Quantum noise in current biased Josephson junction

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Abstract

Quantum fluctuations in a current biased Josephson junction, described in terms of the RCSJ-model, are considered. The fluctuations of the voltage and phase across the junction are assumed to be initiated by equilibrium current fluctuations in the shunting resistor. This corresponds to low enough temperatures, when fluctuations of the normal current in the junction itself can be neglected. We used the quantum Langevin equation in terms of random variables related to the limit cycle of the nonlinear Josephson oscillator. This allows to go beyond the perturbation theory and calculate the widths of the Josephson radiation lines.

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INTRODUCTION

Fluctuations in an equilibrium system are considered to be “classical” if the fluctuation frequency is lower than the system temperature, $\hbar \omega \ll T$. For higher frequencies the fluctuations are “quantum” and contain a contribution called zero-point fluctuations (ZPFs). Due to the ZPFs there are fluctuations even at zero temperature, when the system is in its ground state [1].

The intriguing question: can these zero-point fluctuations be detected, has recently attracted much attention. The answer depends on what is the detector and what means detection. Measuring Casimir forces between two bodies, both at $T = 0$, is definitely a way to detect ZPFs of the electromagnetic field surrounding the bodies. On the other hand there is no radiation from a body at $T = 0$ and an antenna placed nearby will receive no signal, which means that current ZPFs in the body do not radiate.

The situation is even more complicated if the system is not in equilibrium. Probably the simplest example is a gas of excited atoms in a cavity with no radiation in it (the temperature of the radiation is zero, while the temperature of the gas is not). When de-excited, atoms emit spontaneously cavity photons, which is in fact interaction with ZPFs of the cavity field. The de-excitation of the atom can be considered as a signal of the presence of the ZPFs in the cavity.

This paper is stimulated by experiments with nonequilibrium pumped Josephson junction systems [2,3,4], where ZPFs play an important role at low temperatures. In [2] ZPFs were detected in a dc-current biased Josephson junction, measuring the low frequency noise of the voltage $V(t)$ across the junction. The voltage noise was generated by intrinsic current fluctuations $\delta I(t)$ in a shunting resistor $R$. The frequency of the measured voltage noise (of the order 100 kHz) is much below the temperature $T$ of the resistor (1.6 K and 4.2 K) but quantum phenomena are important because this low frequency is mixed due to nonlinear effects with the Josephson frequency $\omega_J$ (between 10 and 500 GHz), which is comparable or even higher than the resistor temperature.
The theory of quantum fluctuations in a dc-current biased Josephson junction was given in [5], based on the Langevin equation for the phase difference across the junction \( \varphi(t) \), where the random forces are intrinsic current fluctuations \( \delta I(t) \) in the resistor. We decided to revisit the problem because of following reasons. In [5] the authors used the theory of classical fluctuations developed in [6], simply replacing in the classical Langevin equation the classical spectral density of the random forces, \((\delta I^2)_{\omega} = T/\pi R\), by its quantum equivalent, \((\bar{h}\omega/2\pi R) \coth(\bar{h}\omega/2T)\). This quantum spectral density, containing ZPFs, is generated by the symmetrized correlator \( \langle (1/2)[\delta I(t')\delta I(t) + \delta I(t)\delta I(t')] \rangle \), which is, however, not relevant in the problem considered. This is because using the symmetrized correlator for the random forces in the Langevin equation imply that all the correlators calculated from this equation are also symmetrized. On the other hand, the symmetrized correlator of the voltage \( V(t) \) do not represent the measured voltage noise.

This problem was addressed in [7,8,9]. It was shown that if the device, measuring current noise, is a "resonator" at zero temperature, with resonant frequency \( \omega \), the measured signal is proportional to the Fourier component \( S(\omega) \) of the nonsymmetrized current correlator \( \langle j(0)j(t) \rangle \), with \( \omega > 0 \) and the dc component in the current \( j(t) \) being subtracted. \( S(\omega) \) is real, but not symmetric in \( \omega \), and hence the condition \( \omega > 0 \) is essential. If the current carrying system is in equilibrium at zero temperature, one finds \( S(\omega) = 0 \) for \( \omega > 0 \), which means there is no signal created by ZPFs. It is because the signal \( S(\omega) \) is in fact proportional to the power spectral density spontaneously radiated by the system. The situation is similar in quantum optics [10], where ZPFs can create radiation only in nonequilibrium systems, like parametric amplifiers.

The second reason to revisit the theory is as follows. In [5] and [6] the fluctuations were calculated using perturbation theory for the phase \( \varphi(t) \), i.e. assuming that \( (\delta \varphi^2)_{\omega} \) is small and proportional to \( (\delta I^2)_{\omega} \). This theory diverges near \( \omega = 0 \) and the harmonics of Josephson frequency \( \omega = k\omega_J \), \( k = 1, 2, \ldots \), which are exactly the points of interest.

To account for all the above mentioned circumstances we use the quantum Langevin equation for \( \varphi(t) \), formulating it according to the general recipe given in [11]. With no
random forces the solution of the Langevin equation for the phase, $\varphi(t) = f(\omega_J t + \alpha_0)$, contains an arbitrary initial phase $\alpha_0$, since the equation for $\varphi(t)$ is invariant with respect to time shift. Due to random forces this initial phase acquires a fluctuating contribution $\delta \alpha(t)$, and the Langevin equation can be rewritten for $\delta \alpha(t)$. Using this approach we avoid the perturbation theory for $\varphi(t)$, which allows to calculate the shape of the radiation lines at the Josephson frequency harmonics $k\omega_J$.

The width of the Josephson emission lines was considered in the well known papers [12], however a white spectrum was assumed for the random forces. This assumption is employed in almost all papers dealing with noise influence on Shapiro steps [13], on the impedance of microwave driven junctions [14,15] and other phenomena in Josephson junctions (more references are given in [16,17]). There are only few exceptions. In [18,19] the white noise with correlator $\sim \delta(t)$ was replaced by a dichotomous telegraph noise with an exponential correlator $\sim \exp(-\gamma t)$. In [20] the sum of these two was considered. Full consideration for the quantum correlations, considering the rounding of the $I - V$- curve, was given in [21].

**I. QUANTUM LANGEVIN EQUATION**

The equations describing a current biased Josephson junction in the RCSJ-model are as follows [22,16]

$$I_c \sin \varphi(t) + \frac{V(t)}{R} + C \frac{dV(t)}{dt} = I_p + \delta I(t), \quad \frac{d\varphi(t)}{dt} = \frac{2e}{\hbar} V(t).$$

(1)

Here $I_c$ and $I_p$ are the critical and bias currents, $R$ and $C$ are the resistor and capacitor of the model, $\varphi(t)$ and $V(t)$ are the phase and voltage difference across the junction and $\delta I(t)$ are the intrinsic current fluctuations. The phase and voltage differences contain fluctuations generated by $\delta I(t)$ and in what follows we write $\varphi(t) = \varphi(t) + \delta \varphi(t)$ and $V(t) = V(t) + \delta V(t)$, separating explicitly the fluctuating contributions.

We neglect the quasiparticle current in the junction, assuming low enough temperatures, which means in terms of the model that $R$ is just the external shunting resistor. With this
assumption $\delta I(t)$ are the equilibrium current fluctuations in the resistor $R$, the only intrinsic fluctuation source in the RCSJ-model [6,23]. This simplifies the problem enormously, since the current fluctuations in the junction are state dependent (they depend on $\varphi(t)$) and hence are non-stationary for time-dependent voltage $V(t)$ [23,24,25].

We eliminate from Eqs.(1) the potential $V$, multiply it by $\hbar/2e$ and introduce $E_c = (\hbar/2e)I_c$. As a result we have

$$ C\left(\frac{\hbar}{2e}\right)^2 \frac{d^2\varphi}{dt^2} + \frac{1}{R} \left(\frac{\hbar}{2e}\right)^2 \frac{d\varphi}{dt} + E_c \sin \varphi - \frac{\hbar}{2e} I_p = \frac{\hbar}{2e} \delta I. \quad (2) $$

This equation is a standard form of a Langevin equation for the phase $\varphi$ as the particle "coordinate" and a "potential" having the dimension of energy, the current fluctuations being the random forces. Comparing with the quantum Langevin equation in [11] we find the correlator

$$ \langle \delta I(t')\delta I(t) \rangle = \frac{1}{\pi R} \int_0^\infty d\omega \hbar \omega \left[ N(\omega) e^{-i\omega(t-t')} + (N(\omega) + 1) e^{i\omega(t-t')} \right] \quad (3) $$

and the commutator

$$ [\delta I(t'), \delta I(t)] = \frac{2i}{\pi R} \int_0^\infty d\omega \hbar \omega \sin \omega(t-t') = -\frac{2i\hbar}{R} \frac{d}{dt} \delta(t-t'), \quad (4) $$

where $N(\omega) = [\exp(\hbar\omega/T) - 1]^{-1}$. It is important that $\delta I(t)$ is a Gaussian process (in the quantum case one has to preserve the order of operators) and the commutator Eq.(1) is a $c$-number. (When the thermal bath responsible for the random forces is chosen to be a collection of harmonic oscillators [11] the Gaussian properties of the random forces follow from the Gaussian properties of a harmonic oscillator in equilibrium. However, probably, this is a more general property, since many degrees of freedom of the thermal bath contribute to the random force.)

We will use the following notations for Fourier components and spectra of correlators

$$ A(t) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} A_\omega, \quad (A^\dagger)_\omega = (A_\omega)^\dagger = A^\dagger_\omega, \quad (5) $$

$$ \langle A(t')^\dagger A(t) \rangle = \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} (A^2)_\omega, \quad \langle A^\dagger_\omega A_\omega \rangle = \delta(\omega - \omega')(A^2)_\omega, \quad (6) $$
where † means ”hermitian conjugate”. In these notations the spectrum of equilibrium current fluctuations is

\[
(\delta I^2)_{\omega} = \frac{\hbar \omega}{\pi R} N(\omega).
\]  

(7)

The width of the spectrum is \( T/\hbar \) and the corresponding correlation time is \( \hbar/T \). At \( T = 0 \) this spectrum has no Fourier components with \( \omega > 0 \), which means that energy can not be extracted from the thermal bath. The spectral density of the symmetrized correlator, which contains ZPFs,

\[
\frac{1}{2} \left[ (\delta I^2)_{\omega} + (\delta I^2)_{-\omega} \right] = \frac{\hbar \omega}{\pi R} \left( N(\omega) + \frac{1}{2} \right) = \frac{\hbar \omega}{2\pi R} \coth \left( \frac{\hbar \omega}{2T} \right)
\]  

(8)

does not have this property. In the classical case, when \( T \gg \hbar \omega \) and \( |t - t'| \gg \hbar/T \), the random forces have a ”white” spectrum, i.e.

\[
(\delta I^2)_{\omega} = \frac{T}{\pi R}, \quad \langle \delta I(t') \delta I(t) \rangle = \frac{2T}{R} \delta(t - t').
\]  

(9)

II. GENERAL CONSIDERATIONS

We recall first the properties of the junction when one neglects fluctuations. The solution of Eqs.(4) with \( \delta I = 0 \) differs qualitatively in cases when the bias current \( I_p \) is smaller or greater than the critical current \( I_c \). We will consider only the case of the non-stationary Josephson effect, when \( I_p > I_c \). In this case the voltage \( \overline{V}(t) \) and the phase \( \overline{\varphi}(t) \) are periodic functions of time (the last one modulo \( 2\pi \)), the period defining the Josephson frequency \( \omega_J \). These functions can be presented as follows

\[
\overline{\varphi}(t) = f(x), \quad \overline{V}(t) = I_c R \, g(x), \quad x = \omega_J t + \alpha_0,
\]  

(10)

where \( f(x) \) and \( g(x) \) are non-dimensional functions

\[
g(x) = \sum_k g_k e^{-ikx}, \quad f(x) = \sum_k f_k e^{-ikx} + x,
\]  

(11)
α₀ is an arbitrary initial phase, appearing because Eqs.(11) are invariant with respect to time shift. Eqs.(10) define the limit cycle of the nonlinear Josephson oscillator, i.e. the trajectory of the system in the phase space \((\varphi, V)\). The time-average voltage across the junction is 
\[ V_0 = I_c R g_0. \]

As was explained in the Introduction the measured voltage noise (proportional to the radiated energy) is
\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle v(0) v(t) \rangle \tag{12}
\]
with \(v(t) = V(t) - V_0\) and \(\omega > 0\). When fluctuations are neglected, \(v(t)\) is a non-random periodic function. Considering the initial phase \(\alpha_0\) to be random within the interval \((0, 2\pi)\), we convert the periodic function \(v(t)\) into its random equivalent, i.e. a stationary random process with an equidistant discrete spectrum. Using
\[
\exp [-i(k - k')\alpha_0] = \delta_{k,k'}, \tag{13}
\]
we average \(v(t')v(t)\) over \(\alpha_0\) and obtain the relevant correlator
\[
\langle v(t')v(t) \rangle = (I_c R)^2 \sum_{k \neq 0} |g_k|^2 \exp[-ik\omega J(t - t')], \tag{14}
\]
and the noise spectrum
\[
(v^2)_\omega \equiv S(\omega) = (I_c R)^2 \sum_{k \neq 0} |g_k|^2 \delta(\omega - k\omega J), \quad \omega > 0. \tag{15}
\]

We turn now to the effect of fluctuations, when the random forces \(\delta I(t)\) tend to move the system away from the limit cycle and destroy the periodicity of the system dynamics. We assume the random forces to be weak. If the system is shifted by the random force perpendicular to the cycle, a ”restoring force” exist, which tends to bring the system back to the cycle. However no such force exist when the system is shifted along the cycle, since this corresponds to a change of arbitrary initial phase \(\alpha_0\). To account for these two effects we look for a fluctuating solution of Eqs.(11) in the form
\[
\varphi(t) = f(\omega J t + \alpha_0 + \delta\alpha(t)), \quad V(t) = I_c R [g(\omega J t + \alpha_0 + \delta\alpha(t)) + \delta\xi(t)], \tag{16}
\]
with two unknown random functions $\delta \alpha(t)$ and $\delta \xi(t)$ (which are operators in the quantum case). The last one describes the shift of the system perpendicular to the limit cycle and since in this case a restoring force acts against the random forces, $\delta \xi(t)$ is small and proportional to $\delta I(t)$; it contains the same frequencies as $\delta I(t)$. Contrary, $\delta \alpha(t)$ can grow unlimitedly even for small $\delta I(t)$, if the random force contains the resonance frequency $\omega_J$ or its harmonics. Hence, $\delta \alpha(t) = \overline{\delta \alpha} + \tilde{\delta \alpha}(t)$, where $\tilde{\delta \alpha}(t)$ is proportional to $\delta I(t)$ and contains the frequencies of the random force, while $\overline{\delta \alpha}(t)$ is secular and slow, i.e. contains only low frequencies proportional to the amplitude of the random force. The function $\delta \alpha(t)$ describes the so called diffusion of the initial phase and its derivative describes Josephson frequency fluctuations, which are responsible for the broadening of the Josephson radiation lines,

$$\delta \omega_J(t) = \frac{d}{dt} \delta \alpha(t) \equiv \dot{\delta \alpha}(t), \quad \delta \alpha(t) = \int_{-\infty}^{t} dt' \dot{\delta \alpha}(t'). \quad (17)$$

In terms of the new variables the voltage noise is

$$\langle v(t') v(t) \rangle = (I_c R)^2 \times$$

$$\left\langle \left[ \sum_{k' \neq 0} g_{k'}^* \exp[ik'(\omega_J t' + \alpha_0 + \delta \alpha(t'))] + \delta \xi(t') \right] \left[ \sum_{k \neq 0} g_k \exp[-ik(\omega_J t + \alpha_0 + \delta \alpha(t))] + \delta \xi(t) \right] \right\rangle.$$

As we will see later $\{\delta \alpha(t), \delta \xi(t)\}$ is a Gaussian process and to find the voltage noise one has to calculate the correlators $\langle \delta \alpha(t') \delta \alpha(t) \rangle$, $\langle \delta \alpha(t') \delta \xi(t) \rangle$ and $\langle \delta \xi(t') \delta \xi(t) \rangle$.

The effect of the random forces on the noise spectra Eq. (12) is twofold: the monochromatic Josephson lines in Eq. (13) acquire a width, and a background for all frequencies $\omega > 0$ appears. Weak random forces means that the width of the Josephson lines is small compared to the distance between them and that the background contribute to the integrated noise less than the lines do.

Of special interest is the background $S(0)$ at low frequencies $\omega \ll \omega_J$. To calculate the background noise at frequencies far enough from the resonant ones $\omega = k\omega_J, k = 1, 2, ...$ one can employ perturbation theory assuming $\delta \alpha(t)$ to be small, while to find the width of these lines a more sophisticated approach has to be used.
III. LANGEVIN EQUATIONS FOR AN OVERDAMPED JUNCTION

Substituting Eqs. (10) into Eqs. (1) with \( \delta I = 0 \) we find the relations between the functions \( f \) and \( g \) as follows

\[
g(x) + \sin f(x) + \omega_J \tau g'(x) = p, \quad \omega_J f'(x) = \omega_0 g(x),
\]

(19)

where \( p = I_p/I_c, \omega_0 = (2e/h)I_cR \), and \( \tau = RC \). To obtain the Langevin equations for \( \delta \xi(t) \) and \( \delta \alpha(t) \) we substitute Eqs. (16) into Eqs. (1) and use Eqs. (19). As a result we find

\[
\tau \delta \dot{\xi}(t) + \delta \xi(t) + \tau g'(x) \delta \dot{\alpha}(t) = \delta I(t)/I_c, \quad f'(x) \delta \dot{\alpha}(t) = \omega_0 \delta \xi(t)
\]

(20)

with \( x = \omega_J t + \alpha_0 + \delta \alpha(t) \). (Some caution is needed in the quantum case since \( \delta \alpha \) and \( \delta \dot{\alpha} \) do not commute. However the main (secular) part of \( \delta \alpha \) is a slow function of \( t \) and hence \( \delta \alpha(t) \) and \( \delta \dot{\alpha}(t') \) effectively commute when \( t' = t \).)

In what follows we will consider the overdamped Josephson junction, assuming the capacitance \( C \) to be small, i.e. \( \omega_0 \tau \ll 1 \). In this case we can skip in the first of Eqs. (20) the term with \( \delta \dot{\alpha}(t) \), since, according to the second of these equations, it is a small correction to the term \( \delta \xi(t) \). We can also use the functions \( f(x) \) and \( g(x) \) as for a junction with \( C = 0 \), in which case \( \omega_0 \) the Josephson frequency is \( \omega_J = \omega_0 \sigma \) with \( \sigma = (p^2 - 1)^{1/2} \) and

\[
g(x) = (p^2 - 1)/(p + \sin x); \quad g_k = (-i)^k \sigma(p - \sigma)^k, \quad k \geq 0; \quad g_{-k} = g_k^*.
\]

(21)

The functions \( f(x) \) and \( g(x) \) for an overdamped junction are related as follows

\[
\sigma f'(x) = g(x), \quad g(x) + \sin f(x) - p = 0.
\]

(22)

As a result we have the following Langevin equations

\[
\tau \delta \dot{\xi}(t) + \delta \xi(t) = \frac{\delta I(t)}{I_c}, \quad \delta \dot{\alpha}(t) = \frac{\omega_0}{\sigma} [p + \sin(\omega_J t + \alpha_0 + \delta \alpha(t))] \delta \xi(t).
\]

(23)

From the first of these equations we find immediately

\[
\delta \xi_\omega = \frac{1}{I_c} \frac{\delta I_\omega}{1 - i\omega \tau} = \frac{\delta J_\omega}{I_c}, \quad (\delta J^2)_\omega = \frac{(\delta I_\omega)^2}{1 + (\omega \tau)^2},
\]

(24)
where $\delta J$ are current fluctuations shunted by the capacitor. As mentioned already $\delta \alpha(t) = \overline{\delta \alpha}(t) + \tilde{\delta} \alpha(t)$, where $\overline{\delta \alpha}(t)$ is fast oscillating but small, while $\overline{\delta \alpha}(t)$ is slow but generally not small. When performing the Fourier transform of $\delta \dot{\alpha}(t)$ in the second of Eqs.\(23\) one can neglect in the right hand side the small $\tilde{\delta} \alpha(t)$ and replace the slow $\overline{\delta \alpha}(t)$ by a constant, absorbed in $\alpha_0$. With this approximation

$$
\delta \dot{\alpha}_\omega = \frac{\omega_0}{I_c} \left\{ \frac{p}{\sigma} \delta J_\omega + \frac{1}{2i\sigma} \left[ \exp[i\alpha_0] \delta J_{\omega+\omega_J} - \exp[-i\alpha_0] \delta J_{\omega-\omega_J} \right] \right\}.
$$

(25)

If one takes into account the slow variation of $\overline{\delta \alpha}(t)$, then $\delta J_\omega$ is replaced by its convolution with the narrow spectra of $\overline{\delta \alpha}(t)$, but this effect can be neglected for weak enough random forces. In the time domain Eq.\(24\) corresponds to

$$
\delta \dot{\alpha}(t) = \frac{\omega_0}{I_c} \left[ p + \sin(\omega_J t + \alpha_0) \right] \frac{\delta J(t)}{I_c}.
$$

(26)

To calculate $(\delta \dot{\alpha}^2)_\omega$ we average $\langle \delta \dot{\alpha}_{\omega}^\dagger \delta \dot{\alpha}_{\omega} \rangle$ over $\alpha_0$ using Eq.\(13\) and obtain

$$
(\delta \dot{\alpha}^2)_\omega = \left( \frac{\omega_0}{I_c} \right)^2 \left\{ \left( \frac{p}{\sigma} \right)^2 (\delta J^2)_\omega + \frac{1}{4\sigma^2} \left[ (\delta J^2)_{\omega+\omega_J} + (\delta J^2)_{\omega-\omega_J} \right] \right\}.
$$

(27)

It is convenient to present

$$
\frac{1}{f'(x)} = \sum_{k=-\infty}^{\infty} b_k e^{-ikx}, \quad b_0 = \frac{p}{\sigma}; \quad b_{\pm 1} = \pm \frac{1}{2i\sigma}; \quad b_k = 0, \quad (|k| > 1).
$$

(28)

Using this representation and the intrinsic current fluctuation spectrum given by Eq.\(7\) we have

$$
(\delta \dot{\alpha}^2)_\omega = \frac{4}{\pi g} \sum_k |b_k|^2 \left\{ \frac{\omega N(\omega)}{1 + (\omega \tau)^2} \right\}_\omega \Rightarrow \omega - k\omega_J,
$$

(29)

where $g = (\hbar/e^2)R^{-1}$ is the non-dimensional conductance of the junction resistor. It what follows we assume $g \gg 1$, and we will see that this is the main condition for the fluctuations to be weak. We will not consider very strong pumping and assume $p \simeq 1$. Then it follows from $\omega_0 \tau \ll 1$ that also $\omega_J \tau \ll 1$ and one can simplify Eq.\(29\):

$$
(\delta \dot{\alpha}^2)_\omega = \frac{4}{\pi g} \Pi(\omega) \sum_k |b_k|^2 \{\omega N(\omega)\}_\omega \Rightarrow \omega - k\omega_J, \quad \Pi(\omega) = \frac{1}{1 + (\omega \tau)^2},
$$

(30)

where $\Pi(\omega)$ is the shunting factor. It is obvious from Eq.\(27\) that approaching the threshold $p = 1$ of the non-stationary Josephson effect, when $\sigma \to 0$, the fluctuations increase.
IV. PERTURBATION THEORY

We consider shortly the situation when $\delta \alpha(t)$ can be assumed to be small. As we will see no problems appear in perturbation theory if one neglects the capacitance of the junction and put $\tau = 0$. Substituting in the second of Eqs. (16) the Fourier expansion of $g$ from Eqs. (11) and expending in $\delta \alpha(t)$ we find the voltage fluctuations

$$\delta V(t) = V(t) - \nabla(t) = I_c R \sum_k (-ik) g_k \exp[-ik(\omega_J t + \alpha_0)] \delta \alpha(t) + R\delta I(t). \tag{31}$$

Using $\delta \alpha_\omega = \delta \dot{\alpha}_\omega / (-i\omega)$ we have for its Fourier transform

$$\delta V_\omega = I_c R \sum_k k g_k \exp[-ik\alpha_0] \frac{\delta \dot{\alpha}_\omega - k\omega_J}{\omega - k\omega_J} + R\delta I_\omega. \tag{32}$$

We substitute here Eq. (25) and shift the summation over $k$, obtaining

$$\delta V_\omega = \sum_k \exp[-ik\alpha_0] Z_k(\omega) \delta I_{\omega - k\omega_J}, \tag{33}$$

where the impedance of the junction is [3]

$$Z_k(\omega) = R \left\{ \delta_{k,0} + \frac{p}{\sigma} \frac{k g_k}{\omega - k\omega_J} + \frac{1}{2i\sigma} \left[ \frac{(k + 1) g_{k+1}}{\omega - (k + 1)\omega_J} - \frac{(k - 1) g_{k-1}}{\omega - (k - 1)\omega_J} \right] \right\}. \tag{34}$$

Averaging over the initial phase $\alpha_0$ we have the general result of the perturbation theory as follows

$$(\delta V^2)_\omega = \sum_k |Z_k(\omega)|^2 (\delta I^2)_\omega - k\omega_J, \quad \omega > 0. \tag{35}$$

In the classical case one can see, using Eq. (5), that the above result agrees with [5]. However in the quantum case the voltage noise is not given by the symmetrized current spectral density, as it was obtained in [5]. It follows from Eq. (32) that perturbation theory diverges at $\omega = k\omega_J$ for $k = 1, 2, \ldots$, i.e. at the frequencies of the Josephson lines. This is because, as already mentioned, at these frequencies $\delta \alpha(t)$ is not small. Note, that in [5], where perturbation theory was developed for $\delta \varphi(t)$, there was also an artifact divergency for $(\delta \varphi^2)_\omega$ at $\omega = 0$. 

11
Using \( g_k \) from Eq.(21) one can check that \( Z_k(0) = 0 \) when \( |k| > 1 \) and find the low frequency \( \omega \ll \omega_J \) voltage noise to be

\[
S(0) = (\delta V^2)_0 = |Z_0(0)|^2(\delta I^2)_0 + |Z_1(0)|^2[(\delta I^2)_{+\omega_J} + (\delta I^2)_{-\omega_J}]
\]

with \(|Z_0(0)|^2 = R^2(p/\sigma)^2\), \(|Z_1(0)|^2 = R^2/4\sigma^2\), giving

\[
S(0) = \frac{RT p^2}{\pi} \left[ 1 + \frac{1}{2p^2 2T} \coth \left( \frac{h\omega_J}{2T} \right) \right].
\]

In this special case the symmetrized current spectral noise enters, and this is why our result for \( S(0) \) agrees with that obtained in [5].

V. DEPHASING FACTOR OF JOSEPHSON LINES

To find the width of the Josephson lines one has to go beyond the perturbation theory to eliminate the singularities of \((\delta V^2)_{\omega}\) near the Josephson frequency and its harmonics. If \( \delta V_{\omega} \) is calculated from Eq.(15) in the vicinity of \( k\omega_J, (k = 1, 2...) \), the term \( R\delta I_{\omega} \) can be neglected, and \((\delta V^2)_{\omega}\) is a double sum over \( k \) and \( k' \). The strongest singularities \((\omega - k\omega_J)^{-2}\) emerge from the diagonal terms with \( k' = k \), while the non-diagonal terms create weaker singularities \((\omega - k\omega_J)^{-1}\). Comparing the diagonal and non-diagonal terms reveal that when \(|\omega - k\omega_J| \ll \omega_J\) the diagonal terms dominate. Hence, to find the width of the Josephson lines \( k\omega_J \) on can neglect in Eq.(18) the terms \( \delta \xi(t) \) and \( \delta \xi(t') \) and pick-up in the double sum over \( k \) and \( k' \) the terms with \( k' = k \) only, considering

\[
\langle v(t') v(t) \rangle_k = (I_c R)^2 |g_k|^2 \exp[-ik\omega_J(t - t')] \langle \exp[ik\delta \alpha(t') \exp[-ik\delta \alpha(t)] \rangle.
\]

As a result the delta-function in the spectrum Eq.(14) is replaced by a form-factor

\[
\Phi_k(\nu) = \int_0^\infty \frac{dt}{2\pi} \exp[i\nu t] D_k(t) + \text{c.c.}
\]

where \( \nu = \omega - k\omega_J \) is the detuning from the center of the line and the dephasing factor is

\[
D_k(t - t') = \langle \exp[ik\delta \alpha(t')] \exp[-ik\delta \alpha(t)] \rangle = D_k(t' - t)^*. \]
The symmetry relation for the form-factor follows from the invariance of the average with respect to time shift. Since, as assumed, the fluctuations $\delta I(t)$ are small, the dephasing factors decay slowly with $t$.

As follows from Eq.(26) and the properties of $\delta I(t)$ described in sec.I, that $\delta \dot{\alpha}(t)$ and $\delta \alpha(t)$ are Gaussian processes and their commutators are $c$-numbers. Using the well known identity $e^A e^B = e^{A+B + \frac{1}{2}[A,B]}$ one can write the operator product entering the dephasing factor Eq.(40) as

$$
\exp[ik\delta \alpha(0)] \exp[-ik\delta \alpha(t)] = \exp[ik(\delta \alpha(0) - \delta \alpha(t))] \exp \left[ \frac{1}{2} k^2 [\delta \alpha(0), \delta \alpha(t)] \right].
$$

Using the Gaussian properties we calculate the averages and have for the dephasing factor

$$ D_k(t) = \exp \left[ k^2 C(t) \right] \exp \left[ -k^2 K(t) \right], $$

where

$$ C(t) = \frac{1}{2} [\delta \alpha(0), \delta \alpha(t)] = \frac{1}{2} \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t} dt_2 [\delta \dot{\alpha}(t_1), \delta \dot{\alpha}(t_2)], $$

$$ K(t) = \frac{1}{2} \langle [\delta \alpha(0) - \delta \alpha(t)]^2 \rangle = \frac{1}{2} \left\langle \left[ \int_{0}^{t} dt' \delta \dot{\alpha}(t') \right]^2 \right\rangle = \frac{1}{2} \int_{0}^{t} dt_1 \int_{0}^{t} dt_2 \langle \delta \dot{\alpha}(t_1) \delta \dot{\alpha}(t_2) \rangle. $$

Presenting the commutator entering $C(t)$ as its average $\langle \delta \dot{\alpha}(t_1) \delta \dot{\alpha}(t_2) - \delta \dot{\alpha}(t_2) \delta \dot{\alpha}(t_1) \rangle$, and substituting the Fourier representation of the correlator $\langle \delta \dot{\alpha}(t_1) \delta \dot{\alpha}(t_2) \rangle$, we find

$$ K(t) = \int_{0}^{\infty} d\omega \ g(\omega) \frac{1 - \cos \omega t}{\omega^2}, \quad C(t) = i \int_{0}^{\infty} d\omega \ q(\omega) \frac{\sin \omega t}{\omega^2} $$

with

$$ g(\omega) = \left[ (\delta \dot{\alpha}^2)_\omega + (\delta \dot{\alpha}^2)_{-\omega} \right], \quad q(\omega) = - \left[ (\delta \dot{\alpha}^2)_\omega - (\delta \dot{\alpha}^2)_{-\omega} \right]. $$

Using Eq.(30) and the relation $(-\omega)N(-\omega) = \omega(N(\omega) + 1)$ one can check that

$$ (\delta \dot{\alpha}^2)_{-\omega} = \frac{4}{\pi g} \Pi(\omega) \sum_k |b_k|^2 \{ \omega [N(\omega) + 1] \}_{\omega} \Rightarrow \omega - k\omega_j, $$

and as a result,

$$ g(\omega) = \frac{4}{\pi g} \Pi(\omega) \sum_k |b_k|^2 \{ \omega [2N(\omega) + 1] \}_{\omega} \Rightarrow \omega - k\omega_j, $$

13
\[ q(\omega) = \frac{4}{\pi g} \Pi(\omega) \sum_k |b_k|^2 \{\omega\}^\omega \Rightarrow \omega - k\omega = \frac{4}{\pi g} \frac{2p^2 + 1}{2\sigma^2} \omega \Pi(\omega). \] 

\([48]\)

\(q(\omega)\) contains only ZPFs and is temperature independent. This is not the case for \(g(\omega)\), which for high and low temperatures is, correspondingly, (for \(\omega > 0\))

\[ g(\omega) = \frac{4}{\pi g} \frac{2p^2 + 1}{2\sigma^2} \omega \Pi(\omega) \coth \left( \frac{h\omega}{2T} \right), \]

\(\Rightarrow\)

\[ g(\omega) = \frac{4}{\pi g} \frac{1}{2\sigma^2} \Pi(\omega) [\Theta(\omega - \omega)(\omega - \omega) + (2p^2 + 1)\omega], \]

\(\Rightarrow\)

where \(\Theta(\omega)\) is the step function. Because of the ZPFs \(g(\omega) \neq 0\) at \(T = 0\). The shunting cut-off is not crucial in calculating \(C(t)\); this integral converge even if one replace \(\Pi(\omega) \Rightarrow 1\). Contrary, the integral \(K(t)\) converges only due to this cut-off.

**VI. QUANTUM DYNAMICAL NARROWING**

As can be seen from the previous section the form-factor of the Josephson line is given as an integral of the type

\[ \Phi(\nu) = \int_0^\infty dt \frac{dt}{2\pi} \exp[i\nu t] D(t) + c.c. = \int_0^\infty dt \frac{dt}{\pi} \cos[\nu t + H(t)] \exp[-W(t)], \]

\(\Rightarrow\)

where the dephasing factor is \(D(t) = \exp[-W(t) + iH(t)]\). The functions \(W(t)\) and \(H(t)\) are given by their spectral representations

\[ W(t) = \int_0^\infty d\omega G(\omega) \frac{1 - \cos \omega t}{\omega^2}, \quad H(t) = \int_0^\infty d\omega Q(\omega) \frac{\sin \omega t}{\omega^2}, \]

\(\Rightarrow\)

where both spectral densities \(G(\omega), Q(\omega)\) contain a small pre-factor (due to \(g^{-1}\)) and \(G(\omega) > 0\), while \(\text{sign} \ Q(\omega) = \text{sign} \ \omega, \quad Q(0) = 0\). This is the quantum version of spectral line dynamical narrowing. (In the classical theory \(H(t) = 0\), hence the line is symmetric in \(\nu\), and \(G'(0) = 0\). One can see from Eqs.\([49]\) that this property indeed holds for high temperatures but is not valid for zero temperature).

The large time asymptotic of \(W(t)\) is obtained replacing \(G(\omega)\) by \(G(0)\), giving

\[ W(t) = \gamma t, \quad \gamma = \frac{\pi}{2} G(0). \]

\(\Rightarrow\)
Since $Q(0) = 0$ the function $H(t)$ is finite at $t \to \infty$. If one replace $W(t)$ by its large time asymptotic and neglects $H(t)$, the form factor becomes a Lorenzian

$$
\Phi(\nu) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + \nu^2}.
$$

(53)

As we will see, this line shape is valid only for small enough $\nu$. To get an expression valid also for large $\nu$ and to estimate the non-Lorenzian contributions we proceed as follows. We present

$$
W(t) = \gamma t + \tilde{W}(t), \quad \tilde{W}(t) = \int_0^\infty d\omega [G(\omega) - G(0)] \frac{1 - \cos \omega t}{\omega^2},
$$

(54)

and expand $\exp[-\tilde{W}(t)] = 1 - \tilde{W}(t)$. Neglecting second and higher order terms in $\tilde{W}(t)$ and $H(t)$, we present the form-factor as a sum of a symmetric and antisymmetric contributions

$$
\Phi(\nu) = \Phi_s(\nu) + \Phi_a(\nu)
$$

(55)

with

$$
\Phi_s(\nu) = \frac{1}{\pi} \frac{\gamma}{\nu^2 + \gamma^2} - \int_0^\infty \frac{dt}{\pi} \exp[-\gamma t] \cos \nu t \tilde{W}(t),
$$

(56)

$$
\Phi_a(\nu) = -\int_0^\infty \frac{dt}{\pi} \exp[-\gamma t] \sin \nu t H(t).
$$

(57)

To calculate the non-Lorenzian parts of the form-factor introduce functions

$$
\Delta_e(\omega, \nu) = -\frac{2}{\pi} \int_0^\infty dt \cos \nu t \exp[-\gamma t](1 - \cos \omega t) = \frac{2}{\pi (\gamma^2 + \nu^2)(\gamma^2 + (\nu - \omega)^2)(\gamma^2 + (\nu + \omega)^2)},
$$

(58)

$$
\Delta_s(\omega, \nu) = \frac{2}{\pi} \int_0^\infty dt \sin \nu t \exp[-\gamma t] \sin \omega t = \frac{2}{\pi (\gamma^2 + (\nu - \omega)^2)(\gamma^2 + (\nu + \omega)^2)}.
$$

(59)

With this definitions we have

$$
\Phi_s(\nu) = \frac{1}{\pi} \frac{\gamma}{\nu^2 + \gamma^2} + \frac{1}{2} \int_0^\infty \frac{d\omega}{\omega^2} [G(\omega) - G(0)] \Delta_e(\omega, \nu).
$$

(60)

$$
\Phi_a(\nu) = -\frac{1}{2} \int_0^\infty \frac{d\omega}{\omega^2} Q(\omega) \Delta_s(\omega, \nu).
$$

(61)
For small $\gamma$ in the vicinity of $\omega = \nu > 0$ both $\Delta_{c,s}(\omega, \nu)$ are smeared $\delta$-functions,

$$\Delta_{c,s}(\omega, \nu) = \frac{1}{\pi} \frac{\gamma}{(\omega - \nu)^2 + \gamma^2}. \quad (62)$$

The main assumption of the dynamical narrowing theory is that due to the small pre-factor in $G(\omega)$ the Lorenzian width $\gamma$ is small, so that $G(\omega)$ and $Q(\omega)$ are smooth on the scale $\gamma$. With this assumption one can replace $\Delta_{c,s}(\omega, \nu)$ by $\delta(\omega - \nu)$ for $\nu \gg \gamma$ (because of the factor $\omega^{-2}$). Then for $\nu > 0$ one finds

$$\Phi_s(\nu) = \frac{1}{\pi \nu^2 + \gamma^2} + \frac{G(\nu) - G(0)}{2\nu^2}, \quad \Phi_a(\nu) = -\frac{Q(\nu)}{2\nu^2}. \quad (63)$$

On the other hand, at $\nu \gg \gamma$ the Lorenzian and the term with $G(0)$ cancel each other and we find that in the far wings of the line the form-factor follows the spectral densities entering the dephasing factor, i.e.

$$\Phi_s(\nu) = \frac{G(\nu)}{2\nu^2}, \quad \Phi_a(\nu) = -\frac{Q(\nu)}{2\nu^2}, \quad \nu > 0, \nu \gg \gamma. \quad (64)$$

It is important to note, that this is in fact a ”perturbation theory” result and can be obtained expanding the dephasing factor in $W(t)$ and $H(t)$. Within the Lorenzian line, when $\nu \lesssim \gamma$, the estimates for the non-Lorenzian contributions in Eq.(63) can be obtained replacing in the denominators $\nu^2$ by $\gamma^2$. They are negligible for small enough $\nu$, when

$$G(\nu) - G(0), Q(\nu) \ll G(0). \quad (65)$$

If one can define a scale $\Delta\omega$, which is the spectral width of $G(\omega)$ and $Q(\omega)$, the results can be summarized in a simple way. The dynamical narrowing theory is valid when $\gamma \ll \Delta\omega$. For detuning much smaller than the spectral width, $\nu \ll \Delta\omega$, the line form-factor is Lorenzian according to Eqs.(53) and (52). When the detuning is of the order or larger than the spectral width, $\nu \gtrsim \Delta\omega$, the form-factor follows the spectral density according to the perturbation theory result given by Eqs.(64). In this domain the line is asymmetric, the red wing ($\nu < 0$) being enhanced compared to the blue one ($\nu > 0$). This is because of ZPFs, which enhance the probability for the oscillator to lose energy compared to the probability to gain it, due to the interaction with the thermal bath. This asymmetry is related to the perturbation theory result Eq.(63), where the non-symmetrized current fluctuation spectral density enters.
VII. THE WIDTH OF JOSEPHSON LINES

Using the results of the previous section we can find the width and the shape of the Josephson lines. For the line \( k\omega_J \) one has to put

\[
G(\omega) = k^2 g(\omega), \quad Q(\omega) = k^2 q(\omega),
\]

and as a result the Lorenzian width of this line is according to Eqs.(52)

\[
\gamma_k = k^2 \Gamma, \quad \hbar \Gamma = \frac{4}{g\sigma^2} \left[ p^2 T + \frac{1}{4} \hbar \omega_J \coth \left( \frac{\hbar \omega_J}{2T} \right) \right].
\]

As was mentioned already, our basic assumption about the weakness of the current fluctuations means that the line width is small compared to the line separation, which means that the result given by Eqs.(57) is correct when

\[
\gamma_k \ll \omega_J.
\]

We will show now that this restriction, together with \( p \approx 1 \), ensures that in the vicinity of the Josephson frequency, i.e. at \( |\omega - k\omega_J| \ll \omega_J \), the dynamical narrowing theory is valid and the form-factor of the Josephson line is a symmetric Lorenzian with the width given by Eqs.(57). The non-Lorenzian contributions and the asymmetry of the line appears only at \( |\omega - k\omega_J| \approx \omega_J \), but in this domain one has to employ perturbation theory including non-diagonal terms with \( k' \neq k \). Two things have to be checked. First, that the Lorenzian width \( \gamma_k \) is small compared to the scales of the spectral densities \( G(\omega) \) and \( Q(\omega) \), given by Eqs.(60). Second, the conditions Eqs.(63), i.e. that the non-Lorenzian contributions to the form-factor are small.

As can be seen from the results of dynamical narrowing theory, the form-factor at given detuning \( \nu \) do not depend on the behavior of the spectral densities \( G(\omega) \) and \( Q(\omega) \) at \( \omega > \nu \). Since we are interested only in \( |\omega - k\omega_J| \ll \omega_J \), the screening factor \( \Pi(\omega) \) in \( g(\omega) \) and \( q(\omega) \) can be neglected. As a result, it follows from Eq.(49) that the scale of \( G(\omega) \) is the larger of \( T/\hbar \) and \( \omega_J \). The situation is more complicated for \( Q(\omega) \) which is linear in \( \omega \) and has no
scale. However, Eq.(63) for the asymmetric contribution to the form-factor is still correct for $\nu \gg \gamma$, since the effective scale of a linear $Q(\omega)$ is $\omega$.

Consider first the situation of high temperatures, $T \gg \hbar \omega_J$, when the scale of $G(\omega)$ is $\Delta \omega = T/\hbar$, and since $T/\hbar \gg \omega_J$, it follows from Eq.(68), that this scale is large compared to $\gamma_k$. Using Eq.(69) and Eq.(18) one can check that the conditions Eq.(53) are satisfied when $\nu \ll T/\hbar$, which is valid automatically since $\nu \ll \omega_J \ll T/\hbar$. In the case of low temperatures, $T \ll \hbar \omega_J$, the scale of $G(\omega)$ is $\Delta \omega = \omega_J$, which is large compared to $\gamma_k$ because of Eq.(68). The conditions Eq.(35) require $\nu \ll \omega_J$, which is also satisfied.

The width at high temperatures, $\Gamma = (g\sigma^2)^{-1} 2(2p^2 + 1)(T/\hbar)$, was obtained in [3] considering the singular term $A_k/(\omega - k\omega_J)^2$ in $(\delta V^2)_\omega$ calculated from perturbation theory, Eq.(15), as a wing of a Lorenzian line and assuming that the total energy radiated in this line is not influenced by the fluctuations. In the high temperature case the fluctuations can be considered to be weak, Eq.(68), if $g\sigma^3 \gg k^2(T/\hbar \omega_0)$.

At low temperatures $\Gamma = (g\sigma^2)^{-1} \omega_J$, and Eq.(18) reduces to $g\sigma^2 \gg k^2$. Finite dephasing of the Josephson lines exist at zero temperature due to the ZPFs, which are active because the junction is far from equilibrium.

One can see from the above estimates that the fluctuations are weak due to the large conductance $g$ of the shunting resistor. Approaching to the threshold $p = 1$, when $\sigma \to 0$, the fluctuations increase. The effect of the fluctuations is stronger for high harmonics, $k \gg 1$.

It is important to note that the general quantum result for the line width, Eq.(67), cannot be obtained from the high temperature classical one simply replacing the temperature $T$ by its quantum equivalent, $(\hbar \omega/2) \coth(\hbar \omega/2T)$.

We note also that a general relation is valid, $\hbar \Gamma = (4\pi e^2/\hbar)S(0)$, which is a simple relation between the width of main Josephson line at $\omega = \omega_J$ and the low frequency noise. As was mentioned in sec.II, there are two indications, pointing to the smallness of the fluctuations: the ratio of the line width to the line separation $\Gamma/\omega_J$, and the ratio of the background contribution to the contribution of the lines, which can be estimated as $S(0)\omega_J/(g_0 I_c R)^2$. It follows from the relation between $\Gamma$ and $S(0)$ that both ratios are of the same order, i.e.
that both ways to estimate the strength of fluctuations are equivalent.

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