Lax Pairs for the Discrete Reduced Nahm Systems

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Received: 29 June 2020 / Accepted: 11 February 2021 / Published online: 18 March 2021
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Abstract
We discretise the Lax pair for the reduced Nahm systems and prove its equivalence with the Kahan–Hirota–Kimura discretisation procedure. We show that these Lax pairs guarantee the integrability of the discrete reduced Nahm systems providing an invariant. Also, we show that Nahm systems cannot solve the general problem of characterisation of the integrability for Kahan–Hirota–Kimura discretisations.

Keywords Lax pairs · Reduced Nahm systems · Integrable discretisation · Difference equations

Mathematics Subject Classification (2010) 37K10 · 37M15 · 39A10

1 Introduction
W. Nahm in 1982 [1] introduced a model for self-dual multimonopoles in terms of three coupled matrix differential equations:
\[ \dot{T}_i = [T_j, T_k], \quad T_i = T_i(t) \in M_{N,N}(\mathbb{C}), \] (1.1)
where the indices \( i, j, k \) are cyclic permutations of the set \{1, 2, 3\}, and \( N \) is a positive integer. The system of three equations (1.1) nowadays called Nahm’s equations.

In [2] some special cases of Nahm’s equations with particular symmetries were studied in connection with the theory of monopoles. The obtained systems of
coupled two-dimensional differential equations are known as the reduced Nahm systems:

\[ \dot{x} = 2x^2 + \frac{y^2}{8}, \quad \dot{y} = -4xy, \]  
\[ \dot{x} = 2x^2 - 48y^2, \quad \dot{y} = -6xy - 8y^2, \]  
\[ \dot{x} = 2x^2 - y^2, \quad \dot{y} = -10xy + y^2. \]  

Due to the symmetry of the associated Nahm matrices the systems (1.2) are called the tetrahedral Nahm system, octahedral Nahm system, and icosahedral Nahm system respectively. The peculiarity of these systems is the fact that they are algebraically integrable, in the sense that they possess an invariant elliptic curve, i.e. a genus one curve, of degree three, four and six respectively. For more information on the general Nahm equations in the context of the modern theory of integrable systems we refer to [3].

In recent years arose the interest in the problem of finding good discretisation of continuous systems. By good discretisation, here we mean a discretisation, which preserves as much as possible the properties of its continuous counterpart. Within this framework a procedure called Kahan-Hirota-Kimura (KHK) discretisation became popular as a way of producing integrable discrete equations from systems of integrable ODEs. Specifically, given a systems of first-order ordinary differential equations:

\[ \dot{x} = F(x) \]  
its KHK is given by the following formula:

\[ \frac{x_{n+1} - x_n}{h} = 2F\left(\frac{x_{n+1} + x_n}{2}\right) - \frac{F(x_{n+1}) + F(x_n)}{2}, \quad x_n = x(nh), \quad h \to 0^+. \]  

This formula was presented first by W. Kahan in a series of unpublished lecture notes [4], and applied by K. Kimura and R. Hirota to produced an integrable discretisation of the Lagrange top [5]. This result attracted the interest of many scientists working in the field of geometric discretisation theory [6], from the Berlin school [7–9]. In particular in [7] it was noticed that when the function $F$ in (1.3) is quadratic the discretisation rule (1.4) give raise to a birational map. Later, some general integrability properties of the KHK discretisation were unveiled through the work of G. R.W. Quispel and his collaborators [10, 11].

In particular in [7–9], Petrera, Pfadler and Suris developed an algebraic approach for the search of invariants for KHK discretisations, called the Hirota–Kimura bases. With this approach they produced lots of examples. Yet, besides invariants and preserved measures, little was know about additional structures of the discrete integrable systems they found. For instance, in the conclusions of [9] the authors write:

“Of course, it would be highly desirable to find some structures, like Lax representation, bi-Hamiltonian structure, etc., which would allow one to check the conservation of integrals in a more clever way, but up to now no such structures have been found for any of the [K]HK type discretizations.”
In this paper, we give an answer to the above comment made by Petrera, Pfadler and Suris in [9]. That is, using a technique presented in [12, 13], we build the discrete analog of the reduced Nahm system from their Lax representation. Then, we show that this discretisation is equivalent to the KHK discretisation discussed in [9]. These Lax pairs are used to produce invariants, and proving integrability of the discrete Nahm systems.

The plan of the paper is the following: in Section 2 we give a review of the literature on the Lax pair for the continuous and discrete Nahm systems. In Section 3 we use such construction to produce the Lax pairs for the reduced Nahm systems (1.2) and prove integrability. In the final Section 4 we give some conclusions and an outlook for further researches. Moreover, we show that there exists Nahm systems whose Lax pair does not provide integrability, yet the system is KHK discretisable, and both the continuous and discrete systems are algebraically integrable. Moreover, we remark that there exist Nahm systems arising from bona fide Lax pairs which are not suitable for KHK discretisation. This shows, that despite the success obtained in explaining the integrability of the Euler top [12–14], and of the reduced Nahm systems (1.2), the Nahm equation approach cannot solve the general problem of characterisation of the integrability of KHK discretisations.

2 Lax Pair for the Continuous and Discrete Nahm Systems

In the literature several different forms of the Lax pair for the Nahm equations have been proposed. For instance, recently in [12] it was proposed the following form:

\[
A (\lambda) = \begin{pmatrix} T_1 & -T_2 \\ T_2 & T_1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 2T_3 \\ -2T_3 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} T_1 & T_3 \\ -T_3 & T_1 \end{pmatrix},
\]

(2.1a)

\[
B (\lambda) = \begin{pmatrix} 0 & T_3 \\ -T_3 & 0 \end{pmatrix} + \lambda \begin{pmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{pmatrix}.
\]

(2.1b)

The above matrices are such that the system (1.1) is equivalent to the following compatibility condition:

\[
\dot{A} = [A, B].
\]

(2.2)

The Lax pair (2.1) consists of \(2N \times 2N\) matrices. In this paper to avoid too cumbersome formulas we consider the inverse matrix complexification of the matrices in (2.1):

\[
M = \begin{pmatrix} M_1 & -M_2 \\ M_2 & M_1 \end{pmatrix} \implies M = M_1 + \lambda M_2.
\]

(2.3)

That is, we consider the following Lax pair for the Nahm system (1.1):

\[
A (\lambda) = T_1 + \lambda T_2 - 2\lambda^2 (T_1 - \lambda T_2),
\]

(2.4a)

\[
B (\lambda) = -\lambda T_3 + \lambda (T_1 - \lambda T_2).
\]

(2.4b)

The compatibility condition (2.2) gives again the Nahm equations (1.1) taking its real and imaginary part.

Associated to the matrix \(B (\lambda)\) there exists a unique family of unitary matrices resolving the differential equation \(\dot{V} = VB\), with initial condition \(V (0) = I_N\). As
proven in [15] this implies that the spectrum of the matrix $A(\lambda)$ does not depend on the independent variable $t$. So, the coefficients of the characteristic polynomial of $A(\lambda)$:

$$p_A(\mu) = \det(A(\lambda) - \mu I_N), \quad (2.5)$$

do not depend on $t$. That is, the coefficients of the characteristic polynomial (2.5) are first integrals of the Nahm system (1.1). Moreover, since the system (1.1) does not depend on the variable $\lambda$ too, for each coefficient we can have multiple first integrals.

**Remark 1** Equation (2.5) yields at least $N$ first integrals, but there is no *a priori* guarantee that these first integrals are functionally independent and/or non-trivial. This implies that the integrability of the system (1.1) must be proved case by case using the specific form of the matrices $T_i$.

In [14], followed by [12, 13], was introduced a method to discretise the compatibility condition (2.2). Consider the discrete time interval:

$$t_n = nh, \quad h \to 0^+, \quad (2.6)$$

hence we define $f_n \equiv f(t_n)$. Then, the compatibility condition (2.2) can be discretised as:

$$\frac{A_{n+1}(\lambda) - A_n(\lambda)}{h} = A_{n+1}(\lambda)B_n(\lambda) - B_{n+1}(\lambda)A_n(\lambda). \quad (2.7)$$

The corresponding system of difference equations is given by:

$$\frac{T_{i,n+1} - T_{i,n}}{h} = T_{j,n+1}T_{k,n} - T_{k,n+1}T_{j,n}, \quad (2.8)$$

where the indices $i, j, k$ are cyclic permutations of the set $\{1, 2, 3\}$.

**Remark 2** In principle a different discretisation of the compatibility condition (2.2) can be given:

$$\frac{A_{n+1}(\lambda) - A_n(\lambda)}{h} = A_n(\lambda)B_{n+1}(\lambda) - B_n(\lambda)A_{n+1}(\lambda), \quad (2.9)$$

yield the following system of difference equations:

$$\frac{T_{i,n+1} - T_{i,n}}{h} = T_{j,n}T_{k,n+1} - T_{k,n}T_{j,n+1}, \quad (2.10)$$

where the indices $i, j, k$ are cyclic permutations of the set $\{1, 2, 3\}$. However, by direct computation it is possible to show that, in the cases considered in this paper, condition (2.10) is equivalent to (2.8) up to the transformation:

$$x_{n+i} \leftrightarrow x_{n-i}, \quad (2.11)$$

where $x_n$ is the vector of the dynamical variables. That is, the evolution defined from (2.8) is the opposite of the evolution defined by (2.10). We notice that this is a general fact when dealing with KHK discretisation as pointed out in [7].

Equation (2.7) can be rearranged as:

$$A_{n+1}(\lambda)(I_N - hB_n(\lambda)) = (I_N - hB_{n+1}(\lambda))A_n(\lambda). \quad (2.12)$$
Introducing the matrices:

\[ L_n (\lambda) = A_n (\lambda), \quad M_n (\lambda) = I_N - hB_n (\lambda), \]  

which allows us to rewrite (2.12) as:

\[ L_{n+1} (\lambda) M_n (\lambda) = M_{n+1} (\lambda) L_n (\lambda). \]  

Following [16, 17] we have that equation (2.14) implies that the spectral data of the matrix \( L_n (\lambda) M_n^{-1} (\lambda) \) are constant along the evolution. Indeed, from (2.14), the matrices \( L_{n+1} (\lambda) M_{n+1}^{-1} (\lambda) \) and \( L_n (\lambda) M_n^{-1} (\lambda) \) are conjugate, so that they have the same characteristic polynomial. This implies that the that the coefficients of the characteristic polynomial of \( L_n (\lambda) M_n^{-1} (\lambda) \) are conserved quantities (invariants) for the system (2.8). Alternatively, using Binet’s rule, we have that the coefficients of the characteristic polynomial of \( L_n (\lambda) \) with respect to \( M_n (\lambda) \):

\[ p_{L,M} (\mu) = \det (L_n (\lambda) - \mu M_n (\lambda)), \]  

divided by \( \det M_n (\lambda) \) are constants of motions. That is, writing such characteristic polynomial in the following way:

\[ p_{L,M} (\mu) = (-1)^N \det M_n (\lambda) \mu^N + c_{N-1} (\lambda) \mu^{N-1} + \cdots + c_0 (\lambda), \]  

we can write these invariants as:

\[ H_0 = \frac{c_0 (\lambda)}{\det M_n (\lambda)}, \quad H_1 = \frac{c_1 (\lambda)}{\det M_n (\lambda)}, \ldots, \quad H_{N-1} = \frac{c_{N-1} (\lambda)}{\det M_n (\lambda)}. \]  

Finally, we note that the same consideration on functional independence of the invariants (2.17) given in Remark 1 apply.

### 3 Discrete Reduced Nahm Systems

In [2] where considered three special cases of Nahm’s equations (1.1) corresponding to symmetry groups of regular solids, namely tetrahedral, octahedral, and icosahedral symmetry.

Assume we are a given \( G \subset SO (3) \), a symmetry group of regular solid. Then, the \( G \)-invariant Nahm matrices \( T_i \) have the following form:

\[ T_i (t) = x(t) \rho_i + y(t) S_j. \]  

Here\( \rho : \mathbb{R}^3 \rightarrow \mathfrak{so} (k) \) is a representation of \( \mathfrak{so} (3) \) on \( \mathbb{C}^k \), while \( (S_1, S_2, S_3) \) is a \( G \)-invariant vector in the symmetric power space \( S^{2k} V \subset \mathbb{R}^3 \otimes \mathfrak{su} (k) \) where \( V \) is the representation corresponding to \( G \) in \( SU (2) \).

In the following we will consider the tetrahedral, octahedral, and icosahedral symmetry cases, with the definitions of the \( G \)-invariant Nahm matrices \( T_i \) given in [2]. The discretisation of these continuous systems was obtained in [9] using the KHK discretisation procedure and proved to be integrable by constructing the invariant with the so-called Hirota–Kimura bases [7]. Here we will prove that the KHK discretisation follows from the discretisation of the Lax pairs and the invariant can be found using the associated characteristic polynomial (2.15).
Remark 3 We note that the results of [7] on the discrete Nahm systems where generalised simultaneously and independently in [18, 19]. Some comments on the geometry of these systems were given in [20]. Later, in [21] it was proved how to construct the tetrahedral and the octahedral case discrete cases using generalised Manin transform. Finally, in [22] it was pointed out that in the octahedral case the geometric construction of [21] induces non-standard features on the procedure of resolution of singularities, proving the existence of families of particular solutions.

3.1 Tetrahedral Symmetry

Consider the reduced Nahm system with tetrahedral symmetry (1.2a). Its Lax pair is given by:

\[
A(x, y; \lambda) = \begin{pmatrix}
0 & 2i \lambda (2x + \frac{y}{2}) & -i (\lambda^2 - 1) (2x - \frac{y}{2}) \\
-2i \lambda (2x - \frac{y}{2}) & 0 & -i (\lambda^2 + 1) (2x + \frac{y}{2}) \\
i (\lambda^2 - 1) (2x + \frac{y}{2}) & i (\lambda^2 + 1) (2x - \frac{y}{2}) & 0
\end{pmatrix},
\]

(3.2a)

\[
B(x, y; \lambda) = \begin{pmatrix}
0 & i (2x + \frac{y}{2}) & -i \lambda (2x - \frac{y}{2}) \\
-i (2x - \frac{y}{2}) & 0 & -\lambda (2x + \frac{y}{2}) \\
in \lambda (2x + \frac{y}{2}) & \lambda (2x - \frac{y}{2}) & 0
\end{pmatrix}.
\]

(3.2b)

where we underlined the explicit dependence of the matrices on the dynamical variables \(x, y\). Considering the characteristic polynomial (2.5) we obtain the invariant given in [2]:

\[
H = y \left( y^2 + 48x^2 \right).
\]

(3.3)

Note that the level curves of the invariant \(H (3.3)\) are genus one (elliptic) curves.

With reference to formula (2.13), from the Lax pair (3.2) we obtain the following discrete Lax pair:

\[
L_n (\lambda) = A(x_n, y_n; \lambda), \quad M_n (\lambda) = I_2 - B(x_n, y_n; \lambda).
\]

(3.4)

We do not present the explicit form of the matrices in (3.4) since it is quite cumbersome. The corresponding compatibility conditions are:

\[
\frac{x_{n+1} - x_n}{h} = 2x_{n+1}x_n + \frac{1}{8}y_{n+1}y_n,
\]

(3.5a)

\[
\frac{y_{n+1} - y_n}{h} = -2 (x_ny_{n+1} + x_{n+1}y_n),
\]

(3.5b)

and coincide with the KHK discretisation of the reduced Nahm system with tetrahedral symmetry (1.2a), originally presented in [9]. The characteristic polynomial of \(L_n (\lambda)\) with respect to \(M_n (\lambda)\):

\[
p_{L,A} (\mu) = \left( 1 - 4h^2 x_n^2 + \frac{h^2 y_n^2}{4} + 12h^3 x_n^2 \lambda^2 y_n + \frac{h^3 y_n^3}{4} \lambda^2 \right) \mu^3
\]

\[- h^2 \lambda^3 y_n (48x_n^2 + y_n^2) \mu^2 + \frac{h y_n}{4} (48x_n^2 + y_n^2) (5\lambda^4 - 1) \mu
\]

\[- \frac{\lambda y_n}{2} (48x_n^2 + y_n^2) (\lambda - 1) (\lambda + 1) (\lambda^2 + 1).
\]

(3.6)
From (2.17) we could get three different invariants, yet they will be necessarily dependent. So, we choose:

\[ H_0(\lambda) = -\frac{1}{2} \frac{\lambda y_n \left(48x_n^2 + y_n^2\right) (\lambda - 1)(\lambda + 1)(\lambda^2 + 1)}{1 - 4h^2x_n^2 + \frac{h^2y_n^2}{4} + 12h^3x_n^2\lambda^2 y_n + \frac{h^3y_n^3\lambda^2}{4}}. \]  

(3.7)

This invariant is dependent on \( \lambda \), therefore to find a non-trivial invariant we can expand in Taylor series with respect to \( \lambda \), that is \( H_0(\lambda) = \sum_{k=0}^{\infty} h_{0,k}\lambda^k \), and take the first non-constant element:

\[ h_{0,1} = -\frac{1}{2} \frac{y_n \left(48x_n^2 + y_n^2\right)}{1 - 4h^2x_n^2 + \frac{h^2y_n^2}{4}}. \]  

(3.8)

This is an invariant for (3.5).

### 3.2 Octahedral Symmetry

Consider the reduced Nahm system with tetrahedral symmetry (1.2b). Its Lax pair is given by:

\[
A(x,y;\lambda) = \begin{pmatrix}
2\lambda (3x + 4y) & 6\lambda^2 (x - 2y) & 0 & -120y \\
-2x + 4y & 2\lambda (x - 12y) & 8\lambda^2 (x + 3y) & 0 \\
0 & -2x - 6y & -2\lambda (x - 12y) & 6\lambda^2 (x - 2y) \\
10\lambda^2 y/3 & 0 & -2x + 4y & -2\lambda (3x + 4y)
\end{pmatrix},
\]

(3.9a)

\[
B(x,y;\lambda) = \begin{pmatrix}
3x + 4y & 6\lambda (x - 2y) & 0 & 0 \\
0 & x - 12y & 8\lambda (x + 3y) & 0 \\
0 & 0 & -(x - 12y) & 6\lambda (x - 2y) \\
10\lambda y/3 & 0 & 0 & -(3x + 4y)
\end{pmatrix},
\]

(3.9b)

where we underlined the explicit dependence of the matrices on the dynamical variables \( x, y \). Considering the characteristic polynomial (2.5) we obtain the invariant given in [2]:

\[ H = y(x + 3y)(x - 2y)^2. \]  

(3.10)

Note that the level curves of the invariant \( H \) are genus one (elliptic) curves.

With reference to formula (2.13), from the Lax pair (3.9) we obtain a discrete Lax pair of the form (3.4). Again, we do not present the explicit form of the matrices since it is quite cumbersome. The corresponding compatibility conditions are:

\[
\frac{x_{n+1} - x_n}{h} = 2x_{n+1}x_n - 48y_{n+1}y_n, \quad (3.11a)
\]

\[
\frac{y_{n+1} - y_n}{h} = -3 (x_{n+1}y_n + x_ny_{n+1}) + 8y_n y_{n+1}. \quad (3.11b)
\]

and coincide with the KHK discretisation of the reduced Nahm system with octahedral symmetry (1.2b), originally presented in [9].
Proceeding in analogous ways as in Section 3.1, that is taking the first non-constant element of the Taylor series of $H_0(\lambda)$, we obtain the following invariant:

$$h_{0,0} = \frac{-960y_n\left(x_n + 3y_n\right)\left(x_n - 2y_n\right)^2}{1 - h^2\left(x_n - 12y_n\right)^2\left[1 - h^2\left(3x_n + 4y_n\right)^2\right]}. \quad (3.12)$$

### 3.3 Icosahedral Symmetry

Consider the reduced Nahm system with tetrahedral symmetry (1.2c). Its Lax pair is given by:

$$A\left(x, y; \lambda\right) = \begin{pmatrix}
\frac{2}{3}(25x + y) & \lambda^2(10x - y) & 0 & 0 & -168y & 336\lambda y \\
-2x + \frac{2}{3} & 2\lambda(3x - y) & 4x^2(4x + y) & 0 & 0 & 168y \\
0 & -2x - \frac{2}{3} & 2\lambda(x + 2y) & 6x^2(3x - y) & 0 & 0 \\
0 & 0 & -2x + 2/3y & -2\lambda(x + 2y) & 4x^2(4x + y) & 0 \\
\frac{7\lambda^2y}{120} & 0 & 0 & -2x - \frac{2}{3} & -2\lambda(3x - y) & \lambda^2(10x - y) \\
\frac{7\lambda^2y}{120} & 0 & 0 & 0 & -2x + \frac{2}{3} & -\frac{2}{3}(25x + y)
\end{pmatrix}, \quad (3.13a)$$

$$B\left(x, y; \lambda\right) = \begin{pmatrix}
\frac{5x + \frac{2}{3}}{5} & \lambda(10x - y) & 0 & 0 & 0 & 168y \\
0 & 3x - y & 4x^2(4x + y) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2x - \frac{2}{3} & 4\lambda(4x + y) & 0 \\
\frac{7\lambda^2y}{120} & 0 & 0 & 0 & 0 & -\frac{2}{3}(5x - \frac{2}{3})
\end{pmatrix}. \quad (3.13b)$$

Considering the characteristic polynomial (2.5) we obtain the invariant given in [2]:

$$H = y(3x - y)^2(4x + y)^3. \quad (3.14)$$

Note that the level curves of the invariant $H$ (3.14) are genus one (elliptic) curves.

The corresponding compatibility conditions are:

$$\frac{x_{n+1} - x_n}{h} = 2x_{n+1}x_n - y_{n+1}y_n, \quad (3.15a)$$

$$\frac{y_{n+1} - y_n}{h} = -5(x_ny_{n+1} + x_{n+1}y_n) + y_ny_{n+1}. \quad (3.15b)$$

and coincide with the KHK discretisation of the reduced Nahm system with icosahedral symmetry (1.2c), originally presented in [9].

Proceeding in analogous ways as in Section 3.1, that is taking the first non-constant element of the Taylor series of $H_0(\lambda)$, we obtain the following invariant:

$$h_{0,1} = \frac{112y_n\left(3x_n - y_n\right)^2(4x_n + y_n)^3}{1 - h^2\left(25x_n^2 + 2x_ny_n + 2y_n^2\right)\left[1 - h^2(x_n - 2y_n)^2\right]\left[1 - h^2(3x_n - y_n)^2\right]}. \quad (3.16)$$

### 4 Conclusions

In this paper we discretised the reduced Nahm systems [2] using the technique employed for the Euler top presented in [12–14]. Our results shows that the discretisation is analog to the so-called Kahan–Hirota–Kimura discretisation. We proved
that such Lax pairs are “bona fide”. That is, they can be used to produce (all) the invariants of the associated systems, and hence to prove integrability. A Lax pair that cannot be used to produce (all) the invariants of a given system is called a *fake Lax pair* [23–25].

We conclude this paper noting that, unfortunately Nahm systems and their generalisations are not enough to explain the integrability of all KHK discretisable systems. To this end we consider the following system, which is a particular case of the coupled Euler top introduced in [26]:

\[
\begin{align*}
\dot{x}_1 &= x_2 x_3, \\
\dot{x}_2 &= x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 + x_4 x_5, \\
\dot{x}_4 &= x_3 x_5, \\
\dot{x}_5 &= x_3 x_4.
\end{align*}
\] (4.1)

This system is naïvely integrable. We say that system of difference equations is naïvely integrable when it possesses \( N - 1 \) functionally independent first integrals (invariants), where \( N \) is the number of degrees of freedom. In [9] it was proved that the system (4.1) possesses five first integrals, four of which are functionally independent, proving naïve integrability.

It is easy to see that the system (4.1) arises from the following Nahm system:

\[
\begin{align*}
T_1 &= \begin{pmatrix} 1 & -x_1 & 0 \\ x_1 & 1 & x_4 \\ 0 & -x_4 & 1 \end{pmatrix}, \\
T_2 &= \begin{pmatrix} 1 & 0 & x_3 \\ 0 & 1 & 0 \\ -x_3 & 0 & 1 \end{pmatrix}, \\
T_3 &= \begin{pmatrix} 1 & x_5 & 0 \\ -x_5 & 1 & x_2 \\ 0 & -x_2 & 1 \end{pmatrix}.
\end{align*}
\] (4.2)

So, the system (4.1) has a Lax pair given by (2.4):

\[
\begin{align*}
A (\lambda) &= \begin{pmatrix} (1 - i) \lambda^2 - 2i \lambda + 1 + i & -\lambda^2 x_1 - 2i \lambda x_5 - x_1 & -i \lambda^2 x_3 + i x_3 \\ \lambda^2 x_1 + 2i \lambda x_5 + x_1 & (1 - i) \lambda^2 - 2i \lambda + 1 + i & -\lambda^2 x_4 - 2i \lambda x_2 + x_4 \\ i \lambda^2 x_3 - i x_3 & -\lambda^2 x_4 + 2i \lambda x_2 - x_4 & (1 - i) \lambda^2 - 2i \lambda + 1 + i \end{pmatrix}, \\
B (\lambda) &= \begin{pmatrix} -i + (1 - i) \lambda & -i x_5 - \lambda x_1 & -i x_3 \\ i x_5 + \lambda x_1 & -i + (1 - i) \lambda & -i x_2 + \lambda x_4 \\ i \lambda x_3 & i x_2 - \lambda x_4 & -i + (1 - i) \lambda \end{pmatrix}.
\end{align*}
\] (4.3)
and the characteristic polynomial (2.5) of \( L(\lambda) \) is:

\[
p_L(\mu) = \mu^3 + (3 - 3i) \left( i\lambda - \lambda^2 - i - \lambda \right) \mu^2 \\
+ \left[ (\mathcal{J}_1 - 6i) \lambda^4 + 4i (\mathcal{J}_2 - 3 + 3i) \lambda^3 - 4\mathcal{J}_3 \lambda^2 + 4i (\mathcal{J}_2 - 3 - 3i) \lambda + \mathcal{J}_1 + 6i \right] \mu \\
- (1 - i) (\mathcal{J}_1 - 2i) \lambda^6 + 2i [2 (i - 1) \mathcal{J}_2 + \mathcal{J}_1 - 6i] \lambda^5 \\
- \frac{3 - i}{5} \left[ 5\mathcal{J}_1 + (12 + 4i)\mathcal{J}_2 - (8 - 4i)\mathcal{J}_3 - 12 + 6i \right] \lambda^4 \\
+ 4i (4 + \mathcal{J}_1 - 2\mathcal{J}_2 - 2\mathcal{J}_3) \lambda^3 \\
+ \frac{3 + i}{5} (12 + 6i - 5\mathcal{J}_1 - 4(3 - i)\mathcal{J}_2 + 4(2 + i)\mathcal{J}_3) \lambda^2 \\
- 2i [2 (1 + i) \mathcal{J}_2 - \mathcal{J}_1 - 6i] \lambda - (1 + i) (\mathcal{J}_1 + 2i) \\
\tag{4.4}
\]

where:

\[
\mathcal{J}_1 = x_1^2 - x_3^2 + x_4^2, \quad \mathcal{J}_2 = x_1x_5 - x_4x_2, \quad \mathcal{J}_3 = x_2^2 - x_3^2 + x_5^2. \tag{4.5}
\]

Taking the coefficients of (4.4) with respect to \( \lambda \) and \( \mu \) we obtain that the three functions in (4.5) are the only independent invariants given by \( L(\lambda) \). This shows that the Lax pair (4.3) does not prove the naïve integrability of the system (4.1). In this sense the Lax pair (4.1) is fake in the sense of [23–25].

Now consider the discrete Nahm system (2.9) corresponding to the Nahm matrices (4.2):

\[
T_{1,n} = \begin{pmatrix}
1 & -x_{1,n} & 0 \\
x_{1,n} & 1 & x_{4,n} \\
0 & -x_{4} & 1
\end{pmatrix}, \tag{4.6a}
\]
\[
T_{2,n} = \begin{pmatrix}
1 & 0 & x_{3,n} \\
0 & 1 & 0 \\
-x_{3,n} & 0 & 1
\end{pmatrix}, \tag{4.6b}
\]
\[
T_{3,n} = \begin{pmatrix}
1 & x_{5,n} & 0 \\
-x_{5,n} & 1 & x_{2,n} \\
0 & -x_{2,n} & 1
\end{pmatrix}. \tag{4.6c}
\]

Unfortunately, we obtain the compatibility conditions are overdetermined, in the sense that we have more compatibility conditions than independent variables. For instance, let us consider the first discrete Nahm equation \( T_{1,n+1} - T_{1,n} = h (T_{2,n+1}T_{3,n} - T_{3,n+1}T_{2,n}) \) writing down the coefficients explicitly:

\[
x_{1,n+1} - x_{1,n} = h \left( x_{3,n+1}x_{2,n} - x_{5,n} + x_{5,n+1} \right), \tag{4.7a}
\]
\[
x_{3,n+1} - x_{3,n} = 0, \tag{4.7b}
\]
\[
x_{1,n+1} - x_{1,n} = h \left( x_{2,n+1}x_{3,n} - x_{5,n} + x_{5,n+1} \right), \tag{4.7c}
\]
\[
x_{4,n+1} - x_{4,n} = h \left( x_{5,n+1}x_{3,n} + x_{2,n} - x_{2,n+1} \right), \tag{4.7d}
\]
\[
x_{4,n+1} - x_{4,n} = h \left( x_{3,n+1}x_{5,n} + x_{2,n} - x_{2,n+1} \right). \tag{4.7e}
\]
Equation (4.7b) implies that the discrete time evolution of $x_{3,n}$ is trivial. So, the discrete Nahm system (4.7e) cannot be a discretisation of the integrable system (4.1).

On the other hand in [9] it was proven that the KHN discretisation of the system (4.1) exists and it is algebraically integrable. More precisely, using the method of the Hirota–Kimura bases [7], the authors proved that such KHK discretisation preserves all five invariants of its continuous counterpart (4.1). We note that the functionally independent invariants of such KHK discretisation can be found directly, that is without using the Hirota–Kimura bases, with the method of [27]. See also [28] for an explanation of the method in the case of difference equations.

This example shows that, the Nahm’s equations approach might be not enough to explain integrability of quadratic vector equations in both the continuous and the discrete case. This statement is clearly limited to the derivation of the coupled Euler system (4.1), using the Nahm matrices (4.2), and does not exclude the possibility of different “good” representations. For instance, during the final revision of this paper we become aware of a different Nahm matrices representation of the coupled Euler system (4.1), such that the associated Lax pair is bona fide [29]. This again underlines the importance of the concept of “fake” Lax pairs, see [23–25].

To conclude, we notice that not all the Nahm systems yield quadratic differential equations. One example is the four-dimensional Nahm system with so (4) matrices presented in [30], whose Nahm matrices are:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & x_1 x_4 \\ 0 & 0 & -x_2 x_3 & 0 \\ 0 & x_2 x_3 & 0 & 0 \\ -x_1 x_4 & 0 & 0 & 0 \end{pmatrix} \quad (4.8a)$$

$$T_2 = \begin{pmatrix} 0 & 0 & x_1 x_3 & 0 \\ 0 & 0 & 0 & x_2 x_4 \\ -x_1 x_3 & 0 & 0 & 0 \\ 0 & -x_2 x_4 & 0 & 0 \end{pmatrix} \quad (4.8b)$$

$$T_3 = \begin{pmatrix} 0 & -x_3 x_4 & 0 \\ x_3 x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 x_2 \\ 0 & 0 & -x_1 x_2 & 0 \end{pmatrix} \quad (4.8c)$$

The corresponding system

$$\dot{x}_i = \prod_{j \neq i} x_j,$$  \hspace{1cm} (4.9)

is cubic, so the associated KHK discretisation is not birational and fall outside the framework of the algebraic theory of discrete integrable systems. Notice that other discretisation methods like polarisation method [31], can be applied to the system (4.9). However, at present no relationship with integrablity or Lax pairs is known. We reserve the analysis of such questions to future works.
Acknowledgements  The author expresses his gratitude to Prof. N. Joshi, Prof. G. R. W. Quispel and Dr. D. T. Tran for their helpful discussions during the preparation of this paper. Moreover, we would like to thank Prof. K. Kimura for sharing his results and comments on the coupled Euler system and on the so(4) system. Finally, we thank the anonymous referee, whose comments led to a great improvement of the paper.

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