AN INCONSISTENCY IN THE STANDARD MAXIMUM LIKELIHOOD ESTIMATION OF BULK FLOWS

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ABSTRACT

Maximum likelihood estimation of the bulk flow from radial peculiar motions of galaxies generally assumes a constant velocity field inside the survey volume. This assumption is inconsistent with the definition of bulk flow as the average of the peculiar velocity field over the relevant volume. This follows from a straightforward mathematical relation between the bulk flow of a sphere and the velocity potential on its surface. This inconsistency also exists for ideal data with exact radial velocities and full spatial coverage. Based on the same relation, we propose a simple modification to correct for this inconsistency.

Key words: cosmology; observations – cosmology: theory – dark matter – large-scale structure of universe

1. INTRODUCTION

In the standard cosmological paradigm, galaxies share the same peculiar velocity field (deviations from a pure Hubble flow), \( \mathbf{v} \), as the underlying dark matter, at least on large scales away from virialized regions. Thus, although the distribution of galaxies may not be an honest tracer of the underlying density field, their motions offer, in principle, an unbiased probe of the gravitationally dominant dark matter. The bulk flow, defined as the average peculiar velocity in a volume of space, is one of the common statistical measures of the velocity field. Usually, the bulk flow of a sphere of comoving radius \( r \) centered at the observer is considered to be

\[
\mathbf{B}(r) = \frac{3}{4\pi r^3} \int_0^r \mathbf{v}(r') d^3r'.
\]

Estimating \( \mathbf{B} \) from observational data is a non-trivial matter. The relevant observations are very challenging and provide only the radial (line-of-sight component) peculiar motions of a relatively small number (a few \( \times 10^3 \)) of galaxies, within \( \lesssim 200 h^{-1} \text{Mpc} \) (e.g., Masters et al. 2006; Tully et al. 2013). Analyses of these observations could easily be plagued by systematic biases due to sparseness of data and varying quality of distance measurements (Lynden-Bell et al. 1988). The bulk flow is essentially a large-scale moment and hence it is particularly prone to systematics which may masquerade as a real signal. Putting these potential biases aside, we focus here on a single point related to estimating \( \mathbf{B} \) from velocity data by means of a maximum likelihood (ML) estimation (e.g., Kaiser 1988). We address the issue using a very simple relation between the bulk flow of a sphere and the velocity potential on its surface. This inconsistency also exists for ideal data with exact radial velocities and full spatial coverage. Based on the same relation, we propose a simple modification to correct for this inconsistency.

2. BASICS

Let \( \hat{x}, \hat{y}, \) and \( \hat{z} \) be unit vectors in the three axes of a fixed Cartesian system. The radial direction is indicated by the unit vector \( \hat{r} \) and the projections of \( \hat{r} \) onto the Cartesian axes are \( \hat{n}^\alpha \), where \( \alpha \) runs over \( x, y, \) and \( z \), corresponding to \( \hat{n}^x = \hat{r} \cdot \hat{x} = \sin \theta \cos \varphi \), \( \hat{n}^y = \hat{r} \cdot \hat{y} = \sin \theta \sin \varphi \), and \( \hat{n}^z = \hat{r} \cdot \hat{z} = \cos \theta \). These projections can be represented as combinations of the \( l = 1 \) spherical harmonics, \( \hat{n}_\alpha = \sqrt{4\pi/3} Y^\alpha_1 \), \( \hat{n}_x = i \sqrt{2\pi/3} (Y_1^+ + Y_1^-) \), and \( \hat{n}_y = -i \sqrt{2\pi/3} (Y_1^+ - Y_1^-) \), where \( Y_1^\pm \) are spherical harmonics.

We assume a potential flow, i.e., the peculiar velocity can be written as \( \mathbf{v}(r) = -\nabla \phi(r) \), where \( \phi \) is the velocity potential function. In an expansion of the angular dependence of \( \phi \) into spherical harmonics, \( \phi_{\alpha\beta} \), only the dipole term, \( l = 1 \), contributes to \( \mathbf{B} \). The expansion by means of \( l = 1 \) spherical harmonics is entirely equivalent to a representation in terms of the angular functions \( n^\alpha(\theta, \varphi) \):

\[
\phi(r, \mathbf{\hat{r}}) = \sum_{\alpha=x,y,z} \phi^\alpha(r) \hat{n}^\alpha, \quad (2)
\]

where \( \phi^\alpha(r) = \frac{3}{4\pi} \int d\Omega \phi(r) n^\alpha \),

\[
(3)
\]

thanks to orthogonality conditions \( \int d\Omega n^\alpha n^\beta = 4\pi/3\delta_{\alpha\beta} \), where \( \delta_{\alpha\beta} = 1 \) for \( \alpha = \beta \) and zero otherwise. The corresponding representation of the peculiar velocity field is

\[
v = -\sum_{\alpha} \left[ \frac{d\phi^\alpha}{dr} \hat{n}^\alpha \hat{r} + \frac{\phi^\alpha}{r} \nabla_{\alpha\beta} \hat{n}^\beta \right]. \quad (4)
\]

where \( \nabla_{\alpha\beta} \hat{n}^\beta = \partial \hat{n}^\alpha/\partial \theta + (\sin \theta)^{-1} \partial \hat{n}^\alpha/\partial \phi \) is perpendicular to \( \hat{r} \). This representation is equivalent to an expansion of \( \mathbf{v} \) in terms of vector spherical harmonics \( \mathbf{Y} \) and \( \mathbf{\psi} \).

This relation implies that the radial velocity \( u(r) = v \cdot \hat{r} \) and \( d\phi/dr \) are related by

\[
\frac{d\phi^\alpha}{dr} = -\frac{3}{4\pi} \int u(r) \hat{n}^\alpha d\Omega. \quad (5)
\]

As an example for the representation in terms \( \hat{n}^\alpha \), consider a constant velocity field \( \mathbf{v} = B_0 \hat{z} \) in the \( z \)-direction. In this case, \( \phi = \phi^z \), where \( \phi^z(r) = -B_0 r \). A substitution of this potential in Equation (4) gives \( \mathbf{v} = B_0 \cos \theta \hat{\theta} - B_0 \sin \theta \hat{\varphi} \), which gives \( B^\theta \equiv B^z = 0 \) and \( B^\varphi = B_0 \), as expected.

Both terms on the right-hand side in relation (4) contribute to \( \mathbf{B} \) in Equation (1). We could integrate the relation over a sphere in terms of the bulk flow in terms of \( \phi^\alpha \). However, a
much more elegant way of achieving the same thing is via the divergence theorem, which gives (Nusser et al. 2014)\textsuperscript{1}

$$B(r) = -\frac{3}{4\pi r^3} \int_S \phi(r) dS,$$  
(6)

where the integration is over the surface of the sphere of radius $r$, with surface element $dS = d\Omega r^2 \hat{r}$. Substituting Equation (2) for $\phi(r)$, this relation gives

$$B^a = -\frac{\phi^a}{r},$$  
(7)

where $B^a$ correspond to the three Cartesian components $B_x$, $B_y$, and $B_z$. Relation (6) also gives the bulk flow of a thin spherical shell of radius $r$ as

$$B^a_{\text{sh}} = -\left[\frac{d\phi^a}{dr} + 2\frac{\phi^a}{r}\right].$$  
(8)

3. BULK FLOWS FROM ML

Assume we are provided with galaxy positions $r_i$ and radial peculiar motions $u_i$ of $i = 1, \ldots, N$ galaxies. The $1\sigma$ error on $u_i$ is $\sigma_i$ and we assume that the positions are given accurately. The latter assumption can be justified if we use the redshifts, rather than the observed distances, as proxy to $r_i$. ML provides an estimate, $\hat{B}$, of the bulk flow by minimizing

$$\chi^2 = \sum_i \frac{w_i}{\sigma_i^2} \left[u_i - \hat{B} \cdot \hat{r}_i\right]^2,$$  
(9)

where we allow for a weighting of the galaxies by $w_i$ in addition to the usual statistical weights dictated by $\sigma_i$. At a minimum, $\partial \chi^2 / \partial B^a = 0$ yields

$$\sum \frac{w_i}{\sigma_i^2} \left[\sum \frac{w_i}{\sigma_i^2} n_i^a \cdot n_i \hat{B}\right] = \sum \frac{w_i}{\sigma_i^2} n_i^a u_i n_i^\alpha,$$  
(10)

where we used $\hat{B} \cdot \hat{r}_i = \sum \hat{B}^\beta n_i^\beta$. Observational errors typically depend on distance and not the angular position, hence $\sigma_i = \sigma(r_i)$. Further, if no angular selection is imposed on the observed galaxies and $w_i = w(r_i)$, then the continuous limit of Equation (10) is

$$\int \frac{d\Omega r^2}{\sigma(r)^2} d\Omega \frac{u(r) \hat{n}^a(r)}{4\pi r^3} = \frac{3}{4\pi} \int dr' r'^2 d\Omega \frac{u(r') \hat{n}^a(r')}{\sigma(r')}.$$  
(11)

where $N(r)$ is the three-dimensional (3D) number density of observed galaxies. We ignore here the contribution of the underlying clustering of matter and thus the dependence of $N$ on $r$ is entirely due to observational selection strategy.

4. THE INCONSISTENCY AND ITS RESOLUTION

According to Equation (11), if $u(r) = B_0 \cdot \hat{r}$, where $B_0$ is constant throughout the sphere, then we recover $\hat{B} = B_0$. However, the estimate in Equation (11) does not generally agree with the definition of the bulk flow as given in Equation (1). To see this we rewrite Equation (7) as $B^a = -\phi^a / r = -(\int d\Omega' r \phi^a / r') / r$ and note the relation between $u$ and $d\phi^a / dr$ in Equation (5).

Hence, for a general choice of $w$, the mathematical relation (Equation (7)) and the ML estimate (Equation (11)) are inconsistent. Nonetheless, the two equations become consistent for the specific choice

$$w = \frac{\sigma^2}{N r^2}.$$  
(12)

The term $N r^2$ could be identified with the number density per unit radius. Since typically $\sigma \propto r$, the weighting is essentially equivalent to $w \sim 1/N$. Minimizing $\chi^2$ in Equation (9) with the weights, $w$, given by Equation (12) will yield $\hat{B}(r)$, which is consistent (up to the statistical error) with the definition of bulk flow as the average peculiar velocity within the sphere. In the absence of errors, this choice of $w$ guarantees that $\hat{B}(r)$ coincides with the true $B(r)$.

Another way to achieve an estimate that agrees with the definition (Equation (1)) is as follows. Let us divide the space into a finite number, $s = 1, \ldots, N_s$ of spherical shells, each of radius $r_s$ and thickness, $\delta_s \ll r_s$. Define a new $\chi^2$ function

$$\chi^2 \equiv \sum_{s=1}^{N_s} \sum_{i \in s} \frac{\sigma_i^{-2}}{2} \left[u_i - \sum \beta V_s^\beta \tilde{n}_s^\beta\right]^2.$$  
(13)

Here the symbol $i \in s$ implies galaxies lying inside the shell $s$ and $V_s^\beta = V_s^\beta(r_s)$, of the bulk flow by minimizing

$$\sum_{\beta} \sum_{i \in s} \sigma_i^{-2} \tilde{n}_s^\beta \tilde{n}_s^\beta V_s^\beta = \sum_{i \in s} \sigma_i^{-2} u_i \tilde{n}_s^\beta.$$  
(14)

Once $V_s^\beta$ are obtained by solving the last equations, the potential can be computed as $\phi_s^\beta = -\sum r_s V_s^\beta$ and the bulk flow of a sphere of radius $r_s$ is identified as $B^a = -\phi_s^a / r_s$. Note that $V_s^\beta$ coincides with the bulk flow of the shell only if the velocity in the shell is constant, otherwise, it will be missing the term $-2\phi^a / r$ as is seen from Equation (8).

5. A NUMERICAL DEMONSTRATION

We give a demonstration for the case of perfect data with zero errors and uniform spatial coverage. We do that with the help of a random Gaussian realization of a velocity field with the power spectrum of the CDM model with density parameters $\Omega_c = 0.225$, $\Omega_b = 0.045$, and $\Omega_m = 0.73$, respectively, for the top, dark matter, baryons, and the cosmological constant. The field is generated on a 512\textsuperscript{3} uniform grid in a box 500$h^{-1}$ Mpc on the side, with $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. A grid point with a velocity close to the observed motion of the Local Group is chosen as the central “observer.” The 3D velocities at the grid points within a distance of 100$h^{-1}$ Mpc from the observer are used to directly compute the true bulk flow, $B_0$, of spheres centered on the observer. The actual radial velocities at the grid points are used as “observational” data, without any dilution and any added noise. Thus, $N(r)$ is constant and $\sigma_i$ is a constant, which is formally taken to be very close to zero. We then derive two

\textsuperscript{1} The proof is as follows. For any constant vector $a$ we have $a \cdot \nabla \phi = \nabla \cdot (a \phi)$. Hence $a \cdot B = -\int d^3r \nabla \cdot (a \phi)$, where the integration is over a spherical volume of radius $r$. Applying the divergence theorem to transform the integration of the divergence of the vector $a \phi$ into a surface integration, we get $a \cdot B = -\int (4\pi r^3) \int \phi a \cdot dS$. This is correct for any constant vector $a$ and hence Equation (6) must hold.
estimates for bulk flows for spheres around the observer. The first estimate is derived using the standard ML as appropriate for this data (i.e., \( w \sigma^2 = \text{const in Equation (11)} \)) and the second is obtained with the modified weighting in Equation (12; i.e., \( w \sigma^2 = 1/r^2 \) in Equation (11)). The two estimates and the true bulk flow are shown in Figure 1 as a function of the radius. The discrepancy between the standard ML estimate and the true bulk flow is substantial, while the modified weighting almost yields perfect agreement. It is interesting that the two estimates would coincide had the data been diluted to \( N \propto r^{-2} \).

6. GENERAL REMARKS

The inconsistency in the ML estimation pointed out here stems from the assumption of a constant \( \mathbf{B} \) in the survey volume. With this assumption, standard ML estimation yields, by definition, the most likely constant velocity vector which fits the data inside the survey volume. However, this constant velocity does not coincide with the definition of the bulk flow as the mean velocity of the relevant volume.

We do not aim here at quantifying the inconsistency for realistic data. Different data sets have their own characteristics and the results of various weightings should be assessed individually. Further, any weighting scheme could be applied to a given data set as long as the implications are assessed self-consistently within the context of a cosmological model or in comparison with other data sets. However, to avoid confusion, the term “bulk flow” should be reserved for estimates of the mean motion rather than any other moment of the data.

Peculiar velocity data could be analyzed in many ways (e.g., Davis et al. 2011; Feldman et al. 2010; Turnbull et al. 2012) that do not resort to an application of the ML estimation as presented above. The constrained realizations method (e.g., Hoffman & Ribak 1991; Yepes et al. 2014) reconstructs a full 3D velocity field from observed radial velocity data. In this method the bulk flow can be computed directly from the reconstructed 3D velocity field.

We have refrained from making specific assessment of the uncertainty associated with the bulk flow using the modified and standard estimates given above. This greatly depends on the details of the data set used in these estimates and is beyond the scope of the current paper.

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