Blow-up, zero $\alpha$ limit and the Liouville type theorem for the Euler-Poincaré equations

Dongho Chae$^*$ and Jian-Guo Liu$^†$

$^{(*)}$Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
email: chae@skku.edu

$^{(†)}$Department of Physics and Department of Mathematics
Duke University
Durham, NC 27708, USA
email: jliu@phy.duke.edu

Abstract

In this paper we study the Euler-Poincaré equations in $\mathbb{R}^N$. We prove local existence of weak solutions in $W^{2,p}(\mathbb{R}^N), p > N$, and local existence of unique classical solutions in $H^k(\mathbb{R}^N), k > N/2 + 3$, as well as a blow-up criterion. For the zero dispersion equation ($\alpha = 0$) we prove a finite time blow-up of the classical solution. We also prove that as the dispersion parameter vanishes, the weak solution converges to a solution of the zero dispersion equation with sharp rate as $\alpha \to 0$, provided that the limiting solution belongs to $C([0,T); H^k(\mathbb{R}^N))$ with $k > N/2 + 3$. For the stationary weak solutions of the Euler-Poincaré equations we prove a Liouville type theorem. Namely, for $\alpha > 0$ any weak solution $u \in H^1(\mathbb{R}^N)$ is $u = 0$; for $\alpha = 0$ any weak solution $u \in L^2(\mathbb{R}^N)$ is $u = 0$.

Key words: finite time blow-up, zero dispersion limit, Liouville type theorem, Euler-Poincaré equations, Camassa-Holm equation

AMS Subject classification: 35Q35

1 Introduction

We consider the following Euler-Poincaré equations in $\mathbb{R}^N$:

\[
\begin{cases}
\partial_t m + (u \cdot \nabla)m + (\nabla u)^\top m + (\text{div} \, u)m = 0, \\
m = (1 - \alpha \Delta)u, \\
u_0(x) = u_0,
\end{cases}
\]

(EP)
where \( \mathbf{u} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is the velocity, \( \mathbf{m} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) represents the momentum, constant \( \sqrt{\alpha} \) is a length scale parameter, \( (\nabla \mathbf{u})^\top \) is the transpose of \( (\nabla \mathbf{u}) \). The Euler-Poincaré equations arise in diverse scientific applications and enjoy several remarkable properties both in the one-dimensional and multi-dimensional cases.

The Euler-Poincaré equations were first studied by Holm, Marsden, and Ratiu in 1998 as a framework for modeling and analyzing fluid dynamics [18, 19], particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling. There are intensive researches on analogs viscous or inviscid, incompressible Lagrangian averaged models. We refer to [7, 12, 26] for results on Navier-Stokes-\( \alpha \) model in terms of existence and uniqueness, zero \( \alpha \) limit to the Navier-Stokes equations, global attractor, etc. We refer to [2, 20, 23] for results on analysis and simulation of vortex sheets with Birkhoff-Rott-\( \alpha \) or Euler-\( \alpha \) approximation.

For one-dimension, the Euler-Poincaré equations coincide with the dispersion-less case of Camassa-Holm (CH) equation [4]:

\[
(CH) \quad \partial_t m + 2u \partial_x m + \partial_x u m = 0, \quad m = (1 - \alpha \partial_{xx})u.
\]

The solutions to (CH) are characterized by a discontinuity in the first derivative at their peaks and are thus referred to as peakon solutions. (CH) is completely integrable with a bi-Hamiltonian structure and their peakon solutions are true solitary waves that emerge from the initial data. Peakons exhibit a remarkable stability—their identity is preserved through nonlinear interactions, see, e.g. [1, 22]. There are many comprehensive analysis on (CH) in the literature. We refer to a review paper [25] for a survey of recent results on well-posedness and existence of local and global weak solutions for (CH). The existence of a global weak solution and uniqueness was proven in [3, 6, 10, 8, 29]. A class of the so called weak-weak solution was studied in [29]. The breakdown of the solution for (CH) was studied in [24].

The Euler-Poincaré equations have many further interpretations beyond fluid applications. For instance, in 2-D, it is exactly the same as the averaged template matching equation for computer vision (see, e.g., [14, 17, 21]). The Euler-Poincaré equations also has important applications in computational anatomy (see, e.g., [22, 30]). The Euler-Poincaré equations can also be regarded as an evolutionary equation for a geodesic motion on a diffeomorphism group and it is associated with Euler-Poincaré reduction via symmetry [1, 11, 22, 15, 30]. We refer to a recent book [22] for a comprehensive review on the subject.

The organization of the paper is as follows. In Section 2, we give some preliminary discussions of the Euler-Poincaré equations and we state a theorem on local existence of weak solution in \( W^{2,p}(\mathbb{R}^N) \), \( p > N \), and local existence of unique classical solutions in \( H^k(\mathbb{R}^N) \), \( k > N/2 + 3 \).

In Section 3, we prove a theorem on a blow-up criterion, as well as, a theorem on finite time blow-up of the classical solution for the zero dispersion equation. For classic solutions with reflection symmetry, the divergence \( \nabla \cdot \mathbf{u} \) satisfy a Riccati equation at the invariant point under the reflection transformation and hence there is a finite time blow up if the divergence is initially negative.
In Section 4, we prove that as the dispersion parameter \( \alpha \) vanishes, the weak solution converges to a solution of the zero dispersion equation with a sharp rate as \( \alpha \to 0 \), provided that the limiting solution belongs to \( C([0,T); H^k(\mathbb{R}^N)) \) with \( k > N/2 + 3 \).

Finally, for the stationary weak solutions of the Euler-Poincaré equations we prove a Liouville type theorem in Section 5. For \( \alpha > 0 \), we prove that any weak solution \( u \in H^1(\mathbb{R}^N) \) is \( u = 0 \). For \( \alpha = 0 \), any weak solution \( u \in L^2(\mathbb{R}^N) \) is \( u = 0 \). This is a surprising result, as all the previous Liouville type results are for dissipative systems. This is the first Liouville type theorem for non-dissipative systems.

We also give a proof of the local existence and uniqueness theorem in Appendix.

2 Preliminaries and local existence

In this section, we discuss some mathematical structures of (EP) and then we state a local existence theorem for the weak solution and the classic solution. We refer to [17, 22] for more in-depth discussions on (EP).

(EP) can be recast as

\[
\partial_t m + \nabla \cdot (u \otimes m) + (\nabla u)\,^\top \! m = 0.
\]  

The last term above can be written in a conservative/tensor form

\[
\sum_{j=1}^N \partial_i u_j m_j = \sum_{j=1}^N \partial_i u_j u_j - \frac{\alpha}{2} \sum_{i,k=1}^N \partial_i \partial_k u_j^2
\]

\[
= \frac{1}{2} \partial_i |u|^2 - \alpha \sum_{j,k=1}^N \partial_k (\partial_i u_j \partial_k u_j) + \alpha \sum_{j,k=1}^N \partial_i \partial_j \partial_k u_j
\]

\[
= \frac{1}{2} \partial_i |u|^2 - \alpha \sum_{j,k=1}^N \partial_j (\partial_i u_k \partial_j u_k) + \frac{\alpha}{2} \sum_{j,k=1}^N \partial_i (\partial_k u_j)^2
\]

\[
= \sum_{j=1}^N \partial_j \left( \frac{1}{2} \delta_{ij} |u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{\alpha}{2} \delta_{ij} |
\partial \nabla u|^2 \right).
\]

Set stress-tensor

\[
T_{ij} = m_i u_j + \frac{\delta_{ij} |u|^2}{2} - \alpha \partial_i u \cdot \partial_j u + \frac{\alpha \delta_{ij}}{2} |\n \partial \nabla u|^2.
\]

Then (EP) becomes

\[
\partial_t m_i + \sum_{j=1}^N \partial_j T_{ij} = 0.
\]  

The first term in \( T_{ij} \) involves a second order derivative of \( u \) and it can be rewritten as

\[
m_i u_j = u_i u_j + \alpha \sum_{k=1}^N \partial_k u_i \partial_k u_j - \alpha \sum_{k=1}^N \partial_k (u_j \partial_k u_i).
\]
The symmetric part of tensor $T$ is given by

$$T^a = u \otimes u + \alpha \nabla u \nabla u^T - \alpha \nabla u^T \nabla u + \frac{1}{2}(|u|^2 + \alpha |\nabla u|^2) \text{Id}$$

and the remainder terms in $T$ are given by

$$T^b_{i,j} = -\alpha \sum_{k=1}^N \partial_k (u_j \partial_k u_i).$$

Hence $T = T^a + T^b$. In view of this, the natural definition of the weak solution of (EP) would be:

**Definition 1** $u \in L^\infty(0,T; H^1_{\text{loc}}(\mathbb{R}^N))$ is a weak solution of (EP) with initial data $u_0 \in H^1_{\text{loc}}(\mathbb{R}^N)$ if the following equation holds for all vector field $\phi(x,t)$ such that $\phi(\cdot, t) \in C^\infty_0(\mathbb{R}^N)$ for all $t \in [0,T)$ and $\phi(x, \cdot) \in C^1([0,T))$ for all $x \in \mathbb{R}^N$

$$\int_0^T \int_{\mathbb{R}^N} (u \cdot \phi_t + \alpha \nabla u : \nabla \phi_t) dx dt + \int_{\mathbb{R}^N} (u_0 \cdot \phi(\cdot, 0) + \alpha \nabla u_0 : \nabla \phi(\cdot, 0)) dx$$

$$+ \int_0^T \int_{\mathbb{R}^N} T^a : \nabla \phi(x, t) dx dt + \alpha \sum_{i,j,k=1}^N \int_0^T \int_{\mathbb{R}^N} u_j \partial_k u_i \partial_j \partial_k \phi_i dx dt = 0,$$

where $T^a$ is given by (3).

(EP) also has a natural Hamiltonian structure. Set

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^N} u \cdot m \text{ d}x.$$

then $\frac{\delta \mathcal{H}}{\delta m} = u$ and (EP) can be recast as

$$\partial_t m = -A \frac{\delta \mathcal{H}}{\delta m},$$

where $A$ is an anti-symmetric operator defined by

$$A u = \sum_{j=1}^N \partial_j (m_i u_j) + \sum_{j=1}^N \partial_i u_j m_j.$$

Consequently, from (2) and (6), there are two conservation laws

$$\frac{d}{dt} \int_{\mathbb{R}^N} m \text{ d}x = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^N} (|u|^2 + \alpha |\nabla u|^2) \text{ d}x = 0.$$

For the one-dimensional case, (EP) coincides with the dispersion-less case of Camassa-Holm (CH) equation and there is an additional Hamiltonian structure and a Lax-pair.
which leads to a complete integrability of (CH) \[4\]. We refer to \[13\] for a general discussion on bi-Hamiltonian system and complete integrability.

When $\alpha = 0$, the above Hamiltonian structure shows that (EP) is a symmetric hyperbolic system of conservation laws

$$
\begin{cases}
\partial_t u + \div(u \otimes u) + \frac{1}{2} \nabla |u|^2 = 0 \\
u(x,0) = u_0
\end{cases}
$$

which possess a global convex entropy function

$$
\frac{1}{2} \partial_t |u|^2 + \div(|u|^2 u) = 0.
$$

We refer (7) as the zero dispersion equation. Indeed, we can recast it in a usual form of a symmetric hyperbolic system (we state it in $\mathbb{R}^3$):

$$
u_t + A\nu_x + Bu_y + Cu_z = 0
$$

with

$$
u = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, A = \begin{pmatrix} 3u & v & w \\ v & u & 0 \\ w & 0 & u \end{pmatrix}.
$$

$A$ is a symmetric matrix and has three eigenvalues: $u$, $2u+|u|$, $2u-|u|$, corresponding to one linearly degenerate field, and two genuinely nonlinear fields, respectively, when $\nu \neq 0$.

We shall remark that although the high dimensional Burgers equation has a similar structure as (7), it does not possess a global convex entropy. In section 5, we will prove a Liouville type theorem for (7). This theorem does not hold true for the high dimensional Burgers equation.

Now we introduce some notations and then we state a theorem on local existence of the weak solution and local existence and uniqueness of the classical solution.

For $s \in \mathbb{R}$ and $p \in [1, \infty]$ we define the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ as follows

$$L^{s,p}(\mathbb{R}^N) = \{ f \in L^p(\mathbb{R}^N) | \|(1 - \Delta)^{s/2} f\|_{L^p} := \|f\|_{L^{s,p}} < \infty \}.$$

For $s \in \mathbb{N} \cup \{0\}$ it is well-known that $L^{s,p}(\mathbb{R}^N)$ is equivalent to the standard Sobolev space $W^{s,p}(\mathbb{R}^N) (\text{see e.g. } \text{[27]}).$ This, in turn, implies immediately that there exist $C_1, C_2$ such that

$$C_1\|\nu\|_{W^{k+2,p}} \leq \|m\|_{L^{k,p}} \leq C_2\|\nu\|_{W^{k+2,p}}$$

for all $k \in \mathbb{N} \cup \{0\}, p \in (1, \infty)$. As usual we denote $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$.

**Theorem 1** \(i\) Assume $\alpha > 0$ and $\nu_0 \in W^{2,p}(\mathbb{R}^N)$ with $p > N$. Then, there exists $T = T(\|\nu_0\|_{W^{2,p}})$ such that a weak solution to (EP) exists, and belongs to $\nu \in L^{\infty}(0,T; W^{2,p}(\mathbb{R}^N)) \cap Lip(0,T; W^{1,p}(\mathbb{R}^N))$. 5
(ii) Let $\alpha > 0$ and $u_0 \in H^k(\mathbb{R}^N)$ with $k > N/2 + 3$. Then, there exists $T = T(\|u_0\|_{H^k})$ such that a classic solution to (EP) exists uniquely, and belongs to $u \in C([0,T); H^k(\mathbb{R}^N))$.

(iii) For $\alpha = 0$, (EP) is a symmetric hyperbolic system of conservation laws with a convex entropy. Consequently, if $u_0 \in H^k(\mathbb{R}^N)$ with $k > N/2 + 1$. Then, there exists $T = T(\|u_0\|_{H^k})$ such that a classic solution to (EP) exists uniquely, and belongs to $u \in C([0,T); H^k(\mathbb{R}^N))$.

The proof of symmetric hyperbolicity and existence of convex entropy in (iii) are given in (7)-(8). The proof of existence of the unique classic solution for symmetric hyperbolic system is standard, see e.g [16]. The proof of (i) and (ii) is also rather standard and will be given in the Appendix for completeness.

3 Finite time blow up

In this section, we first present a theorem on a blow-up criterion and then we prove a theorem on finite time blow up for the zero dispersion equation.

We denote the deformation tensor for $u$ by $S = (S_{ij})$, where $S_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. We recall the Besov space $\dot{B}^0_{\infty,\infty}$, which is defined as follows. Let $\{\psi_m\}_{m \in \mathbb{Z}}$ be the Littlewood-Paley partition of unity, where the Fourier transform $\hat{\psi}_m(\xi)$ is supported on the annulus $\{\xi \in \mathbb{R}^N | 2^{m-1} \leq |\xi| < 2^m\}$ (see e.g. [25]). Then,

$$f \in \dot{B}^0_{\infty,\infty} \text{ if and only if } \sup_{m \in \mathbb{Z}} \|\psi_m \ast f\|_{L^\infty} := \|f\|_{\dot{B}^0_{\infty,\infty}} < \infty.$$ 

The following is a well-known embedding result,

$$L^\infty(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N). \quad (10)$$

**Theorem 2** For $\alpha \geq 0$, we have the following finite time blow-up criterion of the local solution of (EP) in $u \in C([0, t_\ast); H^k(\mathbb{R}^N))$, $k > N/2 + 3$.

$$\limsup_{t \to t_\ast} \|u(t)\|_{H^k} = \infty \text{ if and only if } \int_0^{t_\ast} \|S(t)\|_{\dot{B}^0_{\infty,\infty}} \, dt = \infty. \quad (11)$$

**Remark 1.1** Combining the embedding relation, $W^{1,N}(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N)$ with the inequality $\|D^2u\|_{L^p} \leq C\|m\|_{L^p}$ for $p \in (1, \infty)$ (see (15) below), we have

$$\|S\|_{\dot{B}^0_{\infty,\infty}} \leq C\|S\|_{BMO} \leq C\|DS\|_{L^N} \leq C\|D^2u\|_{L^N} \leq C\|m\|_{L^N}.$$ 

Therefore we obtain the following criterion as an immediate corollary of the above theorem: for all $p > N$,

$$\limsup_{t \to t_\ast} \|m(t)\|_{L^p} = \infty \text{ if and only if } \int_0^{t_\ast} \|m(t)\|_{L^N} \, dt = \infty. \quad (12)$$
Remark 1.2 In the one dimensional case of the Camassa-Holm equation (CH) the above criterion implies that finite time blow-up does not happen if \( \int_0^t \| u_{xx}(\tau) \|_{L^1} d\tau < \infty \) for all \( t > 0 \). Thanks to the conservation law we have \( \sup_{0 < \tau < t} \| u_x(\tau) \|_{L^2} \leq \| u_0 \|_{H^1} < \infty \) for all \( t > 0 \). Since we have embedding \( W^{2,1}(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \), and we do have finite time blow-up for (CH) \([24]\), our criterion is sharp in this one dimensional case.

Proof of Theorem 2 We only give a proof for the case \( \alpha > 0 \). The proof for the case \( \alpha = 0 \) is similar and simpler hence will be omitted.

Using estimates \([33, 34, 35, 36]\) for \( I_1, I_2, I_3 \) in the proof of Theorem 1 in the Appendix, one has

\[
\frac{d}{dt} \| m(t) \|_{H^k} \leq C(\| \nabla u \|_{L^\infty} + \| m \|_{L^\infty} + \| \nabla m \|_{L^\infty}) \| m(t) \|_{H^k}
\]

\[
\leq C(\| m \|_{L^p} + \| Dm \|_{L^p} + \| D^2 m \|_{L^p}) \| m(t) \|_{H^k}.
\]

Hence,

\[
\| m(t) \|_{H^k} \leq \| m_0 \|_{H^k} \exp \left[ C \int_0^t \left\{ \| m(\tau) \|_{L^p} + \| Dm(\tau) \|_{L^p} + \| D^2 m(\tau) \|_{L^p} \right\} d\tau \right]
\]

for \( k > N/2 + 1 \) and \( p > N \), where we used the Sobolev embedding. Consequently, blow up of \( \| m(t) \|_{H^k} \) as \( t \to t^* \) implies that at least one of \( \| m(t) \|_{L^p} \), \( \| Dm(t) \|_{L^p} \) and \( \| D^2 m(t) \|_{L^p} \) blow up as \( t \to t^* \). In the following three steps, we show blow-up criterion for each of them are all given by (11).

Step 1. We first recall the following logarithmic Sobolev inequality(see e.g. \([28]\)),

\[
\| f \|_{L^\infty} \leq C(1 + \| f \|_{B^0_{s,\infty}})(\log(1 + \| f \|_{W^{s,p}})),
\]

where \( s > 0 \), \( 1 < p < \infty \) and \( sp > N \). From the estimate in (31) in the Appendix we obtain

\[
\frac{d}{dt} \| m \|_{L^p} \leq C(1 + \| S \|_{B^0_{s,\infty}}) \log(1 + \| S \|_{W^{1,p}}) \| m \|_{L^p} \quad \text{(for } p > N)\]

\[
\leq C(1 + \| S \|_{B^0_{s,\infty}}) \log(1 + \| D^2 u \|_{L^p}) \| m \|_{L^p}
\]

\[
\leq C(1 + \| S \|_{B^0_{s,\infty}}) \log(1 + \| m \|_{L^p}) \| m \|_{L^p}
\]

for \( p > N \), where we used the boundedness on \( L^p(\mathbb{R}^N) \) of the pseudo-differential operator

\[
\sigma_{ij}(D) := \partial_i \partial_j (1 - \alpha \Delta)^{-1} = -R_i R_j \Delta (1 - \alpha \Delta)^{-1}
\]

with the Riesz transforms \( \{ R_j \}_{j=1}^N \) on \( \mathbb{R}^N \)(see Lemma 2.1, pp. 133\([27]\)), which provides us with

\[
\| D^2 u \|_{L^p} = \sum_{i,j=1}^N \| \sigma_{ij}(D) m \|_{L^p} \leq C \| m \|_{L^p}
\]
for all $p \in (1, \infty)$. By Gronwall’s lemma we obtain

$$\log (1 + \| m(t) \|_{L^p}) \leq \log(1 + \| m_0 \|_{L^p}) \exp \left( C \int_0^t (1 + \| S(\tau) \|_{B_{\infty, \infty}^0}) d\tau \right)$$  \hspace{1cm} (16)

for $p > N$. This implies that

$$\limsup_{t \to t_*} \| m(t) \|_{L^p} = \infty \text{ if and only if } \int_0^{t_*} \| S(t) \|_{B_{\infty, \infty}^0} dt = \infty.$$  \hspace{1cm} (17)

**Step 2.** Taking derivative of (EP) and $L^2(\mathbb{R}^N)$ inner product it with $Dm|Dm|^{p-2}$, we find that

$$\frac{1}{p} \frac{d}{dt} \| Dm(t) \|_{L^p}^p = \frac{1}{p} \int_{\mathbb{R}^N} (\text{div } u) |Dm|^p dx - \int_{\mathbb{R}^N} (Du \cdot \nabla)m . Dm |Dm|^{p-2} dx$$

$$- \int_{\mathbb{R}^N} D(\nabla u)^\top m . Dm |Dm|^{p-2} dx - \int_{\mathbb{R}^N} (\nabla u)^\top Dm . Dm |Dm|^{p-2} dx$$

$$- \int_{\mathbb{R}^N} (\text{div } u) m . Dm |Dm|^{p-2} dx - \int_{\mathbb{R}^N} (\text{div } u) Dm . Dm |Dm|^{p-2} dx$$

$$\leq \left( 3 + \frac{1}{p} \right) \int_{\mathbb{R}^N} Dm |Dm|^p dx + 2 \int_{\mathbb{R}^N} |D^2u| m |Dm|^{p-1} dx$$

$$\leq \left( 3 + \frac{1}{p} \right) \| Du \|_{L^\infty} \| Dm \|_{L^p}^p + 2 \| D^2u \|_{L^{2p}} \| m \|_{L^{2p}} \| Dm \|_{L^p}^{p-1}$$

$$\leq C \| m \|_{L^p} \| Dm \|_{L^p}^p + C \| m \|_{L^{2p}}^2 \| Dm \|_{L^p}^{p-1}$$

for $p > N$, where we used the Sobolev embedding and (15) to estimate

$$\| Du \|_{L^\infty} \leq C \| D^2u \|_{L^p} \leq C \| Dm \|_{L^p}$$

for $p > N$. Hence, for $p > N$ we have

$$\frac{d}{dt} \| Dm(t) \|_{L^p} \leq C \| m \|_{L^p} \| Dm \|_{L^p} + C \| m \|_{L^{2p}}^2.$$

By Gronwall’s lemma, we have

$$\| Dm(t) \|_{L^p} \leq \exp \left( C \int_0^t \| m(\tau) \|_{L^p} d\tau \right) \left( \| Dm_0 \|_{L^p} + C \int_0^t \| m(\tau) \|_{L^{2p}}^2 d\tau \right)$$  \hspace{1cm} (18)

for $p > N$. From estimate (16), one has

$$\int_0^t \| m(s) \|_{L^p} ds \leq t \max_{0 \leq s \leq t} \| m(s) \|_{L^p}$$

$$\leq t \max_{0 \leq s \leq t} \exp \left( \log(1 + \| m(s) \|_{L^p}) \right)$$

$$\leq t \exp \left( \log(1 + \| m_0 \|_{L^p}) \exp \left( C \int_0^t (1 + \| S(\tau) \|_{B_{\infty, \infty}^0}) d\tau \right) \right).$$  \hspace{1cm} (19)
Similarly,
\[
\int_0^t \| m(s) \|_{L^2} ds \leq t \exp \left( \log(1 + \| m_0 \|_{L^2}) \exp \left( C \int_0^t (1 + \| S(\tau) \|_{B_{\infty, \infty}}) d\tau \right) \right).
\]

Combining (18, 19) and (20), one obtains
\[
\limsup_{t \to t_*} \| Dm(t) \|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \| S(t) \|_{B_{\infty, \infty}} dt = \infty. \quad (21)
\]

**Step 3.** Similarly, taking \(D^2\) of (EP) and \(L^2(\mathbb{R}^N)\) inner product it with \(D^2 m | D^2 m|^{p-2}\), we find that
\[
\frac{1}{p} \frac{d}{dt} \| D^2 m(t) \|_{L^p}^p \leq 4 \int_{\mathbb{R}^N} |Du| |D^2 m|^p \, dx + 3 \int_{\mathbb{R}^N} |D^2 u| |Dm| |D^2 m|^{p-1} \, dx
\]
\[
+ 2 \int_{\mathbb{R}^N} |D^3 u| |m| |D^2 m|^{p-1} \, dx
\]
\[
\leq 4 \| Du \|_{L^\infty} \| D^2 m \|_{L^p}^p + 3 \| D^2 u \|_{L^2} \| Dm \|_{L^2} \| D^2 m \|_{L^p}^{p-1}
\]
\[
+ 2 \| D^3 u \|_{L^2} \| m \|_{L^2} \| D^2 m \|_{L^p}^{p-1}
\]
\[
\leq C \| m \|_{L^p} \| D^2 m \|_{L^p}^p + C \| m \|_{L^2} \| Dm \|_{L^2} \| D^2 m \|_{L^p}^{p-1}
\]
for \(p > N\), where we used the estimate (15) as follows
\[
\| D^3 u \|_{L^q} = \| \{ D^2 (1 - \alpha \Delta)^{-1} \} D(1 - \alpha \Delta) u \|_{L^q}
\]
\[
\leq \sum_{i,j=1}^N \| \sigma_{ij} (D) Dm \|_{L^q} \leq C \| Dm \|_{L^q},
\]
which holds for all \(q \in (1, \infty)\). Hence,
\[
\frac{d}{dt} \| D^2 m(t) \|_{L^p} \leq C \| m \|_{L^p} \| D^2 m \|_{L^p} + C \| m \|_{L^2} \| Dm \|_{L^2}.
\]
By Gronwall’s lemma we have
\[
\| D^2 m(t) \|_{L^p} \leq \exp \left( C \int_0^t \| m(\tau) \|_{L^p} d\tau \right) \left( \| D^2 m_0 \|_{L^p} + C \int_0^t \| m(\tau) \|_{L^2} \| Dm(\tau) \|_{L^p} d\tau \right)
\]
for \(p > N\). Similarly to the estimates in (19) and (20), the right hand side terms in the above inequality can all be controlled
\[
\int_0^t (1 + \| S(\tau) \|_{B_{\infty, \infty}}) d\tau.
\]
Therefore, we have
\[
\limsup_{t \to t_*} \| D^2 m(t) \|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \| S(t) \|_{B_{\infty, \infty}} dt = \infty. \quad (22)
\]
Combination of (13, 17, 21, 22) gives the proof of the theorem. □

We now present a finite time blow-up result for \(\alpha = 0\).
Theorem 3 Let the initial data of the system (7), $u_0 \in H^k(\mathbb{R}^N)$, $k > N/2 + 2$, has the reflection symmetry with respect to the origin, and satisfies $\text{div } u_0(0) < 0$. Then, there exists a finite time blow-up of the classical solution.

Proof Taking divergence of (7), we find

$$\partial_t(\text{div } u) + u \cdot \nabla (\text{div } u) + 2 \sum_{i,j=1}^N S^2_{ij} + \sum_{j=1}^N (\Delta u_j)u_j + (\text{div } u)^2 + \sum_{i,j=1}^N (\partial_i \partial_j u_i)u_j = 0,$$

where we used $S_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i)$, and the fact

$$\sum_{i,j=1}^N \partial_i u_j \partial_j u_i + \sum_{i,j=1}^N \partial_i u_j \partial_i u_j = 2 \sum_{i,j=1}^N \partial_i u_j S_{ij} = \sum_{i,j=1}^N (\partial_i u_j + \partial_j u_i) S_{ij} = 2 \sum_{i,j=1}^N S^2_{ij}.$$

Now we consider the reflection transform:

$$R: x \rightarrow \bar{x} = -x, \quad u(x,t) \rightarrow \bar{u}(x,t) = -u(-x,t).$$

Obviously the system (7) is invariant under this transform. The origin of the coordinate is the invariant point under the reflection transform. We consider the smooth initial data $u_0 \in H^k(\mathbb{R}^N)$, $k > N/2 + 2$, which has the reflection symmetry. Then, by the uniqueness of the local classical solution in $H^k(\mathbb{R}^N)$, and hence in $C^2(\mathbb{R}^N)$, the reflection symmetry is preserved as long as the classical solution persists. We consider the evolution of the solution at the origin of the coordinates. Then, $u(0,t) = 0$ and $D^2 u(0,t) = 0$ for all $t \in [0, T_*)$, where $T_*$ is the maximal time of existence of the classical solution in $H^k(\mathbb{R}^N)$. If $T_* = \infty$, we will show that this leads to a contradiction. The system (23) at the origin is reduced to

$$\partial_t(\text{div } u) + 2 \sum_{i,j=1}^N S^2_{ij} + (\text{div } u)^2 = 0,$$

which implies

$$\partial_t(\text{div } u) = -2 \sum_{i,j=1}^N S^2_{ij} - (\text{div } u)^2 \leq -(\text{div } u)^2.$$

Thus,

$$\text{div } u(0,t) \leq \frac{\text{div } u_0(0)}{1 + \text{div } u_0(0)t},$$

which shows $T_* \leq \frac{1}{|\text{div } u_0(0)|}$ for $\text{div } u_0(0) < 0$. □

4 Zero $\alpha$ limit for weak solutions

In this section, we show the following theorem on the zero dispersion limit $\alpha \rightarrow 0$ for the weak solutions.
Theorem 4 Let $u^\alpha \in L^\infty((0,T); H^1(\mathbb{R}^N))$ be a weak solution with initial data $u_0^\alpha$ to (EP) with $\alpha > 0$, and $u \in L^\infty((0,T); H^k(\mathbb{R}^N)) \cap \text{Lip}((0,T); H^2(\mathbb{R}^N))$, $k > N/2 + 3$, be the classic solution with initial data $u_0$ to (EP) with $\alpha = 0$, i.e., (7). Then, we have

$$
\sup_{0 \leq t \leq T} \left\{ \|u^\alpha(t) - u(t)\|_{L^2} + \sqrt{\alpha} \|\nabla(u^\alpha(t) - u(t))\|_{L^2} \right\} 
\leq C(\alpha + \|u_0^\alpha - u_0\|_{L^2} + \sqrt{\alpha} \|\nabla(u_0^\alpha - u_0)\|_{L^2}),
$$

where $C = C(\|u\|_{L^\infty(0,T; H^k(\mathbb{R}^N))}, \|u\|_{\text{Lip}(0,T; H^2(\mathbb{R}^N))})$ is a constant.

Proof We denote $\bar{m} := u - \alpha \Delta u$. Then $(u, \bar{m})$ satisfy (EP) with a truncation term as below

$$\partial_t \bar{m} + \text{div}(u \otimes \bar{m}) + \left(\nabla u\right)^T \bar{m} = -\alpha \left\{ \Delta u_t + \text{div}(u \otimes \Delta u) + (\nabla u)^T \Delta u \right\}. \quad (26)$$

Subtracting (26) from the first equation of (EP), and setting $\bar{m} := m^\alpha - \bar{m}$ and $\bar{u} := u^\alpha - u$, we find

$$\partial_t \bar{m} + \text{div}(\bar{u} \otimes \bar{m}) + \text{div}(\bar{u} \otimes m) + \text{div}(u \otimes \bar{m}) + \left(\nabla \bar{u}\right)^T \bar{m} + \left(\nabla \bar{u}\right)^T m + \left(\nabla u\right)^T \bar{m}$$

$$= \alpha \left\{ \Delta u_t + \text{div}(u \otimes \Delta u) + (\nabla u)^T \Delta u \right\}. \quad (27)$$

Taking $L^2(\mathbb{R}^N)$ inner product (27) with $\bar{u}$, and integrating by part, and observing

$$\int_{\mathbb{R}^N} \text{div}(\bar{u} \otimes \bar{m}) \cdot \bar{u} \, dx = - \int_{\mathbb{R}^N} \bar{u} \cdot (\nabla \bar{u})^T \bar{m} \, dx$$

$$\int_{\mathbb{R}^N} \text{div}(\bar{u} \otimes m) \cdot \bar{u} \, dx = - \int_{\mathbb{R}^N} \bar{u} \cdot (\nabla \bar{u})^T m \, dx,$$

we obtain that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (|\bar{u}|^2 + \alpha |\nabla \bar{u}|^2) \, dx = - \int_{\mathbb{R}^N} \text{div}(u \otimes \bar{m}) \cdot \bar{u} \, dx - \int_{\mathbb{R}^N} \bar{u} \cdot (\nabla u)^T \bar{m} \, dx$$

$$+ \alpha \int_{\mathbb{R}^N} \left[ \bar{u} \cdot \{ \Delta u_t + \text{div}(u \otimes \Delta u) + (\nabla u)^T \Delta u \} \right] \, dx$$

$$:= I_1 + I_2 + I_3.$$

We estimate

$$I_1 = - \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i u_i (\bar{u}_j - \alpha \Delta \bar{u}_j) \bar{u}_j \, dx - \sum_{i,j=1}^N \int_{\mathbb{R}^N} u_i \partial_\alpha (\bar{u}_j - \alpha \Delta \bar{u}_j) \bar{u}_j \, dx$$

$$= J_1 + J_2,$$

where

$$J_1 = - \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i u_i |\bar{u}_j|^2 \, dx + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i \partial_k u_i (\bar{u}_j) \bar{u}_j \, dx$$

$$+ \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i u_i (\partial_k \bar{u}_j) \partial_k \bar{u}_j \, dx$$

$$\leq C\|u(t)\|_{C^2} (\|\bar{u}\|_{L^2}^2 + \alpha \|\nabla \bar{u}\|_{L^2}^2),$$

11
and

\[
J_2 = - \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u_i (\partial_t \bar{u}_j) \bar{u}_j \, dx + \alpha \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u_i \partial_t (\Delta \bar{u}_j) \bar{u}_j \, dx
\]

\[
= - \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} u_i \partial_t |\bar{u}_j|^2 \, dx - \alpha \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \partial_k u_i \partial_t (\partial_k \bar{u}_j) \bar{u}_j \, dx
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \partial_t u_i |\bar{u}_j|^2 \, dx + \alpha \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \partial_t u_i (\partial_k \bar{u}_j) \partial_i \bar{u}_j \, dx
\]

\[
+ \alpha \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \partial_t u_i (\partial_k \bar{u}_j) \partial_i \bar{u}_j \, dx + \frac{\alpha}{2} \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \partial_t u_i |\partial_k \bar{u}_j|^2 \, dx
\]

\[
\leq C \|u(t)\|_{C^2} (\|\bar{u}\|_{L^2}^2 + \alpha \|\nabla \bar{u}\|_{L^2}^2).
\]

One can estimate \(I_3\) immediately as

\[
I_3 \leq \|\bar{u}\|_{L^2}^2 + \alpha^2 C (\|u\|_{L^{2}(0,T;H^2(\mathbb{R}^N))}^2 + \|u\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}^4).
\]

Summarizing the above estimates, we obtain

\[
\frac{d}{dt} (\|\bar{u}\|_{L^2}^2 + \alpha \|\nabla \bar{u}\|_{L^2}^2) \leq C \|u(t)\|_{C^2} (\|\bar{u}\|_{L^2}^2 + \alpha \|\nabla \bar{u}\|_{L^2}^2) + \alpha^2 C (\|u\|_{L^{2}(0,T;H^2(\mathbb{R}^N))}^2 + \|u\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}^4),
\]

which implies by Gronwall’s lemma that

\[
\|\bar{u}\|_{L^2}^2 + \alpha \|\nabla \bar{u}\|_{L^2}^2 \leq C_1 (\alpha^2 + \|\bar{u}(0)\|_{L^2}^2 + \alpha \|\nabla \bar{u}(0)\|_{L^2}^2)
\]

where constant \(C_1\) depended only on \(\|u\|_{L^{2}(0,T;H^2(\mathbb{R}^N))}\) and \(\|u\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}\). This completes the proof of theorem. \(\square\)
5 Liouville type theorem for stationary solutions

In this section, we prove a Liouville type theorem for stationary solutions. Recall that the stationary weak solution defined in Definition 1 reduces to

**Definition 2** \( u \in H^1(\mathbb{R}^N) \) is a stationary weak solution to (EP) on \( \mathbb{R}^N \), if the following holds

\[
\begin{align*}
\sum_{j=1}^{N} \int_{\mathbb{R}^N} \{u_i u_j + \alpha \nabla u_i \cdot \nabla u_j\} \partial_j \varphi_i \, dx + \alpha \sum_{j=1}^{N} \int_{\mathbb{R}^N} u_j \nabla u_i \cdot \nabla \partial_j \varphi_i \, dx \\
+ \sum_{j=1}^{N} \int_{\mathbb{R}^N} \left\{ \frac{\delta_{ij}}{2} |u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{\alpha \delta_{ij}}{2} |\nabla u|^2 \right\} \partial_j \varphi_i \, dx = 0 \quad (28)
\end{align*}
\]

for \( i = 1, \cdots, N \) and for all \( \varphi \in C_0^\infty(\mathbb{R}^N) \).

**Theorem 5**

(i) Let \( u \in H^1(\mathbb{R}^N) \) be a stationary weak solution to (EP) with \( \alpha > 0 \). Then, \( u = 0 \).

(ii) Let \( u \in L^2(\mathbb{R}^N) \) be a stationary weak solution to (EP) with \( \alpha = 0 \). Then, \( u = 0 \).

**Proof** For \( \alpha > 0 \), one can write (28) in the following form,

\[
\sum_{j=1}^{N} \int_{\mathbb{R}^N} T_{ij}^a \partial_j \varphi_i \, dx + \sum_{j,k=1}^{N} \int_{\mathbb{R}^N} \tilde{T}_{ijk}^b \partial_j \partial_k \varphi_i \, dx = 0, \quad (29)
\]

where \( T_{ij}^a \) is defined in (3) and we recall here

\[
T_{ij}^a = u_i u_j + \alpha \nabla u_i \cdot \nabla u_j + \frac{\delta_{ij}}{2} |u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{\alpha \delta_{ij}}{2} |\nabla u|^2,
\]

and

\[
\tilde{T}_{ijk}^b = \alpha u_j \partial_k u_i.
\]

corresponding to \( T_{ij}^b \) in (4).

Let us consider the radial cut-off function \( \sigma \in C_0^\infty(\mathbb{R}^N) \) such that

\[
\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}
\]

and \( 0 \leq \sigma(x) \leq 1 \) for \( 1 < |x| < 2 \). Then, for each \( R > 0 \), we define

\[
\sigma \left( \frac{|x|}{R} \right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\]
Choosing $\varphi_i(x) = x_i\sigma_R(x)$ in (29), we obtain

$$0 = \sum_{i=1}^{N} \int_{\mathbb{R}^N} T_{ii}^a \sigma_R(x) \, dx + \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} T_{ij}^a x_j \partial_i \sigma_R(x) \, dx + \sum_{i,k=1}^{N} \int_{\mathbb{R}^N} \tilde{T}_{ik}^b \partial_k \sigma_R(x) \, dx$$

$$+ \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \tilde{T}_{ij}^b \partial_j \sigma_R(x) \, dx + \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \tilde{T}_{ijk}^b x_i \partial_j \partial_k \sigma_R(x) \, dx$$

$$= I_1 + I_2 + I_3 + I_4 + I_5. \tag{30}$$

The hypothesis $u \in H^1(\mathbb{R}^N)$ implies that $T \in L^1(\mathbb{R}^N)$. Thus, we obtain

$$|I_2| \leq \frac{1}{R} \int_{\{R \leq |x| \leq 2R\}} |T^a||x||\nabla \sigma| \, dx \leq 2 \|\nabla \sigma\|_{L^\infty} \int_{\{R \leq |x| \leq 2R\}} |T^a| \, dx \to 0$$

as $R \to \infty$ by the dominated convergence theorem. Similarly, $I_3, I_4, I_5 \to 0$ as $R \to \infty$.

Thus, passing $R \to \infty$ in (30), we have

$$0 = \lim_{R \to \infty} \sum_{i=1}^{N} \int_{\mathbb{R}^N} T_{ii}^a \sigma_R(x) \, dx$$

$$= \int_{\mathbb{R}^N} \left\{ \frac{(N+2)}{2} |u|^2 + \frac{\alpha N}{2} |\nabla u|^2 \right\} \, dx,$$

which implies $u = 0$. This gives (i).

For the case $\alpha = 0$. All the terms involving $\alpha$ drop and (ii) holds true. This completes the proof the theorem $\Box$

We remark that the Liouville type results in Theorem 5 is rather surprising, as all the previous Liouville type results are for dissipative systems. For the Liouville type results for the dissipative systems, see, e.g. [5]. Theorem 5 is the first Liouville type theorem for non-dissipative systems.

Acknowledgements: This work was initiated at Duke University when the first author visited there. The authors wish to acknowledge the hospitality of Mathematical Sciences Center of Tsinghua University where this research was completed. The research of D.C was supported partially by NRF Grant no. 2006-0093854. The research of J.-G. L. was partially supported by NSF grant DMS 10-11738.

References

[1] Arnold, V., *Sur un principe variationnel pour les ecoulements stationnaires des liq- uides parfaits et ses applications aux probleme de stabilit?e non lin?eaires*, J. Méc., 5 (1966), 29–43.
[2] Bardos, C., Linshiz, J., and Titi, E.S., Global regularity and convergence of a Birkhoff-Rott-α approximation of the dynamics of vortex sheets of the 2D Euler equations, Comm. Pure and Appl. Math., 63 (2010), 697–746.

[3] Bressan, A. and Constantin, A., Global conservative solutions of the Camassa-Holm equation, Arch. Ration. Mech. Anal., 183 (2007), 215–239.

[4] Camassa, R. and Holm, D.D., An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993), 1661–1664.

[5] Chae, D., On the nonexistence of global weak solutions to the Navier-Stokes-Poisson equations in $\mathbb{R}^N$, Comm. PDE, 35 (2010), 535–557.

[6] Chertock, A., Liu, J.-G., and Pendleton, T., Convergence of a particle method and global weak solutions for a family of evolutionary PDEs, submitted.

[7] Chen, S., Foias, C., Holm, D.D., Olson, E., Titi, E.S., and Wynne, S., Camassa-Holm equations as a closure model for turbulent channel and pipe flow, Phys. Rev. Lett., 81 (1998), 5338–5341.

[8] Constantin, A. and Escher, J., Global weak solutions for a shallow water equation, Indiana Univ. Math. J., 47 (1998), 1527–1545.

[9] Constantin, A. and Escher, J., Global existence and blow-up for a shallow water equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Serie IV, 26 (1998), 303–328.

[10] Constantin, A. and Molinet, L., Global weak solutions for a shallow water equation, Comm. Math. Phys., 211 (2000), 45-61.

[11] Ebin, D. and Marsden, J., Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math, 92 (1970), 102–163.

[12] Foias, C., Holm D.D., and Titi, E.S., The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory, J. Dyn. and Diff. Eqns., 14 (2002), 1–35.

[13] Fuchssteiner, B. and Fokas, A.S, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D, 4 (1981), 47–66.

[14] Hirani, A.N., Marsden, J.E. and Arvo, J., Averaged template matching equations, Lecture Notes in Computer Science, volume 2134, EMMCVPR, Springer, (2001), 528–543.

[15] Khesin, B and Wendt, R, The geometry of infinite-dimensional groups, Springer, 2009.

[16] Majda, A., Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Springer-Verlag 1984.
[17] Holm, D.D. and Marsden, J.E., *Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation*, The breadth of symplectic and Poisson geometry, Progr. Math., VOLUME 232, Birkhäuser Boston, (2005) 203–235.

[18] Holm, D.D., Marsden, J.E. and Ratiu, T.S., *Euler-Poincaré models of ideal fluids with nonlinear dispersion*, Phys. Rev. Lett., 80 (1998), 4173–4177.

[19] Holm, D.D., Marsden, J.E. and Ratiu, T.S., *Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math., 137 (1998), 1–81.

[20] Holm, D.D., Nitsche, M., and Putkaradze, V., *Euler-alpha and vortex blob regularization of vortex filament and vortex sheet motion*, J. Fluid Mech., 555 (2006), 149–176.

[21] Holm, D.D., Ratnanather, J.T., Trouvé, A. and Younes, L., *Soliton dynamics in computational anatomy*, NeuroImage, 23 (2004), S170 - S178.

[22] Holm, D.D., Schmah, T. and Stoica, C., *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*, Oxford University Press, 2009.

[23] Jiu, Q.S., Niu, D.J., Titi, E.S. and Xin, Z.P., *The Euler-α approximations to the 3D axisymmetric Euler equations with vortex-sheets initial data*, preprint, (2009).

[24] McKean, H.P., *Breakdown of the Camassa-Holm equation*, Comm. Pure Appl. Math., 57 (2004), 416–418.

[25] Molinet, L., *On well-posedness results for Camassa-Holm equation on the line: a survey*, J. Nonlinear Math. Phys., 11 (2004), 521–533.

[26] Marsden, J.E. and Shkoller, S., *Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains*, Proc. Roy. Soc. London A, 359 (2001), 1449–1468.

[27] Stein, E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton, NJ, Princeton University Press. 1970.

[28] Taylor, M., *Tools for PDE*, AMS Mathematical Surveys and Monographs 81, (2000).

[29] Xin, Z. and Zhang, P., *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math., 53 (2000), 1411–1433.

[30] Younes, L., *Shapes and Diffeomorphisms*, Springer, 2010.
Appendix: proof of Theorem 1

Proof of Theorem 1 The proof of local existence part is standard, and below we derive the key local in time estimate of $u(t) \in L^\infty([0, T); W^{2,p}(\mathbb{R}^N)) \cap Lip(0, T; W^{1,p}(\mathbb{R}^N))$.

\[
\frac{1}{p} \frac{d}{dt} \|m\|_{L^p}^p = \frac{1}{p} \int_{\mathbb{R}^N} (u \cdot \nabla)|m|^p \, dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_j u_i m_i m_j |m|^{p-2} \, dx \\
+ \int_{\mathbb{R}^N} (\text{div } u)|m|^p \, dx
\]

\[
= \left( 1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} Tr(S)|m|^p \, dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} S_{ij} m_i m_j |m|^{p-2} \, dx \\
\leq C\|S\|_{L^\infty} \|m\|_{L^p}^p \leq C\|\nabla u\|_{L^\infty} \|m\|_{L^p}^p \leq C\|m\|_{L^p}^{p+1},
\]

and therefore

\[
\frac{d}{dt} \|m\|_{L^p} \leq C\|m\|_{L^p}^2.
\]

We thus have the following estimate on $L^\infty(0, T; W^{2,p}(\mathbb{R}^N))$,

\[
\|u(t)\|_{W^{2,p}} \leq \frac{C\|u_0\|_{W^{2,p}}}{1 - Ct\|u_0\|_{W^{2,p}}} \quad \forall t \in [0, T),
\]

where $T = \frac{1}{\|u_0\|_{W^{2,p}}}$. In order to have estimate of $u$ in $Lip(0, T; W^{1,p}(\mathbb{R}^N))$, we take $L^2(\mathbb{R}^N)$ inner product (EP) with the test function $\psi \in W^{1,\frac{2}{p^*}}(\mathbb{R}^N)$ for $p > N$. Then,

\[
\int_{\mathbb{R}^N} \partial_t m \cdot \psi \, dx = \int_{\mathbb{R}^N} m(u \cdot \nabla)\psi \, dx - \int_{\mathbb{R}^N} m \cdot \nabla u \cdot \psi \, dx \\
\leq C\|m\|_{L^p} \|u\|_{L^\infty} \|\nabla \psi\|_{L^{\frac{p^*}{p^*}}} + C\|m\|_{L^p} \|\nabla u\|_{L^\infty} \|\nabla \psi\|_{L^{\frac{p^*}{p^*}}} \\
\leq C\|m\|_{L^p}^2 \|\psi\|_{W^{1,\frac{2}{p^*}}},
\]

which provides us with the estimate,

\[
\|\partial_t u\|_{L^\infty(0, T; W^{1,p}(\mathbb{R}^N))} \leq C\|\partial_t m\|_{L^\infty(0, T; W^{-1,p}(\mathbb{R}^N))} \leq C\|m\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.
\]

Hence, for all $0 < t_1 < t_2 < T$ we have

\[
\|u(t_2) - u(t_1)\|_{W^{1,p}} \leq \int_{t_1}^{t_2} \|\partial_t u(t)\|_{W^{1,p}} \, dt \leq C(t_2 - t_1)\|m\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.
\]

Namely,

\[
\|u\|_{Lip(0, T; W^{1,p}(\mathbb{R}^N))} \leq C\|m\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.
\]

This gives (i). Next we prove local in time persistency of regularity for $u(t)$ in $H^k(\mathbb{R}^N)$ with $k > N/2 + 3$. Let $\beta = (\beta_1, \cdots, \beta_N)$ be the standard multi-index.
notation with $|\beta| = \beta_1 + \cdots + \beta_N$. Taking $H^k(\mathbb{R}^N)$ inner product (EP) with $m$, we find
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\beta| \leq k} \|D^\beta m\|^2_{L^2} = - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(u \cdot \nabla)m\} \cdot D^\beta m \, dx
\]
\[
- \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(\nabla u)^\top m\} \cdot D^\beta m \, dx
\]
\[
- \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(\text{div } u)m\} \cdot D^\beta m \, dx
\]
\[
:= I_1 + I_2 + I_3. 
\]

We write
\[
I_1 = - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} \{D^\beta (u \cdot \nabla)m - (u \cdot \nabla) D^\beta m\} \cdot D^\beta m \, dx
\]
\[
+ \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} (u \cdot \nabla) D^\beta m \cdot D^\beta m \, dx
\]
\[
:= J_1 + J_2,
\]
and using the standard commutator estimate, we deduce
\[
J_1 \leq \sum_{|\beta| \leq k} \|D^\beta (u \cdot \nabla)m - (u \cdot \nabla) D^\beta m\|_{L^2} \|D^\beta m\|_{L^2}
\]
\[
\leq C(\|\nabla u\|_{L^\infty} \|m\|_{H^k} + \|u\|_{H^k} \|\nabla m\|_{L^\infty}) \|m\|_{H^k}
\]
\[
\leq C(\|u\|_{H^{N/2+1+\epsilon}} \|m\|_{H^k} + \|m\|_{H^{k-2}} \|m\|_{H^{N/2+1+\epsilon}}) \|m\|_{H^k} \quad (\forall \epsilon > 0)
\]
\[
\leq C \|m\|^3_{H^k}
\]
if $k > N/2 + 1$, where we used the fact $u = (1 - \alpha \Delta)^{-1} m$, and therefore $\|u\|_{H^s} \leq \|m\|_{H^{s-2}}$ for all $s \in \mathbb{R}$.
\[
J_2 = \frac{1}{2} \sum_{|\sigma| \leq k} \int_{\mathbb{R}^N} (u \cdot \nabla)|D^\sigma m|^2 \, dx = - \frac{1}{2} \sum_{|\sigma| \leq k} \int_{\mathbb{R}^N} (\text{div } u)|D^\sigma m|^2 \, dx
\]
\[
\leq C \|\nabla u\|_{L^\infty} \|m\|^2_{H^k} \leq C \|m\|_{H^{N/2-1+\epsilon}} \|m\|^2_{H^k} \quad (\forall \epsilon > 0)
\]
\[
\leq C \|m\|^3_{H^k}
\]
if $k > N/2 - 1$. The estimates of $I_2, I_3$ are simpler, and we have
\[
I_2 + I_3 \leq (\|\nabla u\|^T m\|_{H^k} \|m\|_{H^k} \leq C(\|\nabla u\|_{L^\infty} \|m\|_{H^k} + \|u\|_{H^{k+1}} \|m\|_{L^\infty}) \|m\|_{H^k}
\]
\[
\leq C(\|m\|_{H^{N/2+1+\epsilon}} \|m\|_{H^k} + \|m\|_{H^{k-2}} \|m\|_{H^{N/2+1+\epsilon}}) \|m\|_{H^k}
\]
\[
\leq C \|m\|^3_{H^k}
\]
if $k > N/2$. Summarizing the above estimates, we obtain
\[
\frac{d}{dt} \|m\|^2_{H^k} \leq C \|m\|^3_{H^k}
\]
for $k > N/2 + 1$, which implies
\[
\| u(t) \|_{H^k} \leq \frac{C \| u_0 \|_{H^k}}{1 - C \| u_0 \|_{H^k} t} \quad \forall t \in [0, T), \text{ where } T = \frac{1}{C \| u_0 \|_{H^k}},
\]
where $k > N/2 + 3$.

We now prove uniqueness of solution in this class. Let $(u_1, m_1), (u_2, m_2)$ two solution pairs corresponding to initial data $(u_{1,0}, m_{1,0}), (u_{2,0}, m_{2,0})$. We set $u = u_1 - u_2$, and so on. Subtracting the equation for $(u_2, m_2)$ from that of $(u_1, m_1)$, we find that
\[
\partial_t m + \text{div} (u \otimes m) + \text{div} \left( u \otimes m_2 \right) + (\nabla u)^\top m + (\nabla u)^\top m_2 = 0.
\]
Let $p > N$. Taking $L^2(\mathbb{R}^N)$ product of (37) with $m|m|^{p-2}$, we obtain
\[
\frac{1}{p} \frac{d}{dt} \| m(t) \|_{L^p}^p = - \left( 1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} (\text{div} u_1) |m|^p dx - \int_{\mathbb{R}^N} (\text{div} u) m_2 \cdot m |m|^{p-2} dx
\]
\[
- \int_{\mathbb{R}^N} (u \cdot \nabla) m_2 \cdot m |m|^{p-2} dx - \int_{\mathbb{R}^N} (\nabla u_1)^\top m \cdot m |m|^{p-2} dx
\]
\[
\leq C (\| \text{div} u_1 \|_{L^\infty} \| m \|_{L^p}^p + \| \nabla u \|_{L^\infty} \| m_2 \|_{L^p} \| m \|_{L^p}^{p-1} + \| u \|_{L^p} \| \nabla m_2 \|_{L^\infty} \| m \|_{L^p}^{p-1})
\]
\[
\leq C (\| u_1 \|_{H^k} + \| u_2 \|_{H^k} ) \| m \|_{L^p}^p
\]
for $k > N/2 + 3$. Hence,
\[
\| m(t) \|_{L^p} \leq \| m_0 \|_{L^p} \exp \left( C \int_0^t \left( \| u_1(\tau) \|_{H^k} + \| u_2(\tau) \|_{H^k} \right) d\tau \right).
\]
This inequality implies the desired uniqueness of solutions in the class $L^1(0, T; H^k(\mathbb{R}^N))$ with $k > N/2 + 3$. This gives (ii). The proof of (iii) was explained at the end of Section 2. This completes the proof of Theorem 1. □