PHYSICAL MEASURES FOR MOSTLY SECTIONALLY EXPANDING FLOWS

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ABSTRACT. We prove that a partially hyperbolic attracting set for a $C^2$ vector field supports an ergodic physical/SRB measure if, and only if, the trapping region admits non-uniform sectional expansion on a positive Lebesgue measure subset. Moreover, in this case, the attracting set supports at most finitely many ergodic physical/SRB measures, which are also Gibbs states along the central-unstable direction.

This extends to continuous time systems a similar well-known result obtained for diffeomorphisms, encompassing the presence of equilibria accumulated by regular orbits within the attracting set, since it is shown that slow recurrence to hyperbolic saddle-type equilibria is automatic for Lebesgue almost every point. We present several examples of application, including the existence of physical measures for asymptotically sectional hyperbolic attracting sets.

CONTENTS

1. Introduction and statement of results 2
   1.1. Statements of the results 3
   1.2. Mostly asymptotically sectional expansion 5
   1.3. Discrete time versus continuous time 7
   1.4. Nonuniformly sectional hyperbolic flows 10
   1.5. Comments and conjectures 11
   1.6. Organization of the text 13

2. More examples 14
   2.1. Mostly asymptotically sectional expanding and non-sectional hyperbolic 14
   2.2. Mostly asymptotically sectional expanding and not singular-hyperbolic 18
   2.3. Non-uniform weak expansion without slow recurrence nor physical measure 20

3. Auxiliary results 22

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1. Introduction and statement of results

Much of the recent progress in Dynamics is a consequence of a probabilistic approach to the understanding of complicated dynamical systems, where one focuses on the statistical properties of “typical orbits”, in the sense of large volume in the ambient space. We deal here with flows $\phi_t : M \to M$ on compact manifolds. The most basic statistical data are the time averages $T^{-1} \int_0^T \delta_{\phi_t(z)} dt$, where $\delta_w$ represents the Dirac measure at a point $w$. Birkhoff’s Ergodic Theorem asserts that time averages admit asymptotic limits in the weak* topology at almost every point $z$ with respect to any invariant probability $\mu$. That is, for every continuous observable $\psi : M \to \mathbb{R}$ there exists a subset $E \subset M$ of full measure $\mu(E) = 1$ so that $\lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(\phi_t(z)) dt = \tilde{\psi}(z)$ is well-defined for each $z \in E$. Moreover, if the measure is ergodic, then the time average coincides with the space average, that is, $\tilde{\psi}(z) = \int \psi \, d\mu$ for $z \in E$. However, many invariant measures are singular with respect to volume in general, and so the Ergodic Theorem is not enough to understand the behavior of positive volume (Lebesgue measure) sets of orbits.

A physical measure is an invariant probability measure for which time averages exist and coincide with the space average, for a set of initial conditions with positive Lebesgue measure, i.e. in the weak* topology of convergence of probability measures we have

$$B(\mu) := \left\{ z \in M : \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{\phi_t(z)} dt = \mu \right\} \text{ with } \text{Leb}(B(\mu)) > 0.$$  

This set is the basin of the measure. Sinai, Ruelle and Bowen introduced this notion about fifty years ago, and proved that, for uniformly hyperbolic (Axiom A) diffeomorphisms and flows, time averages exist for Lebesgue almost every point and coincide with one of finitely many physical measures; see [29, 65, 66].
The problem of existence and finiteness of physical measures, beyond the Axiom A setting, remains a main goal of Dynamics. The construction of the so called Gibbs \( u \)-states, by Pesin and Sinai in [56], was the beginning of the extension of the Sinai, Ruelle and Bowen ideas to partially hyperbolic systems, a fruitful generalization of the notion of uniform hyperbolicity, which more recently was shown to encompass Lorenz-like and singular-hyperbolic flows [51, 52, 68] and to be a consequence of robust transitivity [24]. We refer the reader to [15, 25] for surveys on much of the progress obtained so far and the recent extensions to higher-dimensional flows [8, 41].

The papers of Alves, Bonatti and Viana [4, 27], and Dolgopyat [33] are of special interest to us here since they prove existence and finiteness of physical measures for partially hyperbolic diffeomorphisms, which are also \( u \)-Gibbs states, under the assumption that the central direction is either “mostly contracting” [27, 33] or “non-uniformly expanding” [4]. For local diffeomorphisms with a non-flat critical or singular set, the authors in [4] show that a slow recurrence condition is sufficient to obtain an absolutely continuous invariant probability measure when the system is non-uniformly expanding.

Here, we extend the results of [4] to attracting sets for smooth vector fields with a dominated splitting and non-uniform sectional expansion on a positive volume subset. In this setting partial hyperbolicity is natural, and we obtain finitely many physical/SRB measures for the flow, which are also \( cu \)-Gibbs states, whose ergodic basins cover the set of non-uniform sectional expanding orbits except for a subset of volume zero.

Known examples satisfying these conditions are, besides hyperbolic (Axiom A) flows [29], all singular-hyperbolic attracting sets for \( C^2 \) smooth flows [15] (including the Lorenz attractor), the contracting Lorenz attractor (also known as the Rovella attractor [64]) and all sectional-hyperbolic attracting sets [8] (including the multidimensional Lorenz attractor [26]).

The properties of continuous time dynamics enable us to show that non-uniform sectional expansion in a necessary and sufficient condition for existence of ergodic physical/SRB measures for partially hyperbolic attracting sets; and that slow recurrence to hyperbolic equilibria of saddle-type always holds for Lebesgue almost every trajectory not converging to a sink. We present several examples of application, including the existence of physical measures for asymptotically sectional hyperbolic attracting sets.

1.1. Statements of the results. Let \( M \) be a compact connected manifold with dimension \( \dim M = m \), endowed with a Riemannian metric, induced distance \( d \) and volume form \( \text{Leb} \). Let \( \mathcal{X}^r(M) \), \( r \geq 1 \), be the set of \( C^r \) vector fields on \( M \) endowed with the \( C^r \) topology and denote by \( \phi_t \) the flow generated by \( G \in \mathcal{X}^r(M) \).

1.1.1. Preliminary definitions. We say that \( \sigma \in M \) with \( G(\sigma) = 0 \) is an equilibrium or singularity. In what follows we denote by \( \text{Sing}(G) \) the family of all such points. We say that an equilibrium \( \sigma \in \text{Sing}(G) \) is hyperbolic if all the eigenvalues of \( DG(\sigma) \) have non-zero real part.

An invariant set \( \Lambda \) for the flow \( \phi_t \), generated by the vector field \( G \), is a subset of \( M \) which satisfies \( \phi_t(\Lambda) = \Lambda \) for all \( t \in \mathbb{R} \). Given a compact invariant set \( \Lambda \) for \( G \in \mathcal{X}^r(M) \), we say
that $\Lambda$ is isolated if there exists an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \text{Closure } \phi_t(U)$. If $U$ can be chosen so that $\text{Closure } \phi_t(U) \subset U$ for all $t > 0$, then we say that $\Lambda$ is an attracting set and $U$ a trapping region (or isolating neighborhood) for $\Lambda = \Lambda_G(U) = \cap_{t>0} \text{Closure } \phi_t(U)$.

An attractor is a transitive attracting set, that is, an attracting set $\Lambda$ with a point $z \in \Lambda$ so that its $\omega$-limit

$$\omega(z) := \left\{ y \in M : \exists t_n \nearrow \infty \text{ s.t. } \phi_{t_n}z \xrightarrow{n \to \infty} y \right\}$$

coincides with $\Lambda$.

1.1.2. Partial hyperbolic attracting sets for vector fields. Let $\Lambda$ be a compact invariant set for $G \in \mathfrak{X}(M)$. We say that $\Lambda$ is partially hyperbolic if the tangent bundle over $\Lambda$ can be written as a continuous $D\phi_t$-invariant Whitney sum $T\Lambda = E^s \oplus E^{cu}$, where $d_s = \dim E^s_x \geq 1$ and $d_{cu} = \dim E^{cu}_x \geq 2$ for $x \in \Lambda$, and there exists a constant $\lambda \in (0, 1)$ such that for all $x \in \Lambda$, $t \geq 0$, we have\footnote{For some choice of the Riemannian metric on the manifold, see e.g. [50]. Changing the metric does not change the rate $\lambda$ but might introduce the multiplication by a constant.}

- domination of the splitting: $\|D\phi_t|E^s_x\| \cdot \|D\phi_{-t}|E^{cu}_x\| \leq \lambda^t$;
- uniform contraction along $E^s$: $\|D\phi_t|E^s_x\| \leq \lambda^t$.

We refer to $E^s$ as the stable bundle and to $E^{cu}$ as the center-unstable bundle.

Lemma 1.1. [17, Lemma 3.2] Let $\Lambda$ be a compact invariant set for $G$.

1. Given a continuous splitting $T\Lambda = E \oplus F$ such that $E$ is uniformly contracted, then $G(x) \in F_x$ for all $x \in \Lambda$.

2. Assuming that $\Lambda$ is non-trivial and has a continuous and dominated splitting $T\Lambda = E \oplus F$ such that $G(x) \in F_x$ for all $x \in \Lambda$, then $E$ is a uniformly contracted subbundle.

A partially hyperbolic attracting set is a partially hyperbolic set that is also an attracting set.

Remark 1.2. In the flow setting, a dominated splitting becomes partially hyperbolic whenever the flow direction is contained in the central-unstable bundle $G \in E^{cu}$, from Lemma 1.1. Thus, this inclusion is equivalent to partial hyperbolicity. Since the flow direction is invariant, then partial hyperbolicity is the natural setting to study invariant sets (which are not composed only of equilibria) for flows with a dominated splitting.

Remark 1.3. We assume without loss of generality that $\|G\| \leq 1$ in what follows, which is equivalent to a time reparametrization of the corresponding flow on a compact manifold.

1.1.3. Singular/sectional-hyperbolicity. The center-unstable bundle $E^{cu}$ is volume expanding if there exists $K, \theta > 0$ such that $|\det(D\phi_t|E^{cu}_x)| \geq Ke^{\theta t}$ for all $x \in \Lambda$, $t \geq 0$.

A point $p \in M$ is periodic for the flow $\phi_t$ generated by $G$ if $G(p) \neq 0$ and there exists $\tau > 0$ so that $\phi_\tau(p) = p$; its orbit $O_G(p) = \phi_x(p) = \phi_{[0,\tau]}(p) = \{\phi_t p : t \in [0, \tau]\}$ is a periodic orbit, an invariant simple closed curve for the flow. An invariant set is nontrivial if it is not a finite collection of periodic orbits and equilibria.
We say that a compact nontrivial invariant set \( \Lambda \) is a \textit{singular hyperbolic set} if all equilibria in \( \Lambda \) are hyperbolic, and \( \Lambda \) is partially hyperbolic with volume expanding center-unstable bundle. A singular hyperbolic set which is also an attracting set is called a \textit{singular hyperbolic attracting set}.

We say that \( E^{cu} \) is \textit{(2-)sectionally expanding} if there are positive constants \( K, \theta \) such that for every \( x \in \Lambda \) and every 2-dimensional linear subspace \( L_x \subset F_x \) one has \( |\det(D\phi_t|_{L_x})| \geq K e^{\theta t} \) for all \( t \geq 0 \). A \textit{sectional-hyperbolic (attracting) set} is a partially hyperbolic (attracting) set whose central subbundle is sectionally expanding.

1.1.4. \textit{Asymptotical sectional-hyperbolicity}. A compact invariant partially hyperbolic set \( \Lambda \) of a vector field \( G \) whose equilibria are hyperbolic, is \textit{asymptotically sectional-hyperbolic} if the center-unstable subbundle is eventually asymptotically sectional expanding outside the stable manifold of the equilibria. That is,

\[
\limsup_{T \to \infty} \log |\det(D\phi_T|_{F_x})|^{1/T} \geq c_0 > 0
\]  

for every \( x \in \Lambda \setminus \cup \{W^s_\sigma : \sigma \in \text{Sing}_\Lambda(G)\} \) and each 2-dimensional linear subspace \( F_x \) of \( E^{cu}_x \), where we write \( \text{Sing}_\Lambda(G) = \text{Sing}(G) \cap \Lambda \) and \( W^s_\sigma = \{x \in M : \lim_{t \to +\infty} \phi_t x = \sigma\} \) is the \textit{stable manifold} of the hyperbolic equilibrium \( \sigma \). It is well-known that \( W^s_\sigma \) is a immersed submanifold of \( M \); see e.g. [54].

\textbf{Lemma 1.4} (Hyperbolic Lemma). \textit{Every compact invariant subset \( \Gamma \) without equilibria contained in a (asymptotically) sectional-hyperbolic set is uniformly hyperbolic.}

\textit{Proof.} See e.g. [52] Proposition 1.8 for sectional-hyperbolic sets; and [45] Theorem 2.2 for the asymptotically sectional-hyperbolic case. \hfill \Box

1.2. \textit{Mostly asymptotically sectional expansion}. Let us fix \( G \in \mathcal{X}^2(M) \) endowed with a partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \) with a trapping region \( U \). Then we can take a continuous extension \( T_U M = \mathcal{E} \oplus \mathcal{E}^{cu} \) of \( T_\Lambda M = E^s \oplus E^{cu} \) and for small \( a > 0 \) find center unstable and stable cones

\[
C^a(x) = \{v = v^s + v^c : v^s \in \mathcal{E}^s_x, v^c \in \mathcal{E}^{cu}_x, x \in U, \|v^s\| \leq a\|v^c\|\}, \quad \text{and} \quad C^s_a(x) = \{v = v^s + v^c : v^s \in \mathcal{E}^s_x, v^c \in \mathcal{E}^{cu}_x, x \in U, \|v^s\| \leq a\|v^c\|\},
\]

which are invariant in the following sense

\[
D\phi_t(x) \cdot C^a(x) \subset C^a(\phi_t(x)) \quad \text{and} \quad D\phi_{-t} \cdot C^a(x) \supset C^a(\phi_{-t}(x)),
\]

for all \( x \in \Lambda \) and \( t > 0 \) so that \( \phi_{-s}(x) \in U \) for all \( 0 < s \leq t \); see Subsection 3.1.1.

We can assume, without loss of generality, that the continuous extension of the stable direction \( E^s \) of the splitting is still \( D\phi_t \)-invariant. In what follows, we keep the notation \( T_U M = E^s \oplus E^{cu} \) and write \( N^a_x = N^{cu}_x \cap G^+_x, x \in U \).

A partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \) for a vector field \( G \) is \textit{mostly asymptotically sectional expanding} if the flow is \textit{asymptotically sectional expanding} on a positive Lebesgue measure subset, i.e., there exists \( \Omega \subset U \) with \( \text{Leb}(\Omega) > 0 \) and \( c_0 > 0 \) such that

\[
\limsup_{T \to \infty} \log \|\lambda^2 (D\phi_T|_{E^u})^{-1}\|^{1/T} \leq -c_0, \quad x \in \Omega.
\]
Remark 1.5. The compactness of the Grassmanian of 2-subspaces of $E^c_u$ ensures that \(^1\) at every $x \in U \setminus \cup \{ W^s_\sigma : \sigma \in \text{Sing}_\Lambda(G) \}$ see e.g. [20].

It is easy to see that this notion does not depend on the particular continuous extension of $E^c_u$ to $U$ chosen before, due to the domination of the splitting; see Proposition 3.3.

Theorem A. Let a partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ for a vector field $G \in \mathcal{X}(M)$ be given. Then $\Lambda$ is mostly asymptotically sectionally expanding on a positive volume subset $\Omega \subset U$ if, and only if, there exists an ergodic hyperbolic physical/SRB measure $\mu$, which is also a cu-Gibbs state with $\text{Leb}(B(\mu) \cap \Omega) > 0$.

Here hyperbolicity of a probability measure means non-uniform hyperbolicity. That is, the tangent bundle over $\Lambda$ splits into a sum $T_x M = E^s_x \oplus \mathbb{R} \cdot G_x \oplus F_x$ of invariant subspaces defined for $\mu$-a.e. $x \in \Lambda$ and depending measurably on the base point $z$, where $E^s_x$ is any physical/SRB measure in the statement of Theorem A; $\mathbb{R} \cdot G_z$ is the flow direction (with zero Lyapunov exponent); $\mathbb{R} \cdot G_z \oplus F_z = E^c_u$ and $F_z$ is the direction with positive Lyapunov exponents, that is $\lim_{t \to +\infty} \frac{1}{t} \log \| (D\phi_t |_{F_z})^{-1} \| < 0$.

1.2.1. Physical/SRB measures and cu-Gibbs states. In the uniformly hyperbolic setting, it is well known that physical measures $\mu$, for hyperbolic attractors of $C^2$ diffeomorphisms $g$, admit a disintegration into conditional measures along the unstable manifolds of almost every point which are absolutely continuous with respect to the induced Lebesgue measure on these submanifolds, see [28, 29, 56, 69]. By Ledrappier-Young characterization of measures satisfying (Pesin’s) Entropy Formula [40], this is equivalent to

$$h_\mu(g) = \int \chi^+ d\mu = \int \log | Dg|_{E^u} | d\mu > 0,$$

where $E^u$ is the unstable invariant subbundle over the hyperbolic attractor, and $\chi^+ = \sum \lambda^+(z) \cdot \dim E^s(z)$ is the sum of positive Lyapunov exponents with multiplicities. In the hyperbolic setting for diffeomorphisms, condition $\chi^+$ means that $\mu$ is a $u$-Gibbs state. These measures are known as Sinai-Ruelle-Bowen (SRB) measures.

In our setting, existence of unstable manifolds is guaranteed by the hyperbolicity of physical measures: the strong-unstable manifolds $W^s_z$ are the “integral manifolds” tangent to $F_z$ defined by $W^s_z = \{ y \in M : \lim_{t \to -\infty} d(\phi_{-t}(y), \phi_{-t}(z)) = 0 \}$ and exist for $\mu$-a.e. $x$. The weak-unstable manifolds $W^c_u$ are embedded sub-manifolds in a neighborhood of $z$ which, in general, depend only measurably (including its size) on the base point $z \in \Lambda$. We note that, since $\Lambda$ is an attracting set, then $W^c_u \subset \Lambda$ where defined\(^2\).

\(^2\)Since the stable manifold of a hyperbolic critical element in a immersed submanifold [54], then it has zero volume as a subset of the ambient manifold, and so the condition $x \in \Omega \setminus \cup \{ W^s_\sigma : \sigma \in \text{Sing}_\Lambda(G) \}$ becomes superfluous, unless there are sinks among the equilibria of $\Lambda$ — which is indeed possible for asymptotic sectional-hyperbolic non-transitive attracting sets; see e.g. [44, Remark 4].

\(^3\)For if $y \in W^s_z \cap U$ and $z \in \Lambda$, then $d(\phi_{-t}(y), \phi_{-t}(z)) \to 0$ for $t \nearrow \infty$. Thus $\phi_{-t} y \subset U$ for all $t \geq 0$, that is, $y \in \cap_{t \geq 0} \phi_{-t}(U) = \Lambda$. 
The arguments of our proofs, adapted from [4], enable us to obtain not only ergodic hyperbolic physical invariant probability measures, but also the condition corresponding to (5) in the flow setting

\[ h_\mu(\phi_1) = \int \chi^+ \, d\mu = \int \log |\det D\phi_1|_{E^{cu}} \, d\mu > 0, \]

that is, the physical measures are \textit{cu-Gibbs states}.

**Example 1** (Hyperbolic examples). Anosov flows and hyperbolic attractors (attracting basic pieces of the spectral decomposition of Smale) for smooth vector fields admit a physical/SRB probability measure whose basin covers the trapping region except a zero volume subset [38], and are mostly asymptotically sectional expanding, with no equilibria.

In fact, the central-unstable bundle splits \( E^{cu}_\Lambda = \mathbb{R} \cdot G \oplus E^u \) into a pair of continuous subbundles: the direction of the flow and an unstable bundle \( E^u \) (uniformly contracting in negative time).

Using an adapted metric, we can assume without loss of generality that \( T_{\Lambda}M = E^s \oplus \mathbb{R} \cdot G \oplus E^u \) is an orthogonal invariant splitting; see e.g. [36]. Then, for all \( x \in \Lambda \) and also in \( U \), the backward contraction on \( E^u \) ensures that there exists \( \lambda \in (0,1) \) so that, for any bivector \( u \wedge v \) with \( u, v \in E^u_x \), we may assume without loss of generality that \( \langle u, v \rangle = 0 \) and obtain

\[ \| \wedge^2 D\phi_{-t} \cdot (u \wedge v) \| \leq \| D\phi_{-t}^{-1} u \| \cdot \| D\phi_{-t}^{-1} v \| \leq \lambda^{2t} \| u \| \cdot \| v \| = \lambda^{2t} \| u \wedge v \|. \]

If we instead consider a bivector \( G_x \wedge v \) for \( v \in E^u_x \) we obtain

\[ \| \wedge^2 D\phi_{-t} \cdot (G_x \wedge v) \| \leq \lambda^t \| v \| \cdot \| G_{\phi_{-t}^{-1} x} \| / \| G_x \| \leq K \lambda^t \| G_x \wedge v \| \]

since \( \| G_x \| \) is bounded above and also bounded away from zero on \( M \). We thus obtain (4) for all \( x \in M \).

**Example 2** (Singular-hyperbolic attracting sets). We recall that all singular-hyperbolic attracting sets, as the (geometric) Lorenz attractor [15], for \( C^2 \) smooth flows admit finitely many (one only if transitive) physical/SRB probability measures, which are \textit{cu-Gibbs states} and whose basins cover Leb-a.e. point of the trapping region of these attracting sets; see e.g. [8]. These are clearly partially hyperbolic and mostly asymptotically sectional expanding attracting sets.

**Example 3** (Sectional-hyperbolic attracting sets). The multidimensional Lorenz attractor [26] is a sectional-hyperbolic attractor with a generalized Lorenz-like equilibrium. Sectional-expansion on an attracting set naturally implies sectional expansion on a trapping neighborhood \( U \) which, in turn, clearly ensures mostly asymptotically sectional expansion. Our results provide an alternative proof of existence of an ergodic physical/SRB measure for this family of attractors, complementing the arguments given in [26] and the proof presented in [41] for the general smooth sectional-hyperbolic attractor (which was extended in [8] for smooth sectional-hyperbolic attracting sets).

1.3. **Discrete time versus continuous time.** To construct the physical probability measure in the presence of equilibria for a mostly asymptotically sectional expanding partially hyperbolic attracting set, we need to control the recurrence near the equilibria.
A partially hyperbolic attracting set $\Lambda = \Lambda_G(U)$ which is mostly asymptotically sectional expanding, whose equilibria are hyperbolic for a vector field $G$, has continuous slow recurrence to equilibria if on the positive Lebesgue measure subset $\Omega \subset U$, for every $\varepsilon > 0$, we can find $\delta > 0$ so that

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T -\log d_\delta(\phi_t(x), \text{Sing}_\Lambda(G)) \, dt < \varepsilon, \quad x \in \Omega, \quad (7)$$

where $d_\delta(x, S)$ $\delta$-truncated distance from $x \in M$ to a subset $S$, that is

$$d_\delta(x, S) = \begin{cases} 
    d(x, S) & \text{if } 0 < d(x, S) \leq \delta; \\
    (1 - \frac{\delta}{\delta}) d(x, S) + 2\delta - 1 & \text{if } \delta < d(x, S) < 2\delta; \\
    1 & \text{if } d(x, S) \geq 2\delta.
\end{cases}$$

The slow recurrence to equilibria is a natural consequence of $\mu$-integrability of the function $|\log d_\delta(x, \text{Sing}_\Lambda(G))|$ whenever $\mu$ is a physical measure; see e.g. the comments in [5]. However, either this integrability property, or condition (7), are hard to obtain for transformations; see e.g. [7, 18] for a setting where this was deduced from global properties of the transformation.

**Example 4** (Rovella attractors). Another class of partially hyperbolic and mostly asymptotically sectional expanding attracting sets are the Rovella attractors [64] (also known as contracting Lorenz attractors) which are asymptotically singular-hyperbolic; see [45]. These attractors admit a physical/SRB probability measure whose basin covers the trapping region except a zero volume subset and, moreover, exhibit slow recurrence to the equilibrium at the origin\(^4\); see [47].

We can formulate these notions and results in terms of the time-1 map of the flow, or some other fixed time after a time reparametrization, which corresponds to a time discretization of the dynamics. This enables us to relate the continuous time notions and statements to equivalent notions for diffeomorphisms presented in [4]. We need the following preliminary concept.

1.3.1. **Linear Poincaré Flow.** If $x$ is a regular point of the vector field $G$ (i.e. $G(x) \neq \vec{0}$), denote by $N_x = \{v \in T_xM : \langle v, G(x) \rangle = 0\}$ the orthogonal complement of $G(x)$ in $T_xM$. Denote by $O_x : T_xM \to N_x$ the orthogonal projection of $T_xM$ onto $N_x$. For every $t \in \mathbb{R}$ define, see Figure [1]

$$P^t_x : N_x \to N_{\phi tx} \quad by \quad P^t_x = O_{\phi tx} \circ D\phi_t(x).$$

It is easy to see that $P = \{P^t_x : t \in \mathbb{R}, G(x) \neq 0\}$ satisfies the cocycle relation $P^{s+t}_x = P^t_\phi x \circ P^s_x$ for every $t, s \in \mathbb{R}$. The family $P = P_G$ is called the Linear Poincaré Flow of $G$.

---

\(^4\)The known proof of asymptotical sectional expansion for this family of attractors intertwines with the control of recurrence to the equilibrium point.
1.3.2. Asymptotical sectional expansion on average. We write \( f = \phi_t \) for the time-one map of the flow of the vector field \( G \). A partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \) for a vector field \( G \) is non-uniformly sectional expanding if the time-one map of the Linear Poincaré flow along the central unstable direction asymptotically expands on average on a positive Lebesgue measure subset: there are \( \Omega \subset U \) with \( \text{Leb}(\Omega) > 0 \) and \( c_0 > 0 \) so that

\[
\limsup_{n \to \infty} \sum_{i=0}^{n-1} \log \left( \left( P^i \big|_{N^{\text{cu}}_{P^i_x}} \right)^{-1} \right)^{1/n} \leq -c_0 < 0, \quad x \in \Omega.
\]  

(8)

Again, the above notion does not depend on the particular continuous extension of \( E^{\text{cu}}_\Lambda \) to \( U \) chosen before, due to the domination of the splitting; see Proposition 3.3.

A partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \) which is non-uniform sectional expanding, and whose equilibria are hyperbolic for a vector field \( G \), has slow recurrence to equilibria if on the positive Lebesgue measure subset \( \Omega \subset U \), for every \( \varepsilon > 0 \), we can find \( \delta > 0 \) so that

\[
\limsup_{n \to \infty} \sum_{i=0}^{n-1} - \log d_{\delta}(f^i(x), \text{Sing}_\Lambda(G))^{1/n} < \varepsilon, \quad x \in \Omega,
\]  

(9)

Theorem B (Equivalence between discrete and continuous time versions). Let \( G \in \mathcal{X}^2(M) \) be given admitting a partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \). Then

1. the slow recurrence condition \((9)\) holds for \( x \in U \) if, and only if, continuous slow recurrence \((7)\) holds for \( x \); and both hold automatically in \( U \setminus \{W^s : \sigma \in \text{Sing}_\Lambda(G)\} \).

2. the following pair of conditions are equivalent:

   (A) there exists a hyperbolic physical/SRB invariant probability measure for the flow, which is a cu-Gibbs state with \( \text{supp} \mu \subset \Lambda \);

   (B) there exists \( T > 0 \) and \( c_0 > 0 \) and a positive volume subset \( E \subset \Omega \) so that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left( \left( P^T \big|_{N^{\text{cu}}_{P^i_x}} \right)^{-1} \right)^{1/T} < -c_0 < 0, \quad x \in E.
\]  

(10)

3. if either condition of item (2) is met, then

   (a) mostly asymptotically sectional expansion \((4)\) holds Leb-a.e. in \( E \); and

   (b) if, additionally, \( \Lambda \) is transitive, then there exists one ergodic physical/SRB measure such that \( \text{Leb}(B(\mu) \setminus \Omega) = 0 \).
This means that slow recurrence automatically holds on \( \Omega \cup \{ W^s_\sigma : \sigma \in \text{Sing}_\Lambda(G) \} \). Moreover, we reobtain (8) by either reparametrizing the flow \((\phi_t)_{t \in \mathbb{R}}\) to \((\phi_{Tt})_{t \in \mathbb{R}}\) (or, equivalently, replacing \(G\) by a multiple \(T \cdot G\)) whenever we have hyperbolic physical/SRB measures.

Remark 1.6. Contrasting to the general slow recurrence result of Theorem B(1), Pedreira and Pinheiro [55] show that for the Lorenz interval map there are invariant probability measures with fast recurrence, positive entropy and infinite positive Lyapunov exponent, which correspond to invariant measures for the flow with infinite mass.

Using the previous main results we can deduce the following extension of the main statement from [46].

Corollary C. Let a \(C^2\) vector field \(G\) on \(M\) and a trapping region \(U\) containing an asymptotically sectional-hyperbolic attracting set \(\Lambda = \Lambda G(U)\) be given. If \(\Lambda\) is transitive, then \(\Lambda\) supports a unique physical/SRB probability measure. If \(\Lambda\) contains only saddle-type hyperbolic equilibria, then \(\Lambda\) supports finitely many ergodic physical/SRB measures whose basins cover \(U\) except a zero volume measure subset.

1.4. Nonuniformly sectional hyperbolic flows. The assumption (4) is close to the notion of non-uniform sectional hyperbolicity on critical elements (equilibria and periodic orbits) defined by Arbieto-Salgado [21, Definition 2.5 & Remark 2.6] to obtain sectional hyperbolicity for the non-wandering set on a \(C^1\) residual subset of vector fields, among those with non-uniform sectional hyperbolic critical elements.

More precisely, let us assume that the attracting set \(\Lambda\) admits a continuous invariant splitting \(E^{cs} \oplus E^{cu}\) for the flow of \(G\). We say that the positive trajectory \((\phi_t(x))_{t \geq 0}\) of \(x\) is non-uniformly hyperbolic if there exists a positive number \(\omega\) so that

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \log \| D\phi_1 \|_{E^{cs}_{\phi_t(x)}} \| dt < -\omega, \quad \text{and} \quad (11)
\]

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \log \| \Lambda^2 (D\phi_1 \|_{E^{cu}_{\phi_t(x)}})^{-1} \| dt < -\omega. \quad (12)
\]

These conditions on a total probability set (i.e. every \(x \in E\) satisfy both (11) and (12) with the same \(\omega\) and \(\mu(E) = 1\) for each invariant probability measure \(\mu\) supported in \(\Lambda\)) ensure sectional-hyperbolicity. Moreover, a strong form of the property (12) is enough to get weak asymptotic sectional expansion on average, as follows.

Theorem 1.7. If a compact invariant subset \(\Lambda\) admits an invariant continuous splitting \(T\Lambda M = E^{cs} \oplus E^{cu}\) and there exists a total probability subset \(E\) of \(U\) which is non-uniformly hyperbolic, then the splitting is sectional-hyperbolic.

Moreover, if the partially hyperbolic attracting set \(\Lambda = \Lambda_G(U)\) admits \(\omega > 0\) and a positive volume subset of points of \(U\) satisfying (12), then there exists a positive volume subset of points of \(U\) satisfying weak asymptotical sectional expansion on average

\[
\liminf_{n \to \infty} \sum_{i=0}^{n-1} \log \left( \| (P^1 \|_{N^{cu}_{\phi_i(x)}})^{-1} \|^{1/n} \right) < -c_0 < 0, \quad x \in \Omega. \quad (13)
\]
In addition, if points in this positive Lebesgue measure subset satisfy
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \log \| A^2 (D\phi_1 |_{E^{cu}_{\phi_1, t}})^{-1} \| \, dt < -\omega,
\]
then there exists a positive volume subset of points of \( U \) satisfying asymptotic sectional expansion on average \([8]\).

This, together with Theorem \([3]\), enables us to restate Theorem \([1]\) with the assumption \((14)\) in the place of \((4)\).

1.5. Comments and conjectures. Recently, Crovisier et al. \([32]\) obtained physical measures for \( C^1\)-generic \( C^\infty \) multisingular vector fields. However this class of results fails to take into account non-transitive attracting singular sets as well as Rovella-like attractors, which are encompassed by our main statements.

The proof of Theorem \([1]\) relies on reducing to non-uniform sectional expansion and slow recurrence to the time-1 map, in order to apply the following.

**Theorem D.** A non-uniform sectional expanding partially hyperbolic attracting set \( \Lambda = \Lambda_G(U) \) with slow recurrence to the equilibria on \( \Omega \subset U \), with \( \text{Leb}(\Omega) > 0 \), for a vector field \( G \in \mathfrak{X}^2(M) \) admits finitely many ergodic physical/SRB measures, which are cu-Gibbs states, and whose basins cover \( \text{Leb}\)-a.e. point of \( \Omega \).

The proof of Theorem \([1]\) relies on carefully constructing hyperbolic times and pre-disks\(^5\) needed to apply the main technical results from \([1]\).

We adapt the main ideas from \([1]\) to allow for the neutral (non-expanding nor contracting) flow direction along the center-unstable subbundle. The goal is to build, for each hyperbolic time of a given trajectory, a center-unstable disk of uniform inner radius which is uniformly backward contracted along the sectional center-unstable directions, and deduce a distortion bound to control the density of push-forwards of Lebesgue measure along these disks.

This reliance on discrete dynamics, through the reduction to asymptotic properties of iterations of the time-1 map \( f = \phi_1 \), suggests the classical alternative of reducing the flow dynamics to a global Poincaré return map to a well-chosen finite collection of cross-sections, as done in \([15, 18, 19, 47]\) to obtain physical/SRB measures for contracting Lorenz attractors and sectional-hyperbolic attractors. This strategy however is technically challenging:

1. non-uniform sectional expanding partial hyperbolic attractors may admit different hyperbolic non-Lorenz-like equilibria, especially with \( d_{cu} > 2 \), while sectional-hyperbolic or contracting Lorenz attractors admit only a well-controlled family of (generalized) Lorenz-like equilibria;
2. continuous slow recurrence or slow recurrence for the time-1 map do not play a role in the construction of physical measures for sectional-hyperbolic or contracting Lorenz attractors because, in this special partial hyperbolic setting, the dynamics of the global Poincaré map can be further reduced to a one-dimensional quotient map over the stable foliation, which demands that

\(^5\)See Sections \([4, 5]\) together with e.g. \([2, \text{Chap. 7}]\) and compare \([1]\) for many more details.
• $d_{cu} = 2$ in order to obtain the one-dimensional quotient;
• holonomies along the stable foliation exhibit some smoothness on the trapping region, to ensure that the quotient transformation is at least piecewise Hölder-$C^1$.

Circumventing the problems posed by the previous items in the non-uniform sectional expanding partially hyperbolic setting, especially with $d_{cu} > 2$, with similar strategies to e.g. [19, 47] may be impossible, since higher dimensional invariant foliations are in general only Hölder-continuous [37, 63]. However, we can adapt the construction from [4] to the vector field setting, as presented in this text, and also show that slow recurrence is automatic for trajectories not converging to equilibria.

1.5.1. Extensions of the results. On the one hand, using the same techniques from [50], we may replace the domination condition by Hölder continuity of the splitting over $\Lambda$, keeping the conclusions of the main theorems.

On the other hand, from Remark [1.2] we can replace the partial hyperbolic assumption on $\Lambda$ by the assumption that $\Lambda$ be a non-trivial attracting set with a dominated splitting $T_{\Lambda}M = E^s \oplus E^{cu}$, such that the vector field $G$ is contained in $E^{cu}$, and keep the same conclusions of the main results. More precisely, we obtain the following statements with small adaptations of our arguments.

**Theorem 1.8.** Let $\Lambda = \Lambda_G(U)$ be an attracting set for a vector field $G \in \mathcal{X}^2(M)$ admitting an invariant splitting $T_{\Lambda} = E^s \oplus E^{cu}$ such that

- either the splitting is Hölder-continuous and $E^s$ is uniformly contracted;
- or the splitting is dominated and the flow direction is contained in $E^{cu}$.

Then $\Lambda$ is mostly asymptotically sectional expanding on a positive volume subset $\Omega$ if, and only if, there exists an ergodic hyperbolic physical/SRB measure $\mu$, which is a $cu$-Gibbs state with $\text{Leb}(B(\mu) \cap \Omega) > 0$.

It follows from the proof of the main theorems that, if the non-uniform sectional expansion condition (8) is not bounded away from zero, then the conclusion of the main theorems remains with an at most denumerable family of ergodic physical/SRB measures.

It is natural to consider the weaker non-uniform sectional expansion condition (13) obtained by using $\liminf$ in (8); or the weak asymptotic sectional expansion condition

$$\liminf_{T \to \infty} \log |(\Lambda^2 D\phi_T |_{E^{cu}})^{-1}|^{1/T} < 0.$$ (15)

Analogously to Theorem 1.7, if, in the setting of Theorem A, we have weak asymptotic sectional expansion on total probability, that is, the relation (15) for all $x$ on a total probability subset $\Omega \subset \Lambda$, then $\Lambda$ becomes a sectional-hyperbolic attracting set and the conclusion of Theorem A still holds true by the result from [8].

The analogous condition for (local) diffeomorphisms was shown by Alves, Dias, Luzzatto and Pinheiro [1, 58] to be enough to obtain the same conclusions of the main results from [4].

**Conjecture 1.** In the same setting of Theorem [8], either the weak asymptotical sectional expanding on average condition (13), or the weak asymptotical sectional expansion...
condition (15) is enough to obtain the same conclusion on existence and finiteness of physical/SRB measures.

As for the number of distinct ergodic physical/SRB measures supported in the attracting set, motivated by the recent result [10] we propose the following.

**Conjecture 2.** In the setting of Theorem A, the number $s$ of ergodic physical/SRB measures supported in the attracting set satisfies $s \leq 2 \cdot s_L$, where $s_L$ is the number of generalized Lorenz-like equilibria contained in $\Lambda$.

The natural path after obtaining a physical/SRB measure is to study its statistical properties. Motivated by what has already been achieved for singular-hyperbolic attracting sets [11, 13, 14, 19]; for sectional-hyperbolic attracting sets [8, 49]; for contracting Lorenz attractors [48]; and also in the discrete time case [2, 5, 6, 59], we propose the following.

**Conjecture 3.** Given a non-uniformly sectional expanding partial hyperbolic attracting set $\Lambda_G(U)$ with hyperbolic singularities for a $C^2$ vector field $G$, then

- modulo an arbitrary small perturbation of the norm of $G$, the field is topologically equivalent to a $C^2$ nearby vector field so that each ergodic physical/SRB is exponential mixing with respect to smooth observables; and
- both the flow $\phi_t$ of $G$ and its time-1 map $f = \phi_1$ satisfy the Central Limit Theorem, the Law of the Iterated Logarithm and the Almost Sure Invariance Principle (for a comprehensive list, see e.g. [57]) with respect to $\mu$.

Moreover, motivated by [9, 18, 49], the physical/SRB measures supported on $\Lambda$ should be statistically stable and also stochastically stable.

**Remark 1.9.** Recently, Bruin-Farias [30] (see Example 8 in Section 2), polynomial speed of mixing was proved for a neutral geometrical Lorenz-like attractor, so the assumption of hyperbolic equilibria should be necessary in Conjecture 3.

### 1.6. Organization of the text.

In Section 2 we present mostly asymptotically sectional expanding examples which are either non-sectional hyperbolic or non-singular hyperbolic, with or without hyperbolic equilibria, as well as counter-examples failing some of our assumptions and having no physical measure.

In Section 3 we present preliminary results that are needed as tools for the overall construction. In Section 4 we start the proof of Theorem D using the domination of the splitting to obtain bounded distortion on u-disks at hyperbolic times. Using this tool, in Section 5 we study the push-forward of Lebesgue measure at hyperbolic times along u-disks, obtaining the main tool to construct physical/SRB measures which are cu-Gibbs states. We then provide an overview of the construction of physical/SRB measures, citing the relevant results from Alves, Bonatti and Viana [4], and complete the proof of Theorem D.

In Section 6 we prove Theorem B and through this we obtain Theorem A assuming the statements of Theorem D. We also prove Theorem 1.7 and deduce Corollary C in this section.
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2. More examples

Here, we present mostly asymptotically sectional expanding examples which are either non-sectional hyperbolic or non-singular hyperbolic, with or without hyperbolic equilibria; as well as counter-examples failing some of our assumptions and having no physical measure.

2.1. Mostly asymptotically sectional expanding and non-sectional hyperbolic.

Example 5 (Mostly asymptotically sectional expanding, singular-hyperbolic and not sectional-hyperbolic, with no equilibria). We consider the hyperbolic (Anosov) automorphism $f_0$ of the 3-torus $\mathbb{T} = (\mathbb{S}^1)^3$ induced by the linear map defined by

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

with $\text{sp}(A) = \{\lambda_3 \approx 0.198062 < 1 < \lambda_2 \approx 1.55496 < \lambda_1 \approx 3.24698\}$.

Let $p$ be the fixed point at the class of the origin $(0,0,0) \in \mathbb{R}^3$ and a small neighborhood $V$ of $p$ with a choice of basis $\{v_1, v_2, v_3\}$ where $v_i$ is a unit eigenvector corresponding to the eigenvalue $\lambda_i$, $i = 1, 2, 3$. In $V$ the map $f_0$ has the expression $(x, y, z) \mapsto (\lambda_1 x, \lambda_2 y, \lambda_3 z)$.

We consider the one-parameter family of maps of the real line $f_\mu(x) = \psi(x)\lambda_2 x + (1-\psi(x))(1-\mu)\lambda_3 x + \mu \cdot h(x)$

where $0 \leq \mu \leq 1$, $\psi : \mathbb{R} \rightarrow [0, 1]$ is a $C^\infty$ bump function so that for some small $0 < b < a < 1$

- $\text{supp} \psi \subset \mathbb{R} \setminus [-1+b, 1-b]$ and $\psi(x) = 1$, $\forall x : |x| \geq 1+b$; see the left hand side of Figure 2.
- $h(x) = (1-b)x(1+x^2(x^2-a^2))$ so that
  - $h$ has 3 fixed points at 0, $\pm \xi$ with $a < \xi < 1$; and
  - $h'(0) = 1-b < 1$ and $h'(\pm \xi) = (1-b)(1 \pm \xi(4\xi^2 - 2a^2)) > 1$,

which holds if $b > 0$ is small enough; see the right hand side of Figure 2. Moreover, we can also assume that

$$\lambda_3 < f'_\mu(x) < \lambda_1, \quad x \in \mathbb{R} \quad \text{and} \quad \mu \in [0, 1].$$

In addition, since $f'_\mu(0) = \mu h'(0) = \mu(1-b)$ is the minimum of $f'_\mu(x)$, then

$$\lambda_1 + f'_\mu(x) > 1, \quad x \in \mathbb{R}.$$  

We replace the second coordinate map $y \mapsto \lambda_2 y$ by the one-parameter family $y \mapsto \varepsilon_1 \cdot f_\mu(y/\varepsilon_1)$ for $\varepsilon_1 > 0$ small enough so that the ball of radius $\varepsilon_1(1+b)$ around $p$ is
PHYSICAL MEASURES FOR FLOWS

15

contained in $V$, and the properties stated above are preserved at corresponding points after scaling.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The graph of a continuous bump function on the left hand side together with $y = x$ and, on the right hand side, graphs of the maps $y = x$ and $y = f_\mu(x)$ for $\mu = 1$ and $\mu$ close to 1.}
\end{figure}

For $\mu = 0$ we have the original map $f_0$. For $\mu = 1$ we have a map $f_1$ coinciding with $f_0$ outside of $V$ and having inside $V$ three fixed hyperbolic saddle points: $p$ with index 2; and $p_\pm$ with index 1, symmetrically placed with respect to $p$ along the line segment $[-\varepsilon_1(1+b), \varepsilon_1(1+b)]v_2$ inside $V$; see Figure 3

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Depiction of the eigenvalue directions at the origin, on the left hand side; and, on the right hand side, the dynamical behavior of the (un)stable directions in a neighborhood of the origin for $f_1$.}
\end{figure}

We note that $f_1$ has a partially hyperbolic splitting $E^s \oplus E^{cu}$ defined on all of $\mathbb{T}$ which is volume hyperbolic:

- $E^s$ coincides with the stable bundle of $f_0$ spanned everywhere by $v_3$ and is uniformly contracted $\|Df_1|_{E^s}\| = \lambda_3$;
- $E^{cu}$ coincides with the unstable bundle of $f_0$ spanned everywhere by $\{v_1, v_2\}$, domination of the splitting is a consequence of (16); and
- $\det(Df_1|_{E^{cu}}) > 1$ as a consequence of (17).

The invariant bundle $E^{cu}$ further decomposes into the continuous splitting $E^c \oplus E^u$, where $E^u$ is spanned everywhere by $v_1$ and uniformly expanded: $\|Df|_{E^u}\| = \lambda_1$; and $E^c$ is spanned everywhere by $v_2$ and dominated by $E^u$. We claim that $f_1$ is non-uniformly expanding,
that is
\[
\limsup_{n \to \infty} \sum_{i=0}^{n-1} \log \| (Df_1 |_{E^{cu}_{f_1}})^{-1} \|^{1/n} < 0, \quad \text{Leb-a.e. } x \in \mathbb{T}.
\] (18)

But \( f = f_1 \) is a perturbation of the Anosov automorphism \( f_0 \) on the 3-torus around the fixed point satisfying all the conditions stated in [4, Appendix], namely:

1. \( f \) admits invariant cone fields \( C^{cu} \) and \( C^{cs} \), with small width containing, respectively, the unstable and stable bundle of the Anosov diffeomorphism \( f_0 \);
2. there are \( 0 < \sigma_2 < 1 < \sigma_1 \) and \( \delta_0 > 0 \) so that for disks \( D^{cu} \) and \( D^{cs} \) through \( x \) tangent, respectively, to the centre-unstable cone field \( C^{cu} \) and to centre-stable cone field \( C^{cs} \), we have
   \begin{align*}
   (a) & \quad \min\{| \det(Df |_{T_xD^{cu}}) | \} > \sigma_1, \quad \text{for } x \in M; \\
   (b) & \quad \max\{| (Df |_{T_xD^{cu}})^{-1} \|, | (Df |_{T_xD^{cs}})] \| < \sigma_2, \quad \text{for } x \in M \setminus V; \\
   (c) & \quad \max\{| (Df |_{T_xD^{cu}})^{-1} \|, | (Df |_{T_xD^{cs}})] \| < (1 + \delta_0), \quad \text{for } x \in V.
   \end{align*}

Then it follows that \( f_1 \) satisfies (18); see e.g. [4, Appendix] or [2, Section 7.6].

We observe that the flow becomes singular-hyperbolic but not sectional-hyperbolic: the splitting \( T\mathbb{T} = F^s \oplus (F^c \oplus R \cdot G \oplus F^u) \) where \( F^s, F^c, F^u \) are respectively spanned by \( v_1, v_2, v_3 \) everywhere on \( \mathbb{T} \) is such that \( F^{cu} = F^c \oplus R \cdot G \oplus F^u \) is volume expanding, since the action of the flow \( \phi_t \) of \( G \) along \( G \) is a translation. However, at the point \( p_0 = p \times \{0\} \) we have \( \det(D\phi_t |_{F^c \oplus R \cdot G}) = 1 - b < 1 \), contradicting sectional-expansion, since \( p_0 \) belongs to a periodic orbit of \( G \) with period 1.

We claim that this flow is mostly asymptotically sectional expanding.

Indeed, we note that since each submanifold \( \Sigma_s = \mathbb{T} \times \{s\}, 0 \leq s < 1 \) is a global cross-section for the flow \( \phi_t \) with constant return time equal to \( 1 \), then \( \phi_1 |_{\Sigma_s} : \Sigma_s \to \Sigma_s \) is the Poincaré First Return Map to \( \Sigma_s \), and such return maps all coincide with \( f_1 \) by construction of the suspension flow as a translation on the last coordinate. In addition, we get \( P^1 = D(\phi_1 |_{\Sigma_s}) \). Hence, since \( f_1 \) is a partially hyperbolic non-uniformly expanding diffeomorphism, we obtain [8] for Leb-a.e. \( x \in \Sigma_s \) for each \( 0 \leq s < 1 \). Thus by Fubini’s Theorem, we get [8] for Leb-a.e. point of \( \mathbb{T} \), because \( \{\Sigma_s : 0 \leq s < 1\} \) is a smooth foliation of \( \mathbb{T} \). We deduce mostly asymptotically sectional expansion from item (3) of Theorem B.

We can then ensure the existence of finitely many ergodic physical/SRB measures for this class of systems.

**Example 6** (Mostly asymptotically sectional expanding, with equilibria and not sectionally hyperbolic). We adapt the construction of the multidimensional Lorenz attractor, first presented by Bonatti, Pumariño and Viana in [26], to obtain an example of a mostly asymptotically sectional expanding attracting set with an equilibrium.
We consider a “solenoid” constructed over a uniformly expanding map \( g : \mathbb{T} \to \mathbb{T} \) of the \( k \)-dimensional torus \( \mathbb{T} \), for some \( k \geq 2 \). That is, let \( D \) be the unit disk on \( \mathbb{R}^2 \) and consider a smooth embedding \( F_0 : N \cap \{z \} \times \mathbb{D} : z \in \mathbb{T} \}. \) The natural projection \( \pi : N \to \mathbb{T} \) on the first factor conjugates \( F_0 \) to \( g \circ \pi \). We assume that the initial expanding map \( g \) has simple spectrum \( \{ \lambda_1 > \lambda_2 > \cdots > \lambda_k \} \) and that \( F_0 \) admits two distinct fixed points \( p \) and \( q \).

We have that \( D\left. F_0(q) : TqN \to TqN \right| \) is hyperbolic with a 2-dimensional contracting invariant subspace, and a complementary \( k \)-dimensional expanding invariant subspace. Let \( \{v_1, \ldots, v_k, u_1, u_2\} \) be a basis of \( TN = \mathbb{R}^k \times \mathbb{R}^2 \) formed by unit vectors so that \( v_i \) is an eigenvector corresponding to \( \lambda_i \), \( i = 1, \ldots, k \). We choose coordinates on a neighborhood \( V \) of \( q \) in \( N \) so that \( F_0 \mid V \) has the expression \((x, y_1, \ldots, y_k) \mapsto (Ax, \lambda_1 y_1, \ldots, \lambda_k y_k) \) with \( x = x_1 u_1 + x_2 u_2 \) and \( A \) a linear contraction on \( \mathbb{R}^2 \).

We perform the same perturbation as in Example 5 replacing the weakest expanding coordinate map \( y_k \mapsto \lambda_k y_k \) by \( y_k \mapsto \varepsilon_1 f_\varepsilon(y_k/\varepsilon_1) \) obtaining a new base map \( F : N \to N \).

We note that \( F \) is a partially hyperbolic map with an invariant splitting \( E^s \oplus E^c \oplus E^u \), where \( E^s = \{0\} \times \mathbb{R}^2 \), \( E^c = \mathbb{R} \times v_k \) and \( E^u \) is everywhere spanned by \( v_1, \ldots, v_{k-1} \), with \( \| (\wedge^2 DF) \|_{E^c \oplus E^u} \| < 1 \) with respect to the standard product metric in \( N \). That is, we have uniform area expansion along any two-dimensional subspace contained the central-unstable subbundle \( E^{cu} = E^c \oplus E^u \). We also have an attracting subset \( \Lambda_0 = \cap_{n \geq 0} F^n(N) \) with \( N \) as topological basin of attraction.

We further consider the constant vector field \( X = (0, 1) \) on \( M = N \times [0, 1] \) and modify this field on the cylinder \( C = U \times \mathbb{D} \times [0, 1] \) around the periodic orbit of the point \( p = (z, 0) \in N \times \{0\} \), where \( U \) is a neighborhood of \( z \) in \( \mathbb{T} \) such that \( V \cap (U \times \mathbb{D}) = \emptyset \), in such a way as to create a hyperbolic (generalized Lorenz-like) equilibrium \( \sigma \) of saddle-type with \( k \) expanding and 3 contracting eigenvalues, as depicted in Figure 4. The eigenspace of one of the contracting eigenvalues lies along the direction of \( X \), the other two-dimensional contracting directions still lie on the direction of \( D \), and the remaining expanding eigenspaces are transversal to the \( X \) direction.

![Figure 4](image)

This vector field \( Y \) defines a transition map from \( \Sigma_\varepsilon = N \times \{\varepsilon\} \) to \( \Sigma_{1-\varepsilon} = N \times \{1 - \varepsilon\} \) for some fixed small \( \varepsilon > 0 \), which is the identity in the first coordinate when restricted to \( \Sigma_\varepsilon \setminus (U \times \mathbb{D} \times \{\varepsilon\}) \).

We assume that the standard inner product satisfies \( \langle Y, X \rangle > 0 \) on \( \Sigma_\varepsilon \cup \Sigma_{1-\varepsilon} \) and take a \( C^\infty \) bump function \( \psi : [0, 1] \to \mathbb{R} \) so that \( \psi \mid_{[\varepsilon/2, 1-\varepsilon/2]} \equiv 0 \) and \( \psi \mid_{[0, \varepsilon/3) \cup (1-\varepsilon/3, 1]} \equiv 1 \). We define the vector field \( G(x, t) = \psi(t) \cdot X + (1 - \psi(t)) \cdot Y(x, t), \) \( (x, t) \in M \) which generates
a smooth transition map $L$ from $\Sigma_0 = (N \setminus \{p\}) \times \{0\}$ to $\Sigma_1 = N \times \{1\}$. Together with the identification $(x,0) \sim (F_0x,1), x \in N$ we obtain a smooth parallelizable manifold $\widetilde{M} = M/ \sim$ where $G$ induces a $C^\infty$ vector field which we denote by the same letter.

We may assume that the splitting $E^s \oplus E^{cu}$ is still preserved by $L$; this is clear outside of the cylinder $C$, inside $C$ this is obtained by the choice of $Y$ and, moreover, in $C$ the $E^{cu}$ bundle in uniformly expanded.

We may now induce invariant bundles for the flow $\phi_t$ of $G$ on $\widetilde{M}$ by parallel transport: $F^{cu} = \mathbb{R} \times G \oplus E^{cu}$ and $F^s = E^s$ and consider the maximal invariant subset $\Lambda = \cap_{t>0} \phi_t(\widetilde{M})$ which is a attracting set with basin $\widetilde{M}$. Since $q \in N$ becomes a periodic point with period 1 for $G$ and $p \in W^s_{\sigma}$, we still have uniform area expansion along $F^{cu}$ and non-sectional-expansion along $F^s$. But, considering the cone fields $C^{cs}$ and $C^{cu}$ of small width around $F^s$, $F^{cu}$ respectively, we obtain the sufficient conditions (1-2) presented in the previous Example 5 for non-uniform sectional expansion of $f = \phi_1$. Most asymptotically sectional expansion is obtained again as in Example 5 and, consequently, we can apply Theorem A to obtain finitely many ergodic physical/SRB measures whose basins cover $\widetilde{M}$ Leb mod0.

2.2. Mostly asymptotically sectional expanding and not singular-hyperbolic.

Example 7 (Geometric Lorenz-like attractor with non-hyperbolic periodic orbit). We start with a one-dimensional Lorenz-like transformation with two expanding fixed repellers at the boundary of the interval, which is an adaptation of the “intermittent” Manneville map into a local homeomorphism of the circle; see [61]. We consider $I = [-1, 1]$ and the map $f : I \to I$ (see the left hand side of Figure 5) given by

$$x \mapsto \begin{cases} 
2\sqrt{x} - 1 & \text{if } x \geq 0, \\
1 - 2\sqrt{|x|} & \text{otherwise}.
\end{cases}$$

Then we perform the geometric Lorenz construction in such a way to obtain this map as the quotient over the stable leaves of the Poincaré first return map to the global cross-section of a vector field $G_0$; see the right hand side of Figure 5.

As usual in the geometric Lorenz construction, we assume that in the cube $I^3$ the flow is linear $\dot{G}_0 = A \cdot G_0$ with $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and a Lorenz-like equilibrium at the origin $\sigma_0$ satisfying $\lambda_1 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$; see e.g. the detailed description in [15, Chap. 3, Sec. 3].

The map $f$ preserves Lebesgue measure $\lambda$ on $I$ which is $f$-ergodic; see [3, Sec. 5]. In particular, $f$ is transitive (in fact, it is locally eventually onto, and so topologically mixing).

We thus obtain an attractor $\Lambda$ for the flow of the vector field $G$ depicted in the right hand side of Figure 5 which is partially hyperbolic and admits two periodic orbits $\mathcal{O}(p_{\pm})$ corresponding to the indifferent fixed points of $f$ which are not hyperbolic. Indeed, the Poincaré first return map $R : S^* \to S$ to the cross-section $S = I^2 \times \{1\}$, with domain $S^* = S \setminus \{(0) \times I \times \{1\}\}$ given by all the points of $S$ away from the singular line, is a skew-product map $R(x, y) = (f(x), g(x, y))$, where $g$ is a contraction on the second coordinate.
The non-hyperbolicity of $O(p_{\pm})$ ensures that the attractor $\Lambda$ of the 3-vector field $G$ is not singular-hyperbolic.

Following the standard construction described in [15, Chap. 7, Sec. 3.4], there exists an ergodic physical $R$-invariant probability measure $\nu$ on $S$ whose marginal $\pi_*\nu$ is $\lambda$, where $\pi : S \simeq I^2 \to I$ is the natural projection on the first coordinate. Finally, we obtain a physical ergodic invariant probability measure $\mu$ for the flow of $G$ by considering the suspension flow with base map $R$ and roof function provided by the Poincaré first return time $\tau : S^* \to \mathbb{R}^+$ to $S$.

Moreover, $|f'(x)| > 1$ for all $x \in I \setminus \{0, \pm 1\}$ and so, if $(\phi_t)_{t \in \mathbb{R}}$ is the flow of $G$, then since $\tau$ is constant on the fibers of the skew-product and $\lambda$-integrable

$$\int \log |\det D\phi_1| \, d\mu \geq h_\mu(\phi_1) = \frac{h_\nu(R)}{\mu(\tau)} \geq \frac{h_\lambda(f)}{\mu(\tau)} = \frac{1}{\mu(\tau)} \int \log |f'| \, d\lambda > 0,$$

we conclude that $\Lambda$ is mostly asymptotically sectional expanding while not being singular-hyperbolic. From Theorem A, these attractors admit a unique physical/SRB measure due to transitivity.

**Example 8** (Geometric Lorenz-like attractor with non-hyperbolic equilibrium). In the recent work [30] Bruin-Farias construct (similarly to the previous example) and study a geometric Lorenz-like attractor with a neutral equilibrium replacing the hyperbolic Lorenz-like equilibrium from the classical (geometrical) Lorenz attractor. This neutral equilibrium is neither Lorenz-like nor Rovella-like.

The authors show that there exists a unique physical/SRB measure and proceed to study its mixing rate (obtaining polynomial upper bounds). This implies mostly asymptotic sectional expansion without singular-hyperbolicity.

**Remark 2.1.** Example 8 shows, in particular, that the assumption of hyperbolic equilibria is not necessary for the existence of a physical/SRB measure and so also not necessary to
obtain asymptotical sectional expansion. Hence, this assumption should be regarded as a simplifying general assumption which is used in our line of proof.

2.3. Non-uniform weak expansion without slow recurrence nor physical measure.

**Example 9** (Non-uniform expanding and no physical measure). We consider the well-known vector field $X$ generating the flow $(\phi_t)_{t \geq 0}$ of the cylinder $N := S^1 \times \mathbb{R}$ with a double heteroclinic connection (the “Bowen’s eye” flow), e.g., from Takens work [67] showing that *Birkhoff averages may not exist almost everywhere*; see the left hand side of Figure 6 and also [55]. In this system time averages exist only for the sources $C, D$ and for the set of separatrixes and saddle equilibria $W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup \{A, B\}$. Moreover the orbit $(\phi_t(x))_{t \geq 0}$ of each $x$ not in $W$ and different from $C, D$ tends to $W$ as $t \nearrow \infty$.

Letting $f := \phi_1$ denote the time 1 map of the flow, we see that $W$ is a compact $f$-invariant attracting set, since $W = \cap_{n \geq 0} f^n(U)$ for all sufficiently small neighborhoods $U$ of $W$.

![Figure 6. The double heteroclinic connection with non-existing time averages for a full Lebesgue measure subset; on the right hand side. On the left hand side, the North-South flow on the circle.](image)

Moreover, we may choose the saddles eigenvalues and adapted coordinates near $A$ and $B$ to obtain the following for every $x \in N \setminus \{C, D\}$

$$\limsup_{T \to \infty} \frac{\log |\det D\phi_T(x)|^{1/T}}{T} < 0 < \limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} \log \|Df(f^ix)\|^{1/n}}{n}. \quad (19)$$

This shows that this system, although with some average asymptotic expansion, is asymptotically sectional contracting on an open and full Lebesgue measure subset – which shows that these trajectories are not Oseledets regular; see e.g. [22] and [31]. Since physical measures cannot exist in this system due to the non-existence of Birkhoff time averages, then we obtain a weak counterexample to the following conjecture.

**Conjecture 4.** (Viana [70] & [25, Conjecture 12.37]) If a smooth map $f$ has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some SRB measure.

The proof of (19) is a consequence of the following.

---

6Confer also Kiriki et al. [39] and Ott-Yorke [53].
Theorem 2.2. [67] Theorem 1] If $g$ is a continuous function on $N$ with $g(A) > g(B)$ and the positive trajectory of $x$ accumulates $W$, then

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T g(\phi_t x) \, dt = \frac{\sigma g(A) + g(B)}{1 + \sigma}$$

and

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T g(\phi_t x) \, dt = \frac{\lambda g(B) + g(A)}{1 + \lambda},$$

where $\lambda := \alpha^- / \beta^+$ and $\sigma := \beta^- / \alpha^+$ from spectra $\text{sp}(D\phi_t(x)) = \{\alpha^+, -\alpha^-\}$; $\text{sp}(D\phi_t(B)) = \{\beta^+, -\beta^-\}$ with $\alpha^+, \beta^+ > 0$.

Indeed, to ensure that $W$ is attracting it is enough to have $\lambda \sigma > 1$ and we can set this together with $\delta_A := \alpha^+ - \alpha^- < 0$ and $\delta_B := \beta^+ - \beta^- < 0$. Since $|\det D\phi_t(x)| = \exp \int_0^T \text{Tr}(D\phi_t(x)) \, ds$ we set $g(x) = \text{Tr}(D\phi_t(x))$ to get $\log |\det D\phi_t(x)| = \int_0^T g(\phi_s x) \, ds$ and both $g(A) = \delta_A$ and $g(B) = \delta_B$ strictly negative. Thus, the left hand side inequality from (19) follows from Theorem 2.2.

For the right hand side inequality, we set $g(x) = \log \|Df(x)\|$ and note that $t \mapsto g(\phi_t x)$ is $C^1$. Hence, we can write $\int_0^T g(\phi_t x) \, dt = \sum_{i=0}^{n-1} \int_0^1 g(\phi_t f^i x) \, dt$ and $g(\phi_t f^i x) = g(f^i x) + t \cdot \partial_s (\phi_t f^i x) |_{s=s(t)}$ by the Mean Value Theorem for some $s(t) \in (0, t)$. Moreover,

$$\partial_s (\phi_t f^i x) |_{s=s(t)} = \nabla g(\phi_s f^i x) \cdot D\phi_s f^i x X(\phi_s f^i x)$$

is uniformly bounded from above and below, so we can find $\bar{L}$ so that

$$\int_0^1 g(\phi_t f^i x) \, dt \leq \int_0^1 (g(f^i x) + t \bar{L}) \, dt \leq g(f^i x) + \frac{1}{2} \bar{L}.$$ 

This ensures that $(1/n) \sum_{i=0}^{n-1} g(f^i x) \geq (1/n) \int_0^T g(\phi_s x) \, ds - \bar{L}/2n$ and so the right hand inequality of (19) follows again from Theorem 2.2, since for our choice of $g$ we have both $g(A) = \log \|e^{DX(A)}\| = \alpha^+$ and $g(B) = \log \|e^{DX(B)}\| = \beta^+$ stricly positive.

Example 10 (Partially hyperbolic nonuniform sectional expanding with no physical measure). Continuing from the previous example, we consider the compactification $S^2$ of $N$ with a source at infinity and the direct product $M = S^2 \times S^1$ with the “North-South flow” on the circle; see right hand side of Figure 6.

We get a flow $(\psi_t : M \to M)_{t \in \mathbb{R}}$ with an attracting set $\mathcal{A} := S^2 \times \{s\}$ so that $d(\psi_t(z), \mathcal{A}) \to 0$ when $t \nearrow \infty$ for all $z \in M \setminus \mathcal{A}$, where $d$ is any Riemannian distance on $M$. If we let the contraction rate at the sink $S$ of the North-South flow to be stronger than the contracting rates of the saddles $A, B$ from $(\psi_t)_{t \geq 0}$, then $\mathcal{A}$ becomes a partially hyperbolic attracting set with splitting $T\mathcal{A}M = E^s \oplus E^c$ given by $E^s = \{0\} \times T_S S^1$ and $E^c = TS^2 \times \{0\}$.

We note that the region between the saddle connections $W_1$ and $W_4$ containing $C$ has a closure $F$ which is invariant and $K := F \times V_S$ becomes also a partially hyperbolic forward invariant set, where $V_S$ is any compact positively invariant neighborhood of the sink $S$ in

\footnote{Here $\text{Tr}(L)$ is the trace of the linear operator $L$.}
with respect to the North-South flow, with the same splitting as above since we have a direct product.

Moreover, because all future trajectories starting in $K$ accumulate $W_1 \cup W_4 \cup \{A, B\}$, from (19) we obtain

$$\limsup_{T \to \infty} \log |\det D\psi_T|_{E_s}^{1/T} < 0 < \limsup_{n \to \infty} \sum_{i=0}^{n-1} \log \|D\tilde{f}|_{E_{fix}}\|^{1/n}$$

(20)

for all $x \in K\setminus\{C\} \times \mathbb{S}$. Thus, for an open and full Leb-measure subset of the partially hyperbolic forward invariant set $K$ we have average asymptotic expansion along the central bundle together with asymptotic sectional contraction, and no physical/SRB measure.

Moreover, we do not have slow recurrence. Indeed, for any given $\delta, L > 0$ the continuous function $g(x) := \min\{L, -\log d_\delta(x, \{A, B, C\})\}$ is such that $\limsup_{T \to \infty} \frac{1}{T} \int_0^T g(\phi_t x) dt = L$ since $g(A) = g(B) = L$ for all $x$ whose future trajectory accumulates $W$, as a direct consequence of Theorem 2.2. Hence, for these trajectories we arrive at

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T -\log d_\delta(\phi_t x, \{A, B, C\}) dt = +\infty$$

for each small $\delta > 0$. Analogously, since $\|G(x)\|$ is comparable to $d_\delta(\phi_t x, \text{Sing}(X))$ (see Lemma 4.3) we obtain the same results replacing the distance to the equilibria with the norm of the vector field.

**Remark 2.3.** The proof of the existence of a physical measure for asymptotic sectional hyperbolic attractors presented in [46] — in the case when the attractor contains non-Lorenz-like equilibria — is based on the assumption that the right hand side inequality of (20) on a positive Lebesgue measure subset of points $x \in U$ implies the existence of some physical measure. From Examples 9 and 10 we see that the proof in [46] is incomplete.

### 3. Auxiliary results

The following results will be used in our arguments.

#### 3.1. Partial hyperbolic attracting sets

The following properties of partial hyperbolic attracting sets will be used as tools in our arguments.

**3.1.1. Extension of the stable bundle and center-unstable cone fields.** Let $\mathcal{D}^k$ denote the $k$-dimensional open unit disk and let $\text{Emb}^r(\mathcal{D}^k, M)$ denote the set of $C^r$ embeddings $\psi: \mathcal{D}^k \to M$ endowed with the $C^r$ distance. We say that the image of any such embedding is a $C^r$ $k$-dimensional disk.

**Proposition 3.1.** [12, Proposition 3.2, Theorem 4.2 and Lemma 4.8] Let $\Lambda$ be a partially hyperbolic attracting set.

1. The stable bundle $E^s$ over $\Lambda$ extends to a continuous uniformly contracting $D\phi_t$-invariant bundle $E^s$ on an open positively invariant neighborhood $U$ of $\Lambda$.
2. There exists a constant $\lambda \in (0, 1)$, such that
(a) for every point $x \in U$ there is a $C^r$ embedded $d_s$-dimensional disk $W^s_x \subset M$, with $x \in W^s_x$, such that $T_xW^s_x = E^s_x$; $\phi_t(W^s_x) \subset W^s_{\phi_t x}$ and $d(\phi_t x, \phi_t y) \leq \lambda^t d(x, y)$ for all $y \in W^s_x$, $t \geq 0$ and $n \geq 1$.

(b) the disks $W^s_x$ depend continuously on $x$ in the $C^0$ topology: there is a continuous map $\gamma : U \to \text{Emb}^0(D^{d_s}, M)$ such that $\gamma(x)(0) = x$ and $\gamma(x)(D^{d_s}) = W^s_x$.

Moreover, there exists $L > 0$ such that $\text{Lip} \gamma(x) \leq L$ for all $x \in U$.

(c) the family of disks $\mathcal{F}^s = \{W^s_x : x \in U\}$ defines a topological foliation $\mathcal{W}^s$ of $U$: every $x_0 \in U$ admits a neighborhood $V \subset U$ and a homeomorphism $\psi : V \to \mathbb{R}^{d_s} \times \mathbb{R}^{d_u}$ so that $\psi(W^s_x) = \pi_s^{-1}\{\pi_s(\psi(x))\}$ where $\pi_s : \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \to \mathbb{R}^{d_s}$ is the canonical projection.

Remark 3.2. For any two close enough $d_{cu}$-disks $D_1, D_2$ contained in $U$ and transverse to $\mathcal{F}^s$ there exists an open subset $D_1'$ of $D_1$ so that $W^s_x \cap D_2$ is a singleton. This defines the holonomy map $h : D_1 \to D_2, D_1 \ni x \mapsto W^s_x \cap D_2$ and Proposition 3.1 ensures that $h$ is continuous.

The splitting $T_\Lambda M = E^s \oplus E^{cu}$ extends continuously to a splitting $T_U M = E^s \oplus E^{cu}$ where $E^s$ is the invariant uniformly contracting bundle in Proposition 3.1 – however $E^{cu}$ is not invariant in general, but the center-unstable cone field satisfies the following.

Proposition 3.3. Let $\Lambda$ be an attracting set with a dominated splitting so that the flow direction is contained in the center-unstable bundle $G \subset E^{cu}$. Then, for any $a > 0$, after possibly shrinking $U$, we can find $\kappa > 0$ so that $D\phi_t \cdot \mathcal{C}^{cu}_{a\lambda}(\phi_t x) \subset \mathcal{C}^{cu}_{a\lambda}(\phi_t x)$ for all $t > 0$, $x \in U$.

Proof. See [12] Proposition 3.1] considering the choice of adapted Riemannian metric as defined in Subsection 1.1.2 we estimate for $v \in \mathcal{C}^{cu}_{a\lambda}(x)$ (using only the domination of the splitting)

$$\frac{\|D\phi_t(x) \cdot v^s\|}{\|D\phi_t(x) \cdot v^c\|} \leq \frac{\|D\phi_t|_{E^s_x}\| \cdot \|v^s\|}{\|D\phi_t|_{E^{cu}_x}\|^{-1} \cdot \|v^c\|} \leq \lambda^t a.$$

However, since $E^{cu}$ extended to $U$ is not necessarily $D\phi_t$-invariant, we need to project $D\phi_t(x) \cdot v^c$ to $E^{cu}_{\phi_t x}$ parallel to $E^s_{\phi_t x}$ to decompose $D\phi_t(x) \cdot v$ into stable/center-unstable components. Because both $E^{cu}_{\phi_t x}$ and $D\phi_t \cdot E^s_{\phi_t x}$ are contained in $\mathcal{C}^{cu}_{a\lambda}(\phi_t x)$, then we can find $\kappa = \kappa(a) > 0$ so that $\kappa\|\pi^{cu} \cdot D\phi_t(x) \cdot v^c\| \geq \|D\phi_t(x) \cdot v^c\|$, and then

$$\frac{\|D\phi_t(x) \cdot v^c\|}{\|\pi^{cu} \cdot D\phi_t(x) \cdot v^c\|} \leq \kappa\lambda^t a,$$

which completes the proof of the statement. \[\Box\]

3.1.2. Partial hyperbolicity of Poincaré maps. Let $\Sigma, \Sigma'$ be a small cross-sections to $G$ contained in $U$ and let $R : \text{dom}(R) \to \Sigma'$ be a Poincaré map $R(y) = \phi_t(y)(y)$ from an open $^{8}$Note that $R$ needs not correspond to the first time the orbits of $\Sigma$ encounter $\Sigma'$ nor it is defined everywhere in $\Sigma$.}
subset $\text{dom}(R)$ of $\Sigma$ to $\Sigma'$ (possibly $\Sigma = \Sigma'$). The splitting $E^s \oplus E^{cu}$ over $U$ induces a continuous splitting $E^s_{\Sigma} \oplus E^{cu}_{\Sigma}$ of the tangent bundle $T\Sigma$ to $\Sigma$ (analogously for $\Sigma'$) as

$$E^s_{\Sigma}(y) = E^s_y \cap T_y \Sigma \quad \text{and} \quad E^{cu}_{\Sigma}(y) = E^{cu}_y \cap T_y \Sigma.$$  \hspace{1cm} (21)

The splitting (21) is partially hyperbolic for $R$, as follows.

**Proposition 3.4.** [13 Proposition 4.1]\& [15 Lemma 8.25] Let $R: \Sigma \to \Sigma'$ be a Poincaré map with Poincaré time $t(\cdot)$. Then $DR \cdot E^s_{\Sigma}(x) = E^s_{\Sigma}(R(x))$ at every $x \in \Sigma$ and $DR \cdot E^{cu}_{\Sigma}(x) = E^{cu}_{\Sigma}(R(x))$ at every $x \in \Lambda \cap \Sigma$. Moreover, for $x \in \Sigma$ we have $\|DR \cdot E^s_{\Sigma}(x)\| < \lambda^{t(x)}$ and $\|DR \cdot E^{cu}_{\Sigma}(x)\| < \lambda^{t(x)}$.

Given a cross-section $\Sigma$, $b > 0$ and $x \in \Sigma$, the unstable cone of width $b$ at $x$ is

$$C^u_b(\Sigma, x) = \{ v = v^s + v^u : v^s \in E^s_{\Sigma}(x), v^u \in E^{cu}(x) \text{ and } \| v^s \| \leq b \| v^u \| \}.$$  \hspace{1cm} (22)

**Corollary 3.5.** There exists $b > 0$ small enough so that, for each $R : \Sigma \to \Sigma'$ as in Proposition 3.4, we have $\text{DR}(x) \cdot C^u_b(\Sigma, x) \subset C^u_{b\lambda^{t(x)}}(\Sigma', R(x))$ for all $x \in \Sigma$.

**Proof.** Cf. proof of [12 Proposition 3.1] which is similar to the proof of Proposition 3.1. \hfill $\square$

### 3.2. Hölder control of the tangent bundle in the center-unstable direction.

We recall that we have continuous extensions of the two subbundles $E^s$ and $E^{cu}$ defined on an isolating neighborhood $U$ of $\Lambda$, and the respective cone fields $C^u_\lambda(x), C^{cu}_\lambda(x), x \in U$ for a small $0 < a < 1$ which are invariant in the sense of [3].

We may assume without loss of generality that, up to increasing the value of $\lambda < 1$ by a small amount and reducing the neighborhood $U$ of $\Lambda$, a “bunched domination condition” holds true for vectors in these cone fields: there exists $\zeta \in (0, 1)$ so that

$$\|D\phi_t \cdot u\| \cdot \|D\phi_{-t} \cdot v\|^{1+\zeta} \leq \lambda^t \cdot \|u\| \cdot \|v\|, \quad \text{for } t > 0, x \in U, u \in C^u_\lambda(x) \text{ and } v \in C^{cu}_\lambda(\phi_t x).$$

A $C^1$ disk $D$ on $M$, that is, the image of a $C^1$ embedding $\psi : B(0, 1) \subset \mathbb{R}^{\text{dim} \Sigma} \to \Sigma$ defined on the unit ball of an Euclidean space, is a $u$-disk if $T_y D \subset C^{cu}(y), y \in D$.

We fix $\rho_0 > 0$ so that the inverse of the exponential map $\exp_x$ is defined on the $\rho_0$ neighborhood of each point $x \in U$, which we identify with the corresponding neighborhood $V_x$ of the origin 0 in $T_x M$, and $x$ with 0.

We may assume without loss of generality that $E^s_x \subset C^u_\rho(y)$ for all $y \in V_x$ so that, in particular, $E^s_x \cap C^{cu}_\rho(x) = \{0\}$. If $x \in D$ then $T_y D$ is given by the graph of the linear map $A_x(y) : T_y D \to E^s_x$ for each $y \in V_x \cap D$.

We say that the tangent bundle $TD$ is $(C, \zeta)$-Hölder if there are constants $C > 0$ such that $\|A_x(y)\| \leq C \delta(y)^{\zeta}$ for $y \in D \cap V_x$, where $\delta(y)$ is the intrinsic distance from $x$ to $y$ within $D \cap V_x$.

Given a $u$-disk $D$ we write $\kappa(D)$ for the least $C > 0$ so that the tangent bundle of $D$ is $(C, \zeta)$-Hölder.

We recall the notation $f = \phi_1$ for the time-1 map of the flow of $G$. Then we can prove the following, since $\Lambda$ is also a partially hyperbolic attracting set for $f$.

**Proposition 3.6.** There exists $C_1 > 0$ so that each $u$-disk $D \subset U$ satisfies

\footnote{The length of the shortest curve connecting $x$ to $y$ inside $D \cap V_x$}
4. Hyperbolic (Pliss) times for the Linear Poincaré Flow. The following is a very useful tool introduced by Pliss in [60] which enables us to use hyperbolic times.

**Lemma 4.1.** Let \( A \geq c_2 > c_1 \) be real numbers and \( \zeta = (c_2 - c_1)/(A - c_1) \). Given real numbers \( a_1, \ldots, a_N \) satisfying

\[
\sum_{j=1}^N a_j \geq c_2 N \quad \text{and} \quad a_j \leq A \quad \text{for all} \quad 1 \leq j \leq N,
\]

there are \( \ell > \zeta N \) and \( 1 < n_1 < \ldots < n_{\ell} \leq N \) such that

\[
\sum_{j=n_i+1}^{n_{i+1}} a_j \geq c_1 \cdot (n_i - n) \quad \text{for each} \quad 0 \leq n < n_i, \quad i = 1, \ldots, \ell.
\]

**Proof.** See e.g. [60], [42, Section 2] or [4, Lemma 3.1]. \( \square \)

4. Hyperbolic times and center-unstable disks

In this section we start the proof of Theorem D. We only use the domination of the splitting and hyperbolic times along the sectional center-unstable direction.

4.1. Hyperbolic (Pliss) times for the Linear Poincaré Flow. The following is a very useful tool introduced by Pliss in [60] which enables us to use hyperbolic times.

**Lemma 4.1.** Let \( A \geq c_2 > c_1 \) be real numbers and \( \zeta = (c_2 - c_1)/(A - c_1) \). Given real numbers \( a_1, \ldots, a_N \) satisfying

\[
\sum_{j=1}^N a_j \geq c_2 N \quad \text{and} \quad a_j \leq A \quad \text{for all} \quad 1 \leq j \leq N,
\]

there are \( \ell > \zeta N \) and \( 1 < n_1 < \ldots < n_{\ell} \leq N \) such that

\[
\sum_{j=n_i+1}^{n_{i+1}} a_j \geq c_1 \cdot (n_i - n) \quad \text{for each} \quad 0 \leq n < n_i, \quad i = 1, \ldots, \ell.
\]

**Proof.** See e.g. [60], [42, Section 2] or [4, Lemma 3.1]. \( \square \)

Let us fix \( x \in \Omega \) satisfying (8).

Since \( (P^1|_{N^c_w})^{-1} = O_x \cdot Df^{-1} \mid_{P^1(N^c_w)} \), then \( \|(P^1|_{N^c_w})^{-1}\| \leq \|Df^{-1}\| \leq e^L \) with \( L = \sup_{x \in U} \|DG_x\| \), which is finite because \( U \) is relatively compact. We also have \( \|(P^1|_{N^c_w})^{-1}\| \geq \|P^1|_{N^c_w}\|^{-1} \geq e^{-L} \), and thus \( A_0 = \sup_{x \in U} \log \|(P^1|_{N^c_w})^{-1}\| \leq e^L \). Then we apply Lemma 4.1 to \( a_i = -\log \|(P^1|_{N^c_w})^{-1}\| \) for \( i = 1, \ldots, N \) so that \( \sum_{i=1}^N a_i \geq c_0 N/2 \)

– this inequality holds for all large enough \( N = N(x) > 1 \).

We obtain \( \ell > \zeta_1 N \) with \( \zeta_1 = (c_0/2 - c_0/4)/(A_0 - c_0/4) = c_0/(4A_0 - c_0) > 0 \) and times \( 1 < n_1 < \ldots < n_{\ell} \leq N \) such that

\[
\prod_{j=n_i}^{n_{i+1}-1} \|(P^1|_{N^c_{j+1}})^{-1}\| \leq e^{-c_0(n_i-n)/4}, \quad 0 \leq n < n_i, \quad i = 1, \ldots, \ell.
\]

(23)

We say that \( n_i \) is a hyperbolic time for \( x \) if \( \text{Sing}_{A}(G) = \emptyset \). In the presence of equilibria, we need to control the visits of the future orbit of \( x \) near these fixed points where the Linear Poincaré flow is not defined. For that, we reapply Lemma 4.1 to (9), as follows.

For \( \varepsilon_0 \in (0, \zeta_1 c_0/32) \) we take \( \delta_0 > 0 \) satisfying (9) for all \( x \in \Omega \), so that for some \( N(x) > 1 \) we have

\[
n \geq N(x) \implies \sum_{i=0}^{n-1} \log d_{\delta_0}(f^i(x), \text{Sing}_{A}(G)) \geq -2\varepsilon_0 \cdot n.
\]
Since the summands are non-positive, we can take $A = 0$, $c_2 = -2\varepsilon_0$ and $c_1 = -c_0/16$ to obtain $\zeta_2 = (c_2 - c_1)/(A - c_1) = 1 - c_2/c_1 > 1 - \zeta_1$. Hence $\zeta_1 + \zeta_2 > 1$ and for $\zeta = \zeta_1 + \zeta_2 - 1 > 0$ we have $\ell \geq \zeta N$ and times $1 < n_1 < \cdots < n_\ell \leq N$ simultaneously satisfying (23) and the conclusion of Pliss' Lemma for the last summation. We have proved the following.

**Proposition 4.2.** For each sufficiently small $\varepsilon_0 > 0$, we can find a small enough $\delta_0 > 0$ such that there are $\theta, \varepsilon_0$ and for $x \in \Omega$ we can find $N = N(x) \in \mathbb{Z}^+$ so that for any given integer $T \geq N$, there exists $\ell \geq \theta T$ and times $1 < n_1 < \cdots < n_\ell \leq T$ satisfying (23) and

$$d_{\delta_0}(f^j x, \text{Sing}_G(G)) > e^{-\omega_0(n_i-j)/16}, \quad 0 \leq j < n_i, \quad i = 1, \ldots, \ell. \quad (24)$$

The times $n_i$ satisfying the conclusion of Proposition 4.2 will be referred to as hyperbolic times for $x$ when $\text{Sing}_G(G) \neq \emptyset$.

### 4.2. Estimates for nearby points and roughness of hyperbolic times.

We observe that the map $x \in U \mapsto E^{cu}_x$ is H"older-continuous, by the domination of the splitting, see e.g. [12] Subsection 4.2. In addition, both $x \mapsto Df(x)$ and $x \in U^* \mapsto O_x$ are Lipschitz, where $U^* = U \setminus \text{Sing}(G)$, because $G$ is of class $C^2$ and the unit vector field $\hat{G} := G/\|G\|$ defined in $U^*$ has derivative $\hat{G}'^i = O_x \circ D G_x \cdot \hat{G}^i_x$ whose norm is uniformly bounded from above.

Hence $\Psi : U^* \times U^* \mapsto \mathbb{R}, (x, y) \mapsto \log \|(P^i|_{N_y})^{-1}\|$ is H"older-continuous and $\Psi(x, x) = 0$ for all $x \in U^*$. Therefore, there exists a constant $C_2 > 0$ and an exponent $\omega \in (0, 1)$ so that $\Psi(x, y) \leq C_2 \cdot d(x, y)^\omega$.

We recall that $\rho_0 > 0$ is such that $\left( \exp_x \mid_{B(0, \rho_0)} \right)^{-1}$ is well-defined at every $x \in U$.

For $z \in M$ with $G_z \neq \emptyset$ we define the cone

$$C^+_a(z) = \{v + \lambda G_z : v \in G^+_z, \lambda \in \mathbb{R} \& \|v\| \leq a\|\lambda G_z\| \}. \quad (25)$$

We let $a > 0$ be small enough so that for $y \in U^*$

$$\|O_y \cdot D f^{-1}(f(y))v\| \leq e^{na/16} \left\| \left( P^1 \mid_{N^a_y} \right)^{-1} \right\| \cdot \|v\|, \quad v \in C^a_{cu}(f(y)) \cap C^+_a(f(y)).$$

We choose $0 < \rho_1 \leq \min\{\delta_0, \rho_0, 1\}$ such that $C_2 \rho_1^\omega < c_0/16$ and, for each $x, y \in U^*$ with both $d(x, y) < \rho_1$ and $d(x, y) < d(x, \text{Sing}_G(G))/2$, then together with the H"older condition on $\Psi$ we get for $v \in C^a_{cu}(f(y)) \cap C^+_a(f(y))$

$$\|O_y \cdot D f^{-1}(f(y))v\| \leq e^{na/8} \cdot \left\| \left( P^1 \mid_{N^a_y} \right)^{-1} \right\| \cdot \|v\|. \quad (26)$$

In what follows, we need to assume that $\delta_0$ is small enough depending on $G$ to obtain the needed estimates, subject to finitely many conditions. This can be done without loss of generality because of the slow-recurrence condition [9].

**Lemma 4.3.** There exists $b_0 > 0$ so that for each $x \in U^*$, if $d(x, \text{Sing}_G(G)) < \delta_0$, then

$$b_0 L \leq \frac{\|G_x\|}{d(x, \sigma)} \leq L \quad \text{and} \quad 2 \cdot d(y, x) < d(x, \text{Sing}_G(G)) \quad \implies \quad \|G_y\| \geq b_0 \|G_x\|. \quad (27)$$
Proposition 4.4. There exists \( \phi \) time for \( \text{Sing}_\alpha(G) \) are hyperbolic, there are at most finitely many and those accumulated by the orbit of \( x \in \Omega \) are of saddle type. Thus, there exists \( b > 0 \) so that \( \|(DG_\sigma)^{-1}\| < 1/b \) for all \( \sigma \in \omega_G(x) \cap \text{Sing}_\alpha(G) \).

Moreover, we have \( \|(DG_y - DG_\sigma)\| \leq \kappa_0 d(y, \sigma) \beta \) for all \( y \in B(\sigma, 2\delta_0) \) and some constants \( \kappa_0 > 0 \) and \( 0 < \beta \leq 1 \) since \( G \) is of class \( C^{1+} \) — here and in the following estimates, we identify \( B(\sigma, \rho_0) \) with the \( \rho_0 \)-ball on \( T_\sigma M \). Hence, if \( 2\kappa_0 \delta_0^\beta \) \( < b \), then by the Mean Value Inequality

\[
\|G_x - G_\sigma - DG_\sigma(x - \sigma)\| \leq \kappa d(x, \sigma)\|x - \sigma\| \leq \kappa_0 \delta_0^\beta d(x, \sigma)
\]

and so \( \|G_x\| \geq \|(DG_\sigma(x - \sigma)) - \kappa_0 \delta_0^\beta d(x, \sigma)\| \geq (b - \kappa_0 \delta_0^\beta)d(x, \sigma) \). On the other hand, by the smoothness of \( G \) we have that \( \|G_x\| = \|G_x - G_\sigma\| \leq L \cdot d(x, \sigma) \). Finally, if \( 2d(y, x) < d(x, \sigma) \), then

\[
\|G_y\| \geq (b - \kappa_0 \delta_0^\beta)d(y, \sigma) \geq \frac{b - \kappa_0 \delta_0^\beta}{2} d(x, \sigma) \geq \frac{b - \kappa_0 \delta_0^\beta}{L} \|G_x\|
\]

which completes the proof after setting \( b_\ast = (b - \kappa_0 \delta_0^\beta)/L \). \( \square \)

Now we show that hyperbolic times are rough along a trajectory, in the following sense.

Proposition 4.4. There exists \( s_0 > 0 \) small so that, if \( n > 1 \) is a hyperbolic time for \( x \in U^\ast \), then \( n \) is also a hyperbolic time for \( \phi_n x \) for all \( |s| < s_0 \).

Proof. First choose \( s_0 > 0 \) small enough so that \( d(z, \phi_n z) < \rho_1 \) for all \( z \in U \) and \( |s| < s_0 \), and use \((25)\) to obtain

\[
\prod_{i=n-k}^{n-1} \left\| \left( P^1_{N^\ast_{\phi_i f^i x}} \right)^{-1} \right\| = \prod_{i=n-k}^{n-1} \left( \left\| \left( P^1_{N^\ast_{\phi_i f^i x}} \right)^{-1} \right\| \cdot \left\| \left( P^1_{N^\ast_{\phi_i f^i x}} \right)^{-1} \right\| \right) \leq e^{kc_0/8} \cdot e^{-kc_0/4} = e^{-kc_0/8}; \quad k = 1, \ldots, n; \quad |s| < s_0.
\]

Then, if \( d(f^i x, \text{Sing}_\alpha(G)) \leq \delta \), we note that \( d(\phi_n f^i x, \text{Sing}_\alpha(G)) \) is bounded from below by

\[
d(f^i x, \text{Sing}_\alpha(G)) - d(\phi_n f^i x, \phi_n f^i x) \geq d(f^i x, \text{Sing}_\alpha(G)) - |s| \cdot \sup_{|t|<\delta} \|G_{\phi_n f^i x}\|.
\]

Lemma 4.3 provides \( \|G_{\phi_n f^i x}\| \geq b_\ast \|G_x\| \) whenever \( 2d(\phi_n f^i x, \phi_n f^i x) < d(f^i x, \text{Sing}_\alpha(G)) \), which holds for \( |t| < d(f^i x, \text{Sing}_\alpha(G))/(2\|G_x\|) \leq (2b_\ast L)^{-1} \). In this case, we obtain

\[
d(\phi_n f^i x, \text{Sing}_\alpha(G)) \geq d(f^i x, \text{Sing}_\alpha(G)) - |s| \cdot b_\ast \|G_x\| \geq (1 - b_\ast L \cdot |s|) \cdot d(f^i x, \text{Sing}_\alpha(G)).
\]

Hence, choosing \( s_0 \in (0, (2b_\ast L)^{-1}) \) small enough so that \( d(z, \phi_n z) < \rho_1 \) for all \( z \in U \) we can assume without loss of generality that

\[
d_2(\phi_n f^i x, \text{Sing}_\alpha(G)) \geq e^{-L} d_2(f^i x, \text{Sing}_\alpha(G)) \geq e^{-L} e^{-(n-1)\alpha/16}; \quad i = 0, \ldots, n; \quad |s| < s_0.
\]

The above conclusions show that, modulo a small change of rates, \( n \) is still a hyperbolic time for \( \phi_n x \) for each \( |s| < s_0 \). \( \square \)
4.3. Distortion bounds at hyperbolic times along the sectional center-unstable direction. We fix a \( u \)-disk \( D \subset U \) so that \( x \in D \) admits \( n > 1 \) as a hyperbolic time with a choice of \( \varepsilon_0, \delta_0 > 0 \) satisfying Proposition 1.2.

We set \( \Sigma_z = \exp_z(B(z, \rho_1) \cap G^+_{z}) \) as a cross-section to \( G \) through \( z \in U^* \); \( \Sigma^+_z = \exp_z(B(z, (\rho_1/2)e^{-\|x\|_{16-L}}) \cap G^+_{z}) \) a scaled cross-section; \( D_n = f^n(D) \) and \( D^+_n(z) = D_n \cap \Sigma_z \) for each \( z \in D_n \), that is, a section of \( D_n \) through \( z \) in the direction orthogonal to the vector field.

We then consider the Poincaré first hitting maps \( R_i : \text{dom}(R_i) \subset \Sigma^{n-i}_{f^{-i}(x)} \rightarrow \Sigma^{n-i-1}_{f^{i+1}(x)} \) and note that \( DR_i(f^i(x)) = P^1_{f^i_x} \cdot N_{f^i_x} \rightarrow N_{f^{i+1}_x} \), for \( i = 0, \ldots, n-1 \); see Figure 8.

We note that these maps are well-defined even if \( U \) contains equilibria, by the distance bound provided by the condition (24).

Lemma 4.5 (Local sectional continuity). For each \( i = 0, \ldots, n-1 \), let \( D^+_i = f^i(D) \cap \Sigma^{n-i-1}_{f^{i+1}x} \). Then the Poincaré maps satisfy \( \|DR_i(R_iy)^{-1}|_{T_{R_iy}D^+_i} \| \leq e^{\|x\|/8} \cdot \|\|P^1_{f^i_x}|_{n} \| \), for y \( \in \text{dom}(R_i) \).

**Proof.** We have \( y \in \Sigma^{n-i}_{f^{-i}x} \) by definition of \( R_i \) and so \( 2 \cdot d(y, f^i x) \leq \rho_1 e^{-(n-i)\|x\|_{16-L}} \). Moreover, \( d_{s_{\phi_x}}(f^i_x, \text{Sing}_\Lambda(G)) > e^{-(n-i)\|x\|_{16-L}} \) with \( \delta_0 < \rho_1 \leq 1 \). Hence \( 2 \cdot d(y, f^i x) \leq d_{s_{\phi_x}}(f^i_x, \text{Sing}_\Lambda(G)) \) and thus \( 2 \cdot d(y, \text{Sing}_\Lambda(G)) > d_{s_{\phi_x}}(f^i_x, \text{Sing}_\Lambda(G)) \). At this point, we divide the argument into two cases.

**Away from equilibria:** If \( d(f^i x, \text{Sing}_\Lambda(G)) \geq \delta_0 \geq \rho_1 \), then \( \Sigma^{n-i}_{f^{-i}x} \) is away from \( \text{Sing}(G) \) and the Poincaré time from \( \text{dom}(\Sigma^{n-i}_{f^{-i}x}) \) to \( \Sigma^{n-i-1}_{f^{i+1}x} \) is between \( 1 - \xi \) and \( 1 + \xi \) for some uniform small \( \xi > 0 \) depending on \( \rho_1 \).

This ensures that \( R_i y = (\phi_x \circ f)y \) with \( s = s(y) \) such that \( |s-1| < \xi \) and so for \( y \in \text{dom}(R_i) \) and \( v \in T_{R_iy}D^+_{i+1} \)

\[
\|DR_i(R_iy)^{-1}v\| = \|O_y \cdot [D(\phi_x \circ f)(R_iy)]^{-1}v\| \\
= \|O_y \cdot Df^{-1}(\phi_x S f) \cdot D(\phi_x)^{-1}(R_iy)v\| \\
= \|O_y \cdot Df^{-1}(f y) \cdot D(\phi_x)^{-1}(R_iy)v\|.
\]

The time \( s = s(y) \) can be seen as the Poincaré first visit time from the cross-section \( S = f(S_{f^{-i}x}) \) to \( \Sigma^{n-i-1}_{f^{i+1}x} \), and so \( D(\phi_x)^{-1}(R_iy)v \in T_{f y}(S \cap f^{i+1}(D)) \subset C^\Lambda_a(f y) \cap C^\Lambda_a(f y) \) by the proximity between \( R_i y, f y \) and \( f^{i+1} x \). Then the statement of the lemma follows from (25).

**Close to equilibria:** Otherwise, \( d(f^i x, \text{Sing}_\Lambda(G)) < \delta_0 \) and \( \Sigma^{n-i}_{f^{-i}x} \) is close to an equilibrium \( \sigma \in \text{Sing}_\Lambda(G) \).

We show that we can repeat the above argument by obtaining a flow box from \( \text{dom}(R_i) \) to \( \Sigma^{n-i-1}_{f^{i+1}x} \) with flight time bounded from above.

Reducing \( \delta_0 \) if necessary, we may assume, without loss of generality, that the flow on \( B(\sigma, 2\delta_0) \) is topologically conjugated to the flow of \( \dot{X} = DG_\sigma \cdot X \), because \( \sigma \) is hyperbolic.

That is, there exists a bi-Hölder homeomorphism \( h : B(\sigma, 2\delta_0) \rightarrow \mathbb{R}^m \) so that \( h \circ \phi_x(z) = e^{t DG_\sigma} h(z) \) for \( t > 0 \) such that \( \phi_x|_{(0,t)} z \subset B(\sigma, 2\delta_0) \); see e.g. (23). We arrange so that
\( \mathbb{R}^m = \mathbb{R}^u \times \mathbb{R}^s \) is the decomposition into stable and unstable subspaces of \( DG_\sigma \), which decomposes in block form as \( \text{diag}\{A_u, A_s\} \). We identify \( f^i x \) with \( (v, w) \in \mathbb{R}^u \times \mathbb{R}^s \) and then \( f^{i+1} x \) becomes \( (v_1, w_1) = (e^{A_u} v, e^{A_s} w) \).

\[
\text{if} \quad \| \delta \| \leq \| A \| \quad \text{and} \quad \text{we obtain} \quad \| \delta \| \text{if} \quad a \text{point} \quad (v, w) \quad \text{becomes} \quad (v_1, w_1) = (e^{A_u} v, e^{A_s} w).
\]

Going back to the original coordinates, the cross-sections \( h^{-1}(W_i), i = 0, 1 \) touch \( \Sigma_{f_i}^{n-i} \) and \( \Sigma_{f_i+1}^{n-i} \) at \( f^i x, f^{i+1} x \), respectively; see Figure 7. The vector field in between these sections has norm uniformly bounded away from zero and close to \( G_{f_i} \), by the estimate (26). Hence, the flight time is also bounded above depending only on \( G \) in a neighborhood of \( \sigma \).

We have recovered a flow box with bounded flight time from \( \text{dom}(R_i) \) to \( \Sigma_{f_i+1}^{n-i-1} \). We can thus finish repeating the argument as before, using (27) and (25). \( \square \)
We write $g_k = R_{n-1} \circ \cdots \circ R_{n-k}, 1 \leq k \leq n$ in what follows, and $\text{dist}_D(x,y)$ for the distance between two points $x,y$ in the disk $D$, measured along $D$ as the least length of smooth curves from $x$ to $y$ within $D$.

**Lemma 4.6** (Local sectional backward contraction). *Given any $u$-disk $D \subset U$ tangent to the centre-unstable cone field, $x \in D$ and $n \geq 1$ a hyperbolic time for $x$*

$$\text{dist}_{D^{-1}}(f^{-k}(x), g_k(y)) \leq e^{-k\alpha_0/8} \cdot \text{dist}_{D^{-1}}(f^n(x), g_n(y)), \quad k = 1, \ldots, n;$$

for each $y \in D_0^-(x)$ satisfying $\text{dist}_{D^+}(f^n, g_n y) \leq \rho_1$. Moreover, there exists $\tau_\ast > 0$ such that, defining $t_k$ as the least positive real so that $g_k(y) = \phi_{t_{n-k}}(y)$, then $|t_{n-k} - (n-k)| \leq \tau_\ast$.

**Proof.** We follow the proof of [3] Lemma 2.7. Let $\gamma_0$ be a curve of minimal length in $D_n^+(x)$ from $f^n x$ to $g_n y$ and let $\gamma_k = (g_k)^{-1}(\gamma_0), k = 1, \ldots, n$. Arguing by induction, let us assume that for some $k = 1, \ldots, n$ we have the following bound for the length: $\ell(\gamma_k) \leq e^{-k\alpha_0/8} \ell(\gamma_0)$. The choice of $\rho_1$ in (25) together with the definition of hyperbolic times and Lemma 4.5 ensure that

$$\|D(R_{n-1} \circ \cdots \circ R_{n-k-1})^{-1}_{\gamma_0}(z)\| \leq e^{(k+1)\alpha_0/8} \|\gamma'_0(z)\| \prod_{j=n-k}^{n} \|P^1 |_{\gamma_j}\|^{-1} \leq e^{-(k+1)\alpha_0/8} \|\gamma'_0(z)\|$$

where $\gamma'_0(z)$ denotes the tangent vector to $\gamma_0$ at $z$. Therefore

$$\ell(\gamma_{k+1}) \leq e^{-(k+1)\alpha_0/8} \ell(\gamma_0) = e^{-(k+1)\alpha_0/8} \text{dist}_{D^+}(f^n x, g_n y) \leq \rho_1 e^{-(k+1)\alpha_0/8} \leq \rho_1,$$

which shows that the maps are well-defined on their domains and completes the induction. Finally, as a standard consequence of Gronwall’s Inequality, if $y \in D_0^-(x)$ is such that $\text{dist}_{D^+}(f^n x, g_n y) \leq \rho_1$ and $k = 1, \ldots, n$, then

$$d(f^{n-k+1} x, f g_k y) \leq e^{\ell} d(f^{n-k} x, g_k y) \leq e^{\ell} \text{dist}_{D_{n-k}^+}(f^{n-k} x, g_k y) \leq \rho_1 e^{\ell} e^{-k\alpha_0/8}.$$

Hence, the time $\tau_k$ it takes for $f(g_k y)$ to arrive at $\phi_{t_{n-k+1}} y = g_{k-1} y \in D_{n-k+1}^-(x)$ is bounded from above by the above distance divided by the speed of flow. Thus, since for $z$ in a $\rho_1 e^{-(n-k+1)\alpha_0/8}$-neighborhood of $f^{n-k} x$, we have either $\|G_z\| \geq \gamma_0$, or $\|G_z\| \geq \gamma_0.$
by Remark 3.7, then
\( S \) and both summands are restrictions of \( (\text{continuous smooth map; see the proof of Lemma 4.6 and Figure 8). Since } \Sigma \nrightarrow \text{to its domain. Moreover, } \)

\( Chain \ Rule \) \ Follow \[4, \ Proposition \ 2.8\] we write \( Proof. \) \( \)  

\[ |\tau_k| \leq \frac{\rho_1 e^{L-k\alpha}/8}{\|G_{f^{n-k+1}}\|} \leq \frac{\rho_1 e^{L-k\alpha}/8}{\|G_{f^{n-k+1}}\|} \leq \frac{\rho_1 e^{L-k\alpha}/8}{\|G_{f^{n-k+1}}\|} = \frac{\rho_1 e^{L+\alpha/16}}{b^2 L} e^{-(k-1)\alpha/16}. \]

This shows that \( t_{n-k+1} - t_{n-k} = 1 + \tau_k. \) Since \( t_0 = 0, \) then

\[ |t_{n-k} - (n - k)| = \left| \sum_{i=k}^{n-1} (t_{n-i+1} - t_{n-i}) - (n - k) \right| \leq \sum_{i=k}^{n-1} |\tau_i| \leq \frac{\rho_1 e^{L+\alpha/16}}{b^2 L} \sum_{i=k}^{n-1} e^{-(k-1)\alpha/16}. \]

which completes the proof after setting \( \tau_* = \frac{\rho_1 e^{L+\alpha/16}}{b^2 L} (1 - e^{-\alpha/16})^{-1}. \)

**Proposition 4.7** (Sectional bounded distortion). There exists \( C_2 > 1 \) so that, given a \( u\)-disk \( D \) tangent to the centre-unstable cone field with \( \kappa(D) \leq C_1, \) and given \( x \in D \) and \( n \geq 1 \) a hyperbolic time for \( x, \) then

\[ \frac{1}{C_2} \leq \frac{|\det Dg_n|_{T_yD^+_n(x)}}{|\det Df^n|_{T_yD^+_n(x)}} \leq C_2 \]

in the notation of Lemma 4.6 for every \( y \in D^+_0(x) \) such that \( \operatorname{dist}_{D^+_n(x)}(g_n y, f^n x) \leq \rho_1. \)

**Proof.** Follow \[4, \ Proposition \ 2.8\] we write \( J_i(y) = |\det DR_i|_{T_yD^+_i(x)} | \) and so by the Chain Rule

\[ \log \frac{|\det Dg_n|_{T_yD^+_n(x)}}{|\det Df^n|_{T_yD^+_n(x)}} = \sum_{i=0}^{n-1} (\log J_i(y) - \log J_i(x)). \]

We recall that \( R_i = R_t \circ f, \) where \( R_t : S = f(\Sigma_{f, x}^{n-i}) \rightarrow \Sigma_{f, x}^{n-i-1} \) is the Poincaré first visit map; see the proof of Lemma 4.6 and Figure 8. Since \( \Sigma_{f, x}^{n-i} \) is a restriction of a \( u\)-disk with curvature bounded by \( C_1, \) and contained in the \( u\)-disk \( W_i = \phi(-\varepsilon \varepsilon_i) \Sigma_{f, x}^{n-i} \) with \( \kappa(W_i) \leq C_1 \) by Remark 3.7, then \( S \) has bounded curvature by Proposition 3.6. By construction, \( S \) is also tangent to \( \Sigma_{f, x}^{n-i-1} \) at \( f^{i+1}x. \) Hence, we can see \( S \) as a graph of a \((L_1, \zeta)\)-Hölder continuous smooth map \( h : \Sigma_{f, x}^{n-i-1} \rightarrow \mathbb{R} \cdot G_{f, x} \) and \( R_t \) as the projection from this graph to its domain. Moreover, \( f : W_i \rightarrow \phi(-\varepsilon \varepsilon_i) S \) is a \( C^2 \) diffeomorphism from a flat submanifold to a manifold whose curvature is bounded by \( C_1. \) Thus

\[ \log J_i(y) = \log |\det DR_i|_{T_yD^+_i(x)} | + \log |\det Df|_{T_yD^+_i(x)} | \]

and both summands are restrictions of \((L_1, \zeta)\)-Hölder continuous maps.

Therefore, the sum is bounded above by \( \sum_{i=0}^{n-1} 2L_1(e^{-i\alpha}/8) \) \( \leq \frac{2L_1\rho_1^2}{(1 - e^{-\alpha/8})}. \) The proof is complete after setting \( C_2 = \exp(2L_1\rho_1^2/(1 - e^{-\alpha/8})). \)

\[ \square \]
5. LEBESGUE MEASURE AND HYPERBOLIC TIMES

We extend the construction of backward contraction to a full neighborhood of points in a \(u\)-disk at hyperbolic times in Subsection 5.1.

This provides the tools needed to construct the physical/SRB measure, outlined in Subsection 5.2, leading to the proof of Theorem D in Subsection 5.3.

5.1. Distortion bounds and central-unstable disks at hyperbolic times. In what follows, we set \(\text{dist}_D(x, \partial D) = \inf_{y \in \partial D} \text{dist}_D(x, y)\) for the distance from a point \(x \in D\) to the boundary of \(D\). We assume without loss of generality that \(U\) contains a \(\rho_1\)-neighborhood of \(\Lambda\).

Let \(z \in U^*, \; N_z^u := E_X^u \cap G_z^\perp\) and \(W = \exp_z(N_z^u \cap B(0, \rho_0))\) be such that the \(u\)-disk \(D = \phi_{(-\rho, \rho)} W\) for some \(\rho = \rho(z) > 0\) satisfies \(\ell(\phi_{(-\rho, \rho)} z) > 2\rho_1\) and \(\text{Leb}_D(\Omega) > 0\), where \(\text{Leb}_D\) is the volume measure induced in the embedded disk \(D\) by the Riemannian metric of \(M\). Remark 3.7 ensures that \(\kappa(D) \leq C_1\).

We note that this disk is an union of segments of trajectories of the flow – we say that this is a \(cu\)-disk. Moreover, since there exists \(\gamma_s > 0\) such that \(\|G_x\| \leq \gamma_s\) for all \(x \in U\), we necessarily have that \(2\rho_1 < \ell(\phi_{(-\rho, \rho)} z) \leq 2\rho\gamma_s\) and so \(\rho > \rho_1\gamma_s^{-1}\). We set

\[A = A(D, \rho_1) = \{x \in D \cap \Omega : \text{dist}_D(x, \partial D) \geq \rho_1\}\]

so that \(\text{Leb}_D(A) > 0\), reducing \(\rho_1\) if necessary.

Let \(\gamma_0 > 0\) be such that \(\|G_x\| \geq \gamma_0\) for all \(x \in U^*\) with \(d(x, \text{Sing}_\Lambda(G)) > \delta_0\).

Next result states robust local sectional backward contraction and bounded distortion, together with the consequence for the push-forward of Lebesgue measure along \(cu\)-disks.

**Proposition 5.1.** Let \(x \in A\) and \(n > 1\) be a hyperbolic time for \(x\). Then there exists an open neighborhood \(V_n(x)\) of \(x\) in \(D\), a \(\delta_0\)-ball \(W_n(x)\) inside \(f^n(D)\) centered at \(f^n x\) such that

1. \(f^n |_{V_n(x)} : V_n(x) \to W_n(x)\) is a diffeomorphism, and
2. there exists \(s_0 > 0\) such that \(\phi(-s_0, 0) x \subset V_n(x)\) and \(f^n(\phi(-s_0, 0) x)\) has length at least \(\rho_1\), and for all \(-s_0 < s < s_0\)
   (a) \(n\) is a hyperbolic time for \(\phi_s x\), \(D^\perp_1(\phi_s x) \subset V_n(x)\), and
   (b) the translated Poincaré maps \(R^i_t : \text{dom}(R^i_t) \subset \Sigma_{\phi_{sf}^i(x)} \to \Sigma_{\phi_{sf}^i(x)}^n, i = 0, \ldots, n - 1\) composed to form \(g^*_n = R_{n-1}^\circ \cdots \circ R^0\) satisfy:
      (i) \((g^*_n |_{D^\perp_1(\phi_s,x)})^{-1} : D^\perp_1(\phi_s f^n) \to D^\perp_1(\phi_s x)\) is an \(e^{-\gamma_0/8}\)-contraction, and
      (ii) for every \(y \in D^\perp_1(\phi_s x)\) such that \(\text{dist}_D(\phi_s x, y) < \rho_1\) we get
      \[
      C_2^{-1} \leq \frac{|\det Dg_n^*|^{|T_{\phi_{sf}^i(x)} D^\perp_1(\phi_s x)|}}{|\det Df^n|^{|T_{\phi_{sf}^i(x)} D^\perp_1(\psi_s x)|}} \leq C_2.
      \]
3. there exists \(C_3 > 0\) so that \(f^n(\text{Leb}|_{V_n(x)}) \leq C_3 \cdot (\text{Leb}|_{W_n(x)}\).

**Proof.** Fixing \(x \in A\) and \(n\) a hyperbolic time for \(x\), then \(n\) is also a hyperbolic time for \(\phi_s(x) \in D\) for \(-s_0 < s < s_0\) with \(s_0\) given by Proposition 4.4. Moreover, \(D^\perp_1(\phi_s x) \subset D\) and we obtain item (2a).

Hence, \( d(\phi_s f^n x, \text{Sing}_\Lambda(G)) > \delta_0 \), which implies that \( \| G_{\phi_s f^n x} \| \geq \gamma_0 \) for all \( |s| < s_0 \). Thus \( \ell(\phi_{(-s_0,s_0)} f^n x) \geq 2s_0 \gamma_0 \).

In addition, since \( \| G_y - G_x \| \leq Ld(x, y) \), if \( d(x, \text{Sing}_\Lambda(G)) < \delta_0 \), then \( s_0 b_s \| G_x \| \leq \ell(\phi_{(-s_0,s_0)} x) \leq s_0 \cdot 2Ld(x, \text{Sing}_\Lambda(G)) < \frac{1}{2} d_{s_0}(x, \text{Sing}_\Lambda(G)) \), by Lemma 4.3.

To obtain item (2b), we apply Lemma 4.6 together with Proposition 4.7 as follows; see Figure 9. We write

\[
\text{FIGURE 9. Sectioning the trajectory of } V_n(x) \text{ and } W_n(x) \text{ through normal cross-sections to the trajectory of } x, \text{ to then apply Fubini’s Theorem.}
\]

\[
W_n(x), \text{ we use Lemma 4.6 together with Proposition 4.7 as follows; see Figure 9. We write}
\]

\[
v(\phi_s x) = \text{Leb}^+ |V_n(x) \cap D_n^\perp(\phi_s x) \quad \& \quad w(\phi_s x) = \text{Leb}^+ |W_n(x) \cap D_n^\perp(\psi_s x), \quad |s| < s_0 ;
\]

for the normalized volume measures induced on \( D_n^\perp(\psi_s x) \cap V_n(x) \) and \( W_n(x) \cap D_n^\perp(\psi_s x) \), respectively. Since \( \text{Leb} |V_n(x) \cap D_n^\perp(\phi_s x) \cap W_n(x) \cap D_n^\perp(\psi_s x) | \), then we can apply Fubini’s Theorem.

We have that \( g_n^s(y) = \phi_{\tau(s,y)} \circ f^n \) for each \( y \in D_n^\perp(\psi_s x) \) with \( |\tau(s,y)| \leq \tau_s \), so \( f^n = h \circ g_n^s \) and \( h : V_n(x) \rightarrow W_n(x) \) is a diffeomorphism with bounded derivatives. Thus, item (2b)(ii) ensures that \( (g_n^s)_* v(\phi_s x) \leq C_2 w(\phi_s x) \), and there exists a constant \( K > 0 \) so that

\[
f_n^s \left( \text{Leb} |V_n(x) \cap D_n^\perp(\phi_s x) \right)(E) = f_n^s \left( \int_{-s_0}^{s_0} v(\phi_s x)(E) \frac{ds}{\|G_{\phi_s x}\|} \right) = \int_{-s_0}^{s_0} \left[ (h \circ g_n^s)_* v(\phi_s x)(E) \right] \frac{ds}{\|G_{\phi_s x}\|}
\]

\[
\leq C_2 \int_{-s_0}^{s_0} \left[ h \circ g_n^s \right](E) \frac{ds}{\|G_{\phi_s x}\|} \leq C_2K \int_{-s_0}^{s_0} w(\phi_s x)(E) \frac{ds}{\|G_{\phi_s x}\|}
\]

\[
\leq C_2K \int_{\phi_{(-s_0,s_0)} f^n x} \left( \text{Leb} |W_n(x) \cap D_n^\perp(\phi_s x) \right)(E) dz
\]

\[
= C_2K \left( \text{Leb} |W_n(x) \right)(E).
\]

This completes the proof after setting \( C_3 = K C_2 \).\(\square\)
5.2. Construction of a physical probability measure. We now have all the basic tools needed to follow the construction presented in [4] Sections 3 & 4] to obtain a physical/SRB probability measure for the flow. We present a step by step overview in what follows.

For each \( n > 1 \) we set

\[
H_n = \{ x \in A(D, \rho_1) : n \text{ is a hyperbolic time for } x \}.
\]

From Proposition 5.1 if \( x \in H_n \), then \( f^n x \) is \( \rho_1 \)-away from the boundary of \( f^n D \). For \( \delta > 0 \), we denote \( \Delta_n(x, \delta) \) the \( \delta \)-neighborhood of \( f^n x \) inside \( f^n(D) \). If \( \text{Leb}_D \) is the probability measure \( \text{Leb}_D(E) = \text{Leb}(D \cap E)/\text{Leb}(D) \) for every Borel subset \( E \subset D \), obtained by normalizing the Riemannian induced volume measure on \( D \), then \( (f^n \text{Leb}_D)(\Delta_n(x, \delta_1)) \leq C_\delta \text{Leb}(f^n(D) \cap \Delta_n(x, \delta_1)) \), again from Proposition 5.1.

The following is a geometrical consequence of the finite dimensionality and bounded curvature of \( u \)-disks.

**Proposition 5.2.** There is \( \tau > 0 \) so that for \( n > T_2 \) there exists a finite subset \( \hat{H}_n \) of \( H_n \) such that the balls \( \Delta_n(z, \rho_1/4) \) in \( f^n(D) \) centered at \( z \in f^n(\hat{H}_n) \) are pairwise disjoint, and their union \( \Delta_n \) satisfies \( (f^n \text{Leb}_D)(\Delta_n \cap H) \geq (f^n \text{Leb}_D)(\Delta_n \cap f^n(H_n)) \geq \tau \text{Leb}_D(H_n) \).

**Proof.** See [4] Proposition 3.3 & Lemma 3.4. \( \square \)

Let \( D_n = \{ \Delta_n(z, \rho_1/4) : z \in f^n(\hat{H}_n) \} \) be the collection of balls that form \( \Delta_n \). We note that all these balls are \( \delta_0 \)-away from \( \text{Sing}_A(G) \), and we define

\[
\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} f_j^* \text{Leb}_D; \quad \nu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f_j^* \text{Leb}_D) |_{\Delta_j} \quad \text{and} \quad \eta_n := \mu_n - \nu_n. \tag{28}
\]

**Proposition 5.3.** There is \( \alpha > 0 \) so that both \( \nu_n(H) \geq \alpha \) and \( \nu_n(\cup_{i=0}^{n-1} f^i(D \cap H(\sigma))) \geq \alpha \) for all sufficiently large \( n > T_2 \).

**Proof.** Just follow [4] Proposition 3.5], using Proposition 5.2 together with the positive asymptotic frequency of hyperbolic times for each \( x \in \Omega \), given by Proposition 4.2. \( \square \)

We consider weak* accumulation points \( \mu \) and \( \nu \) of \( (\mu_n), (\nu_n) \) respectively, along some subsequence \( (n_k)_k \). It is standard that \( \mu \) is a \( f \)-invariant probability measure and that \( \tilde{\mu} = \int_0^1 (\phi_t)_* \mu \, dt \) is a flow invariant probability measure; see e.g. [17]. In addition, \( \nu(U) \geq \limsup_k \nu_k(U) \geq \alpha > 0 \).

We claim that \( \nu \) has a property of absolute continuity along certain disks contained in its support. We define the collection of these disks in what follows.

Note that \( \nu_n \) is supported on the union \( \cup_{j=0}^{n-1} \Delta_j \) of disks with uniform size and \( \delta_0 \)-away from \( \text{Sing}_A(G) \). Then \( \text{supp} \nu \) is contained in \( \Delta_\infty = \cap_{n=1}^\infty \text{Closure}(\cup_{j \geq n} \Delta_j) \), the family of accumulation points of such disks. That is, for \( y \in \Delta_\infty \) there are \( (j_i)_i \to \infty \), disks \( \Delta_i = \Delta_{j_i}(x_i, \delta_1/4) \subset \Delta_j \), and points \( y_i \in \Delta_i \) so that \( y_i \to y \) when \( i \nearrow \infty \).

We may assume without loss of generality, taking subsequences if necessary, that \( x_i \) converges to some point \( x \). By uniform size and bounded curvature, we can use the Ascoli-Arzelà Theorem to conclude that \( \Delta_i \) converge to a \( u \)-disk \( \Delta(x) \) with radius \( \rho_1/4 \) centered at \( x \). Then \( y \in \text{Closure} \Delta(x) \subset \Delta_\infty \).
Lemma 5.4. Every $y \in \tilde{\Delta}(x)$ is such that $N^u_y$ is uniformly expanding: $\|(P^k |_{N^u_y})^{-1}\| \leq e^{-kco/8}$ for all $k \geq 1$. The disk $\tilde{\Delta}(x)$ is contained $\Lambda$ and also in the unique\(^{10}\) center-unstable manifold $W^c_{x}(\rho_1)$ tangent to $E^c_x$ containing a $\rho_1$-ball around $x$.

Proof. Let $j_i \nearrow \infty$, $x_i \to x$, and $\tilde{\Delta}_i \to \tilde{\Delta}(x)$ be as in the construction described previously. We have that $\tilde{\Delta}_i$ is contained in the $j_i$th-iterate of $D$, which is a $u$-disk. The domination of the splitting on $U$ ensures that $\La(\tilde{\Delta}_i, E^c_{x}) \to 0$ as $i \to \infty$, uniformly on $\Lambda$; this is a consequence of Proposition 3.3.

By Proposition 5.1, $f^{-k}$ is an $e^{-kco/8}$-contraction on $\tilde{\Delta}_i \cap \exp_{\phi_s f^{j_i-x}}(N^u_{\phi_s f^{j_i-x}})$ for every large $i$ and any given fixed $k \geq 1$ and $|s| \leq s_0$. Passing to the limit in $i$, we get that $f^{-k}$ is an $e^{-kco/8}$-contraction on $\tilde{\Delta}(x) \cap \exp_{\phi_{s}}(N^u_{\phi_{s}})$, and $\tilde{\Delta}(x)$ is a $u$-disk in $\Lambda$ by continuity of the splitting on $U$.

We have shown that the subspace $E^c_{x}$ is uniformly sectionally expanding for $Df$ on $\tilde{\Delta}(x)$. Since $Df |_{E^u}$ is uniformly contracted, we are in the setting of \cite{22} Section 3 of Chapter 7 with respect to $f^{-1}$. Then there exists a unique center-unstable manifold $W^c_{x}(\rho_1)$ tangent to $E^c_{x}$ containing a $\rho_1$-ball around $x$. \hfill \Box

5.2.1. Absolute continuity. The same arguments in \cite{1} Section 4.1 imply the following result, where we write cylinder for any diffeomorphic image of $\mathbb{D}^{d_{cu}} \times \mathbb{D}^{d_{s}}$ into $U$.

Proposition 5.5. There exist $C_3 > 1$ and a cylinder $C \subset M$, with a family $K_{\infty}$ of disjoint disks, contained in $C \cap \Delta_{\infty}$, which are graphs over $\mathbb{D}^{d_{cu}}$, such that

(1) the union $K_{\infty}$ of all disks in $K_{\infty}$ has positive $\nu$-measure;
(2) the restriction $\nu |_{K_{\infty}}$ has absolutely continuous conditional measures $\nu_{\gamma}$ with respect to the induced volume $\text{Leb}_{\gamma}$ along the disks $\gamma \in K_{\infty}$, whose density is bounded: $C_3^{-1} \leq d\nu_{\gamma}/d\text{Leb}_{\gamma} \leq C_3$.

Proof. See \cite{1} Proposition 4.1 & Lemma 4.4], whose proof uses the properties of $\Delta_{\infty}$ already obtained. \hfill \Box

5.2.2. Ergodicity and ergodic basin. Following \cite{1} Section 4.2 we obtain the next result.

Lemma 5.6. The $f$-invariant probability measure $\mu = \nu + \eta$ has an ergodic component $\mu_*$ whose Lyapunov exponents are all non-zero, except along the direction of the vector field, and whose conditional measures along local unstable manifolds are absolutely continuous with respect to Lebesgue measure. Moreover, $\text{supp} \mu_* \subset \Lambda$ and $\text{Leb}_{D}(B(\mu_*) \cap H) > 0$.

Proof. This is \cite{1} Lemma 4.5], whose proof uses the properties of $\Delta_{\infty}$ already obtained in the previous arguments. \hfill \Box

5.3. Finitely many physical/SRB measures for the flow. The following completes the proof of Theorem D

\(^{10}\)The center-unstable manifold might depend on the radius, but it is uniquely defined given the radius.
Corollary 5.7. There exist finitely many ergodic hyperbolic physical/SRB invariant probability measures \( \eta_1, \ldots, \eta_k \) for \( f \) and \( \mu_1, \ldots, \mu_k \) for the flow \( \phi_t \) of \( G \), supported on \( \Lambda \), such that \( \text{Leb} \left( \bigcup_{i=1}^k B(\mu_i) \right) = 0 \) and \( \text{Leb} \left( \Omega \cap (B(\eta_i) \triangle B(\mu_i)) \right) = 0, \forall i. \)

Proof. The existence of finitely many ergodic hyperbolic physical/SRB measures \( \eta_1, \ldots, \eta_k \) with respect to \( f \) supported in \( \Lambda \) and satisfying \( \text{Leb} \left( \bigcup_{i=1}^k B(\eta_i) \right) = 0 \) follows by [11 Corollary 4.6] using the properties already obtained.

We are left to obtain the \( G \)-invariant ergodic physical probability measures. The probability measures \( \mu_i = \int_0^1 (\phi_t)_* \eta_i \, dt \) are \( \phi_t \)-invariant for every \( t > 0 \) and \( \mu_i \) are ergodic for the flow, \( i = 1, \ldots, k. \)

Moreover, if \( \varphi : M \to \mathbb{R} \) is continuous and \( x \in B(\eta_i) \), then \( \psi = \int_0^1 \varphi \circ \phi_s \, ds \) is also continuous, and since \( \phi_t \) and \( f \) commute

\[
\int \varphi \, d\mu_i = \int \psi \, d\eta_i = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j x)
\]

\[
= \lim_{n \to +\infty} \frac{1}{n} \int_0^n \varphi(\phi_s x) \, ds = \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(\phi_s x) \, ds,
\]

where the last equality follows from boundedness of \( \varphi \). This shows that \( B(\eta_i) \subset B(\mu_i) \) and so \( \mu_i \) becomes a physical measure and also \( \text{Leb} \left( \Omega \cap (B(\eta_i) \triangle B(\mu_i)) \right) = 0 \) for \( i = 1, \ldots, k. \)

Hyperbolicity of \( \mu_i \) follows from partial hyperbolicity coupled with\(^{11}\)

\[
\frac{\log \left\| (\wedge^2 D\phi_T) \big|_{E_{tx}^u} \right\|^{-1}}{T} \leq \frac{\log \left\| (\wedge^2 D\phi_{T-[T]} |_{E_{tx}^u})^{-1} \right\|}{T} + \sum_{i=0}^{[T]-1} \frac{\log \left\| (\wedge^2 Df) \big|_{E_{tx}^u} \right\|^{-1}}{T}
\]

\[
\leq \frac{[T]}{T} \cdot \frac{1}{[T]} \sum_{i=0}^{[T]-1} \log \left\| (\wedge^2 Df) \big|_{E_{tx}^u} \right\|^{-1} + (2/T) \log L
\]

so that for \( x \in \Omega \cap B(\mu_i) \) we obtain from [8]

\[
\limsup_{T \to \infty} \frac{1}{T} \log \left\| (\wedge^2 D\phi_T) \big|_{E_{tx}^u} \right\|^{-1} \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left\| (\wedge^2 Df) \big|_{E_{tx}^u} \right\|^{-1} \leq -c_0.
\]

By smoothness of the flow, the absolute continuity of conditional measures of \( \eta_i \) along unstable manifolds implies absolute continuity of conditional measures of \( \mu_i \) along weak-unstable (or center-unstable) manifolds, so that each \( \mu_i \) is also a \( cu \)-Gibbs state. That is, each \( \mu_i \) is an ergodic hyperbolic physical/SRB measure for the flow, completing the proof. \( \square \)

6. Proof of equivalence between discrete and continuous notions

To easily deduce Theorems A and B from Theorem D, we recall some general properties of Gibbs \( cu \)-states.

\(^{11}\)We write \( [t] = \sup\{\ell \leq t : \ell \in \mathbb{Z}^+\} \) for the integer part of \( t \in \mathbb{R} \).
6.1. Properties of Gibbs \textit{cu}-states. We collect some useful results here.

**Theorem 6.1.** Let \( \Lambda = \Lambda_G(U) \) be a partially hyperbolic attracting set for a \( C^2 \) vector field \( G \) which is non-uniformly sectional expanding on \( \Omega \subset U \) with \( \text{Leb}(\Omega) > 0 \). Then

(1) the family \( \mathcal{E} \) of all \( G \)-invariant physical probability measures \( \mu \) such that \( \text{Leb}(\Omega \cap B(\mu)) > 0 \) is the convex hull \( \mathcal{E} = \{\sum_{i=1}^{k} t_i \mu_i : \sum_i t_i = 1, t_i \geq 0, i = 1, \ldots, k\} \). The same holds replacing \( G \)-invariance by \( \phi_t \)-invariance, for some fixed value of \( t > 0 \).

(2) for a \( G \)-invariant (or \( f \)-invariant) hyperbolic probability measure \( \mu \) supported in \( \Lambda \), with \( \mu(\Omega) > 0 \), the following are equivalent

\( a \) the Entropy Formula: \( h_\mu(f) = \int \log |\det Df|_{E^{cu}} \, d\mu \);

\( b \) \( \mu \) is a \( cu \)-Gibbs state, that is, admits an absolutely continuous disintegration along center-unstable manifolds;

\( c \) \( \mu \) is a physical measure, i.e., its basin \( B(\mu) \) has positive Lebesgue measure.

(3) the basin \( B(\mu) \) of a physical measure \( \mu \) supported in \( \Lambda \), with \( \text{Leb}(B(\mu) \cap \Omega) > 0 \), admits an open subset \( V \) which intersects \( \Lambda \) and is contained in the ergodic basin except a zero volume subset, that is, \( \text{Leb}(V \setminus B(\mu)) = 0 \) and \( V \cap \Lambda \neq \emptyset \).

(4) if \( \Lambda \) is transitive, then there exists only one physical probability measure which is also a Gibbs-\( cu \)-state such that \( m(B(\mu) \setminus \Omega) = 0 \).

**Proof.** For item (1), Theorem \[\text{D}\] (cf. Corollary 5.7) ensures the existence of finitely many ergodic hyperbolic physical/SRB measures \( \mu_1, \ldots, \mu_k \) such that the union of their ergodic basins covers \( \Omega \) Lebesgue almost everywhere: \( \text{Leb}(\Omega \setminus \bigcup_{i=1}^{k} B(\mu_i)) = 0 \). We note that if there are no equilibria, then we can take \( \Omega = U \). The measures considered can either be invariant for the flow, or \( f \)-invariant, or even \( \phi_t \)-invariant for any fixed \( t > 0 \).

Since \( \text{Leb}(B(\mu) \cap \Omega) > 0 \), it follows that \( B(\mu) \cap \Omega = \Omega \setminus \left( \bigcup_{i=1}^{k} B(\mu) \cap B(\mu_i) \right) \) Lebesgue modulo zero. By definition of ergodic basin, for each continuous observable \( \varphi : U \to \mathbb{R} \) we get

\[
\int \varphi \, d\mu = \frac{1}{\text{Leb}(\Omega \cap B(\mu))} \int_{\Omega \cap B(\mu)} \frac{1}{\text{Leb}(B(\mu) \cap \Omega)} \int \varphi \, d\left( \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x} \right) \, d\text{Leb}(x) = \sum_{i=1}^{k} \frac{\text{Leb}(B(\mu) \cap B(\mu_i) \cap \Omega)}{\text{Leb}(B(\mu) \cap \Omega)} \int \varphi \, d\mu_i,
\]

where the limit is in the weak* topology of the probability measures of the ambient space \( M \). Hence, we deduce \( \mu = \sum_{i=1}^{k} \frac{\text{Leb}(B(\mu) \cap B(\mu_i) \cap \Omega)}{\text{Leb}(B(\mu) \cap \Omega)} \mu_i \) and \( \mu \) as a convex linear combination of the ergodic physical/SRB measures provided by Theorem \[\text{D}\].

For item (2), since \( G \) is contained is \( E^{cu} \) and has zero Lyapunov exponent, then domination of the splitting \( E^s \oplus E^{cu} \) ensures that all Lyapunov exponents along \( E^s \) are strictly negative and so \( \int \log |\det Df|_{E^{cu}} \, d\mu = \int \chi^+ \, d\mu \) by Oseledets’ Multiplicative Ergodic Theorem. This holds either for \( G \)-invariant of \( f \)-invariant probability measures, or even \( \phi_t \)-invariant for a fixed value of \( t > 0 \).
Then, assumption (2a) means $h_\mu(f) = \int \chi^+ \, d\mu > 0$. In particular, $\mu$ is non-atomic\[13\] and this becomes the necessary and sufficient condition for absolutely continuous disintegration along unstable manifolds $W_{x}^{uu}$ for $\mu$-a.e. $x$, by the characterization of measures satisfying the Entropy Formula [10] for $C^2$ smooth dynamics. This means, more precisely, that for $\mu$-a.e. $x \in \Lambda$ there exists $\rho = \rho(x) > 0$ so that

$$
\Pi_x = \{ W_{x}^{uu} : y \in B(x, \rho) \cap W_{x}^{uu} \}
$$

and\[13\] the normalized restriction $\hat{\mu} = \mu |_{\Pi_x}$ disintegrates along the leaves of $\Pi_x$ as $\hat{\mu} = \int \mu_y \, d\hat{\mu}(y)$. Here $\mu_y$ is a probability measure supported on $\gamma^y$ equivalent to the restriction $\text{Leb}_y$ of Lebesgue measure on this submanifold and $\hat{\mu} = \pi \ast \hat{\mu}$, where $\pi : \Pi_x \to \Pi_x$ is the quotient map associating a point of $\Pi_x$ to the corresponding leaf of $\Pi_x$.

Each manifold $W_{x}^{uu}$ is contained in $\Lambda$ with dimension $\dim E^{cu} - 1$ and tangent bundle in $E^{cu}$. The center-unstable (or weak-unstable) manifolds $W_{x}^{cu} = \phi_{(-1,1)}(W_{x}^{uu})$ are tangent to the center-unstable bundle $E^{cu}$ at each point and also the disjoint union of strong-unstable leaves transported by the flow. By smoothness of the flow, the disintegration of $\mu$ along the center-unstable leaves is also absolutely continuous.

Indeed, for small enough $\rho$, $W_{x}^{cu}$ crosses $B(x, \rho)$. Considering $\Pi_x^c = \{ W_{x}^{cu} : y \in B(x, \rho) \cap W_{x}^{cu} \}$, then $\cap \Pi_x = \cap \Pi_x^c$ and $\hat{\mu} = \int \nu_z \, d\hat{\nu}(z)$, with $\hat{\nu} = \tilde{\pi} \ast \hat{\mu}$ where $\tilde{\pi} : \Pi_x \to \Pi_x^c$ is the corresponding quotient map, and $\nu_z = \int \mu_y \, d(\pi_y \nu_z)$ is equivalent to Leb induced on the connected component $\gamma_{z}^{cu}$ of $W_{z}^{cu} \cap B(x, \rho)$ containing $z$. This is the property stated in item (2b).

Assuming condition (2b), the Ergodic Theorem provides a full $\mu$-measure subset $B$ of Birkhoff generic points for $\mu$ which is also a full $\mu$-measure subset. Hence, $B$ has full $\nu_z$-measure for $\nu_z$-a.e. $z$. If we fix a center unstable disk $\gamma_z^{cu}$ for a $\nu_z$-generic $z$, then $\nu_z(B) = 1$ and $B$ is also a full $\text{Leb}_z$-measure subset of $\gamma_z^{cu}$. Since the stable foliation is defined at all points of $\Lambda$, tangent to the stable bundle $E^{s}$ which makes an angle with the center-unstable bundle uniformly bounded away from zero, then the subset $W_{x}^{s}(\gamma_{z}^{cu}) = \{ W_{y}^{s} : w \in \gamma_{z}^{cu} \}$ is an open neighborhood of $z$ for small enough $\varepsilon > 0$, where $W_{y}^{s}(\varepsilon)$ is the connected component of $W_{w}^{s} \cap B(w, \varepsilon)$ containing $w$. Moreover, the stable foliation is absolutely continuous [13 Section 6], and so the subset $W = \{ W_{y}^{s}(\varepsilon) : w \in B \cap \gamma_{z}^{cu} \}$ has full $\text{Leb}$-measure in $W_{x}^{s}(\gamma_{z}^{cu})$:

$$
\text{Leb}(W \setminus W_{x}^{s}(\gamma_{z}^{cu})) = 0.
$$

In addition, each $y \in W$ is such that $d(\phi_{t}y, \phi_{t}w) \to 0$ when $t \to \infty$ for some $w \in B \cap \gamma_{z}^{cu}$. Hence, for any given continuous observable $\varphi : U \to \mathbb{R}$ we obtain

$$
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varphi(\phi_{t}y) \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varphi(\phi_{t}w) \, dt = \int \varphi \, d\mu
$$

\[29\]

\[12\] For otherwise by Ergodic Decomposition, Jacob’s Theorem [73] Chpt. 9, Sec. 6] and Ruelle’s Inequality [43 Chap. IV, Sec. 10] we would obtain the Entropy Formula for each ergodic component $\nu$ of $\mu$. In particular, if $\nu$ is supported on a critical element of $\Lambda$, either an equilibrium or a periodic orbit, then $0 = h_\nu(f) = \int \log | det | E| \, dv = \int \chi^+ \, dv$, contradicting the hyperbolicity assumption on $\mu$.

\[13\] We say that $W_{x}^{uu}$ crosses $B(x, \rho)$ if the connected component $\gamma_y$ of $W_{y}^{uu} \cap B(x, \rho)$ containing $y$ projects into the corresponding connected component $\gamma_x$ of $W_{x}^{uu} \cap B(x, \rho)$ containing $x$, through the stable holonomy map $\pi_y$ in a one-to-one way, i.e., $\pi_y \mid \gamma_y : \gamma_y \to \gamma_x$ is injective.
and thus $W \subset B(\mu)$ with $\text{Leb}(W) > 0$ and $\mu$ becomes a physical measure, as stated in item (2c). For an $f$-invariant measure, we replace (29) by $\lim n^{-1} \sum_{i=0}^{n-1} \varphi(f^i x)$ an argue in the same way.

We note that we immediately obtain item (3) from the previous construction, since $\text{Leb}(V \setminus W) = 0$, once we show that (2c) implies (2a).

Moreover, from item (3) we easily obtain item (4). Indeed, if there are two physical measures $\mu_1, \mu_2$, then from item (3) there exist open subsets $V_i$ such that $\text{Leb}(V_i \setminus B(\mu_i)) = 0$ and $V_i \cap \Lambda \neq \emptyset, i = 1, 2$. Transitivity ensures that there exist $x_1 \in V_1$ and $t > 0$ so that $\phi_t x_1 \in V_2$. Smoothness of the flow ensures that $\phi_t B(\mu_2)$ has positive volume in $V_1$, thus by flow invariance of the ergodic basin we find $y \in B(\mu_1) \cap B(\mu_2)$, which implies that $\mu_1 = \mu_2$.

We are left with showing that condition (2c) implies (2a). But this is an easy consequence of item (1), since a physical probability measure $\mu$ is a linear convex combination of the finitely many ergodic physical/SRB measures provided by Theorem $D$ which are $cu$-Gibbs states, that is, satisfy (2a). The proof is complete. \(\square\)

### 6.2. Proof of the main theorems.

We are now ready to deduce Theorems $A$ and $B$ assuming the statement of Theorem $D$.

**Proof of Theorem $A$.** We start by showing that asymptotic sectional expansion implies non-uniform sectional expansion which, by Theorem $D$ is enough to construct an ergodic physical/SRB probability measure, assuming slow recurrence.

In its turn, to obtain this, it is enough to show that mostly asymptotic sectional expansion $H$ implies the non-uniform sectional expanding condition $[3]$. We recall that the extension of $E_A^c$ to $U$ is not necessarily $D\phi_t$-invariant.

We note that, given any 2-subspace $F_x$ of $E^c_x$, the map $t \mapsto |\det(D\phi_t | F_x)|$ is a multiplicative function in the following sense

$$ |\det(D\phi_{t+s} | F_x)| = |\det(D\phi_s | D\phi_t F_x)| \cdot |\det(D\phi_t | F_x)|, \quad t, s \geq 0, \ x \in U. $$

In addition, by Proposition $3.3$ we get $D\phi_t \cdot E^c_x \subset C^c_{\kappa A}(\phi_t x)$ for constants $\kappa, a > 0$. Then the stable direction $E_{\phi_t x}^s$ is complementary to both the $D\phi_t \cdot E^c_x$ and $E^c_{\phi_t x}$ directions at $\phi_t x$. Therefore there exists a natural isomorphism $\pi^s : (D\phi_t \cdot E^c_x) \to E^c_{\phi_t x}$ given by the projection parallel to $E^c_{\phi_t x}$. Hence, $\pi^s(D\phi_t \cdot F_x) = \tilde{F}_{\phi_t x} \subset E^c_{\phi_t x}$ and since the width $\kappa A \alpha$ of the center-unstable cone around $E^c_{\phi_t x}$ is small for large $t > 0$, then we obtain $\xi_t \to 1$ when $t \to \infty$ such that for any fixed $s \geq 0$

$$ \xi_t^{-1} |\det(D\phi_s | \tilde{F}_{\phi_t x})| \leq \frac{|\det(D\phi_{t+s} | F_x)|}{|\det(D\phi_t | F_x)|} \leq \xi_t |\det(D\phi_s | \tilde{F}_{\phi_t x})|. \quad (30) $$

We note that the assumption $I$ implies that for every $\varepsilon > 0$ and $x \in \Omega$, there exists $N = N(\varepsilon, x) \in \mathbb{Z}^+$ so that for all $n > N$ and all 2-subspace $F_x \subset E^c_x$ we have

$$ \log |\det(D\phi_n | F_x)|^{-1} \leq \log \| (D\phi_n | E^c_x)^{-1}\| \leq -(c_0 - \varepsilon) n. $$
Since $|\det(D\phi_n|_{F_x})| = \prod_{i=0}^{n-1} \frac{|\det(D\phi_{i+1} |_{F_x})|}{|\det(D\phi_i|_{F_x})|}$ we get from the above estimates

$$\left|\log|\det(D\phi_n|_{F_x})|^{-1}\right| + \sum_{i=0}^{n-1} \log|\det(D\phi_1|_{F_{\hat{x}}})| \leq \sum_{i=0}^{n-1} \log|\log\xi_i|.$$  

Hence, for all $n > N(\varepsilon, x)$ and 2-subspace $F_x \subset E^c_{x}$, we can write

$$\sum_{i=0}^{n-1} \log|\det(D\phi_1|_{F_{\hat{x}}})|^{-1} = -(c_0 - \varepsilon)n + \sum_{i=0}^{n-1} \log|\log\xi_i|.$$  

For a regular point $x \in U$, let us choose $F_x$ with orthonormal basis $\{G(x)/\|G(x)\|, n(x)\}$ and so obtain $\hat{F}_{\hat{x}}$ with orthonormal basis $\{G(f^i x)/\|G(f^i x)\|, n(f^i x)\}$ for $i \geq 1$, which ensures that

$$|\det(D\phi_1|_{F_{\hat{x}}})| = \det\left(\frac{\|G(f^{i+1} x)\|/\|G(f^i x)\|}{0} \frac{\|P^1 \cdot n(f^i x)\|}{\|G(x)\|}\right).$$  

Thus, we obtain $\sum_{i=0}^{n-1} \log\|P^1 \cdot n(f^i x)\|^{-1} = \sum_{i=0}^{n-1} \log|\det(D\phi_1|_{F_{\hat{x}}})|^{-1} + \log\frac{\|G(n(x))\|}{\|G(f^i x)\|}$ and so for $n > N(\varepsilon, x)$ we get

$$\sum_{i=0}^{n-1} \log\|P^1|_{\mathbb{R} \cdot n(f^i x)}\|^{-1/\|n(x)\|} \leq -(c_0 - \varepsilon) + \sum_{i=0}^{n-1} \log\xi_i^{1/\|n(x)\|} + \log\left(\frac{\|G(f^i x)\|}{\|G(x)\|}\right)^{1/\|n(x)\|}.$$  

Moreover, since $\pi^\ast$ is a diffeomorphism, the one-dimensional subspaces $\mathbb{R} \cdot n(f^i x)$ span $N^c_{f^i x}$ when $n(x)$ sweeps the unit sphere in $N^c_x$. Hence, taking the supremum over all 2-subspaces $F_x \subset E^c_{x}$ containing $G(x)$, we conclude

$$\sum_{i=0}^{n-1} \log\|P^1|_{N^c_{N^c x}}\|^{-1/\|n(x)\|} \leq -(c_0 - \varepsilon) + \sum_{i=0}^{n-1} \log\xi_i^{1/\|n(x)\|} + \log\left(\frac{\|G(f^i x)\|}{\|G(x)\|}\right)^{1/\|n(x)\|}.$$  

We recall that $\log\xi_i \to 0$ when $i \not\rightarrow \infty$ and $\|G(f^i x)\|$ is bounded above, therefore

$$\limsup_{n \not\rightarrow \infty} \sum_{i=0}^{n-1} \log\|P^1|_{N^c_{N^c x}}\|^{-1/\|n(x)\|} \leq -(c_0 - \varepsilon).$$  

Since $\varepsilon > 0$ and $x \in \Omega$ where arbitrary, we obtain the non-uniform sectional expansion condition (\ref{sectional-expansion}). Hence we have schematically: $\mathbb{I} \implies \mathbb{E} \implies \mathbb{F} \implies$ existence of finitely many ergodic physical/SRB probability measures by Theorem D assuming slow recurrence throughout.

Reciprocally, let $\mu$ be an hyperbolic ergodic physical/SRB measure for the partial hyperbolic attracting set $\Lambda = \Lambda_G(U)$ with $\text{Leb}(B(\mu) \cap \Omega) > 0$, and let us deduce mostly asymptotic sectional expansion.

We start by noting that hyperbolicity of $\mu$ together with Kingman’s Subadditive Ergodic Theorem ensures that there exists $c_0 > 0$ so that

$$\inf_{t > 0} \int \log \|A^2(D\phi_t|_{E^c x})^{-1}\|^{-1/t} \, d\mu = \lim_{t \not\rightarrow \infty} \log \|A^2(D\phi_t|_{E^c x})^{-1}\|^{-1/t} < -c_0, \quad \mu \text{- a.e. } x.$$  

\footnote{Since $\text{Sing}_A(G)$ is formed by hyperbolic equilibria, it is a finite and zero volume subset, so such $x$ is \text{Leb}-generic.}
In addition, the ergodicity of $\mu$ for a flow implies that, for a co-countable set of times $t_* \in \mathbb{R}$, we have that $\mu$ is $\phi_{t_*}$-ergodic\footnote{This property is not true for transformations, i.e., if $\mu$ is $g$-ergodic, then not necessarily $\mu$ is $g^k$-ergodic for some $k > 1$. Hence the analogous to Theorem A is absent from [4].}, see e.g. [22]. That is, if a measurable set $A$ is $\phi_{t_*}$-invariant $\phi_{-t_*}(A) = A$ for this fixed value of $t_*$, then $\mu(A) \cdot \mu(M \setminus A) = 0$.

For a given small $\varepsilon > 0$ let us fix $g = \phi_T$ with some $T > 0$ so that $\int \log \| \mathcal{A}^2 (Dg |_{E^{cu}})^{-1} \|^1/T \, d\mu \leq -c_0 + \varepsilon$ and $\mu$ is $g$-ergodic. Note that $g$ is a partially hyperbolic diffeomorphism with respect to the same splitting $E^s \oplus E^{cu}$ over $\Lambda$. Since $\mu$ is a hyperbolic $cu$-Gibbs state even for the dynamics of $g$, then $\mu$ is a physical measure for $g$ also; see the proof that (2b) implies (2c) in Theorem [6.1].

Using that $\mu$ is $g$-ergodic and physical, together with the subadditive property of the continuous function $(x, t) \mapsto \log \| \mathcal{A}^2 (D\phi_t |_{E^{cu}})^{-1} \|$, we obtain for Leb-a.e. $x \in B_g(\mu)$, since $g^nx = \phi_{nt}(x)$ for $n \geq 0$

$$-c_0 + \varepsilon \geq \int \log \| \mathcal{A}^2 (Dg |_{E^{cu}})^{-1} \| \, d\mu = \lim_{n \to \infty} (1/n) \sum_{i=0}^{n-1} \log \| \mathcal{A}^2 (D\phi_T |_{E^{cu}})^{-1} \|^1/T$$

$$\geq \lim_{n \to \infty} \log \| \mathcal{A}^2 (D\phi_{nt} |_{E^{cu}})^{-1} \|^1/nT = \lim_{T \to \infty} \log \| \mathcal{A}^2 (D\phi_T |_{E^{cu}})^{-1} \|^1/T.$$  

Since $\varepsilon > 0$ was arbitrary, we conclude that we have (4) on the positive volume subset $B_g(\mu) \subset B(\mu)$ which satisfies $\text{Leb}(B_g(\mu) \cap \Omega) > 0$, completing the proof.  

\textbf{Proof of Theorem [7].} For item (1), to obtain slow recurrence from continuous slow recurrence, we note that, since $G$ is $L$-Lipschitz, where $L = \sup_{x \in U} \| DGx \|$, we have for $\varphi(x) = d(x, \text{Sing}(G))$

$$\left| \frac{d}{dt} \varphi(t) \right| \leq \| G(\phi_tx) - G(\sigma) \| \leq L \cdot d(x, \sigma) = L \cdot \varphi(t), \quad (32)$$

whenever $x$ is near $\sigma \in \text{Sing}(G)$. Hence, $e^{-Ls} \leq \varphi(\phi_s x)/\varphi(x) \leq e^{Ls}$ for $|s|$ small enough so that $\varphi(\phi_s x) = d(\phi_s x, \sigma)$. Therefore, setting $\varphi_\delta(x) = d_\delta(x, \text{Sing}_\Lambda(G))$, given $\delta > 0$ we can take $s > 0$ so that $Ls < -\log \delta^{1/2}$ and if $d(x, \sigma) < \delta$, then for $0 \leq t \leq s$

$$-\log \varphi_\delta(\phi_t x) = -\log \varphi(\phi_t x) \geq -\log \varphi(x) - Lt \geq -(1/2) \log \varphi(x). \quad (33)$$

Thus, from (7): for any $\varepsilon > 0$ we can find $\delta$ and $k \geq 2, k \in \mathbb{Z}^+$ so that $L/k < -\log \delta$ and for all $x \in \Omega$ there exists $N = N(x) > 1$ so that for each $n \in \mathbb{Z}^+$ satisfying $n \geq N$ we have

$$\varepsilon n \geq \int_0^n \varphi_\delta(\phi_s x) \, ds = \sum_{i=0}^{nk-1} \int_0^{1/k} \varphi_\delta(\phi_{s+i/k} x) \, ds \geq \frac{1}{2} \sum_{i=0}^{nk-1} \varphi_\delta(\phi_{i/k} x).$$

Setting $g := \phi_{1/k}$, this ensures that for $m \in \mathbb{Z}^+$, if $n = [m/k] + 1$, then

$$\sum_{i=0}^{m-1} -\log \varphi_\delta(g^i x) \leq \sum_{i=0}^{nk-1} -\log \varphi_\delta(\phi_{s+i/k} x) \leq 2 \int_0^n -\log \varphi_\delta(\phi_s x) \, ds < 2n\varepsilon$$

if $x \in \Omega$ and $m > k \cdot N(x)$. Thus we obtain the next time reparametrization of (9):

$$\quad (1/m) \sum_{i=0}^{m-1} -\log \varphi_\delta(g^i x) < 2(1/k + 1/m) \varepsilon. \quad (34)$$
Noting that from (32) we may likewise deduce the reverse inequality to (33), then a similar argument to the previous one enables us to reciprocally obtain continuous slow recurrence (4) from slow recurrence (3). This completes the proof of the first statement of item (1).

To obtain slow recurrence, from Lemma 4.3 we can replace \( \varphi_\delta(x) \) by \( \|G(x)\| \) in the previous estimates. Indeed, we obtain for small \( \delta > 0 \) that if \( d(x, \text{Sing}_A(G)) \leq \delta \), then
\[
1 - \frac{\log(b_s L)}{\ln \|G(x)\|} \leq - \frac{\log d_\delta(x, \text{Sing}_A(G))}{\ln \|G(x)\|} \leq 1 - \frac{\log L}{\log \|G(x)\|},
\]
and so \( \log d_\delta(x, \text{Sing}_A(G)) \leq (1 + h(\delta)) \cdot \|G(x)\| \) for a function \( h \) such that \( h(\delta) \downarrow 0 \) when \( \delta \downarrow 0 \). Hence, since \( \|G\| \leq 1 \) (recall Remark 1.3) we get
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \varphi_\delta(g^i x) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log \|G(g^i x)\| + \log(1 + h(\delta)).
\] (35)

The following lemma ensures the slow recurrence condition on any positive Lebesgue measure subset \( \Omega \subset U \).

**Lemma 6.2.** We have \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log \|G(g^i x)\| = 0 \) for every \( x \in U \) whose future trajectory does not converge to any equilibrium. In particular, it holds for Leb-a.e \( x \in \Omega \).

**Proof.** Since \( \mathbb{R} \cdot G \) defines an invariant subbundle over \( U \setminus \text{Sing}_A(G) \) and \( D\phi_t \cdot G(x) = G(\phi_t x) \) by definition of flow of \( G \), we can rewrite the sum on the right hand side of (35) as follows
\[
\sum_{i=0}^{m-1} \log \|G(g^i x)\| = \sum_{i=1}^{m} \log \frac{\|G(g^i x)\|}{\|G(g^{i-1} x)\|} + \log \|G(x)\| \quad \text{and}
\]
\[
\sum_{i=1}^{m} \log \frac{\|G(g^i x)\|}{\|G(g^{i-1} x)\|} = \sum_{i=1}^{m} \log \left\|Dg \cdot \frac{G(g^{i-1} x)}{\|G(g^{i-1} x)\|}\right\| = \sum_{i=1}^{m} \log \left\|Dg^{i-1} \cdot G(x)\right\|
\]
\[
= \log \|Dg^m \cdot G(x)\| - \log \|G(x)\|.
\]
Therefore, we arrive at
\[
\limsup_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} \log \|G(g^i x)\| = \limsup_{m \to \infty} \log \|Dg^m \cdot G(x)\| - \frac{1}{m} = \limsup_{m \to \infty} \log \|G(g^m x)\|^{-1/m}
\]
and so it is enough to find an upper bound for the last limit superior.

We claim that this upper bound is zero if the future trajectory of \( x \) does not converge to an equilibrium, and hence the left hand side limit superior of positive numbers is also zero. We argue by contradiction: if there exists \( \varepsilon_0 > 0 \) so that
\[
\varepsilon_0 < \limsup_{m \to \infty} \log \|G(g^m x)\|^{-1/m},
\]
then we can find \( N_x > 1 \) so that for all \( m > N_x \) we get \( \|G(g^m x)\| \leq e^{-m\varepsilon_0} \). This implies that \( G(g^m x) \) is converging to \( \text{Sing}_A(G) \) and so converges to some equilibrium of this set. This contradiction proves the first statement of the lemma.
Let $S(\Omega) \subset \text{Sing}_\Lambda(G)$ be the collection of equilibria accumulated by the future trajectories of points of $\Omega$. Then $\sigma \in S(\Omega)$ cannot be a sink (otherwise trajectories converging to $\sigma$ cannot have sectional expansion) nor a source. Since $S(G)$ consists of finitely many hyperbolic fixed points of saddle-type, then every approach of the trajectory to any element $\sigma \in S(G)$ is followed by a departure from $\sigma$, except for the points of the stable manifold of $\sigma$. But, in this setting, the union $\bigcup_{\sigma \in S(G)} W^s_\sigma$ of the local stable manifolds has zero volume. Thus, the previous convergence of $g^n x$ to $\text{Sing}_\Lambda(G) \supset S(\Omega)$ is impossible on a positive Lebesgue measure subset. This completes the proof of the lemma.

The previous arguments altogether show that for every small $\varepsilon > 0$ we can find $\delta > 0$ so that for $x \in \Omega$ we have

$$\limsup_{n \to \infty} \sum_{i=0}^{n-1} \log d_\delta(g^i x, \text{Sing}_\Lambda(G))^{1/n} \leq \log (1+h(\delta)) \leq h(\delta) < \varepsilon,$$

obtaining slow recurrence and completing the proof of item (1).

For item (2): Theorem D shows that $(B)$ implies $(A)$. For the reciprocal, let $\mu$ be a hyperbolic physical/SRB measure for the flow, which is also a $\text{cu}$-Gibbs state supported on the partial hyperbolic attracting set $\Lambda = \Lambda_G(U)$.

To obtain the non-uniform sectional expanding condition \cite{10}, we note that by hyperbolicity of $\mu$, all central-unstable directions transversal to the vector field have positive Lyapunov exponents, thus $\chi(x) = \lim_{t \to +\infty} \log \| \wedge^2 (D\phi_t |_{E_{cu}})^{-1} \|^i / t \leq -c_0 < 0$ for $\mu$-a.e. $x$ and some constant $c_0 > 0$, by Oseledets’ Theorem. By Fatou’s Lemma for bounded sequences of functions we get

$$-c_0 > \int \chi d\mu \geq \limsup_{t \to \infty} \int \log \| \wedge^2 (D\phi_t |_{E_{cu}})^{-1} \|^i / t d\mu(x).$$

Thus, we find $T > 0$ so that

$$\int \log \| \wedge^2 (D\phi_t |_{E_{cu}})^{-1} \|^i / t d\mu \leq -c_0/2$$

for all $t \geq T$. Hence, there exists an ergodic component $\nu$ of $\mu$ so that

$$\int \log \| \wedge^2 (D\phi_t |_{E_{cu}})^{-1} \|^i / t d\nu \leq -c_0/2.$$

Moreover, $\nu$ is also a physical/SRB measure for the flow; see Theorem 6.1. Since $\nu$ is flow invariant, we can assume without loss of generality that $\nu$ is ergodic with respect to $g = \phi_T$, by the same arguments in the proof of Theorem A using the flow invariance and ergodicity of $\nu$. Hence, because $U \ni x \mapsto \log \| (D\phi_T |_{E_{cu}})^{-1} \|$ is continuous and $\nu$ is also physical/SRB with respect to $g$, we get for all $x \in B_g(\mu)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \wedge^2 (D\phi_{iT} |_{E_{cu}})^{-1} \|^{1/T} = \int \log \| \wedge^2 (D\phi_T |_{E_{cu}})^{-1} \|^{1/T} d\nu \leq -c_0/2,$$

where $B_g(\mu)$ is the ergodic basin of $\mu$ as a $g$-invariant probability measure.

From Theorem 6.1 this shows that \cite{10} holds on the positive Lebesgue measure subset $B_g(\nu)$. We also have $B_g(\nu) \subset B(\nu)$ and so $\text{Leb}(B_g(\nu) \cap \Omega) > 0$, completing the proof that $(A)$ implies $(B)$ with $E = B_g(\nu)$. This finishes the proof of item (2).

Item (3b) is a straightforward consequence of item (4) of Theorem 6.1.

For item (3a), we assume from now on that we are in the setting of the statement of Theorem D and its conclusion, i.e., we have both properties $(A)$ and $(B)$ and take $\mu$ an ergodic hyperbolic physical/SRB measure for the flow which is also $f = \phi_T$-ergodic (reparametrizing the flow if needed) and satisfying $\text{Leb}(E \cap B(\mu)) > 0$. 
To obtain mostly asymptotically sectional expansion \([4]\) we note that from \([31]\) we get
\[
\sup_{\mathcal{F}_{f^i x}} |\det(D\phi_1 |_{\mathcal{F}_{f^i x}})|^{-1} = \left(\|G(f^{i+1}x)\|/\|G(f^ix)\|\right) \cdot \sup_{\mathcal{F}_{f^i x}} \|P^1 \cdot \overrightarrow{n}\|^{-1},
\]
where the supremo is taken with respect to all 2-subspaces of \(E^c_{f^i x}\) generated by the orthonormal basis \(\{G(f^ix)/\|G(f^ix)\|, \overrightarrow{n}(f^ix)\}\) and \(i \geq 1\). Since this is a part of the family of all 2-subspaces of \(E^c_{f^i x}\) and the 2-subspaces not accounted for in the supremo above are all contained in \(N^c_{f^i x}\), then
\[
\frac{\|G(f^{i+1}x)\|}{\|G(f^ix)\|} \cdot \|(P^1 |_{N^c_{f^i x}})^{-1}\| \leq \|\wedge^2 (D\phi_1 |_{E^c_{f^i x}})^{-1}\| \leq \|(P^1 |_{N^c_{f^i x}})^{-1}\|^2. \tag{36}
\]
Moreover, because Leb-a.e. \(x\) does not belong to the stable manifold of an equilibrium, we also have \(\lim_{s \to \infty} \log \|G_{\phi_s x}\|^{1/s} = 0\) for Leb-a.e. \(x \in U\). This altogether ensures that
\[
\lim_{n \to \infty} \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \log \|\wedge^2 (D\phi_1 |_{E^c_{f^i x}})^{-1}\| \leq \lim_{n \to \infty} \sup_n \frac{n \sum_{i=0}^{n-1} \log \|(P^1 |_{N^c_{f^i x}})^{-1}\|}{n} < -2c_0. \tag{37}
\]
for all \(x \in E\). Since \((t, x) \mapsto \psi_\ell(x) = \log \|\wedge^2 (D\phi_t |_{E^c_{\psi_\ell x}})^{-1}\|\) is continuous and \(\text{Leb}(E \cap B(\mu)) > 0\)
\[-2c_0 > \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|\wedge^2 (D\phi_1 |_{E^c_{\psi_\ell x}})^{-1}\| = \int \log \|\wedge^2 (D\phi_1 |_{E^c_{\psi_\ell x}})^{-1}\| \, d\mu, \quad x \in E \cap B(\mu),
\]
holds for a positive Lebesgue measure subset of points. Because \(\mu\) is flow invariant and ergodic, then \(\mu\) is \(g\)-ergodic for \(g = \phi_t\) with \(t\) on a co-countable subset \(S \subset \mathbb{R}^+\) by \([62]\). Since \(t \mapsto \psi_\ell x\) is subadditive, then for \(t \in S \cap (0, 1)\) and each \(x \in E \cap B(\mu)\)
\[
\lim_{n \to \infty} \sup_n \log \|\wedge^2 (D\phi_{nt} |_{E^c_{\psi_\ell x}})^{-1}\|^{1/nt} \leq \lim_{n \to \infty} \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \log \|\wedge^2 (D\phi_1 |_{E^c_{\psi_\ell x}})^{-1}\|^{1/t} \leq \int \log \|\wedge^2 (D\phi_1 |_{E^c_{\psi_\ell x}})^{-1}\| \, d\mu < -2c_0,
\]
where \(n \in \mathbb{Z}^+\) and we used subadditivity in the last estimate. Because \(\psi_\ell(x)\) is continuous and \(\{nt : n \in \mathbb{Z}^+, t \in S \cap (0, 1)\}\) is dense in \(\mathbb{R}^+\), then the estimate above is enough to conclude that
\[
\limsup_{n \to \infty} \log \|\wedge^2 (D\phi_T |_{E^c_{\psi_\ell x}})^{-1}\|^{1/T} \leq -2c_0, \quad x \in E \cap B(\mu).
\]
Since \(E = \bigcup_{i=1}^k (E \cap B(\mu))\) except perhaps a subset of zero Lebesgue measure, we can repeat this argument for each \(\mu_i\) to cover Leb-a.e. point of \(E\). This completes the proof of item (3a) and of Theorem \([4]\) \[\square\]

Proof of Theorem \([\Box]\) Let \(E := \{x \in U : \Box\}\) holds for \(x\) and let us denote \(\Gamma(x) := \log \|\wedge^2 (D\phi_1 |_{E^c_{\psi_\ell x}})^{-1}\|\). If \(\mu(E) = 1\) for some ergodic invariant measure \(\mu\) for the flow, we obtain
\[
\mu(\Gamma) = \lim_{t \to \infty} \frac{1}{t} \int \Gamma(\phi_t x) \, d\mu(x) = \int \lim_{t \to \infty} \frac{1}{t} \Gamma(\phi_t x) \, d\mu(x) < -\omega
\]
from Kingman’s Subadditive Ergodic Theorem, since the limit exists and coincides with the limit inferior for \( \mu \)-a.e. \( x \). But by subadditivity \( \int \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \, d\mu(x) \) is bounded above by

\[
\int \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \, d\mu(x) + \int \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \, d\mu(x)
\leq \sup_{x \in U} \sup_{s \in [0, 1]} \log \| \wedge^2 (D\phi_s |_{E^u_x})^{-1} \| + \sum_{t=0}^{[t]-1} \int \Gamma(f^t x) \, d\mu(x) = C + [t]\mu(\Gamma)
\]

where we write \( f := \phi_1 \), \( C \) is a constant depending on the flow, and we used that \( \mu \) is \( f \)-invariant. Therefore, again by the Subadditive Ergodic Theorem

\[
\int \lim_{t \to \infty} \frac{1}{t} \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \, d\mu(x) = \lim_{t \to \infty} \frac{1}{t} \int \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \, d\mu(x)
\leq \limsup_{t \to \infty} \frac{[t]}{t} \cdot \mu(\Gamma) < -\omega.
\]

Since we assume that the above holds for any invariant ergodic probability measure \( \mu \) for the flow, we are in the conditions of the following result.

**Proposition 6.3.** [21 Corollary 4.2] Let \( \{t \mapsto f_t : \Lambda \to \mathbb{R}\}_{t \in \mathbb{R}} \) be a continuous family of continuous function which is subadditive and suppose that, for every invariant probability measure \( \mu \) for the flow, the limit \( \tilde{f}(x) := \lim_{t \to \infty} \frac{1}{t} f_t(x) \) defined for \( \mu \)-a.e. \( x \) satisfies \( \int \tilde{f} \, d\mu < 0 \). Then there exist constants \( T > 0 \) and \( \lambda > 0 \) such that for every \( x \in \Lambda \) and every \( t > T \) we have \( f_t(x) \leq -\lambda \cdot t \).

If we set \( f_t(x) := \log \| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \) we obtain \( T_u, \lambda_u > 0 \) such that

\[
\| \wedge^2 (D\phi_t |_{E^u_x})^{-1} \| \leq e^{-\lambda_u t} \quad \text{for } t \geq T_u \text{ and all } x \in \Lambda.
\]

Analogously, if we instead set \( f_t(x) = \log \| D\phi_t |_{E^s_x} \| \) and reapply the same reasoning, we conclude the existence of \( T_s, \lambda_s > 0 \) so that \( \| D\phi_t |_{E^s_x} \| \leq e^{-\lambda_s t} \) for all \( x \in \Lambda \) and \( t \geq T_s \).

This ensures that the compact set \( \Lambda \) with the invariant continuous splitting \( E^c_x \oplus E^u_x \) is a sectional-hyperbolic set, and conclude the proof of the first statement of the theorem.

For the second statement, we write \( \psi_\delta(x) := \int_0^\delta \Gamma(\phi_t x) \, dt \) where \( \delta = 1/m > 0 \) for some \( m > 1 \) so large that

\[
\log \bigl( \| \wedge^2 (D\phi_1 |_{E^u_{\phi_1 x}})^{-1} \| \bigr) \leq \log(1 + \zeta) \leq \zeta < \frac{\omega}{4}.
\]

Then \( \psi_\delta(x) = \Gamma(x) \pm \xi(x) \) where \( |\xi(x)| < \zeta \).

**Lemma 6.4.** Let us consider a point \( x \in U \) and a sequence \( T_n \nearrow \infty \) satisfying

\[
\int_0^{T_n} \Gamma(\phi_t x) \, dt < -\omega T_n \quad (38)
\]

for all large \( n > 1 \). Then there exists \( 0 \leq \tilde{j} < m \) so that for \( n \) large enough, with \( \ell = \ell_n = [T_n/\delta]/m \), we have \( \sum_{i=0}^{\ell-1} \Gamma(f^{i\tilde{j}} x) < -\ell m \omega/4 \).
Since the integrand $\Gamma$ is uniformly bounded by a constant $C$ as follows: we let $T_n - \delta [T_n/\delta] < \delta$, we obtain

$$\sum_{i=0}^{[T_n/\delta]-1} \psi_\delta (g^i x) < -\omega T_n, \quad \text{where } g := \phi_\delta.$$ 

The integrand $\Gamma$ is uniformly bounded by a constant $C$ as follows: we let $T_n - \delta [T_n/\delta] < \delta$, we obtain

$$\sum_{i=0}^{[T_n/\delta]-1} \psi_\delta (g^i x) < -\omega T_n + C\delta \leq (-\omega + C\delta/T_n) T_n < -[T_n] \omega/2$$ 

for all $n$ large enough. We group iterates that are time-1 apart rewriting the summation as follows: we let $[T_n/\delta] = \ell m$ for some $\ell = \ell_n > 1$ and

$$\sum_{j=0}^{m-1} \sum_{i=0}^{\ell-1} \psi_\delta (f^i g^j x) < -[T_n/\delta] \delta \omega/2 = -\ell \omega/2.$$ 

So, one of the inner sums is negative. More precisely, there exists $\hat{j} = j(x) \in \{0, \ldots, m-1\}$ so that $\sum_{i=0}^{\ell-1} \psi_\delta (f^i g^j x) < -\ell \omega/2$. By the choice of $\delta$ we obtain

$$\sum_{i=0}^{\ell-1} \left( \Gamma(f^i g^j x) \pm \xi(f^i g^j x) \right) < -\ell \omega/2 \implies \sum_{i=0}^{\ell-1} \Gamma(f^i g^j x) < -\ell \omega/2 + \zeta < -\ell \omega/4$$ 

for $n$ large enough, with $\ell = \ell_n = [T_n/\delta]/m$, as in the statement of the lemma.

We set $F := \{x \in U : \text{[12]} \text{ holds for } x\}$ and, for each $j = 0, \ldots, m-1$, we define

$$F_j := \left\{ x \in U : \liminf_{\ell \to \infty} \sum_{i=0}^{\ell-1} \Gamma(f^i g^j x)^{1/\ell} < -m \omega/4 \right\}.$$ 

If we assume that $\text{Leb}(F) > 0$, then from Lemma 6.4 we have that $F \subset \bigcup_{j=0}^{m-1} F_j$ and so there exists $\hat{j} \in \{0, \ldots, m-1\}$ such that $\text{Leb}(F_{\hat{j}}) > 0$. By smoothness of $f$, the set $\Omega = g^{-j} F_{\hat{j}}$ satisfies $\text{Leb}(\Omega) > 0$ and every $x \in \Omega$ is such that

$$\liminf_{n \to \infty} \sum_{i=0}^{n-1} \log \| \wedge^2 (D\phi_1 |_{F_{\hat{j}}^n} )^{-1} \|^{1/n} < -m \omega/4.$$ 

From $\text{(36)}$ and the fact the $\text{Leb}$-a.e. point of $\Omega$ does not belong to the stable manifold of an equilibrium, we obtain

$$\liminf_{n \to \infty} \sum_{i=0}^{n-1} \log \| (P^1 |_{N_{\text{fix}} F_{\hat{j}}^n} )^{-1} \|^{1/n} \leq \liminf_{n \to \infty} \sum_{i=0}^{n-1} \log \| \wedge^2 (D\phi_1 |_{F_{\hat{j}}^n} )^{-1} \|^{1/n},$$ 

and we conclude that $\text{Leb}(F) > 0$ implies the weak asymptotic expansion on average $\text{(13)}$ on a positive Lebesgue measure subset $\Omega$.

Exchanging $\liminf$ by $\limsup$ in $F$ and $F_j$ enables us to follow the same reasoning to show that $\text{(14)}$ on a positive Lebesgue measure subset of $U$ implies the asymptotic expansion on average $\text{(8)}$ on a positive Lebesgue measure subset $\Omega$ of $U$. This completes the proof of Theorem 1.7. \qed

Proof of Corollary C. We show that an asymptotic sectional hyperbolic attracting set is mostly asymptotic sectional expanding, when $\text{Sing}_A(G)$ contains only saddle-type hyperbolic equilibria, to then apply Theorem A to arrive at the conclusions of the statement of Corollary C.
We start by observing that if $\Lambda$ is transitive, then $\text{Sing}_\Lambda(G)$ cannot contain sinks or sources. We are left to find a positive Lebesgue measure subset $\Omega \subset U$ whose trajectories are asymptotically sectional expanding.

In order to find $\Omega$, we note that, from the proof of Theorem A, we have that the asymptotic sectional expansion (4) at any point of $\Lambda' := \Lambda \setminus \bigcup\{W^s_\sigma : x \in \text{Sing}_\Lambda(G)\}$ implies (8). Moreover, from Theorem B(1) points in $\Lambda'$ also satisfy the slow recurrence condition.

Hence, from Proposition 5.1 each $x \in \Lambda'$ admits an increasing sequence of hyperbolic times $n_k \nearrow \infty$ together with the associated neighborhoods $V_{n_k}(x)$ of $x$ in a $u$-disk $D$, and $\delta_0$-balls $W_{n_k}(x)$ inside $f^{n_k}(D)$. From Lemma 5.4 we known that any accumulation disk $\Delta$ of $W_{n_k}(x)$ is a center-unstable manifold tangent to $E^{cu}$ contained in $\Lambda$.

Moreover, any compact part of the stable manifold $W^s_\sigma$ of each saddle-type hyperbolic equilibrium $\sigma \in \text{Sing}_\Lambda(G)$ is a compact submanifold transversal do $\Delta$. Therefore, since $\text{Sing}_\Lambda(G)$ is a finite set, the intersection $\Delta \cap \bigcup\{W^s_\sigma : \sigma \in \text{Sing}_\Lambda(G)\}$ has zero measure with repect to the induced volume measure $\text{Leb}_\Delta$ on $\Delta$.

Thus, $\Delta' := \Delta \cap \Lambda'$ has $\text{Leb}_\Delta$-full measure. Since each $x \in \Delta'$ has a local stable manifold, we have that $y \in W^s_x$ also satisfies non-uniform sectional expansion (8) and slow recurrence (9), since $d(\phi_t y, \phi_t x) \to 0$ as $t \nearrow \infty$.

Therefore, we are left to check that the subset $\Omega := \bigcup\{W^s_x : x \in \Delta'\}$ has positive volume to complete the conditions of Theorem D and conclude the existence of physical/SRB measures supported on $\Lambda$.

The flow is Hölder-$C^1$ (it is in fact of class $C^2$ as the vector field $G$) and so, from standard results, the stable leaves form an absolutely continuous lamination; see e.g. [13, Section 6] & [34, Appendix B.7] and references therein. This ensures that $\text{Leb}(\Omega)$ is positive as a consequence of the positivity of $\text{Leb}_\Delta(\Delta')$. The proof of the corollary is complete. \[\square\]
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