NOT EVERY OBJECT IN THE DERIVED CATEGORY OF A RING IS BOUSFIELD EQUIVALENT TO A MODULE

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ABSTRACT. We consider the derived category of a specific non-Noetherian ring Λ, and show that there are objects in \( D(Λ) \) that are not Bousfield equivalent to any module. This answers a question posed in [DP08].

1. Introduction

Let \( k \) be a countable field, and consider the graded ring
\[
Λ := \frac{k[x_1, x_2, x_3, \ldots]}{(x_1^2, x_2^2, x_3^2, \ldots)}, \quad \text{where} \ deg(x_i) = 2^i.
\]

The unbounded derived category \( D(Λ) \) was studied extensively in [DP08]. There the authors showed that \( D(Λ) \) behaves very differently than the derived category of a commutative Noetherian ring.

For example, while \( |\text{Spec}(Λ)| = 1 \), the Bousfield lattice of \( D(Λ) \) has cardinality \( 2^{2^{2^0}} \). On the other hand, with a commutative Noetherian ring \( R \), the Bousfield lattice is isomorphic to the lattice of subsets of \( \text{Spec} R \). Much work has been done in understanding the Bousfield lattice, and thick and localizing subcategories, in the Noetherian case [Nee92, HPS97], and in extending these results to other tensor-triangulated categories [Bal05, IK, BIK11].

Thomason [Tho97] classified the thick subcategories of finite objects in the derived category of a non-Noetherian ring, but besides this, little is known about the non-Noetherian case. With a view towards the stable homotopy category [HP99], derived categories of non-Noetherian rings may give insight into structure within the Bousfield lattice \( BL \); in particular, the Boolean algebra \( BA \) of complemented Bousfield classes and the distributive lattice \( DL \) of objects \( X \) with \( \langle X \rangle = \langle X \land X \rangle \). In the Noetherian case, \( BA = DL = BL \); in the stable homotopy category \( BA \subseteq DL \subseteq BL \).

The relative simplicity of the Bousfield lattice in the derived category of a Noetherian ring comes, in part, from the fact that every object \( X \) is Bousfield equivalent to a module [Nee92]. (By module, we mean an object in the derived category that has nonzero homology only in degree zero; every module can be thought of as an object in the derived category, in this way.) Specifically,
\[
\langle X \rangle = \left\langle \bigoplus_{p \in \text{supp}(X)} k_p \right\rangle,
\]
where \( k_p \) is the image in \( D(R) \) of the residue field \( k_p \) of \( p \), and
\[
\text{supp}(X) = \{ p \in \text{Spec} R \mid X \land k_p \neq 0 \}.
\]

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In Question 5.8 of \[DP08\], the authors ask if, given a commutative ring \( R \), every object in \( D(R) \) is Bousfield equivalent to a module. We answer this question in the negative. Our main result is the following.

**Theorem 2.1.** In the category \( D(\Lambda) \), there are objects that are not Bousfield equivalent to any module.

2. Tel and modules in \( D(\Lambda) \)

Let \( C \) in \( D(\Lambda) \) be represented by the (homologically graded) chain complex

\[
\begin{align*}
0 & \longrightarrow \Lambda \\
& \quad \xrightarrow{x_1} \Lambda \\
& \quad \xrightarrow{x_1x_2} \Lambda \\
& \quad \xrightarrow{x_2x_3} \Lambda \\
& \quad \xrightarrow{x_3x_4} \Lambda \\
& \quad \xrightarrow{x_4x_5} \Lambda \\
& \quad \xrightarrow{x_5x_6} \cdots
\end{align*}
\]

Define \( f : C \to \Sigma^2C \) to be the following chain map.

\[
\begin{align*}
0 & \longrightarrow \Lambda \\
& \quad \xrightarrow{x_1} \Lambda \\
& \quad \xrightarrow{x_1x_2} \Lambda \\
& \quad \xrightarrow{x_2x_3} \Lambda \\
& \quad \xrightarrow{x_3x_4} \Lambda \\
& \quad \xrightarrow{x_4x_5} \Lambda \\
& \quad \xrightarrow{x_5x_6} \cdots
\end{align*}
\]

One can check that with the module grading \( \deg(x_i) = 2^i \), this is in fact a chain map.

Now define \( \text{Tel} \) to be the sequential colimit

\[
\text{Tel} = \text{colim} \left( C \xrightarrow{f} \Sigma^2C \xrightarrow{\Sigma^2f} \Sigma^4C \longrightarrow \cdots \right).
\]

This is a minimal weak colimit, so it is not unique, but does satisfy

\[
H_n(\text{Tel}) = \text{colim} \left[ H_n(C) \longrightarrow H_n(\Sigma^2C) \longrightarrow \cdots \right].
\]

**Proposition 2.1.** For all \( n \in \mathbb{Z} \), the object \( \text{Tel} \) satisfies

\[
H_n(\text{Tel}) = I\Lambda.
\]

**Proof.** We defer the proof to Section 3. \( \square \)

Let \( IA = \Lambda^* = \text{Hom}_k(\Lambda, k) \) denote the graded vector space dual of \( \Lambda \); \( IA \) is a \( \Lambda \)-module, concentrated in non-positive module degrees. Note that \( \Lambda \) is a finite vector space in each module degree, so \( \Lambda^{**} = (IA)^* = \Lambda \). For an arbitrary element \( X \) in \( D(\Lambda) \), define

\[
IX = \mathbb{R}\text{Hom}_\Lambda(X, IA).
\]

Then Lemma 3.4 in \[DP08\] shows that \( IX \cong \mathbb{R}\text{Hom}_k(X, k) \), thought of as an object in \( D(\Lambda) \). This functor \( I(\cdot) \) is analogous to Brown-Comenetz duality in the stable homotopy category \[BC76, DP08\]. In particular, since \( k \) is an injective cogenerator in the category of \( k \)-modules, \( IX = 0 \) if and only if \( X = 0 \).

We will prove two lemmas about the Bousfield class of \( \text{Tel} \), and use them to prove the Theorem.
Lemma 2.2. $\langle \text{Tel} \rangle \neq \langle IA \rangle$.

Proof. Let $K$ be the cofiber of $f : C \to \Sigma^2 C$. We know that $K$ is not zero, because Proposition 2.1 implies that $f$ is not an equivalence. The following are known about $C$, $K$, and $\text{Tel}$ (see ASHT, Prop. 3.6.9):

\[ \langle C \rangle = \langle K \rangle \vee \langle \text{Tel} \rangle \text{ and } \langle 0 \rangle = \langle K \rangle \wedge \langle \text{Tel} \rangle. \]

Furthermore, [DP08 Cor. 7.3] shows that $\langle I_\Lambda \rangle \leq \langle X \rangle$ for all nonzero $X$ in $D(\Lambda)$. Suppose, towards a contradiction, that $\langle \text{Tel} \rangle = \langle I_\Lambda \rangle$. Then $\langle \text{Tel} \rangle \leq \langle K \rangle$, so $\langle C \rangle = \langle K \rangle$. This implies $\langle 0 \rangle = \langle C \rangle \wedge \langle \text{Tel} \rangle$, so $C \wedge \text{Tel} = 0$. This would force $C \wedge IA = 0$.

But we will now show that $C \wedge IA \neq 0$. Using tensor-hom adjointness in the derived category, we have

\[ I(C \wedge IA) = R\text{Hom}_\Lambda(C \wedge IA, IA) \]
\[ \cong \text{Hom}\Lambda(C, R\text{Hom}_\Lambda(IA, IA)) \]
\[ = \text{Hom}\Lambda(C, I(IA)) \]
\[ \cong \text{Hom}\Lambda(C, \Lambda). \]

The module $\Lambda$ is self-injective, because $\Lambda$ is a $P$-algebra [Mar83 Thm. 13.12]. So

\[ H_0(\text{Hom}_\Lambda(C, \Lambda)) = \text{Hom}_{D(\Lambda)}^0(C, \Lambda) = \text{Hom}_{K(\Lambda)}^0(C, \Lambda), \]
and this is nonzero because there are nontrivial classes of chain maps from $C$ to $\Lambda$.

\[ \begin{array}{cccccccc}
\cdots & 0 & \longrightarrow & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_2} & \Lambda & \longrightarrow \\
\cdots & & & \downarrow f & & & \downarrow 0 & & \\
0 & 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda & \longrightarrow & \cdots
\end{array} \]

Therefore $I(C \wedge IA) \neq 0$, and $C \wedge IA \neq 0$. \hfill \Box

In Section 5 of [DP08], the authors ask if every object is Bousfield equivalent to the direct sum of its homology groups. The last two results show that this is not true. This was also shown recently in [IK, 4.8].

Let $\mathcal{M}$ denote the replete subcategory of $D(\Lambda)$ of all modules; i.e. all objects with nonzero homology only in degree zero. In the following lemma, we think of a Bousfield class $\langle X \rangle$ as the localizing subcategory of $X$-acyclics.

Lemma 2.3.

\[ \mathcal{M} \cap \langle \text{Tel} \rangle = \mathcal{M} \cap \langle IA \rangle. \]

Proof. Since $\langle IA \rangle$ is minimum among nonzero Bousfield classes, we know $\langle IA \rangle \leq \langle \text{Tel} \rangle$, so we already have the $\subseteq$ direction. We will show that if $M$ is a module in $D(\Lambda)$ and $M \wedge IA = 0$, then $M \wedge \text{Tel} = 0$.

In [KM95 Thm. 4.7] the authors construct a strongly convergent Eilenberg-Moore spectral sequence in the category of ($\mathbb{Z}$-graded, so unbounded) modules over a DGA. If we consider $\Lambda$ as a DGA concentrated in chain degree zero, then this spectral sequence is

\[ E^2_{p,q} = H_*(H_*A \wedge H_*B) \Longrightarrow H_*(A \wedge B), \]
Lemma 3.1. Suppose, towards a contradiction, that every object $Y$ in $D(\Lambda)$ is Bousfield equivalent to some module, $M_Y$. Take $X$ with $X \wedge I\Lambda = 0$. Then $M_X \wedge I\Lambda = 0$. Using Lemma 2.3, this says that

$$M_X \in \mathcal{M} \cap (I\Lambda) = \mathcal{M} \cap \langle I\Lambda \rangle.$$ 

Thus $M_X \wedge \text{Tel} = 0$, so $X \wedge \text{Tel} = 0$. This implies that $(I\Lambda) \geq \langle \text{Tel} \rangle$. Since we already have $(I\Lambda) \leq \langle \text{Tel} \rangle$, we conclude that $(I\Lambda) = \langle \text{Tel} \rangle$. But this contradicts Lemma 2.2. □

3. Proof of Proposition 2.1

Our goal is to show that for all $n \in \mathbb{Z}$,

$$H_n(\text{Tel}) = \text{colim} \left[ H_n(C) \rightarrow H_n(\Sigma^2 C) \rightarrow \cdots \right] \cong I\Lambda.$$ 

For concreteness, we will compute $H_{-2}(\text{Tel})$, and then indicate the general case. We will split the computation into several lemmas.

Because of the shift, we are trying to compute

$$H_{-2}(\text{Tel}) = \text{colim} \left[ H_{-2}(C) \rightarrow H_{-4}(C) \rightarrow H_{-6}(C) \rightarrow \cdots \right].$$

We have

$$H_{-2}(C) = \frac{\ker(x_2 x_3)}{\text{im}(x_1 x_2)} = \frac{x_2 x_3}{x_1 x_2}, \quad \text{and generally } H_{-n}(C) = \frac{x_n x_{n+1}}{x_{n-1} x_n}, \quad \text{for } n \geq 2.$$ 

Define

$$M_{-2} = \frac{(x_3)}{(x_2, x_4, x_5, x_6, \ldots)}, \quad \text{and in general } M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)}.$$ 

Lemma 3.1.

$$H_{-2}(\text{Tel}) = \text{colim} \left[ \frac{(x_2, x_3)}{(x_1 x_2)} \rightarrow \frac{(x_4, x_5)}{(x_3 x_4)} \rightarrow \frac{(x_6, x_7)}{(x_5 x_6)} \rightarrow \cdots \right] \cong \text{colim} \left[ M_{-2} \xrightarrow{x_2 x_3} M_{-4} \xrightarrow{x_4 x_5} M_{-6} \xrightarrow{x_6 x_7} \cdots \right].$$

Proof. This uses the universal property of colim. For all $n \geq 2$, we have surjective projection maps

$$\phi_{-n} : H_{-n}(C) = \frac{(x_n, x_{n+1})}{(x_{n-1} x_n)} \rightarrow \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)} = M_{-n}.$$ 

Thus we get maps $H_{-n}(C) \rightarrow \text{colim} M_i$, which induces $\Phi : \text{colim} H_i(C) \rightarrow \text{colim} M_i$. We will show that $\Phi$ is surjective and injective.

where $A$ and $B$ are elements of $D(\Lambda)$. Let $A = M$ be a $\Lambda$-module (concentrated in chain degree zero), and $B = \text{Tel}$. Then using Proposition 2.1, we have

$$E^2 = H_* \left( M \wedge \left( \bigoplus_{i \in \mathbb{Z}} \Sigma^i I\Lambda \right) \right) \cong H_* \left( \bigoplus_{i \in \mathbb{Z}} \Sigma^i (M \wedge I\Lambda) \right).$$

If $M \wedge I\Lambda = 0$ then the $E^2$ page collapses to zero, and $H_*(M \wedge \text{Tel}) = 0$. □

Theorem 2.4. In the category $D(\Lambda)$, there are objects that are not Bousfield equivalent to any module.

Proof. Suppose, towards a contradiction, that every object $Y$ in $D(\Lambda)$ is Bousfield equivalent to some module, $M_Y$. Take $X$ with $X \wedge I\Lambda = 0$. Then $M_X \wedge I\Lambda = 0$. Using Lemma 2.3, this says that

$$M_X \in \mathcal{M} \cap (I\Lambda) = \mathcal{M} \cap \langle I\Lambda \rangle.$$ 

Thus $M_X \wedge \text{Tel} = 0$, so $X \wedge \text{Tel} = 0$. This implies that $(I\Lambda) \geq \langle \text{Tel} \rangle$. Since we already have $(I\Lambda) \leq \langle \text{Tel} \rangle$, we conclude that $(I\Lambda) = \langle \text{Tel} \rangle$. But this contradicts Lemma 2.2. □
onto: We will use standard properties of colimits, that hold for weak colimits as well (see e.g. [Mar83 App. 1.2, Prop. 7]). Take \( \tilde{x} \in \text{colim} M_i \). So \( \tilde{x} \) is represented by \( x \in M_{-r} \) for some \( r \). Since \( \phi_{-r} \) is surjective, we can pick a \( y \in H_{-r}(C) \) such that \( \phi_{-r}(y) = x \). By the definition of a colimit, this factors through \( \Phi \). So, letting \( \tilde{y} \) be the image of \( y \) in \( \text{colim} H_i(C) \), we get \( \Phi(\tilde{y}) = \tilde{x} \).

one-to-one: Suppose \( \Phi(\tilde{y}) = 0 \). Then \( \tilde{y} \) is represented by \( y \in H_{-r}(C) \) for some \( r \). We have a commuting diagram

\[
\begin{array}{ccc}
H_{-r}(C) & \longrightarrow & \text{colim} H_i(C) \\
\phi_{-r} \downarrow & & \downarrow \Phi \\
M_{-r} & \longrightarrow & \text{colim} M_i
\end{array}
\]

Therefore \( x = \phi_{-r}(y) \in M_{-r} \) maps to zero in \( \text{colim} M_i \). This means that either \( x = 0 \), or \( x \) becomes zero eventually in the sequence \( M_{-r} \rightarrow M_{-r-2} \rightarrow M_{-r-4} \rightarrow \cdots \). Suppose that \( x \) becomes zero at \( M_{-r-s} \), where it could be that \( s = 0 \). We claim that the following square commutes

\[
\begin{array}{ccc}
H_{-r}(C) & \longrightarrow & H_{-r-s}(C) \\
\phi_{-r} \downarrow & & \downarrow \phi_{-r-s} \\
M_{-r} & \longrightarrow & M_{-r-s}
\end{array}
\]

Suppose for a moment that this is the case. Since \( \phi_{-r}(y) = x \), this implies that the image of \( y \) in \( H_{-r-s}(C) \), call it \( z \), maps to zero in \( M_{-r-s} \).

If \( z = 0 \), then we’re done - this implies that \( \tilde{y} = 0 \). So consider the case that \( z \neq 0 \), but \( \phi_{-r-s}(z) = 0 \). Now, \( \phi_{-r-s} \) is the map

\[
\begin{pmatrix}
(x_{r+s}, x_{r+s+1}) \\
(x_{r+s-1}, x_{r+s})
\end{pmatrix} \mapsto \begin{pmatrix}
(x_{r+s+1}) \\
(x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, \ldots)
\end{pmatrix}.
\]

Therefore \( z \in (x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, \ldots) \). But from \( H_{-r-s}(C) \), the maps encountered in \( \text{colim} H_i(C) \) are precisely \( x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, \ldots \), so we are guaranteed that eventually \( z \) will be sent to zero. This implies that \( \tilde{y} = 0 \), so \( \Phi \) is injective.

To see that the above square commutes, it suffices to show it for \( s = 2 \). By definition, this is the square

\[
\begin{array}{ccc}
(x_{r+s}, x_{r+s+1}) & \xrightarrow{\phi} & (x_{r+s+1}, x_{r+s+2}) \\
(x_{r+s-1}, x_{r+s}) & & (x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, \ldots)
\end{array}
\]

and it’s straightforward to check that this commutes. \( \square \)

Lemma 3.2.

\[
\text{colim} \left[ M_{-2} \to_{x_2} M_{-1} \to_{x_2} M_0 \to_{x_2} \cdots \right] 
\cong \text{colim} \left[ \begin{pmatrix} k[x_1]^{x_2} \\ (x_1^2) \end{pmatrix} \leftrightarrow \begin{pmatrix} k[x_1, x_2]^{x_3} \\ (x_1^2) \end{pmatrix} \leftrightarrow \begin{pmatrix} k[x_1, x_2, x_3, x_4, x_5]^{x_6} \\ (x_1^2) \end{pmatrix} \leftrightarrow \cdots \right].
\]
Proof. First consider \( M_{-4} = \frac{(x_3)}{(x_4, x_5, x_6, x_7, \ldots)} \). As a \( \Lambda \)-module, this has generator \( x_3 \), and top degree element \( x_1 x_2 x_3 x_5 \).

Let \( \overline{x_1} \) denote the dual of \( x_i \). As a \( \Lambda \)-module, \( \left( \frac{k[x_1, x_2, x_3]}{(x_7^2)} \right)^* \) is generated by \( \overline{x_1} x_2 x_3 \), and has top degree element \( x_1 x_2 x_3 x_5 \). In fact, we can define a \( \Lambda \)-isomorphism from \( \left( \frac{x_3}{x_4, x_5, x_6, x_7, \ldots} \right) \) to \( \left( \frac{k[x_1, x_2, x_3]}{(x_7^2)} \right)^* \), by sending \( x_3 \mapsto x_1 x_2 x_3 x_5 \).

Similarly, for all \( n \geq 2 \), we have \( \Lambda \)-isomorphisms

\[
M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)} \longrightarrow \left( \frac{k[x_1, x_2, \ldots, x_{n-1}]}{(x_7^2)} \right)^*,
\]

defined by sending

\[
x_{n+1} \mapsto x_1 x_2 \cdots x_n.
\]

This map has degree

\[-|x_1| - |x_2| - \cdots - |x_{n-1}| - |x_{n+1}|.
\]

Now, to see that the \( f_i \) among the \( M_i \)'s become inclusions, we will illustrate with an example. Consider

\[
M_{-2} = \frac{(x_3)}{(x_4, x_5, x_6, x_7, \ldots)} \twoheadrightarrow \frac{x_2 x_3}{(x_4, x_5, x_6, x_7, \ldots)} = M_{-4}.
\]

In the bottom left, the generator \( \overline{x_1} \) goes up to the generator \( x_3 \), then right to \( x_2 x_3 x_5 \), which gets sent down to

\[
x_2 x_3 (\overline{x_1} x_2 x_3) = \overline{x_1}
\]

in the bottom right. The degree of this composition is

\[
(|x_1| + |x_3|) + (|x_2| + |x_5|) + (-|x_1| - |x_2| - |x_3| - |x_5|) = 0.
\]

This shows that each map becomes a degree-zero inclusion under the isomorphisms just described. \( \square \)

Lemma 3.3.

\[
\text{colim} \left[ \left( \frac{k[x_1]}{(x_7^2)} \right)^* \hookrightarrow \left( \frac{k[x_1, x_2, x_3]}{(x_7)} \right)^* \hookrightarrow \left( \frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_7^2)} \right)^* \hookrightarrow \cdots \right] = \left( \lim_{\rightarrow} \left[ \cdots \rightarrow \frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_7^2)} \rightarrow \frac{k[x_1, x_2, x_3]}{(x_7^2)} \rightarrow \frac{k[x_1]}{(x_7^2)} \rightarrow 0 \right] \right)^* \cong I \Lambda.
\]

Proof. Let \( V_i^* = \left( \frac{k[x_1, x_2, \ldots, x_i]}{(x_7^2)} \right)^* \). Since these are locally finite, we have \( (V_i^*)^* = V_i \) for all \( i \). The definition of a sequential colimit gives a certain exact sequence

\[
\prod V_i^* \xrightarrow{G} \prod V_i^* \twoheadrightarrow \text{(colim} V_i^* \text{)} \longrightarrow 0,
\]
which dualizes to an exact sequence

\[ 0 \rightarrow (\operatorname{colim} V_i^*)^* \rightarrow \prod (V_i^*)^* \overset{G^*}{\rightarrow} \prod (V_i^*)^*. \]

One can check that in fact \( G^* \) is the map used in the definition of the sequential limit, so we have

\[ \lim V_i = (\operatorname{colim} V_i^*)^*. \]

Since \( \lim V_i \cong \Lambda \), this shows that \( \operatorname{colim} V_i^* \) is the thing that dualizes to \( \Lambda \). In other words \( \operatorname{colim} V_i^* \cong I\Lambda \).

\[ \square \]

**Proof of Proposition 2.1.** Combining the three previous lemmas, we get that \( H_{-2}(\text{Tel}) \cong I\Lambda \). Because the map \( f : C \to \Sigma^2 C \) has degree two, and sequential colimits are determined by their long-term behavior, it’s easy to see that \( H_i(\text{Tel}) = H_{-2}(\text{Tel}) = I\Lambda \) for all even \( i \).

Additionally, a computation of \( H_{-3}(\text{Tel}) \), for example, would proceed as above, but with all indices incremented/decremented by one. The result is the same: \( H_{-3}(\text{Tel}) = H_i(\text{Tel}) = I\Lambda \) for all odd \( i \). Therefore, the object \( \text{Tel} \) has \( H_i(\text{Tel}) = I\Lambda \) for all \( i \).

\[ \square \]

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