THE OSEEN-FRANK LIMIT OF ONSAGER’S MOLECULAR THEORY
FOR LIQUID CRYSTALS

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Abstract. We study the relationship between Onsager’s molecular theory and the Oseen-Frank theory for nematic liquid crystals. Under the molecular setting, we consider the free energy that includes the effects of nonlocal molecular interactions. By imposing the strong anchoring boundary condition on the second moment of the number density function, we prove the existence of global minimizers for the free energy. Moreover, when the re-scaled interaction distance tends to zero, the corresponding global minimizers will converge to an uniaxial distribution whose orientation is described by a minimizer of Oseen-Frank energy.

1. Introduction

The liquid crystal state is a distinct phase of matter that is between those of ordinary liquid and solid crystal. They may flow as a liquid while the molecules are oriented in a crystal-like way. A classification of liquid crystals based on their structural properties was first proposed by G. Friedel in 1922 and they are generally divided into three main classes: in the nematic phase, the molecules tend to have the same alignment but their positions are not correlated. In the cholesteric phase, the molecules tend to have the same alignment which varies regularly through the medium with a periodicity distance. In the smectic phase, the molecules are arranged in layers and exhibit some correlations in their positions in addition to the orientational ordering.

In this work, we shall restrict ourselves to the nematic case. Since the long planar molecules usually involved are symbolized by ellipses, they can be characterized by long-range orientational order: the long axis of the molecules tend to align along a preferred direction. There are mainly three kinds of continuum theories, which use different order parameters in physics, to capture such anisotropic behavior of liquid crystals. Among them, the most intuitive one is the vector theory, which uses a unit-vector field \( n(x) \) to represent the locally preferred direction that the liquid crystal molecules self-orient themselves near \( x \). On this direction, the most well-known model is the Oseen-Frank theory based on curvature elasticity theory, in which the distortion energy of the liquid crystals is characterized by the following Oseen-Frank energy:

\[
E_{OF}[\n] = \frac{k_1}{2}(\nabla \cdot \n)^2 + \frac{k_2}{2}(n(\nabla \times \n))^2 + \frac{k_3}{2}|n \times (\nabla \times n)|^2 + \frac{k_2+k_4}{2}(\text{tr}(\nabla n)^2 - (\nabla \cdot n)^2) 
\]

where \( k_1, k_2, k_3, k_4 \) are elasticity constants and \( \times \) denotes the wedge product for two vectors in \( \mathbb{R}^3 \). The first three terms in (1.1) correspond to the three typical pure deformations: splay, twist and bend while the last component is actually a null lagrangian due to Ericksen [9]. Various analytic results related to the global minimizers of (1.1) under Dirichlet boundary condition is investigated in [15]. In the simplest setting: \( k_1 = k_2 = k_3 = k_F, k_4 = 0 \), Oseen-Frank energy (1.1) reduces to the Dirichlet energy

\[
E_{OF}[\n] = \frac{k_F}{2}|\nabla \n|^2. 
\]

(1.2)
Minimizing (1.2) among mappings from $\Omega$ into $S^2$ under certain boundary conditions leads to harmonic maps into $S^2$, which are widely studied in the past few decades, see for example [19]. For the purpose of describing the hydrodynamics of liquid crytals, Ericksen and Leslie formulated the full system in [8, 18]. This system also attracts the interests of analysts, especially those working on geometric analysis, for its relationship with harmonic map heat flow. See [20] for a review of recent progresses on the mathematics of Ericksen-Leslie system.

The second theory, which will be investigated in this work, is the molecular theory. This is a microscopic theory which uses a number density function $f(x, m)$ to characterize, at each point $x$, the number density of molecules whose orientations are parallel to the direction $m \in S^2$. It was first presented by Onsager in [24] to model the isotropic-nematic phase transition and later developed by Maier and Saupe in a series of paper, see for example [22]. In [6], this theory is developed by Doi et al. for the sake of studying the hydrodynamics of liquid crystals. In Onsager’s theory, each spatial position $x$ of material occupying $\Omega \subset \mathbb{R}^d$ is associated with a number density function $f(x, m) : \Omega \times S^2 \mapsto \mathbb{R}^+$ which indicates the fraction of molecules per unit solid angle having various orientations.

Onsager proposed a mean-field model to describe isotropic-nematic phase transition for liquid crystals. His expression for the free energy, at each fixed $x \in \Omega$ takes the following form:

$$A[f](x) = \int_{S^2} (f(x, m) \log f(x, m) + f(x, m)U[f](x, m)) \, dm,$$

where $U[f](x, m)$ is the mean-field interaction potential defined by

$$U[f](x, m) = \int_{S^2} B(m, m') f(x, m') \, dm' \text{ with } B(m, m') = \alpha |m \wedge m'|^2. \quad (1.4)$$

Here $B(m, m')$ is the interaction potential between two molecules with orientations $m$ and $m'$ respectively and $\alpha$ is a parameter that measures the intensity of the potential. In Onsager’s original treatment [24], $B(m, m')$ is chosen to be

$$B(m, m') = \alpha |m \wedge m'|,$$

calculated from the excluded-volume potential for hard rods. The form (1.4) was introduced later by Maier and Saupe and is widely employed for it shares qualitatively the same features as Onsager’s original one (1.5) at the same time easier to handle analytically. Especially, energy (1.3) with (1.4) is closely related to the variant second moment of $f$, which is usually called $Q$-tensor and defined by

$$Q[f](x) = \int_{S^2} (m \otimes m - \frac{1}{3}I_3) f(x, m) \, dm.$$  

More precisely,

$$A[f] = \int_{S^2} f \log f \, dm + \alpha \left( \frac{2}{3} - |Q[f]|^2 \right). \quad (1.7)$$

In order to characterize the distortion elasticity energy, each family of number density function $f(x, m)$ is associated with a nonlocal free energy, proposed in [7, 31]

$$A_e[f] = \int_{\Omega} \int_{S^2} (f(x, m) \log f(x, m) + f(x, m)U_e[f](x, m)) \, dm \, dx.$$

(1.8)

Here $U_e$ is a spatial nonhomogeneous mean-field potential, chosen to be

$$U_e[f](x, m) = \int_{\Omega} \int_{S^2} B(x, m; x', m') f(x', m') \, dx' \, dm'.$$

(1.9)
where $B(x, m; x', m')$ is the interaction kernel between two molecules with different configuration $(x, m)$ and $(x', m')$ respectively. In this paper, we will follow [7, 31] and choose $B$ like

$$B(x, m; x', m') = \alpha |m \wedge m'|^2 g_{\epsilon}(x - x')$$

with

$$g_{\epsilon}(x) := \frac{1}{\sqrt{\epsilon}} g\left(\frac{x}{\sqrt{\epsilon}}\right)$$

where the parameter $\sqrt{\epsilon}$ denotes the re-scaled length of the molecule as well as the typical molecular interaction distance. Throughout this work, the interaction kernel $g$ will be a non-negative, smooth, radial function with exponential decay and satisfies $\int_{\mathbb{R}^d} g(x) = 1$.

The third continuum theory for nematic liquid crystal is called Landau-de Gennes theory, or $Q$-tensor theory. In this framework, the order parameter is a $3 \times 3$ traceless symmetric matrix-valued function $Q(x)$ characterizing the orientation of molecules near $x$, see [5] for instance. As a phenomenological theory, it was derived based on the thermodynamical consideration of the Gibbs free energy (see [17]) of the system and it gives a phenomenological description of the nematic-isotropic phase transition of liquid crystals. See [23] for a mathematical approach.

These three theories all have made great successes on describing either the static or the dynamics of liquid crystals. In the meantime, efforts have been made to establish the relationships between these three theories. In the static case, for instance, the attempt on determining the elastic constants in Oseen-Frank energy (1.1) from molecular theories can be found in [25]. In [2], a singular bulk energy for $Q$-tensor model is derived from the entropy term in Onsager’s molecular theory. A systematic way of deriving macroscopic models, namely, $Q$-tensor models and vector models, from Onsager’s molecular theory is developed in [14].

For the hydrodynamics, Kuzuoo-Doi [16] formally derive the Ericksen-Leslie equation from the homogenous Doi-Onsager equation and determine the Leslie coefficients by taking Deborah number to be 0. For the sake of recovering Ericksen stress tensor from microscopic theory, in [7] and [28], a kinetic model for nonhomogeneous liquid crystalline polymers is presented and the full Ericksen-Leslie equation is derived. In [30], the authors rigorously proved that, when Deborah number tends to 0, the solution of Doi-Onsager equations will converge to the smooth solution of Ericksen-Leslie equations. On the other hand, in [11], various dynamical $Q$-tensor model is derived from Doi’s kinetic theory by using different moment closure methods. See also [14] for a dynamical $Q$-tensor model derived from kinetic theory which satisfies energy dissipation law.

Concerning the connection between $Q$-tensor theory and vector theory, one can consult [4, 12, 23] for the static case and [29] for the dynamics. In these works, the asymptotic behavior of small elasticity coefficients of Landau-de Gennes model is rigorously analyzed and the Oseen-Frank model or the Ericksen-Leslie model is recovered in the limit. These results show that the $Q$-tensor theory and vector theory agree well away from the singularities.

For the sake of understanding the liquid crystal defects predicted by Oseen-Frank model or Ericksen-Leslie system, the weak solution provides a suitable framework. However, the connections between weak solutions of the molecular models and that of the vector models are not fully explored and this article is intended to contribute to this direction. We shall study the asymptotic behaviors of the minimizers of (1.8)-(1.10) under strong anchoring boundary condition. Our main result is, when the parameter $\epsilon$ tends to 0, the global minimizers will converge to uniaxial distributions that are parameterized by harmonic maps.
Through this work, $\Omega$ will be a simply-connected, bounded domain of class $C^0$ in $\mathbb{R}^d$ with $d = 2, 3$. We shall consider the number density function in the following function space

$$\mathcal{H}(\Omega) = \{ f \in L^1(S^2 \times \Omega), f(x, m) \geq 0, \| f(\cdot, x) \|_{L^1(S^2)} = 1, \text{ a.e. } x \in \Omega \}. \quad (1.11)$$

For each fixed $\epsilon > 0$, we consider the minimizing problem

$$\inf_{f \in \mathcal{A}} A_\epsilon[f] \quad (1.12)$$

in the admissible space

$$\mathcal{A} := \left\{ f \in \mathcal{H}(\Omega) \mid Q[f](x) = Q[h_{n_b}](x) \text{ in } \Omega^\delta \right\}. \quad (1.13)$$

Recall that $Q[f]$ is the variant second moment of $f$ defined by (1.6). In (1.13), $h_{n_b}$ is defined via

$$h_{n_b} := \frac{e^{\eta (m \cdot n_b)^2}}{\int_{S^2} e^{\eta (m \cdot n_b)^2} dm}, \quad (1.14)$$

for some $n_b$ satisfying

$$n_b \in H^1(\mathbb{R}^d, \mathbb{R}^3) \text{ with compact support and } |n_b(x)| = 1 \text{ a.e. for } x \in \Omega. \quad (1.15)$$

The parameter $\eta$ in (1.14) is a constant depending on $\alpha$ and will be precised later. Loosely speaking, we shall choose $\eta$ such that (1.14) is the only global minimizer of the homogeneous Maier-Saupe energy (1.7). The “thin shell” $\Omega^\delta$ is defined as

$$\Omega^\delta := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \delta \}$$

where $\delta > 0$ is the parameter characterizing the strength of boundary effect. For a liquid crystal material in a bounded domain $\Omega$, the study of its static configuration predicted by Onsager’s energy (1.8) should take into account the boundary effect. In our case, we assume a strong anchoring boundary condition in (1.13) by prescribing the orientation of the molecules in a boundary layer that is slightly wider than $\sqrt{\epsilon}$, the re-scaled length of the molecular. More precisely, we assume

$$\delta = \epsilon^{1/2-\sigma} \quad (1.16)$$

for any fixed $\sigma \in (0, \frac{1}{2})$. As the reader will see in the sequel, the non-local boundary condition in (1.13) will reduce to the ‘usual’ strong anchoring boundary condition when $\epsilon \to 0$.

Throughout this work, for any $f \in \mathcal{A}$, we shall denote

$$\tilde{f}(x, m) = \begin{cases} h_{n_b}(x, m), & x \in \mathbb{R}^d \setminus \Omega, \\ f(x, m), & x \in \Omega. \end{cases} \quad (1.17)$$

Our first result is concerned with the critical point of (1.8)-(1.10):

**Theorem 1.** Let $\alpha > 7.5$ and $\eta$ be the largest root of equation

$$\alpha = \frac{\int_0^1 e^{\eta z^2} dz}{\int_0^1 z^2 (1 - z^2) e^{\eta z^2} dz}.$$ 

Assume that $g$ satisfies (2.3). For each $\epsilon > 0$, let $f^\epsilon \in \mathcal{A}$ be the critical point corresponding to (1.8)-(1.10) and $\delta$ is chosen as (1.16). If there exists an $\epsilon$-independent constant $C$ such that

$$\frac{1}{\epsilon} \int_{\Omega} (A[f^\epsilon] - A[h_{n_b}]) \, dx + \frac{\alpha}{2\epsilon} \int_{\mathbb{R}^d \times \mathbb{R}^d} |Q[\tilde{f}^\epsilon](x) - Q[\tilde{f}^\epsilon](y)|^2 g_\epsilon(x - y) \, dx dy \leq C, \quad (1.18)$$

then, modulo extraction of a subsequence,

$$f^\epsilon \rightharpoonup f \text{ weakly in } L^1(\Omega \times S^2),$$

for any $f \in \mathcal{A}$.
where \( f(x,m) \) is given by
\[
f(x,m) = \frac{e^{\eta(m-n(x))^2}}{\int_{S^2} e^{\eta(m-n(x))^2} \, dm}
\]
for some weakly harmonic map \( n(x) \in H^1(\Omega; S^2) \) with boundary condition \( n|_{\partial \Omega} = \pm n_b|_{\partial \Omega} \).

**Remark 1.1.** Weakly harmonic map into \( S^2 \) is defined to be the weak solution to the Euler-Lagrange equation for (1.2). More precisely, it satisfies
\[
\sum_{j=1}^d \int_{\Omega} \partial_j \varphi(x) \cdot (n(x) \wedge \partial_j n(x)) \, dx = 0
\]
for any \( \varphi \in C_0^\infty(\Omega; \mathbb{R}^3) \).

The conclusion in Theorem 1 can be strengthened if we assume \( f^\epsilon \) to be the solutions of (1.12):

**Theorem 2.** Let \( \alpha, \eta, g \) be the same as in Theorem 1. For each \( \epsilon > 0 \), the minimizing problem (1.12) has a solution \( f^\epsilon \in A \) which satisfies (1.18) if additionally \( \delta \) satisfies (1.16). Moreover, if \( \delta \) is chosen as (1.16) and \( n_b|_{\partial \Omega} \in H^1(\Omega; S^2) \) is a minimizing harmonic map
\[1\]
then modulo extraction of a subsequence,
\[
f^\epsilon \rightharpoonup f \text{ weakly in } L^1(\Omega \times S^2),
\]
where \( f(x,m) \) is given by
\[
f(x,m) = \frac{e^{\eta(m-n(x))^2}}{\int_{S^2} e^{\eta(m-n(x))^2} \, dm}
\]
for some minimizing harmonic map \( n(x) \in H^1(\Omega; S^2) \) with boundary condition \( n|_{\partial \Omega} = \pm n_b|_{\partial \Omega} \).

In contrast to the Ginzburg-Landau energy or the Landau-De Gennes energy, the Onsager’s energy (1.8) is non-local by nature and its global minimizers do not possess nature energy estimate that is independent of \( \epsilon \). However, by incorporating in it with the strong anchoring boundary condition (1.13), we prove that the global minimizers \( f^\epsilon \) satisfies (1.15). On the other hand, in order to recover the Oseen-Frank energy in the \( \epsilon \)-limit, we shall work with macroscopic parameter \( Q[f^\epsilon] \) instead of the number density \( f^\epsilon \) itself for the later might not possess compactness. The key step to obtain the strong compactness of \( Q[f^\epsilon] \) is to deduce from (1.18) that, \( g_\epsilon \ast Q[f^\epsilon] \) has a uniform in \( \epsilon \) bound in \( H^1(\mathbb{R}^d) \). As the reader shall see, during the process of passing to the \( \epsilon \)-limit, the \( Q \)-tensor will serve as an intermediate parameter connecting the number density function in molecule theory and the unit vector field in vector theory.

The rest of the article is organized as follows. In Section 2, we introduce the notations and conventions that will be adopted throughout this work. In the meanwhile, some assumptions will be made, for example, on the kernel function \( g \) in (1.10) and the parameter \( \eta \) in (1.14). Moreover, several analytic results concerning Maier-Saupe mean-field theory, especially the isotropic-nematic phase transition will be discussed. Section 3 serves as a preparation for the proof of Theorem 1 where the Euler-Lagrange equation of (1.8) is derived and is recast in terms of the macroscopic variable \( Q[f] \). Moreover, in Proposition 3.1 various weak and strong convergence results are obtained based on the a priori estimate (1.15). Section 4 is

\[1\] In this case, the boundary condition is prescribed by \( n_b|_{\partial \Omega} \).
denote the convolution of two functions \( u \) and \( v \). Note that \( u \ast v \) does not hold in general. On the other hand, we shall also use it to denote \( \nabla \cdot \mathbf{a} \) when it is multiplied by some quantities.

2.2. Convolution Operator and Interaction Kernel. We shall use the notation \( u \ast _\Omega v \) to denote the non-commutative convolution of \( u, v \) in a domain \( \Omega \):

\[
 u \ast _\Omega v := \int _\Omega u(x')v(x-x')dx'.
\]

Note that \( u \ast _\Omega v = v \ast _\Omega u \) does not hold in general. On the other hand, we shall use \( \ast \) to denote the convolution of two functions \( u \) and \( v \) in \( \mathbb{R}^d \):

\[
 u \ast v := \int _{\mathbb{R}^d} u(x-x')v(x')dx' = \int _{\mathbb{R}^d} u(x')v(x-x')dx'.
\]

For any function \( v \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), \( \hat{v} \) will denote its Fourier transform:

\[
 \hat{v}(\xi) = \int _{\mathbb{R}^d} v(x)e^{-2\pi ix \cdot \xi}dx.
\]
Then we have the following formulas (see [27])

\[ \hat{v}(x) = \int_{\mathbb{R}^d} \hat{v}(\xi) e^{2\pi i x \cdot \xi} d\xi, \]
\[ \hat{\nabla} u(\xi) = 2\pi i \xi \hat{u}(\xi), \]
\[ \hat{u} \hat{v}(\xi) = \hat{u}(\xi) \ast \hat{v}(\xi), \]
\[ \hat{u} \ast \hat{v} = \hat{u}(\xi) \hat{v}(\xi), \]
\[ \|u\|_{L^2(\mathbb{R}^d)} = \|\hat{u}\|_{L^2(\mathbb{R}^d)}. \]

Now we turn to the assumptions on the kernel function \( g \) in (1.10). We shall assume \( g(x) \) be a non-negative, smooth, radial function with exponential decay and satisfies

\[ \int_{\mathbb{R}^d} g(x) = 1. \]
\[ \text{Moreover, we assume there exists some } c_0 > 0 \text{ such that } 0 \leq \hat{g}(\xi) \leq 1 \text{ and } c_0 |\xi|^2 \hat{g}^2(\xi) \leq 1 - \hat{g}(\xi), \quad \forall \xi \in \mathbb{R}^d. \quad (2.3) \]

Under these assumptions, one can easily verify that

\[ \hat{g}(0) = 1, \quad \nabla \hat{g}(0) = 0, \quad \nabla^2 \hat{g}(0) = -\frac{4\pi^2}{d} \mathbb{1}_d \text{ where } \mu = \int_{\mathbb{R}^d} |x|^2 g(x) dx. \quad (2.4) \]

Example of \( g \) satisfying the above assumptions is given by

\[ g(x) = \left( \frac{a}{\pi} \right)^{\frac{d}{2}} e^{-a|x|^2} \text{ with } a \in (0, \pi). \]

Actually, as \( \hat{g}(\xi) = e^{-\frac{\pi^2|\xi|^2}{a}} \), (2.3) holds with \( c_0 \leq \frac{\pi^2}{a} \).

We shall re-scale \( g \) by \( g_\varepsilon(x) = \frac{1}{\varepsilon} g(x/\varepsilon) \). Thus, we have

\[ \int_{\mathbb{R}^d} g_\varepsilon(x) dx = 1, \quad \hat{g}_\varepsilon(\xi) = \hat{g}(\sqrt{\varepsilon} \xi). \]

We give a few more words about the boundary condition introduced in (1.15). Any \( n_b \in H^1(\Omega; S^2) \) can be extend to be

\[ n_b \in H^1(\mathbb{R}^d; \mathbb{R}^3), \quad \text{with } n_b(x) \equiv 0 \text{ for } |x| \geq R \text{ where } \Omega \subset B_{\frac{R}{2}}(0), \]
\[ \|n_b\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^3)} \leq C\|n_b\|_{H^1(\Omega)} \text{ with } 1 \leq p \leq 2. \quad (2.5) \]

The existence of \( n_b \) can be justified by first employing the extension theorem of Sobolev space (see for instance [26 page 181]) and then multiplying by a cutoff function

\[ \chi(x) = \begin{cases} 1, & |x| \leq \frac{R}{2}, \\ 0, & |x| \geq R. \end{cases} \]

2.3. The Maier-Saupe energy. Now we turn to the study of the minimizers of the Maier-Saupe energy (1.3). It is proved in [10, 21] that, all the critical points of (1.3) can be explicitly given by

\[ h_\nu := \frac{e^{\eta (\nu \cdot \nu)^2}}{\int_{\mathbb{R}^2} e^{\eta (\nu \cdot \nu)^2} dm} \quad (2.6) \]

for any \( \nu \in S^2 \) where \( \eta \) is a constant depending on the interaction strength \( \alpha \):

\[ \frac{3e^{3\eta}}{\int_0^1 e^{\eta z^2} dz} = 3 + 2\eta + \frac{4\eta^2}{\alpha}. \quad (2.7) \]
The trivial solution $\eta = 0$ to (2.7) corresponds to the isotropic distribution $h \equiv \frac{1}{4\pi}$. If $\alpha > \alpha^* = \min_{\eta \in \mathbb{R}} \alpha(\eta) \approx 6.7314$, then (2.7) has nontrivial solutions satisfying

$$
\alpha = \alpha(\eta) := \frac{\int_0^1 e^{\eta z^2} \, dz}{\int_0^1 z^2 (1 - z^2) e^{\eta z^2} \, dz}.
$$

In [30], it is proved that there exists a unique $\eta^*$ such that $\alpha(\eta^*) = \alpha^*$, and $\alpha(\eta)$ increases monotonically when $\eta > \eta^*$ and decreases monotonically when $\eta < \eta^*$. Thus, for $\alpha > \alpha^*$, there exists two values, denoted by $\eta_1(\alpha)$ and $\eta_2(\alpha)$, such that $\eta_1 > \eta^* > \eta_2$. In addition, $\eta_1(\alpha)$ is an increasing function of $\alpha$, while $\eta_2(\alpha)$ is a decreasing function. It is also proved that, for $\alpha < 7.5$, the critical point corresponding to $\eta = 0$ is stable while for $\alpha > 7.5$ it is unstable. For $\alpha > \alpha^*$, the critical points corresponding to $\eta_1$ are always stable and the ones corresponding to $\eta_2$ is unstable.

**Lemma 2.1.** For $\alpha > 7.5$, the global minimizer of (1.3) is achieved only by the uniaxial distribution (2.6) with $\eta \neq 0$.

The proof will make use of the following classical result concerning the weakly lower-semicontinuity of entropy:

**Lemma 2.2.** Let $f_k \in \mathcal{H}(\Omega)$ (defined by (1.11)) be a sequence of functions such that

$$
\int_{\Omega \times \mathbb{S}^2} f_k \log f_k < \infty \text{ uniformly for } k \in \mathbb{N}^*.
$$

Then there exists $f \in \mathcal{H}(\Omega)$ such that $f_k \rightharpoonup f$ weakly in $L^1(\mathbb{S}^2 \times \Omega)$ and

$$
\int_{\Omega \times \mathbb{S}^2} f \log f \, dxdm \leq \liminf_{k \to \infty} \int_{\Omega \times \mathbb{S}^2} f_k \log f_k \, dxdm. \tag{2.9}
$$

**Proof.** The weakly $L^1$-compactness of $\{f_k\}_{k \geq 1}$ and the almost everywhere inequality $f \geq 0$ follow from [13, page 47 and page 53]. In order to show that $f \in \mathcal{H}(\Omega)$, we choose any test function $\varphi = \varphi(x)$ and the weak convergence of $f_k$ leads to

$$
\int_{\Omega} \varphi(x) \, dx = \int_{\mathbb{S}^2 \times \Omega} f_k(x, m) \varphi(x) \, dxdm \to \int_{\mathbb{S}^2 \times \Omega} f(x, m) \varphi(x) \, dxdm.
$$

Since $\varphi(x)$ is arbitrary, we have that

$$
\int_{\mathbb{S}^2} f(x, m) \, dm = 1 \text{ a.e. } x \in \Omega.
$$

□

**Proof of Lemma 2.1.** We first note that the space $\mathcal{H}$ in the statement of Lemma 2.1 is non-empty since it includes the isotropic distribution $h = \frac{1}{4\pi}$. Choosing any minimizing sequence $f_k \in \mathcal{H}$ such that

$$
\lim_{k \to \infty} A[f_k] = \inf_{f \in \mathcal{H}} A[f],
$$

Then it follows from (1.7) that

$$
\int_{\mathbb{S}^2} f_k \log f_k \, dm \leq C
$$
where $C$ is independent of $k$. Note that, we can consider $f_k$ to be elements in $\mathcal{H}(\Omega)$ that remain constant with respect to $x \in \Omega$. So we can apply Lemma 2.2 and this leads to
\[ f_k \rightharpoonup f \text{ weakly in } L^1(S^2) \]
where $f \in \mathcal{H}$. This proves the existence of global minimizer $f$. Now we investigate its precise form. For any $\tilde{f} \in \mathcal{H}$ and $t \in [0, 1]$, since $(1 - t)f + t\tilde{f} \in \mathcal{H}$, the series $A[(1 - t)f + t\tilde{f}]$ is well defined. Moreover, since $f$ is a global minimizer, we have
\[ 0 \leq \lim_{t \to 0^+} \frac{A[f + t(\tilde{f} - f)] - A[f]}{t} = \int_{S^2} (\log f + U[f])(\tilde{f} - f)dm. \]
Since $\tilde{f}$ is arbitrary in $\mathcal{H}$, there exists some $\lambda \in \mathbb{R}$ such that
\[ \log f + U[f] = \lambda, \text{ a.e. on } S^2. \]
One can easily determine $\lambda$ and deduce
\[ f = e^{-U[f](m)} \int_{S^2} e^{-U[f](m)}dm. \]
This implies that $f$ is also a critical point of (1.3) and this together with the discussion before leads to the desired result. \qed

In the sequel, we will work with $\alpha$ and $\eta$ as in Lemma 2.1:
\[ \alpha > 7.5, \quad \eta = \eta_1(\alpha) \neq 0. \]
The relationship between the uniaxial distribution (2.6) and its $Q$-tensor is nicely summarized by the following formulas:

**Lemma 2.3.** For any uniaxial distribution (2.6) with $\nu \in S^2$, we have
\[ Q[h_\nu] = s_2(\nu \otimes \nu - \frac{1}{3}I_3) \text{ where } s_2 \neq 0. \tag{2.10} \]
For isotropic distribution $h = \frac{1}{4\pi}$, it holds
\[ Q[h] = 0. \]

**Proof.** From [30, Lemma 6.6], we have that
\[ Q[h_\nu] = \int_{S^2} (m \otimes m - \frac{1}{3}I_3)h_\nu(m)dm = s_2(\nu \otimes \nu - \frac{1}{3}I_3). \]
In the above formula, the parameter $s_2$ is called degree of orientation and is defined via
\[ s_2 = \int_{S^2} P_2(m \cdot \nu) e^{\eta(m \cdot \nu)^2} dm = \frac{\int_{-1}^{1} P_2(z)e^{\eta z^2}dz}{\int_{-1}^{1} e^{\eta z^2}dz}, \tag{2.11} \]
with $P_2(x) = \frac{1}{2}(3x^2 - 1)$ being the 2-th Legendre polynomial.
Concerning the sign of $s_2$, note that
\[ \int_{0}^{1} z(1 - z^2)de^{\eta z^2} + \int_{0}^{1} e^{\eta z^2}d(z(1 - z^2)) = e^{\eta z^2}z(1 - z^2)|_{0}^{1} = 0, \]
we deduce that
\[ s_2 = \frac{\eta}{\alpha} \neq 0 \]
by recalling (2.8), (2.11) and the choice of $\eta$. The case for $h = \frac{1}{4\pi}$ is evident. \qed
where $Q^i[f]$ denotes the $i$-th row vector of $Q[f]$.

Proof. The proof consists of three parts:

- **Deriving the Euler-Lagrange equation:** First note that $\mathcal{A}$ is a convex set: if $f, g \in \mathcal{A}$, so does $(1 - t)f + tg \in \mathcal{A}$. Since $f$ is a critical point to (1.12), it holds that

$$0 \leq \lim_{t \to 0^+} \frac{1}{t} (A_t[f + t(g - f)] - A_t[f]) = \int_{\Omega \times S^2} (\log f + U_t[f]) (g - f) dmdx.$$ 

Following the same argument as in the proof of Lemma 2.1 we get

$$\log f(x, m) + U_t[f](x, m) = \lambda(x), \quad \forall x \in \Omega_\delta,$$

where $\lambda(x)$ is the Lagrange multiplier corresponding to the constrain. Thus we obtain the following formula

$$f = \frac{e^{-U_t[f]}}{\int_{S^2} e^{-U_t[f]} d\mu} \text{ a.e. } x \in \Omega_\delta. \quad (3.3)$$

This implies that, the global minimizer of (1.12) is actually smooth function since $U_\epsilon$ is a regular convolution operator.

- **Macroscopic equation:** It follows from (2.1) that

$$\mathcal{R}U_t[f] = -2\alpha \int_{\Omega} \int_{S^2} m \cdot m'(m \wedge m') g_\epsilon(x - x') f(x', m') dx' dm'.$$

Then we have by (3.2) that

$$\mathcal{R} f = -f \mathcal{R} U_t[f]$$

$$= 2\alpha \int_{S^2 \times \Omega} m \wedge m'(m \cdot m') g_\epsilon(x - x') f(m', x') dm' dx'$$

$$= 2\alpha \int_{S^2 \times \Omega} m_i m'_j e^{ij\ell} m_k m'_k g_\epsilon(x - x') f(m', x') dm' dx'$$

$$= 2\alpha \int_{\Omega} m_i m_k e^{ij\ell} g_\epsilon(x - x') \left( \frac{1}{2} \delta_{jk} + \int_{S^2} (m'_j m'_k - \frac{1}{3} \delta_{jk}) f(m', x') dm' \right) dx'$$

$$= 2\alpha f(x, m) m_i m_k e^{ij\ell} Q_{jk}[f] *_{\Omega} g_\epsilon.$$ 

Integrate this identity over $S^2$ and use (2.1) lead to

$$e^{ij\ell} Q_{ik}[f](x) (Q_{jk}[f] *_{\Omega} g_\epsilon)(x) = 0,$$

which is equivalent to (3.1), according to (2.2). \quad \square

**Proposition 3.1.** Let $f^\epsilon \in \mathcal{A}$ be extended to $\tilde{f}^\epsilon$ via (1.17) and satisfies

$$\frac{1}{\epsilon} \int_{\Omega} (A[f^\epsilon] - A[h_{n_\delta}]) dx + \frac{\alpha}{2\epsilon} \int_{\mathbb{R}^d \times \mathbb{R}^d} |Q[\tilde{f}^\epsilon](x) - Q[\tilde{f}^\epsilon](y)|^2 g_\epsilon(x - y) dxdy \leq C, \quad (3.4)$$

for some $\epsilon$ independent constant $C$. Then modulo the extraction of a subsequence,

$$u_\epsilon(x) := Q[\tilde{f}^\epsilon](x) \quad (3.5)$$
has the following properties

\[
\begin{align*}
\{ u_\epsilon \rightarrow & \quad \Psi \text{ strongly in } L^2(\mathbb{R}^d), \\
\nabla (u_\epsilon * g_\epsilon) \rightarrow & \quad \nabla \Psi \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^d)
\end{align*}
\]

(3.6)

where \( \Psi \) satisfies

\[
\Psi \in H^1(\mathbb{R}^d) \text{ with compact support.}
\]

Moreover, up to the extraction of a subsequence,

\[
f^\epsilon \rightharpoonup \bar{f} \text{ weakly in } L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{S}^2),
\]

(3.7)

where \( \bar{f}(x, m) \) is given by

\[
\bar{f}(x, m) = \begin{cases} 
\frac{\phi_n(x)^2}{\int_{S^2} \phi_n(x)^2 \, dm} & \text{for } x \in \Omega, \\
\frac{1}{\int_{S^2} \phi_n(x)^2 \, dm} & \text{for } x \in \mathbb{R}^d \setminus \Omega
\end{cases}
\]

(3.8)

for some \( n(x) \in H^1(\Omega; \mathbb{S}^2) \) and

\[
\Psi(x) = Q[f](x) \text{ a.e. } x \in \mathbb{R}^d.
\]

(3.9)

**Proof.** The proof will be separated into several parts.

- **Proof of (3.6):** The assertion follows if we can prove the following estimate:

\[
\frac{1}{\epsilon} \int_{\mathbb{R}^d} |u_\epsilon * g_\epsilon - u_\epsilon|^2 \, dx + \int_{\mathbb{R}^d} |\nabla (u_\epsilon * g_\epsilon)|^2 \, dx \leq C.
\]

(3.10)

Actually, it follows from (3.10) and compact imbedding theorem of Sobolev space that, up to the extraction of a subsequence, \( \{ u_\sigma * g_\sigma \}_{\sigma > 0} \) is a Cauchy sequence in \( L^2_{\text{loc}}(\mathbb{R}^d) \) and this together with the following inequality implies the strong convergence of \( u_\epsilon \) in \( L^2_{\text{loc}}(\Omega) \):

\[
|u_\epsilon - u_\sigma| \leq |u_\epsilon - g_\epsilon * u_\epsilon| + |u_\sigma - g_\sigma * u_\sigma| + |u_\sigma * g_\sigma - u_\epsilon * g_\epsilon|.
\]

Moreover, it follows from (1.15), (1.14) and (2.5) that, \( \tilde{f}^\epsilon(x, m) \equiv \frac{1}{4\pi} \) for \( x \in \mathbb{R}^d \setminus B_R(0) \). This together with Lemma 2.3 implies that

\[
u_\epsilon(x) := Q[\tilde{f}^\epsilon](x) \equiv 0, \quad \forall x \in \mathbb{R}^d \setminus B_R(0).
\]

So \( u_\epsilon \rightarrow \Psi \) strongly in \( L^2(\mathbb{R}^d) \) where \( \Psi \in L^2(\mathbb{R}^d) \) with compact support.

The second part of (3.6) follows from the weak compactness of \( L^2_{\text{loc}}(\mathbb{R}^d) \) and (3.10): on one hand, we have \( \nabla (u_\epsilon * g_\epsilon) \rightharpoonup \Phi = \{ \Phi_k \}_{1 \leq k \leq d} \) weakly in \( L^2_{\text{loc}}(\mathbb{R}^d) \). Then for any test function \( \varphi(x) \in C^\infty_0(\Omega) \),

\[
- \int_{\mathbb{R}^d} \partial_k (u_\epsilon * g_\epsilon) \varphi = \int_{\mathbb{R}^d} (u_\epsilon * g_\epsilon) \partial_k \varphi = \int_{\mathbb{R}^d} u_\epsilon \cdot (g_\epsilon \ast \partial_k \varphi).
\]

Taking \( \epsilon \rightarrow 0 \) leads to

\[
- \int_{\mathbb{R}^d} \varphi \Phi_k = \int_{\mathbb{R}^d} \Psi \partial_k \varphi \text{ which implies } \nabla \Psi = \Phi \in L^2(\mathbb{R}^d).
\]

Now we are in position to prove (3.10). The reader can consult [1] for an approach without using Fourier transform. First, we have from Plancherel theorem that

\[
\int_{\mathbb{R}^d} |u_\epsilon * g_\epsilon - u_\epsilon|^2 \, dx = \| (1 - \hat{g}_\epsilon) \hat{u}_\epsilon \|_{L^2}^2 \leq 2 \| 1 - g_\epsilon \hat{u}_\epsilon \|_{L^2}^2
\]

(3.11)

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} g_\epsilon(x - y) |u_\epsilon(x) - u_\epsilon(y)|^2 \, dx \, dy.
\]
On the other hand, it follows from (2.3) that we have
\[
\int_{\mathbb{R}^d} |\nabla (g_* u_\epsilon)|^2 dx = 4\pi^2 \|\xi \hat{g}_\epsilon \hat{u}_\epsilon\|_{L^2}^2 \leq C \varepsilon^{-1} \sqrt{(1 - \hat{g}_\epsilon)\|\hat{u}_\epsilon\|_{L^2}^2}
\]
(3.12)

Then we can combine (3.11) with (3.12) together with (3.4) to get (3.10).

- **Proof of (3.7):** It suffices to show
  \[
  \bar{f}_\epsilon \rightarrow \bar{f} \quad \text{weakly in } L^1(\Omega \times S^2)
  \]
  (3.13)
  for some number density function \( \bar{f}(x, m) \) which is uniaxial on \( \Omega \times S^2 \) and is extended to be \( h_{v_b} \) in \( \Omega^c \times S^2 \). Actually, it follows from (1.17), (3.8), and (3.13) that, for any test function \( \varphi(x, m) \),
  \[
  \int_{\mathbb{R}^d \times S^2} \bar{f}_\epsilon(x, m)\varphi(x, m)dm
  = \int_{\Omega \times S^2} \bar{f}_\epsilon(x, m)\varphi(x, m)dm + \int_{\Omega^c \times S^2} h_{v_b}(x, m)\varphi(x, m)dm
  \rightarrow \int_{\Omega \times S^2} \bar{f}(x, m)\varphi(x, m)dm + \int_{\Omega^c \times S^2} h_{v_b}(x, m)\varphi(x, m)dm
  = \int_{\mathbb{R}^d \times S^2} \bar{f}(x, m)\varphi(x, m)dm.
\]

To prove (3.13), we first deduce from (1.7) and (3.4) that
\[
\int_{\Omega \times S^2} \bar{f}_\epsilon \ln \bar{f}_\epsilon dm + \frac{2}{3} \alpha |\Omega| - \alpha \int_{\Omega} |Q[f^\epsilon](x)|^2 dx = \int_{\Omega} A[f^\epsilon](x)dx \leq C
\]

Owing to the fact that
\[
|Q[f^\epsilon](x)| := \left| \int_{S^2} (m \otimes m - \frac{1}{3} I) f(x, m)dm \right| \leq 1, \quad \forall x \in \Omega,
\]
we obtain the entropy estimate
\[
\int_{\Omega \times S^2} \bar{f}_\epsilon \ln \bar{f}_\epsilon dm \leq \bar{C}.
\]

This together with Lemma 2.2 leads to (3.13) and thus (3.7) is proved. It remains to show that \( \bar{f}(x, m) \) is uniaxial on \( \Omega \).

- **Proof of (3.8):** To show that \( \bar{f}(x, m) \) is uniaxial on \( \Omega \), we deduce from (3.4) that
  \[
  \liminf_{\epsilon \to 0} \int_{\Omega} A[\bar{f}^\epsilon](x)dx \leq \int_{\Omega} A[h_{v_b}](x)dx.
  \]

This together with strong compactness of \( Q[\bar{f}^\epsilon](x) \) (see (3.6)) and Lemma 2.2 lead to
\[
\int_{\Omega} A[\bar{f}](x)dx \leq \int_{\Omega} A[h_{v_b}](x)dx.
\]

Since \( n_b \) is unit vector field on \( \Omega \), \( h_{v_b} \) is an uniaxial distributions on \( \Omega \), which minimize the Maier-Saupe energy (1.3) owning to Lemma 2.1. So there exists some function \( n(x) : \Omega \rightarrow S^2 \) such that
\[
\bar{f}(x, m) = \frac{e^{n(m-n(x))^2}}{\int_{S^2} e^{n(m-n(x))^2} dm} \text{ for } x \in \Omega.
\]
On the other hand, (3.7) also implies that
\[ Q[\hat{f}] (x) \rightarrow Q[\hat{f}] (x) \quad \text{weakly in} \quad L^1_{\text{loc}} (\mathbb{R}^d). \]
This together with (3.6) implies that
\[ Q[\hat{f}] (x) = \Psi (x) \in H^1 (\Omega). \]
So \( \hat{f} \big|_{\Omega \times S^2} \) is a uniaxial distribution whose \( Q \)-tensor belongs to \( H^1 (\Omega) \). This together with the assumption that \( \Omega \) is simply-connected enable us to apply the orientability theorem in [3, Theorem 2] and deduce that \( n(x) \in H^1 (\Omega; S^2) \).

For any function \( u(x) \in L^2 (\mathbb{R}^d) \), we define \( A_\epsilon u \) by
\[ A_\epsilon u = \frac{1}{\epsilon} (u - u * g_\epsilon). \quad (3.14) \]
The operator \( A_\epsilon \) is a pseudo-differential operator with non-negative symbol
\[ \tilde{A}_\epsilon u (\xi) = \frac{\hat{g}(0) - \hat{g}(\sqrt{\epsilon} \xi)}{\epsilon} \hat{u}, \]
as is seen from (2.3) that \( \hat{g}(0) - \hat{g}(\sqrt{\epsilon} \xi) \geq 0 \). As a result we can define \( h(\xi) \) as
\[ h(\xi) := \begin{cases} \xi \sqrt{\frac{\hat{g}(0) - \hat{g}(\xi)}{|\xi|^2}}, & \xi \neq 0, \\ 0, & \xi = 0. \end{cases} \quad (3.15) \]

**Lemma 3.2.** The function \( h(\xi) \) defined by (3.15) is globally Lipschitz in \( \mathbb{R}^d \).

**Proof.** It follows from (2.4) that \( h(\xi) \) is continuous at \( \xi = 0 \) since \( \lim_{\xi \to 0} h(\xi) = 0 \), according to (2.4). On the other hand, \( h(\xi) \) is smooth in \( \mathbb{R}^d \setminus \{0\} \) and decays to zero when \( \xi \to \infty \). So \( h \in L^\infty (\mathbb{R}^d) \cap C (\mathbb{R}^d) \). We compute the derivative of \( h \) by
\[ \nabla h (\xi) = \mathbb{I}_d \sqrt{\frac{1 - \hat{g}(\xi)}{|\xi|^2}} - \frac{\xi}{2 \sqrt{1 - \hat{g}(\xi)}} \otimes \left( \frac{\nabla \hat{g}(\xi)}{|\xi|} + \frac{2 \xi}{|\xi|^3} (1 - \hat{g}(\xi)) \right) = \sum_{k=1}^3 A_i (\xi), \forall \xi \neq 0. \]
It is evident that \( A_1, A_3 \in L^\infty (\mathbb{R}^d) \cap C (\mathbb{R}^d) \). Moreover, \( A_2 \in L^\infty (B_1) \cap C^\infty (\mathbb{R}^d \setminus B_1) \) and decays to zero as \( \xi \to \infty \). These all together imply the statement. \( \square \)

Therefore, we shall define a vector valued operator \( T_\epsilon = \{ T_\epsilon^i \}_{1 \leq i \leq d} \) by
\[ \overline{T}_\epsilon u = \xi \sqrt{\frac{\hat{g}(0) - \hat{g}(\sqrt{\epsilon} \xi)}{\epsilon |\xi|^2}} \hat{u} = \frac{1}{\sqrt{\epsilon}} h (\sqrt{\epsilon} \xi) \hat{u}, \quad (3.16) \]
and we have
\[ A_\epsilon = \sum_{k=1}^d T_\epsilon^k \circ T_\epsilon^k. \quad (3.17) \]
The symbol of \( T_\epsilon \) will approach \( \xi \) as \( \epsilon \to 0 \). The following lemma says that the pseudo-differential operator \( T_\epsilon \) will approach, as \( \epsilon \to 0 \), to \( \frac{1}{\sqrt{\frac{d}{2}} \nabla} \).

**Lemma 3.3.** If \( u \in H^1 (\mathbb{R}^d) \), then it holds
\[ T_\epsilon u \rightarrow -i \frac{1}{\sqrt{\frac{d}{2}} \nabla} u \quad \text{strongly in} \quad L^2 (\mathbb{R}^d). \]
Proof. It can be verified from (2.4) that
\[
\sqrt{1 - \hat{g}(\sqrt{\epsilon} \xi)} \frac{\epsilon}{|\xi|^2}
\] is uniformly bounded with respect to \( \epsilon > 0 \) and \( \xi \in \mathbb{R}^d \setminus \{0\} \) and
\[
\lim_{\epsilon \to 0} \sqrt{1 - \hat{g}(\sqrt{\epsilon} \xi)} \frac{\epsilon}{|\xi|^2} = 2 \pi \sqrt{\frac{\mu}{d}}
\] for point-wise \( \xi \in \mathbb{R}^d \setminus \{0\} \).

On the other hand, since \( u \in H^1(\mathbb{R}^d) \), we have \( \|\xi \hat{u}(\xi)\|_{L^2(\mathbb{R}^d)} < \infty \). Therefore, Lebesgue’s dominant convergence theorem implies
\[
\lim_{\epsilon \to 0} \left\| \left( T_\epsilon \sqrt{1 - \hat{g}(\sqrt{\epsilon} \xi)} - 2 \pi \sqrt{\frac{\mu}{d}} \right) \xi \hat{u}(\xi) \right\|_{L^2(\mathbb{R}^d)} = 0.
\]
The proof is finished. \( \square \)

Lemma 3.4. Under the assumptions of Proposition 3.1, up to the extraction of a subsequence,
\[
T_\epsilon Q[\bar{f}] \rightharpoonup -i \sqrt{\frac{\mu}{d}} \nabla Q[\bar{f}] \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^d),
\]
where \( \bar{f} \) is defined by (3.7).

Proof. First note that, the uniform bound (3.4) and the definition of \( T_\epsilon \) at (3.16) imply
\[
\alpha \| T_\epsilon Q[\bar{f}] \|_{L^2(\mathbb{R}^d)}^2 \leq C.
\]
Then from weakly compactness of \( L^2 \)-space, there exists \( \tilde{Q} \) such that
\[
T_\epsilon Q[\bar{f}] \rightharpoonup \tilde{Q}(x) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^d),
\]
or equivalently,
\[
\int_{\mathbb{R}^d} T_\epsilon Q[\bar{f}] (x) : \Phi(x) dx \to \int_{\mathbb{R}^d} \tilde{Q}(x) : \Phi(x) dx, \quad \forall \Phi \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^{d \times d \times d}).
\]
On the other hand, the strong convergence of \( Q[\bar{f}](x) \) stated in (3.6) and Lemma 3.3 imply
\[
\int_{\mathbb{R}^d} T_\epsilon Q[\bar{f}](x) : \Phi(x) dx \to \int_{\mathbb{R}^d} \tilde{Q}(x) : \Phi(x) dx
\]
\[= \int_{\mathbb{R}^d} Q[\bar{f}](x) : T_\epsilon \Phi(x) dx \to -i \sqrt{\frac{\mu}{d}} \int_{\mathbb{R}^d} Q[\bar{f}](x) : (\nabla \cdot \Phi(x)) dx.
\]
The above two formulae together imply \( \tilde{Q}(x) = -i \sqrt{\frac{\mu}{d}} \nabla Q[\bar{f}](x) \). The proof is finished. \( \square \)

4. Asymptotic behavior of critical points

This section is devoted to the proof of Theorem 1. We start from a commutator estimate:

Lemma 4.1. For any \( \varphi \in C^\infty_0(\Omega) \), there exists a constant \( C \) depending on \( \varphi(x) \) such that
\[
\| [T_\epsilon, \varphi(x)] u \|_{L^2(\mathbb{R}^d)} \leq C \| u \|_{L^2(\mathbb{R}^d)}. \quad (4.1)
\]
Proof. By the definition of the commutator, we have
\[ [T_\epsilon, \varphi(x)]u = T_\epsilon(\varphi(x)u(x)) - \varphi(x)T_\epsilon u(x). \]
Then it follows from Plancherel’s theorem, Lemma 3.2 and Young’s inequality that
\[
\| [T_\epsilon, \varphi(x)]u \|_{L^2(\mathbb{R}^d)}
\leq \frac{1}{\sqrt{\epsilon}} \left\| h(\sqrt{\epsilon}\xi) \hat{\varphi} * \hat{u} - \hat{\varphi} * (h(\sqrt{\epsilon}\xi)\hat{u}(\xi)) \right\|_{L^2(\mathbb{R}^d)}
\leq \frac{1}{\sqrt{\epsilon}} \left\| h(\sqrt{\epsilon}\xi) \int_{\mathbb{R}^d} \hat{\varphi}(\xi - \zeta)\hat{u}(\zeta)d\zeta - \int_{\mathbb{R}^d} \hat{\varphi}(\xi - \zeta)h(\sqrt{\epsilon}\zeta)\hat{u}(\zeta)d\zeta \right\|_{L^2(\mathbb{R}^d)}
\leq \frac{C}{\sqrt{\epsilon}} \left\| \int_{\mathbb{R}^d} \hat{\varphi}(\xi - \zeta)\sqrt{\epsilon} |\xi - \zeta|\hat{u}(\zeta)d\zeta \right\|_{L^2(\mathbb{R}^d)}
\leq C \frac{\| \hat{\varphi} (\xi) \|_{L^1(\mathbb{R}^d)} \| \hat{u} \|_{L^2(\mathbb{R}^d)}}{\sqrt{\epsilon}}.
\]
\[ \square \]

Lemma 4.2. Under the assumption of Proposition 3.1, for any \( \varphi \in C^\infty_0(\Omega) \)
\[ [T_\epsilon, \varphi(x)]Q[\bar{f}] \to -i\sqrt{\frac{4}{\epsilon}} [\nabla, \varphi(x)]Q[\bar{f}] \quad \text{strongly in} \ L^2(\mathbb{R}^d). \] (4.2)

Proof. We have
\[
[T_\epsilon, \varphi(x)]Q[\bar{f}] + [i \sqrt{\frac{4}{\epsilon}} \nabla, \varphi(x)]Q[\bar{f}]
= [T_\epsilon, \varphi(x)](Q[\bar{f}] - Q[\bar{f}]) + [T_\epsilon, \varphi(x)]Q[\bar{f}] + [i \sqrt{\frac{4}{\epsilon}} \nabla, \varphi(x)]Q[\bar{f}]
= [T_\epsilon, \varphi(x)](Q[\bar{f}] - Q[\bar{f}]) + \left( T_\epsilon + i \sqrt{\frac{4}{\epsilon}} \nabla \right) (\varphi(x)Q[\bar{f}](x)) - \varphi(x)\left( T_\epsilon + i \sqrt{\frac{4}{\epsilon}} \nabla \right) Q[\bar{f}](x)
\triangleq I_1 + I_2 + I_3.
\]
The estimate of \( I_1 \) follows from the commutator estimate (4.1) and Proposition 3.1 there exists constant \( C \) depending on \( \varphi(x) \) such that
\[
\| [T_\epsilon, \varphi(x)](Q[\bar{f}] - Q[\bar{f}]) \|_{L^2(\mathbb{R}^d)} \leq C \| Q[\bar{f}] - Q[\bar{f}] \|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
To treat \( I_2 \), it follows (3.6) that \( Q[\bar{f}](x) = u(x) \in H^1(\mathbb{R}^d) \) and then \( \varphi Q[\bar{f}] \in H^1(\mathbb{R}^d) \). Thus we deduce from Lemma 3.3 that
\[
\lim_{\epsilon \to 0} \| I_2 \|_{L^2(\mathbb{R}^d)} = 0.
\]
The term \( I_3 \) can be estimated in the same way and proof is completed. \[ \square \]

Now we are ready to prove Theorem 1. Note that we choose \( \delta \) as (1.16).

Proof of Theorem 1. Note that, owning to Proposition 3.1
\[ \bar{f}^\epsilon \rightharpoonup \bar{f} \quad \text{weakly in} \ L^1_{loc}(\mathbb{R}^d \times \mathbb{S}^2), \]
where \( \bar{f}(x,m) \) is given by

\[
\bar{f}(x,m) = \left\{ \begin{array}{ll}
\frac{1}{\Delta_x e^{n(m - n(x))^2}} & \text{for } x \in \Omega, \\
\frac{1}{\Delta_x e^{n(m - n(x))^2} dm} & \text{for } x \in \mathbb{R}^d \setminus \Omega
\end{array} \right.
\]

for some \( n(x) \in H^1(\Omega; \mathbb{S}^2) \).

- **Proof of that \( n(x) \) is a weakly harmonic map:**

We shall work with test function \( \varphi(x) \in C_0^\infty(\Omega; \mathbb{R}^d) \). It is easy to see that, there exists some \( \epsilon_0 > 0 \) such that \( \varphi(x) \in C_0^\infty(\Omega_{\delta(\epsilon)}) \) for all \( \epsilon < \epsilon_0 \). We shall assume in the sequel that \( \epsilon < \epsilon_0 \). We will prove that, \( Q[\bar{f}] \), with \( \bar{f} \) given in (3.7), is a weak solution to the following equation:

\[
\text{div}(Q^i[\bar{f}](x) \wedge \nabla Q^i[\bar{f}](x)) = 0, \quad \forall x \in \Omega.
\]

Here and in the sequel, we adopt the Einstein’s convention on the summation over repeat subscript. Since \( f^x \) are assumed to be critical points, it follows from Lemma 3.1 that

\[
Q^i[f^x](x) \wedge \frac{1}{\epsilon}(Q^i[f^x](x) - (Q^i[f^x] \ast g_\epsilon)(x)) = 0, \quad \forall x \in \Omega_{\delta(\epsilon)},
\]

or equivalently, due to the convention (1.17),

\[
\begin{align*}
Q^i[\bar{f}^x](x) & \wedge \frac{1}{\epsilon} \left( Q^i[\bar{f}^x](x) - (Q^i[\bar{f}^x] \ast g_\epsilon)(x) \right) \\
& = - Q^i[f^x](x) \wedge \frac{1}{\epsilon} \int_{\mathbb{R}^d \setminus \Omega} g_\epsilon(x - x') Q^i[f^x](x') dx', \quad \forall x \in \Omega_{\delta(\epsilon)}.
\end{align*}
\]

According to (1.17), we get

\[
Q^i[\bar{f}^x](x) \wedge \mathcal{A}_x Q^i[\bar{f}^x](x)
\]

\[
= - \frac{1}{\epsilon} Q^i[f^x](x) \wedge \int_{\mathbb{R}^d \setminus \Omega} g_\epsilon(x - x') Q^i[h_{n_b}](x') dx', \quad \forall x \in \Omega_{\delta(\epsilon)}.
\]  

(4.3)

Denote

\[
D_\epsilon := - \frac{1}{\epsilon} \int_{\mathbb{R}^d} \varphi(x) \cdot \left( Q^i[f^x](x) \wedge \int_{\mathbb{R}^d \setminus \Omega} g_\epsilon(x - x') Q^i[h_{n_b}](x') dx' \right) dx.
\]  

(4.4)

From the exponential decay of \( g_\epsilon \), we have

\[
g_\epsilon(x - x') \leq \frac{C}{\sqrt{\epsilon}} e^{-\frac{\delta}{\epsilon^2}} \leq C \epsilon \frac{1}{\epsilon^2}, \quad \text{for } |x - x'| \geq \delta(\epsilon).
\]  

(4.5)

As a result, for any \( x \in \Omega_{\delta(\epsilon)} \) and any \( x' \in \mathbb{R}^d \setminus \Omega \), we have \( |x - x'| \geq \delta(\epsilon) \) and thus (4.3) implies

\[
|D_\epsilon| = \frac{1}{\epsilon} \int_{\Omega_{\delta(\epsilon)}} \varphi(x) \cdot \left( Q^i[f^x](x) \wedge \int_{\mathbb{R}^d \setminus \Omega} g_\epsilon(x - x') Q^i[h_{n_b}](x') dx' \right) dx
\]

\[
\leq C \epsilon \frac{1}{\epsilon^2} \int_{\mathbb{R}^d \setminus \Omega} Q^i[h_{n_b}](x) dx \left\| \int_{\Omega_{\delta}} \varphi(x) Q^i[f^x](x) dx \right\|.
\]

On the other hand, it follows from (2.5) and the second part of Lemma 2.3 that \( Q^i[h_{n_b}] \) has compact support. Thus

\[
\lim_{\epsilon \to 0} |D_\epsilon| = 0
\]
and this together with (4.3), (4.4) lead to
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \varphi(x) \cdot (Q_i^i [\tilde{f}] (x) \wedge A_i Q_i^i [\tilde{f}] (x)) \, dx = 0. \tag{4.6}
\]

In order to obtain harmonic map, we need to manipulate the integrand inside (4.6):
\[
\int_{\mathbb{R}^d} \varphi(x) \cdot (Q_i^i [\tilde{f}] (x) \wedge A_i Q_i^i [\tilde{f}] (x)) \, dx
= \int_{\mathbb{R}^d} \varphi(x) \cdot \left( Q_i^i [\tilde{f}] (x) \wedge S^k \circ S^k Q_i^i [\tilde{f}] (x) \right) \, dx
= \int_{\mathbb{R}^d} \left[ S^k \varphi(x) \right] \cdot \left( Q_i^i [\tilde{f}] (x) \wedge S^k Q_i^i [\tilde{f}] (x) \right) \, dx + \int_{\mathbb{R}^d} \varphi(x) \cdot \left( S^k Q_i^i [\tilde{f}] (x) \wedge S^k Q_i^i [\tilde{f}] (x) \right) \, dx
= \int_{\mathbb{R}^d} \left[ S^k \varphi(x) \right] \cdot \left( Q_i^i [\tilde{f}] (x) \wedge S^k Q_i^i [\tilde{f}] (x) \right) \, dx.
\]

The above formula can also be verified tediously using components of various tensors as well as formula (2.2). Taking \( \varepsilon \to 0 \) in the above formula and employing Lemma 3.4 and Lemma 4.2, we obtain
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \varphi(x) \cdot Q_i^i [\tilde{f}] (x) \wedge A_i Q_i^i [\tilde{f}] (x) \, dx = -\frac{1}{d} \int_{\mathbb{R}^d} [\partial_j, \varphi(x)] \cdot (Q_i^i [\tilde{f}] \wedge \partial_j Q_i^i [\tilde{f}]) \, dx.
\]

The together with (4.6) lead to
\[
\int_{\mathbb{R}^d} \partial_j \varphi(x) \cdot (Q_i^i [\tilde{f}] \wedge \partial_j Q_i^i [\tilde{f}]) = 0,
\]
or equivalently, in terms of the components of matrix \( Q = \{ Q_{ij} \}_{1 \leq i,j \leq d} \):
\[
\int_{\Omega} \partial_j \varphi(x) \epsilon^{jk} Q_{ik} [\tilde{f}] \partial_j Q_{lj} [\tilde{f}] = 0. \tag{4.7}
\]

Since \( \tilde{f} \) is uniaxial in \( \Omega \) according to (3.8), it follows from (2.10) that
\[
Q_{ik} (x) = s_2 \left( n_i (x) n_k (x) - \frac{1}{3} \delta_{ik} \right).
\]

Plugging this formula into (4.7) leads to
\[
s_2 \int_{\Omega} \partial_j \varphi(x) \cdot (n (x) \wedge \partial_j n (x)) = 0
\]
which is the weak formulation of the harmonic map equation since \( s_2 \neq 0 \) (see Lemma 2.3).

- **Boundary Condition:** It follows from (3.6), (3.9) that
\[
\nabla (Q[\tilde{f}] \ast g_\varepsilon) \rightharpoonup \nabla Q[\tilde{f}] \text{ weakly in } L^2_{loc}(\Omega)
\]
and continuous embedding \( H^1_{loc}(\mathbb{R}^d) \hookrightarrow H^{\frac{1}{2}} (\partial \Omega) \) implies that
\[
Q[\tilde{f}] \ast g_\varepsilon (x) \rightharpoonup Q[\tilde{f}](x) \text{ weakly in } H^{\frac{1}{2}} (\partial \Omega). \tag{4.8}
\]
On the other hand, it follows from (1.17) and (1.13) that
\[ Q[\bar{f} \ast g_\epsilon(x)] = \int_{\mathbb{R}^d} Q[\bar{f}](x') g_\epsilon(x - x') dx' \]
\[ = \int_{\Omega \cap \Omega_\delta} Q[\bar{f}](x') g_\epsilon(x - x') dx' \]
\[ = \int_{\Omega \cap \Omega_\delta} Q[h_{n_0}](x') g_\epsilon(x - x') dx' + \int_{\Omega_\delta} Q[\bar{f}](x') g_\epsilon(x - x') dx' \]
\[ = \int_{\Omega} Q[h_{n_0}](x') g_\epsilon(x - x') dx' + \int_{\Omega_\delta} (Q[\bar{f}](x') - Q[h_{n_0}](x')) g_\epsilon(x - x') dx'. \]

Given \( x \in \partial \Omega \), we have \( |x - x'| \geq \delta(\epsilon) \) for any \( x' \in \Omega_\delta \). Thus, from the decay estimate (4.5) for \( g_\epsilon \), we get
\[ \lim_{\epsilon \to 0} \int_{\Omega_\delta(x)} (Q[\bar{f}](x') - Q[h_{n_0}](x')) g_\epsilon(x - x') dx' = 0. \]
So we obtain from (4.9) that
\[ \lim_{\epsilon \to 0} Q[\bar{f} \ast g_\epsilon(x)] = Q[h_{n_0}] \text{ a.e. } x \in \partial \Omega. \] (4.10)

It follows from (4.8) and (4.10) that
\[ Q[\bar{f}](x) = Q[h_{n_0}](x) = s_2 (n_0(x) \otimes n_0(x) - \frac{1}{3}I_3), \quad \forall x \in \partial \Omega. \]
This together with the fact that
\[ Q[\bar{f}](x) = s_2 (n(x) \otimes n(x) - \frac{1}{3}I_3), \quad \forall x \in \Omega \]
implies the boundary condition
\[ n(x) = \pm n_0(x), \quad \text{a.e. } x \in \partial \Omega. \]

5. Asymptotic Behavior of Global Minimizers

The task of this section is to prove Theorem 2. We shall first prove the existence of solutions to the minimizing problem:

**Theorem 3.** For each \( \epsilon > 0 \), the minimizing problem (1.12) has a solution \( f_\epsilon \) in the admissible class \( \mathcal{A} \).

**Proof.** The proof will be divided into several parts:

- **Minimizing sequence:** We first note that the space \( \mathcal{A} \) defined by (1.13) is non-empty since the function \( h_{n_0}(x, m) \) defined by (2.6) is in \( \mathcal{A} \). Choosing any minimizing sequence \( f_k \in \mathcal{A} \) such that
\[ \lim_{k \to \infty} A_\epsilon[f_k] = \inf_{f \in \mathcal{A}} A_\epsilon[f]. \]
Then it follows from (1.8) that
\[ \int_{\Omega} \int_{S^2} f_k \log f_k dm dx \leq C \]
and this together with Lemma 2.2 leads to, up to the extraction of a subsequence,
\[ f_k \rightharpoonup f \text{ weakly in } L^1(\Omega \times S^2) \] (5.1)
where \( f \in \mathcal{H}(\Omega) \). To prove that \( f \in \mathcal{A} \), it follows from (5.1) and the continuity of the operator \( Q \) that, 

\[
Q[f_k] \rightarrow Q[f] \quad \text{weakly in } L^1(\Omega).
\] (5.2)

In order to verify the condition in (1.13), we note that, for fixed \( \epsilon > 0 \), \( g_\epsilon(x-x') \) is a smooth kernel. Combining this fact with \( f_k \in \mathcal{A} \) as well as (5.2), we arrive at

\[
\int_{\Omega^d} s_2 (n_0(x) \otimes n_0(x) - \frac{1}{3} \mathbb{I}_3) : \varphi(x) dx = \lim_{k \to \infty} \int_{\Omega^d} Q[f_k](x) : \varphi(x) dx
\]

\[
= \int_{\Omega^d} Q[f](x) : \varphi(x) dx, \quad \forall \varphi \in C^\infty_0(\Omega^d; \mathbb{R}^{d \times d})
\]

and thus \( f \in \mathcal{A} \).

**Proof that \( f \) is a global minimizer:** Note that, for fixed \( \epsilon \), the operator (1.9) is a regular integral operator. So up to the extraction of a subsequence,

\[
\lim_{k \to \infty} U_\epsilon[f_k](x, m) = U_\epsilon[f](x, m) \quad \text{strongly in } C(\overline{\Omega} \times \mathcal{S}^2).
\]

This together with (2.10) implies the weakly lower-semi-continuity of \( A_\epsilon \):

\[
A_\epsilon[f] = \int_{\Omega} \int_{\mathcal{S}^2} (f \log f + fU_\epsilon[f]) dx dm
\]

\[
\leq \liminf_{k \to \infty} \int_{\Omega \times \mathcal{S}^2} f_k \log f_k dx dm + \lim_{k \to \infty} \int_{\Omega} \int_{\mathcal{S}^2} f_k U_\epsilon[f_k] dx dm
\]

\[
\leq \liminf_{k \to \infty} A_\epsilon[f_k].
\]

The following lemma implies that the interaction energy in (1.8) can be expressed in terms of the \( Q \)-tensor of the number density function:

**Lemma 5.1.** The convolution type potential (1.8) can be written by

\[
A_\epsilon[f] = \int_{\Omega \times \mathcal{S}^2} f \log f dx dm - \alpha \int_{\Omega} |Q[f](x)|^2 dx
\]

\[
+ \alpha \int_{\Omega} M[f](x) : \left( Q[f](x) - (Q[f] * \Omega g_\epsilon)(x) \right) dx + C_1(g_\epsilon, \Omega),
\] (5.3)

where \( C_1(g_\epsilon, \Omega) \) are explicit constants that are independent of \( f \).

**Proof.** It suffices to consider the interaction part:

\[
\alpha^{-1} \int_{\Omega \times \mathcal{S}^2} f(x', m') U_\epsilon[f](x, m) dm dm' dx dx'
\]

\[
= \int_{\Omega \times \Omega} \int_{\mathcal{S}^2 \times \mathcal{S}^2} f(x', m') f(m, m) |m \times m'|^2 g_\epsilon(x-x') dm dm' dx dx'
\]

\[
= \int_{\Omega \times \Omega} \int_{\mathcal{S}^2 \times \mathcal{S}^2} f(x', m') f(x, m) (1 - |m \cdot m'|^2) g_\epsilon(x-x') dm dm' dx dx'
\]

\[
= \int_{\Omega \times \Omega} \int_{\mathcal{S}^2 \times \mathcal{S}^2} f(x', m') f(x, m) \left( \frac{2}{3} - (m \otimes m - \frac{1}{3} \mathbb{I}_3) : (m' \otimes m' - \frac{1}{3} \mathbb{I}_3) \right) g_\epsilon(x-x') dm dm' dx dx'
\]

\[
= \frac{2}{3} \int_{\Omega} g_\epsilon(x-x') dx dx' - \int_{\Omega} |Q[f](x)|^2 dx
\]

\[
+ \int_{\Omega} Q[f](x) : (Q[f](x) - Q[f](x')) g_\epsilon(x-x') dx dx'.
\]
This together with (1.8) implies
\[
A_r[f] = \int_{\Omega \times \mathbb{S}^2} f \log f dx dm - \alpha \int_{\Omega} |Q[f](x)|^2 dx \\
+ \alpha \int_{\Omega} Q[f](x) : (Q[f](x) - (Q[f] * \Omega g_e)(x)) dx + C_1(g_e, \Omega)
\]
with
\[
C_1(g_e, \Omega) = \frac{2\alpha}{3} \int_{\Omega \times \Omega} g_e(x - x') dx dx'.
\]

Lemma 5.2. Let \(\alpha, \eta, g\) be the same as in Theorem 1 and \(\delta\) satisfy (1.16) for some \(\sigma \in (0, \frac{1}{2})\). Then there exists some \(\epsilon\)-independent constant \(C > 0\) such that
\[
\frac{2}{\epsilon} \int_{\Omega} (A[f^\epsilon] - A[h_{n_b}]) dx + \alpha \| \mathcal{T}_r Q[\tilde{f}^\epsilon] \|_{L^2(\mathbb{S}^2)}^2 \leq \alpha \| \mathcal{T}_r Q[h_{n_b}] \|_{L^2(\mathbb{S}^2)}^2 + O(\sqrt{\epsilon}) \leq C.
\]

Proof. Since \(f^\epsilon\) is a global minimizer and \(h_{n_b}|_\Omega\) belongs to the class \(\mathcal{A}\) defined in (1.13), we have
\[
A_r[f^\epsilon] \leq A_r[h_{n_b}].
\]
Thus it follows from (5.3) and (1.3) that
\[
\int_{\Omega} A[f^\epsilon](x) dx + \alpha \int_{\Omega} Q[f^\epsilon](x) : (Q[f^\epsilon](x) - Q[f^\epsilon] * \Omega g_e(x)) dx \\
= \int_{\Omega \times \mathbb{S}^2} f^\epsilon \log f^\epsilon dx dm + \alpha \left( \frac{2}{3} |\Omega| - \int_{\Omega} |Q[f^\epsilon](x)|^2 dx \right) \\
+ \alpha \int_{\Omega} Q[f^\epsilon](x) : (Q[f^\epsilon](x) - Q[f^\epsilon] * \Omega g_e(x)) dx \\
\leq \int_{\Omega \times \mathbb{S}^2} h_{n_b} \ln h_{n_b} dx dm + \alpha \left( \frac{2}{3} |\Omega| - \int_{\Omega} |Q[h_{n_b}](x)|^2 dx \right) \\
+ \alpha \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}](x) - Q[h_{n_b}] * \Omega g_e(x)) dx \\
= \int_{\Omega} A[h_{n_b}](x) dx + \alpha \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}](x) - Q[h_{n_b}] * \Omega g_e(x)) dx.
\]

On the other hand, it follows from the definition of \(\tilde{f}^\epsilon\) in (1.17) that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} Q[\tilde{f}^\epsilon](x) : Q[\tilde{f}^\epsilon](x') g_e(x - x') dx dx' - \int_{\Omega^c \times \Omega^c} Q[h_{n_b}](x) : Q[h_{n_b}](x') g_e(x - x') dx dx' \\
= \int_{\Omega \times \Omega} Q[f^\epsilon](x) : Q[f^\epsilon](x') g_e(x - x') dx dx' + 2 \int_{\Omega^c} Q[\tilde{f}^\epsilon](x) : \int_{\Omega} Q[\tilde{f}^\epsilon](x') g_e(x - x') dx' dx \\
= \int_{\Omega} Q[f^\epsilon](x) : (Q[f^\epsilon] * \Omega g_e)(x) dx + 2 \int_{\Omega^c} Q[h_{n_b}](x) : \int_{\Omega} Q[f^\epsilon](x') g_e(x - x') dx' dx.
\]
Likewise,
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} Q[h_{n_b}](x) : Q[h_{n_b}](x') g_e(x - x') dx dx' - \int_{\Omega^c \times \Omega^c} Q[h_{n_b}](x) : Q[h_{n_b}](x') g_e(x - x') dx dx' \\
= \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}] * \Omega g_e)(x) dx + 2 \int_{\Omega^c} Q[h_{n_b}](x) : \int_{\Omega} Q[h_{n_b}](x') g_e(x - x') dx' dx.
\]
Subtracting the above two identities leads to
\[
\int_{\mathbb{R}^d} Q[\tilde{f}'](x) : Q[\tilde{f}'](x') g_e(x-x') dx dx' - \int_{\mathbb{R}^d} Q[h_{n_b}](x) : Q[h_{n_b}](x') g_e(x-x') dx dx'
= \int_{\Omega} Q[\tilde{f}'](x) : (Q[\tilde{f}'] * \Omega g_e)(x) dx - \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}] * \Omega g_e)(x) dx
+ 2 \int_{\Omega^c} Q[h_{n_b}](x) : \int_{\Omega} (Q[\tilde{f}'](x') - Q[h_{n_b}](x')) g_e(x-x') dx' dx
= \int_{\Omega} Q[\tilde{f}'](x) : (Q[\tilde{f}'] * \Omega g_e)(x) dx - \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}] * \Omega g_e)(x) dx
+ 2 \int_{\Omega^c} Q[h_{n_b}](x) : \int_{\Omega} (Q[\tilde{f}'](x') - Q[h_{n_b}](x')) g_e(x-x') dx' dx. \tag{5.6}
\]

In the last step, we used (1.13). Applying (1.5) again, the last integral in (5.6) can be estimated by
\[
\left| \int_{\Omega^c} Q[h_{n_b}](x) : \int_{\Omega} (Q[\tilde{f}'](x') - Q[h_{n_b}](x')) g_e(x-x') dx' dx \right|
\leq C e^{-\frac{\delta}{2}} \int_{\Omega^c} |Q[h_{n_b}](x)| dx \int_{\Omega} |Q[\tilde{f}'](x') - Q[h_{n_b}](x')| dx' \tag{5.7}
\leq C e^{\frac{3}{2}} \int_{\mathbb{R}^d} |Q[h_{n_b}](x)| dx \leq C \epsilon^2.
\]

In the last step of above estimate, we employed Lemma 2.3 together with 2.5 to show that
\[
\int_{\mathbb{R}^d} |Q[h_{n_b}](x)| dx \leq C < \infty.
\]
Combining (5.6) and (5.7)
\[
\int_{\mathbb{R}^d} Q[\tilde{f}'](x) : (Q[\tilde{f}'] * g_e)(x) dx - \int_{\mathbb{R}^d} Q[h_{n_b}](x) : (Q[h_{n_b}] * g_e)(x) dx
= \int_{\Omega} Q[\tilde{f}'](x) : (Q[\tilde{f}'] * \Omega g_e)(x) dx - \int_{\Omega} Q[h_{n_b}](x) : (Q[h_{n_b}] * \Omega g_e)(x) dx + C \epsilon^2. \tag{5.8}
\]
Substituting (5.8) into (5.5) and then employing (1.17) as well as the symmetric property of convolution *, we get
\[
\frac{1}{\epsilon} \int_{\Omega} A[\tilde{f}'] - A[h_{n_b}]) dx + \frac{\alpha}{\epsilon} \int_{\mathbb{R}^d} Q[\tilde{f}'](x) : \left( Q[\tilde{f}'](x) - (Q[\tilde{f}'] * g_e)(x) \right) dx
\leq \frac{\alpha}{\epsilon} \int_{\mathbb{R}^d} Q[h_{n_b}](x) : \left( Q[h_{n_b}](x) - (Q[h_{n_b}] * g_e)(x) \right) dx + C \epsilon^2. \tag{5.9}
\]
In view of (3.17) and (3.14), this implies the first inequality of (5.4). To prove the second part, we shall estimate the right hand side of (5.9). For the sake of simplicity, we shall denote
Moreover, it follows from Lemma 5.2 that $f$ according to (3.16). So the hypotheses for applying Proposition 3.1 is fulfilled for the global $f$ minimizers where $\bar{n}$ for some $n$. It remains to show that Theorem 1, that Complete the proof of Theorem 2. The existence of solution $f^\epsilon$ to (1.12) is proved in Theorem 3. Moreover, it follows from Lemma 5.2 that $f^\epsilon$ satisfies (5.4), which is equivalent to (1.18), according to (3.16). So the hypotheses for applying Proposition 3.1 is fulfilled for the global minimizers $f^\epsilon$:

$$
\tilde{f}^\epsilon \rightharpoonup \tilde{f} \text{ weakly in } L^1_{loc}(\mathbb{R}^d \times S^2),
$$

where $\tilde{f}(x, m)$ is given by

$$
\tilde{f}(x, m) = \begin{cases} 
\frac{e^n(m, n(x))^2}{\int_{S^2} e^n(m, n(x))^2 \, dm} & \text{for } x \in \Omega, \\
\frac{1}{h_{n_b}(x, m)} & \text{for } x \in \mathbb{R}^d \setminus \Omega
\end{cases}
$$

for some $n(x) \in H^1(\Omega; S^2)$. Likewise, we can show, as in the second part of the proof of Theorem 1 that

$$
n(x)|_{\partial \Omega} = \pm n_b. \quad (5.10)
$$

It remains to show that $n(x)$ is a minimizing harmonic map provided that $n_b|_{\partial \Omega}$ is a harmonic map with boundary condition $n_b|_{\partial \Omega}$. It follows from (5.3) that

$$
\frac{2}{\epsilon} \int_\Omega (A[f^\epsilon] - A[h_{n_b}]) \, dx + \alpha \|\mathcal{T}_cQ[\tilde{f}^\epsilon]\|_{L^2(\mathbb{R}^d)}^2 \leq \alpha \|\mathcal{T}_cQ[h_{n_b}]\|_{L^2(\mathbb{R}^d)}^2 + O(\sqrt{\epsilon}). \quad (5.11)
$$

On the other hand, Lemma 3.3 and Lemma 3.4 state that

$$
\mathcal{T}_cQ[\tilde{f}^\epsilon] \rightharpoonup -i\sqrt{\frac{\mu}{\sigma}} \nabla Q[f] \text{ weakly in } L^2_{loc}(\mathbb{R}^d),
$$

$$
\mathcal{T}_cQ[h_{n_b}] \rightharpoonup -i\sqrt{\frac{\mu}{\sigma}} \nabla Q[h_{n_b}] \text{ strongly in } L^2(\mathbb{R}^d).
$$

These enable us to pass to the limit in (5.11) using lower semi-continuity

$$
\|\nabla Q[\tilde{f}]\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla Q[h_{n_b}]\|_{L^2(\mathbb{R}^d)}^2,
$$

In the last step, we employed Lemma 2.3 and (2.5) successively. □
which is equivalent to
\[
\int_{\Omega} |\nabla Q[f]|^2(x) dx + \int_{\mathbb{R}^4 \setminus \Omega} |\nabla Q[h_{nb}]|^2(x) dx \leq \int_{\Omega} |\nabla Q[h_{nb}]|^2(x) dx.
\]

Owing to (2.10), we obtain
\[
\|\nabla n(x)\|_{L^2(\Omega)} \leq \|\nabla n_b(x)\|_{L^2(\Omega)}.
\]
This combined with (5.10) implies that \(n(x)\) is a minimizing harmonic map. \(\square\)

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