THE STATISTICAL DISCREPANCY BETWEEN THE INTERGALACTIC MEDIUM AND DARK MATTER FIELDS: ONE-POINT STATISTICS

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ABSTRACT

We investigate the relationship between the mass and velocity fields of the intergalactic medium (IGM) and dark matter. Although the evolution of the IGM is dynamically governed by the gravity of the underlying dark matter field, some statistical properties of the IGM inevitably decouple from those of the dark matter once the nonlinearity of the dynamical equations and the stochastic nature of the field is considered. With simulation samples produced by a hybrid cosmological hydrodynamic/N-body code, which is effective in capturing shocks and complicated structures with high precision, we find that the one-point distributions of the IGM field are systematically different from that of dark matter as follows: (1) the one-point distribution of the IGM peculiar velocity field is exponential at least at redshifts less than 2, while the dark matter velocity field is close to a Gaussian field; (2) although the one-point distributions of the IGM and dark matter are similar, the point-by-point correlation between the IGM and dark matter density fields significantly differs on all scales and redshifts analyzed; (3) the one-point density distributions of the difference between IGM and dark matter fields are highly non-Gaussian and long tailed. These discrepancies violate the similarity between the IGM and dark matter and cannot be explained simply as Jeans smoothing of the IGM. However, these statistical discrepancies are consistent with the fluids described by stochastic-force–driven nonlinear dynamics.

Subject headings: cosmology: theory — intergalactic medium — large-scale structure of universe

1. INTRODUCTION

The mass field of the universe is dominated by dark matter with only a tiny fraction of cosmic matter in the form of baryonic particles. Gravitational clustering of dark matter is the major impetus for cosmic structure formation. Collapsed dark matter halos host various baryonic light-emitting and absorbing objects and information about dark matter is available only via baryons. The major component of baryonic matter is in the form of gas, the intergalactic medium (IGM). (Here we use IGM to mean baryonic gas both before and after the galaxy formation.) Therefore, the dynamical relationship between baryonic gas and the underlying dark matter field is essential in order to understand the origin and evolution of cosmic structures.

Since the IGM is only a tiny fraction of cosmic matter, it is usually assumed that its clustering behavior on scales larger than the Jeans length traces the underlying dark matter field. The density and velocity distribution of the IGM is considered to be the same as the dark matter field point-by-point on scales larger than the Jeans length scales. That is,

\[ \delta_{\text{igm}}(x, t) = \delta_{\text{dm}}(x, t), \quad \mathbf{v}_{\text{igm}}(x, t) = \mathbf{v}_{\text{dm}}(x, t), \]  

where \( \delta_{\text{igm}} \) and \( \delta_{\text{dm}} \) are the mass density contrasts and \( \mathbf{v}_{\text{igm}} \) and \( \mathbf{v}_{\text{dm}} \) are the velocity fields, smoothed on the Jeans length scales. Equation (1) applies during the linear regime. Even if the IGM is initially distributed differently than the dark matter, linear growth modes will lead to equation (1) on scales larger than the Jeans length (Bi et al. 1992; Fang et al. 1993; Nusser 2000; Nusser & Haehnelt 1999; see also Appendix B). In this case, all statistical properties of the IGM at scales greater than the Jeans length are completely determined by the dark matter field. In other words, the dynamical behavior of the IGM field can be obtained from the dark matter field via a similarity mapping (e.g., Kaiser 1986).

Because the nonlinear evolution of the IGM is also driven by the gravity of the underlying dark matter, it is often assumed that the relation (1) will hold even in the nonlinear regime. However, this assumption is probably not valid. It has been shown in hydrodynamic studies that a passive substance generally decouples from the underlying field during nonlinear evolution (for a review, see Shraiman & Siggia 2000). For instance, a passive substance might be highly non-Gaussian even when the underlying field is Gaussian (Kraichnan 1994). This nonlinear decoupling is generic to systems consisting of a “passive substance” and an underlying stochastic mass field.

The IGM and dark matter fields interact stochastically since the initial perturbations of the cosmic mass and velocity fields are randomly distributed. Their mass and velocity fields act as random variables. The importance of the stochastic nature for the evolution of the cosmic mass and velocity field has been emphasized in many studies (e.g., Berera & Fang 1994; Jones 1999; Buchert et al. 1999; Coles & Spencer 2003; Ma & Bertschinger 2003). In this paper, we will address the statistical discrepancy between the random fields of the IGM and dark matter in nonlinear regime.

Statistical discrepancies between the IGM and dark matter mass fields may have already been detected in previous studies. The adhesion model of the IGM shows that the deterministic relation \( \delta_{\text{igm}}(x, t) = \delta(x, t) \) no longer holds (Jones 1999). The stochastic nature of the cosmic mass field indicates that the IGM should be described by a random-force–driven Burgers equation (Matrarese & Mohayee 2002), which does not always yield the deterministic solution \( \delta_{\text{igm}}(x, t) = \delta(x, t) \). Cosmological hydrodynamic simulations also find that \( \rho_{\text{igm}} \) does not tightly

\[ \rho_{\text{igm}}(x, t) = \rho(x, t) + \eta(x, t), \]  

where \( \eta(x, t) \) is the stochastic field, which is non-Gaussian and long tailed.
correlate with $\rho_{\text{dm}}$ but is largely scattered around the line $\delta_{\text{igm}} = \delta_{\text{dm}}$, with this scatter not due to noise (Gnedin & Hui 1998). It has also been found that the scatter defined by

$$\Delta \delta(x, t) = \delta_{\text{dm}}(x, t) - \delta_{\text{igm}}(x, t), \Delta \mathbf{v}(x, t) = \mathbf{v}_{\text{dm}}(x, t) - \mathbf{v}_{\text{igm}}(x, t)$$

is highly non-Gaussian (Feng et al. 2003). This strengthens the conclusion that the discrepancy between $\delta_{\text{igm}}$ and $\delta_{\text{dm}}$ is not due to computational noise or other Gaussian processes, but probably arises from the nonlinear evolution of the random fields.

Here we study this discrepancy at a more fundamental level. As a first step, we concentrate on the density and velocity field one-point IGM distributions, and their discrepancy from dark matter. The outline of this paper is as follows. Section 2 addresses the dynamical mechanisms and gives predictions on the statistical discrepancy between the IGM and dark matter density and velocity fields. Section 3 presents the cosmological hydro simulation scheme and samples. The relevant one-point statistics are developed in § 4. Section 5 shows the discrepancy and tests the predictions with one-point statistics of the density and velocity fields. Finally, conclusions and discussions are in § 6.

### 2. STATISTICAL DISCREPANCY BETWEEN THE INTERGALACTIC MEDIUM AND DARK MATTER FIELDS

During the linear regime, thermal diffusion leads to a discrepancy between the IGM field and dark matter field on scales up to the Jeans length. That is, the IGM mass density perturbations are suppressed on scales less than the Jeans length. The IGM density perturbations on scales larger than the Jeans length are Gaussian if the dark matter density perturbations are Gaussian. However, in the nonlinear regime, a statistical discrepancy very different from a simple thermal diffusion can appear. This can be illustrated by considering the isothermal model of the IGM. In this case, the IGM density is given by

$$\rho_{\text{igm}}(x) \propto \exp \left[-m \phi(x)/k_B T\right]$$

where $\phi(x)$ is gravitational potential, $T$ is the local temperature, $m$ is the mass of the IGM particles, and $k_B$ is the Boltzmann constant. If $\phi(x)$ is a Gaussian random field, $\rho_{\text{igm}}$ will be a lognormal random field (Zel’dovich et al. 1990) and a statistical discrepancy between $\rho_{\text{igm}}(x)$ and dark matter arises. Although the isothermal model is not realistic on large scales in general, it reveals that the statistical discrepancy between the IGM and dark matter might arise if (1) the IGM evolution is nonlinear and (2) the relevant fields, like $\phi(x)$, are stochastic.

#### 2.1. The Statistical Discrepancy of Velocity Fields

A more realistic discrepancy appears when considering the differences between the dynamics of the peculiar velocity fields of the IGM $\mathbf{v}_{\text{igm}}$ and dark matter $\mathbf{v}_{\text{dm}}$. Since dark matter particles are collisionless, the intersection of the dark matter particle trajectories will lead to a multivalued velocity field. On the other hand, as a fluid, the velocity field of the IGM will always be single-valued. At the intersection of dark matter particle trajectories, the IGM velocity field will be discontinuous and yield shocks or complicated structures (Shandarin & Zel’dovich 1984). Shocks in the IGM can significantly change the mass density and velocity of the baryon gas but will exert no direct effect on the dark matter field. It is at this point that the dynamical similarity between the IGM and dark matter is broken.

We analyze this situation using the dynamical equations for dark matter and the IGM. For growth modes, the peculiar velocity field of the dark matter is vortex-free. One can define a velocity potential $\varphi$ as $\mathbf{v}_{\text{dm}} = -(1/\alpha)\nabla \varphi_{\text{dm}}$, where $\alpha$ is the cosmic scale factor. The dynamical equation of the velocity potential is (see Appendix A)

$$\frac{\partial \varphi_{\text{dm}}}{\partial t} - \frac{1}{2a^2} (\nabla \varphi_{\text{dm}})^2 = \phi,$$

where $\phi$ is the gravitational potential and depends on density perturbation $\delta_{\text{dm}} = (\rho_{\text{dm}} - \bar{\rho}_{\text{dm}})/\bar{\rho}_{\text{dm}}$ via the Poisson equation (A3). Generally, the field $\phi$ is Gaussian, or only slightly deviates from a Gaussian field with $\langle \phi \rangle = 0$ and variance $\langle \phi^2 \rangle$. Equation (3) is valid till the intersection or shell crossing of dark matter particle trajectories has occurred.

For the IGM growth modes, one can also define a velocity potential $\varphi_{\text{igm}}$, by $\mathbf{v}_{\text{igm}} = -(1/\alpha)\nabla \varphi_{\text{igm}}$. The dynamical equation (3) are approximately given by (see Appendix B)

$$\frac{\partial \varphi_{\text{igm}}}{\partial t} - \frac{1}{2a^2} (\nabla \varphi_{\text{igm}})^2 - \nu \Delta^2 \varphi_{\text{igm}} = \phi.$$

The coefficient $\nu$ is given by

$$\nu = \frac{\gamma k_B T_0}{m_p a \ln (D(t)/\bar{D})},$$

where $D(t)$ describes the linear growth behavior. The diffusion term $\nu$ in equation (4) is given by the Jeans smoothing. The wavenumber of the comoving Jeans scale is $k_J = (a^2/r^2)(m_p/\gamma k_B T_0)$, where $m_p$ is proton mass. The parameters $\gamma$ and $\mu$ are, respectively, the polytropic index and molecular weight of the IGM.

The nonlinear equation (4) is actually the stochastic-force–driven Burgers equation or the KPZ equation (Kardar et al. 1986; Berera & Fang 1994). Fields governed by equation (5) have been extensively studied to model structure formation (Barabási & Stanley 1995). The solution to equation (4) depends on two characteristic scales: (1) the correlation length $r_c$ of the random field $\phi$ and (2) the dissipation length $d = \nu^{3/4} r_c^{1/2} (\langle \phi^2 \rangle)^{-1/4} (\ln D/\bar{D})^{1/3}$, which is due to Jeans smoothing. The behavior of the field $\varphi_{\text{igm}}$ governed by equation (4) is determined by its Reynolds number defined as $R \equiv (r_c/d)^{1/3}$ (e.g., Lässig 2000) or

$$R = (k_J r_c)^{3/2} \left( \frac{k_J^2}{k} \right)^{1/3} \left( \frac{k_J^2}{k} \right)^{1/3} \langle \delta_{\text{dm}}^2(k) \rangle^{1/3},$$

where $r_c$ is the comoving correlation length and $\delta_{\text{dm}}(k)$ is the Fourier component of the density contrast on wavenumber $k$. To derive equation (6), we assume that the gravitational potential $\phi$ is only given by the dark matter mass perturbation. When the Reynolds number is larger than 1, Burgers turbulence occurs in the $\varphi_{\text{igm}}$ field.

The correlation length $r_c$ of the gravitational potential $\phi$ is larger than the Jeans length and we have $k_J r_c > 1$. Therefore, $R$ can be larger than 1 on scales larger than the Jeans length, even when $\delta_{\text{dm}}(k)$ is of order 1. That is, Burgers turbulence develops in the IGM $\varphi_{\text{igm}}$ field, while the dark matter mass density perturbations are still quasi-linear or weakly nonlinear. We can conclude that the evolution of the probability distribution
function of \( r_{\text{igm}}(x, t) \) and \( r_{\text{dm}}(x, t) \) should be significantly different, the former becoming non-Gaussian earlier than the latter.

When Burgers turbulence develops in the IGM, \( \varphi_{\text{igm}} \) is characterized by strong intermittency and contains discontinuities, or shocks. The probability distribution function (PDF) of \( \varphi_{\text{igm}} \) is long tailed (Lässig 2000). The intermittent spikes are the events that populate the long tail. This feature is supported by observations of the Ly\( \alpha \) flux transmission (Jamkhedkar et al. 2000, 2003; Pandol et al. 2002). That is, the randomly distributed shocks and intermittent spikes lead to the statistical discrepancy between the velocity fields of the IGM and dark matter. One can then expect that the PDF of the statistical discrepancy between the velocity fields of the IGM will lead to a decay of \( \Delta \delta(x) \), which is long tailed. Moreover, the nontrivial difference between the IGM and dark matter holds on the scales larger than the Jeans length of the IGM.

2.2. The Second Moment of the Density Distributions

The nontrivial velocity difference \( \Delta v(x, t) \) will lead to a nontrivial density difference between the IGM and dark matter. The linearized continuity equations for dark matter (eq. [A1]) and IGM (eq. [B1]) are, respectively,

\[
\frac{\partial \delta_{\text{dm}}}{\partial t} + \frac{1}{a} \nabla \cdot \mathbf{v}_{\text{dm}} = 0, \quad (7)
\]

\[
\frac{\partial \delta_{\text{igm}}}{\partial t} + \frac{1}{a} \nabla \cdot \mathbf{v}_{\text{igm}} = 0. \quad (8)
\]

Thus, using the definition for \( \Delta \delta(x, t) \), we have

\[
\Delta \delta(x) = \delta_{\text{dm}} - \delta_{\text{igm}} = -\int dt \nabla \cdot \Delta \mathbf{v}. \quad (9)
\]

This result shows that the IGM mass field \( \delta_{\text{igm}}(x) \) does not trace the dark matter field \( \delta_{\text{dm}}(x) \) point-by-point owing to the discrepancy in their velocity fields.

The discrepancy in the mass fields can be measured by the second moments of the density fields defined as

\[
I_{\text{dm}} = \frac{\langle (\Delta \delta_{\text{dm}}(x))^2 \rangle^{1/2}}{\langle \delta_{\text{dm}}^2(x) \rangle^{1/2}}, \quad (10)
\]

\[
I_{\text{igm}} = \frac{\langle (\Delta \delta_{\text{igm}}(x))^2 \rangle^{1/2}}{\langle \delta_{\text{igm}}^2(x) \rangle^{1/2}}, \quad (11)
\]

and

\[
I = \frac{\langle (\Delta \delta(x))^2 \rangle^{1/2}}{\langle \delta_{\text{igm}}(x) + \delta_{\text{dm}}(x) \rangle^{2/3}}. \quad (12)
\]

If \( \delta_{\text{igm}}(x) \) is perfect tracer of \( \delta_{\text{dm}}(x) \), we have \( I_{\text{dm}} = I_{\text{igm}} = I = 0 \). If both \( \delta_{\text{igm}}(x) \) and \( \delta_{\text{igm}}(x) \) are statistically independent, we have \( I = 1 \). Thus, from equation (9), we expect that \( I, I_{\text{dm}}, \) and \( I_{\text{igm}} \) are smaller at higher redshifts, and larger at lower redshifts.

2.3. The Density Probability Distribution Function Discrepancy

From the continuity equations (eqs. [A1] and [B1]) and the definitions in equation (2), we have

\[
\frac{\partial \Delta \delta(x)}{\partial t} = -\frac{1}{a} \nabla \cdot [\Delta \mathbf{v} + (\Delta \delta) \mathbf{v}_{\text{dm}} + (\Delta \mathbf{v}) \mathbf{v}_{\text{igm}}]. \quad (13)
\]

We define a window sampling given by

\[
\Delta^R(x_0) = \int W_R(x' - x_0) \Delta \delta(x') dx', \quad (14)
\]

where the normalized window function \( W_R(x' - x_0) \) is nonzero around \( x_0 \) on spatial scale \( R \). Since \( \Delta \delta \) and \( \Delta \delta_{\text{igm}} \) are statistically isotropic, for an isotropic window function \( W_R(x') \) the terms containing \( \nabla \Delta \delta \) and \( \nabla \delta_{\text{igm}} \) are negligible. Thus, equation (13) becomes a Langevin equation for \( \Delta^R(x) \)

\[
\frac{d \Delta^R}{dt} = -\frac{1}{a} \lambda \Delta^R + \frac{1}{a} \eta, \quad (15)
\]

The first term on the right-hand side of equation (15) is a friction term with friction coefficient given by

\[
\lambda = g_w \int W_R(x' - x_0) (\nabla \cdot \mathbf{v})(1 + \delta_{\text{igm}}). \quad (16)
\]

Since \( |\delta_{\text{igm}}| < 1 \) at most places, we can drop \( \delta_{\text{igm}} \) in equation (17).

Equation (15) contains an additive stochastic force, \( \eta \), and a multiplicative stochastic force, \( \lambda \). Both stochastic forces are given by the random velocity field. The Langevin equation (eq. [15]) with both additive and multiplicative noise is typically used to model intermittent fields (Graham et al. 1982; Platt et al. 1994; Nakao 1998). In the nonlinear regime, most locations in the dark matter density field have \( \delta_{\text{dm}} < 0 \), and therefore, \( \nabla \cdot \mathbf{v}_{\text{dm}} \) and \( \lambda \) are larger than zero. The friction term \( \lambda \) leads to the decay of \( \Delta^R \). That is, the density perturbation difference \( \Delta^R \) caused by the noise \( \eta \) tends to zero on average. Although \( \lambda \) is positive in most cases, it can go negative as a result of fluctuations. In these cases, \( \Delta^R \) will be amplified exponentially and attain large values, which are the spikes in the field. This is intermittency and the PDF of \( \Delta^R \) will be long-tailed. It has been shown when the stochastic terms \( \eta \) and \( \lambda \) are Gaussian, the PDF of \( \Delta^R \) is a power law in general (Appendix C). Although \( \eta \) and \( \lambda \) for the dark matter given by equations (16) and (17) are not Gaussian, we still can conclude that the PDF of \( \Delta^R \) is generally long tailed, as long tails are a common result of a multiplicative stochastic force. For instance, if one ignores the additive noise term \( \eta \), the solution for equation (15) is \( \Delta^R \propto \exp[-(1/\lambda)\delta_{\text{igm}}] \). Thus, the PDF tail \( \Delta^R \) is longer than that of \( \lambda \) so that, for instance, when \( \lambda \) is Gaussian, \( \Delta^R \) is lognormal. Using equation (C7), we see that the PDF of \( \Delta^R \) is generally highly non-Gaussian on scales larger than the Jeans length. It is flat in the central part and gradually becomes a power law with index not lower than \(-1\).

By closely studying the dynamics of the velocity fields we have found that the statistical discrepancy between the dark matter and IGM velocity fields will manifest itself as follows:

1. The evolution of the PDFs of \( r_{\text{igm}} \) and \( r_{\text{dm}} \) will be significantly different. The former becomes non-Gaussian earlier than the latter. Further, the PDF of the difference \( \Delta v(x) \) will be long tailed.

2. The second moments \( I, I_{\text{dm}}, \) and \( I_{\text{igm}} \) (eqs. [10]–[12]) will be smaller at higher redshifts and larger at lower redshifts.
3. The PDF of $A^2$ are generally highly non-Gaussian on scales larger than the Jeans length.

In the following sections we will test these predictions.

3. HYDRODYNAMIC SIMULATIONS

3.1. WENO Hydrodynamic Simulations

To simulate the IGM, we use the hydrodynamic equations of the IGM in the form of conservation laws, equations (B4)–(B6). Although the momentum equation (B5) is the typical Navier-Stokes equation, gravitational instability leads to a system dominated by growth modes and the dynamical equations essentially become a Burgers equation if only the growth modes are considered (Berera & Fang 1994). It is well known that the Burgers equation does not reduce initial chaos, but increases it (Kraichnan 1968). That is, when the Reynolds number is high, an initially random field always yields a collection of shocks with a smooth and simple variation of the field between the shocks. We conclude that an optimal simulation scheme should capture shock and discontinuity transitions as well as to calculate piecewise smooth functions with a high resolution.

For these reasons we do not use schemes based on smoothed particle hydrodynamic (SPH) algorithms. It is well known that one of the main challenges to the SPH scheme is how to handle shocks or discontinuities because SPH schemes smooth the fields. This problem is not yet well settled (e.g., Borve et al. 2001; Omang et al. 2003). Instead, we will take a Eulerian approach to simulating the IGM. However, there is a basic problem in Eulerian based codes in that they cause unphysical oscillations near a discontinuity. An effective method to reduce the spurious oscillations is given by designed limiters, such as the total-variation diminishing (TVD) schemes (Harten 1983). However, TVD accuracy degenerates to first order near smooth transitions as well as to calculate piecewise smooth functions with a high resolution. For these reasons we do not use schemes based on smoothed particle hydrodynamic (SPH) algorithms. It is well known that one of the main challenges to the SPH scheme is how to handle shocks or discontinuities because SPH schemes smooth the fields. This problem is not yet well settled (e.g., Borve et al. 2001; Omang et al. 2003). Instead, we will take a Eulerian approach to simulating the IGM. However, there is a basic problem in Eulerian based codes in that they cause unphysical oscillations near a discontinuity. An effective method to reduce the spurious oscillations is given by designed limiters, such as the total-variation diminishing (TVD) schemes (Harten 1983). However, TVD accuracy degenerates to first order near smooth transitions as well as to calculate piecewise smooth functions with a high resolution.

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WENO has been successfully applied to hydrodynamic problems containing shocks and complex structures, such as shock-vortex interaction (Grasso & Pirozzoli 2000a, 2000b), interacting blast waves (Liang & Chen 1999; Balsara & Shu 2000), Rayleigh-Taylor instability (Shi et al. 2003), and magnetohydrodynamics (Jiang & Wu 1999). WENO has also been used to study astrophysical hydrodynamics, including stellar atmospheres (Zanaa et al. 1998), high Reynolds number compressible flows with supernova (Zhang et al. 2003), and high Mach number astrophysical jets (Carrillo et al. 2003). In the context of cosmological applications, WENO has proved especially adept at handling the Burgers equation (Shu 1999). Recently, a hybrid hydrodynamic/N-body code based on the WENO scheme was developed and passed typical reliability tests including the Sedov blast wave and the formation of the Zel’dovich pancakes (Feng et al. 2004). This code has been successful in producing the QSO Lyα transmitted flux, including the high-resolution sample HS1700+6416 (Feng et al. 2003). The statistical features of these samples are in good agreement with observed features not only on second-order measures, like the power spectrum, but also to orders as high as eighth order for the intermittent behavior. The code also has been shown to be effective in capturing gravitational shocks during the large-scale structure formation (He et al. 2004). Hence, we believe that for the purposes of this work the Eulerian code based on the WENO scheme is the best approach.

3.2. Samples

For the present application, we run the hybrid N-body/hydrodynamic code to trace the cosmic evolution of the coupled system of dark matter and baryonic gas in a flat low-density CDM model ($\Lambda$CDM), which is specified by the cosmological parameters ($\Omega_m, \Omega_{\Lambda}, h, \sigma_8, \Omega_b$) = (0.3, 0.7, 0.7, 0.9, 0.026). The baryon fraction is fixed with the constraint from primordial nucleosynthesis as $\Omega_b = 0.0125 h^{-2}$ (Walker et al. 1991). The linear power spectrum is taken from the fitting formulae presented by Eisenstein & Hu (1999).

Atomic processes including ionization, radiative cooling, and heating are modeled similarly as in Cen (1992) in a plasma of hydrogen and helium of primordial composition ($X = 0.76, Y = 0.24$). Processes such as star formation, and feedback due to SN and AGN activities, are not taken into account as yet. A uniform UV-background of ionizing photons is assumed to have a power-law spectrum of the form $J(\nu) = J_{21} \times 10^{-21} (\nu/\nu_{L1})^{-\alpha} \text{ergs s}^{-1} \text{cm}^{-2} \text{sr}^{-1} \text{Hz}^{-1}$, where the photoionizing flux is normalized by the parameter $J_{21}$ the Lyman limit frequency $\nu_{L1}$ and is suddenly switched on at $z > 10$ to heat the gas and reionize the universe.

The simulations are performed in a periodic, cubic box of size $25 h^{-1}$ Mpc with a 192$^3$ grid and an equal number of dark matter particles. The simulations start at a redshift $z = 49$, and the results are output at redshifts $z = 4.0, 3.0, 2.0, 1.0, 0.5$, and 0.0. At each output stage, we read $\rho_{dm}$, $\rho_{igm}$ (both are, respectively, in the units of $\rho_{dm}$ and $\rho_{igm}$), $v_{dm}$ and $v_{igm}$ in each cell.

The time step is chosen by the minimum value among the following three timescales. The first is from the Courant condition given by

$$\delta t \leq \frac{c}{\max \{|v_x| + c_s, |v_y| + c_s, |v_z| + c_s\}},$$

where $c$ is the speed of light, $v_x$, $v_y$, and $v_z$ are the components of the velocity, and $c_s$ is the sound speed.
The scatter of the IGM versus dark matter has been noted in SPH simulations (Gnedin & Hui 1998). They found that two samples with different resolutions give about 0.01 to 102. The last timescale comes from the requirement that a particle moves not more than a fixed fraction of the cell size.

The effect of the numerical resolution of these samples has been tested to higher order statistics (Feng et al. 2003). It was found that the scatter is smaller at higher redshifts. This would seem to contradict the gas pressure as an explanation for the scatter. On the other hand, the higher the nonlinearity, the higher the Reynolds number (eq. [6]), and the higher the discrepancy. Therefore, the gravitational nonlinear evolution of the IGM and dark matter system provides a plausible explanation of the scatter of Figure 1.

4. ONE-POINT STATISTICS

4.1. One-Point Variables with the Discrete Wavelet Transform Decomposition

To calculate the one point distribution, we use the discrete wavelet transform (DWT). The scaling functions of the DWT analysis serve as the sampling window function. For the details of the mathematical properties of the DWT, see Mallat (1989a, 1989b), Meyer (1992), and Daubechies (1992), and for cosmological applications see Fang & Thews (1998) and Fang & Feng (2000).

Let us briefly introduce the DWT-decomposition for a random field. Consider a one-dimensional density fluctuation $\delta(x)$ on a spatial range from $x = 0$ to $L$. We divide the space into $2^j$ segments labeled by $l = 0, 1, \ldots, 2^j - 1$ each of size $L/2^j$. The index $j$ is a positive integer and gives the length scale $L/2^j$. The larger $j$ is, the smaller the length scale. Any reference to a property as a function of scale $j$ below must be interpreted as the property at length scale $L/2^j$. The index $l$ represents position, and it corresponds to the spatial range $lL/2^j < x < (l + 1)L/2^j$. Hence, the space $L$ is decomposed into cells $(j, l)$.

The discrete wavelet is constructed such that each cell $(j, l)$ supports a compact function, the scaling function $\phi_{j,l}(x)$, which satisfies the orthonormal relation

$$\int \phi_{j,l}(x)\phi_{j',l'}(x)dx = \delta^K_{j,l'},$$

where $\delta^K$ is Kronecker delta function. The scaling function $\phi_{j,l}(x)$ is a window function on scale $j$ centered around the segment $l$. 

Fig. 1.—Relation between $\rho_{\text{igm}}(x)$ and $\rho_{\text{dm}}(x)$ for redshifts $z = 0, 1, 2$, and 4. The data consist of $\simeq 7000$ randomly drawn points from the simulation sample.
For a field $F(x)$, its mean in cell $(j, l)$ can be estimated by

$$F_{j,l} = \frac{\int_0^L F(x) \phi_{j,l}(x) dx}{\int_0^L \phi_{j,l}(x) dx} = \frac{1}{\int_0^L \phi_{j,l}(x) dx} \epsilon_{j,l}^F$$

(20)

where $\epsilon_{j,l}^F$ is called scaling function coefficient (SFC), given by

$$\epsilon_{j,l}^F = \int_0^L F(x) \phi_{j,l}(x) dx.$$ (21)

Thus, a one-dimensional field $F(x)$ can be decomposed into

$$F(x) = \sum_{l=0}^{2^j-1} \epsilon_{j,l}^F \phi_{j,l}(x) + O(\leq j).$$ (22)

The term $O(\geq j)$ in equation (22) contains only the fluctuations of the field $F(x)$ on scales equal to and less than $L/2^j$. This term does not have any contribution to the window sampling on scale $j$. Thus, for a given $j$, the one-point variables $F_{j,l}$ or $\epsilon_{j,l}^F (l = 0, 1, \ldots, 2^j - 1)$ give a complete description of the field $F(x)$ smoothed on scale $L/2^j$. As one-point variables the $\epsilon_{j,l}^F$ are similar to the measure given by count-in-cell technique. However, the orthonormality equation (19) insures that the set of $F_{j,l}$ or $\epsilon_{j,l}^F$ do not cause false correlations. In the calculations below, we use the Daubechies 4 (D4) wavelet (Daubechies 1992).

4.2. One-Point Variables of Density Fields

We first calculate the one-point distributions of variables $\rho_{j,1-igm}$ and $\rho_{j,1-dm}$, which are the one-point variables given by equation (20) replacing $F(x)$ by the density fields $\rho_{igm}(x)$ and $\rho_{dm}(x)$. We then plot in Figure 2 $\rho_{j,1-igm}$ versus $\rho_{j,1-dm}$ for $j = 5, 6, 7, 8$, corresponding to comoving length scales 1.03, 0.516, 0.258, and 0.129 $h^{-1}$ Mpc, respectively. Figure 2 shows that the scatter around the line $\rho_{j,1-igm} = \rho_{j,1-dm}$ is little smaller for smaller $j$. That is, the discrepancy between the IGM and dark matter is smaller on larger scales. Nevertheless, the discrepancy is still substantial on scale $j = 5$ or $1.03 h^{-1}$ Mpc, which is larger than the Jeans length of the IGM.

When the “fair sample hypothesis” (Peebles 1980) holds, each set of $2^j (l = 0, \ldots, 2^j - 1)$ points form an ensemble from which the one-point distribution can be studied. Figure 3 gives the one-point distribution of $\rho_{j,1-igm}$ on scale $j = 5, 6, 7, 8$ at $z = 1$. These distributions are generally non-Gaussian, having a longer tail at higher $j$, i.e., smaller scales. These distributions show significant $j$-dependence. The distribution at $j = 8$ has a tail longer than 9 times the variance, while the tail of the distribution of $j = 5$ is only about four times of the variance. We also see from Figure 3 that the distribution has large change from $j = 5$ to $j = 6$, and from $j = 6$ to $j = 7$, but a smaller change from $j = 7$ to $j = 8$. This is because the scale $j = 8$ is $0.129 h^{-1}$ Mpc, which is close to the Jeans length. On these scales the perturbations in the IGM field are weak. We show the redshift-evolution of the $\rho_{j,1-igm}$ one-point distributions in Figure 4. The one-point distributions also undergo significant redshift evolution. However, a long-tail PDF is already pronounced at redshift $z = 4$.  

5. STATISTICAL DISCREPANCY BETWEEN THE INTERGALACTIC MEDIUM AND DARK MATTER

5.1. One-Point Distributions of Velocity Fields

To demonstrate the statistical discrepancy addressed in § 2, we first analyze the one-point distributions of the one-dimensional velocity fields $v_{igm}(x)$ and $v_{dm}(x).$ For the linear solution $v_{igm}(x) = v_{dm}(x)$ (eq. [1]), we have $v_{j,1-igm} = v_{j,1-dm}$, which are the one-point variables given by equation (20) replacing $F(x)$ by the velocity distribution. It is clear from Figure 5 that the one-point distributions of $v_{j,1-igm}$ and $v_{j,1-dm}$ at
redshift $z = 0$ are very different on all scales; $j = 5, 6, 7,$ and 8. This difference is smaller at higher redshifts. At $z = 4$, the one-point distributions of both $v_{j, t-igm}$ and $v_{j, t-dm}$ are basically the same on all scales.

The nature of the statistical difference between the $v_{j, t-igm}$ and $v_{j, t-dm}$ one-point distributions is not only quantitative, but also qualitative. The $v_{igm}(x)$ one-point distributions are small deviations from a Gaussian PDF on all scales and all redshifts considered. This result is consistent with previous studies using $N$-body simulation samples (e.g., Yang et al. 2001). On the other hand, the one-point distributions of the IGM velocity field shown in Figure 5 are generally exponential at redshifts $z < 2$. Even when the scale is as large as $j = 5$ or $\sim 1$ h$^{-1}$ Mpc, the distribution of $v_{igm}(x)$ is still exponential (Fig. 5a). This emphatically shows that the statistical discrepancy between the IGM and dark matter depends on dynamical evolution.

The dynamical equation (3) for the dark matter looks very similar to equation (4) for the IGM. So why do the two one-point distributions have such different shapes? The reason is that the $\phi$ term of equation (4) is truly an external force, as the gravity potential is independent of the gravitational energy and velocity, dependent only on the dark matter. Therefore, when its Reynolds number is large at lower redshifts, shocks or Burgers turbulence will develop in the IGM field owing to the external driving force and the IGM velocity field will be highly non-Gaussian. On the other hand, the dark matter mass density in equation (3) is not independent of the gravitational potential, and there is no external driving force. Thus, when the gravity potential $\phi$ is Gaussian in weakly nonlinear evolution, the dark matter velocity PDF will approximately be Gaussian too.

The statistical discrepancy of the velocity fields can also be seen with Figure 6, which gives the one-point distribution of the difference $\Delta v_{j, t} = v_{j, t-dm} - v_{j, t-igm}$ on scales $j = 5, 6, 7,$ and 8, and redshifts 0, 1, 2, and 4. These one-point distributions are non-Gaussian on all scales and redshift considered. Although the one-point distributions of $v_{igm}(x)$ and $v_{igm}(x)$ at redshift $z = 4$ are not very different (Fig. 5d), their differences (Fig. 6d) are highly non-Gaussian. Therefore, the discrepancy between $v_{j, t-dm}$ and $v_{j, t-igm}$ is not due to noise, but arises from the nonlinear evolution of the IGM fluid.

5.2. One-Point Distributions of Density Fields

Figure 7 presents the one-point distributions for $\rho_{j, t-igm}$ and $\rho_{j, t-dm}$ on scale $j = 7$ and redshifts $z = 1, 2,$ and 3. The horizontal axes are the variance-normalized densities, $\rho_{j, t-igm}/\langle \rho_{j, t-igm}^2 \rangle^{1/2}$ and $\rho_{j, t-dm}/\langle \rho_{j, t-dm}^2 \rangle^{1/2}$. Unlike the velocity fields, both the IGM and dark matter show highly non-Gaussian behavior even at redshift $z = 3$. Additionally, the two curves in Figure 7 are not significantly different. That is, the mass density one-point distribution is not as sensitive to the statistical discrepancy as the velocity one-point distribution. This is probably because the statistical discrepancy between a “passive substance” and underlying field is significant only for nonconserved quantities (temperature, velocity, etc.; Shraiman & Siggia 2000). Nevertheless, one can see from Figure 7 that the IGM distributions are always a little higher than the corresponding distribution for the dark matter at the range around $\rho_{j, t-igm}/\langle \rho_{j, t-igm}^2 \rangle^{1/2} \approx 1$. This indicates that the tail of the IGM one-point distribution should be shorter than that of dark matter. Therefore, we can expect that the high-order moments of the one-point distributions will more clearly show the discrepancy between the IGM and dark matter.

In this paper, we use only the second-order moment. The correlation between $\rho_{j, t-igm}$ and $\rho_{j, t-dm}$ can be measured with the ratio $I_{d}$, $I_{igm}$ (eqs. [10] and [11]), or $I$ (eq. [12]). Using the DWT one-point variables $\rho_{j, t-igm}$ and $\rho_{j, t-dm}$, we can redefine, respectively, the ratios $I_{d}$ and $I_{igm}$ as

$$I_{j-dm} = \frac{\langle (\rho_{j,t-dm} - \rho_{j,t-igm})^2 \rangle^{1/2}}{\langle \rho_{j,t-dm}^2 \rangle^{1/2}},$$

$$I_{j-igm} = \frac{\langle (\rho_{j,t-dm} - \rho_{j,t-igm})^2 \rangle^{1/2}}{\langle \rho_{j,t-igm}^2 \rangle^{1/2}}.$$  

From the one-point distributions of Figure 7, we have $\langle \rho_{j,t-dm}^2 \rangle \approx \langle \rho_{j,t-igm}^2 \rangle$. Thus, if $\rho_{j,t-igm}$ perfectly traces $\rho_{j,t-dm}$, we have the correlation $\langle \rho_{j,t-igm} \rho_{j,t-dm} \rangle \approx \langle \rho_{j,t-igm}^2 \rangle \approx \langle \rho_{j,t-dm}^2 \rangle$, and therefore, $I_{j-dm} \approx I_{j-igm} \approx 0$. If $\rho_{j,t-igm}$ is fully independent of $\rho_{j,t-dm}$, we have $I_{j-dm} \approx I_{j-igm} \approx \sqrt{2}$. 

![Figure 3](image3.png)

![Figure 4](image4.png)
Fig. 5.—One-point distributions of the velocity fields of the IGM $v_{j,l}^{\mathrm{igm}}$ and dark matter $v_{j,l}^{\mathrm{dm}}$ on scales $j = 5, 6, 7, 8$ and for redshifts (a) $z = 0$, (b) $z = 1$, (c) $z = 2$, and (d) $z = 4$. 

Fig. 5a

Fig. 5b

Fig. 5c

Fig. 5d
Fig. 6.—One-point distributions of the difference $v_{jl \text{-} dm} - v_{jl \text{-} igm}$ on scales $j = 5, 6, 7, \text{and} 8$ and for redshifts (a) $z = 0$, (b) $z = 1$, (c) $z = 2$, and (d) $z = 4$. 

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Fig. 6c

Fig. 6d
5.3. One-Point Distribution of $\Delta \delta(x)$

The DWT one-point variable for the density difference $\Delta \delta(x)$ (eq. [2]) is given by

$$\Delta_{j,l} = \int \Delta \delta(x) \phi_{j,l}(x) dx.$$  (26)

Since $\phi_{j,l}(x)$ is localized in cell $(j,l)$, the integral $\int \phi_{j,l} \phi_{j',l'} dx$ is generally nonzero only for $l \neq l'$ for the Haar wavelet, this is exactly true. For other wavelets, the integral with $l \neq l'$, $l''$ is much less than that of $l = l' = l''$. Thus, for two fields $A(x)$ and $B(x)$, we have $\int A(x) B(x) \psi_{j,l}(x) dx \approx g_{j} c_{j} c_{j}'$, where $c_{j}$ and $c_{j}'$ are the WFCs of $A(x)$ and $B(x)$. The factor $g_{j}$ is

$$g_{j} = \int \psi_{j}^{2}(x)dx,$$  (27)

which is the factor $g_{j}$ used in equation (16).

With $\Delta_{j,l}$, equation (15) can be rewritten as

$$\frac{d\Delta_{j,l}}{dt} = -\frac{1}{a} \lambda_{j} \Delta_{j,l} + \frac{1}{a} \eta_{j},$$  (28)

where

$$\eta_{j} = \int dx' \psi_{j}(x') (\nabla \cdot \Delta \mathbf{r}(x)), $$  (29)

and

$$\lambda_{j} = g_{j} \int x' \psi_{j}(x') (\nabla \cdot \mathbf{r}_{\text{dm}}).$$  (30)

As discussed in § 2.3, the PDF of $\Delta_{j,l}$ should be long tailed. The normalized one-point distributions of $\Delta_{j,l}/(\Delta_{j,l}^{2})^{1/2}$ for $j = 7$ and redshifts 1, 2, and 3 are plotted in Figure 9. All the tails of the distributions are remarkably long. They have 6 $\sigma$
The decoupling of the IGM from the dark matter mass field is important for problems in large-scale structure formation. Some observations have already implied that the IGM and dark matter have decoupled. For instance, X-ray measurements find no evidence for the baryon fraction of clusters to be equal to the universal value from cosmological nucleosynthesis (White & Fabian 1995; David 1997; Ettori et al. 1997; White et al. 1997; Ettori & Fabian 1999). This result violates the similarity between the IGM and dark matter if typical galaxy clusters formed from linear fields on scales of a few ten (comoving) Mpc, which is much larger than the Jeans length of the IGM. Moreover, the X-ray observed luminosity-temperature relation for groups and clusters is found to be inconsistent with the prediction given by the similarity between the IGM and dark matter. The high entropy floor observed in nearby groups and low-mass clusters directly violates the dynamical self-similar scaling (Ponman et al. 1999; Lloyd-Davies et al. 2000). Although at present we cannot attribute all these discrepancies to the decoupling shown in this paper, we now understand that the dynamics of the IGM and dark matter fields will lead to a qualitatively different evolution of those fields. While we did not consider stellar formation and its feedback to the IGM in our present simulations, we believe that the statistical discrepancy will still be a common feature even when these effects are considered because the reason for discrepancy we have uncovered arises from the nonlinear evolution of the random fields of the IGM and dark matter.

Finally, we should mention the effect of the size of the simulated sample. Since the simulation box is only 25 \( h^{-1} \) Mpc, the use of the fair sample hypothesis may not be appropriate when considering perturbations on scales larger than this. We can estimate the effect of these larger perturbations by adding long-wavelength modes (Tormen & Bertschinger 1996). Since the IGM and dark matter fields are linear or quasi-linear on scales larger than 25 \( h^{-1} \) Mpc, the effect of long-wavelength perturbations can be analyzed by adding a displacement to each particle in the simulation box. The displacement is given by the linear field or Zel'dovich approximation of the field consisting of modes on scales larger than 25 \( h^{-1} \) Mpc. As was discussed in §1, there is no discrepancy between the IGM and dark matter in the linear or quasi-linear regime. Thus, the displacement for the IGM is the same as that of the dark matter, meaning that long-wavelength modes are not a source of the discrepancy between the IGM and dark matter.

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APPENDIX A

HYDRODYNAMIC EQUATIONS FOR DARK MATTER FIELDS

Let us consider a flat universe having cosmic scale factor \( a(t) \propto t^{2/3} \) and dominated by dark matter. In a hydrodynamic description, the dark matter is described by a mass density field \( \rho_{dm}(x, t) \) and a peculiar velocity field \( v_{dm}(x, t) \), where \( x \) is the comoving coordinate. The field is described by the equations of continuity, momentum, and gravitational potential as (Wasserman 1978)

\[
\frac{\partial \delta_{dm}}{\partial t} + \frac{1}{a} \nabla \cdot (1 + \delta_{dm}) v_{dm} = 0,
\]  

(A1)
The mean density is \( \rho_{\text{dm}}(t) = 1/6\pi G t^2 \propto a^{-3} \). The gravitational potential \( \phi \) is zero (or constant) when the density perturbation \( \delta_{\text{dm}} = 0 \). The operator \( \nabla \) acts on the comoving coordinate \( x \).

For growth modes in the perturbations, velocity is irrotational. We can then define a velocity potential by

\[
\mathbf{v}_{\text{dm}} = -\frac{1}{a} \nabla \varphi_{\text{dm}}. \tag{A4}
\]

The momentum equation (A2) can then be rewritten as

\[
\frac{\partial \varphi_{\text{dm}}}{\partial t} - \frac{1}{2a^2} (\nabla \varphi_{\text{dm}})^2 = \phi. \tag{A5}
\]

This is the Bernoulli equation.

APPENDIX B

HYDRODYNAMIC EQUATIONS FOR THE INTERGALACTIC MEDIUM

As usual, the IGM is assumed to be an ideal fluid with polytropic index \( \gamma = 5/3 \). The hydrodynamic equations of the IGM are (Peebles 1980)

\[
\frac{\partial \delta_{\text{igm}}}{\partial t} + \frac{1}{a} \nabla \cdot (1 + \delta_{\text{igm}}) \mathbf{v}_{\text{igm}} = 0, \tag{B1}
\]

\[
\frac{\partial \mathbf{v}_{\text{igm}}}{\partial t} + (\mathbf{v}_{\text{igm}} \cdot \nabla) \mathbf{v}_{\text{igm}} = -\frac{1}{\rho_{\text{igm}}} \nabla p - \nabla \phi, \tag{B2}
\]

\[
\frac{\partial \mathbf{E}}{\partial t} + \frac{5}{2} \frac{\dot{a}}{a} \mathbf{E} + \frac{1}{a} \nabla \cdot (\mathbf{E} \mathbf{v}_{\text{igm}}) = -\frac{1}{a} \nabla \cdot (p \mathbf{v}_{\text{igm}}) - \frac{1}{a} \rho_{\text{igm}} \mathbf{v}_{\text{igm}} \cdot \nabla \phi - \Lambda_{\text{rad}}, \tag{B3}
\]

where \( \rho_{\text{igm}}, \mathbf{v}_{\text{igm}}, \mathbf{E} \) and \( p \) are, respectively, the mass density, peculiar velocity, energy density, and pressure of the IGM. The term \( \Lambda_{\text{rad}} \) in equation (2) is given by the radiative heating-cooling of the baryonic gas per unit volume. The gravitational potential \( \phi \) in equations (B2) and (B3), can still be given by equation (A3). That is, the gravity of the IGM is negligible. The evolution of the IGM mass field \( \rho_{\text{igm}}(x, t) \) is governed by the gravity of dark matter only.

The hydrodynamic equations for the IGM, equations (B1)–(B3) can be written in the form of conservation laws for mass, momentum, and energy in a comoving volume as

\[
\frac{\partial a^3 \rho}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x_i} (a^3 \rho v_i) = 0, \tag{B4}
\]

\[
\frac{\partial a^3 \rho v_i}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x_j} (a^3 \rho v_i v_j + a^3 \rho \delta_{ij}) = -a \dot{a} \rho v_i - a^2 \rho \nabla \phi, \tag{B5}
\]

\[
\frac{\partial a^3 \mathbf{E}}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x_i} [a^3 (\mathbf{E} + p) v_i] = -2a \dot{a} \mathbf{E} - a^2 \rho \mathbf{v} \cdot \nabla \phi - a^3 \Lambda_{\text{rad}}. \tag{B6}
\]

In equations (B4)–(B6), we dropped the subscript “igm” for simplicity.

To sketch the gravitational clustering of the IGM, it is not necessary to consider the details of heating and cooling. Thermal processes are generally highly localized, and therefore it is reasonable to describe all thermal processes by a polytropic relation \( \rho(x, t) \propto \rho_{\text{igm}}(x, t) \). Thus, equation (B2) becomes

\[
\frac{\partial \mathbf{v}_{\text{igm}}}{\partial t} + (\mathbf{v}_{\text{igm}} \cdot \nabla) \mathbf{v}_{\text{igm}} = -\frac{\gamma k_B T}{\mu m_p} \frac{\nabla \delta_{\text{igm}}}{(1 + \delta_{\text{igm}})} - \nabla \phi, \tag{B7}
\]

where the parameter \( \mu \) is the mean molecular weight of the IGM particles and \( m_p \) the proton mass. Here we do not need the energy equation and the IGM temperature evolves as \( T \propto \rho^{-1/2} \), or \( T = T_0 (1 + \delta_{\text{igm}})^{-1/2} \).

Equation (B7) differs from equation (A2) only by the temperature-dependent term. If we treat this term in the linear approximation, we have

\[
\frac{\partial \varphi_{\text{igm}}}{\partial t} - \frac{1}{2a^2} (\nabla \varphi_{\text{igm}})^2 - \frac{\nu}{a^2} \nabla^2 \varphi_{\text{igm}} = \phi, \tag{B8}
\]
where $\varphi_{\text{igm}}$ is the velocity potential for the IGM field defined by

$$v_{\text{igm}} = -\frac{1}{a} \nabla \varphi_{\text{igm}}. \quad (B9)$$

The coefficient $\nu$ is given by

$$\nu = \frac{\gamma k_b T_0}{\mu m_p (d \ln D(t)/dt)}, \quad (B10)$$

where $D(t)$ describes the linear growth behavior. The term with $\nu$ in equation (B8) acts like a viscosity (due to thermal diffusion) characterized by the Jeans length $k_J^2 = (a^2/\gamma^2)(\mu m_p/\gamma k_b T_0)$.

The unperturbed solutions of the density and velocity fields of both dark and baryonic matter are $\tilde{\rho}_b = (\Omega_b/\Omega_{\text{dm}}) \tilde{v} \propto a^{-1}$, and $v_b = v = 0$, where $\Omega_b$ and $\Omega_{\text{dm}}$ are, respectively, the density parameters of the IGM and dark matter. Therefore, the linearization of equations (B1) and (B7) yields

$$\frac{\partial \delta_{\text{igm}}}{\partial t} + \frac{1}{a} \nabla \cdot v_{\text{igm}} = 0 \quad (B11)$$

and

$$\frac{\partial v_{\text{igm}}}{\partial t} = -\frac{\gamma k_b T}{\mu m_p} \nabla \delta_{\text{igm}} - \nabla \phi, \quad (B12)$$

where the mean temperature $\overline{T} \propto \overline{\rho}_b^{-1} \propto a^{-3(\gamma-1)}$. In Fourier space, we have

$$\frac{\partial^2 \delta_{\text{igm}}(k, t)}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial \delta_{\text{igm}}(k, t)}{\partial t} + \frac{1}{k^2} k^2 \delta_{\text{igm}}(k, t) = 4\pi G \tilde{\rho}_b \delta_{\text{dm}}(k, t), \quad (B13)$$

and

$$\frac{\partial v_{\text{igm}}(k, t)}{\partial t} + \frac{\dot{a}}{a} v_{\text{igm}}(k, t) = -\frac{1}{k^2} k^2 \delta_{\text{igm}}(k, t) + \frac{4\pi G \tilde{\rho}_b}{k^2} \delta_{\text{dm}}(k, t), \quad (B14)$$

where $v_{\text{igm}}(k, t) = ik v_{\text{igm}}(k, t)$ and the Jeans wavenumber $k_J^2 = (a^2/\gamma^2)(\mu m_p/\gamma k_b T)$ $\propto a^{-2}$. If $\gamma = 4/3$, $k_J$ is time-independent.

In solving equations (B13) and (B14), we consider only the growth mode of the perturbation of dark matter, i.e., $\delta(k, t) \propto a$. In the case of $\gamma = 4/3$, the solution of equations (B13) and (B14) is (Bi et al. 1992)

$$\delta_{\text{igm}}(k, t) = \frac{\delta_{\text{dm}}(k, t)}{1 + 3k^2/2k_J^2} + c_1 t^{-1+\epsilon(1-\gamma)/6} + c_2 t^{-(1-\gamma)/6}, \quad (B15)$$

where $\epsilon = (1 - 4k^2/9k_J^2)^{1/2}$, and constants $c_1$ and $c_2$ depend on the initial condition $\delta_{\text{igm}}(0, 0)$ and $v_{\text{igm}}(0, 0)$. Therefore, regardless the initial condition of IGM, after a long evolution we have

$$\delta_{\text{igm}}(k, t) = \delta_{\text{dm}}(k, t), \quad v_{\text{igm}}(k, t) = v_{\text{dm}}(k, t), \quad \text{if } k \ll k_J. \quad (B16)$$

These solutions mean that the initial conditions of the IGM are unimportant. The IGM will, in the end, follow the same trajectory as the dark matter on scales larger than the Jeans length.

The solutions of linear equations (B11) and (B12) have also been found by using assumptions of the IGM thermal processes other than that used in equations (B13) and (B14) (e.g., Nusser 2000; Matarrese & Mohayee 2002). A common feature of these solutions is

$$\delta_b(k, t) = (1 + \text{decaying terms})\delta(k, t) + \text{decaying terms}, \quad \text{if } k \ll k_J. \quad (B17)$$

The initial conditions of the IGM field affects only the decaying terms, and therefore, the linear solutions (eq. [B16]) hold in general regardless specific assumptions of IGM thermal processes.

APPENDIX C

PROBABILITY DISTRIBUTION FUNCTION OF $\Delta_{j,i}$ IN EQUATION (27)

Using $d\tau = dt/a$, equation (28) can be rewritten as

$$\frac{d\Delta_{j,i}}{d\tau} = -\dot{\epsilon}_j \Delta_{j,i} + \eta_j. \quad (C1)$$
If the stochastic forces $\lambda_j$ and $\eta_j$ are Gaussian, then

$$\langle \lambda_j \rangle = 0,$$

$$\langle (\lambda_j(\tau) - \bar{\lambda}_j)[\lambda_j(\tau')] - 2D_{\lambda,j}\delta(\tau - \tau') \rangle,$$

$$\langle \eta_j \rangle = 0,$$

$$\langle \eta_j(\tau)\eta_j(\tau') \rangle = 2D_{\eta,j}\delta(\tau - \tau').$$

The Fokker-Planck equation corresponding to equation (C1) is (Venkataramani et al. 1996)

$$\frac{\partial}{\partial \tau} P(\Delta_{j,t}, \tau) = -\frac{\partial}{\partial \Delta_{j,t}} j(\Delta_{j,t}, \tau),$$

where $P(\Delta_{j,t}, \tau)$ is the PDF of $\Delta_{j,t}$, and the flux $j(\Delta_{j,t}, \tau)$ is given by

$$j(\Delta_{j,t}, \tau) = (-\bar{\lambda}_j + D_{\lambda,j})\Delta_{j,t} P(\Delta_{j,t}, \tau) - \frac{\partial}{\partial \Delta_{j,t}} [D_{\lambda,j}|\Delta_{j,t}|^2 + D_{\eta,j}] P(\Delta_{j,t}, \tau).$$

For a stationary solution, $\partial P(\Delta_{j,t}, \tau)/\partial \tau = 0$, and therefore, $\partial j(\Delta_{j,t}, \tau)/\partial \Delta_{j,t} = 0$. We have set $j(\Delta_{j,t}, \tau) =$ constant. However, when $\Delta_{j,t}$ is very large, we should have $j(\Delta_{j,t}) = 0$, and therefore $j(\beta^2_0) = 0$. Thus, we have the equation as

$$(-\bar{\lambda}_j + D_{\lambda,j})\Delta_{j,t} P(\Delta_{j,t}, \tau) - \frac{\partial}{\partial \Delta_{j,t}} [D_{\lambda,j}|\Delta_{j,t}|^2 + D_{\eta,j}] P(\Delta_{j,t}, \tau) = 0.$$

The solution of equation (C5) is

$$P(\Delta_{j,t}) = C(D_{\lambda,j}|\Delta_{j,t}|^2 + D_{\eta,j})^{-\beta/2D_{\lambda,j}+1/2},$$

where $C$ is a normalization constant. The PDF is then

$$P(\Delta_{j,t}) \propto \begin{cases} 
\text{constant}, & 0 < \Delta_{j,t} \ll s, \\
(\Delta_{j,t})^{-\beta}, & |\Delta_{j,t}| \gg s,
\end{cases}$$

where

$$s = \frac{D_{\eta,j}}{D_{\lambda,j}},$$

and

$$\beta = \frac{\bar{\lambda}_j}{D_{\lambda,j}} + 1.$$
