Phase Structure of Lattice $\mathcal{N} = 4$ Super Yang-Mills

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Abstract: We make a first study of the phase diagram of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory regulated on a space-time lattice. The lattice formulation we employ is both gauge invariant and retains at all lattice spacings one exactly preserved supersymmetry charge. Our numerical results are consistent with the existence of a single deconfined phase at all observed values of the bare coupling.

Keywords: Lattice Field Theory, Supersymmetric Gauge Theory, Topological Field Theories, Extended Supersymmetry.
1. Introduction

Supersymmetric Yang-Mills theories are interesting from a variety of perspectives; as four-dimensional toy models for understanding theories such as QCD, as potential theories of Beyond Standard Model physics, and because of the AdS/CFT correspondence, they may reveal connections to quantum gravity and string theory. Many features of these theories, such as, for example, dynamical supersymmetry breaking, are inherently non-perturbative in nature. This serves as strong motivation to attempt to study such theories numerically (and perhaps also analytically) when regularized by a space-time lattice.

Unfortunately, it has historically proven almost prohibitively difficult to discretize supersymmetric gauge theories using traditional methods. This stems from the fact that the supersymmetry algebra is an extension of the usual Poincaré algebra and hence is broken completely by naïve discretization on a space-time lattice. Even a discretized subalgebra of the full supersymmetry algebra where translations are restricted to discrete steps cannot be retained in general on discretized space-time. This is directly tied to the failure of the Leibniz rule to hold for finite difference operators. However, recently the development of a series of new theoretical tools have enabled us to construct certain supersymmetric theories on the lattice while retaining absolute preservation of a very small subset of the continuum supersymmetries - see the reviews [1, 2, 3, 4] and references therein. Other recent complementary approaches to the problem of exact lattice supersymmetry can be found in [5, 6, 7, 8, 9, 10, 11, 12, 13] while complementary approaches to $\mathcal{N} = 4$ Yang-Mills
using large N reduction techniques have been studied in [14, 15, 16, 17, 18]. Retaining only a small subset of all continuum supersymmetries may not have as many limitations as one might naively have guessed, since the combination of exact gauge invariance and some exactly preserved supersymmetries turn out to put severe constraints on the lattice theories. Perhaps a useful analog is the breaking of full Euclidean rotational invariance on a hypercubic lattice: in the continuum limit full Lorentz symmetry is recovered, even though only a very small discrete subgroup is preserved at any finite lattice spacing.

One way to understand the new constructions is to realize that they correspond to discretizations of topologically twisted forms of the target continuum theories. The associated lattice theories are possible only if the continuum supersymmetric Yang-Mills theories possess sufficient extended supersymmetry; the precise requirement is that the number of supercharges must be an integer multiple of $2^D$ where $D$ is the space-time dimension. In four dimensions this singles out a unique theory which can be studied using these lattice constructions: $\mathcal{N} = 4$ super Yang-Mills theory, the focus of the current work.

We do not wish to hide the fact that the lattice theories with exactly preserved supersymmetries have features that, superficially, seem unusual to lattice practitioners. The fermionic and bosonic degrees of freedom are distributed in a somewhat unorthodox manner on the lattice. But once this thin layer of disguise has been peeled off, we are here dealing with a quite standard lattice gauge theory, amenable to the usual tools of the field. In many ways, this lattice theory is quite simple. In fact, as compared to essentially all other lattice formulations with fermions, absence of fermionic doublers is trivial. More precisely: the lattice beautifully arranges itself so that the doublers are needed and essential for the exactly preserved supersymmetry. The number of degrees of freedom, including the doublers, match perfectly. The fermion action that arises in these constructions can be interpreted as a Dirac-Kähler action which has long been known to be equivalent, at the level of free field theory, to a (reduced) staggered fermion action. It is the twisting process that leads to the emergence of Dirac-Kähler fermions. This lattice construction is also interesting from a mathematical point of view since it corresponds to a finite system with an exact Q-cohomology which in turn may allow for rigorous results to be derived for supersymmetric gauge theories.

At the intuitive level we can understand these lattice constructions from the fact that they are invariant under only one of the supersymmetry charges. Because of its Grassmann nature, it squares to zero. Indeed, this charge can be viewed as a nilpotent BRST-charge associated with arbitrary field deformations, just as in the continuum topological theory\(^1\). The anticommutator of two supersymmetry charges is by virtue of the supersymmetry algebra a generator of space-time translations, and this is the basic obstacle of lattice-regularized supersymmetry, where only discrete translations are retained. By preserving only one supersymmetry charge on the lattice this problem is removed. However what still needs to be demonstrated is that the conservation of just one exact lattice supersymmetry is sufficient to recover full supersymmetry in the continuum limit. Although only global symmetries are at stake here, this is not as trivial as it may sound since counte-

\(^1\)In special cases, the supersymmetric lattice can, correspondingly, also be invariant under another nilpotent charge, the anti-BRST generator. This is not the case for the $\mathcal{N} = 4$ theory.
erms permitted by the single preserved charge may violate invariance under the remaining supersymmetries one would hope to recover in the continuum. There are also numerical issues: not all lattice-discretized theories are amenable, at present, to numerical simulations due to sign problems. Indeed, the theory we wish to study here potentially has a sign problem, and it is crucial to investigate its severity.

There are many other interesting issues. For example, the theory we study here has flat directions associated with the scalar fields at the classical level. Do these flat directions cause numerical instabilities in the actual simulations? And if so, do they indicate that the functional integral even of the continuum theory may be ill-defined?

For the first time we are now in a position where we can take seriously the path integral of supersymmetric field theories in the sense that we can consider its discretized version in a setting that is entirely well-defined. It is a theory defined in a finite four-volume $V$ and with an ultraviolet cut-off given by the lattice spacing $a$. In a certain sense, we evaluate observables in this supersymmetric theory by explicit integration in the functional integral rather than relying on formal manipulations that ignore issues such as convergence.

Motivation for studying this particular supersymmetric theory comes, of course, also from the fact that it is a truly remarkable four-dimensional gauge theory; it has highly non-trivial interactions, but it is scale-free and it retains (super)conformal invariance even at the quantum level. With this new lattice formulation we can explicitly study this theory for all values of the bare lattice coupling. In the formal continuum theory, there are well-known analytical predictions for, e.g., Wilson loops at strong coupling based on the AdS/CFT correspondence \[19\]. It is clearly of interest to compare these predictions with a lattice-regularized analog of this theory which can be studied numerically. A striking fact that we will have to face in the simulations is that the lattice theory may also be conformal at all couplings. This is a unique situation, which has not been encountered before in the history of this field. Indeed, intuition might suggest that the lattice theory could develop a strongly coupled phase at some finite bare coupling, a phase that would have no continuum limit, would have a mass gap, would perhaps be confining and could undergo spontaneous chiral symmetry breaking.

In this first exploratory study of the lattice-regularized theory we wish to probe these fundamental questions: What is the lattice phase diagram of this theory? Is it really scale-free and conformal at all bare couplings? Does the phase of the Pfaffian resulting from integration over the fermionic degrees of freedom lead to a sign problem? Do the flat directions for the scalars cause numerical instabilities? On the lattice we will also need to regularize both the scalar flat directions (a soft breaking of supersymmetry by means of a mass term) and the fermions (they have zero modes that can be removed by supersymmetry breaking anti-periodic boundary conditions or, alternatively, by fermionic mass terms). To what extent is supersymmetry nevertheless conserved in our actual numerical simulations? As we shall see, we find encouraging results in all directions. Our results indicate that the $U(2)$ theory has no genuine sign problem and that, therefore, phase quenched simulations are justified. We see no evidence of instabilities associated with the flat directions. The bosonic regulator mass can be tuned to zero, and simple supersymmetric Ward Identities are obeyed to high numerical accuracy. Moreover, and most surprisingly,
we find no evidence of a strongly coupled phase in this lattice theory. It appears that the perfect pairing between bosonic and fermionic degrees of freedom imposed by just one supersymmetry generator is sufficient to keep the theory in a scale-free deconfined phase throughout, as in the continuum.

A brief outline of our paper is as follows. In the next section, we briefly review the topological twisting of $\mathcal{N} = 4$ super Yang-Mills theory in the continuum. We discuss the analogous lattice procedure in Section 3, with emphasis on how easily this twisted theory fits into the well-known and existing framework of lattice gauge theory. We present our numerical results in Section 4. Here, because our lattices are still relatively small, we focus on local observables, all of which are well known in the lattice context for studying the phase diagram of a gauge theory. We end with some concluding remarks in Section 5.

2. Twisted Supersymmetric $\mathcal{N} = 4$ Yang-Mills Theory

As discussed in the Introduction, it is possible to discretize a class of continuum supersymmetric Yang-Mills theories using ideas based on topological twisting\(^2\). Though the basic idea of twisting goes back to Witten in his seminal paper on topological field theory [24], it had actually been anticipated in earlier work on staggered fermions on the lattice [25]. In our context, the idea of twisting is to decompose the fields of a Euclidean supersymmetric Yang-Mills theory in $D$ space-time dimensions in representations not of the original (Euclidean) rotational symmetry $SO_{\text{rot}}(D)$, but a twisted rotational symmetry, which is the diagonal subgroup of this symmetry and an $SO_R(D)$ subgroup of the R-symmetry of the theory, that is,

$$SO(D)' = \text{diag}(SO_{\text{Lorentz}}(D) \times SO_R(D)).$$

(2.1)

The continuum twist of $\mathcal{N} = 4$ that is the starting point of the twisted lattice construction was first written down by Marcus in 1995 [26]. It now plays an important role in the Geometric Langlands program and is hence sometimes called the GL-twist [27]. In the case of $\mathcal{N} = 4$ super Yang-Mills this amounts to treating the original four Majorana fermions as a $4 \times 4$ matrix and subsequently expanding this matrix on products of Dirac gamma matrices

$$\Psi = \eta I + \psi_\mu \gamma_\mu + \chi_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\psi}_\mu \gamma_5 \gamma_\mu + \overline{\eta} \gamma_5$$

(2.2)

The sixteen component fields $(\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})$ $(\chi_{\mu\nu}$ is antisymmetric) are the twisted fermions. In a similar fashion, four of the scalars which originally transformed as a vector under the $SO(4)$ flavor subgroup become vectors $B_\mu$ under the twisted rotational symmetry and combine with the usual gauge fields $A_\mu$ to produce complexified gauge fields $A_\mu = A_\mu + iB_\mu$ in the twisted theory. The remaining two scalars remain as singlets under twisted rotations.

It is actually possible to pack these twisted fields into a more compact structure by replacing the Greek index $\mu$ running from 1 \ldots 4 with a Roman index running from 1 \ldots 5.

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\(^2\)Note that the lattice actions constructed using the orbifolding and twisted methods are equivalent [28, 29, 30, 31]. Indeed the original orbifold construction of this theory constitutes an independent UV complete construction of the Marcus/GL twist of $\mathcal{N} = 4$ Yang-Mills
The sixteen twisted fermions then comprise the set \((\eta, \psi_a, \chi_{ab})\) while the bosons can be packed into five complex gauge fields \(A_a\). The rationale for this final change of variables is that the twisted action can then be written in the very simple form

\[
S = \frac{1}{g^2} Q \int \text{Tr} \left( \chi_{ab} F_{ab} + \eta [\overline{D}_a, D_b] - \frac{1}{2} \eta d \right) + S_{\text{closed}}
\]

(2.3)

where \(Q\) represents a supersymmetry transformation that transforms as a scalar under the twisted rotation group (its appearance in the theory parallels that of the scalar fermion \(\eta\)). Furthermore the original supersymmetry algebra implies that this charge will be nilpotent with \(Q^2 = 0\) so that the first term appearing in the action Eqn. (2.3) is trivially invariant under \(Q\) transformations. The second \(Q\)-closed term takes the form

\[
S_{\text{closed}} = -\frac{1}{8} \int \text{Tr} \epsilon_{mnpqr} \chi_{qr} \overline{D}_p \chi_{mn}.
\]

(2.4)

The supersymmetric invariance of this term then relies on the Bianchi identity

\[
\epsilon_{mnpqr} \overline{D}_p F_{qr} = 0.
\]

(2.5)

The nilpotent transformations associated with the scalar supersymmetry \(Q\) are given explicitly by

\[
\begin{align*}
Q A_a &= \psi_a \\
Q \psi_a &= 0 \\
Q \overline{A}_a &= 0 \\
Q \chi_{ab} &= -\overline{F}_{ab} \\
Q \eta &= d \\
Q d &= 0,
\end{align*}
\]

(2.6)

where the complexified field strength \(F_{ab}\) is given by

\[
F_{ab} = [D_a, D_b], \quad \overline{F}_{ab} = [\overline{D}_a, \overline{D}_b],
\]

and the complex covariant derivatives are given by

\[
D_a = \partial_a + A_a, \quad \overline{D}_a = \partial_a + \overline{A}_a.
\]

(2.7)

(2.8)

It is important to recognize that the five-dimensional look of the theory is nothing to be afraid of; in fact, it simply reflects the fact that this four-dimensional field theory can be viewed as the dimensional reduction of \(\mathcal{N} = 1\) super Yang-Mills theory in D=10 dimensions. The five complexified gauge connections are the ten gauge fields of that theory. In the next section we will review how easily this picture translates into the lattice formulation.
3. $\mathcal{N} = 4$ Super Yang-Mills Theory on the Lattice

The prescription for discretization is actually quite natural. The complex gauge fields are represented as Wilson gauge fields which take their values in the algebra of a complexified $U(N)$ gauge group

$$A_a(x) \rightarrow U_a(n) = \sum_{C=1}^{N^2} T^C U_a^C(n)$$

(3.1)

Since we need five links in four dimensions we can simply place these Wilson link fields on a hypercubic lattice with an additional body diagonal

$$\hat{\mu}_1 = (1, 0, 0, 0)$$
$$\hat{\mu}_2 = (0, 1, 0, 0)$$
$$\hat{\mu}_3 = (0, 0, 1, 0)$$
$$\hat{\mu}_4 = (0, 0, 0, 1)$$
$$\hat{\mu}_5 = (-1, -1, -1, -1).$$

(3.2)

Thus while $U_a$, $a = 1 \ldots 4$ are associated with the usual unit vectors of a hypercubic lattice the field $U_5$ is then placed on the body diagonal link. Notice that the basis vectors sum to zero, consistent with the use of such a linearly dependent basis. However, it should also be clear that a more symmetrical choice would be preferable in which the five basis vectors are treated in an entirely equivalent manner. A four dimensional lattice with this higher $S^5$ point group symmetry exists and is called the $A^*_4$ lattice. It is constructed from the set of five basis vectors $\hat{e}_a$ pointing from the center of a four-dimensional equilateral simplex out to its vertices together with their inverses $-\hat{e}_a$. It is the four-dimensional analog of the two-dimensional triangular lattice. A specific basis for the $A^*_4$ lattice is given in the form of five lattice vectors

$$\hat{e}_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right)$$
$$\hat{e}_2 = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right)$$
$$\hat{e}_3 = \left( 0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right)$$
$$\hat{e}_4 = \left( 0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right)$$
$$\hat{e}_5 = \left( 0, 0, 0, -\frac{4}{\sqrt{20}} \right).$$

(3.3)

(3.4)

(3.5)

(3.6)

(3.7)

The basis vectors satisfy the relations

$$\sum_{m=1}^{5} \hat{e}_m = 0; \quad \hat{e}_m \cdot \hat{e}_n = \left( \delta_{mn} - \frac{1}{5} \right); \quad \sum_{m=1}^{5} (\hat{e}_m)_\mu (\hat{e}_m)_\nu = \delta_{\mu\nu}; \quad \mu, \nu = 1, \cdots, 4.$$

(3.8)

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The generators are normalized as $\text{Tr}(T^A T^B) = -\delta^{AB}$
It is not hard to see that the basis vectors of $A_4^*$ are a simple deformation of the those used in the hypercubic representation and indeed a simple Gram matrix allows one to map between the coordinates of some field in hypercubic representation to the physical coordinates relative to the $A_4^*$ lattice (see [3] for details). Indeed for the action and other local quantities it is not necessary to explicitly perform this mapping; the hypercubic lattice representation furnishes a simple arena in which one can calculate the action, check gauge invariance and carry out supersymmetry variations without explicit reference to the $A_4^*$ lattice. Only when we consider questions associated with rotational invariance or space-time dependent correlation functions do we need to map the coordinates of lattice fields into their positions relative to the “physical” $A_4^*$ lattice. We stress this point because it means that simulations can be performed in a quite standard hypercubic lattice set-up, without concerns about details of the $A_4^*$ lattice.

The Wilson links transform in the usual way under ordinary non-complexified $U(N)$ lattice gauge transformations

$$U_a(n) \rightarrow G(n)U_a(n)G^\dagger(n + \hat{\mu}_a). \quad (3.9)$$

Supersymmetric invariance then precisely implies that $\psi_a(n)$ live on the same links and transform identically. A local scalar fermion $\eta(n)$ must clearly live on a site. It transforms accordingly,

$$\eta(n) \rightarrow G(n)\eta(n)G^\dagger(n). \quad (3.10)$$

In a similar fashion we place the fermionic fields $\chi_{ab}$ on new links leading from the origin out to $\hat{\mu}_a + \hat{\mu}_b$. In the hypercubic representation these would correspond to links on two and three dimensional faces associated with the hypercube. However, there is one crucial difference from the fields $\psi_a$ - the fields $\chi_{ab}$ are chosen with opposite orientation on these links as encoded from their gauge transformation property:

$$\chi_{ab}(n) \rightarrow G(n + \hat{\mu}_a + \hat{\mu}_b)\chi_{ab}(n)G^\dagger(n). \quad (3.11)$$

This shows how naturally the supersymmetric degrees of freedom can be distributed on the lattice - the sixteen fermionic degrees of freedom at a site can all be associated with the sixteen distinct links that can be drawn in the unit four dimensional hypercube located at that site.

To complete the discretization we need to describe how continuum derivatives are to be replaced by difference operators. A natural technology for accomplishing this in the case of adjoint fields was developed many years ago. It yields expressions for the derivative operator applied to arbitrary lattice p-forms [28], and is thus very naturally tied to geometry. In the case discussed here, we need just two derivatives given by the expressions

$$D_a^{(+)} f_b(n) = U_a(n)f_b(n + \hat{\mu}_a) - f_b(n)U_a(n + \hat{\mu}_a), \quad (3.12)$$

$$D_a^{(-)} f_a(n) = f_a(n)U_a(n) - U_a(n - \hat{\mu}_a)f_a(n - \hat{\mu}_a). \quad (3.13)$$

These difference operators appeared automatically as a result of orbifold projection in the original constructions of supersymmetric lattice Yang-Mills theories from matrix models.
A beautiful feature has appeared here: the construction of supersymmetric lattice gauge theories by means of orbifolding is in one-to-one correspondence with a simple geometrical principle. Indeed, using this geometrical prescription is by far the easiest way to see how this lattice theory emerges. The lattice field strength is given by the gauged forward difference acting on the link field: \( F_{ab}(n) = D_a^{(+)} U_b(n) \). It is automatically antisymmetric in its indices. Furthermore, as hoped for it transforms like a lattice 2-form and yields a gauge invariant loop on the lattice when contracted with \( \chi_{ab}(n) \) (this is precisely the reason that the field \( \chi \) is chosen to have opposite orientation relative to \( \psi_a \)).

Similarly, the covariant backward difference appearing in \( D_a^{(-)} U_a(n) \) transforms as a 0-form or, correspondingly, as a site field. It can hence can be contracted with the site field \( \eta(n) \) to yield a gauge invariant combination. Thus, the twin requirements of gauge invariance and supersymmetry naturally places strong constraints on the whole construction.

Furthermore, this use of forward and backward difference operators guarantees that the solutions of the lattice theory map one-to-one with the solutions of the continuum theory and the fermion doubling problems are hence evaded. Another way to understand this is to see that by introducing a lattice with half the lattice spacing one can map this Kähler–Dirac fermion action into the action for staggered fermions. We emphasize that, unlike the case of two-flavor or three-flavor QCD, there is no rooting problem in this supersymmetric construction since the additional lattice fermion degeneracy is precisely as already required in the continuum theory.

Just like the continuum theory, the lattice action again contains a \( Q \)-exact term:

\[
S = \sum_n \text{Tr} \left( \chi_{ab}(n) D_a^{(+)} U_b(n) + \eta(n) D_a^{(-)} U_a(n) - \frac{1}{2} \eta(n) d(n) \right). \tag{3.14}
\]

Acting with the \( Q \) transformation on the lattice fields and integrating out the auxiliary field \( d \), we obtain the gauge and \( Q \)-invariant lattice action:

\[
S_0 = \sum_n \text{Tr} \left( F_{ab}^+(n) F_{ab}(n) + \frac{1}{2} \left( D_a^{(-)} U_a(n) \right)^2 - \chi_{ab}(n) D_a^{(+)} \psi_b(n) - \eta(n) D_a^{(-)} \psi_a(n) \right). \tag{3.15}
\]

As in the continuum theory, the \( Q \)-exact action must be augmented by an additional piece that is only \( Q \)-closed,

\[
S_{\text{closed}} = \frac{1}{2} \epsilon_{abcd} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) D_c^{(-)} \chi_{ab}(n + \hat{\mu}_c), \tag{3.16}
\]

which is the direct analog of Eqn. (2.4) in the continuum theory. Remarkably, and as shown in [22], an exact lattice analog of the Bianchi identity,

\[
\epsilon_{abcd} D_c^{(-)} F_{ab}(n + \hat{\mu}_c) = 0, \tag{3.17}
\]

guarantees that the above term is invariant under \( Q \)-transformations on the lattice. Note, incidentally, that the coefficient in front cannot be chosen freely. Only with the specific coefficient shown will we recover the correct naive continuum limit with full supersymmetry and Lorentz invariance. It is intriguing to speculate what happens to this relative coefficient under radiative corrections.
To show that the full action correctly reproduces the continuum theory in the naïve continuum limit, one must (in some suitable gauge) expand the gauge fields around the unit matrix,

$$\mathcal{U}_a(n) = I + aA_a(n)$$

(3.18)

The following interesting phenomenon occurs: Usually the unit matrix appearing here arises trivially once one expands the group element $$U_\mu = e^{aA_\mu}$$ in powers of the lattice spacing. However supersymmetry requires that the bosons and the fermions be treated on an equal footing. Since the fermions are expanded in the algebra this necessitates doing the same for the bosons. Usually this would be a disaster since it would make it impossible to introduce the expansion seen in Eqn. (3.18) without breaking gauge invariance. However, in the case of a complexified $$U(N)$$ gauge group we have another option: the unit matrix can arise from the acquired vacuum expectation value of a dynamical field in the theory – here the trace mode of the imaginary part of the connection or, equivalently, the trace mode of the scalars in the original (untwisted) theory.

This expectation value can be achieved by adding to the supersymmetric action a gauge invariant potential of the form

$$S_M = \mu_L^2 \sum_n \left( \frac{1}{N} \text{Tr}(U_\alpha(n)U_\alpha(n)) - 1 \right)^2.$$  

(3.19)

Here $$\mu_L$$ is a tunable mass parameter, which can be used to control the fluctuations of the lattice fields. Notice that such a potential obviously breaks supersymmetry – however because of the exact supersymmetry at $$\mu_L = 0$$ all supersymmetry breaking counterterms induced via quantum effects will possess couplings that vanish as $$\mu_L \to 0$$ and so can be removed by sending $$\mu_L \to 0$$ at the end of the calculation. By adopting the polar parametrization $$\mathcal{U}_a = e^{A_a + iB_a}$$ it should be clear that the leading effect of this term is to set the expectation value of the trace mode of $$B_a$$ to unity as required. Furthermore, fluctuations of this trace mode are governed by the mass $$\mu_L$$ while all traceless scalar modes feel only a quartic potential. Thus the limit $$\mu_L \equiv \mu a \to 0$$ restores the usual flat directions associated with the $$SU(N)$$ sector as the lattice spacing $$a \to 0$$. A finite mass remains for the $$U(1)$$ mode but since this naively decouples in the continuum limit our expectation is that this should not lead to any observable effects in the $$SU(N)$$ sector. This is one of the key issues we wish to investigate in this paper.

The above discussion illustrates the subtle way in which the continuum limit of this theory must be reached. Without a vacuum expectation value of the scalar trace mode, even the notion of a four dimensional continuum limit with canonically propagating degrees of freedom cannot be introduced.

Once one has such a lattice action an obvious thing to do is to perform a strong-coupling expansion. Normally, such an expansion around infinitely strong bare gauge coupling reveals a phase of the theory that is non-universal, confining, chirally broken and with a mass gap that is given in terms of the strong coupling string tension. Remarkably, such a standard strong-coupling expansion is not easily implemented in this theory. It is exact.

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A recent construction employing only $$SU(N)$$ gauge symmetry is discussed in [35].
supersymmetry, or rather exact $Q$-symmetry that gets in the way: this theory is massless and has only one coupling to all fields. The (inverse) bare coupling multiplies all terms of the lattice action. This suggests that the only consistent strong-coupling expansion will be based on expanding the full Boltzmann factor and bringing down powers of the action. However, there is then no damping term of the functional integral. The bosonic degrees of freedom have non-compact support, and the Grassmann integrals provide the heuristic 'zero' that nevertheless could give formal meaning to such an expansion. However, the precise way in which such an expansion scheme could be implemented seems, at best, to be unclear. It is tempting to view the lack of a natural strong-coupling expansion as evidence that this theory indeed may have no strong-coupling, confining, phase at all.

One further complication should be discussed: the potential sign problem in the lattice theory. To simulate the theory requires carrying out an integration over the fermions. This process generates a Pfaffian which is generically complex. This invalidates the usual Monte Carlo method for computing observables since the measure is no longer real and positive definite. However, in earlier numerical work it has been shown that the phase is actually very small for this theory, at least after dimensional reduction to two dimensions and contrary to the naive expectation \cite{37, 38, 39}. To understand this result, one can compute the partition function at one loop. This was done in Ref. \cite{40} with the result that the exact supersymmetry leads to a perfect cancelation between bosons and fermions and no phase appears in the final effective action. This is equivalent to the statement that the Pfaffian is in fact real and positive definite when evaluated on the moduli space corresponding to constant complex commuting matrices\footnote{At least in the four supercharge case, this phenomenon can be related to the so-called Neuberger 0/0 problem which presents a hurdle to constructing a BRST transformation in lattice gauge theories where the fields are defined on a finite group manifold. For a discussion of this connection see Ref. \cite{52}.} Furthermore since the partition function is a topological invariant it can be calculated exactly at one loop – so this result holds to all orders in perturbation theory. Since the expectation value of the Pfaffian phase factor in the phase quenched ensemble is proportional to this full partition function, this argument suggests that the phase should play no role in the lattice theory. Of course, these arguments require exact $Q$ supersymmetry, which is broken by the mass term we use to control the fluctuations of the scalar trace mode. Since the arguments given above do not depend on this dimensional reduction, one could expect it should also be true in the full four-dimensional theory. As we will show in the next section, our numerical results for the bosonic action (related to a derivative of the partition function) back up this conclusion – it approaches the exact supersymmetric value as $\mu_L \to 0$ in the phase quenched ensemble.

One final wrinkle occurs when we contemplate doing simulations with periodic fermion boundary conditions in all four dimensions - as is natural in an exactly supersymmetric Euclidean theory. The form of the fermion action then allows for an exact zero mode of the form $(\eta^A, \bar{\psi}_a^A \chi^A_{ab}) = (\delta^0, 0, 0)$ on any background gauge field ($A$ is an adjoint index here). This zero mode can be lifted either by use of a thermal boundary condition or by the addition of a supersymmetric term

$$S_{extra} = \mu_F Q \left[ \text{Tr} \left( \eta \right) \text{Tr} \left( U_a U_a^\dagger \right) \right].$$

\hspace{1cm} (3.20)
Performing the $Q$ variation leads to two new contributions to the action

$$\text{Tr} \left[ \mathcal{D}_a^{(-)} U_a \right] \text{Tr} \left( U_a \dagger U_a \right) - \text{Tr} \left( \eta \right) \text{Tr} \left( \psi_a U_a \dagger \right),$$

(3.21)

The second of these removes the fermion zero mode. Thus the complete action to be simulated is

$$S = S_0 + S_{\text{closed}} + S_M + S_{\text{extra}}.$$  

(3.22)

Although $Q$-exact, we should emphasize that the last piece $S_{\text{extra}}$ has no analog in the full $\mathcal{N} = 4$ super Yang-Mills theory. Thus also this term must be tuned to zero before continuum results can be extracted. In practice we have confined our study to systems with antiperiodic boundary conditions and this additional term $S_{\text{extra}}$ is set to zero.

As usual in Monte Carlo simulation, the fermion variables are integrated out and their effect in the simulation is represented by a set of pseudofermion fields. Notice though that the integration measure involves only the fields $(\eta, \psi_a, \chi_{ab})$ and not their complex conjugates. Thus it is a Pfaffian rather than a determinant that is generated. Up to a phase this in turn can be produced with a pseudofermion action of the form

$$S_{PF} = \Phi \dagger (M \dagger M)^{-\frac{1}{4}} \Phi,$$  

(3.23)

where $M$ is the antisymmetric twisted fermion bilinear in $S$. The fractional power of the matrix is approximated by a partial fraction (multimass) expansion implemented using the Rational Hybrid Monte Carlo (RHMC) algorithm.

Thus for the lattice practitioner we have a system of

- A set of bosonic variables appearing as noncompact complex gauge fields
- A set of (twisted) fermions whose effect can be encoded using the usual RHMC algorithm.
- Both sets of fields are defined over a hypercubic lattice with additional face and body links.

This is somewhat Baroque, but it is simple and it is completely manageable. It might be useful to list what can be computed at this stage:

- We can simulate with periodic or antiperiodic fermionic boundary conditions so that we can do either zero or finite temperature (supersymmetry-breaking) simulations
- We can dial in various masses $(\mu_L, \mu_F)$ to explicitly break various symmetries. This will ultimately be useful for computing critical exponents.
- We can compute eigenvalues of the fermion (Dirac) operator.
- We can measure Wilson and Polyakov lines to extract, for example, the static quark-antiquark potential and look for confinement/deconfinement.
• We can monitor the distribution of gauge invariant scalar eigenvalues extracted from the observable \( U^a \) which gives us a handle on possible problems associated with integration over the flat directions.

Finally, we should stress the following. To obtain physical correlation functions from this twisted theory, one must perform the appropriate un-twisting on observables. In terms of our twisted variables, physical quantities will generically appear in rather complicated combinations of the variables described here. However, the map is straightforward and can easily be implemented in measurements. And, in the cases of spectral observables, we do not need to perform the un-twisting: operators with the same sets of space-time symmetries couple to the same set of physical states; only the relative coupling coefficients will be different.

4. Simulation Results

4.1 Introduction to the simulations

We begin with a few words about lattice observables. As usual, gauge invariance implies quite strict limitations on the observables we can construct out of our lattice variables. What is new in this theory as compared to ordinary lattice gauge theory is the natural appearance of link variables that live in the algebra of the gauge group rather than in the group itself. This also implies that the integration measure naturally is over anti-Hermitian gauge variables rather than being the invariant gauge group (Haar) measure. The reason is that the measure must remain invariant under an arbitrary shift symmetry, as is clear from Eqn. 2.4. This brings to the open an important point regarding the ordinary Yang-Mills gauge symmetry of this theory: The gauge transformations of the gauge links (which live in the algebra of the gauge group) are defined by the multiplication rule (3.9). At first glance it is not obvious that the flat integration measure associated with the gauge links is invariant under these gauge transformations: the non-linear transformation begs for the left and right invariant Haar measure instead. However, since the links are complexified one must integrate over both the field and its complex conjugate and this saves the day; the Jacobians arising after a gauge transformation cancelling against each other leaving the final measure invariant as required. However this argument fails for the fermion link fields since they do not appear with their complex conjugates in the measure. Remarkably, however one does find that the ordinary flat measure is invariant after taking the product over all lattice points. This is not totally surprising from the point of view of the orbifolding construction, and it is instructive to see how it arises in detail.

The gauge transformation for a typical fermion link variable such as \( \psi_a(n) \) is written in eq. (3.9),

\[
\psi_a(n) \rightarrow G(n)\psi_a(n)G^\dagger(n + \hat{\mu}_a),
\]

where \( \psi_a(n) = T^A\psi^A_a(n) \) and we integrate over the flat measure in the variables \( \psi^A_a(n) \).

\[\text{[}\text{We thank Issaku Kanamori for pointing this out}\text{]}\]
On a finite lattice, $G^\dagger(n + \vec{\mu}_a)$ is a function different from $G(n)$. It is still sufficient to check gauge invariance for infinitesimal (but different) transformations. Let us choose

$$G(n) = 1 + \alpha^A T^A$$

$$G(n + \vec{\mu}_a) = 1 + \beta^A T^A .$$  \hspace{1cm} (4.2)

Expanding and collecting terms we get the same cancellations as in the continuum plus two new terms in the transformation law for $\psi_a(n)$:

$$T^A \psi^A_a(n) \rightarrow \alpha^A T^A T^B \psi^A_a(n) - \beta^A T^A T^B \psi^A_a(n)$$  \hspace{1cm} (4.3)

In the naive continuum limit, where $\alpha^A - \beta^A \sim a$, these terms can be ignored and the usual gauge invariance of the continuum is recovered. But for finite lattice spacing $a$ the new terms remain. However, on the group $U(N)$ we can always expand a product $T^A T^B$ in the generators of the group:

$$T^A T^B = i 2 f^{ABC} T^C + d^{ABC} T^C ,$$  \hspace{1cm} (4.4)

where $d^{ABC}$ are the symmetric structure constants. We can now read off the additional terms in the transformation of the components $\psi^A_a$. The first new piece vanishes because of $f^{AAC} = 0$, and only the second piece remains. For the link fermion $\psi_a(n)$ it is of the form $(\alpha - \beta)^C d^{AAC}$, which does not vanish. However, the measure is the product $\prod d\psi^A_a$ over all links on the lattice. Link for link the leftover pieces cancel among each other because of the conjugation involved in the gauge transformation $\text{(3.9)}$. It is interesting to see how gauge invariance is not insured for a single link, but recovered once the transformations of the neighboring links are included. From the orbifolding construction one could perhaps have guessed that such a mechanism would need to be invoked.

It should be noted here that even for conventional $U(1)$ lattice gauge theories decompactified gauge-fields obtained via stereographic projection of the group manifold can be used in order to construct a lattice BRST symmetry, see $[41, 42, 43]$. The choice of gauge group is clearly not very essential for a first set of simulations. For simplicity, we have here simulated the (phase quenched) $U(2)$ theory on lattices of size $L = 4^4, 6^4, 8^4$ for a wide range of bare ’t Hooft couplings $\lambda = 0.2 - 2.6$ and values of the regulator mass in the range $\mu_L = 0.1 - 1.0$. The simulations have mostly been performed using anti-periodic (thermal) boundary conditions for the fermions. This evidently breaks supersymmetry, but it also removes an exact fermionic zero momentum mode associated the trace mode of fermions that is otherwise present. The breaking due to anti-periodic boundary conditions turns out to be tiny, and will of course disappear as larger volumes are being considered.

An RHMC algorithm has been used for the simulations. It has been described in detail in ref. $[44]$. The use of a GPU accelerated solver $[45]$ has allowed us to reach larger lattices than have thus far been studied. It is important to recognize that the supersymmetric fermion operator defined on a lattice of size $L$ is equivalent, in terms of counting degrees of freedom, to a staggered operator on a lattice of size $2L$. 

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Figure 1: $\frac{1}{N} \text{Tr} \left( U_a^\dagger U_a \right)$ vs mass parameter $\mu_L$ at 't Hooft coupling $\lambda = 1.0$.

4.2 Lattice moduli stabilization

As in continuum $N = 4$ super Yang-Mills theory, the lattice theory possesses flat directions corresponding to the continuum of classical vacuum states in which the bosonic fields take values in the space of constant, mutually commuting, complex matrices. This continuum of vacuum states is called the moduli space of the theory and is determined by the expectation values of the scalar fields appearing as imaginary parts of gauge fields in the twisted formulation. Potential divergences appear in the partition function of the theory when integrating over these flat directions.

In the lattice theory we stabilize these moduli by the addition of the term Eq. 3.19 to the action. This term certainly lifts the moduli space of the theory but in the lattice theory with $U(N)$ gauge symmetry it plays an even more important role by generating a gauge invariant vacuum expectation value for the complexified Wilson link field $\text{Tr} \, U_a^\dagger U_a = 1$. This allows one to argue that in the limit $a \to 0$ and in a fixed gauge $U_a \sim I + A_a + \ldots$. This latter expansion is required if the naive continuum limit of the lattice theory is to target a four dimensional field theory. Furthermore, the unit matrix that appears in this expression can then be interpreted as corresponding to giving a fixed vev to the trace mode of the scalar field. However, it is not clear that this vev survives quantum corrections and in principle one needs to check this in the simulations.

Clearly the correct vacuum state is picked out uniquely as $\mu_L \to \infty$. But the supersymmetric limit lies in the opposite direction where $\mu_L \to 0$. It is important to be
Figure 2: Eigenvalues of the traceless part of $U_a^\dagger U_a$ averaged over the Monte Carlo ensemble for $\mu_L = 0.5$ and $\lambda = 0.2, 1.0, 2.0$.

able to locate which regions in the bare parameter space are consistent with such a link expectation value and simultaneously possess small supersymmetry breaking. Figure 1 shows a plot of the spatial link and temporal link expectation values versus $\mu_L$ at ’t Hooft coupling $\lambda = 1.0$ on lattices of size $L = 4$ and $L = 6$. For small enough $\mu_L$ the link vev can become destabilized either running to zero or large values. In such regions of the bare parameter space we claim that there is no possible four dimensional continuum limit. Our data indicates that this region of instability is pushed to smaller values of $\mu_L$ for larger lattices so it is likely a finite size artifact. All of the data we show in the following sections corresponds to regions of the phase diagram where the link vacuum expectation value is close to unity.

Beyond leading order, the added potential term also lifts and stabilizes the regular $SU(N)$ flat directions and one might also worry that as $\mu_L \rightarrow 0$ this stabilization mechanism would also prove ineffective. To see that it does not consider the distribution of eigenvalues of $U_a^\dagger U_a$ for several values of $\mu_L$ on a $L = 6$ lattice. Fig. 2 shows the distribution of the eigenvalues of the traceless part of this quantity for several ’t Hooft couplings $\lambda$ and for $\mu_L = 0.5$. For this value of $\mu_L$ the scalar eigenvalues do not wander down the flat directions but remain localized close to the origin in field space. The width of the resulting distributions does however increase as the gauge coupling is increased. Of course the most interesting issue is whether the scalar fields remain bounded as we send the supersymmetry
breaking mass term to zero. The answer seems to be in the affirmative; Fig. 3 shows the distributions for fixed ’t Hooft coupling $\lambda = 1.0$ as the mass parameter $\mu_L$ is decreased. The plots show a very weak dependence on $\mu_L$ consistent with the distributions approaching a well defined limit as $\mu_L \to 0$. However, it is important to note that this limit must be performed carefully; as we have seen we should send $L \to \infty$ before we can truly set $\mu_L$ to zero. If we don’t do this we will encounter instabilities associated with the flat directions.

At first sight the apparent localization of the scalar eigenvalues close to the origin seems to indicate that the classical moduli space is in fact lifted by quantum corrections. Such a conclusion would disagree with the perturbative calculation carried out in [40] which shows that the single exact supersymmetry is sufficient to ensure that the effective potential in the lattice theory vanishes to all orders in perturbation theory in a fashion analogous to the continuum.

We thus do not believe that this is the correct interpretation of the results but instead that the observed localization is connected to the treatment of the zero modes in the theory. First notice that the Pfaffian vanishes on the flat directions since in the presence of a constant commuting bosonic background there appear exact fermion zero modes. In the full path integral these would formally cancel against the corresponding bosonic zero modes corresponding to fluctuations in the flat directions. However the supersymmetry breaking potential we have added lifts these bosonic zero modes. The net effect is that the configurations corresponding to the exact flat directions do not contribute to the lattice path integral. Furthermore, since the valleys corresponding to the flat directions possess increasingly steep sides as we move away from the origin in field space we expect that the contribution of field configurations corresponding to fluctuations away from the flat directions will yield a distribution in the scalar eigenvalues that has a peak close to the origin - as we observe. These effects have been observed before by Staudacher at al [46] in the context of supersymmetric matrix models. We think that this is the correct interpretation of our eigenvalue distributions too - the zero mode sector of $\mathcal{N} = 4$ on a finite lattice corresponding to the corresponding supersymmetric matrix model.

4.3 Bosonic Action and Polyakov lines

In this initial study we have focused on understanding of the phase diagram of the lattice theory. First let us examine the bosonic action. This quantity is related to $\frac{\partial \ln Z}{\partial \lambda}$, which vanishes on account of the topological character of the partition function in the supersymmetric limit\(^7\). We see in fig. 4 that the measured value for $\langle S_B \rangle$ is indeed approximately $\lambda$-independent for small $\mu_L$ and agrees very well with the exact value $S_B / (9L^4N^2/2) = 1$. Notice that this result is both consistent with exact supersymmetry in the lattice theory and additionally lends strength to the claim that a genuine sign problem is absent in this theory, and that the phase quenched ensemble hence is adequate for studying the theory.

We now turn to the Polyakov lines. Since $\mathcal{Q} L_a = 0$ we expect that the Polyakov line is both gauge invariant and supersymmetric. Indeed as for the bosonic action the latter

\(^7\)The fermions appear quadratically in the action and hence their expectation value can be computed via a simple scaling argument
would guarantee that the Polyakov line would take a value which was independent of $\lambda$ in the limit $\mu_L \to 0$. Figure 5 shows the (absolute value of the) temporal Polyakov line versus bare coupling for $L = 8$ and $\mu_L = 0.5, 1.0$. The spatial line agrees with the temporal line within statistical errors. Unlike the bosonic action we see a dependence on the coupling $\lambda$ and little indication that taking $\mu_L$ to zero will regain the supersymmetric result.

Insight into this problem can be gained by plotting a related quantity; the Polyakov line projected to the traceless $SU(2)$ sector. This is easily accomplished by taking the traceless part of $U_a(x)$ and exponentiating the result to achieve a matrix in $SL(2, \mathbb{C})$. The corresponding temporal Polyakov line computed from this link is shown in fig. 6.

In this case the value of the line is approximately independent of coupling $\lambda$ as one would expect for an observable invariant under the exact supersymmetry. We deduce that the supersymmetry breaking we are seeing is associated with the U(1) sector. Perhaps one should not be too surprised by this; after all the potential term we add to stabilize the moduli space gives an explicit mass to the U(1) scalars and hence supplies a strong source of supersymmetry breaking in this sector. Intriguingly we have also computed the Polyakov line from the unitary projection of $U_a$ and find a behavior similar to that in fig. 5.

This is evidence that the breaking is actually associated not with the trace mode of the scalars but the additional massless U(1) gauge field that appears in the theory.

Let us make a final comment. Both the bosonic action and Polyakov lines show only
smooth behavior as we scan in the ’t Hooft coupling even as $\mu_L \to 0$. Over the entire range we have explored, $\lambda \leq 2.6$, there are no hints of phase transitions in the system associated with a two phase structure as one might have naively expected. We will present additional evidence in favor of a single lattice phase in the next section.

4.4 Wilson loops and the static potential

Finally we turn to the static potential which we compute using the “supersymmetric” Wilson loops $W(L, M)$ which include the six scalars. Denoting such Wilson loops by $W(r, t)$ where the second index indicates that we align the loop along the temporal direction, we can define the potential $V(r)$ to be

$$W(r, t) = \exp(-V(r)t)$$

(4.5)

or, equivalently, we make an “effective mass” determination of the potential from

$$V(r) = -\log \frac{W(r, t+1)}{W(r, t)}. \quad (4.6)$$

This is a standard technique from the point of lattice QCD simulations. Examples of this analysis from our $8^4$ data sets are shown in Fig. 7. The fact that the data from different $t$ values are not coincident is a sign that $t$ is not large enough that Eq. 4.5 is obtained; higher-energy excitations of the Wilson loop still contribute to $W(r, t)$. Nevertheless, the figures already indicate that the potential flattens to a constant at large $r$. 

Figure 4: Expectation value of the bosonic action vs ’t Hooft coupling $\lambda$ for $\mu_L = 0.25, 0.5, 1.0$. The data is normalized so that the supersymmetric result is unity.
Figure 5: Absolute value of the temporal Polyakov line vs $\lambda$ for $\mu L = 1.0, 0.5$ on a lattice of size $L = 8$.

We can make this statement a bit more quantitative by taking the largest-$t$ data (Wilson loops at $t = 3$ and 4), extracting the potential by fitting Eq. 4.6, and performing a fit to

$$V(r) = -\frac{C}{r} + A + \sigma r.$$  \hspace{1cm} (4.7)

For our data sets, with four values of $r$, we have one degree of freedom. We observe that $\sigma \approx 0$ and that the fits uniformly have a $\chi^2/\text{DoF}$ smaller than unity. Of course, the quantities in the fit are highly correlated since they come from the same underlying configurations. Therefore, we fold the whole fit into a jackknife. The extracted string tension $\sigma$ is shown in Fig. 8. Again, it is clearly consistent with zero. Observing zero string tension raises the possibility that $V(r)$ is, in fact, Coulombic. We thus repeat the fit, but this time with $V(r) = A + C/r$. Again over the observed range of couplings we have good fits with $\chi^2/\text{DoF}$ again less than unity, now for two degrees of freedom. Fig. 9 shows the coefficient of the Coulomb term as a function of the ‘t Hooft coupling. It is remarkably linear. The naive expectation of perturbation theory (one gauge boson exchange) is

$$C = \frac{g^2N}{4\pi}.$$  \hspace{1cm} (4.8)

This seems to describe the data well, and suggests that the strong coupling regime is above $\lambda \geq 2.5$.

Thus the Wilson loop analysis lends support to the hypothesis of a single phase structure with vanishing string tension for all bare couplings $\lambda$. 


Figure 6: Absolute value of the traceless part of the temporal Polyakov line vs $\lambda$ for $\mu L = 1.0, 0.5$ on a lattice of size $L = 8$.

4.5 Fermion eigenvalues and chiral symmetry breaking

If the system is really conformal for all gauge couplings then it should not support a chiral condensate. To understand this better we have studied the spectrum of the twisted fermion operator on a small lattice. Fig. 10 shows a scatter plot of the fermion eigenvalues coming from a run with $\lambda = 0.8$ and $\mu L = 1$ on a small $L = 3$ lattice. The most obvious feature is that no eigenvalues are found close to the origin. This is a robust statement; at all couplings $\lambda$ a gap appears in the spectrum independent of $\mu L$. This, by virtue of the Banks-Casher theorem, means that chiral symmetry is not spontaneously broken in this lattice theory.

We have also measured the Pfaffian on this small lattice as an explicit check that of possible sign problems. Fig. 11 shows the expectation value of both the cosine and sine of the Pfaffian phase as a function of $\lambda$. Rather reassuringly we see that the fluctuations in $\alpha$ are relatively small which provides a concrete numerical justification of the use the phase quenched approximation in our calculations independent of the measurement of the bosonic action or analytic arguments based on the topological character of the lattice partition function.

4.6 The continuum limit

Finally, we should address the issue of a continuum limit. If indeed this theory is conformal at all values of the bare coupling, the beta function vanishes to all orders in lattice
Figure 7: Potentials from Wilson loops, from $8^4 \mu_L = 1$ simulations. Octagons label potentials from $t = 1 - 2$, squares from $t = 2 - 3$ and diamonds from $t = 3 - 4$. (a) $\lambda = 0.25$; (b) $\lambda = 0.45$; (c) $\lambda = 0.6$; (d) $\lambda = 0.9$; (e) $\lambda = 1.2$; (f) $\lambda = 1.6$.

perturbation theory, just as in the continuum. If correct, this means that the notion of a “bare” gauge coupling takes on a new meaning: a renormalization group flow is not induced by changing the lattice spacing. Instead, the continuum limit can be reached anywhere on the real positive bare $g^2$-axis. What about lattice spacing artifacts? The simple way to eliminate these ultraviolet effects is to go to large distances (volume). In this sense, detailed simulations of this theory will be highly unusual, much like the classical solution of differential equations by means of finite differences. This of course will not mean that the theory is free: there will be anomalous dimensions and logarithmic behavior beyond classical scaling.

Perhaps it is useful to contrast this situation with ones which are more familiar to the lattice practitioner. Begin with a pure non-Abelian gauge theory, defined with an ultraviolet cutoff, the lattice spacing $a$. It possesses a Gaussian fixed point at $g^2 = 0$ which is marginally relevant or unstable under flows towards the infrared. To take the continuum limit, one must tune the bare coupling to zero. In that limit, correlation lengths $\xi$ (inverse masses of bound states) become large compared to $a$. These theories are confining, have a mass gap, and correlation functions always decay exponentially with distance. One can observe the approach to the continuum limit in the value of dimensionless ratios of dimensionful quantities (such as mass ratios). Any lattice discretization of such a system will possess additional irrelevant operators. They will affect the spectrum, and hence the mass ratios. However, these additional irrelevant operators automatically cease to affect observables as the bare coupling is taken arbitrarily close to the Gaussian fixed point.

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In [40] this is shown to be true at one loop.
Because these theories have a mass gap, a finite simulation volume (length $L$) typically affects observables by an amount proportional to $\exp(-mL)$ where $m$ is some characteristic mass.

Next, consider theories of fermions and gauge bosons “inside the conformal window”, where the gauge coupling flows to an infrared fixed point under blocking transformations which remove ultraviolet degrees of freedom. For such theories, the fermion mass is a relevant perturbation and it must be tuned to zero by hand in order that the system approach this fixed point. In the massless limit, and in infinite volume, all correlation functions are power-law and the interesting physical parameters are the critical exponents. The distance of the gauge coupling $g$ from its fixed point value $g_c$ is an irrelevant coupling and the difference $|g - g_c|$ governs power law corrections to scaling, whose size is not universal. These effects – as well as those of all irrelevant operators – die away as the correlation length $\xi$ becomes much greater than the cutoff $a$. Besides the mass, a finite system size (technically, $1/L$) is also a relevant parameter because it converts the power-law fall-off of correlation functions into an exponential fall-off. A combination of simulations done at small but non-zero values of the relevant parameter (here the fermion mass) and finite simulation volume (finite size scaling) can, in principle, elucidate the properties of the system.

A theory with a totally vanishing beta function for all bare couplings is one step further. Let us assume that this is the case for the theory under study here. This means
that when all relevant couplings in the lattice model – presumably a subset of them are \( \mu_L \) and \( 1/L \) – are tuned to their fixed point values, the system will again exhibit algebraic decay of correlation functions at large distance. This time, the appropriate exponents will be functions of the bare ’t Hooft coupling \( \lambda \).

5. Conclusions

We have performed numerical simulations of the phase-quenched \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory in four dimensions. In particular, we have examined standard physical observables such as Wilson loops, Polyakov lines and the bosonic action. We have found no evidence for phase transitions as the bare gauge coupling is varied. Furthermore, the effective string tension is consistent with zero for all bare couplings at the largest distances probed. Indeed we see evidence for Coulomb-like behavior in the static quark potential and a gap opens up in the spectrum of the fermion operator indicating the absence of chiral symmetry breaking. Furthermore, the expectation value of the bosonic action appears to be independent of the gauge coupling as the regulator mass \( \mu_L \) is sent to zero and it equals the value expected on the basis of exact supersymmetry. This gives indirect evidence that the sign problem is indeed absent in this lattice theory, and that for all practical purposes the phase of the Pfaffian can be ignored in actual simulations.

With this first study we have provided ample evidence that it is feasible to study this supersymmetric lattice gauge theory by numerical means. The effects of phase quenching,
Our results presented here are of course only a beginning and should be confirmed by future studies on bigger systems and at stronger coupling. The evaluation of non-trivial correlation functions should be initiated. A study of the broken Ward Identities associated with supercharges that are not exactly conserved on the lattice should made. This will give direct evidence for how full supersymmetry is recovered in the continuum limit. There is obviously much exciting work ahead.
Figure 11: $\cos \alpha$ and $\sin \alpha$ vs $\lambda$ for $\mu_L = 1, L = 3$.

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