TeV Scale Strings and Scattering Amplitudes at the LHC

M.Sc Thesis

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With love to Shuli and Yoash Carmi.

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1 Abstract

We study aspects of TeV string scale models of intersecting D-branes. The gauge bosons arise from strings ending on stacks of D-branes, whereas chiral matter arises from strings stretched between intersecting D-branes. Our focus is on scattering amplitudes (at tree-level), Regge states (string excitations), and collider phenomenology.

Achieving a low string scale is possible in models of Large extra dimensions. At the LHC, a low enough string scale implies that cross sections will deviate from their standard model predictions. Moreover, Regge states as well as Kaluza-Klein states and winding states may be produced.

In a large class of intersecting D-brane models, the quark-gluon amplitudes with at most 2 quarks turn out to be independent of the geometry of the extra dimensions. Therefore these type of amplitudes, which we call universal amplitudes, are model independent. The universal amplitudes involve exchanges of Regge states only, whereas amplitudes with more then 2 quarks also involve exchanges of KK and winding states.

The main computational part of this thesis is concerned with suggesting methods to calculate the decay widths of the Regge states, and with the formalism for treating amplitudes containing exchanges of higher spin particles. Also included is a large review part.
2 Content of thesis

Section 3 is an introduction to the framework and to some of the concepts which arise in this work.

Section 4 is a review of methods to calculate tree level amplitudes in field theory and string theory. Section 4.1 reviews tree level field theory quark-gluon amplitude calculations via the color decomposition and helicity techniques. Section 4.2 reviews lowest order amplitude calculations in string theory. Quark-gluon amplitudes in string theory are introduced. The important concept of universal amplitudes is discussed. Also discussed is the property of the equal form factors of the two classes of universal amplitudes.

Section 5 then focuses on the lowest order scattering amplitudes at the LHC: the 4-point amplitudes. Section 5.1 presents the squared amplitudes in field theory (QCD+EW). Section 5.2 starts with a presentation of the Veneziano amplitude, then the string theory squared amplitudes are presented, and collider phenomenology and LHC constraints discussed.

Section 6 presents a procedure for calculating the decay widths of the exchanged Regge excitations of the gluon and quarks.

Section 7 is basically a review of the 5-point squared amplitudes. Section 7.2 presents the squared amplitudes in field theory (QCD+EW). Section 7.3 presents the squared amplitudes in string theory.

Section 8 discusses the generalization to higher point string theory amplitudes.

Section 9 is basically a review of the 4-point squared amplitudes for direct production of the first excited ($n = 1$) Regge states.

Section 10 is a summary.

Appendices:

Appendix A is important in that it gives the 4-point amplitudes which are used in the calculation of the decay widths of section 6.

Appendix B gives the $\mathcal{M}(ggqq)$ amplitude as an example of a full calculation of a 4-point amplitude of the type that appear in Appendix A.

Appendices C and D list vertex operators and correlation functions which are used in the calculation of the string amplitudes.

Appendix E introduces the helicity formalism used in section 4 and throughout this work.

Appendix F lists formulas related to the color part of the amplitudes.

Appendix G is a short introduction to hadron collider phenomenology.

Appendix H reviews the mathematical functions used in this work.
Appendix I deals with the Wigner $d$-functions which are used often in this work.

Appendix J contains tables of $C$ coefficients which are calculated via approach 1 of section 6.3.1.

The parts which are either original or at least partly original.

- In section 5.2.2 the squared amplitudes from [1] are expanded as a sum of s-channel poles. This exhibits some of the properties of these squared amplitudes, e.g the vanishing of $n = \text{even}$ poles for some of the amplitudes.

- In section 5.2.3 the Breit-Wigner form of the squared amplitudes for exchanges of Regge states with arbitrary $(n, J)$ is given.

- In section 5.2.4 the low energy limit of the string squared amplitudes is taken: the first stringy correction to the standard model squared amplitudes is obtained.

- Section 6 contains many of the calculations of this work. The procedure for calculating the decay widths of the exchanged Regge states is given in sections 6.1-6.2. This is done for arbitrary quantum numbers $n$ and $J$, and generalizes the treatment done in [7], [19] for $n = 1, 2$. The procedure gives formulas for the decay widths in terms of the unknown $C^{n,J}_{m,m'}$ coefficients. Then section 6.3 introduces a number of approaches to calculate these $C$’s.

- In section 8.1.1 a partial treatment of the squaring of the color factor of the 6-point gluon amplitude is presented.

- Appendix A gives the 4-point string amplitudes expanded near a pole. This form is used for extracting the properties of the exchanged Regge states, in particular the decay widths.

- Appendix B (together with section 6) deals with the Wigner $d$-functions and their relation to higher spin scattering amplitudes. Also some $d$’s are calculated and used for the calculation of $C^{n,J}_{m,m'}$ coefficients in the tables of Appendix J.

- Appendix J contains tables of the calculated $C^{n,J}_{m,m'}$ coefficients.
3 Introduction

The LHC has collected about $5 \text{ fb}^{-1}$ of data, and there are first hints of a Higgs boson. So far there aren’t any significant deviations from the standard model. Expectations of discovering new physics in the next year(s) are still very high.

The scenario with which this work deals with, while probably not the most likely scenario, is certainly very interesting. If it is correct, some of the the following astonishing phenomena can be reality:

- String theory at the LHC.
- Discovery of higher spin particles: Regge states.
- Quantum gravity at the LHC.
- Extra dimensions, KK gravitons, Black holes, Hawking radiation.

So these scenarios are extremely interesting.

We will concentrate on issues related to string theory in the open sector, and less on gravity and extra dimensions. In particular, we will focus on scattering amplitudes in string theory with regards to their collider phenomenology.

String theory is a high energy completion of the standard model. It is both a quantum gravity theory and a unification theory. At low energies, the spectrum and interactions of string theory must reduce to the standard model. Indeed we will see that the standard model matter and gauge fields can arise as ground states of the open string. String scattering amplitudes in the low energy limit must equal those of the standard model. The string scale, quite generally, is the scale at which stringy phenomena start to appear. Near this scale, scattering amplitudes begin to deviate from the standard model ones.

We will see that there are classes of string amplitudes which are model independent (for a large class of intersecting D-brane models): they are completely independent of the geometry of the extra dimensions. These amplitudes are the $n$-gluon amplitude, and the $n$-gluon plus two quark amplitudes. From now on we shall call these two types: universal amplitudes. Thus by measuring these amplitudes one can discern string theory regardless of compactification and landscape issues. These universal amplitudes are purely stringy since they contain only exchanges of Regge states (string excitations) and not KK states (caused by the presence of
extra dimensions) or winding states (strings or branes wound around the extra dimensions). Quark-gluon amplitudes with four or more quarks are non-universal since they are dependent on the compactification and can contain exchanges of KK and winding states.

In open string theory the analogues of tree amplitudes are called disk amplitudes. Roughly speaking, when calculating for example the 4 gluon disk amplitude, one obtains the field theory tree result multiplied by a Veneziano amplitude. The Veneziano amplitude $V_t$ is basically a beta function of the Mandelstam variables (see section 5.2.1). $V_t$ goes to 1 in the low energy limit, causing the string amplitudes to match the field theory amplitudes at that limit. When the scattering energy approaches the string scale, $V_t$ deviates from 1 and stringy effects become noticeable. The Veneziano amplitude has an infinite number of poles at a constant interval of $E_{CM}^2$. This gives rise to an infinite tower of resonances called Regge states. The Regge states are excitations of the string. At colliders, these resonances can be discovered directly as peaks in the cross section at equal intervals of the energy squared. The standard model matter and gauge fields occupy the ground state of the string, and each one of them has an infinite tower of Regge excitations.

Regardless of if string theory describes nature, there is no doubt that string amplitude techniques have been extremely fruitful to the understanding and calculation of field theory amplitudes. To name some of the techniques studied over the years: The KLT and BCFW relations, the works of Bern-Kosower-Dixon, and AdS/CFT techniques.

Usually, the string scale and the quantum gravity scale are assumed to be at around the Planck scale ($\sim 10^{19}$ GeV). In this case it is very difficult to discern stringy effects at present collider energies. In the mid nineteen-nineties, studies on D-branes, Large extra dimensions, and related issues, made it possible to consider string and gravity scales even as low as a TeV in type I or II string theory. This makes it possible to observe the wonderful phenomena discussed before, in the near future.

The types of models to be considered are intersecting D-brane models of type II orientifolds with Large extra dimensions. These models can realize the standard model gauge group and matter fields.

3.1 Higher spins, compositeness, and duality

References: [64].

The standard model (plus gravity) contains particles of spin 1/2, spin 1, spin 0, and spin
2. These are the quarks and leptons, gauge bosons, Higgs boson, and graviton, respectively. The last two are yet to be discovered. Hypothesized extensions of the standard model have additional particles. A spin 0 axion is added to solve the strong CP problem. Supersymmetry has SUSY partners of spin 1/2 and spin 0. Supergravity has a spin 3/2 partner to the graviton (the gravitino). Extra dimensional theories have (in the simplest case) KK gravitons of spin 2 and possibly a spin 0 radion.

The point is, that all these particles have spin $J \leq 2$. No consistent theory is known for a finite number of interacting particles with spin larger then 2. From Eq. (3.2) we see that tree level amplitudes for spins $J > 1$ will diverge at high energies, and this creates problems with unitarity.

That being said, composite higher spin particles are abundant. This is because composite particles have orbital angular momentum, which is unbounded and also discrete because of quantum mechanics. For example the electron in a hydrogen atom is electromagnetically bound to the proton. At a given energy state $E = -\frac{E_I}{n^2}$, the orbital angular momentum has the possible values $l = 0, 1, \ldots, n-1$. At the center of atoms there is the nucleus, which is a composite system of nucleons bound by the strong force. The nucleon in turn, and hadrons in general, are composite quark systems bound by the strong force. Not only do quarks tend to form composite systems (as electrons and protons do), but they must. Quarks (and gluons) are confined inside the hadrons, and were never observed as free particles. Quarks (and leptons) are point particles as far as experiments can tell, though there has been theoretical work done on quark compositeness. Another hypothesized theory based on confinement is technicolor, in which confinement generates the electroweak scale.

Returning to hadrons, in the 1950’s and 1960’s a multitude of them, with increasingly higher masses and spins, were discovered at accelerators. It seemed as if more and more particles will be discovered as the energy will increase. Theorists struggled making sense of the results. Quantum field theory, which was so succesfull at explaining electrodynamics, appeared not very useful for explaining the dynamics of these particles. First, a way of dealing with high spin particles was unknown (largely true till this day. String theory is an exception). Second, Putting by hand a large number of different fields in a lagrangian seems awkward. Third, the particles are strongly interacting and QFT calculations were very difficult. These difficulties stimulated different approaches based on an S-matrix or a

1Likewise, we will see later that the Veneziano amplitude has a similar relation between $l$ and $n$. The difference will be that $E \propto \sqrt{n}$.

2We now know that the correct explanation is a non-abelian gauge theory of spin 1/2 quarks and spin 1 gluons, called QCD. There are 6 types of quarks, and they are the fields which enter the lagrangian. The
scattering amplitude, instead of a lagrangian as the starting point. Many of these models, being of phenomenological nature, tried directly to take into account various properties of the measured scattering amplitudes: correct high/low energy behavior, crossing symmetry, duality, Regge trajectories etc.

In 1968 Veneziano attempted a model of meson scattering. He introduced an ansatz for the amplitude, now called the Veneziano amplitude. This model predicted an infinite number of higher spin states: Regge states. The string scale was assumed at the GeV scale for the Regge states to be identified with the discovered mesons. A few years later it was realized that the Veneziano amplitude can be derived from a more fundamental theory: a strange type of field theory in which there are 1-dimensional objects (strings) instead of 0-dimensional particles. Meanwhile two things were realized: the Veneziano model was not very successful in explaining hadrons and the strong force, and a gauge theory named QCD emerged as the correct theory. Soon enough though, string theory was revived as a quantum theory of gravity, and the estimate of the string scale naturally jumped 20 orders of magnitude to Planck scale territory. With the string scale being so high, research on phenomenology of Regge states became sparse.

We now discuss the DHS duality [56] also known as worldsheet duality. Consider a field theory with two types of particles $\phi$ and $\sigma$. If $\sigma$ is a scalar (spin 0), then an interaction may be of the form $\phi^* \phi \sigma$. If $\sigma$ is of spin $J$ type, then it will have $J$ indices and the interaction must be $\phi^* \partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_J} \phi \cdot \sigma^{\mu_1 \ldots \mu_J}$. The lowest order scattering of $\phi$ particles has (inequivalent) $s$ and $t$-channel exchanges, Fig. 1. A scalar $\sigma$ gives the following $t$-channel amplitude

$$A(s, t) = -\frac{g^2}{t - m^2}$$

hadrons are composed of quarks. In experiments we never see quarks since a strong force bounds them together inside hadrons. The dynamics of hadrons is complicated, just like the dynamics of atoms is complicated. Hadrons are numerous because of the different possible combinations of quarks, and because of the existence of excited states.
The exchange of a spin $J$ particle gives $2J$ momenta coming from the derivatives in the interaction vertex. The $t$-channel amplitude is

$$A(s, t) = \frac{g^{2}(-s)^{J}}{t-m^{2}}$$

(3.2)

If we have exchange of particles with different masses and spins, then

$$A(s, t) = \sum_{J} g^{2}_{J}(-s)^{J} \frac{t-m^{2}_{J}}{t-m^{2}}$$

(3.3)

Likewise, the $s$-channel amplitude is

$$A'(s, t) = \sum_{J} g^{2}_{J}(-t)^{J} \frac{s-m^{2}_{J}}{s-m^{2}}$$

(3.4)

_Duality_ is the hypothesis that the $s$ and $t$-channel amplitudes are equal (Fig. 2). This may be done by cleverly choosing the $g_{J}$’s and $m_{J}$’s, and was achieved by Veneziano in 1968.\(^3\) Notice that this is only possible if the sum is infinite since if it was finite, Eq. (3.4) would be an analytic function in the complex $t$-plane meaning that there are no $t$-channel poles, and it couldn’t possibly equal Eq. (3.3).

An infinite sum also has consequences on the high energy behavior of the amplitude. With a finite number of particles, the high energy behavior is determined by the highest spin particle. On the other hand, an infinite sum can make the high energy behavior much better, just like the function $e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^{n}}{n!}$ has a much better high energy behavior then any of it’s individual terms. Indeed, we will see that Veneziano amplitude can be written as in Eq. (3.4), and that at high energies it is exponentially suppressed. This is at the heart of string theory’s ultra soft UV behavior.

\(^3\)Looking at the Veneziano amplitude Eq. (5.29) we see that it can be written as $V_{u} = \sum_{n,J} g^{2}_{n}(-t)^{J} \frac{s-m^{2}}{s-m^{2}}$
3.2 Large extra dimensions

References: [1, 2, 32, 33], see also [30, 31].

The main motivation for the LED scenario is that it offers a solution to the hierarchy problem. The hierarchy problem is the unnaturalness of the large energy gap between the Planck scale \( M_{\text{Planck}} = G^{-1/2} \sim 10^{19} \text{ GeV} \) and the electroweak scale \( M_{\text{EW}} \sim 10^3 \text{ GeV} \).

Extra dimensions are postulated and the D-dimensional quantum gravity scale is assumed to be low near the EW scale \( M_D \sim 1 \text{ TeV} \). So between these two scales there is no hierarchy. The effective 4 dimensional gravity scale is determined after compactifying the extra dimensions. The large hierarchy between the 4-d QG scale and the EW scale is explained by assuming that the compactification volume \( V_\delta \), and hence some of the extra dimensions, are large. In this scenario gravity is allowed to move in the extra dimensions, and this ”leakage” of gravity causes it’s weakness from the point of view of a 4d observer.

In the basic scenario, one assumes the existence of \( \delta \) extra spatial dimensions so that space-time now has \( D = \delta + 4 \) dimensions. Only gravity may propagate in the extra dimensions while the standard model fields are restricted to regular 3d space. The extra dimensions must be of finite extent (compactified) in order to have been avoided in experiments which test Newton’s gravitational force law.

The effective 4d gravity scale \( M_{\text{Planck}} \) is related to \( M_D \) by Gauss’s law:

\[
M_{\text{Planck}}^2 = V_\delta M_D^{2+\delta}
\]  \hspace{1cm} (3.5)

We see that \( V_\delta \) needs to be large, meaning that some of the extra dimensions ought to be large.

We now discuss KK particles. If there exists extra dimensions of finite extent and gravity is allowed to propagate in them, there will be new particles. This is completely analogous to the text book problem of a particle in a box which will have it’s momentum quantized. Similarly, the graviton will have its momentum quantized in the direction of the extra dimensions. The energy-momentum relation for a massless graviton in a space with one extra dimension is:

\[
E^2 = p_x^2 + p_y^2 + p_z^2 + p_{ED}^2 = p_x^2 + p_y^2 + p_z^2 + n^2 \frac{R^2}{R^2}
\]  \hspace{1cm} (3.6)

Where \( R \) is the size of the extra dimension and \( n \) is an integer.

This shows that to an observer in 4-d, the quantized graviton momenta effectively appear as an infinite tower of new particles (KK gravitons) with the same quantum numbers as
the graviton. These spin 2 resonances\(^4\) couple gravitationally with equal strength to all the standard model particles, and are evenly spaced in their mass:

\[
m_{KK} = \frac{n}{R}
\] (3.7)

In the large extra dimension scenario the spacing in mass of the KK states is small (Possibly \(10^{-3} \text{ eV}\)). Regarding collider signals, it turns out that this spacing is smaller than the resolution of the detectors, so that a continuum signal is expected as opposed to resonance peaks. The two most important processes are \textit{virtual exchange} and \textit{direct production} of KK gravitons (These are also the two most important processes for Regge states, as will be discussed). Table 1 lists a few of these processes at hadron colliders. The signal for virtual exchange will be a smooth deviation from the standard model cross sections. In direct production, (at least) one of the final state particles is a KK graviton which is not detected since it is gravitationally interacting. Thus the signal is missing energy.

| Direct production | Virtual exchange |
|-------------------|------------------|
| \(gg \rightarrow gG\) | \(gg \rightarrow q\bar{q}\) |
| \(q\bar{q} \rightarrow gG\) | \(gg \rightarrow \gamma\gamma\) |
| \(q\bar{q} \rightarrow \gamma G\) | \(q\bar{q} \rightarrow q'\bar{q}'\) |
| \(gg \rightarrow qG\) | \(q\bar{q} \rightarrow \gamma\gamma\) |

Table 1: Examples of direct production and virtual exchange of KK gravitons \(G\).

One thing is very clear: The possibility of having quantum gravity effects at LHC energies is exciting.

### 3.3 Brane world and the string scale

\textit{References: } [64, 1, 21, 28, 104].

Extra dimensions necessarily arise in string theory in order to get a consistent and realistic theory. Cancellation of the conformal anomaly requires the spacetime dimension to be equal to the critical dimension. In bosonic string theory the critical dimension is \(D = 26\) (\(\delta = 22\)), and in superstring theory it is \(D = 10\) (\(\delta = 6\)).
In string theory there is an additional scale called the string scale $M_s$, which quite generally is the energy at which stringy phenomena start to appear. The string scale is related to the Regge slope, string length, and string tension through $M_s^2 = \frac{1}{\alpha} = \frac{1}{l_s^2} = 2\pi T$. This parameter enters the string action as:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-hh^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu}$$  \hspace{1cm} (3.8)$$

In the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}e^\phi$, and the action becomes that of D free scalar fields:

$$S = -\frac{T}{2} \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu$$ \hspace{1cm} (3.9)$$

Where $\eta_{\alpha\beta}$ is the flat metric.

The equation of motion is:

$$\frac{\partial^2 X^\mu}{\partial\sigma^2} - \frac{\partial^2 X^\mu}{\partial\tau^2} = 0$$ \hspace{1cm} (3.10)$$

and the constraint equations $\frac{\delta S}{\delta h_{\alpha\beta}} = 0$ are:

$$T_{10} = T_{01} = \dot{X} \cdot X' = 0$$
$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$  \hspace{1cm} (3.11)$$

The energy and angular momentum are:

$$P^\mu = T \int_0^\infty d\sigma \frac{dX^\mu}{d\tau}$$ \hspace{1cm} (3.12)$$

$$J^{\mu\nu} = T \int_0^\infty d\sigma \left( X^\mu \frac{dX^\nu}{d\tau} - X^\nu \frac{dX^\mu}{d\tau} \right)$$  \hspace{1cm} (3.13)$$

As an example, we can now use this information to show (classically) that $\alpha'$ is the Regge slope for the open string. The Regge slope is defined as the maximum angular momentum per energy squared. This is satisfied by a spinning string with a stationary center of mass. The spinning string is described by:

$$x = A \cos \tau \cos \sigma, \quad y = A \sin \tau \cos \sigma, \quad t = A\tau$$  \hspace{1cm} (3.14)$$

Which can easily be shown to satisfy Eqs. (3.10), (3.11). Plugging this solution in Eqs. (3.12), (3.13) gives:

$$E = P^0 = \pi AT$$  \hspace{1cm} (3.15)$$
\[ J_z = \pi A^2 T/2 \quad (3.16) \]

So that,
\[ J_z/E^2 = 1/(2\pi T) = \alpha' \quad (3.17) \]

Note that we will obtain this same result \( J_{\text{max}} < \alpha'E^2 + \text{constant} \), from the poles of the Veneziano amplitude.

We now discuss the brane-world scenario. Following [1], we consider type II superstring theory in \( D = 10 \) space-time with the Dp-branes wrapped around p-3-cycles. The remaining 3 dimensions being the regular three dimensional space. The 6 internal (compactified) dimensions are decomposed into \( d_i = p - 3 \) directions parallel to the Dp-brane, and \( d_j = 9 - p \) directions transverse to the Dp-brane. We denote the transverse and parallel radii as \( R_j^\perp \) and \( R_i^\parallel \) respectively. The total internal volume is:
\[ V_6 = (2\pi)^6 \prod_{i=1}^{d_i} R_i^\parallel \prod_{j=1}^{d_j} R_j^\perp \quad (3.18) \]

The relation between the Planck scale and the string scale is given by\(^5\)
\[ M_{\text{Planck}}^2 = \frac{8}{(2\pi)^6} g_s^{-2} M_s^8 V_6 \quad (3.19) \]

Where \( g_s = e^{\phi_{10}} \) is the string coupling and \( \phi_{10} \) the dilaton field. We can have \( M_s \sim \text{TeV} \) and \( V_6 \sim 10^{32} \), or \( M_s \sim M_{\text{Planck}} \) and \( V_6 \sim 1 \), or any intermediate case. We consider the first case in which the string scale is accessible at present colliders.

From Eqs. (3.5) and (3.19):
\[ M_D = \left( \frac{8}{(2\pi)^6} \right)^{1/8} g_s^{-1/4} M_s \quad (3.20) \]

These two scales are tied together and \( M_s \sim M_D \sim \text{TeV} \).

\(^5\)Compare this with the heterotic string in which \( M_s = g_s M_{\text{Planck}} \). Since there is no dependence on the compactification volume, the string scale cannot be lowered by enlarging the volume. This difference between the type I or II strings and the heterotic string can be traced back to the fact that the type I or II gauge fields arise from open strings whereas in the heterotic theory both the gauge and gravity fields arise from closed strings. These relations for \( M_{\text{Planck}} \) are obtained by considering the \( d = 4 \) low energy effective string action and comparing it to the Einstein-Hilbert action. Similarly, the relation for the gauge coupling constant Eq. (3.21) is obtained by comparing the gauge part of the effective action to the Yang-Mills action.
The gauge fields are confined to the Dp-brane and are free to propagate in the \(d_{||}\) directions. Since colliders measure the gauge interactions, they constrain \(R_{||}\) to be smaller than about a TeV. The gauge couplings will depend on the radii of parallel directions according to:

\[
g_{D_p}^{-2} = \frac{1}{2\pi} g_s^{-1} M_s^{-3} \prod_{i=1}^{d_{||}} R_{||}^i \tag{3.21}
\]

Combining Eqs. (3.19), (3.18) and (3.21) gives:

\[
g_{D_p}^2 M_{\text{Planck}} = 2\frac{5}{2\pi} M_s g_s^{-1} \left( \prod_{j=1}^{d_{\perp}} R_{\perp}^j \right)^{\frac{1}{2}} \left( \prod_{i=1}^{d_{||}} R_{||}^i \right)^{-\frac{1}{2}} \tag{3.22}
\]

For \(p < 7\) we see that enlarging \(R_{\perp}^j\), decreases \(M_s\) (\(M_{\text{Planck}}\) is constant). We can give a rough estimate of \(R_{\perp}\) as follows: We assume that all of the \(R_{\perp}^j\) are equal, and that all of the \(R_{||}^i\) are separately equal. Eqs. (3.19) and (3.18) give:

\[
M_{\text{Planck}}^2 = 8 g_s^{-2} M_s^8 \left( R_{||}^{d_{||}} \right)^{6-d_{\perp}} \left( R_{\perp}^{d_{\perp}} \right)^{d_{\perp}} \tag{3.23}
\]

Further assuming \(R_{||} \approx M_s^{-1} = 1 \text{ TeV}^{-1}\) and \(g_s = 1/25\), we get the estimates in Table 2.

| \(R_{\perp}^{[\text{GeV}^{-1}]}\) | \(d_{\perp} = 1\) | \(d_{\perp} = 2\) | \(d_{\perp} = 3\) | \(d_{\perp} = 4\) | \(d_{\perp} = 5\) | \(d_{\perp} = 6\) |
|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(R_{\perp}^{[\text{m}]\}}\) | \(1.6 \cdot 10^{26}\) | \(4 \cdot 10^{11}\) | \(5.4 \cdot 10^6\) | \(2 \cdot 10^4\) | \(693\) | \(74\) |
| \(E_R[^{\text{MeV}}]\) | \(7.7 \cdot 10^{-24}\) | \(3 \cdot 10^{-9}\) | \(2 \cdot 10^{-4}\) | \(0.06\) | \(1\) | \(16\) |

Table 2: \(R_{\perp}\) estimations. From [1].

Experimental constraints on \(R_{\perp}\) come from Cavendish experiments testing Newton’s inverse square law. These give roughly \(R_{\perp} < 1 \text{ [mm]}\), therefore \(d_{\perp} = 1\) is ruled out. If \(d_{\perp} = 2\), then this estimate gives an \(R_{\perp}\) of the millimeter size.

In the previous section we mentioned KK gravitons, which are excitations of the graviton (closed string state). Their masses will be determined by all the extra dimensions (including the large ones \(R_{\perp}\)) since they move in the bulk. At low energies though, only KK momenta from the large extra dimensions will be excited:

\[
m_{\perp KK} = \frac{n_{\perp}}{R_{\perp}} \tag{3.24}
\]
The open string ground states (the standard model gauge bosons) will have corresponding KK excitations since they propagate in the small extra dimensions $R_\parallel$:

$$m_{KK}^\parallel = \frac{n_{KK}^\parallel}{R_\parallel}$$

(3.25)

In addition to KK states there will also be towers of particles called *winding states*, caused by the wrapping of strings around the extra dimensions:

$$m_{\text{wind.}} = W R_\parallel M_s^2$$

(3.26)

$W$ is an integer called the winding number.

As opposed to Regge states, KK states and winding states are clearly dependent on the geometry of the extra dimensions. Note that closed string states, such as the transverse KK states, interact at 1-loop level and hence their effects are suppressed.

### 3.4 Intersecting D-brane models

*References:* [1, 22, 23, 24, 25, 26, 27, 66, 98, 99].

String phenomenology is the study of how to embed the standard model into superstring theory. Intersecting D-brane models make such an attempt in the framework of type I or II superstring theory. In these models it was shown possible to achieve the following:

1. Contain the standard model gauge group $SU(3) \times SU(2) \times U(1)$.
2. Have chiral fermions: the quarks and leptons.
3. Family replication: the 3 families.
4. $\mathcal{N} = 1$ SUSY or no SUSY.

#### 3.4.1 Generalities

A Dp-brane is a non-perturbative extended object with $p$ space dimensions. The fluctuations of a D-brane are described by a string theory. Open strings attach to D-branes and their ends satisfy Dirichlet boundary conditions transverse to the D-brane, and Neumann boundary

---

6In intersecting brane models, the fermions are placed at the intersection of the D-branes, thus they move in a smaller number of dimensions compared to the gauge bosons which move on the branes.
conditions along the D-brane. D-branes couple to the gauge fields of the R-R sector in type II string theory. Type IIA contains stable $p =$even D$p$-branes that couple to the $n =$odd n-forms of the theory. Likewise type IIB has $p =$odd and $n =$even. These so called BPS branes are stable since they carry a conserved charge. Type I (II) string theory contains $\mathcal{N} = 1(2)$ supersymmetry in 10 dimensions. Introducing D-branes into type I or II string theory breaks some of the supersymmetry. In this way, model building can result in different amounts of supersymmetry.

Closed strings are not attached to D-branes and can propagate in the bulk. Gravity arises from the massless sector of the closed string, so it too can propagate in the bulk. An open string with both ends attached to a single D-brane gives rise to a $U(1)$ gauge field that is confined to the brane, Fig. 3. $N$ copies of this configuration gives of course $U(1)^N$. If the $N$ D-branes are brought close together and stacked on top of each other, the gauge fields will be in the adjoint representation of $U(N)$. It is thus possible to realize the gauge group $SU(N)$ via $U(N) \sim SU(N) \times U(1)$, and necessarily there will be extra $U(1)$'s.

On top of the gauge and gravity interactions, D-branes also make it possible to realize chiral matter. One way to do this is by placing the D-branes on orbifold or conifold singularities. We will consider a different method: D-branes intersecting at angles. Consider a stack of $N$ D-branes intersecting a stack of $M$ D-branes. An open string stretched between the two stacks can give rise to a chiral Weyl fermion in the bi-fundamental representation of $U(N) \times U(M)$, see Fig. 3.

A D-brane is described by the DBI action, and it has tension (a positive contribution to the vacuum energy). Negative tension objects called orientifold planes are introduced to

---

7This is one way to see that D-branes break SUSY. In a supersymmetric theory the vacuum energy is zero.
cancel the D-brane tension. These models are then called orientifold models. A new feature is that now the $SP(2N)$ and $SO(2N)$ gauge groups are possible for the gauge fields, in addition to $U(N)$. $SP(2N)$ and $SO(2N)$ appear if the 3-cycle is invariant under the anti-holomorphic involution $\bar{\sigma}$, whereas $U(N)$ appears if it is not invariant. We consider the second case in this work. In addition to the bi-fundamental, the symmetric and anti-symmetric representations for the chiral fermions are now also possible. These new representations arise from strings stretched between a D-brane and its image. Since these exotic chiral fermions do not appear in the standard model they are usually unwanted, and indeed they do not appear in the model we consider in the next section.

Family replication is achieved as follows. The number of chiral fermions at an intersection of two branes is determined by the intersection number. In a flat non-compact space, the intersection numbers obviously can be only $\pm 1$. But we must consider compact extra dimensions, and this enables multiple intersections between the branes. Consider the compact space to be a 6-torus $T^6 = T^2 \times T^2 \times T^2$ and that D6-branes cover a 1-dimensional cycle on each $T^2$. Each $T^2$ is then described by a pair of wrapping numbers $(n_i^a, m_i^a)$ along the cycles $[x_i^a]$ and $[y_i^a]$. A 3-cycle can then be written as product of three 1-cycles:

$$\pi_a = \prod_{i=1}^{3} (n_i^a [x_i^a] + m_i^a [y_i^a])$$  \hspace{1cm} (3.27)

Since $[x_i^a] \cap [y_i^a] = -1$, the intersection number between branes $a$ and $b$ is:

$$I_{ab} = \pi_a \circ \pi_b \prod_{i=1}^{3} (n_i^a m_i^b - m_i^a n_i^b)$$  \hspace{1cm} (3.28)

The mirror cycles $\pi'_a$ have wrapping numbers $(n_i^a, -m_i^a)$, therefore:

$$I_{a'b} = \pi'_a \circ \pi_b = \prod_{i=1}^{3} (n_i^a m_i^b + m_i^a n_i^b)$$  \hspace{1cm} (3.29)

The chiral spectrum of many orientifold models can be read from Table 3. In the next section the intersection numbers and chiral spectrum of a 4-stack model with D6-branes will be shown.

### 3.4.2 4 stack D-brane models

We now describe the important class of 4 stack D-brane models. These will be our prototype models. Consider type II orientifolds with D6-branes wrapping compact homology 3-cycles
\[
\begin{array}{|c|c|c|}
\hline
\text{sector} & \text{representation} & \text{intersection number } I \\
\hline
a' a & A_a & \frac{1}{2} (\pi'_a \circ \pi_a + \pi_{O6} \circ \pi_a) \\
\hline
a' a & S_a & \frac{1}{2} (\pi'_a \circ \pi_a - \pi_{O6} \circ \pi_a) \\
\hline
a b & (N_a, N_b) & \pi_a \circ \pi_b \\
\hline
a' b & (N_a, N_b) & \pi'_a \circ \pi_b \\
\hline
\end{array}
\]

Table 3: Intersection of 3-cycles. \( \pi_{O6} \) is the 3-cycle of the orientifold plane. The existence of the symmetric and anti-symmetric representations can be seen. From [1].

\[ \begin{aligned}
\pi_a \text{ of the internal space. The massless gauge fields live in the subspaces } & \mathbb{R}^{1,3} \times \pi_a. \text{ There are also } O6\text{-planes } \pi_{O6}, \text{ and for each stack of D6-branes there is an orientifold mirror stack wrapped around the reflected cycles } \pi'_a. \text{ The Intersection numbers } I \text{ fix the chiral spectrum.} \\
\text{Fig. 4 shows the intersection pattern of the four D6-branes.} \\
\text{The gauge group is:} \\
U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d \\
\end{aligned} \]

\[ \text{(3.30)} \]

\[ \text{Which is equivalent to:} \\
SU(3)_a \times SU(2)_b \times U(1)_a \times U(1)_b \times U(1)_c \times U(1)_d \\
\]

\[ \text{(3.31)} \]

The \( SU(3) \) and \( SU(2) \) groups correspond to the strong and weak gauge groups. The four \( U(1) \)'s of Eq. (3.31) generally mix to form the physical particles. Three of these so called \( Z' \) bosons will receive masses of the order of the string scale, via the generalized Green-Schwarz mechanism, see e.g [22] [23]. The remaining \( U(1) \) field stays massless and is identified with the hypercharge.

21
In general, the hypercharge can be written as a linear combination of all the $U(1)'s$:

$$U(1)_Y = \sum_i c_i U(1)_i$$

so that

$$Q_Y = \sum_i c_i Q_i$$

and

$$\frac{1}{\alpha_Y} = \sum_i \frac{N_i c_i^2}{\alpha_i}$$

In our model we assume the so called Madrid hypercharge embedding:

$$Q_Y = \frac{1}{6} Q_a + \frac{1}{2} Q_c + \frac{1}{2} Q_d$$

We see that the abelian gauge boson from the color stack $U(1)_a$ mixes with the hypercharge, and hence with the photon and $Z$ boson. This can be viewed as mixing of the gluon with photon and $Z$. As we will see this gives rise to tree level amplitudes which are forbidden in the standard model (e.g. scattering of gluons into photons).

We now turn to the chiral spectrum of this model. In [27] a general solution for the wrapping numbers $(n^l_a, m^l_a)$ which give the standard model spectrum was found. One example for such a solution is given in Table 4. From Eqs. (3.28) and (3.29) These wrapping numbers give rise to the following intersection numbers:

$$I_{ab} = 1 , \quad I_{ab'} = 2$$
$$I_{ac} = -3 , \quad I_{ac'} = -3$$
$$I_{bd} = 0 , \quad I_{bd'} = -3$$
$$I_{cd} = -3 , \quad I_{cd'} = 3$$

and these intersection numbers give rise to the standard model spectrum (plus extra $U(1)'s$) as shown in Table 5. The 3 quark families comes from $I_{ab} + I_{ab'} = 3$, and so on..

We note the following issues:

- Different types of models are possible. For example, more then 4 stacks of D-branes, D5 instead of D6-branes, gauge groups of grand unified theories, etc.
\[
N_{a} = 3 \quad (n^1, m^1) \quad (n^2, m^2) \quad (n^3, m^3)
\]

\[
N_{b} = 2 \quad (0, -1) \quad (1, 0) \quad (1, \frac{3}{2})
\]

\[
N_{c} = 1 \quad (1, 3) \quad (1, 0) \quad (0, 1)
\]

\[
N_{d} = 1 \quad (1, 0) \quad (0, -1) \quad (1, \frac{3}{2})
\]

Table 4: Wrapping numbers. From [27]

| Intersection | Matter fields | \( Q_a \) | \( Q_b \) | \( Q_c \) | \( Q_d \) | \( Y \) |
|--------------|---------------|-----------|-----------|-----------|-----------|--------|
| \((a, b)\)   | \( Q_L \)     | (3, 2)    | 1         | -1        | 0         | \( \frac{1}{6} \) |
| \((a, b')\)  | \( q_L \)     | 2(3, 2)   | 1         | 1         | 0         | \( \frac{1}{6} \) |
| \((a, c)\)   | \( U_R \)     | 3(\(\bar{3}\), 1) | -1 | 0 | 1 | -\( \frac{2}{3} \) |
| \((a, c')\)  | \( D_R \)     | 3(\(\bar{3}\), 1) | -1 | 0 | -1 | 0 | \( \frac{1}{3} \) |
| \((b, d')\)  | \( L \)       | 3(1, 2)   | 0         | -1        | 0         | -1 | \( -\frac{1}{2} \) |
| \((c, d)\)   | \( E_R \)     | 3(1, 1)   | 0         | 0         | -1        | 1    | 1 |
| \((c, d')\)  | \( N_R \)     | 3(1, 1)   | 0         | 1         | 1         | 0 |

Table 5: Standard model spectrum and \( U(1) \) charges. From [27].

- \( U(1) \) anomalies are canceled by the Green-Schwarz mechanism, see e.g. [22, 23]. The corresponding \( Z' \) bosons receive masses from Chern-Simons terms even if they are not anomalous. The \( U(1) \)'s survive as perturbative global symmetries and can be identified with baryon and lepton number. This leads to proton stability and prevents Majorana neutrino masses. That being said, the condition for the hypercharge to remain massless is (see Eq. (3.32)):

\[
\sum_i c_i N_i (\pi_i - \pi'_i) = 0 \quad (3.37)
\]

- These models contain 3 right handed neutrinos.

- intersecting D-brane models are classified as either supersymmetric or non-supersymmetric. Supersymmetric models usually assume a high string scale near the planck scale, whereas
non-supersymmetric models usually assume a low string scale $\sim 1 - 100$ TeV. Our model is non-supersymmetric.

- It has been argued that the effects of four-fermi operators on FCNC’s, EDM’s (electric dipole moments), and supernova cooling, constrain the string scale to be above $\sim 10^4$ TeV. This implies that non-supersymmetric intersecting brane models suffer a severe fine tuning problem.
4 Review of amplitude calculations

4.1 Field theory

References: Mainly follows [39]. See also [41, 42, 40, 36] and Appendix E where the formalism and notation is presented.

This section is a review of amplitude calculation in field theory via the helicity amplitude technique and the trace color decomposition.

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|----|
| no. of diagrams | 4 | 25 | 220 | 2485 | 34300 | 559405 | 10525900 |

Table 6: The number of Feynman diagrams for the $n$-gluon amplitude at tree level. From [39].

We use the words "quarks" and "gluons" but the gauge symmetry is $SU(N)$ or $U(N)$ and not necessarily $SU(3)$. Evaluation of amplitudes via textbook methods for calculating Feynman diagrams, becomes very complex as one goes to higher loops or as one adds external particles. In this paper we do not deal with loop diagrams. The complexity arises from the large number of Feynman diagrams (see Table 6 and Fig. 5) and from the fact that the non-
abelian vertices give rise to a large number of terms. Usually at the end of the calculation there is a large amount of cancellation between the terms, giving rise to a relatively simple answer. This suggests the existence of a formalism which may simplify the procedure by taking better account of the symmetries of the amplitude. Since the perturbative expansion in Feynman diagrams is not gauge invariant, a major step forward was identifying what combination of Feynman diagrams can give rise to a gauge invariant basis in which to expand. A particularly useful color decomposition was discovered via analogy with the Chan-Paton structure of string amplitudes: The trace color decomposition. Other useful calculation techniques include finding simple representations for the polarization vectors in terms of massless spinors, spinor products, recursion relations among the amplitudes, and Supersymmetric Ward identities.

The helicity amplitude technique consists of calculating the amplitudes with definite helicities for the external particles. There are two types of amplitudes which are particularly simple: the \( n \)-gluon amplitude and the \( n \)-gluon plus a quark anti-quark pair amplitude, Fig. 6. We will refer to these amplitudes as universal amplitudes for reasons which will become clear in the string theory section 4.2. We will see shortly that the universal amplitudes have a simple color decomposition on a color basis which is orthogonal at leading order in \( 1/N \). Written in this basis, a closed formula for the MHV sub-amplitudes exists.

All amplitudes of the form \( (1^+, 2^+ \ldots n^+) \) and \( (1^-, 2^+ \ldots n^+) \) vanish. In the \( ng + q\bar{q} \) case, also the amplitudes with \( q \) and \( \bar{q} \) having the same helicity vanish. Explicitly:

\[
\begin{align*}
m(g_1^+, g_2^+, \ldots, g_n^+) &= 0 \\
m(g_1^-, g_2^+, \ldots, g_n^+) &= 0 \\
m(q^+, q^-, g_1^+, g_2^+, \ldots, g_n^+) &= m(q^-, q^+, g_1^+, g_2^+, \ldots, g_n^+) = 0 \\
m(q^+, q^+, g_1, \ldots, g_n) &= 0
\end{align*}
\]
And of course, amplitudes obtained by reversing all helicities or permuting identical particles, vanish as well.

The Maximally Helicity Violating amplitude \((\text{MHV})\) is an amplitude with 2 particles having a certain helicity, and the rest having the opposite helicity: \((1^+,2^+,3^-,4^-\ldots n^-)\). For the universal amplitudes, the MHV amplitudes do not vanish but they have a simple closed formula for arbitrary \(n\). For \(n = 4\) or \(n = 5\), Eqs. (4.1) and (4.2) imply that the MHV are the only non-vanishing helicity configuration. This trend ends at \(n = 6\) which has also the non-vanishing \(m(1^+,2^+,3^+,4^-,5^-,6^-)\).

4.1.1 \(n\) gluons

An \(n\)-gluon amplitude can be decomposed as follows:\[^8^]

\[
\mathcal{M}(g_1,\ldots,g_n) = \sum_{\{1,\ldots,n\}'} Tr(T^{a_1} T^{a_2} \cdots T^{a_n}) \ m(g_1,\ldots,g_n) \tag{4.3}
\]

Where the color matrices are in the fundamental representation. The subamplitudes (Sometimes called colored ordered amplitudes \(m(1,\ldots,n) \equiv m(g_1,\ldots,g_n) \equiv m(p_1,\epsilon_1,\ldots p_n,\epsilon_n)\) contain the kinematics: polarization vectors from the external legs and momentum vectors from the vertices. It is seen that they multiply a Chan-Paton color factor. The sum \(\{1,\ldots,n\}'\) is over the \((n-1)!\) cyclic inequivalent permutations.

Eq. (4.3), when squared and summed over colors and permutations gives:

\[
|\mathcal{M}|^2(g_1,\ldots,g_n) = \sum_{\lambda,\lambda'} m_{\lambda} S_{\lambda\lambda'} m_{\lambda'}^* \tag{4.4}
\]

Where \(m_{\lambda} \equiv m(g_{1\lambda},\ldots,g_{n\lambda})\) is a given permutation, and

\[
S_{\lambda\lambda'} \equiv \sum_{a_1,\ldots,a_n} Tr(T^{a_1\lambda} \cdots T^{a_n\lambda}) \ [Tr(T^{a_1\lambda'} \cdots T^{a_n\lambda'})]^* \tag{4.5}
\]

[^8]: We saw that the "would be" two most violating helicity amplitudes vanish, hence they are not called MHV.

[^9]: Proof: An \(n\)-gluon feynman diagram contains only gluon lines. A 3-gluon vertex contains \(f^{abc}\) which can be written as \(f^{abc} = -iT^c [T^a T^b T^c - T^b T^a T^c]\), by using \([T^a , T^b] = i f^{abc}\). Now each leg attached to this vertex has a \(T\) attached to it. Each of these legs goes either to an external gluon or to another vertex. In the latter case, the \(f^{cde}\) from the second vertex can be combined with \(T^c\) from the first vertex to give: \(T^c f^{cde} = -i[T^d , T^e}\). So we got for the two vertices: \(f^{abc} f^{cde} = -Tr[T^a T^b [T^d , T^c] - [T^d, T^c] T^b T^a]\), which is of the required form. The 4-gluon vertex has \(f^{abc} f^{cde}\) which is already in the required form. This process can easily be seen to continue by iteration.
The subamplitudes satisfy the following properties:

1. $m(1,\ldots,n)$ is gauge invariant.
2. $m(1,\ldots,n)$ is invariant under cyclic permutations of $1,\ldots,n$.
3. Reflection: $m(n,\ldots,1) = (-1)^n m(1,\ldots,n)$.
4. The dual ward identity (Also called sub-cyclic identity or photon decoupling identity):

$$m(1,2,\ldots,n) + m(2,1,3,\ldots,n) + m(2,3,1,\ldots,n) + \ldots + m(2,3,\ldots,1,n) = 0$$

(4.6)

5. Factorization of $m(1,\ldots,n)$ on multi-gluon poles.

The $MHV$ amplitude has a simple closed formula:

$$m(g_1^+,g_2^+,g_3^-,\ldots,g_n^-) = ig^{n-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

(4.7)

So that,

$$|m|^2(g_1^+,g_2^+,g_3^-,\ldots,g_n^-) = g^{2n-4} \frac{s_{12}^4}{s_{12} \cdots s_{n1}}$$

(4.8)

Where $s_{ij} = (k_i + k_j)^2$, $\langle ij \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}}$, $[ij] = -\sqrt{|s_{ij}|} e^{-i\phi_{ij}}$, see Appendix E.

In most colliders (in particular hadron colliders), the helicities of the particles are not measured. Hence, after the amplitude is squared it should be summed over possible helicities and also over colors. Squaring and summing over colors (see Eq. (F.26)) gives:

$$\sum_{\text{colors}} |M_n|^2 = N^{n-2}(N^2 - 1) \sum_{\{1,\ldots,n\}'} \left[ |m|^2(1,\ldots,n) + O(1/N^2) \right]$$

(4.9)

When summed over colors (see Eq. (4.9)) and MHV configurations one gets the Parke-Taylor amplitudes:

$$\sum_{\text{helicities}} \sum_{\text{colors}} |M|^2(g_1,\ldots,g_n) = 2g^{2n-4}N^{n-2}(N^2 - 1) \sum_{i>j} s_{ij}^4 \sum_{\{1,\ldots,n\}'} \left[ \frac{1}{s_{12} \cdots s_{n1}} + O(1/N^2) \right]$$

(4.10)

---

10. The first two properties follow from the linear independence of the Chan-Paton factor (to leading order in $1/N$, see Eq. (F.26)). Since $Tr[T^{a_1} \cdots T^{a_n}]$ is cyclic invariant, so will the subamplitude be. Since the full amplitude is gauge invariant, so will the subamPLAtudes be.
The factor of 2 comes from the sum over \((+ + - - - -)\) and \((- - + - + +\)), and it is absent for \(n = 4\).

It turns out that for 4 and 5 gluons the \(\mathcal{O}(1/N^2)\) correction vanishes in Eq. (4.9), so that Eq. (4.10) becomes:

\[
\sum_{\text{helicities}} \sum_{\text{colors}} |\mathcal{M}|^2(g_1, g_2, g_3, g_4) = N^2(N^2 - 1)g^4\sum_{i>j} s_{ij}^4 \frac{1}{s_{12}s_{23}s_{34}s_{41}} \quad (4.12)
\]

\[
\sum_{\text{helicities}} \sum_{\text{colors}} |\mathcal{M}|^2(g_1, g_2, g_3, g_4, g_5) = 2g^6N^3(N^2 - 1)\sum_{i>j} s_{ij}^4 \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}} \quad (4.13)
\]

**4.1.2 \(n\) gluons + 2 quarks**

The 2 quark plus \(n\)-gluons amplitude can be decomposed in the following way

\[
\mathcal{M}(q, \bar{q}, g_1, \ldots, g_n) = \sum_{\{1, \ldots, n\}} (T^{a_1} \ldots T^{a_n})_{ij} m(q, \bar{q}, g_1, \ldots, g_n) \quad (4.14)
\]

Eq. (4.14), when squared and summed over colors and permutations gives:

\[
|\mathcal{M}|^2(q, \bar{q}, g_1, \ldots, g_n) = \sum_{\lambda, \lambda'} m_{\lambda} \mathcal{P}_{\lambda\lambda'} m^*_{\lambda'} \quad (4.15)
\]

Where,

\[
\mathcal{P}_{\lambda\lambda'} \equiv \sum_{a_1, \ldots, a_n} (T^{a_1} \ldots T^{a_n})_{ij} \left(T^{a_1\lambda'} \ldots T^{a_n\lambda'}\right)^*_{ij} \quad (4.16)
\]

and \(m_{\lambda} \equiv m(q, \bar{q}, g_{1\lambda}, \ldots, g_{n\lambda})\)

The MHV amplitude has a simple closed formula:

\[
m(q^+, q^-, g_1^+, \ldots, g_n^+) = ig^n \frac{\langle q1 \rangle^3 \langle \bar{q}1 \rangle}{\langle \bar{q}q \rangle} \frac{1}{\langle q1 \rangle \langle 12 \rangle \cdots \langle n\bar{q} \rangle} \quad (4.17)
\]

\[11\]For 6 gluons Eq. (4.9) will be:

\[
\sum_{\text{colors}} |\mathcal{M}_6|^2 = N^4(N^2 - 1) \sum_{\{2,3,4,5,6\}} \left[ |m|^2(1,2,\ldots,6) + \frac{2}{N^2} m^*(123456) \left( m(135264) + m(153624) + m(136425) \right) \right] \quad (4.11)
\]
So that,

$$|m|^2(q^+, q^-, g_1, g_2, \ldots, g_n) = g^{2n} \frac{s_{q_1} s_{\bar{q}_1}}{s_{q_1} s_{12} \cdots s_{n\bar{q}}} \frac{1}{s_{q_{1}} s_{\bar{q}_{1}} s_{q_{2}} s_{\bar{q}_{2}} \cdots s_{n\bar{q}}}$$

(4.18)

Squaring and summing over colors and MHV configurations:

$$\sum_{\text{helicities}} \sum_{\text{colors}} |M|^2(q, \bar{q}, g_1, \ldots, g_n) = 2g^{2n} N^{n-1}(N^2 - 1) \sum_{i=1}^{n} \left( s_{q_i} s_{\bar{q}_i} s_{q_i} s_{\bar{q}_i} \right)$$

$$\times \frac{1}{s_{q\bar{q}} \sum_{\{1, \ldots, n\}} \frac{1}{s_{q_1} s_{12} \cdots s_{n\bar{q}}}} + \mathcal{O}(1/N^2)$$

(4.19)

4.2 String theory

References: We mainly follow [68, 1, 2], see also [64, 65, 66, 67, 100, 101, 102].

4.2.1 Generalities

We review in this section the techniques used to calculate amplitudes in string theory. We obtain the equations which enable us to calculate the leading order amplitudes (disk and sphere amplitudes).

In quantum field theory one calculates correlation functions $\langle \phi(x_1) \cdots \phi(x_n) \rangle$. To get scattering amplitudes, the correlation functions are put on-shell. In a quantum theory of gravity it is not so clear how to deal with off-shell correlation functions. Instead we can calculate the S-matrix by taking the limit $x_i \to \infty$ in the correlation functions. In string theory, a drawing of the lowest order interaction of strings looks as in Fig. 7. Unlike QFT there are no interaction vertices, and locally it is a free theory. Only when observed globally the interactions are seen. Taking $x_i \to \infty$ amounts to taking the legs of the diagram to infinity. The state-operator map says that a state at infinity is equivalent to an insertion of a vertex operator $V_i$ on the world sheet. A conformal transformation can transform our 2 diagrams into a disk and a sphere. A mobius transformation can transform the disk to the upper half plane, and a stereographic projection takes the sphere to the plane (Fig. 8). The vertex operators will be placed on the boundary of the disk and on the sphere (The sphere obviously has no boundary). Weyl invariance enforces the vertex operators to be on-shell. Higher order diagrams are possible by considering holes and handles in the diagrams for open and closed strings respectively. A scattering amplitude will therefore consist of an expansion
in the topology of the world sheet (Fig. 9). Fig. 10 shows the same expansion after performing the conformal transformation.

Figure 7: Open and closed string diagrams with 4 external particles. The legs should be imagined to continue to infinity.

Figure 8: Open strings: transforming to the disk and to the complex half-plane. closed strings: transforming to the sphere and to the complex plane.

We start with the following expression for the scattering amplitude for \( n \) external particles:

\[
M_n = \sum_{\text{topologies}} g_s^{-\chi} \frac{1}{\text{Vol.}} \int DXXDg \ e^{-S_{\text{Poly}}} \ V_1 \ldots V_n \quad (4.20)
\]

Since we are only dealing with tree-level amplitudes, we safely ignored the Faddeev-Popov ghost fields. This equation has the form of an expansion in the string coupling \( g_s \). There is a functional integration over the coordinate \( X \), an integration over all possible worldsheet metrics \( g \), and a sum over the different topologies weighted by \( g_s^{-\chi} \). It is therefore crucial to note that we assume weak coupling.

The Polyakov action in the conformal gauge is:

\[
S_{\text{Poly}} = \frac{1}{2\pi\alpha'} \int d^2z \ \partial X \cdot \bar{\partial} X \quad (4.21)
\]
Figure 9: Perturbation expansion of the 4-point string amplitudes. Top figure: open strings, bottom figure: closed strings.

Figure 10: The previous figure after a conformal transformation. For the open string the vertex operators are on the boundary.

\[ \chi(M) = \frac{1}{4\pi} \int d^2\sigma \sqrt{gR} = 2 - 2n_h - n_b - n_c \]  \hspace{1cm} (4.22)

\( \chi \) is a topological invariant known as the Euler number:

Where \( n_h, n_b, n_c \) are the number of handles, boundaries, and cross-caps of the worldsheet. For sphere topology \( \chi = 2 \), while for the disk \( \chi = 1 \).

We now focus on the lowest order, sphere and disk amplitudes. We need now to integrate over all metrics \( g \). We transform to the flat metric and recall the remnant global transformations: the conformal killing group \( PSL(2; C) \equiv SL(2; C)/Z_2 \) for the sphere, and \( PSL(2; R) \equiv SL(2; R)/Z_2 \) for the disk. Technically this means making the following replacements in the path integral:
\[
\text{Vol.} \int \mathcal{D}g \to \text{Vol.(SL}(2;C) \int \prod_{i=1}^{n} d^2 z_i \nabla \left( \right)  \\
\text{Vol.} \int \mathcal{D}g \to \text{Vol.(SL}(2;R) \sum_{\{1,\ldots,n\}'} \int \prod_{i=1}^{n} dz_i \right)
\]

For open strings the vertex operators are on the boundary of the disk, and hence have a given ordering to them. A cyclic permutation of the vertex operators (which is just a rotation) gives an equivalent configuration because of the reparametrisation invariance of the string action. Hence there should be a sum over the \((n-1)!\) cyclic inequivalent permutations. Also note that for open strings there is a single and not a double integration.

We define \(V_i \equiv V(z_i, k_i)\) and:

\[
\langle V_1 \cdots V_n \rangle \equiv \int \mathcal{D}X \exp \left( -\frac{1}{2\pi\alpha'} \int d^2 z \partial X \cdot \bar{\partial} X \right) \prod_{i=1}^{n} V_1 \cdots V_n
\]

and we note that The \(PSL(2, R)\) and \(PSL(2, C)\) symmetries on the disk and sphere respectively, allow to fix 3 of the insertion points \(z_i\). This leaves \(n - 3\) integrations in Eq. (4.23). The usual choice is \(z_k = 0\), \(z_l = 1\), \(z_m = \infty\).

We can finally write our master formulas for the closed and open leading order string amplitudes:

| Equation | Description |
|----------|-------------|
| (4.25)   | \(\mathcal{M}_n^{\text{closed}} = \frac{g_s^{n-2}}{\text{Vol.(SL}(2;C))} \int \prod_{i=1}^{n} d^2 z_i \langle V_1 \cdots V_n \rangle\) |
| (4.26)   | \(\mathcal{M}_n^{\text{open}} = \sum_{\{1,\ldots,n\}'} \frac{g_s^{n-1}}{\text{Vol.(SL}(2;R))} \int \prod_{i=1}^{n} dz_i \langle V_1 \cdots V_n \rangle\) |

These formulas are valid for any type and any number of external particles. For each particle there corresponds a vertex operator and an integration.

To summarize, the procedure for calculating a scattering amplitude is:

1. Write the vertex functions of the external particles.
2. Calculate the correlator of the vertex functions.
3. Fix 3 of the \(z_i\)'s, and perform the remaining \(n - 3\) integrations.
4. For open strings, sum over permutations of the \(z_i\)'s.
4.2.2 Tachyon amplitudes

A simple example is the tachyon (lowest state scalar) scattering amplitude. The vertex function of a tachyon is \( V(z) \). Defining \( z_i = z_j \), Eq. (D.8) gives the correlation function for open string tachyons:

\[
\langle e^{ik_1 X(z_1)} \cdots e^{ik_n X(z_n)} \rangle = \prod_{i<j} |z_{ij}|^{2\alpha' k_i \cdot k_j} \tag{4.27}
\]

We now use our freedom to fix 3 \( z_i \)'s. We start by choosing \( z_1 = \text{constant} \gg z_i \). This gives:

\[
|z_1|^{2\alpha' k_1 \cdot (k_2 + \cdots + k_n)} \prod_{1<i<j} |z_{ij}|^{2\alpha' k_i \cdot k_j} = \text{const.} \prod_{1<i<j} |z_{ij}|^{2\alpha' k_i \cdot k_j} \tag{4.28}
\]

Where we used \( k_2 + \cdots + k_n = -k_1 \) and \( k_1^2 = m_1^2 = \text{constant} \).

Further choosing \( z_2 = 0, z_3 = 1 \), we get:

\[
\mathcal{M}_n^{\text{(open)}} = \text{const.} \int_{l=4}^{n} dz_l |z_l|^{2\alpha' k_2 \cdot k_l} |1 - z_l|^{2\alpha' k_3 \cdot k_l} \prod_{4 \leq i < j} |z_{ij}|^{2\alpha' k_i \cdot k_j} \tag{4.29}
\]

For closed strings the correlation function is Eq. (D.7), so it is easy to see that:

\[
\mathcal{M}_n^{\text{(closed)}} = \text{const.} \int_{l=4}^{n} d^2 z_l |z_l|^{\alpha' k_2 \cdot k_l} |1 - z_l|^{\alpha' k_3 \cdot k_l} \prod_{4 \leq i < j} |z_{ij}|^{\alpha' k_i \cdot k_j} \tag{4.30}
\]

For \( n = 3 \) tachyons we get \( \mathcal{M}_3 = \text{const} \). Defining \( \hat{s}_{ij} \equiv 2\alpha' k_i \cdot k_j \), we have for \( n = 4 \):

\[
\mathcal{M}_4^{\text{(open)}} \propto \int dz |z|^4 |1 - z|^4 = B(\hat{s} + 1, \hat{\hat{s}} + 1) \tag{4.31}
\]

where \( B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} \) is the Beta function.

For the closed string:

\[
\mathcal{M}_4^{\text{(closed)}} \propto \int d^2 z |z|^\hat{s}/2 |1 - z|^\hat{\hat{s}}/2 = \frac{\Gamma(-1 - \frac{\hat{s}}{4}) \Gamma(-1 - \frac{\hat{\hat{s}}}{4}) \Gamma(3 + \frac{\hat{s}}{4} + \frac{\hat{\hat{s}}}{4})}{\Gamma(2 + \frac{\hat{s}}{4}) \Gamma(2 + \frac{\hat{\hat{s}}}{4}) \Gamma(-2 - \frac{\hat{s}}{4} - \frac{\hat{\hat{s}}}{4})} \tag{4.32}
\]

We can write the last equation in a symmetric form. Closed string tachyons have a negative mass of \( m = -4M_s \), hence \( \hat{s} + \hat{\hat{s}} + \hat{u} = \frac{1}{4\pi^2} \sum_j m_j^2 = -16 \). This gives

\[
\mathcal{M}_4^{\text{(closed)}} \propto \frac{\Gamma(-1 - \frac{\hat{s}}{4}) \Gamma(-1 - \frac{\hat{\hat{s}}}{4}) \Gamma(-1 - \frac{\hat{u}}{4})}{\Gamma(2 + \frac{\hat{s}}{4}) \Gamma(2 + \frac{\hat{\hat{s}}}{4}) \Gamma(2 + \frac{\hat{u}}{4})} \tag{4.33}
\]

34
Eqs. (4.31) and (4.33) are the Veneziano amplitude and the Virasoro-Shapiro amplitude respectively.

4.2.3 Quark-gluon amplitudes

In the previous section we gave tachyons as an example of a scattering amplitude in string theory. Although tachyons are present in bosonic string theory, they are eliminated from the spectrum of string theories which contain world-sheet fermions: superstring theories. In the models we consider, massless fermions and gauge bosons occupy the ground state of the spectrum of the open string. In particular we are interested in quark and gluon scattering amplitudes, and these are calculated quite similarly to tachyons. We recall from section 4.1 that quark-gluon amplitudes may be classified according to whether they are universal or non-universal. Universal amplitudes are defined as those containing 0 or 2 quark (or squark) fields, and non-universal amplitudes are those with more quark fields.

As an example, in order to calculate the $n$-point universal amplitudes we need the following correlation functions.

\begin{equation}
\langle V_A(z_1) \cdots V_A(z_n) \rangle \tag{4.34}
\end{equation}

\begin{equation}
\langle V_A(z_1) \cdots V_A(z_{n-2}) V_\psi(z_{n-1}) V_\bar{\psi}(z_n) \rangle \tag{4.35}
\end{equation}

Appendix C lists the vertex functions of gluons and quarks in terms of the fields from the underlying SCFT. Appendix D lists correlation functions of SCFT fields needed in order to calculate the above correlation functions. In Appendix B we present a full calculation of the $M(ggqq)$ amplitude. The vertex functions contain the color matrices in such a way that a chain of vertex operators (e.g Eq. (4.34)) gives the Chan-Paton color structure. The Chan-Paton structure is identical to the color decomposition that was done for the field theory amplitudes (Eqs. (4.3), (4.14)). Hence the universal string amplitudes will now be written as\textsuperscript{12}

\begin{equation}
\text{Tr} \left[ T^{a_1} T^{a_2} \cdots T^{a_n} \right].
\end{equation}

Historically, this picture of a string with a quark and anti-quark at its ends was introduced as a model for mesons, so that $U(N)$ was the flavor group. This picture remains approximately correct, with the QCD flux tube acting as the string.

\textsuperscript{12}The Chan-Paton structure relates to open string diagrams as shown in Fig. 11. Each string is assumed to carry a "quark" at one end and an "anti-quark" at the other end. The quarks being charged and transform as $N \times \bar{N}$ under a $U(N)$ symmetry. To each string there corresponds an $N \times N$ matrix $T^i_j$. An $n$-point amplitude is obtained by contracting in cyclic order the "anti-quark" index of a string with the "quark" index of the next string, and so will contain the factor: $(T^{a_1})^0_1 (T^{a_2})^1_2 \cdots (T^{a_n})^i_i = \text{Tr} [T^{a_1} T^{a_2} \cdots T^{a_n}].$
Figure 11: Chan-Paton structure.

\[
\mathcal{M}_{\text{string}}(g_1, \ldots, g_n) = \sum_{\{1, \ldots, n\}'} Tr(T^{a_1} \cdots T^{a_n}) \ m_{\text{string}}(g_1, \ldots, g_n)
\]  

(4.36)

\[
\mathcal{M}_{\text{string}}(q, \bar{q}, g_1, \ldots, g_n) = \sum_{\{1, \ldots, n\}} (T^{a_1} \cdots T^{a_n})_{ij} \ m_{\text{string}}(q, \bar{q}, g_1, \ldots, g_n)
\]  

(4.37)

We put the label "string" because shortly these will be compared to the field theory amplitudes. From Eq. (4.26) the subamplitude is:

\[
m_{\text{string}} = g^{n-1}_s \ Vol.(SL(2; R)) \ \int \prod_{i=1}^n dz_i \ \langle \tilde{V}_1 \cdots \tilde{V}_n \rangle
\]  

(4.38)

and \( \tilde{V}_i \) is is just \( V_i \) after stripping it from it’s color matrices.

Following [1, 2], we define \( m_{\text{QCD}} \) as the sub-amplitude in field theory (\( m \) of section 4.1). We can write \( m_{\text{string}} \) as \( m_{\text{QCD}} \) times a function (form factor) that needs to be calculated.

Focusing on the 4 and 5-point helicity amplitudes, the non-MHV subamplitudes vanish also in string theory (Eqs. (4.1) and (4.2)). So we write (and explain afterwords..) for the MHV sub-amplitudes:

- **4 partons**

\[
m_{\text{string}}(g_1, g_2, g_3, g_4) = \mathfrak{B}_4(k_j) \ m_{\text{QCD}}(g_1, g_2, g_3, g_4)
\]  

(4.39)

\[
m_{\text{string}}(g_1, g_2, q_3, \bar{q}_4) = \mathfrak{B}_4(k_j) \ m_{\text{QCD}}(g_1, g_2, q_3, \bar{q}_4)
\]  

(4.40)
\[ m_{\text{string}}(q_1, q_2, q_3, q_4) = \hat{\mathcal{B}}_4(k_j, \theta_i) \ m_{\text{QCD}}(q_1, q_2, q_3, q_4) \] (4.41)

Where\(^{13}\)

\[ \mathcal{B}_4(\hat{s}, \hat{u}) \equiv V_t = \frac{\hat{s}\hat{u}}{t} \int_0^1 dz \ z^{s-1}(1 - z)^{u-1} \] (4.42)

\[ \hat{\mathcal{B}}_4 = \frac{\hat{s}\hat{u}}{t} \int_0^1 dz \ z^{s-1}(1 - z)^{u-1} \ I_\rho(z_i, \theta_j) \] (4.43)

Where \( I_\rho \) is a function which depends on the D-brane setup, and therefore is model dependent.

- **5 partons**

\[ m_{\text{string}}(g_1, g_2, g_3, q_4, \bar{q}_5) = \mathcal{B}_5(k_j) \ m_{\text{QCD}}(g_1, g_2, g_3, q_4, \bar{q}_5) \] (4.44)

\[ m_{\text{string}}(g_1, g_2, g_3, q_4, \bar{q}_5) = \mathcal{B}_5(k_j) \ m_{\text{QCD}}(g_1, g_2, g_3, q_4, \bar{q}_5) \] (4.45)

\[ m_{\text{string}}(g_1, g_2, g_3, q_4, \bar{q}_5) = \hat{\mathcal{B}}_5(k_j, \theta_i) \ m_{\text{QCD}}(g_1, g_2, g_3, q_4, \bar{q}_5) \] (4.46)

\( \mathcal{B}_5 \) will be given in section 7.3.1.

There are two things to be learned from these equations, showing the special properties of the universal amplitudes.

1. **Equality of form factors**: a string helicity amplitude is equal to the corresponding field theory amplitude times a stringy form factor. Eqs. (4.39) and (4.40) have the same form factor \( \mathcal{B}_4 \), and likewise Eqs. (4.44) and (4.45) have the same \( \mathcal{B}_5 \).

2. **Universality**: The form factors \( \mathcal{B}_4 \) and \( \mathcal{B}_5 \) depend only on the kinematics and hence are universal or model independent. On the other hand, the form factors \( \hat{\mathcal{B}}_4 \) and \( \hat{\mathcal{B}}_5 \) depend on the setup of D-branes and geometry of the extra dimensions, hence they are model dependent or non-universal.

\(^{13}\)The kinematical factor in front of the integral ensures that \( \mathcal{B}_4 \rightarrow 1 \) as \( \hat{s}, \hat{u} \rightarrow 1 \) so that at low energies \( m_{\text{string}} \rightarrow m_{\text{QCD}} \).
These two properties generalize to \( n \)-point universal amplitudes. The claim is that an \( n \)-point universal helicity amplitude can be written as:

\[
m_{\text{string}}(g_1 \ldots g_n) = \mathcal{B}_n(k_j) \ m_{\text{QCD}}(g_1 \ldots g_n)
\]

\[
m_{\text{string}}(g_1 \ldots g_{n-2}, q_{n-1}, \bar{q}_n) = \mathcal{B}_n(k_j) \ m_{\text{QCD}}(g_1 \ldots g_{n-2}, q_{n-1}, \bar{q}_n)
\]

(4.47)

(4.48)

with the same form factors which depend only on the external momenta, and not on the compactification. Furthermore, \( \mathcal{B}_n \) can be expressed in terms of generalized hypergeometric functions, and there are \((n-3)!\) independent sub-amplitudes, see Section 8.

Including the supersymmetric gluino and squark (\( \chi \) and \( \phi \)), the \( n \)-point universal amplitudes are:

1. \( m(g^{a_1} \ldots g^{a_n}) \)

2. \( m(\chi^{a_1}, \bar{\chi}^{a_2}, g^{a_3} \ldots g^{a_n}) \)

3. \( m(q^{a_1}, \bar{q}^{a_2}, g^{a_3} \ldots g^{a_n}) \)

4. \( m(\phi^{a_1}, \bar{\phi}^{a_2}, g^{a_3} \ldots g^{a_n}) \)

(4.49)

The universal amplitudes 1 and 2 are related through supersymmetric ward identities, as are amplitudes 3 and 4. In addition, amplitudes 1 and 3 have equal stringy form factors.

4.2.4 Discussion

We further explain the two properties of the universal amplitudes, see [2].

1. Equality of form factors: The explanation is as follows, see Fig. 12. Consider an helicity amplitude with \( k \) quarks and \( n \) gluons: \( m'(q_1 \ldots q_k, g_1 \ldots g_n) \). The quarks arise from strings stretched between between 2 stacks of D-branes intersecting at an angle \( \theta \). If the angle \( \theta \) is gradually changed and taken to zero, the quarks will appear as gluinos of the enhanced gauge group \( T^a \oplus T^b \), since the stacks are on top of each other. This new configuration describes an amplitude with \( k \) gluinos and \( n \) gluons: \( m''(\chi_1 \ldots \chi_k, g_1 \ldots g_n) \). Supersymmetry relates gluinos and gluons, so that \( m'' \) has the same form factor as the all-gluon amplitude \( m'''(g_1 \ldots g_{k+n}) \): \( \mathcal{B}'' = \mathcal{B}''' \). Finally, if \( k = 2 \) then \( m' \) is a universal amplitude and in particular independent of \( \theta \). Hence in this case \( m' = m'' \), and thus \( \mathcal{B}' = \mathcal{B}'' = \mathcal{B}''' \). We have thus proved that the two types of universal amplitudes have equal form factors.

2. Universality: The most interesting property of these amplitudes is that they are universal or model independent. They are the same in many different models of string
theory, because they do not depend on the compactification of the extra dimensions. Universal amplitudes contain only Regge states and not KK or Winding states, which appear in amplitudes with more quarks. KK states arise from the compactification of extra dimensions. Winding states arise when extended objects such as strings or D-branes wrap around the extra dimensions. Regge states are pure string states independent of the extra dimensions.

Mathematically the reason KK and winding states do not appear in the universal amplitudes is the following. KK and winding states only appear in amplitudes constructed from correlators of the boundary changing operators $\Xi^{a\cap b}$, and only when there are 4 or more $\Xi^{a\cap b}$s:

$$\langle \Xi^{a\cap b}(z_1) \Xi^{a\cap b}(z_2) \rangle = z_{12}^{-3/4}$$

$$\langle \Xi^{a\cap b}(z_1) \Xi^{b\cap d}(z_2) \Xi^{d\cap c}(z_3) \Xi^{c\cap a}(z_4) \rangle = \left( \frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}} \right)^{3/4} I_\rho (\{z_i\}; \theta^j) \quad (4.50)$$

$I_\rho$ depends on the compactification and intersections of the D-branes. It includes exchanges of KK and winding states. Since $\Xi^{a\cap b}$ appears only in the quark vertex function and not in the gluon vertex function (see Eqs. (C.2)-(C.5)), KK and winding states will appear only in amplitudes with 4 or more quarks.

This property can also be seen diagramatically. Fig. 13 shows the difference, in this respect, between an amplitude containing 2 quarks and an amplitude with 4 quarks. KK and winding states carry internal charge, and charge conservation requires quark pairs on both sides of a KK/winding state line in the diagram. So an amplitude with one quark pair can not have KK/winding state exchange.
Figure 12: Explanation of the equality of form factors. The 2 quarks of the $n$-gluon plus 2-quark amplitude are shown. From left to right: changing the intersection angle until it is zero.

Figure 13: Explanation of universality. The conservation of internal charge forbids the middle diagram. The same argument holds for the $n$-gluon amplitude.
5 4-point amplitudes

4 particle amplitudes are $2 \rightarrow 2$ processes at colliders. At the LHC, important signals of
$2 \rightarrow 2$ processes include: $pp \rightarrow 2$ jets, $pp \rightarrow$ jet + EW gauge boson and $pp \rightarrow 2$ EW gauge
bosons.

The Mandelstam variables are

$$s \equiv (k_1 + k_2)^2 = 2k_1k_2$$
$$t \equiv (k_1 + k_3)^2 = 2k_1k_3$$
$$u \equiv (k_1 + k_4)^2 = 2k_1k_4$$  (5.1)

The kinematics are shown in Fig. 14.

Energy-momentum conservation for massless quarks and gluons gives

$$s + t + u = \sum m_i^2 = 0$$  (5.2)

The hatted Mandelstam variables $\hat{s}$, $\hat{t}$, $\hat{u}$ are defined as the Mandelstam variables in units
of $M_s$:

$$\hat{s} \equiv s/M_s^2 = \alpha's$$
$$\hat{t} \equiv t/M_s^2 = \alpha't$$
$$\hat{u} \equiv u/M_s^2 = \alpha'u$$  (5.3)

Since this is a scale transformation of the external momenta and since massless QCD/QED are
scale invariant, the QCD/QED amplitudes will be invariant under $s, t, u \rightarrow \hat{s}, \hat{t}, \hat{u}$. Amplitudes
with $W$ and $Z$ bosons are not scale invariant so that their form will change when passing to the
hatted variables. For $s, t, u >> M_Z$, the EW amplitudes become scale invariant as well. The
string amplitudes to be discussed later, will obviously change under this scale transformation.
These properties are easily seen by looking at the squared amplitudes in sections 5.1 and 5.2.2.
5.1 Field theory

References: [74, 75, 76].

Using the techniques and results of section 4.1, the 4-point squared amplitudes may be computed. As an example, consider the \( gg \rightarrow gg \) squared amplitude. Looking at Eq. (4.12) and using 

\[
\sum s_{ij} \propto (s^4 + t^4 + u^4) \quad \text{and} \quad \sum \frac{1}{s_{12}s_{23}s_{34}s_{41}} \propto \frac{1}{s^4t^4 + s^4u^4 + t^4u^4},
\]

we get:

\[
|M|^2 (gg \rightarrow gg) \propto g^4(s^4 + t^4 + u^4) \frac{s^2 + t^2 + u^2}{s^4t^2u^2} = 4g^4 \left[ 3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2} \right] \quad (5.4)
\]

Which is given also in Eq. (5.5). In the last equation we used \( s + t + u = 0 \).

In the next section we list the squared amplitudes for \( 2 \rightarrow 2 \) amplitudes in terms of the Mandelstam variables. Some of the Feynman diagrams are shown in Figs. 15, 16. We consider the processes which are the most important in a hadron collider, namely initial states with a quark or a gluon.

5.1.1 The squared amplitudes

- \( gg \) initial state
\[ |\mathcal{M}|^2(gg \rightarrow gg) = \frac{3}{2}g^4 \left[ 3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2} \right] \] (5.5)

\[ |\mathcal{M}|^2(gg \rightarrow g\gamma) = 0 \] (5.6)

\[ |\mathcal{M}|^2(gg \rightarrow \gamma\gamma) = 0 \] (5.7)

\[ |\mathcal{M}|^2(gg \rightarrow q\bar{q}) = \frac{3}{8}g^4(t^2 + u^2) \left[ \frac{4}{9tu} - \frac{1}{s^2} \right] \] (5.8)

• *gq* initial state

\[ |\mathcal{M}|^2(gq \rightarrow q\bar{q}) = -g^4(s^2 + u^2) \left[ \frac{4}{9su} - \frac{1}{t^2} \right] \] (5.9)

\[ |\mathcal{M}|^2(gq \rightarrow \gamma q) = -\frac{1}{3}g^2e^2s^2 + u^2 \] (5.10)

\[ |\mathcal{M}|^2(gq \rightarrow Wq^\prime) = \frac{\pi\alpha_s}{12\pi} \left[ \frac{s^2 + u^2 + 2M_W^2t}{s^2u} \right] \] (5.11)

• *q\bar{q}* initial state

Figure 16: Feynman diagrams for partons + photons. The same diagrams describe Z bosons.
\[ |\mathcal{M}|^2(qq \rightarrow gg) = \left(\frac{8}{3}\right)^2 |\mathcal{M}|^2(gg \rightarrow q\bar{q}) \]  
\[ \text{(5.12)} \]

\[ |\mathcal{M}|^2(qq \rightarrow \gamma g) = \frac{8}{9} g^2 e^2 \frac{t^2 + u^2}{ut} \]  
\[ \text{(5.13)} \]

\[ |\mathcal{M}|^2(qq \rightarrow \gamma \gamma) = \frac{3}{4} e^4 |\mathcal{M}|^2(q\bar{q} \rightarrow \gamma g) \]  
\[ \text{(5.14)} \]

\[ |\mathcal{M}|^2(qq' \rightarrow Wg) = \frac{2\pi\alpha_s}{9x_W} \left( t - M_W^2 \right) \left( u - M_W^2 \right) \]  
\[ \text{(5.15)} \]

\[ |\mathcal{M}|^2(qq' \rightarrow q\bar{q}') = \frac{4}{9} g^4 s^2 + u^2 \]  
\[ \text{(5.16)} \]

\[ |\mathcal{M}|^2(q\bar{q} \rightarrow q'\bar{q}') = \frac{4}{9} g^4 t^2 + u^2 \]  
\[ \text{(5.17)} \]

\[ |\mathcal{M}|^2(q\bar{q} \rightarrow qq) = \frac{4}{9} g^4 \left[ \frac{t^2 + u^2}{s^2} + \frac{s^2 + u^2}{t^2} - \frac{2u^2}{3st} \right] \]  
\[ \text{(5.18)} \]

- \text{qq initial state}

\[ |\mathcal{M}|^2(qq \rightarrow qq) = \frac{4}{9} g^4 \left[ \frac{s^2 + t^2}{u^2} + \frac{s^2 + u^2}{t^2} - \frac{2s^2}{3tu} \right] \]  
\[ \text{(5.19)} \]

\[ |\mathcal{M}|^2(qq' \rightarrow qq') = \frac{4}{9} g^4 \frac{s^2 + u^2}{t^2} \]  
\[ \text{(5.20)} \]
5.2 String theory

5.2.1 The Veneziano amplitude

The Veneziano amplitude $V_t$, which enters the 4-point open string scattering amplitudes, is a fantastically rich object. $V_t$ is the form factor connecting the string and field theory 4-point universal sub-amplitudes, Eqs. (4.39), (4.40), (4.42):

$$m_{\text{string}} = V_t \cdot m_{\text{QCD}} \quad (5.21)$$

Explicitly, the Veneziano amplitude is:

$$V_t \equiv \frac{\hat{s}\hat{u}}{t} B(-\hat{s}, -\hat{u}) = \frac{\hat{s}\hat{u} \Gamma(-\hat{s})\Gamma(-\hat{u})}{\Gamma(t)} = \frac{\Gamma(1-\hat{s})\Gamma(1-\hat{u})}{\Gamma(1+t)} \quad (5.22)$$

and by crossing:

$$V_u \equiv V_t(\hat{t} \leftrightarrow \hat{u})$$

$$V_s \equiv V_t(\hat{t} \leftrightarrow \hat{s}) \quad (5.23)$$

The beta function has an integral representation:

$$B(a, b) \equiv \int_0^1 dx \ x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (5.24)$$

One thing to notice is that whereas the field theory tree amplitudes are independent of the collision energy (For massless fermions and gauge bosons.), the string amplitudes do depend on it through the Veneziano amplitude. This means e.g. that the angular distribution of the scattered particles changes as the energy changes.

Properties of the Veneziano factor:

1. Low energy expansion ($\hat{s}, \hat{t}, \hat{u} << 1$):

$$V_t = 1 + \frac{\pi^2}{6} \hat{s}\hat{u} + \ldots \quad (5.25)$$

In this limit $V_t$ is 1 plus corrections in inverse powers of the string scale. See also section 5.2.4.
2. High energy limit ($\hat{s} \gg 1$).

There are two types of high energy limits that are usually considered: the fixed scattering angle limit and the Regge limit. In the first case $\theta = \text{constant}$ and $|\hat{s}|, |\hat{t}| \to \infty$. The Regge limit is $|\hat{s}| \to \infty$, $\hat{t} = \text{constant}$. From Eqs. (H.21)-(H.23) we have:

**Fixed angle limit:**

$$V_t, V_u, V_s \to e^{-f(\theta)s} \quad (5.26)$$

**Regge limit:**

$$V_u, V_s \to s^{-t}$$

$$V_t \to e^{-f(\theta)s} \quad (5.27)$$

Where,

$$f(\theta) = \left(\frac{1 - \cos \theta}{2}\right) \ln \left(\frac{1 - \cos \theta}{2}\right) + \left(\frac{1 + \cos \theta}{2}\right) \ln \left(\frac{1 + \cos \theta}{2}\right) \quad (5.28)$$

In the fixed angle limit, $V_t$ is exponentially decreasing. This is extremely soft UV behavior.

3. s-channel pole expansion:

$V_t$ can be expanded on s-channel poles, giving rise to the most useful equation of this work:

$$V_t = \frac{-\hat{s}\hat{u}}{\hat{t}} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{\hat{s} - n} \prod_{K=1}^{n} (\hat{u} + K) \quad (5.29)$$

There are simple s-channel poles at each integer $n$:

$$s = nM_s^2 \quad (5.30)$$

Notice that the residue is a function of $\hat{u}$ only (mind the simple factor in front of the sum..).

4. D.H.S duality:

$$V_t(-\hat{s}, -\hat{u}) = V_t(-\hat{u}, -\hat{s}) \quad (5.31)$$
Thus Eq. (5.29) can then be written as a sum of u-channel poles.

\[ V_t = -\frac{\hat{s}\hat{u}}{t} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{\hat{u} - n} \prod_{K=1}^{n} (\hat{s} + K) \]  

(5.32)

We will almost always use the s-channel pole expansion though.

5. Positivity of the residues: "The no-ghost theorem":

For \( V_t \) to describe a scattering amplitude, the residues of the poles must be positive. This is difficult to prove, and it is correct only if the dimensions of space-time are \( D = 26 \) or \( D = 10 \), for the bosonic and super-string respectively.

6. Polynomial residues and spins:

The residue of the pole contains the angular part of the amplitude, which determines the spins of the exchanged resonances. If the residue is a polynomial in \( \cos \theta \) of degree \( k \), then there can be exchanges of spins from 0 to \( k \). It is seen from Eq. (5.29) that the residue is a polynomial of degree \( n \) in \( \hat{u} \) or equivalently in \( \cos \theta \).

\[ \text{Res}(V_t) \propto \prod_{K=0}^{n-1} (\hat{u} + K) = \hat{u}(\hat{u} + 1) \cdots (\hat{u} + n - 1) = \sum_{p=1}^{n} a_p \hat{u}^p = \sum_{p=1}^{n} b_p \cos^p \theta \]  

(5.33)

Where \( a_p \) and \( b_p \) are constants.

\( V_t \) is the form factor which multiplies \( m_{\text{QCD}} \) (Eq. (5.21)), which itself depends on \( \cos \theta \). The angular dependency in \( m_{\text{QCD}} \) can shift the minimum and maximum spins, so that in general:

\[ 0 + J_0 \leq J \leq n + J'_0 \]  

(5.34)

From Eq. (5.30), there are exchanges of an infinite number of resonances with masses (Fig. 17):

\[ m_n = \sqrt{n} M_s \]  

(5.35)

These are string excitations called Regge states.

We will see that the Regge excitations of the gluon (which we denote by \( g^{nJ} \)) have, at a mass level \( n \), spins in the range:

\[ 0 \leq J \leq n + 1 \]  

(5.36)
Figure 17: $s$-channel Regge state exchange.

Figure 18: Left: The spectrum of Regge excitations of the gluon: $g^{nJ}$. Right: Regge excitations of the quark: $q^{nJ}$. Linear Regge trajectories are also shown.
These $g^{nJ}$ particles are exchanged e.g. in $gg \rightarrow gg$. The spectrum of $g^{nJ}$ states can be plotted on the $n - J$ plane as in Fig. 18.

In sections 4.2.3, 4.2.4 we mentioned that the processes $gg \rightarrow gg$ and $gq \rightarrow gq$ have the same form factor $V_t$. So the amplitude for $gq \rightarrow gq$ will also have an infinity of poles, corresponding to the Regge excitations of the quark $q^{nJ}$ (Fig. 18). The figure also shows Regge trajectories which have the form $J = J_0 + \alpha' m^2 = J_0 + n$, where $J_0 = 0$ or $1/2$ for $g^{nJ}$ and $q^{nJ}$ respectively.

Near a resonance, one term in the sum of Eq. (5.29) is dominant:

$$V_t \stackrel{\hat{s} \rightarrow n}{\rightarrow} \frac{1}{(n-1)!} \frac{1}{\hat{s} - n} \hat{u} \prod_{K=1}^{n-1} (\hat{u} + K)$$

$$V_u \stackrel{\hat{s} \rightarrow n}{\rightarrow} \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\hat{s} - n} \hat{t} \prod_{K=1}^{n-1} (\hat{u} + K)$$

(5.37)

In contrast to $V_t$ and $V_u$, $V_s$ is finite (has no s-channel poles):

$$V_s \stackrel{\hat{s} \rightarrow n}{\rightarrow} \frac{\Gamma(1 - \hat{t}) \Gamma(1 - \hat{u})}{\Gamma(1 + n)} = \text{finite}$$

(5.38)

so near a resonance of the amplitude we can neglect $V_s$ terms.

An interesting property can be seen:

$$\frac{V_u}{V_t} = \frac{\Gamma(1 - \hat{t}) \Gamma(1 + \hat{u})}{\Gamma(1 - \hat{u}) \Gamma(1 + \hat{u})} \frac{\hat{t} \sin(\pi \hat{u})}{\hat{u} \sin(\pi \hat{t})}$$

(5.39)

near a pole this becomes:

$$\frac{V_u}{V_t} = (-1)^{n-1} \frac{\hat{u}}{\hat{t}} \quad \text{when } \hat{s} \rightarrow n$$

(5.40)

From Eq. (5.37) the previous equation is seen to be equivalent to:

$$\prod_{K=1}^{n-1} (\hat{u} + K) = (-1)^{n-1} \prod_{K=1}^{n-1} (\hat{t} + K) \quad \text{when } \hat{s} \rightarrow n$$

(5.41)

5.2.2 The squared amplitudes

References: squared amplitudes from [1].
In Appendix B we give an example of a full calculation of a squared amplitude. In this section we write down the string squared amplitudes just as we did in the field theory case. The Veneziano factors will be written explicitly in terms of the Mandelstam variables. It is then immediately seen that some the processes have only \( n = \text{odd} \) resonances (the residues vanish for \( n = \text{even} \)). This is explained in section 6.2.4 and Appendix A.

We denote by \( A \) the \( U(1) \) gauge boson from the stack \( a \) (the color stack), and by \( B \) the non-abelian gauge boson from stack \( b \).

- \( gg \) initial state

\[
|\mathcal{M}|^2(gg \to gg) = g^4 \left( \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} \right) \frac{g^2}{4} \left( s^2 V_s^2 + t^2 V_t^2 + u^2 V_u^2 \right) - \frac{1}{3} \left( s V_s + t V_t + u V_u \right)^2
\]

\[
= \frac{n^4 + \hat{u}^4 + \hat{t}^4}{n^2(\hat{s} - n)^2} \frac{g^4}{(n-1)!^2} \left[ \prod_{K=1}^{n-1} (\hat{u} + K)^2 \right] \cdot \left\{ \begin{array}{ll} 19/12, & \text{odd } n \\ 9/4, & \text{even } n \end{array} \right\}. \quad (5.42)
\]

\[
|\mathcal{M}|^2(gg \to gA) = \frac{5}{6} g^4 Q_A^2 \left( \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} \right) \left( s V_s + t V_t + u V_u \right)^2
\]

\[
= \frac{5g^4 Q_A^2}{3} \frac{n^4 + \hat{u}^4 + \hat{t}^4}{n^2(\hat{s} - n)^2} \frac{1}{(n-1)!^2} \left[ \prod_{K=1}^{n-1} (\hat{u} + K)^2 \right] \cdot \left\{ \begin{array}{ll} 1, & \text{odd } n \\ 0, & \text{even } n \end{array} \right\}. \quad (5.43)
\]

\[
|\mathcal{M}|^2(gg \to AA) = \frac{12}{5} Q_A^2 |\mathcal{M}|^2(gg \to gA) \quad (5.44)
\]

\[
|\mathcal{M}|^2(gg \to q\bar{q}) = g^4 \frac{t^2 + u^2}{s^2} \left[ \frac{1}{6ut} (t V_t + u V_u)^2 - \frac{3}{8} V_t V_u \right]
\]

\[
= \frac{\hat{u} \hat{t}(\hat{u}^2 + \hat{t}^2)}{n^2(\hat{s} - n)^2} \frac{g^4}{(n-1)!^2} \left[ \prod_{K=1}^{n-1} (\hat{u} + K)^2 \right] \cdot \left\{ \begin{array}{ll} 7/24, & \text{odd } n \\ 25/24, & \text{even } n \end{array} \right\}. \quad (5.45)
\]

- \( gq \) initial state
\[ |\mathcal{M}|^2(gq \to gq) = g^4 s^2 + u^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{t^2} \left[ V_s V_u - \frac{4}{9} \frac{1}{su} (sV_s + uV_u)^2 \right] \]
\[ = -\frac{4g^4}{9} \frac{\hat{u}(\hat{u}^2 + n^2)}{n(s-n)^2} \frac{1}{(n-1)!^2} \prod_{K=1}^{n-1} (\hat{u} + K)^2 \]  
(5.46)

\[ |\mathcal{M}|^2(gq \to Aq) = -\frac{g^4}{3} Q_A^2 \frac{s^2 + u^2}{s u t^2} \left( sV_s^2 + uV_u^2 \right) \]
\[ = \frac{1}{3} g^4 Q_A^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{n(s-n)^2} \frac{1}{(n-1)!^2} \prod_{K=1}^{n-1} (\hat{u} + K)^2 \]  
(5.47)

\[ |\mathcal{M}|^2(gq \to Bq') = -\frac{g^4}{6} |T_{qg}|^2 \frac{s^2 + u^2}{s u} V_t^2 \]
\[ = \frac{1}{6} g^4 |T_{qg}|^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{n(s-n)^2} \frac{1}{(n-1)!^2} \prod_{K=1}^{n-1} (\hat{u} + K)^2 \]  
(5.48)

• \( q\bar{q} \) initial state

\[ |\mathcal{M}|^2(q\bar{q} \to gg) = \left( \frac{8}{3} \right)^2 |\mathcal{M}|^2(gg \to q\bar{q}) \]  
(5.49)

\[ |\mathcal{M}|^2(q\bar{q} \to gA) = g^4 \frac{8}{9} Q_A^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{t^2 u s^2} \left( tV_t + uV_u \right)^2 = \]
\[ \frac{32}{9} g^4 Q_A^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{n^2(s-n)^2} \frac{1}{(n-1)!^2} \left[ \prod_{K=1}^{n-1} (\hat{u} + K)^2 \right] \cdot \left\{ \begin{array}{c} 1, \text{ odd } n \\ 0, \text{ even } n \end{array} \right\} . \]  
(5.50)

\[ |\mathcal{M}|^2(q\bar{q} \to AA) = \frac{3}{4} Q_A^2 |\mathcal{M}|^2(q\bar{q} \to gA) \]  
(5.51)

\[ |\mathcal{M}|^2(q\bar{q} \to gB) = g^4 \frac{4}{9} |T_{qg}|^2 Q_A^2 \frac{\hat{u}(\hat{u}^2 + n^2)}{t u} V_s^2 \]  
(5.52)

\[ |\mathcal{M}|^2(q\bar{q} \to BA) = \frac{3}{4} Q_A^2 |\mathcal{M}|^2(q\bar{q} \to gB) \]  
(5.53)
• *qq* initial state

\[
|M|^2(qq \to qq) = g^4 \left\{ \frac{2}{9} \frac{1}{t^2} \left[ (sF_{tu}^{bb})^2 + (sF_{tu}^{cc})^2 + (uG_{tu}^{'bc})^2 + (uG_{tu}^{'cb})^2 \right] + \right. \\
\left. \frac{2}{9} \frac{1}{t^2} \left[ (sF_{ut}^{bb})^2 + (sF_{ut}^{cc})^2 + (tG_{ut}^{'bc})^2 + (tG_{ut}^{'cb})^2 \right] - \frac{4}{27} \frac{s_2}{tu} \left( F_{tu}^{bb} F_{ut}^{bb} + F_{tu}^{cc} F_{ut}^{cc} \right) \right\} 
\]

(5.54)

\[
|M|^2(qq' \to qq') = \frac{2g^4}{9} \frac{1}{t^2} \left[ (sF_{tu}^{bb})^2 + (sF_{tu}^{cc})^2 + (uG_{tu}^{'bc})^2 + (uG_{tu}^{'cb})^2 \right] 
\]

(5.55)

See [1] for further details on quark-quark scattering.

Let us note a few things that can be seen from these squared amplitudes:

• As seen in Eqs. (5.43), (5.44), (5.50), (5.51), amplitudes with a $gA$ or $AA$ in the final state do not have $n = \text{even}$ poles. This happens because of the vanishing of the following color factors: $f_{0ab} = f_{00a} = f_{000} = 0$. See also section 6.2.4 and Appendix A.

• Taking the leading term near a pole, we see that there are 3 classes of amplitudes:

1. $qq \to gg$, $gg \to Aq$, $gg \to Bq'$ are proportional near a pole.

2. $gg \to gg$, $gg \to gA$, and $gg \to AA$ are proportional near a pole (the latter two vanish at $n = \text{even}$ poles.).

3. $gg \to qq$, $qq \to gg$, $qq \to gA$, and $qq \to AA$ are proportional near a pole (the latter two vanish at $n = \text{even}$ poles.).

Moreover, these 3 classes differ (near a pole) only by a simple kinematic factor: $\hat{u}(\hat{u}^2+n^2)$, $(n^4 + \hat{u}^4 + \hat{t}^4)$, $\hat{u}\hat{t}(\hat{u}^2 + \hat{t}^2)$ for the first, second, and third class respectively.

### 5.2.3 The softened squared amplitudes

The simple poles of the amplitudes are given finite widths via the Breit-Wigner form as in section 6.1.1. Following for example [11], we write the softened squared amplitudes for exchange of Regge states from the first excited state $n = 1$. The Regge states with quantum numbers $(n, J)$ are written as: $g^{n,J}, A^{n,J}, q^{n,J}$, recall also Fig. 18.

• *gg* initial state
\begin{align*}
|M|^2(gg \to gg) &= \frac{19g_4^4}{12} \left\{ W_{gg \to gg}^{g^4} \left[ \frac{1}{(\hat{s} - 1)^2 + (\Gamma_{g,1.0} / M_s)^2} + \frac{\hat{t}^4 + \hat{u}^4}{(\hat{s} - 1)^2 + (\Gamma_{g,1.2} / M_s)^2} \right] \\
&\quad + W_{gg \to gg}^{A^4} \left[ \frac{1}{(\hat{s} - 1)^2 + (\Gamma_{g,1.0} / M_s)^2} + \frac{\hat{t}^4 + \hat{u}^4}{(\hat{s} - 1)^2 + (\Gamma_{g,1.2} / M_s)^2} \right] \right\} \quad (5.56) \\
|M|^2(gg \to gA) &= \frac{5}{3}g_4 Q_A^2 \left[ \frac{1}{(\hat{s} - 1)^2 + (\Gamma_{g,1.0} / M_s)^2} + \frac{\hat{u}^4 + \hat{t}^4}{(\hat{s} - 1)^2 + (\Gamma_{g,1.2} / M_s)^2} \right] \quad (5.57) \\
|M|^2(gg \to AA) &= 4g_4 Q_A^2 \left[ \frac{1}{(\hat{s} - 1)^2 + (\Gamma_{A,1.0} / M_s)^2} + \frac{\hat{u}^4 + \hat{t}^4}{(\hat{s} - 1)^2 + (\Gamma_{A,1.2} / M_s)^2} \right] \quad (5.58) \\
|M|^2(gg \to q\bar{q}) &= \frac{7}{24}g_4 N_f \left\{ W_{gg \to q\bar{q}}^{g^4} \frac{\hat{u}(\hat{u}^2 + \hat{t}^2)}{(\hat{s} - 1)^2 + (\Gamma_{g,1.0} / M_s)^2} + W_{gg \to q\bar{q}}^{A^4} \frac{\hat{u}(\hat{u}^2 + \hat{t}^2)}{(\hat{s} - 1)^2 + (\Gamma_{g,1.2} / M_s)^2} \right\} \quad (5.59)
\end{align*}

- **qq initial state**

\begin{align*}
|M|^2(gg \to gg) &= -\frac{4g_4^4}{9} \left[ \frac{\hat{u}}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} + \frac{\hat{u}^3}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} \right] \quad (5.60) \\
|M|^2(gg \to Aq) &= -\frac{1}{3}g_4 Q_A^2 \left[ \frac{\hat{u}}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} + \frac{\hat{u}^3}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} \right] \quad (5.61) \\
|M|^2(gg \to Bq) &= -\frac{1}{6}g_4 |T_{q\bar{q}}^B|^2 \left[ \frac{\hat{u}}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} + \frac{\hat{u}^3}{(\hat{s} - 1)^2 + (\Gamma_{q,1.2} / M_s)^2} \right] \quad (5.62)
\end{align*}

- **q\bar{q} initial state**
\[
|M|^2(q \bar{q} \rightarrow gg) = \frac{56}{27} g^4 \left\{ \frac{\hat{u} \hat{t} (\hat{u}^2 + \hat{t}^2)}{(s - 1)^2 + (\Gamma_{g^{n,J}}/M_s)^2} + \frac{\hat{u} \hat{t} (\hat{u}^2 + \hat{t}^2)}{(s - 1)^2 + (\Gamma_{A^{n,J}}/M_s)^2} \right\} 
\] (5.63)

\[
|M|^2(q \bar{q} \rightarrow gA) = \frac{32}{9} g^4 Q_A^2 \frac{\hat{u} \hat{t} (\hat{u}^2 + \hat{t}^2)}{(s - 1)^2 + (\Gamma_{g^{n,J}}/M_s)^2} 
\] (5.64)

\[
|M|^2(q \bar{q} \rightarrow AA) = \frac{8}{3} g^4 Q_A^4 \frac{\hat{u} \hat{t} (\hat{u}^2 + \hat{t}^2)}{(s - 1)^2 + (\Gamma_{g^{n,J}}/M_s)^2} 
\] (5.65)

Where the widths are:

\[
\Gamma_{g^{n,J}} = \frac{g^2}{4\pi} N \frac{N}{4} M_s \approx 0.075 M_s 
\] (5.66)

\[
\Gamma_{A^{n,J}} = \frac{g^2}{4\pi} N \frac{N}{2} M_s \approx 0.15 M_s 
\] (5.67)

\[
\Gamma_{g^{n,J}} = \frac{g^2}{4\pi} \left( \frac{N}{10} + \frac{N_f}{40} \right) M_s \approx 0.045 M_s 
\] (5.68)

\[
\Gamma_{A^{n,J}} = \frac{g^2}{4\pi} \left( \frac{N}{5} + \frac{N_f}{40} \right) M_s \approx 0.075 M_s 
\] (5.69)

\[
\Gamma_{q^{n,J}} = \frac{g^2}{4\pi} N \frac{N}{8} M_s \approx 0.038 M_s 
\] (5.70)

and the right hand side was obtained by setting \(\alpha_s = g^2/4\pi \approx 0.1\), \(N = 3\), and \(N_f = 6\).

We now write the relative weights between exchange of an \(SU(N)\) and \(U(1)\) gauge bosons.

\[
W_{g^{n,J}}^{gg \rightarrow gg} = \frac{(N^2 - 1) (\Gamma_{g^{n,J} \rightarrow gg})^2}{(N^2 - 1) (\Gamma_{g^{n,J} \rightarrow gg})^2 + (\Gamma_{A^{n,J} \rightarrow gg})^2} = \begin{cases} \frac{1}{1 + \frac{4N^2 - 1}{(N^2 - 4)^2}} = \frac{25}{57} , & n = odd \\ 1 , & n = even \end{cases} 
\] (5.71)

\[
W_{A^{n,J}}^{gg \rightarrow gg} = 1 - W_{g^{n,J}}^{gg \rightarrow gg} = \frac{(\Gamma_{A^{n,J} \rightarrow gg})^2}{(N^2 - 1) (\Gamma_{g^{n,J} \rightarrow gg})^2 + (\Gamma_{A^{n,J} \rightarrow gg})^2} = \begin{cases} \frac{32}{57} , & n = odd \\ 0 , & n = even \end{cases} 
\] (5.72)
\[
W_{g^nJ}^{gg\rightarrow \bar{q}q} = W_{g^nJ}^{q\bar{q}\rightarrow gg} = \frac{(N^2 - 1) \Gamma_{g^nJ\rightarrow gg} \Gamma_{g^nJ\rightarrow \bar{q}q} + \Gamma_{A^nJ\rightarrow gg} \Gamma_{A^nJ\rightarrow \bar{q}q}}{(N^2 - 1) \Gamma_{g^nJ\rightarrow gg} \Gamma_{g^nJ\rightarrow \bar{q}q} + \Gamma_{A^nJ\rightarrow gg} \Gamma_{A^nJ\rightarrow \bar{q}q}} = \begin{cases} 
\frac{1}{1+\frac{2}{N^2-4}}, & n = \text{odd} \\
1, & n = \text{even} \end{cases} \tag{5.73}
\]

\[
W_{A^nJ}^{gg\rightarrow \bar{q}q} = W_{A^nJ}^{q\bar{q}\rightarrow gg} = 1 - W_{g^nJ}^{gg\rightarrow \bar{q}q} = \frac{\Gamma_{A^nJ\rightarrow gg} \Gamma_{A^nJ\rightarrow \bar{q}q}}{(N^2 - 1) \Gamma_{g^nJ\rightarrow gg} \Gamma_{g^nJ\rightarrow \bar{q}q} + \Gamma_{A^nJ\rightarrow gg} \Gamma_{A^nJ\rightarrow \bar{q}q}} = \begin{cases} 
\frac{2}{5}, & n = \text{odd} \\
0, & n = \text{even} \end{cases} \tag{5.74}
\]

Where we put \( N = 3 \) on the right hand side, and the decay widths are taken from sections 6.2.4 and 6.2.5. The weights are independent of \( n \) and \( J \), they only differ for \( n = \text{odd} \) and \( n = \text{even} \).

Now we that we have seen the \( n = 1 \) case, we jump a little bit ahead of time and give the prescription for arbitrary \( n \) and \( J \). A general helicity amplitude will have the following form near a resonance of mass squared \( m_n^2 = \hat{s} = n \), see Eq. (6.11):

\[
M_{m,m'} = \frac{1}{(\hat{s} - n) + i(\Gamma_{g^nJ}/M_s)} A \sum_{J} C_{m,m'}^{n,J} d_{m,m'}^{J}(\theta) \tag{5.75}
\]

We wrote this with \( g^nJ \), but the same form will hold also for \( q^nJ \).

The amplitude can then be squared:

\[
|M_{m,m'}|^2 = A^2 \sum_{J_1, J_2} C_{m,m'}^{n,J_1} C_{m,m'}^{n,J_2} \frac{d_{m,m'}^{J_1} d_{m,m'}^{J_2}}{(\hat{s} - n)^2 + (\Gamma_{g^nJ}/M_s)^2} \tag{5.76}
\]

Eq. (1.10) shows that \( d's \) are orthogonal, therefore the interference terms vanish in the total cross section:

\[
\sigma_{m,m'} = \int_{-1}^{1} \frac{d\cos \theta}{32\pi s} |M_{m,m'}|^2 = \frac{A^2}{32\pi s} \sum_{J} \left( C_{m,m'}^{n,J} \right)^2 \frac{4}{2J+1} \frac{1}{(\hat{s} - n)^2 + (\Gamma_{g^nJ}/M_s)^2} \tag{5.77}
\]

The amplitudes having exchanges of \( g^nJ \) (as opposed to \( q^nJ \)) have also the opposite helicity configuration \( M_{m,-m'} \). In this case, when the two squared amplitudes are added we get:

\[
|M_{m,m'}|^2 + |M_{m,-m'}|^2 = A^2 \sum_{J_1, J_2} C_{m,m'}^{n,J_1} C_{m,m'}^{n,J_2} \left[ \frac{d_{m,m'}^{J_1} d_{m,m'}^{J_2} + (-1)^{J_1+J_2} d_{m,-m'}^{J_1} d_{m,-m'}^{J_2}}{(\hat{s} - n)^2 + (\Gamma_{g^nJ}/M_s)^2} \right] \tag{5.78}
\]
Where Eqs. (6.28), (6.29) were used.

5.2.4 Low energy limit

References: [11, 71, 72].

When the center of mass energy is significantly lower than the string scale, \( s << M_s^2 \), the string amplitudes coincide with the standard model ones. In this section we calculate the first stringy correction to the standard model amplitudes.

The Veneziano amplitudes can be expanded in powers of \( \alpha' = 1/M_s^2 \). To order \( \alpha'^7 \):

\[
\begin{align*}
\frac{1}{st} V_t &= 1 - \alpha'^2 \frac{\pi^2}{6} - \alpha'^3 (s + t) \zeta(3) - \frac{\pi^4}{360} (4s^2 + st + 4t^2) + \\
\alpha'^5 \left[ \frac{\pi^2}{6} st(s + t) \zeta(3) - (s + t)(s^2 + st + t^2) \zeta(5) \right] - \\
\alpha'^6 \left[ \frac{\pi^6}{15120} 16s^4 + 12s^3t + 23s^2t^2 + 12st^3 + 16t^4 \right] - \frac{1}{2} st(s + t)^2 \zeta(3)^2 \right] + \\
\alpha'^7 \left[ \frac{\pi^4}{360} st(s + t)(4s^2 + st + 4t^2) \zeta(3) + \frac{\pi^2}{6} st(s^2 + st + t^2) \zeta(5) - (s^2 + st + t^2) \zeta(7) \right] + \ldots
\end{align*}
\]

(5.79)

We note that the \( \alpha' \) correction vanishes. We will need only up to \( \alpha'^3 \) order\(^{14}\):

\[
\begin{align*}
V_t &= 1 - \left[ \frac{\pi^2}{6} su \right] \alpha'^2 - [\zeta(3)stu] \alpha'^3 \ldots \\
V_u &= 1 - \left[ \frac{\pi^2}{6} st \right] \alpha'^2 - [\zeta(3)stu] \alpha'^3 \ldots \\
V_s &= 1 - \left[ \frac{\pi^2}{6} tu \right] \alpha'^2 - [\zeta(3)stu] \alpha'^3 \ldots
\end{align*}
\]

(5.81)

\(^{14}\)In this equation, the \( \alpha'^2 \) correction can be shown to arise from the following effective lagrangian:

\[
I_{F^4} = -\frac{\alpha'^2 \pi^2}{6} \text{Tr} \left[ F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\mu_5\mu_6} F_{\mu_7\mu_8} + 2F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\mu_5\mu_6} F_{\mu_7\mu_8} - \frac{1}{4} F_{\mu_1\mu_2} F_{\nu_1\nu_2} F_{\mu_3\mu_4} F_{\nu_3\nu_4} - \frac{1}{2} F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\nu_1\nu_2} F_{\nu_3\nu_4} \right]
\]

(5.80)
We write $|\mathcal{M}|^2 = |\mathcal{M}|^2_{SM} + \Delta |\mathcal{M}|^2$, where $|\mathcal{M}|^2_{SM}$ is the standard model squared amplitude and $\Delta |\mathcal{M}|^2$ is the first correction. The first corrections to the squared amplitudes of section 5.2.2 are:

- **$gg$ initial state**

  $\Delta |\mathcal{M}|^2(gg \rightarrow gg) = -2g^4\zeta(3)stu\left[\frac{9}{2} - \frac{tu}{s^2} - \frac{us}{t^2} - \frac{ts}{u^2}\right] \alpha'^3$ \hspace{1cm} (5.82)

  $\Delta |\mathcal{M}|^2(gg \rightarrow gA) = \frac{5\pi^4g^4Q^2_A}{24}(s^2t^2 + s^2u^2 + u^2t^2) \alpha'^4$ \hspace{1cm} (5.83)

  $\Delta |\mathcal{M}|^2(gg \rightarrow AA) = \frac{\pi^4g^4Q^4_A}{2}(s^2t^2 + s^2u^2 + u^2t^2) \alpha'^4$ \hspace{1cm} (5.84)

  $\Delta |\mathcal{M}|^2(gg \rightarrow q\bar{q}) = \frac{g^4\pi^2}{3}(u^2 + t^2) \alpha'^2$ \hspace{1cm} (5.85)

- **$gq$ initial state**

  $\Delta |\mathcal{M}|^2(gq \rightarrow gg) = \frac{\pi^2g^4Q^4_A}{3}(u^2 + s^2) \alpha'^2$ \hspace{1cm} (5.86)

  $\Delta |\mathcal{M}|^2(gq \rightarrow Aq) = \frac{-2g^4\pi^2}{9}(u^2 + s^2) \alpha'^2$ \hspace{1cm} (5.87)

  $\Delta |\mathcal{M}|^2(gq \rightarrow Bq') = \frac{g^4\pi^2}{18}|T_{q\bar{q}}|^2(u^2 + s^2) \alpha'^2$ \hspace{1cm} (5.88)

- **$q\bar{q}$ initial state**

  $\Delta |\mathcal{M}|^2(q\bar{q} \rightarrow gg) = \frac{64g^4\pi^2}{27}(u^2 + t^2) \alpha'^2$ \hspace{1cm} (5.89)

  $\Delta |\mathcal{M}|^2(q\bar{q} \rightarrow gA) = \frac{8\pi^4g^4Q^2_A}{81}ut(u^2 + t^2) \alpha'^4$ \hspace{1cm} (5.90)

  $\Delta |\mathcal{M}|^2(q\bar{q} \rightarrow AA) = \frac{2\pi^4g^4Q^4_A}{27}ut(u^2 + t^2) \alpha'^4$ \hspace{1cm} (5.91)
\[ \Delta |\mathcal{M}|^2(q\bar{q} \rightarrow gB) = \frac{-4g^4\pi^4|T_{q\bar{q}}^B|^2}{27} \, ut(u^2 + t^2) \, \alpha'^2 \] (5.92)

\[ \Delta |\mathcal{M}|^2(q\bar{q}' \rightarrow BA) = \frac{-g^4\pi^4|T_{q\bar{q}}^B|^2Q_A^2}{9} \, ut(u^2 + t^2) \, \alpha'^2 \] (5.93)

We note that for all the processes with a final state \(A\), the first correction is \(\alpha'^4\). The correction for gluon scattering is \(\alpha'^3\), and all the rest are \(\alpha'^2\).

### 5.3 Collider phenomenology

**References:** [73, 74, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97].

The calculation of a cross section \(d\bar{\sigma}\) is done by convoluting the partonic cross section \(d\sigma(ij \rightarrow kl)\) with the parton distribution functions of the two colliding protons:

\[ d\bar{\sigma} = \int_0^1 \int_0^1 dx_a \, dx_b \sum_{ijkl} f_i(x_a, M) \, f_j(x_b, M) \, d\sigma(ij \rightarrow kl) \] (5.94)

The partonic cross section and the squared amplitude are related:

\[ |\mathcal{M}(ij \rightarrow kl)|^2 = 64\pi^2 s \, \frac{d\sigma}{d\Omega} = 16\pi s^2 \, \frac{d\sigma}{dt} \] (5.95)

A useful form for the dijet cross section is given in Eq. (G.23). The dijet cross section can thus be calculated in field theory and in string theory, using the squared amplitudes written earlier. As is well known, the field theory cross section is a smooth power-law decreasing function. The string theory cross section exhibits bumps at \(s = nM_s^2\), and these are clear signals of new physics. If the string scale is higher then the collision energy then these bumps cannot be seen, but smooth deviations from the field theory cross section can still be searched for (e.g. contact interaction searches).

Another useful type of analysis are dijet angular distributions. Angular distributions are a sensitive probe of new physics since QCD dijets are more central (because of the t-channel poles) whereas new physics tend to be more isotropic. Most importantly, angular distributions are a way to probe exchanges of different spins. Therefore they can be used to differentiate e.g. a bump coming from a spin 2 KK graviton, from Regge state exchange of different spins.

The ratio \(R\) is a useful measure of angular distributions:

\[ R = \frac{\frac{d\sigma}{d\Omega}(|y_1|, |y_2| < 0.5)}{\frac{d\sigma}{d\Omega}(0.5 < |y_1|, |y_2| < 1)} \] (5.96)
It also has the benefit that systematic uncertainties, such as the jet energy scale (JES), tend to cancel in the ratio.

It is very important that the following 4-fermion processes, which are non-universal amplitudes, are suppressed at the LHC:

\[ q \bar{q} \rightarrow q \bar{q} \]
\[ q q \rightarrow q q \]  \hspace{1cm} (5.97)

The first process is suppressed because \( \bar{q} \) has low luminosity in proton collisions, and the second process does not have s-channel Regge state exchange. Therefore, the universal (model independent) amplitudes will dominate the dijet signal.

Looking at the squared amplitudes of section 5.2.2, we note the following things:

1. The process \( p + p \rightarrow \gamma/Z + \text{Jet} \) has even resonances only from the partonic process \( gq \rightarrow Aq \). This means that the even resonances of a \( \gamma/Z + \text{Jet} \) signal are a probe of the Regge excitations of the quarks.

2. The process \( p + p \rightarrow W + \text{Jet} \) has resonances only from the partonic process \( gq \rightarrow Bq' \).

3. The process \( p + p \rightarrow \gamma/Z + \gamma/Z \) has only odd resonances.

Possibilities for collider phenomenology other than virtual exchange of Regge states include:

- Direct production of a Regge state in the final state, section 9 and 20.

- Multi-jets in the final state, beginning with the 3-jet signal. The squared amplitudes of section 2 and section 7.3.3 can be used for this purpose.

- Phenomenology at a lepton collider or photon collider, 8. For example the process \( \gamma + \gamma \rightarrow \gamma + \gamma \) exhibits tree level Regge state exchange.

- Signals other than Regge states include: production of \( Z' \) bosons coming from the extra \( U(1)'s \) of D-brane models (section 3.4 and 10). Also signals arising from the presence of extra dimensions: KK and winding state exchange and production, miniature black hole production and Hawking radiation etc..
5.3.1 Constraints from the LHC

The most directly related limit is from the CMS experiment [92]: exclusion of string resonances from dijet mass distribution with 1 \( fb^{-1} \):

\[
M_{s} > 4 \, TeV
\]  
(5.98)

We list some additional constraints which have some relevance for us.

- **[92]** CMS limits from dijets searches with 1 \( fb^{-1} \):
  - Bound on the mass of excited quarks:
    \[M_{q^*} > 2.49 \, TeV \, .\]
  - Bound on the mass of axigluons:
    \[M > 2.47 \, TeV \, .\]

- **[78]**, **[79]** CMS lower limit on quark contact interaction scale for left handed quarks via dijet angular distributions with 36 \( pb^{-1} \):
  \[\Lambda > 5.6 - 6.7 \, TeV \, .\]

- **[105]** ATLAS limits from dijet searches with 0.81 \( fb^{-1} \):
  - Bound on excited quarks:
    \[M_{q^*} > 2.91 \, TeV \, .\]
  - Bound on axigluons:
    \[M > 3.21 \, TeV \, .\]

- **[80]**, **[81]**, **[82]** ATLAS limits from dijet mass and angular distributions with 36 \( pb^{-1} \) (distributions measured up to \( \sim 3.5 \, TeV \)):
  - Exclusion of quantum gravity scales from Randall-Meade quantum black holes:
    \[0.75 < m_D < 3.67 \, TeV \, .\]
  - Limit on quark contact interactions:
    \[\Lambda > 9.5 \, TeV \, .\]

Another way to discover new physics is by measuring the couplings of the Higgs boson, which (Maybe) was recently discovered at the LHC [107], [108] and Tevatron [109]. See for example [106].
6 Decay widths

References: [1, 7, 19, 73]. After the submission of this article, [103] appeared which deals with related issues.

In this section we suggest several methods to compute decay widths of Regge states. The basic idea is that of [7], in which a tree level amplitude is factorized into two trilinear couplings connected by an s-channel resonance (see e.g the s-channel diagram of Fig. 1). This is e.g similar to the tree level production of a standard model $Z$ boson. In field theory, 1-loop corrections give an imaginary part to the amplitude which causes the resonance to decay. As we will shortly see, the optical theorem enables to compute the decay widths from tree level amplitudes.

This technique is basically field theoretical. 1-loop amplitudes can also be computed in string theory. This is beyond the scope this work.

6.1 Setting the stage

6.1.1 The Breit-Wigner form

In order to compare a squared amplitude to scattering experiments, the decay width of the exchanged particles must be taken into account. Recall that the string amplitudes exhibit an infinite sum of poles of the type $\sim \frac{1}{s - n M^2}$ corresponding to exchange of Regge states with zero decay width. Higher order corrections will produce a finite decay width $\Gamma$, causing a Breit-Wigner softening of the poles:

$$\frac{1}{s - n M^2} \rightarrow \frac{1}{s - n M^2 + i \Gamma M}$$

(6.1)

The squared amplitude will then be

$$\sim \frac{1}{(s - n M^2)^2 + (\Gamma M)^2}$$

(6.2)

We now derive this, following [73]. In field theory (for example $\phi^3$ theory) a tree level s-channel exchange of a resonance is of the form $\frac{1}{s - M^2}$. Radiative corrections will remove the
pole at \( s = M_0^2 \). If \( \Pi(s) \) is the 1PI radiative correction to the tree propagator then the full propagator will be Fig. 19:

\[
\frac{1}{s - M_0^2} \rightarrow \frac{1}{s - M_0^2} + \frac{1}{s - M_0^2} \Pi(s) \frac{1}{s - M_0^2} + \ldots = \frac{1}{s - M_0^2} \Pi(s) \left(1 - \frac{\Pi(s)}{s - M_0^2} + \left(\frac{\Pi(s)}{s - M_0^2}\right)^2 + \ldots\right) = \frac{1}{s - M_0^2} - \Pi(s) \tag{6.3}
\]

The physical mass is determined by:

\[
M^2 - M_0^2 - \text{Re} \, \Pi(M^2) = 0 \tag{6.4}
\]

Using the fact that near the pole we have \( s \approx M^2 \), we get

\[
M^2 - M_0^2 - \Pi(s) = s - M_0^2 - \left[\text{Re} \, \Pi(M^2) + \text{Re} \, \Pi(s)|_{s=M^2} (s - M^2)\right] + \text{Im} \, \Pi(M^2) = \left[1 + \text{Re} \, \Pi(s)|_{s=M^2} (s - M^2) + \text{Im} \, \Pi(M^2) = Z^{-1}(s - M^2) + \text{Im} \, \Pi(M^2) = Z^{-1}[(s - M^2) + iM\Gamma] \tag{6.5}
\]

Where \( Z \) is the field strength renormalization and in the last equation we used:

\[
\text{Im} \, \Pi(M^2) = -\frac{Z^{-1}}{2} \sum_{f_1, f_2} \int \frac{d^3k_1}{(2\pi)^3 2E_1} \frac{d^3k_2}{(2\pi)^3 2E_2} |\bar{M}|^2 (Z \rightarrow f_1 f_2) = -Z^{-1}M\Gamma \tag{6.6}
\]

We finally achieved:

\[
\frac{1}{s - M_0^2} \rightarrow \frac{Z}{s - M^2 + iM\Gamma} \tag{6.7}
\]

### 6.1.2 Amplitudes in terms of the \( d \)-functions

In order to exhibit the exchange of resonances, a given amplitude should be expanded in terms of the physical states, i.e states with a definite spin. Put differently, the amplitude needs to be expanded on the basis of Wigner \( d \)-functions.

Therefore we write an helicity amplitude as:

\[
\mathcal{M}_{m,m'} = \mathcal{M}(12 \rightarrow 34) = \sum_{J^*} \mathcal{M}(12 \rightarrow J^* \rightarrow 34) \tag{6.8}
\]

or,

\[
\mathcal{M}(12 \rightarrow 34) \equiv \langle 34; \theta |\mathcal{M}|12; 0 \rangle = \sum_{a,j} \langle 34; \theta |\mathcal{M}^{aj}|12; 0 \rangle \tag{6.9}
\]

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Figure 20: A particle (with spin in the direction of the blue arrow) decaying to 2 particles.

Where,

\[
\langle 34; \theta | \mathcal{M}^{aJ} | 12; 0 \rangle = \frac{1}{s - n} M_s^2 \ F_{\lambda_3 \lambda_4; a_3a_4}^{aJ} \ F_{\lambda_1 \lambda_2; a_1a_2}^{aJ} \ d_{\lambda_1 - \lambda_2; \lambda_3 - \lambda_4}^J (\theta)
\]

This equation can be viewed as the definition of the \( F \)'s which are called \textit{collinear amplitudes}.

A general helicity amplitude will have the following form near a resonance of mass squared \( m_n^2 = \hat{s} = n \):

\[
\mathcal{M}_{m,m'} = \frac{1}{s - n} \kappa \left( \sum_a \eta_a^{a_3a_4a_1a_2} \right) \sum_J C_{m,m'}^n d_{m,m'}^J (\theta)
\]

Where \( \kappa \) contains constants, and \( \eta \) is the color factor of the amplitude. The tree level string helicity amplitudes will be written in this form in Eqs. (6.30)-(6.36).

### 6.1.3 Decay widths

We now show how the decay width can be calculated from the \( F \)'s, following [7].

Consider a particle at rest with mass \( M \), spin \( J \), and \( J_z = \Lambda \), decaying in to two particles with helicities \( \lambda_3 \) and \( \lambda_4 \) moving in opposite directions along the \( z' \) axis (see Fig. 20).

The S-matrix element for the decay is:

\[
S = i (2\pi)^4 \delta^4 (P - p_3 - p_4) \ \langle \bar{p}_3 \lambda_3 a_3 ; \bar{p}_4 \lambda_4 a_4 | \mathcal{L} | 0, \Lambda, a \rangle
\]

The partial decay width into two particles with definite helicities and colors is:

\[
\Gamma_{\lambda_3 \lambda_4; a_3a_4}^{aJ} = \frac{1}{2M} (2\pi)^4 \int d^4 p_3 d^4 p_4 \ \delta^4 (P - p_3 - p_4) \ \delta^+(p_3^2 - m_3^2) \ \delta^+(p_4^2 - m_4^2) \ \times \ \left| \langle \bar{p}_3 \lambda_3 a_3 ; \bar{p}_4 \lambda_4 a_4 | \mathcal{L} | 0, \Lambda, a \rangle \right|^2
\]

\[
= \frac{p^r}{32\pi^2 M^2} \int d\Omega_3 \left| \langle -\bar{p}_3 \lambda_3 a_3 ; \bar{p}_4 \lambda_4 a_4 | \mathcal{L} | 0, \Lambda, a \rangle \right|^2
\]

(6.13)
Where in the c.m frame:
\[ p^* = |\vec{p}_3| = |\vec{p}_4| = \frac{M}{2} \]  
(6.14)

Now expand \(|0, \Lambda⟩\) on spin states in the \(z'\) direction:
\[ |0, \Lambda⟩ = \sum_{\Lambda'} |0, \Lambda'⟩ \langle 0, \Lambda'|0, \Lambda⟩ = \sum_{\Lambda'} d^I_{\Lambda\Lambda'}(\theta) |0, \Lambda⟩ \]  
(6.15)

From angular momentum conservation:
\[ \Lambda' = \lambda_3 - \lambda_4 \]  
(6.16)

So we get
\[
\Gamma^{aJ}_{\lambda_3 \lambda_4; a_3 a_4} = \frac{1}{64\pi^2 M} |\langle \vec{p}_3 \lambda_3 a_3 ; -\vec{p}_4 \lambda_4 a_4 | \mathcal{L} |0, \Lambda', a⟩|^2 \int d\Omega \ |d^I_{\Lambda,\lambda_3 - \lambda_4}(\theta)|^2
\]
\[
\equiv \frac{1}{64\pi^2 M} |F^{aJ}_{\lambda_3 \lambda_4; a_3 a_4}|^2 \int d\Omega \ |d^I_{\Lambda,\lambda_3 - \lambda_4}(\theta)|^2
\]  
(6.17)

Since
\[ \int d\Omega \ |d^I_{\Lambda,\lambda_3 - \lambda_4}(\theta)|^2 = \frac{4\pi}{2J + 1} \]  
(6.18)

We finally get,
\[
\Gamma^{aJ}_{\lambda_3 \lambda_4; a_3 a_4} = \frac{1}{16(2J + 1)\pi M} |F^{aJ}_{\lambda_3 \lambda_4; a_3 a_4}|^2
\]  
(6.19)

Summing over colors and helicities,
\[
\Gamma^{aJ} = \sum_{\lambda_3, \lambda_4} \sum_{a_3, a_4} \Gamma^{aJ}_{\lambda_3 \lambda_4; a_3 a_4} = \frac{1}{16(2J + 1)\pi M} \sum_{\lambda_3, \lambda_4} \sum_{a_3, a_4} |F^{aJ}_{\lambda_3 \lambda_4; a_3 a_4}|^2
\]  
(6.20)

The total decay width of a particle is the sum, over all allowed final states, of the partial decay widths.

### 6.2 Calculations of decay widths

In this section, expressions for the decay widths of the quark and gluon Regge excitations will be derived in terms of the coefficients \(C^{m,n}_{m',n'}\). In section 6.3, four methods to calculate the \(C\)'s will be suggested.

We note that since we consider only initial and final states which are \(n = 0\) ground states (the standard model particles), the calculations do not include decays of Regge states into lower lying Regge states.
### 6.2.1 Amplitudes in terms of the $d$-functions

Consider the following 7 expansions, which will enable us to write our string helicity amplitudes on a basis of angular functions which exhibit exchanges of particles with a definite spin.

\[
\prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=0}^{n-1} C_{0,0}^{m,J} d_{0,0}^J(\theta) \tag{6.21}
\]

\[
\frac{\hat{t}^2}{s^2} \prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=2}^{n+1} C_{2,-2}^{m,J} d_{2,-2}^J(\theta) \tag{6.22}
\]

\[
\frac{\hat{u}^3}{s^2} \prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=2}^{n+1} C_{2,2}^{m,J} d_{2,2}^J(\theta) \tag{6.23}
\]

\[
\frac{\hat{u}^3}{s^2} \hat{t} \prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=2}^{n+1} C_{2,-1}^{m,J} d_{2,-1}^J(\theta) \tag{6.24}
\]

\[
\frac{\hat{u}^3}{s^2} \prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=2}^{n+1} C_{2,1}^{m,J} d_{2,1}^J(\theta) \tag{6.25}
\]

\[
\frac{\hat{t}^2}{s^2} \hat{u} \prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{J=1}^{n-1} C_{1,0}^{m,J} d_{1,0}^J(\theta) \tag{6.26}
\]

We can immediately obtain the following relations from Eqs. (I.7), (5.41):

\[
C_{2,2}^{m,J} = (-1)^{J+n-1} C_{2,-2}^{m,J} \tag{6.28}
\]

\[
C_{2,1}^{m,J} = (-1)^{J+n-1} C_{2,-1}^{m,J} \tag{6.29}
\]

For this reason, in the following we will not explicitly consider \( C_{2,2}^{m,J} \) and \( C_{2,1}^{m,J} \).

We use the expansions Eqs. (6.21)-(6.27) in order to rewrite the string helicity amplitudes (near a pole \( \hat{s} \to n \)) on the basis of \( d \) functions. Eqs. (A.9), (A.12), (A.15), (A.18), (A.20), (A.24), (A.26) then become:

\[
\mathcal{M}_{0,0} = 4g^2 \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=0}^{n-1} C_{0,0}^{m,J} d_{0,0}^J(\theta) \right] \cdot \left\{ \sum_a f_a^{a_1 a_2 a_3 a_4} \right\} \tag{6.30}
\]
\[ M_{2,-2} = 4g^2 \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=2}^{n+1} C_{2,-2}^J d_{2,-2}^J \right] \left\{ 8 \sum_a \frac{q_{a1a2a} d_{a3a4a}}{\sum_a f_{a1a2a} f_{a3a4a}} \right\} \]  
(6.31)

\[ M_{2,-1} = 2g^2 \delta_{\beta_3} \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=2}^{n+1} C_{2,-1}^J d_{2,-1}^J(\theta) \right] \left\{ 4 \sum_b \frac{q_{a1a2b} T_{a1a2b}}{\sum_a f_{a1a2b} T_{a1a2b}} \right\} \]  
(6.32)

\[ M_{1,\frac{1}{2}} = (-1)^{n-1} 2g^2 \delta_{\beta_3} \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=\frac{3}{2}}^{n+\frac{1}{2}} C_{1,\frac{1}{2}}^J d_{1,\frac{1}{2}}^J(\theta) \right] \left( T_{a1} T_{a2} \right)_{\alpha_1}^{\alpha_3} \]  
(6.33)

\[ M_{2,-\frac{3}{2}} = 2g^2 \delta_{\beta_3} \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=\frac{3}{2}}^{n+\frac{1}{2}} C_{2,-\frac{3}{2}}^J d_{2,-\frac{3}{2}}^J(\theta) \right] \left( T_{a1} T_{a2} \right)_{\alpha_1}^{\alpha_3} \]  
(6.34)

\[ M_{1,\frac{3}{2}}^{(B)} = 2gd_{1,\frac{3}{2}} g \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=\frac{3}{2}}^{n+\frac{1}{2}} C_{1,\frac{3}{2}}^J d_{1,\frac{3}{2}}^J(\theta) \right] \left( T_{a1} T_{a2} \right)_{\alpha_1}^{\alpha_3} \]  
(6.35)

\[ M_{2,-\frac{5}{2}}^{(B)} = 2gd_{1,\frac{3}{2}} g \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \sum_{J=\frac{3}{2}}^{n+\frac{1}{2}} C_{2,-\frac{5}{2}}^J d_{2,-\frac{5}{2}}^J(\theta) \right] \left( T_{a1} T_{a2} \right)_{\alpha_1}^{\alpha_3} \]  
(6.36)

Where in the first three amplitudes, the curly brackets use shorthand notation in which we do not write that the upper row is \( n = \text{odd} \) and the lower row is \( n = \text{even} \). All of these amplitudes contain: the simple pole \( \frac{1}{\hat{s} - n} \), angular dependence in the square brackets, and color factors on the right.

Regarding Eq. (6.21), we would like to note that \( \prod_{K=1}^{n-1} (\hat{u} + K) \) when expanded as a polynomial in \( \cos \theta \) has a definite parity (it is an even (odd) polynomial when \( n \) is odd (even), see e.g. Eq. (6.82)). Likewise, \( d_{00}^J(\theta) \) has a definite parity since it is the Legendre polynomial \( P_J(\cos \theta) \), see Eq. (1.8). It follows that:

\[ C_{0,0}^{n=\text{odd}, J=\text{odd}} = C_{0,0}^{n=\text{even}, J=\text{even}} = 0 \]  
(6.37)
6.2.2 Plan for extracting the $F$’s

As we saw, the amplitudes have the following general form:

$$M_{m,m'} = \frac{1}{\hat{s} - n} \kappa \left( \sum_a \eta_{a_3a_4a_1a_2}^a \right) \sum_J C_{m,m'}^{n,J} d_{m,m'}^J(\theta)$$

(6.38)

Where $\kappa$ contains constants, and $\eta$ is the color factor of the amplitude. In order to calculate the decay widths, we must first calculate the $F$’s defined by:

$$M_{m,m'} = \frac{1}{M_s^2} \frac{1}{\hat{s} - n} \sum_{a_3,a_4} F_{m'a_3a_4}^a F_{m'a_1a_2}^a d_{m,m'}^J(\theta)$$

(6.39)

This was written in Eq. (6.10), but now we use a slightly different notation in terms of $m, m'$ instead of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Comparing the last two equations we get

$$F_{aJ}^{m_1a_3a_4} \eta_{a_3a_4a_1a_2}^a = M_s^2 \kappa \eta_{a_3a_4a_1a_2}^a C_{m,m'}^{n,J}$$

(6.40)

We need to extract the $F$’s from this equation. If we have initial and final states which are identical, then $m = \pm m'$ and the color part factorizes into two equal parts: $\eta_{a_3a_4a_1a_2}^a = \tilde{\eta}_{a_4}^a \tilde{\eta}_{a_1}^a$. Then the two $F$’s are equal and can be extracted:

$$F_{aJ}^{m_1a_3a_4} = M_s \sqrt{\kappa_1} \tilde{\eta}_{a_4}^a \sqrt{C_{m,m}^{n,J}}$$

(6.41)

This formula applies for $M_{0,0}$, $M_{2,-2}$, $M_{1,\frac{1}{2}}$, $M_{3,-\frac{3}{2}}$, $M_{1,\frac{1}{2}}$, $M_{3,\frac{3}{2}}$.

Now we deal with $M_{2,-1}$ which describes $g^- g^+ \rightarrow q^- \bar{q}^+$, and obviously has different initial and final states. From Eq. (6.40) we write:

$$F_{aJ}^{m_1a_3a_4} F_{aJ}^{m_1a_3a_4} = M_s^2 \kappa \eta_{a_3a_4a_1a_2}^a C_{2,-1}^{n,J}$$

(6.42)

In this equation, we know $F_{aJ}^{m_1a_3a_4}$ from Eq. (6.41) when applied to $M_{2,-2}$. So we divide Eq. (6.42) by $F_{aJ}^{m_1a_3a_4}$:

$$F_{-1,a_3a_4}^{aJ} = \frac{M_s^2}{F_{aJ}^{m_1a_3a_4}} \kappa \eta_{a_3a_4a_1a_2}^a C_{2,-1}^{n,J} = M_s \sqrt{\kappa_1} \frac{\eta_{a_3a_4a_1a_2}^a}{\tilde{\eta}_{a_4}^a} \sqrt{C_{m,m}^{n,J}}$$

(6.43)

After we calculate the $F$’s from Eqs. (6.43) and (6.41), the decay widths can then be found using Eq. (6.19):

$$\Gamma_{aJ}^{(m,m')a_3a_4} = \frac{|F_{aJ}^{m,m'a_3a_4}|^2}{16(2J + 1)\pi \sqrt{nM_s}}$$

(6.44)

Where we put: $M = \sqrt{nM_s}$. 

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6.2.3 Extracting the $F$’s

Using the techniques above we now extract the $F$’s and $\Gamma$’s.

All of the decay widths will depend on the following combination of constants:

$$\mathcal{Y} \equiv \frac{g^2 M_s}{(2J + 1)\pi} \frac{\sqrt{n}}{(n - 1)!}$$ (6.45)

We also use the short hand notation: $F_m$ and $\Gamma_m$ thus omitting the obvious dependence on the other indices. Inserting the relevant $\eta$’s and $\kappa$’s from Eqs. (6.30)-(6.36) we get the following $F$’s and $\Gamma$’s:

$$F_{m=0} = 2g M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{C_{0,0}^{mJ}} \left\{ \frac{\sqrt{8} d^{a_1 a_2 a}}{f_{a_1 a_2}} \right\}$$ (6.46)

$$\Gamma_{m=0} = 2\mathcal{Y} C_{0,0}^{mJ} \left\{ \frac{d^{a_3 a_4 a} d^{a_3 a_4 a}}{8 f_{a_1} f_{a_2}} \right\}.$$ (6.47)

$$F_{m=2} = 2g M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{C_{2,-2}^{mJ}} \left\{ \frac{\sqrt{8} d^{a_1 a_2 a}}{f_{a_1 a_2}} \right\}$$ (6.48)

$$\Gamma_{m=2} = 2\mathcal{Y} C_{2,-2}^{mJ} \left\{ \frac{d^{a_3 a_4 a} d^{a_3 a_4 a}}{8 f_{a_1} f_{a_2}} \right\}.$$ (6.49)

$$F_{m=-1} = g M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{\frac{C_{2,-1}^{mJ}}{C_{2,-2}^{mJ}}} \left\{ \frac{\sqrt{2} T^a_{\alpha_1 \alpha_2}}{T^a_{\alpha_1 \alpha_2}} \right\}$$ (6.50)

$$\Gamma_{m=-1} = \frac{\mathcal{Y}}{16} \left( \frac{C_{2,-1}^{mJ}}{C_{2,-2}^{mJ}} \right)^2 \left\{ \frac{2 T^a_{\alpha_1 \alpha_2} T^a_{\alpha_1 \alpha_2}}{T^a_{\alpha_1 \alpha_2} T^a_{\alpha_1 \alpha_2}} \right\}$$ (6.51)

$$F_{m=\frac{1}{2}} = \sqrt{2} g M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{\frac{C_{1/2}^{mJ}}{2}} T^{a_4}_{a_3}$$ (6.52)

$$\Gamma_{m=\frac{1}{2}} = \frac{\mathcal{Y}}{8} C_{1/2}^{mJ} T^{a_4}_{a_3} T^{a_4}_{a_3}$$ (6.53)
\[ F_{m=\frac{3}{2}} = \sqrt{2} g M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{\frac{C_{\frac{3}{2}}^{m,J}}{2 - \frac{3}{2}}} T_{\alpha_3}^{a_4} \]  
(6.54)

\[ \Gamma_{m=\frac{3}{2}} = \frac{\mathcal{Y}}{8} C_{\frac{3}{2}}^{m,J} T_{\alpha_3}^{a_4} T_{\alpha_3}^{a_4} \]  
(6.55)

\[ F_{m=\frac{1}{2}}^{(B)} = \sqrt{2} g D_{P_b} M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{\frac{C_{\frac{1}{2}}^{m,J}}{2 - \frac{1}{2}}} T_{\alpha_3}^{a_4} \]  
(6.56)

\[ \Gamma_{m=\frac{1}{2}}^{(B)} = \frac{g D_{P_b}}{g} \frac{\mathcal{Y}}{8} C_{\frac{1}{2}}^{m,J} T_{\alpha_3}^{a_4} T_{\alpha_3}^{a_4} \]  
(6.57)

\[ F_{m=\frac{1}{2}}^{(B)} = \sqrt{2} g D_{P_b} M_s \sqrt{\frac{n}{(n-1)!}} \sqrt{\frac{C_{\frac{1}{2}}^{m,J}}{2 - \frac{1}{2}}} T_{\alpha_3}^{a_4} \]  
(6.58)

\[ \Gamma_{m=\frac{1}{2}}^{(B)} = \frac{g D_{P_b}}{g} \frac{\mathcal{Y}}{8} C_{\frac{1}{2}}^{m,J} T_{\alpha_3}^{a_4} T_{\alpha_3}^{a_4} \]  
(6.59)

### 6.2.4 Decay widths of the excited gluons

We denote the Regge excitations of the gluon by \( \tilde{g}^{a,nJ} \), see also Fig. 18.

The color index "\( a \)" will be omitted sometimes. The gauge symmetry is \( U(N) \) which is decomposed as \( U(N) \sim SU(N) \times U(1) \), hence we put a tilde in \( \tilde{g}^{a,nJ} \) to remind that it is \( U(N) \).

We denote the Regge excitations of the \( SU(N) \) gluon \( g \) as:

\[ g^{a,nJ} \equiv \tilde{g}^{a,nJ} \text{ (without the tilde), for } a = 1, \ldots, N^2 - 1. \]  
(6.60)

Likewise, we denote the Regge excitations of the \( U(1) \) partner of the gluon \( A \equiv \tilde{g}^{001} \) as:

\[ A^{a,nJ} \equiv g^{0,nJ}, \text{ for } a = 0. \]  
(6.61)

The decay width for the process \( \tilde{g}^{a,nJ} \rightarrow \tilde{g} \tilde{g} \) is the sum from all the helicity states, Eq (6.20):

\[ \Gamma_{\tilde{g}^{a,nJ} \rightarrow \tilde{g} \tilde{g}} = \frac{1}{2} \sum_{\lambda_3, \lambda_4} \sum_{a_3, a_4} \Gamma^{aJ}_{\lambda_3 \lambda_4 a_3 a_4} \]  
(6.62)
Figure 21: A plot in the $n - J$ plane of the gluon Regge excitations $g^{n,J}$ exchanged in $gg \rightarrow gg$. The gray dots are not exchanged in this process.

The $\frac{1}{2}$ is because of double counting or identical final state particles, see [7].

We get from Eqs. (6.47), (6.49), (6.62):

$$\Gamma_{\tilde{g} g^n J \rightarrow \tilde{g} \tilde{g}} = \mathcal{Y} \left[ C_{0,0}^{mJ} + 2C_{2,-2}^{mJ} \right] \cdot \left\{ \sum_{a_3,a_4} d^{a_3 a_4 a} d^{a_3 a_4 a}, \text{ odd } n \right\} \cdot \left\{ \frac{1}{5} \sum_{a_3,a_4} f^{a_3 a_4 a} f^{a_3 a_4 a}, \text{ even } n \right\} \quad (6.63)$$

The factor of 2 in front of $C_{2,-2}^{mJ}$ is because it also counts $C_{2,2}^{mJ}$.

In Eq. (6.63), for a given $n$ it should be understood that: $C_{2,-2}^{mJ=0} = C_{2,-2}^{mJ=1} = 0$ and $C_{0,0}^{mJ=n+1} = C_{0,0}^{mJ=n} = 0$. Therefore, from Eq. (6.37) the exchange of any $J = 0$ particle occurs only at $n = \text{odd}$, and that of $J = 1$ only at $n = \text{even}$, see Figs. 21, 22.

Using the identities from Appendix F we plug in the color factors for the six different combinations of $SU(N)$ and $U(1)$ fields:

$$\Gamma_{g^n J \rightarrow gg} = \mathcal{Y} \left[ C_{0,0}^{mJ} + 2C_{2,-2}^{mJ} \right] \cdot \left\{ \frac{N^2 - 1}{16N} \cdot \frac{\tilde{N}}{N} \right\} \quad (6.64)$$

$$\Gamma_{g^n J \rightarrow gA} = \mathcal{Y} \left[ C_{0,0}^{mJ} + 2C_{2,-2}^{mJ} \right] \cdot \left\{ \frac{1}{4N} \cdot 0 \right\} \quad (6.65)$$
Figure 22: A plot in the $n-J$ plane of the gluon Regge excitations $g^{n,J}$ exchanged in $gg \to gA$. The gray dots are not exchanged.

$$\Gamma_{g^{n,J}\to AA} = 0 \quad (6.66)$$

$$\Gamma_{A^{n,J}\to gg} = \mathcal{Y} \left[ C^{mJ}_{0,0} + 2C^{mJ}_{2,-2} \right] \cdot \left\{ \frac{N^2-1}{8N} \begin{array}{c} \frac{1}{8N} \\ 0 \end{array} \right\} \quad (6.67)$$

$$\Gamma_{A^{n,J}\to gA} = 0 \quad (6.68)$$

$$\Gamma_{A^{n,J}\to AA} = \mathcal{Y} \left[ C^{mJ}_{0,0} + 2C^{mJ}_{2,-2} \right] \cdot \left\{ \frac{1}{8N} \begin{array}{c} \frac{1}{8N} \\ 0 \end{array} \right\} \quad (6.69)$$

Figs. 21, 22 show the spectrum of the gluon Regge excitations $g^{n,J}$ which are exchanged in $gg \to gg$ and $gg \to gA$ respectively. In the process $gg \to gA$, $A^{n,J}$ are not exchanged at all. In the process $gg \to AA$ only $A^{n,J}$ are exchanged and $g^{n,J}$ are not. This happens because $d^{a00} = 0$, Eq. (F.3). Processes with a $gA$ or $AA$ in the final state do not have $n = even$ poles since $f^{0ab} = f^{00a} = f^{000} = 0$, see Appendix A.

Similar reasoning applied to $q\bar{q} \to gg$ and $q\bar{q} \to gA$ will yield a spectrum which is different only by the fact that now there no exchanges of spin $J = 0, 1$ (except for exchange of a massless gluon). This happens of course because of the absence of $C^{mJ}_{0,0}$, Eqs. (6.32) and (6.70).
Proceeding now to the decay into $q\bar{q}$, we have from Eq. (6.51):

$$\Gamma_{g^{nJ}\rightarrow q\bar{q}} = \frac{\mathcal{V}}{8} \left( \frac{C_{2,-1}^{nJ}}{C_{2,-2}^{nJ}} \right)^2 \left\{ \begin{array}{c} 2Tr(T^a T^a) \\ Tr(T^a T^a) \end{array} \right\}$$

(6.70)

For the $SU(N)$ and $U(1)$ fields this gives an equal result since $Tr(T^a T^a) = \frac{1}{2}$ in both cases:

$$\Gamma_{g^{nJ}\rightarrow q\bar{q}} = \Gamma_{A^{nJ}\rightarrow q\bar{q}} = \frac{\mathcal{V}}{8} \left( \frac{C_{2,-1}^{nJ}}{C_{2,-2}^{nJ}} \right)^2 \left\{ \frac{1}{2} \right\}$$

(6.71)

The total $g^{nJ}$ width is the sum from the four channels. After taking into account that there may be $N_f$ quark flavors, we get:

$$\Gamma_{g^{nJ}} = \left[ \mathcal{V} \left( C_{0,0}^{nJ} + 2C_{2,-2}^{nJ} \right) \left\{ \frac{N}{16N} \right\} + \frac{\mathcal{V}}{8} N_f \left( \frac{C_{2,-1}^{nJ}}{C_{2,-2}^{nJ}} \right)^2 \left\{ \frac{1}{2} \right\} \right]$$

(6.72)

Similarly for $A^{nJ}$:

$$\Gamma_{A^{nJ}} = \left[ \mathcal{V} \left( C_{0,0}^{nJ} + 2C_{2,-2}^{nJ} \right) \left\{ \frac{N}{8} \right\} + \frac{\mathcal{V}}{8} N_f \left( \frac{C_{2,-1}^{nJ}}{C_{2,-2}^{nJ}} \right)^2 \left\{ \frac{1}{2} \right\} \right]$$

(6.73)

### 6.2.5 Decay widths of the excited quarks

We denote the Regge excitations of the quark by $q^{nJ}$, see also Fig. 18.

From Eqs. (6.53), (6.55) we have:

$$\Gamma_{q^{nJ}\rightarrow g\bar{q}} = \frac{\mathcal{V}}{8} \left[ C_{\frac{3}{2},\frac{3}{2}}^{nJ} + 2C_{\frac{1}{2},-\frac{1}{2}}^{nJ} \right] \sum_{\alpha_3,\alpha_4} T_{\alpha_3}^{a_4} T_{\alpha_4}^{a_3}$$

(6.74)

It is understood that: $C_{\frac{3}{2},\frac{3}{2}}^{n,J=\frac{1}{2}} = 0$ and $C_{\frac{1}{2},\frac{1}{2}}^{n,J=n+\frac{1}{2}} = 0$. The factor of 2 in front of $C_{\frac{1}{2},-\frac{1}{2}}^{nJ}$ is because it also counts $C_{\frac{1}{2},\frac{1}{2}}^{nJ}$.

The gluon in the decay $q^{nJ} \rightarrow g\bar{q}$ can be either a $SU(N)$ or a $U(1)$. Eqs. (F.17), (F.18) give:

$$\Gamma_{q^{nJ}\rightarrow gg} = \frac{\mathcal{V}}{8} \left[ C_{\frac{1}{2},\frac{1}{2}}^{nJ} + 2C_{\frac{3}{2},-\frac{3}{2}}^{nJ} \right] \frac{N^2 - 1}{2N}$$

(6.75)
\[ \Gamma_{q^n J \to qA} = \frac{Y}{8} \left[ C_{nJ}^{1/2} + 2C_{nJ}^{3/2} \right] \frac{1}{2N} \] (6.76)

Proceeding now to the decay into \( q' \tilde{B} \) (where \( \tilde{B} \) is the \( U(N_b) \) gauge boson from stack \( b \)), we have from Eqs. (6.57), (6.59):

\[ \Gamma_{q^n J \to q' \tilde{B}} = \frac{g_D P_b}{g} \frac{Y}{8} \left[ C_{nJ}^{1/2} + 2C_{nJ}^{3/2} \right] \sum_{a_3, a_4} T^{a_4}_{a_3} T^{a_4}_{a_3} \] (6.77)

For the \( SU(N_b) \) and \( U(1) \) particles

\[ \Gamma_{q^n J \to q'B^0} = \frac{g_D P_b}{g} \frac{Y}{8} \left[ C_{nJ}^{1/2} + 2C_{nJ}^{3/2} \right] \frac{1}{2N_b} \] (6.78)

The total \( q^n J \) width is the sum from all channels:

\[ \Gamma_{q^n J} = \Gamma_{q^n J \to q\bar{q}} + \Gamma_{q^n J \to qA} + \Gamma_{q^n J \to q' \tilde{B}} + \Gamma_{q^n J \to q'B^0} = \]

\[ \frac{Y}{8} \left[ C_{nJ}^{1/2} + 2C_{nJ}^{3/2} \right] \left( \frac{N}{2} + \frac{g_D P_b}{g} \frac{N_b}{2} \right) \] (6.80)

### 6.3 Calculation of the coefficients \( C_{m,J}^{n,m} \)

In this section we suggest four methods to calculate the coefficients \( C_{m,J}^{n,m} \) of Eqs. (6.21)-(6.27). The \( C \)'s can then be plugged into the expressions for the decay widths in sections 6.2.4 and 6.2.5.

#### 6.3.1 Approach 1

In this approach both sides of Eqs. (6.21)-(6.27) are expanded in powers of \( \cos \theta \) for a given value of \( n \). For the left hand side this is done by plugging \( \hat{u} = -\frac{n}{2}[1 + \cos \theta] \) and \( \hat{t} = -\frac{n}{2}[1 - \cos \theta] \), and expanding \( \prod_{K=1}^{n-1} (\hat{u} + K) \). In the right hand side, the \( d \)'s are calculated using Eq. (1.18) and using trigonometric identities are written as a power series in \( \cos \theta \). Then one compares both sides, and the \( C \)'s can be extracted recursively starting from the highest power of \( \cos \theta \) and proceeding to lower powers.
The calculated $d$ functions are given in the tables of Appendix I.

The process of calculating the $C_{m,m}$’s for $n = 1 \ldots 5$ is shown in the tables of Appendix J.

Tables 18, 22, 26, 30, 34 collect all of the $C$’s.

- **Notes on expanding the $\prod_{K=1}^{n-1}(\hat{u} + K)$**.

The factor $\prod_{K=1}^{n-1}(\hat{u} + K)$ appears in each of the 5 equations, and is a polynomial in $\cos \theta$ of degree $n - 1$. For high $n$ it may be helpful to simplify this factor as follows.

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \left( -\frac{n}{2} [1 + \cos \theta] + 1 \right) \left( -\frac{n}{2} [1 + \cos \theta] + 2 \right) \cdots \left( -\frac{n}{2} [1 + \cos \theta] + n - 1 \right) = \\
\left( -\frac{1}{2} \right)^{n-1} (n \cos \theta + [n - 2]) (n \cos \theta + [n - 4]) \cdots (n \cos \theta - [n - 4]) (n \cos \theta - [n - 2])
\]

(6.81)

The first two terms with highest powers are easy to sum, giving:

\[
\left( -\frac{1}{2} \right)^{n-1} \left[ n^{n-1} (\cos \theta)^{n-1} - \frac{(n - 2)(n - 1)n}{6} (\cos \theta)^{n-3} + \ldots \right]
\]

(6.82)

In Eq. (6.81) we can cut the number of terms in half by multiplying the first term with the last, the second term with the next to last, etc...

We get for $n =$odd:

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \left( -\frac{1}{2} \right)^{n-1} (n^2 \cos^2 \theta - [n - 2]^2) (n^2 \cos^2 \theta - [n - 4]^2) \cdots (n^2 \cos^2 \theta - 1^2)
\]

(6.83)

for $n =$even:

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \left( -\frac{1}{2} \right)^{n-1} (n^2 \cos^2 \theta - [n - 2]^2) (n^2 \cos^2 \theta - [n - 4]^2) \cdots (n^2 \cos^2 \theta - 2^2) \cdot n \cos \theta
\]

(6.84)
6.3.2 Approach 2

In essence, this approach similar to the previous approach but takes into account some things that simplify the computations. The procedure for expanding the right hand side is simplified by using known properties of the \(d\) functions, namely that the \(d\) functions are simply proportional to Jacobi polynomials, and the power expansions of the Jacobi polynomials are known. As for the left hand side, the coefficients of the power expansion of \(\prod_{K=1}^{n-1}(\hat{u} + K)\) are known to be the Stirling numbers.

We start by noting that the power expansions of Eqs. (6.21)-(6.27) are now done in terms of \(\hat{u}\) instead of \(\cos \theta\). Plugging Eqs. (I.27)-(I.31) in Eqs. (6.21)-(6.27) we get:

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \sum_{J=0}^{n-1} C_{n,0}^{m,J} P_J^{(0,0)}(\hat{u}) = \sum_{J=2}^{n+1} C_{2,-2}^{m,J} P_{J-2}^{(4,0)}(\hat{u}) = \sum_{J=2}^{n+1} \sqrt{\frac{J+2}{J-1}} C_{2,-1}^{m,J} P_{J-2}^{(3,1)}(\hat{u}) = \sum_{J=\frac{3}{2}}^{n+1} C_{\frac{3}{2},-\frac{3}{2}}^{m,J} P_{J-3/2}^{(3,0)}(\hat{u})
\]

(6.85)

All the factors on the left hand side that multiplied \(\prod_{K=1}^{n-1}(\hat{u} + K)\) got canceled. Therefore, the problem of finding the \(C\)'s reduces to comparing \(\prod_{K=1}^{n-1}(\hat{u} + K)\) with the five different Jacobi polynomials above.

This procedure should be possible for amplitudes with general helicity states \((m, m')\). We have using Eq. (I.23):

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \sum_{J=0}^{n-1+m} \Omega^{(J-m)} C_{m,m'}^{n,J} P_{J-m}^{(m-m', m+m')}(\hat{u})
\]

(6.86)

This can be used to calculate decay widths for a decay of a particle into excited states, as in the direct production of section 9.

Let's continue to show how one can extract the \(C\)'s. The power expansion of the l.h.s of Eq. (6.86) is from Eq. (H.26):

\[
\prod_{K=1}^{n-1}(\hat{u} + K) = \frac{\hat{u}^n}{\hat{u}} = \sum_{i=0}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \hat{u}^i
\]

(6.87)

As for the r.h.s, we recall Eq. (I.25):

\[
P_k^{(m-m', m+m')}(\hat{u}) = \sum_{p=0}^{k} \Delta_{m,m'}^{(p)} \hat{u}^p
\]

(6.88)
Defining, \( k = J - m \), the r.h.s of Eq. (6.86) is now:

\[
\sum_{k=0}^{n-1} \Omega^{(k)} C_{m,m'}^{n,k+m} P_k^{(m-m',m+m')}(\hat{u}) = \sum_{p=0}^{n-1} \sum_{k=0}^{k} \Omega^{(k)} C_{m,m'}^{n,k+m} \Delta_{m,m'}^{(p)} \hat{u}^p
\] (6.89)

Using \( \sum_{k=0}^{n-1} \sum_{p=0}^{n-1} \sum_{k=0}^{n-1} \) we get:

\[
\sum_{p=0}^{n-1} \sum_{k=0}^{k} \Omega^{(k)} C_{m,m'}^{n,k+m} \Delta_{m,m'}^{(p)} \hat{u}^p
\] (6.90)

Comparing this with Eq. (6.87), we get:

\[
\sum_{k=0}^{n-1} \Omega^{(k)} C_{m,m'}^{n,k+m} = (-1)^{n-p-1} \frac{S_n^{(p+1)}}{\Delta_{m,m'}^{(p)}}
\] (6.91)

These are equations for the \( C's \). We can begin with \( p = n - 1 \) and proceed to extract the \( C's \) recursively:

\[
C_{m,m'}^{n,n-1+m} = \frac{S_n^{(n)}}{\Omega(n-1) \Delta_{m,m'}^{(n-1)}} = \frac{1}{\Omega(n-1) \Delta_{m,m'}^{(n-1)}}
\] (6.92)

\[
C_{m,m'}^{n,n-2+m} = \frac{1}{\Omega(n-2)} \left[ -\frac{S_n^{(n-1)}}{\Delta_{m,m'}^{(n-2)}} - \Omega(n-1) C_{m,m'}^{n,n-1+m} \right]
\] (6.93)

And so on... We see that the leading Regge trajectories are obtained first.

So let us write explicitly the \( C's \) for the leading trajectory.

From Eq. (I.26) we have:

\[
\Delta_{m,m'}^{(n-1)} = (-1)^{n-1} \frac{(2n - 2 + 2m)!}{(n-1)! (n - 1 + 2m)! n^{n-1}}
\] (6.94)

Then Eq. (6.92) gives for the 5 combinations of \((m, m')\):

\[
C_{0,0}^{n,n-1} = \frac{1}{\Delta_{0,0}^{(n-1)}} = (-1)^{n-1} \frac{(n-1)!^2 n^{n-1}}{(2n-2)!}
\] (6.95)

\[
C_{2,-2}^{n,n+1} = (-1)^{n-1} \frac{(n-1)! (n + 3)! n^{n-1}}{(2n + 2)!}
\] (6.96)
\[
C_{2,-1}^{n,n+1} = (-1)^{n-1} \frac{(n-1)! (n+3)! n^{n-1}}{(2n+2)!} \sqrt{\frac{n}{n+3}}
\]
(6.97)

\[
C_{\frac{3}{2}, \frac{1}{2}}^{n,n-\frac{1}{2}} = (-1)^{n-1} \frac{(n-1)! n! n^{n-1}}{(2n-1)!}
\]
(6.98)

\[
C_{\frac{3}{2}, \frac{3}{2}}^{n,n+\frac{1}{2}} = (-1)^{n-1} \frac{(n-1)! (n+2)! n^{n-1}}{(2n+1)!}
\]
(6.99)

For consistency, we checked these formulas against the corresponding leading trajectory results (up to \(n = 5\)) from approach 1 (given in Tables 18, 22, 26, 30, 34). Agreement was found in all cases.

Using these \(C\)'s we can write the \(n\) dependence of the decay widths for the leading trajectory resonances. These are given in terms of the combination \(\Gamma \sim \mathcal{Y} C_{m,m'}^{n,J}\), as seen in e.g Eq. (6.63). So all that is needed is to multiply the previous 5 equations by \(\mathcal{Y} \sim \sqrt{\frac{n}{(2J+1)(n-1)!}}\), with \(J = n + \text{const.}\) and the constant is different for each one of the five helicity states. The large \(n\) dependence is seen to be dominated by the \(n^n\) dependence of the \(C\)'s.

We also note the possibility to obtain a formal expression for the \(C\)'s by reversing Eq. (6.86) using the orthogonality of the Jacobi polynomials:

\[
C_{m,m'}^{n,J} = \sqrt{\frac{(2J+1)(J-m)!(J+m)!}{2^{2J+1}(J-m')!(J+m')!}} \times 
\int_{-1}^{1} dx \ (1-x)^{m-m'}(1+x)^{m+m'} P_{j-m,m+m'}^{(m-m',m+m')}(x) \prod_{K=1}^{n-1} \left( -\frac{s}{2} [1-x] + K \right)
\]
(6.100)

It is not clear though if this equation is useful for calculations.

### 6.3.3 Approach 3

This approach was inspired by [54].

Imagine that we have in our disposal the following expansion in which we know the \(\lambda_b\)'s:

\[
\hat{u}^i = \sum_{b=0}^{i} \lambda_b^{(i)} P_b^{(m-m',m+m')} (\hat{u})
\]
(6.101)
This is the reverse of Eq. (6.88). We plug this expansion into Eq. (6.87) and get:

$$\prod_{K=1}^{n-1} (\hat{u} + K) = \sum_{i=0}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \hat{u}^{i} =$$

$$\sum_{b=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \lambda^{(i)}_{b} P_{b}^{(m-m', m+m')} (\hat{u})$$  \hspace{1cm} (6.102)

Using $\sum_{i=0}^{n-1} \sum_{b=0}^{n-1} = \sum_{b=0}^{n-1} \sum_{i=0}^{n-1}$, we get:

$$\sum_{b=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \lambda^{(i)}_{b} P_{b}^{(m-m', m+m')} (\hat{u})$$  \hspace{1cm} (6.103)

Changing dummy variable to $J = b + m$ we get:

$$\sum_{J=m}^{n-1+m} \sum_{i=J-m}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \lambda^{(i)}_{J-m} P_{J-m}^{(m-m', m+m')} (\hat{u})$$  \hspace{1cm} (6.104)

Now this form can finally be compared with the right hand side of Eq. (6.86), yielding:

$$C_{n,J}^{m,m'} = \sqrt{\frac{(J+m')!(J-m)!}{(J+m)!(J-m)!}} \sum_{i=J-m}^{n-1} (-1)^{n-i-1} S_{n}^{(i+1)} \lambda^{(i)}_{J-m}$$  \hspace{1cm} (6.105)

This is a closed expression for $C_{n,J}^{m,m'}$ for any $n, J, m, m'$. Obviously, the question is if we can find an expansion of the form of Eq. (6.101).

### 6.3.4 Approach 4

We take this from [49]. In this approach only the $(m, m') = (0, 0)$ helicity state was taken into account, therefore only the Legendre polynomial appears. It might be possible to generalize such an approach to arbitrary helicity states, but we only have time to write what was done in [49].

Putting $\alpha(s) = as + b$, The partial waves are:

$$a(J, s) = \frac{1}{2} \int_{-1}^{1} dz A(s, t) P_{J}(z) = \pi^{\frac{1}{2}} \alpha^{J-1} \alpha(s)(2aq^2)^{s} \int_{0}^{\infty} dx x^{\alpha(s)-1} f(x) e^{-x}$$  \hspace{1cm} (6.106)

Where the definition

$$f(2aq^2 y) = \left[ \frac{1 - e^{-y}}{y} \right]^{-\alpha(s)-1} (aq^2 y)^{J-\frac{1}{2}} I_{J+\frac{1}{2}} (2aq^2 y) e^{b-1} y$$  \hspace{1cm} (6.107)
and $I_l(x)$ is the modified Bessel function.

The partial width is found by taking the residue:

$$\Gamma(n, J) = -\frac{q_n}{2M_n^2} \text{res} \left[ a(J, s) \right] =$$

$$\pi \frac{1}{2} 2^{-J-1} n(2aq_n^2)^n \frac{1}{(n-J)!} \left( \frac{d}{dx} \right)^{n-J} [f(x)e^{-x}]_{x=0}$$

(6.108)

So that $\Gamma(n, J)$ can be calculated by taking derivatives.

Defining

$$r_n(x) \equiv \frac{n!}{(2n+1)!} (2x)^n$$

(6.109)

The first 4 trajectories were calculated:

$$\Gamma(n, n) = \frac{M_n^2}{q_n} r_n(b_n)$$

(6.110)

$$\Gamma(n, n - 1) = \frac{M_n^2}{q_n} r_{n-1}(b_n) [a_n + \frac{n-1}{2}]$$

(6.111)

$$\Gamma(n, n - 2) = \frac{M_n^2}{q_n} r_{n-2}(b_n) \frac{1}{2} \left[ \frac{b_n^2}{2n-1} + a_n^2 + a_n(n-1) + \frac{(3n-1)(n-2)}{12} \right]$$

(6.112)

$$\Gamma(n, n - 3) = \frac{M_n^2}{q_n} r_{n-3}(b_n) \frac{1}{2} \left[ \frac{b_n^2}{2n-3} (a_n + \frac{n-1}{2}) + \frac{1}{3} \left\{ a_n^3 + \frac{3}{2} a_n^2 (n-1) + \frac{(3n-1)(n-2)}{4} a_n + \frac{n}{8} (n-1)(n-3) \right\} \right]$$

(6.113)
5 point amplitudes

References: [34, 35, 38, 2], [31, 33, 35, 2].

5 particle amplitudes are $2 \rightarrow 3$ processes at colliders. Important signals at the LHC are $pp \rightarrow 3$ jets, $pp \rightarrow 2$ jets + EW gauge boson, $pp \rightarrow 1$ jet + 2 EW gauge bosons and $pp \rightarrow 3$ EW gauge boson. These amplitudes are one order higher than the $2 \rightarrow 2$ processes. As in the 4 amplitude case, it is interesting to study the effect of a low string scale on the energy and angular dependence of such amplitudes. The ultimate goal of a collider is to help uncover the underlying theory. After the initial discovery of a resonance, for instance in the dijet signal, it will be useful to check other types of signals for confirmation, and for measuring the properties of the resonance. The $2 \rightarrow 3$ amplitudes can be helpful in this way. In this section we present the tree squared amplitudes in field theory and string theory, making it comfortable to compare the two. Then we write down the low energy corrections to the string amplitudes.

7.1 Kinematics and definitions

We define

$$s_{ij} \equiv (k_i + k_j)^2 = 2k_i k_j$$

then

$$s_1 \equiv s_{12}, \quad s_2 \equiv s_{23}, \quad s_3 \equiv s_{34}, \quad s_4 \equiv s_{45}, \quad s_5 \equiv s_{51}$$

Introducing the dimensionless units:

$$\hat{s}_{ij} \equiv \alpha' s_{ij}$$
$$\hat{s}_i \equiv \alpha' s_i$$

We define the following kinematic functions that will appear in the squared amplitudes:

$$S_4 \equiv \sum_{i<j} s^4_{ij}$$
$$S_3 \equiv \sum_{i=1,2,3} (s^3_{i4}s_{i5} + s^3_{i4}s^3_{i5})$$

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Figure 23: 5 particle kinematics. Energy-momentum conservation enforces the 3 outgoing particles (in red) to lie in a plane.

Energy-momentum conservation:

$$\sum_{i=1}^{5} k_i = 0 \quad (7.5)$$

The momentum 4-vectors are usually parametrized as (See Fig. 23):

$$k_1 = \frac{\sqrt{s}}{2}(1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

$$k_5 = \frac{\sqrt{s}}{2}(1, -\sin \theta \cos \varphi, -\sin \theta \sin \varphi, -\cos \theta)$$

$$k_4 = \frac{x_1 \sqrt{s}}{2}(1, 1, 0, 0)$$

$$k_2 = \frac{x_2 \sqrt{s}}{2}(1, \cos \theta_{12}, \sin \theta_{12}, 0)$$

$$k_3 = \frac{x_3 \sqrt{s}}{2}(1, \cos \theta_{13}, -\sin \theta_{13}, 0) \quad (7.6)$$

Where

$$\cos \theta_{12} = 1 - \frac{2x_1 + x_2 - 1}{x_1 x_2}$$

$$\cos \theta_{13} = 1 - \frac{2x_1 + x_3 - 1}{x_1 x_3} \quad (7.7)$$
We define the following permutations (see [2]):

\[ \Pi_5 \equiv \left\{ (1, 2, 3, 4, 5)_1, (1, 2, 4, 3, 5)_2, (1, 3, 4, 2, 5)_3, (1, 3, 2, 4, 5)_4, (1, 4, 2, 3, 5)_5, (1, 4, 3, 2, 5)_6, \\
(2, 1, 3, 4, 5)_7, (2, 1, 4, 3, 5)_8, (2, 3, 1, 4, 5)_9, (2, 4, 1, 3, 5)_{10}, (3, 1, 2, 4, 5)_{11}, (3, 2, 1, 4, 5)_{12} \right\} \]

\[ \Pi_q \equiv \left\{ (1, 3, 2, 4, 5)_1, (1, 2, 4, 3, 5)_4, (2, 1, 3, 4, 5)_7, (2, 3, 1, 4, 5)_9, (3, 1, 2, 4, 5)_{11}, (3, 2, 1, 4, 5)_{12} \right\} \]

\[ \Pi_q \equiv \left\{ (1, 2, 4, 3, 5)_2, (2, 1, 4, 3, 5)_8 \right\} \quad (7.8) \]

The index running from 1 to 12 labels the permutations, and will help referring to them in a concise manner. The 12 permutations of \( \Pi_5 \) appear in \( \mathcal{M}(ggggg) \). They come from \((n-1)!/2 = 12\). The 6 permutations of \( \Pi_5 \) appear in \( \mathcal{M}(gggq \overline{q}) \) and they come from \( 3! \).

### 7.2 Field theory

The 5-point amplitudes have historical significance as the gluon was discovered via \( e^+e^- \rightarrow 3 \text{ jets} \) at PETRA. But also for a different reason.

At the end of the 1970’s the 5-point tree level QCD and QED amplitudes have been computed using brute force Feynman diagram techniques. The calculation is very difficult especially due to the large number of terms that have to be controlled. Later it was found that these amplitudes can be algebraically manipulated into a simple form which also exhibits factorization. This fact stimulated research during the 1980’s which led to insights into the cause behind the simplicity, and led to the discovery of powerful methods for calculating amplitudes with loops and higher point amplitudes (See section 4.1).

Considering QCD and QED processes not including leptons, we have the 9 processes in Table 7. The diagrams for the QCD processes are given in Fig. 24. All of the other processes can be obtained by crossing.

Using the techniques and results of section 4.1, the 5-point squared amplitudes may be computed. For example, looking at Eq. (4.13), the \( gg \rightarrow ggg \) squared amplitude can be written at once:

\[ \sum_{\text{helicities}} \sum_{\text{colors}} |\mathcal{M}|^2(g_1, g_2, g_3, g_4, g_5) \sim g^6 S_4 \sum \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}} \quad (7.9) \]
Figure 24: Classes of Feynman diagrams for the 5 particle QCD amplitudes. Each drawing represents all of the diagrams that can be obtained from it by permuting the external legs.

Table 7: 9 QCD and QED processes.

| QCD      | QED      | QCD+QED |
|----------|----------|---------|
| $gg \to ggg$ | $q\bar{q} \to \gamma\gamma\gamma$ | $q\bar{q} \to g\gamma\gamma$ |
| $q\bar{q} \to ggg$ | $q\bar{q} \to q\bar{q}\gamma$ | $q\bar{q} \to g\gamma\gamma$ |
| $q\bar{q}' \to q\bar{q}'\gamma$ | $q\bar{q}' \to q\bar{q}'g$ | - |

In the next section we list the squared amplitudes in the compact form that was achieved in [35].

7.2.1 The squared amplitudes

- QCD amplitudes

We follow [74], and write the QCD amplitudes.

$$|M|^2(g(k_1)\ g(k_2)\ g(k_3)\ g(k_4)\ g(k_5)) \equiv |M_1|^2(k_i, k_j, k_1, k_2, k_3) =$$

$$g^6 \frac{27 \cdot 2}{160} \frac{S_4}{\prod_{m<n} s_{mn}} \sum_{\text{perm.}} s_{ij}s_{j1}s_{12}s_{23}s_{3i}$$  \hspace{1cm} (7.10)
\[ |\mathcal{M}|^2(q(k_i) \ q(k_j) \rightarrow g(k_1) \ g(k_2) \ g(k_3)) \equiv |\mathcal{M}_2|^2(k_i, k_j, k_1, k_2, k_3) = \]

\[ 2g^6 \frac{S_3}{s_{i1}s_{i2}s_{i3}s_{j1}s_{j2}s_{j3}} \left[ 5s_{ij} - 9 \left( \frac{s_{i1}s_{j2} + s_{i2}s_{j1}}{s_{12}} + \frac{s_{i1}s_{j3} + s_{i3}s_{j1}}{s_{13}} + \frac{s_{i2}s_{j3} + s_{i3}s_{j2}}{s_{23}} \right) \right] \]

\[ + \frac{81}{s_{ij}} \left( \frac{2s_{i3}s_{j3}(s_{i1}s_{j2} + s_{i2}s_{j1})}{s_{13}s_{23}} + \frac{2s_{i1}s_{j1}(s_{i2}s_{j3} + s_{i3}s_{j2})}{s_{12}s_{13}} + \frac{2s_{i2}s_{j2}(s_{i1}s_{j3} + s_{i3}s_{j1})}{s_{12}s_{23}} \right) \]

(7.11)

\[ |\mathcal{M}|^2(q(k_i)q'(k_j) \rightarrow g(k_1)q'(k_2)g(k_3)) \equiv |\mathcal{M}_3|^2((k_i, k_j, k_1, k_2, k_3) = g^6 \frac{2(s_{ij}^2 + s_{i2}^2 + s_{i3}^2 + s_{j1}^2)}{s_{i1}s_{j2}s_{i3}s_{j3}s_{i3}s_{23}} \]

\[ \times \left\{ C_1 \left[ (-s_{i2} - s_{j1})(s_{ij}s_{i1} - s_{i2}s_{j1}) - s_{i2}(-s_{ij}s_{i1} - s_{i2}s_{j2}) - s_{i1}(-s_{ij}s_{j2} - s_{i1}s_{12}) \right] \]

\[ -C_2 \left[ (s_{ij} + s_{12})(s_{ij}s_{i1} - s_{i2}s_{j1}) + 2s_{i1}s_{j2}(-s_{i2} - s_{j1}) + 2s_{i2}s_{j1}(-s_{i1} - s_{j2}) \right] \} \]

(7.12)

Where \( C_1 = \frac{16}{27} \) and \( C_2 = \frac{2}{27} \).

\[ |\mathcal{M}|^2(q(k_i) q(k_j) \rightarrow g(k_1) q(k_2) g(k_3)) \equiv |\mathcal{M}_4|^2(k_i, k_j, k_1, k_2, k_3) = \]

\[ |\mathcal{M}_3|^2(k_1, k_2, k_3, k_4, k_5) + |\mathcal{M}_3|^2(k_1, k_2, k_3, k_5) + g^6 \frac{2(s_{ij}^2 + s_{i1}^2)(s_{ij}s_{i1} - s_{i2}s_{j2} - s_{i2}s_{j1})}{s_{i1}s_{j2}s_{i2}s_{j1}s_{i3}s_{j3}s_{i3}s_{23}} \]

\[ \times \left\{ C_3 \left[ (s_{ij} + s_{12})(s_{ij}s_{i1} - s_{i2}s_{j1}) + 2s_{i1}s_{j2}(-s_{i2} - s_{j1}) + 2s_{i2}s_{j1}(-s_{i1} - s_{j2}) \right] \]

\[ + C_4 \left[ (s_{ij} + s_{12})(s_{ij}s_{i1} - s_{i2}s_{j1}) - 2s_{i1}s_{j2}(-s_{i2} - s_{j1}) - 2s_{i2}s_{j1}(-s_{i1} - s_{j2}) \right. \]

\[ -2s_{ij}(s_{i1}s_{i2} + s_{j2}s_{j1}) - 2s_{12}(s_{i1}s_{j1} + s_{j2}s_{i2}) \right] \}

(7.13)
Where $C_3 = \frac{10}{81}$ and $C_4 = \frac{8}{81}$.

All of the QCD 5-point squared amplitudes can be calculated from $|\mathcal{M}_1|^2$, $|\mathcal{M}_2|^2$, $|\mathcal{M}_3|^2$, and $|\mathcal{M}_4|^2$ by crossing, as shown in Table 8.

| Process | $ij \to 123$ | $|\mathcal{M}|^2$ |
|---------|-------------|----------------|
| $qq'q'g$ | $qq' \to q\bar{q}g$ | $|\mathcal{M}_3|^2 (k_i, k_j, k, k_1, k_2, k_3)$ |
|         | $q\bar{q} \to q\bar{q}q'$ | $|\mathcal{M}_3|^2 (k_1, k_2, -k_1, -k_j, k_3)$ |
|         | $qg \to qq'q'$ | $(-\frac{3}{8})|\mathcal{M}_3|^2 (k_i, -k_1, -k, k_2, k_3)$ |
| $qqqq$  | $qg \to qg$ | $|\mathcal{M}_4|^2 (k_i, k_j, k_1, k_2, k_3)$ |
|         | $q\bar{g} \to q\bar{g}g$ | $|\mathcal{M}_4|^2 (k_i, -k_2, k_1, -k_j, k_3)$ |
|         | $gg \to qq\bar{g}$ | $(-\frac{3}{8})|\mathcal{M}_4|^2 (k_i, -k_3, k_1, k_2, -k_j)$ |
| $gggg$  | $gg \to ggg$ | $|\mathcal{M}_1|^2 (k_i, k_j, k_1, k_2, k_3)$ |
| $qqgg$  | $qg \to qgg$ | $|\mathcal{M}_2|^2 (k_i, k_j, k_1, k_2, k_3)$ |
|         | $gg \to qgg$ | $(-\frac{3}{8})|\mathcal{M}_2|^2 (k_i, -k_1, -k_j, k_2, -k_3)$ |

Table 8: The QCD processes which are related by crossing.

- **QED amplitudes**

We follow [34], and write the QED amplitudes.

$$|\mathcal{M}|^2(q(k_i) \bar{q}(k_j) \to \gamma(k_1) \gamma(k_2) \gamma(k_3)) = 2e^6 S_3 \frac{s_{ij}}{s_{i1}s_{i2}s_{i3}s_{j1}s_{j2}s_{j3}}$$ \quad (7.14)

$$|\mathcal{M}|^2(q(k_i) \bar{q}(k_j) \to q'(k_1) \bar{q}'(k_2) \gamma(k_3)) =$$

$$-\frac{e^6}{2} \left( -\frac{s_{12}}{s_{23}s_{13}} - \frac{s_{2j}}{s_{23}s_{j3}} + \frac{s_{21}}{s_{23}s_{i3}} + \frac{s_{j1}}{s_{13}s_{j3}} - \frac{s_{ij}}{s_{j3}s_{i3}} \right) \frac{s_{i2}^2 + s_{j1}^2 + s_{j2}^2}{s_{ij}s_{i2}}$$ \quad (7.15)
\[ |\mathcal{M}|^2(q(k_i) \bar{q}(k_j) \rightarrow q(k_1) \bar{q}(k_2) \gamma(k_3)) = \]
\[-e^6 \left( \frac{s_{12}}{s_{23}s_{13}} - \frac{s_{j2}}{s_{23}s_{j3}} + \frac{s_{21}}{s_{13}s_{23}} + \frac{s_{j1}}{s_{13}s_{j3}} - \frac{s_{ij}}{s_{j3}s_{ij}} \right) \]
\[ \times \frac{s_{i2}s_{j1}(s_{12}^2 + s_{j1}^2) + s_{j1}s_{i2}(s_{j1}^2 + s_{i2}^2) + s_{ij}s_{12}(s_{ij}^2 + s_{12}^2)}{s_{ij}s_{12}s_{i2}s_{j1}} \]
\(7.16\)

- QCD+QED amplitudes

\[ |\mathcal{M}|^2(q q \rightarrow g \gamma \gamma) = 4 g^2 \frac{e^2}{3} |\mathcal{M}|^2(q q \rightarrow \gamma \gamma \gamma) \] (7.17)

\[ |\mathcal{M}|^2(q(k_i) \bar{q}(k_j) \rightarrow g(k_1) \gamma(k_2) \gamma(k_3)) = 2g^4e^2S_3 \frac{1}{s_{i1}s_{12}s_{2j}s_{i3}s_{3j}} \] (7.18)

\[ |\mathcal{M}|^2(g g \rightarrow g \gamma \gamma) = |\mathcal{M}|^2(g g \rightarrow g g \gamma) = 0 \] (7.19)

7.3 String theory

References: We follow [2].

7.3.1 Generalization of the Veneziano amplitude

The 5-point string amplitudes have been calculated (see [2]). The two universal amplitudes were explicitly shown to have the same stringy form factor. These amplitudes are expressed in terms of 2 generalized hypergeometric functions.

The two hypergeometric functions are:

\[ f_1 = \int_0^1 dx \int_0^1 dy \ x^{\delta_2-1}y^{\delta_5-1}(1-x)^{\delta_3}(1-y)^{\delta_4}(1-xy)^{\delta_1-\delta_3-\delta_4} \] (7.20)

\[ f_2 = \int_0^1 dx \int_0^1 dy \ x^{\delta_2}y^{\delta_5}(1-x)^{\delta_3}(1-y)^{\delta_4}(1-xy)^{\delta_1-\delta_3-\delta_4-1} \] (7.21)
These two functions can be written as a sum:

\[ f_1 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(-\hat{s}_{35} + n)}{\Gamma(-\hat{s}_{35})} B(\hat{s}_{23} + n, \hat{s}_{34} + 1) B(\hat{s}_{45} + 1, \hat{s}_{51} + n) \]  

\[ f_2 = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{\Gamma(-\hat{s}_{35} + n)}{\Gamma(-\hat{s}_{35} + 1)} B(\hat{s}_{23} + n, \hat{s}_{34} + 1) B(\hat{s}_{45} + 1, \hat{s}_{51} + n) \]  

The beta functions can be expanded as a sum of their poles (Eq. (5.29)):

\[ f_1 = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{j=0}^{\min\{n,n'\}} \gamma(s_{35}, j) \frac{\gamma(s_{34}, n-j)}{\hat{s}_{23} + n} \frac{\gamma(s_{45}, n'-j)}{\hat{s}_{51} + n'} \]  

\[ f_2 = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{j=1}^{\min\{n,n'\}} \gamma(s_{35} - 1, j - 1) \frac{\gamma(s_{34}, n-j)}{\hat{s}_{23} + n} \frac{\gamma(s_{45}, n'-j)}{\hat{s}_{51} + n'} \]  

Where \( \gamma(s, n) \equiv \frac{1}{n!} \frac{\Gamma(-\hat{s} + n)}{\Gamma(-\hat{s})} \). So there is a sum over double poles implying exchanges of two resonances, Fig. 25.

We make the following definitions:

\[ V^{(5)}(s_j) \equiv s_2 s_5 f_1 + \frac{1}{2} \left( s_2 s_3 + s_4 s_5 - s_1 s_2 - s_3 s_4 - s_1 s_5 \right) f_2 \]  

\[ P^{(5)}(s_j) \equiv f_2 \]  

\[ \epsilon(i, j, m, n) \equiv \alpha^2 \epsilon_{\alpha \beta \mu \nu} k_{i}^{\alpha} k_{j}^{\beta} k_{m}^{\mu} k_{n}^{\nu} \]  

Then a universal sub-amplitude can be written as (see also section 4.2.3):

\[ m_{\text{string}} = B_m m_{\text{QCD}} = \left[ V^{(5)}(s_i) - 2i \epsilon(1, 2, 3, 4) P^{(5)}(s_i) \right] m_{\text{QCD}} \]
Compare this with the 6-gluon case of Eq. (8.6).

One difference from the 4-point amplitude, is the inclusion of the antisymmetric tensor. It will be convenient to write the amplitudes in terms of:

\[
C(k_1, k_2, k_3, k_4, k_5) \equiv \frac{V^{(5)}(s_j) - 2i P^{(5)}(s_j) \epsilon(1, 2, 3, 4)}{[12][23][34][45][51]} \tag{7.30}
\]

\[
C_\lambda \equiv C(k_{1\lambda}, k_{2\lambda}, k_{3\lambda}, k_{4\lambda}, k_{5\lambda}) \tag{7.31}
\]

From Eqs. (7.29), (4.7), (4.17) the two universal amplitudes can then be written as:

\[
m_{\text{string}} = ig^3 \langle IJ \rangle^4 C_\lambda \tag{7.32}
\]

\[
m_{\text{string}} = ig^3 \langle IJ \rangle^4 \langle qI \rangle^3 \langle \bar{q}I \rangle C_\lambda \tag{7.33}
\]

Where \( I, J \) stand for the gluons with negative helicity.

### 7.3.2 Squaring the amplitudes

Written below are the two matrices that arise from squaring and summing over colors. The \(12 \times 12\) matrix \( S_{\lambda\lambda'} \) corresponds to the permutations \( \Pi_5 \), and the \(6 \times 6\) matrix \( P_{\lambda\lambda'} \) to \( \Pi_q \).

The entries for the matrices are given in Tables 9 and 10.

From Eqs. (4.5) and (4.16),

\[
S_{\lambda\lambda'} \equiv \sum_{a_1, \ldots, a_5} t^{a_1, a_2, a_3, a_4, a_5 \lambda} [t^{a_1, a_2, a_3, a_4, a_5 \lambda'}]^* \tag{7.34}
\]

\[
= \begin{bmatrix}
D & X & Y & X & Y & -X & X & -Y & Y & 0 & Y & -X \\
X & D & -X & Y & X & Y & -Y & X & 0 & Y & X & -Y \\
Y & -X & D & X & Y & X & -Y & -Y & -X & -Y & 0 \\
X & Y & X & D & -X & Y & Y & 0 & -X & Y & X & Y \\
Y & X & Y & -X & D & X & 0 & Y & Y & -Y & -X & -X \\
-X & Y & X & Y & D & -Y & X & -Y & 0 & Y & -X & -Y \\
X & -Y & X & Y & 0 & -Y & D & X & 0 & X & -X & Y \\
-Y & X & -Y & 0 & Y & X & X & D & Y & Y & -X & Y \\
Y & 0 & -Y & -X & Y & -X & X & Y & 0 & -X & Y & X \\
0 & Y & -X & Y & -X & Y & X & -X & D & -X & -Y \\
Y & X & -Y & X & -Y & 0 & -X & -Y & Y & -X & D & X \\
-X & -Y & 0 & Y & -X & Y & Y & X & X & -Y & X & D
\end{bmatrix}
\]

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\[ \mathcal{P}_{\lambda'\lambda} \equiv \begin{bmatrix} D_q & X_q & X_q & Y_q & Y_q & Z_q \\ X_q & D_q & Y_q & Z_q & X_q & Y_q \\ X_q & Y_q & D_q & X_q & Z_q & Y_q \\ Y_q & Z_q & X_q & D_q & Y_q & X_q \\ Y_q & X_q & Z_q & Y_q & D_q & X_q \\ Z_q & Y_q & Y_q & X_q & X_q & D_q \end{bmatrix} \] (7.35)

| Group   | \( C_A \) | \( C_F \) | \( N_A \) | \( D/N_A \) | \( X/N_A \) | \( Y/N_A \) |
|---------|-----------|-----------|----------|------------|-------------|-------------|
| \( SU(N) \) | \( N \) | \( \frac{N^2-1}{2N} \) | \( N^2-1 \) | \( \frac{N^4-4N^2+10}{16N} \) | \( \frac{2-N^2}{8N} \) | \( \frac{1}{8N} \) |
| \( SU(3) \) | 3 | \( \frac{4}{3} \) | 8 | 55 | 38 | \( -\frac{7}{21} \) | \( \frac{1}{21} \) |
| \( SU(2) \) | 2 | \( \frac{1}{2} \) | 3 | 5 | \( \frac{16}{1} \) | \( -\frac{1}{8} \) | \( \frac{1}{21} \) |
| \( SO(N) \) | \( \frac{N-2}{2} \) | \( \frac{N-1}{4} \) | \( \frac{N(N-1)}{2} \) | \( \frac{(N-2)(N^2-2N+2)}{128} \) | \( \frac{(N-2)^2}{128} \) | \( \frac{N-2}{64} \) |
| \( SP(N) \) | \( \frac{N+2}{2} \) | \( \frac{N+1}{4} \) | \( \frac{N(N+1)}{2} \) | \( \frac{(N+2)(N^2+2N+2)}{128} \) | \( \frac{(N+2)^2}{128} \) | \( \frac{N+2}{64} \) |

Table 9: Group factors for Eq. (7.34). The \( SO(N) \) and \( SP(N) \) gauge groups are also included. From [2].

| \([N]_a \) , \([N]_b \) | \( D_q \) | \( X_q \) | \( Y_q \) | \( Z_q \) |
|-----------------|--------|--------|--------|--------|
| \([N]_a = 3 \) , \([N]_b = 2 \) | \( \frac{128}{9} \) | \( -\frac{16}{9} \) | \( \frac{2}{9} \) | \( \frac{20}{9} \) |

Table 10: Group factors for Eq. (7.35). From [2].

### 7.3.3 The squared amplitudes

We want to use a notation similar to the 4-point case, so we write the squared amplitudes with the \( U(1) \) gauge boson from the color stack \( A \), whereas [2] used \( \gamma \). We also write \( B \) for the non-abelian gauge boson from stack \( b \).

- 5 gauge bosons
\[ |\mathcal{M}|^2(g_1, g_2, g_3, g_4, g_5) = 64 g_6^6 S_4 \sum_{\lambda, \lambda' \in \Pi_5} C_{\lambda} S_{\lambda \lambda} C_{\lambda'}^* \] (7.36)

See also Eq. (4.4).

\[ |\mathcal{M}|^2(A_1, g_2, g_3, g_4, g_5) = 8 g_6^6 Q_A^2 (N^2 - 1)(N^2 - 4) S_4 \times \left[ |C_1 - C_6 + C_7 + C_9|^2 + |C_2 - C_3 + C_8 + C_{10}|^2 + |C_4 - C_5 + C_{11} + C_{12}|^2 \right] \] (7.37)

\[ |\mathcal{M}|^2(A_1, A_2, g_3, g_4, g_5) = 16 g_6^6 Q_A^4 (N^2 - 1) S_4 \times \left[ |C_1 - C_6 + C_7 + C_9 - C_2 + C_3 - C_8 - C_{10} + C_4 - C_5 + C_{11} + C_{12}|^2 \right] \] (7.38)

\[ |\mathcal{M}|^2(A_1, A_2, A_3, g_4, g_5) = |\mathcal{M}|^2(A_1, A_2, A_3, A_4, g_5) = |\mathcal{M}|^2(A_1, A_2, A_3, A_4, A_5) = 0 \] (7.39)

• 3 gauge bosons + 2 quarks

\[ |\mathcal{M}|^2(g_1, g_2, g_3, q_4, \bar{q}_5) = 16 g_6^6 S_3 \sum_{\lambda \in \Pi_q} C_{\lambda} P_{\lambda \lambda} C_{\lambda}^* \] (7.40)

See also Eq. (4.15).

\[ |\mathcal{M}|^2(A_1, g_2, g_3, q_4, \bar{q}_5) = 4 g_6^6 Q_A^2 [N][N - 1] a S_3 \times \left[ |N - 1| a \left( |C_1 + C_7 + C_9| + |C_4 + C_{11} + C_{12}|^2 \right) - 2 Re \left\{ (C_1 + C_7 + C_9) (C_{11} + C_{12}) \right\} \right] \] (7.41)

\[ |\mathcal{M}|^2(A_1, A_2, g_3, q_4, \bar{q}_5) = 8 g_6^6 Q_A^4 [N][N^2 - 1] a S_3 \sum_{\lambda \in \Pi_q} |C_{\lambda}|^2 \] (7.42)
\[ |\mathcal{M}|^2 (A_1, A_2, A_3, q_4, \bar{q}_5) = 16g^6Q_A^6[N_b][N_a]s_3 \sum_{\lambda \in \Pi_q} C_{\lambda}^2 \]  
(7.43)

\[ |\mathcal{M}|^2 (g_1, g_2, B_3, q_4, \bar{q}_5) = 16g^4g_6^4[NC_F][NC_F]a s_3 \times \left\{ [C_F]_a \left( |C_2|^2 + |C_8|^2 \right) + \left[ C_F - \frac{C_A}{2} \right]_a \left( C_2C_8^* + C_8C_8 \right) \right\} \]  
(7.44)

\[ |\mathcal{M}|^2 (g_1, g_2, B_3, q_4, \bar{q}_5) = |\mathcal{M}|^2 (g_1, g_2, B_3, q_4, \bar{q}_5) |_{NCF \rightarrow NQ^2} \]  
(7.45)

\[ |\mathcal{M}|^2 (A_1, g_2, B_3, q_4, \bar{q}_5) = 16g^4g_6^4Q_A^2[NC_F][NC_F]a s_3 \left| C_2 + C_8 \right|^2 \]  
(7.46)

\[ |\mathcal{M}|^2 (A_1, g_2, B_3, q_4, \bar{q}_5) = |\mathcal{M}|^2 (A_1, g_2, B_3, q_4, \bar{q}_5) |_{NCF \rightarrow NQ^2} \]  
(7.47)

### 7.3.4 Low energy limit

From the expansions

\[ f_1 = \frac{1}{s_2s_5} - \zeta(2) \left( \frac{s_3}{s_5} + \frac{s_4}{s_2} \right) + \zeta(3) \left( -s_1 + s_3 + s_4 + \frac{s_4^2 + s_4s_5}{s_2} + \frac{s_3^2 + s_2s_3}{s_5} \right) + \mathcal{O}(\alpha^2) \]  
(7.48)

\[ f_2 = \zeta(2) - \zeta(3) (s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}(\alpha^2) \]  
(7.49)

we have:

\[ C_1 \equiv C(k_1, k_2, k_3, k_4, k_5) \approx \frac{1 - \frac{\zeta(2)}{2} (s_1s_2 + s_2s_3 + s_3s_4 + s_4s_5 + s_5s_1 + 4i\epsilon(1234))}{[12][23][34][45][51]} \]  
(7.50)

\[ C_1 - C_6 + C_7 + C_9 \xrightarrow{s_i \rightarrow 0} 0 \]  
(7.51)

\[ C_2 - C_3 + C_8 + C_{10} \xrightarrow{s_i \rightarrow 0} 0 \]  
(7.52)

\[ C_4 - C_5 + C_{11} + C_{12} \xrightarrow{s_i \rightarrow 0} 0 \]  
(7.53)

So that Eqs. (7.37) and (7.38) can be seen to vanish in the field theory limit as required.
8 Higher point amplitudes

References: In large parts we follow [14].

From section 4.2 it is seen that higher point universal amplitudes can be computed by generalizing the integral representation of the Veneziano amplitude to multiple integrals. As we have seen for the 5-point case, these integrals have the form of multiple hypergeometric functions. The $n$-gluon amplitude can be expressed in terms of generalized hypergeometric functions of the form:

$$F\left[\begin{array}{c} n_a \\ n_{ab} \end{array}\right] = \int_0^1 dx_1 \ldots \int_0^1 dx_{n-3} \prod_{a=1}^{N-3} x_a^{1+a-n+n_a} \prod_{b=a}^{n-3} x_a^{2\alpha'k_{b+3}(k_1+\sum_{j=a+3}^{\pm 3} k_j)}$$

$$\times \left(1 - \prod_{j=a}^b x_j\right)^{2\alpha'k_{2+a+b+n_{ab}}}$$

(8.1)

There are $(n-3)!$ independent hypergeometric functions. The number $n-3$ comes, due to the $PSL(2,R)$ invariance, from fixing three coordinates.

For example for $n = 4, 5$ we have:

$$F\left[\begin{array}{c} n_1 \\ n_{11} \end{array}\right] = \int_0^1 dx_1 \, x_1^{-2+s_{23}+n_1} (1 - x_1)^{s_{12}+n_{11}} = \frac{2F_1\left[\begin{array}{c} s_{23}+n_1-1,-s_{12}-n_{11} \\ s_{23}+n_1 \end{array}\right]}{s_{23}+n_1-1}$$

(8.2)

\(^{15}(n-3)!\) is also the number of independent string subamplitudes. The relations between the string subamplitudes reduce to the Kleiss-Kuijf and BCJ relations in the field theory limit. This gives a neat explanation to why there are precisely $(n-3)!$ independent field theory subamplitudes. See [16, 17, 2].
and

\[ F\left[ \begin{array}{c} n_1, n_2 \\ n_{11}, n_{12}, n_{22} \end{array} \right] = \]

\[
\int_0^1 \int_0^1 dx_1 dx_2 x_1^{-3+s_{23}+n_1} x_2^{-3+s_{15}+n_2} (1-x_1)^{s_{34}+n_{11}} (1-x_2)^{s_{45}+n_{22}} (1-x_1 x_2)^{s_{35}+n_{12}} = \]

\[
\frac{\Gamma(s_{23} + n_1 - 2)}{\Gamma(s_{23} + s_{34} + n_1 + n_{11} - 1)} \frac{\Gamma(s_{15} + n_2 - 1)}{\Gamma(s_{15} + s_{45} + n_2 + n_{22})} \times 3F_2 \left[ \begin{array}{c} s_{23} + n_1 - 2, s_{15} + n_2 - 1, -s_{35} - n_{12} \\ s_{23} + s_{34} + n_{11} - 1, s_{15} + s_{45} + n_2 + n_{22} - 1 \end{array} ; 1 \right] (8.3) \]

In section 7.3.1 we used the definitions:

\[ f_1 = F\left[ \begin{array}{c} 2,1 \\ 0,0,0 \end{array} \right], \quad f_2 = F\left[ \begin{array}{c} 3,2 \\ 0,-1,0 \end{array} \right] (8.4) \]

### 8.1 6 gluons

Eq. (8.1) gives:

\[
F\left[ \begin{array}{c} n_1, n_2, n_3 \\ n_{11}, n_{12}, n_{22}, n_{13}, n_{23}, n_{33} \end{array} \right] = \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 x_1^{-4+s_{23}+n_1} x_2^{-3+a'(k_2+k_3+k_4)^2+n_2} x_3^{-2+s_{16}+n_3} \]

\[
\times (1-x_1)^{s_{34}+n_{11}} (1-x_2)^{s_{45}+n_{22}} (1-x_3)^{s_{56}+n_{33}} \]

\[
\times (1-x_1 x_2)^{s_{346}+n_{23}} (1-x_2 x_3)^{s_{46}+n_{23}} (1-x_1 x_2 x_3)^{s_{36}+n_{13}} (8.5) \]

As in QCD, the case of 6 gluons is more complicated than 4 and 5 gluons due to the non-vanishing of the non-MHV amplitude \((1^+, 2^+, 3^-, 4^-, 5^-, 6^-)\)

The MHV amplitude can be written as:

\[
m_{\text{string}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = \left[ V^{(6)}(s_i, t_i) - 2i \sum_{k=1}^{k=5} \epsilon_k P_k^{(6)}(s_i, t_i) \right] m_Q^{(6)} (8.6) \]

In other words the stringy form factor is (see Eq. (4.47)):

\[
\mathcal{B}_6^{(MHV)} = V^{(6)}(s_i, t_i) - 2i \sum_{k=1}^{k=5} \epsilon_k P_k^{(6)}(s_i, t_i) (8.7) \]
Compare this with the 5-gluon case of Eq. (7.29).

Where the six functions \( P_k^{(6)} \), \( V^{(6)} \) can be written in terms of six generalized hypergeometric functions:

\[
P_1^{(6)} = s_6 F_2 + (s_6 + s_2 + s_5 - t_1 - t_2) F_4 + (s_2 + s_1 - s_6 - t_1) F_3 \quad (8.8)
\]

\[
P_2^{(6)} = s_6 F_2 - (s_1 - s_3 + s_5 - t_1) F_5 + (s_2 + s_5 + s_3 - t_2) F_4 +
(s_2 - s_3 + s_6 - t_1 + t_3) F_3 + (s_1 + s_3 - s_5 - t_3) F_6 \quad (8.9)
\]

\[
P_3^{(6)} = s_6 F_2 + (s_2 - s_6 - s_3 - t_1 + t_3) F_3 - (s_1 - s_3 + s_5 - t_1) F_5 +
(s_4 + s_5 - t_1) F_4 + (s_3 + s_1 - t_3) F_6 \quad (8.10)
\]

\[
P_4^{(6)} = s_6 F_2 + (-s_6 + s_1 + s_2 - t_1) F_3 + (s_4 + s_5 - t_1) F_4 - (s_1 - s_3 + s_5 - t_1) F_5 \quad (8.11)
\]

\[
P_5^{(6)} = s_6 F_2 + (s_1 + s_4 - t_1 - t_3)(F_3 - F_5) + (s_4 + s_5 - t_1) F_4 + (s_3 + s_4 - s_5 - t_3) F_5 \quad (8.12)
\]

\[
V^{(6)} = t_2 \left[ s_2 s_6 F_1 - s_2 (s_1 - s_5 - t_3) F_3 + (s_4 + s_5 - t_1) \left\{ s_6 (F_2 - F_3) -
(s_3 - s_5 + t_1 - t_3)(F_3 + F_4) - (s_1 - s_3 + s_5 - t_1) F_5 + (s_1 + s_3 - s_5 - t_3) F_6 \right\} +
\frac{1}{2} (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) P_1^{(6)}
\right.
+
\frac{1}{2} (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) P_2^{(6)}
+
\frac{1}{2} (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) P_3^{(6)}
+
\frac{1}{2} (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) P_4^{(6)}
+
\frac{1}{2} (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) P_5^{(6)}
\left. + s_2 s_5 (s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5 + s_2 s_5) F_3^{(6)} + s_2 s_5 P_2^{(6)} + (s_2 s_5 + s_2 s_5) P_3^{(6)} \right]
\quad (8.13)
\]
Where,

\[ F_1 \equiv F \begin{bmatrix} 3,2,1 \\ 0,0,0,0,0,0 \end{bmatrix}, \quad F_3 \equiv F \begin{bmatrix} 4,3,2 \\ 0,0,0,0,0,0 \end{bmatrix}, \quad F_5 \equiv F \begin{bmatrix} 4,3,2 \\ 0,-1,0,-1,0,0 \end{bmatrix}, \]

\[ F_2 \equiv F \begin{bmatrix} 4,3,1 \\ 0,-1,0,0,0,0 \end{bmatrix}, \quad F_4 \equiv F \begin{bmatrix} 4,4,2 \\ 0,-1,0,0,0,0 \end{bmatrix}, \quad F_6 \equiv F \begin{bmatrix} 4,3,2 \\ 0,0,0,-1,-1,0 \end{bmatrix} \quad (8.14) \]

8.1.1 Squaring the amplitude

Having possession of the MHV amplitude we can square it and sum the colors as in the 5 gluon case. This time though, it will not be the full squared amplitude because of the non-MHV part. For \( n \) gluons there are \( \frac{(n-1)!}{2} \) permutations (after using reflection symmetry) giving rise to a \( \frac{(n-1)!}{2} \) \( \times \) \( \frac{(n-1)!}{2} \) matrix:

\[
S_{\lambda\lambda'}^{(n)} = \sum_{a_1,\ldots,a_n} t^{a_1\ldots a_{n}\lambda} [t^{a_1\ldots a_{n}\lambda'}]^* \quad (8.15)
\]

For 5 gluons it was a 12 \( \times \) 12 matrix with 3 independent entries. For 6 gluons it is 60 \( \times \) 60 with 10 independent entries\(^{16}\). We calculate now the diagonal entry for \( n = 5, 6 \). The diagonal is the term of leading order in \( 1/N \), and is relatively simple to calculate. Using the standard notation: \( (a_1\ldots a_n) \equiv Tr(T^{a_1}\ldots T^{a_n}) \), The diagonal term is:

\[
S_{\lambda\lambda}^{(n)} = \sum_{a_1,\ldots,a_n} t^{a_1\ldots a_{n}\lambda} [t^{a_1\ldots a_{n}\lambda}]^* =
\]

\[
\sum_{a_1,\ldots,a_n} [(a_1\ldots a_n) - (a_n\ldots a_1)] [(a_1\ldots a_n) - (a_n\ldots a_1)]^* =
\]

\[
2 \sum_{a_1,\ldots,a_n} [(a_1\ldots a_n)(a_1\ldots a_n)^* - (a_1\ldots a_n)(a_1\ldots a_n)^*] =
\]

\[
2 \sum_{a_1,\ldots,a_n} [(a_1\ldots a_n)(a_n\ldots a_1) - (a_1\ldots a_n)(a_1\ldots a_n)] \quad (8.16)
\]

The first term gives the leading order in \( 1/N \), and has a closed formula for arbitrary \( n \)

\(^{16}\)In field theory, the squaring of the amplitude is very much simplified by the use of the dual Ward identity (DWI) Eq. (4.6). In the squared amplitude, the \( O(1/N^2) \) correction terms vanish completely for 4 and 5 gluons (see Eqs. (4.9), (4.12), (4.13), (4.11)). This simplification does not occur in string amplitudes since these do NOT satisfy the DWI. The DWI follows by replacing one gluon with a photon and demanding that the amplitude vanish. Since in string theory the gluon can mix with the photon, the identity does not hold.
(see Eq. 25 of [43]):

\[
\sum_{a_1, \ldots, a_n} (a_1 \ldots a_n)(a_n \ldots a_1) = \frac{(N^2 - 1)^n + (-1)^n(N^2 - 1)}{(2N)^n}
\] (8.17)

For \( n = 6 \) this gives:

\[
\sum_{a_1, \ldots, a_6} (a_1 \ldots a_6)(a_6 \ldots a_1) = (N^2 - 1) \cdot \frac{N^8 - 5N^6 + 10N^4 - 10N^2 + 5}{64N^4}
\] (8.18)

As for the second term in Eq. (8.16), we calculated it for \( n = 6 \) using Eqs. (F.27), (F.28). We got:

\[
\sum_{a_1, \ldots, a_6} (a_1 \ldots a_6)(a_1 \ldots a_6) = (N^2 - 1) \cdot \frac{N^4 + 10N^2 + 5}{64N^4}
\] (8.19)

Plugging Eqs. (8.18), (8.19) in (8.16) gives:

\[
S^{(6)}_{\lambda\lambda} = 2(N^2 - 1) \cdot \frac{N^8 - 5N^6 + 9N^4 - 20N^2}{64N^4}
\] (8.20)

Notes:

- We checked this procedure for \( n = 5 \) and got the correct value for the diagonal \( S^{(5)}_{\lambda\lambda} \) as in Table 9:

\[
D \equiv S^{(5)}_{\lambda\lambda} = (N^2 - 1) \cdot \frac{N^4 - 4N^2 + 10}{16N}
\] (8.21)

- This procedure can be continued for larger \( n \). In general:

\[
S^{(n)}_{\lambda\lambda'} = \frac{1}{2^{n-1}} N^{n-2}(N^2 - 1) \left[ a_0 + \sum_{i=1}^{2[n/2]-1} \frac{\alpha_i}{N^{2i}} \right]
\] (8.22)

Where \( \alpha_i \) is an integer and \( [a] \) is the entire of \( a \). See [37].

8.2 \( n \) gluons: low energy expansion

In [14] a method for obtaining the leading stringy correction to the \( n \)-gluon MHV amplitude was introduced.
The result is:

\[
m_{\text{string}}(1^-, 2^-, 3^+, 4^+ \ldots, n^+) = \left(1 - \frac{\pi^2}{12} Q^{(n)}\right) m^{(n)}_{\text{QCD}} + O(\alpha'^3) \tag{8.23}
\]

So that the stringy form factor is:

\[
\mathfrak{M}_n^{(MHV)} \approx \left(1 - \frac{\pi^2}{12} Q^{(n)}\right) + O(\alpha'^3) \tag{8.24}
\]

For \( n = 4, 5, 6 \) we have:

\[
Q^{(4)} = s_1 s_2 \tag{8.25}
\]

\[
Q^{(5)} = s_1 s_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + 4i\epsilon(1, 2, 3, 4) \tag{8.26}
\]

\[
Q^{(6)} = s_1 s_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + t_1 t_2 + t_1 t_2 + t_1 t_2 + s_1 s_2 + s_1 s_2 + s_1 s_2 + 4i[\epsilon(1, 2, 3, 4) + \epsilon(1, 2, 3, 4) + \epsilon(1, 2, 3, 4) + \epsilon(1, 2, 3, 4)] \tag{8.27}
\]

See Eqs. (5.81), (7.50).

For any \( n \):

\[
Q^{(n)} = \sum_{k=1}^{E(\frac{n}{2} - 1)} \{[1][k][2]_k\} - \sum_{k=3}^{E(\frac{n}{2} - 1)} \{[1][k][2]_{k-2}\} + C^{(n)} + 4i \sum_{k<l<i<j<N} \epsilon(k, l, i, j) \tag{8.28}
\]

Where

\[
C^{(n)} = \left\{ \begin{array}{l}
-\{[1]_{\frac{n}{2}-2}[\frac{n}{2} + 1]_{\frac{n}{2}-2}\}, \quad n > 4, \text{ even } n \\
-\{[1]_{\frac{n+1}{2}}[\frac{n+1}{2} - 1]_{\frac{n+1}{2}-2}\}, \quad n > 5, \text{ odd } n
\end{array} \right\} \tag{8.29}
\]
9 Direct production of Regge states

References: We follow [20].

Until now we considered amplitudes where the outgoing particles were standard model particles and the string states were internal states. When the center of mass energy of the collision exceeds the mass of the first Regge state, these particles may be directly produced as external particles (Fig. 26). In this section we write down the 4-point squared amplitudes for the direct production of one Regge state from the first excited level ($n = 1$).

The external Regge states from the $n = 1$ level which appear in the amplitudes below are $B(J = 2)$, $\Phi(J = 0)$, $W(J = 1)$, $Q(J = 1/2)$ and $Q^*(J = 3/2)$. $B_0$, $\Phi_0$, and $W_0$ are the corresponding $U(1)$ fields. The $B$ particle should not be confused with the $B$ particles of sections 7.3.3 and 5.2.2. Note that $W$ and $W_0$ do not couple to purely gluonic processes at tree level.

9.1 The squared amplitudes

- $gg$ initial state

$$|\mathcal{M}|^2(gg \rightarrow gB) = \frac{5g^4}{8} \left(V_s^2 + V_t^2 + V_u^2\right) \frac{(\hat{s} - 1)^4 + (\hat{t} - 1)^4 + (\hat{u} - 1)^4}{\hat{s}\hat{t}\hat{u}} \tag{9.1}$$

$$|\mathcal{M}|^2(gg \rightarrow gB_0) = \frac{3}{4} g^4 Q_A^2 \left(V_s + V_t + V_u\right)^2 \frac{(\hat{s} - 1)^4 + (\hat{t} - 1)^4 + (\hat{u} - 1)^4}{\hat{s}\hat{t}\hat{u}} \tag{9.2}$$

\footnote{There is also the particle $\Omega(J = 0)$ which does not couple to purely gluonic processes or to two quarks, thus it does not appear in the amplitudes below. $\Omega$ does however couple to two gluinos.}
\[ |\mathcal{M}|^2(gg \rightarrow g\Phi) = \frac{5g^4}{8} \left( V_s^2 + V_t^2 + V_u^2 \right) \frac{s^4 + t^4 + u^4 + 1}{situ} \]  \hspace{1cm} (9.3)

\[ |\mathcal{M}|^2(gg \rightarrow g\Phi_0) = \frac{3}{4} g^4 Q_A^2 \left( V_s + V_t + V_u \right)^2 \frac{s^4 + t^4 + u^4 + 1}{situ} \]  \hspace{1cm} (9.4)

\[ |\mathcal{M}|^2(gg \rightarrow \bar{q}Q) = \frac{g^4}{4} \left[ \frac{3}{32} (V_t + V_u)^2 + \left( \frac{5}{96} + \frac{Q_A^2}{8} \right) (V_t - V_u)^2 \right] \frac{(s - 1) + (t - 1)u^3 + (u - 1)t^3}{situ} \]  \hspace{1cm} (9.5)

\[ |\mathcal{M}|^2(gg \rightarrow \bar{q}Q^*) = g^4 \left[ \frac{3}{32} (V_t + V_u)^2 + \left( \frac{5}{96} + \frac{Q_A^2}{8} \right) (V_t - V_u)^2 \right] \frac{(s - 1)^2 + (t - 1)^3u + (u - 1)^3t}{situ} \]  \hspace{1cm} (9.6)

- \textit{gq initial state}

\[ |\mathcal{M}|^2(qg \rightarrow qB) = -\frac{g^4}{16} \left[ (V_s - V_u)^2 + \left( \frac{5}{9} + \frac{4Q_A^2}{3} \right) (V_s + V_u)^2 \right] \frac{(s - 1)^2 + (u - 1)^2(t + 4su)}{situ} \]  \hspace{1cm} (9.7)

\[ |\mathcal{M}|^2(qg \rightarrow qB_0) = -\frac{g^4 Q_A^2}{12} \left( V_s + V_u \right)^2 \frac{(s - 1)^2 + (u - 1)^2(t + 4su)}{situ} \]  \hspace{1cm} (9.8)

\[ |\mathcal{M}|^2(qg \rightarrow q\Phi) = -\frac{g^4}{4} \left[ (V_s - V_u)^2 + \left( \frac{5}{9} + \frac{4Q_A^2}{3} \right) (V_s + V_u)^2 \right] \frac{s^2 + u^2}{t} \]  \hspace{1cm} (9.9)

\[ |\mathcal{M}|^2(qg \rightarrow q\Phi_0) = -\frac{g^4 Q_A^2}{3} \left( V_s + V_u \right)^2 \frac{s^2 + u^2}{t} \]  \hspace{1cm} (9.10)
\[ |\mathcal{M}|^2(qg \rightarrow qW) = -\frac{3g^4}{16} \left[ (V_s - V_u)^2 + \left( \frac{5}{9} + \frac{4Q_A^2}{3} \right) (V_s + V_u)^2 \right] \frac{(\hat{s} - 1)^2 + (\hat{u} - 1)^2}{\hat{s}\hat{u}} \]  
(9.11)

\[ |\mathcal{M}|^2(qg \rightarrow qW_0) = -\frac{g^4Q_A^2}{4} (V_s + V_u)^2 \frac{(\hat{s} - 1)^2 + (\hat{u} - 1)^2}{\hat{s}\hat{u}} \]  
(9.12)

\[ |\mathcal{M}|^2(qg \rightarrow gQ) = -\frac{g^4}{16} \left[ (V_s + V_u)^2 + \left( \frac{5}{9} + \frac{4Q_A^2}{3} \right) (V_s - V_u)^2 \right] \frac{(\hat{t} - 1)^3 + (\hat{u} - 1)^3\hat{s} + (\hat{s} - 1)^3\hat{u} - \hat{u}^2}{\hat{s}\hat{t}\hat{u}} \]  
(9.13)

\[ |\mathcal{M}|^2(qg \rightarrow gQ^*) = -\frac{g^4}{4} \left[ (V_s + V_u)^2 + \left( \frac{5}{9} + \frac{4Q_A^2}{3} \right) (V_s - V_u)^2 \right] \frac{(\hat{t} - 1) + (\hat{u} - 1)\hat{s}^3 + (\hat{s} - 1)^3\hat{u}^3}{\hat{s}\hat{t}\hat{u}} \]  
(9.14)

- \text{\textbullet q\bar{q} initial state}

\[ |\mathcal{M}|^2(\bar{q}q \rightarrow gX) = |\mathcal{M}|^2(\bar{q}q \rightarrow gX) = -\frac{8}{3} |\mathcal{M}|^2(qg \rightarrow qX)(s \rightarrow u, u \rightarrow t, t \rightarrow s) \]  
(9.15)

Where \( X = B, \Phi, W \).

\[ |\mathcal{M}|^2(\bar{q}q \rightarrow \bar{q}Q) = g^4 \left\{ \frac{t^2}{4} |su|Q_{su} + \tilde{Q}_{su}|^2 + ut|Q_{su}|^2 + st|\tilde{Q}_{su}|^2 \right\} + \frac{t^2}{12} \left[ su(Q_{su} + \tilde{Q}_{su})(\bar{Q}_{s} + \tilde{Q}_{su})^* + utQ_{su}\bar{Q}_{su} + st\tilde{Q}_{su}Q_{su} \right] - \frac{u^2}{4} \left[ st|R_{st} + \tilde{R}_{st}|^2 + ut|R_{st}|^2 + su|\tilde{R}_{st}|^2 \right] \} \{ s \leftrightarrow u \} \]  
(9.16)
\[ |\mathcal{M}|^2(\bar{q}q \rightarrow \bar{q}'Q') = \]
\[ -\frac{g^4 t^2}{4} \left( su|Q_{su} + \bar{Q}_{su}|^2 + ut|Q_{su}|^2 + st|\bar{Q}_{su}|^2 \right) + (Q \rightarrow R; u \leftrightarrow t) \]
\[ (9.17) \]

\[ |\mathcal{M}|^2(\bar{q}q \rightarrow \bar{q}Q^*) = \]
\[ g^4 \left\{ -\frac{(M_s^2 - t)^2}{4} \left( su|Q_{su} + \bar{Q}_{su}|^2 + ut|Q_{su}|^2 + st|\bar{Q}_{su}|^2 \right) + \frac{t(M_s^2 - t)}{6} |uQ_{su} - s\bar{Q}_{su}|^2 \right. \]
\[ + \frac{(M_s^2 - t)^2}{12} \left[ su(Q_{su} + \bar{Q}_{su})(Q_{su} + \bar{Q}_{su})^* + utQ_{su}\bar{Q}_{su}^* + st\bar{Q}_{su}Q_{su}^* \right] \]
\[ + \frac{t(M_s^2 - t)}{18} (uQ_{su} - s\bar{Q}_{su})(u\bar{Q}_{su} - sQ_{su})^* \]
\[ -\frac{(M_s^2 - u)^2}{4} \left( st|R_{st} + \bar{R}_{st}|^2 + ut|R_{st}|^2 + su|R_{st}|^2 \right) + \frac{u(M_s^2 - u)}{6} |tR_{st} - s\bar{R}_{st}|^2 \} + \{ s \leftrightarrow u \} \]
\[ (9.18) \]

\[ |\mathcal{M}|^2(\bar{q}q \rightarrow \bar{q}Q'^*) = g^4 \left\{ -\frac{(M_s^2 - t)^2}{4} \left( su|Q_{su} + \bar{Q}_{su}|^2 + ut|Q_{su}|^2 + st|\bar{Q}_{su}|^2 \right) + \right. \]
\[ \frac{t(M_s^2 - t)}{6} |uQ_{su} - s\bar{Q}_{su}|^2 + (Q \rightarrow R; u \leftrightarrow t) \} \]
\[ (9.19) \]

Where we defined:

\[ Q_{s,u} \equiv \alpha' e^{\phi_{10}} \int_0^1 dx x^{-s} (1 - x)^{-u-1} Z_{inst}^{ba}(x) I(x, \theta^j) \]
\[ (9.20) \]

\[ \bar{Q}_{s,u} \equiv \alpha' e^{\phi_{10}} \int_0^1 dx x^{-s-1} (1 - x)^{-u} Z_{inst}^{ba}(x) I(x, \theta^j) \]
\[ (9.21) \]

\[ R_{s,u} \equiv \alpha' e^{\phi_{10}} \int_0^1 dx x^{-s} (1 - x)^{-u-1} Z_{inst}^{ba}(x) I(x, \theta^j, \nu^j) \]
\[ (9.22) \]

\[ \bar{R}_{s,u} \equiv \alpha' e^{\phi_{10}} \int_0^1 dx x^{-s-1} (1 - x)^{-u} Z_{inst}^{ba}(x) I(x, \theta^j, \nu^j) \]
\[ (9.23) \]
10 Summary

The main computational part of this thesis is concerned with suggesting methods to calculate the decay widths of the Regge states, and with the formalism for treating amplitudes containing exchanges of higher spin particles. The decay widths have direct relevance to collider phenomenology and the LHC. There is a question regarding the range of validity of the computations, since the methods are field theoretical.

We discussed the property of equal form factors and emphasized the concept of universal amplitudes - string amplitudes which are completely independent of the geometry of the extra dimensions and the D-brane setup. The universality property is one of the most interesting aspects of certain classes of D-brane models. The existence of universal amplitudes is very important, particularly in light of the landscape problem of string theory.

The field of string amplitude calculations is an interesting field of research irrespective of phenomenology. In recent years it has witnessed many developments, together with the field of amplitude calculations in gauge theory, supersymmetry, and gravity. From a different perspective, string theory is a distinctive example of a consistent higher spin theory.

Another exciting feature, is the possibility to have a low string/gravity scale in type I or II string theory. This is done by hypothesizing that some of the extra dimensions are large. This has immediate relevance to LHC physics.

Let us conclude by imagining a possible (highly idealized) discovery scenario: Bumps in the dijet crosssection at a constant spacing of $E_{cm}^2$ are discovered at the LHC. The bumps are shown to arise from the universal amplitudes, and are conjectured to originate from string excitations. The angular distributions are analyzed and are found to correspond to exchanges of higher spin resonances. Further confirmation comes from similar bumps which are found in additional processes: processes with EW gauge bosons, the 3-jet signal, direct production of Regge states etc.. Also $Z'$ bosons are discovered suggesting that the underlying gauge symmetry is $U(N) \sim SU(N) \times U(1)$. The signals are analyzed more closely and effects coming from the model dependent amplitudes (e.g quark scattering) are extracted. For example, traces of KK or winding states are seen. These model dependent signals help in discriminating between different string models. Later on, future electron and photon colliders can likewise discover similar signals, and perform precision measurements.

We are eagerly waiting for results from the LHC and other experiments, and hoping that they correspond to new physics.
A 4-point helicity amplitudes

References: [1].

We write the four point helicity amplitudes near an s-channel pole. We do the algebra of expanding the Veneziano amplitudes and simplifying the group theoretical factors. We recall that the $(+ + + +)$ and $(- + + +)$ helicity amplitudes vanish, which leaves only the MHV amplitudes: $(- + + -)$. The simple kinematics of the process $ij \to kl$ give:

$$
\langle ij \rangle = \langle kl \rangle = \sqrt{s}
$$

$$
\langle ik \rangle = \langle jl \rangle = \sqrt{t}
$$

$$
\langle il \rangle = \langle jk \rangle = \sqrt{u},
$$

(A.1)

In the following, we use the notation $M_{m,m'}$ for the helicity amplitudes.

A.1 $M(g, g, g, g)$

We have three helicity configurations: $(m, m') = (0, 0), (2, -2), (2, 2)$.

- $M_{0,0} = M(g^- g^- \to g^+_3 g^+_4)$

$$
M_{0,0} = 4g^2 \langle 12 \rangle^4 \left\{ \frac{V_t}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \ Tr[T^{a_1}T^{a_2}T^{a_3}T^{a_4} + T^{a_4}T^{a_3}T^{a_2}T^{a_1}] \\
+ \frac{V_s}{\langle 14 \rangle \langle 42 \rangle \langle 23 \rangle \langle 31 \rangle} \ Tr[T^{a_2}T^{a_1}T^{a_3}T^{a_4} + T^{a_4}T^{a_3}T^{a_1}T^{a_2}] \\
+ \frac{V_u}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle} \ Tr[T^{a_1}T^{a_3}T^{a_2}T^{a_4} + T^{a_4}T^{a_2}T^{a_3}T^{a_1}] \right\}
$$

(A.2)

Note the $(4 - 1)! = 6$ permutations, and the reflection symmetry: $m(1, 2, 3, 4) = m(4, 3, 2, 1)$. Eq. (A.1) gives:

$$
M_{0,0} = 4g^2 \left\{ \frac{s V_t}{u} \ Tr[T^{a_1}T^{a_2}T^{a_3}T^{a_4} + T^{a_4}T^{a_3}T^{a_2}T^{a_1}] \\
+ \frac{s V_u}{t} \ Tr[T^{a_2}T^{a_1}T^{a_3}T^{a_4} + T^{a_4}T^{a_3}T^{a_1}T^{a_2}] \\
+ \frac{s^2 V_u}{ut} \ Tr[T^{a_1}T^{a_3}T^{a_2}T^{a_4} + T^{a_4}T^{a_2}T^{a_3}T^{a_1}] \right\}
$$

(A.3)
Using (Eq. (5.37)):

\begin{align*}
V_t & \xrightarrow{s \to n} \frac{1}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \hat{u} \prod_{K=1}^{n-1} (\hat{u} + K) \right] \\
V_u & \xrightarrow{s \to n} \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \hat{t} \prod_{K=1}^{n-1} (\hat{u} + K) \right]
\end{align*}

(A.4)

(A.5)

gives:

\[ M_{0.0} \simeq \frac{\hat{s}}{(n-1)!} \frac{1}{\hat{s} - n} 4g^2 \left[ \prod_{K=1}^{n-1} (\hat{u} + K) \right] \times \]

\[ Tr \left[ T^{a_1} T^{a_2} T^{a_3} T^{a_4} + T^{a_4} T^{a_3} T^{a_2} T^{a_1} + (-1)^{n-1} (T^{a_2} T^{a_1} T^{a_3} T^{a_4} + T^{a_4} T^{a_3} T^{a_2} T^{a_1}) \right] \]

\[ = 4g^2 \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \prod_{K=1}^{n-1} (\hat{u} + K) \right] \cdot \left\{ \begin{array}{ll}
Tr(\{T^{a_1}, T^{a_2}\}\{T^{a_3}, T^{a_4}\}), & \text{odd } n \\
Tr([T^{a_1}, T^{a_2}][T^{a_3}, T^{a_4}]), & \text{even } n
\end{array} \right\} \]

(A.6)

We see that near a pole, the amplitude has the nice property of being factorized as a color part times a kinematical part.

Using (see Appendix F)

\[ \{T^{a_3}, T^{a_4}\} = 4 \sum_a d^{a_3 a_4 a} T^a \]

(A.7)

\[ [T^{a_3}, T^{a_1}] = \sum_a f^{a_3 a_4 a} T^a \]

(A.8)

finally gives:

\[ \mathcal{M}_{0,0} = 4g^2 \frac{n}{(n-1)!} \frac{1}{\hat{s} - n} \left[ \prod_{K=1}^{n-1} (\hat{u} + K) \right] \cdot \left\{ \begin{array}{ll}
8 \sum_a d^{a_1 a_2 a} d^{a_3 a_4 a}, & \text{odd } n \\
\sum_a f^{a_1 a_2 a} f^{a_3 a_4 a}, & \text{even } n
\end{array} \right\} \]

(A.9)

The minus sign in the following equation (which is valid near a pole \( \hat{s} \to n \)), see Eq. (5.40):

\[ V_t = (-1)^{n-1} \frac{u}{t} V_u \]

(A.10)
and the fact that the amplitude contained only the combination \((uV_u \times \text{color} + tV_t \times \text{color})\), caused the even (odd) \(n\) resonances to contain only \(f^{a_1a_2a_3} (d^{a_1a_2a_3})\). This will happen also for the amplitudes: \(M_{2,\pm 2}\) and \(M_{2,\pm 1}\). \(f^{a_1a_2a_3}\) vanishes when at least one of the indices are \(U(1)\) fields, Eq. (F.14). Therefore \(M_{0,0}, M_{2,\pm 2},\) and \(M_{2,\pm 1}\) will not have exchanges of \(n = \text{even}\) resonances for these amplitudes.

- \(M_{2,-2} = \mathcal{M}(g_1^- g_2^+ \rightarrow g_3^- g_4^+)\)

\(M_{2,-2}\) can easily be obtained by crossing \((s \leftrightarrow t)\) of Eq. (A.3):

\[
M_{2,-2} = 4g^2 \left\{ \frac{t}{u} V_s \ T \left[ T^{a_1} T^{a_2} T^{a_3} T^{a_4} + T^{a_2} T^{a_1} T^{a_4} T^{a_3} \right] \\
\frac{t}{s} V_u T \left[ T^{a_2} T^{a_1} T^{a_3} T^{a_4} + T^{a_1} T^{a_2} T^{a_4} T^{a_3} \right] \\
+ \frac{t^2}{us} V_t T \left[ T^{a_1} T^{a_3} T^{a_2} T^{a_4} + T^{a_3} T^{a_1} T^{a_4} T^{a_2} \right] \right\} \quad (A.11)
\]

So that:

\[
M_{2,-2} = 4g^2 \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{t^2}{s^2} \prod_{K=1}^{n-1} \left( \hat{u} + K \right) \right] \left\{ 8 \sum_{a} d_{a_1a_2a}^{a_4a_3a}, \text{ odd } n \right\} \left\{ \sum_{a} f_{a_1a_2a} f_{a_3a_4a}, \text{ even } n \right\} \quad (A.12)
\]

Note that this is equal to Eq. (A.9) multiplied by \(\frac{t^2}{s^2}\).

- \(M_{2,2} = \mathcal{M}(g_1^- g_2^+ \rightarrow g_3^+ g_4^-)\)

\(M_{2,2}\) can easily be obtained from the previous equation by the substitution \(\frac{t^2}{s^2} \rightarrow \frac{u^2}{s^2}\) in the square brackets:

\[
M_{2,2} = 4g^2 \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{u^2}{s^2} \prod_{K=1}^{n-1} \left( \hat{u} + K \right) \right] \left\{ 8 \sum_{a} d_{a_1a_2a}^{a_4a_3a}, \text{ odd } n \right\} \left\{ \sum_{a} f_{a_1a_2a} f_{a_3a_4a}, \text{ even } n \right\} \quad (A.13)
\]

### A.2 \(\mathcal{M}(g,g,q,q)\)

We have four helicity configurations: \((m,m') = (2,-1), (2,1), (\frac{1}{2},\frac{1}{2}), (\frac{3}{2},-\frac{3}{2}).\)

- \(M_{2,-1} = \mathcal{M}(q_3^- g_4^+ \rightarrow g_1^- g_2^+)\)
\[ \mathcal{M}_{2,-1} = 2g^2 \delta_{\beta_3} \frac{(13)^2}{(23)(24)} [(T^{a_1} T^{a_2})_{a_3} \frac{t}{s} V_t + (T^{a_2} T^{a_1})_{a_3} \frac{u}{s} V_u] \]

\[ = 2g^2 \delta_{\beta_3} \frac{t}{u} [(T^{a_1} T^{a_2})_{a_3} \frac{t}{s} V_t + (T^{a_2} T^{a_1})_{a_3} \frac{u}{s} V_u] \]

\[ \simeq \frac{n}{(n-1)! \frac{1}{s-n}} 2g^2 \delta_{\beta_3} \frac{(T^{a_1} T^{a_2})_{a_3} + (-1)^{n-1}(T^{a_2} T^{a_1})_{a_3}}{s^2} \frac{u^2 t^2}{s^2} \left( \prod_{K=1}^{n-1} (\hat{u} + K) \right) \]

\[ \simeq 2g^2 \delta_{\beta_3} \frac{n}{(n-1)! \frac{1}{s-n}} \frac{u^2 t^2}{s^2} \left( \prod_{K=1}^{n-1} (\hat{u} + K) \right) \]

\[ \times \left\{ \begin{array}{l}
\{T^{a_1}, T^{a_2}\}_{a_3}, \text{ odd } n \\
\{T^{a_1}, T^{a_2\prime}\}_{a_3}, \text{ even } n
\end{array} \right\}. \quad (A.14) \]

So we get:

\[ \mathcal{M}_{2,-1} = 2g^2 \delta_{\beta_3} \frac{n}{(n-1)! \frac{1}{s-n}} \frac{u^2 t^2}{s^2} \left( \prod_{K=1}^{n-1} (\hat{u} + K) \right) \left\{ \begin{array}{l}
4 \sum_a \delta^{a_1 a_2 a} T^{a_3 a_4}, \text{ odd } n \\
4 \sum_a \delta^{a_1 a_2 a} T^{a_3 a_4}, \text{ even } n
\end{array} \right\} \quad (A.15) \]

- \( \mathcal{M}_{2,1} = \mathcal{M} (q_3^- \ g_1^+ \rightarrow g_1^+ \ g_3^-) \)

\( \mathcal{M}_{2,1} \) can easily be obtained from the previous equation by the substitution \( \frac{u^2 t^2}{s^2} \rightarrow \frac{t^2 u^2}{s^2} \) in the square brackets:

\[ \mathcal{M}_{2,1} = 2g^2 \delta_{\beta_3} \frac{n}{(n-1)! \frac{1}{s-n}} \frac{t^2 u^2}{s^2} \left( \prod_{K=1}^{n-1} (\hat{u} + K) \right) \left\{ \begin{array}{l}
4 \sum_a \delta^{a_1 a_2 a} T^{a_3 a_4}, \text{ odd } n \\
4 \sum_a \delta^{a_1 a_2 a} T^{a_3 a_4}, \text{ even } n
\end{array} \right\} \quad (A.16) \]

- \( \mathcal{M}_{\frac{1}{2}, \frac{1}{2}} = \mathcal{M} (q_3^- \ g_1^- \rightarrow q_1^+ \ g_2^+) \)

Can be obtained by crossing \( \mathcal{M}_{2,-1} \rightarrow \mathcal{M}_{\frac{1}{2}, \frac{1}{2}} \):

\[ \mathcal{M}_{\frac{1}{2}, \frac{1}{2}} = 2g^2 \delta_{\beta_3} \frac{S}{u} \left( T^{a_1 T^{a_2}}_{a_3} \frac{s}{t} s V_s + (T^{a_2 T^{a_1}}_{a_3} \frac{u}{t} V_u \right] \quad (A.17) \]

This gives:

\[ \mathcal{M}_{\frac{1}{2}, \frac{1}{2}} = (-1)^{n-1} 2g^2 \delta_{\beta_3} \frac{n}{(n-1)! \frac{1}{s-n}} \frac{u^2 t^2}{s^2} \left( \prod_{K=1}^{n-1} (\hat{u} + K) \right) (T^{a_2 T^{a_1}}_{a_3}) \quad (A.18) \]
\[ M_{3/2,-3/2} = M(g_4^- g_1^+ \rightarrow g_3^- g_2^-) \]

Can be obtained by crossing \( M_{2,-1} \xrightarrow{s \leftrightarrow u} M_{3/2,-3/2} : \)

\[
M_{3/2,-3/2} = 2g^2 \delta_{\beta_4}^{\alpha_4} \sqrt{\frac{l}{s}} \left[ (T^{a_1 T^{a_2}})_{\alpha_3}^{\alpha_4} \frac{l}{u} V_i + (T^{a_2 T^{a_1}})_{\alpha_3}^{\alpha_4} \frac{s}{u} V_s \right] \tag{A.19}
\]

\[
M_{3/2,-3/2} = 2g^2 \delta_{\beta_4}^{\alpha_4} \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{l}{s} \prod_{K=1}^{n-1} (\hat{u} + K) \left( T^{a_1 T^{a_2}} \right)_{\alpha_3}^{\alpha_4} \right] \tag{A.20}
\]

Note that this is equal to Eq. (A.18) multiplied by \((-1)^{n-1} \frac{l}{s u^2} \).

A.3 \( M(g, g, q, B) \)

We have four helicity configurations: \((m, m') = (2, -1), (2, 1), (1/2, 1/2), (3/2, -3/2)\).

- \( M_{2,-1}^{(B)} = M(q^- g_1^+ \rightarrow g_1^- B^+_2) \)

\[
M_{2,-1}^{(B)} = 2g_{\rho \sigma} g \frac{(13)^2}{(23)(24)} (T_{a_1}^{a_3} T_{a_2}^{a_4})_{\beta_3}^{\alpha_4} V_s
\]

\[
= 2g_{\rho \sigma} g \sqrt{\frac{l}{u}} (T_{a_1}^{a_3} T_{a_2}^{a_4})_{\beta_3}^{\alpha_4} V_s \tag{A.21}
\]

Which exhibits no poles since \( V_s \) does not.

- \( M_{2,1}^{(B)} = M(q^- q_1^+ \rightarrow g_1^+ B^-_2) \)

\[
M_{2,1}^{(B)} = 2g_{\rho \sigma} g \sqrt{\frac{u}{l}} (T_{a_1}^{a_3} T_{a_2}^{a_4})_{\beta_3}^{\alpha_4} V_s \tag{A.22}
\]

Which exhibits no poles.

- \( M_{1/2,3/2}^{(B)} = M(q^- g_1^+ \rightarrow g_1^+ B^+_2) \)

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Can be obtained by crossing $\mathcal{M}_{2,-1}^{(B)} \xrightarrow{s \leftrightarrow t} \mathcal{M}_{1,1}^{(B)}$:

$$\mathcal{M}_{1,1}^{(B)} = 2g_D p_g \sqrt{\frac{s}{u}} t^{\alpha_1} (T^{a_2})_{\beta_3} V_t$$ (A.23)

$$\mathcal{M}_{1,1}^{(B)} = 2g_D p_g g \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}} \prod_{K=1}^{n-1} (\hat{u} + K) \right] (T^{a_1})_{\alpha_4} (T^{a_2})_{\beta_3}$$

$$\mathcal{M}_{1,1}^{(B)} = 2g_D p_g g \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}} \prod_{K=1}^{n-1} (\hat{u} + K) \right] (T^{a_1})_{\alpha_4} (T^{a_2})_{\beta_3}$$ (A.24)

- $\mathcal{M}_{\frac{1}{2},-\frac{3}{2}}^{(B)} = \mathcal{M}(q_4^{-} g_1^+ \rightarrow q_3^+ B_2^{-})$

Can be obtained by crossing $\mathcal{M}_{2,-1}^{(B)} \xrightarrow{s \leftrightarrow u} \mathcal{M}_{3,\frac{3}{2}}^{(B)}$:

$$\mathcal{M}_{3,\frac{3}{2}}^{(B)} = 2g_D p_g \sqrt{\frac{t}{s}} t^{\alpha_1} (T^{a_2})_{\beta_3} V_u$$ (A.25)

$$\mathcal{M}_{3,\frac{3}{2}}^{(B)} = 2g_D p_g g \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}} \prod_{K=1}^{n-1} (\hat{u} + K) \right] (T^{a_1})_{\alpha_4} (T^{a_2})_{\beta_3}$$

$$\mathcal{M}_{3,\frac{3}{2}}^{(B)} = 2g_D p_g g \frac{n}{(n-1)!} \frac{1}{s-n} \left[ \frac{u^{\frac{1}{2}}}{s^{\frac{1}{2}}} \prod_{K=1}^{n-1} (\hat{u} + K) \right] (T^{a_1})_{\alpha_4} (T^{a_2})_{\beta_3}$$ (A.26)

$\mathcal{M}_{1,1}^{(B)}$ and $\mathcal{M}_{2,-\frac{3}{2}}^{(B)}$ have the same structure as $\mathcal{M}_{1,1}^{(B)}$ and $\mathcal{M}_{3,\frac{3}{2}}^{(B)}$ apart from the color factors and the replacement of $g \rightarrow g_D p_g$. 

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B  Calculation of $\mathcal{M}(ggqq)$: a detailed example

References: This calculation was given in [1]. We fill in some of the details.

The goal of this section is to illustrate the details of a string amplitude calculation using the techniques of sections 4.2.1 - 4.2.3. In section 4.2.2 we showed a derivation of the 4-tachyon amplitude. Some of the new features which will appear in this section are: correlation functions of world sheet fermions, color factors, and polarization vectors.

Recalling Eq. (4.24), we begin by calculating the correlation function of two quarks and two gluons using the equations of Appendix C. In the start we do not write the color matrices and coupling constants, but they will be inserted afterwards.

From Eqs. (C.2)-(C.5) we have:

\[
\langle \ldots \rangle = \langle V_{A'}^{(-)}(z_1, k_1) V_{A'}^{(-)}(z_2, k_2) V_{A'}^{(-1/2)}(z_3, u_3, k_3) V_{A'}^{(-1/2)}(z_4, \bar{u}_4, k_4) \rangle =
\]

\[
\frac{1}{2\alpha'} \xi_{1\mu} \xi_{2\nu} u^\lambda \bar{u}_4 \langle \Xi^{\nu}(z_3) \Xi^{\lambda}(z_4) \rangle \langle e^{-\phi(z_2)} e^{-\phi(z_3/2)} e^{-\phi(z_4/2)} \rangle + \]

\[
x \left( \langle \psi(z_2) S_{\lambda_3}(z_3) S^{\lambda_4}(z_4) \rangle \right) \left( \partial X^\mu(z_1) \prod_{i=1}^4 e^{ik_i \cdot X(z_i)} \right) \]

\[
2\alpha' \langle k_1 \cdot \psi(z_1) \rangle \psi^\mu(z_1) \psi_\nu(z_2) S_{\lambda_3}(z_3) S^{\lambda_4}(z_4) \rangle \langle \prod_{i=1}^4 e^{ik_i \cdot X(z_i)} \rangle \right] (B.1)
\]

From Eqs. (D.13), (D.14), (D.17)- (D.24), this equals:

\[
\langle \ldots \rangle = \frac{1}{2\alpha'} \xi_{1\mu} \xi_{2\nu} u_3 \cdot \left( i \left( 2z_{23} z_{24} \right)^{1/2} \sigma_{\nu} (-2i\alpha') \sum_{r=1}^4 \frac{k_r}{z_1 - z_r} + 2\alpha' k_{1\rho} \left( 2z_{13} z_{14} z_{23} z_{24} \right)^{-1/2} \left\{ \frac{z_{34}}{2} (\sigma^\rho \sigma^\nu) + \eta^{\rho\nu} \sigma^\mu \frac{z_{13} z_{14}}{z_{11}} - \eta^{\rho\mu} \sigma^\nu \frac{z_{13} z_{24}}{z_{12}} + \eta^{\mu\nu} \sigma^\rho \frac{z_{13} z_{24}}{z_{12}} \right\} \right) \cdot \bar{u}_4 \]

(B.2)
\[ \langle \ldots \rangle = \frac{1}{\sqrt{2\alpha'}} 2^{1/2} \alpha' (z_{23} z_{24} z_{34})^{-1} \prod_{i<j} |z_{ij}|^{2\alpha'_i k_j} \times \]

\[ \xi_{1\mu} \xi_{2\nu} u_3 \cdot \left( \sum_{\sigma} \frac{k_{\mu}^\sigma}{z_1 - z_r} \right) + k_{1\rho} \left\{ \begin{array}{l}
\frac{z_{34}}{2 z_{13} z_{14}} (\sigma^a \sigma^b \sigma^c) + \frac{\eta^{\rho \mu} \sigma^\nu}{z_{11}} - \eta^{\rho \mu} \sigma^\nu \frac{z_{24}}{z_{12} z_{14}} + \eta^{\rho \nu} \sigma^\rho \frac{z_{24}}{z_{12} z_{14}} \end{array} \right\} \cdot u_4 \]

The two terms with \( \frac{1}{z_{ii}} \) cancel each other

\[ \langle \ldots \rangle = \sqrt{\alpha'} (z_{23} z_{24} z_{34})^{-1} \prod_{i<j} |z_{ij}|^{2\alpha'_i k_j} \times \]

\[ \left\{ \begin{array}{l}
\left[ \frac{1}{z_1} \xi_{2\rho} (\xi_1 k_2) + \frac{1}{z_{13}} \xi_{2\rho} (\xi_1 k_3) + \frac{1}{z_{14}} \xi_{2\rho} (\xi_1 k_4) - \frac{z_{24}}{z_{12} z_{14}} \xi_{1\rho} (\xi_2 k_1) + \frac{z_{24}}{z_{12} z_{14}} k_{1\rho} (\xi_1 \xi_2) \right] (u_3 \sigma^\rho u_4) \\
+ \frac{1}{2} \frac{z_{34}}{z_{13} z_{14}} k_{1\lambda} \xi_{1\mu} \xi_{2\rho} (u_3 \sigma^\lambda \bar{\sigma}^\mu \sigma^\rho u_4) \end{array} \right\} \]

(B.3)

Now we make the choice \( z_4 = \text{constant} \to \infty \). This has the effect of making the third term vanish, and \( \prod_{i<j} |z_{ij}|^{2\alpha'_i k_j} = \text{const} \cdot |z_{12}|^s |z_{13}|^t |z_{23}|^u \). We get:

\[ \langle \ldots \rangle = \sqrt{\alpha'} |z_{12}|^s |z_{13}|^t |z_{23}|^u \]

\[ \times \left\{ \begin{array}{l}
\left[ \frac{1}{z_1} \xi_{2\rho} (\xi_1 k_2) + \frac{1}{z_{13}} \xi_{2\rho} (\xi_1 k_3) - \frac{1}{z_{12}} \xi_{1\rho} (\xi_2 k_1) + \frac{1}{z_{12}} k_{1\rho} (\xi_1 \xi_2) \right] (u_3 \sigma^\rho u_4) \\
+ \frac{1}{2} \frac{1}{z_{13}} k_{1\lambda} \xi_{1\mu} \xi_{2\rho} (u_3 \sigma^\lambda \bar{\sigma}^\mu \sigma^\rho u_4) \end{array} \right\} \]

(B.4)

Now we can set \( z_1 = 0 \) and \( z_3 = 1 \), and get:

\[ \langle \ldots \rangle = 2 \alpha' g_{D_p g_{D_p}} z_{12} z_{13} z_{23}^{-u_1} \]

\[ \times \left\{ \begin{array}{l}
\left[ \xi_{2\rho} (\xi_1 k_2) - \xi_{1\rho} (\xi_2 k_1) + k_{1\rho} (\xi_1 \xi_2) + z_{12} \xi_{2\rho} (\xi_1 k_3) \right] (u_3 \sigma^\rho u_4) \\
+ \frac{z_{12}}{2} k_{1\lambda} \xi_{1\mu} \xi_{2\rho} (u_3 \sigma^\lambda \bar{\sigma}^\mu \sigma^\rho u_4) \end{array} \right\} \]

(B.5)
Where we have inserted the coupling constants.

Integrating over $z_2$ we get:

$$\langle \ldots \rangle = -2\alpha' g_{D_p x} g_{D_p y} \mathcal{K} B(s, u) \quad (B.7)$$

Where $\mathcal{K}$ is a kinematic factor:

$$\mathcal{K} \equiv \left\{ [k_{1\rho}(\xi_1 \xi_2) - \xi_{1\rho}(\xi_2 k_1) - \frac{s}{t} \xi_{2\rho}(\xi_1 k_2)](u_3 \sigma^\rho \bar{u}_4) - \frac{1}{2} s k_{1\lambda} \xi_{1\rho} \xi_{2\rho}(u_3 \sigma^\lambda \bar{u}_4 \sigma^\rho \bar{u}_4) \right\} \quad (B.8)$$

If both gauge bosons are from stack $a$, then there are two permutations that should be considered. After inserting back the color matrices we have:

$$\mathcal{M}(A^a_1, A^a_2, \psi^{a\beta}_{\beta_3}, \psi^{a\beta}_{\beta_4}) = -2\alpha' g^2 \mathcal{K}$$

$$\times \left[ Tr(T^{a_1} T^{a_2} T^{\alpha_3}_{\beta_3} T^{\alpha_4}_{\beta_4}) B(s, u) + Tr(T^{a_2} T^{a_1} T^{\alpha_3}_{\beta_3} T^{\alpha_4}_{\beta_4}) \frac{t}{u} B(s, t) \right] \quad (B.9)$$

Where $g \equiv g_{D_p x}$.

On the other hand, if the gauge bosons are from two different stacks $a$ and $b$, then there is only one permutation which gives:

$$\mathcal{M}(A^a, A^b, \psi^{a\beta}_{\beta_3}, \psi^{a\beta}_{a_4}) = -2\alpha' g g_{D_p b} \mathcal{K} Tr(T^{a} T^{\alpha_3}_{\beta_3} T^{b}_{\beta_4} T^{\alpha_4}_{a_4}) \frac{t}{s} B(t, u) \quad (B.10)$$

Choosing $k_2$ as the reference momentum for $\xi_1$ and $k_1$ for $\xi_2$, it is seen that $\mathcal{K}$ vanishes if $\xi_1$ and $\xi_2$ have the same helicity. If they have opposite helicity, we use the following identities:

$$\xi_1^+ \xi_2^- = 0 \quad (B.11)$$

$$\xi_2^+(\xi_1^- k_3)(u_3 \sigma^\rho \bar{u}_4) = \frac{1}{\alpha'^2 s^2} \langle 13 \rangle \langle 23 \rangle \langle 24 \rangle \quad (B.12)$$

$$\xi_2^-(\xi_1^+ k_3)(u_3 \sigma^\rho \bar{u}_4) = \frac{1}{\alpha'^2 s^2} \langle 23 \rangle \langle 13 \rangle \langle 14 \rangle \quad (B.13)$$

(see Eqs (E.18), (E.25)), and obtain:

$$\mathcal{M}(g_1^-, g_2^+, q_3^-, q_4^+) = 2 g^2 \delta^{\beta_3}_{\beta_4} \frac{\langle 13 \rangle^2}{\langle 23 \rangle \langle 24 \rangle} \left[ (T^{a_1} T^{a_2})^{\alpha_3}_{\alpha_4} \frac{t}{s} V_t + (T^{a_2} T^{a_1})^{\alpha_3}_{\alpha_4} \frac{u}{s} V_u \right] \quad (B.14)$$

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\[ M(g_1^-, B_2^+, q_3^-, q_4^+) = 2g_D_v g \frac{(13)^2}{(23)(24)} (T^a)_{\alpha_4}^\beta_4 (T^b)_{\beta_3}^\alpha_3 V_s \]  

(B.15)

Where \( B \) is the vector boson from stack \( b \). We clearly see a factorized form of a standard model sub-amplitude times a Veneziano amplitude.

We square the amplitude

\[
|M|^2(g_1^-, g_2^+, q_3^-, q_4^+) = 4g_a^4 N_b \left[ \sum_{a_1,a_2} Tr(T^{a_1}T^{a_1}T^{a_2}T^{a_2}) \frac{t}{u s^2} (tV_t + uV_u)^2 - \sum_{a_1,a_2,i} \frac{1}{2} T^{a_1a_2i} T^{a_1a_2i} \frac{t^2}{s^2} V_s V_u \right] \]  

(B.16)

\[
|M|^2(g_1^-, B_2^+, q_3^-, q_4^+) = 4g_a^2 g_b^2 \sum_{a,b} Tr(T^{a}T^{a}) Tr(T^{b}T^{b}) \frac{t}{u} V_s^2 \]  

(B.17)

Using

\[
\sum_{a} Tr(T^{a}T^{a}) = \frac{N^2 - 1}{2N} 1_N \]  

(B.18)

We get:

\[
|M|^2(gg \rightarrow q\bar{q}) = g^4 N_f \frac{t^2 + u^2}{2N} \left[ \frac{1}{u t} (tV_t + uV_u)^2 - \frac{2N^2}{N^2 - 1} V_t V_u \right] \]  

(B.19)

Which for \( N = 3 \) colors coincides with Eq. (5.45), and

\[
|M|^2(q\bar{q} \rightarrow gB) = \frac{4}{9} g^4 |T_{q\bar{q}}^B|^2 Q_A^2 \frac{t^2 + u^2}{u t} V_s^2 \]  

(B.20)

Eq. (5.52).
C Vertex operators

References: [1, 2, 20, 88].

C.1 Massless particles

We list below some vertex operators for massless fields. We suppress the $z$ dependence of all the fields (for example $\psi_{\mu} \equiv \psi_{\mu}(z)$). The coupling constants are:

$$g \equiv g_{Dpa} \quad , \quad g_A = g_{Dpa}^A = (2\alpha')^{1/2} g_{Dpa} \quad , \quad g_\lambda = (2\alpha')^{1/2} \alpha'^{1/4} g_{Dpa}$$

$$g_\psi = (2\alpha')^{1/2} \alpha'^{1/4} e^{\phi_{10}/2} \quad , \quad g_\phi = (2\alpha')^{1/2} e^{\phi_{10}/2}$$  \hspace{1cm} (C.1)

**Gauge bosons** in the $(-1)$ and $(0)$ ghost picture:

$$V_{A^a}^{(-1)}(z, \xi, k) = g_A [T^a]_{\alpha_1}^{\alpha_2} \xi e^{-\phi_{10} \xi} \psi_{\mu} e^{ik \cdot X}$$  \hspace{1cm} (C.2)

$$V_{A^a}^{(0)}(z, \xi, k) = \frac{g_A}{(2\alpha')^{1/2}} [T^a]_{\alpha_1}^{\alpha_2} [i\partial X^\mu + 2\alpha'(k\psi)\psi_{\mu}] e^{ik \cdot X}$$  \hspace{1cm} (C.3)

These are independent of the internal part of the SCFT.

**Chiral fermions** (These depend on the internal field $\Xi^{n_{\partial 1}}$):

$$V_{\psi_{\beta}}^{(-1/2)}(z, u, k) = g_\psi [T^\beta]_{\alpha_1}^{\delta_1} e^{-\phi/2} u^\lambda S_\lambda \Xi_{\delta 1} e^{ik \cdot X}$$  \hspace{1cm} (C.4)

$$V_{\bar{\psi}}^{(-1/2)}(z, \bar{u}, k) = g_\psi [T^\beta]_{\delta_1}^{\alpha_1} e^{-\phi/2} \bar{u}^\lambda S_\lambda \Xi e^{ik \cdot X}$$  \hspace{1cm} (C.5)

**Gauginos**:

$$V_{\lambda^i}^{(-1/2)}(z, u, k) = g_\lambda [T^\alpha]_{\alpha_2}^{\alpha_1} e^{-\phi/2} u^\lambda S_\lambda S^i e^{ik \cdot X}$$  \hspace{1cm} (C.6)

$$V_{\bar{\lambda}^i}^{(-1/2)}(z, \bar{u}, k) = g_\lambda [T^\alpha]_{\alpha_2}^{\alpha_1} e^{-\phi/2} \bar{u}^\lambda S_\lambda S^i e^{ik \cdot X}$$  \hspace{1cm} (C.7)

**Adjoint scalars**:

$$V_{\phi^{(-1)}}(z, k) = g_A [T^\alpha]_{\alpha_2}^{\alpha_1} e^{-\phi} \Psi e^{ik \cdot X}$$  \hspace{1cm} (C.8)
\begin{align}
V_{\tilde{\phi}^a}^{-1}(z, k) &= g_A^A [T^a]_{\alpha_1}^{\alpha_2} e^{-\phi} \tilde{\psi}^i e^{i k \cdot X} \\
\text{Chiral scalars:} \\
V_{\phi_{\beta}^\dagger}^{-1}(z, k) &= g_\phi [T_{\beta}]^{\beta_1}_{\alpha_1} e^{-\phi} \Pi_{\alpha \beta} e^{i k \cdot X} \\
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) &= g_\phi [T_{\alpha}]^{\alpha_1}_{\beta_1} e^{-\phi} \Pi_{\alpha \beta} e^{i k \cdot X}
\end{align}

\section*{C.2 First excited state}

We list below the vertex operators of the first excited state of the open string. These are used in section 9.

\subsection*{C.2.1 Bosons of the NS sector}

In $D = 10$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_\phi [T_{\alpha}]^{\alpha_1}_{\beta_1} e^{-\phi} \Pi_{\alpha \beta} e^{i k \cdot X} \\
V_{\phi_{\beta}^\dagger}^{-1}(z, k) = g_\phi [T_{\beta}]^{\beta_1}_{\alpha_1} e^{-\phi} \Pi_{\alpha \beta} e^{i k \cdot X} \\
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_\phi [T_{\alpha}]^{\alpha_1}_{\beta_1} e^{-\phi} \Pi_{\alpha \beta} e^{i k \cdot X}
\end{equation}

In $D = 4$ we have:

Spin $J = 0$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = \frac{g_A}{\sqrt{2 \alpha'}} T^a e^{-\phi} \left[ E_{\mu m n} \psi^m \psi^n + B_{m n} i \partial X^m \psi^n + H_m \partial \psi^n \right] e^{i k \cdot X}
\end{equation}

Spin $J = 1$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 2$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 3$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 4$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 0$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 1$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 2$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 3$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

Spin $J = 4$:

\begin{equation}
V_{\phi_{\alpha}^\dagger}^{-1}(z, k) = g_A T^a e^{-\phi} \left[ O_e e^{i k \cdot X} \right]
\end{equation}

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C.2.2 Fermions of the R sector

In $D = 10$:

$$V^{(-1/2)}_{R,a}(z, v, \bar{\rho}, k) = C_A T^a \left[ \bar{v}^A_m i \partial X^m + 2\alpha' \bar{\rho}^m_B \psi_m \psi^n \Gamma^B_n \right] \Theta_A e^{-\phi/2} e^{ik \cdot X} \quad (C.17)$$

In $D = 4$:

Spin $J = 1/2$ and $J = 3/2$:

$$V^{(-1/2)}_{Q^a_{\beta}}(z, v, \bar{\rho}, k) = \alpha'^{1/4} e^{\phi/2} (T^a_{\beta} \alpha^\alpha_1 \left[ i v^\beta_{\mu} \partial X^\mu - \sqrt{\alpha'} \bar{\rho}^\mu_\alpha \psi_\mu \psi_{\nu} \sigma^\alpha_{\nu\beta} \right] S^\beta e^{-\phi/2} e^{ik \cdot b} e^{ik \cdot X} \quad (C.18)$$

$$V^{(-1/2)}_{\bar{Q}^a_{\beta}}(z, \bar{v}, \rho, k) = \alpha'^{1/4} e^{\phi/2} (T^a_{\beta} \alpha^\alpha_1 \left[ i \bar{v}^\mu_{\mu} \partial X^\mu - \sqrt{\alpha'} \bar{\rho}^\mu_\alpha \psi_\mu \psi_{\nu} \sigma^\alpha_{\nu\beta} \right] S^\beta e^{-\phi/2} e^{ik \cdot b} e^{ik \cdot X} \quad (C.19)$$

For $J = 1/2$:

$$\bar{v}_{\bar{a}}^\mu (J = 1/2) = \frac{\sqrt{\alpha'}}{2\sqrt{2}} \bar{\chi}_{\beta} (\bar{\sigma}_{\mu} k)^{\beta}_{\bar{a}} \quad , \quad \rho_{\mu}^a (J = 1/2) = -\frac{1}{6\sqrt{2}} \bar{\chi}_{\beta} \bar{\sigma}_{\mu}^{\beta\alpha} \quad (C.20)$$

While for $J = 3/2$:

$$\bar{v}_{\bar{a}}^\mu (J = 3/2) = \bar{\chi}_{\bar{a}}^\mu \quad , \quad \text{such that} \quad \bar{\chi}_{\bar{a}}^\mu \bar{\sigma}_{\mu}^{\beta\alpha} = k_{\mu} \bar{\chi}_{\bar{a}}^\mu = 0$$

$$\rho^{\mu a} (J = 3/2) = \eta^{\mu a} = \sqrt{\alpha'} \bar{\chi}_{\beta}^{\mu} \bar{\sigma}_{\nu}^{\beta\alpha} k_{\nu} \quad (C.21)$$
D Correlation functions

References: [2, 1, 70, 68].

D.1 An example calculation

We rewrite the definition of the correlation function Eq. (4.24):

$$\langle V(z_1, k_1) \cdots V(z_m, k_m) \rangle \equiv \int \mathcal{D}X \exp \left( -\frac{1}{2\pi\alpha'} \int d^2 \partial X \cdot \bar{\partial} X \right) \prod_{i=1}^{m} V(z_i, k_i) \cdots V(z_m, k_m)$$

(D.1)

We now calculate the correlator for a chain of tachyon vertex operators $V(z_j, k_j) = e^{i k_j \cdot X(z_j)}$:

$$\langle e^{i k_1 \cdot X} \cdots e^{i k_m \cdot X} \rangle = \int \mathcal{D}X \exp \left( -\frac{1}{2\pi\alpha'} \int d^2 \partial X \cdot \bar{\partial} X \right) \exp \left( i \sum_{j=1}^{m} k_j \cdot X(z_j) \right)$$

(D.2)

This is a gaussian integral which is computed as:

$$\int \mathcal{D}X \exp \left( \frac{1}{2\pi\alpha'} \int d^2 \partial X \cdot \bar{\partial} X + iJ \cdot X \right) \approx \exp \left( \frac{\pi\alpha'}{2} \int d^2 z d^2 z' J(z, \bar{z}) \frac{1}{\partial \bar{\partial}} J(z', \bar{z}') \right)$$

(D.3)

The operator $\frac{1}{\partial \bar{\partial}}$ is the propagator $G(z, \bar{z}, z', \bar{z}')$ which obeys the equation:

$$\partial \bar{\partial} G(z, \bar{z}, z', \bar{z}') = \delta(z - z', \bar{z} - \bar{z}')$$

(D.4)

Since we are in two dimensions, the solution for open and closed strings is:

$$G^{(closed)}(z, \bar{z}; z', \bar{z}') = \frac{1}{\pi} \ln |z - z'| = \frac{1}{2} G^{(open)}(z, \bar{z}; z', \bar{z}')$$

(D.5)

Since,

$$J(z, \bar{z}) = \sum_{j=1}^{m} k_j \delta(z - z_j, \bar{z} - \bar{z}_j)$$

(D.6)

Eq. (D.3) gives the result:

$$\langle e^{i k_1 \cdot X} \cdots e^{i k_m \cdot X} \rangle^{(closed)} = \exp \left( \frac{\alpha'}{2} \sum_{j,l} k_j \cdot k_l \ln |z_j - z_l| \right) = \prod_{j<l} |z_j - z_l|^{\alpha' k_j \cdot k_l}$$

(D.7)
\[ \langle e^{ik_1 \cdot X} \ldots e^{ik_m \cdot X} \rangle_{\text{(open)}} = \prod_{j<l} |z_j - z_l|^{2\alpha' k_j k_l} \] (D.8)

### D.2 List of some correlation functions

We list some of the correlation functions of the SCFT fields which appear in the calculation of 4 and 5 point amplitudes. We define \( z_{ij} \equiv z_i - z_j \).

- **\( X^\mu(z) \)**

  \[
  \langle X^\mu(z_1) X^\nu(z_2) \rangle = -2\alpha' \delta^{\mu\nu} \ln(z_{12}) \] (D.9)

  \[
  \langle \partial X^\mu(z_1) X^\nu(z_2) \rangle = -\frac{2\alpha' \delta^{\mu\nu}}{z_{12}} \] (D.10)

  \[
  \langle \partial X^\mu(z_1) \partial X^\nu(z_2) \rangle = -\frac{2\alpha' \delta^{\mu\nu}}{z_{12}^2} \] (D.11)

  \[
  \langle e^{ik_\mu X^\mu(z_1)} e^{ik_\nu X^\nu(z_2)} \rangle = |z_{ij}|^{2\alpha' k_1 k_2} \] (D.12)

  \[
  \langle e^{ik_1 \cdot X(z_1)} \ldots e^{ik_n \cdot X(z_n)} \rangle = \prod_{i<j} |z_{ij}|^{2\alpha' k_i k_j} \] (D.13)

  \[
  \langle \partial X^\mu(z_A) \prod_{i=1}^n e^{ik_i \cdot X(z_i)} \rangle = \left( -2i\alpha' \sum_{r=1}^n \frac{k_r^\mu}{z_{A,r}} \right) \langle \prod_{i=1}^n e^{ik_i \cdot X(z_i)} \rangle \] (D.14)

  \[
  \langle \partial X^\mu(z_A) \partial X^\nu(z_B) \prod_{i=1}^n e^{ik_i \cdot X(z_i)} \rangle = \left( -4\alpha'^2 \alpha' \sum_{r,s=1}^n \frac{k_r^\mu}{z_{A,r} z_{B,s}} - 2\alpha' \eta^{\mu\nu} \right) \langle \prod_{i=1}^n e^{ik_i \cdot X(z_i)} \rangle \] (D.15)

- **\( \phi(z) \)**
\[
\langle e^{-\phi(z_1)} e^{-\phi(z_2)} \rangle = \frac{1}{z_{12}} \quad (D.16)
\]
\[
\langle e^{-\phi(z_1)} e^{-\phi(z_2)/2} e^{-\phi(z_3)/2} \rangle = \frac{1}{z_{12}^{-1/2} z_{13}^{-1/2} z_{23}^{-1/4}} \quad (D.17)
\]
\[
\langle e^{-\phi(z_1)/2} e^{-\phi(z_2)/2} e^{-\phi(z_3)/2} e^{-\phi(z_4)/2} \rangle = \frac{1}{(z_{12} z_{13} z_{14} z_{23} z_{24} z_{34})^{1/4}} \quad (D.18)
\]

• \( \psi^\mu(z) \) and \( S_\alpha(z) \)

\[
\langle \psi^\mu(z_1) \psi^\nu(z_2) \rangle = \frac{\delta^{\mu\nu}}{z_{12}} \quad (D.19)
\]
\[
\langle \psi^\mu(z_1) S_\alpha(z_2) S_\beta(z_3) \rangle = (2z_{12} z_{13})^{-1/2} \sigma^\mu_{\alpha\beta} \quad (D.20)
\]
\[
\langle \psi^\mu(z_1) \psi^\nu(z_2) \psi^\lambda(z_3) S_\alpha(z_4) S_\beta(z_5) \rangle = (2z_{14} z_{15} z_{24} z_{25} z_{34} z_{35})^{-1/2}
\]
\[
\times \left\{ \frac{z_{45}}{2} (\sigma^\mu \sigma^\nu \sigma^\lambda - \eta^\mu\nu \sigma^\lambda z_{14} z_{25} z_{12} z_{13} + \eta^\mu\lambda \sigma^\nu z_{24} z_{35} z_{23} z_{25}) \right\} \quad (D.21)
\]
\[
\langle S_\alpha(z_1) S_\beta(z_2) \rangle = -\frac{\epsilon_{\alpha\beta}}{z_{12}^{1/2}} \quad (D.22)
\]
\[
\langle S_\alpha(z_1) S_\beta(z_2) \rangle = \frac{\epsilon_{\alpha\beta}}{z_{12}^{1/2}} \quad (D.23)
\]

• \( \Xi^{a\cap b}(z) \)

\[
\langle \Xi^{a\cap b}(z_1) \Xi^{a\cap b}(z_2) \rangle = z_{12}^{-3/4} \quad (D.24)
\]
\[
\langle \Xi^{a\cap b}(z_1) \Xi^{b\cap c}(z_2) \Xi^{d\cap c}(z_3) \Xi^{c\cap a}(z_4) \rangle = \left( \frac{z_{13} z_{24}}{z_{12} z_{14} z_{23} z_{34}} \right)^{3/4} I_\rho (\{ z_i \}; \theta^j) \quad (D.25)
\]
\[
\langle J(z_1) \Xi^{a\cap b}(z_2) \Xi^{a\cap b}(z_3) \rangle = \langle J(z_1) \Sigma^{a\cap b}(z_2) \Sigma^{a\cap b}(z_3) \rangle = \frac{3z_{23}^{1/4}}{2z_{12} z_{13}} \quad (D.26)
\]

Where \( J \) is the field in Eq. \( (C.16) \).
E Helicity notation

References: We follow: [39, 41].

We review the spinor helicity formalism.
Massless fermions with a definite helicity, which solve Dirac’s equation are

\[ u_{\pm}(k) = \frac{1}{2}(1 \pm \gamma_5)u(k), \quad v_{\pm}(k) = \frac{1}{2}(1 \mp \gamma_5)u(k) \]

(E.1)

These are chosen as follows

\[ u_+(k) = v_-(k) = \frac{1}{\sqrt{2}}(\sqrt{k^+}, \sqrt{k^-}e^{i\varphi_k}, \sqrt{k^+}, \sqrt{k^-}e^{-i\varphi_k}) \]

(E.2)

\[ u_-(k) = v_+(k) = \frac{1}{\sqrt{2}}(\sqrt{k^+}, -\sqrt{k^-}e^{-i\varphi_k}, -\sqrt{k^+}, \sqrt{k^-}e^{-i\varphi_k}) \]

(E.3)

where

\[ k^\pm = k^0 \pm k^3, \quad e^{\pm i\varphi_k}(k) = \frac{k_1 \pm ik_2}{\sqrt{k^+_i k^-_j}} \]

(E.4)

Introducing the notation

\[ |i^\pm\rangle \equiv u_{\pm}(k_i), \quad \langle i^\pm| \equiv \bar{u}_{\pm}(k_i) = \bar{v}_{\pm}(k_i) \]

(E.5)

we get the products

\[ \langle ij \rangle \equiv \langle i^-|j^+\rangle = \bar{u}_-(k_i)u_+(k_j) = \sqrt{k^-_i k^+_j}e^{i\varphi_{k_i}} - \sqrt{k^+_i k^-_j}e^{i\varphi_{k_j}} = \sqrt{|s_{ij}|}e^{i\phi_{ij}} \]

(E.6)

\[ [ij] \equiv \langle i^+|j^-\rangle = \bar{u}_+(k_i)u_-(k_j) = -\sqrt{k^-_i k^+_j}e^{i\varphi_{k_i}} + \sqrt{k^+_i k^-_j}e^{i\varphi_{k_j}} = -\sqrt{|s_{ij}|}e^{-i\phi_{ij}} \]

(E.7)

where

\[ s_{ij} \equiv (k_i + k_j)^2 = 2k_i \cdot k_j \]

(E.8)

\[ \cos \phi_{ij} = \frac{k^+_i k^+_j - k^+_i k^-_j}{\sqrt{|s_{ij}|}k^+_i k^+_j}, \quad \sin \phi_{ij} = \frac{k^+_i k^-_j - k^-_i k^+_j}{\sqrt{|s_{ij}|}k^+_i k^-_j} \]

(E.9)
Lets write down some useful identities

\[
\langle ij \rangle [ji] = s_{ij} \tag{E.10}
\]

**Antisymmetry**

\[
\langle ij \rangle^* = - [ij] = [ji] \tag{E.11}
\]

**Schouten identity**

\[
\langle ij \rangle \langle kl \rangle = \langle ik \rangle \langle jl \rangle - \langle il \rangle \langle jk \rangle \tag{E.12}
\]

**Fierz rearrangement**

\[
\langle i^+ | \gamma_{\mu} | j^+ \rangle \langle k^+ | \gamma^\mu | l^+ \rangle = 2[ik] \langle lj \rangle \tag{E.13}
\]

Identities involving the anti-symmetric tensor \( \epsilon(i, j, l, m) \equiv \epsilon_{\mu\nu\sigma\rho} k_i^\mu k_j^\nu k_l^\sigma k_m^\rho \):

\[
\langle ij \rangle [jl] \langle lm \rangle [mi] = \frac{1}{2} \left[ s_{ij} s_{lm} - s_{il} s_{jm} + s_{im} s_{jl} - 4i\epsilon(i, j, l, m) \right] \tag{E.14}
\]

\[
[ij] \langle jl \rangle \langle lm \rangle \langle mi \rangle = \frac{1}{2} \left[ s_{ij} s_{lm} - s_{il} s_{jm} + s_{im} s_{jl} + 4i\epsilon(i, j, l, m) \right] \tag{E.15}
\]

Subtracting the last two:

\[
4i\epsilon(i, j, l, m) = [ij] \langle jl \rangle \langle lm \rangle \langle mi \rangle - \langle ij \rangle [jl] \langle lm \rangle [mi] \tag{E.16}
\]

**Momentum conservation**

\[
\sum_{n \neq i,j} \langle in \rangle \langle nj \rangle = 0 \tag{E.17}
\]

A spinor representation for the polarization vector of a massless gauge boson:

\[
\xi^+_{\mu}(k, q) = \frac{\langle q^- | \gamma_{\mu} | k^- \rangle}{\sqrt{2} \langle qk \rangle}, \quad \xi^-_{\mu}(k, q) = -\frac{\langle q^+ | \gamma_{\mu} | k^+ \rangle}{\sqrt{2} \langle qk \rangle} \tag{E.18}
\]

Where \( k \) is the momentum of the gauge boson and \( q \) is called the reference momentum.
We write down some identities involving polarisation vectors. In each one, the expression after the arrow is obtained by choosing the reference vectors \( q_1 = k_2, q_2 = k_1 \) as in Appendix B. This choice causes much simplification.

\[
\xi^-_{\mu}(k_1, q_1) \xi^{\mu-}(k_2, q_2) = \frac{2 \langle q_1^+ | \gamma_\mu | k_1^+ \rangle \langle q_2^+ | \gamma_\mu | k_2^+ \rangle}{2 |q_1k_1| |q_2k_2|} \rightarrow \frac{[q_1q_2]}{|k_1k_2|} \tag{E.19}
\]

Where we used Eq. (E.13).

\[
\xi^+_{\mu}(k_1, q_1) \xi^{\mu+}(k_2, q_2) = \frac{[k_1k_2]\langle q_2q_1 \rangle}{|q_1k_1| |q_2k_2|} \rightarrow \frac{[k_1k_2]}{|k_1k_2|} \tag{E.20}
\]

\[
\xi^-_{\mu}(k_1, q_1) \xi^{\mu+}(k_2, q_2) = \frac{[q_1k_2]\langle q_2k_1 \rangle}{|q_1k_1| |q_2k_2|} \rightarrow 0 \tag{E.21}
\]

\[
p^\mu \xi^+_{\mu}(k_1, q_1) = \frac{\langle q_1p | [p k_1] \rangle}{\sqrt{2}|q_1k_1|} \tag{E.22}
\]

\[
p^\mu \xi^-_{\mu}(k_1, q_1) = -\frac{\langle k_1p | [p q_1] \rangle}{\sqrt{2}|q_1k_1|} \tag{E.23}
\]

\[
\xi^+_{\mu}(k_2, q_2) \left[ \bar{u}(p_4) \gamma_\mu u(p_3) \right] = \sqrt{2} \frac{[k_2p_4] \langle p_3q_2 \rangle}{|q_2k_2|} \rightarrow \sqrt{2} \frac{[k_2p_4] \langle p_3k_1 \rangle}{|k_1k_2|} \tag{E.24}
\]

\[
\xi^-_{\mu}(k_2, q_2) \left[ \bar{u}(p_4) \gamma_\mu u(p_3) \right] = -\sqrt{2} \frac{[q_2p_4] \langle p_3k_2 \rangle}{|q_2k_2|} \rightarrow -\sqrt{2} \frac{[k_1p_4] \langle p_3k_2 \rangle}{|k_1k_2|} \tag{E.25}
\]
F Color factors

References: [1, 2, 7, 86].

We collect in this appendix group theory results and color sums. We write \( f_{abc} \) and \( d_{abc} \) in typewriter text because \( d \) and \( f \) are already in use as the Wigner \( d \)-functions and hypergeometric functions.

The symmetrized trace is defined as:

\[
STr(T^{a_1} \cdots T^{a_n}) \equiv \frac{1}{n!} \sum_{\pi} Tr(T^{a_{\pi1}} \cdots T^{a_{\pi n}})
\]

(F.1)

Proceeding:

\[
d_{a_1a_2a_3} = STr(T^{a_1}T^{a_2}T^{a_3})
\]

(F.2)

\[
d^{000} = \frac{1}{\sqrt{8N}} , \quad d^{00a} = 0 , \quad d^{0AB} = \frac{1}{\sqrt{8N}} \delta^{AB}
\]

(F.3)

\[
d_{a_1a_2a_3a_4} = STr(T^{a_1}T^{a_2}T^{a_3}T^{a_4})
\]

(F.4)

\[
\{T^{a_3}, T^{a_4}\} = 4 \sum_a d_{a_3a_4a} T^a
\]

(F.5)

\[
\sum_{b,c=1}^{N^2-1} d_{abc}d_{abc} = \frac{N^2 - 4}{16N}
\]

(F.6)

\[
\sum_{a,b,c=1}^{N^2-1} d_{abc}d_{abc} = (N^2 - 1) \frac{N^2 - 4}{16N}
\]

(F.7)

\[
\sum_{a_1,a_2,a_3,a_4=1}^{N^2-1} d_{a_1a_2a_3a_4}d_{a_1a_2a_3a_4} = (N^2 - 1)(N^4 - 6N^2 + 18)
\]

(F.8)

\[
2 \sum_{b=1}^{N^2-1} d_{ab0}d_{ab0} = \frac{1}{4N}
\]

(F.9)
\[
\sum_{b,c=1}^{N^2-1} d^{bc0} d^{b0c} = \frac{N^2 - 1}{8N} \quad \text{(F.10)}
\]

\[
d^{000} d^{000} = \frac{1}{8N} \quad \text{(F.11)}
\]

\[
[T^{a_3}, T^{a_4}] = \sum_a f^{a_3 a_4 a} T^a \quad \text{(F.12)}
\]

\[
\sum_{a_1, a_2, a_3=1}^{N^2-1} f^{i_1 a_1 a_2} f^{i_2 a_2 a_3} f^{i_3 a_3 a_1} = \frac{N}{2} f^{i_1 i_2 i_3} \quad \text{(F.13)}
\]

\[
f^{a b 0} = f^{a 00} = f^{000} = 0 \quad \text{(F.14)}
\]

\[
\sum_a (T^a_R T^a_R)_{ij} = C_R \delta_{ij} \quad \text{(F.15)}
\]

For the adjoint representation \((C_R = C_A)\) of \(SU(N)\) this equation gives:

\[
\sum_{b=1}^{N^2-1} \sum_{c=1}^{N^2-1} f^{a_1 a_2 c} f^{a_2 b c} = N \delta^{a_1 a_2} \quad \text{(F.16)}
\]

For the fundamental representation \((C_R = C_F)\) of \(SU(N)\) it gives:

\[
\sum_a (T^a_F T^a_F)_{ij} = \sum_{a=1}^{N^2-1} \sum_{\beta=1}^N T^a_{i\beta} T^a_{j\beta} = \frac{N^2 - 1}{2N} \delta_{ij} \quad \text{(F.17)}
\]

\[
(T^0_F T^0_F)_{ij} = \sum_{\beta=1}^N T^0_{i\beta} T^0_{j\beta} = \frac{1}{2N} \delta_{ij} \quad \text{(F.18)}
\]

\[
Tr(T^a T^b T^c) = d^{abc} + \frac{i}{4} f^{abc} \quad \text{(F.19)}
\]
\[
Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) = d^{a_1a_2a_3a_4} + \frac{i}{2}(d^{a_1a_4n}f^{a_2a_3n} - d^{a_2a_3n}f^{a_1a_4n}) + \frac{1}{12}(f^{a_1a_4n}f^{a_2a_3n} - f^{a_1a_2n}f^{a_3a_4n}) \quad (F.20)
\]

\[
\ell^{a_1a_2a_3a_4} \equiv Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) - Tr(T^{a_4}T^{a_3}T^{a_2}T^{a_1}) = i(d^{a_1a_4j}f^{a_2a_3j} - d^{a_2a_3j}f^{a_1a_4j}) \quad (F.21)
\]

\[
\ell^{a_1a_2a_3a_4a_5} \equiv Tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}) - Tr(T^{a_5}T^{a_4}T^{a_3}T^{a_2}T^{a_1}) =
if^{a_1a_2n}(d^{a_3a_4a_5n} - \frac{1}{12}f^{a_2a_3m}f^{a_4a_5n}m) + if^{a_1a_3n}(d^{a_2a_4a_5n} - \frac{1}{12}f^{a_2a_4m}f^{a_3a_5n}m) +
\]

\[
\ell^{a_4b_5a_1a_2a_3} \equiv (T^{a_1}T^{a_2}T^{a_3})^{a_4}_{\alpha_5}^{\beta_5}_{\beta_4} \quad (F.23)
\]

\[
\ell^{a_4b_5a_1a_2b} \equiv (T^{a_1}T^{a_2})^{a_4}_{\alpha_5}(T^{b})^{\beta_5}_{\beta_4} \quad (F.24)
\]

\[
\sum_{a=1}^{N^2-1} (T^a)_{i_1j_1} (T^a)_{i_2j_2} = \delta_{i_1j_2} \delta_{i_2j_1} - \frac{1}{N} \delta_{i_1j_1} \delta_{i_2j_2} \quad (F.25)
\]

Orthogonality of the Chan-Paton basis to leading order in $1/N$:

\[
\sum_{a_1,\ldots,a_n} Tr(T^{a_1}\ldots T^{a_n}) [Tr(T^{b_1}\ldots T^{b_n})]^* = N^{n-2}(N^2 - 1) \left[ \delta_{\{a\}}\{b\} + O(1/N^2) \right] \quad (F.26)
\]

Reduction of traces:

\[
\sum_{a=1}^{N^2-1} Tr(T^aX)Tr(T^aY) = \frac{1}{2}Tr(XY) - \frac{1}{2N}Tr(X)Tr(Y) \quad (F.27)
\]

\[
\sum_{a=1}^{N^2-1} Tr(T^aXT^aY) = \frac{1}{2}Tr(X)Tr(Y) - \frac{1}{2N}Tr(XY) \quad (F.28)
\]

Where $X$ and $Y$ are general chains of the matrices $T$. 

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G Collider phenomenology

References: [73][74][75][97].

Figs. 27, 28 show the collision of two protons, and two partons in the final state. The rapidities $y_1, y_2$ obey:

\[
\tanh y_1 = \frac{k_{||1}}{E_1}, \quad \sinh y_1 = \frac{k_{||1}}{k_{\perp1}}, \quad \cosh y_1 = \frac{E_1}{k_{\perp1}}, \quad E_1 = \sqrt{k_{||1}^2 + k_{\perp1}^2}
\]

\[
\tanh y_2 = \frac{k_{||2}}{E_2}, \quad \sinh y_2 = \frac{k_{||2}}{k_{\perp2}}, \quad \cosh y_2 = \frac{E_2}{k_{\perp2}}, \quad E_2 = \sqrt{k_{||2}^2 + k_{\perp2}^2}
\] (G.1)

So that

\[
y_1 = \frac{1}{2} \ln \left(\frac{E_1 + k_{||1}}{E_1 - k_{||1}}\right), \quad y_2 = \frac{1}{2} \ln \left(\frac{E_2 + k_{||2}}{E_2 - k_{||2}}\right)
\] (G.2)

In the c.o.m frame the rapidities will be:

\[
y_1 = y_2 = -\ln \left(\tan \frac{\theta}{2}\right) = \frac{1}{2} \ln \left(\frac{1 + |\cos \theta|}{1 - |\cos \theta|}\right)
\] (G.3)

Where $\theta$ is the scattering angle as seen in the c.o.m frame.
We define
\[ y \equiv \frac{1}{2}(y_1 - y_2) \quad , \quad Y \equiv \frac{1}{2}(y_1 + y_2) \quad , \quad k_{\perp} \equiv k_{\perp 1} + k_{\perp 2} \quad \quad (G.4) \]

The variables \( y_i \) and \( p_{\perp i} \) are useful because of their simple boost transformations along the beamline:
\[ y_1 \rightarrow y_1 - \tanh^{-1}(\beta) \quad , \quad k_{\perp 1} \rightarrow k_{\perp 1} \]
\[ y_2 \rightarrow y_2 - \tanh^{-1}(\beta) \quad , \quad k_{\perp 2} \rightarrow k_{\perp 2} \quad \quad (G.5) \]

So that,
\[ Y \rightarrow Y - \tanh^{-1}(\beta) \quad , \quad y \rightarrow y \quad \quad (G.6) \]

We define the Mandelstam variables for the protons: \( \tilde{s}, \tilde{t}, \tilde{u} \). Likewise for the partons: \( s, t, u \).

We have:
\[ \tau \equiv \frac{s}{s} = x_a x_b \quad , \quad s = 4k_{\perp}^2 \cosh^2 y \quad \quad (G.7) \]
\[ x_a = \sqrt{\tau} e^Y = \frac{2k_{\perp}}{\sqrt{s}} \cosh ye^Y \quad , \quad x_b = \sqrt{\tau} e^{-Y} = \frac{2k_{\perp}}{\sqrt{s}} \cosh ye^{-Y} \quad \quad (G.8) \]

The Mandelstam variables:
\[ t = -\frac{s}{2} \frac{e^{-y}}{\cosh y} = -2k_{\perp}^2 \cosh ye^{-y} = -\frac{s}{2}(1 - \cos \theta) \]
\[ u = -\frac{s}{2} \frac{e^{y}}{\cosh y} = -2k_{\perp}^2 \cosh ye^{y} = -\frac{s}{2}(1 + \cos \theta) \quad \quad (G.9) \]

The following variable is useful for angular distributions analysis
\[ \chi \equiv e^{2y} = \frac{1 + \cos \theta}{1 - \cos \theta} \quad \quad (G.10) \]

Then
\[ t = -s \frac{1}{1 + \chi} \quad , \quad u = -s \frac{\chi}{1 + \chi} \]
\[ \cos \theta = -\frac{1 - \chi}{1 + \chi} \quad , \quad \sin \theta = \frac{2\chi^{1/2}}{1 + \chi} \quad \quad (G.11) \]
The calculation of a cross section $d\tilde{\sigma}$ is done by convoluting the partonic cross section $d\sigma(ij \rightarrow kl)$ with the parton distribution functions of the two colliding protons:

$$d\tilde{\sigma} = \int_0^1 \int_0^1 dx_a \, dx_b \sum_{ijkl} f_i(x_a, M) \, f_j(x_b, M) \, d\sigma(ij \rightarrow kl) \quad \text{(G.12)}$$

The partonic cross section and the squared amplitude are related:

$$|\mathcal{M}(ij \rightarrow kl)|^2 = 64\pi^2 s \, \frac{d\sigma}{dt} = 16\pi s^2 \, \frac{d\sigma}{dt} \quad \text{(G.13)}$$

Now we discuss the dijet final state (The formalism for e.g. $\gamma + \text{jet}$ final state is similar.). The mass of a dijet is:

$$M \equiv \sqrt{(E_{j1} + E_{j2})^2 + (\vec{k}_{j1} + \vec{k}_{j2})^2} \quad \text{(G.14)}$$

The dijet cross section is:

$$\frac{d\tilde{\sigma}}{dM^2} = \frac{1}{(2\pi)^2} \int \int \frac{d^3k_1}{2E_1} \frac{d^3k_2}{2E_2} \sum_{ijkl} \int_0^1 \int_0^1 dx_a \, dx_b \times f_i(x_a, M) \, f_j(x_b, M) \, \delta^4(p_a + p_b - k_1 - k_2) \, \delta((p_a + p_b)^2 - M^2) \, 8\pi s \, \frac{d\sigma}{dt} \quad \text{(G.15)}$$

It will be useful, though, to write the cross section in terms of the variables $y, Y$. We can transform this equation by using:

$$\frac{d^3k_1}{2E_1} = \frac{\pi}{2} \, dk_{1\perp} \, dy_1, \quad \frac{d^3k_2}{2E_2} = \frac{\pi}{2} \, dk_{2\perp} \, dy_2 \quad \text{(G.16)}$$

$$s = (p_a + p_b)^2 = 4k_{\perp}^2 \cosh^2 y \quad \text{(G.17)}$$

$$\delta^4(p_a + p_b - k_1 - k_2) = \delta(E_a + E_b - E_1 - E_2) \, \delta(p_{a||} + p_{b||} - k_{1||} - k_{2||}) \delta(\vec{k}_{1\perp} + \vec{k}_{2\perp}) \quad \text{(G.18)}$$

and we get:

$$\frac{d\tilde{\sigma}}{dM^2} = M^2 \int \int dy_1 \, dy_2 \sum_{ijkl} \int \int dx_a \, dx_b \times f_i(x_a, M) \, f_j(x_b, M) \, \frac{1}{4 \cosh^2 y} \, \delta(E_a + E_b - E_1 - E_2) \, \delta(p_{a||} + p_{b||} - k_{1||} - k_{2||}) \, \frac{d\sigma}{dt} \quad \text{(G.19)}$$
Using:

\[ \delta(E_a + E_b - E_1 - E_2) \delta(p_{a||} + p_{b||} - k_{||1} - k_{||2}) = 2 \delta(\sqrt{s}x_a - Me^Y) \delta(\sqrt{s}x_b - Me^{-Y}) \]  

we get:

\[
\frac{d\tilde{\sigma}}{dM} = \frac{1}{2} M \tau \int dy_1 dy_2 \sum_{ijkl} \frac{1}{\cosh^2 y} f_i(\sqrt{\tau e}^Y, M) f_j(\sqrt{\tau e}^{-Y}, M) \frac{d\sigma}{dt} \]  

(G.20)

Using

\[
\int dy_1 \int dy_2 = \int_{-Y_{max}}^0 dY \int_{-(y_{max}+Y)}^{y_{max}+Y} dy + \int_0^{Y_{max}} dY \int_{-(y_{max}-Y)}^{y_{max}-Y} dy \]  

(G.22)

we finally get:

\[
\frac{d\tilde{\sigma}}{dM} = M \tau \sum_{ijkl} \int dY f_i(\sqrt{\tau e}^Y, M) f_j(\sqrt{\tau e}^{-Y}, M) \int dy \frac{1}{\cosh y} \frac{d\sigma}{dt} + 
\]

\[
M \tau \sum_{ijkl} \int dY f_i(\sqrt{\tau e}^Y, M) f_j(\sqrt{\tau e}^{-Y}, M) \int dy \frac{1}{\cosh y} \frac{d\sigma}{dt} \]  

(G.23)
H Mathematical functions

References: [61 63 62 59 60 58].

We review the properties of the mathematical functions which appear in this work.

H.1 Gamma function and Pochammer symbol

The gamma function is

\[ \Gamma(a) = \int_0^{\infty} dx \ x^{a-1} e^{-x} \] (H.1)

It satisfies the following properties

\[ \Gamma(a + 1) = a! \quad \text{if} \quad a = \text{integer} \] (H.2)

\[ \Gamma(a + 1) = a \ \Gamma(a) \] (H.3)

\[ \Gamma(a + n) = a(a + 1) \ldots (a + n - 1) \Gamma(a) \] (H.4)

\[ \Gamma(a) \Gamma(1 - a) = \frac{\pi}{\sin \pi a} \] (H.5)

\[ \Gamma(a) \Gamma(-a) = \frac{-\pi}{a \sin \pi a} \] (H.6)

There is a simple pole at each non-positive integer:

\[ \Gamma(a) \approx \frac{1}{a + n} \frac{(-1)^n}{n!} \] (H.7)

Stirling’s formula is:

\[ \Gamma(a + 1) \approx \sqrt{2\pi a} \left( \frac{a}{e} \right)^a \left[ 1 + \frac{1}{12a} + \ldots \right] \quad \text{for} \quad a >> 1. \] (H.8)

Hence besides having poles, \( \Gamma(a) \) diverges extremely fast for large values of \( a \).

Gauss multiplication formula:

\[ \Gamma(na) = (2\pi)^{\frac{1}{2}(1-n)} n^{na-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma(a + \frac{k}{n}) \] (H.9)
The Pochammer symbol (or rising factorial) is:

\[(a)_n \equiv a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \] (H.10)

Eq. (H.9) applied to the Pochammer symbol gives:

\[(m_1 a)_{m_1 m_2} = m_1^{m_1 m_2} \prod_{k=0}^{m_1-1} \left(a + \frac{k}{m_1}\right)_{m_2} \] (H.11)

Where \(m_1\) and \(m_2\) are integers.

The Pochammer symbol satisfies:

\[(a + b)_n = \sum_{p=0}^{n} \binom{n}{p} (a)_{n-p} (b)_p \] (H.12)

\[(a)_m(a)_n = \sum_{p=0}^{m} \binom{m}{p} \binom{n}{p} p! (a)_{m+n-k} \] (H.13)

**H.2 Beta function**

The Beta function is:

\[B(a, b) \equiv \int_0^1 dx \ x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \] (H.14)

It satisfies the following properties:

\[B(a, b) = B(b, a) \] (H.15)

\[B(a + 1, b) = \frac{a}{a + b} B(a, b) \] (H.16)

\[B(a + 1, b) + B(a, b + 1) = B(a, b) \] (H.17)

The singularities of \(B(a, b)\) are:

\[B(a, b) \approx \frac{1}{b+n} \frac{(-1)^n}{n!} (a-1)(a-2)\ldots(a-n), \text{ for } b \approx -n. \] (H.18)

\(B(a, b)\) has the remarkable property of being the sum of its singularities:

\[B(a, b) = \sum_{n=0}^{\infty} \frac{1}{b+n} \frac{(-1)^n}{n!} (a-1)(a-2)\ldots(a-n), \text{ for } Re(a) > 0. \] (H.19)
Proof: The sum on the right hand side reproduces all of the singularities of \( B(a, b) \), so that it can differ from it only by an entire function of \( b \). It is seen that the sum on the RHS vanishes at \( |b| \to \infty \), and the LHS does too. Thus the entire function must also vanish at this limit. The only asymptotically vanishing entire function is 0, therefore the two sides are equal.

At small values:

\[
B(a, b) \approx \frac{a + b}{ab}, \quad \text{for} \quad a, b << 1. \tag{H.20}
\]

Two limits for large values:

\[
B(a, b) \approx a^{-b}, \quad \text{for} \quad a >> 1, \quad b = \text{const}. \tag{H.21}
\]

and

\[
B(a, \epsilon a) \approx e^{-f(\epsilon)a}, \quad \text{for} \quad a >> 1. \tag{H.22}
\]

Where

\[
f(\theta) = \epsilon \ln \epsilon + (1 - \epsilon) \ln(1 - \epsilon) \tag{H.23}
\]

The first is a polynomial divergence (for \( a > 0, \ b < 0 \)). The second is an exponential vanishing.

H.3 Stirling numbers

Abramowitz and Stegun [59] define the Stirling numbers of the first kind through the expansion of the falling factorial.

\[
x(x-1) \cdots (x-1+n) = \sum_{m=0}^{n} S^{(m)}_{n} x^{m} \tag{H.24}
\]

Since the rising factorial (Pochammer symbol) is related to the falling factorial through:

\[
x(x-1) \cdots (x-1+n) = (-1)^{n}(-x)_{n}, \tag{H.25}
\]

the expansion for the Pochammer symbol will be:

\[
(x)_{n} = \sum_{m=0}^{n} (-1)^{n-m} S^{(m)}_{n} x^{m} \tag{H.26}
\]
Another generating function for the Stirling numbers is:

\[
\left( \ln(1 + x) \right)^m = m! \sum_{m=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}, \quad \text{for } |x| < 1.
\]

(H.27)

Special values:

\[
S_n^{(0)} = \delta_{0,n}, \quad S_n^{(1)} = (-1)^{n-1} (n - 1)!
\]

(H.28)

\[
S_n^{(n-1)} = -\binom{n}{2}, \quad S_n^{(n)} = 1
\]

(H.29)

An explicit form for the Stirling numbers is:

\[
S_n^{(m)} = \sum_{k=0}^{n-m} \sum_{p=0}^{k} \frac{(-1)^p}{k!} \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \binom{k}{p} p^{n-m+k}
\]

(H.30)

The Stirling numbers satisfy the following relations:

\[
S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}
\]

(H.31)

\[
\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)}
\]

(H.32)

\[
\sum_{m=1}^{n} S_n^{(m)} = 0
\]

(H.33)

\[
\sum_{m=0}^{n} (-1)^{n-m} S_n^{(m)} = n!
\]

(H.34)

\[
\sum_{k=m}^{n} S_{n+1}^{(k+1)} n^{k-m} = S_n^{(m)}
\]

(H.35)
H.4 Hypergeometric functions

The generalized hypergeometric function is:

\[ pF_q [ a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q ] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{x^k}{k!} \] (H.36)

Where \((a)_n\) is the Pochammer symbol.

The function

\[ p+1F_p [ a_1, a_2, \ldots, a_{p+1} \mid b_1, b_2, \ldots, b_p \mid x ] \] (H.37)

obeys the differential equation

\[ \vartheta(-x+b_1-1) \cdots (-x+b_p-1) - x \vartheta(-a_1-1) \cdots (-a_{p+1}) y = 0 \] (H.38)

Where

\[ \vartheta \equiv x \frac{d}{dx} \] (H.39)

In particular, the first hypergeometric function

\[ 2F_1 [ a, b \mid c \mid x ] \] (H.40)

obeys:

\[ z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \] (H.41)

Gauss’s hypergeometric theorem is:

\[ 2F_1 [ a, b \mid c \mid 1 ] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \] (H.42)

A recursion relation:

\[ p+1F_{q+1} [ a_1, \ldots, a_p, c \mid b_1, \ldots, b_q, d \mid x ] = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 dt t^{c-1}(1-t)^{d-c-1} pF_q [ a_1, \ldots, a_p \mid b_1, \ldots, b_q \mid tx ] \] (H.43)
H.5 Jacobi polynomials

The Jacobi polynomials are:

\[ P^{(\alpha,\beta)}_k(x) = \frac{1}{2^k} \sum_p \binom{k + \alpha}{p} \binom{k + \beta}{k - p} (x - 1)^{k-p} (x + 1)^p = \]

\[ \frac{(k + \alpha)! (k + \beta)!}{2^k} \sum_p \frac{1}{p! (k + \alpha - p)! (\beta + p)! (k - p)!} (x - 1)^{k-p} (x + 1)^p \]

(H.44)

They are solutions of the following differential equation:

\[(1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0 \quad \text{(H.45)}\]

These polynomials are orthogonal in the following sense:

\[ \int_{-1}^{1} dx \, P^{(\alpha,\beta)}_p(x) P^{(\alpha,\beta)}_k(x)(1 - x)^\alpha (1 + x)^\beta = \frac{2^{\alpha+\beta+1}}{2k + \alpha + \beta + 1} \frac{(k + \alpha)! (k + \beta)!}{k! (k + \alpha + \beta)!} \delta_{kp} \]

(H.46)

They satisfy the recursion relations:

\[ (2k + 1)(k + \alpha + \beta + 1)(2k + \alpha + \beta) P^{(\alpha,\beta)}_{k+1}(x) = \]

\[ \left[ (2k + \alpha + \beta + 1)(\alpha^2 - \beta^2) + (2k + \alpha + \beta)(2k + \alpha + \beta + 1)(2k + \alpha + \beta + 2)x \right] P^{(\alpha,\beta)}_{k}(x) \]

\[ -2(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2) P^{(\alpha,\beta)}_{k-1}(x) \]

(H.47)

Special values:

\[ P^{(\alpha,\beta)}_k(x = 1) = \frac{(k + \alpha)!}{k! \alpha!} \quad \text{(H.48)}\]

\[ P^{(\alpha,\beta)}_k(-x) = (-1)^k P^{(\beta,\alpha)}_k(x) \quad \text{(H.49)}\]

\[ P^{(0,0)}_k(x) = P_k(x) \quad \text{(H.50)}\]

Where \( P_k(x) \) are the Legendre polynomials.
The Jacobi polynomials will be used to express the Wigner $d$ functions, and we will have $x = \cos \theta$. The expansion Eq. (H.44) contains powers of $\frac{1+x}{2} = -\frac{\hat{u}}{\hat{s}}$ and of $\frac{1-x}{2} = -\frac{\hat{t}}{\hat{s}}$. We would prefer having only powers $\hat{u}$ though. Reference [59] gives the following expansion in terms of $\hat{t}$:

$$P^{(\alpha,\beta)}_k(x) = \frac{\Gamma(\alpha + k + 1)}{k! \Gamma(\alpha + \beta + k + 1)} \sum_{p=0}^{k} \binom{k}{p} \frac{\Gamma(\alpha + \beta + k + p + 1)}{\Gamma(\alpha + p + 1)} \left(\frac{x - 1}{2}\right)^p$$  \hspace{1cm} (H.51)

We transform this to an expression with $\hat{u}$ by using Eq. (H.49):

$$(-1)^k \frac{\Gamma(\beta + k + 1)}{k! \Gamma(\alpha + \beta + k + 1)} \sum_{p=0}^{k} \binom{k}{p} \frac{\Gamma(\alpha + \beta + k + p + 1)}{\Gamma(\beta + p + 1)} \left(-\frac{x + 1}{2}\right)^p =$$

$$(-1)^k \frac{\Gamma(\beta + k + 1)}{k! \Gamma(\alpha + \beta + k + 1)} \sum_{p=0}^{k} \binom{k}{p} \frac{\Gamma(\alpha + \beta + k + p + 1)}{\Gamma(\beta + p + 1)} \left(\frac{\hat{u}}{\hat{s}}\right)^p$$  \hspace{1cm} (H.52)

Which can be written as:

$$P^{(\alpha,\beta)}_k(\hat{u}) = (-1)^k \frac{(\beta + k)!}{k! (\alpha + \beta + k)!} \sum_{p=0}^{k} \binom{k}{p} \frac{(\alpha + \beta + k + p)!}{(\beta + p)! \hat{s}^p} \hat{u}^p$$  \hspace{1cm} (H.53)
I Wigner d-functions

In order to deal with collider phenomenology of particles with spin, it is necessary to get acquainted with the Wigner d-functions. These can be viewed as generalizations of the Legendre functions, see Eq. (I.8).

I.1 Review

References: [57, 58, 59, 60, 61, 62].

We review here properties of the Wigner d-functions.

The rotation matrix is:

\[ R = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z} \] (I.1)

Where \( \alpha, \beta, \gamma \) are the Euler angles.

Acting with \( R \) on an angular momentum state \( |Jm\rangle \), gives a linear combination of states with different \( m \):

\[ R|Jm\rangle = \sum_{m'} D^{J}_{m',m}(\alpha\beta\gamma)|Jm'\rangle \] (I.2)

For integral \( J = l \) this takes the form

\[ Y_{lm}(\theta, \phi) = \sum_{m'} D^{J}_{m',m}(\alpha\beta\gamma)Y_{lm'}(\theta', \phi') \] (I.3)

We isolate \( D^{J}_{m',m} \) by operating with \( \langle Jm'| \) on Eq. (I.2),

\[ D^{J}_{m',m}(\alpha\beta\gamma) = \langle Jm'|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|Jm\rangle = e^{-im'\alpha}\langle Jm'|e^{-i\beta J_y}|Jm\rangle e^{-i\gamma m} = e^{-im'\alpha}d^{J}_{m',m}(\beta)e^{-im\alpha} \] (I.4)

In the last line we defined:

\[ d^{J}_{m',m}(\beta) \equiv \langle Jm'|e^{-i\beta J_y}|Jm\rangle \] (I.5)

From now we will use \( \theta \) instead of \( \beta \) as the argument of \( d \). The \( d \)'s satisfy:

\[ d^{J}_{m',m}(\theta) = (-1)^{m-m'}d^{J}_{m,m'}(\theta) = d^{J}_{m,-m'}(\theta) = d^{J}_{m,m'}(-\theta) \] (I.6)

\[ d^{J}_{m',m}(\pi - \theta) = (-1)^{J+m'}d^{J}_{m',-m}(\theta) \] (I.7)
\[ d_{m,0}(\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_{m}(\theta, 0) \] (I.8)

A special case:

\[ d_{J,J}(\theta) = \left( \cos \frac{\theta}{2} \right)^{2J} \] (I.9)

Orthogonality:

\[ \int d \cos \theta \; d_{m',m}(\theta) d_{m',m}^{J'}(\theta) = \frac{2}{2J + 1} \delta^{JJ'} \] (I.10)

This is equivalent to the orthogonality of the Jacobi polynomials Eq. (H.46).

\[ \sum_{\lambda} d_{m',\lambda}(\theta_1) d_{\lambda,m}(\theta_2) = d_{m',m}(\theta_1 + \theta_2) \] (I.11)

The \( d' \)s satisfy the following recursion relations:

\[ 2(m \cos \theta - m') d_{m',m}^{J} = \]
\[ \sqrt{(J - m)(J + m + 1)} \sin \theta d_{m',m+1}^{J} + \sqrt{(J - m')(J + m' + 1)} \sin \theta d_{m',m-1}^{J} \] (I.12)

\[ (m - m') \cot \frac{\theta}{2} d_{m',m}^{J} = \]
\[ \sqrt{(J + m)(J - m + 1)} d_{m',m-1}^{J} + \sqrt{(J + m')(J - m' + 1)} d_{m'-1,m}^{J} \] (I.13)

\[ \sqrt{J - m' + 1} d_{m',m}^{J} = \]
\[ \sqrt{J - m + 1} \cos \frac{\theta}{2} d_{m'-1/2,m-1/2}^{J+1/2} + \sqrt{J + m + 1} \sin \frac{\theta}{2} d_{m'-1/2,m+1/2}^{J+1/2} \] (I.14)

\[ \sqrt{J + m' + 1} d_{m',m}^{J} = \]
\[ -\sqrt{J - m + 1} \sin \frac{\theta}{2} d_{m'+1/2,m-1/2}^{J+1/2} + \sqrt{J + m + 1} \cos \frac{\theta}{2} d_{m'+1/2,m+1/2}^{J+1/2} \] (I.15)
\[
\sqrt{J + m'} d_{m',m}^J = \sqrt{J + m} \cos \frac{\theta}{2} d_{m'-1/2,m-1/2}^{J-1/2} - \sqrt{J - m} \sin \frac{\theta}{2} d_{m'+1/2,m+1/2}^{J-1/2} \tag{I.16}
\]

\[
\sqrt{J - m'} d_{m',m}^J = \sqrt{J + m} \sin \frac{\theta}{2} d_{m'+1/2,m-1/2}^{J-1/2} + \sqrt{J - m} \cos \frac{\theta}{2} d_{m'-1/2,m+1/2}^{J-1/2} \tag{I.17}
\]

The following explicit form is useful for calculating the \(d\)'s. We used this equation to calculate the Tables in section I.3.

\[
d_{m',m}^J(\theta) = A \sum_p B_p \left[ \cos(\theta/2) \right]^{2J + m - m' - 2p} \left[ \sin(\theta/2) \right]^{m' - m + 2p} \tag{I.18}
\]

Where,

\[
A = \sqrt{(J + m')! \ (J - m')! \ (J + m)! \ (J - m)!}
\]

\[
B_p = \frac{(-1)^{m' - m + p}}{(J + m - p)! \ (m' - m + p)! \ (J - m' - p)!} \tag{I.19}
\]

Or alternatively in terms of \(\hat{u}\) and \(\hat{t}\):

\[
d_{m,m'}^J(u,t) = \frac{(-1)^J}{\hat{s}^J} A \sum_p B_p \ \hat{u}^{J + m/2 - m'/2 - p} \ \hat{t}^{m'/2 - m/2 + p} \tag{I.20}
\]

### I.2 \(d\)-functions and Jacobi polynomials

In this section we express the \(d\)-functions in terms of Jacobi polynomials.

A given \(d\)-function is some factor times a Jacobi polynomial:

\[
d_{m',m}^J(\theta) = \Omega^{(J-m)} \left[ \cos(\theta/2) \right]^{m + m'} \left[ \sin(\theta/2) \right]^{m - m'} P_{J-m}^{(m-m', m'+m)}(\cos \theta) \tag{I.21}
\]

Where

\[
\Omega^{(J-m)} = \sqrt{\frac{(J + m)! \ (J - m)!}{(J + m')! \ (J - m')!}} \tag{I.22}
\]

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Compare with Eq. (1.18): We can understand the factor in front of the Jacobi polynomial from Eqs. (H.46) and (I.10): it is the \( \text{"weight function"} \) under which the polynomials are orthogonal. The normalization \( \frac{1}{2J+1} \) is the same as for the Legendre polynomials which are the special cases \( P_{0,0}^0 \).

Writing this in terms of \( \hat{u} \) and \( \hat{t} \) instead of \( \theta \) gives:

\[
d_{m',m}^J(\hat{u}, \hat{t}) = \Omega^{(J-m)} \frac{(-1)^m}{\tilde{s}^m} \hat{u}^\frac{1}{2}(m+m') \hat{t}^\frac{1}{2}(m-m') P_{J-m}^{(m-m', m+m')}(\hat{u})
\]

(I.23)

Eq. (H.53) gives:

\[
P_{J-m}^{(m-m', m+m')}(\hat{u}) = (-1)^{J-m} (J + m')! \sum_{p=0}^{J-m} \binom{J-m}{J-m} \frac{(J + m + p)!}{(m + m' + p)! \tilde{s}^p} \hat{u}^p
\]

(I.24)

We write this equation in shorthand notation:

\[
P_{J-m}^{(m-m', m+m')}(\hat{u}) = \sum_{p=0}^{J-m} \Delta_{m,m'}^{(p)} \hat{u}^p
\]

(I.25)

We will use the coefficient of the highest power in the sum:

\[
\Delta_{m,m'}^{(J-m)} = \frac{(-1)^{J-m} (2J)!}{(J-m)! (J+m)! \tilde{s}^{J-m}}
\]

(I.26)

Notice that this coefficient is independent of \( m' \).

Thus the 5 combinations for \( m \) and \( m' \) can be written as:

\[
d_{0,0}^1(\theta) = P_{0,0}^0
\]

(I.27)

\[
d_{2,-2}^1(\theta) = \left( \frac{1 - \cos \theta}{2} \right)^2 P_{J-2}^{(4, 0)} = \frac{t^2}{s^2} P_{J-2}^{(4, 0)}
\]

(I.28)

\[
d_{2,-1}^1(\theta) = \sqrt{\frac{J+1}{J-1} \sin \theta(1 - \cos \theta)} P_{J-2}^{(3, 1)} = \sqrt{\frac{J+2}{J-1}} \frac{t^{3/2} u^{1/2}}{s^{1/2}} P_{J-2}^{(3, 1)}
\]

(I.29)

\[
d_{1/2,1/2}^1(\theta) = \cos \left( \frac{\theta}{2} \right) P_{J-1/2}^{(0, 1)} = \frac{u^{1/2}}{s^{1/2}} P_{J-1/2}^{(0, 1)}
\]

(I.30)
\[d_{3/2,-3/2}^J(\theta) = \sin\left(\frac{\theta}{2}\right)\left(\frac{1-\cos\theta}{2}\right) P^{(3, 0)}_{J-3/2} = \frac{t^{3/2}}{s^{1/2}} P^{(3, 0)}_{J-3/2}\]  

(I.31)

From Eq. (I.24), the power expansions of the 5 Jacobi polynomials are:

\[P^{(0,0)}_J(\hat{u}) = (-1)^J \sum_{p=0}^{J} \frac{(J+p)!}{(J-p)! \hat{s}^p} \hat{u}^p\]  

(I.32)

\[P^{(4,0)}_{J-2}(\hat{u}) = \frac{(-1)^J (J-2)!}{(J+2)!} \sum_{p=0}^{J-2} \frac{(J+p+2)!}{p!^2 (J-p-2)! \hat{s}^p} \hat{u}^p\]  

(I.33)

\[P^{(3,1)}_{J-2}(\hat{u}) = \frac{(-1)^J}{J(J+1)(J+2)} \sum_{p=0}^{J-2} \frac{(J+p+2)!}{(p+1)! p! (J-p-2)! \hat{s}^p} \hat{u}^p\]  

(I.34)

\[P^{(0,1)}_{J-1/2}(\hat{u}) = (-1)^{J-1/2} \sum_{p=0}^{J-1/2} \frac{(J+p+1/2)!}{p! (p+1)! (J-p-1/2)! \hat{s}^p} \hat{u}^p\]  

(I.35)

\[P^{(3,0)}_{J-3/2}(\hat{u}) = \frac{(-1)^{J-3/2}}{(J-1/2)(J+1/2)(J+3/2)} \sum_{p=0}^{J-3/2} \frac{(J+p+3/2)!}{p!^2 (J-p-3/2)! \hat{s}^p} \hat{u}^p\]  

(I.36)

**I.3 Tables**

From Eq. (I.18) we calculated some \(d's\). For each combination of \(m\) and \(m'\), we calculated up to the fifth \(J\). Since there are 5 combinations of \(m\) and \(m'\), there are 5 tables each containing 5 \(d's\). We used MATHEMATICA.
\[
\begin{align*}
    d_{0,0}^0 &= 1 \\
    d_{1,0}^1 &= \cos \theta \\
    d_{0,0}^2 &= \frac{3}{2} \cos^2 \theta - \frac{1}{2} \\
    d_{0,0}^3 &= \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \\
    d_{0,0}^4 &= \frac{1}{8} (35 \cos^4 \theta - 30 \cos^2 \theta + 3)
\end{align*}
\]

Table 11: \(d_{0,0}^I(\theta)\)

\[
\begin{align*}
    d_{2,-2}^2 &= (\frac{1-\cos \theta}{2})^2 \\
    d_{2,-2}^3 &= (\frac{1-\cos \theta}{2})^2 [3 \cos \theta + 2] \\
    d_{2,-2}^4 &= (\frac{1+\cos \theta}{2})^2 [7 \cos^2 \theta + 7 \cos \theta + 1] \\
    d_{2,-2}^5 &= (\frac{1+\cos \theta}{2})^2 [15 \cos^3 \theta + 18 \cos^2 \theta + 3 \cos \theta - 1] \\
    d_{2,-2}^6 &= (\frac{1+\cos \theta}{2})^2 \left[ \frac{495}{16} \cos^4 \theta + \frac{165}{4} \cos^3 \theta + \frac{45}{8} \cos^2 \theta - \frac{27}{4} \cos \theta - \frac{17}{16} \right]
\end{align*}
\]

Table 12: \(d_{2,-2}^I(\theta)\)

\[
\begin{align*}
    d_{2,-1}^2 &= 2(\frac{1-\cos \theta}{4}) \sin \theta \\
    d_{2,-1}^3 &= \sqrt{\frac{5}{2}} (\frac{1-\cos \theta}{4}) \sin \theta \left[3 \cos \theta + 1 \right] \\
    d_{2,-1}^4 &= \frac{1}{\sqrt{2}} (\frac{1-\cos \theta}{4}) \sin \theta \left[14 \cos^2 \theta + 7 \cos \theta - 1 \right] \\
    d_{2,-1}^5 &= \frac{\sqrt{7}}{2} (\frac{1-\cos \theta}{4}) \sin \theta \left[15 \cos^3 \theta + 9 \cos^2 \theta - 3 \cos \theta - 1 \right] \\
    d_{2,-1}^6 &= \sqrt{\frac{5}{4}} (\frac{1-\cos \theta}{4}) \sin \theta \left[ \frac{99}{4} \cos^4 \theta + \frac{33}{2} \cos^3 \theta - 9 \cos^2 \theta - \frac{9}{2} \cos \theta + \frac{1}{4} \right]
\end{align*}
\]

Table 13: \(d_{2,-1}^I(\theta)\)
\[ d^{1/2}_{1/2, 1/2} = \cos \theta / 2 \]
\[ d^{3/2}_{1/2, 1/2} = \frac{1}{2} \cos \theta / 2 \left[ 3 \cos \theta - 1 \right] \]
\[ d^{5/2}_{1/2, 1/2} = \frac{1}{2} \cos \theta / 2 \left[ 5 \cos^2 \theta - 2 \cos \theta - 1 \right] \]
\[ d^{7/2}_{1/2, 1/2} = \frac{1}{8} \cos \theta / 2 \left[ 35 \cos^3 \theta - 15 \cos^2 \theta - 15 \cos \theta + 3 \right] \]
\[ d^{9/2}_{1/2, 1/2} = \cos \theta / 2 \left[ \frac{63}{8} \cos^4 \theta - \frac{7}{2} \cos^3 \theta - \frac{21}{4} \cos^2 \theta + \frac{3}{2} \cos \theta + \frac{3}{8} \right] \]

Table 14: \( d^{J}_{1/2, 1/2}(\theta) \)

\[ d^{1/2}_{3/2, -3/2} = \left( \frac{1 - \cos \theta}{2} \right) \sin \theta / 2 \]
\[ d^{3/2}_{3/2, -3/2} = \frac{1}{2} \left( \frac{1 - \cos \theta}{2} \right) \sin \theta / 2 \left[ 5 \cos \theta + 3 \right] \]
\[ d^{5/2}_{3/2, -3/2} = \frac{1}{4} \left( \frac{1 - \cos \theta}{2} \right) \sin \theta / 2 \left[ 21 \cos^2 \theta + 18 \cos \theta + 1 \right] \]
\[ d^{7/2}_{3/2, -3/2} = \frac{1}{7} \left( \frac{1 - \cos \theta}{2} \right) \sin \theta / 2 \left[ 21 \cos^3 \theta + 21 \cos^2 - 2 \right] \]
\[ d^{9/2}_{3/2, -3/2} = \left( \frac{1 - \cos \theta}{2} \right) \sin \theta / 2 \left[ \frac{165}{8} \cos^4 \theta + \frac{45}{2} \cos^3 \theta - \frac{9}{4} \cos^2 \theta - \frac{11}{2} \cos \theta - \frac{3}{8} \right] \]

Table 15: \( d^{J}_{3/2, -3/2}(\theta) \)
J  Tables of the coefficients $C_{m,m'}^{n,J}$

The following tables result from the calculation of the coefficients $C_{m,m'}^{n,J}$, by approach 1 of section 6.3.1.

| $n = 1$ | $n = 2$ | $n = 3$ |
|---------|---------|---------|
| $\hat{u} + 1$ | $(\hat{u} + 1)(\hat{u} + 2)$ | $-\cos \theta$ |
| $\frac{1}{4}(9 \cos^2 \theta - 1)$ | $\frac{1}{2}[3d_{0,0}^0 + d_{0,0}^1]$ |

Table 16: $C_{0,0}^{n,J}$

| $n = 4$ | $n = 5$ |
|---------|---------|
| $(\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)$ | $(\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)(\hat{u} + 4)$ |
| $-8 \cos^3 \theta + 2 \cos \theta$ | $\frac{1}{16}(625 \cos^4 \theta - 250 \cos^2 \theta + 9)$ |
| $-\frac{2}{5}[8d_{0,0}^0 + 7d_{0,0}^1]$ | $\frac{125}{14}d_{0,0}^1 + \frac{250}{21}d_{0,0}^2 + \frac{19}{6}d_{0,0}^3$ |

Table 17: $C_{0,0}^{n,J}$

| $n = 1$ | $J = 0$ | 1 | 2 | 3 | 4 |
|---------|---------|---|---|---|---|
| $\frac{19}{6}$ | $\frac{250}{21}$ | 0 | $\frac{125}{14}$ |

Table 18: collection of $C_{0,0}^{n,J}$’s

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\[
\begin{align*}
\text{Table 19: } C^{n,J}_{2,-2} \\
n = 1 & \quad n = 2 & \quad n = 3 \\
\frac{t^2}{s^2} & \quad \frac{t^2}{s^2} (\hat{u} + 1) & \quad \frac{t^2}{s^2} (\hat{u} + 1)(\hat{u} + 2) \\
(1 - \cos \theta)^2 & \quad (1 - \cos \theta)^2 [ - \cos \theta] & \quad (1 - \cos \theta)^2 [\frac{1}{4}(9 \cos^2 \theta - 1)] \\
d^2_{2,-2} & \quad \frac{1}{3} [-d^3_{2,-2} + 2d^2_{2,-2}] & \quad \frac{1}{4}[\frac{9}{7}d^3_{2,-2} - 3d^3_{2,-2} + \frac{26}{7}d^2_{2,-2}] \\
\end{align*}
\]

\[
\begin{align*}
\text{Table 20: } C^{n,J}_{2,-2} \\
n = 4 \\
\frac{t^2}{s^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3) \\
(1 - \cos \theta)^2 [ - 8 \cos^3 \theta + 2 \cos \theta] \\
2 \left[ -\frac{4}{15}d^3_{2,-2} + \frac{24}{35}d^4_{2,-2} - d^3_{2,-2} + \frac{22}{27}d^2_{2,-2} \right] \\
\end{align*}
\]

\[
\begin{align*}
\text{Table 21: } C^{n,J}_{2,-2} \\
n = 5 \\
\frac{t^2}{s^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)(\hat{u} + 4) \\
(1 - \cos \theta)^2 \left[ \frac{1}{16}(625 \cos^4 \theta - 250 \cos^2 \theta + 9) \right] \\
\frac{125}{99}d^5_{2,-2} - \frac{1375}{396}d^2_{2,-2} + \frac{125}{22}d^4_{2,-2} -\frac{125}{18}d^3_{2,-2} + \frac{239}{36}d^2_{2,-2} \\
\end{align*}
\]

\[
\begin{align*}
\text{Table 22: Collection of } C^{n,J}_{2,-2} 's \\
n = 1 & \quad J = 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \\
2 & \quad \frac{2}{3} & \quad -\frac{1}{3} & \quad & \quad & \quad \\
3 & \quad \frac{13}{14} & \quad -\frac{3}{4} & \quad \frac{9}{28} & \quad & \quad \\
4 & \quad \frac{44}{21} & \quad -\frac{2}{3} & \quad \frac{48}{35} & \quad -\frac{8}{15} & \quad \\
5 & \quad \frac{239}{36} & \quad -\frac{125}{18} & \quad \frac{125}{22} & \quad -\frac{1375}{396} & \quad \frac{125}{99} \\
\end{align*}
\]
\[
\begin{array}{|c|c|c|}
\hline
n = 1 & n = 2 & n = 3 \\
\hline
\frac{t^2 u^2}{s^2} & \frac{t^2 u^2}{s^2} (\hat{u} + 1) & \frac{t^2 u^2}{s^2} (\hat{u} + 1)(\hat{u} + 2) \\
\hline
(\frac{1 - \cos \theta}{2}) \sin \frac{\theta}{2} & (\frac{1 - \cos \theta}{2}) \sin \frac{\theta}{2} \left[ - \cos \theta \right] & (\frac{1 - \cos \theta}{2}) \sin \frac{\theta}{2} \left[ \frac{1}{3} (9 \cos^2 \theta - 1) \right] \\
\hline
\frac{1}{2} d_{2,-1}^2 & -\frac{1}{6} \left[ \sqrt{\frac{8}{5}} d_{2,-1}^2 - d_{2,-1}^2 \right] & \frac{1}{2} \left[ \frac{3\sqrt{2}}{28} d_{2,-1}^4 - \frac{3}{4} \sqrt{\frac{2}{5}} d_{2,-1}^2 + \frac{2}{7} d_{2,-1}^2 \right] \\
\hline
\end{array}
\]

Table 23: \( C_{2,-1}^{n,J} \)

\[
\begin{array}{|c|}
\hline
n = 4 \\
\hline
\frac{t^2 u^2}{s^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3) \\
\hline
(\frac{1 - \cos \theta}{2}) \sin \frac{\theta}{2} \left[ - 8 \cos^3 \theta + 2 \cos \theta \right] \\
\hline
-\frac{16}{15 \sqrt{4}} d_{2,-1}^5 + \frac{12 \sqrt{2}}{35} d_{2,-1}^3 - \frac{2 \sqrt{2}}{35} d_{2,-1}^3 + \frac{2 \sqrt{2}}{35} d_{2,-1}^3 \\
\hline
\end{array}
\]

Table 24: \( C_{2,-1}^{n,J} \)

\[
\begin{array}{|c|}
\hline
n = 5 \\
\hline
\frac{t^2 u^2}{s^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)(\hat{u} + 4) \\
\hline
(\frac{1 - \cos \theta}{2}) \sin \frac{\theta}{2} \left[ \frac{1}{16} (625 \cos^4 \theta - 250 \cos^2 \theta + 9) \right] \\
\hline
\sqrt{\frac{2}{5}} \frac{625}{396} d_{2,-1}^5 - \frac{125}{72} \sqrt{\frac{2}{72}} d_{2,-1}^3 + \frac{625}{616} \sqrt{\frac{2}{616}} \sqrt{2} d_{2,-1} - \sqrt{\frac{2}{5}} \frac{125}{72} d_{2,-1}^3 + \frac{596}{63} d_{2,-1}^3 \\
\hline
\end{array}
\]

Table 25: \( C_{2,-1}^{n,J} \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
J = 2 & 3 & 4 & 5 & 6 \\
\hline
n = 1 & \frac{1}{2} & & & & \\
\hline
2 & \frac{1}{6} & -\frac{\sqrt{8}}{6 \sqrt{5}} & & & \\
\hline
3 & \frac{1}{7} & -3 \sqrt{2} & & & \\
\hline
4 & \frac{5}{21} & -2 \sqrt{2} & \frac{3}{8 \sqrt{5}} & 12 \sqrt{2} & -16 & 15 \sqrt{7} \\
\hline
5 & \frac{596}{63} & -25 \sqrt{2} & \frac{625}{616} & \sqrt{2} & -125 & \frac{2}{5 \sqrt{2}} & 625 \sqrt{2} \\
\hline
\end{array}
\]

Table 26: Collection of \( C_{2,-1}^{n,J} \)'s
| $n = 1$ | $n = 2$ | $n = 3$ |
|--------|--------|--------|
| $\frac{u + 1}{s^2}$ | $\frac{u + 1}{s^2}$ | $\frac{u + 1}{s^2}$ |
| $\cos \theta$ | $\cos \frac{\theta}{2} [-\cos \theta]$ | $\cos \frac{\theta}{2} \left[ \frac{1}{4}(9 \cos^2 \theta - 1) \right]$ |
| $d_{1/2,1/2}^{1/2}$ | $-\frac{1}{3} \left[ 2d_{1/2,1/2}^{3/2} + d_{1/2,1/2}^{1/2} \right]$ | $\frac{9}{10} d_{1/2,1/2}^{3/2} + \frac{2}{3} d_{1/2,1/2}^{5/2} + \frac{9}{5} d_{1/2,1/2}^{1/2}$ |

Table 27: $C_{1/2}^{n,j}$

| $n = 4$ |
|--------|
| $\frac{u + 1}{s^2}$ | $(\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)$ |
| $\cos \frac{\theta}{2}$ | $[-8 \cos^3 \theta + 2 \cos \theta]$ |
| $d_{1/2,1/2}^{1/2}$ | $-\frac{2}{35} \left[ 32d_{1/2,1/2}^{3/2} + 24d_{1/2,1/2}^{5/2} + \frac{98}{3} d_{1/2,1/2}^{5/2} + \frac{49}{3} d_{1/2,1/2}^{1/2} \right]$ |

Table 28: $C_{1/2}^{n,j}$

| $n = 5$ |
|--------|
| $\frac{u + 1}{s^2}$ | $(\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)(\hat{u} + 4)$ |
| $\cos \frac{\theta}{2}$ | $\left[ \frac{1}{16} (625 \cos^4 \theta - 250 \cos^2 \theta + 9) \right]$ |
| $d_{1/2,1/2}^{1/2}$ | $\frac{625}{126} d_{1/2,1/2}^{3/2} + \frac{250}{63} d_{1/2,1/2}^{5/2} + \frac{450}{21} d_{1/2,1/2}^{5/2} + \frac{100}{21} d_{1/2,1/2}^{1/2} + \frac{19}{6} d_{1/2,1/2}^{1/2}$ |

Table 29: $C_{1/2}^{n,j}$

| $J = \frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ |
|--------|--------|--------|--------|--------|
| $n = 1$ | 1 | | | |
| 2 | $-\frac{1}{3}$ | $-\frac{2}{3}$ | | |
| 3 | $\frac{1}{3}$ | $\frac{3}{5}$ | $\frac{9}{10}$ | | |
| 4 | $-\frac{14}{15}$ | $-\frac{28}{15}$ | $-\frac{48}{35}$ | $-\frac{64}{35}$ | |
| 5 | $\frac{19}{6}$ | $\frac{100}{21}$ | $\frac{450}{63}$ | $\frac{250}{63}$ | $\frac{625}{126}$ |

Table 30: Collection of $C_{1/2}^{n,j}$'s

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| $n = 1$ | $n = 2$ | $n = 3$ |
|----------|----------|----------|
| $\frac{i\hat{t}^2}{\hat{s}^2}$ | $\frac{i\hat{t}^2}{\hat{s}^2} (\hat{u} + 1)$ | $\frac{i\hat{t}^2}{\hat{s}^2} (\hat{u} + 1)(\hat{u} + 2)$ |
| $(\frac{1-\cos\theta}{2}) \sin \frac{\theta}{2}$ | $(\frac{1-\cos\theta}{2}) \sin \frac{\theta}{2} [-\cos \theta]$ | $(\frac{1-\cos\theta}{2}) \sin \frac{\theta}{2} [\frac{1}{4} (9 \cos^2 \theta - 1)]$ |
| $d_{3/2, -3/2}^{5/2}$ | $-\frac{1}{9} \left[2d_{3/2, -3/2}^{5/2} - 3d_{3/2, -3/2}^{5/2} \right]$ | $\frac{3}{7} d_{3/2, -3/2}^{5/2} - \frac{27}{35} d_{3/2, -3/2}^{5/2} + \frac{4}{5} d_{3/2, -3/2}^{5/2}$ |

Table 31: $C_{\frac{3}{2}, -\frac{3}{2}}^{n, J}$

| $n = 4$ |
|----------|
| $\frac{i\hat{t}^2}{\hat{s}^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)$ |
| $(\frac{1-\cos\theta}{2}) \sin \frac{\theta}{2} [-8 \cos^3 \theta + 2 \cos \theta]$ |
| $-\frac{16}{21} d_{3/2, -3/2}^{11/2} + \frac{32}{21} d_{3/2, -3/2}^{11/2} - \frac{204}{105} d_{3/2, -3/2}^{11/2} + \frac{62}{35} d_{3/2, -3/2}^{11/2}$ |

Table 32: $C_{\frac{3}{2}, -\frac{3}{2}}^{n, J}$

| $n = 5$ |
|----------|
| $\frac{i\hat{t}^2}{\hat{s}^2} (\hat{u} + 1)(\hat{u} + 2)(\hat{u} + 3)(\hat{u} + 4)$ |
| $(\frac{1-\cos\theta}{2}) \sin \frac{\theta}{2} \left[\frac{1}{16} (625 \cos^4 \theta - 250 \cos^2 \theta + 9)\right]$ |
| $\frac{125}{66} d_{3/2, -3/2}^{11/2} - \frac{625}{154} d_{3/2, -3/2}^{11/2} + \frac{125}{21} d_{3/2, -3/2}^{11/2} - \frac{275}{42} d_{3/2, -3/2}^{11/2} + \frac{233}{42} d_{3/2, -3/2}^{11/2}$ |

Table 33: $C_{\frac{3}{2}, -\frac{3}{2}}^{n, J}$
| $n = 1$ | $J = \frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ | $\frac{11}{2}$ |
|---------|------------------|-------------|-------------|-------------|-------------|
| $2$     | $\frac{3}{5}$   | $-\frac{2}{5}$ |             |             |             |
| $3$     | $\frac{4}{5}$   | $-\frac{27}{35}$ | $\frac{3}{7}$ |             |             |
| $4$     | $\frac{62}{35}$ | $-\frac{204}{105}$ | $\frac{32}{21}$ | $-\frac{16}{21}$ |             |
| $5$     | $\frac{233}{42}$ | $-\frac{275}{42}$ | $\frac{125}{21}$ | $-\frac{625}{154}$ | $\frac{125}{66}$ |

Table 34: Collection of $C_{\frac{3}{2}}^{n,J}$'s
Most of the drawings were created using the program JaxoDraw [85].

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