ON THE CAUCHY PROBLEM FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE OVER THE FIELD OF \( p \)-ADIC NUMBERS. I. \(^1\)

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ABSTRACT. For the Cauchy problem for an operator differential equation of the form \( y'(z) = Ay(z) \), where \( A \) is a closed linear operator on a Banach space over the field of \( p \)-adic numbers, the criterion of well-posedness in the class of locally analytic vector-functions is established. It is shown how the Cauchy-Kovalevskaya theorem for \( p \)-adic partial differential equations may be obtained as a particular case from this criterion.

1. Let \( \mathcal{B} \) be a Banach space with norm \( \| \cdot \| \) over the completion \( \Omega = \Omega_p \) of an algebraic closure of the field of \( p \)-adic numbers [1 - 3] (\( p \) is prime), and let \( A \) be a closed linear operator on \( \mathcal{B} \), that is, the convergences \( \mathcal{D}(A) \ni x_n \to x \) and \( Ax_n \to y \ (n \to \infty) \) in \( \mathcal{B} \) imply the inclusion \( x \in \mathcal{D}(A) \) and the equality \( Ax = y \) (\( \mathcal{D}(\cdot) \) is the domain of an operator).

For a number \( \alpha > 0 \), we put
\[
E_\alpha(A) = \left\{ x \in \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(A^n) \mid \exists c = c(x) > 0 \ \forall k \in \mathbb{N}_0 \ \| A^k x \| \leq c \alpha^k \right\}.
\]

The linear space \( E_\alpha(A) \) is a Banach space with respect to the norm
\[
\| x \|_\alpha = \sup_{n \in \mathbb{N}_0} \frac{\| A^n x \|}{\alpha^n}.
\]

The set
\[
E(A) = \bigcup_{\alpha > 0} E_\alpha(A)
\]

is endowed with the inductive limit topology of the Banach spaces \( E_\alpha(A) \):
\[
E(A) = \text{ind lim}_{\alpha \to \infty} E_\alpha(A).
\]

By the closed graph theorem, \( E(A) \) coincides with \( \mathcal{B} \) if and only if \( \mathcal{D}(A) = \mathcal{B} \). The elements of \( E(A) \) are called entire vectors of exponential type for the operator \( A \). Define the type \( \sigma(x; A) \) of a vector \( x \in E(A) \) as
\[
\sigma(x; A) = \inf \{ \alpha > 0 : x \in E_\alpha(A) \} = \lim_{n \to \infty} \| A^n x \|^{\frac{1}{n}}.
\]

Thus, the equality \( \sigma(x; A) = \sigma \) means that for an arbitrary \( \varepsilon > 0 \), there exists a constant \( c_\varepsilon = c_\varepsilon(x) > 0 \) such that
\[
\forall n \in \mathbb{N}_0 \ \| A^n x \| \leq c_\varepsilon (\sigma + \varepsilon)^n,
\]

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and
\[ \lim_{i \to \infty} \frac{\| A^{n_i} x \|}{(\sigma - \varepsilon)^{n_i}} = \infty \]
for some subsequence \( n_i \to \infty \) when \( i \to \infty \). In the case, where the operator \( A \) is bounded, the type of a vector \( x \in \mathfrak{B} = E(A) \) does not exceed the norm of \( A \):
\[ \forall x \in \mathfrak{B} \quad \sigma(x; A) \leq \| A \|. \]

2. The object of consideration now is a power series
\[ y(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathfrak{B}, \quad z \in \Omega. \] (1)

For such a series the convergence radius is determined by the formula
\[ r = r(y) = \frac{1}{\lim_{n \to \infty} \sqrt[n]{|c_n|}}. \] (2)

In the open disk \( d(r^{-}; \Omega) = \{ z \in \Omega : |z|_p < r \} \), the series (1) defines a vector-function \( y(z) \) with values in \( \mathfrak{B} \) (\( | \cdot |_p \) is the \( p \)-adic valuation on \( \Omega \)).

Denote by \( \mathfrak{A}_{loc}(\mathfrak{B}) \) the set of all vector-functions \( y(z) \) which are represented by a series of the form (1) with \( r(y) > 0 \). It is obvious that \( \mathfrak{A}_{loc}(\mathfrak{B}) \) is a vector space over \( \Omega \). We call its elements locally analytic vector-functions. The convergence \( y_n \to y (n \to \infty) \) in \( \mathfrak{A}_{loc}(\mathfrak{B}) \) means that there exists a number \( \delta > 0 \) such that \( r(y_n) \geq \delta \), for any \( n \in \mathbb{N} \), and for an arbitrary \( \varepsilon \in (0, \delta) \),
\[ \sup_{|z|_p \leq \delta - \varepsilon} \| y_n(z) - y(z) \| \to 0, \quad n \to \infty. \]

Let \( y \in \mathfrak{A}_{loc}(\mathfrak{B}) \). Its derivatives are defined as
\[ y^{(k)}(z) = \sum_{n=0}^{\infty} (n+1) \ldots (n+k) c_{n+k} z^n, \quad k \in \mathbb{N}. \]

It follows from (2) that
\[ r(y^{(k)}) \geq r(y). \]

It is easily checked also that if \( z \to 0 \), then
\[ y(z) \to y(0) = c_0, \quad \text{and} \quad \frac{y^{(k)}(z) - y^{(k)}(0)}{z} \to y^{(k+1)}(0) = c_{k+1}(k+1)! \] (3)
in the topology of \( \mathfrak{B} \).

3. Let us consider the Cauchy problem
\[ \begin{cases} \frac{dy(z)}{dz} = Ay(z) \\ y(0) = y_0, \end{cases} \] (4)

where \( A \) is a closed linear operator on \( \mathfrak{B} \). We say that a vector-function \( y(z) \) from \( \mathfrak{A}_{loc}(\mathfrak{B}) \) is a solution of problem (4) if \( y(z) \in \mathcal{D}(A) \) for \( z \in d(r(y)^{-}; \Omega) \) and satisfies (4) in this disk.
Theorem 1. In order that problem (4) have a solution in $\mathfrak{A}_{\text{loc}}(\mathfrak{B})$, it is necessary and sufficient that $y_0 \in E(A)$; moreover $\sigma(y_0; A)r(y) = p^{-\frac{1}{p-1}}$. The problem (4) is well-posed, that is, the solution is unique, and if a sequence of initial data $y_{n,0} \in E(A)$ converges to $y_0$ in $E(A)$, then the sequence of the corresponding solutions $y_n(z) \in \mathfrak{A}_{\text{loc}}(\mathfrak{B})$ converges to $y(z)$ in the space $\mathfrak{A}_{\text{loc}}(\mathfrak{B})$.

Proof. Suppose that

$$y(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k \in \mathfrak{B}, \quad z \in d(r(y)^{-}; \Omega),$$

is a solution of (4). Then $c_k \in \mathcal{D}(A), k \in \mathbb{N}_0$. Indeed, $c_0 = y(0) = y_0 \in \mathcal{D}(A)$. Since

$$\mathcal{D}(A) \ni \frac{y(z) - c_0}{z} = \sum_{k=0}^{\infty} c_k z^{k-1} \to c_1,$$

and

$$\mathcal{A}(y(z) - y_0) = \frac{y'(z) - y'(0)}{z} \to 2c_2$$

as $z \to 0$, we have because of closure of the operator $\mathcal{A}$ that $c_1 \in \mathcal{D}(A)$, and $\mathcal{A}c_1 = 2c_2$. Using (3), we get by induction that

$$\forall k \in \mathbb{N} \quad c_k = \frac{\mathcal{A}c_{k-1}}{k} \in \mathcal{D}(A),$$

hence,

$$\forall k \in \mathbb{N} \quad y_0 \in \mathcal{D}(A^k), \quad A^k y_0 = k! c_k.$$

In view of (2),

$$\frac{1}{r(y)} = \lim_{n \to \infty} \sqrt[n]{\frac{\|A^n y_0\|}{|n!|_p}}.$$

Taking into account the equality

$$\lim_{n \to \infty} \sqrt[n]{|n!|_p} = p^{-\frac{1}{p-1}}$$

(see [1]), we obtain

$$\lim_{n \to \infty} \frac{\|A^n y_0\|}{|n!|_p} = \frac{p^{-\frac{1}{p-1}}}{r(y)},$$

whence

$$\forall n \in \mathbb{N}_0 \quad \|A^n y_0\| \leq c_n^n,$$

where $0 < c = \text{const}$. So, $y_0 \in E(A)$, and $\sigma(y_0; A)r(y) = p^{-\frac{1}{p-1}}$.

Conversely, let $y_0$ be an entire vector of exponential type for the operator $A$ with $\sigma(y_0; A) = \sigma$. Then the series

$$y(z) = \sum_{k=0}^{\infty} \frac{A^k y_0}{k!} z^k \quad (5)$$

is convergent in the disk $d(r^{-}; \Omega)$, where

$$r(r) = \left(\lim_{n \to \infty} \sqrt[n]{\frac{\|A^n y_0\|}{|n!|_p}}\right)^{-1} = \frac{\lim_{n \to \infty} \sqrt[n]{|n!|_p}}{\lim_{n \to \infty} \sqrt[n]{\|A^n y_0\|}} = \frac{p^{-\frac{1}{p-1}}}{\sigma}.$$
We shall prove now that if \( z \in d(r^-; \Omega) \), then \( y(z) \in \mathcal{D}(A) \). Really, since every component of series (5) belongs to \( \mathcal{D}(A) \), the sums \( S_n(z) = \sum_{k=0}^{n} \frac{A^k y_0}{k!} z^k \) belong to \( \mathcal{D}(A) \), too. For \( z \in d(r^-; \Omega) \), the sequence \( S_n(z) \) converges to \( y(z) \) \((n \to \infty)\) in the topology of \( \mathcal{B} \). As \( \sigma(A y_0; A) = \sigma(y_0; A) = \sigma \), the sequence \( A S_n(z) = \sum_{k=0}^{n} \frac{A^{k+1} y_0}{k!} z^k \), \( n \in \mathbb{N} \), converges in \( \mathcal{B} \) \((n \to \infty)\) in the same disk \( d(r^-; \Omega) \). Since the operator \( A \) is closed, we have that \( y(z) \in \mathcal{D}(A) \) when \( z \in d(r^-; \Omega) \).

The formal differentiation of series (5) verifies that \( y(z) \) satisfies (4). Thus, the vector-function (5) is a solution of problem (4).

It remains to check the well-posedness of problem (4). Assume that \( y_{n,0} \to y_0 \) \((n \to \infty)\) in \( E(A) \). This means that there exists a number \( \alpha > 0 \) such that \( y_{n,0} \in E_\alpha(A) \) for sufficiently large \( n \), and \( \|y_{n,0} - y_0\|_\alpha \to 0 \) as \( n \to \infty \). It follows from the above proof of sufficiency that

\[
    r(y_n) \geq \frac{p^{-1}}{\alpha}.
\]

So, we may take \( \delta = \frac{p^{-1}}{\alpha} \), and to complete the proof, we need only show that for an arbitrary fixed \( \varepsilon \in (0, 1) \), \( \|y_n(z) - y(z)\| \to 0 \) \((n \to \infty)\) uniformly in the disk \( d((1 - \varepsilon)\delta^-; \Omega) \). We have

\[
    \|y_n(z) - y(z)\| = \left\| \sum_{k=0}^{\infty} \frac{A^k (y_{n,0} - y_0) z^k}{k!} \right\| \\
    \leq \sum_{k=0}^{\infty} \frac{\|A^k (y_{n,0} - y_0) z^k\|}{k!} = \sum_{k=0}^{\infty} \frac{\|A^k (y_{n,0} - y_0) z^k\|}{k!}.
\]

Taking into account that

\[
    \frac{1}{z} \leq \frac{p}{z}
\]

(see [2]), we arrive at the inequality

\[
    \|y_n(z) - y(z)\| \leq \varepsilon^{-1} \|y_{n,0} - y_0\|_\alpha,
\]

and therefore problem (4) is well-posed. \( \square \)

It is seen from the proof of Theorem 1 that the series \( \sum_{k=0}^{\infty} \frac{A^k y_0}{k!} z^k \) is convergent for all \( z \in \Omega \) if and only if \( y_0 \in \bigcap_{\alpha > 0} E_\alpha(A) \).

**Corollary 1.** If the operator \( A \) is bounded, then for any \( y_0 \in \mathcal{B} \), the Cauchy problem (4) is well-posed in the class \( \mathfrak{A}_{loc}(\mathcal{B}) \).

**Remark 1.** If \( \mathcal{B} \) is a Banach space over the field \( \mathbb{C} \) of complex numbers, then, as was shown in [4], problem (4) is well-posed in the class of locally analytic vector-functions if and only if \( y_0 \) is an analytic vector for the operator \( A \), that is, \( y_0 \in \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(A^n) \), and

\[
    \exists \alpha > 0 \ \exists c > 0 \ \forall k \in \mathbb{N}_0 \ \|A^k y_0\| \leq c \alpha^k k!.
\]

In this case, in order that problem (4) be well-posed in the class of entire vector-functions of exponential type, it is necessary and sufficient that \( y_0 \in E(A) \).
4. In this section we show how the existence and uniqueness theorem for the Cauchy problem for partial differential equations over a non-archimedean field of characteristic zero (see [5]) may be obtained from the above result.

Let $A_\rho$ be the space of $\Omega$-valued functions $f(x)$ analytic on the $n$-dimensional disk

$$d(\rho^+; \Omega^n) = \left\{ x = (x_1, \ldots, x_n) \in \Omega^n : |x|_p = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \leq \rho \right\}.$$ 

This means that

$$f(x) = \sum_\alpha f_\alpha x^\alpha, \quad f_\alpha \in \Omega, \quad \lim_{|\alpha| \to \infty} |f_\alpha|_p \rho^{\alpha} = 0,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The space $A_\rho$ is a non-archimedean Banach space with respect to the norm

$$\|f\|_\rho = \sup_\alpha |f_\alpha|_p \rho^{\alpha}.$$

It is clear that the differential operators

$$\frac{\partial f}{\partial x_j} = \sum_\alpha \alpha_j f_\alpha x_1^{\alpha_1} \cdots x_j^{\alpha_j-1} \cdots x_n^{\alpha_n}, \quad j = 1, \ldots, n,$$

are bounded in $A_\rho$, and

$$\left\| \frac{\partial f}{\partial x_j} \right\|_\rho = \max_\alpha |f_\alpha|_p \rho^{\alpha-1} \leq \frac{1}{\rho} \max_\alpha |f_\alpha|_p \rho^{\alpha} = \frac{1}{\rho} \|f\|_\rho.$$

The multiplication map

$$G : f \mapsto fg,$$

where $g \in A_\rho$, is bounded in $A_\rho$, too, and

$$\|G\| = \|g\|_\rho.$$

Indeed, let $f(x) = \sum_\alpha f_\alpha x^\alpha$, $g(x) = \sum_\alpha g_\alpha x^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then

$$f(x)g(x) = \sum_\alpha c_\alpha x^\alpha,$$

where

$$c_\alpha = \sum_{0 \leq i \leq \alpha} f_i g_{\alpha-i} = \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_n=0}^{\alpha_n} f_{i_1,\ldots,i_n} g_{\alpha_1-i_1,\ldots,\alpha_n-i_n} \quad (i = (i_1, \ldots, i_n)).$$

So,

$$\|fg\|_\rho = \sup_\alpha \max_{0 \leq i \leq \alpha} |f_1|_p |g_{\alpha-i}|_p \rho^{|\alpha|-|i|} \leq \|f\|_\rho \|g\|_\rho.$$

We pass now to the Cauchy problem

$$\begin{cases}
\frac{\partial u(t,x)}{\partial t} = \sum_{|\beta|=0}^n a_\beta(x) D^\beta u(t,x) \\
u(0,x) = \varphi(x),
\end{cases} \quad (6)$$
where \( a_\beta(x) \in \mathcal{A}_\rho, \ \varphi(x) \in \mathcal{A}_\rho, \ D^\beta = \frac{\partial^{\vert \beta \vert}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}. \)

In the space \( \mathcal{A}_\rho \) we define the operator \( A \) as follows:

\[
 f \mapsto Af = \sum_{|\beta|=0}^{n} a_\beta D^\beta f.
\]

The relations

\[
 \left\| \sum_{|\beta|=0}^{n} a_\beta D^\beta f \right\|_\rho \leq \max_{\beta} \|a_\beta D^\beta f\|_\rho \leq \max_{\beta} \|a_\beta\|_\rho \|D^\beta f\|_\rho \leq \max_{\beta} \{\rho^{-|\beta|}\|a_\beta\|_\rho\} \|f\|_\rho
\]

show that the operator \( A \) is bounded, and

\[
 \|A\| \leq \max_{\beta} \{\rho^{-|\beta|}\|a_\beta\|_\rho\}.
\]

It follows from Corollary 1 that problem (6) is well-posed in \( \mathcal{A}_{\text{loc}}(\mathcal{A}_\rho) \) in the disk

\[
 \left\{ t \in \Omega : |t|_p < \frac{p^{-\frac{1}{p-1}}}{\max_{\beta} \{\rho^{-|\beta|}\|a_\beta\|_\rho\}} \right\}.
\]

It should be noted that Theorem 1 is valid also in the case where \( \mathcal{B} \) is a Banach space over an arbitrary non-archimedean field \( K \) of characteristic zero. But then the role of \( p^{-\frac{1}{p-1}} \) is played by a certain constant \( b = b(K) \) such that \( \frac{1}{|n|!_K} \leq b^n \) (see, for instance, [6]).
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