Beating the Rayleigh limit using two-photon interference

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Multiparameter estimation theory offers a general framework to explore imaging techniques beyond the Rayleigh limit. While optimal measurements of single parameters characterizing a composite light source are now well understood, simultaneous determination of multiple parameters poses a much greater challenge that in general requires implementation of collective measurements. Here we show, theoretically and experimentally, that Hong-Ou-Mandel interference followed by spatially resolved detection of individual photons provides precise information on both the separation and the centroid for a pair of point emitters, avoiding trade-offs inherent to single-photon measurements.

A deeper insight rooted in the multiparameter estimation theory reveals a possible solution of the above incompatibility problem. Interestingly, in the strong subdiffraction regime where images of the sources overlap significantly, the problem can be modelled as simultaneous estimation of the length and the rotation angle of a qubit Bloch vector [4]. From the theory of multiparameter estimation it then follows that, provided collective measurement on the photons (or qubits) are allowed, the incompatibility between the optimal individual measurements to estimate the centroid and the sources separation ceases to be an issue [21, 22]. The question is how to realize such a collective measurement in practice.

In this Letter we exploit the advantages offered by the multiphoton interference approach, demonstrating a multiphoton protocol for imaging of two point sources, where the centroid estimation is performed in the optimal way, and at the same time the sources separation parameter is estimated with a superresolution precision. The idea relies on the effect of two-photon interference and does not require pre-estimation of the centroid or fine-tuning of the measurement basis inherent to spatial mode demultiplexing schemes [8–13], where any systematic error in centroid estimation propagates to separation estimation and significantly degrades the imaging protocol.

In Fig. 1(a) we depict a scenario where two photons emitted by a composite source arrive simultaneously at the input ports of the beamsplitter. The proposed protocol exploits both cross-coincidences between the output ports and double events in each port, detected with spatial resolution [23]. The number of cross-coincidences grows with the distinguishability of the two photons and therefore carries information about the separation between point sources. Most importantly, the proposed interferometric scheme does not require prior selection of the measurement basis or the axis of symmetry, as the two photons serve as a reference for each other. Furthermore, thanks to spatially-resolved detection this strategy will be shown to be robust against residual spectral distinguishability. Let us note that previous approaches to collective measurements relied on the fundamental advantage of using photonic entanglement [24], also for superresolution photolithography [25–28], which is essentially different from our technique of simply utilizing the bosonic nature of photons.

The somewhat non-trivial demand of interfering two photons from a realistic classical (thermal) composite source on a beamsplitter could be realized by a photon number quantum nondemolition (QND) measuring device that preserve spatial properties of light, and upon registering single photons delays and redirects them so that they arrive together at the two beamsplitter input ports. Recent advances in storing and controlling single photons in quantum nonlinear media such as Rydberg atoms [29] as well as spatially-multimode quantum memories [30] with processing capabilities [31] could provide a viable way to realize the scheme. In particular, a π phase shift induced by a single photon has already been achieved [32] and current experiments already explore the Rydberg interactions in the transverse spatial domain [33]. The combination of a multimode quantum memory with

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the spatially-resolving QND measurement could follow the steps of experiments demonstrating optical storage in Rydberg media [34, 35], use alternative proposals such as nonlinearities induced by ac-Stark shifts [36] or utilize novel solid-state systems with similar capabilities yet broader spectral bandwidths [37, 38].

To support the intuitions behind the discussed scheme let us compare the two-photon imaging scheme with direct imaging (DI) by modeling a problem of resolving a 1D image formed by two point sources. Let $\psi(x - x_0)$ be a 1D wave function representing the amplitude transfer function of a single source in the image plane centered at point $x_0$. We assume that this function is determined by well characterized properties of the imaging setup. In what follows we denote the corresponding single photon state characterized by $\psi(x - x_0)$ as $|x_0\rangle$.

Consider a situation where the image is produced as a result of an incoherent overlap of images of two point sources separated by a distance $\varepsilon$, located at $x_+ = x_0 + \varepsilon/2$ and $x_- = x_0 - \varepsilon/2$. We may then write the spatial density matrix of a photon emitted from the system as

$$\rho = 1/2(|x_+\rangle \langle x_+| + |x_-\rangle \langle x_-|).$$

In the DI scheme the probability distribution for the position of the detected photon is given by $p_{\rho}(x) = \frac{1}{2} \psi(x - x_+)^2 + \frac{1}{2} \psi(x - x_-)^2$, where $\theta = (x_+ + x_-)/2, x_+ - x_- = (x_0, \varepsilon)$ represents the dependence on the estimated parameters. For any locally unbiased estimator, the covariance matrix for the estimated parameters can be lower bounded using the Cramér-Rao inequality [39]:

$$\text{Cov} \theta \geq \frac{F^{-1}}{N}, \quad F_{ij} = \int_{-\infty}^{\infty} dx \frac{\partial \theta_i \rho(x) \partial \theta_j \rho(x)}{\rho(x)},$$

where $F_{ij}$ is the Fisher information (FI) matrix per single photon, while $N$ represents the total number of photons registered. The bound is asymptotically saturable using e.g., max-likelihood estimator, hence $\lim_{N \to \infty} N \text{Cov} \theta = F^{-1}$. As the FI matrix is diagonal for the given problem, we can easily calculate the variances $\Delta^2 x_0 = (F^{-1})_{11}$, $\Delta^2 \varepsilon = (F^{-1})_{22}$ the respective variances of the estimated parameters per single photon used. In case of DI the FI matrix yields the following precision for estimation in the leading order in $\varepsilon$:

$$\Delta^2 x_0^{-1} = 1 - \frac{\varepsilon^2}{4},$$

$$\Delta^2 \varepsilon^{-1} = \frac{\varepsilon^2}{8},$$

where for concreteness we have assumed a Gaussian-shaped transfer function $\psi(x) = (2\pi)^{-1/4} \exp(-x^2/4)$, yielding intensity profile with standard deviation 1 which can be regarded as a natural unit of distance in the problem. The above expansion is valid for small $\varepsilon$ when source point images are separated by a distance smaller than the transfer function spread, and clearly shows impossibility of precise estimation of $\varepsilon$ in the $\varepsilon \to 0$ limit.

Crucially, as observed in [10], a more fundamental bound based on the quantum FI matrix $F^{Q}$ [40], which does not assume any particular measurement strategy and is based solely on the properties of the quantum state $\rho$ to be measured reads:

$$\Delta^2 x_0^{-1} = 1 - \frac{\varepsilon^2}{4},$$

$$\Delta^2 \varepsilon^{-1} = \frac{1}{4},$$

indicating a potential spectacular robustness of $\varepsilon$ estimation as the $\Delta^2 \varepsilon$ is constant irrespectively of how small $\varepsilon$ is. While the bound (1) with $F$ being replaced by $F^{Q}$ is saturable for the problem considered, it requires collective measurements on many copies of $\rho$ [1, 4, 5, 22].

We are now ready to quantify the precision of estimating $x_0$ and $\varepsilon$ in the two-photon (2P) interferometric scheme and contrast it with the above-mentioned strategies. Given $\rho^{\otimes 2}$ at the input ports of the beam-splitter, we calculate spatially-resolved probabilities for coincidences $p_c(x_1, x_2)$ as well as double events $p_d(x_1, x_2)$, from
which the information about \( x_0 \) and \( \varepsilon \) is drawn. Furthermore, we assume a known two-photon visibility \( \mathcal{V} \) resulting from the operation of the non-demolition photon routing device before the beamsplitter. The resulting precision of estimation per single photon used, see Supplementary Material, expanded up to the second order in \( \varepsilon \) reads:

\[
(\Delta^2 x_0)_{2P}^{-1} = 1 - \frac{\varepsilon^2}{4},
\]

\[
(\Delta^2 \varepsilon)_{2P}^{-1} = \begin{cases} \frac{1}{8} + \frac{5 \varepsilon^2}{16} \mathcal{V} \varepsilon^2 & \mathcal{V} = 1, \\ \frac{1}{32}(1 - \mathcal{V}) \varepsilon^2 & \mathcal{V} < 1 \end{cases}
\]

while the expansion in case of imperfect visibility is valid in the regime where \( \varepsilon^2 \lesssim 1 - \mathcal{V} \). In case of perfect interference, we see that while keeping the optimality of \( x_0 \) estimation, we additionally obtain \( \varepsilon \) estimation with precision reduced by approximately a factor of 2 compared to the fundamental bound given in (5). This shows superiority of 2P over DI, with the added advantage that the measurement setting is fixed and does not require adjusting the measurement for \( \varepsilon \) depending on preestimation of \( x_0 \). Here we would like to stress the importance of spatial information that is available in the experiment: if only the ratio of coincidence and double events was available, there would be no information on \( x_0 \) parameter at all, while the precision of \( \varepsilon \) estimation shows a small reduction for finite \( \varepsilon \) compared with Eq. 7 and reads: \( \frac{1}{8} = \frac{5 \varepsilon^2}{128} + O(\varepsilon^4) \) when the visibility is equal to
one.

The role of spatial information becomes more pronounced for finite visibilities $\mathcal{V}$, for which the spatial information always provides an advantage for all values of $\varepsilon$ compared to the case when we consider only the ratio of cross-coincidences and double-event where the precision reads $\mathcal{V}^2[32(1 - \mathcal{V}^2)]^{-1}\varepsilon^2 + O(\varepsilon^4)$. This is achieved as coincidences that arise due to finite visibility are characterized by a different spatial distribution than coincidences that are due to spatial separation. In both cases we recover the $\varepsilon^2$-scaling and thus for small $\varepsilon$ the advantage of the collective schemes over DI takes the form of a constant factor rather than favorable scaling. Nonetheless, as this factor scales as $(1 - \mathcal{V})^{-1}$ the enhancement can be significant.

For a proof-of-principle experimental demonstration, we generated families of states $\rho^{\otimes 2}$ for a set of separations $\varepsilon$ (see Fig. 2(a) and Supplementary Material for details of the interferometric setup). In Fig. 2(c) and 2(d) we plot the final precision of estimation divided by the total number of photons used as a function of $\varepsilon$ (see Supplementary Material for details of data analysis). The proposed theory (for $\mathcal{V} = 0.92$) accurately predicts the estimation precision for the given experimental parameters demonstrating a significant, over twofold enhancement over the DI scheme. The spatial resolution provides an advantage over the whole range of parameters, as it allows us to distinguish effects of finite visibility versus the reduced mode overlap due to source separation.

In Figure 2(c),(d) we additionally plot the theoretical predictions for $\mathcal{V} = 0.99$ and perfect interference i.e. $\mathcal{V} = 1$. The precision approaches a constant value for $\varepsilon \to 0$ only for $\mathcal{V} = 1$, but offers significant enhancement for realistic visibilities. Note that if information is drawn only from the number of coincidences to double events with no spatial resolution, we can still beat the DI scheme over a broad range of parameters, especially for small $\varepsilon$. This highlights the possibility to perform precise imaging with only single-pixel detectors.

Let us now provide a simple argument for the observed degree of precision enhancement. The approximately two-fold reduction of precision for $\varepsilon$ estimation for $\mathcal{V} = 1$ in the 2P protocol compared to the fundamental bound is due to the fact that the protocol performs collective measurement on two photons only. The essence of the collective measurement is effective projection of $\rho^{\otimes 2}$ on symmetric and antisymmetric subspaces thanks to the properties of the Hong-Ou-Mandel interference. A measurement on a pair of single-photon emitters excited simultaneously emits HBT fringes which can be viewed as the aperture of the measuring system, is fixed, an attempt to retrieve the angular separation between the sources from HBT fringes will suffer from the Rayleigh curse in the limit $|k_1 - k_2| \to 0$. This is because for vanishing $|k_1 - k_2|$ one will observe only a small fraction of the HBT fringe in the vicinity of its maximum.

In the case of the two-photon scheme presented here, we should emphasize the role of the prior QND measurement if superresolution is to be achieved with classical thermal light sources. When HBT interferometry works also with classical light sources, albeit with reduced visibility, the enhancement offers by our scheme stems from realizing two-photon interferometry sufficiently close to the dark fringe, i.e. with high visibility $\mathcal{V}$. In fact, since classical light sources can attain at most 50% visibility of Hong-Ou-Mandel interference, formulas (6) indicate that no significant improvement is possible over the...
DI scheme: for $\mathcal{V} = 0.5$ we get $(\Delta^2 \varepsilon)^{-1}_{2P} = \frac{\varepsilon^2}{2\varepsilon^2}$ vs.
$(\Delta^2 \varepsilon)^{-1}_{DI} = \frac{\varepsilon^2}{8}$ in case of direct imaging.

In conclusion, we have demonstrated both theoretically and experimentally an imaging protocol that circumvents the difficulties in a multi-parameter estimation problem by use of a collective measurement. The presented experimental results conclusively confirm the possibility to exploit the inherent indistinguishability of photons to perform quantum-enhanced simultaneous estimation of source separation and centroid. With this proof-of-principle experiment we have also proposed a set of realistic schemes in which our protocol could be readily applied, even to gain additional information along the traditional single-photon DI scenario or other superresolution techniques. The general theory of super-resolved imaging [1, 4, 7] implies that the same protocol might be directly applied in case of a more general light source distribution provided one would be interested in estimating its first and second moments.

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