IMPROVED HARDY INEQUALITIES WITH A CLASS OF WEIGHTS

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ABSTRACT. In the paper we state conditions on potentials $V$ to get the improved Hardy inequality with weight

$$c_N, \mu \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \mu(x) dx + \int_{\mathbb{R}^N} V \varphi^2 \mu(x) dx$$

$$\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) dx + C_\mu \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx,$$

for functions $\varphi$ in a weighted Sobolev space and for weight functions $\mu$ of a quite general type. Some local improved Hardy inequalities are also given. To get the results we use a generalized vector field method.

1. INTRODUCTION

The paper deals with local and non-local improved Hardy inequalities with a class of weights $\mu$ of a quite general type and with inverse square potentials perturbed by a function $V$. The paper fits into the context of Hardy type inequalities with weight stated in [10].

The classical Hardy inequality was introduced in 1920th [22] in the one dimensional case (see also [23, 24]).

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The classical Hardy inequality is (see, e.g., [18, 25, 26] for historical reviews, and [27]).

\[ c_o(N) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx \quad (1.1) \]

for any functions \( \varphi \in H^1(\mathbb{R}^N) \), where \( c_o(N) = \left( \frac{N-2}{2} \right)^2 \) is the optimal constant.

Weighted Hardy inequalities with optimal constant depending on weight function \( \mu \) have been stated in [21, 14] with Gaussian measures and inverse square potentials with a single pole and in the multipolar case, respectively. In a setting of more general measures we refer to [12, 15, 16], in the last paper with multipolar potentials.

In particular in [15] the authors proved the inequality

\[ c_{N,\mu} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \mu(x) dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(x) dx + C_{\mu} \int_{\mathbb{R}^N} \varphi^2 \mu(x) dx, \quad (1.2) \]

for any functions \( \varphi \) in a weighted Sobolev space with \( c_{N,\mu} = \left( \frac{N+K_{\mu}-2}{2} \right)^2 \), optimal constant, and \( K_{\mu}, C_{\mu} \) constants depending on \( \mu \). For example when \( \mu = \frac{1}{|x|^\gamma}, \gamma < N-2 \), it occurs \( K_{\mu} = -\gamma \) and \( C_{\mu} = 0 \) while if \( \mu = e^{-\delta|x|^2} \), \( \delta > 0 \), we get \( K_{\mu} = 0 \) (see [15]).

In this paper we improve this results by adding a nonnegative correction term in the left-hand side in (1.2).

In particular we state sufficient conditions on \( V \) to get in \( \mathbb{R}^N \) the estimate

\[ c_{N,\mu} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu + \int_{\mathbb{R}^N} V \varphi^2 d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + C_{\mu} \int_{\mathbb{R}^N} \varphi^2 d\mu, \quad (1.3) \]

where \( d\mu = \mu(x) dx \), for any functions \( \varphi \) in a suitable Sobolev space with weight satisfying suitable local integrability assumptions and \( c_{N,\mu} \) the constant in (1.2).

To prove the result we use a method introduced in [11] for a class of weights satisfying the Hölder condition. In this paper we prove the result in the context of more general measures.

Example of weight functions are shown in the paper.

In the case of the Lebesgue measure there is a very huge literature on the extension of Hardy’s inequality. In particular the improved version of
the classical Hardy inequality in bounded domain $\Omega$ in $\mathbb{R}^N$, $N \geq 3$,
\[
\left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{\phi^2}{|x|^2} \, dx + c_\Omega \int_{\Omega} \phi^2 \, dx \leq \int_{\Omega} |\nabla \phi|^2 \, dx
\] (1.4)
has been stated in [3] for all $\phi \in H^1_0(\Omega)$.

Later improvements of the estimate (1.4) of the type
\[
\left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{\phi^2}{|x|^2} \, dx + \int_{\Omega} V \phi^2 \, dx \leq \int_{\Omega} |\nabla \phi|^2 \, dx
\] (1.5)
can be found, for example, in [1, 19, 20, 28].

Reasoning as in the proof of Theorem 3.1, we deduce weighted versions of the local estimate (1.5) for some functions $V$, well-known inequalities when $\mu = 1$. In particular we focus our attention on the inequalities
\[
c_{N,\mu} \int_{B_1} \frac{\phi^2}{|x|^2} \, d\mu + \frac{1}{4} \int_{B_1} \frac{\phi^2}{|x|^2 |\log |x||} \, d\mu \leq \int_{B_1} |\nabla \phi|^2 \, d\mu + C_{\mu} \int_{B_1} \phi^2 \, d\mu
\] (1.6)
and, for $\beta \in (0, 2]$,
\[
c_{N,\mu} \int_{B_1} \frac{\phi^2}{|x|^2} \, d\mu + \beta^2 \int_{B_1} \frac{\phi^2}{|x|^{2-\beta}} \, d\mu \int_{B_1} |\nabla \phi|^2 \, d\mu + C_{\mu} \int_{B_1} \phi^2 \, d\mu
\] (1.7)
for any functions $\phi \in C^\infty_c(B_1)$, where $B_1$ is the unit ball in $\mathbb{R}^N$. For $\beta = 2$ we get the weighted version of (1.4) with 4 in place of $c_\Omega$.

The Hardy inequalities are applied in many fields. From a mathematical point of view, a motivation for us to study Hardy inequalities with weights and related improvements is due to the applications to evolution problems

\[
(P) \quad \begin{cases} 
\partial_t u(x,t) = Lu(x,t) + \tilde{V}(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\
u(\cdot, 0) = u_0 \geq 0 & \in L^2_\mu 
\end{cases}
\]
where $L^2_\mu := L^2(\mathbb{R}^N, d\mu)$ and $L$ is the Kolmogorov operator
\[
Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u,
\] (1.8)
defined on smooth functions, perturbed by singular potentials $\tilde{V}$. 
An existence result can be obtained, reasoning as [12], following Cabré-Martel’s approach based on the relation between the weak solution of \((P)\) and the estimate of the bottom of the spectrum of the operator \(-(L + \tilde{V})\)

\[
\lambda_1(L + \tilde{V}) := \inf_{\varphi \in H^1_0 \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbb{R}^N} \tilde{V} \varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right)
\]

that results from the Hardy inequality. In the case \(\mu = 1\) Cabré and Martel in [4] showed that the boundedness of \(\lambda_1(\Delta + \tilde{V})\) is a necessary and sufficient condition for the existence of positive exponentially bounded in time solutions to the associated initial value problem. Later in [21, 12, 16] similar results have been extended to Kolmogorov operators perturbed by inverse square potentials, in the last paper in the multipolar case. The proof uses some properties of the operator \(L\) and of its corresponding semigroup in \(L^2_{\mu}(\mathbb{R}^N)\).

In the paper we include the existence result for the sake of completeness.

The paper is organized as follows.

In Section 2 we introduce the class of weights and the conditions on the potentials \(V\) with some examples. In Section 3 we state the improved weighted Hardy inequality and some consequences. Section 4 is devoted to the weighted local estimates. Finally, in Section 5, we show an application of the estimates to evolution problems.

2. A CLASS OF WEIGHT FUNCTIONS AND POTENTIALS

Let \(\mu \geq 0\) be a weight function on \(\mathbb{R}^N\). We define the weighted Sobolev space \(H^1_{\mu} = H^1(\mathbb{R}^N, \mu(x) \, dx)\) as the space of functions in \(L^2_{\mu} := L^2(\mathbb{R}^N, \mu(x) \, dx)\) whose weak derivatives belong to \(L^2_{\mu}\).

The class of function \(\mu\) we consider fulfills the conditions

\[
H_1) \quad \begin{align*}
&i) \quad \sqrt{\mu} \in H^1_{\text{loc}}(\mathbb{R}^N); \\
&ii) \quad \mu^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N).
\end{align*}
\]

Let us observe that under the assumption \(i) \ H_1)\) we get \(\mu, \nabla \mu \in L^1_{\text{loc}}(\mathbb{R}^N)\).

The reason we suppose \(H_1)\) is that we need the density of the space \(C^\infty_0(\mathbb{R}^N)\)
in $H^1_\mu$ (see e.g. [29]). So we can regard $H^1_\mu$ as the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the Sobolev norm

$$\| \cdot \|^2_{H^1_\mu} := \| \cdot \|^2_{L^2_\mu} + \| \nabla \cdot \|^2_{L^2_\mu}.$$ 

We introduce, in the proof of the Hardy inequality in the next Section, the function $f = \frac{g}{|x|^{\alpha}}$, $g$ radial function, for suitable values of $\alpha$. We need the following condition on $g$.

$$H_2) \quad i) \quad g > 0, \quad -\frac{1}{g} \frac{\partial g}{\partial x_j} \sqrt{\mu} \in L^2_{loc}(\mathbb{R}^N), \quad \frac{1}{g} \frac{\partial^2 g}{\partial x_j^2} \mu \in L^1_{loc}(\mathbb{R}^N);$$

$$ii) \quad -\frac{\Delta g}{g} + (N - 2) \frac{x}{|x|^2} \cdot \nabla g \geq 0.$$ 

The assumption on the potential $V$ in the estimates is the following

$$H_3) \quad V = V(x) \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad 0 \leq V \leq W := -\frac{\Delta g}{g} + (N - 2) \frac{x}{|x|^2} \cdot \nabla g = -\frac{g''}{g} - \frac{g'}{\rho g},$$

where $g'$, $g''$ are the first and the second derivatives with respect to $\rho = |x|$, respectively.

Under condition $i)$ in $H_2$ we can integrate by parts in the proof of the inequality in the next Section. The class of radial functions $g$ satisfying $ii)$ in $H_2$ is such that

$$(g'\rho)' \leq 0 \quad (2.1)$$

which implies that $g'\rho$ is decreasing, so we have

$$g'(r) r \leq g'(r_0) r_0, \quad r_0 \leq r. \quad (2.2)$$

If $g \in C^2(\mathbb{R}^N \setminus \{0\})$ this condition involves that the function $g(r) - c_1 \log r$ is decreasing

$$g(r) - c_1 \log r < g(r_0) - c_1 \log r_0, \quad c_1 = g'(r_0) r_0,$$

as we can see integrating (2.2) in $[r_0, r]$, $r_0 > 0$.

Functions satisfying condition (2.1) in $(0, R)$, $R < 1$, for example, are the functions $g(r) = |\log r|^\beta$, $\beta \in (0, 1)$, $g(r) = 1 - r^\beta$, $\beta \in (0, 2]$.

The functions $W$ such that there exists a positive radial solution of the Bessel equation associated to the potential $W$

$$g'' + \frac{g'}{r} + W g = 0 \quad (2.3)$$
are good functions. We observe that, if $W = 1$, the Bessel function $J_0$ is a positive solution of the equation (2.3).

The author in [20] proved that, under suitable hypotheses, the equation (2.3) is a necessary and sufficient condition to get an improved Hardy inequality in bounded domains in $\mathbb{R}^N$. A further assumption we need is the following.

$H_4)$

There exist constants $K_1, K_2, K_3 \in \mathbb{R}$, $K_2 > 2 - N$ and $K_3 = K_2$ if $K_2 \neq 0$, $K_3 \leq 0$ if $K_2 = 0$, such that

$$
\left( \nabla g - \alpha \frac{x}{|x|^2} \right) \cdot \frac{\nabla \mu}{g} \leq K_1 - \frac{\alpha K_2}{|x|^2} + K_3 \frac{x}{|x|^2} \cdot \nabla g
$$

or, equivalently, such that the function $g = g(r)$ satisfies the inequality

$$
g' \left( \frac{\mu'}{\mu} - \frac{K_3}{r} \right) \leq K_1 + \frac{\alpha}{r} \left( \frac{\mu'}{\mu} - \frac{K_2}{r} \right).
$$

For $g$ fixed, it is a condition for $\mu$. For example, if $g = 1$, weight functions satisfying $H_4)$ are the functions

$$
\mu(x) = \frac{1}{|x|^\gamma} e^{-\delta|x|^m}, \quad \delta \geq 0, \quad \gamma < N - 2,
$$

for suitable values of $m$ (see [15]). Conversely, for $\mu$ fixed, $H_4)$ represents a condition on $g$.

Finally, we remark that the weights in (2.5) fulfill condition $H_1)$.

## 3. Weighted improved Hardy inequalities

In this Section we state a weighted improved Hardy inequality in the setting of more general measure with respect to [11]. This allow us to improve the results in [15] on weighted Hardy inequalities by adding a nonnegative correction term in the estimates.

The method to get the result has been introduced in [11] for a class of weights satisfying the Hölder condition. We enlarge the class of weights for which we can state the result. For this class a weighted Hardy inequality with a different method has been stated in [15].

The next result states sufficient conditions to get an improved Hardy inequality with weight.
Theorem 3.1. If $H_1$–$H_4$ hold, then we get the estimate

\[
\frac{(N + K_2 - 2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu + \int_{\mathbb{R}^N} V \varphi^2 d\mu \\
\leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K_1 \int_{\mathbb{R}^N} \varphi^2 d\mu
\]

(3.1)

for any functions $\varphi \in H_1^{\mu}$.

Proof. By the density result, we prove (3.1) for any $\varphi \in C_c^\infty(\mathbb{R}^N)$.

We introduce the vector-valued function

\[
F = -\frac{\nabla f}{f} \mu - \frac{\nabla g}{g} \mu + \alpha \frac{x}{|x|^2} \mu,
\]

where $f = \frac{g}{|x|^2}, \alpha \in (0, N + K_2 - 2)$.

We get

\[
\text{div} F = -\Delta f \frac{1}{f} \mu + \left| \frac{\nabla f}{f} \right|^2 \mu - \frac{\nabla f}{f} \cdot \nabla \mu,
\]

where

\[
-\frac{\Delta f}{f} = -|x|^\alpha \Delta \frac{1}{|x|^\alpha} - 2|x|^\alpha \nabla \frac{1}{|x|^\alpha} \frac{\nabla g}{g} - \frac{\Delta g}{g} \\
= \frac{\alpha (N - 2 - \alpha)}{|x|^2} + 2\alpha \frac{x}{|x|^2} \cdot \frac{\nabla g}{g} - \frac{\Delta g}{g}.
\]

Now we observe that $F_j, \frac{\partial F_j}{\partial x_j} \in L^1_{\text{loc}}(\mathbb{R}^N)$, where $F_j$ is the j-th component of $F$. Indeed, for any $K$ compact set in $\mathbb{R}^N$, by the Hölder and the classical Hardy inequalities, using hypotheses i) in $H_1$) on $\mu$ and i) in $H_2$) on $g$, we obtain the following estimate
To obtain the local integrability of the partial derivative of $F_j$

\[
\frac{\partial F_j}{\partial x_j} = \frac{\partial}{\partial x_j} \left( -\frac{1}{g} \frac{\partial g}{\partial x_j} \mu + \alpha \frac{x_j}{|x|^2} \mu \right) = \frac{1}{g^2} \left( \frac{\partial g}{\partial x_j} \right)^2 \mu - \frac{1}{g} \frac{\partial^2 g}{\partial x^2_j} \mu \\
- \frac{1}{g} \frac{\partial g}{\partial x_j} \frac{\partial \mu}{\partial x_j} + \alpha \frac{\mu}{|x|^2} - 2\alpha \frac{x_j^2}{|x|^4} \mu + \alpha \frac{x_j}{|x|^2} \frac{\partial \mu}{\partial x_j}
\]

we estimate the terms on the right-hand side in (3.2). The terms $d_1, d_2$ belong to $L^1_{loc}(\mathbb{R}^N)$ by hypotheses,
$d_4, d_5$ can be estimated using the Hardy inequality and the hypothesis $i)$ in $H_1$) as above. As regards the remaining term we have

$$
\int_K |d_6| dx \leq \alpha \int_K \frac{\sqrt{\mu}}{|x|} \frac{1}{\sqrt{\mu}} \left| \frac{\partial \mu}{\partial x_j} \right| dx
\leq 2 \alpha \left( \int_K \frac{\mu}{|x|^2} dx \right)^{\frac{1}{2}} \left( \int_K |\nabla \sqrt{\mu}|^2 dx \right)^{\frac{1}{2}}.
\leq \frac{4 \alpha}{(N-2)} \left( \int_K |\nabla \sqrt{\mu}|^2 dx \right).
$$

Now we start from the following integral

$$
\int_{\mathbb{R}^N} \text{div} F \phi^2 dx = \int_{\mathbb{R}^N} \left[ \frac{\alpha(N-2) - \alpha}{|x|^2} + 2 \alpha \frac{x}{|x|^2} \frac{\nabla g}{g} - \frac{\Delta g}{g} \right] \phi^2 d\mu + \int_{\mathbb{R}^N} \left| \frac{\nabla g}{g} - \alpha \frac{x}{|x|^2} \right|^2 \phi^2 d\mu - \int_{\mathbb{R}^N} \left( \frac{\nabla g}{g} - \alpha \frac{x}{|x|^2} \frac{\nabla \mu}{\mu} \right) \phi^2 d\mu. \tag{3.3}
$$

The first step is to estimate the integral on the left-hand side in (3.3) from above. To this aim we integrate by parts and use Hölder’s and Young’s inequalities to get

$$
\int_{\mathbb{R}^N} \text{div} F \phi^2 d\mu = -2 \int_{\mathbb{R}^N} \phi F \cdot \nabla \phi d\mu
\leq 2 \left( \int_{\mathbb{R}^N} |\nabla \phi|^2 d\mu \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|\nabla f|^2}{f} \phi^2 d\mu \right)^{\frac{1}{2}} \tag{3.4}
\leq \int_{\mathbb{R}^N} |\nabla \phi|^2 d\mu + \int_{\mathbb{R}^N} \left( \frac{\nabla f}{f} \right)^2 \phi^2 d\mu
= \int_{\mathbb{R}^N} |\nabla \phi|^2 d\mu + \int_{\mathbb{R}^N} \left| \frac{\nabla g}{g} - \alpha \frac{x}{|x|^2} \right|^2 \phi^2 d\mu.
$$

On the other hand, starting from (3.3), by the condition $H_4$) we obtain
\[
\int_{\mathbb{R}^N} \text{div} F \varphi^2 d\mu \geq \alpha (N - 2 - \alpha) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu \\
+ \int_{\mathbb{R}^N} \left( 2\alpha \frac{x}{|x|^2} \cdot \frac{\nabla g}{g} - \frac{\Delta g}{g} \right) \varphi^2 d\mu \\
+ \int_{\mathbb{R}^N} \left( \frac{\nabla g}{g} - \alpha \frac{x}{|x|^2} \right)^2 \varphi^2 d\mu - K_1 \int_{\mathbb{R}^N} \varphi^2 d\mu \\
+ \alpha K_2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu - K_3 \int_{\mathbb{R}^N} \frac{x}{|x|^2} \cdot \nabla g \varphi^2 d\mu = \alpha (N + K_2 - 2 - \alpha) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu \\
+ \int_{\mathbb{R}^N} \left( (2\alpha - K_3) \frac{x}{|x|^2} \cdot \frac{\nabla g}{g} - \frac{\Delta g}{g} \right) \varphi^2 d\mu \\
+ \int_{\mathbb{R}^N} \left( \frac{\nabla g}{g} - \alpha \frac{x}{|x|^2} \right)^2 \varphi^2 d\mu - K_1 \int_{\mathbb{R}^N} \varphi^2 d\mu.
\]

The inequalities (3.4) and (3.5) led us to the estimate

\[
\alpha (N + K_2 - 2 - \alpha) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu + \int_{\mathbb{R}^N} \left[ (2\alpha - K_3) \frac{x}{|x|^2} \cdot \frac{\nabla g}{g} - \frac{\Delta g}{g} \right] \varphi^2 d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K_1 \int_{\mathbb{R}^N} \varphi^2 d\mu.
\]

The maximum value of the first constant on the left-hand side in (3.6) is

\[
\max_{\alpha} \alpha (N + k_2 - 2 - \alpha) = \frac{(N + k_2 - 2)^2}{4},
\]

attained for \( \alpha = \alpha_o = \frac{(N + k_2 - 2)}{4} \).

Observing that \( 2\alpha_o - K_3 \geq N - 2 \) and taking in mind the condition \( H_3 \), we obtain the inequality (3.1).

\[\square\]

**Remark 3.2.** For \( g = 1 \) and, then, \( V = W = 0 \), we obtain a weighted Hardy inequality. For \( g = 1, \mu = 1 \) and, so, when \( K_2 = 0 \), the method to
get the result in the Theorem 3.1 results to be the vector field method used in [27] to prove the Hardy inequality.

An example of weight satisfying condition $H_1$ is the function $\mu = \frac{1}{|x|^{\gamma}}$, for $\gamma < N - 2$. In this case the condition (2.4) is verified for $K_2, K_3 \leq -\gamma$ for any $K_1 \geq 0$. Then, as a consequence of Theorem 3.1 we get the inequality

$$\frac{(N-\gamma-2)^2}{4} \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^{\gamma}} \, dx + \int_{\mathbb{R}^N} V \phi^2 |x|^{-\gamma} \, dx$$

for any functions $\phi \in H^1_{\mu}$. For $V = W = 0$ the inequality above is the Caffarelli-Nirenberg inequality.

We remark that, as a consequence of Theorem 3.1, we deduce the estimate

$$\|\tilde{V}^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^N)} \leq c \|\phi\|_{H^1_{\mu}(\mathbb{R}^N)},$$

where $\tilde{V} = \frac{(N-2)^2}{4} \frac{1}{|x|^2} + V$ and $c$ is a constant independent of $V$ and $\phi$.

For $L^p$ estimates and embedding results of this type with some applications to elliptic equations see [5, 6, 7, 8, 9, 17].

Finally, as a direct consequence of the Theorem 3.1 we deduce the following result concerning a class of general weighted Hardy inequalities for $V$ satisfying $H_3$)

$$\int_{\mathbb{R}^N} V \phi^2 \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 \, d\mu + K_1 \int_{\mathbb{R}^N} \phi^2 \, d\mu$$

for any functions $\phi \in H^1_{\mu}$.

4. LOCAL ESTIMATES

In this Section we state some local weighted estimates by means of the mehtod used to prove Theorem 3.1. These estimates represent the weighted version of well-known improved Hardy inequalities in bounded subset of $\mathbb{R}^N$ (see [1, 19, 20, 28]) and are based on examples of functions $g$ satisfying locally the assumptions of the Theorem 3.1.
The first result is the following.

**Theorem 4.1.** Let $N \geq 3$ and let $B_1$ the unit ball in $\mathbb{R}^N$. Then, under assumptions $H_1$, $i)$ in $H_2$) and $H_4$ on $\mu$, we get

$$
\frac{(N + K_2 - 2)^2}{4} \int_{B_1} \frac{\varphi^2}{|x|^2} d\mu + \frac{1}{4} \int_{B_1} \frac{\varphi^2}{|x|^2 |\log |x||^2} d\mu 
\leq \int_{B_1} |\nabla \varphi|^2 d\mu + K_1 \int_{B_1} \varphi^2 d\mu
$$

(4.1)

for any functions $\varphi \in C^\infty_c(B_1)$.

**Proof.** Reasoning as in the proof of Theorem 3.1, we set $g = |\log |x||^\beta$, $\beta \in (0, 1)$. Then the function $W$ in $H_3$ is given by

$$
W = \frac{\beta (1 - \beta)}{|x|^2 |\log |x||^2}.
$$

To get the integrability required in $i$) in $H_2$), it is sufficient that the weights $\mu$ are such that $g' \frac{g'}{g} \mu$, $g'' \mu \in L^1_{\text{loc}}(\mathbb{R}^N)$. More precisely, pointing out that

$$
- \frac{\partial g}{\partial x_j} = - \frac{x_j}{|x|} g', \quad \frac{\partial^2 g}{\partial x_j^2} = \frac{g'}{|x|} - \frac{x_j^2}{|x|^2} g' + \frac{x_j^2 g''}{|x|^2},
$$

(4.2)

if $K$ is a compact set in $B_1$, we get

$$
\int_K \left\| \frac{1}{g} \frac{\partial g}{\partial x_j} \right\| d\mu \leq \left( \int_K \left\| \frac{1}{g} \frac{\partial g}{\partial x_j} \right\|^2 d\mu \right)^{\frac{1}{2}} \left( \int_K \mu(x) d\mu \right)^{\frac{1}{2}},
$$

$$
\int_K \left\| \frac{1}{g} \frac{\partial g}{\partial x_j} \right\|^2 d\mu \leq \int_K \left\| \frac{g'}{g} \right\|^2 d\mu = \int_K \frac{\beta^2}{|x|^2 |\log |x||^2} d\mu,
$$

$$
\int_K \frac{1}{|x|} \left\| \frac{g'}{g} \right\| d\mu \leq \left( \int_K \frac{\mu}{|x|^2} dx \right)^{\frac{1}{2}} \left( \int_K \left\| \frac{g'}{g} \right\|^2 d\mu \right)^{\frac{1}{2}}
\leq \frac{2}{N - 2} \left( \int_K |\nabla \sqrt{\mu}|^2 d\mu \right)^{\frac{1}{2}} \left( \int_K \frac{\beta^2}{|x|^2 |\log |x||^2} d\mu \right)^{\frac{1}{2}}
$$

and, about the last term on the right-hand side in (4.2),
\[
\int_K \frac{x^2}{|x|^2} |g''| \frac{d\mu}{g} \leq \int_K \left| \frac{g''}{g} \right| d\mu = \int_K \left| \beta(\beta - 1) + \beta \log |x| \right| |x|^2 \log |x|^2 d\mu.
\]

Finally, since \[\max_{\beta \in (0, 1)} [\beta(1 - \beta)] = \frac{1}{4},\]
attained for \(\beta = \frac{1}{2}\), we get the result. \(\Box\)

In the case of weight \(\mu = \frac{1}{|x|^{\gamma}}, \gamma < N - 2\), the inequality (4.1) with \(V = \frac{1}{4} |x|^2 \log |x|^2\).

Another example of weight is given by \(\mu = \frac{1}{|x|^\gamma} e^{-\delta |x|^m}, \gamma < N - 2, \delta, m > 0\). In the last case the condition (2.4) in \(B_1\) is satisfied for \(K_2, K_3 \leq -\gamma - \delta m, K_1 \geq 0\).

For \(\mu = 1\) the inequality (4.1) results to be the improved Hardy inequality with Lebesgue measure in \([1, 19, 28]\).

A further local inequality follows.

**Theorem 4.2.** Let \(N \geq 3\) and let \(B_1\) the unit ball in \(\mathbb{R}^N\). Then, under assumptions H(1), i) in H(2) and H(4) on \(\mu\), we get

\[
\frac{(N + K_2 - 2)^2}{4} \int_{B_1} \frac{\varphi^2}{|x|^2} d\mu + \beta^2 \int_{B_1} \frac{\varphi^2}{|x|^{2-\beta}} d\mu \leq \int_{B_1} |\nabla \varphi|^2 d\mu + K_1 \int_{B_1} \varphi^2 d\mu
\]

for any functions \(\varphi \in C_c^\infty(B_1)\) and \(\beta \in (0, 2]\).

**Proof.** It is enough to consider \(g = 1 - |x|^\beta, \beta \in (0, 2]\), observing that

\[
V = \frac{\beta^2}{|x|^{2-\beta}} \leq W = \frac{\beta^2}{|x|^{2-\beta}(1-|x|^\beta)}.
\]

Finally, we remark that for \(\beta = 2\) and \(\mu = 1\) we get almost the estimate in \([3, 20]\) in the sense that, in place of 4 in the left-hand side in (4.3), the authors obtained \(z_0^2\), where \(z_0\) is the first zero of the Bessel function \(J_0(z)\).
Also in this case the functions $\mu = \frac{1}{|x|^7}$ and $\mu = \frac{1}{|x|^7} e^{-\delta |x|^m}$ are good weights.

5. AN APPLICATION TO EVOLUTION PROBLEMS

In the Section we give a motivation for our interest in Hardy inequalities with weight. These estimates play a crucial role in achieving existence results for solutions to the problem

\[
(P) \quad \left\{ \begin{array}{ll}
\partial_t u(x,t) = Lu(x,t) + \tilde{V}(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\
u(\cdot, 0) = u_0 \geq 0 \in L^2_{\mu},
\end{array} \right.
\]

where $L$ is the Kolmogorov operator

\[
Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u
\]
defined on smooth functions, perturbed by a potential $\tilde{V}(x)$, with $\tilde{V}(x)$ sum of an inverse square potential and $V$ satisfying condition $H_3$.

We say that $u$ is a weak solution to $(P)$ if, for each $T, R > 0$, we have

\[
u \in C([0, T], L^2_{\mu}), \quad Vu \in L^1(B_R \times (0, T), d\mu dt)
\]
and

\[
\int_0^T \int_{\mathbb{R}^N} u(-\partial_t \phi - L\phi) \, d\mu dt - \int_{\mathbb{R}^N} u_0\phi(\cdot, 0) \, d\mu = \int_0^T \int_{\mathbb{R}^N} Vu\phi \, d\mu dt
\]
for all $\phi \in W^{2,1}_2(\mathbb{R}^N \times [0, T])$ having compact support with $\phi(\cdot, T) = 0$, where $B_R$ denotes the open ball of $\mathbb{R}^N$ of radius $R$ centered at 0. For any $\Omega \subset \mathbb{R}^N$, $W^{2,1}_2(\Omega \times (0, T))$ is the parabolic Sobolev space of the functions $u \in L^2(\Omega \times (0, T))$ having weak space derivatives $D^\alpha_t u \in L^2(\Omega \times (0, T))$ for $|\alpha| \leq 2$ and weak time derivative $\partial_t u \in L^2(\Omega \times (0, T))$ equipped with the norm

\[
\|u\|_{W^{2,1}_2(\Omega \times (0, T))} := \left( \|u\|_{L^2(\Omega \times (0, T))}^2 + \|\partial_t u\|_{L^2(\Omega \times (0, T))}^2 \right) + \sum_{1 \leq |\alpha| \leq 2} \|D^\alpha u\|_{L^2(\Omega \times (0, T))}^2.
\]
An additional assumptions on $\mu$ allows us to get semigroup generation on $L^2_\mu$ (see \cite[Corollary 3.7]{2}).

$$H_5) \quad \mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N \setminus \{0\}), \lambda \in (0,1), \mu \in H^1_{loc}(\mathbb{R}^N), \frac{\nabla \mu}{\mu} \in L^r_{loc}(\mathbb{R}^N)$$

for some $r > N$, and $\inf_{x \in K} \mu(x) > 0$ for any compact set $K \subset \mathbb{R}^N$. We remark that the condition $H_5)$ implies ii) in $H_3)$. Indeed if $\mu \in H^1_{loc}(\mathbb{R}^N)$ then $\mu \in L^1_{loc}(\mathbb{R}^N)$ and $\nabla \mu \in L^2_{loc}(\mathbb{R}^N)$. Moreover $\frac{\nabla \mu}{\mu} \in L^r_{loc}(\mathbb{R}^N)$ since $r > 2$. So we get

$$\int_K |\nabla \sqrt{\mu}|^2 \, dx = \frac{1}{4} \int_K \frac{|\nabla \mu|^2}{\mu} \, dx \leq \frac{1}{4} \left( \int_K \frac{|\nabla \mu|}{\mu} \, dx \right)^2 \left( \int_K |\nabla \mu|^2 \, dx \right)^{\frac{1}{2}}.$$

An example of weight function satisfying $H_5)$ is $\mu = e^{-\delta|x|^m}$, $\delta, m > 0$.

In the applications to evolution problems with Kolmogorov operators we need $C_0$-semigroup generation results reasoning as in \cite{21, 12, 16}. Operators of a more general type for which the generation of semigroup was stated can be found, for example, in \cite{13} in the context of weighted spaces.

The bottom of the spectrum of $-(L + \tilde{V})$ is defined as follows

$$\lambda_1(L + \tilde{V}) := \inf_{\phi \in H^1_2(\mathbb{R}^N \setminus \{0\})} \left( \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 \, d\mu - \int_{\mathbb{R}^N} \tilde{V} \phi^2 \, d\mu}{\int_{\mathbb{R}^N} \phi^2 \, d\mu} \right).$$

The authors in \cite{12} stated the following result with a proof similar to the one given in \cite{4}. We include the hypothesis ii) in $H_1)$ to get the density result.

**Theorem 5.1.** Assume that the $\mu$ satisfies ii) in $H_2)\text{ and }H_5)$. Let $0 \leq \tilde{V}(x) \in L^1_{loc}(\mathbb{R}^N)$. Then, if $\lambda_1(L + \tilde{V}) > -\infty$, there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying the estimate

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \geq 0$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

The existence result below relies on the Theorem 3.1 and on the Theorem 5.1.

**Theorem 5.2.** Assume hypotheses ii) in $H_1), H_2)\text{ and }H_5)$. Then there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \geq 0$$
for some constants \( M \geq 1, \omega \in \mathbb{R} \).

**Proof.** The weighted Hardy inequality (3.1) implies that \( \lambda_1 (L + \tilde{V}) > -\infty \). Then the result is a consequence of the Theorem 5.1. \( \square \)

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