Derivative Descendants and Ascendants of Binary Cyclic Codes, and Derivative Decoding

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Abstract—This paper defines derivative descendants and ascendants of extended cyclic codes from the derivative of the Mattson-Solomon polynomials. It proves that the derivative descendants of an extended cyclic code in different directions are the same. It allows us to perform soft-decision decoding on extended cyclic codes based on the soft-decision decoding of their descendants. Simulation results show that its performance over certain extended cyclic codes, including some extended BCH codes, can be close to that of the maximum likelihood decoding.

Index Terms—cyclic codes, soft-decision, derivative decoding, Mattson-Solomon polynomial

I. INTRODUCTION

Cyclic codes, first studied in 1957 [1], form a large class of error-control codes which include many well-known codes, e.g., Bose-Chaudhuri-Hocquenghem (BCH) codes, Reed-Solomon codes, finite geometry codes, punctured Reed-Muller (RM) codes etc. [2]–[5]. Due to the cyclic structure, their encoding and hard-decision decoding can be implemented efficiently. Moreover, their inherent algebraic structure and soft-decision decoding [6]–[14] have always attracted a lot of attention.

This paper starts from the derivative of the Mattson-Solomon (MS) polynomials [2], [15]. Then we define derivative descendants and derivative ascendants of extended cyclic codes, and investigate their dimension and distance. It demonstrates that the derivative descendants of an extended binary cyclic code in different directions result in the same extended cyclic code. Based on these properties, we propose a soft-decision derivative decoding algorithm for extended binary cyclic codes as follows: 1) calculate log-likelihood ratio (LLR) vectors associated with derivative descendants over different directions; 2) carry on soft-decision decoding over these descendants; 3) estimate the final output by aggregating the LLR vectors associated with derivative descendants and their decoding results. Simulation results show that it performs very well over extended cyclic codes, especially those whose derivative descendants can be efficiently decoded with soft-decisions.

The rest of the paper is organized as follows. Section II gives a brief review of cyclic codes and MS polynomials. In Section III, we define the derivative descendants and derivative ascendants of extended cyclic codes. In Section IV, we introduce the derivative decoding algorithm and present simulation results. Section V concludes this paper.

II. CYCLIC CODES AND THEIR DECOMPOSITION

A. Cyclic codes and Mattson-Solomon polynomials

Let \( m \) be a positive integer. A binary cyclic code \( C \) of length \( n = 2^m - 1 \) and dimension \( 0 < k \leq n \) is an ideal in the ring \( \mathbb{F}_2[x]/(x^n - 1) \), which is generated by a generator polynomial \( g(x) \) with degree \( n - k \) such that \( g(x)(x^n - 1) \).

Let \( \alpha \) denote a primitive element of \( \mathbb{F}_{2^m} \). For a codeword \( \alpha = [a_0, a_1, \ldots, a_{n-1}] \) corresponding to a code polynomial \( a(x) = \sum_{i=0}^{n-1} a_i x^i \), the associated Mattson-Solomon polynomial [15] is defined over \( \mathbb{F}_{2^m} \) as follows

\[
A(z) \triangleq \sum_{j=0}^{n-1} A_j z^j,
\]

where

\[
A_j = a(\alpha^{-j}) = \sum_{i=0}^{n-1} a_i \alpha^{-ij}.
\]

The codeword \( a \) can be recovered from \( A(z) \) by

\[
a = [a_i, i \in [n]] = [A(\alpha^i), i \in [n]],
\]

where \([n] \triangleq \{0, 1, 2, \ldots, n - 1\} \). The coefficient \( A_j \) is fixed to 0 if and only if \( \alpha^{-j} \) is a zero of \( g(x) \). In the following, we use \( A(z) \) and \( a \) interchangeably to denote the codeword of cyclic codes.

We define the exponent set (ES) of all the MS polynomials associated with \( C \) as follows

\[
S_C \triangleq \{ j \in [n] : g(\alpha^{-j}) \neq 0 \}.
\]

For brevity, we simply call it the exponent set of \( C \). Please note that its size is the same as the dimension \( k \). Then we can express \( C \) as

\[
C = \{ [A(\alpha^i), i \in [n]] : A(z) = \sum_{j \in S_C} A_j z^j \},
\]
where $A_j \in \mathbb{F}_{2^m}$. The conjugacy constraint [4] i.e. $A_{2j \mod n} = A_j^2$ is required to keep vector $\alpha$ binary. Here, $2j \mod n$ denotes the value of $2j$ modulo $n$.

The cyclic code $C$ can be extended by adding an overall parity-check bit to each codeword. The overall parity-check bit of a codeword $\alpha \in C$ is the evaluation of the corresponding MS polynomial at 0 [2] (we also denote 0 by $\alpha^\infty$) i.e., $A(0)$ (or $A(\alpha^\infty)$). Therefore, the extended cyclic code of $C$ can be also identified by $S_C$. In the later sections, we mainly focus on the extended cyclic codes and we define $I \triangleq \{\infty\} \cup [n]$.

**B. Decomposing cyclic codes as a direct sum of minimal cyclic codes**

For an integer $0 \leq s \leq n - 1$, the cyclotomic coset modulo $n$ containing $s$ is $C_s \triangleq \{s, 2s, 2^2s, ..., 2^{m-1} \mod n\}$, where $m_s$ is the smallest positive integer such that $2^m s = s \mod n$. The smallest entry of $C_s$ is called the coset representative. For a subset $S$ of $[n]$, we denote the smallest and the largest entry of $S$ by $\min(S)$ and $\max(S)$, respectively. We define the set $cc(S) \triangleq \bigcup_{s \in S} C_s$ and the set $cr(S) \triangleq \bigcup_{s \in S}[\min(C_s)]$.

The extended minimal cyclic code [2] associated with the cyclotomic coset $C_s$ is

$$M_s = \{[A(\alpha^s), i \in I] : A(z) = T_m, (A_s z^s) \}
\text{for all } A_s \in \mathbb{F}_{2^{m_s}},\]

where $T_m(z)$ is the trace function

$$T_m(z) \triangleq \sum_{j \in [m_s]} z^{2^j},$$

and $\mathbb{F}_{2^{m_s}}$ is the subfield of $\mathbb{F}_{2^m}$. It is clear that the ES of $M_s$ is $C_s$.

An extended cyclic code with ES $S_C$ can be expressed as the direct sum of the extended minimal cyclic codes

$$C = \bigoplus_{s \in \mathbb{E}(S_C)} M_s
= \{[A(\alpha^s), i \in I] : A(z) = \sum_{s \in \mathbb{E}(S_C)} T_m, (A_s z^s) \}
\text{for all } A_s \in \mathbb{F}_{2^{m_s}},\}

where $\bigoplus$ is the direct sum operator. We call the set $\mathbb{E}(S_C)$ as the representative set of $S_C$. We end this section with the following example.

**Example 1.** Let $\alpha$ denote the primitive element in $\mathbb{F}_{2^4}$. Consider the (16,7) extended cyclic code $C$ associated with the generator polynomial $g(x) = 1 + x^4 + x^6 + x^7 + x^8$. The zeros of $g(x)$ are $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7$. From (1), the ES of $C$ is $S_C = \{0, 1, 2, 4, 5, 10\}$. Then $C$ can be identified by the set

$$\{[A(\alpha^s), i \in I] : A(z) = \sum_{j \in S_C} A_j z^j, \}
\text{where } A_j \in \mathbb{F}_{2^4} \text{ and satisfies } A_{2j \mod 15} = A_j^2. \text{ The representative set of } S_C \text{ is } \{0, 1, 5\}. \text{ Note that } m_0 = 1, m_1 = 4, m_5 = 2. \text{ Then } C \text{ can be expressed as}
$$

$$C = \{[A(\alpha^s), i \in I] : A_0 + T_4(A_1 z) + T_2(A_5 z^5), \}
\text{where } A_0 \in \mathbb{F}_2, A_1 \in \mathbb{F}_2, A_5 \in \mathbb{F}_2.$$

**III. DERIVATIVE DESCENDANTS AND ASCENDANTS**

This section introduces derivative descendants and derivative descendants of extended cyclic codes. Their dimensions and distances are also investigated.

**A. Derivative descendants, and their dimensions and distances**

Let $\beta$ be a power of $\alpha$. The derivative [5] of the MS polynomial $A(z)$ in the direction $\beta$ is defined as

$$\Delta_\beta A(z) \triangleq A(z + \beta) - A(z).$$

With the above definition, we define the derivative descendant of extended cyclic codes.

**Definition 1.** For an extended cyclic code $C$, its derivative descendant in the direction $\beta$ denoted by $D(C, \beta)$ is the extended cyclic code with the smallest dimension which contains

$$\{[\Delta_\beta A(\alpha^i), i \in I] : A(z) \in C\}.$$

In fact, the derivative descendants in all the directions are the same. To prove this, we start with the extended minimal cyclic codes. For an integer $s \in [n]$, we denote its binary expansion by $s = [s_0, s_1, ..., s_{m-1}]$ such that $s = \sum_{j=0}^{m-1} s_j 2^j$. We define the set $W_s \triangleq \{j \in [m] : s_j \neq 0\}$ and the set

$$P(s) \triangleq \{\sum_{j \in W_s} 2^j : V \subseteq W_s\}.$$

**Lemma 1.** Consider the extended minimal cyclic code $M_s$. The ES of $D(M_s, \beta)$ for any $\beta$ is $\mathbb{E}(P(s))$.

**Proof.** For any $A(z) \in M_s$, its derivative in direction $\beta$ is

$$\Delta_\beta A(z) = T_m(A_s(z + \beta)^s) - T_m(A_s z^s)
= T_m(A_s((z + \beta)^s - z^s)).$$

Note that

$$(z + \beta)^s - z^s = \prod_{j \in W_s} (z^{2^j} + \beta^{2^j}) - \prod_{j \in W_s} z^{2^j}
= \sum_{V \subseteq W_s} (z^{\sum_{j \in V} 2^j} \beta^{\sum_{j \in W_s \setminus V} 2^j})
= \sum_{k \in P(s)} z^k \beta^{s-k}.
$$

Then

$$\Delta_\beta A(z) = T_m(A_s \sum_{k \in P(s)} z^k \beta^{s-k}).$$

From the above equation, we see that the exponents of $z$ must be a subset of $\mathbb{E}(P(s))$. Note that the coefficients of $\Delta_\beta A(z)$ must satisfy the conjugacy constraint, because $\Delta_\beta A(\alpha^i) = A(\alpha^i + \beta) - A(\alpha^i)$, and $A(\alpha^i)$ and $A(\alpha^i + \beta)$ are in $\mathbb{F}_2$ for all $i \in I$. Therefore we can write $\Delta_\beta A(z)$ in the form

$$\Delta_\beta A(z) = \sum_{s' \in \mathbb{E}(P(s))} T_{m_s}(A_{s'} z^{s'}).$$
where  
\[ A'_s = \sum_{i \in [m], k \in P(s)} \beta^{(s-k)2^i} A_s^{2^i}. \]

Treat \( A'_s \) as a function of \( A_s \) i.e. \( A'_s(A_s) \). Note that the degree of \( A'_s(A_s) \) is at most \( 2^{m_s-1} \) which implies \( A'_s(A_s) \) has at most \( 2^{m_s-1} \) roots. Therefore, \( A'_s \) is not always zero. As a result, the representative set of the ES of \( D(C, \beta) \) is \( ccr(P(s)) \) and the ES is \( ccc(P(s)) \).

The above lemma shows that the derivative descendants of an extended minimal cyclic code in different directions are the same. Using the fact that the extended cyclic code \( \bar{C} \) is the direct sum of extended minimal cyclic codes, we obtain the following theorem immediately.

**Theorem 1.** For an extended cyclic code \( C \) with ES \( S_C \), its derivative descendants in different directions are the same code denoted by \( D(C) \). The ES of \( D(C) \) denoted by \( S_D \) is \( \bigcup_{s \in ccr(S_C)} ccc(P(s)) \). And the corresponding representative set is \( \bigcup_{s \in ccr(S_C)} ccr(P(s)) \).

**Example 2.** Continuation of Example 1. The representative set of the ES of \( C \) is \( \{0, 1, 5\} \). According to (3), \( P(0) = 0, P(1) = 0 \), \( P(5) = \{0, 1, 4\} \). From Theorem 1, we have \( S_D = \{0, 1, 2, 4, 8\} \). Then

\[ D(C) = \{[A(a^i), i \in I] : A(z) = A_0 + T_4(A_1 z)\}. \]

Now we investigate the dimension and distance of \( D(C) \). For any binary vector \( v \), we denote its Hamming weight by \( wt(v) \). For a subset \( S \) of \([n]\), we define \( \deg(S) \triangleq \max\{\sum_{s \in S} wt(\bar{S})\} \). Let \( d \) denote the minimum distance of \( C \). And let \( k_D \) and \( d_D \) denote the dimension and minimum distance of \( D(C) \), respectively. For any nontrivial binary cyclic code \( \bar{C} \) i.e. \( S_C \neq \emptyset \), we give two propositions about their derivative descendants.

**Proposition 1.** \( k_D \leq \sum_{i=0}^{\deg(S_D)} \binom{m}{i} \).

**Proof.** The dimension of \( D(C) \) satisfies

\[ k_D = |S_D| \leq \sum_{i=0}^{\deg(S_D)} \binom{m}{i}. \]

From Theorem 1,

\[ \deg(S_D) = \deg(\bigcup_{s \in ccr(S_C)} ccc(P(s))). \]

Note that the binary expansion of \( 2s \mod n \) is a cyclic shift of \( \bar{S} \). Thus, \( \deg(C_s) = wt(\bar{S}) \) and \( deg(\text{ccc}(P(s))) = deg(P(s)) \). Then,

\[ \deg(S_D) = \deg(\bigcup_{s \in ccr(S_C)} P(s)). \]

Note that \( \bigcup_{s \in ccr(S_C)} P(s) \subseteq \bigcup_{s \in S_C} P(s) \), then

\[ \deg(S_D) \leq \deg(\bigcup_{s \in S_C} P(s)). \]

For \( s = 0 \), we have \( W_s = \emptyset \) and \( P(s) = \emptyset \). Then \( \deg(P(s)) = 0 \). For any positive \( s \in S_C \), from (3), we have \( \deg(P(s)) = wt(\bar{S}) - 1 \). Then \( \deg(\bigcup_{s \in S_C} P(s)) = \deg(S_C) - 1 \). As a result,

\[ k_D \leq \sum_{i=0}^{\deg(S_D)} \binom{m}{i} \leq \sum_{i=0}^{\deg(S_C) - 1} \binom{m}{i}. \]

**Proposition 2.** \( d_D \leq 2d \).

**Proof.** Consider the codeword \( A(z) \in C \) with \( wt(A(z)) = d \). The Hamming weight of its derivative \( \Delta_\beta A(z) \) satisfies

\[ wt(\Delta_\beta A(z)) = wt(A(z + \beta) - A(z)) \leq wt(A(z + \beta)) + wt(A(z)) = 2d. \]

As a result, \( d_D \leq 2d \).

**B. Derivative ascendant, and their dimensions and distances**

**Definition 2.** For an extended cyclic code \( C \), we define its derivative ascendant denoted by \( A(C) \) as the extended cyclic code with the largest dimension such that \( D(A(C)) \subseteq C \).

We give a proposition to characterize the ES of \( A(C) \).

**Proposition 3.** Let \( S_A \) denote the ES of \( A(C) \). A nonnegative integer \( s \) smaller than \( n \) is in \( S_A \) if and only if

\[ ccc(P(s)) \subseteq S_C. \]

**Proof.** Because the conjugacy constraint is required, \( s \in S_A \) if and only if \( C_s \subseteq S_A \) which is equivalent to \( M_s \subseteq A(C) \).

If \( M_s \subseteq A(C) \), from (2) and Definition 2, we have \( D(M_s) \subseteq D(A(C)) \subseteq C \). From Lemma 1, the ES of \( D(M_s) \) is \( ccc(P(s)) \). Then \( ccc(P(s)) \subseteq S_C \).

If \( ccc(P(s)) \subseteq S_C \), then \( D(M_s) \subseteq C \). From Definition 2, \( A(C) \) is the extended cyclic code with the largest dimension such that \( D(A(C)) \subseteq C \). As a result, \( M_s \subseteq A(C) \).

Let \( k_A \) and \( d_A \) denote the dimension and minimum distance of \( A(C) \), respectively. We give two propositions.

**Proposition 4.** \( k_A \leq \sum_{i=0}^{\deg(S_A) + 1} \binom{m}{i} \).

**Proof.** The dimension of \( A(C) \) satisfies

\[ k_A = |S_A| \leq \sum_{i=0}^{\deg(S_A)} \binom{m}{i}. \]

From Proposition 3, for any \( s \in S_A \), \( P(s) \subseteq S_C \). From (3), \( \deg(P(s)) = wt(\bar{S}) - 1 \). Then

\[ wt(S_A) = \deg(P(s)) + 1 \leq \deg(S_C) + 1 \]

This leads \( \deg(S_A) \leq \deg(S_C) + 1 \). As a result, we have

\[ k_A \leq \sum_{i=0}^{\deg(S_A)} \binom{m}{i} \leq \sum_{i=0}^{\deg(S_C) + 1} \binom{m}{i}. \]
Proposition 5. \(d_A \geq d/2\).

Proof. Let \(D(A(C))\) denote the derivative descendant of \(A(C)\) with minimum distance \(d_{D(A)}\). From Proposition 2, we have \(d_{D(A)} \leq 2d_A\). From Definition 2, we have \(D(A(C)) \subseteq C\) which indicates \(d_{D(A)} \geq d\). As a result, \(d_A \geq d/2\). \(\square\)

IV. DERIVATIVE DECODING FOR EXTENDED CYCLIC CODES

In this section, we are going to introduce the derivative decoding algorithm based on derivative descendants. The algorithm can be utilized for decoding those extended cyclic codes whose derivative descendants have efficient soft-decision decoding algorithms, e.g., extended EG codes and RM codes.

A. Derivative decoding for extended binary cyclic codes

Let \(y = [y_i, i \in I]\) denote the received vector of transmitting a codeword \(a = [A(\alpha^i), i \in I]\) of the extended binary cyclic code \(C\) over a binary-input memoryless symmetric channel (BMS). Let \(W(y|x)\) denote the probability that \(y\) is output by the channel when \(x\) is input to the channel. The LLR vector of the channel output \(y\) is \(L = [L_i : i \in I]\), where \(L_i\) is given by

\[
L_i = \ln\left(\frac{W(y_i|0)}{W(y_i|1)}\right).
\]

The LLR vector associated with the derivative descendant in the direction \(\beta\) is defined as

\[
L_\beta^i \triangleq (L_i^\beta, i \in I)),
\]

where \(L_i^\beta\) is the LLR value associated with \(\Delta_\beta A(\alpha^i) = A(\alpha^i + \beta) - A(\alpha^i)\). We calculate \(L_i^\beta\) as [9], [10]

\[
L_i^\beta = 2 \tanh^{-1}\left(\tanh\left(\frac{L_i}{2}\right) \tanh\left(\frac{L_j}{2}\right)\right),
\]

where \(\alpha^i = \alpha^i + \beta\).

Suppose that there is an efficient soft-decision decoder, denoted by \(\text{decoderDC}\), for the derivative descendant of \(C\). Let \(B\) denote the set of all the nonzero elements in \(\mathbb{F}_{2^m}\) i.e., \(B = \{\alpha^i : i \in [n]\}\). The derivative decoding algorithm can run in an iterative manner. At each iteration, we first calculate the LLR vectors \(L_\beta^i\) for all \(\beta \in B\) according to (4) and (5). Then use \(\text{decoderDC}\) for \(D(C)\) to decode these LLR vectors and obtain the decoding results \(\hat{a}_\beta^i\) for all \(\beta \in B\). Next, use a voting scheme to obtain a new LLR vector \(\hat{L}\). For each entry \(\hat{L}_i\) of \(\hat{L}\), the value is calculated as

\[
\hat{L}_i = \frac{1}{|B|} \sum_{\beta \in B} (1 - 2\hat{a}_\beta^i)L_j.
\]

Please note that, we have used the natural embedding of \(\mathbb{F}_2\) in \(\mathbb{R}\) for the interpretation of \(\hat{a}_\beta^i\) in the above equation. Finally, we compare the new LLR vector \(\hat{L}\) with \(L\). If the relative difference between the two vectors is below a threshold \(\theta\) for every entry, which indicates the LLR vector \(\hat{L}\) converges to a stable value, stop iteration and output the signs of \(\hat{L}\) as the estimate of \(y\). Otherwise, take \(L = \hat{L}\) and proceed into the next iteration until converging or reaching a maximal iteration number \(N_{\text{max}}\). The pseudo code of the above procedure is shown in Algorithm 1.

In fact, the proposed decoding works for cyclic codes of length of \(2^m - 1\) as well. Set \(L_\infty = 0\). Then we can decode them as their extended cyclic codes.

B. Computational complexity

We analyze the computational complexity of the proposed algorithm per iteration according to Algorithm 1. First, it requires \(4|B|n\) floating-point number computations to calculate \(|B|\) LLR vectors associated with \(|B|\) different directions. Next, it takes \(|B|\) times \(\text{decoderDC}\). Thus, the complexity of this step is \(|B|X_{\text{sub}}(n)\), where \(X_{\text{sub}}(n)\) is the complexity of \(\text{decoderDC}\). Last, the aggregation step requires \(|B|n\) floating-point number computations. Note that the maximum iteration number is \(N_{\text{max}}\). As a result, the total number of floating-point number computation is about \(N_{\text{max}}|B|(X_{\text{sub}}(n) + 5n)\). Please note that the complexity can be saved if we use less directions for derivative decoding.

C. Decoding eBCH codes using derivative decoding

The proposed algorithm can be used to decode those extended cyclic codes whose derivative descendants have efficient soft-decision decoding algorithms. In the following we give an example of decoding those extended BCH (eBCH) codes whose derivative descendants are those extended EG codes which can be efficiently decoded as low-density parity-check (LDPC) codes [16] [6] by the sum-product algorithm (SPA) [17], [18]. We denote the derivative decoding based on SPA as DD-SPA.

Example 3. Consider the (64, 24) eBCH code with minimum distance 16 and the (64, 45) eBCH code with minimum distance 8. The corresponding generator polynomials in hexadecimal form are 0xF69AC20921 and 0x782CF, respectively. From (1), the corresponding representative sets are \(S_1 = \{0, 1, 3, 5, 9, 21\}\) with \(\deg(S_1) = 3\) and \(S_2 = \{

Algorithm 1 The Derivative Decoding algorithm

Input: The LLR vector \(L\); the set of directions \(B\); the maximum iteration number \(N_{\text{max}}\); the threshold \(\theta\)

Output: The decoded codeword: \(\hat{a}\)

1: for \(t = 1, 2, \ldots, N_{\text{max}}\) do
2: \(\hat{L}_\beta^i \leftarrow \text{decoderDC}(L_\beta^i)\) for all \(\beta \in B\)
3: \(L_i \leftarrow \frac{1}{|B|} \sum_{\beta \in B} (1 - 2\hat{a}_\beta^i)L_j\) for all \(i \in I\) where \(j\) satisfies \(\alpha^i = \alpha^i + \beta\)
4: if \(|L_i - L_i| \leq \theta|L_i|\) for all \(i \in I\) then
5: \(L \leftarrow L\)
6: \(\text{Break}\)
7: end if
8: \(\hat{a}_i \leftarrow \mathbb{I}[L_i < 0]\) for all \(i \in I\)
9: end for
10: return \(\hat{a}\)
with \( \deg(S_2) = 4 \). According to Theorem 1, the representative sets associated with their derivative descendants are \( \{0, 1, 5\} \) and \( \{0, 1, 3, 5, 9, 11, 13\} \). The corresponding codes are the \((64, 13)\) extended EG code with dimension \( 13 \leq \sum_{i=0}^{2} \binom{9}{i} \) and minimum distance \( 24 \leq 32 \), and a \((64, 34)\) extended cyclic code with dimension \( 34 \leq \sum_{i=0}^{3} \binom{16}{i} \). Please note that the \((64, 34)\) extended cyclic code is a subcode of the \((64, 37)\) extended EG code.

Consider additive white Gaussian noise (AWGN) channels. We decode the two \( eBCH \) codes by our proposed derivative decoding algorithm. The decoders for the \((64, 13)\) extended EG code and the \((64, 34)\) extended cyclic code are the SPA decoders. The parity-check matrix used for decoding the \((64, 13)\) extended EG is a \( 336 \times 64 \) matrix with row weight 4 and column weight 21, and the one used for decoding the \((64, 34)\) extended cyclic code is a \( 72 \times 64 \) matrix with row weight 16 and column weight 17. We set the maximum iteration numbers for SPA and DD as \( N_{SPA,max} = 20 \) and \( N_{DD,max} = 3 \) in the two cases. And we set the threshold \( \theta \) as 0.05. We perform derivative decoding in all the \( |B| = 63 \) directions, denoted by DD(63)-SPA. In addition, we perform derivative decoding in \( |B| = 16 \) directions at random, denoted by DD(16)-SPA. We compare with the performances of decoding \( BCH \) codes using the Berlekamp-Massey (BM) algorithm [5] [19]. Besides, we compare with the performances of decoding 5G CA-polar codes [20] [21] with the same length and dimension. The decoder used for CA-polar codes is the Successive Cancellation List (SCL) decoder [22] with list size 32. And the CRC length is set to 6.

The simulation results are shown in Fig. 1 and the performances of maximum likelihood decoding (MLD) are also provided. We see that at the block error ratio (BLER) of \( 10^{-4} \), DD(63)-SPA outperforms BM about 2.9 dB and 1.9 dB for the \((64, 24)\) \( eBCH \) code and the \((64, 45)\) \( eBCH \) code, respectively. Moreover, the gaps between ML and DD(63)-SPA are 0.5 dB and 0.3 dB in the two cases, respectively. Below the BLER of \( 10^{-4} \), decoding the \((64, 24)\) \( eBCH \) code and the \((64, 45)\) \( eBCH \) code using DD(63)-SPA outperforms decoding the CA-polar codes using SCL decoder, respectively. For decoding \((64, 45)\) \( eBCH \) code, the performance of DD(16)-SPA is 0.1 dB worse than that of DD(63)-SPA at the BLER of \( 10^{-4} \), while DD(16)-SPA can save around 75% computational complexity.

D. Decoding derivative ascendants of EG codes

If an extended cyclic code can be efficiently soft-decision decoded, then its ascendant can be soft-decision decoded by the derivative decoding algorithm. We give an example of decoding the ascendant of the \((256, 175)\) extended EG code.

Example 4. Consider the \((256, 175)\) extended EG code with minimum distance 18 which can be efficiently decoded by the SPA decoder. The corresponding generator polynomial is 0x11377F7700FA55335BA55. Its representative set of ES is \( S = \{0, 1, 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 27, 29, 37, 39, 43, 51, 53, 55, 59, 85, 87, 119\} \) with \( \deg(S) = 6 \). According to Proposition 3, we can construct its derivative ascendant.

It is the extended cyclic code of length 256 and dimension 191 \( \leq \sum_{i=0}^{2} \binom{16}{i} \) with distance \( d \geq 18/2 = 9 \), according to Proposition 4 and Proposition 5. In fact, this code, denoted by DA(256, 191) has minimum distance at least 16 according to the BCH bound [4]. The corresponding generator polynomial of DA(256, 191) is 0x19ACC1AE68A0CEFF.

We provide the simulation result of decoding DA(256, 191) using DD-SPA in Fig. 2. The parity-check matrix used for decoding the \((256, 175)\) extended EG code is a \( 272 \times 256 \) matrix with row weight 16 and column weight 17. The maximum iteration numbers for derivative decoding and SPA are set to \( N_{DD,max} = 4 \) and \( N_{SPA,max} = 20 \), respectively. We perform derivative in 255 directions and set the threshold \( \theta \) as 0.05.
The performance of MLD is also provided. We see that at the BLER of $10^{-4}$, the gap between MLD and DD-SPA is about 0.9dB.

E. Derivative descendants of RM codes and their decoding

Consider the RM code of length $2^m$ and order $r$ denoted by RM($r, m$). The zero set of the generator polynomial associated with RM($r, m$) is $\{\alpha^j : 0 < \text{wt}(j) < m - r\}$. According to (1), the ES of RM($r, m$) is $\{j : 0 \leq \text{wt}(j) \leq r\}$. According to Theorem 1, the derivative descendant of RM($r, m$) is RM($r-1, m$). Note that the dimension and minimum distance of RM($r, m$) is $\sum_{i=0}^{r} \binom{m}{i}$ and $2^{m-r}$, respectively. The equalities in Proposition 1 and Proposition 2 hold for RM codes and their derivative descendants. Similarly, according to Proposition 3, we conclude that the derivative descendant of RM($r, m$) is RM($r+1, m$). The equalities in Proposition 4 and Proposition 5 hold for the RM codes and their derivative descendants.

As a result, we can decode RM($r, m$) codes using derivative decoding based on the decoding algorithms of RM($r-1, m$) codes. From (5), we have $[L^\beta_{\gamma}, \alpha^i \in T] = [L^\beta_{\gamma}, \alpha^i \in T + \beta]$ where $T$ is a subset of $\mathbb{F}_{2^m}$ such that $\beta + T \cup T = \mathbb{F}_{2^m}$. Moreover, considering the $[\alpha^i \in T]$-construction and the automorphism groups of RM codes [2], we conclude that $[L^\beta_{\gamma}, \alpha^i \in T]$ is the LLR vector associated with the codeword in RM($r-1, m-1$). In other words, our derivative decoding started from the MS polynomials can carry on based on RM($r-1, m-1$) codes as state-of-the-art projection decodings [9], [10] started from the $m$-variate polynomials, and obtain the same performance.

V. CONCLUSION

This paper introduces derivative descendants and ascendants for extended cyclic codes. Moreover, derivative decoding is proposed based on the properties of derivative descendants. It allows us not only to construct new extended cyclic codes with efficient decoding, but also to decode some eBCH codes with soft-decision. Simulation results verify that they perform very well over AWGN channels.

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