Remarks on the transformation of Ito’s formula for jump-diffusion processes

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Abstract
In this paper, we give detailed proofs of the transformation of the sum of the jump components that appears in Ito’s formula for jump-diffusion processes into the stochastic integral with respect to a certain counting process. As applications of the transformed Ito’s formula, the Black-Scholes equations in the compound Poisson process model and the jump-diffusion process model are discussed.

Keywords Ito’s formula, jump-diffusion processes, the Black-Scholes equation

Research Activity Group Mathematical Finance

1. Introduction
The aim of this paper is to confirm the following transformation of Ito’s formula for stochastic processes with jumps:

\[ \sum_{0 \leq s \leq t} (f(X(s^-) + \Delta X(s)) - f(X(s^-))) = \int_0^t (f(X(s^-) + \Delta X(s)) - f(X(s^-))) dN(s), \quad (1) \]

where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is any \( C^2 \) function, \( X(t) \) is a stochastic process with jumps, \( \Delta X(t) = X(t) - X(t^-) = X(t^-) - \lim_{h \downarrow 0} X(t - h) \), and \( N(t) \) is the counting process that jumps at the same time as \( X \). This transformation is taken for granted in the papers (for example, [1] or [2]) and its proof is not provided. However, the transformation is not intuitive. Let \( \{t_i\}_{i=0,1,\ldots,n} \) be the partition of the interval \( [0, t] \) such that \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \) for each \( n \in \mathbb{N} \). Then the right-hand side of (1) may be simply approximated by the Riemann sum as

\[ \int_0^t (f(X(s^-) + \Delta X(s)) - f(X(s^-))) dN(s) \]

\[ \simeq \sum_{i=0}^{n-1} (f(X(t_i)) - f(X(t_{i-1}))) (N(t_{i+1}) - N(t_i)). \quad (2) \]

Suppose that \( n \) is sufficiently large. When \( N(t_{i+1}) - N(t_i) \neq 0 \), the jump of \( X \) occurs in \( (t_i, t_{i+1}] \) and so \( X(t_i) = X(t_{i-1}) \). Thus the right-hand side of (2) is equal to zero. This implies that we should investigate the transformation (1) rigorously. In Section 3, we will prove it in the Riemann integral sense when the jump sizes of the integrand are represented by a predictable process and in the Lebesgue-Stieltjes integral sense in more general case.

The rest of the paper is arranged as follows. In Section 2, we present Ito’s formula for jump-diffusion processes and the theorem that will be used in Section 3, where we see the validity of the transformation of Ito’s formula for jump-diffusion processes. In Section 4, as applications, we derive the Black-Scholes equation in the compound Poisson process model and consider the differential equation that the Black-Scholes hedge portfolio must satisfy in the jump-diffusion process models such as the model in [3].

2. Preliminaries
This section presents Ito’s formula for jump-diffusion processes and the theorem used in the subsequent section.

Let \( (\Omega, \mathcal{F}, P) \) be a probability space with a filtration \( (\mathcal{F}(t))_{0 \leq t \leq T} \) where \( \mathcal{F}(t) = \sigma(W(s), N(s); 0 \leq s \leq t), W(t) \) is a standard Brownian motion, and \( N(t) \) denotes a counting process that can be represented as \( N(t) = \sum_{1 \leq i \leq t} 1 \) whose number is finite with probability one. Let \( Y \) be the jump-diffusion process defined by

\[ Y(t) = Y(0) + \int_0^t a(s)ds + \int_0^t b(s)dW(s) + \sum_{i=1}^{N(t)} F_i, \quad t \in [0, T], \quad (3) \]

where \( a(t) \) and \( b(t) \) are continuous adapted processes and \( F_i, \ i = 1, 2, \ldots \), are \( \mathcal{F}(T_i) \)-measurable i.i.d. random variables that denote the size of the \( i \)th jump of \( Y \). Then by Ito’s formula for jump-diffusion processes, which is taken from [4, Proposition 8.14, p.275], for any \( C^2 \) function \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[ f(Y(t)) = f(Y(0)) + \int_0^t a(s)f'(Y(s))ds \]

\[ + \frac{1}{2} \int_0^t b(s)^2 f''(Y(s))ds \]
\[ + \int_0^t b(s)f'(Y(s))dW(s) \]
\[ + \sum_{i \geq 1, T_i \leq t} (f(Y(T_i-)) + F_i) - f(Y(T_i-)) \]

where \( f'(x) \) and \( f''(x) \) denote the first and second order derivatives, respectively. In the subsequent section, we will see that the last term of the right hand side of this equation can be transformed into a certain integral with respect to a counting process like (1).

We present the theorem which is used in the subsequent section. The following definition and theorem are taken from [5]. For details and the proof of the theorem, the reader is referred to [5, Chapter II, Section 5, pp.56-57]. Here the word càdlàg paths and càglàd paths means right continuous paths with left limits and left continuous paths with right limits, respectively.

**Definition 1** Let \( \sigma \) denote a finite sequence of finite stopping times:
\[ \sigma : 0 = S_0 \leq S_1 \leq \cdots \leq S_k < \infty. \]

The sequence \( \sigma \) is called a random partition. A sequence of random partitions \( \sigma_n \)
\[ \sigma_n : S_{0n} \leq S_{1n} \leq \cdots \leq S_{kn} \]
is said to tend to identity if

(i) \( \lim_n \sup_k S_{kn} = \infty \) a.s.

(ii) \( \| \sigma_n \| = \sup_k |S_{kn} - S_{k+1,n} | \to 0 \) (\( n \to \infty \)) a.s.

**Theorem 2** Let \( X \) be a semimartingale, and let \( Y \) be an adapted process with càdlàg or càglàd paths. Let \( (\sigma_n) \)
be a sequence of random partitions tending to identity. Then the processes \( \sum_i Y(S^k_i)(X(S_{i+1}^k \wedge t) - X(S_i^k \wedge t)) \)
tend to the stochastic integral \( \int Y(s-)dX(s) \) uniformly on compacts in probability.

3. The theorems and proofs

In this section, we prove (1) in the case where the jump sizes of the stochastic processes are represented by càdlàg or predictable processes and then consider the transformation for \( Y \) in (3).

First, we prove (1) for the stochastic process whose jump sizes are represented by a càglàd processes. Let \( X \) be a jump-diffusion process defined by
\[ X(t) = X(0) + \int_0^t a(s)ds + \int_0^t b(s)dW(s) + \sum_{i=1}^{N(t)} c(T_i), \]
where \( a, b, W \) and \( N \) are the same as (3) and \( c(t) \) is a càglàd adapted process.

**Theorem 3** For any continuous function \( f : \mathbb{R} \to \mathbb{R} \),
\[ \int_0^t (f(X(s-) + c(s)) - f(X(s-)))dN(s) \]
\[ = \sum_{i=1}^{N(t)} (f(X(T_i-) + c(T_i)) - f(X(T_i-))). \]

We use the following lemma to prove this theorem.

**Lemma 4** For any \( n \in \mathbb{N} \), let \( t^n_j \) be the partition of \([0, n]\) such that
\[ t^n_j = \frac{j}{2^n}, \quad j = 0, 1, \ldots, n2^n. \]

Let \( (S^n_k) \) be a sequence of random times defined by
\[ S^n_k = t^n_k(1 - 1_{\{t^n_k < T_l \leq t^n_{k+1}\}}) \]
\[ + \sum_{i=1}^k T_l 1_{\{t^n_k < T_l \leq t^n_{k+1}\}}, \]

where \( 1_A(\omega) = 1 \) if \( \omega \in A \) and \( = 0 \) if \( \omega \notin A \). Then \((S^n_k)\) is a random partition tending to identity.

**Proof** Since \( S^n_k < t^n_{k+1} \), \( S^n_k \) is finite. Next, we see that \( S^n_k \) is a stopping time. If \( t^n_k < s \leq S^n_k \) then \( S^n_k = \Omega \) and if \( s \geq S^n_k \) then \( S^n_k = \emptyset \). If \( t^n_k < s < t^n_{k+1} \) then \( \{S^n_k \leq s\} = [t^n_k < T_l \leq s] \) and since \( T_l, l = 1, 2, \ldots, k \) are stopping times, \( \{S^n_k \leq s\} \in \mathcal{F}(s) \).

Therefore \( (S^n_k)_{k=0,1,\ldots,n2^n} \) is a random partition. Since \( \sup_k S^n_n > t^n_{n2^n} = n - 1/2^n \) and \( |S^n_{n2^n} - S^n_k| < t^n_k + 1 - t^n_{k+1} = 1/2^{n-1} \), it is clear that the sequence of random partitions \((S^n_k)\) tends to identity.

(QED)

**Proof of Theorem 2** Let \((S^n_k)_{k=0,1,\ldots,n2^n}\) be as in the above lemma. Let \( t^n_k \) be the point in \((S^n_k)_{k=0,1,\ldots,n2^n}\) and just before \( t^n_k \). Then from Theorem 1,
\[ \int_0^t (f(X(s-) + c(s)) - f(X(s-)))dN(s) = \lim_{n \to \infty} \sum_{k=0}^{n2^n} (f(X(S^n_k) + c(S^n_k)) - f(X(S^n_k))) \]
\[ - (N(S^{k+1}_n \wedge t) - N(S^n_k \wedge t)) \]
\[ = \lim_{n \to \infty} \sum_{i \geq 1, T_i \leq t} (f(X(T_i^n) + c(T_i^n)) - f(X(T_i^n))) \]
\[ - (N(T_i^n) - N(t^n_i)) \]
\[ = \lim_{n \to \infty} \sum_{i=1}^{N(t)} (f(X(T_i^n) + c(T_i^n)) - f(X(T_i^n))) \]
\[ = \sum_{i=1}^{N(t)} (f(X(T_i^n) + c(T_i^n)) - f(X(T_i^n))). \]

(QED)

Second, we consider the case where jump sizes are predictable processes. Let \( p(t) \) be a predictable process. Then there is a sequence of left continuous adapted processes \( (c_n(t))_{n=1,2,\ldots} \) that converges to \( p(t) \) in probability (see [5, Chapter IV, Section 2, Theorem 2 and the first definition on p.134]). We define the jump-diffusion process \( Z \) as
\[ Z(t) = Z(0) + \int_0^t a(s)ds + \int_0^t b(s)dW(s) + \sum_{i=1}^{N(t)} p(T_i). \]

Since there is a sequence of left continuous processes \( (f(Z(s-)) + c_n(s)) - f(Z(s-)))_{n=1,2,\ldots} \) that converges to the predictable process \( f(Z(s-)) + p(s) \) - \( f(Z(s-)) \),
Theorem 6
For any continuous function
\[ f : \mathbb{R} \to \mathbb{R}, \]
proof
We fix an \( \omega \in \Omega \) and put
\[ g_n(t) = \sum_{i, T_i \leq t} (f(Y(T_i)) - f(Y(T_i-))) 1_{i(T_i - 1/n, T_i]}(t). \]
Since \( \lim_{n \to \infty} g_n(t) = f(Y(t)) - f(Y(t-)), \)
\[ \int_0^t (f(Y(s)) - f(Y(s-)))dN(s) \]
\[ = \lim_{n \to \infty} \int_0^T g_n(s) dN(s) \]
\[ = \lim_{n \to \infty} \sum_{0 \leq s \leq t} g_n(T_i) \left( N(T_i) - N \left( T_i - \frac{1}{n} \right) \right) \]
\[ = \sum_{0 \leq s \leq t} \left( f(Y(s)) - f(Y(s-)) \right) 1_{0 \leq s \leq t}. \]
We get the result for any interval \([0, t] \):
\[ \int_0^t (f(Y(s)) - f(Y(s-)))dN(s) \]
\[ = \int_0^t (f(Y(s)) - f(Y(s-)))1_{[0,t]}dN(s) \]
\[ = \sum_{0 \leq s \leq t} (f(Y(s)) - f(Y(s-))) 1_{[0,t]} \]
\[ = \sum_{0 \leq s \leq t} (f(Y(s)) - f(Y(s-))) . \]
(\text{QED})

4. Applications
We now derive the Black-Scholes equation for the compound Poisson process model by applying the transformed Itô’s formula. Moreover, we also derive the differential equation that the values of the Black-Scholes hedge portfolios for derivatives must satisfy in the same model as [3].

Let \( S(t) \) be the underlying asset price process defined by
\[ S(t) = S e^{G(t)}, \quad t \in [0, T], \]
\[ G(t) = \mu t + \sum_{i \geq 1, T_i \leq t} p(T_i) - \mu t + \int_0^t p(s) dN(s), \]
where \( S > 0, T > 0, \mu \in \mathbb{R} \) and \( p(t) \) is a predictable process such that \( p(t) \neq 0 \) \( \forall t \in [0, T] \). Let \( B(t) \) be the risk-free asset process:
\[ B(t) = e^{rt}, \quad t \in [0, T] \]
where \( r > 0 \). Applying Itô’s formula to \( S(t) \), we obtain
\[ dS(t) = \mu S(t) dt + S(t-)(e^{p(t)} - 1) dN(t). \]
Let \( V(t, S(t)) \) denote the price of the derivative that cannot be exercised before maturity. By Itô’s formula,
\[ dV(t, S(t)) \]
\[ = \frac{\partial V}{\partial t}(t, S(t)) dt + \mu S(t) \frac{\partial V}{\partial S}(t, S(t)) dt \]
\[ + \left(V(t, e^{p(t)} S(t-)) - V(t, S(t-)) \right) dN(t). \]
We define \( \delta \) as
\[ \delta = \frac{V(t, e^{p(t)} S(t-)) - V(t, S(t-))}{S(t-)(e^{p(t)} - 1)}, \]
so that \( V + \delta S \) will be the risk-free portfolio.

We now derive the Black-Scholes equation. From \( dV + \delta dS = r(V + \delta S) dt \),
\[ \frac{\partial V}{\partial t}(t, S(t)) + \mu S(t) \frac{\partial V}{\partial S}(t, S(t)) - r V(t, S(t)) \]
\[ = (\mu - r) \frac{V(t, e^{p(t)} S(t-)) - V(t, e^{p(t)} S(t-))}{e^{p(t)} - 1} S(t). \]
(4)

Remark 7
It is noted that if \( S(t) \) in (4) is left continuous, the equation will be a little simple:
\[ \frac{\partial V}{\partial t}(t, S(t)) + \mu S(t) \frac{\partial V}{\partial S}(t, S(t)) - r V(t, S(t)) \]
\[ = (\mu - r) \frac{V(t, e^{p(t)} S(t-)) - V(t, e^{p(t)} S(t-))}{e^{p(t)} - 1}. \]
From this, it seems that the modeling of the stock prices with càglàd processes makes our calculation easier.

Next, we derive the differential equation that the Black-Scholes hedge portfolios must satisfy in the same
model as one in [3]. The stock price process in [3] is

\[ S(t) = S + \int_0^t (\alpha - \lambda E[Y - 1])S(s)ds + \int_0^t \sigma S(s)dW(s) + \sum_{i=1}^{N(t)} (Y_i - 1)S(T_i -), \]

where \( \alpha \in \mathbb{R}, Y, Y_i \sim \text{i.i.d.}, i = 1, 2, \ldots \) and \( N(t) \) is a Poisson process with intensity \( \lambda \). Thus it can be also written as

\[ dS(t) = (\alpha - \lambda E[Y - 1])S(t)dt + \sigma S(t)dW + \Delta S(t)dN, \]

where \( \Delta S(t) = (Y_i - 1)S(T_i -) \) if \( t = T_i, i = 1, 2, \ldots \) and \( = 0 \) if \( t \neq T_i, \forall t \). Let \( V(t, S(t)) \) be the price of the derivative that cannot be exercised before maturity. Applying Ito’s formula,

\[
dV(t, S(t)) = \left( \frac{\partial V}{\partial t}(t, S(t)) + (\alpha - \lambda E[Y - 1])S(t) \frac{\partial V}{\partial S}(t, S(t)) \right) dt + \sigma S(t) \frac{\partial^2 V}{\partial S^2}(t, S(t)) dt + \sigma S(t) \frac{\partial V}{\partial S}(t, S(t)) dW + \Delta V(t, S(t))dN.
\]

Assume in the same way as [3] that the Capital Asset Pricing model holds and the jump component of the stock return represents non-systematic risk. Then we have \( E[dV - V S dS|\mathcal{F}] = r(V - V_S S)dt \) and the portfolio satisfies

\[
\frac{\partial V}{\partial t}(t, S(t)) + \frac{\sigma^2}{2} S(t)^2 \frac{\partial^2 V}{\partial S^2}(t, S(t))
+ rS(t) \frac{\partial V}{\partial S}(t, S(t)) - rV(t, S(t))
= \lambda \left( \frac{\partial V}{\partial S}(t, S(t)) \Delta S(t) - \Delta V(t, S(t)) \right). \tag{5}
\]

Since the right hand side of this equation is equal to \( \Delta V \), \( \Delta V = \lambda (V_S \Delta S - \Delta V) \) and (5) can be rewritten as

\[
\frac{\partial V}{\partial t}(t, S(t)) + \frac{\sigma^2}{2} S(t)^2 \frac{\partial^2 V}{\partial S^2}(t, S(t))
+ rS(t) \frac{\partial V}{\partial S}(t, S(t)) - rV(t, S(t))
= \frac{1}{1 + \lambda} \frac{\partial V}{\partial S}(t, S(t)) \Delta S(t).
\]

This equation is slightly different from (14) in [3]. From this equation, we have the following interpretation of the behavior of the price of the derivative \( V \). While the price process of the underlying asset \( S \) is continuous, the price of the derivative \( V \) satisfies the ordinary Black-Scholes partial differential equation. On the other hand, when \( S \) jumps, \( V \) also jumps and the size of the jump is \( (\partial V/\partial S) \Delta S/(1 + \lambda) \), which is equal to the jump size of the price of the underlying asset that is shorted in the Black-Scholes hedge portfolio multiplied by \( 1/(1 + \lambda) \).

5. Conclusion

In this paper, we have verified the transformation of Ito’s formula for the stochastic processes with jumps and by applying the transformed Ito’s formula, the Black-Scholes equation has been derived in the compounded Poisson process model. Moreover, we have also obtained the differential equation that the value of the Black-Scholes hedge portfolios must satisfy in the jump-diffusion process model.

The proof of the transformation is provided in both the Riemann integral sense and the Lebesgue-Stieltjes integral sense. Although the latter proof is valid in more general settings, the former one gives us a valuable suggestion, that is, if we approximate a stochastic integral with respect to a stochastic process with jumps by the corresponding Riemann sum, the partition of the time interval must be constructed of stopping times instead of the deterministic sequence of points in the interval.

It remains to be investigated whether the transformation is correct or not when the probability of the event in which the number of jumps of the stochastic process is infinite is positive. However, from the perspective of the integrability, it is hard to express such a process by stochastic integrals with respect to counting processes. Since it should be written with a Poisson random measure, we must study it in a different way. In any case, this problem will be our future topic for researches.

References

[1] K. K. Aase, Optimum portfolio diversification in a general continuous-time model, Stoc. Proc. Appl., 18 (1984), 81–98.
[2] M. A. Garcia and R. J. Griego, An elementary theory of stochastic differential equations driven by a poisson process, Stoch. Models, 10 (1994), 335–363.
[3] R. C. Merton, Option pricing when underlying stock returns are discontinuous, J. Financ. Econ., 3 (1976), 125–144.
[4] R. Cont and P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC, Boca Raton, 2004.
[5] P. Protter, Stochastic Integration and Differential Equations, Springer, Berlin, 1990.