EXISTENCE OF SOLUTIONS FOR NEUTRAL FUNCTIONAL VOLterra-FREDHOLM INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we study the existence of mild solutions of nonlinear neutral functional Volterra-Fredholm integrodifferential equations with nonlocal conditions. The results are obtained by using fractional power of operators and Sadovskii’s fixed point theorem.

Keywords: Integrodifferential equations, fixed point theorem, Banach space.

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1. INTRODUCTION

Many phenomena in several branches have mathematical model in terms of differential equations. Differential equations are like a bridge which links mathematics and science with applications. It is a rightly considered as a language of sciences. Many branches of science have led to some kind of differential equations.

The study of integrodifferential equations has emerged in recent years as an independent branch of modern research because of its connections to many applied fields such as elasticity, biology, epidemics and other branches of science and engineering. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades.

The advantages of using nonlocal conditions is that measurable at more places can be incorporated to get better models. The nonlocal Cauchy problem for abstract evolution differential equation was first considered by Byszewski (Byszewski, 1991) Subsequently, several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces (Balachandran, 1998; Byszewski and Acka, 1998; Balachandran and Park, 2001a, b; Fu and Ezzinbi, 2004).

In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem have been studied by Byszewski and Lakshmikantham (1990), Byszewski (1997, 1998), Balachandran and Chandrasekar (1996), Lin and Liu (1996) and Murugesu and Suguna (2010).

In this paper, we extend this problem to neutral functional Volterra-Fredholm type integro differential equations with nonlocal conditions and discuss the existence of solutions for nonlinear neutral functional Volterra-Fredholm integro differential equations with nonlocal conditions of the form

\[ \frac{d}{dt} \left[ x(t) + F(t, x(t), x(t), x(t), \ldots, x(t)) \right] = A x(t) + G(t, x(t), x(t), \ldots, x(t)) \]

where \( A \) generates an analytic semigroup and \( F, G, K, k, h \) are given functions to specified later.

This paper has the following subsections. In section 2, we present some preliminary lemmas and definitions which will be used to prove our main results. In section 3, we present the existence of mild solution of the system (1) using Sadovskii’s fixed point theorem (Sodovskii, 1967).

2 PRELIMINARIES

Throughout this work, let \( A \) is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators \( T(t) \) defined in the Banach spaces \( X \). Let \( 0 \in \rho (A) \) then define the fractional power \( A^\alpha \), for \( 0 \leq \alpha \leq 1 \), as a closed linear operator on its domain \( D(A^\alpha) \) which is dense in \( X \). Further, \( D(A) \) is a Banach space under the norm

\[ \| x \| _{\alpha} = \| A^\alpha x \|, x \in D(A^\alpha) \]

Which we denote by \( X_\alpha \). Then for each \( 0 < \alpha \leq 1 \), \( X_\alpha \rightarrow X_\beta \) for \( 0 < \beta < \alpha \leq 1 \) and the imbedding is
compact whenever the resolvent operator of $A$ is compact. We assume that

a) There is a $M \geq 1$ such that $\| T(t) \| \leq M$, for all $0 \leq t \leq a$.

b) For any $a > 0$, there exists a positive constant $C_o$.

\[
\| A A^T(t) \| \leq \frac{C_o}{t^a}, \quad 0 < t \leq a
\]

(2)

for any $0 \leq s, t \leq a$.

\[
\| A^\beta F(s, x, x, \ldots, x) - A^\beta F(s, x_0, x_1, \ldots, x_m) \| \leq \delta \left( \max \{ \| x_i \| : i = 1, 2, \ldots, m \} + 1 \right)
\]

holds for any $(t, x_0, x_1, \ldots, x_m) \in [0, a) \times X^{m+1}$.

**Definition 2.1** (Pazy, 1983)

Let $X$ be a Banach space, a one parameter family $T(t), 0 \leq t < \infty$, of bounded linear operators from $X$ to $X$ is a semigroup of bounded linear operators on $X$, if

(i) $T(0) = 1$, where $1$ is the identity operator on $X$,

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$, (the semigroup property)

A semigroup of bounded linear operator $T(t)$ is uniformly continuous if

\[
\lim_{t \to 0} \| T(t) - 1 \| = 0
\]

**Theorem 2.1. (Sadovski’s Fixed Point Theorem)** (Sadovski, 1967)

Let $\psi$ be a condensing operator on a Banach space $X$, that is $\psi$ is continuous and takes bounded sets into bounded sets and $\mu(\psi(B)) \leq \mu(B)$ for every bounded set $B$ of $X$ with $\mu(B) > 0$. If $\psi(T) \subset T$ for a convex closed and bounded set $\gamma$ of $X$, then $\psi$ has a fixed point in $X$.

**3. Existence of Mild Solution**

**Definition 3.1.**

A continuous function $x(.) : [0, a] \to X$ is said to be a mild solution of the Cauchy problem (1), if the function $AT(t)-sF(s, x(s), x(b(s)), \ldots, x(b_m(s)))$, $s \in [0, a)$ is integrable on $[0, a)$ and the integral equation is satisfied.

\[
\begin{align*}
&x(t) = T(t)x_0 + \int_0^t \left[ \int_0^s f(s, r, x(r))dr \right]ds \\
&\quad + \int_0^t \left[ \int_0^s g(s, r, x(r))dr \right]ds \\
&\quad + \int_0^t \left[ \int_0^s h(s, r, x(r))dr \right]ds \\
&\quad + \int_0^t \left[ \int_0^s i(s, r, x(r))dr \right]ds \\
&\quad + \int_0^t \left[ \int_0^s j(s, r, x(r))dr \right]ds
\end{align*}
\]

(4)

**Theorem 3.1**

If the assumptions (H1) – (H5) are satisfied and $x_0 \in X$, then the Cauchy problem (1) has a mild
solution provided that
\[ L_s \leq M \beta^{-1} \frac{C}{\mu} a^\alpha \epsilon 1 \quad \text{-----------------(5)} \]
\[ (\gamma + \gamma_1 + M \sigma) M + M \sigma L_1 + \frac{1}{p} C \beta^{1-\mu} L_2 \geq 1 \quad \text{-----------------(6)} \]

Where \( M_0 = || A \beta || \).

Proof:

For the sake of brevity, we write that
\[(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) = (t, v(t)) \]
and
\[(t, x(t), x(a_1(t)), \ldots, x(a_m(t))) = (t, u(t)). \]

Define the operator \( Q \) on \( E \) by the formula
\[ (Qx)(t) = T(t) [x_0 + g(x)] - \int_0^t T(t-s) F(s, v(s)) ds + \int_0^t T(t-s) G(s, u(s)) ds \]
\[ + \int_0^t (t-s) K(x, s), \int \{ k(s, \tau, x(\tau)) d\tau \} ds + \int \{ k(s, \tau, x(\tau)) d\tau \} ds \]

For positive integer \( r \), let
\[ B_r = \{ x \in E : \| x(t) \| \leq r, 0 \leq t \leq a \} \]
then for each \( r \), \( B_r \) is clearly a bounded closed convex set in \( E \). Since by (2) and (3) the following relation holds:
\[ \| AT(t-s)F(s, v(s)) \| \leq \frac{C_1}{(t-s)^{\beta}} L \]
then from Bochner's theorem (Marle, 1974) it follows that \( AT(t-s)F(s, v(s)) \) is integrable on \([0, a] \), so \( Q \) is well defined on \( B_n \).

Claim : there exists a positive integer \( r \) such that \( Q B_r \subseteq B_r \):

If it is not true, then for each positive integer \( r \), there is a function \( x \in B_r \), but \( Qx \notin B_r \), that is \( \| Qx(t) \| > r \) for some \( t \in [0, a] \), where \( t(r) \) denotes \( t \) dependent on \( r \). However, on the other hand, we have
\[ t < Qx(t) \]
\[ = T(t) x_0 + g(x) - \int_0^t T(t-s) F(s, v(s)) ds + \int_0^t T(t-s) G(s, u(s)) ds \]
\[ + \int_0^t (t-s) K(x, s), \int \{ k(s, \tau, x(\tau)) d\tau \} ds + \int \{ k(s, \tau, x(\tau)) d\tau \} ds \]
\[ \leq M \| x_0 \| + M M_0 L (r+1) + \frac{1}{p} C_1 \beta^{1-\mu} (r+1) + M \| u(s) \| + M \| v(s) \| \]

Dividing on both sides by \( r \) and taking the limit as \( r \to \infty \), we get,
\[ (\gamma + \gamma_1 + M \sigma) M + M \sigma L_1 + \frac{1}{p} C \beta^{1-\mu} L_2 \geq 1 \]
This contradicts (6). Hence for some positive integer \( r \), \( Q B_r \subseteq B_r \).

Next we will show that the operator \( Q \) has a fixed point on \( B_n \):

Let us decompose \( Q \) as \( Q = Q_1 + Q_2 \), where the operators \( Q_1 \) and \( Q_2 \) are defined on \( B_n \) respectively by
\[ (Q_1x)(t) = -F(t, v(t)) - \int_0^t AT(t-s) F(s, v(s)) ds \]
\[ (Q_2x)(t) = T(t) [x_0 + g(x)] + \int_0^t T(t-s) G(s, u(s)) ds \]
\[ + \int_0^t (t-s) K(x, s), \int \{ k(s, \tau, x(\tau)) d\tau \} ds + \int \{ k(s, \tau, x(\tau)) d\tau \} ds \]
for \( 0 \leq t \leq a \), and we will verify that \( Q_1 \) is contraction and \( Q_2 \) is a compact operator.

Claim : \( Q_1 \) is a contraction

Let \( x, \ x \in B \). Then for each \( t \in [0, a] \) and by condition (H1) and (5), we have
\[ \| (Q_1x)(t) - (Q_1y)(t) \| \leq \| F(t, v(t)) - F(t, v(t)) \| \]
\[ + \| T(t-s) F(s, v(s)) - T(t-s) F(s, v(s)) \| \]
\[ \leq M_0 \| \frac{\phi}{(t-s)^{\beta}} \| \| v(s) - v(s) \| \]
Thus \( Q_1 \) is a contraction.

Claim : \( Q_2 \) is compact

To prove this we have to prove that \( Q_2 \) is continuous on \( B_n \).

Let \( \{ x_n \} \subseteq B_n \), with \( x_n \to x \) in \( B_n \), then by (H2) (i), we have
\[ G(s, u_n(s)) \to G(s, u(s)), \quad n \to \infty \]
\[ \| k(t, x_n(s), x_n(s)) ds - k(t, x(s), x(s)) ds \| \leq 2 \phi_1(s) \]
Since
\[ \| G(s, u_n(s)) - G(s, u(s)) \| \leq 2 \phi_1(s) \]
by the dominated convergence theorem, we have
\[ \| Q_2 - Q_2 \| = \| \int_0^a \| T(t)[g(x) - g(x)] + \int_0^t AT(t-s) G(s, u(s)) - G(s, u(s)) ds \| \]
\[ + \int_0^t \{ k(s, \tau, x(\tau)) d\tau \} ds + \int_0^t \{ k(s, \tau, x(\tau)) d\tau \} ds \]
\[ \leq 0 \quad \text{as } n \to \infty \]

(i.e) \( Q_2 \) is continuous.

Next, we prove that \( \{ Q \} x : x \in B_n \} \) is a family of equicontinuous functions. To see this we fix \( t_1 > 0 \) and \( t_2 > t_1 \) and \( \epsilon > 0 \) be enough small. Then
We can prove that the function $Q_2 x; x \in B_r$ are equicontinuous at $t=0$. Hence $Q_2$ maps $B_r$ into a family of equicontinuous function.

\textbf{Claim :} $V(t) = \{(Q_2 x)(t) : x \in B_r\}$ is relatively compact in $X$.

Let $0 < t \leq a$ be a fixed and $0 < \varepsilon < t$. For $x \in B_r$, we define

$$(Q_2 x)(t) = T(0) x + \int_0^t T(t-s)K(\varepsilon, s, x(s))ds$$

Then from compactness of $T(\varepsilon)$, $\varepsilon > 0$ we obtain $V_\varepsilon = \{(Q_2 x)(t) : x \in B_r\}$ is relatively compact in $X$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in B_r$, we have

$$\| (Q_2 x)(t) - (Q_2 x)(\varepsilon) \| \leq M \int_0^t |g(t, x)|ds + M \int_0^t |q(t, x)|ds$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in $X$.

Thus, by Arzela-Ascoli theorem, $Q_2$ is a compact operator. Those arguments enable us to conclude that $Q=Q_1+Q_2$ is a condensing map $B_r$ and by the Sadovskii’s fixed point theorem there exist a fixed point $x(\cdot)$ for $Q$ on $B_r$. Therefore, the Cauchy problem (1) has a mild solution, and the proof is completed.

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