Achievable efficiencies for probabilistically cloning the states

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Abstract

We present an example of quantum computational tasks whose performance is enhanced if we distribute quantum information using quantum cloning. Furthermore we give achievable efficiencies for probabilistic cloning the quantum states used in implemented tasks for which cloning provides some enhancement in performance.

1. Introduction

Cloning is a type of quantum information processing tool. In 1982 Wootters and Zurek \cite{1} and Dieks \cite{2} independently discovered the no-cloning theorem, one of the first results stressing the peculiarities of quantum information. They showed that unlike classical information, it is impossible to make perfect copies of an unknown quantum state, i.e. qubits can not be copied. Since then quantum cloning has been studied intensively, and much effort has been put into developing optimal cloning processes \cite{3-14}. There are two main approaches to quantum cloning. The first one consists of using ancillary quantum systems and a global unitary operation to obtain multiple imperfect clones of a given, unknown quantum state. These universal quantum cloning machines (UQCM’s) were first invented by Bužek and Hillery \cite{3} and developed by other authors \cite{4-12}. The second kind of cloning procedure first designed by Duan and Guo \cite{13, 14} is nondeterministic, consisting in adding an ancilla, performing unitary operations and measurements, with a postselection of the measurement results. The resulting clones are perfect, but the procedure only succeeds with a certain probability $p < 1$, which depends on the particular set of the states that we are trying to clone. Recently, Galvão and Hardy discuss how quantum information distribution implemented with different types of quantum cloning procedures can improve the performance of some quantum computation tasks \cite{15}. Unfortunately in the second example they obtained the achievable efficiencies for probabilistically cloning the states by a numerical search. Evidently the numerical result is not an exact solution and this is what originally motivated the present work.

Our purpose in this paper is twofold. First we present an example of quantum computation tasks whose performance is enhanced if we distribute quantum information using quantum cloning. The second purpose of the paper is to provide achievable efficiencies for probabilistically cloning the states \cite{15} used in implemented tasks for which cloning provides some enhancement in performance.

2. An example with probabilistic cloning

In this section we give an example of quantum computation tasks that can be better performed if we make use of quantum cloning. The task relies on state-dependent probabilistic quantum cloning discussed by Duan and Guo \cite{13, 14}. Now we present our example by generalizing the second example of Ref. \cite{15} in which they discussed the functions that take two bits to one bit, to the case of three bits to one bit.

The quantum computational task is as follows. Suppose that we are given 3 quantum black-boxes. What each
black-box does is to accept four 2-level quantum systems as an input and apply a unitary operator to it, producing the evolved state as the output. We take the black-boxes to consist of arbitrary quantum circuit that query a given function only once. The query of function \( f \) is the unitary that performs \(|x\rangle\langle y| \rightarrow |x\rangle\langle y \oplus f(x)|\rangle\rangle\), where the symbol \( \oplus \) represent the bitwise XOR operation. Our task will involve determining two functionals, one depending only on \( f_0 \) and \( f_1 \), and the other on \( f_0 \) and \( f_2 \). We will prove that cloning offers an advantage which cannot be matched by any approach that does not resort to quantum cloning.

In order to precisely state our task, we start by considering all functions \( h_i \) which take three bits to one bit. We may represent each such function with eight bits \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, \) and \( a_8 \), writing \( h_{a_1a_2a_3a_4a_5a_6a_7a_8} \) to stand for the function \( h \) such that \( h(000) = a_1, h(001) = a_2, h(010) = a_3, h(011) = a_4, h(100) = a_5, h(101) = a_6, h(110) = a_7, h(111) = a_8 \). Now we define some sets of functions that will be useful in stating our task:

\[
S_{f_0} = \{h_{01000000}, h_{00110011}, h_{11000011}\}, \\
S_1 = \{h_{01010000}, h_{10110000}, h_{10001100}, h_{00011010}, h_{00010101}, h_{10000111}, h_{00011010}, h_{00011010}\}, \\
S_2 = \{h_{00000000}, h_{00001111}, h_{01010101}, h_{00111001}, h_{10001100}, h_{10100101}, h_{10101001}\}, \\
S_{f_1,2} = S_1 \cup S_2, \\
S_0 = S_00000000 \cup S_00011111 \cup S_01010101 \cup S_00110011 \cup S_10001100 \cup S_10100101 \cup S_11000011 \cup S_01101001 \cup S_{10100101}.
\]

Now we first randomly choose a function \( f_0 \in S_{f_0} \), then two other functions \( f_1 \) and \( f_2 \) are picked from the set \( S_{f_1,2} \), also at random but satisfying:

\[
f_0 \oplus f_1, \quad f_0 \oplus f_2 \in S_f. \quad (1)
\]

Here the symbol \( \oplus \) is addition modulo 2. The task will be to find in which of the eight sets \( S_{00000000}, S_{00001111}, S_{01010101}, S_{00110011}, S_{10001100}, S_{10100101} \) lie each of the functions \( f_0 \oplus f_1 \) and \( f_0 \oplus f_2 \), applying quantum circuits that query \( f_0, f_1, \) and \( f_2 \) at most once each. Our score will be given by the average probability of successfully guessing both correctly.

2.1 Score without cloning

Now we will give the attainable score if we do not resort to cloning. Just as \cite{15} the best no-cloning strategy goes as follows. Firstly, from the constraints given by Eq. (1) we note that both \( f_1 \) and \( f_2 \) must be in \( S_1 \) if \( f_0 = h_{01000000} \), and \( f_1 \) and \( f_2 \) must belong to \( S_2 \) if \( f_0 \) is either \( h_{00011001} \) or \( h_{11000011} \). Since \( f_0 \) were drawn from a uniformly random distribution, the probability of both \( f_1 \) and \( f_2 \) in \( S_2 \) is 2/3. Assume that it is the case, then we can discriminate between the two possibilities for \( f_0 \) with a single, classical function call. Furthermore, by using the quantum circuit in Fig.1 twice (once each with \( f_1 \) and \( f_2 \)) we can distinguish the eight possibilities for functions \( f_1 \) and \( f_2 \).

This happens because depending on which function in \( S_2 \) was queried, this quantum circuit results in one of the eight orthogonal states

\[
|\varphi_i\rangle = \frac{1}{2\sqrt{2}} \sum_{x=000}^{111} (-1)^{f_i(x)} |x\rangle. \quad (2)
\]

This allows us to determine functions \( f_0, f_1, \) and \( f_2 \) correctly with probability \( p = 2/3 \), in which case we can determine which sets contain \( f_0 + f_1 \) and \( f_0 + f_2 \) and accomplish our task. Even in the case where the initial
assuming about \( f_0 \) was wrong, we may still have guessed the right sets by chance; the chances of getting both right this way are 1/64. Thus, the best no-cloning average score is
\[
p_1 = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{64} = 0.671875.
\] (3)

### 2.2 Score with cloning

Next we will prove that we can do much better than that with quantum cloning. The idea is similar to Ref. 15, that is, to devise a quantum circuit that queries function \( f_0 \) only once, makes two clones of the resulting state, and then queries functions \( f_1 \) and \( f_2 \), one in each branch of the computation. Since we have some information about the state produced by one query of \( f_0 \), the probabilistic cloning machines investigated by Duan and Guo 13 will suit this task better.

The quantum circuit that we use to solve this problem is depicted in Fig.2.

Immediately after querying function \( f_0 \), we have one of three possible linearly independent states (each corresponding to one of the possible \( f_0 \)’s):
\[
|\Psi_1\rangle \equiv |h_{01000000}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle],
\] (4)
\[
|\Psi_2\rangle \equiv |h_{00110011}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle],
\] (5)
\[
|\Psi_3\rangle \equiv |h_{11000011}\rangle \equiv \frac{1}{2\sqrt{2}}[-(000) - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle].
\] (6)

The probabilistic cloning machines with different cloning efficiencies (defined as the probability of cloning successfully) for each of states 4-6 will be constructed. From Theorem 2 in Ref. 13 we obtain the following exact achievable efficiencies
\[
\gamma_1 \equiv \gamma(|h_{01000000}\rangle) = \frac{7}{127},
\] (7)
\[
\gamma_2 \equiv \gamma(|h_{00110011}\rangle) = \gamma_3 \equiv \gamma(|h_{11000011}\rangle) = \frac{112}{127},
\] (8)

which will be shown in next section.

After the cloning process a measurement on a "flag" subsystem is performed and the result will tell us whether the cloning was successful or not. For this particular cloning process, the probability of success is, on average, \( P_{\text{success}} = (\gamma_1 + \gamma_2 + \gamma_3)/3 = \frac{77}{127} \). If it was successful, then each of the cloning branches goes through the second part of the circuit in Fig.2, to yield one of the eight orthogonal states:
\[
|h_{00000000}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle],
\] (9)
\[
|h_{00001111}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle],
\] (10)
\[
|h_{01010101}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle],
\] (11)
\[
|h_{00111001}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle],
\] (12)
\[
|h_{10011001}\rangle \equiv \frac{1}{2\sqrt{2}}[-(000) + |001\rangle + |010\rangle - |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle],
\] (13)
\[
|h_{11000001}\rangle \equiv \frac{1}{2\sqrt{2}}[-(000) - |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle - |110\rangle - |111\rangle],
\] (14)
\[
|h_{01101001}\rangle \equiv \frac{1}{2\sqrt{2}}[(000) - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle],
\] (15)
\[ |h_{10100010}\rangle \equiv \frac{1}{\sqrt{2}}[-|000\rangle + |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle], \]  
\[ (16) \]

which can be discriminated unambiguously. Therefore, if the cloning process is successful, we manage to accomplish our task.

However, the cloning process may fail with probability \((1 - P_{\text{success}})\). If this happens, it is more likely to be \(h_{01000000}\) than the other two, because of the relatively low cloning efficiency for the state in Eq. (11), in relation to the states in Eqs. (5) and (6) [see Eqs. (7) and (8)]. If we then guess that \(f_0 = h_{01000000}\), we will be right with probability

\[ p_{01000000} = \frac{(1 - \gamma_1)}{(1 - \gamma_1 + (1 - \gamma_2) + (1 - \gamma_3))} = \frac{4}{5}. \]  
\[ (17) \]

What is more, we are still free to design quantum circuits to obtain information about \(f_1\) and \(f_2\), since at this stage we still have not queried them. Given our guess that \(f_0 = h_{01000000}\), only the eight functions in \(S_1\) can be candidates for \(f_1\) and \(f_2\), because of the constraints given by Eq. (11). These eight possibilities can be discriminated unambiguously by run a circuit like that of Fig.1 twice, once with \(f_1\) and once with \(f_2\). The circuit produces one of eight orthogonal states, each corresponding to one of the eight possibilities for \(f_i\). Therefore if our guess that \(f_0 = h_{01000000}\) was correct, we are able to find the correct \(f_1\) and \(f_2\) and therefore accomplish our task. In the case that \(f_0 \neq h_{01000000}\) after all, we may still have guessed the right sets by chance; a simple analysis shows that this will happen with probability \(1/64\).

The above considerations leads to an overall probability of success given by

\[ p_2 = P_{\text{success}} + (1 - P_{\text{success}})[p_{01000000} + (1 - p_{01000000}) \frac{1}{64}] \]
\[ = \frac{22 + 21(\gamma_2 + \gamma_3)}{64} \approx 0.92249 \]
\[ > p_1 = 0.671875, \]  
\[ (18) \]

thus showing that this cloning approach is more efficient than the previous one, which does not use cloning.

### 2.3 Exact achievable efficiencies

Here we present the analytic solution of achievable efficiencies for cloning the state Eqs. (11)-(13). As stated above we use \(\gamma_1 \equiv \gamma(|h_{01000000}\rangle), \gamma_2 \equiv \gamma(|h_{01010011}\rangle), \gamma_3 \equiv \gamma(|h_{11000011}\rangle)\) to express the achievable efficiencies, and let \(|P^{(1)}\rangle, |P^{(2)}\rangle, |P^{(3)}\rangle\) be normalized states of the flag \(P\). \(P_{ij}\) denotes the inner product \(\langle P^{(i)}|P^{(j)}\rangle\) between \(|P\rangle\) and \(|P_j\rangle, i, j = 1, 2, 3\). Clearly, \(|P_{ij}| \leq 1\). Suppose the \(3 \times 3\) matrices \(X^{(1)} = [\langle \Psi_i|\Psi_j\rangle], X^{(2)} = [\langle \Psi_i|\Psi_j\rangle^2 P_{ij}]\) and the diagonal efficiency matrix \(\Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3)\), then

\[ X^{(1)} - \sqrt{\Gamma} X^{(2)} \sqrt{\Gamma} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\gamma_1}{\sqrt{16}} P_{12} & \frac{\sqrt{17}}{\sqrt{16}} P_{13} \\ \frac{\sqrt{17}}{\sqrt{16}} P_{12}^* & \frac{\gamma_2}{16} P_{12}^* + \frac{\gamma_3}{16} P_{13} \\ \frac{\gamma_3}{\sqrt{16}} P_{12}^* & 0 & \frac{1}{\sqrt{16}} P_{13} \end{pmatrix} \]
\[ = \begin{pmatrix} 1 - \frac{\gamma_1}{\sqrt{16}} P_{12} & -\frac{1}{4} - \frac{\sqrt{17}}{\sqrt{16}} P_{12} & \frac{1}{4} - \frac{\sqrt{17}}{\sqrt{16}} P_{13} \\ \frac{1}{4} - \frac{\sqrt{17}}{\sqrt{16}} P_{12}^* & 1 - \frac{\gamma_2}{\sqrt{16}} P_{12}^* & 0 \\ \frac{1}{4} - \frac{\sqrt{17}}{\sqrt{16}} P_{13}^* & 0 & 1 - \gamma_3 \end{pmatrix} \]

Theorem 2 of Ref. [13] provides us with inequalities

\[ 1 - \gamma_1 \geq 0, \]  
\[ (1 - \gamma_1)(1 - \gamma_2) - \frac{1}{4} + \frac{1}{16} \sqrt{\gamma_1 \gamma_2 P_{12}} \geq 0, \]  
\[ (1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3) - (1 - \gamma_3) \frac{1}{4} + \frac{1}{16} \sqrt{\gamma_1 \gamma_2 P_{12}} \geq 0, \]  
\[ (1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3) \frac{1}{4} + \frac{1}{16} \sqrt{\gamma_1 \gamma_2 P_{12}} \geq 0, \]  
\[ (20) \]  
\[ (21) \]
which allow us to derive achievable efficiencies for the probabilistic cloning process. According to the rule stated in above section (see Eq. (15)) the overall probability (score) of success with the help of probabilistic cloning is given by

\[
p_2 = p_{\text{success}} + (1 - p_{\text{success}})[p_{01000000} + (1 - p_{01000000}) \frac{1}{64}]
\]

\[
= \frac{3}{72} + 21 \frac{(2 + y)^2 - 2}{s} + (1 - \frac{3}{72} - \gamma_1 - \gamma_2 - \gamma_3 \frac{1}{64})
\]

From above equation we know that we should find the maximum of \( \gamma_2 + \gamma_3 \) satisfying Eqs. (19) – (21).

In the following, we show that the maximum of \( \gamma_2 + \gamma_3 \) must be greater than or equal to \( \frac{224}{127} \). We consider the case \( \gamma_2 = \gamma_3 \). In this case, there is

\[
(1 - \gamma_1)(1 - \gamma_2) - \frac{1}{4} + 1 \sqrt{\gamma_1 \gamma_2 P_{12}}^2 - \frac{1}{4} - 1 \sqrt{\gamma_1 \gamma_2 P_{13}}^2 \geq 0
\]

which implies that

\[
\frac{7}{8} - qx + sx^2 \geq y \geq 2x \geq 0,
\]

where \( P_{12} = a + bi, P_{13} = c + di, q = \frac{1}{32}(a - c), s = 1 - \frac{1}{256}(a^2 + b^2 + c^2 + d^2), y = \gamma_1 + \gamma_2, \text{ and } x = \sqrt{\gamma_1 \gamma_2}. \) It is not difficult to prove that

\[
127 \leq s \leq 1, \quad -\frac{1}{16} \leq q \leq \frac{1}{16}.
\]

Since \( \frac{7}{8} - q + s \leq 2, \text{ and } 0 \leq x \leq 1, y = \frac{7}{8} - qx + sx^2 \text{ and } y = 2x \) have one intersection point

\[
(x_0, y_0) = \left( \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{2s}, \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{s} \right).
\]

The region in \( x-y \) plane and the region in \( q-s \) plane governed by Eq. (21) are the shaded area in Fig.3 and in Fig.4 respectively.

From \( y = \gamma_1 + \gamma_2 \) and \( x = \sqrt{\gamma_1 \gamma_2} \) we have

\[
\gamma_1 = \frac{1}{2}(y - \sqrt{y^2 - 4x^2}), \quad \gamma_2 = \frac{1}{2}(y + \sqrt{y^2 - 4x^2}).
\]

This implies that \( \gamma_2 \) is a decreasing function of \( x \) when \( y \) is definite, so the maximum of \( \gamma_2 \) should occur in the curve

\[
\frac{7}{8} - qx + sx^2 = y,
\]

that is, the maximum of \( \gamma_2 \) must be the point such that \( \frac{d\gamma_2}{dx} = \frac{\partial\gamma_2}{\partial y} \frac{dy}{dx} + \frac{\partial\gamma_2}{\partial x} \frac{dx}{dx} = 0 \) (i.e. \( x_1 = \frac{5}{8} + q^2 - 4 + \sqrt{(\frac{7}{8} + q^2 - 4)^2 - 144q^2}, \) \( y_1 = \frac{7}{8} - qx_1 + sx_1^2 \)). Thus, the maximum of \( \gamma_2 \) in the plane \( \gamma_2 = \gamma_3 \) is

\[
\gamma_2 = \frac{1}{2} \left\{ \frac{7}{8} - qx_1 + sx_1^2 + \sqrt{(\frac{7}{8} - qx_1 + sx_1^2)^2 - 4x_1^2} \right\}.
\]

where

\[
x_1 = \frac{\frac{7}{8} + q^2 - 4 + \sqrt{(\frac{7}{8} + q^2 - 4)^2 - 144q^2}}{48q}.
\]

Let

\[
w = w(q, s) = \frac{7}{8} - qx_1 + sx_1^2; \quad v = v(q, s) = x_1,
\]

then \( w_{s=1-2q^2} = \frac{9}{10} \sqrt{49 + 32q^2} - \frac{49}{10} \) when \( s = 1 - 2q^2 \); \( v_{s=1-2q^2} = \frac{9}{10} \sqrt{49 + 32q^2} + \frac{49}{10} \) and \( w_{s=1-2q^2} = \frac{7}{8} - \frac{q^2 - 4q^2 + \sqrt{49 + 32q^2} + (\frac{49}{10})^2}{127q^2} + \frac{8q^2 - 4q^2 + \sqrt{49 + 32q^2} + (\frac{49}{10})^2}{127q^2} \) when \( s = \frac{127}{128} \). The \( (v, w) \) region corresponding \( (q, s) \) region in Fig.4 is depicted in Fig.5.
Because $\gamma_2$ is a decreasing function of $v$ while $w$ is definite, the maximum of $\gamma_2$ must be in the left boundary curve $w = \frac{v}{2}$ in $v$-$w$ plane corresponding to the boundary $s = \frac{127}{128}$ in $q$-$s$ plane. By $\frac{dq_2}{dq} < 0$, the maximum of $\gamma_2$ should be at the point

$$q = -\frac{1}{16}, \quad s = \frac{127}{128}. \quad (31)$$

The exact maximum of $\gamma_2$ is

$$\gamma_2 \equiv \gamma(|h_{001100111}\rangle) = \frac{112}{127}, \quad (32)$$

$$\gamma_1 \equiv \gamma(|h_{01000000}\rangle) = \frac{7}{127}. \quad (33)$$

So we do find an exact solution of achievable efficiencies $\gamma_1, \gamma_2, \gamma_3$ satisfying $\gamma_2 = \gamma_3$, and prove that the maximum $\gamma_2 + \gamma_3$ must be greater than or equal to $\frac{224}{127}$.

3. Exact achievable efficiencies for probabilistically cloning the states of Ref. [15]

In this section we will give the exact achievable efficiencies for probabilistically cloning the states in the second example of Ref. [15].

In Ref. [15], the probabilistic cloning quantum states are

$$|h_1\rangle = |h_{0010}\rangle \equiv \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle), \quad (34)$$
$$|h_2\rangle = |h_{0101}\rangle \equiv \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle), \quad (35)$$
$$|h_3\rangle = |h_{1001}\rangle \equiv \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle - |11\rangle). \quad (36)$$

We can build probabilistic cloning machines with different cloning efficiencies for each of the states \[34, 35, 36\]. Let $\gamma_1 \equiv \gamma(|h_{0010}\rangle), \quad \gamma_2 \equiv \gamma(|h_{0101}\rangle), \quad \gamma_3 \equiv \gamma(|h_{1001}\rangle)$ be the achievable efficiencies, and $|P^{(1)}\rangle, \quad |P^{(2)}\rangle, \quad |P^{(3)}\rangle$ be normalized states of the flag $P$. $P_{ij}$ denotes the inner product between $|P_i\rangle$ and $|P_j\rangle$, $i, j = 1, 2, 3$. Clearly, $|P_{ij}| \leq 1$. Suppose

$$X^{(1)} = \begin{pmatrix} \langle h_1|h_1 \rangle & \langle h_1|h_2 \rangle & \langle h_1|h_3 \rangle \\ \langle h_2|h_1 \rangle & \langle h_2|h_2 \rangle & \langle h_2|h_3 \rangle \\ \langle h_3|h_1 \rangle & \langle h_3|h_2 \rangle & \langle h_3|h_3 \rangle \end{pmatrix},$$

$$X^{(2)}_P = \begin{pmatrix} \langle h_1|h_1 \rangle^2 P_{11} & \langle h_1|h_2 \rangle^2 P_{12} & \langle h_1|h_3 \rangle^2 P_{13} \\ \langle h_2|h_1 \rangle^2 P_{21} & \langle h_2|h_2 \rangle^2 P_{22} & \langle h_2|h_3 \rangle^2 P_{23} \\ \langle h_3|h_1 \rangle^2 P_{31} & \langle h_3|h_2 \rangle^2 P_{32} & \langle h_3|h_3 \rangle^2 P_{33} \end{pmatrix},$$

$$\sqrt{T} = \begin{pmatrix} \sqrt{\gamma_1} & 0 & 0 \\ 0 & \sqrt{\gamma_2} & 0 \\ 0 & 0 & \sqrt{\gamma_3} \end{pmatrix}$$

then

$$X^{(1)} - \sqrt{T} X^{(2)}_P \sqrt{T}^T = \begin{pmatrix} 1 - \gamma_1 & -\frac{1}{2} - \sqrt{\frac{\gamma_1}{\gamma_2}} P_{12} & -\frac{1}{2} - \sqrt{\frac{\gamma_1}{\gamma_3}} P_{13} \\ -\frac{1}{2} - \sqrt{\frac{\gamma_2}{\gamma_1}} P_{21} & 1 - \gamma_2 & 0 \\ -\frac{1}{2} - \sqrt{\frac{\gamma_3}{\gamma_1}} P_{31} & 0 & 1 - \gamma_3 \end{pmatrix}$$

Theorem 2 of Ref. [13] provides us with inequalities

$$1 - \gamma_1 \geq 0, \quad (37)$$
\( (1 - \gamma_1)(1 - \gamma_2) - \frac{1}{2} + \frac{1}{4} \sqrt{\gamma_1 \gamma_2} P_{12}^2 \geq 0, \)  
\( (1 - \gamma_1)(1 - \gamma_2)(1 - \gamma_3) - (1 - \gamma_2) \frac{1}{2} + \frac{1}{4} \sqrt{\gamma_1 \gamma_2} P_{12}^2 - (1 - \gamma_3) \frac{1}{2} + \frac{1}{4} \sqrt{\gamma_1 \gamma_3} P_{13}^2 \geq 0, \)  
which allow us to derive achievable efficiencies for the probabilistic cloning process. According to the rule specified in Ref. [15], the overall probability (score) of success with the help of probabilistic cloning is given by

\[
P_2 = p_{\text{success}} + (1 - p_{\text{success}})[p_{0010} + (1 - p_{0010}) \frac{1}{16}]
= \gamma_1 + \gamma_3 + (1 - \gamma_1 + \gamma_3 \gamma_2)\left[\frac{1}{3 - \gamma_1 - \gamma_2 - \gamma_3} + (1 - \gamma_1)\frac{1}{3 - \gamma_1 - \gamma_2 - \gamma_3}\right]
= [6 + 5(\gamma_1 + \gamma_3)]/16.
\]

From above equation we know that we should find the maximum of \( \gamma_2 + \gamma_3 \) satisfying Eqs. \[47-50\].

Our immediate goal is to prove that the maximum of \( \gamma_2 + \gamma_3 \) must be greater than or equal to 8/7. For this purpose, we discuss the problem in the plane \( \gamma_2 = \gamma_3 \). In this plane Eq. \[43\] becomes

\[
(1 - \gamma_1)(1 - \gamma_2) - \frac{1}{2} + \frac{1}{4} \sqrt{\gamma_1 \gamma_2} P_{12}^2 - \left(\frac{1}{2} + \frac{1}{4} \sqrt{\gamma_1 \gamma_2} P_{13}^2\right) \geq 0.
\]

Let

\[
P_{12} = a + bi, \quad P_{13} = c + di, \quad q = \frac{1}{2}(a + c), s = 1 - \frac{1}{16}(a^2 + b^2 + c^2 + d^2), x = \sqrt{\gamma_1 \gamma_2}, \quad y = \gamma_1 + \gamma_2.
\]

Then Eq. \[44\] can be rewritten concisely as

\[
\frac{1}{2} - qx + sx^2 \geq y.
\]

Obviously

\[
\frac{1}{2} - qx + sx^2 \geq y \geq 2x \geq 0.
\]

Here \( y = \frac{1}{2} - qx + sx^2 \) and \( y = 2x \) have one intersection point

\[
x_0 = \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{2s}, \quad y_0 = 2x_0.
\]

The proof is as follows: The intersection points of \( y = \frac{1}{2} - qx + sx^2 = \frac{1}{2} - \frac{1}{4}(c + a)x + \left[1 - \frac{1}{16}(a^2 + b^2 + c^2 + d^2)\right]x^2 \) and \( y = 2x \) are \( x_0 = \frac{2 + q + \sqrt{(2 + q)^2 - 2s}}{2s}, y_0 = 2x_0 \). From \( |P_{12}| \leq 1 \) and \( |P_{13}| \leq 1 \) it is seen \( |a + c| \leq 2 \) and \( 0 \leq a^2 + b^2 + c^2 + d^2 \leq 2 \), which imply that

\[
-\frac{1}{2} \leq q \leq \frac{1}{2}, \quad \frac{7}{8} \leq s \leq 1,
\]

thus \( x_0 = \frac{2 + q + \sqrt{(2 + q)^2 - 2s}}{2s} > 1 \) that contradict with \( x = \sqrt{\gamma_1 \gamma_2} \leq 1 \). Therefore \( y = \frac{1}{2} - qx + sx^2 \) and \( y = 2x \) have one intersection point \( x_0 = \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{2s}, y = 2x_0 \).

The region in \( x - y \) plane governed by Eq. \[44\] is shown in Fig.6, where \( x \) must satisfy

\[
0 \leq x \leq \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{2s} = x_0.
\]

Immediately \( \frac{2x_0}{q y} = \frac{1}{2s} |1 - \frac{2 + q - \sqrt{(2 + q)^2 - 2s}}{2 + q + \sqrt{(2 + q)^2 - 2s}}| \leq 0 \). It follows that when \( s \) is definite \( x_0 \) is a decreasing function of \( q \).

If \( q \) is definite (i.e. \( a + c = k \) is definite), then the maximum \( s \) is to make \( a^2 + b^2 + c^2 + d^2 = (a + c)^2 + b^2 + d^2 - 2ac \) minimum, which imply \( b = d = 0 \) and \( ac = \frac{(a + c)^2}{4} \). Therefore the curve of maximum \( s \) is \( s = 1 - \frac{1}{2} q^2 \) when \( q \) is
definite. While s minimum is to make \( a^2 + b^2 + c^2 + d^2 \) maximum, so minimum s is \( s = \frac{7}{8} \) in the case \( q \) is definite. The boundary of \( s \) and \( q \) is illustrated in Fig.7.

By \( x = \sqrt[3]{\gamma^2} \) and \( y = \gamma_1 + \gamma_2 \) we get

\[
\gamma_1 = \frac{1}{2}(y - \sqrt{y^2 - 4x^2}), \quad \gamma_2 = \frac{1}{2}(y + \sqrt{y^2 - 4x^2}). \tag{48}
\]

It follows that if \( y \) is definite, the smaller \( x \) is, the bigger \( \gamma_2 \) is, so the maximum of \( \gamma_2 \) should take place in the curve

\[
\frac{1}{2} - qx + sx^2 = y, \tag{49}
\]

that is, the maximum of \( \gamma_2 \) must be the point such that \( \frac{d\sigma_2}{dx} = \frac{\partial\sigma_2}{\partial y} + \frac{\partial\sigma_2}{\partial x} \cdot dx = 0 \) (i.e. \( x_1 = \frac{2s + q^2 - 4 + \sqrt{(2s + q^2 - 4)^2 - 8sq^2}}{4sq} \), \( y_1 = \frac{7}{8} - qx_1 + sx_1^2 \)). Thus, the maximum of \( \gamma_2 \) in the plane \( \gamma_2 = \gamma_3 \) is

\[
\gamma_2 = \frac{1}{2}(\frac{1}{2} - qx_1 + sx_1^2 + \sqrt{(\frac{1}{2} - qx_1 + sx_1^2)^2 - 4x_1^2}), \tag{50}
\]

where

\[
x_1 = \frac{2s + q^2 - 4 + \sqrt{(2s + q^2 - 4)^2 - 8sq^2}}{4sq}. \tag{51}
\]

Next we derive the maximum of \( \gamma_2 \). Let

\[
w = w(q, s) = \frac{1}{2} - qx_1 + sx_1^2; \quad v = v(q, s) = x_1. \tag{52}
\]

Now we change \((q, s)\) region to \((v, w)\) region. When \( s = 1 - \frac{1}{8}q^2 \), then \( v = -\frac{q}{2-q^2} \) and \( w = \frac{1}{2} + \frac{3q^2}{2(2-q^2)} \). From \( v = -\frac{q}{2-q^2}, |q| \leq \frac{1}{2} \) and \( v = x_1 \geq 0 \) we know that \( q = \frac{1-v}{2v} \), \( 0 \leq v \leq \frac{2}{7} \). Hence \( w_{s=1-\frac{1}{8}q^2} = -\frac{1}{4} + \frac{3}{7} \sqrt{1+8v^2} \) and \( 0 \leq v = \frac{1}{4} - \frac{1}{8} \leq \frac{1}{8} \) in the case \( s = 1 - \frac{1}{8}q^2 \). Note \( v_{s=\frac{1}{8}} = -\frac{q}{2+q^2+\sqrt{(q^2-\frac{1}{4})^2-7q^2}} \) and \( w_{s=\frac{1}{8}} = \frac{1}{2} - \frac{q}{2+q^2+\sqrt{(q^2-\frac{1}{4})^2-7q^2}} + \frac{7}{8}(\frac{q}{2+q^2+\sqrt{(q^2-\frac{1}{4})^2-7q^2}})^2 \) if \( s = \frac{7}{8} \). The \((v, w)\) region corresponding \((q, s)\) region is shown in Fig.8.

Since \( \gamma_2 \) is a decreasing function of \( v \) \( w \) is definite, from Eq.(50) we obtain that the maximum of \( \gamma_2 \) must appear in the left boundary curve \( w_{s=\frac{1}{8}} \) in \( v - w \) plane corresponding to the boundary \( s = \frac{7}{8} \) in \( q - s \) plane. It can be seen that

\[
\frac{d\gamma_2}{dq} < 0, \tag{53}
\]

while \( s = \frac{7}{8} \). Therefore the maximum of \( \gamma_2 \) should exist at the point

\[
q = -\frac{1}{2}, \quad s = \frac{7}{8}. \tag{54}
\]

The exact maximum of \( \gamma_2 \) is

\[
\gamma_2 = \frac{4}{7} \approx 0.57143 \tag{55}
\]

and

\[
\gamma_1 = \frac{1}{7} \approx 0.14286. \tag{56}
\]

It is clear that our analytic solution is better as compared with the numerical result

\[
\gamma_1 = 0.14165, \quad \gamma_2 = \gamma_3 = 0.57122 \tag{57}
\]

of Ref. [15], since Eqs. [55] and [56] are exact solution. Evidently the maximum of \( \gamma_2 + \gamma_3 \) should be greater than or equal to \( \frac{7}{8} \), although we guess that \( \frac{7}{8} \) should be the maximum of \( \gamma_2 + \gamma_3 \).
However if we make $\gamma_1 + \gamma_2$ to be maximum, under the condition $\gamma_2 = \gamma_3$, it is not difficult to obtain that the probability of cloning success is, on average,

$$P_{\text{success}} = \gamma_1 = \gamma_2 = \gamma_3 = 1 - \frac{2\sqrt{2} + 1}{7} \simeq 0.45308.$$  \hspace{1cm} (58)

We have constructed the quantum logic network for probabilistically cloning the states [15] in [16].

In summary we give achievable efficiencies for probabilistic cloning the quantum states used in implemented tasks for which cloning provides some enhancement in performance, and present an example of quantum computational tasks whose performance is enhanced if we distribute quantum information using quantum cloning. We hope our result will be helpful in the quantum information processing.

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Figure 1: If function $f_i$ is guaranteed to be either in set $S_1$ or in $S_2$, then this quantum circuit can be used to distinguish between the eight possibilities in each set. We can determine $f_i$ by measuring the final state $|\phi_i\rangle = \frac{1}{2\sqrt{2}} \sum_{x=000}^{111} (-1)^{f_i(x)} |x\rangle$ in one of two orthogonal bases, depending on which set contains $f_i$. Here $H$ operations are Hadamard gates.

Figure 2: The cloning procedure in this circuit is probabilistic. After the cloning process we can measure a "flag" subsystem and know whether the cloning was successful or not. If the cloning is successful, we let the clones go through the rest of the circuit, yielding output states $|\phi_i\rangle = \frac{1}{2\sqrt{2}} \sum_{x=000}^{111} (-1)^{f_0(x)+f_i(x)} |x\rangle$ ($i = 1, 2$). These states can be measured in the basis defined by Eqs.(9)-(16) to unambiguously decide which of the eight sets $S_{00000000}, S_{00001111}, S_{01010101}, S_{00110011}, S_{10011001}, S_{11000011}, S_{01101001}, S_{10100101}$ contains $f_0 \oplus f_i$.

Figure 3: The $(x,y)$ region. Note $\frac{7}{8} - q + s \leq 2$. 
Figure 4: The \((q, s)\) region.

\[
s = 1 - 2q^2
\]

Figure 5: The \((v, w)\) region.
Figure 6: The \((x, y)\) region. Note \(\frac{1}{2} - q + s \leq 2\).

Figure 7: The \((q, s)\) region.
Figure 8: The \((v, w)\) region.