Spherical varieties and non-ordinary families of cohomology classes

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Abstract

We give a construction of non-ordinary \( p \)-adic families of cohomology classes. We give criteria for \( \acute{e} \text{tale} \) classes to be mapped into into Galois cohomology and show how this can be used to interpolate the Lemma-Flach Euler system in Coleman families.

1 Introduction

Let \( p \) be a prime. In this paper we give a construction of non-ordinary \( p \)-adic families of cohomology classes interpolating classes constructed by pushing forward classes from a ‘small’ reductive group to a larger one, such as those constructed in [Loe21]. Such cohomology classes arise naturally in the study of Iwasawa theory as Euler systems and in the construction of \( p \)-adic \( L \)-functions. Our construction provides a vast generalisation of previous works such as [LZ16] and [BSV20] and acts a sequel to [LRZ21] in which ordinary families of cohomology classes were considered.

The non-ordinary case requires very different methods than the ordinary case. We use a method inspired by work of Greenberg-Seveso [GS20] and Bertolini-Seveso-Venerucci [BSV20] on balanced diagonal classes. Our method utilises modules of analytic functions in place of the usual modules of analytic distributions to interpolate branching maps between algebraic representations.

Suppose we have an inclusion of reductive groups \( H \hookrightarrow G \) satisfying the conditions of [LRZ21]. Let \( \mu, \lambda \) be dominant weights of \( H, G \) and let \( V_H^{\mu}, V_G^{\lambda} \) be the respective irreducible representations. For a wide-open disc \( U \subset W_G \) containing \( \lambda \), where \( W_G \) is a suitable weight space, we define modules of locally analytic functions \( A^{\lambda}_G, A^{\mu}_U \) which are modules for parahoric subgroups \( J_H, J_G \). There is a natural inclusion

\[ V_G^{\lambda} \hookrightarrow A^{\lambda}_G \]

and a specialisation map

\[ A^{\mu}_U \rightarrow A^{\lambda}_G. \]

Let \( Y_H(J_H), Y_G(J_G) \) denote the parahoric level locally symmetric spaces (of some suitably small tame level) associated to \( H, G \). Given an \( H \)-inclusion \( V_H^{\mu} \rightarrow V_G^{\lambda} \), Loeffler [Loe21] constructs a map

\[ H^i(Y_H(J_H), V_H^{\mu}) \rightarrow H^i(Y_G(J_G), V_G^{\lambda}) \]

for \( i \geq 0 \) as a key ingredient in his construction of norm-compatible classes. We can lift this map to a map

\[ H^i(Y_H(J_H), V_H^{\mu}) \rightarrow H^i(Y_G(J_G), A^{\lambda}_G). \] (1)

In Section 2.3 we construct a ‘big branching map’ map in Betti cohomology

\[ H^i(Y_H(J_H), D_U^H) \rightarrow H^{i+c}(Y_G(J_G), A^{\lambda}_G), \] (2)

where \( D_U^H \) is a module of distributions over \( U \) specialising to \( V_H^{\mu} \) at \( \lambda \). Our main result is the following:
Theorem 1.0.1. There is a commutative diagram of Betti cohomology groups

\[
\begin{array}{ccc}
H^i(Y_H(J_H), D_H^i) & \longrightarrow & H^{i+c}(Y_G(J_G), A_{\ell}^G) \\
\downarrow & & \downarrow \\
H^i(Y_H(J_H), V_{\ell}^i) & \longrightarrow & H^{i+c}(Y_G(J_G), A_{\lambda}^G),
\end{array}
\]

where the top map is (3), the bottom is (1) and \( c = \dim_{\bar{\mathbb{Q}}} Y_G - \dim_{\mathbb{Q}} Y_H - \text{rk}_{R}(\frac{Z_i}{\mu_i}) \).

When \( Y_H, Y_G \) admit compatible Shimura data we construct an analogous map in étale cohomology. This requires a construction of new objects: profinite modules \( A^w_U, A^w_{\lambda} \) which we dub ‘Iwasawa analytic functions’. Roughly speaking, these consist of functions on \( J_G \) which extend to a wide-open rigid analytic neighbourhood.

Theorem 1.0.2. There is a commutative diagram of étale cohomology groups

\[
\begin{array}{ccc}
H^i_{\text{ét}}(Y_H(J_H)_{\overline{\mathbb{Q}}}, D_{\ell}^i(j)) & \longrightarrow & H^{i+2c}_{\text{ét}}(Y_G(J_G)_{\overline{\mathbb{Q}}}, A^w_{\ell}(j+e)) \\
\downarrow & & \downarrow \\
H^i_{\text{ét}}(Y_H(J_H)_{\overline{\mathbb{Q}}}, V_{\ell}^i(j)) & \longrightarrow & H^{i+2c}_{\text{ét}}(Y_G(J_G)_{\overline{\mathbb{Q}}}, A^w_{\lambda}(j+e))
\end{array}
\]

where \( j \in \mathbb{Z} \) and \( c = 2e \).

Let \( \Sigma \) be a finite set of primes containing \( p \) and the primes at which the level of \( G \) ramifies and suppose \( Y_H(J_H), Y_G(J_G) \) admit \( \Sigma \)-integral models \( Y_G(J_H)_{\Sigma}, Y_H(J_H)_{\Sigma} \). The diagram of Theorem 1.0.2 also holds for étale cohomology of integral models. Let \( q = \dim_{\overline{\mathbb{Q}}} Y_H(J_G) \). In Section 4 we show how, under certain hypotheses (small slope, vanishing outside the middle degree after localisation at a maximal ideal of the Hecke algebra) we can push-forward classes in \( H^{q+1}(Y_G(J_G)_{\Sigma}, A^w_{\ell}) \) to obtain classes in Galois cohomology.

Theorem 1.0.3. Suppose \( m \) is a ‘nice’ (in a precise sense) maximal ideal of the unramified Hecke algebra acting on \( H^\bullet(Y_G(J_G)_{\overline{\mathbb{Q}}}, V_{\lambda}^\Sigma) \). Then there is an affinoid \( V \subset W_G \) containing \( \lambda \) and a map

\[ A_{\ell}^U : H^{q+1}(Y_G(J_G)_{\Sigma}, A^w_{\ell}) \to H^1(O_E,\Sigma, W_U) \]

where \( E \) is the reflex field of \( Y_G \), \( O_E,\Sigma \) is its ring of \( \Sigma \)-integers, and \( W_U \) is a family of Galois representations over \( U \) specialising to \( H^q(Y_G(J_G)_{\overline{\mathbb{Q}}}, V_{\lambda}^\Sigma) \) at \( \lambda \).

The proof of this theorem uses in an essential way the profiniteness of the modules \( A^w_U \). In Section 6 we show how this construction can be used to construct a Galois cohomology class \( z_U \in H^1(\mathbb{Q}, W_U) \) interpolating the Lemma-Flach classes constructed in [LSZ21] (or, more precisely, their Iwahori-variants constructed in [LRZ21], Section 7). We expect this class to play a role in proving new cases of Bloch-Kato and Birch–Swinnerton-Dyer, much as their ordinary counterparts have in [LZ21].

2 Branching laws in non-ordinary families

2.1 Setup

We give a general construction of branching laws varying in \( p \)-adic families.

Fix a prime \( p \). We recall the following setting from [LRZ21]:

- Let \( G \) be a connected reductive group over \( \mathbb{Q} \) satisfying Milne’s axiom (SV5), that is, the centre contains no \( \mathbb{R} \)-split torus which is not \( \mathbb{Q} \)-split \( \mathbb{R} \).

\footnote{As noted in [LRZ21] we can relax this condition by considering, for example, representations of the adjoint group \( G^{\text{ad}} = G/\mathbb{Z}G \).}
• \(G\) is a reductive group scheme over \(\mathbb{Z}_p\) whose base-extension to \(\mathbb{Q}_p\) coincides with that of \(G\).

• \(Q_G\) is a choice parabolic subgroup of \(G\) and \(\bar{Q}_G\) is the opposite parabolic, so that \(L_G = Q_G \cap \bar{Q}_G\) is a proreductive group scheme and \(\mathbf{C}_F\) is a collision of \(\mathbf{C}_F\) and \(\mathbf{C}_S\).

• As in \([LRZ21]\) we fix a neat prime-to-2 level \(\mathcal{O}\) and whose \(G\)-level \(\mathcal{O}\) is given by the Frobenius element \(f\), and whose \(G\)-level \(\mathcal{O}\) is given by the Frobenius element \(f\).

• \(\mathcal{L}_G\) denote the locally symmetric space \(\mathcal{G}\), where \(\mathcal{L}_G\) is the unipotent radical of \(G\) and \(\mathcal{H}_G\) is its opposite.

• Let \(S_G\) denote the torus \(S_A\) with \(S_{\mathcal{O}}\) and \(\bar{S}_G\) denote the maximal torus quotient of \(G\).

• Let \(J_G \subset G(\mathbb{Z}_p)\) be the parahoric subgroup associated to \(Q_G\). This group admits an Iwahori decomposition

\[J_G = (J_G \cap \bar{N}_G(\mathbb{Z}_p)) \times L_G(\mathbb{Z}_p) \times N_G(\mathbb{Z}_p)\]

• Let \(\mathcal{A}_G\) be the maximal \(\mathcal{Q}_p\)-split torus in the centre of \(L\).

We choose a subtorus \(S^0_\mathcal{G} \subset S_G\) and let \(L^0_\mathcal{G}\) and \(Q^0_\mathcal{G}\) be its preimages under the quotient maps \(L_G \to S_G\) and \(Q_G \to S_G\) respectively. Let \(\Phi_\mathcal{G}\) denote the roots of \(G\), \(\Delta_\mathcal{G}\) the simple roots and \(\Phi^+_\mathcal{G}\) the positive roots. Set \(\delta_G = \dim N_G\).

### 2.1.1 Algebraic representations

Let \(K/\mathbb{Q}_p\) be a finite unramified extension over which \(G\) splits and let \(\mathcal{O}\) be the ring of integers of \(K\). For \(\lambda \in X^*_G(S_G)\) we define

\[V_\lambda = \{ f \in K[G] : f(n\ell g) = \lambda(\ell) f(g) \ \forall n \in \bar{N}_G, \ell \in L_G, g \in G \}\]

with \(g\) acting by right-translation. By the Borel-Weil-Bott theorem this is the irreducible \(G\)-representation of highest weight \(\lambda\) with respect to a Borel subgroup \(B_G \subset Q_G\). We let \(f_\lambda\) denote the unique choice of highest weight vector satisfying \(f(n\ell n) = \lambda(\ell)\) for \(n\ell n \in U_{\text{Bru}}^G\). We also write

\[P^G_\lambda = \{ f \in K[G] : f(n\ell g) = \lambda^{-1}(\ell) f(g) \ \forall n \in N_G, \ell \in L_G, g \in G \}\]

so that \((P^G_\lambda)^\vee \cong V_\lambda\) as \(G\)-representations. A canonical choice of highest weight vector of \((P^G_\lambda)^\vee\) with respect to \(B_G\) is given by the functional \(\delta_1 : f \mapsto f(1)\).

### 2.1.2 Integral lattices

**Definition 2.1.1.** An admissible lattice in \(V_\lambda\) is an \(\mathcal{O}\)-lattice \(\mathcal{L} \subset V_\lambda\) invariant under \(G/\mathcal{O}\) and whose intersection with the highest weight subspace is \(\mathcal{O} \cdot f_\lambda\).

We refer to \([LRZ21]\) Section 2.3 for properties of admissible lattices. Let \(V_{\lambda,\mathcal{O}}\) denote the maximal admissible lattice in \(V_{\lambda}\), given in the Borel-Weil-Bott presentation as

\[V_{\lambda,\mathcal{O}} = \{ f \in \mathcal{O}[G] : f(n\ell g) = \lambda(\ell) f(g) \ \forall n \in \bar{N}_G, \ell \in L_G, g \in G \}\]

Similarly defining \(P^G_{\lambda,\mathcal{O}}\), then \((P^G_{\lambda,\mathcal{O}})^\vee\) is isomorphic to the minimal admissible lattice in \(V_{\lambda}\).

### 2.1.3 Cohomology of locally symmetric spaces

As in \([LRZ21]\) we fix a neat prime-to-\(p\) level group \(K_p\) for an open compact subgroup \(U \subset G(\mathbb{Z}_p)\) let \(Y_G(U)\) denote the locally symmetric space of level \(K_pU\). We let \(H^i(Y_G(U), \mathcal{F})\) refer to one of the following cohomology theories on \(Y_G\):

• Betti cohomology of the locally symmetric space \(Y_G(U)\) viewed as a real manifold, with coefficients in a locally constant sheaf \(\mathcal{F}\).

Suppose now that \(Y_G\) admits a Shimura datum with reflex field \(E\).
• étale cohomology of $\bar{Y}_G(U)_{\bar{G}}$ with coefficients in a lisse étale sheaf $\mathcal{F}$.

• Let $\Sigma$ be a sufficiently large finite set of primes containing those dividing $p$. We consider the étale cohomology of an integral model $Y_G(U)_{\Sigma}$ of $Y_G(U)$ defined over $\mathcal{O}_E[\Sigma^{-1}]$ with coefficients in $\mathcal{F}$. Here we assume the Shimura datum is of Hodge type.

2.1.4 Branching laws for algebraic representations

We now work in the situation of [Loc21] and [LRZ21] and consider an embedding $\mathcal{H} \to \mathcal{G}$ of reductive $\mathbb{Q}$-groups, extending to an embedding $H \to G$ of reductive group schemes over $\mathbb{Z}_p$. Denote by $\mathcal{F} := \bar{Y}_G \setminus G$ the flag variety associated to the parabolic $\bar{Q}_G$. We assume that there is $u \in \mathcal{F}(\mathbb{Z}_p)$ satisfying:

(A) The $Q^0_H$-orbit of $u$ is Zariski open in $\mathcal{F}$,

(B) The image of $\bar{Q}_G \cap uQ^0_Hu^{-1}$ under the projection $\bar{Q}_G \to S_G$ is contained in $S^0_G$.

Under the above assumptions, the space $U_{\text{sph}} = \bar{Q}_GuQ^0_H$ is a Zariski open subset of $G$. We refer to it (and its image in the flag variety) as the spherical cell.

Remark 2.1.2. Note that since the flag variety is connected, $U^\text{Br}_G \cap U_{\text{sph}} \neq \emptyset$ so we can always take $u \in U^\text{Br}_G(\mathbb{Z}_p)$ and in particular we can take $u \in N_G(\mathbb{Z}_p)$, which we assume from now on.

By [LRZ21] Proposition 3.2.1, for $\lambda \in X^*_H(S_G)$ the space $V^Q_H(\lambda)$ has dimension $\leq 1$.

Definition 2.1.3. We call weights satisfying $\dim V^Q_H(\lambda) = 1$ $Q^0_H$-admissible weights and denote the cone of such weights by $X^*_H(S_G)^{Q^0_H}$.

For $\lambda \in X^*_H(S_G)^{Q^0_H}$ the space $V^Q_H(\lambda)$ is spanned by the polynomial function $f^\text{sph}_\lambda \in K[G]$ uniquely defined by setting $f^\text{sph}_\lambda(u) = 1$. The $H$-orbit of $f^\text{sph}_\lambda$ generates an irreducible representation of $H$ of some highest weight $\mu \in X^*_H(S_H)$ with respect to a Borel $B_H \subset Q_H$. By [LRZ21] Proposition 3.2.6 $f^\text{sph}_\lambda \in V^\mu_{\lambda, \mathcal{O}}$.

Definition 2.1.4. Define the twig $\phi \in \mathcal{O}(G \times H)$ by

$$\phi(g, h) = f^\text{sph}_\lambda(gh^{-1}).$$

Lemma 2.1.5. The twig $\phi$ has the following properties:

• For fixed $g \in G$, the function $h \mapsto \phi(g, h)$ is in $\mathcal{P}^H_{\mu, \mathcal{O}}$.

• For fixed $h \in H$, the function $g \mapsto \phi(g, h)$ is in $V^\mu_{\lambda, \mathcal{O}}$.

Proof. The function $\phi$ is clearly algebraic in both variables. Fix $g \in G$, then for $\ell_h \in L_H$

$$\phi(g, \ell_h h) = f^\text{sph}_\lambda(gh^{-1}\ell_h^{-1}) = \mu(\ell_h)^{-1}\phi(g, h),$$

Now fix $h \in H$. Let $\ell_g \in L_G$, then

$$\phi(\ell_g g, h) = f^\text{sph}_\lambda(\ell_g gh^{-1}) = \lambda(\ell_g)\phi(g, h),$$

whence we are done.

Proposition 2.1.6. Let $\lambda \in X^*_H(S_G)^{Q^0_H}$ and let $\mu \in X^*_H(S_H)$ be the associated weight of $H$. Then there is a unique $H$-equivariant map

$$\text{br}^\lambda_\mu : (\mathcal{P}^H_{\mu, \mathcal{O}})^\vee \to V^\mu_{\lambda, \mathcal{O}}$$

sending $\delta_1$ to $f^\text{sph}_\lambda$. Under this map a functional $\varepsilon \in (\mathcal{P}^H_{\mu, \mathcal{O}})^\vee$ is mapped to the function on $G_{/\mathcal{O}}$ given by

$$g \mapsto \langle \varepsilon, \phi(g, -) \rangle.$$
2.1.5 Weight spaces

Write $G = L_G(Z_p) / L_G^0(Z_p) \subset (S_G / S_G^0)(Z_p)$. The torus $G$ splits into a direct product $G = G^\text{tor} \times G_{1}$ with the logarithm map identifying $G_{1}$ for any ideal.

**Definition 2.1.7.** Write $W_G$ for the rigid analytic space over $\mathbb{Q}_p$ parameterising continuous characters of $G$. The space $W_G$ admits a formal model $\mathfrak{M}_G := \text{Spf} \Lambda \text{(G)}$ over $\mathbb{Z}_p$, where $\Lambda(G)$ is the Iwasawa algebra associated to $G$.

**Definition 2.1.8.** For $i = 1, \ldots, n$ let $s_i \in G_{1}$ be a $\mathbb{Z}_p$-basis. For an integer $m \geq 0$ we denote by $W_m \subset W_G$ the wide-open subspace consisting of weights $\lambda$ satisfying

$$v_p(\lambda(s_i) - 1) > \frac{1}{p^m(p-1)}$$

for all $i$.

Fix a finite extension $E / \mathbb{Q}_p$ and for some $m \geq 0$ let $U \subset W_m$ be a wide-open disc defined over $E$. Let $\Lambda_G(U) \cong \mathcal{O}_E[t_1, \ldots, t_n]$ denote the $\mathcal{O}_E$-algebra of bounded-by-one rigid functions on $U$ with $m$ its maximal ideal.

Let

$$S^\text{stab} = \frac{Q_H \cap u^{-1} Q_G u}{Q_H^0 \cap u^{-1} Q_G u},$$

the quotient of the stabilisers of $[u] \in \mathcal{F}(Z_p)$ in $Q_H$ and $Q_H^0$. As remarked in [LRZ21, Remark 3.2.4] the natural map $S^\text{stab} \rightarrow Q_H / Q_H^0$ is an isomorphism. It turns out more is true:

**Lemma 2.1.9.** The natural inclusion

$$(Q_H \cap u^{-1} Q_G u)(Z_p) \rightarrow Q_H(Z_p)$$

induces an isomorphism

$$\frac{(Q_H \cap u^{-1} Q_G u)(Z_p)}{(Q_H^0 \cap u^{-1} Q_G u)(Z_p)} \cong Q_H(Z_p) / Q_H^0(Z_p).$$

**Proof.** For $g \in Q_H(Z_p)$, write $U_g := U_{\text{sph}} g$. Each $U_g$ is Zariski open and thus $U_g \cap U_1 \neq \emptyset$ for all $g$. Thus for any $g \in Q_H(Z_p)$ there is $r_1, r_g \in Q_G(Z_p), q_1, q_g \in Q_H^0(Z_p)$ such that

$$r_1 u q_1 = r_g u g q_g$$

and so $g g^{-1} = u^{-1} r_g^{-1} r_1 u \in (u^{-1} Q_G u \cap Q_H)(Z_p)$. In particular, for any $g \in Q_H(Z_p)$ there is $g \in Q_H^0(Z_p)$ such that $g q \in (u^{-1} Q_G u)(Z_p)$, whence the claim follows.

**Definition 2.1.10.** We define a group scheme homomorphism

$$\omega : S_H / S_H^0 \rightarrow Q_H / Q_H^0 \rightarrow S_H^\text{stab} \rightarrow S_G / S_G^0$$

where the last map is given by conjugation by $u$ and projection to $S_G$. This is well defined by assumption (B).

Given a $Q_H^0$-admissible weight $\lambda$ the associated $H$-weight $\mu$ is given by $\lambda \circ \omega$. By Lemma 2.1.9 $\omega$ induces a homomorphism

$$\times \rightarrow G$$

of profinite abelian groups.
Lemma 2.1.11. There is a morphism of affine formal schemes:
\[ \Omega : \mathcal{W}_G \to \mathcal{W}_H \]
given on points by \( \Omega(\lambda)(s) = \lambda(\omega(s)) \).

Proof. The map
\[ \Lambda(\mathcal{S}_H) \to \Lambda(\mathcal{S}_G) \]
given by linearly extending \([s] \mapsto [\omega(s)]\) is a map of topological rings since \( \omega \) is algebraic and therefore continuous.

Define the universal character for the torus \( \mathcal{S}_G \)
\[ k^G_{\text{univ}} : \mathcal{S}_G \to \Lambda(\mathcal{S}_G)^\times \]
such that \( k^G_{\text{univ}} \in \mathcal{W}_G(\Lambda_G(U)) \).

Define the universal character for the torus \( \mathcal{S}_H \)
\[ k^H_{\text{univ}} : \mathcal{S}_H \to \Lambda(\mathcal{S}_G)^\times \]
such that \( k^H_{\text{univ}} = \Omega(k^G_{\text{univ}}) \).

Lemma 2.1.12. If \( U \subset \mathcal{W}_m \) then the characters \( k^G_U, k^H_U \) are \( m \)-analytic on \( \mathcal{S}_G \), (resp. \( \mathcal{S}_H \)), viewed as a disjoint union of copies of \( \mathbb{Z}_p^n \) (resp. \( \mathbb{Z}_p^{n'} \)) indexed by \( \mathbb{S}_{tor} G \).

Proof. Since \( k^H_U \) is the pullback of \( k^G_U \) by an algebraic map, it suffices to prove the lemma for the latter character. Moreover, since \( k^G_U \) is a character it suffices to show \( m \)-analyticity on \( \mathcal{S}_G \). If we let \( \{ s_i \} \) be a basis for \( \mathcal{S}_G \equiv (1 + p\mathbb{Z}_p)^n \) then we need to show that for any \( z \in \mathbb{Z}_p \) \( z \mapsto k^G_U(s_i^z) \) is \( m \)-analytic on \( \mathbb{Z}_p \). We can then proceed much as in [LZ16, Lemma 4.1.5].

2.1.6 Hecke algebras

Let \( K^p \subset G(A_f(p)) \) be a prime-to-\( p \) level group and let \( S \) be a finite set of primes containing those at which \( K^p \) is ramified and not containing \( p \). Define
\[ T_S := C^\infty_c(K^p \backslash G(\mathbb{A}^{S(p)} \backslash \mathbb{A}_{f(p)})) / K^p, \mathbb{Z}_p) \]
the space of \( \mathbb{Z}_p \)-valued compactly supported locally constant \( K^p \)-biinvariant functions on \( G(\mathbb{A}^{S(p)} \backslash \mathbb{A}_{f(p)}) \). This is a commutative \( \mathbb{Z}_p \)-algebra.

Set
\[ A^- = \{ a \in A : \nu_p(\alpha(a)) \geq 0 \ \forall \ \alpha \in \Delta_G \backslash \Delta_L \} \]
and define \( A^{-\infty} \subset A^- \) with a strict inequality. We then define the double coset algebra
\[ 
\mathcal{U}_p^- = \mathbb{Z}_p[J_GaJ_G : a \in A^-],
\]
and similarly \( \mathcal{U}_p^{-\infty} \). There is a \( \mathbb{Z}_p \)-algebra isomorphism
\[ 
\mathbb{Z}_p[A^- / A(\mathbb{Z}_p)] \cong \mathcal{U}_p^-.
\]

Definition 2.1.13. Define the unramified \( Q_G \)-parahoric Hecke algebra:
\[ T_{S,p}^- := T_S \otimes \mathcal{U}_p^- . \]
2.2 Analytic coefficient modules

2.2.1 Locally analytic function spaces

Let $B$ be one of the local rings $\mathcal{O}$ or $\Lambda_G(\mathcal{U})$ for a wide-open disc $\mathcal{U} \subset \mathcal{W}$ and let $\mathfrak{m}_B$ be its maximal ideal.

**Definition 2.2.1.** For $m \geq 0$ define

$$\text{LA}_m(\mathbb{Z}_p^d, B) = \{ f : \mathbb{Z}_p^d \to B : \forall \mathbf{a} \in \mathbb{Z}_p^d, \exists f_{\mathbf{a}} \in B(T_1, \ldots, T_d) \text{ s.t. } f(\mathbf{a} + p^m \mathbf{x}) = f_{\mathbf{a}}(\mathbf{x}) \forall \mathbf{x} \in \mathbb{Z}_p^d \}.$$  

This space is isomorphic to $\prod_{\mathbf{a}} B[p^{-m}T_1, \ldots, p^{-m}T_d]$ as a $B$-module, where $\mathbf{a}$ runs over $(\mathbb{Z}/p^m\mathbb{Z})^d$.

The exponential map gives an isomorphism $N_G(\mathbb{Z}_p) \cong (\text{Lie}N_G)(\mathbb{Z}_p) \cong \mathbb{Z}_p^{dG}$, where for the second isomorphism we choose a basis such that $N_G(p\mathbb{Z}_p) = p\mathbb{Z}_p^{dG}$.

**Definition 2.2.2.** For $m \geq 0$, define

$$A_{\kappa, m}^{\text{an}} = \{ f : U_{\text{Bru}}^G(\mathbb{Z}_p) \to B : f(\bar{\kappa}tn) = \kappa(t)f(n), \text{ and } f|_{N_G(\mathbb{Z}_p)} \in \text{LA}_m(\mathbb{Z}_p^{dG}, B) \}$$

equipped with the $\mathfrak{m}_B$-adic topology. If $B = \Lambda_G(\mathcal{U})$ for a wide open disc $\mathcal{U} \subset \mathcal{W}_m$ and $\kappa = k^G_{\mathcal{U}}$, we write $A_{\kappa, m}^{\text{an}} := A_{k^G_{\mathcal{U}}, m}^{\text{an}}$.

Restriction to $N_G(\mathbb{Z}_p)$ gives a $B$-module isomorphism $A_{\kappa, m}^{\text{an}} \cong \text{LA}_m(\mathbb{Z}_p^{dG}, B)$ with inverse $f \mapsto (\bar{\kappa}tn \mapsto \kappa(t)f(n))$. We give these spaces an action of $a \in A^-$ via

$$(a \cdot f)(\bar{\kappa}tn) = f(\bar{\kappa}t\kappa_n^{-1}).$$

The following proposition is well-known and the proof is very similar to that of Proposition 2.2.13 so we omit it.

**Proposition 2.2.3.** Suppose $\kappa : G(\mathbb{Z}_p) \to B^\times$ is an $m$-analytic character. The modules $A_{\kappa, m}^{\text{an}}$ are invariant under right translation by $J_G$ and the action of $A^-$. 

Adapting the proof of [Urb11 3.2.8] to our situation we see that for $a \in A^-$

$$aA_{\kappa, m+1}^{\text{an}} \subset A_{\kappa, m}^{\text{an}}$$

and thus the action of $A^-$ is by compact operators since the inclusions $A_{\kappa, m}^{\text{an}} \subset A_{\kappa, m+1}^{\text{an}}$ are compact.

If $\lambda \in X^+_c(S_G)$ then there is a natural $J_G$-equivariant inclusion

$$V_\lambda^G \hookrightarrow A_{\lambda, m}^{\text{an}},$$

which also preserves the $A^-$ action.

**Definition 2.2.4.** If $\lambda \in \mathcal{U} \subset \mathcal{W}_m$ is the restriction to $G(\mathbb{Z}_p)$ of an algebraic character of $(S_G/S^0_G)(\mathbb{Q}_p)$, then there is a natural specialisation map

$$\rho_\lambda : A_{\mathcal{U}, m}^{\text{an}} \to A_{\lambda, m}^{\text{an}}$$

given by post-composing with the map

$$\Lambda_G(\mathcal{U}) \to \Lambda_G(\mathcal{U}) \otimes_\lambda \mathcal{O}_E \cong \mathcal{O}_E.$$ 

The space $A_{\mathcal{U}, m}^{\text{an}}$ is not the unit ball in a Banach algebra, but we can define a basis $\{e_i\}_{i \in I}$ for a countable indexing set $I$ such that for any $f \in A_{\mathcal{U}, m}^{\text{an}}$ there are $a_i \in \Lambda_G(\mathcal{U})$ such that

- $a_i \to 0$ in the cofinite filtration on $I$.
- We have $f = \sum_{i \in I} a_i e_i$. 


This can be seen easily from the identification of $A_{\mathcal{U},m}^{an}$ with a product of Tate algebras. This basis is sufficient to define Fredholm determinants of compact operators.

**Definition 2.2.5.** Let $\mathcal{V} \subset \mathcal{U} \subset \mathcal{W}$ be an affinoid contained in a wide open disc $\mathcal{U}$. Define 

$$A_{\mathcal{V},m}^{an} := A_{\mathcal{U},m}^{an} \hat{\otimes} \mathcal{O}(\mathcal{V})$$

where $\mathcal{O}(\mathcal{V})$ are the bounded-by-1 global sections of $\mathcal{V}$.

The space $A_{\mathcal{V},m}^{an}[1/p] = A_{\mathcal{U},m}^{an} \hat{\otimes} \mathcal{O}(\mathcal{V})$ is an orthonormalisable Banach $\mathcal{O}(\mathcal{V})$-module with unit ball $A_{\mathcal{V},m}^{an}$.

Let $H \hookrightarrow G$ be an embedding and let $\tilde{N}_H(p\mathbb{Z}_p) \cong \mathbb{Z}_p^d$ be the algebraic isomorphism given by the logarithm map.

**Definition 2.2.6.** For $\kappa \in \{k_H^U, \mu\}$ where $\mu = \lambda \circ \omega$ for some $\lambda \in X^*_G$, define

$$\mathcal{P}_{\kappa,m}^H := \{ f : J_H \to B : f(nt\tilde{n}) = \kappa(t)^{-1}f(\tilde{n}), \text{ and } f|_{\tilde{N}_H(p\mathbb{Z}_p)} \in \text{LA}(\mathbb{Z}_p^d, B) \}.$$ 

These modules clearly satisfy all the properties of $A_{\kappa,m}^{an}$ and admit a natural $J_H$-equivariant inclusion

$$\mathcal{P}_\lambda \hookrightarrow \mathcal{P}_{\lambda,m}^H.$$ 

### 2.2.2 Locally analytic distribution modules

Suppose we have an embedding of reductive $\mathbb{Q}$-groups $H \hookrightarrow G$ satisfying the conditions outlined in the introduction.

**Definition 2.2.7.** For $\kappa \in \{k_H^U, \mu\}$ where $\mu = \lambda \circ \omega$ for some $\lambda \in X^*_G$, define

$$D_{\kappa,m}^H = \text{Hom}_{\text{cont}, B}(\mathcal{P}_{\kappa,m}^H, B),$$ 

given the weak topology.

These spaces are topologically generated as a $B[J_H]$-modules by evaluation-at-1 distribution $\delta_1$. For $m \geq 1$ let $\mathcal{X}_{\kappa,m}^{(i)}$ be the image of $D_{\kappa,m}^H$ in $D_{\kappa,m-1}^H/p^i$.

**Definition 2.2.8.** Define

$$\text{Fil}_{\kappa,m}^n \subset D_{\kappa,m}^H$$

to be the kernel of the composition

$$D_{\kappa,m}^H \to \mathcal{X}_{\kappa,m}^{(i)} \to \mathcal{X}_{\kappa,m}^{(i)} \otimes_{B/p^i} B/\mathfrak{m}_B.$$ 

By generalising the argument of Hansen [Han15] we see that $\{\text{Fil}_{\kappa,m}^n\}_{n \geq 0}$ is a decreasing filtration invariant under $J_H$ such that $D_{\kappa,m}^H/\text{Fil}_{\kappa,m}^n$ is finite for all $n \geq 0$ and

$$D_{\kappa,m}^H = \lim_{\leftarrow n} D_{\kappa,m}^H/\text{Fil}_{\kappa,m}^n,$$

which allows us to define a lisse étale sheaf $\mathcal{D}_{\kappa,m}^H$ over our symmetric spaces for $H$.

**Definition 2.2.9.** Define a $\Lambda_G(\mathcal{U})$-linear specialisation map

$$\text{sp}_\mu : D_{\mathcal{U},m}^H \to (\mathcal{P}_{\mu,0}^H)^\vee.$$ 

as the map uniquely characterised by preserving the evaluation-at-1 map $\delta_1$. 

8
2.2.3 Locally Iwasawa functions

Let $B, d$ be as in the previous section.

**Definition 2.2.10.** For $m \geq 0$ define

$$LI_{m+1}(\mathbb{Z}_p^d, B) = \{ f: \mathbb{Z}_p^d \to B : \forall a \in \mathbb{Z}_p^d, \exists f_a \in B[[T_1, \ldots, T_d]] \text{ s.t. } f(a + p^m a) = f_a(p^m a) \forall a \in \mathbb{Z}_p^d \}. $$

This space admits a natural structure as a $B[[T_1, \ldots, T_d]]$-module, is isomorphic to $\prod_a B[[p^{-m}T_1, \ldots, p^{-m}T_d]]$.

**Definition 2.2.11.** Letting $n_B$ be the maximal ideal of $B[[T_1, \ldots, T_d]]$ and set $Fil^n_{m+1} := n_B LI_{m+1}(\mathbb{Z}_p^d, B)$.

We see that

$$LI_{m+1}(\mathbb{Z}_p^d, B) \approx \lim_n LI_{m+1}(\mathbb{Z}_p^d, B)/Fil^n_{m+1}. $$

so that the modules $LI_{m+1}(\mathbb{Z}_p^d, B)$ are profinite and in particular they are $n_B$-adically complete and separated.

For $m \geq 0$ there is a chain of inclusions

$$LA_m(\mathbb{Z}_p^d, B) \subset LI_{m+1}(\mathbb{Z}_p^d, B) \subset LA_{m+1}(\mathbb{Z}_p^d, B)$$

given by restriction. This corresponds to

$$\prod_{a \text{ mod } p^m} B[p^{-m}T_1, \ldots, p^{-m}T_d] \to \prod_{a \text{ mod } p^m} B[[p^{-m}T_1, \ldots, p^{-m}T_d]] \to \prod_{a \text{ mod } p^{m+1}} B(p^{-(1+m)}T_1, \ldots, p^{-(1+m)}T_d).$$

Write $\lim_m LA_m(\mathbb{Z}_p^d, B) = LA(\mathbb{Z}_p^d, B)$ for the space of all $B$-valued locally analytic functions on $\mathbb{Z}_p^d$. The inductive systems $\{LA_m(\mathbb{Z}_p^d, B)\}_{m \geq 0}$ are cofinal in

$$\ldots \to LA_m(\mathbb{Z}_p^d, B) \to LI_m(\mathbb{Z}_p^d, B) \to LA_{m+1}(\mathbb{Z}_p^d, B) \to \ldots,$$

thus

$$\lim_m LI_m(\mathbb{Z}_p^d, B) = LA(\mathbb{Z}_p^d, B).$$

Let $\kappa: S_G(\mathbb{Z}_p) \to B$ be an $m$-analytic character.

**Definition 2.2.12.** For $m \geq 0$ define

$$A_{n,m}^w := \{ f: U^G_{\text{Bru}}(\mathbb{Z}_p) \to B : f(\tilde{n}t^n) = \kappa(t)f(n), \text{ and } f|_{N_G(\mathbb{Z}_p)} \in LI_m(\mathbb{Z}_p^d, B) \},$$
given the $m_B$-adic topology. As in the locally analytic case we write $A_{n,m}^w := A_{n,m}^w$ when $U \subset W_m$.

These spaces are isomorphic to $LI_m(\mathbb{Z}_p^d, B)$ as $B$-modules and the filtration on $LI_m(\mathbb{Z}_p^d, B)$ defines a filtration $Fil^n_{n,m} \subset A_{n,m}^w$.

**Proposition 2.2.13.** For $n \geq 1$ the modules $A_{n,m}^w, Fil^n_{n,m}$ are invariant under the actions of $J_G$ and $A^-$ inherited from those on $A_{\infty, m+1}^w$.

**Proof.** For $m \in \mathbb{Q}_{\geq 0}$, let $\mathcal{G}_m$ be the rigid fibre of the formal scheme $\mathcal{G}_m$ defined for admissible $\mathbb{Z}_p$-algebras $R$ as

$$\mathcal{G}_m(R) = \ker (G(R) \to G(R/p^m R))$$

and set $\mathcal{G}_m^\circ = \cup_{m' \geq m} \mathcal{G}_{m'} \subset \mathcal{G}_0$. Define a rigid analytic group $\mathcal{J}_{G,m}^\circ = \mathcal{G}_m^\circ \cdot J_G$ (the group generated by $J_G$ and $\mathcal{G}_m^\circ$ in $\mathcal{G}_0$). This group admits an Iwahori decomposition

$$\mathcal{J}_{G,m}^\circ = (\mathcal{N}_m^\circ \cdot \mathcal{N}_G(\mathbb{Z}_p)) \times (\mathcal{L}_m^\circ \cdot L_G(\mathbb{Z}_p)) \times (\mathcal{N}_m^\circ \cdot N_G(\mathbb{Z}_p)).$$
Choosing a set $\Gamma$ of representatives for $J_G \bmod p^{m+1}$ then
\[ J_G^\circ = \bigcup_{\gamma \in \Gamma} G_m^\circ \gamma \]
The space $A_{w,m}^I$ is identified with the module of $B$-valued bounded-by-1 rigid functions $F$ on $J_G^\circ$ such that for $\bar{n} \ell \in \bar{Q}_G(Z_p) \cdot G_m^\circ, \ell \in J_G^\circ$, we have
\[ F(\bar{n} \ell j) = \kappa(\ell)F(j). \]
Viewed through this optic, it is clear that $A_{w,m}^I$ is stable under right translation by $J_G$.

Let $1$ be the trivial character. The space $A_{w,m}^I$ is a subring of $O(J_G^\circ)^{-1} \otimes_{Z_p} B$ and acts on $A_{w,m}^I$ via multiplication of functions in $O(J_G^\circ)_B$. Then $\text{Fil}_1^n$ is an ideal of $A_{w,m}^I$ and $\text{Fil}_1^n = (\text{Fil}_1^n)^n$. Furthermore,
\[ \text{Fil}_1^n = \text{Fil}_1^n A_{w,m}^I. \]
Since the action of $J_G$ respects the ring structure of $O(J_G^\circ)_B$ it suffices to show that $\text{Fil}_1^n$ is preserved by $J_G$.

We note that
\[ \text{Fil}_1^n = \{ F \in A_{w,m}^I : F(\gamma) \equiv 0 \bmod p \forall \gamma \in \Gamma \}, \]
so taking $F \in \text{Fil}_1^n, j \in J_G$ then for $\gamma \in \Gamma$ there is $\gamma j \in \Gamma$ and $\varepsilon \equiv 1 \bmod p^{m+1}$ such that $\varepsilon \gamma j = \gamma j$ and thus
\[ (j \cdot F)(\gamma) = F(\varepsilon \gamma j) \equiv 0 \bmod p. \]

The action of $A^{-}$ on $J_G^\circ$ is by
\[ a \star J_G^\circ = (N_m^\circ \cdot \bar{N}_G(Z_p)) \times (L_m^\circ \cdot L_G(Z_p)) \times a(N_m^\circ \cdot N_G(Z_p))a^{-1} \]
We can see that this is well-defined noting that for a set of representatives $\Gamma'$ of $N_G(Z_p)$ mod $p^{m+1}$ we have
\[ N_m^\circ \cdot N_G(Z_p) = \bigsqcup_{n \in \Gamma'} N_m^\circ \cdot n \]
and each $G_m^\circ \cdot n$ is isomorphic to a direct product of balls $U_{a,m}$ of radius $p^{-m}$ and centre 0 contained in the root spaces $U_{a,m}$:
\[ N_m^\circ \cdot n = \prod_{a \in \Phi_L} U_{a,m}, \]
where $\Phi_L = \Phi_G \setminus \Phi_L$. Thus we have an isomorphism
\[ N_m^\circ \cdot N_G(Z_p) = \prod_{a \in \Phi_L} U_{a,m} \times N_G(Z/p^{m+1}Z) \]
with the action of $a \in A^{-} \{ a \}$ given by
\[ a N_m^\circ \cdot N_G(Z_p) a^{-1} = \prod_{a \in \Phi_L} p^{v_p(\alpha(a))} U_{a,m} \times a N_G(Z/p^{m+1}Z) a^{-1} \]
\[ = \prod_{a \in \Phi_L} U_{a,m+v_p(\alpha(a))} \times a N_G(Z/p^{m+1}Z) a^{-1} \]
and $U_{a,m+v_p(\alpha(a))} \subset U_{a,m}$ since $v_p(\alpha(a)) \geq 0$ by definition of $A^{-}$, so $a N_m^\circ \cdot N_G(Z_p) a^{-1} \subset N_m^\circ \cdot N_G(Z_p)$.

We can prove the filtration $\text{Fil}_1^n$ is invariant under the action of $A^{-}$ in a similar way to that of $J_G$.

\textbf{Corollary 2.2.14.} The modules $A_{w,m}^I$ induce lisse étale sheaves $\mathcal{F}_{w,m}^I$ over $Y_G$.  

10
2.3 Locally analytic branching laws

2.3.1 The Big Twig

We consider functions on the set \((U_{\text{sph}} \cdot Q_H)(\mathbb{Z}_p) = (\breve{Q}_G u Q_H)(\mathbb{Z}_p)\).

**Lemma 2.3.1.** The function

\[
 f_U : (U_{\text{sph}} \cdot Q_H)(\mathbb{Z}_p) \to \Lambda_G(U)
\]

\[
 n_\ell u q g \mapsto k^H_U(g) k^G_U(\ell)
\]

is well-defined.

**Proof.** For \(i = 1, 2\), let \(g_i \in Q_H(\mathbb{Z}_p), q_i \in Q_H(\mathbb{Z}_p)\) and \(\ell \in \breve{Q}_G(\mathbb{Z}_p)\), and suppose \(n_1 \ell_1 u g_1 q_1 = n_2 \ell_2 u g_2 q_2\), then rearranging we get

\[
 (n_2 \ell_2 u g_2)^{-1} n_1 \ell_1 u g_1 \in Q_H(\mathbb{Z}_p),
\]

so \(\omega\) applied to the left hand side is trivial. Since \(Q_H(\mathbb{Z}_p) \subset Q_H(\mathbb{Z}_p)\) we have that

\[
 u^{-1} \ell_2^{-1} n_2^{-1} n_1 \ell_1 u \in Q_H(\mathbb{Z}_p) \cap u^{-1} \breve{Q}_G(\mathbb{Z}_p) u,
\]

whose image under

\[
 Q_H(\mathbb{Z}_p) \cap u^{-1} \breve{Q}_G(\mathbb{Z}_p) u \to S^\text{ab}_G(\mathbb{Z}_p) \to (S_G/S_G^0)(\mathbb{Z}_p)
\]

is equal to the image of \(\ell_2^{-1} \ell\) under the projection \(L_G(\mathbb{Z}_p) \to (S_G/S_G^0)(\mathbb{Z}_p)\). Applying \(\omega\) to \((\dagger)\), we get

\[
 1 = \omega((n_2 \ell_2 u g_2)^{-1} n_1 \ell_1 u g_1) = \omega(g_2)^{-1} \omega(g_1) \omega(u^{-1} n_2^{-1} n_1 \ell_1 u) = \ell_2^{-1} \ell \omega(g_2)^{-1} \omega(g_1),
\]

so

\[
 \ell_2 \omega(g_2) = \ell \omega(g_1) \mod S_G^0(\mathbb{Z}_p)
\]

and applying \(k^G_U\) allows us to conclude. \(\square\)

**Remark 2.3.2.** The above proof shows that we can view \(f_U\) as the pullback of \(k^G_U\) via the map

\[
 (U_{\text{sph}} \cdot Q_H)(\mathbb{Z}_p) \to \mathfrak{S}_G,
\]

\[
 n_\ell u q \mapsto \ell \omega(q) \mod L^0_G(\mathbb{Z}_p).
\]

**Definition 2.3.3.** Define a \(\ast\)-action of \(A^-\) on \(U_{\text{Bru}}(\mathbb{Z}_p)\) by

\[
 a \ast U_{\text{Bru}}(\mathbb{Z}_p) = \breve{Q}_G(\mathbb{Z}_p) a N_G(\mathbb{Z}_p) a^{-1}.
\]

**Lemma 2.3.4.** For any \(a \in A^-\) we have

\[
 (a \ast U_{\text{Bru}}(\mathbb{Z}_p)) u \subset U_{\text{sph}}(\mathbb{Z}_p).
\]

**Proof.** It suffices to check this on \(F(\mathbb{Z}_p)\). Furthermore, as the image of \(U_{\text{sph}}\) in \(F\) is Zariski open it suffices to show that the inclusion holds mod \(p\). We compute

\[
 (a \ast U_{\text{Bru}}(\mathbb{Z}_p)) u = \breve{Q}_G(\mathbb{Z}_p) a N_G(\mathbb{Z}_p) a^{-1} u
\]

\[
 \equiv \breve{Q}_G(F_p) u \mod p
\]

\[
 \in U_{\text{sph}}(F_p),
\]

where the second equality follows from the fact that \(a N_G(\mathbb{Z}_p) a^{-1} \equiv 1 \mod p\). \(\square\)

Fix \(\tau \in A^-\).
Definition 2.3.5. Define
\[ f_{\ell U}^{\text{ph}} : (\tau \ast \text{UBru}(\mathbb{Z}_p)) u \rightarrow \Lambda_G(U) \]
by restricting \( f_U \).

Definition 2.3.6. In order to keep track of the various actions we define some auxiliary function spaces for an \( m \)-analytic character \( \kappa : \mathfrak{S}_G \rightarrow B^* \).

- A \( u^{-1}(J_G \cap \tau J_G \tau^{-1})u \)-module:
  \[ A_{\kappa,m}^1 := \{ f : \tilde{N}_G L_G \tau N_G \tau^{-1} u \rightarrow B : f(\tilde{n}\ell t n\tau^{-1} u) = \kappa(\ell) f(\tau n\tau^{-1} u) \text{ and } n \mapsto f(\tau n\tau^{-1} u) \in \Lambda_m(Z_p^{d_G}, B) \} \]
- A \( \tilde{J}_G \cap \tau J_G \tau^{-1} \)-module:
  \[ A_{\kappa,m}^2 := \{ f : \tilde{N}_G L_G \tau N_G \tau^{-1} \rightarrow B : f(\tilde{n}\ell t n\tau^{-1} u) = \kappa(\ell) f(\tau n\tau^{-1} u) \text{ and } n \mapsto f(\tau n\tau^{-1}) \in \Lambda_m(Z_p^{d_G}, B) \} \]
- A \( \tau^{-1}(J_G \tau \cap J_G) \)-module
  \[ A_{\kappa,m}^3 := \{ f : \tau^{-1} \tilde{N}_G \tau L_G N_G \rightarrow B : f(\tilde{n}\ell t n\tau^{-1} u) = \kappa(\ell) f(\tau n\tau^{-1} u) \text{ and } n \mapsto f(n) \in \Lambda_m(Z_p^{d_G}, B) \} \]

The last module can be easily seen to be isomorphic to \( A_{U,m}^{\text{ph}} \) in such a way that the action of \( \tau^{-1}\tilde{J}_G \tau \cap J_G \) extends to an action of \( J_G \).

These modules are related by the maps
\[ A_{U,m}^{\text{ph}} \xrightarrow{u^{-1}} A_{U,m}^1 \xrightarrow{\tau^{-1}} A_{U,m}^3 \]
which intertwine the various group actions. For \( j \in J_H \) define \( j \cdot f_{\ell U}^{\text{ph}} \) to be the restriction of \( j \cdot f_U \) to \( (\tau \ast \text{UBru}(\mathbb{Z}_p)) u \).

Lemma 2.3.7. When \( U \subset \mathcal{W}_m \) and for all \( j \in J_H \), the function \( j \cdot f_{\ell U}^{\text{ph}} \) is contained in \( A_{U,m}^1 \).

Proof. By Lemma 2.3.1 and the proceeding remark there is a well-defined map
\[ (U_{\text{ph}} \cdot Q_H)(\mathbb{Z}_p) \rightarrow \mathfrak{S}_G, \quad \tilde{n}\ell t u q \mapsto \ell \omega(g) \text{ mod } L_G^0(Z_p). \]
Let \( n \in N_G(\mathbb{Z}_p), j \in J_H \) and let \( \tilde{n} \in \tilde{N}_G(\mathbb{Z}_p), \ell \in L_G(Z_p), q \in Q_H(\mathbb{Z}_p) \) be such that
\[ \tau n\tau^{-1} u j = \tilde{n}\ell t u q. \]

Then by a mod \( p \) variation of the proof of Lemma 2.3.1 we see that \( \ell \omega(q)/\omega(j) \text{ mod } p \in L_G^0(\mathbb{F}_p) \), so in particular if \( \omega(j)_{\text{tor}} \) is the torsion part of \( \omega(j) \) (where we pull back \( \omega \) to \( J_H \) via the projection \( J_H \rightarrow Q_H \) given by the Iwahori decomposition) then \( \tilde{n}\ell t u q \) maps into \( \omega(j)_{\text{tor}} \times \mathfrak{S}_{G,1} \cong \mathbb{Z}_{p}^{d_G} \) under the above map. Thus the restriction of \( j \cdot f_{\ell U}^{\text{ph}} \) to \( (\tau \ast N_G(\mathbb{Z}_p)) u \) is given by the composition of an algebraic map \( Z_p^{d_G} \rightarrow Z_p^{d_G} \cong \omega(j)_{\text{tor}} \times \mathfrak{S}_{G,1} \) with \( k_G^{\text{ph}} \) which is \( m \)-analytic when \( U \subset \mathcal{W}_m \) by Lemma 2.1.12.

Proposition 2.3.8. For \( \lambda \in U \)
\[ \rho_{\lambda}([u^{-1}\tau^{-1}]f_{\ell U}^{\text{ph}}) = [u^{-1}\tau^{-1}] \left( f_{\lambda}^{\text{ph}} \big|_{(\tau \ast U_{\text{ph}}(\mathbb{Z}_p)) u} \right). \]

Proof. This is by construction.

Definition 2.3.9. Define the Big Twig \( \Phi_U : (\tau \ast \text{UBru}(\mathbb{Z}_p)) u \times J_H \rightarrow \Lambda_G(U) \) by
\[ \Phi_U(g, j) = f_{\ell U}^{\text{ph}}(gj^{-1}). \]
Writing $\Phi(g, j) = (j^{-1} \cdot f_U^{\text{ph}})(g)$ Lemma 2.3.7 shows that the Big Twig is well defined.

**Lemma 2.3.10.** Suppose $U \subset W_m$. The Big Twig satisfies the following properties:

- For fixed $j \in J_H$
  
  $$g \mapsto \Phi_U(g, j) \in A^1_{U,m}$$

- For fixed $g \in (\tau \ast U_{\text{Bru}}(\mathbb{Z})_p)$
  
  $$j \mapsto \Phi_U(g, j) \in P^H_{U,m}.$$  

**Proof.** Easy checks. 

2.3.2 The Big Branch

**Definition 2.3.11.** For $m \geq 0$, define the $(m$-analytic) Big Branch

$$\mathcal{BR}_m : D^H_{U,m} \to A^1_{U,m}$$

defined for $g \in (\tau \ast U_{\text{Bru}}(\mathbb{Z})_p)$ by

$$\mathcal{BR}_m(\varepsilon)(g) = \langle \varepsilon, \Phi(g, -) \rangle.$$  

These maps are equivariant for the action of $J_H \cap u^{-1}(\tau J_G \tau^{-1} \cap J_G)u$ on both sides.

**Proposition 2.3.12.** Let $\lambda \in X^*_G(S_G/S^0_G)^{G_m}$ and set $\mu = \Omega(\lambda)$. Let $U \subset W_G$ be a wide-open disc containing $\lambda$. There is a commutative diagram

\[
\begin{array}{cccccc}
D^H_{U,m} & \xrightarrow{\mathcal{BR}_m} & A^1_{U,m} & \xrightarrow{u^{-1}} & A^2_{U,m} & \xrightarrow{\tau^{-1}} & A^3_{U,m} \\
\downarrow & & \downarrow & & \downarrow & & \\
(P^H_\mu)^\vee & \xrightarrow{\text{br}_\lambda^U} & A^1_{\lambda,m} & \xrightarrow{u^{-1}} & A^2_{\lambda,m} & \xrightarrow{\tau^{-1}} & A^3_{\lambda,m}
\end{array}
\]

**Proof.** We claim that it suffices to check that $\mathcal{BR}_m$ is uniquely defined by where it sends $\delta_1$. Indeed, $D^H_{U,m}$ is topologically generated by $\delta_1$ as a $\Lambda_G(U)[J_H]$-module, so we need to show that the image of $\mathcal{BR}_m$ is contained in a closed subspace of $A^1_{U,m} \cap u^{-1}(\tau J_G \tau^{-1} \cap J_G)u$.

Consider the linear subspace $X \subset A^1_{U,m}$ given by the closure of the $\Lambda_G(U)$-linear space spanned by $j \cdot f_U^{\text{ph}}$ for $j \in J_H$. This satisfies the above requirements, thus $\mathcal{BR}_m(\delta_1)$ uniquely determines $\mathcal{BR}$.

Then for $g = \bar{n}t \bar{n} \in U_{\text{Bru}}(\mathbb{Z})_p$,

$$\rho_\lambda(([u^{-1}\tau] \mathcal{BR}(\delta_1))(g) = f_\lambda^{\text{ph}}(\bar{n}t \bar{n}^{-1} u) = ([u^{-1}\tau] f_\lambda^{\text{ph}})(g)$$

by Lemma 2.3.4 whence it is clear that the diagram commutes. 

**Corollary 2.3.13.** There is a commutative diagram of Betti cohomology groups

\[
\begin{array}{cccccc}
H^1(Y_H(J_H \cap (u^{-1}\tau J_G \tau^{-1} u \cap J_G)), D^H_{U,m+1}) & \xrightarrow{T\rho[u^{-1}\tau] \circ \mathcal{BR}} & H^{1+c}(Y_G(J_G), A^{1w}_{U,m+1}) \\
\downarrow & & \downarrow \\
H^1(Y_H(J_H \cap (u^{-1}\tau J_G \tau^{-1} u \cap J_G)), (P^H_\mu)^\vee) & \xrightarrow{T\rho[u^{-1}\tau] \circ \text{br}_\lambda^U} & H^{1+c}(Y_G(J_G), A^{1w}_{\lambda,m+1})
\end{array}
\]

where $\text{Tr}$ is the trace map of the inclusion

$$J_G \cap \tau^{-1} J_G \tau \to J_G.$$
Example 2.3.14. We compute the Big Twig in some familiar situations, using the techniques of [BSV20]. Let $H = GL_2$ and suppose for simplicity that $p \neq 2$. In this case $F_H \cong P^1_{\mathbb{Z}_p}$ and for an integer $k \geq 0$ we can identify the representation $V_k$ of highest weight $k$ with the global sections of the sheaf $O_{ps}(k)$, the space degree $k$ homogeneous polynomials on $\mathbb{Z}_p^2$.

We consider the setting of [LZ10], given by taking $G = GL_2 \times GL_2$ and $H = GL_2$ embedded diagonally. Taking $u = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ and $Q_H^u = \{ (x, y) \}$ then $Q_H^u$ has an open orbit on $F_G = (\mathbb{P}^1)^2$. We have $S_G = T_G$, the standard diagonal torus, and $S^G_G = \{ 1 \}$ so the weight space $W_G$ parameterises characters of $T_G$. The cocharacters

$$
\lambda^1_\nu \colon x \mapsto \left( \begin{array}{c} x \\ x^{-1} \end{array} \right) \times \left( \begin{array}{c} 1 \\ 1 \end{array} \right),
\lambda^2_\nu \colon x \mapsto \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \times \left( \begin{array}{c} x \\ x^{-1} \end{array} \right),
\lambda^3_\nu \colon x \mapsto \left( \begin{array}{c} 1 \\ x \end{array} \right) \times \left( \begin{array}{c} 1 \\ x \end{array} \right)
$$

determine a decomposition

$$W_G = W_{GL_1} \times W_{GL_1}$$

where $W_{GL_1}$ is the standard weight space parameterising characters of $\mathbb{Z}_p^\times$. Write $k_{U,i} = k^G_U \circ \lambda^i_\nu$ and further define

$$k^i_{U,1} = k_{U,1} - k_{U,3},
k^i_{U,2} = k_{U,2} - k_{U,3}.
$$

Let $k, k' \geq 0$ be integers, then for $0 \leq j \leq \min\{ k, k' \}$ there is an $H$-equivariant map

$$V_{k+k'-2j} \otimes \det^j \to V_k \otimes V_{k'}.
$$

Set

$$F_{k,k',j}(x_1, x_2, y_1, y_2) = x_1^{k-j} y_1^{-j} \det(y_1, y_2)^j,$$

then $F_{k,k',j} \in (V_k \otimes V_{k'} \otimes \det^{j-k-k'}) Q_H^u$ and $F_{k,k',j}(u) = 1$ so $F_{k,k',j}$ is a highest weight vector for the action of $H$ of highest weight $(\begin{smallmatrix} x \end{smallmatrix}, x^{-1}) \mapsto x^{k+k'-2j} \det^{j}$. Let $U \subset W_G$ be a wide-open disc with universal character $k^G_U : T_G(\mathbb{Z}_p) \to \Lambda_G(U)^\times$. Then $F_{k,k',j}$ restricted to $(1, 0)^2 \cdot U_{sp}(\mathbb{Z}_p)$ takes values in $\mathbb{Z}_p^\times$ so the function

$$F_U(x_1, x_2, y_1, y_2) = x_1^{k_{U,1}} y_1^{k_{U,2}} \det(x_1, x_2)^{k_{U,3}}
$$

is well defined for $(x_1, x_2, y_1, y_2) \in (1, 0)^2 \cdot U_{sp}(\mathbb{Z}_p)$ and is homogeneous of weight $k^G_U$. Setting $\tau = (p_{\text{}}^p_{1})$ then for $n = (\begin{smallmatrix} 1 & z_1 \\ 0 & 1 \end{smallmatrix}) \times (\begin{smallmatrix} 1 & z_2 \\ 0 & 1 \end{smallmatrix}) \in N_G(\mathbb{Z}_p)$ and $\dot{i}^{-1} = (i_1 i_2 i_3) \in J_H$ the Big twig is given by

$$\Phi_U(\tau n \tau^{-1} \dot{i}^{-1} u i^{-1}) = (i_1 + p^2 i_3 z_1)^{k_{U,1}} (i_1 + p i_3 (1 + p z_2))^k_{U,2} \det(i)^{-1} (1 + p(z_2 - z_1))^{k_{U,3}}.
$$

Example 2.3.15. We consider the situation of [GS20] and [BSV20]. Let $G = GL_2 \times GL_2$ and $H = GL_2 \times GL_2 \times GL_2$. We consider the diagonal embedding of $H$ into $G$. Taking $u = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \times (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \times (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ we see that $H$ has an open orbit on $F_G = (\mathbb{P}^1)^3$. For a triple of non-negative integers $r = (r_1, r_2, r_3)$ let $V_r = V_{r_1} \boxtimes V_{r_2} \boxtimes V_{r_3}$. We have $u H u^{-1} \cap B_G = Z_H = H \cap u^{-1} B_G u = S^G_G$ from which it follows that a necessary condition for $V_r$ to have an $H$-invariant element is for there to exist an integer $r$ such that $r_1 + r_2 + r_3 = 2r$. If we further suppose that for each permutation $\sigma \in S_3$ we have $r_{\sigma(1)} + r_{\sigma(2)} \geq r_{\sigma(3)}$ then the function $\det_r \in V_r$ given by

$$\det(x_1, x_2 \times (y_1, y_2) \times (z_1, z_2) \mapsto 2^{-r} \det(x_1 x_2 \begin{array}{c} r-r_3 \\ \end{array}) \det(x_1 y_2 \begin{array}{c} r-r_3 \\ \end{array}) \det(y_1 z_2 \begin{array}{c} r-r_1 \\ \end{array}),
$$

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is well defined and $\text{Det}_r \in \left(V_r \otimes \det^{-1}\right)^H$. Writing, for $g \in G$, $\text{Det}_r(g) := \text{Det}_r((1,0)^3 g)$, we see that $\text{Det}_r(u) = 1$ (this is the reason for the factor of $2^{-r}$) and thus for $\tilde{nt} \in \tilde{B}, h \in H$ that:

$$
\text{Det}_r(\tilde{nt} h) = \det(h)^r t^{\lambda_r}
$$

where $\lambda_r$ is the highest weight of $V_r$ using additive notation for characters.

Recall that $\mathcal{W}_G$ parameterises weights of the rank 3 torus $\mathbb{T}_G = (T_G/Z_H)(\mathbb{Z}_p)$. For $i = 1, 2, 3$ we define characters $\lambda_i^\vee \in X_*(S_G/S_G^0)$ for the cocharacter given by composing

$$
x \mapsto \begin{pmatrix} x & \cdot \\ \cdot & x^{-1} \end{pmatrix}
$$

with the inclusion into the $i$th $\text{GL}_2$-component of $S_G$ and reduction modulo $S_G^0$. These cocharacters determine a decomposition of $\mathcal{W}_G$:

$$
\mathcal{W}_G = \mathcal{W}_{\text{GL}_1} \times \mathcal{W}_{\text{GL}_1} \times \mathcal{W}_{\text{GL}_1}
$$

and we define $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \subset \mathcal{W}_G$ as a product of wide-open discs $\mathcal{U}_i \subset \mathcal{W}_{\text{GL}_i}$ contained in each factor of the decomposition (7). Suppose that each $\mathcal{U}_i$ is centred on an integer $k_i \in \mathbb{Z}$; the condition that our weights be trivial on $Z_H$ forces $k_1 + k_2 + k_3$ to be even. We define

$$
k_{\mathcal{U}, i} := k_{\mathcal{U}}^G \circ \lambda_i^\vee.
$$

These characters take the form $k_{\mathcal{U}, i}(z) = \omega(z)^{k_1+k_2+k_3} \omega_i^k$ where $\omega(z)$ is the Teichmüller representative of $z \in \mathbb{Z}_p^\times$. We further define

$$
k_{\mathcal{U}, i}(z) = \omega(z)^{k_1+k_2+k_3} \omega_i^k
$$

and similarly for $i = 2, 3$ to obtain characters $k_{\mathcal{U}, i}$ satisfying $k_{\mathcal{U}, \sigma(1)} + k_{\mathcal{U}, \sigma(2)} = k_{\mathcal{U}, \sigma(3)}$ for all permutations $\sigma \in S_3$ and write

$$
k_{\mathcal{U}, 123}^* = k_{\mathcal{U}, 1}^* + k_{\mathcal{U}, 2}^* + k_{\mathcal{U}, 3}^*.
$$

Set $\tau = (p_1) \times (p_1) \times (p_1)$. Retaining additive notation for characters, we then have for $\tilde{nt} \in \tilde{B}_G(\mathbb{Z}_p)$, $n = (1 \ z_1) \times (1 \ z_2) \times (1 \ z_3) \in N_G(\mathbb{Z}_p)$:

$$
\Phi_{\mathcal{U}}(\tilde{nt} \tau \eta^{-1} u, h) = (2\det(h))^{-k_{\mathcal{U}, 123}} \text{det} \begin{pmatrix} 1 & p_{z_1} \\ 1 & 1 + p_{z_2} \end{pmatrix}^{k_{\mathcal{U}, 3}} \text{det} \begin{pmatrix} 1 & p_{z_1} \\ 1 & 1 + p_{z_2} \end{pmatrix} \text{det} \begin{pmatrix} 1 & -1 + p_{z_2} \\ 1 & 1 \end{pmatrix}^{k_{\mathcal{U}, 1}}
$$

defining an element of $A_{\mathcal{U}, m}^\text{an} \otimes (k_{\mathcal{U}, 123}^* \circ \text{det})^{-1}$ for suitably large $m$.

### 2.3.3 étale cohomology

**Lemma 2.3.16.** For $\mathcal{U} \subset \mathcal{W}_{m+1}$ there is a commutative diagram

\[
\begin{array}{ccc}
D_{\mathcal{U}, m}^H & \longrightarrow & A_{\mathcal{U}, m+1}^w \\
\downarrow & & \downarrow \\
(P_{\mathcal{U}}^H)^\vee & \longrightarrow & A_{\mathcal{U}, m+1}^w 
\end{array}
\]

**Proof.** Just compose the diagram with the inclusions

$$
A_{\mathcal{U}, m}^3 \cong A_{\mathcal{U}, m}^\text{an} \hookrightarrow A_{\mathcal{U}, m+1}^w
$$

for appropriate $\kappa$. \qed

**Lemma 2.3.17.** Suppose $\mathcal{U} \subset \mathcal{W}_m$, $m \geq 0$. The Big Branch $\mathcal{BR}_{m+1}$ is continuous for the profinite topologies on $D_{\mathcal{U}, m+1}^H$ and $A_{\mathcal{U}, m+1}^w$.  


Proof. The assumption on \( \mathcal{U} \) means that \( h \mapsto \Phi(g,h) \) is an element of \( T_{\mathcal{U},m}^H,\text{an} \) so the \((m+1)\)-analytic Big Branch \( \mathcal{B}_{m+1} \) factors through restriction to this module. We have a commutative diagram

\[
\begin{array}{c}
D_{\mathcal{U},m+1}^H & \xrightarrow{\mathcal{B}_{m+1}} & A_{\mathcal{U},m+1}^I \xleftarrow{\mathcal{A}_{\mathcal{U},m+1}} A_{\mathcal{U},m}^I \\
\downarrow & & \downarrow \\
X_{\mathcal{U},m+1}^{(i)} & \xrightarrow{\Lambda_{\mathcal{U}/m_{\mathcal{U}}}} & A_{\mathcal{U},m}^I \otimes \Lambda_{\mathcal{U}/m_{\mathcal{U}}} \\
\downarrow & & \downarrow \\
D_{\mathcal{U},m}/p & \xrightarrow{\mathcal{B}_{m}} & A_{\mathcal{U},m}^I \\
\end{array}
\]

We need to show that for each \( n \geq 0 \) there is \( i = i(n) \) such that \( \mathcal{B}_{m+1}(\mathcal{R}_{\mathcal{U},m+1}) \subset \mathcal{G}^n \). From the diagram it suffices to show that there is \( i = i(n) \) such that \( m_{\mathcal{U}}^i A_{\mathcal{U},m}^I \subset \mathcal{G}^n_{m+1} \) but this is easily verifiable. \( \square \)

Corollary 2.3.18. Suppose \( Y_H, Y_G \) admit compatible Shimura data. In this case there is \( e \in \mathbb{Z} \) such that \( 2e = c \) For any integer \( j \) there is a commutative diagram of étale cohomology groups

\[
\begin{array}{c}
H^i(Y_H(J_H \cap (u^{-1}\tau J_G T^{-1}u \cap J_G)), \mathcal{B}_{\mathcal{U},m+1}^H(j)) \xrightarrow{\text{Tro}[\phi],\mathcal{B}_{m+1}} H^{i+2e}(Y_G(J_G), \mathcal{A}_{\mathcal{U},m+1}^I(j + e)) \\
\downarrow & & \downarrow \\
H^i(Y_H(J_H \cap (u^{-1}\tau J_G T^{-1}u \cap J_G)), (\mathcal{P}_{H}^\vee)(j)) \xrightarrow{\text{Tro}[\phi],\text{br}} H^{i+2e}(Y_G(J_G), \mathcal{Z}_{\lambda,m+1}^I(j + e)),
\end{array}
\]

where \( (j) \) denotes a cyclotomic twist.

2.3.4 Eisenstein classes

Let \( H = \text{GL}_2, Q_H = B_H \) and as in Example 2.3.14 let \( V_k \) denote the irreducible \( H \)-rep of highest weight \( k \geq 0 \). A large proportion of examples of Euler systems arise from Eisenstein classes in the cohomology of modular curves, for example [KLZ15, LSZ21]. We briefly discuss how these classes fit into our framework.

Let \( S_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \) be the space of \( \mathbb{Z}_p \)-valued Schwartz functions \( \phi \) on \( (\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2 \) satisfying \( \phi(0,0) = 0 \) and for an integer \( c \) coprime to 6, let \( cS_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \) denote the subspace of \( S_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \) of the form \( \phi^{(c)} \otimes \text{ch}(\mathbb{Z}_p^2) \), where \( \phi \) is a \( \mathbb{Z} \)-valued Schwartz function on \( (\mathbb{A}_f^{(c)})^2 \) and \( \mathbb{Z}_c = \prod_{\ell|c} \mathbb{Z}_\ell \). We equip these spaces with the natural right translation action of \( H \).

Proposition 2.3.19. Let \( k \geq 0 \). For \( c \) as above and \( U \subset H(\mathbb{A}_f) \) a neat open compact subgroup then there is a map

\[
cS_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \to H^1_{\text{et}}(Y_H(U)_{\Sigma}; V_k, \mathbb{Z}_p(1))
\]

\[
\phi \mapsto c \text{Eis}^{k}_{\text{et},\phi}
\]

whose image in \( H^1(Y_H(U)_{\Sigma}; V_k) \) satisfies

\[
c \text{Eis}^{k}_{\text{et},\phi} = \left( c^2 - c^{-k} \begin{pmatrix} c \\ -c \end{pmatrix}^{-1} \right) r_{\text{et}}(\text{Eis}^{k}_{\text{mot},\phi}),
\]

where \( \text{Eis}^{k}_{\text{mot},\phi} \) is Beilinson’s motivic Eisenstein class, and \( r_{\text{et}} \) is the étale regulator.

Let \( K^p \subset H(\mathbb{A}_f^{(p)}) \) be a choice of tame subgroup such that \( K^p J_H \) is neat and let \( \phi \) be a \( \mathbb{Z}_p \)-valued Schwartz function on \( (\mathbb{A}_f^{(p)})^2 \) invariant under \( K^p \). By [KLZ15 Section 4] there is an Eisenstein-Iwasawa class

\[
c \mathcal{E}^{\phi} \in H^2(Y_H(J_H)_{\Sigma}; \Lambda(N_H(\mathbb{Z}_p)\backslash J_H)),
\]

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where \( \Lambda(N_H(\mathbb{Z}_p) \setminus J_H) \) is the étale sheaf associated to the space of \( \mathbb{Z}_p \)-valued measures on \( N_H(\mathbb{Z}_p) \setminus J_H \). for all \( k \geq 0 \) There is a \( k \)th moment map
\[
\text{mom}^k : H^1(Y_H(J_H)_{\Sigma}, \Lambda(N_H(\mathbb{Z}_p) \setminus J_H)(1)) \to H^1(Y_H(J_H)_{\Sigma}, V_{k,\mathbb{Z}_p}^\vee(1))
\]
satisfying
\[
\text{mom}^k(\cdot, \mathcal{E}_{\lambda}^\phi) = \cdot \text{Eis}^k_{\epsilon},
\]
where \( \phi_1 = \phi \otimes \chi(p\mathbb{Z}_p \times \mathbb{Z}_p) \).

**Remark 2.3.20.** The space \( V_{k,\mathbb{Z}_p}^\vee \) is \( H \)-isomorphic to the space \( T\text{Sym}^k(\mathbb{Z}_2^3) \) of weight \( k \) symmetric tensors on \( \mathbb{Z}_2^3 \).

Let \( \mathcal{U} \subset \mathcal{W}_m \). We define a map
\[
\psi_{\mathcal{U}} : \Lambda(N_H(\mathbb{Z}_p) \setminus J_H) \to D_{\mathcal{U},m}^H
\]
given, for \( j \in N_H(\mathbb{Z}_p) \setminus J_H \), by sending the Dirac measure \( \delta_j \) to the ‘evaluation-at-\( j \)’ distribution. The \( k \)th moment map factors through \( \psi_{\mathcal{U}} \) as
\[
\text{mom}^k : \Lambda(N_H(\mathbb{Z}_p) \setminus J_H) \xrightarrow{\psi_{\mathcal{U}}} D_{\mathcal{U},m}^H \xrightarrow{\rho_k} V_{k,\mathbb{Z}_p}^\vee,
\]
and so we define
\[
\mathcal{E}_{\lambda}^{\text{H}}_{\epsilon,\phi} := \psi_{\mathcal{U},*}(\cdot, \mathcal{E}_{\lambda}^\phi) \in H^1(Y_H(J_H), \mathcal{P}_{\mathcal{U}}^H(1)).
\]

**Proposition 2.3.21.** For \( k \in \mathbb{Z}_{\geq 0} \cap \mathcal{U} \), the class \( \mathcal{E}_{\lambda}^{\text{H}}_{\epsilon,\phi} \) satisfies
\[
\rho_k(\cdot, \mathcal{E}_{\lambda}^{\text{H}}_{\epsilon,\phi}) = \cdot \text{Eis}^k_{\epsilon,\phi_1}.
\]

**Proof:** Clear from the above discussion.

We will use these classes in Section 6 to interpolate the Lemma-Flach Euler system of [LSZ21] in Coleman families.

### 3 Complexes of Banach modules

#### 3.1 Slope decompositions

**Definition 3.1.1.** Let \( F \in A\{\{T\}\} \) and let \( h \in \mathbb{R}_{\geq 0} \). We say that \( F \) has a slope \( \leq h \) factorisation if we have a factorisation
\[
F = Q \cdot S
\]
where \( Q \) is a polynomial and \( S \) is Fredholm, such that

1. Every slope of \( Q \) is \( \leq h \)
2. \( S \) has slope \( > h \)
3. \( p^h \) is in the interval of convergence of \( S \).

Such a factorisation is unique if it exists.

**Definition 3.1.2.** Let \( M \) be an \( R \)-module equipped with an \( R \)-linear endomorphism \( u \). For \( h \in \mathbb{Q} \) we say that \( M \) has a \( \leq h \)-slope decomposition if it decomposes as a direct sum
\[
M = M^{u \leq h} \oplus M^{u > h}
\]
such that
Both summands are $u$-stable.

$M^{u \leq h}$ is finitely generated over $A$.

For every $m \in M^{u \leq h}$ there is a polynomial $Q \in R[t]$ of slope $\leq h$ with $Q^*(0)$ a multiplicative unit, such that $Q^*(u)m = 0$.

For any polynomial $Q \in R[t]$ of slope $\leq h$ with $Q^*(0)$ a multiplicative unit, the map
\[
Q^*(u) : M^{u > h} \to M^{u > h}
\]
is an isomorphism.

If such a decomposition exists it is unique and $u$ acts invertibly on $M^{u \leq h}$. Let $R$ be a Banach $\mathbb{Q}_p$-algebra and let $M$ be a Banach $R$-module with an action of $\mathcal{U}_p^-$ by compact operators. For $u \in \mathcal{U}_p^-$ let $F_u \in R\{\{t\}\}$ denote the Fredholm determinant of $u$ acting on $M$. The following theorem follows directly from results of Coleman, Serre and Buzzard:

**Theorem 3.1.3.** Let $R, M$ be as above and suppose that $M$ is projective as a Banach module with $R$-linear compact operator $u$.

If we have a prime decomposition $F_u(T) = Q(T)S(T)$ in $R\{\{T\}\}$ with $Q$ a polynomial such that $Q(0) = 1$ and $Q^*(T)$ invertible in $R$ then there exists $R_Q(T) \in TR\{\{T\}\}$ whose coefficients are polynomials in the coefficients of $Q$ and $S$ and we have a decomposition of $M$:
\[
M = N_u(Q) \oplus F_u(Q)
\]
of closed $R$ submodules satisfying

- The projector on $N_u(Q)$ is given by $R_Q(u)$.
- $Q^*(u)$ annihilates $N_u(Q)$.
- $Q^*(u)$ is invertible on $F_u(Q)$.

If $A$ is Noetherian then $N_u(Q)$ is projective of finite rank and
\[
\det(1 - tu | N_u(Q)) = Q(t).
\]

When the decomposition $F_u = QS$ is a slope $\leq h$ factorisation then the decomposition in the above theorem is a slope $\leq h$ factorisation and $M^{u \leq h} = N_u(Q)$.

### 3.2 Slope decompositions on cohomology

Let $K = K_p K^p \subset G(A_f)$ be a neat open compact subgroup with $K_p \subset J_G$ and let $R$ be a $\mathbb{Q}_p$ Banach algebra. From now on we assume that $G, \mathcal{H}$ admit compatible Shimura data and set $d = \frac{1}{2}\dim Y_G$.

**Definition 3.2.1.** For an $R[K]$-module $M$ let
\[
\mathcal{C}^\bullet(\tilde{Y}_G(K), M)
\]
be the ‘Borel-Serre’ complex defined in [Han17] Section 2.1 whose cohomology computes $H^\bullet(\tilde{Y}_G(K), M)$ (as $R$-modules). We let $RT(\tilde{Y}_G(K), M)$ be the image of the above complex in the derived category of Banach $R$-modules.

**Remark 3.2.2.** We won’t always have defined an étale sheaf associated to $M$ so by abuse of notation we let $H^\bullet(\tilde{Y}_G(K), M)$ denote the Betti cohomology of the locally constant sheaf of $R$-modules induced by $M$ in this case, noting that this is isomorphic as an $R$-module to the étale cohomology when $M$ has an associated étale sheaf (we might think of this as bequeathing the Betti cohomology with a Galois action).
Suppose $M$ is an orthonormalisable Banach $R$-module and with a continuous action of $A^-$. Then we can define an action of the Hecke algebra $\mathbb{T}_{S,p}$ on the complex $\mathcal{C}^\bullet(\tilde{Y}_G(K), M)$ via its interpretation as an algebra of double coset operators. Suppose further that $A^-$ acts compactly on $M$. Then the action of $\U_p^-$ on $\mathcal{C}^\bullet(\tilde{Y}_G(K), M)$ acts compactly on the total complex $\bigoplus \mathcal{C}^\bullet(\tilde{Y}_G(K), M)$. We refer to the following proposition from [Han17 2.3.3]:

**Proposition 3.2.3.** Let $R$ be an affinoid algebra. If $C^\bullet$ is a complex of projective Banach $R$-algebras equipped with an $R$-linear compact operator $u$, then for any $x \in \text{Sp}(R)$ and $h \in \mathbb{Q}_{\geq 0}$ there is an affinoid subdomain $\text{Sp}(R') \subset \text{Sp}(R)$ such that $x \in \text{Sp}(R')$ and such that the complex $C^\bullet \hat{\otimes}_R R'$ admits a slope $\leq h$ decomposition for $u$ and $\bigotimes (C^\bullet \hat{\otimes}_R R')^{w \leq h}$ is a complex of finite flat $R'$-modules.

Let $V \subset W_G$ be a connected affinoid open and let $x_0 \in V$. Let $\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{fs}$ denote the localisation at any $u \in \U_p^-$ (this is independent of the choice of $u$). Since

$$u\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an}) \subset \mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})$$

it is immediate that

$$\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{fs} \cong \mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an})^{fs}.$$  

By Proposition 3.2.3 we can shrink $V$ to an affinoid $V'$ also containing $x_0$ and such that the complex admits a slope $\leq h$ decomposition for any $u \in \U_p^-$, $h \in \mathbb{Q}_{\geq 0}$ with projector $e^{\leq h} \in \mathcal{O}(V)\{\{u\}\}$. In this case

$$\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an})^{w \leq h} \cong \mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{w \leq h}.$$  

**Lemma 3.2.4.** For any $h \in \mathbb{Q}_{\geq 0}$ the slope $\leq h$ total cohomology $H^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an})^{w \leq h}$ is a Galois-stable direct summand of $H^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an}).$

**Proof.** It’s clear from the above discussion that

$$\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{fs} \cong \mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m+1}^{an})^{fs}$$

for all $m$ and moreover since the idempotent $e^{\leq h}$ preserves $\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})$ we have

$$e^{\leq h}\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an}) = \mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{w \leq h}.$$  

and the right-hand side is a direct summand of the left hand side. This implies that $H^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})^{w \leq h}$ is a direct summand of $H^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})$ since the Hecke operators commute with the Galois action this summand is Galois stable.

**Lemma 3.2.5.** Let $U \subset W_G$ be a wide open disc and let $x_0 \in U$. There is a wide open disc $x_0 \in U' \subset U$ such that the complex $\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})$ admits a slope $\leq h$ decomposition.

**Proof.** By Proposition 3.2.3 there is an affinoid $V \subset U$ containing $x_0$ over which $\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{V,m}^{an})$ admits a slope $\leq h$ decomposition. By shrinking we can assume that $V$ is a closed disc centered on $x_0$. Let $U'$ be the wide-open disc given by taking the ‘interior’ of this affinoid. Since an orthonormal basis of $A_{V,m}^{an}$ gives an orthonormal basis of $A_{U',m}^{an}$ and the Banach norm on $\Lambda_G(U')[1/p]$ restricts to the Gauss norm on $\mathcal{O}(V)$ we get a slope $\leq h$ decomposition on $\mathcal{C}^\bullet(\tilde{Y}_G(K), A_{U',m}^{an})$ and all the above results hold in this case. Note in particular that the projector $e^{\leq h}$ is still in $\mathcal{O}(V)\{\{u\}\}$ when computing the decomposition for $U'$.

### 3.3 Refined slope decompositions and classicality

As in [SW21 Section 3.5] we consider a more refined slope decomposition. For $i = 1, \ldots, n$ let $Q_{G,i}$ denote the maximal parabolic subgroups of $G$ containing $Q_G$. These correspond to a subset $\{a_1, \ldots, a_n\}$ of the simple roots of $G$ and by taking $a_i \in A^-$ such that $\nu(a_i(a_j)) > 0$ and $\nu(a_j(a_i)) = 0$ for $j \neq i$ we can associate Hecke operators $U_i \in \U_p^-$ as the image of $a_i$ under the isomorphism $\mathbb{Z}_p[A^-/\mathbb{A}(\mathbb{Z}_p)] \cong \U_p^-$. 

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Definition 3.3.1. Set $h^\text{crit} := -((\lambda, \alpha_i) + 1)e(\alpha(\alpha_i))$.

Let $h := (h_1, \ldots, h_n) \in \mathbb{Q}_{\geq 0}^n$. Suppose a Banach $R$ module $M$ admits a slope $\leq h_{\text{aux}}$ decomposition with respect to the operator $U_0 := U_1 \cdots U_n \in \mathfrak{U}_p^-$ for some $h_{\text{aux}} > \prod h_i$, so that $M \leq h_{\text{aux}}$ is a finite projective Banach $R$-module. In particular, the whole of $A^-$ now acts compactly and supposing that the Fredholm series $F_i$ admit slope $h_i$ decompositions for each $i$ then we can define

$$M \leq h = \cap_i (M \leq h_{\text{aux}}) \leq h_i.$$  

**Lemma 3.3.2.** For $0 \leq i \leq n$ let $h^{(i)} = (h_1, \ldots, h_i)$ and set

$$M \leq h^{(i)} = (M \leq h^{(i)})_{U_i+1 \leq h_i+i}.$$  

Then

$$M \leq h^{(n)} = M \leq h.$$  

**Proof.** It suffices to prove the following statement: Suppose $M$ is a Banach module equipped with two compact operators $U_1, U_2$ whose Fredholm determinants $F_1, F_2$ admit slope $h_1, h_2$ factorisations respectively. Then

$$M_{U_1 \leq h_1} \cap M_{U_2 \leq h_2} = (M_{U_1 \leq h_1})_{U_2 \leq h_2}$$  

Suppose $F_2 = Q_2S_2$ is the slope $\leq h_2$ factorisation and $\tilde{F}_2 = \tilde{Q}_2\tilde{S}_2$ is a slope factorisation of the Fredholm determinant $F_2$ of $U_2$ restricted to $M_{U_1 \leq h_1}$. Then $Q_2$ divides $Q_2$ so for $m \in (M_{U_1 \leq h_1})_{U_2 \leq h_2}$ we have $Q_2^2m = 0$ and thus $m \in M_{U_1 \leq h_1} \cap M_{U_2 \leq h_2}$.

Conversely suppose $m \in M_{U_1 \leq h_1} \cap M_{U_2 \leq h_2}$. Then in particular $m \in M_{U_1 \leq h_1}$ and so we can write $m = m_2 + n$ where $m_2 \in (M_{U_1 \leq h_1})_{U_2 \leq h_2}$ and $n$ is in the complement $(M_{U_1 \leq h_1})_{U_2 > h_2}$. Then $Q_2^2(U_2)m = 0$ but also $Q_2^2(U_2)n = 0$ by the same argument as in the first inclusion, so $Q^2(U_2)n = 0$ and since $Q_2^2$ is a slope $\leq h_2$ polynomial and $Q^2(0)$ is a multiplicative unit then $n = 0$.  

**Corollary 3.3.3.** The module $M \leq h$ is a finite projective $R$-module and a direct summand of $M$ with projector $e \leq h \in R\{\{U_1, \ldots, U_n\}\}$.

We say $h$ is **non-critical** if for all $i = 1, \ldots, n$ we have $h_i < h_i^\text{crit}$.

**Theorem 3.3.4.** For $h \in \mathbb{Q}_{\geq 0}^n$ non-critical and $\lambda \in \mathcal{X}^*_+ (S_G)$ there is a quasi-isomorphism:

$$R\Gamma(Y_G(K), \mathcal{O}_{\mathfrak{X}^\text{an}_{\lambda,m}}^{\leq h}) \cong R\Gamma(Y_G(K), V_{\lambda,m}^G)^{\leq h}.$$  

**Proof.** This is proved for compactly supported cohomology with coefficients in modules of distributions in [SW21, Theorem 4.4] using the dual of a parabolic locally analytic BGG complex. The same proof (sans dualising) can easily be adapted to our setting using this complex.

We end with a variation of Proposition 3.2.3.

**Lemma 3.3.5.** Let $R$ be an affinoid algebra. If $C^\bullet$ is a complex of projective Banach $R$-algebras equipped with a continuous $R$-linear action of $A_p^-$ such that $A_p^-$ acts via compact operators. Then for any $x \in \text{Sp}(R)$ and $h \in \mathbb{Q}_{\geq 0}^n$ there is an affinoid subdomain $\text{Sp}(R') \subset \text{Sp}(R)$ such that $x \in \text{Sp}(R')$ and such that the complex $C^\bullet \hat{\otimes}_R R'$ admits a slope $\leq h$ decomposition and $(C^\bullet \hat{\otimes}_R R')^{\leq h}$ is a complex of finite flat $R'$-modules.

**Proof.** We know that by Proposition 3.2.3 there is an affinoid subdomain $\text{Sp}(R_0)$ containing $x$ and such that $U_0$ admits a slope $h_{\text{aux}}$ decomposition on $\hat{\otimes} C^\bullet$ and affinoids $\text{Sp}(R_i)$ such that $U_i$ admits a slope decomposition on $(\hat{\otimes} C^\bullet)^{\leq h_{\text{aux}}}$ and $x \in \text{Sp}(R_i)$ for each $i$. Taking the intersection of these subsets gives us the required affinoid. The finite flatness follows from the above corollary.

**Definition 3.3.6.** If $\mathcal{U}$ is a wide-open disc such that $H^\bullet(Y_G(K), A_{\mathcal{U},m}^\text{an})$ admits a slope $\leq h$ decomposition we say that $\mathcal{U}$ is $h$-adapted.
3.4 Control theorem

We prove control results for the cohomology of locally symmetric spaces.

Lemma 3.4.1. Let \( \lambda \in \mathcal{W}_m \) with residue field \( L \) and \( \mathcal{U} \subset \mathcal{W}_m \) a wide-open disc containing \( \lambda \), then there is a quasi-isomorphism

\[
R\Gamma(\bar{Y}_G(K), A_{an}^{\mathcal{U},m}]^1[/p] \otimes_{\Lambda_G(\mathcal{U})[1/p]} L \sim R\Gamma(\bar{Y}_G(K), A_{an}^{\lambda,m] \otimes_{Q_p} L}).
\]

Proof. This follows easily from the fact that

\[
A_{an}^{\mathcal{U},m}]^1[/p] \otimes_{\Lambda_G(\mathcal{U})[1/p]} L = A_{an}^{\lambda,m] \otimes_{Q_p} L}.
\]

Corollary 3.4.2. For \( \mathbf{h} \in \mathbb{Q}_{\geq 0}^n \) non-critical, an \( \mathbf{h} \) adapted wide-open disc \( \mathcal{U} \subset \mathcal{W}_m \) and algebraic \( \lambda \in \mathcal{U} \), there is a quasi-isomorphism

\[
R\Gamma(\bar{Y}_G(K), A_{an}^{\mathcal{U},m}]^1[/p] \leq_{\mathbf{h}} \otimes_{\Lambda_G(\mathcal{U})[1/p]} L \sim R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}}[1/p]^{\leq_{\mathbf{h}}} \otimes_{Q_p} L).
\]

Proof. This is an immediate corollary of Theorem [3.3.4] and the previous lemma.

3.5 Vanishing results

Let \( \mathcal{U} \) be an \( \mathbf{h} \)-adapted wide-open disc.

Definition 3.5.1. For \( \mathbf{h} \in \mathbb{Q}_{\geq 0}^n \) set

\[
\mathcal{T}_{\mathcal{U},\mathbf{h}} = \text{im} \left( \mathcal{T}_{S,p}^1 \to \text{End}_{\Lambda_G(\mathcal{U})[1/p]}(R\Gamma(\bar{Y}_G(K), A_{an}^{\mathcal{U},m}]^1[/p] \leq_{\mathbf{h}}) \right)
\]

\[
\mathcal{T}_{\lambda,\mathbf{h}} = \text{im} \left( \mathcal{T}_{S,p}^1 \to \text{End}_{\Lambda_G(\mathcal{U})[1/p]}(R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}}[1/p]^{\leq_{\mathbf{h}}}) \right)
\]

Lemma 3.5.2. The natural map

\[
\mathcal{T}_{\mathcal{U},\mathbf{h}} \to \mathcal{T}_{\lambda,\mathbf{h}}
\]

induces a bijection

\[
\text{Spec}(\mathcal{T}_{\lambda,\mathbf{h}}) \to \text{Spec}(\mathcal{T}_{\mathcal{U},\mathbf{h}})
\]

(8)

Proof. This is [AS08, Theorem 6.2.1(ii)].

Lemma 3.5.3. Suppose \( \lambda \in \mathcal{U} \) and \( \mathfrak{m}_{\lambda} \subset \mathcal{T}_{\lambda,\mathbf{h}} \) is a maximal ideal such that

\[
R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}}[1/p]^{\leq_{\mathbf{h}}}
\]

is quasi-isomorphic to a complex concentrated in degree \( q = \dim Y_G \). Then if \( \mathfrak{m}_{\mathcal{U}} \) is the image of \( \mathfrak{m}_{\lambda} \) under the identification \( \mathbf{N} \) then

\[
R\Gamma(\bar{Y}_G(K), A_{an}^{\mathcal{U},m}]^{\leq_{\mathbf{h}}} \mathfrak{m}_{\mathcal{U}}
\]

is quasi-isomorphic to a complex of projective \( (\mathcal{T}_{\mathcal{U},\mathbf{h}})_{\mathfrak{m}_{\mathcal{U}}} \) modules concentrated in degree \( q \).

Proof. This follows from a standard argument using the Tor spectral sequence arising as a consequence of Corollary [3.4.2] and Nakayama’s lemma.

4 Classes in Galois cohomology

We give a recipe for mapping étale classes into Galois cohomology.
4.1 Bits of eigenvarieties and families of Galois representations

Let \( h \in \mathbb{Q}_{\geq 0} \). Consider the total étale cohomology \( H^\bullet(\bar{Y}_G(K), \mathcal{A}_{U,m}|[1/p])^{\leq h} \) for an \( h \)-adapted wide-open disc \( U \).

**Definition 4.1.1.** For \( U, h \) as above define \( \mathcal{E}_{U,h} \) to be the quasi-Stein rigid space such that

\[
\mathcal{O}(\mathcal{E}_{U,h}) := T^\bullet_{U,h} \otimes_{\Lambda_G(U)} [1/p]^{\leq h} \mathcal{O}(U),
\]

The structure morphism

\[
w : \mathcal{E}_{U,h} \to \mathcal{W}_G
\]
is finite and we refer to it as the weight map.

For \( L/\mathbb{Q}_p \) a point \( x \in \mathcal{E}_{U,h}(L) \) corresponds to an eigensystem of \( T_{S,p} \) acting on \( H^\bullet(\bar{Y}_G(K), \mathcal{A}_{w(x)}|[1/p])^{\leq h} \mathcal{O}(U) \).

**Definition 4.1.2.** We call a point \( x \in \mathcal{E}_{U,h} \) classical if \( w(x) \) is the restriction of a dominant algebraic character and the associated eigensystem occurs in \( H^\bullet(\bar{Y}_G(K), V_{w(x)}) \).

**Remark 4.1.3.** Theorem 3.3.4 says that non-critical slope eigensystems of classical weight are classical.

**Definition 4.1.4.** We say a classical point \( x \in \mathcal{E}_{U,h} \) is really nice if

\[
H^\bullet(\bar{Y}_G(K), V_{w(x)})[1/p]_x = H^q(\bar{Y}_G(K), V_{w(x)})[1/p]_x,
\]
is a free module and the weight map is étale at \( x \).

**Definition 4.1.5.** Define a complex of coherent sheaves \( M^\bullet_{U,h} \) over \( \mathcal{E}_{U,h} \) as that induced by the complex of \( \mathcal{O}(\mathcal{E}_{U,h}) \)-modules

\[
\mathcal{C}^\bullet(\bar{Y}_G(K), \mathcal{A}_{U,m}|[1/p])^{\leq h} \otimes_{\Lambda_G(U)} [1/p] \mathcal{O}(U).
\]

**Proposition 4.1.6.** Let \( x \in \mathcal{E}_{U,h} \) be a really nice point. Then there is an affinoid neighbourhood \( x \in \mathcal{V} \subset \mathcal{E}_{U,h} \) such that for non-critical \( h \) the complex of sheaves

\[
M^\bullet_{U,h}|_{\mathcal{V}}
\]
is quasi-isomorphic to a complex of locally free sheaves concentrated in degree \( q \).

**Proof.** By Lemma 3.5.3 the stalk of \( M^\bullet_{U,h} \) at \( x \) is quasi-isomorphic to a complex concentrated in degree \( q \). By coherence we can find an affinoid \( \mathcal{V} \subset \mathcal{U} \) containing \( x \) such that

\[
M^\bullet_{U,h}|_{\mathcal{V}}
\]
is quasi-isomorphic to a complex concentrated in degree \( q \).

**Definition 4.1.7.** We say an affinoid \( \mathcal{V} \subset \mathcal{E}_{U,h} \) is pretty sweet if the restriction of the weight map to \( \mathcal{V} \) is an isomorphism onto its image and \( M^\bullet_{U,h}|_{\mathcal{V}} \) is quasi-isomorphic to a complex of locally free sheaves concentrated in degree \( f \).

Clearly a subaffinoid of a pretty sweet affinoid is also pretty sweet.

**Lemma 4.1.8.** If \( x \in \mathcal{E}_{U,h} \) is really nice then it has an affinoid neighbourhood \( \mathcal{V} \) which is pretty sweet.

**Proof.** This follows immediately from the weight map being étale at really nice points and Proposition 4.1.6.
Suppose now that $x \in \mathcal{E}_{t,h}$ is really nice with pretty sweet neighbourhood $\mathcal{V}$. Take a wide-open disc $U' \subset \mathcal{V}$ containing $x$ so that $w^{-1}(w(U'))$ is a finite disjoint union of wide-open discs $U_i \subset \mathcal{E}_{t,h}$. Let $\tilde{f} \in \mathcal{T}_{w(U'),h}$ be an idempotent satisfying
\[
\tilde{f} \cdot \mathcal{T}_{w(U'),h} \cong \mathcal{O}(U')
\]
and $\tilde{f} \cdot H^\bullet(\tilde{Y}_G(K),\mathcal{A}_{w(U')})[1/p]^{\leq h} = H^q(\tilde{Y}_G(K),\mathcal{A}_{w(U')})[1/p]^{\leq h}|_{U'}$ and let $f \in \mathbb{T}_{S,p} \mathcal{O}(U')$ be a lift of $\tilde{f}$.

**Definition 4.1.9.** Define an $\mathcal{O}(U')$-linear Galois representation
\[
W_{U'} := H^q(\tilde{Y}_G(K),\mathcal{A}_{w(U')})[1/p]^{\leq h} \otimes_{\mathcal{T}_{U',h}} \mathcal{O}(U') = f \cdot H^q(\tilde{Y}_G(K),\mathcal{A}_{w(U')})[1/p]^{\leq h}.
\]

This Galois representation is a direct summand of $H^q(\tilde{Y}_G(K),\mathcal{A}_{U})[1/p]^{\leq h}$ with projector $f$.

**Lemma 4.1.10.** For $x \in \mathcal{E}_{t,h}$ really nice with residue field $L(x)$ and $U'$ we have
\[
W_{U'} \otimes_{\mathcal{O}(w(U))} L(x) \cong f \cdot H^q(\tilde{Y}_G(K),V_{w(x)}|_x) =: W_x.
\]

**Proof.** The left hand side is isomorphic to the stalk of $H^\bullet(\tilde{Y}_G(K),\mathcal{A}_{U})^{\leq h}$ at $x$ and this is equal to the right hand side by standard control results. \qed

### 4.2 Abel-Jacobi maps

Let $\Sigma$ be a set of primes of the reflex field $E$ such that we have an integral model $Y_G,\Sigma$ of $Y_G$ over $\mathcal{O}_E[\Sigma^{-1}]$ and let $K = K^{p}K^p \subset G_{(\kappa_f)}$ be a neat open compact subgroup with $K_p \subset J_G$. In this section we construct a weight $\kappa$ Abel-Jacobi map
\[
AJ_{\kappa}^{\leq h} : (f \cdot \varepsilon^{\leq h})H^{q+1}(Y_G(K),A_{\kappa,m}^{1w})[1/p] \to H^1(\mathcal{O}_E[\Sigma^{-1}],W_{\kappa})
\]
for weights $\kappa : G_{\Sigma} \to B^\times$, where $W_{\kappa}$ is a Galois representation defined below.

By the Hochschild-Serre spectral sequence there is an Abel-Jacobi map
\[
AJ : H^{q+1}(Y_G(K)_{\Sigma},A_{\kappa,m}^{1w})[1/p]_0 \to H^1(\mathcal{O}_{E,S},H^{q}(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p])
\]
where $H^{q+1}(Y_G(K)_{\Sigma},A_{\kappa,m}^{1w})[1/p]_0$ is the kernel of the base-change map
\[
H^{q+1}(Y_G(K)_{\Sigma},A_{\kappa,m}^{1w}) \to H^{q+1}(\tilde{Y}_G(K),A_{\kappa,m}^{1w}).
\]

Let $h \in \mathbb{Q}_{\geq 0}$.

**Lemma 4.2.1.** Let $\varepsilon^{\leq h} \in B\{\{U_p\}\}$ be the slope $\leq h$ projector on $\mathcal{E}^*(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p]$, then
\[
\varepsilon^{\leq h}H^\bullet(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p] = H^\bullet(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p]^{\leq h}.
\]

**Proof.** The cocycles $Z(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p]$ are closed and so $\varepsilon^{\leq h}$ converges and $\varepsilon^{\leq h}Z(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p] = Z(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p]^{\leq h}$. Since the differentials $d$ are continuous and $U_p$-equivariant and since $\varepsilon^{\leq h}$ converges on each $\mathcal{E}^*(\tilde{Y}_G(K),A_{\kappa,m}^{1w})$ then $d(\varepsilon^{\leq h}x) = \varepsilon^{\leq h}d(x)$ and so $\varepsilon^{\leq h}$ preserves coboundaries and $\varepsilon^{\leq h}B(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p] = B(\tilde{Y}_G(K),A_{\kappa,m}^{1w})[1/p]^{\leq h}$ whence the result follows. \qed

We note that for $B = \mathcal{O},\Lambda_{G}(U)$, the series $\varepsilon^{\leq h}$ is a $p$-adic limit of polynomials. This is clear for $B = \mathcal{O}$ and also holds when $B = \Lambda_{G}(U)$. Indeed, by construction $\varepsilon^{\leq h}$ converges on an affinoid $\mathcal{Y} \times \mathbb{A}_{\text{rig}}^n$ containing $U \times \mathbb{A}_{\text{rig}}^n$ and thus we can write $\varepsilon^{\leq h}$ as a $p$-adic limit of $O(\mathcal{Y})$-coefficient polynomials in the operators $U_1,\ldots,U_n$. We can assume without loss of generality that $\varepsilon^{\leq h}$ is optimally integrally normalised in the sense that $\varepsilon^{\leq h} \in O(\mathcal{Y})^\circ \{\{U_1,\ldots,U_n\}\}$ and $\varepsilon^{\leq h} \neq 0 \bmod p$. Since $p^r$ vanishes on $H^1(\tilde{Y}_G(K),A_{\kappa,m}^{1w}/\Fil^r)$ we
have a well-defined action of $e^\Sigma \hbar$ which mod $p^r$ is represented by a polynomial $e^\Sigma \hbar_r \in B[U_1, \ldots, U_n]$ and this sequence satisfies $e^\Sigma \hbar = \lim_r e^\Sigma \hbar_r$. We can arrange it such that

$$e^\Sigma \hbar_{r+1} \equiv e^\Sigma \hbar_r \mod p^r.$$  

Since $\Omega_\hbar$ acts on $H^\bullet(Y_G(K)_{\Sigma, A_{\kappa, m}/\Fil^r})$ for all $r$ the collection of elements \{\$e^\Sigma \hbar_r\$\}_{r \geq 0} map to a compatible system of endomorphisms of the inverse system $H^\bullet(Y_G(K)_{\Sigma, A_{\kappa, m}/\Fil^r})$ and thus we get an action of $e^\Sigma \hbar$ on $\lim H^\bullet(Y_G(K)_{\Sigma, A_{\kappa, m}/\Fil^r}) = H^\bullet(Y_G(K)_{\Sigma, A_{\kappa, m}})$. Moreover, since the base-change map

$$BC_n : H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}/\Fil^r}) \to H^{q+1}(Y_G(K), A_{\kappa, m}/\Fil^r)$$

is $\Omega_\hbar$-equivariant for $0 \leq r \leq \infty$ and $p$-adically continuous it commutes with $e^\Sigma \hbar$ and thus induces a map

$$e^\Sigma \hbar H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}}) \to H^{q+1}(Y_G(K), A_{\kappa, m}) \leq \hbar.$$  

**Remark 4.2.2.** There is no reason for us to believe that the image of $e^\Sigma \hbar$ in $\End(H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}}))$ is an idempotent.

For $h$ non-critical and $V \subset \mathcal{E}_{\Omega, h}$ a pretty sweet affinoid containing algebraic $\lambda \in X_\bullet^*(\Sigma_G/S_G)$, we have

$$H^{q+1}(Y_G(K), A_{\kappa, m}/\Fil^r)[1/p] \leq \Sigma_\hbar V = 0$$

and

$$H^{q+1}(Y_G(K), A_{\kappa, m}/\Fil^r)[1/p] \leq \Sigma_\hbar V = 0$$

and thus

$$e^\Sigma \hbar H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}})[1/p] \subset H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}})[1/p]0.$$  

Recall in the previous section we defined an element $f \in T_{\Sigma, \hbar} \otimes \Lambda(GU)$.  

**Definition 4.2.3.** We define the weight $\kappa$ slope $\leq \hbar$ Abel-Jacobi map

$$AJ_{\kappa \leq \hbar}(f, e^\Sigma \hbar)H^{q+1}(Y_G(K)_{\Sigma, A_{\kappa, m}})[1/p] \to H^1(\mathcal{E}_{\Sigma}^{[S]}(K), W_{\kappa})$$

to be $AJ$ precomposed with $f \cdot e^\Sigma \hbar$.

## 5 Criterion for $Q^0_H$-admissibility and superfluous variables

### 5.1 Criterion for $Q^0_H$-admissibility

Let $\mu \in X_\bullet^*(S_H)$, $\lambda \in X_\bullet^*(S_G)$ and suppose there is an injective $H$-map

$$V_\mu^H \to V_\lambda^G.$$  

Is there a character $\chi \in X^\bullet(G)$ such that $\lambda + \chi \in X^\bullet^*(S_G)Q_H^\mu$?

**Lemma 5.1.1.** Suppose $\mu$ is trivial on $Q^0_H \cap G^{\text{der}}$ and the maximal torus quotient $C_G$ of $G$ is split. Then there is $\chi_\mu \in X^\bullet(G)$ such that $(\lambda \otimes \chi_\mu^{-1})Q_H^\mu \neq \{0\}$.

**Proof.** We have an injection

$$Q^0_H/(Q^0_H \cap G^{\text{der}}) \to C_G$$

and thus $Q^0_H/(Q^0_H \cap G^{\text{der}})$ is a split torus. The the restriction of $\mu$ to $Q^0_H$ lifts (non-uniquely) to a character $\chi_\mu$ of $G$ whence the result follows.  

\[\square\]
The assumptions in Lemma 5.1.1 won’t hold in every case, so we describe a process (that feels very much like cheating) to widen the number of $Q^0_H$ admissible weights by substituting the pair $(G, H)$ for a slightly modified pair $(G, H)$.

Assume $S^0_H$ satisfies Milne’s assumption (SV5). This is equivalent to the real points of the subgroup

$$(S^0_H)^a = \cap_{\chi \in \mathcal{X}^*(S_H)} \text{Ker}(\chi)$$

being compact. For an algebraic group $M$ define $\tilde{M} := M \times S^0_H$ and let $Q^0_H := \{(h, \tilde{h}) \in Q^0_H \times S^0_H\}$

where $\tilde{h}$ denotes the image of $h$ in $S^0_H$. Since this is the kernel of the map $\tilde{Q}_H \to S^0_H$ given by $(q, s) \mapsto \tilde{q}s^{-1}$ it is a mirabolic subgroup of $\tilde{Q}_H$. We have that $\tilde{F} := \tilde{Q}_G \backslash \tilde{G} \cong F$ and its easy to see that $Q^0_H$ has an open orbit on $\tilde{F}$. A character $\mu \in \mathcal{X}^*(S_H)$ induces a character $\tilde{\mu} \in \mathcal{X}^*(Q^0_H)$ given by

$$(h, \tilde{h}) \mapsto \mu(\tilde{h})$$

which corresponds to $\mu$ under the isomorphism $Q^0_H \cong Q^0_H$ sending $h$ to $(h, \tilde{h})$. What’s more, $\tilde{\mu}$ admits an extension to a character $\chi_\mu \in \mathcal{X}^*(\tilde{G})$ by simply taking for $(g, s) \in \tilde{G}$

$$\chi_\mu(g, s) = \mu(s).$$

Thus

$$Q^0_H \otimes \chi_{\mu} \neq \{0\}$$

i.e. the weight $\lambda - \chi_{\mu}$ is $Q^0_H$-admissible.

### 5.2 Superfluous variables

For this section we suppose that there is a central torus $T_Z \subset Z_G$ such that

$$S_G/S^0_G = T_Z/(S^0_G \cap T_Z) \times S^Z = S_Z \times S^Z$$

for some complementary torus $S^Z$.

**Remark 5.2.1.** This will notably occur when the group itself is of of the form $G \times T_Z$ as in, for example, the construction given in Section 5.1.

In this case we get a product decomposition

$$\mathcal{W}_G = \mathcal{W}_Z \times \mathcal{W}^Z$$

where the components correspond to those in the decomposition \[2\].

**Lemma 5.2.2.** Let $K_G = K^pJ_G \subset G(\mathbb{A}_f)$ be a neat open compact subgroup. Let $U_Z \subset \mathcal{W}_Z, U^Z \subset \mathcal{W}^Z$ be wide-open discs and set $U = U_Z \times U^Z$. For any $h \in \mathbb{Q}^{n}_{\geq 0}$ there is an isomorphism of $\Lambda_U = \Lambda_{U^Z} \otimes \Lambda_{U_Z}$-modules

$$H^\bullet(Y_G(K_G), A^w_{U,m}) \cong \otimes h \otimes \Lambda_{U_Z}$$

**Proof.** There are finitely many $g \in G(\mathbb{A}_f)$ and arithmetic subgroups $\Gamma_g \subset G(\mathbb{Q})$ such that

$$\mathcal{C}^\bullet(Y_G(K_G), A^w_{U,m}) = \otimes \mathcal{C}^\bullet(\Gamma_g, A^w_{U,m})$$

\[2\] We emphasise again that we are only imposing this assumption so that we don’t have to quotient out by central subgroups every five seconds.
as $\Lambda_{U}$-modules. Moreover, since $K_G$ is assumed neat, the groups $\Gamma_g$ have trivial centre so in particular for any $g$ we have a $\Gamma_g$-module isomorphism

$$A_{U,m}^{lw} \cong A_{U,m}^{lw} \otimes \Lambda_{U}$$

and

$$\mathcal{E}^\bullet_2(\Gamma_g, A_{U,m}^{lw}) = \mathcal{E}^\bullet_2(\Gamma_g, A_{U,m}^{lw}) \otimes \Lambda_{U}.$$

Recall from Section 3.3 that $a_i \in A^-$ are elements defining the Hecke operators $U_i$ used in our slope decompositions. Taking the image of $a_i$ in $S_G/S^0_G$ we have a decomposition $a_i = a_iZ \times a_i^Z$ corresponding to (9) and it’s easy to see that $a_iZ$ acts trivially on $A_{U,m}^{lw}$ from which we can infer that for $h \in \mathbb{Q}_{\geq 0}$:

$$\mathcal{E}^\bullet_2(\Gamma_g, A_{U,m}^{lw}) \otimes h \cong \mathcal{E}^\bullet_2(\Gamma_g, A_{U,m}^{lw}) \otimes h \otimes \Lambda_{U}$$

and thus

$$\mathcal{E}^\bullet_2(\tilde{Y}_G(K_G), A_{U,m}^{lw}) \cong \mathcal{E}^\bullet_2(\tilde{Y}_G(K_G), A_{U,m}^{lw}) \otimes h \otimes \Lambda_{U}.$$

Since $\Lambda_{U}$ is a flat $\mathcal{O}$-module we deduce the result.

The lemma has a global geometric interpretation:

**Lemma 5.2.3.** Let $\mathcal{U} = U^Z \times W_Z$, then

$$\mathcal{E}^\bullet_2 \otimes h \cong \mathcal{E}^\bullet_2 \otimes h \times W_Z.$$

**Proof.** This follows from the fact that

$$\text{End}_{\Lambda_{U} \otimes \mathcal{O}_W} (\mathcal{E}^\bullet_2(\tilde{Y}_G(K_G), A_{U,m}^{lw}) \otimes h \otimes \mathcal{O}_W) = \text{End}_{\Lambda_{U}} (\mathcal{E}^\bullet_2(\tilde{Y}_G(K_G), A_{U,m}^{lw}) \otimes h) \otimes \mathcal{O}_W.$$

Throughout this paper there have been a number of times where we have had to shrink our subspace $\mathcal{U} \subset W_G$. The upshot of the above discussion is that if $\mathcal{U}$ decomposes as a product over the decomposition (9) then we only need to shrink the $W^Z$-component. If $D_{\mathcal{U}}(S_Z, M)$ denotes the space of locally analytic distributions on $S_Z$ with values in a Banach-module $M$ then there is a canonical isomorphism

$$D_{\mathcal{U}}(S_Z, M) \cong \mathcal{O}_{W_Z} \otimes M.$$

For $\mathcal{U} = U_Z \times W^Z$ the Galois representation of Definition 4.1.9 looks like

$$W_{\mathcal{U}} \cong D_{\mathcal{U}}(S_H, W_{U_Z})$$

echoing Theorem A [LZ16].

**Example 5.2.4.** In Section 5.1 we showed that we could modify a spherical pair $(G, H)$ to get a pair $(\tilde{G}, \tilde{H})$ such that any weight $\lambda \in X^*_+(S_G)$ is $Q^0_H$-admissible up to a twist by a character $\chi$ of $\tilde{G}$. Recall that $\tilde{G} = G \times S^0_H$ so there is an obvious decomposition $S_{\tilde{G}}/S^0_{\tilde{G}} = S_G/S^0_G \times S^0_H$ of the form (9) and thus a decomposition

$$W_{\tilde{G}} = W_G \times W_Z.$$

Moreover, as showed in Section 5.1 the character $\chi$ can be chosen to factor through the $S^0_H$ component, i.e. $\chi \in W_Z$. Therefore, using the results of this section, taking $\mathcal{U} = U \times W_Z$ with $\mathcal{U} \subset W_G$ we can construct a class $f_{\mathcal{U}}^{\text{ph}} \in A_{\mathcal{U},m}^{lw}$ such that for any $\lambda \in \mathcal{U}$, $f_{\mathcal{U}}^{\text{ph}}$ interpolates the $Q^0_H$-invariant vectors $f_{\lambda, -\chi}^{\text{ph}} \in A_{\lambda, m}^{lw}(-\chi)$. 

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6 Example: \((\text{GSp}_4, \text{GL}_2 \times \text{GL}_1 \text{ GL}_2)\)

We show how the above theory can be used to construct a class interpolating non-ordinary variants of the Lemma-Flach Euler system constructed in \cite{LSZ21}. Let \(G = \text{GSp}_4\) and \(H = \text{GL}_2 \times \text{GL}_1 \text{ GL}_2\). These groups admit a natural embedding

\[ H \hookrightarrow G. \]

as used in \cite{LSZ21}. As in \cite{LRZ21} Section 7.1 the get full weight variation we need to modify \(G\) and \(H\). Set

\- \(\tilde{G} = G \times \text{GL}_1 \times \text{GL}_1, \quad \tilde{H} = (\text{GL}_2 \times \text{GL}_1 \text{ GL}_2) \times \text{GL}_1 \times \text{GL}_1,\)

\- \(Q_{\tilde{G}} = B_{\tilde{G}} = B_G \times \text{GL}_1 \times \text{GL}_1, \quad Q_{\tilde{H}} = B_{\tilde{H}} \times \text{GL}_1 \times \text{GL}_1\), where \(B_G = T_G \times N_G, B_H = T_H \times N_H\) are the respective upper triangular Borels.

\- For a \(\mathbb{Z}_p\)-algebra \(R\) define \(Q_{\tilde{H}}^0(R) = \{(z_1) \times (\begin{pmatrix} x & y \\ y & -1 \end{pmatrix}) \times (y) : x, y \in R^\times \}\).

\- Set \(L_{\tilde{G}} = T_{\tilde{G}} \times \text{GL}_1 \times \text{GL}_1\) and \(L_H = T_H \times \text{GL}_1 \times \text{GL}_1\).

There is a natural embedding

\[ \tilde{H} \hookrightarrow \tilde{G} \]

extending the embedding of \(H\) into \(G\) and \(Q_{\tilde{H}}^0\) has an open orbit on the flag variety \(F\) with trivial stabiliser.

Let \(0 \leq q \leq a, 0 \leq r \leq b\). Consider the following dominant character of \(T_{\tilde{G}}\)

\[ \lambda^{[a,b,q,r]} : \begin{pmatrix} x_1 & x_2 \\ x_2^{-1} & x_3 \\ x_1^{-1} & x_3 \end{pmatrix} \times (x_4) \times (x_5) \rightarrow x_1^{a+b}x_2^aq-b-2x_3^{a-b}x_4^r-q+a^r x_5^q \]

and write \(V^{[a,b,q,r]}\) for the irreducible \(\tilde{G}\)-representation of highest weight \(\lambda^{[a,b,q,r]}\) with maximal admissible lattice \(V^{[a,b,q,r]}_{\mathbb{Z}_p}\). Note that \(V^{[a,b,0,-a]} = D^{a,b}\) in the notation of \cite{LSZ21}. An easy computation using the branching law for \(H \hookrightarrow G\) \cite[Proposition 4.3.1]{LSZ21} shows that

\[ (V^{[a,b,q,r]}_{\mathbb{Z}_p})^{Q_{\tilde{H}}^0} \neq 0 \]

and thus that there is an \(\tilde{H}\)-map

\[ (P^{[c,d]}_{\mathbb{Z}_p})^{\vee} \rightarrow V^{[a,b,q,r]}_{\mathbb{Z}_p} \]

where \(P^{[c,d]}_{\mathbb{Z}_p} := P_{\lambda^{[c,d]},\mathbb{Z}_p}^H\) for the weight

\[ \lambda^{[c,d]} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \times (y_3) \times (y_4) \rightarrow y_1^{a+c}y_2^{-c+d}y_3^{-d} \]

and \(c = a + b - q - r, d = a - q + r\).

Fix prime-to-\(p\) open compact subgroups \(K^p_G \subset G(\mathbb{A}_f^{(1)})\) and \(K^p_H = K^p_G \cap H\) such that \(K^p_GJ_G\) and \(K^p_HJ_H\) are neat. Define \(K^p_{\tilde{G}} = K^p_G \times (\mathbb{A}_f^{(1)})^2\) and \(K^p_{\tilde{H}} = K^p_G \cap \tilde{H}(\mathbb{A}_f^{(1)}).\) We see from \cite{LRZ21} Section 7.1] (but using parahoric test data) that there is a class

\[ c_1, c_2^{[a,b,q,r]}_{\text{et}} \in H^4(Y_{\tilde{G}}(J_G)_{\Sigma}, V^{[a,b,q,r]}_{\mathbb{Z}_p}(3)) \]

obtained by pushing forward a cup-product of Eisenstein classes

\[ c_1, c_2^{[a,b,q,r]}_{\text{et}} = c_1^{[a,b,q,r]} \cup c_2^{[a,b,q,r]} \in H^2(Y_H(J_H)_{\Sigma}, (P^{[c,d]}_{\mathbb{Z}_p})^{\vee}(2)) \]

where the auxiliary values \(c_1, c_2\) are chosen to ensure integrality of the classes as in Section 2.3.4 and \(\phi = \phi_1 \otimes \phi_2\).
By the results of Section 4 if $\Pi$ is a cohomological cuspidal automorphic representation of $G$ of weight $(a, b)$ and non-critical slope $\leq h$ giving a really nice point on the $G$ eigenvariety, then there is an Abel-Jacobi map

$$AJ_\Pi : (f \cdot e^{\leq h}) H^4(Y_G(J_G)\Sigma; V_{\mathbb{Z}_p}^{[a, b, q, r]}(3))_0 \to H^1(\mathbb{Q}, W_\Pi)$$

where $W_\Pi$ is the 4-dimensional Galois representation constructed by Taylor and Weissauer [Tay91, Wei05].

**Theorem 6.0.1.** Let $U \subset W_m$ be a wide-open disc. There is a class

$$c_{1, c_2 z_{U,m}} \in D^{\text{frob}}\left((\mathbb{Z}_p^\times)^2, H^4(Y_G(J_G)\Sigma; E_{Iw}^{[w]}(1))\right)$$

such that for any cohomological cuspidal automorphic representation $\Pi$ of weight $(a, b)$ and non-critical slope $\leq h$ at $p$ giving a really nice point on the $G_{Sp_4}$ eigenvariety and for any $0 \leq q \leq a, 0 \leq r \leq b$, up to shrinking $U$, the image of $AJ_{U}^{\leq h}(c_{1, c_2 z_{U,m}})$ under the specialisation map $\rho^{[a, b, q, r]}$ is $AJ_\Pi(c_{1, c_2 z_{U,m}}^{[a, b, q, r]})$.

**Proof.** The weight space $W_G$ decomposes as

$$W_G = W_G \times W_{GL_1} \times W_{GL_1}.$$ Let $\tilde{U} = U \times W_{GL_1} \times W_{GL_1}$.

Define

$$c_{1, c_2} \tilde{E}_{f, \phi(p)} \in H^2(Y_{\tilde{H}}(J_{Iw}), (N_{H}(\mathbb{Z}_p)\backslash J_{Iw})(2)) \cong H^2(Y_{\tilde{H}}(J_{Iw}), (N_{H}(\mathbb{Z}_p)\backslash J_{Iw})(2)) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times)$$

for prime-to-$p$ Schwartz functions $\phi_1(p), \phi_2(p) = \phi_1(p) \otimes \phi_2(p)$ as the image of $c_{1, c_2} \tilde{E}_{f, \phi_1(p)} \cup c_{1, c_2} \tilde{E}_{f, \phi_2(p)}$. These classes interpolate the classes $c_{1, c_2} \tilde{E}_{f, \phi_1(p)}$ for varying $c, d$. Similar to Section 2.3.4 define

$$c_{1, c_2} \tilde{E}_{f, \phi(p)} \in H^2(Y_{\tilde{H}}(K_{Iw}), (\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times)$$

by pushing forward $c_{1, c_2} \tilde{E}_{f, \phi(p)}$ along the natural map

$$\Lambda(N_{H}(\mathbb{Z}_p)\backslash J_{Iw}) \to D_{\tilde{H}_{U,m}}^{\text{frob}}.$$ We have an isomorphism $H^4(Y_G(J_G), E_{Iw}^{[w]}(1)) \cong H^4(Y_G(J_G), E_{Iw}^{[w]}(1)) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times)$.

Applying the machinery of this paper to $c_{1, c_2} \tilde{E}_{f, \phi}$ we obtain

$$c_{1, c_2} z_{U,m}^J \in H^4(Y_G(J_G)\Sigma; E_{Iw}^{[w]}(1)) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times).$$

Shrinking $U$ if necessary, the slope $\leq h$ Abel-Jacobi map $AJ_{U}^{\leq h}$ is well-defined (after inverting $p$), and we have an equality of classes

$$\rho^{[a, b, q, r]}(AJ_{U}^{\leq h}(c_{1, c_2 z_{U,m}^J})) = AJ_{\Pi}(c_{1, c_2 z_{U,m}^{[a, b, q, r]}}) \in H^1(\mathbb{Q}, W_\Pi),$$

whenever $[a, b, q, r] \in U \times W_{GL_1} \times W_{GL_1}$.

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