Squeezed Atom Laser for Bose-Einstein Condensate with Minimal Length

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Abstract
We study a protocol for constructing a squeezed atom laser for a model originating from
the generalized uncertainty principle. We show that the squeezing effects arising from such
systems do not require any squeezed light as an input, but the squeezing appears automatic-
ically because of the structure of the model it owns. The output atom laser beam becomes
squeezed due to the nonlinear interaction between the Bose-Einstein condensate and the
deformed radiation field created due to the noncommutative structure. We analyze several
standard squeezing techniques based on the analytical expressions followed by a numerical
analysis for further insights.

Keywords Squeezed states · Minimal length · Noncommutative space · Atom laser ·
Generalized uncertainty principle

1 Introduction

After many decades since its inception, quantum theory yet serves as an endless source of
contemporary phenomena in the physical world as well as it keeps enriching in its mathe-
matical structure. With the advent of laser and quantum optics, the theory has become more
popular, since it gives rise to the opportunity of testing many fundamental properties of
quantum mechanics through simple experiments [1–3]. A crucial prerequisite for many of
these tests is the ability to create squeezed states, as they are the only source of continuous
variable quantum entanglement [2, 4]. There exist several methods of generating squeezed
states by using the standard techniques of quantum optics, which do assist the procedure of
verifying the fundamental properties. For some recent development in this regard one may
refer, for instance [5, 6]. However, the atom laser [7] provides the freedom to revisit many of
these tests using massive particles rather than the photons. For instance, in [8], the authors

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have shown that the continuous variable entanglement between the amplitude and phase of spatially separated atomic beams for the Einstein-Podolsky-Rosen (EPR) tests can be generated by dissociating a molecular Bose-Einstein condensate (BEC), or by outcoupling from a BEC using a Raman transition with squeezed light [9]. The tests of fundamental quantum theories by using atom laser not only provide an alternative avenue over quantum optics, but also it promises improved accuracy in interferometry [10, 11], quantum shot-noise limit for the measurement of quadrature squeezing [12], etc.

The squeezed atom laser is created usually by transferring the squeezed state of light onto the atoms by coupling the atoms out of the BEC and into the atom laser beam [13–16], and indeed there have been partial success in generating them in recent experiments [17–20]. But, the drawback of such a scheme is that it is required to construct a squeezed light at a particular frequency at which the atoms make up the BEC, which is certainly challenging. Meanwhile, an alternative scheme has also been suggested by utilizing a nonlinear interaction caused by atom-atom scattering to create a Kerr squeezing effect [21, 22], thus, removing a significant source of complexity. Besides, some other designs for constructing atom laser have been proposed, such as, from q-deformed [23] and f-deformed boson [24]. For an elaborate description on BEC in a similar context one may refer, for instance, [25–29].

In this article, we propose yet another alternative way to obtain a squeezed atom laser by using a model, which not only has a strong mathematical ground originating from the quantum group, but also it demonstrates ample interesting results in generating squeezed states and in other aspects of quantum optics [30–34]. Previously, it was shown that the optical squeezing properties are inherited in these models by construction, here we show that it is true for the atom laser case also. The models are motivated by the noncommutative structure of space-time, which were introduced to solve the issue of ultraviolet divergence in quantum field theory [35, 36]. A much simpler form of these models is familiar as minimal length now a days, which are one of the most promising candidates in explaining the Planck scale phenomena and quantum gravity.

In what follows, we shall discuss the detailed technicalities for the construction of the atom laser in the given scenario and then analyze its squeezing properties. In Section 2, we recall the introductory notions of the origin of the model and detailed mathematical and physical consistencies. Section 3 consists of the detailed discussion for the construction of atom laser for the models that are introduced in Section 2. In Section 4, we analyze the quantum statistical properties in order to gain insights on the squeezing properties of the atom laser. Finally, our conclusions are stated in Section 5.

2 Minimal Length and Generalised Uncertainty Principle

The concept of minimal length originally follows from the Snyder’s noncommutative version of space-time [35], which was introduced in the essence of implementing a natural cut-off in the ultraviolet domain to renormalize the theory of quantum fields. Today, the theory is about 70 years old and, it has evolved from time to time to reveal its usefulness in different contexts [36–42]. A more recent and simpler version that modifies the Heisenberg uncertainty relation to a generalized version, so-called generalised uncertainty principle (GUP) is of great interest now a days. In such cases, the space-time commutation relation often involves higher power of momentum and, thus, leads to the existence of nonzero minimal uncertainties in position coordinate, which is familiar as minimal length in the literature [43–48]. These models provide a wide-range of applications in several areas of modern physics, particularly in gravitation and black holes [49–54], cosmology [55, 56],
string theory [57–61], path integral quantum gravity [62, 63], doubly-special relativity [64, 65], quantum optics and information theory [31, 32, 34] and many more. Moreover, there have been several experimental proposal to test such theories by using opto-mechanical setup [66–68]. For further interest on the subject, one may follow some review articles in the context, for instance [38, 69, 70].

Here we consider one such minimal length model in 1D obeying the commutation relation [43, 46, 71]

\[ [X, P] = i\hbar(1 + \bar{\tau}P^2), \quad X = (1 + \bar{\tau}p^2)x, \quad P = p, \] (2.1)

with \( \bar{\tau} = \tau/m\omega\hbar \) having the dimension of inverse squared momentum and \( \tau \) being dimensionless real positive number. The corresponding observables \( X \) and \( P \) are represented in terms of the usual canonical variables \( x, p \) satisfying \([x, p] = i\hbar\). The model (2.1) has been explored extensively to construct coherent states, squeezed states, etc., as well as their physical implications have been discussed [31, 32, 34]. One interesting aspect of such model is that it can be associated with standard nonlinear generalized models by using some mathematical techniques. However, unlike the usual nonlinear generalization, there are some interesting consequences that follow from such minimal length model. For instance, it has been shown that the squeezed states constructed out of these models are intrinsically more squeezed than the usual quantum optical models, for further details one may refer to a review article in the context [30]. Quantum mechanically the parameter \( \tau \) brings the signature of the nonlinearity providing additional degrees of freedom in the quantum optical systems. More importantly, the minimal length for the model (2.1) turns out to be of the order of \( \hbar\sqrt{\bar{\tau}} \) [71]. Therefore, the standard definition of the smallest unit of length in quantum mechanics, i. e. the Planck length is replaced by a length scale belonging to the sub-Planckian regime depending on the value of the parameter \( \tau \).

Nevertheless, in order to apply these models in the construction of atom laser, let us discuss the associated mathematical detail briefly. Let us introduce a set of ladder operators \( A, A^\dagger \) [32, 34, 72]

\[
A = af(\hat{n}) = f(\hat{n} + 1)a, \\
A^\dagger = f(\hat{n})a^\dagger = a^\dagger f(\hat{n} + 1),
\] (2.2)

with \( f(\hat{n}) \) being an operator valued function of the number operator \( \hat{n} = a^\dagger a \) so that their action on the Fock state are given by

\[
A|n\rangle = \sqrt{n}f(n)|n - 1\rangle, \quad A^\dagger|n\rangle = \sqrt{n + 1}f(n + 1)|n + 1\rangle.
\] (2.3)

Thus, if a Hamiltonian can be factorized as \( A^\dagger A = a^\dagger a f^2(a^\dagger a) \), the corresponding eigenvalues can be expressed as \( e_n = \hbar\omega f^2(n) \). So far, the ladder operators have been defined generally. We now introduce a specific model, namely the harmonic oscillator in the minimal length scenario

\[
H = \frac{p^2}{2m} + \frac{m\omega^2}{2}X^2 - \hbar\omega \left( \frac{1}{2} + \frac{\tau}{4} \right),
\] (2.4)

which satisfies (2.1). In order to find the ladder operators of this oscillator, one needs to find the eigenvalues. However, since the Hamiltonian (2.4) is non-Hermitian, its solution is not so straightforward. Nevertheless, it is possible to obtain real eigenvalues if one applies the standard techniques of \( PT \)-symmetric non-Hermitian systems [73, 74] and the notions of pseudo-Hermiticity [75, 76]. The idea is to construct an isospectral Hermitian Hamiltonian \( h \) corresponding to the non-Hermitian Hamiltonian \( H \) given by (2.4), by considering the non-Hermitian Hamiltonian \( H \) to be pseudo-Hermitian. More precisely, if the Hermitian and the non-Hermitian Hamiltonians are related by the similarity transformation \( h = \eta H\eta^{-1} \), and
one finds the corresponding metric $\eta$ such that $\eta^\dagger \eta$ is a positive definite operator, then the eigenvalues of $h$ belong to the similar class to those of $H$. In our case, the metric is found to be $\eta = (1 + \tau p^2)^{-1/2}$ so that the corresponding Hermitian Hamiltonian turns out to be

$$h = \eta H \eta^{-1} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \frac{\omega\tau}{4\hbar} (p^2 x^2 + x^2 p^2 + 2xp^2x) - \hbar\omega \left(\frac{1}{2} + \frac{\tau}{4}\right) + \mathcal{O}(\tau^2).$$

(2.5)

The eigenvalues of $h$ (2.5) and $H$ (2.4) were computed in [31, 46, 71] as $E_n = \hbar\omega n (C + Dn) + \mathcal{O}(\tau^2)$ upto the first order of $\tau$ in Rayleigh-Schrödinger perturbation theory. For more details in this regard, we refer the readers to [30]. A general extension of pseudo-Hermitian systems for higher dimensional models is discussed in [77], which the readers may look at for their general interest. Coming back to our discussion, the ladder operators of the harmonic oscillator associated with the minimal length scenario can be defined by (2.2) and (2.3) for

$$f(n) = \sqrt{C + Dn}, \quad C = 1 + D, \quad D = \frac{\tau}{2},$$

(2.6)

where $\tau$ is the deformation parameter with the degree of noncommutativity being controlled by it. Having understood the ladder operators of the harmonic oscillator in the minimal length case, we can now set out for the construction of atom laser in the given scenario.

### 3 Atom Laser in Minimal Length Scenario

Before discussing the case corresponding to the minimal length, first let us briefly recall the original linear model for the atom laser. The simplest case is to consider a two-level atom having states, say, $|g\rangle$ and $|e\rangle$, where the initial Bose-Einstein condensation (BEC) occurs in the trapped state $|g\rangle$. Whereas, the state $|e\rangle$ remains unconfined from the magnetic trap, but is coupled to the trapped state $|g\rangle$ by a one-mode squeezed optical field tuned near the $|g\rangle \rightarrow |e\rangle$ transition. Within this setup an experiment was performed at MIT [78], where a BEC in a superposition of trapped and untrapped hyperfine states by a radio frequency (rf) pulse was created. The corresponding second quantized Hamiltonian for a Bose gas containing $N$ identical two-level atoms in the rotating wave approximation acquires the form [79]

$$H = \omega_x a^\dagger a + \omega_0 b_g^\dagger b_e + \Omega \left(ab_g^\dagger b_e + a^\dagger b_g b_e\right),$$

(3.1)

with $\hbar = 1$. Here, $b_g^\dagger (b_g)$ and $b_e^\dagger (b_e)$ are the creation (annihilation) operators of the atoms for the magnetically trapped state $|g\rangle$ and the untrapped state $|e\rangle$, respectively, with the level difference being $\omega_0$. Whereas, $a^\dagger$ and $a$ are the creation and annihilation operators of the rf pulse of frequency $\omega_x$, which couples $b_g^\dagger (b_g)$ with $b_e^\dagger (b_e)$ through the matrix element $\Omega = \sqrt{\omega_x/2\epsilon_0 V}$, where $\epsilon_0$ is the permittivity of the vacuum and $V$ is the effective mode volume. The Hamiltonian (3.1) can be simplified further if we ignore the slow change of the large number of the initial condensate atoms $N_c$ in the ground state, i.e. if we take the Bogoliubov approximation [80]. Under this approximation, one can replace both of the ladder operators of the ground state $b_g^\dagger, b_g$ by the $c$-number $\sqrt{N_c}$, so that the Hamiltonian (3.1) is transformed to

$$H = \omega \left(a^\dagger a + b^\dagger b\right) + \Omega\sqrt{N_c} \left(ab_g^\dagger + a^\dagger b_g\right),$$

(3.2)

where $\omega_x = \omega_0 = \omega$ as well as $b_e$ has been redefined to $b$ to simplify our notation. The minimal length scenario can restructure the Hamiltonian further, if we consider that the rf
pulse is deformed according to the notions of the minimal length model. This is equivalent to replacing the usual ladder operators $a, a^\dagger$ by the deformed operators $A, A^\dagger$ defined by (2.3) along with (2.6). Physically this means that one needs to first create a deformed electromagnetic field and then let it interact with the atomic Hamiltonian under consideration. Note that the type of deformation we obtain from the minimal length structure is nothing but a nonlinear deformation, as was also discussed before. Therefore, by using standard techniques of generating nonlinear light, one can produce a deformed rf pulse for our purpose. Nevertheless, within this framework, the effective Hamiltonian takes the form

$$H_d = \omega \left( A^\dagger A + b^\dagger b \right) + \Omega N \left( a b^\dagger + A^\dagger b \right).$$  \hspace{1cm} (3.3)$$

Now, if we replace the generalized ladder operators $A, A^\dagger$ in terms of $a, a^\dagger$ as defined by (2.2) followed by a substitution of the function $f(n)$ from (2.6), we obtain

$$H = \omega \left[ \left( C + Da^\dagger a \right) a^\dagger a + b^\dagger b \right] + \Omega N \left[ a b^\dagger \sqrt{C + Da^\dagger a} + \sqrt{C + Da^\dagger aa^\dagger b} \right].$$ \hspace{1cm} (3.4)

In order to achieve an approximate solution of the Hamiltonian (3.4), let us introduce the slowly varying polariton operators $M(t)$ and $N(t)$ \cite{21, 81}

$$M(t) = \frac{1}{\sqrt{2}} e^{-2i\Omega \sqrt{N_c} t} [a(t) + b(t)], \quad N(t) = \frac{1}{\sqrt{2}} e^{2i\Omega \sqrt{N_c} t} [a(t) - b(t)],$$ \hspace{1cm} (3.5)

where the operators $M(t), N(t)$ satisfy the conventional commutation relations $[M, M^\dagger] = [N, N^\dagger] = 1$, with the rest of the commutators being vanished. The inverse transformation of (3.5) reads as

$$a(t) = \frac{1}{\sqrt{2}} \left[ e^{2i\Omega \sqrt{N_c} t} M(t) + e^{-2i\Omega \sqrt{N_c} t} N(t) \right],$$

$$b(t) = \frac{1}{\sqrt{2}} \left[ e^{2i\Omega \sqrt{N_c} t} M(t) - e^{-2i\Omega \sqrt{N_c} t} N(t) \right],$$ \hspace{1cm} (3.6)

which when replaced in (3.4) and the oscillating terms like $M^\dagger N, N^\dagger M$, etc. are being neglected, we arrive at

$$H = \left\{ \frac{\omega}{4} (2C + D + 2) + \frac{\alpha_1}{2D} (D + 4C) \right\} M^\dagger M + \left\{ \frac{\omega}{4} (2C + D + 2) - \frac{\alpha_1}{2D} (D + 4C) \right\} N^\dagger N$$

$$+ \frac{D}{4} \left( \omega + \frac{2\alpha_1}{D} \right) (M^\dagger M)^2 + \frac{D}{4} \left( \omega - \frac{2\alpha_1}{D} \right) (N^\dagger N)^2 + 2\alpha_2 M^\dagger M N^\dagger N,$$ \hspace{1cm} (3.7)

with $\alpha_1 = \Omega D \sqrt{N_c}/(2\sqrt{C}), \alpha_2 = \omega D/2$. Here, $M(t)$ and $N(t)$ have been abbreviated to $M$ and $N$ for our convenience and, thus, wherever $M$ and $N$ appear they are taken to be time dependent. Note that, we consider our system to be a closed one which has no interaction with the environment. Although the effects of the environment which appears as fluctuation and dissipation are very important in any realistic model but we can ignore all of these effects since we study the system in a period of time which is much smaller than the inverse of all damping rates of the system. Besides, in order to avoid any complication that may arise at this stage of the study, we take the rotating wave approximation to neglect the oscillating terms like $M^\dagger N, N^\dagger M$, etc. This is also the reason why we can use the Heisenberg equation of motion to solve our system. The solution of the Heisenberg equation of motion for the operators $M(t)$ and $N(t)$

$$\frac{dM(t)}{dt} = i[H, M(t)], \quad \frac{dN(t)}{dt} = i[H, N(t)],$$ \hspace{1cm} (3.8)
are given by
\[ M(t) = M(0)e^{-i\left(\frac{\omega}{2}(C + D + 1) + \frac{2\alpha_1}{D}(C + D) + \frac{D}{2}\left(\omega + \frac{2\alpha_1}{D}M^\dagger M\right) + \omega DN^\dagger N\right)t}, \]

(3.9)
\[ N(t) = N(0)e^{-i\left(\frac{\omega}{2}(C + D + 1) - \frac{2\alpha_1}{D}(C + D) + \frac{D}{2}\left(\omega - \frac{2\alpha_1}{D}M^\dagger M\right) + \omega DN^\dagger N\right)t}. \]

(3.10)

By replacing (3.9) and (3.10) into (3.6), we obtain the expression of the light field operator as follows
\[ a(t) = \frac{1}{\sqrt{2}}\left[M(0)e^{-i\Theta_M t} + N(0)e^{-i\Theta_N t}\right], \]

(3.11)
where
\[ \Theta_M = \frac{\omega}{2}(C + D + 1) + \alpha_3 + (\alpha_1 + \alpha_2)M^\dagger M + 2\alpha_2N^\dagger N, \]

(3.12)
\[ \Theta_N = \frac{\omega}{2}(C + D + 1) - \alpha_3 + (\alpha_2 - \alpha_1)M^\dagger M + 2\alpha_2N^\dagger N, \]

(3.13)
with \( \alpha_3 = \alpha_1 + \Omega\sqrt{N_e}(2 + \sqrt{C}) \). Therefore, we obtain the atom laser solution of the radiation field given by (3.11), which can be analyzed further to verify its squeezing properties.

4 Quantum Statistical Squeezing Properties

In this section we shall explore the squeezing properties of the atom laser constructed in the minimal length scenario. First, we shall study the effects analytically and, then, we shall illustrate them numerically. Let us assume that at the beginning the atoms are in the ground state so that the radiation field is initially in a Glauber coherent state \( |\psi(0)\rangle_{\text{field}} = |\alpha\rangle_a \), while the initial untrapped excited state is a vacuum state \( |\psi(0)\rangle_{\text{atom}} = |0\rangle_b \). Thus, the initial state of the atoms-radiation system is described as
\[ |\psi(0)\rangle = |\alpha\rangle_a \otimes |0\rangle_b, \]
where \( |\alpha\rangle_a \) and \( |\alpha\rangle_b \) are the coherent states associated with the operators \( a(t) \) and \( b(t) \), respectively. In order to compute the quantum statistical properties, we first require to calculate the expectation value of \( a(t) \), which is computed as
\[ \langle \psi(0)|a(t)|\psi(0)\rangle = \frac{\alpha}{2}e^{-|\alpha|^2}\left(\beta_+e^{-i\delta_+} + \beta_-e^{-i\delta_-}\right), \]

(4.1)
where
\[ |\alpha|^2 \cos\left(\frac{3\alpha_2 + \alpha_1}{2}t\right) \cos\left(\frac{\alpha_2 \pm \alpha_1}{2}t\right), \]

(4.2)
\[ \delta_\pm = \left[\frac{\omega}{2}(C + D + 1) \pm \alpha_3\right]t + |\alpha|^2 \sin\left(\frac{3\alpha_2 + \alpha_1}{2}t\right) \cos\left(\frac{\alpha_2 \pm \alpha_1}{2}t\right). \]

(4.3)
A similar type of computations yield the other useful expectation values, such as
\[ \langle a^\dagger(t)a(t) \rangle = \frac{|\alpha|^2}{2}\left[1 + e^{-|\alpha|^2[1 - \cos(\alpha_1 t) \cos(\alpha_2 t)]} \cos\left(2t\alpha_3 - |\alpha|^2 \sin(\alpha_1 t) \cos(\alpha_2 t)\right)\right]. \]

(4.4)
\[ \langle a^\dagger(t)a(t)a^\dagger(t)a(t) \rangle = |\alpha|^2 \langle a^\dagger(t)a(t) \rangle - \frac{|\alpha|^4}{8} \left[ 1 - e^{-|\alpha|^2 \{ 1 - \cos(2\alpha_1 t) \cos(2\alpha_2 t) \} } \right] \times \cos \left( 2\alpha_3 - |\alpha|^2 \sin(\alpha_1 t) \cos(\alpha_2 t) \right) \]. \quad (4.5) \]

Now, using the result of (4.1) it is easy to calculate
\[ \langle a(t)^2 \rangle = \frac{\alpha^2}{4} e^{-|\alpha|^2} \left( \gamma_+ e^{-i\epsilon} + \gamma_- e^{-i\epsilon} + \xi e^{-i\lambda} \right), \quad (4.6) \]

where
\[ \gamma_\pm = e^{-|\alpha|^2/2} \left[ \cos(4\alpha_2 t) + \cos \{ 2t(\alpha_2 \mp \alpha_1) \} \right], \quad (4.7) \]
\[ \epsilon_\pm = \frac{1}{2} \left[ 4t \left\{ \frac{\omega}{2} (C + D + 1) \mp \alpha_3 \right\} + |\alpha|^2 \{ \sin(4t\alpha_2) + \sin(2t(\alpha_2 \mp \alpha_1)) \} \right], \quad (4.8) \]
\[ \xi = e^{-|\alpha|^2/2} \left[ \cos \left\{ 2t(\alpha_1 - 3\alpha_2) \right\} + \cos \left\{ 2t(\alpha_1 + 3\alpha_2) \right\} \right], \quad (4.9) \]
\[ \lambda = 4t \left\{ \frac{\omega}{2} (C + D + 1) \mp \alpha_3 \right\} - \frac{|\alpha|^2}{2} \{ \sin(2t(\alpha_1 - 3\alpha_2)) - \sin(2t(\alpha_1 + 3\alpha_2)) \}. \quad (4.10) \]

Using (4.4) and (4.5), we can calculate the Mandel parameter [82]
\[ Q = \frac{\langle (a^\dagger(t)a(t))^2 \rangle - \langle a^\dagger(t)a(t) \rangle^2}{\langle a^\dagger(t)a(t) \rangle} - 1, \quad (4.11) \]

which determines the photon statistics of the quantized radiation field. \( Q = 0 \) corresponds to the case of Poissonian distribution, whereas \( Q > 0 \) and \( Q < 0 \) imply to the super-Poissonian and sub-Poissonian cases, respectively. Thus, for a squeezed state of light one should expect the sub-Poissonian (\( Q < 0 \)) nature of the Mandel parameter. Fig. 1 shows the numerical behavior of the time evolution of the Mandel parameter for two different values of the deformation parameter. It is evident from the plots that the Mandel parameter is always negative and, thus, indicating the squeezed behavior of the radiation field.

In order to study the quadrature squeezing properties, we first define the quadrature operators
\[ y = \frac{\langle a^\dagger(t) + a(t) \rangle}{\sqrt{2}} \quad \text{and} \quad z = i \frac{\langle a^\dagger(t) - a(t) \rangle}{\sqrt{2}}, \]

which can be interpreted as the dimensionless position and momentum operators, respectively. It is well-known that for a well-behaved coherent state of light the Heisenberg uncertainty relation is minimized, i.e. \( (\Delta y)^2(\Delta z)^2 = 1/4 \). Whereas, for squeezed light one of the quadratures must be squeezed, so that either \( (\Delta y)^2 < 1/2 \) or \( (\Delta z)^2 < 1/2 \) should hold. These squeezing conditions can also

**Fig. 1** Time evolution of the Mandel parameter for \( |\alpha| = 0.5, \Omega = 0.1, \omega = 1, N_c = 10^5 \) (a) \( D = 0.1 \) (b) \( D = 0.4 \)
be expressed as $2(\Delta y)^2 - 1 < 0$ and $2(\Delta z)^2 - 1 < 0$, so that we can write the relations elegantly for our case as follows

$$S_1 = 2(\Delta y)^2 - 1 = 2 \left\{ \langle a(t) a(t) \rangle + \text{Re}[\langle a^2(t) \rangle] - \text{Re}[\langle a(t) \rangle^2] - |\langle a(t) \rangle|^2 \right\}, \quad (4.12)$$

$$S_2 = 2(\Delta z)^2 - 1 = 2 \left\{ \langle a(t) a(t) \rangle - \text{Re}[\langle a^2(t) \rangle] + \text{Re}[\langle a(t) \rangle^2] - |\langle a(t) \rangle|^2 \right\}. \quad (4.13)$$

Therefore, the radiation field can be said to be quadrature squeezed if either $S_1$ or $S_2$ is negative. A numerical analysis can be performed easily with the help of the following expressions

$$\text{Re}\left[ \langle a(t) \rangle^2 \right] = \frac{|\alpha|^2}{4} e^{-2|\alpha|^2} \left[ 2\beta_+ \beta_- \cos(\delta_+ + \delta_- - 2\theta) + \beta_+^2 \cos(2\delta_+ - 2\theta) \
+ \beta_-^2 \cos(2\delta_- - 2\theta) \right], \quad (4.14)$$

$$\text{Re}\left[ \langle a^2(t) \rangle \right] = \frac{|\alpha|^2}{4} e^{-|\alpha|^2} \left[ \gamma_+ \cos(2\theta - \epsilon_+) + \gamma_- \cos(2\theta - \epsilon_-) + \xi \cos(2\theta - \lambda) \right], \quad (4.15)$$

which have been obtained from (4.1) and (4.6), respectively. Here the parameter of the coherent state $\alpha$ is decomposed as $\alpha = |\alpha| e^{i\theta}$. Thus, by replacing (4.14) and (4.15) into (4.12) and (4.13) we can study the quadrature squeezing behavior of the radiation field easily, which is demonstrated in Fig. 2. In panel (a), we demonstrate the time-evolution of the squeezing parameter $S_1$ corresponding to the $y$ quadrature, whereas panel (b) shows the behavior of that of $S_2$, which is associated to the $z$ quadrature. We notice that both of the quadratures become squeezed almost periodically, of course, the uncertainty principle restricts them not to be squeezed simultaneously. The same phenomena happen for other values of the deformation parameter $D$, which we do not present here. The straight lines passing through the origin in Fig. 2 represent the undeformed case, which indicates that there is no quadrature squeezing in such a scenario. This is also the case for Fig. 1, which we have not shown in the plots, since in that case we need to expand the scale, and in the large scale one may not be able to see the original behavior of the deformed case clearly.

5 Concluding Remarks

The existence of minimal length is supported by many quantum theories of gravity and, thus, noncommutative theories have become popular as one of the candidates of quantum
gravity. However, it is also known by now that the noncommutative theories are not directly compatible with standard quantum mechanics, but with its slight deformed version, without violating any basic laws of quantum mechanics. Therefore, it has become an important area of research to study the deformed quantum mechanical phenomena in different areas of quantum optics and information theory, where the effects of quantum mechanics are easily visible. As it turns out that people do not find entirely new phenomena, but some interesting consequences follow up. For instance, people often end up with systems having more degrees of freedom or control by which one can deal with the system in a much better way. In this paper, we have explored the atom laser for the deformed quantum mechanical structure emerging from the noncommutative framework. The most interesting outcome of this study is to find a squeezed atom laser without the help of any external squeezed light signal as its input. This can not be achieved with the ordinary quantum mechanical atom laser. In order to obtain a squeezed atom laser, people have to take either the nonlinear interaction between the atom and the beam, or one has to send the squeezed light in its input. However, in our system we do not require any of them, but the squeezing properties are inherited by the model by its construction. On top of that, our model is compatible with quantum gravity theories.

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