Covariant Currents in $N = 2$ Super-Liouville Theory

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ABSTRACT

Based on a path integral prescription for anomaly calculation, we analyze an effective theory of the two-dimensional $N = 2$ supergravity, i.e., $N = 2$ super-Liouville theory. We calculate the anomalies associated with the BRST supercurrent and the ghost number supercurrent. From those expressions of anomalies, we construct covariant BRST and ghost number supercurrents in the effective theory. We then show that the (super-)coordinate BRST current algebra forms a superfield extension of the topological conformal algebra for an arbitrary type of conformal matter or, in terms of the string theory, for an arbitrary number of space-time dimensions. This fact is very contrast with $N = 0$ and $N = 1$ (super-)Liouville theory, where the topological algebra singles out a particular value of dimensions. Our observation suggests a topological nature of the two-dimensional $N = 2$ supergravity as a quantum theory.

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1. Introduction

The $N = 2$ string or the two-dimensional $N = 2$ supergravity introduced by Ademollo et al [1,2,3,4,5,6,7] has critical dimension $d = 2$ and there is no transverse degree of freedom. Very recently, it has been argued that the no-ghost theorem can be established [8]. The $N = 2$ subcritical strings or $N = 2$ super-Liouville theory has also been analyzed. Distler, Hlousek, and Kawai [9] noticed that the local ansatz for the Jacobian, that relates the interacting measure with the free measure, works for any kind of conformal matter or, in terms of strings, for an arbitrary number of space-time dimensions. All those special features of $N = 2$ string suggest that this theory is a topological quantum field theory [10,11]. In a previous work [12], we proved that critical and subcritical $N = 2$ strings are topological field theories in the sense that the (super-)coordinate BRST current algebra gives a realization of an $N = 2$ superfield extension of the topological conformal algebra [10,13] for arbitrary type of conformal matter.

In this paper we want to analyze in detail the appearance of this topological conformal algebra in the case of $N = 2$ super-Liouville theory by constructing an effective theory based on anomalous identities associated with the BRST and ghost number symmetries. The finite renormalization of the coupling constant will be interpreted as a one-loop order effect of the BRST invariant measure. The relation with the critical string will be also commented.

The organization of the paper is as follows. In Section 2, we compute the BRST and ghost number anomalies in the $N = 2$ supergravity in the superconformal gauge, by using the path integral representation [14]. The $N = 2$ super-Liouville theory is constructed in Section 3 with a consideration on the covariant BRST and ghost number supercurrents. In Section 4 we derive the topological conformal algebra. In section 5 there are some conclusions and there is an Appendix about some basic facts of the $N = 2$ superfield formalism.
2. BRST and ghost number anomalies in $N = 2$ supergravity

In this section, we compute anomalies associated with the BRST and the ghost number supercurrents in the two-dimensional $N = 2$ supergravity. We follow the path integral prescription for anomaly calculation [14]. In the case of $N = 1$ supergravity, the superfield path integral is known to be an appropriate tool to compute those anomalies [15,16]. For $N = 2$ case, however, it is well-known that the action of the matter multiplet (A.4) cannot be written directly in terms of the real scalar superfield (A.8) [1]. Therefore it is not clear how the superfield path integral is useful in the present context of the anomaly calculation.*

Here we consider the path integral in the component fields. The classical gauge symmetries of the $N = 2$ supergravity are the general coordinate, the local Lorentz, the $N = 2$ supersymmetry, the Weyl, the super-Weyl, and the chiral transformations [3]. We assume under our regularization, the Weyl, the super-Weyl and the chiral transformations are anomalous at quantum level. Using the non-anomalous transformations, we can fix the $N = 2$ supergravity multiplet in the superconformal gauge as

\[ e^a_\mu = e^{\sigma/2} \delta^a_\mu, \]
\[ \chi^\pm_\mu = \gamma_\mu \phi^\pm, \]
\[ A_\mu = \epsilon_{\mu\nu} \partial^\nu \phi, \]

where $\sigma$ is the Liouville mode and, $\phi^\pm$ and $\phi$ are their $N = 2$ superpartners.

The BRST supercurrent and ghost number supercurrent anomalies will depend on the Liouville mode $\sigma$ and their $N = 2$ superpartners. As we are working in a path integral in the components, we should construct the integration variables depending on $\sigma$, $\phi^\pm$, and $\phi$ such that the integration measure is invariant under the supercoordinate transformations. This program have some difficulties, for $N = 1$, see for example [15]. Here instead we are going to use the following strategy.

* For the ghost and anti-ghost multiplet, the action can be written directly by the superfield as in (A.12). Thus the superfield path integral may work.
Let us forget about the anomalous character of the super-Weyl and the chiral transformations. By using these symmetries, we can fix the gauge

\[ e^a_\mu = e^{\sigma/2} \delta^a_\mu, \]
\[ \chi^{\pm}_\mu = 0, \]
\[ A_\mu = 0. \] (2.2)

In this way, the BRST supercurrent and ghost number supercurrent anomalies will depend only on the Liouville mode. When we consider the effect of the super-Weyl and the chiral anomalies, the remaining components of the gravitino and the U(1) gauge field should appear in various anomalies. Here we assume that we are actually using a regularization that is invariant under the super-coordinate and the gauge transformations. Especially, we assume the invariance under the global super transformation:

\[ \delta \sigma = i \left( \alpha^+ \phi^- - \alpha^- \phi^+ \right), \]
\[ \delta \phi = \alpha^- \phi^+ + \alpha^+ \phi^-, \]
\[ \delta \phi^{\pm} = \alpha^{\pm} (\partial \phi \pm i \partial \sigma). \] (2.3)

To get the dependences on the remaining components of the Liouville superpartners, we will use the invariance (or covariance) under (2.3). This strategy is the same as the one for a calculation of the super-Liouville action in [3].

The partition function of the \( N = 2 \) supergravity in the superconformal gauge is defined by

\[
\int d\bar{\mu} \exp \left\{ -\frac{1}{2\pi} \int \left[ \frac{1}{2} \left( -\partial X^\mu \bar{\partial} X^\mu - \partial Y^\mu \bar{\partial} Y^\mu + \psi^+ \bar{\partial} \psi^- + \psi^- \bar{\partial} \psi^+ \right) \right. \\
\left. + b \bar{\partial} c + \beta^+ \bar{\partial} \gamma^- + \beta^- \bar{\partial} \gamma^+ + \eta \bar{\partial} \xi \right] \right\},
\] (2.4)
where we defined the integration measure $d\tilde{\mu}$ by

$$
d\tilde{\mu} = D(e^{\sigma} c) D(e^{\sigma} \tau) D(e^{-\sigma/2} b) D(e^{-\sigma/2} \tilde{b}) D(e^{3\sigma/4} \gamma^+) D(e^{3\sigma/4} \tilde{\gamma}^+) \\
\times D(e^{3\sigma/4} \gamma^-) D(e^{3\sigma/4} \tilde{\gamma}^-) D(e^{-\sigma/4} \beta^+) D(e^{-\sigma/4} \tilde{\beta}^+) D(e^{-\sigma/4} \tilde{\beta}^-) D(e^{-\sigma/4} \beta^-) \\
\times D(e^{\sigma/2} \xi) D(e^{\sigma/2} \tilde{\xi}) D(\eta) D(\eta) D(e^{\sigma/4} \psi^+ \mu) D(e^{\sigma/4} \tilde{\psi}^+ \mu) \\
\times D(e^{\sigma/4} \psi^- \mu) D(e^{\sigma/4} \tilde{\psi}^- \mu) D(e^{\sigma/2} X^\mu) D(e^{\sigma/2} Y^\mu) \\
\equiv D\tilde{\sigma} D\tilde{\beta} D\gamma D\tilde{\gamma} D\tilde{\psi} D\tilde{\psi} D\tilde{\mu} \partial \phi + D\tilde{\beta} D\tilde{\gamma} D\tilde{\psi} D\tilde{\psi} D\tilde{\mu} \\
\times D\tilde{\sigma} D\tilde{\beta} D\gamma D\tilde{\gamma} D\tilde{\psi} D\tilde{\psi} D\tilde{\mu} D\tilde{X} D\tilde{Y}.
$$

(2.5)

The various weight factors ($\exp \sigma$) in the integration measure are determined from the general coordinate (BRST) invariance of the integration measure [17,18]. In the above expression, we did not include the integration of the Liouville (or Weyl) mode $\sigma$. We will turn this point in the next section. Our starting point (2.4) and (2.5) are the same as the one in [4].

We also note the above integration measure is invariant under the conformal transformation as noted in [16]:

$$
\delta \phi = V \partial \phi + h (\partial V) \phi, \quad \delta \phi = V \partial \phi,
$$

(2.6)

where $h$ is the conformal weight of the generic field $\phi$.

In the path integral formulation [14], the anomaly is generally ascribed to a non-invariance of the integration measure and the Jacobian factor associated with the anomalous transformation gives rise to the anomaly. The Jacobian factor of general conformal fields in a conformally flat background, is analyzed in [16]. According to [16], under an infinitesimal change of the integration variable, $\phi \rightarrow \phi + \varepsilon(x)\phi$, a logarithm of the Jacobian factor $J$ is given by

$$
\ln J = \pm \frac{1}{2\pi} \int d^2 x \varepsilon(x) \left[ \left( \frac{a - b}{3} \right) \partial \bar{\phi} \partial \sigma + M^2 e^{-2(a+b)\sigma} \right],
$$

(2.7)
and, for $\tilde{\phi} \to \tilde{\phi} + \varepsilon(x) \partial \tilde{\phi}$,

$$
\ln J = \pm \frac{1}{24\pi} \int d^2 x \varepsilon(x) \left[ (b^2 - 4ab) \partial \sigma \overline{\partial} \partial \sigma + (2a - 3b) \overline{\partial} \partial^2 \sigma 
- 12M^2 (a + b) \partial \sigma e^{-(2a+b)\sigma} \right].
$$

(2.8)

In (2.7) and (2.8), the double sign ($\pm$) corresponds to the statistics of the field $\phi$. The trace operation that is necessary to evaluate the above Jacobian factors is regularized by using an exponential type damping factor $e^{-H/M^2}$ with the regulator $H$, which is defined by

$$
H \equiv -D\overline{D} - e^{a\sigma} \partial e^{b\sigma} \overline{e}^{a\sigma}.
$$

(2.9)

where $D\overline{D}$ is the kinetic operator of the field $\tilde{\phi}$. By rewriting the action in (2.4) in terms of the integration variables in (2.5), we can read off the various values of $a$ and $b$ in (2.9) for the each fields (see Table 1).

To see how our present formulation works, let us first consider the ghost number anomaly [19]. In $N = 2$ case, the ghost number current is known to be anomaly free, due to a cancellation of the background charge [20]. The ghost number supercurrent [6] is defined by

$$
j_{gh}(Z) \equiv -BC(Z)
= i\eta e + \theta^- (-\eta \gamma^+ + i\beta^+ c) + \theta^+ (\eta \gamma^- + i\beta^- c)
+ \theta^- \theta^+ (-\eta \xi - \beta^- \gamma^+ - \beta^+ \gamma^- - bc)
\equiv j_{gh}^0(z) + \theta^- j_{gh}^+(z) + \theta^+ j_{gh}^-(z) + \theta^- \theta^+ j_{gh}^{++}(z).
$$

(2.10)

We can immediately see

$$
\overline{\mathcal{J}} \langle j_{gh}^0(z) \rangle = \overline{\mathcal{J}} \langle j_{gh}^+(z) \rangle = \overline{\mathcal{J}} \langle j_{gh}^-(z) \rangle = 0,
$$

(2.11)

i.e., these currents are anomaly free. To show this, let us consider the following
infinitesimal change of variables in (2.4):

\[
b \to b + i\varepsilon(x)\eta, \quad \xi \to \xi - i\varepsilon(x)c. \tag{2.12}\]

Note that the partition function itself does not change under a change of the integration variables. Therefore the variation of the action and the variation of the integration measure should be canceled each other, and we have the following identity:

\[
-\frac{1}{2\pi} \int d^2x \varepsilon(x) \bar{\mathcal{D}} \left< j_{gh}^0(z) \right> + \langle \ln J \rangle = 0, \tag{2.13}\]

where \( J \) is a Jacobian factor associated with the change of variables (2.12). However, for (2.12), the Jacobian is trivial, i.e., \( J = 1 \) and \( \ln J = 0 \), because the variation of the field is not proportional to the field itself. Therefore \( \bar{\mathcal{D}} \left< j_{gh}^0(z) \right> \) is anomaly free. Similar considerations show other relations in (2.11).

A potential anomalous term in a vacuum expectation value is thus a product of the equation of motion and the conjugate field, because such a combination is proportional to the Jacobian factor of a change of variable whose variation is proportional to the field itself. In this sense the final combination \( \bar{\mathcal{D}} \left< j_{gh}^{+-}(z) \right> \) is potentially dangerous. Let us consider the following change of variables:

\[
\delta c = \varepsilon(x)c, \quad \delta b = -\varepsilon(x)b,
\]
\[
\delta \gamma^\pm = \varepsilon(x)\gamma^\pm, \quad \delta \beta^\pm = -\varepsilon(x)\beta^\pm,
\]
\[
\delta \xi = \varepsilon(x)\xi, \quad \delta \eta = -\varepsilon(x)\eta, \tag{2.14}\]

(more precisely, we should write down the variation of the tilded integration variables, but for (2.14), the variation of the tilded variables is proportional to the one of the untilded variables). Then we have

\[
-\frac{1}{2\pi} \int d^2x \varepsilon(x) \bar{\mathcal{D}} \left< j_{gh}^{+-}(z) \right> + \langle \ln J \rangle = 0, \tag{2.15}\]

where \( J \) is the Jacobian factor associated with the variation (2.14). From the
master formula (2.7), we have

\[
\ln J = -\frac{1}{4\pi} \int d^2 x \varepsilon(x)(-3 + 2 \times 2 - 1) \overline{\mathcal{D}} \partial \sigma = 0,
\] (2.16)

where the contributions from the different sector \([(b,c), (\beta^\mp, \gamma^\pm), \text{and } (\eta, \xi) \text{ respectively}] are separately indicated. We can see that the ghost number anomaly vanishes due to a cancellation of the background charges. Our formulation reproduces the desired answer as is expected.

Let us now turn to the anomaly associated with the conservation of the BRST supercurrent. We define the BRST supercurrent as [12]

\[
j_B(Z) \equiv C \left(T^X + \frac{1}{2} T^{gh}\right) + \frac{1}{4} D^- [C (D^+ C) B] + \frac{1}{4} D^+ [C (D^- C) B] \\
\equiv J_B(Z) + \hat{j}_B(Z).
\] (2.17)

To see the structure of the BRST anomaly, we call the first term in the first line in (2.17) as \(J_B(Z)\) and the total divergence parts as \(\hat{j}_B(Z)\). We should note here that we chose the total divergence parts \(\hat{j}_B(B)\) (which do not affect to the BRST charge \(Q_B\)) by

\[
j_B(Z) = - \{Q_B, j_{gh}(Z)\},
\] (2.18)

to make the BRST current manifestly BRST invariant if \(Q_B^2 = 0\). In this sense the above choice is the most symmetric one and this choice of \(\hat{j}_B(Z)\) is crucial for our conclusion.

We also define the components of the BRST supercurrent as

\[
J_B(Z) \equiv J^0_B(z) + \theta^- J^+_B(z) + \theta^+ J^-_B(z) + \theta^- \theta^+ J^{+-}_B(z),
\]

\[
\hat{j}_B(Z) \equiv \hat{j}^0_B(z) + \theta^- \hat{j}^+_B(z) + \theta^+ \hat{j}^-_B(z) + \theta^- \theta^+ \hat{j}^{+-}_B(z).
\] (2.19)
The explicit form of the component currents of $J_B(Z)$ becomes:

\[
J_B^0(z) = \frac{1}{2} c \left[ \psi^{-\mu} \psi^{+\mu} - i \partial (c\eta) + \frac{1}{2} \gamma^+ \beta^- - \frac{1}{2} \gamma^- \beta^+ \right],
\]
\[
J_B^+(z) = -\frac{1}{2} c \left[ -i \partial Y^{\mu} \psi^{+\mu} + \partial X^{\mu} \psi^{+\mu} - \partial \left( \gamma^+ \eta \right) + i \partial \left( c\beta^+ \right) - \frac{1}{2} i \gamma^+ b + \frac{1}{2} \gamma^+ \partial \eta \right.
\]
\[
\left. + \frac{1}{2} \xi \beta^+ + \frac{1}{2} i \partial \epsilon \beta^+ \right]
\]
\[
- \frac{1}{2} i \gamma^+ \left[ \psi^{-\mu} \psi^{+\mu} - i \partial (c\eta) - \frac{1}{2} \gamma^+ \beta^- - \frac{1}{2} \gamma^- \beta^+ \right],
\]
\[
J_B^-(z) = -\frac{1}{2} c \left[ -i \partial Y^{\mu} \psi^{-\mu} + \partial X^{\mu} \psi^{-\mu} + \partial \left( \gamma^- \eta \right) + i \partial \left( c\beta^- \right) - \frac{1}{2} i \gamma^- b - \frac{1}{2} \gamma^- \partial \eta \right]
\]
\[
+ \frac{1}{2} i \gamma^- \left[ \psi^{-\mu} \psi^{+\mu} - i \partial (c\eta) + \frac{1}{2} \gamma^+ \beta^- - \frac{1}{2} \gamma^- \beta^+ \right],
\]
\[
J_B^{+-}(z) = \tilde{J}_B^{+-}(z) + \frac{3}{4} \partial \left( c\gamma^+ \beta^- + c\gamma^+ \beta^- \right) - \frac{1}{2} \partial (c\xi \eta)
\]
\[
+ \frac{i}{2} \xi \left( \psi^{-\mu} \psi^{+\mu} + \gamma^+ \beta^- - \gamma^- \beta^+ \right) + \frac{i}{2} \gamma^- \partial \left( \gamma^+ \eta \right) - \frac{i}{2} \gamma^+ \partial \left( \gamma^- \eta \right) - \frac{1}{2} \gamma^- \gamma^+ b.
\]

In the last expression, $\tilde{J}_B^{+-}(z)$ is defined by

\[
\tilde{J}_B^{+-}(z) \equiv \frac{1}{2} c \left( \partial X^{\mu} \partial X^{\mu} + \partial Y^{\mu} \partial Y^{\mu} + \partial \psi^{-\mu} \psi^{+\mu} + \partial \psi^{+\mu} \psi^{-\mu} \right)
\]
\[
+ c \left( \partial cb - \frac{1}{2} \gamma^- \partial \beta^+ - \frac{3}{2} \partial \gamma^- \beta^+ - \frac{1}{2} \gamma^+ \partial \beta^- - \frac{3}{2} \partial \gamma^+ \beta^- + \partial \xi \eta \right).
\]

Similarly, the components of the hat supercurrent $\hat{j}_B(Z)$ are given by
\( \mathcal{J}_B^+ (z) = - \frac{1}{2} \gamma^{-} \gamma^{+} \eta + \frac{1}{4} c \left( \gamma^{+} \beta^{-} - \gamma^{-} \beta^{+} \right) \),

\( \mathcal{J}_B^- (z) = - \frac{1}{4} \partial \left( c \gamma^{+} \eta \right) - \frac{1}{4} c \partial \gamma^{+} \eta + \frac{i}{2} \gamma^{+} \xi \eta + \frac{1}{4} \gamma^{+} \partial c \eta - \frac{i}{4} \gamma^{+} \gamma^{+} \beta^{-}
- \frac{i}{4} \xi \beta^{+} + \frac{i}{4} c \partial c \beta^{+} - \frac{i}{4} \gamma^{+} \gamma^{-} \beta^{-} + \frac{i}{4} \gamma^{+} \gamma^{+} \beta^{-}, \)  

\( \mathcal{J}_B^{-} (z) = - \frac{1}{2} \partial \left( c \gamma^{-} \eta \right) + \frac{1}{4} c \partial \gamma^{-} \eta + \frac{i}{2} \gamma^{-} \xi \eta - \frac{1}{4} \gamma^{-} \partial c \eta - \frac{i}{4} \gamma^{-} \gamma^{+} \beta^{+}
+ \frac{i}{4} c \xi \beta^{-} + \frac{i}{4} c \partial c \beta^{-} - \frac{i}{4} \gamma^{-} \gamma^{-} \beta^{-} + \frac{i}{4} \gamma^{-} \gamma^{+} \beta^{-}, \)  

\( \mathcal{J}_B^{-} (z) = - \frac{1}{2} \partial (c \xi \eta) + \frac{1}{4} \partial \left( c \gamma^{+} \beta^{-} \right) + \frac{1}{4} \partial \left( c \gamma^{-} \beta^{+} \right). \)  

Let us start the calculation of the BRST anomaly from the first part, \( \overline{\partial} \langle \mathcal{J}_B^0 (z) \rangle. \) Noting the dangerous combination, i.e., a product of the equation of motion and the conjugate field, we find

\[ \overline{\partial} \langle \mathcal{J}_B^0 (z) \rangle = \left\langle \frac{1}{2} c \overline{\partial} \left( \psi^{-} \mu \psi^{+ \mu} \right) + \frac{1}{4} c \overline{\partial} \left( \gamma^{+} \beta^{-} - \gamma^{-} \beta^{+} \right) \right\rangle. \]  

Therefore we use the following variation of the integration variables

\[ \delta \psi^{+ \mu} = - \frac{1}{2} \varepsilon (x) c \psi^{+ \mu}, \quad \delta \psi^{- \mu} = \frac{1}{2} \varepsilon (x) c \psi^{- \mu}, \]  

(2.24)

to get

\[ - \frac{1}{2 \pi} \int d^2 x \varepsilon (x) \left\langle \frac{1}{2} c \overline{\partial} \left( \psi^{-} \mu \psi^{+ \mu} \right) \right\rangle + \langle \ln J \rangle = 0, \]  

(2.25)

where \( J \) is the Jacobian factor associated with the above transformation (2.24). However if we note the fact that, in our present formulation, the Jacobian factor does not depend on the \( U(1) \) charge (the superscript \( \pm \)) but only on the conformal weight, we can see the contributions from \( \psi^{+ \mu} \) and \( \psi^{- \mu} \) cancel each other, i.e., \( \ln J = 0 \) in (2.25). Therefore

\[ \left\langle \frac{1}{2} c \overline{\partial} \left( \psi^{-} \mu \psi^{+ \mu} \right) \right\rangle = 0. \]  

(2.26)
From the same reason, we have

\[
\left\langle \frac{1}{4} i \partial \gamma^+ \beta^- \right\rangle = \left\langle \frac{1}{4} i \partial \gamma^- \beta^+ \right\rangle.
\]  

(2.27)

Combining (2.26) and (2.27),

\[
\overline{\partial} \left\langle J_0^B(z) \right\rangle = 0,
\]

(2.28)

i.e., \( J_0^B(z) \) is anomaly free.

For \( \overline{\partial} \left\langle J_B^+(z) \right\rangle \), since \( \left\langle -\frac{1}{2} i \gamma^+ \overline{\partial} (\psi^- \psi^+) \right\rangle = 0 \), we can see,

\[
\overline{\partial} \left\langle J_B^+(z) \right\rangle = \left\langle \frac{1}{4} i \gamma^+ \overline{\partial} (cb) - \frac{1}{4} i \overline{\partial} (\gamma^+ \gamma^+ \gamma^-) + \frac{1}{4} i \gamma^+ \overline{\partial} (\gamma^- \beta^+) \right\rangle.
\]  

(2.29)

We consider the following variations:

\[
\delta c = \frac{1}{4} i \varepsilon(x) \gamma^+ c, \quad \delta b = -\frac{1}{4} i \varepsilon(x) \gamma^+ b,
\]

\[
\delta \gamma^+ = \frac{1}{4} i \varepsilon(x) \gamma^+ \gamma^+, \quad \delta \gamma^- = -\frac{1}{4} i \varepsilon(x) \gamma^+ \gamma^-,
\]

\[
\delta \beta^+ = \frac{1}{4} i \varepsilon(x) \gamma^+ \beta^+, \quad \delta \beta^- = -\frac{1}{4} i \varepsilon(x) \gamma^+ \beta^-,
\]

(2.30)

and obtain the following identity:

\[
-\frac{1}{2\pi} \int d^2 x \varepsilon(x) \overline{\partial} \left\langle J_B^+(z) \right\rangle + \langle \ln J \rangle = 0.
\]  

(2.31)

From the master formula (2.7), we have

\[
\ln J = \frac{1}{2\pi} \int d^2 x \varepsilon(x) \frac{i}{8} \gamma^+ \overline{\partial} \partial \sigma,
\]

(2.32)

and then

\[
\overline{\partial} \left\langle J_B^+(z) \right\rangle = \frac{i}{8} \left\langle \gamma^+ \overline{\partial} \partial \sigma \right\rangle = \frac{i}{8} \overline{\partial} \left\langle \gamma^+ \partial \sigma \right\rangle.
\]  

(2.33)

In deriving the last expression, we used a safe equation of motion, \( \left\langle \overline{\partial} \gamma^+ \right\rangle = 0 \). The equation of motion alone is always safe, i.e., the Schwinger–Dyson equation always is valid.
Similarly, for \( \overline{\vartheta} \langle J_B^- (z) \rangle \) (by interchanging \(+ \leftrightarrow -\)), we have

\[
\overline{\vartheta} \langle J_B^- (z) \rangle = \frac{i}{8} \langle \gamma^- \overline{\vartheta} \partial \sigma \rangle = \frac{i}{8} \overline{\vartheta} \langle \gamma^- \partial \sigma \rangle.
\]  
(2.34)

To evaluate \( \overline{\vartheta} \langle J_B^+ (z) \rangle \), let us first consider \( \overline{\vartheta} \langle \tilde{J}_B^+ (z) \rangle \). We take the following variations:

\[
\delta \sigma = 0,
\delta X^\mu = \varepsilon (x) c \partial X^\mu, \quad \delta Y^\mu = \varepsilon (x) c \partial Y^\mu,
\delta \psi^\pm \mu = \varepsilon (x) \left[ c \partial \psi^\pm \mu + \frac{1}{2} (\partial c) \psi^\pm \mu \right],
\delta c = \varepsilon (x) c \partial c,
\delta b = \varepsilon (x) \left[ c \partial b + 2 (\partial c) b \right]
\]
\[
+ \frac{1}{2} \left( \partial X^\mu \partial X^\mu + \partial Y^\mu \partial Y^\mu + \partial \psi^- \mu \psi^\mu - \partial \psi^\mu \psi^- \mu \right),
\]
\[
- \frac{1}{2} \gamma^- \partial \beta^+ - \frac{3}{2} \gamma^+ \partial \beta^- - \frac{3}{2} \partial \gamma^+ \beta^- + \partial \xi \eta \right],
\]
\[
\delta \gamma^\pm = \varepsilon (x) \left[ c \partial \gamma^\pm - \frac{1}{2} (\partial c) \gamma^\pm \right],
\delta \beta^\pm = \varepsilon (x) \left[ c \partial \beta^\pm + \frac{3}{2} (\partial c) \beta^\pm \right],
\delta \xi = \varepsilon (x) c \partial \xi,
\delta \eta = \varepsilon (x) [ c \partial \eta + (\partial c) \eta ].
\]

Then we have the following identity:

\[
- \frac{1}{2 \pi} \int d^2 x \left\{ \varepsilon (x) \overline{\vartheta} \langle \tilde{J}_B^+ (z) \rangle \right.
\]
\[
- \partial \varepsilon (x) c \left[ \frac{1}{2} (\psi^\dag \psi^- \mu + \psi^- \mu \psi^\dag) + b \overline{d} \overline{c} + \overline{\eta} \overline{d} \overline{\xi} \right]
\]
\[
+ \frac{3}{2} \beta^+ \overline{\gamma}^- + \frac{1}{2} \overline{\beta}^+ \gamma^- + \frac{3}{2} \beta^- \overline{\gamma}^+ + \frac{1}{2} \overline{\beta}^- \gamma^+ \}
\]
\[
+ \langle \ln J_1 \rangle = 0,
\]

where \( J_1 \) is the Jacobian factor associated the variations (2.35). The variations
cause the following variations of the integration variables:

\[ \delta \tilde{X}^\mu = \varepsilon(x) \left[ c \partial \tilde{X}^\mu - \frac{1}{2} c(\partial \sigma) \tilde{X}^\mu \right], \]
\[ \delta \tilde{Y}^\mu = \varepsilon(x) \left[ c \partial \tilde{Y}^\mu - \frac{1}{2} c(\partial \sigma) \tilde{Y}^\mu \right], \]
\[ \delta \tilde{\psi}^{\pm \mu} = \varepsilon(x) \left[ c \partial \tilde{\psi}^{\pm \mu} - \frac{1}{4} c(\partial \sigma) \tilde{\psi}^{\pm \mu} + \frac{1}{2} (\partial c) \tilde{\psi}^{\pm \mu} \right], \]
\[ \delta \tilde{c} = \varepsilon(x) e^{-\sigma} \partial \tilde{c}, \]
\[ \delta \tilde{b} = \varepsilon(x) \left[ c \partial \tilde{b} + \frac{1}{2} c(\partial \sigma) \tilde{b} + 2(\partial c) \tilde{b} \right], \]
\[ \delta \tilde{\gamma}^{\pm} = \varepsilon(x) \left[ c \partial \tilde{\gamma}^{\pm} - \frac{3}{4} (\partial \sigma) \tilde{\gamma}^{\pm} - \frac{1}{2} (\partial c) \tilde{\gamma}^{\pm} \right], \]
\[ \delta \tilde{\beta}^{\pm} = \varepsilon(x) \left[ c \partial \tilde{\beta}^{\pm} + \frac{1}{4} c(\partial \sigma) \tilde{\beta}^{\pm} + \frac{3}{2} (\partial c) \tilde{\beta}^{\pm} \right], \]
\[ \delta \tilde{\xi} = \varepsilon(x) \left[ c \partial \tilde{\xi} - \frac{1}{2} c(\partial \sigma) \tilde{\xi} \right], \]
\[ \delta \eta = \varepsilon(x) [c \partial \eta + (\partial c) \eta]. \]

From the master formula (2.7) and (2.8) we have

\[ \ln J_1 = \frac{1}{2\pi} \int d^2 x \varepsilon(x) \left[ -\frac{d}{3} c \partial^2 \sigma + \frac{d-2}{4} c \sigma \partial \sigma - \frac{d+6}{12} \partial c \partial \sigma - dM^2 \partial (ce^\sigma) \right]. \] (2.38)

For the remaining part in (2.36), we can see

\[ \frac{1}{2\pi} \int d^2 x (-\partial \varepsilon(x)) \left\langle c \left[ \frac{1}{2} (\psi^+ \partial \psi^- + \psi^- \partial \psi^+) + b \partial c + \eta \partial \xi \right. \right. \]
\[ \left. \left. + \frac{3}{2} \beta^+ \partial \gamma^- + \frac{1}{2} \partial \beta^+ \gamma^- + \frac{3}{2} \beta^- \partial \gamma^+ + \frac{1}{2} \partial \beta^- \gamma^+ \right] \right\rangle \] (2.39)
\[ + \langle \ln J_2 \rangle = 0, \]
where the Jacobian $J_2$ is associated with

$$
\delta \psi^{\pm \mu} = -\frac{1}{2} \partial \varepsilon(x) c \psi^{\pm \mu},
\delta b = -\partial \varepsilon(x) c b,
\delta \gamma^{\pm} = \frac{1}{2} \partial \varepsilon(x) c \gamma^{\pm},
\delta \beta^{\pm} = -\frac{3}{2} \partial \varepsilon(x) c \beta^{\pm},
\delta \eta = -\partial \varepsilon(x) c \eta.
$$

By using the master formula,

$$
\ln J_2 = \frac{1}{2\pi} \int d^2 x \varepsilon(x) \left[ -\frac{d-12}{12} \partial (c \bar{\partial} \partial \sigma) - dM^2 \partial (c e^\sigma) \right].
$$

(2.41)

Combining the above results (2.38) and (2.41), we have

$$
\bar{\partial} \langle \bar{J}_B^{+-}(z) \rangle = \frac{d-2}{4} \langle c (\partial \sigma \bar{\partial} \partial \sigma - \bar{\partial} \partial^2 \sigma) \rangle - \frac{3}{2} \langle \partial (c \bar{\partial} \partial \sigma) \rangle.
$$

(2.42)

Finally we have to calculate:

$$
\bar{\partial} \langle J_B^{+-}(z) - \bar{J}_B^{+-}(z) \rangle
= \bar{\partial} \left\langle \frac{3}{4} \partial (c \gamma^+ \beta^- + c \gamma^- \beta^+) - \frac{1}{2} \partial (c \xi \eta) \right\rangle
= \partial \left\langle \frac{3}{4} \bar{\partial} (\gamma^- \beta^+ + \gamma^+ \beta^-) - \frac{1}{2} \bar{\partial} (\xi \eta) \right\rangle
= \frac{5}{4} \langle \partial (c \bar{\partial} \partial \sigma) \rangle.
$$

(2.43)

(The calculation is similar to the ghost number anomaly.) Collecting the above considerations, we finally get

$$
\bar{\partial} \langle J_B^{+-}(z) \rangle = \frac{d-2}{4} \langle c (\partial \sigma \bar{\partial} \partial \sigma - \bar{\partial} \partial^2 \sigma) \rangle - \frac{1}{4} \langle \partial (c \bar{\partial} \partial \sigma) \rangle
= \frac{d-2}{4} \bar{\partial} \left\langle c \left( \frac{1}{2} \partial \sigma \bar{\partial} \partial \sigma - \partial^2 \sigma \right) \right\rangle - \frac{1}{4} \bar{\partial} \langle \partial (c \bar{\partial} \partial \sigma) \rangle,
$$

(2.44)

where, in the final step, we used a safe equation of motion $\langle \bar{\partial} c \rangle = 0$. 

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For the hat currents in (2.22), similar calculations show,

\[ \overline{\partial} \langle \hat{J}_B^0 \rangle = 0, \]
\[ \overline{\partial} \langle \hat{J}_B^+ \rangle = -\frac{i}{8} \langle \gamma^+ \overline{\partial} \partial \sigma \rangle = -\frac{i}{8} \overline{\partial} \langle \gamma^+ \partial \sigma \rangle, \]
\[ \overline{\partial} \langle \hat{J}_B^- \rangle = -\frac{i}{8} \langle \gamma^- \overline{\partial} \partial \sigma \rangle = -\frac{i}{8} \overline{\partial} \langle \gamma^- \partial \sigma \rangle, \]
\[ \overline{\partial} \langle \hat{J}_B^{+-} \rangle = \frac{1}{4} \langle \partial \left( \overline{c} \overline{\partial} \partial \sigma \right) \rangle = \frac{1}{4} \overline{\partial} \langle \partial \left( \overline{c} \overline{\partial} \partial \sigma \right) \rangle, \]  

where, in the final step, we used safe equations of motion, \( \langle \overline{\partial} \gamma^\pm \rangle = \langle \overline{\partial} c \rangle = 0 \).

One may summarize those identities (2.28), (2.33), (2.34), and (2.44), and (2.45) in supercurrent forms:

\[ \overline{\partial} \langle J_B(Z) \rangle = \overline{\partial} \left( \frac{i}{8} \theta^- \gamma^+ \partial \sigma + \frac{i}{8} \theta^+ \gamma^- \partial \sigma \right. \]
\[ + \theta^- \theta^+ \left[ \frac{d-2}{4} c \left( \frac{1}{2} \overline{\partial} \sigma \overline{\partial} \sigma - \partial^2 \sigma \right) - \frac{1}{4} \partial \left( \overline{c} \overline{\partial} \partial \sigma \right) \right] \left. \right) \],

\[ \overline{\partial} \langle \hat{J}_B(Z) \rangle = \overline{\partial} \left( -\frac{i}{8} \theta^- \gamma^+ \partial \sigma - \frac{i}{8} \theta^+ \gamma^- \partial \sigma + \frac{1}{4} \theta^- \theta^+ \partial \left( \overline{c} \overline{\partial} \partial \sigma \right) \right). \]

Now, as noted previously, we assumed that our regularization actually preserves the global supersymmetry (2.3). In terms of the superfield, this assumption requires the right hand sides of (2.46) should behave as covariant supercurrents. Thus here we introduce the super-Liouville field by

\[ \Phi(Z) = \phi(z) + \theta^- \phi^+(z) + \theta^+ \phi^-(z) + i \theta^- \theta^+ \partial \sigma(z), \]

where \( \phi \) and \( \phi^\pm \) are the remaining components of the U(1) gauge field and the gravitino respectively, and \( \sigma \) is the Liouville mode. The covariant combinations which reproduce (2.46) under a condition \( \phi = \phi^\pm = 0 \) are

\[ \overline{\partial} \langle J_B(Z) \rangle = \frac{d-2}{4} \overline{\partial} \left\langle C \left( \frac{1}{2} D^- \Phi D^+ \Phi + i \partial \Phi \right) \right\rangle \]
\[ - \frac{i}{8} \overline{\partial} \left\langle D^+ \left( CD^- \Phi \right) + D^- \left( CD^+ \Phi \right) \right\rangle, \]
\[ \overline{\partial} \langle \hat{J}_B(Z) \rangle = \frac{i}{8} \overline{\partial} \left\langle D^+ \left( CD^- \Phi \right) + D^- \left( CD^+ \Phi \right) \right\rangle. \]
The above covariantizations are unique ones. Therefore if we take our definition of the BRST supercurrent (2.17), we have

\[ \overline{\partial} \langle j_B(Z) \rangle = \frac{d-2}{4} \overline{\partial} \left\langle C \left( \frac{1}{2} D^- \Phi D^+ \Phi + i\Phi \right) \right\rangle. \] (2.49)

Here we should emphasize that the BRST anomaly in (2.49) vanishes for \( d = 2 \). In the case of \( N = 0 \) and \( N = 1 \) (super-)gravity, on the other hand, even if one take a BRST invariant BRST current as in (2.18), the BRST anomaly contains a additional total divergent piece [21,16], which does not proportional to \( d - 26 \) and \( d - 10 \) respectively. As a consequence, the BRST anomaly in \( N = 0 \) and \( N = 1 \) (super-)gravity does \textit{not} vanish even in the critical dimension \( d = 26 \) and \( d = 10 \). The origin of the total divergent BRST anomaly is the fact that the BRST invariant path integral measure is invariant under the global BRST transformation, up to a total divergence [18]. Therefore it generates a total divergent anomaly in general under a \textit{localized} BRST transformation, like (2.35). In this sense, the absence of a total divergent anomaly in (2.49) is an intrinsic feature of the \( N = 2 \) theory and suggests the topological nature of \( N = 2 \) theory as a quantum theory.

We can also summarize the ghost number anomaly in terms of the supercurrent:

\[ \overline{\partial} \langle j_{gh}(Z) \rangle = 0. \] (2.50)

The anomalous identities, (2.49) and (2.50) will play an important role when we construct the effective covariant supercurrents in \( N = 2 \) super-Liouville theory.
3. \( N = 2 \) super-Liouville theory and the covariant supercurrents

In the previous section, we saw that the various anomalies appear through the \( \sigma \)-dependences in the integration measure (2.5). Here we try to construct an effective theory which is supposedly equivalent with the original \( N = 2 \) supergravity (2.4), by incorporating the effect of anomalies. Firstly, following a standard procedure to produce the Wess–Zumino term in the string theory [19], we repeat an infinitesimal transformation of the integration variables. For example, we change \( \tilde{X}^\mu \) as

\[
\tilde{X}^\mu \rightarrow \left( 1 + \frac{\sigma}{2} dt \right) \tilde{X}^\mu. \tag{3.1}
\]

By repeating this infinitesimal transformation up to a finite \( t \), the kinetic operator of \( \tilde{X}^\mu \) changes to

\[
e^{-\sigma(1-t)/2} \overline{\partial} \partial e^{-\sigma(1-t)/2}, \tag{3.2}
\]

thus all the \( \sigma \) dependences in (2.4) and (2.5) disappear at \( t = 1 \).

On the other hand, from the master formula (2.7), the change of variable in (3.1) generates the following Jacobian:

\[
\ln J(t)_{\tilde{X}^\mu} = \frac{d}{2\pi} dt \int d^2 x \sigma \left[ -\frac{1}{12} (1-t) \overline{\partial} \partial \sigma + \frac{1}{2} M^2 e^{(1-t)\sigma} \right]. \tag{3.3}
\]

Summing over all the contributions from various fields and by integrating \( \ln J(t) \) from \( t = 0 \) to \( 1 \), we have the so-called Liouville action:

\[
\int_0^1 \ln J(t) = -\frac{2-d}{16\pi} \int d^2 x \partial \sigma \overline{\partial} \sigma. \tag{3.4}
\]

Note that the “Liouville term,” \( e^\sigma \), disappears in (3.4) because of the supersymmetry of the original model.
As noted in the previous section, the Liouville action should be invariant under the global super transformation (2.3). Therefore, under a supersymmetric regularization, the Liouville action (3.4) should have the form [3],

\[ S_{\text{Liouville}} \equiv -\frac{2-d}{16\pi} \int d^2x \left[ \partial \sigma \overline{\partial} \sigma + \partial \phi \overline{\partial} \phi - \phi^+ \overline{\partial} \phi^- - \phi^- \overline{\partial} \phi^+ + (\text{c. c.}) \right]. \quad (3.5) \]

In this stage, since we have extracted the \( \sigma \)-dependences in the integration measure as the Liouville action (3.5), our partition function of the matter and the ghost multiplets in a fixed metric background has the following form:

\[
\int d\mu \exp \left\{ -\frac{1}{2} \int \left[ \frac{1}{2} \left( -\partial X^\mu \overline{\partial} X^\mu - \partial Y^\mu \overline{\partial} Y^\mu + \psi^{+\mu} \overline{\partial} \psi^{-\mu} + \psi^{-\mu} \overline{\partial} \psi^{+\mu} \right) + b \overline{\partial} c + \beta^+ \overline{\partial} \gamma^- + \beta^- \overline{\partial} \gamma^+ + \eta \overline{\partial} \xi + (\text{c. c.}) \right] \right\}, \quad (3.6)
\]

where \( d\mu \) is a “naive” integration measure:

\[
d\mu = D c D \sigma D b D \overline{\sigma} D \gamma^+ D \overline{\sigma} D \gamma^- D \overline{\sigma} D \beta^+ D \overline{\beta}^+ D \beta^- D \overline{\beta}^- \times D \xi D \overline{\xi} D \eta D \overline{\sigma} D \psi^{+\mu} D \overline{\psi}^{-\mu} D \psi^{-\mu} D \overline{\psi}^{+\mu} D X^\mu D Y^\mu. \quad (3.7)
\]

From (3.6) and (3.7), we have the following correlation functions of \( X^\mu(Z) \), \( C(Z) \), and \( B(Z) \):

\[
\langle X^\mu(Z_a) X^\nu(Z_b) \rangle = \eta^{\mu\nu} \ln Z_{ab},
\]

\[
\langle C(Z_a) B(Z_b) \rangle = \langle B(Z_a) C(Z_b) \rangle = \frac{\theta^-_{ab} \theta^+_{ab}}{Z_{ab}}, \quad (3.8)
\]

where \( Z_{ab} \) and \( \theta^\pm_{ab} \) are defined by

\[
Z_{ab} = z_a - z_b - \left( \theta^+_{a} \theta^-_{b} + \theta^+_{a} \theta^-_{b} \right),
\]

\[
\theta^\pm_{ab} = \theta^\pm_{a} - \theta^\pm_{b}. \quad (3.9)
\]

Our next question is the following: What is the correct expression of the ghost number supercurrent and the BRST supercurrent in the partition function (3.6)?
Note that, in the partition function (3.6), we do not have any anomalies and we can always use the naive equations of motion. From the expressions of the BRST anomaly (2.48) and the ghost number anomaly (2.49), we may take
\[
j_B(Z) \equiv C \left( T^X + \frac{1}{2} T^{\text{gh}} \right) + \frac{d - 2}{4} C \left( \frac{1}{2} D^- \Phi D^+ \Phi + i \partial \Phi \right) + \frac{1}{4} D^- \left[ C \left( D^+ C \right) B \right] + \frac{1}{4} D^+ \left[ C \left( D^- C \right) B \right],
\]
(3.10)
\[
j_{\text{gh}}(Z) \equiv -BC,
\]
as the effective covariant supercurrents in the partition function (3.6). In (3.10), we determined the \( \Phi \)-dependences as to reproduce (2.49) and (2.50) under uses of the naive equations of motion of the matter and the ghost fields. This prescription was also applied to \( N = 0 \) and \( N = 1 \) (super-)gravity [21,16]. We emphasize that we obtained the anomalous identities (2.49) and (2.50) in the BRST invariant path integral framework [17,18], thus those expressions should reflect the (super-)coordinate covariance in the quantum theory.

In order to have a complete description of the \( N = 2 \) quantum supergravity, we should quantize the Liouville supermultiplet. We define the partition function of the Liouville supermultiplet as
\[
\int \mathcal{D} \left( e^{\sigma/2} \right) \mathcal{D} \left( e^{\sigma/4} \phi^+ \right) \mathcal{D} \left( e^{\sigma/4} \phi^- \right) \mathcal{D} \left( e^{\sigma/4} \phi^- \right) \mathcal{D} \left( e^{\sigma/4} \phi^- \right) \mathcal{D} \left( e^{\sigma/2} \phi \right) \times \exp \left\{ -\frac{2 - d}{16\pi} \int d^2 x \left[ \partial \sigma \partial \sigma + \partial \phi \partial \phi - \phi^+ \partial \phi^- - \phi^- \partial \phi^+ + \left( \text{c. c.} \right) \right] \right\},
\]
(3.11)
where we have chosen the weight factors \((\exp \sigma)\) following the prescription in [17,18]. The full partition function is given by a product of (3.6) and (3.11).

If we apply the same procedure of the derivation of (3.4) also to the gravitinos \( \phi^\pm \) and the gauge field \( \phi \), the partition function (3.11) changes to
\[
\int \mathcal{D} \left( e^{\sigma/2} \right) \mathcal{D} \phi^+ \mathcal{D} \phi^- \mathcal{D} \phi^- \mathcal{D} \phi \times \exp \left[ -\left( \frac{2 - d}{16\pi} - \frac{1}{24\pi} \right) \int d^2 x \partial \sigma \partial \sigma - \frac{1}{4\pi} M^2 \int d^2 x e^\sigma + \cdots \right],
\]
(3.12)
where we only indicated the \( \sigma \)-dependence of the action. The integration of the
Liouville field in (3.12) is, on the other hand, highly non-linear because the integration variable is $e^{\sigma/2}$, not simply $\sigma$. To avoid this problem, here we apply the background field method \cite{21,16} and include the one-loop renormalization effect arising from the non-trivial measure $e^{\sigma/2}$.

To do this, we set $e^{\sigma/2} \equiv e^{\sigma_0/2} + \varphi$ and expand the Liouville action in (3.12) with respect to $\varphi$ up to the second order. If we assume the $M^2e^{\sigma}$ term in (3.12) is canceled by a suitable counter term, the resulting action for the quantum fluctuation $\varphi$ has the same form of the action of $\tilde{X}^{\mu}(z)$ with replacement $\sigma \rightarrow \sigma_0$. Thus, up to the one-loop order, we have additional contribution from the Liouville part itself,

$$\frac{1}{16\pi} \int d^2x \partial \sigma_0 \bar{\partial} \sigma_0. \quad (3.13)$$

We regard this factor as the one-loop finite renormalization effect. Adding this effect to the original contribution from (3.12), and after a covariantization, we finally have

$$\int \mathcal{D}\sigma \mathcal{D}\phi^+ \mathcal{D}\bar{\phi}^+ \mathcal{D}\phi^- \mathcal{D}\bar{\phi}^- \mathcal{D}\phi \times \exp \left\{ -\frac{1}{16\pi} \int d^2x \left[ \partial \sigma \bar{\partial} \sigma + \partial \phi \bar{\partial} \phi - \phi^+ \bar{\partial} \phi^- - \phi^- \bar{\partial} \phi^+ + (c. c.) \right] \right\}, \quad (3.14)$$

where we have rewritten $\sigma_0$ as $\sigma$ and taken a naive integration measure $\sigma$ for the Liouville field, since we already include the (one-loop) quantum effect of $e^{\sigma/2}$. We should note here the coefficient in (3.11), $2 - d$, changes to $1 - d$ in (3.14). Since the action in (3.14) has the same form as the matter supermultiplet $X^{\mu}(Z)$, the correlation function of the Liouville superfield $\Phi$ is given by

$$\langle \Phi (Z_a) \Phi (Z_b) \rangle = \frac{4}{d - 1} \ln Z_{ab}. \quad (3.15)$$

As the effective covariant supercurrents in the partition function (3.14), we
may take (3.10) with a replacement $2 - d \rightarrow 1 - d$, i.e.,

$$
j_B(Z) \equiv C \left( T^X + \frac{1}{2} T_{\text{gh}} \right) + \frac{d - 1}{4} C \left( \frac{1}{2} D^- \Phi D^+ \Phi + i \partial \Phi \right)
+ \frac{1}{4} D^- [C (D^+ C) B] + \frac{1}{4} D^+ [C (D^- C) B],
$$

(3.16)

$$
j_{\text{gh}}(Z) \equiv -BC.
$$

We comment on the differences of (3.16) from the analogous construction for the $N = 0$ and $N = 1$ (super-)Liouville cases [21,16]. The differences are i) no appearance of a correction term due to the Liouville field in the ghost number supercurrent $j_{\text{gh}}(Z)$ in (3.16) and, ii) no appearance of a divergence correction term in the expression of BRST supercurrent $j_B(Z)$ in (3.16). The origins of these facts are respectively, i) a vanishing of the ghost number anomaly in $N = 2$ theory (2.50), ii) no appearance of the BRST anomaly which is not proportional to $d - 2$ in (2.49). In fact, as is discussed in the following section, these two facts might be related each other.

We regard the whole set of the partition function (3.6) and (3.14), and the effective supercurrents in (3.16) as the effective theory of the two-dimensional $N = 2$ supergravity, i.e., $N = 2$ super-Liouville theory. The advantage of this effective theory is that we can use propagators in a flat space-time, like (3.8). Although the replacement $2 - d \rightarrow 1 - d$ in the current operator construction in (3.16) is *ad hoc*, we will check the covariance of those supercurrents by using the operator product expansion (OPE) in the next section. This shift of the parameter, $2 - d \rightarrow 1 - d$ also appeared as the ansatz in [9].
4. BRST supercurrent algebra and
the topological conformal algebra

In this section, we show that our effective supercurrents in (3.16) forms a
topological conformal algebra [13,22], which appears in two-dimensional topological
(conformal) field theories [10,11]. This observation was reported in our previous
communication [12].

Firstly we change the normalization of the Liouville superfield as

\[ \Phi(Z) \rightarrow \frac{2}{\sqrt{d-1}} \Phi(Z). \] (4.1)

Thus the correlation function in (3.15) changes to

\[ \langle \Phi(Z_a) \Phi(Z_b) \rangle = \ln Z_{ab}, \] (4.2)

and the effective BRST supercurrent changes to

\[ j_B(Z) = C(Z) \left( T^X + T_{\text{Liouville}} + \frac{1}{2} T^{gh} \right) \]
\[ + \frac{1}{4} D^- [C (D^+ C) B] + \frac{1}{4} D^+ [C (D^- C) B]. \] (4.3)

In the above expression, we defined the Liouville energy-momentum tensor:

\[ T_{\text{Liouville}} = \frac{1}{2} D^- \Phi D^+ \Phi + \kappa \partial \Phi, \] (4.4)

where \( \kappa \) satisfies

\[ \kappa^2 = \frac{1 - D}{4}. \] (4.5)

The BRST charge in the \( N = 2 \) super-Liouville theory thus is given by

\[ Q_B = \int DZ C \left( T^X + T_{\text{Liouville}} + \frac{1}{2} T^{gh} \right), \] (4.6)
and, as we will see, it satisfies $Q_B^2 = 0$ for any $d$. Furthermore we will also see

$$T(Z) = \{Q_B, B(Z)\},$$ (4.7)

where the total energy momentum tensor $T$ is defined by

$$T = T^X + T^{gh} + T^{\text{Liouville}}.$$ (4.8)

At this point we examine the BRST supercurrent algebra in $N = 2$ super-Liouville theory. We change the notation as

$$T(Z) \equiv T(Z),$$
$$G(Z) \equiv j_B(Z),$$
$$\overline{G}(Z) \equiv B(Z),$$
$$J(Z) \equiv j_{\text{ghost}}(Z).$$ (4.9)

For the superconformal properties, the relevant operator product expansion is (for any $d$),

$$T(Z_a) \Psi(Z_b) \sim h \frac{\theta^- a \theta^+ b}{Z_{ab}^2} \Psi(Z_b) + \frac{1}{2Z_{ab}} (\theta^- a D^+_b - \theta^+ b D^-_a) \Psi(Z_b)$$
$$+ \frac{\theta^- a \theta^+_b}{Z_{ab}} \partial_{z_b} \Psi(Z_b),$$ (4.10)

where $\Psi = T, G, \overline{G},$ and $J$ with $h = 1, 0, 1,$ and $0$ respectively. This expression implies those operators are primary fields with the $U(1)$ charge 0 and the superconformal weight $h$. Especially the case $\Psi = T$ implies a vanishing of the total central charge for any $d$. Moreover the BRST supercurrent $j_B(Z)$ and the ghost number supercurrent $j_{gh}(Z)$ in (3.16) are primary fields. In this sense, the supercurrents (3.16) in this effective theory are covariant in the quantum level and this desired feature suggests our construction in (3.16) is reliable.
For another relations between various operators, we have (also for any \(d\)),

\[
G(Z_a)G(Z_b) \sim \frac{1}{2Z_{ab}} \left( \theta_{ab}^+ D_b^- - \theta_{ab}^- D_b^+ \right) J(Z_b) + \frac{\theta_{ab}^- \theta_{ab}^+}{Z_{ab}} T(Z_b),
\]

\(G(Z_a)G(Z_b) \sim 0,\)

\(\overline{G}(Z_a)\overline{G}(Z_b) \sim 0,\)

\(J(Z_a)J(Z_b) \sim 0,\)

\(J(Z_a)G(Z_b) \sim \frac{\theta_{ab}^- \theta_{ab}^+}{Z_{ab}} G(Z_b),\)

\(J(Z_a)\overline{G}(Z_b) \sim -\frac{\theta_{ab}^- \theta_{ab}^+}{Z_{ab}} \overline{G}(Z_b).\)  

(4.11)

Surprisingly, in the above operator algebra, no quantum anomalous term appears and it coincides with the classically expected form. In the case of \(N = 0\) and \(N = 1\) cases \([16,23]\), on the other hand, the quantum anomalous terms vanish only at \(d = -2\) and \(d = \pm \infty\) respectively. The algebra in (4.10) and (4.11) form a kind of the topological conformal algebra or the twisted \(N = 4\) superconformal algebra \([13,22]\). Therefore, in the \(N = 2\) super-Liouville theory, the super-coordinate BRST supercurrent algebra gives a representation of a \(N = 2\) superfield extension of the topological conformal algebra for any \(d\). Our observation thus suggests the topological nature of the \(N = 2\) super-Liouville theory (or the \(N = 2\) fermionic string theory) as the quantum theory.

We note that the first relation in (4.11) implies (4.7), and the second relation in (4.11) implies the BRST invariance of the BRST supercurrent, therefore the BRST charge is nilpotent.

The anomaly-free property of the operator algebra in (4.10) and (4.11) might be understood from the absence of the ghost number anomaly in \(N = 2\) theory: In our construction, which is analogous to the one in \([21,16]\), the correction of

\* By comparing the conformal weight and the statistics of the each component fields, we can see that our algebra in (4.10) and (4.11) are not the same as the twisted \(N = 4\) superconformal algebras analyzed by Nojiri \([22]\).
the ghost number current due to the Liouville mode is determined from the ghost number anomaly. In $N = 2$ case, since we have no ghost number anomaly in (2.50), the form of the effective ghost number supercurrent in (3.16) has no correction due to the Liouville mode. From experiences on the $N = 0$ and $N = 1$ (super-)Liouville theories [21,16,23], we know the following relation for the effective currents:

\[ j_B = - \{ Q_B, j_{gh} \}, \quad (4.12) \]

and we can see that the $(d - \text{critical dimensions} + 1)$ non-proportional correction of the BRST (super-)current in the left hand side is generated from the Liouville correction of the ghost number (super-)current in the right hand side. If we expect the general validity of the relation (4.12) in our construction, the BRST super-current in the $N = 2$ super-Liouville theory will not have $d - 1$ non-proportional correction. Actually, we can check (4.12) from the explicit form (3.16). In the BRST current algebra like (4.11) in the $N = 0$ and $N = 1$ case [21,16,23], we can also observe that the anomalous terms in the algebra arise from the above mentioned non-trivial correction of the BRST and ghost number (super-)currents. In the case of $N = 2$, therefore, we may expect the anomaly free property of the operator algebra.

It is also useful to see how the critical string can be considered as a subcritical string in dimension $1$ plus the Liouville superfield, in fact in this situation, $\kappa = 0$ and all the operators of the effective theory coincide with the ones of the critical string, since in this case there is no restriction for the possible values of $d$. 

25
5. Conclusion

We computed the BRST and the ghost number anomalies in the $N = 2$ supergravity in the superconformal gauge, working in a path integral in terms of the component fields. The dependences of the Liouville mode in the anomalies were directly calculated while the dependences of the Liouville superpartners were determined by using the global supersymmetry (2.3). The final results were written in terms of superfields.

The effective $N = 2$ super-Liouville theory was constructed at one loop level and there is a finite renormalization of the coupling constant, i.e., from $2-d$ to $1-d$. The algebra of the operators in (4.9) gives rise to an $N = 2$ superfield extension of the topological conformal algebra for any value of dimensions $d$ without any anomalous terms. The crucial points for this property are the vanishing of the ghost number anomaly and the definition of the BRST supercurrent. Our observation shows an appearing of a quite simplification in the $N = 2$ case and also suggests a topological nature of the $N = 2$ super-Liouville theory.

The $N = 2$ critical string can be considered as an $N = 2$ subcritical string in dimension 1 plus the Liouville superfield. All the above features distinguish $N = 2$ string from the $N = 0$ and $N = 1$ strings. The physical relevance of the topological algebra is under study.

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APPENDIX

Let us recall some basic facts of $N = 2$ string in the superfield formalism,* The
$N = 2$ superspace is described in terms of the bosonic $(z, \bar{z})$ and the fermionic
$(\theta^{\pm}, \bar{\theta}^{\pm})$ coordinates. We define covariant derivatives by

$$D^\pm = \frac{\partial}{\partial \theta^{\pm}} + \theta^{\pm} \partial, \quad \bar{D}^\pm = \frac{\partial}{\partial \bar{\theta}^{\pm}} + \bar{\theta}^{\pm} \bar{\partial}. \quad (A.1)$$

The action can be written in terms of two superfields $S^\mu(z, \bar{z}, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$ and $S^{\mu*}(z, \bar{z}, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$ satisfying two constraints ($\mu$ runs over 1 to $d$):

$$D^+ S^\mu = \bar{D}^- S^\mu = 0, \quad (A.2)$$

and

$$D^+ S^{\mu*} = \bar{D}^- S^{\mu*} = 0. \quad (A.3)$$

The action is given by [1]

$$A = \int dz d\bar{z} \int d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^- S^{\mu*} S^\mu. \quad (A.4)$$

The solution of the equations of motion

$$D^+ \bar{D}^+ S^\mu = 0, \quad \bar{D}^- D^- S^{\mu*} = 0, \quad (A.5)$$

can be written as

$$S^\mu = S^\mu_1 + S^\mu_2, \quad (A.6)$$

where

$$D^- S^\mu_1 = \bar{D}^- S^\mu_1 = \bar{D}^+ S^\mu_1 = 0, \quad D^- S^\mu_2 = \bar{D}^- S^\mu_2 = D^+ S^\mu_2 = 0. \quad (A.7)$$

* We follow the notation of [6].
A real superfield $X^\mu$ is constructed via

$$X^\mu(z, \theta^+, \theta^-) = S^\mu_1 (z + \theta^- \theta^+, \theta^-) + S^{\mu*}_1 (z + \theta^+ \theta^-, \theta^+). \quad (A.8)$$

The components of $X^\mu(Z)$ are

$$X^\mu(Z) = X^\mu(z) + \theta^- \psi^\mu(z) + \theta^+ \psi^-\mu(z) + i\theta^- \theta^+ \partial Y^\mu(z), \quad (A.9)$$

where $X^\mu(z)$ and $Y^\mu(z)$ are free bosonic fields and $\psi^{\pm\mu}(z)$ are free fermions.

The contribution to the energy momentum tensor from $X^\mu$ is

$$T^X(Z) = \frac{1}{2} D_- X^\mu D_+ X^\mu(Z). \quad (A.10)$$

The $N = 2$ string action is invariant under several local gauge transformations. We are working in the superconformal gauge. The gauge fixing generates a Faddeev–Popov determinant expressible as a superfield action using $N = 2$ superfield ghost $C$ and antighost $B$:

$$C \equiv c + i\theta^+ \gamma^--i\theta^- \gamma^+ + i\theta^- \theta^+ \xi,$$

$$B \equiv -i\eta - i\theta^+ \beta^- - i\theta^- \beta^+ + \theta^- \theta^+ b. \quad (A.11)$$

The ghosts $c$ and $b$ are for the $\tau$-$\sigma$ general coordinate invariances, $\gamma^\pm$ and $\beta^\pm$ are the super ghosts for the two local supersymmetry transformations and $\xi$ and $\eta$ are the ghosts associated with the local U(1) symmetry. Their Lagrangians are the first order systems with background charge $Q$ [20] and statistics $\epsilon$ of $(Q, \epsilon) = (-3, +)$, $(2, -)$ and $(-1, +)$ respectively. Notice that the total background ghost charge vanishes. The ghost action in terms of superfield is given by

$$A_{gh} = \frac{1}{\pi} \int d^2zd\theta^+ d\theta^- B\overline{\partial} C + (c. c.). \quad (A.12)$$

The ghost energy momentum tensor becomes

$$T^{gh}(Z) = \partial(CB)(Z) - \frac{1}{2} D^+ C D^- B(Z) - \frac{1}{2} D^- C D^+ B(Z). \quad (A.13)$$
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| statistics | $a$ | $b$ |
|------------|-----|-----|
| $\tilde{c}$ | $-$ | $-1$ | $1$ |
| $\tilde{b}$ | $-$ | $\frac{1}{2}$ | $-2$ |
| $\tilde{\gamma}^{\pm}$ | $+$ | $-\frac{3}{4}$ | $\frac{1}{2}$ |
| $\tilde{\beta}^{\pm}$ | $+$ | $\frac{1}{4}$ | $-\frac{3}{2}$ |
| $\tilde{\xi}$ | $-$ | $-\frac{1}{2}$ | $0$ |
| $\eta$ | $-$ | $0$ | $-1$ |
| $\tilde{\psi}^{\mu}$ | $-$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ |
| $\tilde{X}^{\mu}, \tilde{Y}^{\mu}$ | $+$ | $-\frac{1}{2}$ | $0$ |

Table 1