Surface defects and symmetries

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Abstract. In quantum field theory, defects of various codimensions are natural ingredients and carry a lot of interesting information. In this contribution we concentrate on topological quantum field theories in three dimensions, with a particular focus on Dijkgraaf-Witten theories with abelian gauge group. Surface defects in Dijkgraaf-Witten theories have applications in solid state physics, topological quantum computing and conformal field theory. We explain that symmetries in these topological field theories are naturally defined in terms of invertible topological surface defects and are thus Brauer-Picard groups.

1. Introduction

In this contribution we discuss the relation between two important structures in quantum field theory: symmetries on the one hand, and defects of various codimensions on the other. While it is hardly necessary to emphasize the importance of symmetries, defects in quantum field theories or, in a different formulation, quantum field theories on stratified spaces, and their relation to symmetries are a more recent topic of interest.

Our exposition is organized as follows. We first discuss a few general aspects of topological codimension-1 defects, both in two-dimensional and in three-dimensional quantum field theories. A special emphasis is on the relation between invertible topological defects and symmetries. Afterwards we consider defects for a particular class of three-dimensional topological field theories, Dijkgraaf-Witten theories. An advantage of these theories is that they admit a mathematically precise formulation as gauge theories, see [7, 10, 23] and references therein. Apart from providing insights in a language that is close to standard field theoretical formulations, this allows one to uncover interesting relations between categories of (relative) bundles and recent results in representation theory.

2. Topological defects in quantum field theories

It has been recently realized in several different contexts that boundary conditions and defects of various codimensions constitute important parts of the structure of a quantum field theory. (Alternatively, one might say that it is instructive to consider the quantum field theory not only on smooth manifolds, but also on stratified spaces.) Codimension-1 defects, also called interfaces, separate regions that can support different quantum field theories. Such defects arise naturally in applications, ranging from condensed matter systems, where they appear as interfaces between different phases of matter, to domain walls in cosmology. The aspect we wish
to emphasize in this contribution is the fact that such defects provide a lot of structural insight into quantum field theories, including in particular their symmetry structures.

Similarly as in the case of boundary conditions, there are various different types of defects one can consider. In particular, one can impose various conservation conditions of physical quantities on defects. An important subclass of codimension-1 defects are topological defects. They are characterized by the property that the values of correlators for configurations with defects do not change when the location of a defect is only slightly changed, that is, changed by a homotopy without crossing any field insertions or other defects. Examples of such defects have been known for a long time (see e.g. [8]). Consider, for instance, the two-dimensional Ising model, defined by a $\mathbb{Z}_2$-valued variable on the vertices of a two-dimensional CW-complex. Select a line that crosses bonds transversally, and change the coupling on each bond that is crossed by the line from ferromagnetic to anti-ferromagnetic. In the continuum limit this provides a topological defect line in the critical Ising model.

2.1. Symmetries from invertible topological defects

A particularly important subclass of topological defects are the invertible topological defects. Owing to their topological nature, two topological defects can be brought to coincidence, leading to a fusion product of defects. The precise mathematical formulation of the relevant monoidal structure depends on the dimension in which the quantum field theory is defined. For the moment, let us consider two-dimensional quantum field theories. All statements made in the sequel have the status of theorems [12, 13] in the case of two-dimensional rational conformal field theories; the general picture should be much more widely applicable, though. For a two-dimensional rational conformal field theory, the topological line defects form a monoidal category, with morphisms provided by field insertions which can change the type of defect. In particular, there is an “invisible defect”, which is a monoidal unit $\mathbf{1}$ in the category of defects.

An invertible defect $D$ is characterized by the fact that there exists another defect $D^\vee$ with the property that the fusion of $D$ and $D^\vee$ gives the invisible defect,

$$D \otimes D^\vee \cong \mathbf{1} \cong D^\vee \otimes D.$$ 

(Thus in particular for every theory the invisible defect is invertible.) This behaviour leads to the relation $\dim(D)$

\[ \begin{array}{ccc}
D & \downarrow & D^\vee \\
\uparrow & \quad & \quad \\
\quad & \quad & \dim(D) \\
\end{array} \]

which is to be understood as an identity of correlators when applied locally in any configuration of fields and defect lines. With this relation, it is immediate to deduce a connection with symmetries: one has equalities of the form

1 Here $\dim(D)$ is the (quantum) dimension of the defect, which for an invertible defect can take only the values $+1$ or $-1$. This factor is suppressed in the sequel. In unitary theories one has $\dim(D) = 1$ for every invertible defect.
between correlators of different bulk fields.

Let us discuss this issue explicitly for the case of the two-dimensional critical Ising model. There are then three primary fields, commonly referred to as the identity field \(1\), the spin field \(\sigma\) and the energy field \(\epsilon\). Their scaling dimensions are \(0\), \(\frac{1}{16}\) and \(\frac{1}{2}\), respectively. Indecomposable topological line defects of the critical Ising model turn out to be in bijection with these fields: in particular, \(1\) corresponds to the invisible defect and \(\epsilon\) to an invertible defect. (It is in fact the latter defect that amounts to changing the couplings from ferromagnetic to antiferromagnetic.)

The action of this specific invertible defect on bulk fields is as follows: it leaves the bulk fields corresponding to \(1\) and to \(\epsilon\) invariant and changes the sign of \(\sigma\). Inside RCFT correlators for the critical Ising model we thus have the equalities

\[
\begin{align*}
\epsilon &= 1 \\
\epsilon &= \epsilon \\
\epsilon &= -\sigma
\end{align*}
\]

As can be seen from these pictures, there is a natural action of invertible topological line defects on field insertions: wrapping such a defect around a bulk field insertion yields another bulk field.

A further advantage of having a realization of symmetries in terms of invertible topological defects is seen by studying what happens when the defect is moved to a boundary with boundary condition \(M\): this process yields another boundary condition \(M'\), according to

\[
D \quad M \quad = \quad M'
\]

In a similar way, by wrapping around various structures in an obvious way, the group of invertible topological defects acts as a symmetry group on all data of the field theory, including field insertions of bulk, boundary and disorder fields, boundary conditions and types of defects. In the framework of the TFT construction of correlators of two-dimensional rational conformal field theories [25], isomorphism classes of invertible topological defects can be explicitly classified [12, 13]. In the example of the critical Ising model one finds a symmetry group \(\mathbb{Z}_2\), consisting of the invisible defect and the \(\epsilon\)-defect described above, while e.g. for the critical three-state Potts model one obtains a non-abelian symmetry group \(S_3\).

For the present purposes, it is important to realize that topological codimension-1 defects can also exist in more general classes of quantum field theories. Moreover, in the general case geometric considerations suggest a natural action of such defects on field theoretic structures, like boundary conditions and defects of various codimensions, as well. In short, defects can wrap around field theoretic structures.

2.2. T-dualities and Kramers-Wannier dualities from topological line defects

Before turning to higher-dimensional field theories, we point out that topological line defects are indeed of much wider use: they can also implement Kramers-Wannier dualities and T-dualities.
If a general topological defect wraps around a bulk field, the following situation is created:

\[
\phi D = \sum_{\text{intermediate defects } D_i} \phi D_i \phi
\]

In this way a bulk field \( \phi \) is turned by the defect \( D \) into a disorder field. To obtain an order-disorder duality, one also needs the opposite process, turning disorder fields into ordinary local bulk fields. It can be shown [12, 13] that to this end the dual defect \( D^\vee \) must be used, and that in this case one turns the disorder field back into a bulk field if and only if the fused defect \( D \otimes D^\vee \) is a direct sum of invertible defects. This condition can be examined in concrete models. It is in particular satisfied for the defect corresponding to the spin field \( \sigma \) in the critical Ising model, thanks to the well known fusion rule \( \sigma \otimes \sigma \cong 1 \oplus \epsilon \); this defect indeed produces the action of the Kramers-Wannier duality in the critical model. Again, we have a natural action on correlators,

\[
\sigma \sigma \sigma \sigma = \frac{1}{2} \sigma
\]

which shows how correlators involving only bulk fields are related to correlators involving disorder fields. Specifically, through the action of the \( \sigma \)-defect the correlator of four spin fields on a sphere can be seen to be equal to the correlator of four disorder fields, according to

\[
\sigma \sigma \sigma \sigma = \frac{1}{\sqrt{2}} \sigma
\]

and the correlator of two spin fields on a torus can be expressed as

\[
\frac{1}{2} + \frac{1}{2}
\]

T-duality of compactified free bosons can be understood as a special type of order-disorder duality, see Section 5.4 of [14]. Indeed, the chiral data of two T-dual full conformal field theories
can be described by the same theory, namely the one describing a $\mathbb{Z}_2$-orifold of the free boson theory. The line defect implementing the duality is then constructed from twist fields of this orifold theory.

2.3. Relative field theories

Before turning to defects in the specific class of three-dimensional Dijkgraaf-Witten theories, we briefly explain two applications of defects. The first is in the context of relative field theories. By definition, a $(d-1)$-dimensional relative field theory is a field theory on the boundary of a $d$-dimensional field theory or, more generally, on a codimension-1 defect. There are interesting cases in which the latter is a topological field theory; we mention two particular situations:

- The case that the $d$-dimensional topological field theory is even an invertible theory. This provides a geometric setup for describing anomalous theories; see [11].
- The case $d = 3$, with the three-dimensional topological field theory being of Reshetikhin-Turaev type and built from a modular tensor category that is the representation category of a vertex algebra. In this case the relative theory on a topological surface defect is a local rational conformal field theory whose underlying chiral theory is the one built from the vertex algebra. This provides a geometric interpretation [21] of the TFT construction [25] of RCFT correlators.

2.4. Quantum codes

Another application of surface defects in three-dimensional topological field theories are quantum codes. It is well known that the space of qubits of the toric code on a surface $\Sigma$ can be described by a topological field theory of Turaev-Viro type. Indeed, the vector space assigned by the topological field theory to the surface $\Sigma$ contains the information on a vector space together with an action of mapping class groups which is used to construct quantum gates. Two problems are apparent: First, the surfaces $\Sigma$ of concrete samples typically have a low genus, and thus, according to the Verlinde formula, the TFT state space $\text{tft}(\Sigma)$ has low dimension. Moreover, the representation of the mapping class group might be too small to allow for universal gates. To circumvent these problems the idea has been put forward [3, 2, 1] to consider samples with a bilayer (or multi-layer) system and twist defects that create branch cuts. They effectively lead to systems defined on surfaces of higher genus and thus to higher-dimensional quantum codes with richer representations of mapping class groups. Such systems have been analyzed mathematically [17] by exploiting results on permutation equivariant categories.

3. Defects and boundary conditions in three-dimensional topological field theories

We now turn our attention to a particularly accessible subclass of three-dimensional topological field theories, Dijkgraaf-Witten theories. These are topological field theories of Turaev-Viro type, and they can be constructed explicitly in a gauge-theoretic setting with finite gauge group. We treat them as 3-2-1-extended topological field theories.

3.1. Construction of extended Dijkgraaf-Witten theories from $G$-bundles

As a first input datum, we select a finite group $G$. The space of field configurations on a closed oriented compact smooth three-manifold $M$ is then taken to be the groupoid $\text{Bun}_G(M)$ of $G$-bundles on $M$. By considering formally a path integral with vanishing action, we obtain a partition function that evaluates to

$$\text{tft}_G(M) = \int_{\text{Bun}_G(M)} DA e^0 = |\text{Bun}_G(M)|.$$
Here we use the fact that the groupoid $\text{Bun}_G(M)$ is essentially finite, so that there is a well-defined counting measure, its groupoid cardinality. The groupoid cardinality of an (essentially) finite groupoid $\Gamma$ is

$$|\Gamma| := \sum_{\gamma \in \pi_0(\Gamma)} \frac{1}{|\text{Aut}_\Gamma(\gamma)|},$$

where the summation is over the set $\pi_0(\Gamma)$ of isomorphism classes of objects of $\Gamma$, each of which is counted with the inverse of the cardinality of its automorphism group.

Trivially, the number $\text{tft}_G(M) = |\text{Bun}_G(M)|$ defines an invariant of the three-manifold $M$. It is non-trivial, though, that this invariant is local in the following sense. Let us cut the closed three-manifold into two pieces, consisting of three-manifolds with boundaries which are closed oriented surfaces. Suppose that the topological field theory can associate meaningful quantities to such manifolds as well, which would allow us to reduce the problem of computing a three-manifold invariant to computing invariants of simpler manifolds. To see what the topological field theory $\text{tft}_G$ should associate to a closed oriented two-manifold $\Sigma$, we take a three-manifold $M$ with boundary $\Sigma$. For any $G$-bundle $P$ on $\Sigma$, consider the groupoid $\text{Bun}_G(M, P)$ of $G$-bundles on $M$ restricting to the $G$-bundle $P$ on the boundary $\partial M$. Then $\text{Bun}_G(M, P)$ is an essentially finite groupoid: its groupoid cardinality provides a function

$$\Psi_M : \text{Bun}_G(\Sigma) \to [\text{Bun}_G(M, P)]$$

that depends only on the isomorphism class of the bundle $P$. To the surface $\Sigma$ itself, we should associate the recipient for all these functions, the vector space $[\pi_0(\text{Bun}_G(\Sigma))]$ of complex-valued functions on the set $\pi_0(\text{Bun}_G(\Sigma))$ of isomorphism classes of $G$-bundles on $\Sigma$. We thus recover the well known feature that a three-dimensional topological field theory assigns vector spaces to surfaces. In the case of Dijkgraaf-Witten theories, these vector spaces are obtained by linearization of (isomorphism classes of objects of) categories of bundles.

Closed two-manifolds can, in turn, be decomposed, e.g. in a pair-of-pants decomposition, by cutting them along circles. Hence let $S$ be a closed oriented one-manifold and ask what an extended topological field theory should associate to $S$. Following the same type of analysis as above, we choose a surface $\Sigma$ with boundary $S$. Fixing a $G$-bundle $P$ on the boundary $\Sigma$, we assign to it a vector space

$$\Psi_S : P \mapsto [\pi_0(\text{Bun}_G(\Sigma, P))].$$

The so obtained map $\Psi_S : \text{Bun}_G(S) \to \text{Vect}$ is a vector bundle on the space of field configurations on $S$. Thus the extended Dijkgraaf-Witten theory based on $G$ should associate to $S$ the $\text{C}$-linear category of vector bundles over the space of field configurations. Denoting, for an essentially finite groupoid $\Gamma$, the functor category from $\Gamma$ to a category $\text{C}$ by $[\Gamma, \text{C}]$, this is the functor category

$$\text{tft}_G(S) = [\text{Bun}_G(S), \text{Vect}].$$

We thus learn that a 3-2-1-extended topological field theory assigns to a one-manifold a $\text{C}$-linear category. Moreover, the assignment consists of two steps: one first assigns to $S$ the groupoid $\text{Bun}_G(S)$, and in a second step to this groupoid its linearization $[\text{Bun}_G(S), \text{Vect}]$.

Hereby we naturally arrive at the general definition of 3-2-1-extended topological field theories as a symmetric monoidal 2-functor

$$\text{tft} : \text{cobord}_{3,2,1} \to \text{2-Vect( }\text{),}$$

which we discuss in the rest of this subsection. This definition naturally generalizes Atiyah’s classical definition of (non-extended) topological field theory. Here $\text{2-Vect( }\text{)}$ is the symmetric
monoidal bicategory of finitely semisimple \(-\)linear abelian categories, with the Deligne product as a monoidal structure. The bicategory \(\text{cobord}_{3,2,1}\) is an extended cobordism category whose objects are closed oriented one-manifolds \(S\), whose 1-morphisms \(S \to S'\) are two-manifolds \(\Sigma\) with boundary together with a decomposition \(\partial \Sigma \simeq S' \sqcup -S\), and whose 2-morphisms are three-manifolds with corners, up to diffeomorphism, again with an appropriate decomposition of the boundary.

According to the definition, the 2-functor \(\text{tft}\) assigns to the oriented circle \(S^1\) a finitely semisimple \(-\)linear category \(C := \text{tft}(S^1)\). The trinion (pair of pants), regarded as a cobordism \(S^1 \sqcup S^1 \to S^1\) gives a functor \(\otimes : C \boxtimes C \to C\). Finally, suitable three-manifolds with corners provide natural isomorphisms that endow this functor with an associativity constraint and the category \(C\) with the structure of a braiding. The constraints on these natural transformations, expressed by pentagon and hexagon diagrams, can be deduced from homotopies between three-manifolds with corners. In this way, the category \(C = \text{tft}(S^1)\) is expected to be endowed with the structure of a modular tensor category (see [4] for some recent progress). The Reshetikhin-Turaev construction can be seen as a converse, constructing a 3-2-1-extended topological field theory from a modular tensor category.

3.2. Extended three-dimensional topological field theories with boundaries and defects

Incorporating defects and boundaries into a topological field theory amounts to consider an enlarged bicategory \(\text{cobord}_{3,2,1}\) of cobordisms. Recently several definitions have been proposed for this enlarged category, see e.g. [22, Sect 4.3]; we do not present a formal definition, but rather formulate some of the physical requirements that such a category should satisfy. Again we start from oriented three-manifolds, but this time we also allow them to have oriented boundaries that are actual physical boundaries to which boundary conditions need to be assigned, rather than the cut-boundaries that were already present in Section 3.1, which are instead a device for incorporating some aspects of locality into the theory. We also admit codimension-1 submanifolds, called surface defects. Such a defect separates two three-dimensional regions each of which supports a topological field theory, and the theories associated to the two regions may be different. Labels must be assigned to such defects, too. In this way, we get in particular invariants of three-manifold with embedded surfaces. In the present setting of 3-2-1-extended topological field theories, it is natural to go one step further and allow for codimension-2 defects as well. In particular, there will be generalized Wilson lines separating two surface defects from each other, and likewise boundary Wilson lines that are confined to a boundary component.

In a next step one then imposes locality by cutting such a three-manifold to produce three-manifolds with cut-boundaries. It is natural to impose the condition that the cut-surface intersects all structure in the three-manifold, e.g. surface defects and boundaries, transversally. Boundary manifolds now carry a decoration consisting of lines labeled by surface defects and boundaries labeled by boundary conditions. Imposing further locality, as in Section 3.1, requires to cut such decorated surfaces transversally. This yields new types of labeled one-manifolds, which are to be taken as the objects in the bicategory \(\text{cobord}_{3,2,1}\). Let us describe these objects in detail: There are oriented intervals and circles, with finitely many marked points in the interior that divide them into subintervals. Each subinterval is to be marked with a topological field theory. Points adjacent to two subintervals are to be marked by a surface defect, boundary points by a boundary condition. The possible boundary conditions and types of surface defects have to be determined by a field-theoretic analysis. For the case of Dijkgraaf-Witten theories we present this analysis in the next subsection.

3.3. Defects and boundaries in Dijkgraaf-Witten theories

We now describe an explicit gauge-theoretic construction [19] of boundary conditions and surface defects for Dijkgraaf-Witten theories. A key idea is to keep the two-step procedure outlined in
Section 3.1: first we assign to manifolds of various dimension groupoids of (generalized) bundles, and then we linearize those categories.

Our ansatz is inspired by the notion of a relative bundle. Given a morphism \( j : Y \to X \) of smooth manifolds and a group homomorphism \( \iota : H \to G \), the category of relative bundles has as objects triples consisting of a \( G \)-bundle \( P_G \to X \), an \( H \)-bundle \( P_H \to Y \), and an isomorphism

\[
\alpha : \text{Ind}_H^G(P_H) \xrightarrow{\cong} j^* P_G
\]

of \( G \)-bundles on \( Y \). A morphism is a pair consisting of a morphism \( P_G \xrightarrow{\varphi_G} P_G' \) of \( G \)-bundles on \( X \) and a morphism \( P_H \xrightarrow{\varphi_H} P_H' \) on \( Y \) of \( H \)-bundles, obeying the obvious compatibility constraint

\[
\begin{align*}
\text{Ind}_H^G(P_H) & \xrightarrow{\alpha} j^* P_G \\
\text{Ind}_H^G(\varphi_H) & \xrightarrow{\text{Ind}_H^G(\alpha)} j^* \varphi_G \\
\text{Ind}_H^G(P_H') & \xrightarrow{\alpha'} j^* P_G'
\end{align*}
\]

with the isomorphisms \( \alpha \) and \( \alpha' \).

In our construction, relative bundles are the appropriate tool for describing boundaries. In the case of a surface defect \( \Sigma \to M \) the situation is somewhat more subtle. Suppose that \( \Sigma \) divides the three-manifold \( M \) into two disjoint connected components \( \Sigma_+ \) and \( \Sigma_- \). We consider the situation that Dijkgraaf-Witten theories for two, possibly different, finite groups \( G_\pm \) are associated to the components \( \Sigma_+ \) and \( \Sigma_- \). Denote by \( M_\pm := M_\pm \cup \Sigma \) the closure of \( M_\pm \) in \( M \), and by \( j_\pm : \Sigma \to M_\pm \) the corresponding embeddings. Then we consider, for a given group homomorphism \( H \to G_+ \times G_- \) as field configurations the groupoid whose objects consist of two \( G_\pm \)-bundles \( P_\pm \to M_\pm \), an \( H \)-bundle \( P_H \to \Sigma \) and an isomorphism

\[
\beta : \text{Ind}_H^G(P_H) \xrightarrow{\cong} j_+^* P_+ \times j_-^* P_-
\]

of \( G_+ \times G_- \)-bundles on \( \Sigma \).

We have to add yet one further datum to Dijkgraaf-Witten theories. Namely, typically, the linearization of groupoids is twisted by a 2-cocycle on the relevant groupoid of generalized bundles. This requires the choice of additional data on the finite groups involved. We first discuss the situation without boundaries and without surface defects, for a Dijkgraaf-Witten theory based on a finite group \( G \). Then the additional datum is a three-cocycle \( \omega \in Z^3(G, \times) \) on the group. A more geometrically inclined reader is invited to think about this cocycle as a three-cocycle on the stack \( \text{Bun}_G \) of \( G \)-bundles, and thus as a (Chern-Simons) 2-gerbe on \( \text{Bun}_G \). Such a three-cocycle leads to a holonomy on closed three-manifolds and thus furnishes a topological Lagrangian, yielding a three-dimensional topological field theory \( \text{tft}_{G, \omega} \).

To determine the category \( \text{tft}_{G, \omega}(S^4) \), we need a 2-cocycle \( \tau(\omega) \) on the groupoid \( \text{Bun}_G(S^4) \cong G//G \), the action groupoid for the adjoint action of \( G \) on itself. The transgressed 2-cocycle \( \tau(\omega) \) can be obtained [26] from the group cocycle \( \omega \) by evaluating \( \omega \) on a suitable triangulated three-manifold. This indeed produces the well-known 2-cocycle [6] for Dijkgraaf-Witten theories.

We now illustrate the generalization of this prescription to boundaries and defects in an example. Let \( \Sigma \) be an interval with a marked point in its interior, corresponding to a surface defect. To each of the two subintervals \( I_{1,2} \) we assign a finite group \( G_{1,2} \) and three-cocycles \( \omega_{1,2} \in Z^3(G_{1,2}, \times) \), of which we think of as topological bulk Lagrangians. To the end point adjacent to the first interval, we assign a group homomorphism \( \iota_1 : H_1 \to G_1 \) and a boundary Lagrangian as follows. We think about the three-cocycle \( \omega_1 \) on \( G_1 \) as a Chern-Simons 2-gerbe on \( \text{Bun}_{G_1} \). The group homomorphism \( \iota_1 \) induces a morphism \( \text{Bun}_{H_1} \to \text{Bun}_{G_1} \) of 2-gerbes. A
similar situation, one categorical dimension lower, is familiar from the study of D-branes, see e.g. [15]: for D-branes, one has a gerbe on the bulk manifold and a morphism of gerbes on the boundary, from the trivial gerbe to the restriction of the bulk gerbe. This is an instance of the general principle that in gauge theories, holonomies of manifolds with boundary, and thus topological actions on such manifolds, can only be defined if a trivialization on the boundary is chosen.

In the situation at hand, what is to be trivialized is the restriction of the Chern-Simons 2-gerbe to \( \text{Bun}_{H_1} \). Thus in terms of group cochains, we are looking for a 2-cochain \( \theta_1 \in C^2(H_1), \) that represents a morphism from the trivial 2-gerbe on \( \text{Bun}_{H_1} \) to the 2-gerbe described by the restriction of the three-cocycle \( \omega_1 \) on \( G_1 \) to \( \text{Bun}_{H_1} \). Accordingly we impose the condition \( d\theta_1 = \iota_1^*(\omega_1) \). We think about \( \theta_1 \) as a topological boundary Lagrangian. The situation at the other end point is, in complete analogy, described by a group homomorphism \( \iota_2 : H_2 \to G_2 \) and a 2-cochain \( \theta_2 \in C^2(H_2), \) such that \( d\theta_2 = \iota_2^*(\omega_2) \).

In a similar vein, for the situation of the surface defect is a higher-categorical analogue of bibranes [20], which describe the target space physics of topological defects: We now have a group homomorphism \( \iota_{12} : H_{12} \to G_1 \times G_2 \) and a 2-cochain \( \theta_{12} \in C^2(H_{12}), \) satisfying \( d\theta_{12} = \iota_{12}^*(\omega_2) \cdot \iota_1^*(\omega_1)^{-1} \), with the group homomorphisms \( \iota_i : H_{12} \to G_1 \times G_2, \) obtained from \( \iota_{12} \) and the canonical projections. Indecomposable boundary conditions and defects correspond to group homomorphism \( \iota_{1,2} \), respectively \( \iota_{12} \), that are injective, i.e. are subgroup embeddings.

In [19], an explicit prescription was given for transgressing these cochains to a 2-cocycle on the relevant category of (generalized) relative bundles. This prescription is a generalization of the one given in [26] for Dijkgraaf-Witten theories without boundaries or defects. It makes the categories associated to one-manifolds with defects and boundaries quite explicitly computable.

On the other hand, there are also general results for boundary conditions and defects in three-dimensional topological field theories of Reshetikhin-Turaev type. Suppose the theory is based on \((G,\omega)\), this condition is satisfied automatically: the category \( \text{Vect}(G) \) of \( G \)-graded vector spaces with associator twisted by \( \omega \).

The general analysis [18] then shows that boundary conditions are in bijection with module categories over the fusion category \( \mathcal{A} \). For the fusion category \( \text{Vect}(G) \) relevant for Dijkgraaf-Witten theories, indecomposable module categories are, in turn, known [24] to be in bijection with subgroups \( i : H \to G \) and 2-cochains \( \theta \in C^2(H,\omega) \) such that \( d\theta = i^*\omega \). This matches exactly the gauge-theoretic description of types of boundary conditions. According to the general results [18], the topological field theory should assign to an interval with Dijkgraaf-Witten theory based on \((G,\omega)\) and boundaries with boundary conditions \((H_1,\theta_1)\) and \((H_2,\theta_2)\) the category

\[
\text{Fun}_{\text{Vect}(G)}(\mathcal{M}(H_1,\theta_1), \mathcal{M}(H_2,\theta_2))
\]

of module functors from the indecomposable module category \( \mathcal{M}(H_1,\theta_1) \) over the fusion category \( \text{Vect}(G) \) to the indecomposable module category \( \mathcal{M}(H_2,\theta_2) \). A non-trivial calculation [19] shows that this category coincides, as a finitely semisimple abelian category, with the category obtained by linearizing categories of generalized relative bundles.

3.4. Symmetries from invertible topological surface defects
We now turn to the relation between symmetries of three-dimensional topological field theories of Turaev-Viro type and invertible surface defects. For such a theory, the modular tensor category \( \mathcal{C} \) of bulk Wilson lines is the Drinfeld center of a fusion category \( \mathcal{A} \), i.e. \( \mathcal{C} = \mathcal{Z}(\mathcal{A}) \).
According to the paradigm discussed in Section 1, symmetries should correspond to invertible topological surface defects, and their action should be described by wrapping those defects. By the analysis presented in the preceding subsection, invertible surface defects are described by invertible bimodule categories over the fusion category $\mathcal{A}$. These form a monoidal bicategory; restricting to only invertible 1-morphisms and 2-morphisms in this bicategory, one obtains a categorical 2-group, the Brauer-Picard 2-group [9]. Its isomorphism classes form a finite group $\text{BrPic}(\mathcal{A})$; in the sequel work with this group, rather than the underlying 2-group.

Applying an extended topological field theory to a two-manifold with boundary yields a functor. Take this two-manifold to be a cylinder with a surface defect of type $D$ wrapping the non-contractible cycle:

This yields an endofunctor $F_D: \mathcal{C} \to \mathcal{C}$. A detailed analysis [16] shows that if the defect $D$ is invertible, then the functor $F_D$ has the structure of a braided monoidal autoequivalence. In the case of Dijkgraaf-Witten theories, the modular tensor category of bulk Wilson lines is $\mathcal{C} = Z(\text{Vect}(G))$, and an invertible (and thus indecomposable) defect is given by a subgroup $\iota: H \hookrightarrow G \times G$ and a 2-cochain $\theta \in C^2(H, \times)$. The functor $F_D \equiv F_{(H,\theta)}$ can then be explicitly obtained by linearizing the span of action groupoids

for adjoint actions which describe the relevant categories of (relative) bundles. Here $\hat{\pi}_i$ is the functor on groupoids that is induced from the group homomorphism $\iota: H \to G \times G$. One can then compute a pull-push functor on the linearizations to arrive at the functor

$$F_{(H,\theta)}: Z(\text{Vect}(G)) = [G//G, \text{Vect}] \to [G//G, \text{Vect}],$$

which comes with a natural monoidal structure and which is braided. This construction provides an explicit group homomorphism

$$\text{BrPic}(\mathcal{A}) \to \text{brdEq}(Z(\text{Vect}(G)))$$

from the Brauer-Picard group to the group $\text{brdEq}(Z(\text{Vect}(G)))$ of braided autoequivalences of the Drinfeld center of $\text{Vect}(G)$. A detailed comparison [16] shows that this group homomorphism is exactly the one considered in [9]; this embeds the construction of that paper naturally into the framework of topological field theories with defects. It is also shown in [9] that the group homomorphism is actually an isomorphism. According to the considerations above, this representation theoretic result can be reinterpreted as the statement that symmetries of topological field theories of Turaev-Viro type can already be detected from their action on the category of bulk Wilson lines.
3.5. Symmetries for abelian Dijkgraaf-Witten theories

We now restrict our attention to the class of Dijkgraaf-Witten theories with vanishing three-cocycle and based on a finite abelian group $A$. In this case the Brauer-Picard group is explicitly known [9]:

$$\text{BrPic}(\text{Vect}(A)) \cong O_q(A \oplus A^*),$$

where $A^*$ is the character group of $A$, $q: A \oplus A^* \rightarrow \times$ is the natural quadratic form determined by $q(g, \chi) = \chi(g)$ for $g \in A$ and $\chi \in A^*$, and $O_q(A \oplus A^*)$ is the subgroup of those automorphisms of the finite group $A \oplus A^*$ that preserve the quadratic form $q$.

There are three obvious types of symmetries the Dijkgraaf-Witten theory should possess:

(i) Symmetries of $\text{Bun}_A$.

As explained in Section 3, we can think about the stack $\text{Bun}_A$ as a target space for Dijkgraaf-Witten theories. Accordingly, symmetries of $\text{Bun}_A$, i.e. group automorphisms of $A$, can be expected to be kinematical symmetries of the Dijkgraaf-Witten theory. Indeed, for $\varphi \in \text{Aut}(A)$ the graph $\text{graph}(\varphi) \subset A \oplus A$ is a subgroup; together with the trivial 2-cochain $\theta = 1 \in C^2(\text{graph}(\varphi), \times)$ it describes an invertible surface defect. Following the prescription given in Section 3.4, we compute the corresponding braided autoequivalence to be the element

$$\varphi \oplus (\varphi^*)^{-1} : A \oplus A^* \rightarrow A \oplus A^*$$

of $O_q(A \oplus A^*)$.

(ii) Automorphisms of the Chern-Simons 2-gerbe on $\text{Bun}_A$.

Recall that the three-cocycle $\omega \in C^3(A, \times)$ has the interpretation of a Chern-Simons 2-gerbe on $\text{Bun}_A$. In the case at hand this 2-gerbe is trivial. Nevertheless its automorphisms are not trivial; rather, they are given by 1-gerbes on $\text{Bun}_A$. Their isomorphism classes are given by the cohomology group $H^2(A, \times)$. By transgression [26], any cohomology class gives an alternating bihomomorphism $\beta: A \times A \rightarrow \times$, which in physical terms can be interpreted as a “B-field”. (Indeed, for a finite abelian group, transgression provides a group isomorphism between the cohomology group $H^2(A, \times)$ and the group of alternating bihomomorphisms.)

Again we can explicitly identify an invertible topological surface defect: the one given by the diagonal subgroup $A_{\text{diag}} \subset A \oplus A$ together with a representative of a cohomology class in $H^2(A, \times)$, which we identify with the associated alternating bihomomorphism $\beta$. The corresponding braided equivalence can be computed to act as

$$A \oplus A \rightarrow A \oplus A$$

$$(g, \chi) \mapsto (g, \chi + \beta(g, -)).$$

Thus this type of automorphism shifts the character by the contraction of the B-field $\beta$ with the group element.

(iii) Electric-magnetic dualities.

It is finally expected that electric-magnetic dualities form a class of symmetries of Dijkgraaf-Witten theories (see e.g. [5] for a discussion of such symmetries for Turaev-Viro theories based on finite-dimensional semisimple Hopf algebras).

To present a simple example, assume that $A$ is a cyclic group and fix a group isomorphism $\delta: A \rightarrow A^*$. Then there is a natural braided equivalence

$$A \oplus A^* \rightarrow A \oplus A^*$$

$$(g, \chi) \mapsto (\delta^{-1}(\chi), \delta(g)).$$
This can again be described by an invertible surface defect, namely the one given by the diagonal subgroup $A_{\text{diag}} \subset A \oplus A$ and the alternating bihomomorphism $\beta$ defined by

$$
\beta(g_1, g_2) := \frac{\delta(g_1)(g_2)}{\delta(g_2)(g_1)} \text{ for } g_1, g_2 \in A.
$$

A careful investigation [16] of the structure of the finite group $O_q(A \oplus A^*)$ shows that in fact the three types of symmetries listed above generate this group, i.e. they already provide all symmetries of the theory.

4. Conclusions
We conclude by briefly summarizing the crucial insights presented in this contribution:

- Topological defects are important structures in quantum field theories.
- By general principles they describe symmetries and dualities of quantum field theories in such a way that the action of these symmetries and dualities on all different structures of a quantum field theory becomes apparent. This includes the action on field insertions as well as the action on boundary conditions and defects of various codimensions.
- Topological codimension-1 defects and boundaries provide a natural setting for studying relative field theories. This includes in particular anomalous field theories and the TFT construction of RCFT correlators.
- Applications of topological defects virtually concern all fields of physics to which quantum field theory is applied.

There are several classes of quantum field theories in which topological defects can be treated quite explicitly. Besides two-dimensional rational conformal field theories, these include three-dimensional topological field theories of Reshetikhin-Turaev type. We have demonstrated, for the subclass of Dijkgraaf-Witten theories, how defects and boundary conditions can be formulated in a gauge-theoretic setting using (generalizations of) relative bundles. This provides a natural embedding of structures of categorified representation theory, in particular of the theory of module and bimodule categories over monoidal categories, into the more comprehensive setting of (extended) topological field theory.

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