Gravitational Bremsstrahlung with tidal effects in the post-Minkowskian expansion

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We compute the mass and current quadrupole tidal corrections to the four-momentum and energy flux radiated during the scattering of two spinless bodies, at leading order in $G$ and at all orders in the velocities, using the effective field theory worldline approach. In particular, we derive the conserved stress-energy tensor linearly coupled to gravity generated by the two bodies, including tidal fields, and the waveform in direct space. The integral is solved using scattering amplitude techniques. We show that our expressions are consistent with existing results up to the next-to-next-to-leading order in the post-Newtonian expansion.

Introduction – The direct detection of gravitational waves from binary black holes [1] and neutron stars [2] has opened a new way to test gravity in the strong-field regime [3] and explore fundamental physics [4]. An important target of current and future observations is the measurement of tidal deformations during the coalescence of compact objects [5–15], which may shed light on the internal structure of neutron stars [16], the nature of black holes [17] or the existence of more exotic astrophysical objects [18–20].

Tidal deformations affect the conservative two-body dynamics as well as the emitted energy in gravitational waves. They have been studied utilising different analytical techniques, most notably the post-Newtonian (PN) expansion [21–26], the effective-one-body approach [27–29], Non-Relativistic-General-Relativity (NRGR) [30–35] and the self-force formalism [36–39] (see [40] for a review).

Another technique that has been employed to study the gravitational two-body problem is the post-Minkowskian (PM) method [41–49], consisting in expanding the gravitational dynamics for small interactions, while keeping the velocities fully relativistic. It has been recently subject of great interest and activity, in particular in association with the effective-one-body approach [48–54], scattering amplitude techniques [55–72], and worldline approaches [73–87]. Tidal effects have been studied with the PM expansion in [88–99]. These developments concern the scattering of two bodies moving on unbounded orbits but computed observables can be extended to the case of bound orbits by applying the so-called “boundary-to-bound” (B2B) dictionary, consisting in an analytic continuation between hyperbolic and elliptic motion [100–103].

A long-standing and, until recently, unsolved problem was the calculation of the four-momentum radiated in gravitational waves—the so-called gravitational Bremsstrahlung—during the scattering of two spinless bodies, at leading PM order, i.e. at $\mathcal{O}(G^5)$. This was finally obtained very recently in [104, 105] via the amplitude-based method of [59], in [69] using the eikonal approach and in [106] by a classical effective field theory (EFT) worldline approach. (See also [96, 107–116] for previous work on radiation effects. Earlier pioneering studies include [47, 117–122]. Moreover, see [123, 124] for conservative and radiative effects in QED.) Crucially, these calculations strongly benefited from several computational tools developed in the high-energy community [125], such as reduction to master integrals by Integration-by-Parts (IBP) identities [126–128] and differential equations [129–132] to solve the latter using the near-static regime as initial conditions.

In particular, in [106] two of us showed that it is possible to use these tools to directly compute radiated observables in the PM expansion without going through the classical limit of scattering amplitudes. Indeed, the emitted four-momentum was obtained by phase-space integration of the graviton momentum weighted by the modulo squared of the classical radiation amplitude [115, 116], the latter being derived by matching to the conserved stress-energy tensor linearly coupled to gravity, generated by localized sources. The phase-space integral was then recasted as a 2-loop integral that we solved with the aforementioned techniques.

In this letter we use the same approach but we go beyond the minimally coupled case, and we compute for the first time the effect of tidal deformations on the four-momentum radiated into gravitational waves during the scattering of the two bodies. From this, extending the technique recently developed in [103], we also compute the tidal corrections to the emitted energy flux, which is valid for both open and closed orbits. We focus on the leading tidal contributions to the orbital dynamics, i.e. to quadrupolar deformations, but the extension to higher multipoles can be straightforwardly obtained using the same approach.

The article is organized as follows. We first define the Feynman rules in the case of tidal couplings, which will allow us to derive the stress-energy tensor linearly coupled to gravity, and the waveform in direct space, at leading PM order. From the stress-energy tensor, we compute, using reverse unitarity, the total four-momentum radiated into gravitational waves, and from this the emitted flux. We then use the B2B dictionary [100–103] to
check our results with PN derivations [26].

**Leading PM tidal effects** – We consider the scattering of two gravitationally interacting spinless bodies with mass \(m_1\) and \(m_2\), approaching each other from infinity. Using the mostly minus metric signature, setting \(\hbar = c = 1\) and defining the Planck mass as \(m_{\text{Pl}} \equiv 1/\sqrt{32\pi G}\), the total action describing the dynamics with tidal effects reads

\[
S = -2m_{\text{Pl}}^2 \int d^4x \sqrt{-g} R + S_{\text{pp}} + S_{\text{tidal}} .
\]  

At leading order in their size, the bodies are described by point-particle actions,

\[
S_{\text{pp}} = -\sum_{a=1,2} \frac{m_a}{2} \int d\tau_a g_{\mu\nu}(\tau_a) U_\mu^a(\tau_a) U_\nu^a(\tau_a) ,
\]  

where \(\tau_a\) and \(U_\mu^a\equiv dx_\mu^a/d\tau_a\) (with \(U_\mu^aU_\nu^a=1\)) are, respectively, the proper time, and the four-velocity of body \(a\). Note that we have used the Polyakov-like form of the einbein to unity, which simplifies the gravitational coupling to the matter sources [76, 133, 134].

Tidal effects are included by augmenting the point-particle action with non-minimal worldline couplings involving higher-order derivatives of the gravitational field [30]. At leading PM order, only linear tidal deformations, i.e., those whose response is linear in the external gravitational field, are relevant. These are described by couplings quadratic in the Weyl tensor \(C_{\mu\alpha\nu\beta}\) evaluated at the particle position. The Weyl tensor can be decomposed in terms of the gravito-electric and gravito-magnetic fields, defined as

\[
E_{\mu\nu} \equiv C_{\mu\alpha\nu\beta} U^\alpha U^\beta , \quad B_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu} C_{\alpha\beta\gamma\delta} U^\nu U^\delta ,
\]

where \(\epsilon_{\alpha\beta\gamma\mu}\) is the Levi-Civita tensor. At lowest-order in derivatives, and restricting to parity-even operators for symmetry reasons, the action describing tidal deformations is given by

\[
S_{\text{tidal}} = \sum_{a=1,2} \int d\tau_a \left( c_{E_2} E_{\mu\nu}^a E_{\mu\nu}^a + c_{B_2} B_{\mu\nu}^a B_{\mu\nu}^a \right) ,
\]

where \(c_{E_2}\) and \(c_{B_2}\) are Wilson coefficients related to the relativistic Love numbers \(k_0^{(2)}\) and \(j_0^{(2)}\) [28], respectively as \(c_{E_2} = \frac{1}{3} j_0^{(2)} R_a^2 / G\), \(c_{B_2} = \frac{1}{2} j_0^{(2)} R_a^2 / G\), with \(R_a\) the radius of the object \(a\). Tidal operators can be equally defined by replacing the the Weyl tensor in eq. (3) with the Riemann tensor: the difference can be removed by field redefinitions, see e.g. [24, 30, 135]. Here we will use the Riemann tensor because it leads to simpler calculations. In full generality, one could also add to eq. (4) operators including spatial derivatives, orthogonal to the worldline of the body, of the gravito-electric or gravito-magnetic field, as well as time derivatives along the worldline [28]. Higher spatial derivatives describe higher-order multipolar deformations of the objects while time derivatives account for the time dependence of the Wilson coefficients, see e.g. [7, 34].

Following [106, 116], our first goal is to compute the stress-energy tensor \(T^{\mu\nu}\) defined as the linear term sourcing the gravitational field in the effective action [30, 136, 137], i.e.,

\[
\Gamma[x_a, h_{\mu\nu}] = -\frac{1}{2m_{\text{Pl}}} \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x) ,
\]

with \(h_{\mu\nu} \equiv m_{\text{Pl}}(g_{\mu\nu} - \eta_{\mu\nu})\), which includes contributions from both the bodies and the gravitational self-interactions. To do so, we use a matching procedure consisting in expanding the action (1) for small \(h_{\mu\nu}\) and computing perturbatively all Feynman diagrams with one external graviton. The stress-energy tensor is obtained by matching this result with the one computed using eq. (5). To proceed, we need to introduce the Feynman rules.

Adding the usual de Donder gauge-fixing term to eq. (1), from the quadratic part of the gravitational action one can extract the graviton propagator,

\[
\frac{\delta_{\mu\nu}}{k^2} = \frac{i}{\eta_{\mu\nu}} F_{\mu\nu\rho\sigma} ,
\]

where \(F_{\mu\nu\rho\sigma} \equiv \eta_{\mu\nu}(\partial_{\rho} b_{\sigma} - \partial_{\sigma} b_{\rho})\). (The boundary conditions that specify the contour of integration in the complex \(k^0\)-plane are discussed in [116].) Furthermore, expanding the Einstein-Hilbert action in (1) at cubic order we can extract the cubic graviton vertex.

We also need to find the Feynman rules coming from the interaction of gravity with the external sources, i.e. the two bodies. These are of two types: minimal and tidal. For the former, from eq. (2) one sees that there is only one linear interaction vertex. As discussed in [116] (see also [76, 77]), we isolate the powers of \(G\) by expanding the position and velocity of the bodies around straight trajectories, i.e.,

\[
x_{\mu}(\tau_a) = b_{\nu}^a \tau_a + \delta^{(1)} x_{\mu}^a(\tau_a) + \ldots ,
\]

\[
U_{\mu}^a(\tau_a) = u_{\mu}^a + \delta^{(1)} u_{\mu}^a(\tau_a) + \ldots ,
\]

where \(u_{\mu}^a\) is the (constant) asymptotic incoming velocity and \(b_{\nu}^a\) is the body displacement orthogonal to it, \(b_{\nu}^a \cdot u_{\mu}^a = 0\), while \(\delta^{(1)} x_{\mu}^a\) and \(\delta^{(1)} u_{\mu}^a\) are respectively the deviation from the straight trajectory and constant velocity of body \(a\) at order \(G\), induced by the gravitational interaction. With this expansion we obtain the usual Feynman rules for the leading and next-to-leading PM-order graviton coupling in the point-particle case [116], respectively represented by the diagrams

\[
\begin{align}
\begin{tikzpicture}[baseline=(a.base),scale=0.8]
  \node[circle,draw,inner sep=1pt] (a) at (0,0) {$a$};
  \node[circle,draw,inner sep=1pt] (b) at (0,-1) {$\tau_a$};
  \node[circle,draw,inner sep=1pt] (c) at (1,-1) {$b_a$};
  \draw[->] (a) to (b);
  \draw[->] (b) to (c);
\end{tikzpicture}
\end{align}
\]

where a filled dot denotes a minimally-coupled particle evaluated using the straight worldline and the cross attached to the wiggly line is there to remind us that there is no propagator attached to the straight worldline. Their explicit expressions can be found in [106, 116].
Moreover, we need to provide the Feynman rules from tidal contributions. In this case, from eq. (4) there is no tidal coupling linear in the graviton. Tidal couplings of two gravitons to the body can be directly computed from the action using that

\[ M_{\mu\nu\alpha\beta}^{E}(\ell) = \frac{2\delta E_{\mu\nu}^{\ell}}{8\pi G \ell} \eta_{\mu\rho} u_{\alpha}^{\ell} u_{\beta}^{\ell} \]

\[ + (\ell \cdot u_{\alpha}) \eta_{\rho(\mu} \eta_{\nu)\beta} - 2(\ell \cdot u_{\alpha}) u_{\rho}^{\ell} \eta_{\mu(\rho} \eta_{\nu)\beta} \]

\[ M_{\mu\nu\alpha\beta}^{B}(\ell) = \frac{2\delta B_{\mu\nu}^{\ell}}{8\pi G \ell} \left[ \frac{1}{2} \rho_{\mu\rho\alpha\beta} \eta_{\beta}(\ell \cdot u_{\alpha}) - \eta_{\beta} u_{\rho}^{\ell} (\alpha \leftrightarrow \beta) \right], \]

where we use the flat metric \( \eta_{\mu\nu} \) to raise and lower indices. At leading PM order one obtains

\[ \tau_{a}^{\nu} = - \frac{i\delta^{2} S_{\text{tidal}}}{\delta h_{\mu\nu}(\ell_{1}) \delta h_{\alpha\lambda}(\ell_{2})} \equiv V_{\mu,\nu,\alpha,\lambda}(\ell_{1}, \ell_{2}), \]

where

\[ V_{\mu,\nu,\alpha,\lambda} = i \sum_{\chi = E, B} \sum_{a = 1, 2} \frac{c_{a}^{2}}{4m_{\chi}^{2}} \int d\tau_{\alpha} e^{i(\ell_{1} + \ell_{2}) \cdot (b_{a} + u_{\nu} \tau_{\alpha})} \Pi_{X_{a}}^{X_{\alpha,\beta},\nu,\mu,\lambda}, \]

with

\[ \Pi_{X_{a}}^{X_{\alpha,\beta},\nu,\mu,\lambda}(\ell_{1}, \ell_{2}) \equiv M_{\mu\nu\alpha\beta}(\ell_{1}) A_{X_{a}}^{X_{\alpha,\beta}}(\ell_{2}). \]

On the left-hand side of eq. (11), the square denotes a tidally-coupled particle evaluated using the straight worldline. We have verified that our expression agrees with that that can be read off from the 4-point amplitude at leading PM order obtained in Ref. [88].

**Stress-energy tensor with tidal effects** – The stress-energy tensor needed to compute the emitted four-momentum is given by the sum of the point-particle and tidal contributions, i.e.,

\[ \tilde{T}^{\mu\nu} = \tilde{T}_{pp}^{\mu\nu} + \tilde{T}_{\text{tid}}^{\mu\nu}, \]

where the tilde denotes the Fourier transform, \( \tilde{T}_{\text{tid}}^{\mu\nu} \) is the tidal contribution, which does not enter the calculation, the leading-order diagrams are represented in Fig. 1. Using the notation \( \int_{q} \equiv \int d^{4} q / (2\pi)^{4} \) and \( \delta^{(n)}(x) \equiv (2\pi)^{n} \delta(x) \), it can be written as

\[ \tilde{T}_{\text{tid}}^{\mu\nu}(k) = \frac{m_{1} m_{2}}{4m_{2}^{2}} \int_{q_{1}, q_{2}} \delta(q_{1} \cdot u_{1}) \delta(q_{2} \cdot u_{2}) \delta^{(4)}(k - q_{1} - q_{2}) \]

\[ \times \frac{e^{i\eta_{1} \cdot b_{1} + i\eta_{2} \cdot b_{2}}}{q_{1}^{2} q_{2}^{2}} \left[ \frac{1}{2} \left[ \tilde{t}_{\text{tid}}^{\mu\nu}(q_{1}, q_{2}) + t^{\mu\nu}_{\text{part}}(q_{1}, q_{2}) + t^{\mu\nu}_{\text{part}}(q_{1}, q_{2}) \right] \right], \]

where \( t_{\text{tid}}^{\mu\nu} \) is the contribution from diagram (a), \( t_{\text{part}}^{\mu\nu} \) from the same diagram but with the two particles exchanged and \( t_{\text{part}}^{\mu\nu} \) from (b). We defer the reader to Ref. [106, 116] for their explicit expressions.

The contribution of the tidal operators to the stress-energy tensor has no static piece. The leading PM term can be obtained from the diagram (c) in Fig. 1 and it is symmetric under exchange of the two particles. We obtain

\[ \tilde{T}_{\text{tid}}^{\mu\nu} = \frac{m_{1} m_{2}}{4m_{2}^{2}} \int_{q_{1}, q_{2}} \delta(q_{1} \cdot u_{1}) \delta(q_{2} \cdot u_{2}) \delta^{(4)}(k - q_{1} - q_{2}) \]

\[ \times \frac{e^{i\eta_{1} \cdot b_{1} + i\eta_{2} \cdot b_{2}}}{q_{1}^{2} q_{2}^{2}} \sum_{a = 1, 2} \sum_{X = E, B} \frac{m_{2}^{4} \delta^{2} S_{\text{tidal}}}{4m_{2}^{2} \Pi_{X_{a}}^{X_{\alpha,\beta},\nu,\mu,\lambda}(q_{1}, q_{2})}, \]

with \( t_{\text{tid}}^{\mu\nu} = -2 \frac{c_{a}^{2}}{m_{a}} \eta^{\mu\nu} \eta^{\alpha\beta} \Pi_{\alpha,\beta,\mu,\lambda}^{X_{a}}(u_{\nu}^{1} u_{\alpha}^{2} - \eta^{\alpha\lambda} / 2) \), and an analogous formula for \( (1 \leftrightarrow 2) \). The explicit expressions of \( t_{\text{tid}}^{\mu\nu} \) and \( t_{\text{part}}^{\mu\nu} \) after use of eq. (13), as well as the calculation of the waveform in direct space, are given in the Supplemental Material (SM).

Note that, because of scaling arguments, the tidal contribution vanishes in the soft limit \( \omega \rightarrow 0 \). As a consequence, at this order in \( G \) tidal fields leave no trace on the gravitational wave memory. (See the SM for details.) Since the emitted angular momentum, \( J_{\text{rad}} \), at \( O(G^{2}) \) is proportional to the gravitational wave memory \[112], we conclude that there are no tidal effects on the emission of angular momentum at this order. On the other hand, at leading-PM-order the radiation reaction on the scattering angle is related to the radiated angular momentum by \( \chi_{\text{rad}} = \frac{1}{2} \chi_{\text{cons}}^{\alpha,\beta} J_{\text{rad}}^{\alpha,\beta} \) \[112, 114, 138], where the leading-order conservative contribution to the scattering angle, \( \chi_{\text{cons}}^{\alpha,\beta} \), is of \( O(G) \). As a consequence, \( \chi_{\text{rad}} \) is unaffected by tidal effects at \( O(G^{3}) \).

**Radiated four-momentum** – The derivation of the emitted linear momentum closely follows the procedure presented in Ref. [106]. In particular, the emitted four-momentum is given as an integral over phase space of the outgoing graviton momentum \( k^{\mu} \) weighted by the probability of one graviton emission, which here is given by the square of the total stress-energy tensor from eq. (14). Although we use a quantum mechanical language, this
quantity is well defined classically [31, 106]. Defining \( \delta_+(k^2) \equiv \theta(\pm k^0)\delta(k^2) \) we obtain, for the leading-order contribution from the tidal effects to the radiated momentum,

\[
P_{\text{tid}}^\mu = \frac{1}{2m^2} \int_k \delta_+(k^2) k^\mu \text{Re} \left[ \mathcal{T}_{\mu \alpha \beta \rho}^e \mathcal{P}_{\alpha \beta \rho} \mathcal{T}_{\text{tid}}^\mu \right] .
\]

From the relation with tidal Love numbers below eq. (4), for \( R_a \sim Gm_a \), the contribution quadratic in \( \mathcal{T}_{\text{tid}} \) is further suppressed by \( \mathcal{O}(G^4) \) and is thus neglected.

Following [106], we can interpreted the phase-space delta function as a cut propagator, so that the integrand reproduces a vacuum-to-vacuum diagram with a cut, pictorially represented as

\[
\begin{array}{c}
\delta_+(k^2) \text{Re} \left[ \mathcal{T}_{\mu \alpha \beta \rho}^e \mathcal{P}_{\alpha \beta \rho} \mathcal{T}_{\text{tid}}^\mu \right] = 0.1. \\
\end{array}
\]

where, using eqs. (15) and (16) for the stress-energy tensor, the three topologies come from considering the contributions from \( \text{Re} \left[ \mathcal{F}_{\mu \alpha \beta \rho}^T_{\rho \sigma} \mathcal{P}_{\alpha \beta \rho} \mathcal{T}_{\text{tid}}^\mu \right] \) and \( \text{Re} \left[ \mathcal{F}_{\mu \alpha \beta \rho}^T_{\rho \sigma} \mathcal{P}_{\alpha \beta \rho} \mathcal{T}_{\text{tid}}^\mu \right] \), respectively. Notice that the \( H \) diagram is absent, because there are no tidal interations linear in \( h_{\mu\nu} \).

We can now recast the problem of computing the emitted momentum as evaluating a cut 2-loop integral followed by a 2d Fourier transform. In particular, the emitted four-momentum can be decomposed without loss of generality along \( u_1^\mu \equiv (u_1^0 - \gamma u_1^0) / (1 - \gamma^2) \), \( u_2^\mu \equiv (u_2^0 - \gamma u_2^0) / (1 - \gamma^2) \) (satisfying \( u_1 \cdot u_2 = \delta_{ab} \)), with \( \gamma \equiv m_1 / m_2 \) and \( b^\mu \). By the symmetries of the integrand, one can show that the component along \( b^\mu \) vanishes, so that the momentum can be written as

\[
P_{\text{tid}}^\mu = \frac{15\pi G^3 m_1^2 m_2^2}{64|b|^2} \sum_X \left[ \frac{C_X^2}{m_1} \left( \mathcal{E}_X^\mu u_1^\mu + \mathcal{F}_X^\mu u_2^\mu \right) + (1 \leftrightarrow 2) \right].
\]

The functions \( \mathcal{E}_X \) and \( \mathcal{F}_X \) inside the brackets depend only on \( \gamma \) and can be expressed as 2d Fourier transforms of cut 2-loop integrals, \( \mathcal{I}_X^\mu(\gamma) \) and \( \mathcal{J}_X^\mu(\gamma) \). For instance,

\[
\mathcal{E}_X(\gamma) = |b|^2 \int_q \delta(q \cdot u_1) \delta(q \cdot u_2) e^{iq \cdot b} (-q^2)^{5/2} \mathcal{I}_X(\gamma),
\]

and analogously for \( \mathcal{F}_X(\gamma) \). The explicit expressions of the 2-loop integrals are given in the SM.

Making use of reverse unitarity [139–142], we can use IBP identities to express the 2-loop integrals \( \mathcal{I}_X^\mu, \mathcal{J}_X^\mu \) as linear combinations of simpler master integrals. We perform this reduction using the Mathematica package LiteRed [143, 144], finding that the three integrals defined in eqs. (4.13)–(4.15) of Ref. [106] form a complete base. (In the minimally coupled case we need a fourth integral, defined in eq. (4.16) of this reference. This comes from the \( H \) diagram, which here is absent.) These integrals can be solve using differential equation methods [125, 129–132, 145, 146]. Eventually, we find that

\[
\mathcal{E}_X = f_1 X + f_2 X \log \left( \frac{\gamma + 1}{2} \right) + f_3 X \frac{\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}},
\]

with \( f_1^X, f_2^X, f_3^X \) and \( \mathcal{F}_X \) given in Table I.

From eq. (19), one can compute the radiated energy in the center-of-mass frame from tidal effects, \( \Delta E_{\text{tid}} = P_{\text{tid}} \cdot u_{\text{c.m.}} \). Defining the total mass \( m \equiv m_1 + m_2 \), the symmetric mass ratio \( \nu \equiv m_1 m_2 / m^2 \) and \( \kappa(\nu, \gamma) \equiv E / m = \sqrt{1 + 2\nu(\gamma - 1)} \), where \( E \) is the incoming energy of the two-body system, this reads

\[
\Delta E_{\text{tid}} = \frac{15\pi G^7 m^8 \nu^2}{64|b|^2} \mathcal{G}(\mathcal{E}_X, \mathcal{F}_X),
\]

where

\[
\mathcal{G}(\mathcal{E}_X, \mathcal{F}_X) \equiv \sum_X \left[ \kappa_{X^2} \mathcal{E}_X + \lambda_{X^2}(\mathcal{F}_X - \mathcal{E}_X) \right],
\]

and we have introduced the dimensionless parameters [94]

\[
\begin{align*}
\lambda_{X^2} & \equiv \frac{1}{G^4 m^3} \left( \frac{c_{X^2} m_2}{m_1} + \frac{c_{X^2} m_1}{m_2} \right), \\
\kappa_{X^2} & \equiv \frac{1}{G^4 m^4} \left( \frac{c_{X^2}^2}{m_1^2} + \frac{c_{X^2}^2}{m_2^2} \right).
\end{align*}
\]
Expanding for small velocities \( v \equiv \sqrt{\gamma^2 - 1/\gamma} \), we find
\[
\mathcal{E}^E = 288 v^3 + \frac{2143}{7} v^5 + \frac{14542}{21} v^7 + \mathcal{O}(v^9),
\]
\[
\mathcal{E}^B = -98 v^5 + \frac{585}{4} v^7 + \mathcal{O}(v^9),
\]
\[
\mathcal{F}^E = 288 v^3 + 336 \gamma^5 + \frac{3027}{4} v^7 + \mathcal{O}(v^9),
\]
\[
\mathcal{F}^B = -210 v^3 - \frac{669}{4} v^7 + \mathcal{O}(v^9),
\]
(26)

which shows that the current (magnetic) quadrupole is 1PN order higher than the mass (electric) one, as expected.

Finally, the emitted energy from a two-body encounter can be used to derive the energy loss for closed orbits by the use of the B2B relation \([100–103], \Delta E^{(\text{closed})}(\gamma, J) = \Delta E^{(\text{open})}(\gamma, J) - \Delta E^{(\text{open})}(\gamma, -J)\), where the emitted energy on the right-hand side must be expressed in terms of the angular momentum \( J = |b|m v\sqrt{\gamma^2 - 1/\gamma} \) (with \( h = E/m \)) and analytically continued to bound orbits with \( h < 1 \), corresponding to \( \gamma = 1 + \frac{h^2 - 1}{2h} < 1 \). This yields
\[
\Delta E^{(\text{closed})} = \frac{15\pi G^7 m^1 \nu^3 (1 - \gamma^2)^{7/2}}{64J^7h^8} \bar{g}(\mathcal{E}^X, \mathcal{F}^X)
\]
(27)
where
\[
\bar{g}(\mathcal{E}^X, \mathcal{F}^X) = \frac{\gamma + 1/2}{\sqrt{1 - \gamma^2}},
\]
(28)

This expression is consistent with known results in the PN approximation.

**Radiated Flux** – The instantaneous flux is defined as \( F \equiv dE/dt \). Focusing on the tidal correction, \( F_{\text{tid}} \), and integrating this relation for half of the scattering trajectory, we obtain
\[
\Delta E_{\text{tid}}(\gamma) = 2 \int_{|b|}^{\infty} \frac{dr}{r} F_{\text{tid}}(r, \gamma).
\]
(29)

We have assumed that the expression of the flux is in isotropic gauge; thus, we have dropped the dependence on \( J \) in \( F_{\text{tid}} \). From eq. (22), the leading-order tidal contribution to the flux scales as \( G^7 \) so that its dependence on \( r \) is fully determined: \( F_{\text{tid}}(r, \gamma) \propto r^{-8} \). By integrating the right-hand side of eq. (29) with this ansatz, and using \( \dot{r} \) for straight orbits at this PM order, we find
\[
F_{\text{tid}}(r, \gamma) = \frac{G^7 m^8 \sqrt{\gamma^2 - 1}}{4h^8} \bar{g}(\mathcal{E}^X, \mathcal{F}^X),
\]
(30)

where \( \xi \equiv E_1 E_2 / E^2 \), and \( E_0 \) is the initial asymptotic energies of body \( a = 1, 2 \). This result extends the one for point-particles computed in \([103]\). As discussed there, due to the absence of a term higher in \( G \), the leading PM computation is insufficient to reconstruct the leading PN flux but it provides the full velocity–or reduced-energy–series to order \( G^3 \).

**Consistency check** – We can compare our result for small velocities to the emitted flux and energy in one period derived in the PN expansion in the large eccentricity limit, i.e. to leading order in large \( J \).

The tidal effects on the gravitational wave energy flux for spinless bodies has been computed up to the next-to-next-to-leading PN order in \([26]\) (see \([24, 25]\) for a derivation of the equations of motion and Hamiltonian in this case, respectively; see also \([96]\) for a calculation of the PM Hamiltonian and the emitted energy for quasi-circular orbits at leading PN order, with interactions cubic in the curvature and tidal effects). Although in that reference the results were given only for quasi-circular orbits, their authors have kindly provided us with an expression of the flux \( F^{(\text{PN})}_{\text{tid}} \) and the conserved energy \( E \) and angular momentum \( J \) for generic orbits, written in terms of \( r, \dot{r} \) and \( \phi \), respectively the two-body distance, the radial velocity and the angular velocity in the center-of-mass frame. We have used the expressions for \( E \) and \( J \) to replace \( \dot{r} = \dot{r}(r, E, J) \) and \( \phi = \phi(r, E, J) \) in the flux and we have computed the emitted energy for generic closed orbits by integrating it in the variable \( r \) over one period.

The resulting energy reduces to that given in \([26]\) for circular orbits. Moreover, it is consistent with the expansion eq. (26) (taking into account the factor of -2 according to eq. (27)). Since all the powers of \( \gamma \) in Table I intervene in this expansion, this is a rather nontrivial check of our calculation. Moreover, the PN flux \( F^{(\text{PN})}_{\text{tid}} \) coincides with the low-velocity expansion of eq. (30), up to total derivatives in the balance equations—the so-called Schott terms. Although the two fluxes are written in different gauges (in harmonic and isotropic gauge, respectively in Ref. \([26]\) and in eq. (30)) the gauge difference is 2PM orders higher and can be neglected. For the reader’s convenience, we report the explicit expression of the PM flux in the ancillary file submitted with the arXiv version of this article.

**High-energy limit** – Going back to the energy loss for hyperbolic-like orbits, eq. (22), for large \( \gamma \) we find \( E_{\gamma}^{\text{HE}} = (a_X + b_X \log \gamma) \gamma^5 + \mathcal{O}(\gamma^3) \) and \( F_{\gamma}^{\text{HE}} = c_X \gamma^6 + d_X \gamma^4 + \mathcal{O}(\gamma^2) \), with \( a_E = 937/2 - 945 \log 2, b_E = 1559/4 - 945 \log 2, \ c_E = 126, \ c_B = 0, \ d_E = -504 \) and \( d_B = -315 \). While \( E_{\gamma}^{\text{HE}} \) and \( E_{\gamma}^{\text{HE}} \) scale in the same way with \( \gamma \), \( F_{\gamma}^{\text{HE}} \) and \( F_{\gamma}^{\text{HE}} \) behave differently. Our perturbative expansion is valid for \( \gamma (\mathcal{G} / |b|) \ll 1 \) \([53, 147, 148]\) (see also \([121]\)). In this regime \( \Delta E_{\text{tid}} \ll \Delta E \sim (\mathcal{G} / |b|)^3 (m/h) \gamma^3 \ll E \).

**Conclusion** – We have computed the four-momentum and the flux emitted in gravitational waves by the scattering of tidally interacting bodies at leading order in the post-Minkowskian approximation. Our computation uses the worldline effective field theory approach and the results obtained are, up to our knowledge, new. We focused on electric and magnetic-type quadrupolar effects.
but our computations can be straightforwardly extended to higher multipoles or to higher-orders in the curvature fields.

We have derived the emitted energy for bound orbits using the B2B dictionary and verified that it is consistent with PN results for eccentric orbits. Considering the ultra-relativistic limit of the energy loss, we observe that the contributions of the electric and magnetic component scale differently unlike the case of the conservative scattering angle. It would be interesting to use the derived PM flux to study the corresponding modifications of the waveform.

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Supplemental Materials

TIDAL STRESS-ENERGY TENSOR

Here we give the explicit expression of the tidal stress-energy tensor defined in eq. (16). Introducing \( \beta \equiv 2\gamma^2 - 1 \) and \( \omega_\alpha = k \cdot u_\alpha \), we have

\[
\frac{m_1 \epsilon_{\mu \nu}^{(1)} E_1^\nu}{q_1^2 c E_1^\mu} = \omega_1 \epsilon_{\mu \nu}^{(1)\nu} - \beta \omega_1 q_2^\mu q_2^\nu - 2\omega_1 q_1^\mu u_1^\nu + \omega_1 (\beta q_1^\mu + 4\gamma q_1^\nu) q_2^\mu q_2^\nu + 4\gamma q_2^\nu q_2^\mu u_1^\nu
\]

\[
- \omega_1 (\beta q_1^\mu + 4\gamma q_1^\nu) q_2^\mu q_2^\nu + 2\omega_1^2 q_2^\mu q_2^\nu + \omega_1 \left( \frac{\gamma q_2^\mu}{2} + 2\omega_1 q_1^\nu \right) q_2^\mu q_2^\nu - (\beta q_1^\mu + 4\gamma q_1^\nu) q_2^\mu q_2^\nu + 2\omega_1 q_2^\mu q_2^\nu + \omega_1 \left( \frac{\gamma q_2^\mu}{2} + 2\omega_1 q_1^\nu \right) q_2^\mu q_2^\nu
\]

and analogous expressions for \( \epsilon_{\mu \nu}^{(1)} E_2^\nu \) with \( (1 \leftrightarrow 2) \).

WAVEFORM

The asymptotic waveform in direct space, i.e. \( h_\lambda \equiv \epsilon_\lambda E_\mu / m_1 \), with \( \lambda = \pm \), evaluated at distances \( r \) much larger than the interaction region, reads

\[
h_\lambda (x) = \frac{4G}{r} \int \frac{dk_0}{2\pi} e^{-ik_0 u} e^{i\lambda x} \tilde{T}^{\alpha \beta} |_{k^\mu = k^0 n^\mu}
\]

where \( u = t - r \) is the retarded time. The stress-energy tensor on the right-hand side is evaluated on-shell, i.e. \( k^\mu = k^0 n^\mu \), with \( n^\mu = (1, n) \) and \( n^2 = 1 \).

In the point-particle case, the waveform was computed using our approach in [116] (see also [115]), in agreement with earlier calculations [118]. Here we focus on the tidal contribution, obtained by replacing (16) in the above expression. Integrating first in \( k^0 \), we obtain

\[
h_\lambda^{\text{tid}} = \sum_{X=E,B} \frac{G m_2}{r m_1} c X^\mu (n) \int_q \frac{\delta(q \cdot u_2)}{q^2 (k - q)^2} \frac{d^3 q}{(2\pi)^3} \chi(q) \left( q^2 \right) \left( q \cdot u_2 \right)
\]

where \( \bar{b} = b + \frac{u_2}{n} (u \cdot n) \). We can then choose a frame and remove the remaining delta function by integrating in \( q^2 \). Regardless of the chosen frame, the remaining integral can be put in the form of the master integral

\[
I = \int_q \frac{c q^\mu b^\nu}{q \cdot M \cdot q} = \frac{1}{4\pi} \left( \frac{b \cdot M^{-1} \cdot b}{|\det(M)|} \right)^{1/2}
\]

where \( M \) is a 3 \times 3 matrix, or of its tensorial generalization, \( I^{1 \pm \cdots i_n} = \int_q \frac{a^{i_1 \cdots i_n} c^{q \cdot \alpha}}{q \cdot M \cdot q} \), which can be solved by taking derivatives of the master integral, \( I^{1 \pm \cdots i_n} = \frac{\partial^{i_1 \cdots i_n}}{\beta} \).

Performing the calculation in the rest frame of particle 2, i.e. for \( a_i = \gamma(1, n \cdot v) \) and \( b_i = \delta_1 a_i \), and choosing \( b_i = (0, b) \) and \( b_1 = 0 \) for simplicity, we obtain

\[
h_\lambda^{\text{tid}} = 120 \frac{G^2 m_1 m_2 \gamma^2 \gamma^2}{r |b|^2} \sum_{a=1,2} \sum_{X=E,B} \sum_{\lambda} \frac{c X^\mu c^\nu a^\sigma a^\tau A_{\lambda X}^{a \mu \nu \sigma \tau}}{m_2 (n \cdot u_2)^3 c_0^9}
\]

where we have defined \( e_\lambda \equiv (\hat{v}, \hat{b}) \) (with \( I = v, b \)) and the functions \( c_1 = 1 + \frac{\gamma^2}{(n \cdot u_2)^2} \left( \frac{u}{|b|} + n \cdot b \right)^2 \) and \( c_2 = 1 + \gamma^2 v^2 \frac{|b|^2}{|b|^2} \). Defining \( f_a = 6 c_a - 7, g_a = \sqrt{c_a - 1} (4 c_a - 7) \), the explicit expressions for \( A_{\lambda X}^{a \mu \nu \sigma \tau} \) are

\[
A_{E2}^{bb} = (n \cdot u_2)^2 \beta f_a,
\]

\[
A_{E2}^{vb} = \gamma(n \cdot u_2) f_a v \delta_1 b \cdot n + g_a,
\]

\[
A_{E2}^{v\tau} = \frac{(2 f_a^2 - 7 f_a - 10)}{30} + f_a (q^2 - 1) + v \gamma \delta_1 b \cdot n [f_a v \delta_1 b \cdot n + 2 g_a]
\]

\[
A_{B2}^{bb} = 2 \gamma(n \cdot u_2) [\gamma n \cdot u_2 - \frac{(n \cdot u_1) (n \cdot u_2)}{n \cdot u_2}] f_a,
\]

\[
A_{B2}^{vb} = f_a \gamma^2 (2 \gamma \delta_1 n \cdot u_2 - 1) e b \cdot n
\]
\[ A^{\nu}_{\mu 2} = 2\gamma^2 v \mathbf{b} \cdot \mathbf{n} \left[ f_a \gamma^2 v \delta_{a1} \mathbf{b} \cdot \mathbf{n} + \frac{\gamma (n \cdot u_1) (n \cdot u_2)}{n \cdot u_a} \right], \]
\[ A^{\nu}_{\mu 2} = 2\gamma^2 v \mathbf{b} \cdot \mathbf{n} \left[ f_a \gamma^2 v \delta_{a1} \mathbf{b} \cdot \mathbf{n} + \frac{\gamma (n \cdot u_1) (n \cdot u_2)}{n \cdot u_a} \right]. \] (37)

One can verify that, upon PN expanding, the contribution of the current (magnetic) quadrupole enters at 1PN higher than the mass (electric) one, as expected.

Note, also, that the gravitational wave memory, i.e., the difference in the waveform between asymptotic past and future, defined as
\[ \Delta h_\lambda(x) = \int_{-\infty}^{+\infty} du \, \hat{h}_\lambda(u, x), \] (38)
is not affected by tidal deformations at this order in G. Indeed, from eq. (33) the contribution of tidal effects to the memory reads
\[ \Delta h_\lambda^{\text{tid}} = -\frac{4G}{r} \frac{d}{d\tau} \left( k^0 \right) \delta_{\alpha\beta} \hat{\mathcal{T}}_{\alpha\beta}^{\text{tid}} | k^\mu = k^\mu |. \] (39)

From eqs. (31) and (32) above, \( \hat{T}_{\alpha\beta}^{\text{tid}} \sim O((k^0)^4) \) and thus the right-hand side of this equation vanishes. This conclusion can be extended to higher-order tidal fields, which contribute to \( \hat{T}_{\alpha\beta} \) higher in \( k^0 \).

\section*{2-LOOP INTEGRALS}

Introducing the following basis of propagators
\[ \rho_1 = 2 \ell_1 \cdot u_1, \quad \rho_2 = -2 \ell_1 \cdot u_2, \quad \rho_3 = -2 \ell_2 \cdot u_1 \]
\[ \rho_4 = 2 \ell_2 \cdot u_2, \quad \rho_5 = \ell_1^2, \quad \rho_6 = \ell_2^2, \] (40)
\[ \rho_7 = (\ell_1 + \ell_2)^2, \quad \rho_8 = (\ell_1 - q)^2, \quad \rho_9 = (\ell_2 - q)^2, \]
we can explicitly write the 2-loop integrals in eq. (20) as
\[ T_2^X = \frac{2\gamma^2}{15(-q^2)} \frac{m_1}{c X_1} \int_{\ell_1, \ell_2} \delta_{(\rho_7)} \delta_{(\rho_1)} \delta_{(\rho_4)} \frac{N X_1}{\rho_5 \rho_6 \rho_7}, \] (41)
\[ T_2^X = \frac{2\gamma^2}{15(-q^2)} \frac{m_2}{c X_2} \int_{\ell_1, \ell_2} \delta_{(\rho_7)} \delta_{(\rho_1)} \delta_{(\rho_4)} \frac{N X_2}{\rho_5 \rho_6 \rho_7}, \] (42)
where \( N_{X_1} = \text{Re} \left[ \left( \sigma_{\alpha^0 \beta} + t_{\alpha^0}^{\beta} + t_{\alpha^0}^{\beta} \right) P_{\alpha^0}^{\sigma^0} t_{\sigma^0} X^\mu \right] \).

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