EXPLICIT CONSTRUCTION OF COMPLETE KÄHLER METRICS OF SAPER TYPE BY DESINGULARIZATION

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Abstract. We construct complete Kähler metrics of Saper type on the nonsingular set of a subvariety $X$ of a compact Kähler manifold using (a) a method for replacing a sequence of blow-ups along smooth centers, used to resolve the singularities of $X$, with a single blow-up along a product of coherent ideals corresponding to the centers and (b) an explicit local formula for a Chern form associated to this single blow-up. Our metrics have a particularly simple local formula, involving essentially a product of distances to the centers of the blow-ups used to resolve the singularities of $X$. Our proof of (a) uses a generalization of Chow’s theorem for coherent ideals, proved using the Direct Image Theorem.

Introduction

Let $X$ be a singular subvariety of a compact Kähler manifold $M$. In [GM] we showed how to construct a particular type of complete Kähler metric on the nonsingular set of $X$. These metrics grow less rapidly than Poincaré metrics near the singular set $X_{\text{sing}}$ of $X$, and are of interest because in certain cases it is known that their $L^2$-cohomology equals the intersection cohomology of $X$, while the $L^2$-cohomology of a Poincaré metric is usually not equal to the intersection cohomology of $X$. We call our metrics “Saper-type” metrics after Leslie Saper, who first drew our attention to this subject. Saper proved that on any variety with isolated singularities there is a complete Kähler metric whose $L^2$-cohomology equals its intersection cohomology (see [Sa1], [Sa2]). Our metrics agree with Saper’s in the case of isolated singularities, but our construction requires no restriction on the type of singularities.

The construction of Saper-type metrics in [GM] used the geometry of a finite sequence of blow-ups along smooth centers which resolves the singularities of $X$. In this paper we show how to replace a finite sequence of blow-ups along smooth centers by a single blow-up along one center (perhaps singular), which we describe in terms of its coherent sheaf of ideals $\mathcal{I}$. Blowing up $M$ along $\mathcal{I}$ desingularizes $X$. The support of $\mathcal{I}$ is the singular locus $X_{\text{sing}}$ of $X$. We construct $\mathcal{I}$ as a product of coherent ideals $\mathcal{I}_j$ corresponding to the smooth centers $C_j$. Each $\mathcal{I}_j$ is the direct...
image on $M$ of a product of the ideal sheaf of $C_j$ with a sufficiently high power of the exceptional ideal of the previous blow-ups. Our proof that the blow-up of $M$ along $\mathcal{I}$ is equivalent to the blow-up along the centers $C_j$ uses a generalization of Chow’s Theorem for ideals, which we prove using the Direct Image Theorem. We then give a simple and explicit construction of a Chern form associated to the blow-up along $\mathcal{I}$, in terms of local generators of $\mathcal{I}$. Finally we use this Chern form to obtain a simpler and more explicit form of our Saper-type metrics. We also give an example in which we compute $\mathcal{I}$ explicitly in a neighborhood of a singular point.

The Saper-type metric which we obtain can be described in terms of its Kähler (1,1)-form as

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log F)^2,$$

where $\omega$ is the Kähler (1,1)-form of a Kähler metric on $M$ and $F$ is a $C^\infty$ function on $M$, vanishing on $X_{\text{sing}}$. We first construct local $C^\infty$ functions $F_\alpha$, on small open sets $U_\alpha$ in $M$, by setting

$$F_\alpha = \sum_{j=1}^r |f_j|^2$$

where $f_1, \ldots, f_r$ are local holomorphic generating functions on $U_\alpha$ for the coherent ideal sheaf $\mathcal{I}$ described above. To construct a global metric on $M - X_{\text{sing}}$ (and consequently on $X - X_{\text{sing}}$), we patch with a $C^\infty$ partition of unity on $M$. It is crucial that this patching takes place on $M$, rather than on a blow-up of $M$, which might add unwanted elements to the $L^2$-cohomology.

The motivation for our construction is that it may be easier to keep track of a single $C^\infty$ function $F$ in a coordinate neighborhood of $p$ in $M$, rather than to keep track of the many coordinate neighborhoods associated to successive blow-ups which resolve the singularities of $X$. Similarly, it may be more convenient to work with a single ideal sheaf $\mathcal{I}$ on $M$, rather than a sequence of centers and blow-ups.

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I. Outline and Main Results

In sections II and III we give some background and basic results about coherent sheaves of ideals and blow-ups. We begin by describing the direct and inverse images of sheaves, and in particular, direct and inverse images of coherent sheaves of ideals. Then we describe the blow-up $\pi : \tilde{M} \to M$ of a complex manifold $M$ along a coherent sheaf of ideals $\mathcal{I}$. The analytic subset $C = V(\mathcal{I})$ of $M$ determined by $\mathcal{I}$ is called the center of the blow-up. If $C$ is smooth and of codimension at least 2, then $\tilde{M}$ is smooth. The blow-up map $\pi$ is proper and is a biholomorphism except along its exceptional divisor $E = \pi^{-1}(C)$. Even though the direct image of an ideal sheaf may not be an ideal sheaf in general, the direct image of an ideal sheaf under a blow-up map is an ideal sheaf.

Section IV is devoted to a proof of Chow’s Theorem for Ideals using the Direct Image Theorem, which states that the direct image of a coherent sheaf under a proper map is coherent. Section V contains some corollaries for blow-up maps which are useful in constructing single-step blow-ups from a sequence of blow-up maps.

Chow’s Theorem for Ideals. Let $U$ be an open neighborhood of 0 in $\mathbb{C}^n$ and let $X$ be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{J}$ be a coherent sheaf of ideals on $X$. Then $\mathcal{J}$ is relatively algebraic in the following sense: $\mathcal{J}$ is generated (after shrinking $U$ if necessary) by a finite number of homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, with analytic coefficients in $U$-coordinates.

Chow’s Theorem for Ideals helps to describe the relatively algebraic structure of blow-ups. The most useful corollary for the purposes of this paper is the following, which shows that, even though the inverse image of the direct image of an ideal sheaf may not be the original ideal sheaf in general, on a blow-up of a compact complex manifold we can ensure that the two are equal by first multiplying by a high enough power of the ideal sheaf $\mathcal{I}_E$ of the exceptional divisor. This corollary is similar to results of Hironaka and Rossi in [HR] but our proof uses simpler and more explicit methods and is more constructive in nature. We go on to apply this corollary repeatedly to get an explicit description of a coherent sheaf for single-step blow-ups, as a product of coherent sheaves corresponding to a sequence of blow-ups along smooth centers.
Corollary. Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold $M$ along a coherent sheaf of ideals $J_1$ and let $E$ be the exceptional divisor of $\pi$. Let $J_2$ be a coherent sheaf of ideals on $\tilde{M}$. Then there exists an integer $d_0$ such that

$$\pi^{-1}\pi_*(J_2^d E) = J_2^d E$$

for all $d \geq d_0$.

For the purposes of this paper and to apply Hironaka’s theorem on embedded resolution of singularities, we need only consider blow-ups of smooth spaces. If $\tilde{M}$ is smooth, the blow-up of $\tilde{M}$ along $J_2$ is isomorphic to the blow-up of $\tilde{M}$ along $J_2^d E$. Furthermore, the blow-up of $M$ along $J_2^d E$ is isomorphic to the blow-up of the base space $M$ along $J_1\pi_*(J_2^d E)$. Thus we can replace the pair of blow-ups, first along $J_1$ and then along $J_2$, by a single blow-up along $J_1\pi_*(J_2^d E)$. Repeating this procedure for a finite sequence of smooth centers enables us to construct a coherent sheaf of ideals $I$ on $M$ such blowing up $M$ along $I$ is equivalent to blowing up successively along smooth centers. Section VI contains a more detailed version of the proof of the following proposition.

Proposition (Single-Step Blow-ups). Let $M$ be a compact complex manifold and let

$$M_m \xrightarrow{\pi_m} M_{m-1} \to \ldots \to M_2 \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = M$$

be a finite sequence of blow-ups along smooth centers $C_j \subset M_{j-1}$ of codimension at least 2. Then there is a coherent sheaf of ideals $I$ on $M$ such that the blow-up of $M$ along $I$ is isomorphic to the blow-up of $M$ along the sequence of smooth centers $C_j$. Furthermore, we may construct $I$ to be of the form

$$I = I_1 I_2 \ldots I_m,$$

where each $I_j$ is a coherent sheaf of ideals on $M$ and

i. $I_j$ is the direct image on $M$ of the ideal sheaf of $C_j$ multiplied by a high enough power of the ideal sheaf of the exceptional divisor of the first $j-1$ blow-ups,

ii. the inverse image of $I_j$ on $M_{j-1}$ is the ideal sheaf of $C_j$ multiplied by the same power of the exceptional ideal sheaf as in (i), and

iii. the blow-up of $M_{j-1}$ along the inverse image of $I_j$ is isomorphic to the blow-up of $M_{j-1}$ along $C_j$.

This result is related to Theorem II.7.17 of [Ha1], but our proposition is much more explicit and constructive in nature.

We are particularly interested in the case of a sequence of blow-ups along smooth centers which resolves the singularities of a singular subvariety $X$ of $M$. In this case, the proposition gives us a coherent ideal sheaf $I$ on $M$, supported on the singular locus of $X$, such that blowing up along $I$ desingularizes $X$, and also gives a factorization of $I$ in terms of ideals corresponding to the original sequence of blow-ups. This factorization of $I$ is essentially unique for curves.

In section VII we give a simple and explicit construction of a Chern form associated to a blow-up. Suppose that $\pi : \tilde{M} \to M$ is the blow-up of a complex manifold $M$ along a coherent sheaf of ideals $I$ such that $\tilde{M}$ is smooth. Let $E$ be
the exceptional divisor and $L_E$ the line bundle on $\tilde{M}$ associated to $E$. Let $f_1, \ldots, f_r$ be local holomorphic generating functions for $I$ on a small open set $U \subset M$. We construct a Chern form for $L_E$ on $\tilde{U} = \pi^{-1}(U)$ by pulling back the negative of a Fubini-Study form on projective space. This Chern form is strictly negative on the fibres of the map $E \to C = V(I)$, and is given on $\tilde{U} - \tilde{U} \cap E$ by

$$c_1(L_E) = \pi^*(-\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \sum_{j=1}^r |f_j(z)|^2).$$

If $M$ is compact, we may patch together local Chern forms using a $C^\infty$ partition of unity on $M$, in such a way that the negativity on fibres is preserved.

Now consider in more detail a singular subvariety $X$ of a compact Kähler manifold $M$. Hironaka’s famous theorem tells us that the singularities of $X$ may be resolved by a finite sequence of blow-ups of $M$ along smooth centers, such that the total exceptional divisor of the composite of all the blow-ups is a normal crossings divisor $D$ in $\tilde{M}$ which has normal crossings with the desingularization $\tilde{X}$ in $\tilde{M}$ and such that $\tilde{M} - D \cong M - X_{\text{sing}}$ and $\tilde{X} - \tilde{X} \cap D \cong X - X_{\text{sing}}$ (cf. [BM]). By the Single-Step Blow-up Proposition, we may resolve the singularities of $X$ by blowing up $M$ along a single coherent sheaf of ideals $I$ on $M$, whose blow-up is isomorphic to the blow-up obtained using the sequence of smooth centers. The inverse image ideal sheaf of $I$ in the blow-up $\tilde{M}$ determines the normal crossings divisor $D$ and the support of $I$ in $M$ is $X_{\text{sing}}$. We construct a Chern form for the blow-up along $I$, using local holomorphic generating functions of $I$ as above and patching with a $C^\infty$ partition of unity on $M$. This Chern form is negative definite on the fibres of the map from $D$ to $X_{\text{sing}}$. Subtracting this Chern form from the Kähler $(1,1)$-form of a Kähler metric on $M$ gives the $(1,1)$-form of a Kähler metric on $\tilde{M}$, our “desingularizing metric.” The completion of $X - X_{\text{sing}}$ with respect to this metric is nonsingular. Similarly, we use the local holomorphic generators of $I$ to construct our complete Kähler Saper metric on $\tilde{M} - D \cong M - X_{\text{sing}}$. Both metrics are described in more detail below and in section VIII.

**Theorem.** Let $X$ be a singular subvariety of a compact Kähler manifold $M$ and let $\omega$ be the Kähler $(1,1)$-form of a Kähler metric on $M$. Then there exists a $C^\infty$ function $F$ on $M$, vanishing on $X_{\text{sing}}$, such that for $l$ a large enough positive integer,

i. the $(1,1)$-form

$$\tilde{\omega} = l\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log F$$

is the Kähler form of a desingularizing Kähler metric for $X$, i.e. the completion of $X - X_{\text{sing}}$ with respect to $\tilde{\omega}$ is a desingularization of $X$ and

ii. the $(1,1)$-form

$$\omega_S = l\omega - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} (\log F)^2$$

is the Kähler form of a complete Kähler modified Saper metric (in the terminology of [GM]) on $M - X_{\text{sing}}$ and hence on $X - X_{\text{sing}}$.

Furthermore, the function $F$ may be constructed to be of the form

$$F = \prod_\alpha F_\alpha^{p_\alpha}.$$
where \( \{\rho_\alpha\} \) is a \( C^\infty \) partition of unity subordinate to an open cover \( \{U_\alpha\} \) of \( M \),

\[
F_\alpha = \sum_{j=1}^r |f_j|^2,
\]

and \( f_1, \ldots, f_r \) are holomorphic functions on \( U_\alpha \), vanishing exactly on \( X_{\text{sing}} \cap U_\alpha \). More specifically, \( f_1, \ldots, f_r \) are local holomorphic generators of a coherent ideal sheaf \( I \) on \( M \) such that blowing up \( M \) along \( I \) desingularizes \( X \), \( I \) is supported on \( X_{\text{sing}} \), and the exceptional divisor of the blow-up along \( I \) has normal crossings with itself and with the desingularization of \( X \).

The coherent ideal sheaf \( I \) is constructed as a product \( I_1 I_2 \cdots I_m \) of coherent ideal sheaves corresponding to a sequence of blow-ups along smooth centers \( C_j \) which resolves the singularities of \( X \). This factorization of \( I \) gives a corresponding factorization of \( F_\alpha \), as essentially a product of distances to the centers,

\[
F_\alpha = \prod_{j=1}^m \sum_{i=1}^{r_j} |v_{ji}|^2
\]

where, for each \( j \), the functions \( \{v_{ji}\} \) are local holomorphic functions on \( U_\alpha \) whose pullbacks to the preimage of \( U_\alpha \) under the first \( j - 1 \) blow-ups generate an ideal sheaf with the same blow-up as \( C_j \).

The idea behind the metric constructions in this paper is to first find simple and explicit formulas locally on \( M \), then patch by \( C^\infty \) partitions of unity on \( M \). We wish to avoid formulas which are local only on blow-ups of \( M \) and we also wish to avoid introducing \( C^\infty \) partition-of-unity functions on the blow-ups.

We conclude, in section IX, by constructing \( I \) for the cuspidal cubic \( y^2 - x^3 \). The method used generalizes to the case of any singular locally toric complex analytic variety. The details will be given elsewhere.

II. Direct and Inverse Images of Coherent Sheaves of Ideals

Coherent Sheaves

We first review the important concept of coherence (see e.g. [GrR1], [GuR]).

Let \( M \) be a complex space and let \( \mathcal{S} \) be an analytic sheaf on \( M \), i.e. a sheaf of \( \mathcal{O}_M \)-modules. For example, consider an ideal sheaf of \( \mathcal{O}_M \) or the sheaf of holomorphic sections of a holomorphic vector bundle on \( M \).

Definition. The sheaf \( \mathcal{S} \) is of finite type at \( x \in M \) if there exists an open set \( U \) of \( x \) such that the restriction \( \mathcal{S}_U \) of \( \mathcal{S} \) to \( U \) is generated by a finite number of sections of \( \mathcal{S} \) over \( U \). This means that there exist sections \( s_1, \ldots, s_r \) of \( \mathcal{S} \) over \( U \) such that for each point \( y \in U \) and for each germ \( g_y \in \mathcal{S}_y \), there exist \( a_{1y}, \ldots, a_{ry} \in \mathcal{O}_{M,y} \) such that

\[
g_y = \sum_{i=1}^r a_{iy}s_{iy}.
\]

The sheaf \( \mathcal{S} \) is of finite type on \( M \) if \( \mathcal{S} \) is of finite type at \( x \) for all \( x \in M \).
Remark. Note that if $s$ and $t$ are sections of $S$ on a neighborhood of a point $y$ such that $s_y = t_y$ (i.e., they have the same germs at $y$), then $s = t$ in an open neighborhood of $y$, by fundamental properties of sheaves. In particular, in the definition above, if $g_y, a_1y, ..., a_ry$ are the germs of $g, a_1, ..., a_r$ at $y$ then there exists a neighborhood $V \subset U$ of $y$ such that

$$g = \sum_{i=1}^{r}a_is_i$$

on $V$.

Each finite collection $s = (s_1, ..., s_r)$ of sections of $S$ over $U$ determines a sheaf homomorphism

$$\psi_s : \mathcal{O}_U \to S_U$$

given by

$$(f_1, ..., f_r) \mapsto \sum_{i=1}^{r} f_is_i.$$  

Definition. The sheaf $S$ is of relation finite type at $x \in M$ if $\ker \psi_s$ is of finite type at $x$ for all finite collections $s$ of sections of $S$ over an open neighborhood $U$ of $x$. $S$ is of relation finite type on $M$ if $S$ is of relation finite type at $x$ for all $x \in M$.

Definition. The sheaf $S$ is coherent on $M$ if

(1) $S$ is of finite type on $M$, and

(2) $S$ is of relation finite type on $M$.

Since coherent sheaves are always finite type, by definition, it follows that if $S$ is a coherent sheaf on a complex space $X$ and $s_1, ..., s_r$ are sections of $S$ on a neighborhood $U$ of a point $x$ such that the germs $s_1x, ..., s_rx$ generate $S_x$, then there exists a neighborhood $V \subset U$ of $x$ such that $s_1, ..., s_r$ generate $S_V$.

We refer the reader to [F], [GrR1], [GrR2], [GuR], and [W] for background on the following and other fundamental properties of coherent sheaves:

i. The sheaf $\mathcal{O}_M$ is coherent.

ii. A subsheaf of a coherent sheaf is coherent if and only if it is of finite type. In particular, an ideal sheaf of $\mathcal{O}_M$ is coherent if and only if it is of finite type.

iii. A coherent ideal sheaf $I$ on a complex space determines a closed complex analytic subspace $V(I)$, and the ideal sheaf $I_Y$ of a closed complex analytic subspace $Y$ of a complex space is coherent.

Lemma II.1. If $I_1$ and $I_2$ are coherent sheaves of ideals on a complex space $M$, then the product ideal sheaf $I_1I_2$ is also coherent.

Proof. Since both $I_1$ and $I_2$ are of finite type, their product is of finite type and is thus coherent. □

We define direct images and inverse images of coherent sheaves of ideals, and give some conditions under which these sheaves are themselves coherent ideal sheaves (in general they may be only sheaves of modules). We show that direct and inverse images of composite maps are composites of the direct and inverse image maps (functoriality). We also show that the inverse image of a product of ideals is the product of the inverse image ideals. Direct and inverse images of ideal sheaves under blow-up maps are discussed in Lemmas III.9 and V.8.
**Direct Images**

**Direct Image.** Let \( f : M \to N \) be a holomorphic map of complex spaces and let \( S \) be a sheaf on \( M \). The direct image sheaf \( f_*S \) on \( N \) is the sheaf associated with the presheaf given by \( f_*S(U) = S(f^{-1}(U)) \), for \( U \) any open set in \( N \).

If \( S \) is coherent, the direct image \( f_*S \) is not necessarily coherent. However \( f_*S \) is coherent if \( f \) is proper, by the Direct Image Theorem. We recall the Direct Image Theorem in our context (see e.g. [GrR1], pp 207, 227, and 36).

**Direct Image Theorem.** Let \( f : M \to N \) be a holomorphic map of complex spaces and let \( S \) be a coherent sheaf on \( M \). If \( f \) is proper then \( f_*S \) is coherent.

In particular, if \( f \) is a blow-up map (see section III), then \( f \) is proper and \( f_*S \) is coherent if \( S \) is.

If \( J \) is a sheaf of ideals on \( M \), then \( f_*J \) is a sheaf of \( O_N \)-modules but not, in general, an ideal sheaf on \( N \). We will show (Lemma III.9) that if \( f \) is a blow-up map then \( f_*J \) is an ideal sheaf.

**Inverse Images**

Once again, let \( f : M \to N \) be a holomorphic map of complex spaces. Let \( S \) be a sheaf of \( O_N \)-modules.

**Topological Inverse Image.** We define the topological inverse image \( f'^*S \) to be the fibre product \( S \times_N M \), i.e. the stalk of \( f'^*S \) over a point \( m \in M \) is the stalk of \( S \) over \( f(m) \in N \): \[
(f'^*S)_m = S_{f(m)}.\]

Note that \( f'^*S \) is a sheaf of \( f'^*O_N \)-modules. If \( S \) is coherent then so is \( f'^*S \).

**Pullback Sheaf.** We define the pullback sheaf as

\[
f^*S = f'^*S \otimes_{f'^*O_N} O_M.
\]

Note that \( f^*S \) is a sheaf of \( O_M \)-modules and once again, if \( S \) is coherent then so is \( f^*S \). Also

\[
f^*O_N = f'^*O_N \otimes_{f'^*O_N} O_M = O_M.
\]

If \( I \) is an ideal sheaf on \( N \), we have an exact sequence

\[
0 \to I \to O_N.
\]

Since tensoring is not in general left exact, the induced map

\[
f^*I \to f^*O_N = O_M
\]

is not necessarily injective, so \( f^*I \) is not necessarily an *ideal sheaf* on \( M \). The *image* of \( f^*I \) in \( O_M \) is an ideal sheaf, which we call the inverse image ideal sheaf and will describe in more detail later in this section.
Flat Maps. A holomorphic map $f : M \to N$ of complex spaces is flat if

$$O_{M,m} \text{ is } O_{N,f(m)}\text{-flat}$$

for all $m \in M$. Equivalently, $f$ is flat if for every exact sequence

$$0 \to S_1 \to S_2$$

of $O_{N,f(m)}$-modules, the induced sequence

$$0 \to S_1 \otimes_{O_{N,f(m)}} O_{M,m} \to S_2 \otimes_{O_{N,f(m)}} O_{M,m}$$

is also exact.

There are many references on flat maps, e.g. ([F], p. 147 and p. 155).

Examples. If $X$ and $Y$ are complex spaces, the canonical projection $X \times Y \to Y$ is flat. Every locally trivial holomorphic map is flat. In particular, if $f : L \to X$ is a line bundle over a complex space $X$ (or more generally, a vector bundle), then $f$ is flat.

Lemma II.2. If $f : M \to N$ is a flat holomorphic map of complex spaces and $0 \to S_1 \to S_2$ is an exact sequence of sheaves of $O_N$-modules, then $0 \to f^*S_1 \to f^*S_2$ is an exact sequence of sheaves of $O_M$-modules.

Proof. Suppose that

$$0 \to S_1 \to S_2$$

is an exact sequence of sheaves of $O_N$-modules, i.e.

$$0 \to S_{1,n} \to S_{2,n}$$

is an exact sequence of $O_{N,n}$-modules for each $n \in N$. Then in particular,

$$0 \to S_{1,f(m)} \to S_{2,f(m)}$$

is an exact sequence of $O_{N,f(m)}$-modules for all $m \in M$. If $f : M \to N$ is flat, then

$$0 \to S_{1,f(m)} \otimes_{O_{N,f(m)}} O_{M,m} \to S_{2,f(m)} \otimes_{O_{N,f(m)}} O_{M,m}$$

is exact for all $m \in M$, i.e.

$$0 \to (f^*S_1)_m \otimes (f^*O_N)_m O_{M,m} \to (f^*S_2)_m \otimes (f^*O_N)_m O_{M,m}$$

is exact for all $m \in M$. These tensor products can be rewritten as

$$0 \to (f^*S_1) \otimes f^*O_N O_M \to (f^*S_2) \otimes f^*O_N O_M$$

showing that

$$0 \to f^*S_1 \otimes f^*O_N O_M \to f^*S_2 \otimes f^*O_N O_M$$

is exact. By the definition of $f^*$, this means that

$$0 \to f^*S_1 \to f^*S_2$$

is exact. □
Lemma II.3. If $\mathcal{L}$ is the sheaf of holomorphic sections of a line bundle (or more generally of a vector bundle) over a complex space $M$, and

$$0 \to S_1 \to S_2$$

is an exact sequence of sheaves of $\mathcal{O}_M$-modules, then

$$0 \to S_1 \otimes \mathcal{L} \to S_2 \otimes \mathcal{L}$$

is also exact.

Proof. A finitely generated module over a local noetherian ring is flat if and only if it is free ([Ma], Proposition 3.G, p. 21). Therefore $\otimes_{\mathcal{O}_M} \mathcal{L}$ preserves exact sequences. □

Inverse Image Ideal. Let $f : M \to N$ be a holomorphic map of complex spaces. If $\mathcal{I}$ is an ideal sheaf on $N$, the image of $f^*\mathcal{I}$ in $\mathcal{O}_M$ is an ideal sheaf which we define to be the inverse image ideal sheaf $f^{-1}\mathcal{I}$.

The ideal sheaf $f^{-1}\mathcal{I}$ is sometimes written $f^*\mathcal{I} \cdot \mathcal{O}_M$ or $f^{-1}\mathcal{I} \cdot \mathcal{O}_M$. If $\mathcal{I}$ is coherent, then $f^{-1}\mathcal{I}$ is also coherent.

If $\mathcal{I}$ is a coherent ideal, the subscheme of $M$ determined by $f^{-1}\mathcal{I}$ is the inverse image scheme of the subscheme of $N$ determined by $\mathcal{I}$, i.e.

$$V(f^{-1}\mathcal{I}) = f^{-1}(V(\mathcal{I})).$$

Lemma II.4. If $f : M \to N$ is a flat holomorphic map of complex spaces and $\mathcal{I}$ is an ideal sheaf on $N$, then $f^{-1}\mathcal{I} \cong f^*\mathcal{I}$.

Proof. By Lemma II.2 above, if $f$ is flat, then the map $f^*\mathcal{I} \to f^*\mathcal{O}_N = \mathcal{O}_M$ is injective. □

Corollary II.5. If $f : L \to X$ is a line bundle (or more generally a vector bundle) and $\mathcal{I}$ is an ideal sheaf on $X$, then $f^{-1}\mathcal{I} = f^*\mathcal{I}$.

Proof. As noted in the discussion of flat maps above, the projection of a line bundle (or vector bundle) onto its base space is a flat map. □

Composites

Next we describe the behavior of direct and inverse images under composites. The proofs are straightforward, using the definitions above.

Lemma II.6 (The Composite of Direct Images is the Direct Image of the Composite). Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $\mathcal{S}$ be a sheaf on $M_1$. Then

$$g_*(f_*\mathcal{S}) \cong (g \circ f)_*\mathcal{S}.$$

Proof. Let $U$ be an open set in $M_3$. Then

$$g_*(f_*\mathcal{S})(U) = (f_*\mathcal{S})(g^{-1}(U))$$

$$= \mathcal{S}(f^{-1}g^{-1}(U))$$

$$= \mathcal{S}((g \circ f)^{-1}(U))$$

$$= (g \circ f)_*(U).$$ □
Lemma II.7 (The Composite of Topological Inverse Images is the Topological Inverse Image of the Composite). Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $S$ be a sheaf on $M_3$. Then

$$f'(g'S) \cong (g \circ f)'S.$$ 

Proof. We will prove the statement on stalks. Let $m$ be a point in $M_1$. Then

$$(f'(g'S))_m = (g'S)_{f(m)}$$

$$= S_{g \circ f(m)}$$

$$= ((g \circ f)'S)_m. \Box$$

Lemma II.8 (The Composite of Pullbacks is the Pullback of the Composite). Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $S$ be a sheaf on $M_3$. Then

$$f^*(g^*S) \cong (g \circ f)^*S.$$ 

Proof. For convenience, let $O_i$ represent $O_{M_i}$ for $i = 1, 2, 3$. Recall that

$$g^*S = g'S \otimes g'\mathcal{O}_3 \mathcal{O}_2.$$ 

Similarly

$$f^*(g^*S) = f'(g^*S) \otimes f'\mathcal{O}_2 \mathcal{O}_1$$

$$= f'(g'S \otimes g'\mathcal{O}_3 \mathcal{O}_2) \otimes f'\mathcal{O}_2 \mathcal{O}_1.$$ 

Looking at stalks over $m \in M_1$ we have

$$(f^*(g^*S))_m = f'(g'S \otimes g'\mathcal{O}_3 \mathcal{O}_2)_m \otimes (f'\mathcal{O}_2)_m \mathcal{O}_1,m$$

$$= (g'S \otimes g'\mathcal{O}_3 \mathcal{O}_2)_{f(m)} \otimes (f'\mathcal{O}_2)_m \mathcal{O}_1,m$$

$$= (g'S)_{f(m)} \otimes (g'\mathcal{O}_3)_{f(m)} \mathcal{O}_2,f(m) \otimes (f'\mathcal{O}_2)_m \mathcal{O}_1,m$$

$$= S_{g(f(m))} \otimes (g'\mathcal{O}_3)_{f(m)} \mathcal{O}_2,f(m) \otimes (f'\mathcal{O}_2)_m \mathcal{O}_1,m$$

$$= S_{g(f(m))} \otimes (g'\mathcal{O}_3)_{f(m)} \mathcal{O}_1,m$$

$$= ((g \circ f)'S)_m \otimes ((g \circ f)'\mathcal{O}_3)_m \mathcal{O}_1,m$$

$$= ((g \circ f)'S)_m. \Box$$

The following lemma is more naturally understood in terms of subschemes determined by coherent sheaves of ideals. Its interpretation in term of subschemes is that the inverse image subscheme under a composite map is the composite of the inverse images. Briefly, $f^{-1}$ is functorial on ideals and their corresponding subschemes.

Lemma II.9 (The Composite of Inverse Images is the Inverse Image of the Composite). Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $I$ be a sheaf of ideals on $M_3$. Then

$$f^{-1}(g^{-1}I) \cong (g \circ f)^{-1}I.$$
Proof. As in the previous proof, let $O_i = O_{M_i}$. Recall that $g^{-1}\mathcal{I}$ is defined to be the image of $g^*\mathcal{I}$ in $O_2$, so there is a surjective map

$$g^*\mathcal{I} \mapsto g^{-1}\mathcal{I}.$$ 

The map of topological inverse images

$$f'g^*\mathcal{I} \mapsto f'g^{-1}\mathcal{I}$$

is also surjective.

Tensoring over $f'\mathcal{O}_2$ by $\mathcal{O}_1$ we obtain the map

$$f^*g^*\mathcal{I} \mapsto f^*g^{-1}\mathcal{I},$$

which is surjective since tensoring is right exact.

Finally we note that

$$f^{-1}g^{-1}\mathcal{I} = \text{image of } f^*g^{-1}\mathcal{I} \text{ in } \mathcal{O}_1 \text{ by definition}$$

$$= \text{image of } f^*g^*\mathcal{I} \text{ in } \mathcal{O}_1 \text{ by surjectivity}$$

$$= \text{image of } (g \circ f)^*\mathcal{I} \text{ in } \mathcal{O}_1 \text{ by Lemma II.8}$$

$$= (g \circ f)^{-1}\mathcal{I} \text{ by definition} \qed$$

**Products of Ideals**

The following lemma is also more naturally understood in terms of subschemes determined by coherent sheaves of ideals. The subscheme of $M$ determined by $(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2)$ is the union of the subschemes determined by $f^{-1}\mathcal{I}_1$ and $f^{-1}\mathcal{I}_2$, which are the inverse images of the subschemes determined by $\mathcal{I}_1$ and $\mathcal{I}_2$. The subscheme of $M$ determined by $f^{-1}(\mathcal{I}_1\mathcal{I}_2)$ is the inverse image of the union of the subschemes determined by $\mathcal{I}_1$ and $\mathcal{I}_2$, which is the same as the union of the inverse images.

**Lemma II.10 (The Inverse Image Ideal of a Product of Ideal Sheaves is the Product of the Inverse Image Ideal Sheaves).** Let $f : M \to N$ be a holomorphic map of complex spaces and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be sheaves of ideals on $N$. Then

$$(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2) \cong f^{-1}(\mathcal{I}_1\mathcal{I}_2).$$

Proof. Note that both $f^{-1}(\mathcal{I}_1\mathcal{I}_2)$ and $(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2)$ are generated as ideals in $\mathcal{O}_M$ by products of the form $f^*w_1f^*w_2$ where $w_1$ is a germ of $\mathcal{I}_1$ and $w_2$ a germ of $\mathcal{I}_2$. \qed

The direct image of a product of ideal sheaves is not necessarily equal to the product of the direct images, but we will show later (Lemma V.8) that the two are equal if the map is a blow-up of a smooth center and the ideal sheaves are first multiplied by a high enough power of the ideal sheaf of the exceptional divisor.
III. Blowing up a Complex Manifold along a Coherent Sheaf of Ideals

Let $M$ be a complex manifold and let $I$ be a coherent sheaf of ideals on $M$. Here and throughout the paper we will always assume that $I$ is not the zero sheaf. Since $I$ is coherent, for each point $p \in M$ we may choose a coordinate neighborhood $U$, centered at $p$, such that $I(U)$ is generated by a finite number of global sections over $U$. We first define the blow-up of $M$ along $I$ locally over such an open set $U$, using a collection of generators of $I(U)$. We then show that the result is independent of the collection of generators chosen, so that the blow-up may be defined globally over $M$.

Blow-ups may also be defined for singular complex spaces but we do not need such generality here.

Local Description of Blow-ups

Let $M$ be a complex manifold and $I$ a coherent sheaf of ideals on $M$ as above. Let $U$ be a small enough coordinate neighborhood in $M$ that $I(U)$ is generated by a finite collection of global sections $f_1, ..., f_r$ on $U$. Set

$$V(I) = \{ z \in U : h(z) = 0 \text{ for all } h \in I \}.$$ 

We define a map

$$\phi_f : U - V(I) \rightarrow \mathbb{P}^{r-1}$$

by setting $\phi_f(z) = [f_1(z) : ... : f_r(z)]$. Let $\Gamma(\phi_f)$ be the graph of $\phi_f$ in $U \times \mathbb{P}^{r-1}$, i.e.

$$\Gamma(\phi_f) = \{ (z, [\xi]) : z \in U - V(I) \text{ and } [\xi] = [f_1(z) : ... : f_r(z)] \}$$

$$= \{ (z, [\xi]) : z \in U - V(I) \text{ and } f_i(z)\xi_j = f_j(z)\xi_i, \ 1 \leq i, j \leq r \}.$$

We define $\tilde{U}_f$ to be the smallest reduced complex analytic subspace of $U \times \mathbb{P}^{r-1}$ containing the graph $\Gamma(\phi_f)$. The support of $\tilde{U}_f$ is the closure of $\Gamma(\phi_f)$ in the usual topology.

The blow-up map of $U$ along $I$ is the projection $\pi : \tilde{U}_f \rightarrow U$, which is a proper map.

We will now show that the complex space $\tilde{U}_f$ is independent of the generators $f$ chosen for $I$.

Lemma III.1. If $\{f_1, ..., f_r\}$ and $\{g_1, ..., g_s\}$ are two collections of generators of $I$ on $U$ then

$$\tilde{U}_f \cong \tilde{U}_g.$$

Proof. Define a map $\psi : \Gamma(\phi_f) \rightarrow \Gamma(\phi_g)$ by

$$\psi(z, [\xi]) = (z, [g_1(z) : ... : g_s(z)]).$$

The map $\psi$ is well-defined because $g_1(z), ..., g_s(z)$ are not all 0 for $z \in U - V(I)$, (since $g_1, ..., g_s$ are generators of $I$). Furthermore $\psi^{-1}$ exists and is given by

$$\psi^{-1}(z, [\xi]) = (z, [f_1(z) : ... : f_r(z)].$$
Both $\psi$ and $\psi^{-1}$ are clearly holomorphic so $\Gamma(\phi_f) \cong \Gamma(\phi_g)$. We will now show that they extend to holomorphic maps on $\tilde{U}_f$ and $\tilde{U}_g$.

Since $\{f_1, ..., f_r\}$ and $\{g_1, ..., g_s\}$ both generate $I$, there exist $\alpha_{ij}, \beta_{ij} \in \mathcal{O}(U)$ such that

$$g_i(z) = \sum_{j=1}^{r} \alpha_{ij}(z)f_j(z)$$

and

$$f_i(z) = \sum_{j=1}^{s} \beta_{ij}(z)g_j(z)$$

for all $z$ in $U$. Briefly,

$$f(z) = \beta(z)g(z) = \beta(z)\alpha(z)f(z)$$

for all $z \in U$. The functions $\alpha$ and $\beta$ might not define maps on all of $\mathbb{P}^{r-1}$ and $\mathbb{P}^{s-1}$ but they do define maps on $\Gamma(\phi_f)$ and $\Gamma(\phi_g)$.

Suppose that $(z', [\xi']) \in U \times \mathbb{P}^{r-1}$ is the limit of points $(z_{\gamma}, [\xi_{\gamma}]) \in \Gamma(\phi_f)$, i.e. there is a sequence of points $\{z_{\gamma}\} \in U$ such that

$$z_{\gamma} \to z' \quad \text{and} \quad [\xi_{\gamma}] = [f_1(z_{\gamma}) : \ldots : f_r(z_{\gamma})] \to [\xi']$$

Some component of $[\xi']$ is nonzero, say the first component, so that we may assume that $\xi' = (1, \xi'_2, \ldots, \xi'_{r})$. Then we may also assume that the sequence $\{z_{\gamma}\}$ has the property that $f_1(z_{\gamma}) \neq 0$ for all $\gamma$ and that the sequence $\xi_{\gamma}$ is of the form

$$\xi_{\gamma} = (1, \xi_{\gamma,2}, \ldots, \xi_{\gamma,r}) = \left(1, \frac{f_2(z_{\gamma})}{f_1(z_{\gamma})}, \ldots, \frac{f_r(z_{\gamma})}{f_1(z_{\gamma})}\right)$$

where

$$\xi_{\gamma} \to \xi'.$$

We will use this description to show that $\alpha(z')\xi' \neq 0$. We have

$$\beta(z_{\gamma})\alpha(z_{\gamma})\xi_{\gamma} = \beta(z_{\gamma})\alpha(z_{\gamma})\frac{f(z_{\gamma})}{f_1(z_{\gamma})} \quad \text{by (**)}$$

$$= \frac{f(z_{\gamma})}{f_1(z_{\gamma})} \quad \text{by (*)}$$

$$= \xi_{\gamma} \quad \text{by (**).}$$

Thus

$$\beta(z')\alpha(z')\xi' = \lim_{\gamma \to \infty} \beta(z_{\gamma})\alpha(z_{\gamma})\xi_{\gamma}$$

$$= \lim_{\gamma \to \infty} \xi_{\gamma}$$

$$= \xi'$$

by continuity of $\alpha$ and $\beta$. In particular, $\alpha(z')\xi' \neq 0$ so $[\xi] = [\alpha(z')\xi']$ exists as a point of $\mathbb{P}^{s-1}$ (and is independent of the choices of representatives $\xi'$ and $\xi_{\gamma}$).

We define $\psi$ on $(z', [\xi'])$ to be

$$\psi(z', [\xi']) = (z', [\alpha(z')\xi']).$$

The definition of $\psi^{-1}$ is similar. Clearly these extensions of $\psi$ and $\psi^{-1}$ to the closures of $\Gamma(\phi_f)$ and $\Gamma(\phi_g)$ are holomorphic and their compositions are the identity, so we obtain the required isomorphism $\tilde{U}_f \cong \tilde{U}_g$. $\square$
**Local Blow-up.** From the preceding lemma we see that it makes sense to define the blow-up of $U$ along $\mathcal{I}$ as $\text{Bl}_I U = \tilde{U} = \tilde{U}_f$ for any set of generators $f$.

If $\mathcal{I}$ is the ideal of a **smooth** subspace $C$ of $U$ then $\tilde{U}$ is also smooth. The set $C$ is called the **center** of the blow-up. If $\mathcal{I}$ is the ideal of a singular subset of $U$ then $\tilde{U}$ may be singular.

**Lemma III.2.** Let $\mathcal{I}$ and $\mathcal{J}$ be nonzero coherent ideal sheaves on $U$ which are generated by global sections on $U$. Suppose that $\mathcal{J}$ is principal, i.e. generated by a single function on $U$. Then $\text{Bl}_{\mathcal{I},\mathcal{J}} U \cong \text{Bl}_I U$.

**Proof.** Suppose that $\mathcal{J}$ is generated locally by the single function $h$. Then $[hf_1 : ... : hf_r] = [f_1 : ... : f_r]$ on $U - V(\mathcal{I}, \mathcal{J})$. □

We will use this lemma a little later to prove a corresponding statement about line bundles (Lemma III.4).

**Global Description of Blow-ups**

Let $\mathcal{I}$ be a coherent sheaf of ideals on a complex manifold $M$. By Lemma III.1, we may extend the local definition of the blow-up canonically, to define a global blow-up

$$\pi : \tilde{M} = \text{Bl}_I M \to M.$$  

The blow-up map $\pi$ is proper and the restriction of $\pi$ from $\tilde{M} - \pi^{-1}(V(\mathcal{I}))$ to $M - V(\mathcal{I})$ is biholomorphic.

If $\mathcal{I}$ is the ideal sheaf of a **smooth** submanifold $C$ of $M$, then $\tilde{M}$ is smooth.

**Ideals, Divisors, Line Bundles, and Sections**

Let $M$ be a complex manifold and let $D$ be a divisor on $M$. We denote by $L_D$ or $[D]$ the corresponding line bundle on $M$. Let $\mathcal{L}_D$ be the invertible sheaf of holomorphic sections of $[D]$.

Let $s_D$ be a meromorphic section of $[D]$ whose divisor $(s_D)$ is $D$. Such a section always exists: if $D$ is defined on an open covering $\{U_i\}$ of $M$ by meromorphic functions $\{f_i\}$, the functions $\{f_i\}$ themselves define such a section $s_D$.

If $s$ is any other meromorphic section of $[D]$ then $\frac{s}{s_D}$ is a meromorphic function on $M$. Let $\mathcal{K}_M$ be the sheaf of meromorphic functions on $M$. We may embed $\mathcal{L}_D$ into $\mathcal{K}_M$ by the map

$$s \mapsto \frac{s}{s_D},$$

i.e. if $U$ is any open set in $M$ and $s \in \mathcal{L}_D(U)$, we map $s$ to $\frac{s}{s_D} \in \mathcal{K}_M(U)$.

Now suppose that $Y$ is an effective divisor (codimension one subscheme) of $M$ with ideal sheaf $\mathcal{I}_Y$, and that $Y$ is given on an open cover $\{U_i\}$ of $M$ by holomorphic functions $\{f_i\}$. Let $s_Y$ be the corresponding holomorphic section of $[Y]$. Then $\frac{1}{s_Y}$ is a meromorphic section of $[-Y]$. We may embed $\mathcal{L}_{-Y}$ into $\mathcal{K}_M$ by the map

$$s \mapsto ss_Y.$$
The image of $\mathcal{L}_{-Y}$ in $\mathcal{K}_M$ is just the ideal $\mathcal{I}_Y$ in $\mathcal{O}_M \subset \mathcal{K}_M$. Therefore

$$\mathcal{L}_{-Y} \cong \mathcal{I}_Y.$$ 

Suppose that $\mathcal{I}$ is any coherent ideal in $\mathcal{O}_M$. Tensoring the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_M$$

by $\mathcal{L}_{-Y}$ gives an exact sequence

$$0 \to \mathcal{I} \otimes \mathcal{L}_{-Y} \to \mathcal{O}_M \otimes \mathcal{L}_{-Y} = \mathcal{L}_{-Y}$$

by Lemma II.3 above. The image of $\mathcal{I} \otimes \mathcal{L}_{-Y}$ in $\mathcal{L}_{-Y}$ is just $\mathcal{I} \mathcal{L}_{-Y}$ (see e.g. [Ma], p. 18). The image of $\mathcal{I} \mathcal{L}_{-Y}$ under the embedding $\mathcal{L}_{-Y} \hookrightarrow \mathcal{K}_M$ is then $\mathcal{I} \mathcal{I}_Y$. Therefore

**Lemma III.3.** Let $\mathcal{I}$ be a coherent sheaf of ideals on a complex manifold $M$ and let $Y$ be an effective divisor on $M$. Then

$$\mathcal{I} \otimes \mathcal{L}_{-Y} \cong \mathcal{I} \mathcal{I}_Y.$$ 

**Lemma III.4.** If $\mathcal{I}$ is a coherent sheaf of ideals on a complex manifold $M$, and $\mathcal{L}$ is the sheaf of holomorphic sections of a holomorphic line bundle on $M$, then the blow-up of $M$ along $\mathcal{I}$ is biholomorphic to the blow-up of $M$ along $\mathcal{I} \otimes \mathcal{L}$. In particular, if $Y$ is an effective divisor on $M$, then the blow-up of $M$ along $\mathcal{I}$ is biholomorphic to the blow-up of $M$ along $\mathcal{I} \otimes \mathcal{L}_{-Y} \cong \mathcal{I} \mathcal{I}_Y$.

**Proof.** Apply Lemma III.2. 

**Lemma III.5.** Let $M$ be a complex manifold and let $\mathcal{I}$ be a coherent sheaf of ideals on $M$. Let $\pi: \tilde{M} = Bl_M I \to M$ be the blow-up of $M$ along $\mathcal{I}$. Then $\mathcal{I}$ is a sheaf of principal ideals on $M$ (i.e. an invertible sheaf). The complex subspace of $\tilde{M}$ corresponding to $\pi^{-1} \mathcal{I}$ is a hypersurface.

**Proof.** Suppose that $\mathcal{I}$ is generated locally on an open set $U$ in $M$ by $f_1, \ldots, f_r$. Since $\tilde{U}$ is contained in the subset of $U \times \mathbb{P}^{r-1}$ given by the equations $f_i(z)\xi_j = f_j(z)\xi_i$, it is enough to prove that the inverse image ideal of $\mathcal{I}$ on this set is principal. But this is clear since on the set $U_i = \{\xi_i \neq 0\}$, we have

$$f_j = \frac{\xi_j}{\xi_i} f_i$$

so $f_i$ generates the inverse image ideal of $\mathcal{I}$ on $U_i$. 

**Exceptional Divisors of Blow-ups**

The hypersurface in $\tilde{M}$ corresponding to $\pi^{-1} \mathcal{I}$, described in Lemma III.5 above, is called the **exceptional divisor** $E$ of $\pi$, i.e.

$$E = V(\pi^{-1}(I)) = \pi^{-1} V(I).$$

The proof of Lemma III.5 above gives us a local description of $E$. Suppose that $f_1, \ldots, f_r$ generate $\mathcal{I}$ on an open set $U$ in $M$. Cover $U \subset U \times \mathbb{P}^{r-1}$ by sets $U_i = \{\xi_i \neq 0\}$. Then $E$ is given on $U_i$ by $f_i = 0$.

The map $\pi: \tilde{M} \to M$ is a proper map which is biholomorphic from $\tilde{M} - E$ to $M - V(\mathcal{I})$. If $\mathcal{I}$ is the ideal sheaf of a smooth center $C$, i.e. $C = \mathcal{I}_C$, then $\tilde{M}$ is smooth, $E = \pi^{-1}(C)$ is a smooth submanifold of $\tilde{M}$, and for each $p \in C$ the inverse image $E_p = \pi^{-1}(p)$ is biholomorphic to $\mathbb{P}^{k-1}$, where $k$ is the codimension of $C$ in $M$. 
Exceptional Line Bundles of Blow-ups

Corresponding to the exceptional divisor $E$ on $\tilde{M}$ is an exceptional line bundle $L_E = [E]$. Both $E$ and $L_E$ are independent of the local generators of $\mathcal{I}$ used to construct the blow-up.

In terms of local generators $f_1, ..., f_r$ of $\mathcal{I}$, transition functions for $L_E$ are

$$g_{ij} = \frac{f_i}{f_j} = \frac{\xi_i}{\xi_j}$$

i.e. if $s$ is a holomorphic section of $L_E$ over $\tilde{U}$ then $s$ is represented by holomorphic functions $s_i$ on $U_i = \{\xi_i \neq 0\}$ with

$$s_i = g_{ij} s_j \quad \text{on } U_i \cap U_j.$$ 

Since local transition functions for $L_E$ on the set $\tilde{U}$ are of the form $g_{ij} = \frac{\xi_i}{\xi_j}$, the line bundle $L_E$ on $\tilde{U}$ is the restriction of the universal bundle $\mathcal{O}(\mathbb{P}^{r-1})$ on $U \times \mathbb{P}^{r-1}$. More precisely, let $\sigma_1 : U \times \mathbb{P}^{r-1} \to U$ and $\sigma_2 : U \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ be the first and second projection maps, as shown below.

$$\xymatrix{ \text{Bl}_I U = \tilde{U} \ar[r] & U \times \mathbb{P}^{r-1} \ar[r]^-{\sigma_2} & \mathbb{P}^{r-1} \ar[d]^-{\sigma_1} \ar[r] & U }$$

Let $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ be the universal bundle on $\mathbb{P}^{r-1}$. Then the restriction to $\tilde{U}$ of the line bundle $\sigma_2^* \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ is $L_E$ on $\tilde{U}$.

We may interpret the fibre of $L_E$ over $(z, [\xi]) \in \tilde{U}$ as the line through $\xi$ in $\mathbb{C}^r$.

Universal Property of Blow-ups

**Lemma III.6 (Universal Property of Blow-ups).** Let $M$ be a complex manifold and let $\mathcal{I}$ be a coherent sheaf of ideals on $M$. Let $\pi : \tilde{M} = \text{Bl}_I M \to M$ be the blow-up of $M$ along $\mathcal{I}$. Suppose that $\phi : N \to M$ is a holomorphic map of a complex space $N$ to $M$, such that the inverse image ideal $\phi^{-1} \mathcal{I}$ is principal (i.e. an invertible sheaf). Then there exists a unique holomorphic lifting

$$\tilde{\phi} : N \to \tilde{M}$$

such that $\pi \circ \tilde{\phi} = \phi$.

**Proof.** Suppose that $f_1, ..., f_r$ are generators for $\mathcal{I}$ over a small open set $U \subset M$. Then $f_1 \circ \phi, ..., f_r \circ \phi$ are generators for $\phi^{-1} \mathcal{I}$ over $\phi^{-1}(U)$ in $N$. Since $\phi^{-1} \mathcal{I}$ is assumed to be a principal ideal sheaf, all of the functions $f_i \circ \phi$ are multiples of one of them, so we have a well-defined map

$$\tilde{\phi} : \phi^{-1}(U) \to U \times \mathbb{P}^{r-1}$$

given by

$$v \mapsto (\phi(v), [f_1 \circ \phi(v) : ... : f_r \circ \phi(v)]).$$
By construction, the image of $\tilde{\phi}$ lies in the blow-up $\tilde{U}$ in $U \times \mathbb{P}^{r-1}$ and $\pi \circ \tilde{\phi}(v) = \phi(v)$.

By an argument similar to the proof of Lemma III.1 above, which showed that the blow-up $\tilde{U}$ is independent of the collection of generators $\{f_i\}$ used to construct it, we see that the map $\phi$ is independent of the generators $\{f_i\}$. Thus we can extend our local construction to a well-defined holomorphic map $\phi : N \to M$.

Finally we check the uniqueness of $\tilde{\phi}$. Suppose that $\tilde{\phi}'$ is any holomorphic map from $N$ to $\tilde{M}$ such that $\pi \circ \tilde{\phi}' = \phi = \pi \circ \tilde{\phi}$. Since $\pi$ is a biholomorphism away from the exceptional set, $\tilde{\phi}'$ and $\tilde{\phi}$ must agree on $\phi^{-1}(M - V(I)) = N - V(\phi^{-1}I)$. But $\phi^{-1}I$ was assumed to be a principal ideal, so $V(\phi^{-1}I)$ is a hypersurface in $N$. This means that $\tilde{\phi}'$ and $\tilde{\phi}$ agree on a dense set of $N$, so they must agree everywhere. □

**Blow-up of a Product of Ideals**

We will show that the blow-up of a product of two ideals looks like the composite of two blow-ups. Since we have defined blow-ups only for smooth manifolds, we will restrict ourselves to the case in which the blow-up along one ideal is smooth, for example if that ideal is the ideal of a smooth submanifold.

**Proposition III.7.** Let $M$ be a complex manifold and $I_1$ and $I_2$ coherent sheaves of ideals on $M$. Let $\pi : Bl_{I_1}M \to M$ be the blow-up of $M$ along $I_1$ and suppose that the blow-up space $Bl_{I_1}M$ is smooth. Then

$$Bl_{I_1,I_2}M \cong Bl_{\pi^{-1}I_1}Bl_{I_1}M,$$

i.e. the blow-up of $M$ along the product ideal $I_1I_2$ is isomorphic to the blow-up of $M$ along $I_1$ followed by the blow-up along the inverse image ideal of $I_2$.

**Proof.** We will apply the universal mapping property of blow-ups (Lemma III.6). Let $N = Bl_{\pi^{-1}I_1}Bl_{I_1}M$ and let $\phi : N \to M$ be the composite of the blow-up maps. Then $\phi^{-1}I_1$ and $\phi^{-1}I_2$ are principal ideal sheaves on $N$ so $\phi^{-1}(I_1I_2)$ is also principal. By the universal mapping property, $\phi$ lifts to a holomorphic map $\bar{\phi} : N \to Bl_{I_1I_2}M$. This map is a biholomorphism away from the exceptional sets.

Similarly, if $\psi : Bl_{I_1I_2}M \to M$ is the blow-up of $M$ along $I_1I_2$, then $\psi^{-1}I_1$ is a principal ideal sheaf on $Bl_{I_1I_2}M$ and we can lift $\psi$ to a map $\tilde{\psi}_1 : Bl_{I_1I_2}M \to Bl_{I_1}M$. Next we check that $\psi_1^{-1}(\pi^{-1}I_2)$ is again a principal ideal sheaf, so that we can lift $\psi_1$ to a map $\tilde{\psi} : Bl_{I_1I_2}M \to Bl_{\pi^{-1}I_1}Bl_{I_1}M = N$.

Since the maps $\bar{\phi}$ and $\tilde{\phi}$ are holomorphic everywhere and are inverses of each other on open dense sets, they must be inverses of each other everywhere. □

**Corollary III.8.** Let $M$ be a complex manifold, $C$ a smooth center in $M$, and $I_C$ the ideal sheaf of $C$. Then the blow-up of $M$ along $I_C$ is isomorphic to the blow-up along $I_C^d$ for any integer $d > 1$, i.e.

$$Bl_{I_C}M \cong Bl_{I_C^d}M.$$

**Proof.** Apply Proposition III.7, noting that $\pi^{-1}I_C$ is principal and that blowing-up along a principal ideal sheaf leaves a space unchanged. □
Direct Images under Blow-up Maps

We conclude section III by showing that the direct image of an ideal sheaf under a blow-up map is an ideal sheaf. As always, we assume that the ideal sheaf $\mathcal{I}$ for our blow-up is not the zero sheaf, so that $C = V(\mathcal{I})$ has codimension at least 1.

**Lemma III.9.** Let $\pi : \tilde{M} \to M$ be the blow-up of a complex manifold $M$ along a coherent sheaf of ideals $\mathcal{I}$ on $M$. Let $\mathcal{J}$ be a sheaf of ideals on $\tilde{M}$. Then the direct image $\pi_*\mathcal{J}$ is a sheaf of ideals on $M$. If $\mathcal{J}$ is coherent then so is $\pi_*\mathcal{J}$.

**Proof.** We wish to define a map $\pi_*\mathcal{J} \to \mathcal{O}_M$ and show that it is injective. To define a sheaf map $\pi_*\mathcal{J} \to \mathcal{O}_M$, it is enough to define presheaf maps $\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U)$ for all open sets $U$ in $M$. To show that a map of sheaves $\pi_*\mathcal{J} \to \mathcal{O}_M$ is injective, it is enough to show that $\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U)$ is injective for all open sets $U$ in $M$.

Recall that $\pi_*\mathcal{J}(U) = \mathcal{J}(U)$, where $\tilde{U} = \pi^{-1}(U)$. If $U$ does not intersect $C = V(\mathcal{I})$, then $\tilde{U} \cong U$ and $\pi_*\mathcal{J}(U)$ may be identified naturally as an ideal in $\mathcal{O}_M(U)$. Now suppose that $U$ does intersect $C$ and consider $g \in \pi_*\mathcal{J}(U) = \mathcal{J}(\tilde{U})$. Let $E$ be the exceptional divisor of $\pi$ in $\tilde{M}$. Since

$$\tilde{U} - \tilde{U} \cap E \cong U - U \cap C,$$

we may define a holomorphic function $G$ on $U - U \cap C$ whose pullback to $\tilde{U} - \tilde{U} \cap E$ is $g$. For each $p \in U \cap C$, the fibre $\pi^{-1}(p)$ is compact, since $\pi$ is proper. Therefore $g$ is constant on $\pi^{-1}(p)$ and bounded on a neighborhood of $\pi^{-1}(p)$ in $\tilde{U}$. Thus the function $G$ is locally bounded in $\tilde{U}$, so $G$ extends uniquely to a holomorphic function on $U$ by Riemann's Removable Singularity Theorem. Since $\pi^*G$ and $g$ are holomorphic on $\tilde{U}$ and equal on the dense set $\tilde{U} - \tilde{U} \cap E$, they must be equal on all of $\tilde{U}$, i.e. $\pi^*G = g$ on $\tilde{U}$. For each $g \in \mathcal{J}(\tilde{U})$ there is a unique such $G \in \mathcal{O}_M(U)$, so we have a well-defined map

$$\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U).$$

Clearly $G$ is identically zero if and only if $g$ is identically zero, so the map is injective.

By the Direct Image Theorem, $\pi_*\mathcal{J}$ is coherent if $\mathcal{J}$ is, since $\pi$ is proper. \qed

**IV. Chow’s Theorem for Ideals**

This section is devoted to the proof of Chow’s Theorem for Ideals, using the Direct Image Theorem. References for the usual Chow’s theorem are [F] and [M].

In section V we will state some applications to blow-ups.

**IV.1 Chow’s Theorem for Ideals.** Let $U$ be an open neighborhood of $\{0\}$ in $\mathbb{C}^r$ and let $X$ be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{I}$ be a coherent sheaf of ideals on $X$. Then $\mathcal{I}$ is relatively algebraic in the following sense: $\mathcal{I}$ is generated (after shrinking $U$ if necessary) by a finite number of homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, with analytic coefficients in $U$-coordinates.

Since a sheaf on $X \subset U \times \mathbb{P}^n$ may be considered as a sheaf on $U \times \mathbb{P}^n$, we will ignore $X$ and prove the theorem for a coherent sheaf of ideals $\mathcal{I}$ on $U \times \mathbb{P}^n$. Although we have assumed that $U$ is an open neighborhood of $\{0\}$ in $\mathbb{C}^r$, the same methods could be used for any complex space $U$. When we say that $\mathcal{I}$ is generated by homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, we mean that the
dehomogenizations of these polynomials generate the ideal locally. We will show at
the end of this section that we may choose all the polynomial generators of \( \mathcal{I} \) to be
of the same degree \( d \), for \( d \) sufficiently large.

The usual Chow’s theorem follows directly from Theorem IV.1: if \( Y \) is an analytic
subset of \( U \times \mathbb{P}^n \) and \( \mathcal{I} = \mathcal{I}_Y \) is the ideal sheaf of \( Y \) on \( X = U \times \mathbb{P}^n \), then
(after shrinking \( U \) if necessary) \( Y \) is cut out by a finite number of homogeneous
polynomials in \( \mathbb{P}^n \)-coordinates with analytic coefficients in \( U \)-coordinates.

**Outline of Proof of Chow’s Theorem for Ideals.** Let \( \mathcal{X}^{n+1} \) be the blow-up of \( \mathbb{C}^{n+1} \) at the origin and let \( \sigma_1 \) and \( \sigma_2 \) be the two projection maps of \( U \times \mathcal{X}^{n+1} \) as shown:

\[
U \times \mathcal{X}^{n+1} \xrightarrow{\sigma_2} U \times \mathbb{P}^n \\
\sigma_1 \downarrow \\
U \times \mathbb{C}^{n+1}
\]

The map \( \sigma_2 \) is flat since \( U \times \mathcal{X}^{n+1} \) is a line bundle over \( U \times \mathbb{P}^n \), the product of the
identity on \( U \) with the universal line bundle on \( \mathbb{P}^n \). Thus \( \sigma_2^{-1}\mathcal{I} = \sigma_2^*\mathcal{I} \) (Lemma II.4).
This inverse image ideal sheaf is coherent (see facts on inverse image ideals, section II). The map \( \sigma_1 \) is proper, so the direct image \( J = \sigma_1^*(\sigma_2^{-1}\mathcal{I}) \) is also coherent,
by the Direct Image Theorem. Furthermore, \( J \) is a sheaf of ideals on \( U \times \mathbb{C}^{n+1} \),
not merely a sheaf of modules, since \( \sigma_1 \) is a blow-up (Lemma III.9). We will show
(Lemmas IV.2 - IV.5) that \( J \) is generated by homogeneous polynomials in \( \mathbb{C}^{n+1} \)-coordinates on a neighborhood of \((0, 0)\), and that the corresponding polynomials in
homogeneous \( \mathbb{P}^n \)-coordinates generate \( \mathcal{I} \).

More specifically, let \( x = (x_1, ..., x_r) \) and \( y = (y_0, ..., y_n) \) be coordinates for \( U \) and
\( \mathbb{C}^{n+1} \). If \( F(x, y) \) is a holomorphic function in a neighborhood of \((0, 0)\) in \( U \times \mathbb{C}^{n+1} \) and \( \lambda \in \mathbb{C}^* \), let \( F(\lambda) \) be the holomorphic function given by

\[
F(\lambda)(x, y) = F(x, \lambda y).
\]

We will now prove Lemmas IV.2 - IV.5 to complete the proof of Chow’s Theorem
for Ideals. As above, let \( x = (x_1, ..., x_r) \) and \( y = (y_0, ..., y_n) \) be coordinates for
\( U \subset \mathbb{C}^r \) and \( \mathbb{C}^{n+1} \), and let \( F(\lambda)(x, y) = F(x, \lambda y) \).
Lemma IV.2. A holomorphic function $F$ is a section of $\mathcal{J} = \sigma_1(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0) \in U \times \mathbb{C}^{n+1}$ if and only if $F^{(\lambda)}$ is a section of $\mathcal{J}$ in a neighborhood of $(0,0)$ for each $\lambda \in \mathbb{C}^*$.

Proof. A holomorphic function is a section of $\mathcal{J} = \sigma_1(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ if and only if its pullback by $\sigma_1$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $\sigma_1^{-1}(0,0) = \{0\} \times \mathbb{P}^n$ in $U \times \mathbb{C}^{n+1}$. Suppose that $F$ is a section of $\mathcal{J}$ on a neighborhood of $(0,0)$. To show that $F^{(\lambda)}$ is a section of $\mathcal{J}$ on a neighborhood of $(0,0)$, it is enough to show that $\sigma_1^*F^{(\lambda)}$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $p$ for each $p \in \sigma_1^{-1}(0,0)$. This reduces the proof to a simple calculation in local coordinates near $p$ and $q = \sigma_2(p)$.

Choose homogeneous coordinates $[\xi_0 : \ldots : \xi_n]$ on $\mathbb{P}^n$ such that the point $q = \sigma_2(p)$ in $U \times \mathbb{P}^n$ is given by $q = (0, [1 : 0 : \ldots : 0])$. Let $W \subset \{\xi_0 \neq 0\} \subset \mathbb{P}^n$ be a neighborhood of $[1 : 0 : \ldots : 0]$ and let $w_i = \frac{\xi_i}{\xi_0}$, for $1 \leq i \leq n$, be nonhomogeneous coordinates for $W$. The preimage $\sigma_2^{-1}(U \times W) \cong U \subset \mathbb{C} \times W$ is a neighborhood of $p$ in $U \times \mathbb{C}^{n+1}$ with coordinates $(x, y_0, w) = (x_1, \ldots, x_r, y_0, w_1, \ldots, w_n)$ in which $p = (0,0,0)$. The maps $\sigma_1$ and $\sigma_2$ are given by

$$\sigma_1(x, y_0, w) = (x, y_0, y_0w) \quad \text{and} \quad \sigma_2(x, y_0, w) = (x, w).$$

Since the ideal sheaf $\mathcal{I}$ is coherent, $\mathcal{I}$ is generated on a neighborhood of $q$ by a finite collection of holomorphic functions $G_1, \ldots, G_s$. The pullbacks $\sigma_2^*G_1, \ldots, \sigma_2^*G_s$ generate $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $p$. Since $\sigma_1^*F$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $p$, there exist holomorphic functions $A_1, \ldots, A_s$ on a neighborhood of $p$ such that

$$\sigma_1^*F(x, y_0, w) = \sum_{i=1}^s A_i(x, y_0, w)\sigma_2^*G_i(x, y_0, w).$$

Fix $\lambda \neq 0$. Then for $y_0$ close enough to 0, $(x, \lambda y_0, w)$ is in the domain of the functions $\sigma_1^*F$ and $A_1, \ldots, A_s$ and

$$\sigma_1^*F^{(\lambda)}(x, y_0, w) = \sigma_1^*F(x, \lambda y_0, w)$$

$$= \sum_{i=1}^s A_i(x, \lambda y_0, w)\sigma_2^*G_i(x, \lambda y_0, w)$$

$$= \sum_{i=1}^s A_i(x, \lambda y_0, w)G_i(x, w)$$

$$= \sum_{i=1}^s A_i(x, \lambda y_0, w)\sigma_2^*G_i(x, y_0, w).$$

Let $A_i^{(\lambda)}(x, y_0, w) = A_i(x, \lambda y_0, w)$ for $1 \leq i \leq s$. Then each $A_i^{(\lambda)}$ is holomorphic on a neighborhood of $p$ and

$$\sigma_1^*F^{(\lambda)}(x, y_0, w) = \sum_{i=1}^s A_i^{(\lambda)}(x, y_0, w)\sigma_2^*G_i(x, y_0, w),$$

i.e. $\sigma_1^*F^{(\lambda)}$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $p$. □
Lemma IV.3. If $F^{(\lambda)}(x, y)$ is a section of $\mathcal{J}$ on a neighborhood of $(0, 0) \subset U \times \mathbb{C}^{n+1}$ for all $\lambda \in \mathbb{C}^*$, then each homogeneous term in $y$ of $F(x, y)$ is a section of $\mathcal{J}$ on a neighborhood of $(0, 0)$.

Proof. For any holomorphic function $F$ on a neighborhood of $(0, 0)$, let

$$F(x, y) = \sum_\alpha a_\alpha(x) y^\alpha$$

be the expansion of $F(x, y)$ in terms of monomials $y^\alpha = y_0^{\alpha_0} y_1^{\alpha_1} ... y_n^{\alpha_n}$ in $y$ with analytic coefficients $a_\alpha(x)$ in $x$. Let $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n$. The homogeneous term in $y$ of degree $k$ in $F$ is

$$F_k(x, y) = \sum_{|\alpha|=k} a_\alpha(x) y^\alpha.$$

Then

$$F = \sum_{k=0}^\infty F_k \quad \text{and} \quad F^{(\lambda)} = \sum_{k=0}^\infty \lambda^k F_k.$$ 

We wish to show that if $F$ is a section of $\mathcal{J}$ on a neighborhood of $(0, 0)$, then each $F_k$ is also a section of $\mathcal{J}$ on a neighborhood of $(0, 0)$. To minimize the use of subscripts, we will also use $F$ and $F_k$ to represent the germs of these functions at $(0, 0)$.

Let $A = \mathcal{O}_{U \times \mathbb{C}^{n+1}, (0, 0)}$ (a Noetherian local ring), $(y) = (y_0, ..., y_n)$ (an ideal contained in the unique maximal ideal in $A$), and $J = \mathcal{J}_{(0, 0)}$ (also an ideal in $A$). Let

$$\text{Jet}_m(F) = \sum_{k=0}^m F_k$$

be the $m$-jet of $F$ with respect to $y$. Note that $F - \text{Jet}_m(F) \in (y)^{m+1}$.

By a corollary of Krull’s Theorem (see e.g. [K], Corollary 5.7, p. 151),

$$J = \cap_{m \geq 0} (J + (y)^m),$$

where $(y)^0$ is defined to be $A$. Since

$$A = J + (y)^0 \supset J + (y)^1 \supset J + (y)^2 \supset ...$$

it follows that

$$J = \cap_{m \geq m_0} (J + (y)^m)$$

for any $m_0 \geq 0$.

Suppose that $F^{(\lambda)} \in J$ for all $\lambda \in \mathbb{C}^*$. Then since

$$F^{(\lambda)} - \text{Jet}_m(F^{(\lambda)}) \in (y)^{m+1}$$

we have

$$\text{Jet}_m(F^{(\lambda)}) \in J + (y)^{m+1}$$

for all $\lambda \in \mathbb{C}^*$. Since $\text{Jet}_m(F^{(\lambda)}) = \sum_{k=0}^m \lambda^k F_k$ for all $\lambda \in \mathbb{C}^*$, by taking $m + 1$ values of $\lambda$ it follows that

$$F_k \in J + (y)^{m+1}$$
for $0 \leq k \leq m$. Fixing $k$, we have

$$F_k \in J + (y)^{m+1} \quad \text{for } m \geq k$$

or

$$F_k \in J + (y)^m \quad \text{for } m \geq k + 1,$$

i.e.

$$F_k \in \cap_{m \geq k+1} (J + (y)^m).$$

By the corollary of Krull’s Lemma mentioned above, $F_k \in J$ for all $k$. □

**Lemma IV.4.** If $\mathcal{J}_{(0,0)}$ is generated by a collection of elements of $\mathcal{O}_{U,0}[y_0, ..., y_n]$ which are homogeneous in $y$, then $\mathcal{J}_{(0,0)}$ is generated by a finite collection of elements of $\mathcal{O}_{U,0}[y_0, ..., y_n]$ which are homogeneous in $y$.

**Proof.** Throughout the proof, whenever we refer to homogeneous functions, we mean functions which are homogeneous in $y$. The ring $\mathcal{O}_{U \times \mathbb{C}^{n+1},(0,0)}$ is Noetherian. As an ideal of $\mathcal{O}_{U \times \mathbb{C}^{n+1},(0,0)}$, the ideal $\mathcal{J}_{(0,0)}$ must be finitely generated, but we want generators which are in $\mathcal{O}_{U,0}[y_0, ..., y_n]$ and homogeneous. In order to keep track of the rings and ideals involved, we use the following notation:

$$A = \mathcal{O}_{U \times \mathbb{C}^{n+1},(0,0)} \quad \text{(a Noetherian ring)}$$

$$B = \mathcal{O}_{U,0}[y_0, ..., y_n] \quad \text{(a Noetherian subring of } A)$$

$$J = \mathcal{J}_{(0,0)} \quad \text{(an ideal in } A)$$

$$J' = J \cap B \quad \text{(an ideal in } B).$$

Suppose that there is a collection $H$ (perhaps infinite) of homogeneous generators of $J$ over $A$ such that $H \subset B$. Then $H \subset J'$. Since $J'$ is an ideal in $B$ and $B$ is Noetherian, there exists a finite set $H' \subset B$ such that $H'$ generates $J'$ over $B$.

Each element of $H'$ must be a linear combination of a finite number of homogeneous generators in $H$. Thus $J'$ is generated over $B$ by a finite number of homogeneous generators in $H$, i.e. we may choose $H'$ to be a finite set of homogeneous elements.

Since $H \subset J'$, $H'$ also generates $H$ over $B$. Since $B \subset A$, $H'$ generates $H$ over $A$. Finally, since $H'$ generates $H$ over $A$ and $H$ generates $J$ over $A$, $H'$ generates $J$ over $A$, i.e. there exists a finite set $H' \subset \mathcal{O}_{U,0}[y_0, ..., y_n]$ of homogeneous polynomials in $J$ such that $H'$ generates $\mathcal{J}_{(0,0)}$ over $\mathcal{O}_{U \times \mathbb{C}^{n+1},(0,0)}$. □

Note that each homogeneous element of $\mathcal{O}_{U,0}[y_0, ..., y_n]$ is represented on a neighborhood of $(0,0)$ by a homogeneous polynomial in $y$ with analytic coefficients in $x$.

**Lemma IV.5.** The same polynomials that generate $\mathcal{J}$ over a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$, generate $\mathcal{I}$ over a neighborhood of $\{0\} \times \mathbb{P}^n$ in $U \times \mathbb{P}^n$.

**Proof.** Suppose that $\mathcal{J}$ is generated in a neighborhood of $(0,0)$ by $F_1(x,y), ..., F_s(x,y)$, where $F_i(x,y)$ is a homogeneous polynomial of degree $d_i$ in $y$ with analytic coefficients in $x$. We will show that $\mathcal{I}$ is generated on a neighborhood of $\{0\} \times \mathbb{P}^n$ in $U \times \mathbb{P}^n$ by the corresponding polynomials $F_i(x,\xi)$, where $[\xi] = [\xi_0 : ... : \xi_n]$ are homogeneous coordinates for $\mathbb{P}^n$. More precisely, we will show that $\mathcal{I}$ is generated
on a neighborhood of any point \( q \in \{0\} \times \mathbb{P}^n \) by dehomogenizations of \( F_1, \ldots, F_s \) near \( q \).

Choose homogeneous coordinates \( \xi \) on \( \mathbb{P}^n \) such that \( q = (0, [1 : 0 : \ldots : 0]) \). Nonhomogeneous coordinates on the set \( W = \{ \xi_0 \neq 0 \} \subset \mathbb{P}^n \) are \( w_i = \frac{x_i}{\xi_0} \) for \( 1 \leq i \leq n \). We will check that \( \mathcal{I} \) is generated in a neighborhood of \( q \) by the polynomials

\[
\frac{F_i(x, \xi)}{\xi_0^d} = F_i \left( x, \frac{\xi}{\xi_0} \right) = F_i(x, 1, w_1, \ldots, w_n).
\]

First we look at the maps \( \sigma_1 \) and \( \sigma_2 \) in local coordinates. We may use \((x, y_0, w)\) as local coordinates in \( \sigma_2^{-1}(U \times W) \cong U \times \mathbb{C} \times W \). Local coordinates for \( U \times \mathbb{C}^{n+1} \) are \((x, y_0, y_1, \ldots, y_n)\), where \( y_i = y_0w_i \) for \( 1 \leq i \leq n \). The maps \( \sigma_1 \) and \( \sigma_2 \) are given by

\[
\sigma_1(x, y_0, w) = (x, y_0, y_0w) \quad \text{and} \quad \sigma_2(x, y_0, w) = (x, w).
\]

Suppose that \( G \) is a holomorphic section of \( \mathcal{I} \) on a neighborhood of \( q \) in \( U \times \mathbb{P}^n \). Then \( \sigma_2^* G \) is a holomorphic section of \( \sigma_2^{-1} \mathcal{I} \) in a neighborhood of \( \sigma_2^{-1}(q) = \{(0, y_0, 0) : y_0 \in \mathbb{C} \} \). Since the homogeneous polynomials \( F_1, \ldots, F_s \) generate \( \mathcal{J} = \sigma_1^*(\sigma_2^{-1} \mathcal{I}) \) on a neighborhood of \((0, 0) \in U \times \mathbb{C}^{n+1}\), their pullbacks \( \sigma_1^* F_1, \ldots, \sigma_1^* F_s \) generate \( \sigma_2^{-1} \mathcal{I} \) on a neighborhood of \( \sigma_1^{-1}(0, 0) \in U \times \mathbb{C}^{n+1} \). In particular, there exist holomorphic functions \( A_1, \ldots, A_s \) on a neighborhood of the point \((x = 0, y_0 = 0, w = 0)\) in \( U \times \mathbb{C}^{n+1} \) such that

\[
\sigma_2^* G(x, y_0, w) = \sum_{i=1}^{s} A_i(x, y_0, w) \sigma_1^* F_i(x, y_0, w)
\]

on that neighborhood. But \( \sigma_2^* G(x, y_0, w) = G(x, w) \) is independent of the value of \( y_0 \) and \( \sigma_1^* F_i(x, y_0, w) = F_i(x, y_0, y_0w) = y_0^d F_i(x, 1, w) \) since \( F_i \) is homogeneous of degree \( d_i \) in \( y \). Therefore

\[
G(x, w) = \sum_{i=1}^{s} A_i(x, y_0, w) y_0^d F_i(x, 1, w).
\]

Choose some fixed nonzero value of \( y_0 \), close enough to 0 that \((x, y_0, w)\) is in the domain of all the functions \( A_i \) for \( x \) and \( w \) close enough to 0. Define

\[
a_i(x, w) = A_i(x, y_0, w) y_0^{-d_i}.
\]

Then

\[
G(x, w) = \sum_{i=1}^{s} a_i(x, w) F_i(x, 1, w).
\]

Since the functions \( a_i \) are holomorphic on a neighborhood of the point \( q = (x = 0, w = 0) \), and the functions \( F_i(x, 1, w) \) are the local dehomogenizations of the homogeneous polynomials \( F(x, \xi) \), we are done. \( \square \)

This completes the proof of Chow’s Theorem for Ideals. We now show that the homogeneous polynomial generators of the ideal sheaf \( \mathcal{I} \) can be chosen to be of the same degree \( d \), for large enough \( d \).
Corollary IV.6. Let \( U \) be an open neighborhood of \( \{0\} \) in \( \mathbb{C}^r \) and let \( X \) be an analytic subset of \( U \times \mathbb{P}^n \). Let \( \mathcal{I} \) be a coherent sheaf of ideals on \( X \). Then (possibly after shrinking \( U \)) there exists a positive integer \( d_0 \) such that for all \( d \geq d_0 \) the ideal \( \mathcal{I} \) is generated by a finite number of degree \( d \) homogeneous polynomials in homogeneous \( \mathbb{P}^n \)-coordinates with analytic coefficients in \( U \)-coordinates.

Proof. As before, we may treat \( \mathcal{I} \) as a sheaf on \( U \times \mathbb{P}^n \). By Chow’s Theorem for Ideals, we may choose a finite collection of homogeneous polynomials generating \( \mathcal{I} \). We wish to show that we can choose homogeneous polynomials which are all of the same degree. Suppose that \( F_1, \ldots, F_s \) are homogeneous polynomials of degrees \( d_1, \ldots, d_s \) generating \( \mathcal{I} \) on \( U \times \mathbb{P}^n \). Let \( d_0 \) be any integer at least as large as the largest of \( d_1, \ldots, d_s \). Then replace each \( F_i \) with the set of all \( \xi^a F_i \) as \( \xi^a \) runs through all degree \( d_0 - d_i \) monomials in homogeneous coordinates \( [\xi] = [\xi_0 : \ldots : \xi_n] \) on \( \mathbb{P}^n \), i.e. use all monomials of the form \( \xi_0^{\alpha_0} \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n} \) where \( \alpha_0 + \alpha_1 + \ldots + \alpha_n = d_0 - d_i \). At every point in \( U \times \mathbb{P}^n \), the dehomogenizations of the polynomials \( \xi^a F_i \) generate the same ideal as the dehomogenization of the polynomial \( F_i \). \( \square \)

Degree \( d \) homogeneous polynomials on \( \mathbb{P}^n \) may be viewed as sections of \( \mathcal{O}(d) \), the sheaf of holomorphic sections of the \( d \)-th power of the hyperplane bundle on \( \mathbb{P}^n \). By abuse of notation, we will also use \( \mathcal{O}(d) \) to refer to the corresponding sheaf on \( U \times \mathbb{P}^n \), obtained by pullback from \( \mathbb{P}^n \) under the projection map \( U \times \mathbb{P}^n \to \mathbb{P}^n \). If \( \mathcal{I} \) is a coherent sheaf of ideals on \( U \times \mathbb{P}^n \), holomorphic sections of \( \mathcal{I} \otimes \mathcal{O}(d) \) may be represented by homogeneous polynomials of degree \( d \) in homogeneous \( \mathbb{P}^n \)-coordinates with analytic coefficients in \( U \)-coordinates, whose local dehomogenizations are sections of \( \mathcal{I} \).

We can thus restate Corollary IV.6 as follows.

Corollary IV.7. Let \( U \) be an open neighborhood of \( \{0\} \) in \( \mathbb{C}^r \) and let \( X \) be an analytic subset of \( U \times \mathbb{P}^n \). Let \( \mathcal{I} \) be a coherent sheaf of ideals on \( X \). Then (possibly after shrinking \( U \)) there exists a positive integer \( d_0 \) such that for all \( d \geq d_0 \) the ideal \( \mathcal{I} \otimes \mathcal{O}(d) \) is generated by a finite number global sections on \( X \subset U \times \mathbb{P}^n \).

V. Chow’s Theorem Applied to Blow-ups

In this section we consider some consequences of Chow’s Theorem for Ideals for blow-ups.

Corollary V.1 (Blow-ups are Relatively Algebraic). Let \( \pi : \tilde{M} \to M \) be the blow-up of a complex manifold \( M \) along a coherent sheaf of ideals \( \mathcal{I} \). Then for each point \( p \) in \( M \), there exists a neighborhood \( U \) of \( p \) in \( M \) and an embedding of \( \tilde{U} = \pi^{-1}(U) \) into \( U \times \mathbb{P}^{r-1} \), for some \( r \), such that \( \tilde{U} \) is cut out by a finite number of homogeneous polynomials in homogeneous \( \mathbb{P}^{r-1} \)-coordinates with analytic coefficients in \( U \)-coordinates. Furthermore, we may choose all the homogeneous polynomial generators to be of the same degree \( d \) for \( d \) sufficiently large.

Proof. Choose \( U \) small enough that \( \mathcal{I} \) is generated by global sections \( f_1, \ldots, f_r \) on \( U \) and let \( \tilde{U} \to U \times \mathbb{P}^{r-1} \) be the induced embedding. Then use Corollary IV.6 of Chow’s Theorem for Ideals, with \( X = U \times \mathbb{P}^{r-1} \) and the ideal \( \mathcal{I} = \mathcal{I}_0 \). \( \square \)

Now consider a coherent sheaf of ideals \( \mathcal{J} \) on \( \tilde{M} \). Corollary IV.7 tells us that if \( U \) is a small enough open set in \( M \) and \( d \) is a large enough positive integer, the sheaf \( \mathcal{J} \otimes \mathcal{O}(d) \) is generated by a finite number of global sections on \( \tilde{U} \subset U \times \mathbb{P}^{r-1} \).
Recall from section III that the restriction of \( \mathcal{O}(d) \) to \( \tilde{U} \) is just \( \mathcal{L}^d_E \), the sheaf of holomorphic sections of the \( d \)th power of the dual of the exceptional line bundle. From this observation and from Lemma III.3, we have
\[
\mathcal{J} \otimes \mathcal{O}(d) \cong \mathcal{J} \otimes \mathcal{L}^d_E \cong \mathcal{J}^d_E.
\]

**Corollary V.2.** Let \( \pi : \tilde{M} \to M \) be the blow-up of a complex manifold \( M \) along a coherent sheaf of ideals \( \mathcal{J} \) and let \( \mathcal{I} \) be the exceptional divisor of \( \pi \). Let \( \mathcal{J} \) be a coherent sheaf of ideals on \( \tilde{M} \). Then for each point \( p \) in \( M \) there exists a neighborhood \( U \) of \( p \) in \( M \), an embedding of \( \tilde{U} = \pi^{-1}(U) \) into \( U \times \mathbb{P}^{r-1} \), for some \( r \), and an integer \( d_0 \) such that the ideal \( \mathcal{J}^d_E \) is generated by a finite number of global sections on \( \tilde{U} \) for all \( d \geq d_0 \).

**Proof.** Construct an embedding \( \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1} \) using local generators of \( \mathcal{I} \), as usual. Then use Corollary IV.7 of Chow’s Theorem for Ideals, with \( X = \tilde{U} \) and the coherent sheaf of ideals \( \mathcal{J}^d_E \) on \( \tilde{U} \). \( \square \)

Alternatively, the existence of these global generators over \( \tilde{U} \) can be proved using the positivity of the line bundle \( L^{-1}_E \) along fibres of the map from \( E \) to its image in \( M \), as in Hironaka and Rossi [HR], using results of Grauert. Except for the use of the Direct Image Theorem, our method is more explicit. We show not only that global sections exist on \( \tilde{U} \), but how they are related to homogeneous polynomials in \( \mathbb{P}^{r-1} \)-coordinates generating \( \mathcal{I} \) locally.

In the special case of compact projective manifolds, these constructions can be made global, using an ample line bundle on the original manifold.

Applying the previous corollary and noting that homogeneous polynomials on \( U \times \mathbb{P}^{r-1} \) determine hypersurfaces of \( \tilde{U} \), we obtain the following.

**Corollary V.3.** Let \( \pi : \tilde{M} \to M \) be the blow-up of a complex manifold \( M \) along a coherent sheaf of ideals \( \mathcal{I} \) and let \( \mathcal{J} \) be a coherent sheaf of ideals on \( M \). Then for each point \( p \) in \( M \) there exists a neighborhood \( U \) of \( p \) in \( M \), such that the complex space \( V(\mathcal{J}) \) determined by \( \mathcal{J} \) is cut out by a finite number of hypersurfaces in \( \tilde{U} = \pi^{-1}(U) \). In particular, if \( C \) is a smooth center in \( \tilde{M} \) and \( \mathcal{J} = \mathcal{I}_C \), then \( C \) is cut out by hypersurfaces, not only locally in \( \tilde{M} \), but over the pre-images \( \tilde{U} \) of small open sets \( U \) in \( M \).

The next corollary will be instrumental in constructing single-step blow-ups.

**Corollary V.4.** Let \( \pi : \tilde{M} \to M \) be the blow-up of a compact complex manifold \( M \) along a coherent sheaf of ideals \( \mathcal{I} \) and let \( \mathcal{E} \) be the exceptional divisor of \( \pi \). Let \( \mathcal{J} \) be a coherent sheaf of ideals on \( \tilde{M} \). Then there exists an integer \( d_0 \) such that
\[
\pi^{-1} \pi_* (\mathcal{J}^d_E) = \mathcal{J}^d_E
\]
for all \( d \geq d_0 \).

**Proof.** By compactness it is enough to prove the statement locally over neighborhoods of points in \( M \). By Corollary V.2, for each point \( p \) in \( M \) there exists a neighborhood \( U \), an embedding \( \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1} \), for some \( r \), and an integer \( d_0 \) such that \( \mathcal{J}^d_E \) is generated by a finite number of global sections on \( \tilde{U} \), for \( d \geq d_0 \). These sections are holomorphic functions, vanishing on \( E \) for \( d > 0 \). By the Riemann Extension Theorem, they determine holomorphic functions on \( U \). These functions on \( U \) generate \( \pi_* (\mathcal{J}^d_E) \) and their pullbacks to \( \tilde{U} \) generate \( \pi^{-1} \pi_* (\mathcal{J}^d_E) \). Therefore \( \pi^{-1} \pi_* (\mathcal{J}^d_E) = \mathcal{J}^d_E \). \( \square \)
Remark V.5. Using local coordinates and local generators of $\mathcal{I}$, we can describe more concretely the relationship between homogeneous polynomials generating $\mathcal{J}$ over $\tilde{U}$ and holomorphic functions generating $\mathcal{J}\mathcal{T}_E^d$ over $\tilde{U}$.

Since $\mathcal{I}$ is coherent, $\mathcal{I}$ is generated by a finite collection of holomorphic functions $f_1, ..., f_r$ on $U$, for $U$ small enough. Let $z$ represent $U$-coordinates and $[\xi] = [\xi_1 : ... : \xi_r]$ homogeneous $\mathbb{P}^{r-1}$-coordinates. By Chow's Theorem for Ideals, $\mathcal{J}$ is generated by a finite collection of homogeneous polynomials $F(z, \xi)$ (homogeneous in $\xi$ and analytic in $z$). The ideal sheaf $\mathcal{I}_E$ of the exceptional divisor is generated by the pullbacks of $f_1, ..., f_r$ to $\tilde{U}$. For simplicity we will also refer to these pullbacks as $f_1, ..., f_r$. The sheaf $\mathcal{I}_E$ is generated by all monomials of degree $d$ in $f_1, ..., f_r$. The sheaf $\mathcal{J}\mathcal{T}_E^d$ is generated by all products of the form $f^\alpha F(z, \xi)$, where $f^\alpha$ represents a degree $d$ monomial in $f_1, ..., f_r$. The function $F(z, \xi)$ is of the form

$$F(z, \xi) = \sum_{\beta} c_\beta(z)\xi^\beta$$

where $\xi^\beta$ is a monomial of degree $d$ in $\xi_1, ..., \xi_r$ and $c_\beta(z)$ is a holomorphic function of $z$. Then

$$f^\alpha F(z, \xi) = \sum_{\beta} c_\beta(z)\xi^\beta f^\alpha$$

$$= \sum_{\beta} c_\beta(z)\xi^\beta f^\alpha \quad \text{since } f_i\xi_j = f_j\xi_i.$$

Thus

$$f^\alpha F(z, \xi) = \xi^\alpha F(z, f).$$

The sheaf $\mathcal{J}\mathcal{T}_E^d$ is generated by all such products as $\xi^\alpha$ ranges over all degree $d$ monomials in $\xi_1, ..., \xi_r$. Since these monomials in $\xi$ cannot all be zero simultaneously, the collection $\{\xi^\alpha F(z, f)\}_\alpha$ is generated by $F(z, f)$.

We now see explicitly the holomorphic generators of $\mathcal{J}\mathcal{T}_E^d$ described in the previous corollary - they are the functions $F(z, f)$. These functions are holomorphic on $\tilde{U}$ and vanish on $E$ for $d > 0$, so they define holomorphic functions on $U$. As functions on $U$, they generate $\pi_* (\mathcal{J}\mathcal{T}_E^d)$. Their pullbacks to $\tilde{U}$ generate $\pi^{-1} \pi_* (\mathcal{J}\mathcal{T}_E^d)$ and are once again the functions $F(z, f)$.

Example V.6. Let $\mathcal{I}$ be ideal sheaf of the origin in $\mathbb{C}^3$ (i.e. $V(\mathcal{I}) = C = \{Z_1 = Z_2 = Z_3 = 0\}$), let $\pi : \tilde{\mathbb{C}}^3 \to \mathbb{C}^3$ be the blow-up along $\mathcal{I}$, and let $E = \pi^{-1}(C)$ be the exceptional divisor. Let $\mathcal{J}$ be the ideal on $\tilde{\mathbb{C}}^3$ generated by the homogeneous polynomial $\xi_1\xi_2 - \xi_3^2$. Let $F(Z) = Z_1Z_2 - Z_3^2$ be the corresponding polynomial on $\mathbb{C}^3$. Then $\pi^* F$ is a holomorphic section of $\mathcal{J}\mathcal{T}_E^2$. We have

$$\mathcal{J} \supset \mathcal{J}\mathcal{T}_E \supset \mathcal{J}\mathcal{T}_E^2 \supset ...$$

and

$$\pi^{-1} \pi_* (\mathcal{J}\mathcal{T}_E^d) = \begin{cases} \mathcal{J}\mathcal{T}_E^d & d < 2 \\ \mathcal{J}\mathcal{T}_E^d & d \geq 2. \end{cases}$$

Note that although we refer to $\xi_1\xi_2 - \xi_3^2$ as a generator of $\mathcal{J}$, it is not a function on $\tilde{\mathbb{C}}^3$. If $U$ is any neighborhood of 0 in $\mathbb{C}^3$, the only nonzero holomorphic sections
of $\mathcal{J}$ on $\tilde{U} = \pi^{-1}(U)$ are those generated by homogeneous polynomials of degree at least 2, which must be vanishing on $E$ to degree at least 2.

Once again, the next result could be proved using properties of positive line bundles with methods similar to those of Hironaka and Rossi in [GR] and results of Grauert. In the algebraic setting it could be proved using ample line bundles. We restrict ourselves to the case in which the blow-up $\tilde{M}$ is smooth, since this is the only case we require and since we have defined the blow-up of $\tilde{M}$ along $\mathcal{J}$ only in the case in which $\tilde{M}$ is smooth.

**Corollary V.7.** Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth, and let $E$ be the exceptional divisor of $\pi$. Let $\mathcal{J}$ be a coherent sheaf of ideals on $\tilde{M}$. Then there exists an integer $d_0$ such that the blow-up of $\tilde{M}$ along $\mathcal{J}$ is isomorphic to the blow-up of $M$ along $\pi^{-1}\pi_*(\mathcal{J}_E^d)$ for all $d \geq d_0$.

**Proof.** By Corollary V.4 there exists a $d_0$ such that $\pi^{-1}\pi_*(\mathcal{J}_E^d) = \mathcal{J}_E^d$ for all $d \geq d_0$. By Lemmas III.3 and III.4, the blow-up along $\mathcal{J}$ is isomorphic to the blow-up along $\mathcal{J}_E^d$. □

The direct image of a product is not always the product of the direct images. In the next lemma we give a condition under which products of ideal sheaves behave well under direct images of blow-up maps.

**Lemma V.8.** Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold $M$ along a coherent sheaf of ideals $\mathcal{I}$ and let $E$ be the exceptional divisor. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be coherent sheaves of ideals on $M$. Then for $d_1$ and $d_2$ large enough,

$$\pi_*(\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_E^{d_1+d_2}) = \pi_*(\mathcal{J}_1 \mathcal{I}_E^{d_1}) \pi_*(\mathcal{J}_2 \mathcal{I}_E^{d_2}).$$

**Proof.** Since $M$ is compact, it is enough to prove the lemma locally, on a blow-up $\pi : \tilde{U} \to U$ of an open set $U$. We use the notation of remark V.5 above. By Corollary IV.6, if $\mathcal{J}$ is a coherent sheaf of ideals on $\tilde{U}$, then for $d$ large enough and possibly after shrinking $U$, the ideal $\mathcal{J}$ is generated on $\tilde{U} \subset U \times \mathbb{P}^{r-1}$ by a finite number of degree $d$ homogeneous polynomials $F(z, \xi)$ in homogeneous coordinates $\xi$ on $\mathbb{P}^{r-1}$. As was shown in remark V.5, the functions $F(z, f)$ generate the direct image $\pi_*(\mathcal{J}_E^d)$.

If a finite collection $\{F(z, \xi)\}$ of degree $d_1$ polynomials generates $\mathcal{J}_1$ and a finite collection $\{G(z, \xi)\}$ of degree $d_2$ polynomials generates $\mathcal{J}_2$, then the collection $\{F(z, f)\}$ generates $\pi_*(\mathcal{J}_1 \mathcal{I}_E^{d_1})$ and the collection $\{G(z, f)\}$ generates $\pi_*(\mathcal{J}_2 \mathcal{I}_E^{d_2})$. The collection of all products $F(z, f)G(z, f)$ generates $\pi_*(\mathcal{J}_1 \mathcal{I}_E^{d_1})\pi_*(\mathcal{J}_2 \mathcal{I}_E^{d_2})$. Similarly, the collection of all products $F(z, \xi)G(z, \xi)$ generates $\mathcal{J}_1 \mathcal{J}_2$, and since these products are degree $d_1 + d_2$ homogeneous polynomials in $\xi$, the collection of all products $F(z, f)G(z, f)$ generates $\pi_*(\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_E^{d_1+d_2})$. Thus $\pi_*(\mathcal{J}_1 \mathcal{J}_2 \mathcal{I}_E^{d_1+d_2}) = \pi_*(\mathcal{J}_1 \mathcal{I}_E^{d_1}) \pi_*(\mathcal{J}_2 \mathcal{I}_E^{d_2})$. □

**Remark V.9.** To see that the direct image of a product is not always the product of the direct images, we refer to Example V.6. In that example, we described a sheaf of ideals $\mathcal{J}$ on $\mathbb{C}^3$ generated by a degree 2 homogeneous polynomial and such that

$$\pi^{-1}\pi_*(\mathcal{J}_E^d) = \begin{cases} \mathcal{J}_E^d & d < 2 \\ \mathcal{J}_E^d & d \geq 2. \end{cases}$$
Suppose that \( \pi^* (J^d E) = (\pi^* J)(\pi^* I^d E) \). Then
\[
\pi^{-1}\pi^*(J^d E) = (\pi^{-1}\pi^* J)(\pi^{-1}\pi^* I^d E) \quad \text{by Lemma II.10}
\]
\[
= (J^d E_I^d E)
\]
\[
= J^3 I^3 E
\]
which is impossible since
\[
\pi^{-1}\pi^*(J^d E) = J^d E
\]
by the example.

**VI. Replacing a Sequence of Blow-ups by a Single Blow-up**

Let \( X \) be a singular subvariety of a compact complex manifold \( M \). In this section we show how to replace a sequence of blow-ups along smooth centers, which resolves the singularities of \( X \), by a single blow-up of \( M \) along a coherent sheaf of ideals \( I \), which is a product of coherent ideals corresponding to the centers. The support of \( I \) is the singular locus of \( X \), the proper transform of \( X \) in the blow-up of \( M \) along \( I \) is nonsingular, and the exceptional divisor of the blow-up along \( I \) is a normal crossings divisor which has normal crossings with the desingularization of \( X \).

**Proposition VI.1.** Let \( M \) be a compact complex manifold and let
\[
M'' \xrightarrow{\pi''} M' \xrightarrow{\pi'} M
\]
be a sequence of blow-ups such that
a. \( \pi: M' \rightarrow M \) is the blow-up of \( M \) along a coherent sheaf of ideals \( I \) such that \( M' \) is smooth and \( V(I) \) has codimension at least 2 and
b. \( \pi': M'' \rightarrow M' \) is the blow-up of \( M' \) along a smooth center \( C \) of codimension at least 2.

Let \( E \) be the exceptional divisor of \( \pi \) in \( M' \). Then the sequence of blow-ups \( M'' \rightarrow M' \rightarrow M \) is equivalent to a single blow-up along a coherent sheaf of ideals \( J \) on \( M \) given by
\[
J = II'
\]
where \( I' = \pi_*(I_C I^d E) \) and \( d \) is a large enough positive integer that \( \pi^{-1}\pi_*(I_C I^d E) = I_C I^d E \). Furthermore
i. the blow-up of \( M' \) along \( \pi^{-1}I' = I_C I^d E \) is isomorphic to the blow-up along \( C \), i.e. the blow-up of \( M' \) along \( \pi^{-1}I' \) is isomorphic to \( M'' \), and
ii. the complex space \( V(J) \) determined by \( J \) has codimension at least 2 in \( M \).

**Proof.** By Corollary V.4
\[
\pi^{-1}\pi_*(I_C I^d E) = I_C I^d E
\]
for all sufficiently large \( d \). We apply Proposition III.7 to \( J = II' \) to show that blowing up \( M \) along \( J \) is equivalent to first blowing up \( M \) along \( I \) to obtain \( M' \), and then blowing up \( M' \) along \( I_C I^d E \). But the blow-up along \( I_C I^d E \) is equivalent to the blow-up along \( I_C \) by Lemma III.4.
Finally we note that
\[ V(J) = V(I) \cup V(\pi_*(I_C T^d_E)) \]
\[ = V(I) \cup \pi(V(I_C) \cup V(I^d_E)) \]
\[ = V(I) \cup \pi(C) \]

which has codimension at least 2. \( \square \)

We apply Proposition VI.1 inductively to obtain

**Proposition VI.2.** Let \( M_0 \) be a compact complex manifold and let

\[ M_m \xrightarrow{\pi_m} M_{m-1} \xrightarrow{\pi_{m-1}} \ldots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 \]

be a sequence of blow-ups along smooth centers \( C_j \subset M_{j-1} \) of codimension at least 2. Then the composite \( \pi_1 \circ \cdots \circ \pi_m : M_m \to M_0 \) is equivalent to a single blow-up along a coherent sheaf of ideals

\[ \mathcal{I} = \mathcal{I}_1 \mathcal{I}_2 \ldots \mathcal{I}_m \]

where \( \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m \) are coherent sheaves of ideals on \( M \) such that

i. the blow-up of \( M_{j-1} \) along the inverse image ideal of \( \mathcal{I}_j \) on \( M_{j-1} \) is isomorphic to the blow-up of \( M_{j-1} \) along \( C_j \), and

ii. the complex space \( V(\mathcal{I}) \) has codimension at least 2 in \( M_0 \).

**Proof.** We construct the ideal sheaves \( \mathcal{I}_1, \ldots, \mathcal{I}_m \) inductively, using Proposition VI.1, and noting that all the spaces \( M_j \) are smooth, since the centers of the blow-ups are smooth. We may construct an ideal sheaf \( \mathcal{I}_j \) from \( \mathcal{I}_{C_j} \) either step-by-step, going down one level at a time, or all in one step, using the composite of the first \( j - 1 \) blow-ups. We use the second method in this proof, because it is notationally simpler. The first method is computationally simpler, so we use it in our example in section IX.

Start by letting \( \mathcal{I}_1 = \mathcal{I}_{C_1} \), the ideal sheaf of the first center \( C_1 \), and construct \( \mathcal{I}_2 \) as in Proposition VI.1. The blow-up of \( M_1 \) along \( \pi_1^{-1} \mathcal{I}_2 \) is isomorphic to \( M_2 \) and the complex space \( V(\mathcal{I}_1 \mathcal{I}_2) \) has codimension at least 2. Next suppose that we have constructed \( \mathcal{I}_1, \ldots, \mathcal{I}_{j-1} \) satisfying condition (i), and such that \( V(\mathcal{I}_1 \ldots \mathcal{I}_{j-1}) \) has codimension at least 2 in \( M_0 \). Condition (i) implies that the blow-up of \( M_0 \) along the product \( \mathcal{I}_1 \ldots \mathcal{I}_{j-1} \) is isomorphic to \( M_{j-1} \). Let

\[ \tau = \pi_1 \circ \cdots \circ \pi_{j-1} : M_{j-1} \to M_0 \]

be this blow-up map and let \( D \) be the exceptional divisor of \( \tau \) in \( M_{j-1} \). Pick \( d \) large enough such that \( \tau^{-1} \tau_*(\mathcal{I}_j T^d_D) = \mathcal{I}_j T^d_D \) and set

\[ \mathcal{I}_j = \tau_*(\mathcal{I}_j T^d_D) \]

Then apply Proposition VI.1. \( \square \)

Using Hironaka’s theorem on the existence of embedded resolutions of singularities we obtain
Corollary VI.3. Let $M$ be a compact complex manifold and let $X$ be a singular subvariety of $M$. Let

$$M_n \xrightarrow{\pi} M_{n-1} \xrightarrow{\pi} \ldots \xrightarrow{\pi} M_1 \xrightarrow{\pi} M_0 = M$$

be a sequence of blow-ups along smooth centers $C_j \subset M_{j-1}$ of codimension at least 2 which resolves the singularities of $X$, and such that the total exceptional divisor of the composite map has normal crossings and has normal crossings with the desingularization of $X$ in $M_n$. Then there exists a coherent sheaf of ideals $\mathcal{I}$ on $M$ of the form

$$\mathcal{I} = \mathcal{I}_1 \mathcal{I}_2 \ldots \mathcal{I}_m$$

such that for each $j$, the blow-up map of $M$ along $\mathcal{I}_1 \mathcal{I}_2 \ldots \mathcal{I}_j$ is equivalent to the composite map $\pi_1 \circ \pi_2 \circ \ldots \circ \pi_j : M_j \to M_0$. In particular,

i. the proper transform $\tilde{X}$ of $X$ in the blow-up $\tilde{M}$ of $M$ along $\mathcal{I}$ is nonsingular,

ii. $V(\pi^{-1}\mathcal{I})$ is a normal crossings divisor in $\tilde{M}$ which has normal crossings with $\tilde{X}$, and

iii. the support of $\mathcal{I}$ is the singular locus of $X$ in $M$.

VII. Chern Forms and Metrics for Exceptional Line Bundles

Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold $M$ along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth. Let $E$ be the exceptional divisor of $\pi$ and $L_E = [E]$ the associated line bundle on $\tilde{M}$. In this section we describe explicitly the construction of a Chern form on $L_E$ which is negative definite on the fibres of the map $E \to C = V(\mathcal{I})$.

We first construct local Chern forms on sets of the form $\tilde{U} = \pi^{-1}(U)$, where $U$ is a small open set in $M$. An embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ induces a local metric and local Chern form on the line bundle $L_E$ over $\tilde{U}$, using the Fubini-Study form on $\mathbb{P}^{r-1}$. Different embeddings of $\tilde{U}$ corresponding to different choices of local generators of $\mathcal{I}$ may give different Chern forms in the same Chern class. This type of local Chern form has a particularly simple formula in terms of the local generators of $\mathcal{I}$. It is negative definite on the fibres of the map $E \to C$ and negative semi-definite on $\tilde{U}$, since it is the pullback of the negative of the Fubini-Study form on $\mathbb{P}^{r-1}$. We then patch globally using $C^\infty$ partitions of unity on $M$, to obtain global metrics and Chern forms for $L_E$.

Chern Forms on Line Bundles

We begin with some background material on Chern forms. Let $L \to N$ be a holomorphic line bundle on a complex manifold $N$. Choose a cover of $N$ by open sets $V_i$ such that $L$ is trivial on $V_i$, and let $\{g_{ij}\}$ be holomorphic transition functions for a trivialization of $L$ over $\{V_i\}$. A holomorphic section $s$ of $L$ over $N$ may be given by a collection of holomorphic functions $s_i$ on $V_i$ which transform on $V_i \cap V_j$ by the rule

$$s_i = g_{ij} s_j.$$

A hermitian metric $h$ on $L$ may be described by a collection of positive $C^\infty$ functions $h_i$ on $V_i$ such that the norm of $s$ is given on $V_i$ by

$$|| s ||^2 = | s_i |^2 h_i.$$
The functions \( h_i \) transform by the rule

\[
h_{ij} = |g_{ij}|^2 h_i.
\]

Local description of a Chern form. The Chern form of \( L \) with respect to \( h \) is given on \( V_i \) by

\[
c_1(L, h) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_i.
\]

Note that this \((1,1)\)-form is well-defined on \( N \), because

\[
\partial \partial \log h_{ij} = \partial \partial (\log |g_{ij}|^2 h_i) = \partial \partial (\log g_{ij} + \log \nabla g_{ij} + \log h_i)
\]

since \( g_{ij} \) is holomorphic.

Formula for a Chern form off the zero locus of a section \( s \). On the set on which \( s \neq 0 \) we may write

\[
c_1(L, h) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log ||s||^2.
\]

Chern forms via pullbacks. Chern forms behave well under pullbacks. Suppose that \( \phi : N_1 \to N_2 \) is a holomorphic map of complex manifolds and \( L \) is a line bundle on \( N_2 \) with metric \( h \). Then \( \phi^* L \) is a line bundle on \( N_1 \) with an induced metric \( \phi^* h \), and the Chern form of \( \phi^* L \) with respect to \( \phi^* h \) is the pullback of the Chern form of \( L \) with respect to \( h \), i.e.

\[
c_1(\phi^* L, \phi^* h) = \phi^* c_1(L, h).
\]

Local Chern Forms for Blow-ups

Let \( U \) be an open set in \( \mathbb{C}^n \) and let \( \pi : \tilde{U} \to U \) be the blow-up of \( U \) along a coherent sheaf of ideals \( \mathcal{I} \) such that \( \tilde{U} \) is smooth. We will assume that \( U \) is small enough that \( \mathcal{I} \) is generated by global sections \( f_1, ..., f_r \) over \( U \). Let \( E \) be the exceptional divisor and \( L_E \) the associated line bundle on \( \tilde{U} \).

If \( \mathcal{I} \) is generated by a single function over \( U \), then the sheaf \( \mathcal{I} \) is principal, the line bundle \( L_E \) is trivial on \( \tilde{U} \), and we may choose a metric \( h \) on \( L_E \) such that \( c_1(L_E, h) = 0 \).

We assume from now on that the blow-up is non-trivial, i.e. that \( \mathcal{I} \) has support of codimension at least 1 in \( U \) and is not principal on \( U \). In this case \( r > 1 \) and the generators \( f_1, ..., f_r \) of \( \mathcal{I} \) give an embedding

\[
\iota_f : \tilde{U} \leftrightarrow U \times \mathbb{P}^{r-1},
\]

as described in section III. Let \( [\xi_1 : ... : \xi_r] \) be homogeneous coordinates for \( \mathbb{P}^{r-1} \). The blow-up \( \tilde{U} \) is covered by open sets

\[
\tilde{U}_i = U \cap \{ \xi_i \neq 0 \}
\]
on which \( L_E \) is trivial. Transition functions for \( L_E \) on the intersections \( \tilde{U}_i \cap \tilde{U}_j \) are the functions \( g_{ij} = \frac{\xi_i}{\xi_j} \). To distinguish between a generating function \( f_i \) on \( U \) and its pullback to \( \tilde{U} \subset U \times \mathbb{P}^{r-1} \), we will let

\[
\tilde{f}_i = \pi^* f_i.
\]
The exceptional divisor \( E \) is given on \( \tilde{U}_i \) by \( \tilde{f}_i = 0 \). The collection of functions \( \tilde{f}_i \) on the sets \( \tilde{U}_i \) determines a section of \( L_E \) over \( \tilde{U} \), vanishing exactly on \( E \).
Lemma VII.1. Let $U$ be an open set in $\mathbb{C}^n$ and let $\pi : \tilde{U} \to U$ be the blow-up of $U$ along a coherent sheaf of ideals $\mathcal{I}$ which is generated by global sections $f_1, ..., f_r$ on $U$. Suppose that the blow-up is non-trivial and that $\tilde{U}$ is smooth. Let $E$ be the exceptional divisor and $L_E$ the associated line bundle on $\tilde{U}$. Then the embedding $\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ induces a metric $h$ on $L_E$ whose Chern form $c_1(L_E, h)$ is negative semi-definite on $\tilde{U}$, negative definite on the fibres of the map $E \to C = V(\mathcal{I})$, and given on $\tilde{U} - E$ by

$$c_1(L_E, h) = \pi^* \left( -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \sum_{j=1}^{r} | f_j(z) |^2 \right).$$

Proof. We will construct the Chern form by pullback, without using an explicit formula for the metric $h$. For an explicit local formula for $h$, see the remark following this proof.

Recall that the exceptional line bundle $L_E$ on $\tilde{U}$ is the pullback of the universal bundle $O_{\mathbb{P}^{r-1}}(-1)$. The Fubini-Study form $\omega_{\text{Fub-St}}$ on $\mathbb{P}^{r-1}$ gives a Chern form for $O_{\mathbb{P}^{r-1}}(1)$ and $-\omega_{\text{Fub-St}}$ gives a Chern form for $O_{\mathbb{P}^{r-1}}(-1)$. Pulling back to $\tilde{U}$, we obtain a Chern form for $L_E$ (with respect to an induced metric $h$) given by

$$c_1(L_E, h) = \iota_f^* \sigma_2^*(-\omega_{\text{Fub-St}}),$$

where $\sigma_2$ is the second projection map $\sigma_2 : U \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ and $\iota_f$ is the inclusion map $\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$. The negativity properties of $c_1(L_E, h)$ stated in the lemma follow directly from the fact that $\omega_{\text{Fub-St}}$ is positive on $\mathbb{P}^{r-1}$.

Now recall the formula for the Fubini-Study form on projective space. If $\xi_1, ..., \xi_r$ are homogeneous coordinates on $\mathbb{P}^{r-1}$, then $w_{ij} = \frac{\xi_i}{\xi_j}$ for $j \neq i$ are nonhomogeneous coordinates on $U_i = \{ \xi_i \neq 0 \}$. The Fubini-Study form $\omega_{\text{Fub-St}}$ is given on $U_i$ by

$$\omega_{\text{Fub-St}} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (1 + \sum_{j \neq i} w_{ij}^2) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (1 + \sum_{j \neq i} \left| \frac{\xi_j}{\xi_i} \right|^2).$$

We continue to use the notation $\hat{f}_i = \pi^* f_i$ to distinguish between the function $f_i$ on $U$ and its pullback to $\tilde{U}$. On $\hat{U}_i = \tilde{U} \cap U_i$ we have $\xi_i^* = \frac{\hat{f}_i}{f_i}$ which gives us

$$c_1(L_E, h) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (1 + \sum_{j \neq i} \left| \frac{\hat{f}_j}{f_i} \right|^2) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \sum_{j=1}^{r} \left| \frac{\hat{f}_j}{f_i} \right|^2.$$
On $\tilde{U}_i - \tilde{U}_i \cap E$ we have $\tilde{f}_i(z) \neq 0$ so
\[
c_1(L_E, h) = -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} (\log \sum_{j=1}^{r} |\tilde{f}_j|^2 - \log |\tilde{f}_i|^2) \]
\[
= -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \sum_{j=1}^{r} |\tilde{f}_j(z)|^2 \]
\[
= \pi^* \left( -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \sum_{j=1}^{r} |f_j(z)|^2 \right).
\]
This formula is independent of $i$, so is valid on all of $\tilde{U} - E$. □

Remark. Local defining functions for the metric $h$ on $L_E$ induced from the embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ may also be given explicitly. Let $s$ be the section of $L_E$ given on $\tilde{U}_i$ by $f_i = 0$. The norm of $s$ under the metric $h$ is given by
\[
|| s ||^2 = \sum_{j=1}^{r} |\tilde{f}_j|^2.
\]
The metric $h$ is described locally by positive $C^\infty$ functions $h_i$ on $\tilde{U}_i$ satisfying
\[
|| s ||^2 = |\tilde{f}_i|^2 h_i.
\]
Thus
\[
h_i = \frac{\sum_{j=1}^{r} |\tilde{f}_j|^2}{|\tilde{f}_i|^2}.
\]

**Global Chern Forms for Blow-ups**

**Proposition VII.2.** Let $\pi : \hat{M} \to M$ be the blow-up of a compact complex manifold $M$ along a coherent sheaf of ideals $\mathcal{I}$ such that $\hat{M}$ is smooth. Let $E$ be the exceptional divisor and $L_E$ the associated line bundle.

Then there is a metric $h$ on $L_E$ whose Chern form $c_1(L_E, h)$ on $\hat{M}$ is negative definite along the fibres of the map $E \to C$ and is given on $\hat{M} - E$ by
\[
c_1(L_E, h) = \pi^* \left( -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log F \right),
\]
where $F$ is a global $C^\infty$ function on $M$, vanishing on the support of $\mathcal{I}$. Furthermore, $F$ may be constructed to be of the form
\[
F = \prod_\alpha F_\alpha^{\rho_\alpha},
\]
where $\{\rho_\alpha\}$ is a $C^\infty$ partition of unity subordinate to an open cover $\{U_\alpha\}$ of $M$, $F_\alpha$ is a function on $U_\alpha$ of the form
\[
F_\alpha = \sum_{j=1}^{r} |f_j|^2,
\]
and $f_1, \ldots, f_r$ are local holomorphic generators of the coherent ideal sheaf $\mathcal{I}$ on $U_\alpha$.

**Proof.** Let $\{U_\alpha\}$ be a finite open cover of $M$ by open sets small enough that $\mathcal{I}$ is generated by global sections on each $U_\alpha$. If the support of $\mathcal{I}$ does not intersect some $U_\alpha$ or if $\mathcal{I}$ is generated by a single generator on $U_\alpha$, then $L_E$ is trivial on the set $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$. In this case we may choose $F_\alpha$ to be a constant and the local Chern form will be 0. Otherwise, in the nontrivial case, suppose that $f_1, \ldots, f_r$ are local generating functions for $\mathcal{I}$ on $U_\alpha$ and let

$$F_\alpha = \sum_{j=1}^r |f_j|^2 \quad \text{and} \quad \tilde{F}_\alpha = \pi^* F_\alpha.$$

By Lemma VII.1, there is a local $C^\infty$ metric $h_\alpha$ for $L_E$ on $\tilde{U}_\alpha$ which is negative definite on the fibres of the map $E \to C$ and is given on $\tilde{U}_\alpha - \tilde{U}_\alpha \cap E$ by

$$c_1(L_E, h_\alpha) = \pi^* \left(- \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log F_\alpha\right) = - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \tilde{F}_\alpha.$$ 

Now choose a $C^\infty$ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and let $\tilde{\rho}_\alpha$ be the pullback of $\rho_\alpha$ to $\tilde{M}$. Then $\{\tilde{\rho}_\alpha\}$ is a partition of unity on $\tilde{M}$ subordinate to the open sets $\{\tilde{U}_\alpha\}$. Note that each function $\tilde{\rho}_\alpha$ is constant along the fibres of the map $E \to C$.

We define a global $C^\infty$ metric for $L_E$ as follows. For any section $s$ of $L_E$, let $\|s\|_\alpha^2$ be the norm-squared of $s$ with respect to the metric $h_\alpha$ on $\tilde{U}_\alpha$ and let

$$\|s\|^2 = \prod_\alpha \|s\|_{\tilde{\rho}_\alpha}^2.$$ 

If $\{V_i\}$ is a cover of $\tilde{M}$ by open sets on which $L_E$ is trivial, and $h_{\alpha i}$ is the positive $C^\infty$ function representing $h_\alpha$ on $V_i$, then the positive $C^\infty$ function for $h$ on $V_i$ is

$$h_i = \prod_\alpha h_{\alpha i}^{\tilde{\rho}_\alpha}.$$ 

If $s$ is given on $V_i$ by the holomorphic function $s_i$, then on $U_\alpha \cap V_i$ we have

$$\|s\|_{\alpha i}^2 = |s_i|^2 h_{\alpha i}$$

and on $V_i$,

$$\|s\|^2 = |s_i|^2 h_i.$$ 

The global form $c_1(L_E, h)$ associated with this metric preserves the property of the local forms of being negative definite on the fibres of the map $E \to C$ because the partition of unity functions $\{\tilde{\rho}_\alpha\}$ are constant on fibres of the map $E \to C$. On $V_i$, this Chern form is given by

$$c_1(L_E, h) = - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_i = - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \prod_\alpha h_{\alpha i}^{\tilde{\rho}_\alpha} = - \frac{\sqrt{-1}}{2\pi} \sum_\alpha \partial \bar{\partial} \tilde{\rho}_\alpha \log h_{\alpha i}.$$
Let $s$ be a global holomorphic section of $L_E$ on $\tilde{M}$ whose associated divisor is $E$. Such a section always exists - just choose local holomorphic defining equations of $E$ to determine $s$ locally. For example, on $\tilde{U}_{\alpha i} = \tilde{U}_\alpha \cap \{\xi_i \neq 0\} \subset U_\alpha \times \mathbb{P}^{r-1}$, take $s_{\alpha i} = \tilde{f}_i = \pi^* f_i$, where $f_1, ..., f_r$ are local holomorphic generators of $\mathcal{I}$ on $U_\alpha$. Then

$$||s||^2_{\alpha} = \sum_{j=1}^{r} |\tilde{f}_j|^2 = \tilde{E}_\alpha$$

and

$$||s||^2 = \prod_\alpha \tilde{E}_{\alpha}^\rho = \pi^*(\prod_\alpha F_{\alpha}^\rho).$$

Thus the Chern form $c_1(L_E, h)$ is given on $\tilde{M} - E$ by

$$c_1(L_E, h) = -\frac{1}{2\pi} \partial \bar{\partial} \log ||s||^2 = \pi^*(-\frac{1}{2\pi} \partial \bar{\partial} \log F),$$

where

$$F = \prod_\alpha F_{\alpha}^\rho \quad \Box$$

**VIII. Construction of Saper-Type Metrics**

Let $X$ be a singular subvariety of a compact Kähler manifold $M$ and let $X_{\text{sing}}$ be the singular locus of $X$. We will construct Saper-type metrics on $M - X_{\text{sing}}$, first locally, then globally using a $C^\infty$ partition of unity on $M$. These metrics are complete Kähler metrics on $M - X_{\text{sing}}$ which grow less rapidly than Poincaré metrics near the singular locus. More details on the growth rate of Saper-type metrics and their relationship to intersection cohomology may be found in [GM], [Sa1], and [Sa2].

We also construct a non-complete Kähler metric on $M - X_{\text{sing}}$ with the property that the completion of $X - X_{\text{sing}}$ with respect to this metric is a desingularization of $X$. We call this metric a “desingularizing metric” for $X$.

The constructions of both metrics are based on resolution of singularities using a single coherent ideal sheaf $\mathcal{I}$ on $M$ (see Corollary VI.3) and the explicit formula for a Chern form for the blow-up of $M$ along $\mathcal{I}$ given in Proposition VII.2

**Local Construction of Metrics**

Before constructing Saper-type metrics, we will describe a Kähler metric for a local blow-up.

Let $U$ be an open set in $\mathbb{C}^n$ and let $\pi : \tilde{U} \rightarrow U$ be the blow-up of $U$ along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{U}$ is smooth. Let $E$ be the exceptional divisor of $\pi$. Assume that $U$ is small enough that $\mathcal{I}$ is generated by global sections on $U$ and let

$$\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$$

be the embedding associated with a collection of generators $f$. Let $\sigma_1$ and $\sigma_2$ be the projection maps

$$U \times \mathbb{P}^{r-1} \xrightarrow{\sigma_2} \mathbb{P}^{r-1} \xrightarrow{\sigma_1} U$$
Suppose that $\omega$ is the Kähler form of a Kähler metric on $U$ and let $\omega_{\text{Fub-St}}$ be the Kähler form of the Fubini-Study metric on $\mathbb{P}^{r-1}$.

**Lemma VIII.1.** The embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ induces a Kähler metric on $\tilde{U}$ whose Kähler form is

$$\omega' = \pi^* \omega - c_1(L_E, h),$$

where $c_1(L_E, h)$ is a Chern form of the line bundle $L_E$ (with respect to a metric $h$) of the type described in Lemma VII.1.

**Proof.** The Kähler form on $\tilde{U}$ given by the restriction of the product metric on $U \times \mathbb{P}^{r-1}$ is

$$\omega' = \pi^* \omega + \sum_{j=1}^{r} f_j^* \omega_{\text{Fub-St}},$$

where $f_1, \ldots, f_r$ were holomorphic generators for $\mathcal{I}$ on $U$. Thus the $(1,1)$-form

$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \sum_{j=1}^{r} |f_j|^2$$

on $U - V(\mathcal{I})$ determines a Kähler metric on $U - V(\mathcal{I})$. This Kähler metric is essentially the local model of our desingularizing metric.

The function $F = \sum_{j=1}^{r} |f_j|^2$ can also be used to construct a Saper-type metric on $U - V(\mathcal{I})$. We are particularly interested in the case of a coherent sheaf of ideals $\mathcal{I}$ which determines a resolution of singularities of a singular variety and which is supported on the singular locus of the variety. Theorems VIII.2 and VIII.3 describe local and global constructions, respectively, of Saper and desingularizing metrics for a singular variety. The main differences between the two theorems are that we must patch with a $C^\infty$ partition of unity in the global case, and that our desingularizing metric may require a multiple of the original metric in that case.

**Theorem VIII.2. Local Metrics.** Let $X$ be a singular subvariety of a compact Kähler manifold $M$ with singular locus $X_{\text{sing}}$. Let $\omega$ be the Kähler $(1,1)$-form of a Kähler metric on $M$. Let $p$ be any point in $X_{\text{sing}}$. Then there exists a neighborhood $U$ of $p$ and a $C^\infty$ function $F$ on $U$, vanishing on $U \cap X_{\text{sing}}$, such that

i. the $(1,1)$-form

$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log F$$

is the Kähler form of an incomplete metric on $U - U \cap X_{\text{sing}}$ which determines a local embedded resolution of singularities and

ii. the $(1,1)$-form

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (\log F)^2$$

on $U - U \cap X_{\text{sing}}$ is the Kähler form of a modified Saper metric on $U - U \cap X_{\text{sing}}$ (in the sense of [GM]) for $l$ a large enough positive integer.
Furthermore, the function \( F \) may be constructed to be of the form

\[
F = \sum_{j=1}^r |f_j|^2,
\]

where \( f_1, \ldots, f_r \) are holomorphic functions on \( U \) which are local generators of a coherent ideal sheaf \( \mathcal{I} \) on \( M \), such that blowing up \( M \) along \( \mathcal{I} \) desingularizes \( X \), \( \mathcal{I} \) is supported on \( X_{\text{sing}} \), and the exceptional divisor of the blow-up along \( \mathcal{I} \) has normal crossings with itself and with the desingularization of \( X \).

**Proof.** Part (i) is a consequence of Lemma VIII.1 and Lemma VII.1. Part (ii) and its global version follow from Theorem 9.2.1 of [GM], in which we also give estimates of the rate of growth of \( \omega_S \). The idea behind the proof of that theorem is that we can decompose the term involving \( F \) in our Saper-type metric as

\[
-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log F)^2 = \frac{\sqrt{-1}}{2\pi} \left( \frac{1}{|\log F|^2} \partial \bar{\partial} \log F + \frac{\partial F \wedge \bar{\partial} F}{|F|^2 (\log F)^2} \right).
\]

The first term gives positivity of \( \omega_S \), since it is a multiple of the negative of the Chern form \( c_1(L_E, h) \). The second term is similar to the Poincaré metric on the punctured disc and becomes unbounded near the singular locus. \( \square \)

**Global Construction of Metrics**

To construct global metrics we patch together our local metrics using \( C^\infty \) partitions of unity on \( M \). As described in section VII, this patching does not affect negativity of our Chern forms along fibres of the maps \( E \to C \) from the exceptional divisors to their corresponding centers. However the Chern forms may not remain negative semidefinite elsewhere, so that it may be necessary to introduce multiples of the original metric.

**Theorem VIII.3. Global Metrics.** Let \( X \) be a singular subvariety of a compact Kähler manifold \( M \) with singular locus \( X_{\text{sing}} \). Let \( \omega \) be the Kähler \((1,1)\)-form of a Kähler metric on \( M \). There exists a global \( C^\infty \) function \( F \) on \( M \), vanishing exactly on \( X_{\text{sing}} \), such that for \( l \) a large enough positive integer

i. the \((1,1)\)-form

\[
\tilde{\omega} = l \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log F
\]

is the Kähler form of an incomplete Kähler metric on \( M - X_{\text{sing}} \) which is a desingularizing metric for \( X \) (i.e. the completion of \( M - X_{\text{sing}} \) with respect to \( \tilde{\omega} \) is nonsingular), and

ii. the \((1,1)\)-form

\[
\omega_S = l \omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log F)^2
\]

on \( M - X_{\text{sing}} \) is the Kähler form of a complete Kähler modified Saper metric (in the sense of [GM]).
Furthermore, the function $F$ may be constructed to be of the form

$$F = \prod_{\alpha} F^{\rho_{\alpha}},$$

where $\{\rho_{\alpha}\}$ is a $C^{\infty}$ partition of unity subordinate to an open cover $\{U_{\alpha}\}$ of $M$, $F_{\alpha}$ is a function on $U_{\alpha}$ of the form

$$F_{\alpha} = \sum_{j=1}^{r} |f_{j}|^{2},$$

and $f_{1},...,f_{r}$ are holomorphic functions on $U_{\alpha}$, vanishing exactly on $X_{\text{sing}} \cap U_{\alpha}$. More specifically, $f_{1},...,f_{r}$ are local holomorphic generators of a coherent ideal sheaf $I$ on $M$ such that blowing up $M$ along $I$ desingularizes $X$, $I$ is supported on $X_{\text{sing}}$, and the exceptional divisor of the blow-up along $I$ has normal crossings with itself and with the desingularization of $X$.

Proof. Part (i) follows from Proposition VII.2. Part (ii) follows from our description of $F$ in section VII.2 and Theorem 9.2.1 of [GM].

IX. Example

The cuspidal cubic. Let $M = \mathbb{P}^{2}$ and let $X$ be the cuspidal cubic given in homogeneous coordinates by $\xi_{0}^{2} - 2\xi_{2}^{3} = 0$. In local coordinates $x,y$ in a neighborhood $U \cong \mathbb{C}^{2}$ of the singular point, $X$ is given by

$$y^{2} - x^{3} = 0.$$

The singularity may be resolved by three blow-ups of points, in such a way that the components of the total exceptional divisor have normal crossings with each other and with the desingularization of $X$. We will show that these three blow-ups are equivalent to a single blow-up along the ideal sheaf given locally by

$$I = (x,y)(x^{2},y)(x^{3},x^{2}y,y).$$

First blow-up $\pi_{1}$. The center $C_{1}$ for the first blow-up is the point $x = y = 0$ and its ideal is $I_{C_{1}} = (x,y)$. The blow-up $U_{1} = \pi_{1}^{-1}(U)$ may be covered by two coordinate charts, which we will call the $x$- and $y$-coordinate charts, according to whether the chart is a complement in $U_{1}$ of the strict transform of $x = 0$ or $y = 0$. (The exceptional divisor is given by the vanishing of the $x$-coordinate in the $x$-chart and the $y$-coordinate in the $y$ chart.) On the $x$-coordinate chart, $\pi_{1}$ is given by

$$\pi_{1}(x_{1},y_{1}) = (x_{1},x_{1}y_{1}) = (x,y)$$

and the exceptional divisor $E_{1}$ is given by $x_{1} = 0$. The inverse image $\pi_{1}^{-1}(X)$ is given by $x_{1}^{2}y_{1}^{2} - x_{1}^{3} = 0$. The strict transform $X_{1}$ of $X$ is obtained from the inverse image by removing all copies of $E_{1}$, i.e. by dividing by the highest possible power of $x_{1}$, which gives

$$y_{1}^{2} - x_{1} = 0.$$

Although $X_{1}$ is smooth, it does not have normal crossings with the divisor $E_{1}$ at the point $x_{1} = y_{1} = 0$, so we must blow up again at this point. Before doing so, we note that in the $y$-coordinate chart, the strict transform $X_{1}$ is smooth and has normal crossings with $E_{1}$, so there is no need to blow up further at any points in that chart.
Second blow-up \(\pi_2\). The center \(C_2\) for the second blow-up is the point \(x_1 = y_1 = 0\) in the \(x\)-coordinate chart of \(U_1\), and its ideal is \(\mathcal{I}_{C_2} = (x_1, y_1)\). In the \(x\)-coordinate chart of \(\pi_2\) we have normal crossings, so there is no need to blow up further at any points in that chart. In local coordinates \((x_2, y_2)\) for the \(y\)-coordinate chart of \(\pi_2\), we have

\[
\pi_2(x_2, y_2) = (x_2 y_2, y_2) = (x_1, y_1)
\]

and \(\mathcal{I}_{E_2} = (y_2)\). The strict transform \(X_2\) of \(X_1\) is given by

\[
y_2 - x_2 = 0
\]

and the strict transform \(\tilde{E}_1\) of \(E_1\) by \(x_2 = 0\). The total exceptional divisor of the first two blow-ups, which is the union of \(E_2\) and \(\tilde{E}_1\), does not have normal crossings with \(X_2\) so we blow up again.

Third blow-up \(\pi_3\). The center \(C_3\) for the third blow-up is the point \(x_2 = y_2 = 0\) with ideal \(\mathcal{I}_{C_3} = (x_2, y_2)\). After this third blow-up, the strict transform of \(X\) and all three components of the total exceptional divisor have normal crossings.

Construction of \(\mathcal{I}\). We will construct \(\mathcal{I}\) as a product \(\mathcal{I} = \mathcal{I}_1 \mathcal{I}_2 \mathcal{I}_3\) of ideals corresponding to the centers of the blow-ups. We begin by choosing \(\mathcal{I}_1 = \mathcal{I}_{C_1} = (x, y)\).

To obtain \(\mathcal{I}_2\), we start with \(\mathcal{I}_{C_2}\) and multiply by a high enough power of \(\mathcal{I}_{E_1}\), such that taking the direct image under \(\pi_1\) and then the inverse image does not change the ideal. We define \(\mathcal{I}_2\) to be the direct image of the resulting product under the map \(\pi_1\).

Locally, in the \(x\)-coordinate chart of \(\pi_1\), \(\mathcal{I}_{C_2}\) is given by \((x_1, y_1)\) and \(\mathcal{I}_{E_1}\) by \((x_1)\), where \(x_1 = x\) and \(y_1 = \frac{y}{x}\). Thus \(\mathcal{I}_{C_2}\) is not the inverse image of an ideal sheaf, but \(\mathcal{I}_{C_2} \mathcal{I}_{E_1}\) is, since

\[
\pi_1^{-1}(x^2, y) = \mathcal{I}_{C_2} \mathcal{I}_{E_1}.
\]

The direct image \(\pi_1(\mathcal{I}_{C_2} \mathcal{I}_{E_1})\) is the largest ideal sheaf whose inverse image is contained in \(\mathcal{I}_{C_2} \mathcal{I}_{E_1}\), so \(\pi_1(\mathcal{I}_{C_2} \mathcal{I}_{E_1})\) contains \((x^2, y)\). It is easily checked that \(x^2\) and \(y\) generate \(\pi_1(\mathcal{I}_{C_2} \mathcal{I}_{E_1})\), since they are the only monomials whose pullbacks are sections of \(\mathcal{I}_{C_2} \mathcal{I}_{E_1}\). Thus

\[
\mathcal{I}_2 = \pi_1(\mathcal{I}_{C_2} \mathcal{I}_{E_1}) = (x^2, y).
\]

Similarly, to obtain \(\mathcal{I}_3\) we start with \(\mathcal{I}_{C_3}\), given locally by \((x_2, y_2)\), and recall that \(x_2 = \frac{x}{y_1}\) and \(y_2 = y_1\). Hence \(\mathcal{I}_{C_3} \mathcal{I}_{E_2}\) is the inverse image of an ideal sheaf \(\mathcal{J}\) given locally on \(U_1\) by \((x_1, y_1^2)\), and \(\mathcal{J} \mathcal{I}_{E_2}\) is the inverse image of the ideal sheaf \((x^3, y^2)\). Since \(\pi_2^{-1}(\mathcal{I}_{E_1}) = \mathcal{I}_{E_1} \mathcal{I}_{E_2}\), it follows that

\[
\pi_2^{-1} \pi_1^{-1}(x^3, y^2) = \mathcal{I}_{C_3} \mathcal{I}_{E_1} \mathcal{I}_{E_2}^3.
\]

In local coordinates, \(\pi_2^{-1} \pi_1^{-1}(x^3, y^2) = (x_2, y_2)(x_2^2)(y_2^2)\). We define \(\mathcal{I}_3\) to be the direct image \(\pi_1 \pi_2(\mathcal{I}_{C_3} \mathcal{I}_{E_1}^2 \mathcal{I}_{E_2}^3)\), and note that \(\mathcal{I}_3\) contains \((x^3, y^2)\), since \(\mathcal{I}_3\) is the largest ideal sheaf whose inverse image is contained in \(\mathcal{I}_{C_3} \mathcal{I}_{E_1}^2 \mathcal{I}_{E_2}^3\). To find any remaining generators of \(\mathcal{I}_3\), we test monomials not generated by \(x^3\) or \(y^2\) to see which pull back to sections of \(\mathcal{I}_{C_3} \mathcal{I}_{E_1}^2 \mathcal{I}_{E_2}^3\). It is easily checked that \(x, y, x^2\), and \(xy\) are not in \(\mathcal{I}_3\), but \(x^2 y\) is in \(\mathcal{I}_3\) since \(x^2 y = x^3 y_1 = x^3 y_2^4\). Thus

\[
\mathcal{I}_3 = \pi_1 \pi_2(\mathcal{I}_{C_3} \mathcal{I}_{E_1}^2 \mathcal{I}_{E_2}^3) = (x^3, x^2 y, y^2).
\]
We define the ideal $I$ to be the product of $I_1$, $I_2$, and $I_3$:

$$I = (x, y)(x^2, y)(x^3, x^2y, y^2).$$

Blowing up along $I$ is equivalent to blowing up sequentially along the centers $C_1$, $C_2$, and $C_3$.

The method used in this example may be generalized to any locally toric complex analytic variety. Details will be given elsewhere.

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