Unfocused beam patterns in nonattenuating and attenuating fluids

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Abstract. The most important aspect of an ultrasound measuring system is knowledge of the transducer beam pattern. At all depths accurate single integral equations have been derived for the full beam pattern of steady state unfocused circular flat piston sources radiating into nonattenuating and attenuating fluids. The axial depth of the beginning of the unattenuated beam pattern far field is found to be at $6.41Y_0$. The unattenuated single integral equations are identical to a Jinc function directivity term at this and deeper depths. For attenuating fluids values of $\alpha$ and $z$ are found that permit the attenuated axial pressure to be represented by a plane wave multiplicative exponential attenuation factor. This knowledge will aid in the experimental design of highly accurate attenuation measurements. Accurate single integral equations for the attenuated full beam pattern are derived using complex Bessel functions.

1. Introduction

To aid in the use of unfocused circular plane piston sources in fluids, new easy to use approximate single integral expressions have been derived [1]. Human tissue, like fluids, only supports longitudinal wave propagation and will be used in the presented numerical examples.

2. Nonattenuating fluids

Figure 1. Beam geometry of a radiating circular flat piston of radius $a$. 

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The radiating surface moves uniformly with speed $u_0 \exp(i \omega t)$ normal to its surface. The fluid is homogeneous, isotropic, non-scattering and non-attenuating. Figure 1 presents the beam geometry. The circular plane piston source is in the $XY$ plane centered at the origin. Without loss of generality, the field point $Q$ is in the $XZ$ plane a lateral distance $x$ from the $Z$ axis (beam axis) at an axial depth $z$. Each point $dS$ on the source surface emits spherical waves. Integration over all of these point sources, each with strength $dq = u_0 dS = u_0 dxdy = u_0 \sigma d\sigma d\phi$, produces a pressure amplitude $P(r, \theta, t)$ at $Q$ a distance $r$ from the origin at an angle $\theta$ from the $Z$ axis. The pressure at point $Q$ is [1]

$$P(r, \theta, t) = \frac{i \rho_0 c u_0}{2\pi} \int_0^{2\pi} \int_0^a \frac{e^{i(ar-k' r')}}{r'} \sigma d\sigma d\phi,$$

(1)

where $k = 2\pi/\lambda = \omega/c$, $\omega$ is the angular frequency, $\lambda$ is the wavelength, $a$ is the piston radius, $c$ is the fluid acoustic velocity, and $\rho_0$ is the mean physical mass density of the fluid.

2.1. Axial pressure
Along the beam axis $r = z$ and $\theta = 0$. Equation (1) reduces to an exact integral that can be rewritten in the form

$$P(z, 0) = \rho_0 c u_0 e^{i(\omega z - k'a)} \left[ 1 - e^{-ik\left(\sqrt{z^2 + a^2} - z\right)} \right].$$

(2)

At any time $t$ the pressure magnitude in the unfocused beam pattern may be determined by calculating the magnitude of the product of the first and third terms in equation (2)

$$|P(z, 0)| = 2\rho_0 c u_0 \sin\left(\frac{k}{2} \left(\sqrt{z^2 + a^2} - z\right)\right).$$

(3)

To calculate the far zone axial depth of the beginning of the far field of an unfocused plane piston source start with equation (3). At large $z$ the argument of the sine term goes to zero and $\sin(x)$ may be replaced by $x$. Expanding the square root in the argument of the sine using a binomial series (which is valid when $z > a$)

$$\sin\left(\frac{k}{2}\sqrt{z^2 + a^2} - z\right) \approx \sin\left(\frac{ka^2}{2z}\right) \approx \frac{\pi a^2}{2\lambda z}$$

(4)

producing the required spherical waves. The shortest axial depth where the right hand side of equation (4) is valid may be found using the first two terms of the series expansion of $\sin(x)$

$$\sin(x) = x - \frac{x^3}{6} + \ldots$$

(5)

and normalizing $z$ with respect to the distance $Y_0$

$$z = n\frac{a^2}{\lambda} = nY_0,$$

(6)

where $n$ is a positive number. $\sin(x) = x$ is true to 1% when the second term in equation (5) is 1% of the first term. In that case $n = 6.41$. So the far field exists at axial depths greater than 6.41 times $Y_0$.

2.2. Full beam pattern
Calculating the pressure lateral to the beam axis at all depths requires evaluating equation (1) at each field point $Q$. $r'$ is replaced with the geometric identity

$$r' = \left(r^2 + a^2 - 2r a \sigma \sin \theta \cos \phi \right)^{1/2}$$

(7)

to obtain the exact full beam pattern pressure. The Fresnel approximation used to evaluate equation (1) involves replacing $r'$ in the denominator of the integral’s kernel with $r$ and taking it out of the integral. Then the square root in equation (7) is expanded using a binomial series keeping only the first two terms.
and this expression is used in the exponent of the integral kernel.

A standard approximation is to neglect the second term in the brackets in equation (8) at considerable distances from the source [2]. From Figure 1 \( \sin \theta = x/r \). For field points within a distance \( a \) of the beam axis, the ratio \( \sigma / r \) is roughly equal to \( \sin \theta \). And for field points further than \( a \) from the beam axis, the ratio \( \sigma / r \) is less than \( \sin \theta \). So the second and third terms inside the brackets in equation (8) have approximately the same magnitude regardless of distance from the source.

Keeping all three terms the approximate pressure is

\[
P_{ap}(r, \theta, t) = i \frac{\rho_0 \omega u_0}{2\pi r} e^{i(\omega t - kr)} \int_0^a \sigma e^{-\frac{k\sigma^2}{2r}} \int_0^{2\pi} e^{ik\sigma \sin \theta \cos \phi} d\phi d\sigma.
\]  

(9)

Using the identity

\[
2\pi J_m(x) = (-i)^m \int_0^{2\pi} e^{i\cos \phi} \cos(mx) d\phi,
\]

where \( m = 0, 1, 2, 3 \ldots \) equation (9) becomes

\[
P_{ap}(r, \theta, t) = i \frac{\rho_0 \omega u_0}{r} e^{i(\omega t - kr)} \int_0^a \sigma e^{-\frac{k\sigma^2}{2r}} J_0(k\sigma \sin \theta) d\sigma.
\]  

(11)

At any time \( t \) the magnitude of the unfocused beam pattern may be determined by calculating the magnitude of the product of the beam pattern diffraction terms in equation (11)

\[
\left| P_{ap}(r, \theta) \right| = \frac{\rho_0 \omega u_0}{r} \left[ I_C^2(r, \theta) + I_S^2(r, \theta) \right]^{1/2},
\]

(12)

where

\[
I_C(r, \theta) = \int_0^a \sigma \cos \left( \frac{k\sigma^2}{2r} \right) J_0(k\sigma \sin \theta) d\sigma \quad \text{and}
\]

\[
I_S(r, \theta) = \int_0^a \sigma \sin \left( \frac{k\sigma^2}{2r} \right) J_0(k\sigma \sin \theta) d\sigma.
\]

(13)

(14)

Acoustics texts represent the unfocused circular piston far zone beam pattern by a Jinc function directivity term

\[
\frac{2J_1(ka \sin \theta)}{ka \sin \theta}.
\]

(15)

The appropriateness of using this Jinc function expression for the lateral beam profile in the far zone may now be examined. equation (12) was used to compute the lateral beam profile for \( a = 1 \) cm, \( f = 5 \) MHz and \( c = 0.154 \) cm/\( \mu \)sec at beam pattern depths of \( nY_0 \) with \( n = 1, 2, 5 \) and 6.41. The results are shown in figures 2 (a) and (b) where the square of the Jinc function is compared to the square of the pressure.

The lateral beam profile at each depth was normalized to its central, axial amplitude and, for convenience, the independent variable is the angle theta. Figure 2 demonstrates that as \( n \) increases from unity the lateral beam profile rapidly approaches a Jinc Function. Figures 2 (a) and (b) demonstrate that the \( n = 6.41 \) lateral beam profile is identical to the Jinc Function directivity term.

3. Attenuating fluids

Equation (1) may be generalized to propagation in an attenuating fluid by replacing \( k \) with the complex wave vector

\[
k_c = k - i\alpha,
\]

(16)
whose imaginary component is the fluid’s attenuation coefficient in units of Nepers/cm.

Figure 2. Comparison of lateral beam profile to Jinc function.

3.1. Axial Pressure

Attenuation measurements performed along the beam axis of a circular aperture piston source are interpreted by assuming the existence of a multiplicative exponential attenuation factor. It is important to determine under what conditions this simple multiplicative exponential attenuation factor is an accurate representation of the attenuated pressure.

Using equation (16) the axial pressure becomes

$$P_a(z,0,t) = \frac{\rho_0 \omega u_0}{k^2 + \alpha^2} e^{i(kz - \rho_0 \omega u_0)} e^{-i(\alpha_0 - \rho_0 \omega u_0)} e^{-\alpha z} e^{-\alpha_0(z^2 + a^2)}.$$

(17)

An exact expression for the attenuated axial pressure magnitude is obtained from the magnitude of the product of the beam pattern diffraction terms in equation (17)

$$|P_{\alpha EX}(z,0)| = \frac{\rho_0 \omega u_0}{\sqrt{k^2 + \alpha^2}} e^{-\alpha z} \left[ 1 + e^{-2\alpha(z^2 + a^2 - z)} - 2e^{-\alpha(z^2 + a^2 - z)} \cos(kz^2 + a^2 - 2z) \right]^{1/2}.$$  

(18)

For low attenuation and/or large axial distances the two exponential terms inside the curly brackets go to unity producing an approximate expression for the axial pressure magnitude in an attenuating fluid

$$|P_{\alpha APP}(z,0)| = \frac{2\rho_0 \omega u_0}{\sqrt{k^2 + \alpha^2}} e^{-\alpha z} \sqrt{\sin^2 \left( \frac{kz^2 + a^2 - z}{2} \right)}.$$  

(19)

When the approximate equation (19) is valid the fluid attenuation appears in the usually assumed form of a multiplicative exponential attenuation factor. If equation (6) is substituted into the exponential terms in either equation (18) or equation (19) it can easily be shown that the normalized representation valid for an unattenuated beam pattern is not valid when the fluid is attenuating.

By considering the exponential terms in the exact equation (18) it can be determined what values of $\alpha$ and $z$ permit use of the approximate equation (19). The first two terms in the binomial series expansion of $e^x$ are $1 + x$, so for $\kappa\%$ accuracy in assuming that an exponential term goes to unity $x$ must be less than $\kappa/100$. The second term in the curly brackets in equation (18) has the larger exponential argument, so for $\kappa\%$ accuracy in replacing this exponential term (and $\kappa/2\%$ accuracy in replacing the other exponential term) by unity the condition on $\alpha$ and $z$ is

$$\alpha \leq \frac{\kappa}{200(z^2 + a^2 - z)} \approx \frac{\kappa z}{100a^2}.$$  

(20)
with the binomial series approximation on the right valid when \( z > a \).

When the exponential terms in equation (18) are replaced with \( 1 - \kappa/100 \) and \( 1 - \kappa/200 \), respectively, it is found that

\[
|P_{aEX}(z,0)| = \sqrt{1 - \frac{\kappa}{200}} |P_{aAPP}(z,0)|. \tag{21}
\]

For small \( \kappa \) values the square root in equation (21) may be replaced with the first two terms of its binomial series expansion and the percentage error in replacing equation (18) by equation (19) is then \( \kappa/4\% \).

When there is an equals sign on the left of equation (20) \( \alpha \) has its largest value consistent with equation (19) being within \( \kappa/4\% \) of equation (18). Figure 3 presents this maximum \( \alpha \) value as a function of \( z \) for three different circular aperture piston diameters. It demonstrates that as \( z \) increases, larger maximum values of \( \alpha \) are permitted without causing more than a 1% error. Since \( \alpha \) is proportional to \( \kappa \) in equation (20), to represent the exact axial pressure to 10% accuracy both ordinates in figure 3 should be multiplied by 10.

![Figure 3. Maximum attenuation for 1% accuracy of equation (17) for \( P_{aAPP} \) (i.e. \( \kappa = 4 \)).](image)

### 3.2. Full beam pattern

The attenuated full beam pattern is derived from equation (1) with the substitution of equation (16) and then applying a Fresnel approximation for \( r' \). Or the Fresnel approximation may be applied first and then the substitution indicated in equation (16) made without error. So equation (16) may be substituted into the result of the Fresnel approximation (equation (9)) to obtain

\[
P_{\alpha APP}(r,\theta,t) = i\frac{\rho_0 \omega_0}{2\pi r} e^{i(\omega t - kr)} e^{-\sigma r} \int_0^a \!\! \sigma e^{\frac{k\sigma^2}{2r}} e^{\frac{\alpha \sigma^2}{2r}} \int_0^{2\pi} \!\! e^{i\sigma \sin \theta \cos \phi} e^{i\alpha \sigma \sin \theta \cos \phi} d\phi d\sigma. \tag{22}
\]

Equation (22) may be rewritten in the form

\[
P_{\alpha APP}(r,\theta,t) = i\frac{\rho_0 \omega_0}{2\pi r} e^{i(\omega t - kr)} e^{-\sigma r} \int_0^a \!\! \sigma e^{\frac{k\sigma^2}{2r}} e^{\frac{\alpha \sigma^2}{2r}} \int_0^{2\pi} \!\! e^{i(k-\alpha \sigma)\sigma \sin \theta \cos \phi} d\phi d\sigma \tag{23}
\]

and equation (10) used to obtain
\begin{equation}
P_{a,\text{APP}}(r,\theta,t) = \frac{i \rho_0 \omega u_0}{r} e^{i (\sigma t - kr)} e^{-i \frac{kr}{2r}} \int_0^\sigma e^{-\frac{a \sigma^2}{2r}} e^{-\frac{a \sigma^2}{2r}} J_0(\sigma \sin \theta (k - i \alpha)) d\sigma. \tag{24}
\end{equation}

The zeroth order complex Bessel function in equation (24) can be represented by a real part, \(U_0(\rho,\phi)\), and an imaginary part, \(V_0(\rho,\phi)\). Substituting them into equation (24) and calculating the magnitude of the product of the beam pattern diffraction terms

\[|P_{a,\text{APP}}(r,\theta)| = \frac{\rho_0 \omega u_0}{r} e^{-\sigma r} \left[ I_{a,\text{RE}}^2 (r,\theta) + I_{a,\text{IM}}^2 (r,\theta) \right]^{1/2}, \tag{25}\]

where

\begin{equation}
I_{a,\text{RE}}(r,\theta) = \int_0^\sigma e^{-\frac{a \sigma^2}{2r}} \left[ U_0(\rho(\theta),\phi) \sin \left( \frac{k \sigma^2}{2r} \right) - V_0(\rho(\theta),\phi) \cos \left( \frac{k \sigma^2}{2r} \right) \right] d\sigma \tag{26}
\end{equation}

and

\begin{equation}
I_{a,\text{IM}}(r,\theta) = \int_0^\sigma e^{-\frac{a \sigma^2}{2r}} \left[ U_0(\rho(\theta),\phi) \cos \left( \frac{k \sigma^2}{2r} \right) + V_0(\rho(\theta),\phi) \sin \left( \frac{k \sigma^2}{2r} \right) \right] d\sigma. \tag{27}
\end{equation}

At any depth the attenuated unfocused beam pattern will be much lower in magnitude than the unattenuated unfocused beam pattern. But will the shape of the lateral beam profile change with attenuation? To answer this question lateral beam profile calculations were performed for \(f = 3.5\) MHz, \(a = 1\) cm and \(c = 0.154\) cm/\(\mu\)sec at the depth \(0.25 Y_0 = 5.692\) cm. The calculations were performed with attenuations \(\eta_0\) of 0, 0.014, 1.41 and 10 dB/(cm-MHz). Figure 4 shows the results of these calculations with all peak pressures normalized to unity.

![Figure 4. Change of near zone lateral beam profile with attenuation.](image)

The 0.014 dB/(cm-MHz) lateral beam profile is identical with the unattenuated lateral beam profile. The 1.41 dB/(cm-MHz) lateral beam profile is slightly different from the unattenuated beam profile. The lateral beam profile oscillations are smoothed out at 10 dB/(cm-MHz).

[1] Goldstein A 2004 Steady state unfocused circular aperture beam patterns in nonattenuating and attenuating fluids J. Acoust. Soc. Am. 115 99-110
[2] Kinsler L E, Frey A R, Coppens A B and Sanders J V 1982 Fundamentals of Acoustics, Third Edition (John Wiley & Sons, New York) pp 176-182