The Distortion-Rate Function of Sampled Wiener Processes

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Abstract

The minimal distortion attainable in recovering the waveform of a continuous-time Wiener process from an encoded version of its uniform samples is considered. We first introduce a combined sampling and source coding problem for the Wiener process under uniform sampling, and prove an associated source coding theorem that characterizes the minimal distortion from samples. We then derive a closed-form expression for this minimal distortion, given in terms of a reverse waterfilling expression involving the asymptotic eigenvalue density of a sequence of finite-rank operators. This expression allows to determine the minimal distortion attainable for any prescribed sampling rate and number of bits per time unit. In addition, it can be used to determine the ratio between bitrate and sampling rate when the distortion due to sampling is of the same magnitude as the distortion due to lossy compression. For example, we show that using approximately 1 bit per sample, the distortion is about 1.12 times that of the distortion-rate function of the Wiener process at the same bitrate. We also derive an upper bound on the minimal distortion, that is obtained by a sub-optimal compress-and-estimate remote source coding scheme. This bound implies that as the sampling rate goes to infinity, compress-and-estimate is sub-optimal up to a linear factor in the sampling interval.

Index Terms

Source coding; Brownian motion; Wiener process; Sampling; Remote source coding; Analog to digital conversion; Compress-and-estimate;

I. INTRODUCTION

Consider the task of encoding a continuous-time random process with some constraint on the bitrate at the output of this encoder. The optimal tradeoff between bitrate and distortion in the recovery of the process from its encoded version is described by the distortion-rate function (DRF) of the process. Modeling a phenomena using a continuous-time function has the advantage that the model can be adjusted to any scale in time resolution. When the random process is bandlimited, each of its realizations assumes a discrete-time representation obtained by uniformly sampling above its Nyquist rate. This implies that a source coding theorem for bandlimited processes that provides a characterization of their DRF can be obtained directly from their discrete-time representations [2], [3]. Moreover, the sample path of a random bandlimited process is almost surely smooth, implying that two consecutive time samples are highly correlated provided the time lag between them is small enough. This property implies that
oversampling a bandlimited signal above its Nyquist rate would not lead to an increase in compression or coding
performance. In fact, in some systems the exact opposite occurs: when memory is limited oversampling increases
distortion [4], [5], [6].

In contrast to a bandlimited signal model, the Wiener process describes a continuous time phenomena whose time
fluctuations scale with the time resolution, a property known as scale-invariance or self-similarity. In particular, it
is impossible to obtain an equivalent discrete-time representation of this process by sampling its waveform at any
finite sampling rate [7].

In theory, it is possible to map each realization of a continuous-time Wiener process over a finite interval to a
discrete sequence using the Karhunen-Loeve (KL) transform. This approach has been used to develop a source
coding theorem for the Wiener process in [8]. In practice, however, the computation of the KL transform involves
integration of the sample path of the process with respect to the KL basis elements. This integration is difficult to
realize using purely analog devices, while any digital approximation is subject to inaccuracies as a result of the
sample path fluctuation of the Wiener process. Instead, we suggest evaluating the performance limits by assuming
the practical setup of Fig. 1. In this model, the continuous-time Wiener process $W(\cdot)$ is first uniformly sampled at
rate $f_s$, resulting in the discrete-time process $\bar{W}[\cdot]$. The process $\bar{W}[\cdot]$ is then encoded using no more than $R$ bits per
time unit, and ultimately the original waveform $W(\cdot)$ is reconstructed as accurately as possible from this encoded
version. The parameters in this problem are the sampling rate $f_s$, the bitrate $R$ and the average distortion $D$. The
tradeoff among the three is described by the function $D(f_s, R)$, which is defined as the minimal distortion that can
be attained in reconstructing the original waveform when the sampling rate is $f_s$ and the bitrate is $R$.

In this work we consider this combined sampling and source coding model and derive a closed form expression for
the function $D(f_s, R)$. We show that $D(f_s, R)$ is attained by first obtaining the minimal mean square error estimate
of the source from its samples, which, due to the Markov property of the Wiener process, is given by linearly
interpolating between these samples. This interpolation results in a continuous-time process, which is then encoded
using an optimal source code applied to its KL coefficients. We note that these KL coefficients are functions of the
samples, so that this the last encoding does not require any analog operation. The expression for $D(f_s, R)$ allows
us to compare the distortion-rate performance at any finite sampling rate $f_s$ to the unconstrained distortion-rate
performance given by the DRF $D_W(R)$ of $W(\cdot)$ from [8]. We show that $D(f_s, R)$ converges to $D_W(R)$ at a rate that
is inversely quadratic in the sampling rate (quadratic in the sampling interval). Moreover, we determine, in terms
of the number of bits per sample, the equilibrium point where distortion due to sampling equals distortion due to
lossy compression; When the number of bits per sample is above this point, the error due to sampling dominates
the expression for $D(f_s, R)$ and any increase in bitrate reduces distortion only slightly. When the number of bits
per sample is below the equilibrium point, any increase in sampling rate without increasing the bitrate accordingly
has a negligible effect in reducing distortion. Finally, we conclude that the optimal distortion-rate performance of
the Wiener process can only be attained as the sampling rate goes to infinity, implying that encoding with a finite
number of bits per sample always leads to sub-optimal performance compared to the DRF $D_W(R)$.
Next, we explore the performance when the order of interpolation and compression in the achievable scheme of $D(f_s, R)$ is reversed. That is, the samples are first compressed in a way which is optimal for recovering the discrete-time process defined by these samples. Only then the original continuous-time waveform is estimated from the encoded version of these samples. We denote this scheme by compress-and-estimate (CE). Such a scheme may arise as a sub-optimal design of an encoder for the continuous-time process, or when the origin of a discrete-time sequence as a sampled continuous-time phenomena is known in retrospect: initially the samples are encoded in an optimal manner as a discrete-time process, and only later is their continuous-time origin discovered.

The CE scheme is sub-optimal for the combined sampling and source coding setting, and therefore leads to an upper bound on $D(f_s, R)$. The difference between this upper bound and $D(f_s, R)$ gives the penalty on sub-optimal system design or lack of source knowledge that leads to the CE scheme. We show that this penalty is small in low bitrates and never goes above $1/6$ times the intensity coefficient of the Wiener process.

A. Related Works

The DRF of the Wiener process was computed by Berger in [8]. Berger has also shown that this DRF is achievable even though the Wiener process is not stable, as its variance grows linearly with time. Gray established similar results for the more general family of unstable auto-regressive processes [9], but he did not consider the continuous-time case. The ideas behind Berger’s and Gray’s works were generalized to other non-stable signal models in [10].

The use of sampling in communication originates to Shannon [11], who showed how Whittaker’s cardinal series can be used to study the transmission of bandlimited continuous-time signals. Shannon’s result was extended to stationary processes in [12] and [13], and to multitude other settings including nonuniform sampling [14], processes with sparse spectral content [15], and cyclostationary processes [16], to name a few.

While both sampling and source coding play an important role in communication theory, the interplay between the two has received much less attention. Berger’s classical book [2, Ch 4.5.3] outlines an approach which is based on sampling to derive Shannon’s DRF of a continuous-time Gaussian source. A full proof of a source coding theorem which is based on sampled data is given in [3], but only for Gaussian stationary processes as the sampling rate goes to infinity.

In this work we are interested in the behavior of the optimal source coding performance of the source at a fixed sampling frequency, rather than its asymptotic behavior as the samples become infinitely dense. That is, we ask for
the minimal distortion that can be attained in estimating a realization of the process from its sampled and rate-limited version at a finite sampling rate and a finite bitrate. When the original process can be perfectly recovered from its samples, this question reduces back to the standard source coding problem \[2\] and the answer is given by the DRF of the process. Hence, it is only when this sampling introduces error that a non-trivial setting arise, and leads to an interplay between two different sources for the distortion: sampling error and lossy compression (quantization) error. In \[17\], this interplay was studied for the case of second-order Gaussian stationary processes using a combined sampling and source coding model similar to Fig. 1. The main result of \[17\] is a full characterization of the minimal distortion as a function of the sampling rate \(f_s\) and the bitrate \(R\) in terms of the power spectral density of the source.

When pre-processing before sampling is allowed, it was shown that even when the source is not bandlimited, its DRF can be attained by sampling above a critical finite rate which depends on the bitrate \[18\]. The current work can be seen as an extension of \[17\] to a particular non-stationary source, namely the Wiener process. However, in contrast to \[17\], here finite sampling rate is never sufficient to attain the optimal source coding performance without a sampling constraint. Due to the inability to completely eliminate the distortion due to either sampling or lossy compression, it is important to understand how to couple the sampling rate and the bitrate to find an operating point where the two sources of error are balanced: increasing only the sampling rate or only the bitrate beyond this operating point would make one of the distortions more dominant than the other, thus has only a negligible effect on the overall distortion.

**B. Contributions**

The first result of this paper is a source coding theorem for a combined sampling and source coding problem with a continuous-time Wiener process as the source. This theorem asserts that the minimal distortion in recovering a realization of the continuous-time Wiener process from its rate-limited encoded uniform samples over any finite time horizon is given by the solution to an optimization problem involving only probability density functions of limited mutual information. We denote the resulting minimal distortion by \(D(f_s, R)\), where \(R\) is the code-rate (or the encoding bitrate) and \(f_s\) is the sampling rate. We show that the minimal distortion converges to the DRF of the Wiener process as the sampling rate goes to infinity. This last fact settles an interesting observation made in \[8\] about the equivalence of the DRF of a discrete-time Wiener process at low bit-resolution (high distortion) to the DRF of the continuous-time Wiener process.

Next, we derive a closed-form expression for \(D(f_s, R)\). This expression allows the determination of the sampling rate and bitrate required in order to attain a prescribed target distortion. Our characterization of \(D(f_s, R)\) proves that the DRF of the Wiener process is strictly smaller than the performance achieved by uniform sampling and encoding the samples for any finite sampling rate \(f_s\). This last fact is in contrast to the case of other Gaussian signal models that, although non-bandlimited, admit a finite critical sampling rate above which their DRF is attained \[13\].

In addition to a coding theorem and a closed-form expression for \(D(f_s, R)\), we also provide an upper bound to \(D(f_s, R)\). This upper bound is the result of the following sub-optimal encoding scheme for the combined sampling and source coding problem: the encoder first applies an optimal distortion-rate code to the samples and
then interpolates the compressed samples. This is in contrast to the optimal coding scheme that attains $D(f_s, R)$, in which the encoder first interpolates between the samples before applying an optimal distortion-rate code to represent the result of this interpolation. Therefore, the comparison of this upper bound to the optimal performance described by $D(f_s, R)$ demonstrates the loss in performance as a result of an encoding mismatch: the rate-distortion code employed on the samples of the continuous-time Wiener process is tuned to attain the DRF of the resulting discrete-time Wiener process, rather than an optimal coding scheme designed to attain the DRF of the process resulting from linearly interpolating between the samples. Nevertheless, we show that the difference between this sub-optimal scheme and the optimal scheme that leads to $D(f_s, R)$ is bounded by $1/6$ times the variance of the Wiener process at time $t = 1$.

The rest of this paper is organized as follows: in Section II we define a combined sampling and source coding problem for the Wiener process. In Section III we discuss previous results with respect to sampling and encoding the Wiener process. Our main results are given in Sections IV, V and VI. Concluding remarks can be found in Section VII.

II. Problem Formulation

Our signal model is a continuous-time Gaussian process $W(\cdot) = \{W(t), t > 0\}$ with zero mean and covariance function

$$K_W(t, s) \triangleq \mathbb{E}[W(t)W(s)] = \sigma^2 \min\{t, s\}, \quad t, s \geq 0.$$  

The common definition of the Wiener process also requires that each realization of $W(\cdot)$ has an almost surely continuous path [19]. In our setting, however, only the weaker assumption of Riemann integrability of the sample path is needed, and other properties of the Wiener process that follow from the continuity of its path will not be used.

We consider the system depicted in Fig. 1 to describe the random waveform $W(0 : T) \triangleq \{W(t), t \in [0, T]\}$ using a code of rate $R$ bits per time unit. Unlike in the regular source coding problem for the Wiener process considered in [8], our setting assumes that $W(0 : T)$ is first uniformly sampled at frequency $f_s$. This sampling leads to the finite random vector

$$\tilde{W}[0 : Tf_s] \triangleq \{W(n/f_s), n \in \mathbb{N} \cap [0, f_sT]\}. \quad (1)$$

Next, the vector $\tilde{W}[0 : Tf_s]$ is mapped by the encoder to an index $M_T \in \{1, \ldots, 2^{RT}\}$. The decoder, upon receiving $M_T$, provides an estimate $\hat{W}(0 : T) = \{\hat{W}(t), t \in [0, T]\}$ which is only a function of $M_T$.

The optimal performance theoretically achievable (OPTA) in terms of the distortion in estimating $W(\cdot)$ from its samples $\tilde{W}[\cdot]$ is defined as

$$D^{OPTA}(R) = \inf_T D_T^{OPTA}(R)$$

where

$$D_T^{OPTA}(R) = \inf_T \frac{1}{T} \int_0^T \mathbb{E} \left( W(t) - \hat{W}(t) \right)^2 dt,$$
and the infimum is taken over all encoder-decoder mappings
\[ \hat{W}(0:Tf_s) \rightarrow M_T \rightarrow \hat{W}(0:T). \]

The characterization of \( D_{OPTA}^R \) in terms of an optimization problem involving mutual information between probability distributions is one of our preliminary results, given in the next section.

### III. Background

In this section we review previous results on sampling and encoding the Wiener process, and discuss their relation to the combined sampling and source coding problem presented above.

#### A. The Distortion-Rate Function of Wiener Processes

The DRF of the Wiener process \( W(\cdot) \) is given by \([8]\) as

\[ D_W(R) = \frac{2\sigma^2}{\pi^2 \ln 2} R^{-1} \approx 0.292 \sigma^2 R^{-1}. \tag{2} \]

That is, the minimal averaged distortion attainable in recovering a realization of the Wiener process from its encoded version is proportional to its variance parameter \( \sigma^2 \) and inversely proportional to the number of bits per time unit of this encoding. As explained in \([8]\), \( D_W(R) \) can be attained using the following procedure: first consider a finite time interval \( T \) and divide it into \( N \) identical sub-intervals of length \( T' = T/N \). For each \( n = 0, \ldots, N-1 \), represent \( W(t - nT') - W(nT') \) according to its KL expansion over the interval \([0, T']\). Properties of the Wiener process and the KL expansion \([20]\) imply that the coefficients of this expansion constitute a set of \( N \) independent and identically distributed infinite sequences of independent Gaussian random variables. These sequences are encoded using a single code of \( RT \) bits which is optimal with respect to their Gaussian distribution under the \( \ell_2(\mathbb{N}) \) norm. The reconstruction waveform is obtained by using the decoded KL coefficients in the expansion for \( W(t - nT') - W(nT') \), \( n = 0, \ldots, N-1 \). The procedure above is guaranteed to attain distortion close to \( D_W(R) \) as desired by an appropriate choice of \( N \) as \( T \) goes to infinity. Finally, it can be shown that with this choice of \( N \), the average code-rate required to encode the sequence of endpoints \( \{W(nT')\}_{n=1}^N \) vanishes as \( T \) goes to infinity, so the latter can be provided to the decoder without increasing the bitrate used by the scheme above.

We note for each \( n \), the \( k \)th KL coefficient \( A_k^{(n)} \) in the procedure above is obtained by

\[ A_k^{(n)} = \frac{1}{T'} \int_0^{T'} f_k(t) \left[ W(t - nT') - W(nT') \right] dt, \tag{3} \]

where \( f_k(t) \) is the \( k \)th KL eigenvalue of the Fredholm integral equation of the second kind \([21]\) over the interval \([0, T']\) with Kernel \( K_W(t, s) \). While an integral of the form \((3)\) is well-defined, its realization using analog components may be challenging, where the reason is twofold. First, the Wiener process has equal energy in all its frequency components, whereas electronic devices tend to smooth or attenuate signal components beyond a certain frequency. It is due to this reason that most KL algorithms operate in discrete-time or resort to some sort of time-discretization \([22]\). Second, the analog components of the transform must be adjusted to the statistics of the Wiener process as well as the time horizon \( T \). The KL transform may not be robust to these changes in the statistics compared to
other transform techniques \cite{23} or to digital realizations that can adjust their parameters to these variations. It is for these reasons that this work explores an alternative encoding of $W(\cdot)$ which assumes that only the samples $\tilde{W}[:]$ are available at the encoder rather than the entire continuous-time waveform.

Note that since

$$E(\tilde{W}[n]\tilde{W}[k]) = E(W(n/f_s)W(k/f_s)) = \sigma^2 \min\{n,k\} / f_s,$$

the process $\tilde{W}[:]=\{W(n/f_s), n = 0, 1, \ldots\}$ defines a discrete-time Wiener process with intensity $\sigma^2/f_s$. Using Berger’s expression from [8] for the DRF of a discrete-time Wiener process, we conclude that $D_{\tilde{W}}(\tilde{R})$, the DRF of $\tilde{W}[:],$ is given by the following parametric form

$$D(\theta) = \frac{\sigma^2}{f_s} \int_0^1 \min\{S_{\tilde{W}}(\phi), \theta\} d\phi,$$

$$\tilde{R}(\theta) = \frac{1}{2} \int_0^1 \log^+ [S_{\tilde{W}}(\phi)/\theta] d\phi,$$

where $\tilde{R}$ is measured in bits per sample and

$$S_{\tilde{W}}(\phi) \triangleq \frac{1}{4\sin^2(\pi\phi/2)}$$

is the asymptotic density of the eigenvalues in the KL expansion of $\tilde{W}[:]$ as the time horizon goes to infinity. Expression (5) gives the distortion as a function of the rate, or the rate as a function of the distortion, through a joint dependency in the parameter $\theta$. Such parametric representation is said to be of a waterfilling form, since only the part of $S_{\tilde{W}}(\phi)$ below the waterlevel parameter $\theta$ contributes to the distortion.

Keeping the bitrate $R = f_s\tilde{R}$ fixed and increasing $f_s$, we see that the asymptotic behavior of the DRF of $\tilde{W}[:]$ as $f_s$ goes to infinity is given by (4) when $\tilde{R}$ goes to zero or, equivalently, when $\theta$ goes to infinity. The latter can be obtained by expanding both expressions in (4) according to $\theta^{-1}$, which, after eliminating $\theta$ leads to

$$D_{\tilde{W}}(\tilde{R}) \sim \frac{\sigma^2}{f_s} \left( \frac{2}{\pi^2 \ln 2} \tilde{R} + \frac{\tilde{R} \ln 2}{12} + O(\tilde{R}^{-2}) \right)$$

$$= \frac{2\sigma^2}{\pi^2 \ln 2} R^{-1} + \frac{\sigma^2 \ln 2}{12} \frac{R}{f_s^2} + O(f_s^{-3}).$$

The first term in (6) is the DRF of the continuous-time Wiener process \cite{2}. Thus, we have proved the following proposition:

**Proposition 1:** Let $\tilde{W}[:]$ be the process obtained by uniformly sampling the Wiener process $W(\cdot)$ at frequency $f_s$. We have

$$\lim_{f_s \to \infty} D_{\tilde{W}}(R/f_s) = D_W(R).$$

In fact, it can easily be shown that $D_{\tilde{W}}(R/f_s)$ is monotonically increasing in $f_s$ so that

$$\sup_{f_s>0} D_{\tilde{W}}(R/f_s) = D_W(R).$$

\footnote{This is consistent with our previous notations since $\tilde{W}[:]$ is a discrete-time process and the code rate is measured in bits per source symbol.}
Proposition 1 provides an intuitive explanation to an interesting fact observed in [8], which says that the DRF of a discrete time Wiener process in high distortion behaves as the DRF of a continuous-time Wiener process. Proposition 1 shows that this fact is simply the result of evaluating the DRF of the discrete-time Wiener process $\bar{W}[: \cdot]$ at high sampling rates, while holding the bitrate $R$ constant. As a result of the high sampling rate, the number of bits per sample $\bar{R} = R / f_s$ goes to zero and the DRF of the discrete-time Wiener process is evaluated at the large distortion (low bit) limit. The fact that $D_{\bar{W}}(R/f_s)$ is monotonically increasing in $f_s$ implies that the sample path of the sampled Wiener process gets harder to describe as the frequency at which those samples are obtained increases. An illustration of the convergence of $D_{\bar{W}}(R/f_s)$ to $D_W(R)$ can be seen in Fig. 5-(b) below.

The rest of the paper is devoted to the non-asymptotic regime of the sampling frequency $f_s$. That is, we are interested to characterize the distortion-rate tradeoff when $W(\cdot)$ is described from its sampled version $\bar{W}[: \cdot]$ at a finite sampling frequency $f_s$. Before proceeding, we explain what may be a source of confusion with respect to characterizing $D(f_s, R)$ when $f_s$ is finite. Berger’s source coding theorem from [8] implies that the distortion in reconstructing the continuous-time waveform by encoding the discrete-time samples is always larger than $D_W(R)$. Note, however, that the quadratic distortion with respect to the discrete-time samples can be made arbitrarily close to $D_{\bar{W}}(R/f_s)$, which, by (7), is always smaller than $D_W(R)$. Therefore, the right way to analyze the convergence in the above coding scheme as $f_s$ goes to infinity is by a close examination of an estimation of $W(0 : T)$ from the encoded version of the discrete-time samples $W[0 : T f_s]$.

B. MMSE Estimation from Sampled Data

On the other extreme of the combined sampling and source coding problem of Fig. 1 is the relaxation of the bitrate constraint by letting $R \to \infty$. Under this relaxation, the function $D_w(f_s, R)$ reduces to the MMSE in estimating $W(\cdot)$ from its uniform samples $\bar{W}[: \cdot]$ at sampling frequency $f_s = T_s^{-1}$, denoted as $\text{mmse}(W|\bar{W})$. For $t > 0$, denote by $t^+$ and $t^-$ the two points on the grid $\mathbb{Z}T_s$ closest to $t$, namely, $t^- = \lfloor t/T_s \rfloor T_s$ and $t^+ = \lceil t/T_s \rceil T_s$. Due to the Markov property of $W(\cdot)$, the MMSE estimation of $W(t)$ from $W[: \cdot]$ is given by a linear interpolation between these two points:

$$\tilde{W}(t) \triangleq \mathbb{E}[W(t)|\bar{W}[: \cdot]] = \mathbb{E}[W(t)|W(t^+), W(t^-)] = \frac{t^+ - t}{T_s} W(t^-) + \frac{t - t^-}{T_s} W(t^+).$$

A typical realization of the processes $W(\cdot)$, $\bar{W}[: \cdot]$ and $\tilde{W}(\cdot)$ is illustrated in Fig. 2.

The estimation error $B(t) \triangleq W(t) - \tilde{W}(t)$ defines a Brownian bridge on any interval whose endpoints are on the
grid $Z T_s$. The correlation function of $B(\cdot)$ is given by

$$K_B(t,s) = \mathbb{E}[B(t)B(s)]$$

$= \sigma^2 \frac{T_s}{T_s} \begin{cases} (t^+ - t^\wedge s)(t^\vee s - t^-) & nT_s \leq t, s \leq (n+1)T_s, \\ 0 & \text{otherwise}, \end{cases}$

where $t^\vee s$ and $t^\wedge s$ denote the maximum and minimum of $\{t, s\}$, respectively. In particular, we conclude that

$$\text{mmse}(W(t)|\bar{W}) = K_B(t, t) = \frac{\sigma^2}{T_s} (t^+ - t)(t - t^-),$$

and

$$\text{mmse}(W|\bar{W}) = \frac{1}{T_s} \int_{nT_s}^{(n+1)T_s} K_B(t, t) dt = \frac{\sigma^2 T_s}{6} = \frac{\sigma^2}{6 f_s}.$$  \hfill(10)

From the definition of $D_{OPTA}(R)$, it immediately follows that

$$D_{OPTA}(R) \geq \max \{ \text{mmse}(W|\bar{W}), D_W(R) \}.$$  \hfill(11)

In the next two sections we provide a closed form expression for the function $D_{OPTA}(R)$.

IV. COMBINED SAMPLING AND SOURCE CODING THEOREM

In this section we provide an information representation for the OPTA in encoding a realization of the Wiener process from its samples as defined in Section II

**Theorem 2:** Let

$$D_{W|\bar{W}}(R) = \liminf_{T \to \infty} D_T(R),$$

where

$$D_T(R) = \inf \frac{1}{T} \int_0^T \mathbb{E} \left( W(t) - \hat{W}(t) \right)^2 dt,$$

and the infimum is taken over all joint distributions $P \left( \bar{W}[0 : T f_s], \hat{W}(0 : T) \right)$ such that the mutual information rate $f_s I \left( \bar{W}[0 : T f_s]; \hat{W}(0 : T) \right)$

$$f_s I \left( \bar{W}[0 : T f_s]; \hat{W}(0 : T) \right) \hfill(12)$$
is limited to \( R \) bits per time unit. Then

\[
D^{OPTA}(R) = D_{W|\bar{W}}(R).
\]

**Proof:** For \( T > 0 \), define the following distortion measure on \( \mathbb{R}^{[Tf_s]} \times L_2[0,T] \):

\[
\tilde{d}(\bar{w}[0:Tf_s],\tilde{w}(0:T)) = \frac{1}{T} \int_0^T (W(t) - \tilde{w}(t))^2 dt|W[0:Tf_s] = \bar{w}[0:Tf_s]|.
\]

Here \( L_2[0,T] \) is the space of square integrable functions over \( [0,T] \), \( w(\cdot) \) is an element of this space, and the relation between \( W(0:T) \) and \( \tilde{W}[0:Tf_s] \) is the same as in (1). In words, \( \tilde{d} \) is the averaged quadratic distortion between the reconstruction waveform \( \tilde{w}(0:T) \) and all possible realizations of the random waveform \( W(0:T) \), given its values on the set \( 0, T_s, \ldots, |Tf_s|T_s \). Note that by properties of conditional expectation we have

\[
\mathbb{E}\tilde{d}(\tilde{W}[0:Tf_s],\tilde{W}(0:T)) = \frac{1}{T} \int_0^T \mathbb{E}(W(t) - \tilde{W}(t))^2 dt.
\]

A source coding result for the finite random i.i.d vector \( \tilde{W}[0:Tf_s] \) using an arbitrary distortion measure over any alphabet can be found in [24]. This theorem implies that the OPTA distortion for such a vector can be obtained by minimizing over all joint probability distributions of \( \tilde{W}[0:Tf_s] \) and \( \tilde{W}(0:T) \), such that the mutual information rate (12) is limited to \( R \) bits per time unit. In the context of our problem, this proves a source coding theorem under sampling at rate \( f_s \) of an information source consisting of multiple independent realizations of the waveform \( W(0:T) \). Since we are interested in describing a single realization as \( T \) goes to infinity, what is required is an argument that allows us to separate finite sections of the entire realization \( W(\cdot) \) and consider the joint encoding thereof as multiple realizations over a finite interval.

When the continuous source is stationary or asymptotic mean stationary, such an argument is achieved by mixing properties of the joint distribution [25]. In our case, however, \( W(\cdot) \) is not stationary. The way to separate the waveform into multiple i.i.d finite interval sections was proposed by Berger [8]: encode the endpoint of each interval of length \( T \) using a separate bitstream. It can be shown that this task is equivalent to an encoding of a discrete-time Wiener process of variance \( T\sigma^2 \). The distortion \( \delta \) in an encoding of the latter using a delta modulator with \( \bar{R} = RT \) bits per sample is smaller than \( 0.586\sigma^2/\bar{R} \). For any finite \( R \), this number can be made arbitrarily small by enlarging \( T \). This means that the endpoints \( \tilde{W}[1f_sT], \tilde{W}[2f_sT], \ldots \), can be described using an arbitrarily small number of bits per time unit. Since the increments of \( W(\cdot) \) are independent, its statistics conditioned on the sequence of endpoints is the same as of multiple i.i.d realizations of \( W(0:T) \), and Theorem [2] follows from the first part of the proof.

Note that \( D_{W|\bar{W}}(R) \) implicitly depends on the sampling rate \( f_s \) through the definition of the process \( \tilde{W}[\cdot] \) in (1). In order to make this dependency explicit, we henceforth use the notation

\[
D(f_s,R) = D_{W|\bar{W}}(R).
\]
V. THE DISTORTION-RATE FUNCTION OF SAMPLED WIENER PROCESSES

In this section we provide an exact expression for $D(f_s, R)$. This is done by evaluating the DRF of the process $\tilde{W}(\cdot)$ and using (14).

A. Relation to Indirect Source Coding

Note that in the setting of Fig. 1 it is required to provide an estimate of $W(\cdot)$, but without directly observing its realizations. Therefore, our sampling and source coding problem can be seen as an instance of the indirect or remote source coding problem [26], [2], and the function $D(f_s, R)$ is denoted as the indirect distortion-rate function (iDRF) of $W(\cdot)$ given $\bar{W}[:].$ It follows from [27] that an achievable scheme for $D(f_s, R)$ is obtained by first obtaining $\tilde{W}(\cdot)$ of (8), which is the MMSE estimation of $W(\cdot)$ from $\bar{W}[:].$ With $\tilde{W}(\cdot)$ at hand, the encoder applies an optimal rate-distortion code to $\tilde{W}(\cdot)$, i.e., as in a regular source coding problem with $\tilde{W}(\cdot)$ as the source. Such an achievable scheme also implies the following decomposition [27], [28]:

$$D(f_s, R) = \text{mmse}(W | \tilde{W}) + D_{\tilde{W}}(R), \quad (14)$$

where $D_{\tilde{W}}(R)$ is the (direct) DRF of the process $\tilde{W}(\cdot).$ The encoding scheme that leads to (14) and the relationships among the various other distortion functions defined thus far are illustrated in the diagram in Fig. 3.

The representation (14) implies that $D(f_s, R)$ can be found by evaluating the function $D_{\tilde{W}}(R).$ This evaluation is addressed in the next subsection.

B. Exact Expression for $D(f_s, R)$

Note that since we have

$$W(t) = \tilde{W}(t) + B(t), \quad t \geq 0,$$

Fig. 3: Relations among the processes $W(\cdot)$, $\bar{W}[:], \tilde{W}(\cdot)$ and their associated distortion functions. Each reconstruction operation is associated with a distortion quantity. In this paper we show that $D_{\tilde{W}}(R) \leq D_{\bar{W}}(R/f_s) \leq D_W(R) \leq D(f_s, R) \leq D(f_s, R),$ where $D(f_s, R) = \text{mmse}(W | \bar{W}) + D_{\tilde{W}}(R)$.
and since \( B(\cdot) \) and \( W(\cdot) \) are independent processes, the covariance function of \( \bar{W}(\cdot) \) equals

\[
K_{\bar{W}}(t,s) = K_{W}(t,s) - K_{B}(t,s).
\]

The function \( K_{\bar{W}}(t,s) \) is illustrated for a fixed \( t \in (0,T) \) in Fig. 4. The KL eigenvalues \( \{ \lambda_k \} \) and their corresponding eigenfunctions \( \{ f_k \} \) satisfy the Fredholm integral equation of the second kind \( [21] \)

\[
\lambda_k f_k(t) = \int_0^T K_{\bar{W}}(t,s)f_k(s)ds, \quad k = 1,2,\ldots.
\]

We show in Appendix A that there are at most \( N \triangleq [T f_s] \) eigenvalues \( \{ \lambda_k \} \) that satisfy \( (16) \), and these are given by

\[
\lambda_k = \frac{\sigma^2 T^2}{6} \left( 2 \cos(k\pi) - \sin \left( \frac{(2k-1)(N-1)\pi}{2N} \right) \right), \quad k = 1,\ldots,N.
\]

We also show in this appendix that as \( T \) goes to infinity with the ratio \( f \triangleq k/T = f_s k/N \) constant, \( 0 < f < f_s \), the density of the eigenvalues converges to the function

\[
T_s^2 \left( S_{\bar{W}} \left( \frac{\pi f}{2 f_s} \right) - \frac{1}{6} \right), \quad 0 < f < f_s,
\]

which leads to the following theorem:

**Theorem 3:** The DRF of the process \( \bar{W}(\cdot) \), obtained by linearly interpolating the samples of a Wiener process at sampling rate \( f_s \), is given by the following parametric expression:

\[
D_{\bar{W}}(\theta) = \frac{\sigma^2}{f_s} \int_0^1 \min \left\{ \theta, S_{\bar{W}}(\phi) - \frac{1}{6} \right\} d\phi,
\]

\[
R(\theta) = \frac{f_s}{2} \int_0^1 \log^+ \left( S_{\bar{W}}(\phi) - \frac{1}{6} \right) / \theta d\phi,
\]

where \( S_{\bar{W}}(\phi) \) is the density of the eigenvalues in the KL expansion of \( \bar{W}[\cdot] \) given by \( (15) \).

**Proof:** The density function \( (18) \) satisfies the conditions of \([2] \) Thm. 4.5.4 \( \) (note that the stationarity property of the source is only needed in \([2] \) Thm. 4.5.4] to establish the existence of a density function, which is given explicitly in our case by \( (18) \)). This theorem implies that the waterfilling expression over the eigenvalues \( \{ \lambda_k \} \) converges, as \( T \) goes to infinity, to the waterfilling expression over the density \( S_{\bar{W}}(f) \).

It is important to emphasize that although \( \bar{W}(\cdot) \) is a continuous-time process, its KL coefficients can be obtained directly from the discrete-time samples \( \bar{W}[\cdot] \) and without performing any analog integration as opposed to the KL coefficients of \( W(\cdot) \) in \( (3) \). Indeed, for any function \( g(t) \in L_2[0,T] \) it follows from the definition \((8) \) that

\[
\int_0^T g(u)\bar{W}(u)du = \sum_{n=0}^{f_s T - 1} \{ \bar{W}[n]A_n + \bar{W}[n+1]B_n \},
\]

where

\[
A_n = \frac{1}{T_s} \int_{nT_s}^{(n+1)T_s} g(u) ((n+1)T_s - u) du,
\]

\[
B_n = \frac{1}{T_s} \int_{nT_s}^{(n+1)T_s} g(u) (u - nT_s) du,
\]
and where we assumed for simplicity that $T f_s$ is an integer. By taking $g(t)$ to be the $k$th eigenfunction in the KL decomposition of $\tilde{W}(\cdot)$ as given in Appendix A, we see that the $k$th KL coefficient of $\tilde{W}(\cdot)$ over $[0,T]$ can be expressed as a linear function of the process $\bar{W}[\cdot]$. This last fact implies that a source code which is based on the KL transform of $\tilde{W}(\cdot)$ can be applied directly to a linear transformation of $\bar{W}[\cdot]$ and does not require analog integrations as in (3).

We are now ready to derive a closed-form expression for $D(f_s,R)$ as per the following theorem.

**Theorem 4:** The iDRF of the Wiener process $W(\cdot)$ from its uniform samples at rate $f_s$ and bitrate $R$ is given by the following parametric form:

\begin{align}
D(\theta) &= \frac{\sigma^2}{6 f_s} + \frac{\sigma^2}{f_s} \int_0^1 \min \left\{ \theta, S_{\bar{W}}(\phi) - \frac{1}{6} \right\} d\phi, \\
R(\theta) &= \frac{f_s}{2} \int_0^1 \log \left( S_{\bar{W}}(\phi) - \frac{1}{6} / \theta \right) d\phi.
\end{align}

**Proof:** The parametric expression (21) follows from (10), (14), and Theorem 3.

We note that the kernel $K_{\bar{W}}(t,s)$ converges to the kernel $K_W(t,s)$ in the $L_2[0,T]$ sense as $f_s$ goes to infinity. This convergence implies that the corresponding sequence of Hertgloz operators defined by (16) converges, in the strong operator norm, to the operator defined by the kernel $K_W(t,s)$. Since a Hertgloz operator is compact, it follows that the eigenvalues $\lambda_k$ converge to the eigenvalues of the KL expansion for the Wiener process. Moreover, since this convergence is uniform in $T$, convergence of the eigenvalues implies convergence of the corresponding DRFs. In fact, as $f_s$ goes to infinity it can be shown that the eigenvalues of the discrete-time Wiener process converges to the eigenvalues of the continuous-time process. This provides a slightly stronger result than Proposition 1 since it shows that the convergence of the two DRFs occurs for any finite $T$. Similar results were derived for cyclostationary Gaussian stationary processes in [29].
C. Discussion

It follows from Theorem 4 that the function $D(f_s, R)$ is monotonically decreasing in $f_s$ and converges to $D_w(R)$ as $f_s$ goes to infinity. We remark that although intuitive, monotonicity of $D(f_s, R)$ in $f_s$ should not be assumed in view of [17, Exm. VI.2], which shows that the iDRF of a Gaussian stationary process given its samples may not be monotone in the sampling rate. The function $D(f_s, R)$ is illustrated, along with other distortion functions defined in the figure.
in this paper, in Figure 5. Note that we can rewrite (21a) as

$$D(f_s, R) = \frac{\sigma^2}{6f_s} + \frac{\sigma^2}{f_s} \tilde{D}(\bar{R}),$$  \hspace{1cm} (22)$$

where \( \tilde{D}(\bar{R}) \) is given by the parametric equation

$$\tilde{D}(\theta) = \int_0^1 \left\{ \theta, S_\theta(\phi) - \frac{1}{6} \right\} d\phi,$$  \hspace{1cm} (23a)

$$\bar{R} = \frac{1}{2} \int_0^1 \log^+ \left( \left[ S_\theta(\phi) - \frac{1}{6} \right] / \theta \right) d\phi,$$  \hspace{1cm} (23b)

which only depends on the number of bits per sample \( \bar{R} = R/f_s \), and has the waterfilling interpretation illustrated in Fig. 7.

It follows from (22) that for high values of \( \bar{R} \) the distortion is dominated by the term \( \text{mmse}(W|\hat{W}) \), whereas for low values of \( \bar{R} \) the distortion is dominated by the lossy compression error \( D_{\hat{W}}(R) \). The transition between the two regimes roughly occurs at some \( \bar{R}_0 \approx \frac{R}{f_s} \) that satisfies \( \tilde{D}(\bar{R}_0) = 1/6 \). In order to measure the relative distortion due to sampling, we consider the ratio between \( D(f_s, R) \) and \( D_{\hat{W}}(R) \) for a fixed bitrate \( R \) as a function of \( \bar{R} \). This ratio is given by

$$\frac{D(R/\bar{R}, R)}{D_{\hat{W}}(R)} = \frac{\pi^2 \ln 2}{2} \tilde{D}(\bar{R}) \left( \frac{1}{6} + \tilde{D}(\bar{R}) \right),$$  \hspace{1cm} (24)$$

and is illustrated in Fig. 6. For example, at \( \bar{R}_0 \approx 0.98 \) we have

$$D(R/\bar{R}_0, R) = \frac{\sigma^2}{3f_s} = \frac{\bar{R}_0}{3} R^{-1} \approx 0.327 R^{-1},$$

which, from (4), yields that \( D(R/\bar{R}_0, R) \approx 1.1185 D_{\hat{W}}(R) \). The ratio (24) between \( D(f_s, R) \) and \( D_{\hat{W}}(R) \) as a function of \( \bar{R} \) is illustrated in Fig. 6.

One conclusion from (24) is that for any non-zero number of bits per sample \( \bar{R} \), the performance with a sampling constraint \( D(R/\bar{R}, R) \) is always sub-optimal compared to the performance without any sampling constraint \( D_{\hat{W}}(R) \). Alternatively, the distortion at any finite sampling rate is strictly larger than the distortion at an infinite sampling rate. This fact is in contrast to distortion-rate performance of other signal models, for which there exists a finite sampling frequency \( f_R \) above which the iDRF from the samples equals the DRF with an unconstrained sampling rate [18]. In other words, for other signal models, above the sampling rate \( f_R \) distortion increases since, due to the finite bitrate, fewer bits are available to describe these additional samples and the resulting source code distortion increases total distortion. For example, it was shown in [18 Exm. 4] that it is possible to attain the DRF of the normalized Gauss-Markov process with cutoff frequency \( f_0 \) at bitrate \( R \) by sampling at rate \( f_R \) that satisfies

$$R = \frac{f_R}{\ln 2} \left( 1 - \frac{2f_0}{\pi f_R} \arctan \left( \frac{\pi f_R}{2f_0} \right) \right),$$

that is, using \( R/f_R \) bits per sample.

In view of (22) and (24), in the remainder of this section we will study the expression (21) for \( D(f_s, R) \) in the two regimes of high and low bits per sample \( \bar{R} \), corresponding to low and high sampling rate \( f_s \), respectively.
Fig. 6: The ratio between $D(f_s, R)$ and $\overline{D}(f_s, R)$ (dashed) to $D_W(R)$, as a function of the number of bits per sample $\bar{R} = R / f_s$. This ratio describes the excess distortion due to sampling compared to the unconstrained optimal distortion-rate performance at bitrate $R$. The asymptotic behavior is described by the line $y = \frac{\pi^2 \ln 2}{12} x$.

Fig. 7: Left: the function $\tilde{D}(\bar{R})$ of (23a). Right: waterfilling interpretation of the parametric equation (23a).

1) Low Sampling Rates: As shown in Fig. 7, the minimal value of $S_W(\phi) - 1/6$, the integrand in (21), is $1/12$. As a result, it is possible to eliminate $\theta$ from (21) when the sampling rate $f_s$ is small compared to the bitrate. This implies that

\[
D(f_s, R) = \frac{\sigma^2}{f_s} \left( \frac{1}{6} + \frac{2 + \sqrt{3}}{6} 2^{-2R/f_s} \right)
\]

whenever

\[
\frac{R}{f_s} \geq \frac{1 + \log(\sqrt{3} + 2)}{2} \approx 1.45.
\]
For example, with the number of bits per sample that attains equality in (26), leads to
\[
\frac{D(R/\bar{R}_0,R)}{D_W(R)} = 1 + (2 + \sqrt{3})e^{-\frac{2\theta}{\overline{\theta}^2}} \approx 1.328,
\]
as shown in Fig. 6

2) High Sampling Rates: When \( R/f_s \ll 1, \theta \) is large compared to \( S_\theta(\phi) - 1/6, \phi \in (0,1) \), and the integral in (19b) is non-zero only for small values of \( \phi \). In this range, the function \( \sin^{-2}(x) \) can be written as \( 1/3 + x^{-2} \) plus higher order terms in \( x^{-1} \). We therefore have
\[
D(f_s,R) = \frac{2\sigma^2}{\pi^2 \ln 2} R^{-1} + \frac{\sigma^2 \ln 2}{18} \frac{R}{f_s^2} + O\left( f_s^{-4} \right).
\]
We conclude that as \( f_s \to \infty \), \( D(f_s,R) \) converges to \( D_W(R) \) with rate inversely quadratic in \( f_s \). In the next section we will study an encoding scheme for the combined sampling and source coding problem which converges to \( D_W(R) \), but with rate only inversely linear in \( f_s \).

VI. COMPRESS-AND-ESTIMATE UPPER BOUND

In this section we provide an upper bound for the function \( D(f_s,R) \) obtained by the following achievable scheme for encoding the process \( \bar{W}(\cdot) \) of (8): encode the samples using a rate-\( R \) code that approaches the DRF of the process \( \bar{W}[^{\cdot}] \), and consequently recover the continuous-time process \( \bar{W}(\cdot) \) by a linear interpolation similar to (8).

We denote this achievable scheme as compress-and-estimate (CE), since the samples \( \bar{W}[^{\cdot}] \) are first compressed, and later the source \( W(\cdot) \) is estimated from the compressed version of the samples. In comparison, the optimal source coding scheme that attains the iDRF \( D(f_s,R) \) can be seen as an estimate-and-compress scheme: as explained in Subsection V-A, the source \( W(\cdot) \) is first estimated from \( \bar{W}[^{\cdot}] \) and the estimated process \( \tilde{W}(\cdot) \) is later lossy compressed in an optimal manner that attains the DRF of \( \bar{W}(\cdot) \).

The CE scheme is motivated by an encoder design which is unaware of the continuous-time origin of the samples, or otherwise cannot adjust itself to varying sampling rates. Instead, the encoder assumes that the task is to recover the discrete-time samples under the same bitrate constraint. That is, the optimal code is obtained by solving a standard (and not indirect) source coding problem with the process \( \bar{W}[^{\cdot}] \) as the input and the \( \ell_2(\mathbb{N}) \) norm as the distortion criterion.

Using this CE scheme we obtain the following result:

**Proposition 5:** The distortion-rate function of the estimator \( \bar{W}(\cdot) \) of the Wiener process \( W(\cdot) \) from its uniform samples at rate \( f_s \) satisfies
\[
D_{\bar{W}}(R) \leq D_W(R/f_s),
\]
where \( D_W(R) \) is the DRF of the discrete-time process obtained by uniformly sampling \( W(\cdot) \) at rate \( f_s \).

**Proof sketch:** The bound (28) is obtained taking the reconstruction of \( \bar{W}(\cdot) \) in the interval \([nT_s,(n+1)T_s]\) to an interpolation of the form (8) between \( \bar{W}[n] \) and \( \bar{W}[n+1] \), where \( \bar{W}[\cdot] \) is a reconstruction of \( \bar{W}[^{\cdot}] \) that approaches its distortion-rate function at rate \( R = R/f_s \). The details can be found in Appendix B.

\[
\text{Proof sketch:}
\]
We now define the following function

\[ D(f_s, R) \triangleq \text{mmse}(W|\bar{W}) + D_{\bar{W}}(R/f_s) \]

\[ = \frac{\sigma^2}{6f_s} + \frac{\sigma^2}{f_s} \int_0^1 \min \{ S_{\bar{W}}(\phi), \theta \} d\phi, \]  
(29)

where \( \theta \) is determined by

\[ R(\theta) = \frac{f_s}{2} \int_0^1 \log^+ [S_{\bar{W}}(\phi)/\theta] d\phi. \]

For this function we obtain the following result:

**Theorem 6:** For any \( R > 0 \) and \( f_s > 0 \), the indirect distortion-rate function of the Wiener process \( W(\cdot) \) from its samples \( \bar{W}[\cdot] \) satisfies

\[ D(f_s, R) \leq \bar{D}(f_s, R), \]

where \( \bar{D}(f_s, R) \) is given by (29).

**Proof:** The theorem follows from Proposition 5 and (4) and (10) applied to (14).

The bound \( \bar{D}(f_s, R) \) is illustrated in Fig. 5. It is monotonically decreasing in \( f_s \) and converges to \( D_W(R) \) as \( f_s \) or \( R \) go either to zero or to infinity. Moreover, from (21) and from (22) we conclude that

\[ \bar{D}(f_s, R) - D(f_s, R) = D_{\bar{W}}(\bar{R}) - \frac{\sigma^2}{6} \tilde{D}(R). \]  
(30)

That is, the degree to which the bound is sub-optimal is only a function of the number of bits per sample \( \bar{R} = R/f_s \).

Moreover, the maximal value of (30) is \( \sigma^2/6 \), which implies that the bound is tight up to this factor.

As in the case of \( D(f_s, R) \) in Section V, it is convenient to analyze this bound in the two regimes of low and high sampling rate separately.

**Low sampling rates:** When \( R \geq f_s \) the bound (29) reduces to

\[ \bar{D}(f_s, R) = \frac{\sigma^2}{f_s} \left( \frac{1}{6} + 2^{-2R/f_s} \right), \]  
(31)

and for \( R \geq \frac{1}{2} \left( \log(\sqrt{3} + 2) + 1 \right) \) the difference (30) becomes

\[ \bar{D}(f_s, R) - D(f_s, R) = \frac{4 \sqrt{3}}{6} 2^{-2R/f_s}. \]

For example, with \( f_s = R \) samples per seconds we obtain

\[ \bar{D}(f_s, R) \leq \frac{5}{12} \sigma^2 R^{-1} \approx 1.43D_W(R). \]  
(32)

Other choices of \( f_s < R \) lead to higher values in (31).

**High sampling rates:** It follows from (6) that in the limit of large sampling rate,

\[ \bar{D}(f_s, R) = \frac{\sigma^2}{6f_s} + \frac{2\sigma^2}{\pi^2 \ln 2} R^{-1} + O(f_s^{-2}). \]  
(33)

Comparing (33) to (27), we see that the high sampling rate behavior of the CE upper bound is sub-optimal compared to \( D(f_s, R) \) by a \( f_s^{-1} \) term.
The reason why the CE scheme that leads to $\mathcal{D}(f_s, R)$ is not tight can be understood as a form of mismatched encoding: $\mathcal{D}(f_s, R)$ is obtained by encoding $\bar{W}[]$ in a way which is optimal to minimize the quadratic distortion between $\bar{W}[]$ and its reconstruction. However, remote source coding theory tells us that an optimal coding scheme for $\bar{W}[]$ is obtained by first estimating (non-causally) $\tilde{W}(\cdot)$ from $\bar{W}[]$, and then applying an optimal code with respect to this estimator [2]. In the CE scheme the order of encoding and estimation is reversed, what leads to sub-optimal performance in general [30]. Since the sample paths of the two processes converge to each other as $f_s$ increases, this mismatch in encoding does not play a significant role at high sampling rates where the bound is indeed tight. At low sampling rates the MMSE term dominates both $D(f_s, R)$ and $\mathcal{D}(f_s, R)$, hence the bound is also tight in the low sampling rate regime. These two phenomena can be observed in Fig. 5-(b).

VII. Conclusions

We considered the minimal distortion in estimating the path of a continuous-time Wiener process from a bitrate-limited version of its uniform samples, taken at a finite sampling rate. We derived a closed form expression for this minimal distortion, and determined the excess distortion due to uniform sampling compared to minimal distortion-rate performance at an infinite sampling rate. In addition, we derived an upper bound to the minimal distortion from the samples by bounding the distortion in a compress-and-estimate achievability scheme in remote source coding [30].

This characterization allows us to determine the sampling rate and the number of bits per time unit required to achieve a prescribed target distortion. For example, we concluded that for a fixed bitrate $R$, it is possible to attain the optimal distortion performance to a factor of less than 1.12 by sampling at rate higher than $f_s = 1.03R$, which is equivalent to allocating less than 0.98 bits per sample on average. In addition, we showed that the number of bits per sample must go to zero in order to approach the distortion-rate function of the Wiener process, and that for any finite sampling rate the distortion-rate performance with sampling is strictly worse than without a sampling constraint. These last two facts are in contrast to other continuous-time signal models for which the optimal distortion-rate performance can be attained by sampling at a finite sampling rate and using a fixed number of bits per sample [18].

APPENDIX A

In this appendix we prove that the eigenvalues of the KL integral (16) are given by (17).
Equation (16) can be written as
\[
\frac{\lambda}{\sigma^2} f(t) = \int_0^t s f(s) ds + \int_{t-T_s}^{t-T_s} f(s) ds + \int_{t-T_s}^t f(s) ds. \tag{34}
\]
Differentiating the last expression leads to
\[
\frac{\lambda}{\sigma^2} f'(t) = \int_{t-T_s}^{t-T_s} f(s) ds + \int_{t-T_s}^t f(s) ds, \tag{35}
\]
which implies
\[
\frac{\lambda}{\sigma^2} f''(t) = 0. \tag{36}
\]
We conclude that the solution to (34) is a piece-wise linear functions on intervals of the form \([nT_s, (n+1)T_s]\) for \(n = 0, \ldots, N\), where \(N = \lfloor T/T_s \rfloor\). Since the DRF is obtained by evaluating the solution as \(T\) goes to infinity, and since this limit exists, there is no loss in generality by assuming \(T/T_s\) is an integer.

By imposing the initial conditions (34), (35) it follows that the eigenfunctions in the KL transform are of the form
\[
f_k(t) = \sqrt{A_k} \left( \frac{t^+ - t}{T_s} \sin \left( \frac{2k-1}{2T} \pi t \right) + \frac{t^+ - t}{T_s} \sin \left( \frac{2k-1}{2T} \pi t^+ \right) \right),
\]
where \(A_k\) is a normalization constant. The corresponding eigenvalues can be found by evaluating (35), which leads to
\[
\lambda_k = \frac{\sigma^2 T_s^2}{6} \left( \frac{2 \cos(k \pi) - \sin \left( \frac{(2k-1)(N-1)\pi}{2N} \right)}{\cos(k \pi) + \sin \left( \frac{(2k-1)(N-1)\pi}{2N} \right)} \right), \quad k = 1, \ldots, N. \tag{37}
\]

APPENDIX B

In this section we prove Proposition [5]. Consider the DRF of the discrete-time process \(\bar{W}[\cdot]\) denoted as \(D_{\bar{W}}(\bar{R})\). Fix \(R > 0\) and \(\epsilon > 0\). Take \(N\) large enough such that the distortion achieved by a reconstruction process \(\hat{\bar{W}}[\cdot]\) using a code of rate \(\bar{R} = R/f_s\) is smaller than \(D_{\bar{W}}(R) + \epsilon\). Without loss of generality, we can assume that \(\hat{\bar{W}}[n]\) is the minimal MSE estimate of the random variable \(\bar{W}[n]\) given the reconstruction sequence \(\hat{\bar{W}}[\cdot]\) (otherwise, we improve the performance of the decoding by considering the reconstruction sequence \(\hat{W}/[n] = \mathbb{E} \left[ \bar{W}[n]|\hat{\bar{W}}[\cdot] \right] \)). As a result we get
\[
\mathbb{E} \left[ \hat{\bar{W}}(t) | \hat{\bar{W}}[\cdot] \right] = \frac{t^+ - t}{T_s} \mathbb{E} \left[ \bar{W}(t^-) | \hat{\bar{W}}[\cdot] \right] + \frac{t^+ - t}{T_s} \mathbb{E} \left[ \bar{W}(t^+) | \hat{\bar{W}}[\cdot] \right]
\]
\[
= \frac{t^+ - t}{T_s} \hat{\bar{W}}[f_s t^-] + \frac{t^+ - t}{T_s} \hat{\bar{W}}[f_s t^+] \tag{37}
\]
Consider

\[\text{mmse}(\tilde{W} | \hat{W}) \leq \frac{1}{NT_s} \int_0^{NT_s} \mathbb{E} \left( \tilde{W}(t) - \mathbb{E} \left[ \tilde{W}(t) \mid \hat{W} \right] \right)^2 dt \]

\[= \frac{1}{NT_s} \sum_{n=0}^{N-1} \int_{nT_s}^{(n+1)T_s} \mathbb{E} \left( \tilde{W}(t) - \mathbb{E} \left[ \tilde{W}(t) \mid \hat{W} \right] \right)^2 dt \]

\[\leq \frac{1}{NT_s} \int_{nT_s}^{(n+1)T_s} \mathbb{E} \left( \tilde{W}(t) - \frac{t-nT_s}{T_s} \hat{W}[n+1] - \frac{T_s(n+1)-t}{T_s} \hat{W}[n] \right)^2 dt \]

(38)

where \((a)\) follows from the finite time horizon, \((b)\) follows from (37), and \((c)\) follows since

\[\tilde{W}(t) = \frac{t-T_s}{T_s} W(t^+) + \frac{T_s-t}{T_s} W(t^-),\]

and by introducing the notation

\[\Delta_n \equiv W(nT_s) - \hat{W}[n] = \hat{W}[n] - W[n].\]

Note that by using the same notation, we get

\[\frac{1}{N} \sum_{n=0}^{N-1} \Delta_n \leq D_{\tilde{W}}(R) + \varepsilon.\]

By evaluating the integral in (39) we obtain

\[\text{mmse}(\tilde{W} | \hat{W}[]) \leq \frac{1}{N} \sum_{n=0}^{N-1} \left( \frac{1}{3} \mathbb{E} \Delta_n^2 + \frac{1}{3} \mathbb{E} \Delta_{n+1}^2 + \frac{1}{3} \mathbb{E} \Delta_n \Delta_{n+1} \right).\]

Since \(2|\Delta_{n+1} \Delta_n| \leq \Delta_n^2 + \Delta_{n+1}^2\), we have

\[\text{mmse}(\tilde{W} | \hat{W}[]) \leq \frac{1}{N} \sum_{n=0}^{N-1} \left( \frac{1}{2} \mathbb{E} \Delta_{n+1}^2 + \frac{1}{2} \mathbb{E} \Delta_n^2 \right)\]

\[= \frac{1}{N} \sum_{n=1}^{N-1} \mathbb{E} \Delta_n^2 + \frac{\mathbb{E} \Delta_0^2}{2N} + \frac{\mathbb{E} \Delta_N^2}{2N} \]

\[\leq \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \Delta_n^2 \]

(40)

\[\leq \frac{N+1}{N} (D_{\tilde{W}}(\hat{R}) + \varepsilon).\]

(41)

Since \(\varepsilon > 0\) is arbitrary and \(N\) can be taken as large as needed, (41) implies that \(D_{\tilde{W}}(R/f_s)\) is an achievable distortion.

**APPENDIX C**

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