Linearization stability of reflection-asymmetric thin-shell wormholes with double photon spheres

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Wormholes are hypothetical objects which can be black hole mimickers with strong gravitational fields. Recently, Wielgus et al. have discussed the shadows of reflection-asymmetric thin-shell wormholes which are composed of the parts of a Schwarzschild spacetime and a Reissner-Nordstrom spacetime with double photon spheres in [M. Wielgus, J. Horak, F. Vincent, and M. Abramowicz, Phys. Rev. D 102, 084044 (2020)]. In this paper, we study the linearization stability of the reflection-asymmetric thin-shell wormholes with the double photon spheres.

I. INTRODUCTION

Recently, LIGO and VIRGO Collaborations have detected gravitational waves from black hole binaries [1] and Event Horizon Telescope Collaboration has reported the shadow image of a black hole candidate at the center of a giant elliptical galaxy M87 [2]. The theoretical and observational aspects of compact objects in general relativity will be more important than before.

Wormholes are hypothetical objects with a non-trivial topology in general relativity [3, 4]. Morris and Thorne have discussed passability of wormholes and they have also shown that energy conditions violate at least at a throat of static and spherically symmetric wormholes if we assume general relativity without a cosmological constant [5]. Wormholes can be black hole mimickers because they can have strong gravitational fields. For example, spherically symmetric wormholes with strong gravitational fields can have unstable (stable) circular light orbits named photon spheres (antiphoton spheres) [6, 7].

Thin-shell wormholes whose energy condition is broken only at the throat which is supported by a thin shell [8, 9] were considered by Visser [10] with Darmois-Israel matching [11, 12, 13]. Linearization stability of a thin shell of a Schwarzschild wormhole was investigated by Poisson and Visser [14] and then stability of thin-shell wormholes such as Reissner-Nordstrom wormholes [15, 16, 17], the other static and spherically symmetric wormholes [18, 19, 20], plane symmetric wormholes [21, 22], cylindrical symmetric wormhole [23, 24], higher-dimensional wormholes [25, 26], and lower-dimensional wormholes [27, 28] have been discussed. Passability [29] and nonlinear stability [30] of thin-shell wormholes have been studied.

Usually two copied manifolds are used to construct the thin-shell wormholes while reflection-asymmetric wormholes also have been investigated in Refs. [31, 32, 33]. Recently, a shadow of a reflection-asymmetric thin-shell wormhole which is composed of the parts of two Schwarzschild spacetime with different masses were discussed by Wang et al. [32]. Wielgus et al. have investigated a shadow by double photon spheres of reflection-asymmetric thin-shell wormholes which are composed of the parts of the Schwarzschild manifold and a Reissner-Nordstrom manifold under a constraint on the metric tensors not to jump at the throat [33]. It would be difficult to distinguish a reflection-symmetric wormhole constructed by the parts of black hole spacetimes from the black hole by astronomical observations if the wormhole has photon spheres and if there are few light sources in the other side of the throat. On the other hand, the asymmetric thin-shell wormholes with the asymmetric double photon spheres can be distinguished from the black holes by the observations since light rays can be reflected by a potential wall near the throat [34].

In this paper, we investigate the linearization stability of the reflection-asymmetric thin-shell wormholes which are composed of the parts of the Schwarzschild spacetime and the Reissner-Nordstrom spacetime with the double photon spheres constructed in [33].

This paper is organized as follows. We review the Reissner-Nordstrom spacetime very briefly in Sec. II and we construct the reflection-asymmetric thin-shell wormholes with the double photon spheres in Sec. III. We investigate the linearization stability of the reflection-asymmetric thin-shell wormholes in Sec. IV and we conclude our results in Sec. V. In this paper we use the units in which a light speed and Newton’s constant are unity.

II. REISSNER-NORDSTROM SPACETIME

The Reissner-Nordstrom spacetime has a line element

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( f(r) \) is given by

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]
and where $M > 0$ and $Q$ are a mass and a charge, respectively. It is a black hole spacetime with an event horizon at $r = r_{\text{EH}} \equiv M + \sqrt{M^2 - Q^2}$ for $0 \leq Q^2 \leq M^2$ while it has naked singularity for $M^2 < Q^2$. There is a photon sphere at
\[ r = r_{\text{PS}} \equiv \frac{3M + \sqrt{9M^2 - 8Q^2}}{2} \]  
for $Q^2 \leq 9M^2/8$ and there is an antiphoton sphere at
\[ r = r_{\text{APS}} \equiv \frac{3M - \sqrt{9M^2 - 8Q^2}}{2} \]  
for $M^2 \leq Q^2 \leq 9M^2/8$.

### III. REFLECTION-ASYMMETRIC THIN-SHELL WORMHOLE

By the Darmois-Israel matching, we construct a wormhole spacetime without reflection symmetry or $\mathbb{Z}_2$ symmetry. We consider two Reissner-Nordstrom spacetimes with line elements
\[ ds^2_\pm = -f_\pm(r)dt^2_\pm + \frac{dr^2_\pm}{f_\pm(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  
where $f_\pm(r)$ are given by
\[ f_\pm(r) = 1 - \frac{2M_\pm}{r} + \frac{Q^2_\pm}{r^2}. \]  
Here, we have assumed that we can choose the same coordinates $r$, $\theta$, and $\phi$ in the spacetimes. We make two manifolds $\mathcal{M}_\pm \equiv \{ r > a \}$, where $a$ is a constant satisfying $a > r_{\text{EH} \pm}$ by removing $\Omega_\pm \equiv \{ r \leq a \}$ from the Reissner-Nordstrom spacetimes. The boundaries of the manifolds $\mathcal{M}_\pm$ are timelike hypersurfaces $\Sigma_\pm \equiv \{ r = a \}$ and we identify the hypersurfaces $\Sigma \equiv \Sigma_+ = \Sigma_-$. As a result, we obtain a manifold $\mathcal{M}$ by gluing the manifolds $\mathcal{M}_\pm$ at a throat located at $\Sigma$. The hypersurface $\Sigma$ is filled with a Dirac distribution matter and it is called thin shell. We permit $a = a(\tau)$, where $\tau$ is the proper time of the thin shell since we are interested in the stability of the thin shell $\Sigma$.

We assume that we can set the coordinates $y^i$ on the both sides of the hypersurface $\Sigma$. The induced metric $h_{ij} \equiv g_{\mu \nu}e^\mu_i e^\nu_j$ on the hypersurface $\Sigma$ is given by
\[ ds^2_\Sigma = h_{ij}dy^i dy^j = -dr^2 + a^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  
Here $e^i_\mu$ are basis vectors given by $e^i_\mu \equiv \partial x^\mu / \partial y^i$. We introduce a bracket $[T]$ denoting the jump of a function $T$ across $\Sigma$,
\[ [T] \equiv T_+|_{\Sigma} - T_-|_{\Sigma}, \]  
where $T_+$ and $T_-$ are $T$ in $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively. The thin shell satisfies Einstein equations
\[ S^i_j = -\frac{1}{8\pi} ([K^i_j] - [K] \delta^i_j), \]  
where $S^i_j$ is the surface stress-energy tensor of the thin shell given by
\[ S^i_j = (\sigma + p)U^i U_j + p\delta^i_j, \]  
where $U_i$ is given by $U_i dy^i \equiv u_\mu e^\mu_i dy^i = dr$, where $u_\mu$ is the four velocity of the thin shell, and where $\sigma = -S^T_t$ and $p = S^\theta_\theta = S^\phi_\phi$ are the surface energy density and the surface pressure of the thin shell, respectively. Here, $K_{ij}$ is the extrinsic curvature given by
\[ K_{ij} = (n_\mu \nu - \Gamma^\rho_{\mu \nu} n_\rho)e^\mu_i e^\nu_j, \]  
where $n_\nu$ is the unit normal to the hypersurface and where $\Gamma^\rho_{\mu \nu}$ is Christoffel symbols.

The four velocity of the thin shell at $t_\pm = T_\pm(\tau)$ and $r = a(\tau)$ is given by $u^\mu \partial_\mu = \dot{T}_\pm \partial_{t_\pm} + \dot{a} \partial_a$, where the overdot is differentiation with respect to $\tau$ and where $T_\pm = \sqrt{f_\pm + \dot{a}^2 / f_\pm}$. The unit normals to the hypersurface in $\mathcal{M}_\pm$ are obtained as $n_{\mu \pm} dx^\mu_\pm = \pm (\dot{a} dt_\pm + \dot{T}_\pm dr_\pm)$ and the basis vectors are given by $e^\mu_\pm \partial_\mu = \ddot{T}_\pm \partial_{t_\pm} + \dot{a} \partial_a$, $e^\mu_\pm \partial_\mu = \partial_\theta$, and $e^\mu_\pm \partial_\mu = \partial_\phi$. The extrinsic curvatures of the hypersurfaces in $\mathcal{M}_\pm$ are given by
\[ K^T_\pm = \frac{\pm 1}{\sqrt{a^2 + f_\pm}} \left( \ddot{a} + \frac{f'_\pm}{2} \right), \]  
and the traces are
\[ K_\pm = \frac{\pm 1}{\sqrt{a^2 + f_\pm}} \left( \ddot{a} + \frac{f'_\pm}{2} \right) \pm \frac{2}{a} \sqrt{a^2 + f_\pm}. \]  
From $(\tau, \tau)$ and $(\theta, \theta)$ components of the Einstein equations \[ (3.5). \]  
we obtain
\[ \sigma = -\frac{\sqrt{a^2 + f_+}}{4\pi a} - \frac{\sqrt{a^2 + f_-}}{4\pi a}, \]  
and
\[ p = \frac{1}{8\pi \sqrt{a^2 + f_+}} \left( \ddot{a} + \frac{a^2 + f_+}{a} + \frac{f'_+}{2} \right) \]  
\[ + \frac{1}{8\pi \sqrt{a^2 + f_-}} \left( \ddot{a} + \frac{a^2 + f_-}{a} + \frac{f'_-}{2} \right), \]  
and then we obtain, from Eqs. \[ (3.11) \]  
and \[ (3.12), \]  
\[ \frac{d(\sigma A)}{d\tau} + p \frac{dA}{d\tau} = 0, \]  
where $A = 4\pi a^2$ is the area of the throat. Equation \[ (3.13) \]  
can be expressed by
\[ a\dot{\sigma}' + 2(\sigma + p) = 0, \]  
where $\dot{\sigma}' = \dot{\sigma} / \dot{a}$ and the prime denotes the differentiation with respect to $a$. We assume that the thin shell is filled.
with a barotropic fluid with \( p = p(\sigma) \). From Eq. (3.14), we notice that the surface density of the barotropic fluid is expressed as a function of \( a \) or \( \sigma = \sigma(a) \). The equation of motion of the thin shell is given by, from Eq. (3.11),

\[
\ddot{a}^2 + V(a) = 0, \tag{3.15}
\]

where \( V(a) \) is an effective potential defined by

\[
V(a) \equiv \bar{f} - \left( \frac{\Delta}{4\pi a\sigma} \right)^2 - (2\pi a\sigma)^2, \tag{3.16}
\]

where \( \bar{f} \) and \( \Delta \) are given by

\[
\bar{f} \equiv \frac{f_- + f_+}{2}, \tag{3.17}
\]

and

\[
\Delta \equiv \frac{f_+ - f_-}{2}, \tag{3.18}
\]

respectively. The derivative of \( V \) with respect to \( a \) is obtained as

\[
V' = \bar{f}' - \frac{\Delta}{8\pi^2 a^2\sigma^3} \left[ 4\Delta' \sigma + 4\Delta (\sigma + 2p) \right] - 8\pi^2 a \sigma (\sigma + 2p). \tag{3.19}
\]

By using Eq. (3.14) again, the second derivative of \( V \) is obtained as

\[
V'' = \bar{f}'' - \frac{\Delta^2}{8\pi^2 a^2\sigma^2} - \frac{\Delta}{8\pi^2 a^2\sigma^3} \left[ 4\Delta' \sigma + 4\Delta (\sigma + 2p) \right] - 8\pi^2 \left[ (\sigma + 2p)^2 + 2\sigma (\sigma + 2p) (1 + 2\beta^2) \right], \tag{3.20}
\]

where \( \beta^2 \equiv dp/d\sigma = p'/\sigma' \).

Here and hereafter, we impose a constraint

\[
f_+(a) = f_-(a) \tag{3.21}
\]

and we concentrate on the case that the manifold \( \mathcal{M}_- \) is the part of the Schwarzschild black hole spacetime, i.e., \( Q_- = 0 \), as well as Ref. [63]. The constraint is expressed as

\[
Q^2_+ = 2aM_-(\xi - 1), \tag{3.22}
\]

where \( \xi \) is an asymmetry parameter defined by \( \xi = M_+/M_- \). The reflection-asymmetric thin-shell wormholes with double photon spheres must have the throat in domains \( r_{EH-} < a < r_{PS-} \) and \( r_{EH+} < a < r_{PS+} \). Permitted parameters \( (\xi, a/M_-) \) for the reflectional-asymmetry thin-shell wormholes with double photon spheres are shown in Fig. 1.

**IV. STABILITY OF THIN-SHELL WORMHOLE**

We consider the linearization stability of a static wormhole with a thin shell at \( a = a_0 \) under the constraint \( Q_- = 0 \) and \( a = a_0 \). The surface energy density \( \sigma_0 \) and pressure \( p_0 \) of the thin shell are given by

\[
\sigma_0 = -\frac{\sqrt{f_0}}{2\pi a_0}, \tag{4.1}
\]

and

\[
p_0 = \frac{1}{8\pi \sqrt{f_0}} \left( \frac{2f_0}{a_0} + \bar{f}_0 \right), \tag{4.2}
\]

respectively. Here and hereafter a function with subscript 0 means the function at \( a = a_0 \). Since \( V_0' = 0 \) is satisfied, the effective potential can be expanded around \( a = a_0 \) as

\[
V(a) = \frac{V_0''}{2}(a - a_0)^2 + O \left( (a - a_0)^3 \right), \tag{4.3}
\]

where \( V_0'' \) is given by

\[
V_0'' = A_0 - B_0 (1 + 2\beta_0^2), \tag{4.4}
\]

where \( A_0 \) and \( B_0 \) are defined by

\[
A_0 \equiv \bar{f}_0'' - \frac{\Delta_0^2}{2f_0} - \frac{\bar{f}_0'^2}{2f_0^2}, \tag{4.5}
\]

and

\[
B_0 = \frac{2f_0}{a_0^2} - \frac{\bar{f}_0'}{a_0}, \tag{4.6}
\]

FIG. 1. Permitted parameters \((\xi, a/M_-)\) of the asymmetry wormholes with the double photon spheres are in shaded zones. \( \mathcal{M}_+ \) is the part of the Reissner-Nordstrom black hole (BH) spacetime in a deep blue shaded zone while it is the part of the Reissner-Nordstrom naked singularity (NS) spacetime in light blue shaded zones. \( \mathcal{M}_- \) is the subset of the Schwarzschild black hole spacetime. A blue dotted line denotes \( a = r_{PS+} \). A red dashed line is explained in Sec. IV.
respectively. The thin shell is stable (unstable) when $V''_0 > 0$ ($V''_0 < 0$). Thus, the thin shell is stable when
\[ \beta^2_0 < \frac{1}{2} \left( \frac{A_0}{B_0} - 1 \right) \] (4.7)
and $B_0 > 0$ hold or
\[ \beta^2_0 > \frac{1}{2} \left( \frac{A_0}{B_0} - 1 \right) \] (4.8)
and $B_0 < 0$ hold. Figure 2 shows the parameter zone of $(\xi, a_0/M_-)$ for $B_0 > 0$ and the one for $B_0 < 0$ and their boundary $B_0 = 0$. The boundary of the two zones $B_0 = 0$ is given by
\[ a_0/M_- = \frac{(7 - \xi)}{2} \text{ for } \frac{32 + 6\sqrt{2}}{17} \leq \xi \leq 2 + \sqrt{\frac{2}{3}} \] (4.9)
On the boundary, the thin shell is unstable for any $\beta^2_0$.

The parameters $(a_0/M_-, \beta^2_0)$ for the stable thin shell are shown in Fig. 3.

**V. CONCLUSION**

We have investigated the linearization stability of the reflection-asymmetric thin-shell wormhole which is composed of parts of the Schwarzschild and Reissner-Nordstrom manifolds with double photon spheres. We have imposed the constraint (3.22) as well as Ref. [63]. The linearization stability given by Fig. 3 are characterized by the boundary $B_0 = 0$ or $a_0/M_- = \frac{(7 - \xi)}{2}$ for $(32 + 6\sqrt{2})/17 \leq \xi \leq 2 + \sqrt{\frac{2}{3}}$ shown as the red line in Fig. 2. In the boundary case $B_0 = 0$, the throat with $a_0/M_- = 2.25$ is unstable for any $\beta^2_0$ as shown in the left bottom panel with $\xi = 2.5$ in Fig. 3. We also note the throat at $a_0/M_- = r_{EH-}/M_- = 2$ for $1 \leq \xi \leq 2$ is unstable for any $\beta^2_0$ as shown in the panels in Fig. 3.

We comment on the Schwarzschild thin-shell wormhole with reflection symmetry or $\xi = 1.0$. In Fig. 2, $B_0$ vanishes at a point $(\xi, a_0/M_-) = (1.0, 3.0)$. Thus, the throat of the Schwarzschild reflection-symmetric wormhole with $\xi = 1.0$ at $a_0/M_- = 3$ is unstable for any $\beta^2_0$ as shown Fig. 3.

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FIG. 3. The static thin shell is stable if it is in blue shaded zones of \((a_0/M_-, \beta^2_0)\). Left top, right top, left middle, right middle, left bottom, and right bottom panels show the cases of \(\xi = 1.0\) for \(2 < a_0/M_- < 3\), \(\xi = 1.2\) for \(2 < a_0/M_- < 2.8\), \(\xi = 1.6\) for \(2 < a_0/M_- < 2.4\), \(\xi = 1.8\) for \(2 < a_0/M_- < 729/320\), \(\xi = 2.5\) for \(25/12 < a_0/M_- < 75/32\), and \(\xi = 3.5\) for \(49/20 < a_0/M_- < 441/160\), respectively.

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