Generalization of adaptive cross approximation for time-domain boundary element methods

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A numerical approach to the solution of the wave equation is performed by means of the boundary element method. In the interest of increasing the efficiency of this method a low-rank approximation such as the adaptive cross approximation is carried out. We discuss a generalization of the adaptive cross approximation to approximate a three-dimensional array of data. In particular, we perform an approximation of an array of boundary element matrices in the Laplace domain. The proposed scheme is illustrated by preliminary numerical experiments.

1 Numerical approach to the wave equation

We consider the homogeneous wave equation with vanishing initial conditions and given Dirichlet data on the boundary. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \Gamma := \partial \Omega, T > 0 \) the final time and \( c > 0 \) the wave velocity. Find the displacement function \( u \) such that

\[
\frac{\partial^2}{\partial t^2} u(x, t) - c^2 \Delta u(x, t) = 0 \quad (x, t) \in \Omega \times (0, T),
\]

\[
u(x, 0) = \frac{\partial}{\partial t} u(x, 0) = 0 \quad x \in \Omega,
\]

\[
u(x, t) = g(x, t) \quad (x, t) \in \Gamma \times (0, T).
\]

We solve this initial boundary value problems by means of the boundary element method. By using the fundamental solution of the underlying differential operator, the partial differential equation is restated as an integral equation. In addition, by applying the trace operator to this integral equation we obtain the boundary integral equation in time domain. The convolution quadrature method (CQM) is used for the temporal discretization, and the spatial discretization is carried out by the collocation method. In conclusion, we get an array of system matrices depending on the frequency. This array of matrices is interpreted as a three-dimensional array of data which we intend to approximate by a data-sparse representation.

2 Generalization of the adaptive cross approximation

The basic idea of a generalization of adaptive cross approximation (ACA) has been proposed in [2] and this is illustrated in Fig. 1. To briefly explain the idea, assume we are given an array of frequencies where the discretized integral operator has to be evaluated. This generates a three-dimensional array of data \( C_{ij,k} \), where the first index is related to the collocation point and the second one to the basis function, but both refer to spatial discretization. The third index corresponds to the frequency or temporal discretization, respectively. The essential items of the data-sparse approximation of \( C_{ij,k} \) are given in Algorithm 1.

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Algorithm 1: basic idea of generalized ACA
For \( \ell = 1, 2, 3 \ldots \)
1. compute face \( H_{ij}^\ell \) via low-rank approximation
\[ h_{ij}^\ell = C_{ij,k} \]
2. define pivot position
\[ (i_{\ell,j}, j_{\ell}) := \arg \max_{i,j} |h_{ij}^\ell| \]
3. compute fiber \( F_{ij}^\ell \)
\[ f_{k}^\ell = C_{i_{\ell,j}, k} \]
Stop if \( \| \tilde{H}_{\ell}^\ell \|_F \| F_{ij}^\ell \|_F \leq \varepsilon \| S_{ij} \|_F \)
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The face \( H_{ij}^\ell \) denotes the system matrix assembled at a fixed frequency, which is approximated by the standard ACA [1]. The maximum entry of the matrix determines the pivot position. Finding this position in a low-rank approximated face in

Fig. 1: Illustration of generalized ACA.
appropriate time is the crucial task of the generalization. The low-rank approximation of a matrix requires a decomposition into a hierarchical scheme, so that the matrix is partitioned into admissible and non-admissible blocks. Subsequently, all admissible blocks are approximated, for instance, by the standard ACA. In order to avoid the execution of the product of the low-rank matrices, the index $i$ or $j$, respectively, is defined by the maximum of the absolute row sum of the low rank matrices. To get an idea, assume that an admissible block $A$ is approximated by the truncated singular value decomposition. Thus we get

$$A \approx A_r = U \Sigma V^H,$$

where the low-rank matrices $U$ and $V$ consist of orthonormal columns. Consequently, the index $i_\ell$ of the pivot position is defined by

$$\arg \max_{i,j} |a_{ij}| = \arg \max_{i,j} \sum_{k=1}^r |u_{ik}\sigma_k| = \arg \max_{i} \sum_{k=1}^r |u_{ik}\sigma_k| \sim i_\ell,$$

and an analogous argumentation holds to $j_\ell$. This concept is adopted for the low-rank approximation by ACA. First, a QR-decomposition of the low-rank matrices $U$ and $V$ is performed. Second, the SVD is used for the smaller inner matrix $R_U R_V^H$, and this ultimately results in low-rank matrices $U$ and $V$ with orthonormal columns,

$$A \approx A_r = U V^H = Q_U (R_U R_V^H) Q_V^H = Q_U U \Sigma V^H Q_V^H = U \Sigma V^H.$$

Regarding to this pivot position the fiber $F_\ell$ is computed by keeping the indices related to the spatial discretization fixed and evaluating the integral operator at all frequencies. Thus, the first cross of the generalized ACA with an approximated face and a fiber is generated, which has to be updated afterwards. With each further iteration, the residual of the face and of the fiber has to be determined. Furthermore, the current approximation is defined as $S_\ell = \sum_{d=1}^{\ell} \tilde{H}_d \otimes \tilde{F}_d$, and finally the algorithm terminates successfully if a suitable stopping criterion with given accuracy $\epsilon$ is satisfied, see Algorithm 1.

### 3 Discussion

The performance of the introduced algorithm is shown by a numerical experiment. In this example the discretized integral operator at 11 different frequencies is considered as the three-dimensional array of data. In Fig. 2 the error of the approximation in the squared Frobenius norm against the iteration counter of the algorithm is plotted. The numerical experiment is performed without any low-rank approximation of the face, with low-rank approximation by SVD and by ACA. The Algorithm 1 terminates if the proposed criterion with the given accuracy of $\epsilon = 10^{-4}$ is fulfilled. This is satisfied after six iterations for all three alternatives. At the sixth iteration, the ACA approximated face exhibits a slightly different approximation error. This indicates that the low-rank approximation error of the faces already has an effect. As a result the computation time and the memory consumption is reduced. However, further experiments have to be carried out to verify that the determination of the pivot position of the face is appropriate and whether the approximation of the face is sufficient.

### References

[1] S. Rjasanow and O. Steinbach, The Fast Solution of Boundary Integral Equations (Springer, New York, 2007), chap. 3.

[2] M. Bebendorf, A. Kühnemund, and S. Rjasanow, Appl. Numer. Math. 74, pp. 1–16 (2013).