Chameleon Field in a Spherical Shell System

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We investigate the validity of screening mechanism of the fifth force for chameleon field in highly inhomogeneous density profile. For simplicity, we consider a static and spherically symmetric system which is composed of concentric infinitely thin shells. We calculate the fifth force profile by using a numerical method for a relatively large Compton wavelength of the chameleon field. An approximate solution is also derived for the small Compton wavelength limit. Our results show that, if the thin-shell condition for the corresponding smoothed density profile is satisfied, the fifth force is safely screened outside the system irrespective of the configuration of the shells inside the system. In contrast to the screening outside the system, we find that the fifth force can be comparable to the Newtonian gravitational force inside the system. If the system is highly inhomogeneous, the chameleon field cannot trace the potential minimum varying with the density and repeats being kicked, climbing up and rolling down the potential even when the effective mass of the chameleon field is sufficiently large in the system on average. One should not feel complacent about the wellbehavedness of the fifth force field with an averaged density distribution when we consider inhomogeneous objects.

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I. INTRODUCTION

General relativity is the most successful theory of gravity, which can explain various gravitational phenomena including gravitational waves recently observed for the first time [1, 2]. There are no experimental results which clearly contradict the prediction from general relativity so far. Nevertheless, many people are fascinated by the fundamental question of how accurately general relativity describes our universe, and attracted by seeking for alternative gravitational theory as a clue of quantum gravity or to elucidate the dark side of our universe.

Scalar-tensor theories of gravity are simple examples of the modified gravity, which were originally proposed by Brans and Dicke in 1961 [3]. It contains an additional scalar field other than the Einstein-Hilbert term and the standard matter term. One of the motivations to consider scalar-tensor theories is to explain the accelerated expansion of the universe by adding the scalar degree of freedom. However, the scalar field usually couples to the standard model particles and causes the difference in motion of them, which is seen as the so-called fifth force. Experimental tests of gravity in the solar system can give strong constraints on the fifth force and parameters of scalar-tensor theories [4].

In order to accord with the fifth force constraints, scalar-tensor theories must have a mechanism that screens the fifth force mediated by the scalar field on small scales. There are several types of scalar fields with such a mechanism [5]. The chameleon field introduced by Khoury and Weltman is an example of these fields [6, 7]. The chameleon field has a large value of the effective mass in a sufficiently high density region such as on the Earth or in the solar system, so that the fifth force mediated by the chameleon field becomes an unobservable short-range force. In contrast, the chameleon field may have a smaller mass or long Compton wavelength in cosmological low density regions so that it could accelerate the expansion of our universe. This chameleon mechanism can be also applied to other types of modified gravity theories such as $f(R)$ gravity (see, e.g. Ref. [8]). A lot of experimental tests have been proposed and performed in order to seek such a field, e.g. astrophysical tests such as using distance indicators [9], Galaxy rotation curve [10] and laboratory tests such as torsion pendulum [11], atom interferometer [12], and so on.

The calculations of the fifth force have been mainly done with a spherically symmetric smooth density profile surrounded by a cosmological low density region as the environment. For instance, for a compact object, we can estimate the scalar charge and show the fifth force can be much weaker than the Newtonian gravitational force (see, e.g. Ref. [7] and Refs. [13–17] for relativistic stars). Recently, the chameleon mechanism in more general situations has been started to be investigated in numerical ways. The screening effect on the structure formation is investigated by generalized N-body simulations [18–22] and strong constraints on $f(R)$ parameters are obtained from the modified gravity effects on galaxy clusters [23, 24]. Also, the screening for non-spherical sources is investigated in Ref. [25]. In this paper, we focus on a fact that has been overlooked in the above analyses. It has been usually assumed that it is adequate to use the smoothed density. However, actual objects in the universe are not smooth but inhomogeneous. For example, a galaxy is composed of many stars
as well as gas, dust, and dark matter. If the Compton wavelength of the field is shorter than scales of the inhomogeneities, the smoothing may not be justified and effects of the inhomogeneities should be taken into account. For example, in our galaxy, the upper bound on the Compton wavelength of the chameleon field can be obtained as $\lambda_\phi < 10^{7-12}\text{m} \ [27]$ by rescaling the terrestrial experimental upper bound, which is smaller than the average interstellar distance. Also in globular clusters, the Compton wavelength $\sim 10^4\text{m}$, is much less than distances between the stars in the cluster. This indicates that the chameleon field varies rapidly and be kicked by the inhomogeneity. Then, a significant fifth force can be mediated inside an inhomogeneous object like a galaxy.

In order to extract the essence of effects of inhomogeneities, as a first step, in this paper, we keep the system as simple as possible with extremely large density contrasts. Concretely, we assume a static spherically symmetric system composed of a set of infinitely thin shells at regular intervals of radius, where the inhomogeneities are controlled by the number of the shells. The shell intervals corresponds to the scales of the inhomogeneities in this system. Thus, if we choose the parameters such that Compton wavelength shorter than the shell intervals, the scalar field is perceptible to the inhomogeneities and a significantly large fifth force may appear inside the system. Moreover, the fluctuations of the field inside the shells may also affect the scalar charge of the system. Therefore, we calculate the field profile and the fifth force strength, and investigate those dependence on the parameters of the system.

This paper is organized as follows. In the section II, we introduce the chameleon field and the fifth force. A brief review of the constant density case is given in the section III for comparison with our case. Then, we introduce our model, the spherical shell system in the section IV. The results of the fifth force is shown in the section V. Section VI is devoted to a summary and conclusion. In this paper, we use natural units in which both the speed of light $c$ and the reduced Planck constant $\hbar$ are one.

II. CHAMELEON FIELD

A prototype of the chameleon field is given by a scalar field with a conformal coupling and a runway-type potential [7],

$$\nabla_\mu \nabla^\mu \phi - \frac{\beta}{M_{\text{pl}}} \rho - V'(\phi) = 0 ,$$

(1)

where $\beta$ represents a dimensionless conformal coupling and the potential $V(\phi)$ is typically assumed to be the inverse power-law potential: $V(\phi) = M^{4+n}/\phi^n$. The prime means the derivative with respect to $\phi$. Here, $M_{\text{pl}}$ is the Planck mass and $M$ is bounded above as $M \lesssim 10^{-3}\text{eV}$ to evade laboratory constraints on the fifth force [7], where $\beta$ is assumed as $O(1)$. The second and the third terms can be combined into derivative of the following effective potential:

$$V_{\text{eff}}(\phi) \equiv \frac{\beta}{M_{\text{pl}}} \rho \phi + \frac{M^{4+n}}{\phi^n} .$$

(2)
This effective potential has the minimum at

\[ \phi_{\text{min}}(\rho) \equiv M \left( \frac{n M^3 M_{\text{pl}}}{\beta \rho} \right)^{\frac{1}{n+1}}, \tag{3} \]

and the mass around this minimum is

\[ m^2_{\text{eff}}(\rho) \equiv V''_{\text{eff}}(\phi_{\text{min}}) = \frac{(n+1)\beta \rho}{M M_{\text{pl}}} \left( \frac{\beta \rho}{n M^3 M_{\text{pl}}} \right)^{\frac{1}{n+1}}. \tag{4} \]

The effective mass increases with \( \rho \) and the chameleon field becomes massive in the dense region.

In the static and spherically symmetric case, the equation of motion (EoM) becomes

\[ \frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d \phi}{dr} - \rho \frac{\beta}{M_{\text{pl}}} + n \frac{M^{4+n}}{\phi^{n+1}} = 0. \tag{5} \]

For later convenience, we rewrite the above equation by the following dimensionless variable:

\[ \hat{\phi} \equiv \frac{\phi}{\phi_c}, \tag{6} \]

where \( \phi_c \) is the field value at the potential minimum (3) for the central density \( \rho_c \), that is, \( \phi_c \equiv \phi_{\text{min}}(\rho_c) \). In addition, we introduce a length scale \( L \) and use the normalized radius \( x \) defined by \( x \equiv r/L \). Then, we obtain

\[ \frac{d^2 \hat{\phi}}{dx^2} + \frac{2}{x} \frac{d \hat{\phi}}{dx} - \hat{\rho} \hat{m}^2_c L^2 + \frac{\hat{m}^2_c L^2}{\hat{\phi}^{n+1}} = 0, \tag{7} \]

where \( \hat{m}^2_c \equiv \beta \rho_c/M_{\text{pl}} \phi_c \) and \( \hat{\rho} \equiv \rho/\rho_c \). Note that \( m^2_{\text{eff}}(\rho_c) = (n+1)\hat{m}^2_c \).

As reviewed in the next section, the large effective mass (4) can enforce the chameleon field to be approximately fixed at the minimum \( \phi_c \) in the interior of the star and the matter does not contribute to the scalar charge. Therefore, the fifth force

\[ F_\phi = \frac{\beta}{M_{\text{pl}}} \frac{d \phi}{dr} = \frac{\beta \phi_c}{M_{\text{pl}}} \frac{d \hat{\phi}}{dr}, \]

is screened by the \( \rho \)-dependent mass. However, this argument is based on the smoothed density. When density contrasts are high, the potential minimum (3) and the effective mass (4) will vary rapidly. For such a system, it will not be appropriate to solve the EoM with the smoothed density and the inhomogeneities should be taken into account. Unlike the smooth density case, the field value may vary also in the interior of the system, which may cause the appearance of a significant fifth force.
III. FIELD PROFILE OF CONSTANT DENSITY STARS

We review how the chameleon field is sourced by a star with the constant density $\rho_c$ surrounded by the cosmological density $\rho_\infty$. We take the radius of the star as the unit of the length scale $L$. If the effective mass (4) for the constant density $\rho_c$ is sufficiently large, the field value stays near the potential minimum $\phi_c$ around the center of the star. Then, we assume that there is a radius from which the field value starts to change and denote this radius as $x_{\text{roll}}$. We can divide the whole region into the following three pieces.

1. $x < x_{\text{roll}}$
   In this region, the chameleon field does not change much, and its value $\hat{\phi}$ and the first derivative $d\hat{\phi}/dx$ can be approximated by one and zero, respectively.

2. $x_{\text{roll}} < x < 1$ ($r_{\text{roll}} < r < L$)
   The chameleon field rolls down the effective potential toward the outside. Then, the first term in the effective potential is dominant, so that the EoM becomes
   \[
   \frac{d^2\hat{\phi}}{dx^2} + \frac{2d\hat{\phi}}{x} = \tilde{m}_c^2 L^2.
   \]  
   The solution of the equation (8) with the boundary conditions $\hat{\phi} = 1$ and $d\hat{\phi}/dx = 0$ at $x = x_{\text{roll}}$ is given by
   \[
   \hat{\phi} = 1 + \frac{\tilde{m}_c^2 L^2}{6} \left( \frac{2x_{\text{roll}}^3}{x} + x^2 - 3x_{\text{roll}}^2 \right).
   \]

3. $x > 1$ ($r > L$)
   In the outside of the star, the chameleon field quickly falls into the value sufficiently close to the minimum for the cosmological background $\hat{\phi}_\infty \equiv (\rho_c/\rho_\infty)^{\frac{1}{n+1}}$. Then, the approximate solution is obtained by linearizing the EoM (7) and we obtain
   \[
   \hat{\phi} = \hat{\phi}_\infty + A \frac{e^{-m_\infty Lx}}{x},
   \]
   where $m_\infty^2 \equiv m_{\text{eff}}^2(\rho_\infty)$.

Matching $\hat{\phi}$ and $d\hat{\phi}/dx$ at $x = 1$ by using the equations (9) and (10), we obtain
   \[
   A = -\frac{\tilde{m}_c^2 L^2}{3} \frac{1 - x_{\text{roll}}^3}{m_\infty L + 1} e^{m_\infty L},
   \]
   and
   \[
   \hat{\phi}_\infty - 1 = \frac{\tilde{m}_c^2 L^2}{6} \left( \frac{2}{m_\infty L + 1} + 1 + 2x_{\text{roll}}^3 - 3x_{\text{roll}}^2 \right).
   \]

If the density of the object is sufficiently large, it is expected that the chameleon field stays near the minimum $\hat{\phi} \approx 1$ up to the surface of the star and then $x_{\text{roll}} \approx 1$. This limit is so-called the thin shell regime because the only thin shell part of the star ($x_{\text{roll}} <
\( x < 1 \) contributes to the exterior field profile. Then, the equations (11) and (12) can be approximated as follows:

\[
A \simeq -m_c^2 L^2 \frac{1 - x_{\text{roll}}}{m_\infty L + 1} e^{m_\infty L},
\]

\[
1 - x_{\text{roll}} \simeq \frac{1}{m_c^2 L^2} (m_\infty L + 1) (\hat{\phi}_\infty - 1).
\]

We obtain the approximate form of \( \hat{\phi} \) by substituting equations (13) and (14) into the equation (10) as follows:

\[
\hat{\phi} = \hat{\phi}_\infty - (\hat{\phi}_\infty - 1) \frac{e^{-m_\infty L(x-1)}}{x}.
\]

We can check that the equation (14) is consistent with the assumption \( x_{\text{roll}} \simeq 1 \) if the following condition is satisfied:

\[
\frac{\hat{\phi}_\infty - 1}{m_c^2 L^2} (m_\infty L + 1) \ll 1,
\]

which is satisfied when the Compton wavelength \( \lambda_\phi \equiv 1/m_c (m_c \equiv \sqrt{n + 1} m_c) \) is much shorter than the radius of the star \( L \).

From the equation (8), the fifth force for the constant density star is calculated as,

\[
F_\phi = F_{\phi \text{con}} := \frac{\beta \phi_c}{M_{\text{pl}} L} (\hat{\phi}_\infty - 1) (m_\infty L x + 1) \frac{e^{-m_\infty L(x-1)}}{x^2}.
\]

The chameleon field has a sufficiently small effective mass (4) in the cosmological background unless \( M \) is too small. Then, the Compton wavelength of the chameleon field in the cosmological background is much longer than the radius of the star \( L \), e.g. \( \lambda_\phi \sim 1 \) Mpc for \( M \sim 10^{-3} \) eV. Taking the limit \( m_\infty L \to 0 \), we obtain the following expression:

\[
\lim_{m_\infty L \to 0} F_{\phi \text{con}} = \frac{\beta \phi_c}{M_{\text{pl}} L} (\hat{\phi}_\infty - 1) \frac{1}{x^2}.
\]

Since the Newtonian gravitational force made by the constant density star is given by

\[
F_{\text{Newton}} = \frac{1}{8\pi M_{\text{pl}}^2} \frac{4\pi L^3 \rho_c}{3 L^2 x^2} = \frac{\rho_c L}{6 M_{\text{pl}}^2} \frac{1}{x^2},
\]

we can evaluate the ratio \( \mathcal{R} \) between the fifth force and the Newtonian gravitational force, which corresponds to the scalar charge in units of the mass, as

\[
\mathcal{R} := \left| \frac{F_\phi}{F_{\text{Newton}}} \right| = 6 \frac{\beta M_{\text{pl}} \phi_c}{\rho_c L^2} (\hat{\phi}_\infty - 1) (m_\infty L x + 1) e^{-m_\infty L(x-1)}
\]

\[
\simeq 6 \beta^2 \frac{\phi_c}{m_c^2 L^2} (\hat{\phi}_\infty - 1),
\]

where we have taken the limit \( m_\infty L \to 0 \) in the second line. We can find that, if the thin-shell assumption is valid, that is, the equation (16) is satisfied, the value of \( |F_\phi/F_{\text{Newton}}| \) is suppressed.
On the other hand, in the case $x_{\text{roll}} \simeq 0$, which is called the thick-shell limit, the field value and the fifth force are given by, respectively,

\[ \hat{\phi} = \hat{\phi}_\infty - \frac{\tilde{m}_\infty L^2}{3(m_\infty L + 1)} e^{-m_\infty L(x-1)} x, \]

\[ R = 2\beta^2 \frac{m_\infty L x + 1}{m_\infty L + 1} e^{-m_\infty L(x-1)} = O(\beta^2), \]

where we have assumed $m_\infty L \simeq 0$.

**IV. SPHERICAL SHELL SYSTEM**

In the previous section, we have considered the case in which the field value almost stays in the minimum of the effective potential $V_{\text{eff}}$ inside the object. In contrast, as is mentioned in the section I, we consider a system with $N$ pieces of concentric spherical shells separated by vacuum regions with regular intervals (see Fig. 1). The shells are assumed to be infinitely thin, that is, they can be treated as delta-function sources along the radial direction. We named an inner region of the innermost shell as region 0, a region between the innermost shell and second innermost shell as region 1, and similarly until region N as shown in Fig 1. In these vacuum regions, there is no minimum of the effective potential. Therefore, in this system, neither the potential minimum nor the effective mass is defined at any radius and the previous intuitive argument cannot be applied. In reality, we would need to introduce a small density between the shells. Nevertheless, if the field value does not reach the minimum of the effective potential in the interval, the situation is similar to our
setting, and our results are applicable to such cases. In this section, we investigate how the field profile in this system can differ from that obtained for the smoothed density. In order to avoid running away of the chameleon field to infinity, we assume that the shell system is surrounded by the cosmological density $\rho = \rho_\infty$ as usual.

For simplicity, we assume that the surface densities of the shells and intervals between shells are identical. Then, denoting the total mass of the system by $M_{\text{tot}}$, the surface density $\sigma$ is given by,

$$\sigma = \frac{M_{\text{tot}}}{4\pi (L/N)^2(1^2 + 2^2 + \cdots + N^2)} = \frac{3M_{\text{tot}}N}{2\pi L^2(N + 1)(2N + 1)},$$

where we have used the parameter $L$ in the section II as the radius of the outermost shell.

The smoothed density $\rho_c$ can be written as $\rho_c = 3M_{\text{tot}}/4\pi L^3$ and is related to the surface density as,

$$\sigma = \rho_c L \frac{N}{(N + 1)(N + 1/2)}.$$

With these setup, the field equation (7) becomes

$$\frac{d^2 \hat{\phi}}{dx^2} + \frac{2}{x} \frac{d \hat{\phi}}{dx} + \frac{\tilde{m}_c^2 L^2}{\hat{\phi}^n+1} = 0,$$

in the vacuum regions, and

$$\frac{d^2 \hat{\phi}}{dx^2} + \frac{2}{x} \frac{d \hat{\phi}}{dx} + \frac{\tilde{m}_c^2 L^2}{\hat{\phi}^n+1} - \hat{\rho}_\infty \tilde{m}_c^2 L^2 = 0,$$

in the outer cosmological region $r > L$. The junction condition at each shell is given by [26]

$$[\phi]^- = 0,$$

$$\left[ \frac{d \phi}{d r} \right]^- = \beta \frac{\sigma}{M_{\text{pl}}}.$$

The symbol $[\cdot]^\pm$ on the left hand side of the equations is defined by

$$[f(x)]^\pm \equiv \lim_{x \to x_{\text{shell}}+0} f(x) - \lim_{x \to x_{\text{shell}}-0} f(x).$$

It will be more suggestive to rewrite the surface density in the second junction condition in terms of the smoothed density $\rho_c$ or the effective mass $\tilde{m}_c (\equiv m_c/\sqrt{n + 1})$:

$$\left[ \frac{d \phi}{d x} \right]^- = \beta \frac{\rho_c L^2}{M_{\text{pl}} \phi_c} \frac{N}{(N + 1)(N + 1/2)}$$

$$= \tilde{m}_c^2 L^2 \frac{N}{(N + 1)(N + 1/2)}.$$
The Newtonian gravitational force generated by the shells is given by

\[ F_{\text{Newton}} = \frac{1}{8\pi M^2_{\text{pl}}} M_{\text{tot}} \frac{i(i+1)(2i+1)}{N(N+1)(2N+1) r^2} \]

\[ = \frac{\rho_c L}{6M^2_{\text{pl}}} \frac{i(i+1)(2i+1)}{N(N+1)(2N+1) x^2} \]

\[ \frac{1}{\rho_c L} \quad (i/N < x < (i+1)/N), \quad (30) \]

where \( i \) runs over 0 to \( N \), which represents the force in the \( i \)-th region shown in Fig 1. As is well known, the Newtonian gravitational force \( F_{\text{Newton}} \) depends only on the enclosed mass at a given radius irrespective of its internal structures. Then, from the equation (8), we obtain

\[ R = \frac{6\beta M_{\text{pl}} N(N+1)(2N+1) d\phi}{\rho_c L^2} \]

\[ \frac{i(i+1)(2i+1)}{x^2} \]

\[ \frac{1}{\rho_c L} \quad (i/N < x < (i+1)/N). \quad (31) \]

Here, it is worthy to be noted that there are three length scales, the size of the system \( L \), the shell interval \( \Delta x = L/N \) and the Compton wavelength of the field \( \lambda_\phi = 1/m_c \). For the smoothed density, the chameleon field does not vary rapidly, \( x^2 d\phi/dx \sim \phi_\infty - 1 \), and the small factor \( 1/\tilde{m}_c^2 L^2 \simeq (\lambda_\phi/L)^2 \) ensures the screening. On the other hand, when the density contrasts are high, which is controlled by \( \Delta x \), the chameleon field is expected to vary rapidly and the ratio \( R \) might become large. \(^1\) To see the impact of the inhomogeneities, in the next section, we will evaluate the value of \( d\phi/dx \) for various values of the parameters \( \tilde{m}_c L \) and \( N = L/\Delta x \).

V. SCREENING IN SPHERICAL SHELL SYSTEM

Here, we solve the field equations (24) and (25) with the junction conditions (26) and (29) taking into account the thin-shell condition (16) for the smoothed density. For concreteness, we use the averaged density of our galaxy for \( \rho_c \), \( \rho_c \simeq 10^7 \rho_\infty \), which corresponds to

\[ \hat{\phi}_\infty = 10^{7}\pi. \]

\[ (32) \]

Then, in the case of smoothed density, the thin-shell condition is given by \( \tilde{m}_c^2 L^2 > 10^{7/(n+1)} \).

A. Numerical analysis

First, we consider marginal cases \( \tilde{m}_c^2 L^2 \gtrsim 10^{7/(n+1)} \) with a numerical method, while the thin-shell condition is well satisfied in the actual our galaxy \( \tilde{m}_c^2 L^2 \gtrsim 10^{28-22/(n+1)} \) with \( L \sim 10 \text{kpc} \) [27]. For simplicity, we choose the power \( n \) of the potential as \( n = 2 \) in the analysis here. Then, the thin-shell condition is given by \( \tilde{m}_c^2 L^2 \gtrsim 10^{7/3} \sim 200 \). We calculate

\[^1\] Note that, at this point, the small factor \( 1/\tilde{m}_c^2 L^2 \) appears just as a result of the normalization for the variables.
the field profile by numerically solving the field equations using the shooting method with the junction conditions at each shell as well as the boundary conditions \( \frac{d\hat{\phi}}{dx}|_{x=0} = 0 \) and \( \lim_{x\to\infty} \hat{\phi}(x) = \hat{\phi}_\infty \). We show the field profile and the value of \( R \) as functions of \( x \) in Fig. 2 for \( \tilde{m}_c^2 L^2 = 10^2, 10^3, \) and \( 10^4 \).

In Fig. 2, it is clearly shown that, in the \( \tilde{m}_c^2 L^2 = 10^2 \) case, the fifth force is comparable to the Newtonian gravitational force everywhere. In contrast, in the \( \tilde{m}_c^2 L^2 = 10^4 \) case, the fifth force is suppressed compared with the Newtonian gravitational force outside the shell system. Therefore, the criterion of the thin-shell condition is applicable to the spherical shell system. We also check the dependence on the number of shells \( N \). In Fig. 3, \( R \) is depicted as a function of \( x \) outside the system for \( N = 1, 5, \) and \( 10 \) with \( \tilde{m}_c^2 L^2 = 10^4 \). The behavior of \( R \) is similar to the smoothed-density case \( \rho = \rho_c \) irrespective of the number of shells as is shown in Fig. 3. Therefore, our results suggest that, even when a gravitating system is highly inhomogeneous inside it, the fifth force is screened outside it when the thin-shell condition is satisfied for the smoothed density.

### B. Analytic approximate solution

As is mentioned in the previous subsection, the value of \( \tilde{m}_c^2 L^2 \) is very large in the actual our galaxy. Numerical analyses for such a huge value of \( \tilde{m}_c^2 L^2 \) are very difficult \([13, 14]\). Instead of solving the EoM numerically, here, following Ref. \([27]\), we use an approximation which is valid for a sufficiently large value of \( \tilde{m}_c^2 L^2 \). We suppose that, for a large value of \( \tilde{m}_c^2 L^2 \), the potential term is much larger than the friction term \( (2/x)\frac{d\hat{\phi}}{dx} \) between the shells. Then, the EoM can be approximated as follows:

\[
\frac{d^2 \hat{\phi}}{dx^2} + \frac{\tilde{m}_c^2 L^2}{\hat{\phi}^{n+1}} \simeq 0.
\] (33)
FIG. 3. This figure shows the variation of the fifth force for different numbers of shells $N = 1, 5,$ and 10 with $\tilde{m}_c^2 L^2 = 10^4$. We show an enlarged figure of the outside part in the right panel to show the dependence on the number of shells. Each line almost coincides with the one for the smoothed density case in the outside of the outermost shell.

The solution for the above equation is given by

$$\frac{d\phi}{dx} \simeq \pm \sqrt{C + \frac{2\tilde{m}_c^2 L^2}{n\hat{\phi}^n}},$$

(34)

where $C$ is an integration constant. As is shown in Fig. 4, the same shape is repeated between the shells. The first derivative $d\phi/dx$ vanishes at the middle point and the profile of $\hat{\phi}$ has a symmetric shape with respect to this middle point. Assuming a similar repeating structure in the solution for a large value of $\tilde{m}_c^2 L^2$, the first derivative at the shell positions can be estimated as

$$\left.\frac{d\phi}{dx}\right|_{x=\pm\Delta x+0} \simeq \frac{1}{2} \left[\left.\frac{d\phi}{dx}\right|_{x=\Delta x+0}\right]_0^+.$$  

(35)

From the junction condition at a shell (29), we can determine the constant $C$ in terms of the field value at the shell position $\phi_s$ as follows:

$$\tilde{m}_c^2 L^2 \frac{N}{(N+1)(2N+1)} \simeq \sqrt{C + \frac{2\tilde{m}_c^2 L^2}{n\hat{\phi}_n^s}}.$$  

(36)

The constant $C$ is written in simpler form by using the field value at the middle point, $\phi_0$, as $C = -2\tilde{m}_c^2 L^2/(n\hat{\phi}_0^n)$. Then, in order for the above approximation to be valid, we need to impose the following condition:

$$\frac{1}{x} \frac{d\phi}{dx} / \left(\frac{\tilde{m}_c^2 L^2}{\hat{\phi}^{n+1}}\right) \simeq \sqrt{\frac{2}{n} \hat{\phi}^{n/2+1} \tilde{m}_c L x} \sqrt{1 - \left(\frac{\hat{\phi}}{\hat{\phi}_0}\right)^n} \ll 1$$  

(37)

Our numerical results in Fig. 2 show that the field value $\hat{\phi}$ varies at most by $\Delta\hat{\phi}/\hat{\phi} = \mathcal{O}(1)$ in the inner regions. Therefore, our approximation is valid for a sufficiently large value of $\tilde{m}_c L$. 
FIG. 4. We plot the numerical result and analytical approximate one for $\tilde{m}_c^2 L^2 = 10^4$, $N = 10$ on the same figure. The blue line corresponds to the numerical result and the green one to the analytical approximate one.

The approximation (37) cannot be applied to the region near the center $\tilde{m}_c L x \ll 1$ and then the solution (34) too. In this region, we use the following asymptotic expansion of $\hat{\phi}$ inside the innermost shell:

\[
\hat{\phi} = c_0 - \frac{1}{6} c_0^{-n-1} \tilde{m}_c^2 L^2 x^2 - \frac{1}{120} (n + 1) c_0^{-2n-3} \tilde{m}_c^4 L^4 x^4 + \cdots,
\]

where $c_0$ is the field value at the origin. This expansion is valid for sufficiently small $\tilde{m}_c L x$. We can construct the field profile by jointing the approximate solutions (38), (34) and (10) at each shell with the junction condition (26). In Fig. 4, we show that the analytic approximate solution agrees well with the numerical result for $\tilde{m}_c^2 L^2 = 10^4$ and $N = 10$. The deviation between the analytic approximation and the numerical result is less than several percents.

C. Fifth Force outside the System

Let us evaluate the value of $R$ in the outside of the shell system by using the analytic approximate solution constructed in the previous subsection. For simplicity, we concentrate on a specific form of the potential with $n = 2$, then Eq. (34) can be easily solved as

\[
\hat{\phi} = \sqrt{C_1 - \frac{\tilde{m}_c^2 L^2}{C_1} (x + C_2)^2},
\]

where $C_1$ and $C_2$ are integration constants. The integration constants can be determined from $d\hat{\phi}/dx = 0$ at the middle point $x = x_0$ and $\hat{\phi} = \hat{\phi}_s$ at the shell positions $x = x_0 \pm 1/(2N)$. As a result, we obtain

\[
\hat{\phi}(x) = \sqrt{\frac{\hat{\phi}_s^2 + \sqrt{\hat{\phi}_s^4 + \tilde{m}_c^2 L^2/N^2}}{2}} - \frac{2\tilde{m}_c^2 L^2 (x - x_0)^2}{\hat{\phi}_s^2 + \sqrt{\hat{\phi}_s^4 + \tilde{m}_c^2 L^2/N^2}}.
\]
According to the junction condition (36), we can determine the field value at the shell position \( \hat{\phi}_s \) by the following equation:

\[
\hat{\phi}_s \left( \hat{\phi}_s^2 + \sqrt{\hat{\phi}_s^4 + \bar{m}_c^2 L^2/N^2} \right) = \frac{(N + 1)(2N + 1)}{N^2}.
\]  

(41)

From the above equation, we obtain the following behavior depending on the value of the parameter \( m_c L/N = \Delta x/\lambda_\phi \):

\[
\hat{\phi}_s \sim \begin{cases} 
1 & (\bar{m}_c^2 L^2/N^2 \ll 1) \\
2N/(\bar{m}_c L) & (\bar{m}_c^2 L^2/N^2 \gg 1)
\end{cases}
\]

(42)

with estimating the right-hand side of Eq. (41) to be \( \mathcal{O}(1) \). Therefore, it is assured that \( \hat{\phi}_s \) is less than \( \mathcal{O}(1) \). If we assume the approximation is kept to be valid, the field value at the outermost shell is also given by \( \hat{\phi}_s \). Then, we can estimate the fifth force outside the object from the equation (10) as

\[
\lim_{m_\infty L \to 0} F_\phi = \frac{\beta}{M_{\text{pl}} L} (\hat{\phi}_\infty - \hat{\phi}_s) \frac{1}{x^2} \phi_c,
\]

(43)

in the limit \( m_\infty L \to 0 \). The effect of the inhomogeneities on the fifth force outside the object can be calculated by taking the difference between Eqs. (43) and (18) as follows:

\[
\lim_{m_\infty L \to 0} \left( F_\phi - F_{\phi, \text{con}} \right) = \frac{\beta}{M_{\text{pl}} L} (1 - \hat{\phi}_s) \frac{1}{x^2} \phi_c.
\]

(44)

We see that, from Eq. (42), the value of \( \phi_s \) approaches to unity for \( \Delta x/\lambda_\phi \to 0 \). However, if the scale of the inhomogeneities \( \Delta x \) is comparable or larger than the Compton wavelength \( \lambda_\phi \), we may observe the deviation of the fifth force from the smoothed density case although it is suppressed by the factor \( 1/(\bar{m}_c^2 L^2) \) compared to the Newtonian gravitational force as follow:

\[
\lim_{m_\infty \to 0} \left| \frac{F_\phi - F_{\phi, \text{con}}}{F_{\text{Newton}}} \right| = \frac{6\beta^2}{\bar{m}_c^2 L^2} (1 - \hat{\phi}_s).
\]

(45)

Of course, the above approximation is violated for large \( \hat{\phi} \), which occurs for large Compton wavelength compared to \( \Delta x \). Then, we cannot use the approximation for \( \bar{m}_c^2 L^2 = 10^2 \) as shown in Fig. 2. However, the approximation works well in the parameter region of our interest, that is, where the inhomogeneity may have a significant effect.

D. Fifth Force inside the System

The value of \( \mathcal{R} \) at each shell can be easily evaluated as follows:

\[
\mathcal{R} = \frac{6\beta^2 i}{(i + 1)(2i + 1)}.
\]

(46)
FIG. 5. The value of $\mathcal{R}$ is depicted as a function of $x$. The spiky blue lines show the result of numerical integration for $N = 10$ and $\tilde{m}_c^2 L^2 = 10^4$. The red points shows the analytic approximation at the position of each shell given by Eq. (46) with substituting $i = x/N$.

The maximum value $\mathcal{R}_{\text{max}} = \beta^2$ is realized at the innermost shell for $i = 1$ irrespective of the parameters $\tilde{m}_c^2$ or $L$. We also note that this result is independent of the value of $n$, which specifies the potential form. Although we need explicit potential form to calculate the fifth force in the vacuum regions, it suffices to estimate the maximum values. We confirm the validity of the approximation (46) comparing it with the numerical result (see Fig. 5). This result is very suggestive in the following sense: even if the Compton wavelength is sufficiently smaller than the size of the object, so that the fifth force is screened outside the object, the value of the fifth force can be comparable to the Newtonian gravitational force near the center of the object.

VI. CONCLUSION

We investigated the chameleon screening mechanism for inhomogeneous density profiles. For a density profile with high density contrasts, it is expected that the chameleon field cannot trace the minimum of the varying potential and the smoothing of the density may not be justified. To verify it, we considered one of the simplest examples, the spherical shell system composed of a set of concentric shells, where there is no potential minimum at any radius and the chameleon field cannot be stable by a large mass as usually assumed for a successful screening of the fifth force. The results show that the fifth force can be screened outside the system if the so-called thin-shell condition is satisfied for the smoothed density as in the case of a constant density profile. However, the field profile inside the system can be significantly different from the smoothed density case. We derived an analytic approximate expression for the fifth force inside the system with the help of insights from the numerical results. In our simple toy model, irrespective of the other model parameters, the maximum value of the ratio between the fifth force and the Newtonian gravitational force is given by $\beta^2$ with $\beta$ being the dimensionless coupling constant for the conformal coupling between
the standard matter and the chameleon field. Since the value of $\beta$ is usually assumed to be in the order of 1, our result suggests the possibility that the fifth force can be significantly large inside an object with a highly inhomogeneous density profile even when the shin-shell condition is well satisfied for the smoothed density. Due to the fact that this result is irrelevant to the property of the effective Compton wavelength, the same concern may exist in other fifth force models which have a circumstance dependent screening mechanism, such as symmetron [28] and environmentally dependent dilaton [29]. One should not feel complacent about the wellbehavedness of the fifth-force field with an averaged density distribution. A significant fifth force strength can be induced inside an inhomogeneous object. Since it does not follow the inverse square law, unlike the case of Newtonian gravitational force, the configuration of outer shell affects the fifth force inside it. As is shown in Fig. 2, the fifth force works in the direction of collecting matters to each shell. Then, it may cause new instability other than the one caused by the usual gravitational attraction and should be investigated more carefully as a factor that may affect the structure formation.

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