Thermodynamical instability of black holes

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Abstract. In contrast to Hawking radiation of black hole with a given spacetime structure, we consider a competitive transition due to a heat transfer from a hotter inner horizon to a colder outer horizon of Kerr black hole, that results in a stable thermodynamical state of extremal black hole. In this process, by supposing an emission of gravitational quanta, we calculate the mass of extremal black hole in the final state of transition.

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1. Introduction

The black hole radiation, discovered by S. Hawking [1, 2], deals with the quantum field theory in curved spacetime [3], which is held adiabatically constant, while matter particles are created from virtual pairs due to the action of gravitational force near the black hole horizon, that separates the particle from its antiparticle of virtual pair, when particles outside the horizon form the almost black body radiation spectrum, for the distant observer. From the quantum theory point of view, the Hawking radiation corresponds to diagonal matrix element of transition for the gravitational degrees of freedom, since the spacetime structure is not changed, and particles are created in the constant gravitational background.

However, we could challenge to investigate non-diagonal transitions between gravitational states, that should change the spacetime structure, of course. Such the quantum processes would be inherently different from the Hawking radiation. But a strict consideration should rigorously involve a true theory of quantum gravity, which is still not at our hands now (though see [4]). Therefore, we cannot calculate probabilities of non-diagonal gravitational transitions between black holes.

Nevertheless, even in the classical formulation, the gravity establishes the thermodynamical conception of black holes [1, 2, 5, 6, 7] regarding the horizons in terms of entropy, temperature etc. In this way, the temperature of black hole horizon $T_{BH}$ equals the temperature of Hawking radiation, while the energy conservation dictates to introduce the entropy $S_{BH}$, ascribed to the horizon because of the thermodynamical relation between the variations $\delta M_{BH} = T_{BH} \delta S_{BH}$, wherein the entropy obeys the connection to the horizon area $A_{BH}$,

$$S_{BH} = \frac{1}{4G} A_{BH},$$

with $G$ being the Newton constant, while the temperature is given by the surface gravity, i.e. the acceleration of free falling at the horizon $\kappa$,

$$T_{BH} = \frac{\kappa}{2\pi}. \tag{2}$$

To concretize, the rotating black hole is described by the Kerr solution of Einstein equations. Then, it has two horizons, the inner and outer ones, with appropriate temperatures,

$$T_{\pm} = \frac{r_{\pm} - r_{-}}{A_{\pm}}, \tag{3}$$

with relevant areas $A_{\pm}$ at radiuses $r_{\pm} \geq r_{-}$. There are two limits:

(i) at $r_{-} \to 0$ one gets the Schwarzschild black hole with zero orbital momentum and zero area of inner horizon $A_{-} \to 0$ giving the infinitely hot inner horizon contracted to the singularity at $r = 0$;

(ii) at $r_{-} \to r_{+}$ one gets the extremal black hole, which horizons are degenerated at absolute zero of temperature.

‡ In the text below we put $G = 1$. 
Then, holding the constant spacetime structure, we require the inner and outer spatial thermostats outside the horizons to be almost unchanged. It means that virtual particles behind the horizons would transfer the heat from the hotter inner horizon to the colder outer horizon, that takes place due to the Hawking radiation outside the outer horizon. Therefore, the temperature of outer horizon will increase during the Hawking radiation. For the case of Schwarzschild black hole, the increase of outer horizon temperature at fixed infinite temperature of inner horizon contracted to the singularity means the decrease of black hole mass, i.e. the evaporation of black hole, since \( T_+^{-1} = 8\pi M_{\text{BH}} \). Evidently, that is the infinitely hot singularity causes the evaporation of Schwarzschild black hole, and that was our constraint to hold the spacetime structure fixed in order to derive the Hawking radiation.

However, in thermodynamics, two free thermal systems at different temperatures will tend to equaling their temperatures due to the heat transfer between them. Therefore, the black hole with different temperatures of its horizons is thermodynamically instable. It will tend to a state with the common temperature of two horizons, as the thermodynamics claims. Such the process involves the change of spacetime structure of black hole, i.e. the non-diagonal transition for the gravitational field, of course. The transition has to result in the emission of gravitational quanta. The gravitational transition could be accompanied with the radiation of non-gravitational field, too.

Further we will derive the thermal stability criteria for the Kerr black hole and calculate the mass of stable black hole in the final state by the gravitational transition from the initial state of given spacetime structure.

2. Argumentation and results

The condition of thermodynamical stability of black hole means that the temperatures of two horizons are equal to each other. The difference of temperatures is expressed in terms of horizon areas,

\[
T_+ - T_+ = \frac{1}{\sqrt{4\pi}} \frac{(A_+ - A_-)^2}{A_+ A_-} \frac{1}{\sqrt{A_+ + A_-}},
\]

hence, the stability, i.e. \( T_+ - T_+ = 0 \), certainly gives the following generic constraint:

\[
A_+ = A_-. \tag{5}
\]

Thus, any non-extremal black hole will transfer to the extremal one in order to reach the thermodynamical stability.

However, the constraint of thermal stability in (5) does not fix the value of extremal area \( A_e = A_+ = A_- \). In order to find \( A_e \) for the given initial state of black hole, we should specify the mechanism of transition. Let us consider the transition, when the gravitational fields are emitted, while virtual particles do not produce the Hawking radiation.
Thermodynamical instability of black holes

Trajectories of virtual particles, confined behind the horizons, have been considered in [8]. For instance, the spacetime interval of virtual massive particle in the Schwarzschild background with gravitational radius \( r_g \) is given by

\[
ds^2 = \frac{r_c}{r_g} \frac{r}{r_c - r} dr^2 > 0,
\]

where \( r_c \leq r_g \) is the maximal distance from the singularity, i.e. \( r \leq r_c \). We have found that such the trajectories periodically evolve with the imaginary time for the outer distant observer. This fact means that virtual particles form the thermal ensemble, and the period defines the inverse temperature of system \( \beta = 1/T_{BH} \). The periodicity determines the discrete spectrum of \( r_c \). The ground state corresponds to \( r_c = r_g \).

In classical approximation, the logarithm of partition function is given by the sum of euclidian actions for the virtual particles per the period. For the ground state, we find

\[
\ln Z = -\beta \frac{1}{2} \sum m,
\]

where the sum of masses \( m \) for the virtual particles is fixed by the following condition: the black hole mass is equal to the energy of thermal system, viz. straightforwardly \( M_{BH} = -\partial \ln Z/\partial \beta \). Then,

\[
\frac{1}{2} \sum m = M_{BH} - TS,
\]

which equals the Helmholtz free energy \( F \).

Notice that eqs. (7) and (8) is valid for the ground state of Kerr black hole too [8]. Moreover, the product of temperature to the entropy is invariant with respect to the horizon prescription, \( T_+ S_+ = T_- S_- \). Thus, the free energy ascribed to the black hole is irrelative to the choice of inner or outer horizon.

If the transition between black holes is determined by the emission of gravitational quanta, then the sum of masses of virtual particles behind the horizons should be constant, since the process should be given by the diagonal matrix element of transition for these particles, i.e.

\[
\sum m = \text{const}.
\]

Condition (9) means that the free energy of black hole \( F \) is conserved during the non-diagonal gravitational transition changing the spacetime structure. In the final state we get the extremal black hole with the horizon temperature equal to zero, \( T_e = 0 \),

\[\S\] A similar picture is valid for the Kerr black hole: we can specify appropriate trajectories of virtual particles confined between inner and outer horizons. The difference is reduced to the introduction of two inverse temperatures, which ratio can be consistently fixed in order to preserve the periodicity behind two horizons.

\[\parallel\] For ground level of massive particle moving at the equator trajectory, the interval is given by the following expression: \( ds^2 = r^2 dr^2/(r_+ - r)(r - r_-) \), in consistency with [3] in limits of \( r_- \to 0 \), \( r_c \to r_g \to r_+ \).
hence, $\mathcal{F} = M_e$ and the mass of black hole after the transition is given by the following expression:

$$M_e = M_{\text{BH}} - \frac{1}{4} T_{\text{BH}} A_{\text{BH}}.$$  \hfill (10)

For instance, if the initial state is the Schwarzschild black hole possessing $J = 0$, then $T_{\text{BH}} = 1/8\pi M_{\text{BH}}$, while the horizon area equals $A_{\text{BH}} = 16\pi M_{\text{BH}}^2$, and hence,

$$M_e = \frac{1}{2} M_{\text{BH}}|_{J=0}.$$ \hfill (11)

Remember, that the orbital momentum of extremal black hole equals $J = M_e^2$.

In addition, let us comment on the condition of ground level for the virtual particles used above in our consideration. In [9] we have shown, that the black hole emits Hawking radiation until the virtual particles occupy excited levels, while the radiation stops, if all particles fall to the ground level. In this sense, our argumentation above becomes more rigorous, since at the ground level the Hawking radiation is certainly switched off, and the non-diagonal gravitational processes provide the only mechanism of transitions between black holes.

Next, as concerns for the quantum spectrum of black holes, usually one puts $M_{\text{BH}}^2|_{J=0} = n/2$ with $n$ being large integer \[10, 11, 12, 13\]. Therefore, the transition to extremal black hole in accordance with \[11\] would give $J = n/8$, that is not half-integer or integer at arbitrary integer $n$. It means, that the transition would result in a non-extremal black hole, say, with an acceptable value of $J < n/8$, so that some particles will be probably excited from the ground level. Further, the excitations will decay due to the Hawking radiation, which will decreases the mass of black hole to reach the limit of extremal black hole at given $J$, for instance. Similar notes could be done for the case of initial state given by a non-extremal black hole at $J \neq 0$, of course.

Finally, our consideration suggests that all candidates to inactive black holes, observed in astronomy, are extremal. As for active black hole candidates, i.e. those of interacting with the ordinary matter, which surrounds the black hole, we suppose that the interaction is not related with the transitions described in the paper, since the interaction changes the mass of extremal black hole due to the accretion of matter or by emission of matter. The accretion of matter by the extremal black hole with a given orbital momentum $J$ has to conserve the thermodynamic stability, i.e. the extremal structure. Therefore, an accreted orbital momentum $\delta J$ has to adjust an appropriate amount of accreted mass $\delta M^2 = \delta J$, but if the real amount of mass $\delta' M^2$ producing the accretion of orbital momentum exceeds the adjusted value, then the difference $\Delta M^2 = \delta' M^2 - \delta M^2$ has to be ejected by the black hole at the conserved orbital momentum $\tilde{J} = J + \delta J$. To our opinion, such the ejection prefers for the balanced matter jets in opposite directions along the axis of black hole rotation.
3. Conclusion

The non-extremal rotating black hole is thermodynamically instable because its two horizons have different temperatures, while the heat transfer between them will tend to equalize these temperatures due to transitions changing the spacetime structure. The stable state is the extremal black hole having single horizon at zero temperature. Supposing the transition caused by the emission of gravitational quanta, we have calculated the mass of extremal black hole in the final state. The process inherently differs from the evaporation of black hole due to the Hawking radiation.

Acknowledgments

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