A Periodicity-Induced Generalized Fourier Transform Pair

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Abstract—The field radiated by an infinite periodic structure can be expressed in terms of Floquet waves (FWs), both in the frequency domain (FD) and time domain (TD) [1]. A new periodicity-induced generalized Fourier transform (FT) pair is derived relating FD-FWs to TD-FWs and vice versa, based on tabulated transforms and physical conditions at infinity. The new FTs are directly related to the simple canonical problem of a line array of sequentially excited dipoles that is a basic building block for more general phased periodic structures.

Index Terms—Arrays, Fourier transforms (FTs), Green function, periodic structures, transient scattering.

I. INTRODUCTION

Floquet waves (FWs) generated by one-dimensional (1-D) phased periodicity along a rectilinear coordinate \( z \) are parameterized by the dispersion relation

\[
k_z(\omega) = \omega \gamma_z + \alpha_q, \quad \alpha_q = 2\pi q/d, \quad q = 0, \pm 1, \pm 2, \ldots
\]

where \( \omega \) is the radian frequency, \( k_z \) is the \( z \)-domain wavenumber, \( \gamma_z \) is the interelement phase gradient, \( d \) is the interelement spacing, and \( q \) is the FW index [1]. The dispersion relation for \( q \neq 0 \) differs from the nondispersive case \( q = 0 \), i.e., \( k_{0z} = \omega \gamma_z \), only through the constant term \( \alpha_q \). Closed form relations between frequency-domain (FD) and time-domain (TD) FWs can be established by conventional tabulated Fourier transforms (FT) when \( q = 0 \) [2, pp. 277]. However, no corresponding direct tabulations seem to exist for \( q \neq 0 \). This has motivated the study of a generalized FT pair for a class of functions that differs from those listed in the mathematical tables by involving Hankel functions with an \( \omega \) dependence of the form \( k_{pq}(\omega) = \sqrt{k^2 - k_{0z}^2} \) instead of \( k_{0z}(\omega) = \sqrt{k^2 - \alpha_q^2} \), with \( k = \omega/c \) (\( c \) being the ambient wave speed) and \( k_{0z}(\omega) \) given in (1). The periodicity-induced FT will establish direct relations between FD-FW and TD-FW with \( q \neq 0 \).

The important nondimensional parameter

\[
\eta = \gamma_z c
\]

permits distinction between two cases depending on \( |\eta| \gtrless 1 \), in which the phase velocity \( \nu_p = \gamma_z^{-1} \) is \( c/\eta \) of the excitation of the periodic structure along \( z \) can be larger (\( |\eta| < 1 \)) or smaller (\( |\eta| > 1 \)) than the ambient wave speed \( c \).

This periodicity-induced FT is directly related to simple radiating systems such as the sequentially excited periodic line array of dipoles that has been studied in detail in [1]. There, TD-FWs have been defined and found via various analytic methods. Here, we prove this generalized periodicity-induced Fourier transform pair going from TD to FD and vice versa, in a direct manner. This constitutes the basic building block for more complicated periodic structures with sequentially excited periodicity cells.

II. A PERIODICITY-INDUCED GENERALIZED FOURIER TRANSFORM PAIR

Defining the Fourier forward and inverse transforms as

\[
I(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{I}(t) e^{-j\omega t} dt, \quad \hat{I}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{j\omega t} d\omega
\]

with the caret denoting time-dependent quantities, it is demonstrated in what follows that the periodicity-induced FT pair

\[
\frac{1}{j} \mathcal{H}_0(2)[k_{pq}(\omega)\eta] \leftrightarrow \frac{e^{\frac{c}{2}\eta^2 \gamma_z^2}}{\pi \sqrt{f^2 - \tau_0^2}} f(t)
\]

obeys the definitions (3).

On the FD side of (4), the top Riemann sheet of the radial wavenumber \( k_{pq}(\omega) \) in (5) is chosen to render \( 3m k_{pq} \leq 0 \), consistent with the radiation condition at \( \rho = \infty \) (\( \rho \) is the cylindrical coordinate perpendicular to \( z \)). Furthermore, \( \Re k_{pq} \geq 0 \) or \( \leq 0 \) for \( \omega > 0 \) or \( < 0 \), respectively, in order to satisfy the radiation condition for positive and negative real frequencies. The conditions \( |k_{pq}| \geq |k| \) determine an exponentially decaying or oscillating function along \( \rho \), respectively. On the TD side of (4), \( U(\tau) = 1 \) or \( 0 \) if \( \tau > 0 \) or \( \tau < 0 \), respectively. In (4)-(7) \( \tau_0 \) is positive real for \( |\eta| < 1 \), \( \tau_0^* \) is negative for \( |\eta| > 1 \), and it is convenient to define the branch of the root in (7) as \( \tau_0 = -j|\tau_0| \) (this is in accord with the root of \( k_{pq} \), since \( \sqrt{k^2(1-\eta^2)} = k_{pq} \) when \( \eta = 0 \)).

A. TD → FD

In order to verify (4) with (3) directly, we derive the FD-FW by Fourier inversion of its TD counterpart. Using in the first integral in (3), a frequency shift \( \omega' = \omega - \eta z_0 \) where the \( q \)-dependent \( z_0 \) defined in (7) is a function of the nondimensional parameter \( \eta \), the first integral in (3) is rewritten as

\[
I(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{f(t)}{\sqrt{f^2 - \tau_0^2}} e^{-j\omega t}.
\]

Depending on \( |\eta| \gtrless 1 \), two different procedures are applied. Case \( |\eta| < 1 \): Inserting the expression for \( f(t) \) from (6), and noting that the unit step function \( U(t - \tau_0) \) truncates the domain of integration in (8) we have

\[
I(\omega) = \frac{2}{\pi} \int_{\tau_0}^{\infty} dt \frac{\cos(z_0 \sqrt{f^2 - \tau_0^2})}{\sqrt{f^2 - \tau_0^2}} e^{-j\omega t}.
\]
Rewriting $\exp(-j\omega t) = \cos(\omega t) - j\sin(\omega t)$ yields the two tabulated integrals when $\omega' > 0$, [2, pp. 28,86]

\[
\int_{-\infty}^{\infty} \frac{d\tau' \cos(\omega' \tau')}{\sqrt{\tau'^2 - \tau_0^2}} = \left\{ \begin{array}{ll}
K_0(\tau_0 \sqrt{\omega^2 - \omega'^2}) & \text{if } 0 < \omega' < |\omega_0|,

\frac{\omega_0}{\sqrt{\tau'^2 - \tau_0^2}} & \text{if } |\omega_0| < \omega'.
\end{array} \right.
\]

(10)

\[
\int_{-\infty}^{\infty} \frac{d\tau' \sin(\omega' \tau')}{\sqrt{\tau'^2 - \tau_0^2}} = \left\{ \begin{array}{ll}
0, & \text{if } \frac{\omega_0}{\tau_0} < |\omega_0|,

\frac{\omega_0}{\sqrt{\tau'^2 - \tau_0^2}} & \text{if } |\omega_0| \leq \omega'.
\end{array} \right.
\]

(11)

where, in both (10) and (11), the upper equality apply when $0 < \omega' < |\omega_0|$, while the lower applies when $|\omega_0| < \omega'$. Also, $J_0$ and $Y_0$ are the zero-order Bessel functions of the first and second kind, respectively, and $K_0$ is the modified Bessel function [3, p. 358]. Using the relations $K_0(\zeta) = -j(\pi/2)H_0^{(2)}(\zeta)$ and $Y_0(\zeta) + jJ_0(\zeta) = jH_0^{(2)}(\zeta)$, (10) and (11) can be combined and (9) becomes

\[
I(\omega) = \frac{1}{j} \left\{ H_0^{(2)}(\omega') \left( -j\tau_0 \sqrt{\omega^2 - \omega'^2} \right) + \frac{\omega_0^2}{\tau_0^2} \left( \tau_0 \sqrt{\omega^2 - \omega'^2} \right) \right\},
\]

(12)

Substituting the value of $\omega' = \omega - \eta \sigma_0$ and $\sigma_0$ from (7), we have

\[
\omega^2 - \omega_0^2 = \omega^2 - 2\omega \eta \sigma_0 + \frac{\omega_0^2}{\eta^2} = \frac{\omega^2 - \omega_0^2}{\eta^2} (k^2 - k_0^2).
\]

(13)

Recalling that $\tau_0 = \rho \sqrt{1 - \eta^2}/c$, the arguments of the Hankel functions in (12) are re-expressed as follows:

\[
k_\rho \rho = \left\{ \begin{array}{ll}
-\rho \sqrt{k_0^2 - k^2} = -\rho \tau_0 \sqrt{\omega^2 - \omega'^2}, & \text{if } 0 < \omega' < |\omega_0|,

\rho \sqrt{k_0^2 - k^2}, & \text{if } |\omega_0| < \omega'.
\end{array} \right.
\]

(14)

where we have chosen the branch of the square root so that for $k_\rho(\omega)$ in (5) [defined after (7)] thus establishing the result in (4). For $\omega' < 0$, the same considerations apply, after noting that (10) is independent of the sign of $\omega'$ while in (11), the substitution $\omega' = -|\omega'|$ relates the result to the one for $\omega' > 0$.

Case $|\eta| > 1$: We recall that in this case we have $\tau_0^2 = -|\tau_0|^2$, with $|\tau_0| = \rho \sqrt{\eta^2 - 1}/c$, in (6) and (8), resulting in

\[
I(\omega) = \int_{-\infty}^{\infty} dt e^{-j \omega t} \frac{e^{j \pi n (\eta |\tau_0|)\sqrt{\omega^2 + |\tau_0|^2}}}{\pi \sqrt{\omega^2 + |\tau_0|^2}}.
\]

(15)

This transform is given in [4, pp. 493] as

\[
I(\omega) = \frac{1}{j} H_0^{(2)}(0) \left( \tau_0 \sqrt{\omega^2 - \omega'^2} \right).
\]

(16)

Using the equality in (13), substituting for $|\tau_0|$ and choosing the branch of the square root in (16) as that for $k_\rho$, we again obtain the FD part in (4).

B. $FD \rightarrow TD$

Using in the second integral in (3) with (4), a frequency shift $\omega' = \omega - \eta \sigma_0$, with the $q$-dependent $\sigma_0$ defined in (7), leads to

\[
\tilde{I}(t) = \frac{e^{j \omega q}}{2\pi j} \int_{-\infty}^{\infty} d\omega' e^{j \omega' t} H_0^{(2)} \left( \frac{\sqrt{(\omega'^2 - \sigma_0^2)(1 - \eta^2)}}{\eta^2} \right)
\]

(17)

in which we have used (13) for $k_\rho$ in the argument of the Hankel function. The integrand has branch points at $\omega' = \pm\sigma_0$, shown in Fig. 1. Due to the frequency shift $\omega' = \omega - \eta \sigma_0$, the vertical dashed line at $\omega = -\eta \sigma_0 (\omega = 0)$ separates positive and negative $\omega'$ frequencies (here $\eta \sigma_0 > 0$ for simplicity). The dashed region denotes the side of the cuts where $\Re(\omega^2 - \sigma_0^2)^{1/2} > 0$, according to the choice of the root for $k_\rho$ in the text after (6). In order to FT the outgoing/decaying FD function, the integration path is moved to the real axis and indented accordingly with respect to the singularities.

Case $|\eta| < 1$: Using the definition $\tau_0 = \rho \sqrt{1 - \eta^2}/c$, (17) is rewritten as

\[
\tilde{I}(t) = \frac{e^{j \omega q}}{2\pi j} \int_{-\infty}^{\infty} d\omega' e^{j \omega' t} H_0^{(2)} \left( \frac{\eta \sqrt{\omega'^2 - \sigma_0^2}}{\eta^2} \right)
\]

(18)

in which the square root is defined as $3\pi n \sqrt{\omega^2 - \sigma_0^2} < 0$ and $\Re(\omega^2 - \sigma_0^2) \geq 0$ or $\leq 0$ for $\omega > 0$ or $\omega < 0$, respectively, in accord with that for $k_\rho$ in the text after (7). Separation of positive and negative $\omega'$ occurs at $\omega' = -\eta \sigma_0$ between the two branch points. In order to Fourier-invert the FD function that satisfies the radiation condition at $\infty$ [see text after (7)] for any $\omega'$, the integration path from $-\infty$ to $+\infty$ is shifted below the branch cuts [see Fig. 1(a)] where the sign of the square root is in accord with the radiation condition at $\infty$. This choice is also in agreement with [4, pp. 35] where to ensure the existence of the Fourier pair in (3) the $\omega'$ variable in (3) and, therefore, the $\omega'$ contour of integration in (18), is shifted slightly from the real $\omega'$ axis into $3\pi n \sqrt{\omega^2 - \sigma_0^2} < 0$.

Using the large argument asymptotic approximation $H_0^{(2)}(\zeta) \sim (2/\pi \zeta)^{1/4} \exp(-j \zeta - j \pi/4)$, it is easy to see that for $t < \tau_0$ the integrand decays exponentially for $3\pi n \omega' < 0$. Therefore, for $t < \tau_0$, the integration contour can be deformed onto $P_{-\infty}$ where the integral vanishes by Jordan’s lemma, and since no singularities are included in the deformation, the integral in (18) vanishes by Cauchy’s theorem. For $t > \tau_0$, the integration contour is deformed onto $P_1 + P_2 + P_\infty$, with the integral on $P_\infty$ vanishing. The integration of the even part of the integrand on the symmetric integration path $P_1 + P_2$ vanishes.
The integration of the odd part on $P_1$ is equal to the contribution from $P_2$, and $\hat{I}(t)$ in (18) can be evaluated as twice the integral on $P_2$

$$\hat{I}(t) = \frac{e^{\sqrt{\omega^2 - \omega'^2}}}{\pi} \int_{\omega_1}^{\omega_2} d\omega' \sin(\omega't) H_0^{(2)} \left( \tau_0 \sqrt{\omega'^2 - \omega'^2} \right).$$

(19)

Since on the upper and lower side of the branch cut the square root assumes opposite negative/positive values, then using the relation $H_0^{(2)}(\xi e^{-i\pi}) = -H_0^{(1)}(\xi)$, reversing the integration path above the real $\omega'$ axis, and combining $H_0^{(2)}(\xi) + H_0^{(1)}(\xi) = 2J_0(\xi)$, leads to

$$\hat{I}(t) = e^{\sqrt{\omega^2 - \omega'^2}} \frac{2}{\pi} \int_{\omega_1}^{\omega_2} d\omega' e^{i\omega't} J_0 \left( \tau_0 \sqrt{\omega'^2 - \omega'^2} \right)$$

(20)

in which the square root assumes positive values. This sine transform is given in [2, pp. 113], yielding directly the right-hand side of (4).

Case $|\eta| > 1$: Using the definition $|\tau_0| = \rho \sqrt{\eta^2 - 1}/c$, (17) is rewritten as

$$\hat{I}(t) = \frac{e^{\sqrt{\omega^2 - \omega'^2}}}{2\pi \rho} \int_{-\infty}^{\infty} d\omega' e^{i\omega't} H_0^{(2)} \left( |\tau_0| \sqrt{\omega'^2 - \omega'^2} \right).$$

(21)

Since $|\tau_0| > 0$, the square root in (21) is still defined as $3\rho \sqrt{\omega'^2 - \omega'^2} < 0$, and $\Re \sqrt{\omega'^2 - \omega'^2} \geq 0$ or $\leq 0$ for $\omega > 0$ or $\omega < 0$, respectively. Branch points are still located at $\omega' = \pm \omega_0$, with the only difference that the branch cuts are now located as in Fig. 1(b). Note that now the separation of positive and negative $\omega$ frequencies at $\omega' = \pm \omega_0$ occurs outside the branch point region, as shown in Fig. 1(b), for the case $\eta \omega_0 > 0$; similarly, at $\omega' = \pm \omega_0$ for the case $\eta \omega_0 < 0$. Moreover, from Fig. 1(b), the whole region $-|\omega_0| < \omega' < |\omega_0|$ corresponds to $\omega > 0$. Therefore, in order to Fourier-invert the FD function that satisfies the radiation condition at $\infty$ [see text after (7)] for any $\omega$, the integration path from $-\infty$ to $+\infty$ is indented between the cuts as in Fig. 1(b), where $\Re \sqrt{\omega'^2 - \omega'^2} \geq 0$ (for $\eta \omega_0 > 0$). In the case of $\eta \omega_0 < 0$, the whole region $-|\omega_0| < \omega' < |\omega_0|$ corresponds to $\omega < 0$, and the integration path is still defined in between the cuts, where now $\Re \sqrt{\omega'^2 - \omega'^2} \leq 0$. Since for any deformation a branch point singularity is included, the integral is nonvanishing for any value of $t \gtrsim |\tau_0|$, and its evaluation is given in [4, pp. 481] which leads directly to the right-hand side of (4).

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