Abstract—In this paper, the classical algebraic regulator problem is studied in a data-driven context. The endosystem is assumed to be an unknown system that is interconnected to a known exosystem that generates disturbances and reference signals. The problem is to design a regulator so that the output of the (unknown) endosystem tracks the reference signal, regardless of its initial state and the incoming disturbances. In order to do this, we assume that we have a set of input-state data on a finite time-interval. We introduce the notion of data informativity for regulator design, and establish necessary and sufficient conditions for a given set of data to be informative. Also, formulas for suitable regulators are given in terms of the data. Our results are illustrated by means of two extended examples.

I. INTRODUCTION

Recently, the paradigm of data-driven control has gained a lot of attention in analysis and controller design of linear systems [1], [3], [4], [7], [8], [11], [16]–[18], [20], [22]–[24]. Instead of using an explicit mathematical model, the data-driven approach uses only data obtained from the unknown system for verifying its system theoretic properties and for constructing controllers. Recently, it was argued in [23] that the data-driven approach can also be useful in cases where the given data do not give sufficient information to identify the ‘true’ model for the system, for example due to the fact that the data are not persistently exciting. Indeed, in [23] the notion of informativity of data was introduced to cover situations in which a given set of data gives rise to a whole family of system models that are compatible with the data. In other words, situations in which it is impossible to distinguish between models on the basis of the given data. A set of data is called informative for a given system property if the property holds for all systems compatible with the data. In [23], the notion of informativity was also developed in the context of controller design. In particular, conditions for informativity of data for the following control problems were given: state feedback stabilization, deadbeat control, linear quadratic optimal control, and stabilization by dynamic output feedback. Also, formulas (in terms of the data) were given to compute suitable controllers.

The aim of the present paper is to extend the framework of informativity to the classical algebraic regulator problem (see e.g [9], [12], [13], [15] and the textbooks [19], [21]). This is the problem of finding a feedback controller (called a regulator) that makes the output of the controlled system track some a priori given reference signal, regardless of the disturbance input entering the system, and the initial state. In the context of the algebraic regulator problem, the relevant reference signals and disturbances (such as step functions, ramps or sinusoids) are signals that are generated as solutions of suitable autonomous linear systems. Given such reference signal and class of disturbance signals, one first constructs a suitable generating autonomous system (called the exosystem). Next, this exosystem is interconnected to the control system (called the endosystem), and a new output is defined as the difference between the original system output and the reference signal. A regulator should then be designed to make the output of the interconnection converge to zero for all disturbances and initial states.

In this paper, the ‘true’ endosystem is assumed to be unknown, and therefore no mathematical model is available. Instead, we have collected data on the input, endosystem state, and exosystem state in the form of samples on a finite time-interval. The exosystem is assumed to be known, since this system models the reference signals and possible disturbance inputs. Also, the matrices in the output equations are assumed to be known, since these specify the design specification (namely the output that should converge to zero) on the controlled system. A given set of data will then be called informative for regulator design if the data contain sufficient information to design a single regulator for the entire family of systems that are compatible with this set of data. We will establish necessary and sufficient conditions for a given set of data to be informative for regulator design. In particular, it will be shown how to replace the characteristic regulator equations by their data-driven counterparts, and to compute suitable regulators.

We note that data-driven regulator design was studied before in [10] and [6], albeit from a rather different perspective. We also mention alternative methods that deal with tracking objectives, such as iterative feedback tuning (IFT) and virtual reference feedback tuning (VRFT) as developed in [14] and [5], respectively. These methods do however not address the classical regulator problem, and are thus quite different from the work that will be presented in this paper.

The main contributions of the present paper are the following:

1) We give a definition of the problem of data-driven tracking and regulation using the concept of informativity.
2) We give necessary and sufficient conditions for data to be informative for regulator design, i.e., for the existence of a single regulator for all systems compatible with the given data.
We establish formulas for computing these regulators, entirely in terms of the data. It should be noted that these regulators may be called robust, in the sense that a single regulator works for the whole set of systems that are compatible with the given data, see also [10].

The outline of this paper is as follows. In Section II, we illustrate the data-driven problem of tracking and regulation using an extended example. Subsequently we put the problem in a general framework, and define the concept of informativity for regulator design. In Section III, we review some classical basic material on the regulator problem. Then, in Section IV we formulate our main result, giving necessary and sufficient conditions for informativity for regulator design, and formulas to compute regulators. The main result is illustrated by means of two extended examples. Finally, in Section V, we formulate our conclusions.

II. DATA-DRIVEN TRACKING AND REGULATION

We will first illustrate the problem to be considered in this paper by means of an extended example.

Example 1. Consider the scalar linear time-invariant discrete-time system

\[ x(t + 1) = a_s x(t) + b_s u(t) + d(t), \]  

where \( x \) is the state, \( u \) the control input, and \( d \) a disturbance input. The values of \( a_s \) and \( b_s \) in this system representation are unknown. We assume that the disturbance can be any constant signal of finite amplitude. Suppose that we want the state \( x(t) \) to track the given reference signal \( r(t) = \cos \frac{\pi}{2} t \), for any constant disturbance input, regardless of the initial state of the system. We want to design a control law for (1) that achieves this specification. We assume that \( r, x \) and \( d \) are available for feedback and allow control laws of the form

\[ u(t) = k_1 r(t) + k_2 r(t + 1) + k_3 d(t) + k_4 x(t). \]  

Interconnecting (1) and (2) results in the controlled system

\[ x(t + 1) = (a_s + b_s k_4) x(t) + (b_s k_3 + 1)d(t) + b_s k_1 r(t) + b_s k_2 r(t + 1), \]

where the gains \( k_i \) should be designed such that \( x(t) - r(t) \to 0 \) as \( t \to \infty \) for any constant disturbance input \( d \) and initial state \( x(0) \). It is also required that the controlled system is internally stable, in the sense that \( a_s + b_s k_4 \) is stable \(^1\).

The values of \( a_s \) and \( b_s \) that represent the true system are unknown, but in the data-driven context it is assumed that we do have access to certain data. In particular, it is assumed that we have finite sequences of samples of \( x(t), u(t) \) and \( d(t) \) on a given time interval \( \{0, 1, \ldots, \tau \} \), given by

\[ U_- := [u(0) \ u(1) \ \cdots \ u(\tau - 1)], \]

\[ X := [x(0) \ x(1) \ \cdots \ x(\tau)], \]

\[ D_- := [d(0) \ d(1) \ \cdots \ d(\tau - 1)], \]  

\(^1\)We say that a matrix is stable if all its eigenvalues are contained in the open unit disk.

where, in this particular example, by assumption \( d(t) = d(0) \) for \( t = 1, 2, \ldots, \tau - 1 \). Define

\[ X_+ := [x(1) \ x(2) \ \cdots \ x(\tau)], \]

\[ X_- := [x(0) \ x(1) \ \cdots \ x(\tau - 1)]. \]

It is assumed that these data are generated by the true system, so we must have \( X_+ = a_s X_+ + b_s U_+ + D_- \). For this example, the problem of data-driven control design is now to use the data (3) to determine whether a suitable controller (2) exists, and to compute the associated gains \( k_1, k_2, k_3 \) and \( k_4 \) using only these data.

Note that in the above, both the reference signal and the disturbance signals are generated by the autonomous linear system

\[ \begin{bmatrix} r_1(t + 1) \\ r_2(t + 1) \\ d(t + 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ d(t) \end{bmatrix} \]  

with initial state \( r_1(0) = 1 \) and \( r_2(0) = 0 \), and \( d(0) \) arbitrary. Indeed, it can be seen that the reference signal \( r(t) = \cos \frac{\pi}{2} t \) is equal to \( r_1(t) \). In addition, the solutions \( d(t) \) are all constant signals of finite amplitude. The autonomous system (4) is called the exosystem.

The interconnection of the (unknown) to be controlled system (1) (called the endosystem) with the exosystem (4), is represented by

\[ \begin{bmatrix} r_1(t + 1) \\ r_2(t + 1) \\ d(t + 1) \\ x(t + 1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_s \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ r_1(t) \\ r_2(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_s \\ 0 \end{bmatrix} u(t). \]  

In this representation, the part corresponding to the exosystem is known, but the part corresponding to the endosystem (specifically: \( a_s \) and \( b_s \)) is unknown. We now also specify a (known) output equation

\[ z(t) = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ d(t) \\ x(t) \end{bmatrix}. \]

Then the problem of our example can be rephrased as: design a full state feedback control law

\[ u(t) = k_1 r_1(t) + k_2 r_2(t) + k_3 d(t) + k_4 x(t) \]

for the system (5) such that in the controlled system we have \( z(t) \to 0 \) as \( t \to \infty \) for the initial states \( r_1(0) = 1, r_2(0) = 0, \) and \( d(0) \) arbitrary, while internal stability is achieved in the sense that \( a_s + b_s k_4 \) is a stable matrix. In order to allow tracking of signals from the richer class of all reference signals of the form \( r(t) = A \cos \left( \frac{1}{2} \pi t + \omega \right) \), \( A \) and \( \omega \) are determined by the initial states \( r_1(0) = 1 \) and \( r_2(0) \), we may slightly relax the problem formulation and require \( z(t) \to 0 \) as \( t \to \infty \) for all initial states \( r_1(0), r_2(0) \) and \( d(0) \).

After having introduced our problem set up by means of the above example, we will now formulate it in a general framework.
Consider an endosystem represented by
\[ x_2(t + 1) = A_2 x(t) + B_2 u(t) + A_3 x_1(t). \]
Here, \( x_2 \) is the \( n_2 \)-dimensional state, \( u \) the \( m \)-dimensional input, and \( x_1 \) the \( n_1 \)-dimensional state of the exosystem
\[ x_1(t + 1) = A_1 x_1(t). \]
that generates all possible reference signals and disturbance inputs. The matrices \( A_2, B_2 \) are unknown, but the matrix \( A_1 \) is known. Also \( A_3 \) is a known matrix that represents how the endosystem interconnects with the exosystem. The output to be regulated is specified by
\[ z(t) = D_1 x_1(t) + D_2 x_2(t) + E u(t), \]
where the matrices \( D_1, D_2 \) and \( E \) are known. By interconnecting the endosystem with the state feedback controller
\[ u(t) = K_1 x_1(t) + K_2 x_2(t), \]
we obtain the controlled system
\[
\begin{bmatrix}
    x_1(t + 1) \\
    x_2(t + 1)
\end{bmatrix} =
\begin{bmatrix}
    A_1 & 0 \\
    A_3 + B_2 K_1 & A_2 + B_2 K_2
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix},
\]
\[ z(t) = (D_1 + E K_1) x_1(t) + (D_2 + E K_2) x_2(t). \]
If \( z(t) \to 0 \) as \( t \to \infty \) for all initial states \( x_1(0) \) and \( x_2(0) \), we say that the controlled system is output regulated. If \( A_2 + B_2 K_1 \) is stable, we call the controlled system endo-stable. If the control law (9) makes the controlled system both output regulated and endo-stable, we call it a regulator.

As illustrated in the example above, we assume that we do not know the true endosystem (6), and therefore the design of a regulator can only be based on available data. In the general framework, these are finite sequences of samples of \( x_1(t), x_2(t) \) and \( u(t) \) on a given time interval \( \{0, 1, \ldots, \tau\} \) given by
\[
U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(\tau - 1) \end{bmatrix},
\]
\[
X_{1-} := \begin{bmatrix} x_1(0) & x_1(1) & \cdots & x_1(\tau - 1) \end{bmatrix},
\]
\[
X_2 := \begin{bmatrix} x_2(0) & x_2(1) & \cdots & x_2(\tau) \end{bmatrix}.
\]
An endosystem with (unknown) system matrices \( (A_2, B_2) \) is called compatible with these data if \( A_2 \) and \( B_2 \) satisfy the equation
\[ X_{2+} = A_2 X_{2-} + A_3 X_{1-} + B_2 U_-, \]
where we denote
\[
X_{2-} := \begin{bmatrix} x_2(0) & x_2(1) & \cdots & x_2(\tau - 1) \end{bmatrix},
\]
\[
X_{2+} := \begin{bmatrix} x_2(1) & x_2(2) & \cdots & x_2(\tau) \end{bmatrix}.
\]
The set of all \( (A_2, B_2) \) that are compatible with the data is denoted by \( \Sigma_D \), i.e.,
\[ \Sigma_D := \{ (A_2, B_2) | (10) \text{ holds} \}. \]
We assume that the true endosystem \( (A_{2s}, B_{2s}) \) is in \( \Sigma_D \), i.e., the true system is compatible with the data. In general, the equation (10) does not specify the true system uniquely, and many endosystems \( (A_2, B_2) \) may be compatible with the same data.

Now we turn to controller design based on the data \( (U_-, X_{1-}, X_2) \). Note that, since on the basis of the given data we can not distinguish between the true endosystem and any other endosystem compatible with these data, a controller will be a regulator for the true system only if it is a regulator for any system with \( (A_2, B_2) \) in \( \Sigma_D \). If such regulator exists, we call the data informative for regulator design. More precisely:

**Definition 2.** We say that the data \((U_-, X_{1-}, X_2)\) are informative for regulator design if there exists \( K_1 \) and \( K_2 \) such that the control law \( u(t) = K_1 x_1(t) + K_2 x_2(t) \) is a regulator for any endosystem with \((A_2, B_2)\) in \( \Sigma_D \).

The problem that will be considered in this paper is to find necessary and sufficient conditions on the data \((U_-, X_{1-}, X_2)\) to be informative for regulator design. Also, in case that these conditions are satisfied, we will explain how to compute a regulator using only these data. Before addressing this problem, in the next section we will review some basic material on the regulator problem.

**III. THE REGULATOR PROBLEM**

In this section, we briefly review some basic material on the regulator problem. Following [21], we distinguish between analysis and design.

We first consider the analysis question under what conditions a controlled system is endo-stable and output regulated. Consider the autonomous linear system represented by
\[
\begin{align*}
    x_1(t + 1) &= A_1 x_1(t), \\
    x_2(t + 1) &= A_2 x_2(t) + A_3 x_1(t), \\
    z(t) &= D_1 x_1(t) + D_2 x_2(t).
\end{align*}
\]
In accordance with the terminology introduced in Section II, we call this system endo-stable if \( A_2 \) is a stable matrix. We call it output regulated if \( z(t) \to 0 \) as \( t \to \infty \) for all initial states \( x_1(0) \) and \( x_2(0) \). The following is the discrete-time version of Lemma 9.1 in [21]

**Proposition 3.** Assume that \( A_1 \) is anti-stable. Then the system (12) is endo-stable and output regulated if and only if \( A_2 \) is stable and there exists a matrix \( T \) satisfying the equations
\[
TA_1 - A_2 T = A_3, \quad D_1 + D_2 T = 0.
\]
In this case, \( T \) is unique.

Next, we consider the design problem and review conditions under which, for a given interconnection of an endosystem and exosystem, there exists a regulator, i.e., a controller that makes the controlled system endo-stable and output regulated. For the endosystem \( x_2(t + 1) = A_2 x_2(t) + B_2 u(t) + A_3 x_1(t) \) together with the exosystem (7) and output equation (8), the following is well-known and can be proven easily by extending results from [21] to the discrete-time case:

**Proposition 4.** Assume that \( A_1 \) is anti-stable. There exists a regulator of the form (9) if and only if \( (A_2, B_2) \) is stabilizable,

\[ \text{We say that a matrix is anti-stable if all its eigenvalues } \lambda \text{ satisfy } |\lambda| \geq 1. \]
and there exist matrices $T$ and $V$ satisfying the regulator equations
\[ TA_1 - A_2 T - B_2 V = A_3, \quad D_1 + D_2 T + EV = 0. \] (14)

In this case, a regulator is obtained as follows: choose any $K_2$ such that $A_2 + B_2 K_2$ is stable, and define $K_1 := -K_2 T + V$.

IV. THE DATA-DRIVEN REGULATOR PROBLEM

Clearly, a necessary condition for the data $(U_-,X_1-,X_2)$ to be informative for regulator design is that they are informative for endo-stabilization:

**Definition 5.** We call the data $(U_-,X_1-,X_2)$ are informative for endo-stabilization if there exists $K_2$ such that $A_2 + B_2 K_2$ is a stable matrix for all $(A_2, B_2)$ in $\Sigma_D$.

In order to obtain necessary and sufficient conditions for informativity for endo-stabilization we formulate:

**Proposition 6.** Let $\tau$ be a positive integer. Let $Z,X$ be real $n \times \tau$ matrices and let $U$ be a real $m \times \tau$ matrix. Consider the set $\Sigma_{(Z,X,U)} := \{(A,B) \mid Z = AX + BU\}$. Then the following hold:

1) There exists a matrix $K$ such that $A + BK$ is stable for all $(A,B) \in \Sigma_{(Z,X,U)}$ if and only if $X$ has full row rank, and there exists a right-inverse $X^\dagger$ such that $ZX^\dagger$ is stable. In that case, by taking $K := UX^\dagger$ we have $A + BK$ is stable for all $(A,B) \in \Sigma_{(Z,X,U)}$.

2) For any $K$ such that $A + BK$ is stable for all $(A,B) \in \Sigma_{(Z,X,U)}$ there exists a right-inverse $X^\dagger$ such that $K = UX^\dagger$, and, moreover, $A + BK = ZX^\dagger$ for all $(A,B) \in \Sigma_{(Z,X,U)}$.

**Proof.** The proof can be given by slightly adapting the proof of Theorem 16 in [23].

This immediately gives the following conditions for informativity for endo-stabilization:

**Lemma 7.** The data $(U_-,X_1-,X_2)$ are informative for endo-stabilization if and only if $X_{2-}$ has full row rank, and there exists a right inverse $X^\dagger_{2-}$ of $X_{2-}$ such that $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable. In that case, by taking $K_2 := U_- X^\dagger_{2-}$ we have $A_2 + B_2 K_2$ is stable for all $(A_2, B_2) \in \Sigma_D$.

The following theorem is the main result of this paper. It gives necessary and sufficient conditions on the data to be informative for regulator design, and explains how suitable regulators are computed using only these data.

**Theorem 8.** Assume that $A_1$ is anti-stable and suppose, for simplicity, that it is diagonalizable. Then the data $(U_-,X_1-,X_2)$ are informative for regulator design if and only if at least one of the following two conditions hold 3:

1) $X_{2-}$ has full row rank, and there exists a right-inverse $X^\dagger_{2-}$ of $X_{2-}$ such that $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable and $D_2 + EU_- X^\dagger_{2-} = 0$. Moreover, $im D_1 \subseteq im E$. In this case, a regulator is found as follows: choose $K_1$ such that $D_1 + EK_1 = 0$ and define $K_2 := U_- X^\dagger_{2-}$.

2) $X_{2-}$ has full row rank and there exists a right-inverse $X^\dagger_{2-}$ of $X_{2-}$ such that $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable. Moreover, there exists a solution $W$ to the linear equations
\[ X_{2-} W A_1 - (X_{2-} + A_3 X_{1-}) W = A_3, \] (15a)
\[ D_1 + (D_2 X_{2-} + EU_-) W = 0. \] (15b)

In this case, a regulator is found as follows: choose $K_1 := U_- (I - X^\dagger_{2-} X_{2-}) W$ and $K_2 := U_- X^\dagger_{2-}$.

Before turning to the proof, we will explain how to apply this theorem. What we know about the system are the system matrices $A_1,A_3,D_1,D_2$ and $E$ and the data $(U_-,X_1-,X_2)$. The aim is to use this knowledge to compute a single regulator $(K_1,K_2)$ that works for all endosystems $(A_2,B_2)$ in the set $\Sigma_D$ defined by (11).

In order to check the existence of such regulator, we verify the two conditions 1) and 2) in Theorem 8. If neither of the two conditions holds, then the data are not informative. On the other hand, if condition 1) holds then a regulator $(K_1,K_2)$ is computed as follows:

- find a right-inverse $X^\dagger_{2-}$ of $X_{2-}$ such that the matrix $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable and $D_2 + EU_- X^\dagger_{2-} = 0$,
- compute $K_1$ as a solution of $D_1 + EK_1 = 0$,
- define $K_2 := U_- X^\dagger_{2-}$.

If condition 2) holds then a regulator is computed as follows:

- find a right-inverse $X^\dagger_{2-}$ of $X_{2-}$ such that the matrix $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable,
- find a solution $W$ of the data-driven regulator equations (15),
- define $K_1 := U_- (I - X^\dagger_{2-} X_{2-}) W$,
- define $K_2 := U_- X^\dagger_{2-}$.

**Proof.** ($\Rightarrow$) We first prove sufficiency. Assume that the condition 1) holds. Since $(X_{2-} + A_3 X_{1-}) X^\dagger_{2-}$ is stable, the data are informative for endo-stabilization and by taking $K_2 := U_- X^\dagger_{2-}$ we have $A_2 + B_2 K_2$ is stable for all $(A_2, B_2) \in \Sigma_D$. Since $A_1$ is assumed to be anti-stable, this implies that for all $(A_2, B_2) \in \Sigma_D$ there exists a unique solution $T$ to the Sylvester equation $TA_1 - (A_2 + B_2 K_2) T = A_3 + B_2 K_1$. By the fact that $D_1 + EK_1 = 0$ and $D_2 + EK_2 = 0$, this solution $T$ also satisfies $D_1 + EK_1 + (D_2 + EK_2) T = 0$. Thus, for all $(A_2, B_2) \in \Sigma_D$, there exists a matrix $T$ that satisfies the equations (13). It follows from Proposition 3 that for all $(A_2, B_2) \in \Sigma_D$ the controlled system is endo-stable and output regulated.

Next, assume that condition 2) holds. By Lemma 7, the data are informative for endo-stabilization and by taking $K_2 := U_- X^\dagger_{2-}$ we have $A_2 + B_2 K_2$ is stable for all $(A_2, B_2) \in \Sigma_D$. Let $W$ satisfy the equations (15). Define $T := X_{2-} W$ and $V := U_- W$. Then the pair $(T,V)$ satisfies the regulator equations (14) for all $(A_2, B_2) \in \Sigma_D$. Then, by Proposition 4, for each such $(A_2, B_2)$ a regulator is given by the pair $(K_1,K_2)$, with $K_1 := -K_2 T + V = -K_2 X_{2-} W + U_- W = U_- (I - X^\dagger_{2-} X_{2-}) W$. This completes the proof of the sufficiency part.

We will now turn to the necessity part. Assume that the data are informative for regulator design. By Proposition 3,
there exist $K_1$ and $K_2$ and for any $(A_2, B_2) \in \Sigma_{D}$ a matrix $T_{(A_2, B_2)}$ such that $A_2 + B_2K_2$ is stable and

$$T_{(A_2, B_2)}A_1 - (A_2 + B_2K_2)T_{(A_2, B_2)} = A_3 + B_2K_1,$$

$$D_1 + EK_1 + (D_2 + EK_2)T_{(A_2, B_2)} = 0.$$  

We emphasize that $T_{(A_2, B_2)}$ may depend on the choice of $(A_2, B_2) \in \Sigma_{D}$. However, since $A_2 + B_2K_2$ is stable for all $(A_2, B_2) \in \Sigma_{D}$, by Proposition 6 there exists a right-inverse $X_{2-}$ of $X_{2-}$ such that $A_2 + B_2K_2 = (X_{2-} - A_3X_{1-})X_{2-}^T$ for all $(A_2, B_2) \in \Sigma_{D}$. The latter matrix is independent of $(A_2, B_2)$. Call it $M$. Define

$$\Sigma_{D}^0 := \{(A_0, B_0) \mid [A_0 \ B_0] \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} = 0\}.$$  

Note that $\Sigma_{D}^0$ is the solution space of the homogeneous version of the defining equation (10) for $\Sigma_{D}$ (see (11)). We now distinguish two cases, namely (i) $B_0K_1 = 0$ for all $(A_0, B_0) \in \Sigma_{D}^0$, and (ii) $B_0K_1 \neq 0$ for some $(A_0, B_0) \in \Sigma_{D}^0$.

First consider case (i). Then for all $(A_2, B_2), (A_2, B_2) \in \Sigma_{D}$ we have $B_2K_1 = B_2K_1$. Thus, there exists a common matrix $T$ that solves the equations

$$TA_1 - MT = A_3 + B_2K_1,$$

$$D_1 + EK_1 + (D_2 + EK_2)T = 0,$$

for all $(A_2, B_2) \in \Sigma_{D}$. From this, we obtain

$$TA_1 - [A_2 \ B_2] \begin{bmatrix} T \\ K_2T + K_1 \end{bmatrix} = A_3$$

for all $(A_2, B_2) \in \Sigma_{D}$, and therefore

$$[A_0 \ B_0] \begin{bmatrix} T \\ K_2T + K_1 \end{bmatrix} = 0$$

for all $(A_0, B_0) \in \Sigma_{D}^0$. This implies

$$\text{im} \left[ \begin{bmatrix} T \\ K_2T + K_1 \end{bmatrix} \right] \subseteq \text{im} \left[ \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} \right].$$

As a consequence, there exists a matrix $W$ such that

$$\begin{bmatrix} T \\ K_2T + K_1 \end{bmatrix} = \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} W.$$

Clearly, $W$ satisfies the equations (15), showing that condition 2) holds.

Next, consider case (ii). Let $S$ be a real $(n_2 + m) \times r$ matrix such that

$$\ker \left[ \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}^T \right] = \text{im} \ S.$$  

Partition $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$. Then $(A_0, B_0) \in \Sigma_{D}^0$ if and only if $A_0 = NS_1^T$ and $B_0 = NS_2^T$ for some $n_2 \times r$ matrix $N$. Note that, by hypothesis, $S_2^T K_1 \neq 0$.

Let $(A_2, B_2) \in \Sigma_D$. Recall that for any such $(A_2, B_2)$ there exists a unique $T_{(A_2, B_2)}$ such that

$$T_{(A_2, B_2)}A_1 - MT_{(A_2, B_2)} = A_3 + B_2K_1,$$

$$D_1 + EK_1 + (D_2 + EK_2)T_{(A_2, B_2)} = 0.$$  

Now let $N$ be any real $n_2 \times r$ matrix. Then also $(A_2 + NS_1^T, B_2 + NS_2^T) \in \Sigma_{D}$. Define $T_N := T_{(A_2, B_2)} - T_{(A_2 + NS_1^T, B_2 + NS_2^T)}$. Then clearly $T_N$ is the unique solution to

$$T_NA_1 - MT_N = NS_2^T K_1,$$  

which in addition satisfies $(D_2 + EK_2)T_N = 0$. Consider now a spectral decomposition $A_1 = Q^{-1} \Lambda Q$, where $\Lambda$ is the diagonal matrix $\Lambda = \text{diag}((\lambda_1, \ldots, \lambda_n))$ and

$$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_{n_1} \end{bmatrix}, \ Q^{-1} = [\hat{q}_1 \ldots \hat{q}_{n_1}].$$

Then, for fixed $N$, the unique solution $T_N$ to the Sylvester equation (17) can be expressed as

$$T_N = \sum_{i=1}^{n_1} (\lambda_i I - M)^{-1} NS_2^T K_1 \hat{q}_i \hat{q}_i^T$$

(see [2]), which implies that $T_NQ^{-1}$ is equal to

$$\left[ (\lambda_1 I - M)^{-1} NS_2^T K_1 \hat{q}_1 \ldots (\lambda_n I - M)^{-1} NS_2^T K_1 \hat{q}_{n_1} \right].$$

Note that the matrices $\lambda_i I - M$ are indeed invertible since $M$ is stable and the eigenvalues $\lambda_i$ of $A_1$ satisfy $|\lambda_i| \geq 1$. Since, in addition, $(D_2 + EK_2)T_N = 0$, we see that for all $i = 1, \ldots, n_1$ we have

$$(D_2 + EK_2)(\lambda_i I - M)^{-1} NS_2^T K_1 \hat{q}_i = 0.$$  

Since $S_2^T K_1 \neq 0$, there must exist an index $i$ such that $S_i^T K_1 \hat{q}_i \neq 0$. For this $i$, let $z$ be a real vector such that $z^T S_i^T K_1 \hat{q}_i \neq 0$. Now choose $N := e_j z^T$, where $e_j$ denotes the $j$th standard basis vector in $\mathbb{R}^{n_1}$. By the discussion above we obtain $(D_2 + EK_2)(\lambda_i I - M)^{-1} e_j = 0$. Since this holds for any $j$, we actually find $(D_2 + EK_2)(\lambda_i I - M)^{-1} = 0$, so $D_2 + EK_2 = 0$. Using (16), we must also conclude that $D_1 + EK_1 = 0$, which implies $\text{im} D_1 \subseteq \text{im} E$. Since $K_2$ is stabilizing it must be of the form $U_- X_{2-}^T$ for some right-inverse $X_{2-}$. This implies that $(X_{2-} - A_3X_{1-})X_{2-}^T$ is stable and $D_2 + EU_- X_{2-}^T = 0$, that is, condition 1) holds. This completes the proof of Theorem 8.



**Remark 9.** In order to avoid technicalities, in Theorem 8 we have assumed that the matrix $A_1$ is diagonalizable. The theorem however also holds if we drop this assumption. We omit the proof here.

**Remark 10.** According to Theorem 8, the data are informative for regulator design if and only if at least one of the conditions 1) or 2) holds. Condition 2) is in terms of solvability of the ‘data driven regulator equations’ (15a) and (15b). These equations hold for all $(A_2, B_2)$ compatible with the data. In the end a matrix $T$ is defined as $T := X_{2-}W$ and together with $V := U_-W$ the classical regulator equations (14) are then satisfied for all $(A_2, B_2)$ compatible with the data. This is then ‘the classical design’, and the difference $x_2(t) - TX_1(t)$ converges to 0 as $t$ runs off to infinity (see [21], page 199).

If Condition 2) does not hold, but instead Condition 1) holds, then the only way to get output regulation is to make the entire output $z = (D_1 + EK_1)x_1 + (D_2 + EK_2)x_2$ equal...
to 0 pointwise. This is done by making $D_1 + EK_1 = 0$ (possible because $\text{im } D_1 \subseteq \text{im } E$) and $D_2 + EK_2 = 0$, where $K_2 = U_-.X_{2-}^\dagger$ also makes the system endo-stable.

Note that Theorem 8 gives a characterization of all data that are informative for regulator design, and gives a method to design a suitable regulator. Nonetheless, the procedure to compute this regulator is not entirely satisfactory. Indeed, in the case that condition 2) holds it is not clear how to find a right inverse of $X_{2-}$ such that $(X_{2+} - A_2X_{1-})X_{2-}^\dagger$ is stable.

In the case of condition 1), the additional constraint $D_2 + EU_.X_{2-}^\dagger = 0$ needs to be satisfied. In general, $X_{2-}$ has many right inverses, and $(X_{2+} - A_2X_{1-})X_{2-}^\dagger$ can be stable, with or without $D_2 + EU_.X_{2-}^\dagger = 0$, depending on the choice of the particular right inverse $X_{2-}^\dagger$. To deal with this problem and to solve the problem of regulator design, we formulate the problem of finding a suitable right inverse in terms of feasibility of linear matrix inequalities (LMIs).

**Theorem 11.** Let $(U_-, X_{1-}, X_2)$ be given data. Then the following hold:

1) $X_{2-}$ has full row rank and has a right inverse $X_{2-}^\dagger$ such that $(X_{2+} - A_2X_{1-})X_{2-}^\dagger$ is stable if and only if there exists a matrix $\Theta \in \mathbb{R}^{T \times n}$ such that

$$X_{2-} - \Theta = (X_{2-} - \Theta)^\top$$

and

$$\Theta^\top (X_{2+} - A_2X_{1-}) \Theta > 0.$$  \hspace{1cm} (18)

2) $X_{2-}$ has full row rank and has a right inverse $X_{2-}^\dagger$ such that $(X_{2+} - A_2X_{1-})X_{2-}^\dagger$ is stable with, in addition, $D_2 + EU_.X_{2-}^\dagger = 0$ if and only if there exists a solution $\Theta \in \mathbb{R}^{T \times n}$ of (18) and (19) that satisfies the linear equation

$$(D_2 X_{2-} + EU_.) \Theta = 0.$$  \hspace{1cm} (19)

In both cases, a suitable right-inverse is given by $X_{2-}^\dagger := \Theta(X_2 - \Theta)^{-1}$.

**Proof.** The proof can be given by adapting the proof of Theorem 17 in [23].

**Example 12.** We will now apply Theorem 8 to Example 1. Putting the example in our general framework we have

$$x_1 = \begin{bmatrix} r_1 \\ d \end{bmatrix}, \quad x_2 = x, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = -1, \quad E = 0.$$  \hspace{1cm}

Assume $\tau = 3$, and that the data on the disturbance input are $D_\tau = [d(0) \quad d(1) \quad d(2)] = [\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}]$. Since the signal to be tracked is $\cos \frac{\pi}{2} \tau t$, we must have $r_1(0) = 1$, $r_2(0) = 0$ so $r_1(t) = \cos \frac{\pi}{2} \tau t$ and $r_2(t) = \cos \frac{\pi}{2}(t+1)$. This leads to

$$X_{1-} = \begin{bmatrix} r_1(0) & r_1(1) & r_1(2) \\ r_2(0) & r_2(1) & r_2(2) \\ d(0) & d(1) & d(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$  \hspace{1cm}

Assume that $U_- = [u(0) \quad u(1) \quad u(2)] = [1 \quad 0 \quad 0]$ and $X_2 = [x_2(0) \quad x_2(1) \quad x_2(2) \quad x_2(3)] = [0 \quad \frac{3}{2} \quad 2 \quad \frac{3}{2}]$. It can be checked that condition 2) of Theorem 8 holds. Indeed, a solution $W$ to the linear equations (15) is given by

$$W = \begin{bmatrix} -1 & 1 & -1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm}

Furthermore, $X_{2-}^\dagger = [-\frac{1}{2} \quad \frac{3}{2} \quad 0]^T$ is a right-inverse of $X_{2-}$ and $(X_{2+} - A_2X_{1-})X_{2-}^\dagger = \frac{1}{2}$ is stable. A regulator is then given by $K_1 = U_-(I - X_{2+}^\dagger X_{2-})W = [-\frac{1}{2} \quad 1 \quad -1]$ and $K_2 := U_-X_{2-}^\dagger = -\frac{1}{2}$.

It can be checked that the above data are compatible with the true endosystem $a_s = 1, b_s = 1$. In fact, in this particular example, the true system is uniquely determined by the data. Indeed, this follows from the fact that

$$X_{2+} = \begin{bmatrix} a_s & b_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} + D_-, \quad \text{in which } \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} \text{ has full row rank.}$$

Thus, a regulator could also have been computed directly from the regulator equations (14) after first identifying the true endosystem $a_s = 1, b_s = 1$. It can indeed be verified that $T = [1 \quad 0 \quad 0]$ together with $V = [-1 \quad 1 \quad -1]$ satisfy the regulator equations (14) for the true endosystem. By choosing $K_2 = -\frac{1}{2}$, this would lead to the same regulator as above with $K_1 = -K_2T + V = [-\frac{1}{2} \quad 1 \quad -1]$.

We note that, in general, the true endosystem may not be uniquely determined by the data. This is illustrated by the following example.

**Example 13.** Consider the two-dimensional endosystem

$$x_2(t+1) = A_2x_2(t) + B_2x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t),$$

where $A_2$ and $B_2$ are unknown $2 \times 2$ and $2 \times 1$ matrices, respectively. Let $x_2 = [x_{21} \quad x_{22}]^T$. The disturbance input $d$ is assumed to be a constant signal with finite amplitude, so is generated by $d(t+1) = d(t)$. We want to design a regulator so that $2x_{21} + \frac{1}{2}x_{22}$ tracks a given reference signal. In this example, the reference signals $r$ are assumed to be generated by a given autonomous linear system with state space dimension, say, $n_1$. Its representation will be irrelevant here. The total exosystem will then have state space dimension $n_1 + 1$, and our output equation is given by $z(t) = D_1x_1(t) + D_2x_2(t) + Eu(t)$, with $D_1$ a $1 \times (n_1 + 1)$ matrix such that $D_1x_1 = -r$ and $D_2 = [2 \quad \frac{1}{2}]$. We take $E = 2$. Also note that $A_3 = [0_{1\times n_1} \quad 0_{1\times n_1}]$. Here, $0_{1\times n_1}$ denotes $1 \times n_1$ zero matrix. Suppose that $\tau = 2$ and assume we have the following data:

$$U_- = [-1 \quad -1], \quad D_- = [1 \quad 1], \quad X_2 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$  \hspace{1cm}

These data can be seen to be generated by the true endosystem $A_2s = [\frac{2}{4} \quad \frac{1}{4}], B_2s = [\frac{1}{2} \quad \frac{1}{2}]$. We now check condition 1) of Theorem 8. First note that, indeed, $\text{im } D_1 \subseteq \text{im } E$. Also, $X_{2-}$ is non-singular and $(X_{2+} - A_2X_{1-})X_{2-}^\dagger = \begin{bmatrix} \frac{1}{2} \quad -\frac{1}{2} \\ 1 \quad \frac{1}{2} \end{bmatrix}$. This
matrix has eigenvalues $\frac{1}{2} \pm \frac{1}{2}i$, so is stable. Finally, $D_2 + EU_-X_2^{-1} = 0$. According to Theorem 8, a regulator for all endosystems compatible with the given data is given by

$$K_2 = U_1X_2^{-1} = [-1 -\frac{1}{4}], \quad K_1 = -\frac{1}{2}D_1.$$  \hspace{1cm} (20)

It can be verified that the set of endosystems compatible with our data is equal to the affine set

$$\Sigma_D = \{ (\begin{bmatrix} a & b \\ \frac{a}{2} - \frac{1}{4} & \frac{b}{4} + \frac{1}{4} \end{bmatrix}, \begin{bmatrix} a - \frac{1}{2} \\ b - 1 \end{bmatrix}) \mid a, b \in \mathbb{R} \}.$$  

The controller given by (20) is a regulator for all these endosystems.

**Remark 14.** It is also possible to consider the situation that, in addition to $A_2$ and $B_2$, also the matrix $A_3$ (representing how the exosignal $x_1$ enters the endosystem) is unknown. In that case, the set all endosystems compatible with the data $(U_-,X_2,X_-)$ is defined as follows:

$$\Sigma_D = \{ (A_2, B_2, A_3) \mid X_{2+} = A_2X_{2-} + B_2U_- + A_3X_{1-} \}.$$  

The data are then called informative for regulator design if there exists a single regulator $u = K_1x_1 + K_2x_2$ for all endosystems in $\Sigma_D$. The analogue of Theorem 8 for this situation is as follows. Both in Conditions 1) and 2), an additional condition $X_1X_{2-} = 0$ should be imposed on a suitable right-inverse of $X_{2-}$. In addition, in Condition 2), the old data-driven regulator equations (15) should be replaced by:

$$X_{2-}WA_1 - X_{2+}W = 0,$$ \hspace{1cm} (21a)

$$X_{1-}W = I,$$ \hspace{1cm} (21b)

$$D_1 + (D_2X_{2-} + EU_-)W = 0.$$ \hspace{1cm} (21c)

Note that, as expected, $A_3$ no longer appears in the equations (it is unknown). In both cases, the formulas for $K_1$ and $K_2$ are the same as in Theorem 8. Due to space limitations, the proof is omitted.

**V. CONCLUSIONS**

We have introduced the notion of data informativity in the context of the classical algebraic regulator problem. Our main results are necessary and sufficient conditions for a given set of data to be informative for regulator design, and formulas to compute regulators using only this set of data. We have recasted the computation of suitable regulators in terms of feasibility of LMI’s. Our results have been illustrated by means of two extended examples. In the present paper, only static state feedback regulators have been considered. As an open problem for future research we mention the extension to dynamic output feedback regulators. Results obtained in [23] on the problem of stabilization by dynamic output feedback (both in terms of input-state-output data and input-output data) are expected to be relevant here. Another possible venue for future research is to consider the situation that, in addition to $A_2$, $A_3$ and $B_2$, also the matrix $A_3$ is unknown. Finally, it would be interesting to include noise in the problem formulation, and to consider the situation in which, in addition to the modeled disturbances, bounded noise may enter the unknown endosystem (see also [22]).