EMERGENCE OF TIME

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March 24, 2022

Abstract

In the groupoid approach to noncommutative quantization of gravity, gravitational field is quantized in terms of a $C^*$-algebra $A$ of complex valued functions on a groupoid $G = E \times \Gamma$, where $E$ is a suitable space and $\Gamma$ a group of fundamental symmetries. In the noncommutative quantum gravitational regime the concepts of space and time are meaningless. We study the “emergence of time” in the transitions process from the noncommutative regime to the standard space-time geometry. Precise conditions are specified under which modular groups of the von Neumann algebra generated by $A$ can be defined. These groups are interpreted as representing a state depending time flow of a quantum gravitational system. If the above conditions are further refined one obtains a state independent time flow. We show that quantum gravitational dynamics can be expressed in terms of modular groups.

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1 Introduction

The issue of time is one of the most discussed problems in the research domain known as “quantum gravity”. Very often the opinion is expressed that the correct formulation of questions concerning the time issue could pave the way towards the looked for theory of quantum gravity (for a sample of references see [1]). In many working models, the standard (coordinate) time of macroscopic physics either becomes “imaginary” (i.e., acquires fully spatial properties) [2], or entirely loses its meaning (see [1]). Carlo Rovelli, in the series of papers ([3, 4, 5]), not only has propagated the idea that at the fundamental level a well defined concept of time is totally absent, but also has obtained some interesting results showing how could physics be done without the usual notion of time.

In [6] we have proposed a scheme for quantizing gravity in which the quantum gravity regime is modelled by a noncommutative geometry. In this geometry all local concepts (such as time instant, point location and their neighbourhoods) are meaningless. Space-time appears only in the transition process from the noncommutative regime to the macroscopic physics. (It is a scheme rather than a full theory since some its main ingredients, such as the group of fundamental symmetries, remain unspecified; their choice is left for future developments). The aim of the present paper is to study the emergence of time when gradually going from the noncommutative regime to the usual space-time geometry. We prove some mathematically rigorous results responsible for the main stages of this process. Our results are based on the work by Connes and Rovelli [7]; it is surprising how the postulates of these authors, regarding the future theory of quantum gravity, agree with our scheme.

To make the present paper self-contained, in Section 2, we briefly summarize our approach to noncommutative quantization of gravity. The algebra $\mathcal{A}$ determining the noncommutative geometry, which models the quantum gravity regime, is an algebra of complex valued functions on a groupoid of fundamental symmetries with convolution as multiplication. The first important result is that if we make a suitable identifications of elements of the algebra $\mathcal{A}$ (if we make a quotient $\mathcal{A}/N_{\omega_q}$ where $N_{\omega_q}$ is an ideal of $\mathcal{A}$ and $\omega_q$ is a state on $\mathcal{A}$), the one-parameter group $\alpha_t$, $t \in \mathbb{R}$, of automorphisms of the von Neumann algebra $\mathcal{R}$ generated by the algebra $\mathcal{A}$, the so-called modular group, can be defined. This group is interpreted as a time flow (dependent
on the state \(\omega_q\). This is proved in Section 3. It turns out that if we further identify elements of \(A\) (with the help of the inner equivalence relation), the modular group can be made state independent. This is explained and proved in Section 4. In Section 5, we show that the dynamical equation of our scheme can be expressed in terms of modular groups (when these groups are available). Some interpretative comments are given in Section 6.

## 2 A noncommutative scheme for quantization of gravity

In this Section we briefly summarize our approach to quantum gravity. It is based on the direct product \(G = E \times \Gamma\), regarded as a smooth groupoid (for definition see [8]), where \(E\) is an \(n\)-dimensional smooth manifold (or a structured space of constant dimension, see [9]) and \(\Gamma\) a Lie group acting on \(E\) (to the right). Elements of \(\Gamma\) are “fundamental symmetries” of our theory (heuristically, we could think of \(E\) as of the total space of the fibre bundle of frames over space-time \(M\), and of \(\Gamma\) as of its structural group, i.e., a connected component of the Lorentz group). The correct choice of \(\Gamma\) is left for the future development of the proposed scheme.

Now, we define the algebra \(A = C_c^\infty(G, \mathbb{C})\) of smooth compactly supported complex-valued functions on \(G\) with the convolution

\[(a \ast b)(\gamma) := \int_{G_p} a(\gamma_1)b(\gamma_2),\]

as multiplication, where \(G_p\) is a fiber of \(G\) over \(p \in E\), \(a, b \in A\), \(\gamma = \gamma_1 \circ \gamma_2\), \(\gamma_1, \gamma_2 \in G_p\), \(g \in \Gamma\). \(A\) is also an involutive algebra with involution defined as \(a^*(\gamma) = a(\gamma^{-1})\).

Now, we develop a noncommutative geometry as determined by the algebra \(A\) in terms of derivations of this algebra (see [10]).

Let \(\text{Der}A\) be the set of all derivations of the algebra \(A\). It is a \(Z(A)\)-module where \(Z(A)\) denotes the center of \(A\). \(\text{Der}A\) can be thought of as a noncommutative counterpart of the module of vectors fields. The pair \((A, V)\), where \(V\) is a \(Z(A)\)-submodule of \(\text{Der}A\), is called differential algebra. In our case \(A = C_c^\infty(G, \mathbb{C})\), and as \(V\) we choose those derivations of \(A\) which are naturally adapted to the structure of \(G = E \times \Gamma\) (as a direct product), i.e., all those \(v \in \text{Der}A\) which can be presented in the form \(v = v_E + v_{\Gamma}\) where \(v_E\)
is the "component" of \( v \) parallel to \( E \), and \( v_\Gamma \) the "component" of \( v \) parallel to \( \Gamma \).

By a \textit{metric} on the \( \mathcal{Z}(\mathcal{A}) \)-submodule \( V \) we understand a \( \mathcal{Z}(\mathcal{A}) \)-bilinear non-degenerate symmetric mapping \( g : V \times V \to \mathcal{A} \). We chose the metric of the form

\[
g = pr_E^*g_E + pr_\Gamma^*g_\Gamma
\]

where \( g_E \) and \( g_\Gamma \) are metrics on \( E \) and \( \Gamma \), respectively, and \( pr_E \) and \( pr_\Gamma \) are the obvious projections. This choice of \( V \) and \( g \) is naturally adapted to the product structure of \( G \) but, if necessary, we could try other choices as well.

It turns out that now we can define the linear connection (essentially by using the Koszul formula), curvature and the Ricci operator \( \mathbf{R} : V \to V \) (the counterpart of the Ricci tensor with one index up and one index down). For detail the reader should consult reference [6]. This allows us to define the \textit{noncommutative Einstein equation} in the operator form

\[
\mathbf{R} + 2\Lambda \mathbf{I} = 0
\]  

(1)

where \( \Lambda \) is a constant related to the usual cosmological constant, and \( \mathbf{I} \) is the identity operator (the factor 2 appears as the result of our conventions, see [6]). It should be expected that "at the fundamental level" there is only "pure noncommutative geometry", therefore we assume that at this level there is no "matter source" (such as a counterpart of the energy-momentum tensor), but for the sake of generality we keep \( \Lambda \) in the equation (if necessary we can always put \( \Lambda = 0 \)).

The set \( \text{ker} \mathbf{G} := \{ v \in V : \mathbf{G}(v) = 0 \} \), where \( \mathbf{G} = \mathbf{R} + 2\Lambda \mathbf{I} \), is a \( \mathcal{Z}(\mathcal{A}) \)-submodule of \( V \); it gives a solution of eq. (1). The differential algebra \( (\mathcal{A}, \text{ker} \mathbf{G}) \), where \( \mathcal{A} = C^\infty_c(G, \mathcal{C}) \) is called \textit{Einstein algebra} (strictly speaking only \( \text{ker} \mathbf{G} \) is determined by eq. (1)).

Now, we introduce the representation of the algebra \( \mathcal{A} \), the so-called \textit{Connes representation}, in the Hilbert space \( \mathcal{H} = L^2(G_q) \)

\[
\pi_q : \mathcal{A} \to \mathcal{B}(\mathcal{H}),
\]

where \( \mathcal{B}(\mathcal{H}) \) denotes an algebra of bounded operators on \( \mathcal{H} \), with the help of the formula

\[
(\pi_q(a)\psi)(\gamma) = \int_{G_q} a(\gamma_1)\psi(\gamma_1^{-1}\gamma),
\]  

(2)
with \( \gamma = \gamma_1 \circ \gamma_2, \gamma, \gamma_1, \gamma_2 \in G_q, q \in E, \psi \in \mathcal{H}, a \in \mathcal{A} \) (see [11, p. 102]). It can be shown that the completion of \( \mathcal{A} \) with respect to the norm \[
 \| a \| = \sup_{q \in E} \| \pi_q(a) \|
\]
is a \( C^* \)-algebra (let us notice that \( q \) can formally be understood as the pair \((q, e) \in G \) where \( e \) is the unit of \( \Gamma \)). We shall denote this algebra by \( \mathcal{E} \) and call \textit{Einstein} \( C^* \)-\textit{algebra}.

The next natural step is to perform quantization with the help the usual \( C^* \)-algebraic method (see, for instance, [12]). A quantum gravitational system is represented by an Einstein \( C^* \)-algebra \( \mathcal{E} \), and its observables by Hermitian elements of \( \mathcal{E} \). If \( a \) is a Hermitian element of \( \mathcal{E} \), and \( \phi \) a state on \( \mathcal{E} \) then \( \phi(a) \) is the expectation value of the observable \( a \) when the system is in the state \( \phi \). The essentially new ingredient of our approach is the postulate according to which the dynamical equation of a quantum gravitational system is

\[
 i\hbar \pi_q(v(a)) = [\pi_q(a), F] \quad (3)
\]

for every \( q \in E, \psi \in L^2(G_q) \). Here \( v \in \ker G \) and in this way generalized Einstein’s equation (1) is coupled to quantum dynamical equation (3). \( F \) is a Fredholm operator, i.e. an operator \( F : \mathcal{H} \to \mathcal{H} \) such that \( F(\mathcal{H}) \) is closed and the dimensions of its kernel and cokernel are finite. The Planck constant \( \hbar \) should be regarded as measuring a deformation from commutativity. Eq. (3) is a noncommutative counterpart of the Schrödinger equation in the Heisenberg picture of the usual quantum mechanics. Equation (3) acts on that Hilbert space \( L^2(G_q) \) which should be regarded as a counterpart of the Hilbert space in the position representation in quantum mechanics. Accordingly, the quantity \( |\psi(\gamma)|^2 \) is the probability density of the “fundamental symmetry” \( \gamma \in G_q \) to occur.

In [13] it has been shown that the above sketched gravity quantization scheme correctly reproduces the usual general relativity (on space-time) and quantum mechanics (in the Heisenberg picture) when the algebra \( \text{Der} \mathcal{A} \) is suitably restricted to its center \( \mathcal{Z}(\mathcal{A}) \) (or to some subset of \( \mathcal{Z}(\mathcal{A}) \)).

3 State dependent flow of time

We shall assume that \( \mathcal{A} = C^\infty_c(G, \mathbb{C}) \) is already suitably completed to form a \( C^* \)-algebra. Let us consider a state \( \omega : \mathcal{A} \to \mathbb{C} \) on \( \mathcal{A} \), i.e., a positive, linear
and normed functional on $\mathcal{A}$. From the Gelfand-Naimark-Segal theorem (see [13]) it follows that there exists exactly one representation $\pi_\omega : \mathcal{A} \to \text{End}\mathcal{H}_\omega$ of the algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$, and the vector $\xi_\omega \in \mathcal{H}_\omega$ such that

(i) $\text{Lin}(\pi_\omega(\mathcal{A})\xi_\omega) = \mathcal{H}_\omega,$

(ii) $\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega)$

for every $a \in \mathcal{A}$.

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R} = (\pi_\omega(\mathcal{A}))''$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. We shall consider a one-parameter group of automorphisms $\alpha_t : \mathcal{R} \to \mathcal{R}, t \in \mathbb{R}$. Let the vector $\xi_\omega$ be a cyclic and separating vector in the Hilbert space $\mathcal{H}_\omega$ ($\xi$ is separating in $\mathcal{A}$ if it is cyclic in $\mathcal{A}'$). We define the operator $S : \mathcal{R} \to \mathcal{R}$ by

$$S(b)(\xi_\omega) = b^*(\xi_\omega)$$

for $b \in \mathcal{R}$. It can be shown [11, p. 43] that $S$ has the following properties

(i) $S = S^{-1}$,

(ii) the operator $J = S|S|^{-1}$ satisfies the condition $J\mathcal{R}J^{-1} = \mathcal{R}'$,

(iii) the operator $\Delta = |S|^2 = S^*S$ satisfies the condition $\Delta^{it}\mathcal{R}\Delta^{-it} = \mathcal{R}$ for every $t \in \mathbb{R}$.

From these properties it follows that

$$S = J \cdot \Delta^{1/2}$$

where $J$ is an antiunitary operator, and $\Delta$ a self-adjoint, positive operator. The Tomita-Takesaki theorem [14] asserts that the mappings $\alpha_t : \mathcal{R} \to \mathcal{R}, t \in \mathbb{R}$, given by

$$\alpha_t(b) = \Delta^{-it}b\Delta^{it}, \quad (4)$$

$b \in \mathcal{R}$, form a one-parameter group of automorphisms of the von Neumann algebra $\mathcal{R}$. It is called the modular group of modular automorphisms of the state $\omega$ on the von Neumann algebra $\mathcal{R}$, or the modular group for brevity. Connes and Rovelli [7] have interpreted this one-parameter group as a state dependent time in the framework of noncommutative geometry, provided that the state $\omega$ is of the form $\omega(a) = \text{Tr}[a\omega]$, for every $a \in \mathcal{A}$, where $\mathcal{A}$
is any C*-algebra of bounded linear operators on a Hilbert space. The last assumption was necessary in order to connect the state dependent time flow with statistical (thermodynamical) properties of the considered system (see Sec. 6 below). In the following, we shall interpret \( \alpha_t \) in terms of time, but we shall remain strictly within the algebraic approach.

The question arises: what is the relationship between the GNS representation of \( A \) and the Connes representation of \( A \) as far as time properties are concerned? The answer to this question is given by the following theorem:

**Theorem 1:** Let \( \pi_q : A \to \text{End} \mathcal{H} \) be the Connes representation of the algebra \( A = C^\infty_c(G, \mathbb{C}) \) in the Hilbert space \( \mathcal{H} = L^2(G_q) \) in which there is a cyclic vector \( \xi_0 \). There exists the unique state \( \omega_q = (\pi_q(a)\xi_0, \xi_0) \), for every \( a \in A \), and

\[
\pi_{\omega_q}(a)[b] = [\pi_q(a)(b)]
\]

is the GNS representation of \( A \). Here \([...]\) denotes an element of the quotient space \( A/N_{\omega_q} \), \( N_{\omega_q} \) being the ideal \( N_{\omega_q} = \{ a \in A : \omega_q(aa^*) = 0 \} \) of the algebra \( A \).

**Proof:** For a cyclic vector \( \xi_0 \in \mathcal{H} = L^2(G_q) \), we define the state \( \omega_q(a) = (\pi_q(a)\xi_0, \xi_0) = (a \ast \xi_0, \xi_0) \), \( a \in A \). Now, \( N_{\omega_q} = \{ a \in A : \omega_q(aa^*) = 0 \} = \{ a \in A : (\pi_q(aa^*)\xi_0, \xi_0) = 0 \} \). A suitable completion of \( A/N_{\omega_q} \) gives us the Hilbert space \( \mathcal{H}_{\omega_q} \). We define the GNS representation \( \pi_{\omega_q} : A \to \text{End} \mathcal{H}_{\omega_q} \) by

\[
(\pi_{\omega_q}(a))[b] = [a \ast b] = [\pi_{\omega_q}(a)(b)]
\]

with \( a \in A, [b], [a \ast b] \in \mathcal{H}_{\omega_q} \). \( \square \)

It is now evident that we can construct the modular group \( \alpha_t \) of the state \( \omega_q \) on the von Neumann algebra \( \mathcal{R} = (\pi_{\omega_q}(A))'' \). If we interpret this one-parameter group as a time flow, we have the interesting conclusion: To have a (state dependent) time flow in the noncommutative space, determined by the algebra \( A = C^\infty_c(G, \mathbb{C}) \), we must form the quotient space \( A/N_{\omega_q} \), i.e., we must glue together some elements of \( A \). In other words, in the original noncommutative regime (as determined by \( A \), in principle, there is no time; a state dependent time flow emerges only in the process of a suitable coarse graining of the original space (i.e., in the process of forming the quotient \( A/N_{\omega_q} \)).
4 State independent time flow

Let us consider the subset

\[ U = \{ u \in \mathcal{A} : uu^* = u^*u = I \} \]

of the algebra \( \mathcal{A} \); it forms a group, called the unitary group of \( \mathcal{A} \).

**Lemma 2:** Let \( \mathcal{U} \) be the unitary group of \( \mathcal{A} = C^\infty_c(G, \mathbb{C}) \) and let \( u \in \mathcal{A} \) implies \( u_q \in L^2(G_q) \), then the Connes representation \( \pi_q : \mathcal{U} \to \text{End}\mathcal{H}, \mathcal{H} = L^2(G_q) \), given by

\[ \pi_q(u)\xi = u_q^\star \xi, \]

for \( u \in \mathcal{U}, \xi \in \mathcal{H} \), is a unitary representation.

**Proof:** A representation \( \pi \) is unitary if it preserves scalar products, or equivalently if: \( \pi(u)^* = \pi(u^{-1}) = \pi(u)^{-1} \), for \( u \in \mathcal{U} \). Let \( \xi, \eta \in \mathcal{H}, u \in \mathcal{U} \). One has

\[ (\pi_q(u)\xi, \pi_q(u)\eta) = (u_q^\star \xi, u_q^\star \eta) = \int_{G_q} (u_q^\star \xi)^* (u_q^\star \eta) = \xi^\star u_q^\star u_q^\star \eta = (\xi, \eta). \]

Since the Connes representation of \( \mathcal{U} \) is unitary it is also irreducible (see [15, p. 127]) and, consequently, there exists a one-to-one correspondence between such representations and pure states on \( \mathcal{A} \).

Let \( \mathcal{R} \) be a von Neumann algebra. An automorphism \( \alpha : \mathcal{R} \to \mathcal{R} \) is called inner if there exists a nontrivial element \( u \) of the unitary group \( \mathcal{U} \) (i.e., \( u \neq I \)) such that

\[ \alpha(a) = u^*au \]

for every \( a \in \mathcal{R} \). Let us consider two automorphisms \( \alpha' \) and \( \alpha'' \) of \( \mathcal{R} \). These automorphisms are said to be inner equivalent, \( \alpha' \sim \alpha'' \), if there is an inner automorphism \( \alpha_{\text{inner}} \) such that

\[ \alpha'' = \alpha_{\text{inner}} \alpha' \]

or, equivalently,

\[ u\alpha''(a) = \alpha'(a)u \]

for every \( a \in \mathcal{R} \) and some \( u \in \mathcal{U} \). In other words, two automorphisms of \( \mathcal{R} \) are inner equivalent if they differ by an inner automorphism. The set of all
equivalence classes of this relation is called a group of outer automorphisms of \( R \), denoted by \( \text{Out}(R) \).

In general, the modular group \( \alpha_t \) of a state \( \omega \) on the von Neumann algebra \( R \) does not consist of inner automorphisms, but it projects down to a non-trivial one-parameter group \( \tilde{\alpha}_t, t \in \mathbb{R} \), in \( \text{Out}(R) \). According to the cocycle Radon-Nikodym theorem (see [11, p. 44]) two modular automorphisms \( \omega_1 \) and \( \omega_2 \) on \( R \) are inner equivalent. Consequently, the one-parameter group \( \tilde{\alpha}_t \) does not depend on the choice of \( \omega \) on \( R \), and \( \tilde{\alpha}_t \) can be interpreted as a state independent, “canonical time” in the considered noncommutative space (see [7]).

Lemma 2 allows us to apply the above construction to our case. We thus obtain a one-parameter group in \( \text{Out}(R) \), where \( R = (\pi_\omega(q(A)))'' \) (see remarks following theorem 1), which is independent of the choice of the state \( \omega_q \). This can be interpreted in the following way. If we make a suitable “coarse graining” of our state dependent time \( \alpha_t \) (i.e. if we change from \( R \) to \( \text{Out}(R) \)), we obtain a “time flow” in our noncommutative space which depends only on the algebra \( A = C^\infty_c(G, \mathbb{C}) \).

5 Noncommutative dynamics

An interesting feature of our scheme for quantizing gravity is the fact that although in the noncommutative regime there is no time (in the usual sense), dynamics can be done in terms of derivations of the algebra \( A = C^\infty_c(G, \mathbb{C}) \) (see eq. (3)). If the modular groups of Section 3 are interpreted as a sort of (state dependent) time, the question arises whether these groups can be related to the noncommutative dynamics. To answer this question is the aim of the present section.

From eq. (4), defining modular groups, we have

\[
\alpha_t(a) = e^{-it \ln \Delta} a e^{it \ln \Delta}.
\]

By differentiating this expression we obtain

\[
\frac{d}{dt} \alpha_t(a)|_{t=0} = i[a, \ln \Delta].
\]

On the other hand, let us notice that the modular group \( \alpha_t \) determines the derivation of the von Neumann algebra \( R \). Indeed, if \( v \in \text{Der}R \), and the
system is in a state $\omega$, we have

$$v(\pi_\omega(a)) = \frac{d}{dt}|_{t=0}(\alpha_t(\pi_\omega(a))) =$$

$$= i[\pi_\omega(a), \ln \Delta] = i\text{ad}_{\ln \Delta}(\pi_\omega(a)).$$

(5)

When we change from the Connes representation $\pi_q$ of the algebra $\mathcal{A}$ to its CNS representation $\pi_{\omega_q}$ [by forming the quotient $\mathcal{A}/N_{\omega_q}$ (see theorem 1)] the algebra becomes more coarse, and one-parameter modular groups emerge. Let us assume that both the derivation $v$ and the Fredholm operator $F$, appearing in eq. (3), are invariant with respect to the equivalence relation $\sim$ defined by: $a \sim b$ iff $a - b \in N_{\omega_q}$, for every $a, b \in \pi_q(\mathcal{A})$; i.e., we assume that $a \sim b$ implies $F(a) \sim F(b)$, for every $a, b \in \pi_q(\mathcal{A})$, and similarly for derivation $v$. Then eq. (3) takes the form

$$i\hbar \bar{v}([\pi_q(a)]) = [[\pi_q(a)], \bar{F}]$$

where $\bar{v}([a]) = [v(a)], \bar{F}([a]) = [F(a)]$, or taking into account (5),

$$\hbar \frac{d}{dt}|_{t=0}(\alpha_t(\pi_{\omega_q}(a))) = [\pi_{\omega_q}(a), \bar{F}]$$

where $\bar{F} = \ln \Delta$. We can see that if the “modular time” $\alpha_t$ is available, it can serve as a dynamical time.

6 Interpretation

To define the modular group $\alpha_t$, Connes and Rovelli have distinguished the state on $\mathcal{A}$ of the form $\omega(a) = \text{Tr}[a\omega]$ for every $a \in \mathcal{A}$ (which, in the language used by physicists is a density matrix). Owing to this choice they were able to argue that the time flow has a statistical (thermodynamic) origin. They emphasize that it is not only the arrow of time that emerges in this way, but also the time flow itself. Indeed, the exegesis of the mathematical formalism presented in the preceding sections allows us to interpret the one-parameter groups $\alpha_t$ and $\tilde{\alpha}_t$ as a state dependent and state independent time flows, respectively (possibly even without a sharply determined arrow of time). The non-local character of the algebra $\mathcal{A} = C^\infty_c(G, \mathbb{C})$ suggests that in the noncommutative regime there is no multiplicity (there are no
individuals), and consequently there can be neither statistics (in the usual sense) nor thermodynamics. Only after suitable identifications of elements of $\mathcal{A}$ have been made (with the help of the corresponding equivalence relations) the groups $\alpha_t$ and $\tilde{\alpha}_t$ appear, and one can speak of a certain temporal succession. It is not that time has its origin in thermodynamics, but rather both thermodynamics and time have their origin in a transition from the noncommutative geometry as determined by the full algebra $\mathcal{A}$ to a geometry in which the groups $\alpha_1$ and $\tilde{\alpha}_1$ are meaningful.

The fact that modular groups lead to correct classical limits remains in agreement with this interpretation. For instance, as shown by Connes and Rovelli [7], the modular group of the Gibbs state gives the standard time flow modulo the choice of the time unit, and in the case of classical mechanics it goes onto the usual Hamiltonian evolution.

Moreover, we could even claim that the probabilistic nature of the standard quantum mechanics has its origin in the “phase transition” from the noncommutative regime to the usual physics. For what does it mean that if $a$ is a Hermitian element of the Einstein algebra $\mathcal{E}$, and $\phi$ a state on $\mathcal{E}$, then $\phi(a)$ is the expectation value of the observable $a$ when the system is in the state $\phi$ (see Sec. 2 above)? In the noncommutative regime the terms such as “expectation value” seem to be meaningless, and $\phi(a)$ is just a value of the functional $\phi$ at $a$. The above phrase could mean only that the expression $\phi(a)$ (and similar expressions as well) acquires its quantum-probabilistic interpretation if we go from the noncommutative regime to the standard quantum mechanics (as described in [6]).

It is worth mentioning that Rovelli [3, 4] has demonstrated that there exists a natural extension of canonical Heisenberg-picture of quantum mechanics which remains valid even if time is not defined. This is exactly the case in our scheme in which the quantum sector of noncommutative gravity is such an extension of the Heisenberg picture. The usual time evolution appears only within a “non-quantum gravity approximation”.

ACKNOWLEDGMENT: We thank Professor Piotr Hajac for turning our attention to paper [4].

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