Quantum gravity, dynamical triangulations and higher derivative regularization

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Abstract

We consider a discrete model of euclidean quantum gravity in four dimensions based on a summation over random simplicial manifolds. The action used is the Einstein-Hilbert action plus an $R^2$-term. The phase diagram as a function of the bare coupling constants is studied in the search for a sensible continuum limit. For small values of the coupling constant of the $R^2$ term the model seems to belong to the same universality class as the model with pure Einstein-Hilbert action and exhibits the same phase transition. The order of the transition may be second or higher. The average curvature is positive at the phase transition, which makes it difficult to understand the possible scaling relations of the model.

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1 Introduction

Understanding the theory of quantum gravity remains one of the greatest challenges in theoretical physics. One can try to circumvent the problem by embedding gravity in a larger theory like string theory. This is in many respects an appealing approach, but it seems to have lost some of its momentum and there is not much hope that it will be possible in the near future to arrive in a natural way to an effective theory of gravity (and matter) in four dimensions. Discussing basic principles one should maybe not be so worried about our technical inability to deduce the consequences of string theory since this is not the first time in theoretical physics we are unable to extract anything but the simplest perturbative consequences of an otherwise healthy and probably correct theory. It is more worrisome that string theory is not (yet) well defined beyond the loop expansion. Again this might not seem so disastrous since the same was (and to some extent still is) true for ordinary field theory. However, in the last three years there has been a significant progress in our understanding of the problems connected with the summation of all loops in string theory. The message has not been encouraging. At the moment we have no general principles which allow us to define in an unambiguous way the summation over all genera in string theory. From this point of view it might be somewhat premature to announce string theory as the fundamental theory. Strictly speaking it is not yet a theory but a set of rules which allow us to calculate certain perturbative quantities.

If we decide to drop string theory as the theory which will teach us the nature of quantum gravity it might be (good ?) conservative policy to stay entirely within the ordinary field theoretical framework. At first glance it does not look too promising either. The four dimensional theory is hampered by being non-renormalizable and we do not at present know any example where such a theory can be defined non-perturbatively, is non-trivial and at the same time satisfies what we usually view as the axioms of quantum field theory. At the same time one of the lessons of the last thirty years is that field theory is deeply connected to the theory of critical phenomena via the path integral and has a natural formulation in Euclidean space-time. But precisely for gravity the Einstein-Hilbert action is unbounded from below due to the conformal mode and the Euclidean path integral is ill-defined. One could try to make sense of the unboundedness of the action either by a contour rotation associated with the conformal mode as suggested by Hawking and others[1] or by stochastic regularization (the so-called fifth time action) as advocated by Greensite and Halpern [2, 3]. It is not the purpose here to enter into a discussion of the virtues and drawbacks of these interesting suggestions. Let us only mention that they do
not really have the flavour of general descriptions based on ordinary field theory. Of course it is wise to bear in mind that if any field theory should depart from basic axiomatic principles it is quantum gravity, but lacking a general alternative we have decided to return to analysis of quantum gravity in the context of ordinary field theory.

Field theory suggests one rather simple minded way out of the above mentioned problems. This was already discussed long ago by Weinberg who called it asymptotic safety. The idea is simply that when we, by means of the renormalization group equations, work our way back from the infrared fixed point where the Einstein-Hilbert action seems a good effective description we will at some point reach a non-trivial ultraviolet fixed point. In addition the associated critical surface is assumed to be finite dimensional, which means that only a finite number of parameters are left arbitrary in the theory, which from this point of view can be said not to differ much from ordinary renormalizable field theories. The effective Lagrangian description of the theory by means of fields suitable for the infrared fixed point might then be an infinite series

$$L_G = \sqrt{g} \left[ \Lambda - \frac{1}{16\pi G} R + f_2 R^2 + f'_2 R_{\mu\nu} R^{\mu\nu} + \cdots \right]$$

(1.1)

which might even be non-polynomial, but which might now (and we will assume this is the case) make sense if we make a formal rotation to Riemannian space where the metric has the signature \{+1, +1, +1, +1\} (the generalization of the rotation to Euclidean space in ordinary field theory).

Of course one weakness in this scenario is that the existence of the ultraviolet fixed point has been entirely hypothetical. Further we have not exactly been flooded with examples of this kind in field theory, as already mentioned. Finally, and we agree that this point is an annoyance, such a solution does not have the appeal and aesthetical beauty of the original theory which we want to quantize. If the quantum theory of gravity offers to us a solution like (1.1) the next task must be to find a simpler description in terms of other variables, maybe somewhat like the switch in hadronic physics from hadrons to quarks and gluons.

A few things have happened since Weinberg outlined the above strategy. There exist now regularizations of the path integral which allow us to define theories like (1.1) non-perturbatively and further the progress in computer science has made it possible to calculate approximately these path integrals. It is therefore possible to explore, by numerical means, the phase diagram of the regularized theory and try to locate phase transition points in the coupling constant space. If the transitions are of second order one can attempt to define a continuum limit. The approach has
one virtue: it requires only a finite amount of work to verify whether the idea of a
non-trivial ultraviolet fixed point is viable or not.

The rest of this article is organized as follows: In section 2 we define the
discretized model to be used to regularize the path integral. Section 3 provides some
details about the numerical method used, while section 4 describes the results of ex-
tensive numerical simulations. Section 5 contains a discussion of the results obtained
so far.

2 The discretized model

The continuum theory of gravity is reparametrization invariant. If we discretize the
theory in order to regularize it we will have to break this invariance provided the
action depends on the metric. An alternative is to consider theories which depend
only on topology. A very interesting approach in this direction in three dimensions
has recently been followed by many people following the work of Turaev and Viro [5]
and it has now been generalized to four dimensions [6]. Unfortunately the precise
connection with the usual continuum version of Einstein gravity is not yet clear,
especially in four dimensions.

If we restrict ourselves to the conservative approach of discretizing Einstein’s
theory of gravity we will break reparametrization invariance since an action like
(1.1) depends explicitly on the metric and without having done anything yet we can
already now say that the most important question to answer in case one manages
to carry out successfully the program outlined in the introduction will be to check
that the theory defined by approaching in a well defined way the ultraviolet fixed
point is really reparametrization invariant. This is by no means obvious since the
regularization has broken this invariance explicitly.

2.1 Quantization of Regge calculus

Two rather different regularization schemes have been suggested. The oldest one
goes back to Regge [7] and we will call it Regge calculus. It was originally invented
as a means to approximate a given smooth manifold by a piecewise flat Riemann-
nian manifold, obtained by a suitable triangulation of the smooth manifold. Regge
showed that it was still possible to assign in a sensible way a concept of curvature
to such a piecewise flat manifold. For $d$-dimensional manifolds the building blocks
would be $d$-dimensional simplices and the curvature assigned to the $d-2$-dimensional
sub-simplices. In this way one has both volume and curvature assigned to the piece-
wise flat manifold and it is possible to approximate the continuum Einstein-Hilbert
action which reads

\[ S[g] = \lambda \int d^d \xi \sqrt{g} - \frac{1}{16\pi G} \int d^d \xi \sqrt{g} R \]  

(2.1)

by the following discretized expressions:

\[ \int d^d \xi \sqrt{g} \to \sum_j V_j(d) \]  

(2.2)

\[ \int d^d \xi \sqrt{g} R \to \sum_j V_j(d) \left[ 2\delta_j \frac{V_j(d-2)}{V_j(d)} \right] \]  

(2.3)

In (2.2)-(2.3) the summation is over all \( d - 2 \)-dimensional sub-simplices \( j \) with volume \( V_j(d-2) \). \( \delta_j \) is the so-called deficit angle associated with the \( d - 2 \) dimensional sub-simplex \( j \), while the \( d \)-dimensional volume \( V_j(d) \) associated with the \( d - 2 \) dimensional sub-simplex \( j \) is defined as

\[ V_j(d) = \frac{2}{d(d+1)} \sum_{i \ni j} V_i(d) \]  

(2.4)

where the summation is over all \( d \)-dimensional simplices \( i \) which contain the subsimplex \( j \). It is possible to show that by a suitable refinement of the triangulation of the smooth manifold the discretized expressions (2.2) and (2.3) will actually converge to the continuum value. In this way Regge calculus provides a geometrical coordinate-independent description of gravity where it is natural to use the length of the links (the geodesic length of the edges in the triangulation of the given manifold) as dynamical variables since they completely specify the flat \( d \)-simplices used as building blocks. Originally the method was used mainly in a classical context where there are no conceptional problems connected with the approach. However, already as early as in 1968 Regge and Ponzano in an impressive paper, which contains also the seed to recent development in topological gravity mentioned above, pointed out that quantum mechanical amplitudes in three-dimensional Regge calculus can be defined by a functional integral and maybe computed non-perturbatively. However, if we seriously want to apply the Regge calculus directly in the functional integral it looses some of its beauty. In the classical context the geometry was specified by the length of the links of the building blocks and the incidence matrix which specified how the building blocks were glued together. This incidence matrix which determines the topology was fixed and not considered a dynamical variable. When we use the Regge formalism in the path integral the situation is opposite. The task is not to approximate a given continuum Riemannian manifold but (at least) to sum over equivalence classes of metrics associated with a given manifold. Unfortunately there
is no simple one-to-one correspondence between link length and equivalence classes of metrics, as is easily seen by considering triangulations of the two-dimensional plane. Obviously many triangulations correspond to the same Riemannian geometry. This means that a nasty jacobian is involved if we want to use link lengths as our integration variables. In addition one has to choose an integration measure which ensures that the link lengths satisfy the triangle inequalities and their higher dimensional analogues, which express that the $k$-dimensional volume of $k$-dimensional sub-simplices in the given triangulation must be positive. A great deal of work has gone into understanding and repairing these shortcomings of conventional Regge calculus. For a recent excellent review and references we refer to [8]. While the classical Regge calculus gives a coordinate independent geometrical description of gravity it of course has nothing like reparametrization invariance\footnote{One might try to define the analogue of local coordinate transformations, see for instance [10].}. It is therefore necessary to prove that this invariance is recovered at the point in coupling constant space where the continuum limit is taken. Unfortunately the “quantum Regge calculus” has not, in our opinion, been so successful in this respect. For instance computer simulations in [11] seemingly give the wrong coupling to Ising spins in two dimensions where the coupled spin-gravity system can be solved explicitly in the continuum. Hopefully this is due to problems with the simulations rather than a basic flaw in the approach, but we do not know for sure.

2.2 Dynamical triangulation gravity

Due to the above mentioned problems with the translation of classical Regge calculus to quantum theory we will here use another approach which has been extensively used in the last few years in the study of two-dimensional gravity and non-critical strings (which are nothing but two-dimensional gravity coupled to special matter fields) [12, 13, 14, 15]. We will call it dynamically triangulated gravity or (interchangeably) simplicial gravity.

In this approach the fundamental building blocks are regular simplices. In $d = 2$ this means equilateral triangles, in $d = 3$ regular tetrahedra. One now constructs the manifolds by gluing together the regular $d$-dimensional simplices along their $d - 1$ dimensional sub-simplices, in such a way that they form a piecewise flat manifold. The dynamics is shifted from the length of the links to the connectivity of the piecewise linear manifold and as we shall see there will not be the over-counting present in the quantum Regge prescription.

The assignment of volume and curvature for a given triangulation $T$ created by
gluing together the regular simplices is in this case very simple. Let us introduce the following notation: An \( i \)-dimensional (sub)-simplex is denoted \( n_i \), i.e. vertices are denoted \( n_0 \), links \( n_1 \) etc. The total number of such (sub)-simplices in \( T \) is denoted \( N_i(T) \). By the order \( o(n_i) \) we understand the number of \( d \)-dimensional simplices which share the sub-simplex \( n_i \). We will usually consider only the class of regular simplicial manifolds where we have put the following restrictions on \( o(n_i) \):

\[
o(n_{d-1}) = 2, \quad o(n_i) \geq d - i + 1, \quad (i \leq d - 2).
\] (2.5)

(2.2)-(2.3) reduce to

\[
\int d^d \xi \sqrt{g} \propto N_d
\] (2.6)

and

\[
\int d^d \xi \sqrt{g} R \propto \sum_{n_{d-2}} (c_d - o(n_{d-2})).
\] (2.7)

The constant \( c_d \) in (2.7) should be adjusted in such a way that for a hypothetical triangulation of flat space the sum should give zero. For \( d = 2 \) one can triangulate flat space with regular triangles. The order of each vertex is 6 and consequently \( c_2 = 6 \). Higher dimensional flat space does not admit a regular tessellation, but one can still ask for the average value of \( o(n_{d-2}) \) required to fill up \( d \)-dimensional flat space. The angle \( \theta_d \) between two \( d - 1 \) dimensional simplices belonging to the same \( d \)-simplex is given by

\[
\cos \theta_d = 1/d
\] (2.8)

and in order to fill up \( d \)-dimensional space we need to have

\[
o(n_{d-2}) = 2\pi/\theta_d \equiv c_d
\] (2.9)

This is the constant which enters in (2.7). We find

\[c_2 = 6, \quad c_3 = 5.104, \quad c_4 = 4.767.\] (2.10)

Let us further note that

\[
\sum_{n_{d-2}} o(n_{d-2}) = \frac{(d + 1)d}{2} N_d
\] (2.11)

since there are \( \binom{d + 1}{d - 1} \) \( d-2 \) dimensional sub-simplices in a \( d \)-dimensional simplex.

The discretized version of the continuum action can now, for a given triangulation \( T \), be written as

\[
S_d[T] = \kappa_d N_d(T) - \kappa_{d-2} N_{d-2}(T).
\] (2.12)
A more general action would be the following:

\[ S_d[T] = \sum_{i=0}^{d} \kappa_i N_i(T) \]  

involving the fugacities for all different \( i \)-dimensional sub-simplices. Of course one can choose to consider actions which cannot be expressed entirely in terms of the \( N_i \)'s. The higher derivative terms which can be added to the continuum action (2.11) will in general be of this kind, and we are going to consider them later, but let us for the moment discard such terms. Not all \( N_i \)'s are independent. The relations between the \( N_i \)'s can be worked out by the requirement that the triangulation should be locally homeomorphic to \( \mathbb{R}^d \). This means for instance that all \( n_d \)'s sharing a given vertex \( n_0 \) should be homeomorphic to the unit ball in \( \mathbb{R}^d \). Similar restrictions hold for the neighbours to an \( i \)-dimensional simplex \( n_i \) in the triangulation, and the relations this imposes on the \( N_i \)'s are summarized in the so-called Dehn-Sommerville[16] relations

\[ N_i = \sum_{k=i}^{d} (-1)^{k+d} \binom{k+1}{i+1} N_k, \]  

valid for all \( i \geq 0 \). These relations are not independent, but allow us to eliminate all \( N_{2i+1} \)'s if \( d \) is even and all \( N_{2i} \)'s if \( d \) is odd. In the case of even dimensions we have for a given triangulation \( T \) in addition Euler’s relation

\[ \sum_{i=0}^{d} (-1)^{i+d} N_i(T) = \chi_d(T). \]  

where \( \chi_d(T) \) denotes the Euler characteristic of the piecewise linear manifold which corresponds to the triangulation \( T \). Of course this relation is only useful if we know the Euler characteristic of the triangulation \( T \). As we shall see the restriction of topology is very important, and in case we fix the topology of \( T \) we can use (2.13) to eliminate for instance \( N_0 \). In case the topology is not fixed we can trade \( N_0(T) \) for \( \chi_d(T) \). For odd dimensions (2.15) follows from the Dehn-Sommerville relations with

\[ \chi_{d=2n+1} = 0. \]  

In odd dimensions the Euler characteristic is identically zero for any simplicial manifold, and it follows just from the requirement of local homeomorphism to \( \mathbb{R}^d \).

The recipe for going from the continuum functional integral to the discretized one is now:

\[ \int \mathcal{D}[g_{\mu\nu}] \rightarrow \sum_{T \sim T} \]  

\[ \int \mathcal{D}[g_{\mu\nu}] e^{-S[g]} \rightarrow \sum_{T \sim T} e^{-S[T]} \]
The formal integrations on the lhs of (2.17) and (2.18) are over all equivalence classes of metrics, i.e. the volume of the diffeomorphism group is divided out. $\mathcal{T}$ denotes a suitable class of triangulations. One class of constraints is given by (2.5), but it should be stressed that such short distance restrictions are not expected to be important in the scaling limit.

Since different triangulations give rise to different curvature assignment one can view the above summation as a summation over different Riemannian manifolds. There is no problem with over-counting in this formulation. The idea of the continuum functional integral is precisely to perform such a sum with weight $e^{-S[g]}$. Of course the discretized sum on the rhs of (2.17) and (2.18) can only be viewed as an approximation to the continuum expression which hopefully “converges” in the scaling limit to the correct expression. The questions which are difficult to answer are whether the class of piecewise flat manifolds is “close” to the class of Riemannian manifolds and whether the piecewise linear manifolds are selected sufficiently uniformly with respect to Riemannian manifolds that (2.17) and (2.18) are good approximations. Unfortunately there is no weak coupling expansion where one can check this, but it is very encouraging that the formalism is known to work in the two-dimensional case, even if one couples conformal matter with central charge $c \leq 1$ to the system. In this case it is possible to solve both the continuum and the discretized system. In particular we see that reparametrization invariance is recovered in the scaling limit.

Since $d = 4$ has our main interest in this work we can write our partition function as

$$Z(\kappa_2, \kappa_4) = \sum_{T \in \mathcal{T}} e^{-\kappa_4 N_4 + \kappa_2 N_2}$$

(2.19)

This is the grand canonical partition function, where the volume of the universe can vary. It is sometimes convenient to change from the grand canonical ensemble described by (2.19) to the canonical ensemble where the volume $N_4$ is kept fixed. The corresponding partition function will be

$$Z(\kappa_2, N_4) = \sum_{T \sim \mathcal{T}(N_4)} e^{\kappa_2 N_2(T)}$$

(2.20)

$$Z(\kappa_2, \kappa_4) = \sum_{N_4} Z(\kappa_2, N_4) e^{-\kappa_4 N_4}.$$  

(2.21)

If the entropy, i.e. the number of configurations for a given $N_4$, is exponentially bounded it is easy to prove that there is a critical line $\kappa_4 = \kappa_4^c(\kappa_2)$ in the $(\kappa_2, \kappa_4)$-coupling constant plane. For a given $\kappa_2$ the partition function (2.19) will then be well defined for $\kappa_4 > \kappa_4^c(\kappa_2)$. Let us call this domain in the coupling constant plane $\mathcal{D}$. 

Critical behaviour can be found only when we approach the boundary $\partial D$ which we denote the critical line, but in general we only expect interesting critical behaviour at certain critical points on $\partial D$ (i.e. at certain values of $\kappa_2$). These are the points we are looking for in the numerical simulations.

Let us end this subsection by discussing a point which is worth emphasizing. We have been deliberately vague defining the class of triangulations $\mathcal{T}(N_4)$ over which the summation is to be performed in a formula like (2.20). Already in two dimensions where the classification of topology is so simple (it is defined by the Euler number $\chi$) an unrestricted summation over manifolds of different topology does cause problems. In fact the two-dimensional analogue of (2.20) and (2.21) which by means of Euler’s relation can be written:

\[
Z(\tilde{\kappa}_0, N_2) = \sum_{\mathcal{T}(N_2)} e^{\tilde{\kappa}_0 \chi} \quad (2.22)
\]

\[
Z(\tilde{\kappa}_0, \tilde{\kappa}_2) = \sum_{N_2} Z(\tilde{\kappa}_0, N_2) e^{-\tilde{\kappa}_2 N_2} \quad (2.23)
\]

does not make any sense. The well known reason is that the number of triangulations, $\mathcal{N}(N_2)$, which one can make by gluing together a given number $N_2$ of equilateral triangles to a two-dimensional surface is too large. It grows factorially fast: $\mathcal{N}(N_2) \geq N_2!$. This means that the two-dimensional analogue of (2.21) is never convergent. This is not a spurious result of a perverse discretization. An analogous result has been proven in the continuum two-dimensional theory where the volume of moduli space grows at least factorially with the genus [17]. It is the same effect we observe in the discretized case: As long as we fix the topology, i.e. the Euler characteristic $\chi$ of the surface, we have: $\mathcal{N}_\chi(N_2) \sim N_2^{\gamma_\chi - 3} \exp(\tilde{\kappa}_2 N_2)$, i.e. only an exponential growth of the number of surfaces. In this case (2.23) is well defined for a certain range of $\tilde{\kappa}_2$’s. However, an unrestricted summation over topologies makes the subleading pre-exponential factor $N_2^{\gamma_\chi}$ dominant when $|\chi| \sim N_2$ since $\gamma_\chi \sim -5\chi/4$.

In higher dimensions the situation is of course only worse and the best we can hope for is a well defined expression for a fixed (or at least restricted) topology. In the following we will restrict ourselves to four-dimensional manifolds with the topology of $S^4$.

The above outlined non-perturbative definition of gravity has nothing to add to our understanding (or rather lack or understanding) of the question of whether or not to sum over topologies in quantum gravity. Apart from the problem that the topologies of non-simply connected four-dimensional manifolds cannot be classified in a sensible way, the partitionfunction does not even make sense if we only restrict
ourselves to the sub-class of topologies which one can construct by simple analogy to two-dimensional surfaces of genus $g$. In the rest of this article we will only be interested in the search for non-trivial fixed point of the above defined theory (modified with higher curvature terms) where the class of triangulations $\mathcal{T}$ corresponds to manifolds with the topology of $S^4$.

### 2.3 Higher curvature terms

There is no straightforward generalization of Regge’s work to theories of gravity which involve higher derivative terms like $R^2$ in the action. The reason is that Regge viewed the piecewise flat manifold, not as a discrete approximation to an underlying continuum surface, but as one where curvature could be defined in a mathematical stringent way, entirely in terms of the geometrical concepts involved in parallel transportation. In this way the curvature occurs in $\delta$-functions on the lattice geometry, with support on the $d-2$-dimensional sub-simplices. Anything other than the Einstein action (and the cosmological term) will then involve higher powers of $\delta$-functions and this means that for piecewise flat manifolds, interpreted as by Regge, terms like $\int \sqrt{g} R^2$ are infinite. In order to make sense of higher derivative terms one has to change the perspective on Regge calculus somewhat as advocated by Hamber and Williams [18] and view the lattice geometry as representing an approximation to some smooth geometry and the local curvature as some average curvature for a small volume. In fact our formulas for Regge calculus have already hinted this interpretation in the sense that we have assigned a volume density $V_{n_{d-2}}(d)$ to each $d-2$-dimensional sub-simplex $n_{d-2}$ which can be viewed as an appropriate share of the volumes of the $d$-dimensional simplices to which the sub-simplex $n_{d-2}$ belongs. In the same way we have written the curvature density $R$ as $\delta_{n_{d-2}}/V_{n_{d-2}}(d)$ viewing it formally as representing some average value in the volume $V_{n_{d-2}}(d)$. With such an interpretation one can of course write

$$\int d^d \xi \sqrt{g} R^2 \sim \sum_{n_{d-2}} V_{n_{d-2}}(d) \left[ \frac{2 \delta_{n_{d-2}} V_{n_{d-2}}(d-2)}{V_{n_{d-2}}(d)} \right]^2 \quad (2.24)$$

This definition must be interpreted with some care if we want convergence to the continuum value for a smooth manifold by successive subdivision [19]. We do not have to worry too much about these subtleties here since our task in the functional integral is not to approximate a given smooth manifold but to select some class of manifolds which can be used in an (approximate) evaluation of the integral. From this point of view we will use the $R^2$ term as representing typical higher derivative terms which one would have to insert in order to stabilize the Euclidean path integral.
as explained in the introduction. As is well known from for instance lattice gauge theories discretized versions of higher derivative terms are by no means universal. It is clear that this point of view is not as beautiful as the original geometrical way that Regge viewed piecewise flat manifolds, but our perspective is that a term like (2.24) will probe a universality class of theories which have an effective expansion in terms of higher derivative actions like (2.24).

In the case where we consider the piecewise flat manifolds which can be obtained by the process of dynamical triangulation as described above (2.24) simplifies and we get

$$\int d^d \xi \sqrt{g} R^2 \sim \sum_{n_{d-2}} o(n_{d-2}) \left[ \frac{c_d - o(n_{d-2})}{o(n_{d-2})} \right]^2$$

(2.25)

This formula has a slight problem with the continuum interpretation since flat space does not have a regular tessellation except for $d = 2$. This means that the term can never scale to zero:

$$\sum_{n_{d-2}} o(n_{d-2}) \left[ \frac{c_d - o(n_{d-2})}{o(n_{d-2})} \right]^2 \geq \text{const.} \ N_d.$$

(2.26)

If we introduce a scaling parameter $a$, which is to be identified with the link length, and which is going to be scaled to zero, the physical volume $V \equiv \int d^d \xi \sqrt{g}$ being kept fixed, and if we assume canonical scaling of the terms involved of both sides of (2.26) we get (restricting ourselves to four dimensions which have our main interest):

$$\int d^d \xi \sqrt{g} R^2 \bigg|_\text{DT} > \text{const.} \frac{a^4}{a^4} \int d^d \xi \sqrt{g} \bigg|_\text{DT}.$$

(2.27)

This means that the leading term on the lhs (2.26) is just a cosmological constant term and by expanding the lhs we see that it also contains an Einstein-Hilbert term etc.. Under the assumption of naive scaling we have to write instead of (2.27)

$$\sum_{n_{d-2}} o(n_{d-2}) \left[ \frac{c_d - o(n_{d-2})}{o(n_{d-2})} \right]^2 \sim \int d^d \xi \sqrt{g} \left[ \frac{c_0}{a^4} + \frac{c_1}{a^2} R + c_2 R^2 + \cdots \right].$$

(2.28)

Our lattice “$R^2$"-term is thus to be considered as a generalized higher derivative term which, when added to the lattice Einstein-Hilbert term, in addition will lead to a redefinition of the bare cosmological coupling constant and the bare gravitational coupling constant.

One interesting aspect of the dynamical triangulation approach is that for a finite lattice volume it automatically provides a cut off for the Einstein action. This is not the case for the conventional Regge calculus where the action can go to infinity without the volume diverging. The reason is that the volume (for $d > 2$) of the
$d - 2$-dimensional sub-simplices can diverge without the corresponding volume of the $d$-dimensional simplices going to infinity. In the case of dynamical triangulations we have (restricting again ourselves to four dimensions)

$$-\text{const.} \cdot N_4 < \sum_{n_2} (c_4 - o(n_2)) < \text{const.} \cdot N_4.$$  \hspace{1cm} \text{(2.29)}

If we assume a conventional scaling in the tentative continuum limit we can rewrite (2.29) as

$$\left| \int d^4\xi \sqrt{g} R \right|_{DT} \leq \frac{\text{const.}}{a^2} \left( \int d^4\xi \sqrt{g} \right)_{DT}$$  \hspace{1cm} \text{(2.30)}

where again we have have introduced the link length $a$, which is going to be scaled to zero while the physical volume $V \equiv \int d^4\xi \sqrt{g}$ is kept fixed.

### 2.4 Observables

Let us here discuss the observables (see also [20] for a more general discussion). Due to diffeomorphism invariance and the fact that we in quantum gravity have to integrate over all Riemannian manifolds the observables which are most readily available are averages of invariant local operators like the curvature $R(x)$ and suitable contractions of powers of the curvature tensor, like $R_{\mu\nu}^2$ or $R_{\mu\nu\lambda\sigma}^2$, and the fluctuation of these averages. In addition one can discuss and measure so-called fractal properties of space-time and finally with some effort define the concept of correlators of local invariant operators.

The simplest observable is the average curvature. If we consider the discretized partition function we have

$$\int d^4\xi \sqrt{g(\xi)} R(\xi) \propto \frac{2}{d(d+1)} \sum_{n_2} o(n_2) \frac{c_4 - o(n_2)}{o(n_2)} = \frac{c_4}{10} N_2 - N_4$$  \hspace{1cm} \text{(2.31)}

and since the volume is

$$\int d^4\xi \sqrt{g(\xi)} \propto \frac{2}{d(d+1)} \sum_{n_2} o(n_2) = N_4$$  \hspace{1cm} \text{(2.32)}

we can define the average curvature per volume as

$$\langle R \rangle = \frac{\int d^4\xi \sqrt{g(\xi)} R(\xi)}{\int d^4\xi \sqrt{g(\xi)}} \propto \frac{c_4}{10} \frac{N_2}{N_4} - 1$$  \hspace{1cm} \text{(2.33)}

In (2.33) $\langle R \rangle$ is defined for a single manifold. We get of course the quantum version by calculating the functional average of $\langle R \rangle$ over all Riemannian manifolds, weighted by $e^{-S}$. This is what we will do numerically. The average curvature is a bulk quantity
which will allow us to get a quick survey of the phase diagram of four-dimensional quantum gravity. In a similar way we can define (with the drawbacks described in the last subsection) \( \langle R^2 \rangle \) by

\[
\langle R^2 \rangle = \frac{\int d^4\xi \sqrt{g(\xi)} R^2(\xi)}{\int d^4\xi \sqrt{g(\xi)}} \propto \sum_{n_2} o(n_2) \left( \frac{(c_4 - o(n_2))}{o(n_2)} \right)^2. \tag{2.34}
\]

Again this average is defined over a single manifold and we have eventually to take the weighted average over all manifolds in our ensemble. A quantity which will have our interest will be \( \langle R^2 \rangle - \langle R \rangle^2 \).

A more refined, but related observable is the integrated curvature-curvature correlation. In a continuum formulation it would be

\[
\chi(\kappa^2) \equiv \langle \int d^d\xi_1 d^d\xi_2 \sqrt{g(\xi_1)} R(\xi_1) \sqrt{g(\xi_2)} R(\xi_2) \rangle - \langle \int d^d\xi \sqrt{g(\xi)} R(\xi) \rangle^2. \tag{2.35}
\]

In a lattice regularized theory one would expect that away from the critical points short range fluctuations will prevail, while approaching the critical point long range fluctuation might be important and would result in an increase in \( \chi(\kappa^2) \). The observable \( \chi(\kappa^2) \) is the second derivative of the free energy \( F = -\ln Z \) with respect to the gravitational coupling constant \( G^{-1} \). In the case where the volume \( N_4 \) is kept fixed we see that

\[
\chi(\kappa^2, N_4) \sim \langle N_2^2 \rangle_{N_4} - \langle N_2 \rangle_{N_4}^2 = -\frac{d^2 \ln Z(\kappa^2, N_4)}{d\kappa^2}. \tag{2.36}
\]

From the above discussion we have to look for points along the critical line \( \partial \mathcal{D} \) where \( \chi(\kappa^2, N_4)/N_4 \) diverges in the infinite volume limit \( N_4 \to \infty \).

Another observable is the Hausdorff dimension. One can define the Hausdorff dimension in a number of ways, which are not necessarily equivalent. Here we will simply measure the average volume \( V(r) \) contained within a radius \( r \) from a given point. In [20] the concept of a \textit{cosmological Hausdorff dimension} \( d_{ch} \) was defined. It essentially denotes the power which relates the average radius of the ensemble of universes of a fixed volume to this volume:

\[
\langle \text{Radius} \rangle_{N_4} \sim N_4^{1/d_{ch}}. \tag{2.37}
\]

From the distribution \( V(r) \) we can try to extract \( d_{ch} \). If for large \( r \) we have the behaviour

\[
V(r) \sim r^{d_h} \tag{2.38}
\]

we can identify \( d_{ch} \) and \( d_h \).
By the use of Regge calculus it is straightforward to convert these continuum formulas to our piecewise flat manifolds. Between two points in the piecewise flat manifold there is a geodesic which is a piecewise linear path. Rather than using this definition we will use approximations which are much more convenient from a numerical point of view and which one expects should be sufficient for the purpose of extracting general scaling behaviour and fractal properties: We define the distance between two vertices as the shortest path along links which connects the two vertices, i.e. it is essentially the number of links of the shortest path since all links have the same length. We call this distance the “$n_1$”-distance between vertices and denote its value by $d_1$. In the same way we can define a shortest path between two 4-simplices as the shortest path obtained by moving between the centers of neighbour 4-simplices. We call this distance the “$n_4$”-distance between 4-simplices and denote its value by $d_4$. The dual graph to a given triangulation, obtained by connecting the centers of neighbour 4-simplices, will be a $\phi^5$-graph and the $n_4$-distance on the triangulation will be the $n_1$-distance on the dual $\phi^5$-graph. A priori these distances are not related and it is easy to find triangulations where they differ vastly for specific choices of vertices and associated 4-simplices. But for averages over vertices and over triangulations one would expect that they carry the same information about the geometry and we shall see that this is indeed the case.

With the $n_1$- and $n_4$-definitions of geodesic distances on the triangulations it becomes trivial to measure numerically relations like (2.38). As we shall see it is less trivial to extract in a reliable way the Hausdorff dimension $d_h$.

Let us finally discuss the measurements of correlation functions in quantum gravity. If we have local invariant operators $O_1(x)$ and $O_2(x)$ (like for instance $R(x)$) we can define a correlation function of geodesic distance $d$ by

$$\tilde{G}(d) = \int \int d^4\xi_1\sqrt{g(\xi_1)}d^4\xi_2\sqrt{g(\xi_2)}O_1(\xi_1)O_2(\xi_2)\delta(d(\xi_1, \xi_2) - d).$$

(2.39)

In this definition $d(\xi_1, \xi_2)$ denotes the geodesic distance between $\xi_1$ and $\xi_2$ for a given manifold, but (2.39) makes sense also as a functional average, and therefore in principle in quantum gravity. When we discuss numerical data we will always have in mind the functional average. It might be convenient to divide by a volume element to get the dimension of a point-point correlation function. If $V(r)$ denotes the volume inside a “ball” of radius $r$ we can write

$$dV(r) = V'(r)dr$$

and we define

$$G(d) = \tilde{G}(d)/V'(d).$$

(2.40)
In the case where we have a finite Hausdorff dimension the exponential fall off of \( G(d) \) and \( \tilde{G}(d) \) are identical, but it needs not be the case if we have an infinite Hausdorff dimension.

3 The numerical method

Unfortunately the analytic methods of two dimensional simplicial gravity have not yet been extended to higher dimensions. The numerical method of “grand canonical” Monte Carlo simulation, which is well tested in two dimensions \([21, 22, 23]\), has recently been applied to three dimensions \([24, 25, 26, 27, 28]\) and in four dimensions \([31, 32, 33, 34]\). A necessary ingredient for Monte Carlo simulations in simplicial quantum gravity is a set of so-called “moves”, i.e. local changes of the triangulations, which are ergodic in the class of triangulations we consider. A general set of moves in any dimension has been known for a long time \([35]\). They are however not optimal for numerical simulations. A more convenient set of moves for higher dimensional gravity was suggested in \([25]\). The ergodicity of these moves in three dimensions was proved in \([26]\), and the generalization of this proof to \( d = 4 \) was given in \([29]\). In \( d \) dimensions there are \( d + 1 \) moves. Their general description is as follows: remove an \( i \)-dimensional simplex of order \( d+1-i \) and the higher dimensional simplices of which it is part, and replace it by a \( d-i \)-dimensional simplex (“orthogonal” to the removed \( i \)-dimensional simplex) plus the appropriate higher dimensional simplices such that we still have a triangulation. Such moves will be allowed provided they do not violate \([27, 29]\) and provided they do not create simplices already present, i.e. which already have the same vertices as the newly created simplices.

Let us consider four dimensions where there are five moves. The first move consists of removing a four-dimensional simplex \( n_4(\text{old}) \) and inserting a vertex \( n_0(\text{new}) \) in the void interior and adding links (and the induced higher dimensional simplices) which connect \( n_0(\text{new}) \) to the five vertices of \( n_4(\text{old}) \). In this way \( n_4(\text{old}) \) is replaced by five new \( n_4 \)’s and the total change of \( N_i \)’s is

\[
\Delta N_0 = 1, \quad \Delta N_1 = 5, \quad \Delta N_2 = 10, \quad \Delta N_3 = 10, \quad \Delta N_4 = 4. \tag{3.1}
\]

The second move consists of removing a three-dimensional simplex \( n_3 \) and the two \( n_4 \)’s sharing it, and then inserting a link (“orthogonal” to \( n_3 \)) and associated \( n_2 \)’s, \( n_3 \)’s and \( n_4 \)’s. The total change of \( N_i \)’s is in this case

\[
\Delta N_0 = 0, \quad \Delta N_1 = 1, \quad \Delta N_2 = 4, \quad \Delta N_3 = 5, \quad \Delta N_4 = 2. \tag{3.2}
\]

The third move is a “self-dual” move where \( \Delta N_i = 0 \). It consist of removing a triangle of order three and associated higher dimensional simplices and inserting the
“orthogonal” triangle and its associated higher dimensional simplices such that we still have a triangulation. The fourth move is the inverse of the second move, while the fifth move is the inverse of the first.

The change in the action induced by these moves can now readily be calculated and we are in a position to use the standard Metropolis updating procedure. The weights required for detailed balance are easily determined. Let us only remark here that the nature of the problem naturally suggests to use indirect addressing by pointers since there is no rigid lattice structure. In addition we found it most efficient to keep pointers to vertices of order five, links of order four and triangles of order three since these are the ones used in the updating. Since programs of the above nature are not well suited for vectorization it is optimal to run them on fast workstations.

Since we are forced to use a grand canonical updating where the volume of the universe $N_4$ is changing, it is convenient to use the technique first introduced in [30] and used successfully in the simulations in three-dimensional gravity [24, 27]. It allows us to get as close to a canonical simulation as possible and it provides at the same time an estimate of the critical point $\kappa_4^c(\kappa_2)$ for a given value of the coupling constant $\kappa_2$. The idea is the following: Assume we want to perform a measurement at some fixed value $N_4(F)$. The task is to constrain the fluctuations of $N_4$ to the neighbourhood of $N_4(F)$ without violating ergodicity. First we make an approximate estimate of the critical point $\kappa_4^c$, which we denote $\kappa_4^c(N_4(F))$. It can in principle depend on $N_4(F)$. Next we choose the actual $\kappa_4$ used in the simulation as a function of the value of $N_4$ in the following way:

$$\kappa_4(N_4) = \begin{cases} 
\kappa_4^c(N_4(F)) - \Delta\kappa_4 & \text{for } N_4 < N_4(F) \\
\kappa_4^c(N_4(F)) + \Delta\kappa_4 & \text{for } N_4 > N_4(F).
\end{cases} \quad (3.3)$$

For sufficiently large values of $N_4(F)$ and small values of $\Delta\kappa_4$ we will get an exponential distribution of $N_4$'s peaked at $N_4(F)$:

$$P(N_4) \sim e^{(N_4-N_4(F))(\kappa_4^c-N_4(F))}. \quad (3.4)$$

By monitoring $\Delta\kappa_4$ we can effectively control the width of the distribution of $N_4$ around $N_4(F)$ without violating the principle of ergodicity. A measurement of the exponential distribution also allows us to determine $\kappa_4^c$. If the exponentially fall off is different above and below $N_4(F)$ it means that $\kappa_4^c$ is different from our guess $\kappa_4^c(N_4(F))$ and we can use the optimal value in the next run. Further the measurements of $\kappa_4^c$ for different values of $N_4(F)$ allow us to extract the subleading
correction to the distribution. Assume that the partition function has the form:

$$Z(\kappa_2, \kappa_4) = \sum_{N_4} Z(\kappa_2, N_4)e^{-\kappa_4 N_4}$$  \hspace{1cm} (3.5)$$

where

$$Z(\kappa_2, N_4) \sim N_4^{\gamma(\kappa_2)-2}e^{\kappa_4^e(\kappa_2)N_4} (1 + O(1/N_4)).$$  \hspace{1cm} (3.6)$$

By our method the critical point $\kappa_4^c$ determined by measurements in the neighbourhood of various $N_4(F)$'s would lead to:

$$\kappa_4^c(\text{measurement}) = \kappa_4^c(\kappa_2) + \frac{\gamma(\kappa_2) - 2}{N_4(F)}$$  \hspace{1cm} (3.7)$$

and a determination of the entropy exponent $\gamma(\kappa_2)$, which in fact governs the volume fluctuations of the system.

4 Numerical results

The measurements of average curvature etc. were performed for different values of $N_4$: 4000, 9000, 16000 and 32000. The number of attempted updatings were of the order $10,000 \times N_4$ (sometimes considerably longer at critical points where thermalization was slow). The phase diagram was scanned varying the (inverse) bare gravitational coupling constant $\kappa_2$ and the “$R^2$” coupling constant which we denote $h$. For each value of $\kappa_2$ and $h$ and each value of $N_4$ this implies a fine-tuning of the value of the bare “cosmological” coupling constant $\kappa_4$ to its critical value $\kappa_4^c(\kappa_2, h, N_4)$. As explained in the last section it is convenient to perform the simulations in the neighbourhood of some fixed value of the volume so we choose a specific $N_4(F)$ and limit the fluctuations in volume to some neighbourhood of $N_4(F)$. If we decide to perform the measurements after a given number, $n$, of Monte Carlo sweeps, in practise we perform the actual measurement the first time, after the $n$’th sweep, the system passes a state where the value of $N_4$ is equal to $N_4(F)$. This is what we mean when we say that the measurements were performed for a given value of $N_4$.

The result for the average curvature is shown in fig. 1 for $h = 0$ over a large range of $\kappa_2$, while fig. 2 shows the average curvature for different values of $h$, ranging from $h = 0$ to $h = 20$ and $N_4 = 16000$. It is not possible with our choice of $R^2$ term to increase $h$ further since the acceptance rates in the Metropolis updating become too small.

We observe the following: In case there is no coupling constant except the cosmological coupling constant the average curvature $\langle R \rangle$ is zero, which is a nice result
since it shows that the selection of manifolds in the context of dynamical triangulations has no bias towards positive or negative curvature. In case we take $\kappa_2$ positive the average curvature will be positive (this corresponds to the conventional sign of the gravitational coupling constant). If we take $\kappa_2$ negative (“anti-gravity”) the average curvature will be negative.

For $h = 0$ and $\kappa_2 \approx 1.1$ we see a change towards large positive curvature. The same change is seen for $h > 0$, only we have to go to larger values of $\kappa_2$ when $h > 10$. For a fixed positive value of $\kappa_2$ the curvature decreases with increasing $h$ as expected. However, as discussed above, the limit $h \to \infty$ does not really correspond to zero local curvature due to the special form of the our discretized “$R^2$”-term. The absolute minimum value of the discretized term corresponds to a constant negative curvature: $R = -0.046$.

In fig. 3 we have shown the susceptibility defined by (2.36). It can be measured directly, or as the derivative of the average curvature. We have used the second method, but have also measured the susceptibility directly, with comparable results. One sees a clear peak which grows somewhat with volume. This could be taken as a sign that the system becomes critical in this region, although the system size is not big enough to exclude the possibility of a phase transition of higher order.

An independent signal of criticality is found by looking at the observable $\langle R^2 \rangle - \langle R \rangle^2$. In fig. 4a we show its behaviour as a function of $\kappa_2$ for $h$ between 0 and 20. For small $h$ values we see a clear peak with a volume dependence in the region where the $\kappa_2$ susceptibility has a peak too. We note the clear asymmetry between the two sides of the peak, especially for $h = 0$ (fig. 4b). This explains why the position of the peak seems to be shifted towards smaller values of $\kappa_2$ when compared to the susceptibility curve. Here again our system is not big enough to exclude the possibility that the increase with volume is only a finite size effect and that eventually we shall observe only a discontinuity in the derivative at a critical point, which again could signal a phase transition of higher (perhaps 3rd) order. The signal deteriorates somewhat for large values of $h$, contrary to the susceptibility signal. We can draw a critical line in the $(\kappa_2, h)$-coupling constant plane, fig. 5, and fig. 6 shows the average curvature at the transition point as a function of $h$ for $N_4 = 4000, 9000$ and $16000$. We see that the average curvature gets smaller when $h$ increases and also when $N_4$ increases, but for all the values we have been able to probe we have

$$\langle R \rangle_c \equiv \langle R(\kappa_2^c, h) \rangle > 0 \tag{4.1}$$

and we can not conclude that $\langle R \rangle_c > 0$ is a finite volume effect. The data indicate rather that $\langle R \rangle_c$ remains positive even for $h \to \infty$. For $h > 12$ there seems to be a
qualitative change in the distributions, which might indicate a different transition, but we have not found that it cured the problems of the \( h = 0 \) situation.

The computer simulations in three dimensions revealed a similar situation: A transition and a \( \langle R \rangle_c > 0 \). In three dimensions there was a very strong hysteresis in the same transition, favouring a first order transition. Here we have not seen the same strong hysteresis. For \( N_4 = 4000 \) there was no problem moving from one phase to the other. For larger \( N_4 \) we have observed very slow thermalization and huge fluctuations in geometry close to \( \kappa_2 = \kappa_{2c} \), but it did not present itself as clear hysteresis.

Let us now explore the change in geometry along the critical line. Above we have defined the “geodesic” link distance \( d_1 \) between vertices and the four-simplex distance \( d_4 \) between four-simplices. The average values describe typical radii of our universes. They are shown in fig. 7 (\( \langle d_1 \rangle \)) and fig. 8 (\( \langle d_4 \rangle \)). Although the \( d_4 \) distances are approximately six times larger that the \( d_1 \) distances they clearly behave qualitatively in the same way and reveal a drastic change in the geometry as we pass the critical region of \( \kappa_2 \). The nature of the change seems to be independent of \( h \).

The typical universes generated by the computer simulations have small radii, almost independent of the volume if we are below the critical \( \kappa_2 \) region. After we have passed the critical region the radii become quite large and show a very clear volume dependence. In fact it seems as if the radius grows almost linearly with volume. *Qualitatively this implies that the Hausdorff dimension is large below the critical region and small (in fact close to one) above the critical region.* We have not attempted to determine the larger Hausdorff dimension. The growth in radius with volume is so small that one has to go to much larger volumes in order to do it in a reliable way.

We get a nice representation of the change in geometry between the two phases by showing the actual distribution of geodesic length in the universes. This is done in fig. 9 for the \( d_4 \) distances for \( h = 0 \). If space-time has a fractal structure with some Hausdorff dimension \( d_h \) the distribution should be like

\[
P(d) \sim d^{d_h-1}
\]

In fig. 9 we have shown four curves which correspond to \( \kappa_2 = 0.9, 1.0, 1.1 \) and \( 1.2 \) in the critical region. Since the transition is smooth and extrapolates from large to

---

3It has been argued that one should not use the “geometrical method” advocated here as a measure of the Hausdorff dimension, but extract it from the random walk representation of the massive propagator, since this seems to give more “reasonable” values. We disagree with this point of view. The two methods are mathematically equivalent and disagreement reflects in our opinion the fact that it is not possible to extract the Hausdorff dimensions with the desired precision.
small $d_h$ it is of course possible to find a $\kappa_2$ in the transition region where we get a curve quite similar to (4.2) with $d_h \approx 4$ which is of course amusing, but we do not consider the value as especially well determined by the numerical simulations.

Finally the difference in geometry between the highly connected phase for $\kappa_2$ below the critical region and very extended phase above the critical region is reflected in the curvature distribution. In fig. 10 we have shown the distribution of the average curvature per simplex, defined as an average over all ten triangles forming a 4d simplex. In addition we plot the values obtained by blocking the value of curvature over larger and larger regions (3 and 5) in the $d_4$ distance. The value of $h$ is 0 and the $\kappa_2$ values are chosen in the neighborhood of the phase transition. The same qualitative behaviour is observed for larger values of $h$. In all the cases the distribution seems to consist of two parts: one sharply peaked at small curvature values and the second rather broad, shifted towards the positive curvature. In the highly connected phase the broad part disappears after blocking and already after one or two steps the distribution approaches a $\delta$-function. On larger scales we simply have a space with constant curvature $R$. In the other phase it is the peaked part, which disappears after blocking and it seems as if the distribution approaches some non-trivial limit.

One disturbing aspect of our results if we want to give the above mentioned phase transition a continuum interpretation is the fact that $\langle R \rangle_c > 0$. If we take this result literally it is difficult to attribute any sensible naive continuum scaling to the system. If we introduce a scaling parameter $a$, which conveniently can be identified with the link length in the triangulation the simplest scaling would be one where $a \to 0$ while the volume $a^4 N_4$ was kept fixed. One would then expect the following relation between the “bare” lattice curvature and the continuum curvature:

$$\langle R \rangle_{(\text{lattice})} = R_{(\text{continuum})} a^2$$

which shows that if $R_{(\text{continuum})}$ should remain finite in the scaling limit $a \to 0$ $\langle R \rangle$ must scale to zero. It does not. We have found no way to repair this and although it is possible to find a scaling

$$\langle R(\kappa_2, h) \rangle - R_c(\kappa_2^c, h) \sim |\kappa_2 - \kappa_2^c|^{\delta - 1}$$

its significance is not clear to us due to $R_c > 0$. One possible explanation is that our formalism does not admit a tessellation of flat space and that this in some way reflects itself in an expectation value of $R$? Another possibility is, as mentioned above, that the observed phase transition is in fact higher order and the naive scaling
relations need not hold although it is very difficult to imagine an alternative scenario which would correspond to a physical scaling.

The appearance of $R_c > 0$ for $h = 0$ was one of the motivations to look at theories with higher curvature terms. The other motivation was to investigate the question of universality in the spirit outlined by Weinberg, as explained in the introduction. However, our results are negative in the sense that even if $R_c$ indeed decreases with increasing $h$, it does not go to zero and for not too large values of $h$ we clearly are in the same universality class as for $h = 0$. Distributions, the nature of the transition etc. seem to be the same except for a displacement in $\kappa_4$ and $\kappa_2$. As mentioned before, for $h > 12$ we see a qualitative change in the distributions, which might indicate a different transition. This point requires however further investigation.

5 Discussion

The notation of a “hot” and a “cold” phase in quantum gravity in $d = 3$ was introduced in [27]. In the hot phase the large entropy of “quantum” universes was dominant. These quantum universes were characterized by a large Hausdorff dimension and a high connectivity and the hot phase was continuously connected to “anti-gravity”, where the (bare) gravitational coupling constant is negative. In the cold phase extended structures dominated. In fact the Hausdorff dimension seemed close to one, suggesting some kind of linear structure. This phase was interpreted as representing the dominance of the conformal mode. In the regularized theory the action is not unbounded from below, but instead some lattice configurations which are pure artifacts without any connection to the continuum will dominate. These were the extended structures observed in three dimensions. The interesting question was asked, whether it was possible at the transition point to have truly extended structures, relevant for continuum physics. In [28] it was shown that the transition in three dimensions was of first order, and a continuum limit was ruled out from this point of view.

From a superficial point of view the situation does not look so different in four dimensions. We have two phases which we again can call hot and cold. The hot phase is continuously connected to the “anti-gravity” region where the bare gravitational coupling constant is negative. As in three dimensions the cold phase is characterized by an almost linear, extended structure, while the hot phase has a larger Hausdorff dimension and much larger connectivity. In the hot phase the average order of vertices is much larger and the average curvature changes from being large positive in the cold phase to small positive or even negative in the hot phase. However, the
nature of the transition seems different in four dimensions. We have not seen any true hysteresis, but there are very long thermalization times in the cold phase where the linear structures developed. This is in contrast to three dimensions where a very pronounced hysteresis was observed \[28\]. Our data are not incompatible with a second order transition, and this opens for the possibility that a continuum limit can be associated with the transition. The scenario is from this point of view quite nice: In three dimensions a first order transition rules out a continuum limit, but we do not really want the continuum limit in a usual sense in three dimensions since we would be confronted with the embarrassing question of a three dimensional graviton. The physical Hilbert space of pure three-dimensional quantum gravity is most likely finite dimensional \[4\] and does not allow for true dynamical fields. The situation in four dimensions is probably very different and it is interesting that the discretized model seems to hint at such a difference.

It is also encouraging that our data share some similarities with the results obtained by Regge calculus (\[9\] and references therein). As explained in the introduction the philosophy of the two methods is quite different and it would be a strong argument in favour of universality if one manages to obtain the same results by the two methods. In Regge calculus one also observes the two phases, and the phase with large positive curvature is characterized by very singular spiky configurations. They seem similar to our linear structures, which however cannot arise by single points moving away from the rest, as is the case in the Regge formalism. The “hot” phase is in the Regge formalism characterized by a small negative curvature which however (contrary to our results) scales to zero at the critical point. The latest results \[9\] indicate that one actually has a first order transition for \(h = 0\) and only for a finite \(h > 0\) it changes to a second order transition. This transition for finite \(h\) might have some similarity with the change we have seen for large \(h\), but we still have \(R_c > 0\). We postpone the discussion of this point to later publication.

At this point we should emphasize again that we see one major obstacle to taking a continuum limit at the critical point and that is the fact that the average curvature does not scale to zero. As is seen from the fig.1 we have \(\langle R \rangle \approx 0\) for \(N_4\) large in the case where there is no gravitational coupling constant. This is a nice result as it implies that the measure \(\sum_{T \sim S^4}\) selects positive and negative curvature with the same weight, as already noticed. As soon as we add a gravitational coupling constant \(\kappa_2\) we get an expectation value \(\langle R \rangle = R_0(\kappa_2)\), which is essentially a linear function of \(\kappa_2\) for \(\kappa_2\) not too large. Since the only interesting critical behaviour takes place for \(\kappa_2 > 0\) we need at least a reinterpretation of the scaling limit in order to

\[4\] The situation might be different if we include matter fields.
be able to claim we can make contact with continuum physics. One possibility is that the expectation value of $R_c$ has its root in the missing tessellation of flat space, but we feel that would be a surprise in a quantum theory of gravity. A more radical point of view would be that there is no four-dimensional quantum theory of gravity. Maybe quantum gravity needs matter, as the quantum theory of matter might need gravity? 

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**Figure Captions**

Fig. 1 The average curvature $\langle R \rangle(\kappa^2)$ for $h = 0$ and $N_4 = 4000(\bigcirc)$, $N_4 = 9000(\triangle)$, $N_4 = 16000(\times)$ and $N_4 = 32000(\times)$.

Fig. 2 The average curvature $\langle R \rangle(\kappa^2)$ for $h = 0, 10$ and 20 and $N_4 = 16000$.

Fig. 3 Susceptibility $\chi(\kappa^2)$ for $h = 0$ and 20 and $N_4 = 4000(\bigcirc)$, $N_4 = 9000(\triangle)$ and $N_4 = 16000(\times)$.

Fig. 4a $\langle R^2 \rangle - \langle R \rangle^2$ as a function of $\kappa^2$ for $h = 0, 10$ and 20 and $N_4 = 4000(\bigcirc)$, $9000(\triangle)$, $16000(\times)$ and 32000(\times).

Fig. 4b $\langle R^2 \rangle - \langle R \rangle^2$ as a function of $\kappa^2$ for $h = 0$ and $N_4 = 4000(\bigcirc)$, $9000(\triangle)$, $16000(\times)$ and 32000(\times).

Fig. 5 Critical line $\kappa_c^2(h)$.

Fig. 6 Average curvature at the phase transition for $N_4 = 4000(\bigcirc)$, $9000(\triangle)$ and $16000(\times)$.

Fig. 7 Average $d_1$ distance for $h = 0$ and 20 and for $N_4 = 4000(\bigcirc)$, $9000(\triangle)$ and $16000(\times)$.

Fig. 8 Average $d_4$ distance for $h = 0$ and 20 and for $N_4 = 4000(\bigcirc)$, $9000(\triangle)$ and $16000(\times)$.

Fig. 9 Distribution of the $d_4$ distances for $h = 0$ and $\kappa^2 = 0.9, 1.0, 1.1$ and 1.2.

Fig. 10 Distribution of the curvature per simplex and blocked at a $d_4$ distance 3 and 5.