Deformed Richardson-Gaudin model

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Abstract. The Richardson-Gaudin model describes strong pairing correlations of fermions confined to a finite chain. The integrability of the Hamiltonian allows the algebraic construction of its eigenstates. In this work we show that the quantum group theory provides a possibility to deform the Hamiltonian preserving integrability. More precisely, we use the so-called Jordanian $r$-matrix to deform the Hamiltonian of the Richardson-Gaudin model. In order to preserve its integrability, we need to insert a special nilpotent term into the auxiliary $L$-operator which generates integrals of motion of the system. Moreover, the quantum inverse scattering method enables us to construct the exact eigenstates of the deformed Hamiltonian. These states have a highly complex entanglement structure which require further investigation.

The Richardson-Gaudin model \cite{1, 2} is an integrable spin-$\frac{1}{2}$ periodic chain with Hamiltonian

\[ H = \sum_{j=1}^{N} \epsilon_j S_j^z + g \sum_{j, k=1}^{N} S_j^- S_k^+ \]  

(1)

where $g$ is a coupling constant and $S_j^\pm = S_j^x \pm iS_j^y$, with $N$ copies of the Lie algebra $su(2)$ generators $S_l^\alpha$,

\[ [S_l^\alpha, S_l'^\beta] = i\varepsilon^\alpha\beta\gamma S_l^\gamma \delta_{l'l'}, \quad \alpha, \beta = x, y, z \]

As shown by Cambiaggio et al \cite{3}, by introducing the fermion operators $c_{lm}^\dagger$ and $c_{lm}$ related to the $sl(2)$ generators by

\[ S_l^z = 1/2 \sum_m c_{lm}^\dagger c_{lm} - 1/2, \quad S_l^+ = 1/2 \sum_m c_{lm}^\dagger c_{lm}^\dagger = (S_l^-)^\dagger \]

the Richardson-Gaudin model in Eq. (1) gets mapped onto the pairing model Hamiltonian

\[ H_P = \sum_l \epsilon_l n_l + g/2 \sum_{l, l'} A_l^\dagger A_{l'} \]  

(2)

Here $c_{lm}^\dagger$ ($c_{lm}$) creates (annihilates) a fermion in the state $|lm\rangle$ (with $|l\bar{m}\rangle$ in the time reversed state of $|lm\rangle$) and

\[ n_l = \sum_m c_{lm}^\dagger c_{lm}, \quad A_l^\dagger = (A_l)^\dagger = \sum_m c_{lm}^\dagger c_{lm}^\dagger \]
are the corresponding number- and pair-creation operators. The pairing strengths \( g_{\mu \nu} \) are here approximated by a single constant \( g \), with \( \epsilon \) the single-particle level corresponding to the \( m \)-fold degenerate states \( |lm\rangle \).

As it is well-known, the pairing model in Eq. (2) is central in the theory of superconductivity. Richardson’s exact solution of the model [1], exploiting its integrability, has been important for applications in mesoscopic and nuclear physics where the small number of fermions prohibits the use of conventional BCS theory [4]. Moreover, its (pseudo)spin representation in the guise of the Richardson-Gaudin model, Eq. (1), provides a striking link between quantum magnetism and pairing phenomena, both central concepts in the physics of quantum matter.

The eigenstates of the Richardson-Gaudin Hamiltonian, eq. (1), can be constructed algebraically using the quantum inverse scattering method (QISM) [5, 6]. The main objects of this method are the classical \( r \)-matrix

\[
   r(\lambda, \mu) = 4 \lambda - \mu \sum_{\alpha} s^{\alpha} \otimes s^{\alpha} \bigg|_{s=1/2} \approx \frac{1}{\lambda - \mu} \begin{pmatrix}
   1 & 0 & 0 & 0 \\
   0 & -1 & 2 & 0 \\
   0 & 2 & -1 & 0 \\
   0 & 0 & 0 & 1
\end{pmatrix} \tag{3}
\]

where \( h(\lambda) \), \( X^+(\lambda) \), \( X^-(\lambda) \) are the generators of the loop algebra \( \mathcal{L}(sl(2)) \) whereas the \( L \)-matrix is

\[
   L(\lambda) = \begin{pmatrix}
   h(\lambda) & 2X^-(\lambda) \\
   2X^+(\lambda) & -h(\lambda)
\end{pmatrix}
\]

The commutation relations (CR) of loop algebra generators are given in compact matrix form

\[
   [L_1(\lambda), L_2(\mu)] = -[r_{12}(\lambda, \mu), L_1(\lambda) + L_2(\mu)]
\]

where

\[
   L_1(\lambda) = L(\lambda) \otimes 1, \quad L_2(\mu) = 1 \otimes L(\mu)
\]

and \( r(\lambda, \mu) \) is the \( 4 \times 4 \) \( c \)-number matrix in Eq. (3). A consequence of this form is the commutativity of transfer matrices,

\[
   t(\lambda) = \frac{1}{2} \text{tr}_0(L^2(\lambda)) \in \mathcal{L}(sl(2)), \quad [t(\lambda), t(\mu)] = 0 \tag{4}
\]

The corresponding mutually commuting operators extracted from the decomposition of \( t(\lambda) \) define a Gaudin model [2, 7]. However, to get Richardson Hamiltonian a mild change of the \( L \)-operator is necessary

\[
   L(\lambda) \to L(\lambda; c) := c h_0 + L(\lambda)
\]

where \( h_0 = \sigma_0^z \) in auxiliary space \( \mathbb{C}^2_0 \) of spin \( 1/2 \). This transformation does not change the CR of matrix elements of this matrix \( L(\lambda; c) \) due to the symmetry of the \( r \)-matrix (3):

\[
   [Y \otimes 1 + 1 \otimes Y, r(\lambda, \mu)] = 0, \; Y \in sl(2)
\]

The resulting transfer matrix obtains some extra terms

\[
   t(\lambda; c) = \frac{1}{2} \text{tr}_0(L(\lambda; c))^2 = c^2 1 + c h(\lambda) + h^2(\lambda) + 2 (X^+(\lambda)X^-(\lambda) + X^-(\lambda)X^+(\lambda))
\]

Let us consider a spin-\( 1/2 \) representation on auxiliary space \( V_0 \simeq \mathbb{C}^2 \) and spin \( \ell_k \) representations on quantum spaces \( V_k \simeq \mathbb{C}^{\ell_k+1} \) with extra parameters \( \epsilon_k \) corresponding to site \( k = 1, 2, \ldots, N \).
The whole space of quantum states is \( H = \otimes_1^N V_k \) and the highest weight vector (highest spin, "ferromagnetic state") \( | \Omega_+ \rangle \) satisfies
\[
X^+ (\lambda) | \Omega_+ \rangle = 0, \quad h(\lambda) | \Omega_+ \rangle = \rho(\lambda) | \Omega_+ \rangle
\] (5)
where
\[
\rho(\lambda) = \sum_{k=1}^N k/(\lambda - \epsilon_k)
\]
It is useful to introduce notation for global operators of \( sl(2) \)-representation \( Y_{gl} := \sum_{k=1}^N Y_k \). To find the eigenvectors and spectrum of \( t(\lambda) \) on \( H \) one requires that vectors of the form
\[
| \mu_1, \ldots, \mu_M \rangle = \prod_{j=1}^M X^- (\mu_j) | \Omega_+ \rangle
\]
are eigenvectors of \( t(\lambda) \),
\[
t(\lambda) | \{ \mu_j \}_{j=1}^M \rangle = \Lambda (\lambda; \{ \mu_j \}_{j=1}^M) | \{ \mu_j \}_{j=1}^M \rangle
\]
provided that the parameters \( \mu_j \) satisfy the Bethe equations:
\[
2c + \sum_{k=1}^N k/(\mu_i - \epsilon_k) - \sum_{j \neq i}^M 2/(\mu_i - \mu_j) = 0, \quad i = 1, \ldots, M
\] (6)
The realization of the loop algebra generators on the space \( H \) takes the form
\[
h(\lambda) = \sum_{k=1}^N \frac{h_k}{\lambda - \epsilon_k}, \quad X^- (\lambda) = \sum_{k=1}^N \frac{X^-_k}{\lambda - \epsilon_k}, \quad X^+ (\lambda) = \sum_{k=1}^N \frac{X^+_k}{\lambda - \epsilon_k}
\] (7)
The coupling constant \( g \) of (1) is connected with parameter \( c = 1/g \) while the Hamiltonian (1) is obtained as operator coefficient of term \( 1/\lambda^2 \) in the expansion of \( t(\lambda; c) \) at \( \lambda \to \infty \).
The quantum group theory provides a possibility to deform a Hamiltonian preserving integrability \([8, 9]\). Specifically, we use the so-called Jordanian \( r \)-matrix to quantum deform the Hamiltonian of Richardson-Gaudin model (1). We add to \( sl(2) \) symmetric \( r \)-matrix (3) the Jordanian part
\[
r^{J}(\lambda, \mu) = \frac{C_2^2}{\lambda - \mu} + \xi (h \otimes X^+ - X^+ \otimes h)
\]
with Casimir element \( C_2^2 \) in the tensor product of two copies of \( sl(2) \),
\[
C_2^2 = h \otimes h + 2 (X^+ \otimes X^- + X^- \otimes X^+)
\]
After Jordanian twist the \( r \)-matrix (14) is commuting with the generator \( X^+_0 \) only
\[
\left[ X^+_0 \otimes \mathbb{I} + \mathbb{I} \otimes X^+_0 , r^{(J)}(\lambda, \mu) \right] = 0
\]
Hence, one can add the term $cX_0^+ + L(\lambda, \xi)$ to the $L$-operator. This yields the twisted transfer-matrix

$$t^{(J)}(\lambda) = \frac{1}{2} \text{tr}_0(cX_0^+ + L(\lambda, \xi))^2 = cX^+(\lambda) + h(\lambda)^2 - 2h'(\lambda) + 2(2X^- (\lambda) + \xi)X^+(\lambda)$$  \hspace{1cm} (8)

The corresponding commutation relations between the generators of the twisted loop algebra are explicitly given by

$$[h(\lambda), h(\mu)] = 2\xi(X^+(\lambda) - X^+(\mu)), \quad [X^-(\lambda), X^-(\mu)] = -\xi(X^-(\lambda) - X^-(\mu))$$ $$[X^+(\lambda), X^-(\mu)] = -\frac{h(\lambda) - h(\mu)}{\lambda - \mu} + \xi X^+(\lambda), \quad [X^+(\lambda), X^+(\mu)] = 0$$

$$[h(\lambda), X^-(\mu)] = 2\frac{X^-(\lambda) - X^-(\mu)}{\lambda - \mu} + \xi h(\mu), \quad [h(\lambda), X^+(\mu)] = -2\frac{X^+(\lambda) - X^+(\mu)}{\lambda - \mu}$$ \hspace{1cm} (9)

The realization of the Jordanian twisted loop algebra $\mathcal{L}_j(sl(2))$ with CR (9) is given similar to (7) with extra terms proportional to the deformation parameter $\xi$

$$h(\lambda) = \sum_{k=1}^{N} \left( \frac{h_k}{\lambda - \epsilon_k} + \xi X^+_k \right), \quad X^-(\lambda) = \sum_{k=1}^{N} \left( \frac{X^-_k}{\lambda - \epsilon_k} - \frac{\xi}{2} h_k \right), \quad X^+(\lambda) = \sum_{k=1}^{N} \frac{X^+_k}{\lambda - \epsilon_k}$$ \hspace{1cm} (10)

To construct eigenstates for the twisted model one has to use operators of the form $[9, 10]$

$$B_M(\mu_1, \ldots, \mu_M) = X^-(\mu_1) \left( X^-(\mu_2) + \xi \right) \ldots \left( X^-(\mu_M) + \xi (M - 1) \right)$$

acting by these operators on the ferromagnetic state $|\Omega_+\rangle$.

The deformed Richardson-Gaudin model Hamiltonian can now be extracted from the transfer-matrix $t^{(J)}(\lambda)$ as the operator coefficient in its expansion $\lambda \to \infty$.

According to (4) and (8) one can also extract quantum integrals of motion $J_k$ using the realization (10). It would yield rather cumbersome expressions for $J_k$

$$t^{(J)}(\lambda) = J_0 + \frac{1}{\lambda} J_1 + \frac{1}{\lambda^2} J_2 + \ldots$$

The corresponding quantum deformed Hamiltonian reads

$$H \simeq J_2 = c \sum_{j=1}^{N} \epsilon_j X^+_j + 2\xi \left\{ \sum_{j=1}^{N} \epsilon_j h_j \right\} X^+_j - h_{gl} \sum_{j=1}^{N} \epsilon_j X^+_j + \left( h_{gl}^2 + 2h_{gl} + 4X^-_{gl}X^+_{gl} \right)$$

It is instructive to write down a simplified case without the Jordanian twist: $\xi = 0$. One thus obtains

$$J_0 = 0, \quad J_3 = X^+_{gl}, \quad J_2 \simeq \sum_{k=1}^{N} \epsilon_k X^+_k + g/2 \left( h_{gl}^2 + 2h_{gl} + 4X^-_{gl}X^+_{gl} \right)$$

The case $\xi = 0$ can also be obtained by taken off from the inhomogeneous $XXX$ spin chain. The model can be described by a $2 \times 2$ monodromy matrix $[5]$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$
and entries of this matrix satisfy quadratic relations

\[ R(\lambda, \mu) T(\lambda) \otimes T(\mu) = (I \otimes T(\mu)) (T(\lambda) \otimes I) R(\lambda, \mu) \tag{11} \]

If we multiply by a constant 2 \times 2 matrix \( M(\varepsilon) \) resulting matrix \( \tilde{T}(\lambda) = M(\varepsilon) \cdot T(\lambda) \) will satisfy the same relation (11). Choosing a triangular matrix \( M(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \) the entries of monodromy matrices become simply related:

\[ \tilde{A} = A + \varepsilon C, \quad \tilde{B} = B + \varepsilon D, \quad \tilde{C} = C, \quad \tilde{D} = D. \]

This choice of \( M(\varepsilon) \) (of the same type as considered in [11]) permits us to use the same reference state \(|\Omega_+\rangle \in \mathcal{H} (5)\) and \( \tilde{B} \) as a creation operator of the algebraic Bethe ansatz [5].

Bethe states are given by the same action of product operators \( \tilde{B}(\mu_j) = B(\mu_j) + \varepsilon D(\mu_j) \) although operators \( \tilde{B}(\mu_j) \) do not commute with \( D(\mu_j) \):

\[ D(\lambda) B(\mu) = \alpha(\lambda, \mu) B(\mu) D(\lambda) + \beta(\lambda, \mu) B(\lambda) D(\mu) \]

where

\[ \alpha(\lambda, \mu) = (\lambda - \mu + \eta)/(\lambda - \mu), \quad \beta(\lambda, \mu) = -\eta/(\lambda - \mu) \]

For a 3 magnon state one gets due to B-D ordering

\[
\prod_{j=1}^{3} \tilde{B}(\mu_j) = \prod_{j=1}^{3} B(\mu_j) + \varepsilon \sum_{s=1}^{3} \alpha(\mu_k, \mu_s) \alpha(\mu_s, \mu_l) B(\mu_k) B(\mu_l) D(\mu_s) \\
+ \varepsilon^2 \sum_{s=1}^{3} \alpha(\mu_k, \mu_s) \alpha(\mu_l, \mu_s) B(\mu_k) D(\mu_l) D(\mu_s) + \varepsilon^3 \prod_{j=1}^{3} D(\mu_j)
\]

Similar formula is valid for \( \tilde{M} \)-magnon state. Hence, acting on ferromagnet state \(|\Omega_+\rangle\), we obtain filtration of states with eigenvalues of \( S^2 : N \frac{T}{T} + \frac{N}{T} - 1, N \frac{T}{T} - 2, N \frac{T}{T} - 3 \).

More complicated deformations of the Richardson-Gaudin model can be obtained using \( r \)-matrices related to the higher rank Lie algebras [12]. The structure of the eigenstates of the transfer matrix and their entanglement properties [13] are under investigation.

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**References**

[1] Richardson R W 1963 J. Math. Phys. 6 1034

[2] Gaudin M 1983 *La fonction d’onde de Bethe* (Paris: Masson) chapter 13

[3] Cambiaggi M C, Rivas A M F and Saraceno M 1997 *Nucl. Phys. A* 624 157

[4] Dukelsky J, Pittel S and Sierra G 2004 *Rev. Mod. Phys.* 76 643

[5] Faddeev L D 1998 How algebraic Bethe ansatz works for integrable models *Quantum symmetries, Proceedings of the Les Houches summer school, session LXIV* eds A Connes, K Gawedzki and J Zinn-Justi (North-Holland) p 149

[6] Kulish P P and Sklyanin E K 1982 Quantum spectral transform method. Recent developments *Lecture Notes in Phys.* 151 (Springer-Verlag) p 61

[7] Sklyanin E K 1999 *Lett. Math. Phys.* 47 275

[8] Kulish P P and Stolin A A 1997 *Czech. J. Phys.* 12 207

[9] Kulish P P 2002 *Twisted sl(2) Gaudin model* Preprint PDMI 08/2002

[10] Antonio N C and Manojlovic N 2005 *J. Math. Phys.* 46 102701

[11] Mukhin E, Tarasov V and Varchenko A 2010 *Contemp. Math.* 506 187

[12] Stolin A A 1991 *Math. Scand.* 69 57

[13] Dunning C, Links J and Zhou H-Q 2005 *Phys. Rev. Lett.* 94 227002