On Dress Codes with Flowers

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Abstract Fractional Repetition (FR) codes are well known class of Distributed Replication-based Simple Storage (Dress) codes for the Distributed Storage Systems (DSSs). In such systems, the replicas of data packets encoded by Maximum Distance Separable (MDS) code, are stored on distributed nodes. Most of the available constructions for the FR codes are based on combinatorial designs and Graph theory. In this work, we propose an elegant sequence based approach for the construction of the FR code. In particular, we propose a beautiful class of codes known as Flower codes and study its basic properties.

key words: Distributed storage systems, Fractional Repetition Codes, Flower Codes, Sequences, Dress Codes, Codes for distributed storages.

1. Introduction

In Distributed Storage Systems (DSSs), data file is encoded into certain packets and those packets are distributed among n nodes. A data collector has to collect packets from any k (called reconstruction degree) nodes among the n nodes to reconstruct the whole file. In the case of node failure, system is allowed to reconstruct a new node to replace the failed node. The repair is of two types viz. exact and functional. In the repair process, the new node is constructed by downloading β packets from each node (helper node) of a set of d (repair degree) nodes. Thus total bandwidth for a repairing a node is dβ. For an (n, k, d) DSS, such regenerating codes are specified by the parameters \{n, k, d\}, where B is the size of the file and α is the number of packets on each node. One has to optimize both α and β, hence we get two kind of regenerating codes viz. Minimum Storage Regenerating (MSR) codes useful for archival purpose and Minimum Bandwidth Regenerating (MBR) codes useful for Internet applications [1], [2]. Some of the MBR codes studied by the researchers fails to optimize other parameters of the system such as disk I/O, computation and scalability etc. Towards this goal, a class of MBR codes called Dress codes were introduced and studied by researchers [1], [3]–[5] to optimize disk I/O. These codes have a repair mechanism known as encoded repair or table based repair. Dress codes consisting of an inner code called fractional repetition (FR) code and outer MDS code. Construction of FR codes has been an important research problem and many constructions of FR codes are known based on graphs [1], [6]–[10], combinatorial designs [8], [11]–[15] and other combinatorial configuration [16]–[19]. Existence of FR code is discussed in [20]. FR codes have been studied in different directions such as Weak FR codes [21], [22], Irregular FR codes [23], Variable FR codes [24]. A new family of Fractional Repetition Batch Code is studied in [9], [25]. For a given FR code, algorithms to calculate repair degree d and upper-bound of reconstruction degree k, is given in [26]. In this paper, a more general definition of FR code (Definition 1) is considered for a realistic practical scenario. Further, a sequence based construction for FR code, is given. In particular, construction of Flower code is studied in detail.

The structure of the paper is as follows. In Section 2, we define a general FR codes and collects relevant background material. In Section 3, construction of flower code with single and multiple ring is given. We also study its relations with sequences. In particular, we show how to construct them using arbitrary binary sequences. Final section concludes the paper with general remarks.

2. Background

Distributed Replication-based Simple Storage (Dress) Codes consists of an inner Fractional Repetition (FR) code and an outer MDS code (see Figure 1). In this, 5 packets are encoded into 6 packets using a MDS code and then each packet is replicated twice and distributed among 4 nodes. In case of one node failure data can be recovered easily in such a system. We formally define FR code as follows.

Definition 1. (Fractional Repetition Code): Given a DSS with n nodes \( U_i (i \in \{1, 2, \ldots, n\}) \) and θ packets \( P_j (j \in \Omega = \{1, 2, \ldots, \theta\}) \), one can define FR code \( \mathcal{C}(n, \theta, \alpha, \rho) \) as a collection \( \mathcal{C} \) of n subsets \( \{U_i : i \in \Omega\} \), which satisfies the following conditions:

- For each \( j \in \Omega \), packet \( P_j \) appears exactly \( \rho_j \) times in the collection \( \mathcal{C} \).
- For every \( i = 1, 2, \ldots, n \), \( |U_i| = \alpha_i \) (\( \alpha_i \in \mathbb{N} \)).

where \( \alpha_i \) denotes the number of packets on the node \( U_i \), \( \rho = \max\{\rho_j\}_{j=1}^{\theta} \) is the maximum replication among all packets. The maximum number of packets on any node is given by \( \alpha = \max\{\alpha_i\}_{i=1}^{n} \). Clearly \( \sum_{j=1}^{\theta} \rho_j = \sum_{i=1}^{n} \alpha_i \).

Remark 2. When a node fails it can be repaired by a set of different nodes. Number of nodes contacted for repairing the fail node is known as repair degree. Hence each node \( U_i \) has a set of different repair degrees. Let \( \{d_i\}_{i=1}^{\theta} \) denotes the typical repair degree of the node \( U_i \), then the set of repair degrees is \( \{d_i\}_{i=1}^{\theta} : j \in \mathbb{N} \). If \( d_i = \max_{j=1}^{\theta} \{d_i\} \) is the maximum repair degree of the node \( U_i \), then \( d = \max_{i=1}^{n} \{d_i\} \) denotes...
the maximum repair degree of any node.

An example of FR code \( C : (7, 5, 3, 4) \) is shown in Table 1. In this example, packet 1 is replicated 4 times and all other packets are replicated 3 times. The set of repair degrees for each node are shown in the last column. Note that node \( U_7 \) has two different repair degrees.

![Table 1](image)

### Remark 3.
Each FR code \( C : (n, \theta, \alpha, \rho) \) considered in this paper from now on has same replication factor \( \rho \) for each packet i.e., \( \rho_j = \rho \) for every \( j = 1, 2, \ldots, \theta \).

For every FR code one can generate a node-packet distribution incidence matrix denoted by \( M_{n \times \theta} \). The distribution matrix is unique representation of FR code. The formal definition of node-packet distribution incidence matrix is as follows.

### Definition 4. (Node-Packet Distribution Incidence Matrix):
For an FR code \( C : (n, \theta, \alpha, \rho) \), a node-packet distribution incidence matrix is a matrix \( M_{n \times \theta} = [a_{ij}]_{n \times \theta} \) s.t.

\[
a_{ij} = \begin{cases} 
1 & \text{if packet } P_j \text{ appears } a_{ij} \text{ times on node } U_i; \\
0 & \text{if packet } P_j \text{ is not on node } U_i.
\end{cases}
\]

Clearly, if the packet \( P_j \) appears exactly ones on node \( U_i \) then the matrix is binary. For example, the node packet distribution incidence matrix \( M_{7 \times 5} \) for the FR code \( C : (7, 5, 3, 4) \) as given in Table 1, will be

\[
M_{7 \times 5} = \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

In order to construct an FR code \( C : (n, \theta, \alpha, \rho) \), one has to drop \( \theta \) packets on \( n \) nodes such that an arbitrary packet \( P_j \) is replicated \( \rho \) times in the code. The size of node \( U_i \) is \( \alpha_i \). This motivates us to define cycle and jump.

### Definition 5. (Cycle):
For every \( 1 \leq m \leq \rho \), a cycle is defined as one complete dropping of packets on \( n \) nodes such that all \( \theta \) packets are exhausted from \( \Omega_\theta = \{1, \ldots, \theta\} \) without any replication. Note that \( \rho \) is the replication factor of each packets in FR code \( C : (n, \theta, \alpha, \rho) \).

An example of such cycle is shown in Table 2.

### Definition 6. (Jump):
A jump is defined as the number of null packets between two consecutive packets from \( \Omega_\theta = \{1, \ldots, \theta\} \) while dropping them on \( n \) nodes \( U_1, U_2, \ldots, U_n \).

An example of such jump is shown in Table 2. In the table, dash between two packets (such as packets indexed by 3 and 4) represents jump for the certain cycle.

![Table 2](image)

One can associate a binary characteristic sequence (dropping sequence), node sequence (ordering the node sequence where the packet is dropped) and node-packet incidence matrix with any fractional repetition code. We now formally collect these definitions.

### Definition 7. (Dropping Sequence):
An FR code \( C : (n, \theta, \alpha, \rho) \) can be characterized by a binary characteristic sequence (weight of the sequence is \( \rho \theta \)) which is one whenever a packet is dropped on a node and zero whenever no packet is dropped.

### Example 8.
For a given \( n = 4 \) and \( \theta = 6 \) a possible dropping sequence for the FR code \( C : (4, 6, 3, 2) \) as shown in Figure 1, is \( \{1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1\} \). Using the sequence one can generate the FR code by dropping packet \( P_3 \) on node \( U_1 \), since \( d(1) = 1 \) and so on.

Dropping sequence \( (d(m))_{m=1}^t \) of an FR code \( C : (n, \theta, \alpha, \rho) \) has the following properties.

1. \( d(m) = \begin{cases} 
1 & : \text{if packet is dropped on node } U_t; \\
0 & : \text{if packet is not dropped on node } U_t,
\end{cases} \)

where

\[
t = \begin{cases} 
\{ m \mod n \} & : \text{if } n \uparrow m; \\
\{ n \} & : \text{if } n = m.
\end{cases}
\]

2. WLOG one can set \( d(l) = 1 \) for dropping sequence \( (d(m))_{m=1}^t \), where \( l \in \mathbb{N} \) s.t. weight of \( (d(m))_{m=1}^t \) is \( w((d(m))_{m=1}^t) = \rho \theta \) and \( d(m) \in \{0, 1\} \).

3. It is clear that \( l \in \{\rho \theta, \rho \theta + 1, \ldots, \rho \theta + (n-1)\theta - 1\} \) but one can reduce the value of \( l \) such that \( \rho \theta \leq l \leq \rho \theta + (n-1)\theta - 1 \) by puncturing \( n \) consecutive 0's in \( (d(m))_{m=1}^t \). It can be observed easily that if \( l > \rho \theta + (n-1)\theta - 1 \) then \( \exists q \geq n \) consecutive 0’s in \( (d(m))_{m=1}^t \).

4. If \( d(r) = 1 (1 \leq r \leq l) \) then packet \( P_\lambda \) is distributed on node \( U_t \), where

\[
\lambda = \begin{cases} 
\eta(r) & : \text{if } \eta(r) \neq 0; \\
\theta & : \text{if } \eta(r) = 0,
\end{cases}
\]

where \( \eta(r) = w((d(m))_{m=1}^r) \mod \theta \).

### Definition 9. (Node Sequence):
An FR code \( C : (n, \theta, \alpha, \rho) \) can be characterized by a finite sequence \( (s_i)_{i=1}^t \) defined from \( \{1, 2, \ldots, \rho \theta \} \subset \mathbb{N} \) to \( \Omega_\theta = \{1, 2, \ldots, n\} \subset \mathbb{N} \) such as packet \( P_t \) (\( t \in \Omega_\theta \)) is dropped on node \( s_i \in (\Omega_\theta) \), where
3. Constructions of Flower Codes

A generalized ring construction of Flower codes was described in [21] which give rise to Weak FR codes with \( \rho = 2 \) depending upon weather \( \theta \) is a multiple of \( n \) or not. In this section, we generalize \( \rho = 2 \) construction to \( \rho > 2 \). In order to construct the FR code of replication factor \( \rho \), we first place \( n \) nodes on a circle. Now we can place \( \theta \) packets on each of them one by one till a cycle is complete. One has to place the packets till we complete all \( \rho \) cycles. This will give rise to an FR code since all \( \theta \) packets are replicated \( \rho \) times in the system. Now we can vary the packet dropping mechanism by introducing jumps within a cycle or after every cycle. To do so first we define different kind of jumps. This process yields several classes of interesting FR codes.

Definition 12. (Internal & External Jumps): An internal jump is a jump applied within a cycle \( m, 1 \leq m \leq \rho \). Similarly a jump is called external jump if it is applied between two consecutive cycles \( m \) and \( m + 1, 1 \leq m \leq \rho \). In particular, internal (external) jump function is denoted by \( f_{\text{in}}(f_{\text{ex}}) \) with domain \( \{1, 2, \ldots, \rho\} \setminus \{\theta, 2\theta, \ldots, \rho\} \) \( \{\{1, 2, \ldots, \rho\}\} \). Both jump functions have common co-domain \( \mathbb{N} \cup \{0\} \).

Example 13. For given nodes \( n = 3 \), packets \( \theta = 4 \), internal jump function \( f_{\text{in}} = 1 \) and external jump function \( f_{\text{ex}} \), one can construct node sequence \( (1, 3, 2, 1, 2, 1, 3, 2) \) for some FR code \( \mathcal{F}_{\mathcal{F}} : (3, 4, 3, 2) \).

Remark 14. A special kind of jump, (for example, see Definition 15), can be described by a characteristic function \( \xi(i) = 1 \) (drop) or 0 (do not drop) which tells us when to drop a packet at a position \( i, 1 \leq i \leq n \).

Definition 15. (Subset Type Jumps): Let \( \Omega_n = \{1, 2, \ldots, n\} \) be an index set of \( n \) nodes and let \( A \subseteq \Omega_n \). A jump is called subset type jump if it’s characteristic function \( \xi(i) \) is given by

\[
\xi(i) = \begin{cases} 
1 & \text{if } i \in A; \\
0 & \text{if } i \not\in A.
\end{cases}
\]

Next we are ready to define a Flower code with single ring having a subset type jump within its \( \rho \) cycles.

Definition 16. (Flower code with single ring): A Flower code \( \mathcal{F}_{\mathcal{F}} : (n, \theta, \alpha, \rho) \) with single ring and having a subset type jumps can be defined by first placing \( n \) nodes along a single ring and then dropping packets as per a subset jump \( A_m(1 \leq m \leq \rho) \) (see Definition 15) within every cycle

\[
m(1 \leq m \leq \rho) \text{ till we drop all } \rho \text{ packets on the ring. An example for such jump, is illustrate in Table 3.}
\]

Table 3 A possible distribution for a Flower code \( \mathcal{F}_{\mathcal{F}} : (8, 7, 4, 3) \) with subset type jumps \( A_1 = \{1, 2, 4\} \), \( A_2 = \{5, 6, 7, 8\} \) and \( A_3 = \{2, 3, 5, 6, 7\} \).

| Node | Packet distribution | Node Capacity |
|------|---------------------|---------------|
| \( U_i \) | For \( A_1 \) | For \( A_2 \) | For \( A_3 \) | \( \alpha_i \) |
| \( U_1 \) | \( P_3 \) | \( P_4 \) | \( P_5 \) | - | - | - | 3 |
| \( U_2 \) | \( P_2 \) | \( P_6 \) | \( P_7 \) | - | - | - | 3 |
| \( U_3 \) | - | - | - | - | - | - | 2 |
| \( U_4 \) | \( P_3 \) | \( P_6 \) | - | - | - | - | 2 |
| \( U_5 \) | - | - | - | - | - | - | 3 |
| \( U_6 \) | \( P_4 \) | \( P_5 \) | \( P_3 \) | - | - | - | 3 |
| \( U_7 \) | - | - | - | - | - | - | 3 |
| \( U_8 \) | - | - | - | - | - | - | 1 |

Lemma 17. Consider a Flower code \( \mathcal{F}_{\mathcal{F}} : (n, \theta, \alpha, \rho) \) with single ring and having subset type jumps on node subsets \( A_1, A_2, \ldots, A_\rho \). If total number of packets distributed on node \( U_i, i_i \in A_m; 1 \leq m \leq \rho \) and \( t = 1, 2, \ldots, |A_m| \) for subset type jump with single ring on node subset \( A_m \) is \( P(U_i, A_m) \) then

\[
P(U_i, A_m) = \begin{cases} \frac{\theta}{|A_m|} & \text{if } t \leq \theta \mod |A_m|; \\
\frac{|A_m|}{|A_m|} & \text{otherwise.}
\end{cases}
\]

Proof. Suppose \( A_1, A_2, \ldots, A_\rho \) are the subsets of \( \Omega_n \). For the given \( n \) nodes, \( \theta \) packets and \( A_m (m = 1, 2, \ldots, \rho) \) one can construct flower code \( \mathcal{F}_{\mathcal{F}} : (n, \theta, \alpha, \rho) \) with single ring and subset type jump on \( A_m \), \( \forall m \). Since subset type jump on particular subset \( A_m \), is assigning \( \theta \) distinct packets on nodes \( U_i, i_i \in A_m \) \( 1 \leq t \leq |A_m| \) in some specific order. If \( |A_m| \) divides \( \theta \) then number of packets dropped on a particular node \( U_i \) is \( \frac{\theta}{|A_m|} \). If \( |A_m| \) does not divide \( \theta \) then after dropping \( \left\lfloor \frac{\theta}{|A_m|} \right\rfloor \) \( |A_m| \) packets on each node of \( A_m \), there will remain \( \theta \mod |A_m| \) \( |A_m| \) packets to assign nodes. Hence, there are \( \theta \mod |A_m| \) number of nodes \( U_i \in A_m \) with \( \left\lfloor \frac{\theta}{|A_m|} \right\rfloor + 1 \) packets each. Remaining nodes in the set \( A_m \) have \( \left\lfloor \frac{\theta}{|A_m|} \right\rfloor \) packets each. Hence, the lemma is proved.

Lemma 18. For a Flower code \( \mathcal{F}_{\mathcal{F}} : (n, \theta, \alpha, \rho) \) with single ring and having subset type jumps on node subsets \( A_1, A_2, \ldots, A_\rho \), the total number of packets stored on a particular node \( U_i \) is

\[
\alpha_i = \sum_{m=1}^{\rho} P(U_i, A_m),
\]

where, \( P(U_i, A_m) = 0 \) for \( U_i \notin A_m \).
Proof. For a Flower code $\mathcal{F}_F : (n, \theta, \alpha, \rho)$ with single ring and having subset type jumps, all packets stored on a particular node are equal to the sum of total number of packets dropped on the node for each subset type jump of $A_m$ ($1 \leq m \leq \rho$). Hence proved.

Remark 19. For a Flower code $\mathcal{F}_F : (n, \theta, \alpha, \rho)$ with a subset type jump $A_m (1 \leq m \leq \rho)$ has the following properties

- If $|A_{max}| = \max \{|A_1|, |A_2|, \ldots, |A_\rho|\}$ then
  $$\left\lfloor \frac{\theta}{|A_{max}|} \right\rfloor \leq \alpha \leq \left\lfloor \frac{\rho}{|A_{max}|} \right\rfloor.$$ 
- If $\exists U_i \text{ s.t. } U_i \in \bigcap_{m=1}^{\rho} A_m \text{ then the node } U_i \text{ has maximum number of distributed distinct packets i.e. } |U_i| = \alpha$ but converse is not true.
- If $\exists U_p (p \in A_m) \text{ s.t.}$
  $$U_p \notin \bigcup_{i=1}^{\rho} A_i, \text{ then } \alpha_{max} \geq \left\lfloor \frac{\theta}{|A_m|} \right\rfloor$$

and vice versa, where $1 \leq i, j, p \leq n$.

To construct FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$, one can concatenate $\rho$ distinct cycles with some internal and external jumps, where each cycle defined on node n.

Definition 20. (Flower code with multiple rings): A system with $\rho$ cycles of $\theta$ packets in which packets are distributed among $n$ nodes arranged on a circle with internal jump function $f_{in} : \{1, 2, \ldots, \rho\} \setminus \{\theta, 2\theta, \ldots, \rho\} \rightarrow \mathbb{N} \cup \{0\}$ and external jump function $f_{ex} : \{1, 2, \ldots, \rho\} \rightarrow \mathbb{N} \cup \{0\}$ is called Flower code $\mathcal{F}_\theta$ with parameters $n, \theta, \alpha, \rho$, where $\alpha$ is the maximum collective frequency of appearance of a node in all cycles.

Example 21. A Flower code $\mathcal{F}_\theta : (n, \theta, \alpha, \rho)$ with internal jump function $f_{in}(x) = 1$ and external jump function $f_{ex}(x) = 0$, is illustrated in Table 4.

| Node | Packet distribution | Node Capacity |
|------|---------------------|--------------|
| $U_1$ | $P_1$ - | $P_3$ - | - |
| $U_2$ | - | $P_3$ - | $P_1$ - |
| $U_3$ | - | - | $P_3$ - |
| $U_4$ | - | $P_3$ - | $P_2$ - |
| $U_5$ | - | $P_3$ - | - |

Note that the Flower code $\mathcal{F}_\theta : (n, \theta, \alpha, \rho)$ has internal and external jump functions ($f_{in}$ and $f_{ex}$ respectively) and each Flower code $\mathcal{F}_\theta : (n, \theta, \alpha, \rho)$ is an FR code $\mathcal{C} : (n, \theta, \alpha, \rho)$ so terms of node sequence $s_m$ in $(s_m)_{m=1}^\rho$ can be represented in terms of $f_{in}$ and $f_{ex}$ as described in Theorem 22.

Theorem 22. Consider a Flower code $\mathcal{F}_\theta : (n, \theta, \alpha, \rho)$ having an internal jump function $f_{in} : \{1, 2, \ldots, \rho\} \setminus \{\theta, 2\theta, \ldots, \rho\} \rightarrow \mathbb{N} \cup \{0\}$ and an external jump function $f_{ex} : \{1, 2, \ldots, \rho\} \rightarrow \mathbb{N} \cup \{0\}$. If node sequence of the Flower code $\mathcal{F}_\theta : (n, \theta, \alpha, \rho)$ is $(s_m)_{m=1}^\rho$ then

$$s_m = \begin{cases} 1 & : \text{ if } m = 1; \\ \theta(m) & : \text{ if } \theta(m) \neq 0 \text{ and } 1 \leq m \leq \rho; \\ n & : \text{ if } \theta(m) = 0, 1 < m \leq \rho; \\ 0 & : \text{ if } m > \rho, \end{cases}$$

where

$$\theta(m) = \left[ \frac{m}{\rho} \right] \cdot \frac{m - 1}{\theta} \pmod{n}.$$
Flower code can also be represented by node sequence, dropping sequence and incidence matrix. The following lemmas establish the relation among those collectively. The lemmas are as follows.

**Lemma 23.** Consider a dropping sequence \( (d(m))^t_{m=1} \) for a Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \). If \( d(t) = 1 \) for any \( t (1 \leq t \leq t) \) then binary node-packet distribution incidence matrix \( M_{n \times \theta} = [a_{ij}]_{n \times \theta} \) is given by \( a_{ij} = 1 \), where

\[
\begin{align*}
  j &= \begin{cases} 
  \text{wt} (d(m))^t_{m=1} (\text{mod } \theta) & \text{if } \theta | \text{wt} (d(m))^t_{m=1}; \\
  \theta & \text{if } \theta \text{ does not divide } \text{wt} (d(m))^t_{m=1}; \\
  \end{cases} \\
  i &= \begin{cases} 
  t (\text{mod } n) & \text{if } n | t; \\
  n & \text{if } n \nmid t; \\
  \end{cases}
\end{align*}
\]

(6)

**Proof.** Let \( (d(m))^t_{m=1} \) be a dropping sequence for an FR code \( \mathcal{C} : (n, \theta, \alpha, \rho) \) with \( n \) nodes and \( \theta \) packets. For \( d(m) = 1 \) \( d(t) \in (d(m))^t_{m=1} \), index of the packet associated with the \( d(t) \) is mapped to weight of subsequence \( (d(m))^t_{m=1} \). Hence the lemma.

\[\square\]

**Remark 24.** If Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) has a non-binary incidence matrix then one can calculate the incidence matrix by the Algorithm 1 using dropping sequence.

**Algorithm 1.** Algorithm to compute node-packet distribution incidence matrix \( M_{n \times \theta} \) for Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \).

**REQUIRE** Dropping sequence \( (d(m))^t_{m=1} \) of Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \).

**ENSURE** Node-packet distribution incidence matrix \( M_{n \times \theta} \).

1. Initially set \( a_{ij} = 0 \) and \( t = 1 \) where \( a_{ij} \) is the element of \( M_{n \times \theta} \), \( 1 \leq i \leq n \), \( 1 \leq j \leq \theta \) and \( 1 \leq t \leq t \).
2. If \( d(t) = 0 \) then jump to step 3 and if \( d(t) = 1 \) then calculate \( i,j \) and set \( a_{ij} = 1 \) and go to step 3.
3. If \( t < \theta \) then set \( t = t + 1 \) and go to step 2 otherwise stop.

It is clear that Algorithm 1 is the generalization of Lemma 23.

**Lemma 25.** Consider a Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) with node sequence \( (s_i)^{\rho_0}_{i=1} \). Its dropping sequence is given by \( (d(m))^t_{m=1} \) s.t. \( 1 \leq i \leq \rho_0 \)

\[
d(m) = \begin{cases} 
  1 & \text{if } m = \sum_{j=1}^{s_i} (s_j - s_{j-1}) (\text{mod } n); \\
  0 & \text{if } m \neq \sum_{j=1}^{s_i} (s_j - s_{j-1}) (\text{mod } n),
\end{cases}
\]

where \( s_0 = 0 \).

**Proof.** For a Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \), consider a dropping sequence \( (d(m))^t_{m=1} \). By Definition 9, two packets with consecutive indexes are associated with \( s_i \) and \( s_{i+1} \) for some \( i \). Using Definition 7, one can find that \( s_{i+1} - s_i (\text{mod } n) \) number of zeros exist between two consecutive 1’s. In particular, the 1’s are associative with the two packets. It proves the lemma.

\[\square\]

**Lemma 26.** Consider a Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) with dropping sequence \( (d(m))^t_{m=1} \). Its node sequence is given by \( (s_i)^{\rho_0}_{i=1} \) s.t. \( 1 \leq t \leq t \) with \( d(t) = 1 \),

\[
i = \text{wt} (d(m))^t_{m=1}
\]

and

\[
s_i = \begin{cases} 
  t (\text{mod } n) & \text{if } n | t; \\
  n & \text{if } n \nmid t.
\end{cases}
\]

**Proof.** Using Definition 9 and Definition 7, one can easily prove the lemma by observing that the weight of subsequence \( (d(m))^t_{m=1} \) is associated with the packet for \( d(t) = 1 \).

\[\square\]

These sequence construction approaches suggest that one can construct an FR code from any arbitrary finite binary sequence by treating it as a characteristic sequence of dropping packets on the nodes. This can be done in different ways by choosing appropriate sequences of symbols. Hence, an arbitrary finite binary sequence \( (\chi(m))^\ell_{m=1} \) \((\chi(m) \in \mathcal{Z}_2)\) with length \( \ell (\in \mathbb{N}) \) can be defined as a characteristic sequence for an FR code. In the sequence, value \( \chi(m) = 1 \) represents to drop (value \( \chi(m) = 0 \) not to drop) a packet on certain node. Hence a more general Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) can be defined as follows.

**Definition 27.** (Flower code): For \( n \) nodes, \( \theta \) packets and an arbitrary binary sequence \( (\chi(m))^\ell_{m=1} \) \((\chi(m) \in \mathcal{Z}_2)\) of length \( \ell (\in \mathbb{N}) \) (treated as characteristic sequence indexed by \( m \)), Flower code \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) can be defined as a system in which packet indexed by \( m(\text{mod } \theta) \) is dropped on node indexed by \( m(\text{mod } \theta) \) if \( \chi(m) = 1 \), where \( P_0 \) and \( U_0 \) are mapped to packet \( P_0 \) and node \( U_0 \), respectively. Clearly \( \alpha_i = \sum_{m=i}^{i+\theta-1} \chi(i + nm) \) and \( \rho_j = \sum_{m=0}^{\theta} \chi(j + tm) \).

**Example 28.** For given \( n = 4, \theta = 5 \) and a binary characteristic sequence \( (\chi(m))^16_{m=1} = \{1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1\} \), one can find Node packet distribution of Flower code \( \mathcal{F}_\theta : (4, 5, 3, 2) \) as shown in Table 5.

| Nodes \( U_i \) | Packets distribution | Repair degree \( d_i \) |
|----------------|----------------------|------------------------|
| 1              | \( P_1, P_4 \)       | 2                      |
| 2              | \( P_2, P_3 \)       | 3                      |
| 3              | \( P_1, P_2, P_3 \)  | 3                      |
| 4              | \( P_1, P_2, P_3, P_4 \) | 4                  |

**Remark 29.** Flower codes \( \mathcal{F}_\theta : (n, \theta, \alpha, \rho) \) (Definition 27) with different packet replication factor \( \rho_j \) \((j \in \Omega_d)\) can also be constructed using any binary sequence \( (\chi(m))^\ell_{m=1} \).
4. Conclusion
This paper introduces a novel class of FR codes based on sequences. It will be an interesting future task to find a condition when Flower codes are optimal (in sense of the capacity). Our work opens a nice connection of FR codes with well known area of sequences. It would be an interesting future task to study FR codes by using well known class of sequences.

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