New exact solutions of the standard pairing model for well-deformed nuclei

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A new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model is proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively. This new approach draws upon an observation that the original one-plus two-body problem in a k-particle Hilbert subspace can be mapped unto a one-body grand hard-core boson picture that can be solved step by step with a simple Bethe ansatz known from earlier work. Based on this new procedure, it is further shown that the extended pairing model for deformed nuclei [Phys. Rev. Lett. 92, 112503 (2004)] is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis show that the standard pairing model with the first step approximation displays similar pair structures of first few exact low-lying states of the model, which, therefore, provides a link between the two models.

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I. INTRODUCTION

Pairing is an important residual interaction when a mean-field approach is used as a starting approximation for a description of nuclear structure. In particular, pairing correlations are essential for a description of binding energies, odd-even effects, single-particle occupancies, excitation spectra, electromagnetic transition rates, beta-decay probabilities, transfer reaction amplitudes, low-lying collective modes, level densities, and moments of inertia, etc. [1–3] Two commonly used methods, the Bardeen-Cooper-Schrieffer (BCS) [4] and Hartree-Fock-Bogolyubov (HFB) [5] technique for finding approximate solutions are well known. The limitations of these methods, when applied in nuclear physics, are well understood, [6] which is also the case when using these methods to determine the energy spectra of nano-scale metallic grains. [6-7] Various procedures have been used to correct the approximation deficiencies, such as particle-number projected mean-field treatments, [8–10] the use of coherent states, [11] stochastic number projection techniques, [12] statistical descriptions, [13] treatments of the residual parts of the Hamiltonian in the random phase approximation, [14–15] and various recursive approaches. [16–17] Typically, however, these procedures have been found only limited applicability because the results usually yield insufficient accuracy. Other methods are likewise limited by their own sets of complications.

On the other hand, an exact treatment of the nuclear mean-field plus pairing type Hamiltonian was initiated by Richardson and Gaudin, known as the Richardson-Gaudin method. [18–21] Recently, extensions to the Richardson-Gaudin theory have been made using the Bethe ansatz methodology, [22–31] especially, an application of the Richardson-Gaudin solution to Sm isotopes with less than seven pairs of valence nucleons was made. [32] Though these approaches show that the mean-field plus pairing model is exactly solvable, the solutions are generally not simple and normally require extensive numerical work, especially when the number of levels and valence pairs are large in spite of the recent efforts in improving the procedure. [33] In [34], an extended pairing model with many-particle interaction terms was proposed, which can be solved based on a simpler Bethe ansatz, and describes even-odd mass differences in 154–171 Yb isotopes rather well. However, it was not clear why the pairing interaction strength in the model should be drastically reduced with increasing of the number of valence nucleon pairs in order to fit experimental data of even-odd mass differences. Moreover, since the standard form of pairing Hamiltonian is commonly adopted, it should be interesting to see whether the extended pairing model in some aspects covers main features of the standard pairing model.

In the following, a new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model will be proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively, which is shown in Sec. II. In Sec. III, as a demonstration of the procedure, a system with $p = 10$ levels will be analyzed. In Sec. IV, it will be shown that the extended pairing model for deformed nuclei is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis reveals that the standard pairing Hamiltonian with the first step approximation displays similar pair structures of first few exact low-lying states of the model, which, therefore, provides a link between the two models. A short discussion regarding implications of our results is given in Sec. V.
II. A NEW DIAGONALIZATION PROCEDURE FOR THE MEAN-FIELD PLUS STANDARD PAIRING HAMILTONIAN

The deformed mean-field plus standard pairing model Hamiltonian is given by

\[ \hat{H} = \sum_{j=1}^{p} \epsilon_j \hat{n}_j - GS^{+}S^{-} = \sum_{j=1}^{p} \epsilon_j \hat{n}_j - G \sum_{ij} S_{ij}^{+}S_{ij}^{-}, \]

(1)

where \( p \) is the total number of levels considered, \( G > 0 \) is the overall pairing strength, \( \{ \epsilon_j \} \) are single-particle energies taken from any deformed mean-field, such as the Nilsson model\(^{[35-37]} \) or the relativistic mean-field theory,\(^{[38]} \) \( \hat{n}_j = c_{j\uparrow}^{\dagger}c_{j\uparrow} + c_{j\downarrow}^{\dagger}c_{j\downarrow} \) is the fermion number operator for the \( j \)-th level, and \( S_{ij}^{+} = c_{i\uparrow}^{\dagger}c_{j\uparrow}^\dagger \) (\( S_{ij}^{-} = c_{i\downarrow}c_{j\downarrow}^\dagger \)) are pair creation (annihilation) operators. The up and down arrows in these expressions refer to time-reversed states. Since the formalism for even-odd systems is similar, in the following, we only focus on the even-even seniority zero case.

Since each deformed level can be occupied by no more than a single pair due to the Pauli principle, Hamiltonian (1) is also equivalent to a finite-site hard-core Bose-Hubbard model with infinite range hopping and infinite on-site repulsion. Let \( B_{i_1i_2\cdots i_k}^{\Gamma} = S_{i_1}^{+}S_{i_2}^{+}\cdots S_{i_k}^{+} \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq p \). In the following, we set \( \mu = (i_1i_2\cdots i_k) \) to be the \( \mu \)-th normal order sequence with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq p \). The operator \( B_{\mu}^{\dagger} \) can be regarded as a grand hard-core boson creation operator.\(^{[34]} \) The total number of these operators is \( p!/((p-k)!k!) \) which exactly equals the dimension of the Hilbert subspace of \( k \) pairs with no double occupancy.

For \( k \)-pair excitations, by using the standard second quantization formalism, the Hamiltonian (1) can effectively be reduced to the following ‘one-body’ Hamiltonian in the grand hard-core boson picture:

\[ \hat{H}_k = \sum_{\mu\nu} \langle \mu | \sum_{j=1}^{p} \epsilon_j \hat{n}_j | \nu \rangle B_{\mu}^{\dagger}B_{\nu} - G \sum_{\mu\nu} \langle \mu | S^{+}S^{-} | \nu \rangle B_{\mu}^{\dagger}B_{\nu}, \]

(2)

where \( B_{\mu} = (B_{\mu}^{\dagger})^\dagger \), because any \( k \)-pair eigenstate of (1) can be expanded in terms of single grand hard-core boson states \( \{ |\mu\rangle = B_{\mu}^{\dagger} |0\rangle \} \), where \( |0\rangle \) is the pair vacuum state. In (2), the mean-field one-body term in the Hilbert subspace spanned by \( \{ |\mu\rangle = B_{\mu}^{\dagger} |0\rangle \} \) is diagonal with the matrix elements

\[ \langle \mu | \sum_{j=1}^{p} \epsilon_j \hat{n}_j | \nu \rangle = \delta_{\mu\nu} 2\epsilon_{\mu} = \delta_{\mu\nu} 2 \sum_{t=1}^{k} \epsilon_{t}, \]

(3)

while the matrix elements of the pairing interaction term are

\[ \langle \mu | S^{+}S^{-} | \nu \rangle = \sum_{qp} \langle \mu | p/2-q \rho k \rangle Q \langle p/2-q \rho k | S^{+}S^{-} | p/2-q \rho k \rangle Q \langle p/2-q \rho k | \nu \rangle, \]

(4)

in which

\[ Q \langle p/2-q \rho k | S^{+}S^{-} | p/2-q \rho k \rangle Q = h_{k}^{(q)} = (k-q)(p-k-q+1) \]

(5)

is matrix element of \( S^{+}S^{-} \) in the Racah quasi-spin formalism\(^{[38]} \) with the total quasi-spin \( S_Q = p/2 - q \), where \( q = 0, 1, \cdots, \min[p,k,p-k] \), and \( \rho \) is an additional quantum number needed in distinguishing from different states with the same quasi-spin for given \( p \) and \( k \). For given \( p \), the total number of different quasi-spin states with the same \( S_Q = p/2 - q \) is given by\(^{[39]} \)

\[ \omega_q = \frac{(p-2q+1)!p!}{(p-q+1)(p-q)q!}. \]

(6)

Furthermore, \( \alpha_{\mu\nu}^{qp} = \langle \mu | p/2-q \rho k \rangle Q \) used in (4) is an overlap of the quasi-spin state \( |p/2-q \rho k\rangle Q \) with \( \mu \)-th single grand hard-core boson state \( |\mu\rangle \), which can be chosen as real. It can easily be verified that the total number of different quasi-spin states equals exactly to the dimension of the Hilbert subspace of \( k \) pairs with no double occupancy:
Thus, (2) can explicitly be written as

\[
\hat{H}_k = \sum_{\mu=1}^{d} 2\tilde{\epsilon}_\mu B_\mu^+ B_\mu - G \sum_{q=0}^{\min[k,p-k]} \omega_q \sum_{\rho=1}^{k} \sum_{\mu \nu} a_{\mu \rho}^q a_{\nu \rho}^q B_\mu^+ B_\nu^+.
\]

(8)

In order to diagonalize the Hamiltonian (8), let us consider a simpler Hamiltonian with only first term and \( q = 0 \) part in the second term of (8):

\[
h_0 = \sum_{\mu=1}^{d} 2\tilde{\epsilon}_\mu B_\mu^+ B_\mu - Gh_k^{(0)} \sum_{\mu \nu} a_{\mu \nu}^{01} B_\mu^+ B_\nu^+.
\]

(9)

As shown in [34, 40], The Hamiltonian (9) can be digonalized into the following form:

\[
h_0 = \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)})
\]

(10)

with

\[
D^+(E^{(\tau_0)}) = \sqrt{\frac{1}{N_{\tau_0}}} \sum_{\mu} \frac{\alpha_{\mu}^{01} \alpha_{\mu}^{01}}{2\tilde{\epsilon}_\mu - E^{(\tau_0)}} B_\mu^+.
\]

(11)

where \( E^{(\tau_0)} \) is the \( \tau_0 \)-th root of the following equation:

\[
Gh_k^{(0)} \sum_{\mu} \frac{(\alpha_{\mu}^{01})^2}{2\tilde{\epsilon}_\mu - E^{(\tau_0)}} = 1,
\]

(12)

and \( N_{\tau_0} \) is the normalization constant obtained from

\[
\sum_{\mu} \frac{(\alpha_{\mu}^{01})^2}{(E^{(\tau_0)} - 2\tilde{\epsilon}_\mu)(E^{(\tau_0)} - 2\tilde{\epsilon}_\mu)} = \delta_{\tau_0 \tau_0} N_{\tau_0}.
\]

(13)

In this case, (12) will provide with exactly \( d \) different roots \( E^{(\tau_0)} \) as long as all combinations of the single-particle energies \( \tilde{\epsilon}_\mu = \sum_{j=1}^{k} \epsilon_j \) are different for all \( k \)-pair excitation cases. Fortunately, this is always the case when single-particle energies \( \{\epsilon_j\} \) are generated from any deformed mean-field theory.

Since (8) and (9) should be diagonalized within the single grand hardcore boson subspace spanned by \( \{B_\mu^+ | 0 \} \), the effective commutation relations needed to prove that (9) can indeed be expressed in the form shown in (10) are

\[
[B_\nu^+, B_\mu^+] = \delta_{\mu \nu},
\]

(14)

which are only valid when they are applied onto the vacuum state. Using (9) and the ansatz (11), we have

\[
[h_0, D^+(E^{(\tau_0)})] = E^{(\tau_0)} D^+(E^{(\tau_0)}) - \sqrt{\frac{1}{N_{\tau_0}}} \left(1 - Gh_k^{(0)} \sum_{\mu} \frac{(\alpha_{\mu}^{01})^2}{2\tilde{\epsilon}_\mu - E^{(\tau_0)}} \right) \sum_{\nu} \alpha_{\nu}^{01} B_\nu^+.
\]

(15)
Though a direct proof is in demand, it can be checked numerically with any set of parameters, \(G\), \((\alpha^0_\mu, \alpha^{01}_\mu, \alpha^1_\mu)\), and \(\{\bar{\epsilon}_\mu\}\) with \(\bar{\epsilon}_1 \neq \bar{\epsilon}_2 \cdots \neq \bar{\epsilon}_d\), that the orthonormal condition (13) is automatically satisfied when \(E^{(\tau_0)}\) satisfies (12). Therefore, (9) can indeed be expressed as that shown in (10) as long as (12) is satisfied.

In order to simplify our expression, in the following, the indices \((q, \rho)\) are relabeled by \(r\) with \(r = (q, \rho)\). Thus, the Hamiltonian (8) can be rewritten as

\[
\hat{H}_k = \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - G \sum_{r=1}^{d-1} h^{(r)}_k \sum_{\mu\nu} \alpha^r_\mu \alpha^r_\nu B^+_{\mu \nu} B_{\nu}.
\]

(16)

Then, we similarly have

\[
\sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - G h^{(1)}_k \sum_{\mu\nu} \alpha^1_\mu \alpha^1_\nu B^+_{\mu \nu} B_{\nu} = \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - G h^{(1)}_k \sum_{\tau_0 \tau_0^r_0} \Lambda_{\tau_0} \Lambda_{\tau_0^r_0} D^+(E^{(\tau_0)}) D(E^{(\tau_0)_r}) = \sum_{\tau_1} E^{(\tau_1)} D^+(E^{(\tau_1)}) D(E^{(\tau_1)}),
\]

(17)

where

\[
\Lambda_{\tau_0} = \sqrt{\frac{1}{N_{\tau_0}}} \sum_{\mu} \frac{\alpha^0_\mu \alpha^1_\mu}{2\bar{\epsilon}_\mu - E^{(\tau_0)}},
\]

(18)

\[
D^+(E^{(\tau_1)}) = \sqrt{\frac{1}{N_{\tau_1}}} \sum_{\tau_0} \frac{\Lambda_{\tau_0}}{E^{(\tau_0)} - E^{(\tau_1)}} D^+(E^{(\tau_0)}),
\]

(19)

\(E^{(\tau_1)}\) is the \(\tau_1\)-th root of the following equation:

\[
G h^{(1)}_k \sum_{\tau_0} \frac{(\Lambda_{\tau_0})^2}{E^{(\tau_0)} - E^{(\tau_1)}} = 1,
\]

(20)

and \(N_{\tau_1}\) is the normalization constant obtained from

\[
\sum_{\tau_0} \frac{(\Lambda_{\tau_0})^2}{(E^{(\tau_0)} - E^{(\tau_1)})(E^{(\tau_0)} - E^{(\tau_1))})} = \delta_{\tau_1 \tau_1} N_{\tau_1}.
\]

(21)

Thus, using the results shown in (17)-(21) and following the above procedure consecutively, we finally have

\[
\hat{H}_k = \sum_{\tau_0} E^{(\tau_0)} D^+(E^{(\tau_0)}) D(E^{(\tau_0)}) - G \sum_{r=1}^{d-1} h^{(r)}_k \sum_{\mu\nu} \alpha^r_\mu \alpha^r_\nu B^+_{\mu \nu} B_{\nu} = \sum_{\tau_{d-1}} E^{(\tau_{d-1})} D^+(E^{(\tau_{d-1})}) D(E^{(\tau_{d-1})}),
\]

(22)

where

\[
D^+(E^{(\tau_{d-1})}) = \sqrt{\frac{1}{N_{\tau_{d-1}}} \sum_{\tau_{d-2}} \frac{\Lambda_{\tau_{d-2}}}{E^{(\tau_{d-2})} - E^{(\tau_{d-1})}}} D^+(E^{(\tau_{d-2})})
\]

(23)

with

\[
\Lambda_{\tau_s} = \sqrt{\frac{1}{N_{\tau_s}}} \sum_{\tau_0 \tau_1 \cdots \tau_{s-1}} \prod_{\nu=0}^{s-1} \frac{\Lambda_{\tau_\nu}}{E^{(\tau_\nu)} - E^{(\tau_{\nu+1})}} \sum_{\mu} \frac{\alpha^{s+1}_\mu \alpha^{01}_\mu}{2\bar{\epsilon}_\mu - E^{(\tau_0)}},
\]

(24)
\[ D^+(E^{(\tau_{s+1})}) = \sqrt{\frac{1}{N^{\tau_{s+1}}}} \sum_{\tau_s} A_{\tau_s} E^{(\tau_s)} - E^{(\tau_{s+1})} D^+(E^{(\tau_s)}), \]  
(25)

\[ G h^{(s+1)}_k \sum_{\tau_s} \frac{(A_{\tau_s})^2}{E^{(\tau_s)} - E^{(\tau_{s+1})}} = 1, \]  
(26)

and

\[ \sum_{\tau_s} \frac{(A_{\tau_s})^2}{E^{(\tau_s)} - E^{(\tau_{s+1})}} = \delta_{\tau_{s+1}} N^{\tau_{s+1}}, \]  
(27)

for \( s = 0, 1, 2, \cdots, d - 2 \). Hence, after \( d \) steps, the Hamiltonian (8) is diagonalized as shown in (22), of which eigenstate is

\[ |k, \tau_{d-1}\rangle = D^+(E^{(\tau_{d-1})})|0\rangle \]  
(28)

with the corresponding eigen-energy \( E^{(\tau_{d-1})} \).

This new step-by-step diagonalization procedure needs at most \( d \) steps to get final exact results, but in each step the corresponding Bethe ansatz equation (26) contains only one variable, of which roots can easily be obtained numerically similar to what is required in the extended pairing model proposed previously, and in the TDA and RPA approximations with separable potentials. Though this method may be unpractical for large size systems because one needs to get all \( d \) roots from the equation (26) in each step, this procedure can also be used to check contributions from pairing potential in the Racah quasi-spin formalism for different \( q \) of the second term in (8), and is certainly applicable to relatively small systems. Actually, for \( k \) pair excitation, though each term with different \( q \) from the second term of (8) will contribute to the final eigen-energy and correlate with eigenstates, the first few of these terms are key to determine properties of the first few low-lying states of the model as will be shown in the next section.

III. A NUMERICAL EXAMPLE FOR \( p = 10 \)

In this section, we will apply this new step-by-step diagonalization procedure to the deformed mean-field plus standard pairing model for \( p = 10 \) levels with number of pairs \( k = 1, 2, \cdots, 10 \), in which the single particle energies are given by \( \epsilon_i = i + \chi_i \) for \( i = 1, 2, \cdots, 10 \), where \( \chi_i \) are random numbers within the interval \((0, 1)\) to avoid accidental degeneracy, and the pairing strength is set to be \( G = 0.5 \). Since \( h^{(0)}_k > h^{(1)}_k > \cdots > h^{d-1}_k \) is always satisfied, the lowest quasi-spin term with \( q = 0 \) from the pairing potential should be most important to the first few of eigenstates of the model, which is indeed the case as can be seen from results shown in Table I.

In each step, we need to calculate the overlaps of the quasi-spin states \(|p/2 - q \rho k\rangle_Q\) with \( \mu \)-th single grand hard-core boson states \(|\mu\rangle\). For \( q = 0 \), the \( k \)-pair state

\[ |p/2 \ k\rangle_Q = \sqrt{(p-k)!/(k!)^2} (S^+)^k|0\rangle = \sqrt{k!(p-k)!/p!} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} S^+_{i_1} S^+_{i_2} \cdots S^+_{i_k}|0\rangle = \sqrt{1/d} \sum_{\mu} B^+_{\mu}|0\rangle \]  
(29)

is the eigen-state of the operator \( S^+S^- \). Thus we have

\[ \alpha_{01}^0 = \langle \mu | p/2k \rangle_Q = \sqrt{1/d}. \]  
(30)

For \( q \geq 1 \), the quasi-spin states \(|p/2 - q \rho k\rangle_Q\) can be obtained by directly diagonalizing \( S^+S^- \) as shown in (5), or by using representation theory of \( SU(2) \times S_p \) summarized in [39]. Then, one can use them to calculate the overlaps \( \alpha_{\mu q}^0 = \langle \mu | p/2 - q \rho k \rangle_Q \).

In Table I, we list first 5 eigenenergies and overlaps of the eigenstates with the corresponding exact ones for number of pairs \( k = 1, 2, \cdots, 10 \) calculated with only \( h^{(0)} \) term involved, which is called the first step approximation. With the first step approximation, it can be seen from Table I that results of \( k = 1 \) and \( k = 10 \) cases are exact because
states mainly correlate with high excited states of the model and keep low part of the spectrum less affected. Diagonalization steps though there is still deviation in eigenenergies, which shows that the first few from Table II that the overlaps of the first 4 eigenstates with the corresponding exact ones will reach 99% after three steps obtained from the first step approximation with the corresponding exact ones, in which the single-particle energies $\epsilon_1 = 1.706$, $\epsilon_2 = 2.754$, $\epsilon_3 = 3.440$, $\epsilon_4 = 4.349$, $\epsilon_5 = 5.743$, $\epsilon_6 = 6.604$, $\epsilon_7 = 7.591$, $\epsilon_8 = 8.959$, $\epsilon_9 = 9.335$, $\epsilon_{10} = 10.125$, and pairing strength $G = 0.5$, where the single-particle energies and $G$ are given in arbitrary units.

| $k$ | exact | approx. | olp | exact | approx. | olp | exact | approx. | olp | exact | approx. | olp |
|-----|-------|--------|-----|-------|--------|-----|-------|--------|-----|-------|--------|-----|
| $E_1$ | 2.255 | 2.255 | 100% | 6.662 | 8.133 | 98% | 12.873 | 15.423 | 93% | 21.101 | 24.272 | 91% |
| $E_2$ | 4.797 | 4.797 | 100% | 8.999 | 9.808 | 97% | 15.508 | 17.213 | 92% | 24.293 | 26.909 | 90% |
| $E_3$ | 6.438 | 6.438 | 100% | 10.756 | 11.392 | 78% | 16.961 | 18.576 | 89% | 26.161 | 28.312 | 65% |
| $E_4$ | 8.272 | 8.272 | 100% | 13.128 | 12.261 | 75% | 17.973 | 19.832 | 82% | 26.176 | 29.058 | 67% |
| $E_5$ | 10.969 | 10.969 | 100% | 13.375 | 13.625 | 85% | 18.936 | 20.598 | 65% | 27.621 | 29.882 | 62% |

| $k$ | exact | approx. | olp | exact | approx. | olp | exact | approx. | olp | exact | approx. | olp |
|-----|-------|--------|-----|-------|--------|-----|-------|--------|-----|-------|--------|-----|
| $E_6$ | 44.638 | 48.868 | 91% | 59.532 | 63.713 | 96% | 76.870 | 79.982 | 96% | 95.625 | 97.091 | 95% |
| $E_7$ | 47.415 | 50.152 | 93% | 62.610 | 65.911 | 93% | 78.949 | 82.641 | 97% | 101.540 | 105.221 | 98% |
| $E_8$ | 49.165 | 52.261 | 81% | 63.852 | 67.440 | 82% | 80.277 | 83.861 | 81% | 116.212 | 116.212 | 100% |
| $E_9$ | 51.728 | 54.227 | 74% | 65.263 | 69.651 | 55% | 83.148 | 86.404 | 75% | 130.440 | 130.440 | 100% |

TABLE II: First 5 eigenenergies of the standard pairing model with $p = 10$ levels for $k = 0, 1, 2, \cdots, 10$ obtained from the first step approximation (appro.) and compared with the corresponding exact results (exact), and overlap (olp) of the eigenstates obtained from the first step approximation with the corresponding exact ones, in which the single-particle energies $\epsilon_1 = 1.706$, $\epsilon_2 = 2.754$, $\epsilon_3 = 3.440$, $\epsilon_4 = 4.349$, $\epsilon_5 = 5.743$, $\epsilon_6 = 6.604$, $\epsilon_7 = 7.591$, $\epsilon_8 = 8.959$, $\epsilon_9 = 9.335$, $\epsilon_{10} = 10.125$, and pairing strength $G = 0.5$, where the single-particle energies and $G$ are given in arbitrary units.

| $k$ | 1st step | 2nd step | 3rd step | 4th step | exact |
|-----|--------|--------|--------|--------|------|
| $q=0$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
| $E_1$ | 35.748 | 88% | 33.862 | 98% | 32.481 | 99% | 32.176 | 99% | 31.856 | 100% |
| $E_2$ | 37.447 | 88% | 36.566 | 97% | 35.631 | 99% | 35.102 | 99% | 34.528 | 100% |
| $E_3$ | 39.209 | 83% | 38.398 | 90% | 37.451 | 99% | 36.847 | 99% | 36.407 | 100% |
| $E_4$ | 40.167 | 81% | 39.422 | 89% | 37.942 | 99% | 37.637 | 99% | 37.382 | 100% |
| $E_5$ | 41.206 | 55% | 40.821 | 57% | 39.779 | 97% | 38.819 | 97% | 38.828 | 100% |

$h^{(q)} = 0$ for $q \geq 1$. The approximate energy eigenvalues will gradually greater than the corresponding exact ones with increasing of the number of pairs $k$ since pairing potential terms $h^{(q)}$ with $q = 1, 2, \cdots, \min[p - k, k]$ will contribute more and more to the final eigenenergies. However, the overlaps of the eigenstates with the corresponding exact ones are always greater than 88% for the ground and the first excited states for any number of pairs $k$. Therefore, $h^{(0)}$ term of the pairing potential is dominant in determining pairing structure of the first two excitation states in the model though the corresponding energy eigenvalues are different from the exact ones.

Since the largest deviation of the energy eigenvalues from the exact ones occurs at the half-filling case, using the procedure shown in the previous section, we calculated energy eigenvalues step by step for $k = 5$ case with $q = 0, 1, \cdots, 4$ since $h^{(5)} = 0$, of which the results are shown in Table II. For given $q$, there are actually $\omega_q$ sub-steps involved in the diagonalization process according to the procedure shown in the previous section. It can be seen from Table II that the overlaps of the first 4 eigenstates with the corresponding exact ones will reach 99% after three diagonalization steps though there is still deviation in eigenenergies, which shows that the first few $h^{(q)}$ terms with $q = 0, 1, 2$ are key to determine pair structure of the first few low-lying states of the model. While high-lying quasi-spin states mainly correlate with high excited states of the model and keep low part of the spectrum less affected.

IV. COMPARISON WITH THE EXTENDED PAIRING MODEL

For $k$ pair excitations, if only $h^{(0)}$ term from the standard pairing potential is considered for the standard pairing Hamiltonian (8), namely
\[ \hat{H}^{(1)}_k = \sum_{\mu=1}^{d} 2\epsilon_\mu B^+_\mu B_\mu - G(k(p-k+1)/d) \sum_{\mu\nu} B^+_\mu B_\nu, \]  

(31)

following (9)-(13), eigenstate of (31) can be written as

\[ |k; \tau\rangle = \sum_{\mu=1}^{d} \frac{1}{2\epsilon_\mu - E(\tau)} B^+_\mu |0\rangle, \]

(32)

where \( \tau \) is an additional quantum number used to distinguish different excitation states, and \( E(\tau) \) is an unknown variable to be determined in diagonalizing (31). In solving the following eigen-equation

\[ \hat{H}^{(1)}_k |k; \tau\rangle = E^{(\tau)}_k |k; \tau\rangle \]

(33)

with \( B_\mu B^+_\mu |0\rangle = \delta_{\mu\nu} |0\rangle \), it shows that the variable \( E(\tau) \) must satisfy the following equation:

\[ G(k(p-k+1)/d) \sum_{\mu=1}^{d} \frac{1}{2\epsilon_\mu - E(\tau)} = 1, \]

(34)

and the eigen-energy \( E^{(\tau)}_k = E^{(\tau)} \). Thus, the additional quantum number \( \tau \) labels different roots of (34). This is the so-called the first step approximation shown in section II. The solution is complete so long as all combinations of the single-particle energies \( \sum_{\tau=1}^{k} \epsilon_\tau \) are different for all \( k \)-pair excitation cases. Fortunately, this is always the case when single-particle energies \( \{\epsilon_\tau\} \) are generated from any deformed mean-field theory. Since the single grand particle energies \( 2\epsilon_\mu \) are all different, there are exactly \( p!/(p-k)!k!l! \) distinct roots in (34). The resultant eigenstates (32), which are mutually orthogonal but not normalized, satisfy

\[ (k; \tau |k; \tau') = \delta_{\tau\tau'} N_\tau, \]

(35)

where

\[ N_\tau = \sum_{\mu=1}^{d} \frac{1}{(2\epsilon_\mu - E(\tau))^2}. \]

(36)

It follows that the normalized eigenstate can be expressed as \( |k; \tau\rangle = \sqrt{1/N_\tau} |k; \tau\rangle \).

As shown in [34], a Nilsson mean-field plus extended pairing interaction Hamiltonian

\[ \hat{H}_{ex} = \sum_{j=1}^{p} \epsilon_j n_j - G_{ex} \sum_{i,j} S^+_i S_j - G_{ex} \left( \sum_{\rho=2}^{\infty} \frac{1}{(\rho!)^2} \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2\rho}} S^+_{i_1} S^+_{i_2} \cdots S^+_{i_{2\rho}} S_{i_{2\rho+1}} S_{i_{2\rho+2}} \cdots a_{i_{2\rho}} \right), \]

(37)

where no pair of indices among \( \{i_1, i_2, \cdots, i_{2\rho}\} \) are the same for any \( \rho \), can also be solved exactly by using a simple Bethe ansatz that is similar to what is proposed in this work. Besides the usual Nilsson mean-field and the standard pairing interaction, this form includes many-pair hopping terms that allow nucleon pairs to simultaneously scatter (hop) between and among different Nilsson levels. Furthermore, the extended pairing interaction Hamiltonian (37) can be used to describe even-odd mass differences rather well as long as the extended pairing interaction strength \( G_{ex} \) decreases with an increasing number of pairs \( k \). It follows from this that it is interesting to compare results of the deformed mean-field plus standard pairing Hamiltonian (1) with those from the extended pairing model [34]. And indeed, it is not difficult to show that the expressions for eigenstates of the extended pairing Hamiltonian and those of the standard pairing Hamiltonian (1) in the first step approximation are the same. For \( k \)-pair excitations, the eigenenergies \( E^{(\tau)}_k(\text{ex}) \) of the extended pairing Hamiltonian (37) are given by

\[ E^{(\tau)}_k(\text{ex}) = E^{(\tau)}_k - (k-1)G_{ex}, \]

(38)
where $E_{ex}^\tau$ is the $\tau$-th root of the Bethe ansatz equation,

$$G_{ex} \sum_{\mu=1}^{d} \frac{1}{2\epsilon_{\mu} - E_{ex}^{(\tau)}} = 1. \quad (39)$$

A comparison of (39) with the Bethe ansatz equation (32) for the standard pairing Hamiltonian in the first step approximation (31) shows that the two Hamiltonians yield exactly the same excitation energies and the corresponding eigenstates so long as the parameter $G_{ex}$ in the extended pairing model and the parameter $G$ in the standard pairing Hamiltonian (1) satisfy the following relation:

$$G_{ex} = ((p - k)!/(p - k + 1)k/p)! G. \quad (40)$$

Furthermore, while the ground states of the two Hamiltonians are also the same, the ground-state energies are different. However, once the overall pairing strength $G$ is fixed, and the parameter $G_{ex}$ is chosen according to (40), it is easy to show that the difference between the ground-state energy of the extended model and that of the standard pairing model in the first step approximation is given by

$$E_k^{(g)}(ex) - E_k^{(g)} = -(k - 1)G_{ex}. \quad (41)$$

This expression shows that the extended pairing interaction contributes a little more attraction among valence pairs than the standard pairing interaction in the first step approximation, but reproduces excitation energies exactly the same as those in the standard pairing model with first step approximation. Since $G_{ex}$ decreases drastically with increasing of $k$ toward the half-filling, the ground state energy difference of the two Hamiltonians becomes negligible with increasing number of pairs $k$ with $k \leq |p/2| - 1$ when $p$ is even, and $k \leq |p/2|$ when $p$ is odd, where $|x|$ denotes integer part of $x$.

As an example, the ground-state energy difference (41) of the two Hamiltonians in the sixth (82-126) major shell with the standard pairing strength $G = 0.2\text{MeV}$, which is a typical parameter value for describing deformed nuclei in this region, shows that the ground-state energy difference of the two Hamiltonians are rather small in this case. The largest deviation of the ground-state energy of the two Hamiltonians is at $k = 2$ with $E_k^{(g)}(ex) - E_k^{(g)} = -36.3636\text{keV}$. Notwithstanding, since the only difference between the two Hamiltonians, so long as $G_{ex}$ is taken to be related to $G$ by prescription (40), is in the overall binding energy, and since an analytic expression for this difference in also known in terms of $G_{ex}$ through (41), for practical purposes the two Hamiltonians yield the same results, even though the Hamiltonians are quite different. This in itself is interesting, since it shows that a many-pair interaction Hamiltonian can have identical solutions to the two-pair interaction with truncations. Obviously, it follows that for such systems the structure of fixed-Z (isotopic) and fixed-N (isotonic) chains follow solely from the structure the simplest single-pair member of the chain and simple “pair-counting” factors related to the pairing interaction strength and single-particle energies.

Thus, we conclude that, basically, the extended pairing model is different from the standard pairing model. However, if only a first few eigenstates are considered, the pair structure of these states in the two models are similar, especially in ground state, as can be seen from analysis of the overlaps in the previous section. It can be expected that the difference of the two models will be negligible when number of pairs $k$ or pairing interaction strength $G$ is small. In addition, since the extended pairing model can be solved exactly with a single one variable equation (39), which is simpler than the Richardson-Gaudin equations with $k$ variables for the standard pairing Hamiltonian, the extended pairing model can be applied to relative large systems, especially when one only want to know a first few low-lying eigenstates and corresponding eigenenergies.

V. CONCLUSION

A new step-by-step diagonalization procedure for evaluating exact solutions of the nuclear deformed mean-field plus pairing interaction model is proposed via a simple Bethe ansatz in each step from which the eigenvalues and corresponding eigenstates can be obtained progressively. This new approach draws upon an observation that the original one- plus two-body problem in a $k$-particle Hilbert subspace can be mapped unto a one-body grand hard-core boson picture that can be solved step by step with a simple Bethe ansatz known from earlier work, in which one only needs to solve a single variable non-linear equation instead of a set of coupled non-linear equations with $k$ variables as is required, for example, within the framework of the well-known Richardson-Gaudin method. Though this method
may be unpractical for large size systems because one needs to get all $d$ roots from the Bethe ansatz equation in each step, this procedure can be used to check contributions from pairing potential in the Racah quasi-spin formalism, and is certainly applicable to relatively small systems.

As is shown in the example with $p = 10$ levels, though each term with different $q$ from the pure pairing interaction will contribute to the final eigen-energy and correlate with eigenstates, a first few of these terms are key to determine a first few low-lying states of the model. While high-lying quasi-spin states mainly correlate with high excited states of the model and keep low part of the spectrum less affected.

Based on this new procedure, it is further shown that the extended pairing model for deformed nuclei\cite{34} is similar to the standard pairing model with the first step approximation, in which only the lowest energy eigenstate of the standard pure pairing interaction part is taken into consideration. Our analysis show that the standard pairing Hamiltonian with the first step approximation displays similar pair structures of a first few low-lying states of the standard pairing model, which, therefore, provides a link between the two models.

Furthermore, the new method proposed is not limited to the deformed mean-field plus pairing problem only, as it should also prove useful for solving a much large class of quantum many-body problems in which model Hamiltonians are described by

$$
\hat{H} = \hat{H}_0 + \lambda \hat{H}_1,
$$

where $\lambda$ is a real parameter, and $\hat{H}_0$ and $\hat{H}_1$ do not commute, $[\hat{H}_0, \hat{H}_1] \neq 0$. According to our procedure, if the particle number is a conserved quantity, and $\hat{H}_0$ and $\hat{H}_1$ can be diagonalized independently in a $k$-particle basis, then the Hamiltonian (42) is exactly solvable by using the step-by-step exact diagonalization procedure. Moreover, the method can also be extended to deal with Hamiltonians with more than two non-commutative terms by using a similar procedure consecutively. Research in this direction is in progress.

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