Charged scalar waves from the RN/CFT correspondence

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Abstract

We examine new tests for (non-)extremal Reissner-Nordström/Conformal field theory correspondences (RN/CFT) in this paper. The decay rate of the charged scalar wave sourced by an orbiting star around the black hole is computed and is compared with the decay rate computed in the corresponding CFT. We find that precise matches are achieved.

1 Introduction

As examples of AdS/CFT correspondence [1], it is pointed out, e.g. [2–6], that the near-horizon dynamics of the charged (RN) black hole is equivalent to that of the boundary CFT. For the extremal black hole, the corresponding CFT is a chiral one [2–4, 6]. For the near-extremal black hole, the corresponding CFT has two sectors (left and right) [5].

Recently, new checks of AdS/CFT correspondences are suggested by studying the decay rates of particles excited by some star orbiting the black holes, see [7, 8] etc. In this paper, along this direction, we compute the decay rates of the charged scalar particles in two kinds of RN/CFT correspondences mentioned above [2–6]. For these two kinds of correspondences, on the gravity side, the four dimensional theory is lifted to a five dimensional one with an extra compact direction introduced. So, the $U(1)$ gauge symmetry in the four dimensional theory becomes an isometry of the five dimensional RN spacetime metric. It is interesting to investigate how this geometrization of the 4D gauge symmetry appear in the computation of the decay rates on the CFT sides.
In the section 2, we consider the near-horizon extremal RN/Chiral CFT correspondence (NHERN/χCFT). In the section 3, we consider the near-horizon near-extremal RN/CFT correspondence (NHNERN/CFT). For both kinds of correspondences, decay rates computed on the gravity side match precisely to those computed on the CFT side, respectively. We give some remarks on the difference of these two kinds of correspondences in the final section 4.

2 Near horizon extremal RN/χCFT

In this section, we will consider the correspondence between the dynamics of the near-horizon extremal RN black hole and that of the boundary chiral CFT. Decay rates of a charged scalar wave are computed on both sides.

2.1 Decay rate on the gravity side

The extremal RN black hole is the vacuum solution of the Einstein-Maxwell theory and is described by the following metric and the gauge field strength,

\[ ds^2 = -(1 - Q/\hat{r})^2 d\hat{t}^2 + (1 - Q/\hat{r})^{-2} d\hat{r}^2 + \hat{r}^2 d\Omega_2^2 \]

\[ F = \frac{Q}{\hat{r}^2} d\hat{t} \wedge d\hat{r} \]  

(1)

where \( d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the round metric of the two sphere, and one has set the Newton constant \( G_N = 1 \), the position of the event horizon is located at \( \hat{r} = M = Q \).

To discuss the near horizon dynamics, taking the limiting process by defining \[ \hat{r} = \frac{\hat{r} - Q}{\epsilon}, \quad t = \frac{\epsilon}{Q^2} \hat{r} \]  

(2)

with \( r, t \) fixed when \( \epsilon \to 0 \). In terms of this new coordinate system, the metric of black hole and Maxwell field strength becomes,

\[ ds^2 = Q^2 \left( -r^2 dt^2 + \frac{dr^2}{r^2} + d\Omega_2^2 \right) \]

\[ F = Q dt \wedge dr \]

(3)  

(4)

which shows that the near-horizon geometry is \( AdS_2 \times S^2 \) and the isometry group is \( SL(2, \mathbb{R}) \times SO(3) \). The gauge potential one-form can be chosen as

\[ A = -Q r dt \]  

(5)
by taking a proper gauge.

Let us consider a star carrying electric charge $e_0$ with mass $\mu_0$, which is moving around the black hole along a circular orbit at the near horizon region. For the two Killing vectors, $\xi_{(t)} = \partial_t$ and $\xi_{(\varphi)}$, there are two conserved quantities \[9\],

$$E_0 = -g_{\alpha\beta}(u^\alpha + q_0 A^\alpha)\xi_{(t)}^\beta = -Q^2 r_*^2 \left(i_* + \frac{q_0}{Q r_*}\right)$$

$$L_0 = g_{\alpha\beta}(u^\alpha + q_0 A^\alpha)\xi_{(\varphi)}^\beta = Q^2 \sin^2 \theta_* \dot{\varphi}_*$$

where $E_0$ and $L_0$ are energy and angular momentum per unit mass of the star respectively, $q_0 = e_0/\mu_0$, constants $r_*$ and $\theta_* = \pi/2$ are radial and angular position of the star respectively. $L_0$ and $E_0$ are constrained by the on-shell condition, $-1 = Q^2(-r_*^2 \dot{t}_*^2 + \sin^2 \theta_* \dot{\varphi}_*)$. In the following, without loss of generality, we will set $E_0 = 0$, then one have solution,

$$i_* = -\frac{q_0}{Q r_*}, \quad \ddot{\varphi}_* = \frac{L_0}{Q^2}$$

with $L_0 = Q\sqrt{q_0^2 - 1}$. We will consider real angular momentum $L_0$, i.e., $q_0 > 1$.

Assume that the coupling of the star with a charged scalar field $\Phi$ is defined by the following action,

$$S_\Phi = -\int d^4x \sqrt{-g} \left( \nabla_\mu - ieA_\mu \right) \Phi^* \left( \nabla^\mu - ieA^\mu \right) \Phi + 4\pi \lambda \int d\tau \int d^4x \delta^4(x - x_*) \Phi^* + \text{c.c.}$$

where $e$ is the electric charge of the scalar field, $\lambda$ is the coupling constant, ‘c.c.’ indicates complex conjugate of the previous term. The equation of motion reads

$$\sqrt{-g} \left( \nabla_\mu - ieA_\mu \right) \left( \nabla^\mu - ieA^\mu \right) \Phi = -4\pi \lambda \int d\tau \delta^4(x - x_*)$$

$$= -\frac{4\pi \lambda r_* Q}{q_0} \delta \left( \varphi + \frac{r_* L_0}{q_0 Q} \right) \delta(\theta - \pi/2) \delta(r - r_*)$$

The source (star) respects the Killing symmetry, $\chi = \partial_\varphi - \frac{q_0 Q}{r_* \mu_0} \partial_t$. The variable separation respects this Killing symmetry is

$$\Phi = \sum_{\ell,m} e^{i m \varphi + \ell \omega t} S_{\ell m}(\theta) R_{\ell m}(r) \equiv \sum_{\ell,m} e^{i(m \varphi - \ell \omega t)} S_{\ell m}(\theta) R_{\ell m}(r)$$
where, \( m \in \mathbb{Z}, \omega = -m\frac{e_0Q}{q_0} \). The equation of motion separates into radial equations:

\[
r^2R''_{\ell m}(r) + 2rR'_{\ell m}(r) + \left( \frac{\omega^2}{r^2} - \frac{2eQ\omega}{r} + e^2Q^2 - \mu^2 \right) R_{\ell m}(r) = -\tilde{\lambda}\delta(r - r_*)
\]  

where \( \tilde{\lambda} = 2\frac{\lambda r_0q_0}{q_0e_0} S_m^{\ell}(\frac{\pi}{2})^* \), \( \mu^2 = \ell(\ell-1) (\ell = 0, 1, 2, \ldots) \). \( S_m^{\ell}(\theta) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{(2\ell+m)!}} P_m^{\ell}(\theta) \) is the normalized associated Legendre function of the first kind,

\[
\frac{d^2}{d\theta^2}P_m^{\ell}(\theta) + \cot\theta \frac{d}{d\theta}P_m^{\ell}(\theta) - \frac{m^2}{\sin^2\theta} P_m^{\ell}(\theta) = -\mu^2 P_m^{\ell}(\theta)
\]

and the orthonormal condition reads,

\[
\int d\theta \sin\theta S_m^{\ell}(\theta)^* S_m^{\ell}(\theta) = \delta_{\ell\ell'}
\]

Two linearly independent solution of the homogeneous radial equation, set \( \lambda = 0 \) in (11), are given by Whittaker functions,

\[
R^{(1)}(r) = M(-ieQ, h - \frac{1}{2}, -2i\omega/r)
\]

\[
R^{(2)}(r) = W(-ieQ, h - \frac{1}{2}, -2i\omega/r)
\]

where the weight \( h \) is defined by

\[
h = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\mu^2 - 4e^2Q^2}
\]

Near the horizon,

\[
R^{(1)}(r \to 0) \sim Ae^{-i\omega/r} - ieQ + Be^{i\omega/r} + ieQ
\]

\[
R^{(2)}(r \to 0) \sim (-2i\omega)^{-ieQ}e^{i\omega/r} r^{+ieQ}
\]

which shows that \( R^{(2)} \) is purely ingoing. Near the boundary,

\[
R^{(1)}(r \to \infty) \sim (-2i\omega)^h r^{-h}
\]

\[
R^{(2)}(r \to \infty) \sim Cr^{-h} + Dr^{h-1}
\]

We will take \( h > 1/2 \), so \( R^{(1)} \) satisfies purely Neumann boundary condition. Constants \( A, B, C, D \) are defined by

\[
A = (-2i\omega)^{+ieQ} \frac{\Gamma(2h)}{\Gamma(h + ieQ)}, \quad B = (2i\omega)^{-ieQ} e^{izh} \frac{\Gamma(2h)}{\Gamma(h - ieQ)}
\]

\[
C = (-2i\omega)^h \frac{\Gamma(1 - 2h)}{\Gamma(1 - h + ieQ)}, \quad D = (-2i\omega)^{1-h} \frac{\Gamma(2h - 1)}{\Gamma(h + ieQ)}
\]
Since we want to compute the ingoing flux near the horizon, we need a solution which is purely ingoing at the horizon and satisfies purely Neumann boundary condition on the boundary. This kind of solution can be constructed by [7]

\[ R_{\ell m}(r) = \theta(r_* - r)C_2 R^{(2)}(r) + \theta(r - r_*)C_1 R^{(1)}(r) \]  

Substituting into the equation of motion, the constants \(C_1, C_2\) can be determined to be

\[ C_1 = \frac{\tilde{\lambda} R^{(2)}(r_*)}{W}, \quad C_2 = \frac{\tilde{\lambda} R^{(1)}(r_*)}{W} \]  

where \(W\) is \(r\)-independent Wronskian,

\[ W = r^2(R^{(1)}(r)R^{(2)'}(r) - R^{(1)'}(r)R^{(2)}(r)) = -2i\omega \frac{\Gamma(2h)}{\Gamma(h + ieQ)} \]  

Now we can compute the Klein-Gordon particle number flux per unit time,

\[ \mathcal{F} = -\int d\theta d\phi \sqrt{-g} J^r \]  

where the current is defined by

\[ J_\mu = \frac{i}{8\pi}(\Phi^*(\nabla_\mu - ieA_\mu)\Phi - h.c.) = -\frac{1}{4\pi} \text{Im}(\Phi^*(\nabla_\mu - ieA_\mu)\Phi) \]  

Then the flux is

\[ \mathcal{F} = \frac{Q^4 r^2}{2} \sum_{\ell m} \text{Im}(R^*_{\ell m}(r)R'_{\ell m}(r)) \]  

Near the boundary,

\[ \mathcal{F}_{\ell m}(r \to 0) = \frac{Q^4 |\omega|}{2} e^{-\pi eQ} |C_2|^2 \]  

for \(m < 0\), i.e., \(\omega > 0\). Near the boundary,

\[ \mathcal{F}_{\ell m}(r \to \infty) = 0 \]  

for real \(h\).

### 2.2 Decay rate on CFT side

In this subsection, we will compute the decay rate in the CFT on the boundary dual to the quantum gravity on the near-horizon region of the extremal RN spacetime. This
problem is studied in Refs. [2,3,6]. Before performing the computation, let us review some facts here.

The symmetry of the near-horizon-extremal RN black hole includes the isometry of the metric and the gauge symmetry,

$$SL(2, R) \times SO(3) \times U(1)_{\text{em}}$$

To find out the CFT dual, the four dimensional RN black hole is lifted to five dimensions. Especially, the metric becomes,

$$ds_5^2 = ds_{4D}^2 + Q^2(dy + A)^2$$

where $ds_{4D}^2$ is the 4D RN metric (3), the compact coordinate has period $2\pi$, $y \approx y + 2\pi$, $A = A/Q$. In the 5D language, the 4D $U(1)_{\text{em}}$ gauge symmetry is geometrized as the isometry of the $S^1$-fibration of 4D RN black hole. That is, the 4D gauge symmetry transformation, $A \to A - d\alpha$, appears in 5D as

$$A \to A - \frac{1}{Q} d\alpha, \quad y \to y + \frac{1}{Q} \alpha, \quad \Phi \to e^{-i\alpha} \Phi$$

where $\alpha$ is real function depends only on 4D coordinates.

In this 5D setting, the near horizon dynamics will be dual to a chiral 2D CFT on the boundary: only left-movers are excited and the right-movers are frozen to the ground state. For these two sectors, their temperatures are

$$T_L = \frac{1}{2\pi}, \quad T_R = 0$$

Specifically, on the gravity side, non-zero conserved charge of the generator $\partial_t$ will indicate the deviation of the extremal state, on the CFT side, non-zero charge of $SL(2, R)_R$ indicate the excitation of the right-movers [10]. So we have a correspondence: the time translation $\partial_t$ on gravity side is mapped to the right-translation $\partial_{\sigma^{-}}$ on CFT side, or

$$t = \sigma^{-}$$

On the other hand, the non-zero conserved charge (electric charge) of the generator $\partial_y$ on the gravity side corresponds to the non-zero $SL(2, R)_L$-charge, and $\partial_y$ is mapped to the left-translation, $\partial_y \sim \partial_{\sigma^{+}}$, or

$$y = -\sigma^{+}$$
where we have used the fact that both $y$ and $\sigma^+$ have period $2\pi$, so we identify these two coordinate directly, the extra minus sign is found to be necessary for the matching of the decay rates in NHERN/$\chi$CFT.

Now, adding an orbiting star into the 4D RN black hole will, on the CFT side, lead to perturbation of the CFT. Assume that the perturbed action is

$$\mathcal{S}^{\text{int}}_{\text{CFT}} = \sum_{\ell} \int d\sigma^+ d\sigma^- J_\ell(\sigma^+, \sigma^-) \mathcal{O}_\ell(\sigma^+, \sigma^-)$$  \hspace{1cm} (37)

The source $J_\ell$ should be determined by the asymptotic behavior of the scalar field near the boundary, since only scalar field is assumed to be coupled to the orbiting star. The conformal weight of $\mathcal{O}$ can also be read off from the asymptotic behavior of the scalar field, $h_L = h_R = h$.

Now let us look at $J_\ell$. At first, since the star respect the symmetry $\chi$, so

$$J_\ell(\sigma^+, \sigma^-) \sim e^{i m(\varphi + \frac{\alpha}{Q} t)} = e^{i (m \varphi - \omega t)} = e^{i (m \varphi - \omega \sigma^-)}$$  \hspace{1cm} (38)

Secondly, the source $J_\ell$ should have the same gauge behavior as the charged scalar field. Since a gauge transformation is given by (33), $\Phi \rightarrow e^{-i e\alpha} \Phi$, we must require the source transform in the same way,

$$J_\ell \rightarrow e^{-i e\alpha} J_\ell$$  \hspace{1cm} (39)

and since $y$ is identified as $\sigma^+$ and under gauge transformation, $y \rightarrow y + \alpha/Q$, the dependence of $J_\ell$ on the $y$-coordinate should be

$$J_\ell(\sigma^+, \sigma^-) \sim e^{-i e\alpha y} = e^{i e\alpha \sigma^+}$$  \hspace{1cm} (40)

In all, we have expansion of the source as follows,

$$J_\ell(\sigma^+, \sigma^-) = \sum_m e^{i m(\varphi - \omega \sigma^- + i e\alpha \sigma^+)} J_{\ell m}$$  \hspace{1cm} (41)

where $J_{\ell m}$ are constants. To determine $J_{\ell m}$, one can extend the solution $R_{\ell m}$ at the near horizon region to the whole spacetime [7],

$$R_{\ell m}^{\text{ext}}(r) = C_2 R^{(2)}(r), \quad 0 < r < \infty$$  \hspace{1cm} (42)

The asymptotic behavior of $R_{\ell m}^{\text{ext}}(r)$ is

$$R_{\ell m}^{\text{ext}}(r \rightarrow \infty) \sim C_2 (C r^{-h_1} + D r^{h_1})$$  \hspace{1cm} (43)
The coefficient $J_{\ell m}$ can be read off from the Dirichlet mode as,

$$J_{\ell m} = C_2 D$$  \hspace{1cm} (44)$$

The decay rate of vacuum-to-vacuum per unit time is \cite{10},

$$\mathcal{R} = 2\pi \sum_{\ell m} |J_{\ell m}|^2 \int d\sigma^+ d\sigma^- e^{i\omega \sigma^- - ieQ\sigma^+} \langle \mathcal{O}_\ell(\sigma^+, \sigma^-) \mathcal{O}_\ell(0, 0) \rangle$$

$$= 2\pi \sum_{\ell m} C_O^2 |C_2|^2 |D|^2 \frac{(2\pi T_R)^{2h_R - 1}}{\Gamma(2h_R)} e^{\frac{2\pi \omega}{T_R}} \left| \Gamma \left( h_R + i \frac{\omega}{2\pi T_R} \right) \right|^2$$

$$\times \frac{(2\pi T_L)^{2h_L - 1}}{\Gamma(2h_L)} e^{-\frac{eQ}{2\pi T_L}} \left| \Gamma \left( h_L + i \frac{eQ}{2\pi T_L} \right) \right|^2$$  \hspace{1cm} (45)$$

Substituting $T_L = \frac{1}{2\pi}$, $h_L = h_R = h$, and taking the limit, $T_R \to 0$,

$$\mathcal{R}_{\ell m} = C_O^2 Q^{2h + \frac{3}{2}(2h - 1)} F_{\ell m}(r \to 0)$$  \hspace{1cm} (46)$$

for $m < 0$, i.e., $\omega > 0$. This decay rate is precisely equal to that computed in the gravity side \cite{29} if we take $C_O^2 = \frac{Q^{2h + \frac{3}{2}(2h - 1)}{2^{5 - 2h} \pi^2}}{2}.$

3 Near horizon near-extremal RN/CFT

There is a correspondence between the dynamics on the near-horizon near-extremal RN black hole and a 2d CFT, e.g. \cite{5, 6}. In this section we will compute the decay rates of a charged scalar wave from both sides, and compare the results.

3.1 Decay rate on the gravity side

Similar to the extremal RN case discussed in the previous section, in this subsection, we will consider a charged scalar wave sourced by a charged star which is orbiting the near-extremal RN black hole.

The non-extremal (general) RN black hole can be described by the metric and gauge field,

$$ds^2 = -\frac{\hat{r} - r_+)(\hat{r} - r_-)}{\hat{r}^2} dt^2 + \frac{\hat{r}^2}{(\hat{r} - r_+)(\hat{r} - r_-)} d\hat{r}^2 + \hat{r}^2 d\Omega_2^2,$$  \hspace{1cm} (47)$$

$$\hat{F} = \frac{Q}{\hat{r}^2} d\hat{t} \wedge d\hat{r}$$  \hspace{1cm} (48)$$
Define new coordinates, \[ r = \hat{r} - \frac{Q}{\epsilon}, \quad t = \frac{i}{Q^2 \epsilon}, \quad r_0 = \frac{\sqrt{2QE}}{\epsilon} \] (49)

where \( E = Q - M \) is the deviation from the extremality. The near-horizon near-extremal limit can be obtained by letting \( \epsilon \to 0 \) and keeping \( r, t, r_0 \) fixed. In this new coordinate system, the metric and the gauge field become

\[
\begin{align*}
\text{ds}^2 &= Q^2 \left[ -(r^2 - r_0^2) dt^2 + \frac{dr^2}{r^2 - r_0^2} + d\Omega^2_2 \right] \\
F &= Qdt \wedge dr
\end{align*}
\] (50) (51)

where \( r = r_0 \) is the horizon position. The gauge potential can be chosen as

\[ A = -Qr dt \] (52)

The Hawking temperature of the near-horizon spacetime (50) can be found to be

\[ T_H = \frac{r_0}{2\pi} \] (53)

It is different from the Hawking temperature corresponding to the original coordinate with hats, \( \hat{T}_H = \frac{\xi}{Q} T_H \), which vanishes under the limit \( \epsilon \to 0 \).

Similar to the extremal case, corresponding to the two Killing vectors, \( \xi(t) = \partial_t \) and \( \xi(\phi) \), we define conserved quantities,

\[
\begin{align*}
E_0 &= -g_{\alpha\beta}(u^\alpha + q_0 A^\alpha)\xi^\beta_\alpha = Q^2(r^2 - r_0^2) \left( \dot{r}_* + \frac{q_0 r_*}{Q(r^2 - r_0^2)} \right) \\
L_0 &= g_{\alpha\beta}(u^\alpha + q_0 A^\alpha)\xi^\beta_\phi = Q^2 \sin^2 \theta_x \dot{\phi}_*
\end{align*}
\] (54) (55)

where \( x_*^\mu \) are coordinates of the orbiting star, and we will let \( r = r_* > r_0 \) be constant and \( \theta = \theta_* = \pi/2 \) in the following. The normalization condition is \(-1 = Q^2[-(r_*^2 - r_0^2) \dot{t}_*^2 + \dot{\phi}_*^2] \). For convenience, take \( E_0 = 0 \) and then \( \dot{t}_* = -\frac{q_0 r_*}{Q(r_*^2 - r_0^2)} \), \( \dot{\phi}_* = L_0/Q^2 \), or

\[
\dot{t}_* = -\frac{q_0 r_*}{Q(r_*^2 - r_0^2)} \tau, \quad \dot{\phi}_* = \frac{L_0}{Q^2 \tau}
\] (56)

where the integral constants are set to zero. The equation of motion of the scalar field is

\[
\sqrt{-g} \left( \nabla_\mu - ieA_\mu \right)(\nabla^\mu - ieA^\mu)\Phi = -\frac{4\pi \lambda}{|t_*|} \delta \left( \varphi - \frac{\dot{\phi}_*}{\dot{t}_*} \right) \delta(\theta - \pi/2) \delta(r - r_0)
\] (57)
The source (star) respects the Killing symmetry, $\chi = \partial_\varphi + \frac{i}{\varphi} \partial_t$. The variable separation respects this Killing symmetry is

$$\Phi = \sum_{\ell,m} e^{im(\varphi - \frac{2\pi}{\ell})} S^{m}_\ell(\theta) R^{r}_{\ell m}(r) \equiv \sum_{\ell,m} e^{i(m\varphi - \omega t)} S^{m}_\ell(\theta) R^{r}_{\ell m}(r)$$

(58)

where $\omega = m\dot{\varphi}/\dot{t}$, $m \in \mathbb{Z}$. In the following we will take $m > 0$, i.e., $\omega < 0$, in the concrete calculation. The equation of motion separates into radial equations:

$$((r^2 - r_0^2)R^{r}_{\ell m})' + \left[\frac{(\omega - eQr)^2}{r^2 - r_0^2} - \mu^2\right] R^{r}_{\ell m} = -\ddot{\lambda}\delta(r - r_*)$$

(59)

where $\mu^2 = \ell(\ell - 1)$ ($\ell = 0, 1, 2, \ldots$), $R^{r}_{\ell m} = \partial_r R^{r}_{\ell m}$, $\ddot{\lambda} = \frac{2\lambda}{|eQr^2} S^{m}_\ell(\frac{\pi}{2})^*$. $S^{m}_\ell(\theta)$ is defined in the previous section. The two linearly independent solutions of the radial homogeneous equation (when $\lambda = 0$) are the following hyper-geometric functions,

$$R^{(1)}_{\ell m} = (r + r_0)^{-i\alpha}(r - r_0)^{i\beta} F_1(1 - h - i\alpha + i\beta, h - i\alpha + i\beta, 1 - 2i\alpha; \frac{r + r_0}{2r_0})$$

(60)

$$R^{(2)}_{\ell m} = (r + r_0)^{i\alpha}(r - r_0)^{i\beta} F_1(1 - h + i\alpha + i\beta, h + i\alpha + i\beta, 1 + 2i\alpha; \frac{r + r_0}{2r_0})$$

(61)

where the constants are defined by

$$\alpha = \frac{1}{2}|eQ + \frac{\alpha}{r_0}|, \quad \beta = \frac{1}{2}(eQ - \frac{\alpha}{r_0}), \quad h = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4e^2Q^2 + 4\mu^2}$$

(62)

The asymptotic behaviors near the boundary of $R^{(1,2)}$ are

$$R^{(1)}_{\ell m}(r \to \infty) \sim A_{\infty} r^{-h} + B_{\infty} r^{-h}$$

(63)

$$R^{(2)}_{\ell m}(r \to \infty) \sim C_{\infty} r^{-h} + D_{\infty} r^{-h}$$

(64)

where the coefficient $A_{\infty}$ is defined by

$$A_{\infty} = \frac{\Gamma(1 - 2i\alpha) \Gamma(2h - 1)(-2r_0)^{-i\alpha + i\beta - h + 1}}{\Gamma(h - i\alpha + i\beta) \Gamma(h - i\alpha + i\beta)}$$

(65)

and the other coefficients are related to $A_{\infty}$: $B_{\infty} = A_{\infty}(h \to 1 - h)$, $C_{\infty} = A_{\infty}(\alpha \to -\alpha)$ and $D_{\infty} = A_{\infty}(h \to 1 - h, \alpha \to -\alpha)$. The near horizon behaviors are

$$R^{(1)}_{\ell m}(r \to r_0) \sim A_0 (r - r_0)^{i\beta} + B_0 (r - r_0)^{-i\beta}$$

(66)

$$R^{(2)}_{\ell m}(r \to r_0) \sim C_0 (r - r_0)^{i\beta} + D_0 (r - r_0)^{-i\beta}$$

(67)

where the coefficient $A_0$ is defined by

$$A_0 = \frac{i\pi (2r_0)^{-i\alpha} \text{csch}(2\pi \beta) \Gamma(1 - 2i\alpha)}{\Gamma(2i\beta + 1) \Gamma(1 - h - i\alpha - i\beta) \Gamma(h - i\alpha - i\beta)}$$

(68)
and the other coefficients are related to \( A_0 \): \( B_0 = e^{2\pi \beta} (2r_0)^{2i\beta} A_0(\beta \to -\beta) \), \( C_0 = A_0(\alpha \to -\alpha) \) and \( D_0 = B_0(\alpha \to -\alpha) \). From these results, one can see that the mode which is purely ingoing near horizon (near region) can be constructed as

\[
R^{(n)}_{\ell m} = D_0 R^{(1)}_{\ell m} - B_0 R^{(2)}_{\ell m}
\]

and the mode which is purely Neumann at infinity (far region) is

\[
R^{(f)}_{\ell m} = C_\infty R^{(1)}_{\ell m} - A_\infty R^{(2)}_{\ell m}
\]

We have

\[
R^{(n)}_{\ell m}(r \to r_0) \sim \frac{\alpha}{\beta} (2r_0)^{2i\beta} e^{2\pi \beta} (r - r_0)^{i\beta}
\]

\[
R^{(f)}_{\ell m}(r \to \infty) \sim \frac{2i\alpha}{1 - 2\hbar} (2r_0)^{1+2i\beta} e^{2\pi \beta} r^{-h}
\]

Now the solution of the inhomogeneous equation (11) is constructed by the Green’s method as

\[
R_{\ell m} = \theta(r_\ast - r) C_{(n)} R^{(n)}_{\ell m} + \theta(r - r_\ast) C_{(f)} R^{(f)}_{\ell m}
\]

where

\[
C_{(n)} = \frac{\tilde{\lambda} R^{(f)}(r_\ast)}{2 W}, \quad C_{(f)} = \frac{\tilde{\lambda} R^{(n)}(r_\ast)}{2 W}
\]

where \( W \) is \( r \)-independent Wronskian,

\[
W = (r^2 - r_0^2)(R^{(n)}(r) R^{(f)'}(r) - R^{(n)'}(r) R^{(f)}(r))
\]

\[
= 2\pi i\alpha^2 (2r_0)^{2-h+5i\beta} e^{2\pi \beta} \frac{(1 + \coth \pi \beta)^2 \tanh(\pi \beta) \Gamma(2h - 1)}{\Gamma(1 - 2i\beta) \Gamma(h - i\alpha + i\beta) \Gamma(h + i\alpha + i\beta)}
\]

The Klein-Gordon particle number flux per unit time is still given by the equation (26) and for the mode with quantum numbers \( \ell, m \),

\[
\mathcal{F}_{\ell m} = \frac{Q^2}{2} (r^2 - r_0^2) \text{Im}(R^*_{\ell m}(r) R'_{\ell m}(r))
\]

At infinity, simple calculation shows that \( \mathcal{F}_{\ell m}(r \to \infty) = 0 \). Near the horizon,

\[
\mathcal{F}_{\ell m}(r \to r_0) = \frac{\alpha^2}{\beta} Q^2 r_0 e^{4\pi \beta} |C_{(n)}|^2
\]
Since the spacetime has non-zero temperature \([53]\), the decay rate of the scalar particle into the horizon is

\[
\mathcal{R}_{\ell m} = \frac{1}{e^{-\frac{\Phi_H}{r_H}} - 1} \mathcal{F}_{\ell m}(r \to r_0)
\]  

(78)

where \(\Phi_H = A_t(r = r_0) = -Q r_0\) is the gauge potential at the horizon (chemical potential). Using the definition of \(\beta\) in \([62]\), we have

\[
\mathcal{R}_{\ell m} = \frac{1}{1 - e^{-4\pi \beta \alpha^2 Q^2 r_0 |C(m)|^2}}
\]  

(79)

We will see that this Planck factor is important for the matching of the decay rates in NHNERN/CFT correspondence.

### 3.2 Decay rate on the CFT side

Now turn to the computation from the CFT viewpoint. The analysis of the dictionary in the near-horizon near-extremal RN/CFT correspondence is very similar to the one in the near-horizon extremal RN/\(\chi\)CFT correspondence, which is discussed in the previous section. For details, one can refer to the original references, [5, 6] et al. In the following we will only sketch points necessary in the computation.

Similar to the extremal case discussed in the previous section, we have the following matching of coordinates,

\[
t \leftrightarrow \sigma^- , \quad y \leftrightarrow -\sigma^+
\]  

(80)

Corresponding to this identification, one can derive the left and right temperatures of the boundary CFT as [5],

\[
T_L = \frac{1}{2\pi} , \quad T_R = \frac{r_0}{2\pi}
\]  

(81)

Furthermore, the conformal weights of the operator corresponding to the scalar field \(\Phi\) is \((h_L, h_R) = (h, h)\). Similar to the extremal case, the source \(J_\ell\) depends on the compact coordinate through \(J_\ell(\sigma^+, \sigma^-) = \sum_m e^{im\phi - i\omega r_0 + i\epsilon Q \sigma^+} J_{\ell m}\). Now using the expression \([15]\) for the decay rate, for the mode with quantum numbers \(\ell, m\),

\[
\mathcal{R}_{\ell m}^{(\text{CFT})} = 2\pi e^{-2\pi \beta} \left(\frac{r_0}{Q}\right)^{2h-1} \frac{|\Gamma(h - i\omega r_0)|^2 |\Gamma(h + i\epsilon Q)|^2}{\Gamma(2h)^2} C_\ell^2 |J_{\ell m}|^2
\]  

(82)
The constant $J_{\ell m}$ is determined by the Dirichlet term of $R_{\ell m}^{(n)}$ near infinity. That is, it is the coefficient of $r^{\ell-1}$ in $c_{(n)} R_{\ell m}^{(n)}$ when $r \to \infty$,
\[
J_{\ell m} = \frac{4\pi \alpha (2r_0)^{1-h+3i\beta}}{1-e^{-4\pi\beta}} \frac{\Gamma(2h-1)}{\Gamma(1-2i\beta)\Gamma(h-i\alpha+i\beta)\Gamma(h+i\alpha+i\beta)} C_{(n)} \tag{83}
\]

Substituting into the decay rate, we have
\[
R_{\ell m}^{(CFT)} = \frac{1}{1-e^{-4\pi\beta}} \frac{\alpha^2 2^{5-2h} \pi^2 r_0 Q^{1-2h}}{(2h-1)^2} C_2^2 \left| C_{(n)} \right|^2 \tag{84}
\]
If we choose $C_2^2 = \frac{Q^{2h+1}(2h-1)^2}{2^{5-2h} \pi^2}$, this decay rate is completely the same as that computed in the gravity side (79).

4 Discussions

If we compare the computations of the decay rates in the previous two sections, one will find that if the extremal limit, $E = M - Q \to 0$, is taken before the near horizon limit, $\epsilon \to 0$, the near-extremal case reduced to the extremal one ($r_0 = 0$). However, if the near horizon limit is taken before the extremal limit, we will obtain the near-extremal case ($r_0 \neq 0$). It should be pointed out that one could not obtain the results of the extremal case by taking the limit $r_0 \to 0$ on the results of the near-extremal case: $r_0 = 0$ is the essential singularity of the hypergeometric functions (60) and (61).

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