We demonstrate non-equilibrium scaling laws for the aging dynamics in glass formers that emerge from combining two recent theoretical approaches to describe the dynamics of glass-forming fluids. The non-equilibrium self-consistent generalized Langevin equation (NE-SCGLE) provides a starting point linking the waiting-time evolution of static properties to the relaxation dynamics of the system, while a recent extension of MCT, the stochastic β-relaxation theory (SBR) provides scaling laws for this relaxation dynamics that also include the effect of non-mean-field fluctuations in the ideal glass of MCT. We demonstrate the scaling laws by a quantitative comparison to recent computer-simulation results for the evolution following density-quenches in hard-sphere-like systems that elucidate three different scaling regimes predicted by the theory.

MCT is a microscopic theory that very successfully describes the liquid-state dynamics close to the glass transition. In its original form it is restricted to the equilibrium ensemble, although recent extensions allow to treat nonlinear response to various external fields. Its application to aging dynamics has been proposed 20 years ago by Latz, but the complexity of that theory has so far only allowed to obtain some results linked to the seminal work by Cugliandolo et al. on the p-spin model. The complexity stems from the fact that in absence of the equilibrium fluctuation-dissipation theorem (FDT), correlation and response functions are not straightforwardly connected, and are described by coupled integral equations that are not readily evaluated.

To cut this Gordian knot, the NE-SCGLE invokes an assumption of “local stationarity” for the relaxation process, reducing the complexity of the full problem considerably. Essentially, it partially decouples the
evolution of the correlation functions from that of the underly-
ing static response functions. The resulting theory tests favorably against both simulation [28, 29, 41] and experi-
mental data [45, 47].

NE-SCGLE in fact refers to two separate ingredients: an evolution equation for the static observable, and an underly-
ing kinetic theory for the mobility of rearrange-
ments, the SCGLE [48]. The latter is, for the present purposes, structurally identical to MCT. In particular, it provides the same asymptotic scaling laws for the equi-
librium structural relaxation [49]. We will use those well-established scaling laws to describe the asymptotic waiting-time dependence after a quench.

The non-equilibrium extension of the SCGLE is usu-
ally derived by referencing Onsager’s laws of linear irre-
versible thermodynamics and the corresponding stochas-
tic theory of thermal fluctuations (see Refs. [50, 51]). Un-
der certain assumptions, it leads to an innocuous look-
ing relaxation equation for the waiting-time evolution of the non-equilibrium static structure factor \( S(k; t_w) \). We demonstrate that this equation can also be rationalized in a spirit closer to MCT employing the integration-
through transients (ITT) formalism [32]: writing the evolu-
tion equation of the non-equilibrium distribution function \( p(t) \) of a system as \( \partial_t p(t) = \Omega(t)p(t) \), with some linear differential operator \( \Omega(t) \), a formal solution is \( p(t) - p(t_w) = \int_0^t \frac{dt'}{P_2} \exp \left[ \int_{t'}^t \Omega(r) dr \right] P_2 \Omega(t')p_{t_w}(t) \) where \( P_2 = 1 \) is the identity operator. Note that \( S(k; t_w) = (\langle \delta_{-k} \rho_k \rangle, \rho_{-k} p(t_w) \rangle \) where \( \rho_k \) are the microscopic number-density fluctuations and \( f, g \) is the usual \( L_2 \) scalar product in Hilbert space. For a sudden quench, \( \Omega(t) = \Omega_{f} \) for \( t < 0 \) and \( \Omega(t) = \Omega_{f} \) for \( t > 0 \), we can make use of the relation \( \Omega(t')p(t_w) = \partial_{t_w} p(t_w) \) for all \( t' \geq t_w > 0 \), which avoids the need to formulate the effect of the quench in the time-evolution operator ex-
plicitly. Projecting onto density-pair modes as the re-
levant variables, \( P_2 = \langle \delta_{-k} \delta_{-k} \rho_k \rho_{-k} \rangle \rangle^{-1} \langle \delta_{-k} \delta_{-k} \rho_k \rho_{-k} \rangle \) (suitably normalized), and neglecting memory effects, we obtain \( S(k; t) - S(k; t_w) \approx \int_0^{t_w} \frac{dt'}{C_4(k; t, t')} \delta_{t_w} S(k; t_w) \) with some four-point density correlation function \( C_4(k; t, t') \), and thus for \( t \rightarrow \infty \),

\[
\frac{\partial S(k; t_w)}{\partial t_w} = -\mu(k; t_w) (S(k; t_w) - S_f(k)),
\]

where \( \mu(k; t_w) \) is a mobility factor that is slaved to the structural relaxation dynamics [32, 42]. The initial state before the quench is \( S(k; 0) = S_i(k) \), and \( S_f(k) \) characterizes the quenched-to-final state. Equation 1 es-
entially is a formalized extension of the empirical Tool-
narayanaswamy model of physical aging [43].

Equation 1 already predicts universal scaling laws for the aging dynamics to be encoded in the equilibrium dynamics: since the glass transition is a dynamical phe-
nomenon, in its vicinity the static structure functions remain regular, and we can linearize \( S(k; t_w) \) for small control-parameter distances \( \varepsilon(t_w) \) to the transition. The temporal evolution is thus asymptotically governed by the evolution of the distance parameter along the relevant direction in \( k \)-space (MCT’s critical eigenvector [30, 44],

\[
\partial_{t_w} \varepsilon(t_w) = -\mu(\varepsilon(t_w)) (\varepsilon(t_w) - \varepsilon_f).
\]

Now enter the scaling laws for \( \mu(\varepsilon) \): close to the critical point of MCT, \( \mu(\varepsilon) \sim 1/\tau(\varepsilon) \sim (|\varepsilon|) \gamma \) for liquid states \( (\varepsilon < 0) \), and \( \mu(\varepsilon) = 0 \) in the ideal-glass state \( (\varepsilon \geq 0) \). The non-trivial exponent \( \gamma \) is related to the equilibrium structure of the system at its glass transition through the MCT exponent parameter \( \lambda [30, 49] \). The fact that \( \mu \) approaches zero, allows for non-equilibrium stationary solutions of Eq. 2, where the relaxation towards equi-
brilibrium gets “stuck”.

We immediately get two important scaling laws from Eq. 2: (i) for quenches close to the glass-transition point \( (|\varepsilon_f| \ll |\varepsilon_i|) \), there exists a growing window in \( t_w \), where \( \partial_{t_w} \varepsilon \sim |\varepsilon|^{\gamma - 1} \), which results in \( |\varepsilon| \sim t_w^{1/\gamma} \) and, thus, simple or full aging, \( \tau \sim t_w \rightarrow \infty \).

(ii) for a deep quench into the ideal glass, \( \varepsilon_f \gg |\varepsilon(t_w)| \) holds in the limit of \( t_w \rightarrow \infty \), because the relaxation gets stuck around values close to zero. Then, \( \partial_{t_w} \varepsilon \sim |\varepsilon| \gamma \), resulting in the asymptotic law \( \tau \sim t_w^{\gamma/(\gamma - 1)} \). Since \( \gamma > 1 \), the exponent \( \delta = \gamma/(\gamma - 1) \) is also larger than unity, and we find hyper-aging or super-aging, \( \tau \sim t_w^{\delta} \) for \( t_w \rightarrow \infty \).

These scaling laws describe the idealized indefinite aging of a system that is quenched to a state with infinite relaxation time. In reality, the ultimate MCT-like diver-
gence of the relaxation time is not observed; this one can attribute to long-wavelength fluctuations that cause deviations from the mean-field like scenario [55, 57]. It will provide a cut-off for the scaling laws, rendering them transient rather than truly infinite-waiting-time asympto-
totes, as we shall discuss below.

For quenches to liquid states close to the glass transi-
ton, \( \varepsilon_f < 0 \), the mobility always remains positive, and the corresponding long-time asymptote is then (iii) \( \tau \sim \text{const. for } t_w \rightarrow \infty \). For the typical slow evolution of the structural relaxation time, this implies a broad cross-over where \( \tau \) grows sublinearly with \( t_w \), and hence sub-aging. Although not a rigorous asymptote, an empir-
ical power law, \( \tau \approx t_w^{\delta} \) with \( \delta < 1 \), typically fits well in this regime [40].

To elucidate the emergence of the three regimes – sim-
ple, sub- and hyper-aging – we devise a schematic model of aging. Qualitatively, the mobility is the inverse of an integrated friction memory kernel; in the spirit of MCT schematic models, we assume that the slow dynamics of all such microscopic correlation functions is governed by a single-mode (density) correlation function \( \phi(t; t_w) \),

\[
\mu(t_w) = 1 \int_0^\infty dt \phi(t; t_w).
\]
The latter obeys a Mori-Zwanzig type integral equation,
\[ \partial_t \phi(t; t_w) + \phi(t; t_w) + \int_0^t m(t-t'); t_w) \partial_{t'} \phi(t'; t_w) dt' = 0. \] (3b)

In Eq. (3b) we anticipate that \( t_w \) only enters parametrically in determining the coupling coefficients of the memory kernel \( m(t; t_w) \). This encodes the assumption of local stationarity, and is in the spirit of the ITT framework \(^3\) that relates non-equilibrium transport coefficients to such “transient” correlation functions.

We complete the schematic model by the closure
\[ m(t; t_w) = v_1(t_w) \phi(t; t_w) + v_2(t_w) \phi(t; t_w)^2, \] (3c)

with two coupling parameters \( v_1 \) and \( v_2 \) that describe the current \( t_w \)-dependent state of the system. For fixed \( t_w \), the model specified by Eqs. (3b) and (3c) is the widely studied schematic F\(_{12} \) model of MCT. It has a line of glass transitions \( (v_1', v_2') \) where \( \epsilon = 0 \).

Equations (3) define our schematic model. Together with the (mean-field) assumption \( \tau(t_w) \propto 1/\mu(t_w) \), and \( v_1 = v_1', v_2(t_w) = v_2'(1 + \epsilon(t_w)) \) to define the distance to the glass transition point, it allows to fit available computer-simulation data for \( \tau(t_w) \) after mapping \( \epsilon_i = \epsilon(0) \) and \( \epsilon_f \) to the simulation’s control parameters.

Results for \( \tau(t_w) \) from the schematic model for quenches to various final states close to the MCT transition give a consistent description of computer-simulation data for density-quenched quasi-hard spheres (Fig. 1).

FIG. 1. Structural relaxation time \( \tau \) as a function of waiting time \( t_w \) after an instantaneous quench. Solid lines: schematic model, quenches from \( \epsilon_i = -0.5 \) to \( \epsilon_f = -0.34, -0.22, -0.12, -0.07, -0.04, -0.02, -0.01, -0.007, 0.01, \) and 0.02 (bottom to top). A dashed line indicates simple aging, \( \tau \sim t_w \), a dotted line hyper-aging, \( \tau \sim t_w^\delta \) with \( \delta = 1.684 \), and a dash-dotted line sub-aging, \( \tau \approx t_w^\delta \), with \( \delta = 0.9 \). Thick dashed lines: stochastic \( \beta \)-relaxation theory (SBR) for \( \epsilon = 0.01 \) and 0.02. Symbols: simulation results for quasi-hard spheres from Ref. \(^2\), quenched to various final packing fractions \( \epsilon_f \) (related to \( \epsilon_f \) as shown in the inset), translated to schematic-model units (\( \tau \rightarrow 2\tau, t_w \rightarrow 100t_w \)).

For the fit, we have allowed to adjust a global time scale and the proportionality factor between \( \mu \) and \( 1/\tau \), and we have chosen a transition point \( (v_1', v_2') \) such that the exponent parameter of MCT matches a value usually found for hard-sphere like systems, \( \lambda = 0.735 \). This determines the exponent \( \gamma = 1/2a + 1/2b \) with \( \Gamma(1-a)^2/\Gamma(1-2a) = \lambda = \Gamma(1+b)^2/\Gamma(1+2b) \), and thus the exponent \( \delta' \approx 1.684 \).

The schematic model elucidates the three aging regimes of the ideal-glass theory: empirical sub-aging is found as a cross-over for quenches to final states in the liquid, \( \epsilon_f < 0 \), while hyper-aging emerges from the model as the asymptote for quenches to the glass, \( \epsilon_f > 0 \). A growing intermediate-\( t_w \) window that extends to \( t_w \rightarrow \infty \) at the critical point of MCT, \( \epsilon_f = 0 \), displays simple aging.

The evolution of \( \tau \) after the quench relates to the well-known problem of determining a diverging relaxation time at fixed waiting time \( t_w \) (corresponding to a typical experiment duration or probing time scale): approaching the transition, the power-law divergence of \( \tau \) as a function of quenched-to-state \( \epsilon_f \) that is predicted by the idealized theory, is cut off at any finite \( t_w \), and replaced by a cross-over to a slower growth (Fig. 2). In our model, we obtain \( \tau \sim |\epsilon|^\delta' \), with a prefactor that diverges with increasing \( t_w \) (dash-dotted lines in Fig. 2).

Divergences from the ideal theory are noted in the simulation data for quenches to the highest final densities and at large \( t_w \). We attribute this to the avoidance of the ideal MCT transition, that also causes the hyper-aging regime to be interrupted.

To understand this, we turn to the SBR \(^3\) \(^4\), a recent extension of MCT that includes fluctuations in the local glassiness, viewing \( \sigma \sim \epsilon \) as a dynamical fluctuating order parameter. SBR predicts scaling laws that re-
The results link the hyper-aging exponent $\delta'$ to the exponent characterizing the equilibrium relaxation time. Hence, they link a non-equilibrium dynamical exponent of the system to a non-trivial equilibrium exponent, and through this to the equilibrium static structure of the system. Sub-aging on the other hand, emerges only as an effective cross-over, i.e., as a finite-$t_w$ deviation from the mathematically rigorous simple-aging asymptote.

Interrupted hyper-aging versus sub-aging emerges as a clear indicator of the separation between ideal-glass like dynamics, and the dynamics that arises from the avoidance of the ideal glass transition. It could in principle be used to determine more precisely the position of the ideal glass transition.

This separation leads us to speculate that models with a non-avoided MCT-like glass transition might show clear hyper-aging asymptotes. High-dimensional systems of hard spheres, approaching the expected mean-field-like behavior in $d = \infty$ \cite{58, 59}, might be suitable candidates. On the other hand, in the context of spin glasses with MCT transitions, e.g., the spherical $p$-spin model, numerical solutions so far favor sub- and normal aging \cite{41, 40, 61}. But the analytical determination of the scaling laws is still a critical open issue \cite{40, 61}. Hyper-aging in a trapped phase has been discussed very recently in the context of decision-making models that incorporate reinforcement by memory effects \cite{62}. Our Eq. (2) predicts weak ergodicity breaking and aging that gets stuck at the MCT-critical point; it will be interesting to explore the connection to the strong ergodicity breaking discussed in spin glasses \cite{63} and the loss of ultrametricity connected with the hyper-aging asymptote in suitably enhanced models. We thank L. Berthier, M. Fuchs, and G. Szamel for their valuable comments, and A. Meyer and M. Medina-Noyola for continued support. Part of this work has benefited from discussions at the CECAM Flagship Workshop “Memory Effects in Dynamical Processes” of the Erwin-Schrödinger Institut (ESI) in Vienna. Th.V. also thanks the Glass & Time group at Roskilde University and specifically Jeppe Dyre for their kind hospitality during a research visit where this manuscript was finalized.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Ratio $\tau/t_w$ as a function of waiting time $t_w$. Simulation data of Ref. \cite{29} (filled symbols; data divided by 50; $\varphi_c \approx 0.585$) and of Ref. \cite{28} (open symbols); and on a more polydisperse system from Ref. \cite{14} (grey; $\varphi_c \approx 0.605$). Exemplary SBR results are shown for $\varepsilon_f = -0.1, 0$, and 0.1 (solid lines; bottom to top). Dashed lines indicate the corresponding asymptotic expressions for the ideal-glass MCT.}
\end{figure}

place the divergent power law with a cross-over between a power law on the liquid side and exponential growth on the glassy side of the transition. Specifically \cite{58}, for the structural relaxation time

$$\tau \sim \left[ \int_{-\infty}^{0} \frac{ds}{2\pi \Delta \sigma} e^{-\frac{(s-s_0)^2}{2\Delta \sigma^2}} |s|^b \right]^{-1/b} \quad (4a)$$

and for the mobility

$$\mu \sim \left[ \int_{-\infty}^{0} \frac{ds}{2\pi \Delta \sigma} e^{-\frac{(s-s_0)^2}{2\Delta \sigma^2}} |s|^7 \right]^{1/b} \quad (4b)$$

where we have identified $\sigma = \varepsilon$. Here, $\Delta \sigma$ is a material parameter that quantifies the strength of long-wavelength order-parameter fluctuations. Using Eqs. (4) to evaluate $\mu$ in Eq. (2) and to calculate $\tau$, we obtain an improved asymptotic description of the $\tau$-vs-$t_w$ curves (colored dashed lines in Fig. 3) that account for the cross-over from hyper-aging to a constant $\tau$ as the system finally equilibrates even in the ideal-MCT glass.

Interestingly, the hyper-aging law predicted by the ideal theory still survives as a transient. In the simulation data, this is best seen as a non-monotonic variation of the ratio $\tau/t_w$ as a function of $t_w$ that is present for all quenches to $\varphi_f > \varphi_c$ (Fig. 3). This transient hyper-aging signature fits well the corresponding SBR prediction (solid lines in Fig. 3).

In conclusion, we present scaling laws for the evolution of the structural relaxation time $\tau$ as a function of system age $t_w$ after the quench of a glass-forming fluid to states close to the ideal glass-transition point of MCT. Based on the NE-SCGLE to describe the evolution of static quantities after such quenches, the scaling laws delineate regimes of simple and transient hyper- and sub-aging.

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