Pricing of options on stocks driven by multi-dimensional operator stable Lévy processes

Przemysław Repetowicz
Department of Physics, Trinity College Dublin 2, Ireland

Mark M. Meerschaert
Department of Physics 211 Leifson Physics Building University of Nevada Reno NV 8955

Peter Richmond
Department of Physics, Trinity College Dublin 2, Ireland

Received (29/09/2004)
Revised (Day Month Year)

We model the price of a stock via a Langévin equation with multi-dimensional fluctuations coupled both in the price in time. We generalize previous models in that we assume that the fluctuations conditioned on the time step are compound Poisson processes with operator stable jump intensities. We derive exact relations for Fourier transforms of the jump intensity in case of different scaling indices $\alpha$ of the process. We express the Fourier transform of the joint probability density of the process to attain given values at several different times and to attain a given maximal value in a given time period through Fourier transforms of the jump intensity. Then we consider a portfolio composed of stocks and of options on stocks and we derive the Fourier transform of a random variable $D_t$ (deviation of the portfolio) that is defined as a small temporal change of the portfolio diminished by the the compound interest earned. We show that if the price of the option at time $t$ is an inverse of a positive, larger than unity, power of a logarithm of the price of the stock at time $t$ then the deviation of the portfolio has a zero mean $E[D_t] = 0$ and the option pricing problem may have a solution.

1. Introduction

Early statistical models of financial markets assumed a Gaussian distribution [2]. However, there is evidence [1] that stock price returns are not Gauss distributed and that the distributions diminish slowly in the high end (fat tails). A better statistical description is provided by a model where the logarithm of the stock price is a one dimensional compound Poisson process with jumps being asymptotically power-law distributed (stable Lévy process [3]). That model, however, does not eliminate discrepancies between experimental facts and the theory and drawbacks of the theory.

The exponents of the power laws (tail indices) in distributions appear to be outside of the range of validity predicted by the theory [1]. Furthermore models in the literature [5,4] assume that jumps in the price are independent of waiting times.
between jumps (no price-temporal coupling).

Fat tails can be explained by models basing on operator stable Lévy processes [3]. Here the price of the stock is an exponential from a sum of a large number of projections of many-dimensional jumps whose distributions are invariant under auto-convolution. This implies that the distributions are, in the high end, mixtures of power laws with different exponents, each one describing a walk in one of the dimensions.

We briefly describe development of models that account for a price-temporal coupling (coupled random walks) below. [6,7] formulate an analytical model that yields probability distributions of price jumps under an “ad hoc” ansatz about the form of price-temporal coupling. [8] derives a long time scaling limit for the jump probability distribution in coupled random walks for an anomalous fractional diffusion. [10] obtains results for master equations of coupled random walks in the long time scaling limit and show their connection to fractional derivative calculus. [11] considers one-dimensional continuous-time coupled random walks (Weierstrass flights) and derives the long time scaling limit of the mean squared displacement of the walker as a function of two parameters $\alpha$ and $\beta$ which are fractional dimensions that represent the flight and the waiting. An extension of Weierstrass flights to a model where the walker, after random waiting times, moves with arbitrary finite velocities to new positions is formulated in [12] and formulas are derived for the sojourn probability density and for likelihoods of extreme events. Finally [13, 6] construct contingency tables between logarithmic price returns and waiting times for stocks on the New York Stock Exchange and for United States Dollar – Japanese Yen exchange rates and reject the hypothesis of the independence of returns on waiting times at a high confidence level.

We formulate a multi-dimensional random walk model that recognizes that the jumps are not independent of the waiting times. The model has a two-level structure with the higher level being a walk with random fluctuations preceded by random waiting times and coupled to those and the lower level describing the fluctuations as sums of random numbers of random jumps with a jump intensity (Lévy density) $\varpi(x|t)$. The levels of the model have a subordinated time horizon that will be explained in the next section. Our model works with two arbitrary functions, viz the Lévy density $\varpi(x|t)$ with a scaling index $\alpha$ and a waiting time probability density function (pdf) $\rho_{\Delta T}(t)$ with a scaling index $0 < \beta \leq 1$ and generalizes previous models proposed in the literature in this subject matter.

2. The model

In this section we formulate an assumption about the price evolution of the stock and we recollect some definitions from the theory of stochastic processes. In section 2.1 we recapitulate some known facts concerned with limit distributions of sums of independent random vectors [3]. In sections 2.2 and 2.3 we obtain exact relations for Fourier transforms of jump intensities and for Fourier transforms of distribu-
tions of integer powers of one dimensional projections of jumps that are sums of large numbers of independent random vectors. In section 2.4 we derive the Fourier transform of a joint probability density of an event that the stochastic process attains given values at given times and, in addition to that, that it attains a given maximal value in some predefined time interval. In section (2.5) we define an option on the stock, define a portfolio that is composed of stocks and of options on stocks, define a random variable (deviation of portfolio) that describes the temporal change of that portfolio in a small time span diminished by a compound rate of interest earned on the portfolio in that time span, and we derive conditions on the deviation of the portfolio to have a zero mean.

2.1. The stock price

Let \( \log(S_t) \) be the logarithm of the price of the stock (log-price) at time \( t \). We assume that the log-price has a drift \( \alpha \). In a small time interval \( \delta t \) the log-price may or may not change with respect to this drift. The probability of that change depends on a presumed waiting time pdf \( \rho_{\Delta T}(t) \). This means that:

\[
\log(S_{t+\delta t}) - \log(S_t) = \alpha(\delta t)\beta + \left\{ \begin{array}{ll}
\sum_{i=1}^{d} \sigma_i \delta L^{(i)}_t & \text{with probability } \int_0^{\delta t} \rho_{\Delta T}(\xi)d\xi \\
0 & \text{with probability } 1 - \int_0^{\delta t} \rho_{\Delta T}(\xi)d\xi
\end{array} \right. (2.1)
\]

where \( \beta \) is a real number that will be specified later, the parameters \( \alpha \) and \( \vec{\sigma} := (\sigma_1, \ldots, \sigma_d) \) (volatility) are assumed to be constant as a function of time \( t \) and of the stock price \( S_t \). We assume that the random vector \( \vec{\delta L}_t \) (fluctuation) is infinitely divisible (see e.g. [3] for the precise definition). This implies that the logarithm of the Fourier transform (log-characteristic function) of it has a unique representation [3] in terms of the Lévy-Khintchin formula. This means that:

\[
\chi_{\vec{\delta L}_t}(k) := E \left[ \exp(i\vec{k} \cdot \vec{\delta L}_t) \right] = i\vec{a} \cdot \vec{k} - \frac{1}{2} Q(\vec{k}, t) + \int_{\vec{x} \neq 0} \left( e^{i\vec{k} \cdot \vec{x}} - 1 - \frac{i\vec{k} \cdot \vec{x}}{1 + ||\vec{x}||^2} \right) \phi(\vec{x}|t)d\vec{x} \quad (2.2)
\]

where \( a \) is a constant, \( Q(\vec{k}, t) \) is a quadratic form that describes Gaussian fluctuations and the integral represents a compound Poisson process with some rate \( c > 0 \) and with a jump pdf \( \phi(\vec{x}|t) \). Here we do not use the measure-theory notation but instead we replace the measure by Riemann integrals with some probability densities. We denote \( \delta \vec{L}_t = ID[0, 0, \phi] \). Since in our model (2.1) the drift of the stock price is already accounted for by the parameter \( \alpha \) and we neglect the Gaussian part \( Q(\vec{k}, t) \) in the fluctuations. We also assume that \( \phi(\vec{x}|t) \) is symmetric in \( \vec{x} \). Therefore we have:

\[
\delta \vec{L}_t = ID[0, 0, \phi] \quad (2.3)
\]

The probability density function of fluctuations conditioned on the random waiting time \( \delta T \) reads:

\[
P \left( \delta \vec{L}_t = \delta \vec{L}_t | \delta T = \delta t \right) = \omega_{\delta \vec{L}_t | \delta t} \left( \delta \vec{L}_t | \delta t \right) = \exp(-c) \sum_{n=0}^{\infty} \frac{e^n}{n!} \phi^{\otimes n}(\delta \vec{L}_t | \delta t) \quad (2.4)
\]
The joint pdf of a fluctuation \( \delta \vec{L}_t = \delta \vec{l}_t \) preceded by a waiting time \( \delta T = \delta t \) (the joint fluctuation pdf)

\[
\omega_{\delta \vec{L}_t, \delta T}(\delta \vec{l}_t, \delta t)
\]

The pdf of a fluctuation \( \delta L_t = \delta l_t \) conditioned on a random waiting time \( \delta T = \delta t \) (the conditional fluctuation pdf)

\[
\omega_{\delta L_t, |\delta T}(\delta l_t | \delta t)
\]

The joint pdf of a jump \( \delta \vec{L}_t = \delta \vec{l}_t \) preceded by a waiting time \( \delta T = \delta t \) (the joint jump pdf)

\[
\phi(\delta \vec{l}_t, \delta t)
\]

The pdf of a jump \( \delta \vec{L}_t \) conditioned on a random waiting time \( \delta T = \delta t \) (the conditional jump pdf)

\[
\phi(\delta \vec{l}_t | \delta t)
\]

The pdf of the \( n \)th power \( \Xi^{(n)} := (\sigma \cdot \delta \vec{L}_t)^n \) of the fluctuation term in (2.1) conditioned on the waiting time \( \delta T = \delta t \) (the conditional \( n \)-marginal pdf)

\[
\nu^{(n)}(z | \delta t)
\]

The Levy measure of jumps (the jump intensity)

\[
\varpi(\delta \vec{l}_t)
\]

The pdf of a waiting time \( \delta T = \delta t \) (the waiting time pdf)

\[
\rho_{\delta T}(\delta t)
\]

Table 1. Summary of definitions

where \( \otimes \) means an \( n \)-times auto-convolution. The last equality in (2.4) represents a probability density function of a compound Poisson process with rate \( c \) and jump intensity \( \phi(\delta \vec{l}_t | \delta t) \).

The joint pdf \( \omega_{\delta \vec{L}_t, \delta T}(\delta \vec{l}_t, \delta t) \) of a fluctuation \( \delta \vec{L}_t = \delta \vec{l}_t \) preceded by a waiting \( \delta T = \delta t \) reads:

\[
\omega_{\delta \vec{L}_t, \delta T}(\delta \vec{l}_t, \delta t) = \omega_{\delta \vec{L}_t, |\delta T}(\delta \vec{l}_t | \delta t) \rho_{\delta T}(\delta t)
\]  

(2.5)

Before proceeding further we summarize the definitions in Table 1.

Note that the quantities [A], [B] differ from their counterparts [C], [D] in that, since the fluctuations are compound Poisson random variables, a fluctuation \( \delta \vec{L}_t \) consists, in general, of several jumps \( \delta \vec{L}_t = \sum_{j=1}^{N_t} \delta \vec{L}^{(j)}_t \) where the number of jumps \( N_t \) is a Poisson random number \( N_t = \text{Poisson}(c) \) with rate \( c \). Therefore the quantities [A], [B] are expressed via sums of auto-convolutions of [C], [D] according to formula (2.4). We also stress that all statements in this paper are concerned with large \( \bar{x} \) scaling limits (the high-end of the distribution) of distributions in question.
We assume that the time horizon $\delta T$ and the price fluctuations $\delta \hat{L}_t$ are small at the higher level of the model and large at the lower level. Note that such an assumption, accompanied by additional ad hoc hypotheses, was tacit in previous works (equation (16) in [7] cond-mat/0308017 preprint). This assumption will enable us to use the scaling limit of the jump probability density function in a coupled random walk: This is the Theorem 2.2 on page 733 in [8] that reads:

**Theorem 2.1.** Assume that the waiting time is $\beta$-stable with index $0 < \beta < 1$, meaning that the Laplace transform of the waiting time pdf has a following form:

\[ L_t[\rho](s) = \exp(-K \Gamma(1-\beta) s^\beta) \]

where $K = K(a)$ is a normalization constant of the waiting time pdf ($K(a) = \int_0^\infty \rho(t) dt$) for some $a >> 0$.

The large $\vec{x}$ and large $t$ scaling limit of the joint jump pdf in a coupled random walk reads

\[ \phi(\vec{x}, t) = \det(t^{-\beta \vec{E}}) \varpi(t^{-\beta \vec{E} \vec{x}}) \frac{K(a)\beta}{t^{\beta+1}} \]  

(2.6)

where $\varpi : \mathbb{R}^d \ni \vec{x} \rightarrow \varpi(\vec{x}) \in \mathbb{R}_+$ is the probability density in $\vec{x}$ (termed as the jump intensity), and the matrix $\vec{E}$ is an exponent of the operator stable law (stable index) corresponding to the jump $\vec{x} \in \mathbb{R}^d$ random variable and $\det(t^{-\beta \vec{E}})$ is a normalization constant of the jump pdf.

The eigenvalues $\lambda$ of $\vec{E}$ satisfy $\text{Re}[\lambda] \geq 1/2$ and the eigenvalues $\text{Re}[\lambda] = 1/2$ are simple (not degenerate).

**Corollary 2.1.** The scaling limit of the conditional jump pdf is:

\[ \phi(\vec{x}|t) = \int_{\vec{x} \in \mathbb{R}^d} \frac{\phi(\vec{x}, t)}{\phi(\vec{x}, t) d\vec{x}} = \det(t^{-\beta \vec{E}}) \varpi(t^{-\beta \vec{E} \vec{x}}) \]  

(2.7)

This follows in a straightforward manner by integrating over $\vec{x}$ in (2.6).

**Corollary 2.2.** The jump intensity $\varpi$ satisfies a functional equation:

\[ \exists \vec{E} \in L(\mathbb{R}^d) \forall \vec{x} \neq \vec{0} \quad t \neq 0 \quad \varpi^{\otimes}(\vec{x}) = \varpi(t^{-E \vec{E}}) \det(t^{-\vec{E}}) \]  

(2.8)

where $\varpi^{\otimes}$ is the $t$th auto-convolution defined as a limit of $t_n$-times auto-convolutions for some rational $t_n$ such that $\lim_{n \rightarrow \infty} t_n = t$. Since the random process is infinitely divisible the definition is correct.

The corollary follows from the self-similarity of the random walk. This means that there exists a sequence of linear operators $\mathbb{R}(n) \in L(\mathbb{R}^d)$ and a sequence of real numbers $b_n$ and a sequence of integer numbers $k_n$ such that for some independent, identically distributed random variables $\vec{Y}_i \in \mathbb{R}^d$ we have:

\[ \mathbb{R}(n) \left( \vec{Y}_1 + \ldots + \vec{Y}_{k_n} \right) - b_{k_n} \xrightarrow{n \rightarrow \infty} \vec{Y} \quad \text{with a pdf } \varpi \]  

(2.9)

On the other hand if we take the sequence of rational numbers $t_n$ we have

\[ \mathbb{B}(n) \left( \vec{Y}_1 + \ldots + \vec{Y}_{tn,k_n} \right) - b_{tn,k_n} \xrightarrow{n \rightarrow \infty} \vec{Y}_1 \quad \text{with a pdf } \varpi^{\otimes} \]  

(2.10)
Now if we take $n$ large enough we can write $k_n t_n = k_m(n)$ for a certain subsequence $m(n)$. Therefore the random variable $Y_1$ has the same pdf $\varpi$ except that the argument of the pdf is linearly transformed $\vec{x} \to t^{-E^T} \vec{x}$.

**Corollary 2.3.** The Fourier transform $\hat{\varpi}(\vec{k}) := \mathcal{F}_{\vec{x}}[\varpi](\vec{k})$ of the jump intensity $\varpi$ satisfies a functional equation:

$$\hat{\varpi}^t(\vec{k}) = \hat{\varpi}(tE^T \vec{k})$$

(2.11)

Where $E^T$ is the transpose of $E$. The right hand side of the equality (2.11) follows in a straightforward manner by taking a Fourier transform of the right hand side of (2.8). The left hand side is also straightforward.

$$\mathcal{F}_{\vec{x}} [\varpi^{t \otimes}] (\vec{k}) = \lim_{n \to \infty} \mathcal{F}_{\vec{x}} [\varpi^{t_n \otimes}] (\vec{k}) = \lim_{n \to \infty} \hat{\varpi}^{t_n}(\vec{k}) = \hat{\varpi}^t(\vec{k})$$

(2.12)

see [3] for details.

We illustrate **Theorem 2.1** in following examples:

**Example 1.** Let $d = 1$ and $\vec{Y} = \vec{Y}$ is normal with mean zero and variance 1. Assume that $\beta = 1$. Then $E = E = 1/2$ and $\varpi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

**Example 2.** Let $d = 1$ and let the log-characteristic function of $Y$ be $\log(E[\exp(ikY)]) = -|k|^{\mu}$ where $0 \leq \mu \leq 2$. Then $E = 1/\mu$ and the function $\varpi$ can has an asymptotic expansion:

$$\varpi(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{2} \mu n \right) \frac{\Gamma(\mu n + 1)}{n!} \frac{1}{x^{\mu n + 1}}$$

(2.13)

In the next section we are going to analyze the Langévin equation (2.1) for the log-price of the stock driven by operator stable fluctuations. For this purpose we need the time dependence of the conditional jump pdf on the waiting time for small waiting times and we need the probability distribution of the $n$th power of a scalar product $\Xi^{(n)} := (\vec{\sigma} \cdot \delta L_t)^n$ for $n \in \mathbb{N}$ (the conditional $n$-marginal pdf). We have:

**Proposition 2.1.** The conditional jump pdf $\phi(\vec{x}|t)$ depends on time $t$ as follows:

$$\phi(\vec{x}|t) = t^{-\beta T^t} \mathbb{E} e^{(t^{-\beta E^T} \vec{x})}$$

(2.14)

This is a straightforward conclusion from (2.7) and from the fact that $\det(\vec{A}) = e^{\vec{T}^t \vec{A}}$ for every linear operator $\vec{A} \in L(\mathbb{R}^d)$.

**Proposition 2.2.** The Fourier transform of the conditional $n$-marginal probability density function $\hat{\nu}^{(n)}(k|\delta t) := \mathcal{F}_{\vec{x}} [\nu^{(n)}] (k)$ as a function of the Fourier transform of the jump intensity $\hat{\varpi}(\vec{\lambda}) := \mathcal{F}_{\vec{x}} [\varpi](\vec{\lambda})$ takes following form:

$$\hat{\nu}^{(n)}(k|\delta t) = \int_{-\infty}^{\infty} d\vec{z} \hat{\varpi}(\vec{z}|(\delta t)^{\beta E^T} \vec{\sigma}|(1)) K^{(n)}(\vec{t}, 1)$$

(2.15)
where

\[ K^{(n)}(t, \ell) := \begin{cases} 
2Re \left[ e^{i \tilde{\varpi} t} \int_0^\infty d\xi e^{-\xi t} + e^{-\frac{n-1}{2m} \pi i \xi} \right] & \text{if } n \text{ is odd} \\
2e^{i \tilde{\varpi} t} \int_0^\infty d\xi e^{-\xi t} \cos \left( \cos \left( \frac{\pi (n-1)}{2m} \right) \xi \right) & \text{if } n \text{ is even}
\end{cases} \]  

(2.16)

and \( \ell := k \sigma^m \). For small values of \( k \) the conditional \( n \)-marginal pdf reads:

\[ \varphi^{(n)}(k|\delta t) = \tilde{\varphi} \left( (\delta t)^{\frac{\beta}{\pi}} \left( \Theta k^{1/n} \sigma \right) \right) \]  

(2.17)

where

\[ \Theta := \frac{\mathcal{C} \pi}{2 \cos(\pi (n-1)/2m)} \]  

(2.18)

and \( \mathcal{C} \) is defined in Appendix A. The proof of the proposition is in Appendix A.

**Corollary 2.4.** The mean value of the \( n \)th power of the fluctuation term in (2.1) conditioned on the waiting time \( \delta T = \delta t \) is infinite if \( n \) is even and \( n \geq 2 \) and is zero if \( n \) is odd.

\[ E \left[ (\tilde{\sigma} \cdot \delta \tilde{L}_t)^n \right] | \delta T = \delta t = \begin{cases} 
\infty & \text{for } n \in 2\mathbb{N} \text{ and } n \geq 2 \\
0 & \text{for } n \in \mathbb{N} \setminus 2\mathbb{N}
\end{cases} \]  

(2.19)

The first statement follows from \( \partial \varphi^{(n)}(k|\delta t)/\partial k \bigg|_{k=0} = \infty \) (see (2.17)) and the second statement follows from the assumption that the conditional jump pdf \( \phi(z|\delta t) \) and hence the jump intensity \( \varpi(z) \) are symmetric in \( z \) (see beginning of section 2.1).

**Corollary 2.5.** The \( n \)th power of the fluctuation term in (2.1) has a form:

\[ Z := (\tilde{\sigma} \cdot \delta \tilde{L}_t)^n = \mathcal{R}(\delta t)^\beta + O ((\delta t)^{2\beta}) \]  

(2.20)

where \( \mathcal{R} \) is a random number that depends neither on time \( t \) nor on \( \delta t \).

We have:

\[ \frac{1}{dz} P \left( z \leq Z \leq z + dz \mid \delta T = \delta t \right) = \nu^{(n)}(z|\delta t) = \mathcal{F}^{-1}_k \left[ \varphi^{(n)} \right] (z) = \mathcal{F}^{-1}_k \left[ \tilde{\varphi} \left( (\delta t)^{\frac{\beta}{\pi}} \left( \Theta k^{1/n} \sigma \right) \right) \right] (z) = \mathcal{F}^{-1}_k \left[ \tilde{\varphi} (\delta t)^\beta \left( \Theta k^{1/n} \sigma \right) \right] (z) = \mathcal{F}^{-1}_k \left[ 1 - (\delta t)^\beta (\tilde{\varphi} - 1) \Theta k^{1/n} \sigma + O ((\delta t)^{2\beta}) \right] (z) = \delta (z-z_0) - (\delta t)^\beta \mathcal{F}^{-1}_k \left[ (\tilde{\varphi} - 1) \Theta k^{1/n} \sigma \right] (z) + O ((\delta t)^{2\beta}) \]  

(2.21)

(2.22)

(2.23)

(2.24)

for some \( z_0 \in \mathbb{R} \). (2.24) is equivalent to (2.20) for \( z > z_0 \). The random variable \( \mathcal{R} \) conforms to a distribution whose characteristic function is \( (\tilde{\varphi} - 1) \Theta k^{1/n} \sigma \).

The dependence of \( \nu^{(n)}(k|\delta t) \) on time \( \delta t \) depends on the form of the matrix \( \Theta \) (stable index). We investigate that dependence in next sections.

Now we are going to investigate the functional form of the jump intensity \( \varpi(z) \) in the limit of large jumps \( z \). It is more convenient to analyze the Fourier transform
\( \tilde{\varpi}(\tilde{k}) \) for small values of \( \tilde{k} \). This function is a solution of equation (2.11) and is determined in a unique way via the stable index \( E \subseteq L(\mathbb{R}^d) \). Since, however, on the grounds of the Jordan decomposition theorem, every matrix \( E \) has a unique representation as a block diagonal matrix where every block is of the form:

\[
\begin{pmatrix}
a & 0 & 0 & \cdots & 0 \\
1 & a & 0 & \cdots & 0 \\
0 & 1 & a & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & a
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\mathcal{B} & 0 & 0 & \cdots & 0 \\
I & \mathcal{B} & 0 & \cdots & 0 \\
0 & I & \mathcal{B} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & I & \mathcal{B}
\end{pmatrix}
\]

(2.25)

where \( a \) is a real eigenvalue of \( E \) in the first case, and in the second case

\[
\mathcal{B} = \begin{pmatrix} a - b \\ b & a \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(2.26)

where \( a \pm ib \) is a complex conjugate pair of eigenvalues of \( E \) the set of all possible jump intensities \( \varpi \) is narrowed down to few classes of solutions only, each one corresponding to a particular Jordan decomposition of the matrix \( E \). In the following we investigate these classes of solutions as a function of \( E \). We proceed as follows:

1. Take a Jordan decomposition of \( E \subseteq L(\mathbb{R}^d) \),
2. Find the Fourier transform \( \tilde{\varpi}(\tilde{k}) \) of the jump intensity by solving the functional equation (2.11)
3. Find, from (2.15) the Fourier transform \( \tilde{\nu}(n)(k|\delta t) \) of the conditional \( n \)-marginal pdf (see Table 1) as a function \( \delta t \) and of \( \tilde{k} \) for small \( \delta t \) and small \( \tilde{k} \).

In following sections we seek for solutions \( \varpi \) of the equation (2.8) for different types of blocks in the Jordan representation. The names of the sections refer to the type of Jordan decompositions of \( E \).

2.2. The pure scaling

Let \( E = (d\mu)^{-1}I \) where \( I \) is a \( d \) dimensional identity matrix and \( \mu > 0 \) is a constant. Here \( TrE = \mu^{-1} \). The mapping:

\[
T^{\mathcal{E}} : \mathbb{R}^d \ni \tilde{k} \rightarrow t^{(d\mu)^{-1}} \tilde{k} \in \mathbb{R}^d
\]

(2.27)

changes the length of \( \tilde{k} \) only. We therefore assume the jump intensity \( \varpi(\tilde{k}) = \varpi(|\tilde{k}|) \) to depend on the length of the vector only. Substituting that assumption into (2.11) and taking \( t^{(d\mu)^{-1}} = |\tilde{k}|^{-1} \) we get:

\[
\tilde{\varpi}(|\tilde{k}|)|\tilde{k}|^{-d\mu} = \tilde{\varpi}(|\tilde{k}^{-1}|\tilde{k})
\]

(2.28)

and

\[
\tilde{\varpi}(\tilde{k}) = \tilde{\varpi}(1)|\tilde{k}|^{d\mu} = \exp \left( C|\tilde{k}|^{d\mu} \right)
\]

(2.29)
where $C := \log(\tilde{x}(1))$.

The conditional $n$-marginal pdf (see Table 1) from (2.15) reads:

$$
\tilde{\nu}^{(n)}(k|\delta t) = \int_{-\infty}^{\infty} d\ell \exp \left\{ C(\delta t)^{\beta} |k|^{\mu} \right\} K^{(n)}(k\sigma^n, \ell)
$$

(2.30)

where the kernel $K$ is defined in (4.11) and has a series expansion in $\ell$ given in (4.15). From (2.17) we get the $n$-marginal pdf for even $n$ and for small values of $k$. We have:

$$
\tilde{\nu}^{(n)}(k|\delta t) = \exp \left\{ C \left( \frac{C\pi}{2a_n} \right) (\delta t)^{\beta} k^{\mu} \right\}
$$

(2.31)

where $a_n := \cos(\pi(n-1)/2n)$ and the constant $C$ is defined in Appendix A.

### 2.3. The scaling & rotation

Take $d = 2$ and let:

$$
E = \begin{pmatrix} (2\mu)^{-1} & -b \\ b & (2\mu)^{-1} \end{pmatrix}
$$

(2.32)

The trace $\text{Tr}[E] = \mu^{-1}$. We denote by $O_\beta := \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$ a two dimensional rotation by an angle $\beta$. The mapping:

$$
t^{t^{(2\mu)^{-1}}} E O_{-\log(t)} \tilde{k} \in \mathbb{R}^2
$$

(2.33)

changes the length of $\tilde{k}$ by a multiplicative factor $t^{(2\mu)^{-1}}$ and rotates the vector by an angle $-\log(t)$.

We seek solutions of (2.11) in the form:

$$
\tilde{x}(\tilde{k}) = \exp \left( \tilde{v}(|\tilde{k}|, \theta) \right)
$$

(2.34)

where the function $\tilde{v}(\tilde{k}) = \tilde{v}(|\tilde{k}|, \theta)$ is assumed to depend on the length $|\tilde{k}|$ and on the angle $\theta := \varepsilon(\tilde{k}, (1, 0))$. Inserting (2.34) into (2.11) we obtain:

$$
t^{t^{(2\mu)^{-1}}} \tilde{v}(|\tilde{k}|, \theta) = \tilde{v}(t^{(2\mu)^{-1}}|\tilde{k}|, \theta - b\log(t))
$$

(2.35)

Now we assume that the function $\tilde{v}$ factorizes:

$$
\tilde{v}(\tilde{k}) = \tilde{v}_1(|\tilde{k}|) \Theta(\theta)
$$

(2.36)

Inserting (2.36) into (2.35) and taking $t^{(2\mu)^{-1}} = |\tilde{k}|^m$ with $m \in \mathbb{N}$ we get:

$$
|\tilde{k}|^{2\mu m} \tilde{v}_1(|\tilde{k}|) \Theta(\theta) = \tilde{v}_1(|\tilde{k}|^m) \Theta(\theta - b\log(t)) = \tilde{v}_1(|\tilde{k}|^m) \Theta(\theta - 2\mu b m \log(k))
$$

(2.37)

After separating variables and assuming that the jump intensity $\omega(\tilde{x})$ is even in $\tilde{x}$ we obtain $\tilde{v}$ in the following form:

$$
\tilde{v}(\tilde{k}) = \tilde{v}_1(1) k^{2\mu} \sum_{q=-\infty}^{\infty} \text{Re} \left[ \frac{\pi n_q}{\mu b m_q \log(k)} \right]
$$

(2.38)
where \((n_q, m_q) \in \mathbb{N}\) for \(q \in \mathbb{Z}\) are some numbers that depend on \(t\). We find only solutions for \(t = |\vec{k}|^{2m}\) for \(m \in \mathbb{N}\) and leave the generic case for future work. We check that (2.38) is indeed a solution of (2.35) for \(t = |\vec{k}|^{2\mu}\) under following assumptions:

\[
\forall q \in \mathbb{Z} \quad \exists p' \in \mathbb{N} \quad \frac{n_q}{m_q(m + 1)} = \frac{n_{q'}}{m_{q'}} \quad \text{and} \quad \frac{n_q}{m_q} m + 1 = 2N
\]

(2.39)

This leads to a following solution for the jump intensity in the Fourier domain:

\[
\tilde{\omega}(\vec{k}) = \exp \left[ Dk^{2\mu} \sum_{q=-\infty}^{\infty} \cos \left( \frac{2\pi (m + 1)|q|+1}{m} \frac{\theta}{\log(k)} \right) \right]
\]

(2.40)

where \(D := \tilde{\nu}_1(1)\). For small \(k\) the conditional \(n\)-marginal pdf reads:

\[
\tilde{\nu}(n|\delta t) = \exp \left[ D \left( \Theta \sigma k^{1/n} \right)^{2\mu} (\delta t)^{\beta} \sum_{q=-\infty}^{\infty} \cos \left( \frac{2\pi (m + 1)|q|+1}{m} \Omega(\delta t, \sigma, k) \right) \right]
\]

(2.41)

where

\[
\Omega(\delta t, \sigma, k) := \left( \frac{\theta - b\beta \log(\delta t)}{\log(\Theta \sigma k^{1/n}) + \frac{\beta}{2\mu} \log(\delta t)} \right)
\]

(2.42)

where the number \(\Theta\) is defined in (2.18). We summarize the results for the Fourier transforms of the jump intensity and of the conditional \(n\)-marginal pdf in both cases of pure scaling and of scaling \& rotation:

\[
\tilde{\omega}(\vec{k}) = \begin{cases} 
\exp \left( C|\vec{k}|^{d\mu} \right) & \text{pure scaling} \\
\exp \left[ Dk^{2\mu} \sum_{q=-\infty}^{\infty} \cos \left( \frac{2\pi (m + 1)|q|+1}{m} \frac{\theta}{\log(k)} \right) \right] & \text{scaling \& rotation}
\end{cases}
\]

(2.43)

\[
\tilde{\nu}(n|\delta t) = \begin{cases} 
\exp \left( C \left( \Theta \sigma k^{1/n} \right)^{d\mu} (\delta t)^{\beta} \right) & \text{pure scaling} \\
\exp \left[ D \left( \Theta \sigma k^{1/n} \right)^{2\mu} (\delta t)^{\beta} \sum_{q=-\infty}^{\infty} \cos \left( \frac{2\pi (m + 1)|q|+1}{m} \Omega(\delta t, \sigma, k) \right) \right] & \text{scaling \& rotation}
\end{cases}
\]

(2.44)

where \(\Theta\) is defined in (2.18) and the results for the conditional \(n\)-marginals hold for small \(k\) only.

2.4. The joint transition probability

The purpose of this section is to derive probability distributions of financial instruments like barrier European style options with multiple exercise times and American style options (exotic options) [15] as a function of the conditional 1-marginal pdf. We derive a joint probability distribution of log-prices \(X_t = \log(S_t)\) of the stock at certain times and of the maximal log-price in some time interval.
We fix \( l + 1 \) values of time \( t_0 < t_1 < t_2 < \ldots < t_l \) and we analyze the joint cumulative distribution function \( F_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_{t_1}}(x_1,x_2,\ldots,x_l;y) \) related to the log-prices \( X_{t_i} \) for \( i = 1,\ldots,l \) of the stock and to the maximal value of the log-price \( M_{t_1} := \sup_{0 \leq s \leq t} X_s \). We set \( X_{t_0} = 0 \), we define \( \delta t = (t_l - t_0)/N \) and we define integers \( 1 \leq i_1 < \ldots < i_l \leq N \) such that \( t_j - t_0 = i_j \delta t \). From the definition of the joint cumulative distribution function and from the Langevin equation (2.1) we have:

\[
F_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_{t_1}}(x_1,x_2,\ldots,x_l;y) := P(X_{t_1} < x_1,\ldots,X_{t_l} < x_l;M_{t_1} < y) = P \left( \bigcup_{j=1}^{l} X_{t_0+i_j \delta t} < x_j; \bigcup_{j=1}^{N} X_{t_0+j \delta t} < y \right) \tag{2.45}
\]

\[
= P \left( \bigcup_{j=1}^{l} \left( \sum_{p=1}^{i_j} \vec{\sigma} \cdot \vec{\delta} \hat{L}_{t_0+p \delta t} \right) < (x_j - t_j \mu); \bigcup_{j=1}^{N} \left( \sum_{p=1}^{j} \vec{\sigma} \cdot \vec{\delta} \hat{L}_{t_0+p \delta t} \right) < (y - (t_0 + j \delta t) \mu) \right) \tag{2.46}
\]

\[
= \int_{\mathbb{R}^N} 1_{\Delta(x_1,\ldots,x_l;y)} \prod_{j=1}^{N} \nu(\xi_j|\delta t) d\xi_j \tag{2.47}
\]

where \( \bigcup_{j=1}^{l} \) denotes a union of \( l \) conditions, the integration region

\[
\Delta(x_1,\ldots,x_l;y) := \begin{cases} \bigcup_{j=1}^{l} \left( \sum_{p=1}^{i_j} \xi_p \right) < (x_j - t_j \mu) \\ \bigcup_{j=1}^{N} \left( \sum_{p=1}^{j} \xi_p \right) < (y - (t_0 + j \delta t) \mu) \end{cases} \tag{2.49}
\]

is an infinite region in \( \mathbb{R}^N \) bounded by \( l + N \) hyper-planes of dimension \( N - 1 \) and \( \nu(\xi|t) := \nu^{(1)}(\xi|t)/\delta \hat{L} \) is the conditional probability density of the random variable \( \vec{\sigma} \cdot \vec{\delta} \hat{L} \). The Fourier transform of that probability density is given in (2.44). Now the procedure consists of following three steps.

(a) Derive the joint probability density function \( f_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_{t_1}}(x_1,x_2,\ldots,x_l;y) \) by differentiating the cumulative density function,

(b) Derive the joint characteristic function by taking Fourier transforms of all \( l + 1 \) random variables and expressing the result through Fourier transforms \( \tilde{\nu}^{(1)}(k|\delta t) := \mathcal{F}_z [\nu^{(1)}(z|\delta t)](k) \) of the pdf of the random variable \( \vec{\sigma} \cdot \vec{\delta} \hat{L} \)

(c) Derive the joint pdf in (a) by inverting expressions containing the Fourier transforms \( \tilde{\nu}^{(1)}(k|\delta t) \) in obtained (b).

We describe steps (a) and (b) in the following, leave step (c) for future work and give an example.

(a) \[
f_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_{t_1}}(x_1,x_2,\ldots,x_l;y) := \tag{2.50}
\]
\[
\frac{\partial^{l+1}}{\partial x_1 \ldots \partial x_l \partial y} \left[ F_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_t} (x_1, x_2, \ldots, x_l; y) \right] \tag{2.51}
\]
\[
\sum_{j=1}^{l} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \delta \left( x_j - t_j \mu - \sum_{p=1}^{i} \xi_p \right) \delta \left( y - (t_0 + i \delta t) \mu - \sum_{p=1}^{i} \xi_p \right) \prod_{j=1}^{N} \nu(\xi_j | \delta t) d\xi_j \tag{2.52}
\]

The sum on the right-hand side in (2.52) corresponds to intersections of all possible pairs of hyper-planes that bound the region (2.49).

(b) The joint characteristic function of the \(l+1\) random variables in question reads:
\[
\chi_{X_{t_1},X_{t_2},\ldots,X_{t_l};M_t}(k_1, k_2, \ldots, k_l; w) := \tag{2.53}
\]
\[
E \left[ \exp(\imath (k_1 X_{t_1} + \ldots + k_l X_{t_l} + w M_t)) \right] \tag{2.54}
\]
\[
\frac{1}{N} \sum_{j=1}^{l} \sum_{i=1}^{N} \left\{ \begin{array}{cl}
(\hat{\nu}(k_j + w|\delta t))^{i} (\hat{\nu}(k_j |\delta t))^{l-i} & \text{if } i \leq i_j \\
(\hat{\nu}(k_j + w|\delta t))^{i} (\hat{\nu}(w |\delta t))^{l-i} & \text{otherwise}
\end{array} \right\} \tag{2.55}
\]

where in (2.55) we replaced the delta functions in the integrand by exponentials \(\exp(\imath k_j x_j)\) and \(\exp(\imath w y)\) and we factored out expressions depending on \(\xi_p\).

**Example:**

(1) We take \(l = 1\) and we obtain the joint characteristic function of the log-price and of the maximum of the log-price. We have:
\[
E \left[ \exp(\imath (k X_t + w M_t)) \right] = \frac{1}{N} (\hat{\nu}(k|\delta t))^{i} \sum_{i=1}^{N} \left( \frac{\hat{\nu}(k + w|\delta t)}{\hat{\nu}(k|\delta t)} \right)^{i} \tag{2.56}
\]

The result (2.55) will be used in future work for pricing exotic options and for deriving variants of the Wiener-Hopf factorization formula [21] for the operator stable Lévy process in question.

### 2.5. The option price

An option on a financial asset is an agreement settled at time \(t\) to purchase (call) or to sell (put) the asset at some time \(T\) (maturity) in the future. For the sake of simplicity we consider European style options, i.e., such that can be only exercised at maturity (this means that boundary conditions are imposed on the option price at maturity \(t = T\)). Extending the analysis to American style options (to be exercised at any time) is possible and can be done by considering European style options with \(m\) different maturities [14] and performing the limit \(m \to \infty\).

In order to minimize the risk we diversify the portfolio by dividing the money available in to \(N_S\) stocks \(S_t\) and \(N_C\) options \(C(S_t; t)\). This means that the portfolio \(V(t)\) reads:
\[
V(t) = N_S S_t + N_C C(S_t; t) \tag{2.57}
\]

where the coefficients \(N_S\) and \(N_C\) are customarily chosen [15] as \(N_S = -\partial C/\partial S N_C\) and in the following we take \(N_C = 1\).
The investment strategy consists in replicating the portfolio, i.e., investing a copy of it into a safe central-bank account and let it earn a compound interest at a rate $r$ independent of $t$. We analyze the distribution of deviations

$$D_t := V(t + \delta t) - e^{r(\delta t)^\beta} V(t) \quad (2.58)$$

between the compound interest $e^{r(\delta t)^\beta} V(t) - V(t)$ earned in the safe central-bank account and changes $V(t + \delta t) - V(t)$ of the price of the portfolio and we check if a self-financing strategy exists, i.e., if it is possible to choose $C = C(S_t; t)$ subject to a condition $C_T = \max(S_T - K, 0)$ such that the average value of the portfolio deviation equals zero

$$E[D_t] = 0 \quad (2.59)$$

(the deviations have no drift) and the variance of this deviation $\text{Var}[D_t]$ is minimal. Since we assumed in (2.6) that the waiting times between stock price changes are power law distributed with index $\beta$ we need to take the infinitesimal interest earned in time period $[t, t + \delta t]$ equal to $r(\delta t)^\beta$ in order to ensure that the deviation variable $D_t$ exists in the limit $\delta t \to 0$.

Our approach differs here from the approach [20] in financial mathematics where one assumes in the first place that the deviations $D_t$ have no drift (there are no arbitrage opportunities on the market) and one imposes conditions (semi-martingale condition) on the high end of the jump intensity $\varpi(\vec{x}) \sim e^{-|\vec{x}|}$ in order to ensure this assumption. We cannot make such an assumption for operator stable Lévy processes. The processes that we are using are sums of a large number of independent, identically distributed random vectors where the distribution of the latter diminish according to a power law and not to an exponential in the high end. The processes are invariant under auto-convolutions (2.8) (self-similar) and in our opinion it is much more appealing from the physical point of view to use them in financial modeling rather than to use the abstract notion of semi-martingales from the metric space of Lévy processes with right continuous and left bounded paths endowed with a Skorokhod topology.

Equation (2.1) defines the price of a stock in terms of an exponential from a Lévy process. Thus a small change of the stock price reads:

$$S_{t + \delta t} - S_t = S_t \left( e^{\alpha(\delta t)^\beta} \exp \left( (\vec{\sigma} \cdot \delta \vec{L}_t) \right) - 1 \right) \quad (2.60)$$

$$= S_t \left( \alpha(\delta t)^\beta + \sum_{m=1}^{\infty} \frac{(\vec{\sigma} \cdot \delta \vec{L}_t)^m}{m!} \right) \quad (2.61)$$

where in (2.61) we expanded the exponential in a Taylor series and retained all terms of the expansion since all these terms are on the grounds of (2.20) proportional to $(\delta t)^\beta$ for large values of $\delta \vec{L}_t$.

We understand the price formation process as an accumulation of a large number of shocks that, once they exceed some threshold value, give rise to a change of the stock price (see discussion under Table 1)
From (2.60) and (2.61) we see that the price of the stock $S_t$ is not a martingale. It is also not possible to construct a martingale from the price of the stock by subtracting the sum of means $E[S_{t+\delta t} - S_t | S_t]$ over $t$ (the compensator) from $S_t$ as it is done in [18]. Indeed, the mean value of the left hand side of (2.60) conditioned on the history of the stock price up to time $t$ is a sum of all possible positive powers of the fluctuation term $\left(\vec{\sigma} \cdot \delta \vec{L}_t\right)^m$ and is, on the grounds of (2.19), infinite. It is therefore readily seen that, in the framework of our model, satisfying the condition (2.59) and thus solving the option pricing problem in its classical formulation is not straightforward. We will therefore analyze the probability distribution of the deviation variable $D_t$ and work out conditions under which it is possible to minimize the drift $E[D_t]$ of the portfolio (2.57).

We expand a change of the price of an option $C(S_t; t)$ on the stock in a Taylor series in powers of the change of the stock price:

$$C_{t+\delta t} - C_t = \frac{\partial C}{\partial t} (\delta t)^\beta + \frac{\partial C}{\partial S} (S_{t+\delta t} - S_t) + \sum_{n=2}^{\infty} \frac{\partial^n C}{\partial S^n} \frac{(S_{t+\delta t} - S_t)^n}{n!}$$

(2.62)

where the partial derivatives $\partial C/\partial t$ an $\partial^n C/\partial S^n$ are assumed to be independent of time $t$ and they will be determined later.

From (2.60) and (2.61) we compute the $n$th powers of the changes of the stock price and we retain only the terms that are proportional to $(\delta t)^\beta$. We have

$$(S_{t+\delta t} - S_t)^n = S_t^n \left(\exp \left(\vec{\sigma} \cdot \delta \vec{L}_t\right) - 1\right)^n$$

for $n \geq 2$  (2.63)

where we have neglected all terms $(\mu \delta t)^n \left(\vec{\sigma} \cdot \delta \vec{L}_t\right)^{n_2}$ on the grounds of (2.20). We insert (2.62) and (2.63) into (2.58) and we get the deviation random variable:

$$D_t = (\partial_t C + r S_t \partial_S C - r C_t) (\delta t)^\beta + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n C}{\partial S^n} S_t^n \left(\exp \left(\vec{\sigma} \cdot \delta \vec{L}_t\right) - 1\right)^n$$

(2.64)

Note that all terms on the right hand side of (2.64) are proportional to $(\delta t)^\beta$. Now we derive the Fourier transform of the probability density function of $D_t$ conditioned on the value of the price of the stock at time $t$ (conditional deviation characteristic function). We have:

$$\chi_{D_t | S_t}(k) := E\left[e^{ikD_t} | S_t\right] =$$

$$\exp(D_t (\delta t)^\beta) \int_{-\infty}^{\infty} d\lambda \phi^{(1)}(\lambda | \delta t) \mathcal{M}(k, \lambda)$$

(2.65)

(2.66)

(2.67)

(2.68)
where $D_t := (\partial_t C + r S_t \partial S C - r C_t)$ and

$$\mathcal{M}(k, \lambda) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \exp \left( ik \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n C}{\partial S^n} S_t^n (\exp(z) - 1)^n \right) \exp(-i\lambda z)$$ (2.69)

and in (2.68) we made use of (2.15) and of (4.10). In further calculations we make use of the following proposition.

**Proposition 2.3.** The integral

$$\mathcal{M}(k, \lambda) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \exp \{ ik P(e^z - 1) \} e^{-i\lambda z}$$ (2.70)

where $P(z)$ is a real function such that the only root of $P$ is $P(0) = 0$ and $P(\infty) = \infty$ tends, for small values of $k$, to a following delta function:

$$\lim_{k \to 0} \mathcal{M}(k, \lambda) = \delta \left( \lambda + \frac{2\pi \zeta_1}{|\log(P^{-1}(\frac{\lambda}{T}))|} \right)$$ (2.71)

where $\zeta_1 \in \mathbb{R}$ is some constant, $P^{-1}(z)$ is the inverse function to $P(z)$ in some vicinity of $z = 0$ and $\epsilon > 0$ is a positive real number. The proof is given in Appendix B.

Inserting (2.71) into (2.68) we get the conditional deviation characteristic function in the limit of small $k$. We have:

$$\chi_{D_t | S_t}(k) = \exp(D_t(\delta t)^3) \bar{\omega} \left( \frac{\delta t}{\bar{\sigma}} \frac{\bar{\sigma}^3}{|\bar{\sigma}|} \frac{-2\pi \zeta_1}{|\log(P^{-1}(\frac{\lambda}{T}))|} \right)$$ (2.72)

where

$$P(x) := \sum_{n=2}^{\infty} \frac{1}{n!} \partial^n C \partial S^n x^n$$ (2.73)

and we check by direct substitution that a class of functions:

$$P(x) := \sum_{n=2}^{\infty} \frac{1}{n!} \partial^n C \partial S^n x^n = \frac{\bar{\xi}}{(\log(x))^{\alpha}}$$ (2.74)

with $\alpha > 1$ and $\bar{\xi} \in \mathbb{R}$ ensures that the drift of deviations

$$E[D_t | S_t] = \left. \frac{\partial \chi_{D_t | S_t}(k)}{\partial k} \right|_{k=0} = 0$$ (2.75)

is zero. Therefore functions from the class (2.74) are suitable candidates for the solution of the option pricing problem in our model. In future work we will analyze the uniqueness of the solution of the option pricing problem and we will derive conditions for the solution to have:

$$C_T = C(S_T, T) = \max(S_T - K, 0)$$ (2.76)

a given initial value at maturity $T$. 
3. Conclusions

We have applied the technique of characteristic functions to the problem of pricing an option on a stock that is driven by operator stable fluctuations. We have shown that it is possible to construct an option on a stock as a function of the price of the stock in such a way that deviations of the portfolio are compensated by the compound interest earned on the portfolio. This requires the option to diminish very slowly, viz. as a positive power of an inverse logarithm, with the price of the stock. In future work we will price exotic options with different exercise times by computing the unconditional, joint characteristic function of the deviations of the portfolio related to values of stock price at these exercise times and by using initial conditions on the value of the portfolio at these exercise times. Testing of the theory in Monte Carlo simulations and fitting the theory to market data will also be pursued in future work.

4. Appendix A

We prove formula (2.15) for the conditional \( n \)-marginal probability function:

\[
\tilde{\nu}^{(n)}(k|\delta t) = \int_{-\infty}^{\infty} e^{ikz} \nu^{(n)}(z|\delta t) dz = (4.1)
\]

\[
\int_{\mathbb{R}^d} \exp ik \left( \bar{\sigma} \cdot \delta \tilde{l}_t \right)^n \omega_{\delta \tilde{l}_t|\delta T} \left( \delta \tilde{l}_t|\delta t \right) d(\delta \tilde{l}_t) = (4.2)
\]

\[
\int_{\mathbb{R}^d} \exp ik \left( \bar{\sigma} \cdot \delta \tilde{l}_t \right)^n \phi \left( \delta \tilde{l}_t|\delta t \right) d(\delta \tilde{l}_t) = (4.3)
\]

\[
(\delta t)^{-\beta} \int_{\mathbb{R}^d} \exp ik \left( \bar{\sigma} \cdot \delta \tilde{l}_t \right)^n \varpi(\delta t)^{-\beta} \delta \tilde{l}_t d(\delta \tilde{l}_t) = (4.4)
\]

\[
\frac{(\delta t)^{-\beta} \mathcal{E}_\mathbb{R}^d}{(2\pi)^d} \int_{\mathbb{R}^d} d\tilde{\chi}(\bar{\lambda}) \int_{\mathbb{R}^d} d(\delta \tilde{l}_t) \exp(\varpi(\delta t)^{-\beta} \delta \tilde{l}_t d(\delta \tilde{l}_t)) = (4.5)
\]

\[
\frac{(\delta t)^{-\beta} \mathcal{E}_\mathbb{R}^d}{(2\pi)^d} \int_{\mathbb{R}^d} d\tilde{\chi}(\bar{\lambda}) \int_{\mathbb{R}^d} d(\delta \tilde{l}_t) \exp(\varpi(\delta t)^{-\beta} \delta \tilde{l}_t d(\delta \tilde{l}_t)) = (4.6)
\]

\[
\frac{(\delta t)^{-\beta} \mathcal{E}_\mathbb{R}^d}{(2\pi)^d} \int_{\mathbb{R}^d} d\tilde{\chi}(\bar{\lambda}) \left( \prod_{i=2}^{d} \frac{2\pi \delta(t_i)}{2\pi} \right) \int_{-\infty}^{\infty} dw e^{t(w^n-w)} e^{t(1-t_i)w} = (4.7)
\]

\[
\int_{-\infty}^{\infty} dt_1 \tilde{\chi}(\varpi(\delta t)^{-\beta} \delta \tilde{l}_t) \frac{\bar{\sigma}}{|\bar{\sigma}|}(t_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{t(w^n-1-w)} = (4.8)
\]

\[
\int_{-\infty}^{\infty} dt_1 \tilde{\chi}(\varpi(\delta t)^{-\beta} \delta \tilde{l}_t) \frac{\bar{\sigma}}{|\bar{\sigma}|}(t_1) e^{t(1-t_i)w} = (4.9)
\]

In (4.2) we conditioned on the fluctuation \( \delta \tilde{l}_t \in \mathbb{R}^d \) and we integrated over \( z \in \mathbb{R} \).
In (4.3) we assumed that the time \( \delta t \) is small enough that at most one jump occurs during it and therefore we replaced the conditional fluctuation pdf by the conditional jump pdf and in (4.4) we used (2.7). In (4.5) we replaced the jump pdf \( \varpi \)
through an integral over $\tilde{X} \in \mathbb{R}^d$ from the Fourier transform $\hat{\omega}$ and we changed the order of integration integrating first over $\tilde{X}$ and then over $\delta \tilde{t}$. In (4.6) we defined $\tau := k\sigma^n$, $\tilde{t} := \hat{\omega}(\delta t)^{-\beta/\alpha} \lambda$ we introduced an orthogonal operator $\hat{\pi}$ such that $\mathbb{R} \bar{\sigma} = (|\bar{\sigma}|, 0, \ldots, 0)$ and we substituted for $\mathbb{R} \delta \tilde{t}$. In (4.7) we performed the integrations over $(\delta \tilde{t})_2, \ldots, (\delta \tilde{t})_d$ and in the remaining integral over $(\delta \tilde{t})_1$ we substituted for $w = (\delta \tilde{t})_1$. In (4.8) we eliminated the delta functions and transformed the d-dimensional integral over $\tilde{X}$ into a one-dimensional integral over $\tilde{t}_1$. In (4.9) we defined a function (kernel) as:

$$K^{(n)}(\tau, t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{i(tw^n - \tau w)} = (4.10)$$

and we evaluated it by rotating the range of integration in the complex plane by an angle $\pi/(2n)$ (Wick rotation). In doing that we applied the Cauchy theorem to a contour composed of an interval $w \in [-R, R]$ in the real axis, two arches $w = Re^{i\phi}$ for $\phi \in [0, \pi/(2n)]$ and $\phi \in [\pi, \pi + \pi/(2n)]$ respectively and a line $w = \xi e^{i\pi/(2n)}$ for $\xi \in [-R, R]$. The integrals over arches are bounded from above by $R \int_{0}^{\pi/(2n)} d\phi \exp[-\tau\xi^n \sin(n\phi) \pm iR \sin(\phi)]$ for the arch in the upper (+) and the lower (−) complex half-plane respectively and they tend to zero for $R \to \infty$ if $n$ is even and positive.

Now we compute a small $\tau$ expansion of the kernel $K^{(n)}(\tau, t)$ for $n$ even. We denote $a_n := \cos(\pi(n-1)/2n)$, we substitute $z := \tau \xi^n$ in (4.11) and get:

$$K^{(n)}(\tau, t) := \frac{e^{i\tau \pi}}{n \xi^{1/n}} \int_{0}^{\infty} dz z^{1/n-1} \exp(-z) \cos \left( \frac{a_n \xi^{1/n}}{\xi^{1/n}} z^{1/n} \right) = (4.12)$$

$$\frac{e^{i\tau \pi}}{n \xi^{1/n}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left( \frac{a_n \xi^{1/n}}{\xi^{1/n}} \right)^{2m} \int_{0}^{\xi} dz z^{2m+1} \exp(-z) = (4.13)$$

$$\frac{e^{i\tau \pi}}{n \xi^{1/n}} \left( \frac{\pi}{\xi} \right)^{1/n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (m\pi)^{2m+1} \frac{\Gamma(2m+1, \tau)}{\xi^{2m+1}} = (4.14)$$

$$\frac{e^{i\tau \pi}}{m \pi} \exp(-\tau) \left( \frac{\pi^{1/n+1}}{\xi^{1/n}} \right) \sum_{p=0}^{\infty} \frac{\alpha_{p+1} \tau^p}{\xi^{(p+1)/n}}$$

In (4.13) we defined $\tau := (m\pi/(2a_n))^{1/n} \tau$, we expanded the cosine into a Maclurin series and, owing to the fact that the period of the cosine is small for small $\tau$, the integrand oscillates rapidly and only the vicinity of the singularity of the integrand contributes meaningfully to the integral, we truncated the integration at $z = m\pi/2$ for some $m \in \mathbb{N}$. In (4.14) we expressed the result through the truncated Gamma.
function. In (4.15) we defined

\[ \mathcal{A}_p := \left( \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)!} \frac{\pi^{2m+1}}{m!} \frac{\Gamma\left(\frac{m}{n} + q\right)}{\Gamma\left(\frac{m}{n}\right)} \right) \]

(4.16)

we made use of a series expansion (4.17) of the truncated Gamma function.

\[ \Gamma(\alpha, x) = \frac{x^{\alpha-1}}{\alpha} \sum_{p=0}^{\infty} \frac{\Gamma(p)}{p!} \Gamma(\alpha + p) \]

(4.17)

We note that for \( x \) small enough the kernel satisfies:

\[ |K^{(n)}(t, l)| := \mathcal{A}_1 e^{-q^{1+1/n}} \frac{1}{\pi^{1/n}} \sum_{m} \frac{1}{2\alpha_m} \]

(4.18)

for \( x = q \leftrightarrow l_q = (m\pi)/(2\alpha_m^{1/n}) \cdot (t)^{1/n} \) and for \( q^{-1} = q(m)^{-1} \in \mathbb{N} \) and \( q \leq 1 \) and thus we have

\[ \lim_{q \to 0} K^{(n)}(t, l) \simeq \delta \left( 1 - \mathcal{C} \frac{\pi}{2\alpha_m} (t)^{1/n} \right) \]

(4.19)

where \( \mathcal{C} := (m)/(q(m)^{1/n}) \) is some number. It is readily seen from (4.10) and (4.11) that \( \int_{-\infty}^{\infty} dK^{(n)}(t, l) = 1 \) and the same holds for the right hand side of (4.19). Inserting (4.19) into (4.9) we obtain (2.17) \textit{q.e.d.}

5. Appendix B

We calculate the integral (2.70) for small values of \( k \). We have:

\[ \mathcal{H}(k, \lambda) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \exp \{ i k P(e^{z} - 1) \} e^{-i\lambda z} = \]

(5.1)

\[ \frac{1}{2\pi} \sum_{\alpha_i} \int_{r_i}^{r_{i+1}} \frac{dw}{k} \left( \frac{P(-1,i)(w)}{1 + P(-1,i)(w)} \right)^{\lambda} e^{i w} \exp \left( -i\lambda \log \left( 1 + P(-1,i)(w) \right) \right) \]

(5.2)

\[ = \frac{1}{2\pi} \sum_{\alpha_i} \int_{r_i}^{r_{i+1}} \frac{dw}{k} \left( \frac{P(-1,i)(w)}{P(-1,i)(w) + 1} \right)^{\lambda} e^{-w} \]

(5.3)

\[ = \delta + \theta \int_{0}^{r} \frac{dw}{k} \left( \frac{P(-1,i)(w)}{P(-1,i)(w) + 1} \right)^{\lambda} \]

(5.4)

\[ = \delta + \theta \int_{0}^{r} \frac{dw}{k} \left( \frac{P(-1,i)(w)}{P(-1,i)(w) + 1} \right)^{\lambda} \]

(5.5)

\[ = \delta + \frac{\theta}{\lambda} \int_{0}^{r(1,i)(w)} \frac{dz}{z^{1+\lambda}} \]

(5.6)

\[ = \delta + \frac{\theta}{\lambda} \left( \frac{P(1,i)(w)}{P(1,i)(w) + 1} \right)^{-\lambda} \]

(5.7)

\[ = \delta + \frac{\theta}{\lambda} \exp(-\lambda \log \left( P(-1,i)(w) \right)) \]

(5.8)
In (5.2) we substituted \( w = kP(e^z - 1) \), we denoted by \( z_i(y) := P^{(-1)}(y) \) the \( i \)th solution (inverse function) of the equation \( P(e^z - 1) = y \) and we divided the integration range over \( z \) into intervals \([u_i, v_{i+1})\) in which the function \( P(e^z - 1) \) is monotone and thus invertible. We also denoted \( r_i := kP(e^{u_i} - 1) \). In (5.3) we used the fact that \( P(\infty) = \infty \) and thus \( P^{(-1)}(\infty) = \infty \) and we approximated the integrand by its large \( w/k \) values.

In (5.4) we rotated the integration line by \( \pi/2 \) in the anti-clockwise direction making use of the fact that the integral over a quarter of a circle \( w = Re^{i\phi} \) for \( \phi \in [0, \pi/2] \) is bounded from the above by \( \int_0^{\pi/2} d\phi e^{-R\sin(\phi)} \) and it vanishes for \( R \to \infty \). In (5.5) we used the fact that the only root of \( P^{(-1)}(z) \) is at \( z = 0 \) and thus we separated the integral into two parts the first one equal to \( R \) over the integration range with all singularities excluded \( U = \bigcup [r_i, r_{i+1}] \setminus [0, \epsilon] \) and the second one over \([0, \epsilon] \). Note that since \( \lim_{k \to 0} |r_{i+1} - r_i| = 0 \) and the integrand has no singularities in \( U \) the number \( \mathbb{F} \) is finite when \( k \to 0 \). The number \( \theta \) counts intervals \([r_i, r_{i+1}] \) such that \( 0 \in [r_i, r_{i+1}] \). In (5.5) we substituted \( z = P^{(-1),i}(\mathbb{F}) \) and in (5.5) we performed the integral assuming that \( \lambda < 0 \). Now we take \( m \in \mathbb{N} \) small \( k \) and \( \lambda_m := -(2\pi m)/|\log(P^{(-1),i}(\mathbb{F}))| \) and we evaluate the integral (5.8) for this value.

\[
\mathbb{M}(k, \lambda_m) := \mathbb{F} + \frac{\exp(-i2\pi m)}{2\pi m |\log(P^{(-1),i}(\mathbb{F}))|} = \mathbb{F} + \theta \frac{|\log(P^{(-1),i}(\mathbb{F}))|}{2\pi m}
\]

(5.9)

Since from (5.1) we have \( \int_0^\infty d\lambda \mathbb{M}(k, \lambda) = 1 \) we conclude that

\[
\lim_{k \to 0} \mathbb{M}(k, \lambda) = \delta \left( \lambda + \frac{2\pi \mathcal{C}_1}{|\log(P^{(-1),i}(\mathbb{F}))|} \right)
\]

(5.10)

where \( \mathcal{C}_1 \) is some constant \( q.e.d. \).

References

[1] Gopikrishnan P. et al., Inverse cubic law for the distribution of stock price variations, Eur. Phys. J. B 3, 139–140 (1998)
[2] Bachelier L., Theory of Speculation, Ann. Sci. Ecole Norm. Sup. 3, 21 (1900); reprint from P.H. Cootner (editor), The random character of stock prices, second edition (MIT Press Cambridge, 1969)
[3] Meerschaert M M, Scheffler H P, Limit Distributions for Sums of Independent Random Vectors: Heavy tails in Theory and Practice John Wiley & Sons, Inc. 2001
[4] Fama E.F., Efficient Capital Markets: A Review of Theory and Empirical Work, J. of Finance 25, 383–417 (1970)
[5] Mandelbrot B., The variations of certain speculative prices, J. of Business 36 392–417 (1963)
[6] Repetowicz P, Richmond P, Modeling share price evolution as a continuous time random walk (CTRW) with non-independent price changes and waiting times, Physica A, In Press, Corrected Proof, Available online 1 September 2004
[7] Masoliver J, The CTRW in finance: Direct and inverse problems, preprint cond-mat/0308017; Masoliver J, Montero M, Continuous Time Random Walk model for financial distributions, Phys Rev. E 67, 021112 (2003)
[8] Becker-Kern P, Meerschaert M M, Scheffler H P, Limit Theorems for Coupled Continuous Time Random Walks, The Annals of Probability 32, No. 1B, 730–756 (2004)
[9] Meerschaert M M, Scheffler H P, Limit Theorems for Continuous Time Random Walks with Slowly varying waiting times, preprint
[10] Meerschaert M M, Benson D A, Governing equations and solutions of anomalous random walk limits, Phys. Rev. E 66, 060102(R) (2002)
[11] Kutner R, Hierarchical spatio-temporal coupling in fractional wanderings. (I) Continuous-time Weierstrass flights, Physica A 264, 84–106, 1999
[12] Kutner R, Extreme events as foundations of lévy walks with varying velocity, Chemical Physics 284, 481–505 (2002)
[13] Raberto M et al., Waiting times and returns in high-frequency financial data: an empirical study, Physica A 314, 749–755 (2002)
[14] Dash Jan W, Path Integrals and Options - I, preprint available on-line at http://www.physik.fu-berlin.de/~kleinert/b3/papers/ by courtesy of H. Kleinert
[15] Musiela M, Rutkowski M, Applications of Mathematics: Stochastic Modelling and applied probability: Martingale Methods in Financial Modelling, Springer-Verlag Berlin Heidelberg (1997)
[16] Karatsas I, Shreve S, Brownian Motion and Stochastic Calculus, 2nd ed. New York: Springer-Verlag. 1997; WWW: http://mathworld.wolfram.com/ItosLemma.html
[17] Eberlein Raible, Term Structure Models Driven by General Levy Processes, Mathematical Finance, 9, iss. 1, pp. 31-53 (1999)
[18] Eberlein E, Oezkan F, The Default-able Levy Term Structure: Rating and Restructuring private communication with E. Eberlein
[19] Kleinert H, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, World Scientific Publishing Co., Singapore 3rd edition (alpha version), pp. 1375-1380, (2002)
[20] Rama C, Integro-differential equations and numerical methods, in:Financial Modelling with Jump Processes Chapman & Hall, CRC Financial Mathematics Series, pages: 381–430
[21] Sato K-I, Wiener-Hopf factorizations, in: Levy Processes and Infinitely Divisible Distributions Cambridge University Press 1999
[22] Samko S G, Kilbas A A, Marichev O I, Fractional Integrals and Derivatives Theory and Applications Gordon and Breach Science Publishers S.A. 1993
[23] Dzherbashyan M M, Nersesyan A B, The criterion of the expansion of the functions to the Dirichlet series, Izv. Akad. Nauk Armyan. SSR Ser. Fiz.-Mat. Nauk, 11, no 5, 85–108