Normalized ground states for the fractional nonlinear Schrödinger equations

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Abstract

In this paper, we study the existence and instability of standing waves with a prescribed $L^2$-norm for the fractional Schrödinger equation

$$i\partial_t \psi = (-\Delta)^s \psi - f(\psi),$$

(0.1)

where $0 < s < 1$, $f(\psi) = |\psi|^p \psi$ with $\frac{4}{N} < p < \frac{4s}{N-2s}$ or $f(\psi) = (|x|^{-\gamma} * |\psi|^2)\psi$ with $2s < \gamma < \min\{N, 4s\}$. To this end, we look for normalized solutions of the associated stationary equation

$$(-\Delta)^s u + \omega u - f(u) = 0.$$  

(0.2)

Firstly, by constructing a suitable submanifold of a $L^2$-sphere, we prove the existence of a normalized solution for (0.2) with least energy in the $L^2$-sphere, which corresponds to a normalized ground state standing wave of (0.1). Then, we show that each normalized ground state of (0.2) coincides a ground state of (0.2) in the usual sense. Finally, we obtain the sharp threshold of global existence and blow-up for (0.1). Moreover, we can use this sharp threshold to show that all normalized ground state standing waves are strongly unstable by blow-up.

Keywords: Fractional Schrödinger equation; Normalized ground states; Sharp threshold; Strong instability

1 Introduction

In recent years, there has been a great deal of interest in using fractional Laplacians to model the physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [36, 37] used the theory of functionals over functional measure generated by the Lévy stochastic process to deduce the following fractional nonlinear Schrödinger equation (NLS)

$$i\partial_t \psi = (-\Delta)^s \psi - f(\psi), \quad \psi(0, x) = \psi_0(x),$$

(1.1)

where $0 < s < 1$, $f(\psi) = |\psi|^p \psi$ or $f(\psi) = (|x|^{-\gamma} * |\psi|^2)\psi$. The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s \psi = \mathcal{F}^{-1}[|\xi|^{2s} \mathcal{F}(\psi)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. The fractional NLS also appears in the continuum limit

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of discrete models with long-range interactions (see e.g. [35]) and in the description of Boson stars as well as in water wave dynamics (see e.g. [27]).

The intention of this paper is to study (1.1) from a variational perspective. To this end, it is of great interest to consider standing waves to (1.1), which are solutions of the form $e^{i\omega t}u$, where $\omega \in \mathbb{R}$ is a frequency and $u$ is complex-valued. This ansatz yields

$$(-\Delta)s u + \omega u - f(u) = 0,$$

where $f(u) = |u|^p u$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$.

At this moment, our intention is reduced to explore (1.2). To do this, we would like to mention two substantially distinct options in terms of the frequency $\omega$. The first one is to fix the frequency $\omega \in \mathbb{R}$. In this situation, every solution to (1.2) corresponds to a critical point of the action functional $J(u)$ on $H^s$, where

$$J(u) := \frac{1}{2} \|u\|^2_{H^s} + \frac{\omega}{2} \|u\|^2_{L^2} - \frac{1}{p+2} \|u\|^{p+2}_{L^{p+2}}, \quad \text{if } f(u) = |u|^p u,$$

$$J(u) := \frac{1}{2} \|u\|^2_{H^s} - \frac{1}{4} \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2)(x)|u(x)|^2 dx, \quad \text{if } f(u) = (|x|^{-\gamma} * |u|^2)u. \quad (1.3)$$

In this case particular attention is devoted to least action solutions, namely solutions minimizing $J(u)$ among all non-trivial solutions.

Alternatively, it is interesting to study solution to (1.2) having prescribed $L^2$-norm, namely, for any given $c > 0$, to consider solution to (1.2) satisfying the $L^2$-norm constraint

$$S(c) = \{u \in H^s : \|u\|^2_{L^2} = c\}, \quad c > 0. \quad (1.5)$$

Physically, such a solution is so-called normalized solution to (1.2), which formally corresponds to a critical point of the energy functional $E(u)$ restricted on $S(c)$, where

$$E(u) := \frac{1}{2} \|u\|^2_{H^s} - \frac{1}{p+2} \|u\|^{p+2}_{L^{p+2}}, \quad \text{if } f(u) = |u|^p u,$$

$$E(u) := \frac{1}{2} \|u\|^2_{H^s} - \frac{1}{4} \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2)(x)|u(x)|^2 dx, \quad \text{if } f(u) = (|x|^{-\gamma} * |u|^2)u. \quad (1.6)$$

It is worth pointing out that, in this situation, the frequency $\omega \in \mathbb{R}$ is an unknown part, which is determined as the Lagrange multiplier associated to the constraint $S(c)$.

From a physical point of view, it is quite meaningful to consider normalized solution to (1.2). This is not only because the $L^2$-norm of solution to the Cauchy problem of (1.1) is conserved along time, that is, for any $t > 0$

$$\int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = \int_{\mathbb{R}^N} |\psi_0(x)|^2 dx,$$
see Proposition 2.1 but also because the mass has often a clear physical meaning; for instance, it represents the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation, two main fields of application of the NLS. Moreover, this approach turns out to be useful also from the purely mathematical perspective, since it gives a better insight of the properties of the stationary solutions for (1.1), such as stability or instability (this was already evident in the seminal contributions by H. Berestycki and T. Cazenave [7], and by T. Cazenave and P.-L. Lions [14]). For these reasons, here we focus on existence and properties of solutions to (1.2) with prescribed mass and the $L^2$-supercritical nonlinearity, a problem which was, up to now, essentially unexplored.

The existence of normalized stationary states can be formulated as the following problem: given $c > 0$, we aim to find $(u_c, \omega_c) \in H^s \times \mathbb{R}$ solving (1.2) together with the normalization condition (1.5). When $f(u) = |u|^p u$ with $0 < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $0 < \gamma < \min\{N, 4s\}$, it is standard that $E(u)$ is of class $C^1$ in $H^s$, and any critical point $u$ of $E|_{S(c)}$ corresponds to a solution to (1.2) satisfying (1.3), with the parameter $\omega \in \mathbb{R}$ appearing as Lagrange multiplier. We are particularly interested in ground state solutions, defined as follows:

**Definition 1.1.** (Ground states) We write that $u_c$ is a ground state of (1.2) on $S(c)$ if it is a solution to (1.2) having minimal energy among all the solutions which belongs to $S(c)$:

$$E|_{S(c)}(u_c) = 0 \quad \text{and} \quad E(u_c) = \inf \{ E(v_c) : v_c \in S(c), \ E'|_{S(c)}(v_c) = 0 \}.$$ 

Before stating our main results, let us recall known results related to the normalized solutions for some Schrödinger type equations and systems. It is well known that, when dealing with the Schrödinger equations, the $L^2$-critical exponent plays a special role. This is the threshold exponent for many dynamical properties such as global existence vs. blow-up, and the stability or instability of ground states. From the variational point of view, if the problem is purely $L^2$-subcritical, then $E(u)$ is bounded from below on $S(c)$. Thus, for every $c > 0$, ground states can be found as global minimizers of $E|_{S(c)}$, see [13, 14]. Moreover, the set of ground states is orbitally stable. In the $L^2$-supercritical case, on the contrary, $E|_{S(c)}$ is unbounded from below. By exploiting the mountain pass lemma and a smart compactness argument, L. Jeanjean [32] showed that a normalized ground state does exist for every $c > 0$ also in this case. For quite a long time the paper [32] was the only one dealing with existence of normalized solutions in cases when the energy is unbounded from below on the $L^2$-constraint. More recently, however, problems of this type received much attention, see [5, 0, 10, 33, 39, 43, 44] for normalized solutions to scalar equations in the whole space $\mathbb{R}^N$, see [1, 2, 3, 128, 29, 38] for normalized solutions to systems in $\mathbb{R}^N$.

For the fractional Schrödinger equation (1.2), in the $L^2$-subcritical case, i.e. $0 < p < \frac{4s}{N} \text{ or } 0 < \gamma < 2s$, $E(u)$ is bounded from below on $S(c)$. Thus, for every $c > 0$, ground states can be found as global minimizers of $E|_{S(c)}$. Moreover, the set of ground states is orbitally stable.
Recently, these problems have been studied by using the concentration compactness principle in [8, 15, 20, 22, 30, 48], using the profile decomposition theory in [23, 24, 41, 50, 52]. In the $L^2$-supercritical case, i.e. $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $2s < \gamma < 2s$, on the contrary, $E|_{S(c)}$ is unbounded from below. To the best of our knowledge, there are no any results in this case.

The aim of this paper is to consider the existence and properties of normalized ground states to (1.2), the sharp threshold of global existence and blow-up, and the strong instability of normalized ground state standing waves for (1.1) in the $L^2$-supercritical case. Our main results are as follows:

**Theorem 1.2.** Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then for any $c > 0$, there exists a couple of weak solution $(u_c, \omega_c) \in H^s \times \mathbb{R}^+$ to problems (1.2)-(1.5). Moreover, we have

$$
\begin{cases}
\|u_c\|_{H^s} \to +\infty, \\
\omega_c \to +\infty, \\
E(u_c) \to +\infty,
\end{cases}
$$

as $c \to 0^+$ and

$$
\begin{cases}
\|u_c\|_{H^s} \to 0, \\
\omega_c \to 0, \\
E(u_c) \to 0,
\end{cases}
$$

as $c \to +\infty$.

To the best of our knowledge, this seems to be the first contribution regarding existence of normalized ground states for the fractional NLS in the $L^2$-supercritical case. The proof of this theorem is based on a constrained minimization method. In the mass-supercritical case, i.e., $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $2s < \gamma < \min\{N, 4s\}$, the functional $E(u)$ is no longer bounded from below on $S(c)$, the minimization method on $S(c)$ used in [21, 41, 50] does not work. Motivated by minimization method on Pohozaev manifold, we try to construct a submanifold of $S(c)$, on which $E(u)$ is bounded from below and coercive, and then we look for minimizers of $E(u)$ on such a submanifold. Precisely, we introduce an auxiliary functional

$$Q(u) := s\|u\|_{H^s}^2 - \frac{Np}{2(p+2)}\|u\|_{L^{p+2}}^{p+2}, \text{ if } f(u) = |u|^p u, \quad (1.8)$$

$$Q(u) := s\|u\|_{H^s}^2 - \frac{\gamma}{4} \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2)(x)|u(x)|^2 dx, \text{ if } f(u) = (|x|^{-\gamma} * |u|^2)u. \quad (1.9)$$

and construct a submanifold $V(c)$ as follows

$$V(c) := \{ u \in S(c) : Q(u) = 0 \}. \quad (1.10)$$
By considering the minimization problem

$$m(c) := \inf_{u \in V(c)} E(u),$$

we find a critical point of $E$ restricted to $V(c)$ and prove that it is indeed a critical point of $E$ restricted to $S(c)$. Let us denote the set of minimizers of $E$ on $V(c)$ as

$$\mathcal{M}_c := \{ u \in V(c) : E(u) = \inf_{v \in V(c)} E(v) \}.$$  

Then we prove the existence part of Theorem 1.2 by showing a simple property of $\mathcal{M}_c$.

Compared with [32], we use a constrained minimization method instead of a mini-max procedure. Although these two methods both work on finding a normalized ground state of problem (1.2)-(1.5), we believe that the constrained minimization method is more convenient in getting the normalized ground state solution to problem (1.2)-(1.5). In particular, in order to solve the minimization problem (1.11), we consider an equivalent minimization problem (3.3), which can be easily solved by using Brezis-Lieb’s lemma. Moreover, it is easier to obtain the sharp threshold of global existence and blow-up for (1.1) by using the minimization problem (3.3).

For any $\omega > 0$, the existence of ground state solution $u_\omega$ to problem (1.2) has been studied in [11, 18, 31, 42, 50]. Next, we analyze the connection between the couple of weak solution $(u_c, \omega_c)$ to (1.2) obtained in Theorem 1.2 and $u_\omega$.

**Theorem 1.3.** Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then for any $u_c \in \mathcal{M}_c$, there exists $\omega_c > 0$ such that $(u_c, \omega_c) \in H^s \times \mathbb{R}$ is a couple of weak solution to problem (1.2). Furthermore, $u_c$ is a ground state solution to problem (1.2) with $\omega = \omega_c$.

**Remark.** $u_c$ is a ground state solution to problem (1.2) with $\omega = \omega_c$ means that

$$J(u_c) = \inf\{ J(u) : J'(u) = 0 \text{ and } u \neq 0 \},$$

where $J(u)$ is defined in (1.3) or (1.4). Theorem 1.3 indicates that every normalized ground state of problem (1.2) coincides a ground state of problem (1.2). This information is interesting itself. For example, it is well-known that the ground state solution $u_\omega$ of (1.2) with $f(u) = |u|^p u$ is unique up to translation, see [25, 26]. We consequently obtain that for every $c > 0$, the solution of minimizing problem (1.11) is unique up to translation. Moreover, based on the minimizing problems (1.11) and (3.3), To this end, we introduce the following invariant sets.

$$\mathcal{A}_c := \{ u \in S(c) : \, Q(u) > 0 \text{ and } E(u) < m(c) \},$$

$$\mathcal{B}_c := \{ u \in S(c) : \, Q(u) < 0 \text{ and } E(u) < m(c) \}. $$
Theorem 1.4. (Global versus blow-up dichotomy) Let $N \geq 2$, $\frac{N^2}{2N-1} \leq s < 1$, $\psi_0 \in H^s$, $f(\psi) = |\psi|^p \psi$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$, or $f(\psi) = (|x|^{-\gamma} * |\psi|^2) \psi$ with $2s < \gamma < \min\{N, 4s\}$. Then, $A_{\|\psi_0\|_{L^2}^2}$ and $B_{\|\psi_0\|_{L^2}^2}$ are two invariant manifolds of (1.1). More precisely, if $\psi_0 \in A_{\|\psi_0\|_{L^2}^2}$ or $\psi_0 \in B_{\|\psi_0\|_{L^2}^2}$, then the solution $\psi(t)$ satisfies $\psi(t) \in A_{\|\psi_0\|_{L^2}^2}$ or $\psi(t) \in B_{\|\psi_0\|_{L^2}^2}$ for any $t \in [0, T^*)$, respectively. Moreover, we can obtain the following sharp threshold of global existence and blow-up for (1.1).

1. If $\psi_0 \in A_{\|\psi_0\|_{L^2}^2}$, then the solution $\psi(t)$ of (1.1) with initial data $\psi_0$ exists globally in time.

2. When $f(\psi) = |\psi|^p \psi$, assume further that $p < 4s$, $\psi_0 \in B_{\|\psi_0\|_{L^2}^2}$ and $\psi_0$ is radial, then the solution $\psi(t)$ of (1.1) blows up in finite time.

3. When $f(\psi) = (|x|^{-\gamma} * |\psi|^2) \psi$, assume further that $x \psi_0 \in H^{s_0}$ with $s_0 = \max\{2s, \frac{2s+1}{1}\}$, $x \psi_0 \in L^2$, $x \cdot \nabla \psi_0 \in L^2$, $\psi_0 \in B_{\|\psi_0\|_{L^2}^2}$ and $\psi_0$ is radial, then the solution $\psi(t)$ of (1.1) blows up in finite time.

Remark 1. Note that the condition $p < 4s$ is technical due to the localized virial estimate, see Lemma 2.9. However, this only leads to a restriction in the two dimensional case. Indeed, for $N \geq 3$ and $2s < 2$, we have $p < \frac{4s}{N-2s} < 4s$.

Remark 2. For the classical NLS, i.e., $s = 1$ in (1.1), it follows from the virial identity and (5.3) with $s = 1$ that

$$\frac{d^2}{dt^2} \|x \psi(t)\|_{L^2}^2 = 4Q(\psi(t)) \leq 8(E(\psi_0) - m(\|\psi_0\|_{L^2}^2)) < 0,$$

where $Q(\psi(t))$ is defined by (1.8) or (1.9) with $s = 1$. This implies that the solution $\psi(t)$ of (1.1) with $s = 1$ blows up in finite time.

But for the fractional NLS (1.1) with $f(\psi) = |\psi|^p \psi$, it follows from Lemma 2.9 and (5.3) that

$$\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq 4Q(\psi(t)) + C\eta \|\psi(t)\|_{H^s}^2 + \circ(1)$$

$$\leq 8s(E(\psi_0) - m(\|\psi_0\|_{L^2}^2)) + C\eta \|\psi(t)\|_{H^s}^2 + \circ(1)$$

$$\leq C\eta \|\psi(t)\|_{H^s}^2 + \circ(1),$$

where $\eta > 0$, $\circ(1) \to 0$ as $R \to \infty$, $\|\psi(t)\|_{H^s}^2$ may be unbounded. Therefore, there exist some essential difficulties in proving Theorem 1.4 between the fractional NLS and the classical NLS. In this paper, we will develop some new ideas to solve these problems.

Notice that $B_c$ contains functions arbitrary close to $u_c$ in $H^s$. Indeed, letting $u^\lambda_c(x) = \lambda^{N/2} u_c(\lambda x)$ with $\lambda > 1$, it easily follows that $u^\lambda_c \in B_c$ and $u^\lambda_c \to u_c$ as $\lambda \to 1$. Therefore, as an immediate corollary of Theorem 1.4, we can derive the strong instability of normalized ground states to (1.1).
Corollary 1.5. Let $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$, $c > 0$, $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ and $p < 4s$. Assume that $u_\gamma \in \mathcal{M}_c$, the standing wave $\psi(t, x) = e^{i\omega t} u_\gamma(x)$ is strongly unstable in the following sense: there exists $\{\psi_{0,n}\} \subset H^s$ such that $\psi_{0,n} \to u$ in $H^s$ as $n \to \infty$ and the corresponding solution $\psi_n$ of (1.1) with initial data $\psi_{0,n}$ blows up in finite time for any $n \geq 1$.

Corollary 1.6. Let $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$, $c > 0$, $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then for any $u_\gamma \in \mathcal{M}_c$ radial, the standing wave $\psi(t, x) = e^{i\omega t} u_\gamma(x)$ is strongly unstable in the following sense: there exists $\{\psi_{0,n}\} \subset H^s$ such that $\psi_{0,n} \to u$ in $H^s$ as $n \to \infty$ and the corresponding solution $\psi_n$ of (1.1) with initial data $\psi_{0,n}$ blows up in finite time for any $n \geq 1$.

Remark. It is well-known that the ground state solution $u_\omega$ of (1.2) with $f(u) = |u|^p u$ is unique up to translation, see [25, 26]. Based on this fact and the translation invariance of (1.1), we can prove that for every $u_\gamma \in \mathcal{M}_c$, the ground state standing wave $\psi(t, x) = e^{i\omega t} u_\omega(x)$ is strongly unstable. But, to the best of our knowledge, the uniqueness of ground state solution $u_\omega$ of (1.2) with $f(u) = (|x|^{-\gamma} * |u|^2)u$ is unknown, so we only prove the instability of radial normalized ground states.

This paper is organized as follows: in Section 2, we firstly collect some lemmas such as the local well-posedness theory of (1.1), Brezis-Lieb’s lemma, a compactness lemma, a sharp Gagliardo-Nirenberg type inequality and the localized virial estimate related to (1.1). In Section 3, 4 and 5, we will prove Theorems 1.2, 1.3 and 1.4 respectively.

Notations. Throughout this paper, we use the following notations. $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion. For any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} (1 + |\xi|^{2s})|\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

dowered with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|\hat{u}\|_{H^s(\mathbb{R}^N)},$$

where up to a multiplicative constant

$$\|u\|_{\tilde{H}^s(\mathbb{R}^N)} = \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right\}^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of $u$. In this paper, we often use the abbreviations $L^r = L^r(\mathbb{R}^N)$, $H^s = H^s(\mathbb{R}^N)$.

2 Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem (1.1). The local well-posedness for (1.1) in the energy
space $H^s$ was first studied by Hong and Sire in [34]. The proof is based on Strichartz estimates and the contraction mapping argument. Note that for non-radial data, Strichartz estimates have a loss of derivatives. Fortunately, this loss of derivatives can be compensated by using Sobolev embedding. However, it leads to a weak local well-posedness in the energy space compared to the classical nonlinear Schrödinger equation. We refer the reader to [17, 34] for more details.

One can remove the loss of derivatives in Strichartz estimates by considering radially symmetric data. We refer the reader to [17, 34] for more details.

Proposition 2.1 (Radial $H^s$ LWP). Let $N \geq 2$, $\frac{N}{2N-1} \leq s < 1$, $f(\psi) = |\psi|^p \psi$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(\psi) = (|x|^{-\gamma} * |\psi|^2)\psi$ with $2s < \gamma < \min\{N, 4s\}$. Then for any $\psi_0 \in H^s$ radial, there exist $T^* \in (0, +\infty]$ and a unique solution to (1.1) satisfying $\psi \in C([0, T^*), H^s)$. Moreover, the following properties hold:

- $\psi \in L_{loc}^p([0, T^*), W^{s,b})$ for any fractional admissible pair $(a, b)$.
- If $T^* < +\infty$, then $\|\psi(t)\|_{H^s} \to \infty$ as $t \uparrow T^*$.
- The solution $\psi(t)$ enjoys the following conservations of mass and energy, i.e., for all $t \in [0, T^*)$

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

$$E(\psi(t)) = E(\psi_0),$$

where $E(\psi(t))$ is defined by (1.6) or (1.7).

In this paper, we also need the so-called Brezis-Lieb’s lemma, see [9, 40].

Lemma 2.2. Let $0 < p < \infty$. Suppose that $f_n \to f$ almost everywhere and $\{f_n\}$ is a bounded sequence in $L^p$, then

$$\lim_{n \to \infty} \left(\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p\right) = 0.$$

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}^N} (|x|^{-\gamma} * |u_n|^2)|u_n|^2 \, dx - \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u - u|^2)|u_n - u|^2 \, dx\right) = \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2)|u|^2 \, dx.$$

The following compactness lemma is vital in our discussion, see [19, 21].

Lemma 2.3. Let $N \geq 1$, $0 < s < 1$, $0 < p < \frac{4s}{N-2s}$. Let $\{u_n\}$ be a bounded sequence in $H^s$ and satisfy that

$$\liminf_{n \to \infty} \|u_n\|_{L^{p+2}} \geq m,$$

for some $m > 0$. Then there exist a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^N$ and $U \in H^s \setminus \{0\}$ such that up to a subsequence,

$$u_n(\cdot + x_n) \rightharpoonup U \text{ weakly in } H^s.$$
Next, we recall a sharp Gagliardo-Nirenberg type inequality established in [11, 52].

**Lemma 2.4.** Let $N \geq 2$, $0 < s < 1$ and $0 < p < \frac{4s}{N - 2s}$. Then, for all $u \in H^s$,

$$
\|u\|_{L^{p+2}}^{p+2} \leq C_{\text{opt}} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} \|u\|_{L^2}^{(p+2) - \frac{2s}{pN}},
$$

(2.3)

where the optimal constant $C_{\text{opt}}$ given by

$$
C_{\text{opt}} = \left( \frac{2s(p + 2) - pN}{pN} \right)^{\frac{Np}{4s}} \frac{2s(p + 2)}{(2s(p + 2) - pN)\|R\|_{L^2}^p},
$$

and $R$ is a ground state solution of the following elliptic equation

$$
(-\Delta)^s R + R = |R|^p R \text{ in } \mathbb{R}^N.
$$

(2.4)

In particular, in the $L^2$-critical case $p = \frac{4s}{N}$, $C_{\text{opt}} = \frac{2s+N}{N\|R\|_{L^2}}$.

**Lemma 2.5.** [51] Let $2s < \gamma < \min\{N, 4s\}$ and $N \geq 2$. Then for any $u \in H^s$,

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2|u(y)|^2}{(x - y)^\gamma} dxdy \leq \left( \frac{4s - \gamma}{\gamma} \right)^{\frac{2s}{\gamma}} \frac{4s}{(4s - \gamma)\|R\|_{L^2}^2} \|u\|_{H^s} \|R\|_{L^2} \|u\|_{H^s}^2
$$

(2.5)

where $R$ is ground state solution of

$$
(-\Delta)^s R + R - \left( \frac{1}{|x|} * |R|^2 \right) R = 0.
$$

**Lemma 2.6.** [11, 57] (Pohozaev identity) Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N - 2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$, and $u \in H^s$ is a weak solution of problem (1.2), then

$$(N - 2s)\|u\|_{H^s}^2 + N\omega\|u\|_{L^2}^2 = \frac{2N}{p + 2}\|u\|_{L^{p+2}}^{p+2} \quad \text{if } f(u) = |u|^p u,$$

$$(N - 2s)\|u\|_{H^s}^2 + N\omega\|u\|_{L^2}^2 = \frac{2N - \gamma}{2} \int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2)(x)|u|^2(x)dx \quad \text{if } f(u) = (|x|^{-\gamma} * |u|^2)u.$$

Finally, we recall the localized virial estimate related to [11] with $f(u) = |u|^p u$, which is the main ingredient in the proof of the sharp threshold of global existence and blow-up. The localized virial estimate was used by Bouloenger-Himmelsbach-Lenzmann [11] to show the existence of finite time blow-up radial solutions to (1.1) in the $L^2$-critical and $L^2$-supercritical cases. Let us start with the following estimate.

**Lemma 2.7.** [11]. Let $N \geq 1$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{1,\infty}$. Then for all $u \in H^{1/2}$,

$$
\left| \int_{\mathbb{R}^N} \pi(x) \nabla \varphi(x) \cdot \nabla u(x) dx \right| \leq C \left( \|\nabla|^{1/2} u\|_{L^2} + \|u\|_{L^2} \|\nabla|^{1/2} u\|_{L^2} \right),
$$

for some $C > 0$ depending only on $\|\nabla \varphi\|_{W^{1,\infty}}$ and $N$. 

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Let $N \geq 1$, $1/2 \leq s < 1$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{3,\infty}$. Assume $\psi \in C([0,T^*), H^s)$ is a solution to (1.1). The localized virial action of $\psi$ is defined by

$$M_\varphi(\psi(t)) := 2 \int_{\mathbb{R}^N} \nabla \varphi(x) \cdot Im(\overline{\psi(t,x)} \nabla \psi(t,x))dx. \quad (2.6)$$

It follows from Lemma 2.7 that $M_\varphi(\psi(t))$ is well-defined. Indeed, by Lemma 2.7

$$|M_\varphi(\psi(t))| \lesssim C(\|\nabla \varphi\|_{L^\infty}, \|\Delta \varphi\|_{L^\infty}) \|\psi(t)\|^2_{H^{1/2}} \lesssim C(\|\varphi\|\psi(t)\|^2_{H^s} < \infty.$$  

To study the time evolution of $M_\varphi(\psi(t))$, we need the following auxiliary function

$$\psi_m(t,x) := c_s \frac{1}{-\Delta + m} \psi(t,x) = c_s \mathcal{F}^{-1} \left( \frac{\hat{\psi}(t,\xi)}{|\xi|^2 + m} \right), \quad m > 0, \quad (2.7)$$

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$  

Remark that since $\psi(t) \in H^s$, the smoothing property of $(-\Delta + m)^{-1}$ implies that $\psi_m(t) \in H^{s+2}$ for any $t \in [0,T^*)$.

**Lemma 2.8** ([11]). For any $t \in [0,T^*)$, the following identity holds true

$$\frac{d}{dt} M_\varphi(\psi(t)) = -\int_0^t m^s \int_{\mathbb{R}^N} \Delta^2 \varphi |\psi_m(t)|^2 dxdm + 4 \sum_{j,k=1}^N \int_0^t m^s \int_{\mathbb{R}^N} \partial_j^2 \varphi \partial_k \psi_m(t) \partial_k \psi_m(t) dxdm$$

$$- \frac{2p}{p+2} \int_{\mathbb{R}^N} \Delta \varphi |\psi(t)|^{p+2} dx, \quad (2.8)$$

where $\psi_m$ is defined in (2.7).

Using Plancherel’s and Fubini’s theorem, it follows that

$$\int_0^t m^s \int_{\mathbb{R}^N} |\nabla \psi_m| dxdm = \int_{\mathbb{R}^N} \left( \frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s dm}{(|\xi|^2 + m)^s} \right) |\xi|^2 |\hat{\psi}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^N} (s|\xi|^{2s-2})|\xi|^2 |\hat{\psi}(\xi)|^2 d\xi = s\|\psi\|^2_{H^s}. \quad (2.9)$$

If we make formal substitution and take the unbounded function $\nabla \varphi(x) = 2x$, then we have $\partial_j^2 \varphi = 2\delta_{jk}$ and $\Delta^2 \varphi = 0$. Using (2.9), we find formally the virial identity

$$\frac{d}{dt} M_{|x|^2}(\psi(t)) = 8s\|\psi(t)\|^2_{H^s} - \frac{4Np}{p+2} \|\psi(t)\|_{L^{p+2}}^{p+2}$$

$$= 4NpE(\psi(t)) - 2(Np - 4s)\|\psi(t)\|^2_{H^s}. \quad (2.10)$$

Now let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be as above. We assume in addition that $\varphi$ is radially symmetric and satisfies

$$\varphi(r) := \begin{cases} r^2 & \text{for } r \leq 1, \\
\text{const.} & \text{for } r \geq 10, \end{cases} \quad \text{and } \varphi''(r) \leq 2 \text{ for } r \geq 0. \quad (2.11)$$
Here the precise constant is not important. For $R > 0$ given, we define the rescaled function $\varphi_R : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi_R(x) = \varphi_R(r) := R^2 \varphi(r/R).$$

It is easy to see that

$$2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi'_R(r)}{r} \geq 0, \quad 2N - \Delta \varphi_R(x) \geq 0, \quad \forall r \geq 0, \forall x \in \mathbb{R}^N. \quad (2.13)$$

Moreover,

$$\|\nabla^j \varphi_R\|_{L^\infty} \lesssim R^{2-j}, \quad j = 0, \cdots, 4,$$

and

$$\text{supp}(\nabla^j \varphi_R) \subset \begin{cases} \{ |x| \leq 10R \} & \text{for } j = 1, 2, \\ \{ R \leq |x| \leq 10R \} & \text{for } j = 3, 4. \end{cases}$$

Finally, we recall the following virial estimate for the time evolution of $M_{\varphi_R}(\psi(t))$, see [11].

**Lemma 2.9** ($H^s$ radial virial estimate). Let $N \geq 2, \frac{N}{2N-1} \leq s < 1, 0 < p < \frac{4s}{N-2s}, \varphi_R$ be as in (2.12) and $\psi \in C([0, T^*), H^s)$ be a radial solution to (1.1). Then for any $t \in [0, T^*)$,

$$\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq 4s \|\psi(t)\|_{H^s}^2 - \frac{2Np}{p+2} \|\psi(t)\|_{L^{p+2}}^{p+2} \quad + O \left( R^{-2s} + R^{-\frac{2(N-1)}{2} + \varepsilon s} \|\psi(t)\|_{H^s}^{p+\varepsilon} \right)$$

$$= 2Np E(\psi(t)) - (Np - 4s) \|\psi(t)\|_{H^s}^2 \quad + O \left( R^{-2s} + R^{-\frac{2(N-1)}{2} + \varepsilon s} \|\psi(t)\|_{H^s}^{p+\varepsilon} \right),$$

for any $0 < \varepsilon < \frac{(2N-1)p}{2s}$. Here the implicit constant depends only on $\|\psi_0\|_{L^2}, N, \varepsilon, s$ and $p$.

### 3 Existence of normalized ground states

In this section, we will prove Theorem 1.2. Firstly, we establish some preliminaries.

**Lemma 3.1.** Let $u^\lambda(x) = \lambda^{N/2} u(\lambda x)$, $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} + |u|^2) u$ with $2s < \gamma < \min\{ N, 4s \}$. Then for any $u \in S(c)$, there exists a unique $\lambda_0 > 0$ such that

1. when $f(u) = |u|^p u$,

$$E(u^{\lambda_0}) := \max_{t > 0} E(u^\lambda) = \left( \frac{Np - 4s}{2Np} \right) \left( \frac{2s(p+2)}{Np} \right) \frac{4s}{p+2} \frac{\|u\|_{H^s}^{2Np}}{\|u\|_{L^{p+2}}^{2Np-4s}} \frac{\|u\|_{H^s}^{2Np-4s}}{\|u\|_{L^{p+2}}^{2Np-4s}}$$

2. when $f(u) = (|x|^{-\gamma} + |u|^2) u$,
where $Q(u)$ is given in (1.8) or (1.9).

**Proof.** We only prove the case $f(u) = |u|^p u$. The case $f(u) = (|x|^{-\gamma} * |u|^2)u$ is similar. Firstly, we define

$$g(\lambda) := E(u^\lambda) = \frac{\lambda^{2ns}}{2} \|u\|_{H^s}^2 - \frac{\lambda^{Np}}{p+2} \int_{\mathbb{R}^N} |u(x)|^{p+2} dx.$$  

Then, $g(\lambda) > 0$ for sufficiently small $\lambda > 0$ and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. This implies that $g(\lambda)$ has a unique critical point $\lambda_0 > 0$ corresponding to its maximum on $(0, \infty)$, and

$$E(u^{\lambda_0}) = \max_{\lambda > 0} E(u^\lambda), \quad g'(\lambda_0) = s\lambda_0^{2s-1} \|u\|_{H^s}^2 - \frac{Np}{2(p+2)} \lambda_0^{\frac{Np}{2}} \int_{\mathbb{R}^N} |u(x)|^{p+2} dx = 0,$$

which yields

$$Q(u^{\lambda_0}) = s\lambda_0^{2s} \|u\|_{H^s}^2 - \frac{Np}{2(p+2)} \lambda_0^{\frac{Np}{2}} \int_{\mathbb{R}^N} |u(x)|^{p+2} dx = 0.$$

We consequently obtain $u^{\lambda_0} \in V(c)$. Moreover,

$$Q(u) = s\|u\|_{H^s}^2 - \frac{Np}{2(p+2)} \int_{\mathbb{R}^N} |u(x)|^{p+2} dx = s\|u\|_{H^s}^2 (1 - \frac{4s-Np}{4})$$

which concludes (i) and (ii). (iii) and (iv) follow from the fact that $g'(\lambda) = \lambda^{-1} Q(u^\lambda).$ \hfill $\Box$

**Lemma 3.2.** Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. If $u \in H^s$ is a weak solution of problem (1.2), then $Q(u) = 0$. Moreover, $u = 0$ if $\omega \leq 0$.

**Proof.** When $f(u) = |u|^p u$, by Lemma 2.6, the following Pohozaev identity holds for $u \in H^s$,

$$(N - 2s)\|u\|_{H^s}^2 + N\|u\|_{L^2}^2 = \frac{2N}{p+2} \|u\|_{L^{p+2}}^{p+2}.$$  

Multiplying (1.2) by $u$ and integrating over $\mathbb{R}^N$, we derive a second identity

$$\|u\|_{H^s}^2 + \omega \|u\|_{L^2}^2 = \|u\|_{L^{p+2}}^{p+2}.$$  

Thus we have immediately

$$Q(u) = \|u\|_{H^s}^2 - \frac{Np}{2s(p+2)} \|u\|_{L^{p+2}}^{p+2} = 0.$$
Also after simple calculations, we obtain

$$\omega \|u\|_{L^2}^2 = \left( \frac{2s(p+2)}{NP} - 1 \right) \|u\|_{H^s}^2.$$  

(1) If $\omega < 0$, we get $u \equiv 0$ immediately;
(2) If $\omega = 0$, $\|u\|_{H^s}^2 = 0$ then $u \equiv 0$.

The proof for $f(u) = (|x|^{-\gamma} \ast |u|^2)u$ is similar, so we omit the details. \hfill \Box

**Lemma 3.3.** Let $f(u) = |u|^pu$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} \ast |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. If $u$ is a critical point of $E|_{S(c)}$, then $E'(u) + \omega_c u = 0$ in $H^{-s}$ for some $\omega_c > 0$.

**Proof.** Since $u$ is a critical point of $E|_{S(c)}$, there exists $\omega_c \in \mathbb{R}$ such that $E'(u) + \omega_c u = 0$ in $H^{-s}$. Thus

$$\langle E'(u) + \omega_c u, u \rangle = \|u\|_{H^s}^2 + \omega_c \|u\|_{L^2}^2 - \|u\|_{L^{p+2}}^{p+2} = 0. \tag{3.1}$$

By Lemma 2.6, $u$ satisfies

$$(N - 2s)\|u\|_{H^s}^2 + N\omega \|u\|_{L^2}^2 = \frac{2N}{p+2} \|u\|_{L^{p+2}}^{p+2}. \tag{3.2}$$

Combining (3.1) with (3.2), we have

$$\omega_c = \frac{(4s + 2ps - pN)\|u\|_{H^s}^2}{Npc} > 0$$

for $\frac{4s}{N} < p < \frac{4s}{N-2s}$. The proof for $f(u) = (|x|^{-\gamma} \ast |u|^2)u$ is similar, so we omit the details. \hfill \Box

Next, we analyze the property of the function $c \rightarrow m(c)$.

**Lemma 3.4.** Let $f(u) = |u|^pu$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} \ast |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then

$$m(c) = \inf_{u \in S(c)} \max_{t > 0} E(u^\lambda),$$

where $u^\lambda = \lambda^{N/2}u(\lambda x)$.

**Proof.** Firstly, we notice that the minimizing problem in (1.11) is well-defined. Indeed, when $f(u) = |u|^pu$ and $u \in S(c)$, we have

$$E(u) = E(u) - \frac{2}{NP}Q(u) = \frac{Np - 4s}{2Np} \|u\|_{H^s}^2 > 0.$$ 

When $f(u) = (|x|^{-\gamma} \ast |u|^2)u$ and $u \in S(c)$, it follows that

$$E(u) = E(u) - \frac{1}{\gamma}Q(u) = \frac{\gamma - 2s}{2\gamma} \|u\|_{H^s}^2 > 0.$$
Thus, we denote $m(c) := \inf_{u \in V(c)} E(u)$. By Lemma 3.1 for any $u \in V(c)$,

$$E(u) = \max_{\lambda > 0} E(u^\lambda) \geq \inf_{v \in S(c)} \max_{\lambda > 0} E(u^\lambda),$$

then

$$\inf_{u \in V(c)} E(u) \geq \inf_{u \in S(c)} \max_{\lambda > 0} E(u^\lambda).$$

On the other hand, by Lemma 3.1 for any $u \in S(c)$, there exists a unique $\lambda_0 > 0$ such that $u^{\lambda_0} \in V(c)$ and

$$\max_{\lambda > 0} E(u^\lambda) = E(u^{\lambda_0}) \geq \inf_{u \in V(c)} E(u).$$

This implies that

$$\inf_{u \in S(c)} \max_{\lambda > 0} E(u^\lambda) \geq \inf_{u \in V(c)} E(u).$$

Thus, we have $m(c) = \inf_{u \in S(c)} \max_{\lambda > 0} E(u^\lambda)$. The proof for $f(u) = (|x|^{-\gamma} * |u|^2)u$ is similar, so we omit the details.

**Lemma 3.5.** Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then the function $c \to m(c)$ is strictly decreasing on $(0, \infty).

**Proof.** When $f(u) = |u|^p u$, for any $0 < c_1 < c_2 < +\infty$, there exists $u_1 \in S(c_1)$ such that

$$\max_{\lambda > 0} E(u_1^\lambda) < \theta \frac{2ps-Np+4s}{Np-4s} m(c_1),$$

where $\theta = \frac{c_2}{c_1}$. Set

$$u_2(x) = \theta \frac{2s-N}{4s} u_1(\theta^{-\frac{1}{2}} x),$$

then

$$\|u_2\|_{H^s} = \|u_1\|_{H^s}^{\frac{2}{p+2}} and \|u_2\|_{L^2} = \theta \|u_1\|_{L^2} = c_2,$$

$$\|u_2\|_{L^p+2}^{p+2} = \theta \frac{2ps-Np+4s}{4s} \|u_1\|_{L^p+2}^{p+2}.$$  

By Lemma 3.4 we have

$$m(c_2) \leq \max_{\lambda > 0} E(u_2^\lambda)$$

$$= \left( \frac{Np-4s}{2Np} \right) \left( \frac{2s(p+2)}{Np} \right) \frac{4s}{Np-4s} \|u_2\|_{H^s}^{\frac{2Np}{Np-4s}} \|u_2\|_{L^p+2}^{\frac{4s(p+2)}{Np-4s}}$$

$$= \left( \frac{Np-4s}{2Np} \right) \left( \frac{2s(p+2)}{Np} \right) \frac{4s}{Np-4s} \theta \frac{Np-4s-2ps}{Np-4s} \|u_1\|_{H^s}^{\frac{2Np}{Np-4s}} \|u_1\|_{L^p+2}^{\frac{4s(p+2)}{Np-4s}}$$

$$= \theta \frac{Np-4s-2ps}{Np-4s} \max_{\lambda > 0} E(u_1^\lambda)$$

$$< \theta \frac{Np-4s-2ps}{Np-4s} \theta \frac{2ps-Np+4s}{Np-4s} m(c_1)$$

$$= m(c_1)$$

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holds for $0 < c_1, c_2 < +\infty$. The proof for $f(u) = (|x|^{-\gamma} * |u|^2)u$ is similar, so we omit the details.

Now, we solve the minimization problem $\text{(3.3)}$. To this end, we consider the following minimization problem:

$$\tilde{m}(c) = \inf \{ \tilde{E}(v) : v \in S(c), \ Q(v) \leq 0 \},$$

where

$$\tilde{E}(v) := E(v) - \frac{2}{Np} Q(v) = \frac{Np - 4s}{2Np} \| v \|^2_{H^s}, \ \text{if} \ f(u) = |u|^p u,$$

$$\tilde{E}(v) := E(v) - \frac{1}{\gamma} Q(v) = \frac{\gamma - 2s}{2\gamma} \| v \|^2_{H^s}, \ \text{if} \ f(u) = (|x|^{-\gamma} * |u|^2)u.$$  \hfill (3.4)

**Proposition 3.6.** Let $N \geq 2$, $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N - 2s}$ or $f(u) = (|x|^{-\gamma} * |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then there exists $u \in V(c)$ and $\tilde{E}(u) = \tilde{m}(c)$.

**Proof.** We only prove this result for $f(u) = |u|^p u$. We first show that $\tilde{m}(c) > 0$. By $Q(v) \leq 0$, we have

$$\| v \|^2_{H^s} \leq \| v \|^p_{L^{p+2}} \leq C_{\text{opt}} \| v \|^\frac{pN}{2s} \| v \|^\frac{(p+2) - \frac{pN}{2s}}{2},$$

which implies that

$$\frac{pN}{2s} - (p+2) \leq C_{\text{opt}} \| v \|^\frac{pN}{2s} - 2.$$  \hfill (3.5)

Taking the infimum over $v$, we get $\tilde{m}(c) > 0$.

We now show the minimizing problem $\text{(3.3)}$ is attained. Let $\{v_n\}$ be a minimizing sequence for $\text{(3.3)}$, i.e., $\{v_n\} \subset S(c)$, $Q(v_n) \leq 0$ and $\tilde{E}(v_n) \to \tilde{m}(c)$ as $n \to \infty$. Thus, there exists $C_0 > 0$ such that

$$\liminf_{n \to \infty} \| v_n \|^p_{L^{p+2}} \geq s \liminf_{n \to \infty} \| v_n \|^2_{H^s} \geq C_0 > 0.$$  \hfill (3.6)

Applying Lemma $\text{(2.3)}$, there exist a subsequence, still denoted by $\{v_n\}$ and $u \in H^s \setminus \{0\}$ such that

$$u_n := \tau_{x_n} v_n \rightharpoonup u \neq 0 \ \text{weakly in} \ H^s,$$

for some $\{x_n\} \subset \mathbb{R}^N$. Moreover, we deduce from Lemma $\text{(2.1)}$ that

$$Q(u_n) - Q(u_n - u) - Q(u) \to 0,$$

$$\tilde{E}(u_n) - \tilde{E}(u_n - u) - \tilde{E}(u) \to 0,$$

$$\| u_n \|^2_{L^2} - \| u_n - u \|^2_{L^2} - \| u \|^2_{L^2} \to 0.$$  \hfill (3.7)

Now, we show that $Q(u) \leq 0$ and $\| u \|^2_{L^2} = c$ by excluding the other possibilities:

(1) If $Q(u) > 0$ and $\| u \|^2_{L^2} < c$, it follows from $\text{(3.6)}$ and $Q(u_n) \leq 0$ that $Q(u_n - u) \leq 0$ for sufficiently large $n$. Set $c_1 = c - \| u \|^2_{L^2}$ and $w_n = \sqrt{c_1} (u_n - u)$, then we have

$$\| u_n - u \|^2_{L^2} \to \sqrt{c_1}, \ w_n \in S(c_1), \ \text{and} \ Q(w_n) \leq 0.$$
Thus, by the definition of $\bar{m}(c_1)$, it follows that

$$\bar{E}(u_n) \geq \bar{m}(c_1) \quad \text{and} \quad \bar{E}(u_n - u) \geq \bar{m}(c_1).$$

Applying $\bar{m}(c_1) = m(c_1) > m(c)$, we can obtain $\bar{E}(u) = \frac{Np - 4s}{2Np} \|u\|^2_{H^s} \leq 0$ which is a contradiction with $u \neq 0$.

(2) If $Q(u) > 0$ and $\|u\|^2_{L^2} = c$, then $u_n \to u$ in $L^2$ as $n \to \infty$. This implies that $u_n \to u$ in $L^{p+2}$ as $n \to \infty$. On the other hand, we deduce from $Q(u) > 0$ that $Q(u_n - u) \leq 0$ for sufficiently large $n$. Thus, we can obtain $u_n \to u$ in $H^s$ as $n \to \infty$. This yields $Q(u_n - u) \to 0$ as $n \to \infty$. Thus, it follows from (3.7) and $Q(u) > 0$ that $Q(u_n) > 0$ for sufficiently large $n$, which is a contradiction with $Q(u_n) \leq 0$.

(3) If $Q(u) \leq 0$ and $\|u\|^2_{L^2} < c$, then we conclude from (3.7) and $\bar{m}(\|u\|^2_{L^2}) = m(\|u\|^2_{L^2}) > m(c) = \bar{m}(c)$ that $\bar{E}(u_n - u) = \frac{Np - 4s}{2Np} \|u_n - u\|^2_{H^s} < 0$, which is a contradiction.

Therefore, we have $Q(u) \leq 0$ and $\|u\|^2_{L^2} = c$. It follows from the definition of $\bar{m}(c)$ and the weak lower semicontinuity of norm that

$$\bar{m}(c) \leq \bar{E}(u) \leq \liminf_{n \to \infty} \bar{E}(u_n) = \bar{m}(c).$$

This yields that

$$\bar{E}(u) = \bar{m}(c).$$

Finally, we show that $Q(u) = 0$. Suppose that $Q(u) < 0$ and set

$$f(\lambda) := Q(u^\lambda) = \lambda^{2s} \|u\|^2_{H^s} - \frac{Np}{2} \frac{\lambda^{p+2}}{2 + p} \|u\|^2_{L^{p+2}},$$

then $f(\lambda) > 0$ for sufficiently small $\lambda > 0$ and $f(1) = Q(u) < 0$. Therefore, there exists $\lambda_0 \in (0, 1)$ such that $Q(u^{\lambda_0}) = 0$. Then, it follows that

$$\bar{E}(u^{\lambda_0}) = \frac{Np - 4s}{2Np} \|u\|^2_{H^s} \lambda_0^{2s} < \bar{E}(u) = \bar{m}(c),$$

which contradicts the definition of $\bar{m}(c)$. Hence, we have $Q(u) = 0$. \qed

By the fact $\bar{m}(c) = m(c)$ and this proposition, we can obtain the following Corollary.

**Corollary 3.7.** Let $N \geq 2$, $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-2s} + |u|^2)u$ with $2s < \gamma < \min\{N, 4s\}$. Then there exists $u \in V(c)$ and $E(u) = m(c)$.

**Lemma 3.8.** ([12]) Let $X$ be a real Banach space, $U \subset X$ be an open set. Suppose that $f, g_1, \cdots, g_m : U \to \mathbb{R}^1$ are $C^1$ functions and $x_0 \in M$ is such that $f(x_0) = \inf_{x \in M} f(x)$ with

$$M = \{x \in U : g_i(x) = 0, i = 1, 2, \cdots, m\}.$$ 

If $\{g_i'(x_0)\}_{i=1}^m$ is linearly independent, then there exists $k_1, \cdots, k_m \in \mathbb{R}$ such that

$$f'(x_0) + \sum_{i=1}^m k_ig_i'(x_0) = 0.$$
Lemma 3.9. Let $f(u) = |u|^p u$ with $\frac{4s}{N} < p < \frac{4s}{N-2s}$ or $f(u) = (|x|^{-\gamma} |u|^2) u$ with $2s < \gamma < \min\{N, 4s\}$. Then each critical point of $E|_{V(c)}$ is a critical point of $E|_{S(c)}$.

Proof. We only prove the case for $f(u) = |u|^p u$. Suppose that $u$ is a critical point of $E|_{V(c)}$, then by Lemma 3.8, we have an alternative: either (i) $Q'(u)$ and $(\|u\|_{L^2}^2)'$ are linearly dependent, or (ii) there exists $\omega_1, \omega_2 \in \mathbb{R}$ such that

$$E'(u) + \omega_1 Q'(u) + \omega_2 u = 0 \quad \text{in } H^s. \quad (3.9)$$

If (i) holds, then $u$ satisfies

$$2s(-\Delta)^s u + \omega^* u - \frac{Np}{2} |u|^p u = 0 \quad \text{in } H^s,$$

for some $\omega^* \in \mathbb{R}$. Multiplying the above equation by $u$ and integrating, we get

$$2s\|u\|_{H^s}^2 + \omega^* \|u\|_{L^2}^2 - \frac{Np}{2} \|u\|_{L^{p+2}}^{p+2} = 0.$$

By Pohozaev identity, we derive

$$(N - 2s)\|u\|_{H^s}^2 + \frac{N}{2s} \omega^* \|u\|_{L^2}^2 - \frac{N^2 p}{2s(p+2)} \|u\|_{L^{p+2}}^{p+2} = 0.$$

Thus we have

$$2s\|u\|_{H^s}^2 - \frac{N^2 p^2}{4s(p+2)} \|u\|_{L^{p+2}}^{p+2} = 0.$$

Notice that $Q(u) = 0$ and $\frac{4s}{N} < p < \frac{4s}{N-2s}$, then we have immediately $\|u\|_{H^s}^2 = 0$, which is a contradiction with $u \in S(c)$. This implies that (i) does not occur and (ii) is true. It is enough to show that $\omega_1 = 0$. By (3.9) we have

$$\langle E'(u) + \omega_1 Q'(u) + \omega_2 u, u \rangle = (1 + 2s \omega_1)\|u\|_{H^s}^2 - \frac{Np}{2} \omega_1 \|u\|_{L^{p+2}}^{p+2} + \omega_2 \|u\|_{L^2}^2 = 0. \quad (3.10)$$

By Pohozaev identity corresponding to equation (3.9),

$$(1 + 2s \omega_1)(N - 2s)\|u\|_{H^s}^2 - \frac{2N}{p+2} \omega_1 \|u\|_{L^{p+2}}^{p+2} + N \omega_2 \|u\|_{L^2}^2 = 0. \quad (3.11)$$

Combining (3.10) with (3.11) we have

$$\|u\|_{H^s}^2 = \frac{Np(1 + 2s \omega_1)}{2s(p+2)(1 + 2s \omega_1)} \|u\|_{L^{p+2}}^{p+2}. \quad (3.12)$$

Since $u \in V(c)$, $\|u\|_{H^s}^2 = \frac{Np}{2s(p+2)} \|u\|_{L^{p+2}}^{p+2}$, then by (3.12) we have $\omega_1 = 0$. Finally, by Lemma 3.3 we get $\omega_2 > 0$. \qed
Proof of Theorem 1.2. By Corollary 3.7, there exists a couple of weak solution \((u_c, \omega_c) \in \mathcal{M}_c \times \mathbb{R}^+\) to problems (1.2)-(1.5). If \(v \in S(c)\) satisfies \(E'(S_c(v)) = 0\), then by Lemma 3.2, 3.3, we have \(Q(v) = 0\), which implies that \(v \in V(c)\). Hence, \(E(v) \geq E(u_c)\) and \(u_c \in S(c)\) is a normalized ground state of problems (1.2)-(1.5).

By Lemma 3.2, we have \(Q(u_c) = s\|u_c\|_{H^s}^2 - \frac{Np}{2p} \|u_c\|_{L^{p+2}}^{p+2} = 0\). Applying the inequality (2.3), we have
\[
\frac{2s(p + 2)}{Np} \|u_c\|_{H^s}^2 \geq \|u_c\|_{L^{p+2}}^{p+2} \leq \frac{Np}{2s} \frac{\|u_c\|_{H^s}^2}{\|u_c\|_{L^2}^{(p+2)-\frac{2N}{2s}}},
\]
then
\[
\|u_c\|_{H^s}^2 \geq \frac{2s(p + 2)}{Np} \left( C_{opt} \|u_c\|_{H^s}^{\frac{2N}{2s} - (p+2)} \right) \to +\infty
\]
as \(c \to 0^+\), i.e. \(\|u_c\|_{H^s} \to +\infty\) as \(c \to 0^+\). Moreover,
\[
m(c) = E(u_c) = \frac{Np - 4s}{2Np} \|u_c\|_{H^s}^2 \to +\infty
\]
as \(c \to 0^+\). From equation (1.2), we have \(\|u_c\|_{H^s}^2 + \omega_c \|u_c\|_{L^{p+2}}^2 = \|u_c\|_{L^{p+2}}^{p+2}\), then
\[
\omega_c = \frac{1}{c} \left( \|u_c\|_{L^{p+2}}^{p+2} - \|u_c\|_{H^s}^2 \right)
= \frac{1}{c} \left( \frac{2s(p + 2)}{Np} \|u_c\|_{H^s}^2 - \|u_c\|_{H^s}^2 \right)
= \frac{2s(p + 2) - Np}{Np} \|u_c\|_{H^s}^2
\to +\infty
\]
as \(c \to 0^+\), for \(\frac{4s}{N} < p < \frac{4s}{N-2s}\).

Next, we consider the case \(c \to +\infty\). Let \(u_1 \in V(1)\), \(\bar{u}(x) = c^{-\frac{2s}{Np-4s}} u_1 \left( c^{-\frac{p}{Np-4s}} x \right)\). By some simple calculations, we have
\[
\|\bar{u}\|_{H^s}^2 = c^{\frac{Np-2ps-4s}{Np-4s}} \|u_1\|_{H^s}^2 \quad \text{and} \quad \|\bar{u}\|_{L^{p+2}}^{p+2} = c^{\frac{Np-2ps-4s}{Np-4s}} \|u_1\|_{L^{p+2}}^{p+2}.
\]
These imply that \(Q(\bar{u}) = 0\) and
\[
E(\bar{u}) = \frac{1}{2} \|\bar{u}\|_{H^s}^2 - \frac{1}{p+2} \|\bar{u}\|_{L^{p+2}}^{p+2}
= \frac{Np - 4s}{2Np} \|\bar{u}\|_{H^s}^2
= \frac{Np - 4s}{2Np} c^{\frac{Np-2ps-4s}{Np-4s}} \|u_1\|_{H^s}^2
\to 0
\]
as \(c \to +\infty\), for \(\frac{4s}{N} < p < \frac{4s}{N-2s}\). Therefore, \(0 < m(c) = E(u_c) \leq E(\bar{u}) \to 0\) as \(c \to +\infty\). So
\[
\|u_c\|_{H^s}^2 = \frac{2Np}{Np - 4s} m(c) \to 0
\]
and
\[
\omega_c = \frac{1}{c} \frac{2s(p+2)-Np}{Np} \|u_c\|^2_{H^s} \to 0
\]
as \(c \to +\infty\). Thus the proof is completed.

## 4 Proof of Theorem 1.3

**Proof of Theorem 1.3** It is standard that if \(\frac{4s}{N} < p < \frac{4s}{N-2s}\) or \(2s < \gamma < \min\{N,4s\}\)
\[
d(c) := \inf_{v \in N_{\omega_c}} J_{\omega_c}(v)
\]
is attained by a function \(\tilde{u}\), which is a ground state solution to problem (1.2) with \(\omega = \omega_c\), where
\[
N_{\omega_c} = \{v \in H^s : \langle J'_{\omega_c}(v),v \rangle = 0, \ v \neq 0\}.
\]
Then, by Lemma 3.3 and Lemma 3.9 \((u_c, \omega_c)\) and \((\tilde{u}, \omega_c)\) are two couples of weak solution to problem (1.2). By Lemma 3.2, we have \(\omega_c > 0\) and
\[
Q(u_c) = Q(\tilde{u}) = 0,
\]
which implies that
\[
\|u_c\|^2_{H^s} = \frac{Np}{2s(p+2)} \|u_c\|^2_{L^{p+2}}, \quad \text{and} \quad \|\tilde{u}\|^2_{H^s} = \frac{Np}{2s(p+2)} \|\tilde{u}\|^2_{L^{p+2}}.
\]
Hence, we have
\[
\omega_c \|u_c\|^2_{L^2} = \frac{2s(p+2) - Np}{Np} \|u_c\|^2_{H^s}, \quad \text{and} \quad \omega_c \|\tilde{u}\|^2_{L^2} = \frac{2s(p+2) - Np}{Np} \|\tilde{u}\|^2_{H^s}.
\]
By (4.2)-(4.3), we have
\[
J_{\omega_c}(u_c) = \frac{1}{2} \|u_c\|^2_{H^s} + \frac{\omega_c}{2} \|u_c\|^2_{L^2} - \frac{1}{p+2} \|u_c\|^2_{L^{p+2}} = \frac{s}{N} \|u_c\|^2_{H^s}, \quad \text{and} \quad J_{\omega_c}(\tilde{u}) = \frac{s}{N} \|\tilde{u}\|^2_{H^s},
\]
and
\[
E(u_c) = \left(\frac{1}{2} - \frac{2s}{Np}\right) \|u_c\|^2_{H^s} + \frac{Np - 4s}{2ps} J_{\omega_c}(u_c) \quad \text{and} \quad E(\tilde{u}) = \frac{Np - 4s}{2ps} J_{\omega_c}(\tilde{u}).
\]
Since \(\tilde{u}\) is a ground state solution to problem (1.2) with \(\omega = \omega_c\), then \(J_{\omega_c}(u_c) \geq J_{\omega_c}(\tilde{u})\). Thus
\[
m(c) = E(u_c) = \frac{Np - 4s}{2ps} J_{\omega_c}(u_c) \geq \frac{Np - 4s}{2ps} J_{\omega_c}(\tilde{u}) = E(\tilde{u}) \geq m(\|\tilde{u}\|^2_{L^2}).
\]
By Lemma 3.5 we have \(\|u_c\|^2_{L^2} \leq \|\tilde{u}\|^2_{L^2}\), then \(\|u_c\|^2_{H^s} \leq \|\tilde{u}\|^2_{H^s}\), which implies that \(J_{\omega_c}(u_c) \leq J_{\omega_c}(\tilde{u})\). So, \(J_{\omega_c}(\tilde{u}) = J_{\omega_c}(u_c) = d(c)\). This completes the proof of Theorem 1.3.

## 5 Proof of Theorem 1.4

**Proof of Theorem 1.4** Firstly, we show that the set \(A_c\) and \(B_c\) are not empty. Indeed, for arbitrary but fixed \(u \in S(c)\), set \(u^\lambda(x) = \lambda^\frac{N}{2} u(\lambda x)\). Then we have \(u^\lambda \in S(c)\), for all \(\lambda > 0\);
\[
E(u^\lambda) \to 0 \quad \text{as} \quad \lambda \to 0 \quad \text{and} \quad Q(u^\lambda) > 0 \quad \text{for sufficiently small} \quad \lambda > 0.
\]
This proves that \(A_c \neq \emptyset\). In addition, \(E(u^\lambda) \to -\infty\) and \(Q(u^\lambda) \to -\infty\) as \(\lambda \to +\infty\). Thus, \(B_c \neq \emptyset\).
In the following, we will prove that $A_{\|\psi_0\|_{L^2}^2}$ and $B_{\|\psi_0\|_{L^2}^2}$ are two invariant manifolds of (1.1). Let $\psi_0 \in A_{\|\psi_0\|_{L^2}^2}$, by Proposition 2.1, we see that there exists a unique solution $\psi \in C([0, T^*), H^s)$ with initial data $\psi_0$. We deduce from the conservations of energy that
\[ E(\psi(t)) = E(\psi_0) < m(\|\psi_0\|_{L^2}^2), \]  
for any $t \in [0, T^*)$. In addition, by the continuity of the function $t \mapsto Q(\psi(t))$ and Corollary 3.7, if there exists $t_0 \in [0, T^*)$ so that $Q(\psi(t_0)) = 0$, then $E(\psi(t_0)) \geq m(\|\psi_0\|_{L^2}^2)$, which contradicts with (5.1). Therefore, we have $Q(\psi(t)) > 0$ for any $t \in [0, T^*)$. Similarly, we can prove that $B_{\|\psi_0\|_{L^2}^2}$ is invariant under the flow of (1.1).

Now, we prove (1). Let us prove (1) by contradiction. If not, there exists $T^* > 0$ such that
\[ \lim_{t \to T^*} \|\psi(t)\|_{H^s} = +\infty. \]  
(5.2)
Applying the conservation of energy, we have
\[ E(\psi_0) - \frac{2}{Np} Q(\psi(t)) = E(\psi(t)) - \frac{2}{Np} Q(\psi(t)) = \left(1 - \frac{2s}{Np}\right)\|\psi(t)\|_{H^s}^2, \]
which implies that
\[ \lim_{t \to T^*} Q(\psi(t)) = -\infty, \]
if (5.2) happens. Since $Q(\psi_0) > 0$, by continuity there exists $t_0 \in (0, T^*)$ such that
\[ Q(\psi(t_0)) = 0 \text{ and } E(\psi(t_0)) = E(\psi_0) < m(\|\psi_0\|_{L^2}^2). \]
This contradicts the fact that $m(\|\psi_0\|_{L^2}^2) = \inf_{u \in V(\|\psi_0\|_{L^2}^2)} E(u)$. Thus, if $\psi_0 \in A_{\|\psi_0\|_{L^2}^2}$, then the solution $\psi(t)$ of (1.1) exists globally.

Next, we prove (2). If $\psi_0 \in B_{\|\psi_0\|_{L^2}^2}$, then $Q(\psi(t)) < 0$ for any $t \in [0, T^*)$. We deduce from Proposition 3.6 that
\[ m(\|\psi_0\|_{L^2}^2) = \bar{m}(\|\psi_0\|_{L^2}^2) \leq \bar{E}(\psi(t)) = E(\psi(t)) - \frac{2}{Np} Q(\psi(t)) < E(\psi_0) - \frac{Q(\psi(t))}{2s}, \]
for all $t \in [0, T^*)$. This implies that
\[ Q(\psi(t)) \leq 2s(E(\psi_0) - m(\|\psi_0\|_{L^2}^2)) < 0, \]  
(5.3)
for all $t \in [0, T^*)$.

Now, we claim that there exists $C_1 > 0$ such that
\[ \frac{d}{dt}M_{\varphi_R}(\psi(t)) \leq -C_1 \|\psi(t)\|_{H^s}^2, \]  
(5.4)
for $f(u) = |u|^pu$ and any $t \in [0, T^*)$, where $M_{\varphi_R}(\psi(t))$ is defined by (2.6). Firstly, we prove that there exists $C_2 > 0$ such that
\[ \|\psi(t)\|_{H^s} \geq C_2, \]  
(5.5)
for every \( t \in [0, T^*) \). Indeed, suppose this bound is not true, then there exists \( \{ t_k \} \subseteq [0, T^*) \) such that \( \| \psi(t_k) \|_{H^s} \to 0 \). However, we deduce from mass conservation and the sharp Gagliardo-Nirenberg inequality \((2.3)\) that

\[
\| \psi(t_k) \|_{L^{p+2}}^{p+2} \leq C_{opt} \| \psi(t_k) \|^n_{H^s} \| \psi(t_k) \|_{L^2}^{(p+2)-n} \to 0
\]
as \( k \to \infty \). Therefore, we have

\[
Q(\psi(t_k)) := s \| \psi(t_k) \|_{H^s}^2 - \frac{Np}{2(p+2)} \| \psi(t_k) \|_{L^{p+2}}^{p+2} \to 0,
\]
as \( k \to \infty \), which contradicts to \((5.3)\).

We now prove \((5.4)\). Since the solution \( \psi(t) \) is radial, we apply Lemma 2.9 to have

\[
\frac{d}{dt} M_{\varphi R}(\psi(t)) \leq 4s \| \psi(t) \|_{H^s}^2 - \frac{2Np}{p+2} \| \psi(t) \|_{L^{p+2}}^{p+2} + O\left( R^{-2s} + R^\frac{p(n-1)}{2} + s \| \psi(t) \|_{H^s}^p + \| \psi(t) \|_{H^s}^{p+\varepsilon} \right),
\]

for all \( t \in [0, T^*) \) and \( R > 1 \). Thanks to the assumption \( p < 4s \), we can apply the Young inequality to obtain for any \( \eta > 0 \),

\[
R^{-\frac{p(n-1)}{2} + s \| \psi(t) \|_{H^s}^p} \leq C \eta \| \psi(t) \|_{H^s}^2 + \eta^{-\frac{p+2s}{4s-p-2s}} R^{-\frac{2s(p(n-1)-2s)}{4s-p-2s}}.
\]

We thus obtain

\[
\frac{d}{dt} M_{\varphi R}(\psi(t)) \leq 4s \| \psi(t) \|_{H^s}^2 - \frac{2Np}{p+2} \| \psi(t) \|_{L^{p+2}}^{p+2} + C \eta \| \psi(t) \|_{H^s}^2 + O\left( R^{-2s} + \eta^{-\frac{p+2s}{4s-p-2s}} R^{-\frac{2s(p(n-1)-2s)}{4s-p-2s}} \right),
\]

for all \( t \in [0, T^*) \), any \( \eta > 0 \), any \( R > 1 \) and some constant \( C > 0 \).

We fix \( t \in [0, T^*) \) and denote

\[
\mu := \frac{4Np|E(\psi_0)| + 2}{Np - 4s}.
\]

We consider two cases.

**Case 1.**

\[
\| \psi(t) \|_{H^s}^2 \leq \mu.
\]

Since

\[
4s \| \psi(t) \|_{H^s}^2 - \frac{2Np}{p+2} \| \psi(t) \|_{L^{p+2}}^{p+2} = 4Q(\psi(t)) \leq 8s(E(\psi_0) - m(\| \psi_0 \|_{L^2}^2))
\]

for all \( t \in [0, T^*) \), we have

\[
\frac{d}{dt} M_{\varphi R}(\psi(t)) \leq 8s(E(\psi_0) - m(\| \psi_0 \|_{L^2}^2)) + C \mu
\]

\[
+ O\left( R^{-2s} + \eta^{-\frac{p+2s}{4s-p-2s}} R^{-\frac{2s(p(n-1)-2s)}{4s-p-2s}} \right).
\]
By choosing $\eta > 0$ small enough and $R > 1$ large enough depending on $\eta$, it follows that
\[
\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq 8s(E(\psi_0) - m(\|\psi_0\|_{L^2})) \leq \frac{8s(E(\psi_0) - m(\|\psi_0\|_{L^2}))}{\mu} \|\psi(t)\|_{H^s}^2, \tag{5.7}
\]

Case 2.
\[
\|\psi(t)\|_{H^s}^2 > \mu.
\]

In this case, it follows from conservation of energy that
\[
4s\|\psi(t)\|_{H^s}^2 - \frac{2Np}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2} = 2NpE(\psi(t)) - (Np - 4s)\|\psi(t)\|_{H^s}^2 \\
\leq \frac{\mu}{2}(Np - 4s) - 1 - (Np - 4s)\|\psi(t)\|_{H^s}^2.
\]

We thus obtain
\[
\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq -\frac{Np - 4s}{2}\|\psi(t)\|_{H^s}^2 + C\eta\|\psi(t)\|_{H^s}^2 \\
+ O\left(R^{-2s} + \eta^{-\frac{p+2s}{4s-p-2s}} R^{-\frac{2s(p(N-1)-2s)}{4s-p-2s}}\right).
\]

Since $Np - 4s > 0$, we choose $\eta > 0$ small enough so that
\[
\frac{Np - 4s}{2} - C\eta \geq \frac{Np - 4s}{4}.
\]

We next choose $R > 1$ large enough depending on $\eta$ so that
\[
-1 + O\left(R^{-2s} + \eta^{-\frac{p+2s}{4s-p-2s}} R^{-\frac{2s(p(N-1)-2s)}{4s-p-2s}}\right) \leq 0.
\]

We thus obtain
\[
\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq -\frac{Np - 4s}{4}\|\psi(t)\|_{H^s}^2.
\]

We are now able to show that the solution $\psi(t)$ blows up in a finite time. Assume by contradiction that $T^* = \infty$. It follows from (5.4) and (5.5) that $\frac{d}{dt} M_{\varphi_R}(\psi(t)) \leq -C$ with some constant $C > 0$. Integrating this bound, we conclude that $M_{\varphi_R}(\psi(t)) < 0$ for all $t \geq t_1$ with some time sufficiently large time $t_1 \gg 1$. Thus, integrating (5.4) on $[t_1, t]$, we obtain
\[
M_{\varphi_R}(\psi(t)) \leq -c \int_{t_1}^t \|(-\Delta)^{\frac{s}{2}} \psi(\tau)\|_{L^2}^2 d\tau \quad \text{for all} \quad t \geq t_1. \tag{5.8}
\]

On the other hand, we use Lemma 2.5 and $L^2$-mass conservation to find that
\[
| M_{\varphi_R}(\psi(t)) | \leq C(\varphi_R)(\|(-\Delta)^{\frac{s}{2}} \psi(t)\|_{L^2}^{\frac{1}{2}} + \|(-\Delta)^{\frac{s}{2}} \psi(t)\|_{L^2}^{\frac{1}{2}}), \tag{5.9}
\]

where we used the interpolation estimate $\|\nabla (-\Delta)^{\frac{s}{2}} \psi\|_{L^2} \leq \|\psi\|_{L^2}^{\frac{1}{4}} \|(-\Delta)^{\frac{s}{2}} \psi\|_{L^2}^{\frac{3}{4}}$ for $s > \frac{1}{2}$.

So, we deduce from (5.5) and (5.9) that
\[
| M_{\varphi_R}(\psi(t)) | \leq C(\varphi_R)(\|(-\Delta)^{\frac{s}{2}} \psi(t)\|_{L^2}^{\frac{1}{2}}). \tag{5.10}
\]
This, together with (5.8), implies that
\[ \mathcal{M}_\varphi[\psi(t)] \leq -C(\varphi_R) \int_{t_1}^t |\mathcal{M}_\varphi(\psi(\tau))|^{2s}d\tau \quad \text{for} \quad t \geq t_1. \] (5.11)
This yields \( \mathcal{M}_\varphi[\psi(t)] \leq -C(\varphi_R)|t - t_*|^{-2s} \) for \( s > \frac{1}{2} \) with some \( t_* < +\infty \). Therefore, we have \( \mathcal{M}_\varphi[\psi(t)] \to -\infty \) as \( t \to t_* \). Hence the solution \( \psi(t) \) cannot exist for all time \( t \geq 0 \) and consequently we must have that \( T^* < +\infty \) holds.

Finally, we prove (3). By a similar argument as the case \( f(\psi) = |\psi|^p\psi \), we can also obtain (5.3) for (1.1) with \( f(\psi) = (|x|^{-\gamma} * |\psi|^2)\psi \). It follows from [16, 51] that \( x\psi(t) \in L^2, x \cdot \nabla \psi(t) \in L^2 \) for all \( t \in [0, T^*) \). Moreover, \( \int_{\mathbb{R}^N} \tilde{\psi}(t, x)(-\Delta)^{1-s}x\psi(t, x)dx \) is non-negative and
\[ \frac{d^2}{dt^2} \int_{\mathbb{R}^N} \tilde{\psi}(t, x)(-\Delta)^{1-s}x\psi(t, x)dx \leq 2Q(\psi(t)) \leq 4s(E(\psi_0) - m(\|\psi_0\|_{L^2}^2)) < 0, \]
where we use (5.3). This implies that there exists \( 0 < T^* < \infty \) such that
\[ \int_{\mathbb{R}^N} \tilde{\psi}(T^*, x)(-\Delta)^{1-s}x\psi(T^*, x)dx = 0. \]
Now, using the conservation of mass and the inequality
\[ \|u\|^2_{L^2} \leq \frac{2}{N} \left( \int_{\mathbb{R}^N} \bar{u}x(-\Delta)^{1-s}xudx \right)^{1/2} \left( \int_{\mathbb{R}^N} \bar{u}(-\Delta)^sudx \right)^{1/2}, \quad \text{for} \ u \in H^s, \]
we see that for all \( t \in [0, T^*) \)
\[ \|\psi_0\|^2_{L^2} = \|\psi(t)\|^2_{L^2} \leq \frac{2}{N} \left( \int_{\mathbb{R}^N} \tilde{\psi}(t, x)(-\Delta)^{1-s}x\psi(t, x)dx \right)^{1/2} \left( \int_{\mathbb{R}^N} \tilde{\psi}(t, x)(-\Delta)^s\psi(t, x)dx \right)^{1/2} \]
\[ \leq \frac{2}{N} \left( \int_{\mathbb{R}^N} \tilde{\psi}(t, x)(-\Delta)^{1-s}x\psi(t, x)dx \right)^{1/2} \|\psi(t)\|_{H^s}. \]
This yields that \( \lim_{t \to T^*} \|\psi(t)\|_{H^s} = \infty \). This completes the proof of Theorem 1.3.

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