INTERMEDIATE PLANAR ALGEBRA REVISITED, II

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Abstract. We describe the subfactor planar algebra of an intermediate subfactor \( N \subset Q \subset M \) of an extremal subfactor \( N \subset M \) of finite Jones index which is not necessarily irreducible.

1. Introduction

Given the fact that modern subfactor theory, as pioneered by Vaughan Jones, deals with the relative position of a subfactor inside an ambient factor, it is a very fundamental question to consider relative positions of an intermediate subfactor and this turns out to be a very active area of research. Given a finite-index irreducible subfactor \( N \subset M \) (that is, \( N' \cap M = \mathbb{C} \)) and an intermediate subfactor \( Q \), one may explicitly describe (as in [1], see also [2]) the planar algebra of \( N \subset Q \) in terms of the planar algebra of \( N \subset M \). The proof in [1] is technical which involves lots of beautiful pictorial calculations involving the so-called ‘biprojections’. We must mention that in the proof we have crucially used the fact that \( N \subset M \) is irreducible. However, in the non-irreducible case the description of the planar algebra of \( N \subset Q \) has not yet appeared in the literature. In this paper we consider an intermediate subfactor \( N \subset Q \subset M \) of an extremal subfactor \( N \subset M \) of finite Jones index which is not necessarily irreducible and describe the subfactor planar algebra of \( N \subset Q \) (which we denote by \( P_{N \subset Q} \)) in terms of the subfactor planar algebra \( P_{N \subset M} \) in such a way that in the irreducible case, it recovers the description of \( P_{N \subset Q} \) as expounded in [1].

We briefly mention related work. In [6], Hartglass has constructed an object called an N-P-M planar algebra which is an algebra over an operad of three-shaded tangles and shown that if \( Q \) is the standard invariant of \( N \subset M \) that contains an intermediate subfactor \( P \), then \( Q \) can be faithfully realized inside a natural N-P-M planar algebra \( \mathcal{P} \) associated to the triple \( N \subset P \subset M \). It is also worth mentioning that D. Bisch gave a partial description of the standard invariant of \( N \subset Q \) in [4] by giving the standard invariant of the inclusion \( N \subset Q_1 \), where \( Q_1 \) is the first step in the basic construction of \( N \subset Q \). Our main result is an explicit description of the planar algebra of \( N \subset Q \) in terms of the original planar algebra. We fix an arbitrarily chosen minimal projection \( f \) in \( Q' \cap M \) which gives rise to

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a projection \( f_n \) in \( P_n^{NC} \) and a new partially labelled tangle \( fTf \) associated to any tangle \( T \). The biprojection \( q \) (namely, the Jones projection \( e_Q \)) corresponding to the intermediate subfactor \( Q \) and the projection \( f \) yield naturally a mapping \( F = \{ F_m \} \) from tangles of any colour (say \( m \)) to partially labelled tangles of the same colour and carefully chosen scalar-valued functions \( \alpha_{NCQ<CM} \) and \( c \) defined on the collection of all tangles (see Definition \( \text{I.4} \)), such that \( P^{NCQ} \) may be identified with a planar algebra, call it \( P' \), with \( P'_n = \text{range}(Z_{Nf(T_n f)}^{NC}) \), where \( T_n \) is the identity tangle of colour \( n \) and the multilinear map \( Z' \) associated to a tangle \( T := T_{k_0 \cdots k_b} \) is given by

\[
Z'_T = (\text{tr} f)^{(T)} \alpha_{NCQ<CM}(T) Z^{NC}_{F(Tf)},
\]

with inputs from \( P' \). In order to verify that \( (P', Z'_T) \) is indeed a planar algebra we need to check that the operation of tangles is compatible with composition of tangles and to this end we verify the following crucial ‘co-cycle type’ equation holds for any two tangles \( T \) and \( \tilde{T} \) so that \( T \circ \tilde{T} \) makes sense:

\[
Z_{F(T \circ \tilde{T})}^{NC} = \left( \frac{(\text{tr} f)^{(T)}}{(\text{tr} f)^{(T \circ \tilde{T})}} \alpha_{NCQ<CM}(T) \alpha_{NCQ<CM}(\tilde{T}) \right) Z_{F(T \circ \tilde{T})}^{NC}.
\]

We divide the proof of the verification of the Equation \( \text{I.1} \) into two cases as follows:

Case (I): \( Q' \cap M = \mathbb{C} \); and Case (II): \( Q' \cap M \neq \mathbb{C} \).

In view of the fact that in Case (I) we must have \( f = 1 \) it is easy to see that \( (P', Z'_T) \) coincides with the description of the intermediate planar algebra for the irreducible case as in \( \text{I.1} \). The verification of the Equation \( \text{I.1} \) involves a suitable adaptation of Theorem 3.4 in \( \text{I.1} \) using a cute pictorial relation involving \( q \). For the proof of Case (II) we have taken a two-fold strategy. First for the chosen minimal projection \( f \) in \( Q' \cap M \) we consider the inclusions \( Nf \subset Qf \subset fMf \) and then we observe that the description of \( P^{NCfQf} \) reduces to the Case (I). In other words, using Case (I) we may describe \( P^{NCfQf} \) in terms of \( P^{NCfMf} \). As a final strategy, we simply observe that \( P^{NCfQf} \) and \( P^{NCQ} \) are isomorphic as planar algebras and recall from Corollary 4.2.14 in \( \text{I.1} \) that the planar algebra of the subfactor \( Nf \subset fMf \) is isomorphic to the reduced planar algebra \( fP^{NCMf} \). Therefore, we obtain a description of \( P^{NCfQf} \) (and hence of \( P^{NCQ} \)) in terms of \( P^{NCM} \).

2. Notation and basic facts

In this paper, all factors we will be considering are of type \( II_1 \) and all subfactors \( N \subset M \) will have finite Jones index \( [M : N] \). By \( \text{tr}_M \) we mean the unique normal faithful trace defined on \( M \). \( E^M_N \) denotes the trace preserving conditional expectation from \( M \) onto \( N \); we may omit the subscript \( M \) and instead write \( E_N \) and \( \text{tr} \) if it is clear from the context. We will assume throughout that the reader is familiar with planar algebras introduced by
Jones in [7]. To fix the notations and definitions for the version of planar algebras that we use, we refer to [8]. We will briefly describe the notations here for the reader’s convenience. Let $P = (P_k)_{k \geq 0}$ be a planar algebra where $P_k$ denotes the $k$-box space $N' \cap M_{k-1}$, $\delta = [M : N]^{-1/2}$ and write $Z_T$ for the multilinear operator corresponding to a planar tangle $T$. We dispense with shading figures since the shading is uniquely determined by the sub- and superscripts of the tangle.

Now given a planar algebra $P$ and an $f \in P_1$ a projection, we can produce a new planar algebra $fPf$ called the reduced planar algebra which we briefly describe now. More details can be found in [7]. First define the projections $f_k \in P_k$ as in Figure 1. Given a planar tangle $T$, define the partially labelled tangle $fTf$ by inserting $f$ in each string of $T$. Now let us describe the planar algebra $fPf$ as follows. First define the spaces of $fPf$ as $(fPf)_k = f_k P_k f_k$ and the tangle action by $Z_{fTf} = Z_{fPf}^{TPf}$. It is easy to check that $(fPf, Z_{fPf})$ is a subfactor planar algebra and we have the following:

**Proposition 1** ([7], Corollary 4.2.14). Let $N \subset M$ be an extremal II$_1$ subfactor with $[M : N] < \infty$ and $f$ a projection in $N' \cap M$. Then the reduced planar algebra $fP^N \subset fMf$ is naturally isomorphic to the planar algebra of the reduced subfactor $Nf \subset fMf$.

![Figure 1. The projection $f_k \in P_k$.](image)

It is well-known from [3, 9, 5] that if $N' \cap M = \mathbb{C}$, there is a bijective correspondence between biprojections $q$ (corresponding to the Jones projection of $L^2(M)$ onto $L^2(Q)$) and the intermediate subfactor $Q$, where $N \subset Q \subset M$. More precisely, we have the following (reformulation of) Theorem 3.2 of [3].

**Theorem 2** ([3, 9, 5]). Let $N \subset M$ be an extremal II$_1$ subfactor and $P = P^{(N \subset M)}$ be the planar algebra of $N \subset M$. Suppose there exists an intermediate subfactor $Q$, $N \subset Q \subset M$ and $q \in P_2$ denotes the biprojection corresponding to $Q$. Then $q$ satisfies the relations in Figure 2 with $c = [M : N]^{1/2}[M : Q]^{-1}$. Furthermore, in the case $N' \cap M = \mathbb{C}$, the converse is also true. More precisely, $q \in P_2$ satisfying the relations (a)-(d) in Figure 2 implies the existence of an intermediate subfactor $Q$, $N \subset Q \subset M$ corresponding to $q$.

Throughout the paper let us fix a II$_1$ subfactor $N \subset M$ having an intermediate $Q$ with the corresponding biprojection $q \in P_2$ (i.e $q = e_Q$) where
$P = P_{N \subset M}$. Our goal is to determine $P_{N \subset Q}$ in terms of the planar algebra $P_{N \subset M}$. Now let $f \in Q' \cap M$ be a minimal projection. Consider the projection $f_2$ defined as in the Figure 1. Then we have,

Lemma 3. Let $N \subset Q \subset M$ be an intermediate subfactor with the corresponding biprojection $q$ and $f \in Q' \cap M$ be a minimal projection. Then $\frac{1}{M}_f f_2 q f_2$ is the biprojection corresponding to the intermediate subfactor $N f \subset Q \subset f M f$.

Proof. This essentially follows from Lemma 2.2 from [3]. Indeed in that lemma replace $N$ by $Q$, $p$ by $f$, $q$ by $\tilde{f} = \frac{1}{M}_f$ and hence $M_1$ is replaced by $Q_1$ where $Q_1$ is the basic construction of $Q \subset M$. It is easy to observe that $f \tilde{f} = \tilde{f} f = f_2$. Then $Q f_2 \subset (f M f) \tilde{f} \subset f_2 Q_1 f_2$ is the basic construction of $(Q f_2 \subset (f M f) \tilde{f}) \cong (Q f \subset f M f)$ with the corresponding biprojection $\frac{1}{M}_f f_2 q f_2$.

Lemma 4. The subfactor $N f \subset Q f$ is isomorphic to $N \subset Q$.

Proof. It is easy to see that the map $x \mapsto x f$ is the required isomorphism. We leave the details to the interested reader.

Let $E_n$ denote the tangle as in Figure 3. For a tangle $T$, define the
partially labelled tangle $F_n(T)$ by inserting $\frac{1}{(\text{trf})^n} f_2q f_2$ in $D_1, \cdots D_n$ and $T$ in $D_{n+1}$ of $E_n$ respectively. That is,

$$F_n(T) = E_n \circ_{\{D_1, \cdots, D_n, D_{n+1}\}} \left( \frac{1}{(\text{trf})^n} f_2q f_2, \cdots, \frac{1}{(\text{trf})^n} f_2q f_2, T \right).$$

We write $E$ in place of $E_n$ if it is clear from the context. Now for $x \in P_n$, define $F_n(x) = Z_{F_n(I_n)}(x)$. Pictorially we have $F_n(x)$ as in Figure 4. Thus $F_n$ can be viewed as a map from $P_n$ into $P_n$ given by $F_n(x) = Z_{E_n}(\frac{1}{(\text{trf})^n} f_2q f_2 \otimes \cdots \otimes \frac{1}{(\text{trf})^n} f_2q f_2 \otimes x)$. 

**Figure 3.** The defining tangle $E_n$.

**Figure 4.** The element $F_n(x)$. 
3. Main result

Throughout this section we fix an extremal finite index subfactor $N \subset M$ having an intermediate subfactor $Q$. The main result of this section is the description of the planar algebra $P^N \subset Q$ in terms of $P = P^N \subset M$ by explicitly specifying the spaces and tangle action in terms of $P$. We begin by recalling a definition and proposition from \cite{1}.

**Definition 6.** (Definition 3.1, \cite{1}) Let $T$ be a $k$-tangle with $b \geq 1$ internal discs $D_1, \ldots, D_b$ of colours $k_1, \ldots, k_b$. Then define $\alpha_{N \subset Q \subset M}(T) = [M : Q]^{\frac{1}{2}c(T)}$, where

$$c(T) = \left(\lfloor k_0/2 \rfloor + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_b/2 \rfloor\right) - l(T)$$

with $l(T)$ being the number of closed loops after capping the black intervals of the external disc of $T$ and cupping the black intervals of all internal discs of $T$.

**Proposition 7.** (Proposition 3.2, \cite{1}) If $T = T_{k_0, k_1, \ldots, k_b}$ and $\tilde{T} = \tilde{T}_{\tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_b}$ are tangles with discs of indicated colours such that $\tilde{k}_0 = k_i$ for some $1 \leq i \leq b$, then

$$\frac{\alpha_{N \subset Q \subset M}(T)\alpha_{N \subset Q \subset M}(\tilde{T})}{\alpha_{N \subset Q \subset M}(T \circ_i \tilde{T})} = [M : Q]^{\frac{1}{2}(k_0 - l(T) - l(\tilde{T}) + l(T_0, \tilde{T}))}.$$ 

Recall that for a planar tangle $T$ and an arbitrarily chosen minimal projection $f \in Q' \cap M$, $F(T)$ denotes the partially labelled tangle obtained from $T$ by “surrounding it with $1_{(trf)^2}$” where $f_2$ is the projection given in the Figure \cite{1} and $fTf$ denotes the partially labelled tangle obtained by inserting $\boxed{f}$ in each string of $T$.

**Definition 8.** Define $P'_n \subset P_n$ by $P'_n = F_n(f_n P_n f_n)$ and

$$Z_{P'}^T = (trf)^{c(T)}\alpha_{N \subset Q \subset M}(T)Z_{F(T)}^P|_{P'}.$$ 

**Remark 9.** It is easy to observe that $Z_{F(T)}^P|_{F(T)} = Z_{F(T)}^P$ with inputs coming from $P'$. We illustrate this fact with an example and it should be clear that the proof of the general case is similar. Let $T$ be the tangle given in Figure \cite{2}.

The tangles $fF(T)f$ and $F(fTf)$ are shown in Figure \cite{2}. Now the assertion is clear from the fact that $f$ is a projection in $P_1$ and inputs are coming from $P'$.

Now we prove the main theorem of this article.

**Theorem 10.** Keeping the foregoing notations, $(P', Z_{P'}^T|_{P'})$ is a subfactor planar algebra which is isomorphic to $P^N \subset Q$. 

**Figure 5.**

**Figure 6.** The tangles $fF(T)f$ and $F(fTf)$.

**Proof.** Our proof strategy is as follows. We divide the proof into two cases (I) $Q' \cap M = \mathbb{C}$ and (II) $Q' \cap M \neq \mathbb{C}$. The proof of the Case (I) is similar to the proof of the irreducible case as in [1] with appropriate modifications, and for the proof of Case (II) we will choose a minimal projection $f \in Q' \cap M$ and consider the intermediate $Nf \subset Qf \subset fMf$ which reduces to Case (I), thereby finishing the proof.

Case (I): In this case $f = 1$ and hence the operation $F(T)$ for a triangle $T$ is nothing but surrounding $T$ by “$q$”. So our description of the spaces $P'_n = F_n(P_n)$ and the tangle action $Z^{P''}_{T'} = \alpha_{N \subset Q \subset M}(T)Z^{P''}_{F(T)}|_{P''}$, where $\alpha_{N \subset Q \subset M}(T) = [M : Q]^{1+\nu(T)}$ coincides with the description in the irreducible case. Therefore, in view of Proposition 7 the key ingredient in proving that $(P', Z^{P''}_{T'})$ to be a planar algebra is the verification of the Equation 3.2 (see [1] for details) for inputs coming from $P'$ where $T = T_{k_1, \ldots, k_b}$.
and \( \tilde{T} = \tilde{T}_{k_0} \) be tangles with \( k_i = \tilde{k}_0 \):

\[
Z_{F(T) \circ F(\tilde{T})} = \tau(q)^{\frac{1}{2}(k_i + l(T(\tilde{T})-l(T)-l(\tilde{T}))} Z_{F(T), \tilde{T}}.
\]

Observe that the Equation 3.2 is obtained by putting \( f = 1 \) in the Equation 1.1. Following [1] (see Theorem 3.4), the proof of the verification of the Equation 3.2 with inputs from \( P' \) is divided into various steps and subcases until the proof is obvious.

Step 1: We reduce to the case that \( T \) is a 0-tangle. So the equation that must be seen to hold on \( P' \) is

\[
Z_{F(T) \circ F(\tilde{T})} = \tau(q)^{\frac{1}{2}(k_i + l(T(\tilde{T})-l(T)-l(\tilde{T}))} Z_{F(T), \tilde{T}}.
\]

(since \( F(T) = T \) for a 0-tangle \( T \)). See [1] for details.

Step 2: As in [1], appealing to sphericality we may assume that the tangle \( T \) has the form given in Figure 7 where \( \hat{T} \) is some tangle of colour \( k_1 \) and \( i = 1 \).

Step 3: It is sufficient to prove the Equation 3.2 holds for \( \tilde{T} \) being a Temperley-Lieb tangle as explained in [1].

Step 4: We settle the Temperley-Lieb case by induction on \( k_1 \) by dividing into two different subcases. In each of the subcases, we will show that the statement for a suitably chosen \( S \) and \( \tilde{S} \) with \( k_0(\tilde{S}) < k_0(\tilde{T}) \) implies it for \( T \) and \( \tilde{T} \). The only case where we use the irreducibility of \( N \subset M \) is the subcase 4.2(b), see [1] for details. We will modify the proof of this subcase accordingly and all other cases remain the same.

So without loss of generality, we can assume that \( \tilde{T} \) is a Temperley-Lieb tangle such that some 2\( i \) and 2\( i + 1 \) are joined and \( T \) is a 0-tangle having the form as in Figure 7 with the black intervals \([2i - 1, 2i]\) and \([2i + 1, 2i + 2]\) are part of the same black region in \( \tilde{T} \).

![Figure 7. The tangle T.](image)

Now \( \tilde{T} \) has the form in Figure 8 for some Temperley-Lieb tangle \( \tilde{S} \) of colour \( k_1 - 1 \). Draw a dotted line from the midpoint of the interval \([2i - 1, 2i]\) to the midpoint of the interval \([2i + 1, 2i + 2]\) in \( \tilde{T} \) that lies entirely in the black region that these are both part of. This line does not intersect any
string of $\hat{T}$ (by the definition of a region) and so the part of $\hat{T}$ that lies inside this dotted line is a 1-box that joins the points $2i$ and $2i + 1$. Hence $\hat{T}$ has the form given as in Figure 9 where $W$ is some tangle of colour $k_1 - 1$.

Now by assumption we have $Q' \cap M = C$. This is equivalent to the pictorial relation as in Figure 10. Indeed it follows from Corollary 3.3 and Lemma 4.2 of [2]. Furthermore, Figure 10 together with the exchange relation in the Figure 2(d) implies the pictorial relation in Figure 11 for $x \in P_1$.

**Claim:** Let $\hat{S}$ be the tangle given in Figure 12. We claim that the validity of the Equation 3.2 for the pair $(S, \hat{S})$ implies the validity for the pair $(T, \hat{T})$.

So suppose that Equation 3.2 holds for the pair $(S, \hat{S})$. Then we have

$$Z_{S_01F(\hat{S})} = \tau(q)^{\frac{1}{2}(k_1-1+l(S_01\hat{S})-l(\hat{S})-l(\hat{S})))} Z_{S_01\hat{S}}. \quad (3.3)$$
The tangles $T \circ_1 \tilde{T}$ and $S \circ_1 \tilde{S}$ are shown in Figure 13. It can be easily

seen that $T \circ_1 \tilde{T}$ has an extra floating one box than $S \circ_1 \tilde{S}$. Hence $Z_{T_1 \tilde{T}} = \begin{array}{c} x \\ \end{array} Z_{S_1 \tilde{S}}$ for $x \in P_1$ and $l(T \circ_1 \tilde{T}) = l(S \circ_1 \tilde{S}) + 1$. Also we have, $l(T) = l(\tilde{T}) = l(W) = l(S)$ and $l(\tilde{T}) = l(\tilde{S})$. Now it remains to compare $T \circ_1 F(\tilde{T})$ and $S \circ_1 F(\tilde{S})$ given in Figure 14. Applying the relation in Figure 11 we have $Z_{T \circ_1 F(\tilde{T})} = \tau(q) \begin{array}{c} x \\ \end{array} Z_{S \circ_1 F(\tilde{S})}$ for $x \in P_1$ and this together with the Equation 3.3 justifies the claim.
Combining all the steps we may conclude that \((P', Z_{T'})\) is a planar algebra. To finish the proof of Case (I) we observe that the planar algebra \((P', Z_{T'})\) is isomorphic to the planar algebra \(P_{N \subset M}^{\mathbb{Q}}\) by Theorem 3.7 of [1].

Case (II): \(Q' \cap M \neq \mathbb{C}\). Let \(f \in Q' \cap M\) be a minimal projection. Consider the intermediate \(Nf \subset Qf \subset fMf\) with corresponding biprojection \(\frac{1}{\pi f}f_2qf_2\), see Lemma 3. Observe that

\[
\alpha_{Nf \subset Qf \subset fMf}(T) = [fMf : Qf]^{1/2 c(T)} = (\text{tr} f)^{c(T)}[M : Q]^{1/2 c(T)} = (\text{tr} f)^{c(T)} \alpha_{Nf \subset Qf \subset fMf}(T).
\]

Now since \(f\) is minimal, by Lemma 5, \((Qf)' \cap fMf = f(Q' \cap M)f = \mathbb{C}\). So this reduces to Case (I). By Lemma 4 we have the subfactor \(Nf \subset Qf\) isomorphic to \(N \subset M\). Hence we have,

\[
P_{n}^{\mathbb{Q}} = P_{n}^{Nf \subset Qf} = F_n(P_{n}^{Nf \subset fMf}) \quad \text{(by Case (I))}
\]

\[
= F_n((fPf)_n) \quad \text{(reduced planar algebra, \S 2)}
\]

\[
= F_n(f_nP_n f_n)
\]

\[
= P_n
\]

and

\[
Z_T^P \cong Z_T^{P_{Nf\subset Qf}} = \alpha_{Nf \subset Qf \subset fMf}(T)Z_{F(T)}^{P_{Nf \subset fMf}}|_{P_{n}^{Nf \subset Qf}} \quad \text{(by Case (I))}
\]

\[
= (\text{tr} f)^{c(T)}\alpha_{Nf \subset Qf \subset fMf}(T)Z_{F(T)}^{P} |_{P_n'} \quad \text{(reduced planar algebra, \S 2)}
\]

\[
= (\text{tr} f)^{c(T)}\alpha_{Nf \subset Qf \subset fMf}(T)Z_{F(T)}^{P} \quad \text{(follows from Remark 9)}
\]

\[
= Z_T^{P'}.
\]

This completes the proof of the theorem. \(\square\)

**Remark 11.** The planar algebra of \(Q \subset M\) may also be described in terms of planar algebra of \(N \subset M\) by considering the ‘dual planar algebra’ \(P_{M' \subset M_1}'\)

\[\text{Figure 14. The tangles } T \circ_1 F(\tilde{T}) \text{ and } S \circ_1 F(\tilde{S}).\]
(see [7]) since the type $\text{II}_1$ factor $Q_1$ is an intermediate subfactor of $M \subset M_1$. The details are easy and omitted.

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