots & 0 & -k_n + \frac{\sum_{i=1}^{n} k_i}{n+1} & 0 \\
0 & \ldots & \ldots & 0 & +\frac{\sum_{i=1}^{n} k_i}{n+1} \\
\end{array}\right)$$
in it. Hence

\[ <\kappa,\kappa'> = -2 \left( \sum_j k_j k'_j - \frac{1}{n+1} \sum_j k_j \sum_i k'_i \right). \]

On the other hand

\[ 2\rho_{\text{su}(n+1)} = \sum_{1 \leq i < j \leq n+1} (\epsilon_i - \epsilon_j) \]
\[ = n\epsilon_1 + (n-2)\epsilon_2 + \ldots + (-n)\epsilon_{n+1} \]
\[ = 2n\epsilon_1 + 2(n-1)\epsilon_2 + \ldots + 2\epsilon_n \]

and

\[ 2\rho = \sum_{2 \leq i < j \leq n+1} (\epsilon_i - \epsilon_j) \]
\[ = (n-1)\epsilon_1 + 2(n-1)\epsilon_2 + 2(n-2)\epsilon_3 + \ldots + 2\epsilon_n. \]

### 4.3.4 Kernels of the Dirac–Dolbeault operators \( D' \) and \( D'' \) on \( \mathbb{C}P^n \).

We consider the basis of \( p \) defined by \( X_k := q(e_k) = E_{k0} - E_{0k}, X_{n+k} := jX_k = q(ie_k) = iE_{0k} + iE_{k0} \) for \( 1 \leq k \leq n \), with \( e_1, \ldots, e_n \) the standard basis of \( \mathbb{C}^n \). Observe that \( X_k - ijX_k = q(e_k) - iq(ie_k) = 2E_{k0} \) and similarly \( X_{n+k} - ijX_{n+k} = 2iE_{k0} \). Furthermore \( \Omega(q(e_k), q(e_l)) = \Omega(q(ie_k), q(ie_l)) = 0 \) and \( \Omega(q(e_k), q(ie_l)) = \delta_{kl} \). Hence, for any element \( L \in \oplus_l \text{Hom}_{K,(k)}(V_{\lambda}, S^l(\mathbb{C}^n)) \) the operator \( D'^{(\lambda,k)} \) can be written as in (78)

\[ D'^{(\lambda,k)}L = \frac{1}{2} \sum_{rs=1}^{2n} \Omega^r s c(X_r) \circ L \circ \pi_{\lambda*}(X_s - ijX_s) \]
\[ = \frac{1}{2} \sum_{k=1}^{n} c(e_k) \circ L \circ \pi_{\lambda*}(2iE_{k0}) - \frac{1}{2} \sum_{k=1}^{n} c(ie_k) \circ L \circ \pi_{\lambda*}(2E_{k0}) \]
\[ = 2i \sum_{k=1}^{n} c(e_k) \circ L \circ \pi_{\lambda*}(E_{k0}) \]

where \( (c(e_k)f)(z) := \frac{1}{2\pi} z^k f(z) \) following (51). Similarly \( X_k + ijX_k = -2E_{k0} \) and \( X_{n+k} + ijX_{n+k} = 2iE_{k0} \) and (79) becomes

\[ D''^{(\lambda,k)}L = -2i \sum_{k=1}^{n} a(e_k) \circ L \circ \pi_{\lambda*}(E_{k0}) \]

where \( (a(e_k)f)(z) := \frac{\partial}{\partial z^k} f(z) \). In view of Theorem 4.8 the space of spinors for the \( Mp^c \)-structure on \( \mathbb{C}P^n \) defined by \( \chi_k \) (as in (72)) is given by

\[ \oplus_{r \geq 0, 0 \leq b \leq r} V_{\lambda} = \otimes_{r=b}^{2r+b-k} V_{(2r+b-k, r+1)} \otimes \left( \oplus_{l=b}^{r+b} \text{Hom}_{K,(l)}(V_{\lambda}, S^l(\mathbb{C}^n)) \right). \]
and each $\text{Hom}_{K,(k)}(V_\lambda, S^l(C^n))$ in this sum is 1-dimensional. Since $D'$ preserves each $\left(\bigoplus_{r=b}^r \text{Hom}_{K,(k)}(V_\lambda, S^l(C^n))\right)$ and increases the degree of polynomials, it is clear that

$$\bigoplus_{l \geq 0,0 \leq b \leq l, l+b \geq k} V_\lambda = (2l+b-k)\epsilon_1 + l\epsilon_2 + \ldots + l\epsilon_{n-1} + (l-b)\epsilon_n \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(C^n))$$

is included in the kernel of $D'$. Furthermore, for any $\lambda = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-b)\epsilon_n$, and for any $0 \leq l < r$, the operator $D'$ maps a generator $L_{(k,\lambda,l)}$ of $\text{Hom}_{K,(k)}(V_\lambda, S^l(C^n))$ on a multiple of a generator $L_{(k,\lambda,l+1)}$ of $\text{Hom}_{K,(k)}(V_\lambda, S^{l+1}(C^n))$; we shall show that this multiple is not zero. Using [34] we have

$$D'(\lambda,k) L_{(k,\lambda,l)} w = 2i \sum_{s=1}^n c(s) \circ L_{(k,\lambda,l)} \circ \pi_{\lambda s}(E_{\beta s}) w$$

and we choose the vector $w = w_{(l+1)}$ in $V_\lambda$ so that it is a highest weight vector for $K$ and $L_{(k,\lambda,l+1)} w_{(l+1)} = z_{n}^{l+1}$. It has weight

$$(2l+2-k)\epsilon_1 + (l+1)\epsilon_2 + \ldots + (l+1)\epsilon_n = \lambda + (r-l-1)(\epsilon_{n+1} - \epsilon_1) + b(\epsilon_n - \epsilon_1).$$

Since $\pi_{\lambda s}(E_{\beta 0}) w_{(l+1)}$ has weight

$$(2l+2-k)\epsilon_1 + (l+1)\epsilon_2 + \ldots + (l+1)\epsilon_n + \epsilon_{k+1} - \epsilon_1$$

for any $k < n$ and since $L_{(k,\lambda,l)}$ vanish on all vectors of weight higher than $(2l-k)\epsilon_1 + l\epsilon_2 + \ldots + l\epsilon_n$, we have

$$D'(\lambda,k) L_{(k,\lambda,l)} w_{(l+1)} = 2ic(\epsilon_n) \circ L_{(k,\lambda,l)} \circ \pi_{\lambda s}(E_{\beta 0}) w_{(l+1)}.$$ 

On the other hand, if $\nu_\lambda$ denotes a highest weight vector in $V_\lambda$, any vector in $V_\lambda$ is a linear combination of vectors of the form

$$(\Pi_{j \geq 1}(\pi_{\lambda s}(E_{jk}))^{r_{jk}})(\pi_{\lambda s}(E_{\beta 0}))^{r_{\beta 0}} \ldots (\pi_{\lambda s}(E_{\beta 10}))^{r_{\beta 10}} \nu_\lambda.$$ 

Since $L_{(k,\lambda,l+1)}$ is $K$-equivariant and vanishes on all vectors of weight higher than $(2l+2-k)\epsilon_1 + (l+1)\epsilon_2 + \ldots + (l+1)\epsilon_n$, the element $w_{(l+1)}$ has a non-zero component in

$$(\pi_{\lambda s}(E_{\beta 0}))^{r_{l-1}}(\pi_{\lambda s}(E_{\beta (n-1)0}))^{b} v_\lambda$$

$$w_{(l+1)} = c_{l+1}(\pi_{\lambda s}(E_{\beta 0}))^{r_{l-1}}(\pi_{\lambda s}(E_{\beta (n-1)0}))^{b} v_\lambda + w'_{(l+1)}$$

with $w'_{(l+1)} \in \bigoplus_{j \geq 1}(\pi_{\lambda s}(E_{jk}))^{r_{jk}}(\pi_{\lambda s}(E_{\beta 0}))^{r_{\beta 0}} \ldots (\pi_{\lambda s}(E_{\beta 10}))^{r_{\beta 10}} \nu_\lambda$.

Now, $L_{(k,\lambda,l)}$ is also $K$-invariant, and $\pi_{\lambda s}(E_{\beta 0})$ commutes with $\pi_{\lambda s}(E_{jk})$ for $j > k > 0$ so that

$$L_{(k,\lambda,l)} \circ \pi_{\lambda s}(E_{\beta 0}) w_{(l+1)} = c_{l+1} L_{(k,\lambda,l)} \circ \pi_{\lambda s}(E_{\beta 0})(\pi_{\lambda s}(E_{\beta 0}))^{r_{l-1}}(\pi_{\lambda s}(E_{\beta (n-1)0}))^{b} v_\lambda \neq 0.$$ 

Similarly, since $D''$ decreases the degree of the polynomials,

$$\oplus_{r \geq 0, 0 \leq l \leq r+l \geq k} V_{\lambda} = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-l)\epsilon_n \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))$$

is the kernel of $D''$ and we have

**Proposition 4.11** For the $Mp^c$-structure on $\mathbb{C}P^n$ defined by $\chi_k$ (as in (72)), the kernel of $D'$ is given by

$$\oplus_{l \geq 0, 0 \leq b \leq l+b \geq k} V_{\lambda} = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-b)\epsilon_n \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))$$

and, for each $\lambda = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-b)\epsilon_n$ with $r \geq 0, 0 \leq b \leq r, r+b \geq k$, the operator $D'$ induces a bijection from $\text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))$ to $\text{Hom}_{K,(k)}(V_\lambda, S^{l+1}(\mathbb{C}^n))$ for each $b \leq l < r$.

The kernel of $D''$ is given by

$$\oplus_{r \geq 0, 0 \leq l \leq r+l \geq k} V_{\lambda} = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-l)\epsilon_n \otimes \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))$$

and, for each $\lambda = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-b)\epsilon_n$ with $r \geq 0, 0 \leq b \leq r, r+b \geq k$, the operator $D''$ induces a bijection from $\text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))$ to $\text{Hom}_{K,(k)}(V_\lambda, S^{l+1}(\mathbb{C}^n))$ for each $b < l \leq r$.

The kernel of $D$ in each $V_\lambda = (2r+b-k)\epsilon_1 + r\epsilon_2 + \ldots + r\epsilon_{n-1} + (r-b)\epsilon_n \otimes \left(\oplus_{l=b}^{r} \text{Hom}_{K,(k)}(V_\lambda, S^l(\mathbb{C}^n))\right)$ is isomorphic to $V_\lambda$ if $r-b$ is even (and only involves polynomials of the same parity as $r$) and this kernel is $0$ if $r-b$ is odd.

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