Universal collective modes in 2 dimensional chiral superfluids

WeiHan Hsiao
Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA
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In this work, we utilize semi-classical kinetic equations to investigate the order parameter collective modes of a class of 2 dimensional superfluids. Extending the known results for \( p \)-wave superfluids, we show for any chiral ground states, there exists at least a pair of modes with mass \( \sqrt{2}\Delta \) in the weak-coupling limit. We further investigate effects that may modify these universal massive modes. It is found that they receive corrections from fermionic vacuum polarization and gap anisotropy. Moreover, for each chiral superfluid, we determine the most significant Landau parameter which contributes to the mass renormalization and show explicitly renormalized modes become massless at the Pomeranchuk instability of the fermion vacuum. These results provide potential diagnostics for distinguishing 2 dimensional chiral ground states of different angular momenta with order parameter collective modes and reveal another low energy degrees of freedom near nematic transition.

I. INTRODUCTION

In the studies of interacting quantum many-body systems, collective modes allow us to explore the correlated motions of underlying microscopic degrees of freedom. Should a superfluid phase exist, the order parameter component enriches the nature of collective excitations. To name a few instances, the number of gapless modes is related to the symmetry breaking pattern of the ground state because of Goldstone theorem. Paradigmatic examples include the A and B phases of superfluid \(^{3}\)He in \((3+1)\)D [1], in which massive sub-gap modes exist owing to the triplet pairing structure. Coupling with particle-hole channel, order parameter collective modes may also result in resonant signature in transport properties [2]. On top of these, it was pointed out by Nambu [3] that a class of models, including the Nambu-Jona-Lasinio model in nuclear physics, whose sums of the squares of order parameter collective modes satisfy a sum rule at the level of random-phase approximation.

Developments in the last paragraph permit various extensions. Natural questions that may follow include: (i) Do sub-gap massive collective modes also exist in pairing channels over than \( p \)-wave pairing and in \((2+1)\)D spacetime? (ii) Do these bosonic degrees of freedom acknowledge the underlying fermionic ground state or the property of the Fermi surface?

These 2 questions have been partially addressed. Regarding (i), it is known that the 2 dimensional analog of B-phase hosts 4 modes of mass \( \omega^2 = 2\Delta^2 \) with angular momenta \( \ell = \pm 2 \), and the analog of A-phase, whose fermionic spectrum is fully gapped in 2 dimensions, hosts six modes of mass \( m^2 = 2\Delta^2 \) [4–6]. On the other hand, (ii) has been investigated in the context of \((3+1)\)D \(^{3}\)He superfluid with Fermi liquid theory [1, 7–10], where the corrections to the masses of massive sub-gap modes and the sound speeds of Goldstone modes can be expressed in terms of Landau parameters. In addition, for \( \text{Sr}_2\text{RuO}_4 \) [11], it has been shown that strong coupling effect and gap anisotropy are able to modify the magnitude of the masses and break the spectrum degeneracy.

This work is intended to address the complementary faces of (i) and (ii). We specifically focus on superfluids in \((2+1)\)D with general pairing channels \( L = 0, 1 \cdots \). For \( L = 1 \), the 2 dimensional analogs of A and B phases are considered, whereas for higher \( L \) we concentrate on chiral ground states.

We look for massive sub-gap modes, and investigate mechanisms that may correct their masses in long wavelength limit \( q = 0 \).

We find that in the weak-coupling limit, there is a least a pair of bosonic modes of universal mass \( m_L = \sqrt{2}\Delta \) for all \( L \geq 1 \). We investigate corrections to these degenerate modes owing to fermionic vacuum and gap anisotropy in a phenomenological manner and determine the angular momentum channels substantial for mass renormalization and degeneracy breaking. For a given chiral ground states of angular momentum \( L \), the order parameter fluctuations longitudinal to the ground state are renormalized by the Landau parameter \( F_{2L} \). In the presence of ground state anisotropy, only the angular components of angular momenta \( 2L \) and \( 4L \) can substantially break the degeneracies of the bare mass \( \sqrt{2}\Delta \).

The Fermi liquid correction is especially intriguing in \((2+1)\)D. As we will show shortly in Sec.IV, it implies the sub-gap modes soften when the pivotal Landau parameter is negative. In the regime \( \omega/\Delta \ll 1 \),

\[
\frac{m_L^2}{4\Delta^2} = \frac{3(1 + F_{2L})}{6 + F_{2L}} \rightarrow 1, \text{ as } F_{2L} \rightarrow -1. \quad (1)
\]

The massive sub-gap modes in chiral superfluids are able to detect Pomeranchuk instability [12, 13] by becoming massless. In particular, when \( L = 1 \), \( F_{2L} = F_2 \). The limit \( F_2 \rightarrow -1 \) serves as one of the mechanisms behind nematic electronic phases [14, 15] and the result presented here, on top of previous studies on unconventional superconductors [16] and quantum Hall nematic phases [17], is another example where approaching the Pomeranchuk instability in quadrupolar channel or a nematic phase can influence the nature of a paired phase [18].

This paper is organized as follows. In Sec. II, we review the semi-classical equation approach for the computation of collective excitations. The equations derived are used in Sec. III to compute collective excitations for various ground states. In Sec. IV, we calculate the Fermi liquid ground state effect upon the bare bosonic spectra. Finally, Sec. V studies the effect of anisotropic perturbation of the superconducting ground state with a phenomenological polar mode decomposition. A summary and several open directions are composed at the end to close the main text. The full solutions to the kinetic equa-
tion (2) without assuming \( q = 0 \) and \( \Delta \in \mathbb{R} \) are present along with the method in appendix.

II. FORMALISM

Bosonic collective modes in superfluids or superconductors [19] can be computed with various approaches. In this work, we adopt the time-dependent mean field approximation to include the Fermi liquid corrections. This approach can be formulated in terms of generalized Landau-Boltzmann kinetic equations [1], or the linearized non-equilibrium Eliasson-Eilenberger equation [9]. Though we will not provide a derivation starting from defining Green’s functions, which we refer the readers to Ref.[9], we will give a complete elaboration of the machinery.

In the semi-classical limit, physical quasi-particle distribution is related to the Keldysh Green’s function \( \tilde{g}(\varepsilon, \mathbb{p}; \omega, \mathbf{q}) \). In its argument \( (\varepsilon, \mathbb{p} = p \mathbb{p}) \) are the Fourier transformed variables of the fast coordinates, where as \( (\omega, \mathbf{q}) \) are ones of the center of mass coordinates. In clean limit, the linear response involves the molecular mean field at equilibrium and the linearized non-equilibrium Eilenberger equations [1], or the linearized non-equilibrium Eilenberger equation [9].

\[
\varepsilon + \tau_3 \delta \tilde{g} - \delta \tilde{g} \tau_3 \varepsilon = -v_F \mathbb{p} \cdot \mathbf{q} \delta \tilde{g} - [\sigma_0, \delta \tilde{g}],
\]

where \( \varepsilon \pm \) denotes \( \varepsilon \pm \omega/2 \). The operators \( \sigma_0 \) and \( \delta \tilde{g} \) are the molecular mean field at equilibrium and the linear perturbation respectively. Similarly, \( \tilde{g}_0(\varepsilon, \mathbb{p}) \) represents the Keldysh propagator at equilibrium and is related to retarded and advanced propagators via \( \tilde{g}_0 = (\tilde{g}_0^R - \tilde{g}_0^A) \tanh(\varepsilon/2T) \), which yields

\[
\tilde{g}_0 = \frac{-2\pi i (\tau_3 \varepsilon - \Delta)}{\sqrt{\varepsilon^2 - |\Delta|^2}} \Theta(\varepsilon^2 - |\Delta|^2) \text{sgn}(\varepsilon) \tanh(\varepsilon/2T) \tag{3}
\]

The low energy fluctuation of quasi-particles and the deduced physical quantities are given by the \( \varepsilon \)-integrated \( \tilde{g} \)

\[
\int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi i} \tilde{g}(\varepsilon, \mathbb{p}; \omega, \mathbf{q}).
\tag{4}
\]

For example, the particle-current is given by

\[
\delta j = N(0) \int \frac{d\theta}{2\pi} \int \frac{d\varepsilon}{2\pi i} \frac{1}{2} \mathbb{p} \text{tr}[\tau_3 \delta \tilde{g}(\varepsilon; \omega, \mathbf{q})].
\tag{5}
\]

In particular, the molecular field, or self-energy, \( \delta \tilde{g} \) is self-consistently determined by the convolution of inter-particle potentials and \( \delta \tilde{g} \).

To further elaborate, we note that \( \delta \tilde{g} \) has a general structure

\[
\delta \tilde{g} = \left( \begin{array}{c} \delta g + \delta \mathbf{g} \cdot \mathbf{\sigma} \\ i\sigma_2(\delta f' + \delta \mathbf{f}' \cdot \mathbf{\sigma}) \sigma_2 \end{array} \right) ,
\]

and accordingly so does \( \delta \tilde{g} \),

\[
\delta \tilde{g} = \left( \begin{array}{c} \delta \mathbf{g} + \delta \mathbf{g} \cdot \mathbf{\sigma} \\ i\sigma_2(\delta d' + \delta \mathbf{d}' \cdot \mathbf{\sigma}) \sigma_2 \end{array} \right) ,
\]

where the primed variables are

\[
\begin{align*}
\delta g'(\mathbf{p}; \omega, \mathbf{q}) &= \delta g(-\mathbf{p}; \omega, \mathbf{q}) \\
\delta \varepsilon'(\mathbf{p}; \omega, \mathbf{q}) &= \delta \varepsilon(-\mathbf{p}; \omega, \mathbf{q}) \\
\delta f'(\mathbf{p}; \omega, \mathbf{q}) &= \delta f^*(\mathbf{p}; -\omega, -\mathbf{q}) \\
d'(\mathbf{p}; \omega, \mathbf{q}) &= d^*(\mathbf{p}; -\omega, -\mathbf{q}).
\end{align*}
\tag{8}
\]

Physical observables are usually expressed in terms of the symmetric and anti-symmetric combination of \( \delta g \) as diagram (1) and their primed partners. In this work, we define \( (+) \) and \( (-) \) combinations of a function \( f \) as

\[
f^{(\pm)} = f \pm f'.
\tag{9}
\]

The eigenvalues \( \pm 1 \) represent the parity under charge conjugation. As we will see, the charge density and energy stress tensor correspond to the scalar and quadrupole modes of \( \delta g^{(\pm)} \) respectively, whereas the current density is proportional to the vector mode of \( \delta g^{-} \). Similarly, \( d^{(+)} \) and \( d^{-} \) stand for the amplitude and phase fluctuations of the pairing fields.

At one-loop, the correction to the self-energy is determined by the two-body vertex. Evaluating internal momentum integral over the Fermi surface, we have, in the Landau channel,

\[
\delta \varepsilon(\mathbf{p}; \omega, \mathbf{q}) = \delta \varepsilon_{\text{ext}}(\mathbf{p}; \omega, \mathbf{q})
\tag{10a}
\]

\[
+ \int \frac{d\theta'}{2\pi} A^s(\theta, \theta') \int \frac{d\varepsilon'}{4\pi i} \delta g(\varepsilon', \mathbf{p}'; \omega, \mathbf{q}),
\]

\[
\delta \varepsilon(\mathbf{p}; \omega, \mathbf{q}) = \delta \varepsilon_{\text{ext}}(\mathbf{p}; \omega, \mathbf{q})
\tag{10b}
\]

\[
+ \int \frac{d\theta'}{2\pi} A^s(\theta, \theta') \int \frac{d\varepsilon'}{4\pi i} \delta g(\varepsilon', \mathbf{p}'; \omega, \mathbf{q}).
\]

where \( A^s \) is the spin-independent (exchange) forward scattering amplitude which can be rewritten in terms of Landau parameters via

\[
A(\mathbf{p}, \mathbf{p}') = F(\mathbf{p}, \mathbf{p}') - \frac{d\sigma'^e}{2\pi} F(\mathbf{p}, \mathbf{p}'') A(\mathbf{p}'', \mathbf{p}').
\tag{11}
\]

Similarly in Cooper channel, the off-diagonal components are related by the linearized gap equations.

\[
d(\mathbf{p}; \omega, \mathbf{q}) = \int \frac{d\theta'}{2\pi} V_c(\theta, \theta') \int \frac{d\varepsilon'}{4\pi i} \delta f(\varepsilon', \mathbf{p}'; \omega, \mathbf{q})
\tag{12a}
\]

\[
d(\mathbf{p}; \omega, \mathbf{q}) = \int \frac{d\theta'}{2\pi} V_o(\theta, \theta') \int \frac{d\varepsilon'}{4\pi i} \delta f(\varepsilon', \mathbf{p}'; \omega, \mathbf{q})
\tag{12b}
\]

where \( V_c \) and \( V_o \) are the pairing potentials in even (odd) angular momentum channel.

Analogous to 3 dimensions, where scattering amplitudes and pairing potentials are expanded in terms of spherical harmonics, in 2 dimensions we have

\[
A = \sum_{\ell=\infty} A_{\ell} e^{-i\ell(\theta-\theta')}, \quad A_{\ell} = A_{-\ell},
\tag{13a}
\]

\[
V_c = \sum_{\ell (\text{even})} V_c e^{-i\ell(\theta-\theta')} + \text{h.c.}
\tag{13b}
\]

\[
V_o = \sum_{\ell (\text{odd})} V_c e^{-i\ell(\theta-\theta')} + \text{h.c.}.
\tag{13c}
\]
from which we can derive \( A_\ell = \frac{F_\ell}{F_p} \), where \( F_\ell \) is the conventional dimensionless Landau parameter of channel \( \ell \). For other functions \( f(\bar{p}) \) evaluated on Fermi surface \( \bar{p} \), the angular decomposition is defined as

\[
f = \sum_{\ell = -\infty}^{\infty} e^{-i\ell \theta} f_\ell.
\]

We can then provide a recipe for the computation. We first invert (2) to obtain \( \delta \sigma \) as a function of \( \delta \tau \). Taking the convolution as in (10a), (10b), (12a), and (12b) establishes integral equations for \( \delta \sigma \). Projecting equations (12a) and (12b) to different angular modes \( \ell \) gives us the dynamical equations of \( d_\ell \) and \( d_\ell \), which are sourced by \( \delta \varepsilon \) and \( \delta \varepsilon \). The bare bosonic collective modes are defined by the normal modes of the homogeneous part of the equations. To include the Fermi liquid corrections, we project (10a), and (10b) to their \( \ell \)th angular modes as well and solve \( \delta \varepsilon \) and \( \delta \varepsilon \) in terms of \( \delta \varepsilon_{\text{ext}} \), \( \delta \varepsilon_{\text{ext}} \) and \( \delta \varepsilon \). Plugging the results back into the equations for \( d_\ell \) and \( d_\ell \) yields inhomogeneous equations sourced solely by external fields. The renormalized mass spectrum is solved as the poles of the solution kernels.

In the rest of this section, we use the above formulation to derive the integral equation for 2 dimensional spin-singlet and spin-triplet superfluids and compute the collective modes and Fermi liquid corrections in the sections following.

### A. spin-singlet pairing

In a spin-singlet pairing channel, the equilibrium self-energy is characterized by a complex gap field \( \Delta \).

\[
\hat{\sigma}_0 = \hat{\Delta} = \begin{pmatrix} 0 & \Delta i\sigma_2 \\ \Delta^* i\sigma_2 & 0 \end{pmatrix}.
\]

The fluctuation of the spin-singlet order parameter can be parametrized by a complex number \( d \). It transforms as a scalar under spin rotation \( SO_3(3) \) and can have internal structures, i.e., tensor indices under orbital rotation \( SO_3(2) \) depending on pairing symmetries. In the absence of magnetic field, spin-triplet fluctuations \( \delta \varepsilon \) decouple from \( d \). Hence we consider them separately in the present work. Plugging (15) into (2), inverting it using the variables defined in (6) and (7), and taking the convolution as in (12a) and (12b) give us, in the long-wavelength limit, the off-diagonal components of the molecular fields

\[
d(\bar{p}; \omega) = \int \frac{d\theta'}{2\pi} V_\omega(\theta, \theta') \left[ \left( \gamma + \frac{1}{4} \hat{\lambda} (\omega^2 - 2|\Delta|^2) \right) d \right. \\
\left. - \frac{\hat{\lambda}}{2} \Delta^2 d' - \frac{\omega}{4} \hat{\lambda} \Delta \delta \varepsilon^{(+)} \right],
\]

\[
d'(\bar{p}; \omega) = \int \frac{d\theta'}{2\pi} V_\omega(\theta, \theta') \left[ \left( \gamma + \frac{1}{4} \hat{\lambda} (\omega^2 - 2|\Delta|^2) \right) d' \right. \\
\left. - \frac{\hat{\lambda}}{2} (\Delta^*)^2 d + \frac{\omega}{4} \hat{\lambda} \Delta^* \delta \varepsilon^{(+)} \right].
\]

The definition of \( \gamma \) is given in Appendix (A1). The function \( \lambda \), often called the Tsunedo function, whose complete form is given in appendix A. In \( q \to 0 \) limit,

\[
\check{\Delta} = \frac{\lambda(\bar{p}; \omega)}{|\Delta|^2} = \int_{|\Delta|}^\infty \frac{d\varepsilon}{\sqrt{\varepsilon^2 - |\Delta|^2}} \frac{\tanh \frac{\omega}{2\varepsilon}}{\varepsilon^2 - \omega^2 / 4}.
\]

We see there could be some residual angular dependence through the anisotropy in \(|\Delta|^2\) even in the long wavelength limit. This point will be crucial as we examine the effect of gap anisotropy. Suppose only a single pairing channel \( L \) is significant, i.e., that \( V = V_L(e^{-iL(\theta - \theta')} + h.c.) \). Taking \( \int d\theta^d e^{iL\theta} \) on both sides of (16a) and (16b) eliminates \( \gamma \)s with their right-hand sides. Thus, we obtain the dynamical equations of motion

\[
\langle e^{iL\theta} \hat{\lambda} \left[ |\omega^2 - 2|\Delta|^2| d - 2\Delta^2 d' - \omega \Delta \delta \varepsilon^{(+)} \right] \rangle = 0
\]

\[
\langle e^{iL\theta} \hat{\lambda} \left[ |\omega^2 - 2|\Delta|^2| d' - 2(\Delta^*)^2 d + \omega \Delta^* \delta \varepsilon^{(+)} \right] \rangle = 0,
\]

where we use the angle bracket \( \langle \cdots \rangle \) to denote the angular average \( \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \cdots \).

### B. spin-triplet pairing

In a spin-triplet pairing channel, the ground state self-energy is characterized by the vector-valued gap function \( \Delta \)

\[
\hat{\Delta} = \begin{pmatrix} 0 & \Delta \cdot i\sigma_2 \\ \Delta^* \cdot i\sigma_2 & 0 \end{pmatrix}.
\]

The fluctuation is encoded in the dynamics of the \( d \) vector, which transforms as a vector under \( SO_3(3) \), and could contain internal structure depending on pairing symmetry as well. For 2 dimensional \( p \)-wave superfluids or superconductors, it can be expanded as \( d_\mu(\bar{p}) = d_\mu \bar{p}_i \), where \( i = x, y \). Similarly inverting the kinetic equations, the dynamical equations for \( d \) in \( q \to 0 \) limit are

\[
d = \int \frac{d\theta'}{2\pi} V_\omega(\theta, \theta') \left[ \left( \gamma + \frac{1}{4} \hat{\lambda} (\omega^2 - 2|\Delta|^2) \right) d \\
+ \frac{\hat{\lambda}}{2} (\Delta \cdot \Delta^*) d - 2(\Delta \cdot d') \Delta \\
- \frac{\omega}{4} \hat{\lambda} (\Delta \delta \varepsilon^{(+)} - i\Delta \times \delta \varepsilon^{(+)} \right].
\]

\[
d' = \int \frac{d\theta'}{2\pi} V_\omega(\theta, \theta') \left[ \left( \gamma + \frac{1}{4} \hat{\lambda} (\omega^2 - 2|\Delta|^2) \right) d' \\
+ \frac{\hat{\lambda}}{2} (\Delta^* \cdot \Delta^*) d - 2(\Delta^* \cdot d) \Delta \\
+ \frac{\omega}{4} \hat{\lambda} (\Delta^* \delta \varepsilon^{(+)} + i\Delta^* \times \delta \varepsilon^{(+)} \right].
\]
Again we multiply (20a) and (20b) by $V = V_L[e^{-iL(\theta-\theta')} + e^{iL(\theta-\theta')}]$ and integrate over $\theta$. The regularized integral $\gamma$s again cancel out the right-hand sides and we obtain

$$
\left\langle e^{iL\theta} \hat{\lambda} \left( \omega^2 - 2|\Delta|^2 \right) d + 2(\Delta \cdot \Delta) d' - 4(\Delta \cdot d') \Delta \right\rangle = \omega \left\langle e^{iL\theta} \hat{\lambda} (\Delta \delta (\theta^+) - i \Delta \times \delta (\theta^+)) \right\rangle.
$$

(21a)

$$
\left\langle e^{iL\theta} \hat{\lambda} \left( \omega^2 - 2|\Delta|^2 \right) d' + 2(\Delta^* \cdot \Delta^*) d - 4(\Delta^* \cdot d) \Delta^* \right\rangle = -\omega \left\langle e^{iL\theta} \hat{\lambda} (\Delta^* \delta (\theta^+) + i \Delta^* \times \delta (\theta^+)) \right\rangle.
$$

(21b)

In the following section, we will solve (18a), (18b), (21a), and (21b) for ground states of different pairing channels and symmetries.

### III. COLLECTIVE MODES

In this section, we utilize the equations derived in the last section to compute the bare bosonic spectra for various superconducting ground states. More specifically, we will focus on unitary gaps with isotropic gap amplitudes. The dynamical equations are considerably simplified as the Tsunedo function $\lambda$ drops out of all angular averages. The masses of order parameter collective modes appear as normal modes of the homogeneous part of (18a), (18b), (21a), and (21b). The particle-hole self-energy $\delta \varepsilon$ and $\delta \hat{\varepsilon}$ are treated as external sources at the zeroth order, and they will be renormalized in the next section as we conclude Fermi liquid effects.

#### A. $s$-wave pairing

For $s$-wave pairing, it is possible to choose a gauge such that $\Delta \in \mathbb{R}$, in which limit the amplitude mode $d(+)\text{ and phase mode }d^(-)$ decouple. The bosonic field has no internal structure and is simply a complex scalar, implying 2 order parameter collective modes obeying the following equations

$$
(\omega^2 - 4\Delta^2) d(+) = 0 \quad (22a)
$$

$$
\omega^2 d(-) = 2\omega \Delta \delta \varepsilon(+) \quad (22b)
$$

The normal modes have $m^2 = 4\Delta^2$ and $m^2 = 0$ corresponding to the simplest example of Higgs and Goldstone bosons respectively. Note that if we compute (22b) to the leading non-vanishing order in $q^2$, we would have obtain $(\omega^2 - \frac{1}{2}(v_Fq)^2)d(-)$, entailing the Goldstone boson moves at the speed $v_F/\sqrt{2}$. Another observation is that the Higgs mode receives to external force and consequently it would not be renormalized by particle-hole self-energy. On the other hand, the Goldstone boson is sourced by the density mode $\delta \varepsilon(+)\text{, which would trigger Higgs mechanism in the presence of Coulomb interaction.}$

### B. $p$-wave pairing

The $p$-wave pairing states have more degrees of freedom, and thus more collective modes, owing to triplet-pairing and orbital structure. In 2 dimensions, the fluctuation of $p$-wave superconductors can be represented by the complex tensor $d_{\mu\nu}\text{, which contains }3 \times 2\text{ complex degrees of freedom, leading to 12 collective modes in total. The number of the massless modes }N_G\text{, as we will see shortly, can be determined by ground state symmetry breaking pattern. The rest }((6-N_G)\times 2\text{ is number of sub-gap collective modes.)}$

#### a. $B$-phase

We first consider the 2-dimensional analog of $^3$He B-phase, where the gap function assumes the form

$$
\Delta = \Delta (\hat{x}p_x + \hat{y}p_y), \quad \Delta \in \mathbb{R}.
$$

(23)

In this phase, the global symmetry breaks following the pattern $SO_3(3) \times SO_2(2) \times U(1) \rightarrow SO(2)$, which immediately indicates the existence of 4 Goldstone modes. Besides, the residual symmetry is $SO(2)$ rotation and we expect the fluctuations can be characterized by total angular momentum $J$. Owing to this fact, it is also convenient to choose another basis for $d_{\mu} = \sum_{m=\pm 1} d_{\mu m} e^{i\mu \theta}$, where $\theta$ is the polar angle of $\hat{p}$, as follows

$$
D_{\pm m} = d_{xm} \pm id_{ym} \quad (24)
$$

$$
D_{0m} = d_{zm}. \quad (25)
$$

Moreover, as the gap function is real, modes transforming differently under charge conjugation again decouple. That is to say, we can further separate $d(\pm) = d\pm d'$ degrees of freedom. We first look at the $d(\pm)$ modes governed by the equation

$$
(\omega^2 - 4\Delta^2) d(\pm) + 4(\Delta \cdot d(\pm)) \Delta = 2\omega \Delta \delta \varepsilon(\pm). \quad (26)
$$

Organizing the dynamical equations using the basis $D_{\alpha\sigma'}$, we could find

$$
(\omega^2 - 4\Delta^2) D_{\alpha\pm}^0 = 0 \quad (27a)
$$

$$
(\omega^2 - 2\Delta^2) D_{\pm \pm}^0 = 2\omega \Delta \delta \varepsilon_{\pm \pm} \quad (27b)
$$

$$
(\omega^2 - 4\Delta^2) (D_{\alpha \pm}^+ - D_{\alpha \pm}^-) = 0 \quad (27c)
$$

$$
\omega^2 (D_{\alpha \pm}^+ + D_{\alpha \pm}^-) := \omega^2 D_{\alpha \pm}^0 = 4\omega \Delta \delta \varepsilon_{\pm \pm}. \quad (27d)
$$

Consequently, $d(\pm)$ has 2 sub-gap massive modes $J = \pm 2$ of the same mass $\sqrt{2} \Delta$, and they are sourced by the quadrupolar molecular field.

Next we look at $d(\pm)$, which obeys

$$
[\omega^2 d(\pm) - 4\Delta (\Delta \cdot d(\pm))] = -2i\omega \Delta \times \delta \varepsilon(\pm). \quad (28)
$$

Following the same procedure to project each component to different $J$ sectors, we would obtain

$$
\omega^2 d_{00}^\pm = -2i\omega (\Delta \times \delta \varepsilon(\pm)) \cdot \hat{z} \quad (29a)
$$

$$
(\omega^2 - 2\Delta^2) D_{\pm \pm}^0 = \mp 2\omega \Delta \delta \varepsilon_{\pm \pm} \cdot \hat{z} \quad (29b)
$$

$$
\omega^2 (D_{\alpha \pm}^+ - D_{\alpha \pm}^-) = 4\Delta \omega \delta \varepsilon_{\pm \pm} \cdot \hat{z} \quad (29c)
$$

$$
(\omega^2 - 4\Delta^2) (D_{\alpha \pm}^+ + D_{\alpha \pm}^-) = 0. \quad (29d)
$$
It can be seen that the sums of the mass squares of (+) and
(−) modes with the same quantum numbers all equal 4\Delta^2,
which serves an example of Nambu’s sum rule.

In spite of the pure phenomenological view adopted in this
work, if the underlying UV physics possesses D_4 lattice sym-
mmetry, there are other permitted unitary and fully gapped real
ground states, \Delta \sim \hat{x}p_y - \hat{y}p_x, \hat{x}p_x - \hat{y}p_y, and \hat{x}p_y + \hat{y}p_x.
They differ from the B-phase by either a rotation, spatial
reflection, or a combination of these two operations. It is
straightforward to check and find unsurprisingly all of them
have the same mass spectrum characterized by different quan-
tum numbers.

b. A-phase 2 dimensional A-phase has a different
symmetry breaking pattern SO_3 \otimes SO_2 \otimes U(1) \rightarrow
U_{L-\mathcal{N}/2(1)\rightarrow U_{L-Q}(1)}. Consequently, there are 3 Goldstone
bosons and the bosonic fluctuations should be characterized
by eigenvalues Q.

Let us now consider a p + ip ground state described by
\[ \Delta = \frac{p_x + i p_y}{p_F} \Delta \hat{z} = e^{i \theta} \Delta \hat{z}, \Delta \in \mathbb{R}. \] (30)
The dynamic equations for d_\ell and d'_\ell (\ell = \pm 1) are now
coupled and given as follows.
\[ \langle e^{i \theta} \hat{x} (\omega^2 - 2 \Delta^2) d + 2 \Delta^2 e^{2i \theta} - 4 \Delta^2 \hat{z} \cdot d' e^{i(\theta)} \rangle
= \omega \Delta \langle e^{i \theta} e^{i \theta} [\hat{z} \delta \epsilon^{(+) - i \hat{z} \times \delta \epsilon^{(+)}}]. \] (31a)
\[ \langle e^{i \theta} \hat{x} (\omega^2 - 2 \Delta^2) d + 2 \Delta^2 e^{-2i \theta} - 4 \Delta^2 \hat{z} \cdot d' e^{-i(\theta)} \rangle
= - \omega \Delta \langle e^{i \theta} e^{-i \theta} [\hat{z} \delta \epsilon^{(+) + i \hat{z} \times \delta \epsilon^{(+)}}]. \] (31b)

We first look at d_{\ell=1} and d'_{\ell=-1}. They obey the equations
\[ (\omega^2 - 2 \Delta^2) d_1 = \omega \Delta (\hat{z} \delta \epsilon^{(+) - i \hat{z} \times \delta \epsilon^{(+)}}) \] (32a)
\[ (\omega^2 - 2 \Delta^2) d'_{-1} = - \omega \Delta (\hat{z} \delta \epsilon^{(+) + i \hat{z} \times \delta \epsilon^{(+)}}) \] (32b)
and have the same mass \sqrt{2} \Delta. These six modes are the mas-
sive Nambu partners of one another in A-phase. On the other
hand, equations for d_{\ell=1} and d'_{\ell=1} are coupled. Solving
them, one can find 3 massless modes and 3 modes of mass
m^2 = 4 \Delta^2.

Quite intuitively, the spectrum would be the same if even
we have considered the ground state aligned in other direc-
tions, e.g. \Delta = \hat{x} \Delta e^{i \theta}. Microscopically, a different choice of
ground state polarization direction corresponds to a different
choice of orientation in which the spins are equally aligned.
For instance, in the ground state (30), which is conjectured to
be the ground state of Sr_2RuO_4, the orbital angular momen-
tum of the Cooper pair points in \hat{z} direction, whereas the spins
of constituent fermions lies in plane.

C. d-wave pairing

The d-wave gap fluctuation is captured by the complex field
d_{ij} p_i p_j with irreducible complex degrees of freedom 1 \times 2,
represented by the modes d_{\pm 2} e^{\pm i(\theta)}. In this work, we consider
the ground state
\[ \frac{\Delta}{p_F} (p_x + i p_y)^2 = \Delta e^{i(\theta)}, \Delta \in \mathbb{R}. \] (33)

Equations (18a), and (18b) then become
\[ (\omega^2 - 2 \Delta^2) d_2 = \omega \Delta \delta \epsilon^{(+) 4} \] (34a)
\[ (\omega^2 - 2 \Delta^2) d'_{-2} = - \omega \Delta \delta \epsilon^{(+) 4} \] (34b)
\[ (\omega^2 - 4 \Delta^2) d_{-2} + d'_2 = 0 \] (34c)
\[ \omega^2 (d_{-2} - d'_2) = 2 \omega \Delta \epsilon^{(+) 4}. \] (34d)
Clearly d_2 and d'_{-2} have masses m^2 = 2 \Delta^2.

D. Higher L chiral ground states

Extending the analyses for equations (31a), (31b), (34a),
and (34b), we could actually consider a more general ground
state
\[ \text{singlet} : \delta \Delta e^{iL_0 \theta}, L_s = \text{even} \] (35a)
\[ \text{triplet} : \hat{z} \delta \Delta e^{iL_0 \theta}, L_t = \text{odd}. \] (35b)
Modes d_{L_s}, d'_{L_s}, d_{L_t}, and d'_{L_t} would automatically satisfy
\[ (\omega^2 - 2 \Delta^2) d_{L_s} = \omega \Delta \delta \epsilon^{(+) 4 L_s} \] (36a)
\[ (\omega^2 - 2 \Delta^2) d'_{-L_s} = - \omega \Delta \delta \epsilon^{(+) -2L_s} \] (36b)
\[ (\omega^2 - 2 \Delta^2) \hat{z} \cdot d_{L_t} = \omega \Delta \delta \epsilon^{(+) 4 L_t} \] (36c)
\[ (\omega^2 - 2 \Delta^2) \hat{z} \cdot d'_{-L_t} = - \omega \Delta \delta \epsilon^{(+) 2L_t}, \] (36d)
In this sense \omega = \sqrt{2} \Delta is a universal order parameter collective
mode for any chiral ground state of angular momentum \textit{L},
each of which is sourced by quasi-particle self-energy \delta \epsilon^{2L}.
As one can notice from either the dynamical equations or
the arguments presented, the eigenfrequency can be universal
because all chiral ground states follow similar symmetry
breaking patterns. In 2 dimensions, all ground state consid-
ered in the present work are equivalently left with a SO(2)
or U(1) rotational symmetry. For the B-phase of superfluid,
if it refers to the total angular momentum \textit{J}, whereas for other
chiral superconductors, it is a combination of SO(2) orbital
rotation and residual U(1) gauge transformation and thus can
be labeled formally by an angular momentum number. These
massive sub-gap modes are those with the highest angular mo-
menta in their pairing channels. While other modes of smaller
angular momenta are able to interfere and break their degener-
cacies into Goldstone \omega = 0 and Higgs \omega = 2 \Delta bosons, they
decouple from each other simply because of selection rules
imposed by the rules of angular momenta addition.

IV. FERMI LIQUID CORRECTIONS

In the previous section we found for chiral ground states of
given \textit{L}, bosonic modes d_{L_s} and d'_{L_t} have finite mass \sqrt{2}\Delta.
in mean field approximation. In this section we compute the Fermi liquid corrections to the mass spectra. Before presenting quantitative details, we point out some general features. Those modes with mass \( m^2 = 4\Delta^2 \) in general are not sourced by fermionic self-energy, and consequently these modes are not renormalized. On the other hand, for those massless modes, short-range fermionic self-energy can at most renormalize the sound speed and the magnitude of external source fields instead of generating a gap. We will demonstrate this with some specific examples and in the rest of the section we will focus on the sub-gap mode \( m = \sqrt{2}\Delta \).

A. Massless Modes

Let us first look at (22b). The right hand side \( \delta \varepsilon_0^{(+)} \) consists of pure external perturbation \( \delta u_{\text{ext}} \) and the renormalization coming form the integral part of (10a). In long wavelength limit,

\[
\delta \varepsilon^{(+)}(\theta) = \delta u^{(+)}_{\text{ext}} + \int \frac{d\theta'}{2\pi} F^\ast(\theta, \theta')[-\lambda \delta \varepsilon^{(+)} + \frac{\omega \lambda}{2\Delta} d^{(-)}]. \tag{37}
\]

Projecting out \( \ell = 0 \) component, we obtain

\[
(1 + \lambda(\omega) F_0^\ast) \delta \varepsilon^{(+)}_0 = \delta u^{(+)}_0 + \frac{\omega \lambda}{2\Delta} d^{(-)}, \tag{38}
\]

plugging which back into (22b) yields

\[
\omega^2 d^{(-)} = 2\omega \Delta \delta u_0. \tag{39}
\]

It entails that \( d^{(-)} \) remains massless. To demonstrate a triplet-pairing example, we look at B-phase (23) and (27c). For triplet-pairing states, the diagonal term of (10a) reads

\[
\delta \varepsilon^{(+)}(\theta) = \delta u^{(+)}_{\text{ext}} + \int \frac{d\theta'}{2\pi} F^\ast(\theta, \theta')[-\lambda \delta \varepsilon^{(+)} + \frac{1}{2}\omega \lambda \Delta \cdot d^{(-)}], \tag{40}
\]

whose projection to \( \ell \)th mode is

\[
\delta \varepsilon^{(+)}_\ell = \frac{\delta u^{(+)}_{\text{ext}} + \frac{\lambda}{2} \omega F_0^\ast (\Delta \cdot d^{(-)})_\ell}{1 + \lambda(\omega) F_\ell^\ast}. \tag{41}
\]

For \( \ell = 0 \),

\[
B : \delta \varepsilon_0^{(+)} = \frac{\delta u_0^{(+)} + \frac{\lambda}{2\Delta} F_0^\ast D^{(-)}_{0(+)}}{1 + \lambda F_0^\ast}. \tag{42}
\]

and we again find

\[
\omega^2 D^{(-)}_{0(+)} = 4\omega \Delta \delta u_0. \tag{43}
\]

In these 2 examples, we see the dynamical equations for \( d_0 \) and \( D^{(-)}_{0(+)} \) are not even modified by \( F_0^\ast \). By dimension counting, it could be seen that the renormalization of self-energy are given by the following relation

\[
(1 + c_1 F_\lambda) \delta \varepsilon = \delta u + c_2 \omega^2 \lambda F D, \tag{44}
\]

where \( c_1 \) and \( c_2 \) are dimensionless numbers of order 1. It can be implied gapless modes cannot be gapped by short-ranged interactions parametrized by Landau parameters.

B. Massive Sub-gap Modes

In this section let us continue to examine how Landau parameters renormalize massive modes. We start with equation (27b). Take \( \ell = \pm 2 \) component of (41).

\[
B : \delta \varepsilon_{\pm 2}^{(+)} = \frac{\delta u_{\pm 2} + \frac{\omega \lambda}{2\Delta} F_2^\ast D^{(-)}_{\pm 2}}{1 + \lambda F_2^\ast}. \tag{45}
\]

Plugging this back into (27b) renormalizes the solutions as

\[
D^{(-)}_{\pm 2} = \frac{2\omega \Delta \delta u_{\pm 2}}{(\omega^2 - 4\Delta^2)} + \frac{1}{2} \lambda F_2^\ast (\omega^2 - 4\Delta^2). \tag{46}
\]

The new mass is given by the zero of the denominator. In the limit \( |F_2^\ast| \ll 1 \),

\[
\omega^2 \simeq 2\Delta^2 (1 + \frac{1}{2} \lambda F_2^\ast). \tag{47}
\]

\( \lambda \) is a positive number of order 1. From this expression we can see modes get heavier for repulsive interactions \( F_2^\ast > 0 \) and soften for attractive interactions \( F_2^\ast < 0 \).

Next let us look at the amplitude mode (29b) sourced by spin-dependent quasi-particle energy.

\[
\delta \varepsilon^{(+)}_z(\theta) = \delta h_z + \int \frac{d\theta'}{2\pi} F^a(\theta, \theta') \left[ -\lambda \delta \varepsilon^{(+)} + \frac{i\omega}{2\lambda} (\Delta \times d^{(+)})_z \right]. \tag{48}
\]

Projecting it to \( \ell = \pm 2 \) modes,

\[
\delta \varepsilon^{(+)\pm 2} = \frac{\delta h_{\pm 2} + F_0^\ast \frac{\lambda}{2\Delta} D^{(+)\pm 2}}{1 + \lambda F_0^\ast}. \tag{49}
\]

Substituting this back into (29b) yields

\[
D^{(+)\pm 2} = \frac{\pm 2\omega \Delta \delta h_{\pm 2}}{(\omega^2 - 4\Delta^2)} + \frac{1}{2} \lambda F_2^\ast (\omega^2 - 4\Delta^2). \tag{50}
\]

Therefore, the mass correction is given by the same transcendental equation with the replacement \( F_2^\ast \rightarrow F_2^\ast \).

We are now ready to repeat the above computation for general chiral ground states. As it can be inferred from the previous analyses, the equations for singlet-pairing states are identical to ones for the longitudinal components \( (d \cdot \Delta) \) of the triplet-pairing states. Hence, we will concentrate on triplet-pairing states and take \( L = 1 \) without loss of generality since higher \( L \) states have the same algebraic forms.

The main difference between the preceding analyses and the one for chiral states is that the gap function can no longer be chosen real by a gauge transformation. Consequently, the scalar self-energy would satisfy the equation

\[
\delta \varepsilon^{(+)} = \delta u^{(+)}_{\text{ext}} + \int \frac{d\theta'}{2\pi} F(\theta, \theta') \left[ -\lambda \delta \varepsilon^{(+)} + \frac{1}{2}\omega \lambda (\Delta \cdot d - \Delta \cdot d') \right]. \tag{51}
\]
Let us take the $z$ component of (32a) and (32b)
\[
\begin{align*}
(\omega^2 - 2\Delta^2)d_{1z} &= \omega \Delta \delta \varepsilon_z^{(+)} \tag{52} \\
(\omega^2 - 2\Delta^2)d'_{-1z} &= -\omega \Delta \delta \varepsilon_{-2}^{(+)} \tag{53}
\end{align*}
\]
Renormalizing $\delta \varepsilon_z^{(+)}$ with (51), we find
\[
\begin{align*}
d_{1z} &= \frac{\omega \Delta \delta u_{z}^{(+)}}{(\omega^2 - 2\Delta^2) + \frac{1}{2} \lambda F_0^2 (\omega^2 - 4\Delta^2)} \tag{54a}
d'_{-1z} &= \frac{\omega \Delta \delta u_{-2}^{(+)}}{(\omega^2 - 2\Delta^2) + \frac{1}{2} \lambda F_0^2 (\omega^2 - 4\Delta^2)} \tag{54b}
\end{align*}
\]
Finally we look at the transverse fluctuation by looking at the $x$ component.
\[
\begin{align*}
(\omega^2 - 2\Delta^2)d_{1x} &= -i \omega \Delta (\hat{z} \times \delta \varepsilon_z^{(+)})_x \tag{55a}
(\omega^2 - 2\Delta^2)d'_{-1x} &= -i \omega \Delta (\hat{z} \times \delta \varepsilon_{-2}^{(+)})_x \tag{55b}
\end{align*}
\]
The spin-dependent self-energy now takes the form
\[
\hat{z} \times \delta \varepsilon^{(+)} = \hat{z} \times \delta \mathbf{h}^{(+)} + \int \frac{d\theta '}{2\pi} F^n(\theta, \theta ') \left[ -\lambda \hat{z} \times \delta \varepsilon^{(+)} + i \frac{\omega}{2} \hat{z} \times (\mathbf{d}^* \times \mathbf{d}) \right] . \tag{56}
\]
Projecting it to $\ell = \pm 2$ allows to solve
\[
d_{1x} = \frac{-i \omega \Delta (\hat{z} \times \delta \mathbf{h}^{(+)}_2)}{(\omega^2 - 2\Delta^2) + \frac{1}{2} \lambda F_0^2 (\omega^2 - 4\Delta^2)} . \tag{57}
\]
To sum up, the analyses in this section have shown the following: (i) The Goldstone modes are not gapped by short-range interaction parametrized by Landau parameters. (ii) For $p$-wave superconductors in both B-phase and A-phase, the sub-gap modes $\omega^2 = 2\Delta^2$ receives renormalization from quadrupolar Landau parameters $F_2^2$ or $F_2^s$. (iii) For all chiral ground states of finite orbital momenta $L$, the sub-gap modes longitudinal to their ground states receive mass renormalization from the channel $F_{2L}^2$. The mass corrections referred to in (ii) and (iii) are all determined by the following equation.
\[
(\omega^2 - 2\Delta^2) + \frac{1}{2} \lambda (\omega) F (\omega^2 - 4\Delta^2) = 0 . \tag{58}
\]
In figure 1 we plot the numerical solution to (58) as a function of $F$, which stands for the Landau parameter of the channel of interest. In accord with the intuition we acquired from the small $F$ expansion, a strong repulsive interaction in particle-hole channel increases the magnitude of the gap, which asymptotically approaches pair-breaking threshold $2\Delta$. On the other hand, an attractive interaction softens the mass of order parameter. In particular, we see the mode would become massless as $F = -1$, at which Pomeranchuk instability of 2 dimensional Fermi liquid is triggered.

We can then look at the region $F = -1 + \epsilon$ with $\epsilon \ll 1$. At $T = 0$ we can expand the equation (58) around $\omega^2 \approx 0$ and extract its dependence on $F$ near the instability. Using the closed form (A5a), we can deduce
\[
\omega^2 = \frac{3(1 + F)}{6 + F} \times 4\Delta^2 \approx \frac{12}{5} \epsilon \Delta^2. \tag{59}
\]
This expression allows us to study how this mode becomes massless as we approaches the instability.

Knowing the mass corrections are expressed in terms of $F_l$, where $l \geq 2$. To complete the argument we would like to seek means of determining Landau parameters with other physical predictions or some specific microscopic model. The cleanest prediction involving $F_2^2$ is the difference between the speeds of zero sound and first sound, which, in the 2 dimensional normal state, can be shown to be
\[
\frac{c_0^2}{c_1^2} = \frac{1}{2} \frac{1 + F_2^2}{2 + F_0^2}. \tag{60}
\]
with corrections of order $(v_F q / \omega)^4$.

\section{V. GAP ANISOTROPY}

At the end of last section we have shown the 6-fold degeneracy in the A-phase of $p$-wave superconductor can be lifted with finite $F_2^2$ and $F_2^s$ and similar conclusions apply to other chiral ground states. In this section we consider another mechanism, gap anisotropy that is able to break the degeneracy in the bosonic spectrum in chiral ground states.

Microscopically, gap isotropy can be destroyed as there are multiple active bands with substantial superconducting instabilities. Taking Sr$_2$RuO$_4$ again as the prototype, the substantial hybridization of quasi-1 dimensional $\alpha$ and $\beta$ bands breaks cylindrical symmetry of the $\gamma$ band and results in strong anisotropy in gap amplitude [20].

In this work we consider a phenomenological approach, examining the following model
\[
\Delta e^{iL \theta} I(\epsilon; \theta), \tag{61}
\]
where $\lim_{\epsilon \to 0^+} I(\epsilon; \theta) = 1$ and study the fluctuation longitudinal to ground state. To further simplify, we also assume
The normal mode can be identified as the roots to the equation \( \det \Lambda = 0 \). Suppose we measure the energy in terms of the scaled frequency \( \tilde{\omega}^2 = \omega^2/(4\Delta^2) \). The characteristic equations are then

\[
\tilde{\omega}^2 = 0, \quad 2\tilde{\omega}^2(\tilde{\lambda}_{00}^2 - \tilde{\lambda}_{0L}^2) = \tilde{\lambda}_{00}(\tilde{\lambda}_{10} - \tilde{\lambda}_{1(2L)}). \tag{64a}
\]

\[
2(\tilde{\omega}^2)(\tilde{\lambda}_{00}^2 - \tilde{\lambda}_{0L}^2) + \tilde{\omega}^2(4\tilde{\lambda}_{0L}\tilde{\lambda}_{1L} - 3\tilde{\lambda}_{00}\tilde{\lambda}_{10} - \tilde{\lambda}_{00}\tilde{\lambda}_{1(2L)}) + (\tilde{\lambda}_{10}^2 - 2\tilde{\lambda}_{1L}^2 + \tilde{\lambda}_{10}\tilde{\lambda}_{1(2L)}) = 0, \tag{64c}
\]

where

\[
\tilde{\lambda}_{nm} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (I^2)^n \cos 2m\theta \cdot \tilde{\lambda}(\theta; \omega). \tag{65}
\]

In the limit \( \epsilon \to 0 \), the gap recovers isotropy. Every \( \tilde{\lambda}_{nm} \) with non-vanishing column index vanishes and \( \tilde{\lambda}_{00} = \tilde{\lambda}_{L0} \). The above equations reduce to

\[
\tilde{\omega}^2 = 0, \tag{66a}
\]

\[
\tilde{\omega}^2 = 1/2, \tag{66b}
\]

\[
(2\tilde{\omega}^2 - 1)(\tilde{\omega}^2 - 1) = 0. \tag{66c}
\]

One resumes the results fund in section III. We see that the Goldstone mode persists, while the Higgs boson receives correction owing to the change in the pairing breaking threshold. Naively thinking, sub-gap massive modes engage in different equations and the breakdown of their degeneracy is anticipated. In the rest of this section, we call the sub-gap mode satisfying (64b) mode 1, and the one satisfying (64c) mode 2.

To investigate the effect of gap anisotropy more concretely, we imagine \( I^2 \) is decomposed into different angular modes and first look at the leading harmonics.

\[
I^2 = (1 + \epsilon \cos n\theta), \quad n \in \mathbb{N}. \tag{67}
\]

This model preserves the average amplitude \( \langle |\Delta|^2 \rangle \) but locally reduces the gap up to the amount ratio of \( (1 - \epsilon) \), which in turn reduces the threshold of Cooper pairing breaking. In these models, \( \langle \sqrt{T^2} \rangle \leq 1 \) for \( 0 < |\epsilon| < 1 \). The sub-gap modes can be regarded as a generalized class of Higgs modes which comes from the Hybridization of original Goldstone and Higgs bosons. The scale of their masses should therefore be determined by the amplitude of the ground state condensate. As a consequence, we expect the reduction of \( \langle |\Delta| \rangle \) would soften at those sub-gap modes.

We computed the mass corrections to masses \( \sqrt{2}\Delta \) for \( L = 1 \) and \( L = 2 \) and we see the previous two intuitions are not entirely accurate.

For \( L = 1 \) (spin-singlet pairing), we consider harmonics \( 1 \leq n \leq 8 \) and plot the numerical solutions of \( \tilde{\omega}^2 = (\omega/(2\Delta))^2 \) as a function of \( \epsilon \) in figure 2. For \( n \neq 2, 4 \), the magnitudes of \( \tilde{\omega}^2 \)'s decrease. However, the effect of degeneracy breaking is extremely weak. For \( n = 1 \), the energy split is only of order \( 10^{-4} \times (4\Delta^2) \). Moreover for \( n = 3, 5, 6, 7 \) and 8, all solutions collapse to the same curve as we can see from the mid panel in figure 2.

On the other hand, in the right panel of figure 2, we find the degeneracy breaking is strong only for \( n = 2 \) and \( n = 4 \). The magnitude of the renormalized masses is a result of competition between level repulsion and reduced mass scale \( \sqrt{T^2} \). The former would increase the magnitude of mode 1 and the latter makes it decrease. In particular, for \( n = 4 \) mode 1 receives strong level repulsion effect at small anisotropy and acquires an increase in its mass. For other modes, the degeneracy breaking effect is either small or numerically indistinguishable.

Similar results are found for the \( L = 2 \) chiral ground state, for which we also perform the computation for \( 1 \leq n \leq 8 \). For \( n = 1 \) and 2, the magnitudes of decrease and degeneracies are broken weakly. For \( n = 3, 5, 6, 7 \), the magnitudes reduce, but the degeneracies are not broken. Only \( n = 4 \) and \( n = 8 \) perturbations exhibit strong level repulsion effects.

That many angular modes are not able to break the degeneracy between mode 1 and mode 2 can be seen perturbatively from (64b) and (64c). Since we are looking at \( \tilde{\omega} \sim 1/2, \tilde{\lambda}_{ab} \) can safely be expanded as a power series of \( \epsilon \). It then can be shown that if the perturbation mode \( n \) does not equal \( 2L \) or \( 4L \), \( \tilde{\lambda}_{0L}, \tilde{\lambda}_{1L} \) and \( \lambda_{1(2L)} \sim \epsilon^2 \). Consequently, to first order in \( \epsilon \) (64c) can be factorized. One of the roots is essentially (64b) and the degeneracy is not broken at this order. From this perspective, we can also generalize the numerical results for \( p \)-wave and \( d \)-wave chiral states to a general chiral ground state of angular momentum \( L \). Strong level repulsion and degeneracy lifting shall occur owing to angular perturbations carrying \( 2L \) and \( 4L \). Heuristically, they can be understood as the difference of momenta between \( d_L \) and \( \Delta^2 d'_{L} \) and \( \Delta^2 d'_{L} \). As a consequence, the only circumstance that the latter 2 terms could interfere \( d_L \) is when they couple respectively to \( n = 2 \) and \( n = 4 \) harmonics.

VI. CONCLUSION

In conclusion, we revisit a class of 2 dimensional superfluid/superconductor Hamiltonians. Using the semi-classical kinetic equation, we compute the order parameter collective modes for 2 dimensional B-phase and general chiral ground states of angular momenta \( L \geq 0 \). Extending the known results for \( L = 1 \), we show that the sub-gap modes of the uni-
FIG. 2. Solutions to (64b) and (64c) for the $p$-wave chiral ground state $L = 1$ and the model (67) with $1 \leq n \leq 8$. Inset: In the right panel, the splitting of degeneracy is shown after zoom in. In the middle panel, no splitting is found.

FIG. 3. Solutions to (64b) and (64c) for the $d$-wave chiral ground state $L = 2$ perturbed by the model (67) with $1 \leq n \leq 8$. Inset: In the top left and the top right panels the splitting of degeneracies is shown after zoom in. In the bottom left panel, no degeneracy splitting is found.

versal mass value $m = \sqrt{2} \Delta$ exist for all chiral ground states $L \geq 1$. By renormalizing the fermionic self-energy, we calculate the correction of these sub-gap modes from Fermi-liquid corrections and discover those sub-gap modes are sourced by $F_{2L}$, where $L$ is the angular momentum of their underlying ground state. The masses increase for repulsive fermionic interactions and soften for attractive ones. Remarkably, renormalized sub-gap modes become gapless when the Pomeranchuk instability in the corresponding channel is triggered.

We further investigate the degeneracy breaking effect owing to ground state gap anisotropy led by potential microscopic band mixing using a phenomenological approach. The results suggest that not all harmonics are pivotal to level repulsion. In the class of the perturbations considered, given a chiral ground state characterized by angular momentum $L$, level repulsion effect is significant only when the perturbation is $2L$th harmonics or $4L$th harmonics.

To the author’s knowledge, whether these modes are damped at finite wavelength and how they participate in the low energy dynamics in the presence of gauge fields or near a quantum critical point are all appealing but open questions. The author hopes the approach and conclusion drawn from
The author thanks E. Berg, K. Levin and J. A. Sauls for valuable suggestions, and grateful for Dam Thanh Son, Yu-Ping Lin and Chien-Te Wu for comments on the manuscript. This work could provide the studies of quantum Hall nematic physics and nematic phase in High-Tc superconductivity a complementary perspective and new insights.

ACKNOWLEDGMENTS

In the main text, we found that for p-wave superfluids, the quadrupolar Landau parameter $F_2^\pi$ plays an important role in renormalizing the masses of the order parameter collective modes. In this section we consider the sound propagation in a 2 dimensional Fermi liquid and derive observables which also depend on aforementioned Landau parameters, especially $F_2^\pi$, such that one measurement could provide predictive power for other computations.

We will first generalize the fact that $F_2^\pi$ appears in the difference between the speeds of the zero sound and the first sound to 2 dimensions. We consider an idealized case where only $F_0$, $F_1$, and $F_2$ are significant, i.e., that $F_\ell = 0$ for $\ell \geq 3$. In normal phase, we parametrize the fluctuation around Fermi surface by $\delta n_k(\omega, q) = -\frac{\partial n_k^0}{\partial \omega} \nu_k$ and $\nu_k(\omega, q) = \sum_m e^{-im\theta} \nu_m$. In terms of the variable $\nu_k$, the kinetic equation assumes the form

$$(s - \cos \theta) \nu_k - \cos \theta \sum_m e^{-im\theta} F_m \nu_m = \cos \theta U,$$ (B1)

where $s = \omega/q$, and $U = U(\omega, q)$ denotes the external potential. Suppose only the scalar channel of $U$ is turned on. We project the kinetic equation (B1) to $\ell = 0$ and $\ell = 1$, and eliminate $\nu_{\pm 1}$. These operations give us

$$\left(s^2 - \frac{1}{2}(1 + F_0^\pi)(1 + F_1^\pi)\right) \nu_0$$ (B2)

$$- \frac{1}{4}(1 + F_1^\pi)(1 + F_2^\pi)(\nu_2 + \nu_{-2}) = \frac{1 + F_1^\pi}{2} U.$$

$s^2 = \frac{1}{2}(1 + F_0^\pi)(1 + F_1^\pi)$ is the square of 2 dimensional first sound speed. To proceed, we reformulate the kinetic equation as

$$\nu_n + \sum_{m=-\infty}^\infty \Omega_{n-m} F_m \nu_m = -\Omega(n) U,$$ (B3)

where the integral $\Omega$ is

$$\Omega(n-m) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi \cos \theta - s} e^{i(n-m)\theta}.$$ (B4)

It is straightforward to see $\Omega_n = \Omega_{-n}$, $\Omega_1 = s\Omega_0$ and $\Omega_2 + \Omega_1 = 2s\Omega_2$. Using the equations of $n = 1$ and $\pm 2$ in Eqn. (B3), we are able to solve $\nu_2 + \nu_{-2}$ in terms of $\nu_0$.

$$\nu_2 + \nu_{-2} = \frac{2\Omega_2 \nu_0}{\Omega_0 + F_2^\pi (\Omega_0 + \Omega_4) \Omega_0 - 2\Omega_2^2}.$$ (B5)

Plugging it back to (B2), we obtain

$$\left(s^2 - \frac{1}{2}(1 + F_0^\pi)(1 + F_1^\pi) - \frac{1}{2}(1 + F_1^\pi) \right) \nu_0$$ (B6)

$$\times (1 + F_2^\pi)\frac{\Omega_2}{\Omega_0 + F_2^\pi (\Omega_0 + \Omega_4) \Omega_0 - 2\Omega_2^2} \nu_0$$

$$= \frac{1 + F_1^\pi}{2} U.$$ (B7)

Appendix A: $\gamma$ and the Tsunedo function $\lambda$

The integral $\gamma$ is

$$\int_0^\infty \frac{dx}{x^2} \frac{1}{2} \tanh \left( \frac{x}{2T} \right) = \frac{2\pi}{\sqrt{\Delta^2 - |\Delta|^2}} \gamma,$$ (A1)

It is formally divergent, but can be regularized and identified by $1/V_i$ (or $1/(2V_0)$) using linearized gap equation.

The function $\lambda$ was first introduced by Tsunedo as a kind of Cooper pair susceptibility.

$$\lambda(p; \omega, q) = \frac{\lambda}{|\Delta(p)|^2} = \int_0^\infty \frac{dx}{2\pi i} \frac{n(x-)}{2\pi x} \frac{(2\pi x - \eta^2)}{(2\pi x + \eta^2)} + 4|\Delta|^2 \eta^2,$$ (A2)

where $\eta = v_F q \cdot \hat{p}$ and

$$n(x) = \frac{2\pi i s n(x)}{\sqrt{\Delta^2 - |\Delta|^2}} \Theta(\Delta^2 - |\Delta|^2) \tanh \frac{x}{2T}.$$ (A3)

In $q \to 0$ limit, the integral reduces to

$$\lambda = |\Delta|^2 \int_0^\infty \frac{dx}{2\pi i} \frac{\sqrt{\Delta^2 - \omega^2/4}}{\Delta^2 \omega^2 - \omega^2/4}.$$ (A4)

These expressions can be used in numerical evaluation. This function actually has an analytic closed form in the limit $T \to 0$. Writing $x = \omega/(2|\Delta|)$,

$$\lambda(\omega) = \frac{\sin^{-1} x}{x \sqrt{1 - x^2}}, \ |x| < 1,$$ (A5a)

where as for $|x| > 1$,

$$\lambda(\omega) = \frac{1}{2x \sqrt{x^2 - 1}} \left[ \log \left( \frac{x^2 - 1}{x^2 + 1} \right) + i \pi n(x) \right].$$ (A5b)
In high frequency limit, to order \( s^{-2} \) we have \( \Omega_1 = -\frac{1}{2} s^{-2} \), \( \Omega_2 = -\frac{1}{2} s^{-2} \), and \( \Omega_4 = 0 \), the dispersion relation can be solved to be
\[
\frac{s_0^2 - s_i^2}{s_i^2} = \frac{1 + E_2}{2 + E_0}.
\]
(B8)

This is a natural generalization of the known textbook result.

**Appendix C: Full Dynamical Equations**

In this section we sketch the steps for inverting the kinetic equation and give the full dynamical equations at finite wavelength. Expanding (2) with the ground state of interest, we could find components of the Keldysh Green’s function satisfy a general equation
\[
\Omega |\tilde{g}\rangle = M |\tilde{\sigma}\rangle.
\]
(C1)

The quasi-classical Green’s functions can thus be obtained via
\[
\int \frac{d\varepsilon}{2\pi i} |\tilde{g}\rangle = \int \frac{d\varepsilon}{2\pi i} \Omega^{-1} M |\tilde{\sigma}\rangle.
\]
(C2)

We note that when performing \( \varepsilon \) integral in this work, the particle-hole symmetry \( \varepsilon \leftrightarrow -\varepsilon \) is assumed.

The defined in (C1) the matrices are
\[
\Omega = \begin{bmatrix}
-\eta & \omega & 2i\Delta_I & -2\Delta_R \\
\omega & -\eta & 0 & 0 \\
2i\Delta_I & 0 & -\eta & 2\varepsilon \\
2\Delta_R & 0 & 2\varepsilon & -\eta \\
\end{bmatrix}
\]
(C3)

and
\[
M = \begin{bmatrix}
0 & -m_a & -in_s\Delta_I & n_s\Delta_R \\
-m_a & 0 & -n_s\Delta_R & in_a\Delta_I \\
-i\Delta_I n_s & \Delta_R n_a & 0 & -m_s \\
-n_s\Delta_R & in_a\Delta_I & -m_s & 0 \\
\end{bmatrix}.
\]
(C4)

\( \Delta_R \) and \( \Delta_I \) are the real and imaginary parts of the gap function. In terms of the \( n \) defined by (A3), the elements in \( M \) are
\[
n_s = n(\varepsilon_+) + n(\varepsilon_-) \quad \text{(C5a)}
\]
\[
n_a = n(\varepsilon_+) - n(\varepsilon_-) \quad \text{(C5b)}
\]
\[
m = \varepsilon n(\varepsilon) \quad \text{(C5c)}
\]
\[
m_s = m(\varepsilon_+) + m(\varepsilon_-) \quad \text{(C5d)}
\]
\[
m_a = m(\varepsilon_+) - m(\varepsilon_-). \quad \text{(C5e)}
\]

In the rest of the section we give the proper combinations \( |\tilde{g}\rangle \) and \( |\tilde{\sigma}\rangle \) and the complete dynamical equations.

1. **singlet-pairing ground state**

For a singlet-pairing state, the bosonic fluctuation couples only to spin independent fermionic self-energies, and the relevant equations are those which \( \delta \tilde{g}, \delta \tilde{g}', d \) and \( d' \) obey. These equations can be easily solved by taking
\[
|\tilde{g}\rangle = \begin{pmatrix}
\delta \tilde{g}(-) \\
\delta \tilde{g}'(+), \delta \tilde{\sigma} \\
\delta f(+), \delta f'(-)
\end{pmatrix} \quad |\tilde{\sigma}\rangle = \begin{pmatrix}
\delta \tilde{g}'(-) \\
\delta \tilde{g}' (+) \\
\delta f'(-), \delta f'(+)
\end{pmatrix}.
\]
(C6)

Expressing \( |\tilde{g}\rangle \) in terms of \( |\tilde{\sigma}\rangle \) and performing convolutions with suitable potentials would imply the following equations.

\[
\delta \varepsilon(-)(\hat{p}; \omega, q) - \delta \varepsilon(+)_{\text{ext}} = \int \frac{d\theta'}{2\pi} A^*(\theta, \theta') \left\{ \left( 1 + (1 - \lambda(\hat{p}')) - \frac{\eta^2}{\omega^2 - \eta^2} \right) \delta \varepsilon(-)(\hat{p}') + \frac{\lambda(\hat{p}')\eta'}{2} \right\}. \quad \text{(C7a)}
\]
\[
\delta \varepsilon(+)(\hat{p}; \omega, q) - \delta \varepsilon(+)_{\text{ext}} = \int \frac{d\theta'}{2\pi} A^*(\theta, \theta') \left\{ \frac{\omega\eta'}{\omega^2 - \eta^2} (1 - \lambda(\hat{p}')) \delta \varepsilon(-)(\hat{p}') \\
+ \frac{\omega^2}{\omega^2 - \eta^2} (1 - \lambda(\hat{p}')) \delta \varepsilon(-)(\hat{p}') - \frac{1}{2} \omega \lambda(\hat{p}') [d(\hat{p}') \Delta^*(\hat{p}') - d(\hat{p}') \Delta^*(\hat{p}')] \right\}. \quad \text{(C7b)}
\]
\[
d(\hat{p}; \omega, q) = \int \frac{d\theta'}{2\pi} V_e(\theta, \theta') \left\{ \left( \gamma + \frac{1}{4} \lambda(\hat{p}')(\omega^2 - \eta^2 - 2|\Delta(\hat{p}')|^2) \right) d(\hat{p}') \\
- \frac{\lambda(\hat{p}')}{2} \Delta^2(\hat{p}') d'(\hat{p}') - \frac{\Delta(\hat{p}')\lambda(\hat{p}')}{4} (\eta^2 \delta \varepsilon(-)(\hat{p}') + \omega \delta \varepsilon(+)\hat{p}')) \right\}. \quad \text{(C7c)}
\]
\[ d' (p'; \omega, q) = \int \frac{d\theta'}{2\pi} V_c (\theta, \theta') \left\{ \left( \gamma + \frac{1}{4} \bar{\lambda} (p') (\omega^2 - \eta'^2 - 2|\Delta (p')|^2) \right) d' (p') - \frac{\bar{\lambda} (p') (\Delta^* (p'))^2 d(p')}{2} + \frac{\Delta' (p') \bar{\lambda} (p')}{4} \right\} \]

\[ \frac{i \delta f (-)}{\delta g (-)} , \left\{ \frac{\delta f (-)}{\delta \bar{g} (-)} , \frac{\delta f (+)}{\delta \bar{g} (+)} \right\} \]. (C8a)

The part coupled with spin-dependent \( \delta g \), on the other hand, includes the transverse and binormal parts of the anomalous Green's function.

\[ \{ \hat{n} \times \delta g (-), \hat{n} \times \delta g (+), i \hat{\delta} f (-), i \hat{\delta} f (+) \}. \] (C8b)

The above 2 sets of vectors give only the binormal and transverse information about \( \delta g \). It turns out the spin-singlet components \( \delta f \) and \( d \) are required to access the longitudinal information of \( \delta g \) using the combination below.

\[ \{ \delta g_L (-), \delta g_L (+), \delta e_L (-), \delta e_L (+), \delta f (-), \delta f (+), d (-), d (+) \}. \] (C8c)

These spin-singlet degrees of freedom \( \delta f \) and \( d \) are treated as external sources and turned off at the end of computation. After solving all above \( \vec{g} \) in terms of \( \vec{\sigma} \), we could again make use of (10a), (10b), (12a), and (12b) to obtain the following equations.
\[ \delta \epsilon^{(+)}(\mathbf{p}; \omega, \mathbf{q}) = \int \frac{d\theta'}{2\pi} A^a(\theta, \theta') \left\{ \frac{\omega^2}{\omega^2 - \eta^2} (1 - \lambda(\mathbf{p}')) \delta \epsilon^{(+)}(\mathbf{p}') + \frac{\eta' \omega}{\omega^2 - \eta'^2} (1 - \lambda(\mathbf{p}')) \delta \epsilon^{(-)}(\mathbf{p}') \\
+ \lambda(\mathbf{p}') [\delta \epsilon^{(+)}(\mathbf{p}') \cdot \mathbf{n}(\mathbf{p}')] \mathbf{n}(\mathbf{p}') - \frac{i \omega}{2} \lambda(\mathbf{p}') [\Delta^{*}(\mathbf{p}') \times d(\mathbf{p}') + \Delta(\mathbf{p}') \times d'(\mathbf{p}')] \right\}. \quad (C9d) \]

\[ d(\mathbf{p}; \omega, \mathbf{q}) = \int \frac{d\theta'}{2\pi} V_o(\theta, \theta') \left\{ \left( \gamma + \frac{1}{4} \lambda(\mathbf{p}') (\omega^2 - \eta'^2 - 2|\Delta(\mathbf{p}')|^2) \right) d(\mathbf{p}') \\
- \frac{1}{4} \lambda(\mathbf{p}') \Delta(\mathbf{p}') [\eta' \delta \epsilon^{(-)}(\mathbf{p}') + \omega \delta \epsilon^{(+)}(\mathbf{p}')] + \frac{1}{4} \lambda(\mathbf{p}') i \Delta(\mathbf{p}') \times (\eta' \delta \epsilon^{(-)}(\mathbf{p}') + \omega \delta \epsilon^{(+)}(\mathbf{p}')) \\
+ \frac{1}{2} \lambda(\mathbf{p}') [(\Delta(\mathbf{p}') \cdot \Delta(\mathbf{p}')) d'(\mathbf{p}') - 2(\Delta(\mathbf{p}') \cdot d'(\mathbf{p}')) \Delta(\mathbf{p}')] \right\}. \quad (C9e) \]

\[ d'(\mathbf{p}; \omega, \mathbf{q}) = \int \frac{d\theta'}{2\pi} V_o(\theta, \theta') \left\{ \left( \gamma + \frac{1}{4} \lambda(\mathbf{p}') (\omega^2 - \eta'^2 - 2|\Delta(\mathbf{p}')|^2) \right) d'(\mathbf{p}') \\
+ \frac{1}{4} \lambda(\mathbf{p}') \Delta^{*}(\mathbf{p}') [\eta' \delta \epsilon^{(-)}(\mathbf{p}') + \omega \delta \epsilon^{(+)}(\mathbf{p}')] + \frac{1}{4} \lambda(\mathbf{p}') \Delta^{*}(\mathbf{p}') \times (\eta' \delta \epsilon^{(-)}(\mathbf{p}') + \omega \delta \epsilon^{(+)}(\mathbf{p}')) \\
+ \frac{1}{2} \lambda(\mathbf{p}') [(\Delta^{*}(\mathbf{p}') \cdot \Delta^{*}(\mathbf{p}')) d(\mathbf{p}') - 2(\Delta^{*}(\mathbf{p}') \cdot d(\mathbf{p}')) \Delta^{*}(\mathbf{p}')] \right\}. \quad (C9f) \]