Transport Properties of a One-Dimensional Two-Component Quantum Liquid with Hyperbolic Interactions

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Abstract

We present an investigation of the sinh-cosh (SC) interaction model with twisted boundary conditions. We argue that, when unlike particles repel, the SC model may be usefully viewed as a Heisenberg-Ising fluid with moving Heisenberg-Ising spins. We derive the Luttinger liquid relation for the stiffness and the susceptibility, both from conformal arguments, and directly from the integral equations. Finally, we investigate the opening and closing of the ground state gaps for both SC and Heisenberg-Ising models, as the interaction strength is varied.

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In Ref. \[1\] and \[2\], to be called I and II in the following, we solved the integrable one-dimensional (1D) SC-model defined by the Hamiltonian

\[
H = -\frac{1}{2} \sum_{1 \leq j \leq N} \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} v_{jk}(x_j - x_k). \tag{1}
\]

The pair potential is given as

\[
v_{jk}(x) = s(s+1) \left[ \frac{1 + \sigma_j \sigma_k}{2 \sinh^2(x)} - \frac{1 - \sigma_j \sigma_k}{2 \cosh^2(x)} \right], \quad s > -1, \tag{2}
\]

and the quantum number \(\sigma = \pm 1\) distinguishes the two kinds of particles. We may usefully think of it as either representing charge or spin. For values of the interaction strength \(s\) in the range \(-1 < s < 0\), the system exhibits two gapless excitation branches with different Fermi velocities as does the repulsive 1D Hubbard model \[3\], and thus may be classified as a typical two-component 1D Luttinger liquid \[4\]. The asymptotic behavior of the correlation functions is given by finite-size arguments of conformal field theory. A Wiener-Hopf type calculation \[2\] shows that the spin-spin part of the dressed charge matrix is essentially identical to the dressed charge scalar in the Heisenberg-Ising (H-I) model \[5\].

In this Letter, we will further explore the connection of the SC model with the H-I model by examining the response of the system to a flux \(\Phi\). The addition of a flux is compatible with integrability and allows the study of the transport properties by an adiabatic variation of \(\Phi\). For the H-I model, this has already been done \[6,7\] for the interaction strength range \(-1 \leq \Delta \leq 1\). We will show that the spin degrees of freedom of the SC model for \(0 > s > -1\) may be usefully viewed as a H-I model with moving H-I spins. The presence of the translational degrees of freedom will simply renormalize the spin-spin coupling.

We thus restrict ourselves in what follows to the unbound case \(-1 < s < 0\), such that there are two gapless excitations corresponding to a particle-hole and a two spin-wave continuum with excitation velocities \(v\) and \(v_s\), respectively. Let us then modify the Bethe ansatz equations of Eq.(II.7) by threading them with a flux \(\Phi\). We have two coupled equations for \(N\) particles with pseudo-momenta \(k = (k_1, \ldots, k_N)\) and \(M\) spin waves with rapidities \(\lambda = (\lambda_1, \ldots, \lambda_M)\) on a ring of length \(L\). The energy of a given state is \(E(k) = \)
\[ \frac{1}{2} \sum_{j=1}^{N} k_j^2 \] and the total momentum is \( P(k) = \sum_{j=1}^{N} k_j \). Boosting the system by \( \Phi \) will accelerate the two kinds of particles in opposite directions due to the two components being of equal but opposite charge. Therefore, we have no center-of-mass motion and \( P = 0 \). The energy of a given state will change as a function of \( \Phi \), and the energy shift of the ground state may be written as \( \Delta E_0(\Phi) \equiv E_0(\Phi) - E_0(0) \equiv D\Phi^2/2L + O(\Phi^4) \), where \( D \) is called the stiffness constant and can be specified by perturbation arguments for \( \Phi \) up to \( \pi \). Note that since we do not have any center-of-mass motion, we can call \( D \) either spin or charge stiffness depending on what interpretation of \( \sigma \) we adopt. We choose the spin language for comparison with the H-I model. However, the charge interpretation is probably more natural to describe transport properties. We furthermore caution the reader that the term charge stiffness has been previously used in lattice models to describe center-of-mass motion.

The twisted Bethe ansatz equations are given by

\[ - Lk_j = 2\pi I_j(k_j) - \frac{M}{N} \Phi + \sum_{a=1}^{M} \theta_{0,-1}(k_j - \lambda_a) + \sum_{l=1}^{N} \theta_{0,0}(k_j - k_l), \quad (3a) \]

\[ 0 = 2\pi J_a(\lambda_a) + \Phi + \sum_{b=1}^{M} \theta_{1,-1}(\lambda_a - \lambda_b) + \sum_{j=1}^{N} \theta_{0,-1}(\lambda_a - k_j). \quad (3b) \]

The two-body phase shifts for particle-particle, particle-spin wave and spin wave-spin wave scattering, \( \theta_{0,0}(k) \), \( \theta_{0,-1}(k) \) and \( \theta_{1,-1}(k) \) respectively, have been given in I. The particle quantum numbers \( I_j \) and the spin-wave quantum numbers \( J_a \) are integers or half-odd integers depending on the parities of \( N, M \) as well as on the particle statistics. For simplicity, we use bosonic selection rules, although a purely fermionic or a mixed bose-fermi system may be studied along similar lines. In the ground state of the bosonic system, we have

\[ I_1, I_2, \ldots, I_N = -\frac{(N-1)}{2}, -\frac{(N-3)}{2}, \ldots, \frac{(N-1)}{2}, \]

\[ J_1, J_2, \ldots, J_M = -\frac{(M-1)}{2}, -\frac{(M-3)}{2}, \ldots, \frac{(M-1)}{2}, \quad (4) \]

for both \( N \) and \( M \) even.

We start with some general considerations. Let us denote by \( E_{\{I,J\}}(\Phi) \) the energy of a state specified by the \( \Phi = 0 \) set of quantum numbers \( \{I,J\} \). We then adiabatically turn
on the flux until we return to our initial state. The energy will also return to its initial
value, although, it may return sooner; so, the period of the wave function will be an integer
multiple of the period of the energy. We can define a topological winding number $n$ to be
the number of times the flux $\Phi$ increases by $2\pi$ before the state returns to its initial value.
As Sutherland and Shastry have shown, the ground state winding number of the H-I model
with $S_z = 0$ in the parameter range $-1 < -\cos(\mu) \equiv \Delta < 1$ is 2, implying charge carriers
with half the quantum of charge, except at isolated points $\Delta = \cos(\pi/Q)$, where $M \geq Q \geq 2$
is an integer. In particular, at $\Delta = 0$, the free particle wave function has periodicity $2\pi N_{HI}$,
where $N_{HI}$ is the number of H-I sites, implying free acceleration in the thermodynamic limit.

We now note the following important fact: Choosing $\mu \equiv -\pi s$, the spin wave-spin wave
phase shift $\theta_{-1,-1}$ is identical to the spin-spin phase shift in the H-I model, and we may
rewrite the equation for the rapidities as

$$N \bar{\theta}_{0,-1}(\lambda_a, \mu) \equiv N \sum_{j=1}^{N} \theta_{0,-1}(\lambda_a - k_j, \mu)/N = 2\pi J_a(\lambda_a) + \Phi + \sum_{b=1}^{M} \theta_{-1,-1}(\lambda_a - \lambda_b, \mu). \quad (5)$$

which nearly is identical to the Bethe Ansatz equation of the H-I model, as can be readily
seen when we use the standard transformation for the H-I momenta $p = f(\alpha, \mu)$. We then
merely have to identify $\alpha \equiv \pi \lambda$. The sole effect of the pseudo-momenta $k$ is an averaging
on the left hand side. Let us now restrict ourselves in what follows to the neutral (spin
zero) sector such that we have $M$ particles with $\sigma = -1$ and $M$ particles with $\sigma = +1$
for a total of $N = 2M$. Then, a discussion of the behavior of the rapidities $\lambda$ for varying
$\Phi$ exactly mimics the discussion of the H-I momenta $p$ in Ref. [6] at $S_z = 0$: As long as
$|\Phi| \leq 2\pi (s + 1)$, all $\lambda$’s stay on the real axis. At $\Phi = 2\pi (s + 1)$, the largest root $\lambda_M$ goes to
infinity. For $\Phi$ increasing beyond this point, $\lambda_M$ will reappear from infinity as $i\pi + \gamma_1$ until
exactly at $\Phi = 2\pi$, $\lambda_M = i\pi$ ($\gamma_1 = 0$) and the remaining $M-1$ rapidities have redistributed
themselves symmetrically around 0 on the real axis. However, as mentioned above, this
behavior is different at the threshold values $s = (1 - Q)/Q$. The momenta $k$ are always real
and distributed about the origin. Eq. (3b) simplifies at $\Phi = 2\pi (s + 1)$ (and thus $\lambda_M = \infty$)
and is in fact just the equation for $M - 1$ rapidities in the ground state. So as in Ref. [6]
using simple thermodynamical arguments, we may write

$$\Delta E_0(2\pi(s+1)) = E_0(N, M-1) - E_0(N, M) = 1/2L \cdot \chi^{-1},$$

(6)

where $\chi$ is the susceptibility. Comparing with the definition of the stiffness constant $D$, we find $D = \chi^{-1}/4\pi^2(s+1)^2$.

On the other hand, we can read off the finite-size energy corrections for the SC model, and then finite-size arguments of conformal field theory give an expression for $\Delta E_0(2\pi(s+1))$ in terms of the conformal weights, the dressed charge matrix $\Xi$ and the spin wave velocity $v_s$. The neutral sector dressed charge matrix is given in Eq.(II.35) and thus we have $\chi^{-1} = 2\pi v_s(s+1)$. We may therefore express the stiffness $D$ in terms of the spin wave velocity as

$$D = v_s/2\pi(s+1).$$

(7)

We emphasize that this formula for $D$ is true also for a system of purely fermionic particles. Shastry and Sutherland [6] have given an exact formula for the stiffness constant in the H-I model, by using the known expression for the H-I model spin wave velocity $v_s = \pi \sin(\mu)/\mu$ [8]. No such expression is known for the SC model and we can only give $v_s$ as

$$v_s = \frac{1}{2\pi} \int_{-B}^{B} \frac{e^{-\pi k/2s} \epsilon'(k) dk}{\int_{-B}^{B} e^{-\pi k/2s} \rho(k) dk}.$$ 

(8)

Here, we use the definitions of II, Section II. However, written in terms of spin velocities the stiffness formulas are identical and only the values of the respective spin wave velocities are different. Thus the presence of the translational degrees of freedom in the SC model simply renormalizes the spin-wave velocity.

We have iterated the Bethe Ansatz equations [3] in the neutral sector for reasonably large systems and density $d \equiv N/L = 1/2$ as a function of $\Phi$. By our correspondence between the H-I model, and the spin wave part of the SC model, we expect free spin waves at $s = -1/2$. In the thermodynamic limit, we would thus expect the periodicity of the ground state energy to be infinite. For a finite system, this will be reduced to a periodicity that scales with the system size. For the SC model we have indeed found that at $s = -1/2$
the periodicity of the ground state energy is given as $2\pi N$. We may then speak of $s \to -1^+$ as the ferromagnetic critical point and $s \to 0^-$ as the antiferromagnetic critical point of the SC model. In Fig.(1), we show the full spectrum of low-lying states with zero momentum at $s = -1/2$ for $L = 12$, $N = 6$ and $M = 3$. The ground state curve is emphasized and its periodicity is $6 \cdot 2\pi$.

Note that at $\Phi = 2\pi$ there is a level crossing between ground state and first exited state in Fig.(1). When the interaction strength changes from $s = -1/2$, immediately a gap opens between the ground state and first exited state. Just as in the H-I model the periodicity is reduced to $4\pi$. Note that a perturbation theory argument can not describe this behavior. Fig.(2) shows the behavior of the ground state energy variation $L[1 - E(\Phi)/E(2\pi)]$ for $s = -1/3$ near $\Phi = 2\pi$ for different lattice sizes. The rounding is well pronounced and does not vanish as we increase the size.

Thus the behavior of the low-lying states in the SC and H-I models is qualitatively the same, up to the renormalization of quantities such as the spin wave velocity $v_s$. Let us briefly describe the behavior of the gaps in the H-I model, keeping in mind the correspondence $\mu = -\pi s$. Increasing $\mu$ beyond $\pi/2$ ($\Delta = 0$), we see that the gap continues to widen up to a maximum value at $\mu \sim 7\pi/12$ ($\Delta \sim 0.26$). It then closes up again exactly at $\mu = 2\pi/3$ ($\Delta = 1/2$). As has been noted before, this value of $\mu$ coincides with the appearance of a $Q = 3$ string. Further increase of $\mu$ again opens, and then closes the gap at the threshold for the next-longer $Q = 4$ string. This behavior continues, and the threshold values accumulate as $\mu \to \pi$ ($\Delta \to 1$). In Fig.(3), we show the ground state and the first exited state of the H-I model on a ring of $N_{HI} = 12$. Note that due to the finite size of the ring, we can only observe strings up to length $Q = 6$. We will present a more detailed finite-size study of the behavior of the gaps in H-I and SC model in another publication. We only mention that for fixed $\mu$ the gap scales with the system size as a negative power of $N_{HI}$, with variable exponent depending on the coupling constant $\mu$.

The stiffness constant $D$ is the curvature of the ground state energy $E_0(\Phi)$ as a function of $\Phi$. In Fig.(4), we show $D$ for systems of 12, 24 and 32 lattice sites. We also show the
behavior of $D$ as given by Eq.(7). As $s \to 0^-$, the spin wave velocity approaches the velocity of a non-interacting single-component model, i.e. $v_s \to \pi d/2$ [3]. Thus $D$ approaches the non-zero value $1/8$ which is compatible with the result of Ref. [6]. Furthermore, the SC model exhibits a gap for $s > 0$ and so $D$ is zero. Thus $D$ exhibits a jump discontinuity at $s = 0$ just as in the H-I model for $\Delta = -1$.

Note that Eq.(7) may also be written as $D\chi^{-1} = v_s^2$. This is nothing but the Luttinger relation for the spin wave excitations [4]. Let us briefly explain how to derive this formula without using arguments of conformal field theory. In the thermodynamic limit, we convert Eq.(3) into a set of coupled integral equations as in Eq.(II.10). Here $\rho(k)$ and $\sigma(\lambda)$ are the distribution functions of particles and down-spins, respectively. The density $d$ and the magnetization $M$ are then given parametrically in terms of the integral limits $B$ and $C$. We now use an iteration scheme, i.e., first, for $B = \infty$ and $M = 0$, i.e. at half-filling and zero magnetization, we calculate $\rho(k) \equiv \rho_0(k)$. We then use this $\rho_0(k)$ in the equation for $\sigma(\lambda)$ with $C$ finite and $B$ nearly $\infty$. Finally, we use this $\sigma(\lambda)$ to calculate $\rho(k)$ and thus the effect on the momenta and the energy. Since we are only interested in the leading order correction terms, we may stop. Then the corrections to the energy are

$$
\frac{\Delta E}{L} = \frac{1}{2} \left[ D\left(\frac{\Phi}{L}\right)^2 + \chi^{-1}M^2 \right]
= \frac{v_s}{4\pi(1+s)} \left[ \left(\frac{\Phi}{L}\right)^2 + [2\pi(1+s)]^2M^2 \right],
$$

where $M = \frac{1}{2}d(1-2M/N)$. A complete account of this calculation can be found in Ref. [8].

The derivation of the Luttinger relation uses integral equations and as such is valid in the thermodynamical limit. Most of the other results given above have been derived using Eq.(3). These equations, however, have been derived by the asymptotic Bethe Ansatz (AsymBA). This method is only correct in the thermodynamic limit [10]. Thus all our finite-size results should exhibit correction terms. From the hyperbolic form of the pair potential (4), we may expect these corrections to be exponentially small in $L$. Indeed, a log-log plot of the ground state energy versus $L$ at fixed interaction strength shows a simple power law behavior already for $L \geq 6$. Thus the $L \to \infty$ behavior of the finite-size Bethe Ansatz
equations for the SC model does not seem to differ in any significant respect from usual finite size behavior for short ranged models. This further supports our use of the AsymBA in the present study.

In conclusion, we have shown that the SC model exhibits all the rich structure of the H-I model for $-1 < s < 0$. In particular, there is a Luttinger relation for the spin waves just as in the H-I model, that can be derived from (i) conformal arguments, (ii) an exact calculation in the thermodynamic limit and (iii) is furthermore supported by numerical results for finite systems. Thus this yields credibility to both the conformal and the Luttinger approach in models solved by the AsymBA. Finally, we have reported an interesting behavior of the gaps in H-I and SC models.

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FIGURES

FIG. 1. The low-lying states for the SC model at $L = 12$, $N = 6$ and $M = 3$. The bold curve corresponds to the ground state and the winding number is $n = 6 = N$. Note the various level crossing in this free spin wave case, especially the crossing of ground state and first exited state at $\Phi = 2\pi$.

FIG. 2. Plot of the ground state energy variation $L[1 - E(\Phi)/E(2\pi)]$ for the SC-model at $s = -1/3$ for $L = 12, 20$ and 28.

FIG. 3. The charge stiffness $D(s)$ for the SC model. The dashed curves correspond to $L = 12, 24$ and 32 and converge to $D(0) = 1/8$ at $s \to 0^-$. The solid curve comes from Eq.(3), which can be derived by conformal methods or from the Luttinger relation. (Note that as $s \to 0^-$, the solid curve does not converge to $1/8$. This is due to a buildup of numerical errors in the integration routine.)

FIG. 4. Energy of the ground state and first exited state and their difference in the H-I model for $N_{HI} = 12$. Note the closing of the gap at $\Delta = \cos(\pi/Q)$ for $Q = 2, 3, 4, 5$. 