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A novel integrated quasi-zero stiffness vibration isolator for coupled translational and rotational vibrations

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ABSTRACT

Quasi-zero stiffness (QZS) vibration isolators can provide better isolation performance in the low frequency range than linear vibration isolators. Currently, most of the designed QZS isolators perform vibration isolation only in one direction and few papers are focused on simultaneously isolating the vibrations in two directions. In this paper, an integrated translational-rotational QZS vibration isolator is designed by using the cam-roller mechanism. The proposed QZS system is able to provide the high-static-low-dynamic stiffness in two directions simultaneously. The excitations in both translational and rotational directions are considered independent but with mutual interaction to their induced vibration response. The workable ranges of the QZS system and its limitations are first numerically identified. Then the static characteristics and typical nonlinear dynamic response with jump phenomena are theoretically investigated. The jump-down frequencies for small amplitude oscillations are determined from their amplitude-frequency relationships. Furthermore, the force transmissibility and moment transmissibility of the proposed QZS system are compared with those of the corresponding linear system without the cam-roller mechanism, which clearly demonstrate better isolation performance in both translational and rotational directions.

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1. Introduction

Quasi-zero stiffness (QZS) vibration isolators possess the high-static-low-dynamic stiffness characteristics, which use the negative stiffness structures to reduce the dynamic stiffness of the system for a desired payload. The QZS characteristics have found their applications in the platform vibration isolators [1,2], beam vibration systems [3,4], and rotor systems [5,6].

The structures with negative stiffness can be achieved by applying different mechanisms. Euler buckled beam was frequently used for the negative stiffness in the early stage [7–9]. Then coil spring was developed for horizontal-spring type or oblique-spring type [10–12]. Cam-roller-spring mechanism was also used for generating QZS characteristic for isolating vertical vibrations [13–16], torsional vibrations [6,17], and for adaptive pneumatic vibration isolator [1]. Other negative stiffness elements, such as bio-inspired structures [18,19] and electromagnetic structures [20–22], were investigated for generating QZS feature. Recently, metamaterials and origami structures were also applied to achieve the QZS characteristics [23–27].

The QZS system can be divided into two types; the passive type and the semi-active (or active) type [28–30]. Passive QZS isolators have simple structures but require precise combination of the positive stiffness and negative stiffness structures.
Few passive QZS systems were discussed for mitigation of vibrations of multiple degree-of-freedom (DOF) systems. A three-DOF QZS system was developed by applying the symmetrically scissor-like structures \[31\], to achieve the vibration isolation in both vertical and horizontal directions. Two six-DOF isolation platforms were designed by assembling multiple QZS struts to form either a pyramidal platform \[14\] or a hexapod platform \[15\]. The QZS strut used in both isolation platforms was constructed based on the cam-roller-spring mechanism. A Stewart isolator using multiple QZS struts was also proposed to realise high-static-low-dynamic stiffness in each single-DOF system of six directions \[32\]. Several QZS vibration isolators were used for isolating vibrations in multiple directions \[3,33,34\], but the interaction between the DOFs was seldom discussed. A torsion–translational QZS isolator was recently proposed and two independent excitations were considered in two DOF systems \[6\]. The interactions of the vibrations induced by these two excitations were discussed.

Compared to one-DOF QZS isolators proposed for practical engineering applications, such as vehicle seat \[35\] and transport incubator \[36\], an integrated translational-rotational QZS vibration isolator is proposed in this paper to isolate the vibrations in two directions simultaneously. The high-static-low-dynamic stiffness characteristics can be provided in two DOFs, not only in the translational direction, but also in the rotational direction. The workable range of the QZS system and its limitations are numerically identified by studying the static responses and the physical connection conditions. Typical nonlinear dynamic responses are theoretically discussed by considering the external excitations in both translational and rotational directions, which are independent but with dynamic interactions of their responses generated. The jump-down frequencies for small-amplitude oscillations are determined from their amplitude-frequency relationships. Furthermore, the force transmissibility and moment transmissibility of the QZS system are compared with those of the corresponding linear system (i.e. the system without incorporating the cam-roller mechanism), which clearly demonstrate better isolation performance in both translational and rotational DOFs.

This paper is organized as follows. Section 2 presents the theoretical model of the proposed QZS system and studies its static characteristics. Section 3 numerically discusses the nonlinear dynamic behaviour with jump phenomena under two independent excitations. Section 4 investigates the effects of the design parameters on the isolation performance of the QZS system. Concluding remarks are summarized in Section 5.

2. The model of a coupled two-DOF QZS system

The coupled translational-rotational QZS system proposed includes two main support springs and one cam-roller structure as shown in Fig. 1. The stiffness for either of main springs is \(k_l\), and the distance from either of the main springs to the platform mass centre is \(d_v\). Without the cam-roller structure, the corresponding system can be considered as a linear vibration isolation system which provides the linear stiffness in the translational (vertical) or rotational direction. With the designed unsymmetric cam-roller structure, the negative stiffness in both directions can be provided. The cam structure is located on one end of the platform and the distance from the cam centre to the centre of mass of the platform is \(L\). The platform limits its motion in the translational (vertical) and rotational directions by using a pin-and-slot device. One end of the roller structure is joined to the base (or the fixed frame) with a horizontal spring, while the other end is in contact with the cam. The stiffness of the horizontal spring is \(k_h\). The motion of the roller and the force applied by the horizontal spring are restricted in the horizontal direction. At the position of the metastable equilibrium for the platform when the desired payload is applied, the horizontal spring, roller centre, cam centre and the centre of geometry of the platform are all in the same horizontal plane. The roller position (its centre) is \((r + R + L)\) away from the mass centre of the platform with

![Fig. 1. Mechanism of the proposed translational-rotational QZS structure: (a) the initial condition, (b) with force and moment applied.](image-url)
a per-compression $\delta$ of the horizontal spring. An example of a payload placed on the platform is shown in Figure (1b), which could result in a force in the vertical direction and a moment about the mass centre of the platform due to the eccentric effect.

2.1. Static analysis

The relationship between the response force and moment generated from the QZS system, the vertical motion $y$, and the rotational motion $\theta$ of the platform can be derived according to the geometrical relations of the structure. The resultant displacement of the cam (roller side) in the vertical direction due to the motion of the platform can be found according to Fig. 1(b) as:

$$y_t = y + L \sin \theta$$

(1)

For the corresponding linear system without the cam-roller structure, the resultant force and moment produced from the spring deformations can be expressed as:

$$F_{\text{linear}} = 2k_y y$$

(2a)

$$M_{\text{linear}} = 2d_y k_t \sin \theta$$

(2b)

For the proposed QZS system, due to the displacement of the cam and the relative motion of the cam and roller, the roller will be pushed by the horizontal spring, and the force in the horizontal direction is given by:

$$F_h = k_h (\delta - \Delta x_r)$$

(3)

where $\delta$ represents the per-compression of the horizontal spring, $\Delta x_r = (r + R + L) - P_r$, is the roller’s relative displacement in the horizontal direction, and $P_r = \sqrt{(r + R)^2 - (y)^2} + L \cos \theta$, is the distance of geometric centre of the roller to the mass centre of the platform.

The resultant force from the cam-roller structure in the vertical direction and the resultant moment about the mass centre of the platform can be calculated as:

$$F_N = -N \times F_{\text{lin}} \sin \phi$$

(4a)

$$M_N = N \times F_{\text{lin}} \times P_r \sin \phi = F_N \times P_r$$

(4b)

where $F_{\text{lin}} = k_h (\delta - (r + R + L) + P_r) \sin \phi$ is the interaction force between the roller and the cam, $\phi$ is the angle between the line connecting the geometric centres of the cam and roller and the horizontal axis. From Fig. 1(b), it can be obtained that

$$\sin \phi = \frac{y + L \sin \theta}{\sqrt{(r + R)^2 - (y + L \sin \theta)^2}}.$$

By introducing the non-dimensional form of parameters and substituting Eq. (1) and Eq. (3) into Eq. (4), the resultant force in the vertical direction and the moment acting on the platform can be expressed as:

$$F = 2y - \frac{\delta - (1 + \lambda) + \sqrt{1 - (y + \lambda \theta)^2} + L \sqrt{1 - \theta^2}}{1 - (y + \lambda \theta)^2} \frac{y + \lambda \theta}{y + \lambda \theta}$$

(5a)

$$M = \frac{2d_y \theta - \frac{\delta - (1 + \lambda) + \sqrt{1 - (y + \lambda \theta)^2} + L \sqrt{1 - \theta^2}}{1 - (y + \lambda \theta)^2} \left( \sqrt{1 - (y + \lambda \theta)^2} + L \sqrt{1 - \theta^2} \right) \frac{y + \lambda \theta}{y + \lambda \theta}}{1 - (y + \lambda \theta)^2}$$

(5b)

where $\hat{y} = \frac{y}{R \sin \theta}$, $F = \frac{\hat{y}}{6(\hat{R}\theta)}$, $\alpha = \frac{R}{\hat{R} \theta}$, $\lambda = \frac{R}{(\hat{R} \theta)^2}$, $\frac{\delta}{\hat{R} \theta} = \frac{\delta}{(\hat{R} \theta)^2} \frac{\delta}{(\hat{R} \theta)^2}$, $\frac{\delta}{\hat{R} \theta} = \frac{\delta}{(\hat{R} \theta)^2}$, and here it has been assumed that $\theta$ is small by considering small amplitude vibrations, thus $\sin \theta \approx \theta$.

Differentiating Eq. (5a) and (5b) with respect to the vertical displacement or angular displacement yields the translational stiffness and the torsional stiffness:

$$K_v = 2 + \frac{\alpha \left( \frac{y + \lambda \theta}{1 - (y + \lambda \theta)^2} - \frac{y + \lambda \theta}{1 - (y + \lambda \theta)^2} \left( \frac{\delta - (1 + \lambda) + \sqrt{1 - (y + \lambda \theta)^2} + L \sqrt{1 - \theta^2}}{1 - (y + \lambda \theta)^2} \right)^2 \right)}{1 - (y + \lambda \theta)^2}$$

(6a)
The response force and moment corresponding to the translational and rotational motions are illustrated in Fig. 2. It is easy to notice from the figure that both force and moment loads near the equilibrium position (0, 0) have a typical QZS characteristic. However, when away from the equilibrium position, the system stiffness would change according to its motion condition $(y, \theta)$. In this regard, the system stiffness in the translational and rotational directions are of interest for the QZS system. The stiffness ratio of the stiffness in the translational and rotational directions of the QZS system to the corresponding linear system is shown in Fig. 3 by means of the contour maps. Only when the ratio is less than 1, which means the QZS system stiffness is lower than that of the linear system, the proposed QZS system can have better isolation performance than the corresponding linear system in the lower frequency range. This requirement can be easily satisfied by properly designing the system parameters.
In order to keep the cam and roller in contact and gain the physical insights to make the system workable, the absolute displacement of the cam in the vertical direction should be not larger than the sum of the cam and roller radius, which can be expressed as:

$$|y| = |y + L \sin \theta| \leq R + r$$

By imposing the QZS condition, the upper and lower limits of the equilibrium position due to the translational and rotational motion of the platform are shown in Fig. 4, where both engagement and disengagement regions are labelled. The influence of the design parameter $d_v$ on the engagement region is also shown. The engagement region of the system becomes larger when reducing $d_v$.

### 2.2. Approximation of the force and moment generated

In order to investigate the dynamical behaviour of the proposed QZS system, analytical approximated solutions for the force and moment generated are simplified by using Taylor series expansion for the two-DOF system with keeping up to the third-order about the equilibrium position $(0,0)$, which are succinctly written as:

$$F_{(y,\theta)}^{(3)} = \lambda_1 + \lambda_2 y + \lambda_3 y^2 + \lambda_4 y^3 + O^3_y$$

$$M_{(y,\theta)}^{(3)} = \gamma_1 + \gamma_2 \theta + \gamma_3 \theta^2 + \gamma_4 \theta^3 + O^4_\theta$$

Fig. 3. Stiffness ratio of the translational stiffness (a) and rotational stiffness (b) of the QZS system to that of the linear system.

Fig. 4. Upper and lower motion limits of the platform for the engagement region.
where

\[
\begin{align*}
\lambda_1 &= \left[ -\alpha L^2 \theta + \frac{1}{2} \alpha L^2 (1 + L - L^2) \theta^2 + O_\theta^4 \right] \\
\lambda_2 &= \left[ (2 - \alpha \delta) + \frac{1}{2} \alpha L (1 + 3L - 3L^2) \theta^2 + O_\theta^4 \right] \\
\lambda_3 &= \left[ \frac{3}{2} \alpha L \left( 1 - \frac{\delta}{L} \right) \theta + \frac{3}{4} \alpha L^2 (1 + 5L - 5L^2) \theta^3 + O_\theta^4 \right] \\
\lambda_4 &= \left[ \frac{1}{2} \alpha \left( 1 - \frac{\delta}{L} \right) + \frac{1}{4} \alpha L \left( 1 + 15L - 15L \delta \right) \theta^2 + O_\theta^4 \right] \\
\gamma_1 &= \left[ -\alpha L^2 \left( 1 + \frac{\delta}{L} \right) \psi + \frac{1}{2} \alpha \left( 1 - \frac{L}{L^2} \right) \psi^2 + O_\psi^4 \right] \\
\gamma_2 &= \left[ \frac{3}{2} \alpha \left( 1 - \frac{\delta}{L} \right) \psi + \frac{3}{4} \alpha L \left( 1 + L - L \delta \right) \psi^2 + O_\psi^4 \right] \\
\gamma_3 &= \left[ \frac{1}{2} \alpha L \left( 1 + 4L + 3L^2 + \delta - 3L^2 \delta \right) \psi + \frac{1}{4} \alpha \left( -1 + 6L + 15L^2 + \delta - 15L^2 \delta \right) \theta^2 + O_\theta^4 \right] \\
\gamma_4 &= \left[ \frac{1}{2} \alpha L^2 \left( 1 + 2L + L^2 + \delta - L^2 \delta \right) \psi + \frac{1}{4} \alpha L^2 \left( -3 + 8L + 15L^2 + 3\delta - 15L^2 \delta \right) \psi^2 + O_\psi^4 \right]
\end{align*}
\]

and \(O_\theta^4, O_\psi^4\) present the higher order components.

The exact values and approximated values of the force and moment corresponding to different translational and rotational conditions are shown in Fig. 5(a) and (b), respectively. By using the difference between the exact and approximation values over the exact value, the relative fitting errors for the force and moment are shown in Fig. 6 by means of the contour maps. It is easy to observe from the figure that the relative fitting error of the response force has a narrow range when \(\theta = 0\), meanwhile the relative fitting error of the response moment has a narrow range when \(\psi = 0\). It should be mentioned that the Taylor series expansion is only valid when both translational and rotational vibrations are small. Thus the motions within a low relative fitting error range (normally less than 5%) are simulated for dynamic analysis.

It is worth noting that the workable ranges of the proposed QZS vibration isolator should take into account all the limitations including the system stiffness, the engagement region, and the relative fitting error for analysis issue. By implementing the limitations due to effective stiffness range, as shown in Fig. 4, and the relative fitting error caused by Taylor series expansion, as shown in Fig. 6, only the scenario that the motions fall within a specified range will be considered and discussed in the following sections, that is, \(\psi \leq 0.2\) and \(\theta \leq 0.05\).
3. Dynamic analysis

The dynamic behaviours of the coupled translational-rotational QZS system are investigated in this section. It is assumed that a desired payload is located at the mass centre of the platform, and the platform is at its equilibrium position when no external excitations are applied. The payload mass and its moment of inertia about the axis passing through the mass centre of the platform are \( m \) and \( I \), respectively. The equivalent harmonic force and moment are applied to the mass centre of the platform due to the external excitations applied to the base of the platform. The equations of motion of the two-DOF QZS system can be established by applying Newton’s second law of motion as:

\[
\begin{align*}
\ddot{y} + c_y \dot{y} + F_y = F_y \cos (\omega_y t + \varphi_y) \\
\ddot{\theta} + c_\theta \dot{\theta} + M = M \cos (\omega \theta t + \varphi_\theta)
\end{align*}
\]  

(10a)  

(10b)

where \( c_y \) and \( c_\theta \) are the damping coefficients, \( F_y, \omega_y, \varphi_y \) and \( M, \omega_\theta, \varphi_\theta \) are the amplitudes, frequencies and initial phases of the external excitations in the translational and rotational directions, respectively. Symbols \( \dot{y}, \ddot{y}, \dot{\theta} \) and \( \ddot{\theta} \) denote the differentiations with respect to the time \( t \).

By introducing the non-dimensional notations, Eqs. (10a) and (10b) can be rewritten as:

\[
\begin{align*}
\ddot{\gamma} + 2 \xi_y \dot{\gamma} + \gamma = \gamma \cos (\Omega_y \tau + \varphi_y) \\
\ddot{\beta} + 2 \xi_\theta \dot{\beta} + \beta = \beta \cos (\Omega_\theta \tau + \varphi_\theta)
\end{align*}
\]  

(11a)  

(11b)

where

\[
\begin{align*}
\Omega_y &= \sqrt{\frac{k_y}{m}}, \quad \Omega_\theta = \sqrt{\frac{k_\theta}{I}} \\
c_y &= \frac{\xi_y}{\sqrt{k_y/m}}, \quad c_\theta = \frac{\xi_\theta}{\sqrt{k_\theta/I}} \\
F_y &= \frac{F}{\sqrt{k_y/m}}, \quad M &= \frac{M}{\sqrt{k_\theta/I}}, \quad \gamma &= \frac{\dot{y}}{(R_x R_\phi)}, \quad \beta = \frac{\dot{\theta}}{\Omega_\theta}, \quad \dot{\gamma} = \frac{\ddot{y}}{(R_x R_\phi)}
\end{align*}
\]

Since the corresponding linear system without the cam-roller structure uses a two-spring set for isolating vibrations in both translational and rotational directions, the natural frequency of the linear system is different from a typical one-spring system. According to the system parameters, Eq. (2a) and (2b), the natural frequency of the corresponding linear system in the translational and rotational directions can be calculated as \( \sqrt{2\Omega_y} \) and \( \sqrt{2\Omega_\theta} \), respectively.

The Harmonic Balance Method is used to solve Eq. (11a) and Eq. (11b). The solutions to these equations are assumed to be:

\[
\begin{align*}
y &= A_y \cos (\Omega_y \tau) \\
\theta &= A_\theta \cos (\Omega_\theta \tau)
\end{align*}
\]  

(12a)  

(12b)

where \( A_y \) and \( A_\theta \) are the amplitudes of the vibration response in the translational and rotational directions, respectively.

Substituting the solutions Eqs. (12a) and (12b) into Eqs. (11a) and (11b), and replacing the force and moment expressions of Taylor series expansion Eqs. (9a) and (9b), yields:

\[
\begin{align*}
\Omega_y^2 A_y \cos (\Omega_y \tau) - 2 \xi_y A_y \sin (\Omega_y \tau) + F_y (A_y \cos (\Omega_y \tau), A_y \sin (\Omega_y \tau)) \\
= F_y \cos (\varphi_y) \cos (\Omega_y \tau) - F_y \sin (\varphi_y) \sin (\Omega_y \tau)
\end{align*}
\]  

(13a)
\[-\Omega_v^2 A_v \cos (\Omega_v \tau_v) - 2 \xi_v \Omega_v A_v \sin (\Omega_v \tau_v) + \bar{M}(A_v \cos (\Omega_v \tau_v), A_v \cos (\Omega_v \tau_v))\]
\[= \bar{M} \cos (\phi_v) + \bar{M} \sin (\phi_v) \sin (\Omega_v \tau_v) \]

where
\[F_A(\Omega_v \cos (\Omega_v \tau_v), A_v \cos (\Omega_v \tau_v)) = \left[ (x_1 + x_2 A_v^2)A_v + (x_3 + x_4 A_v^2)A_v^3 \right] \cos (\Omega_v \tau_v), \]
\[x_1 = 2 + x_3, x_2 = 0.5 x_3(1 - x_3), x_4 = 0.5 x_3(1 + 3 L - 3 L^2), x_5 = 0.5 x_3(1 - 3 L), x_6 = 0.5 x_3(1 + 3 L - 3 L^2), \]
\[\beta_1 = 0.5 x_3(1 + 3 L + L^2 - L^2), \beta_2 = 0.5 x_3(1 + 3 L + L^2 - L^2).\]

It should be noted that by considering a small-amplitude periodic motion about the equilibrium position, the coefficients of the trigonometric functions \(\cos (\Omega_v \tau_v)\) and \(\sin (\Omega_v \tau_v)\) on both sides of Eq. (13a), and \(\cos (\Omega_v \tau_0)\) and \(\sin (\Omega_v \tau_0)\) on both sides of Eq. (13b) should be equal, which leads to the following resultant equations:

\[-\Omega_v^2 A_v + \left[ (x_1 + x_2 A_v^2)A_v + (x_3 + x_4 A_v^2)A_v^3 \right] = \bar{F} \cos (\phi_v) \]
\[-2 \xi_v \Omega_v A_v = - \bar{F} \sin (\phi_v) \]
\[-\Omega_v^2 A_v + \left[ (\beta_1 + \beta_2 A_v^2)A_v + (\beta_3 + \beta_4 A_v^2)A_v^3 \right] = \bar{M} \cos (\phi_v) \]
\[-2 \xi_v \Omega_v A_v = - \bar{M} \sin (\phi_v) \]

From Eqs. (14a) and (14b), the initial phase of the resultant excitations can be obtained as:

\[\tan (\phi_v) = \frac{2 \xi_v \Omega_v}{(x_3 + x_4 A_v^2)A_v^2 + (x_1 + x_2 A_v^2) - \Omega_v^2} \]

\[\tan (\phi_0) = \frac{2 \xi_v \Omega_v}{(\beta_1 + \beta_2 A_v^2)A_v^2 + (\beta_3 + \beta_4 A_v^2) - \Omega_v^2} \]

Squaring both sides of Eqs. (14a) and (14b), and then adding the resultant equations, leads to the amplitude-frequency equations as:

\[\Omega_v^4 - 2 C_11 \Omega_v^2 + C_{12} = 0 \]
\[\Omega_v^4 - 2 C_21 \Omega_v^2 + C_{22} = 0 \]

where
\[C_11 = (x_1 + x_2 A_v^2) - 2 \xi_v^2 + (x_3 + x_4 A_v^2)A_v^2, \]
\[C_{12} = (x_1 + x_2 A_v^2)^2 + 2 \left( x_1 + x_2 A_v^2 \right) \left( x_3 + x_4 A_v^2 \right) A_v^2 + (x_3 + x_4 A_v^2)^2 A_v^4 \]
\[C_21 = (\gamma_2 + 2 \xi_v^2 \gamma_4 - 2 \xi_v^2)^2, \]
\[C_{22} = \frac{(\gamma_2 + 2 \xi_v^2 \gamma_4 - 2 \xi_v^2)^2 \bar{M}}{A_v^2}. \]

Solving the quadratic equations of \(\Omega_v\) and \(\Omega_0\) gives rise to

\[\Omega_{12} = \sqrt{C_{11} \pm \sqrt{C_{11}^2 - C_{12}}} \]
\[\Omega_{21} = \sqrt{C_{21} \pm \sqrt{C_{21}^2 - C_{22}}} \]

The maximum transitional amplitude \(A_v\) and rotational amplitude \(A_0\) would appear when \(\Omega_{v1} = \Omega_{v2}\) and \(\Omega_{01} = \Omega_{02}\), i.e.

\[C_{11}^2 - C_{12} = 0 \]
\[C_{21}^2 - C_{22} = 0 \]

Eqs. (16a) and (16b) can be used to obtain typical nonlinear resonance curves with stable and unstable regions, which are also known as jump phenomenon. The jump-down frequency can be calculated by substituting the maximum transitional amplitude \(A_v\) and rotational amplitude \(A_0\) into Eqs. (17a) and (17b), respectively.

The transitional force and rotational moment responses can be determined by

\[F_t(t) = 2 \xi_v \ddot{y} + \bar{F}(A_v \cos (\Omega_v \tau_v), A_v \cos (\Omega_v \tau_v)) \approx \sqrt{\left( x_1 + x_2 A_v^2 \right) A_v + \left( x_3 + x_4 A_v^2 \right) A_v^3 + (2 \xi_v \Omega_v A_v)^2 \cos (\Omega_v \tau_v + \phi_v) \]
\[M_t(t) = 2 \xi_v \ddot{\theta} + \bar{M}(A_v \cos (\Omega_v \tau_v), A_v \cos (\Omega_v \tau_v)) \approx \sqrt{\left( \beta_1 + \beta_2 A_v^2 \right) A_v + \left( \beta_3 + \beta_4 A_v^2 \right) A_v^3 + (2 \xi_v \Omega_v A_v)^2 \cos (\Omega_0 \tau_v + \phi_v) \]

\(8\)
where \( \tan(\phi_v) = \frac{2\zeta_v \Omega_v A_v}{[(\alpha_1 + \alpha_2 A_v^2) A_v + (\alpha_3 + \alpha_4 A_v^2) A_v^3]^2} \) and \( \tan(\phi_\theta) = \frac{2\zeta_\theta \Omega_\theta A_\theta}{[(\beta_1 + \beta_2 A_\theta^2) A_\theta + (\beta_3 + \beta_4 A_\theta^2) A_\theta^3]^2} \).

According to the definition of transmissibility, the force transmissibility and moment transmissibility of the QZS system can be calculated as:

\[
T_v = \sqrt{\frac{1}{F_e} \left( \frac{X_v}{A_v^2} + \left( \frac{X_3 + X_4 A_v^2}{A_v^3} \right)^2 + \left( \frac{2\zeta_v \Omega_v A_v}{A_v^3} \right)^2 \right)}
\]

(20a)

\[
T_\theta = \sqrt{\frac{1}{M_e} \left( \frac{Y_\theta}{A_\theta^2} + \left( \frac{Y_3 + Y_4 A_\theta^2}{A_\theta^3} \right)^2 + \left( \frac{2\zeta_\theta \Omega_\theta A_\theta}{A_\theta^3} \right)^2 \right)}
\]

(20b)

4. Parameter analysis and discussion

According to the specified working range discussed in Section 2 and two amplitude-frequency relationships given in Section 3, the engagement and disengagement regions of the coupled QZS system with respect to the excitations are shown in Fig. 7 for the platform’s initial condition at the equilibrium position. The workable zone of the force and moment excitations found from the figure are limited to 0.040 and 0.026 for the proposed QZS system.

4.1. Amplitude-frequency relationship

The numerical solution of the nonlinear amplitude-frequency relationship can be solved from Eqs. (16a) and (16b) to investigate the dynamic behaviour of the proposed QZS system. However, some characteristics of the nonlinear behaviour, such as jump-down frequency and large amplitude excitations, cannot be captured in the semi-analytic solutions by using the Harmonic Balance Method. The transitional force response and rotational moment response with respect to the frequencies of the excitations applied within a workable regions are shown in Fig. 8. Since the excitations in the translational and rotational DOFs are considered to be dynamically coupled, two jump phenomena can be found in the amplitude-frequency response curves, which represent the coexistence of stable and unstable solutions. Fig. 8(a) shows that the secondary jump-down frequency of the force response is decreased when the frequency of the rotational excitation is increased.

Under different excitation amplitudes, it can be found that for the translational response as shown in Fig. 9(a–c), the primary jump-down frequency is induced by the translational excitation and the secondary jump-down frequency is generated by the rotational excitation. While for the rotational responses as shown in Fig. 9(d–f), the primary jump-down frequency is caused by the excitations in both DOFs meanwhile the secondary jump-down frequency is mainly induced by the translational excitation, which is hard to be recognised from the figure as it is too small.

The unstable regions of the translational and rotational responses with respect to the excitation frequencies are illustrated in Fig. 10. Since the responses in both DOFs are mutually coupled, the unstable region of the QZS system should be combined by integrating Fig. 10(a) and (b). As shown in Fig. 9(a–f), the small-amplitude oscillations of the QZS system in both DOFs appear immediately after the primary jump-down, which means the QZS system can attenuate vibrations only in the upper right corner of the excitation frequency region as shown in Fig. 10(a) and (b).
4.2. Force and moment transmissibility

From Fig. 8, there are coupled effects under different excitation frequencies in both DOFs. The force and moment transmissibility of the proposed QZS system are calculated and compared with those of the corresponding linear system by removing the cam-roller structure as shown in Fig. 11(a) and (b). As the undesirable vibration of the QZS system in the unstable regions is not of interest to vibration isolation, only the force and moment transmissibility in the stable regions (after jump-down frequencies) are discussed in this section. For the force transmissibility, as illustrated in Fig. 11(a), the variation of the rotational excitation frequency has an insignificant effect on the vibration in the translational direction. On the contrary, for the moment transmissibility, as illustrated in Fig. 11(b), the variation of the translational excitation frequency has a significant influence on the rotation performance of the system, not only on the isolation performance but also on its bandwidth. Overall, it indicates that the proposed translational-rotational QZS system could have significant improvement on the isolation performance in the lower frequency range than the corresponding linear system, where the linear system would demonstrate vibration isolation only after $\sqrt{2}$ times of its natural frequency. Additionally, different damping ratios are also simulated and presented in Fig. 12(a–f). An increase of the damping ratio in both DOF systems can generally improve their vibration isolation performances.

According to Fig. 10 and Eqs. (20), the force transmissibility and moment transmissibility of the translational and rotational DOFs under different excitation conditions are illustrated in the contour maps, as shown in Fig. 13, in which different colours indicate the amplitude levels of the transmissibility and the blank (white) area implies the system transmissibility is higher than 1. Since the QZS characteristic is mainly effective in the low frequency range, only the frequency ranges of interest are displayed. The colour change of the texture patterns shown in the figures illustrates that the transmissibility varies nonlinearly according to the excitation frequencies.

5. Conclusion

In this paper, a coupled translational-rotational QZS vibration isolator has been proposed by using the cam-roller mechanism, which includes two main support springs, one unsymmetric cam-roller structure and motion restriction structure on
Fig. 9. Transitional force response (a–c) and rotational moment response (d–f) with respect to different excitation amplitudes when (black) Fe = 0.01 & Me = 0.002; (blue) Fe = 0.01 & Me = 0.004; (red) Fe = 0.02 & Me = 0.004. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 10. Unstable regions of the translational amplitudes (a) and rotational amplitudes (b) with respect to excitation frequencies.
its platform. The proposed QZS system has been proved to be able to provide the high-static-low-dynamic stiffness simultane-ously in the translational and rotational directions.

The static characteristics of the nonlinear QZS vibration isolator were numerically identified by considering the structure limitation and system stiffness. The quasi-zero stiffness condition was ensured about the equilibrium position in both trans-lational and rotational directions. Engagement region of the cam-roller structure and the effective QZS region were discussed under different system parameters. Subsequently, the equations of motion of the proposed QZS system were established as a two-DOF nonlinear system. Excitations in both translational and rotational directions were applied to study the dynamic behaviour of the two-DOF QZS nonlinear system. Both excitations were considered to be independent but with interactions between the responses they induced to the system. By using the Harmonic Balance Method, typical nonlinear dynamic responses with jump phenomena were theoretically discussed and two jump-down frequencies were determined from the amplitude-frequency relationships. Furthermore, the force and moment transmissibility of the QZS system were

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**Fig. 11.** Force transmissibility (a) and moment transmissibility (b) beyond the jump-down frequency for the QZS system under different excitation frequencies, in comparison to the corresponding linear system with different damping ratios.

**Fig. 12.** Force transmissibility (a–c) and moment transmissibility (d–f) beyond the jump-down frequency for the QZS system under damping conditions ($\zeta$, $\dot{\zeta}$ = 0.05, 0.08, 0.10) when (a–c) $\Omega_r$ = 0.3, 0.6 and 0.9; (d–f) $\Omega_r$ = 0.3, 0.6 and 0.9.
compared to those of the corresponding linear system without the cam–roller structure, to clearly demonstrate the desired isolation performances in the low frequency range in both translational and rotational directions.

A prototype of the proposed QZS isolator will be fabricated and the validation of the theoretical results will be conducted in our future work. Due to the COVID-19, the fabrication of the prototype has been significantly delayed.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Fig. 13. Contour maps of the force transmissibility $T_v$ (upper row) and moment transmissibility $T_h$ (lower row) with respect to the excitations when 1) (a, d) $F_e = 0.01$ & $M_e = 0.002$; 2) (b, e) $F_e = 0.01$ & $M_e = 0.004$; 3) (c, f) $F_e = 0.02$ & $M_e = 0.004$. 

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