On durability of a hydraulic fracture filled with proppant particles

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Abstract. On the basis of conservation laws and basic principles of thermodynamics, a mathematical model is developed for flows of a two-phase granular fluid. The phases consist of a viscoplastic granular Bingham fluid and a viscous Newtonian fluid. As an application, one-dimensional flows are studied in a channel to address the stability of the proppant pack which fills a hydro-fracture. We find correlations between the phase flow rates and the pressure gradient. Such correlations are similar to a Darcy law. We determine a criterion for the initiation of motion of a granular phase in a porous medium. Given a yield stress of the granular phase, it is proved that this phase does not flow if either the pressure gradient or the channel thickness is small. The phase flow rates are studied numerically at various input parameters such as the phase viscosities, yield stresses and etc. The factors slowing down the penetration of the solid phase into the porous medium are revealed.

1. Introduction

There are rocks containing viscoplastic components [1]. Under some conditions like elevated temperature or regional stresses such rocks can fill cavity voids and cracks. Particularly, the efficiency of hydraulic fracturing technology is to be evaluated as far as the kerogen-containing oil rocks are concerned. The problem is that the pore space formed by proppant particles which fix the hydraulic fracture can be saturated with plastic rock in time. Thus, the question of a hydro-fracture durability arises.

The goal of the present paper is to develop a mathematical model which gives an insight into viscoplastic rock stability near cracks. Particularly, we are motivated by estimation of the proppant porous layer durability in case of mobility of the nearby rock.

We treat the rock as a two-phase granular fluid, with the pore fluid velocity and the velocity of the solid part of rock being different. The solid part of rock itself is considered as a granular material with an assigned viscosity and a yield stress. To estimate how fast the proppant pack looses the pore space, we study the rate of flows of the two-phase granular fluid through the proppant pack which forms a porous medium with some effective permeability. We argue like in the fluid mechanics theory [2] and consider the flows of the two-phase fluid in a narrow layer like the Hele-Shaw cell using the similarity of the governing equations (averaged over the cross section) with the Darcy law. On this way one can estimate flow rates through the proppant pack (see formulas (21)).
In addition to viscosity many granular materials are characterized by yield stress as well. Such a stress is due to colloidal forces between the smallest suspended particles [3]. Because of the scales separation between colloidal particles and large noncolloidal particles, the material can be considered as a suspension of noncolloidal particles immersed in a viscoplastic fluid. It is assumed that on this way a progress in the understanding of granular materials can be achieved by studying the effect of noncolloidal particle additives on a viscoplastic fluid with known properties [4].

In this study, we use a different approach based on the Cosserat continuum, when each material particle is treated as a rigid body. In [5], on the basis of thermodynamic principles, a two-phase medium model is proposed for the joint flow of Newtonian and granular fluids. The latter is described in the framework of the Cosserat continuum as in the theory of micropolar fluid [6]. Here we consider a more general case then in [5] by assuming that the granular phase has the properties of a viscoplastic granular Bingham material; i.e., it is characterized by two viscosities and two yield stresses: for shear and rotation of particles. Within the Cosserat continuum, each material point is characterized not only by velocity and angular velocity but by micro-inertia as well.

2. Basic equations
We consider a joint motion of a Newtonian fluid and a granular fluid. In what follows we use the $f$ and $s$ indexes to label these fluids. Starting from basic thermodynamic principles, the following momentum equations are derived in [5]:

$$\frac{\partial (\rho_s \mathbf{v}_s)}{\partial t} + \text{div} (\rho_s \mathbf{v}_s \otimes \mathbf{v}_s) = \frac{-\rho_s \nabla p}{\rho} - \frac{\rho_s \rho f}{2\rho} \nabla (\mathbf{v}_s - \mathbf{v}_f)^2 - k (\mathbf{v}_s - \mathbf{v}_f) + \text{div} \mathbf{T}_s,$$ \hspace{1cm} (1)

$$\frac{\partial (\rho_f \mathbf{v}_f)}{\partial t} + \text{div} (\rho_f \mathbf{v}_f \otimes \mathbf{v}_f) = \frac{-\rho_f \nabla p}{\rho} + \frac{\rho_s \rho f}{2\rho} \nabla (\mathbf{v}_s - \mathbf{v}_f)^2 + k (\mathbf{v}_s - \mathbf{v}_f) + \text{div} \mathbf{T}_f.$$ \hspace{1cm} (2)

Here, $\mathbf{v}_i$, $\rho_i$, $T^i$, $p$, $\rho$, $k$ are the velocity, the partial density, the stress tensor, the pressure, the total density and the resistivity. We use the following tensor notations. Given vectors $\mathbf{a}$ and $\mathbf{b}$, we denote their tensor product by $\mathbf{a} \otimes \mathbf{b}$: $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. Given a matrix $T$, we introduce the vector $\text{div} T$ as follows: $(\text{div} T)_i = \partial T_{ij}/\partial x_j$.

Let us remind that $\rho_s = \tilde{\rho}_s \phi_s$, with $\tilde{\rho}_s$ standing for the material density of the granular phase and $\phi_s$ being the volume fraction of this phase. Observe that

$$\rho_f = \tilde{\rho}_f \phi_f, \quad \phi_s + \phi_f = 1.$$

The mass conservation laws are

$$\frac{\partial \rho_f}{\partial t} + \text{div} (\rho_f \mathbf{v}_f) = 0, \quad \frac{\partial \rho_s}{\partial t} + \text{div} (\rho_s \mathbf{v}_s) = 0.$$

As for the granular phase, it is assumed to be a Cosserat continuum. It implies that each granular particle is characterised by the angular velocity $\mathbf{\omega}$ and by the internal specific spin $\mathbf{s}$. Moreover, particles rotation is governed by the angular momentum equation

$$\frac{\partial (\rho_s \mathbf{s})}{\partial t} + \text{div} (\rho_s \mathbf{v}_s \otimes \mathbf{s}) = \text{div} N^s - \epsilon : T^s + \rho_s \mathbf{l}.$$ \hspace{1cm} (3)

Here, $N^s$ and $\mathbf{l}$ stand for the couple stress tensor and the mechanical couple density respectively; $\epsilon$ is the Levi-Civita third order tensor,

$$\epsilon_{sij} = \mathbf{e}_s \cdot (\mathbf{e}_i \times \mathbf{e}_j), \quad (\epsilon : T^s)_i = \epsilon_{ijk} T^s_{jk},$$
where \( \{\mathbf{e}_i\}^3 \) is any orthogonal basis in \( \mathbb{R}^3 \).

Let us describe rheology of the granular phase. We introduce the rate of strain tensors

\[
B = \nabla \mathbf{v}_s - \Omega, \quad A = \nabla \omega,
\]

where \( \nabla \mathbf{v}_s \) is any orthogonal basis in \( \mathbb{R}^3 \), \( \mathbf{e}_i \) is the unit vector in the \( x_i \)-direction, and \( \Omega \) is the angular velocity. The sub-differential inclusion (4) implies that

\[
\{ \nabla \mathbf{v}_s \}_{ij} = \partial (\mathbf{v}_s)_{ij}/\partial x_j, \quad \Omega_{ij} = \mathbf{e}_i \cdot (\omega \times \mathbf{e}_j).
\]

Observe that both \( B \) and \( A \) are objective relative to change of frame of references [6]. With \( S^a \) standing for the viscous part of the stress tensor \( T^s \), we write the representation formula \( T^s = -pI + S^a \). In the present paper, we consider a more general case then in [5]. We assume that the granular phase enjoys the properties of a viscoplastic granular Bingham material described in [7]. To this end, we introduce the viscosities \( \mu_s \), \( \mu_a \) and the yield stresses \( \tau_s \) and \( \tau_n \). Let \( B_s \) and \( B_a \) stand for the symmetric and skew-symmetric parts of the matrix \( B \). We denote \( \varepsilon = \mu_a/\mu_s \) and postulate that

\[
S^a \in \partial V(B_0), \quad \text{where} \quad B_0 = B_s + \varepsilon B_a, \quad V(B_0) = \mu_s |B_0|^2 + \tau_s |B_0|.
\]  

(4)

By definition, the sub-differential inclusion (4) implies that \( S^a : (X - B_0) \leq V(X) - V(B_0) \) for any \( 3 \times 3 \)-matrix \( X \). Similarly, we postulate that

\[
N^a \in \partial V_n(A), \quad \text{where} \quad V_n(A) = \gamma |A|^2 + \tau_n |A|,
\]

(5)

with the prescribed parameters \( \gamma \) and \( \tau_n \).

3. Poiseuille-like flows

Let us consider steady flows between two parallel planes in the \( x \)-direction under a constant pressure gradient, \( p_x = \text{const} < 0 \), \( p_y = p_z = 0 \). The velocities \( \mathbf{v}_s = (v, 0, 0)^T \) \( \mathbf{v}_f = (u, 0, 0)^T \) depend on the vertical variable \( y \) only, \( -H < y < H \). The same is true for the angular velocity \( \omega = (0, 0, \omega)^T \). Equations (1)-(3) become

\[
0 = -\frac{\rho_s}{\rho} p_x - k(v - u) + \frac{\partial}{\partial y} S^a_{12}, \quad 0 = -\frac{\rho_f}{\rho} p_x + k(v - u) + \frac{\partial}{\partial y} S^f_{12},
\]

\[
0 = \frac{\partial}{\partial y} N_{32} + S^a_{21} - S^a_{12}.
\]

(6)

(7)

It is explained in [7] that the constitutive law (4) can be written as follows:

\[
S^a_{12} = 2 \mu_s \left[ \frac{v_y}{2} + \varepsilon \left( \frac{v_y}{2} + \omega \right) \right] + \tau_s \left( \frac{v_y}{2} + \omega \right), \quad b \neq 0,
\]

\[
S^a_{21} = 2 \mu_s \left[ \frac{v_y}{2} - \varepsilon \left( \frac{v_y}{2} + \omega \right) \right] + \tau_s \left( \frac{v_y}{2} - \omega \right), \quad b \neq 0,
\]

\[
|S^a_{12}|^2 + |S^a_{21}|^2 \leq \tau_s^2, \quad b = 0,
\]

(8)

(9)

(10)

where

\[
b^2 \equiv \left( \frac{v_y}{2} \right)^2 + \varepsilon^2 \left( \frac{v_y}{2} + \omega \right)^2.
\]

Similarly, the law (5) becomes

\[
N_{32} = 2 \gamma \omega_y + \tau_n \text{sign} \omega_y \quad \text{while} \quad \omega_y \neq 0 \quad \text{and} \quad |N_{32}| \leq \tau_n \quad \text{if} \quad \omega_y = 0.
\]

(11)

As for the fluid phase, we have that

\[
S^f_{12} = \mu_f u_y.
\]

(12)
Because of symmetry we consider the flow in the domain $0 < y < H$ with the following boundary conditions:
\[
\frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{\partial v}{\partial y} \bigg|_{y=0} = 0, \quad u|_{y=H} = v|_{y=H} = \omega|_{y=H} = 0, \quad \omega|_{y=H} = -\frac{\alpha_0}{2} \frac{\partial v}{\partial y} \bigg|_{y=H}.
\]  
(13)

The latter condition implies that $\omega = \alpha_0 \text{rot} v_s / 2$ at $y = H$, i.e. micro-rotation is proportional to macro-rotation [5], with $0 \leq \alpha_0 \leq 1$.

Let us consider the case when the Newtonian phase flows while the granular phase is motionless. It implies that
\[
v = 0, \quad \omega = 0, \quad |S_{12}^s|^2 + |S_{21}^s|^2 \leq \tau_2^2.
\]

It follows from (7) that $S_{12}^s = S_{21}^s$ hence $-\tau_s \leq \sqrt{2} S_{12}^s \leq 0$. Equations (6) become
\[
0 = -\rho \rho_s p_x + ku + \frac{\partial}{\partial y} S_{12}^s, \quad 0 = -\rho \rho_s p_x - ku + \mu_s \frac{\partial^2 u}{\partial y^2}.
\]  
(14)

One can verify easily that the velocity $u$ is given by the formula
\[
u = \beta (e^{\alpha V} + e^{-\alpha V}) - \beta \frac{\alpha^2}{\alpha^2} \quad \alpha^2 = \frac{k}{\mu_f}, \quad \beta = \rho \rho_s p_x - \rho \mu_f, \quad \alpha = e^{\alpha V} + e^{-\alpha V}.
\]

Next, we integrate the first equation in (14) to find the no-flow condition for the granular phase:
\[
\frac{\rho_s + \rho \rho_f F(t)}{\rho} \leq Bn = \frac{\tau_s}{\sqrt{2} |p_x| H}, \quad \text{where} \quad t = H \sqrt{\frac{\mu_f}{k}}, \quad F(t) = 1 - \frac{e^t - e^{-t}}{t(e^t + e^{-t})}.
\]  
(15)

Here, $Bn$ is the Bingham number.

Now, we consider the case when both the phases flow. Let us introduce dimensionless variables
\[
v = V v', \quad u = U u', \quad y = H y', \quad \omega = \omega_0 \omega', \quad S_{ij}^s = \frac{2 \mu_s V}{H} S_{ij}^s, \quad N_{32}^s = \frac{2 \gamma V}{H^2} N_{32}^s, \quad R_s = \frac{\rho_s}{\rho}, \quad R_f = \frac{\rho_f}{\rho},
\]
with $\omega_0 = V / H$, and perform calculations provided the following dimensionless parameters are fixed:
\[
\Pi = \frac{|p_x| H^2}{2 V \mu_s}, \quad \tau_1 = \frac{\tau_s H}{2 \mu_s V}, \quad \tau_n = \frac{\tau_n H^2}{2 \gamma V}, \quad \gamma_1 = \frac{\gamma}{2 \mu_s H^2}, \quad k_1 = \frac{k H^2}{2 \mu_s}, \quad \varepsilon = \frac{\mu_a}{\mu_s}, \quad \varepsilon_1 = \frac{\mu_f}{2 \mu_s}.
\]

To solve the problem (6)-(13) numerically, we apply regularization. Given a small number $\delta \rightarrow 0$, we omit the prime indexes and formulate the regularized equations in the dimensionless variables as follows:
\[
0 = R_s \Pi - k_1 (v - u) + \frac{\partial}{\partial y} S_{12, \delta}^s, \quad 0 = R_f \Pi + k_1 (v - u) + \varepsilon \frac{\partial^2 u}{\partial y^2},
\]  
(16)

\[
0 = 2 \varepsilon \gamma_1 - N_{32, \delta} + S_{21, \delta}^s - S_{12, \delta}^s,
\]  
(17)

with
\[
S_{12, \delta}^s = \left[ \frac{v_y}{2} + \varepsilon \left( \frac{v_y}{2} + \omega \right) \right] \left( 1 + \frac{\tau_1}{\sqrt{b^2 + \delta^2}} \right), \quad S_{21, \delta}^s = \left[ \frac{v_y}{2} - \varepsilon \left( \frac{v_y}{2} + \omega \right) \right] \left( 1 + \frac{\tau_1}{\sqrt{b^2 + \delta^2}} \right),
\]  
(18)

\[
N_{32, \delta} = \omega_y \left( 1 + \frac{\tau_n}{\sqrt{\omega_y^2 + \delta^2}} \right).
\]  
(19)

\[
\frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{\partial v}{\partial y} \bigg|_{y=0} = 0, \quad u|_{y=1} = v|_{y=1} = \omega|_{y=1} = 0, \quad \omega|_{y=1} = -\frac{\alpha}{2} \frac{\partial v}{\partial y} \bigg|_{y=1}.
\]  
(20)
4. Numerical results
With the use of Wolfram Mathematica, we perform calculations starting from the following list of data:

\[ \varepsilon = 0.6, \tau_1 = 0.95, \tau_{n1} = 0.5, Rf = 0.3, \Pi = 1.425, k_1 = 0.6, \varepsilon_1 = 0.05, \gamma_1 = 0.2, \alpha = 0.1. \]

Figure 1 depicts the velocity profiles. As for the granular phase velocity, one can observe clearly a plug zone where the velocity is constant. Such an effect is due to the fact that the granular phase enjoys a yield stress.

Let us define mean values

\[ Q_f = \frac{1}{0} u \, dy, \quad Q_s = \frac{1}{0} v \, dy, \quad W = \frac{1}{0} \omega \, dy. \]

It is well known in the theory of lubrication that the average velocity \( Q \) of a viscous Newtonian fluid (with the viscosity \( \mu \)) over the channel cross section (of the size \( 2H \)) satisfies the Darcy formula \( Q = -\frac{H^2}{2\mu} \frac{p_y}{3H} \). We develop a numerical code which is equivalent to the formulas

\[ Q_f = -\frac{\Pi H^2}{3} F_f(\Pi, \tau_1, \tau_{n1}, k_1, \varepsilon, \varepsilon_1, \gamma_1, \alpha), \quad Q_s = -\frac{\Pi H^2}{3} F_s(\Pi, \tau_1, \tau_{n1}, k_1, \varepsilon, \varepsilon_1, \gamma_1, \alpha), \quad (21) \]

which enable us, particularly, to evaluate the flow rate \( Q_s \) of the granular phase when \( H \) stands for an effective pore throat size of the proppant pack.

In what follows, we use the code (21). Because of the yield stress in the viscoplastic granular phase, there is a critical pressure gradient effect. It implies that the granular phase flows provided the applied pressure gradient is great enough, Figure 3. The same effect holds for the angular velocity of the granular phase, Figure 4. That the Bingham number \( Bn \) defines initiation of the granular phase flow is clear from the no-flow criterion (15). Figures 5 and 6 illustrate this criterion: great values of \( \tau^* \) forbids not only the particles motion but their rotation as well.

Little is known about the rotational yield stress \( \tau_n \). It is explained in [7] and [8] that it is due to \( \tau_n \) that a weak plug zone does appear. In such a zone only the rate of strain tensor \( A \) becomes zero in contrast to the strong plug zone where both \( A \) and \( B \) vanish. Figure 2 shows the velocity profiles in the case \( \tau_s = 0 \) and \( \tau_n > 0 \). One can see weak plug layers close to the boundaries \( y = \pm 1 \). In the weak plug zone all the particles have the same angular velocity. It implies that particles cling to each other in such a zone.

5. Conclusions
A mathematical model is developed for mobility of the viscoplastic rock close to a crack. The problem is motivated by durability of a hydrofracture proppant pack embedded in the rock with viscoplastic components. The granular properties of the solid rock’s part are described within the framework of the Cosserat continuum when each infinitesimal volume of material is characterized not only by velocity but by rotation as well. We formulate a criterion for the immobility of the solid rock part in terms of its yield stress. We perform calculations to determine the rock mobility for different pressure drops, the crack permeabilities and different yield stress values. The theoretical results imply that it is the yield stress that defines durability of the proppant pack. Hence there is a need to perform laboratory experiments for measurement of such a parameter.

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Figure 1. Dimensionless velocity profiles. Solid, dashed and dashed-dotted lines correspond to fluid phase velocity, granular phase velocity (increased 5 times) and angular velocity of granular phase (increased 10 times).

Figure 2. Dimensionless velocity profiles with a weak plug zone. Solid, dashed and dashed-dotted lines correspond to fluid phase velocity, granular phase velocity and angular velocity of granular phase. The data list is changed as follows: \( \tau_1 = 0, \tau_{n1} = 50, \gamma_1 = 0.2 \).

Figure 3. Averaged dimensionless velocities \( Q_f \) and \( Q_s \) versus dimensionless pressure gradient \( \Pi \). Solid and dashed lines correspond to fluid and granular phases respectively.

Figure 4. Dimensionless mean angular velocity \( W \) of the granular phase versus dimensionless pressure gradient \( \Pi \).

Figure 5. Averaged dimensionless velocities \( Q_f \) and \( Q_s \) versus dimensionless yield stress \( \tau_1 \). Solid and dashed lines correspond to fluid and granular phases respectively.

Figure 6. Dimensionless mean angular velocity \( W \) of the granular phase versus dimensionless yield stress \( \tau_1 \).
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