Fokker-Planck models of NGC 6397 – B. The globular cluster

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ABSTRACT

This is the second of two papers presenting a detailed examination of Fokker-Planck models for the globular cluster NGC 6397 and is concerned with extracting information on the cluster from the models. The models give a current cluster mass of \( 6.6 \pm 0.5 \times 10^4 M_\odot \) of which about 2500 \( M_\odot \) is in neutron stars. This mass gives a \( V \)-band mass-to-light ratio of 1.2 in solar units. The models and data provide weaker estimates of the structural parameters, but suggest that the core radius is less than 0.3 pc \( (11'' \) ) and the tidal radius is \( 17 \pm 4 \) pc. In turn, by assuming a flat rotation curve for the galaxy, the mass and tidal radius suggest that the latter was set at a distance of 2.5 kpc from the galactic center.

1. Introduction

This is the second part of a binary paper discussing Fokker-Planck models matched with observations of the globular cluster NGC 6397. In the first paper (Drukier 1994; Paper A), the details of the modeling and comparison techniques were discussed, as was an overview of results of the over 1000 models run. Briefly, the models solve the isotropic, orbit-averaged form of the Fokker-Planck equation, where the distribution functions are functions of energy and mass. An energy source in the form of a statistical treatment of binaries formed in three-body reactions is used to reverse core collapse. The models also include a tidal boundary and the effects of mass-loss due to stellar evolution. More details are given in Paper A together with definitions for many of the symbols used here. The data used for the comparisons are the surface density profile (SDP) and two mass functions (MFs) from Drukier et al. (1993), the intermediate-distance mass-function from Fahlman et al. (1989) and the velocity dispersion profile from Meylan & Mayor (1991). Again, I follow the naming convention of Drukier et al. (1993) and refer to the three mass functions as the du Pont:if, FRST, and du Pont:out MFs in order of distance from the cluster center.

It was found in Paper A that neither the initial mass function (IMF), nor any of the other initial parameters were uniquely constrained by the observations. As a consequence, there are a

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1 Postscript figures for this paper are available by anonymous FTP from ftp.ast.cam.ac.uk in the directory /pub/drukier or by email to the author at drukier@mail.ast.cam.ac.uk. This paper has been submitted for publication in The Astrophysical Journal.
number of models which give satisfactory matches. In this paper I will describe several of these
models and use them to discuss the stellar content of NGC 6397, its current mass and mass-to-light
ratio, and its dynamical history. I will also look at the tidal radius of the cluster, which is related
its galactocentric distance and orbit, and the core radius.

The next section discusses information extracted from the entire ensemble of models, ie. the
current mass of the cluster and its core and tidal radii. The first is well constrained; the latter
two less so. Section presents comparisons of nine models drawn from the various model sets.
The model sets are defined as in Paper A by their IMF and tidal radius $r_t$. The definitions of the
IMFs are given in Table 1. The tidal radius defines the galactocentric radius at which the model is
assumed to orbit and hence the strength of the tidal stripping. It is quoted for a cluster with mass
$10^5 M_\odot$. The true tidal radius scales as $M^{1/3}$ (see §3.2 in Paper A). These have been chosen as
the best overall models from each. The differences between the best matches from each IMF give
information on the relative proportions of high-mass remnants, visible stars, and faint low-mass
stars. The final section summarizes the conclusions of this paper and binds this binary.

2. Global constraints

2.1. Mass

One of the initial results from the preliminary “hunting” stage described in §3.3 of Paper A,
was the observation that there was a preferred mass for matching the mass functions. Figure. 1
shows $\chi^2_{MF}$ (the quality of fit to the mass functions as defined in §3.1 of Paper A) vs. total
mass for a sample of 21 U20 models with varying initial parameters including $r_t$. In all cases the
$\chi^2_{MF}$ vs. mass curve is parabolic and has a minimum near a mass of $7 \times 10^4 M_\odot$. For this IMF,
the mass of the model must be approximately $7 \times 10^4 M_\odot$ in order to match the observed mass
functions. If NGC 6397 did have this IMF, then this result implies that the current mass of the
cluster is also $7 \times 10^4 M_\odot$. I fitted the $\chi^2_{MF}$ vs. mass curves with a parabola in the mass range
$4 \times 10^4 M_\odot$ to $1.5 \times 10^5 M_\odot$ in order to refine the estimate of the optimal mass. Figure 2 shows this
minimum in $\chi^2_{MF}$ as a function of mass for all the U20 models having such minima. The models
with extremely poor $\chi^2_{MF}$ tend to have other problems which make them unacceptable matches in
any case. Usually, they started off with long relaxation times and show little dynamical evolution.
As a result, the best fitting mass is a compromise between all the poor fits. That an adequate
estimate is possible from even such poor models strengthens my confidence in this method of
estimating the mass of the cluster. The models show a range of optimal masses, but the mode,
mean and median all indicate a typical value of $6.9 \times 10^4 M_\odot$. The apparent correlation between
the minimum value of $\chi^2_{MF}$ and the optimal mass is related to the correlation between $\chi^2_{MF}$ and
the age of the model as discussed in Paper A.
With the exception of some very poor fitting models, the other model sets also give a consistent mass for NGC 6397 even though most of the models do not give very good matches for the shapes of the individual mass functions, let alone the surface density profile. This shows that we can get a current mass for NGC 6397 without knowing the IMF to any great precision. This is so, provided that the IMF being used does give present day MFs consistent with the observations. By using a range of IMFs, as is done below, we can increase our confidence in the results.

In Paper A, I showed that the requirement for a model to have its best matches to both $\chi^2_{MF}$ and $\chi^2_{SDP}$ (the quality of fit to the surface density profile as defined in §3.1 of Paper A) simultaneously defines a surface in the $(W_0, M_0, r_l/r_t)$ parameter space. I refer to this as the “$\Delta t = 0$ surface” and the models on it are referred to as being “well-fitting”. This surface actually has a thickness of about $\pm 0.4$ Gyr which arises from the interval between records of the state of the models (See §5 in Paper A). By considering the $\Delta t = 0$ surfaces for the various model sets, we can approach the mass question from a slightly different angle. Figure 3 shows the mass at minimum $\chi^2_{MF}$ as estimated on the $\Delta t = 0$ surface as a function of time for each of the model sets. The format of this diagram is the same as for similar figures in Paper A. For the U20, $r_t = 18.5$ pc model set, the time dependence of the optimal mass is similar to that seen for the individual models, but with less scatter in mass. The remaining scatter derives from the effects of variations in the other initial parameters. This model set covered the largest range of the $(W_0, M_0)$ parameter space (recall from Paper A that $r_l/r_t$ is a function of $(W_0, M_0)$ on the $\Delta t = 0$ surface) and shows the largest variation in the optimal mass at a given time. This dependence on the other parameters is confirmed by the apparent bimodality in the U20, $r_t = 21$ pc model set. The models in this set were run in two separated “islands” in $(W_0, M_0)$ space, and these are reflected in the diagram. The lower optimal mass models started with $M_0 < 5 \times 10^5 M_\odot$ and $W_0 < 5.75$ and the higher mass group originate in an island with $M_0 > 5 \times 10^5 M_\odot$ and $W_0 > 6.00$. The other model sets covered much smaller regions of $(W_0, M_0)$ space. From the two U20 model sets covering a large range of parameters (ie. those with $r_t = 18.5$ pc and $r_t = 21.$ pc) I estimate the intrinsic scatter in a mass estimate at a given age to be $\pm 1,000 M_\odot$. For all but the U10 model sets, there is an increase of 800 to 900 $M_\odot$/Gyr in the optimal mass over the time range used. Adopting the isochrone age of 16±2.5 Gyr (Anthony-Twarog, Twarog, & Suntzeff 1992), this becomes a $\pm 2000 M_\odot$ systematic uncertainty in the optimal mass. Table 2 gives the estimated optimal mass for each of the model sets for both 16 Gyr and for the age of the best models in that model set. The U20, $r_t = 17$ pc mass is based on an extrapolation to 16 Gyr. For the X2 and NNS model sets, there was not a sufficient number of useful models to produce such a graph. For these model sets the masses in Table 2 are averages over all the models showing minima in $\chi^2_{MF}$. In the X2 model set, no systematic trend with time was seen while for the NNS model set there may be a trend of the same size as in the majority of the other model sets. The model sets which give high mass estimates are those which do not fit the surface density profile at 16 Gyr, but only at ages

$^{2}W_0$ is the dimensionless central potential of the King model used for the initial state of the model, $M_0$ is the initial mass, and $r_l/r_t$ is the ratio of the initial limiting radius of the model to the tidal radius defined above.
closer to 12 Gyr. At the earlier time the masses are lower and are more consistent with the U10 masses. Based on this I conclude that the current mass of NGC 6397 is $6.6 \pm 0.5 \times 10^4 M_\odot$.

### 2.2. Tidal and core radii

The computation of $\chi^2_{MF}$ involves detailed comparison with the observed mass functions at three radii. We already know that the IMF is approximately right for the cluster, otherwise the shapes of the mass functions would give very poor matches. We also know that we have approximately the correct tidal radius because we can fit the surface density profile fairly well. As I have shown, the mass at which $\chi^2_{MF}$ reaches a minimum reflects the optimal fitting of the three mass functions combined. This point is illustrated by the most discordant point in Fig. 2. This model started with $r_t = 21$ pc, $W_0 = 4$, $M_0 = 6.97 \times 10^5 M_\odot$ and $r_l/r_t = 0.66$. It suffered a large stellar evolution expansion and lost a lot of mass through the tidal boundary, so it did not re-collapse very much before its mass went to zero. The model never really fit any of the observed MFs, but at a mass of $7.6 \times 10^4 M_\odot$ the fit to the mass functions was the least poor, giving, even for this very poor model, a minimum in $\chi^2_{MF}$, at $\chi^2_{MF} = 3.6$. The bulk of the models have minima in $\chi^2_{MF}$ much closer to unity and match the observed MFs much better. That all the minima are not at the same point reflects the detailed differences in the mass function comparisons.

One type of difference depends on the history of the model and the small effects that changes in the initial conditions have on the degree of mass segregation. There can be differences both with radius at a given time and with respect to the IMF. The size of these differences warrants further study, but will not be dealt with here.

The other type of difference is related to the density profile. If we compare models GM031 and GK079 (Figs. 5 and 7), which have $r_t = 17$ and 21 pc respectively, it is obvious that the latter has a higher density at large radii and, correspondingly, a du Pont:out MF which, unlike most of the rest of the models, is not systematically smaller than the observed MF. In principle we can use this effect to try and extract an estimate of the current tidal radius of NGC 6397. Unfortunately, only weak limits can be applied in the case of NGC 6397. The primary problem is that the outermost mass function is still not far enough out to be a strong constraint on the tidal radius. But, as was discussed in Drukier et al. (1993), at the radial distance of the du Pont:out mass function, only half of the objects counted belong to the cluster. Fields further from the cluster center will be even more heavily contaminated with field stars. Within any set of models there is a wide range of minima both in the composite $\chi^2_{MF}$ and in $\chi^2$ for just the du Pont:out MF because of the other factors discussed above. Until there is some way to disentangle these effects it will be difficult to draw any conclusion on the tidal radius of the cluster. From an examination by eye of specific models it appears to me that $r_t \sim 20 \pm 4$ pc. Assuming a flat rotation curve at 220 km s$^{-1}$ for the galaxy, this gives, using eq. (5) of Paper A, a galactocentric distance of $2.5 \pm 0.7$
kpc for NGC 6397. The observed galactocentric distance of the cluster is 6 kpc (Djorgovski 1993). Cudworth & Hanson (1993) found its space velocity to be $(\Pi, \Theta, Z) = (24 \pm 6, 126 \pm 12, -105 \pm 12)$ km s$^{-1}$. From the $\Pi$ and $Z$ velocities for this southern cluster ($b = -12^\circ$), NGC 6397 is receding from perigalacticon. The present tidal radius $(17 \pm 4$ pc) may reflect the distance of its most recent perigalactic passage.

For almost all the models run the optimal fits to the surface density profile occurred in the collapse phase. The change in $\chi^2_{SDP}$ as a function of time shown in Fig. 1 of Paper A is quite typical. The dependence on the size of the core is shown here in Fig. 4 for several U20 models. The “core radius” used is the empirical one given by the radius at which the surface density of the mass species used for the SDP comparison reaches half its central value. Clearly, the minima in $\chi^2_{SDP}$ occur before core bounce and then becomes somewhat larger both as core collapse proceeds and subsequently in the post-collapse phase. The root of this difference lies in the details of the matching between the data and models in the central part of the cluster. It is well to bear in mind that the core is not resolved in the star count data, so the observational core radius is not well defined. For the same reason, the model core radius is not constrained very well by the observations. The central cusp is also seen in the surface brightness profile but that is dominated by a few bright stars. Lauzeral et al. (1992) manage to obtain a core radius of 0.6 pc by removing the bright stars, but, as was shown in Drukier (1993), the small number of stars in the core still precludes the exclusion of a smaller core radius. Meylan & Mayor (1991) compiled all the surface brightness data onto one system and measured a core radius of 0.22 pc by fitting the data to multi-mass King models. This $r_c$ is consistent with those listed in Table 4. As discussed in Drukier et al. (1993), the SDP within 1 pc is somewhat flatter than beyond this radius, but it never does flatten out. In the models, as the core continues to collapse, the region outside the core shows a single power-law slope and the straight region around 0.5 pc occupies the region where the observed SDP has a bump. Closer to core collapse the models fit the central density better, but the match around 0.5 pc is much poorer. As the cluster continues to evolve, the central slope continues to flatten. This gives the increase in $\chi^2_{SDP}$ during the post-collapse phase.

3. Details of some good matches

I will now turn to the presentation of some specific models which come the closest to satisfying all the observational constraints. None is perfect, but the differences between them provide further information with regard to the stellar content of NGC 6397. The models selected are from the model sets described in Paper A. For the model sets showing large variations in $\chi^2_{SDP}$ and $\chi^2_{MF}$ with the age of the model, I have chosen the well-fitting model closest to the point of intersection between the the two $\chi^2$ curves. The one exception is model U20-C where a somewhat older model has been used. For the U10, $r_t = 18.5$ pc model set, the one without a time dependence to the quality of the matches, two models have been chosen, one at the young limit and one at the old. I
have also included one model from the poorly fitting model set X2 with $r_t = 18.5$ pc. Table 3 lists the initial parameters for each of the nine models. Details about the state of each model at its best matching time are listed in Table 4. The comparisons between some of the models and the observations are shown in Figs. 6 to 12.

Figure 5 shows the match to the data of model U20-A. The mass functions are all consistent with the general shapes of the mass functions, although the fine details are missed by the models. This is as expected since the IMFs are simply combinations of power laws and the true MFs appear to have more structure than can be represented by this model. The number of stars in each mass function is also well fit for the du Pont:if and FRST mass functions; in common with all the other models the number of stars in the du Pont:out field is underestimated. The surface density profile is well matched at all radii with the possible exception of the very center. Other models, as will be seen, deviate by much larger amounts from the observed central value, so in comparison model U20-A is quite successful. The velocity dispersions are also well matched. The main problem with this model is that it requires the turn-off mass to be 0.86 $M_\odot$ and hence the age to be 12 Gyr. As discussed in Paper A, this problem is generic to any model set with too high a central mean mass, i.e. one with lots of heavy remnants.

Comparing Fig. 5 (model U20-A, U20 with $r_t = 17$ pc model set) with Fig. 6 (model U20-B, U20 with $r_t = 18.5$ pc model set) Fig. 7 (model U20-D, U20 with $r_t = 21$ pc model set) shows the effect of increasing the tidal radius for the U20 IMF. Several systematic trends are visible in the comparison of these models. First, the modeled du Pont:if mass function becomes steeper with increasing $r_t$ for $m > 0.4 M_\odot$. This suggests that the amount of mass segregation within the half-mass radius decreases with increasing $r_t$. On the other hand, the shapes of the outer two mass functions are much the same for all values of $r_t$. The number of stars in the modeled du Pont:out MF increases with increasing $r_t$, as would be expected, and the quality of the match to this mass function similarly improves. On the other hand, the central surface density decreases with increasing tidal radius, as does the velocity dispersion. The U20, $r_t = 20$ pc model set model U20-C is consistent with these trends. The trends in the central SDP, the du Pont:if MF and the velocity dispersion are all consistent with a decrease in the total mass in the inner part of the cluster with increasing tidal radius. Considering the appropriate lines in Table 4, the total mass varies by about 6% for the four models, but the mass within the inner parsec decreases by 1/4 as the tidal radius increases. This trend is much less pronounced in models U10-B2 and U10-C, but the mass difference and mass fraction in heavy remnants is also smaller with these two models. As is often the case in this modeling, post-facto explanations are possible, but they are not easy to generalize in the face of differences in the IMF.

Until here the models shown have all had the same IMF, U20. The effects of changing the upper mass limit, and hence the fraction of neutron stars, is shown by comparing Fig. 8 with Fig. 9 (model U30 IMF and Figs. 10 and 11 for the U10 IMF. The details of these IMFs are given in Table 1 and more fully in Table 2 of Paper A. Unusually, model U10-B1 (Fig. 11) has been caught deep in core collapse. This demonstrates that such a state is not excluded, but is just usually not
favored by the matching procedure. The age of the model is also compatible with the isochrone age. The velocity data is equally consistent for both U10 models and the differences in the mass function matches are not significant.

In a slightly different vein, Fig. 11 shows the comparison for model L05-B, a model with the same upper limit as the U20 IMF (20 $M_\odot$) but which extends down to 0.05 $M_\odot$. This model shows some larger deviations from the observations. The central surface density is much flatter in shape than observed and the velocity dispersion, while within the observational errors, is systematically low. The mass function at the high mass end is somewhat steeper than observed. All three of these deviations indicate that this IMF has too few massive stars relative to the number of low mass stars. Model L05-B is, of all the models shown, the largest compromise between the SDP and the MFs. As was shown in Paper A, the L05 model set showed the strongest time dependence for both $\chi^2_{MF}$ and $\chi^2_{SDP}$. At the young extreme, there is a model which gives a better fit to the SDP and at a much older limit there are models which give better matches to the MFs. At both extremes the deficiencies in the match to the other data are accentuated, and in all the L05 models the velocity dispersions are systematically low. This all suggests that the L05 IMF has too high a proportion of low mass stars and that NGC 6397 contains only about 20% by mass of stars with masses less than 0.2 $M_\odot$. On the positive side, the age of this model, 14 Gyr, is consistent with the isochrone age of NGC 6397.

Unlike the model sets discussed until now, the X2 and NNS model sets produced no models which adequately matched the observations. The primary problem lay in the surface density profile as I show in Fig. 12. I show this model at the time of the minimum in $\chi^2_{MF}$ at about the same time before core collapse as most of the other models discussed here. However, for the X2 and NNS model sets the time of core collapse produces a maximum in $\chi^2_{SDP}$. At this time the SDP profile has a central logarithmic slope of about $-1.3$, much steeper than the observed value of $-0.9$. This indicates that the turn-off stars have about the same mass as the stars dominating the core. If we consider the radius at which the enclosed mean mass equals the mean mass of the main-sequence turn-off (as defined by the age of the model), then for a typical X2 model this radius is about 0.2 pc compared with 0.4 to 0.8 pc for the better matching model sets. Interestingly, the best matching U10 model, U10-B1, has the smallest of these radii, the X2 model set has just taken this trend too far. The model mass functions are somewhat too steep at the high mass end of the mass functions. This suggest that a flatter IMF is required. The conclusion I draw from this is that NGC 6397 contains about 2500 $M_\odot$ in neutron stars. Model X2-B contains only 1400 $M_\odot$ and this is clearly not adequate, but model U10-B1, with 2600 $M_\odot$ is quite satisfactory.

4. Conclusions

These detailed comparisons lead to the following conclusions about NGC 6397.
1. The total mass of the cluster is $6.6 \pm 0.5 \times 10^4 M_\odot$.

2. Approximately 2500 $M_\odot$ of this is in neutron stars.

3. Given that the absolute integrated V magnitude of NGC 6397 is $M_V = -7.02$ (Djorgovski 1993) the global mass-to-light ratio is $1.2 \pm 0.1 \,(M_\odot/L_\odot)_V$.

4. The mass function probably flattens for stars less massive than the observed limit. The mass fraction in stars with masses less than 0.2 $M_\odot$ is probably less than $1/3$ and is more like $1/5$.

5. The core radius is unresolved in these data, but is probably less than 0.3 pc ($11''$).

6. The tidal radius of the cluster is $17 \pm 4$ pc reflecting a probable perigalactic distance of $2.5 \pm 0.7$ kpc. These numbers are somewhat uncertain as they come from the mass function fits and not from the surface density profile. Crowding by field stars prevents direct observation of the tidal cut off.

There are a couple of systematic problems with these matches. One is that the models usually underestimate the number of stars in the du Pont:out mass function and the second is that the model velocity dispersion is lower than the observed velocity dispersions. The mass-to-light ratio derived here is substantially lower than the mass-to-light ratio of 2, which Meylan & Mayor (1991) derived from fitting King models. The King models give a higher mass, $10^5 M_\odot$, for the cluster, and thus a higher $M/L$. The fit of Meylan & Mayor required increasing anisotropy in the outer part of the cluster and it could be that the deficiencies in the models here also indicate a requirement for anisotropy. An anisotropic velocity tensor would result in a somewhat higher line-of-sight velocity dispersion and in higher stellar densities at radii where anisotropy is important when compared with an isotropic distribution.

Weinberg (1994) has run some similar models to those run here; excluding stellar evolution, but including disk shocking. At the galactocentric radii of the models discussed here, continuous disk shocking destroys all clusters with $W_0 < 6.5$ in less than a Hubble time. While these are only preliminary results, they do suggest that this is an important effect for clusters like NGC 6397. The requirement for a high initial concentration would further restrict the range of initial parameters which lead to acceptable models. Somewhat higher initial masses may also be possible. The strength of tidal shocking will depend on the cluster’s orbit, so it is difficult to extrapolate from Weinberg’s results.

In this binary paper I have demonstrated that the type of Fokker-Planck model described here can produce models capable of matching a set of detailed observations of a globular cluster. The caveats are that anisotropy in the velocity dispersion may be required and that disk shocking may have an important role to play. In the continued absence of an accurate procedure for including velocity anisotropy in Fokker-Planck models (but see Takahashi 1993), the present approach is still useful. A large investment of computer time is required to test various possible IMFs, but once the general nature of the relationship between the models and the observations is clear, a more faster
and more systematic study is possible. I have also shown how to make the comparisons and how to extract specific information from the ensemble of models. A study such as this one can give definite values for the mass of a cluster and can serve as a good guide to the relative abundances of both the heavy and light unobserved stars. Adequate data sets would also give information on the radial structure of the cluster including the core and limiting radii. Information on the latter can be extracted from the mass functions even when field star contamination prevents direct observation of the boundary. Due to the large number of parameters available for constructing the models, as many constraints as possible are desirable. This first extensive comparison between Fokker-Planck modeling and detailed observations of a single cluster is quite encouraging both for confidence in the models and for the ability to use the models to interpret the observations.

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Table 1: IMF definitions (with masses in $M_\odot$)

| IMF | $m_{\text{min}}$ | $m_{\text{max}}$ | Comments$^a$ |
|-----|------------------|------------------|--------------|
| U10 | 0.1              | 10.              | $x = 1.5, m < 0.4; x = 0.9, m > 0.4$ |
| U20 | 0.1              | 20.              | $x = 1.5, m < 0.4; x = 0.9, m > 0.4$ |
| U30 | 0.1              | 30.              | $x = 1.5, m < 0.4; x = 0.9, m > 0.4$ |
| L05 | 0.05             | 20.              | $x = 1.5, m < 0.4; x = 0.9, m > 0.4$ |
| X2  | 0.1              | 20.              | $x = 1.5, m < 0.4; x = 0.9, 0.4 < m < 2.; x = 2, m > 2.$ |
| NNS | 0.1              | 20.              | as U20, but all stars with initial masses > 8 are assumed to escape |

$^a$$x$ is the mass spectral index for a power-law mass function of the form $dN \propto m^{-(1+x)} dm$.

Table 2: Preferred model masses

| IMF | $r_t$ (pc) | $M$ at 16 Gyr ($10^5 M_\odot$) | $M$ at best time$^a$ ($10^5 M_\odot$) |
|-----|------------|-------------------------------|-------------------------------------|
| U30 | 18.5       | 0.73                          | 0.68                                |
| U20 | 17.0       | 0.71$^b$                      | 0.68                                |
| U20 | 18.5       | 0.70                          | 0.66                                |
| U20 | 20.0       | 0.70                          | 0.65                                |
| U20 | 21.0       | 0.69                          | 0.66                                |
| L05 | 18.5       | 0.71                          | 0.69                                |
| U10 | 18.5       | 0.63                          | 0.63                                |
| U10 | 20.0       | 0.63                          | 0.62                                |
| X2  | 18.5       | 0.59$^c$                      |                                     |
| NNS | 18.5       | 0.61$^c$                      |                                     |

$^a$time of minima in the mean $\chi^2$ (see Table 3 below and Fig. 7 of Paper A.)

$^b$extrapolated

$^c$average over all models
### Table 3: Initial parameters of models

| Model | IMF | \( r_t \) (pc) | \( W_0 \) (10^5 \( M_\odot \)) | \( M_0 \) (10^5 \( M_\odot \)) | \( r_t/r_0 \) | \( r_h \) (pc) | \( t_rh \) (Gyr) |
|-------|-----|----------------|---------------------|---------------------|-------------|-------------|----------------|
| U30-B | U30  | 18.5           | 5.50                | 5.50                | 0.66        | 3.6         | 1.6           |
| U20-A | U20  | 17.0           | 5.12                | 5.19                | 0.66        | 3.6         | 1.7           |
| U20-B | U20  | 18.5           | 6.36                | 5.79                | 1.00        | 4.5         | 2.5           |
| U20-C | U20  | 20.0           | 5.12                | 4.59                | 0.66        | 4.0         | 1.9           |
| U20-D | U20  | 21.0           | 6.69                | 5.19                | 1.10        | 5.0         | 2.8           |
| L05-B | L05  | 18.5           | 4.50                | 4.50                | 0.76        | 4.8         | 4.9           |
| U10-B1| U10  | 18.5           | 5.75                | 4.00                | 0.84        | 3.9         | 2.0           |
| U10-B2| U10  | 18.5           | 5.50                | 5.00                | 0.90        | 4.7         | 3.0           |
| U10-C | U10  | 20.0           | 5.75                | 5.00                | 0.96        | 5.2         | 3.4           |
| X2-B  | X2   | 18.5           | 5.50                | 4.00                | 1.02        | 5.0         | 3.4           |

### Table 4: Parameters of models at the best matching time

| Model | Age (Gyr) | \( M_{TO} \) (\( M_\odot \)) | \( r_m^a \) (pc) | \( \chi_{MF}^2 \) | \( \chi_{SDP}^2 \) | Mass (10^5 \( M_\odot \)) | \( r_h \) (pc) | \( r_c \) (pc) | \( M (< 1 \text{pc}) \) (10^5 \( M_\odot \)) | NS mass (\( M_\odot \)) | \( f_{<2}^b \) |
|-------|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|-------------|-------------|----------------|----------------|-------------|
| U30-B | 12.2      | 0.86            | 0.79            | 1.4             | 1.9             | 0.70            | 3.6         | 0.29        | 0.104          | 9700           | 0.20        |
| U20-A | 12.1      | 0.86            | 0.68            | 1.4             | 1.4             | 0.70            | 3.3         | 0.24        | 0.114          | 8600           | 0.19        |
| U20-B | 12.0      | 0.86            | 0.69            | 1.4             | 1.5             | 0.68            | 3.6         | 0.26        | 0.100          | 7800           | 0.20        |
| U20-C | 13.7      | 0.83            | 0.81            | 1.4             | 2.0             | 0.66            | 3.9         | 0.32        | 0.089          | 7600           | 0.21        |
| U20-D | 12.2      | 0.86            | 0.69            | 1.6             | 1.7             | 0.67            | 4.1         | 0.33        | 0.083          | 6700           | 0.23        |
| L05-B | 13.8      | 0.83            | 0.56            | 1.9             | 1.9             | 0.72            | 4.1         | 0.42        | 0.081          | 6300           | 0.34        |
| U10-B1| 14.1      | 0.82            | 0.39            | 1.6             | 1.1             | 0.66            | 3.2         | 0.02        | 0.101          | 2600           | 0.20        |
| U10-B2| 16.2      | 0.79            | 0.43            | 1.6             | 1.1             | 0.64            | 3.2         | 0.35        | 0.098          | 3000           | 0.18        |
| U10-C | 17.8      | 0.77            | 0.52            | 1.4             | 1.1             | 0.63            | 3.5         | 0.25        | 0.093          | 2900           | 0.19        |
| X2-B  | 18.1      | 0.77            | 0.23            | 2.2             | 2.7             | 0.58            | 2.8         | 0.14        | 0.112          | 1400           | 0.18        |

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\( a \)Radius at which the mean mass of the enclosed stars equals the turn-off mass \( M_{TO} \).

\( b \)Mass fraction in stars with \( m < 0.2M_\odot \).
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Fig. 1.— Plot of the quality of fit to the mass functions ($\chi^2_{MF}$) vs. total mass for a representative sample of 21 U20 models. The sharp corners indicate the sampled times for each run. Mass decreases with time. Note the cluster of minima around $7 \times 10^4 M_\odot$.

Fig. 2.— Plot of minima in $\chi^2_{MF}$ for all the U20 models. Models with large $\chi^2_{MF}$ are a poor match to the data on other grounds and can be excluded from the overall mass estimate. The typical optimal mass is $6.9 \times 10^5 M_\odot$ for the U20 IMF.

Fig. 3.— Optimal mass for all the models showing such minima plotted against the model age for the eight model sets with well-fitting models in Paper A. The arrangement of the model sets is the same as in Fig. 7 in Paper A. The IMF of the model set is listed across the top and $r_t$ (in pc) is listed at the right. The U20, $r_t = 18.5$ pc model set has the best sampled parameter space and the thickness of the distribution should reflect the uncertainty in the mass estimator.

Fig. 4.— Plot of $\chi^2_{SDP}$ vs. the empirical core radius for the same models as in Fig. 1. The models start at top-right and evolve downwards and then to the left. The minima in $r_c$ are at core collapse. Post collapse evolution takes them upwards and to the right. Note that the minima occur before core collapse and that $\chi^2_{SDP}$ increases in the post collapse phase.

Fig. 5.— Comparison diagram for model U20-A in the U20, $r_t = 17$ pc model set. Clockwise from upper left: The surface density profile; the FRST mass function; (top) the du Pont:if mass function and (bottom) the du Pont:out mass function; and the velocity dispersion profile. The dashed line in the mass function panels indicates the shape of the IMF.

Fig. 6.— As Fig. 5 for model U20-B in the U20, $r_t = 18.5$ pc model set.

Fig. 7.— As Fig. 5 for model U20-D in the U20, $r_t = 21$ pc model set.

Fig. 8.— As Fig. 5 for model U30-B in the U30, $r_t = 18.5$ pc model set.

Fig. 9.— As Fig. 5 for model U10-B1 in the U10, $r_t = 18.5$ pc model set.

Fig. 10.— As Fig. 5 for model U10-B2 in the U10, $r_t = 18.5$ pc model set.

Fig. 11.— As Fig. 5 for model L05-B in the L05, $r_t = 18.5$ pc model set.

Fig. 12.— As Fig. 5 for model X2-B in the X2, $r_t = 18.5$ pc model set.