Convergence of stochastic approximation via martingale and converse Lyapunov methods

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Abstract
In this paper, we study the almost sure boundedness and the convergence of the stochastic approximation (SA) algorithm. At present, most available convergence proofs are based on the ODE method, and the almost sure boundedness of the iterations is an assumption and not a conclusion. In Borkar and Meyn (SIAM J Control Optim 38:447–469, 2000), it is shown that if the ODE has only one globally attractive equilibrium, then under additional assumptions, the iterations are bounded almost surely, and the SA algorithm converges to the desired solution. Our objective in the present paper is to provide an alternate proof of the above, based on martingale methods, which are simpler and less technical than those based on the ODE method. As a prelude, we prove a new sufficient condition for the global asymptotic stability of an ODE. Next we prove a “converse” Lyapunov theorem on the existence of a suitable Lyapunov function with a globally bounded Hessian, for a globally exponentially stable system. Both theorems are of independent interest to researchers in stability theory. Then, using these results, we provide sufficient conditions for the almost sure boundedness and the convergence of the SA algorithm. We show through examples that our theory covers some situations that are not covered by currently known results, specifically Borkar and Meyn (2000).

Keywords Stochastic approximation · martingale methods · converse Lyapunov theory · global asymptotic stability

Dedicated to Prof. Eduardo Sontag on his 70th birthday and to Prof. Rajeeva L. Karandikar on his 65th birthday.

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1 Introduction

The stochastic approximation (SA) algorithm, originally introduced by Robbins and Monro [19], is a widely used method for finding a zero of a function \( f : \mathbb{R}^d \to \mathbb{R}^d \), when only noisy measurements of \( f \) are available. Since its introduction, it has been a workhorse of probability theory and has found many applications. An early survey can be found in [13], and a more recent and broader survey can be found in [8].

Here is a brief description of the algorithm. Suppose \( f : \mathbb{R}^d \to \mathbb{R}^d \), and it is desired to find a solution \( \theta^\ast \) to the equation \( f(\theta) = 0 \), when one can access only noisy measurements of \( f(\cdot) \). In broad terms, the algorithm proceeds as follows: One begins with an initial guess \( \theta_0 \). In the original version of SA proposed in [19], at time \( t + 1 \geq 1 \), one has access to a noisy measurement

\[
y_{t+1} = f(\theta_t) + \xi_{t+1},
\]

where \( \theta_t \) is the current guess and \( \xi_{t+1} \) is the uncertain disturbance. Then, the guess \( \theta_t \) is updated according to

\[
\theta_{t+1} = \theta_t + \alpha_t y_{t+1}, = \theta_t + \alpha_t (f(\theta_t) + \xi_{t+1}),
\]

where \( \{\alpha_t\}_{t \geq 1} \) is a prespecified deterministic sequence of step sizes, with \( \alpha_t \in (0, 1) \) for all \( t \). It is desired to study the limit behavior of the sequence \( \{\theta_t\} \). Define

\[
S_f := \{ \theta \in \mathbb{R}^d : f(\theta) = 0 \}.
\]

The aims are to determine conditions under which (i) the sequence \( \{\theta_t\} \) is bounded almost surely, and (ii) \( d(\theta_t, S_f) \to 0 \) as \( t \to \infty \), where

\[
d(\theta, S) := \inf_{\phi \in S} \|\theta - \phi\|_2.
\]

A well-established method for analyzing the limit behavior of \( \{\theta_t\} \) is known as the ODE method, whereby the sample paths of the recursion (2) are related to the (deterministic) solution trajectories of the ODE \( \dot{\theta} = f(\theta) \). This method was introduced in [7, 15]. A very readable summary of this method is found in [17], while book-length treatments of the method are available in [1, 4, 12]. In the original paper [19], the step sizes \( \alpha_t \) are positive and satisfy the conditions below, generally referred to as the Robbins–Monro conditions. The two conditions are often shown together, but we display them separately for reasons that will become clear later.

\[
\sum_{t=1}^{\infty} \alpha_t^2 < \infty,
\]

\[
\sum_{t=1}^{\infty} \alpha_t = \infty.
\]
Until 2000, the available results had the following general format: If the function $f(\cdot)$ satisfies some conditions, and if the sequence of iterates $\{\theta_t\}$ is bounded almost surely, then $\theta_t \to S_f$ almost surely, provided some other conditions are satisfied. Some authors refer to the almost-sure boundedness of the iterates $\{\theta_t\}$ as “stability,” and $\theta_t \to S_f$ almost surely as “convergence.” Thus, the typical result stated that stability plus other conditions imply convergence. A major breakthrough was achieved in [3], in which the stability of the iterations is a conclusion and not a hypothesis. Thus, under suitable conditions on the function $f$, both stability and convergence follow. In that paper, it is assumed that there is a unique solution $\theta^*$ to $f(\theta) = 0$. It is shown that, under suitable conditions (which are spelled out precisely later; see Theorem 2), $\theta_t \to \theta^*$ almost surely as $t \to \infty$.

The present paper has the same objective, namely to make the stability of the iterations a conclusion and not a hypothesis. It is ironic that, as far back as 1965, there is a paper by Gladyshev [9] that established both stability and convergence of the SA algorithm, using martingale methods. Moreover, there is a clear “division of labor” whereby (3) leads to the almost sure boundedness of the iterations, while the addition of (4) leads to convergence. This division of labor is not found in any other paper until now. However, the results in [9] are restricted to a special class of functions $f$. Thus, the present author was motivated by a desire to extend the martingale methods of [9] to the same class of functions $f$ as are covered by the ODE method in [3], or perhaps a more general class. Subsequent to the publication of [9], Robbins and Siegmund [18] introduced a very general theorem that they called an “almost supermartingale convergence theorem,” in which they rediscovered some of the basic ideas in [9]. In [18], the authors mention that they were unaware of [9] at the time of writing their paper.

This is the starting point of the present paper. Except in Theorem 5, we study the case where $\theta^*$ is the unique solution of $f(\theta) = 0$ and is thus the unique equilibrium of the associated ODE $\dot{\theta} = f(\theta)$. It is assumed that $\theta^*$ is a globally attractive equilibrium of the ODE. The same assumptions are made in [3]. Then, we state and prove a new sufficient condition for global asymptotic stability, namely Theorem 4, which is less restrictive than the standard results found in, for example, Hahn [10], Khalil [11] and Vidyasagar [21]. The new sufficient condition allows us to conclude that the systems studied in [9] are globally asymptotically stable—a conclusion that does not follow from existing Lyapunov theory. Then we present new sufficient conditions in Theorem 5, for the iterations $\{\theta_t\}$ to (i) remain bounded almost surely and (ii) converge to $\theta^*$ almost surely. These conditions require the existence of a Lyapunov function that is bounded both above and below by multiples of $\|\theta\|^2$, whose derivative can approach zero arbitrarily slowly. The proof of Theorem 5 is based on the Robbins–Siegmund theorem in [18], stated here as Theorem 3. Then, it is shown by example that Theorem 5 is applicable to situations where the Borkar–Meyn result (Theorem 2) does not apply. However, in Theorem 5, the need to assume the existence of a suitable Lyapunov function is still a bottleneck. To overcome this, we strengthen the hypothesis to the requirement that $\theta^*$ is a globally exponentially stable equilibrium. Then we state and prove a new converse theorem, namely Theorem 6, on the existence of a Lyapunov function that has a globally bounded Hessian matrix. Combining these two theorems leads to a new “self-contained” result, namely Theorem 7, where all assumptions are
only on \( f(\cdot) \). The converse theorem proved here builds on an earlier theorem in [6], which uses a significantly different Lyapunov function compared to its predecessors. Both Theorems 4 and 6 are new and are possibly of independent interest within the stability theory community. In each case (namely, \( \theta^* \) is either globally asymptotically stable, or globally exponentially stable), the “division of labor” as in [9] continues to hold: If (3) is satisfied, then the sequence of iterates \( \{\theta_t\} \) is bounded almost surely. Further, if (4) also holds, then \( \theta_t \rightarrow \theta^* \) almost surely.

The paper is organized as follows: In Sect. 2, we introduce the list of various assumptions made and then state the main results of the paper. For the convenience of the reader, we also state the relevant results from [3, 9]. This section also includes a discussion of the various assumptions and the conclusions of the various theorems. The actual proofs of the various theorems are given in Sect. 3. Section 4 contains several illustrative examples that highlight the advances made in our results when compared to known results. Finally, Sect. 5 contains a discussion of some problems for future research.

## 2 Statements of main results

### 2.1 List of assumptions

Throughout the remainder of the paper, within each theorem we choose from the following list of “standing” assumptions: Note that not every assumption is needed in every theorem.

Let \( \theta^t_0 = (\theta_0, \ldots, \theta_t) \), \( \xi^t_1 = (\xi_1, \ldots, \xi_t) \), and let \( \mathcal{F}_t \) denote the \( \sigma \)-algebra generated by \( \theta^t_0, \xi^t_1 \). Then,

(F1) The equation \( f(\theta) = 0 \) has a unique solution \( \theta^* \).
(F2) The function \( f \) is globally Lipschitz-continuous with constant \( L \).

\[
\| f(\theta) - f(\phi) \|_2 \leq L \| \theta - \phi \|_2, \quad \forall \theta, \phi \in \mathbb{R}^d.
\]  

(F2’) The function \( f \) is twice continuously differentiable and is globally Lipschitz-continuous with constant \( L \).
(F3) The equilibrium \( \theta^* \) of the ODE \( \dot{\theta} = f(\theta) \) is globally exponentially stable. Thus, there exist constants \( \mu \geq 1, \gamma > 0 \) such that

\[
\| s(t, \theta) - \theta^* \|_2 \leq \mu \| \theta - \theta^* \|_2 \exp(-\gamma t), \quad \forall t \geq 0, \forall \theta \in \mathbb{R}^d.
\]  

(F4) There is a finite constant \( K \) such that

\[
\| \nabla^2 f_i(\theta) \|_S \cdot \| \theta - \theta^* \|_2 \leq K, \quad \forall i \in [d], \forall \theta \in \mathbb{R}^d,
\]  

where \([d]\) denotes the set \( \{1, \ldots, d\} \), and \( \| \cdot \|_S \) denotes the spectral norm of a matrix, i.e., its largest singular value.
(N1) The noise sequence \( \{ \xi_t \} \) satisfies:

\[
E(\xi_{t+1} | \mathcal{F}_t) = 0 \text{ a.s., } \forall t \geq 0.
\]  

(8)

(N2) The noise sequence \( \{ \xi_t \} \) satisfies:

\[
E(\|\xi_{t+1}\|_2^2 | \mathcal{F}_t) \leq \sigma^2 (1 + \|\theta_t - \theta^*\|_2^2), \text{ a.s. } \forall t \geq 0,
\]  

for some finite constant \( \sigma^2 \).

Note that, as a consequence of Assumption (F2), for each \( \theta \in \mathbb{R}^d \) there is a unique function \( s(\cdot, \theta) \) that satisfies the ODE

\[
\frac{ds(t, \theta)}{dt} = f(s(t, \theta)), s(0, \theta) = \theta.
\]  

(10)

A consequence of Assumption (F4) is that

\[
\left| \frac{\partial^2 f_i(\theta)}{\partial \theta_j \partial \theta_k} \right| \cdot \|\theta - \theta^*\|_2 \leq K, \quad \forall i, j, k \in [d], \quad \forall \theta \in \mathbb{R}^d.
\]  

(11)

2.2 Restatement of the theorems of Gladyshev and Borkar–Meyn

In this subsection, we restate the theorems of Gladyshev [9] and Borkar–Meyn [3] to facilitate comparison with the contributions of the present paper.

Theorem 1 ([9]) Suppose assumptions (F1), (N1) and (N2) hold. In addition, \( f(\cdot) \) is a passive function; that is, for each \( 0 < \epsilon < M < \infty \),

\[
\sup_{\epsilon < \|\theta - \theta^*\|_2 < M} \langle \theta - \theta^*, f(\theta) \rangle < 0,
\]

\[
\|f(\theta)\|_2 \leq K \|\theta - \theta^*\|_2, \quad K < \infty.
\]

With these assumptions,

1. If \( \sum_{t=0}^{\infty} \alpha_t^2 < \infty \), then \( \{ \theta_t \} \) is bounded almost surely.
2. If in addition \( \sum_{t=0}^{\infty} \alpha_t = \infty \), then \( \theta_t \to \theta^* \) almost surely as \( t \to \infty \).

Remark Note that the second assumption implies that \( f(\cdot) \) is continuous at \( \theta^* \). However, it need not be continuous anywhere else.

Theorem 2 ([3]) Suppose assumptions (F1) and (N1) and (N2) hold, and in addition:

- The Robbins–Monro conditions (3) and (4) hold.
- \( f(\cdot) \) is globally Lipschitz continuous.
- There is a “limit function” \( f_\infty : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\frac{f(r \theta)}{r} \to f_\infty(\theta) \text{ as } r \to \infty,
\]
uniformly over compact subsets of $\mathbb{R}^d$.

- $0$ is a globally exponentially stable equilibrium\(^1\) of 
  \[ \dot{\theta} = f_\infty(\theta). \]

Under the stated assumptions,

1. $\{\theta_t\}$ is bounded almost surely.
2. $\theta_t \to \theta^* \text{ as } t \to \infty$.

At present, Theorem 2 is the only convergence result based on the ODE method, which has the almost sure boundedness of the iterations as a conclusion, and not a hypothesis. The Lyapunov approach presented here is based on Assumption (F1), namely that the equation $f(\theta)$ has a unique solution $\theta^*$. One of the perceived advantages of the ODE method is that it is applicable even to the case where this equation has multiple solutions. However, as shown in [4, 17], when the equation under study has multiple solutions, the available results once again have the almost sure boundedness of the iterations as a hypothesis and not a conclusion. The only result based on the ODE method where this is a conclusion and not a hypothesis is [3], and in this paper also it is assumed that the equation $f(\theta)$ has a unique solution. In this sense, the results presented here are comparable to those obtained using the ODE method.

2.3 Preliminaries

The proofs of various theorems are based on the following well-known theorem from [18]. The original reference [18] is somewhat inaccessible. However, the same theorem is stated as Lemma 2 in [2, Section 5.2]. A recent survey of many results along similar lines is found in [8], where Theorem 3 is stated as Lemma 4.1.

**Theorem 3** ([18]) Suppose $\{z_t\}, \{\eta_t\}, \{\gamma_t\}, \{\psi_t\}$ are nonnegative stochastic processes adapted to some filtration $\{\mathcal{F}_t\}$, that satisfy

\[ E(z_{t+1}|\mathcal{F}_t) \leq (1 + \eta_t)z_t + \gamma_t - \psi_t \text{ a.s., } \forall t. \]  
(12)

Define the set $\Omega_0 \subseteq \Omega$ by

\[ \Omega_0 := \left\{ \omega : \sum_{i=0}^{\infty} \eta_i(\omega) < \infty \right\} \cap \left\{ \omega : \sum_{i=0}^{\infty} \gamma_i(\omega) < \infty \right\}, \]  
(13)

Then, for all\(^2\) $\omega \in \Omega_0$, we have that (i) $\lim_{t \to \infty} z_t(\omega) \text{ exists and is finite, and } (ii)\]

\[ \sum_{t=0}^{\infty} \psi_t(\omega) < \infty, \quad \forall \omega \in \Omega_0. \]  
(14)

\(^1\) In [3], only global asymptotic stability is assumed. However, as shown therein, because the vector field $f_\infty$ is “scale-free” in that $f_\infty(\lambda \theta) = \lambda f_\infty(\theta)$ for all $\lambda > 0$, the two assumptions are equivalent.

\(^2\) Here and elsewhere, “for all” really means “for almost all.”
In particular, if \( P(\Omega_0) = 1 \), then \( \{ z_t \} \) is bounded almost surely, in the sense that
\[
P\{ \omega \in \Omega : \sup_t z_t(\omega) < \infty \} = 1,
\]
and
\[
\sum_{t=0}^{\infty} \psi_t(\omega) < \infty \text{ a.s.}
\]
Note that, while applying the above theorem here, it is always assumed that \( P(\Omega_0) = 1 \).

To state our first theorem on the convergence of the SA algorithm, we begin by recalling some notation and concepts from nonlinear stability theory. The reader is referred to [10, 11, 21] for further details.

**Definition 1** A function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is said to belong to class \( K \), denoted by \( \phi \in K \), if \( \phi(0) = 0 \), and \( \phi(\cdot) \) is strictly increasing. A function \( \phi \in K \) is said to belong to class \( K R \), denoted by \( \phi \in K R \), if in addition, \( \phi(r) \rightarrow \infty \) as \( r \rightarrow \infty \). A function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is said to belong to class \( B \), denoted by \( \phi \in B \), if \( \phi(0) = 0 \), and in addition, for all \( 0 < \epsilon < M < \infty \) we have that
\[
\inf_{\epsilon \leq r \leq M} \phi(r) > 0.
\]
See Example 2 for an example of a function that belongs to class \( B \), but not to class \( K \).

Observe that if \( \phi \in K R \), then \( \phi^{-1} \) exists and also belongs to \( K R \).

Next we recall a standard concept from stability theory. Consider the ODE
\[
\dot{\theta} = f(\theta),
\]
and suppose \( V : \mathbb{R}^d \rightarrow \mathbb{R}^+ \) is \( C^1 \). Let \( \nabla V : \mathbb{R}^d \rightarrow \mathbb{R}^d \) denote the gradient of \( V \), viewed as a column vector. Then, the function \( \dot{V} : \mathbb{R}^d \rightarrow \mathbb{R} \) defined by
\[
\dot{V}(\theta) = \langle \nabla V(\theta), f(\theta) \rangle
\]
is called the **derivative** of \( V \) along the trajectories of (18).

**2.4 Theorem statements**

We begin with a theorem that gives a sufficient condition for global asymptotic stability. The definition of global asymptotic stability is quite standard and can be found in any standard reference, such as [10, 11, 21]. Because Theorem 4 represents a weakening of the best-known sufficient conditions, it should be of independent interest to researchers in nonlinear stability theory. Theorem 4 is the basis for extending Theorem 1 to more
general families of functions $f(\cdot)$. In the process, we are able to prove the convergence of the SA algorithm in situations that are not covered by Theorem 2.

**Theorem 4** Suppose Assumptions (F1) holds, and that there exists a function $V: \mathbb{R}^d \to \mathbb{R}_+$ and functions $\eta, \psi \in \mathcal{K}\mathcal{R}, \phi \in \mathcal{B}$ such that

$$
\eta(\|\theta - \theta^*\|_2) \leq V(\theta) \leq \psi(\|\theta - \theta^*\|_2), \quad \forall \theta \in \mathbb{R}^d,
$$

(20)

$$
\dot{V}(\theta) \leq -\phi(\|\theta - \theta^*\|_2), \quad \forall \theta \in \mathbb{R}^d,
$$

(21)

Then, $\theta^*$ is a globally asymptotically stable equilibrium of the ODE (18).

We now state the first result on the almost-sure boundedness and convergence of the stochastic approximation algorithm (2). As is a recurring theme throughout the paper, the conditions for almost sure boundedness of the iterations are separate from and weaker than those for convergence.

**Theorem 5** Suppose $f(\theta^*) = 0$, and Assumptions (F2) and (N1) and (N2) hold. Suppose in addition that there exists a $C^2$ Lyapunov function $V: \mathbb{R}^d \to \mathbb{R}_+$ that satisfies the following conditions:

- There exist constants $a, b > 0$ such that

  $$
a\|\theta - \theta^*\|_2^2 \leq V(\theta) \leq b\|\theta - \theta^*\|_2^2, \quad \forall \theta \in \mathbb{R}^d.
$$

(22)

- There is a finite constant $M$ such that

  $$
  \|\nabla^2 V(\theta)\|_S \leq 2M, \quad \forall \theta \in \mathbb{R}^d.
  $$

(23)

With these hypotheses, we can state the following conclusions:

1. If $\dot{V}(\theta) \leq 0$ for all $\theta \in \mathbb{R}^d$, and if (3) holds, then the iterations $\{\theta_t\}$ are bounded almost surely.

2. Suppose further that there exists a function $\phi \in \mathcal{B}$ such that

  $$
  \dot{V}(\theta) \leq -\phi(\|\theta - \theta^*\|_2), \quad \forall \theta \in \mathbb{R}^d,
  $$

(24)

and in addition, both (3), (4) hold. Then $\theta_t \to \theta^*$ almost surely as $t \to \infty$.

**Theorem 6** Suppose Assumptions (F1), (F2'), (F3) and (F4) hold. Under these hypotheses, there exists a $C^2$ function $V: \mathbb{R}^d \to \mathbb{R}_+$ such that $V$ and its derivative $\dot{V}: \mathbb{R}^d \to \mathbb{R}$ defined by

$$
\dot{V}(\theta) = \langle \nabla V(\theta), f(\theta) \rangle
$$

(25)

together satisfy the following conditions: There exist positive constants $a, b, c$ and a finite constant $M$ such that

$$
a\|\theta - \theta^*\|_2^2 \leq V(\theta) \leq b\|\theta - \theta^*\|_2^2, \quad \dot{V}(\theta) \leq -c\|\theta - \theta^*\|_2^2, \quad \forall \theta \in \mathbb{R}^d.
$$

(26)

$$
\|\nabla^2 V(\theta)\|_S \leq 2M, \quad \forall \theta \in \mathbb{R}^d.
$$

(27)
Combining Theorems 5 and 6 gives the following “self-contained” theorem:

**Theorem 7** Suppose Assumptions (F1), (F2'), (F3) and (F4) as well as Assumptions (N1)–(N2) hold. Under these hypotheses,

1. If the step size sequence \( \{\alpha_t\} \) satisfies (3), then the stochastic process \( \{\theta_t\} \) is bounded almost surely.
2. If the step size sequence \( \{\alpha_t\} \) satisfies (4) in addition to (3), then the stochastic process \( \{\theta_t\} \) converges almost surely to \( \theta^* \) as \( t \to \infty \).

### 2.5 Discussion of the theorems

#### 2.5.1 Theorem 4

The usual theorems on global asymptotic stability look similar to Theorem 4, except that the function \( \phi(\cdot) \) in (21) is assumed to belong to the Class \( \mathcal{K} \), not class \( \mathcal{B} \); see for example [10, Theorem 26.2] or [21, Theorem 5.3.56]. The change here is that the assumption on \( \phi(\cdot) \) is weakened to \( \phi \in \mathcal{B} \) from \( \phi \in \mathcal{K} \). See Example 2 for a function that belongs to class \( \mathcal{B} \) but cannot be bounded below by any function of class \( \mathcal{K} \). Example 3 has a system whose global asymptotic stability can be deduced using Theorem 4, but not by traditional theorems, using any Lyapunov function of the form \( V(\theta) = \theta^{2m}, \ m \geq 1 \).

#### 2.5.2 Theorem 5

- In the first part of the theorem that deals with the almost sure boundedness of the iterations, there is no assumption that \( \theta^* \) is the unique solution of \( f(\theta) = 0 \). In other words, Assumption (F1) is not made. Moreover, the assumptions imply only that \( \theta^* \) is a stable equilibrium of the ODE \( \dot{\theta} = f(\theta) \). This is in sharp contrast to existing theorems in SA theory, where it is assumed that \( \theta^* \) is globally asymptotically stable. So far as the author is able to determine, there is no predecessor to Theorem 5, which establishes the almost-sure boundedness of the iterations under a set of hypothesis that guarantee only that the equilibrium \( \theta^* \) of the ODE \( \dot{\theta} = f(\theta) \) is stable—not globally asymptotically stable.
- In the second part of the theorem that deals with the convergence of the iterations, assumption (24) ensures that \( \theta^* \) is a globally asymptotically stable equilibrium of the ODE \( \dot{\theta} = f(\theta) \); this follows from Theorem 4. Therefore, Assumption (F1) is implicit in the second part of the theorem.

#### 2.5.3 Theorem 6

Theorem 5 requires the existence of a suitable Lyapunov function \( V \) that satisfies various conditions. Therefore, verifying whether or not such a function exists is a bottleneck. It would be highly desirable to provide sufficient conditions that involve only the function \( f(\cdot) \) that guarantee the existence of a suitable Lyapunov function. This is the objective of Theorem 6.
As shown in Theorem 4, the conditions on $V$ in Theorem 5 ensure that the equilibrium $\theta^*$ of the ODE (18) is globally asymptotically stable. By strengthening the assumption to the global exponential stability of $\theta^*$ and adding a few other assumptions, it is possible to prove a “converse” Lyapunov theorem that establishes the existence of a suitable $V$ function. This is done in Theorem 6.

Note that there is already a well-developed “converse Lyapunov theory” that establishes all the requirements on $V$, except for the global boundedness of the Hessian of $V$; see for example [21, Section 5.7] or [10, Sections 48–51]. Therefore, the contribution of Theorem 6 is in establishing that $V$ has a globally bounded Hessian. This theorem is new and possibly of independent interest to researchers in nonlinear stability theory.

2.5.4 Theorem 7

The closest available results to Theorem 7 in the current literature are from [3]. In that paper, it is also assumed that there is a unique solution $\theta^*$ to the equation $f(\theta) = 0$. In addition, it is assumed that the functions

$$f_r(\theta) : \theta \mapsto \frac{f(r \theta)}{r}$$

converge uniformly over compact sets to a limit function $f_\infty$ as $r \to \infty$ and that $\theta$ is a globally exponentially stable equilibrium of the ODE $\dot{\theta} = f_\infty(\theta)$. There is no requirement of the limit function $f_\infty$ here. Instead, we have Assumption (F4), which requires that the spectral norm of the Hessian matrix of each component $f_i(\theta)$ decays at least as fast as $1/\|\theta - \theta^*\|_2$. It is now shown that Condition (F4) is only a slight strengthening of the assumption that $f(\cdot)$ is globally Lipschitz continuous.

2.5.5 Significance of assumption (F4)

Suppose $f : \mathbb{R}^d \to \mathbb{R}^d$ is $C^2$ and globally Lipschitz-continuous with constant $L$. Thus,

$$\|f(\theta) - f(\phi)\|_2 \leq L\|\theta - \phi\|_2, \quad \forall \theta, \phi \in \mathbb{R}^d.$$ 

Then, $f(\cdot)$ is absolutely continuous everywhere, and it follows that the Jacobian $\nabla f(\cdot) \in \mathbb{R}^{d \times d}$ exists almost everywhere is globally bounded, i.e.,

$$\|\nabla f(\theta)\|_S \leq L, \quad \forall \theta \in \mathbb{R}^d,$$

where $\| \cdot \|_S$ is the spectral norm of a matrix, induced by the $\ell_2$-vector norm. Now observe that, for each index $i \in [d]$, we have

$$\nabla f_i(\theta) = \nabla f_i(\theta^*) + \left[ \int_0^1 \nabla^2 f_i(\theta^* + \lambda(\theta - \theta^*)) \, d\lambda \right](\theta - \theta^*)$$

$$= \nabla f_i(\theta^*) + M_i(\theta, \theta^*)(\theta - \theta^*),$$

where $M_i(\cdot, \cdot)$ is the Hessian matrix of $f_i(\cdot)$.
where

\[ M_i(\theta, \theta^*) = \int_0^1 \nabla^2 f_i(\theta^* + \lambda(\theta - \theta^*)) \, d\lambda. \]

Hence, it follows that

\[ \|M_i(\theta, \theta^*)(\theta - \theta^*)\|_S \leq 2L, \quad \forall i \in [d], \quad \forall \theta, \phi \in \mathbb{R}^d. \quad (28) \]

Now note that in effect Assumption (F4) consists of replacing the integrand in the definition of \( M_i(\theta, \theta^*) \) as follows:

\[ \|M_i(\theta, \theta^*)\|_S \leftarrow \|\nabla^2 f_i(\theta)\|_S. \]

Almost all the literature on stochastic approximation assumes that the function under study is globally Lipschitz-continuous. In turn this imposes some restrictions on \( \nabla^2 f(\cdot) \), as shown in (28). The above argument shows Assumption (F4) is not too much stronger than the consequence of the global Lipschitz continuity assumption.

**Example 1** As a specific example, any vector field of the form

\[ f(\theta) = A(\theta - \theta^*) + h(\theta)\|\theta - \theta^*\|_2^{-2r} \]

would satisfy Assumption (F4) if (i) each component of \( h(\cdot) \) has a globally bounded Hessian, and (ii) \( r \geq 0.5 \). To see this, fix any index \( i \in [d] \), and observe that

\[ f_i(\theta) = a^{(i)}(\theta - \theta^*) + h_i(\theta)\|\theta - \theta^*\|_2^{-2r}, \]

where \( a^{(i)} \) denotes the \( i \)-th row of \( A \). Thus,

\[
\frac{\partial f_i}{\partial \theta_j} = a_{ij} + \frac{\partial h_i}{\partial \theta_j}(\theta)\|\theta - \theta^*\|_2^{-2r} - 2r h_i(\theta)\|\theta - \theta^*\|_2^{-2r-2}(\theta_j - \theta_j^*), \\
\frac{\partial^2 f_i}{\partial \theta_j \partial \theta_k} = \frac{\partial^2 h_i}{\partial \theta_j \partial \theta_k}(\theta)\|\theta - \theta^*\|_2^{-2r} - 2r \frac{\partial h_i}{\partial \theta_j}(\theta)\|\theta - \theta^*\|_2^{-2r-2}(\theta_k - \theta_k^*) - 2r h_i(\theta)\|\theta - \theta^*\|_2^{-2r-2} \delta_{jk},
\]

where in the last equation \( \delta_{jk} \) denotes the Kronecker delta. Now observe that, because \( h(\cdot) \) has a globally bounded Hessian, we have that

\[ \|\nabla^2 h(\theta)\|_{jk} = O(1), \quad \|\nabla h_i(\theta)\|_2 = O(\|\theta - \theta^*\|_2), \quad |h_i(\theta)| = O(\|\theta - \theta^*\|^2_2), \]

\[ (2 \cdot 10^3)^2 \leq 2 \cdot 10^6. \]
where $O(1)$ means bounded globally. Thus, every term in the expression for $\partial^2 f_i/\partial \theta_j \partial \theta_k$ is $O(\|\theta - \theta^*\|_2^{-2r})$. Therefore,

$$\|\nabla^2 f_i(\theta)\|_S = O(\|\theta - \theta^*\|_2^{-2r}), \|\nabla^2 f_i(\theta)\|_S \cdot \|\theta - \theta^*\|_2 = O(\|\theta - \theta^*\|_2^{-2r+1}).$$

Since $-2r + 1 \leq 0$ whenever $r \geq 0.5$, condition (F4) holds whenever $r \geq 0.5$. \hfill \Box

The existence of a Lyapunov function $V$ that satisfies (26) is quite standard. Indeed, the usual choice is

$$V(\theta) := \int_0^\infty \|s(t, \theta)\|_2^2 \, dt. \quad (29)$$

However, for this choice of $V$, no conclusions can be drawn about the behavior of the gradient $\nabla V$ nor the Hessian $\nabla^2 V$. In [6], the authors introduce a completely different Lyapunov function of the form

$$V(\theta) := \int_0^T e^{2\kappa \tau} \|s(\tau, \theta) - \theta^*\|_2^2 \, d\tau, \quad (30)$$

where $0 < \kappa < \gamma$ is arbitrary, and $T$ is any finite number such that

$$\frac{\ln \mu}{\gamma - \kappa} \leq T < \infty,$$

where $\mu, \gamma$ are defined in (6). For this choice of Lyapunov function, it is shown in [6] that there exists a finite constant $L'$ such that

$$\|\nabla V(\theta)\|_2 \leq L' \|\theta - \theta^*\|_2. \quad (31)$$

Now Theorem 6 extends the theory further by showing that, if (F4) holds, then $\nabla^2 V$ is globally bounded. This in turn implies (31). As we shall see, (F4) is the key assumption that allows us to extend the converse Lyapunov theory of [6] and prove that $V$ has a globally bounded Hessian.

3 Proofs of the theorems

3.1 Proof of Theorem 4

Proof Let $\theta(\cdot)$ denote a solution trajectory of the ODE (18). Then, (21) implies that $V(\theta(t))$ is a nonincreasing function of $t$ and therefore has a limit as $t \to \infty$, call it $V_\infty$. It is now shown that $V_\infty = 0$. To see this, suppose that $V_\infty > 0$. Then, the right-side bound in (20) implies that

$$\|\theta - \theta^*\|_2 \geq \psi^{-1}(V_\infty) =: c_1 > 0, \forall t,$$
while the left-side bound in (20) implies that
\[ \|\theta - \theta^*\|_2 \leq \eta^{-1}(V(\theta(0))) =: c_2 < \infty. \]
In turn, this implies that
\[ -\dot{V}(\theta(t)) \geq \inf_{c_1 \leq r \leq c_2} \phi(r) > 0, \]
because \( \phi(\cdot) \) is a function of class \( B \). This is a contradiction because
\[ V(\theta(t)) = V(\theta(0)) + \int_0^t \dot{V}(\theta(\tau)) \, d\tau, \]
and if \(-\dot{V}(\theta(t))\) is bounded away from zero, then eventually \( V(\theta(t)) \) would become negative. Therefore, \( V_\infty = 0 \), and \( V(\theta(t)) \to 0 \) as \( t \to \infty \). Now the left inequality in (20) shows that \( \|\theta(t)\|_2 \to 0 \) as \( t \to \infty \). \( \square \)

3.2 Proof of Theorem 5

**Proof** We begin by observing that, by Taylor’s theorem, we have
\[ V(\theta + \eta) = V(\theta) + \langle \nabla V(\theta), \eta \rangle + \frac{1}{2} \langle \eta, \nabla^2 V(\theta + \lambda \eta) \eta \rangle, \]
for some \( \lambda \in [0, 1] \). Since \( \|\nabla^2 V(\theta + \lambda \eta)\|_S \leq 2M \), it follows that
\[ V(\theta + \eta) \leq V(\theta) + \langle \nabla V(\theta), \eta \rangle + M \|\eta\|_2^2, \ \forall \theta, \eta \in \mathbb{R}^d. \]
Now apply the above bound with \( \theta = \theta_i \) and \( \eta = \alpha_i f(\theta_i) + \alpha_i \xi_{i+1} \). This gives
\[
V(\theta_{i+1}) \leq V(\theta_i) + \alpha_i \langle \nabla V(\theta_i), f(\theta_i) \rangle + \alpha_i \langle \nabla V(\theta_i), \xi_{i+1} \rangle + \alpha_i^2 M \|f(\theta_i)\|_2 + \xi_{i+1}\|^2_2 \\
= V(\theta_i) + \alpha_i \dot{V}(\theta_i) + \alpha_i \langle \nabla V(\theta_i), \xi_{i+1} \rangle \\
+ \alpha_i^2 M \|f(\theta_i)\|_2^2 + \|\xi_{i+1}\|_2^2 + 2 \langle f(\theta_i), \xi_{i+1} \rangle. 
\]
Now we bound \( E(V(\theta_{i+1})|F_i) \) using Assumptions (F2), (N1), (N2) and the bounds in (22). This gives
\[
E(V(\theta_{i+1})|F_i) \leq V(\theta_i) + \alpha_i^2 M \left[ L^2 \|\theta_i - \theta^*\|_2^2 + \sigma^2 (1 + \|\theta_i - \theta^*\|_2^2) \right] + \alpha_i \dot{V}(\theta_i). 
\] (32)

To prove the first conclusion of the theorem, recall the hypotheses that \( \dot{V}(\theta) \leq 0 \) for all \( \theta \) and \( \sum_{i=1}^\infty \alpha_i^2 < \infty \), and apply (22). Recall also that (3) holds. Using these bounds in (32) gives

\[ E(V(\theta_{i+1})|F_i) \leq V(\theta_i) + \alpha_i^2 M \left[ L^2 \|\theta_i - \theta^*\|_2^2 + \sigma^2 (1 + \|\theta_i - \theta^*\|_2^2) \right] + \alpha_i \dot{V}(\theta_i). \]

\[ \square \] Springer
\[ E(V(\theta_{t+1})|\mathcal{F}_t) \leq V(\theta_t) + \alpha_t^2 M \left[ L^2 \| \theta_t - \theta^* \|_2^2 + \sigma^2 (1 + \| \theta_t - \theta^* \|_2^2) \right] \]
\[ \leq \left[ 1 + \alpha_t^2 M \left( L^2 + \sigma^2 \right) \right] V(\theta_t) + \alpha_t^2 M \sigma^2. \quad (33) \]

Now apply Theorem 3 with
\[ z_t = V(\theta_t), \eta_t = \alpha_t^2 M \left( L^2 + \sigma^2 \right), \gamma_t = \alpha_t^2 M \sigma^2, \psi_t = 0. \]

Then it follows that \( \lim_{t \to \infty} V(\theta_t) \) exists almost surely and is finite. Combined with (22), this shows that \( \{\theta_t\} \) is bounded almost surely.

To prove the second conclusion, we restore the term \( \dot{V}(\theta_t) \) in (32) and use (24). This gives
\[ E(V(\theta_{t+1})|\mathcal{F}_t) \leq \left[ 1 + \alpha_t^2 M \left( L^2 + \sigma^2 \right) \right] V(\theta_t) + \alpha_t^2 M \sigma^2 - \alpha_t \phi(\| \theta_t - \theta^* \|_2). \]

Now we again apply Theorem 3 with
\[ z_t = V(\theta_t), \eta_t = \alpha_t^2 M \left( L^2 + \sigma^2 \right), \gamma_t = \alpha_t^2 M \sigma^2, \psi_t = \alpha_t \phi(\| \theta_t - \theta^* \|_2). \]

This time, the conclusions are that (i) there exists a random variable \( \zeta \) such that \( V(\theta_t) \to \zeta \) almost surely, and (ii)
\[ \sum_{t=0}^{\infty} \alpha_t \phi(\| \theta_t - \theta^* \|_2) < \infty \text{ a.s.} \quad (34) \]

Let \( \Omega_1 \subseteq \Omega \) denote the values of \( \omega \) for which
\[ \sup_t V(\theta_t(\omega)) < \infty, V(\theta_t(\omega)) \to \zeta(\omega), \text{ and } \sum_{t=0}^{\infty} \alpha_t \phi(\| \theta_t - \theta^* \|_2) < \infty. \]

Note that \( P(\Omega_1) = 1 \). It is now shown that \( \zeta(\omega) = 0 \) for all \( \omega \in \Omega_1 \). Suppose by way of contradiction that, for some \( \omega \in \Omega_1 \), we have that \( \zeta(\omega) = 2\epsilon > 0 \). Choose a \( T \) such that \( V(\theta_t(\omega)) \geq \epsilon \) for all \( t \geq T \) and also define \( V_M := \sup_t V(\theta_t(\omega)) \). Then, we have that
\[ \sqrt{\epsilon/b} \leq \| \theta_t \|_2 \leq \sqrt{V_M/a}, \forall t \geq T. \]

Define
\[ \delta := \inf_{\sqrt{\epsilon/b} \leq r \leq \sqrt{V_M/a}} \phi(r), \]
and observe that $\delta > 0$ because $\phi$ belongs to the class $\mathcal{B}$. Therefore,

$$\sum_{t=T}^{\infty} \alpha_t \phi (\|\theta_t - \theta^*\|_2) \geq \sum_{t=T}^{\infty} \alpha_t \delta = \infty,$$

provided (4) holds. But this contradicts (34). Hence, no such $\omega \in \Omega_1$ can exist. In other words, $\zeta = 0$ almost surely, and $V(\theta_t) \to 0$ almost surely. Finally, it follows from (22) that $\theta_t \to \theta^*$ almost surely as $t \to \infty$, which is the second conclusion.

### 3.3 Proof of Theorem 6

**Proof** Following [6], define the Lyapunov function candidate $V$ as in (30). Then, as shown in [6], $V$ satisfies (26) and (31). The latter is not of any concern to us. So we focus on proving (27).

Note that the solution function $s(\cdot, \theta)$ satisfies

$$s(t, \theta) = \theta + \int_0^t f(s(\tau, \theta)) \, d\tau. \quad (35)$$

Therefore,

$$\nabla_{\theta} s(t, \theta) = I + \int_0^t \nabla_{\theta} f(s(\tau, \theta)) \, d\tau. \quad (36)$$

Next, the chain rule gives

$$\nabla_{\theta} f(s(\tau, \theta)) = \nabla_{\phi} f(\phi) \big|_{\phi = s(\tau, \theta)} \nabla_{\theta} s(\tau, \theta).$$

Now the global Lipschitz continuity of $f$ implies that

$$\|\nabla_{\phi} f(s(\tau, \phi))\|_S \leq L, \forall \phi, \forall \tau.$$ 

Therefore, (36) leads to (after dropping the subscript $\theta$)

$$\|\nabla s(t, \theta)\|_S \leq 1 + \int_0^t L \|\nabla s(\tau, \theta)\|_S \, d\tau.$$ 

Now Gronwall’s inequality leads to the bound

$$\|\nabla s(t, \theta)\|_S \leq \exp(Lt), \forall t, \forall \theta. \quad (37)$$

---

3 Note that dropping a finite number of terms does not affect the validity of (4).
Next we proceed to find a bound on the second partial derivatives. It follows from (35) that
\[
\frac{\partial s_i(t, \theta)}{\partial \theta_j} = \delta_{ij} + \int_0^t \frac{\partial f_j(s(\tau, \theta))}{\partial \theta_j} \, d\tau,
\]
where \(\delta_{ij}\) is the Kronecker delta. Next,
\[
\frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_k} = \int_0^t \frac{\partial^2 f_j(s(\tau, \theta))}{\partial \theta_j \partial \theta_k} \, d\tau. \tag{38}
\]
We will use (38) later. Next, expand \(V(\theta)\) as
\[
V(\theta) = \int_0^T e^{2\kappa \tau} \sum_{i=1}^d \left[ s_i(\tau, \theta) - \theta_i^* \right]^2 \, d\tau.
\]
Thus,
\[
\frac{\partial V(\theta)}{\partial \theta_j} = \int_0^T 2e^{2\kappa \tau} \sum_{i=1}^d \left[ s_i(\tau, \theta) - \theta_i^* \right] \frac{\partial s_i(\tau, \theta)}{\partial \theta_j} \, d\tau,
\]
\[
\frac{\partial^2 V(\theta)}{\partial \theta_j \partial \theta_k} = I_1 + I_2,
\]
where
\[
I_1 = \int_0^T 2e^{2\kappa \tau} \sum_{i=1}^d \frac{\partial s_i(\tau, \theta)}{\partial \theta_k} \frac{\partial s_i(\tau, \theta)}{\partial \theta_j} \, d\tau, \tag{39}
\]
\[
I_2 = \int_0^T 2e^{2\kappa \tau} \sum_{i=1}^d \left[ s_i(\tau, \theta) - \theta_i^* \right] \frac{\partial^2 s_i(\tau, \theta)}{\partial \theta_j \partial \theta_k} \, d\tau. \tag{40}
\]
We will prove the boundedness of each integral separately. Note that, as a consequence of (37), we have
\[
\left| \frac{\partial s_i(\tau, \theta)}{\partial \theta_k} \right|, \left| \frac{\partial s_i(\tau, \theta)}{\partial \theta_j} \right| \leq \| \nabla s(\tau, \theta) \|_S \leq \exp L \tau, \quad \forall \tau, i, j, k.
\]
So the first integral is bounded by
\[
|I_1| \leq \int_0^T 2de^{2\kappa \tau} e^{2L \tau} \, d\tau =: C_1 < \infty
\]
for some constant \(C_1\), whose precise value need not concern us. So we concentrate on showing that, under Assumption (F4), \(I_2\) is also bounded globally.
Towards this end, we begin by observing that
\[ \| s(t, \theta) - \theta^* \|_2 \geq e^{-Lt} \| \theta - \theta^* \|_2, \quad \forall t \geq 0. \]

The proof is elementary and can be found in [5, Theorem 8]. In particular,
\[ \| s(t, \theta) - \theta^* \|_2 \geq e^{-LT} \| \theta - \theta^* \|_2, \quad \forall t \in [0, T]. \tag{41} \]

Now we estimate the entity \( \partial^2 f_i(s(\tau, \theta))/\partial \theta_j \partial \theta_k \) in (38). Note that
\[
\frac{\partial f_i(s(\tau, \theta))}{\partial \theta_j} = \sum_{l=1}^{d} \frac{\partial f_i(\phi)}{\partial \phi_l} \bigg|_{\phi=s(\tau, \theta)} \frac{\partial s_l(\tau, \theta)}{\partial \theta_j},
\]
\[
\frac{\partial^2 f_i(s(\tau, \theta))}{\partial \theta_j \partial \theta_k} = \sum_{l=1}^{d} \frac{\partial f_i(\phi)}{\partial \phi_l} \bigg|_{\phi=s(\tau, \theta)} \frac{\partial^2 s_l(\tau, \theta)}{\partial \theta_j \partial \theta_k} + \sum_{l=1}^{d} \frac{\partial}{\partial \theta_k} \left[ \frac{\partial f_i(\phi)}{\partial \phi_l} \bigg|_{\phi=s(\tau, \theta)} \right] \frac{\partial s_l(\tau, \theta)}{\partial \theta_j}. \tag{42}
\]

The second term can be expanded as:
\[
\sum_{l=1}^{d} \sum_{r=1}^{d} \frac{\partial^2 f_i(\phi)}{\partial \phi_l \partial \phi_r} \bigg|_{\phi=s(\tau, \theta)} \frac{\partial s_r(\tau, \theta)}{\partial \theta_k} \frac{\partial s_l(\tau, \theta)}{\partial \theta_j}.
\]

Now Assumption (F4) and the bound (41) together imply that
\[
\left| \frac{\partial^2 f_i(\phi)}{\partial \phi_l \partial \phi_r} \bigg|_{\phi=s(\tau, \theta)} \right| \leq \| \nabla^2 f_i(s(\tau, \theta)) \|_S \leq \frac{K}{\| s(\tau, \theta) - \theta^* \|_2} \leq \frac{K e^{LT}}{\| \theta - \theta^* \|_2}, \quad \forall \tau \in [0, T].
\]

Also, as shown in (37),
\[
\left| \frac{\partial s_r(\tau, \theta)}{\partial \theta_k} \right|, \left| \frac{\partial s_l(\tau, \theta)}{\partial \theta_j} \right| \leq \| \nabla s(\tau, \theta) \|_S \leq e^{L\tau} \leq e^{LT}, \quad \forall \tau \in [0, T].
\]

Next, the global Lipschitz continuity of \( f \) implies that
\[
\left| \frac{\partial f_i(\phi)}{\partial \phi_l} \right| \leq L.
\]
Substituting all of these bounds including (42) into (38) gives

\[
\left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_l} \right| \leq \int_0^t L \sum_{l=1}^d \left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_k} \right| \, d\tau + \int_0^t \sum_{l=1}^d \sum_{r=1}^d \frac{K e^{LT} e^{L \tau} e^{L \tau}}{\| \theta - \theta^* \|_2} \, d\tau
\]

\[
\leq C_2 + \int_0^t L \sum_{l=1}^d \left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_k} \right| \, d\tau,
\]

where

\[
C_2 = \frac{d^2 T K e^{3LT}}{\| \theta - \theta^* \|_2}
\]

is inversely proportional to \( \| \theta - \theta^* \|_2 \). Now define

\[
h_{jk}(t, \theta) := \sum_{i=1}^d \left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_k} \right|.
\]

Note that the right side of (43) does not depend on \( i \). Therefore, (43) implies that

\[
h_{jk}(t, \theta) \leq \sum_{i=1}^d \left[ C_2 + \int_0^t L \sum_{l=1}^d \left| \frac{\partial^2 s_i(\tau, \theta)}{\partial \theta_j \partial \theta_k} \right| \right]
\]

\[
\leq C_2 d + \int_0^t Ld h_{jk}(\tau, \theta) \, d\tau.
\]

So by Gronwall’s inequality

\[
h_{jk}(t, \theta) \leq C_2 d e^{LdT}, \quad \forall t \in [0, T].
\]

Since \( h_{jk} \) is a sum, each individual component must also be smaller than \( h_{jk} \) in magnitude. Thus,

\[
\left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_l} \right| \leq C_2 d e^{LdT} \leq \frac{C_3}{\| \theta - \theta^* \|_2}
\]

for a suitable constant \( C_3 \). Therefore, we have established that the Hessian of each \( s_i \) decays as \( \theta \) gets farther away from \( \theta^* \). Now we return to \( I_2 \) as defined in (40), and observe that, as a consequence of Assumption (F3) of global exponential stability, we have

\[
|s_i(t, \theta) - \theta_i^*| \leq \|s(t, \theta) - \theta^*\|_2 \leq \mu \|\theta - \theta^*\|_2, \quad \forall t \geq 0.
\]
Now in the definition of $I_2$, we get the bound

$$|s_i(t, \theta) - \theta^*_i| \cdot \left| \frac{\partial^2 s_i(t, \theta)}{\partial \theta_j \partial \theta_k} \right| \leq \mu \|\theta - \theta^*\|_2 \cdot \frac{C_3}{\|\theta - \theta^*\|_2} = \mu C_3.$$ 

Since the integrand in (40) is bounded and $T$ is finite, it follows that $I_2$ is also bounded. This finally leads to the desired conclusion that $\|\nabla^2 V\|_S$ is globally bounded. \hfill \(\square\)

Note that in the above proof, the finiteness of the constant $T$ is crucial. Therefore, the traditional infinite integral type of Lyapunov function defined in (29) is not directly amenable to such analysis. It is perhaps possible to replace the Lyapunov function candidate of (29) by another function of the form

$$V(\theta) := \left[ \int_0^\infty \|s(t, \theta)\|^{2p} \, dt \right]^{1/p}.$$ 

However, something similar to Assumption (F4) would still be required.

### 4 Examples

In this section, we give a few examples of the results presented thus far. We also discuss the implications of Assumption (F4) in (7).

**Example 2** Observe that every $\phi$ of class $\mathcal{K}$ also belongs to class $\mathcal{B}$. However, the converse is not true. Define

$$\phi(r) = \begin{cases} r, & \text{if } r \in [0, 1], \\ e^{-(r-1)}, & \text{if } r > 1. \end{cases}$$

Then, $\phi$ belongs to Class $\mathcal{B}$. However, since $\phi(r) \to 0$ as $r \to \infty$, $\phi$ cannot be bounded below by any function of class $\mathcal{K}$.

**Example 3** This example illustrates how Theorem 4 goes beyond currently available theorems in Lyapunov stability theory. The current theorems, of which [10, Theorem 26.2] and [21, Theorem 5.3.56] are typical, require $-\dot{V}(\theta)$ to be a function of Class $\mathcal{K}$.

Recall the function $\phi(\cdot)$ defined in Example 2. Now consider the ODE $\dot{\theta} = f(\theta)$, where

$$f(\theta) = \begin{cases} -\phi(\theta), & \theta \geq 0, \\ -f(-\theta), & \theta < 0. \end{cases}$$

Thus, $f(\cdot)$ is just an odd extension of $-\phi(\cdot)$. If we choose the Lyapunov function $V(\theta) = \theta^2$, then

$$\dot{V}(\theta) = -|\theta| \cdot \phi(|\theta|).$$
Therefore, $-\dot{V}$ is a function of class $\mathcal{B}$, and the global asymptotic stability of the equilibrium $\theta^* = 0$ follows from Theorem 4. However, $-\dot{V}(\theta)$ is not a function of class $\mathcal{K}$, nor can it be bounded below by a function of Class $\mathcal{K}$ because $|\theta| \cdot \phi(|\theta|) \to 0$ as $|\theta| \to \infty$. More generally, for every function of the form $V(\theta) = \theta^{2m}$, $m \geq 1$, $-\dot{V}$ cannot be bounded below by a function of Class $\mathcal{K}$. Hence, the traditional theorems fail to apply for any such function $V$.

**Example 4** Using Theorem 1, one can infer the convergence of the SA algorithm in the one-dimensional case, when the measurement $y_{t+1}$ is of the form $\theta^{(t)}$, with $f(0) = 0$, and

$$\sup_{\epsilon \leq \theta \leq M} \theta f(\theta) < 0, \quad \forall 0 < \epsilon < M < \infty. \quad (44)$$

Such a function $f(\cdot)$ is called a “passive” function in circuit theory. The objective of this example is to demonstrate that Gladyshev’s result does not in general follow from those in [3], but does follow from Theorem 5.

Consider the one-dimensional ODE $\dot{\theta} = -f(\theta)$, where $f(0) = 0$ and satisfies (44), $|f(\theta)| \to 0$ as $|\theta| \to \infty$. Then, $f(\cdot)$ satisfies the hypotheses of Theorem 1. In this case, the scale-free function defined in [3], namely

$$f_\infty(\theta) := \lim_{r \to \infty} \frac{f(r \theta)}{r} \equiv 0, \quad \forall \theta.$$ 

Hence, the ODE $\dot{\theta} = f_\infty(\theta)$ cannot be globally asymptotically state. On the other hand, if we use the Lyapunov function $V(\theta) = \theta^2$ (which is in effect what is done in [9]), then

$$-\dot{V}(\theta) = -\theta f(\theta),$$

which is a function of Class $\mathcal{B}$. Hence, it follows from Theorem 5 that $\{\theta_t\}$ is bounded almost surely if (3) holds. If, in addition, (4) also holds, then $\theta_t \to 0$ almost surely as $t \to \infty$.

Note that, whenever $f(\cdot)$ remains bounded as $\|\theta\|_2 \to 0$, we get $f_\infty \equiv 0$ for all $\theta$. Hence, Theorem 2 cannot be applied to such a situation.

**Example 5** As an illustration of Theorem 5 for the case where $f(\theta) = 0$ has multiple solutions, consider the following function $f : \mathbb{R} \to \mathbb{R}$:

$$f(\theta) = \begin{cases} -1 + \sin(\theta + \pi/2), & \theta \geq 0, \\ -f(\theta), & \theta < 0. \end{cases}$$

The solutions of this equation are $\theta_n = 2\pi n$, for every integer $n$.

Now define $\theta^* = 0$ to be the solution of interest to us and define the Lyapunov function $V(\theta) = \theta^2$. Then, $\dot{V}(\theta) = \theta f(\theta) \leq 0$ for all $\theta$. Therefore, all assumptions of the first part of Theorem 5 are satisfied, and we can infer that the iterations $\{\theta_t\}$ are almost surely bounded whenever the step sizes $\{\alpha_t\}$ are square summable.
It is worth noting that the equilibrium $\theta^* = 0$ is asymptotically stable but not globally asymptotically stable.

**Example 6** A standard problem in Reinforcement Learning is known as “value evaluation,” wherein one wants to solve a linear equation of the form

$$v^* = r + \gamma Av^*,$$

where $v^* \in \mathbb{R}^d$ is called the “value” vector, $r \in \mathbb{R}^d$ is called the “reward” vector, $\gamma \in (0, 1)$ is called the “discount factor,” and $A \in \mathbb{R}^{d \times d}$ is the state transition matrix of a Markov chain. Hence, if we define

$$\|M\|_{\infty \rightarrow \infty} := \sup_{v \neq 0} \frac{\|Mv\|_{\infty}}{\|v\|_{\infty}},$$

then $\|A\|_{\infty \rightarrow \infty} = 1$, and $\|\gamma A\|_{\infty \rightarrow \infty} = \gamma < 1$.

To apply SA to this problem, let us switch notation to be consistent with that in the paper, and rewrite as

$$\theta = r + \gamma A \theta.$$

If we define the function $f$ via

$$f(\theta) = r + \gamma A \theta - \theta,$$

then the unique equilibrium of the associated ODE $\dot{\theta} = f(\theta)$ is indeed the desired solution $v^*$. If it happens that the $\ell_2$-induced norm $\|\gamma A\|_S > 1$, then there exists a $v \in \mathbb{R}^d$ such that

$$\langle v - v^*, f(v) - f(v^*) \rangle > 0.$$

Thus, the map $f$ does not satisfy the hypotheses of Theorem 1. However, the convergence of the SA algorithm can still be inferred using Theorem 5, as follows: Note that, since $\gamma < 1$ and $\rho(A) \leq 1$, the eigenvalues of the matrix $\gamma A - I$ all have negative parts. Hence, $v^*$ is a globally exponentially stable equilibrium. Then, it follows from [21, Theorem 5.4.42] that, whenever $Q$ is a symmetric, positive definite matrix, the so-called Lyapunov matrix equation

$$P(\gamma A - I) + (\gamma A - I)^\top P = -Q$$

has a unique positive-definite solution for $P$. Thus, $V(\theta) = \theta^\top P \theta$ satisfies the hypotheses of Theorem 5 (because the Hessian of $V(\cdot)$ is constant and thus bounded). Hence, we can conclude that the SA algorithm will converge to the desired solution $v^*$, provided (3) and (4) hold.
5 Conclusions and future work

In this paper, we have presented some simple proofs for the almost sure boundedness and convergence of the stochastic approximation (SA) algorithm, based on martingale methods and converse Lyapunov theory. Two new results have been presented in Lyapunov stability: The first is a new sufficient condition for global asymptotic stability, which is weaker than currently known conditions. The second is a “converse” theorem that ensures the existence of a Lyapunov function with a globally bounded Hessian matrix for globally exponentially stable systems. Each of these theorems is coupled with the well-known Robbins–Siegmund theorem to provide some simple proofs for the convergence of the stochastic approximation (SA) algorithm. The results presented here in Lyapunov theory are new and may be of independent interest to researchers in nonlinear stability theory. The fact that the convergence proofs of SA are based on Lyapunov theory, and not the ODE method discussed in [7, 12, 15], opens the possibility that the same approach can be used to prove the convergence of the SA algorithm in more general settings, such as two-time scale SA [14] or projected gradient SA [20]. These lines of research are currently under investigation.

A different class of stochastic algorithms is studied in [16]. Specifically, the basic recursion is

\[
\theta_{t+1} = \theta_t + \alpha_t (f(\theta_t) + \xi_{t+1} + \eta_{t+1}),
\]

where the function \(f(\cdot)\) is a \textit{gradient vector field}, that is,

\[
f(\theta) = -\nabla J(\theta)
\]

for some function \(J : \mathbb{R}^d \to \mathbb{R}\) with compact level sets. This means that, for every constant \(c \in \mathbb{R}\), the level set

\[
S_J(c) := \{\theta \in \mathbb{R}^d : J(\theta) \leq c\}
\]

is compact. In addition, there are now \textit{two} measurement noise sequences \(\{\xi_t\}\) and \(\{\eta_t\}\). The sequence \(\{\eta_t\}\) converges to zero almost surely as \(t \to \infty\), while the sequence \(\{\xi_t\}\) \textit{is not required to satisfy} the conditional zero mean assumption (8). There are other minor differences, but these are the main differences between the set-up studied here and that studied in [16]. A similar extension is also presented in [4, p. 17]. It is worth noting that, in this more general setting, the almost sure boundedness of the iterations is \textit{assumed and not inferred}. See for example Assumption (A4) on [4, p. 11]. Our current research includes an extension of the martingale method to the case where Assumptions (N1) and (N2) on the noise \textit{namely (8) and (9)} \textit{are not assumed}. In particular, the noise is not assumed to have zero conditional mean, and the constant \(\sigma^2\) is replaced by a time-varying number \(\sigma^2_t\) that is allowed to be unbounded. Despite these relaxations, the almost sure boundedness of the iterations is inferred and not assumed, and the convergence is established. Those results will be presented elsewhere.

In [2], specifically Chapter 1 of Part II, the authors introduce a class of SA algorithms that are more general than those studied here. In broad terms, the iterations are driven by
a Markov process. Specifically (see [2, Eq. (1.1.1)]), the general formulation, converted to the present notation, is

$$
\theta_{t+1} = \theta_t + \alpha_t H(\theta_t, X_{t+1}) + \alpha_t^2 \rho_t(\theta_t, X_{t+1}),
$$

(46)

where $\theta_t \in \mathbb{R}^d$ and $X_t \in \mathbb{R}^n$. Assumption (A2) in [2] then requires that there exists a family $\pi_{\theta}(x, A)$ of transition probabilities on $\mathbb{R}^n$ such that, for every Borel subset $A \subseteq \mathbb{R}^n$, we have

$$
\Pr\{X_{t+1} \in A | \mathcal{F}_t\} = \pi_{\theta_t}(X_t, A),
$$

where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\theta'_0$, $X'_1$. The authors observe that if $\pi_{\theta}(x, d\mu) = \mu_{\theta}(dz)$ for some probability measure $\mu$ (that is, the transition probability does not depend on $x$), then the above formulation reduces to the Robbins–Monro formulation studied here. It would be of interest to explore whether martingale-based methods can be extended to this more general situation.

As a final comment, it might be possible to apply Lyapunov methods to establish the almost sure boundedness of the iterations and then to use ODE methods to derive more detailed estimates about the convergence than is possible using Lyapunov methods. This approach merits further study.

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