Yangian in the Twistor String

John Corn, Thomas Creutzig and Louise Dolan

Department of Physics and Astronomy
University of North Carolina, Chapel Hill, NC 27599, USA

Abstract

We study symmetries of the quantized open twistor string. In addition to global PSL(4|4) symmetry, we find non-local conserved currents. The associated non-local charges lead to Ward identities which show that these charges annihilate the string gluon tree amplitudes, and have the same form as symmetries of amplitudes in $\mathcal{N} = 4$ super conformal Yang Mills theory. We describe how states of the open twistor string form a realization of the PSL(4|4) Yangian superalgebra.

E-mail: jrcorn@physics.unc.edu, creutzig@physics.unc.edu, ldolan@physics.unc.edu
Contents

1 Introduction 2

2 PSL(4|4) Noether currents of the open twistor string 3

3 How the PSL(4|4) symmetry acts 8
   3.1 Action of PSL(4|4) Currents and Charges on Fields 8
   3.2 Action of ordinary Charges on the Tree Amplitudes 10

4 Non-local Noether currents and charges 14

5 More symmetry in the open twistor string 20

6 PSL(4|4) Yangian superalgebra symmetry 23

A Some super analysis 26
   A.1 The Lie superalgebra psl(N|N) 26
   A.2 Lie supergroups 27

B Structure constants and their properties 27

C Real form psl(4|4) 29

D Vanishing of the right-hand side of the Serre relation (6.4) on one field 31
1 Introduction

Twistor string theory \[1\]-\[3\] is equivalent to a massless field theory and provides a string structure to analyze four-dimensional massless scattering amplitudes with \(N = 4\) supersymmetry. At tree level, the spectrum can describe states of both super Yang Mills theory and conformal supergravity. The world sheet theory has a target in super twistor space, and has been argued to have a Yangian extension of its \(PSL(4|4)\) global symmetry \[1\].

In this paper we construct non-local conserved currents which lead to a Hopf algebra coproduct, show the associated charges annihilate the string gluon tree amplitudes and give a realization of the Yangian symmetry algebra for this world sheet action. We leave the conformal graviton amplitudes for future analysis. Explicit constructions of the amplitudes have revealed some of the string’s framework \[4\]-\[12\]. How the Yangian acts on the various building blocks of the theory \[13\]-\[17\] should help further unravel the theory, at least in the planar limit. The string also relates to other twistor methods \[18\]-\[22\].

The Yangian symmetry is expected because of the string’s close connection to Yang Mills theory, for which an infinite-dimensional symmetry for the planar theory has been displayed \[23\]-\[39\], and its appearance as dual conformal invariance was found explicitly for the amplitudes \[40\]-\[49\].

We review Yangians and their nature in field theory. A Yangian superalgebra is a superalgebra, that is an algebra that has bosonic and fermionic elements, and it is graded by the non-negative integers. The subalgebra of level zero is a Lie superalgebra and the grade one piece is its adjoint representation. The higher level parts of the superalgebra are generated by the algebra multiplication of elements of level one. They are subject to some restrictions which arise from the comultiplication. The comultiplication is an important additional structure that specifies how the elements of the Yangian superalgebra act on products of states. For an ordinary Lie superalgebra this is trivial

\[ Q_0(A B) = Q_0(A) B + (-1)^{|Q_0| A} A Q_0(B). \]  

But the elements of level one of the Yangian have non-trivial comultiplication. They act schematically as

\[ Q_1(A B) = Q_1(A) B + (-1)^{|Q_1| A} A Q_1(B) + Q_0(A) Q_0(B). \]  

Now one needs to require that this operation is compatible with the algebra product.
This can be ensured by the Serre relations. The question is how symmetries with such non-trivial action on products of states can arise in field theory. For a model specified by a Lagrangian plus a path integral measure, with both invariant under some global symmetry, usually we have corresponding conserved local Noether currents and charges. But the action of charges on products of states is then of the form (1.1). The way to realize a non-trivial action on products of fields is by introducing non-local conserved charges. Non-locality is then reflected in the comultiplication (1.2).

The world-sheet theory of the open twistor string is a two-dimensional conformal field theory, that is a local model. Introducing non-local fields thus seems strange. Nonetheless, in [50], it was shown how Yangian symmetry arises via non-local currents in Lie group Wess-Zumino-Witten models. We will follow their analysis in spirit, but we will use the non-local currents only as an intermediate step. We can do that due to the special nature of the global symmetry supergroup $\text{PSL}(4|4)$ of the twistor string. Our strategy is as follows. We start by introducing non-local currents as in [50], [51]. We show that the associated charges satisfy the comultiplication of the level one elements of the $\text{psl}(4|4)$ Yangian. Then we derive Ward identities for these charges. They simplify considerably due to the vanishing of the Killing form of $\text{psl}(4|4)$. These Ward identities lead us to define new operators acting on the fields of the twistor string. These operators form a realization of the Yangian. In addition they form a symmetry because of the Ward identities we found, and the new level one charges do not have the properties of the old charges, namely construction from a non-local current.

In section 2 we review the twistor string, describe the ordinary $\text{PSL}(4|4)$ Noether currents and discuss their canonical quantization in the boundary conformal field theory. In section 3 we give the action of these ordinary currents and their charges on the fields and the tree amplitudes. The non-local Noether currents are described in section 4, and their action on the fields and the string tree amplitudes is shown in section 5. In section 6, we discuss the Serre relations and the representation of the Yangian of $\text{PSL}(4|4)$. In the appendices we review superalgebras and give the $\text{PSL}(4|4)$ structure constants and their properties, including the vanishing of the adjoint quadratic Casimir (the Killing form).

2 $\text{PSL}(4|4)$ Noether currents of the open twistor string

The open twistor string can be described by the action

$$S = S_{YZ} + S_G + S_{\text{ghost}}, \quad (2.1)$$
with \( S_G \) given by a conformal field theory of central charge \( c = 28 \), \( S_{\text{ghost}} \) is the standard \( c = -26 \) ghost system of the bosonic string, and \( S_{YZ} \) is the world sheet action for fields with a target of twistor superspace,

\[
S_{YZ} = \int i \left[ Y^{\bar{z}I} D_{\bar{z}} Z_I + Y^{\bar{z}I} D_{\bar{z}} Z_I \right] g^{1/2} d^2 x
\]

(2.2)

where \( 1 \leq I \leq 8 \), \( D_\alpha = \partial_\alpha - i A_\alpha \), and \( g \) is the determinant of the world sheet metric with Euclidean signature, \( z = x_1 + i x_2 \), \( \bar{z} = x_1 - i x_2 \). The world sheet can be described as the upper half-plane, and thus has a boundary.

The equations of motion are

\[
(\partial_{\bar{z}} - i A_{\bar{z}}) Z_I = 0, \quad (\partial_z - i A_z) Z_I = 0, \quad Y^{\bar{z}I} + i A_{\bar{z}} Y^{\bar{z}I} = 0, \quad Y_{\bar{z}I} + i A_z Y_{\bar{z}I} = 0.
\]

(2.3)

The constraints from varying the world sheet gauge fields are

\[
Y^{\bar{z}I} Z_I = Y^{\bar{z}I} Z_I = 0,
\]

and the boundary conditions in upper half-plane world sheet coordinates are

\[
\overline{Z}_I = U Z_I, \quad Y^{\bar{z}I} = U^{-1} Y_{\bar{z}I},
\]

(2.4)

where \( U = e^{2i\alpha} \), for some function \( \alpha \), which varies and is real on the boundary, and is continuous up to multiples of \( \pi \). The reality conditions are \( \overline{Y^{\bar{z}I}} = -Y^{\bar{z}I} \) for bosonic components \( (1 \leq I \leq 4) \), and \( \overline{Y_{\bar{z}I}} = Y^{\bar{z}I} \) for fermionic components \( (5 \leq I \leq 8) \), and \( \overline{A_{\bar{z}}} = -A_{\bar{z}} \).

The twistor string has global PSL(4|4) symmetry. To write the Noether currents for PSL(4|4), we consider the following symmetry transformations of the world sheet action (2.1),

\[
Z_I \to e^{\rho_a T^a_I} Z_I, \quad \overline{Z}_I \to e^{\rho_a T^a_I} \overline{Z}_I,
\]

\[
Y^{\bar{z}I} \to Y^{\bar{z}J} e^{-\rho_a T^{aJ}} Y_{\bar{z}I}, \quad Y_{\bar{z}I} \to Y_{\bar{z}J} e^{-\rho_a T^{aJ}} Y_{\bar{z}I},
\]

(2.5)

where the generators of the Grassman envelope of the superalgebra psl(4|4), \( \rho_a T^a_I \), are real and have zero trace and zero supertrace. The other fields in (2.1) are singlets of psl(4|4) and ultimately singlets of the Yangian, so from now on we will discuss symmetries of
(2.2) See appendix A for some background on the supergroup PSL(4|4) and superalgebra psl(4|4). The infinitesimal transformations are

\[ \delta Z_I = \rho_a T^a_J Z_J, \quad \delta \bar{Z}_I = \rho_a \bar{T}^a_J \bar{Z}_J, \]

leaving the Lagrangian invariant. The Noether current is

\[ J^z = g^{z \bar{z}} j^z, \quad J^{\bar{z}} = g^{z \bar{z}} j^{\bar{z}}, \]

with \( j^z = i Y^z \rho_a T^a_J Z_J, \quad j^{\bar{z}} = i Y^{\bar{z}} \rho_a \bar{T}^a_J \bar{Z}_J, \) (2.7)

where \( \partial_\alpha J^\alpha = 0 \) and \( j^\alpha \) satisfies \( j^\alpha = 0 \) with use of the equations of motion (2.3), and also separately \( j^z_z = 0 \) and \( j^{\bar{z}}_{\bar{z}} = 0 \). So we also have

\[ dj = 0, \] (2.8)

In addition to (2.5), the identity matrix \( T^J_I = \delta^J_I \) and the \( U(1) \) R-symmetry matrix \( T^J_I = (-1)^{\deg I} \delta^J_I \) also generate symmetries of the classical action. Here \( \deg I = 0 \) when \( I \) is a bosonic index, and \( \deg I = 1 \) when \( I \) is a fermionic index. For simplicity we will denote the degree by the index, \( (-1)^{\deg I} \equiv (-1)^I \). The transformation associated with the identity matrix obeys the boundary condition (2.4) for any complex function \( \rho \), and generates the \( gl(1,\mathbb{C}) \) gauge invariance which we use to choose \( A_z = A_{\bar{z}} = 0 \). The \( U(1) \) R-symmetry corresponds to a current that has a conformal anomaly and is not conserved in the quantum theory.

We review the canonical quantization [52] of the conformal field theory (2.2). The commutation relations are found from

\[ Z_I(z) Y^J(\zeta) =: Z_I(z) Y^J(\zeta) + \frac{\delta^J_I}{z - \zeta}, \] (2.9)

for \( |z| > |\zeta| \), and where \( Y^J \equiv g^{J \bar{J}} Y^{\bar{J}} \) is holomorphic in the gauge where \( A_z = A_{\bar{z}} = 0 \). The Virasoro current relevant for the twistor fields is given by

\[ L_{YZ}(z) = - \sum_J Y^J(z) \partial Z_J(z) \] (2.10)

and its operator product expansion with the chiral currents \( Y^I(z) Z_J(z) : \) is

\[ L_{YZ}(z) : Y^I(\zeta) Z_J(\zeta) : \sim - \frac{\delta^J_I}{(z - \zeta)^3} + \frac{Y^I(\zeta) Z_J(\zeta)}{(z - \zeta)^2} + \frac{\partial : Y^I(\zeta) Z_J(\zeta) :}{(z - \zeta)}. \] (2.11)
Here $\sim$ denotes equality up to the regular part. The anomalous term vanishes for the supertraceless currents, that is to say the psl(4|4) currents and the $gl(1, \mathbb{C})$ gauge current, so they are true primary fields and are conserved in the quantum theory.

In order to define Noether charges, we consider the boundary theory explicitly. In a boundary conformal field theory one specifies how fields are identified at the boundary, such as in \( (2.4) \),

$$\frac{(-1)^I}{1} \bar{Y}^I = U^{-1} Y, \quad \bar{Z} = U Z.$$  \hspace{1cm} (2.12)

Our theory possesses a current superalgebra symmetry in addition to Virasoro invariance (see, e.g., \[54]-[58].)

In such a situation one usually has boundary conditions of the form

$$J(z) = \Omega \bar{J}(\bar{z}) \quad \text{for} \quad z = \bar{z}. \hspace{1cm} (2.13)$$

Here $\Omega$ is a map on the space of fields specifying the boundary conditions. If this map is an automorphism of the Lie superalgebra, then these gluing conditions preserve the current algebra symmetry. In this case, we can analytically continue the current $J(z)$ on the entire plane

$$J(z) = \begin{cases} J(z) & \text{z in upper half plane} \\ \Omega \bar{J}(z) & \text{z in lower half plane} \end{cases} \hspace{1cm} (2.14)$$

Then the charge is defined by an integral that runs from boundary (a) to boundary (b) of the open string, where

$$Q = \int d\sigma j^0 = \epsilon_{\mu\nu} \int dx^\mu j^\nu = \int_a^b dz j^z - \int_a^b d\bar{z} j^{\bar{z}} = \oint dz j^z,$$

and we include a factor of $\frac{1}{2\pi i}$ in $\oint$. See Figure 1.

In our case, $\Omega = \frac{(-1)^I}{1} J^I + I$ which is an automorphism of psl(4|4), and the conserved Noether charges associated with the transformations \( (2.6) \) can be written as

$$Q_{0J}^I = \oint dz J_{0J}^I(z) \hspace{1cm} (2.15)$$
where the contour encircles the origin, and the Noether currents for the psl(4|4) charges are

\[ J^I_0(z) \equiv Y^I(z) Z_J(z) - \frac{1}{8} (-1)^{I+E} \delta^I_E Y^E(z) Z_E(z) - \frac{1}{8} \delta^I_J Y^E(z) Z_E(z). \] (2.16)

They have zero trace and zero supertrace and, from (2.9), the current algebra satisfies

\[ J^I_0(z) J^K_0(\zeta) \sim (z - \zeta)^{-1} \left( f_{JKLM} N J^M_0(\zeta) + \frac{1}{8} \left( 1 - (-1)^{I+J} \right) \delta^I_L \delta^K_J Y^E(\zeta) Z_E(\zeta) \right) \]

\[ + (z - \zeta)^{-2} \left( -(-1)^I \delta^I_L \delta^K_J + \frac{1}{8} ((-1)^I + (-1)^K) \delta^I_J \delta^K_L \right), \] (2.17)

where the psl(4|4) structure constants \( f_{JKLM}^N \) have vanishing Killing form

\[ f_{NJMN}^L f_{JKLR}^S = 0, \] (2.18)

and are given in appendix B together with some of their properties. The charges (2.15)
satisfy
\[ [Q^I_{0J}, Q^K_{0L}] = f^I_{JKLM} Q^M_N + \frac{1}{8} (1 - (-1)^{I+J}) \delta^I_L \delta^K_J \oint dz Y^E(z) Z_E(z) \] (2.19)

Since the central generator \( \oint dz Y^E(z) Z_E(z) \) acts as zero on gauge invariant quantities like the vertex operators, the twistor string has \( \text{psl}(4|4) \) global symmetry, and

\[ [Q^I_{0J}, Q^K_{0L}] = f^I_{JKLM} Q^M_N \] (2.20)
on gauge invariant states. The mixed brackets in (2.19) denote an (anti) commutator for (odd) even generators,

\[ [Q^I_{0J}, Q^K_{0L}] \equiv Q^I_{0J} Q^K_{0L} - (-1)^{(I+J)(K+L)} Q^K_{0L} Q^I_{0J}. \] (2.21)

See appendix C for a familiar basis.

3 How the PSL(4|4) symmetry acts

3.1 Action of PSL(4|4) Currents and Charges on Fields

The ordinary charges generate infinitesimal transformation on the fields \( Y^K(z) \) and \( Z_K(z) \) and act on products of more general conformal fields via standard Lie algebra comultiplication, as follows.

The operator product of the current \( J^I_{0J}(z) \) with fields \( Z_K(\zeta) \) is

\[ J^I_{0J}(z) Z_K(\zeta) \sim (z - \zeta)^{-1} \left( (-1)^{I+J+K} \delta^I_J Z_J(z) + \frac{1}{8} (-1)^I \delta^I_J Z_K(\zeta) + \frac{1}{8} \delta^I_J Z_K(\zeta) \right) \]

\[ \equiv (z - \zeta)^{-1} t^I_J[\zeta] Z_K(\zeta), \] (3.1)

for \(|z| > |\zeta|\) where we have defined the differential operations

\[ t^I_J[\zeta] \equiv -(-1)^{I+J} Z_J(\zeta) \frac{\partial}{\partial Z_I(\zeta)} + \frac{1}{8} \delta^I_J (-1)^{I+E} Z_E(\zeta) \frac{\partial}{\partial Z_E(\zeta)} + \frac{1}{8} \delta^I_J Z_E(\zeta) \frac{\partial}{\partial Z_E(\zeta)} + Y^I(\zeta) \frac{\partial}{\partial Y^I(\zeta)} - \frac{1}{8} \delta^I_J (-1)^{I+E} Y^E(\zeta) \frac{\partial}{\partial Y^E(\zeta)} - \frac{1}{8} \delta^I_J Y^E(\zeta) \frac{\partial}{\partial Y^E(\zeta)}. \] (3.2)
Since $J_{0,j}^I(z)$ and $Z_K(\zeta)$ are respectively local, we find using the standard contour argument,

$$[Q_{0,j}^I, Z_K(\zeta)] = \oint_{\Gamma_0^>} dz (z - \zeta)^{-1} t_j^I[\zeta] Z_K(\zeta) - \oint_{\Gamma_0^<} dz (z - \zeta)^{-1} t_j^I[\zeta] Z_K(\zeta)$$

$$= \oint_{\Gamma_C} dz t_j^I[\zeta] Z_K(\zeta) = t_j^I[\zeta] Z_K(\zeta), \quad (3.3)$$

where the contours $\Gamma_0^>, \Gamma_0^<$ encircle the origin clockwise with $|z| > |\zeta|, |z| < |\zeta|$ respectively, and $\Gamma_C$ encircles $\zeta$ but not the origin. Also\[9\]

$$[Q_{0,j}^I, Y^K(\zeta)] = t_j^I[\zeta] Y^K(\zeta) = \delta^K_J Y^I(\zeta) - \frac{1}{8} (-1)^{I+K} \delta^K_J Y^K(\zeta) - \frac{1}{8} \delta^K_J Y^K(\zeta). \quad (3.4)$$

(3.3), (3.4) correspond to infinitesimal field transformations that leave the twistor string world sheet action invariant, and are generated by the super commutator of the Noether charge with the field.

Similarly, for any field $V(\zeta) \equiv V(Z(\zeta), Y(\zeta))$ that is a function of $Z_K(\zeta)$ and or $Y^K(\zeta)$, such as the vertex operators, the operator product with the ordinary current is

$$J_{0,j}^I(z)V(\zeta) \sim (z - \zeta)^{-1} t_j^I[\zeta] V(\zeta), \quad (3.5)$$

From (3.2), it follows that

$$\left[ t_j^I[\zeta], t_K^L[\zeta] \right] = f^K_J L_M^N t_M^N[\zeta],$$

where the structure constants $f^K_J L_M^N$ are given (B.1). The charge acts\[9\]

$$[Q_{0,j}^I, V(\zeta)] = t_j^I[\zeta] V(\zeta), \quad (3.6)$$

and on a product of fields as

$$[Q_{0,j}^I, V_1(\zeta_1)V_2(\zeta_2)]$$

$$= [Q_{0,j}^I, V_1(Z(\zeta_1))] V_2(Z(\zeta_2)) + (-1)^{(I+J)\text{deg}(V_1)} V_1(Z(\zeta_1)) \{ Q_{0,j}^I, V_2(Z(\zeta_2)) \}$$

$$= t_j^I[\zeta_1] V_1(\zeta_1) V_2(\zeta_2) + (-1)^{(I+J)\text{deg}(V_1)} V_1(\zeta_1) t_j^I[\zeta_2] V_2(\zeta_2). \quad (3.7)$$
This can be written as the standard Lie superalgebra comultiplication,

$$\Delta Q^I_{0,J} = Q^I_{0,J} \otimes 1 + (-1)^{(I+J)F} Q^I_{0,J}, \quad (3.8)$$

which instructs one how to build the action of the charge on two fields from knowledge of the action of the charge on a single field, etc. The operator $F$ is the fermion number operator. Clearly the standard comultiplication follows from the fact that the action of the charge on a single site is given by a commutator as in (3.6).

### 3.2 Action of Ordinary Charges on the Tree Amplitudes

It follows directly that the psl(4|4) symmetry charges annihilate the tree amplitudes, as we now show. The charges, which can be expressed in term of the modes as

$$Q^I_{0,J} = \sum_n Y^I_{-n} Z^J_n - \frac{1}{8} (-1)^{I+J} \delta^I_J \sum_n Y^E_n Z^E_n - \frac{1}{8} \delta^I_J \sum_n Y^E_{-n} Z^E_n, \quad (3.9)$$

annihilate the vacuum,

$$Q^I_{0,J} |0\rangle, \quad \langle 0 | (Q^I_{0,J})^\dagger = \pm \langle 0 | Q^I_{0,J} = 0, \quad (3.10)$$

since the modes $Y^I_n, Z^I_n$ satisfy the vacuum conditions [52]

$$Y^I_n |0\rangle = 0, \quad n \geq 0, \quad Z^I_n |0\rangle = 0, \quad n \geq 1, \quad (3.11)$$

where $Z(z) = \sum Z_n z^{-n}$, and $Y(z) = \sum Y_n z^{-n-1}$, and the hermiticity conditions are

$$(Z^I_{-n})^\dagger = Z^I_n, \quad (Y^I_n)^\dagger = -(-1)^I Y^I_{-n}. \quad (3.12)$$

This is in contrast to the $U(1)_R$ charge which does not annihilate the vacuum.

The tree amplitudes in twistor string theory are given by

$$A_n^{\text{tree}} = \int \langle 0 | e^{d\gamma_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S \quad (3.13)$$

where $d\gamma_M$ is the invariant measure on the Möbius group, $d\gamma_S$ is the invariant measure on the group of scale transformations, $d$ is the instanton number which is equal to one.
less than the number of negative helicities, $q_0 = \sum_{I=1}^{8} q_0^I$ the sum of the zero modes of $X^I$ relevant for bosonizing $Y^I(z), Z_J(z)$; and the homogeneous conformal fields $V_i(z_i)$ are the vertex operators of the gluon or graviton supermultiplets \[3, 52, 59\].

Under a charge $Q_0$, the vertex operators transform as

$$V(z) \to V'(z) = e^{\rho Q_0} V(z) e^{-\rho Q_0} \simeq V(z) + \rho [Q_0, V(z)] + O(\rho^2).$$  \hspace{1cm} (3.14)$$

The psl(4|4) charges $Q_0$ commute with $e^{d\rho_0}$,

$$[Q_0, e^{d\rho_0}] = 0,$$

since the modes satisfy

$$Y^I_{n-d\rho_0} = e^{d\rho_0} Y^I_n, \quad Z^I_{n+d\rho_0} = e^{d\rho_0} Z^I_n \quad \text{for } 1 \leq I \leq 8. \hspace{1cm} (3.15)$$

Therefore the tree amplitudes are invariant under the symmetry transformations,

$$A_{\text{tree}}^n = \int \langle 0| e^{d\rho_0} V'_1(z_1) V'_2(z_2) \ldots V'_n(z_n) |0\rangle \prod_{r=1}^{n} d\gamma_r / d\gamma_M d\gamma_S$$

$$= \int \langle 0| e^{d\rho_0} e^{\rho Q_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} d\gamma_r / d\gamma_M d\gamma_S$$

$$= \int \langle 0| e^{d\rho_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} d\gamma_r / d\gamma_M d\gamma_S$$  \hspace{1cm} (3.17)$$

from (3.9), (3.14), (3.15). Thus working to first order in $\rho$, we see that the charges $Q_0$ annihilate the tree amplitudes,

$$\int \langle 0| e^{d\rho_0} Q_0 V_1(z_1) V_2(z_2) \ldots V_n(z_n) |0\rangle \prod_{r=1}^{n} d\gamma_r / d\gamma_M d\gamma_S = 0.$$  \hspace{1cm} (3.18)$$

Furthermore we can evaluate the action of the charges on the amplitude as follows.

For simplicity, we consider the gluon tree amplitudes. The vertices for the negative and
positive helicity gluons are

\[ V_+^A(z) = \int \frac{dk}{k} \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} J^A(z) \psi^1(z) \psi^2(z) \psi^3(z) \psi^4(z), \]

\[ V_-^A(z) = \int dk \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) e^{ik\mu^a(z)\bar{\pi}_a} J^A(z). \] (3.19)

These are functions of \( Z_J(z) \), whose components are labeled as \( Z^I = (\lambda^a, \mu^a, \psi^1, \psi^2, \psi^3, \psi^4) \).

As an example, consider the action of the momentum operator on the vertices,

\[ [p_a^\dot{\alpha}, V_\pm^A(z)] = t_a^\dot{\alpha}[z] V_\pm^A(z) = -\lambda_a(z) \frac{\partial}{\partial \mu_a(z)} V_\pm^A(z) = i k\pi^\dot{\alpha} \lambda_a(z) V_\pm^A(z) \]

\[ = i \pi^\dot{\alpha} V_\pm^A(z). \] (3.20)

Then from (3.18),

\[ \int \langle 0 | e^{d\phi_0} p_a^\dot{\alpha} V_1(z_1) V_2(z_2) \ldots V_n(z_n)|0 \rangle \prod_{r=1}^{n} dz_r / d\gamma_M d\gamma_S \]

\[ = \int \langle 0 | e^{d\phi_0} V_1(z_1) \ldots \sum_{r=1}^{n} [p_a^\dot{\alpha}, V_r(z_r)] \ldots V_n(z_n)|0 \rangle \prod_{r=1}^{n} dz_r / d\gamma_M d\gamma_S \]

\[ = i \left( \sum_{r=1}^{n} \pi^\dot{\alpha}_r \right) \int \langle 0 | e^{d\phi_0} V_1(z_1) V_2(z_2) \ldots V_n(z_n)|0 \rangle \prod_{r=1}^{n} dz_r / d\gamma_M d\gamma_S \]

\[ = 0, \] (3.21)

which is momentum conservation.

If we Fourier transform the gluon vertex operators from \( \bar{\pi}_\dot{\alpha} \) back to \( \omega^\dot{\alpha} \) which is a twistor space coordinate, and introduce the superspace coordinate \( \theta^M \) to include all fields in the gluon supermultiplet, they become the wavefunction for the world sheet fields \( Z^I(z) \) to be at a point \( Z'^I = (\pi^a, \omega^\dot{\alpha}, \theta^M) \) in super twistor space,

\[ W^A(z) = \int \prod_{a=1}^{2} \delta(k\lambda^a(z) - \pi^a) \delta(k\mu^a(z) - \omega^\dot{\alpha}) \prod_{M=1}^{4} (k\psi^M(z) - \theta^M) \frac{dk}{k} J^A(z). \] (3.22)
See for example [52]. Then the action of the charges on $W^A$ is

$$[Q^I_J, W^A(z)] = t^I_J[z] W^A(z) = t^I_J W^A(z)$$  \hspace{1cm} (3.23)$$

where $t^I_J$ is now expressed in terms of points in twistor space rather than world sheet fields,

$$t^I_J \equiv -(-1)^{I+J+1} Z^I_J \frac{\partial}{\partial Z^I} + \frac{1}{8} \delta^I_J \rho^E \frac{\partial}{\partial Z^E} + \frac{1}{8} \delta^I_J \rho^E \frac{\partial}{\partial Z^E}.$$  \hspace{1cm} (3.24)$$

So if we consider the superspace tree amplitude, that is the tree constructed with vertices (3.22), then the psl(4|4) charges act as

$$\int \langle 0 | e^{dq_0} Q^I_J W_1(z_1) W_2(z_2) \ldots W_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r/d\gamma_M d\gamma_S = \int \langle 0 | e^{dq_0} W_1(z_1) \ldots \sum_{r=1}^n [Q^I_J, W_r(z_r)] \ldots \ldots W_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r/d\gamma_M d\gamma_S$$

$$= \sum_{r=1}^n t^I_{r,J} \int \langle 0 | e^{dq_0} W_1(z_1) W_2(z_2) \ldots W_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r/d\gamma_M d\gamma_S,$$

$$= 0,$$  \hspace{1cm} (3.25)$$

which is zero because the charge annihilates the vacuum as in (3.18). A similar formula for the gauge field theory appears in [47]. Indeed the transformation on generators $t^I_{r,J},$

$$-(-1)^{I+J+1} Z^I_J \frac{\partial}{\partial Z^I} \rightarrow Z^I_J \frac{\partial}{\partial Z^I}$$  \hspace{1cm} (3.26)$$

(with trace and supertrace removed) leaves the psl(4|4) commutation relations invariant, and is the transformation of a matrix $M^I_J$, $M \rightarrow -(M)^{\text{supertranspose}}$ which is an outer automorphism of the superalgebra psl(4|4).
Note that the tree amplitude $M_n(\pi^a, \omega^a, \theta^M)$ is the Fourier transform of $M_n(\pi^a, \pi^a, \eta^M)$, 

\[
M_n(\pi^a, \omega^a, \theta^M) \equiv \int \langle 0| e^{dq_0} W_{A_1}(z_1) W_{A_2}(z_2) \ldots W_{A_n}(z_n) |0 \rangle \prod_{r=1}^{n} dz_r / d\gamma_d d\gamma_S 
\]

\[
= \int \prod_{r=1}^{n} \prod_{a=1}^{2} \prod_{M=1}^{4} d\eta_r M d\pi_{ra} e^{-i\omega^a_{\pi} \pi^a_{\eta^M}} e^{\theta^M_{\eta^M} M_n(\pi^a, \pi^a, \eta^M)}, \tag{3.27}
\]

where $M_n(\pi^a, \pi^a, \eta^M)$ is of Grassmann degree $4(d + 1)$ and is expressed in terms of some of the conjugate twistor space points $W_i' = (\bar{\omega}_a, \pi^a, \eta^M)$, e.g., for MHV trees $M_n^{\text{MHV}}(\pi^a, \pi^a, \eta^M) = \delta^4(\sum_{r=1}^{n} \pi^a_r \pi^a_r) \delta^8(\sum_{r=1}^{n} \pi^a_r \pi^a_r) f_{12}^{A_1 \ldots A_n}$.

### 4 Non-local Noether currents and charges

In this section we compute non-local currents whose associated charges will also annihilate the scattering amplitudes. We show how these charges have the same comultiplication rules as the level one generators of the Yangian superalgebra of $\mathfrak{psl}(4|4)$. In order to realize the coproduct of the Yangian, which is a Hopf algebra, it will be necessary to define currents in terms of a field that is not local with respect to the conformal fields [50], [51]. We introduce the non-local field $\chi^I_J(z)$,

\[
\chi^I_J(z) \equiv \int_P dw J^I_J(w), \tag{4.1}
\]

which not only is defined as an integral over a local field, but also is not local with respect to local fields, e.g.

\[
(-1)^{R^{(I+J)}} Z_R(\zeta) \chi^I_J(z) \cong \chi^I_J(z) Z_R(\zeta) + 2\pi i \{Q^I_J, Z_R(\zeta)\}, \tag{4.2}
\]

where the left hand side of (4.2) is defined for $|\zeta| > |z|, |z| > |P|$, the right hand side is defined for $|z| > |\zeta|, |P| > |\zeta|$ for an arbitrary point $P$, and the equality $\cong$ is meant in the sense of analytic continuation, see Figure 2.

The operator product of $\chi^I_J(z)$ with $Z_R(\zeta)$ is

\[
\chi^I_J(z) Z_R(\zeta) \sim \int_P^z dw(w - \zeta)^{-1} J^I_J(\zeta) Z_R(\zeta) = \ln \left( \frac{(z - \zeta)}{(P - \zeta)} \right) \frac{t^I_J[\zeta]}{t^I_J[\zeta]} Z_R(\zeta). \tag{4.3}
\]
We define the non-local currents $J_{1N}^M(z)$ by

$$J_{1N}^M(z) = f_{NK}^M \int_P dw J_0^I(w) J_0^K(z) = f_{NK}^M \int_P \chi^I_J(z) J_0^K(z). \quad (4.4)$$

It is not necessary to normal order because the singular terms vanish due to the properties of the structure constants. The $J_{1N}^M(z)$ are not local with respect to the fields,

$$(-1)^{R(M+N)} Z_R(\zeta) J_{1N}^M(z) \cong J_{1N}^M(z) Z_R(\zeta) +$$

$$+ f_{NK}^M \int_P (-1)^{R(K+L)} 2\pi i [Q_0^I, Z_R(\zeta)] J_0^K(z). \quad (4.5)$$

Figure 2: The integral that defines $\chi^I_J$ is along $C_{w_1}$ for the right-hand side of (4.2) and along $C_{w_2}$ for the left-hand side.
The operator product with fields $Z_R(\zeta)$ is

$$J^M_{1N}(z) \ Z_R(\zeta) \sim (z - \zeta)^{-1} \ f^M_{NK} \ I^J : \chi^I_J(\zeta) \ t^K_L[\zeta] \ Z_R(\zeta) :$$

$$+ \ln \left(\frac{z - \zeta}{P - \zeta}\right) \ f^M_{NK} \ I^J (-1)^{(I+J)(K+L)} : J^K_L(\zeta) \ t_I^J[\zeta] \ Z_R(\zeta) :. \quad (4.6)$$

The action of the non-local charge on the field $Z_R(\zeta)$, is

$$Q^M_{1N}(Z_R(\zeta)) = \oint_{C_\zeta} dz J^M_{1N}(z) \ Z_R(\zeta) = 2 \ f^M_{NK} \ I^J : \chi^I_J(\zeta) \ t^K_L[\zeta] \ Z_R(\zeta) :, \quad (4.7)$$

where the cut for the logarithm extends from $\zeta$ passing through the point $P$, and the contour $C_\zeta$ starts just above $P$, circles around $\zeta$ and stops just below $P$. See Figure 3.

![Figure 3](z-plane)

**Figure 3:** The contour $C_\zeta$ starts just above $P$, circles around $\zeta$ and stops just below $P$.

To derive (4.7) we have used $\oint_{C_\zeta} dz f(z) \ln \left(\frac{z - \zeta}{P - \zeta}\right) = -\int_P^\zeta dw f(w)$, for any $f(w)$ analytic on and inside the contour $C_\zeta$ [50].

The expression (4.7) is the infinitesimal transformation on the field $Z_R(\zeta)$ generated by the non-local charge [51]. It is not merely $\{Q^M_{1N}, Z_R(\zeta)\}$ due to the lack of locality in
\[ Q^M_{1N}(Z_R(\zeta)) = [Q^M_{1N}, Z_R(\zeta)] + f^M_{NK} L^J_i (-1)^{R(K+L)} 2\pi i [\tilde{Q}^I_{0J}, Z_R(\zeta)] Q^K_{0L}. \] (4.8)

This mismatch of the commutator and the infinitesimal field transformation follows from

\[ Q^M_{1N} Z_R(\zeta) = \oint_{C_1:|z|>|z|>1} dz J^M_{1N}(z) Z_R(\zeta), \]

\[ (-1)^{R(M+N)} Z_R(\zeta) Q^M_{1N} = \oint_{C_2:|\zeta|>|z|>1} dz (-1)^{R(M+N)} Z_R(\zeta) J^M_{1N}(z) \]

\[ = \oint_{C_2:|\zeta|>|z|>1} dz J^M_{1N}(z) Z_R(\zeta) + f^M_{NK} L^J_i (-1)^{R(K+L)} 2\pi i [Q^I_{0J}, Z_R(\zeta)] Q^K_{0L}, \] (4.9)

so that

\[ [Q^M_{1N}, Z_R(\zeta)] = \oint_{C_2} dz J^M_{1N}(z) Z_R(\zeta) - f^M_{NK} L^J_i (-1)^{R(K+L)} 2\pi i [Q^I_{0J}, Z_R(\zeta)] Q^K_{0L}, \] (4.10)

which is (4.8).

Therefore the operator product of the non-local current with any field \(V(Z(\zeta))\) that is a function of \(Z_R(\zeta)\), such as the gluon vertex operator, is

\[ J^M_{1N}(z) V(Z(\zeta)) \sim (z-\zeta)^{-1} f^M_{NK} L^J_i \chi^I_J(\zeta) t^K_L[Z(\zeta)] V(Z(\zeta)) : \]

\[ + \ln \left( \frac{z-\zeta}{P-\zeta} \right) f^M_{NK} L^J_i (-1)^{(I+J)(K+L)} : J^K_L(z) t^I_J[Z(\zeta)] V(Z(\zeta)) :. \]

(4.11)

The action of the first level Yangian charge on the field \(V(Z(\zeta))\), is

\[ Q^M_{1N} (V(Z(\zeta))) = \oint_{C_2} dz J^M_{1N}(z) V(Z(\zeta)) \]

\[ = 2 f^M_{NK} L^J_i \chi^I_J(\zeta) t^K_L[Z(\zeta)] V(Z(\zeta)) : \] (4.12)
For simplicity, we consider a bosonic vertex operator $V(Z(\zeta))$, where
\begin{align*}
V(Z(\zeta)) \chi^I_J(z) &\equiv \chi^I_J(z) V(Z(\zeta)) + 2\pi i [Q^I_{0J}, V(Z(\zeta))], \\
V(Z(\zeta)) J^M_{1N}(z) &\equiv J^M_{1N}(z) V(Z(\zeta)) + f^M_{NK} J^I_{L} 2\pi i [Q^I_{0J}, V(Z(\zeta))] J^K_{0L}(z), \\
Q^M_{1N}(V(Z(\zeta))) &=[Q^M_{1N}, V(Z(\zeta))] + f^M_{NK} J^J_{L} 2\pi i [\tilde{Q}^I_{0J}, V(Z(\zeta))] Q^K_{0L}.
\end{align*}
(4.13, 4.14, 4.15)

Since the infinitesimal transformation (4.7) is not a simple (anti)commutator (4.8), the action of the non-local charge on a product of fields will not be as simple as for the ordinary generators (3.8), and leads to the Hopf superalgebra coproduct.

Comultiplication

The action of the non-local charge on two fields is
\begin{align*}
Q^M_{1N} \left( V_1(Z(\zeta_1)) V_2(Z(\zeta_2)) \right) &= \oint_{C_{\zeta_1,\zeta_2}} dz J^M_{1N}(z) V_1(Z(\zeta_1)) V_2(Z(\zeta_2)),
\end{align*}
(4.16)

where the contour $C_{\zeta_1,\zeta_2}$ starts at $P$ above both cuts (cut from $\zeta_1$ to $P$, and cut from $\zeta_2$ to $P$), encircles both $\zeta_1, \zeta_2$ and stops below both cuts at $P$, see Figure 4. For simplicity,
we take both fields $V_1$ and $V_2$ to be bosonic. We have for $|z|, |P| > |\zeta_1|, |\zeta_2|$

$$J^M_{1N}(z) V_1(Z(\zeta_1)) V_2(Z(\zeta_2)) = f^M_{NK} L^I J^J \chi^I_0 J(z) J^K_0 L(z) V_1(Z(\zeta_1)) V_2(Z(\zeta_2)), \quad (4.17)$$

and for $|\zeta_1|, |\zeta_2| > |z|, |P|$

$$V_1(Z(\zeta_1)) V_2(Z(\zeta_2)) J^M_{1N}(z) = f^M_{NK} L^I J^J \chi^I_0 J(z) J^K_0 L(z) V_1(Z(\zeta_1)) V_2(Z(\zeta_2)),$$

- $f^M_{NK} L^I J^J \chi^I_0 J(z) V(Z(\zeta_1)) V(Z(\zeta_2)) J^K_0 L(z)$

- $f^M_{NK} L^I J^J V_1(Z(\zeta_1)) 2\pi i [Q^I_0 J, V_2(Z(\zeta_2))] J^K_0 L(z), \quad (4.18)$

using (4.13) twice. Then

$$[Q^M_{1N}, V_1(Z(\zeta_1)) V_2(Z(\zeta_2))] = \oint_{\zeta_1, \zeta_2} dz J^M_{1N}(z) V_1(Z(\zeta_1)) V_2(Z(\zeta_2))$$

- $f^M_{NK} L^I J^J \chi^I_0 J(z) V(Z(\zeta_1)) V(Z(\zeta_2)) Q^K_0 L$

- $f^M_{NK} L^I J^J V_1(Z(\zeta_1)) 2\pi i [Q^I_0 J, V_2(Z(\zeta_2))] Q^K_0 L, \quad (4.19)$

So from (4.16),

$$Q^M_{1N}\left(V_1(Z(\zeta_1)) V_2(Z(\zeta_2))\right) = [Q^M_{1N}, V_1(Z(\zeta_1)) V_2(Z(\zeta_2))]$$

- $f^M_{NK} L^I J^J \chi^I_0 J(z) V(Z(\zeta_1)) V(Z(\zeta_2)) Q^K_0 L$

- $f^M_{NK} L^I J^J V_1(Z(\zeta_1)) 2\pi i [Q^I_0 J, V_2(Z(\zeta_2))] Q^K_0 L$

$$= Q^M_{1N}\left(V_1(Z(\zeta_1))\right) V_2(Z(\zeta_2)) + V_1(Z(\zeta_1)) Q^M_{1N}\left(V_2(Z(\zeta_2))\right)$$

- $f^M_{NK} L^I J^J \chi^I_0 J(z) V(Z(\zeta_1)) V(Z(\zeta_2)) Q^K_0 L, \quad (4.20)$

which can be written as the Hopf algebra comultiplication,

$$\Delta Q^M_{1N} = Q^M_{1N} \otimes 1 + (-1)^{(M+N)F} \otimes Q^M_{1N} - 2\pi i f^M_{NK} L^I J^J Q^K_0 J \otimes Q^K_0 L. \quad (4.21)$$
We can reexpress the action of the non-local charge on the product of two fields as

\[ Q_{M}^{1N} \left( V(\zeta_{1}) V(\zeta_{2}) \right) = [Q_{M}^{1N}, V(\zeta_{1}) V(\zeta_{2})] + 2\pi i f_{M}^{KL} J^{J} [Q_{0}^{I}, V(\zeta_{1}) V(\zeta_{2})] Q_{0}^{K}_{L}, \quad (4.22) \]

and will use this form to demonstrate how these non-local charges annihilate the tree amplitudes.

5 More symmetry in the open twistor string

We consider the action of the non-local charges on the tree amplitudes. We show that the non-local charges annihilate the super gluon amplitudes of the open twistor string, and hence give rise to interesting Ward identities which we compare with \( \mathcal{N} = 4 \) Yang Mills field theory.

The non-local charges formally annihilate the vacuum using (3.11),

\[ Q_{M}^{1N} |0\rangle, \quad \langle 0| (Q_{1N}^{M})^{\dagger} = \pm \langle 0| Q_{1N}^{M}, \quad [Q_{1N}^{M}, e^{q_{0}}] = 0. \quad (5.1) \]

In analogy with the ordinary \( PSL(4|4) \) case, we consider the action of the non-local charge on the tree amplitude using (4.22),

\[
\int \langle 0| e^{q_{0}} Q_{M}^{1N} (W_{1}(z_{1}) W_{2}(z_{2}) \ldots W_{n}(z_{n})) |0\rangle \prod_{r=1}^{n} dz_{r} / d\gamma_{M} d\gamma_{S} \\
= \int \langle 0| e^{q_{0}} [Q_{M}^{1N}, W_{1}(z_{1}) W_{2}(z_{2}) \ldots W_{n}(z_{n})] |0\rangle \prod_{r=1}^{n} dz_{r} / d\gamma_{M} d\gamma_{S} \\
+ \int \langle 0| e^{q_{0}} f^{M}_{N} J^{J} \left[ 2\pi i [Q_{0}^{I}, W_{1}(z_{1}) W_{2}(z_{2}) \ldots W_{n}(z_{n})] Q_{0}^{K}_{L} |0\rangle \prod_{r=1}^{n} dz_{r} / d\gamma_{M} d\gamma_{S} \\
= 0, \quad (5.2)\]

where we have suppressed the group index. To display the action of the non-local charge
on the amplitudes, we compute (5.2) directly, defining

$$W(z_1, \ldots, z_n) \equiv \langle 0 | e^{dq} W_1(Z(z_1)) \cdots W_n(Z(z_n)) | 0 \rangle.$$  \hfill (5.3)

We use (4.20) to express (5.2),

$$\int \langle 0 | e^{dq} Q_{1 \prime N}^M (W_1(z_1) W_2(z_2) \cdots W_n(z_n)) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

$$= \int \langle 0 | e^{dq} \sum_{j=1}^n W_1(z_1) \cdots Q_{0 \prime j}^I (W_j(z_j)) \cdots W_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

$$- \int \langle 0 | e^{dq} 2\pi i f^{M L J}_{NK} \sum_{1 \leq i < j \leq n} W_1(z_1) \cdots W_{i-1}(z_{i-1}) Q_{i \prime j}^I (W_i(z_i)) W_{i+1}(z_{i+1}) \cdots$$

$$\times W_{j-1}(z_{j-1}) Q^K_{0 \prime j} (W_j(z_j)) W_{j+1}(z_{j+1}) \cdots W_n(z_n) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

$$= -2 f^{M L J}_{NK} \sum_{1 \leq i < j \leq n} t^I_{i j} t^K_{i j} \int \left( \ln \frac{(z_i - z_j)}{(P - z_j)} - \ln \frac{(z_j - z_i)}{(P - z_i)} \right) W(z_1, \ldots, z_n) \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

$$+ (2\pi i) f^{M L J}_{NK} \sum_{1 \leq i < j \leq n} t^I_{i j} t^K_{i j} \int W(z_1, \ldots, z_n) \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

(5.4)

where we have replaced the $t^I_{i j} [z_i]$ which appear in evaluating the operator products, by $t^I_{i j}$ defined in (3.24), since the vertex operators $W(z)$ are delta functions given in (3.22).

To show the independence of $P$, we write (5.4) as

$$-2 f^{M L J}_{NK} \sum_{1 \leq i < j \leq n} t^I_{i j} t^K_{i j} \int \left( \ln \frac{(z_i - z_j)}{(z_j - z_i)} - \ln \frac{(P - z_j)}{(P - z_i)} \right) W(z_1, \ldots, z_n) \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S$$

$$+ 2\pi i f^{M L J}_{NK} \sum_{1 \leq i < j \leq n} t^I_{i j} t^K_{i j} \int W(z_1, \ldots, z_n) \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S.$$  \hfill (5.5)
where the $P$-dependent part is evaluated as

$$f_{NK}^{M L J} \sum_{1 \leq i < j \leq n} t^I_{i J} t^K_{j L} \ln \frac{P - z_j}{P - z_i} W(z_1, \ldots, z_n)$$

$$= - \ln(P - z_1) f_{NK}^{M L J} t^I_{1 J} \sum_{j=2}^{n} t^K_{j L} W(z_1, \ldots, z_n)$$

$$+ \sum_{\ell=2}^{n-1} \ln(P - z_\ell) f_{NK}^{M L J} \left( \sum_{i=1}^{\ell-1} t^I_{i J} t^K_{\ell L} - t^I_{\ell J} \sum_{j=\ell+1}^{n} t^K_{j L} \right) W(z_1, \ldots, z_n)$$

$$+ \ln(P - z_n) f_{NK}^{M L J} \sum_{i=1}^{n-1} t^I_{i J} t^K_{n L} W(z_1, \ldots, z_n).$$  \quad (5.6)$$

We use that the ordinary symmetry annihilates the integrand of the amplitude (3.25), and

$$f_{NK}^{M L J} \left( t^I_{i J} t^K_{j L} + t^I_{j J} t^K_{i L} \right) = 0 \quad \text{for } i \neq j,$$  \quad (5.7)$$

and for each $i$,

$$f_{NK}^{M L J} \left( t^I_{i J} t^K_{i L} \right) = \frac{1}{2} f_{NK}^{M L J} f^{I K}_L R S t^R_{i S} = 0,$$  \quad (5.8)$$

since the Killing form (B.2) vanishes. Then (5.6) becomes

$$= \ln(P - z_1) f_{NK}^{M L J} t^I_{1 J} \sum_{j=2}^{n} t^K_{j L} W(z_1, \ldots, z_n)$$

$$+ \sum_{\ell=2}^{n-1} \ln(P - z_\ell) f_{NK}^{M L J} \left( \sum_{i=1}^{\ell-1} t^I_{i J} t^K_{\ell L} - t^I_{\ell J} \sum_{j=\ell+1}^{n} t^K_{j L} \right) W(z_1, \ldots, z_n)$$

$$+ \ln(P - z_n) f_{NK}^{M L J} \sum_{i=1}^{n-1} t^I_{i J} t^K_{n L} W(z_1, \ldots, z_n)$$

$$= 0,$$  \quad (5.9)$$

Therefore the $P$ dependent part of (5.5) vanishes, and using $\ln \frac{z_i - z_j}{z_j - z_i} = \ln(-1) = \pi i + 2 \pi i p$
for any integer $p$, we find (5.4) leads to

$$
\int \langle 0 | e^{i \varphi} Q_1^I (W_1(z_1) \ldots W_n(z_n)) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S
$$

$$
= -4\pi ip f_{NK}^{I} L_{i}^{J} \sum_{1 \leq i < j \leq n} t_{i}^{I} t_{j}^{K} L \int \langle 0 | e^{i \varphi} W_1(Z(z_1)) \ldots W_n(Z(z_n)) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S
$$

$$
= 0,
$$

(5.10)

for arbitrary integer $p$, so along with the ordinary symmetries (5.25), we have

$$
f_{NK}^{I} L_{i}^{J} \sum_{1 \leq i < j \leq n} t_{i}^{I} t_{j}^{K} L \int \langle 0 | e^{i \varphi} W_1(Z(z_1)) \ldots W_n(Z(z_n)) | 0 \rangle \prod_{r=1}^n dz_r / d\gamma_M d\gamma_S = 0,
$$

(5.11)

which reflects the non-local symmetry of the tree amplitudes of the twistor string. We note that up to central terms that separately annihilate the amplitudes, a similar formula appears in [47]-[49] as the first level Yangian generators acting on gauge theory amplitudes.

6 PSL(4|4) Yangian superalgebra symmetry

In this section, we show how the open twistor string carries a realization of the Yangian of $\text{psl}(4|4)$. This realization will be independent of $P$, and is inspired from the observation that the Ward identities for the non-local charges are independent of $P$, (5.11).

Let us start by computing the OPE of our non-local currents (4.4) with the Virasoro field (2.10),

$$
L_{YZ}(z) J_{1N}^{M}(\zeta) = \frac{J_{1N}^{M}(\zeta)}{(z - \zeta)^2} + \frac{\partial J_{1N}^{M}(\zeta)}{(z - \zeta)} + f_{NK}^{M} L_{i}^{J} J_{0I}^{I}(P) J_{0K}^{K}(w).
$$

(6.1)

Hence, the non-local current is a Virasoro primary only in the limit $P \to \infty$. Note that taking this limit does not commute with the integrals we take. But we take it as a hint for a $P$-independent representation of the Yangian.

We keep the action of the level zero charges as before and choose the level one generators
to act trivially on a single field

$$Q_{1N}^M (V(z)) = 0, \quad (6.2)$$

while the action of $Q_{1N}^M$ on products of fields is defined by the Yangian comultiplication (4.21), so that on a product of two fields,

$$Q_{1N}^M (V(z_1)V(z_2)) = -2\pi i f_{MK}^{NL} Q_{0L}^K (V(z_1)) \otimes Q_{0L}^K (V(z_2)). \quad (6.3)$$

The crucial observation is that this representation of the Yangian also provides a symmetry of the super gluon amplitudes of the open twistor string, as can be seen by (7.11).

We prove that the tree level representation in terms of twistor string fields for the PSL(4|4) charges (2.15), (2.16), (3.8), and the level one charges, (6.2), (6.3), (4.21), is consistent with the Serre relation, because it satisfies a useful criterion [24] as we now show.

**Serre relation and the useful criterion**

The Serre relation for the Yangian of $\text{psl}(4|4)$ can be written as

$$f_{NM}^R f_{SK}^L [Q_{1J}^I, Q_{1L}^K] + (-1)^{(I+J)(M+N)+R+S} f_{RS}^I f_{JK}^L [Q_{1N}^M, Q_{1L}^K] +$$

$$+ (-1)^{(R+S)(I+J)+M+N} f_{MJ}^I f_{NK}^L [Q_{1S}^R, Q_{1L}^K] =$$

$$= h (-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H} \times$$

$$\times f_{JC}^I f_{DP}^Q f_{NE}^M f_{K}^L f_{HS}^R f_{U}^{V} f_{DG}^C f_{FG}^H \left\{ Q_{0Q}^P Q_{0L}^K Q_{0V}^U \right\}, \quad (6.4)$$

where $\{ \ldots \}$ is the graded totally symmetrized product and $h$ is a constant that depends on the normalization.

It is sufficient to check the Serre relation acting on one field, since then the coproduct (4.21) assures the relation for all higher sites. Clearly the left-hand side of (6.4) vanishes for one field from (6.2).

We show in appendix D that the right-hand side of the Serre relation acting on one field also vanishes, using the fact that our representation for the $gl(4|4)$ charge $\widetilde{Q}_{0J}^I$ on a single
field,

\[ [\tilde{Q}_0^I J, V(Z(\zeta))] = \oint dz : Y^I(z) Z_J(z) : , V(Z(\zeta))] \]

\[ = \left( -(-1)^{IJ+I} \left( Z_J(\zeta) \frac{\partial}{\partial Z_I(\zeta)} \right) + Y^I(\zeta) \frac{\partial}{\partial Y^J(\zeta)} \right) V(Z(\zeta)) \equiv \tilde{t}^I_J[\zeta] V(Z(\zeta)), \]  

(6.5)

satisfies a useful criterion [24]:

\[ (-1)^E \tilde{Q}_0^M E \tilde{Q}_0^E J (V(Z(\zeta))) = \tilde{Q}_0^M (V(Z(\zeta))). \]  

(6.6)

Here we consider \( V(Z(z)) \) as a homogeneous function \( Z_E(z) \frac{\partial}{\partial Z_E(z)} V(Z(z)) = 0 \) which describes conformal fields \( V(z) \) such as the vertex operators \( W(z) \). This will simplify our discussion, although is not necessary since the central terms we drop would cancel in what follows. To derive (6.6),

\[ (-1)^E \tilde{Q}_0^M E \tilde{Q}_0^E J (V(Z(\zeta))) = (-1)^E \tilde{t}_0^M E \tilde{t}_0^E J V(Z(\zeta)) \]

\[ = (-1)^{M+M+EJ} Z_E(\zeta) \frac{\partial}{\partial Z_M(\zeta)} \left( Z_J(\zeta) \frac{\partial}{\partial Z_E(\zeta)} V(Z(\zeta)) \right) \]

\[ = (-1)^{M+M+EJ} Z_E(\zeta) Z_J(\zeta) \frac{\partial}{\partial Z_E(\zeta)} \frac{\partial}{\partial Z_M(\zeta)} V(Z(\zeta)) = -(-1)^{M+M} Z_J(\zeta) \frac{\partial}{\partial Z_M(\zeta)} V(Z(\zeta)) \]

\[ = \tilde{Q}_0^M (V(Z(\zeta))). \]  

(6.7)

From (6.2) and (4.21) we also have that

\[ [Q_0^I J, Q_1^K L] = f^I_J K L R S Q_R^S, \]  

(6.8)

acting on gauge invariant states. Therefore with the proof of (6.4) in appendix E, the Yangian symmetry algebra now follows from the defining relations (2.20), (6.8) and (6.4).
Acknowledgments

We are grateful to Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, James Drummond, Peter Goddard, Johaness Henn, Hubert Saleur, David Skinner and Edward Witten for discussions. This work was partially supported by the U.S. Department of Energy, Grant No. DE-FG02-06ER-4141801, Task A.

A Some super analysis

We provide some properties of Lie superalgebras and Lie supergroups we use in our analysis. A good reference for Lie supergroups is [60], in this appendix we use the notation of [54].

A.1 The Lie superalgebra \( \mathfrak{psl}(N|N) \)

The Lie superalgebra \( \mathfrak{sl}(N|N) \) can be represented by matrices

\[
(T)_{IJ} = \begin{pmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{pmatrix},
\]

with \( \text{Str} (T) \equiv \text{Tr} M_1 - \text{Tr} M_4 = 0 \). This superalgebra possesses a one-dimensional ideal generated by the identity matrix. The quotient of \( \mathfrak{sl}(N|N) \) by this ideal is the Lie superalgebra \( \mathfrak{psl}(N|N) \). It is convenient to work with the Lie superalgebra \( \mathfrak{sl}(N|N) \), we simply have to remember that we divided out the central element.

We define the \( \mathfrak{psl}(4|4) \) structure constants by

\[
\{T^a, T^b\} = f^{abc} g_{cd} T^d,
\]

where the brackets denote either commutators or anticommutators. There is an invariant, nondegenerate metric \( g_{ab} \) that is used to raise and lower indices, \( g^{ab} = \frac{1}{2} \text{Str} T^a T^b \) and \( g_{ab} = \frac{1}{2} \text{Str} T^b T^a \). The structure constants \( f_{abc} \) are totally antisymmetric with an additional minus sign under the interchange of two odd nearest neighbor indices.

It can be useful to rewrite the single index superalgebra generators \( J^a \) with a double index as \( (E_{AB})_{IJ} = \delta_{AI} \delta_{BJ} \), where

\[
[E_{AB}, E_{CD}] = \delta_{CB} E_{AD} - \delta_{AD} E_{CB},
\]

\[
[E_{AB}, E_{CD}] = \delta_{CB} E_{AD} - (-1)^{\text{deg} E_{AB} \text{deg} E_{CD}} \delta_{AD} E_{CB}.
\]
where $\text{deg } E_{AB}$ is 0 for bosonic generators, and 1 for fermionic generators.

### A.2 Lie supergroups

The Lie supergroup is a Lie group over a Grassmann ring and it is obtained by exponentiating the Grassmannn envelope of the Lie superalgebra. An element of the Grassmann envelope has the following form

$$\rho = \rho_a T^a$$  \hspace{1cm} (A.2)

where the $\rho_a$ are Grassmann-even when $T^a$ is bosonic and Grassmann-odd when $T^a$ is fermionic. Hermitian conjugation is complex conjugation of the transpose in the bosonic case. For matrices generating a Lie superalgebra, it is super complex conjugation of the supertranspose,

$$\rho^\dagger = \overline{\rho}_a (T^a)^{st}$$

where the bar is ordinary complex conjugation for bosons, and satisfies\footnote{Lifting complex conjugation to the ring of Grassmann numbers is not unique. Our choice ensures that the adjoint operation for Lie supergroups is involutive.}

$$\overline{c\theta} = \overline{c}\overline{\theta}, \quad \overline{\theta} = -\theta, \quad \overline{\theta_1\theta_2} = \overline{\theta}_1\overline{\theta}_2,$$

for any Grassmann elements $\theta, \theta_i$ and any complex number $c$. The supertranspose of $T$ is defined

$$
\begin{pmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{pmatrix}^{st} =
\begin{pmatrix}
M^t_1 & -M^t_3 \\
M^t_2 & M^t_4
\end{pmatrix}
$$

so that $(TS)^{st} = S^{st}T^{st}$.

### B Structure constants and their properties

The $\text{psl}(4|4)$ structure constants are given by

$$f^I_{JK}L^N_M = \delta^K_J \delta^N_L \delta^I_M - (-1)^{(I+J)(K+L)} \delta^N_J \delta^K_L \delta^I_M + \frac{1}{8}((-1)^L - (-1)^K)(-1)^I \delta^I_J \delta^K_L \delta^N_M$$

$$- \frac{1}{8}((-1)^J - (-1)^K)(-1)^I \delta^I_J \delta^K_L \delta^N_M - \frac{1}{8}(1 - (-1)^{I+J}) \delta^I_J \delta^K_L \delta^N_M.$$  \hspace{1cm} (B.1)
These have vanishing Killing form

$$f_{MNK}^L J_I f_J^{K LR} S = 0,$$  \hspace{1cm} (B.2)

and are traceless and supertraceless in all indices,

$$\delta_I^J f^I_{J L M} N = 0, \quad (-1)^J \delta_I^J f^I_{J L M} N = 0, \quad f^I_{J L M} N \delta^M_N = 0, \quad f^I_{J L M} N (-1)^M \delta^M_N = 0,$$  \hspace{1cm} (B.3)

as well as totally anti supersymmetric in any pair of indices, for eg.

$$f^I_{J L M} N = (-1)^{(I+J)(K+L)} f^K_{I J M} N.$$

They satisfy the Jacobi identity

$$f^I_{J K S} J L R f^{R W V} S X U + f^K_{I W S} L X R f^{R I V} S J U (-1)^{(I+J)(R+S)} + f^W_{I S} X J R f^{R K V} S L U (-1)^{(W+X)(I+J+K+L)} = 0.$$  \hspace{1cm} (B.4)

The inverted structure constants are

$$f_{M L J}^I NK I = (-1)^K \left( \delta^L_N \delta^M_J \delta^K_I - (-1)^{(M+N)(M+I)} \delta^M_K \delta^L_J \delta^K_I \right)$$

$$+ \frac{1}{8} \left( 1 - (-1)^{I+J} \right) (-1)^M \delta^M_N \delta^K_I \delta^K_J + \frac{1}{8} \left( (-1)^I - (-1)^J \right) \delta^K_N \delta^K_J \delta^K_I$$

$$+ \frac{1}{8} \left( (-1)^N - (-1)^M \right) \delta^K_M \delta^K_N \delta^K_I.$$  \hspace{1cm} (B.5)

We ‘raise’ and ‘lower’ the index pairs on the structure constants by

$$g^A_B C_D = (-1)^A \delta^A_D \delta^B_C, \quad (g^{-1})^A_B C_D = (-1)^B \delta^B_A \delta^C_D,$$  \hspace{1cm} (B.6)

$$f^M_{NK} J I = f^M_N Q I J \left( g^{-1} \right)^P Q L = f^M_N Q I J (-1)^Q \delta^K_P \delta^K_L,$$

$$f^M_{NK} J I = f^M_N Q I J g^L_K P = f^M_{NK} I J (-1)^K \delta^P_L \delta^P_Q.$$  \hspace{1cm} (B.7)
C Real form psl(4|4)

The Noether charge generators of the real form psl(4|4) can be written as

\[ p_{\dot{a}} = Q^\dot{a}_{\dot{0} a} = - \oint dz \ Y^{\dot{a}}(z) Z_a(z), \]

\[ k^{\dot{a}}_{\dot{a}} = Q^\dot{a}_{\dot{0} \dot{a}} = - \oint dz \ Y^{\dot{a}}(z) Z_{\dot{a}}(z), \]

\[ m^a_{\dot{b}} = Q^a_{\dot{0} \dot{b}} = - \oint dz \ \frac{1}{2} (Y_b(z) Z^a(z) + Y^a(z) Z_b(z)) \]

\[ = - \oint dz \ (Y^a(z) Z_b(z) - \frac{1}{2} \delta^a_b Y^c(z) Z_c(z)) \]

\[ \dot{m}^a_{\dot{b}} = Q^a_{\dot{0} \dot{b}} = - \oint dz \ \frac{1}{2} (\dot{Y}_b(z) Z^a(z) + \dot{Y}^a(z) Z_b(z)) \]

\[ = - \oint dz \ (Y^a(z) Z_b(z) - \frac{1}{2} \delta^a_b \dot{Y}^c(z) Z_c(z)), \]

\[ d = Q^d_{\dot{0} a} = - \oint dz \ \frac{1}{2} (Y^a(z) Z_b(z) - Y^a(z) Z_a(z)), \]

\[ r^A_B = Q^A_{\dot{0} B} = - \oint dz \ (Y^A(z) Z_B(z) - \frac{1}{4} \delta^A_B Y^C(z) Z_C(z)), \]

\[ q^A_a = Q^A_{\dot{0} a} = \oint dz \ Y^A(z) Z_a(z), \]

\[ \dot{q}^A_A = Q^A_{\dot{0} \dot{A}} = - \oint dz \ Y^{\dot{a}}(z) Z_A(z), \]

\[ s^a_A = Q^a_{\dot{0} A} = - \oint dz \ Y^a(z) Z_A(z), \]

\[ \dot{s}^A_{\dot{a}} = Q^A_{\dot{0} \dot{a}} = \oint dz \ Y^{\dot{a}}(z) Z_\dot{a}(z), \]  \hspace{1cm} (C.1)

where \( I = a, \dot{a}, A, \) so \( 1 \leq a, \dot{a} \leq 2 \) and \( 1 \leq A \leq 4 \). All the generators in (C.1) are antihermitian, except for \( q^A_a \) and \( s^A_{\dot{a}} \), which are hermitian.
The commutation relations for the generators (C.1) satisfy the real form $\text{psl}(4|4)$,

\[
[m^a_b, J_c] = \delta^a_c J_b - \frac{1}{2} \delta^a_b J_c, \quad [m^a_b, J^c] = -\delta^c_b J^a + \frac{1}{2} \delta^c_a J^b,
\]

\[
[\tilde{m}^a_b, J_c] = \delta^a_c J_b - \frac{1}{2} \delta^a_b J_c, \quad [\tilde{m}^a_b, J^c] = -\delta^c_b J^a + \frac{1}{2} \delta^c_a J^b,
\]

\[
[r^A_B, J_C] = \delta^A_C J_B - \frac{1}{2} \delta^A_B J_C, \quad [r^A_B, J^C] = -\delta^C_B J^A + \frac{1}{2} \delta^C_A J^B,
\]

\[
[s^a_A, p^b_b] = \delta^a_b \tilde{q}^b_A, \quad [k^a_A, \tilde{q}^b_C] = -\delta^b_a s^a_C \Rightarrow [k^{a\tilde{a}}, \tilde{q}_b^C] = \delta^b_\tilde{a} s^a_C,
\]

\[
[s^A_a, p^b_b] = -\delta^b_a \tilde{q}^A_a \Rightarrow [s^A_a, p^b_b] = \delta^b_a \tilde{q}^A_a, \quad [k^a_A, q^B_b] = \delta^A_a \tilde{q}^B_b,
\]

\[
\{\tilde{q}^A_a, q^B_b\} = \delta^B_A p^a_a, \quad \{s^A_a, s^B_B\} = \delta^A_B k^a_a,
\]

\[
[k^a_A, p^b_b] = -\delta^b_a m^a_b + \delta^b_a m^a_b - \delta^b_a \delta^a_b d \Rightarrow [k^{a\tilde{a}}, p^b_b] = \delta^a_b m^a_b + \delta^a_b \tilde{m}^a_b + \delta^b_a \delta^a_b d,
\]

\[
\{s^a_A, q^B_b\} = \delta^B_A m^a_b + \delta^a_B r^B_A + \frac{1}{2} \delta^a_B \delta^B_A (d - C),
\]

\[
\{\tilde{s}^a_A, \tilde{q}^B_b\} = \delta^B_A \tilde{m}^a_b - \delta^a_B r^A_B + \frac{1}{2} \delta^a_B \delta^A_B (d + C),
\]

\[
[d, J] = \text{dim}J, \quad [C, J] = 0,
\]

(C.2)

where the central generator $C \equiv \frac{1}{2} \oint dZ(z) Z_f(z)$ acts as zero on gauge invariant states.

We raise and lower the bosonic indices as $Z^a = e^{ab} Z_b$, $Z_a = e_{ab} Z^b$, and $Z^{\tilde{a}} = e^{\tilde{a}b} Z_b$, $Z_{\tilde{a}} = e_{\tilde{a}b} Z^b$, with $\epsilon^{12} = 1 = -\epsilon^{21}$, $\epsilon_{12} = -1 = -\epsilon_{21}$ for both the dotted and undotted indices.
D  Vanishing of the right-hand side of the Serre relation \((6,4)\) on one field

We evaluate the right hand side of the Serre relation first for \(gl(4|4)\),

\[
(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H} \sum_{JCP}^{I} \sum_{NEK}^{L} \sum_{SHU}^{V} \sum_{DFG}^{H} \left\{ \tilde{Q}_{0Q}^{P} \tilde{Q}_{0L}^{K} \tilde{Q}_{0V}^{U} \right\} \\

= \left( -(-1)^{M+R+N+S(M+J+R+N)+S+E} \delta_{I}^{L} \delta_{M}^{J} \delta_{N}^{K} \delta_{S}^{U} \right) \left\{ \tilde{Q}_{0E}^{I} \tilde{Q}_{0N}^{J} \tilde{Q}_{0R}^{M} - \tilde{Q}_{0J}^{I} \tilde{Q}_{0E}^{M} \tilde{Q}_{0N}^{R} \right\} \\

+ (-1)^{(M+J)(N+R)+N+E} \delta_{N}^{I} \left\{ \tilde{Q}_{0E}^{I} \tilde{Q}_{0N}^{J} \tilde{Q}_{0R}^{M} - \tilde{Q}_{0J}^{I} \tilde{Q}_{0E}^{M} \tilde{Q}_{0N}^{R} \right\} \\

+ (-1)^{(J+S)(R+N)+S+E} \delta_{S}^{I} \left\{ \tilde{Q}_{0E}^{I} \tilde{Q}_{0N}^{J} \tilde{Q}_{0R}^{M} - \tilde{Q}_{0J}^{I} \tilde{Q}_{0E}^{M} \tilde{Q}_{0N}^{R} \right\} \\

+ (-1)^{(J+M+S+J+S+E) \delta_{J}^{R} \left\{ \tilde{Q}_{0E}^{I} \tilde{Q}_{0N}^{J} \tilde{Q}_{0R}^{M} - \tilde{Q}_{0J}^{I} \tilde{Q}_{0E}^{M} \tilde{Q}_{0N}^{R} \right\} \\

+ (-1)^{(J+S+N+J+M+M+I+R+N+S+N) \delta_{J}^{M+J+M+I+R+N+S+N+M+S+J+S+E} \delta_{J}^{R} \left\{ \tilde{Q}_{0E}^{I} \tilde{Q}_{0N}^{J} \tilde{Q}_{0R}^{M} - \tilde{Q}_{0J}^{I} \tilde{Q}_{0E}^{M} \tilde{Q}_{0N}^{R} \right\} \\

\]

which vanishes for our single site representation from \((6,6)\). This holds as well for \(gl(n|n)\). Here the \(gl(4|4)\) structure constants are those of \(psl(4|4)\) but without the traces and supertraces removed,

\[
\tilde{f}_{JKL}^{I} = \delta_{J}^{K} \delta_{L}^{I} - (-1)^{(I+J)(K+L)} \delta_{J}^{N} \delta_{L}^{M} \delta_{I}^{K}, \quad \tilde{g}_{JKI}^{M} = (-1)^{K} \left( \delta_{J}^{L} \delta_{K}^{M} \delta_{I}^{J} - (-1)^{(M+N)(M+I)} \delta_{K}^{M} \delta_{J}^{L} \delta_{I}^{N} \right). \quad (D.1)
\]

\((D.1)\) implies that the \(psl(4|4)\) expression

\[
-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H} \sum_{JCP}^{I} \sum_{NEK}^{L} \sum_{SHU}^{V} \sum_{DFG}^{H} \left\{ \tilde{Q}_{0Q}^{P} \tilde{Q}_{0L}^{K} \tilde{Q}_{0V}^{U} \right\} \\

\]

also vanishes on one site, for if we view \((D.1)\) as a tensor \(M_{JN}^{IMR} \), then up to terms containing the central generator \(\sum_{C} \tilde{Q}_{0C}^{C} \), \((D.3)\) is just the tensor \(M_{JN}^{IMR} \) with the trace
and supertrace removed in each pair of indices $IJ, MN, RS$.

This follows from the properties of the $\text{psl}(n|n)$ structure constants given in Appendix B,

$$
(-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H} \int_{GC}^{D} P Q \int_{NE}^{M} F L \int_{SH}^{R G} U V \int_{D E}^{F H} Q \{ Q_{0 Q}^P Q_{0 L}^K Q_{0 V}^U \}
$$

$$
= (-1)^{(C+D)(K+L)+(G+H)(U+V)+G+H} \int_{GC}^{D} P Q \int_{NE}^{M} F L \int_{SH}^{R G} U V \int_{D E}^{F H} Q \{ \tilde{Q}_{0 Q}^P \tilde{Q}_{0 L}^K \tilde{Q}_{0 V}^U \}
$$

$$
\sim (\tilde{f} + h_1 + h_3)(\tilde{f} + h_1 + h_3)(\tilde{f} + h_1 + h_3) \tilde{f} \tilde{Q} \tilde{Q} \tilde{Q},
$$

(D.4)

where we have given separate labels to the supertrace and trace terms in (B.1) and (B.5) for the three pair of indices as $h_1, h_2, h_3$, $f = \tilde{f} + h_1 + h_2 + h_3$, and the expression in the final term of (D.4), $(\tilde{f} + h_1)(\tilde{f} + h_1)(\tilde{f} + h_1) \tilde{f} \tilde{Q} \tilde{Q} \tilde{Q}$ is the tensor $\mathcal{M}_{IJNS}^{MB}$ with the trace and supertrace removed in each pair of indices $IJ, MN, RS$.

Thus we have shown that the right hand side of the Serre relation (6.4) vanishes for charges acting on a single field, which together with (6.2) and the coproduct (4.21) implies that the Serre relation holds for any product of gauge invariant fields.

References

[1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. 252, 189 (2004) [arXiv:hep-th/0312171].

[2] N. Berkovits, *An alternative string theory in twistor space for $N = 4$ super-Yang-Mills*, Phys. Rev. Lett. 93, 011601 (2004) [arXiv:hep-th/0402045].

[3] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, JHEP 0408, 009 (2004) [arXiv:hep-th/0406051].

[4] L. Dolan and P. Goddard, *Gluon tree amplitudes in open twistor string theory*, JHEP 0912, 032 (2009) [arXiv:0909.0499 [hep-th]].

[5] L. Dolan and P. Goddard, *General split helicity gluon tree amplitudes in open twistor string theory*, JHEP 1005, 044 (2010) [arXiv:1002.4852 [hep-th]].

[6] M. Spradlin and A. Volovich, *From twistor string theory to recursion relations*, Phys. Rev. D 80, 085022 (2009) [arXiv:0909.0229 [hep-th]].
[7] J. L. Bourjaily, J. Trnka, A. Volovich and C. Wen, *The Grassmannian and the twistor string: connecting all trees in N=4 SYM*, arXiv:1006.1899 [hep-th].

[8] D. Nandan, A. Volovich and C. Wen, *A Grassmannian etude in NMHV minors*, JHEP 1007, 061 (2010) [arXiv:0912.3705 [hep-th]].

[9] R. Roiban, M. Spradlin and A. Volovich, *Dissolving N = 4 loop amplitudes into QCD tree amplitudes*, Phys. Rev. Lett. 94, 102002 (2005) [arXiv:hep-th/0412265].

[10] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, JHEP 0404, 012 (2004) [arXiv:hep-th/0402016].

[11] R. Roiban and A. Volovich, *All googly amplitudes from the B-model in twistor space*, Phys. Rev. Lett. 93, 131602 (2004) [arXiv:hep-th/0402121].

[12] R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, Phys. Rev. D 70, 026009 (2004) [arXiv:hep-th/0403190].

[13] N. Arkani-Hamed, F. Cachazo and C. Cheung, *The Grassmannian origin of dual superconformal invariance*, JHEP 1003, 036 (2010) [arXiv:0909.0483 [hep-th]].

[14] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *The S-Matrix in twistor space*, JHEP 1003, 110 (2010) [arXiv:0903.2110 [hep-th]].

[15] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality for the S Matrix*, JHEP 1003, 020 (2010) [arXiv:0907.5418 [hep-th]].

[16] N. Arkani-Hamed, J. Bourjaily, F. Cachazo and J. Trnka, *Local spacetime physics from the Grassmannian*, arXiv:0912.3249 [hep-th].

[17] N. Arkani-Hamed, J. Bourjaily, F. Cachazo and J. Trnka, *Unification of residues and Grassmannian dualities*, arXiv:0912.4912 [hep-th].

[18] L. Mason and D. Skinner, *Dual superconformal invariance, momentum twistors and grassmannians*, JHEP 0911, 045 (2009), [arXiv:0909.0250 [hep-th]].

[19] L. Mason and D. Skinner, *Heterotic twistor-string theory*, Nucl. Phys. B 795 (2008) 105 [arXiv:0708.2276 [hep-th]].

[20] D. Skinner, *A direct proof of BCFW recursion for twistor-strings*, arXiv:1007.0195 [hep-th].
[21] A. Brandhuber, P. Heslop and G. Travaglini, *A note on dual superconformal symmetry of the N=4 super Yang-Mills S-matrix*, Phys. Rev. D **78**, 125005 (2008) [arXiv:0807.4097 [hep-th]].

[22] M. Wolf, *On hidden symmetries of a super gauge theory and twistor string theory*, JHEP **0502**, 018 (2005) [arXiv:hep-th/0412163].

[23] L. Dolan, C. Nappi, and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, JHEP **0310**, 017 (2003) [arXiv:hep-th/0308089].

[24] L. Dolan, C. Nappi, and E. Witten, *Yangian symmetry in D = 4 superconformal Yang-Mills theory*, Contributed to 3rd International Symposium on Quantum Theory and Symmetries (QTS3), Cincinnati, Ohio, 10-14 Sep 2003. Published in *Cincinnati 2003, Quantum theory and symmetries* 300-315, [arXiv:hep-th/0401243].

[25] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[26] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the AdS(5) x S^5 super-string*, Phys. Rev. D **69**, 046002 (2004) [arXiv:hep-th/0305116].

[27] N. J. MacKay, *Introduction to Yangian symmetry in integrable field theory*, Int. J. Mod. Phys. A **20**, 7189 (2005) [arXiv:hep-th/0409183].

[28] N. J. MacKay, *On the classical origins of Yangian symmetry in integrable field theory*, Phys. Lett. B **281** 90 (1992); Erratum-ibid. B **308**, 444 (1993).

[29] V. Drinfel’d, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. **32** 254 (1985).

[30] V. Drinfel’d, *A new realization of Yangians and quantized affine algebras*, Sov. Math. Dokl. **36** 212 (1988).

[31] L. Dolan, *Yangians and Kac-Moody loop algebras in superconformal gauge theory*, Proceedings of Science, 27th Johns Hopkins Workshop on Current Problems in Particle Theory: Symmetries and Mysteries of M-Theory, Goteborg, Sweden, 24-26 Aug 2003, [http://pos.sissa.it/cgi-bin/reader/conf.cgi?confid=11](http://pos.sissa.it/cgi-bin/reader/conf.cgi?confid=11)

[32] L. Dolan, *Kac-Moody algebra is hidden symmetry of chiral models*, Phys. Rev. Lett. **47** 1371 (1981).
[33] L. Dolan and A. Roos, *Nonlocal currents as Noether currents*, Phys. Rev. D **22**, 2018 (1980).

[34] L. Dolan, *Kac-Moody algebras and exact solvability in hadronic physics*, Phys. Rept. **109**, 1 (1984).

[35] E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, *Remarks about the existence of non-local charges in two-dimensional models*, Phys. Lett. B **82** (1979) 442.

[36] A. M. Polyakov, *Gauge fields as rings of glue*, Nucl. Phys. B**164**, 1971 (1980).

[37] M. Luscher and K. Pohlmeyer, *Scattering of massless lumps and nonlocal charges in the two-dimensional classical non-linear sigma model*, Nucl. Phys. B**137**, 46 (1978).

[38] M. Luscher, *Quantum non-local charges and absence of particle production in the two-dimensional non-linear σ model*, Nucl. Phys. B**135**, 1 (1978).

[39] A. D. Popov and C. R. Preitschopf, *Conformal symmetries of the self-dual Yang-Mills equations*, Phys. Lett. B **374**, 71 (1996) [arXiv:hep-th/9512130].

[40] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory*, Nucl. Phys. B **828**, 317 (2010) [arXiv:0807.1095 [hep-th]].

[41] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, *Magic identities for conformal four-point integrals*, JHEP **0701**, 064 (2007) [arXiv:hep-th/0607160].

[42] L. F. Alday and J. M. Maldacena, *Gluon scattering amplitudes at strong coupling*, JHEP **0706**, 064 (2007) [arXiv:0705.0303 [hep-th]].

[43] N. Berkovits and J. Maldacena, *Fermionic T-Duality, dual superconformal symmetry, and the amplitude/Wilson loop connection*, JHEP **0809**, 062 (2008) [arXiv:0807.3196 [hep-th]].

[44] N. Beisert, R. Ricci, A. A. Tseytlin and M. Wolf, *Dual superconformal symmetry from AdS5 x S5 superstring integrability*, Phys. Rev. D **78**, 126004 (2008) [arXiv:0807.3228 [hep-th]].

[45] N. Beisert, *On Yangian symmetry in planar N=4 SYM*, [arXiv:1004.5423 [hep-th]].

[46] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, [arXiv:0905.1473 [hep-th]].
[47] J. M. Drummond, J. M. Henn and J. Plefka, Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory, JHEP 0905, 046 (2009) [arXiv:0902.2987 [hep-th]].

[48] J. M. Drummond and L. Ferro, Yangians, Grassmannians and T-duality, JHEP 1007, 027 (2010) [arXiv:1001.3348 [hep-th]].

[49] J. M. Drummond and L. Ferro, The Yangian origin of the Grassmannian integral, [arXiv:1002.4622 [hep-th]].

[50] D. 2. Bernard, Z. Maassarani and P. Mathieu, Logarithmic Yangians in WZW models, Mod. Phys. Lett. A 12, 535 (1997) [arXiv:hep-th/9612217].

[51] D. Bernard, Hidden Yangians in 2D massive current algebras, Comm. Math. Phys. 137, 191 (1991).

[52] L. Dolan, and P. Goddard, Tree and loop amplitudes in open twistor string theory, JHEP 0706, 005 (2007) [arXiv:hep-th/0703054].

[53] L. Dolan and P. Goddard, Current algebra on the torus, Commun. Math. Phys. 285, 219 (2009) [arXiv:0710.3743 [hep-th]].

[54] T. Creutzig, Branes in supergroups, arXiv:0908.1816 [hep-th].

[55] T. Creutzig and Y. Hikida, Branes in the OSP(1|2) WZNW model, arXiv:1004.1977 [hep-th].

[56] T. Creutzig, Geometry of branes on supergroups, Nucl. Phys. B 812, 301 (2009) [arXiv:0809.0468 [hep-th]].

[57] T. Creutzig and V. Schomerus, Boundary correlators in supergroup WZNW models, Nucl. Phys. B 807, 471 (2009) [arXiv:0804.3469 [hep-th]].

[58] T. Creutzig, T. Quella and V. Schomerus, Branes in the GL(1|1) WZNW-model, Nucl. Phys. B 792, 257 (2008) [arXiv:0708.0583 [hep-th]].

[59] L. Dolan and J. N. Ihry, Conformal supergravity tree amplitudes from open twistor string theory, Nucl. Phys. B 819, 375 (2009) [arXiv:0811.1341 [hep-th]].

[60] F. A. Berezin, edited by A.A. Kirillov and D. Leites, Introduction to superanalysis, Dordrecht, Netherlands: Reidel (1987) (Mathematical Physics and Applied Mathematics, vol. 9).