Background driving distribution functions and series representations for log-gamma selfdecomposable random variables

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Abstract. We identify the background driving distribution functions (BDDF) for selfdecomposable distributions (random variables). For log-gamma variables and their background driving variables we find their series representations. An innovation variable for Bessel-K distribution is given as a compound Poisson variable.

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0. INTRODUCTION.

The Lévy class, \( L \), of selfdecomposable probability distributions (or random variables or characteristic functions) is obtained as the limit of a linear normalization of partial sums of sequences of independent random variables. It constitutes a proper subclass of the class, \( ID \), of all infinitely divisible distributions; see the books by Gnedenko and Kolmogorov (1954), Sections 29-30, Feller (1966), Chapter XVII or Loéve (1963), Section 23. More recently in Bradley and Jurek (2016) it was proved that \( L \) can be obtained by assuming only that we have strongly mixing sequences, keeping in mind that sequences of independent random variables are strongly mixing ones.

\( L \) is a very large class and includes among others: stable, gamma, log-gamma, \( \chi^2 \), Student-t, \( \log |t| \), Snedecor F, hyperbolic variables and many other probability measures; cf. for instance Jurek (1997) and references therein. Moreover, the Thorin class, EGGC, of extended generalized gamma convolutions, is a subset of the class \( L \); cf. Bondesson (1992), Chapter 7.

The class \( L \) has the following two equivalent characterizations. Namely, a selfdecomposable random variable \( X \) admits:

(a) a random integral representation

\[
X := \int_0^\infty e^{-s}dY(s) \quad \text{with} \quad \mathbb{E}[1 + \log |Y(1)|] < \infty, \tag{1}
\]

with respect to some (uniquely determined) Lévy process \( Y(t), t \geq 0 \); and,

(b) a decomposition property

\[
\forall (0 < c < 1) \exists (X_c \perp X) \quad X \overset{d}{=} cX + X_c, \tag{2}
\]

where \( \perp \) means independence, and \( X_c \) is referred to as an innovation in the context of autoregressive sequences. Cf. Jurek and Vervaat (1983) or Jurek and Mason (1993), Chapter 3.

Terminology. We refer to the process \( (Y(t), t \geq 0) \) in (1) as the background driving Lévy process (BDLP) of \( X \), to \( Y(1) \) – as the background driving random variable (BDRV) of \( X \), and to the probability distribution function \( G_X(a) := P(Y(1) \leq a), a \in \mathbb{R} \), as the background driving distribution function (BDDF) of \( X \). Finally, the characteristic function \( \psi \) of the random variable \( Y(1) \) is called the background driving characteristic function (BDCF) of the characteristic function \( \phi \) of \( X \).

Remark 1. a) In Jurek and Vervaat (1983), pp. 252-253, using the remainders \( \{X_c : 0 \leq c \leq 1\} \) from (2), a cadlag process \( Z(t), 0 \leq t < \infty \), with
independent increments was constructed such that \( Y(t) := \int_{[0,t]} e^{s}dZ(s) \) is the BDLP of \( X \) from (1).

b) In Jeanblanc, Pitman and Yor (2002), Theorem 1, identified the BDLP of \( X \) as \( Y(t) := \int_{1}^{t} e^{-r}dV(r) \), \( t \geq 0 \), \( V(1) := X \) (\( X \) is from (1)), for an additive 1-self-similar process \( V(r), r \geq 0 \). In other words, selfdecomposable \( X \) can be inserted into an additive 1-similar process. In fact, we could have any \( H \)-self-similar process. [One should keep in mind that the process \( V(r) \) from above is denoted as \( X_r \) in Jeanblanc, Pitman and Yor (2002).]

For the identification of BDDFs \( G_X(a) := P(Y(1) \leq a) \), in principle, we may use \( Y(1) \) from Remark 1 a) or b). However, in this note we will utilize the characteristic functions \( \phi \) of \( X \) and \( \psi \) of \( Y(1) \), respectively, cf. Proposition 1, below. The proposed method of finding BDDF is illustrated by some explicit examples; (cf. Lemma 1 and Section III).

Finally, for log-gamma and Bessel-K variables, the corresponding BDDFs and innovation variables are described as infinite series of some exponential variables (Propositions 2 - 4 and Corollary 5). These potentially may help in numerical simulations of selfdecomposable variables; cf. discussions in Hosseini (2019) and some references therein. In Corollary 4 there is a formula for the error function.

I. THE RESULTS.

I. 1. Background driving distribution functions (BDDF).

Here is the BDDF formula for the selfdecomposable variable \( X \) with the characteristic function \( \phi \).

**Proposition 1.** Let a selfdecomposable \( X \) have the integral representation (1) and \( \phi(t) := E[e^{itX}] \) be its characteristic function. Then \( \phi \) is differentiable \((t \neq 0)\) and the function

\[
G_X(a) := P(Y(1) \leq a) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \Re(\exp[-ita + t\phi'(t) / \phi(t)]) \frac{dt}{t}, \quad a \in C_G,
\]

is the BDDF of \( X \). [Above \( C_G \) denotes the continuity points of the function \( G_X \) and \( \Re \) stands for imaginary part of \( z \).]

In the case of symmetric selfdecomposable variables we get the following:

**Corollary 1.** Suppose that a selfdecomposable random variable \( X \) is symmetric random variable, then its BDDF is given by

\[
G_X(a) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \exp[t\phi'(t) / \phi(t)] \sin(ta) \frac{dt}{t}, \quad a \in C_G.
\]
Using b) from Remark 1 and the relation between the characteristic function of a selfdecomposable variable and its background driving characteristic function (see (10) below) we have

$$
E[\exp(it\int_{e^{-1}}^{1}r^{-1}dV(r))] = \exp(t(\log \phi_{V(1)}(t))^\prime), \quad t \in \mathbb{R}.
$$

As an application of Proposition 1 we find three expressions for the BDDF of the gamma $\gamma_{\alpha,\lambda}$ distribution, i.e., the probability distribution with the density $\lambda^\alpha/\Gamma(\alpha)x^{\alpha-1}e^{-\lambda x}1_{(0,\infty)}(x)$, $(\alpha > 0, \lambda > 0)$. Recall that,

$$
\phi_{\gamma_{\alpha,\lambda}}(t) = E[e^{it\gamma_{\alpha,\lambda}}] = (1 - it/\lambda)^{-\alpha} = \exp[\alpha \int_{0}^{\infty} \left(e^{itx} - 1\right) \frac{e^{-\lambda x}}{x} dx]
$$

$$
= \exp \int_{0}^{1} \alpha \left[ \int_{0}^{\infty} \left(e^{itxv} - 1\right) \lambda e^{-\lambda x} dx \right] \frac{dv}{v}, \quad (\phi_{\gamma_{\alpha,\lambda}} \text{ in terms of its BDCF})
$$

where the last equality follows from the random integral representation of the gamma distribution; cf. Jurek (1996), Corollary 1 and Remark 1.

**Lemma 1.** For a gamma variable $\gamma_{\alpha,\lambda}$ its BDRV equals $Y(1) = \sum_{k=1}^{N_{\alpha}} E_{k}(\lambda)$, where $N_{\alpha}$, $E_{k}(\lambda), k = 1, 2, ...$ are independent, $N_{\alpha}$ is Poisson variable with a parameter $\alpha$ and $E_{k}(\lambda)$ are exponential identically distributed with parameter $\lambda$. Moreover, the BDDF for the selfdecomposable $X = \gamma_{\alpha,\lambda}$ is given as follows:

$$
G_{X}(a) = P(\sum_{k=1}^{N_{\alpha}} E_{k}(\lambda) \leq a)
$$

$$
= \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \exp(-\frac{\alpha (t/\lambda)^2}{1+(t/\lambda)^2}) \sin (ta - \frac{\alpha (t/\lambda)}{1+(t/\lambda)^2}) \frac{1}{t} dt
$$

$$
= e^{-\alpha} + e^{-\alpha} \int_{0}^{2\sqrt{\alpha t}} I_{1}(w)e^{-w^2/4\alpha} dw, \quad a \in C_{G_{X}} \cap (0, \infty);
$$

where $I_{1}(x)$ is a Bessel function; cf. Appendix c) below.

More examples illustrating Proposition 1 are given in Section III, below.

**I. 2. Series representations for log-gamma variables.**

From Shanbhag, Pestana and Sreehari (1977), Remark 1, p.291, we know that log $\gamma_{\alpha,\lambda}$ random variables are selfdecomposable, so they have the representation (1). However, they also admit the following series representation:
Proposition 2. Let $C$ be the Euler constant and $\gamma_{1,a+n}, n = 0, 1, 2, ...$ be a sequence of independent exponential random variables with scale parameters $\alpha + n$. Then the series

$$S := -\log \lambda - C - \gamma_{1,a} - \sum_{n=1}^{\infty} (\gamma_{1,a+n} - 1/n) = -\log \lambda + \Psi(\alpha) - \sum_{n=0}^{\infty} (\gamma_{1,a+n} - 1/(\alpha + n)),$$  \hspace{1cm} (8)

converges almost surely (in probability, in distribution, in $L^2$). Moreover, $S \overset{d}{=} \log \gamma_{a,\lambda}$ and $\phi_{\log \gamma_{a,\lambda}}(t) = \lambda^{-it} \frac{\Gamma(a+it)}{\Gamma(a)}$. (\(\Gamma\) denotes the Gamma function).

From the above series representation we conclude

Corollary 2. For log-gamma variables we have

$$\mathbb{E}[\log \gamma_{a,\lambda}] = -\log \lambda + \Psi^{(0)}(\alpha); \ \text{Var}[\log \gamma_{a,\lambda}] = \Psi^{(1)}(\alpha) = \int_{0}^{\infty} \frac{x e^{-\alpha x}}{1 - e^{-x}} \, dx.$$

($\Psi^{(0)}$ and $\Psi^{(1)}$ denote the digamma and its first derivative function, respectively; cf. Gradsteyn and Ryzhik (1994), Sec.8.36.

Our next aim is to find the BDCF $\psi_{\log \gamma_{a,\lambda}}(t)$ for the selfdecomposable characteristic function $\phi_{\log \gamma_{a,\lambda}}(t)$.

Proposition 3. The background driving characteristic function (BDCF) for the log-$\gamma_{a,\lambda}$ variable is as follows

$$\psi_{\log \gamma_{a,\lambda}}(t) = \exp \left[ it \left( -\log \lambda + \Psi^{(0)}(\alpha) \right) + \int_{0}^{\infty} \left( e^{-itx} - 1 + itx \right) e^{-\alpha x} h_\alpha(x) \, dx \right],$$

where $h_\alpha(x) := [\alpha + (1 - \alpha)e^{-x}][(1 - e^{-x})^{-2} - 1] \text{ and } \int_{0}^{\infty} x^2 e^{-\alpha x} h_\alpha(x) \, dx < \infty$ and $\Psi^{(0)}(z)$ is the digamma function.

Since $\log \gamma_{a,\lambda}$ is selfdecomposable, it has a background driving distribution function (BDDF) $G_{\log \gamma_{a,\lambda}}$ for which we have:

Corollary 3. Let $G_{\log \gamma_{a,\lambda}}$ be the BDDF of the log-gamma variable. Then its expected value and the variance are as follows:

(a) $\mathbb{E}[G_{\log \gamma_{a,\lambda}}(x)] = -\log \lambda + \Psi^{(0)}(\alpha);$  
(b) $\text{Var}[G_{\log \gamma_{a,\lambda}}] = \int_{0}^{\infty} x^2 e^{-\alpha x} \left[ \alpha + (1 - \alpha)e^{-x} \right] \left( 1 - e^{-x} \right)^{-2} \, dx.$

(c) For $\beta > 0$ all moments $\mathbb{E}[|x|^\beta dG_{\log \gamma_{a,\lambda}}(x)]$ are finite.
Finally, here is a series representation for the innovation variable $X_c$ from (2) for the log-gamma variable:

**Proposition 4.** For independent binomial $b_c^{(k)}$ and gamma random variables $\gamma_{1,\alpha+k}$, $k = 0, 1, 2, ...$ where $P(b_c^{(k)} = 1) = 1 - c$ and $P(b_c^{(k)} = 0) = c$ with $0 < c < 1$, the random series

$$Z_c(\alpha) := -(1 - c)C + b_c^{(0)}(-1)\gamma_{1,\alpha} + \sum_{n=1}^{\infty} [b_c^{(n)}(-1)\gamma_{1,\alpha+n} + \frac{1-c}{n}], \quad (9)$$

converges almost surely (in probability, in distribution). Moreover, $Z_c(\alpha)$ has characteristic function $E[e^{itZ_c(\alpha)}] = \lambda^{-it(1-c)} \frac{\Gamma(\alpha+it)}{\Gamma(\alpha+ict)} = \phi_{\log \gamma_{\alpha,\lambda}}(t)/\phi_{c\log \gamma_{\alpha,\lambda}}(t)$.

**II. PROOFS.**

**Proof of Proposition 1.**

If $\phi$ and $\psi$ are characteristic functions of $X$ and $Y(1)$, respectively, then

$$\phi(t) = \exp \int_0^t \log \psi(u) \frac{du}{u}, \text{ equivalently } \psi(t) = \exp\left[t \frac{\phi'(t)}{\phi(t)}\right] \quad t \neq 0; \quad (10)$$

cf. Jurek (2001), Proposition 3. If $Y(1)$ is the BDRV of selfdecomposable random variable $X$ then $Y(1)$ has finite logarithmic moment; cf. Jurek and Vervaat (1983), Theorem 2.3. Consequently, by Gil-Pelaez (1951), Wendel (1961) and Ushakov (1999), Theorem 1.2.4 (cf. also Appendix (b)) we have that the cumulative distribution function of $Y(1)$ can be obtained from $\psi$ via the following inversion formula:

$$P(Y(1) \leq a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im(e^{-ita} \cdot \psi(t)) \frac{dt}{t}, \quad (a \text{ is a continuity point);}$$

which with (10) completes the argument for the proof of Proposition 1.

**Proof of Lemma 1.**

From (6), $\phi_{\gamma(\alpha,\lambda)}(t) = (1 - it/\lambda)^{-\alpha}$ and therefore from (10) we find that their BDCF are given as follows

$$\psi_{\gamma(\alpha,\lambda)}(t) = \exp[\alpha \frac{it/\lambda}{1 - it/\lambda}] = \exp \alpha \left[\frac{1}{1 - it/\lambda} - 1\right]. \quad (11)$$

Thus they correspond to compound Poisson distributions. More explicitly, if $E_k(\lambda), k = 1, 2, ..$ are i.i.d (exponentially distributed with scale parameter $\lambda$) and independent of $N_\alpha$, Poisson variable with parameter $\alpha$, then for

$$Y(1) := \sum_{k=1}^{N_\alpha} E_k(\lambda) \quad \text{we have} \quad E[e^{itY(1)}] = \exp \alpha \left[\frac{1}{1 - it/\lambda} - 1\right], \quad (12)$$
that is, $Y(1)$, has a compound Poisson distribution and it is the BDRV for the selfdecomposable $\gamma_{\alpha,\lambda}$ random variable. (If $N_\alpha = 0$ then the sum in (12) is zero.) This completes a proof of the first equality in (7).

For the second equality in (7) we use Proposition 1. Namely, for $a > 0$,

$$P\left( \sum_{k=1}^{N_\alpha} E_k(\lambda) \leq a \right) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im\left( \exp(-ita + \alpha \frac{it/\lambda}{1-it/\lambda}) \right) \frac{dt}{t}$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp\left( -\alpha \frac{(t/\lambda)^2}{1+(t/\lambda)^2} \right) \sin\left( ta - \frac{\alpha (t/\lambda)}{1+(t/\lambda)^2} \right) \frac{1}{t} dt. \quad (13)$$

For the third equality in (7) we use the additive property of gamma function with respect to the shape parameter and the identity

$$\sum_{k=1}^{\infty} \frac{b^{k-1}}{k!(k-1)!} = \frac{I_1(2\sqrt{b})}{\sqrt{b}}, \quad (14)$$

(cf. Gradshteyn-Rhyzik(1994), 8.445 or Appendix below), as follows

$$P\left( \sum_{k=1}^{N_\alpha} E_k(\lambda) \leq a \right) = \sum_{k=0}^{\infty} P\left[ \left( \sum_{j=1}^{N_\alpha} E_j(\lambda) \leq a \right) \cap (N_\alpha = k) \right]$$

$$= e^{-\alpha} + \sum_{k=1}^{\infty} e^{-\alpha} \frac{\alpha^k}{k!} P(\gamma_k,\lambda \leq a) = e^{-\alpha} + e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \frac{\lambda^k}{\Gamma(k)} \int_0^a x^{k-1} e^{-\lambda x} dx$$

$$= e^{-\alpha} + e^{-\alpha} \alpha \int_0^a \sum_{k=1}^{\infty} \frac{(\alpha x)^{k-1}}{k!(k-1)!} dx = e^{-\alpha} + e^{-\alpha} \alpha \int_0^a \frac{I_1(2\sqrt{\alpha x})}{\sqrt{\alpha x}} e^{-\lambda x} dx,$$

and changing variable we conclude the proof of Lemma 1.

**Proof of Proposition 2.**

Firstly, recall that series of centered independent gamma variables $\sum_{n=1}^{\infty} (\gamma_{1,\alpha+n} - 1/(\alpha + n))$ converges in all above mentioned modes, because $\inf_n (n+\alpha) = \alpha + 1 > 0$ and $\sum_n (n+\alpha)^{-2} < \infty$; cf. Jurek (2000), Proposition 1 and Corollary 2.

Secondly, we have two converging numerical series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)} = \alpha^{-1}(\Psi^{(0)}(\alpha + 1) + C); \quad \sum_{n=1}^{\infty} \frac{1}{(n+\alpha)^2} = \Psi^{(1)}(\alpha + 1); \quad (15)$$

cf. Appendix part c). Above $\Psi(z) \equiv \Psi^{(0)}(z) := \frac{d}{dz} \log \Gamma(z)$ is the digamma function and $\Psi^{(1)}(z)$ is its first derivative.
Finally, since \( \gamma_{1,\alpha+n} - 1/n = \gamma_{1,\alpha+n} - \mathbb{E}[\gamma_{1,\alpha+n}] - \alpha/n(\alpha + n) \) we conclude that the infinite random series in (8) converges in all three modes.

Since, in particular, series (8) converges weakly its characteristic function is given as an infinite product:

\[
\mathbb{E}[e^{itS}] = \lambda^{-it} \exp(-itC) (1 + it/\alpha)^{-1} \prod_{n=1}^{\infty} e^{it/n(1 + it/(\alpha + n))}^{-1} \quad \text{(by (6))}
\]

\[
= \lambda^{-it} \exp \left[ -itC + \int_{0}^{\infty} (e^{-itx} - 1) \frac{e^{-ax}}{x} dx 
+ \sum_{n=1}^{\infty} \left( \frac{it}{n} + \int_{0}^{\infty} (e^{-itx} - 1) \frac{e^{-x(\alpha+n)}}{x} dx \right) \right] \quad \text{(by (15))}
\]

\[
= \lambda^{-it} \exp \left[ -itC - it\alpha^{-1} + \int_{0}^{\infty} (e^{-itx} - 1 + itx) \frac{e^{-ax}}{x} dx 
+ \sum_{n=1}^{\infty} \left( \frac{it}{n} - \frac{1}{\alpha + n} \right) + \int_{0}^{\infty} (e^{-itx} - 1 + itx) \left( \sum_{n=0}^{\infty} \frac{e^{-x(\alpha+n)}}{x} \right) dx \right] \quad \text{(by (15))}
\]

\[
= \lambda^{-it} \exp \left[ it\Psi(0) + \int_{0}^{\infty} (e^{-itx} - 1 + itx) \frac{e^{-ax}}{x(1 - e^{-x})} dx \right]
\]

\[
= \lambda^{-it} \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)} = \mathbb{E}[e^{it\log \gamma_{\alpha,\lambda}}]. \quad (16)
\]

The last two equalities are from Jurek (1997), example (c) on p. 98, where the correct reference should be to Whittaker and Watson (1920) p. 249 and/or Shanbhag, Pestana and Sreehari (1977), Lemma 1. This gives a proof of the first equality in (8).

For the second equality in (8), note that by (15) and Appendix (c),(iv) we have

\[
\sum_{n=1}^{\infty} (\gamma_{1,\alpha+n} - 1/n) = \sum_{n=1}^{\infty} (\gamma_{1,\alpha+n} - \frac{1}{n + \alpha}) - \Psi(\alpha) - 1/\alpha - C,
\]

which concludes the proof of Proposition 2.

[In Bondesson (1992), Chapter 7, p. 113 there is a series representation of \( \log \gamma_{\alpha,1} \) as in the second part of (8), without specifying the constant. More importantly, it was obtained as an example of analytically introduced class of extended generalized Gamma convolution (EGGC); cf. Definition on p. 103.]
Remark 2. (a) For \( \alpha > 0 \) and \( t \in \mathbb{R} \) we have
\[
\frac{e^{-itC} \prod_{n=1}^{\infty} \exp(it/n)}{1 + it/\alpha} = \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)} = \exp[it\Psi(0)(\alpha) + \int_{-\infty}^{0} (e^{itx} - 1 - itx) \frac{e^{ax}}{|x|(1 - e^x)} dx]. \tag{17}
\]

(b) \( \mathbb{E}[(\exp it \log \gamma_{\alpha,1})] = \frac{\Gamma(\alpha + it)}{\Gamma(\alpha)}, \ t \in \mathbb{R}. \)

For part (a) : the first equality coincides with GR 8.326(2) and with Erdelyi (1953), p. 5, formula (3). The second one is from Shanbhag, Pestana and Sreehari (1977), Lemma 1; note that their constant \( c_\alpha = \Psi(0)(\alpha) \) via (16).

Part (b) is from last line in (16).

Proof of Corollary 2.

Using the representation (8) and formula (15) we have
\[
\mathbb{E}[\log \gamma_{\alpha,\lambda}] = -\log \lambda - C - \alpha^{-1} + \alpha \sum_{n=1}^{\infty} \frac{1}{n(\alpha + n)} = -\log \lambda + \Psi(0)(\alpha + 1) - \alpha^{-1}
\]
which gives the first identity. For the second one, using independence of the summands in (8) and the equality \( \sum_{n=1}^{\infty} \frac{1}{(\alpha + n)^2} = \Psi^{(1)}(\alpha + 1) \) we get
\[
\text{Var}[\log \gamma_{\alpha,\lambda}] = 1/\alpha^2 + \sum_{n=1}^{\infty} \frac{1}{(\alpha + n)^2} = \Psi^{(1)}(\alpha + 1) + 1/\alpha^2 = \Psi^{(1)}(\alpha),
\]
which completes the first formula for the variance. Second one follows from (16) viewed as Kolmogorov representation of infinitely divisible variables with finite second moments. This completes the proof of Corollary 2.

Proof of Proposition 3.

From (8) and first line in (6) we have
\[
\phi_{\log \gamma_{\alpha,\lambda}}(t) = e^{-it(C+\log \lambda)} \prod_{n=1}^{\infty} \mathbb{E}[e^{-it(\gamma_{\alpha,n+1}-1/n)}] = e^{-it(C+\log \lambda)}(1 + it/\alpha)^{-1} \prod_{n=1}^{\infty} e^{it/n}(1 + it/(\alpha + n))^{-1}.
\]

From second line in (6), for \( \beta > 0 \), we have identity
\[
(1 + it/\beta)^{-1} = \exp \left[ -it/\beta + \int_{0}^{1} \left[ \int_{0}^{\infty} (e^{-itux} - 1 + itux) e^{-\beta x} dx \right] du \right].
\]
Applying that identity for $\beta := \alpha + n, n = 0, 1...$ in the first formula above and then using (15) we get

$$\lambda^t \phi_{\log \gamma_{a, \lambda}}(t) = \exp \left[ -it \mathbf{C} - it\alpha^{-1} + \int_0^1 \int_0^\infty (e^{-itux} - 1 + itux)\alpha e^{-\alpha x} \, dx \right] \frac{du}{u}$$

$$+ \sum_{n=1}^{\infty} \left( \frac{it}{n} - \frac{1}{\alpha + n} \right) + \int_0^1 \int_0^\infty (e^{-itux} - 1 + itux)(\alpha + n)e^{-x(\alpha + n)} \, dx \right] \frac{du}{u} \right]$$

$$= \exp \left[ it\Phi(0) + \int_0^1 \int_0^\infty (e^{-itux} - 1 + itux) \sum_{n=0}^{\infty} \frac{e^{-x(\alpha + n)}}{1 - e^{-x}} \, dx \right] \frac{du}{u}$$

$$= \exp \int_0^1 \left[ it\Phi(0) + \int_0^\infty (e^{-itux} - 1 + itux)e^{-ax} h_a(x) \, dx \right] \frac{du}{u}. \tag{18}$$

From (10), the relations between self-decomposable $\phi$ and its BDCF $\psi$, are of the form $\phi(t) = \exp \int_0^1 \log \psi(tw) \frac{dw}{w}$. Hence we infer the BDCF $\psi_{\log \gamma_{a, \lambda}}(t)$ is in the square bracket in (18) by putting $u=1$.

Since for $\alpha > 0$ and $x > 0$

$$\alpha + (1 - \alpha)e^{-x} \leq \max(\alpha, 1 - \alpha) + \max(\alpha, 1 - \alpha)e^{-x} \leq 2 \max(\alpha, 1 - \alpha);$$

$$(\frac{x}{1 - e^{-x}})^2 \leq (1 + x)^2 \text{ and } \int_0^\infty e^{-\alpha x}(1 + x)^2 \, dx < \infty \text{ because}$$

$$\int_0^1 (1 + x)^2 e^{-\alpha x} \, dx = -\alpha^{-3} e^{-\alpha x}[(\alpha(x + 1) + 1)^2 + 1] + \text{const};$$

we infer that the integral is finite, which completes the proof of Proposition 3.

Proof of Corollary 3.
Parts (a) and (b) are from the second to last line in (16) and the Kolmogorov’s representation of infinitely divisible variables with finite second moments. Part (c) follows from the fact that $\int_1^\infty x^\beta e^{-\alpha x}h_a \, dx < \infty$ and from Jurek and Mason(1993), Proposition 1.8.13 on p. 36.

Proof of Proposition 4.
Step 1. Let us put $\xi_n := [b(c)(-1)\gamma_{1, \alpha + n} + \frac{1 - c}{n}]$. Then

$$\mathbb{E}[\xi_n] = \frac{\alpha - 1 - c}{\alpha + n} + \frac{\alpha(1 - c)}{n(\alpha + n)} \sum_{n=1}^{\infty} \mathbb{E}[\xi_n] = (1 - c)(\Psi^0(\alpha + 1) + C);$$
(see Appendix, Section c). or use WolframAlpha) so, series of expected values of $\xi_n$ converges. Moreover, note that

$$\text{Var}[\xi_n] = \mathbb{E}[(b_c^{(n)}(1)\gamma_{1,\alpha+n})^2] - \left(\mathbb{E}[b_c^{(n)}(1)\gamma_{1,\alpha+n}]\right)^2 = \frac{1-c}{(\alpha+n)^2} - \frac{(1-c)^2}{(\alpha+n)^2}$$

$$= \frac{c(1-c)}{(\alpha+n)}; \quad \sum_{n=1}^{\infty} \text{Var}[\xi_n] = c(1-c)\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)^2} = c(1-c)\Psi(\alpha + 1)$$

so the series of variances converges as well. All in all, Three Series Theorem gives convergence almost surely (and in probability) of the series in (9).

Step 2. Since for independent variables binomial $b_c$ ($P(b_c = 1) = 1-c$) and $\gamma_{1,\beta}$ we have that

$$\mathbb{E}[e^{ibt\gamma_{1,\beta}}] = \mathbb{E}\left[\mathbb{E}[e^{ibt\gamma_{1,\beta}}|b_c]\right] = c + \frac{1-c}{1-it/\beta} = \frac{1-ict/\beta}{1-it/\beta},$$

therefore for $\xi_n$ defined above

$$\mathbb{E}[e^{it\xi_n}] = e^{it(1-c)/n} \left[1 - \frac{1-c}{1+it/\alpha}\right] = e^{it(1-c)/n} \frac{1+ict/\alpha}{1+it/\alpha} = \left[e^{it/n} \frac{1}{1+it/\alpha}\right] \cdot \left[e^{ict/n} \frac{1}{1+ict/\alpha}\right]^{-1}.$$  \quad (19)

Step 3. Using the definition of $Z_c(\alpha)$ and then (19) and Remark 2 we have

$$\mathbb{E}[e^{itZ_c(\alpha)}] = e^{-itC(1-c)} \mathbb{E}\left[\prod_{n=1}^{\infty} e^{it\xi_n}\right]$$

$$= e^{-itC(1-c)} \frac{1+it\alpha}{1+it/\alpha} \prod_{n=1}^{\infty} e^{it(1-c)/n} \frac{1+ict/\alpha}{1+it/\alpha} = [\Gamma(\alpha + it)/\Gamma(\alpha)]/[\Gamma(\alpha + ict)/\Gamma(\alpha)] = \mathbb{E}[e^{it\log\gamma_{1,\lambda}}]/\mathbb{E}[e^{itc\log\gamma_{1,\lambda}}],$$

which completes the proof.

III. MORE EXAMPLES AND NEW FORMULAS.

1. Lévy distributions $L(m, c)$.

For a location parameter $m \in \mathbb{R}$ and a scale parameter $c > 0$, Lévy variable $\text{Lévy}(m, c)$ has the probability density function

$$f(m, c; x) := \sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{x}{2(c-m)}}}{(x-m)^{3/2}}, \quad \text{for} \; x > m.$$  \quad (20)
If one defines the error function $erf(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$, ($erf(\infty) = 1$) then the cumulative probability distribution function $F \equiv F(m, c; x)$ of $L(m, c)$ is given as

$$F(m, c; x) = erf\left(\sqrt{\frac{c}{2(x-m)}}\right) := 1 - erf\left(\sqrt{\frac{c}{2(x-m)}}\right)$$

$$= \frac{2}{\sqrt{\pi}} \int_\infty^\infty e^{-s^2} ds, \quad \text{for } x > m. \quad (21)$$

Finally the characteristic function $\phi_F$ is equal to

$$\phi_F(t) = e^{imt - |ct|^{1/2}(1 - i \text{sign } t)}, \quad t \in \mathbb{R}. \quad (22)$$

Hence $F = \text{Lévy}(m, c)$ are the stable distributions with the exponent 1/2. Consequently, they are selfdecomposable and thus admit the random integral representation (1). From (22), via (10), we get

$$\psi_F(t) = \exp[imt - |(c/2)t|^{1/2}(1 - i \text{ sign } t)], \quad \text{i.e., } Y(1) \overset{d}{=} \text{Lévy} (m, c/2). \quad (23)$$

So, the BDPD for 1/2-stable $L(m, c)$ is another 1/2-stable distribution but with the scale parameter $c/2$, that is, $L(m, c/2)$.

From Proposition 1, the cumulative probability distribution function $G$ of $Y(1)$ is

$$G(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im[\exp(-itx + imt - ((c/2)t)^{1/2}(1 - i))] \frac{dt}{t}$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty e^{-(c/2)t)^{1/2}} \sin(t(x-m)) - ((c/2)t)^{1/2} dt \quad (s := ((c/2)t)^{1/2})$$

$$= \frac{1}{2} + \frac{2}{\pi} \int_0^\infty e^{-s} \sin((2/c)(x-m)s^2 - s) \frac{ds}{s} = erf\left(\sqrt{\frac{c/2}{2(x-m)}}\right), \quad (24)$$

where the last equality is justified by (21).

As a by-product of the above we get the identity:

**Corollary 4.** For $a \in \mathbb{R}$ we have identity

$$\int_0^\infty e^{-x} \sin((a^2 x^2 - x) dx = \frac{\pi}{2} \left( erf\left(|a|\sqrt{2}\right)^{-1} - \frac{1}{2} \right) = \frac{\pi}{2} \left( 1 - erf\left(|a|\sqrt{2}\right)^{-1} \right)$$

**2. Stable distributions.** The above straightforward calculations showed that the BDDF of 1/2-stable distribution is 1/2-stable distribution. That fact
is not surprising one as the stable laws are the fixed points of the integral mapping (1); cf. Jurek and Vervaat (1983), Theorem 4.1.

3. **Bessel-K distributions**

Bessel-K distribution is the symmetrization of $\gamma_{\alpha,\lambda}$, so it is selfdecomposable and its characteristic function is $\phi_{BK(\alpha,\lambda)}(t) = (1 + t^2/\lambda^2)^{-\alpha}$. $K$ stands here for the Bessel function $K_\alpha$ which appears in the formula for the probability density function of Bessel-K distributions; cf. Johnson, Kotz and Balakrishnan (1994), Sect. 4.4, p. 50]. Moreover, we have

$$BK(\alpha, \lambda) = \sqrt{2\gamma_{\alpha,\lambda^2}} Z,$$

where the gamma variable and the standard normal $Z$ are independent. This is so, because

$$E[e^{it\sqrt{2\gamma_{\alpha,\lambda^2}} Z}] = E[E[e^{it\sqrt{2\gamma_{\alpha,\lambda^2}} Z} | \gamma_{\alpha,\lambda^2}]] = \frac{\lambda^{2\alpha}}{\Gamma(\alpha)} \int_0^\infty e^{-t^2 x \alpha} x^{-\alpha} e^{-\lambda^2 x} dx = \frac{\lambda^{2\alpha}}{\Gamma(\alpha)} \int_0^\infty e^{-(t^2 + \lambda^2) x} x^{\alpha-1} dx = [\frac{\lambda^{2\alpha}}{\lambda^2 + t^2}]^\alpha = \phi_{BK(\alpha,\lambda)}(t),$$

which concludes the proof of (26).

Hence and from (10) the BDCF of $BK(\alpha, \lambda)$ is equal to

$$E[e^{itY(1)}] = \psi_{BK(\alpha,\lambda)}(t) = \exp[2\alpha\left(\frac{1}{1 + t^2/\lambda^2} - 1\right)].$$

Consequently, the BDLP $(Y(t), t \geq 0)$ for Bessel-K distribution is the compound Poisson process.

**Corollary 5.** If a random variable $BK$ has the Bessel-K distribution then its innovation $BK_c$ has the following representations:

$$E[\exp(itBK_c)] = [e^{c^2}(1-e^2)]^{\frac{1}{1 + t^2/\lambda^2}} = (b_0^2\mathcal{E}(\lambda_0^c)) = \sum_{k=1}^{N_{-\alpha\log c^2}} e^{\eta_k} \mathcal{E}(\lambda_k^c),$$

where $\eta_k$ are i.i.d. uniform on $(0,1)$, $\mathcal{E}(\lambda_k^c)$ are i.i.d. symmetric exponential variables and $d$ means the equality in distribution.

**Proof.** Since $BK_c := \int_0^{\log c} e^{-s} dY(s)$, with $Y(1)$ given in (27), therefore it...
can be written as follows
\[
\mathbb{E}[\exp(it BK_c)] = \exp \int_0^{-\log c} \log \psi_{BK(\alpha,\lambda)}(e^{-st}) ds
\]
\[
= \exp \int_0^{-\log c} (-2\alpha) \frac{t^2 e^{-2s}/\lambda^2}{1 + t^2 e^{-2s}/\lambda^2} ds = (1 + c^2 t^2 / \lambda^2)^\alpha
\]
\[
= [c^2 1 + (1 - c^2)] 1 + t^2 / \lambda^2] \overset{d}{=} (b c^2 \mathcal{E}(\lambda o))^{\alpha} \overset{d}{=} \sum_{k=1}^{N_{-\alpha}} e^{\eta_k \mathcal{E}(\lambda o)},
\]

To see the last two equality one needs to compute the characteristic functions of \( b c^2 \mathcal{E}(\lambda) \) and of the random sum (compound Poisson distribution).

**Note:** analogous formulas are valid for \( \gamma_{\alpha,\lambda} \) variables. Also compare Lawrance (1982) and Hosseini (2019).

**IV. APPENDIX.**

**a).** Besides those two equivalent characterizations (formulas (1) and (2)) of the selfdecomposability, also the following limit laws give the class L variables \( X \).

Namely for \( X \in L \), there exist a strong mixing sequence \((V_n, n = 1, 2, \ldots)\) of random variables, there exist deterministic sequences \( a_n > 0, b_n \in \mathbb{R}, n = 1, 2, \ldots \) such that

(i) the triangular array \((a_n V_j : 1 \leq j \leq n; n \geq 1)\) is infinitesimal; and

(ii) \( a_n (V_1 + V_2 + \ldots + V_n) + b_n \Rightarrow X, n \to \infty \) (weak convergence);

cf. Bradley and Jurek (2016). [Of course, sequences of independent variables are strongly mixing.]

**b).** For the proof of Proposition 1 the following inversion formula was essential.

Let distribution function \( G \), with its characteristic function \( \psi \), has finite logarithmic moment, that is, \( \int_{-\infty}^{\infty} \log(1 + |x|) dG(x) < \infty \). Then

\[
G(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \text{Im}(e^{-iux} \psi(u)) \frac{du}{u}, \quad x \in C_G
\]  

(27)

cf. Gil-Paleaz(1951), Wendel(1961) and Ushakov(1999), Theorem 1.2.4.
c). For an ease of reference let us recall some special functions and formulas:

\[(i) \quad \Gamma(z) := \int_0^\infty x^{z-1}e^{-x} \, dx, \ \Re z > 0; \ (ii) \quad \Psi(z) := \frac{d}{dz} \log \Gamma(z), \ \Re z > 0; \]

\[(iii) \quad I_\alpha(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{\alpha+2j}}{j!\Gamma(\alpha+j+1)}, \ \alpha > 0, \ (\text{cf.8.445}); \]

\[(iv) \quad \Psi(x) = -C - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(k+x)}; \ (\text{cf. 8.362.1}); \]

\[(v) \quad \Psi(x+1) = \Psi(x) + \frac{1}{x}; \ (\text{cf. 8.365.1}); \]

\[(vi) \quad \Psi^{(n)}(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}, \ n \geq 1; \ (\text{cf. 8.363.8}); \]

(All bold face formulas are taken from Gradshteyn-Ryzhik(1994)).

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