Discovering Classification Rules for Interpretable Learning with Linear Programming

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Abstract: Rules embody a set of if-then statements which include one or more conditions to classify a subset of samples in a dataset. In various applications such classification rules are considered to be interpretable by the decision makers. We introduce two new algorithms for interpretability and learning. Both algorithms take advantage of linear programming, and hence, they are scalable to large data sets. The first algorithm extracts rules for interpretation of trained models that are based on tree/rule ensembles. The second algorithm generates a set of classification rules through a column generation approach. The proposed algorithms return a set of rules along with their optimal weights indicating the importance of each rule for classification. Moreover, our algorithms allow assigning cost coefficients, which could relate to different attributes of the rules, such as; rule lengths, estimator weights, number of false negatives, and so on. Thus, the decision makers can adjust these coefficients to divert the training process and obtain a set of rules that are more appealing for their needs. We have tested the performances of both algorithms on a collection of datasets and presented a case study to elaborate on optimal rule weights. Our results show that a good compromise between interpretability and accuracy can be obtained by the proposed algorithms.

Keywords: Rule generation; linear programming; interpretability

1. Introduction. Medical diagnosis, educational and juridical decisions often have important consequences for the society. Therefore, both the accuracy and interpretability of these decisions are of crucial importance. In particular, these decisions should be understandable by the decision makers. Rule sets consisting of a few intuitively coherent rules have shown to accomplish this purpose in such domains (Lakkaraju et al., 2016). Here, we first aim at rule extraction from powerful ensemble methods and then solely focus on rule generation. In both cases, our objective is to obtain a set of classification rules that balances the trade-off between accuracy and interpretability. Our main tools in this effort are linear optimization and column generation.

A rule is an independent if-then statement, which contains one or more conditions that assign a class to a set of samples. For example in binary classification, “if (Clump Thickness is greater than six) and (Single Epithelial Cell Size is less than four) then the tumor is malignant” is such a rule that can be used for breast cancer diagnosis. When a sample satisfies this rule, then it receives a label corresponding to one of the two classes. In case a sample is covered by more than one rule, then majority voting among the assigned labels is used to determine the class of the sample.

Growing decision trees is closely related to rule-based learning. A Decision Tree (DT) naturally results with a set of leaves, where each leaf corresponds to a different rule. On one hand, rule learning is considered to be more flexible for interpretability than tree-based learning approaches. Fürnkranz (1999) lists superiorities of rule-based learning over tree-based learning. Leaves (rules) of a DT are not independent from each other. For example, in a binary tree when a splitting is performed at a node, the left child grows a rule, while its right sibling grows its negation. As a result, every sample obeys to exactly one rule and it is classified according to the corresponding leaf. Thus, inaccurate rules, which are negations of their siblings after splitting, can be created. This may render false classification of samples and reduce both accuracy and interpretability. However, independently constructed rules do not need such a negation rule, and hence, rule-based learning can be considered more flexible for interpretability. On the other hand, unlike DTs, an independent set of rules does not necessarily cover the entire sample space. This may result in a state where a test sample can not be classified with the proposed rule set. This drawback is often handled by assigning a default label to those uncovered samples (Fürnkranz et al., 2012). To minimize the number of uncovered samples during testing, the training data is required to be fully covered. Separate-and-conquer algorithm by Fürnkranz (1999) achieves this heuristically by fitting
rules on uncovered training samples, and new rules are generated until each training sample is covered by at least one rule. Instead of such a sequential covering, we take advantage of our linear programming (LP) approach and explicitly impose a covering constraint on the training set.

In this paper, we propose two algorithms for interpretability and learning that are based on mathematical programming. We first give an LP model that is used by our Rule Extraction (RUX) algorithm, which selects rules from tree or rule ensembles for interpretation. Then, we develop a Rule Generation (RUG) algorithm, which generates classification rules using column generation (CG).

Rule extraction methods attempt to select a set of rules from accurate complex or black-box models to interpret the predictions of these models (Hayashi and Oishi, 2018). For instance, several works in the literature aim at interpreting Random Forest (RF) models by extracting rules from the trees in the forest (Liu et al., 2012; Lu Thi et al., 2015; Adnan and Islam, 2017; Wang et al., 2020; Birbil et al., 2020). Most of these studies use heuristic approaches to select the desired set of rules. Birbil et al. (2020) suggest a set covering formulation to extract interpretable rules from RFs. Other applications include extraction of rules from artificial neural networks (Andrews et al., 1995) and support vector machines (Barakat and Bradley, 2010).

There are several studies that are closely related to ours, since they also employ mathematical programming for rule learning. Malioutov and Varshney (2013) propose a rule-based binary classifier solving a binary program that minimizes the number of rules for a boolean compressed sensing problem. Wang and Rudin (2015) present a mixed integer linear programming (MILP) formulation to learn decision rules for binary classification (e.g., patterns). They give a discussion on how their classifier is equivalent to DTs and RFs. A three-phase framework, which relaxes the MILP problem, is offered to solve larger instances. Pattern selecting phase solves the MILP formulation over the generated rules with an objective of minimizing the number of misclassified samples, the number of patterns generated, and the total length of each pattern. Dash et al. (2020) offer a CG-based framework to find optimal rule set for binary classification, where the objective is to find a trade-off between classification rule simplicity and accuracy. One-hot encoding is used to binarize categorical data, and numerical data is also discretized with sample deciles as thresholds. For large instances, the pricing subproblem is either solved with time limits, or the model columns are generated by a greedy heuristic. Malioutov and Meel (2018) solve a Max-Sat formulation by constraint programming to construct interpretable classification rules. Ghosh and Meel (2019) also propose a framework based on MaxSAT formulation that can be applied to binary classification problems with binary features. Their approach is incremental and takes advantage of partitioning the data set into several clusters. Their objective is to minimize the number of generated rules and the number of misclassified samples.

With our current work, we make the following contributions.

- RUX and RUG are based on an LP model and thus, both algorithms are scalable for large datasets. This is different than existing studies that use MILP formulations.
- The proposed algorithms directly address multi-class problems whilst existing studies using optimization-based approaches are for binary classification. To that end, the objective function in our LP formulation minimize classification error using a loss function instead of explicitly counting misclassified samples.
- Both RUX and RUG can work with continuous or categorical features, and hence, they do not require encoding of the data.
- Along with the set of rules, our algorithms also return the optimal weights for the rules. These weights allow to attach importance to each rule for interpreting the classification.
- Both algorithms admit assigning cost coefficients to the rules. These coefficients could relate to different attributes of the rules, such as; rule lengths, estimator weights, number of false negatives, and so on. The objective function also allows penalizing rules that may have undesired outcomes (like long rule lengths). Thus, the decision makers can use these coefficients to lead the training
process to obtain a set of rules more appealing for their needs.

- The novelty in our column generation approach is the use of a regular decision tree (DT) with sample weights as the pricing subproblem. Training trees with sample weights is very fast, and also standard in all machine learning packages as boosting methods also rely on sample weights.
- We present our algorithms for multi-class classification problems. The proposed ideas can also be extended to discovering regression rules for interpretable learning with linear programming.

2. Rule Extraction. We consider a classification problem with $K$ classes and denote the set of class labels by $\mathcal{K}$. The training dataset consists of samples with features $x_i \in \mathbb{R}^p$ for $i \in \mathcal{I}$ and labels $y_i$ for $i \in \mathcal{I}$. To work with multiple classes, we define a vector-valued mapping $y(x_i) \in \mathcal{K} \subset \mathbb{R}^K$ as in Zhu et al. (2009). That is, if $y_i = k$, then

$$y(x_i) = (-\frac{1}{K-1}, \ldots, -\frac{1}{K-1}, 1, \ldots, -\frac{1}{K-1})^T,$$

where the value one appears only at the $k^{th}$ component of the vector.

Suppose that we have a collection of rules indexed by $\mathcal{J}$. A rule $j \in \mathcal{J}$ assigns the vector $R_j(x_i) \in \mathcal{K}$ to input $x_i$, only if rule $j$ covers sample $i$. This vector is also formed in the same manner as in (1). To predict the class of a given sample $x_i$, with this collection of rules, we use a set of nonnegative weights $w_j$, $j \in \mathcal{J}$ associated with the rules and evaluate

$$\hat{y}(x_i) = \sum_{j \in \mathcal{J}} a_{ij} R_j(x_i) w_j,$$

where $a_{ij} \in \{0,1\}$ indicates whether rule $j$ covers sample $i$ or not. Then, the index of the largest component of the resulting vector $\hat{y}(x_i)$ is assigned as the predicted label $\hat{y}_i$ of sample $i \in \mathcal{I}$. Note that (2) is similar to the weighting of the classifiers in standard boosting methods. Here, instead of classifiers, we use rules for classifying only the covered samples.

In order to evaluate the classification error, we use the hinge loss and define the total classification loss by

$$\sum_{i \in \mathcal{I}} \max \{ 1 - \sum_{j \in \mathcal{J}} \hat{a}_{ij} w_j, 0 \},$$

where $\hat{a}_{ij} = \kappa a_{ij} R_j(x_i)^T y(x_i)$ with $\kappa = (K - 1)/K$. This loss function allows us to write a linear programming model, where the objective is to find the set of rules that minimizes the total loss. To this end, we introduce the auxiliary variables $v_i$, $i \in \mathcal{I}$ standing for $v_i \geq \max \{ 1 - \sum_{j \in \mathcal{J}} \hat{a}_{ij} w_j, 0 \}$, and obtain our master problem

$$\begin{align*}
\text{minimize} & \quad \sum_{i \in \mathcal{I}} v_i + \sum_{j \in \mathcal{J}} c_j w_j \\
\text{subject to} & \quad \sum_{j \in \mathcal{J}} \hat{a}_{ij} w_j + v_i \geq 1, \quad i \in \mathcal{I}; \\
& \quad \sum_{j \in \mathcal{J}} a_{ij} w_j \geq \varepsilon, \quad i \in \mathcal{I}; \\
& \quad v_i \geq 0, \quad i \in \mathcal{I}; \\
& \quad w_j \geq 0, \quad j \in \mathcal{J},
\end{align*}$$

where $c_j \geq 0$, $j \in \mathcal{J}$ are the cost coefficients. These coefficients and the second set of constraints require further explanation. The cost coefficients serve two important roles: First, solutions involving many rules with nonzero weights are avoided. The less the number of rules in the resulting set, the easier it is to interpret a solution. In other words, we prefer sparse solutions and $c_j$ serves to that preference. Second, in many different application domains, rules have actual costs that need to be taken into consideration. As we also highlight in our title, when interpretability is of concern, one could try to obtain rules with few features because shorter rules are considered to be easier to interpret (Lakkaraju et al., 2016). In this case, the number of conditions in a rule can be set as the rule cost. As another example, consider a classification problem in medical diagnosis, where false negatives are likely to be perceived as more costly than false positives. Such an evaluation could also be easily incorporated with a rule cost in the proposed
model. We should point out that there exists a trade-off between model accuracy and the rule set size. This can also be handled with our formulation by introducing a fixed cost coefficient, i.e., using the same value for all \( c_j, j \in \mathcal{J} \).

The master problem (3) also involves a set of covering constraints with the fixed right-hand-side, \( \varepsilon > 0 \). These constraints make sure that each sample is covered by at least one rule. The need for these covering constraints is exemplified as follows:

Consider a binary classification problem, where we select three samples \( \{x_i, x_k, x_l\} \) along with their labels \( y_i = 1, y_k = -1 \) and \( y_l = -1 \). Suppose that we have two rules \( j \) and \( j' \) such that the former covers all three, whereas the latter covers only the last two samples. Here, the rules use majority voting as it is applied to the leaves in trained DTs. The labels assigned by the rules \( j \) and \( j' \) are as follows: \( R_j(x_l) = R_j(x_k) = R_j(x_1) = -1 \) and \( R_{j'}(x_k) = R_{j'}(x_1) = -1 \). The first set of constraints in (3) then becomes

\[
-a_{ij} w_j + v_i \geq 1, \\
 a_{kj} w_j + a_{kj'} w_{j'} + v_k \geq 1, \\
 a_{lj} w_j + a_{lj'} w_{j'} + v_l \geq 1.
\]

For simplicity, if we also assume that \( c_j = c_{j'} \), then the optimal solution becomes \( w_j = v_k = v_l = 0 \) and \( w_{j'} = v_i = 1 \). With this solution, sample \( x_i \) is not covered.

We remark that covering each sample in a dataset is crucial from the perspective of giving a reliable interpretation with the obtained rules. This is particularly important when the resulting set of rules are is to interpret the classification of individual samples in the dataset. Clearly, varying \( \varepsilon \) may lead to a change in the set of optimal rules. For instance a large \( \varepsilon \) value may impose larger rule weights on those samples covered with only few rules. However, we point out that the role of this constraint is just coverage, and hence, setting \( \varepsilon \) to a small strictly positive value is sufficient*

Up until this point, we have not specified any details about the rule set \( \mathcal{J} \). On one hand, this rule set can be static in the sense that it can be obtained by extracting the rules available through a trained tree ensemble algorithm or by using the rules resulting from a rule-based method (Cohen and Singer, 1999). For instance, consider a Random Forest model trained on a given dataset. Then, the rule set \( \mathcal{J} \) can be obtained from the leaves of the trees in the forest, since each leaf corresponds to a rule. Solving (3) with such a rule set allows us to extract the rules that were most critical for classification. As we have mentioned before, we can assign the length of a rule as its cost and try to obtain rules with desirable lengths for interpretation. In a similar vein, consider another example with a trained AdaBoost (Freund and Schapire, 1997) model for which the base estimators are set as DTs. Again, the leaves of the trees from AdaBoost can be used to construct the rule set \( \mathcal{J} \) in (3), which is then solved to extract an interpretable set of rules. The costs of the rules in this case could be the inverse of the estimator weights assigned by the AdaBoost algorithm to the trees. In this way, the obtained set of rules is more likely to inherit the importance of the DTs from the AdaBoost model. Using these trained models to construct our master problem leads to our first algorithm RUX, which is based on tree ensembles. We show in Section 4 that RUX can indeed extract only a selection of rules from the trained models without significantly sacrificing accuracy. The computational complexity for RUX is determined by the underlying LP solver used. In practice, most of the LP solvers often use interior point methods that work in polynomial-time.

3. Rule Generation. Suppose now that we do not have the entire set of rules \( \mathcal{J} \) explicitly or the size of this collection is too large to be constructed by an existing learning framework. Thus, the rules should be generated in an iterative fashion. Since (3) is a linear programming problem and rules correspond to columns, this iterative scheme leads to the well-known column generation approach in

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*In case one insists on treating \( \varepsilon \) as a hyperparameter and proceeds with tuning, it is important to know that there will be intervals of \( \varepsilon \) values where the optimal set of rules remains the same. We have given this parametric analysis in Appendix B. Using this analysis, the tuning can be conducted in a computationally efficient manner.
optimization (Desaulniers et al., 2006). At each iteration of column generation, a linear programming model is constructed with a subset of the columns (column pool) of the overall model. This model is called the restricted master problem. After solving the restricted master problem, the dual optimal solution is obtained. Then using this dual solution, a pricing subproblem is solved to identify the columns with negative reduced costs. These columns are the only candidates for improving the objective function value when they are added to the column pool. The next iteration continues after extending the column pool with the negative reduced cost columns.

In order to apply column generation to our problem, we define a subset of rules \( J_t \subset J \) at iteration \( t \) and form the restricted master problem by replacing \( J \) with \( J_t \) in (3). Let us denote the dual variables associated with the first set of constraints by \( \beta_i, i \in I \). Likewise, let \( \gamma_i, i \in I \) be the dual variables corresponding to the coverage constraints. In vector notation, we simply use \( \beta \) and \( \gamma \). Then, the dual restricted master problem at iteration \( t \) becomes

\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in I} (\beta_i + \varepsilon \gamma_i) \\
\text{subject to} & \quad \sum_{i \in I} (\hat{a}_{ij}\beta_i + a_{ij}\gamma_i) \leq c_j, \quad j \in J_t; \\
& \quad 0 \leq \beta_i \leq 1, \quad i \in I; \\
& \quad \gamma_i \geq 0, \quad i \in I.
\end{align*}
\]

If we denote the optimal dual solution at iteration \( t \) by \( (\beta^{(t)}, \gamma^{(t)}) \), then improving the objective function value of problem (4) requires finding at least one rule \( j' \in J/J_t \) such that

\[
\bar{c}_{j'} = c_{j'} - \sum_{i \in I} (\hat{a}_{ij'}\beta^{(t)}_i + a_{ij'}\gamma^{(t)}_i) < 0,
\]

where \( \bar{c}_{j'} \) is the reduced cost of column \( j' \). In fact, this condition simply checks whether \( j' \in J/J_t \) violates the dual feasibility. To find those rules with negative reduced costs, we formulate the pricing subproblem as

\[
\max_{j \in J/J_t} \left\{ \sum_{i \in I} (\hat{a}_{ij}\beta^{(t)}_i + a_{ij}\gamma^{(t)}_i) \right\}.
\]

Linear programming theory ensures that if the pricing subproblem does not return any rule with a negative reduced cost, then we have the optimal solution to our master problem (3) with the current set of rules, \( J_t \). Otherwise, we have at least one rule with a negative reduced cost. After adding one or more rules with negative reduced costs to \( J_t \), we proceed with \( J_{t+1} \) and solve the restricted master problem or, equivalently, its dual (4). Algorithm 1 shows the steps of the exact rule generation approach. It is important to note that the sole purpose of solving pricing subproblem (6) is to return a subset of rules with negative reduced costs. We denote this subset by \( J_- \subseteq J/J_t \).

**Algorithm 1** Exact Rule Generation

Input: training data, \((x_i, y_i)_{i \in I}\)  
\( t = 0 \)  
Construct initial rule pool, \( J_0 \)  
\( {\text{while True do}} \)  
\( (\beta^{(t)}, \gamma^{(t)}) \leftarrow \text{Solve (4)} \)  
\( J_- \leftarrow \text{Solve pricing subproblem, (6)} \)  
\( \text{if } J_- = \emptyset \text{ then} \)  
\( \quad \text{return } J_t \)  
\( \text{end if} \)  
\( t \leftarrow t + 1 \)  
\( J_t = J_{t-1} \cup J_- \)  
\( \text{end while} \)

We have overlooked two important steps in Algorithm 1. The first one is the construction of the initial rule pool \( J_0 \) (line three). One possible solution for this step is to train a DT and use its leaves as the
starting set of rules. The second step is solving the pricing subproblem (line six). Note that solving (6) is associated with finding a rule with the highest accuracy when the samples have weights $\beta_i^{(t)}$, $i \in I$. As constructing an optimal binary decision tree is known to be \textit{NP}-complete (Laurent and Rivest, 1976), solving problem (6) is rather difficult even when $\gamma_i^{(t)} = 0$. This also allows us to come up with the proxy pricing subproblem of column generation as shown in Figure 1. Here, treating $\gamma_i = 0$ allows the construction of a DT using the dual values $\beta_i^{(t)}$ as sample weights. DTs can be solved very quickly using standard libraries that are available in all machine learning packages. Therefore, we propose to train a DT as a heuristic pricing approach, where the sample weights correspond to dual variables $\beta_i^{(t)}$, $i \in I$. Then, we can check whether any leaf of the DT satisfies (5) before adding those rules to the current rule pool. Notice that a negative reduced cost column (rule) with $\gamma_i = 0$ would also have a negative reduced cost when the pricing subproblem is solved exactly.

![Figure 1: Proposed rule generation algorithm.](https://github.com/sibirbil/RuleDiscovery)

Figure 1 shows the proposed heuristic approach for rule generation. The procedure DecisionTree takes a vector of sample weights as an input and returns a set of rules $\mathcal{J}$ (leaves of the decision tree). The vector $e$ is the vector of ones. The subset $\mathcal{J}_{-} \subseteq \mathcal{J}$ contains those rules with negative reduced costs satisfying (5).

As we use the dual solution $\gamma^{(t)}$ only for checking the reduced cost, training decision trees with sample weights in this manner boils down to solving a proxy pricing subproblem. Indeed, our rule generation algorithm RUG does not guarantee to solve the overall problem (3) to optimality. Fortunately, this may be to our advantage since a suboptimal solution can rectify the problem of overfitting to the training data. In fact, overfitting due to exact optimality is a common concern also in recent optimization-based approaches to learning (Günlük et al., 2018; Verwer and Zhang, 2019). Avid readers of optimization-based learning would also notice that there are some connections between RUG and the boosting methods. We elaborate on this point in Appendix C.

We remark once again that growing decision trees with sample weights is a standard tool in every library on machine learning. Likewise, there are various options to solve linear programs. Both growing DTs and solving LPs can be achieved in polynomial-time. Therefore, the proposed algorithm in Figure 1 is extremely easy to implement with the existing packages of popular programming languages.

4. Computational Study. We have evaluated the performances of the proposed algorithms on a collection of frequently used datasets. The details of these datasets are given in Appendix A. Both the rule extraction (RUX) and the rule generation (RUG) algorithms are implemented in Python. Since our computation times do not exceed one minute even on a standard laptop computer, we do not report them here. We have used stratified $10 \times 3$ nested cross validation to estimate the generalization performance of the tested methods. In all experiments, the right-hand-side value of each covering constraint is set to $\varepsilon = 0.01$.

†Our implementation is available at https://github.com/sibirbil/RuleDiscovery.
4.1 Numerical Experiments. First, we apply our RUX algorithm to trained random forest (RF) and AdaBoost (ADA) models. For both RF and ADA, we have selected the maximum-depth parameter from the set \{1, 2, 3\} and the number-of-estimators parameter from the set \{100, 200\}. After obtaining the best parameter set, we have used the corresponding trained model to apply RUX. We denote the models after applying RUX to RF and ADA by RUX-RF and RUX-ADA, respectively. In RUX-RF, we have used the rule-length as the cost coefficients \(c_j, j \in J\) (in (3)). For RUX-ADA, the cost coefficients are assigned as the inverse of the estimator weights of the trees in the trained ADA model. That is, the rules coming from the same tree receive the same cost coefficient.

Table 1 summarizes our results. Overall, we observe that RUX-RF and RUX-ADA give accuracies on par with RF and ADA. In fact, RUX-RF can even outperform RF on several datasets. These are BANKNOTE, TICTACTOE, ADULT, BANK-MKT, MAGIC, MUSK, PHONEME and MAMMOGRAPHY. On average, RUX-RF and RUX-ADA obtain far fewer number of rules than their counterparts RF and ADA, respectively. When we compare RUX variants, we observe that RUX-ADA generally obtains smaller sets with shorter rule lengths than RUX-RF. This is due to the larger lengths preferred by RF models as the average number of rules obtained with RF is higher than ADA. Take for instance glass, for which ADA returns more rules than RF. For those instances, we clearly see that the average rule length resulting from RUX-RF is smaller than those from RUX-ADA.

| Datasets   | ACC (0.03) | # Rules (0.03) | ACC (0.03) | # Rules (0.03) | ACC (0.03) | # Rules (0.03) | ACC (0.03) | # Rules (0.03) | ACC (0.03) | # Rules (0.03) |
|------------|------------|----------------|------------|----------------|------------|----------------|------------|----------------|------------|----------------|
| HEARTS     | 0.812 (0.03) | 1512.4 (480.7) | 0.917 (0.03) | 1250.2 (388.4) | 0.847 (0.03) | 1172.2 (351.7) | 0.872 (0.03) | 1172.2 (351.7) | 0.847 (0.03) | 1172.2 (351.7) |
| ADULT      | 0.953 (0.03) | 1600.3 (485.5) | 0.812 (0.03) | 1512.4 (480.7) | 0.812 (0.03) | 1512.4 (480.7) | 0.812 (0.03) | 1512.4 (480.7) | 0.812 (0.03) | 1512.4 (480.7) |
| TICTACTOE  | 0.803 (0.03) | 1024.2 (414.7) | 0.700 (0.03) | 671.8 (482.6)  | 0.780 (0.03) | 757.8 (452.8)  | 0.780 (0.03) | 757.8 (452.8)  | 0.780 (0.03) | 757.8 (452.8)  |
| WINE       | 0.962 (0.03) | 1104.2 (414.7) | 0.978 (0.03) | 1104.2 (414.7) | 0.978 (0.03) | 1104.2 (414.7) | 0.978 (0.03) | 1104.2 (414.7) | 0.978 (0.03) | 1104.2 (414.7) |

We report our results with the proposed RUG algorithm in Table 2. In addition to RF and ADA, we have also added a DT algorithm for comparison. The same set of values \{1, 2, 3\} is used for tuning the maximum-depth parameter both in DT and RUG (for constructing the initial column pool and the proxy pricing subproblem; see Figure 1). In RUG, we have used the rule length as cost coefficients. When we compare the accuracies of different methods, we observe that the average performance of RUG is better than those of DT and RF, and close to ADA. Since the resulting models from DT are interpretable, we also report the rule numbers as well as the rule lengths for DT. RUG outperforms DT in terms of accuracy at the expense of generating more rules. However, the average lengths of the rules obtained with RUG are shorter than those obtained with DT in all instances.

In their recent work, Dash et al. (2020) propose a rule learning approach based on column generation. They also give a comparison against other studies and present that their algorithm shows one of the best performances. We compare their results against a variant of RUG in Table 4 of Appendix D. We remark that their numerical study is limited to binary classification instances, and hence, the benchmarking is conducted only on a subset of datasets that we have used in our previous two tables. On average, our results are on par with the results obtained with their method in terms of accuracy and interpretability. It is important to note that both RUX variants and RUG are an order of magnitude faster on large datasets than the approach of Dash et al. (2020). Appendix D involves a further discussion.

Even though RUX variants and RUG obtain far fewer number of rules with short lengths for many datasets,
is their interdependent structure. For example, the first three rules shown in Figure 2. Even with fewer number of rules, the accuracy of RUG is better than the "Breast Cancer Wisconsin (Original)" dataset (Dua and Graff, 2017). This dataset has nine features with integer values from 1 to 10 and two diagnosis classes (benign or malignant). RUG reaches the accuracy level of 0.96 and generates 18 rules that we denote as $R_1$ to $R_{18}$. Figure 2 shows that RUG can also attain accuracy level of 0.95 using only the first 12 rules. The plot given in Figure 2 be used by decision makers (DMs) to interpret the rules and their weights obtained with RUG. The horizontal axis lists the rules in descending order of their optimal weights assigned by RUG. Here, the rule weights shown on the left side of the vertical axis are normalized by dividing the weights with the largest among them. On the right side, we present the accuracy of RUG on the test data cumulatively adding one rule at a time in the model. In other words, using only the first rule results with the accuracy of 0.92. With the addition of second and third rules, the accuracy of RUG increases to 0.93. This can be traced following the blue line and the corresponding point for each rule. When rule twelve is added, the accuracy of RUG catches that of the RF. As a remark, there is a slight deterioration in the accuracy when rule four is included and it rebounds back with addition of rule eleven. This can be expected since the performance is evaluated on the test data. Above each bar in the plot, we give the cumulative percentage of test instances covered by the rules. That is, rule one covers 73% of the test data and coverage gradually increases with the addition of subsequent rules. The coverage of samples reaches to 95% with only three rules and when the first nine rules are used, the coverage becomes full. Overall, interpretability plot is a playground for DMs to choose the best trade-off between accuracy and interpretability depending on the sensitivity of the application.

We next compare the rules extracted by RUG against the rules (leaves) of a DT trained on the same dataset with the same setting (tree depth of three). Figure 3 lists the sets of rules resulting from RUG and DT in blocks (a) and (b), respectively. DT yields seven rules (leaves) with an accuracy of 0.92. To make a fair comparison, only the first three rules of RUG is shown where the accuracy of the RUG reaches 0.93. Rules $R_1$ to $R_3$ are in descending order of their weights. As a reminder, these are the same first three rules shown in Figure 2. Even with fewer number of rules, the accuracy of RUG is better than DT.

We also observe that RUG extracts more interpretable rules than DT. A drawback of DT rules ($D_1$ to $D_7$) is their independent structure. For example $D_3$ is constructed as the complement of the statement

| DATASET | RF | ADA | DT | RUG |
|---------|----|-----|----|-----|
| BANKNOTE | 0.955 (0.018) | 0.999 (0.003) | 0.937 (0.017) | 0.827 (0.003) |
| HEARTS | 0.832 (0.059) | 0.815 (0.069) | 0.775 (0.041) | 0.74 (0.009) |
| ILP | 0.713 (0.014) | 0.700 (0.059) | 0.715 (0.009) | 0.688 (0.006) |
| IONOSPHERE | 0.929 (0.041) | 0.934 (0.041) | 0.897 (0.051) | 0.816 (0.042) |
| LIVER | 0.605 (0.060) | 0.737 (0.077) | 0.675 (0.064) | 0.682 (0.060) |
| ID3 | 0.759 (0.033) | 0.760 (0.034) | 0.736 (0.032) | 0.723 (0.033) |
| TACTACTOE | 0.764 (0.049) | 1.000 (0.000) | 0.714 (0.034) | 0.714 (0.034) |
| TRANSFUSION | 0.766 (0.025) | 0.794 (0.030) | 0.738 (0.035) | 0.738 (0.035) |
| WDBC | 0.953 (0.030) | 0.970 (0.020) | 0.919 (0.030) | 0.919 (0.030) |
| ADULT | 0.798 (0.004) | 0.866 (0.004) | 0.844 (0.005) | 0.844 (0.005) |
| BANK_MKT | 0.781 (0.014) | 0.844 (0.012) | 0.775 (0.008) | 0.775 (0.008) |
| MAGIC | 0.804 (0.007) | 0.872 (0.004) | 0.767 (0.009) | 0.767 (0.009) |
| MUSHROOM | 0.981 (0.008) | 1.000 (0.000) | 0.985 (0.004) | 0.985 (0.004) |
| MUSK | 0.904 (0.007) | 0.994 (0.003) | 0.915 (0.011) | 0.915 (0.011) |
| OILSPILL | 0.957 (0.000) | 0.970 (0.011) | 0.963 (0.013) | 0.963 (0.013) |
| PHONEME | 0.800 (0.017) | 0.858 (0.030) | 0.769 (0.020) | 0.769 (0.020) |
| MAMMOGRAPHY | 0.983 (0.002) | 0.987 (0.003) | 0.984 (0.002) | 0.984 (0.002) |
| SECKS | 0.910 (0.014) | 0.933 (0.078) | 0.890 (0.081) | 0.890 (0.081) |
| WINE | 0.978 (0.039) | 0.955 (0.051) | 0.904 (0.071) | 0.904 (0.071) |
| GLASS | 0.897 (0.072) | 0.794 (0.066) | 0.687 (0.036) | 0.687 (0.036) |
| ECOLOGY | 0.854 (0.055) | 0.836 (0.050) | 0.804 (0.041) | 0.804 (0.041) |
| SENSORLESS | 0.768 (0.008) | 0.892 (0.014) | 0.425 (0.002) | 0.425 (0.002) |

*Table 2: The performances of Random Forest (RF), AdaBoost (ADA), Decision Tree (DT) and Rule Generation (RUG). Here, ACC, # RULES, and RULE LEN. stand for accuracy, number of rules, and rule length, respectively. The first value in each cell gives the mean, whereas the value in parenthesis gives the standard deviation.*
“Bare Nuclei $\leq 5$” in $D_1$ and $D_2$. However, $D_3$ has two observations; one originally labeled as malignant and the other as benign. This implies a false negative classification of a patient in one out of two cases (50%). Similar structure can be observed for DT rules $D_5$ to $D_7$, where they are created as the complement of “Uniformity of Cell Shape $\leq 3$” in rules $D_1$ to $D_4$. Such an unbalanced structure is not observed for the rules extracted by RUG. More importantly, a false negative classification can be compensated in two ways by RUG: First, multiple rules covering a sample gives more chance to correct classification of the sample. In case there are two or more rules that do not agree, the rule weights can compromise the final class of a sample and correctly classify it based on the loss function’s structure. An example of such a case can be illustrated using the false negative classification of the sample with $D_3$. Using RUG, the patient data is covered by four rules; $R_4$, $R_9$ and $R_{15}$ with weights 0.43, 0.29 and 0.14 as malignant, and $R_{13}$ with weight 0.14 as benign. Using the labels assigned by the rules and their weights, this sample is classified as malignant. This implies that, RUG has detected the tumor and made the correct diagnosis for the patient. Second way comes from the flexibility of the LP model behind RUG. The rules that yield false negative classifications can be explicitly penalized in the objective function. That is, each rule yielding a false negative classification of a sample can be assigned a high cost coefficient (penalty), and consequently, model (3) can be enforced to choose another rule. As an example, $R_{13}$ can be penalized for this case and left outside the solution to promote correct classification.

5. Conclusion. We have proposed a linear programming (LP) approach to discover a set of critical rules in multi-class classification. The objective of this LP is to find a set of rule weights such that the sum of the classification error and the total rule cost is minimized. Using the optimal weights of the LP, a set of interpretable rules is obtained that can be used for classification. This approach has led to two algorithms that are both simple, fast and easy-to-implement. The numerical experiments have shown that the proposed algorithms may be used for hitting the right balance between accuracy and interpretability.

The first algorithm works with a predefined set of rules. To demonstrate our algorithm, we have used the models trained by Random Forest and AdaBoost algorithms. These two algorithms are frequently used for their remarkable accuracy. However, it is very difficult to interpret the trained models as both algorithms return tree ensembles involving thousands of rules. With Random Forest, we have used the rule lengths as cost coefficients in our algorithm to obtain shorter rules for interpretation. As AdaBoost returns also a set of estimator weights, we have used these weights as cost coefficients to mimic the performance of the trained AdaBoost model. Our numerical results show that the performance of our
Figure 3: Rules generated by RUG versus the rules (leaves) obtained with DT. RUG rules are shown in descending order of their weights.

rule extraction algorithm is close to –for several datasets even better than– its ensemble counterparts. More importantly, this performance comes with much fewer rules than the rules produced with both Random Forest and AdaBoost. On one hand, the set of rules required to apply our rule extraction algorithm may result from tree ensembles. On the other hand, this set may also come from other learning methods such as rule ensembles that produce a large collection of rules. Testing our algorithm further with those rule ensembles is an interesting future research direction.

Instead of a given collection of rules, our second algorithm uses a rule generation mechanism that iteratively solves LPs and builds up a set of rules. This algorithm has a solid mathematical basis as its core idea comes from the well-known column generation approach in large-scale linear programming. A significant advantage of this algorithm is having more control on the generation of the rules. However, there is a challenge that lies within the difficult exact pricing subproblem. As a remedy, we have proposed to solve a proxy subproblem by training decision trees with sample weights. These sample weights are obtained from the dual of the LPs at that iteration. Decision trees can be easily trained with the existing software packages, and hence, this proxy pricing problem can be implemented quite efficiently. We have compared the performance of our rule generation algorithm with the highly interpretable Decision Tree algorithm, and with the highly accurate tree ensemble algorithms, Random Forest and AdaBoost. At the expense of more but shorter rules than Decision Trees, our rule generation algorithm performs on par with tree ensemble algorithms in terms of accuracy.

This study is conducted for multi-class classification. With a piecewise-linear loss function (e.g. mean absolute deviation), our LP model can also be reformulated for regression problems. An alternative approach can be to incorporate the dual values corresponding to coverage constraints within the solution of the proxy pricing subproblem. This might require developing a specially tailored DT that depends on the usage of $\gamma$ values. These are other lines of research that we plan to pursue in the future.
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Appendix A. Properties of the Test Instances. We have used 22 datasets in our experiments. The properties of these datasets are summarized in Table 3. The first 17 datasets are for binary classification, whereas the remaining datasets are for multi-class classification.

Table 3: The properties of the datasets. Here, SAMPLE SIZE, CLASSES and FEATURES stand for the number of observations, the number of classes and the number of features, respectively. The datasets are from the following sources: OILSPILL from Kubat et al. (1998), PHONEME from an online repository, remaining from (Dua and Graff, 2017)

| INSTANCE         | SAMPLE SIZE | CLASSES | FEATURES |
|------------------|-------------|---------|----------|
| BANKNOTE         | 1372        | 2       | 4        |
| HEARTS           | 363         | 2       | 13       |
| ILPD             | 583         | 2       | 10       |
| IONOSPHERE       | 351         | 2       | 34       |
| LIVER            | 345         | 2       | 6        |
| PIMA             | 768         | 2       | 8        |
| TIC-TAC-TOE      | 958         | 2       | 9        |
| TRANSFUSION      | 748         | 2       | 4        |
| WDBC             | 569         | 2       | 31       |
| ADULT            | 32561       | 2       | 14       |
| BANK-MKT         | 11162       | 2       | 16       |
| MAGIC            | 19020       | 2       | 10       |
| MUSHROOM         | 8124        | 2       | 22       |
| MUSK             | 6598        | 2       | 166      |
| OILSPILL         | 937         | 2       | 49       |
| PHONEME          | 5404        | 2       | 5        |
| MAMMOGRAPHY      | 11183       | 2       | 6        |
| SEEDS            | 210         | 3       | 7        |
| WINE             | 178         | 3       | 13       |
| GLASS            | 214         | 6       | 9        |
| ECOLE            | 336         | 8       | 7        |
| SENSORLESS       | 58509       | 11      | 48       |

Appendix B. Parametric Model. Before we analyze the parametric model, let us first rewrite problem (3) in canonical form using matrix notation. To simplify our exposition, we define

\[
\mathbf{z} = \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{e} \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{e} \\ \mathbf{e} \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \hat{\mathbf{A}} & \mathbf{I} & 0 \\ \mathbf{A} & 0 & \mathbf{I} \end{bmatrix},
\]

where \( \mathbf{w} = [w_j]_{j \in \mathcal{J}}, \mathbf{v} = [v_i]_{i \in \mathcal{I}}, \mathbf{c} = [c_j]_{j \in \mathcal{J}}, \hat{\mathbf{A}} = [\hat{a}_{ij}]_{i \in \mathcal{I}, j \in \mathcal{J}}, \mathbf{A} = [a_{ij}]_{i \in \mathcal{I}, j \in \mathcal{J}} \) along with \( \mathbf{e} \) and \( \mathbf{0} \) denoting vector of ones and zeros, respectively. The auxiliary vector \( \mathbf{u} \) is used as a slack variable to obtain a model in canonical form. With this notation, the master problem and its dual become

\[
\min \{ \mathbf{d}^\top \mathbf{z} : \mathbf{Mz} = \mathbf{b}, \mathbf{z} \geq 0 \} \tag{7}
\]

and

\[
\max \{ \mathbf{b}^\top \mathbf{\alpha} : \mathbf{M}^\top \mathbf{\alpha} \leq \mathbf{d} \}, \tag{8}
\]

respectively. We note that the first set of constraints in (3) is also written as an equality. This is still a valid formulation, since the vector \( \mathbf{v} \) plays the role of slack variables. Recall that the second set of constraints in the master problem guarantees coverage of the samples. Without loss of generality, we have set the right-hand-side of these constraints to one, i.e. \( \varepsilon = 1 \). Therefore, one may define a hyperparameter \( \delta > -1 \) and consider a set of alternative right-hand-side values \( \mathbf{b}(\delta) = \mathbf{b} + \delta \bar{\mathbf{b}} \) with \( \bar{\mathbf{b}}^\top = [\mathbf{0}^\top, \mathbf{e}^\top] \). Such a change in the right-hand-side value leads to a parametric linear program. Next, we give an interval of \( \delta \) values so that as long as \( \delta \) remains in this interval, the optimal basis does not change and there is no need to solve the master problem again.

Suppose that we have solved problem (7) and obtained the optimal solutions \( \mathbf{z}^* \) and \( \mathbf{\alpha}^* \). Let us denote the indices of the basic and nonbasic variables with \( \mathcal{B} \) and \( \mathcal{N} \), respectively. By rearranging the corresponding columns, we can write

\[
\mathbf{Mz}^* = \mathbf{M}_B \mathbf{z}_B^* + \mathbf{M}_N \mathbf{z}_N^* = \mathbf{b}
\]
and
\[ d^T z^* = d^T_B z^*_B + d^T_N z^*_N. \]
Hence, we obtain \( z^*_B = M^{-1}_B b, z^*_N = 0 \) and \( \alpha^* = d_B M^{-1}_B \). Clearly, replacing \( b \) with \( b(\delta) \) does not affect the feasibility of the dual solution. The primal solution, however, becomes
\[
z^*_B(\delta) = M^{-1}_B b(\delta) = M^{-1}_B b + \delta M^{-1}_B b = z^*_B + \delta z^*_B.
\]
We observe for \( z^*_N(\delta) = 0 \) that
\[
M_B z^*_B(\delta) = b(\delta), \\
d^T z^*_B(\delta) = b(\delta)^T \alpha^*.
\]
This shows that current basis remains optimal as long as \( z^*_B(\delta) \geq 0 \). This leads to the following simple interval for \( \delta \) values:
\[
\max_{\delta \in B, \delta > 0} \{-1, -\frac{z^*_i}{\beta_i} (\text{at sample } i) \} \leq \delta \leq \min_{\delta \in B, \delta < 0} \{-\frac{z^*_i}{\beta_i} \},
\]
where \([z^*_i]_{i \in B} = z^*_B \) and \([\bar{z}^*_i]_{i \in B} = \bar{z}^*_B \). During hyperparameter tuning, if \( \delta \) goes out of this interval, then problem (8) can be solved after replacing \( b \) with \( b(\delta) \). Consequently, the same line of parametric analysis can be applied to find the next interval of values.

**Appendix C. Relation to Boosting Methods.** The relation between the proposed rule generation algorithm (RUG) and the boosting methods can be summarized in three parts:

(i) In boosting methods, a sequence of weak classifiers are trained in tandem, and each classifier depends on the classifiers that come before it. To improve accuracy with every new classifier, the weights of the misclassified points are increased. Moreover, each classifier in the sequence receives a weight according to its classification performance, while the weights of the previous classifiers are fixed. In our master problem (3), we also assign weights to the rules and these rules receive a weight according to their classification performance, while the weights of the previous rules are fixed. In our master problem (3), we also assign weights to the rules and these rules receive a weight according to their classification performance, while the weights of the previous rules are fixed.

(ii) The duality discussion with our linear programs also lends itself to an interesting discussion about our approach and the margin maximization idea in boosting methods as presented by Schapire et al. (1998), Grove and Schuurmans (1998) and Demiriz et al. (2002). These authors establish that a sample with a large margin is likely to be classified correctly. Thus, margin maximization is about assigning larger weights to those samples with small margins. Clearly, the first set of constraints in (3) always holds as equalities due to the nonnegative costs of the variables \( v_i, i \in I \). Using complementary slackness conditions in linear programming, we have the following facts:

- \( \sum_{j \in J_i} \hat{a}_{ij} w_j > 1 \) implies \( v_i = 0 \) and \( \beta_i = 0 \).
- \( \beta_i > 0 \) only if \( \sum_{j \in J_i} \hat{a}_{ij} w_j \leq 1 \).
- \( 0 < \beta_i < 1 \) only if \( \sum_{j \in J_i} \hat{a}_{ij} w_j = 1 \) (marginal accuracy for sample \( i \)).
- \( \sum_{j \in J_i} \hat{a}_{ij} w_j < 1 \) implies \( \beta_i = 1 \) (misclassification of sample \( i \)).

This shows that the optimal dual variable \( \beta_i^{(t)} \) becomes positive only if sample \( i \) is misclassified, or it is correctly classified but remains on the boundary. So at the next iteration of rule generation, only those samples that have small margins are considered.

(iii) Mason et al. (2000) relate boosting algorithms to gradient descent. It is well-known that the dual optimal solution plays the role of gradient vector for the primal problem. This points out yet another connection between our approach and the boosting methods.
Table 4: The performances of CG Dash et al. (2020), BRS Wang et al. (2017), AM and BCD Su et al. (2016), RIPPER Cohen (1995), and RUG-T. Here, ACC., and COMP. stand for accuracy and complexity, respectively. Complexity is defined as the summation of the number of generated rules and the number of conditions in the rules.

| Dataset       | CG Acc. | CG Comp. | BRS Acc. | BRS Comp. | AM Acc. | AM Comp. | BCD Acc. | BCD Comp. | RIPPER Acc. | RIPPER Comp. | RUG-T Acc. | RUG-T Comp. |
|---------------|---------|----------|----------|-----------|---------|----------|----------|-----------|-------------|--------------|------------|------------|
| BANNOTE       | 0.99    | 25       | 0.99     | 30.4      | 0.99    | 24.2     | 0.99     | 21.3      | 0.99        | 28.6         | 0.99       | 48.7       |
| HEARTS        | 0.79    | 11.3     | 0.79     | 24        | 0.73    | 11.5     | 0.74     | 15.4      | 0.79        | 16           | 0.82       | 19.7       |
| ILPD          | 0.70    | 10.9     | 0.70     | 4.4       | 0.72    | 0        | 0.72     | 0         | 0.7         | 9.5          | 0.70       | 10.6       |
| IONOSPHERE    | 0.90    | 12.3     | 0.87     | 12        | 0.91    | 16       | 0.92     | 14.6      | 0.88        | 14.6         | 0.91       | 24.9       |
| LIVER         | 0.60    | 5.2      | 0.54     | 15.1      | 0.56    | 8.7      | 0.52     | 4         | 0.57        | 5.4          | 0.69       | 58.6       |
| PIMA          | 0.74    | 4.5      | 0.74     | 17.4      | 0.73    | 2.7      | 0.73     | 2.1       | 0.73        | 17           | 0.74       | 30.9       |
| TIC-TAC-TOE   | 1.00    | 32       | 1.00     | 32        | 0.84    | 24.9     | 0.82     | 12.6      | 0.98        | 32.9         | 0.91       | 70         |
| TRANSFUSION   | 0.78    | 5.6      | 0.77     | 6         | 0.76    | 0        | 0.76     | 0         | 0.79        | 6.8          | 0.76       | 26.9       |
| WDBC          | 0.94    | 13.9     | 0.95     | 16        | 0.96    | 11.6     | 0.96     | 17.3      | 0.93        | 16.8         | 0.94       | 28.8       |
| ADULT         | 0.84    | 88       | 0.82     | 39.1      | 0.83    | 15       | 0.82     | 13.2      | 0.84        | 133.3        | 0.85       | 43.8       |
| BANK_MKT      | 0.90    | 9.9      | 0.87     | 13.2      | 0.90    | 6.8      | 0.90     | 2.1       | 0.90        | 56.4         | 0.82       | 85.6       |
| MAGIC         | 0.85    | 93       | 0.83     | 97.2      | 0.81    | 11.5     | 0.80     | 9         | 0.85        | 177.3        | 0.83       | 72.8       |
| MUSHROOM      | 1.00    | 17.8     | 1.00     | 17.5      | 1.00    | 15.4     | 1.00     | 14.6      | 1.00        | 17           | 1.00       | 31         |
| MUSK          | 0.96    | 123.9    | 0.93     | 33.9      | 0.97    | 101.3    | 0.92     | 24.4      | 0.96        | 143.4        | 0.93       | 87.2       |

The figures in columns 2-11 are taken from Dash et al. (2020).

Appendix D. Comparison with Recent Binary Classification Methods for Interpretation.

Table 4 presents a comparison of RUG against existing studies from the literature. In the table, “CG” stands for the column generation framework by Dash et al. (2020), “BRS” is the Bayesian Rule Set approach by Wang et al. (2017), “AM” and “BCD” are alternating minimization and block coordinate descent algorithms by Su et al. (2016), respectively. The last one “RIPPER” is the well-known rule generation heuristic of Cohen (1995). The acronym “ACC.” stands for the mean accuracy and “COMP.” denotes the complexity. Dash et al. (2020) define complexity as the sum of the number of rules and the number of conditions for each rule. Although complexity is not directly addressed in our study, it is calculated for RUG for the sake of completeness. To make a fair comparison, we have additionally considered RUG with threshold values (RUG-T), for which the number of subproblem calls is restricted to five, and the rules satisfying the weight threshold of $w_j \geq 0.05$, $j \in J$ are selected for testing. The last two columns Table 4 give the mean accuracies and the complexity values for RUG-T, respectively. Dash et al. (2020) consider only binary classification. Thus, the multi-class classification datasets in the previous tables are excluded. Moreover, two datasets from Dash et al. (2020) are excluded since we have failed to access those instances. The average accuracy of RUG-T is on par with the CG which has superior accuracy than BRS, AM, BCG and RIPPER. For HEARTS, ILPD, IONOSPHERE, LIVER and ADULTS datasets, RUG-T outperforms CG in accuracy. However, the accuracy is in favor of CG when compared to RUG-T on datasets TIC-TAC-TOE, TRANSFUSION, BANK-MKT, MAGIC and MUSK. The average complexity of RUG-T is slightly worse than that of CG except MUSK dataset. RUG-T exposes a worthwhile example on how to fine-tune parameters of RUG so that a good balance can be achieved for interpretability. We also observe that RUG is robust in the sense that reducing the number of RMP calls does not reduce the accuracy drastically. Besides, applying a threshold to select rules contributes to interpretability without a severe damage on accuracy. Fortunately, RUG-T is significantly faster than the CG. Indeed, all our algorithms run in less than a minute, while CG is reported to take around 20 minutes for all datasets except MUSK and TIC-TAC-TOE.

‡This is the JRip implementation in Weka.