A superpolynomial version of nonsymmetric Jack polynomials

Charles F. Dunkl

Dedicated to the memory of Dick Askey, who was my special functions teacher, and who made it respectable to find exact answers to analysis problems.

Received: 1 September 2020 / Accepted: 12 February 2021 / Published online: 6 May 2021
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Abstract
Superpolynomials consist of commuting and anti-commuting variables. By considering the anti-commuting variables as a module of the symmetric group, the theory of vector-valued nonsymmetric Jack polynomials can be specialized to superpolynomials. The theory significantly differs from the supersymmetric Jack polynomials introduced and studied in several papers by Desrosiers et al. (Nucl Phys B606:547–582, 2001). The vector-valued Jack polynomials arise in standard modules of the rational Cherednik algebra and were originated by Griffeth (Trans Am Math Soc 362:6131–6157, 2010) for the family $G(n, \ell, N)$ of complex reflection groups. In the present situation there is an orthogonal basis of anti-commuting polynomials which corresponds to hook tableaux arising in Young’s representations of the symmetric group. The basis is then used to construct nonsymmetric Jack polynomials by specializing the machinery set up in a paper by Luque and the author (SIGMA 7, 2011). There is an inner product for which these polynomials form an orthogonal basis, and the squared norms are explicitly found. Supersymmetric polynomials are obtained as linear combinations of the nonsymmetric Jack polynomials contained in a submodule; this is based on an idea of Baker and Forrester (Ann Comb 3:159–170, 1999). The Poincaré series for supersymmetric polynomials graded by degree is obtained and is interpreted in terms of certain minimal polynomials. The squared norms of a special subset of these minimal polynomials are polynomials in the parameter. There is a brief discussion of antisymmetric polynomials and an application to wavefunctions of the Calogero–Moser quantum model on the circle.

Keywords Vector-valued Jack polynomials · Supersymmetric polynomials · Hook tableaux

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1 Introduction

Superpolynomials involve both commuting and anti-commuting variables. By interpreting the latter as modules of the symmetric group $S_N$, one can adapt the structure of vector-valued nonsymmetric Jack polynomials to this setting. Desrosiers et al. [2–5] constructed Jack superpolynomials with the use of differential operators motivated by equations of supersymmetric quantum mechanics. These are significantly different from the operators to be set up in this work, however, the definition of supersymmetric polynomials is the same. Since our approach uses polynomials taking values in $S_N$-modules the paper begins with details on the Young-type construction applied to polynomials in anti-commuting variables of some fixed degree. Sections 2 and 3 contain the basic definitions and construction of tableau-like polynomials forming orthogonal bases of the two irreducible modules. Section 4 is a brief overview of the vector-valued nonsymmetric Jack polynomials and provides details of how the group acts on these polynomials. The theory has one free parameter $\kappa$. As is usual in this area there are numerous arguments using induction with simple reflections (adjacent transpositions). The polynomials are a special case of the construction by Griffeth [10] which is defined for the family $G(n, p, N)$ of complex reflection groups. There is a natural inner product for the space of Jack polynomials for which they are mutually orthogonal. The formula for the squared norm of a Jack polynomial as a rational function of the parameter $\kappa$ is stated in this section. The paper of Dunkl and Luque [9] is used as background reference.

Section 5 uses a technique of Baker and Forrester [1] and ideas from [9] to construct supersymmetric Jack polynomials and determine the norms. In Sect. 6 the Poincaré–Hilbert series which gives the number of linearly independent supersymmetric polynomials of given degree is found. The series gives information about the minimal supersymmetric polynomials which generate all the polynomials over the ring of ordinary symmetric polynomials. Section 7 concerns certain minimal polynomials whose norms are found explicitly as polynomials in $\kappa$. The concluding Sect. 8 briefly discusses the construction of antisymmetric polynomials and the application of vector-valued Jack polynomials as wavefunctions of the Hamiltonian coming from the Calogero–Sutherland quantum-mechanical model of identical particles on a circle with $1/r^2$ interactions.

2 Preliminaries

All the subsets appearing here are subsets of $\{1, 2, \ldots, N\}$. The complement $E^C := \{ j \notin E : 1 \leq j \leq N \}$. The symbol $#$ denotes the cardinality of a set. Let $\sigma(n) := (-1)^n$ for $n \in \mathbb{Z}$. 
\begin{definition}
For a set $E$ and $1 \leq j \leq N$ let
\begin{align*}
\text{inv} (E) &= \# \left\{ (i, j) \in E \times E^C : i < j \right\}, \\
\text{inv}' (E) &= \# \left\{ (i, j) : i \in E, j \notin E, i < j < N \right\}, \\
\text{s} (j, E) &= \# \left\{ i \in E : j < i \right\}, \\
E^\perp &= \left\{ j \in E^C : j < N \right\}.
\end{align*}

The symmetric group $S_N$ is the group of permutations of $\{1, 2, \ldots, N\}$. For $i \neq j$ the transposition $(i, j)$ fixes $k \neq i, j$ and interchanges $i$ and $j$. The transpositions $s_i := (i, i + 1)$ for $1 \leq i < N$ generate $S_N$, and are fundamental in proofs by induction.

The fermionic variables $\theta_1, \theta_2, \ldots, \theta_N$ satisfy $\theta_i \theta_j = -\theta_j \theta_i$ for all $i, j$. The symmetric group $S_N$ acts by permutation: suppose $p (\theta)$ is a polynomial in $\theta_i$ and $w \in S_N$ then $w p (\theta) = p (\theta_{w(1)}, \theta_{w(2)}, \ldots, \theta_{w(N)})$. As basis elements for polynomials in $\{\theta_i\}$ we use
\begin{equation*}
\phi_E := \theta_{i_1} \cdots \theta_{i_m}, \quad E = \{i_1, i_2, \ldots, i_m\}, \quad 1 \leq i_1 < i_2 < \cdots < i_m \leq N.
\end{equation*}

Our first step is to analyze the representation of $S_N$ acting on (with $1 \leq m \leq N - 1$)
\begin{equation*}
P_m = \text{span} \{\phi_E : \#E = m\}.
\end{equation*}

The space is equipped with the inner product defined by declaring $\{\phi_E\}$ to be an orthonormal basis. Clearly $\dim P_m = \binom{N}{m}$. We list key properties for the action of $S_N$ on $\{\phi_E\}$.

\begin{proposition}
If $j \notin E$ then $\phi_E \theta_j = \sigma (s (j, E)) \phi_{E \cup \{j\}}$. Suppose $(i, j)$ is a transposition then (1) $(i, j) \phi_E = \phi_E$ if $i, j \in E^C$; (2) $(i, j) \phi_E = -\phi_E$ if $i, j \in E$; (3) $(i, j) \phi_E = \sigma (s (i, E) + s (j, E \setminus \{i\})) \phi_{(E \setminus \{i\}) \cup \{j\}}$ if $i \in E, j \notin E$.
\end{proposition}

\begin{proof}
Suppose $(i, j) \subset E$ and $i < j$, then
\begin{equation*}
(i, j) \phi_E = \sigma (s (j, E)) \phi_{E \setminus \{j\}} \theta_j = \sigma (s (j, E)) \sigma (s (i, E \setminus \{i, j\})) \phi_{E \setminus \{i, j\}} \theta_i \theta_j,
\end{equation*}
\begin{equation*}
(i, j) \phi_E = -\sigma (s (j, E)) \sigma (s (i, E \setminus \{i, j\})) \phi_{E \setminus \{i, j\}} \theta_i \theta_j = -\phi_E.
\end{equation*}

Suppose $i \in E$ and $j \notin E$, then
\begin{equation*}
(i, j) \phi_E = (i, j) \sigma (s (i, E)) \phi_{E \setminus \{i\}} \theta_i = \sigma (s (i, E)) \phi_{E \setminus \{i\}} \theta_j = \sigma (s (i, E) + s (j, E \setminus \{i\})) \phi_{(E \setminus \{i\}) \cup \{j\}}.
\end{equation*}

Note $s (j, E \setminus \{i\}) = s (j, E)$ if $j > i$ and $s (j, E \setminus \{i\}) = s (j, E) - 1$ if $j < i$. \hfill \Box
Denote $(E \setminus \{i\}) \cup \{j\}$ by $(i, j) E$ when $i \in E$, $j \notin E$. Suppose # $(\{i, i + 1\} \cap E) = 1$, then $s_i \phi_E = \phi_{s_i E}$ and

$$(i, i + 1) \in E \times E^C : s_i E = (E \setminus \{i\}) \cup \{i + 1\}, \quad \text{inv} (s_i E) = \text{inv} (E) - 1,$$

$$(i, i + 1) \in E^C \times E : s_i E = (E \setminus \{i\}) \cup \{i\}, \quad \text{inv} (s_i E) = \text{inv} (E) + 1.$$

We set up a duality map from $\mathcal{P}_m$ to $\mathcal{P}_{N-m}$.

**Definition 2** Suppose $# E = m$ then define $\delta \phi_E := \sigma (\text{inv} (E)) \phi_{E^C}$. Extend $\delta$ to $\mathcal{P}_m$ by linearity, so that $\delta$ is a linear isomorphism from $\mathcal{P}_m$ to $\mathcal{P}_{N-m}$.

**Proposition 2** Suppose $1 \leq i < N$ then $\delta s_i = -s_i \delta$.

**Proof** Suppose $i, i + 1 \in E$ then $s_i \phi_E = -\phi_E$ and $s_i \phi_{E^C} = \phi_{E^C}$; hence $\delta s_i \phi_E = -s_i \delta \phi_E$. Suppose $i, i + 1 \in E^C$ then $s_i \phi_E = \phi_E$ and $s_i \phi_{E^C} = -\phi_{E^C}$. If $(i, i + 1) \in E \times E^C$ then $s_i E = (E \setminus \{i\}) \cup \{i + 1\}$, $\text{inv} (s_i E) = \text{inv} (E) - 1$ and $(s_i E)^C = s_i E^C = (E^C \setminus \{i + 1\}) \cup \{i\}$. Thus

$$\delta s_i \phi_E = \delta \phi_{s_i E} = \sigma (\text{inv} (s_i E)) \phi_{(s_i E)^C} = \sigma (\text{inv} (s_i E)) \phi_{s_i E^C} = \sigma (\text{inv} (E) - 1) \phi_{s_i E^C} = -s_i \delta \phi_E.$$

A similar argument applies to the case $(i, i + 1) \in E^C \times E$. \hfill \Box

The length of $w \in S_N$ is the number of factors of the shortest product of $\{s_i\}$ required to express $w$ and equals $\ell (w) := \# \{(i, j) : i < j, w (i) > w (j)\}$.

**Corollary 1** Suppose $w \in S_N$ then $\delta w = \sigma (\ell (w)) w \delta$.

The map $\delta$ can also be interpreted in the direction $\mathcal{P}_{N-m} \to \mathcal{P}_m$, and $\delta^2 = \sigma (m (N - m))$ since $\delta^2 \phi_E = \sigma (\text{inv} (E^C) + \text{inv} (E)) \phi_E$ and $\text{inv} (E^C) + \text{inv} (E) = \# (E \times E^C)$.

### 3 Representation theory for hook tableaux

The irreducible representation $(N - m, 1^m)$ of $S_N$ is realized on the span of reverse standard Young tableaux (RSYT) of shape $(N - m, 1^m)$. Each such tableau $T$ has the entries $N, N - 1, \ldots, 1$ in decreasing order in row 1 and in column 1. For background on representations of $S_N$, see James and Kerber [11]. In the examples and diagrams we will display column 1 as the second row, (to cut down on blank space); suppose $N = 9$ and $m = 3$ then the entries of $T$ are indexed as

$$T [1, 1] \quad T [1, 2] \quad T [1, 3] \quad T [1, 4] \quad T [1, 5] \quad T [1, 6].$$

Here is a typical tableau:

$$\begin{bmatrix}
9 & 8 & 6 & 4 & 3 & 2 \\
\circ & 7 & 5 & 1 \\
\end{bmatrix}.$$
Notice that the entries in column 1 determine the tableau, so in this case there are \( \binom{8}{3} \) different tableaux. The content of the entry at \( T[i,j] \) is defined to be \( j - i \) (in the hook case the content values are \(-m, 1 - m, \ldots, 0, 1, \ldots, N - m - 1\)). The content vector of \( T \) is denoted \( \{c(i, T)\}_{i=1}^{N} \) where \( c(i, T) \) is the content of the cell containing \( i \). For the above example the content vector is \([-3, 5, 4, 3, -2, 2, -1, 1, 0] \).

**Definition 3** The Jucys–Murphy elements of \( S_N \) are the elements of the group algebra \( \mathbb{Q}S_N \) defined by

\[
(\omega_i := \sum_{j=i+1}^{N} (i, j), 1 \leq i \leq N)
\]

and they commute pairwise (note \( \omega_N = 0 \)). The Jucys–Murphy elements and the simple reflections satisfy the following commutation relations:

\[
s_i \omega_j = \omega_j s_i, \quad j = 1, \ldots, i - 1, \quad i + 2, \ldots, N,
\]

\[
s_i \omega_i s_i = \omega_{i+1} + s_i, \quad 1 \leq i < N.
\]

The representation of \( S_N \) on the span of the RSYT’s of a given shape (partition of \( N \)) is defined in such a way that \( \omega_i T = c(i, T) T \) for each \( i \). The details of the action are given later.

### 3.1 The submodule of isotype \((N-m, 1^m)\)

In this subsection we construct elements of \( P_m \) which correspond to RSYT’s of shape \((N-m, 1^m)\) and have the appropriate eigenvalues for \( \{\omega_i\} \).

**Definition 4** For \( \#E = m + 1 \) define a polynomial in \( P_m \) by

\[
\psi_E = \sum_{j \in E} \sigma(s(j, E)) \phi_{E \setminus \{j\}}.
\]

The following is the starting point for the construction, the steps of which are ordered by \( \text{inv}(E) \).

**Definition 5** Let \( E_0 = \{N-m, N-m+1, \ldots, N\} \).

**Theorem 1** If \( 1 \leq i \leq N-m-1 \) then \( \omega_i \psi_{E_0} = (N-m-i) \psi_{E_0} \) and if \( N-m \leq i \leq N \) then \( \omega_i \psi_{E_0} = -(N-i) \psi_{E_0} \).

**Proof** Set \( g = \psi_{E_0} \) and \( g_i = \phi_{E_0 \setminus \{i\}} \) for \( N-m \leq i \leq N \); thus \( g = \sum_{i=N-m}^{N} (-1)^{N-i} g_i \). It is clear that \( 1 \leq i < j < N-m \) implies \((i, j) g = g\). If \( N-m \leq i < N \) then \( s_i g_j = -g_j \) when \( N-m \leq j < i \) or \( i+1 < j \leq N \) while \( s_i g_i = g_{i+1} \) and \( s_i g_{i+1} = g_i \). Thus \( s_i g = -g \) for all \( N-m \leq i < N \).
It follows that \((i, j) g = -g\) for all \(N - m \leq i < j \leq N\) \((i, j)\) is a product of an odd number of \(s_k\), with \(N - m < k < N\). Thus if \(N - m \leq i < N\) then \(\omega_i g = \sum_{j=i+1}^{N} (i, j) g = -(N-i) g\). It remains to consider \(\sum_{j=N-m}^{N} (i, j) g\) for \(1 \leq i < N - m\). For \(N - m \leq j < k \leq N\) let \(g_{ijk} = \phi_i(E)\psi_j(E)\psi_k(E)\). Then \((i, j) g_k = (-1)^{j-N+m} g_{ijk}\) and \((i, k) g_j = (-1)^{k-N+m+1} g_{ijk}\), and

\[
\sum_{j=N-m}^{N} (i, j) g = \sum_{j=N-m}^{N} \sum_{k=N-m}^{N} (-1)^{N-k} (i, j) g_k = \sum_{j=N-m}^{N} (-1)^{N-j} (i, j) g_j + \sum_{N-m \leq j < k \leq N} \left\{ (-1)^{N-k} (i, j) g_k + (-1)^{N-j} (i, k) g_j \right\}
\]

because \((i, j) p_j = p_j\) and

\[
(-1)^{N-k} (i, j) g_k + (-1)^{N-j} (i, k) g_j = (-1)^m \left\{ (-1)^{j-k} + (-1)^{k+1-j} \right\} g_{ijk} = 0.
\]

Finally, if \(1 \leq i < N - m\) then

\[
\omega_i g = \sum_{j=i+1}^{N-m-1} (i, j) g + \sum_{j=N-m}^{N} (i, j) g = \{(N-m-1-i) + 1\} g.
\]

This completes the proof. \hfill \square

**Corollary 2** The respective \(\{\omega_i\}\)-eigenvalues of \(\psi_{E_0}\) agree with the content vector of the tableau \(T_0\) of shape \((N-m, 1^m)\) given by \(T_0[i, 1] = N + 1 - i\) for \(1 \leq i \leq m+1\) and \(T_0[1, j] = N - m + j - 1\) for \(2 \leq j \leq N - m\). The polynomial \(\psi_{E_0}\) is of isotype \((N-m, 1^m)\) and this representation is a summand of \(P_m\).

Next we show that \(\text{span}\{\psi_E : \#E = m+1\}\) is closed under \(S_N\) and thus is an irreducible module of isotype \((N-m, 1^m)\).

**Proposition 3** Suppose \(\#E = m+1\) and \(1 \leq i < N\) : \((1)\) if \(i, i + 1 \notin E\) then \(s_i \psi_E = \psi_E\); \((2)\) if \(i, i + 1 \in E\) then \(s_i \psi_E = -\psi_E\); \((3)\) if \(\#(\{i, i + 1\} \cap E) = 1\) then \(s_i \psi_E = \psi_{S_i E}\).
Proof If \( i, i + 1 \notin E \) then \( s_i \phi_{E \setminus \{ j \}} = \phi_{E \setminus \{ j \}} \) for each \( j \in E \). If \( i, i + 1 \in E \) then \( s(i, E) = s(i + 1, E) + 1 \) and

\[
\begin{align*}
s_i \psi_E &= \sum_{j \in E, j \neq i, i + 1} \sigma(s(j, E)) s_i \phi_{E \setminus \{ j \}} \\
&\quad + \sigma(s(i, E)) s_i \phi_{E \setminus \{ i \}} + \sigma(s(i + 1, E)) s_i \phi_{E \setminus \{ i + 1 \}} \\
&= -\sum_{j \in E, j \neq i, i + 1} \sigma(s(j, E)) \phi_{E \setminus \{ j \}} + \sigma(s(i, E)) \phi_{E \setminus \{ i \}} \\
&\quad + \sigma(s(i + 1, E)) \phi_{E \setminus \{ i + 1 \}} \\
&= -\psi_E
\end{align*}
\]

because \( \sigma(s(i, E)) = -\sigma(s(i + 1, E)) \). Suppose \( (i, i + 1) \in E \times E^C \) then \( s_i E = (E \setminus \{ i \}) \cup \{ i + 1 \} \), \( s(j, s_i E) = s(j, E) \) for all \( j \neq i, i + 1 \), and \( s(i + 1, s_i E) = s(i, E) \); thus

\[
\begin{align*}
s_i \psi_E &= \sum_{j \in E, j \neq i, i + 1} \sigma(s(j, E)) s_i \phi_{E \setminus \{ j \}} + \sigma(s(i, E)) s_i \phi_{E \setminus \{ i \}} \\
&= \sum_{j \in E, j \neq i} \sigma(s(j, s_i E)) \phi_{s_i E \setminus \{ j \}} + \sigma(s(i, E)) \phi_{s_i E \setminus \{ i \}} \\
&= \psi_{s_i E}
\end{align*}
\]

because \( (s_i E) \setminus \{ i + 1 \} = E \setminus \{ i \} \). If \( (i, i + 1) \in E^C \times E \) then use the fact that \( s_i (s_i E) = E \) and \( s_i^2 = 1 \). \( \square \)

Definition 6 Let \( \mathcal{P}_{m,0} = \text{span} \{ \psi_E : \#E = m + 1 \} \). Let \( \mathcal{E}_0 = \{ E : \#E = m + 1, N \in E \} \).

Theorem 2 The subspace \( \mathcal{P}_{m,0} \) is an irreducible \( S_N \)-module isomorphic to the module \( (N - m, 1^m) \), the span of RSYT’s with this shape.

Proof By Theorem 1 and Proposition 3 \( \mathcal{P}_{m,0} \) is the \( S_N \)-module generated by \( \psi_{E_0} \), which is of isotype \( (N - m, 1^m) \). \( \square \)

We set up a correspondence between sets and hook tableaux which illuminates the meaning of content vectors and \( \{ \omega_1 \} \)-eigenvalues.

Definition 7 Suppose \( E \in \mathcal{E}_0 \) and \( E = \{ i_1, \ldots, i_m, i_{m+1} \} \), \( E^C = \{ j_1, \ldots, j_{N-m-1} \} \) with \( i_1 < i_2 < \cdots < i_{m+1} = N \) and \( j_1 < j_2 < \cdots \) then \( Y_E \) is the RSYT of shape \( (N - m, 1^m) \) given by \( Y_E[k,1] = i_{m+2-k} \) for \( 1 \leq k \leq m + 1 \), and \( Y_E[1,k] = j_{N-m+1-k} \) for \( 2 \leq k \leq N - m \).

Example: let \( N = 8, m = 3, E = \{ 2, 5, 7, 8 \} \) then

\[
Y_E = \begin{bmatrix}
8 & 6 & 4 & 3 & 1 \\
7 & 5 & 2 & & \\
\end{bmatrix}
\]

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We will construct $T_E \in \mathcal{P}_{m,0}$ satisfying $\omega_i T_E = c (i, Y_E) T_E$ for each $i$ and $E \in \mathcal{E}_0$. Here is an explicit definition of the content vector satisfying $[c (i, E)]_{i=1}^N = [c (i, Y_E)]_{i=1}^N$.

**Definition 8** For $E \in \mathcal{E}_0$ the content vector is $[c (i, E)]_{i=1}^N$ where $c (i, E) = -s (i, E)$ if $i \in E$ and $c (i, E) = s (i, E^C) + 1$ if $i \notin E$.

**Theorem 3** For each $E \in \mathcal{E}_0$ there exists $T_E \in \mathcal{P}_{m,0}$ such that $\omega_i T_E = c (i, E) T_E$. If $i \notin E$ and $i + 1 \in E \setminus \{N\}$ then $T_{s_i E} = s_i T_E - (c (i, E) - c (i + 1, E))^{-1} T_E$.

**Proof** We proceed by induction on $\text{inv} (E)$ and start with $E = E_0$ with $\text{inv} (E_0) = 0$ and $T_{E_0} = \psi_{E_0}$; this is valid by Theorem 1. Suppose $T_E$ has been constructed for every $E \in \mathcal{E}_0$ with $\text{inv} (E) \leq n$ and $F \in \mathcal{E}_0$ satisfies $\text{inv} (F) = n + 1$. Then there exists $i < N - 1$ such that $i \notin F$ and $i + 1 \in F$, so that $\text{inv} (s_i F) = \text{inv} (F) - 1 = n$. Set $E = s_i F$ so that $F = s_i E$. Let $b = (c (i, E) - c (i + 1, E))^{-1}$ and define $T_F := s_i T_E - b T_E$. If $j < i$ or $j > i + 1$ then $\omega_j s_i = s_i \omega_j$ and so $\omega_j T_{s_i E} = c (j, E) T_{s_i E}$, also $c (j, E) = c (j, F)$. Then

$$\omega_i T_F = \omega_i (s_i T_E - b T_E) = (1 + s_i \omega_{i+1}) T_E - \omega_i b T_E = T_E + c (i + 1, E) s_i T_E - c (i, E) b T_E = c (i + 1, E) (s_i T_E - b T_E).$$

Similarly $\omega_{i+1} T_F = c (i, E) T_F$. Since $c (i, E) = c (i + 1, F)$ and $c (i + 1, E) = c (i, F)$ this completes the proof. \(\square\)

It may seem that there is a uniqueness problem in the construction of $T_{s_i E}$ (possibly $s_i E = s_j F$) but the eigenvalues of $\{\omega_i\}$ do determine $T_E$ up to a multiplicative constant.

There is a useful triangularity property for $\{\psi_E\}$ which completes the uniqueness proof.

**Definition 9** For $0 \leq n \leq m (N - 1 - m)$ let $\mathcal{P}_{m,0}^{(n)} = \text{span} \{\psi_E : E \in \mathcal{E}_0, \text{inv} (E) \leq n\}$.

The extreme cases are $\text{inv} ([N - m, \ldots, N]) = 0$ and $\text{inv} ([1, 2, \ldots, m, N]) = m (N - 1 - m)$. There is an important relation to Gaussian binomial coefficients:

$$\sum_{E \in \mathcal{E}_0} q^{\text{inv} (E)} = \left[\begin{array}{c} N - 1 \\ m \end{array}\right]_q = \frac{(q; q)_{N-1}}{(q; q)_m (q; q)_{N-1-m}},$$

where $(a; q)_n := \prod_{i=0}^{n-1} (1 - a q^i)$. There are representation-theoretic meanings of this series which will be discussed later (Sect. 6). For example if $N = 6$, $m = 2$ then

$$\left[\begin{array}{c} 5 \\ 2 \end{array}\right]_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$
then $F = s_i E$ with $(i, i + 1) \in E^C \times E$ and inv$(E) = n$. Let $T_E = \psi_E + p_E$ with $p_E \in \mathcal{P}_{m,0}^{(n-1)}$, and $b = (c(i, E) - c(i + 1, E))^{-1}$ then

$$T_F = s_i T_E - b T_E = s_i \psi_E + s_i p_E + b T_E,$$

and $s_i p_E + b T_E \in \mathcal{P}_{m,0}^{(n)}$. By Proposition 3 (3) $s_i \psi_E = \psi_{s_i E} = \psi_F$. \qed

**Corollary 3** Suppose $E \in \mathcal{E}_0$ and $c(i, E) - c(i + 1, E) = \epsilon$ with $\epsilon = \pm 1$, for some $i < N$, then $s_i T_E = \epsilon T_E$.

**Proof** Let $p = s_i T_E - \epsilon T_E$. By an equation similar to those in the proof of Theorem 1 one finds that $\omega j p = c(i + 1, E) p , \omega j p = c(i, E) p$, and $\omega j p = c(j, E) p$ for $j \neq i, i + 1$. These eigenvalues are impossible because $i, i + 1$ are in the same row of $Y_E$ if $\epsilon = 1$, or the same column if $\epsilon = -1$. Thus $p = 0$. \qed

We have shown that $\{\psi_E : E \in \mathcal{E}_0\}$ is a basis, implicitly based on the degree of $(N - m, 1^m)$ being $(N - m - 1, 1^m)$. Explicitly it can be shown (with straightforward computation) that if $# E = m + 1$ and $N \notin E$ then $\psi_E = \sum_{j \in E} \sigma(s(j, E)) \psi_{(j,N)}E$ (recall $(j,N) E = (E \setminus \{j\}) \cup \{N\}$). Also $\{T_E : E \in \mathcal{E}_0\}$ is an orthogonal basis.

### 3.2 The submodule of isotype $(N - m + 1, 1^{m-1})$

In order to complete the proof that $\mathcal{P}_m \cong (N - m, 1^m) \oplus (N - m + 1, 1^{m-1})$ we will use the duality map $\delta$ from $(m, 1^{N-m})$ to $(N - m + 1, 1^{m-1})$. Begin by transferring the above results to $(m, 1^{N-m})$ by interchanging $m$ and $N - m$. We use $\perp$ to mark the corresponding objects: $\mathcal{E}_0^\perp = \{E : # E = N - m + 1, N \in E\}$, $E_0^\perp = \{m, m + 1, \ldots, N\}$. Thus $T_{E_0^\perp} = \psi_{E_0^\perp} = \sum_{j=m}^{N} (-1)^{N-j} \mathcal{E}_0^\perp \phi_{E_0^\perp}(j)$ and

$$\delta T_{E_0^\perp} = \sum_{j=m}^{N} (-1)^{N-j} \sigma \left( \text{inv} \left( E_0^\perp \setminus \{j\} \right) \right) \phi_{(E_0^\perp \setminus \{j\})}^C.$$

Let $E_1 = (E_0^\perp)^C = \{1, 2, \ldots, m - 1\}$ and $(E_0^\perp \setminus \{j\})^C = E_1 \cup \{j\}$ for $m \leq j \leq N$. Also $\text{inv} \left( E_0^\perp \setminus \{j\} \right) = j - m$ and thus

$$\delta T_{E_0^\perp} = (-1)^{N-m} \sum_{j=m}^{N} \phi_{E_1 \cup \{j\}}.$$

Since $s_i \delta = -\delta s_i$ for all $i < N$ we see (from Theorem 1) that $\omega i \left( \delta T_{E_0^\perp} \right) = -c(i, E_0^\perp) \delta T_{E_0^\perp}$ for $1 \leq i \leq N$ and $-c(i, E_0^\perp) = -(m - i)$ for $1 \leq i \leq m - 1$ and $-c(i, E_0^\perp) = (N - i)$ for $m \leq i \leq N$. Thus the respective $\{\omega i\}$ eigenvalues of $\delta T_{E_0^\perp}$ coincide with the content vector of $Y_{E_0^\perp}$. So the $\mathcal{S}_N$-module generated by $\delta T_{E_0^\perp}$
is of isotype \((N - m + 1, 1^{m-1})\), of degree \(\binom{N-1}{m-1}\) and this establishes the decomposition of \(P_m\) (the sum of the degrees of \((N - m, 1^m)\) and \((N - m + 1, 1^{m-1})\) is \(\binom{N-1}{m-1} + \binom{N-1}{m-1} = \binom{N}{m}\)). Motivated by the formula for \(\delta T_{E_0}^\perp\) we make the following:

**Definition 10** Suppose \(\#E = m - 1\) then

\[
\eta_E := \sum_{j \notin E} \sigma \left( s (j, E) \right) \phi_{E \cup \{j\}}.
\]

Thus \(\delta T_{E_0}^\perp = (-1)^{N-m} \eta_{E_1}\). Suppose \(E \in E_0^\perp\) then \(\#E^C = m - 1\) and \(N \notin E\), accordingly define \(E_1 = \{E : \#E = m - 1, N \notin E\}\) and \(P_{m,1} = \text{span}\{\eta_E : E \in E_1\}\). The corresponding definition of \(Y_E\) (Definition 7) is as follows:

**Definition 11** Suppose \(E \in E_1\) and \(E = \{i_1, \ldots, i_{m-1}\}\), \(E^C = \{j_1, \ldots, j_{N-m+1}\}\) with \(i_1 < i_2 < \cdots \) and \(j_1 < j_2 < \cdots < j_{N-m+1} = N\) then \(Y_E\) is the RSYT of shape \((N - m + 1, 1^{m-1})\) given by \(Y_E[k, 1] = \ell_{m+1-k} - \ell_{m+1-k}^1\) for \(2 \leq k \leq m\), \(Y_E[1,k] = j_{N-m+2-k}\) for \(1 \leq k \leq N - m + 1\).

Example: let \(N = 9, m = 3, E = \{1, 2\}\) then

\[
Y_E = \begin{bmatrix}
8 & 7 & 6 & 5 & 4 & 3 \\
0 & 1 & & & & \\
\end{bmatrix}.
\]

The following is used to find the formula relating \(\delta \psi^F\) to \(\eta_E\) for \(F \in E_0^\perp\) with \(F = E^C\).

**Lemma 1** Suppose \(j \in E\), then

\[
\text{inv} \left( E \setminus \{j\} \right) + s \left( j, E \right) + s \left( j, E^C \right) = \text{inv} \left( E \right) + (\#E - 1).
\]

**Proof** Since \(\text{inv} \left( E \right) = \# \{(i,k) \in E \times E^C : i < k\}\) we see that the set of pairs being counted for \(\text{inv} \left( E \setminus \{j\} \right)\) omits \(\{(j,k) : k \in E^C\}\) and includes \(\{(i,j) : i \in E, i < j\}\). The cardinalities of these two sets are \(s \left( j, E^C \right)\) and \((\#E - 1 - s \left( j, E \right))\), respectively. \(\square\)

**Proposition 5** Suppose \(E \in E_1\) then \(\delta \psi^F_{E^C} = (-1)^{N-m} \sigma \left( \text{inv} \left( E^C \right) \right) \eta_E\).

**Proof** Let \(F = E^C\) then

\[
\delta \psi^F_{E^C} = \delta \sum_{j \in F} \sigma \left( s \left( j, F \right) \right) \phi_{F \setminus \{j\}} = \sum_{j \in E} \sigma \left( s \left( j, F \right) + \text{inv} \left( F \setminus \{j\} \right) \right) \phi_{(F \setminus \{j\})^C}
\]

\[
= \sum_{j \notin E} \sigma \left( -s \left( j, E \right) + \text{inv} \left( F \right) + \#F - 1 \right) \phi_{E \cup \{j\}}
\]

\[
= (-1)^{N-m} \sigma \left( \text{inv} \left( F \right) \right) \sum_{j \notin E} \sigma \left( s \left( j, E \right) \right) \phi_{E \cup \{j\}}
\]

by the Lemma applied to \(F\). \(\square\)
We are ready to define the basis elements of $\mathcal{P}_{m,1}$ corresponding to the RSYT of shape $(N - m + 1, 1^{m-1})$.

**Definition 12** For $E \in \mathcal{E}_1$ let $T_E := (-1)^{N-m} \sigma \left( \text{inv} \left( E^C \right) \right) \delta T_{E_1}^{-}$.

**Proposition 6** Suppose $E \in \mathcal{E}_1$ and $1 \leq i \leq N$ then $\omega_i T_E = -c \left( i, E^C \right) T_E$.

Thus $T_E$ corresponds to the transpose of $Y_{E_1}$. The content vector $\left[ c \left( i, E \right) \right]_{i=1}^{N}$ of $E$ is given by $c \left( i, E \right) = -1 - s \left( i, E \right)$ if $i \in E$ and $c \left( i, E \right) = s \left( i, E^C \right)$ if $i \notin E$ (so that $c \left( i, E \right) = c \left( i, Y_E \right)$). From Definition 8 it follows that $c \left( i, E \right) = -c \left( i, E^C \right)$ for all $i$. This subsection concludes with the transformation properties of $T_E$ derived from $\delta$.

**Proposition 7** Suppose $E \in \mathcal{E}_1$, $i \in E$, and $i + 1 \in E^C \setminus \{N\}$ then $T_{siE} = s_i T_E - (c \left( i, E \right) - c \left( i + 1, E \right))^{-1} T_F^\perp$.

**Proof** Set $F = E^C$ then $T_{siF}^{-} = s_i T_F^{-} - (c \left( i, F \right) - c \left( i + 1, F \right))^{-1} T_F^\perp$ by Theorem 3. Apply $\delta$ to the equation:

$$\delta T_{siE}^{-} = -s_i \delta T_F^{-} - (c \left( i, F \right) - c \left( i + 1, F \right))^{-1} \delta T_F^{-}.$$

$$(-1)^{N-m} \sigma \left( \text{inv} \left( siF \right) \right) T_{siE} = (-1)^{N-m} \sigma \left( \text{inv} \left( F \right) \right)$$

$$\times \left( -s_i - (c \left( i, F \right) - c \left( i + 1, F \right))^{-1} \right) T_E,$$

$$T_{siE} = \left( s_i + (c \left( i, F \right) - c \left( i + 1, F \right))^{-1} \right) T_E,$$

because $\text{inv} \left( siF \right) = \text{inv} \left( F \right) + 1$. Furthermore $c \left( i, F \right) - c \left( i + 1, F \right) = -c \left( i, E \right) + c \left( i + 1, E \right)$.

Notice that the direction $E \rightarrow siE$ decreases $\text{inv} \left( E \right)$ and $E_1$ maximizes $\text{inv} \left( E \right)$ in $\mathcal{E}_1$. The set $\{ T_E : E \in \mathcal{E}_1 \}$ is an orthogonal basis for $\mathcal{P}_{m,1}$.

### 3.3 Projections

Define $\partial \theta_i$ by $\partial \theta_i \phi_E = 0$ and $\partial \theta_i \left( \theta_i \phi_E \right) = \phi_E$ where $i \notin E$, thus for $i \in E$

$$\partial \theta_i \phi_E = \sigma \left( \# \left\{ j \in E : j < i \right\} \right) \phi_{E\setminus\{i\}}.$$

Let $\tilde{\theta}_i$ denote the multiplication operator $\phi_E \mapsto \theta_i \phi_E = \sigma \left( \# \left\{ j \in E : j < i \right\} \right) \phi_{E\setminus\{i\}}$ if $i \notin E$, otherwise $\tilde{\theta}_i \phi_E = 0$. If $\#E = k$ then $\delta \theta_i \phi_E = (-1)^{N-k} \theta_i \delta \phi_E$ (straightforward proof). Let $M := \sum_{i=1}^{N} \tilde{\theta}_i$, $D := \sum_{i=1}^{N} \partial \theta_i$. With the usual calculations it can be shown that
\[ MD + DM = N, \]
\[ \delta D\phi_E = \sigma \left( #E^C \right) M\delta\phi_E, \]
\[ (\ker D) \cap \mathcal{P}_m = \mathcal{P}_{m,0}, \]
\[ (\ker M) \cap \mathcal{P}_m = \mathcal{P}_{m,1}, \]
\[ \delta (\ker D) \subset \ker M. \]

Also \((MD)^2 = N(MD)\), \((DM)^2 = N(DM)\), so the eigenvalues of \(MD, DM\) are \(N, 0\). Thus \(\frac{1}{N}MD, \frac{1}{N}MD\) are the projections \(\mathcal{P}_m \to \mathcal{P}_{m,0}, \mathcal{P}_m \to \mathcal{P}_{m,1}\), respectively.

### 3.4 Norms

Recall that \(\{\phi_E : #E = m\}\) is an orthonormal basis for \(\mathcal{P}_m\), and each \(s_i\) is a self-adjoint isometry (This can be implemented by defining dual variables \(\widehat{\theta}_i\) with the property \(\widehat{\theta}_i\theta_j = \delta_{ij}\), and if \(E = \{i_1, \ldots, i_k\}\) with \(i_1 < i_2 < \cdots < i_k\) then define \(\phi_E = \widehat{\theta}_{i_k} \cdots \widehat{\theta}_{i_2} \theta_{i_1}\); thus \(\langle \phi_F, \phi_E \rangle = \phi_F^* \phi_E\)) Hence each \(\omega_i\) is self-adjoint and two eigenvectors with at least one different \(\{\omega_i\}\)-eigenvalue are orthogonal to each other. Suppose for some \(f \in \mathcal{P}_m\) there are \(i, b\) such that \(f' = s_i f - bf\) satisfies \(\langle f, f' \rangle = 0\) then \(\langle s_i f, f \rangle = b |f|^2\) and \(|f'|^2 = |s_i f|^2 - 2b \langle s_i f, f \rangle + b^2 |f|^2 = (1 - b^2) |f|^2\) since \(|s_i f|^2 = |f|^2\). In the context of the previous subsections \(|T_{s_i E}|^2 = (1 - b^2) |T_E|^2\).

By definition \(|T_{E_0}|^2 = m + 1\) and \(|T_{E_1}|^2 = N - m + 1\).

**Proposition 8** Suppose \(E \in \mathcal{E}_0\) then
\[
|T_E|^2 = (m + 1) \prod_{1 \leq i < j < N \atop (i, j) \in E \times E^C} \left( 1 - \frac{1}{(c(i, E) - c(j, E))^2} \right).
\]

**Proof** Argue implicitly by induction on inv \((E)\). The product for \(E = E_0\) is empty (= 1). For each \(E\) let \(h(i, j; E) = 1 - ((c(i, E) - c(i + 1, E))^2).\) Suppose \((k, k + 1) \in E^C \times E\), then inv \((s_k E) = \text{inv}(E) + 1,\) and \(h(i, j; E) = h(i, j; s_k E)\) for \(i, j \neq k, k + 1\). Also \(h(i, k + 1; E) = h(i, k; s_k E)\) for \(i < k\) and \(i \in E; h(k + 1, j; E) = h(k, j; s_k E)\) for \(j > k + 1\) and \(j \neq E.\) \(|T_{s_k E}|^2\) differs from \(|T_E|^2\) by the extra factor \(h(k, k + 1; s_k E) = h(k, k + 1, E)\) \(\square\)

**Proposition 9** Suppose \(E \in \mathcal{E}_1\) then
\[
|T_E|^2 = (N - m + 1) \prod_{1 \leq i < j < N \atop (i, j) \in E^C \times E} \left( 1 - \frac{1}{(c(i, E) - c(j, E))^2} \right).
\]

**Proof** This follows from the duality map \(\delta\) being an isometry and using the previous formula for \(E^C \in \mathcal{E}_0\). \(\square\)
4 Superpolynomials and nonsymmetric Jack polynomials

Here we extend the polynomials in \( \{ \theta_i \} \) by adjoining \( N \) commuting variables \( x_1, \ldots, x_N \) (that is \( [x_i, x_j] = 0, [x_i, \theta_j] = 0, \theta_i \theta_j = -\theta_j \theta_i \) for all \( i, j \)). Each polynomial is a sum of monomials \( x^\alpha \phi_E \) where \( E \subset \{ 1, 2, \ldots, N \} \) and \( \alpha \in \mathbb{N}_0^N \), \( x^\alpha = \prod_{i=1}^N x_i^{a_i} \). The partitions in \( \mathbb{N}_0^N \) are denoted by \( \mathbb{N}_0^{N,+} \) (\( \lambda \in \mathbb{N}_0^{N,+} \) if and only if \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \)). The fermionic degree of this monomial is \#\( E \) and the bosonic degree is \( |\alpha| = \sum_{i=1}^N \alpha_i \). Let \( s\mathcal{P}_m = \text{span} \{ x^\alpha \phi_E : \alpha \in \mathbb{N}_0^N, \#E = m \} \). Then using the decomposition \( \mathcal{P}_m = \mathcal{P}_{m,0} \oplus \mathcal{P}_{m,1} \) let

\[
\begin{align*}
s\mathcal{P}_{m,0} &= \text{span} \left\{ x^\alpha \psi_E : \alpha \in \mathbb{N}_0^N, E \in \mathcal{E}_0 \right\}, \\
s\mathcal{P}_{m,1} &= \text{span} \left\{ x^\alpha \eta_E : \alpha \in \mathbb{N}_0^N, E \in \mathcal{E}_1 \right\}.
\end{align*}
\]

The symmetric group \( S_N \) acts on \( s\mathcal{P}_m \) by \( wp (x, \theta) = p (xw, \theta w) \) (recall \( (xw)_i = x_{w(i)} \) and \( (\theta w)_i = \theta_{w(i)} \) for \( 1 \leq i \leq N \)).

**Definition 13** The Dunkl and Cherednik–Dunkl operators are \( (1 \leq i \leq N, p \in s\mathcal{P}_m) \)

\[
\begin{align*}
\mathcal{D}_i p (x; \theta) &= \frac{\partial p (x; \theta)}{\partial x_i} + \kappa \sum_{j \neq i} \frac{p (x; \theta (i, j)) - p (x (i, j); \theta (i, j))}{x_i - x_j}, \\
\mathcal{U}_i p (x; \theta) &= \mathcal{D}_i (x_i p (x; \theta)) - \kappa \sum_{j=1}^{i-1} p (x (i, j); \theta (i, j)).
\end{align*}
\]

The same commutation relations as for the scalar case hold, that is,

\[
\begin{align*}
\mathcal{D}_i \mathcal{D}_j &= \mathcal{D}_j \mathcal{D}_i, \quad \mathcal{U}_i \mathcal{U}_j = \mathcal{U}_j \mathcal{U}_i, \quad 1 \leq i, j \leq N \\
wp \mathcal{D}_i &= \mathcal{D}_w(i) w, \forall w \in S_N; \quad s_j \mathcal{U}_i = \mathcal{U}_i s_j, \quad j \neq i - 1, i; \\
s_i \mathcal{U}_i s_i &= \mathcal{U}_i + 1 + \kappa s_i, \quad \mathcal{U}_i s_i = s_i \mathcal{U}_i + 1 + \kappa, \quad \mathcal{U}_i + 1 s_i = s_i \mathcal{U}_i - \kappa.
\end{align*}
\]

The three relations in the third line are mutually equivalent. The simultaneous eigenfunctions of \( \{ \mathcal{U}_i \} \) are called (vector-valued) nonsymmetric Jack polynomials (NSJP). They are the special case for the \( S_N \)-representations \( (N - m, 1^m) \) and \( (N - m + 1, 1^m-1) \) of the polynomials constructed by Griffith [10] for the complex reflection groups \( G (n, p, N) \). For generic \( \kappa \) these eigenfunctions form a basis of \( s\mathcal{P}_m \) (generic means that \( \kappa \neq \frac{m}{n} \) where \( m, n \in \mathbb{Z} \) and \( 1 \leq n \leq N \)). They have a triangularity property with respect to the partial order \( \triangleright \) on compositions, which is derived from the dominance order:

\[
\alpha < \beta \iff \sum_{j=1}^i \alpha_j \leq \sum_{j=1}^i \beta_j, \quad 1 \leq i \leq N, \quad \alpha \neq \beta,
\]

\[
\alpha < \beta \iff (|\alpha| = |\beta|) \land \left( (\alpha^+ < \beta^+) \lor (\alpha^+ = \beta^+ \land \alpha < \beta) \right).
\]
The rank function on compositions is involved in the formula for an NSJP.

**Definition 14** For $\alpha \in \mathbb{N}_0^N$, $1 \leq i \leq N$

$$r_\alpha (i) = \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \},$$

then $r_\alpha \in S_N$.

A consequence is that $r_\alpha \alpha = \alpha^+$, the nonincreasing rearrangement of $\alpha$, for any $\alpha \in \mathbb{N}_0^N$. For example if $\alpha = (1, 2, 0, 5, 4, 5)$ then $r_\alpha = [5, 4, 6, 1, 3, 2]$ and $r_\alpha \alpha = \alpha^+ = (5, 5, 4, 2, 1, 0)$ (recall $w_\alpha i = \alpha_{w^{-1}(i)}$). Also $r_\alpha = I$ if and only if $\alpha \in \mathbb{N}_0^{N,+}$.

For each $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{E}_0$ or $E \in \mathcal{E}_1$ there is a NSJP $J_{\alpha,E}$ with leading term $x^\alpha (r_\alpha^{-1}T_E)$, that is,

$$J_{\alpha,E} (x; \theta) = x^\alpha \left( r_\alpha^{-1}T_E \right) + \sum_{\alpha > \beta} x^\beta v_{\alpha,\beta,T} (\kappa; \theta), \quad (4.1)$$

where $v_{\alpha,\beta,T} (\kappa; \theta) \in \mathcal{P}_{m,0}$ or $\mathcal{P}_{m,1}$, respectively. The coefficients of the polynomials $v_{\alpha,\beta,T} (\kappa; \theta)$ are rational functions of $\kappa$. Note $r_\alpha^{-1}T_E (\theta) = T_E (\theta r_\alpha^{-1})$. These polynomials satisfy

$$U_i J_{\alpha,E} = \zeta_{\alpha,E} (i) J_{\alpha,E},$$

$$\zeta_{\alpha,E} (i) := \alpha_i + 1 + \kappa c (r_\alpha (i), E), \ 1 \leq i \leq N.$$

For detailed proofs see [9].

**Example 1** $N = 4, m = 2$, $\alpha = (0, 1, 1, 0)$, $E = \{2, 3, 4\} \in \mathcal{E}_0$, $[c (j, E)]_{j=1}^4 = [1, -2, -1, 0]$

$$J_{\alpha,E} = \left(x_2 x_3 - \frac{\kappa x_2 x_4}{1 - 2\kappa}\right) (-\theta_1 \theta_3 + \theta_1 \theta_4 - \theta_2 \theta_4)$$

$$+ \frac{\kappa x_2 x_4}{(1 - 2\kappa)(1 + \kappa)} \{(1 - \kappa) \theta_1 \theta_2 - (1 - 2\kappa) (\theta_1 \theta_3 - \theta_2 \theta_3) - \kappa (\theta_1 \theta_4 - \theta_2 \theta_4)\},$$

$$\zeta_{\alpha,E} = [1 - \kappa, 2 + \kappa, 2 - 2\kappa, 1].$$

We collect formulas for the action of $s_i$ on $J_{\alpha,E}$. They will be expressed in terms of the spectral vector $\zeta_{\alpha,E} = [\alpha_i + 1 + \kappa c (r_\alpha (i), E)]_{i=1}^N$ and (for $1 \leq i < N$)

$$b_{\alpha,E} (i) = \frac{\kappa}{\zeta_{\alpha,E} (i) - \zeta_{\alpha,E} (i + 1)} = \frac{\kappa}{\alpha_i - \alpha_{i+1} + \kappa (c (r_\alpha (i), E) - c (r_\alpha (i + 1), E))}.$$

The formulas are consequences of the commutation relationships: $s_j U_i = U_i s_j$ for $j < i - 1$ and $j > i$; $s_i U_i s_i = U_{i+1} + \kappa s_i$ for $1 \leq i < N$. We examine the
action of $s_i$ on $J_{\alpha,E}$ with $i < N$. Observe that the formulas manifest the equation
\[(s_i + b_{\alpha,E}(i)) (s_i - b_{\alpha,E}(i)) = 1 - b_{\alpha,E}(i)^2.\]
Suppose that $\alpha_i \neq \alpha_{i+1}$, then
(1) $\alpha_i < \alpha_{i+1}$ implies
\[(s_i - b_{\alpha,E}(i)) J_{\alpha,E} = J_{s_i\alpha,E}
\]
\[(s_i + b_{\alpha,E}(i)) J_{s_i\alpha,E} = \left(1 - b_{\alpha,E}(i)^2\right) J_{\alpha,E}.\]
(2) $\alpha_i > \alpha_{i+1}$ implies
\[(s_i - b_{\alpha,E}(i)) J_{\alpha,E} = \left(1 - b_{\alpha,E}(i)^2\right) J_{s_i\alpha,E},
\]
\[(s_i + b_{\alpha,E}(i)) J_{s_i\alpha,E} = J_{\alpha,E}.\]

Suppose that $\alpha_i = \alpha_{i+1}$ then let $j = r_\alpha(i)$ (thus $r_\alpha(i + 1) = j + 1$). By definition
\[b_{\alpha,E}(i) = (c(j,E) - c(j + 1, E))^{-1}.\]
Then
(1) \{$j, j + 1\} \subset E$ \(b_{\alpha,E}(i) = -1\) implies $s_i J_{\alpha,E} = -J_{\alpha,E},$
(2) \{$j, j + 1\} \cap E = \emptyset$ \(b_{\alpha,E}(i) = 1\) implies $s_i J_{\alpha,E} = -J_{\alpha,E},$
(3) \(j, j + 1\) $\in E^C \times E$ and $E \in \mathcal{E}_0$, or \(j, j + 1\) $\in E \times E^C$ and $E \in \mathcal{E}_1$ implies
\[(s_i - b_{\alpha,E}(i)) J_{\alpha,E} = J_{s_i\alpha,E},
\]
\[(s_i + b_{\alpha,E}(i)) J_{s_i\alpha,E} = (1 - b_{\alpha,E}(i)^2) J_{\alpha,E},\]
(4) \(j, j + 1\) $\in E \times E^C$ and $E \in \mathcal{E}_0$, or \(j, j + 1\) $\in E^C \times E$ and $E \in \mathcal{E}_1$ implies
\[(s_i - b_{\alpha,E}(i)) J_{\alpha,E} = (1 - b_{\alpha,E}(i)^2) J_{s_i\alpha,E},
\]
\[(s_i + b_{\alpha,E}(i)) J_{s_i\alpha,E} = J_{\alpha,E}.\]

The reason for the difference between the \(\mathcal{E}_0\) and \(\mathcal{E}_1\) cases is that the directions of inductively defining \(T_E\) are opposite. One other construction enables the determination of any $J_{\alpha,E}$ in a finite number of steps starting from $T_{E_0}$ for $E \in \mathcal{E}_0$, or from $T_{E_1}$ for $E \in \mathcal{E}_1$; this refers to the affine step. Let \(w_N = s_1 s_2 \cdots s_{N-1}\) (a cyclic shift) and define \(\Psi\) acting on $N$-vectors:
\[\Psi(a_1, a_2, \ldots, a_N) = (a_2, a_3, \ldots, a_N, a_1 + 1).\]

Then $J_{\Psi_\alpha,E} = x_N w_N^{-1} J_{\alpha,E}$ and $[\xi_{\Psi_\alpha,E}(i)]_{i=1}^N = \Psi [\xi_{\alpha,E}(i)]_{i=1}^N$. By a simple calculation we find $r_{\Psi_\alpha}(i) = r_\alpha(i + 1)$ for $1 \leq i < N$ and $r_{\Psi_\alpha}(N) = r_\alpha(1)$, that is, $r_{\Psi_\alpha}(i) = r_\alpha(w_N(i))$ for all $i$. Furthermore
\[x_N w_N^{-1} J_{\alpha,E} = x_N J_{\alpha,E} (x_N, x_1, x_2, \ldots, x_{N-1}; \theta_N, \theta_1, \ldots, \theta_{N-1}).\]

For example, if $\alpha = (2, 1, 4)$ then $x^\alpha = x_1^2 x_2 x_3^4$ and $x^{\Psi_\alpha} = x_1 x_2^4 x_3^3$, also $r_\alpha = [2, 3, 1]$ and $r_{\Psi_\alpha} = [3, 1, 2].$
Theorem 5
Suppose \( (\alpha, E) \) and there are arrows from \((\alpha, E)\) to \((\alpha \Psi, E)\), called affine steps, arrows from \((\alpha, E)\) to \((s_i \alpha, E)\) when \( \alpha_i < \alpha_{i+1} \), called steps, and arrows from \((\alpha, E)\) to \((\alpha, s_i E)\) when \((i, i+1) \in E \times E^C\) and \(E \in \mathcal{E}_0\), or \((i, i+1) \in E^C \times E\) and \(E \in \mathcal{E}_1\); the latter are called jumps (jumping from one set \(E\) to another). The graphs for \(s \mathcal{P}_{m,0}, s \mathcal{P}_{m,1}\) have the roots \((0, E_0)\) and \((0, E_1)\), respectively.

The graph makes it possible to define a symmetric bilinear form on \(s \mathcal{P}_m\), extending the inner product defined above for \( \mathcal{P}_m\), having the following properties \((f, g \in s \mathcal{P}_m)\)

\[
\langle w f, w g \rangle = \langle f, g \rangle, \quad w \in S_N, \\
\langle x_i f, g \rangle = \langle f, D_i g \rangle, \quad 1 \leq i \leq N, \\
\deg f \neq \deg g \implies \langle f, g \rangle = 0,
\]

where \(\deg f\) is the bosonic degree \((\deg x^\alpha = |\alpha|)\). As a consequence the operators \(\mathcal{U}_i\) are self-adjoint for this form and \((\alpha, E) \neq (\beta, F)\) implies \(\{ J_{\alpha,E}, J_{\beta,F} \} = 0\), because of the eigenvector property. To allow more concise statements of the theorems establishing the value of \(\| J_{\alpha,E} \|^2\) we first define two products.

Definition 15
For \(\lambda \in \mathbb{N}_0^{N,+}, \alpha \in \mathbb{N}_0^N, E \in \mathcal{E}_0 \cup \mathcal{E}_1\), and \(z = 0, 1\) let

\[
\mathcal{P}(\lambda, E) = \prod_{i=1}^{N} (1 + \kappa c(\lambda, i,E))^{\lambda_i} \prod_{1 \leq i < j \leq N}^{\lambda_i - \lambda_j} \prod_{l=1}^{\lambda_i - \lambda_j} \left( 1 - \frac{\kappa}{l + \kappa (c(i,E) - c(j,E))} \right)^2,
\]

\[
\mathcal{R}_z(\alpha, E) = \prod_{1 \leq i < j \leq N}^{\alpha_j - \alpha_i + \kappa (c(r_\alpha(j),E) - c(r_\alpha(i),E))} \left( 1 + \frac{\kappa}{\alpha_j - \alpha_i} \right)^{-1},
\]

and let \(\mathcal{R}(\alpha, E) = \mathcal{R}_0(\alpha, E) \mathcal{R}_1(\alpha, E)\) (\(\mathcal{R}\) is for “rearrangement”).

Theorem 4
Suppose \(\lambda \in \mathbb{N}^{N,+}\) and \(E \in \mathcal{E}_0 \cup \mathcal{E}_1\) then

\[
\| J_{\lambda,E} \|^2 = \| T_E \|^2 \mathcal{P}(\lambda, E).
\]

The norm of \(J_{\alpha,E}\) uses the auxiliary product.

Theorem 5
Suppose \(\alpha \in \mathbb{N}_0^N, E \in \mathcal{E}_0 \cup \mathcal{E}_1\) then

\[
\| J_{\alpha,E} \|^2 = \mathcal{R}(\alpha, E)^{-1} \| J_{\alpha,+,E} \|^2.
\]

These formulas show that the bilinear form is positive-definite for \(-\frac{1}{N} < \kappa < \frac{1}{N}\); the typical term in the product has the form \((l \geq 1)\)

\[
\frac{(l + \kappa (c(i,E) - c(j,E) + 1)) (l + \kappa (c(i,E) - c(j,E) - 1))}{(l + \kappa (c(i,E) - c(j,E)))^2}
\]
and \(|c(i, E) - c(j, E)| \leq N-1\) (maximum value with the cells \([[1, N-m], [m+1, 1]]\) for \(E_0\) and with \([[1, N-m+1], [m, 1]]\) for \(E_1\); thus each factor is positive. The formulas are the type-A specialization of Griffith’s results [10].

**Example 2** \(N = 4, m = 2, \alpha = (0, 1, 1, 0), E = \{2, 3, 4\} \in E_0, [c(j, E)]_j^4 = [1, -2, -1, 0], \zeta_{\alpha, E} = [1 - \kappa, 2 + \kappa, 2 - 2\kappa, 1]\)

\[
J_{\alpha, E} = \left(\frac{\kappa x_2 x_3 - \frac{\kappa x_2 x_4}{1-2\kappa}}{1-2\kappa}\right) (-\theta_1 \theta_3 + \theta_1 \theta_4 - \theta_3 \theta_4)
\]

\[
\|J_{\alpha, E}\|^2 = \frac{3(1 - 3\kappa)(1 + 2\kappa)(1 - \kappa)}{(1 + \kappa)(1 - 2\kappa)}.
\]

5 Supersymmetric polynomials

A supersymmetric polynomial of fermionic degree \(m\) is a polynomial \(p \in sP_m\) which satisfies \(wp = p\) for all \(w \in S_N\); the minimal equivalent condition is \(p(xs_i; \theta s_i) = p(x; \theta)\) for \(1 \leq i < N\). In this section, we consider such polynomials which arise as \(\sum wJ_{\alpha, E}\) for some fixed \(\alpha, E\). These polynomials are simultaneous eigenfunctions of the commutative set \(\left\{\sum_{i=1}^N U_i^k : 1 \leq k \leq N\right\}\). The tableaux \(Y_E\) are useful for labeling \(S_N\)-orbits of \(J_{\alpha, E}\).

**Definition 16** For \(\alpha \in N_N^N, E \in E_0 \cup E_1\) let \([\alpha, E]\) denote the tableau obtained from \(Y_E\) by replacing \(i\) by \(\alpha_i^{+}\) for \(1 \leq i \leq N\). Let \(M(\alpha, E) = \text{span}\{J_{\beta, F} : [\beta, F] = [\alpha, E]\}\).

Example: let \(N = 8, m = 3, E = \{2, 5, 7, 8\}, \alpha = (3, 5, 6, 2, 1, 4, 4), \alpha^{+} = (6, 5, 4, 4, 3, 2, 2, 1), and \)

\[
[\alpha, E] = \begin{bmatrix}
1 & 2 & 4 & 4 & 6 \\
\circ & 2 & 3 & 5
\end{bmatrix}.
\]

**Theorem 6** ([9, Prop. 5.2]) Suppose \(\alpha \in N_N^N, E \in E_0 \cup E_1\), then there is a series of transformations of the form \(as_i + b\) mapping \(J_{\alpha, E}\) to \(J_{\beta, F}\) if and only if \([\beta, F] = [\alpha, E]\).

It is a consequence of the transformation rules that if \([\beta, F] = [\alpha, E]\) then the spectral vector \(\zeta_{\beta, F}\) is a permutation of \(\zeta_{\alpha, E}\). Furthermore \(M(\alpha, E)\) is an \(S_N\)-module.

**Theorem 7** ([9, Thm. 5.9]) Suppose \(\alpha \in N_N^N, E \in E_0 \cup E_1\) and \([\alpha, E]\) is column-strict (the entries in column 1 are strictly decreasing) then there is a unique symmetric polynomial (up to constant multiplication) in \(M(\alpha, E)\) otherwise there is no nonzero \(S_N\)-invariant.

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The paper [3] defined a superpartition with $N$ parts and fermionic degree $m$ as an $N$-tuple $(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)$ which satisfies $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m$ and $\Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_N$. Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{E}_0$ and $[\alpha, E]$ is column strict, then $\Lambda_i = [\alpha, E][m + 2 - i, 1]$ for $1 \leq i \leq m$ and $\Lambda_i = [\alpha, E][1, N + 1 - i]$ for $m + 1 \leq i \leq N$, and also $\Lambda_m > \Lambda_N$. Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in \mathcal{E}_1$ and $[\alpha, E]$ is column strict, then $\Lambda_i = [\alpha, E][m + 1 - i, 1]$ for $1 \leq i \leq m$ and $\Lambda_i = [\alpha, E][1, N + 2 - i]$ for $m + 1 \leq i \leq N$, and also $\Lambda_m \leq \Lambda_N$ (because $\Lambda_m = [\alpha, E][1, 1]$ and $\Lambda_N = [\alpha, E][1, 2]$). Thus the inequalities $\Lambda_m > \Lambda_N$ and $\Lambda_m \leq \Lambda_N$ distinguish $\mathcal{E}_0$ from $\mathcal{E}_1$.

Suppose $[\alpha, E]$ is column-strict. We will determine formulas for the supersymmetric polynomial in $\mathcal{M}(\alpha, E)$ and its norm. This material is the specialization of results in [9]. The idea is based on a technique of Baker and Forrester [1]. Some auxiliary concepts are required. Consider the set of $F$ such that $[\alpha, F] = [\alpha, E]$. Among these there is one minimizing $\text{inv}(F)$ and one maximizing $\text{inv}(F)$.

**Definition 17** Suppose $E \in \mathcal{E}_0$ then the root $E_R$ and the sink $E_S$ satisfy

$$\text{inv}(E_R) = \min \{ \text{inv}(F) : [\alpha, F] = [\alpha, E] \},$$

$$\text{inv}(E_S) = \max \{ \text{inv}(F) : [\alpha, F] = [\alpha, E] \}.$$ 

The root and the sink are produced by minimizing the entries of $F$ in row 1, respectively, minimizing the entries of $F$ in column 1. For $E \in \mathcal{E}_1$ the definitions of $E_R$ and $E_S$ are reversed.

Example: Let $N = 10$, $m = 3$, $E = \{1, 4, 7, 10\}$, $\alpha = (3322221100)$ so that

$$Y_E = \begin{bmatrix} 10 & 9 & 8 & 6 & 5 & 3 & 2 & 0 & 7 & 4 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad [\alpha, E] = \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix};$$

there are 16 sets $F$ with $[\alpha, F] = [\alpha, E]$, and $E_R = \{2, 6, 8, 10\}$, $E_S = \{1, 3, 7, 10\}$.

There is an analog of the product $\mathcal{R}_z(\alpha, E)$ for content vectors:

**Definition 18** For $E \in \mathcal{E}_0$, $F \in \mathcal{E}_1$, and $z = 0$, 1 let

$$C_z^{(0)}(E) = \prod_{1 \leq i < j \leq N} \left( 1 + \frac{(-1)^z}{c(i, E) - c(j, E)} \right),$$

$$C_z^{(1)}(F) = \prod_{1 \leq i < j \leq N} \left( 1 + \frac{(-1)^z}{c(i, F) - c(j, F)} \right).$$

**Theorem 8** Suppose $\lambda \in \mathbb{N}_0^{N,+}$, $E \in \mathcal{E}_k$, $k = 0$ or 1, and $[\lambda, E]$ is column-strict then

$$p_{\lambda,E} = \sum_{[\alpha, F] \in T(\lambda, E)} \frac{\mathcal{R}_1(\alpha, F)}{C_1^{(k)}(F)} J_{\alpha,F}.$$
is the supersymmetric polynomial in \( M(\alpha, E) \), unique when the coefficient of \( x^\lambda \) is
\[
\sum_{(\lambda, F) \in T(\lambda, E)} \frac{1}{C_1^{(k)}(F)} T_F.
\]

**Proof** We show that \( s_i p_{\lambda, E} = p_{\lambda, E} \) for each \( i < N \). The space \( M(\alpha, E) \) decomposes as a sum of one and two-dimensional spaces invariant under the action of \( \alpha \).

This leads to adjacency relations for the coefficients in \( \sum (\alpha, F) A(\alpha, F) J_{\alpha, F} \). There are 3 cases (1) \( \alpha_i < \alpha_{i+1} \) (2) \( \alpha_i = \alpha_{i+1} \), \( |c r_\alpha (i) - c r_\alpha (i + 1) - F| \geq 2 \), (3) \( \alpha_i = \alpha_{i+1} + c (r_\alpha (i) - c r_\alpha (i + 1) - F) = 1 \). The case \( \alpha_i = \alpha_{i+1} + c (r_\alpha (i) - c r_\alpha (i + 1) - F) = 1 \) does not occur because \( [\lambda, E] \) is column-strict. In case (3) \( s_i J_{\alpha, F} = J_{\alpha, F} \). Now suppose \( \alpha_i < \alpha_{i+1} \) and \( b_{\alpha, F} (i) = \frac{\xi_i (\alpha, F) - \xi_{i+1} (\alpha, F)}{\alpha} \), so that
\[
\frac{\xi_i (s_i \alpha, F) - \xi_{i+1} (s_i \alpha, F)}{\alpha} = -b_{\alpha, F} (i).
\]

The transformation formula
\[
(s_i - b_{\alpha, E} (i)) J_{\alpha, F} = J_{s_i \alpha, F}
\]
implies \( A(\alpha, F) = (1 + b_{\alpha, F} (i)) A(s_i \alpha, F) \). Combining this with
\[
\frac{R_\zeta (\alpha, F)}{R_\zeta (s_i \alpha, F)} = 1 - (-1)^{\zeta} b_{\alpha, F} (i)
\]
(which follows from the product for \( a \) having the extra pair \( (i, i + 1) \) with \( \alpha_i < \alpha_{i+1} \)) leads to
\[
A(s_i \alpha, F) = A(\alpha, F) = A(\lambda, E),
\]

since there is a series of steps transforming \( \alpha \) to \( \lambda = \alpha^+ \).

Suppose \( \alpha_i = \alpha_{i+1} \) and \( j = r_\alpha (i) \), then let \( b_F (j) = (c (j, F) - c (j + 1, F))^{-1} \).

If \( E \in E_0 \) suppose \( (j, j + 1) \in F^C \times (F \setminus \{N\}) \) or if \( E \in E_1 \) then suppose \( (j, j + 1) \in F \times (F^C \setminus \{N\}) \); in both cases \( s_i - b_{\alpha, F} (j) \) \( J_{\alpha, F} = J_{s_j \alpha, F} \). This implies \( A(\alpha, F) = (1 + b_F (j)) A(\alpha, s_j F) \). Combine with \( k = 0, 1 \)
\[
\frac{C^{(k)}_\zeta (s_j E)}{C^{(k)}_\zeta (E)} = 1 - (-1)^{\zeta} b_F (j)
\]
to obtain
\[
\frac{A(\alpha, F)}{A(\alpha, s_j F)} = 1 + b_F (j) = \frac{C^{(k)}_1 (s_j F)}{C^{(k)}_1 (F)},
\]
for \( E \in E_k \). So every coefficient in \( \sum (\alpha, F) A(\alpha, F) J_{\alpha, F} \) can be expressed in terms of \( A(\lambda, E) \), which is now the unique undetermined constant.

Besides the supersymmetry property the polynomial \( p_{\lambda, E} \) satisfies
\[
\sum_{j=1}^N U_j p_{\lambda, E} = \sum_{j=1}^N \left( \lambda_j + 1 + \kappa c (i, E_R) \right)^s p_{\lambda, E}
\]
for $s \geq 1$; this relation holding for $1 \leq s \leq N$ together with supersymmetry defines $p$ (up to a multiplicative constant).

Next we describe the process for computing the norm of $p_{\lambda, E}$. Recall that $\|J_{\lambda, E}\|^2 = |T_E|^2 \mathcal{P}(\lambda, E)$ and $\|J_{\alpha, E}\|^2 = |T_E|^2 \mathcal{P}(\alpha^+, E) (\mathcal{R}_0(\alpha, E) \mathcal{R}_1(\alpha, E))^{-1}$ (Theorem 5). If $[\lambda, F] = [\lambda, E]$ then $\mathcal{P}(\lambda, F) = \mathcal{P}(\lambda, E)$ (some of the factors in the product may be permuted). To determine $\|p\|^2$ fix some $(\alpha, F) \in [\lambda, E_R]$ then

$$\sum_{w \in S_N} w J_{\alpha, F} = c p_{\lambda, E}$$

for some constant $c$ (because there is only one invariant in $\mathcal{M}(\alpha, E)$). For now suppose $E \in \mathcal{E}_0$. Then

$$c \|p_{\lambda, E}\|^2 = \sum_{w \in S_N} \{w J_{\alpha, F}, p_{\lambda, E}\} = N! \{J_{\alpha, F}, p_{\lambda, E}\}$$

(5.1)

$$= N! \frac{\mathcal{R}_1(\alpha, F)}{C_1^{(0)}(F)} \|J_{\alpha, F}\|^2 = N! \frac{\mathcal{R}_1(\alpha, F)}{C_1^{(0)}(F)} \frac{|F|^2 \mathcal{P}(\lambda, F)}{\mathcal{R}_1(\alpha, F) \mathcal{R}_0(\alpha, F)}$$

$$= N! \mathcal{P}(\lambda, E_R) (m + 1) \frac{C_0^{(0)}(F)}{\mathcal{R}_0(\alpha, F)}.$$

The constant $c$ can be determined for $\alpha = \lambda^-$ (the nondecreasing rearrangement of $\lambda$) and $F = E_R$. Let $w_0 = r_{\lambda^-}$ so that $w_0 \lambda^- = \lambda$. By the triangular property of $\triangleright$ and the minimal property of $\lambda^-$ it follows that $J_{\lambda^-, E_R} = x^{\lambda^-} w_0^{-1} T_{E_R} + \sum_{\beta \triangleleft \lambda^-} x^{\beta} v_{\lambda^- \beta, E_R}(\kappa; \theta)$ (see Formula (4.1)) where $\beta^+ \neq \lambda (\beta^+ < \lambda)$. Thus $x^\lambda$ appears in $\sum_{w \in S_N} w J_{\lambda^-, E_R}$ exactly when $w = w_1 w_0$ and $w_1 \in G_\lambda$, the stabilizer of $\lambda$, and so $\sum_{w \in G_\lambda} w T_{E_R}$ becomes the relevant quantity.

**Lemma 2** The coefficient of $T_{E_S}$ in $\sum_{w \in G_\lambda} w T_{E_R}$ is $\# G_{E_R}$, the order of the stabilizer of $T_{E_R}$.

**Proof** The stabilizer of $\lambda$ is a direct product of symmetric groups of intervals $(S[a, b] = \{w \in S_N : w(i) = i, i < a, i > b\})$, thus if $\lambda_{a-1} > \lambda_a = \lambda_{a+1} = \cdots = \lambda_b > \lambda_{b+1}$ then $S[a, b]$ is a factor of $G_\lambda$. There are two possibilities for such an interval (2) $a, a + 1, \ldots, b$ are all in row 1 of $Y_{E_R}$ then $w \in S[a, b]$ implies $w T_{E_R} = T_{E_R}$ and $w J_{\lambda, E_R} = J_{\lambda, E_R}$ (2) one of $a, a + 1, \ldots, b$ is in column 1 (excluding $[1, 1]$) of $Y_{E_R}$. In case (2) $b$ is in column 1 for $E_R$ and $a$ is in column 1 for $E_S$ (reversed for $E \in E_1$).

For simplification of the argument assume there is only one interval in case (2): then by the transformation laws (Theorem 3)

$$s_a s_{a+1} \cdots s_{b-1} T_{E_R} = T_{E_S} + \sum \{b E T_E : \text{inv}(E_R) \leq \text{inv}(E) < \text{inv}(E_E)\}$$

and $w(b) = a$ implies $w = s_a s_{a+1} \cdots s_{b-1} w_1$ where $w_1(b) = b$, that is, there are $(b - a)!$ permutations taking $b$ to $a$. In general the coefficient of $T_{E_S}$ in $\sum_{w \in G_\lambda} w T_{E_R}$ is a product to which the intervals in case (1) or case (2) contribute $(b_k - a_k + 1)!$, $(b_k - a_k)!$, respectively. The product is exactly $\# G_{E_R}$. \hspace{1cm} \Box
Theorem 9 Suppose $p_{\lambda, E}$ is as defined in Theorem 8 and $E_R \in \mathcal{E}_k$ with $k = 0, 1$ then

$$
\| p_{\lambda, E} \|^2 = v_k \frac{N!}{\#G_{ER}} \frac{C^{(k)}_0 (E_R) \mathcal{P} (\lambda, E_R)}{R_0 (\lambda^-, E_R) C^{(k)}_1 (E_S)},
$$

with $v_0 = m + 1, v_1 = N - m + 1$.

Proof From formula (5.1) we find

$$
c p_{\lambda, E} = \sum_{w \in S_N} w J_{\lambda^-, E_R} = c \sum_{[\alpha, F] \in T(\lambda, E)} \frac{R_1 (\alpha, F)}{C^{(k)}_1 (F)} J_{\alpha, F}
$$

$$
c \| p_{\lambda, E} \|^2 = v_k N! \mathcal{P} (\lambda, E_R) \frac{C^{(k)}_0 (E_R)}{R_0 (\lambda^-, E_R)}.
$$

In the first line the equation for the coefficient of $x^\lambda T_{E_S}$ is $\#G_{ER} = \frac{c}{C^{(k)}_1 (E_S)}$ (by Lemma 2). Dividing the second line by $c = (\#G_{ER}) C^{(k)}_1 (E_S)$ leads to the stated formula for $\| p_{\lambda, E} \|^2$.

Example 3 $N = 4, \lambda = (2, 1, 1, 0), E = \{2, 3, 4\} \in \mathcal{E}_0$ (isotype $(2, 1, 1)$)

$$
[\lambda, E] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},
$$

$$
p_{\lambda, E} = \theta_1 \theta_2 (x_1 - x_2) \{(x_1 x_2 + x_3 x_4) (x_3 + x_4) - 2 x_3 x_4 (x_1 + x_2)\} + \ldots
$$

$$
= (\theta_1 \theta_2 + \theta_1 \theta_3 - 2 \theta_1 \theta_4 + \theta_2 \theta_4 + \theta_3 \theta_4) x_1^2 x_2 x_3 + \ldots
$$

and

$$
\| p_{\lambda, E} \|^2 = * (1 - 2 \kappa) (2 - 3 \kappa) (1 - 4 \kappa),
$$

$$
\sum_{i=1}^4 U_i^2 p_{\lambda, E} = 6 \left( \kappa^2 - 2 \kappa + 3 \right) p_{\lambda, E}.
$$

There is a simplification for the ratio $\mathcal{P} (\lambda, E_R) \mathcal{R}_0 (\lambda^-, E_R)$ by use of

$$
\mathcal{R}_0 (\lambda^-, E_R) = \prod_{1 \leq \ell < k \leq N} \left( 1 + \frac{\kappa}{\lambda_\ell - \lambda_k + \kappa (c (\ell, E_R) - c (k, E_R))} \right),
$$

which follows from Definition 15 by setting $i = w^{-1} (k), j = w^{-1} (\ell)$ where $w = r_{\lambda^-}$; then $\lambda^-_i = \lambda_\ell, \lambda^-_j = \lambda_k$. It is possible that $i < j$ does not imply $k > \ell$ (if $\lambda_i = \lambda_j$) but $\lambda_\ell > \lambda_k$ does imply $\ell < k$ and $i < j$. In the product replace $\ell$ by $i$ and...
k by j, and cancel out part of the factor in $\mathcal{P}(\lambda, E_R)$ for $l = \lambda_i - \lambda_j$. Introduce a utility product to allow a more concise formula:

$$\Pi_0 (n, d) := \left(1 - \frac{\kappa}{n + d\kappa}\right) \prod_{l=1}^{n-1} \left(1 - \frac{\kappa}{l + d\kappa}\right)^2,$$

then

$$\frac{\mathcal{P}(\lambda, E_R)}{\mathcal{R}_0(\lambda^-, E_R)} = \prod_{i=1}^{N} (1 + \kappa c(i, E_R)\lambda_i) \prod_{1 \leq i < j \leq N} \prod_{\lambda_i > \lambda_j} \Pi_0 \left(\lambda_i - \lambda_j, c(i, E_R) - c(j, E_R)\right).$$

As Griffeth [10] pointed out this is a complicated product that conceals possible cancellations of numerator and denominator factors, because the content vector need not have consecutive entries differing by 1, as happens in the scalar case where $[c(i, T)]_{i=1}^{N} = [N - 1, N - 2, \ldots, 1, 0]$. The norm formulas are qualitatively different from those constructed in [5] because here the zeros and poles can occur on both positive and negative rational values of $\kappa$. There are some special cases of low degree where the norm formula reduces to a polynomial in $\kappa$; to be discussed in the next section.

### 6 Poincaré series and generators

This section concerns combinatorial aspects and generating supersymmetric polynomials by multiplying basis elements by symmetric polynomials. We showed that the supersymmetric polynomials in $s\mathcal{P}_m$ are labeled by column-strict tableaux of shape $(N - m, 1^m)$ and $(N - m + 1, 1^{m-1})$. Stanley [14, p. 379] found the formula for the number $t_n$ of tableaux of a given shape $\tau$, with entries nondecreasing in each row and each column, and with the sum of the entries equalling $n$

$$\sum_{n=0}^{\infty} t_n q^n = H_\tau(q) = \prod_{(i, j) \in \tau} \left(1 - q^{h(i, j; \tau)}\right)^{-1},$$

where the product is over the boxes of the Ferrers diagram of $\tau$ and $h(i, j; \tau)$ is the length of the hook at $(i, j)$. To modify this series for column-strict tableaux add $j - 1$ to each entry of row $j$ for $1 \leq j \leq \ell(\tau)$, thus adding $n_\tau := \sum_{j=1}^{\ell(\tau)} (j - 1) \tau_j$ to $n$. Recall the shifted $q$-factorial: $(a; q)_n = \prod_{i=1}^{n} \left(1 - aq^{i-1}\right)$. For $\tau = (N - m, 1^m)$ it is easy to find $H_\tau(q) = \{(1 - q^N)(q; q)_m(q; q)_{N-m-1}\}^{-1}$ and $n_\tau = \frac{m(m+1)}{2}$. Replace $m$ by $m - 1$ to obtain these quantities for $(N - m + 1, 1^{m-1})$. Thus the number of supersymmetric polynomials of isotype $(N - m, 1^m)$ and bosonic degree $n$ is the coefficient of $q^n$ in
described above. The idea is to involve two other meanings of the Gaussian coefficient.

For example let \( \tau = (2, 1, 1) \) then the series begins \( q^3 + 2q^4 + 4q^5 + 6q^6 + 10q^7 + \ldots \).

Suppose there are supersymmetric polynomials \( p_1, p_2, \ldots, p_k \in \mathcal{P}_{m,0} \) of bosonic degrees \( n_1, n_2, \ldots, n_k \) such that every supersymmetric polynomial \( f \in s\mathcal{P}_{m,0} \) has a unique expansion \( f (x; \theta) = \sum_{i=1}^{k} h_i (x) p_i (x; \theta) \) where each \( h_i \) is symmetric \( (h_i (xw) = h_i (x) \) for all \( w \in S_N) \). Define the generating function \( Q_{N,m} (q) = \sum_{i=1}^{k} q^{n_i} \) then

\[
Q_{N,m} (q) \prod_{i=1}^{N} \left( 1 - q^i \right)^{-1} = q^{m(m+1)/2} \left\{ (1 - q^N) (q; q)_m (q; q)_{N-m-1} \right\}^{-1},
\]

because \((q; q)^{-1}_N\) is the Poincaré series for symmetric polynomials in \( N \) variables, graded by degree. Thus

\[
Q_{N,m} (q) = q^{m(m+1)/2} \frac{(q; q)_{N-1}}{(q; q)_m (q; q)_{N-m-1}} = q^{m(m+1)/2} \left[ \begin{array}{c} N-1 \\ m \end{array} \right]_q. \tag{6.1}
\]

The Gaussian coefficient is defined by \([n]_q = \frac{(q; q)_n}{(q; q)_{n-b}}\). Considering the decomposition \( s\mathcal{P}_m = s\mathcal{P}_{m,0} \oplus s\mathcal{P}_{m,1} \) we find

\[
Q_{N,m} (q) + Q_{N,m-1} (q) = q^{m(m-1)/2} \left[ \begin{array}{c} N \\ m \end{array} \right]_q.
\]

Since \( \lim_{q \to 1} [n]_q = [n] \) we note that \( Q_{N,m} (1) + Q_{N,m-1} (1) = \left( \frac{N}{m} \right) = \dim \mathcal{P}_m \); this is a consequence of a reciprocity theorem for representations. With a similar calculation we find the Poincaré series for the supersymmetric polynomials in \( s\mathcal{P}_{m,1} \) to be \( q^{m(m-1)/2} \left\{ (q; q)_m - (q; q)_{N-m} \right\}^{-1} \).

This information leads to determining the set \( \{ p_j (x; \theta) \} \) of supersymmetric polynomials which are eigenfunctions of \( \sum_{i=1}^{N} \mathcal{U}_i^s \) for \( s \geq 1 \) and which are generators as described above. The idea is to involve two other meanings of the Gaussian coefficient. Fix two integers \( k, \ell \geq 1 \) and consider two (well-known) counting problems:

1. \( A (k, \ell, n) = \# \left\{ (j_1, \ldots, j_k) : 0 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq k, \sum_{i=1}^{k} j_i = n \right\} \) (the number of partitions of \( n \) with length \( \leq \ell \)), and let \( a (k, \ell) = \sum_{n=0}^{\ell} A (k, \ell, n) q^n \);
2. \( B (k, \ell, n) = \# \{ F \subset \{ 1, 2, \ldots, k+\ell \} : \# F = \ell, \text{ inv} (E) = n \} \), and \( b (k, \ell) = \sum_{n=0}^{\ell} B (k, \ell, n) q^n \).

Then \( a (k, \ell) = \left[ \frac{k+\ell}{k} \right]_q = b (k, \ell) \). We sketch a proof for the first equation and set up a bijection to prove the second. Break up the sum for \( a (k, \ell) \) by summing over the values of \( j_\ell \): \( A (k, \ell, n) = \sum_{s=0}^{k} A (s, \ell - 1, n-s) \); this implies \( a (k, \ell) = \sum_{s=0}^{k} q^s a (s, \ell - 1) \) and \( a (k, \ell) - a (k-1, \ell) = q^k a (k, \ell - 1) \). Induction on \( m + k \) together with \( a (k, 0) = 1 = a (0, \ell) \) proves that \( a (k, \ell) = \left[ \frac{k+\ell}{k} \right]_q \).
For the bijection let $F = \{i_1, i_2, \ldots, i_\ell\}$ with $i_1 < \cdots < i_\ell$ and $\text{inv} (E) = n$, then for $1 \leq u \leq \ell$ define $j_u = \# \{v \in F^C : v > i_{\ell+1-u}\}$; thus $0 \leq \cdots \leq j_u \leq j_{u+1} \leq \cdots \leq k$ and $\sum_{u=1}^{\ell} j_u = n$. In the other direction given $[j_i]_{i=1}^{\ell}$ define $i_u = k + u - j_{\ell+1-u}$ for $1 \leq u \leq \ell$. This implies $i_u < i_{u+1}$. Let 
\[
F_u = \big\{ s \in F^C : s > i_u \big\} = \{i_u + 1, \ldots, k + \ell\} \setminus \{i_{u+1}, \ldots, i_\ell\}, 
\]
\[
\#F_u = (k + \ell - 1 - i_u) - (\ell - u) = k + u - i_u = j_{\ell+1-u}.
\]
Thus $\text{inv} (F) = \sum_{u=1}^{\ell} j_u$, and this proves the bijection.

In the present situation the set $F = \{Y_E[i,1]\}_{i=2}^{N-m-1}$ where $E \in \mathcal{E}_0$ (or $2 \leq i \leq N - m$ for $E \in \mathcal{E}_1$). Suppose the supersymmetric polynomial is in $\mathcal{M}(\lambda, E)$ (Definition 16) with $E \in \mathcal{E}_0$; the minimal condition for a column-strict tableau is $[\lambda, E][i,1] = i - 1$ for $1 \leq i \leq m$ and $[\lambda, E][1,i] \leq [\lambda, E][1,i+1]$ for $1 \leq i \leq N - m - 1$. Imposing the additional condition that $[\lambda, E][1,i] \leq m$ leads to the situation discussed above, namely a one-to-one correspondence between sets $E$ in $\mathcal{E}_0$ and $(N - m)$-tuples $[0, j_1, \ldots, j_{N-m-1}]$ such that $\text{inv} (E^C) = \sum_{u=1}^{\ell} j_u$. There is an analogous statement for $\mathcal{E}_1$.

**Example 4** Let $N = 8$, $m = 3$, $\lambda = (3, 2, 2, 1, 1, 1, 1, 0)$ and

$$[\lambda, E] = \begin{bmatrix} \circ & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

then $k = 3$, $\ell = 4$, $F = \{3 + 1 - 2, 3 + 2 - 2, 3 + 3 - 1, 3 + 4 - 1\} = \{2, 3, 5, 6\}$ and $E = \{1, 4, 7, 8\}$. Thus

$$Y_E = \begin{bmatrix} 8 & 6 & 5 & 3 & 2 \\ \circ & 7 & 4 & 1 \end{bmatrix}$$

and $\text{inv} (E^C) = 6$ ($= 12 - \text{inv} (E)$). Note that $E = E_S$, the sink.

The set $E$ arising this way is the sink $E_S$ for $\mathcal{E}_0$, and the root $E_R$ for $\mathcal{E}_1$. There are as many tableaux $[\lambda, E]$ satisfying $[\lambda, E][i,1] = i - 1$ for $1 \leq i \leq m$ and $[\lambda, E][1,i] \leq m$ as generators of the supersymmetric polynomials (over the ring of symmetric polynomials) in $\mathcal{P}_{m,0}$; the degrees also match up with the generating function (6.1). It seems plausible that these are generators (there is no nontrivial linear relation with symmetric polynomial coefficients), considering that multiplying one of them by a nontrivial symmetric polynomial produces a polynomial containing $x_i^{m+1}$ but more is needed to prove this. In the next section we look at a minimal (in the sense of the dominance order on partitions) subset of the polynomials.

### 7 Norms of certain minimal polynomials

This section concerns the norms of the supersymmetric polynomial $p_{\lambda, E} \in \mathcal{M}(\lambda, E)$ where $[\lambda, E][i,1] = i - 1$ for $1 \leq i \leq m + 1$ and $[\lambda, E][1,j] \leq m$ for $1 \leq j \leq M$. Springer
(with $M = N - m$). To facilitate computation with $[\lambda, E_R]$ use $\mu_i$ for row 1, and $\tilde{\mu}_i$ for column 1 as described by

$$
\begin{pmatrix}
\tilde{\mu}_{m+1} & \mu_{M-1} & \mu_{M-2} & \ldots & \mu_3 & \mu_2 & \mu_1 \\
\circ & \tilde{\mu}_m & \tilde{\mu}_{m-1} & \ldots & \tilde{\mu}_2 & \tilde{\mu}_1 & \circ
\end{pmatrix}.
$$

Thus $[\mu_i]_{i=1}^{M-1}$ and $[\tilde{\mu}_i]_{i=1}^{m+1}$ are both partitions, and the latter is strictly decreasing. (This is essentially the same as the superpartition notation except for $\tilde{\mu}_{m+1}$.) The associated content values are $c_i = M - i$ for $1 \leq i \leq M - 1$ and $\tilde{c}_i = i - m - 1$ for $1 \leq i \leq m + 1$. Pick parameters $s, k$ with $0 \leq s \leq m - 1$ and $1 \leq k \leq M - 2$ and define $[\lambda, E]$ by $\tilde{\mu}_i = m + 1 - i$ for $1 \leq i \leq m + 1$, and $\mu_i = s + 1$ for $1 \leq i \leq k$, $\mu_i = s$ for $k + 1 \leq i \leq M - 1$. That is,

$$
[\lambda, E] = \begin{pmatrix}
(M) & (M-1) & \ldots & (k+1) & (k) & \ldots & (1) \\
0 & s & \ldots & s & s+1 & \ldots & s+1 \\
\circ & 1 & 2 & \ldots & m-1 & m & \circ
\end{pmatrix}.
$$

We will prove a denominator-free formula for $\|p_{\lambda,E}\|^2$. Heuristically this is possible since $\lambda$ is $\prec$-minimal among $\lambda'$ with the same column 1 and the same sum of entries in row 1 (for example if $M = 5$ and the row-total is 6 then $(2, 2, 1, 1)$ is $\prec$-minimal).

**Theorem 10** Suppose $[\lambda, E]$ is given by (7.2) and $0 \leq s < m$ then

$$
\|p_{\lambda,E}\|^2 = **s! \prod_{i=1}^{k-1} (1 + i \kappa) \prod_{j=1}^{M-k-2} (1 + j \kappa)_s \prod_{l=M-k-1}^{M-2} (2 + l \kappa)_s \\
\times \prod_{i=2}^{m} (1 - i \kappa)_{i-1} (1 - \kappa N)_{m-s-1} (m - s + 1 - \kappa (m + 1))_s \\
\times (m - s - \kappa (N - k)),
$$

and the asterisk $*$ stands for $v_k \frac{NC_0^{(k)}(E_R)}{\#G_{E_R\prec_1}^{(k)}(E_S)}$, which is independent of $\kappa$.

The proof involves several lemmas. When $k = 0$ so that $\mu_i = s$ for $1 \leq i \leq M - 1$ the formula reduces to

$$
**s! \prod_{j=1}^{M-2} (1 + j \kappa)_s \prod_{i=2}^{m} (1 - i \kappa)_{i-1} (1 - \kappa N)_{m-s} (m - s + 1 - \kappa (m + 1))_s.
$$

There are $|\tilde{\mu}|$ factors of the form $(a - b \kappa)$ and $|\mu| - 1$ factors $(a + b \kappa)$ with $a, b \geq 1$. The following are used to compute telescoping products:

**Lemma 3** Suppose $u \leq v$, $n \geq 1$ and $a$ is arbitrary then

$$
\prod_{i=u}^{v} \Pi_0(n, a + i) = \frac{(1 + \kappa (a + u - 1))_n (1 + \kappa (a + v + 1))_{n-1}}{(1 + \kappa (a + u))_{n-1} (1 + \kappa (a + v))_n}.
$$
**Proof** The first part of the $i$-product (where $l = n$) equals

$$\prod_{i=u}^{v} \left( \frac{n + \kappa (a + i - 1)}{n + \kappa (a + i)} \right) = \frac{n + \kappa (a + u - 1)}{n + \kappa (a + v)}$$

and the second part ($1 \leq l \leq n - 1$) equals

$$\prod_{\ell=1}^{n-1} \prod_{i=u}^{v} \left( \frac{\ell + \kappa (a + i - 1)}{(\ell + \kappa (a + i))^2} \right) \left( \frac{\ell + \kappa (a + u)}{\ell + \kappa (a + u))^2} \right) = \frac{(1 + \kappa (a + u - 1))_{n-1} (1 + \kappa (a + v + 1))_{n-1}}{(1 + \kappa (a + u))^2 (1 + \kappa (a + v))^2}.$$ 

Multiply the two parts together. \hfill \Box

**Lemma 4** Let $n \geq 1$ then

$$\prod_{j=1}^{n} \Pi_0 (j, -j) = \frac{1}{n (1 - \kappa)} \frac{(1 - \kappa (n + 1))_n}{(1 - \kappa n)^n-n}.$$ 

**Proof** Proceed by induction. The formula is true for $n = 1$ since $\Pi_0 (1, -1) = \frac{1-2\kappa}{1-\kappa}$. Suppose the formula is valid for $n$ and evaluate

$$\Pi_0 (n + 1, -n - 1) = \left( 1 - \frac{\kappa}{n + 1} (1 - \kappa) \right) \prod_{l=1}^{n} \frac{(l - \kappa n) (l - \kappa (n + 2))}{(l - \kappa (n + 1))^2} = \frac{(n + 1 - (n + 2) \kappa) (1 - \kappa n)_n (1 - \kappa (n + 2))}{(n + 1) (1 - \kappa (n + 1))^2} = \frac{n (1 - \kappa n)_{n-1} (1 - \kappa (n + 2))_{n+1}}{(n + 1) (1 - \kappa (n + 1))^2}.$$ 

This telescopes with the product over $1 \leq j \leq n$. \hfill \Box

**Lemma 5** Suppose $n \geq 1$ and $F (r)$ is a function of $r$ then

$$\prod_{r=0}^{n-1} \frac{(1 + F (r))_{n-r}}{(1 + F (r + 1))_{n-1-r}} = (1 + F (0))_n.$$ 

The formula in Lemma 3 can be used for $\prod_{i=u}^{v} \Pi_0 (n, a - i) = \prod_{j=-v}^{-u} \Pi_0 (n, a + j)$. 

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A superpolynomial version of nonsymmetric Jack polynomials

We split the computation of \( \frac{P(\lambda, E_R)}{R_0(\lambda^-, E_R)} \) into (1a) \((\lambda_{i'}, \lambda_{j'}) = (\mu_i, \mu_j)\), (1b) \((\lambda_{i'}, \lambda_{j'}) = (s + 1, \tilde{\mu}_j)\) and \(\tilde{\mu}_j > s + 1\), (1c) \((\lambda_{i'}, \lambda_{j'}) = (s, \tilde{\mu}_j)\) and \(\tilde{\mu}_j > s\); (2a) \((\lambda_{i'}, \lambda_{j'}) = (\tilde{\mu}_i, \tilde{\mu}_j)\), (2b) \((\lambda_{i'}, \lambda_{j'}) = (s + 1, \tilde{\mu}_j)\) and \(s + 1 > \tilde{\mu}_u\), (2c) \((\lambda_{i'}, \lambda_{j'}) = (s, \tilde{\mu}_j)\) and \(s > \tilde{\mu}_u\). The partial products are denoted \(P_{1a}, P_{1b}\) etc.

**Lemmas 3 and 5**

**Part (1a):**

\[
P_{1a} = \prod_{i=1}^{k} (1 + (M - i) \kappa)_{s+1} \prod_{j=k+1}^{M-1} (1 + (M - j) \kappa) \prod_{i=1}^{k} \prod_{j=k+1}^{M-1} \Pi_0(1, j - i)
\]

\[
= \prod_{i=1}^{k} (1 + (M - i) \kappa)_{s+1} \prod_{j=k+1}^{M-1} (1 + (M - j) \kappa) \prod_{i=1}^{k} \prod_{j=k+1}^{M-1} \frac{1 + \kappa (k - i)}{1 + \kappa (M - 1 - i)}
\]

\[
= \prod_{i=1}^{k-1} (1 + i \kappa) \prod_{i=1}^{k} (2 + \kappa (M - i)) \prod_{j=k+1}^{M-1} (1 + \kappa (M - j)) \frac{1 + \kappa (M - 1)}{1 + \kappa (M - k - 1)}.
\]

If \(k = 0\) then \(P_{1a} = \prod_{i=1}^{M-1} (1 + \kappa i)_{s}\).

**Part (1b):** \(\mu_i = s + 1 > \tilde{\mu}_u\), then \(m + 1 - u < s + 1, u > m - s\), set \(j = M + 1 - u\)

\[
P_{1b} = \prod_{j=0}^{s} \prod_{i=M-k}^{M-1} \Pi_0(s + 1 - j, i + j) = \prod_{j=0}^{s} \frac{(1 + \kappa (j + M - k - 1))_{s+1-j} (1 + \kappa (j + M))_{s-j}}{(1 + \kappa (j + M - k))_{s-j} (1 + \kappa (j + M - 1))_{s+1-j}} = \frac{(1 + \kappa (M - k - 1))_{s+1}}{(1 + \kappa (M - 1))_{s+1}}.
\]

**Part (1c):** \(\mu_i = s > \tilde{\mu}_u\), set \(j = M + 1 - u\)

\[
P_{1c} = \prod_{j=0}^{s-1} \prod_{i=1}^{M-k-1} \Pi_0(s - j, i + j) = \prod_{j=0}^{s-1} \frac{(1 + \kappa j)_{s-j} (1 + \kappa (j + M - k))_{s-j-1}}{(1 + \kappa (j + 1))_{s-j-1} (1 + \kappa (j + M - k - 1))_{s-j}} = \frac{(1)_s}{(1 + \kappa (M - k - 1))_s}.
\]
Then

\[
P_{1a} P_{1b} P_{1c} = \prod_{i=1}^{k-1} (1 + i \kappa) \prod_{i=1}^{k} (2 + \kappa (M - i)) s
\]

\[
\times \prod_{j=k+1}^{M-1} (1 + \kappa (M - j)) s \frac{1 + \kappa (M - 1)}{1 + \kappa (M - k - 1)}
\]

\[
\times s! (s + 1 + \kappa (M - k - 1))
\]

\[
\times \prod_{l=1}^{M-k-1} (1 + \kappa (M - 1)) s_{s+1} = s! \prod_{i=1}^{k-1} (1 + i \kappa) \prod_{j=1}^{M} (1 + j \kappa) s \prod_{l=1}^{M-k-1} (2 + l \kappa)
\]

Part (2a):

\[
P_{2a} = \prod_{i=1}^{m} (1 - i \kappa) \prod_{1 \leq i < j \leq m+1} \Pi_0 (j - i, i - j)
\]

\[
= \prod_{i=1}^{m} (1 - i \kappa)_{i-1} \left( m! (1 - \kappa)^m \right) \frac{1 - (m+1) \kappa}{m! (1 - \kappa)^m}
\]

\[
= \prod_{i=1}^{m+1} (1 - i \kappa)_{i-1}
\]

To prove this assume the formula

\[
\prod_{1 \leq i < j \leq n+1} \Pi_0 (j - i, i - j) = \Pi_n = \frac{(1 - (n+1) \kappa)_n}{n! (1 - \kappa)^n},
\]

which is valid for \( n = 1 \) by Lemma 4. Then

\[
\frac{\Pi_{n+1}}{\Pi_n} = \prod_{i=1}^{n+1} \Pi_0 (n + 2 - i, i - n - 2) = \prod_{j=1}^{n+1} \Pi_0 (j, -j)
\]

\[
= \frac{1}{(n+1) (1 - \kappa)} \frac{(1 - (n+2) \kappa)_{n+1}}{(1 - (n+1) \kappa)_n}
\]

This proves the formula for \( \Pi_n \) by induction and thus also the formula for \( P_{2a} \).
Part (2b): $\mu_i = s + 1 < \tilde{\mu}_u$, thus $s + 1 < m + 1 - u \leq m$ and

$$P_{2b} = \prod_{j=s+2}^m \prod_{i=M-k}^{M-1} \Pi_0 (j - s - 1, -j - i)$$

$$= \prod_{j=s+2}^m \frac{(1 - \kappa (M + j))_{j-s-1} (1 - \kappa (M + j - k - 1))_{j-s-2}}{(1 - \kappa (M + j - 1))_{j-s-2} (1 - \kappa (M - k + j))_{j-s-1}}$$

$$= \frac{(1 - \kappa (M + m))_{m-s-1}}{(1 - \kappa (M + m - k))_{m-s-1}}.$$

Part (2c): $\mu_i = s < \tilde{\mu}_u$, thus $s < m + 1 - u \leq m$ and (by Lemma 3 with $a = -j$, $u = 1 + k - M$, $v = -1$, and then by Lemma 5)

$$P_{2c} = \prod_{j=s+1}^m \prod_{i=1}^{M-k-1} \Pi_0 (j - s, -j - i)$$

$$= \prod_{j=s+1}^m \frac{(1 - \kappa (M + j - k))_{j-s} (1 - j \kappa)_{j-s-1}}{(1 - \kappa (M + j - k - 1))_{j-s} (1 - \kappa (j + 1))_{j-s}}$$

$$= \frac{(1 - \kappa (M + m - k))_{m-s}}{(1 - \kappa (m + 1))_{m-s}}.$$

Then

$$P_{2a} P_{2b} P_{2c} = \prod_{i=1}^{m+1} (1 - i \kappa)_{i-1} \frac{(1 - \kappa (M + m))_{m-s-1}}{(1 - \kappa (M + m - k))_{m-s-1}} \frac{(1 - \kappa (M + m - k))_{m-s}}{(1 - \kappa (m + 1))_{m-s}}$$

$$= \prod_{i=1}^m (1 - i \kappa)_{i-1} (m - s + 1 - \kappa (m + 1))_s (1 - \kappa (M + m))_{m-s-1}$$

$$\times (m - s - \kappa (M + m - k)).$$

This concludes the proof of Theorem 10, up to a constant independent of $\kappa$.

The content products for the $(N - m, 1^m)$ situations are computed with the usual telescoping (with $s \geq 1$)

$$C_0^{(0)} (E_R) = \frac{m!}{s! (M + s + 1)_{m-s-1} (M + s - k)},$$

$$C_1^{(0)} (E_S) = \frac{(M + s + 1)_m (M + s - k)_s!}{(m + 1)!},$$

$$\frac{C_0^{(0)} (E_R)}{C_1^{(0)} (E_S)} = \left\{ \frac{m!}{s! (M + s + 1)_{m-s-1} (M + s - k)} \right\}^2 \frac{m + 1}{m + M}.$$
If $s = 0$ then

$$C_0^{(0)} (E_R) = \frac{m!}{(M + 1)_{m-1} (M - k)}, C_1^{(0)} (E_S) = \frac{(M + 1)_m}{(m + 1)!}.$$  

Similar calculations can be used to find $C_0^{(1)} (E_R), C_1^{(1)} (E_R)$ for the $(N - m + 1, 1^{m-1})$ case (replace $m$ by $m - 1$ in the $(\mu, \tilde{\mu})$-notation). The order of the stabilizer group of $E_R$ is $\#G_{E_R} = k! (M - k - 1)!$ when $s \geq 1$ and $k! (M - k)!$ when $s = 0$.

The idea of polynomial-type norm formulas for minimal degree symmetric polynomials was studied for arbitrary irreducible $S_N$-modules in [6, Theorem 8] and for $G(n, r, N)$-modules by Griffith and the author [8, Sect. 3]. There is a connection between these minimal degree norm formulas and the concept of asphericity. A rational number $\kappa_0$ is said to be aspherical if the rational Cherednik algebra specialized to $\kappa = \kappa_0$ has a module which contains no symmetric (invariant) elements (this is the type-$A$ version). Suppose $M_0$ is a submodule of $s \mathcal{P}_{m,0}$ (that is, closed under $w \in S_N$, multiplication by $x_i$ and application of $\mathcal{D}_i$ for $1 \leq i \leq N$) and $M/M_0$ is aspherical then $p_{\lambda,E} \in M_0$ with $\lambda = (m, m - 1, \ldots, 0 \cdots, E = \{1, 2, \ldots, m, N\}$ and $\|p_{\kappa,E}\|^2 = 0$ (or else the definition of the inner product implies $1 \in M_0$ and $M_0 = s \mathcal{P}_{m,0}$). Thus $\kappa_0$ is a zero of $\prod_{i=2}^{m} (1 - \kappa x_i)_{i-1} (1 - \kappa N)_{m-1}$. namely $\kappa_0 = \frac{k}{n}$ with $1 \leq k < n \leq m$ or $\kappa_0 = \frac{k}{N}$ with $1 \leq k < m$. Any aspherical parameter associated with the module $s \mathcal{P}_{m,0}$ (of isotype $(N - m, 1^m)$) must have one of these values. Note that our choice of sign is the opposite of the convention used in much of the literature on aspherical parameters. There is a stronger concept called total asphericity, see Losev [13].

8 Further results

8.1 Antisymmetric polynomials

Recall the duality map $\delta$ from Definition 2. When this map is applied to $\mathcal{P}_{m,0}$ the NSJP’s transform to NSJP’s in $\mathcal{P}_{N-m,1}$ but with the parameter $\kappa$ changed to $-\kappa$. We use the notation $\mathcal{D} (\kappa), \mathcal{U} (\kappa)$ to indicate the parameter. Apply $\delta$ to $\mathcal{D} (\kappa), \sum_E p_E (x) \phi_E (\theta)$ to obtain

$$\delta \mathcal{D} (\kappa) p = \sum_E \frac{\partial}{\partial x_i} p_E (x) \delta \phi_E + \kappa \sum_{j \neq i} \sum_E \frac{p_E (x) - p_E (x (i,j))}{x_i - x_j} \delta (i, j) \phi_E$$

$$= \left\{ \sum_E \frac{\partial}{\partial x_i} p_E (x) \delta \phi_E - \kappa \sum_{j \neq i} \sum_E \frac{p_E (x) - p_E (x (i,j))}{x_i - x_j} (i, j) \delta \phi_E \right\}$$

$$= \mathcal{D} (-\kappa) \delta p.$$
Similarly $\delta U(\kappa); \; p = U(-\kappa); \; \delta p$. Suppose $\alpha \in \mathbb{N}_0^N$ and $E \in E_0$ then by (4.1)

$$\delta J_{\alpha,E}(x; \theta) = x^\alpha \delta \left( r^{-1}_{\alpha} T_E \right) + \sum_{\alpha > \beta} x^\beta \delta v_{\alpha,\beta,T}(\kappa; \theta),$$

and $\delta \left( r^{-1}_{\alpha} T_E \right) = \sigma \left( \ell \left( r^{-1}_{\alpha} \right) \right) r^{-1}_{\alpha} \delta T_E$ and $\delta T_E = (-1)^{e} T_{E^C}$ (the power of $-1$ can be determined from Proposition 5. Further $\ell \left( r^{-1}_{\alpha} \right) = \# \{ (i, j): i < j, w(i) > w(j) \}$. Thus $\delta J_{\alpha,E}(x; \theta) = (-1)^{b} J_{\alpha,E^C}(x; \theta)$, a NSJP for $-\kappa$. The spectral vector stays the same because $\alpha_i + 1 + \kappa c \left( r_{\alpha}(i), E \right) = \alpha_i + 1 - \kappa c \left( r_{\alpha}(i), E^C \right)$. Similar considerations apply to $E \in E_1$.

Suppose $p(x; \theta) = \sum_{E} p_E(x) \phi_E(\theta)$ is supersymmetric, that is $s_i p(x; \theta) = \sum_{E} p_E(xs_i) s_i \phi_E(\theta) = p(x; \theta)$ for $1 \leq i < N$ then $s_i \delta p = -\delta s_i p = -\delta p$ and $\delta p$ is antisymmetric. Thus by defining a supersymmetric Jack polynomial $p_{\lambda,E^C}$ in $\mathcal{P}_{N-m,1}$ using the appropriate modification of Theorem 8 with $-\kappa$ we obtain an antisymmetric polynomial $\delta p_{\lambda,E^C}$ which is an eigenfunction of $\sum_{i=1}^{N} U(\kappa_i)$. Suppose $E \in E_0$ and the tableau $[\lambda, E]$ is as in Definition 16 then $\mathcal{M}(\lambda, E)$ contains a unique (up to a constant multiple) nonzero antisymmetric polynomial if and only if $[\lambda, E]$ is row-strict (this applies only to row 1, of course; see [9, Theorem 5.12]).

**Example 5** let $N = 9, m = 3, E = \{3, 4, 6, 9\}$ and $\lambda = (5, 4, 3, 3, 3, 2, 2, 1, 0)$ then

$$Y_{EC} = \begin{bmatrix} 9 & 6 & 4 & 3 & 2 & 1 \\ 0 & 8 & 7 & 5 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} \lambda, E^C \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix},$$

Then $p_{\lambda,E^C} \in \mathcal{P}_{6,1}$ and is of isotype $(4, 1^5)$ while $\delta p_{\lambda,E^C}$ is antisymmetric and in $\mathcal{M}(\lambda, E)$, of isotype $(6, 1^3)$.

### 8.2 Wavefunctions on the torus

This is a sketch of how the polynomials $s \mathcal{P}_m$ can be interpreted on the unit circle. The quantum Calogero–Sutherland model for $N$ identical particles with $1/r^2$ interactions on the circle has the Hamiltonian

$$\mathcal{H} = - \sum_{i=1}^{N} \left( \frac{\partial}{\partial \phi_i} \right)^2 + \frac{1}{2} \sum_{1 \leq i < j \leq N} \frac{\kappa (\kappa - 1)}{\sin^2 \left( \frac{1}{2} (\phi_i - \phi_j) \right)}$$

$$= \sum_{i=1}^{N} \left( x_i \frac{\partial}{\partial x_i} \right)^2 - 2\kappa \sum_{1 \leq i < j \leq N} \frac{x_i x_j (\kappa - 1)}{(x_i - x_j)^2},$$
where \( x \in \mathbb{T}^N \); the torus and its surface measure in terms of polar coordinates are

\[
\mathbb{T}^N := \left\{ x \in \mathbb{C}^N : |x_j| = 1, 1 \leq j \leq N \right\},
\]

\[
dm(x) = (2\pi)^{-N} d\phi_1 \cdots d\phi_N, \quad x_j = \exp(i\phi_j), \quad -\pi < \phi_j \leq \pi, \quad 1 \leq j \leq N.
\]

The time-independent Schrödinger equation \( \mathcal{H}\psi = E\psi \) has solutions expressible as the product of the base state

\[
\psi_0(x) = \prod_{1 \leq i < j \leq N} \left| 2\sin \frac{\phi_i - \phi_j}{2} \right|^\kappa = \prod_{1 \leq i < j \leq N} \left( -\frac{(x_i - x_j)^2}{x_i x_j} \right)^{\kappa/2}
\]

with Jack polynomials (Lapointe and Vinet [12]). We generalized this result by introducing a matrix-valued base state [7]. In the present context this is a \( (N^m) \times (N^m) \)-matrix-valued function \( L \) of \( x \) defined on \( \mathbb{C}^N \setminus \bigcup_{i < j} \{ x : x_i = x_j \} \). Recall the dual basis \( \{ \hat{\phi}_E \} \) defined in 3.4 and express \( L \) as

\[
L(x) = \sum_{E,F} L_{EF}(x) \phi_E \otimes \hat{\phi}_F,
\]

\[
L(x) \sum_F a_F \phi_F = \sum_E \left( \sum_F L_{EF}(x) c_F \right) \phi_E.
\]

Because \( \mathcal{P}_m \) splits into two irreducible \( S_N \)-modules the matrix \( L(x) \) is also split into \( L_0(x), L_1(x) \) with

\[
(DM) L_0(x) = L_0(x) (DM) = NL_0(x),
\]

\[
(MD) L_1(x) = L_1(x) (MD) = NL_1(x).
\]

Let \( \gamma_0 = \frac{N - 2m - 1}{2} \) and \( \gamma_1 = \frac{N - 2m + 1}{2} \), the average contents for \( (N - m, 1^m) \), \( (N - m + 1, 1^{m-1}) \), respectively. The differential system for \( L(x) \) is (for \( s = 0, 1 \))

\[
\frac{\partial}{\partial x_i} L_s(x) = \kappa L_s(x) \left\{ \sum_{j \neq i} \frac{1}{x_i - x_j} (i, j) - \frac{\gamma_s}{x_i} I \right\}, \quad 1 \leq i \leq N,
\]

where \( L_s(x)(i, j) = \sum_{E,F} L_{EF}(x) \phi_E \otimes \hat{(i, j)} \phi_F \), and \( I \) is the identity matrix. The effect of the term \( \frac{\gamma_s}{x_i} I \) is to make \( L_s(x) \) homogeneous of degree 0, that is, \( L_s(cx) = L_s(x) \) for \( c \in \mathbb{C} \setminus \{0\} \); indeed

\[
\sum_{i=1}^N x_i \frac{\partial}{\partial x_i} L_s(x) = \kappa L_s(x) \left\{ \sum_{1 \leq i < j \leq N} (i, j) - N\gamma_s I \right\}.
\]
and if \( p(\theta) \in P_{m,0} \) then \( \sum_{i<j} (i, \ j) \ p(\theta) = \sum_{k=1}^{N} c(k, E_0) = \frac{(N-m)(N-m-1)}{2} - \frac{m(m+1)}{2} = \gamma_0 N \) (and similarly for \( P_{m,1} \)). Let \( T^N_{reg} = T^N \setminus \bigcup_{i<j} \{ x : x_i = x_j \} \), then \( T^N_{reg} \) has \((N-1)!\) connected components and each component is homotopic to a circle. Let \( x_0 := (1, e^{2\pi i}/N, e^{4\pi i}/N, \ldots, e^{2(N-1)\pi i}/N) \) and denote the connected component of \( T^N_{reg} \) containing \( x_0 \) by \( C_0 \), called the fundamental chamber. Thus \( C_0 \) is the set consisting of \((e^{i\theta_1}, \ldots, e^{i\theta_N})\) with \( \theta_1 < \theta_2 < \ldots < \theta_N < \theta_1 + 2\pi \). The homogeneity \( L(ux) = L(x) \) for \(|u| = 1\) shows that \( L(x) \) has a well-defined analytic continuation to all of \( C_0 \) starting from \( x_0 \). Briefly, there is a method to define \( L_s(x) \) on the other connected components of \( T^N_{reg} \) so that when restricted to supersymmetric polynomials \( L_s(x) \) is the supersymmetric Jack polynomial associated with \( \lambda, E \) is column-strict then

\[
L_s(x) \sum_{i=1}^{N} (u_i - 1 - \kappa \gamma_s)^2 L_s(x)^{-1} = \sum_{i=1}^{N} (x_i \partial_i)^2 - 2\kappa \sum_{1 \leq i < j \leq N} x_i x_j (\kappa - 1) (x_i - x_j)^2 = \mathcal{H}.
\]

Thus if \( p_{\lambda,E} \) is the supersymmetric Jack polynomial associated with \( \lambda, E \) where \( E \in E_0 \) and \( \lambda, E \) is column-strict then

\[
\mathcal{H} \left( L_0(x) \ p_{\lambda,E}(x; \theta) \right) = \sum_{i=1}^{N} (\lambda_i + \kappa (c(i, E) - \gamma_0))^2 L_0(x) \ p_{\lambda,E}(x; \theta).
\]

This is not the same Hamiltonian defined in [2]; in that paper the coupling constant satisfies \( \kappa > 1 \). In terms of the \((\mu, \tilde{\mu})\) notation from (7.1) the eigenvalue is

\[
\sum_{i=1}^{m+1} \left( \tilde{\mu}_i + \kappa \left( i - \frac{N+1}{2} \right) \right)^2 + \sum_{i=1}^{N-m-1} \left( \mu_i + \kappa \left( \frac{N+1}{2} - i \right) \right)^2.
\]

The supersymmetric polynomial of lowest degree has \( \mu_i = 0 \) for \( 1 \leq i \leq N-m-1 \) and \( \tilde{\mu}_i = m+1-i \) for \( 1 \leq i \leq m+1 \) and its eigenvalue is

\[
\frac{1}{6} m (m+1) \{(2m+1)(1+\kappa) - 3\kappa N\} + \frac{\kappa^2}{12} N \left( N^2 - 1 \right).
\]

To get the eigenvalues for the isotype \((N-m+1, 1^{m-1})\) replace \( m \) by \( m-1 \) in the formulas.

The adjoint map is defined by \( \left\{ \sum_E f_E(x) \phi_E(\theta) \right\}^* \sum_F g_F(x) \phi_F(\theta) = \sum_E f_E(x) g_E(x) \). There is a normalization of \( L_s(x) \) so that for \( f, g \)

\[
\int_{T^N} \langle L_s(x) \ f(x; \theta) \rangle^* L_s(x) \ g(x; \theta) \ dm(x) = \langle f, g \rangle_T,
\]
where \( \langle f, g \rangle_T \) is a conjugate-linear inner product on \( sP_m \) satisfying \( 1 \leq i \leq N \)

\[
\langle w f, w g \rangle_T = \langle f, g \rangle_T, \quad w \in S_N,
\]

\[
\langle x_i D_i f, g \rangle_T = \langle f, x_i D_i g \rangle_T,
\]

\[
\langle x_i f, x_i g \rangle_T = \langle f, g \rangle_T.
\]

This inner product is different from those studied in [5]. These properties imply that multiplication by \( x_i \) is an isometry and that each \( U_i \) is self-adjoint, thus \( (\alpha, E) \neq (\beta, F) \) implies \( \langle J_{\alpha, E}, J_{\beta, F} \rangle_T = 0 \). Also \( \langle J_{\alpha, E}, J_{\alpha, E} \rangle_T = \left\{ \prod_{i=1}^{N} (1 + \kappa c (i, E)) \right\}_{\alpha_i}^{-1} \| J_{\alpha, E} \|^2 \) (from Theorems 4 and 5). The details can be found in [7].

**Declarations**

**Conflict of interest** the author declares no conflict of interest.

**Funding** There is no external funding.

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