A well-conditioned integral equation for electromagnetic scattering from composite inhomogeneous bi-anisotropic material and closed perfect electric conductor objects

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Abstract
A well-conditioned volume-surface integral equation, called the volume integral equation-combined field integral equation, is applied to analyse electromagnetic (EM) scattering from arbitrarily shaped three-dimensional composite objects comprising both inhomogeneous bi-anisotropic material and closed perfect electric conductors (PECs). The equivalent surface and volume currents are respectively expanded using the commonly used RWG and SWG basis functions, while a matrix equation is derived by the method of moments. Because the magnetic field integral equation is involved in modelling the surface electric current, and the constitutive parameters are all tensors, some new kinds of singularities are encountered and properly handled in the filling process of the impedance matrix. Several numerical results of EM scattering from composite bi-anisotropy and closed PEC objects are shown to illustrate the accuracy and efficiency of the proposed scheme. The validity of the continuity condition of electric flux enforced on the bi-anisotropy-PEC interfaces, which can be used to eliminate the volumetric electric unknowns, is also verified.

1 | INTRODUCTION

With the rapid development of materials science, various applications of bi-anisotropic materials have been developed for the design of antennas [1], waveguide mode converters [2], radar absorbing materials [3], antireflection coatings [4], microwave devices [5,6], and so on. Consequently, in the field of computational electromagnetics, EM scattering from composite objects containing anisotropic or bi-anisotropic materials and perfect electric conductors (PECs) has inspired substantial research. However, because the constitutive relations of bi-anisotropy are enforced as an additional coupling between electric and magnetic fields, and because all their constitutive parameters are tensors, it is quite a challenge to accurately analyse such composite objects. Among the many numerical methods, the integral equation method, in conjunction with method of moments (MoM) [7,8], is one of the most competitive choices for analyse the composite complex material-PEC objects and discussed in several articles [9–17]. In [9], the volume integral equation (VIE) was presented to analysing EM scattering from inhomogeneous anisotropic objects and has been further extended to analyse electric anisotropy-PEC objects [10]. In [11], scattering from bi-anisotropic objects were analysed by the hybrid finite element-boundary integral (FEBI) method. A PEC object coated with homogenous bi-isotropic materials was modelled using the surface integral equation (SIE) method in [12], from which its authors later proposed a VIE method with solenoidal basis functions for objects with inhomogeneous bi-isotropy [13]. Article [14] used the adaptive integral method to accelerate the MoM solution of the VIE for inhomogeneous bi-anisotropic objects. Using the volume-surface integral equation (VSIE), the scattering problem from composite electric anisotropy-PEC objects was solved in [15]. With the equivalent currents
expanded by piecewise constant basis functions, article [16] used the VIE to calculate extremely anisotropic objects. In [17], the VIE was applied to analyse bi-anisotropic objects, while the eigenvalue spectrum was derived and analysed.

On the choice of integral equations, compared with pure SIE-based methods, the VSIE is more robust and generalized in modelling composite objects containing thin inhomogeneous materials because only the Green’s function of the background medium is needed [8]. This generality owes to the fact that according to the equivalence principle, the VSIE implementation simultaneously retains two kinds of generally applicable integral equations, that is, the VIE to model the field superposition in the material regions and the SIE to impose the boundary condition on the PECs. For the SIE part, the electric field integral equation (EFIE) is widely adopted because it can be used to model both open and closed PECs. However, the EFIE is a first-kind Fredholm integral equation, usually resulting in an ill-conditioned impedance matrix. For the closed PEC part, if the EFIE is used alone, it will also encounter the interior resonance problem [8]. On the other hand, in practical applications, metal objects having finite thickness can be modelled as closed PECs. To avoid the interior resonance problem as well as improve the matrix condition, one can linearly combine the magnetic field integral equation (MFIE) with the EFIE to form a combined field integral equation (CFIE) that is second-kind [18]. For a closed PEC object with a finite thickness, the CFIE can provide acceptable accuracy at a not very low frequency with faster convergence. Furthermore, the CFIE can be combined with the VIE to yield a new generalized VSIE, called the volume integral equation-combined field integral equation (VIE-CFIE), that is expected to make the matrix equation easier to iteratively solve than the conventional VIE-EFIE. In the authors’ previous work, we have successfully adopted the VIE-CFIE to analyse EM scattering from composite inhomogeneous bi-isotropy and closed PEC objects [19] and accelerated the MoM solution of the VIE from inhomogeneous bi-anisotropic objects via the spherical harmonics expansion-based multilevel fast multipole algorithm (SE-MLFMA) [20].

To the authors’ best knowledge, no article has thus far shown in detail how to use MoM to solve the VSIE with composite bi-anisotropy-PEC objects, especially when the VIE-CFIE is conditionally applied. In this paper, the VIE-CFIE is presented to analyse EM scattering from arbitrarily shaped composite objects comprising both inhomogeneous bi-anisotropic materials and closed PECs. By discretising the equivalent surface and volume currents using the commonly used RWG and SWG basis functions that are defined on the triangular and tetrahedral cells [21,22], the VIE-CFIE yields a well-conditioned matrix equation. Introducing the MFIE during modelling of the closed PEC surface, compared with the condition under existing articles such as [10,15], resulted in the appearance of new singularities in the process of matrix filling that will be properly handled and elaborated.

2 | DERIVATION OF THE VOLUME-SURFACE INTEGRAL EQUATION FOR BI-ANISOTROPY-PERFECT ELECTRIC CONDUCTOR OBJECTS

Consider a composite object that contains both inhomogeneous bi-anisotropic material occupying a region \( V \) and PEC surface \( S \) in the free space. Assume this object is illuminated by an incident EM plane wave \( \vec{E}_0, \vec{H}_0 \) from an arbitrary direction, radiating the scattered fields \( \vec{E}, \vec{H} \) into space. In the bi-anisotropic region \( V \), the coupled constitutive relations between the electric flux density \( \vec{D} \), magnetic flux density \( \vec{B} \), and electric field \( \vec{E} \), magnetic field \( \vec{H} \) are written as

\[
\begin{bmatrix}
\vec{D}(\vec{r}) \\
\vec{B}(\vec{r})
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon(\vec{r}) & \mu(\vec{r}) \\
\mu(\vec{r}) & \varepsilon(\vec{r})
\end{bmatrix}
\begin{bmatrix}
\vec{E}(\vec{r}) \\
\vec{H}(\vec{r})
\end{bmatrix}
\forall \vec{r} \in V
\]  

(1)

where all the constitutive parameters (permittivity \( \varepsilon \), permeability \( \mu \), and coupling parameters \( \varepsilon, \mu \) are \( \vec{r} \)-dependent tensors. Equation (1) can also be rewritten as

\[
\begin{bmatrix}
\vec{E} \\
\vec{H}
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon & -\mu \\
-\mu & \varepsilon
\end{bmatrix}^{-1}
\begin{bmatrix}
\vec{D} \\
\vec{B}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} \\
\tilde{a}_{21} & \tilde{a}_{22}
\end{bmatrix}
\begin{bmatrix}
\vec{D} \\
\vec{B}
\end{bmatrix}
\]  

(2)

while the variable \( \vec{r} \) is omitted for the purpose of easy reading. According to the volume equivalence principle, the scattered fields from the bi-anisotropic material can be seen as being produced by both the equivalent volume electric currents \( \vec{f}_V \) and the magnetic currents \( \vec{M}_V \). From the two curl equations of Maxwell’s equations, \( \vec{f}_V \) and \( \vec{M}_V \) for bi-anisotropy are further derived as

\[
\begin{bmatrix}
\vec{f}_V \\
\vec{M}_V
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_0 & 0 \\
0 & \mu_0
\end{bmatrix}
\begin{bmatrix}
\vec{0} & \frac{1}{\varepsilon_0} \tilde{a}_{11} \\
\frac{1}{\mu_0} \tilde{a}_{22} & 0
\end{bmatrix}
\begin{bmatrix}
\vec{D} \\
\vec{B}
\end{bmatrix}
\]  

(3)

where \( \vec{I} \) denotes the identity tensor, and the time-harmonic factor is \( e^{j\omega t} \) with the imaginary unit \( j = \sqrt{-1} \).

Due to \( \vec{f}_V \) and \( \vec{M}_V \) in \( V \) as well as the equivalent surface electric current \( \vec{j}_S \) on \( S \), the scattered fields are cast in terms of auxiliary potentials as

\[
\begin{align*}
\vec{E} &= -j\omega \left( \vec{A}_S + \vec{A}_V \right) - \nabla (\varphi^S + \varphi^V) - \frac{1}{\varepsilon_0} \nabla \times \vec{A}_V \\
\vec{H} &= \frac{1}{\mu_0} \nabla \times \left( \vec{A}_S + \vec{A}_V \right) - j\omega \vec{A}_V - \nabla \varphi^M
\end{align*}
\]  

(4)
The vector and scalar potentials are expressed as the convolutions of currents or their divergences and the Green's function as

\[
\begin{align*}
\vec{A}_f(\vec{r}) &= \mu_0 \int_T \nabla \cdot \vec{J}(\vec{r}') G(\vec{r}, \vec{r}') dT' \\
\phi_f(\vec{r}) &= \frac{j}{\omega \varepsilon_0} \int_T \nabla \times \vec{E}(\vec{r}') G(\vec{r}, \vec{r}') dT'
\end{align*}
\]

\(T = S, V\)

\[
\begin{align*}
\mathbf{A}^M(\vec{r}) &= \varepsilon_0 \int_V \mathbf{M}_V(\vec{r}') G(\vec{r}, \vec{r}') dV' \\
\mathbf{E}^M(\vec{r}) &= j \omega \mu_0 \int_V \mathbf{J}_V(\vec{r}') G(\vec{r}, \vec{r}') dV'
\end{align*}
\]

\(5\)

The Green's function of free space is expressed as

\[
G(\vec{r}, \vec{r}') = G = \frac{e^{-j k_0 |\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|}
\]

(6)

with free space wavenumber \(k_0\).

In the region \(V\), the VIE is formed by making the incident fields equal to the total fields \((E, H)\) minus the scattered fields as

\[
\begin{align*}
\vec{E}(\vec{r}), \vec{H}(\vec{r}) - \begin{bmatrix} \vec{E}^i(\vec{r}), \vec{H}^i(\vec{r}) \\ \vec{E}^s(\vec{r}), \vec{H}^s(\vec{r}) \end{bmatrix} = \begin{bmatrix} \vec{E}(\vec{r}), \vec{H}(\vec{r}) \end{bmatrix} \quad \forall \vec{r} \in V
\end{align*}
\]

(7)

On PEC surface \(S\), by vanishing the tangential \((\text{tan})\) component of the total electric field, the EFIE is formed as

\[
\left. \vec{E}(\vec{r}) \right|_{\text{tan}} = \left. \begin{bmatrix} \vec{E}^i(\vec{r}), \vec{H}^i(\vec{r}) \end{bmatrix} + \left. \begin{bmatrix} \vec{E}^s(\vec{r}), \vec{H}^s(\vec{r}) \end{bmatrix} \right|_{\text{tan}} = 0 \quad \forall \vec{r} \in S
\]

(8)

The VIE can be combined with the EFIE to form the commonly used VIE-EFIE, a first-kind integral equation that is usually ill conditioned [8]. Furthermore, on the closed PECs, the MFIE

\[
\frac{1}{2} \nabla \times \left( \vec{n}(\vec{r}) \times \vec{H}(\vec{r}) \right) = \vec{n}(\vec{r}) \times \nabla \phi(\vec{r}) \quad \forall \vec{r} \in S
\]

(9)

can be modelled and linearly added to the EFIE to form the well-conditioned CFIE as [18]

\[
\text{CFIE} = a \text{EFIE} + (1-a)\eta_0 \text{MFIE}
\]

(10)

where \(\vec{n}\) is the outwardly directed normal, \(a (0 \leq a \leq 1)\) is a real constant, and \(\eta_0\) is the intrinsic impedance of free space. We can combine the VIE and CFIE together to build the VIE-CFIE to solve the EM scattering from composite objects comprising bi-anisotropic materials and closed PECs. Generally, when the constitutive parameter tensors have the same order of magnitude, the VIE-CFIE is well conditioned.

3 | METHOD OF MOMENTS

3.1 | Discretisation of the volume integral equation-combined field integral equation

By dispersing the equivalent currents or flux densities, the MoM discretises the VSIE into a matrix equation. In the implementation, \(\vec{J}_S\) on \(S\) and \(\vec{D}\) and \(\vec{B}\) in \(V\) are respectively expanded using the set of RWG basis functions \(f_i\) [21] defined in the domain \(S\) and SWG basis functions \(\tilde{f}_i^V\) [22] defined in the domain \(V_i\) as

\[
\begin{align*}
\vec{J}_S &= \sum_{i=1}^{N_S} I^S_i f_i \\
\vec{D} &= \sum_{i=1}^{N_V} I^D_i \tilde{f}_i^V \\
\vec{B} &= \eta_0 \sum_{i=1}^{N_V} I^B_i \tilde{f}_i^V
\end{align*}
\]

(11)

In Equation (11), \(N_S\) and \(N_V\) are numbers of RWG and SWG basis functions, while the total number of unknowns is \(N_S + 2N_V\). \(I^S_i, I^D_i,\) and \(I^B_i\) are the corresponding unknown expansion coefficients, respectively. To hold the continuity of the normal component consistent with the boundary condition of material interface, we disperse \(\vec{D}\) and \(\vec{B}\) instead of \(\vec{J}_V\) and \(\vec{M}_V\). It is further assumed that \(\bar{\varepsilon}, \bar{\mu}, \bar{\varepsilon}, \bar{\zeta}\) are approximately constant tensors inside each tetrahedron, which is a generalisation of that presented in [22]. As a consequence, the tensors \(\bar{a}_{pq}\) and \(\tilde{\rho}_{pq}\) with \(p/q = 1\) or 2 defined in Equations (2) and (3) over a single tetrahedron are also considered constant.

Substituting Equation (11) into Equations (3)–(10) and combining with the Galerkin's testing results in a generalized impedance matrix equation that can be represented succinctly as

\[
\begin{bmatrix}
Z_{SS} & Z_{SD} & Z_{SB} \\
Z_{DS} & Z_{DD} & Z_{DB} \\
Z_{BS} & Z_{BD} & Z_{BB}
\end{bmatrix}
\begin{bmatrix}
I_S \\
I_D \\
I_B
\end{bmatrix}
= \begin{bmatrix}
V_S \\
V_D \\
V_B
\end{bmatrix}
\]

(12)
\[ [Z_{SS}] = [\alpha [Z_{SS}^2] + (1 - \alpha) \eta_0 [Z_{SS}^M] \]
\[ [Z_{SD}] = [\alpha [Z_{SD}^2] + (1 - \alpha) \eta_0 [Z_{SD}^M] \]
\[ [Z_{SB}] = [\alpha [Z_{SB}^2] + (1 - \alpha) \eta_0 [Z_{SB}^M] \]
\[ \{V_S\} = [\lambda [V_S^2] + (1 - \lambda) \eta_0 [V_S^M] \]

(13)

where \( I_S, I_D \) and \( I_B \) are the vectors of unknown expansion coefficients, \( V_S, V_D \) and \( V_B \) are the excitation vectors, \( [Z_{PQ}] \) (\( P, Q = S, D, \) or \( B \)) denotes the impedance submatrix representing the interactions between various types of testing and basis functions, and the superscript \( E \) or \( M \) means the corresponding submatrix or subvector is from EFIE or MFIE, respectively. For convenience, three linear vector operators are defined beforehand as

\[
\begin{align*}
\hat{P}_T(X) &= \int f_T(X')GdT' \\
\hat{Q}_T(X) &= \int f_T \cdot \overline{X}(X')GdT' \\
\hat{K}_T(X) &= \int f_T \times \overline{X}(X')GdT'
\end{align*}
\]

(14)

Each submatrix entry denoting the interaction between the \( j \)th testing function and \( i \)th basis function in Equations (12) and (13) is then given by

\[
[Z_{SS}^E]_{ij} = j\mu_0 \int f_j^S \overline{P}_S [f_i^S] + j \int \frac{1}{\omega_0} \left< f_j^S, Q_S [f_i^S] \right>
\]

(15)

\[
[Z_{SS}^M]_{ij} = \frac{1}{2} \left< f_j^S, f_i^S \right> + \left< f_j^S, f_i^S \right> \]

(16)

\[
[Z_{DS}]_{ij} = j\mu_0 \int f_j^V \overline{P}_S [f_i^S] + j \int \frac{1}{\omega_0} \left< f_j^V, Q_S [f_i^S] \right>
\]

(17)

\[
[Z_{BS}]_{ij} = \left< f_j^V, K_S [f_i^S] \right>
\]

(18)

and

\[
\begin{align*}
[Z_{SD}^E]_{ij} &= [Z_{SD}^E]_{ij} \left< \overline{\Phi}_{11}, \overline{\Phi}_{21} \right> \\
[Z_{SD}^M]_{ij} &= [Z_{SD}^M]_{ij} \left< \overline{\Phi}_{12}, \overline{\Phi}_{22} \right> \\
[Z_{SB}^E]_{ij} &= [Z_{SB}^E]_{ij} \left< \overline{\Phi}_{11}, \overline{\Phi}_{21} \right> \\
[Z_{SB}^M]_{ij} &= [Z_{SB}^M]_{ij} \left< \overline{\Phi}_{12}, \overline{\Phi}_{22} \right> \\
[Z_{DD}^E]_{ij} &= [Z_{DD}^E]_{ij} \left< \overline{\Phi}_{11}, \overline{\Phi}_{11}, \overline{\Phi}_{21}, \varepsilon_0, \mu_0 \right> \\
[Z_{DD}^M]_{ij} &= [Z_{DD}^M]_{ij} \left< \overline{\Phi}_{12}, \overline{\Phi}_{12}, \overline{\Phi}_{22}, \varepsilon_0, \mu_0 \right> \\
[Z_{BD}^E]_{ij} &= [Z_{BD}^E]_{ij} \left< \overline{\Phi}_{11}, \overline{\Phi}_{21}, \overline{\Phi}_{11}, \varepsilon_0, \mu_0 \right> \\
[Z_{BD}^M]_{ij} &= [Z_{BD}^M]_{ij} \left< \overline{\Phi}_{12}, \overline{\Phi}_{12}, \overline{\Phi}_{22}, \varepsilon_0, \mu_0 \right> \\
[Z_{BB}^E]_{ij} &= [Z_{BB}^E]_{ij} \left< \overline{\Phi}_{12}, \overline{\Phi}_{12}, \overline{\Phi}_{22}, \varepsilon_0, \mu_0 \right>
\end{align*}
\]

(19)

with

\[
[Z_{ji}^E]_{ji} = j\omega \mu_0 \left< f_j^S, P_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right> + j \omega \mu_0 \left< f_j^S, Q_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right> - j \omega \mu_0 \left< f_j^S, K_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right>
\]

(20)

\[
[Z_{ji}^M]_{ji} = \left< f_j^S, P_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right> + j \omega \mu_0 \left< f_j^S, Q_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right> - j \omega \mu_0 \left< f_j^S, K_V \left( \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right) \right>
\]

(21)

\[
[Z_{ji}^V]_{ji} = \frac{1}{\omega} \left< f_j^V, \overline{\Phi}_{ip, q} \cdot \overline{f}_i^V \right>
\]

(22)

where \( (gg) \) denotes the \( L^2 \)-inner product, \( \hat{n}_i \) denotes the outer-normal direction of the triangle containing the \( j \)th RWG test function, \( \overline{\Phi}_{ip, q} \) and \( \overline{\Phi}_{pq} \) are constant tensors over the tetrahedron containing the \( i \)th SWG basis function, and \( p_1/p_2/q = 1 \) or 2 with \( p_1 + p_2 + q = 3 \).

Furthermore, the continuity condition (CC) of electric flux can be explicitly enforced on the material-PEC interfaces to reduce the number of volumetric electric unknowns [19]. The CC establishes the relation between \( \overline{D} \) and \( \overline{J}_S \), which can be written as [19]

\[
\hat{n}(\overline{r}) \cdot \overline{D}(\overline{r}) = - \frac{\nabla \cdot \overline{J}_S(\overline{r})}{\omega}
\]

(23)

Because the CC comes from the current continuity equation that is independent of the material type, it can be safely adopted to the bi-anisotropy-PEC interfaces theoretically.

### 3.2 Matrix filling

In the following, details of the matrix filling process, especially how to handle singularities, are described. The \( i \)th RWG basis function is defined over a common side of length \( l_i \) shared by two triangles \( S_i^+ \) of areas \( S_i^+ \) as [21]
\[\tilde{f}_i^V(\vec{r}^') = \pm \frac{a_i}{3\epsilon_i^V} (\vec{r}^' - \vec{r}_i) \quad \forall \vec{r}^' \in V_i^\pm \]  

(24)

Similarly, the ith SWG basis function is defined over a common face of area \( a_i \) shared by two tetrahedrons \( V_i^\pm \) of volumes \( \epsilon_i^\pm \) as \([22]\)

\[\tilde{f}_i^V(\vec{r}^') = \pm \frac{a_i}{3\epsilon_i^V} (\vec{r}^' - \vec{r}_i) \quad \forall \vec{r}^' \in V_i^\pm \]  

(25)

In Equations (24) and (25), the sign \( \pm \) means the current flowing direction of the \( i \)th basis function is outward or inward relative to \( T_i^\pm \) \((T = S \) for RWG or \( V \) for SWG\), and \( \vec{r}_i \) is the free vertex of the \( i \)th basis function in \( T_i^\pm \). If the field point \( \vec{r} \) is far from the source point \( \vec{r}^' \), all of the matrix entries in (12) can be easily evaluated using a universal quadrature rule. The Gaussian quadrature rule with four-fifths sampling points is recommended for integrations over the triangle/tetrahedron domain while calculating the interactions between the testing and basis functions, which ensures accurate integration of up to third-order polynomials \([23,24]\). Conversely, when \( \vec{r} \) approaches \( \vec{r}^' \), because of the Green's function (6) or its gradient, singularities or near singularities will appear and need special attention. How to evaluate the values of \([Z_{\text{SWG}}]^i_j\) and \([Z_{\text{VIE}}]^i_j\) generated by the SIE part can be employed as proposed in \([21]\) or \([18]\), while that of \([Z_{\text{VIE}}]^i_j\) \((P, Q = B \) or \( D)\) generated by the VIE part can be found in \([9]\). The evaluations of \([Z_{\text{DS}}]^i_j\) and \([Z_{\text{BS}}]^i_j\) are shown in \([19]\), both of which are independent of the material type contained by the calculated object.

For \([Z_{\text{SWG}}]^i_j\), three types of integrals involving the linear vector operators \( P_{V_i}, Q_{V_i}, \) and \( K_{V_i} \), must be handled. During \( \vec{r} \rightarrow \vec{r}^' \), the inner product calculation relating \( P_{V_i}, Q_{V_i}, \) or \( K_{V_i} \), can be easily transformed into a specific form with one order singularity that can be expediently handled using either the singularity extraction \([25]\) or Duffy transformation \([26]\) method. For the calculation relating \( K_{V_i} \), one can exchange the integral order as

\[
\left\langle f_j^S, K_{V_i} \left( \vec{b}_i \cdot \tilde{f}_i^V \right) \right\rangle = \int_{S_i^V} \int_{S_i^V} f_j^S(\vec{r}) \cdot f_i^V(\vec{r}) \times \nabla GdV' dS
\]

\[= -\frac{l_{d_i}}{6\xi_j^V \epsilon_i^V} f_{V_i} \tilde{b}_i G \left( \vec{r}^' - \vec{r}_j \right) \]

\[\cdot \int_{S_i^V} \left[ \left( \vec{r}^' - \vec{r}_j \right) \right] \times \nabla GdS' \]

\[= -\frac{l_{d_i}}{6\xi_j^V \epsilon_i^V} f_{V_i} \tilde{b}_i G \left( \vec{r}^' - \vec{r}_j \right) \]

\[\cdot \int_{S_i^V} \left[ \frac{\left( \vec{r}^' - \vec{r}_j \right)}{\epsilon_i^V} \int_{S_i^V} GdS \right] dV'
\]

(26)

where \( \tilde{b}_i \) denotes an arbitrary tensor that is constant in single tetrahedron. The above derivation takes the nature of \( \left( \vec{r} - \vec{r}^' \right) \times \nabla G \equiv 0 \). In this way, the singularity in the gradient of Green's function over a plane triangle can be handled in \([27]\).

For \([Z_{\text{SWG}}]^i_j\) introduced by the MFIE, there are also three types of integrals. The singularity involving \( P_{V_i} \) is easy to handle. For the inner product involving \( Q_{V_i} \), according to the two-dimensional Gauss theorem and the identity \( \nabla \cdot (\vec{a} \cdot \vec{b}) = [\nabla \cdot (\vec{a})] \cdot \vec{b} + \text{Tr}(\vec{a}^\top \cdot \nabla \vec{b}) \), we can translate it as

\[
\langle f_j^S, n_j^\perp Q_{V_i} \left( \tilde{b}_i \cdot f_i^V \right) \rangle
\]

\[= \int_{S_i^V} \nabla \cdot \left( f_j^S(\vec{r}) \times n_j^\perp \right) \cdot f_i^V(\vec{r}) \times \nabla GdV' dS
\]

\[= \int_{S_i^V} \nabla \cdot \left( \int_{S_i^V} f_j^S(\vec{r}) \times n_j^\perp f_i^V(\vec{r}) \times \nabla GdV' dS \right)
\]

\[= \int_{S_i^V} f_j^S(\vec{r}) \times n_j^\perp f_i^V(\vec{r}) \times \nabla GdV' dS
\]

\[= \int_{S_i^V} f_j^S(\vec{r}) \times n_j^\perp f_i^V(\vec{r}) \times \nabla GdV' dS
\]

(27)

where \( \text{Tr}(\tilde{b}_i) \) denotes the trace of \( \tilde{b}_i \), \( \tilde{n}_j^\perp \) and \( \tilde{n}_{V_i} \) denote the outer-normal direction of triangle \( S_i^V \) and that of the four triangular faces of tetrahedron \( V_i^\pm \), respectively. Then the order of singularities occurring in Equation (27) is reduced to one.

For the inner product involving \( K_{V_i} \), we transform it as
Therefore, the two-order singularity appeared in the first term of the last right-hand side (RHS) can be handled [18]. However, how to deal with the second one, which is also order two and first encountered, is not obvious. Here we deal with it using the singularity extraction method and simply summarise the key steps. We translate the inner integral of the second term and extract the singularity as

\[
f_{S_{\gamma}} \hat{n}_j^\pm \times \vec{r} \times \nabla' G dS = f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' G dS - \hat{n}_j^\pm f_{S_{\gamma}} \vec{r} \cdot \nabla' G dS \]

\[
= f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \left( G - \frac{1}{4\pi R} \right) dS - \frac{1}{4\pi} f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \frac{1}{R} dS
\]

\[
+ \frac{1}{4\pi} f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \frac{1}{R} dS
\]

\[
(29)
\]

where \( R = |\vec{r} - \vec{r}'| \). Thus, the first two RHS terms in the last equation can be evaluated numerically using a Gaussian quadrature rule, while the last two terms must be further analysed.

Assume that a coordinate transformation is applied so that the triangle locates on the \( u-v \) plane having a normal in the direction \( \hat{\omega} \), where \( \hat{\omega} \cdot (\vec{r} - \vec{r}_0) = u_0 \), \( \hat{\omega} \cdot (\vec{r} - \vec{r}_0) = v_0 \) and \( \hat{\omega} \cdot (\vec{r} - \vec{r}_0) = 0 \), as shown in Figure 1. Using matrix notation and operation, \( \vec{r} \) can be expressed as

\[
\vec{r} = \begin{bmatrix} \hat{u}^T \\ \hat{v}^T \end{bmatrix}^{-1} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}
\]

\[
(30)
\]

where \( \hat{u}^T \), \( \hat{v}^T \) and \( \hat{\omega}^T \) are the row vector expressions of \( \hat{u} \), \( \hat{v} \) and \( \hat{\omega} \). Further, according to [25], three different types of inner integrals can be analytically evaluated as

\[
f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \frac{1}{R} dS = \begin{bmatrix} \frac{1}{u_0} \\nabla' \frac{1}{R} \end{bmatrix} dS = \begin{bmatrix} \frac{\vec{T}_1}{\vec{T}_v} \\vec{T}_v \end{bmatrix}
\]

\[
(31)
\]

Then according to Equations (30) and (31), the third term of Equation (29) is handled as

\[
f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \frac{1}{R} dS = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cdot \nabla' \frac{1}{R} dS
\]

\[
+ \frac{\vec{T}_0 \vec{n}_j^\pm \cdot \nabla' \frac{1}{R} dS}{R} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cdot \nabla' \frac{1}{R} dS
\]

\[
+ \vec{T}_0 \vec{n}_j^\pm \cdot \vec{T}_1
\]

\[
(32)
\]

\[
f_{S_{\gamma}} \hat{n}_j^\pm \cdot \nabla' \frac{1}{R} dS = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cdot \nabla' \frac{1}{R} dS
\]

\[
+ \frac{\vec{T}_0 \vec{n}_j^\pm \cdot \nabla' \frac{1}{R} dS}{R} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \cdot \nabla' \frac{1}{R} dS
\]

\[
+ \vec{T}_0 \vec{n}_j^\pm \cdot \vec{T}_1
\]

\[
(33)
\]

Substituting Equation (30) into the fourth term of Equation (29), we have [25].

According to Equations (30)–(33), all the kinds of singularity in Equation (29) can be handled analytically.

### 4 NUMERICAL RESULTS

In this section, the bi-static or monostatic radar cross sections (RCSes) of several composite objects are calculated. When the VIE-CFIE is adopted to model the objects, \( a = 0.5 \). GMRES with a restart number of 100 is used as the iterative solver to reach convergence with a relative residual error of 0.001 [28]. A simple diagonal preconditioner is used to accelerate the
iterative solving process. A zero vector is taken as the initial guess for all calculations, which are serially carried out on a workstation with 2.4 GHz CPU and 384 GB RAM in single precision.

The first object is a bi-anisotropic cylinder whose radius and height are 0.5λ and 0.2λ, respectively. The constitutive parameters of the bi-anisotropy are

\[
\begin{align*}
\bar{\varepsilon}_r &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, & \bar{\mu}_r &= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\bar{\varepsilon}_r &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \bar{\mu}_r &= \begin{bmatrix} 0 & j\Omega & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

where Ω is variable. The cylinder is meshed into 7490 tetrahedrons with respect to 31,872 unknowns. Illuminated by an EM plane wave from +z-axis, the θθ- and φφ-polarized bi-

\begin{figure}[h]
\centering
(a) \quad \theta\theta\text{-polarization:} \\
(b) \quad \phi\phi\text{-polarization:}
\caption{Bi-static radar cross sections at \( \phi = 0^\circ \) plane for a bi-anisotropic cylinder of radius 0.5λ and height 0.2λ, illuminated by an electromagnetic plane wave from +z-axis. (a) \( \theta\theta \)-polarization, (b) \( \phi\phi \)-polarization}
\end{figure}

static RCSes at \( \phi = 0^\circ \) plane with different values of Ω are calculated and given in Figure 2 (denoted by cal.). For comparison, the results using FEBI from [11, Figure 5] are also shown (ref.). Good agreements are observed for each Ω.

The second object is a coated PEC sphere. The radius of the inner PEC sphere and the thickness of the coating electric anisotropy are 1 m and 0.42 m, respectively. The relative permittivity tensors of the coated electric anisotropy are defined under the spherical coordinate as \( \bar{\varepsilon}_r = \text{diag}(\varepsilon_r, \varepsilon_r, \varepsilon_t) \) and \( \bar{\mu}_r = \text{diag}(\mu_r, \mu_r, \mu_t) \). In this case, \( \varepsilon_r = 4, \varepsilon_t = 2 \) and \( \mu_r = I \). When the object is illuminated by a \( \theta \)-polarized plane wave from -z-axis at 22 MHz, the bi-static RCS at \( \phi = 0^\circ \) plane is calculated using the VIE-CFIE, while the numbers of triangles and tetrahedrons are 112 and 587 with respect to 1422 unknowns after discretising. During the calculation, the CC is alternatively enforced and combined with the VIE-CFIE (denoted by CC-VIE-CFIE) to reduce 112 volumetric unknowns. The numerical results are shown in Figure 3, while for comparison, the exact result from Mie series [10, Figure 1] is also given. It is seen that the numerical results with or without the CC agree well with the exact result everywhere, indicating that the VIE-CFIE has high calculation accuracy. Because this object is small, the computational details are not reported.

The third object is also a coated PEC sphere. The radius of the inner PEC sphere is 0.3λ, and the thickness of coating material is 0.05λ. The coating material is bi-anisotropic, whose relative permittivity and permeability tensors are defined under the spherical coordinate while \( \varepsilon_r = \mu_r = 4 \) and \( \varepsilon_t = \mu_t = 2 \). Another two coupling parameters are defined as \( \xi_r = \xi_t = (0.5 - j0.5)I \), while the asterisk denotes the complex conjugate. This object is illuminated by a plane wave from +z-axis. It is impossible to analyse this object using the SIE-based schemes but the VSIE can do. Both the VIE-CFIE and VIE-CFIE are used to calculate the co-polarization of bi-
While the observation range is $0^\circ \leq \theta \leq 180^\circ$ and $\phi = 0^\circ$. In this calculation, four types of mesh size are adopted to discretize the object as shown in Table 1, where the numbers of triangles, tetrahedrons and unknowns are listed, respectively.

To investigate the numerical accuracy, we set the VIE-EFIE result from the mesh size of $0.017\lambda$ as the benchmark, while the root-mean-square (RMS) error is applied to analyze the accuracy, defined as

$$RMS = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\sigma_i^{cal} - \sigma_i^{ben})^2}$$  \hspace{1cm} (35)$$

where $M$ is the number of observation angles, $\sigma_i^{cal}$ and $\sigma_i^{ben}$ denote the calculated and benchmark RCSes measured in dB in

| Mesh Size | 0.017\lambda | 0.03\lambda | 0.055\lambda | 0.1\lambda |
|-----------|--------------|-------------|--------------|------------|
| Number of unknowns | 159,896 | 53,961 | 20,410 | 4916 |
| Number of triangles | 3716 | 1702 | 896 | 272 |
| Number of tetrahedrons | 36,382 | 11,846 | 4236 | 996 |
| Equation type | VIE-EFIE | VIE-CFIE | VIE-EFIE | VIE-CFIE |
| RMS error (dB) | — | 0.018 | 0.023 | 0.037 |
| MAX error (dB) | — | 0.024 | 0.095 | 0.11 |
| Number of iterations | 103 | 33 | 105 | 33 |

Abbreviations: EFIE, electric field integral equation; MAX, maximal; RMS, root-mean-square; VIE-CFIE, volume integral equation-combined field integral equation.

**FIGURE 4** Bi-static radar cross sections for a perfect electric conductor sphere of radius $0.3\lambda$ coated with $0.05\lambda$ thick bi-anisotropic material with the constitutive parameters 

$$\varepsilon = 4\varepsilon_0 = \mu_0 = 2$$ (defined in spherical coordinate), and 

$$\varepsilon_r = \mu_r = (0.5-j0.5)$$, illuminated by a $\theta$-polarized electromagnetic plane wave from $+z$-axis. EFIE, electric field integral equation; VIE-CFIE, volume integral equation-combined field integral equation.

**FIGURE 5** Monostatic radar cross sections for a 252.4 mm length perfect electric conductor almond coated with 10 mm thick bi-anisotropic material, illuminated by a $\theta$-polarized electromagnetic plane wave at 1 GHz: (a) $H=0$ and $H=0.5$, (b) $H=0.25$. EFIE, electric field integral equation; VIE-CFIE, volume integral equation-combined field integral equation.
the \(i\)th observation angle, respectively. Table 1 also shows when the incident plane wave is \(\theta\)-polarized, the RMS and maximal (MAX) errors from different implementations.

as well as the number of iterations during the iterative solution. It is observed that when the mesh size is fixed, the RMS and MAX errors from the VIE-EFIE and VIE-CFIE are quite close, which means the new matrix entries as well as singularities introduced by the MIE have been handled properly, and the VIE-CFIE can also give reliable results. On the other hand, due to the improvement of matrix condition, the convergence speed of the VIE-CFIE is always several times faster than that of the VIE-EFIE. For different mesh sizes, the errors are tolerable if the mesh size is smaller than 0.055\(\lambda\). However, if the mesh size is set as 0.1\(\lambda\), the errors are unacceptable. The reason is that for this coating material, 0.1\(\lambda\) is roughly equivalent to 0.4 times of the wavelength in the coating material, which is too large. The numerical results from 0.017\(\lambda\) and 0.1\(\lambda\) are shown in Figure 4, while the maximal errors arise over the valley range (in this case, about 45° for the \(\theta\theta\)-polarization and 75° for the \(\varphi\varphi\)-polarization).

The fourth object is a coated PEC almond containing sharp tips [29], shown as an inset in Figure 6. The length of the PEC almond is 252.4 mm, and the coating thickness is 10 mm.

The frequency of the incident \(\theta\)-polarized EM wave is 1 GHz. A moderate mesh size is chosen to generate a total of 25,611 unknowns with respect to 1983 triangles and 5188 tetrahedrons. The constitutive parameters of the coating material are

\[
\begin{align*}
\epsilon_r & = \mu_r = \begin{bmatrix} 2 & j & 0 \\ -j & 2 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \\
\zeta_r & = \zeta_r^* = \nu(1-j)I 
\end{align*}
\]

(36)

where \(\nu\) is a variable. Figure 5 shows the monostatic RCS with \(\theta_0\), 0.25, or 0.5, and the observation range is \(0^\circ \leq \varphi \leq 360^\circ\) and \(\theta = 90^\circ\) with 181 observation angles, while co and cross denote co- and cross-polarization, respectively. Besides, the cross RCS of the coated almond with \(\nu = 0\) (in this case, the coating material degrades into anisotropy) is below −80 dBsm everywhere and not shown. Excellent agreements are observed between the results from the VIE-CFIE and those from the VIE-EFIE, indicating that despite the sharp structures are contained, the results from the VIE-CFIE are also dependable. Meanwhile, with different values of \(\nu\), the magnitude of the copolarization RCS is almost the same in most angles, while that of the cross-polarization one is largely varied. Besides, the larger the value of \(\nu\), the larger the cross-polarization RCS. This phenomenon demonstrates the effect of the coupled parameters on the electric and magnetic fields inside the biaxial isotropic coating.

In order to investigate how \(\nu\) influences the condition of the impedance matrix, Figure 6 shows the numbers of iterations for different values of \(\nu\), while the relative residual error is fixed to 0.001. In this process, the incident wave is from +\(x\)-axis, illuminating the tip of the coated almond. It is observed

![Figure 6](image_url)

**Figure 6** Numbers of iterations with respect to different values of \(\nu\), illuminated by a \(\theta\)-polarized electromagnetic plane wave from +\(x\)-axis. EFIE, electric field integral equation; VIE-CFIE, volume integral equation-combined field integral equation

![Figure 7](image_url)

**Figure 7** Bi-static radar cross sections of a multi-tablet containing five layers of different materials, illuminated by a \(\theta\)-polarized electromagnetic plane wave from +z-axis. (a) \(\varphi=0^\circ\) plane, (b) \(\varphi=90^\circ\) plane. EFIE, electric field integral equation; VIE-CFIE, volume integral equation-combined field integral equation.
that with different values of \( n \), the convergence of the VIE-CFIE is always several times faster than that of the VIE-EFIE, illustrating the robustness and efficiency of the proposed scheme. Another finding is that when \( n \) is small, both the VIE-EFIE and VIE-CFIE can reach the target convergence after dozens of iterations. Along with the increase of \( n \), the iterations will slightly increase, followed by a sharp increase. When \( n \) is larger than about 0.7, none can reach the target convergence after 2000 iterations. The reason is that under the fixed values of \( \tilde{\varepsilon}_r \) and \( \tilde{\mu}_r \), if \( \tilde{\varepsilon}_r > 0.7 \), the main diagonal elements of \( \tilde{a}_{11} \) and \( \tilde{a}_{22} \) in Equation (19) will be negative, which leads to a particularly ill-conditioned impedance matrix.

The fifth object is a multi-tablet containing five layers of different materials, as shown inside Figure 7a. The size of each tablet is \( 0.5 \lambda \times 0.5 \lambda \times 0.05 \lambda \). Besides, both the second and fourth layers are PEC, while the others are penetrable material. The first layer is bi-anisotropic, whose constitutive parameters are

\[
\tilde{\varepsilon}_{1r} = \tilde{\mu}_{1r} = \begin{bmatrix} 2 & j & 0 \\ -j & 2 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad \tilde{\varepsilon}_{1r}^* = \tilde{\mu}_{1r}^* = (0.5 + j0.5)I
\]

(37)

The third layer is isotropic with scalar relative permittivity and permeability as \( \varepsilon_{3r} = \mu_{3r} = 2 - j \). The fifth is uniaxial anisotropic, the parameters of which are

\[
\tilde{\varepsilon}_{5r} = \tilde{\mu}_{5r} = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\]

(38)

This object is illuminated by an EM plane wave from +z-axis. After discretisation, the numbers of triangles and tetrahedrons are 1024 and 3626 with 17,576 total unknowns. In this example, the CC of electric flux is attempted to be explicitly enforced on the material-PEC interfaces to reduce the number of volumetric electric unknowns, while 872 volumetric electric unknowns are eliminated due to the use of CC. The numerical results of the bi-static RCSes calculated using the VIE-EFIE, VIE-CFIE, and those enforced by the CC (denoted by CC-EFIE-VIE and CC-VIE-CFIE) are shown in Figure 7 for the normal incidence. It is observed that the numerical results from the VIE-EFIE or VIE-CFIE with and without CC are almost in excellent agreement everywhere, indicating that the CC is valid when the bi-anisotropic materials are involved.

Table 2 shows the computational details, containing the number of iterations, the peak memory usage and the total CPU time. It is observed that when the CC is used in either the VIE-EFIE or VIE-CFIE, all of the peak memory usages, the number of iterations and total CPU time are reduced. However, the reduction is very limited, because the eliminated number of unknowns (872) occupies quite a small proportion of the total number of unknowns (17,576). This phenomenon illuminates that the CC is more suitable for the calculation of the thin-coated objects [19]. On the other hand, the CC does not deteriorate the matrix condition for the objects containing complex material blocks. In other words, the CC can always be adopted safely and reliably.

## 5 | CONCLUSIONS

In this paper, a second-kind integral equation, called VIE-CFIE, has been proposed and applied in modelling EM scattering from composite objects involving inhomogeneous bi-anisotropic materials and closed PECs. In the process of an MoM solution, some new kinds of singularities occur and have been properly handled using the singularity extraction method. The accuracy and efficiency of the proposed VIE-CFIE are demonstrated by the calculation of various objects with different constitutive parameters and geometry structures. Numerical experiment shows that even if the calculated object contains fine structures, complex inhomogeneous materials and multilayers, the VIE-CFIE can always give reliable results and be several times faster than the VIE-EFIE during the iterative solution. The validity of the continuity condition of electric flux explicitly enforced on the bi-anisotropy-PEC interfaces has also been investigated and can be reliably used to reduce the number of unknowns for the volumetric electric current.

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## CONFLICT OF INTEREST

All authors declare no conflict of interest.

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