Dimensional regularization of the free-electron self-energy and vertex correction in Coulomb gauge

Ingvar Lindgren
Physics Department, University of Gothenburg, Göteborg, Sweden

July 26, 2011

Abstract
There is presently a great interest in studying static and dynamic properties of highly charged ions that can be produced in large particle accelerators, like that at GSI in Darmstadt. To perform corresponding theoretical calculations with great accuracy requires a highly developed machinery of computational methods that have not until recently been available. In order to combine many-body perturbation theory with quantum electrodynamics, the calculations have generally to be performed in the Coulomb gauge, where applications have not been so developed as in, for instance, the Feynman gauge. Formulas for the free-electron self-energy and vertex correction have been given without derivation by Adkins (Phys. Rev. D27, 1814 (1983); Phys. Rev. D34, 2489 (1986)). In the present paper the formulas of Adkins are verified with detailed derivations.

1 Introduction
There is now an increased interest in studying the effects of quantum-electrodynamics in combination with electron correlation in electronic systems, particularly in connection with experiments on highly charged ions (see ref. [1] and references therein). In order to take full advantage of the development in atomic many-body theory [2], it is necessary to perform the calculations in the Coulomb gauge. Dimensional regularization in that gauge is more complicated than in, for instance, the Feynman gauge and has to date to our knowledge not been used in practical calculations. Such calculations have now been performed at our department [3], and as a background we have reconsidered the formulas derived by Adkins some time ago [4, 5]. Adkins gives only the final results without any derivation, and we have found that it might be useful to produce full derivations of the formulas. One derivation is reproduced here, and an alternative treatment is being published separately [6].

In the book cited above [1] the dimensional regularization of the free-electron self-energy and vertex correction are treated in the Feynman
gauge and the self-energy also in the Coulomb gauge, while the vertex correction in the latter gauge was found to be too complex to include in the book. For that reason it is instead reproduced here.

2 Free-electron self energy in Coulomb gauge

We shall mainly follow Adkins [4] in regularizing the free-electron self-energy in the Coulomb gauge. We start from the expressions for the self-energy (c = 1)

\[ \Sigma_{\text{free}}(p) = i e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2 + i\eta} \gamma^\nu D_{\nu\mu}(k) \]  

and for the photon propagator

\[ D_{\mu\nu}^{C}(k; k) = \frac{1}{\epsilon_0} \left[ \frac{\delta_{\mu,0}\delta_{\nu,0}}{k^2} - \frac{\delta_{\mu,1}\delta_{\nu,1}}{k^2} \left( g_{ij} + \frac{k_i k_j}{k^2} \right) \frac{1}{k^2 + i\eta} \right] \]

The three terms in the propagator correspond to the Coulomb, Gaunt and scalar-retardation parts.

2.1 Coulomb contribution

The Coulomb part of the self energy becomes

\[ \frac{i e^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^0(\not{p} - \not{k} + m)\gamma^0}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta} \]

\[ = \frac{i e^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta} \]

using the commutation rules in Appendix A (Eq. 55). With q = −p and s = p^2 − m^2 the denominator is of the form k^2 + 2kq + s and we can apply the formulas (64) and (65) in Appendix B for dimensional regularization with D = 4 − ε (n = 1). This gives with k^0 → −q^0 = p^0, k_i → −q_i = p_i y, \( \gamma \cdot k = −\gamma^0 k_0 \rightarrow \gamma \cdot p y \) and \( w = p^2 y^2 + (1 - y)yp_0^2 - (p^2 - m^2)y \)

\[ \frac{i e^2}{\epsilon_0} \int \frac{d^Dk}{(2\pi)^D} \frac{\not{p} - \not{k} + m}{k^2 + 2kq + s + i\eta} \frac{1}{k^2 + i\eta} \]

\[ = (i)^2(-1)^n \frac{e^2}{\epsilon_0 (4\pi)^{D/2}} \int_0^1 \frac{dy}{\sqrt{y}} \left( \gamma \cdot p (1 - y) + m \right) \frac{\Gamma(\epsilon/2)}{w^{\epsilon/2}} \]

The Gamma function can be expanded as

\[ \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma_k + \ldots \]

\[ ^{1}\text{This part is essentially reproduced from the above-mentioned book [1] sect. 12.5 with due permission from the publisher.} \]
where $\gamma_E = 0.5772\ldots$ is Euler’s constant, and furthermore

$$\left(\frac{m^2}{w}\right)^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln\left(\frac{w}{m^2}\right) + \cdots$$

$$\frac{1}{(4\pi)^{D/2}} = \frac{1}{(4\pi)^2}\left(1 + \frac{\epsilon}{2} \ln 4\pi + \cdots\right)$$

This yields

$$\frac{\Gamma(\epsilon/2)}{(4\pi)^{D/2}} \left(\frac{m^2}{w}\right)^{-\epsilon/2} = \frac{1}{(4\pi)^2}\left(2/\epsilon - \gamma_E + \cdots\right) \left(1 + \frac{\epsilon}{2} \ln 4\pi + \cdots\right) \left(1 - \frac{\epsilon}{2} \ln (w/m^2) + \cdots\right)$$

$$= \frac{1}{(4\pi)^2} \left[\Delta - \ln\left(\frac{w}{m^2} + \cdots\right)\right]$$

(5)

where

$$\Delta = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \cdots$$

(6)

This leads to the Coulomb contribution (3), omitting the factor $m^{-\epsilon/2}$,

$$K \int_0^1 \frac{dy}{\sqrt{y}} \left(\gamma \cdot p (1 - y) + m\right) \left(\Delta - \ln(yX)\right)$$

where in relativistic units\(^2\)

$$K = \frac{e^2}{\epsilon_0 (4\pi)^2} = \frac{\alpha}{4\pi}$$

(7)

and $w = m^2 y X$, $X = 1 + (p^2/m^2)(1 - y)$. This leads to

$$K \int_0^1 \frac{dy}{\sqrt{y}} \left((\gamma \cdot p (1 - y) + mc) \left(\Delta - \ln y - \ln X\right)\right)$$

and the Coulomb part of the free-electron self energy becomes (times $K$)

$$\left(\frac{4}{3\gamma} \gamma \cdot p + 2m\right) \Delta + \left(\frac{32}{9\gamma} \gamma \cdot p + 4m\right) - \int_0^1 \frac{dy}{\sqrt{y}} \left((\gamma \cdot p (1 - y) + m) \ln X\right)$$

(8)

### 2.2 Gaunt contribution

The *Gaunt term* becomes, using Eq. (1) and the second term of Eq. (2),

$$-\frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_i(p - k + m)\gamma^i}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta}$$

(9)

\(^2\epsilon = \epsilon_0 = 1, e^2 = 4\pi\alpha (\alpha \text{ fine-structure constant})\)
Using the Feynman integral \[ (59) \] (second version) in Appendix \[ 4 \] with 
\( a = k^2 \) and \( b = (p - k)^2 - m^2 c^2 \), this can be expressed\[ 3 \]

\[
- \frac{ie^2}{\varepsilon_0} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_{\ell}(\tilde{p} - \tilde{k} + m)}{[k^2 + (p^2 - 2pk - m^2)c^2]}
\]

\[
= - \frac{ie^2}{\varepsilon_0} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{(3 - \epsilon)\cdot m - (2 - \epsilon)(\tilde{p} - \tilde{k}) - \tilde{p} + \tilde{k}}{[k^2 + (p^2 - 2pk - m^2)c^2]} \]

\[ (10) \]

after applying the commutation rules in Eq. \[ 56 \].

With the substitutions \( k \rightarrow -q = px \) and \( s = (p^2 - m^2)x \) we can apply the equations \[ 61 \] and \[ 62 \] in Appendix \[ 5 \] leading to

\[
- \frac{ie^2}{\varepsilon_0} \int_0^1 dx \int \frac{D^k}{(2\pi)^4} \frac{(3 - \epsilon)\cdot m - (2 - \epsilon)(\tilde{p} - \tilde{k}) - \tilde{p} + \tilde{k}}{[k^2 + 2kq + s]} \]

\[
= \frac{e^2}{\varepsilon_0(4\pi)^{D/2}} \int_0^1 dx \left[ (3 - \epsilon)\cdot m - (2 - \epsilon)\tilde{p}(1 - x) - \tilde{p}(1 - x) \right] \Gamma(\epsilon/2) \frac{\eta^2}{w^{\epsilon/2}}
\]

\[
= \frac{e^2}{\varepsilon_0(4\pi)^{D/2}} \int_0^1 dx \left[ - (1 - x)(3\gamma^0p_0 - \gamma \cdot p) + 3m + \epsilon((1 - x)\tilde{p} - m) \right] \Gamma(\epsilon/2) \frac{\eta^2}{w^{\epsilon/2}}
\]

where \( w = q^2 - s = p^2x^2 - (p^2 - m^2)x = m^2xY \). This yields the Gaunt contribution \[ 43 \] (times K)

\[
- \int_0^1 dx \left\{ [(1 - x)(3\gamma^0p_0 - \gamma \cdot p) - 3m] \Delta - \ln(xY) \right\} - 2[(1 - x)\tilde{p} - m]
\]

using the relation \[ 44 \] and the fact that \( \epsilon \Delta \rightarrow 2 \) as \( \epsilon \rightarrow 0 \). Then the Gaunt part of the free-electron self energy becomes\[ 3 \]

\[
\left[ - \frac{1}{2} (3\gamma^0p_0 - \gamma \cdot p) + 3m \right] \Delta - \frac{5}{4} \gamma^0p_0 - \frac{1}{4} \gamma \cdot p + m + \int_0^1 dx \left\{ [(1 - x)(3\gamma^0p_0 - \gamma \cdot p) - 3m] \ln Y \right\}
\]

\[ (11) \]

\[ 2.3 \] Scalar-retardation contribution

Finally, the scalar-retardation part becomes similarly, using the third term of Eq. \[ 2 \] and the commutation rules \[ 53 \],

\[
- \frac{ie^2}{\varepsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma\cdot k_i(\tilde{p} - \tilde{k} + m)\gamma\cdot k_j + 1}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta}
\]

\[
= \frac{ie^2}{\varepsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma\cdot k_i\gamma\cdot k_j(\tilde{p} - \tilde{k} - m) - 2\gamma\cdot k_i(k_j p_j - k_j k_j)}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta}
\]

\[
= - \frac{ie^2}{\varepsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{p} - \tilde{k} - m + 2\gamma\cdot k_i k_j p_j / k^2}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2 + i\eta}
\]

\[ (12) \]

\[ 3 \] We use the convention that \( \mu, \nu, \ldots \) represent all four components (0,1,2,3), while \( i, j, \ldots \) represent the vector part (1,2,3).

\[ 4 \] Note misprint in first bracket of Eq. (12.113) of ref. \[ 1 \].
with $\gamma^i k_i \gamma^j k_j = -k^2 = -k_i k_i$. With the same substitutions as in the Gaunt case this becomes

$$-\frac{ie^2}{\epsilon_0} \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \gamma^i \frac{\gamma^j k_i k_j}{k^2 - 2pkx + (p^2 - m^2)x}$$

(13)

With the substitutions $k \rightarrow -q = px$ and $s = (p^2 - m^2)x$ the first part is of the form Eq. (61) and Eq. (62) in Appendix B and becomes

$$\frac{e^2c}{\epsilon_0(4\pi)^{D/2}} \int_0^1 dx \left[ \gamma^i - \tilde{p} x - m \right] \frac{\Gamma(e/2)}{u^{\sigma/2}}$$

(14)

and with Eq. (5)

$$K \int_0^1 dx \left[ \gamma^i - \tilde{p} x - m \right] \left( \Delta - \ln(xY) \right)$$

(15)

with $K = e^2/(\epsilon_0 (4\pi)^2)$ and $w$ being the same as in the Gaunt case, $w = q^2 - s = p^2 x^2 - p^2 x + m^2 x = m^2 x Y$.

The second part of Eq. (13) is of the form Eq. (66) and becomes

$$(k_ik_j \rightarrow q_i q_j y^2 = p_i p_j x^2 y^2 \text{ in first term, } \rightarrow -\frac{1}{2} \gamma' = -\frac{1}{2} \delta_{ij} \text{ in second})$$

$$K \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ 2\gamma^* p_i p^j p^j \frac{\Gamma(1 + \epsilon/2)}{u^{\+ \epsilon/2}} - \gamma^* p_j \frac{\Gamma(e/2)}{u^{\sigma/2}} \right\}$$

$$K \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{2\gamma^* p_i p^j p_j}{m^2} xy Z - \gamma^* p_j \left( \Delta - \ln(xyZ) \right) \right\}$$

$$= K \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{2\gamma \cdot \vec{p}^2}{m^2} \frac{xy Z}{Z} + \gamma \cdot \vec{p} \left( \Delta - \ln(xyZ) \right) \right\}$$

with $w = xy \left[ -p^2 x^2 + p^2 y^2 - p^2 + m^2 \right] = xy \left[ p^2 (1 - x) - p^2 (1 - y) + m^2 \right] = m^2 ZXY$

Integration by parts of the first term yields (times $K$), noting that $dZ/dy = -p^2 x$,

$$-\int_0^1 dx \left[ \sqrt{y} \gamma 2 \gamma \cdot \vec{p} \ln Z \right] + 3 \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot \vec{p} \ln Z$$

The total scalar-retardation part then becomes (with $Z(y = 1) = Y$)

$$\int_0^1 dx \left[ \frac{\gamma^*}{\gamma^i} - \tilde{p} x - m \right] \left( \Delta - \ln(xY) \right) - \int_0^1 dx 2 \gamma \cdot \vec{p} \ln Y$$

$$+ 3 \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot \vec{p} \ln Z + \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot \vec{p} \left( \Delta - \ln(xyZ) \right)$$
or
\[
\int_0^1 dx \left( \gamma^0 p_0 (1 - x) - \gamma \cdot p (1 + x) - m \right) \left( \Delta - \ln x \right) + \int_0^1 dy \sqrt{y} \gamma \cdot p \Delta
\]
\[
- \int_0^1 dx \left( \gamma^0 p_0 (1 - x) - \gamma \cdot p (1 - x) - m \right) \ln Y
\]
\[
- \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot p \ln(xy) + 2 \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot p \ln(xy)
\]
\[
- 3 \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot p \ln Z
\]
Then the scaler-retardation part of the free-electron self energy becomes
\[
\left( \frac{1}{2} \gamma^0 p_0 - \frac{5}{6} \gamma \cdot p - m \right) \Delta + \frac{3}{4} \gamma^0 p_0 - \frac{5}{36} \gamma \cdot p - m
\]
\[
- \int_0^1 dx \left( \gamma^0 p_0 (1 - x) - \gamma \cdot p (1 - x) - m \right) \ln Y - 3 \int_0^1 dx \int_0^1 dy \sqrt{y} \gamma \cdot p \ln Z
\]

Summarizing all contributions yields the mass-renormalized free-electron self energy in the Coulomb gauge
\[
\frac{e^2}{\epsilon_0 (4\pi)^2} \left[ - \left( \not{p} - m \right) \Delta - \frac{1}{2} \gamma^0 p_0 + \frac{19}{6} \gamma \cdot p - \int_0^1 dy \sqrt{y} \left( \gamma \cdot p (1 - y) + m \right) \ln X
\]
\[
+ 2 \int_0^1 dx \left[ (1 - x) \not{p} - m \right] \ln Y + \int_0^1 dx \int_0^1 dy \sqrt{y} 2\gamma \cdot p \ln Z \right] \quad (16)
\]

where we have subtracted the on-shell (\( \not{p} = m \)) value, \( K \mu (3\Delta + 4) \). (which is the same as in the Feynman gauge \[1\] Eq. (12.103)). The expressions for \( X, Y, Z \) are given in the text. The result is in agreement with that of Adkins \[4\].

3 Free-electron vertex function in Coulomb gauge

Next, we consider the free-electron vertex function in the Coulomb gauge and start from the expression \[5\] Eq. (1)]
\[
\Lambda_\sigma(p, p') = i e^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\sigma \frac{p' - k + m}{(p' - k)^2 - m^2 + i\eta} \gamma^\alpha \frac{p - k + m}{(p - k)^2 - m^2 + i\eta} \gamma^\mu D_{\nu\mu} \quad (17)
\]

We restrict ourselves here to the case, where \( \gamma^\sigma = \gamma^0 \), which implies that the interaction at the vertex is scalar and can be the Coulomb interaction with the nucleus or with another electron.
3.1 Coulomb contribution

Inserting the **Coulomb part** of the photon propagator \( \frac{1}{\epsilon_0} \) into the vertex expression \( (2) \), yields

\[
\frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^0}{(p' - k + m)} \frac{\gamma^0}{(p - k)^2 - m^2 + i\eta} \frac{1}{k^2}
\]

Using the relation \( \gamma^0 = \gamma^0 \gamma^0 \) (see Appendix A) and the Feynman parametrization \( (69) \) in Appendix C, this leads to

\[
\frac{ie^2}{\epsilon_0} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^0(\gamma^0 - \gamma^0)(\gamma^0 - \gamma^0)}{(k^2 - 2pk + p^2 - m^2 - (p^2 - p^2 - 2kp + 2kp')x^2)^2} \frac{1}{k^2}
\]

With

\[
q = -(1 - x)p - p'x = -(p - dx); \quad d = p - p'
\]

the denominator is of the form \( k^2 + 2kq + s \). Introducing the dimension \( D = 4 - \epsilon \), we then have

\[
\frac{ie^2}{\epsilon_0} \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma^0(\gamma^0 - \gamma^0)(\gamma^0 - \gamma^0)}{(k^2 + 2pq + s)^2} \frac{1}{k^2}
\]

and we can then apply the formulas Eq. (64) to Eq. (66) in Appendix E. In Eq. (65) and the first part of Eq. (66) we make the substitution \( k^\mu \rightarrow -q^\mu y - \delta_{\mu,0} q_0 (1 - y) \) or

\[
k \rightarrow -qy = py - dx \quad k^0 \rightarrow -q_0 = p_0 - dx
\]

\[
\gamma \cdot q y - \gamma^0 q_0 = -(\gamma \cdot p - \gamma \cdot dx)y + \gamma^0(p_0 - dx)
\]

\[
\gamma y - \gamma^0 q_0 = (\gamma \cdot p - \gamma \cdot dx)y + \gamma^0(p_0 - dx)
\]

and in the second part of Eq. (66)

\[
k^\mu k^\nu \rightarrow -\frac{1}{2} [g^{\mu\nu} + \delta_{\mu,0} \delta_{\nu,0}(1 - y)/y]
\]

\[
k^\mu k^\nu \rightarrow -\frac{1}{2} [g^{\mu\nu} + \delta_{\mu,0} \delta_{\nu,0}(1 - y)/y]
\]

\[
k \rightarrow \gamma \cdot q y - \gamma^0 q_0 (1 - y) + \gamma^0 (p_0 - dx)
\]

This yields in analogy with the self energy

\[
(1)^2 \frac{e^2}{\epsilon_0} \frac{m^{-\epsilon/2}}{(4\pi)^{D/2}} \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \left[ p' + (\gamma \cdot p - \gamma \cdot dx)y - \gamma^0 (p_0 - dx) + m \right] \left[ \frac{1}{w} \right] + \frac{1}{2} \left( 3 - \epsilon - 1/y \right) \frac{\Gamma(\epsilon/2)}{w^{\epsilon/2}} \right\}
\]
With Eqs (5) and (7) we have in the limit $\epsilon \to 0$

$$
- K\gamma^0 \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{NumC}{w} - \frac{1}{2} (3 - \epsilon - 1/y) \left( \Delta - \ln(w/m^2) \right) \right\}
$$

$$
= -K\gamma^0 \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{NumC}{w} + \left( \frac{1}{2} (3 - 1/y) \ln(w/m^2) - 1 \right) \right\} \quad (18)
$$

and after partial integration

$$
- K\gamma^0 \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{NumC}{w} - (1 - y) \frac{m^2}{w} \frac{dw}{dy} - 1 \right\} \quad (19)
$$

Here,

$$
NumC = \left[ \vec{p}' + (\gamma \cdot \vec{p} - \gamma \cdot d\vec{x}) y - \gamma^0 (p_0 - d_0 x) + m \right] \left[ \vec{p} - (\gamma \cdot \vec{p} - \gamma \cdot d\vec{x}) y - \gamma^0 (p_0 - d_0 x) + m \right]
$$

$$
w = (q^2 y^2 + (1 - y) q_0^2 y - sy = q_0^2 y - q^2 y^2 - sy
$$

$$
y[(1 - x)p_0 + p_0'x]^2 - y^2 [(1 - x)p + p'x]^2
$$

$$
- [(1 - x)(p_0^2 - p^2) + p_0'^2 x - p'^2 x - m^2] y
$$

and

$$
\frac{dw}{dy} = [(1 - x)p_0 + p_0'x]^2 - 2y [(1 - x)p + p'x]^2
$$

$$
- [(1 - x)(p_0^2 - p^2) + p_0'^2 x - p'^2 x - m^2] = \Delta_x - y [(1 - x)p + p'x]^2 \quad (20)
$$

with

$$
\Delta_x = w/y = [(1 - x)p_0 + p_0'x]^2 - y [(1 - x)p + p'x]^2
$$

$$
- [(1 - x)(p_0^2 - p^2) + p_0'^2 x - p'^2 x - m^2]
$$

$$
\Delta_x = m^2 - x(1 - x)d_0^2 + (1 - x)xyd^2 + (1 - x)(1 - y)p^2 + x(1 - y)p'^2 \quad (21)
$$

The expression Eq. (19) then becomes

$$
- K\gamma^0 \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{NumC - (1 - y) \left( \Delta_x - y [(1 - x)p + x p']^2 \right) - w}{w} \right\}
$$

$$
= -K\gamma^0 \int_0^1 dx \int_0^1 dy \sqrt{y} \left\{ \frac{NumC - \Delta_x + y(1 - y) [(1 - x)p + x p']^2}{\Delta_x} \right\} \quad (22)
$$
From Eq. (19)

\[ NumC = \left[ \gamma^0 p'_0 - \gamma \cdot p' + (\gamma \cdot p - \gamma \cdot dx) y - \gamma^0 (p_0 - d_0 x) + m \right] \times \left[ \gamma \cdot p - (\gamma \cdot p - \gamma \cdot dx) y + \gamma^0 d_0 x + m \right] \]

\[ = \left[ - \gamma^0 d_0 (1 - x) + m + \gamma \cdot p(1 - x) y - \gamma \cdot p'(1 - xy) \right] \times \left[ \gamma^0 d_0 x + m + \gamma \cdot p(1 - y + xy) - \gamma \cdot p' xy \right] \]

This can also be expressed

\[ NumC = m^2 - \gamma^0 m d_0 (1 - 2x) + m \gamma \cdot d - d_0^2 x (1 - x) \]

\[ - \gamma^0 d_0 (1 - x) [\gamma \cdot p(1 - y + xy) - \gamma \cdot p' xy] + [\gamma \cdot p(1 - x) y - \gamma \cdot p'(1 - xy)] \gamma^0 d_0 x \]

\[ - \gamma \cdot p \gamma \cdot p'(1 - x) xy^2 - \gamma \cdot p' \gamma \cdot p(1 - xy)(1 - y + xy) \]

\[ + (\gamma \cdot p)^2 (1 - x) y (1 - y + xy) + (\gamma \cdot p')^2 (1 - xy) y xy \]

With

\[ [(1 - x) p + p' x]^2 = -x(1 - x) d^2 + (1 - x) p^2 + x p'^2 \]

the numerator in Eq. (19) becomes

\[ -\gamma^0 R^0_x = NumC - \Delta x + y (1 - y) [(1 - x) p + x p']^2 \]

\[ = -\gamma^0 m d_0 (1 - 2x) + m \gamma \cdot d \]

\[ - \gamma^0 d_0 \gamma \cdot p (1 - x) (1 - y + 2xy) + \gamma^0 d_0 \gamma \cdot p' x (1 + y - 2xy) \]

\[ - \gamma \cdot p' \gamma \cdot p (1 - y) - d^2 2xy (1 - x) - p^2 (1 - x) (1 - y) - p'^2 x (1 - y) \]

The Coulomb contribution to the free-electron vertex function then becomes

\[ K \int_0^1 dx \int_0^1 \frac{dy}{\sqrt{\Delta x}} \frac{R^0_x}{\Delta x} \]

(23)

which agrees with the result of Adkins [5] \( R^0_x \) with

\[ d = p - p' \rightarrow k; \ x \rightarrow u; \ y \rightarrow x \]
3.2 Gaunt contribution

For the Gaunt part we have

$$-\frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \gamma^i (p' - k + m) \gamma_0 (p - k) \gamma^i \frac{1}{k^2 + i\eta}$$

(24)

and, using the parametrization (34) in Appendix C and the commutation rules in Appendix A, this becomes

$$-\frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \gamma^i \gamma_0 (p' - k + m) \gamma^i \frac{1}{k^2 + i\eta}$$

(25)

with $q = -(x - y)p - p'y$ and $s = (p^2 - m^2)x - (p'^2 - p^2)y$.

Applying the commutation rule $A\gamma^i = -\gamma^i A + 2A'\gamma^i$ and $\tilde{A}\gamma^i = -\gamma^i \tilde{A} - 2A'$, the numerator becomes

$$\text{Num} = \gamma_i (p' - k + m)(\gamma^i)\gamma^j$$

(26)

We can now apply the formulas (31) to (33) in Appendix B with $k \to -q = (x - y)p + p'y$ in Eq. (22) and first part of Eq. (35) and $k^\mu k'^\nu \to -g^{\mu\nu}/2$ in the second part and

$$w = q^2 - s = [(x - y)p + p'y]^2 - (p^2 - m^2)x + (p^2 - p'^2)y \quad (y \to xu)$$

$$w'/x = \Delta_y = x[(1 - u)p + p'u]^2 - (p^2 - m^2) + (p^2 - p'^2)u = w'x + w''$$

$$w'' = [(1 - u)p + p'u]^2 \quad w' = -(p^2 - m^2) + (p^2 - p'^2)u$$

$$\Delta_y = p'^2(1 - u)(x - xu - 1) - p'^2x(1 - xu) + (p^2 + p'p)xu(1 - u) + m^2$$

$$= m^2 - ux(1 - u)d^2 - p'^2(1 - u)(1 - x) - p'^2u(1 - x)$$

(27)

This agrees with Adkins' $\Delta_y$ [5].

The Gaunt part (25) then becomes with the substitution $y \to xu$

$$2K\gamma^0 \frac{1}{\Gamma(3)} \int_0^1 x dx \int_0^1 du \left\{ \frac{[\text{Num}]_{k \to -q}}{w} + \frac{[\text{Num}]_{k^\mu k'^\nu \to -g^{\mu\nu}/2}}{w'^2} \right\}$$

(28)

The evaluation will be made below together with the Gaunt-like part of the scalar retardation.
3.3 Scalar-retardation contribution

The scalar-retardation part becomes similarly

\[-\frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \gamma^4 k_i \frac{p' - \vec{k} + m}{(p' - k)^2 - m^2 + i\eta} \gamma^0 \frac{\vec{p} - \vec{k} + m}{(p - k)^2 - m^2 + i\eta} \frac{\gamma^j k_j}{k^2} \frac{1}{k^2 + i\eta}\]

\[= \frac{ie^2}{\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \gamma^0 \gamma^4 k_i \frac{p' - \vec{k} + m}{(p' - k)^2 - m^2 + i\eta} \frac{\vec{p} - \vec{k} + m}{(p - k)^2 - m^2 + i\eta} \frac{\gamma^j k_j}{k^2} \frac{1}{k^2 + i\eta}\]

\[= \frac{2ie^2}{\epsilon_0} \gamma^0 \int \frac{dDk}{(2\pi)^D} \int_0^1 dx \int_0^1 dy \frac{\gamma \cdot k (\vec{p}' - \vec{k} + m)(\vec{p} - \vec{k} + m) \gamma \cdot k}{[k^2 + 2kq + s]^3} \frac{1}{k^2 + i\eta}\]

with (as in the Gaunt case) \(q = -(x - y)p - p'y\) and \(s = (p^2 - m^2)x - (p^2 - p'^2)y\). With the commutation rule \(A \gamma^i = -\gamma^i A + 2A^i\) we have

\[A \gamma^j k_j = -\gamma^j k_j A + 2A^i k_j; \quad A \gamma \cdot k = -\gamma \cdot k A + 2A \cdot k; \quad \vec{A} \gamma \cdot k = -\gamma \cdot k \vec{A} - 2\gamma \cdot k\]

\[\gamma \cdot k A \gamma \cdot k = -\gamma \cdot k \gamma \cdot k A + 2\gamma \cdot k A \cdot k = k^2 A + 2\gamma \cdot k A \cdot k\]

\[\gamma \cdot k \vec{k} \gamma \cdot k = k^2 \vec{k} + 2\gamma \cdot k k^2 = k^2 \vec{k}\]

and the numerator in Eq. (29) becomes

\[
\text{Num} = -\gamma \cdot k (\vec{p}' - \vec{k} + m) \left[ \gamma \cdot k (\vec{p} - \vec{k} - m) - 2(\vec{p} \cdot k - k^2) \right]
\]

\[= -k^2 (\vec{p}' - \vec{k} - m)(\vec{p} - \vec{k} - m) + 2\gamma \cdot k (\vec{p}' \cdot k - k^2)(\vec{p} - \vec{k} - m)
\]

\[+ 2\gamma \cdot k (\vec{p}' - \vec{k} + m)(\vec{p} \cdot k - k^2)\] (30)

3.3.1 Gaunt-like part

The parts involving \(k^2\) can be evaluated as the Gaunt part,

\[-(\vec{p}' - \vec{k} - m)(\vec{p} - \vec{k} - m) - 2\gamma \cdot k (\vec{p} - \vec{k} - m) - 2\gamma \cdot k (\vec{p}' - \vec{k} + m) - 2\vec{p}' \cdot k + 2\vec{p} \cdot k\]

which together with the Gaunt part Eq. (26)

\[[(3 - \epsilon)(\vec{p}' - \vec{k} - m) - 2(\gamma \cdot \vec{p}' - \gamma \cdot k)](\vec{p} - \vec{k} - m)
\]

\[+ 2(\vec{p}' - \vec{k} - m)(\gamma \cdot k - \gamma \cdot p + \gamma \cdot k) + 4(\vec{p} \cdot \vec{p}' - \vec{p} \cdot k - \vec{p}' \cdot k + k^2)\] (31)
gives (for the time being omitting $\epsilon$)

$$
NumG = 2 \left[ (\gamma^0 p_0' - \tilde{k} - m)(p - \tilde{k} + m) + (p' - \tilde{k} - m)\gamma \cdot p \right. \\
- \left. \gamma \cdot k (p' - \tilde{k} + m) - (p' - \tilde{k} + m)\gamma \cdot k - p' \cdot k + p \cdot k + 2(p \cdot p' - p \cdot k - p' \cdot k + k^2) \right]
$$

and adding the contribution due to $\epsilon$

$$
\text{Partial integration of the log term yields (where according to Eq. 27 the Gaunt-like contribution (33)}
$$

where

$$
\int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}u \left\{ \frac{[NumG]_{k \rightarrow -q}}{w} + \left[ NumG \right]_{k^\mu k^\nu \rightarrow \frac{1}{2} g^{\mu\nu}} \frac{\Gamma(\epsilon/2)}{w^{\epsilon/2}} \right\} \quad (33)
$$

and adding the contribution due to $\epsilon$ in Eq. (31). The substitution in the second part gives

$$
k^2 \rightarrow - \frac{1}{2} ; \quad \epsilon \rightarrow - \gamma^i k_i \gamma^j k_j \rightarrow \frac{1}{2} \gamma^i \gamma_i = \frac{1}{2} (3 - \epsilon)
$$

and

$$
\left[ NumG \right]_{k^\mu k^\nu \rightarrow \frac{1}{2} g^{\mu\nu}} = 2 - \epsilon
$$

$$
\frac{\Gamma(\epsilon/2)}{(w/m^2)^{\epsilon/2}} = \Delta - \ln \left( \frac{w}{m^2} + \cdots \right)
$$

The omitted $\epsilon$-dependent contribution in Eq. (32) is $- \epsilon$. This gives the Gaunt-like contribution (33)

$$
K\gamma^0 \int_0^1 \mathrm{d}u \int_0^1 x \mathrm{d}x \left[ \frac{[NumG]_{k \rightarrow q}}{w} + 2(1 - \epsilon) \left[ \Delta - \ln \left( \frac{w}{m^2} \right) \right] \right] \quad (34)
$$

Partial integration of the log term yields (where according to Eq. 27)

$$
w = w' x^2 + w'' x \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}x} = 2xw' + w''
$$

$$
\int_0^1 x \mathrm{d}x \ln \left( \frac{w}{m^2} \right) = \left[ \frac{x^2}{2} \ln \left( \frac{w}{m^2} \right) \right]_0^1 - \int_0^1 x^2 \frac{m^2}{w} \frac{\mathrm{d}w}{\mathrm{d}x} \mathrm{d}x
$$

$$
= \frac{1}{2} \ln \left( \frac{w' + w''}{m^2} \right) - \int_0^1 x^2 \frac{m^2}{2w} (2xw' + w'') \quad (35)
$$

12
The result Eq. (34) then becomes

\[ K\gamma^0 \int_0^1 du \left[ \int_0^1 dx \frac{[\text{NumG}_{k \rightarrow xq_A} + x(2xw' + w'')]}{w} + \Delta - 2 - \ln \left( \frac{w' + w''}{m^2} \right) \right] \]

(36)

The constant factor can be moved to the numerator as \(-4w = -4x^2w' - 4xw''\), which yields with \(w = x\Delta_y\)

\[ K\gamma^0 \int_0^1 du \left[ \int_0^1 dx \frac{[\text{NumG}_{k \rightarrow xq_A} - 2x^2w' - 3xw'']}{\Delta_y} + \Delta - 2 - \ln \left( \frac{w' + w''}{m^2} \right) \right] \]

(37)

Here, the numerator becomes

\[ 2R_y^0 = [\text{NumG}_{k \rightarrow xq_A} - x(2xw + 3w'')] \]

\[ = 2 \left[ (p_0' - k_0)(p_0 - k_0) - m(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 k_0 - m) + \gamma \cdot p' \gamma \cdot p \right. \]

\[ - \gamma \cdot k(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 k_0) + 2p \cdot p' - p \cdot k + p' \cdot k + k^2 \] \(k \rightarrow xq_A\)

- \(x(2xw + 3w'') \)

(38)

or with

\[ w' + w'' = [(1 - u)p + p'u]^2 - (p^2 - m^2) + (p^2 - p'^2)u = m^2 - u(1 - u)(p - p')^2 \]

(39)

\[ R_y^0 = (p_0' - xq_A^0)(p_0 - xq_A^0) - m(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 xq_A^0 - m) \]

+ \(\gamma \cdot p' \gamma \cdot p - x\gamma \cdot q_A(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 xq_A^0) \)

+ \(2p \cdot p' - xp \cdot q_A - x^2q_A^2 \)

- \(x^2[(\gamma^0 q_A^2) - q_A^2] + \frac{3x}{2} [p^2(1 - u) + p'^2u - m^2] \)

(40)

Here, \(q_A = (1 - u)p + up'\) and

\[ xp \cdot q_A + xp' \cdot q_A = x(p + p') \cdot [(1 - u)p + up'] = x(1 - u)p^2 + xup'^2 + xp \cdot p' \]

The scalar part of the last line in Eq. (40) becomes

\[-x^2[(1 - u)p_0 + up_0]^2 + \frac{3x}{2} [(1 - u)p_0^2 + up_0^2 - m^2] \]

\[= \frac{3x}{2} m^2 + x^2u(1 - u)d_0^2 + \left[ -x^2(1 - u)^2 + \frac{3x}{2}(1 - u) - x^2u(1 - u) \right]p_0^2 \]

\[+ \left[ -x^2u^2 + \frac{3x}{2}u - x^2u(1 - u) \right]p_0^2 \]

\[= \frac{3x}{2} m^2 + x^2u(1 - u)d_0^2 + \frac{1}{2} x(1 - u)(3 - 2x)p_0^2 + \frac{1}{2} xu(3 - 2x)p_0^2 \]
Here, with an expression of the type after the substitution (29) we apply the formulas (66) and (67) in Appendix B, leading to

$$
- [1 - x/2 + 2x^2u(1-u)]d^2 + [2x^2(1-u)^2 - \frac{5x}{2}(1-u) + 1 - x/2 + 2x^2u(1-u)]p^2
+ [2x^2u^2 - \frac{5x}{2}u + 1 - x/2 + 2x^2u(1-u)]p^2
\]

$$

= - [1 - x/2 + 2x^2u(1-u)]d^2 + [1 - x/2 - \frac{x}{2}(1-u)(5-4x)]p^2
+ [1 - x/2 - \frac{1}{2} xu(5-4x)]p^2
\]

This gives (times $\gamma^0$)

$$
R_y^0 = (1 - \frac{3x}{2}m^2) + (p_0 - xq_A^0)(p_0 - xq_A^0) - m(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 xq_A^0)
+ x^2u(1-u)d_0^2 + \frac{1}{2} x(1-u)(3-2x)p_0^2 + \frac{1}{2} xu(3-2x)p_0^2
+ \gamma \cdot p' \gamma \cdot p - x\gamma \cdot q_A(\gamma^0 p_0 + \gamma^0 p_0' - 2\gamma^0 xq_A^0)
- [1 - x/2 + 2x^2u(1-u)]d^2 + [1 - x/2 - \frac{x}{2}(1-u)(5-4x)]p^2
+ [1 - x/2 - \frac{1}{2} xu(5-4x)]p^2
\]

(41)

The Gaunt-like contribution then becomes

$$
K\gamma^0 \left[ \Delta + \int_0^1 dx \int_0^1 dy \frac{2R_y^0}{\Delta y} - \int_0^1 du \ln \frac{m^2 - w - w'}{m^2} \right]
\]

(42)

where $w + w'$ is given by Eq. (39). This agrees with the result of Adkins [5].

3.3.2 Non-Gaunt-like part

For the remaining "non-Gaunt-like" part of the scalar retardation (29) we apply the formulas (66) and (67) in Appendix B leading to an expression of the type after the substitution $y \rightarrow xu$

$$
2K\gamma^0 \int_0^1 dx \int_0^1 du \int_0^1 dz \int_{3/2}^1 \frac{1}{\Gamma(3)} \left[ \Gamma(2) \frac{N_z}{w^2} + \frac{N_z}{w} \right]
\]

$$

= K\gamma^0 \int_0^1 dx \int_0^1 du \int_0^1 dz \int_{3/2}^1 \frac{N_z}{w^2} + \frac{N_z}{w} \left[ \frac{N_z}{w^2} + \frac{N_z}{w} \right]
\]

(43)

Here, with $y = xu$ and $q = x[(1-u)p + up']$

$$
w = q^2 z^2 + (1-z)x q_0^2 - sz = -z^2 q^2 + q_0^2 - sz
\]

$$
s = (p^2 - m^2)x - (p^2 - p'^2)y
\]

$$
w = -x^2 z^2 \left[ (1-u)p + up' \right]^2 + x z \left[ (1-u)p_0 + up_0' \right]^2
- xz (p^2 - m^2) + z xu (p^2 - p'^2) = xz \Delta_z
\]

(44)
This gives

\[
\Delta z = -xz[(1-u)p + up']^2 + x[(1-u)p_0 + up_0']^2 \\
= (p^2 - m^2) + u(p^2 - p'^2) \\
= m^2 + xuz(1-u)d^2 - xu(1-u)d_0^2 + (1-u)(1-xz)p^2 + u(1-xz)p'^2 \\
= (1-x)(1-u)p_0^2 - u(1-x)p_0^2
\]

which agrees with Adkins with \( x \to s; u \to u; z \to x \) \[5\].

We can now express the non-Gaunt part \[43\] as

\[
K \gamma^0 \int_0^1 dx \int_0^1 du \int_0^1 dz \frac{N_z'}{xz(\Delta z)^2} + \frac{N_z}{\Delta z} \tag{46}
\]

The non-Gaunt-like part of the numerator Eq. \[30\] is

\[
\text{NumNG} = 2 \left[ \gamma \cdot k p' \cdot k (\gamma^0 k_0 - m) + \gamma \cdot k p \cdot k (\tilde{p}' - \gamma^0 k_0 + m) \right]
\]

In the first part of Eq. \[66\] and Eq. \[67\] we then make the replacements (c.f. Coulomb contribution)

\[
k^4 \to -q^4 z = q_A^4 xz; \quad k \to q_A xz; \quad k_0 \to -q_0 = q_A^0 x
\]

\[
\gamma \cdot k \gamma^0 k_0 p \cdot k \to \gamma \cdot q_A \gamma^0 q_A^0 p \cdot q_A x^2 z^2
\]

Then

\[
N_z' = \left[ \text{NumNG} \right]_{k_0 \to -q_0; k \to q_A; k_0 \to -q_0} = 2x^2 z^2 R_2^0 \tag{47}
\]

which gives

\[
R_2^0 = \gamma \cdot q_A p' \cdot q_A (\gamma^0 - m) + \gamma \cdot q_A p \cdot q_A (\tilde{p}' + m)
\]

\[
- x \gamma \cdot q_A \gamma^0 q_A^0 (p' \cdot q_A + p \cdot q_A)
\]

or

\[
R_2^0 = m \gamma \cdot q_A (p \cdot q_A - p' \cdot q_A) + m \gamma \cdot q_A d \cdot q_A
\]

\[
+ \gamma \cdot q_A (p' \cdot q_A \gamma^0 p_0 + p \cdot q_A \gamma^0 p_0)
\]

\[
+ \gamma \cdot q_A (p \cdot q_A \gamma \cdot p' - p' \cdot q_A \gamma \cdot p)
\]

\[
- x \gamma \cdot q_A \gamma^0 q_A^0 (p' \cdot q_A + p \cdot q_A) \tag{48}
\]

The third line can be expressed

\[
\gamma \cdot q_A (p \cdot q_A \gamma \cdot p' - p' \cdot q_A \gamma \cdot p)
\]

\[
= [(1-u)\gamma \cdot p + u\gamma \cdot p'] \left( \gamma \cdot p' [(1-u)p^2 + up \cdot p'] \right)
\]

\[
= C_1 + C_2 + C_3
\]
where
\[
\begin{align*}
C_1 &= \gamma \cdot p \gamma \cdot p' [(1-u)^2 p^2 + u(1-u) p \cdot p'] = \gamma \cdot p \gamma \cdot p' A \\
C_2 &= -\gamma \cdot p' \gamma \cdot p [u(1-u) p \cdot p' + u'^2 p'^2] = -\gamma \cdot p' \gamma \cdot p B \\
C_3 &= p \cdot p' [(1-u)^2 p^2 - u'^2 p \cdot p'] = p \cdot p' (A - B)
\end{align*}
\]
with \( A + B = q_A^2 \). \( C_1 \) can be reexpressed as
\[
C_1 = -(\gamma \cdot p' \gamma \cdot p + 2p \cdot p') A
\]
which leads to
\[
C_1 + C_2 + C_3 = -\gamma \cdot p' \gamma \cdot p (A + B) - p \cdot p' (A + B)
\]
or
\[
\gamma \cdot q_A (p \cdot q_A \gamma \cdot p' - p' \cdot q_A \gamma \cdot p) = -(\gamma \cdot p' \gamma \cdot p + p \cdot p') q_A^2
\]  
(49)

In the second part of Eq. (66) we make the replacement
\[
k^i k^j \rightarrow -\frac{i}{2} g^{ij}; \quad \gamma \cdot k p \cdot k \rightarrow \frac{i}{2} \gamma \cdot p
\]
and in Eq. (67)
\[
k^i k^0 k^j \rightarrow -\frac{1}{2} g^{ij} q_0
\]
or
\[
\gamma \cdot k\gamma_0 k_0 p \cdot k = \gamma^i k_i \gamma^0 k_0 p^j k_j \rightarrow -\frac{i}{2} \gamma^j \gamma^0 q_0 = +\frac{i}{2} \gamma \cdot p \gamma_0 q_0^0 x
\]
The numerator in the second part of Eq. (49) is then
\[
N_z = \left[ N_{numNG} \right]_{k^i k^j \rightarrow -\frac{i}{2} g^{ij}; k^i k^0 k^j \rightarrow -\frac{1}{2} g^{ij} q_0} = R_0^0
\]  
(50)
which gives
\[
R_z = \gamma \cdot p' (\hat{\beta} - m) + \gamma \cdot p (\hat{\beta}' + m) - \gamma \cdot p \gamma_0 q_A^0 x - \gamma \cdot p' \gamma_0 q_0^0 x
\]
\[
= m \gamma \cdot d + (\gamma \cdot p' \bar{p}_0 + \gamma \cdot p \bar{p}_0') + (\gamma \cdot p + \gamma \cdot p') \gamma_0 q_A^0 x
\]
\[
+ \gamma \cdot p \gamma \cdot p' - \gamma \cdot p' \gamma \cdot p
\]
\[
= m \gamma \cdot d + (\gamma \cdot p' \bar{p}_0 + \gamma \cdot p \bar{p}_0') + (\gamma \cdot p + \gamma \cdot p') \gamma_0 q_A^0 x
\]
\[
+ 2p \cdot p' + 2\gamma \cdot p \gamma \cdot p' 
\]  
(51)
The non-Gaunt-like contribution to the free-electron vertex function then becomes
\[
K \gamma^0 \int_0^1 dx \int_0^1 du \int_0^1 dz \sqrt{z} \left[ \frac{2xR_z^0 + R_0^0}{(\Delta z)^2} \right] (52)
\]
which agrees with Adkins [5] with
\[
x \rightarrow s; z \rightarrow x
\]
We have now verified the formulas of Adkins for dimensional regularization in the Coulomb gauge of the free-electron self energy and vertex function for \( \mu = 0 \).
Acknowledgements
The author is much obligated to his collaborators, Sten Salomonson, Daniel Hedendahl, and Johan Holmberg, for many valuable discussions.

A Relations for the alpha and gamma matrices

Taken from ref. [1, App. D3].

The gamma matrices satisfy the following anti-commutation rule:

\[ \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 2g^{\mu\nu} \]
\[ \mathcal{A} \mathcal{B} + \mathcal{B} \mathcal{A} = 2\mathcal{A}\mathcal{B} \]  

(53)

where \( \mathcal{A} \) is defined

\[ \mathcal{A} = \gamma^\mu \hat{A}_\mu \]  

(54)

This leads to

\[ \gamma^\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu \]
\[ \gamma^\mu \hat{\mathcal{A}} \gamma^\mu = -2\hat{\mathcal{A}} \]
\[ \gamma^\mu \gamma^\mu = 4 \]
\[ \gamma^\mu \gamma^\nu \gamma^\mu = -2\gamma^\mu \]
\[ \gamma^\mu \hat{\mathcal{A}} \gamma^\mu = -2\hat{\mathcal{A}} \]
\[ \gamma^0 \gamma^0 = \gamma^0 \gamma^0 = 1 \]
\[ \gamma^\sigma \gamma^0 = \gamma^0 \gamma^\sigma \]
\[ \hat{\mathcal{A}} \gamma^0 = \gamma^0 \hat{\mathcal{A}} \]
\[ \gamma^0 \gamma^\sigma \gamma^0 = \bar{\gamma}^\sigma \]
\[ \gamma^0 \hat{\mathcal{A}} \gamma^0 = \bar{\mathcal{A}} \]
\[ \gamma^0 \gamma^\sigma \gamma^0 \gamma^0 = \bar{\gamma}^\sigma \bar{\gamma} \gamma^0 \]
\[ \gamma^0 \hat{\mathcal{A}} \hat{\mathcal{B}} \gamma^0 = \bar{\mathcal{A}} \bar{\mathcal{B}} \]
\[ \gamma^0 \hat{\mathcal{A}} \hat{\mathcal{B}} \gamma^0 = \bar{\mathcal{A}} \bar{\mathcal{B}} \]
\[ \gamma^0 \hat{\mathcal{A}} \hat{\mathcal{B}} \gamma^0 = \bar{\mathcal{A}} \bar{\mathcal{B}} \gamma^0 \]

(55)

where \( \hat{\mathcal{A}} = \gamma^0 A_0 - \gamma^i A_i = \gamma^0 + \gamma \cdot dA \) and

\[ \tilde{\gamma}^\sigma = \begin{cases} 
\gamma^\sigma (\sigma = 1) \\
-\gamma^\sigma (\sigma = 1, 2, 3)
\end{cases} \]

\( ^5 \)Note misprint in the first formula of Eq. (D.58).
With the number of dimensions being equal to $4 - \epsilon$, to be used in dimensional regularization, the relations become

\[
\begin{align*}
\gamma^\mu \gamma_\mu &= 4 - \epsilon \\
\gamma^\mu \gamma^\sigma \gamma_\mu &= -(2 - \epsilon) \gamma^\sigma \\
\gamma^\mu \tilde{A} \gamma_\mu &= -(2 - \epsilon) \tilde{A} \\
\gamma^\mu \gamma^\sigma \gamma^\tau \gamma_\mu &= 4 \gamma^\sigma \gamma^\tau - \epsilon \gamma^\sigma \gamma^\tau \\
\gamma^\mu \tilde{A} \tilde{B} \gamma_\mu &= 4 \tilde{A} \tilde{B} - \epsilon \tilde{A} \tilde{B} \\
\gamma^\mu \tilde{A} \tilde{B} \gamma_\mu &= -2 \gamma^\tau \gamma^\sigma \gamma^\beta + \epsilon \gamma^\beta \gamma^\sigma \gamma^\tau \\
\gamma_\tau \gamma^\tau &= 3 - \epsilon \\
\gamma_i \gamma^\tau \gamma^\rho &= -(2 - \epsilon) \gamma^\tau - \tilde{\gamma}^\tau \\
\gamma_i \tilde{A} \gamma^\tau &= -(2 - \epsilon) \tilde{A} - \tilde{\tilde{A}} \\
\gamma_i \gamma^\tau \gamma^\rho \gamma^\sigma &= 4 \gamma^\rho \gamma^\sigma - \tilde{\gamma}^\rho \tilde{\gamma}^\sigma - \epsilon \gamma^\rho \gamma^\sigma \\
\gamma_i \tilde{A} \tilde{B} \gamma^\tau &= 4 \tilde{A} \tilde{B} - \tilde{\tilde{A}} \tilde{B} - \epsilon \tilde{A} \tilde{B} \\
\gamma_i \gamma^\tau \gamma^\beta \gamma^\gamma &= -2 \gamma^\gamma \gamma^\beta - \tilde{\gamma}^\gamma \tilde{\gamma}^\beta - \epsilon \gamma^\beta \gamma^\gamma \\
\gamma_i \tilde{A} \tilde{B} \tilde{C} \gamma^\gamma &= -2 \tilde{C} \tilde{B} \tilde{A} - \tilde{\tilde{A}} \tilde{B} \tilde{C} + \epsilon \tilde{A} \tilde{B} \tilde{C} \\
\end{align*}
\]

(56)

**B Formulas for dimensional regularization**

Taken from ref. [1, App. G2].

Following the book by Peskin and Schroeder [7], we can by means of Wick rotation evaluate the integral

\[
\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - \Delta)^m} = i(-1)^m \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2_E + \Delta)^m} = i(-1)^m \int \frac{d\Omega_D}{(2\pi)^D} \int_0^\infty \frac{dE_0}{(E_0^2 + \Delta)^m} \int_0^{l^0_E} \frac{d^{D-1} \tilde{l}_E}{(l^2_E + \Delta)^m}
\]

We have here made the replacements $l^0 = il^0_E$ and $l = l_E$ and rotated the integration contour of $l_E$ 90°, which with the positions of the poles should give the same result. The integration over $d^D l_E$ is separated into an integration over the D-dimensional sphere $\Omega_D$ and the linear integration over the component $l_0^0_E$. This corresponds in three dimensions to the integration over the two-dimensional angular coordinates and the radial coordinate (see below).

\[
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + s + i\eta)^n} = i(-1)^n \frac{\Gamma(n - D/2)}{4\pi^{D/2} \Gamma(n)} \frac{1}{s^{n-D/2}} \]

(57)

\[
\int d^4 k \frac{k^\mu}{(k^2 + s + i\eta)^n} = 0 \]

(58)
\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^{\nu}}{(k^2 + s + i\eta)^n} = \frac{i(-1)^n \Gamma(n-D/2-1)}{4\pi^{D/2} \Gamma(n)} \frac{1}{s^{n-D/2-1}} \quad (59)
\]

**Covariant gauge**

Compared to Adkins [4] Eqs (A1a), (A3), (A5a):

\[
p \rightarrow -q; \ M^2 \rightarrow -s; \ \omega \rightarrow D/2; \ \alpha \rightarrow n; \ \xi \rightarrow n; \ Q = p \rightarrow -q; \ A_{\mu\nu} \rightarrow g_{\mu\nu};
\]

\[
\Delta \rightarrow w = q^2 - s
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + 2kq + s + i\eta)^n} = \frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n)} \frac{1}{\Gamma(n-D/2)} \frac{\Gamma(n-D/2)}{u^{n-D/2}} \quad (60)
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^n} = -\frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n)} \frac{1}{\Gamma(n-D/2)} \Gamma(n-D/2) \frac{\Gamma(n-D/2)}{u^{n-D/2}}
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^{\nu}}{(k^2 + 2kq + s + i\eta)^n} = \frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n)} \Gamma(n-D/2) \frac{\Gamma(n-D/2)}{u^{n-1-D/2}} \quad (61)
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^n} = -\frac{g^{\mu\nu}}{2} \frac{\Gamma(n-1-D/2)}{u^{n-1-D/2}} \quad (62)
\]

**Non-covariant gauge**

Compared to Adkins [4] Eqs (A1b), (A4), (A5b):

\[
p \rightarrow -q; \ M^2 \rightarrow -s; \ \omega \rightarrow D/2; \ \alpha \rightarrow n; \ \beta \rightarrow 1; \ \xi \rightarrow n + 1; \ k^2 \rightarrow -k^2;
\]

\[
Q = pq \rightarrow -qy; \ A_{\mu\nu} \rightarrow g_{\mu\nu} + \delta_{\mu,0} \delta_{\nu,0} \frac{1-y}{y}; \ (AQ)_\mu \rightarrow -q^\mu y - \delta_{\mu0} (1-y) q_0
\]

\[
\Delta \rightarrow w = q^2 y^2 + (1-y)yq_0^2 - sy + \lambda^2 (1-y) = -q^2 y^2 + yq_0^2 - sy + \lambda^2 (1-y)
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + 2kq + s + i\eta)^n} \frac{1}{k^2 - \lambda^2} = \frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n+1)} \frac{1}{\Gamma(n-D/2)} \frac{\Gamma(n+1-D/2)}{u^{n+1-D/2}} \quad (63)
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^n} \frac{1}{k^2 - \lambda^2} = -\frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n)} \frac{1}{\Gamma(n)} \times \frac{\Gamma(n+1-D/2)}{u^{n+1-D/2}} \quad (64)
\]

\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^n} \frac{1}{k^2 - \lambda^2} = -\frac{i(-1)^n}{(4\pi)^{D/2} \Gamma(n)} \frac{1}{\Gamma(n)} \times \frac{\Gamma(n+1-D/2)}{u^{n+1-D/2}} \quad (65)
\]

\[
k \rightarrow -\gamma^\mu y - \gamma^0 q_0 (1-y) = \gamma \cdot q y - \gamma^0 q_0
\]
\[
\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k'^\nu}{(k^2 + 2kq + s + i\eta)^n} \frac{1}{k^2 - \lambda^2} = \frac{i(-1)^n}{(4\pi)^{D/2}} \frac{1}{\Gamma(n)}
\]
\[
\times \int_0^1 dy \ y^{n-1/2} \left[ \left\{ [q^\mu y + \delta_{\mu,0} q_0(1-y)] [q'^\nu y + \delta_{\nu,0} q_0(1-y)] \right\} \frac{\Gamma(n + 1 - D/2)}{w^{n+1-D/2}} \right.
\]
\[
- \frac{1}{2} \left[ \left\{ [q^\mu y + \delta_{\mu,0} q_0(1-y)/y] \right\} \frac{\Gamma(n - D/2)}{w^{n-D/2}} \right] \]  
\]  
\]
\[
\left[ \left\{ [q^\mu y + \delta_{\mu,0} q_0(1-y)/y] \right\} \frac{\Gamma(n - D/2)}{w^{n-D/2}} \right] \]  
\]
\[
\left(66\right)
\]

\[\text{C} \quad \text{Feynman integrals}\]

Taken from ref. [1]. App. J1]

In this Appendix we shall give some integrals, which simplify many QED calculations considerably (see the books of Mandl and Shaw [8, Ch. 10] and Sakurai [9, App. E], and we shall start by deriving some formulas due to Feynman.

We start with the identity

\[
\frac{1}{ab} = \frac{1}{b-a} \int_a^b \frac{dt}{t^2}
\]  
\]
\[
(68)
\]

With the substitution \( t = b + (a - b)x \) this becomes

\[
\frac{1}{ab} = \int_0^1 \frac{dx}{[b + (a - b)x]^2} = \int_0^1 \frac{dx}{[a + (b - a)x]^2}
\]  
\]
\[
(69)
\]

Differentiation with respect to \( a \), yields

\[
\frac{1}{a^2 b} = 2 \int_0^1 \frac{xdx}{[b + (a - b)x]^3}
\]  
\]
\[
(70)
\]

Similarly, we have

\[
\frac{1}{abc} = 2 \int_0^1 \frac{dx}{[a + (b - a)x + (c - b)y]^3}
\]  
\]
\[
= 2 \int_0^1 \frac{dx}{\left[ a + (b - a)x + (c - a)y \right]^3}
\]  
\]
\[
(71)
\]
Next we consider the integral
\[ \int \frac{1}{(k^2 + s + i\eta)^3} = 4\pi \int |k|^2 |d|k| \int_{-\infty}^{\infty} \frac{dk_0}{(k^2 + s + i\eta)^3} \]

The second integral can be evaluated by starting with
\[ \int_{-\infty}^{\infty} \frac{dk_0}{k_0^2 - |k|^2 + s + i\eta} = \frac{i\pi}{\sqrt{|k|^2 - s}} \]
evaluated by residue calculus, and differentiating twice with respect to \(s\). The integral then becomes
\[ \int d^4k \frac{1}{(k^2 + s + i\eta)^3} = \frac{3i\pi^2}{2} \int \frac{|k|^2 |d|k|}{(|k|^2 + s)^{5/2}} = \frac{i\pi^2}{2s} \quad (72) \]

The second integral can be evaluated from the identity
\[ \frac{x^2}{(x^2 + s)^{5/2}} = \frac{1}{(x^2 + s)^{3/2}} - \frac{s}{(x^2 + s)^{5/2}} \]
and differentiating the integral
\[ \int \frac{dx}{\sqrt{x^2 + s}} = \ln(x + \sqrt{x^2 + s}) \]
yielding
\[ \int \frac{x^2}{(x^2 + s)^{5/2}} = \frac{1}{3s} \]

For symmetry reason we find
\[ \int d^4k \frac{k_\mu}{(k^2 + s + i\eta)^3} = 0 \quad (73) \]
Differentiating this relation with respect to \(k_\nu\), leads to
\[ \int d^4k \frac{k_\mu k_\nu}{(k^2 + s + i\eta)^3} = \frac{g^{\mu\nu}}{3} \int d^4k \frac{1}{(k^2 + s + i\eta)^3} = \frac{i\pi^2 g^{\mu\nu}}{6s} \quad (74) \]
using the relation \[\text{[1], Eq. (A4)}. \]

By making the replacements
\[ k \Rightarrow k + q \quad \text{and} \quad s \Rightarrow s - q^2 \]
the integrals (72) and (73) lead to
\[ \int d^4k \frac{1}{(k^2 + 2kq + s + i\eta)^3} = \frac{i\pi^2}{2(s - q^2)} \quad (75) \]
\[ \int d^4k \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^3} = - \int d^4k \frac{q^\nu}{(k^2 + 2kq + s + i\eta)^3} = -\frac{i\pi^2 q^\mu}{2(s - q^2)^2} \tag{76} \]

Differentiating the last relation with respect to \( q_\nu \), leads to
\[ \int d^4k \frac{k^\mu k^{\nu}}{(k^2 + 2kq + s + i\eta)^4} = \frac{i\pi^2}{12} \frac{2g^{\mu\nu} + q^{\mu}q^{\nu}}{(s - q^2)^2} \tag{77} \]

Differentiating the relation (75) with respect to \( s \), yields
\[ \int d^4k \frac{1}{(k^2 + 2kq + s + i\eta)^n} = \frac{i\pi^2}{6(s - q^2)^2} \tag{78} \]

which can be generalized to arbitrary integer powers \( \geq 3 \)
\[ \int d^4k \frac{1}{(k^2 + 2kq + s + i\eta)^n} = \frac{i\pi^2}{(n - 1)! (s - q^2)^{n-2}} \tag{79} \]

This can also be extended to non-integral powers
\[ \int d^4k \frac{k^\mu}{(k^2 + 2kq + s + i\eta)^n} = -\frac{i\pi^2}{\Gamma(n)} \frac{q^\mu}{(s - q^2)^{n-2}} \tag{80} \]

and similarly
\[ \int q^4k \frac{k^\mu k^{\nu}}{(k^2 + 2kq + s + i\eta)^n} = \frac{i\pi^2}{2\Gamma(n)} \frac{[2(n - 3) q^{\mu}q^{\nu} + g^{\mu\nu}]}{(s - q^2)^{n-2}} \tag{82} \]

**References**

[1] I. Lindgren, *Relativistic Many-Body Theory: A New Field-Theoretical Approach* (Springer-Verlag, New York, 2011).

[2] I. Lindgren and J. Morrison, *Atomic Many-Body Theory* (Second edition, Springer-Verlag, Berlin, 1986, reprinted 2009).

[3] D. Hedendahl and J. Holmberg, (to be submitted to Phys. Rev. Lett.) .

[4] G. Adkins, Phys. Rev. D 27, 1814 (1983).

[5] G. Adkins, Phys. Rev. D 34, 2489 (1986).

[6] J. Holmberg and D. Hedendahl, (to appear in ArXiv:quant.ph) .

[7] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publ. Co., Reading, Mass., 1995).

[8] F. Mandl and G. Shaw, *Quantum Field Theory* (John Wiley and Sons, New York, 1986).

[9] J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley Publ. Co., Reading, Mass., 1967).