FURSTENBERG SYSTEMS OF HARDY FIELD SEQUENCES AND APPLICATIONS

By

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Dedicated to the memory of Michael Boshernitzan

Abstract. We study measure preserving systems, called Furstenberg systems, that model the statistical behavior of sequences defined by smooth functions with at most polynomial growth. Typical examples are the sequences \((n^2), (n \log n),\) and \((\lfloor n^2/2 \rfloor, \alpha)\), \(\alpha \in \mathbb{R} \setminus \mathbb{Q},\) where the entries are taken mod 1. We show that their Furstenberg systems arise from unipotent transformations on finite-dimensional tori with some invariant measure that is absolutely continuous with respect to the Haar measure and deduce that they are disjoint from every ergodic system. We also study similar problems for sequences of the form \((g(S(n^2/2)y)),\) where \(S\) is a measure preserving transformation on the probability space \((Y, \nu), g \in L^\infty(\nu),\) and \(y\) is a typical point in \(Y.\) We prove that the corresponding Furstenberg systems are strongly stationary and deduce from this a multiple ergodic theorem and a multiple recurrence result for measure preserving transformations of zero entropy that do not satisfy any commutativity conditions.

1 Introduction and main results

1.1 Introduction A well known observation of Furstenberg is that the statistical behavior of the sequence \((p(n))\) on \(\mathbb{T}\) (or the sequence \((e^{2\pi i p(n)})\) on \(\mathbb{S}^1),\) where \(p \in \mathbb{R}[t]\) is an arbitrary polynomial with real coefficients, can be modeled by dynamical systems of algebraic nature (see [24, Theorem 3.13]). For instance, the statistical behavior of the sequence \((n^2\alpha)\) on \(\mathbb{T},\) where \(\alpha \in \mathbb{R},\) can be modeled by the measure preserving system \((\mathbb{T}^2, m_{\mathbb{T}^2}, S)\) where \(m_{\mathbb{T}^2}\) is the Haar measure and \(S: \mathbb{T}^2 \rightarrow \mathbb{T}^2\) is defined by

\[
S(x, y) := (x + \alpha, y + x), \quad x, y \in \mathbb{T}.
\]

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To be more precise, for every $f \in C(\mathbb{T})$, there exists $g \in C(\mathbb{T}^2)$ (in fact, we can choose $g(x, y) := f(y), x, y \in \mathbb{T}$) such that if $a(n) := f(n^2 \alpha), n \in \mathbb{N}$, then

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{\ell} a(n + nj) = \int \prod_{j=1}^{\ell} g(S^{nj}(x, y)) \, dm_{\mathbb{T}^2}(x, y)
$$

holds for all $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}$.

Constructing similar statistical models of dynamical nature, which we will later call “Furstenberg systems”, for other sequences of interest in analytic number theory, is an intriguing problem that has recently attracted a lot of attention. For instance, it is conjectured that the Liouville function $\lambda$ can be modeled by a Bernoulli system (this is equivalent to a conjecture of Chowla) and the Möbius function $\mu$ can be modeled by the direct product of a procyclic system and a Bernoulli system (see [1, 32, 33]). At the moment only partial information about the structure of such measure preserving systems is available; see, for example, [19] for some related progress and [20, 25] for some recent results related to more general bounded multiplicative functions.

In this article we seek to construct dynamical models for sequences arising from smooth non-oscillating functions with at most polynomial growth, and for convenience we take them to belong to some Hardy field. Typical examples include the sequence $(n^2)$ and the sequence $([n^2] \alpha), \alpha \in \mathbb{R} \setminus \mathbb{Q}$, which are thought of as sequences on $\mathbb{T}$. We will see that the first sequence can be modeled by the non-ergodic measure preserving system $(\mathbb{T}^2, m_{\mathbb{T}^2}, S)$ where $S: \mathbb{T}^2 \to \mathbb{T}^2$ is defined by $S(x, y) := (x, y + x), \quad x, y \in \mathbb{T}$.

The second sequence can be modeled by the direct product of two systems of the previous form. We get similar results when $n^2$ is replaced by $n^a$ with $a \in \mathbb{R}_+ \setminus \mathbb{Z}$, but with $S$ replaced by a (non-ergodic) unipotent transformation on $\mathbb{T}^d$ where $d := \lfloor a \rfloor + 1$ (see (1) for the exact form). We also obtain results when $n^{\frac{1}{2}}$ is replaced by $n \log n$, or $n(\log n)^{\frac{1}{2}}, n^{2/3} + n^{2/3}$, where $\alpha$ is irrational, and, perhaps surprisingly, it turns out that these four sequences have different dynamical models that are representative for general Hardy field sequences with at most polynomial growth. The reader will find comprehensive results in Theorems 1.1 and 1.5. Using these results and a disjointness argument, we deduce in Corollary 1.3 that for all $a \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi in^a} \omega(n) = 0 \quad \text{and} \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi in^a \alpha} \omega(n) = 0
$$

for every ergodic sequence $\omega: \mathbb{N} \to \mathbb{U}$ (a notion defined in Section 2.3). Interestingly, in the previous statement the sequence $n^a$ can be replaced by $n \log n$ but
not by \( n(\log n)^b \) for any \( b < 1 \), the reason being that all dynamical models of the first sequence are disjoint from all ergodic systems but some models of the second sequence are ergodic.

Moreover, we study similar problems for sequences of the form \((g(S^{[na]}))\) where \( a \in \mathbb{R}^+ \setminus \mathbb{Z} \), \((Y, \nu, S)\) is an arbitrary measure preserving system, \( g \in L^\infty(\nu) \), and \( y \) is a typical point in \( Y \). Although it seems hard to determine the exact structure of the dynamical models of such sequences, we show in Theorem 1.6 that they enjoy a dilation invariance property called “strong stationarity” (defined in Section 2.2), a property that is not always shared by dynamical models of the above sequences when \( a \) is a positive integer. An important point is that strongly stationary systems have trivial spectrum and their ergodic components are direct products of infinite-step nilsystems and Bernoulli systems, and these structural properties imply disjointness from all ergodic zero-entropy systems. This allows us to deduce in Corollary 1.7 that if \( T, S \) are arbitrary ergodic measure preserving transformations acting on a probability space \((X, \mathcal{X}, \mu)\) and the transformation \( T \) has zero entropy, then for all \( a \in \mathbb{R}^+ \setminus \mathbb{Z} \) and \( f, g \in L^\infty(\mu) \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{[na]} S^{[nw]} g = \int f d\mu \int g d\mu
\]

where the limit is taken in \( L^2(\mu) \). We stress that we impose no commutativity assumptions on \( T, S \). Such multiple ergodic theorems are rather rare because the usual toolbox that enables us to study convergence problems for such averages requires that \( T \) and \( S \) generate a nilpotent group. Lastly, we note that for \( a \in [1, +\infty) \) the previous averages may diverge if we drop the zero-entropy assumption on \( T \) (see [2, Example 7.1] and [21, Section 4]).

### 1.2 Definitions and notation.

In order to facilitate exposition, we introduce some definitions and notation.

For \( N \in \mathbb{N} \) we let \([N] := \{1, \ldots, N\} \). Let \( a : \mathbb{N} \to \mathbb{C} \) be a bounded sequence. If \( A \) is a non-empty finite subset of \( \mathbb{N} \) we let

\[
\mathbb{E}_{n \in A} a(n) := \frac{1}{|A|} \sum_{n \in A} a(n).
\]

We also use a similar notation for finite averages of measures. If \( A \) is an infinite subset of \( \mathbb{N} \) we let

\[
\mathbb{E}_{n \in A} a(n) := \lim_{N \to \infty} \mathbb{E}_{n \in A \cap [N]} a(n)
\]

if the limit exists.
If \((M_k)_{k \in \mathbb{N}}\) is a strictly increasing sequence of positive integers we denote with \(M\) the sequence of intervals \((M_k)_{k \in \mathbb{N}}\). If \(a: \mathbb{N} \to \mathbb{C}\) is a bounded sequence we let

\[
E_{n \in M} a(n) := \lim_{k \to \infty} \frac{1}{M_k} \sum_{n \in [M_k]} a(n)
\]
if the limit exists.

If \(a, b: \mathbb{R}_+ \to \mathbb{R}\) are functions we write
- \(a(t) \prec b(t)\) if \(\lim_{t \to +\infty} a(t)/b(t) = 0\);
- \(a(t) \sim b(t)\) if \(\lim_{t \to +\infty} a(t)/b(t)\) exists and is non-zero;
- \(a(t) \ll b(t)\) if there exists \(C > 0\) such that \(|a(t)| \leq C|b(t)|\) for all large enough \(t \in \mathbb{R}\).

In particular, \(a(t) \prec 1\) means that \(\lim_{t \to +\infty} a(t) = 0\). We say that the function \(a: \mathbb{R}_+ \to \mathbb{R}\) has at most polynomial growth if there exists \(d \in \mathbb{N}\) such that \(a(t) \prec t^d\).

With \(\mathbb{N}\) we denote the set of positive integers and with \(\mathbb{Z}_+\) the set of non-negative integers.

We often denote sequences on \(\mathbb{N}\) or on \(\mathbb{Z}\) by \((a(n))\), instead of \((a(n))_{n \in \mathbb{N}}\) or \((a(n))_{n \in \mathbb{Z}}\); the domain of the sequence is going to be clear from the context.

With \(\mathbb{R}_+\) we denote the set of non-negative real numbers. For \(t \in \mathbb{R}\) we let \(e(t) := e^{2\pi i t}\). With \([t]\) we denote the integer part of \(t\) and with \(\{t\}\) the fractional part of \(t\).

We denote with \(S^1\) the complex unit circle and with \(\mathbb{U}\) the complex unit disc. With \(\mathbb{T}\) we denote the one-dimensional torus and we often identify it with \(\mathbb{R}/\mathbb{Z}\) or with \([0, 1)\). We often denote elements of \(\mathbb{T}\) with real numbers but we are implicitly assuming that these real numbers are taken mod \(1\).

### 1.3 Results about Hardy field sequences.

We start with a result that describes the possible dynamical systems that model the statistical behavior of Hardy field sequences (see definition in Section 3) with at most polynomial growth taken mod \(1\). The role of these “dynamical models” play the “Furstenberg systems” that are associated with these sequences via a variant of a correspondence principle due to Furstenberg; we refer the reader to Section 2.3 for the definition and basic facts regarding these systems.

It turns out that the possible Furstenberg systems admit an algebraic characterization and have the form \(X_d := (\mathbb{T}^{d+1}, \lambda \times m_{\mathbb{R}_+}, S_d)\), where \(d\) is the “degree” of the sequence, \(\lambda\) is a probability measure on \(\mathbb{T}\), and \(S_d\) is the unipotent homomorphism of \(\mathbb{T}^{d+1}\) defined by

\[
S_d(y_0, \ldots, y_d) := (y_0, y_1 + y_0, \ldots, y_d + y_{d-1}), \quad y_0, \ldots, y_d \in \mathbb{T}.
\]
Note that the measure $\lambda \times m_T$ is $S_d$-invariant and the system $X_d$ is non-ergodic unless $\lambda$ is a point mass (in which case it is ergodic if and only if $\lambda = \delta_0$ for some irrational $\alpha \in \mathbb{T}$). For a given Hardy field function $a : \mathbb{R}_+ \to \mathbb{R}$ with at most polynomial growth, the following result determines the structure of all possible Furstenberg systems of the sequence $(a(n))$ on $\mathbb{T}$ and related sequences.

**Theorem 1.1.** Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function with at most polynomial growth and $b : \mathbb{N} \to \mathbb{T}$ or $\mathbb{S}^1$ be defined by $b(n) := a(n) \mod 1$ or $b(n) := e(a(n)), n \in \mathbb{N}$.

(i) If $t^d \log t < a(t) < t^{d+1}$ for some $d \in \mathbb{Z}_+$, then $(b(n))$ has a unique Furstenberg system that is isomorphic to the system $X_d$ defined above with $\lambda := m_T$.

(ii) If $a(t) \sim t^d \log t$ for some $d \in \mathbb{Z}_+$, then $(b(n))$ does not have a unique Furstenberg system, and any Furstenberg system of $(b(n))$ is isomorphic to the system $X_d$ defined above for some probability measure $\lambda \ll m_T$.

(iii) If $t^d \prec a(t) \prec t^d \log t$ for some $d \in \mathbb{Z}_+$, then $(b(n))$ does not have a unique Furstenberg system, and any Furstenberg system of $(b(n))$ is isomorphic to the system $X_d$ defined above with $\lambda = \delta_t$ for some $t \in \mathbb{T}$ (and for any such $b$ all measures $\delta_t$, $t \in \mathbb{T}$, arise).

(iv) If $a(t) = t^d a + \bar{a}(t)$ for some $d \in \mathbb{Z}_+$ where $\bar{a}(t) \prec t^d$ and $\bar{a}$ is irrational, then $(b(n))$ has a unique Furstenberg system that is isomorphic to the system $X_d$ defined above where $\lambda = \delta_{\bar{a}}$, in particular, it is isomorphic to a totally ergodic affine transformation on $\mathbb{T}^d$ with the Haar measure.

(v) If none of the above applies, then $a(t) = p(t) + \epsilon(t) + \bar{a}(t)$ where $p \in \mathbb{Q}[t]$, $\epsilon(t) \to 0$, and $\bar{a}$ is a Hardy field function that is covered in cases (i)–(iv). In particular, there exists $r \in \mathbb{N}$ such that for $k = 0, \ldots, r - 1$ the sequence $b(n + k)$ is covered in cases (i)–(iv).

**Remarks.**

- If $\phi : \mathbb{T} \to \mathbb{C}$ is Riemann-integrable, combining the previous result with Proposition 2.3 below we get similar results for the sequence $\phi(a(n))$.

- The systems described in Part (i) turn out to be strongly stationary (see definition in Section 2.2). For a related result covering Hardy field sequences on nilmanifolds see Theorem 5.2 below.

In order to prove the previous result we show in Lemmas 4.3 and 4.4 below that the sequence $(e(a(n)))$ has the same statistical behavior as the sequence $(S_d f)$ where $S_d : \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ is given by (1) and $f : \mathbb{T}^{d+1} \to \mathbb{C}$ is defined by $f(y) := e(y_d)$ for $y = (y_0, \ldots, y_d) \in \mathbb{T}^{d+1}$. A key tool that we use in the proof of this fact is an equidistribution result of Boshernitzan (see Theorem 3.2) that helps us compute the correlations of the first sequence.
A consequence of the previous structural result is the following disjointness statement:

**Corollary 1.2.** Let \( a : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a Hardy field function such that

\[
t^d \log t < a(t) < t^{d+1} \quad \text{or} \quad a(t) \sim t^d \log t
\]

for some \( d \in \mathbb{Z}_+ \) and \( b : \mathbb{N} \rightarrow \mathbb{T} \) or \( \mathbb{S}^1 \) be defined by \( b(n) := a(n) \mod 1 \) or \( b(n) := e(a(n)), n \in \mathbb{N} \). Then all Furstenberg systems of the sequence \( b \) are disjoint from all ergodic systems.

**Remark.** If \( t^d < a(t) < t^d \log t \) for some \( d \in \mathbb{Z}_+ \), then as shown in Part (iii) of Theorem 1.1 some of the Furstenberg systems of the sequence \( (b(n)) \) are ergodic.

Using the previous result and a disjointness argument we get the following:

**Corollary 1.3.** Let \( a : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a Hardy field function such that

\[
t^d \log t < a(t) < t^{d+1}
\]

for some \( d \in \mathbb{Z}_+ \) or \( a(t) \sim t^d \log t \) for some \( d \in \mathbb{N} \), and let \( b(n) := e(a(n)) \) or \( b(n) := e([a(n])a), n \in \mathbb{N} \), where \( a \in \mathbb{R} \setminus \mathbb{Z} \). Then

\[
(2) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b(n) w(n) = 0
\]

for every ergodic sequence \( w : \mathbb{N} \rightarrow \mathbb{U} \).

**Remarks.**

- Examples of ergodic sequences are all nilsequences, all bounded generalized polynomial sequences (see [6]), or more generally, sequences of the form \( (\phi(S^n) y) \) where \((Y, \nu, S)\) is a uniquely ergodic system, \( y \in Y \), and \( \phi : \mathbb{T} \to \mathbb{C} \) is Riemann-integrable with respect to \( \nu \). Also several multiple correlation sequences are known to be ergodic, for example sequences of the form \( \int \prod_{j=1}^{\ell} T_j^{p_j(n)} f_j d\mu \), where \( T_1, \ldots, T_\ell \) are commuting measure preserving transformations acting on a probability space \((X, \mu), f_1, \ldots, f_\ell \in L^\infty(\mu), \) and \( p_1, \ldots, p_\ell : \mathbb{Z} \to \mathbb{Z} \) are polynomials (for a proof see [30, Section 2.2 and Proposition 3.3]).

- If \( a(t) \sim \log t \), then our argument gives for

\[
b(n) := e(a(n)) \quad \text{or} \quad b(n) := e([a(n])a), \quad n \in \mathbb{N},
\]

with \( a \) irrational, that

\[
\lim_{N \to \infty} \left( \mathbb{E}_{n \in [N]} b(n) w(n) - \mathbb{E}_{n \in [N]} b(n) \cdot \mathbb{E}_{n \in [N]} w(n) \right) = 0.
\]
• If \( t^d \prec a(t) \prec t^d \log t \) for some \( d \in \mathbb{Z}_+ \), then it can be shown that (2) fails for some ergodic sequence \( w: \mathbb{N} \to \mathbb{U} \). We briefly sketch the argument when \( b(n) := e(a(n)) \), \( n \in \mathbb{N} \), and \( d = 1 \). In this case we have \( a(t) := ta_1(t) \) for some \( a_1: \mathbb{R}_+ \to \mathbb{R} \) with \( 1 \prec a_1(t) \prec \log t \). We can choose \( M_k \to +\infty \) such that \( \{ a_1(M_k) \} \to a \). We let \( w(n) := e(-a(n)) \) if \( n \in [M_k/2, M_k] \) for some \( k \in \mathbb{N} \), and \( w(n) := e(-na) \) otherwise. Then it can be shown that the sequence \( w \) has a unique Furstenberg system and it is isomorphic to the system \( (\mathbb{T}, m_\mathbb{T}, S) \), where \( Sx := x - a, x \in \mathbb{T} \) (the argument is similar to the one used in the proof of Part (iii) of Theorem 1.1), hence it is ergodic. But (2) fails since \( E_{M_k/2 \leq n \leq M_k} b(n) w(n) = 1 \) for every \( k \in \mathbb{N} \).

Another consequence of Theorem 1.1 is that under certain growth conditions, equidistribution properties of Hardy field sequences remain valid even if one samples the sequence along an arbitrary ergodic subsequence.

**Corollary 1.4.** Let \( a: \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function such that

\[
 t^d \log t \prec a(t) \prec t^{d+1} 
\]

for some \( d \in \mathbb{Z}_+ \) or \( a(t) \sim t^d \log t \) for some \( d \in \mathbb{N} \). Then for every ergodic sequence \( b: \mathbb{N} \to \mathbb{N} \) and \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), the sequences \( ((a \circ b)(n)) \) and \( \{(a \circ b)(n)) \alpha \) are equidistributed mod 1, and the sequence \( \{(a \circ b)(n)) \) is equidistributed mod \( q \) for every \( q \in \mathbb{N} \).

**Remarks.** • The case where \( b(n) = n, n \in \mathbb{N} \), follows from the equidistribution result of Boshernitzan stated in Theorem 3.2. Other examples of ergodic sequences of integers include the sequences \( b(n) = [na + \beta] \), \( n \in \mathbb{N} \), where \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). More generally, if \( (Y, \nu, S) \) is a uniquely ergodic system, \( U \) is a set of positive measure with boundary of measure zero, \( y_0 \in Y \), and \( E := \{ n \in \mathbb{N}: S^n y_0 \in U \} \), then \( E \) has positive density and the sequence formed by taking the elements of \( E \) in increasing order is an ergodic sequence of integers.

• The conclusion fails if \( t^d \prec a(t) \prec t^d \log t \) for some \( d \in \mathbb{Z}_+ \), for reasons similar to those described in the third remark after Corollary 1.3.

Finally, using Theorem 1.1 we can also describe the structure of Furstenberg systems of sequences of the form \( ([a(n)]) \alpha \) on \( \mathbb{T} \), where \( a: \mathbb{R}_+ \to \mathbb{R} \) is a Hardy field function with at most polynomial growth and \( \alpha \in \mathbb{R} \). For simplicity we restrict our analysis to the special case where \( t^d \log t \prec a(t) \prec t^{d+1} \) for some \( d \in \mathbb{Z}_+ \) and irrational \( \alpha \).
Theorem 1.5. Let \( a: \mathbb{R}^+ \to \mathbb{R} \) be a Hardy field function such that
\[
t^d \log t \prec a(t) \prec t^{d+1}
\]
for some \( d \in \mathbb{Z}_+ \). Let \( \phi: \mathbb{T} \to \mathbb{C} \) be Riemann-integrable, and \( b(n) := \phi([a(n)]\alpha) \), \( n \in \mathbb{N} \), for some \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( S_d: \mathbb{T}^{d+1} \to \mathbb{T}^{d+1} \) be given by (1). Then \( b(n) \) has a unique Furstenberg system and it is a factor of the system \((\mathbb{T}^{2(d+1)}, m_{\mathbb{T}^{2(d+1)}}, S_d \times S_d)\).

1.4 Results about Hardy field iterates. Let \((Y, \nu, S)\) be a measure preserving system and \( g \in L^\infty(\nu) \). The next result gives structural information on the Furstenberg systems of sequences of the form \((g(S^{[a(n)]}y))\) for typical values of \( y \in Y \) (we refer the reader to Sections 2.1 and 2.3 for explanations regarding the terminology used).

Theorem 1.6. Let \( a: \mathbb{R}^+ \to \mathbb{R} \) be a Hardy field function such that
\[
t^{d+\varepsilon} \prec a(t) \prec t^{d+1}
\]
for some \( d \in \mathbb{Z}_+ \) and \( \varepsilon > 0 \). Furthermore, let \((Y, \nu, S)\) be a measure preserving system. Then every strictly increasing sequence of positive integers \((N_k)\) has a subsequence \((N'_k)\) such that for almost every \( y \in Y \) and for every \( g \in L^\infty(\nu) \) the sequence \((g(S^{[a(n)]}y))\) admits correlations on \( N' := ([N'_k])_{k \in \mathbb{N}} \) and the corresponding Furstenberg system has trivial spectrum,\(^1\) and its ergodic components are isomorphic to direct products of infinite-step nilsystems and Bernoulli systems.

Remarks. • It is expected that for almost every \( y \in Y \) for every \( g \in L^\infty(\nu) \) the sequence \((g(S^{[a(n)]}y))\) has a unique Furstenberg system; but this is equivalent to a pointwise convergence result for multiple ergodic averages that at the moment seems out of reach.

• If \((Y, \nu, S)\) is a weak mixing system and \( d \in \mathbb{N} \), then using Theorem 5.6 below (or [4, Theorem A]) it is not hard to show that in the conclusion of Theorem 1.6 all the Furstenberg systems can be taken to be Bernoulli systems. On the other hand, if \((Y, \nu, S)\) is a non-trivial infinite-step nilsystem, then it is possible to show that the corresponding Furstenberg systems are non-ergodic and their ergodic components are infinite-step nilsystems.

• At the expense of using a different averaging scheme we can relax the growth assumption on \( a(t) \) to the assumption \( t^d \prec a(t) \prec t^{d+1} \) for some \( d \in \mathbb{Z}_+ \). To prove this, two modifications are needed in our argument: The first is in the definition of Furstenberg systems, one has to use the weighted averages \( \mathbb{E}_{n \in \mathbb{N}}^{\nu} \) for \( \nu := d(\theta) \) (see

\(^1\) We say that a system \((X, \mu, T)\) has trivial spectrum if \( Tf = e^{2\pi i \alpha} f \) for some \( \alpha \in [0, 1) \) and non-zero \( f \in L^2(\mu) \), implies that \( \alpha = 0 \).
Section 5.1 for their definition) in place of the usual Cesàro averages. The second is in Section 5.1, one has to use the corresponding equidistribution results from [7] for the weighted averages. The same comment applies for Corollary 1.7.

- When \( a(n) = n^2 \), \( n \in \mathbb{N} \), the corresponding Furstenberg systems may have non-trivial spectrum. For example, let \( S: \mathbb{T} \to \mathbb{T} \) be given by \( Sx := x + \alpha, x \in \mathbb{T} \), for some irrational \( \alpha \), and \( g(y) := e^{2\pi iy}, y \in \mathbb{T} \). Then it is not hard to show that for every \( y \in \mathbb{T} \) the sequence \( (g(S^{n^2}y)) \) has a unique Furstenberg system and it is isomorphic to the system \( (\mathbb{T}^2, m_{\mathbb{T}^2}, R) \) where \( R \) is defined by \( R(z, w) := (z + \alpha, w + z) \), \( z, w \in \mathbb{T} \). We also remark that this system is not strongly stationary; this is in contrast with Theorem 5.1 below (which covers the case of fractional powers).

The proof of Theorem 1.6 is less direct than the one of Theorem 1.1 because it appears to be hard to exhibit precise systems that model the statistical behavior of the sequence \( (g(S^{a(n)}y)) \). Instead, we proceed by showing in Theorem 5.1 that the Furstenberg systems of such sequences are strongly stationary (a property that fails when the sequence \( (a(n)) \) is polynomial). The proof of this fact follows from the multiple ergodic theorem of Proposition 5.7, which in turn is proved using recent deep results of Bergelson, Moreira, and Richter [7], using the theory of characteristic factors of Host–Kra [27] and equidistribution results on nilmanifolds. The structure of strongly stationary systems was determined in [21, 29] and we use these structural results as a black box in order to complete the proof of Theorem 1.6.

Using the structural result of Theorem 1.6 and a disjointness argument we deduce the following multiple ergodic theorem and a corresponding multiple recurrence result:

**Corollary 1.7.** Let \( a: \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function such that

\[
t^{d+\epsilon} < a(t) < t^{d+1}
\]

for some \( d \in \mathbb{Z}_+ \) and \( \epsilon > 0 \). Furthermore, let \((X, \mathcal{X}, \mu)\) be a probability space and \( T, S: X \to X \) be measure preserving transformations (not necessarily commuting). Suppose that the system \((X, \mu, T)\) has zero entropy. Then:

(i) For every \( f, g \in L^\infty(\mu) \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{[a(n)]} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)
\]

where the limit is taken in \( L^2(\mu) \).

(ii) For every \( A \in \mathcal{X} \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^n A \cap S^{[a(n)]} A) \geq (\mu(A))^3.
\]
Remarks. • Using weighted averages and the second remark after Theorem 1.6 we can get a variant of (3) and use it to deduce that if a Hardy field function \( a : \mathbb{R}_+ \to \mathbb{R} \) satisfies \( t^d < a(t) < t^d+1 \) for some \( d \in \mathbb{Z}_+ \), then for every \( A \in \mathcal{X} \) and \( \varepsilon > 0 \) we have

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^n A \cap S^{a(n)} A) \geq (\mu(A))^3 - \varepsilon.
\]

• The conclusion of Corollary 1.7 fails if we do not assume that \((X, \mu, T)\) has zero entropy, since for every strictly increasing sequence of integers \((b(n))\) and every \( c : \mathbb{N} \to [-1, 1] \) there exist Bernoulli systems \((X, \mu, T)\) and \((X, \mu, S)\), and \( f, g \in L^\infty(\mu) \), such that \( c(n) := \int T^nf \cdot S^{b(n)} g \ d\mu, \ n \in \mathbb{N} \), (see [21, Section 4]), and as a consequence the averages

\[
\frac{1}{N} \sum_{n=1}^{N} \int T^nf \cdot S^{b(n)} g \ d\mu
\]

do not always converge.

• If we assume that the transformations \( T, S \) commute, then it is known by [14] that the Pinsker factor is characteristic for pointwise convergence of the averages in (3) and as a consequence for mean convergence. Hence, in this case, we get mean convergence in (3), (4) without the assumption that the system \((X, \mu, T)\) has zero entropy (but in the commutative case this can also be obtained by using the method of [18]).

1.5 Open problems For a given ergodic system \((Y, \nu, S)\) and function \( g \in L^\infty(\nu) \), it is also natural to study the possible Furstenberg systems of sequences of the form \( g(S^{p(n)} y) \) where \( p \) is a polynomial with integer coefficients and \( y \) is a typical point in \( Y \). When \( p(n) = n \), it is an easy consequence of the pointwise ergodic theorem that for almost every \( y \in Y \) the sequence \( g(S^{n} y) \) has a unique Furstenberg system and it is a factor of the system \((Y, \nu, S)\) (see Proposition 2.5). The situation is dramatically different when one considers non-linear polynomials in which case one expects severe restrictions on the structure of the possible Furstenberg systems. Furthermore, different Furstenberg systems arise than those arising in Theorem 1.6.

**Problem 1.** Let \((Y, \nu, S)\) be a system, \( p \in \mathbb{Z}[t] \) be a non-linear polynomial, and \( g \in L^\infty(\nu) \). Show that for almost every \( y \in Y \) the sequence \( g(S^{p(n)} y) \) has a unique Furstenberg system that is ergodic and isomorphic to a direct product of an infinite-step nilsystem and a Bernoulli system.
If one assumes in Problem 1 uniqueness of the Furstenberg system, then using the multiple ergodic theorem from [3] it is easy to deduce that for \((Y, \nu, S)\) weak mixing, and \(p \in \mathbb{Z}[t]\) non-linear, for almost every \(y \in Y\) the Furstenberg system of the sequence \((g(S^{p(n)}y))\) is a Bernoulli system. On the other hand, proving uniqueness of the Furstenberg system seems very hard as this amounts to proving a pointwise convergence result for multiple ergodic averages that currently seems out of reach. So as a first step for an unconditional result, one probably has to compromise with a result in the spirit of Theorem 1.6 that describes some of the possible Furstenberg systems of the sequence \((g(S^{p(n)}y))\).

Lastly, it would be interesting to know if a variant of Corollary 1.7 holds when \(a(t)\) is a polynomial; this is the context of the next problem.

**Problem 2.** Let \((X, \mathcal{X}, \mu)\) be a probability space, and \(T, S: X \to X\) be measure preserving transformations. Suppose that the system \((X, \mu, T)\) has zero entropy and \(f, g \in L^\infty(\mu)\).

(i) Is it true that the averages

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{n}f \cdot S^{p(n)}g
\]

converge in \(L^2(\mu)\) when \(p(n) = n\) or \(p(n) = n^2\)?

(ii) Is it true that for every \(A \in \mathcal{X}\) with \(\mu(A) > 0\) there exists \(n \in \mathbb{N}\) such that

\[
\mu(A \cap T^nA \cap S^{p(n)}A) > 0
\]

when \(p(n) = n\) or \(p(n) = n^2\)?

The method used to prove Corollary 1.7 does not give a positive result in this case. The reason is that for typical \(x \in X\) for every \(f \in L^\infty(\mu)\) the Furstenberg systems of the sequences \((f(T^n x))\) and \((g(S^{p(n)}x))\) are not always disjoint. We also remark that Questions (i) and (ii) have a negative answer if one drops the zero entropy assumption (see [2, Example 7.1] and [24, Page 40], or [5], for \(p(n) = n\), and [21, Section 4] for general polynomial \(p\)).

One can also ask similar questions for averages of the form

\[
\frac{1}{N} \sum_{n=1}^{N} T^{[an]}f \cdot S^{[bn]}g
\]

where \(a, b > 1\) are distinct non-integers. We remark that if either \(a\) or \(b\) is in \((0, 1)\), then a relatively simple argument gives mean convergence without any assumption on \(T\) and \(S\) (for \(a, b \in (0, 1)\) see [17, Proposition 6.4] or [15]).
2 Background in ergodic theory

2.1 Measure preserving systems. Throughout the article, we make the standard assumption that all probability spaces \((X, \mathcal{X}, \mu)\) considered are Lebesgue, meaning, \(X\) can be given the structure of a compact metric space and \(\mathcal{X}\) is its Borel \(\sigma\)-algebra. A measure preserving system, or simply a system, is a quadruple \((X, \mathcal{X}, \mu, T)\) where \((X, \mathcal{X}, \mu)\) is a probability space and \(T: X \to X\) is an invertible, measurable, measure preserving transformation. We typically omit the \(\sigma\)-algebra \(\mathcal{X}\) and write \((X, \mu, T)\). The system is **ergodic** if the only sets that are left invariant by \(T\) have measure 0 or 1. It is **totally ergodic** if the system \((X, \mu, T^n)\) is ergodic for every \(n \in \mathbb{N}\). It is **weak mixing** if the system \((X \times X, \mu \times \mu, T \times T)\) is ergodic. We say that \(\lambda \in S^1\) is an **eigenvalue** of \((X, \mu, T)\) if there exists non-zero \(f \in L^2(\mu)\) such that \(Tf = \lambda f\). Throughout, for \(n \in \mathbb{N}\) we denote by \(T^n\) the composition \(T \circ \cdots \circ T\) (\(n\) times) and let \(T^{-n} := (T^n)^{-1}\) and \(T^0 := \text{id}_X\). Also, for \(f \in L^1(\mu)\) and \(n \in \mathbb{Z}\) we denote by \(T^nf\) the function \(f \circ T^n\).

In order to avoid unnecessary repetition, we refer the reader to [19, 28] for some other standard notions from ergodic theory. In particular, the reader will find in Section 2 and in Appendix A of [19] the definition of the terms factor, conditional expectation with respect to a factor, isomorphism, inverse limit, finite-step nilsystem, infinite-step nilfactor, Bernoulli system, ergodic decomposition, joining, and disjoint systems; all these notions are used in this article.

2.2 Strong stationarity We define here a notion that plays a crucial role in the proof of Theorem 1.6 and Corollary 1.7.

**Definition.** Let \((X, \mu, T)\) be a system. We say that
- a conjugation closed sub-algebra \(\mathcal{F}\) of \(L^\infty(\mu)\) is **T-generating**, if the linear span of elements of the form \(T^n f\), with \(f \in \mathcal{F}\) and \(n \in \mathbb{N}\), is dense in \(L^2(\mu)\);
- the system \((X, \mu, T)\) is **strongly stationary**, if there exists a T-generating set \(\mathcal{F}\) such that for every \(r \in \mathbb{N}\) we have

\[
\int \prod_{j=1}^\ell T^{n_j} f_j \, d\mu = \int \prod_{j=1}^\ell T^{rn_j} f_j \, d\mu
\]

for all \(\ell \in \mathbb{N}, n_1, \ldots, n_\ell \in \mathbb{Z}\), and \(f_1, \ldots, f_\ell \in \mathcal{F}\).

**Remark.** It follows from [29] that a system \((X, \mu, T)\) is strongly stationary if and only if there exists a \(T\)-generating set \(\mathcal{F}\) and measure preserving maps \(\tau_n\) on \((X, \mu)\), \(n \in \mathbb{N}\), such that \(f(\tau_n x) = f(x), f \in \mathcal{F}\), and \((T\tau_n)(x) = (\tau_n T^n)(x)\) for every \(x \in X\) and \(n \in \mathbb{N}\).
It is easy to verify that Bernoulli systems are strongly stationary. It is shown in [29] that if an ergodic system is strongly stationary, then it is necessarily Bernoulli. An example of a non-ergodic strongly stationary system is given by the transformation $T: \mathbb{T}^2 \to \mathbb{T}^2$ with the Haar measure $m_{\mathbb{T}^2}$, defined by

$$T(x, y) := (x, y + x), \quad x, y \in \mathbb{T}.$$

The reader can verify that the set $\mathcal{F} := \{f(y): f \in L^\infty(m_{\mathbb{T}})\}$ is $T$-generating and (5) is satisfied. In a similar fashion, it can be shown that the systems $X_d$, defined by the transformation $S_d$ in (1), are strongly stationary when we take the Haar measure $m_{\mathbb{T}^{d+1}}$ on $\mathbb{T}^{d+1}$.

The structure of general strongly stationary systems was determined in [16]. We will use the following structural consequence of the main results in [16, 29]:

**Theorem 2.1.** Strongly stationary systems have trivial spectrum and their ergodic components are direct products of infinite-step nilsystems and Bernoulli systems.

2.3 Furstenberg systems of sequences. In this subsection we reproduce the notion of a Furstenberg system from [19] in a slightly more general context and record some basic related facts that will be used later.

**Definition.** Let $(Y, d)$ be a compact metric space and $M := ([M_k])_{k \in \mathbb{N}}$ be a sequence of intervals with $M_k \to \infty$. We say that a finite collection of bounded sequences $a_1, \ldots, a_\ell: \mathbb{Z} \to Y$ admits joint correlations on $M$, if the limits

$$\lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} \prod_{j=1}^s f_j(\tilde{a}_j(m + n_j))$$

exist for all $s \in \mathbb{N}$, all $n_1, \ldots, n_s \in \mathbb{Z}$ (not necessarily distinct), all $f_1, \ldots, f_s \in C(Y)$, and all $\tilde{a}_1, \ldots, \tilde{a}_s \in \{a_1, \ldots, a_\ell\}$.

**Remarks.** • Given $a_1, \ldots, a_\ell: \mathbb{Z} \to Y$, since $C(Y)$ is separable, using a diagonal argument, we get that every sequence of intervals $M = ([M_k])_{k \in \mathbb{N}}$ has a subsequence $M' = ([M'_k])_{k \in \mathbb{N}}$, such that the sequences $a_1, \ldots, a_\ell$ admit joint correlations on $M'$.

• Let $X := (Y^\ell)^\mathbb{Z}$. Note that the algebra generated by functions of the form $x \mapsto h(x_j(k)), x \in X$, for $j \in \{1, \ldots, \ell\}, k \in \mathbb{Z}$, and $h \in C(Y)$, separates points in $C(X)$, where $x := (x(n))_{n \in \mathbb{Z}} = (x_1(n), \ldots, x_\ell(n))_{n \in \mathbb{Z}}$. We conclude that if the sequences $a_1, \ldots, a_\ell: \mathbb{Z} \to Y$ admit joint correlations on $M$, then for all $f \in C(X)$ the following limit exists:

$$\lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} f(T^m a).$$
where \( a := (a_1, \ldots, a_\ell) \in X \) and \( T \) is the shift transformation on \( X \), which is defined by \((Tx)(n) := x(n+1), n \in \mathbb{Z}, x \in X\). Hence, the weak-star limit \( \lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} \delta_{T^m a} \) exists.

If a finite collection of sequences admits joint correlations on a given sequence of intervals, then we use a variant of the correspondence principle of Furstenberg [23, 24] in order to associate a measure preserving system that captures the statistical properties of these sequences.

**Definition.** Let \((Y, d)\) be a compact metric space \( \ell \in \mathbb{N} \) and \( a_1, \ldots, a_\ell : \mathbb{Z} \to Y \) be sequences that admit joint correlations on \( M := ([M_k])_{k \in \mathbb{N}} \). We let \( A := \{a_1, \ldots, a_\ell\} \), \( X := (Y^\ell)^\mathbb{Z} \), \( T \) be the shift transformation on \( X \), defined by \((Tx)(n) := x(n+1), n \in \mathbb{Z}, x \in X\), and \( \mu \) be the weak-star limit \( \lim_{M \to \infty} \mathbb{E}_{m \in [M]} \delta_{T^m a} \) where \( a := (a_1, \ldots, a_\ell) \) is thought of as an element of \( X \).

- We call \((X, \mu, T)\) the **joint Furstenberg system associated with** \( A \) on \( M \), or simply, the **F-system of** \( A \) on \( M \).
- We say that the finite collection \( A \) has a **unique** Furstenberg system, if the weak-star limit \( \lim_{M \to \infty} \mathbb{E}_{m \in [M]} \delta_{T^m a} \) exists, or equivalently, if \( A \) admits joint correlations on \( ([M])_{M \in \mathbb{N}} \).

**Remarks.**

- If we are given sequences \( a_1, \ldots, a_\ell : \mathbb{N} \to Y \), we extend them to \( \mathbb{Z} \) in an arbitrary way; then the measure \( \mu \) will not depend on the extension.
- A sequence may not admit correlations on \( ([M])_{M \in \mathbb{N}} \), so with our definition it may not have a unique Furstenberg system, but nevertheless all its Furstenberg systems could be measure theoretically isomorphic. This happens, for example, when \( a(n) := \{ \log \log n \}, n \in \mathbb{N} \); in this case all Furstenberg systems are isomorphic to the trivial one point system, but \( (a(n)) \) does not admit correlations on \( ([M])_{M \in \mathbb{N}} \).
- It follows from [13, Proposition 3.8] that if a collection of sequences does not have a unique Furstenberg system on \( ([M])_{M \in \mathbb{N}} \), then it has uncountably many Furstenberg systems.
- A collection of sequences \( a_1, \ldots, a_\ell : \mathbb{Z} \to \mathbb{U} \) may have several non-isomorphic Furstenberg systems depending on which sequence of intervals \( M \) we use in the evaluation of their joint correlations. We call any such system a **(joint) Furstenberg system of** \( a_1, \ldots, a_\ell \).

If \( Y = S^1, \ell = 1, a_1 = a, \) and \( F_0 \in C(X) \) is defined by \( F_0(x) := x(0), x \in X\), then letting \( z^1 := z, z^{-1} := \overline{z} \) for \( z \in \mathbb{C} \), we get that the following identities hold (and in fact characterize the measure \( \mu \)):

\[
\mathbb{E}_{m \in \mathbb{M}} \prod_{j=1}^s a^{\epsilon_j}(m + n_j) = \int \prod_{j=1}^s T^{n_j} F_0^{\epsilon_j} d\mu
\]
for all \( s \in \mathbb{N}, n_1, \ldots, n_s \in \mathbb{Z}, \epsilon_1, \ldots, \epsilon_s \in \{-1, 1\} \). Moreover, if all the limits on the left hand side of (7) exist, then the sequence \((a(n))\) admits correlations on \( \mathcal{M} \).

In practice, in order to describe the structure of the Furstenberg system of a sequence \( a: \mathbb{N} \to U \) on \( \mathcal{M} \), we try to find a closed formula for the correlations on the left-hand side of (7) (see, for example, Lemma 4.3) and then try to figure out a simple system and a function that has the same correlations (see, for example, Lemma 4.4). If this is not feasible, then we try to obtain some partial information about these correlations that gives us useful feedback for the structure of the Furstenberg systems (see, for example, Theorem 5.1, which is based on Proposition 5.7).

Using the previous definition we can associate ergodic properties to arbitrary bounded sequences of complex numbers and also to strictly increasing sequences of integers with range a set of positive density.

**Definition.** With \( U \) we denote the complex unit disc. We say that:

- A sequence \( a: \mathbb{N} \to U \) is **ergodic** (or has **zero entropy**), if all its Furstenberg systems have the corresponding property.

- A sequence \( a: \mathbb{N} \to \mathbb{N} \) is **ergodic** (or has **zero entropy**), if it is strictly increasing, its range \( E := a(\mathbb{N}) \) is a set of positive density, and the \( \{0, 1\} \)-valued sequence \( 1_E \) is ergodic (respectively, has zero entropy).

The following lemma is a simple consequence of the definitions and will be used in order to establish strong stationarity for certain bounded sequences.

**Lemma 2.2.** (i) Suppose that the sequence \( a: \mathbb{N} \to U \) admits correlations on \( \mathcal{M} \) and for every \( s \in \mathbb{N} \), and \( n_1, \ldots, n_s \in \mathbb{Z} \), the correlations

\[
\mathcal{E}_{m \in \mathcal{M}} \prod_{j=1}^{s} a_j(m + r n_j)
\]

are independent of \( r \in \mathbb{N} \), for all \( a_1, \ldots, a_s \in \{a, \overline{a}\} \). Then the Furstenberg system of the sequence \( (a(n)) \) on \( \mathcal{M} \) is strongly stationary.

(ii) Let \((Y, d)\) be a compact metric space and suppose that the sequence \( a: \mathbb{N} \to Y \) admits correlations on \( \mathcal{M} \) and for every \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s \in \mathbb{Z} \), the correlations

\[
\mathcal{E}_{m \in \mathcal{M}} \prod_{j=1}^{s} f_j(a(m + r n_j))
\]

are independent of \( r \in \mathbb{N} \) for all \( f_1, \ldots, f_s \in C(Y) \). Then the Furstenberg system of the sequence \( (a(n)) \) on \( \mathcal{M} \) is strongly stationary.
2.4 Furstenberg systems of images of sequences. For a given sequence \( a: \mathbb{N} \to Y \) and “regular” function \( \phi: Y \to \mathbb{C} \) we would like to relate Furstenberg systems of sequences of the form \((\phi(a(n)))\) to those of the sequence \((a(n))\).

**Definition.** Let \( \nu \) be a Borel measure on the compact metric space \((Y, d)\).

- We say that \( \phi: Y \to \mathbb{R} \) is **Riemann-integrable with respect to** \( \nu \), if it is Borel measurable and for every \( \varepsilon > 0 \) there exist \( \phi^-, \phi^+ \in C(Y) \) such that
  \[
  \phi^-(y) \leq \phi(y) \leq \phi^+(y), \quad y \in Y, \quad \text{and} \quad \int (\phi^+ - \phi^-) \, d\nu \leq \varepsilon.
  \]

- We say that a complex valued function \( \phi: Y \to \mathbb{C} \) is **Riemann-integrable with respect to** \( \nu \) if its real and imaginary parts are Riemann-integrable.

**Remark.** It can be shown that \( \phi: Y \to \mathbb{C} \) is Riemann-integrable if the set of discontinuity points of \( \phi \) has \( \nu \)-measure 0.

The next result gives information about the possible Furstenberg systems of images of sequences under Riemann-integrable functions. Its proof is based on some rather standard approximation arguments; for the reader’s convenience we include some details.

**Proposition 2.3.** Let \((Y, d)\) be a compact metric space. Suppose that the sequences \( a_1, \ldots, a_\ell: \mathbb{Z} \to Y \) admit joint correlations on \( M := ([M_k])_{k \in \mathbb{N}} \) and let \((X, \mu, T)\) be their joint Furstenberg system on \( M \). For \( a := (a_1, \ldots, a_\ell) \) let

\[
\nu := \lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} \delta_{a(m)},
\]

where the limit is taken in the weak-star sense, and suppose that the function \( \phi: Y^\ell \to \mathbb{C} \) is Riemann-integrable with respect to the measure \( \nu \). Then the sequence

\[
b(n) := \phi(a_1(n), \ldots, a_\ell(n)), \quad n \in \mathbb{N},
\]

admits correlations on \( M := ([M_k])_{k \in \mathbb{N}} \), the corresponding Furstenberg system is a factor of the system \((X, \mu, T)\), and if \( \phi \) is injective, it is isomorphic to the system \((X, \mu, T)\).

**Proof.** We first remark that the existence of the weak-star limit in (8) follows from our assumption that the sequences \( a_1, \ldots, a_\ell \) admit joint correlations on \( M \).

Let \((X', \mu', T')\) be the Furstenberg system of \( b \) (recall that \( X' = \biguplus \mathbb{Z} \)). We first show that \( b \) admits correlations on \( M \), or equivalently, that

\[
\lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} \prod_{j=1}^s f_j(\phi(a_1(n + n_j), \ldots, a_\ell(n + n_j)))
\]
exists for all \( s \in \mathbb{N} \), all \( n_1, \ldots, n_s \in \mathbb{Z} \) (not necessarily distinct), and all \( f_1, \ldots, f_s \in C(\mathbb{U}) \). By density with respect to the uniform norm we can assume that the functions \( f_1, \ldots, f_s \) are Lip-continuous on \( \mathbb{U} \). Using this and by approximating \( \phi \) in \( L^1(\nu) \) we get that in order to show that (10) holds it suffices to show that the averages in (9) converge since by our assumption the sequences \( a_1, \ldots, a_\ell \) admit joint correlations on \( \mathcal{M} \).

Next, we define the map \( \Phi : X \to X' \) by

\[
\Phi((x_1(n), \ldots, x_\ell(n))_{n \in \mathbb{Z}}) := (\phi(x_1(n), \ldots, x_\ell(n)))_{n \in \mathbb{Z}}.
\]

Since \( \phi \) is Borel measurable, the map \( \Phi \) is a measurable map and we clearly have that \( T' \circ \Phi = \Phi \circ T \). Note also that if \( \phi \) is one to one, then so is \( \Phi \). It remains to show that \( \mu' = \mu \circ \Phi^{-1} \). To this end, let \( f \in C(X') \).

If \( \phi \) is continuous, then \( f \circ \Phi \in C(X) \), hence

\[
\int f \, d(\mu \circ \Phi^{-1}) = \int f \circ \Phi \, d\mu = E_{m \in \mathcal{M}} (f \circ \Phi)(T^m a) = E_{m \in \mathcal{M}} f(T^m a) = \int f \, d\mu'.
\]

Hence, \( \mu' = \mu \circ \Phi^{-1} \).

To get a similar identity when \( \phi \) is Riemann-integrable with respect to \( \nu \), the only part that needs justification is that the identity

\[
(10) \quad \int f \circ \Phi \, d\mu = E_{m \in \mathcal{M}} (f \circ \Phi)(T^m a)
\]

holds for every \( f \in C(X') \). Using uniform approximation and linearity we can assume that \( f \) is a cylinder function, meaning, of the form \( f(x') = \prod_{j=1}^{s} F_{n_j}(x_j) \), for some \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s \in \mathbb{Z} \), where for \( i \in \mathbb{Z} \) we let \( F_i(x') := x'(i), x' \in X' \). Writing elements \( x \in X \) as \( x = (x_1, \ldots, x_\ell) \), where \( x_1, \ldots, x_\ell \in X' \), we have

\[
f(\Phi(x)) = \prod_{j=1}^{s} F_{n_j}(\Phi(x)) = \prod_{j=1}^{s} \phi(x_1(n_j), \ldots, x_\ell(n_j)).
\]

Hence, in order to verify that (10) holds it suffices to show that

\[
(11) \quad \int \prod_{j=1}^{s} \phi_j(x_1(n_j), \ldots, x_\ell(n_j)) \, d\mu(x) = E_{m \in \mathcal{M}} \prod_{j=1}^{s} \phi_j(a_1(m + n_j), \ldots, a_\ell(m + n_j))
\]

whenever \( \phi_1, \ldots, \phi_s : Y^\ell \to \mathbb{C} \) are Riemann-integrable with respect to \( \nu \) (it is convenient to prove this more general version with \( s \) different functions). Furthermore,
for \( j = 1, \ldots, s \), writing \( \phi_j \) as a linear combination (over \( \mathbb{C} \)) of four non-negative real valued functions that are Riemann-integrable with respect to \( \nu \), and using linearity, we see that it suffices to verify the previous identity when the functions \( \phi_1, \ldots, \phi_s \) are real valued and take values in \([0, 1]\). Since (11) holds for continuous functions \( \phi_1, \ldots, \phi_s \), using a standard approximation argument from above and below by continuous functions and (8), we get that (11) holds for Riemann-integrable functions with respect to \( \nu \) as well. This completes the proof. \[ \square \]

We will use the previous result in the proof of Theorem 1.5 in order to show that under suitable assumptions on the sequence \( a : \mathbb{N} \to \mathbb{R} \), all Furstenberg systems of the sequence \( (e([a(n)]a)) \) are factors of joint Furstenberg systems of the sequences \( (a(n)) \) and \( (a(n)a) \) (thought of as sequences on \( \mathbb{T} \)). These three sequences are linked via the identity \( e([a(n)]a) = \phi(a(n)a, a(n)), n \in \mathbb{N} \), where \( \phi : \mathbb{T}^2 \to \mathbb{S}^1 \) is defined by \( \phi(x, y) = e(x - \{y\}a) \), which is Riemann-integrable with respect to \( m_{\mathbb{T}^2} \).

One way to establish uniqueness and determine the structure of the Furstenberg system of a sequence, is to represent it as the image under a “regular” function of an orbit of a point in a uniquely ergodic system. This is the context of the next result (which is well known to experts).

**Corollary 2.4.** Let \( (Y, \nu, S) \) be a uniquely ergodic system and \( g : Y \to \mathbb{C} \) be Riemann-integrable with respect to \( \nu \). Then for every \( y \in Y \) the sequence \( (g(S^ny)) \) has a unique Furstenberg system that is a factor of the system \( (Y, \nu, S) \). Furthermore, if \( g \) is injective, then we have an isomorphism.

**Proof.** Let \( y \in Y \). By unique ergodicity we get that the sequence \( (S^n y) \) has a unique Furstenberg system that is isomorphic to the system \( (Y, \nu, S) \). Moreover, the weak-star limit defined in equation (8) of Proposition 2.3 is equal to \( \nu \). The result now follows from Proposition 2.3. \[ \square \]

It is easy to deduce from the previous result that the sequence \( (\sin n) \) has a unique Furstenberg system that is an ergodic rotation on the circle. Moreover, for \( c_1, c_2, \alpha, \beta \in \mathbb{R} \), the sequences

\[
(c_1 \cos(na) + c_2 \sin(n\beta)) \quad \text{and} \quad (c_1 \mathbf{1}_{[0,1/2]}(\{na\}) + c_2 \mathbf{1}_{[1/2,1/3]}(\{n\beta\}))
\]

have unique Furstenberg systems and they are both factors of rotations on the 2-dimensional torus.

We will also use the following result (again, well known to experts):

**Proposition 2.5.** Let \( (Y, \nu, S) \) be a system and suppose that \( \nu = \int \nu_y \, dv(y) \) is the ergodic decomposition of \( \nu \). Then for every \( g \in L^\infty(v) \) and for almost every \( y \in Y \), the sequence \( (g(S^ny)) \) has a unique Furstenberg system that is a factor of the system \( (Y, \nu_y, S) \) (and as a consequence it is ergodic).
Proof. Let $g \in L^\infty(\nu)$ be bounded by 1. By the pointwise ergodic theorem for almost every $y \in Y$ the sequence $(g(S^ny))$ has a unique Furstenberg system, call it $(X, \mu_y, T)$, where $X = \mathbb{U}^\mathbb{Z}$. Let $\Phi : Y \rightarrow X$ be defined by $\Phi(y) = (g(S^ny))_{n \in \mathbb{Z}}, \quad y \in Y.$ Then $T \circ \Phi = \Phi \circ S$ and the pointwise ergodic theorem easily implies that for almost every $y \in Y$ we have $\nu_y = \mu_y \circ \Phi^{-1}$. This completes the proof.

3 Background on Hardy fields

Let $B$ be the collection of equivalence classes of real valued functions defined on some half line $[c, +\infty)$, where we identify two functions if they agree eventually. A Hardy field $\mathcal{H}$ is a subfield of the ring $(B, +, \cdot)$ that is closed under differentiation (the term Hardy field was first used by the Bourbaki group in [11]). A Hardy field function is a function that belongs to some Hardy field. We are going to assume throughout that all Hardy fields mentioned are translation invariant, meaning, if $a(t) \in \mathcal{H}$, then $a(t + h) \in \mathcal{H}$ for every $h \in \mathbb{R}$.

A particular example of such a Hardy field is the set $\mathcal{LE}$ that was introduced by Hardy in [26] and consists of all logarithmic-exponential functions, meaning all functions defined on some half line $[c, +\infty)$ by a finite combination of the symbols $+, -, \times, :, \log, \exp$, operating on the real variable $t$ and on real constants. For example, the functions $t^a(\log t)^b$ where $a, b \in \mathbb{R}$ are all elements of $\mathcal{LE}$.

Every Hardy field function is eventually monotonic and hence has a limit at infinity (possibly infinite). If one of the functions $a, b : [c, +\infty) \rightarrow \mathbb{R}$ belongs to a Hardy field and the other function belongs to the same Hardy field or to $\mathcal{LE}$, then the limit $\lim_{t \rightarrow +\infty} a(t)/b(t)$ exists (possibly infinite). This property is key and will often justify our use of l’Hospital’s rule. We are going to freely use all these properties without any further explanation in the sequel. The reader can find more information about Hardy fields in [8, 9] and the references therein.

Recall that $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ has at most polynomial growth if $a(t) \prec t^d$ for some $d \in \mathbb{N}$. The most important property of Hardy field functions of at most polynomial growth that will be used throughout this article, is that we can relate their growth rates with the growth rates of their derivatives. The next lemma illustrates this principle and will be used frequently: 

---

2The equivalence classes just defined are often called germs of functions. We choose to use the word function when we refer to elements of $B$ instead, with the understanding that all the operations defined and statements made for elements of $B$ are considered only for sufficiently large values of $t \in \mathbb{R}$.\[\]
Lemma 3.1. Let \( a : \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function with at most polynomial growth.

(i) If \( t^\varepsilon < a(t) \) for some \( \varepsilon > 0 \), then for every \( r \in \mathbb{N} \) we have

\[
a'(t) \sim a(t)/t \quad \text{and} \quad a(t + r) - a(t) \sim a(t)/t.
\]

(ii) If \( a(t) \prec t \), then for every \( r \in \mathbb{N} \) we have

\[
\lim_{t \to +\infty} a'(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} (a(t + r) - a(t)) = 0.
\]

Proof. We prove (i). Applying l’Hospital’s rule (note that all limits below are well defined because \( a(t) \) is a Hardy field function) we get

\[
\lim_{t \to +\infty} \frac{ta'(t)}{a(t)} = \lim_{t \to +\infty} \frac{(\log |a(t)|)'}{(\log t)'} = \lim_{t \to +\infty} \frac{\log |a(t)|}{\log t}.
\]

Since \( a(t) \) has at most polynomial growth and \( t^\varepsilon < a(t) \), the last limit is a positive real number. This proves that \( a'(t) \sim a(t)/t \). Using the mean value theorem we deduce that \( a(t + r) - a(t) \sim a(t)/t \) for every \( r \in \mathbb{N} \).

We prove (ii). Arguing by contradiction suppose that the limit \( \lim_{t \to +\infty} a'(t) \) is non-zero (the limit exists since \( a(t) \) is a Hardy field function). Then an easy application of the mean value theorem gives that the limit \( \lim_{t \to +\infty} a(t)/t \) cannot be zero, contradicting our assumption. Finally, using the mean value theorem we deduce that \( \lim_{t \to +\infty} (a(t + r) - a(t)) = 0 \) for every \( r \in \mathbb{N} \).

We will also use the following equidistribution result:

Theorem 3.2 (Boshernitzan [9]). Let \( a : \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function with at most polynomial growth. Then the sequence \( (a(n)) \) is equidistributed on \( \mathbb{T} \) if and only if

\[
\lim_{t \to +\infty} \frac{|a(t) - p(t)|}{\log t} = +\infty
\]

for every polynomial \( p \in \mathbb{Q}[t] \).

4 Proof of results concerning Hardy field sequences

In this section we will prove the results stated in Section 1.3.

4.1 A preliminary result. We start with a preliminary equidistribution result for Hardy field sequences of sublinear growth. It will be used to define the measure \( \lambda \) that appears in the description of the systems \( X_d \) that are used in Theorem 1.1.
Lemma 4.1. Let \( a : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a Hardy field function such that \( a(t) \prec t \). Let \( \mathcal{P}_a \) denote the set of probability measures on \( \mathbb{T} \) that are weak-star limit points of the sequence of probability measures \( \mathbb{E}_{n \leq N} \delta_{a(n)} \), \( N \in \mathbb{N} \). Then the following hold:

(i) If \( a(t) \succ \log t \), then \( \mathcal{P}_a = \{ m_T \} \).

(ii) If \( a(t) \sim \log t \), then \( \mathcal{P}_a \) is not a singleton and all its elements are absolutely continuous with respect to \( m_T \).

(iii) If \( 1 \prec a(t) \ll \log t \), then \( \mathcal{P}_a = \{ \delta_\alpha, \alpha \in \mathbb{T} \} \).

(iv) If none of the above applies, then \( \lim_{t \rightarrow +\infty} a(t) = \alpha \), for some \( \alpha \in \mathbb{R} \) and \( \mathcal{P}_a = \{ \delta_\alpha \} \).

Proof. Note that (i) follows from Theorem 3.2 and that (iv) is trivial.

We prove (ii). We first show that \( \mathcal{P}_a \) is not unique. In this step we will only use that \( 1 \prec a(t) \ll \log t \). It suffices to show that the averages \( \mathbb{E}_{n \in [N]} e(a(n)) \) do not converge as \( N \rightarrow \infty \). Arguing by contradiction suppose that

\[
\lim_{N \rightarrow \infty} \mathbb{E}_{n \in [N]} e(a(n)) = \alpha
\]

for some \( \alpha \in \mathbb{C} \). Let \( \beta \in \mathbb{S}^1 \) with \( \beta \neq \alpha \). Our assumptions and the mean value theorem imply that \( a(t + 1) - a(t) \rightarrow 0 \) as \( t \rightarrow +\infty \). Using this and that \( |a(t)| \rightarrow +\infty \) as \( t \rightarrow +\infty \) we get that there exist \( N_k \rightarrow +\infty \) such that \( e(a(N_k)) \rightarrow \beta \). Moreover, our assumptions and the mean value theorem easily imply that \( \lim_{c \rightarrow 1^{-}} \sup_{t \in [c,t]} |a(t) - a(s)| = 0 \), hence

\[
\lim_{c \rightarrow 1^{-}} \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [cN_k,N_k]} e(a(n)) = \beta \neq \alpha,
\]

which contradicts (13).

Next we show that if \( \lambda \in \mathcal{P}_a \), then \( \lambda \ll m_T \). Without loss of generality we can assume that \( a(t) \rightarrow +\infty \) as \( t \rightarrow +\infty \) and for simplicity we assume that \( a(t) \) (and hence \( a^{-1}(t) \)) is strictly increasing on \( \mathbb{R}_+ \) and \( a'(t) \) is positive and strictly decreasing on \( \mathbb{R}_+ \). It suffices to show that there exists a constant \( C > 0 \) that depends only on \( a(t) \), such that for every \( c, d \in [0, 1) \) with \( c < d \) we have

\[
\limsup_{N \rightarrow \infty} \frac{\left| \{ n \in [N] : a(n) \in [c, d] \} \right|}{N} \leq C(d - c).
\]

If we prove this, then \( \lambda \leq C m_T \), hence \( \lambda \ll m_T \).

To this end, let

\[
A_N := \{ n \in [N] : a(n) \in [c, d] \}
\]

and note that

\[
A_N = \bigcup_{k=0}^{[a(N)]} \{ n \in [N] : k + c \leq a(n) \leq k + d \} = \bigcup_{k=0}^{[a(N)]} \{ [a^{-1}(k + c), a^{-1}(k + d)] \cap [N] \}.
\]
Since
\[ ||a^{-1}(k + c), a^{-1}(k + d)] \cap [N] - (a^{-1}(k + d) - a^{-1}(k + c))|| \leq 1 \]
for \( k = 0, \ldots, [a(N)] - 1 \), we have
\[ |A_N| = \sum_{k=0}^{[a(N)]-1} (a^{-1}(k + d) - a^{-1}(k + c)) + r_N + O(a(N)) \]
where
\[ r_N := \|[a^{-1}([a(N)] + c), a^{-1}(R_N)] \cap [N]|| \quad \text{and} \quad R_N := \min\{ [a(N)] + d, a(N) \}. \]
Hence, using the mean value theorem we get that
\begin{equation}
|A_N| = (d - c) \sum_{k=0}^{[a(N)]-1} (a^{-1})'(\xi_k) + r_N + O(a(N))
\end{equation}
where \( \xi_k \in [k + c, k + d] \) for \( k = 0, \ldots, [a(N)] - 1 \).

Next, note that our assumption gives \( a(t) = C_1 \log t + e_1(t) \) for some \( C_1 > 0 \) and \( e_1(t) \prec \log t \). Since \( a(t) \) is a Hardy field function, using l’Hospital’s rule we deduce that \( a'(t) = \frac{C_1}{t} + e_2(t) \) where \( e_2(t) \prec \frac{1}{t} \). It follows from this that
\begin{equation}
(a^{-1})'(t) = \frac{1}{a'(a^{-1}(t))} = C_1^{-1} a^{-1}(t) + e_3(t)
\end{equation}
where \( e_3(t) \prec a^{-1}(t) \). Hence, for some \( C_2 > 0 \) that depends only on \( a(t) \), we have
\[ C_2 \cdot (a^{-1})'(\xi_k) \leq a^{-1}(\xi_k) \leq a^{-1}(k + 1) \]
for \( k = 0, \ldots, [a(N)] - 1 \). Using this we get that
\[ C_2 \sum_{k=0}^{[a(N)]-1} (a^{-1})'(\xi_k) \leq \sum_{k=0}^{[a(N)]-1} (a^{-1})(k + 1) \leq \int_1^{a(N)} a^{-1}(t) dt + N \sim N, \]
where to get the asymptotic for the integral we use l’Hospital’s rule, the fundamental theorem of calculus, and that \( a'(t) \sim \frac{1}{t} \).

Finally, we treat the term \( r_N \). First note that if \( a(N) \leq [a(N)] + c \), then \( r_N = 0 \). So we can assume that \( a(N) > [a(N)] + c \) in which case we have \( R_N > [a(N)] + c \). We have that
\[ |r_N - (a^{-1}(R_N) - a^{-1}([a(N)] + c))] \leq 1, \]
and as before, using the mean value theorem and (16), we get that there is a constant \( C_3 > 0 \) such that
\[ C_3(a^{-1}(R_N) - a^{-1}([a(N)] + c)) \leq (R_N - ([a(N)] + c)) a^{-1}(a(N)) \leq (d - c)N, \]
where we used that $R_N \leq a(N)$ to justify the first estimate and that $R_N \leq [a(N)] + d$ to justify the second estimate.

Inserting these estimates in (15) and using that $a(N)/N \to 0$ as $N \to \infty$, we deduce that (14) holds for some $C > 0$.

We prove (iii). First note that since $1 \prec a(t) \prec t$ and $a(t + 1) - a(t) \to 0$ as $t \to +\infty$ (by Lemma 3.1) we have that the sequence $(a(n))$ is dense in $T$.

Let $\alpha \in [0, 1]$. It suffices to show that if $N_k \to \infty$ is such that $\{a(N_k)\} \to \alpha$, then

$$\lim_{k \to \infty} \mathbb{E}_{n \in [N_k]} \delta_{c(n)} = \delta_\alpha$$

where the limit is a weak-star limit. Suppose that $\alpha \in (0, 1)$ (the argument is similar if $\alpha = 0$ or 1) and $0 < c < \alpha < d < 1$. It suffices to show that

$$\lim_{k \to \infty} \frac{|\{n \in [N_k]: \{a(n)\} \in [c, d]\}|}{N_k} = 1. \tag{17}$$

We first claim that for every $r \in (0, 1)$ we have that

$$\lim_{t \to +\infty} (a(t) - a(rt)) = 0. \tag{18}$$

To see this, notice first that our assumption $a(t) \prec \log t$ and l’Hospital’s rule imply that $|a'(t)|$ is eventually decreasing and $a'(t) \prec \frac{1}{t}$. Using this and the mean value theorem, we deduce that for all large enough $t \in \mathbb{R}$ we have

$$|a(t) - a(rt)| \leq |(1 - r)a'(rt)| \to 0 \quad \text{as} \quad t \to +\infty.$$

Since $\{a(N_k)\} \to \alpha \in (c, d)$, it follows from (18) that for large enough $k \in \mathbb{N}$ we have that

$$[rN_k, N_k] \subset \{n \in [N_k]: \{a(n)\} \in [c, d]\}.$$

Hence,

$$\liminf_{k \to \infty} \frac{|\{n \in [N_k]: \{a(n)\} \in [c, d]\}|}{N_k} \geq 1 - r.$$

Since $r \in (0, 1)$ is arbitrary, letting $r \to 0^+$ we deduce (17). This completes the proof.

\[\square\]

### 4.2 Proof of Theorem 1.1.

We are going to deduce Theorem 1.1 from the following result:

**Proposition 4.2.** Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function such that for some $d \in \mathbb{Z}_+$ one has $t^d \prec a(t) \prec t^{d+1}$ or $a(t) = t^d \alpha + \tilde{a}(t)$ where $\tilde{a}(t) \prec t^d$ and $\alpha$ is irrational. We define the measure $\lambda$ on $T$ by

$$\lambda := \lim_{k \to \infty} \mathbb{E}_{n \in [M_k]} \delta_{c(n)} \quad \text{where} \quad c(t) := a^{(d)}(t)/d!,$$
assuming that the previous weak-star limit exists for the sequence $M_k \to +\infty$. Then the sequence $b(n) := e(a(n))$, $n \in \mathbb{N}$, admits correlations on $M := ([M_k])_{k \in \mathbb{N}}$ and the $F$-system of $b$ on $M$ is isomorphic to the system $(\mathbb{T}^{d+1}, \lambda \times m_{\mathbb{T}^d}, S_d)$ where $S_d: \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ is defined by

$$S_d(y_0, \ldots, y_d) = (y_0 + y_1, y_0 + y_2, \ldots, y_0 + y_{d-1}), \quad y_0, \ldots, y_d \in \mathbb{T}.$$ 

Let us first see how we deduce Theorem 1.1 from Proposition 4.2.

**Proof of Theorem 1.1 assuming Proposition 4.2.** First note that since $\phi: \mathbb{S}^1 \to \mathbb{T}$ given by $\phi(e(t)) := t \mod 1$ is well defined, continuous, injective, and $\phi(e(a(n))) = a(n) \mod 1$, $n \in \mathbb{N}$, using Proposition 2.3 we get that it suffices to prove the stated properties for the sequence $b(n) := e(a(n))$, $n \in \mathbb{N}$.

We move now to the proof of the four parts of the theorem for the sequence $b(n) = e(a(n))$, $n \in \mathbb{N}$. Recall that $c(t) = a^{(d)}(t)/d!$, $t \in \mathbb{R}_+$. We establish Part (i). First note that our assumption $r^d \log t < a(t) < r^{d+1}$ and Lemma 3.1 imply that $\log t < c(t) < t$. It follows from Theorem 3.2 that the sequence $(c(n))$ is equidistributed on $\mathbb{T}$. So in this case $\lambda = m_{\mathbb{T}}$ and Proposition 4.2 gives that the sequence $(b(n))$ has a unique $F$-system that is isomorphic to the system $(\mathbb{T}^d, m_{\mathbb{T}^{d+1}}, S_d)$. We establish Parts (ii) and (iii). Let $(X, \mu, T)$ be an $F$-system of $b$ on some sequence of intervals $M$. Then Proposition 4.2 gives that $(X, \mu, T)$ is isomorphic to the system $(\mathbb{T}^d, \lambda \times m_{\mathbb{T}^d}, S)$. Lastly, we show that the sequence $(b(n))$ does not have a unique $F$-system. For this, it suffices to show that the sequence of measures $E_{m \in [M]} \delta_{T^n b}$, $M \in \mathbb{N}$, on $(\mathbb{S}^1)^\mathbb{Z}$ does not converge weak-star as $M \to \infty$. This would follow if we show that for some $k \in \mathbb{Z}$ the limit

$$\lim_{M \to \infty} E_{m \in [M]} e(k \Delta^d a(m))$$

(which is a correlation of the sequence $b$) does not exist, where for $a: \mathbb{R}_+ \to \mathbb{R}$ we let $(\Delta^0 a)(t) := a(t)$, $t \in \mathbb{R}_+$, and for $i \in \mathbb{Z}_+$ we let

$$(\Delta^{i+1} a)(t) := a(t + 1) - a(t), \quad t \in \mathbb{R}_+.$$ 

Note that since $r^d \sim a(t) \ll r^d \log t$, Lemma 3.1 gives that $1 \ll \Delta^d a(t) \ll \log t$. Hence, Part (iii) of Lemma 4.1 gives that for some $k \in \mathbb{Z}$ the limit

$$\lim_{M \to \infty} E_{m \in [M]} e(k \Delta^d a(m))$$

does not exist.

We establish Part (iv). In this case Lemma 3.1 gives that

$$c(t) = a^{(d)}(t)/d! = a/d! + e(t), \quad t \in \mathbb{R}_+,$$
where \( e(t) := \tilde{a}^{(d)}(t) \to 0 \) as \( t \to +\infty \). By Part (iv) of Lemma 4.1 the weak-star limit \( \lim_{M \to \infty} E_{m \in [M]} \delta_{\tilde{m}} \) exists and is equal to the point mass \( \delta_{\tilde{m}} \). Hence, Proposition 4.2 gives that the sequence \( b \) has a unique F-system that is isomorphic to the system \((\mathbb{T}^{d+1}, \delta_{\tilde{m}} \times m_{\mathbb{T}^d}, S_d)\). This system is easily shown to be isomorphic to the system \((\mathbb{T}^d, m_{\mathbb{T}^d}, S'_d)\), where \( S'_d : \mathbb{T}^d \to \mathbb{T}^d \) is defined by

\[
S'_d(y_1, \ldots, y_d) := (y_1 + a/dl!, y_2 + y_1, \ldots, y_d + y_{d-1}), \quad y_1, \ldots, y_d \in \mathbb{T}.
\]

Lastly, since \( \alpha \) is irrational, it is well known that this system is totally ergodic.

We establish Part (v). Suppose that \( a(t) \) does not satisfy any of the conditions in (i)–(iv). We first claim that \( a(t) = p(t) + \epsilon(t) + \tilde{a}(t) \) for some \( p \in \mathbb{Q}[t] \), \( \epsilon : \mathbb{R}_+ \to \mathbb{R} \) with \( \epsilon(t) \to 0 \), and \( \tilde{a} \) is a Hardy field function that is covered in cases (i)–(iv). To see this, let \( d \) be the largest non-negative integer such that \( t^d \ll a(t) \) (then \( a(t) \ll t^{d+1} \)). If \( t^d \ll a(t) \), then \( a(t) \) is covered by parts (i)–(iii). If \( t^d \ll a(t) \) is not satisfied, since \( t^d \ll a(t) \), we have that \( \lim_{t \to +\infty} a(t)/t^d =: a_d \in \mathbb{R} \). If \( a_d \) is irrational, then \( a(t) \) is covered by Part (iv). If \( a_d \) is rational, then \( a(t) = t^d a_d + a_1(t) \) where \( a_1(t) := a(t) - t^d a_d \) satisfies \( a_1(t) \ll t^d \). Continuing like that, we find that there exists \( k \in \{0, \ldots, d\}, a_k, \ldots, a_d \in \mathbb{Q} \), \( \epsilon : \mathbb{R}_+ \to \mathbb{R} \), \( \epsilon(t) \to 0 \), and \( a_k : \mathbb{R}_+ \to \mathbb{R} \) such that \( a_k(t) \ll t^d \) and \( a_0(t) := a(t) - (\epsilon(t) + t^d a_k + \ldots + t^d a_d) \) is covered by parts (i)–(iv) (note that \( \epsilon(t) \) is needed only when \( k = 0 \)). Then \( a(t) = p(t) + \epsilon(t) + \tilde{a}(t) \) with \( p(t) := t^d a_k + \ldots + t^d a_d \) and \( \tilde{a}(t) = a_k(t) \). Lastly, let \( r \) be the least common multiple of the coefficients of \( p(t) \) and \( k \in \{0, \ldots, r - 1\} \). Then \( p(rn + k) \in \mathbb{Z}[t] \), hence \( b(rn + k) - e(\tilde{a}(rn + k)) \to 0 \). It follows that the sequences \( b(rn + k) \) and \( (e(\tilde{a}(rn + k))) \) have the same F-systems. Note also that \( a(t) \) satisfies conditions (i)–(iv) if and only if the same holds for \( \tilde{a}(rt + k) \).

This completes the proof. \( \square \)

Next we move to the proof of Proposition 4.2, which will be based on the following two lemmas. The first one allows us to compute correlations of sequences of the form \( (e(a(n))) \) where \( a(t) \) is any Hardy field function with at most polynomial growth, and the second one correlations of the sequence \( (S''_d(f)) \) where \( S'_d : \mathbb{T}^{d+1} \to \mathbb{T}^{d+1} \) is as in Proposition 4.2 and \( f \in C(\mathbb{T}^{d+1}) \) is suitably chosen. Our aim is to show that the correlations of the two sequences coincide.

**Lemma 4.3.** Let \( a(t), \epsilon(t), M, \) and \( \lambda \) be as in the statement of Proposition 4.2. Then for every \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s \in \mathbb{Z} \), \( k_1, \ldots, k_s \in \{-1, +1\} \), the limit

\[
E_{m \in M} \prod_{j=1}^{s} e(k_j a(n + n_j))
\]

exists. Furthermore, if \( l_d := \sum_{j=1}^{s} k_j n_j \), then this limit is equal to \( \int e(l_d t) d\lambda(t) \) if \( \sum_{j=1}^{s} k_j n_j = 0 \) for \( i = 0, \ldots, d - 1 \), and is equal to 0 otherwise.
Proof. By our assumptions and Lemma 3.1 we have that \( \lim_{t \to +\infty} a^{(d+1)}(t) = 0 \). Using this and Taylor expansion, we get that for every \( h \in \mathbb{Z} \) we have

\[
a(t+h) = \sum_{i=0}^{d} a^{(i)}(t) \frac{h^i}{i!} + \epsilon_h(t), \quad t \in \mathbb{R}_+,
\]

where the function \( \epsilon_h : \mathbb{R}_+ \to \mathbb{R} \) satisfies \( \lim_{t \to +\infty} \epsilon_h(t) = 0 \). So if

\[
A(t) := \sum_{j=1}^{s} k_j a(t + n_j), \quad t \in \mathbb{R}_+,
\]

we have that

\[
\prod_{j=1}^{s} e(k_j a(n + n_j)) = e(A(n)), \quad n \in \mathbb{N},
\]

and

\[
A(t) = \sum_{i=0}^{d} c_i a^{(i)}(t) + \epsilon(t), \quad t \in \mathbb{R}_+
\]

(note that \( A(t) \) is again a Hardy field function), where

\[
c_i := \frac{1}{i!} \sum_{j=1}^{s} k_j n_j^i, \quad i = 0, \ldots, d,
\]

\[
\epsilon(t) := \sum_{j=1}^{s} k_j \epsilon_n(t), \quad t \in \mathbb{R}_+.
\]

Note that \( \lim_{t \to +\infty} \epsilon(t) = 0 \). Recall that \( c := a^{(d)}/d! \). If \( c_0 = \cdots = c_{d-1} = 0 \), then \( \lim_{t \to +\infty}(A(t) - l_d c(t)) = 0 \), where \( l_d := \sum_{j=1}^{s} k_j n_j^d \). Hence,

\[
\mathbb{E}_{n \in \mathbb{M}} e(A(n)) = \mathbb{E}_{n \in \mathbb{M}} e(l_d c(n)) = \int e(l_d t) d\lambda(t).
\]

Otherwise, let \( i_0 \) be the smallest \( i \in \{0, \ldots, d-1\} \) such that \( c_i \neq 0 \). Using Lemma 3.1 and our assumptions on \( a(t) \), we deduce that either

\[
t^{d-i_0} < A(t) < t^{d-i_0+1} \quad \text{or} \quad |A(t) - t^{d-i_0} \beta| < t^{d-i_0}
\]

where \( \beta := i_0! \alpha \) is irrational. Since \( d-i_0 \geq 1 \), in both cases we have by Theorem 3.2 that \( \mathbb{E}_{n \in \mathbb{N}} e(A(n)) = 0 \). This completes the proof.

In the statement below we use the convention \( 0^0 = 1 \).
**Lemma 4.4.** Let \( a(t), c(t), M, \lambda, d, \) and \( S_d \) be as in the statement of Proposition 4.2. Let also \( f : \mathbb{T}^{d+1} \to \mathbb{S}^1 \) be defined by \( f(y) := e(y_0) \) for \( y = (y_0, \ldots, y_d) \in \mathbb{T}^{d+1} \). Then for every \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s \in \mathbb{Z}, k_1, \ldots, k_s \in \{-1, +1\}, \) we have that the integral

\[
\int \prod_{j=1}^{s} S_d^{n_j} f^{k_j} \, d(\lambda \times m_{\mathbb{T}^d})
\]

is equal to

\[
\int e(l_d t) \, d\lambda(t)
\]

if \( \sum_{j=1}^{s} k_j n_j^i = 0 \) for \( i = 0, \ldots, d - 1 \), where \( l_d := \sum_{j=1}^{s} k_j n_j^d \), and is equal to 0 otherwise.

**Proof.** For \( y = (y_0, \ldots, y_d) \in \mathbb{T}^{d+1} \) direct computation gives that

\[
f(S_d^n y) = e \left( \sum_{i=0}^{d} \binom{n_i}{i} y_{d-i} \right), \quad n \in \mathbb{N}.
\]

Hence, for \( y \in \mathbb{T}^{d+1} \) we have

\[
\prod_{j=1}^{s} f^{k_j}(S_d^n y) = e \left( \sum_{i=0}^{d} c_i y_{d-i} \right)
\]

where

\[
c_i := \sum_{j=1}^{s} k_j \binom{n_j}{i}, \quad i = 0, \ldots, d.
\]

It follows that the integral in (19) is equal to \( \int e(c_d y_0) \, d\lambda(y) \) if \( c_i = 0 \) for \( i = 0, \ldots, d - 1 \), and is equal to 0 otherwise. Lastly, one easily verifies that \( c_i = 0 \) for \( i = 0, \ldots, d - 1 \) if and only if \( \sum_{j=1}^{s} k_j n_j^i = 0 \) for \( i = 0, \ldots, d - 1 \), which implies that \( c_d = \sum_{j=1}^{s} k_j n_j^d = l_d \). This completes the proof. \( \square \)

Combining the previous two results we can now prove Proposition 4.2.

**Proof of Proposition 4.2.** By the first part of the statement of Lemma 4.3 we get that the sequence \( b \) admits correlations on \( M \). Let \( (X, \mu, T) \) be the F-system of \( b \) on \( M \) where, as usual, \( X = (\mathbb{S}^1)^\mathbb{Z} \), \( T \) is the shift transformation on \( X \), and \( \mu := \lim_{k \to \infty} \mathbb{E}_{m \in [M_1]} \delta_{T^n b} \). It remains to establish the asserted isomorphism. To this end, we define the map \( \Phi : \mathbb{T}^{d+1} \to X \) by

\[
\Phi(y) := (f(S_d^n y))_{n \in \mathbb{Z}}, \quad y \in \mathbb{T}^{d+1},
\]
where \( y = (y_0, \ldots, y_d) \) and \( f(y) := e(y_d), y \in \mathbb{T}^{d+1} \). We clearly have \( \Phi \circ S_d = T \circ \Phi \). Moreover, it is easy to check that the map \( \Phi \) is injective. It remains to verify that 
\[ \mu = (\lambda \times m_{\mathbb{T}^d}) \circ \Phi^{-1}. \]
Let \( F_0(x) := x(0), x \in X \). For every \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s \in \mathbb{Z} \), \( k_1, \ldots, k_s \in \{-1, +1\} \), we have 
\[ \int \prod_{j=1}^{s} T^{n_j} F_0^{k_j} d\mu = E_{n \in \mathcal{M}} \prod_{j=1}^{s} e(k_j d(n + n_j)) = \int \prod_{j=1}^{s} S_d^{n_j} f^{k_j} d(\lambda \times m_{\mathbb{T}^{d+1}}), \]
where the first identity follows from (7) and the second identity follows by combining Lemma 4.3 and Lemma 4.4. Using this and the fact that \( f = F_0 \circ \Phi \), we get that a linearly dense subset of \( C(X) \) has the same integral with respect to the measures \( \mu \) and \( (\lambda \times m_{\mathbb{T}^d}) \circ \Phi^{-1} \), hence, the two measures coincide. This completes the proof. 

\[ \Box \]

**4.3 Proof of Corollaries 1.2, 1.3, 1.4.** We start with the proof of Corollary 1.2, which is a consequence of the structural result of Theorem 1.1 and an ergodic theorem from [31].

**Proof of Corollary 1.2.** Let \( (X, \mu, T) \) be an F-system of \( b \). By Parts (i) and (ii) of Theorem 1.1 the system \( (X, \mu, T) \) is isomorphic to the system 
\[ X_d := (\mathbb{T}^{d+1}, \nu := \lambda \times m_{\mathbb{T}^d}, S_d) \]
where \( \lambda \) is a continuous probability measure on \( \mathbb{T} \) and \( S_d \) is the unipotent homomorphism of \( \mathbb{T}^{d+1} \) defined by 
\[ S_d(y_0, \ldots, y_d) := (y_0, y_1 + y_0, \ldots, y_d + y_{d-1}), \quad y_0, \ldots, y_d \in \mathbb{T}. \]
Hence, it remains to show that \( X_d \) is disjoint from every ergodic system \( (Z, \rho, R) \). So let \( \sigma \) be a joining of these two systems. In order to show that \( \sigma = \nu \times \rho \) it suffices to show that for every \( f \in C(\mathbb{T}^{d+1}) \) with \( \int f d\nu = 0 \) and every \( g \in L^\infty(\rho) \) we have 
\[ \int f(y) g(z) d\sigma(y, z) = 0. \]
Since \( \sigma \) is \( (T \times R) \)-invariant it suffices to show that 
\[ \lim_{N \to \infty} E_{n \in [N]} \int f(S_d^n y) g(R^n z) d\sigma(y, z) = 0. \]
Using uniform approximation of \( f \) by trigonometric polynomials we can assume that \( f \) is a complex exponential of \( \mathbb{T}^{d+1} \).

Let \( y = (y_0, \ldots, y_d) \) and suppose first that \( f(y) = e(k_0) \) for some \( k \in \mathbb{Z} \). Then \( f(S_d^n y) = e(k_0) \) for every \( n \in \mathbb{N} \) and using the ergodicity of the system \( (Z, \rho, R) \)
we get that the limit in (20) is equal to $\int f \, d\mu \cdot \int g \, d\nu = 0$. So we can assume that there exists $d' \in \{1, \ldots, d\}$ such that $f(y) = e(\sum_{k=0}^{d'} l_k y_k)$ for some $l_0, \ldots, l_{d'} \in \mathbb{Z}$ with $l_{d'} \neq 0$. In this case, a simple computation gives that

$$f(S_n y) = e(qy_0 n^{d'} + p_y(n)), \quad y \in \mathbb{T}^{d+1},$$

where $q := l_{d'}/d'$! and for every $y \in \mathbb{T}^{d+1}$ we have that $p_y$ is a polynomial with real coefficients and degree strictly smaller than $d'$.

Using again the ergodicity of the system $(\mathbb{Z}, \rho, R)$ and [31, Theorem 4], we get that there exists a subset $Z'$ of $\mathbb{Z}$ with $\rho(Z') = 1$ such that the following holds: For every $\alpha \in \mathbb{R}$ such that $e(ka)$ is not an eigenvalue of $(\mathbb{Z}, \rho, R)$ for every non-zero $k \in \mathbb{Z}$, we have

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} e(an^{d'} + p(n)) g(R^n z) = 0$$

for every polynomial $p$ of degree smaller than $d'$ and every $z \in Z'$. Let $Y'$ be the set of all $y = (y_0, \ldots, y_d) \in \mathbb{T}^{d+1}$ such that $e(ky_0)$ is not an eigenvalue of $(\mathbb{Z}, \rho, R)$ for every non-zero $k \in \mathbb{Z}$. Obviously the projection of $Y'$ on the $y_0$ coordinate differs from $\mathbb{T}$ on a countable set. Since $\nu = \lambda \times m_{\mathbb{T}^d}$ and the measure $\lambda$ is continuous, we have that $\nu(Y') = 1$. From the above we get that for all $(y, z) \in Y' \times Z'$ we have

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} f(S_n y) g(R^n z) = 0.$$

Since $\rho(Y' \times Z') = 1$, the bounded convergence theorem implies that (20) holds. This completes the proof. \hfill $\square$

Corollary 1.3 is a consequence of Corollary 1.2 and some pretty standard maneuvers that enables us to pass orthogonality statements from the sequences $(e(a(n)t))$, $t \in \mathbb{R} \setminus \{0\}$, to the sequence $(e([a(n)]a))$ where $a \in \mathbb{R} \setminus \mathbb{Z}$.

**Proof of Corollary 1.3.** Suppose first that $b(n) = e(a(n))$, $n \in \mathbb{N}$. Arguing by contradiction, suppose that (2) fails for some ergodic sequence $w: \mathbb{N} \to \mathbb{U}$. Then there exists a sequence of intervals $M = ([M_k])_{k \in \mathbb{N}}$, with $M_k \to \infty$, such that the sequences $b, w$ admit joint correlations on $M$ and

$$(21) \quad \mathbb{E}_{n \in M} b(n) w(n) \neq 0.$$

By Corollary 1.2 the $F$-systems of $b, w$ on $M$ are disjoint, hence their joint $F$-system (which is a joining of the two systems) is the direct product of these systems. This easily implies that

$$\mathbb{E}_{n \in M} b(n) w(n) = \mathbb{E}_{n \in M} b(n) \cdot \mathbb{E}_{n \in M} w(n) = 0,$$

where the last equality holds since $\mathbb{E}_{n \in \mathbb{N}} e(a(n)) = 0$ by Theorem 3.2. This contradicts (21) and completes the proof in the case where $b(n) = e(a(n))$, $n \in \mathbb{N}$.
Suppose now that $b(n) = e([a(n)]\alpha), n \in \mathbb{N}$, for some $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. First note that $b(n) = e(a(n)\alpha)\phi(a(n)), n \in \mathbb{N}$, where $\phi: \mathbb{T} \to \mathbb{S}^1$ is given by $\phi(t) := e(-\{t\} \alpha), t \in \mathbb{T}$, is Riemann-integrable with respect to the measure $m_\mathbb{T}$. Hence, it suffices to show that under our assumptions on the sequences $a, w$, for every $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ we have that

$$E e(a(n)\alpha) \phi(a(n)) w(n) = 0$$

for every $\phi: \mathbb{T} \to \mathbb{C}$ that is Riemann-integrable (with respect to $m_\mathbb{T}$). Suppose first that $\phi(t) := e(kt), t \in \mathbb{T}$, for some $k \in \mathbb{Z}$. Then $e(a(n)\alpha) \phi(a(n)) = e(a(n)(\alpha + k))$, and since by assumption $\alpha + k \neq 0$, we get by the previous case (for $a(n)(\alpha + k)$ in place of $a(n)$) that (22) holds. Using linearity and uniform approximation by trigonometric polynomials, we deduce that (22) also holds when $\phi \in C(\mathbb{T})$. Finally, let $\varepsilon > 0$ and $\phi_\varepsilon \in C(\mathbb{T})$ be such that $\|\phi - \phi_\varepsilon\|_{L^1(m_\mathbb{T})} \leq \varepsilon$. Then using that (22) holds for $\phi_\varepsilon$ in place of $\phi$ we get that

$$\limsup_{N \to \infty} |E e(a(n)\alpha) \phi(a(n)) w(n)| \ll \limsup_{N \to \infty} \|\phi - \phi_\varepsilon\|_{L^1(m_\mathbb{T})} \leq \varepsilon,$$

where to justify the last identity we used that the sequence $(a(n))$ is equidistributed on $\mathbb{T}$ by Theorem 3.2. Since $\varepsilon$ is arbitrary we get that (22) holds, completing the proof. \hfill \square

**Corollary 1.4** is a simple consequence of Corollary 1.3.

**Proof of Corollary 1.4.** If $E$ is the range of the sequence $b$, then for every $k \in \mathbb{Z}$ one easily verifies that

$$d(E) \cdot E e(k\cdot b(n)) = E e(k\cdot (a(n)) \cdot 1_E(n)).$$

By assumption the sequence $1_E(n)$ is ergodic, hence if $k \neq 0$ we have that Corollary 1.3 applies and gives that the last average is 0. Since $d(E) > 0$ we deduce that

$$E e(k\cdot b(n)) = 0$$

for every non-zero $k \in \mathbb{Z}$. Hence, the sequence $((a \circ b)(n))_{n \in \mathbb{N}}$ is equidistributed on $\mathbb{T}$.

Similarly, one verifies that for every $k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ such that $k\alpha \notin \mathbb{Z}$ we have

$$E e(k[a\cdot b(n)])\alpha = 0.$$ 

This implies the other two equidistribution properties and completes the proof. \hfill \square
4.4 Proof of Theorem 1.5. The proof of Theorem 1.5 will be based on the next result:

**Proposition 4.5.** Let \( a : \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function such that

\[
t^d \log t < a(t) < t^{d+1}
\]

for some \( d \in \mathbb{Z}_+ \). Then for every \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) the pair of sequences \((a(n)), (a(n)\alpha)\) (with elements on \( \mathbb{T} \)) has a unique joint F-system that is isomorphic to the system \((\mathbb{T}^{2(d+1)}, m_{\mathbb{T}^{2(d+1)}}, S_d \times S_d)\) where \( S_d \) is given by (1).

**Proof.** By Proposition 2.3 the F-system of the sequences \((a(n))\) (with elements on \( \mathbb{T} \)) and \((e(a(n)))\) are isomorphic. Moreover, a similar argument gives that a joint F-system on \( \mathbf{M} \) of the pair of sequences \((a(n)), (a(n)\alpha)\) and the corresponding one for the pair of sequences \((e(a(n))), (e(a(n)\alpha))\) are isomorphic. Hence, it suffices to establish the asserted statement with the sequences \((e(a(n)))\) and \((e(a(n)\alpha))\) in place of the sequences \((a(n))\) and \((a(n)\alpha)\) respectively.

Let \((X, \mu, T)\) be the F-system of the sequence \((e(a(n)))\) and \((X \times X, \nu, S)\), where \( S = T \times T \), be a joint F-system of the pair of sequences \((e(a(n))), (e(a(n)\alpha))\) on \( \mathbf{M} \). It suffices to show that \( \nu = \mu \times \mu \). For \( i = 1, 2 \) let \( F_{i,0} \in C(X \times X) \) be defined by \( F_{i,0}(x_1, x_2) := x_i(0) \), where \((x_1, x_2) \in X \times X \). Since the collection of functions of the form \( \prod_{i=1}^2 \prod_{j=1}^s S_{n_{ij}} F_{i,0}^{k_{ij}} \), where \( n_{ij} \in \mathbb{Z}, k_{ij} \in \{-1, +1\} \), for \( i \in \{1, 2\}, j \in \{1, \ldots, s\}, s \in \mathbb{N} \), is linearly dense in \( C(X \times X) \), it suffices to show that

\[
\prod_{i=1}^2 \prod_{j=1}^s x_i^{k_{ij}}(n_{ij}) \nu = \prod_{i=1}^2 \prod_{j=1}^s x_i^{k_{ij}}(n_{ij}) \mu
\]

for every \( s \in \mathbb{N} \) and \( n_{ij} \in \mathbb{Z}, k_{ij} \in \{-1, +1\}, i \in \{1, 2\}, j \in \{1, \ldots, s\} \), we have the identity

(23) \[ \mathbb{E}_{n \in \mathbb{M}} \prod_{i=1}^2 \prod_{j=1}^s e(k_{ij}a_i(n + n_{ij})) = \prod_{i=1}^2 \left( \mathbb{E}_{n \in \mathbb{M}} \prod_{j=1}^s e(k_{ij}a_i(n + n_{ij})) \right). \]

By assumption we have that \( t^d \log t < a(t) < t^{d+1} \) for some \( d \in \mathbb{Z}_+ \). By Lemma 4.3 the right-hand side in (23) is 1 if

(24) \[ \sum_{j=1}^s k_{ij}n_{ij}^r = 0 \quad \text{for } i = 1, 2 \text{ and } r = 0, \ldots, d, \]

and is 0 otherwise.
Next we deal with the left hand side in (23). Using Taylor expansion and arguing exactly as in the proof of Lemma 4.3 we find that if
\[ A(t) := \sum_{i=1}^{2} \sum_{j=1}^{s} k_{i,j} a_{i}(t + n_{i,j}), \quad t \in \mathbb{R}_{+}, \]
then for some \( e : \mathbb{R}_{+} \rightarrow \mathbb{R} \) that satisfies \( \lim_{t \to +\infty} e(t) = 0 \) we have
\[ A(t) = \sum_{i=1}^{2} \sum_{r=0}^{d} c_{i,r} a_{i}^{(r)}(t) + \epsilon(t) = \sum_{r=0}^{d} (c_{1,r} + c_{2,r} a_{i}^{(r)}(t) + \epsilon(t), \]
where for \( i = 1, 2 \) we have
\[ c_{i,r} := \frac{1}{r!} \sum_{j=1}^{s} k_{i,j} n_{i,j}^{r}, \quad r = 0, \ldots, d. \]
We deduce that if (24) holds, then \( A(t) = \epsilon(t) \to 0 \) as \( t \to +\infty \). Therefore, we have \( \mathbb{E}_{n \in \mathbb{N}} e(A(n)) = 1 \) and the left-hand side in (23) is 1. Suppose now that (24) does not hold. Then \( \sum_{r=0}^{d} c_{1,r} n_{i,j}^{r} \neq 0 \) for some \( i \in \{1, 2\} \) and \( r \in \{0, \ldots, d\} \).

Let \( r_{0} \) be the smallest \( r \in \{0, \ldots, d\} \) such that \( |c_{1,r} + |c_{2,r}| \neq 0 \). Since \( c_{1,r}, c_{2,r} \) are rational and \( \alpha \) is irrational, we have that \( c_{1,r} + c_{2,r} \alpha \neq 0 \). Using Lemma 3.1 we get that \( A \sim a^{(r_{0})} \) and deduce that \( A(i) \sim a(t)/t^{r_{0}} \) for some \( r \in \{0, \ldots, d\} \). Combining this with Theorem 3.2 we get that \( \mathbb{E}_{n \in \mathbb{N}} e(A(n)) = 0 \). We deduce that in all cases (23) holds. This completes the proof.\[ \square \]

We can now proceed to the proof of Theorem 1.5.

**Proof of Theorem 1.5.** First we carry out a reduction. Let \( c(n) := [a(n)]\alpha, \quad n \in \mathbb{N} \). Theorem 3.2 gives that for every non-zero \( t \in \mathbb{R} \) the sequence \( (a(n)t) \) is equidistributed on \( \mathbb{T} \), and using a standard argument (see, for example, the proof of [10, Theorem 6.3]) we deduce that for every irrational \( \alpha \) the sequence \( (c(n)) \) is equidistributed on \( \mathbb{T} \). Since \( b(n) = \phi(c(n)), \quad n \in \mathbb{N}, \) and the function \( \phi \) is Riemann-integrable (with respect to \( m_{\mathbb{T}} \)), Proposition 2.3 applies and gives that in order to get the asserted properties for the sequence \( b \) it suffices to get them for the sequence \( c \).

We thus turn our attention to the sequence \( c \). Note first that if \( \psi : \mathbb{T}^{2} \to \mathbb{T} \) is defined by
\[ \psi(x, y) := y - \{x\} \alpha \mod 1, \]
then \( \psi \) is Riemann-integrable with respect to \( m_{\mathbb{T}^{2}} \) (the set of discontinuities of \( \psi \) has \( m_{\mathbb{T}^{2}} \)-measure 0) and we have the identity
\[ c(n) = \psi(a(n), a(n)\alpha), \quad n \in \mathbb{N}. \]
Next, note that by Part (i) of Theorem 1.1 for every non-zero $\alpha \in \mathbb{R}$ the sequences $(a(n))$ and $(a(n)\alpha)$ on $\mathbb{T}$ have unique F-systems, and they are both isomorphic to the system $(\mathbb{T}^{d+1}, m_{\mathbb{T}^{d+1}}, S_d)$ where $S_d : \mathbb{T}^{d+1} \to \mathbb{T}^{d+1}$ is given by (1). By Proposition 4.5 for every irrational $\alpha \in \mathbb{R}$ the pair of sequences $(a(n)), (a(n)\alpha)$ has a unique joint F-system, and it is isomorphic to the system $(\mathbb{T}^{2(d+1)}, m_{\mathbb{T}^{2(d+1)}}, S_d \times S_d)$. The needed conclusion for the sequence $c$ now follows from Proposition 2.3, assuming that we verify that the sequence $(a(n)), (a(n)\alpha)$ is equidistributed on $\mathbb{T}^2$ with respect to the Haar measure $m_{\mathbb{T}^2}$. Since $\alpha$ is irrational, this easily follows from Theorem 3.2, our assumption $t^d \log t \prec a(t) \prec t^{d+1}$ for some $d \in \mathbb{Z}_+$, and Weyl’s equidistribution theorem. This completes the proof. □

5 Proof of results concerning Hardy field iterates

The proof of Theorem 1.6 is a direct consequence of the next result, which establishes strong stationarity for the Furstenberg systems defined in Theorem 1.6, and Theorem 2.1 that describes the structure of strongly stationary systems.

**Theorem 5.1.** Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function such that

$$t^{d+\varepsilon} \prec a(t) \prec t^{d+1}$$

for some $\varepsilon > 0$. Furthermore, let $(Y, \nu, S)$ be a system. Then every strictly increasing sequence $(N_k)$ has a subsequence $(N'_k)$ such that for almost every $y \in Y$ and for every $g \in L^\infty(\nu)$ the sequence $(g(S^{[a(n)]}y))$ admits correlations on $N' := (\lfloor N'_k \rfloor)_{k \in \mathbb{N}}$ and the corresponding Furstenberg system is strongly stationary.

The remainder of this section is devoted to the proof of Theorem 5.1.

5.1 Strong stationarity of Hardy-field nilsequences. If $G$ is a group we let $G_1 := G$ and $G_{j+1} := [G, G_j], j \in \mathbb{N}$. We say that $G$ is nilpotent if $G_s$ is the trivial group for some $s \in \mathbb{N}$. A nilmanifold is a homogeneous space $X = G/\Gamma$, where $G$ is a nilpotent Lie group and $\Gamma$ is a discrete cocompact subgroup of $G$. With $e_X$ we denote the image in $X$ of the unit element of $G$. A nilsystem is a system of the form $(X, X, m_X, T_b)$, where $X = G/\Gamma$ is a nilmanifold, $b \in G$, $T_b : X \to X$ is defined by $T_b(g \cdot e_X) := (bg) \cdot e_X$ for $g \in G$, $m_X$ is the normalized Haar measure on $X$, and $X$ is the completion of the Borel $\sigma$-algebra of $G/\Gamma$. If $G$ is connected and simply connected and $b \in G$, then $b'$ is well defined for every $t \in \mathbb{R}$.

The first step in the proof of Theorem 1.6 is to establish strong stationarity in the case where the system $(Y, \nu, S)$ is a nilsystem. Although in the proof of Theorem 1.6 we only use Proposition 5.5, we state and prove the following result that is of independent interest:
**Theorem 5.2.** Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function such that
\[ t^{d + \varepsilon} \prec a(t) \prec t^{d + 1} \]
for some $d \in \mathbb{Z}_+$ and $\varepsilon > 0$. Let $X = G/\Gamma$ be a nilmanifold and $b \in G$. Then for every $x \in X$ the sequences $(b^{a(n)}x)$ and $(b^{[a(n)]}x)$ have unique F-systems that are strongly stationary (in the first case we assume that $G$ is connected and simply connected).

**Remark.** It can be shown that the symbolic systems of the above sequences have zero topological entropy. If we combine this with Theorem 2.1, we get that the ergodic components of the corresponding F-systems are finite-step nilsystems (we expect, but it does not follow from our arguments that they are finite-step nilsystems). It would be interesting to verify that a similar property holds for all Hardy field functions $a(t)$ with at most polynomial growth.

The proof of Theorem 5.2 will be based on an equidistribution result from [7] that was proved for certain weighted averages that we define next. For $r \in \mathbb{N}$, let
\[ \Delta_r^1 a := a(n + r) - a(n), \quad n \in \mathbb{N}, \]
and for $i \in \mathbb{N}$ define inductively $\Delta_r^{i+1} a := \Delta_r^1 (\Delta_r^i a)$.

If $w : \mathbb{N} \to \mathbb{R}_+$ is an eventually increasing sequence and $\lim_{n \to \infty} w(n) = +\infty$, then for every $a : \mathbb{N} \to \mathbb{U}$ we let (for those $N \in \mathbb{N}$ for which $w(N) \neq 0$)
\[ E_{n \in [N]}^w a(n) := \frac{1}{w(N)} \sum_{n=1}^N (w(n+1) - w(n)) a(n). \]
For example, if $w(n) = n, n \in \mathbb{N}$, then we get the Cesàro averages, and if $w(n) = \log n, n \in \mathbb{N}$, then we get an averaging scheme equivalent to logarithmic averages. The next result is a direct consequence of results proved in [7].

**Proposition 5.3 ([7]).** Let $k, r \in \mathbb{N}$, $d \in \mathbb{Z}_+$, and $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function such that $t^{d + \varepsilon} \prec a(t) \prec t^{d + 1}$ for some $d \in \mathbb{Z}_+$ and $\varepsilon > 0$. For $j = 1, \ldots, k$ let $n_{j,r} := \sum_{i=0}^d c_{i,j} \Delta_r^i a$ for some $c_{0,j}, \ldots, c_{d,j} \in \mathbb{R}$. Let $X = G/\Gamma$ be a nilmanifold and $b \in G$. Then for every $x \in X$ and $h_1, \ldots, h_k \in C(X)$ the limits
\[ \lim_{N \to \infty} \mathbb{E}_{n \in [N]}^{w_r} \prod_{j=1}^k h_j(b^{n_{j,r}}(n) \cdot x), \quad \lim_{N \to \infty} \mathbb{E}_{n \in [N]}^{w_r} \prod_{j=1}^k h_j(b^{[n_{j,r}]}(n) \cdot x) \]
exist and do not depend on $r$, where $w_r := |\Delta_r^d a|$ (in the first case we assume that $G$ is connected and simply connected).
Proof. For $x := e_X$, it is proved in [7, Theorem 4.12 and Corollary 4.13] (for $r = 1$ but the same argument works for general $r \in \mathbb{N}$) that the two limits exist and it follows from the proof that the limit does not depend on $r \in \mathbb{N}$. For general $x \in X$, one writes $x = g \cdot e_X$ for some $g \in G$ and applies the previous result for $b' := g^{-1}bg$ and $h'_j(x) := h_j(gx), x \in X$, for $j = 1, \ldots, k$. □

The next lemma enables us to deduce from Proposition 5.3 a similar result for Cesàro averages.

Lemma 5.4. Let $r \in \mathbb{N}$ and $d \in \mathbb{Z}^+$. Let $a: \mathbb{R}_+ \to \mathbb{R}_+$ be a Hardy field function such that $t^{d+\varepsilon} \prec a(t) \prec t^{d+1}$ for some $d \in \mathbb{Z}_+$ and $\varepsilon > 0$, and let $w := |\Delta^d t^d a|$. If $(X, \|\cdot\|)$ is a normed space and $b: \mathbb{N} \to X$ is a bounded sequence such that

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} b(n) = L,$$

then

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} b(n) = L.$$

Proof. Note that by Lemma 3.1 we have that $t^\varepsilon \prec w(t) \prec t$ and $w$ is eventually increasing. Hence,

$$\lim_{t \to +\infty} \log(w(t))/\log t \neq 0$$

and using l’Hospital’s rule we get that $\lim_{t \to +\infty} tu'(t)/w(t) \neq 0$ (all limits exist since $a$ is a Hardy field function). Using the mean value theorem twice, that $w''$ is eventually monotonic, and that $w''(t) \prec w'(t)$, we get that

$$\lim_{t \to +\infty} (w(t+1) - w(t))/w'(t) = 1.$$ We deduce that

$$\lim_{t \to +\infty} t(w(t+1) - w(t))/w(t) \neq 0.$$ Hence, if we let $u(t) := w(t+1) - w(t), t \in \mathbb{R}_+$, we have that the sequence $U(n)/(nu(n))$ is bounded, where $U(n) := u(1) + \cdots + u(n), n \in \mathbb{N}$. Therefore, we have reduced matters to proving the following elementary statement: Let $(X, \|\cdot\|)$ be a normed space and $b: \mathbb{N} \to X$ be a bounded sequence. Let also $u: \mathbb{N} \to \mathbb{R}_+$ be eventually decreasing, $U(n)/(nu(n))$ be bounded, where $U(n) := u(1) + \cdots + u(n), n \in \mathbb{N}$, and suppose that

$$\lim_{N \to \infty} \frac{1}{U(N)} \sum_{n=1}^N u(n)b(n) = L.$$ Then

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} b(n) = L.$$ This is a straightforward exercise in partial summation. □
Combining the previous two results we get the following:

**Proposition 5.5.** Let \( k, r \in \mathbb{N}, d \in \mathbb{Z}^+, \) and \( a: \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function such that \( t^{d+\varepsilon} \prec a(t) \prec t^{d+1} \) for some \( d \in \mathbb{Z}_+ \) and \( \varepsilon > 0. \) Let also \( X = G/\Gamma \) be a nilmanifold, \( b \in G, \) and \( h_1, \ldots, h_k \in C(X). \) Then for every \( x \in X \) and \( n_1, \ldots, n_k \in \mathbb{N} \) the limits

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \prod_{j=1}^k h_j(b^{a(n+m_j)} \cdot x), \quad \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \prod_{j=1}^k h_j(b^{[a(n+m_j)]} \cdot x)
\]

exist and do not depend on \( r \) (in the first case we assume that \( G \) is connected and simply connected).

**Proof.** First note that if \( a: \mathbb{N} \to \mathbb{U} \) is a sequence, then for every \( k, n, r \in \mathbb{N} \) we have that

\[
a(n + kr) = (1 + \Delta r)^k a(n).
\]

Hence, by Proposition 5.3 we get that the limits

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \prod_{j=1}^k h_j(b^{a(n+m_j)} \cdot x), \quad \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \prod_{j=1}^k h_j(b^{[a(n+m_j)]} \cdot x)
\]

exist and do not depend on \( r \in \mathbb{N} \) where \( w := |\Delta^d a|. \) Using Lemma 5.4, we get the asserted statement. \( \Box \)

**Proof of Theorem 5.2.** It follows from Proposition 5.5 (take \( r = 1 \)) that the sequences \( (b^{a(n)} x) \) and \( (b^{[a(n)]} x) \) admit correlations on \( N := (\mathbb{N})_{N \in \mathbb{N}}; \) hence these sequences have unique F-systems. Moreover, since the limits in (25) do not depend on \( r, \) we get by Lemma 2.2 that these F-systems are strongly stationary. \( \Box \)

### 5.2 Strong stationarity of Hardy field iterates

We will use the following result that follows from Theorem D and Theorem 4.5 in [7]:

**Theorem 5.6 ([7]).** Let \( a: \mathbb{R}_+ \to \mathbb{R} \) be a Hardy field function such that

\[
t^{d+\varepsilon} \prec a(t) \prec t^{d+1}
\]

for some \( d \in \mathbb{Z}_+ \) and \( \varepsilon > 0. \) Then for every ergodic system \( (X, \mu, T), \ell \in \mathbb{N}, \) functions \( f_1, \ldots, f_\ell \in L^\infty(\mu), \) and \( n_1, \ldots, n_\ell \in \mathbb{Z}, \) the following limit exists:

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \prod_{j=1}^\ell T^{[a(n+n_j)]} f_j
\]

in \( L^2(\mu). \) Furthermore, for \( d \in \mathbb{N} \) and \( n_1, \ldots, n_\ell \) distinct, if \( \mathbb{E}(f_j | Z) = 0, \) for some \( j \in \{1, \ldots, \ell\}, \) where \( Z \) is the infinite-step nilfactor of the system \( (X, \mu, T), \) then the limit is 0.
Remark. Mean convergence is proved in [7] with the averages \( \lim_{N \to \infty} \mathbb{E}_{n \in \mathbb{N}}^{w_{N}} \) in place of the averages \( \lim_{N \to \infty} \mathbb{E}_{n \in \mathbb{N}} \). One gets the asserted statement by combining this result with Lemma 5.4.

Proposition 5.7. Let \( a: \mathbb{R}_{+} \to \mathbb{R} \) be a Hardy field function such that
\[
t^{d+\varepsilon} \prec a(t) \prec t^{d+1}
\]
for some \( d \in \mathbb{Z}_{+} \) and \( \varepsilon > 0 \). Then for every system \((X, \mu, T)\), \( \ell \in \mathbb{N} \), functions \( f_1, \ldots, f_\ell \in L^\infty(\mu) \), and \( n_1, \ldots, n_\ell \in \mathbb{Z} \), the following limit exists
\[
\lim_{N \to \infty} \mathbb{E}_{n \in \mathbb{N}} \prod_{j=1}^{\ell} T^{[a(n+r_{N})]} f_j
\]
in \( L^2(\mu) \) and is independent of \( r \in \mathbb{N} \).

Proof. If \( d = 0 \) we have \( \lim_{t \to +\infty} (a(t + 1) - a(t)) = 0 \), and since \( a(t) \) is eventually monotonic, we get for every \( h \in \mathbb{Z} \) that \([a(n+h)] = [a(n)]\) for a set of \( n \in \mathbb{N} \) with density 1. Therefore, the result is obvious in this case.

Suppose now that \( d \in \mathbb{N} \). A standard ergodic decomposition argument allows us to assume that the system is ergodic. Using Theorem 5.6 we can assume that all functions are \( \mathbb{Z} \)-measurable where \( \mathbb{Z} \) is the infinite-step nilfactor of the system. Using the Host–Kra theory of characteristic factors [27] (see also [28, Theorem 4.2]) and a standard approximation argument we can assume that the system is an ergodic nilsystem and the functions are continuous. In this case the result follows from Proposition 5.5.

Proof of Theorem 5.1. Let \( N_k \to \infty \) be a strictly increasing sequence of integers, \((Y, \nu, S)\) be a system, and \( \mathcal{G} \subset L^\infty(\mu) \) be a countable collection of functions that is dense in \( L^\infty(\mu) \) with the \( L^2(\mu) \) norm. Recall that mean convergence of a sequence of functions implies pointwise convergence along a subsequence. With this in mind, using the convergence result of Theorem 5.6 and a diagonal argument, we get that there exists a subsequence \( (N'_k) \) of \( (N_k) \) such that for \( \nu \)-almost every \( y \in Y \) and for every \( g \in \mathcal{G} \) the sequence \( (g(S^{[a(n)]} y)) \) admits correlations on \( N' := ([N'_k]) \). Hence, for almost every \( y \in Y \) for every \( g \in \mathcal{G} \) the limit
\[
\mathbb{E}_{n \in \mathbb{N}} \prod_{j=1}^{\ell} g_j(S^{[a(n+r_{N})]} y)
\]
exists for all \( \ell, r \in \mathbb{N}, n_1, \ldots, n_\ell \in \mathbb{Z}, \) and \( g_1, \ldots, g_\ell \in \{ g, \bar{g} \} \). Since this limit coincides with the \( L^2(\mu) \)-limit, Proposition 5.7 implies that for almost every \( y \in Y \)
it is independent of \( r \in \mathbb{N} \). Furthermore, using an approximation argument we get that a similar property holds with the set \( \mathcal{G} \) replaced with \( L^\infty(\nu) \). Using Lemma 2.2 we get that for almost every \( y \in Y \), for every \( g \in L^\infty(\nu) \) the sequence \( (g(S^{[a(n)]}y)) \) admits correlations on \( \mathcal{N}' \) and the corresponding F-system is strongly stationary. This completes the proof. \( \square \)

5.3 Proof of Corollary 1.7. We prove Part (i). Suppose that the conclusion fails. Then for some \( f, g \in L^\infty(\mu) \) there exist \( \varepsilon > 0 \) and \( N_k \to \infty \) such that

\[
\| \mathbb{E}_{n \in [N_k]} T^n f \cdot g(S^{[a(n)]}x) - \mathbb{E}(f|\mathcal{J}_T) \cdot \mathbb{E}(g|\mathcal{J}_S) \|_{L^2(\mu)} \geq \varepsilon
\]

for every \( k \in \mathbb{N} \) where \( \mathcal{J}_T := \{ h \in L^2(\mu) : Th = h \} \) and \( \mathcal{J}_S \) is defined similarly.

By Theorem 1.6 there exists a subsequence \((N'_k)\) of \((N_k)\) such that for almost every \( x \in X \) the sequence \((g(S^{[a(n)]}x))\) admits correlations on \( \mathcal{N}' := ([N'_k]) \) and the corresponding F-systems have trivial spectrum and their ergodic components are isomorphic to direct products of infinite-step nilsystems and Bernoulli systems. Note also that by Proposition 2.5 for almost every \( x \in X \) the sequence \((f(T^n x))\) admits correlations on \(([N])_{N \in \mathbb{N}}\) and the corresponding F-systems are ergodic and have zero entropy (by assumption).

It follows from [19, Proposition 3.12] that for almost every \( x \in X \) the F-system of the sequence \((g(S^{[a(n)]}x))\) on \( \mathcal{N}' \) and the F-system of the sequence \((f(T^n x))\) on \( \mathcal{N}' \) are disjoint. Using a standard disjointness argument we deduce from this that for almost every \( x \in X \) we have

\[
\mathbb{E}_{n \in \mathcal{N}'} f(T^n x) \cdot g(S^{[a(n)]}x) = \mathbb{E}_{n \in \mathcal{N}'} f(T^n x) \cdot \mathbb{E}_{n \in \mathcal{N}'} g(S^{[a(n)]}x).
\]

Lastly, note that by the ergodic theorem and [10] we have for almost every \( x \in X \) that

\[
\mathbb{E}_{n \in \mathbb{N}} f(T^n x) = \mathbb{E}(f|\mathcal{J}_T)(x), \quad \mathbb{E}_{n \in \mathbb{N}} g(S^{[a(n)]}x) = \mathbb{E}(g|\mathcal{J}_S)(x).
\]

Combining these facts, and using the bounded convergence theorem, we get a contradiction from (26), completing the proof.

Part (ii) follows immediately from Part (i) and the estimate in [12, Lemma 1.6].

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