ON INTERACTION OF CIRCULAR CYLINDRICAL SHELLS WITH A POISEUILLE TYPE FLOW

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Abstract. We study dynamics of a coupled system consisting of the 3D Navier–Stokes equations which is linearized near a certain Poiseuille type flow between two unbounded circular cylinders and nonlinear elasticity equations for the transversal displacements of the bounding cylindrical shells. We show that this problem generates an evolution semigroup $S_t$ possessing a compact finite-dimensional global attractor.

1. Introduction.

1.1. Description of the model. Let a domain between two concentric circular cylinders of radii $R_1$ and $R_2$ respectively and infinite length with the axes directed along the X-line be denoted by $\Omega \subset \mathbb{R}^3$. We denote the surface of the inner cylinder by $\partial\Omega_1$ and the surface of the outer cylinder by $\partial\Omega_2$. Let $\partial\Omega_1 = \Gamma_0^1 \cup \Gamma_1^1$ and $\partial\Omega_2 = \Gamma_0^2 \cup \Gamma_1^2$ where $\Gamma_0^i = \partial\Omega_i \setminus \Gamma_1^i$, $i = 1, 2$ are finite flexible parts of the inner and outer cylindrical surfaces respectively whose projections on the X-line are the intervals $(l_i, L_i)$. The projections of the rigid parts $\Gamma_1^i$ of the inner and outer cylinders on the X-line are $(-\infty, l_1) \cup (L_1, +\infty)$ and $(-\infty, l_2) \cup (L_2, +\infty)$ respectively. We denote by $n_i^j(x, y, z) = (n_i^1(x, y, z), n_i^2(x, y, z), n_i^3(x, y, z))$ the outward unit normals to $\partial\Omega_i$.

We consider the image $\Pi = (R_1, R_2) \times (0, 2\pi) \times (-\infty, \infty)$ of the domain $\Omega$ after the coordinates transformation into cylindrical coordinates $T(r, \phi, x) = (x, y, z) = (x, r \cos \phi, r \sin \phi)$. We use the following notations $T\partial\Omega_i = \Upsilon_i = \{R_i\} \times (0, 2\pi) \times (-\infty, \infty)$, $TT_0^1 = \Upsilon_0^1 = \{R_i\} \times (0, 2\pi) \times (l_i, L_i)$, $TT_1^1 = \Upsilon_1^1 = \Upsilon_1 \setminus \Upsilon_0^1$.

Let $z = (r, \phi, x)$ be the point in $\mathbb{R}^3$ in the cylindrical coordinates. We consider the following linear Navier–Stokes equations in $\Omega$ for the fluid velocity field $V(z, t) = (v(z, t), u(z, t), w(z, t))$ and for the pressure $p(z, t)$:

$$V_t - \mu \Delta V + L_0 V + \nabla p = G_f(t) \quad \text{in} \quad \Pi \times (0, +\infty),$$

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\[
\text{div} \, V = 0 \quad \text{in} \quad \Pi \times (0, +\infty),
\]
where \( \mu > 0 \) is the dynamical viscosity, \( G_f(z, t) = (g_1(z, t), g_2(z, t), g_3(z, t)) \) is a volume force (which may depend on \( t \)). The functions \( v(z, t) \) and \( u(z, t) \) denote the radial and angular velocities, while the function \( w(z, t) \) represents the longitudinal velocity of the fluid. We remind that the Laplace, divergence, and gradient operators in the cylindrical coordinates have the form

\[
\Delta V = \begin{pmatrix}
\frac{\nu}{r} + v_r + v_{xx} - \frac{v}{r^2} + \frac{v_\phi}{r} - \frac{2\nu_\phi}{r^2} \\
\frac{\nu}{r} + u_{rr} + u_{xx} - \frac{u}{r^2} + \frac{u_\phi}{r} + \frac{2\nu_\phi}{r^2} \\
u_r + w_{rr} + w_{xx} + \frac{w_\phi}{r}
\end{pmatrix},
\]

\[
\text{div} \, V = \frac{v}{r} + v_r + \frac{u_\phi}{r} + w_x,
\]

and

\[
\nabla p = \left( p_r, \frac{p_\phi}{r}, p_x \right)^T.
\]

The linear first order operator \( L_0 \) has the form \( L_0 V = (0, 0, a(r)w_x)^T \), where

\[
a(r) = k/4 \left( \frac{R_2^2 \log R_2 - R_2^2 \log R_1}{\log(R_2/R_1)} + \frac{R_2^2 - R_1^2}{\log(R_2/R_1)} \log r - r^2 \right)
\]

and \( k \) is a positive constant.

We supplement 1 and 2 with the boundary conditions imposed on the velocity field \( V = V(z, t) \):

\[
V(r, 0, x, t) = V(r, 2\pi, x, t), \quad V \equiv (v(R_i); u(R_i); w(R_i)) = \begin{cases} (\eta_i; 0; 0) & \text{on} \quad \Upsilon_i, \\ (0; 0; 0) & \text{on} \quad \Upsilon_i', \quad i = 1, 2. \end{cases}
\]

Here \( \eta_i = \eta_i(\phi, x, t) \) is the transversal displacement of the elastic cylindrical shell containing the fluid and satisfying the following equation (see, e.g., [18]):

\[
\rho_i \eta_{i\alpha\alpha\alpha\alpha} + \frac{h_i E_i}{12(1 - \sigma_i)} \Delta^2_{R_i} \eta_i = \langle \eta_i, F_i \rangle - p|_{\Upsilon_i} + G_i^r(t), \quad \text{in} \quad \Upsilon_i \times (0, \infty),
\]

supplemented with boundary conditions

\[
\eta_i(\phi, l_i, t) = \eta_i(\phi, L_i, t) = \eta_i(\phi, L_i, t) = 0, \quad (\eta_i(0, x, t) = \eta_i(2\pi, x, t)) \quad (\eta_i(0, x, t) = \eta_i(2\pi, x, t)).
\]

Here \( G_i^r(t) \) are given external (non-autonomous) loads, the operators \( \Delta^2_{R_i} \) are given by the formulas

\[
\Delta^2_{R_i} \eta_i = \eta_{i\alpha\alpha\alpha\alpha\alpha\alpha} + \frac{1}{R_i^2} \eta_{i\phi\phi\phi} + \frac{2}{R_i^2} \eta_{i\alpha\alpha\phi}, \quad i = 1, 2,
\]

and correspondingly \( \Delta_{R_i} \eta_i = \eta_{i\alpha\alpha\alpha\alpha\alpha\alpha} + \frac{1}{R_i^2} \eta_{i\phi\phi\phi} \). The bracket \( \langle \cdot, \cdot \rangle \) of von Karman type contrary to the classical von Karman bracket for functions in the Cartesian coordinates

\[
[u, v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2 \cdot \partial^2_{x_1 x_2} u \cdot \partial^2_{x_1 x_2} v,
\]

is defined in the cylindrical coordinates and has the form
\[ \langle \eta_i, F_i \rangle = \frac{1}{R_i} F_{ixx} + \frac{1}{R_i^2} [\eta_i, F_i] \]
\[ = \frac{1}{R_i} F_{ixx} + \frac{1}{R_i^2} (F_{i\phi \phi} \eta_{ixx} - 2F_{ixx} \eta_{ix \phi} + F_{ixx} \eta_{i\phi \phi}) \]

for \( i = 1, 2 \). The functions \( F \) solve the boundary value problems

\[ \frac{1}{E_i \eta_i} \Delta_{R_i}^2 F_i + \langle \eta_i, \eta_i \rangle = 0, \text{ in } \Upsilon_0^1 \times (0, \infty) \]  

\[ F_i(\phi, l_i, t) = F_{ix}(\phi, l_i, t) = F_i(\phi, L_i, t) = F_{ix}(\phi, L_i, t) = 0, \]  

\[ F_i(0, x, t) = F_i(2\pi, x, t), \quad F_{i\phi}(0, x, t) = F_{i\phi}(2\pi, x, t) \]

We supply 1-11 with initial data of the form

\[ v(z, 0) = v_0(z), \quad u(z, 0) = u_0(z), \quad w(z, 0) = w_0(z), \]  

\[ \eta_i(\phi, x, 0) = \eta_i^0(\phi, x), \quad \eta_{ii}(\phi, x, 0) = \eta_{ii}^0(\phi, x), \]

satisfying the condition

\[ \frac{v_0(r, x)}{r} + v_0(r, x) + \frac{u_0(\phi, x)}{r} + w_0(\phi, x) = 0. \]

If we assume that the velocity field \( V \) decays sufficiently fast as \( |x| \to +\infty \) and \( x \in \overline{\Omega} \), then 2 and 3 imply the following compatibility conditions

\[ \int_{\Upsilon_0^1} \eta_{i\phi}(\phi, x, t)dz = \int_{\Upsilon_0^2} \eta_{i\phi}(\phi, x, t)dz \quad \text{for all } t \geq 0, \quad i = 1, 2. \]

which can be interpreted as preservation of the volume of the fluid.

Thus, our general model includes the case of interaction of the Poiseuille (laminar axisymmetric) flow in the unbounded circular cylindrical domain bounded by the (solid) cylindrical walls \( \Gamma_1^0 \) and a circular cylindrical elastic shell \( \Gamma_0^1 \). The motion of the fluid is described by the 3D Navier–Stokes equations linearized around the Poiseuille flow.

In general the Poiseuille flow is a laminar flow in an incompressible and Newtonian fluid flowing through a long cylindrical domain of the form

\[ \Omega = \{(x_1; x_2; x_3) : (x_2; x_3) \in \mathcal{B} \subset \mathbb{R}^2, \quad x_1 \in \mathbb{R} \}, \]

where \( \mathcal{B} \) is a domain in \( \mathbb{R}^2 \), and the Poiseuille velocity field has the form \( a_0 = (a(x_2; x_3); 0; 0) \), where \( a(x_2; x_3) \) solves the elliptic problem

\[ \nu \Delta a = -k \quad \text{in } \mathcal{B}, \quad a = 0 \quad \text{on } \partial \mathcal{B}, \]

where \( k \) is a positive parameter. Linearization of the nonlinear Navier-Stokes equations around the flow \( a_0 \) gives us the model with

\[ L_0V = (a_0, \nabla)V + (V, \nabla)a_0. \]

There are two important special cases of the choice of \( \mathcal{B} \): (i) \( \mathcal{B} \) is a bounded domain in \( \mathbb{R}^2 \) (the Poiseuille flow in cylindrical tubes) and (ii) a flow between two parallel planes.

To describe deformations of the shell we use the Donnell’s nonlinear shallow shell model ([17, 18]) which accounts only for transversal displacements. Since we deal with linearized fluid equations the interaction model considered assumes that large deflections of the shell produce small effect on the corresponding underlying flow.
1.2. Previous work. We note that the mathematical studies of the problem of fluid–structure interaction in the case of viscous fluids and elastic plates/bodies have a long history.

The case of moving elastic bodies [16] and the case of elastic bodies with the fixed interface [2, 4, 6, 22] were studied from the point of view of the well-posedness and stability of the problems.

We refer to [7, 10, 13, 24, 25, 26, 27, 30] and the references therein for the case of shallow plates/membranes. Works [7, 24, 27] are devoted to the well-posedness of the problems of fluid-structure interaction, in papers [25, 26] the stability of linear problems of interaction of a viscous incompressible fluid and (damped) plate equations accounting for the longitudinal displacements. In paper [30] the approximate controllability of a linear model of interaction between a viscous incompressible fluid and a thin elastic structure located on a part of the fluid domain boundary, when the rigid and the elastic parts of the boundary make a rectangular corner and if the control acts on the whole elastic structure. The existence of global attractor for fluid-structure interaction problems was investigated in [10, 13]. While in the first paper a nonlinear system describing the interaction of a viscous incompressible fluid in a bounded vessel with a flat elastic part of the boundary moving in the in-plane directions only is considered, the second one deals with the transversal displacement on a flexible flat part of the boundary.

We also mention the paper [14] which deals with dynamical issues for a model taking into account both transversal and longitudinal deformations. All these sources deal with the case of bounded reservoirs $\Omega$ and a flat elastic shallow shell or plate.

Regarding infinite reservoirs $\Omega$ it can be mentioned, to the best of our knowledge, only work [15] which establishes the existence of a compact global attractor to a linearized around a Poiseuille type flow Navier-Stokes system in an unbounded domain coupled with a nonlinear equation for a bounded flat part of the boundary accounting for the transverse displacements only.

In our paper we consider for the first time, to the best of our knowledge, the interaction of the Newtonian fluid with nonlinear cylindrical (concentric) shells whose dynamics is described by a von Karman-type (Donnell’s) model in the cylindrical coordinates. The peculiarity of the problem considered consists in the presence of the von Karman-type nonlinearity in the polar coordinates which loses the classical properties of the von Karman bracket after transition to the Cartesian coordinates. However, in the cylindrical coordinates the domain is not smooth and there exists no bounded domain with smooth boundary which lies in $\Omega$ and whose boundary contains the flexible wall of the vessel. This renders impossible the direct construction (see [15]) of the extension operators of functions defined on the flexible part of the boundary to the solenoidal fields in the whole domain and the trace operators for the normal components of the fluid velocity lying in $L_2$ (see, e.g. [19, 20, 33]).

To overcome this obstacle we resort to the construction of homeomorphisms between suitable Sobolev classes on the boundary and in the domain in Cartesian and cylindrical coordinates. This method allows also to define trace operators on $\mathcal{T}_0^1$ for functions defined in the Lipschitz domain $\Pi$ in spaces of Sobolev type of order higher than $3/2$. However, the approach used is not applicable in case of one cylindrical tube ($R_1 = 0$) since the above mentioned mappings between spaces in the Cartesian and in the cylindrical coordinates are not homeomorphisms on the axis of the cylinder. This case is the subject for further investigations.
We establish the well-posedness of the system considered and investigate the long-time dynamics of solutions to the coupled problem in 1-11.

In our argument we use the ideas and methods developed in papers [13, 15].

Since we do not assume any kind of mechanical damping in the plate component, this means that dissipation of the energy in the fluid flow due to viscosity is sufficient to stabilize the system. We model considered does not takes into account rotational inertia of the cylindrical filaments. In case then rotational inertia terms are present in the shell equations under absence of mechanical damping it is impossible to establish results on dissipativity or asymptotical smoothness of the system, since the energy terms accounting for the rotational inertia cannot be estimated from above by the terms corresponding to the viscous damping of the fluid (cf. [13, 15]). Though, there is a result on the exponential stability of a linear problem of such a type [3], the methods used in this work are not applicable in the nonlinear case.

1.3. Abstract results on attractors. For the readers’ convenience we recall some basic definitions and results from the theory of attractors.

**Definition 1.1** ([5, 8, 9, 11, 32]). A global attractor of a dynamical system \((S_t, H)\) with the evolution operator \(S_t\) on a complete metric space \(H\) is defined as a bounded closed set \(A \subset H\) which is invariant \((S_tA = A \text{ for all } t > 0)\) and uniformly attracts all other bounded sets:

\[
\lim_{t \to \infty} \sup \{ \text{dist}_H(S_t y, A) : y \in B \} = 0 \quad \text{for any bounded set } B \text{ in } H.
\]

An important characteristic of a global attractor is its fractal dimension:

**Definition 1.2** ([8, 9, 11]). The fractal dimension \(\text{dim}_f^H M\) of a compact set \(M\) in a complete metric space \(H\) is defined as

\[
\text{dim}_f^H M = \lim_{\varepsilon \to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},
\]

where \(N(M, \varepsilon)\) is the minimal number of closed sets in \(H\) of diameter \(2\varepsilon\) which cover \(M\).

To establish the existence of attractor we use the concept of gradient systems. The main features of these systems are: (i) in the proof of the existence of a global attractor we can avoid a dissipativity property (existence of an absorbing ball) in the explicit form and (ii) the structure of the attractor can be described via unstable manifolds ([8]).

**Definition 1.3** ([8, 9, 11]). Let \(Y \subseteq H\) be a forward invariant set of a dynamical system \((S_t, H)\). A continuous functional \(\Phi(y)\) defined on \(Y\) is said to be a Lyapunov function on \(Y\) for the dynamical system \((S_t, H)\) if \(t \mapsto \Phi(S_t y)\) is a nonincreasing function for any \(y \in Y\).

The Lyapunov function is said to be strict on \(Y\) if the equation \(\Phi(S_t y) = \Phi(y)\) for all \(t > 0\) and for some \(y \in Y\) implies that \(S_t y = y\) for all \(t > 0\); that is, \(y\) is a stationary point of \((S_t, H)\).

The dynamical system is said to be gradient if there exists a strict Lyapunov function on the whole phase space \(H\).

We single out a class of the so-called quasi-stable systems that enjoy some kind of stabilizability inequalities written in some general form. These inequalities, although often difficult to establish (most often they are obtained by means of multipliers technic), once they are proved provide a number of consequences that describe various properties of attractors [8, 11].
Definition 1.4 ([8, 11]). A seminorm \( n(x) \) on a Banach space \( H \) is said to be compact if any bounded sequence \( \{x_m\} \subset H \) contains a subsequence \( \{x_{m_k}\} \) which is Cauchy with respect to \( n \), i.e., \( n(x_{m_k} - x_{m_l}) \to 0 \) as \( k, l \to \infty \).

The dynamical system \((S_t, H)\) is said to be quasi-stable on a set \( \mathcal{B} \subset H \) (at time \( t_* \)) if there exist (a) time \( t_* > 0 \), (b) a Banach space \( Z \), (c) a globally Lipschitz mapping \( K : \mathcal{B} \to Z \), and (d) a compact seminorm \( n_Z(\cdot) \) on the space \( Z \), such that
\[
\|S_{t_*}y_1 - S_{t_*}y_2\|_H \leq q \cdot \|y_1 - y_2\|_H + n_Z(Ky_1 - Ky_2) \tag{16}
\]
for every \( y_1, y_2 \in \mathcal{B} \) with \( 0 \leq q < 1 \). The space \( Z \), the operator \( K \), the seminorm \( n_Z \) and the time moment \( t_* \) may depend on \( \mathcal{B} \).

The following statement collects criteria on existence, finite dimensionality and properties of attractors to gradient systems.

Theorem 1.5 ([8, 11]). Assume that \((S_t, H)\) is a gradient quasi-stable dynamical system. Assume its Lyapunov function \( \Phi(y) \) is bounded from above on any bounded subset of \( H \) and the set \( \Phi_R = \{y : \Phi(y) \leq R\} \) is bounded for every \( R \). If the set \( N \) of stationary points of \((S_t, H)\) is bounded, then \((S_t, H)\) possesses a compact global attractor which possesses finite fractal dimension. Moreover, the global attractor \( \mathcal{A} \) consists of full trajectories \( \gamma = \{u(t) : t \in \mathbb{R}\} \) such that
\[
\lim_{t \to -\infty} \text{dist}_H(u(t), N) = 0 \text{ and } \lim_{t \to +\infty} \text{dist}_H(u(t), N) = 0. \tag{17}
\]
and
\[
\lim_{t \to +\infty} \text{dist}_H(S_t x, N) = 0 \text{ for any } x \in H; \tag{18}
\]
that is, any trajectory stabilizes to the set \( N \) of stationary points.

The paper is organized as follows. In Section 2 we introduce Sobolev type spaces we need and prove the result on the well-posedness of the system. In Sections 3 and 4 we deal with extension operators and characterization of the phase spaces for the problem in Cartesian and in cylindrical coordinates respectively. In Section 5 we prove the well-posedness of the problem. Section 6 is devoted to the existence of a compact finite-dimensional global attractor.

2. Spaces and notations. We introduce the Sobolev spaces on domains and compact manifolds following [1, 21, 31, 35].

Remark 2.1. We note that the domain \( \Omega \) is unbounded, \( \Pi \) is unbounded and possesses non-smooth boundary, the domains \( \Pi_0^1 \) are bounded but their boundaries are non-smooth. However, all these domains satisfy the cone property, therefore standard results on the equivalent norms and interpolation theorems used below are applicable (see, e.g. [35] Section 4.2).

Remark 2.2. The boundaries of \( \Pi \) and \( \Pi_0^1 \) are piecewise-smooth, however, one can define the standard traces on each smooth part of the boundary.

Definition 2.3. Let \( B \) be a sufficiently smooth or Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \) equipped with Cartesian coordinates and \( H^s(B) \) be the Sobolev space of order \( s \in \mathbb{R} \) on \( B \) which we define (see [35]) as restriction (in the sense of distributions) of the space \( H^s(\mathbb{R}^d) \) (introduced via Fourier transform). We define the norm in \( H^s(B) \) by the relation
\[
\|\phi\|^2_{H^s(B)} = \inf \left\{ \|\psi\|^2_{H^s(\mathbb{R}^d)} : \psi \in H^s(\mathbb{R}^d), \psi = \phi \text{ on } B \right\}.
\]

We denote by \( H_0^s(B) \) the closure of \( C_0^\infty(B) \) in \( H^s(B) \). We also introduce the spaces
Definition 2.4.

$$H^s_\ast(B) := \{ f \mid_B : f \in H^s(\mathbb{R}^d), \supp f \subset \overline{B} \}, \quad s \in \mathbb{R}.$$  

with the induced norms $\|f\|_{H^s_\ast(B)} = \|f\|_{H^s(\mathbb{R}^d)}$ for $f \in H^s_\ast(B)$.

It is clear that

$$\|f\|_{H^s(\mathbb{R}^d)} \leq \|f\|_{H^s_\ast(B)}, \quad f \in H^s_\ast(B).$$

It is known (see [35, Theorem 4.3.2/1]) that $C_0^\infty$ is dense in $H^s_\ast(B)$ and

$$H^s_\ast(B) \subset C^0(B) \subset H^s(B), \quad s \in \mathbb{R};$$

$$H^s_\ast(B) = H^s(B), \quad -\infty < s \leq 1/2;$$

$$H^s_\ast(B) = H^s(B), \quad -1/2 < s < \infty, \quad s - 1/2 \notin \{0, 1, 2, \ldots\}.$$  

In particular, $H^s_\ast(B) = H^s(\mathbb{R}^d) = H^s(B)$ for $|s| < 1/2$. By [35, Remark 4.3.2/2] we also have that $H^s_\ast(B) \neq H^s(\mathbb{R}^d)$ for $|s| > 1/2$. Note that in the notations of [29] the space $H^{m+1/2}_0(B)$ is the same as $H^{m+1/2}_0(\mathbb{R}^d)$ for every $m = 0, 1, 2, \ldots$.

Definition 2.5. A Riemannian manifold $M$ with metrics $\xi$ is of bounded geometry if the following two conditions are satisfied:

(i) The injectivity radius of $(M, \xi)$ is positive.

(ii) Every covariant derivative of the Riemann curvature tensor of $(M, \xi)$ is bounded.

Remark 2.6. Manifolds $\Gamma^i_0$ are of bounded geometry since all covariant derivatives of the curvature tensor are bounded and the injectivity radius, namely, the distance from a point on the cylinder to the opposite line, is positive.

To describe boundary traces on $\Gamma^i_0$ we use Sobolev spaces on a complete Riemannian manifold (with boundary) $D$ with Riemannian metric $\xi$, defined by means of Laplace-Beltrami operator $\Delta_D$ (see, e.g., [21, 31, 34]).

Definition 2.7. Let $D$ is a connected $d$-dimensional Riemannian manifold of bounded geometry.

(i) The spaces $\mathcal{H}^m(D)$ for $m \in \mathbb{N}_+$ are completions of $C^\infty(D)$ with respect to the norms

$$\|q\|_{\mathcal{H}^m(D)}^2 = \|(I - \Delta_D)^{m/2}q\|_{L^2(D)}^2 = \int_D ((I - \Delta_D)^{m/2}q)^2 \, d\xi < \infty.$$  

(ii) For $s > 0$ $\mathcal{H}^s(D)$ is the collection of all $q \in L^2(D)$ such that $q = (I - \Delta_D)^{-s/2}h$ for some $h \in L^2(D)$ with the norm $\|q\|_{\mathcal{H}^s(D)} = \|h\|_{L^2(D)}$.

For $s < 0$ $\mathcal{H}^s(D)$ is the collection of all $q \in \mathcal{D}'(D)$ having the form $q = (I - \Delta_D)^{k}h$ with $h \in \mathcal{C}^{2k+s}(D)$, where $k \in \mathbb{N}_+$ and $2k + s > 0$ with the norm $\|q\|_{\mathcal{H}^s(D)} = \|h\|_{\mathcal{H}^{2k+s}(D)}$.

Remark 2.8. The spaces $\mathcal{H}^s(D)$ with $s < 0$ are independent of $k$ in the sense of equivalent norms. For $s_1, s_2 \in \mathbb{R}, 0 < \theta < 1$

$$[\mathcal{H}^{s_1}(D), \mathcal{H}^{s_2}(D)]_\theta = \mathcal{H}^{(1-\theta)s_1+\theta s_2}(D).$$  

For fractional Sobolev spaces on manifolds with boundary there exists a bounded surjective trace operator $\gamma_{\partial D}$ for $s > 1/2$ from $\mathcal{H}^s(D)$ to $\mathcal{H}^{s-1/2}(\partial D)$ in case both manifold and its boundary are of bounded geometry (see e.g. [21]). We also denote by $\mathcal{H}^s_0(D)$ the closure of $C^\infty(D)$ with respect to the norm in $\mathcal{H}^s(D)$ for $s > 0$. 


Definition 2.9. If \( D \) is a smooth manifold of bounded geometry contained in a compact smooth manifold \( M \) we can define the spaces
\[
\mathcal{H}_s(D) = \{ q \in \mathcal{H}_s(M) : \text{supp} q \subset D \}.
\]
We endow the classes \( \mathcal{H}_s(D) \) with the induced norms \( \| q \|_\mathcal{H}_s(D) = \| q \|_{\mathcal{H}_s(M)} \)
(see, e.g., [31]).

Remark 2.10. It is known ([31]) that \( C_0^\infty(D) \) is dense in \( \mathcal{H}_s(D) \) and
\[
\mathcal{H}_s(D) = \mathcal{H}_s(M) / \mathcal{H}_s^s(K),
\]
where \( K = M \setminus D \). Moreover, ([23, 31])
\[
\mathcal{H}_s(D) = \mathcal{H}_s^s(D), \quad s \geq 0, \quad s - 1/2 \notin \{0, 1, 2, \ldots\}
\]
and, consequently, do not depend on the choice of \( M \),
\[
\mathcal{H}_0^s(D) = \mathcal{H}^s(D), \quad -\infty < s \leq 1/2
\]
and
\[
(\mathcal{H}_0^s(D))^* = \mathcal{H}^{-s}(D), \quad s \geq 0.
\]

It is easy to see from (20, 22) and from the well-known property of dual spaces that for \( 0 < s < 1/2 \)
\[
(\mathcal{H}_s^s(D))^* = (\mathcal{H}_s^s(M))^*/(\mathcal{H}_s^s(D))^\perp = \mathcal{H}^s(M) / \mathcal{H}_s^s(K)
\]
\[
= \mathcal{H}_s^s(M) / \mathcal{H}_s^s(K) = \mathcal{H}_s^s(D) = \mathcal{H}_0^s(D) = \mathcal{H}_s^s(D).
\]
Since, \( \mathcal{H}_s^s(D) \) is reflexive for \( 0 < s < 1/2 \) i.e.,
\[
(\mathcal{H}_s^s(D))^* = (\mathcal{H}_0^s(D))^* = (\mathcal{H}^{-s}(D))^* = \mathcal{H}_0^s(D) = \mathcal{H}_s^s(D),
\]
assertion (21) holds also for \( -1/2 < s < 0 \).

Definition 2.11. Let \( \mathcal{C}_d(\Omega) \) be the class of \( [C^\infty]^3 \) vector-valued solenoidal (i.e., divergence-free) functions in the Cartesian coordinates \( f(x, y, z) = (f_1(x, y, z); f_2(x, y, z); f_3(x, y, z)) \) on \( \Omega \) which vanish in a neighborhood of \( \partial \Omega \) and also for \( |x| \) large enough.

Definition 2.12. The space \( X_s^s(\Omega) \) is the closure of \( \mathcal{C}_d(\Omega) \) with respect to the \( H^s(\Omega) \)-norm.

Definition 2.13. We introduce the space \( \mathcal{C}_p(\Pi) \) of \( [C^\infty]^3 \) vector-valued solenoidal (i.e., divergence-free), \( 2\pi \)-periodic functions in the cylindrical coordinates vanishing in a neighborhood of \( \partial \Pi \) and also for \( |x| \) large enough
\[
\mathcal{C}_p(\Pi) = \{ V(r, \phi, x) = (v(r, \phi, x); u(r, \phi, x); w(r, \phi, x)) \in C^\infty(\Pi) : \frac{v}{r} + v_r + \frac{u_\phi}{r} + w_x = 0, \partial_\phi^j V(r, 0, x) = \partial_\phi^j V(r, 2\pi, x),
\]
\[
j = 0, 1, 2, V \to 0, |x| \to \infty, V \text{ vanishes in a neighbourhood of } \partial \Pi, i = 1, 2 \}.
\]

Remark 2.14. Due to Remark 1.1 the following properties hold:
(i) The norms in the spaces \( H^k(\Pi) \) for \( k \in N \) are equivalent to
\[
\| v \|_{L^2(\Pi)}^2 = \iint_{\Pi} |v|^2 r \, dr \, d\phi \, dx.
\]
and
\[\|v\|^2_{H^k(\Pi)} = \sum_{|\alpha| \leq k} R^2_2 2\pi \int_{0}^2 \int_{-\infty}^{\infty} |D^\alpha v|^2 rdrd\phi dx.\]

(ii) The equivalent norm in $H^1(\Pi)$ for vector-valued functions $V = (v,u,w)$ is
\[\|V\|^2_{H^1(\Pi)} = \int_{0}^{2\pi} \int_{-\infty}^{\infty} \int_{R_1}^{R_2} \left( w^2_x + \frac{v^2}{r} + v^2_r + \frac{u^2}{r^2} + v^2 \right) rdrd\phi dx.\]

(ii) The following interpolation result holds true for $0 < \theta < 1$ [35].

We will also need the following spaces:

**Definition 2.15.** We denote by $X_s(\Pi)$ the closure of $C_p(\Pi)$ in the $H^s(\Pi)$-norm.

**Definition 2.16.**
\[C(\Omega) = \{ f(x,y,z) \in C^\infty(\Omega) : f \to 0, |x| \to \infty \}\]
\[C(\Pi) = \{ v(r,\phi,x) \in C^\infty(\Pi) : \partial^j v(r,0,x) = \partial^j v(r,2\pi,x), j = 0,1,2..., v \to 0, |x| \to \infty \}\].

**Definition 2.17.**
\[C(\Gamma_0^i) = \{ b(\phi,x) \in C^\infty(\Gamma_0^i) : \partial^j b(0,x) = \partial^j b(2\pi,x), j = 0,1,2..., i = 1,2 \}\].

3. Extension operator and characterization of spaces on $\Omega$. Let $\Omega' \subseteq \Omega$ be a smooth bounded domain such that $\Gamma_0^i \subset \partial \Omega'$ for $i = 1,2$. For instance, it can be constructed by joining bounded parts of $\partial \Omega_1$ and $\partial \Omega_2$ containing $\Gamma_0^1$ and $\Gamma_0^2$ respectively by two halves of a torus. We define the space
\[\hat{L} = \{ \psi = (\psi_1,\psi_2) \in L^2(\Gamma_0^1) \times L^2(\Gamma_0^2) : \int_{\Gamma_0^i} \psi_1 d\xi = \int_{\Gamma_0^i} \psi_2 d\xi \}\]
consider the following Stokes problem
\[-\mu \Delta f + \nabla p = g, \quad \text{div } f = 0 \quad \text{in } \Omega';\]
\[f = 0 \text{ on } \bigcup_{i=1}^{2} \Gamma_i^i, \quad f = (0;\psi_1n_2^i;\psi_2n_3^i) \text{ on } \Gamma_0^i, \quad i = 1,2.\]

(24)

where $g \in [L^2(\Omega)]^3$ and $\psi = (\psi_1,\psi_2) \in \hat{L}$ are given. This type of boundary value problems for the Stokes equation was studied by many authors (see, e.g., [28] and [33] and the references therein). We collect some properties of solutions to 24 in the following assertion.

**Proposition 3.1.** With the reference to problem 24 the following statements hold for the spaces $\mathcal{H}_x^s(\Gamma_0^i)$ defined by means of prolongation to $\partial \Omega'$.
Let \( g \in [H^{-1+\sigma}(\Omega')]^3 \) and \( \psi \in \left( \mathcal{H}^{1/2+\sigma}(\Gamma_0^1) \times \mathcal{H}^{1/2+\sigma}(\Gamma_0^2) \right) \cap \hat{\mathcal{L}}. \) Then for every \( \sigma \geq 0 \) problem 24 has a unique solution \( \{ f ; p \} \) in \([H^{1+\sigma}(\Omega')]^3 \times [H^\sigma(\Omega')]/\mathbb{R}\) such that
\[
\| f \|_{H^{1+\sigma}(\Omega')} + \| p \|_{H^\sigma(\Omega')/\mathbb{R}} \leq c_0 \left\{ \| g \|_{H^{-1+\sigma}(\Omega')}^3 + \| \psi \|_{\mathcal{H}^{1/2+\sigma}(\Gamma_0^1) \times \mathcal{H}^{1/2+\sigma}(\Gamma_0^2)} \right\}.
\] (25)

If \( g = 0, \psi \in \left( \mathcal{H}^{-1/2+\sigma}(\Gamma_0^1) \times \mathcal{H}^{-1/2+\sigma}(\Gamma_0^2) \right) \cap \hat{\mathcal{L}}, \sigma \geq 0, \) then
\[
\| f \|_{H^\sigma(\Omega')} + \| p \|_{H^{-1+\sigma}(\Omega')/\mathbb{R}} \leq c_0 \| \psi \|_{\mathcal{H}^{-1/2+\sigma}(\Gamma_0^1) \times \mathcal{H}^{-1/2+\sigma}(\Gamma_0^2)}.
\] (26)

In particular, we can define a linear operator \( N_0 : \hat{\mathcal{L}} \mapsto [H^{1/2}(\Omega')]^3 \) by the formula
\[
N_0 \psi = w \text{ iff } \begin{cases} -\mu \Delta w + \nabla p = 0, & \text{div } w = 0 \text{ in } \Omega'; \\ w = 0 \text{ on } \bigcup_{i=1}^2 \Gamma_i; \\ w = (0; \psi_i n^i_2; \psi_i n^i_3) \text{ on } \Gamma_i^0, \ i = 1, 2, \end{cases}
\] (27)

for \( \psi \in \hat{\mathcal{L}} \) (\( N_0 \psi \) solves 24 with \( g \equiv 0 \)). It follows from 25 and 26 that \( N_0 : \left( \mathcal{H}^{-1/2+\sigma}(\Gamma_0^1) \times \mathcal{H}^{-1/2+\sigma}(\Gamma_0^2) \right) \cap \hat{\mathcal{L}} \mapsto [H^\sigma(\Omega')]^3 \cap X_0(\Omega) \) continuously for \( \sigma \geq 0. \) Here \( X_0(\Omega') \) is the closure in the \( L^2 \)-norm of the class of \( C^\infty \) solenoidal functions, vanishing at infinity and in a neighborhood of \( \Omega' \setminus (\Gamma_0^1 \cup \Gamma_0^2). \)

Let \( g \in [H^{-1/2+\sigma}(\Omega')]^3 \) and \( \psi \in \left( \mathcal{H}^{\sigma}(\Gamma_0^1) \times \mathcal{H}^{\sigma}(\Gamma_0^2) \right) \cap \hat{\mathcal{L}}, \) with \( 0 < \sigma \leq 1/2. \) Then we can define the traces of the pressure \( p \) on \( \Gamma_i^0, \) which possess the property \( p|_{\Gamma_i^0} \in H^{-1+\sigma}(\Gamma_i^0)/\mathbb{R} \) and
\[
\| p \|_{H^{-1+\sigma}(\Gamma_i^0)/\mathbb{R}} \leq c_0 \left\{ \| g \|_{H^{-1/2+\sigma}(\Omega')}^3 + \| \psi \|_{\mathcal{H}^{\sigma}(\Gamma_0^1) \times \mathcal{H}^{\sigma}(\Gamma_0^2)} \right\}.
\] (28)

For the proof see the Appendix.

Now we adjust the result from 15 to our situation:

**Proposition 3.2.** There exists a linear bounded operator \( \text{Ext} : \hat{\mathcal{L}} \mapsto [L_2(\Omega')]^3 \) such that for \( i = 1, 2 \)
\[
\text{div } \text{Ext}[\psi] = 0 \text{ in } \Omega, \ (\text{Ext}[\psi], n)|_{\Gamma_i^0} = \psi_i, \ (\text{Ext}[\psi], n)|_{\Gamma_i} = 0,
\]
and
\[
\| \text{Ext}[\psi] \|_{H^{1/2}(\Omega')}^3 \leq C \| \psi \|_{L^2}, \forall \psi \in \hat{\mathcal{L}}.
\]

Moreover, if \( \psi \in \mathcal{H}_0^\sigma(\Gamma_0^1) \times \mathcal{H}_0^\sigma(\Gamma_0^2) \) for some \( \sigma \geq 0 \) and \( \sigma \neq 1/2 + Z, \) then \( \text{Ext}[\psi] \in [H^{\sigma+1/2}(\Omega')]^3 \) with the estimate
\[
\| \text{Ext}[\psi] \|_{H^{\sigma+1/2}(\Omega')}^3 \leq C \| \psi \|_{\mathcal{H}_0^\sigma(\Gamma_0^1) \times \mathcal{H}_0^\sigma(\Gamma_0^2)},
\] (29)

and the relations \( \text{Ext}[\psi]|_{\Gamma_i} = (0; 0; 0) \) and \( \text{Ext}[\psi]|_{\Gamma_i} = (0; \psi_i n^i_2; \psi_i n^i_3) \) hold on the boundary \( \partial \Omega', \) where \( n^i = (0, n^i_2, n^i_3), \ i = 1, 2 \) are the unit normals to \( \Gamma_i^0. \)

There exists a smooth bounded subdomain \( \Omega' \) in \( \Omega \) such that \( \Gamma_0^0 \subset \partial \Omega' \) and \( \text{Ext}[\psi]|_{\partial \Omega} = 0, \) and \( \text{Ext}[\psi] \in [H^{s+1/2}(\Omega')]^3 \) provided \( \psi \in \mathcal{H}_0^s(\Gamma_0^1) \times \mathcal{H}_0^s(\Gamma_0^2) \) for \( s \geq 0 \) and \( s \neq 1/2 + Z. \)
The proof of Proposition 3.2 is presented in the Appendix.

**Remark 3.3.** We excluded the cases of semi-integer $\sigma$ to guarantee the independence of the spaces $\mathcal{H}^\sigma_{\gamma}(\Gamma^i_0)$ from the choice of $\Omega'$.

It follows from Propositions 3.1, 3.2 (see [19, 33] for similar arguments) that

$$X_0(\Omega) = \{ f(x, y, z) = (f_1(x, y, z); f_2(x, y, z); f_3(x, y, z)) \in [L_2(\Omega)]^3 : \quad \text{div} f = 0, \ (f, n) = 0 \text{ on } \partial \Omega \}$$

(30)

and

$$X_1(\Omega) = \{ f(x, y, z) = (f_1(x, y, z); f_2(x, y, z); f_3(x, y, z)) \in [H^1(\Omega)]^3 : \quad \text{div} f = 0; \ f = 0 \text{ on } \partial \Omega, \}$$

(31)

Using the extension operator constructed above we introduce the set

$$\mathcal{M}(\Omega) = \left\{ f = f_0 + \text{Ext}[q] : f_0 \in \mathcal{C}_0(\Omega), \ q \in \tilde{L} \cap (H^2_0(\Gamma^1_0) \times H^2_0(\Gamma^2_0)) \right\}.$$  

and denote by $\tilde{X}_s(\Omega)$ the closure of $\mathcal{M}(\Omega)$ with respect to the $H^s$-norm. One can see that

$$\tilde{X}_0(\Omega) = \{ f = (f^1; f^2; f^3) \in [L_2(\Omega)]^3 : \ \text{div} f = 0; \ (f, n) = 0 \text{ on } \Gamma^1_1 \cap \Gamma^2_1 \}$$

and

$$\tilde{X}_1(\Omega) = \left\{ f = (f^1; f^2; f^3) \in [H^1(\Omega)]^3 \bigg| \quad \text{div} f = 0, \ f = 0 \text{ on } \Gamma^1_1 \cap \Gamma^2_1, \ f^3 = 0 \text{ on } \Gamma^1_0 \cap \Gamma^2_0 \right\}.$$  

For details concerning this type of spaces see, e.g., [19, 28, 33].

**Remark 3.4.** In the case of the Poiseuille flow in the tube we deal with a domain satisfying the Friedrichs-Poincare property:

$$\exists d_\Omega > 0 : \int_\Omega |f(x)|^2 dx \leq d_\Omega^2 \int_\Omega |\nabla f(x)|^2 dx, \ \forall f \in H^1_0(\Omega).$$  

(32)

By the localization argument one can show that the inequality in 32 implies a similar property for any $f \in \{ g \in H^1(\Omega) : \ |g|_{\Gamma^1_1 \cap \Gamma^2_1} = 0 \}$ and thus

$$\exists c_\Omega > 0 : \ ||f||_\Omega \leq c_\Omega ||\nabla f||_\Omega, \ \forall f \in \tilde{X}_1(\Omega).$$  

(33)

4. Extension operator and characterization of spaces on $\Pi$.

4.1. **Trace operators on $\Pi$.** Our aim in this subsection is to construct a operators mapping traces from $\Gamma^i_0$ to $\Sigma^i_0$, $i = 1, 2$ and their inverse operators. For this purpose we give the following definition:

**Definition 4.1.** The operators $j_i : C^\infty(\overline{\Gamma^i_0}) \to \mathcal{C}(\overline{\Sigma^i_0})$ act by the formula

$$j_i q(x, y, z) = q(R_i \cos \phi, R_i \sin \phi, x) = b(\phi, x),$$  

(34)

where $y^2 + z^2 = R^2_i$. Now we extend these operators to the Sobolev-type spaces. We denote by $\mathcal{H}^s(\Pi)$ the closure of $\mathcal{C}(\Pi)$ in the $H^s(\Pi)$-norm and by $\mathcal{H}^s(\Sigma^i_0)$ the closure of $\mathcal{C}(\Sigma^i_0)$ in the $H^s(\Sigma^i_0)$-norm.

Let $q(x, y, z) \in \mathcal{H}^s(\Gamma^i_0)$, there exists a sequence $q_n(x, y, z) \in \mathcal{C}(\Sigma^i_0)$ such that $||q_n - q||_{\mathcal{H}^s(\Gamma^i_0)} \to 0$ as $n \to \infty$ and we define operators $j^*_i : \mathcal{H}^s(\Gamma^i_0) \to \mathcal{H}^s(\Sigma^i_0)$ as

$$j^*_i q = \lim_{n \to \infty} j_i q_n, \text{ in } H^s(\Sigma^i_0).$$  

(35)

**Lemma 4.2.** The operators $j_i$ are one-to-one and onto mappings.
Proof. Applying the reverse coordinates transformations
\[ \phi = g(y, z) = \begin{cases} \arctan \frac{z}{y}, & y > 0, z \geq 0, \\ \pi + \arctan \frac{z}{y}, & y < 0, \\ 2\pi + \arctan \frac{z}{y}, & y > 0, z < 0, \\ \frac{3}{2}\pi, & y = 0, z = R_i, \\ \frac{3}{2}\pi, & y = 0, z = -R_i, \end{cases} \tag{36} \]
we get that for any function \( b(\phi, x) \in C(\Omega_0^i) \) there exists
\[ q(x, y, z) = b(g(y, z), x) \in C(\Omega_0^i) \tag{37} \]
such that \( j_i q = b \). Thus, \( j_i \) are surjective.

It is easy to see that \( j_i \) are injections. Consequently, there exist operators \( j_i^{-1} : C(\Omega_0^i) \to C(\Omega_0^i) \) acting by the formula \( j_i^{-1} b(\phi, x) = q(x, y, z) \) on \( \Omega_0^i \) where \( q \) are given by 37.

Lemma 4.3. The operators \( j_i^s \) are well-defined for \( s \geq -1/2 \), i.e. the limits in 35 exist and definitions do not depend on the choice of sequences.

Proof. We can rewrite the function \( g(y, z) \) in 36 in local coordinates on the surfaces \( \Omega_0^i \)
\[ \phi = g(y, z) = g_1(y) = \arccos \frac{y}{R_i}, \tag{38} \]
and
\[ q(x, y, z) = q(x, y, \sqrt{R_i^2 - y^2}) = \tilde{q}_1(x, y) = b(g_1(y), x), \]
for \( 0 \leq \phi \leq \pi \) or
\[ \phi = g(y, z) = g_2(y) = 2\pi - \arccos \frac{y}{R_i}, \tag{39} \]
and
\[ q(x, y, z) = q(x, y, -\sqrt{R_i^2 - y^2}) = \tilde{q}_2(x, y) = b(g_2(y), x), \]
for \( \pi \leq \phi \leq 2\pi \).

The unit normal field on the surfaces of the outer and inner cylinders have the form \( n_i(x, y, z) = \left(0, \pm \frac{y}{\sqrt{y^2 + z^2}}, \pm \frac{z}{\sqrt{y^2 + z^2}}\right) \). The tangential gradient \( \nabla_{\Omega_0^i} \) on the surfaces of the cylinders are defined by
\[ \nabla_{\Omega_0^i} q(x, y, z) = (I - n_i(x, y, z) \otimes n_i(x, y, z)) \nabla q(x, y, z). \tag{40} \]
The Laplace-Beltrami operator on the cylinders in the Cartesian coordinates then has the form
\[ \Delta_{\Omega_0^i} = \nabla_{\Omega_0^i} \cdot \nabla_{\Omega_0^i} = (1 - 2y^2 + y^4 \frac{R_i^4}{R_i^4}) \frac{\partial}{\partial y} \frac{y^2}{R_i^4} \frac{\partial}{\partial y} + (1 - 2y^2 + y^4 \frac{R_i^4}{R_i^4}) \frac{\partial}{\partial y} \frac{y^2}{R_i^4} \frac{\partial}{\partial y} + (4z^2 y^2 \frac{R_i^4}{R_i^4} - 4y^2 z^2 \frac{R_i^4}{R_i^4} + y(2y - z^2)) \frac{\partial}{\partial y} \frac{\partial}{\partial y} + (4z^2 y^2 - 4z^2 y^2 + z(z^2 - y^2)) \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \tag{41} \]
and in the cylindrical coordinates
\[ \Delta_{\Omega_0^i}^p = \frac{1}{R_i^2} \frac{\partial^p}{\partial \phi^2} + \frac{\partial^2}{\partial \phi^2} \]
Using the definition of Sobolev spaces on manifolds 36, and 37 we obtain that for any \( q \in C^\infty(\Gamma_0^1) \) and \( k \in \mathbb{N} \)
\[
\| j_i q \|_{2 \mathcal{H}^k(\Gamma_0^1)}^2 = \int_{\Gamma_0^1} \int_0^{L_i} \left| (I - \Delta q_i)|^{k/2} (j_i q)(\phi, x) \right|^2 d\phi dx \\
\leq C \left( \int_{-R_i}^{R_i} \int_{i_i}^{L_i} \left| (I - \Delta q_i)|^{k/2} \tilde{q}_1(x, y) \right|^2 \frac{R_i}{\sqrt{R_i^2 - y^2}} dy dx \\
+ \int_{-R_i}^{R_i} \int_{i_i}^{L_i} \left| (I - \Delta q_i)|^{k/2} \tilde{q}_2(x, y) \right|^2 \frac{R_i}{\sqrt{R_i^2 - y^2}} dy dx \right) \\
\leq C \int_{\Gamma_0^1} \left| (I - \Delta q_i)|^{k/2} q(x, y, z) \right|^2 d\xi \leq C \| q \|_{2 \mathcal{H}^k(\Gamma_0^1)}^2.
\]

Arguing in the same way we get the reverse estimate and by the interpolation argument for any \( s \geq 0 \)
\[
C_1 \| q \|_{2 \mathcal{H}_s(\Gamma_0^1)}^2 \leq \| j_i q \|_{2 \mathcal{H}^s_0(\Gamma_0^1)}^2 \leq C_2 \| q \|_{2 \mathcal{H}_s^c(\Gamma_0^1)}^2.
\]

For any \( \{ q_m \} \in \mathcal{C}(\Omega) \) such that \( \| q_m - q \|_{\mathcal{H}_s(\Gamma_0^1)} \to 0, \ m \to \infty \)
\[
\| j_i q_k - j_i q_m \|_{\mathcal{H}_s(\Gamma_0^1)} \leq C \| q_n - q_m \|_{\mathcal{H}_s(\Gamma_0^1)} \to 0, \ n, m \to \infty.
\]

Consequently, the limit in 35 exists for \( s \geq 0 \). Let there exist two sequences \( \{ q_m^1 \}, \{ q_m^2 \} \in \mathcal{H}_s(\Gamma_0^1) \) such that \( \| q_m^l - q \|_{\mathcal{H}_s(\Gamma_0^1)} \to 0 \) as \( m \to \infty \) for \( l = 1, 2 \). Then \( \| j_i q_m^1 - j_i q_m^2 \|_{\mathcal{H}_s(\Gamma_0^1)} \leq C \| q_m^1 - q_m^2 \|_{\mathcal{H}_s(\Gamma_0^1)} \to 0 \) and the lemma is proved for \( s \geq 0 \).

To obtain the same result for \(-1/2 \leq s < 0\) we use 22 and 23. The function \( q \in C^\infty(\Gamma_0^1) \) can be treated as an element of \( \mathcal{H}^{-1/2}(\Gamma_0^1) \) and at the same time (by isometry) as an element of the dual space to \( \mathcal{H}^{1/2}(\Gamma_0^1) \). Due to 43 we have
\[
\| q \|_{\mathcal{H}_{-1/2}(\Gamma_0^1)}^2 = \inf_{p \in \mathcal{H}^{1/2}(\Gamma_0^1)} \int_{\Gamma_0^1} \| p \|_{\mathcal{H}_{1/2}(\Gamma_0^1)} \| q \|_{\mathcal{H}_{-1/2}(\Gamma_0^1)} d\xi \\
\leq C \inf_{\| p \|_{\mathcal{H}^{1/2}(\Gamma_0^1)} = 1} \int_{\Gamma_0^1} \int_0^{L_i} |j_i q|^2 \| j_i q \|_{\mathcal{H}^{1/2}(\Gamma_0^1)} d\phi d\xi \\
\leq C \int_{\Gamma_0^1} \| j_i q \|_{\mathcal{H}_{-1/2}(\Gamma_0^1)} \| j_i q \|_{\mathcal{H}_{1/2}(\Gamma_0^1)} d\xi \leq C \| j_i q \|_{\mathcal{H}_{-1/2}(\Gamma_0^1)}.
\]

We can analogously infer the inverse estimate and interpolating 4.1 with 43 and arguing as above for we get the statement of the lemma.

**Lemma 4.4.** The operators \( j_i^s \) define homeomorphisms between spaces \( \mathcal{H}^s(\Gamma_0^1) \) and \( \mathcal{H}_{s}(\Gamma_0^1) \) for \( s \geq -1/2 \).

**Proof.** Due to estimate 43 for any \( s \geq 1/2 \) the remaining task is to show that operators \( j_i^s \) are surjective. Indeed, if \( b(\phi, x) \in \mathcal{H}^s(\Gamma_0^1) \) there exists an approximating
sequence $b_n \in \mathcal{C}(\Gamma'_0)$ and by Lemma 4.3 there exists a sequence $\{q_n\} \in C^\infty(\Gamma'_0)$ such that $j_n q_n = b_n$. Then 43 leads to
\[\|q_n - m\|_{\mathcal{H}^s(\Gamma'_0)} = \|(j_i)^{-1}(b_n - m)\|_{\mathcal{H}^s(\Gamma'_0)} \leq C\|b_n - b_m\|_{\mathcal{H}^s(\Gamma'_0)} \to 0, \ m, m \to \infty,\]
therefore, there exists the limit $q$ of $\{q_n\}$ in $\mathcal{H}^s(\Gamma'_0)$ and
\[j^*_n q = \lim_{n \to \infty} j_n q_n = \lim_{n \to \infty} b_n = b\]
in $\mathcal{H}^s(\Gamma'_0)$. Consequently, the operators $j^*_n$ are surjective. □

We introduce the spaces
\[\mathcal{C}_l(\Gamma'_0) = \{b(\phi, x) \in C^\infty(\Gamma'_0) : \partial^{(j)}_\phi b(0, x) = \partial^{(j)}_\phi b(2\pi, x), \ j = 0, 1, 2, \ldots, \ i = 1, 2, \ \text{supp} \ b \in \Gamma_0 \cup \{\phi = 0\} \cup \{\phi = 2\pi\}\}.
\]
By $\mathcal{H}_l^s(\Gamma'_0)$ we denote the closures of $\mathcal{C}_l(\Gamma'_0)$ in $H^s(\Gamma'_0)$. In particular, by the trace theorems for functions in Lipschitz domains [1],
\[\mathcal{H}^s_l(\Gamma'_0) = \begin{cases} \{b \in H^s(\Gamma'_0) : b(\phi, l_i) = b(\phi, L_i) = 0, \ b(0, x) = b(2\pi, x)\}, & 1/2 < s < 3/2 \\ H^s(\Gamma'_0), & s < 1/2 \end{cases} \tag{44}\]
The following assertion is a corollary of Lemma 4.4.

Lemma 4.5. The operators $j_{i0}$ defined as $j_i$ in 34 but considered as operators from $C^\infty_0(\Gamma'_0)$, $i = 1, 2$ to the spaces $\mathcal{C}_l(\Gamma'_0)$ can be prolonged to the homeomorphisms $j^*_0$ acting from $\mathcal{H}^s_0(\Gamma'_0)$ to $\mathcal{H}^s_l(\Gamma'_0)$ for $s > -1/2$ and $s \neq 1/2 + \mathbb{Z}$. Analogously, the inverse operators $(j_{i0})^{-1} : \mathcal{C}_l(\Gamma'_0) \to C^\infty_0(\Gamma'_0)$ can be prolonged to the homeomorphisms $(j^*_{i0})^{-1} : \mathcal{H}^s_l(\Gamma'_0) \to \mathcal{H}^s_0(\Gamma'_0)$.

4.2. Coordinates transformation operators. Now we define the operator $J : \mathcal{C}(\Omega) \to \mathcal{C}(\Pi)$ by the formula
\[J f(x, y, z) = V(r, \phi, x) = (v(r, \phi, x); u(r, \phi, x); w(r, \phi, x)). \tag{45}\]
\[v(r, \phi, x) = f_2(r \cos \phi, r \sin \phi, x) \cos \phi + f_3(r \cos \phi, r \sin \phi, x) \sin \phi, \]
\[u(r, \phi, x) = -f_2(r \cos \phi, r \sin \phi, x) \sin \phi + f_3(r \cos \phi, r \sin \phi, x) \cos \phi, \]
\[w(r, \phi, x) = f_1(r \cos \phi, r \sin \phi, x) \]
and operators $J_s : H^s(\Omega) \to \mathcal{H}^s(\Pi)$ for $s \geq 0$ as follows:
\[J_s f = \lim_{n \to \infty} J f_n, \ \text{in} \ \mathcal{H}^s(\Pi), \ \text{where}, \ f \in H^s(\Omega), \ \{f_n\} \in \mathcal{C}(\Omega), \ \lim_{n \to \infty} \|f_n - f\|_{H^s(\Omega)} = 0. \tag{47}\]

Lemma 4.6. (i) Operator $J$ is a one-to-one and onto mapping. (ii) Operators $J_s$ are well-defined, i.e. the limit in 47 exists and the definition does not depend on the choice of the sequence $\{f_n\}$. (iii) $J_s$ defines a homeomorphism between $H^s(\Omega)$ and $\mathcal{H}^s(\Pi)$.

Proof. First we prove that operator $J$ is surjective. Indeed, applying the reverse coordinates transformation
\[r = \sqrt{y^2 + z^2} \tag{48}\]
\[\phi = h(y, z) = \begin{cases} \arctan \frac{z}{y}, \ y > 0, z \geq 0, \\ \pi + \arctan \frac{z}{y}, \ y < 0, \\ 2\pi + \arctan \frac{z}{y}, \ y > 0, z < 0, \\ \pi, \ y = 0, z > 0, \\ \frac{3\pi}{2}, \ y = 0, z < 0, \end{cases} \tag{49}\]
we get that for any function \( V(r, \phi, x) = (v(r, \phi, x), u(r, \phi, x), w(r, \phi, x)) \in \mathcal{C}(\Pi) \) there exists \( f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \in \mathcal{C}(\Omega) \) of the form

\[
\begin{align*}
    f_1(x, y, z) &= w(\sqrt{y^2 + z^2}, h(y, z), x) \\
    f_2(x, y, z) &= -u(\sqrt{y^2 + z^2}, h(y, z), x) \sin h(y, z) + v(\sqrt{y^2 + z^2}, h(y, z), x) \cos h(y, z) \\
    f_3(x, y, z) &= u(\sqrt{y^2 + z^2}, h(y, z), x) \cos h(y, z) + v(\sqrt{y^2 + z^2}, h(y, z), x) \sin h(y, z)
\end{align*}
\]

such that \( Jf = g \). Thus, \( J \) is surjective.

It is obvious that operator \( J \) is an injection. Consequently, there exists an inverse operator \( J^{-1} : \mathcal{C}(\Pi) \to \mathcal{C}(\Omega) \) acting by the formula \( J^{-1}V(r, \phi, x) = f(x, y, z) \).

For any \( f \in \mathcal{C}(\Omega) \) \( J_s f = Jf \). Indeed, we can choose in definition 47 \( f_n = f \) for any \( n \in \mathbb{N} \). For any \( f \in \mathcal{C}(\Omega) \)

\[
\|Jf\|_{H^s(\Omega)}^2 = \sum_{|\alpha|=0}^k \int_{0}^{\frac{R_2}{2\pi}} \int_{0}^{2\pi} |D^\alpha Jf(r, \phi, x)|^2 r dr d\phi dx
\]

\[
\leq C \sum_{|\alpha|=0}^k \int_{0}^{R_2} \int_{0}^{2\pi} |D^\alpha f(x, y, z)|^2 (1 + y^2 + z^2)^k dx dy dz \leq C\|f\|_{H^s(\Omega)}^2
\]

Relying on the interpolation theorem we get that for fractional \( s \geq 0 \)

\[
\|Jf\|_{H^s(\Omega)}^2 \leq C\|f\|_{H^s(\Omega)}^2.
\]

For any \( V \in \mathcal{C}(\Pi) \)

\[
\|(J)^{-1}V\|_{H^s(\Omega)}^2 = \sum_{|\alpha|=0}^k \int_{0}^{2\pi} |D^\alpha (J)^{-1}V(r, \phi, x)|^2 dx dy dz
\]

\[
\leq C \sum_{|\alpha|=0}^k \int_{0}^{2\pi} \int_{0}^{R_2} |D^\alpha V(r, \phi, x)|^2 P_k(r \cos \phi, r \sin \phi) r dr d\phi dx
\]

\[
\leq C\|V\|_{H^s(\Omega)}^2,
\]

where \( P_k(r \cos \phi, r \sin \phi) \) are polynomials of the \( k \)-th degree. Relying on the interpolation theorem we get that for fractional \( s \geq 0 \)

\[
\|(J)^{-1}V\|_{H^s(\Omega)}^2 \leq C\|V\|_{H^s(\Omega)}^2.
\]

Arguing as in Lemma 4.4 we obtain from 53, 54 the statement of the lemma. \( \square \)

### 4.3. Extension operator and characterization of spaces.

Let

\[
\tilde{L} = \{ b = (b^1, b^2) \in L^2(\mathcal{T}^1_0) \times L^2(\mathcal{T}^2_0) : \int_{l_1}^{L_1} \int_0^{2\pi} b^1 dx d\phi = \int_{0}^{\frac{2\pi}{2\pi}} \int_0^{b^2 dx d\phi} \}
\]

**Lemma 4.7.** A function \( b = (b^1, b^2) \in \tilde{L} \) can be approximated by a sequence of functions \( b_m = (b_m^1, b_m^2) \in (\mathcal{C}(\mathcal{T}^1_0) \times \mathcal{C}(\mathcal{T}^2_0)) \cap \tilde{L} \).
For the proof see the Appendix below.

Due to Proposition 3.2 and Lemma 4.5 one can define for \( s \geq 0 \) and \( s \neq 1/2 + \mathbb{Z} \) the extension operators \( \text{Ext}^{s}_{\mathcal{T}_{0}^{s}} : (\mathcal{H}^{s}(\mathcal{Y}_{0}) \times \mathcal{H}^{s}(\mathcal{Y}_{0})) \cap \mathbb{L} \to \mathcal{H}^{s+1/2}(\mathbb{L}) \) as

\[
\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}, b^{2}] = J_{s+1/2}\text{Ext}^{s}[(j_{10}^{s})^{-1}b^{1}, (j_{20}^{s})^{-1}b^{2}].
\]

(55)

**Proposition 4.8.** The following properties of operators \( \text{Ext}^{s}_{\mathcal{T}_{0}^{s}} \) hold true for any \( s \geq 0 \) such that \( s \neq 1/2 + \mathbb{Z} \):

(i) \[
\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}, b^{2}]|_{\mathcal{T}_1} = (0; 0; 0), \quad \text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}, b^{2}]|_{\mathcal{T}_0} = (b^{1}; 0; 0).
\]

(ii) Let \( \text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}, b^{2}] = (v, u, w) \), then \( \frac{v}{r} + v_{r} + \frac{u}{r} + w_{x} = 0 \).

Proof. By Lemma 4.7 there exist \( \{b^{i}_{m}\} \in \mathcal{C}(\mathcal{Y}_{0}) \) for \( i = 1, 2 \) such that \( \|b^{i}_{m} - b^{i}\|_{\mathcal{H}^{s}(\mathcal{Y}_{0})} \to 0 \) as \( m \to \infty \) and \( (b^{i}_{m}, b^{i}_{m}) \in \mathcal{L} \). If we denote \( q^{i}_{m} = (j_{io})^{-1}b^{i}_{m} \) and \( \text{Ext}[(q^{i}_{m}, q^{i}_{m})] = f_{m} = (f_{1m}, f_{2m}, f_{3m}) \in \mathcal{C}(\Omega) \) ([33], Proposition 2.2) then, for any \( s \geq 0 \), \( s \neq 1/2 + \mathbb{Z} \)

\[
\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}_{m}, b^{2}_{m}] = J\text{Ext}[(j_{10})^{-1}b^{1}_{m}, (j_{20})^{-1}b^{2}_{m}] = J\text{Ext}[(q^{1}_{m}, q^{2}_{m})]
\]

\[
= (f_{2m}(r \cos \phi, r \sin \phi, x) \cos \phi + f_{3m}(r \cos \phi, r \sin \phi, x) \sin \phi, \nonumber
\]

\[
- f_{2m}(r \cos \phi, r \sin \phi, x) \sin \phi + f_{3m}(r \cos \phi, r \sin \phi, x) \cos \phi, \nonumber
\]

\[
f_{1m}(x, r \cos \phi, r \sin \phi)) = (v_{m}, u_{m}, w_{m})
\]

Consequently,

\[
\|v_{m} + v_{mr} + \frac{u_{m} \phi}{r} + w_{mx}\|_{L^{2}(\Pi)} \leq C \|f_{1m} + f_{2m} + f_{3m}\|_{L^{2}(\Omega)} = 0.
\]

(57)

By Proposition 3.2, Lemmas 4.4, 4.6 we infer the estimate

\[
\|\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}_{m}, b^{2}_{m}] - \text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}, b^{2}]\|_{H^{s+1/2}(\mathbb{L})} \nonumber
\]

\[
\leq C(\|\text{Ext}[(j_{10})^{-1}b^{1}_{m} - b^{1}]_{\mathcal{L}^{2}(\mathbb{L})} + \|j_{10}^{-1}b^{1} - j_{20}^{-1}b^{2}\|_{H^{s+1/2}(\mathbb{L})}) \nonumber
\]

\[
\leq C(\|j_{10}^{-1}b^{1} - b^{1}\|_{\mathcal{L}^{2}(\mathbb{L})}^{2} + \|j_{20}^{-1}b^{2}\|_{\mathcal{L}^{2}(\mathbb{L})}^{2}) \nonumber
\]

\[
\leq C(\|b^{1} - b^{1}\|_{\mathcal{L}^{2}(\mathcal{Y}_{0})}^{2} + \|b^{2} - b^{2}\|_{\mathcal{L}^{2}(\mathcal{Y}_{0})}^{2}) \to 0, \quad m \to \infty.
\]

(58)

Then (ii) follows from 57 and 58.

Obviously, \( \text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}_{m}, b^{2}_{m}]|_{\mathcal{T}_1} = (0, 0, 0) \) on \( \Gamma^{1}_{1} \cap \Gamma^{2}_{2} \). From (56) and Proposition 3.2 we deduce that \( \text{Ext}[(j_{10})^{-1}b^{1}_{m}, (j_{20})^{-1}b^{2}_{m}]|_{\mathcal{T}_0} = (0, q^{1}_{m}, q^{2}_{m}, q^{1}_{m}, q^{2}_{m}) \). Therefore,

\[
\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{1}_{m}, b^{2}_{m}]|_{\mathcal{T}_0} = (b^{1}_{m}, 0, 0).
\]

For \( s > 1 \) we can replace \( b^{i}_{m} \) with \( b^{i} \) in 56 with the same computations, for \( 0 \leq s \leq 1 \) properties in (i) are satisfied owing to the trace theorem for Lipschitz domains (see, [1]).

\[
\text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b^{i}]|_{\mathcal{T}_0} = (0, \tilde{q}^{i}_{m}, \tilde{q}^{i}_{m}, \tilde{q}^{i}_{m})
\]

Using the extension operator constructed above we introduce the set

\[
\mathcal{M}(\Pi) = \left\{ V = V_{0} + \text{Ext}^{s}_{\mathcal{T}_{0}^{s}}[b] : V_{0} \in \mathcal{C}_{p}(\Pi), \quad b \in \mathcal{L} \cap (\mathcal{C}(\mathcal{Y}_{0}) \times \mathcal{C}(\mathcal{Y}_{0}^{2})) \right\}.
\]

Then we denote by \( \tilde{\mathcal{X}}_{s}(\Pi) \) the closure of \( \mathcal{M}(\Pi) \) with respect to the \( \mathcal{H}^{s} \)-norm.
Proposition 4.9. The following characterization of spaces holds true:

\[
\tilde{X}_0(\Pi) = \left\{ V = (v; u; w) \in [L_2(\Pi)]^3 \middle| \begin{array}{l}
\frac{v}{r} + v_r + \frac{u}{r} + w_x = 0; \\
v = 0 \text{ on } \Upsilon_1^1 \cup \Upsilon_1^2; \\
u(r, 2\pi, x) = u(r, 0, x)
\end{array} \right\}
\]

(59)

and

\[
\tilde{X}_1(\Pi) = \left\{ V = (v; u; w) \in [H^1(\Pi)]^3 \middle| \begin{array}{l}
\frac{v}{r} + v_r + \frac{u}{r} + w_x = 0, \\
v = u = w = 0 \text{ on } \Upsilon_1^1 \cup \Upsilon_1^2, \\
u = w = 0 \text{ on } \Upsilon_0^1 \cap \Upsilon_0^2, \\
V(r, 2\pi, x) = V(r, 0, x)
\end{array} \right\}
\]

(60)

Proof. It follows from the definition of the operators \( Ext^s_{l_0} \) and Lemma 4.6 that the operator \( J \) is a bijective mapping from \( M(\Omega) \) to \( M(\Pi) \) and, consequently, \( J_s \) are homeomorphisms between \( X_s(\Omega) \) and \( X_s(\Pi) \). We need only to check the forth property in 59, since the other assertions follow from Proposition 4.8 and the trace theorem. Indeed, for any \( V(r, \phi, x) \in \tilde{X}_0(\Pi) \) where exists a sequence \( V_m(r, \phi, x) \in M(\Pi) \) such that \( \| V - V_m \|_{L_2(\Pi)} \to 0 \) then \( m \to \infty \) and \( V_m = (v_m, u_m, w_m) = V_{0m} + Ext_{l_0}^s[b_m] \), where \( V_{0m} \in C_{lp}(\Pi) \) and \( b_m \in \tilde{L} \cap \tilde{C}_l(\Upsilon_0^1) \times \tilde{C}_l(\Upsilon_0^2) \). By Proposition 3.2, \( (J^{-1}V_m, n_i)_{\Gamma_1^1} = 0 \). Then, by Proposition 4.8 \( j_i(J^{-1}V_m, n_i)_{\Gamma_1^1} = v_m|_{\Gamma_1^1} = 0 \). By Theorem 1.2 [33]

\[
\| (J^{-1}V_m, n_i)_{\Gamma_1^1} - (J_0^{-1}V, n_i)_{\Gamma_1^1} \|_{H^{-1/2}(\Gamma_1^1)} \to 0, \ m \to \infty.
\]

We consider the domain \( \Pi' \) being the image of \( \Omega' \) constructed in Proposition 3.1 after coordinates transformation \( T \). Since for functions from \( H^1 \) in a Lipschitz domain one can define the continuous trace operator acting onto \( H^{1/2} \) on the boundary with the right inverse lifting operator (see, e.g. [1]), then the conditions of Remark 1.3 in [33] are satisfied and the generalized Stokes formula is valid for \( \Pi' \). We note that \( \partial \Pi' = (\Upsilon_0^1 \cup \Upsilon_1^1) \cup (\Upsilon_1^2 \cup \Upsilon_0^2) \cup (\Sigma_1 \cup \Sigma_2) \cup \Lambda \), where \( \Upsilon_1^1 \) are bounded parts of \( \Upsilon_1^1, \Sigma_1 \) and \( \Sigma_2 \) are bounded and belong to planes \( \phi = 0 \) and \( \phi = 2\pi \) (and have the same shape) and \( \Lambda = T(\Omega \setminus \Omega') \). For any \( V \in M(\Pi) \)

\[
\int_{\partial \Pi' \setminus (\Sigma_1 \cup \Sigma_2)} (V, n) ds = 0.
\]

Consequently, by Proposition 4.9 for any \( \Sigma_1 \) belonging to the strip \( R_1 < r < R_2 \)

\[
0 = \int_{\Pi'} \left( \frac{v}{r} + v_r + \frac{u}{r} + w_x \right) r dr dx \phi = \int_{\Sigma_1 \cup \Sigma_2} (V, n) ds
\]

\[
= \int_{\Sigma_1} (u(0, r, x) - u(2\pi, r, x)) r dr dx.
\]

Consequently, \( u(0, r, x) - u(2\pi, r, x) = 0 \) on any bounded smooth subdomain of the strip \( R_1 < r < R_2 \). By the generalized Stokes formula one can show (analogously to [33]) that the same equality holds for \( V \in X_0(\Pi) \) in the sense of distributions in \( H^{-1/2} \). The proposition is proved. \( \square \)

5. Well-posedness. To define weak (variational) solutions to problem 1-13 we need to define the class of test functions. Let

\[
\tilde{H} = (\mathcal{H}^2_l(\Upsilon_0^1) \times \mathcal{H}^2_l(\Upsilon_0^2)) \cap \tilde{L}
\]

(61)
We denote by $\tilde{P}$ the projection on $\tilde{H}$ in $\mathfrak{H}_2^2(\Gamma_0^1) \times \mathfrak{H}_2^2(\Gamma_0^2)$ which is orthogonal with respect to the inner product $(\Delta R_1, \Delta R_1; \bar{\mathfrak{H}}_2^2 + \Delta R_2, \Delta R_2; \bar{\mathfrak{H}}_2^2)$. The subspace $(I - \tilde{P})(\mathfrak{H}_2^2(\Gamma_0^1) \times \mathfrak{H}_2^2(\Gamma_0^2))$ consists of functions $\eta \in \mathfrak{H}_2^2(\Gamma_0^1) \times \mathfrak{H}_2^2(\Gamma_0^2)$ such that $\Delta^2 \eta_1 = \Delta^2 \eta_2 = C$. Similarly to [13] we consider the following class of test functions:

$$\mathcal{L}_T = \left\{ \Psi = (\psi_1, \psi_2, \psi_3) \mid \begin{array}{l}
\Psi \in L_2(0, T; [H^2(\Pi)]^3), \\
\Psi_t \in L_2(0, T; [L_2(\Pi)]^3), \\
\psi_1 + \psi_1 + \psi_2 + \psi_3 = 0, \\
\Psi|_{\Gamma_0^1} = (b^i, 0, 0), \\
\eta_i = 0 = \eta_0 = (\eta_{10}, \eta_{20}), \\
b^i \in L_2(0, T; \tilde{H}), \\
\Psi(T) = 0.
\end{array} \right\}$$

**Definition 5.1.** A pair of functions $(V(t); \eta(t))$, where $\eta(t) = (\eta^1(t), \eta^2(t))$ is said to be a weak solution to 1–13 on a time interval $[0, T]$ if

- $V \in L_\infty(0, T; X_0(\Pi)) \cap L_2(0, T; X_1(\Pi))$;
- $\eta = (\eta_1, \eta_2) \in L_\infty(0, T; \mathfrak{H}_2^2(\Gamma_0^1) \times \mathfrak{H}_2^2(\Gamma_0^2))$, $\eta_t \in L_\infty(0, T; \tilde{L})$, $\eta(0) = \eta_0 = (\eta_{10}, \eta_{20})$;
- for every $\Phi \in \mathcal{L}_T$ the following equality holds:

$$- \int_0^T (V, \Psi_t) dt + \mu \int_0^T (\nabla V, \nabla \Psi) dt + \int_0^T (L_0 V, \Psi) dt$$

$$- \int_0^T (\eta_t, b_1) dt + \int_0^T (\Delta R_1, \eta^1, \Delta R_1, b^1) dt + \int_0^T (\Delta R_2, b_2) dt$$

$$= \int_0^T (G_f, \Psi) dt + \int_0^T (G_b, b_1, \eta) dt - \int_0^T (\mathfrak{F}(\eta(t)), b) dt$$

$$+ (V_0, \Psi(0)) + \int_0^T (\eta_1, b(0)) dt,$$

where $V_0 = (v_0, u_0, w_0)$, $b = (b^1, b^2)$, $G_b = (G_b^1, G_b^2)$, $\eta_1 = (\eta_{11}, \eta_{12})$, $\mathfrak{F}(\eta) = (\mathfrak{F}^1(\eta^1), \mathfrak{F}^2(\eta^2))$, where $\mathfrak{F}^i(\eta^i) = (\eta^i, F^i(\eta^i))$ and $F^1(\eta^1)$ are solutions to 9–11.

- the compatibility condition $V(t)|_{\Gamma_0^1}(\eta_{1t}(t); 0; 0)$ holds for almost all $t$ and $i = 1, 2$.

**Lemma 5.2.** For any $\eta, F, \xi \in \mathfrak{H}_2^2(\Gamma_0^i), i = 1, 2$

$$\int_{\Gamma_0^1} \langle \eta, F \rangle \xi dx d\phi = \int_{\Gamma_0^2} \langle \eta, \xi \rangle F dx d\phi. \quad (62)$$

**Proof.** To justify the subsequent computations we use the approximation procedure by the prolongations by zero to $\bar{\mathfrak{H}}^i$ of functions from $\mathcal{C}_1(\Gamma_0^i)$. Then,

$$\int_{\Gamma_0^i} \langle \eta, F \rangle \xi dx d\phi = \int_{\bar{\mathfrak{H}}^i} \langle \eta, F \rangle \xi dx d\phi$$

$$= \frac{1}{R_i^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \left( R_i \xi_z + \partial^2_{zz} (F \eta_{z\phi}) - 2 \partial^2_{z\phi} (F \eta_{x\phi}) + \partial^2_{\phi\phi} (F \eta_{xx}) \right) \xi dx d\phi$$

$$= - \frac{1}{R_i^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \left( R_i \xi_x + \partial_z (F \eta_{\phi\phi}) \xi \right)$$
and we can use eigenfunctions of the Stokes operator. Consequently, compactness properties of the fluid
Using the properties of functions from $\mathcal{C}_l(\Upsilon_0)$ it easy to see that
\[
\partial_\phi(F_{\eta xx})\xi_x^2 = F_{\phi \eta xx}\xi_0^2 + F_{\eta xx\phi}\xi_0^2 = 0
\]
and
\[
\partial_x(F_{\eta xx})\xi_x^2 = F_x\eta xx\xi_0^2 + F_{\eta xx\phi}\xi_0^2 = 0.
\]
Consequently,
\[
\int_{\Upsilon_0}^\infty \langle \eta, F \rangle d\xi d\phi = \frac{1}{R_i^2} \int_0^{2\pi} \int_{-\infty}^\infty (R_i F_{\xi x} + F_{\eta xx}\phi_0) d\phi
\]
In further considerations we take into account the fact that the domain considered
is bounded in two directions. Consequently, compactness properties of the fluid
velocity variable are valid and we can use eigenfunctions of the Stokes operator.

For the phase spaces we use
\[
\mathcal{H} = \left\{(V_0; \eta_0; \eta_1) \in X_0(P) \times (\mathcal{H}_2^2(\Upsilon_0^1) \times \mathcal{H}_2^2(\Upsilon_0^2)) \times \tilde{L} : v_0 = \eta_1 \text{ on } \Upsilon_0^1 \right\}
\]
or
\[
\tilde{\mathcal{H}} = \left\{(V_0; \eta_0; \eta_1) \in \mathcal{H} : \eta_0 \in \tilde{H} \right\} \subset \mathcal{H}
\]
with the norm $\|(V_0; \eta_0; \eta_1)\|_2^2 = \|V_0\|^2_2 + \|\Delta R_1, \eta_01\|^2_{\mathcal{H}_2^1} + \|\Delta R_2, \eta_02\|^2_{\mathcal{H}_2^2} + \|u_1\|^2_{\mathcal{H}_2^1 \times \mathcal{H}_2^2}$. Applying Galerkin approximations procedure and the arguments on the uniqueness
and smoothness of the solutions similar to those used in [13] and using Lemma 5.2
we come to the following result

**Theorem 5.3.** Assume that $U_0 = (V_0; \eta_0; \eta_1) \in \mathcal{H}$, $G_f(t) \in L_2(0, T; X'_1(\Pi))$ and
$G_h(t) \in L_2(0, T; \mathcal{H}^{-1/2}(\Upsilon_0^1) \times \mathcal{H}^{-1/2}(\Upsilon_0^2))$. Then for any interval $[0, T]$ there exists
a unique weak solution $(V(t); \eta(t))$ to 1-13 with the initial data $U_0$. This solution
possesses the property
\[
U(t) \equiv (V(t); \eta(t); \eta(t)) \in C(0, T; \mathcal{H}),
\]
and satisfies the energy balance equality

\[ E(V(t), \eta(t), \eta_t(t)) + \int_0^t [\mu \|\nabla V\|_{L^2}^2 + (L_0 V, V)_\Omega] \, dt \]

\[ = E(V_0, \eta_0, \eta_1) + \int_0^t (G_f, V)_{\Omega t} \, dt + \int_0^t (G_b, \eta_t)_{\Omega_t^b \times \Omega_0^b} \, dt \]

(68)

for every \( t > 0 \), where the energy functional \( E \) is defined by the relation

\[ E(V, \eta, \eta_t) = \frac{1}{2} \bigl( \| V \|_2^2 + \| \eta_t \|_2^2 + \| \Delta R_i \eta \|_2^2 + \| \Delta R_i \eta_t \|_2^2 \bigr) + \Pi(\eta), \]

here the potential \( \Pi \) has the form \( \Pi(\eta) = \Pi^1(\eta^1) + \Pi^2(\eta^2) \), where

\[ \Pi^i(\eta^i) = \frac{1}{4E_i h_i} \int_{\Gamma_0^i} |\Delta R_i F^i(\eta^i)|^2 \, dx. \]

Moreover, there exists a constant \( a_{C,T} > 0 \) such that for any couple of weak solutions \( \hat{U}(t) = (V(t); \eta(t); \eta_t(t)) \) and \( \hat{U}(t) = (\bar{V}(t); \bar{\eta}(t); \bar{\eta}_t(t)) \) with the initial data possessing the property \( \| \hat{U}_0 \|_{2\chi}, \| \bar{U}_0 \|_{2\chi} \leq C \) we have

\[ \| U(t) - \bar{U}(t) \|_2^2 + \int_0^t \| \nabla (V - \bar{V}) \|_{L^2}^2 \, dt \leq a_{C,T}^2 \| U_0 - \bar{U}_0 \|_2^2, \quad t \in [0, T]. \]  

(69)

If \( U_0 \in \hat{K} \), then \( U(t) \in \hat{K} \) for every \( t > 0 \).

6. Asymptotical behavior. In this section we are interested in global asymptotic behavior of the dynamical system generated in the space \( \hat{K} \). Our main result states the existence of a compact global attractor of finite fractal dimension.

Using the ideas presented in \([12]\) one can prove the following result.

**Lemma 6.1.** There exists \( C(\delta) > 0 \) such that for any \( \eta_0, \eta_1 \in \mathcal{H}_i^2(\Sigma_0^i) \) the estimate

\[ \| \langle \eta_0, F_i(\eta_1) \rangle - \langle \eta_1, F_i(\eta_0) \rangle \|_{2\chi - \sigma(\Sigma_0^i)} \]

\[ \leq C(1 + \| \eta_0 \|_{2\chi(\Sigma_0^i)}^2 + \| \eta_1 \|_{2\chi(\Sigma_0^i)}^2) \| \eta_0 - \eta_1 \|_{2\chi - \sigma(\Sigma_0^i)} \]

holds true for \( i = 1, 2 \) and every \( 0 < \sigma \leq 1/2 \).

**Proof.** To justify the subsequent arguments we prolong the functions \( \eta_k \) by zero to a smooth bounded domain \( \Sigma_0^i \) containing \( \Sigma_0^i \) by adding to \( \Sigma_0^i \) two “semicircles” on each side where Dirichlet boundary conditions hold. The functions \( \tilde{F}_i = F_i(\tilde{\eta}_i) - F_i(\eta_i) \) solve the problems

\[ \frac{1}{E_i h_i} \Delta_0^2 \tilde{F}_i + \langle \eta_i + \tilde{\eta}_i, \tilde{\eta}_i - \eta_i \rangle = 0 \]

(70)

\[ \tilde{F}_i(\phi, x, t) = \nabla \tilde{F}_i(\phi, x, t) = 0, \quad \text{on} \quad \partial \Sigma_0^i \setminus \partial \Sigma_0^i \]

(71)

\[ \tilde{F}_i(0, x, t) = \tilde{F}_i(2\pi, x, t), \quad \tilde{F}_i(\phi, 0, x, t) = \tilde{F}_i(\phi, 2\pi, x, t), \quad x \in (l_i, L_i) \]

(72)

on \( \Sigma_0^i \). Let \( \eta_i = \tilde{\eta}_i - \eta_i \), then it is easy to see that

\[ \| \tilde{F}_i \|_{H^{3-\sigma}(\Sigma_0^i)} \leq \| \langle \eta_i + \tilde{\eta}_i, \eta_i - \tilde{\eta}_i \rangle \|_{H^{-1-\sigma}(\Sigma_0^i)} \]

\[ \leq C(\| \eta_i \|_{H^{3-\sigma}(\Sigma_0^i)} + \| \tilde{\eta}_i \|_{H^2(\Sigma_0^i)} + \| \eta_i \|_{H^2(\Sigma_0^i)} + \| \tilde{\eta}_i \|_{H^2(\Sigma_0^i)}) \| \eta \|_{H^{2-\sigma}(\Sigma_0^i)} \]

\[ \leq C(1 + \| \tilde{\eta}_i \|_{H^2(\Sigma_0^i)} + \| \eta_i \|_{H^2(\Sigma_0^i)}) \| \eta \|_{H^{2-\sigma}(\Sigma_0^i)} \].
Since

\[ \| \hat{\eta}, F_i(\hat{\eta}) \|_{H^{-\sigma}(\Sigma_0)} \leq C(\| F_i \|_{H^{2-\sigma}(\Sigma_0)} + \| [\eta, F_i(\eta)] \|_{H^{-\sigma}(\Sigma_0)} + \| [\hat{\eta}, F_i(\hat{\eta})] \|_{H^{-\sigma}(\Sigma_0)}) \]

using the representation

\[ [\eta, F_i] = \partial_x(F_{ix}\eta_{\phi} - F_{i\phi}\eta_{xx}) + \partial_{\phi}(F_{i\phi}\eta_{xx} - F_{ix}\eta_{\phi}) \]  

(73)

we obtain

\[ \| \hat{\eta}, F_i(\hat{\eta}) \|_{H^{-\sigma}(\Sigma_0)} \leq C \sum_{k,m} \| D_k(\hat{\eta} - \hat{\eta}) D_m F_i(\hat{\eta}) \|_{1-\sigma} \]

\[ \leq C \| \hat{\eta} \|_{H^{2-\sigma}(\Sigma_0)} \| F_i(\hat{\eta}) \|_{H^{3-\sigma}(\Sigma_0)} \leq C \| \hat{\eta} \|_{H^{2-\sigma}(\Sigma_0)}(1 + \| \hat{\eta} \|_{H^2(\Sigma_0)}) \]

for any \( 0 < \sigma \leq 1/2 \). Due to representation

\[ [\eta, F_i] = -\partial_{xx}(F_{i\phi}\eta_{\phi}) - \partial_{\phi}(F_{ix}\eta_{xx}) + \partial_{\phi}(F_{i\phi}\eta_{xx} + F_{ix}\eta_{\phi}) \]  

(74)

we get

\[ \| \hat{\eta}, F_i(\hat{\eta}) \|_{H^{-\sigma}(\Sigma_0)} \leq C \sum_{k,m} \| D_k(\hat{\eta}) D_m F_i(\hat{\eta}) \|_{1-\sigma} \]

\[ \leq C \| \hat{\eta} \|_{H^{2-\sigma}(\Sigma_0)} \| F_i \|_{H^{3-\sigma}(\Sigma_0)} \leq C \| \hat{\eta} \|_{H^{2-\sigma}(\Sigma_0)}(1 + \| \hat{\eta} \|_{H^2(\Sigma_0)}) \]

\[ \| \hat{\eta} - \hat{\eta} \|_{H^{-\sigma}(\Sigma_0)} \leq C(1 + \| \hat{\eta} \|_{H^2(\Sigma_0)}) \| \hat{\eta} \|_{H^{2-\sigma}(\Sigma_0)} \]

Taking into account that the solution to 70-72 equals to zero for \( \Sigma_0 \setminus \Upsilon_0 \) and belongs to \( H^2 \), we get that the restrictions of solutions of 70-72 to \( \Upsilon_0 \) are solutions to 70 with boundary conditions 11-10 with the same estimates. \( \square \)

Relying on Lemma 6.1 we come to the main theorem

**Theorem 6.2.** Assume that \( G_b \equiv g = (g^1, g^2) \in C^{-1/2}(\Upsilon_0) \times C^{-1/2}(\Upsilon_0) \) is autonomous and \( G_f \equiv 0 \). Then

1. Problem 1-13 generates a gradient dynamical system \( (S_f, H) \).
2. The set of stationary points \( N \) to 1-13 is nonempty and bounded and any stationary point has the form \((0, \eta)\), where \((\eta^1, \eta^2) \in \bar{H}\) satisfies

\[
(D_{\eta^1} \eta^1, D_{\eta^2} \eta^2)_{\Upsilon_0} + (D_{\eta^1} \eta^2, D_{\eta^2} \eta^1)_{\Upsilon_0} + \beta_1 \hat{\eta}^1_{\Upsilon_0} + \beta_2 \hat{\eta}^2_{\Upsilon_0} \]
\[
= (g^1, g^2)_{\Upsilon_0} \quad \forall (\beta_1, \beta_2) \in \bar{H}. \]  

(75)

3. System \( (S_f, H) \) possesses a compact global attractor \( \mathcal{A} \) of finite fractal dimension.
4. Any trajectory \( \gamma = \{(V(t); \eta(t); \eta(t)) : t \in \mathbb{R}\} \) from the attractor \( \mathcal{A} \) possesses the properties \((V; \eta; \eta_0) \in L^\infty(\mathbb{R}; X_1(\Omega) \times \bar{H} \times \bar{L}) \) and there is \( R > 0 \) such that

\[
\sup_{\gamma \subset \mathcal{A}} \sup_{t \in \mathbb{R}} \left( \| V_2 \|^2_{\Omega} + \| \eta_0 \|^2_{2, \Upsilon_0} + \| \eta_0 \|^2_{2, \Upsilon_0} \right) \leq R^2. \]

5. The global attractor \( \mathcal{A} \) consists of full trajectories \( \{(V(t); \eta(t); \eta(t)) : t \in \mathbb{R}\} \) which are homoclinic to the set \( N \), i.e.

\[
\lim_{t \to \pm \infty} \inf_{\eta \in \mathcal{N}_0} \left( \| V(t) \|^2_{\Omega} + \| \eta \|_{2, \Upsilon_0}^2 + \| \eta_0 \|_{2, \Upsilon_0}^2 \right) = 0, \]

where \( \mathcal{N}_0 \) consists of all \( \eta^* = (\eta^*, \eta^*) \in \bar{H} \) solving 75. In addition we have

\[
\lim_{t \to \pm \infty} \text{dist}_{\mathcal{F}}(S_f, \mathcal{N}) = 0 \text{ for any initial data } y \in \mathcal{F}. \]  

(76)
Proof. (1) First we show that the dynamical system \((S_t, \hat{H})\) is gradient. Indeed, it follows from energy inequality 68 that the set
\[
W_R = \left\{ U : \Phi(V, \eta, \eta_t) = \mathcal{E}(V, \eta, \eta_t) - (g, \eta)_{\Gamma_0^1} \leq R \right\}
\]
where \(U = (V, \eta, \eta_t)\) is forward invariant with respect to \(S_t\) for each \(R > 0\). It is easy to see that there exist positive constants \(C_i, i = 1, 4\) such that
\[
C_1 \|U\|_{\mathcal{H}}^2 - C_2 \leq \Phi(U) \leq C_3 \|U\|_{\mathcal{H}}^2 + C_4.
\]
Consequently, \(\Phi(U^n) \to +\infty\) if and only if \(\|U^n\|_{\mathcal{H}} \to +\infty\). Therefore, the set \(W_R\) is bounded and any bounded set belongs to \(W_R\) for some \(R\). Moreover, it follows from energy inequality 68 that the continuous functional \(\Phi\) on \(\mathcal{H}\) possesses the properties:
(1) \(\Phi(S_t U) \leq \Phi(U)\) for all \(t \geq 0\) and \(U \in \hat{H}\); (2) the equality \(\Phi(U) = \Phi(S_t U)\) holds for all \(t > 0\) only if \(U\) is a stationary point of \(S_t\). This means that \(\Phi(U)\) is a strict Lyapunov function and \((S_t, \hat{H})\) is a gradient dynamical system.

(2) Now we investigate the set of stationary points of \((S_t, \hat{H})\). It follows from Definition 5.1 that a stationary solution \((V, \eta)\) \(\in X_1(\Pi) \times \hat{H}\) satisfies \(V|_{\Gamma_0^1} = (0, 0, 0)\) and the equality
\[
\mu(\nabla V, \nabla \psi)|_{\Pi} + (L_0 V, \psi) + (\Delta_{R_1} \eta^1, \Delta_{R_1} \beta^1)_{\Gamma_0^1}
+ (\Delta_{R_2} \eta^2, \Delta_{R_2} \beta^2)_{\Gamma_0^2} = (g, \beta)_{\Gamma_0^1 \times \Gamma_0^2} - (F(\eta(t), \beta))_{\Gamma_0^1 \times \Gamma_0^2}
\]
for \(\psi \in X_1(\Pi)\) such that \(\psi|_{\Gamma_0^1} = (\beta^1, 0, 0), i = 1, 2\), where \(\beta = (\beta^1, \beta^2) \in \hat{H}\). Choosing \(\beta \equiv 0\) and \(\psi = V\) we get that \(V \equiv 0\). Consequently, equation 77 turns into 75. Now we need to show the existence of solutions to 75. For this purpose we minimize the functional
\[
\Psi(\eta) = \frac{1}{2} \|\Delta_{R_1} \eta^1\|_{\Gamma_0^1}^2 + \frac{1}{2} \|\Delta_{R_2} \eta^2\|_{\Gamma_0^2}^2 + \int_{\Gamma_0^1} \Pi^1(\eta^1) d\phi dx + \int_{\Gamma_0^2} \Pi^2(\eta^2) d\phi dx
\]
\[
- (g^1, \eta^1)_{\Gamma_0^1} - (g^2, \eta^2)_{\Gamma_0^2}
\]
on \(\hat{H}\).

One can easily see that this functional is bounded from below, namely there exist \(c_1, c_2 > 0\) such that \(\Psi(\eta) \geq c_1 \|\eta\|_{\hat{H}}^2 - c_2\). Thus we can construct appropriate Galerkin approximations \(\{\eta_n\} \subset \hat{H}\) for the global minimum and note that \(\inf_{\eta_n} \Psi(\eta_n) \leq \Psi(0)\). This facts provide us an a priori estimate \(\|\eta_n\|_{\hat{H}}^2 \leq c_2 + \Psi(0)\) which allows to prove the existence of a solution by the same method as in e.g. [9, Lemma 2.5.3]. Therefore, the set of stationary points is nonempty and it follows readily from 75 that it is bounded.

(3) To prove the existence and finite dimensionality of the global attractor it is necessary due to Theorem 1.5 to show that the dynamical system \((S_t, \hat{H})\) is quasi-stable. Having in hand Lemma 6.1 the proof is analogous to that given in [13, 15].

(4) To obtain the result on regularity we apply Theorem 7.9.8 [12].

(5) The results on the structure of the global attractor follow from Theorem 1.5.

\[\square\]

Appendix A.

Proof of Proposition 3.1. We to keep to the scheme of the proof of the similar assertion given in [13]. Since the extension of elements from \(\mathcal{H}^s_0(\Gamma_0^1) \times \mathcal{H}^s_0(\Gamma_0^2)\) by zero to the whole boundary \(\partial \Omega^r\) do not change the smoothness in Sobolev class, i.e.,
leads to elements from $H^\sigma(\partial\Omega')$, we can use the regularity results available for the Stokes problem with the Dirichlet type boundary conditions imposed on the whole $\partial\Omega'$ (see, e.g., [28, 33] and also the paper [20] and the references therein). This observation leads to the following arguments.

1. The existence and uniqueness of solutions along with the bound in 25 follow from Proposition 2.3 and Remark 2.6 on Sobolev norm’s interpolation in [33, Chapter 1].

2. By Theorem 3 [20] (applied for the boundary data $\hat{\psi} \in H^{-1/2}(\partial\Omega') \cap L$, which is extension by zero outside $\Gamma_0^1 \cup \Gamma_0^2$ of the function $\psi \in \mathcal{H}^{-1/2}_s(\Gamma_0^1) \times \mathcal{H}^{-1/2}_s(\Gamma_0^2)$) we have 26 with $\sigma = 0$. Therefore interpolating with 25 for $\sigma = 0$ and $g \equiv 0$ we obtain 26 for all $\sigma \geq 0$.

3. We first represent $f$ in the form $f = \hat{f} + f^*$, where $\hat{f}$ solves 24 with $\psi \equiv 0$ and $f^*$ satisfies 24 with $g \equiv 0$. Let $\hat{p}$ and $p^*$ be the corresponding representatives of the pressure (which are identified with an element in a factor-space). By the first statement we have that $\hat{p} \in H^{1/2+\sigma}(\Omega')$ and thus by the standard trace theorem there exists $\hat{p}|_{\partial\Omega'} \in H^\sigma(\partial\Omega')$. This implies that $\hat{p}|_{\Gamma_0^i} \in H^\sigma(\Gamma_0^i) \subset H^{-1+\sigma}(\Gamma_0^i)$ and

$$\|\hat{p}\|_{H^{-1+\sigma}(\Gamma_0^i)/\mathbb{R}} \leq c\|\hat{p}\|_{H^\sigma(\Gamma_0^i)/\mathbb{R}} \leq c\|\hat{g}\|_{[H^{-1/2+\sigma}(\Omega')]}.$$  

(78)

In the case $g \equiv 0$ the pressure $p^*$ is a harmonic function in $\Omega'$ which belongs to $H^{-1/2+\sigma}(\Omega')$. This allows us to assign a meaning to $p^*|_{\Gamma_0^i}$ in $H^{-1+\sigma}(\Gamma_0^i)$. Indeed, let $\phi \in C_0^\infty(\Gamma_0^i)$ and $\hat{\phi} \in C_0^\infty(\partial\Omega')$ be the extension of $\phi$ by zero. Then by the trace theorem there exists a smooth function $w_\phi^i$ on $\Omega'$ such that

$$w_\phi^i|_{\partial\Omega'} = 0, \quad \frac{\partial w_\phi^i}{\partial n} \bigg|_{\partial\Omega'} = \hat{\phi}, \quad \|w_\phi^i\|_{H^{5/2-\sigma}(\Omega')} \leq C\|\phi\|_{\mathcal{H}^{1-\sigma}(\Gamma_0^i)}.$$  

The application of Green’s formula yields $(p^*, \Delta w_\phi^i)_{\Omega'} = (p^*, \phi)_{\Gamma_0^i}$. Therefore

$$|(p^*, \phi)_{\Gamma_0^i}| = |(p^*, \Delta w_\phi^i)_{\Omega'}| \leq C\|p^*\|_{-1/2+\sigma, \Omega'} \|\hat{\phi}\|_{1-\sigma, \partial\Omega'}.$$  

Since $\|\hat{\phi}\|_{1-\sigma, \partial\Omega'} = \|\phi\|_{\mathcal{H}^{1-\sigma}(\Gamma_0^i)}$ and $C_0^\infty(\Gamma_0^i)$ is dense in $\mathcal{H}^{1-\sigma}(\Gamma_0^i)$, we obtain

$$|p^*|_{H^{-1+\sigma}(\Gamma_0^i)/\mathbb{R}} \leq c\|p^*\|_{H^{-1/2+\sigma}(\Omega')/\mathbb{R}} \leq c\|\psi\|_{H^\sigma(\Gamma_0^i)}.$$  

(79)

Thus relation 28 follows from 78 and 79. 

Proof of Proposition 3.2. We consider Stokes problem 24 on a smoothly bounded sub-domain $\Omega'$ in $\Omega$ such that $\Gamma_0^i \subset \partial\Omega'$, $i = 1, 2$. We can take a solution $f$ to 24 and define $Ext[f]$ as the zero extension of $f$ on the domain $\Omega$. One can see that for this operator $Ext$ all statements of Proposition 3.2 are in force.

Proof of Lemma 4.7. There exists an approximating sequence $d_m = (d^1_m, d^2_m) \in C(\Gamma_0^1) \times C(\Gamma_0^2)$ such that $\|d_m - b\|_{L^2} \to 0$ as $m \to \infty$ and, consequently,

$$\int_{l_1}^{L_1} \int_0^{2\pi} (d^1_m - b^1) d\phi = \varepsilon^1_m \to 0, \quad m \to \infty.$$  

Then,

$$\int_{l_1}^{L_1} \int_0^{2\pi} d^1_m d\phi - \int_{l_2}^{L_2} \int_0^{2\pi} d^2_m d\phi = \varepsilon^1_m - \varepsilon^2_m = \varepsilon_m \to 0, \quad m \to \infty.$$  

Denoting $b^1_m = d^1_m$ and $b^2_m = d^2_m - \varepsilon_m |\Gamma_0^1|$, we infer the assertion of the lemma. 

□
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