IMPURITY-BOUND EXCITONS IN ONE AND TWO DIMENSIONS

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ABSTRACT. We study three-body Schrödinger operators in one and two dimensions modelling an exciton interacting with a charged impurity. We consider certain classes of multiplicative interaction potentials proposed in the physics literature. We show that if the impurity charge is larger than some critical value, then three-body bound states cannot exist. Our spectral results are confirmed by variational numerical computations based on projecting on a finite dimensional subspace generated by a Gaussian basis.

1. Introduction and main results

Three-body complexes, in which one particle is oppositely charged from the other two, play an important role in solid-state physics. Such complexes are typically encountered when excitons (i.e. two-body complexes consisting of a negative electron and a positive hole) interact with a third charge. If the third particle is an additional mobile electron or hole, charged excitons (trions) may form. Alternatively, excitons interacting with an immobile charged impurity may lead to impurity-bound excitons [6, 2]. The latter can be modelled as a light electron-hole pair interacting with an infinitely heavy impurity charge $\kappa$.

In two previous papers we studied in detail one-dimensional impurity-bound excitons where the interactions were modelled by contact potentials [3], as well as trions [9]. In the current manuscript, we extend the analysis to the physically more relevant case of two-dimensional atomically thin semiconductors, in which impurity-bound excitons are frequently observed. We consider interactions given by multiplicative potential operators of the Keldysh form [5, 1, 13], both in one and two dimensions. Hence, in this paper we study the spectral properties of the operators

$$H_{\kappa,\lambda}(V) = -\Delta - \kappa V(x) + \lambda V(y) - V(x-y) \quad \text{in} \quad L^2(\mathbb{R}^{2d}), \quad d = 1, 2,$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a potential function and $\kappa, \lambda$ are positive coupling constants. In the sequel we will adopt the notation $$H_{\kappa}(V) := H_{\kappa,0}(V).$$

The operator $H_{\kappa}(V)$ describes an impurity of infinite mass interacting with an exciton. The impurity charge $\kappa$ controls how the impurity interacts with the electron and the hole. Our main interest here is to show that if $\kappa$ is larger than some critical value, then generically, $H_{\kappa}(V)$ does not have "three-body" bound states. Also, we numerically analyze the asymptotic behavior for $|\kappa| \ll 1$ and demonstrate important differences with respect to the contact potential model. Our analytical findings are supported by numerical results for both critical and asymptotic limits.

Very roughly said, the generic situation is the following: if $\kappa > 0$ is small enough, then one expects at least one discrete eigenvalue (even infinitely many for the class of $2d$ potentials we consider), while when $\kappa$ is larger than some critical value, no discrete eigenvalues can exist.

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In the 1d case we show in Theorem 1.1 and Corollary 1.2 that if the interaction potential is even, localized, smooth enough and with a non-degenerate maximum at zero, then the above "generic" case applies. Nevertheless, in Proposition 1.3 we construct a flat-well potential which has at least one bound state for all \( \kappa > 1 \).

In the 2d case we only consider the "physical" potential proposed by [1] (see also (1.3)), which has a logarithmic divergence near the origin and goes like \(-1/|x|\) at infinity. For this particular potential we show in Theorem 1.4 that \( H_{\kappa}(V) \) has infinitely many discrete eigenvalues for a certain interval of variation for \( \kappa \), but no eigenvalues at all for large enough \( \kappa \).

1.1. **Notation.** Given a set \( M \) and two functions \( f_1, f_2 : M \to \mathbb{R} \), we write \( f_1(m) \lesssim f_2(m) \) if there exists a numerical constant \( c \) such that \( f_1(m) \leq c f_2(m) \) for all \( m \in M \). The symbol \( f_1(m) \gtrsim f_2(m) \) is defined analogously. Moreover, we use the notation

\[
\left\langle f, g \right\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x) : g(x) \, dx
\]

the scalar product of \( f \) and \( g \). Since our operators are real, we will often work with real functions, only, especially when we perform variational arguments. Given a self-adjoint operator \( A \) on a Hilbert space \( \mathcal{H} \) we will denote by \( N(A, \tau)_{\mathcal{H}} \) the number of eigenvalues of \( A \) less than \( \tau \) counted with their multiplicities.

Now we formulate our main results.

1.2. **The one-dimensional case.** Let \( v : \mathbb{R} \to \mathbb{R} \) be a potential function satisfying the following

**Assumption 1.** We have

1. \( v \in C^3_0(\mathbb{R}) \) and \( v(x) \geq 0 \) for all \( x \in \mathbb{R} \).
2. \( v(x) = v(-x) \) for all \( x \in \mathbb{R} \).
3. \( v(x) < v(0) \) for all \( x \neq 0 \) and \( v''(0) < 0 \).

Given such a \( v \), it is easily seen that the operator \( H_{\kappa,\lambda}(v) \) in \( L^2(\mathbb{R}^2) \) is associated with the closed quadratic form

\[
q_{\kappa,\lambda}[u] = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \, dy - \kappa \int_{\mathbb{R}^2} u^2(x, y) v(x) \, dx \, dy + \lambda \int_{\mathbb{R}^2} u^2(x, y) v(y) \, dx \, dy
- \int_{\mathbb{R}^2} u^2(x, y) v(x-y) \, dx \, dy,
\]

with \( d(q) = H^1(\mathbb{R}^2) \). Let \( q_{\kappa,\kappa}[u] = q_{\kappa}[u] \).

**Theorem 1.1.** Let assumption 1  be satisfied. Then to any \( \lambda > 1 \) there exists \( \kappa_c(\lambda) > 0 \) such that

\[
\kappa \geq \kappa_c(\lambda) \quad \Rightarrow \quad \sigma_d(H_{\kappa,\lambda}(v)) = \emptyset.
\]  

(1.2)

**Corollary 1.2.** There exists \( \kappa_c \) such that the discrete spectrum of \( H_{\kappa}(v) \) is empty for all \( \kappa \geq \kappa_c \).

The following example indicates that the non-flatness condition in Assumption 1 cannot be omitted.
Proposition 1.3. Let \( w : \mathbb{R} \to \mathbb{R} \) be given by
\[
w(x) = \begin{cases} 
  w_0 & \text{for } |x| \leq 1, \\
  0 & \text{for } |x| > 1.
\end{cases}
\]

Then there exists \( w_c \in (0, \infty) \) such that for \( w_0 \geq w_c \) the operator
\[
H_\kappa(w) = -\partial^2_x - \partial^2_y - \kappa w(x) + \kappa w(y) - w(x - y)
\]
in \( L^2(\mathbb{R}^2) \) has at least one discrete eigenvalue for all \( \kappa > 1 \).

1.3. The two-dimensional case. It was shown in [1] that the Coulomb potential energy created by a point charge at the origin that electrons feel in a two-dimensional layer is well approximated by the function
\[
V_{cte}(x) = \frac{1}{r_0} \log \left( \frac{|x|}{|x| + r_0} - w(|x|) \right), \quad x \in \mathbb{R}^2,
\]
where \( r_0 > 0 \) is a constant and \( w : \mathbb{R}_+ \to \mathbb{R} \) satisfies Assumption 2.

Assumption 2. We have
(1) \( w \in C^2(\mathbb{R}_+) \) and for all \( r \in \mathbb{R}_+ \) it holds
\[
0 \leq w(r) \leq w(0).
\]
(2) \( w(r) = O(e^{-r}) \) as \( r \to \infty \).

Without loss of generality in what follows we will put
\[
r_0 = 1.
\]

Let us consider the operator
\[
\mathcal{H}_\kappa = -\Delta_x - \Delta_y + \kappa V_{cte}(x) - \kappa V_{cte}(y) + V_{cte}(|x - y|), \quad x, y \in \mathbb{R}^2
\]
in \( L^2(\mathbb{R}^4) \).

Theorem 1.4. The discrete spectrum of \( \mathcal{H}_\kappa \)
(i) is empty for \( \kappa \) larger than some critical value,
(ii) but contains infinitely many eigenvalues for certain values of \( \kappa \in (1/2, 1) \).

1.4. Numerical results. To illustrate and support the exact finding of the present work, we now analyze numerically a concrete model of impurity-bound excitons in \( d = 2 \). To this end, we apply the full Keldysh potential
\[
V(x) = \frac{\pi}{2r_0} \left\{ H_0 \left( \frac{|x|}{r_0} \right) - Y_0 \left( \frac{|x|}{r_0} \right) \right\},
\]
where \( H_0 \) and \( Y_0 \) are Struve and Bessel functions, respectively, and we take \( r_0 = 20 \). We expand eigenstates in a Gaussian basis \( \psi_{nmp} = \exp(-\alpha_n x^2 - \beta_m y^2 - \gamma_p (x - y)^2) \) with exponents between 0 and 7. A total of 320 basis functions are used in the expansion. In the unperturbed case, \( \kappa = 0 \), an exciton binding energy \( \Lambda_0(V) \sim -0.0529 \) is found (see (2.4) for the definition of \( \Lambda_0(V) \)). This value also gives the lower bound of the essential spectrum for small \( \kappa \). In Fig. 1a, the continuum is illustrated by the hatched area. As \( \kappa \) is increased above \( k_e \sim 0.844 \), the bottom of the essential spectrum is given by the two-body electron-impurity complex instead (see also (2.4) for the definition of \( \Lambda_1(\kappa, V) \)).
In Proposition 2.1 we show that for the more general class of potentials we consider, the value of $k_e$ always lies in the interval $(1/2, 1)$.

As illustrated by the colored lines in Fig. 1a, discrete eigenstates exist when $0 < \kappa < \kappa_c \approx 1.029$. Only a single discrete eigenvalue (marked by the blue line) exists in the entire range $0 < \kappa < \kappa_c$ with the others only emerging above a certain lower critical value $\tilde{\kappa}_c$, e.g. $\tilde{\kappa}_c \approx 0.815$ for the second eigenvalue (shown in green).

It is particularly interesting to investigate the $\kappa$-dependence of the fundamental discrete eigenvalue shown in blue in Fig. 1a. Hence, in Fig. 1b, we have shown the difference between this state and the bottom of the continuum. It is immediately clear from the plot that this energy difference has a very weak $\kappa$-dependence in the asymptotic limit $\kappa \to 0$. In the figure, we have fitted the numerical behavior to the analytical form $\Delta E = A \exp(-a\kappa^2)$. A rather satisfactory fit is observed for $\kappa \lesssim 0.25$.

The rigorous analysis of the small $\kappa$ behaviour will be done elsewhere, but let us give a hand-waving argument for why one should expect a binding energy which goes like $\exp(-a\kappa^{-2})$. The explanation...
is that our operator is somehow similar with a one-body 2d-Schrödinger operator with a potential $\kappa W$ where $\int_{\mathbb{R}^2} W(x)dx = 0$. Thus the perturbation is effectively of order $\kappa^2$. Up to a Birman-Schwinger argument, and knowing that the resolvent of the free Laplacian in 2d has a logarithmic threshold behavior, one expects to have a bound state $\lambda < 0$, $|\lambda| \ll 1$, which obeys an estimate of the form $\log(-\lambda)\kappa^2 \sim -1$, \cite{[10]}

1.5. The structure of the paper. After the Introduction, in Section 2 we identify the essential spectrum of this class of operators, a result which is valid for both dimensions. In Section 3 we treat the one-dimensional case, while in Section 4 we deal with 2d. We end with an Appendix.

2. Preliminaries

2.1. The essential spectrum.

**Proposition 2.1.** Let $V \geq 0$ be non-zero and assume that $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p = 1$ if $d = 1$, and $p > 1$ if $d = 2$. Then there exists $k_e \in (\frac{1}{2}, 1)$ such that

$$\sigma_{es}(H_{\kappa, \lambda}(V)) = [\Lambda(\kappa, V), \infty), \quad \forall \lambda > 0,$$

(2.1)

where

$$\Lambda(\kappa, V) = \begin{cases} \inf \sigma(-\Delta - V(x - y)) & \text{if } \kappa < k_e, \\ \inf \sigma(-\Delta - \kappa V(x)) & \text{if } \kappa \geq k_e. \end{cases}$$

(2.2)

**Proof.** Let

$$\Lambda_0(V) = \inf \sigma(-\Delta - V(x - y)), \quad \text{and} \quad \Lambda_1(\kappa, V) = \inf \sigma(-\Delta - \kappa V(x)).$$

Since $V \geq 0$, the HVZ-theorem (see e.g. \cite{[11]}) implies that (2.1) holds true with

$$\Lambda(\kappa, V) = \min \{\Lambda_0(V), \Lambda_1(\kappa, V)\}.$$

(2.3)

By introducing the new variables $s = x - y$ and $t = \frac{x + y}{2}$ we find that $-\Delta - V(x - y)$ is unitarily equivalent to the operator $-2\Delta_s - \frac{1}{2} \Delta_t - V(s)$ in $L^2(\mathbb{R}^{2d})$. Hence if $\kappa > 0$ we have

$$\Lambda_0(V) = \inf \sigma(-2\Delta_s - V(s)) < 0, \quad \Lambda_1(\kappa, V) = \inf \sigma(-\Delta_s - \kappa V(x)) < 0,$$

(2.4)

where the strict inequalities follow from the fact that $V \geq 0$, $V \neq 0$ and $d \leq 2$. On the other hand, $\Lambda_1(0, V) = 0$ and standard spectral theory arguments show that $\Lambda_1(\cdot, V)$ is a continuously decreasing function of $\kappa$ which obeys $\Lambda_1(\kappa, V) \to -\infty$ as $\kappa \to \infty$. This implies that there exists a unique $k_e > 0$ for which $\Lambda_0(V) = \Lambda_1(k_e, V)$. Now if $\kappa \geq 1$ we have the inequalities

$$-2\Delta_s - V(s) > -\Delta_s - V(s) \geq -\Delta_s - \kappa V(x),$$

thus in view of equations (2.3) and (2.4) we conclude that $k_e < 1$. Also, if $0 < \kappa \leq 1/2$ we have

$$0 > \Lambda_1(\kappa, V) \geq \Lambda_1(2^{-1}, V) = 2^{-1} \inf \sigma(-2\Delta - V) = 2^{-1} \Lambda_0(V) > \Lambda_0(V),$$

which shows that $k_e > 1/2$. \hfill $\square$

**Remark 2.2.** The exact value of $k_e$ depends on the profile of $V$. If $d = 1$ and $V$ is replaced by a Dirac delta quadratic form, then $k_e = \frac{1}{\sqrt{2}}$, see \cite{[3]}.
3. Proofs in the one-dimensional case

3.1. Auxiliary results. To prove Theorem 1.1 we will need several auxiliary results. Let us introduce a scaling function $U_\kappa : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$(U_\kappa f)(x, y) = \kappa^{\frac{1}{2}} f(\kappa^{\frac{1}{2}}x, \kappa^{\frac{1}{2}}y).$$ \hfill (3.1)

Then $U_\kappa$ maps $L^2(\mathbb{R}^2)$ unitarily onto itself. We define the operator

$$\mathcal{H}_{\kappa, \lambda} = \frac{1}{\sqrt{\kappa}} U_\kappa^* \mathcal{H}_{\kappa, \lambda}(v) U_\kappa$$ \hfill (3.2)

in $L^2(\mathbb{R}^2)$. Obviously

$$\sigma_d(\mathcal{H}_{\kappa, \lambda}(v)) = \emptyset \iff \sigma_d(\mathcal{H}_{\kappa, \lambda}) = \emptyset. \hfill (3.3)$$

Next we define the operator

$$h_\kappa = -\partial_x^2 - \sqrt{\kappa} v(\kappa^{-\frac{1}{2}} x)$$ \hfill (3.4)

in $L^2(\mathbb{R})$. Clearly

$$\sigma_{es}(h_\kappa) = [0, \infty) \quad \forall \kappa > 0,$$ \hfill (3.5)

and a simple calculation shows that

$$\mathcal{H}_{\kappa, \lambda} = h_\kappa - \partial_y^2 + \frac{\lambda}{\sqrt{\kappa}} v(\kappa^{-\frac{1}{2}} y) - \frac{1}{\sqrt{\kappa}} v \left(\kappa^{-\frac{1}{2}} (x - y)\right).$$ \hfill (3.6)

Similarly as above we use the notation

$$\mathcal{H}_{\kappa, n} = \mathcal{H}_\kappa.$$

Now we turn our attention to the case of large $\kappa$. Let

$$\omega = \sqrt{\frac{-v''(0)}{2}}. \hfill (3.7)$$

We have

**Lemma 3.1.** Let $E_1(\kappa)$ and $E_2(\kappa) > E_1(\kappa)$ be the two lowest eigenvalues of $h_\kappa$. Then

$$\lim_{\kappa \to \infty} (E_j(\kappa) + \sqrt{\kappa} v(0)) = (2j - 1) \omega, \quad j = 1, 2. \hfill (3.8)$$

**Proof.** Let $\chi : \mathbb{R} \to [0, 1]$ and $\tilde{\chi} : \mathbb{R} \to [0, 1]$ be two $C^\infty_0$ functions such that

$$\chi(x) = 1 \quad \forall x \in [-1, 1], \quad \chi(x) = 0 \quad \forall x : |x| > 2,$$

and

$$\tilde{\chi}(x) = 1 \quad \forall x \in [-3, 3], \quad \tilde{\chi}(x) = 0 \quad \forall x : |x| > 4.$$ \hfill (3.10)

Take $0 < \varepsilon < \frac{1}{16}$ and define

$$S_\kappa = \tilde{\chi}(\kappa^{-\varepsilon} x) \left(-\partial_x^2 + \omega^2 x^2 - i\right)^{-1} + (1 - \tilde{\chi}(\kappa^{-\varepsilon} x)) \left(-\partial_x^2 + \Delta_\kappa(x) - i\right)^{-1},$$ \hfill (3.9)

where

$$\Delta_\kappa(x) = \sqrt{\kappa} (v(0) - v(\kappa^{-\frac{1}{2}} x)) + \kappa^{2\varepsilon} \chi(\kappa^{-\varepsilon} x).$$

Let

$$C_\kappa := [h_\kappa, \tilde{\chi}(\kappa^{-\varepsilon} x)] = -\kappa^{-2\varepsilon} \tilde{\chi}''(\kappa^{-\varepsilon} x) - 2\kappa^{-\varepsilon} \tilde{\chi}'(\kappa^{-\varepsilon} x) \partial_x.$$ \hfill (3.10)

Using the fact that

$$\left(1 - \tilde{\chi}\left(\frac{x}{\kappa^{\varepsilon}}\right)\right) \chi(\kappa^{-\varepsilon} x) = 0 \quad \forall x \in \mathbb{R},$$

we have

$$\mathcal{H}_\kappa S_\kappa = C_\kappa S_\kappa.$$
we obtain
\[
(h_\kappa + \sqrt{\kappa} v(0) - i) S_\kappa = \tilde{\chi}\left(\frac{x}{R^\kappa}\right) (h_\kappa + \sqrt{\kappa} v(0) - i) (-\partial_x^2 + \omega^2 x^2 - i)^{-1} \\
+ \left(1 - \tilde{\chi}\left(\frac{x}{R^\kappa}\right)\right) (h_\kappa + \sqrt{\kappa} v(0) - i) (-\partial_x^2 + \Delta_\kappa(x) - i)^{-1} \\
+ C_\kappa \left(-\partial_x^2 + \omega^2 x^2 - i\right)^{-1} + (-\partial_x^2 + \Delta_\kappa(x) - i)^{-1}
\]
\[
= \tilde{\chi}\left(\frac{x}{R^\kappa}\right) (h_\kappa + \sqrt{\kappa} v(0) - i) (-\partial_x^2 + \omega^2 x^2 - i)^{-1} + \left(1 - \tilde{\chi}\left(\frac{x}{R^\kappa}\right)\right) \\
+ C_\kappa \left(-\partial_x^2 + \omega^2 x^2 - i\right)^{-1} + (-\partial_x^2 + \Delta_\kappa(x) - i)^{-1}.
\]
(3.11)

From Taylor’s formula with remainder applied to \(v\), given any \(x \in \mathbb{R}\), one can find \(t_x \in \mathbb{R}\) such that
\[
v(0) - v(\kappa^{-\frac{1}{2}} x) = \omega^2 x^2 \kappa^{-\frac{1}{2}} + \frac{v''(t_x)}{6} x^3 \kappa^{-\frac{3}{4}}.
\]
(3.12)

This together with the definition of \(\chi\) implies that there exists \(c_1 > 0\), independent of \(\kappa\), such that
\[
\Delta_\kappa(x) \geq c_1 \kappa^{2\varepsilon}
\]
(3.13)
holds for all \(x \in \mathbb{R}\) and all \(\kappa \geq 1\), see Lemma [A.1]. Hence in view of (3.10)
\[
\| C_\kappa \left(-\partial_x^2 + \omega^2 x^2 - i\right)^{-1} + (-\partial_x^2 + \Delta_\kappa(x) - i)^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq c_2 \kappa^{-\varepsilon}
\]
holds for all \(\kappa \geq 1\) and some \(c_2 > 0\) independent of \(\kappa\). To control the remaining term in (3.11), we use again the expansion (3.12) and note that
\[
h_\kappa + \sqrt{\kappa} v(0) = -\partial_x^2 + \omega^2 x^2 + \frac{v''(t_x)}{6} x^3 \kappa^{-\frac{3}{4}},
\]
which implies
\[
\tilde{\chi}\left(\frac{x}{R^\kappa}\right) (h_\kappa + \sqrt{\kappa} v(0) - i) (-\partial_x^2 + \omega^2 x^2 - i)^{-1} = \tilde{\chi}\left(\frac{x}{R^\kappa}\right) + T_{\kappa,1}
\]
where
\[
\| T_{\kappa,1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq c_3 \kappa^{-\varepsilon}, \quad \kappa \geq 1,
\]
and \(c_3 > 0\) is a constant independent of \(\kappa\). Putting the above estimates together we conclude that
\[
(h_\kappa + \sqrt{\kappa} v(0) - i) S_\kappa = 1 + T_\kappa,
\]
(3.14)
with \(T_\kappa\) satisfying the estimate
\[
\| T_\kappa \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq (c_2 + c_3) \kappa^{-\varepsilon} \quad \forall \kappa \geq 1.
\]
(3.15)

Hence for \(\kappa\) large enough the operator \(1 + T_\kappa\) is invertible and the Neumann series for \((1 + T_\kappa)^{-1}\) shows that
\[
\lim_{\kappa \to \infty} \| (h_\kappa + \sqrt{\kappa} v(0) - i)^{-1} - S_\kappa \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = 0.
\]
(3.16)

On the other hand, since the multiplication operator \(\tilde{\chi}(\kappa^{-\varepsilon} x) - 1\) converges strongly to zero in \(L^2(\mathbb{R})\) as \(\kappa \to \infty\) and \((-\partial_x^2 + \omega^2 x^2 - i)^{-1}\) is compact, it follows that
\[
\| \tilde{\chi}(\kappa^{-\varepsilon} x) (-\partial_x^2 + \omega^2 x^2 - i)^{-1} - (-\partial_x^2 + \omega^2 x^2 - i)^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \to 0
\]
as \(\kappa \to \infty\). Hence in view of (3.9) and (3.13)
\[
\lim_{\kappa \to \infty} \| S_\kappa - (-\partial_x^2 + \omega^2 x^2 - i)^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = 0.
\]
(3.17)

This in combination with (3.16) implies that \(h_\kappa + \sqrt{\kappa} v(0)\) converges in the norm resolvent sense to \(-\partial_x^2 + \omega^2 x^2\) and the claim follows. □
Lemma 3.2. Let $\psi_\kappa$ be the positive eigenfunction of $h_\kappa$ associated to the eigenvalue $E_1(\kappa)$ and normalized such that $\|\psi_\kappa\| = 1$ for all $\kappa > 0$. Then there exist $\alpha > 0$ and $\kappa_2 \geq 1$ such that

$$\sup_{\kappa \geq \kappa_2} \int_{\mathbb{R}} e^{2\alpha\sqrt{1+x^2}} |\psi_\kappa(x)|^2 \, dx < \infty. \quad (3.18)$$

Proof. For any $z \in \mathbb{C}$ and $f \in C_0^\infty(\mathbb{R})$ we have

$$e^{\alpha\sqrt{1+(\cdot)^2}} (h_\kappa - z) e^{-\alpha\sqrt{1+(\cdot)^2}} f = (h_\kappa - z + W_\alpha) f,$$

where

$$W_\alpha := \frac{2\alpha x}{\sqrt{1+x^2}} \frac{\partial_x}{\partial_x} + \frac{\alpha}{(1+x^2)^{3/2}} - \frac{\alpha^2x^2}{1+x^2}. \quad (3.19)$$

This shows that for any $u \in D(h_\kappa) = H^2(\mathbb{R})$ and every $\alpha \in (0,1)$

$$\|W_\alpha u\|^2 \leq 4\alpha^2 \|\partial_x u\|^2 + 4\alpha^2 \|u\|^2. \quad (3.20)$$

In particular, this shows that $h_\kappa + W_\alpha$ is closed on the domain of $h_\kappa$. Next we define the curve

$$\Gamma := \{ z \in \mathbb{C} : |z + \sqrt{\kappa} v(0) - \omega| = \omega \}. \quad (3.21)$$

By Lemma 3.1 there exists $\kappa_\omega$ such that

$$\sup_{z \in \Gamma} \sup_{\kappa \geq \kappa_\omega} \|(h_\kappa - z)^{-1}\| < \infty. \quad (3.22)$$

On the other hand, since $h_\kappa + \sqrt{\kappa} v(0) \geq -\partial_x^2$ in the sense of quadratic forms, for any $z \in \Gamma$ we have

$$\left| \left\langle (h_\kappa - z)u, u \right\rangle_{L^2(\mathbb{R})} \right| \geq \Re \left\langle (h_\kappa - z)u, u \right\rangle_{L^2(\mathbb{R})} \geq \|\partial_x u\|^2 - 2\omega \|u\|^2.$$

This in combination with (3.20) implies that

$$\|W_\alpha u\|^2 \leq 4\alpha^2 (1+2\omega) \|u\|^2 + 4\alpha^2 \left| \left\langle (h_\kappa - z)u, u \right\rangle_{L^2(\mathbb{R})} \right| \leq 4\alpha^2 (1+2\omega) \|u\|^2 + 4\alpha^2 \|(h_\kappa - z)u\| \|u\| \leq 8\alpha^2 (1+\omega) \|u\|^2 + 2\alpha^2 \|(h_\kappa - z)u\|^2$$

for all $z \in \Gamma$. Hence by [4] Thm. IV.1.16] there exists $\alpha \in (0,1)$ small enough such that the operator $h_\kappa + W_\alpha - z$ is invertible for all $z \in \Gamma$ and all $\kappa \geq \kappa_\omega$, with a bounded inverse. Then one can prove the identity:

$$(h_\kappa - z)^{-1} e^{-\alpha\sqrt{1+(\cdot)^2}} = e^{-\alpha\sqrt{1+(\cdot)^2}} (h_\kappa + W_\alpha - z)^{-1},$$

which shows that

$$\sup_{z \in \Gamma} \sup_{\kappa \geq \kappa_\omega} \left\| e^{\alpha\sqrt{1+(\cdot)^2}} (h_\kappa - z)^{-1} e^{-\alpha\sqrt{1+(\cdot)^2}} \right\| < \infty. \quad (3.23)$$

Now denote by

$$P_\kappa = \psi_\kappa \langle \cdot, \psi_\kappa \rangle_{L^2(\mathbb{R})} \quad (3.24)$$

the projection on the eigenspace of $h_\kappa$ associated to $E_1(\kappa)$. Then by Lemma 3.1 and equation (3.23)

$$\sup_{\kappa \geq \kappa_\omega} \left\| e^{\alpha\sqrt{1+(\cdot)^2}} P_\kappa e^{-\alpha\sqrt{1+(\cdot)^2}} \right\| \leq \sup_{\kappa \geq \kappa_\omega} \frac{1}{2\pi} \int_{\Gamma} \left\| e^{\alpha\sqrt{1+(\cdot)^2}} (h_\kappa - z)^{-1} e^{-\alpha\sqrt{1+(\cdot)^2}} \right\| \, dz \leq \infty. \quad (3.25)$$

To continue we denote by $\phi_1$ the normalized ground state of the harmonic oscillator $-\partial_x^2 + \omega^2 x^2$. Let $\chi_R$ be the characteristic function of the interval $[-R, R]$. Since $\|\phi_1\|_{L^2(\mathbb{R})} = 1$, there exists $R$ large enough
such that \( \langle \phi_1, \chi_R \phi_1 \rangle_{L^2(\mathbb{R})} \geq 3/4 \). On the other hand, from the proof of Lemma 3.1, see equations (3.16) and (3.17), it follows that \( \psi_\kappa \) converges strongly to \( \phi_1 \) in \( L^2(\mathbb{R}) \) as \( \kappa \to \infty \). Therefore

\[
\langle \psi_\kappa, \chi_R \psi_\kappa \rangle_{L^2(\mathbb{R})} \geq \frac{1}{2}
\]

holds true for all \( \kappa \geq \kappa_R \), where \( \kappa_R \) depends only on the (fixed) value of \( R \). Writing

\[
\psi_\kappa = \frac{P_\kappa(\psi_\kappa \chi_R)}{\langle \psi_\kappa, \chi_R \psi_\kappa \rangle_{L^2(\mathbb{R})}}
\]

we thus conclude with the estimate

\[
\| e^{\alpha \sqrt{1 + z^2}} \psi_\kappa \|_{L^2(\mathbb{R})} \leq 2 \| e^{\alpha \sqrt{1 + z^2}} P_\kappa e^{-\alpha \sqrt{1 + z^2}} \| \| e^{\alpha \sqrt{1 + z^2}} \chi_R \psi_\kappa \|_{L^2(\mathbb{R})} \leq 2 e^{\alpha \sqrt{1 + z^2}} \| e^{\alpha \sqrt{1 + z^2}} P_\kappa \| \| e^{-\alpha \sqrt{1 + z^2}} \chi_R \psi_\kappa \|_{L^2(\mathbb{R})},
\]

which holds for all \( \kappa \geq \kappa_R \). Hence in view of (3.25)

\[
\sup_{\kappa \geq \kappa_2} \| e^{\alpha \sqrt{1 + z^2}} \psi_\kappa \|_{L^2(\mathbb{R})} < \infty,
\]

where \( \kappa_2 = \max\{\kappa_R, \kappa_\omega\} \).

### 3.2. Proof of Theorem 1.1

We will prove the absence of discrete eigenvalues of \( \mathcal{H}_{\kappa, \lambda} \) when \( \kappa \) is larger than some critical value depending on \( \lambda > 1 \). By Proposition 2.1 we have

\[
\sigma_{\text{dis}}(\mathcal{H}_{\kappa, \lambda}) = \{E_1(\kappa), \infty\} \quad \forall \kappa \geq 1, \forall \lambda > 0.
\]

Hence from (3.6) and perturbation theory the claim follows if we can show that

\[
\mathcal{H}_{\kappa, \lambda} - z \quad \text{is invertible} \quad \forall z \in \left[ E_1(\kappa) - \frac{v(0)}{\sqrt{\kappa}}, E_1(\kappa) \right).
\]

Define the projection \( \Pi_\kappa \) on \( L^2(\mathbb{R}^2) \) by

\[
\Pi_\kappa = P_\kappa \otimes 1_y,
\]

where \( P_\kappa \) is given by (3.24) and \( 1_y \) denotes the identity operator in \( L^2(\mathbb{R}) \). Let \( \Pi_\kappa^\perp = 1 - \Pi_\kappa \). Then, according to the Feshbach-Schur formula \( [12], (3.27) \) is equivalent to proving that for all

\[
z \in \left[ E_1(\kappa) - \frac{v(0)}{\sqrt{\kappa}}, E_1(\kappa) \right),
\]

the operator

\[
\Pi_\kappa^\perp (\mathcal{H}_{\kappa, \lambda} - z) \Pi_\kappa^\perp \quad \text{is invertible in } \text{Ran}(\Pi_\kappa^\perp),
\]

and at the same time, the operator

\[
\Pi_\kappa (\mathcal{H}_{\kappa, \lambda} - z) \Pi_\kappa - \Pi_\kappa \mathcal{H}_{\kappa, \lambda} \Pi_\kappa \Pi_\kappa^\perp (\Pi_\kappa^\perp (\mathcal{H}_{\kappa, \lambda} - z) \Pi_\kappa^\perp)^{-1} \Pi_\kappa^\perp \mathcal{H}_{\kappa, \lambda} \Pi_\kappa \quad \text{is invertible in } \text{Ran}(\Pi_\kappa).
\]

As for (3.29) we note that by (3.6)

\[
\Pi_\kappa^\perp (\mathcal{H}_{\kappa, \lambda} - z) \Pi_\kappa^\perp \geq (E_2(\kappa) - E_1(\kappa) - v(0) \kappa^{-\frac{1}{2}}) \Pi_\kappa^\perp,
\]

Hence in view of Lemma 3.1, there exists \( C_1 > 0 \), independent of \( \kappa \) and \( z \), such that

\[
\| (\Pi_\kappa^\perp (\mathcal{H}_{\kappa, \lambda} - z) \Pi_\kappa^\perp)^{-1} \|_{\text{Ran}(\Pi_\kappa^\perp) \to \text{Ran}(\Pi_\kappa^\perp)} \leq C_1
\]

holds for all \( \kappa \) large enough and all \( z \) satisfying (3.28).
In order to prove (3.30) we note that
\[ v(\kappa^{-\frac{1}{2}}(x - y)) = v(\kappa^{-\frac{1}{2}}y) + \kappa^{-\frac{1}{2}} \int_0^x v'(\kappa^{-\frac{1}{2}}(t - y)) \, dt, \]
which together with (3.6) yields
\[ \Pi_{\kappa} \mathcal{H}_{\kappa,\lambda} \Pi_{\kappa} = -\partial_y^2 + E_1(\kappa) + \frac{\lambda - 1}{\sqrt{\kappa}} v(\kappa^{-\frac{1}{2}}y) - \kappa^{-\frac{3}{2}} \int_{\mathbb{R}} \psi^2(x) \int_0^x v'(\kappa^{-\frac{1}{2}}(t - y)) \, dt \, dx. \tag{3.32} \]
Denote by
\[ K_z(x, y, x', y') = (\Pi_{\kappa}^1 (\mathcal{H}_{\kappa,\lambda} - z) \Pi_{\kappa}^1)^{-1/2}(x, y, x', y') \tag{3.33} \]
the Schwartz integral kernel of \((\Pi_{\kappa}^1 (\mathcal{H}_{\kappa,\lambda} - z) \Pi_{\kappa}^1)^{-1/2}\). To treat the second term in (3.30) we introduce the bounded operator \(A : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)\) corresponding to:
\[ (Af)(x, y) = \int_{\mathbb{R}^2} K_z(x, y, x', y') v(\kappa^{-\frac{1}{2}}(x' - y')) \psi_n(x') f(y') \, dx' \, dy', \tag{3.34} \]
where the above formal expression has to be first understood as a map from the Schwartz space \(S(\mathbb{R})\) to the dual of \(S(\mathbb{R}^2)\), which can afterwards be extended to a bounded operator between \(L^2(\mathbb{R})\) to \(L^2(\mathbb{R}^2)\). An important role here is played by the estimate:
\[ \int_{\mathbb{R}^2} |v(\kappa^{-\frac{1}{2}}(x' - y')) \psi_n(x') f(y')|^2 \, dx' \, dy' \leq v(0)^2 \int_{\mathbb{R}} f^2(y') \, dy'. \]
Another important observation is that \(A\) is also bounded from \(L^\infty(\mathbb{R})\) to \(L^2(\mathbb{R}^2)\) due to the inequality:
\[ \int_{\mathbb{R}^2} |v(\kappa^{-\frac{1}{2}}(x' - y')) \psi_n(x') f(y')|^2 \, dx' \, dy' \leq \kappa^{1/4} \|v\|_2^2 \|f\|_\infty^2. \]
Hence \(A\) is bounded uniformly with respect to \(z \in [E_1(\kappa) - v(0)/\sqrt{\kappa}, E_1(\kappa)]\) in view of (3.31), and we can write:
\[ \Pi_{\kappa} \mathcal{H}_{\kappa,\lambda} \Pi_{\kappa}^1 (\Pi_{\kappa}^1 (\mathcal{H}_{\kappa,\lambda} - z) \Pi_{\kappa}^1)^{-1} \Pi_{\kappa}^1 \mathcal{H}_{\kappa,\lambda} \Pi_{\kappa} = \frac{1}{\kappa} A^* A. \]
This implies:
\[ \Pi_{\kappa} (\mathcal{H}_{\kappa,\lambda} - z) \Pi_{\kappa} - \Pi_{\kappa} \mathcal{H}_{\kappa,\lambda} \Pi_{\kappa}^1 (\Pi_{\kappa}^1 (\mathcal{H}_{\kappa,\lambda} - z) \Pi_{\kappa}^1)^{-1} \Pi_{\kappa}^1 \mathcal{H}_{\kappa,\lambda} \Pi_{\kappa} = \Pi_{\kappa} (B_{\kappa,\lambda} + E_1(\kappa) - z) \Pi_{\kappa}, \]
where
\[ B_{\kappa,\lambda} := -\partial_y^2 + \frac{\lambda - 1}{\sqrt{\kappa}} v(\kappa^{-\frac{1}{2}}y) - \kappa^{-\frac{3}{2}} \int_{\mathbb{R}} \psi^2(x) \int_0^x v'(\kappa^{-\frac{1}{2}}(t - y)) \, dt \, dx - \frac{1}{\kappa} A^* A. \tag{3.35} \]
To prove (3.27) it thus suffices to show that
\[ B_{\kappa,\lambda} + \xi \quad \text{is invertible in} \quad L^2(\mathbb{R}) \quad \text{for all} \quad \xi \in (0, \kappa^{-\frac{1}{2}} v(0)]. \tag{3.36} \]
To this end we write
\[ B_{\kappa,\lambda} + \xi = -\partial_y^2 + \xi + a_1 a_2 + b_1 b_2 + d_1 d_2, \tag{3.37} \]
where
\[ d_1 = -\frac{1}{\kappa} A^*, \quad d_2 = A, \]
and $a_1, a_2, b_1$ and $b_2$ are multiplication operators in $L^2(\mathbb{R})$ given by

$$
a_1(y) = \frac{\lambda - 1}{\sqrt{\kappa}} \sqrt{v(\kappa^{-\frac{1}{2}} y)}, \quad a_2 = \sqrt{v(\kappa^{-\frac{1}{2}} y)}, \quad b_1(y) = -\kappa^{-\frac{1}{2}} \left| \psi^2(x) \int_0^x v'(\kappa^{-\frac{1}{2}} (t - y)) \, dt \, dx \right|^{\frac{1}{2}}
$$

$$
b_2(y) = \left| \psi^2(x) \int_0^x v'(\kappa^{-\frac{1}{2}} (t - y)) \, dt \, dx \right|^{\frac{1}{2}} \text{sign} \left( \psi^2(x) \int_0^x v'(\kappa^{-\frac{1}{2}} (t - y)) \, dt \, dx \right). \quad (3.38)
$$

Let

$$
Q_{\kappa, \lambda}(\xi) = I + \left( \begin{array}{ccc}
a_2(-\partial_y^2 + \xi)^{-1}a_1 & a_2(-\partial_y^2 + \xi)^{-1}b_1 & a_2(-\partial_y^2 + \xi)^{-1}d_1 \\
b_2(-\partial_y^2 + \xi)^{-1}a_1 & b_2(-\partial_y^2 + \xi)^{-1}b_1 & b_2(-\partial_y^2 + \xi)^{-1}d_1 \\
d_2(-\partial_y^2 + \xi)^{-1}a_1 & d_2(-\partial_y^2 + \xi)^{-1}b_1 & d_2(-\partial_y^2 + \xi)^{-1}d_1
\end{array} \right) \quad (3.39)
$$

be a matrix-valued operator in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}^2)$. By (3.37) and the resolvent equation it follows that if $Q_{\kappa, \lambda}(\xi)$ is invertible for all $\xi \in (0, \kappa^{-\frac{1}{2}} v(0)]$, then (3.36) holds true and

$$(B_{\kappa, \lambda} + \xi)^{-1} = (-\partial_y^2 + \xi)^{-1} - (-\partial_y^2 + \xi)^{-1}(a_1, b_1, d_1) \left( Q_{\kappa, \lambda}(\xi) \right)^{-1}(a_2, b_2, d_2)^T (-\partial_y^2 + \xi)^{-1}.$$

Next we note that $(-\partial_y^2 + \xi)^{-1}$ is an integral operator in $L^2(\mathbb{R})$ with the kernel

$$
(-\partial_y^2 + \xi)^{-1}(y, y') = \frac{e^{-\sqrt{\xi}|y-y'|}}{2\sqrt{\xi}} =: \frac{1}{2\sqrt{\xi}} + m(y, y'). \quad (3.40)
$$

Let $M$ be the integral operator in $L^2(\mathbb{R})$ with the kernel $m(y, y')$ defined above and let

$$
|\Phi| := (a_2(y), b_2(y), \int_\mathbb{R} d_2(x, y; y') \, dy')^T, \quad \langle \Psi | := (a_1(y), b_1(y), \int_\mathbb{R} d_1(x; y, y') \, dx),
$$

where $d_1(x; y, y')$ and $d_2(x, y; y')$ are the integral kernels of $A^*$ and $A$ respectively. The integrals involving the integral kernels make sense because $A$ is bounded from $L^\infty(\mathbb{R})$ to $L^2(\mathbb{R}^2)$. Equations (3.39) and (3.40) then imply that

$$
Q_{\kappa, \lambda}(\xi) = I + R_{\kappa, \lambda} + \frac{1}{2\sqrt{\xi}} |\Phi\rangle\langle\Psi| \quad (3.41)
$$

holds true with

$$
R_{\kappa, \lambda} = \left( \begin{array}{ccc}
a_2Ma_1 & a_2Mb_1 & a_2Md_1 \\
b_2Ma_1 & b_2Mb_1 & b_2Md_1 \\
d_2Ma_1 & d_2Mb_1 & d_2Md_1
\end{array} \right). \quad (3.42)
$$

To prove the invertibility of $Q_{\kappa, \lambda}(\xi)$ we first show that $I + R_{\kappa, \lambda}$ is invertible for $\lambda - 1$ sufficiently small, but positive, and $\kappa$ sufficiently large, uniformly with respect to $\xi > 0$. To do so we estimate the operator norm of all the entries of $R_{\kappa, \lambda}$ keeping in mind that the integral kernel of $M$ satisfies

$$
|m(y, y')| = \frac{1 - e^{-\sqrt{\xi}|y-y'|}}{2\sqrt{\xi}} \leq |y - y'|. \quad (3.43)
$$

To simplify the notation in the sequel we introduce the following shorthands;

$$
m_{j,k} = \int_{\mathbb{R}} x^j v^k(x) \, dx, \quad m'_{j,k} = \int_{\mathbb{R}} |x|^j |v'(x)|^k \, dx, \quad \mu_j = \sup_{\kappa \geq \kappa_2} \int_{\mathbb{R}} |x|^j \psi_\kappa^2(x) \, dx, \quad (3.44)
$$
where \( j, k \in \mathbb{N} \) and \( \kappa_2 \) is given by Lemma 3.2. We start with the first column of \( R_{\kappa, \lambda} \). Using (3.43) we get

\[
\|a_2 M a_1\|_{L^2(\mathbb{R})}^2 \leq \|a_2 M a_1\|_{H^s}^2 \leq \frac{(\lambda - 1)^2}{\kappa} \int_{\mathbb{R}^2} v(\kappa^{-\frac{1}{4}} y) |y - y'|^2 v(\kappa^{-\frac{1}{4}} y') \ dy dy' \\
\leq \frac{(\lambda - 1)^2}{\kappa} \int_{\mathbb{R}^2} v(\kappa^{-\frac{1}{4}} y) (2y^2 + 2y'^2) v(\kappa^{-\frac{1}{4}} y') \ dy dy' \\
= 4(\lambda - 1)^2 m_{0,1} m_{2,1}.
\]

In the same way, using Lemma A.2 it follows that

\[
\|b_2 M a_1\|_{L^2(\mathbb{R})}^2 \leq \frac{(\lambda - 1)^2}{\kappa} \int_{\mathbb{R}^2} b_2(y) (2y^2 + 2y'^2) v(\kappa^{-\frac{1}{4}} y') \ dy dy' \\
= \frac{2(\lambda - 1)^2}{\kappa^{3/4}} \left( m_{0,1} \int_{\mathbb{R}} b_2(y) y^2 dy + m_{2,1} \int_{\mathbb{R}} b_2(y) dy \right) \\
\leq 2(\lambda - 1)^2 m_{0,1} m_{2,1} + \frac{2(\lambda - 1)^2}{\sqrt{\kappa}} (m_{0,1} m_{0,1} + m_{2,1} m_{2,1} m_{1}) \\
\leq C_{ba} (\lambda - 1)^2 \quad \forall \kappa \geq \kappa_2,
\]

where \( C_{ba} \) is a constant independent of \( \lambda \) and \( \kappa \). Similarly,

\[
\|b_2 M b_1\|_{L^2(\mathbb{R})}^2 \leq \kappa^{-\frac{3}{2}} \int_{\mathbb{R}^2} b_2(y) (2y^2 + 2y'^2) b_2(y') \ dy dy' \\
\leq C_{bb} \kappa^{-\frac{1}{2}} \quad \forall \kappa \geq \kappa_2,
\]

where we used Lemma A.2 and (3.38).

To estimate \( d_2 M a_1 \) we first observe that for any \( f \in L^2(\mathbb{R}) \)

\[
(d_2 M a_1 f)(x, y) = \frac{\lambda - 1}{\sqrt{\kappa}} \int_{\mathbb{R}^2} K_z(x, y, x', y') u(x', y') \ dx' \ dy',
\]

where \( K_z(x, y, x', y') \) is given by (3.35) and

\[
u(x', y') = \psi_{\kappa}(x') v(\kappa^{-\frac{1}{4}} (x' - y')) \int_{\mathbb{R}} m(y', y'') \sqrt{v(\kappa^{-\frac{1}{4}} y'')} f(y'') \ dy''
\]

Hence by the Hölder inequality and Lemma A.2

\[
\|u\|_{L^2(\mathbb{R}^2)}^2 \leq \|f\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}^2} \psi_{\kappa}(x') v^2(\kappa^{-\frac{1}{4}} (x' - y')) \left( \int_{\mathbb{R}} (2y'^2 + 2y''^2) v(\kappa^{-\frac{1}{4}} y'') \ dy'' \right) \ dx' \ dy' \\
= 2\|f\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}^2} \psi_{\kappa}(x') v^2(\kappa^{-\frac{1}{4}} (x' - y')) (y'^2 m_{0,1} + \kappa \frac{3}{4} m_{2,1}) \ dx' \ dy' \\
\leq 2\|f\|_{L^2(\mathbb{R})}^2 \left[ 2\kappa \frac{3}{4} m_{0,1} m_{2,2} + \kappa \frac{3}{4} m_{0,2} m_{2,1} \right] + \kappa m_{2,1} m_{0,2}
\]

Therefore, in view of (3.31) and (3.46) there exists a constant \( C_{da} \), independent of \( \kappa \), such that

\[
\|d_2 M a_1\|_{L^2(\mathbb{R})} \leq C_{da} (\lambda - 1) \quad \forall \kappa \geq \kappa_2.
\]

In the same way it follows that

\[
\|d_2 M b_1\|_{L^2(\mathbb{R})} \leq C_{db} \kappa^{-\frac{1}{4}} \quad \forall \kappa \geq \kappa_2.
\]
As for the operator $d_2 M d_1 : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ we note that for any $f \in L^2(\mathbb{R})$ it holds
\begin{equation}
(d_2 M d_1 f)(x, y) = -\frac{1}{\kappa} \Pi_{\kappa}^+ \left( \mathcal{H}_{\kappa, \lambda} - z \right) \Pi_{\kappa}^+ \frac{1}{2} \psi_{\kappa}(x) v(\kappa^{-\frac{1}{4}}(x - y)) u(y),
\end{equation}
where
\begin{equation}
u(y) = \int_{\mathbb{R}^4} \psi_{\kappa}(y') m(y, y') v(\kappa^{-\frac{1}{4}}(y' - t)) K_z(y', t, x, s) f(x', s) \, dx' \, ds \, dt \, dy'.
\end{equation}
From (3.49) and Hölder inequality we then obtain
\begin{align*}
\|d_2 M d_1\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)}^2 & \leq \frac{1}{\kappa^2} \left\| \left( \Pi_{\kappa}^+ \left( \mathcal{H}_{\kappa, \lambda} - z \right) \Pi_{\kappa}^+ \frac{1}{2} \psi_{\kappa}(x) v(\kappa^{-\frac{1}{4}}(x - y)) \right) \| \right\|^2 \\
& \leq 2 C m_{0, 2} \kappa^{-\frac{7}{4}} \int_{\mathbb{R}^3} \psi_{\kappa}^2(x) \psi_{\kappa}^2(t) \left( (s - x)^2 + (x - t)^2 \right) v^2(\kappa^{-\frac{1}{4}}(x - s)) \, dx \, ds \, dt \\
& = 2 C m_{0, 2} \kappa^{-\frac{7}{4}} \left( \kappa^2 m_{2, 2} + \int_{\mathbb{R}^3} \psi_{\kappa}^2(x) \psi_{\kappa}^2(t) \left( x^2 + t^2 \right) v^2(\kappa^{-\frac{1}{4}}(x - s)) \, dx \, ds \, dt \right) \\
& = 2 C m_{0, 2} \kappa^{-\frac{7}{4}} \left( \kappa^2 m_{2, 2} + 2 \kappa^4 m_{0, 2} \mu_2 \right),
\end{align*}
where we have used Lemma A.2 again and the fact that $\psi_\kappa(\cdot)$ is even. Hence there exists a constant $C_d$ such that
\begin{equation}
\|d_2 M d_1\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \leq C_d \kappa^{-\frac{1}{2}} \quad \forall \kappa \geq \kappa_2.
\end{equation}
Concerning the remaining entries of $R_{\kappa, \lambda}$, we notice that by duality, (3.38) and (3.47)
\begin{equation}
\|a_2 M a_1\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} = \frac{1}{(\lambda - 1) \sqrt{\kappa}} \left\| d_2 M a_1\right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_{a_2} \kappa^{-\frac{1}{4}} \quad \forall \kappa \geq \kappa_2.
\end{equation}
Similarly it follows from (3.45) and (3.48) that
\begin{equation}
\|a_2 M b_1\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \frac{1}{(\lambda - 1) \kappa^{1/4}} \|b_2 M a_1\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_{b_2} \kappa^{-\frac{1}{4}} \quad \forall \kappa \geq \kappa_2,
\end{equation}
and
\begin{equation}
\|b_2 M d_1\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} = \kappa^{-\frac{1}{4}} \|d_2 M b_1\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)} \leq C_{b_2} \kappa^{-\frac{1}{2}} \quad \forall \kappa \geq \kappa_2.
\end{equation}
Putting together the above estimates we conclude that
\begin{equation}
\|R_{\kappa, \lambda}\| \leq C_R (|\lambda - 1| + \kappa^{-\frac{1}{4}}) \quad \forall \lambda > 1, \ \forall \kappa \geq \kappa_2,
\end{equation}
hold for some $C_R > 0$, where the norm of $R_{\kappa, \lambda}$ is calculated on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}^2)$. Hence there exists $1 < \lambda_0 < 2$ (which has to be chosen close enough to 1) and some $\kappa_0 \geq \kappa_2$ (independent of $\lambda_0$), such that
\begin{equation}
\|R_{\kappa, \lambda}\| \leq \frac{1}{2} \quad \forall \lambda \in (1, \lambda_0), \ \forall \kappa \geq \kappa_0
\end{equation}
for these values of $\lambda$ and $\kappa$ the operator $\mathbb{I} + R_{\kappa, \lambda}$ is invertible, uniformly in $\xi$, and (3.41) becomes:
\begin{equation}
Q_{\kappa, \lambda}(\xi) = \left( \mathbb{I} + \frac{1}{2 \sqrt{\xi}} |\Phi\rangle \langle\Psi| (\mathbb{I} + R_{\kappa, \lambda})^{-1} \right) (\mathbb{I} + R_{\kappa, \lambda}),
\end{equation}
hence we reduced the invertibility of $Q_{\kappa, \lambda}(\xi)$ to the one of
\begin{equation}
\mathbb{I} + \frac{1}{2 \sqrt{\xi}} |\Phi\rangle \langle\Psi| (\mathbb{I} + R_{\kappa, \lambda})^{-1}.
\end{equation}
After a second Feshbach-Schur reduction with respect to the projection on the vector $|\Phi\rangle$, we notice that this operator is invertible if and only if the function

$$f_{\kappa,\lambda}(\xi) := 1 + \frac{1}{2\sqrt{\xi}} \langle \Psi, (1 + R_{\kappa,\lambda})^{-1}\Phi \rangle, \quad \xi \in (0, \kappa^{-\frac{3}{2}} v(0)]$$

(3.54)

is never zero. The Neumann series for $(1 + R_{\kappa,\lambda})^{-1}$ in combination with (3.52) gives

$$\langle \Psi, (1 + R_{\kappa,\lambda})^{-1}\Phi \rangle = \langle \Psi, \Phi \rangle + \sum_{n=1}^{\infty} (-1)^n \langle \Psi, R_{\kappa,\lambda}^n \Phi \rangle$$

$$\geq \langle \Psi, \Phi \rangle - \|\Phi\| \|\Psi\| \|R_{\kappa,\lambda}\| \frac{1}{1 - \|R_{\kappa,\lambda}\|}$$

$$\geq \langle \Psi, \Phi \rangle - 2 \|\Phi\| \|\Psi\| \|R_{\kappa,\lambda}\|,$$

(3.55)

where all the scalar products and norms are calculated on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}^2)$. Equation (3.31) and a straightforward computation show that

$$\langle \Psi, \Phi \rangle = (\lambda - 1) \kappa^{-\frac{3}{4}} m_{0,1} - \frac{1}{\kappa} \text{tr}(A^*A) \geq (\lambda - 1) \kappa^{-\frac{3}{4}} m_{0,1} - C_1 \kappa^{-\frac{3}{4}} m_{0,2},$$

(3.56)

and that there exists a constant $C_0$ such that

$$\|\Phi\| \|\Psi\| \leq C_0 \sqrt{\kappa^{-\frac{1}{2}} (\lambda - 1)^2 + \kappa^{-1}} \quad \forall \kappa \geq \kappa_0.$$

(3.57)

Hence if we set

$$\kappa_c(\lambda) = \max \{ \kappa_0, (\lambda - 1)^{-4} \},$$

(3.58)

then equations (3.51), (3.55) and (3.56) imply

$$\kappa^{1/4} \langle \Psi, (1 + R_{\kappa,\lambda})^{-1}\Phi \rangle \geq (\lambda - 1) m_{0,1} + O((\lambda - 1)^2) \quad \forall \lambda \in (1, \lambda_0), \quad \forall \kappa \geq \kappa_c(\lambda).$$

This shows that there exists $0 < \lambda_c < \lambda_0 < 2$ such that

$$\langle \Psi, (1 + R_{\kappa,\lambda})^{-1}\Phi \rangle \geq 0, \quad \forall \lambda \in (1, \lambda_c), \quad \forall \kappa \geq \kappa_c(\lambda).$$

Thus $f_{\kappa,\lambda}(\xi)$ in (3.54) is never zero if $\lambda > 1$ is close enough to 1 and, at the same time, $\kappa$ is larger than some $\lambda$-dependent critical value. Since the number of discrete eigenvalues of $\mathcal{H}_{\kappa,\lambda}$ is non-increasing with respect to $\lambda$, we obtain the claim of the theorem for all $\lambda > 1$. \hfill \Box

3.3. **Proof of Corollary 1.2.** We know from Proposition 2.1 that $H_{\kappa}(v)$ and $H_{\kappa,3/2}(v)$ have the same essential spectrum if $\kappa > 1$. Due to Theorem 1.1 the discrete spectrum of $H_{\kappa,3/2}(v)$ is empty if $\kappa$ is larger than some critical value. Since $H_{\kappa}(v) \geq H_{\kappa,3/2}(v)$ for all $\kappa \geq 3/2$, the result follows from the min-max principle. \hfill \Box

3.4. **Proof of Proposition 1.3.** We are interested in the case when $k > k_c$. Let $h_{\kappa,w}$ be the operator in $L^2(\mathbb{R})$ given by

$$h_{\kappa,w} = -\partial_x^2 + \kappa w(x),$$

(3.59)

and let $e_w(\kappa) < 0$ be its lowest eigenvalue. In view of Proposition 2.1 we have

$$\sigma_{\text{ess}}(H_{\kappa}) = [e_w(\kappa), \infty), \quad \forall \kappa \geq k_c.$$

(3.60)

We will construct a test function $u$ in the form

$$u(x, y) = \varphi_{\kappa}(x) f(y),$$

(3.61)
where \( \varphi_\kappa(x) \) is a normalized eigenfunction of the operator \( h_{kw} \) associated to its lowest eigenvalue \( e_w(\kappa) \), and

\[
 f(y) = \begin{cases} 
 0 & \text{if } y \leq 1 \\
 y - 1 & \text{if } 1 < y < 2 \\
 \exp(4 - 2y) & \text{if } 2 \leq y .
\end{cases}
\]

Integration by parts then shows that

\[
 Q_w[u] := \langle u, H_\kappa(w) u \rangle_{L^2(\mathbb{R}^2)} - e_w(\kappa) \| u \|^2
 = \int_{\mathbb{R}^2} \varphi_\kappa^2(x)|f'(y)|^2 \, dx \\ dy + \kappa \int_{\mathbb{R}} w(y) f^2(y) \, dy - \int_{\mathbb{R}^2} w(x - y) \varphi_\kappa^2(x) f^2(y) \, dx \\ dy
 = 2 - \int_{\mathbb{R}^2} w(x - y) \varphi_\kappa^2(x) f^2(y) \, dx \\ dy \leq 2 - w_0 \int_{y=1}^{y=\kappa^2} (y - 1)^2 \varphi_\kappa^2(x) \, dx \\ dy
 \leq 2 - w_0 \int_0^1 \int_1^{x+1} (y - 1)^2 \varphi_\kappa^2(x) \, dx \\ dy = 2 - w_0 \frac{1}{3} \int_0^1 \varphi_\kappa^2(x) x^3 \, dx.
\]

(3.62)

On the other hand, an explicit calculation yields

\[
 \varphi_\kappa(x) = \begin{cases} 
 C_\kappa \cos(\beta_\kappa x) & \text{for } |x| \leq 1, \\
 D_\kappa e^{-|x|\omega_\kappa} & \text{for } |x| > 1,
\end{cases}
\]

(3.63)

where

\[
 \beta_\kappa = \sqrt{\kappa w_0 + e_w(\kappa)} , \quad \omega_\kappa = \sqrt{-e_w(\kappa)} ,
\]

(3.64)

and \( C_\kappa \) and \( D_\kappa \) are constants satisfying

\[
 C_\kappa \cos(\beta_\kappa) = D_\kappa e^{-\omega_\kappa} \quad (3.65) \]

\[
 \beta_\kappa C_\kappa \sin(\beta_\kappa) = \omega_\kappa D_\kappa e^{-\omega_\kappa}. \quad (3.66)
\]

The last two equations imply that \( e_w(\kappa) \) is given by the smallest solution to the implicit equation

\[
 \frac{\sqrt{-e_w(\kappa)}}{\beta_\kappa} = \tan(\beta_\kappa) ,
\]

(3.67)

where

\[
 0 < \beta_\kappa < \frac{\pi}{2}. \quad (3.68)
\]

From equations (3.62), (3.68) and the elementary inequality \( \sin^2(x) \leq x^2 \) we then obtain the upper bound

\[
 Q_w[u] \leq 2 - w_0 \frac{2}{3} C_\kappa^2 \int_0^1 x^3 \, dx \leq 2 - w_0 \frac{2}{3} C_\kappa^2 \left( \int_0^{\frac{\pi}{2}} x^3 \, dx - \beta_\kappa^2 \int_0^{\frac{\pi}{2}} x^5 \, dx \right)
\]

\[
 \leq 2 - 4 w_0 \frac{2}{9 \pi^4} C_\kappa^2 .
\]

(3.69)

To prove that \( Q_w[u] \) is negative for \( w_0 \) large enough we thus need a lower bound on \( C_\kappa \) independent of \( \kappa \). The condition \( \| \varphi_\kappa \| = 1 \) gives

\[
 C_\kappa^2 (1 + \beta_\kappa^{-1} \cos(\beta_\kappa) \sin(\beta_\kappa)) + \frac{D_\kappa^2}{\omega_\kappa} e^{-2\omega_\kappa} = 1 ,
\]
which in view of equations (3.64) and (3.65)-(3.66) implies
\[ C^2_\kappa \left[ 1 + \cos(\beta_\kappa) \sin(\beta_\kappa) \frac{\kappa w_0}{\beta_\kappa \omega_\kappa^2} \right] = 1. \]  (3.70)
Using \( \sin(\beta_\kappa) / \beta_\kappa \leq 1 \) in the above expression together with the identity \( \omega_\kappa^2 = -e_w(\kappa) \) we have:
\[ 1 \leq C^2_\kappa \left( 1 - \frac{\kappa w_0}{e_w(\kappa)} \right). \]  (3.71)
To continue we have to estimate \( e_w(\kappa) \) from above. The choice of the test function
\[ \psi(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ e^{(1-|x|)\kappa w_0} & \text{for } |x| > 1. \end{cases} \]
gives
\[ e_w(\kappa) \leq \frac{\langle \psi, h_{\kappa w_0} \psi \rangle}{\|\psi\|^2} = - \frac{\kappa w_0}{2 + \frac{1}{\kappa w_0}} < 0. \]
Hence we get the upper bound
\[ -\frac{\kappa w_0}{e_w(\kappa)} \leq 2 + \frac{1}{\kappa w_0}. \]
This in combination with (3.71) leads to
\[ C^2_\kappa \geq \left( 3 + \frac{1}{\kappa w_0} \right)^{-1}, \quad \forall \kappa \geq k_e. \]
Because \( k_e > 1/2 \) (see Proposition 2.1), the above estimate implies
\[ C^2_\kappa \geq \left( 3 + \frac{2}{w_0} \right)^{-1}, \quad \forall \kappa \geq k_e. \]
Inserting this back into (3.69) leads to:
\[ Q_w[u] \leq 2 - \frac{4}{9\pi^4} \frac{w_0^2}{3w_0 + 2} \quad \forall \kappa \geq k_e. \]
Hence \( Q_w[u] < 0 \) for \( w_0 \) large enough, uniformly in \( \kappa \geq k_e \), which ends the proof. \( \square \)

4. Proofs in the two-dimensional case

We introduce the scaling function
\[ (U_\kappa f)(x, y) = \kappa f(\kappa^{1/2} x, \kappa^{1/2} y), \]  (4.1)
where \( U_\kappa \) maps \( L^2(\mathbb{R}^4) \) unitarily onto itself, and define the operator
\[ A_\kappa := \frac{1}{\kappa} U_\kappa^* \mathcal{H} \kappa U_\kappa = a_\kappa - \Delta_y - V_{ctr}(\kappa^{-2} y) + \frac{1}{\kappa} V_{ctr}(\kappa^{-2} |x - y|), \]  (4.2)
where
\[ a_\kappa = -\Delta + V_{ctr}(\kappa^{-2} x) \quad \text{in } L^2(\mathbb{R}^2). \]  (4.3)
Next we consider the quadratic form
\[ \int_{\mathbb{R}^2} \left( |\nabla u|^2 + (\log |x|) |u|^2 \right) \, dx, \quad u \in C^0_0(\mathbb{R}^2). \]  (4.4)
By Lemma 4.2 this form is bounded from below. We denote by \( q_0 \) its closure with the domain:
\[ d(q_0) = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\log |x|| |u|^2 \, dx < \infty \right\}. \]
Let $a_0$ be the self-adjoint operator in $L^2(\mathbb{R}^2)$ generated by $q_0$. Then $a_0$ acts on its domain as

$$a_0 = -\Delta + \log |x|,$$  \hfill (4.5)

and the spectrum of $a_0$ is purely discrete because the potential is confining. Let

$$E_1 < E_2 < E_3 < \ldots$$  \hfill (4.6)

be the distinguished eigenvalues of $a_0$ (possibly degenerate, with the exception of $E_1$). As for the operator $a_\kappa$, we notice that $\sigma_{\text{ess}}(a_\kappa) = [0, \infty)$ and that in view of the negativity of $V_{\text{ctr}}$ the discrete spectrum of $a_\kappa$ is non-empty for all $\kappa$. We denote

$$E_1(\kappa) := \inf \sigma(a_\kappa)$$

the lowest eigenvalue of $a_\kappa$. Let $\phi_1$ and $\varphi_\kappa$ be the normalized eigenfunctions of $a_0$ and $a_\kappa$ respectively:

$$a_0 \phi_1 = E_1 \phi_1, \quad a_\kappa \varphi_\kappa = E_1(\kappa) \varphi_\kappa, \quad \|\phi_1\|_{L^2(\mathbb{R}^2)} = \|\varphi_\kappa\|_{L^2(\mathbb{R}^2)} = 1. \hfill (4.7)$$

**Lemma 4.1.** For $\kappa$ large enough it holds

$$\sigma(a_\kappa) \cap \left(-\infty, -w(0) - \log \sqrt{\kappa} + \frac{E_2 + E_1}{2}\right) = \{E_1(\kappa)\}. \hfill (4.8)$$

Moreover, we have

$$\lim_{\kappa \to \infty} \left( E_1(\kappa) + \log \sqrt{\kappa} \right) = E_1 - w(0), \hfill (4.9)$$

and

$$\lim_{\kappa \to \infty} \|\varphi_\kappa - \phi_1\|_{L^2(\mathbb{R}^2)} = 0. \hfill (4.10)$$

**Proof.** Keeping in mind (4.6) we introduce the operators

$$\hat{a}_0 = a_0 - E_3 \quad \text{and} \quad \hat{a}_\kappa = a_\kappa + w(0) + \log \sqrt{\kappa} - E_3, \quad \kappa \geq \kappa_0 := e^{2E_3}. \hfill (4.11)$$

Then

$$W_\kappa(x) := \hat{a}_\kappa - \hat{a}_0 = w(0) - w(\kappa^{-\frac{1}{2}}|x|) - \log(1 + \kappa^{-\frac{1}{2}}|x|).$$

Let $u \in L^2(\mathbb{R}^2)$ and let $f = (a_0 + i)^{-1}u$. Then by the resolvent equation

$$\|(\hat{a}_\kappa + i)^{-1}u - (a_0 + i)^{-1}u\| \leq \|W_\kappa f\|, \hfill (4.12)$$

Since $\log(1+|x|) f \in L^2(\mathbb{R}^2)$ and $W_\kappa \to 0$ uniformly on compact sets in $\mathbb{R}^2$, it follows that $\|W_\kappa f\| \to 0$ as $\kappa \to +\infty$. Hence $\hat{a}_\kappa$ converges to $\hat{a}_0$ in the sense of strong-resolvent convergence as $\kappa \to \infty$. On the other hand, in view of (4.11)

$$\hat{a}_\kappa = -\Delta + \log \frac{|x|}{1 + \kappa^{-1/2}|x|} - E_3 + w(0) - w(\kappa^{-\frac{1}{2}}|x|)$$

$$\geq -\Delta + \log \frac{|x|}{1 + \kappa_0^{-1/2}|x|} - E_3 =: S.$$

The operator $S$ is bounded from below in $L^2(\mathbb{R}^2)$ and its essential spectrum coincides with the half-line $[0, \infty)$. We can thus apply the result of [14]. The latter states that the negative eigenvalues of $\hat{a}_\kappa$ converge (including multiplicities) to the negative eigenvalues of $\hat{a}_0$ as $\kappa \to +\infty$. Since $\hat{a}_0$ has exactly two negative eigenvalues: $E_1 - E_3$ and $E_2 - E_3$, this implies that

$$\lim_{\kappa \to \infty} \left( E_j(\kappa) + \log \sqrt{\kappa} \right) = E_j - w(0), \quad j = 1, 2,$$
where $E_2(\kappa)$ is the second eigenvalue of $a_\kappa$. Hence (4.9) and (4.8). Moreover, the eigenfunctions of $\hat{a}_\kappa$ relative to negative eigenvalues converge in norm to the eigenfunctions of $\hat{a}_0$ relative to its negative eigenvalues, see [14]. As the eigenfunctions of $\hat{a}_\kappa$ coincide with the eigenfunctions of $a_\kappa$, and the eigenfunctions of $\hat{a}_0$ coincide with those of $a_0$, we obtain (4.10).

\[\square\]

4.1. Large coupling: absence of discrete spectrum. In this section we prove the absence of discrete spectrum of the operator $H_\kappa$ for large $\kappa$. We need some preliminary results.

Lemma 4.2. Let $\kappa \geq 1$. Then for every $\varepsilon > 0$ there exists $C_\varepsilon$ independent of $\kappa$ and such that
\[
|\langle V_{c_{\text{tr}}}(\kappa^{-\frac{1}{2}}x) u, u \rangle_{L^2(\mathbb{R}^2)}| \leq (C_\varepsilon + \log \sqrt{\kappa}) \|u\|_2^2 + \varepsilon \|\nabla u\|_2^2
\] (4.13)
holds for all $u \in H^1(\mathbb{R}^2)$.

Proof. Let $u \in H^1(\mathbb{R}^2)$. Since $V_{c_{\text{tr}}} < 0$, $w$ is decreasing and $\kappa \geq 1$, we have
\[
|\langle V_{c_{\text{tr}}}(\kappa^{-\frac{1}{2}}x) u, u \rangle| - w(0) \leq \int_{\mathbb{R}^2} \log \left(1 + \frac{\sqrt{\kappa}}{|x|}\right) u^2(x) \, dx
\]
\[
\leq \log \sqrt{\kappa} \|u\|_2^2 + \int_{\mathbb{R}^2} \log \left(1 + \frac{1}{|x|}\right) u^2(x) \, dx
\]
\[
\leq \log(2\sqrt{\kappa}) \|u\|_2^2 + \int_{\mathbb{R}^2} \log \left(\frac{1}{|x|}\right) u^2(x) \, dx,
\] (4.14)
where $\mathcal{B}_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. From the compactness of the imbedding $H^1(\mathcal{B}_1) \hookrightarrow L^q(\mathcal{B}_1)$ with $2 \leq q < \infty$ it follows that for any $\varepsilon > 0$ there exists $C_\varepsilon'$ such that
\[
\left(\int_{\mathcal{B}_1} |u|^q \, dx\right)^\frac{2}{q} \leq \varepsilon \|\nabla u\|_2^2 + C_\varepsilon' \|u\|_2^2.
\] (4.15)
Since $\log \left(1 + \frac{1}{|x|}\right) \in L^p(\mathcal{B}_1)$ for all $1 \leq p < \infty$, the claim follows by an application of the Hölder inequality to the last term in (4.14).

\[\square\]

Lemma 4.3. Let $\varphi_\kappa$ be given by (4.7). Then there exist $\alpha > 0$ and $\kappa_3 \geq 1$ such that
\[
\sup_{\kappa \geq \kappa_3} \int_{\mathbb{R}^2} e^{2\alpha \sqrt{1+|x|^2}} \varphi_\kappa^2(x) \, dx < \infty
\] (4.16)
and
\[
\sup_{\kappa \geq \kappa_3} \sup_{x \in \mathbb{R}^2} (1 + |x|^3) \varphi_\kappa^2(x) < \infty.
\] (4.17)

Proof. In order to prove (4.16), we proceed as in the proof of Lemma 3.2. By Lemma 4.1 there exist $\delta > 0$ and $\kappa_6$ such that
\[
\sup_{z \in \gamma} \sup_{\kappa \geq \kappa_6} \|((a_\kappa - z)^{-1}\| < \infty,
\] (4.18)
where
\[
\gamma = \{z \in \mathbb{C} : |z - E_1 + w(0) + \log \sqrt{\kappa}| = \delta\}.
\] (4.19)
Next, for any $z \in \mathbb{C}$ it holds
\[
e^{\alpha \sqrt{1+|z|^2}} (a_\kappa - z) e^{-\alpha \sqrt{1+|z|^2}} = a_\kappa - z + \hat{W}_\alpha,
\]
where $\hat{W}_\alpha$ is a first order differential operator which acts the polar coordinates as
\[
\hat{W}_\alpha = \frac{2\alpha r}{\sqrt{1+r^2}} \partial_r + \frac{\alpha}{(1+r^2)^{3/2}} + \frac{\alpha}{1+r^2} - \frac{\alpha^2 r^2}{1+r^2}.
\] (4.20)
For any $u \in H^1(\mathbb{R})$ and any $\alpha \in (0, 1)$ we then have

$$\|\hat{W}_\alpha u\|^2 \lesssim \alpha^2 (\|\nabla u\|^2 + \|u\|^2).$$

(4.21)

Now we note that

$$a_\kappa + \log \sqrt{\kappa} \geq -\frac{1}{2} \Delta - C$$

holds in the sense of quadratic forms on $H^1(\mathbb{R}^2)$ for all $\kappa \geq 1$ and some $C > 0$ independent of $\kappa$, see Lemma 4.2. Therefore

$$\langle (a_\kappa - z)u, u \rangle_{L^2(\mathbb{R})} \geq \text{Re} \langle (a_\kappa - z)u, u \rangle_{L^2(\mathbb{R})} \geq \frac{1}{2} \|\nabla u\|^2 - \hat{C} \|u\|^2.$$ 

holds true for all $\kappa \geq 1$, all $z \in \gamma$ and some constant $\hat{C} > 0$ independent of $\kappa$. This in combination with (4.21) gives

$$\|\hat{W}_\alpha u\|^2 \lesssim \alpha^2 (\|u\|^2 + \|(a_\kappa - z)u\|^2)$$

for all $z \in \gamma$. As in the proof of Lemma 3.2 we conclude that there exists $\alpha \in (0, 1)$ such that the operator

$$e^{\alpha \sqrt{1+\|\cdot\|^2}} (a_\kappa - z) e^{-\alpha \sqrt{1+\|\cdot\|^2}} = a_\kappa + \hat{W}_\alpha - z$$

is invertible for all $z \in \gamma$ and all $\kappa \geq \kappa_\delta$, with a bounded inverse, see [4, Thm. IV.1.16]. In view of the identity

$$e^{\alpha \sqrt{1+\|\cdot\|^2}} (a_\kappa - z)^{-1} e^{-\alpha \sqrt{1+\|\cdot\|^2}} = (a_\kappa + \hat{W}_\alpha - z)^{-1},$$

it follows that

$$\sup_{z \in \gamma} \sup_{\kappa \geq \kappa_\delta} \|e^{\alpha \sqrt{1+\|\cdot\|^2}} (a_\kappa - z)^{-1} e^{-\alpha \sqrt{1+\|\cdot\|^2}}\| < \infty.$$ 

(4.22)

Now let

$$\hat{P}_\kappa = \varphi_\kappa \langle \cdot, \varphi_\kappa \rangle_{L^2(\mathbb{R})}.$$ 

(4.23)

Then by Lemma 4.1 and equation (4.18)

$$\sup_{\kappa \geq \kappa_\delta} \|e^{\alpha \sqrt{1+\|\cdot\|^2}} \hat{P}_\kappa e^{-\alpha \sqrt{1+\|\cdot\|^2}}\| \leq \sup_{\kappa \geq \kappa_\delta} \frac{1}{2\pi} \int_{\gamma} \|e^{\alpha \sqrt{1+\|\cdot\|^2}} (a_\kappa - z)^{-1} e^{-\alpha \sqrt{1+\|\cdot\|^2}}\| \, dz < \infty.$$ 

Since $\varphi_\kappa$ converges strongly to $\phi_0$ in $L^2(\mathbb{R}^2)$ as $\kappa \to \infty$, see (4.10), we can now follow line by line the arguments of the proof of Lemma 3.1 and conclude that (4.16) holds true with some $\kappa_3 \geq \kappa_\delta$.

It remains to prove (4.17). By (4.7)

$$-\Delta \varphi_\kappa = (\mathcal{E}_1(\kappa) + \log \sqrt{\kappa} + w(\kappa^{-\frac{1}{2}}|x|)) \varphi_\kappa + \log \left(\frac{1}{\sqrt{\kappa}} + \frac{1}{|x|}\right) \varphi_\kappa.$$ 

Since $\mathcal{E}_1(\kappa) + \log \sqrt{\kappa} + w(\kappa^{-\frac{1}{2}}|x|)$ is bounded in $\mathbb{R}^2$, uniformly with respect to $\kappa$, see (4.9), it follows that

$$\|\Delta \varphi_\kappa\|^2 \leq C + \int_{\mathbb{R}^2} \log^2 \left(1 + \frac{1}{|x|}\right) \varphi_\kappa^2(x) \, dx \leq C + \log^2 2 + \int_{\mathbb{R}^2} \log^2 \left(1 + \frac{1}{|x|}\right) \varphi_\kappa^2(x) \, dx.$$ 

Now we proceed as in the proof of Lemma 4.2 using (4.15) and the fact that

$$\|\nabla \varphi_\kappa\|^2 \leq \|\Delta \varphi_\kappa\|_2 \|\varphi_\kappa\|_2 \leq \delta \|\Delta \varphi_\kappa\|^2 + \delta^{-1} \quad \forall \delta > 0,$$ 

(4.24)

which follows from integration by parts, we find that $\|\Delta \varphi_\kappa\|_2$ is bounded uniformly in $\kappa$. The continuity of the Sobolev imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ then implies that

$$\sup_{\kappa \geq 1} \|\varphi_\kappa\|_\infty \lesssim \sup_{\kappa \geq 1} \|\varphi_\kappa\|_{H^2(\mathbb{R}^2)} < \infty.$$ 

(4.25)
On the other hand, since $\varphi_\kappa$ is radial, being the ground-state of a Schrödinger operator with a radial potential, an integration by parts in combination with (4.25) shows that
\[
2 \int_0^r t^3 \varphi_\kappa'(t) \varphi_\kappa(t) \, dt = r^3 \varphi_\kappa^2(r) - 6 \int_0^r t^2 \varphi_\kappa^2(t) \, dt.
\] (4.26)

By (4.16)
\[
\sup_{\kappa \geq \kappa_0} \int_0^\infty r^n \varphi_\kappa^2(r) \, dr < \infty \quad \forall \ n \geq 1.
\] (4.27)

Hence the claim follows from the Hölder inequality and equations (4.24)-(4.26).

**Lemma 4.4.** There exists a constant $C > 0$ such that
\[
\int_{\mathbb{R}^2} |V_{\text{ctr}}(\kappa^{-\frac{1}{2}}|x-y|)| \varphi_\kappa^2(x) \, dx \leq -C V_{\text{ctr}}(\kappa^{-\frac{1}{2}}y) \quad \forall \ y \in \mathbb{R}^2,
\] holds true for all $\kappa$ large enough.

**Proof.** Note that $V_{\text{ctr}} < 0$ and that
\[
|V_{\text{ctr}}(\kappa^{-\frac{1}{2}}|x-y|)| = \log \left(1 + \frac{\sqrt{\kappa}}{|x-y|}\right) + w(\kappa^{-\frac{1}{2}}|x-y|),
\]
\[-V_{\text{ctr}}(\kappa^{-\frac{1}{2}}y) = \log \left(1 + \frac{\sqrt{\kappa}}{|y|}\right) + w(\kappa^{-\frac{1}{2}}|y|).
\]

Moreover, from the inequality
\[
\log \left(1 + \frac{\beta}{t}\right) \geq \frac{\beta}{\beta + t} \quad \forall \ t > 0, \ \forall \ \beta > 0,
\] (4.29)

and from the assumptions on $w$ it follows that
\[
w(\kappa^{-\frac{1}{2}}|x-y|) \leq \frac{1}{1 + \kappa^{-\frac{1}{2}}|x-y|} \leq \log \left(1 + \frac{\sqrt{\kappa}}{|x-y|}\right).
\]

Hence in view of the positivity of $w$ to prove the claim it suffices to show that
\[
\int_{\mathbb{R}^2} \log \left(1 + \frac{\sqrt{\kappa}}{|x-y|}\right) \varphi_\kappa^2(x) \, dx \leq c \log \left(1 + \frac{\sqrt{\kappa}}{|y|}\right) \quad \forall \ y \in \mathbb{R}^2.
\] (4.30)

holds for all $\kappa$ large enough and some $c > 0$. To simplify the notation we write $t = |x|$ and $r = |y|$ keeping in mind that $\varphi_\kappa$ is radial. Then by Lemma 4.3
\[
\int_{\mathbb{R}^2} \log \left(1 + \frac{\sqrt{\kappa}}{|x-y|}\right) \varphi_\kappa^2(x) \, dx \leq 2\pi \int_0^{\infty} \log \left(1 + \frac{\sqrt{\kappa}}{t-r}\right) \varphi_\kappa^2(t) \, dt
\]
\[= 2\pi \int_{-r}^{\infty} \log \left(1 + \frac{\sqrt{\kappa}}{|t|}\right) \varphi_\kappa^2(r+t) \, (r+t) \, dt
\]
\[\leq 2\pi \log \left(1 + \frac{2\sqrt{\kappa}}{r}\right) + \int_{-r/2}^{r/2} \log \left(1 + \frac{\sqrt{\kappa}}{|t|}\right) \varphi_\kappa^2(r+t) \, (r+t) \, dt
\]
\[\leq 4\pi \log \left(1 + \frac{\sqrt{\kappa}}{r}\right) + 2 \sup_{|t| \leq r/2} \varphi_\kappa^2(r+t) \, (r+t) \int_{-r/2}^{r/2} \log \left(1 + \frac{\sqrt{\kappa}}{t}\right) \, dt
\]
\[\leq 4\pi \log \left(1 + \frac{\sqrt{\kappa}}{r}\right) + \frac{cr}{1 + r^3} \int_{0}^{r} \log \left(1 + \frac{\sqrt{\kappa}}{t}\right) \, dt.
\] (4.31)
holds for some $c > 0$. Here we have used the fact that $\log(1 + 2x) \leq 2 \log(1 + x)$ holds for any $x > 0$.

A simple calculation shows that
\[
\int_0^r \log \left(1 + \frac{\sqrt{\kappa}}{t}\right) dt = r \log \left(1 + \frac{\sqrt{\kappa}}{r}\right) + \sqrt{\kappa} \log \left(1 + \frac{r}{\sqrt{\kappa}}\right)
\]
\[
\leq r \log \left(1 + \frac{\sqrt{\kappa}}{r}\right) + r.
\]
This in combination with (4.29) and (4.31) proves (4.30) and hence the claim.

\[\square\]

4.2. Proof of Theorem 1.4. We are going to prove the absence of discrete spectrum of the operator $A_\kappa$ defined in (4.2). Proposition 2.1 shows that for $\kappa \geq 1$ it holds

\[
\inf \sigma_{\text{ess}}(A_\kappa) = \mathcal{E}_1(\kappa).
\] (4.32)

Since the form domain of $A_\kappa$ coincides with $H^1(\mathbb{R}^4)$, see Lemma 4.2, it suffices to show that

\[
\langle A_\kappa u, u \rangle_{L^2(\mathbb{R}^4)} \geq \mathcal{E}_1(\kappa) \|u\|_2^2 \quad \forall u \in H^1(\mathbb{R}^4)
\] (4.33)

holds true for $\kappa$ large enough. Given $u \in H^1(\mathbb{R}^4)$ we write

\[
u(x, y) = \varphi_\kappa(x) \psi(y) + f(x, y), \quad \psi(y) = \int_{\mathbb{R}^2} \varphi_\kappa(x) u(x, y) \, dx.
\] (4.34)

Then

\[
\int_{\mathbb{R}^2} \varphi_\kappa(x) f(x, y) \, dx = 0 \quad \forall y \in \mathbb{R}^2,
\] (4.35)

and integrating by parts we obtain

\[
\langle A_\kappa u, u \rangle_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^2} |\nabla_y \psi|^2 \, dy + \mathcal{E}_1(\kappa) \int_{\mathbb{R}^2} |\psi|^2 \, dy + \int_{\mathbb{R}^4} \left(|\nabla f|^2 + V_{\text{ctr}}(\kappa^{-\frac{1}{2}} x) f^2\right) \, dx \, dy
\]
\[
- \int_{\mathbb{R}^4} \left(V_{\text{ctr}}(\kappa^{-\frac{1}{2}} y) - \kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|)\right) (\varphi_\kappa \psi + f^2) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^2} |\nabla_y \psi|^2 \, dy + \mathcal{E}_1(\kappa) \|u\|_2^2 + \int_{\mathbb{R}^4} \left(|\nabla f|^2 + (V_{\text{ctr}}(\kappa^{-\frac{1}{2}} x) - \mathcal{E}_1(\kappa)) f^2\right) \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^2} \psi^2(y) \left(\int_{\mathbb{R}^2} \kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi^2_\kappa(x) \, dx - V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |y|)\right) \, dy
\]
\[
+ 2\kappa^{-1} \int_{\mathbb{R}^4} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi_\kappa(x) \psi(y) f(x, y) \, dx \, dy
\]
\[
- \int_{\mathbb{R}^4} \left(V_{\text{ctr}}(\kappa^{-\frac{1}{2}} y) - \kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|)\right) f^2 \, dx \, dy.
\] (4.36)

Hence

\[
\langle A_\kappa u, u \rangle_{L^2(\mathbb{R}^4)} \geq \mathcal{E}_1(\kappa) \|u\|_2^2
\]
\[
+ \int_{\mathbb{R}^4} \left(|\nabla f|^2 + (V_{\text{ctr}}(\kappa^{-\frac{1}{2}} x) - \mathcal{E}_1(\kappa) + 2\kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) f^2\right) \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^2} \psi^2(y) \left(\int_{\mathbb{R}^2} 2\kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi^2_\kappa(x) \, dx - V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |y|)\right) \, dy,
\] (4.37)

where we have used the inequality

\[
2V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi_\kappa \psi f \geq V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi_\kappa^2 \psi^2 + V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) f^2
\]
and the fact that $V_{cte} < 0$. Note that the last term in (4.37) is positive for $\kappa$ large enough by Lemma 4.4. Moreover, since $\mathcal{E}_1(\kappa)$ is simple, Lemma 4.1 and equation (4.35) ensure that
\[
\int_{\mathbb{R}^2} \left[ |\nabla_x f|^2 + (V_{cte}(\kappa^{-\frac{1}{2}} x) - \mathcal{E}_1(\kappa)) f^2(x, y) \right] \, dx \geq \int_{\mathbb{R}^2} (\mathcal{E}_2(\kappa) - \mathcal{E}_1(\kappa)) f^2(x, y) \, dx \geq \frac{E_2 - E_1}{2} \int_{\mathbb{R}^2} f^2(x, y) \, dx
\] holds for all $y \in \mathbb{R}^2$ and $\kappa$ large enough. Note also that $E_2 - E_1 > 0$ and that $E_1$ is simple. Hence for every $\eta \in (0, 1)$ it holds
\[
\left\langle A_\kappa u, u \right\rangle_{L^2(\mathbb{R}^4)} - \mathcal{E}_1(\kappa) \|u\|^2_2 \geq \frac{E_2 - E_1}{2} \|f\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^4} \left[ (1 - \eta) |\nabla_x f|^2 + V_{cte}(\kappa^{-\frac{1}{2}} x) f^2 - \mathcal{E}_1(\kappa) f^2 \right] \, dx dy + \int_{\mathbb{R}^4} \left( \eta |\nabla f|^2 + 2\kappa^{-1} V_{cte}(\kappa^{-\frac{1}{2}} |x - y|) f^2 \right) \, dx dy
\]
(4.39)

By scaling and Lemma 4.1 it follows that for $\kappa \to \infty$
\[
\inf \sigma \left( - (1 - \eta) \Delta_x + V_{cte}(\kappa^{-\frac{1}{2}} x) - \mathcal{E}_1(\kappa) \right) = \log(1 - \eta) + o_\kappa(1) \quad \forall \eta \in (0, 1),
\]
(4.40)

where $o_\kappa(1)$ denotes a quantity which tends to zero as $\kappa \to \infty$. Hence inserting
\[
\eta = 1 - \exp((E_1 - E_2)/4)
\]
into (4.39) we get
\[
\left\langle A_\kappa u, u \right\rangle_{L^2(\mathbb{R}^4)} - \mathcal{E}_1(\kappa) \|u\|^2_2 \geq \left( \frac{E_2 - E_1}{4} + o_\kappa(1) \right) \|f\|_2^2 + \int_{\mathbb{R}^4} \left( \eta |\nabla f|^2 + 2\kappa^{-1} V_{cte}(\kappa^{-\frac{1}{2}} |x - y|) f^2 \right) \, dx dy
\]
In order to estimate the second term on the right hand side we use again the change of variables $(x, y) \mapsto (s, t) = (x - y, \frac{t + g(s, t)}{2})$. This gives
\[
\int_{\mathbb{R}^4} \left( \eta |\nabla f|^2 + 2\kappa^{-1} V_{cte}(\kappa^{-\frac{1}{2}} |x - y|) f^2 \right) \, dx dy = \int_{\mathbb{R}^4} \left( 2\eta |\nabla_x g|^2 + \frac{1}{2} \eta |\nabla_t g|^2 + 2\kappa^{-1} V_{cte}(\kappa^{-\frac{1}{2}} |s|) g^2 \right) \, ds dt,
\]
where $g(s, t) = f(t + s/2, t - s/2)$. In view of Lemma 4.2 we then obtain the lower bound
\[
\left\langle A_\kappa u, u \right\rangle_{L^2(\mathbb{R}^4)} - \mathcal{E}_1(\kappa) \|u\|^2_2 \geq \left( \frac{E_2 - E_1}{4} - O(\kappa^{-1} \log \kappa) + o_\kappa(1) \right) \|f\|_2^2,
\]
which proves (4.33). \hfill \Box

4.3. Proof of Theorem 4.4 (ii).

**Proposition 4.5.** If $\kappa \in [k_e, 1)$, then $\mathcal{H}_\kappa$ has infinitely many discrete eigenvalues.

**Proof.** Let $\kappa \in [k_e, 1)$. In view of Proposition 2.1 and the variational principle it suffices to show that there exists a subspace $\mathcal{F}_\kappa \subset H^1(\mathbb{R}^4)$ such that
\[
\dim(\mathcal{F}_\kappa) = \infty \quad \land \quad \forall u \in \mathcal{F}_\kappa : \left\langle A_\kappa u, u \right\rangle_{L^2(\mathbb{R}^4)} < \mathcal{E}_1(\kappa) \|u\|^2_2.
\]
By choosing

\[ u(x, y) = \varphi_\kappa(x) \psi(y), \psi \in H^1(\mathbb{R}^2) \]

and using the calculations made in the proof of Theorem 1.4 (with \( f = 0 \)) we obtain the identity

\[ \langle A_\kappa u, u \rangle_{L^2(\mathbb{R}^4)} = E_1(\kappa) \| u \|^2_2 + \int_{\mathbb{R}^2} |\nabla_y \psi|^2 \, dy + \int_{\mathbb{R}^2} U_\kappa(y) \psi^2(y) \, dy, \]

where

\[ U_\kappa(y) = \int_{\mathbb{R}^2} \kappa^{-1} V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |x - y|) \varphi_\kappa^2(x) \, dx - V_{\text{ctr}}(\kappa^{-\frac{1}{2}} |y|). \]

A direct calculation now shows that

\[ \lim_{|y| \to \infty} |y|U_\kappa(y) = \kappa^{1/2} - \kappa^{-1/2} < 0. \]

Hence by standard results of spectral theory it follows that there exists an infinite-dimensional subspace \( G_\kappa \subset H^1(\mathbb{R}^2) \) such that

\[ \int \nabla_y \psi^2 \, dy + \int U_\kappa(y) \psi^2(y) \, dy < 0 \quad \forall \psi \in G_\kappa. \]

Setting \( F_\kappa = \{ u \in H^1(\mathbb{R}^4) : u(x, y) = \varphi_\kappa(x) \psi(y), \psi \in G_\kappa \} \) then completes the proof of (4.41). \( \square \)

4.4. Small coupling.

**Lemma 4.6.** The number of discrete eigenvalues of the operator \( H_\kappa(v) \) is non-decreasing in \( \kappa \) on the interval \( (0, k_e) \).

**Proof.** By Proposition 2.1 the number of discrete eigenvalues of the operator \( H_\kappa(V) \) is equal to \( N(H_\kappa(V), \Lambda_0(V))_{L^2(\mathbb{R}^4)} \) for all \( \kappa \) in the interval \( (0, k_e) \). Let \( 0 < \kappa < k_e \) and assume that \( N(H_\kappa(V), \Lambda_0(V))_{L^2(\mathbb{R}^4)} = N \geq 1 \). Then there exist \( \psi_1, \psi_2 \ldots \psi_N \in H^1(\mathbb{R}^4) \) (which can be chosen real valued) and \( E_1, E_2, \ldots E_N \) such that

\[ H_\kappa(V) \psi_j = E_j \psi_j, \quad E_j < \Lambda_0(v) \quad \forall j = 1, \ldots, N, \]

and \( \langle \psi_j, \psi_k \rangle_{L^2(\mathbb{R}^4)} = \delta_{jk} \). Since by definition of \( \Lambda_0(V) \):

\[ \int |\nabla \psi_j|^2 \, dx \, dy - \int \psi_j^2(x, y) v(x - y) \, dx \, dy \geq \Lambda_0(V) > E_j = \langle H_\kappa(v) \psi_j, \psi_j \rangle \quad \forall j = 1, \ldots, N, \]

it follows that

\[ \int \psi_j^2(x, y) v(y) \, dx \, dy - \int \psi_j^2(x, y) v(x) \, dx \, dy < 0 \quad \forall j = 1, \ldots, N. \tag{4.42} \]

Now let \( \kappa' \in (\kappa, k_e) \). Then in view of (4.42) we have

\[ \langle H_{\kappa'}(V) \psi_j, \psi_j \rangle_{L^2(\mathbb{R}^4)} < \langle H_\kappa(V) \psi_j, \psi_j \rangle_{L^2(\mathbb{R}^4)} < \Lambda_0(v) \quad \forall j = 1, \ldots, N, \]

and since \( \psi_j \) are mutually orthonormal, this implies that \( N(H_{\kappa'}(v), \Lambda_0(v))_{L^2(\mathbb{R}^4)} \geq N \). \( \square \)
Appendix A.

**Lemma A.1.** Let \( v \) satisfy assumption \([\square]\) and let \( 0 < \varepsilon < \frac{1}{3} \). Then there exists a constant \( c_1 > 0 \) such that
\[
\Delta_\kappa(x) = \sqrt{\kappa} (v(0) - v(\kappa^{-\frac{1}{2}} x)) + \kappa^{2\varepsilon} \chi(\kappa^{-\varepsilon} x) \geq c_1 \kappa^{2\varepsilon} \quad \forall x \in \mathbb{R}, \forall \kappa \geq 1. \tag{A.1}
\]

**Proof.** Since \( \Delta_\kappa(x) = \Delta_\kappa(-x) \), it suffices to prove (A.1) for all \( x \geq 0 \). Let \( a > 0 \) such that \( v(x) = 0 \) whenever \( x > a \). Let \( t := x\kappa^{-1/4} \) and fix a \( \kappa \geq 1 \). From assumption \([\square]\) and the definition of \( \Delta_\kappa \) and \( \chi \) it follows that
\[
\Delta_\kappa(x) = \Delta_\kappa(t \kappa^{-\frac{1}{2}}) \geq \min\{v(0), 1\} \kappa^{2\varepsilon} \quad \forall t \in [0, \kappa^{\varepsilon^{-\frac{1}{2}}} \cup (a, \infty)].
\]

Now define
\[
t_0 := \frac{3 \omega^2}{\|v''\|_{\infty}},
\]
keeping in mind that by assumption \([\square]\) we have \( \|v''\|_{\infty} > 0 \). In view of (3.12) it then follows that
\[
\sqrt{\kappa} v(0) - \sqrt{\kappa} v(t) \geq \sqrt{\kappa} \frac{\omega^2}{2} t^2 \geq \frac{\omega^2}{2} \kappa^{2\varepsilon} \quad \forall t \in [\kappa^{\varepsilon^{-\frac{1}{2}}}, t_0].
\]

To complete the proof we note that the function \( v(0) - v(t) \) attains a positive minimum on \([t_0, a]\). Therefore
\[
\sqrt{\kappa} v(0) - \sqrt{\kappa} v(t) \geq c \sqrt{\kappa} \quad \forall t \in [t_0, a]
\]
holds true with some \( c > 0 \) independent of \( \kappa \). \( \square \)

**Lemma A.2.** Let \( b_2 \) and \( \kappa_2 \) be given by equation (3.38) and Lemma 3.2 respectively. Then for all \( \kappa \geq \kappa_2 \) it holds
\[
\int_{\mathbb{R}^2} \psi_\kappa(x') v^2(\kappa^{-\frac{1}{2}} (x' - y')) y^2 \, dx' \, dy' \leq 2 \kappa^{\frac{3}{2}} m_{2,2} + 2 \kappa^{\frac{5}{2}} m_{0,2} \mu_2 \tag{A.2}
\]
\[
\int_{\mathbb{R}} b_2^2(y) \, dy \leq \kappa^{\frac{1}{4}} m'_{0,1} \mu_1 \tag{A.3}
\]
\[
\int_{\mathbb{R}} b_2^2(y) y^2 \, dy \leq \kappa^{\frac{3}{4}} m'_{2,1} \mu_1 + \frac{1}{2} \kappa^{\frac{1}{2}} m'_{0,1} \mu_3 \tag{A.4}
\]
where we adopted notation (3.44).

**Proof.** We have
\[
\int_{\mathbb{R}^2} \psi_\kappa^2(x') v^2(\kappa^{-\frac{1}{2}} (x' - y')) y^2 \, dx' \, dy' = \kappa^{\frac{1}{2}} \int_{\mathbb{R}^2} \psi_\kappa^2(x') (r + x')^2 v^2(r) \, dr \, dx' = 2 \kappa^{\frac{3}{2}} \int_{\mathbb{R}^2} \psi_\kappa^2(x') (r^2 + x'^2) v^2(r) \, dr \, dx' = 2 \kappa^{\frac{3}{2}} m_{2,2} + 2 \kappa^{\frac{1}{2}} m_{0,2} \mu_2.
\]

To prove (A.3) we introduce the new variable \( s = \kappa^{-\frac{1}{2}} (y - t) \) to get
\[
\int_{\mathbb{R}} b_2^2(y) \, dy \leq \int_{\mathbb{R}} \psi_\kappa^2(x) \int_0^x |v'(s)| k^{\frac{3}{2}} \, ds \, dt \leq \kappa^{\frac{1}{2}} m'_{0,1} \int_{\mathbb{R}} \psi_\kappa^2(x) |x| \, dx = \kappa^{\frac{1}{2}} m'_{0,1} \mu_1.
\]
Similarly,

$$\int_R b_2^2(y) \, y^2 \, dy \leq \int_R \psi_\kappa^2(x) \int_0^x \left( \int_R (\kappa^{\frac{1}{2}} s + t)^2 |v'(s)| \kappa^{\frac{1}{2}} \, ds \right) \, dt \, dx$$

$$= \int_R \psi_\kappa^2(x) \int_0^x \left( \int_R (\kappa^{\frac{1}{2}} s^2 + t^2) |v'(s)| \kappa^{\frac{1}{2}} \, ds \right) \, dt \, dx$$

$$\leq \kappa^{\frac{1}{4}} \int_R \psi_\kappa^2(x)(\kappa^{\frac{1}{2}} m_{2,1}' x + m_{0,1}' x^3) \, dx$$

$$= \kappa^{\frac{3}{2}} m_{2,1}' \mu_1 + \kappa^{\frac{1}{4}} m_{0,1}' \mu_3 .$$

Acknowledgements.

T.G.P. is supported by the QUSCOPE Center, which is funded by the Villum Foundation. H.C. was partially supported by the Danish Council of Independent Research — Natural Sciences, Grant No. DFF-4181-00042. H.K. was partially supported by the No. MIUR-PRIN2010-11 grant for the project ”Calcolo delle variazioni.”

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