STRATIFICATION OF ALGEBRAIC QUOTIENTS
AND CHARACTER VARIETIES

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Abstract. In this paper, we study a weaker version of algebraic quotient for the action of an algebraic group on an algebraic variety that is well behaved under stratification. Focusing on the topological properties of these quotients, we obtain a series of results about their structure and uniqueness. As an application, we compute the Deligne-Hodge polynomials of \( \text{SL}_2(\mathbb{C}) \)-character varieties for free groups and surface groups, as well as their counterparts with punctures of Jordan type.

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1. Introduction

Let \( \Gamma \) be a finitely generated group and let \( G \) be a complex algebraic group. The set of representations \( \rho : \Gamma \to G \) can be endowed with the structure of a complex algebraic variety, the so-called representation variety of \( \Gamma \) into \( G \), denoted \( X_G(\Gamma) \). The group \( G \) itself acts on \( X_G(\Gamma) \) by conjugation so, taking the Geometric Invariant Theory quotient of \( X_G(\Gamma) \) by this action, we obtain the moduli space of representations \( R_G(\Gamma) = X_G(\Gamma) \sslash G \), also known as the character variety.

These varieties have been object of studies during the past twenty years, in part due to their prominent role in the non-abelian Hodge theory. In this context, we consider \( \Gamma = \pi_1(\Sigma) \), where \( \Sigma \) is a closed orientable surface, and it is customary to just denote \( R_G(\Sigma) = R_G(\pi_1(\Sigma)) \). For \( G = \text{GL}_n(\mathbb{C}) \) (resp. \( G = \text{SL}_n(\mathbb{C}) \)), the Riemann-Hilbert correspondence \([34, 35]\) gives a real analytic correspondence between \( R_G(\Sigma) \) and the moduli space of flat bundles of rank \( n \) and degree 0 (resp. and trivial determinant). Moreover, we can remove a point \( p \in \Sigma \) and just consider representations sending the generator of \( \pi_1(\Sigma - \{p\}) \) around \( p \) to the primitive root of unit \( e^{2\pi id/n} \) (this is an example of a parabolic structure, see Section 5). In this case, we obtain the so-called twisted character variety which is diffeomorphic to the moduli space of rank \( n \) logarithmic flat bundles of degree \( d \) with a pole at \( p \) with residue \( -\frac{4}{n} \text{Id} \). In the same spirit, the Hitchin-Kobayashi correspondence \([33, 36]\) shows that \( R_{\text{GL}_n(\mathbb{C})}(\Sigma) \) (resp. twisted) is real analytic equivalent to the moduli space of rank \( n \) and degree 0 (resp. degree \( d \)) Higgs bundles (see [17]).

However, these correspondences are far from being algebraic. For this reason, it is important to analyze the Hodge structures on these character varieties. An useful combinatorial tool for this purpose is the so-called Deligne-Hodge polynomial, also referred to as the \( E \)-polynomial, which is an assignment of a polynomial \( e(X) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \) to
every complex algebraic variety $X$. As described in Section 5, this polynomial is constructed as an alternating sum of the Hodge numbers of $X$, in the spirit of a combination of the Poincaré polynomial and the Euler characteristic.

The first strategy for the computation of $E$-polynomials of character varieties was accomplished by [16] by means of a theorem of Katz. Following this method, an expression of these polynomials for twisted character varieties was given in terms of generating functions in [16] for $G = \text{GL}_n(\mathbb{C})$ and in [28] for $G = \text{SL}_n(\mathbb{C})$. In the general parabolic case, much remains to be known, being the most important advance [15] in which the Deligne-Hodge polynomial is computed for $G = \text{GL}_n(\mathbb{C})$ and generic semi-simple marked points.

The other approach to this problem was initiated by Logares, Muñoz and Newstead in [22]. The strategy is to focus on the computation of the $E$-polynomial of $X_G(\Sigma)$ by chopping it into simpler strata for which the $E$-polynomial can be easily computed. Following these ideas, in the case $G = \text{SL}_2(\mathbb{C})$, they computed the $E$-polynomials for a single marked point and genus $g = 1, 2$. Later, it was computed for two marked points and $g = 1$ in [21] and for a marked point and $g = 3$ in [26]. In the case of arbitrary genus and, at most, a marked point, the case $G = \text{SL}_2(\mathbb{C})$ was accomplished in [27] and the case $G = \text{PGL}_2(\mathbb{C})$ in [23]. In the former papers, recursive formulas for the $E$-polynomials are obtained in terms of the genus. Using a mixed method, explicit expressions of the $E$-polynomials have been computed in [1] for orientable surfaces with $G = \text{GL}_3(\mathbb{C}), \text{SL}_3(\mathbb{C})$ and for non-orientable surfaces with $G = \text{GL}_2(\mathbb{C}), \text{SL}_2(\mathbb{C})$.

Inspired by the geometric method, a new framework is described in [13] in which these calculations can be understood. In that paper, it is proven that there exists a lax monoidal Topological Quantum Field Theory that computes the Deligne-Hodge polynomial of the representation variety of any compact manifold with any parabolic structure. Using this TQFT, in [12] were computed explicit expressions of the $E$-polynomials of the representation varieties for $\text{SL}_2(\mathbb{C})$ and parabolic structures with any number of punctures and Jordan type conjugacy classes. However, it is still needed a procedure that, given such a polynomial of the representation variety, computes the one for the corresponding character variety. The purpose of this paper is to fill that gap shedding some light about the structure of the quotient.

In order to do that, we will need to consider weak quotients that go beyond the scope of classical GIT as reviewed in Section 2. More precisely, let $X$ be an algebraic variety with an action of an algebraic group $G$ such that the GIT quotient $\pi : X \to X \sslash G$ is good. Suppose that we can decompose $X = Y \sqcup U$ with $Y \subseteq X$ closed and $U \subseteq X$ open, both invariant for the action. In general, $X \sslash G \neq (Y \sslash G) \sqcup (U \sslash G)$ so the GIT quotient is not well-behaved under decompositions. The problem is that, while $U \to \pi(U)$ is still a good quotient (so $\pi(U) = U \sslash G$), the closed part $\pi|_Y : Y \to \pi(Y)$ may no longer be a good quotient since they could exist $G$-invariant functions on $Y$ that do not factorize through $\pi|_Y$. 

Nonetheless, the closed part $Y \rightarrow \pi(Y)$ holds all the topological properties of good quotients. For this reason, in Section 3 we introduce the notion of pseudo-quotient as a kind of weak quotient regarding only the topological properties of good quotients. Using this approach, we will prove in Section 4 Theorem 4.1 that, if $X = Y \sqcup U$ as above and $\pi : X \rightarrow \overline{X}$ is a pseudo-quotient for the action of $G$, then $\pi : Y \rightarrow \pi(Y)$ and $\pi : U \rightarrow \pi(U)$ are pseudo-quotients. Moreover, pseudo-quotients allow us to obtain a sort of quotient even in the case that $G$ is not reductive. In this sense, pseudo-quotients are a simpler tool than the sophisticated techniques used for constructing non-reductive GIT quotients, as in [3], [9] and [19].

On the other hand, pseudo-quotients are no longer unique but, as we will show in Section 5, their classes in the $K$-theory of complex algebraic varieties are so. In particular, this is enough to ensure that their $E$-polynomials agree. Putting together these two facts, we obtain the following result.

**Theorem 1.1.** Let $X$ be a complex algebraic variety with a linear action of a reductive group $G$. If we decompose $X = Y \sqcup U$ with $Y \subseteq X$ closed and $U$ orbitwise-closed (see Definition 3.3), then

$$e(X \sslash G) = e(Y \sslash G) + e(U \sslash G).$$

As an application to character varieties, we have a decomposition of the representation variety as $X_G(\Gamma) = X_G^r(\Gamma) \sqcup X_G^i(\Gamma)$, where $X_G^r(\Gamma)$ denotes the set of reducible representations and $X_G^i(\Gamma)$ the set of irreducible ones. In that case, the results of Section 6 will imply that

$$e(R_G(\Gamma)) = e(X_G^r(\Gamma) \sslash G) + e(X_G^i(\Gamma) \sslash G).$$

In this way, each stratum can be analyzed separately. For the stratum $X_G^r(\Gamma)$, the situation is quite simple since the action on it is closed and (essentially) free, so $e(X_G^r(\Gamma) \sslash G)$ is just the quotient of the $E$-polynomial of $X_G^r(\Gamma)$ over the $E$-polynomial of $G/G^0$, where $G^0 \subseteq G$ is the center of $G$.

For the stratum $X_G^i(\Gamma)$, the situation is a bit harder. The idea here is that the closures of the orbits of elements of $X_G^i(\Gamma)$ always intersect the subvariety of diagonal representations. This is a geometric situation that we call a core (see Proposition 4.4) and implies that the GIT quotient is isomorphic to the quotient of the diagonal representations under permutation of eigenvalues. Therefore, for this stratum, the calculation reduces to the analysis of a quotient by a finite group.

Using this idea, in Section 7 we will recompute, for $G = \text{SL}_2(\mathbb{C})$, the Deligne-Hodge polynomial of character varieties of free groups and surface groups from the ones of the corresponding representation variety, reproving the results of [26]. Moreover, in Section 8 we will explore the parabolic case and we will compute the Deligne-Hodge polynomial of $\text{SL}_2(\mathbb{C})$-parabolic character varieties of free and surface groups with any number punctures with conjugacy classes of Jordan type. In the case of surface groups, the obtained result is the following.
Theorem 1.2. Fix $G = SL_2(\mathbb{C})$. Let $\Sigma_g$ be the genus $g$ compact orientable surface, let $Q = \{(p_1, \lambda_1), \ldots, (p_s, \lambda_s)\}$ be a parabolic structure with $\lambda_i = [-Id], [J_+], [J_-]$ and let $R_{SL_2(\mathbb{C})}(\Sigma_g, Q)$ be the corresponding $SL_2(\mathbb{C})$-character variety. Let $r_+$ be the number of $J_+$, $r_-$ the number of $J_-$ and $t$ the number of $-Id$ in $Q$. Denote $r = r_+ + r_-$ and $\sigma = (-1)^{t+r_-}$.

- If $\sigma = 1$, then

$$e(R_{SL_2(\mathbb{C})}(\Sigma_g, Q)) = (q^2 - 1)^{2g+r-2} q^{2g-2} + (-1)^r 2^{2g}(q - 1)q^{2g-2}(1 - (1 - q)^{r-1})$$

$$+ \frac{1}{2}(q - 1)^{2g+r-2} q^{2g-2} (2^{2g} + q - 3)$$

$$+ \frac{1}{2}(q + 1)^{2g+r-2} q^{2g-2} (2^{2g} + q - 1).$$

- If $\sigma = -1$, then

$$e(R_{SL_2(\mathbb{C})}(\Sigma_g, Q)) = (-1)^{r-1} 2^{2g-1} (q + 1)^{2g+r-2} q^{2g-2}$$

$$+ (q - 1)^{2g+r-2} q^{2g-2} ((q + 1)^{2g+r-2} + 2^{2g-1} - 1).$$

So far, these polynomials were only known in the case of a single marked point ([26]) or two marked points and genus one ([21]).

As will be clear from the calculations, the general case of any conjugacy class does not present further difficulties than for Jordan type classes. However, as in the previous cases, in order to compute the Deligne-Hodge polynomial of the character variety, the one of the representation variety is needed. Unfortunately, this datum is unknown at the present time, so the calculation cannot be completed. For that reason, future work would be to address the computation of that polynomials using the TQFT methods of [13] and [12] and to compute the one of the character variety using the ideas shown in the present paper. Also, it is necessary to explore the higher rank case in order to envisage how to extend the techniques of this paper to that context.

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2. Review of Geometric Invariant Theory

Except when explicitly said, along this section and sections 3 and 4, we will work over an arbitrary algebraically closed field $k$. In this section, we will review some of the most important notions of Geometric Invariant Theory (GIT for short). The definitions are essentially the ones used in Newstead’s book [31].

In order to fix some notation, given an algebraic group $G$ acting algebraically on a variety $X$ we will denote by $Gx$ (or by $[x]$ when $G$ is clear from the context) the orbit of $x \in X$ and by $\overline{Gx}$ (or by $\overline{[x]}$) its Zariski closure. In general, the space of orbits $X/G$ has no structure of algebraic variety so we need to consider more subtle quotients.

Recall that a categorical quotient of $X$ by $G$ is a $G$-invariant regular morphism $\pi : X \to Y$ onto some algebraic variety $Y$ such that, for any $G$-invariant regular morphism $f : X \to Z$, with $Z$ and algebraic variety, there exists an unique regular morphism $\tilde{f} : Y \to Z$ such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow{\pi} & & \downarrow{\tilde{f}} \\
Y & \nearrow & 
\end{array}
\]

Using this universal property, it follows that the categorical quotient, if exists, is unique up to regular isomorphism.

Remark 2.1. Along this paper, we will always work with categorical quotients within the category of algebraic varieties and regular morphisms. However, sometimes in the literature, larger categories are considered.

Example 2.2. Let $X = \text{Spec}(R)$ be an affine algebraic variety, where $R$ is a finitely generated torsion-free $k$-algebra. By considering $R$ as the algebra of regular functions on $X$, the action of $G$ on $X$ induces an action on $R$. By Nagata’s theorem (see [30] or Theorem 3.4 of [31]), if $G$ is reductive (i.e. if its radical is isomorphic to a torus group) then the ring of $G$-invariant elements of $R$, $R^G$, is finitely generated. Therefore, taking $Y = \text{Spec}(R^G)$, the inclusion $R^G \hookrightarrow R$ induces a regular morphism $\pi : X \to Y$ that can be shown to be a categorical quotient.

The definition of a categorical quotient does not say anything about the geometry of $Y$. Trying to capture these geometric properties, a regular morphism $\pi : X \to Y$ is called a good quotient if it satisfies:

i) $\pi$ is $G$-invariant.

ii) $\pi$ is surjective.

iii) For any open set $U \subseteq Y$, $\pi$ induces an isomorphism

\[
\pi^* : \mathcal{O}_Y(U) \xrightarrow{\cong} \mathcal{O}_X(\pi^{-1}(U))^G \subseteq \mathcal{O}_X(\pi^{-1}(U)),
\]

where $\mathcal{O}_X$ is the sheaf of regular functions on $X$. 
iv) If $W \subseteq X$ is a closed $G$-invariant set, then $\pi(W) \subseteq Y$ is closed.

v) Given two closed $G$-invariant subsets $W_1, W_2 \subseteq X$, $W_1 \cap W_2 = \emptyset$ if and only if $\pi(W_1) \cap \pi(W_2) = \emptyset$.

If $\pi : X \to Y$ is a good quotient, then it is a categorical quotient (see Corollary 3.5.1 of [31]). On the other side, the categorical quotients described in Example 2.2 actually are good quotients (see Theorem 3.5 of [31]).

The last useful standard quotient considered in the literature is the so-called geometric quotient. It is a good quotient $\pi : X \to Y$ such that, for any $y \in Y$, $\pi^{-1}(y) = Gy$. This last condition is sometimes referred to as an orbit space. Obviously, geometric quotients are categorical so they are unique. Observe that, from the properties of good quotients, a geometric quotient is the same as a good quotient in which the action of $G$ on $X$ is closed (i.e. $Gx$ is a closed subset of $X$ for all $x \in X$).

Remark 2.3. Sometimes in the literature, good quotients are referred to as good categorical quotients and geometric quotients are called good geometric quotients. Moreover, some authors (notably [31]) add to the definition of good quotient the requirement that $\pi$ is affine. The inclusion of this hypothesis is purely technical and is only useful for proving some restrictions to the existence of geometric quotients (see Proposition 3.24 and Remark 3.25 of [31]). In particular, it has no influence in the validity of existence results. Actually, the addition of this hypothesis is not too restrictive since most times it can be taken for granted (see Propositions 0.7 and 0.8 of [29]).

In this framework, GIT deals with the problem of existence of these quotients for the action of a reductive group $G$ on an algebraic variety $X$. Suppose also that the action of $G$ is linearizable, meaning that there exists a line bundle $L \to X$ with a fiberwise linear action of $G$ compatible with the one on $X$. Observe that, if $L$ is ample, then some tensor power of $L$ gives an embedding $X \hookrightarrow \mathbb{P}^N$ for $N$ large enough. In this setting, the linearization reduces to a linear representation $G \to \text{GL}_{N+1}(k)$ such that, when restricted to $X \subseteq \mathbb{P}^N$, gives the original action.

If $L \to X$ is a fixed linearization of the action of $G$, then a point $x \in X$ is called semi-stable if there exists a $G$-invariant section $f$ of $L^r$ for some $r > 0$ such that $f(x) \neq 0$ and $X_f = \{x \in X \mid f(x) \neq 0\} \subseteq X$ is affine. If the action of $G$ on $X_f$ is, in addition, closed and $\dim Gx = \dim G$, then $x$ is called stable. The set of semi-stable and stable points are open subsets of $X$ and we will denote them by $X^{SS}$ and $X^S$, respectively.

With these definitions, the most important result of GIT about the existence of quotients says that if $G$ is a reductive group acting via a linearizable action on a (quasi-projective) variety $X$, then there exists a good quotient on $X^{SS}$ that restricts to a geometric quotient on $X^S$. It is customary to call this good quotient the GIT quotient and denote it by $X^{SS} \to X // G$, or $X^{SS} \sslash G$ when we want to emphasize that it is defined only on $X^{SS}$. The proof of this result is just an appropriate gluing of the good quotients constructed in Example 2.2 for an affine covering of $X^{SS}$. Despite that, a priori, the
result of this gluing is an algebraic scheme, if we push forward the ample line bundle \( L|_{X^{ss}} \to X^{ss} \) to \( X \sslash G \) we obtain an ample line bundle there that embeds it into the projective space, turning the scheme into a variety.

3. **Pseudo-quotients**

Let \( \pi : X \to Y \) be a good quotient. If \( U \subseteq Y \) is an open subset, then the restriction \( \pi : \pi^{-1}(U) \to U \) is again a good quotient (see Theorem 3.10 of \[31\]). However, if we take a closed set \( W \subseteq Y \), the restriction \( \pi : \pi^{-1}(W) \to W \) could be no longer a good quotient. However, the topological properties of the good quotient remain valid when restricted to \( \pi^{-1}(W) \), so it is useful to collect them as a kind of weaker quotient.

**Definition 3.1.** Let \( X \) be an algebraic variety with an action of an algebraic group \( G \). A **pseudo-quotient** for the action of \( G \) on \( X \) is a surjective \( G \)-invariant regular morphism \( \pi : X \to Y \) such that, for any disjoint \( G \)-invariant closed sets \( W_1, W_2 \subseteq X \),

\[
\pi(W_1) \cap \pi(W_2) = \emptyset.
\]

**Remark 3.2.** Suppose that \( \pi : X \to Y \) is a pseudo-quotient:

i) Let \( x_1, x_2 \in X \). Since \( \pi \) is \( G \)-invariant, \( \pi \) maps every point of \( Gx_i \) into the same point of \( Y \) (i.e. \( \pi(x_i) \)). Therefore, since the \( Gx_i \) are closed \( G \)-invariant sets, \( Gx_1 \cap Gx_2 = \emptyset \) if and only if \( \pi(x_1) \neq \pi(x_2) \).

ii) Let \( W \subseteq X \) be a \( G \)-invariant closed set and suppose that \( \pi(W) \) were not closed. Then, for any \( y \in \pi(W) \setminus \pi(W) \), we would have that \( \pi^{-1}(y) \) and \( W \) are closed \( G \)-invariant sets so \( \{y\} \cap \pi(W) = \emptyset \), which is impossible. Thus, the image of any \( G \)-invariant closed set is closed. In particular, good quotients are pseudo-quotients.

iii) Let \( U \subseteq Y \) be an open set and let \( \pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U)) \) be the induced ring morphism. Since \( \pi \) is \( G \)-invariant, this morphism factorizes through the inclusion \( \mathcal{O}_X(\pi^{-1}(U))^G \subseteq \mathcal{O}_X(\pi^{-1}(U)) \) so it defines a ring morphism \( \pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))^G \). However, in contrast with good quotients, we require no longer this morphism to be an isomorphism. For all these reasons, a pseudo-quotient is a regular map satisfying conditions i), ii), iv) and v) of a good quotient, but maybe failing iii).

iv) If \( \pi : X \to Y \) is a pseudo-quotient and \( W \subseteq X \) is a closed \( G \)-invariant set, then the restriction \( \pi : W \to \pi(W) \) is also a pseudo-quotient. For open sets, an easy adaptation of Lemma 3.6 of \[31\] shows that, if \( U \subseteq Y \) is an open set, then \( \pi : \pi^{-1}(U) \to U \) is a pseudo-quotient.

**Definition 3.3.** Let \( X \) be a variety and let \( G \) be an algebraic group acting on \( X \). A subset \( A \subseteq X \) is said to be **orbitwise-closed** if, for any \( a \in A \), \( Ga \subseteq A \). Analogously, \( A \) is said to be **completely orbitwise-closed** if both \( A \) and \( X - A \) are orbitwise-closed.

**Example 3.4.** A closed invariant subset is orbitwise-closed. An open orbitwise-closed set is completely orbitwise-closed.
Lemma 3.5. Let $G$ be an algebraic group acting on a variety $X$ and let $\pi : X \to Y$ be a pseudo-quotient.

i) If $A \subseteq X$ is completely orbitwise-closed, then it is saturated for $\pi$, i.e. $\pi^{-1}(\pi(A)) = A$.

ii) If $U$ is open and orbitwise-closed, then $\pi(U)$ is open. Moreover, $\pi|_U : U \to \pi(U)$ is a pseudo-quotient.

Proof. For i) trivially, $A \subseteq \pi^{-1}(\pi(A))$. For the other inclusion, if $x \in \pi^{-1}(\pi(A))$ then $\pi(x) = \pi(a)$ for some $a \in A$. Hence, since $\pi$ is a pseudo-quotient, $Gx \cap Ga \neq \emptyset$ which implies that $x \in A$ since $A$ is orbitwise-closed. For ii) observe that, by Remark 3.2, if we set $W = X - U$, then $\pi(W)$ is closed so $Y - \pi(W)$ is open. But, since $U$ is completely orbitwise-closed and $\pi$ is surjective, $Y = \pi(U) \cup \pi(W)$, so $\pi(U) = Y - \pi(W)$ is open. Finally, the fact that $\pi|_U : U \to \pi(U)$ is a pseudo-quotient follows from Remark 3.2 (iv) since $U$ is saturated.

Example 3.6. In contrast with categorical quotients, pseudo-quotients may not be unique. As an example, let us take $X = \mathbb{A}^2$ and $G = k^*$ acting by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$, for $\lambda \in k^*$ and $(x, y) \in \mathbb{A}^2$. Since the ring of $G$-invariant regular functions on $X$ is $\mathcal{O}_X(X)^G = k[xy]$, standard GIT results show that the inclusion $k[xy] \hookrightarrow k[x, y] = \mathcal{O}_X(X)$ induces a good quotient $\pi : X \to \mathbb{A}^1$. Now, let $C = \{y^2 = x^3\} \subseteq \mathbb{A}^2$ be the nodal cubic curve. The map $\alpha : \mathbb{A}^1 \to C$, $\alpha(t) = (t^2, t^3)$, is a regular bijective morphism so $\alpha \circ \pi : X \to C$ is a pseudo-quotient for $X$. However, $C$ is not isomorphic to $\mathbb{A}^1$ since $C$ is not normal.

The previous example is general in the way that, if $\pi : X \to Y$ is a pseudo-quotient and $\alpha : Y \to Y'$ is any regular bijective morphism, then $\alpha \circ \pi : X \to Y'$ is also a pseudo-quotient. However, this non-uniqueness of pseudo-quotients is, in some sense, the prototype of fail. We are going to devote the rest of this section to study this kind of uniqueness properties. The first result in this direction is the following.

Proposition 3.7. Let $X$ be an algebraic variety acted by an algebraic group $G$. Suppose that $\pi : X \to Y$ and \( \pi' : X \to Y' \) are pseudo-quotients and that $\pi$ is a categorical quotient. Then, there exists a regular bijective morphism $\alpha : Y \to Y'$.

Proof. By definition, the map $\pi' : X \to Y'$ is $G$-invariant so, using the categorical property of $Y$, it defines a regular map $\alpha : Y \to Y'$ such that $\pi' = \alpha \circ \pi$. The surjectivity of $\alpha$ follows from the one of $\pi'$. For the injectivity, suppose that $\alpha(y) = \alpha(y')$ for some $y, y' \in Y$. Then, there exists $x, x' \in X$ such that $y = \pi(x)$ and $y' = \pi(x')$ so $\pi'(x) = \alpha(\pi(x)) = \alpha(\pi(x')) = \pi'(x')$. Thus, since $\pi'$ is a pseudo-quotient, $Gx \cap Gx' \neq \emptyset$ so $y = \pi(x) = \pi(x') = y'$.

If there exists a pseudo-quotient $\pi : X \to Y$ which is also categorical, we will say that $X$ admits a categorical pseudo-quotient. Observe that, if that occurs, any categorical
quotient of $X$ is automatically a pseudo-quotient. This happens, for example, if $G$ is a reductive group acting linearly on $X$ and all the points of $X$ are semi-stable for the action.

In order to obtain stronger results about uniqueness of pseudo-quotients, for the rest of the section we will suppose that $k$ is an algebraically closed field of characteristic zero.

**Remark 3.8.**

1) If there exists a regular bijective morphism $\alpha : X \to Y$ between algebraic varieties, then $X$ and $Y$ define the same element in the $K$-theory of algebraic varieties (see [2] for the definition). This is a consequence of the fact that, in characteristic zero, every dominant injective regular morphism is birational (essentially, because every field extension is separable and, thus, the degree of $\alpha$ is the degree of the field extension $K(X)/K(Y)$). Thus, there exists proper subvarieties $X' \subseteq X$ and $Y' \subseteq Y$ such that $X - X'$ and $Y - Y'$ are isomorphic. Therefore, arguing inductively on $X'$ and $Y'$, there exists a stratification of $X$ and another one for $Y$ with isomorphic pieces by pairs. Hence, in the $K$-theory of algebraic varieties, $X$ and $Y$ define the same object.

2) If $\alpha : X \to Y$ is a regular bijective morphism and $Y$ is normal then, indeed, $\alpha$ is an isomorphism. To check that, observe that, as we argued above, $\alpha$ is a birational equivalence. Then, by Zariski’s main theorem (see [14]), $\alpha$ is a regular isomorphism of $X$ with some open subvariety $V \subseteq Y$. But, since $\alpha$ is surjective, $V = Y$, proving that $\alpha$ was an isomorphism. In particular, if $\pi : X \to Y'$ is a pseudo-quotient with $Y'$ normal and the action on $X$ admits a categorical pseudo-quotient $X \to Y$ then $Y \cong Y'$ and $\pi$ is categorical too.

In the general case, it can happens that the categorical quotient does not exist. Even in this case, in characteristic zero it is possible to compare pseudo-quotients. The key point is the following proposition that adapts Proposition 0.2 of [29] to the context of pseudo-quotients. Also, compare it with Remark 3.8 above.

**Proposition 3.9.** Let $\pi : X \to Y$ be a pseudo-quotient for the action of some algebraic group $G$ on $X$. If $X$ is irreducible and $Y$ is normal, then $\pi$ is a good quotient.

**Proof.** As we mentioned in Remark 3.2(ii), it is enough to prove that, for any open set $U \subseteq Y$, the induced ring morphism $\pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))^G \subseteq \mathcal{O}_X(\pi^{-1}(U))$ is an isomorphism. Recall that, since $\pi$ is $G$-invariant, $\pi^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(\pi^{-1}(U))^G$. Moreover, since $\pi$ is surjective, $\pi^*$ is injective.

In order to prove the surjectivity of $\pi^*$, let $f : \pi^{-1}(U) \to k = \mathbb{A}^1$ be a regular $G$-invariant function. We want to show that $f$ can be lifted to a regular morphism $\tilde{f} : U \to \mathbb{A}^1$ such that $f = \tilde{f} \circ \pi$. The only possible candidate is $\tilde{f}(y) = f(x)$ for any $x \in \pi^{-1}(y)$, which is well-defined since $\pi$ is a pseudo-quotient. Observe that $\tilde{f}$ is continuous in the Zariski topology. To check that, let $W \subseteq \mathbb{A}^1$ a closed set. Then, the set $f^{-1}(W)$ is a closed $G$-invariant set so, by Remark 3.2(ii), $\pi(f^{-1}(W)) \subseteq U$ is closed. But, by construction, $\pi(f^{-1}(W)) = \tilde{f}^{-1}(W)$, proving the continuity of $\tilde{f}$. 


Therefore, it is enough to prove that \( \tilde{f} : U \to \mathbb{A}^1 \) is regular. For this purpose, construct the morphism \( \pi' = f \times \pi : \pi^{-1}(U) \to \mathbb{A}^1 \times U \) and let \( U' \subseteq \mathbb{A}^1 \times U \) be the closure of \( \pi'((\pi^{-1}(U)) \). Denoting by \( p_1 : \mathbb{A}^1 \times U \to \mathbb{A}^1 \) and \( p_2 : \mathbb{A}^1 \times U \to U \) the first and second projections respectively and \( \omega = p_2|_{U'} \) we have a commutative diagram

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\pi'} & U' \subseteq \mathbb{A}^1 \times U \\
\downarrow{\pi} & & \downarrow{\omega} \\
U & & \mathbb{A}^1 \\
\end{array}
\]

Observe that, in order to finish the proof, it is enough to prove that \( \omega \) is an isomorphism, since, in that case \( \tilde{f} = p_1 \circ \omega^{-1} \) would be regular. First of all, \( \omega \) is surjective since \( \pi \) is. Moreover, \( \omega \) is injective on \( \pi'((\pi^{-1}(U)) \subseteq U' \) since, for any \( x, x' \in \pi^{-1}(U) \), if \( \omega(\pi(x)) = \omega(\pi(x')) \) then \( \pi(x) = \pi(x') \) which happens if and only if \( Gx \cap Gx' \neq \emptyset \). In that case, since \( f \) is \( G \)-invariant, \( f(x) = f(x') \) and, thus, \( \pi'(x) = (f(x), \pi(x)) = (f(x'), \pi(x')) = \pi'(x'). \)

Therefore, in order to conclude that \( \omega \) is bijective, it is enough to prove that \( \pi' \) is surjective. Let \( A = U' - \pi'(\pi^{-1}(U)) \) which can be described as the set of points \( z \in U' \) such that, for any \( x \in \pi^{-1}(U) \) with \( \pi(x) = \omega(z) \), it holds \( \pi'(x) \neq z \). Using the continuity of \( \tilde{f} \), we can rewrite \( A \) as the open set

\[
a = \{(l, y) \in U' \mid \tilde{f}(y) \neq l \}.
\]

Thus, if \( A \) was non-empty, then, since \( \pi'(\pi^{-1}(U)) \subseteq U' \) is dense, \( \pi'(\pi^{-1}(U)) \cap A \neq \emptyset \), which is impossible.

Hence, \( \pi' \) is surjective and, thus, \( \omega : U' \to U \) is a regular bijective morphism. But, since the characteristic of \( k \) is zero, as mentioned in Remark 3.8 Zariski's main theorem implies that \( \omega \) is an isomorphism, as desired. \( \square \)

**Corollary 3.10.** Let \( X \) be an algebraic variety with an action of an algebraic group \( G \). For any pseudo-quotients \( \pi_1 : X \to Y_1 \) and \( \pi_2 : X \to Y_2 \), the varieties \( Y_1 \) and \( Y_2 \) define the same element in the \( K \)-theory of complex algebraic varieties.

**Proof.** First of all, observe that, restricting to the irreducible components of \( X \) if necessary, we can suppose that \( X \) is irreducible. Let \( Y_i' \subseteq Y_i \) be the open subset of normal points of \( Y_i \) for \( i = 1, 2 \). Since the \( \pi_i^{-1}(Y_i') \subseteq X \) are open orbitwise-closed sets, \( U = \pi_1^{-1}(Y_1') \cap \pi_2^{-1}(Y_2') \subseteq X \) is an open orbitwise-closed set. Therefore, the restrictions \( \pi_i|_U : U \to \pi_i(U) \subseteq Y_i' \) are pseudo-quotients onto normal varieties so, by Proposition 3.9, they are good quotients. In particular, they are categorical quotients so, by uniqueness, \( \pi_1(U) \) is isomorphic to \( \pi_2(U) \).

Therefore, proceeding inductively on \( X - U \), we find an stratification \( Y_1 = Z_1 \cup \ldots Z_s \) and \( Y_2 = \bar{Z}_1 \cup \ldots \bar{Z}_s \) such that \( Z_j \) is isomorphic to \( \bar{Z}_j \) for \( j = 1, \ldots, s \), so they define the same object in the \( K \)-theory of algebraic varieties. \( \square \)
4. Stratification techniques of quotients

In this section, we will show that the previous results can be used to reconstruct the quotient of an algebraic variety from the corresponding quotients of the pieces of a suitable decomposition. This kind of arguments will be very useful in the following computations. Except when explicitly noted, the arguments of this section works on any algebraically closed ground field \( k \).

**Theorem 4.1.** Let \( X \) be an algebraic variety with an action of an algebraic group \( G \) with a pseudo-quotient \( \pi : X \to \overline{X} \). Suppose that we have a decomposition \( X = Y \sqcup U \) where \( Y \) is a closed subvariety and \( U \) is an open orbitwise-closed subvariety. Then, we have a decomposition
\[
\overline{X} = \pi(Y) \sqcup \pi(U)
\]
where \( \pi(U) \subseteq \overline{X} \) is open, \( \pi(Y) \subseteq \overline{Y} \) is closed and the maps \( \pi|_U : U \to \pi(U) \) and \( \pi|_Y : Y \to \pi(Y) \) are pseudo-quotients.

Furthermore, if \( k \) has characteristic zero then, for any pseudo-quotients \( Y \to \overline{Y} \) and \( U \to \overline{U} \), we have that \( [X] = [Y] + [U] \) in the \( K \)-theory of algebraic varieties.

**Proof.** The decomposition \( \overline{X} = \pi(Y) \sqcup \pi(U) \) and properties of \( \pi(Y) \) and \( \pi(U) \) follows immediately from the surjectivity of \( \pi \) together with Remark 3.2 iv and Lemma 3.5 ii. For the last part, use Corollary 3.10. \( \square \)

**Remark 4.2.** Along this remark, with the notations of Theorem 4.1, suppose that \( \pi : X \to \overline{X} \) is good.

- In that case, \( \pi|_U : U \to \pi(U) \) is also good. However, a priori nothing more can be said about the closed part \( \pi|_Y : Y \to \pi(Y) \).
- Furthermore, if \( \pi(Y) \) is normal and \( k \) has characteristic zero, then, for any categorical quotients \( Y \to \overline{Y} \) and \( U \to \overline{U} \) we have \( \overline{X} = \overline{Y} \sqcup \overline{U} \). It follows immediately from the uniqueness of categorical quotients together with the observation that the pseudo-quotient \( \pi|_Y : Y \to \pi(Y) \) is good by Proposition 3.9.

**Example 4.3.** The hypothesis that \( \pi : X \to \overline{X} \) is a pseudo-quotient is needed in Theorem 4.1 even if \( k = \mathbb{C} \). Consider \( G = \mathbb{C} \) and \( X = \mathbb{A}^2 \) with the action \( \lambda \cdot (x, y) = (x, y + \lambda x) \), for \( \lambda \in \mathbb{C} \) and \( (x, y) \in \mathbb{A}^2 \). Recall that \( G \) is not reductive, so classical GIT theory does not guarantee that a good quotient for the action exists. Actually, the map \( \pi : X \to \overline{X} = \mathbb{A}^1 \), \( \pi(x, y) = x \), is a categorical quotient but is not a pseudo-quotient so it is not good.

Let us take \( U = \{x \neq 0\} \subseteq X \) and \( Y = \{x = 0\} \subseteq X \). Observe that \( U \) is orbitwise-closed (actually, the full action is closed), the restriction \( \pi|_U : U \to \overline{U} = \pi(U) = \mathbb{A}^1 - \{0\} \) is a good quotient and \( \pi(Y) = \{0\} \). On the other hand, the identity map \( Y \to Y \) is a categorical pseudo-quotient for the trivial action of \( G \) on \( Y \) but \( [X] \neq [Y] + [U] \) (it can be checked using the Deligne-Hodge polynomial, see Section 5) so the conclusion of Theorem 4.1 fails.
Another useful application of pseudo-quotients arises when we find a subset of an algebraic variety that concentrates the action. In this case, we can obtain much information of the pseudo-quotients of the whole variety from the ones of these special subvarieties.

**Proposition 4.4.** Let \( X \) be an algebraic variety with an action of an algebraic group \( G \). Suppose that there exists a subvariety \( Y \subseteq X \) and an algebraic subgroup \( H \subseteq G \) such that:

1. \( Y \) is orbitwise-closed for the action of \( H \).
2. For any \( x \in X \), \( Gx \cap Y \neq \emptyset \).
3. For any \( W_1, W_2 \subseteq Y \) closed (in \( Y \)) \( H \)-invariant subsets, we have \( W_1 \cap W_2 \neq \emptyset \) if and only if \( GW_1 \cap GW_2 \neq \emptyset \).

Such a pair \((Y, H)\) is called a core. Suppose that there exists a pseudo-quotient \( \pi : X \to X \) for the action of \( G \) on \( X \). Then, \( \pi \) restricts to a pseudo-quotient \( \pi|_Y : Y \to X \) for the action of \( H \) on \( Y \).

**Proof.** For the surjectivity of \( \pi|_Y \), take \( \pi = \pi(x) \) for some \( x \in X \) and, by hypothesis [ii], there exists \( y \in Y \) such that \( G\pi(y) \cap Gx \neq \emptyset \). Thus, \( \pi(y) = \pi(x) = \pi \).

Now, let \( W_1, W_2 \subseteq Y \) be two disjoint closed \( H \)-invariant subsets. If \( \pi(W_1) \cap \pi(W_2) \neq \emptyset \), then \( \pi^{-1}(W_1) \cap \pi^{-1}(W_2) \neq \emptyset \). But we claim that \( \pi^{-1}(W_1) = GW_i \), so this is impossible. In order to check that, observe that the inclusion \( GW_i \subseteq \pi^{-1}(W_1) \) is trivial. For the other inclusion, since \( GW_i \) is a closed \( G \)-invariant set, then \( \pi(GW_i) \) is a closed subset containing \( \pi(W_1) \) and, thus, \( \pi(W_1) \subseteq \pi(GW_i) \) which implies that \( \pi^{-1}(W_1) \subseteq \pi^{-1}(\pi(GW_i)) = GW_i \), as desired. \( \square \)

5. **Hodge Theory and Algebraic Quotients**

Along this section, we will work on \( k = \mathbb{C} \). Given a complex algebraic variety \( X \), its rational compactly supported cohomology, \( H_c^\bullet(X; \mathbb{Q}) \), carries an additional linear structure, called a mixed Hodge structure.

To be precise, in [7] and [8], Deligne proved that, seen with the analytic topology, the rational compactly supported cohomology has a natural pair of filtrations \( W_\bullet \) of \( H_c^k(X; \mathbb{Q}) \) (increasing, called the weight filtration) and \( F_\bullet \) of \( H_c^k(X; \mathbb{Q}) \otimes \mathbb{C} \) (decreasing, called the Hodge filtration). These filtrations satisfy that, for any \( w \in \mathbb{Z} \), the induced filtration of \( F_\bullet \) on the graded complex \((Gr_w^F H_c^k(X; \mathbb{Q})) \otimes \mathbb{C} \) is \( w \)-orthogonal i.e. \( F_p \oplus F_{w-p+1} = (Gr_w^F H_c^k(X; \mathbb{Q})) \otimes \mathbb{C} \), for all \( p \). For further information, see [22].

Given a complex algebraic variety \( X \), we define the \((p, q)\)-pieces of its \( k \)-th compactly supported cohomology groups by

\[
H_c^{k, p, q}(X) = Gr_p^F (Gr_{w+q}^F H_c^k(X; \mathbb{Q})) \otimes \mathbb{C}.
\]
From them, we define the *Hodge numbers* as $h^{k,p,q}_c(X) = \dim H^{k,p,q}_c(X)$ and the *Deligne-Hodge polynomial*, or *E-polynomial*, as the alternating sum

$$e(X) = \sum_k (-1)^k h^{k,p,q}_c(X) u^p v^q \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].$$

An algebraic variety $X$ is called of *balanced type*, or *Hodge-Tate type*, if $H^{k,p,q}_c(X) = 0$ for $p \neq q$. In that case, $e(X)$ only depends on $uv$ and it is customary to perform the change of variables $q = uv$. As we will see, all the varieties that we will work with along this paper are balanced (see also Proposition 2.8 of [22]).

**Remark 5.1.** Sometimes in the literature, the *E-polynomial* is defined as $e(X)(-u, -v)$. It does not introduce any important difference but it would produce an annoying change of sign.

**Example 5.2.** Suppose that $X$ is smooth and projective so it is a compact Kähler manifold. In that case, classical Hodge theory shows that de Rham cohomology decomposes in terms of Dolbeault cohomology as

$$H^k_d(X; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{Dol}(X).$$

In that case, the mixed Hodge structure is given by $F^p = \bigoplus_{r \geq p} H^{k-r}_{Dol}(X)$ and the weight filtration has the single step $W_{k-1} = 0$ and $W_k = H^k(X; \mathbb{Q})$. Hence, the Hodge pieces are $H^{k,p,q}_c(X) = H^{p,q}_{Dol}(X)$ if $p+q = k$ and 0 if $p+q \neq k$. For further information, see [18], Corollary 3.2.12.

Observe that $e(X)$ is constructed as an alternating sum in the spirit of an Euler characteristic. This algebraic property implies that $e$ can be extended to a ring homomorphism

$$e : K\text{Var}_\mathbb{C} \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}],$$

where $K\text{Var}_\mathbb{C}$ denotes the $K$-theory of complex algebraic varieties. In particular, $e(X)$ only depend on the $K$-theory class of $X$ and we have the equalities $e(X \sqcup Y) = e(X) + e(Y)$ and $e(X \times Y) = e(X) e(Y)$ for $X$ and $Y$ complex algebraic varieties.

**Remark 5.3.** In fact, the later property holds with much more generality. In [22], Proposition 2.4 (see also [12], [REF]), it is proven that, if $\pi : X \to B$ is a regular morphism which is a fiber bundle in the analytic topology with fiber $F$ and with trivial monodromy, then $e(X) = e(F)e(B)$. As we mentioned in Remark 2.5 of [22], these hypothesis hold in several common cases like if $\pi$ is locally trivial in the Zarisky topology with $B$ irreducible, if $F = \mathbb{P}^N$ or if $\pi$ is a principal $G$-bundle with $G$ a connected algebraic group.

**Remark 5.4.** On the other hand, the behaviour of the Deligne-Hodge polynomial under quotients is not so easy, even for very simple groups. In order to illustrate it, let $X$ be a complex algebraic variety with an algebraic action of $\mathbb{Z}_2$ on it. Denote $e(X)^+ = e(X \sslash \mathbb{Z}_2)$ and $e(X)^- = e(X) - e(X \sslash \mathbb{Z}_2)$ so that $e(X) = e(X)^+ + e(X)^-$. In [22],
it is proven that, if $X_1$ and $X_2$ are algebraic varieties with an action of $\mathbb{Z}_2$ then, with respect to the simultaneous action of $\mathbb{Z}_2$ on $X_1 \times X_2$, we have

$$e(X_1 \times X_2)^+ = e(X_1)^+ e(X_2)^+ + e(X_1)^- e(X_2)^-.$$ 

Proceeding recursively with this formula, actually it can be shown that

$$e(X^n)^+ = \frac{1}{2} \left[ e(X)^n + (e(X)^+ - e(X)^-)^n \right]$$

$$e(X^n)^- = \frac{1}{2} \left[ e(X)^n - (e(X)^+ - e(X)^-)^n \right]$$

The general case of any finite group is more involved and is treated in [11].

The results of sections 3 and 4 about the uniqueness of pseudo-quotients are very useful to prove equality of Deligne-Hodge polynomials since, as we mentioned above, they only depend on the $K$-theory class.

- Let $X$ be an algebraic variety with an action of an algebraic group $G$. Then, for any pseudo-quotients $\pi : X \to Y$ and $\pi' : X \to Y'$ (categorical or not) we have $e(Y) = e(Y')$. It follows from Corollary 3.10

- Let $X$ be an algebraic variety with a decomposition $X = Y \sqcup U$ with $Y \subseteq X$ closed and $U \subseteq X$ an open orbitwise-closed subset. Then, for any pseudo-quotients $X \to \overline{X}$, $Y \to \overline{Y}$ and $U \to \overline{U}$ we have $e(\overline{X}) = e(\overline{Y}) + e(\overline{U})$. It follows from Theorem 4.1. In particular, if $G$ is a reductive group acting linearly on $X$ and $X = X^{ss}$, we can consider the usual GIT quotients, which are good, so we have an equality of $E$-polynomials $e(X \sslash G) = e(Y \sslash G) + e(U \sslash G)$.

- If $(Y, H)$ is a core for the action of $G$ on $X$ then, for any pseudo-quotient $Y \to \overline{Y}$ we have $e(\overline{X}) = e(\overline{Y})$. It follows from Proposition 4.4.

Finally, in the following computations, we will need to study how Deligne-Hodge polynomial behaves for geometric quotiens. To do so, have to show that such geometric quotient is actually a principal bundle. This is precisely the content of the so-called Luna’s slice theorem, whose proof can be found in [24] (see also [10], Proposition 5.7).

**Theorem 5.5** (Luna’s slice theorem). Let $X$ be an affine variety with an action of an algebraic reductive group $G$ on it. Let $X_0 \subseteq X$ the set of points where the action of $G$ is free and closed (on $X$). Then, $X_0$ is open and saturated for the GIT quotient $\pi : X \to X \sslash G$ and the restriction $\pi|_{X_0} : X_0 \to \pi(X_0)$ is a principal $G$-bundle of analytic spaces.

**Corollary 5.6.** With the notations and hypothesis of Theorem 5.5 if $U \subseteq X_0$ is a $G$-invariant open set, then $\pi|_U : U \to \pi(U) = U \sslash G$ is a principal $G$-bundle. Moreover, if $G$ is connected, we have $e(U \sslash G) = e(U) / e(G)$.

**Proof.** By Luna’s slice theorem 5.5 $\pi|_{X_0} : X_0 \to \pi(X_0)$ is a principal $G$-bundle. Since $U$ is an orbitwise-closed open set, Lemma 3.5 implies that $U$ is saturated and $\pi(U)$ is open. In that setting, the restriction $\pi|_U : U \to \pi(U)$ is also a principal $G$-bundle.
For the last part, just use Remark 5.3 to observe that \( \pi|_U \) has trivial monodromy, so \( e(U) = e(G) e(U \sslash G) \).

\[\square\]

\section{Representation varieties}

For the sake of completeness, in this section we will review some well-known results about the structure of representation varieties that will be also useful to fix the notation. We will work on an arbitrary algebraically closed field \( k \).

Given a finitely generated group \( \Gamma \) and an algebraic group \( G \), the set of representations \( \text{Hom}(\Gamma, G) \) has a natural algebraic variety structure given as follows. Let \( \Gamma = \langle \gamma_1, \ldots, \gamma_r \mid R_\alpha(\gamma_1, \ldots, \gamma_r) = 1 \rangle \) be a presentation of \( \Gamma \) with finitely many generators, where \( R_\alpha \) are the relations (possibly infinitely many). In that case, we define the injective map \( \psi : \text{Hom}(\Gamma, G) \to G^r \) given by \( \psi(\rho) = (\rho(\gamma_1), \ldots, \rho(\gamma_r)) \). Moreover, the image of \( \psi \) is the algebraic subvariety of \( G^r \)

\[ \text{Im} \psi = \{(g_1, \ldots, g_r) \in G^r \mid R_\alpha(g_1, \ldots, g_r) = 1 \}. \]

Hence, we can impose an algebraic structure on \( \text{Hom}(\Gamma, G) \) by declaring that \( \psi \) is a regular isomorphism over its image. Observe that this algebraic structure does not depend on the chosen presentation. With this algebraic structure, \( \text{Hom}(\Gamma, G) \) is called the representation variety of \( \Gamma \) into \( G \) and is denoted by \( X_G(\Gamma) \).

The variety \( X_G(\Gamma) \) has a natural action of \( G \) by conjugation i.e. \( g \cdot \rho(\gamma) = g \rho(\gamma) g^{-1} \) for \( g \in G, \rho \in X_G(\Gamma) \) and \( \gamma \in \Gamma \). Recall that two representations \( \rho, \rho' \) are said to be isomorphic if and only if \( \rho' = g \cdot \rho \) for some \( g \in G \). For this reason, if \( G \) is reductive, it is interesting to consider the GIT quotient \( R_G(\Gamma) = X_G(\Gamma) \sslash G \) which is usually called the character variety. The aim of this paper is to study the properties of this quotient and to show how to compute the Deligne-Hodge polynomial \( R_G(\Gamma) \) from the one of \( X_G(\Gamma) \). In particular, this paper focus on the following cases:

- \( \Gamma = F_n \), the free group with \( n \) generators. In that case, for short, we will denote the associated representation variety by \( X_n(G) = X_G(F_n) = G^n \) or, when the group \( G \) is understood, just by \( X_n \). The importance of this case comes from the fact that, if \( \Gamma \) is any finitely generated group with \( n \) generators, then the epimorphism \( F_n \to \Gamma \) gives an inclusion \( X_G(\Gamma) \subseteq X_n(G) \).

- \( \Gamma = \pi_1(\Sigma_g) \), the fundamental group of the genus \( g \) compact surface. In that case, we will denote the associated representation variety by \( X_g(G) = X_G(\pi_1(\Sigma_g)) \) or even just by \( X_g \) when the group \( G \) is understood from the context. Since the standard presentation of \( \Gamma \) is

\[ \pi_1(\Sigma_g) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \right\rangle, \]

then \( X_g(G) \subseteq X_{2g}(G) \). Actually, \( X_g(G) \) is given by tuples of \( 2g \) elements of \( G \) satisfying the relation of \( \pi_1(\Sigma_g) \) so it is a closed subvariety of \( X_{2g}(G) \). The GIT
quotient of this representation variety under the conjugacy action will be denoted by $\mathcal{R}_g(G)$ or just $\mathcal{R}_g$.

6.1. Stability of representation varieties. Before studying the general case of representation varieties, let us consider a simpler case. Fixed an integer $m \geq 1$, consider the action of $GL_m(k)$ on itself by conjugation, that is $P \cdot X = PXP^{-1}$ for $P, X \in GL_m(k)$. This action is linearizable since, if $M_m$ is the vector space of $m \times m$ matrices, then, for any $P \in GL_m(k)$, the map $A \in M_m \rightarrow A \cdot P$ given by $\lambda + X \mapsto \lambda + P \cdot X = \lambda + PXP^{-1}$ is linear and commutes with homotheties. Hence, we can see $GL_m(k)$ as a quasi-affine variety of linear one. In other words, $L = \mathcal{O}_{\mathbb{P}^m}(1)|_{GL_m(k)} \rightarrow GL_m(k)$ is an ample $G$-line bundle compatible with the conjugacy action.

Actually, the same holds for any linear algebraic group $G \subseteq GL_m(k)$ and its action by conjugation. For that, let $i : G \rightarrow GL_m(k)$ be the embedding of $G$ as linear group. Since conjugation is just twice product on the group, $i$ is a $G$-equivariant map for the conjugacy action. Hence, if $L \rightarrow GL_m(k)$ is the linearization of the action above, then $L' = i^*L \rightarrow G$ is a linearization for the action of $G$ by conjugation. Moreover, since $L$ is ample, $L'$ is too.

Proposition 6.1. Let $\Gamma$ be a finitely generated group and let $G$ be a linear algebraic group. Then, the action of $G$ on $X_G(\Gamma)$ is linearizable via an ample line bundle.

Proof. As we mentioned above, $X_G(\Gamma) \subseteq X_n(G) = G^n$ for some $n > 0$, so it is enough to prove it in the free case. In that situation, just take the line bundle $\bigotimes_{i=1}^n \pi_i^*L' \rightarrow X_n(G)$ where $\pi_i : G^n \rightarrow G$ is the $i$-th projection and $L' \rightarrow G$ is the line bundle constructed above.

Recall that a linear representation $\rho : \Gamma \rightarrow GL(V)$, where $V$ is a finite dimensional $k$-vector space, is said to be reducible if there exists a proper $\Gamma$-invariant subspace of $V$. If $G$ is a linear group, we can see $G \subseteq GL_m(k)$ for $m$ large enough, so it also make sense to speak about reducible $G$-representations. With this definition, we have a decomposition

$$X_G(\Gamma) = X_G^r(\Gamma) \sqcup X_G^{ir}(\Gamma),$$

where $X_G^r(\Gamma) \subseteq X_G(\Gamma)$ is the closed subvariety of reducible representations and $X_G^{ir}(\Gamma)$ the open set of irreducible ones.

Proposition 6.2. Let $X_{GL_m(k)}(\Gamma)^S$ and $X_{GL_m(k)}(\Gamma)^{SS}$ be the set of stable and semi-stable points, respectively, of $X_{GL_m(k)}(\Gamma)$ under the action of $GL_m(k)$ by conjugation. Then, we have $X_{GL_m(k)}(\Gamma)^{SS} = X_{GL_m(k)}(\Gamma)$ and $X_{GL_m(k)}(\Gamma)^S = X_{GL_m(k)}^{ir}(\Gamma)$.

Proof. We will use the Hilbert-Mumford criterion of stability (see [31], Theorem 4.9). Recall that this criterion says that $A \in X_{GL_m(k)}(\Gamma)^S$ (resp. $A \in X_{GL_m(k)}(\Gamma)^{SS}$) if and only if $\mu(A, \lambda) > 0$ (resp. $\geq 0$) for all 1-parameter subgroups $\lambda : k^* \rightarrow GL_m(k)$, where $\mu(A, \lambda)$ is the minimum $\alpha \in \mathbb{Z}$ such that $\lim_{t \rightarrow 0} t^\alpha \lambda(t) \cdot A$ exists.
Let us prove that all the elements of $\mathfrak{X}_{\text{GL}_m(k)}(\Gamma)$ are semi-stable. Given a 1-parameter subgroup $\lambda : k^* \to \text{GL}_m(k)$, pick a regular function $\alpha : k^* \to k^*$ such that $\alpha(t)^m = \det \lambda(t)$ for all $t \in k^*$. In this case, $\overline{\chi}(t) = \alpha(t)^{-1}\lambda(t) \in \text{SL}_m(k)$ for all $t \in k^*$, so it is a 1-parameter subgroup of $\text{SL}_m(k)$. By [31], Theorem 4.11, there exists $P \in \text{SL}_m(k)$ such that

$$P^{-1}\overline{\chi}(t)P = \begin{pmatrix}
t^{s_1} & 0 & \cdots & 0 \\
0 & t^{s_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & t^{s_m}
\end{pmatrix} =: D_{s_1,\ldots,s_m}(t)$$

for some $s_1 \geq s_2 \ldots \geq s_m \in \mathbb{R}$, not all zero, and $\sum_i s_i = 0$. Now, given $A \in \mathfrak{X}_{\text{GL}_m(k)}(\Gamma)$, we have that

$$\lambda(t) \cdot A = \lambda(t)A\lambda(t)^{-1} = \overline{\chi}(t)A\overline{\chi}(t)^{-1} = PD_{s_1,\ldots,s_m}(t)P^{-1}APD_{s_1,\ldots,s_m}(t)^{-1}P^{-1}.$$ 

Hence, without loss of generality, we can suppose that $\lambda(t) = D_{s_1,\ldots,s_m}(t)$. Let $A = (A_1,\ldots,A_n) \in \mathfrak{X}_{\text{GL}_m(k)}(\Gamma)$, where $A_l = (a_{ij}^l)$ for $1 \leq l \leq n$. A direct computation shows that the $(i,j)$-entry of $\lambda(t) \cdot A_l$ is $a_{ij}^l t^{s_{j-i}}$. In particular, all the elements in the diagonal and under it are multiplied by $t^\beta$ with $\beta \leq 0$. Since at least one of them have to be non-zero, $\lim_{t \to 0} t^\alpha \lambda(t) \cdot A$ cannot exist if $\alpha < 0$. Hence, $\mu(A,\lambda) \geq 0$ for all 1-parameter subgroups, proving that all the points are semi-stable.

For the stable points, suppose that $A = (A_1,\ldots,A_n)$ is reducible with a proper invariant subspace of dimension $k < m$. In that case, maybe after conjugation, we can get that the entries $a_{i,j}^l = 0$ for $i > k$ and $j \leq k$. Taking $\lambda(t) = D_{s_1,\ldots,s_m}(t)$ with $s_i = 1/k$ for $i \leq k$, and $s_i = -1/(m-k)$ for $i > k$, we have that all the non-zero entries of $\lambda(t) \cdot A_l$ are multiplied by $t^\beta$ with $\beta \geq 0$. Hence, $\lim_{t \to 0} \lambda(t) \cdot A$ exists so $\mu(A,\lambda) = 0$ and $A$ cannot be stable.

On the other hand, if $A \in X_n$ is not stable, then $\lim_{t \to 0} \lambda(t) \cdot A$ exists for some 1-parameter subgroup $\lambda(t)$. As we explained above, maybe after conjugate $A$, we can suppose that $\lambda(t) = D_{s_1,\ldots,s_m}(t)$. In that case, $\lambda(t) \cdot A$ cannot have non-zero entries with a factor $t^\beta$ with $\beta < 0$, which is possible if and only if there is $1 \leq k < m$ such that $s_1 = s_2 = \ldots = s_k > s_{k+1} = \ldots = s_m$ and all the entries $a_{i,j}$ with $i > k$ and $j \leq k$ of $A$ vanish, that is, if and only if $A$ has a proper invariant subspace.

**Corollary 6.3.** Let $\Gamma$ a finitely generated group and $G$ a linear algebraic group. Then all the points of $\mathfrak{X}_G(\Gamma)$ are semi-stable for the action of $G$ by conjugation and $\mathfrak{X}_G(\Gamma)^S = \mathfrak{X}_G^r(\Gamma)$

**Proof.** For the first part, observe that the inclusion $G \subseteq \text{GL}_m(k)$ induces a natural inclusion $\mathfrak{X}_G(\Gamma) \subseteq \mathfrak{X}_{\text{GL}_m(k)}(\Gamma)$. Hence, since all the points of $\mathfrak{X}_{\text{GL}_m(k)}(\Gamma)$ are semi-stable for the conjugacy action of $\text{GL}_m(k)$, the same holds for $\mathfrak{X}_G(\Gamma)$. For the stable points, just use the same argument than above taking into account that the reducibility of a representation does not change after a conjugation. □
By definition, the action of $G$ on the stable locus is closed. However, we cannot expect that the action of $G$ there was free because all the element in the center of $G$, $G^0 \subseteq G$, act trivially. However, it turns out that, along the irreducible representations, the action of $G/G^0$ is, indeed, free.

**Proposition 6.4.** Suppose that $k$ is an algebraically closed field. The action of $G/G^0$ on the irreducible representations $\mathfrak{X}^\text{ir}_G(\Gamma)$ is free. In particular $\mathfrak{X}^\text{ir}_G(\Gamma) \to \mathfrak{X}^\text{ir}_G(\Gamma)/\!(G/G^0)$ is a free geometric quotient.

**Proof.** Suppose that $G \subseteq \text{GL}_m(k)$ for $m$ large enough. Let $A = (A_1, \ldots, A_n) \in \mathfrak{X}^\text{ir}_G(\Gamma)$ and suppose that $P \in G$ satisfies $P \cdot A = A$, that is, $PA_iP^{-1} = A_i$ for all $i$ or, equivalently, $PA_i = A_iP$, i.e. $A_i$ and $P$ commutes. We are going to prove that there exists a basis $v_1, \ldots, v_m \in k^m$ of eigenvectors of $P$ with the same eigenvalue. In that case, $P$ would be a multiple of the identity so $P \in G^0$.

To do so, let $v_1 \in k^m - \{0\}$ be an eigenvector of $P$ with eigenvalue $\lambda \in k$. Since $A$ is irreducible, the matrices $A_i$ cannot have a common eigenvector so $A_j v_1 \notin \langle v_1 \rangle$ for some $1 \leq j \leq n$. Moreover, since $P$ and $A_j$ commutes, $A_j v_1$ is also an eigenvector of $P$ of eigenvalue $\lambda$ and we set $v_2 = A_j v_1$. By induction, suppose that we have build $v_1, \ldots, v_l$ a set of linearly independent eigenvectors of $P$ of eigenvalue $\lambda$ with $l < m$. Since $A$ is irreducible, $A_j v_i \notin \langle v_1, \ldots, v_l \rangle$ for some $1 \leq i \leq l$ and $1 \leq j \leq n$. Hence, $A_j v_i$ is an eigenvector of $P$ of eigenvalue $\lambda$ linearly independent with $v_1, \ldots, v_m$, so we can set $v_{l+1} = A_j v_i$. 

**Remark 6.5.**

- If $G \subseteq \text{GL}_2(k)$, then a representation $\rho : \Gamma \to G$ is reducible if and only if all the elements of $\rho(\Gamma)$ have a common eigenvector.

- By [4] Proposition 1.10, every affine group is linear so the previous results automatically holds for affine groups.

## 7. The $\text{SL}_2(\mathbb{C})$-representation variety

For the rest of the paper, we will work within the ground field $k = \mathbb{C}$ and we will focus on the algebraic group $G = \text{SL}_2(\mathbb{C})$. For short, we will just denote by $\text{SL}_2 = \text{SL}_2(\mathbb{C})$ and $\text{PGL}_2 = \text{PGL}_2(\mathbb{C})$.

The center of $\text{SL}_2$ are the matrices $\pm \text{Id}$ so the action of $\text{SL}_2$ by conjugation descends to an action of $\text{SL}_2/\{\pm \text{Id}\} = \text{PGL}_2$. Hence, given a finitely generated group $\Gamma$, we have a decomposition $\mathfrak{X}_{\text{SL}_2}(\Gamma) = \mathfrak{X}^\text{ir}_{\text{SL}_2}(\Gamma) \sqcup \mathfrak{X}^\text{ir}_{\text{SL}_2}(\Gamma)$. The first stratum of this decomposition is a closed set and the second stratum is an open set with a geometric free action of $\text{PGL}_2$. Observe that two representations of $\mathfrak{X}_{\text{SL}_2}(\Gamma)$ are conjugated by an element of $\text{GL}_2$ if and only if they are so by an element of $\text{SL}_2$, so the conjugacy classes agree. In particular, $A \in \mathfrak{X}_{\text{SL}_2}(\Gamma)$ is reducible if and only if it is $\text{SL}_2$-conjugate to an upper triangular element.
Proposition 7.1. Let \( \Gamma \) be a finitely generated group. Suppose that \( A \in \mathcal{X}_{\text{SL}_2}(\Gamma) \) has non trivial isotropy for the action of \( \text{PGL}_2 \). Then, \( A \) is conjugate to an element of the following classes:
\[
\left( \begin{pmatrix} \pm 1 & \alpha_1 \\ 0 & \pm 1 \end{pmatrix}, \ldots, \begin{pmatrix} \pm 1 & \alpha_n \\ 0 & \pm 1 \end{pmatrix} \right) \quad \text{or} \quad \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} \right),
\]
for \( \alpha_i \in \mathbb{C} \) and \( \lambda_i \in \mathbb{C}^* = \mathbb{C} - \{0\} \).

Proof. By Proposition 6.4, \( A \) must be reducible so, after conjugation, we can suppose that
\[
A = (A_1, \ldots, A_n) = \left( \begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_n & \alpha_n \\ 0 & \lambda_n^{-1} \end{pmatrix} \right).
\]
If all \( \lambda_i = \pm 1 \), then \( A \) belongs to the first class. Hence, we can suppose that \( \lambda_j \neq \pm 1 \) for some \( 1 \leq j \leq n \). In that case, conjugating with
\[
P_0 = \begin{pmatrix} 1 & \alpha_j \\ 0 & \lambda_j^{-1} \end{pmatrix}
\]
we obtain that \( \alpha_j = 0 \) so \( A_j \) is a diagonal matrix. We claim that, after such a conjugation, all the \( \alpha_i \) must vanish so \( A \) is of the second class. Otherwise, since \( P_0 \) preserves upper triangular matrices, there must be another upper triangular non-diagonal matrix \( A_k \).

In that case, if \( P \in \text{SL}_2 \) fixes \( A \), we obtain in particular that \( P \) stabilizes both a diagonal matrix and a non-diagonal upper triangular matrix. But this is only possible if \( P = \pm \text{Id} \). \( \square \)

Given a finitely generated group \( \Gamma \), let us denote by \( \mathcal{X}_{\text{SL}_2}^U(\Gamma) \) the representations that has the form of the first class and not of the second class and let \( \mathcal{X}_{\text{SL}_2}^V(\Gamma) \) be the union of the orbits of elements of \( \mathcal{X}_{\text{SL}_2}^U(\Gamma) \). Analogously, we will denote by \( \mathcal{X}_{\text{SL}_2}^D(\Gamma) \) the representations that has the form of only the second class and by \( \mathcal{X}_{\text{SL}_2}^\delta(\Gamma) \) their orbits. Finally, \( \mathcal{X}_{\text{SL}_2}(\Gamma) \) will be the representations that belong to both classes. Observe that the action of \( \text{SL}_2 \) on the later stratum is trivial. Setting \( \mathcal{X}_{\text{SL}_2}(\Gamma) = \mathcal{X}_{\text{SL}_2}^U(\Gamma) - \mathcal{X}_{\text{SL}_2}^D(\Gamma) - \mathcal{X}_{\text{SL}_2}^\delta(\Gamma) \), we obtain an stratification
\[
\mathcal{X}_{\text{SL}_2}(\Gamma) = \mathcal{X}_{\text{SL}_2}^U(\Gamma) \sqcup \mathcal{X}_{\text{SL}_2}^V(\Gamma) \sqcup \mathcal{X}_{\text{SL}_2}^D(\Gamma) \sqcup \mathcal{X}_{\text{SL}_2}^\delta(\Gamma) \sqcup \mathcal{X}_{\text{SL}_2}(\Gamma).
\]
Observe that this stratification also decomposes \( \mathcal{X}_{\text{SL}_2}(\Gamma) \) in terms of the conjugacy classes of the stabilizers for the action of \( \text{SL}_2 \). To be precise, the stabilizer of the points of \( \mathcal{X}_{\text{SL}_2}^U(\Gamma) \) and \( \mathcal{X}_{\text{SL}_2}^\delta(\Gamma) \) are conjugated to the subgroups of \( \text{SL}_2 \)
\[
\text{Stab } J_+ = \left\{ \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}, \beta \in \mathbb{C} \right\} \quad \text{and} \quad \text{Stab } D_\lambda = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \mu \in \mathbb{C}^* \right\},
\]
respectively, where \( D_\lambda \) is the diagonal matrix with eigenvalues \( \lambda \pm 1 \). For the stratum \( \mathcal{X}_{\text{SL}_2}(\Gamma) \), the action is trivial and for the strata \( \mathcal{X}_{\text{SL}_2}^U(\Gamma) \) and \( \mathcal{X}_{\text{SL}_2}^\delta(\Gamma) \) the action of \( \text{PGL}_2 \) is free.
Remark 7.2. The stratification considered in this paper is strongly related to the so-called Luna’s stratification for $\mathfrak{x}_{SL^2}(\Gamma)$ (see [23] and [24], also Section 6.9 of [36]). In this case, the Luna strata are $\mathfrak{x}_{SL^2}(\Gamma)$, $\mathfrak{x}_{SL^2}^{e}(\Gamma)$, $\mathfrak{x}_{SL^2}^{\delta}(\Gamma)$ and $\mathfrak{x}_{SL^2}^{\upsilon}(\Gamma) \cup \mathfrak{x}_{SL^2}^{\nu}(\Gamma)$.

7.1. **Free groups.** Let us fix $\Gamma = F_n$, the free group of $n$ generators, and recall that we abbreviate $X_n = \mathfrak{x}_{SL^2}(F_n)$. In this case, the stratification considered above has the following properties:

- **$X_n^{\upsilon}$**: Given $A = (A_1, \ldots, A_n) \in X_n^{\upsilon}$, let
  \[
  \begin{pmatrix}
  (\epsilon_1 \alpha_1), & \ldots, & (\epsilon_n \alpha_n)
  \end{pmatrix}
  \]
  be the element of $X_n^U$ conjugate to $A$ with $\epsilon_i = \pm 1$ and $\alpha_i \in \mathbb{C}$ not all zero. Observe that such an element of $X_n^U$ is unique up to simultaneous rescaling of the upper triangular components $\alpha_i$. Thus, the $SL^2$-orbit of $A$, $[A]$, is the set of reducible representations $(B_1, \ldots, B_n) \in X_n$ with a double eigenvalue such that, in their upper triangular form,
  \[
  \begin{pmatrix}
  (\epsilon_1 \beta_1), & \ldots, & (\epsilon_n \beta_n)
  \end{pmatrix}
  \]
  there exists $\lambda \neq 0$ such that $(\alpha_1, \ldots, \alpha_n) = \lambda (\beta_1, \ldots, \beta_n)$. Then, taking $\lambda \to 0$, we find that the closure of the orbit, $[A]$, is precisely set of reducible representations with double eigenvalue such that their upper triangular components satisfy $(\alpha_1, \ldots, \alpha_n) = \lambda (\beta_1, \ldots, \beta_n)$ for some $\lambda \in \mathbb{C}$. In particular, for $\lambda = 0$ we have that $(\epsilon_1 \text{Id}, \ldots, \epsilon_n \text{Id}) \in [A]$.

  The Deligne-Hodge polynomial of this stratum can be computed observing that the tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n - \{0\}$ determines the diagonal form up to projectivization. Hence, we obtain a regular fibration
  \[
  \mathbb{C}^* \longrightarrow SL_2/\text{Stab} J_+ \times \{\pm 1\}^n \times (\mathbb{C}^n - \{0\}) \longrightarrow X_n^{\upsilon}.
  \]
  Observe that this fibration is locally trivial in the Zariski topology so, by [22], this fibration has trivial monodromy and, thus
  \[
  e(X_n^{\upsilon}) = e(\{\pm 1\}^n) e(\mathbb{C}P^{n-1}) e(SL_2/\text{Stab} J_+) = 2^n(q^2 - 1) \frac{q^n - 1}{q - 1}.
  \]

- **$X_n^{\nu}$**: Given $A = (A_1, \ldots, A_n) \in X_n^{\nu}$, let
  \[
  \begin{pmatrix}
  (\lambda_1, 0), & \ldots, & (\lambda_n, 0)
  \end{pmatrix}
  \]
  be the element of $X_n^D$ conjugate to $A$ with $\lambda_i \in \mathbb{C}^*$ and not all equal to $\pm 1$. In order to compute the Deligne-Hodge polynomial of this stratum, recall that the diagonal form of an element of $X_n^{\delta}$ is determined up to permutation of the columns, so we have a double covering
  \[
  SL_2/\text{Stab} D_\lambda \times ((\mathbb{C}^*)^n - \{(\pm 1, \ldots, \pm 1)\}) \longrightarrow X_n^{\delta}.
  \]
Therefore, we obtain that
\[ X^\delta_n = \frac{\text{SL}_2/\text{Stab} D_n \times ([\mathbb{C}^\ast]^n - \{(\pm 1, \ldots, \pm 1)\})}{\mathbb{Z}_2} \]

Using Remark 5.4, its Deligne-Hodge polynomial is
\[ e(X^\delta_n) = \frac{q^3 - q}{2} ((q - 1)^{n-1} + (q + 1)^{n-1}) - 2^n q^2. \]

- \( X^\epsilon_n \). Here, the action of \( \text{SL}_2 \) is trivial so, in particular, the action is closed. This stratum consists of \( 2^n \) points, so \( e(X^\epsilon_n) = 2^n \).

- \( X^\rho_n \). This is the set of reducible representations, not completely reducible, such that some matrix has no double eigenvalue. Given \( A \in X^\rho_n \), \( A \) is conjugate to an element of the form
\[
\left( \begin{array}{cc}
\lambda_1 & \alpha_1 \\
0 & \lambda_1^{-1}
\end{array} \right), \ldots, \left( \begin{array}{cc}
\lambda_n & \alpha_n \\
0 & \lambda_n^{-1}
\end{array} \right)
\]
where the vector \((\alpha_1, \ldots, \alpha_n)\) is determined up to the action of \( U = \mathbb{C}^\ast \times \mathbb{C} \) by \((\mu, a) \cdot (\alpha_1, \ldots, \alpha_n) = (\alpha_1\mu^2 - a\mu (\lambda_1 - \lambda_1^{-1}), \ldots, \alpha_n\mu^2 - a\mu (\lambda_n - \lambda_n^{-1}))\)

Observe that, since \( A \not\in X^D_n \), we must have \((\mu, a) \cdot (\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)\) for all \((\mu, a) \in U\). This vanishing can happen if and only if \((\alpha_1, \ldots, \alpha_n) = \lambda(\lambda_1 - \lambda_1^{-1}, \ldots, \lambda_n - \lambda_n^{-1})\) for some \( \lambda \in \mathbb{C} \). Therefore, the allowed values for \((\alpha_1, \ldots, \alpha_n)\) lie in \( \mathbb{C}^n - l \), where \( l \subseteq \mathbb{C}^n \) is the line spanned by \((\lambda_1 - \lambda_1^{-1}, \ldots, \lambda_n - \lambda_n^{-1})\). Thus, we have regular fibration
\[ U \longrightarrow \text{PGL}_2 \times \Omega \longrightarrow X^\rho_n, \]
where \( \Omega \cong ([\mathbb{C}^\ast]^n - \{\pm 1, \ldots, \pm 1\}) \times [\mathbb{C}^n - l] \) is the set of allowed values for the eigenvalues and the antidiagonal component. Hence, the Deligne-Hodge polynomial of this stratum is
\[ e(X^\rho_n) = \frac{q^3 - q}{(q - 1)q} ((q - 1)^{n-1} - 2^n) (q^n - q). \]

From this analysis, it is possible to describe the GIT quotient \( X_n/\text{SL}_2 \) quite explicitly. First of all, recall that we have a decomposition \( X_n = X^r_n \sqcup X^{ir}_n \) with \( X^{ir}_n \) an open subvariety. Since \( \text{SL}_2 \) is a reductive affine group, by the results of Section 5, we have
\[ e(X_n/\text{SL}_2) = e(X^r_n/\text{SL}_2) + e(X^{ir}_n/\text{SL}_2). \]

The action of \( \text{PGL}_2 \) on \( X^{ir}_n \) is closed and free, so the Deligne-Hodge polynomial of the quotient can be computed by Corollary 5.6 obtaining
\[ e(X^{ir}_n/\text{SL}_2) = \frac{e(X^{ir}_n)}{e(\text{PGL}_2)} = \frac{e(X^{ir}_n)}{q^3 - q}. \]
The computation of $e(X_n^{ir})$ can be done using the analysis above since $X_n^{ir} = X_n - X_n - X_n^\delta - X_n^\iota - X_n^\iota$. Using that $e(X_n) = e(\text{SL}_2)^n = (q^3 - q)^n$, we find that
\[ e(X_n^{ir}) = 2^n q^2 - \frac{1}{2} (q^3 - q)((q + 1)^{n-1} + (q - 1)^{n-1}) - (2^n q + (q - 1)^n q^n - (q - 1)^n q - 2^n)(q + 1) - 2^n + (q^3 - q)^n. \]

On the other hand, for $X_n^\iota$ the situation is more involved since the action is very far from being closed. Actually, we are going to show that $(X_n^D, \mathbb{Z}_2)$ is a core for the action of $\text{SL}_2$ on $X_n^\iota$. For that, we need a preliminary result about the behaviour of the orbits.

**Proposition 7.3.** Let $W \subseteq X_n^D(\text{SL}_2)$ be a closed set. For any $A \in \overline{W}$ we have that $[A] \cap W \neq \emptyset$.

**Proof.** Since $W$ is closed, it is the zero set of some regular functions $f_1, \ldots, f_r : X_n^D \cong (\mathbb{C}^*)^n \rightarrow \mathbb{C}$. Since $O((\mathbb{C}^*)^n) \cong \mathbb{C}[^\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}]$, after multiplying by some polynomials we can suppose that $f_i \in \mathbb{C}[\lambda_1, \ldots, \lambda_n]$. Now, let us consider the algebraic variety
\[ \Omega = \left\{ (A_i, \lambda_i, v) \in X_n \times (\mathbb{C}^*)^n \times (\mathbb{C}^2 - \{0\}) \mid A_i v = \lambda_i v, f_i(\lambda_1, \ldots, \lambda_n) = 0 \right\} \subseteq X_n \times \mathbb{C}^{n+2}. \]

Now, let $p : X_n \times \mathbb{C}^{n+2} \rightarrow X_n$ be the projection onto the first factor, and let $W' = p(W)$. Observe that, if $A = (A_1, \ldots, A_n) \in W'$, then the matrices $A_i$ have a common eigenvector so $A$ is reducible. Hence,
\[ QAQ^{-1} = \left( \begin{array}{cc} \lambda_1 & a_1 \\ 0 & \lambda_1^{-1} \end{array} \right), \ldots, \left( \begin{array}{cc} \lambda_n & a_n \\ 0 & \lambda_n^{-1} \end{array} \right) \]
for some $Q \in \text{SL}_2$ and satisfying $f_i(\lambda_1, \ldots, \lambda_n) = 0$. Therefore, taking $P_m = \left( \begin{array}{cc} m^{-1} & 0 \\ 0 & m \end{array} \right)$, we have
\[ (P_m Q) \cdot A \rightarrow \left( \begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{array} \right), \ldots, \left( \begin{array}{cc} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{array} \right) \in X_n^D, \]
for $m \rightarrow \infty$. That diagonal element is in the analytic closure of $[A]$ and, thus, also in the Zariski one. Moreover, $f_i(\lambda_1, \ldots, \lambda_n) = 0$, so this element belongs to $W$. Therefore, $[A] \cap W \neq \emptyset$. Hence, in order to finish the proof, it is enough to show that $[W] \subseteq W'$. Trivially, $[W] \subseteq W'$, so it is enough to prove that $W'$ is closed.

To do so, let us consider the projectivization of $\Omega$ to the closed projective set
\[ \tilde{\Omega} = \left\{ (A_i, [\mu_0 : \mu_1 : \ldots : \mu_n], \bar{v}) \in X_n \times \mathbb{P}^n \times \mathbb{P}^1 \mid A_i \mu_0 v = \mu_i v, \tilde{f}_i(\mu_0, \ldots, \mu_n) = 0 \right\} \subseteq X_n \times \mathbb{P}^n \times \mathbb{P}^1, \]
where $\tilde{f}_i$ denotes the homogenization of the polynomial $f_i$. Observe that any element $(A_i, [\mu_0 : \mu_1 : \ldots : \mu_n], \bar{v}) \in \tilde{\Omega}$ must have $\mu_0 \neq 0$ so there are no points at infinity. Hence, we can also write $W' = \rho(\tilde{\Omega})$ where $\rho : X_n \times \mathbb{P}^n \times \mathbb{P}^1 \rightarrow X_n$ is the first projection.
Recall that projective spaces are universally closed, so \( \rho \) is closed and, thus, \( W' = \rho(\tilde{\Omega}) \) is closed, as we wanted. \( \square \)

**Corollary 7.4.** Consider the action of \( \mathbb{Z}_2 \) on \( X_n^D \) by permuting the columns. Then \((X_n^D, \mathbb{Z}_2)\) is a core for the action of \( \text{SL}_2 \) on \( X_n^r \).

**Proof.** First, observe that the orbit of an element of \( X_n^D \) by the action of \( \mathbb{Z}_2 \) is finite, so the orbit is automatically closed and, thus, part [ii] of Proposition 4.4 holds. For part [iii] observe that, for any \( A \in X_n^r \), taking \( Q \) and \( P_m \) as in the proof of Proposition 7.3, we see that \([A] \cap X_n^D \neq \emptyset\). This intersection consists of, at most, two points uniquely determined by the eigenvalues of \( A \) so, in particular, they are \( \mathbb{Z}_2 \)-equivalent.

In order to prove condition [iii], let \( W_1, W_2 \subseteq X_n^D \) be two \( \mathbb{Z}_2 \)-invariant disjoint closed subsets and suppose that \( A \in [W_1] \cap [W_2] \). By Proposition 7.3 we have \([A] \cap W_1 \neq \emptyset \) and \([A] \cap W_2 \neq \emptyset \). However, this cannot happen since the elements of \([A] \cap X_n^D \) are \( \mathbb{Z}_2 \)-equivalent and \( W_1, W_2 \) are \( \mathbb{Z}_2 \)-invariant and disjoint. Thus, \([W_1] \cap [W_2] = \emptyset\), proving condition [iii]. This proves that \((X_n^D, \mathbb{Z}_2)\) is a core. \( \square \)

**Corollary 7.5.** The Deligne-Hodge polynomial of the GIT quotient of the reducible stratum is

\[
e(X_n^r \sslash \text{SL}_2) = \frac{1}{2}((q - 1)^n + (q + 1)^n).
\]

**Proof.** Observe that \( \mathbb{Z}_2 \) is a reductive group so the GIT quotient of \( X_n^D \) by \( \mathbb{Z}_2 \) exists and is a pseudo-quotient. Hence, by Proposition 4.4 and the results of Section 5 we have that \( e(X_n^r \sslash \text{SL}_2) = e(X_n^D \sslash \mathbb{Z}_2) \). For computing this last polynomial, observe that \( X_n^D = (\mathbb{C}^*)^n \) and use the calculations of [27] (see also Remark 5.4). \( \square \)

After this analysis, we have computed the Deligne-Hodge polynomials of both strata so, adding up the contributions we obtain that

\[
e(X_n \sslash \text{SL}_2) = \frac{1}{2}(q + 1)^{n-1}q + \frac{1}{2}(q - 1)^{n-1}q - (q - 1)^{n-1}q^{n-1} + (q^3 - q)^{n-1}.
\]

**Remark 7.6.** This result agrees with the computations of [20] and [5]. Actually, the argument is, somehow, parallel to the one in the former paper.

**Remark 7.7.** From the analysis of this section, it is also possible to study the case in which \( \Gamma = \mathbb{Z}^n \) is the free abelian group with \( n \) generators and \( G = \text{SL}_m \). Observe that, in this case, \( \mathcal{X}_{\text{SL}_m}(\mathbb{Z}^n) \) is the set of tuples of \( n \) pairwise commutating matrices of \( \text{SL}_m \). Since commutating matrices share a common eigenvector, all the representations of \( \mathcal{X}_{\text{SL}_m}(\mathbb{Z}^n) \) are reducible so \( \mathcal{X}_{\text{SL}_m}(\mathbb{Z}^n) = \mathcal{X}_{\text{SL}_m}^r(\mathbb{Z}^n) \). Hence, analogously to Corollary 7.4, \( (\mathcal{X}_{\text{SL}_m}^D(\mathbb{Z}^n), S_m) \) is a core for \( \mathcal{X}_{\text{SL}_m}^D(\mathbb{Z}^n) \), where \( S_m \) acts on \( \mathcal{X}_{\text{SL}_m}^D(\mathbb{Z}^n) = (\mathbb{C}^*)^{n(m-1)} \) by permutation of the eigenvalues. Hence, we obtain that

\[
\mathcal{X}_{\text{SL}_m}(\mathbb{Z}^n) \sslash \text{SL}_m = (\mathbb{C}^*)^{n(m-1)} \sslash S_m.
\]
Analogously, for $G = \text{GL}_m(\mathbb{C})$, we obtain that $\mathcal{X}_{\text{GL}_m(\mathbb{C})}(\mathbb{Z}^n) / \text{GL}_m(\mathbb{C}) = (\mathbb{C}^*)^{nm} / S_m$. This reproves Theorem 5.1 of [11]. In the case $m = 2$, the Deligne-Hodge polynomial of these character varieties can be computed by means of Remark 5.4, obtaining that

$$e \left( \mathcal{X}_{\text{SL}_2}(\mathbb{Z}^n) / \text{SL}_2 \right) = e \left( (\mathbb{C}^*)^n / \mathbb{Z}_2 \right) = \frac{1}{2} \left( (q - 1)^n + (q + 1)^n \right),$$

$$e \left( \mathcal{X}_{\text{GL}_2}(\mathbb{Z}^n) / \text{GL}_2 \right) = e \left( (\mathbb{C}^*)^{2n} / \mathbb{Z}_2 \right) = \frac{1}{2} \left( (q - 1)^{2n} + (q + 1)^{2n} \right).$$

In the higher rank case, we need to use stronger results about equivariant cohomology in order to compute the corresponding quotient by $S_m$. This is, precisely, the strategy accomplished in [11].

7.2. Surface groups. In this section, we will consider the case $\Gamma = \pi_1(\Sigma_g)$, where $\Sigma_g$ is the genus $g \geq 1$ compact surface. Recall that, for short, we will denote the associated $\text{SL}_2$-representation variety by $\mathcal{X}_g = \mathcal{X}_{\pi_1(\Sigma_g)}$ and its GIT quotient $R_g = \mathcal{X}_g / \text{SL}_2$. We have that $\mathcal{X}_g \subseteq X_{2g}$ as a closed subvariety. In order to identify the elements of $\mathcal{X}_g$, let us denote the set of upper triangular matrices of $X_n$ by $X_n^{UT} \cong (\mathbb{C}^*)^n \times \mathbb{C}^n$ and $\mathcal{X}_g^{UT} = X_{2g}^{UT} \cap \mathcal{X}_g$. Given $A \in X_{2g}^{UT}$, say

$$A = \left( \begin{array}{cc} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{array} \right), \left( \begin{array}{cc} \mu_1^{-1} & \beta_1 \\ 0 & \mu_1 \end{array} \right), \ldots, \left( \begin{array}{cc} \lambda_g & \alpha_g \\ 0 & \lambda_g \end{array} \right), \left( \begin{array}{cc} \mu_g & \beta_g \\ 0 & \mu_g \end{array} \right)$$

with $\lambda_i, \mu_i \in \mathbb{C}^*$ and $\alpha_i, \beta_i \in \mathbb{C}$, a straightforward computation shows that $A \in \mathcal{X}_g^{UT}$ if and only if

$$\sum_{i=1}^g \lambda_i \mu_i \left[ (\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] = 0.$$ 

In particular, this implies that, for the strata given by Proposition 7.1, we have $\mathcal{X}_g^{UT} = X_{2g}^{UT}, \mathcal{X}_g^D = X_{2g}^D$ and $\mathcal{X}_g^v = X_{2g}^v$. Let $\pi \subseteq \mathbb{C}^{2g}$ be the $(\alpha_i, \beta_i)$-plane defined by the previous equation for fixed $(\lambda_i, \mu_i)$.

- For $\mathcal{X}_g^r$, observe that, using the equality $\mathcal{X}_g^r = X_{2g}^r$ and Corollary 7.3, we obtain that $(\mathcal{X}_g^r, \mathbb{Z}_2)$ is a core for the action. Therefore, since $X_{2g}^r = (\mathbb{C}^*)^n$, we have

$$e \left( \mathcal{X}_g^r / \text{SL}_2 \right) = e \left( (\mathbb{C}^*)^{2g} / \mathbb{Z}_2 \right) = \frac{1}{2} \left( (q - 1)^{2g} + (q + 1)^{2g} \right).$$

- For $\mathcal{X}_g^{ir}$, observe that, since $X_{2g}^{ir}$ is an open set of $X_{2g}$ in which the action of $\text{PGL}_2$ is closed and free, and $\mathcal{X}_g \subseteq X_{2g}$ is closed, then $\mathcal{X}_g^{ir} = X_{2g}^{ir} \cap \mathcal{X}_g$ is an open subset of $\mathcal{X}_g$ with a closed and free action. Therefore, Corollary 5.6 gives us that

$$e \left( \mathcal{X}_g^{ir} / \text{SL}_2 \right) = \frac{e \left( \mathcal{X}_g^{ir} \right)}{e \left( \text{PGL}_2 \right)} = \frac{e \left( \mathcal{X}_g^{ir} \right)}{q^3 - q}.$$ 

In order to complete the calculation, it is enough to compute $e \left( \mathcal{X}_g^{ir} \right)$. For this purpose, we will use that $\mathcal{X}_g^{ir} = \mathcal{X}_g - \mathcal{X}_g^v - \mathcal{X}_g^s - \mathcal{X}_g^r$ and count.
- \( X_g^v \). In this case, since \( X_g^U = X_{2g}^v \) and \( X_g \) is \( \text{SL}_2 \)-invariant, we have that \( X_g^v = X_{2g}^v \). Therefore, 
\[
e (X_g^v) = e (X_{2g}^v) = 2^{2g}(q^2 - 1)q^{2g} - 1.
\]

- \( X_g^b \). Again, \( X_g^b = \Xi_{2g}^b \) and, thus, \( X_g^b = X_{2g}^b \). Therefore, 
\[
e (X_g^b) = e (X_{2g}^b) = \frac{q^3 - q}{2} ((q - 1)^{2g-1} + (q + 1)^{2g-1}) - 2^{2g}q^2.
\]

- \( X_g^i \). Again, \( X_g^i = X_{2g}^i \), which are \( 2^{2g} \) matrices so \( e (X_g^i) = 2^{2g} \).

- \( X_g^c \). In this case, any element is conjugated to one of the form 
\[
\begin{pmatrix}
\lambda_1 & \alpha_1 \\
0 & \lambda_1^{-1}
\end{pmatrix}, \quad \begin{pmatrix}
\mu_1^{-1} & \beta_1 \\
0 & \mu_1
\end{pmatrix}, \ldots, \begin{pmatrix}
\lambda_g & \alpha_g \\
0 & \lambda_g
\end{pmatrix}, \quad \begin{pmatrix}
\mu_g & \beta_g \\
0 & \mu_g
\end{pmatrix},
\]

with \( (\lambda_1, \mu_1, \ldots, \lambda_g, \mu_g) \in (\mathbb{C}^*)^{2g} - \{ (\pm 1, \ldots, \pm 1) \} \) and \( (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \in \pi - l \) where \( l \) is the line spanned by \( (\lambda_1 - \lambda_1^{-1}, \mu_1 - \mu_1^{-1}, \ldots, \lambda_g - \lambda_g^{-1}, \mu_g - \mu_g^{-1}) \).

Thus, we have a fibration 
\[
U \longrightarrow \text{PGL}_2 \times \Omega \longrightarrow X_g^c,
\]

where \( U = \mathbb{C} \times \mathbb{C}^* \) and \( \Omega \) is a fibration with trivial monodromy \( \pi - l \rightarrow \Omega \rightarrow (\mathbb{C}^*)^{2g} - \{ (\pm 1, \ldots, \pm 1) \} \). Using that \( e (\pi) = q^{2g-1} \) and \( e (l) = q \), the Deligne-Hodge polynomial is 
\[
e (X_g^c) = \frac{q^3 - q}{(q - 1)q} ((q - 1)^{2g} - 2^{2g}) (q^{2g-1} - q).
\]

Therefore, putting all together we have that 
\[
e (X_g^v) = (q + 1)(q - 1)^{2g} (q^{2g-1} - q) + \frac{q^3 - q}{2} ((q - 1)^{2g-1} + (q + 1)^{2g-1}) - 2^{2g}(q^2 - 1).
\]

From \([27], \text{Proposition 11}\), we know that the Deligne-Hodge polynomial of the total space is 
\[
e (X_g) = 2^{2g-1}(q - 1)^{2g-1}(q + 1)q^{2g-1} + 2^{2g-1}(q + 1)^{2g-1}(q - 1)q^{2g-1}
+ \frac{1}{2}(q + 1)^{2g-1}(q - 1)^{2g-1} + \frac{1}{2}(q - 1)^{2g-1}(q + 1)(q - 3)q^{2g-1}
+ (q + q^{2g-1})(q^2 - 1)^{2g-1}.
\]

So we finally find that 
\[
e (\mathcal{R}_g) = \frac{1}{2} ((2^{2g} + 2(q - 1)^{2g-2} + q - 1)q^{2g-2} + q^2 + 2(q - 1)^{2g-2} + q)(q + 1)^{2g-2}
+ \frac{1}{2} ((2^{2g} - 1)(q - 1)^{2g-2} - (q - 1)^{2g-2}q - 2^{2g+1})q^{2g-2} + \frac{1}{2}(q - 1)^{2g-1}q.
\]

Remark 7.8. This result agrees with the one given in \([27], \text{Theorem 14}\), and in \([1], \text{Theorem 1.3}\).
8. Parabolic representation varieties

In this section, we will discuss a slightly more general setting for representation varieties that will be very useful for applications. As always, let $\Gamma$ be a finitely generated group and let $G$ be an algebraic group. A parabolic structure $Q$ is a finite set of pairs $(\gamma, \lambda)$ where $\gamma \in \Gamma$ and $\lambda \subseteq G$ is a locally closed subset which is closed under conjugation. Given an algebraic structure $Q$, we define the parabolic representation variety, $\mathcal{X}_G(\Gamma, Q)$, as the subset of $\mathcal{X}_G(\Gamma)$

$$\mathcal{X}_G(\Gamma, Q) = \{ \rho \in \mathcal{X}_G(\Gamma) \mid \rho(\gamma) \in \lambda \text{ for all } (\gamma, \lambda) \in Q \}.$$ 

As in the non-parabolic case, $\mathcal{X}_G(\Gamma, Q)$ has a natural algebraic variety structure given as follows. Suppose that $Q = \{(\gamma_1, \lambda_1), \ldots, (\gamma_s, \lambda_s)\}$. Choose a finite set of generators $S$ of $\Gamma$ that contains all the $\gamma_i$, say $S = \{\eta_1, \ldots, \eta_r, \gamma_1, \ldots, \gamma_s\}$. In that case, using $S$ to identify $\mathcal{X}_G(\Gamma)$ with a closed subvariety of $G^{s+r}$ (see Section 6), then we also have a natural identification $\mathcal{X}_G(\Gamma, Q) = \mathcal{X}_G(\Gamma) \cap (G^r \times \lambda_1 \times \ldots \times \lambda_s)$. We impose in $\mathcal{X}_G(\Gamma, Q)$ the algebraic structure inherited from this identification and, as above, such a structure does not depend on $S$.

The conjugacy action of $G$ on $\mathcal{X}_G(\Gamma)$ restricts to an action on $\mathcal{X}_G(\Gamma, Q)$ since the subsets $\lambda_i$ are closed under conjugation. Moreover, as a subvariety of $\mathcal{X}_G(\Gamma)$, all the previous results about stability translate to the parabolic setting. In this way, all the points of $\mathcal{X}_G(\Gamma, Q)$ are semi-stable for this action, the stable points are $\mathcal{X}_G^s(\Gamma, Q) = \mathcal{X}_G(\Gamma) \cap \mathcal{X}_G(\Gamma, Q)$, in which the action of $G/G^0$ is closed and free.

In this paper, we are going to focus in the following cases:

- Fix some elements $h_1, \ldots, h_s \in G$ and let $[h_i]$ be their conjugacy classes. Then, we take $\Gamma = F_{n+s}$ and $Q = \{(\gamma_1, [h_1]), \ldots, (\gamma_s, [h_s])\}$, where $\gamma_1, \ldots, \gamma_s \in F_{r+s}$ is an independent set. Observe that $[h_i] \subseteq G$ is locally closed since, by Lemma 3.7, it is an open subset of $[h_i]$.
- Let $\Sigma = \Sigma_g = \{p_1, \ldots, p_s\}$ with $p_i \in \Sigma_g$ distinct points, called the punctures or the marked points. In that case, we have a presentation of the fundamental group of $\Sigma$ given by

$$\pi_1(\Sigma) = \left\langle \alpha_1, \beta_1 \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_s \mid \prod_{i=1}^g \alpha_i \beta_i \prod_{j=1}^s \gamma_j = 1 \right\rangle,$$

where the $\gamma_i$ are the positive oriented simple loops around the punctures. As parabolic structure, we will take $Q = \{(\gamma_1, [h_1]), \ldots, (\gamma_s, [h_s])\}$ for some fixed elements $h_i \in G$. Observe that the epimorphism $F_{2g+s} \rightarrow \pi_1(\Sigma)$ gives an inclusion $\mathcal{X}_G(\pi_1(\Sigma), Q) \subseteq \mathcal{X}_G(F_{2g+s}, Q)$.

As in the previous section, we will focus on the case $k = \mathbb{C}$ and $G = \text{SL}_2(\mathbb{C})$. In this paper, we are going to study the parabolic character varieties whose parabolic data lie
in the conjugacy classes of the matrices

\[ J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad -\text{Id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \]

As we will see, the most important case to be considered is to take the parabolic structure \( Q_s^+ = \{(\gamma_1, [J_+]), \ldots, (\gamma_s, [J_+])\} \). The rest of cases can be easily obtained from this one, as shown in Section 8.3.

8.1. Free group. For short, let us denote \( X_{n,s} = X_{\text{SL}_2}(F_{n+s}, Q_s^+) \). As before, we have an stratification \( X_{n,s} = X_{n,s}^{ir} \sqcup X_{n,s}^r \). Hence, by the results of Section 5, we have that

\[ e(X_{n,s} \parallel \text{SL}_2) = e(X_{n,s}^{ir} \parallel \text{SL}_2) + e(X_{n,s}^r \parallel \text{SL}_2) . \]

Let us analyze each stratum separately.

- \( X_{n,s}^{ir} \) is again an open subvariety in which the the action of \( \text{PGL}_2 \) is closed and free, so Corollary 5.6 gives us that

\[ e(X_{n,s}^{ir} \parallel \text{SL}_2) = e(X_{n,s}^{ir}) / e(\text{PGL}_2) . \]

In order to compute \( e(X_{n,s}^{ir}) \) we count:

- \( X_{n,s}^{ir} \). In this case, \( X_{n,s}^{ir} = \{\pm 1\}^n \times [\mathbb{C}^n \times (\mathbb{C}^*)^s] \) and the action restricts to projectivizing the second factor. Therefore,

\[ e(X_{n,s}^{ir}) = 2^n (q^2 - 1) \frac{q^n(q - 1)^s}{q - 1} . \]

- \( X_{n,s}^\delta \). In this case, since \( J_+ \) is not diagonalizable, \( X_{n,s}^D = \emptyset \) so \( X_{n,s}^\delta = \emptyset \) and it makes no contribution.

- \( X_{n,s}^{\iota} \). Again, it is empty so it makes no contribution.

- \( X_{n,s}^\kappa \). In this case, any element is conjugated to one of the form

\[ \left( \begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_n & \alpha_n \\ 0 & \lambda_n^{-1} \end{pmatrix}, \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & c_s \\ 0 & 1 \end{pmatrix} \right) \]

with \( (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n - \{(\pm 1, \ldots, \pm 1)\} \) and \( (\alpha_1, \ldots, \alpha_n, c_1, \ldots, c_s) \in \mathbb{C}^n \times (\mathbb{C}^*)^s \). Thus, we have a fibration

\[ U \rightarrow \text{PGL}_2 \times \Omega \rightarrow X_{n,s}^\kappa , \]

where \( \Omega = [(\mathbb{C}^*)^n - \{(\pm 1, \ldots, \pm 1)\}] \times [\mathbb{C}^n \times (\mathbb{C}^*)^s] \). Observe that we do not need to remove any antidiagonal value since the intersection of the line spanned by \( (\lambda_1 - \lambda_1^{-1}, \ldots, \lambda_n - \lambda_n^{-1}, 0, \ldots, 0) \) with \( \mathbb{C}^n \times (\mathbb{C}^*)^s \) is empty. Hence, the Deligne-Hodge polynomial is

\[ e(X_{n,s}^\kappa) = \frac{q^3 - q}{(q - 1)^2} q^n(q - 1)^s . \]
Therefore, putting all together we have that
\[ e \left( X^r_{n,s} \right) = (q - 1)^n(q - 1)^s(q + 1)q^n. \]

Using that \( X_{n,s} = SL_2^n \times [J_+]^s \) and \( e ([J_+]) = (q^2 - 1) \), we obtain that \( e \left( X_{n,s} \right) = (q^3 - q)^n(q^2 - 1)^s \), so we finally find
\[ e \left( X^r_{n,s} \right) = (q - 1)^n(q - 1)^s \left((q^2 + q)^n(q + 1)^s - (q + 1)q^n \right). \]

- For \( X^r_{n,s} \) the situation becomes completely different from the previous ones. As we have shown, \( X^D_{n,s} = \emptyset \) so it can be no longer a core for the action. The key point now is that, precisely for this reason, the action of \( SL_2 \) on \( X^r_{n,s} \) is closed.

To be precise, let \( A \in X^r_{n,s} \subseteq X^s_{n+s} \) and let \([A]\) be the closure of its orbit in \( X^s_{n+s} \). The difference \([A] - [A]\) lies in \( X^D_{n+s} \) so, since \( X^D_{n+s} \cap X_{n,s} = X^D_{n,s} = \emptyset \), the orbit of \( A \) is closed in \( X^s_{n,s} \). However, the action of \( PGL_2 \) on \( X^s_{n,s} \) is not free everywhere so we have to distinguish between two strata:

- \( X^e_{n,s} \). Here, the action of \( PGL_2 \) is free so, by Corollary 5.6 we have
  \[ e \left( X^e_{n,s} \right) / SL_2) = e \left( X^e_{n,s} \right) / q^3 - q = ((q - 1)^n - 2^n) q^{n-1}(q - 1)^{s-1}. \]

- \( X^v_{n,s} \). Here, the action of \( PGL_2 \) is not free, but it has stabilizer isomorphic to \( Stab J_\pm \cong \mathbb{C} \). The fact that \([J_\pm] \cong SL_2 / Stab J_\pm \) implies that the GIT quotient \( X^v_{n,s} \rightarrow X^e_{n,s} / SL_2 \) is a locally trivial fibration with fiber \( SL_2 / Stab J_\pm \) and trivial monodromy. Hence, we have that
  \[ e \left( X^v_{n,s} / SL_2 \right) = e \left( X^e_{n,s} / SL_2 \right) = 2^n q^n(q - 1)^{s-1}. \]

Observe that we have used that \( e (SL_2 / Stab J_+) = e (SL_2) / e (Stab J_+) = q^2 - 1 \).

We have an stratification \( X^r_{n,s} = X^v_{n,s} \cup X^e_{n,s} \), where \( X^e_{n,s} \) is open orbitwise-closed. Thus, we have
\[ e \left( X^r_{n,s} / SL_2 \right) = e \left( X^v_{n,s} / SL_2 \right) + e \left( X^e_{n,s} / SL_2 \right) = (2^n + (q - 1)^{n-1})(q - 1)^s q^{n-1}. \]

Summarizing, the analysis above shows that
\[ e \left( X_{n,s} / SL_2 \right) = 2^n(q - 1)^s q^{n-1} + (q^3 - q)^{n-1}(q^2 - 1)^s. \]

8.2. Surface groups. Let \( \Sigma \) be the genus \( g \) compact surface with \( s \) punctures and let us denote \( \mathfrak{X}_{g,s} = \mathfrak{X}_{SL_2}(\pi_1(\Sigma), Q^+_g) \). As we mentioned in Section 5, the decomposition into reducible representations and irreducible ones gives an equality
\[ e \left( \mathfrak{X}_{g,s} / SL_2 \right) = e \left( \mathfrak{X}^r_{g,s} / SL_2 \right) + e \left( \mathfrak{X}^e_{g,s} / SL_2 \right). \]

In order to understand this stratification, observe that, as closed subvarieties of the one for the free case, we have \( \mathfrak{X}^D_{g,s} = \mathfrak{X}^e_{g,s} = \mathfrak{X}^r_{g,s} = \emptyset \). Let us define \( X^U_{n,s} = X_{n+s} \cap X_{n,s} \) and
Let us analyze each stratum separately.

- $X^r_{g,s}$ is again an open subvariety in which the action of $\text{PGL}_2$ is closed and free so Corollary 5.6 gives us that

$$
\pi_s = \left\{ \sum_{j=1}^s c_j = 0 \mid c_j \neq 0 \right\}.
$$

In order to compute the Deligne-Hodge polynomial of this space, just observe that $\pi_s = (\mathbb{C}^*)^s - \pi_{s-1}$. Therefore, using the base case $\pi_1 = \emptyset$, we have

$$
e (\pi_s) = (q-1)^{s-1} - e (\pi_{s-1}) = \sum_{k=1}^{s-1} (-1)^{k+1} (q-1)^{s-k} = (-1)^s \left( \frac{(1-q)^s - 1}{q} + 1 \right).
$$

Observe that, as in the free case, the action of $\text{SL}_2$ on $X^r_{n,s}$ is just the projectivization on $\mathbb{C}^n \times \pi_s$ so we obtain a fibration with trivial monodromy

$$
\mathbb{C}^* \to \frac{\text{SL}_2}{\text{Stab}_J} \times \{ \pm 1 \}^{2g} \times [\mathbb{C}^{2g} \times \pi_s] \to X^r_{g,s}.
$$

Therefore, we have

$$
e (X^r_{g,s}) = 2^{2g}(q^2 - 1) \frac{q^{2g}}{q-1} \left[ (-1)^s \left( \frac{(1-q)^s - 1}{q} + 1 \right) \right].
$$

- $X^d_{g,s}$. This case makes no contribution since $X^D_{n,s} = \emptyset$.

- $X^e_{g,s}$. Again, it is empty so it makes no contribution.

- $X^g_{g,s}$. In this case, any element is conjugated to one of the form

$$
\left( \begin{pmatrix} \lambda_1 & \alpha_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \mu_g & \beta_n \\ 0 & \mu_g^{-1} \end{pmatrix}, \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & c_s \\ 0 & 1 \end{pmatrix} \right)
$$

with $\lambda_i, \mu_i \in \mathbb{C}^*$, $\alpha_i, \beta_i \in \mathbb{C}$ and $c_i \in \mathbb{C}^*$. Then, we have that $A \in X^g_{g,s}$ if and only if

$$
\sum_{i=1}^{g} \lambda_i \mu_i \left( (\mu_i - \lambda_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right) + \sum_{i=1}^{s} c_i = 0.
$$
with \((\lambda_1, \ldots, \mu_g) \in (\mathbb{C}^*)^{2g} - \{(\pm 1, \ldots, \pm 1)\}\) and \((\alpha_1, \ldots, \beta_g, c_1, \ldots, c_s) \in \Pi_s\), where we denote

\[
\Pi_s = \left\{ \sum_{i=1}^{g} \lambda_i \mu_i \left[ (\mu_i - \mu_i^{-1}) \beta_i - (\lambda_i - \lambda_i^{-1}) \alpha_i \right] + \sum_{i=1}^{s} c_i = 0 \right\},
\]

for fixed \((\lambda_i, \mu_i)\). In order to compute the Deligne-Hodge polynomial of \(\Pi_s\) observe that \(\Pi_s = \mathbb{C}^{2g} \times (\mathbb{C}^*)^{s-1} - \Pi_{s-1}\). Using as base case that \(\Pi_1\) is \(\mathbb{C}^{2g}\) minus a hyperplane, we have

\[
e(\Pi_s) = q^{2g}(q - 1)^{s-1} - e(\Pi_{s-1})
\]

\[
\quad = q^{2g} \sum_{k=1}^{s} (-1)^{k+1}(q - 1)^{s-k} + (-1)^s q^{2g-1} = q^{2g-1}(q - 1)^s.
\]

Thus, we have a fibration

\[
U \rightarrow \text{PGL}_2 \times \Omega \rightarrow \mathcal{X}^{e}_{g,s},
\]

where \(\Pi_s \rightarrow \Omega \rightarrow (\mathbb{C}^*)^{2g} - \{(\pm 1, \ldots, \pm 1)\}\) is a fibration with trivial monodromy. Therefore, the Deligne-Hodge polynomial is

\[
e(\mathcal{X}^{e}_{g,s}) = \frac{q^3 - q}{(q - 1)q} \left[ (q - 1)^{2g} - 2^{2g} \right] \left[ q^{2g-1}(q - 1)^s \right].
\]

Therefore, putting all together, we obtain that

\[
e(\mathcal{X}^{r}_{g,s}) = 2^{2g} (-1)^s (q + 1)q^{2g} \left( \frac{(-q + 1)^s - 1}{q} + 1 \right)
\]

\[
\quad - (2^{2g} - (q - 1)^{2g})(q - 1)^s(q + 1)q^{2g-1}.
\]

In [12], [REF], it is proven that the Deligne-Hodge polynomial of the whole representation variety is

\[
e(\mathcal{X}_{g,s}) = (q^2 - 1)^{2g+s-1} q^{2g-1} + \frac{1}{2} (q - 1)^{2g+s-1} q^{2g-1}(q + 1)(2^{2g} + q - 3)
\]

\[
\quad + \frac{(-1)^s}{2} (q + 1)^{2g+s-1} q^{2g-1}(q - 1)(2^{2g} + q - 1).
\]

Therefore, substracting the contribution of the previous strata, we obtain

\[
e(\mathcal{X}^{sr}_{g,s}) = 2^{2g-1} (-1)^s (q + 1)q^{2g+s-1}(q - 1)q^{2g-1}
\]

\[
\quad - 2^{2g} (-1)^s (q + 1)q^{2g} \left( \frac{(1 - q)^s - 1}{q} + 1 \right)
\]

\[
\quad + \frac{1}{2} (-1)^s (q + 1)^{2g+s-1}(q - 1)^{2g} q^{2g-1} + (2^{2g} - (q - 1)^{2g})(q - 1)^s(q + 1)q^{2g-1}
\]

\[
\quad + \frac{1}{2} (q - 1)^{2g+s-1}(q + 1)(q - 3)q^{2g-1}
\]

\[
\quad + \frac{(2^{2g}q^2 + 2^{2g+1}q + 2^{2g} + 2(q + 1)^{2g+s})(q - 1)^{2g+s-1} q^{2g-1}}{2(q + 1)}.
\]
Summarizing, the analysis above shows that Parabolic data of Jordan type.

8.3. Parabolic data of Jordan type. Let us denote by \( \Gamma_{g,s} \) the fundamental group of the genus \( g \) compact surface with \( s \) removed points. Consider the parabolic structure \( Q = \{ (\gamma_1, [C_1]), \ldots, (\gamma_s, [C_s]) \} \), where \( C_i = J_+ \) or \( \{-\text{Id}\} \). Let \( r_+ \) be the number of \( J_+ \), \( r_- \) the number of \( J_- \) and \( t \) the number of \( \{-\text{Id}\} \) (so that \( r_+ + r_- + t = s \)), and let \( \sigma = (-1)^{r_+ + t} \). Observe that \( J_+ \in [-J_-] \) and \( [-\text{Id}] = \{-\text{Id}\} \) so, depending on \( \sigma \), we have:

- If \( \sigma = 1 \), then we have \( X_{\text{SL}_2}(\Gamma_{g,s}, Q) = X_{\text{SL}_2}(\Gamma_{g,r}, Q_+^r) \) where \( r = r_+ + r_- \). Hence, we have \( e(X_{\text{SL}_2}(\Gamma_{g,s}, Q) \parallel \text{SL}_2) = e(X_{\text{SL}_2}(\Gamma_{g,r}, Q_+^r) \parallel \text{SL}_2) \) and the polynomial follows from the computation above.
- If \( \sigma = -1 \), then we have \( X_{\text{SL}_2}(\Gamma_{g,s}, Q) = X_{\text{SL}_2}(\Gamma_{g,r+1}, Q_-^{r+1}) \) where \( r = r_+ + r_- \) and \( Q_r = \{ (\gamma_1, [J_+]), \ldots, (\gamma_r, [J_+]), (\gamma_{r+1}, \{-\text{Id}\}) \} \). This is the so-called twisted representation variety. This variety does not contain reducible representations. To check that, let \( A = (A_1, B_1, \ldots, C_1, \ldots, C_r, -\text{Id}) \in X_{\text{SL}_2}(\Gamma_{g,r+1}, Q_-^{r+1}) \) so that

\[
\prod_{i=1}^{2g} [A_i, B_i] \prod_{j=1}^{s} C_i = -\text{Id}.
\]
If \( v \in \mathbb{C}^2 - \{0\} \) is a common eigenvector of \( A \), then, since all the eigenvalues of the commutators \([A_i, B_i]\) and the \( C_i \) are equal to 1, the left hand side of the previous equation fixes \( v \) but the right hand side does not. This proves that such a \( v \) cannot exist. Therefore, the action of \( \text{PGL}_2 \) on \( \mathfrak{X}_{\text{SL}_2}(\Gamma_{g,r+1}, Q_r^-) \) is closed and free.

As proven in [12] [REF], the representation variety has Deligne-Hodge polynomial

\[
e(\mathfrak{X}_{\text{SL}_2}(\Gamma_{g,r+1}, Q_r^-)) = (q - 1)^{2g+r-1}(q + 1)q^{2g-1}((q + 1)^{2g+r-2} + 2^{2g-1} - 1) + (-1)^{r+1}2^{2g-1}(q + 1)^{2g+r-1}(q - 1)q^{2g-1}.
\]

Therefore, Corollary 5.6 gives us

\[
e(\mathfrak{X}_{\text{SL}_2}(\Gamma_{g,s}, Q) \sslash \text{SL}_2) = (-1)^{r+1}2^{2g-1}(q + 1)^{2g+r-2}q^{2g-2} + (q - 1)^{2g+r-2}q^{2g-2}((q + 1)^{2g+r-2} + 2^{2g-1} - 1).
\]

Recall that the union of the conjugacy classes of \( J_+, J_-, \text{Id} \in \text{SL}_2 \), together with \( \text{Id} \), is exactly the set of \( C \in \text{SL}_2 \) such that \( \text{tr} \ C = \pm 2 \). Therefore, so far, we have computed the Deligne-Hodge polynomial of the parabolic character variety of a punctured compact orientable surface with any parabolic structure of the form \( Q = \{(\gamma_i, [C_i])\} \) where \( C_i \in \text{SL}_2 \) satisfies that \( \text{tr} \ C_i = \pm 2 \).

From the point of view of stratifying the quotient, there is nothing special in restricting us to the case \( \text{tr} \ C_i = \pm 2 \). Actually, the techniques of this paper can also manage the case of \( C_i \) being any element of \( \text{SL}_2 \) without major modifications. From this analysis, the Deligne-Hodge polynomial of those character varieties can be obtained from the ones of the corresponding representation varieties, as in the cases presented in this paper.

The problem is that the Deligne-Hodge polynomials of general parabolic representation varieties are not known yet. It is a future work to compute them using the almost-TQFT described in [13] by means of the methods of [12]. The extra difficulty of this case is that, when we want to introduce conjugacy classes of semi-simple type, a new interaction phenomenon appears in the TQFT that complicates the calculations. Nonetheless, we expect that a deeper analysis of the interference phenomenon will allow us to accomplish the calculation and, once these polynomials have been computed, the corresponding ones for the character varieties follow using the techniques of this paper.

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