INTEGRAL NON-HYPERBOLIKE SURGERIES

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Abstract. It is shown that a hyperbolic knot in the 3-sphere admits at most
nine integral surgeries yielding 3-manifolds which are reducible or whose funda-
damental groups are not infinite word-hyperbolic.

1. Introduction

The well-known Hyperbolic Dehn surgery Theorem due to Thurston [12] says
that each hyperbolic knot admits only finitely many Dehn surgeries yielding non-
hyperbolic manifolds. A lot of works have been done to study how many, when and
on which knots such exceptional surgeries can occur. See [5] for a survey.

About the number of exceptional surgeries, it is conjectured that they are at
most TEN, and the knot admitting ten is only the figure-eight knot in the 3-
sphere $S^3$. See [9, Problem 1.77] for a detail. In [8], Hodgson and Kerckhoff
achieved the first universal upper bound SIXTY. In [4] and [10], Agol and Lack-
enby independently showed that there are at most TWELVE surgeries yielding
non-hyperbolike 3-manifolds. Here a 3-manifold is called non-hyperbolike (in the
sense of Agol) if it is reducible or does not have infinite word-hyperbolic fundamen-
tal group. Note that hyperbolic implies hyperbolike, equivalently non-hyperbolike
implies non-hyperbolic. Furthermore if the well-known Geometrization Conjecture
is true, hyperbolike and hyperbolic become equivalent.

The aim of this paper is to present the following result, which gives a new
upper bound on the number of non-hyperbolike surgeries under the assumption
that surgery slopes are integral. Remark that if a meridian-longitude system for
$K$ is fixed, then surgery slopes for $K$ are parametrized by $\mathbb{Q} \cup \{1/0\}$ in a standard
way. See [11] for example.

Theorem 1.1. Let $K$ be a hyperbolic knot in a closed orientable 3-manifold. Sup-
pose that an arbitrarily chosen meridian-longitude system for $K$ is fixed, and by
using this, surgery slopes for $K$ are parametrized by $\mathbb{Q} \cup \{1/0\}$. If two 3-manifolds
obtained by Dehn surgeries on $K$ are both non-hyperbolike, and their surgery slopes
are both integers, then the distance between the surgery slopes is at most eight. This
implies that there are at most NINE integral non-hyperbolike surgeries for $K$.

In particular case of the figure-eight knot in $S^3$, with the standard meridian-
longitude system, the exceptional surgery slopes are $-4, -3, -2, -1, 0, 1, 2, 3, 4,$ and
$1/0$. The slopes $-4$ and $4$ have distance eight, and there are nine integral exceptiona-
surgery slopes. Thus our bound is best possible.

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It should be noted that Theorem 1.1 assures that if some knot admits only integral non-hyperbolike surgeries, then the knot have at most TEN non-hyperbolike surgeries, that is, nine integral ones and a trivial one.

For example, it is remarked in [13] that some class of arborescent knots admit only integral surgeries yielding reducible, toroidal or Seifert fibered 3-manifolds. Such 3-manifolds are sometimes called non-hyperbolike in the sense of Gordon, and are shown to be actually non-hyperbolike in the sense of Agol. Precisely we obtain:

**Corollary 1.2.** An arborescent knot admits at most TEN surgeries yielding reducible, toroidal or Seifert fibered 3-manifolds, unless it is a Montesinos knot of type $M(x,1/p,1/q)$ or its mirror image, where $x \in \{-1/2n,-1\pm1/2n,-2+1/2n\}$, and $p, q$ and $n$ are positive integers. □

2. Proof

We start with preparing definitions and notations.

As usual, we call an embedded circle in a 3-manifold a **knot**. A knot $K$ is called **hyperbolic** if its complement $C_K$ admits a complete hyperbolic structure of finite volume.

A **slope** is defined to be the isotopy class of an unoriented non-trivial simple closed curve on the torus. The **distance** between two slopes means the minimal geometric intersection number of their representatives.

The complement of an open tubular neighborhood of $K$ is called the **exterior** $E_K$ of the knot. Given a knot, a new manifold is obtained by taking the exterior of the knot and attaching a solid torus back. This operation is called a **Dehn surgery** on the knot. When one perform a Dehn surgery on a knot, a meridian curve of the attached solid torus determines a slope on the peripheral torus $\partial E_K$ of the knot. This slope is called the **surgery slope** of the Dehn surgery. In the following, we will use $K(r)$ to denote the 3-manifold obtained by Dehn surgery on $K$ with surgery slope $r$.

From now on, we assume that $K$ is a hyperbolic knot in a closed orientable 3-manifold. Since $K$ is hyperbolic, the complement $C_K$ is regarded as a complete hyperbolic 3-manifold with single cusp. The universal cover of $C_K$ is identified with the hyperbolic 3-space $\mathbb{H}^3$. Under the covering projection, an equivariant set of horospheres bounding disjoint horoballs in $\mathbb{H}^3$ descends to a torus embedded in $C_K$, which we call a **horotorus**. As demonstrated in [12], on a horotorus $T$, a Euclidean structure is obtained by restricting the hyperbolic structure of $C_K$. By using this structure, the length of a curve on $T$ can be defined. Also $T$ is naturally identified with the boundary $\partial E_K$ of the exterior $E_K$ of $K$, for the image of the horoballs under the covering projection is topologically $T$ times half open interval. Thus, for a slope $r$ on $\partial E_K$, we can define the **length** of $r$ with respect to $T$ as the minimal length of the simple closed curves on $T$ corresponding to those on $\partial E_K$ with slope $r$.

The following three results will be used in our proof. All notations as above will be still used.

The next proposition was shown by Agol [4] and Lackenby [10], independently.

**Proposition 2.1** ([4, Theorem 6.2], [10, Theorem 3.1]). With respect to some horotorus, if the length of a slope $r$ on $\partial E_K$ is greater than 6, then the surgered manifold $K(r)$ is irreducible and its fundamental group is infinite word-hyperbolic. □
Now let us choose the maximal horotorus $T$, that is, the one bounding the maximal region with no overlapping interior. The next proposition holds for such $T$, which was given in [2]. See also [1] and [3].

**Proposition 2.2.** With respect to the maximal horotorus, every slope on $\partial E_K$ has the length at least $4\sqrt{2}$ if $K$ is neither the figure-eight knot nor the knot $5_2$ in the knot table [11].

The next one was obtained in [7], which is the key to show that the figure eight knot complement has the minimal volume among orientable 1-cusped hyperbolic 3-manifolds.

**Proposition 2.3 ([7, Proposition 5.8]).** For any hyperbolic knot in a closed orientable $3$-manifold, the area of the maximal horotorus must be at least 3.35.

**Proof of Theorem 1.1.** We first assume that $K$ is the figure eight knot in $S^3$. In this case, exceptional surgeries are completely understood, as noted before, the statement holds. Next, in the case that $K$ is the knot $5_2$ in $S^3$, it is also shown in [6] that the statement also holds.

Now, we consider a hyperbolic knot $K$ in general neither the figure eight knot nor the knot $5_2$. Let $r_1$ and $r_2$ be slopes having distance $\Delta$ such that they are both integers with respect to the fixed meridian-longitude system for $K$ and the surgered manifolds $K(r_1), K(r_2)$ are both non-hyperbolike.

Take the maximal horotorus $T$ in the complement of $K$. Let $\mu$ be the closed geodesic on $T$ corresponding to the fixed meridian on the boundary $\partial E_K$ of the exterior $E_K$ of $K$. Let $\gamma_i$ be the closed geodesic on $T$ corresponding to a simple closed curve with slope $r_i$ on $\partial E_K$ for $i = 1, 2$. Up to translations, we may assume that $\gamma_1, \gamma_2$ and $\mu$ have a common intersection point.

Consider a component $\tilde{T}$ of the preimage of $T$ in the universal cover $\mathbb{H}^3$ of the complement of $K$. Since $T$ has a Euclidean structure, $\tilde{T}$ is identified with the Euclidean 2-plane $\mathbb{E}^2$. On $\tilde{T}$, the preimage of the common intersection point of $\gamma_1, \gamma_2$ and $\mu$ give a lattice. By fixing one of the points, say $O$, each primitive lattice point corresponds to a slope on $T$, and the distance between $O$ and a primitive lattice point is equal to the length of the corresponding slope.

We take lattice points $A$ and $B$ such that the paths $OA$ and $OB$ are lifts of $\gamma_1$ and $\gamma_2$ respectively, and the path $AB$ is projected to $\Delta$ times multiple of $\mu$ on $T$. Note that the latter condition can be achieved by integrality of $r_1$ and $r_2$. Also note that the area of the triangle $OAB$ is just the half of $3.35\Delta$.

Then we have:
- the length $\overline{OA}, \overline{OB}$ of the paths $OA, OB$ is at most 6 by Proposition 2.1
- the length $\overline{AB}$ of the path $AB$ is at least $4\sqrt{2}\Delta$ by Proposition 2.2
- the area $\text{Area}(OAB)$ of the triangle $OAB$ is greater than the half of $3.35\Delta$ by Proposition 2.3

Suppose for a contradiction that $\Delta > 8$. Let $\theta$ be the angle between $OA$ and $OB$ so that $0 < \theta < \pi$. Then by
$$\overline{OA}^2 + \overline{OB}^2 \leq 6^2 + 6^2 = 72 < 64\sqrt{2} < (4\sqrt{2}\Delta)^2 \leq \overline{AB}^2$$
the angle $\theta$ is greater than $\pi/2$. 


Now, by the Euclidean cosine law, we have
\[ \frac{AB^2 - OA^2 + OB^2}{2} = -OA \cdot OB \cos \theta , \]
and so,
\[ (2.1) \quad \frac{\Delta^2}{\sqrt{2}} - 36 < \left( \frac{\sqrt{2} \Delta}{2} - 6^2 - 6^2 \right) < 6 \cdot (\cos \theta) = 36 |\cos \theta| \]
holds.

On the other hand, by the formula of the area of a Euclidean parallelogram, we have
\[ 2 \cdot \text{Area}(OAB) = OA \cdot OB \sin \theta , \]
and so,
\[ (2.2) \quad 3.35 \Delta < 6 \cdot 6 \sin \theta = 36 \sin \theta \]
Combining Equations (2.1) and (2.2), we have
\[ \left( \frac{\Delta^2}{\sqrt{2}} - 36 \right)^2 + (3.35 \Delta)^2 < 36^2 (\sin^2 \theta + \cos^2 \theta) = 1296. \]
From this, we have
\[ \left( \frac{\Delta^2}{\sqrt{2}} - 36 \right)^2 + (3.35 \Delta)^2 - 1296 < 0 \]
\[ \frac{\Delta^4}{2} + (11.2225 - 36 \sqrt{2}) \Delta^2 < 0 \]
\[ \Delta^2 < 2(36 \sqrt{2} - 11.2225) < 80 \]
\[ \Delta < \sqrt{80} < 8.95 . \]
However, \( \Delta \) must be an integer, and so we would have \( \Delta \leq 8 \). This implies a contradiction to the assumption that \( \Delta > 8 \). \( \square \)

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