Bohr model without quantum jumps

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Abstract

Omission of Bohr’s second postulate permits a derivation of spectral intensity with transition amplitudes $X_{nn'} = r_B (n'^3 - n^3) / 3$. The transition amplitudes serve as upper bounds to quantum mechanical matrix elements. They also provide insight into the latter in terms of Sommerfeld ellipses and transition trajectories. The speed of a nascent photon in the region of the electron transition is addressed and the orbit concept is reinterpreted.

PACS numbers: 31.10.+z, 32.30.-r, 32.70.-n, 03.65.Sq
I. INTRODUCTION

The Bohr model of the hydrogen atom can be regarded as the greatest coup in quantum physics. With bold assumptions it derives in a few, strikingly simple steps a frequency formula that historically breached the long-standing mystery of the spectral lines and provided a key to the structure of the atom. The Bohr model is not without shortcomings though. Chief among them is its silence on the brightness of spectral lines. The shortcomings have led to the demise of the Bohr model and its elaboration by Sommerfeld—the “old quantum theory”—and the subsequent rise of quantum mechanics.

Despite its limitations the Bohr model is still taught in introductory physics for historical and conceptual reasons and the simple derivation of energy levels and radiation frequencies. The model is based on two postulates—stationary states and quantum leaps—and the specific assumption of circular electron orbits. The stationary states are fixed with quantization conditions, leading to orbit size,

\[ r_n = r_B n^2, \]  

orbit energy,

\[ E_n = -\frac{R_y}{n^2}, \]  

and orbital frequency,

\[ f_n = \frac{2R_y}{\hbar} \frac{1}{n^3}. \]  

all dependent on the quantum number \( n = 1, 2, 3, \ldots \) Here \( r_B = h^2/4\pi^2 m e^2 \) is the Bohr radius, \( R_y = 2\pi^2 m e^4/h^2 \) the Rydberg energy, \( e \) the elementary charge, \( m \) the electron mass, and \( h \) is Planck’s constant.

The Bohr model treats the transition of the electron from orbit \( n \) to \( n' \) as a “quantum leap” with the difference in orbit energy accounting for the energy \( \epsilon \) of an emitted or absorbed photon. The Planck-Einstein relation associates \( \epsilon \) with the radiation frequency \( f_{nn'} \),

\[ E_{n'} - E_n = \epsilon = h f_{nn'}. \]  

Combining Eqs. (4) and (2) gives the Balmer formula,
\[ f_{nn'} = \frac{R_y}{\hbar} \left( \frac{1}{n^2} - \frac{1}{n'^2} \right), \]  

(5)

in terms of quantum numbers and fundamental constants.

The connection of the “quantum-leap world” inside the atom with classical electrodynamics outside is established at the “rim” of the atom, that is, for very large quantum numbers compared to the transition, \( n \) and \( n' \gg \Delta n = n' - n \). In this limit Eq. (5) can be approximated,

\[ f_{nn'} = \frac{R_y n'^2 - n^2}{h n^2 n'^2} = \frac{R_y (n' + n)(n' - n)}{h n^2 n'^2} \approx \frac{R_y 2n\Delta n}{h n^4} = f_n \Delta n. \]  

(6)

Here Eq. (3) has been used to invoke the orbital frequency \( f_n \). In a transition between high neighbor orbits, \( \Delta n = 1 \), the radiation frequency due to the quantum jump of the electron becomes practically equal to the electron’s orbital frequency, \( f_{nn'} \approx f_n \). The limiting procedure in Eq. (6), whereby the quantum realm and the macroscopic regime merge, is called Bohr’s correspondence principle.

**II. NO QUANTUM LEAPS**

What happens if we keep Bohr’s first postulate—the (quantized) stationary states—but drop the second postulate—the quantum leaps? We then assume that a transition from quantum state \( n \) to \( n' \) is a process of continuously changing action, denoted by a continuous quantum variable \( \tilde{n} \). In this view the transition of the electron from orbit \( n \) to \( n' \) is an intermediate process with intermittent orbital frequency \( f(\tilde{n}) = (2R_y/\hbar)/\tilde{n}^3 \) and intermittent radius \( r(\tilde{n}) = r_B \tilde{n}^2 \) between \( r_n \) and \( r_{n'} \). The frequency associated with the quantum transition is obtained by summation of infinitesimal changes of the orbital frequency,

\[ f_{nn'} = \int_n^{n'} f(\tilde{n})d\tilde{n} = \frac{2R_y}{h} \int_n^{n'} \tilde{n}^{-3}d\tilde{n} = -\frac{1}{2} \frac{2R_y}{h} \tilde{n}^{-2} \bigg|_n^{n'} = \frac{R_y}{h} \left( \frac{1}{n^2} - \frac{1}{n'^2} \right). \]  

(5')

The result is the Balmer formula, as in Eq. (5)[II]
For very large quantum numbers, $n$ and $n' \gg \Delta n$, the orbital frequency $f_n$ changes relatively little with increasing $n$ so that the integral of the transition frequency, Eq. (5’), between quantum states $n$ and $n' = n + \Delta n$ can be approximated,

$$f_{nn'} = \int_{n}^{n'} f(\tilde{n})d\tilde{n} \approx f_n \int_{n}^{n'} d\tilde{n} = f_n \Delta n. \quad (6')$$

The result is the correspondence principle, as in Eq. (6).

The projection of the electron motion in orbit $n$ onto an axis through the nucleus can be considered an oscillating dipole,

$$p_n(t) = -eX_n \cos(2\pi f_n t), \quad (7)$$

with amplitude $X_n = r_n$ from Eqs. (1) and frequency $f_n$ from Eq. (3). By classical electrodynamics, the instantaneous radiative power of an oscillating dipole is

$$S(t) = \frac{2\dot{p}^2}{3c^3}, \quad (8)$$

where the double dot indicates the second derivative. No such radiation,

$$S_n(t) = \frac{2}{3c^3} \left(\frac{d^2 p_n}{dt^2}\right)^2 = \frac{32\pi^4 e^2}{3c^3} f_n^4 X_n^2 \cos^2(2\pi f_n t), \quad (9)$$

occurs for the dipole $p_n(t)$, Eq. (7), due to its postulated stationary-state motion. However, radiation of frequency $f_{nn'}$, Eq. (5’), is emitted or absorbed when a quantum transition occurs. To this end we replace in Eq. (9) the orbital quantum number $n$ by the quantum-number pair $nn'$ for the transition. Taking the time average over a period, $\langle \cos^2 ... \rangle = 1/2$, the average radiative power becomes

$$\langle S_{nn'} \rangle = \frac{16\pi^4 e^2}{3c^3} f_{nn'}^4 X_{nn'}^2. \quad (10)$$

We determine the transition amplitude $X_{nn'}$, in analogy to the transition frequency $f_{nn'}$, via infinitesimal increments of the intermittent radius $r(\tilde{n})$,

$$X_{nn'} = \int_{n}^{n'} r(\tilde{n})d\tilde{n} = r_B \int_{n}^{n'} \tilde{n}^2 d\tilde{n} = \frac{1}{3} r_B (n'^3 - n^3). \quad (11)$$
The *quantum-mechanical* expression for the radiative power associated with a transition between quantum states \(nl\) and \(n'l'\) is like Eq. (10) except for the radial matrix element,

\[
\langle nl|r|n'l'\rangle = \int_0^\infty R_{nl}(r)rR_{n'l'}(r)dr,
\]

in place of the transition amplitude \(X_{nn'}\). Here \(R_{nl}(r)\) is a radial wavefunction and \(l\) denotes the angular quantum number.

A pair of quantum numbers, \(nl\), becomes necessary to characterize a quantum state in Sommerfeld’s extension of the Bohr model by elliptical orbit as well as in quantum mechanics. An \((nl)\) Sommerfeld ellipse has the same length of semimajor axis, the same binding energy, and the same orbital frequency as the \(n\)th Bohr orbit, Eqs. (1) - (3). However, its semiminor axis is shorter,

\[
b_{nl} = r_B n \sqrt{l(l+1)},
\]

with \(l = 0, 1, \ldots, n-1\).

### III. RELATIONS WITH MATRIX ELEMENTS

Figure 1 gives a comparison of quantum mechanically calculated matrix elements, listed in Appendix A, with transition amplitudes between *neighboring* Bohr orbits, \(n \leftrightarrow n' = n + 1\). In this case, after expansion and cancellation, Eq. (11) simplifies to

\[
\frac{X_{nn'}}{r_B} = n(n+1) + \frac{1}{3}.
\]

The value of the transition amplitude \(X_{nn'}\) is given on the abscissa; the value of the matrix elements \(\langle nl|r|n'l'\rangle\) on the ordinate, together with a repetition of the \(X_{nn'}\). This places the amplitudes \(X_{nn'}\) of transitions between Bohr orbits, shown by circles, on the diagonal line in the graph. The data align vertically in columns, starting with the transition \(n = 1 \leftrightarrow 2 = n'\) on the left, and continue with transitions \(2 \leftrightarrow 3\), etc., until \(5 \leftrightarrow 6\) on the right.

For a given pair of (principal) quantum-number neighbors, \(n\) and \(n'\), the matrix elements \(\langle nl|r|n'l'\rangle\) turn out to be always less than the corresponding
transition amplitudes $X_{nn'}$. The transition amplitude, Eq. (11), thus serves as an upper bound for the respective matrix elements.

The selection rule for dipole transitions, $\Delta l = \pm 1$, permits several possible transitions between states with the same principal quantum numbers, $n \leftrightarrow n'$, depending on the states’ angular quantum numbers $l$ and $l' = l \pm 1$. In Fig. 1 the matrix elements of such transitions, $(n, l) \leftrightarrow (n + 1, l \pm 1)$, fall beneath each transition amplitude $X_{nn'} = X_{n,n+1}$, forming the columns under the diagonal. Let us call the transitions where both quantum numbers increase or decrease, $(n, l) \leftrightarrow (n + 1, l + 1)$, comutant. Their matrix elements are displayed by pointed area symbols ($\Box$, $\Diamond$, $\triangle$) and connected with dashed trend lines. In contradistinction, we want to call the transitions with oppositely changing quantum numbers, $(n, l) \leftrightarrow (n + 1, l - 1)$, contramutant. Their matrix elements are displayed by line symbols ($\times$, $+, \ast$) and connected with dotted trend lines. All comutant matrix elements end up above the dash-dotted line in Fig. 1—the “separatrix”—all contramutant matrix elements beneath.

The top matrix element in each column—just beneath the diagonal—represents the comutant transition between quantum states of neighboring principal quantum numbers and the maximum angular quantum numbers, $\langle nl| r| n+1, l + 1 \rangle = \langle n, n - 1| r| n+1, n \rangle$. In terms of Sommerfeld orbits those quantum states are neighbor ellipses with the largest possible minor axis, Eq. (13). Their semimajor axes are as long as the radii of the respective Bohr orbits, Eq. (1), but their semiminor axes are slightly shorter. In other words, they have the smallest possible deviation from circularity that quantization permits. Figure 2(a) shows an example.

Proceeding down a given column in Fig. 1—from beneath the diagonal to above the separatrix—we find the comutant matrix elements $\langle nl| r| n+1, l + 1 \rangle$, with $l = n - 1, ..., 0$. The Sommerfeld ellipses involved in that descending order are progressively more slender, see Fig. 2(b), culminating in a line ellipse of the smaller orbit, $(n, 0)$—(not shown in Fig. 2).

The bottom matrix element in each column of Fig. 1 represents the contramutant transition between quantum states of neighboring principal quantum numbers and the maximum angular quantum numbers, $\langle nl| r| n+1, l - 1 \rangle = \langle n, n - 1| r| n+1, n - 2 \rangle$. The corresponding Sommerfeld ellipses again have the smallest possible deviation from circularity that quantization and contramutant transition permit. But now the neighboring Sommerfeld orbits are a short fat ellipse and a long slim ellipse confocally penetrating the former...
one—see Fig. 2(c) for an example.

Proceeding upward a given column in Fig. 1, from the bottom to below the separatrix, we find the contramutant matrix elements \(|nl|r|n+1, l-1\rangle\), with \(l = n - 1, \ldots, 1\). The Sommerfeld ellipses involved in that ascending order are again progressively more slender; this time culminating in a line ellipse of the larger orbit, \((n+1, 0)\) — (not shown in Fig. 2).

Why are the matrix elements \(|nl|r|n'\rangle\) always smaller than the corresponding transition amplitudes \(X_{nn'}\)? What affects their value? And why are the matrix elements for transitions between orbits with maximum circularity at both the top and bottom of the columns in Fig. 1, but those with minimum circularity next to the separatrix? The deviation of the matrix elements from the transition amplitudes between Bohr orbits can conceptually be understood in terms of orbit geometry and transition trajectory. We will find that orbit geometry provides a scaling whereas the transition trajectory gives rise to an interference effect.

Consider an electron orbiting along an \((nl)\) Sommerfeld ellipse. As shown in Appendix B, its instantaneous acceleration consists of a radial part and a "centripetal" part,

\[
\left| \frac{d^2r_{nl}}{dt^2} \right| = \frac{d^2r_{nl}}{dt^2} - r_{nl} \left( \frac{d\alpha_{nl}}{dt} \right)^2. \tag{14}
\]

Furthermore, the electron’s average centripetal acceleration is proportional to the orbit’s semiminor axis,

\[
\langle r_{nl} \left( \frac{d\alpha_{nl}}{dt} \right)^2 \rangle = 4\pi^2 f_n^2 b_{nl}. \tag{15}
\]

We first discuss the top matrix element in each column in Fig. 1, just below the diagonal. It represents a comutant transition between neighbor ellipses with the least deviation from circularity. For these orbits we will, in an approximate treatment, consider only the average centripetal acceleration, neglecting the radial contribution. By Eqs. (15), (13) and (1) the ratio of acceleration in an \((nl)\) Sommerfeld ellipse and the \(n\)th Bohr orbit is

\[
\frac{\langle r_{nl} \left( \frac{d\alpha_{nl}}{dt} \right)^2 \rangle}{\langle r_n \left( \frac{d\alpha_n}{dt} \right)^2 \rangle} = \frac{b_{nl}}{r_n} = \frac{\sqrt{l(l+1)}}{n}. \tag{16}
\]
According to Bohr’s first postulate, no radiation is emitted or absorbed while the electron keeps orbiting along the \((nl)\) or \((n'l')\) Sommerfeld ellipse. However, emission or absorption occurs for an \((nl) \leftrightarrow (n'l')\) transition. It is reasonable to expect that the average acceleration during the transition is some average of the average acceleration in both orbits. In the present approximation we employ the simplest average—the arithmetic mean. It yields for a comutant, \((nl) \leftrightarrow (n + 1, l + 1)\) transition

\[
\langle n, l | r | n + 1, l + 1 \rangle \approx \frac{1}{2} \left\{ \frac{\sqrt{l(l+1)}}{n} + \frac{\sqrt{(l+1)(l+2)}}{(n+1)} \right\}.
\]  

(17a)

For the top matrix element in each column of Fig. 1, Eq. (17a) is 95% accurate or better. The approximation improves with increasing \(n\)—to the right in Fig. 1—but gets worse with decreasing \(l\)—down toward the separatrix. It ceases for transitions that involve orbits with \(l = 0\) (line ellipses)—next to the separatrix.

In Fig. 1 the top matrix elements fall on a slightly concave trend-curve which approaches the diagonal line of the corresponding transition amplitudes \(X_{n,n+1}\). In the large-\(n\) limit where Bohr’s correspondence principle holds, the top matrix elements merge with the transition amplitudes between Bohr orbits, \(\langle n, n - 1 | r | n + 1, n \rangle \cong X_{n,n+1}\). This is also obtained from the approximation (17a) in the large-\(n\) limit where \(l \cong l + 1 = n \cong n + 1\).

A visualization of comutant transitions between neighbor orbits is facilitated by Fig. 2(ab). The electron’s trajectory during a transition between Bohr orbits (dashed) must be some spiral (not shown) between those circles. Similarly, the transition trajectory between the Sommerfeld orbits must be an elliptical spiral, connecting smoothly the outer and inner ellipse. Note that the larger ellipse completely encompasses the smaller ellipse, akin to the larger Bohr orbit’s complete enclosure of the smaller one. Therefore a comutant transition between Sommerfeld ellipses can be considered as essentially a transition between Bohr orbits but geometrically scaled by the ratio of minor axes, Eq. (17a).

The situation is quite different for a contramutant transition, illustrated in Fig. 2(c). What is a simple inward spiral between Bohr orbits, \(4 \rightarrow 3\), now becomes an “exotic” transition from the long slim to the short fat Sommerfeld orbit, \((4, 1) \rightarrow (3, 2)\), where the electron has to move outside the larger ellipse to reach the smaller one. Thus, in contrast to transitions between Bohr orbits,
where the intra-orbital trajectory constructively contributes to the emission or absorption of radiation, the extra-orbital trajectory in a contramutant transition diminishes the radiation through partial cancellation. This leads, qualitatively, to small values of the contramutant matrix elements, falling beneath the separatrix in Fig. 1.

The pattern in Fig. 1, where the matrix elements at both the top and bottom of each column originate from transitions between the fattest ellipses and those toward the separatrix from gradually slimmer ellipses, suggests that the minor-axis scaling holds not only for comutant transitions, Eq. (17a), but also for the contramutant transitions. Both these influences—orbit scaling and cancellation due to extra-orbital transition trajectory—are contained in an empirical approximation for the contramutant matrix elements,

$$\langle n, l | r | n + 1, l - 1 \rangle \approx \kappa \left\{ \sqrt{\frac{l(l+1)}{n}} - \sqrt{\frac{(l-1)l}{n+1}} \right\},$$  

(17b)

with a fudge factor $\kappa = 1/4$. The formula is not derived from any principles; it is devised in analogy to Eq. (17a) but with a negative contribution for the long slim ellipse. It approximates the bottom matrix elements reasonably well—except $(2, 1) \leftrightarrow (3, 0)$ which involves a line ellipse. This finding may lend support to the notion of counter-radiative effects from extra-orbital transition trajectories.

To demonstrate both approximations we compare the transition amplitude $X_{34} \simeq 12.3$, Eq. (11'), with the two largest matrix elements in the third column of Fig. 1, that is, $\langle 32 | r | 43 \rangle$ and $\langle 31 | r | 42 \rangle$, and with the bottom member, $\langle 32 | r | 41 \rangle$. Their fraction of $X_{34}$ is 83%, 61%, and 11%, respectively. The scaled fractions, Eqs. (17ab), are 84%, 54%, and 12%. Figure 2(abc) shows the corresponding elliptical orbits, together with the (dashed) Bohr orbits, for an assessment of intra-orbital and extra-orbital transition trajectories.

While Eqs. (17ab) approximate well the top and bottom matrix elements in the columns of Fig. 1, their accuracy deteriorates for the matrix elements toward the separatrix. The reason is the increasing slenderness of the involved ellipses, culminating in line orbits. The ellipses’ slenderness gives rise to stronger overtones (higher Fourier coefficients) of the radial oscillations whose contribution to the acceleration, Eq. (14), have been neglected in the minor-axis scaling, Eq. (16).

Going beyond the inspection of Sommerfeld ellipses, more quantitative
insight into matrix elements is obtained by the shape of the radial wavefunctions, shown, for the three above cases, in Fig. 3(abc). The heavy curve displays the integrand of Eq. (12); the sum of positive (negative) areas between the radial axis and the curve above (beneath) visualizes the matrix element. In a sense the matrix elements can be regarded as resulting from interference of the weighted wavefunctions—constructive in the case $32 \leftrightarrow 43$, less so for $31 \leftrightarrow 42$, and considerably destructive for $32 \leftrightarrow 41$. It may well be that such wavefunction interference and the scaled trajectory effects, considered above, are merely different manifestations of the same radiation dynamics.

**IV. HISTORICAL PERSPECTIVE**

Max Born\textsuperscript{14} came close to the present approach of continuous changes inside the atom with his observation that differential quotients in the large-$n$ limit of quantum transitions correspond to difference quotients in the small-$n$ regime,

$$\frac{\Delta \Gamma}{\Delta n} \leftrightarrow \frac{d \Gamma}{d \bar{n}}. \quad (18)$$

The quantity $\Gamma$ is differentially related to a classical (continuous) orbital quantity,

$$g(\bar{n}) \equiv k \frac{d \Gamma(\bar{n})}{d \bar{n}}, \quad (19)$$

where $k$ is a coefficient of proportionality. The quantization of $g(\bar{n})$ approaches in the large-$n$ limit the transition quantity

$$g_{nn'} \approx g_n \Delta n = k \frac{d \Gamma(\bar{n})}{d \bar{n}} \bigg|_{\Delta n} \Delta n, \quad n \gg \Delta n = n' - n. \quad (20)$$

This is Bohr’s correspondence principle—a generalization of Eq. (6). With the analogy (18), called “Born’s correspondence rule,”\textsuperscript{15} Eq. (20) generalizes to

$$g_{nn'} = k \frac{\Delta \Gamma}{\Delta n} \Delta n = k \Delta \Gamma \quad (21)$$

for any quantum number $n$. What Born didn’t do was *integrate* Eq. (19) to obtain the numerator of the difference quotient,
\[ \Delta\Gamma = \int_n^{n'} d\Gamma = \int_n^{n'} \frac{d\Gamma(n)}{dn} d\tilde{n} = k^{-1} \int_n^{n'} g(n)d\tilde{n}, \] (22)
and thus the transition property as an integral over the corresponding orbital quantity,
\[ g_{nn'} = \int_n^{n'} g(n)d\tilde{n}. \] (23)
Two specific examples of Eq. (23) are the above Eqs. (5') and (11).

V. BIRTH OF A PHOTON

It is tempting to determine the transition analogues of other orbital quantities of the old Bohr model, such as the period of revolution,
\[ T_n = \frac{1}{2} \frac{h}{R_y} n^3, \] (24)
and the orbital speed,
\[ v_n = \frac{\alpha c}{n}. \] (25)
Here \( \alpha \approx 1/137 \) is the fine-structure constant and \( c \) is the speed of light. By Eq. (23) the corresponding transition period is
\[ T_{nn'} = \frac{1}{8} \frac{h}{R_y} (n'^4 - n^4) \] (26)
and the transition speed
\[ v_{nn'} = \alpha c \ln \left( \frac{n'}{n} \right). \] (27)
What is the meaning of these quantities?

The transition period turns out to be slightly longer than the radiation period, both being bracketed by the period of revolution of the involved orbits, \( T_n < 1/f_{nn'} < T_{nn'} < T_{n'}. \) The largest discrepancy holds for the \( 1 \leftrightarrow 2 \) transition, with a ratio of \( T_{nn'}/f_{nn'}^{-1} \approx 1.4. \) In the limit of transitions
between high-quantum number orbits, \( n \) and \( n' \gg \Delta n \), all those periods merge, in accordance with the correspondence principle.

Classical electrodynamics distinguishes between radiation phenomena near the source of accelerating charges—the so-called “near zone”—and those very far from the source—the “radiation zone.” Near-zone effects are instantaneously caused by changes of the source; far-zone effects are retarded. Clearly, the transition frequency \( f_{nn'} \) and transition amplitude \( X_{nn'} \), which together compose the radiative power \( \langle S_{nn'} \rangle \), Eq. (10), must be quantities of the radiation zone. In contrast, it seems likely that the transition period \( T_{nn'} \) relates to the near zone between orbits \( n \) and \( n' \).

That conclusion can hardly be avoided for the transition speed \( v_{nn'} \). For the \( 2 \rightarrow 1 \) inward electron transition Eq. (27) gives rise to an outward transition speed \(|v_{21}| = |\alpha c \ln (1/2)| \approx |−0.69 \, v_1| \), that is, about 70% of the electron’s ground-state speed \( v_1 = \alpha c \), Eq. (25). The transition speed \( v_{nn'} \) is very slow when the electron transition occurs between high neighbor orbits but very fast when the electron transits from a high orbit to the ground state. However, for all practical purposes \( v_{nn'} \) will not exceed the speed of light \( c \).

These findings suggest that, with an inward transition of the electron, the transition speed \( v_{nn'} \) is the (negative) average radial speed of the nascent photon in the near zone, that is, between electron orbits \( n \) and \( n' \). By this interpretation the photon starts from rest, \( v = 0 \), at the beginning of the electron transition. The transition period \( T_{nn'} \) can be regarded as the time interval during which the nascent photon “peels off” (decouples) from the inward spiraling electron. The fresh photon will keep accelerating beyond the near zone until it reaches the speed of light \( c \) in the radiation zone.

When directions are reversed, the same scenario must describe the “death” of an absorbed photon. An incoming photon of radiation frequency \( f_{12} \), for instance, will decelerate as it approaches the near zone. The transition speed, \( v_{21} = \alpha c \ln (2/1) \approx +0.69 \, v_1 \), represents the (inward) average radial speed of the moribund photon between electron orbits 2 and 1.

VI. EPISTOMOLOGY

Despite its initial successes (Balmer formula, space quantization, fine-structure formula) the orbit-based old quantum theory of Bohr and Sommerfeld had been insufficient in regard to the atom’s magnetic properties (Zeeman effect), the intensity of the spectral lines, the stability of the hydrogen-
molecule ion, \( H^+_2 \), and the \( He \) atom. As we know now, with the benefit of hindsight, one of the reasons for these shortcomings was the ignorance of electron spin. However, in the early 1920s it was suspected, chiefly by Pauli, Heisenberg and Born,\(^{17}\) that the failures of the old quantum theory were caused by the fallacy of the very concept of electron orbits.

In his article on matrix mechanics Heisenberg\(^{18}\) categorically rejected the notion of electron orbits as unobservable in principle. In the spirit of positivist philosophy he instead proposed that any theory in physics should involve only relationships between fundamentally observable quantities, such as frequency and intensity of spectral lines. The present approach—a “new old quantum theory”—obtains both spectral frequencies and intensities from electron orbits. How is that possible?

It has been pointed out\(^ {19} \) that Heisenberg disobeyed his own demand by invoking fundamentally unobservable quantities—virtual oscillators—in his theory. Something similar occurs in Schrödinger’s wave mechanics where wavefunctions \( \Psi \) play a central role but are, by themselves, unobservable. The orbit conundrum is readily resolved, though, if we regard quantum orbits not as observable spatial descriptions—the notion of “ring atoms” in the Bohr model or “needle atoms” for \( l = 0 \) Sommerfeld orbits contradicts all experience—but merely as entities to calculate observable quantities. This interpretation gives such virtual orbits in the new-old quantum theory a status equivalent to the virtual oscillators in matrix mechanics or to the wavefunctions in wave mechanics. It also renders the oft-mentioned incompatibility of quantum orbits with the Heisenberg uncertainty principle immaterial.

The present modification of the Bohr model, with the first postulate in place but the second postulate omitted, regards both the orbital and transitional motion of the electron as continuous processes. They differ merely in the action variable\(^ {20} \) alluded to in the introduction. Motion on an orbit trajectory is accompanied by constant action, \( I_n = \frac{\hbar n}{2\pi} \), and contrary, motion on a transition trajectory by continuously changing action, \( I(\tilde{n}) = \frac{\hbar \tilde{n}}{2\pi} \). In place of Bohr’s two postulates we can rephrase their essence more succinctly: \textit{Electrodynamic phenomena occur only in processes with continuously changing action variable}. This automatically exempts the stationary states from electrodynamics, restricting the latter to quantum transitions. The use of a continuous quantum variable, \( \tilde{n} \), rules out quantum leaps and instead permits calculus and simple quantum electrodynamics right in the heart of the atom.
ACKNOWLEDGMENTS
I thank Ernst Mohler for valuable discussions. I also thank Preston Jones
and Van Katkanant for help with computer integration and graphics.
Appendix A. MATRIX ELEMENTS

TABLE I. Matrix elements $M$ for dipole transitions between quantum states $nl$ and $n'l'$, Eq. (12), here listed as $\langle nl | M | n'l' \rangle$. Values, in the unit of Bohr radius $r_B$, are from Ref. 21 or otherwise calculated by integration of radial wavefunctions from Ref. 22. The dash-dotted line corresponds to the separatrix in Fig. 1, with comutant transitions above and contramutant transitions below.

Appendix B. KEPLER ACCELERATION

Consider Kepler motion of a body along an elliptical orbit. The body’s Cartesian coordinates are $x = r \cos \alpha$ and $y = r \sin \alpha$. The components of its acceleration are

$$\frac{d^2x}{dt^2} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\alpha}{dt} \right)^2 \right] \cos \alpha - \left[ 2 \frac{dr}{dt} \frac{d\alpha}{dt} + r \frac{d^2\alpha}{dt^2} \right] \sin \alpha$$

(28a)

and

$$\frac{d^2y}{dt^2} = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\alpha}{dt} \right)^2 \right] \sin \alpha + \left[ 2 \frac{dr}{dt} \frac{d\alpha}{dt} + r \frac{d^2\alpha}{dt^2} \right] \cos \alpha$$

(28b)

By Kepler’s second law the areal speed, here expressed in relation to angular momentum $L$ and mass $m$ of the body, $2dA/dt = L/m \equiv C$, is a constant of the motion,

$$C = r^2 \frac{d\alpha}{dt}.$$
Its derivative, \( \frac{dC}{dt} = 0 = r [2(\frac{dr}{dt})(\frac{d\alpha}{dt}) + r(\frac{d^2 r}{dt^2})] \), makes the brackets on the far right of Eqs. (28 ab) vanish. We square and add Eqs. (28 ab), then take the root,

\[
\sqrt{\left(\frac{d^2 x}{dt^2}\right)^2 + \left(\frac{d^2 y}{dt^2}\right)^2} = \left|\frac{d^2 r}{dt^2}\right| = \frac{d^2 r}{dt^2} - r \left(\frac{d\alpha}{dt}\right)^2.
\] (30)

This gives the acceleration as the sum of a radial and a “centripetal” term. We square Eq. (29) and solve for the instantaneous centripetal acceleration,

\[
r \left(\frac{d\alpha}{dt}\right)^2 = \frac{C^2}{r^3}.
\] (31)

Combined with the path-average of the inverse cube radial distance, taken over a Kepler orbit,

\[
\langle r^{-3} \rangle_s = \frac{1}{a^2 b},
\] (32)

and the expression for the constant of motion in terms of orbital semiaxes and frequency,

\[
C = 2\pi ab f,
\] (33)

the average centripetal acceleration is

\[
\left\langle r \left(\frac{d\alpha}{dt}\right)^2 \right\rangle = 4\pi^2 f^2 b.
\] (34)
References

1A planetary system of an orbiting electron around a massive central nucleus is, by classical electrodynamics, unstable due to radiative loss according to the (centripetal) acceleration of the electron charge. No such instability of atoms is observed though. Bohr resolved this conflict with his postulate of stationary states: Defying electrodynamics, electrons can move in certain orbits without radiative energy loss. An electron moving in such an orbit is said to be in a stationary state.

2N. Bohr, Phil. Mag. 26, 1-25 (1913). Bohr gives three alternate methods of quantization, one of which—quantization of angular momentum, $L_n = n\hbar/2\pi$—is still used to date in introductory physics. That method fortuitously gives the correct energy levels and radiation frequencies but disagrees with the experimental values of angular momentum (ignoring electron spin) by one unit, $\hbar/2\pi$. The proper method is the quantization of action, $A_n = mn_n^2 = n\hbar$.

3Consistent with integration rules it is natural to associate the suffix $n$ in the radiation frequency $f_{nn'}$ and in other transition quantities—Eqs. (5), (5'), (11), (26) and (27)—with the electron’s initial orbit and $n'$ with its target orbit. On the other hand, it is convenient to regard emission as a positive entity and absorption as negative. Similarly, mathematical convention designates radial outward motion as positive and inward motion as negative. Unfortunately, because of the opposite radial motion of electron and photon during the transition, the sign of the electromagnetic transition quantities, obtained from those equations, is opposite to convenience or convention.

4Note that the Planck-Einstein relation, Eq. (4), is not used in the derivation of Eq. (5').

5J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1962), 2nd ed., p. 396.

6J. R. Reitz, F. J. Milford and R. W. Christy, Foundations of Electromagnetic Theory (Addison-Wesley, Reading, MA, 1979), p. 461.

7If the dipole oscillation, Eq. (7), is expressed in terms of complex exponentials, $exp(\pm i2\pi ft)$, then the rhs of Eqs. (9) and (10) must be multiplied
by 4.

8L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), 2nd ed., p. 261.

9$R_{nl}(r)$ is the “stretched” radial wavefunction, in contrast to the “true” radial wavefunction $R_{nl}(r)$ from $\psi(r, \theta, \varphi) = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\varphi)$. They are related by $R_{nl} = rR_{nl}$.

10A. Sommerfeld, “Zur Quantentheorie der Spektrallinien,” Annalen der Physik, 51, 1-94 (1916). Note that Eq. (13) and the range of $l$ are modified from Sommerfeld’s historical theory to obtain agreement with quantum mechanics.

11Except for the lowest transition; $1, 0 \leftrightarrow 2, 1$.

12The terms “co-mutant” and “contra-mutant” are chosen to signify changes of quantum numbers $n$ and $l$ in the same or opposite direction, respectively.

13The purpose of Eqs. (17ab) is conceptual and only intended to indicate scaling by orbit geometry and constructive or destructive effects from transition trajectories.

14M. Born, “Über Quantenmechanik,” Z. Phys. 26, 379-395 (1924).

15M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966), p. 193.

16Take, for example, an electron transition from a very high orbit $N$ to the ground state, $n' = 1$. By Eq. (27) the transition speed equals the speed of light $c$ when $N = \exp(1/\alpha)$. The radius of that orbit is $r_N \approx 10^{109}m$—much larger than the universe!

17D. Serwer, “Unmechanischer Zwang: Pauli, Heisenberg, and the Rejection of the Mechanical Atom, 1923-1925,” Historical Studies in the Physical Sciences, 8, 189-256 (1977).

18W. Heisenberg, “Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen,” Z. Phys. 33, 879-893 (1925).
E. MacKinnon, “Heisenberg, Models, and the Rise of Matrix Mechanics,” Historical Studies in the Physical Sciences, 8, 137-188 (1977).

Originally defined as a process quantity, the concept of action also serves as a state quantity in the context of action-angle variables \((I, w)\); see M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer, New York, 1990), p. 33.

E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra (Cambridge UP, 1953), p. 133.

L. Pauling and E. B. Wilson, Introduction to Quantum Mechanics (Dover, New York, 1935), pp. 135-136.

The negative sign in Eq. (30) expresses the “centripetal” orientation.

M. Bucher, D. Elm and D. P. Siemens, “Average position in Kepler motion,” Am. J. Phys. 66, 929-930 (1998).

H. Goldstein, Classical Mechanics (Addison-Wesley, Reading MA, 1980), 2nd ed., pp. 70-102.
FIGURE CAPTIONS

Fig. 1. Comparison of dipole matrix elements $\langle n|l|r|n'|l' \rangle$ with transition amplitudes $X_{nn'}$ between neighboring Bohr orbits. The circles on the diagonal give $X_{nn'}$. The dash-dotted line (“separatrix”) divides matrix elements of comutant (above) and contramutant (below) quantum transitions (see text).

Fig. 2. Bohr orbits (dashed) and Sommerfeld ellipses involved in $n \leftrightarrow n' = 3 \leftrightarrow 4$ quantum transitions. The corresponding transition amplitude and matrix elements are displayed in the third column of Fig. 1 with $X_{34}$ on the diagonal, (a) directly beneath, (b) next down, and (c) at the bottom.

Fig. 3. Radial wavefunctions (light curves) of the quantum states in Fig. 2 and matrix-element integrand, Eq. (12), (heavy curve).