Explicit construction of constrained instantons

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Instantons in massless theories do not carry over to massive theories due to Derrick’s theorem. This theorem can, however, be circumvented, if a constraint that restricts the scale of the instanton is imposed on the theory. Constrained instantons are considered in four dimensions in $\phi^4$ theory and $SU(2)$ Yang-Mills-Higgs theory. In each of these theories a calculational scheme is set up and solved in the lowest few orders in the mass parameter in such a way that the need for a constraint is exhibited clearly. Constrained instantons are shown to exist as finite action solutions of the field equations with exponential fall off only for specific constraints that are unique in lowest order in the mass parameter in question.

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I. INTRODUCTION

Instantons have been prominent tools for the computation of nonperturbative effects in classically conformally invariant field theories including gauge theories since the pioneering achievements of Belavin et al. [1] and ’t Hooft [2]. In the presence of mass, including mass generation because of spontaneous symmetry breaking, instantons leading to a finite action do not exist as a consequence of a generalization of Derrick’s theorem [3]. However, as pointed out by Frishman and Yankielowicz [4] and Affleck [5], a finite action solution of the field theory in question can be obtained if a constraint is imposed on the theory restricting the scale of the instantons to be small compared to the inverse mass parameter.

Since then, constrained instantons have enjoyed considerable attention [6], [7], [8], [9], [10], [11]. However, little consideration has been given to a systematic explicit analytic construction of constrained instantons.

In the present paper, a detailed account is given of the explicit construction of constrained instantons in the context of the two models also considered in [6], viz. $\phi^4$ theory with a negative potential, and $SU(2)$ Yang-Mills-Higgs theory. The latter example is especially interesting because of its relevance for the standard model of electroweak unification. The constructions in the two models are carried out recursively in the mass parameters, following the pattern indicated in [6] and in such a way that the constrained instanton solutions at short distances do not contain singularities spoiling the finiteness of the actions, while their large-distance behavior is determined by the modified Bessel function $K_1$, thus ensuring the exponential fall off familiar from massive field theories.

For $\phi^4$ theory we find that the only way to achieve this goal is by means of a constraint cubic in the field or by having a similar constraint through a source term in the field equation, while other constraints only depending on the field are ruled out because they lead to singular behavior of the constrained instanton solution at the origin. For the Yang-Mills-Higgs theory exponential fall off at infinity can be obtained by adjustment of integration constants, but a constraint is necessary for the Yang-Mills field in order to ensure absence of singularities of the constrained instanton at small distances that prevent the action from being finite. No modification of the Higgs field equation is necessary.

The important point about the analysis of the present paper is that the accomplishment of a good constraint is twofold: It should

1. restrict the scale parameter of the instanton solution, and
2. ensure that the instanton solution leads to a finite action

and 1. is a necessary but by no means sufficient condition for 2. In fact, as we shall demonstrate, most constraints that ensure 1. lead to constrained instanton solutions that are singular at the origin in such a way that 2. is violated. In lowest order in the mass parameter it is found that the form of the constraint is uniquely fixed, whereas there is considerable freedom to choose the constraint in higher orders.

The paper is organized in the following way: In Sec. II the scalar $\phi^4$ theory is considered. Lowest-order corrections to the instanton solutions due to a mass parameter and constraint terms are calculated explicitly in Sec. II A and II B and in these sections also a leading-term analysis is carried out to all orders in the mass parameter, showing that the constrained instanton solution leads to a finite action and has exponential fall off at large distances. This argument is completed in Sec. II C and extended to subleading terms in Sec. II D. In Sec. III a corresponding analysis is carried out for the Yang-Mills-Higgs theory. In Sec. III A conditions for finiteness of the

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action are obtained, and in Sec. [11b] explicit solutions of the field equations are found by iteration up to fourth order in the mass parameter. In Sec. [11c] a leading order analysis is carried out on similar lines as in the scalar case. In Secs. [11d], [11e] and [11f] a constraint leading to the desired properties of the solution is constructed. In the course of this construction also the subleading terms are considered. Finally the connection to the ’t Hooft integral measure is established in Sec. [11g].

II. SCALAR $\phi^4$ THEORY

In this section we consider an Euclidean $\phi^4$ theory with negative $\phi^4$ term. The Lagrangian is

$$L = -\frac{1}{g} \left( \frac{1}{2} \langle \partial_{\mu} \phi \rangle^2 + \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \phi^4 \right).$$

(1)

In the massless case the field equation has an instanton solution $\phi_0$:

$$\partial^2 \phi_0 + \frac{1}{6} \phi_0^3 = 0, \quad \phi_0 = \frac{4\sqrt{3} \rho}{\rho^2 + x^2}.$$  

(2)

Here and elsewhere $\rho$ is a scale parameter characterizing the instanton solution. The equation governing small deviations $\delta \phi$ from this solution has the zero mode $\frac{\partial}{\partial \rho} \phi_0$.

A. Mass corrections

Take the mass $m$ to be small but non-zero. This is expected to give rise to a small deviation $\phi_1$ from the massless solution. It is convenient to introduce the variable $t = \frac{\rho}{mx}$ in terms of which the equation obeyed by $\phi_1$ is

$$\left( \frac{d^2}{dt^2} + \frac{6}{t(t+1)} \right) \phi_1 = \sqrt{3} m \rho^2 \frac{1}{t^2(t+1)}.$$  

(3)

The operator on the left-hand side has a zero mode $\frac{6(t-1)}{(t+1)t}$ corresponding to $\frac{\partial}{\partial \rho} \phi_0$. Introducing the Spence function

$$\Phi(x) = \int_0^x \frac{du}{u} \log(1+u)$$  

(4)

a solution of (3) is

$$\phi_1 = \sqrt{3} m \rho^2 \left[ \left( 1 + t - \frac{12t}{1+t} \right) \log \frac{1+t}{t} + 6 \frac{(1-t)(1+t)^2}{(1+t)^2} \Phi \left( \frac{1}{t} \right) - \frac{12}{1+t} + 9 \right]$$  

(5)

To this solution may be added a term proportional to the zero mode $\frac{6(t-1)}{(t+1)t}$.

In the limit $t \to \infty (x \to 0)$ the solution behaves as a constant

$$\phi_1 \simeq 10 \sqrt{3} m \rho^2$$  

(6)

which may be modified by a finite amount by adding to the solution a multiple of the zero mode. In the opposite limit, $t \to 0 (x \to \infty)$, the outcome is

$$\phi_1 \simeq \sqrt{3} m \rho^2 \log \frac{1}{t} - 3 \sqrt{3} m \rho^2$$  

(7)

that is unaffected by the zero mode.

Next consider higher order mass corrections by iteration of the equation

$$\left( \partial^2 - m^2 \right) \phi = -\frac{1}{6} \phi^3$$  

(8)

in the mass parameter $m^2$. We are mainly concerned with the asymptotic behavior in the regimes $x \to 0$ and $x \to \infty$. At $x \to 0$ the mass corrections must be finite while at $x \to \infty$ the leading mass corrections should sum to

$$4\sqrt{3} m \rho^2 K_1(mx),$$  

(9)

with $K_1$ a modified Bessel function, thus ensuring exponential fall off of the instanton solution for large $x$ according to (5). The factor in front is found by comparison with the massless instanton solution (2). Exponential fall off of the subleading terms also has to be achieved somehow. This problem will be considered in Sec. [11d].

Writing the solution of (3) as a power series in $m^2$ with the term proportional to $m^{2n}$ denoted $\phi_n$, we observe from (2) the following behavior of $\phi_0$ at large $x$:

$$\phi_0 \simeq \frac{4\sqrt{3} \rho}{x^2},$$  

(10)

while $\phi_1$ in this limit is given by (6). Thus it stands to reason that the leading terms to power $m^{2n}$ are proportional to $m^{2n}(x^2)^{n-1}$ (for power-counting purposes logarithmic factors of $x$ and $m$ can be disregarded, as will be clear in the course of our argument). That this is indeed the case is proven by induction by means of (9), which is equivalent to:

$$\partial^2 \phi_n - m^2 \phi_{n-1} = -\frac{1}{6} \sum_{\nu_1 + \nu_2 + \nu_3 = n} \nu_1 \nu_2 \nu_3 \phi_{\nu_1+\nu_2+\nu_3}.$$  

(11)
we find that the term on the right-hand side involving \( \phi_n \) is negligible compared to the first term on the left-hand side, whereas the terms not involving \( \phi_n \) are dominated by the second term on the left-hand side. Thus, to leading order the right-hand side of (\ref{eq:1}) can be safely neglected. It then follows that if (\ref{eq:12}) is valid to order \( n-1 \) then it also holds to order \( n \), and hence to all orders by induction.

Eq. (\ref{eq:1}) with the nonlinear term on the right-hand side disregarded is the Klein-Gordon equation in four-dimensional Euclidean space. Consequently the solution is a linear combination of \( \frac{m}{x} I_0(mx) \) and \( \frac{m}{x} K_1(mx) \), with \( I_1 \) and \( K_1 \) modified Bessel functions of the first and the second kind, respectively (see appendix A). By comparison of (\ref{eq:A2}) and (\ref{eq:A3}), which for small values of \( m x \) is expected according to the analysis of [5] and is solved by imposing a constraint on the solution.

At \( x \to 0 \) it was found in (\ref{eq:2}) and (\ref{eq:3}) that \( \phi_0 \) and \( \phi_1 \) are regular. It follows from (\ref{eq:8}) that all higher order terms are regular in this limit order by order in \( m^2 \). To see this we rewrite the differential equation determining \( \phi_n \) in terms of the variable \( t = \frac{s^2}{x} \), obtaining an asymptotic equation of the form

\[
\left( \frac{d^2}{dt^2} + \frac{6}{t(t+1)^2} \right) H(t) = t^{-k}, \quad k \geq 3. \tag{15}
\]

The solution of this equation is

\[
H(t) = \text{const} \times \frac{t(1-t)}{(1+t)^2} + \frac{1}{k(k-1)} t^{2-k} + O(t^{1-k}) \tag{16}
\]

where the first term on the right-hand side originates from the zero-mode. This solution, and consequently \( \phi_n \), approaches a constant for \( t \to \infty \). In this way it follows by induction that the solution of (\ref{eq:8}) is finite for \( x \to 0 \) to all orders in \( m^2 \).

Assuming

\[
\phi_i \propto m^{2i}(t^2)^{i-1}, \quad i < n \tag{12}
\]

B. Constraint corrections

Eq. (\ref{eq:8}) should now be modified in such a way that the \( I_1 \)-term in (\ref{eq:4}) is eliminated while the regular behavior at \( x \to 0 \) is kept. According to the prescription of (\ref{eq:13} this should be achieved by introducing a constraint.

With a constraint

\[
\int d^4x \phi^n(x) = c \rho^{4-n} \tag{17}
\]

with \( n = 3 \) or \( n \geq 5 \) a positive integer, the first order equation corresponding to (\ref{eq:8}) is

\[
\left( \partial^2 + \frac{1}{2} \phi_0^2 \right) \delta \phi_1 = n \sigma (\phi_0)^{n-1}. \tag{18}
\]

Here \( c \) and \( \sigma \) are constants. The case \( n = 4 \) is excluded (the constraint in this case does not break scale invariance, and this is a necessary condition for a finite action instanton solution to exist).

For \( n = 3 \) the solution is trivial:

\[
\delta \phi_1 = 6 \sigma \tag{19}
\]

This solution can be used to modify (\ref{eq:8}) in such a way that the unwanted \( I_1 \)-term of (\ref{eq:4}) disappears, so in this case one indeed obtains a constrained instanton with an exponential fall off at large distances. The details are given below in Sec. IIC.

Introducing the variable \( u = \frac{x}{1+t} \) and defining

\[
F = \frac{1}{n \sigma \rho^2} \left( \frac{\rho}{4 \sqrt{3}} \right)^{n-1} \delta \phi_1 \tag{20}
\]

one converts (\ref{eq:18}) into an inhomogeneous hypergeometric equation

\[
\left( u(1-u) \frac{d^2}{du^2} - 2u \frac{d}{du} + 6 \right) F = u^{n-3}, \quad n \geq 3. \tag{21}
\]

For completeness the solution is determined also in the case \( n = 4 \):

\[
F = \frac{1}{4} u, \tag{22}
\]

and for \( n \geq 5 \) the solution is:

\[
F = \frac{u^{n-2}}{(n-2)(n-3)} u^{n-4} \sum_{i=n-2}^{\infty} \frac{(i+3)(i+2)}{i(i-1)} u^{i+1}. \tag{23}
\]

To these solutions may be added an arbitrary multiple of the zero-mode \( \frac{1}{1+t} = u(1-2u) \).

In the case \( n \geq 5 \) the terms in the infinite series tend from above to a geometric series for large \( i \), and hence the expression diverges in the limit \( u \to 1 \), i.e. \( x \to 0 \):
\[
\lim_{u \to 1} (1 - u) F(u) = \frac{n - 4}{n(n - 1)(n - 2)}. \tag{24}
\]

In consequence, we have found the following behavior of \(\delta \phi_1\) in the case \(n \geq 5\):

\[
\delta \phi_1(x) \simeq 0 \text{ for } x \to \infty, \tag{25}
\]

\[
\delta \phi_1(x) \simeq \frac{\sigma \rho^4}{4} \left( \frac{\rho}{4\sqrt{3}} \right)^{1-n} \frac{n - 4}{(n-1)(n-2)} x^2 \text{ for } x \to 0.
\]

In this case the constraint corrections lead to a singular behavior of the instanton at \(x \to 0\), invalidating Affleck’s equation (2.7) in \([5]\). It is instructive to see this in detail.

In order to compare (23) with eq. (2.7) in Affleck’s paper we use the following identity:

\[
\int d^4x \left( \frac{\partial \delta \phi_{\mu}}{\partial \rho} \partial^2 \delta \phi_1 - \delta \phi_1 \partial^2_\mu \frac{\partial \delta \phi_{\mu}}{\partial \rho} \right) = \int d^4x \partial_\mu \left( \frac{\partial \delta \phi_{\mu}}{\partial \rho} \partial^2 \delta \phi_1 \right) = \sigma \frac{\partial}{\partial \rho} \int d^4x (\phi_0)^n. \tag{26}
\]

To obtain the last version of (24) we used that \(\frac{\partial \delta \phi_{\mu}}{\partial \rho}\) is a zero mode, as well as (18).

The last version of (24) is nonvanishing. Rewriting the middle version by means of Gauss’ theorem and assuming that there is no contribution from a surface near the origin one would conclude, following Affleck, that \(\delta \phi_1\) goes as a nonvanishing constant for \(x \to \infty\).

However, it follows from (23) that a contribution actually arises from a surface near the origin. This means that \(\delta \phi_1\) need not go as a constant for \(x \to \infty\), and indeed it vanishes in this limit according to (24).

In the preceding paragraphs we only considered constraints that are monomials in the field \(\phi\). For more generalized constraints the equation (21) is replaced by

\[
(u(1-u) \frac{d^2}{du^2} - 2u \frac{d}{du} + 6) F(u) = g(u) \tag{27}
\]

where \(g(u)\) is some function. If \(g(u)\) can be expressed as a power series it follows from the analysis of the preceding section that the solution is singular for \(x \to 0\) unless \(g(u)\) only contains terms linear in \(u\) or constant.

One example is the constraint \([4]\)

\[
\int d^4x (\partial_\mu \delta \phi \partial^\mu \phi)^n = c \rho^{4-4n}
\]

which leads to the following inhomogeneous hypergeometric equation

\[
\left( u(1-u) \frac{d^2}{du^2} - 2u \frac{d}{du} + 6 \right) F(u) = -2(1+n)u^{3n-2}(1-u)^{n-1} + 6(n-1)u^{3n-3}(1-u)^n, \ n \geq 2 \tag{28}
\]

replacing (21). Here the right-hand side can be written as a power series in \(u\) with cubic and higher-order terms, and the solution is consequently singular for \(x \to 0\). This situation is similar to what was encountered for the constraint (7) with \(n \geq 5\). Thus all these constraints must be rejected.

Another type of constraint, suggested by Frishman and Yankielowicz \([4]\), corresponds to having an equation of the form

\[
\left( u(1-u) \frac{d^2}{du^2} - 2u \frac{d}{du} + 6 \right) F = \delta(u-u_0), \ n \geq 3
\]

with \(u_0\) a constant between 0 and 1 and \(\delta(u-u_0)\) the Dirac \(\delta\)-function. In this case one also expects \(F(u)\) to be singular at \(u = 1\). In order to prove this one can smear the constraint with a test function \(g(u_0)\) and obtains then the situation considered previously.

C. Construction of finite action constrained instanton

In Sec. II A it was found that the leading terms of the solution sum to the result given in (14) where the last term, having exponential growth at \(x \to \infty\), prevents the solution from being a finite action solution and should be eliminated by means of a constraint.

For a constraint represented by a term in the action of the form

\[
\sigma \left( \int d^4x \phi^3(x) - c\rho \right), \tag{30}
\]

the constraint can be used to modify (7) in such a way that the unwanted \(I_1\)-term of (14) disappears, so in this case one indeed obtains a constrained instanton with an exponential fall off at large distances. Comparing (7) to (3) we see from (13) that if we fix the Lagrange multiplier according to \(\bar{\sigma} = \bar{c}\) with

\[
\bar{\sigma} = \frac{\sqrt{3}}{6} \rho m^2 \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right) \tag{31}
\]

we have

\[
\phi_1 + \delta \phi_1 \simeq \sqrt{3} \rho m^2 \left( \log \frac{m^2 x^2}{4} + 2\gamma - 1 \right). \tag{32}
\]

This is exactly \(4\sqrt{3} \rho \frac{m}{x} K_1(mx)\) to this order.

With (3) added to the action and the Lagrange multiplier \(\sigma\) taking the value (31) one now obtains \(4\sqrt{3} \rho \frac{m}{x} K_1(mx)\) by summation of the leading terms to all orders in the mass parameter. This follows from:

- the analysis of Sec. II A is unaffected by an extra term proportional to \(m^2 \phi^2\) on the right-hand side of the field equation; the leading terms at large \(x\) still obey the Klein-Gordon equation.
The conclusion is that for a constraint \( (30) \) with \( \sigma \) given by (31) the sum of the leading terms has exponential fall off at large distances. The analysis of Sec. II A on the finiteness of the solution at small distances is easily seen to be unaffected by the constraint.

D. Leading vs. subleading terms

Eq. (14) valid for \( x \to \infty \) was found in a leading term approximation, where only leading powers in \( x \) were kept. It should be checked that these leading terms are still leading after summation. As was demonstrated in Sec. II C, the \( I_1 \)-term is removed by the constraint (30). The leading terms by themselves grow faster the higher the order, but they conspire to a sum with exponential fall off. It is thus not a priori clear that the nonleading terms are still nonleading after summation. As we shall demonstrate, the sum of the nonleading terms also have exponential fall off at large distances.

The leading terms by themselves grow faster the higher order, but they conspire to a sum with exponential fall off at large distances. The analysis of Sec. II A on the finiteness of the solution at small distances is easily seen to be unaffected by the constraint. The conclusion is that for a constraint (30) with \( \sigma \) given by (31) the sum of the leading terms has exponential fall off at large distances. The analysis of Sec. II A on the finiteness of the solution at small distances is easily seen to be unaffected by the constraint.

III. YANG-MILLS-HIGGS INSTANTON

An analysis of the SU(2) Yang-Mills-Higgs theory similar to that of the \( \phi^4 \) theory is carried out in this section. The Euclidean Lagrangian is

\[
L = -\frac{1}{g^2} \left[ \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \kappa \left( (D_\mu \Phi)^\dagger D_\mu \Phi + \frac{1}{4} (\Phi^\dagger \Phi - \mu^2)^2 \right) \right]
\]

(36)

where

\[
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c,
\]

(37)

\[
D_\mu = \partial_\mu - i \tau^a A_\mu^a
\]

(38)

with \( \tau^a \) the Pauli matrices, and \( \kappa > 0 \). The Yang-Mills field acquires by the Higgs mechanism the mass

\[
m_{vec} = \sqrt{\frac{\kappa}{2\mu}}.
\]

(39)

For \( \mu = 0 \) the Yang-Mills field equation has the instanton solution in the singular gauge

\[
A_{0\mu}^a = \frac{2\rho^2 \eta_{\mu\nu}^a x_\nu}{x^2 (x^2 + \rho^2)}
\]

(40)

with \( \eta_{\mu\nu}^a \) defined in [2]. With the fields in the form

\[
\sigma \left( \int d^4 x \phi_0^2(x) \phi(x) - c\rho \right)
\]

(35)

clearly allows the same conclusion, but only to leading order. The extra term in the field equation is now a source term instead of an expression quadratic in the field. In the equation corresponding to (34) this means that the constraint induced term contains massless propagators. This in turn means that \( \phi^{(3)} \) has a fall off at infinity according to a power law.

It is clearly desirable that the subleading terms have exponential fall off like the leading terms. This can be obtained also with a constraint leading to a source term in the field equation. The reason is that the constant \( \sigma \) in front of the source term is of second order in the mass. It is thus possible to use a constraint \( \sigma \int d^4 x \hat{\phi}^2(x) \Phi(x) \), with

\[
\hat{\phi} = \phi_0 + \cdots
\]

and where the higher-order terms are adjusted order by order such that \( \hat{\phi} = \phi \) with \( \Phi \) the solution obtained by means of the constraint (30). In this way a source term constraint gives the same constrained instanton as a term corresponding to an extra term in the Lagrangian that is cubic in the scalar field.

Constraints corresponding to a source term in the field equation were originally suggested by Wang [3].
\[ A_\mu^a = -\partial_\mu \phi \log \alpha, \quad \phi = \begin{pmatrix} 0 \\ f \end{pmatrix} \]  
\[ \alpha \partial_\nu \partial^2 \alpha - 3 \partial_\nu \alpha \partial^2 \alpha = \frac{\kappa}{2} f^2 \alpha \partial_\nu \alpha, \]  
(42)

\[ \alpha^2 \partial^2 f - \frac{3}{4} (\partial_\nu \alpha)^2 f + \frac{1}{2} \alpha^2 f (\mu^2 - f^2) = 0. \]  
(43)

These equations are solved order by order in \( \mu \) in such a way that the solution leads to a finite action. It is assumed that terms in \( \alpha \) are of even order, while terms in \( f \) are of odd order in \( \mu \). The exponential decay at infinity can be ensured by tuning of the integration constants in the first few orders, while integrability of the Lagrangian density at the origin requires modification of the above equations corresponding to a constraint.

**A. Conditions for finiteness of the action**

With the fields only depending on the norm \( x = |x| \) and in terms of the parameter \( t = \frac{x}{2} \), the Lagrangian \( L \) is

\[ L = -\frac{1}{g^2} \left[ 3 \left( a^2 + \left( \frac{b \alpha^2}{t} - a \right)^2 \right) + \kappa \frac{4t^3}{\rho^2} \left( \frac{df}{dt} \right)^2 \right. \]

\[ + \kappa \frac{3t^3}{\rho^2} f^2 \left( \frac{d\alpha}{\alpha dt} \right)^2 + \kappa \left( \frac{f^2 - \mu^2}{2} \right) \]  
(44)

where

\[ a = \frac{4}{\rho^2} \left( t^3 \left( \frac{d \log \alpha}{dt} \right)^2 - t^2 \frac{d \log \alpha}{dt} \right), \]  
(45)

\[ b = \frac{2t \alpha}{\rho^2} - 4t^4 \frac{d \alpha}{\alpha dt}. \]  
(46)

The Lagrangian is negative semi-definite since the square bracket in (44) can be expressed as a sum of squares. Thus the condition for a finite action is that each term in the sum gives a finite contribution separately.

\( f \) and \( \log \alpha \) should at large \( x \)-values decrease exponentially. Writing

\[ \alpha = \alpha_0 + \alpha_2 + \alpha_4 + \cdots, \]  
(47)

and similarly

\[ f = f_1 + f_3 + f_5 + \cdots \]  
(48)

with the indices enumerating the power of the mass parameter, and

\[ \alpha_0 = 1 + t \]  
(49)

one obtains the following estimate at small \( t \)-values:

\[ \log \alpha \simeq (t + \alpha_2 + \alpha_4 + \cdots) - \frac{1}{2} (t + \alpha_2 + \alpha_4 + \cdots)^2 + \cdots. \]  
(50)

To ensure the exponential decay at small \( t \)-values one should express the first bracket in this expression by the modified Bessel function \( K_1 \). To make \( \log \alpha \) behave order by order as a power series expansion of \( \frac{m_{vec}}{2} K_1(m_{vec}) \) with some mass parameter we have to require absence of inverse powers of \( t \) in \( \alpha_2 \), while \( \alpha_4 \) only is allowed to have a \( 1/t \)-term etc. Similar considerations apply to \( f \). Thus the leading terms should in the limit \( t \to 0 \) conspire to modified Bessel functions according to

\[ \alpha = \alpha_0 + \alpha_2 + \alpha_4 + \cdots \simeq 1 + \frac{m_{vec} \rho^2}{x} K_1(m_{vec}). \]  
(51)

For \( f \) the appropriate condition turns out to be

\[ f = f_1 + f_3 + f_5 + \cdots \simeq \mu - \frac{\rho^2 \mu^2}{2x} K_1(\mu x), \]  
(52)

as demonstrated explicitly in Sec. 3.C.

To investigate the integrability at the origin of the Lagrangian density (44) one first considers:

\[ a^2 = \alpha_0^2 + 2\alpha_0 \alpha_2 + \alpha_2^2 + 2\alpha_0 \alpha_4 + \cdots \]  
(53)

and the same for \( (\frac{b \alpha^2}{t} - a)^2 \). From (49) follows that in zeroth order:

\[ a_0 = b_0 x^2 - a_0 = -\frac{4}{\rho^2} \left( \frac{t^2}{(1 + t)^2} \right) \]  
(54)

so \( a_0^2 \) and \( (\frac{b \alpha^2}{t} - a_0)^2 \) are bounded for \( t \to \infty \). Next

\[ a_2 = \frac{4}{\rho^2} \left( \frac{2t}{1 + t} - 1 \right) \frac{d \alpha}{d t} \frac{\alpha_2}{1 + t} \]  
(55)

from which it is deduced that to keep \( \alpha_0 \alpha_4 \) integrable at the origin one can at most allow \( \alpha_2 \) to go as \( t \log t \) at large \( t \). If this condition is fulfilled then the second square of (44) is integrable at the origin to second order as well.

In fourth order both squares contain two terms. If there is a term \( t \log t \) present in \( \alpha_2 \) then \( \alpha_2 \) is not integrable at the origin; it contains a term quadratic in \( t \). This must be cancelled by the other fourth order term in \( \alpha_2 \). To determine \( \alpha_0 \alpha_4 \) one finds from (45):

\[ a_4 = \frac{4}{\rho^2} \left( t^3 \left( \frac{d}{dt} \frac{\alpha_2}{1 + t} \right)^2 \right. \]

\[ \left. + \left( \frac{2t}{1 + t} - 1 \right) t^2 \left( \frac{d}{dt} \frac{\alpha_4}{1 + t} - \frac{\alpha_2}{1 + t} \frac{d \alpha_2}{dt} \right) \right) \]  
(56)
To ensure a finite integral of $a^2$ to fourth order there must be present in $\alpha$, a term quadratic in $t$. Now examine the second square of (44) in a similar way. It is found that
\[
\left( \frac{b_0 \rho^2}{t} - a_2 \right)^2
\]
is integrable at the origin, whereas
\[
\left( \frac{b_0 \rho^2}{t} - a_0 \right) \left( \frac{b_1 \rho^2}{t} - a_1 \right)
\]is finite upon integration only if $\alpha_4$ diverges no faster than $t \log t$. Thus we conclude that to have a finite action it is necessary to demand that $\alpha_2$ diverges at most logarithmically for large values of the variable $t$. Continuing this analysis to higher orders reveals that $\alpha_{2n}$ can at most diverge logarithmically at large $t$ for all $n > 0$.

From the third square in (44) it follows immediately that $f$ can also at most grow as $\log t$ for $t \to \infty$. This also ensures the integrability at the origin of the final two terms in (44), and thus of the entire Lagrangian.

### B. Iteration

In this section the field equations are solved order by order in the mass parameter $\mu$. In each order it is examined whether this solution has exponential fall off at infinity, as well as integrability of the Lagrangian density at the origin, following the considerations of the previous section. We start by rewriting (42) and (43) in terms of the parameter $t = \frac{x^2}{2}$.

\[
\frac{d}{dt} \left( \alpha^{-3} t^3 \frac{d^2 \alpha}{dt^2} \right) = \frac{\kappa \rho^2}{8} f^2 \alpha^{-3} \frac{d \alpha}{dt},
\] (58)

\[
\alpha^2 \frac{d^2 f}{dt^2} - \frac{3}{4} \left( \frac{d \alpha}{dt} \right)^2 f = \frac{\alpha^2 f \rho^2}{8 t^3} (f^2 - \mu^2).
\] (59)

The equations can then in each step be solved by quadrature. The integration constants arising this way are denoted $c_{i,j}$ where the first subscript indicates the order and the second subscript is an extra label.

#### 1. Orders zero, one, two and three

To order zero the solution of (58) is (49), corresponding to the massless instanton in the singular gauge, while to first order one obtains the following solution of (59)
\[
f_1 = \frac{c_{1;1}}{1 + t} + c_{1;2}(1 + t)^2.
\] (60)

It is necessary to choose $c_{1;2} = 0$ in order to keep $f_1$ bounded for $x \to 0$. A similar term will arise in each order and must always be chosen equal to zero. Eq. (60) reduces for $c_{1;1} = \mu$ to the isospin $\frac{1}{2}$ zero mode $\overline{2}$. It will be shown that this value of $c_{1;1}$ is enforced by the boundary conditions.

In second order the solution of (58) is
\[
\alpha_2 = \frac{1}{2} \left( c_{2;1} - \frac{1}{3} \kappa \rho^2 c_{1;1} \right) \frac{1}{t} - 3 c_{2;1} \ln t + 3 c_{2;1} \ln t + t + c_{2;3} + (c_{2;2} - 3 c_{2;1}) t + c_{2;3}.
\] (61)

According to the discussion after (57) the first term must vanish, i.e.
\[
c_{2;1} = \frac{1}{3} \kappa \rho^2 c_{1;1}.
\] (62)

The integration constant $c_{2;2}$ is taken equal $3 c_{2;1}$ (a different choice of $c_{2;2}$ corresponds to a different scale of $\rho$). The terms $\frac{1}{2} c_{2;1} t^2$ and $3 c_{2;1} \ln t$ have to be eliminated for a finite action solution according to the discussion after (57). This can be accomplished by modifying the equation determining $\alpha_2$ according to:
\[
\frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha}{dt^2} \right) + \frac{\kappa \rho^2 c_{1;1}^2}{4} \frac{t}{(1 + t)^4} = \frac{\kappa \rho^2 c_{1;1}^2}{8(1 + t)^4}
\] (63)
resulting in
\[
\alpha_2 = - \frac{\kappa \rho^2}{8} c_{1;1} \ln t + c_{2;3}.
\] (64)

At small $t$-values $\log \alpha$ should vanish like $m K_1(mx)$ with $K_1$ a modified Bessel function; see (44). This is matched for
\[
c_{2;3} = \frac{\kappa \rho^2 c_{1;1}^2}{8} (\log \frac{\kappa \rho^2 c_{1;1}^2}{8} + 2 \gamma - 1)
\] (65)
that leads to the following form of $\alpha_2$:
\[
\alpha_2 = \frac{\kappa \rho^2 c_{1;1}^2}{8} (\log \frac{\kappa \rho^2 c_{1;1}^2}{8} + 2 \gamma - 1).
\] (66)

The mass parameter $m$ in the modified Bessel function is for $c_{1;1} = \mu$ actually equal to $m_{\text{vec}}$, the vector mass generated by the Higgs field (58).

The modification of (57) as displayed in (49) is an indication of the necessity of a constraint in the sense of [3]. However, it should be emphasized that the modification is unique to second order in the mass parameter $\mu$. Any other modification will either cause $\alpha_2$ to behave differently from the modified Bessel function $K_1$ at infinity, since it will modify the coefficient of the log $t$-term or introduce more singular terms for $t \approx 0$, or it will give rise to unintegrable singularities of the action density at the origin. This point is further elaborated upon in Sec. 2 III D.

In third order the solution of (59) is
\[
f_3 = - \frac{\kappa \rho^2 c_{3;1}^3}{16(1 + t)^2} \left( \log \frac{\kappa \rho^2 c_{1;1}^2}{8t} + 2 \gamma - 1 \right)
\]
\[
+ \frac{\kappa \rho^2 c_{1;1}^2}{8 \sqrt{1 + t}} (\mu^2 - \frac{\kappa \rho^2 c_{1;1}^2}{8 t}) \left( \frac{3}{2} t - (1 + t)^2 \log(1 + \frac{1}{t}) \right).
\]
\[ + \frac{\rho^2 c_{11}}{16\sqrt{1+t}} \left( c_{11} - \mu^2 \right) \left( \frac{(1+t)^2}{t} + 2(1+t) + \frac{1}{2} \right) \]
\[ - 3(1+t)^2 \log \left( 1 + \frac{1}{t} \right) - \frac{c_{31}}{2\sqrt{1+t}}. \]  
\tag{67}
\]

\( f \) should at \( x \to \infty \) behave as a modified Bessel function \( K_1 \):

\[ f(x) - \mu \propto \frac{m}{x} K_1(mx) \]
\tag{68}

with \( m \) some mass parameter. The small-\( m \)-expansion of the Bessel function \( \left[ \right] \) is used for the determination of \( c_{31} \) and \( c_{11} \). The mass parameter is also fixed this way. Making a small-\( t \) expansion of \( f_1 \) one finds

\[ f_1 = c_{11} - \frac{c_{11}\rho^2}{2\pi^2} + \cdots. \]
\tag{69}

Thus, \( f_3 \) must for \( t \to 0 \) reduce to

\[ f_3 \simeq -\frac{c_{11}\rho^2 m^2}{4} \left( \log \frac{m^2 \rho^2}{4t} + 2\gamma - 1 \right). \]
\tag{70}

From this expression it is first observed that no term proportional to \( \frac{1}{t} \) occurs. Comparing with \( \left( \right) \) one immediately concludes

\[ c_{11} = \mu. \]
\tag{71}

\( \left( \right) \) then reduces in the small-\( t \) limit to

\[ f_3 \simeq -\frac{\kappa \rho^2 \mu^3}{16} \left( \log \frac{\kappa \rho^2 \mu^2}{8} + 2\gamma - 1 \right) \]
\[ + \frac{3\rho^2 \mu^3}{16} \left( 1 - \frac{\kappa}{2} \right) + \frac{\rho^2 \mu^3}{2} \log t - \frac{c_{31}}{2}. \]
\tag{72}

From the term involving \( \log t \) one sees that the mass parameter \( m \) must be identified with the Higgs mass \( \mu \). Notice that this identification is enforced by the boundary condition. Also the following value of the integration constant \( c_{31} \) is found:

\[ c_{31} = \frac{\rho^2 \mu^3}{4} \left( 1 - \frac{\kappa}{2} \right) \left( \log \frac{\rho^2 \mu^2}{4} + 2\gamma + \frac{1}{2} \right) \]
\[ - \frac{\kappa \rho^2 \mu^3}{8} \log \frac{\kappa}{2}. \]
\tag{73}

Hence \( f_3 \) is completely determined.

2. Order four

\( \alpha_4 \) is according to \( \left[ \right] \) a solution of:

\[ \frac{d}{dt} \left( \frac{t}{1+t} \right)^3 \frac{d^2 \alpha_4}{dt^2} - \frac{3\alpha_2 \rho^2}{(1+t)^4} \frac{d^2 \alpha_2}{dt^2} = \frac{\kappa \rho^2 \mu^2}{8} \left( \frac{d\alpha_2}{dt} \frac{1}{(1+t)^4} - \frac{3\alpha_2}{(1+t)^5} + \frac{2f_3}{\mu(1+t)^2} \right) \]
\tag{74}

whence by insertion of \( \alpha_2 \) and \( f_3 \) and performing the first integration:

\[ \frac{d^2 \alpha_4}{dt^2} = \frac{\kappa^2 \rho^4 \mu^4}{64} \left( \log \frac{\kappa \rho^2 \mu^2}{8t} + 2\gamma - 1 \right) \left( \frac{1}{t^3} + \frac{2}{t^2(1+t)} \right) \]
\[ - \frac{\kappa^2 \rho^4 \mu^4}{32} \left( 1 - \frac{\kappa}{2} \right) \left( \frac{1}{t^2} \right)^2 \log \frac{1+t}{t} - \frac{5}{6t} \frac{1}{2} \left( \frac{1}{t^6} - \frac{3}{2} \frac{1}{t^2} \right) \]
\[ + \frac{\kappa^4 \rho^4 \mu^4}{96t^3} \left( \frac{\kappa}{2} - 1 \right) \left( \log \frac{\rho^2 \mu^2}{4} + 2\gamma - 1 \right) + \frac{\kappa}{2} \log \frac{\kappa}{2} \]
\[ + c_{41} \left( \frac{1+t}{t} \right)^3. \]
\tag{75}

Letting \( t \to \infty \) and disregarding all terms which vanish as \( t^{-2} \) or faster results in

\[ \frac{d^2 \alpha_4}{dt^2} \simeq \frac{3}{2} \alpha_4 \left( \frac{3}{t} + 1 \right) \]
\tag{76}

that gives rise to the following terms in \( \alpha_4 \):

\[ \frac{1}{2} c_{41} t^2 + 3c_{41} \alpha t \log t \]
\tag{77}

that grow too fast for \( t \to \infty \) to allow a finite action solution. However, we cannot take the integration constant \( c_{41} \) equal to zero; indeed we find below that it has to have a nonzero value in order that \( \log \alpha \) behaves as a Bessel function for \( t \to 0 \).

A similar problem was encountered for \( \alpha_2 \) where it led to the modified differential equation \( \left[ \right] \). Modifying the differential equation \( \left[ \right] \) in the same way means removing from its right-hand side the terms on the right-hand side of \( \left[ \right] \). Integrating twice the resulting equation one finds the solution

\[ \alpha_4 = \frac{\kappa^2 \rho^4 \mu^4}{64} \left( \log \frac{\kappa \rho^2 \mu^2}{8t} + 2\gamma - 1 \right) \times \]
\[ \times \left( \frac{1}{2t} + 2(1+t) \log \frac{1+t}{t} \right) \]
\[ - \frac{\kappa^2 \rho^4 \mu^4}{64} \left[ \frac{3}{4t} - 2 \log t + 2(1+t) \Phi \left( \frac{1}{t} \right) \right] \]
\[ + \frac{\kappa^2 \rho^4 \mu^4}{32} \left( 1 - \frac{\kappa}{2} \right) \left[ \frac{1}{2} (1+t) (5-t) \log \frac{1+t}{t} \right] \]
\[ - (1-2t) \Phi \left( \frac{1}{t} \right) + \frac{t^2}{2} + \frac{5}{12t} \]
\[ - \frac{\kappa^2 \rho^4 \mu^4}{192t} \left( \frac{\kappa}{2} - 1 \right) \left( \log \frac{\rho^2 \mu^2}{4} + 2\gamma - 1 \right) + \frac{\kappa}{2} \log \frac{\kappa}{2} \]
\[ + c_{41} \left( \frac{1}{2t} - 3 \log t \right) + c_{42} t \alpha_4 \]
\tag{78}

with \( \Phi(x) \) the Spence function defined in \( \left[ \right] \).

The asymptotic behavior at \( t \to 0 \) of \( \left[ \right] \) must be equal to terms of order \( m_{40}^4 \) in \( \rho^2 \frac{m_{40}^4}{x} K_1(m_{40}^4 x) \):

\[ \frac{\kappa^2 \rho^4 \mu^4}{128} \left( \log \frac{\kappa \rho^2 \mu^2}{8t} + 2\gamma + \frac{5}{2} \right) \frac{1}{t}. \]
\tag{79}
While the $\log t$ terms match immediately, the terms of form $\frac{c_{\alpha,1}}{t}$ have to be adjusted by means of the integration constant $c_{4;1}$ according to

\[
c_{4;1} = -\frac{\kappa^2 \rho^4 \mu^4}{96} (1 - \frac{\kappa}{2}) \left( \log \frac{\rho^2 \mu^2}{4} + 2\gamma + \frac{3}{2} \right)
+ \frac{\kappa^2 \rho^4 \mu^4}{192} \log \frac{\kappa}{2}.
\] (80)

For large $t$ the asymptotic form of $\alpha_4$ is according to (83):

\[
\alpha_4 \simeq c_{4;2} t.
\] (81)

Here one should take

\[
c_{4;2} = 0
\] (82)
in order to ensure acceptable behavior of $\alpha_4$ at $t \to \infty$. The constant $c_{4;3}$ is arbitrary.

To summarize, we have found to order $\mu^4$ that a finite action solution exists if (68) is modified to:

\[
\frac{d}{dt} \left( \alpha_3 t^3 \frac{d^2 \alpha}{dt^2} \right) - \frac{3\bar{\sigma} t}{2(1 + t)^2} = \frac{\kappa \rho^2}{8} f^2 \alpha^{-3} \frac{d\alpha}{dt}
\] (83)

with

\[
\bar{\sigma} = -\frac{\kappa^2 \rho^4 \mu^4}{6} + \frac{\kappa^2 \rho^4 \mu^4}{24} (1 - \frac{\kappa}{2}) \left( \log \frac{\rho^2 \mu^2}{4} + 2\gamma + \frac{3}{2} \right)
- \frac{\kappa^2 \rho^4 \mu^4}{48} \log \frac{\kappa}{2} + O(\mu^6),
\] (84)

and here Bessel function behavior of the solution at large distances has been obtained by a suitable choice of the integration constants.

C. Limit considerations

Next (58) and (59) are examined in the limits where $t \to 0$ and $t \to \infty$ in order to reach some conclusion which are valid to all orders in the mass, in the same way as in Sec. IIIA. In these limits the equations simplify sufficiently to allow a leading term analysis.

1. $t \to 0$

Here it is checked by induction that the leading terms of $\alpha$ and $f$ in powers of $t$ conspire to give the modified Bessel function $K_1$ according to (1) and (53). More specifically, it will be checked that these expressions agree with the leading terms in the field equations. $\alpha_0$ is given in (49) while $f_1$ at large distances behaves according to (80). The induction hypothesis is

\[
\alpha_n \propto t^{1-\frac{\kappa}{2}} , \quad f_n \propto t^{\frac{3}{2}-\frac{\kappa}{2}} , \quad n > 1.
\] (85)

This hypothesis is correct for $n = 2, 3, 4$ according to (60) (with $c_{1;1} = \mu$), (67) and (78). We want to prove it by induction for $n > 4$ and to show that the leading terms sum to (51) and (52).

For $n \geq 2$ one of the leading terms includes a logarithmic factor, but this makes no difference in what follows since it does not affect the estimate of the power behavior after differentiation.

Keeping only the leading terms for $t \to 0$ one gets from (86) to order $n$ in the mass parameter, with the induction hypothesis used in orders lower than $n$:

\[
\frac{d}{dt} \frac{d^2 \alpha_n}{dt^2} \simeq \frac{\kappa \rho^2 \mu^2}{8} \frac{d\alpha_{n-2}}{dt}
\] (86)
correct to order $t^{1-\frac{\kappa}{2}}$. This relation proves the estimate for $\alpha_n$ in (85). After summation over $n$ this corresponds exactly to $\left( \partial^2 - \frac{\kappa^2 \rho^2}{4} \right) \partial_n \alpha = 0$ which is solved by (51).

As seen in IIIA this does not guarantee exponential fall off; also exponential rise is possible, unless a particular solution is picked in low orders. However, the requirements necessary for obtaining the desired asymptotic behavior of the full solution have been met in the present case by the choice of integration constants in Sec. IIIB and by modifying the second-order equation according to (63).

For $n = 2$ and $n = 4$ the leading terms are of order $\rho^0$ and $t^{-1}$, respectively. In each of these cases (31) only contains a nonlogarithmic term, and the equation only restricts the logarithmic parts of the leading terms; the nonlogarithmic parts have to be fixed by adjustment of integration constants, as we saw in Sec. IIIA. However, for $n > 4$ the situation is different. Here the left-hand side of (86) is of order $t^{1-\frac{\kappa}{2}}$ and contains in each case both a logarithmic and a nonlogarithmic term, and both the logarithmic and the nonlogarithmic part of $\alpha_n$ is determined by the equation. Consequently, no further adjustment is necessary to produce (31) as the only solution of (86) after summation over $n$, and no restrictions on the integration constants $c_{n;1}$ occur at these orders.

As an example we will use (86) to determine the leading terms of $\alpha_6$. Keeping only the leading terms in $\alpha_4$ (which are of order $t^{-1}$) as given in (78) we find the equation

\[
\frac{d^2 \alpha_6}{dt^2} \simeq \frac{\kappa^3 \rho^6 \mu^6}{1024 t^4} \left( \log \frac{\kappa \rho^2 \mu^2}{8t} + 2\gamma - \frac{5}{2} \right).
\] (87)

This is integrated to give the result

\[
\alpha_6 \simeq \frac{\kappa^3 \rho^6 \mu^6}{6144 t^2} \left( \log \frac{\kappa \rho^2 \mu^2}{8t} + 2\gamma - \frac{10}{3} \right).
\] (88)

Comparing this to (63) we find that $\alpha_6$ is indeed the sixth order term in $\rho^2 \frac{\partial^2 \alpha}{\partial x^2} + K_1(m_{res}x)$. Note that this result is obtained without tuning the integration constants.

Next the same analysis is applied to (54). The equation obeyed by the leading terms is to order $n$:

\[
\frac{d^2 f_n}{dt^2} \simeq \frac{\kappa \rho^2 \mu^2}{4t^3} f_{n-2}
\] (89)
correct to order $t^{-\frac{4}{3} - \frac{2}{3}}$, which matches (52) with (59), and the choice of integration constants in low orders fixes the asymptotic behavior according to (52).

2. $t \to \infty$

In this limit it is checked that $\alpha_n$, $n \neq 0$, and $\sqrt{t} f_n$ diverge at most logarithmically. This has already been proven in Sec. III B in the cases $n = 2, 3, 4$. For $n > 4$ the statement is demonstrated by induction.

For $n > 4$ the induction hypothesis in combination with (88) leads to

$$d \left( \left( \frac{t}{1 + t} \right)^3 \left( \frac{d^2 \alpha_n}{dt^2} - \frac{3\alpha_2}{1 + t} \frac{d^2 \alpha_{n-2}}{dt^2} - \frac{3\alpha_4}{1 + t} \frac{d^2 \alpha_{n-4}}{dt^2} + \frac{6\alpha_2^2}{(1 + t)^2} \frac{d^2 \alpha_{n-4}}{dt^2} + \cdots \right) \right) = O(t^{-4}).$$

The induction hypothesis implies that all terms on the left-hand side excluding the first one are at most $O(t^{-4})$. Thus, rewriting this equation after the first integration:

$$\frac{d^2 \alpha_n}{dt^2} \simeq O(t^{-3}) + c_{n;1} \left( \frac{1 + t}{t} \right)^3$$

one sees that the terms $c_{n;1} \left( \frac{1 + t}{t} \right)^3$ give rise to unwanted terms in $\alpha_n$ of the form $c_{n;1}(3\alpha t \log t + \frac{1}{3}t^2)$. Similar unwanted terms were discarded from $\alpha_2$ and $\alpha_4$, leading to the modified equation (88) instead of (58) for the determination of $\alpha$, with the constant $\sigma$ given by (81). This procedure can also be applied here, with the same conclusion. The argument shows that the $O(\mu^0)$ terms of $\sigma$ are arbitrary since they are given by the arbitrary constants $c_{n;1}$.

Similarly, upon examining equation (59) at order $n$ one obtains from the induction hypothesis that keeping only the leading terms we can disregard all terms of order $t^{-\frac{2}{3}}$ and thus we are left with

$$(1 + t)^2 \frac{d^2 f_n}{dt^2} - \frac{3}{4} f_n = O(t^{-\frac{2}{3}})$$

whence

$$\frac{f_n}{(1 + t)^{\frac{2}{3}}} = O(t^{-2}).$$

Thus, $\sqrt{t} f_n$ can diverge at most logarithmically, and the proof by induction is completed.

D. Modified equations

In Secs. III B and III C it was found that in order to obtain a finite action solution of eqs. (58) and (59) that reduces to the usual instanton in the massless limit, one must modify the field equation (58) to (83), while (59) requires no modification. In the light of this, it should be investigated which types of constraints can lead to this modification.

Several gauge-invariant constraints have been considered in the literature. Two of these are examined and shown to lead to an infinite action. The Yang-Mills field equation with the general expression for the constraint modification $\sigma \int dt^4 x O$ included in the action is:

$$-\gamma_n^2 \alpha^2 \partial_\nu (\alpha^{-3} \partial^2 \alpha) + \sigma \delta \frac{\delta A_\mu}{\delta A_\mu} \int dt^4 x O$$

$$= -\frac{c}{2} f^2 \theta_{\mu \nu} \partial_\nu \log \alpha$$

(94)

with $\sigma$ a suitable constant.

First the constraint used by Klinkhamer [8] is considered:

$$O_K = \left( \frac{1}{8} \epsilon_{\mu \nu \kappa \tau} G^{\mu \nu} G^{\kappa \tau} \right)^2$$

(95)

To zeroth order in the mass one finds:

$$\delta \frac{\delta A_\mu}{\delta A_\mu} \int dt^4 x O_K = -\frac{3072}{\rho^8} \theta_{\mu \nu} x_\nu \left( \frac{t}{1 + t} \right)^7$$

(96)

Inserting this into the field equation (94) gives the equation for $\alpha_2$ where we have substituted $\alpha^2 \to \alpha_0^2$ in the constraint term:

$$\frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_2}{dt^2} + \frac{\delta K}{(1 + t)^9} \right) = \frac{\kappa \rho^2 \mu^2}{8(1 + t)^4}$$

(97)

with $\delta K$ a constant of order $\mu^2$. It is impossible, with this equation determining $\alpha_2$, to eliminate both the term $\frac{1}{2} c_{2;2} t^2$ and $3c_{2;1} t \log t$ in (61), which is necessary for a finite action solution, and still have Bessel function behavior at infinity, so this constraint must be rejected.

Similarly, the constraint used by Aoyama et al. [10] is considered:

$$O_A = \epsilon^{abc} G^{\mu \nu} G_{\mu \nu} G_{\rho \tau}$$

(98)

whence:

$$\frac{\delta}{\delta A_\mu} \int dt^4 x O_A = \frac{768}{\rho^8} \theta_{\mu \nu} x_\nu \left( \frac{t}{1 + t} \right)^5$$

(99)

leading to an equation

$$\frac{d}{dt} \left( \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_2}{dt^2} + \frac{\delta A_\mu}{\delta A_\mu} \right) = \frac{\kappa \rho^2 \mu^2}{8(1 + t)^4}$$

(100)

that again leads to an infinite action.

The question is now whether it is possible at all to obtain the modified equations from an action principle with a Lagrangian density expressed only in terms of the field variables $A^n_\mu$ and $\phi$. The immediate difficulty here is that the equation is formulated in terms of the variable
\( \alpha \) and not the gauge potential \( A_\mu^a \). In order to handle this we decompose \( A_\mu^a \) as follows:

\[
A_\mu^a = \bar{\eta}_\mu^a (\alpha_\nu - \partial_\nu \log \alpha) + A_{\alpha \mu}^a
\]

(101)

with

\[
\partial_\nu \alpha_\nu = 0, A_{ai} = A_{ia}, \delta_{ai} A_{ai} = 0, i = 1, 2, 3.
\]

(102)

The ansatz (11) is regained if one sets \( \alpha_\nu = 0 \) and \( A_{\alpha \mu}^a = 0 \).

The Yang-Mills field equation can be obtained by insertion of (101) into the Lagrangian (86), if one takes variation of (86) with respect to \( \log \alpha \) (or \( \alpha \)), \( \alpha_\nu \), and \( A_{\alpha \mu}^a \), and next sets \( \alpha_\mu = 0 \) and \( A_{\alpha \mu}^a = 0 \). Since the constrained instanton is expressed in terms of \( \alpha \), the field components \( \alpha_\nu \) and \( A_{\alpha \mu}^a \) act as Lagrange multiplier fields as far as the formulation of the constraint goes. Thus the difficult part of the construction of a good constraint concerns the part of the constraint involving only \( \alpha \).

With

\[
S_{YM} = \int L_{YM} d^4 x, L_{YM} = -\frac{1}{g^2} \frac{1}{2} \alpha^a G_{\mu \nu}^a G^{\nu \rho}
\]

one obtains

\[
\delta S_{YM} \over \delta \alpha(x) = -\frac{1}{g^2} \frac{3}{\alpha^3} \partial_\mu \left( \alpha^2 \partial_\mu \left( \frac{\partial^2 \alpha(x)}{\alpha^3} \right) \right).
\]

(103)

This should be compared to the modified equation (83) in terms of the variable \( x \):

\[
\partial_\mu \left( \alpha^{-3} \partial^2 \alpha - \frac{\alpha}{\rho^2} \left( \frac{\rho^2}{x^2 + \rho^2} \right)^3 \left( 1 + \frac{3x^2}{\rho^2} \right) \right)
= \frac{\kappa}{2} f^2 \alpha^{-3} \partial_\mu \alpha.
\]

(104)

Thus we learn that we want to add a term \( S_{const} \) to the action (the subscript "const" indicates constraint) such that

\[
\frac{\delta S_{const}}{\delta \alpha^a} = \frac{1}{\alpha} \partial_\mu \left( \alpha^2 \partial_\mu F(\alpha) \right)
\]

(105)

where to second order in the mass variable

\[
F(\alpha) \simeq \frac{3 \bar{\sigma}_2}{g^2 \rho^2} \left( \frac{\rho^2}{x^2 + \rho^2} \right)^3 \left( 1 + \frac{3x^2}{\rho^2} \right).
\]

(106)

with \( \bar{\sigma}_2 \) the part of \( \bar{\sigma} \) as given in (84) that is of second order in \( \mu^2 \).

Eq. (105) contains two partial derivatives. Consequently \( S_{const} \) can contain the term \( \sigma \) such that

\[
S_\alpha = \int g(\alpha) \partial_\mu \alpha \partial_\mu \alpha dt^4 x
\]

(107)

that is compared with (102) in lowest order in the mass variable, where the procedure leading to (87) and (104) is repeated and where \( \sigma \) is fixed at the value \( \bar{\sigma}_2 \). The two formulas are equivalent for

\[
g(\alpha) = -\frac{18}{g^2 \rho^2} \left( -\frac{2}{3\alpha^3} + \frac{1}{2\alpha^2} \right).
\]

(108)

The fields \( \alpha_\nu \) and \( A_{\alpha \mu}^a \) only have to enter the constraint linearly. This is accomplished if one adds to the action a term \( \sigma S_b \) with

\[
S_b = -\int d^4 x (\bar{\eta}_\mu^a \alpha_\nu + A_{\alpha \mu}^a) \bar{\eta}_\mu^a \alpha_3 \partial_\lambda \left( 1 - \frac{1}{\alpha} \right)^3 \times (1 + \frac{3}{\alpha - 1}).
\]

(109)

The total constraint action is according to this prescription

\[
S_{const} = \sigma (S_\alpha + S_b - c)
\]

(110)

where the Lagrange multiplier \( \sigma \) should be fixed to \( \bar{\sigma} \).

The constraint (110) does not have a very convenient form since it is desirable that the twelve components of the gauge field enter the constraint on the same footing. We have not succeeded in constructing a good constraint with this property. Obvious candidates like (87) and (104) fail to produce a finite action.

E. A constraint that almost works

Instead of a constraint involving the field variables alone it is also possible, as pointed out by Wang [9], to use a source-type constraint, where the whole quantum field enters the constraint linearly. In order to obtain a pure source term in the modified Yang-Mills field equation, one should adjust the extra term in (83), converting it into:

\[
\alpha^2 \frac{d}{dt} \left( \alpha^{-3} t^3 \frac{d^2 \alpha}{dt^2} \right) - \frac{3 \bar{\sigma} t}{2(1 + t)^2} = \frac{\kappa \rho^2}{8} f^2 \alpha^{-1} \frac{d \alpha}{dt}.
\]

(111)

This is accomplished if one adds to the action a term

\[
S_{const} = \sigma (\Sigma_{prov}[A] - c)
\]

(112)

where \( \Sigma_{prov}[A] \) is defined by

\[
\Sigma_{prov}[A] = -12 \int d^4 x A_\mu^a \bar{\eta}_\mu^a \frac{\rho^2 x_\lambda}{dt^4 x} \frac{d^2 \alpha}{dt^2}^2
\]

(113)

and where the Lagrange multiplier \( \sigma \) should be fixed to \( \bar{\sigma} \) (this provisional constraint will be completed in Sec. III F). Comparing \( \Sigma_{prov} \) to the massless instanton solution (40) one finds

\[
\Sigma_{prov}[A] = \frac{3}{\rho} \int d^4 x A_\mu^a (x) \left( \frac{\partial A_\mu^a (x)}{\partial \rho} - \frac{2}{g^2} A_\mu^a (x) \right).
\]

(114)
1. Modified limit considerations

Since we have swapped $\alpha_0$ for $\alpha$ in (111) compared to $\Sigma_3$, we have to check that the limit considerations of Sec. III E still hold true. Orders zero and two are exactly as in Sec. III E but something new appears in the fourth order equation:

\[
\frac{d}{dt} \left( \frac{t}{1+t} \right)^3 \left( \frac{d^2 \alpha_4}{dt^2} - 3 \alpha_2 \frac{t^3}{(1+t)^4} \frac{d^2 \alpha_2}{dt^2} \right) - \frac{3 \bar{\sigma}_4 t}{2(1+t)^2} + \frac{\alpha_2}{3(1+t)^5} = \frac{\kappa \rho^2 \mu^2}{8} \left( \frac{d}{dt} \left( \frac{\alpha_2}{(1+t)^4} \right) + \frac{\alpha_2}{(1+t)^3} + \frac{2 f_3}{\mu(1+t)^2} \right). \tag{115}
\]

The $\bar{\sigma}_2$-term is new, so we must verify that this new term does not ruin the integrability at the origin or the Bessel function behavior at infinity. In the limit $t \to 0$ the new term goes as $t \log t$ which is subleading. In the other limit, $t \to \infty$, this term vanishes as $t^{-3}$ so it can only give allowed logarithmic contributions to $\alpha_4$. This analysis shows again that the constrained instanton solution is uniquely determined up to $O(\mu^3)$ but ambiguous in higher orders. To a general order $n > 4$ one can perform the analysis carried out in connection with (115) with the same conclusion.

Thus, by adding to the action (113) one finds a solution to the modified field equations with a finite action, and which gives the massless instanton in the $\mu \to 0$ limit.

F. Leading vs. subleading terms

Until now the exponential fall off at infinity valid to all orders in the mass only takes into account the leading terms to each order. An analysis similar to that of (114) is carried out in this section showing that when the sum of the leading terms vanishes exponentially in the limit $x \to \infty$, then so does the sum of the subleading terms, provided that the constraint is modified such that the constraint term of the field equation explicitly shows exponential fall off.

The final form of the Yang-Mills field equation (42) including the extra term from Sec. III E multiplied with an exponential factor is taken as:

\[
\alpha \partial_\nu \partial^2 \alpha - 3 \partial_\nu \alpha \partial^2 \alpha - \frac{\kappa}{2} f^2 \partial_\nu \alpha = \alpha^2 S_\nu. \tag{116}
\]

where we define the source $S_\nu$:

\[
S_\nu = -\frac{12 \bar{\sigma}_2 \rho^2 x_\nu}{x^3(x^2 + \rho^2)^2} e^{-km_{vec} x^2}. \tag{117}
\]

with $k$ an arbitrary real positive number. The Higgs field equation (13) is unchanged. This gives the following equations for the nextleading terms

\[
\partial_\nu (\partial^2 - \frac{\kappa \rho^2}{2}) \alpha^{(4)} = 3 \partial_\nu \alpha^{(2)} \partial^2 \alpha^{(2)} + \kappa \mu f^{(2)} \partial_\nu \alpha^{(2)} - \frac{12 \bar{\sigma}_2 \rho^2 x_\nu}{x^6} e^{-km_{vec} x^2}. \tag{118}
\]

\[
(\partial^2 - \mu^2)^2 f^{(4)} = -2 \alpha^{(2)} \partial^2 f^{(2)} + \frac{3}{4} \mu (\partial_\nu \alpha^{(2)})^2 + \frac{1}{2} \left( 4 \mu^2 \alpha^{(2)} f^{(2)} + 3 \mu (f^{(2)})^2 \right). \tag{119}
\]

This modification follows if (112) is added to the action in a form where the integrand in (113) is multiplied by the exponential factor $e^{-km_{vec} x^2}$. The final form of the constraint functional is thus

\[
\Sigma[A] = -12 \int d^2 x A_{\mu \nu} \vec{\eta}_{\mu \nu} \frac{\rho^2 x_\lambda}{x^2(x^2 + \rho^2)^2} e^{-km_{vec} x^2}. \tag{120}
\]

As was the case in Sec. III E one again has to check that the limit considerations of Sec. III C still are valid. Orders zero and two are exactly as in (113) but ambiguous in higher orders. In the other limit, $x \to 0$, the extra exponential factor becomes unity and thus this constraint also ensures a finite action.

\[
\Sigma[A] \text{ can be calculated order by order as the equation for } \alpha \text{ is solved. In zeroth order:}
\]

\[
\Sigma_0 = -36 \pi^2 \tag{121}
\]

while to second order

\[
\Sigma_2 = 12 \pi^2 \kappa \rho^2 \mu^2 \left( \log \frac{\kappa \rho^2 \mu^2}{8} + 2 \gamma + 2 + 2\arctan \frac{8}{\kappa \rho^2 \mu^2} \right). \tag{122}
\]

Picking a value of the constant $c$ obviously fixes the scale parameter $\rho$ according to:

\[
-36 \pi^2 + 12 \pi^2 \kappa \rho^2 \mu^2 \left( \log \frac{\kappa \rho^2 \mu^2}{8} + 2 \gamma + 2 + 2\arctan \frac{8}{\kappa \rho^2 \mu^2} \right) + O(\rho^4 \mu^4) = c. \tag{123}
\]

Taking here

\[
c = -36 \pi^2 (1 + \epsilon) \tag{124}
\]

with $0 < \epsilon \ll 1$ one obtains a transcendental equation the solution of which expresses $\rho$ in terms of $\epsilon$ in such a way that $\rho \mu \ll 1$. In this way the choice of the constant $c$ fixes the scale.

1. Finding the modified $\alpha_4$

For the determination of $\alpha_4$ from (116) one solves the following equation replacing (74):

\[
\partial_\nu (\partial^2 - \frac{\kappa \rho^2}{2}) \alpha^{(4)} = 3 \partial_\nu \alpha^{(2)} \partial^2 \alpha^{(2)} + \kappa \mu f^{(2)} \partial_\nu \alpha^{(2)} - \frac{12 \bar{\sigma}_2 \rho^2 x_\nu}{x^6} e^{-km_{vec} x^2}. \tag{118}
\]
\[
\frac{d}{dt} \left( \frac{t}{1 + t} \right)^3 \frac{d^2 \alpha_4}{dt^2} - 3 \alpha_2 \frac{t^3}{(1 + t)^4} \frac{d^2 \alpha_2}{dt^2} - \tilde{\sigma}_4 \frac{3t}{2(1 + t)^2} + \tilde{\sigma}_2 \left( \frac{3t}{(1 + t)^3} \alpha_2 + \tilde{\sigma}_2 - \frac{2f_3}{\mu(1 + t)^2} \right),
\]

with \( \tilde{\sigma}_4 \) a suitable constant. \( \alpha_4 \) is split according to

\[
\alpha_4 = \hat{\alpha}_4 + \tilde{\alpha}_4
\]

with \( \hat{\alpha}_4 \) given in (78) while \( \tilde{\alpha}_4 \) is

\[
\hat{\alpha}_4 = \frac{\kappa^2 \rho^4 \mu^4}{64} \left( \log \frac{\kappa^2 \rho^2}{8} + 2\gamma - 1 \right) \times
\]

\[
\times \left[ -\frac{1}{6t} - (1 + t) \log \frac{1 + t}{t} - \frac{(16k + 1)\kappa^2 \rho^4 \mu^4}{768t} - \frac{\kappa^2 \rho^4 \mu^4}{64} \left( (1 + t)^{\frac{3}{2}} \log \frac{1 + t}{t} + \log t \right) \right.
\]

\[
\left. + \frac{1}{6} \log \frac{1 + t}{t} - \log (1 + t) - 2 \Phi \left( \frac{1}{t} \right) \right]
\]

which has the following asymptotic behavior at \( t \to 0 \):

\[
\hat{\alpha}_4 \simeq -\frac{\kappa^2 \rho^4 \mu^4}{384t} \left( \log \frac{\kappa^2 \rho^2}{8} + 2\gamma - 1 \right) - \frac{(16k + 1)\kappa^2 \rho^4 \mu^4}{768t}
\]

(128)

that one adds to (78) in order to obtain the asymptotic behavior of the modified \( \alpha_4 \). The outcome should match (78). The constant \( c_{4;1} \) must consequently have an additional term

\[
\frac{\kappa^2 \rho^4 \mu^4}{192} \left( \log \frac{\kappa^2 \rho^2}{8} + 2\gamma + \frac{16k - 1}{2} \right)
\]

(129)

to be added to (80). The resulting value of \( c_{4;1} \) and consequently of the factor \( \tilde{\sigma}_4 \) in front of the fourth-order constraint, vanishes for \( k \) chosen according to

\[
\frac{\kappa^2 \rho^4 \mu^4}{24} k = \frac{\kappa^2 \rho^4 \mu^4}{192} (3 - \kappa) - \frac{\kappa^2 \rho^4 \mu^4}{96} \log \frac{\kappa}{2}
\]

\[
- \frac{\kappa^2 \rho^4 \mu^4}{96} (\kappa - 1) \left( \log \frac{\rho^2}{4} + 2\gamma \right)
\]

(130)

where each term on the right-hand side is positive for \( 1 < \kappa < 2 \).

With the calculation in this subsection it is clear that different constraints producing a finite action solution have different subleading terms. In fact, it is possible to remove the subleading terms from \( \alpha_4 \) if the exponential of (118) also contains a term

\[
-k_1 \frac{\kappa \mu^2 \rho^2 \log t}{2}
\]

in the exponent, with \( k_1 \) a positive constant.

2. An alternative approach

Writing the constraint term of (98) as is suggested in (99)

\[
-\frac{\sigma}{4} \left( 1 - \frac{1}{\alpha} \right)^3 \left( 1 + \frac{3}{\alpha - 1} \right)
\]

(131)

one can check that the subleading terms vanish exponentially. This form of the constraint can also be obtained through a source term. One starts with a modification term

\[
-\frac{\sigma}{4} \left( 1 - \frac{1}{\alpha_0} \right)^3 \left( 1 + \frac{3}{\alpha_0 - 1} \right)
\]

(132)

and uses this to obtain \( \alpha_2 \). Then one modifies the constraint to get a modification term

\[
-\frac{\sigma}{4} \left( 1 - \frac{1}{\alpha + \alpha_2} \right)^3 \left( 1 + \frac{3}{\alpha + \alpha_2 - 1} \right)
\]

(133)

and uses this to find \( \alpha_4 \). Then one again modifies the constraint to find \( \alpha_6, \alpha_8 \) etc. In this way we can effectively work with the modification term (31).

The integration constant \( c_{4;1} \) and thus the fourth-order constraint are with this approach modified compared to (116) with (129) added; also the subleading terms contain additional terms to those found in the previous section. This emphasizes the ambiguity of the constraint beyond lowest order.

3. Effective Lagrangian

A systematic way of representing subleading terms at large distances is like in the scalar case obtained by iteration of field equations. It is here convenient to use a field \( a_\nu = \partial_\nu \log \alpha \), in terms of which the field equations are

\[
\partial^2 a_\nu - 2a_\nu \partial_\lambda a_\lambda - \partial_\nu (a_\mu a_\mu) - 2a_\nu a_\mu a_\mu - \frac{\kappa}{2} f^2 a_\nu = S_\nu,
\]

(134)

\[
\partial^2 f - \frac{3}{4} a_\mu a_\mu f + \frac{1}{2} (\mu^2 - f^2) f = 0,
\]

(135)

with \( S_\nu \) defined in (117). These field equations are obtained from an effective Lagrangian

\[
L_{eff} = -\frac{1}{2} \partial_\mu a_\nu \partial_\nu a_\mu - \partial_\nu a_\lambda a_\lambda a_\mu - \frac{1}{2} a_\nu a_\mu a_\mu a_\mu
\]

\[
- \frac{\kappa}{4} f^2 a_\nu a_\nu - \frac{2}{3} \partial_\mu f \partial_\nu f - \frac{\kappa}{6} (f^2 - \mu^2)^3 + a_\nu S_\nu.
\]

(136)
G. The 't Hooft path integral measure

For completeness we briefly indicate how the 't Hooft path integral measure is obtained from our analysis. This subsection mostly contains well-known results. However, we indicate how our methods can be used to generalize the result of 't Hooft.

The value of the classical action up to second order in the mass parameter when calculated by means of the constrained instanton solution of Sec. III B leads to the result

$$S = -\frac{8\pi^2}{g^2} - \frac{2\pi^2\kappa^2\mu^2}{g^2}$$

(137)

in agreement with [2], [3], [7]. Beyond second order the result will contain ambiguities since it depends on the form of $\alpha_0$ which, as we have seen, can be modified.

The Euclidean path integral

$$Z = \int [dA][d\phi] e^{-S_E[A,\phi]}$$

(138)

with $S_E[A,\phi]$ the Euclidean Yang-Mills-Higgs action is evaluated by the saddle-point method. For this purpose the previously determined constrained instanton solution is used by a Faddeev-Popov trick. We write unity as

$$1 = \int d\rho \Delta[A,\rho] \delta (\Sigma[A] - c) = \int d\rho \Delta[A,\rho] \delta (\Sigma[A - \bar{A}])$$

(139)

where the constrained instanton is denoted $(\bar{A},\bar{\phi})$, and where the constraint it obeys was used in the last step.

Multiplying the path integral by the Faddeev-Popov unity one obtains

$$Z = \int [dA][d\phi] \int d\rho \Delta[A,\rho] \delta (\Sigma[A] - c) e^{-S_E[A,\phi]}$$

(140)

where $\Sigma[A]$ was defined in (120), while

$$\Delta[A,\rho] = \left| 12 \int d^4 x A_{\mu
u} \bar{\eta}_{\mu\nu\lambda} \frac{\partial}{\partial \rho} x_{\lambda} (x^2 + \rho^2)^2 \right|$$

(141)

to order $\mu^0$. It is known from the previous analysis that the modified action

$$\tilde{S}_E[A,\phi] = S_E[A,\phi] + \bar{\sigma} \Sigma, \quad \bar{\sigma} = -\frac{\kappa\rho^2\mu^2}{6} + \cdots$$

(142)

has a finite solution, so using the $\delta$-function we write $S_E = \tilde{S}_E - \bar{\sigma} c$ and the path integral is hence

$$Z = \int [dA][d\phi] \int d\rho \Delta[A,\rho] \delta (\Sigma[A - \bar{A}]) e^{-\tilde{S}_E[A,\phi]} e^{\bar{\sigma} c}.$$  

(143)

New integration variables are introduced through the substitution

$$(A,\phi) \rightarrow (\bar{A} + A,\bar{\phi} + \phi).$$

In the Gaussian approximation

$$\Delta[A + \bar{A},\rho] \simeq \Delta[\bar{A},\rho]$$  

(144)

where terms depending on $A$ have been disregarded in $\delta$, and

$$Z = e^{-S_E[\bar{A},\bar{\phi}]} \int [dA][d\phi] \int d\rho \Delta[\bar{A},\rho] \delta(\Sigma[A]) \times$$

$$\times e^{-\tilde{S}_E[\bar{A} + A,\bar{\phi} + \phi] - \tilde{S}_E[\bar{A},\bar{\phi}]}.$$  

(145)

The path integral (145) is the same as that given by 't Hooft [3] in the approximation where the classical action $S_E[A,\phi]$ is computed up to order $\mu^2$ and the fluctuations (including the Faddeev-Popov determinant $\Delta[A,\rho]$) to order $\mu^0$. In this approximation the integral over $\rho$ is identical to what is obtained by means of the standard method of collective coordinates. This follows from (114), where the term $\frac{\partial S}{\partial \rho}$ in the integrand projects out the dilatation zero mode. Also $\Delta[\bar{A},\rho]$ is easily checked to be the same as the corresponding expression obtained by means of collective coordinates. This argument is actually independent of the detailed form of the constraint, provided the projection of the derivative of the constraint with respect to the gauge field onto the dilatation zero mode is nonvanishing.

IV. CONCLUSION

Our results can be summarized in the following way:

For $\phi^4$ theory a finite action instanton solution of the massive theory without any constraint does not exist because of its large-distance behavior that is characterized by exponential increase instead of fall off. Two types of constraint are considered. If the constraint is required to depend only on the scalar field, the only possible way to cure this defect is by means of a constraint cubic in the field. Other constraints only depending on the field are ruled out because they lead to singular behavior of the constrained instanton solution at the origin. The constraint can also amount to having a source term in the field equation. This type of constraint can be constructed in such a way that it has the same effect as the constraint cubic in the field referred to above.

For the Yang-Mills-Higgs theory the situation is rather different. Here we found that exponential fall off at infinity can be obtained by adjustment of integration constants without imposition of any constraint. On the other hand, a constraint is necessary in order to ensure absence of singularities of the constrained instanton at small distances that prevent the action from being finite. The form of the constraint required for this purpose is uniquely determined to lowest order in the mass variable, and only a special constraint involving field variables only can be constructed. On the other hand, a constraint corresponding to a source term in the Yang-Mills field equation is possible; the explicit form of the
modified field equation is given in (116). The source term in (116) can be modified somewhat, and the constrained instanton is correspondingly not uniquely determined in and above fourth order in the mass. No modification of the Higgs field equation is necessary.

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APPENDIX A: MODIFIED BESSEL FUNCTIONS

The modified Bessel equation of order unity

\[ x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} - (x^2 + 1)f = 0 \]  
(A1)

has as linearly independent solutions the two modified Bessel functions

\[ I_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left( \frac{x^2}{4} \right)^n \]  
(A2)

and

\[ K_1(x) = \left( \log\left(\frac{x}{2}\right) + \gamma \right) I_1(x) + \frac{1}{x} \]  
\[ -\frac{x}{4} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left( \sum_{k=1}^{n+1} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k} \right) \left( \frac{x^2}{4} \right)^n \]  
(A3)

where \( \gamma \) is Euler’s constant and the first term in the sum is \( \frac{x}{4} \). For \( x \to \infty \):

\[ I_1(x) \approx \sqrt{\frac{1}{2\pi x}} e^x \]  
(A4)

\[ K_1(x) \approx \sqrt{\frac{\pi}{x}} e^{-x} \]  
(A5)

The Klein-Gordon equation in four-dimensional Euclidean space

\[ (\partial^2 - m^2)\phi = 0, x \neq 0 \]  
(A6)

is with \( \phi \) only a function of \( x = |x| \) and writing

\[ \phi(x) = \frac{m}{x} G(mx) \]  
(A7)

converted into

\[ \left( \frac{d^2}{dx^2} + \frac{3}{x} \frac{d}{dx} - m^2 \right) \phi(x) = \frac{m^4}{\xi^2} \left( \xi^2 \frac{d^2 G(\xi)}{d\xi^2} + \xi \frac{dG(\xi)}{d\xi} - (1 + \xi^2)G(\xi) \right) \]  
(A8)

with \( \xi = mx \). Here the expression in brackets is recognized as the defining equation of the modified Bessel functions of order one (A1). Consequently the solution of (A6) is a linear combination of \( \frac{m}{x} I_1(mx) \) and \( \frac{m}{x} K_1(mx) \).