Higher derived brackets, strong homotopy associative algebras and Loday pairs

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Abstract

We give a quick method of constructing strong homotopy associative algebra, namely, the higher derived product construction. This method is associative analogue of classical higher derived bracket construction in the category of Loday algebras. We introduce a new type of algebra, Loday pair, which is noncommutative analogue of classical Leibniz pair. We study strong homotopy Loday pairs and the higher derived brackets on the Loday pairs.

1 Introduction

Let \((g, d, [,])\) be a differential graded (dg, for short) Lie algebra. We define a new product by \([x, y]_d := (-1)^{|x|}[dx, y]\). This product is called a derived bracket of Koszul-Kosmann-Schwarzbach ([15]). It is known that the algebra of the derived bracket is a Loday algebra (so-called Leibniz algebra), i.e., \([x, y]_d\) satisfies the Leibniz identity:

\[ [x, [y, z]]_d = [[x, y], z]_d + (-1)^{|x||y|}[y, [x, z]]_d. \]

The derived bracket construction is a method of constructing new algebra structure. It plays important roles in modern analytical mechanics and in differential geometry. For instance, a Poisson bracket on a smooth manifold is a derived bracket of a graded Poisson bracket which is called a Schouten-Nijenhuis bracket.

\[ \{f, g\} = (-1)^{|f|}[\pi, f]_{SN}g_{SN}, \]

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where \( f, g \) are smooth functions, \( \pi \) is a Poisson structure tensor, \([ \cdot, \cdot ]_{SN}\) is the Schouten-Nijenhuis bracket and \( \{ \cdot, \cdot \} \) is the induced Poisson bracket. Since \( d \pi := [\pi, -]_{SN} \) is a differential, the Poisson bracket is a derived bracket. We recall another example of derived brackets. Let \( f = f(p, q) \) and \( g = g(p, q) \) be super functions on a super symplectic manifold, where \( (p, q) \) is a canonical coordinate of the manifold. We consider a Laplacian with odd degree \( \Delta_{BV} := \sum (\pm) \frac{\partial^2}{\partial p \partial q} \) and define a differential \( d_{BV} := [\Delta_{BV}, -] \). It is known that the derived bracket associated with \( d_{BV} \) is a Poisson bracket (so-called BV-bracket): 

\[
(f, g) = \sum \frac{f_0}{\partial p} \frac{\partial g}{\partial q} \cdot \frac{f_0}{\partial q} \cdot \frac{\partial g}{\partial p} = (\pm)[d_{BV}\hat{f}, \hat{g}],
\]

where \( \hat{f}(-) := f \times (-) \) is a scalar multiplier, \( (\pm) \) is an appropriate sign and \([\cdot, \cdot]\) is a Lie bracket (commutator). Thus various bracket products (Lie algebroid brackets, Courant brackets, BV-brackets and so on) are given as derived brackets (see [16]).

The idea of the derived bracket arises in several mathematical areas. We recall a derived bracket in the category of associative algebras. Let \((A, *, d)\) be a dg associative algebra. We define a modified product by \( a *_{d} b := (-1)^{|a|}(da) * b, a, b \in A \). Then it is again associative. This new product is called a **derived product**, which is used in the study of Loday type algebras (cf. Loday [17]).

The derived bracket/product constructions have been extended to any algebra over binary quadratic operad in [22].

We consider \( n \)-fold derived brackets composed of Lie brackets:

\[
[x_1, \ldots, x_n]_d := (\pm)[[\ldots[[d x_1, x_2], x_3]\ldots], x_n].
\]

Such higher brackets were studied by several authors in various contexts (cf. Akman (1996) [1], Vallejo (2001) [24], Roytenberg (2002) [20] and Voronov (2005) [25]). Koszul’s original type higher derived brackets, which are denoted by \( \Phi \), are defined on super commutative algebras by

\[
\Phi^a_D(a_1, \ldots, a_n) := [[[d\hat{a}_1, \hat{a}_2], \ldots], \hat{a}_n](1)
\]

where \( d := [\Delta, -] \) and where \( \Delta \) is a certain differential operator like \( \Delta_{BV} \) above. The higher brackets \( \Phi \) are used to study **higher order** differential operators (cf. [1],[24]). In [23], the author studied a higher derived bracket construction in the category of Loday algebras. We briefly describe the result in [23]. Let \((V, \delta_0)\) be a dg Loday algebra and let

\[
d := \delta_0 + t \delta_1 + t^2 \delta_2 + \cdots
\]
be a formal deformation of $\delta_0$, where $dd = 0$. Define a higher derived bracket on the Loday algebra by

$$l_n(x_1, \ldots, x_n) := (\pm)[[\ldots[[\delta_{n-1}x_1, x_2], x_3]\ldots], x_n],$$

where the binary bracket $[,]$ is a Loday bracket. It was shown that the collection of the higher derived brackets $(l_1, l_2, l_3, \ldots)$ provides a strong homotopy (shortly, sh) Loday algebra structure (also called Loday $\infty$-algebra or sh Leibniz algebra or Leibniz $\infty$-algebra). If each $l_{n \geq 2}$ is skewsymmetric, then the sh Loday algebra is an sh Lie ($L_{\infty}$)-algebra. This proposition is a homotopy version of the binary derived bracket construction in [15].

The first aim of this note is to study a higher version of the derived product construction. Let $(A, \delta_0)$ be a dg associative algebra and $d = \sum_{i \geq 0} t^i \delta_i$ be a formal deformation of $\delta_0$. Define a higher derived products by

$$m_n(a_1, \ldots, a_n) := (\pm)(\delta_{n-1}a_1) * a_2 * \cdots * a_n.$$ 

We show in Theorem 3.1 below that the system with the higher derived products $(A, m_1, m_2, \ldots)$ becomes an sh associative algebra (or $A_{\infty}$-algebra).

The second aim of this note is to unify the higher derived bracket/product constructions. To complete this task we recall Leibniz pairs. The notion of Leibniz pair was introduced by Flato-Gerstenhaber-Voronov [5], motivated by the study of deformation quantization. The Leibniz pairs are defined to be the pairs of Lie and associative algebras $(g, A)$ equipped with derivation representations $rep : g \to \text{Der}(A)$. The representation satisfies the following two derivation relations:

$$[x, [a, b]] = [[x, a], b] + [a, [x, b]],$$

$$[x, [y, a]] = [[x, y], a] + [y, [x, a]],$$

where $x, y \in g$, $a, b \in A$ and where $[x, a]$ is the derivation action of $L$ on $A$ and $[a, b]$ is the associative multiplication on $A$, i.e., $[a, [b, c]] = [[a, b], c]$. We recall two typical examples of Leibniz pairs.

a) The self pair of a Poisson algebra $P$, $(P, P)$, is obviously a Leibniz pair.

b) Let $g \to M$ be a Lie algebroid over a smooth manifold $M$. Then the pair $(\Gamma g, C^\infty(M))$ is a Leibniz pair, where $\Gamma g$ is the space of sections of $g$.

We consider the pairs of Loday algebras and associative algebras satisfying (1) and (2). We call such pairs the Loday pairs. There exists interesting examples of Loday pairs, which are regarded as noncommutative analogues of Examples a),b).

a-1) It is known that a Poisson manifold is a classical solution of a master equation
associated with 2-dimensional topological field theory (cf. Cattaneo-Felder [3, 4]).
In the 3-dimensional cases, the classical solutions are known as Courant algebroids
(Ikeda [10, 11, 12], see also [21]). A Courant algebroid is a vector bundle \( E \to M \)
of which the space of sections is a Loday algebra satisfying some axioms. When \( E \)
is a Courant algebroid, the pair \((\Gamma E, C^\infty(M))\) is a Loday pair.

b-1) Let \( L \to M \) be a vector bundle over \( M \) with a bundle map \( \rho : L \to TM \).
\( L \) is called a Leibniz algebroid (Ibanez and collaborators [9]), if the space of sections \( \Gamma L \)
has a Loday bracket satisfying

\[
[X_1, fX_2] = f[X_1, X_2] + \rho(X_1)(f)X_2,
\]
where \( X_1, X_2 \in \Gamma L \) and \( f \in C^\infty(M) \). When \( L \) is a Leibniz algebroid, the pair \((\Gamma L, C^\infty(M))\) is a
Loday pair.

The place of Loday pairs among other objects may be illustrated by the following
table.

| g \ A | commutative | noncommutative | dimension |
|-------|------------|---------------|-----------|
| commutative | Poisson algebras | classical Leibniz pairs | 2 |
| | Lie algebroids | | |
| noncommutative | Courant algebroids | Loday pairs | 3 |
| | Leibniz algebroids | | |

Loday pairs are noncommutative analogues of Courant/Leibniz algebroids.

We introduce a coalgebra description of Loday pairs, and then study higher derived bracket construction in the category Loday pairs (see Section 4). The Leibniz pairs up to homotopy which are called sh Leibniz pairs are studied by Kajiura-Stasheff [13, 14] and by Hoefel [8] in the context of open-closed string field theory.

We introduce a new type of homotopy algebra, *sh Loday pair*, which is considered as a noncommutative analogue of Leibniz pair. We show that sh associative/Loday algebras are both subalgebras of sh Loday pairs. The higher derived brackets in the category of Loday pairs are defined by

\[
n_{i+j}(x_1, \ldots, x_i, a_1, \ldots, a_j) := (\pm)[[\cdots[[x_{i+j-1}, x_1], x_2], \ldots], a_1, \ldots], a_j],
\]
where \( x \in L, a \in A \) and \( [,] \) is a multiplication on a Loday pair. The higher derived brackets and the higher derived products are both subsystem of \( \{n_{i+j}\} \). The second main result of this note is as follows. Let \((L, A, \delta_0)\) be a Loday pair \((L, A)\) with differential \( \delta_0 \) and let \( d := \sum_{i \geq 0} t^i \delta_i \) be a deformation of \( \delta_0 \). We prove in Proposition 4.12 that the system with the unified higher derived brackets \((n_1, n_2, n_3, \ldots)\) is an sh Loday pair.

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Notations and Assumptions. In the following, we assume that the characteristic of the ground field \( K \) is zero and that a tensor product is defined over the field, \( \otimes := \otimes_\mathbb{K} \). We follow the Koszul sign convention. For instance, a linear map \( f \otimes g : V \otimes V \to V \otimes V \) satisfies, for any \( v_1 \otimes v_2 \in V \otimes V \),
\[
(f \otimes g)(v_1 \otimes v_2) = (-1)^{|g||v_1|} f(v_1) \otimes g(v_2),
\]
where \( |g| \) and \( |v_1| \) are degrees of \( g \) and \( v_1 \). We will use a degree shifting operator \( s \) (resp. \( s^{-1} \)) with degree +1 (resp. −1). The shifting operators satisfy \( s \otimes s = (s \otimes 1)(1 \otimes s) = -(1 \otimes s)(s \otimes 1) \). We denote by \((-1)^o\) the sign \((-1)^{|o|}\) without miss reading.

2 Preliminaries

We consider the tensor space over a graded vector space \( V \):
\[
\bar{T}V := V \oplus V^\otimes 2 \oplus \cdots.
\]
The space \( \bar{T}V \) has an associative coalgebra structure, \( \Delta : \bar{T}V \to \bar{T}V \otimes \bar{T}V \), defined by \( \Delta(V) := 0 \) and
\[
\Delta(v_1, \ldots, v_n) := \sum_{i=1}^{n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n), \tag{3}
\]
where \( v_i \in V \). Then \( (\bar{T}V, \Delta) \) becomes a cofree coalgebra in the category of nilpotent coalgebras. Let Coder(\( \bar{T}V \)) be the space of coderivations, i.e., \( D^c \in \text{Coder}(\bar{T}V) \) satisfies the coderivation rule:
\[
(D^c \otimes 1) \Delta + (1 \otimes D^c) \Delta = \Delta D^c.
\]
It is well-known that Coder(\( \bar{T}V \)) is identified with the space of the endomorphisms on \( V \):
\[
\text{Hom}(\bar{T}V, V) \cong \text{Coder}(\bar{T}V). \tag{4}
\]
We recall an explicit formula of the isomorphism. For a given \( i \)-ary endomorphism \( f : V^\otimes i \to V \), we define a coderivation \( f^c \) by \( f^c(V^{n<i}) := 0 \) and
\[
f^c(v_1, \ldots, v_{n\geq i}) := \sum_{k=0}^{n-i} (-1)^{f(v_k+\cdots+v_k)} (v_1, \ldots, v_k, f(v_{k+1}, \ldots, v_{k+i}), v_{k+i+1}, \ldots, v_n).
\]
The inverse of the mapping \( f \mapsto f^c \) is the restriction (so-called corestriction).

The space Coder(\( \bar{T}V \)) has a canonical Lie bracket of graded commutator. Therefore \( \text{Hom}(\bar{T}V, V) \) has a Lie bracket which is induced by the isomorphism (4). The
induced Lie bracket on $\text{Hom}(\bar{T}V, V)$, which is denoted by $\{f, g\}$, is well-known as a Gerstenhaber bracket on a Hochschild complex. If $sV$ (the shifted space of $V$) is an associative algebra, then $\text{Hom}(\bar{T}V, V)$ becomes a Hochschild complex:
\[
\cdots \xrightarrow{b} \text{Hom}(V^\otimes n, V) \xrightarrow{b} \text{Hom}(V^\otimes n+1, V) \xrightarrow{b} \cdots .
\]
The coboundary map $b$ is induced by the associative structure on $sV$ (see Remark 2.2).

If $f^c$, $g^c$ are coderivations associated with $i$-ary, $j$-ary endomorphisms, respectively, then the Lie bracket $[f^c, g^c]$ is the coderivation associated with the Gerstenhaber bracket of $f$ and $g$, i.e.,
\[
\{f, g\}^c = [f^c, g^c],
\]
where $\{f, g\}$ is an $(i + j - 1)$-ary endomorphism.

In the following, we identify $\text{Coder}(\bar{T}V)$ with $\text{Hom}(\bar{T}V, V)$. Hence we omit the subscript “$c$” from $f^c$ without miss reading.

**Definition 2.1.** Let $sV$ be the shifted space equipped with a collection of $i(\geq 1)$-ary endomorphisms, $m_i : (sV)^\otimes i \to sV$. We assume that the degree of $m_i$ is $2 - i$ for each $i$. We set the shifted map:
\[
\partial_i := s^{-1} \circ m_i \circ (s \otimes \cdots \otimes s).
\]
This is an element in $\text{Hom}(\bar{T}V, V)$ or in $\text{Coder}(\bar{T}V)$ up to the identification. We define a coderivation by
\[
\partial := \partial_1 + \partial_2 + \cdots .
\]
The system $(sV, m_1, m_2, ...)$ is called a strong homotopy (sh) associative algebra, or sometimes called an $A_\infty$-algebra, if $\partial$ is square zero, or equivalently,
\[
\frac{1}{2} [\partial, \partial] = 0.
\]

**Remark 2.2.** The usual associative algebra can be seen as a special sh associative algebra such that $\partial_n \neq 0$. In such a case, we put $b(-) := [\partial_2, -]$. Then $b$ becomes the coboundary map of the Hochschild complex.

### 3 Associative cases

#### 3.1 Derived products

Let $(A, *, \delta_0)$ be a differential graded (dg) associative algebra. We consider a deformation of $\delta_0$:
\[
d := \delta_0 + t\delta_1 + t^2\delta_2 + \cdots .
\]
The deformation $d$ is a square zero derivation on $A[[t]]$. The square zero condition of $d$ is equivalent to the following condition.

$$\sum_{i+j=\text{Const}} \delta_i \delta_j = 0. \quad (5)$$

We define the higher derived products on $sA$ by

$$m_i := (-1)^{(i-1)(i-2)/2} s \circ M_i \circ (s^{-1} \otimes \cdots \otimes s^{-1})(s \delta_{i-1} s^{-1} \otimes 1 \otimes \cdots \otimes 1),$$

where $M_i(a_1, ..., a_i) := a_1 * a_2 * \cdots * a_i$ for any $a_1, ..., a_i \in A$. By a direct computation, we have

$$m_i(sa_1, ..., sa_i) = (\pm)s \left( \delta_{i-1} a_1 * a_2 * \cdots * a_i \right),$$

where

$$\pm = \begin{cases} (-1)^{a_1+a_3+\cdots+2n+1} & i = \text{even}, \\ (-1)^{a_2+a_4+\cdots+2n+1} & i = \text{odd}. \end{cases}$$

The main theorem of this note is as follows.

**Theorem 3.1.** The system with the higher derived products $(sA, m_1, m_2, ...)$ is an sh associative algebra.

We need some lemmas in order to show this theorem.

**Lemma 3.2.** Let $\partial_i$ be the coderivation associated with the higher derived product $m_i$. Then $\partial_i$ has the following form.

$$\partial_i = M_i \circ (\delta_{i-1} \otimes 1 \otimes \cdots \otimes 1).$$

**Proof.**

$$\partial_i := s^{-1} \circ m_i \circ (s \otimes \cdots \otimes s)$$

$$= (-1)^{(i-1)(i-2)/2} M_i \circ (s^{-1} \otimes \cdots \otimes s^{-1})(s \delta_{i-1} s^{-1} \otimes 1 \otimes \cdots \otimes 1)(s \otimes \cdots \otimes s)$$

$$= (-1)^{(i-1)(i-2)/2} M_i \circ (s^{-1} \otimes \cdots \otimes s^{-1})(s \delta_{i-1} \otimes s \otimes \cdots \otimes s)$$

$$= (-1)^{(i-1)(i-2)/2} (-1)^{i-1} M_i \circ (s^{-1} \otimes \cdots \otimes s^{-1})(s \otimes s \otimes \cdots \otimes s)(\delta_{i-1} \otimes 1 \otimes \cdots \otimes 1)$$

$$= (-1)^{(i-1)(i-2)/2} (-1)^{i-1} (-1)^{i(i-1)/2} M_i \circ (\delta_{i-1} \otimes 1 \otimes \cdots \otimes 1).$$

\[ \square \]

Let $\text{Der}(A)$ be the space of derivations on the algebra $(A, *)$. For any $D \in \text{Der}(A)$, we define an $i$-ary map by

$$M_i D := M_i \circ (D \otimes 1 \otimes \cdots \otimes 1),$$

in particular, $M_1 D = D$. One can identify $M_i D$ with a coderivation in $\text{Coder}(\overline{T}A)$. 

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Lemma 3.3. For any $D, D' \in \text{Der}(A)$ and for any $i, j$, the Lie bracket of the coderivations, $[M_i D, M_j D']$, is compatible with the one of the derivations, $[D, D']$, namely,

$$[M_i D, M_j D'] = M_{i+j-1}[D, D'].$$ 

Proof. We assume for the sake of simplicity that the variables have no degree. We put $a = (a_1, \ldots, a_{i+j-1}) \in A^{\otimes (i+j-1)}$. Then we have

$$M_i D \circ M_j D'(a) = \sum_{s=0}^{i-1} Da_1 \cdots D'a_{s+1} \cdots a_{i+j-1}$$

$$= D(D'a_1 \cdots a_j) \cdots a_{i+j-1} + \sum_{s=1}^{i-1} Da_1 \cdots D'a_{s+1} \cdots a_{i+j-1}$$

$$= (DD'a_1) \cdots a_{i+j-1} + (-1)^{DD'} \sum_{t=2}^{j} D'a_1 \cdots Da_t \cdots a_{i+j-1}$$

$$+ \sum_{s=1}^{i-1} Da_1 \cdots D'a_{s+1} \cdots a_{i+j-1}.$$ 

On the other hand, we have

$$M_j D' \circ M_i D(a) = \sum_{t=1}^{j} D'a_1 \cdots Da_t \cdots a_{i+j-1}$$

$$= D'(Da_1 \cdots a_i) \cdots a_{i+j-1} + \sum_{t=2}^{j} D'a_1 \cdots Da_t \cdots a_{i+j-1}$$

$$= (D'Da_1) \cdots a_{i+j-1} + (-1)^{D'D} \sum_{s=1}^{i-1} Da_1 \cdots D'a_{s+1} \cdots a_{i+j-1}$$

$$+ \sum_{t=2}^{j} D'a_1 \cdots Da_t \cdots a_{i+j-1}.$$ 

Hence we obtain

$$[M_i D, M_j D'](a) = (DD'a_1) \cdots a_{i+j-1} - (-1)^{DD'} (D'Da_1) \cdots a_{i+j-1}$$

$$= M_{i+j-1}[D, D'](a).$$

We give a proof of Theorem 3.1:

Proof. The higher derived product $m_i$ corresponds to the coderivation $\partial_i = M_i \delta_{i-1}$. The deformation condition $[d, d]/2 = 0$ corresponds to the homotopy algebra condition,

$$\sum_{i+j=\text{Const}} [\partial_i, \partial_j] = \sum_{i+j=\text{Const}} [M_i \delta_{i-1}, M_j \delta_{j-1}] = M_{i+j-1} \sum_{i+j=\text{Const}} [\delta_{i-1}, \delta_{j-1}] = 0.$$
We consider the special case of $m_{n \neq 2} = 0$, namely, the case of trivial deformation:

$$d = t\delta_1.$$ 

**Corollary 3.4.** Assume that $m_{n \neq 2} = 0$, or equivalently, $sA$ is the usual associative algebra with the binary derived product. Then the collection of $\{M_i \text{Der}(A)\}$ is a subcomplex of the Hochschild complex $\text{Hom}(\bar{T}A, A)$, where

$$M_i \text{Der}(A) := \langle M_i D \mid D \in \text{Der}(A) \rangle.$$ 

**Proof.** The coboundary map on $\text{Hom}(\bar{T}A, A)$ is given by

$$b(-) := [\partial_2, -] = [M_2 \delta_1, -].$$

Hence we obtain $b(M_i D) = M_{i+1} [\delta_1, D]$. 

### 3.2 Deformation theory

We discuss a relationship between deformation theory and sh associative algebras. The main result of this subsection is Proposition 3.5 below. A Loday algebra version of this proposition was shown in [22].

The deformation of $\delta_0$, $d = \delta_0 + t\delta_1 + \cdots$, is considered as a differential on $A[[t]]$ which is an associative algebra of formal series with coefficients in $A$. Let $h(t) := t h_1 + t^2 h_2 + \cdots$ be a derivation on the associative algebra $A[[t]]$ with degree $|h(t)| := 0$. We consider the second deformation $d' = \sum t^n \delta_n'$. The deformations $d$ and $d'$ are equivalent, if they are related via the gauge transformation:

$$d' := \exp(X_{h(t)})(d),$$

where $X_{h(t)} := [-, h(t)]$. We denote by $\partial' = \sum \partial'_n$ the induced sh associative structure associated with $d'$.

**Proposition 3.5.** If $d$ and $d'$ are gauge equivalent, then the sh associative structures $\partial$ and $\partial'$ are equivalent, namely,

$$\partial' = \exp(X_{Mh})(\partial),$$

where $Mh$ is a well-defined infinite sum of coderivations:

$$Mh := M_2 h_1 + M_3 h_2 + \cdots + M_{i+1} h_i + \cdots,$$
and the integral of $M_h$,

$$e^{M_h} := 1 + M_h + \frac{1}{2!}(M_h)^2 + \cdots,$$

is a dg coalgebra isomorphism between $(\bar{T}A, \partial)$ and $(\bar{T}A, \partial')$, namely, (6) and (7) below hold.

\[ \partial' = e^{-M_h} \cdot \partial \cdot e^{M_h}, \quad (6) \]
\[ \Delta e^{M_h} = (e^{M_h} \otimes e^{M_h}) \Delta. \quad (7) \]

Proof. The proof of this proposition is the same as the one in [22]. \qed

In general, an $A_\infty$-morphism is defined to be a dg coalgebra morphism between $(\bar{T}A, \partial)$ and $(\bar{T}A', \partial')$. Hence $e^{M_h}$ is an $A_\infty$-isomorphism.

4 Loday pairs

We introduce the concept “Loday pair” and study its homotopy algebras.

4.1 Sh Loday algebras

We recall sh Loday algebras. Let $L$ be a graded vector space and let $sL$ be the shifted space and let $l_i : (sL)^{\otimes i} \rightarrow sL$ be a multilinear map with degree $2 - i$, for each $i \geq 1$.

Definition 4.1. ([2], and see also [22]) The system with the multiplications, $(sL, l_1, l_2, \ldots)$, is called a strong homotopy (sh) Loday algebra (Loday $\infty$-algebra or sh Loday algebra or Loday $\infty$-algebra), if the collection \{l_i\}_{i \geq 1} satisfies (8) below.

$$\sum_{i+j=\text{Const}} \sum_{k=j}^{i+j-1} \sum_{\sigma} \chi(\sigma)(-1)^{(k+1-j)(j-1)}(-1)^j(sx_{\sigma(1)} + \ldots + sx_{\sigma(k-j)})$$

$$l_i(sx_{\sigma(1)}, \ldots, sx_{\sigma(k-j)}), l_j(sx_{\sigma(k+1-j)}, \ldots, sx_{\sigma(k-1)}, sx_k, sx_{k+1}, \ldots, sx_{i+j-1}) = 0, \quad (8)$$

where $(sx_1, \ldots, sx_{i+j-1}) \in sL^{\otimes (i+j-1)}$, $\sigma$ is a $(k - j, j - 1)$-unshuffle, $\chi(\sigma)$ is an anti-Koszul sign, $\chi(\sigma) := \text{sgn}(\sigma)e(\sigma)$.

Sh Lie algebras are special examples of sh Loday algebras such that all $l_i$ ($i \geq 2$) skewsymmetric. It is easy to show this claim. If each $l_{i \geq 2}$ is skewsymmetric, then

$$l_i(sx_{\sigma(1)}, \ldots, sx_{\sigma(k-j)}), l_j(sx_{\sigma(k+1-j)}, \ldots, sx_{\sigma(k-1)}, sx_k, sx_{k+1}, \ldots, sx_{i+j-1}) =$$

$$\pm l_i(l_j(sx_{\sigma(k+1-j)}, \ldots, sx_{\sigma(k-1)}, sx_k), sx_{\sigma(1)}, \ldots, sx_{\sigma(k-j)}, sx_{k+1}, \ldots, sx_{i+j-1}) =$$

$$\pm l_j(l_i(sx_{\tau(1)}, \ldots, sx_{\tau(j)}), sx_{\tau(j+1)}, \ldots, sx_{\tau(i+j-1)}),$$

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where $\tau$ is an unshuffle permutation. And $\sum_{k=j}^{i+j-1} \sum_{\sigma} l_k (l_j (s \sigma(x_{\tau(1)}), \ldots, s \sigma(x_{\tau(j)}), s \sigma(x_{\tau(j+1)}), \ldots, s \sigma(x_{\tau(i+j-1)})) = 0.

This is the defining relation of sh Lie algebras.

The cofree nilpotent dual-Loday coalgebra over $L$ is the tensor space $\bar{T}L$ with a comultiplication, $\Delta_L : \bar{T}L \to \bar{T}L \otimes \bar{T}L$, defined by $\Delta_L(x) := 0$ and
\[
\Delta_L(x_1, \ldots, x_{n+1}) := \sum_{i=1}^{n} \sum_{\sigma} \epsilon(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}) \otimes (x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}, x_{n+1}),
\]
where $\epsilon(\sigma)$ is a Koszul sign, $\sigma$ is an $(i, n-i)$-unshuffle. Let $\text{Coder}(\bar{T}L)$ be the space of coderivations on $\bar{T}L$. By a standard argument, we obtain an isomorphism:
\[
\text{Coder}(\bar{T}L) \cong \text{Hom}(\bar{T}L, L).
\]

We recall an explicit formula of the isomorphism. Let $f : L^{\otimes i} \to L$ be an $i$-ary map. It is one of the generators in $\text{Hom}(\bar{T}L, L)$. The coderivation associated with $f$ is defined by $f_c(L^{\otimes n < i}) := 0$ and
\[
f_c(x_1, \ldots, x_{n+1}) := \sum_{k=i}^{n} \sum_{\sigma} \epsilon(\sigma)(-1)^{k-i} f(x_{\sigma(1)} + \ldots + x_{\sigma(k-i)})
\]
\[
(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, f(x_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_{\sigma(k+1)}, \ldots, x_{n}),
\]
where $\sigma$ is a $(k-i, i-1)$-unshuffle. The inverse of $f \mapsto f_c$ is the restriction (so-called corestriction). The collection of $i$-ary maps, $\{l_i\}_{i \geq 1}$, induces a collection of coderivations on $\bar{T}L$, $\{\partial_i\}_{i \geq 1}$. We put $\partial_0 := \partial_1 + \partial_2 + \cdots$. The definition (8) is equivalent to the homogenous condition:
\[
\frac{1}{2} [\partial_L, \partial_L] = \partial_L \partial_L = 0.
\]

4.2 Regularization

For the Leibniz identity, $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]]$, one can regard the term “$[x_2, [x_1, x_3]]$” as an associative anomaly, that is, $[x_1, [x_2, x_3]] - [[x_1, x_2], x_3] \equiv 0$ modulus $[x_2, [x_1, x_3]]$. We notice that the sh Loday relation (8) has an sh associative

3The word “dual-” means the Koszul duality ([7]), i.e., the operad of dual-Loday algebras is the Koszul dual of the operad of Loday algebras.
anomaly. We take out the regular subterms from (8):

$$\sum_{i+j=\text{Const}} \sum_{a=0}^{i-1} (-1)^{(a+1)(j-1)} (-1)^j (sx_1+...+sx_a)$$

$$l_i(sx_1,...,sx_a), l_j(sx_{a+1},...sx_{a+j}), sx_{a+j+1},...sx_{i+j-1}),$$

(11)

where $a := k - j$. This is the defining relation of sh associative algebras. Thus the no regular terms can be seen as the obstruction for sh associativity. In the same way, we take out the regular subterms from (9). Then it has the same form as the comultiplication of the associative coalgebra (cf. (3)):

$$\Delta_L(x_1,...,x_{n+1}) \stackrel{\text{regular}}{\sim} \sum_{i=1}^{n} (x_1, x_2, ..., x_i) \otimes (x_{i+1}, ..., x_n, x_{n+1}).$$

(12)

As observed above, the associative world is the regular subsystem in the Loday/Leibniz world. Along this picture, we try to unify the sh Loday/associative algebras.

### 4.3 Unification

We consider a pair of graded vector spaces $(L, A)$. Set a tensor space:

$$LA := \sum_{n \geq 1} \sum_{i+j=n} L^\otimes i \otimes A^\otimes j.$$ 

Define a comultiplication $\Delta$ on $LA$ by the same manner as (9). For instance,

$$\Delta(x, a_1, a_2, a_3) = x \otimes (a_1, a_2, a_3) \pm a_1 \otimes (x, a_2, a_3) \pm a_2 \otimes (x, a_1, a_3) \pm$$

$$(x, a_1) \otimes (a_2, a_3) \pm (x, a_2) \otimes (a_1, a_3) \pm (a_1, a_2) \otimes (x, a_3) \pm$$

$$(x_1, a_1, a_2) \otimes a_3,$$

(13)

where $x \in L$ and $a_1, a_2, a_3 \in A$. The space of coderivations on $(LA, \Delta)$, $\text{Coder}(LA, \Delta)$, is identified with a subspace of $\text{Hom}(LA, L \oplus A)$. This identification is defined by the same rule as (10). For instance, if a binary map $f : \sum_{i+j=2} L^\otimes i \otimes A^\otimes j \to L \oplus A$ corresponds to a coderivation, then it satisfies

$$f^c(x, a_1, a_2, a_3) = (f(x, a_1), a_2, a_3) \pm$$

$$(x, f(a_1, a_2), a_3) \pm (a_1, f(x, a_2), a_3) \pm$$

$$(x, a_1, f(a_2, a_3)) \pm (x, a_2, f(a_1, a_3)) \pm (a_1, a_2, f(x, a_3)).$$

(14)

---

4The word “regular” is used in the sense of $\sigma = id$, i.e., the order of variables is regular.
We notice that \( f(x, a_2) \) and \( f(a_2, a_3) \) are \( A \)-valued, because the elements of \( A \) must be put on the right side of the elements of \( L \). By a simple observation, we obtain

\[
\text{Coder}(LA, \Delta) \cong \text{Hom}(TL, L) \oplus \sum_{i \geq 0, j \geq 1} \text{Hom}(L^{\otimes i} \otimes A^{\otimes j}, A).
\]

(15)

We are regularizing \( \Delta \) with respect to the order of variables of \( A \) (we say this operation an \( A \)-regularization). For instance, the \( A \)-regularization of (13) is

\[
\Delta(x, a_1, a_2, a_3) \mapsto (x \otimes (a_1, a_2, a_3) \pm a_1 \otimes (x, a_2, a_3) \pm (x, a_1) \otimes (a_2, a_3) \pm (a_1, a_2) \otimes (x, a_3) \pm (x_1, a_1, a_2) \otimes a_3,
\]

where the ordering \( a_1 < a_2 < a_3 \) is preserved. We denote by \( \text{reg}(\Delta) \) the regularized comultiplication. It generally has the form:

\[
\text{reg}(\Delta)(x, a) := \sum_{i+k} \sum_{\sigma} (\pm)(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, a_1, \ldots, a_k) \otimes (x_{\sigma(i+1)}, \ldots, x_{\sigma(m)}, a_{k+1}, \ldots, a_n),
\]

where \( x = (x_1, \ldots, x_m) \), \( a = (a_1, \ldots, a_m) \) and \( \sigma \) is an \((i, m-i)\)-unshuffle. Remark that the restrictions \( \text{reg}(\Delta)|_L \) and \( \text{reg}(\Delta)|_A \) coincide with the classical comultiplications above, respectively. Let \( D^c \) be a coderivation on the coalgebra \((LA, \Delta)\). We define the \( A \)-regularization of \( D^c \), \( \text{reg}(D^c) \), by regularizing the order of variables of \( A \). For instance, the regularization of (14) becomes

\[
\text{reg}(f^c)(x, a_1, a_2, a_3) = (f(x, a_1), a_2, a_3) \pm (a_1, f(x, a_2), a_3) \pm (x, a_1, f(a_2, a_3)) \pm (a_1, a_2, f(x, a_3)).
\]

**Lemma 4.2.** The regularization \( \text{reg}(D^c) \) is a coderivation on \((LA, \text{reg}(\Delta))\), that is,

\[
(\text{reg}(D^c) \otimes 1 + 1 \otimes \text{reg}(D^c)) \text{reg}(\Delta) = \text{reg}(\Delta) \text{reg}(D^c).
\]

**Proof.** Apply \( \text{reg} \) on the both side of \((D^c \otimes 1 + 1 \otimes D^c) \Delta = \Delta D^c \). We have \( \text{reg}(D^c \otimes 1) \Delta = \text{reg}((D^c \otimes 1) \text{reg}(\Delta)) = (\text{reg}(D^c) \otimes 1) \text{reg}(\Delta) \) and \( \text{reg}(1 \otimes D^c) \Delta = \text{reg}(1 \otimes D^c) \text{reg}(\Delta) \). Hence we obtain

\[
\text{reg}((D^c \otimes 1 + 1 \otimes D^c) \Delta) = (\text{reg}(D^c) \otimes 1 + 1 \otimes \text{reg}(D^c)) \text{reg}(\Delta).
\]

On the other hand, we obtain \( \text{reg}(\Delta D^c) = \text{reg}(\Delta \text{reg}(D^c)) = \text{reg}(\Delta) \text{reg}(D^c) \). Therefore we get the identity of the lemma.

The space of coderivations on the regularized coalgebra \((LA, \text{reg}(\Delta))\) also corresponds to the same homomorphism space as (15). The above lemma implies that the correspondence is the \( A \)-regularization of the isomorphism in (15).
4.4 Unified derived brackets

Let $L$ be a Loday algebra and let $A$ be an associative algebra. We assume that the degrees of the multiplications on $(L, A)$ are both zero (or even).

**Definition 4.3.** The pair $(L, A)$ equipped with a binary multiplication $[,] : L \otimes A \to A$ is called a left-Loday pair, or simply, Loday pair, if it satisfies

\[
[x, [y, a]] = [[x, y], a] + (-1)^{xy}[y, [x, a]], \quad (16)
\]

\[
[x, [a, b]] = [[x, a], b] + (-1)^{xa}[a, [x, b]], \quad (17)
\]

where $x, y \in L$, $a, b \in A$ and where $[x, y]$ is the Loday bracket on $L$ and $[a, b]$ is the associative multiplication on $A$, i.e., $[a, [b, c]] = [[a, b], c]$ for any $a, b, c \in A$.

The Loday pairs are algebraizations of Leibniz algebroids ([9]). The classical Leibniz pair in [5] is the Lie version of our noncommutative Leibniz pair (i.e. Loday pair). We give two geometric examples of Loday pairs.

**Example 4.4.** (Courant bracket) Let $M$ be a smooth manifold. Consider a bundle $\mathcal{T}M := TM \oplus T^*M$ (so-called generalized tangent bundle). The Courant bracket is defined on the space of sections of $\mathcal{T}M$, $\Gamma \mathcal{T}M$, by $[\xi_1 + \theta_1, \xi_2 + \theta_2] := [\xi_1, \xi_2] + L_{\xi_1} \theta_2 - i_{\xi_2} d\theta_1$, where $\xi_1, \xi_2 \in \Gamma \mathcal{T}M$ and $\theta_1, \theta_2 \in \Gamma T^*M$. Then the pair $(\Gamma \mathcal{T}M, C^\infty(M))$ is a Loday pair.

**Example 4.5.** Let $(M, \pi)$ be a Poisson manifold equipped with a Poisson structure tensor $\pi$. The space of multivector fields $\Gamma \wedge TM$ becomes a graded Poisson algebra of type $(-1, 0)$, whose Poisson bracket is known as Schouten-Nijenhuis (SN) bracket. The Poisson tensor is a solution of Maurer-Cartan equation $[\pi, \pi]_{SN} = 0$. Since the degree of $\pi$ is $+2$, $d := [\pi, -]_{SN}$ becomes a differential with degree $+1$. This differential is the coboundary operator of the Poisson cohomology. We define a Loday bracket by $[X, Y]_\pi := (-1)^{X}[dX, Y]_{SN}$ for any $X, Y \in \Gamma \wedge TM$. The bracket $[,]_\pi$ is the derived bracket of $SN$-bracket by the Poisson structure. Then the self pair $(\Gamma \wedge TM, \Gamma \wedge TM)$ is a Loday pair with multiplications $[,]_\pi$ and $\wedge$. In particular, the sub-pair $(C^\infty(M), C^\infty(M))$ is the self pair of the Poisson algebra on $M$.

We get a natural result.

**Corollary 4.6.** The structure of Loday pair on $(sL, sA)$ is equivalent to a (binary) codifferential on the regularized coalgebra $(LA, \text{reg}(\Delta))$.

This corollary leads us to
**Definition 4.7.** An sh (left) Loday pair is, by definition, a pair \((sL, sA)\) equipped with a codifferential on \((LA, \text{reg}(\Delta))\).

We consider a derived bracket construction in the category of Loday pairs.

**Definition 4.8.** A derivation on a Loday pair \((L, A)\) is, by definition, a pair of derivations, \(D = (D_L, D_A)\), \(D_L \in \text{Der}(L)\) and \(D_A \in \text{Der}(A)\) satisfying

\[ D[o_1, o_2] = [D o_1, o_2] + (-1)^{D o_1}[o_1, D o_2] \]

for any \(o_1, o_2 \in (L, A)\). We assume that the parity of \(D_L\) is equal with the one of \(D_A\).

**Example 4.9.** If \((L, A)\) is a Loday pair, then an adjoint action \([x, -], x \in L\), is a derivation.

A Loday pair \((L, A)\) is called a dg Loday pair, if it has a differential \(\delta\) on \((L, A)\).

It is easy to check that dg Loday pairs are special sh Loday pairs such that the higher homotopies vanish. Given a dg Loday pair \((L, A, \delta)\), define derived brackets by

\[
[sx, sy]_{d} := (-1)^{\delta x} s[\delta x, y], \\
[sx, sa]_{d} := (-1)^{\delta x} s[\delta x, a], \\
[sa, sb]_{d} := (-1)^{\alpha s}[\delta a, b].
\]

Then the triple of the derived brackets provides a new structure of Loday pair on \((sL, sA)\).

**Example 4.10.** In Example 4.5, if \(\pi'\) is a second Poisson tensor which is compatible with \(\pi\), i.e., \([\pi, \pi']_{SN} = 0\), then \(\delta := [\pi', -]_{SN}\) is a differential on the Loday pair.

We consider the higher derived bracket construction for Loday pairs. Let \(D\) be a derivation on a Loday pair \((L, A)\). We put

\[
N_k D(x, a) := [[[...[[D x_1, x_2], ..., x_i], a_{i+1}], ...], a_{i+j}], \quad (18)
\]

where \(x := (x_1, ..., x_i)\), \(a := (a_{i+1}, ..., a_{i+j})\) and \(k := i + j\), in particular, \(N_k D(a) = M_k D(a)\).

**Lemma 4.11.** We regard \(N, D\) as a coderivation on \((LA, \text{reg}(\Delta))\). For any derivations \(D, D'\) on \((L, A)\) and for any \(k, l \geq 1\),

\[
[N_k D, N_l D'] = N_{k+l-1}[D, D'].
\]
We will give a proof of this lemma in the end of this section. The main result of this section is as follows.

**Proposition 4.12.** Let \((L, A, \delta_0)\) be a dg Loday pair. We consider a deformation of \(\delta_0\), \(d := \sum_{i \geq 0} t^i \delta_i\). For each \(k \geq 1\), define a coderivation by

\[
\partial_k := \Delta_k \delta_{k-1}.
\]

Then \(\partial := \sum_k \partial_k\) is a structure of sh Loday pair.

The multilinear map \(N_k \delta_{k-1}(x, a)\) corresponds to the higher bracket on the sifted pair \((sL, sA)\):

\[
n_k(sx_1, \ldots, sx_i, sa_{i+1}, \ldots, sa_{i+j}) := (\pm)s[[[\ldots[\delta_{i+j-1}x_1, x_2], \ldots], x_i], a_{i+1}], \ldots, a_{i+j}],
\]

where \(k := i + j\), \(k \geq 1\) and

\[
\pm := \begin{cases} 
(1)^{01+02+\ldots+02n+1+\ldots} & i + j = \text{even}, \\
(1)^{02+04+\ldots+02n+\ldots} & i + j = \text{odd}.
\end{cases}
\]

and where \(o \in \{x, a\}\). The restrictions \((sL, n|_{sL})\) and \((sA, n|_{sA})\) become an sh Loday algebra and an sh associative algebra, respectively.

We give a proof of Lemma 4.11. To show this lemma we use convenient symbols:

\[
[x_1, \ldots, x_i] := [[[x_1, x_2], \ldots], x_i],
\]

\[
Dx := [Dx_1, \ldots, x_i],
\]

where \(x := (x_1, \ldots, x_i)\). The pure Loday version of the lemma was shown in [22].

We consider the mixed case. Since the adjoint action \(\hat{L} := [L, -]\) is a derivation on \(A\), (18) becomes

\[
N_k D(x, a) = (M_j \hat{D}x)(a),
\]

We denote by \(|\cdot|\) the length of word.

**Lemma 4.13.** Assume that \(|(x, a)| := k + l - 1\) and \(|a| \geq 1\).

\[
[N_k D, N_l D'](x, a) = \sum_{(x_1, x_2)} M_{|a|} \hat{D}x_1, \hat{D}x_2(a),
\]

where \((x_1, x_2)\) runs over the unshuffle-permutations including \((\emptyset, x)\) and \((x, \emptyset)\).

**Proof.** \(N_l D'(x, a)\) is decomposed into the pure Loday term (if it exists) and the mixed term:

\[
N_l D'(x, a) = \sum_{|x_2| = l} (x_1, D'x_2, x_3, a) + \sum_{|x_2| < l} (x_1, a_1, D'(x_2, a_2), a_3)
\]

\[
= \sum_{|x_2| = l} (x_1, D'x_2, x_3, a) + \sum_{|x_2| < l} (x_1, (M_{|x_2|} \hat{D}x_2)(a)).
\]
We have

\[ N_k D \circ N_l D'(x, a) = \sum_{|x_2|=l} [Dx_1, D'x_2, x_3, a] + \sum_{|x_2|<l} \left( M_{k-|x_1|} D\hat{x}_1 \circ M_{l-|x_2|} D'\hat{x}_2 \right)(a), \]

and this gives

\[ [N_k D, N_l D'](x, a) = \sum_{|x_2|=l} [Dx_1, D'x_2, x_3, a] - \sum_{|x_2|=k} [D'x_1, Dx_2, x_3, a] + \sum_{|x_1|<k, |x_2|<l} [M_{k-|x_1|} D\hat{x}_1, M_{l-|x_2|} D'\hat{x}_2](a). \]

The third term has the desired formula:

\[ \sum_{|x_1|<k \text{ and } |x_2|<l} M_{|a|}[D\hat{x}_1, D'\hat{x}_2](a). \]

It is not difficult to show that (see Appendix below)

\[ \sum_{|x_2|=l} [Dx_1, D'x_2, x_3, a] - \sum_{|x_2|=k} [D'x_1, Dx_2, x_3, a] = \sum_{|x_1| \geq k \text{ or } |x_2| \geq l} M_{|a|}[D\hat{x}_1, D'\hat{x}_2](a). \] (20)

Hence we obtain (19).

If \( x = x \) or \(|x| = 1\), then \((x_1, x_2) \in \{(\emptyset, x), (x, \emptyset)\}\) and

\[ \sum_{(x_1, x_2)} M_{|a|}[D\hat{x}_1, D'\hat{x}_2] = M_{|a|}[D, D'x] + M_{|a|}[D\hat{x}, D'] = M_{|a|}[D, D']x. \]

By using induction for the length of \( x \), one can easily prove that

\[ \sum_{(x_1, x_2)} M_{|a|}[D\hat{x}_1, D'\hat{x}_2] = M_{|a|}[D, D']x. \]

Hence we obtain the identity of Lemma 4.11:

\[ [N_k D, N_l D'](x, a) = (M_{|a|}[D, D']x)(a) = N_{l+k-1}[D, D'](x, a). \]

**Appendix.** We show (20). For any \( A, B \in L \) and for any \( y := (y_1, \ldots, y_n) \in L^\otimes n \), by the Leibniz rule, we have

\[ [[A, B], y] = \sum_{(y_1, y_2)} [[A, y_1], [B, y_2]], -] \]
where \((y_1, y_2)\) are the unshuffle permutations of \(y\) including \((\emptyset, y)\) and \((y, \emptyset)\). Now, replace \(A \rightarrow Dx_1, B \rightarrow D'x_2\) and \(y \rightarrow x_3\). Then

\[
\sum_{|x_2|=l} [Dx_1, D'x_2, x_3, a] = \sum_{|x_2|=l} [[Dx_1, y_1], [D'x_2, y_2], a]
\]

\[
= \sum_{|x_2|=l} [D(x_1, y_1), D'(x_2, y_2), a]
\]

\[
= \sum_{|x_2|\geq l} [[Dx_1, D'x_2], a]
\]

\[
= \sum_{|x_2|\geq l} M_{[a]}[\widehat{Dx_1}, \widehat{D'x_2}](a) = \sum_{|x_2|\geq l} M_{[a]}[\widehat{Dx_1}, \widehat{D'x_2}](a)
\]

where \(x_i := (x_i, y_i)\) redefined \(i \in \{1, 2\}\). The other term is, by the same manner,

\[
- \sum_{|x_2|=k} [D'x_1, Dx_2, x_3, a] = \sum_{|x_2|\geq k} M_{[a]}[\widehat{D'x_1}, \widehat{Dx_2}](a).
\]

This implies (20).

References

[1] F. Akman. On Some Generalizations of Batalin-Vilkovisky Algebras. Journal of Pure and Applied Algebra. 120 (1997), no. 2, 105–141.

[2] M. Ammar and N. Poncin. Coalgebraic Approach to the Loday Infinity Category, Stem Differential for 2n-ary Graded and Homotopy Algebras. Preprint Arxive, math/0809.4328.

[3] AS. Cattaneo and G. Felder. A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. 212 (2000), no. 3, 591–611.

[4] AS. Cattaneo and G. Felder. On the AKSZ formulation of the Poisson sigma model. EuroConference Moshe Flato 2000, Part II (Dijon). Lett. Math. Phys. 56 (2001), no. 2, 163–179.

[5] M. Flato, M. Gerstenhaber and A.A. Voronov. Cohomology and deformation of Leibniz pairs. Lett. Math. Phys. 34 (1995), no. 1, 77–90.

[6] M. Doubek, M. Markl and P. Zima. Deformation theory (lecture notes). Arch. Math. (Brno) 43 (2007), no. 5, 333–371.

[7] V. Ginzburg and M. Kapranov. Koszul duality for operads. Duke Math. J. 76 (1994), no. 1, 203–272.

[8] E. Hoefel. On the Coalgebra Description of OCHA. arXiv:math/0607435v2.
[9] R. Ibanez, M. de Leon, J.C. Marrero and E. Padron. Leibniz algebroid associated with a Nambu-Poisson structure. J. Phys. A 32 (1999), no. 46, 8129–8144.

[10] N. Ikeda. Topological Field Theories and Geometry of Batalin-Vilkovisky Algebras. J. High Energy Phys. 0210 (2002) 076.

[11] N. Ikeda. On the construction of topological field theory and quantization. Seminar at Kagoshima University. (2007/11/22).

[12] N. Ikeda. Topological Field Theory, AKSZ-formalism and Higher Poisson Structure. Seminar at Akita University. (2008/11/12).

[13] H. Kajiura and J. Stasheff. Homotopy algebras inspired by classical open-closed string field theory. Comm. Math. Phys. 263 (2006), no. 3, 553–581.

[14] H. Kajiura and J. Stasheff. Homotopy algebras of open-closed strings. Groups, homotopy and configuration spaces, 229–259, Geom. Topol. Monogr., 13, Geom. Topol. Publ., Coventry, 2008.

[15] Y. Kosmann-Schwarzbach. From Poisson algebras to Gerstenhaber algebras. (English, French summary) Ann. Inst. Fourier (Grenoble) 46 (1996), no. 5, 1243–1274.

[16] Y. Kosmann-Schwarzbach. Derived brackets. Lett. Math. Phys. 69 (2004), 61–87.

[17] J-L. Loday. Dialgebras. Lecture Notes in Mathematics, 1763. Springer-Verlag, Berlin, (2001), 7–66.

[18] M. Markl. Homotopy algebras via resolutions of operads. Preprint Arxive, math/9808101.

[19] M. Markl, S. Shnider and J. Stasheff. Operads in algebra, topology and physics. Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, (2002). x+349 pp.

[20] D. Roytenberg. Quasi-Lie bialgebroids and twisted Poisson manifolds. Lett. Math. Phys. 61 (2002), no. 2, 123–137.

[21] D. Roytenberg. AKSZ-BV formalism and Courant algebroid-induced topological field theories. Lett. Math. Phys. 79 (2007), no. 2, 143–159.

[22] K. Uchino. Derived brackets and sh Leibniz algebras. (submitted). arXiv:0904.1961.

[23] K. Uchino. Derived bracket construction and Manin products. (submitted). arXiv:0902.0044.
[24] J-A. Vallejo. Nambu-Poisson manifolds and associated $n$-ary Lie algebroids. J. Phys. A. 34 (2001), no. 13, 2867–2881.

[25] T. Voronov. Higher derived brackets and homotopy algebras. J. Pure Appl. Algebra 202 (2005), no. 1-3, 133–153.

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