Two-Winner Election Using Favorite-Candidate Voting Rule

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Abstract. We investigate two-winner election problem seeking to minimize the social cost. We are interested in strategy-proof mechanisms where each voter only reports a single candidate. In our model, candidates and voters are located in Euclidean space and candidates’ locations are known to the mechanism. The quality of a mechanism is measured by its distortion, defined as the worst-case ratio between the social cost achieved by the mechanism and the optimal one. We find that the ratio between the maximum and minimum distances among every two candidates plays a vital role in the distortion of mechanisms. When there are three candidates, the problem is solved mainly by previous work. We mainly focus on the problem with at least four candidates. When voters and candidates are embedded in 1-dimensional space, we establish several lower bounds of the distortion. When voters and candidates are embedded in at least 3-dimensional space, we give a tight bound of the distortion.

Keywords: Voting · Social Choice · Distortion.

1 Introduction

In social choice theory, mechanisms are designed to aggregate voters’ preferences over a set of candidates and return an outcome. There is a wide range of applications in facility location issues and political issues [Procaccia and Rosenschein (2006), Anshelevich, Bhardwaj, and Postl (2013), Feldman, Fiat, and Golomb (2016)]. In this work, both voters and candidates are embedded in Euclidean space and represented by points. The cost of each voter is the distance between the voter and the nearest candidate elected. The social cost is the sum of each voter’s cost, and we design mechanisms to minimize the social cost. In our model, each candidate’s location is known to the mechanism while each voter’s location is private information. It captures practical situations such as the government plans to build several libraries in a city and solicits preference information over the alternatives from residents.

Feldman, Fiat, and Golomb (2016) initiate the comparison of three types of mechanisms which differ in the granularity of their input space (voting, ranking and location mechanisms). Gross, Anshelevich, and Xia (2017) propose a mechanism that randomly asks voters for their favorite candidates until two
voters agree. Anshelevich and Postl (2017) demonstrate the quality of randomized social choice algorithms in a setting where agents submit their metric preferences. Anshelevich and Zhu (2018) assume that candidates’ locations are known to the mechanisms, and they obtain a new upper bound for single-winner election with this extra information. Fain et al. (2019) study the voting rules that require limited ordinal information and present the Random Referee rule, which achieves a constant squared distortion. Chen, Li, and Wang (2020) consider single-candidate vote mechanisms for multi-winner election. We follow this line of work and focus on single-candidate vote (scv) mechanisms where a voter only submits her favorite candidate. Ordinal preferences and locations of voters are inaccessible. On the one hand, it is a more efficient method to collect the voters’ information. On the other hand, it can protect the privacy of voters as much as possible.

We assume that each voter is strategic and can misreport its favorite candidate against the mechanism to get a better outcome. We are interested in strategy-proof mechanisms where no voter can benefit from lying (Lu et al. 2010; Feldman, Fiat, and Golomb 2016). To evaluate the performance, we use the distortion introduced by Procaccia and Rosenschein (2006), which is the worst-case ratio of social cost between the outcome produced by the mechanism and the optimal solution. The distortion becomes a standard measurement subsequently (Anshelevich et al. 2011; Skowron and Elkind 2013; Goel, Krishnaswamy, and Munagala 2017). Please refer to Anshelevich et al. (2021) for more distortion results in the past 15 years.

In this paper, we mainly focus on two-winner election. The two-facility game is studied by Procaccia and Tennenholtz (2009); Lu et al. (2010); Fotakis and Tzamos (2014). They assume that voters report their locations to the mechanism, and all participants can be candidates. Generally, electing two winners is a hard problem. It is difficult to ensure the strategy-proof property and maintain the quality of the mechanism simultaneously. In particular, due to scv, the information available for designing mechanisms is limited in our model.

The ratio between the maximum and minimum distances among every two candidates, denoted by $\sigma$, turns out to play an essential role in the distortion.

In Section 3, we consider the mechanism where all candidates and voters are located on a line. Although the mechanism design problem in one-dimensional space seems manageable at first glance, it deserves serious study. There are plenty of works investigating the mechanism in one-dimensional space (Lu et al. 2010; Feldman, Fiat, and Golomb 2016; Filos-Ratsikas and Voudouris 2021). We divide the problem into two cases according to the number of candidates. When the number is beyond 3, we observe that the lower bound of distortion is related to the minimum of the root of $\sigma$ and the number of voters $n$. Besides, when the number of candidates is exact 3, this case can be considered as eliminating the least popular candidate. Inspired by Chen, Li, and Wang (2020), we can get the tight bound $7/3$. We also examine the classic mechanism which elects the leftmost and rightmost candidate who receive votes (Procaccia and Tennenholtz 2009; Fotakis and Tzamos 2014). The distortion of this mechanism is $2n - 3$ in
our setting. The lower bound results in one-dimensional space can be naturally extended to two-dimensional space.

In Section 4, we consider the situation where candidates and voters are located in multi-dimensional space. We find a necessary and sufficient condition for a mechanism to be strategy-proof. It says that the probability of every committee in any anonymous randomized strategy-proof $w$-winner election mechanisms can only be related to the number of votes of the corresponding $w$ candidates. The property enables us to get a lower bound of distortion $\Omega(\sigma)$. It is also a useful tool with which we can design strategy-proof mechanisms, not just for two-winner election. We propose a strategy-proof mechanism called “Pair-Independent” that achieves $O(\sigma)$ distortion, that is, the mechanism can achieve asymptotically tight bound of distortion. To our best knowledge, there is no performance-guaranteed mechanism using scv that is strategy-proof for two-winner election before. Besides, we consider the deterministic mechanism and show that there is no bounded anonymous strategy-proof mechanism. If we relax the anonymity restrictions, we propose “Sequential Dictator” mechanism with distortion $O(n\sigma)$. Finally, we have a characterization result for the single-winner election as a byproduct.

We summarize the main results in Table 1, where LB and UB are lower and upper bound abbreviations.

\textbf{Table 1.} A summary of our results

| Dimension | Deterministic | Randomized |
|-----------|---------------|------------|
| One       | UB: $2n - 3$  | UB: $2n - 3$ |
|           | LB: $\frac{1}{2} \min\{n, \sqrt{\sigma}\}$ | LB: $\frac{1}{6} \min\{n, \sqrt{\sigma}\}$ |
| Multi     | UB: $2(n - 2)\sigma + 1$ | UB: $1 + 6\sigma$ |
|           | LB*: $\infty$ | LB: $\sigma/6$ |

*This LB of deterministic mechanism for multi-dimensional space requires the anonymous property.

1.1 Related Work

For facility location problems, the foundation of locating a single facility on the real line is the work of Moulin [1980]. Dokow et al. [2012] consider the setting of discrete lines. For two-facility games, Procaccia and Tennenholtz [2009] give an upper bound of $n - 2$ and a lower bound of 1.5 for any strategy-proof mechanism. In addition, Lu et al. [2010] prove asymptotically tight bound of $\Omega(n)$ for deterministic strategy-proof mechanisms and provide a randomized strategy-proof mechanism with the approximation ratio of 4 for the line metric space. Fotakis and Tzamos [2010] propose the winner-imposing strategy-proof
mechanism. Sui, Boutilier, and Sandholm (2013) consider the percentile mechanisms. Fotakis and Tzamos (2014) show that for instance with \( n \geq 5 \) agents, any strategy-proof mechanism either admits a unique dictator, or places the facilities at the leftmost and rightmost. Xu et al. (2021) consider the two-facility games with minimum distance requirement.

The most relevant previous work to ours is Chen, Li, and Wang (2020). We both consider strategy-proof mechanisms using scv rules for multi-winner election. Their setting can be construed as single-loser election, whereas ours is two-winner election.

2 Model

Voters and candidates are located in Euclidean space \( \Omega \) with distance metric function \( d \). Let \( N = \{1, \ldots, n\} \) be the set of voters. The location of voter \( i \in N \) is denoted by \( x_i \) and the location profile of all voters is \( x = (x_1, \ldots, x_n) \). Let \( M = \{y_1, \ldots, y_m\} \) be the set of candidates. We refer to \( y_k \) as the \( k \)-th candidate and her location interchangeably. Each voter \( i \) is asked to vote for one candidate, called her action and denoted as \( a_i \in M \). The collection of action profile is \( a = (a_1, \ldots, a_n) \). An election in our setting can be referred to a tuple \( \Gamma = (\Omega, M, a) \).

We call a location profile \( x \) consistent with election \( \Gamma \), if each voter reveals the candidate closest to her. The set of location profiles consistent with \( \Gamma \) can be denoted by \( \chi(\Gamma) \).

Given an election, we are concerned with mechanisms that elect two candidates in \( M \). The set of all candidate pairs is denoted by \( M^2 = \{(y_1, y_2), (y_1, y_3), \ldots\} \). Given an action profile \( a \), a randomized mechanism \( f \) outputs a distribution \( f(a) \) on \( M^2 \). The probability of the elected candidate pair \( (y_k, y_l) \in M^2 \) is \( p_{k,l}(a) \). If \( f \) represents a deterministic mechanism, it outputs a specific candidate pair \( f(a) \in M^2 \) with probability 1.

The cost of a voter \( i \in N \) is the distance to the closer of two winners. That is \( \text{cost}((y_k, y_l), x_i) = \min\{d(x_i, y_k), d(x_i, y_l)\} \). The social cost denoted by \( SC((y_k, y_l), x) \) is the sum of every voter’s cost, that is \( SC((y_k, y_l), x) = \sum_{i \in N} \text{cost}((y_k, y_l), x_i) \). We use \( OPT(x) \) to denote the optimal social cost where the candidate pair is elected by an omniscient mechanism. That is \( OPT(x) = \min_{(y_k, y_l) \in M^2} SC((y_k, y_l), x) \). The expected social cost of a (randomized) mechanism \( f \) is \( \mathbb{E}[SC(f(a), x)] \). The performance of a mechanism is measured by distortion. The distortion on an election \( \Gamma \) is the worst-case ratio between the expected social cost achieved by the mechanism and the optimal social cost over all possible location profiles, i.e.,

\[
\text{dist}(f, \Gamma) = \sup_{x \in \chi(\Gamma)} \frac{\mathbb{E}[SC(f(a), x)]}{OPT(x)}.
\]

Then, the distortion of a (randomized) mechanism is the worst-case distortion over all possible elections, that is \( \text{Dist}(f) = \sup_{\Gamma} \text{dist}(f, \Gamma) \).
The distortion is closely related to the ratio between the maximum and minimum distances among every two candidates for two-winner election. We introduce a parameter \( \sigma \). Formally, \( \sigma = \frac{d_{\text{max}}}{d_{\text{min}}} \) where \( d_{\text{max}} = \max_{(y_k, y_l) \in M^2} d(y_k, y_l) \) and \( d_{\text{min}} = \min_{(y_k, y_l) \in M^2} d(y_k, y_l) \).

We are interested in the mechanisms that are "strategy-proof". A mechanism is strategy-proof if any voter votes for any one of the nearest candidates can always optimize her (expected) cost, regardless of the actions of others. Formally, \( \forall i \in N, \) and \( a_i^* \) represents the nearest candidate of voter \( i, \) \( \forall a_i' \neq a_i^* \), it holds that

\[
\text{cost}(f(a_i^*, a_{-i}), x_i) \leq \text{cost}(f(a_i', a_{-i}), x_i).
\]

### 3 One-Dimensional Metric Space

This section focuses on minimizing the social cost when voters and candidates are restricted on a line, known as 1-Euclidean space. We divide the issue into two cases depending on the number of candidates.

When there are more than 3 candidates, we need to assign probabilities to candidate pairs carefully in randomized strategy-proof mechanisms. Otherwise, voters may benefit from misreporting their favorite candidates. In particular, when a voter has two candidates with the same shortest distance, we must ensure that the voter has the same cost when choosing any of them. Based on this fact, we utilize a series of action profiles and derive the corresponding constraints on the probability distribution of candidate pairs. Further, we get a lower bound of distortion. When there are exactly 3 candidates, the strategy-proof property is relatively easy to be satisfied. The lower bound results can be naturally extended to two-dimensional space.

We first investigate the instance with \( m = 4 \) candidates. We prove the lower bound of distortion by constructing a well-defined instance for both deterministic and randomized mechanisms. Intuitively, three candidates are located at 0, 1 and 2 on the real line respectively. Also there is a candidate further away from them. The result for more candidates can be easily extended.

**Instance 1** Four candidates are located at \( y_1 = -\sigma + 2, y_2 = 0, y_3 = 1, y_4 = 2 \) where \( \sigma \geq 3 \).

**Theorem 1.** For two-winner election on the real line and \( m = 4 \), no strategy-proof scv mechanism can achieve a distortion smaller than \( \frac{1}{12} \min\{n, \sqrt{n}\} \).

Proof. We consider the Instance 1. We first define a series of action profiles. For \( 0 \leq t \leq k \), let \( \mathbf{a}^t \) denote \( (a_1 = \ldots = a_{k-t} = y_1, a_{k-t+1} = \ldots = a_k = y_2, a_{k+1} = \ldots = a_{(n+k)/2} = y_3, a_{(n+k)/2+1} = \ldots = a_n = y_4) \). In other words, \( \mathbf{a}^t \) describes the situation where \( y_1 \) gets \( k - t \) votes, \( y_2 \) gets \( t \) votes, \( y_3 \) and \( y_4 \) both get \( \frac{n-k}{2} \) votes. We assume that the parameter \( k \) is chosen from the interval \( \left[ \frac{n}{2\sqrt{n}}, \frac{n}{2} \right] \).

Second, we consider a location profile where voters are located exactly at \( y_1, y_3 \) and \( y_4 \). Precisely, we have \( \mathbf{x}^1 = (x_1 = \ldots = x_k = -\sigma + 2, x_{k+1} = \ldots = x_{(n+k)/2} = 1, x_{(n+k)/2+1} = \ldots = x_n = 2) \). According to the location profile,
the optimal candidate pair is among \((y_3, y_4), (y_1, y_3)\) and \((y_1, y_4)\) with the minimum social cost \(OPT(x^1) = \min\{ (n - k)/2, k(\sigma - 1) \}\). Since \(k \geq \frac{n}{2\sigma - 1}\), we have \(OPT(x^1) = (n - k)/2 \) which implies the optimal candidate pair is \((y_1, y_3)\) or \((y_1, y_4)\). For a strategy-proof mechanism, every voter votes for the nearest candidate. In our case, the action profile is \(a^0\). The expected social cost achieved by the mechanism is at least \(p_{3,4}(a^0) \cdot k(\sigma - 1) + (1 - p_{3,4}(a^0)) \cdot (n - k)/2\). The first term refers to the case where candidate pair \((y_3, y_4)\) is elected and the second term refers to the case where neither \(y_3\) nor \(y_4\) is elected. We define the corresponding election as \(f_1\) and have

\[
dist(f, f_1) \geq p_{3,4}(a^0) \cdot k(\sigma - 1) + (1 - p_{3,4}(a^0)) \cdot \frac{n-k}{2}
\]

\[
= \left( \frac{2k(\sigma - 1)}{n-k} - 1 \right) p_{3,4}(a^0) + 1. \tag{1}
\]

Third, we consider a location profile where voters are located exactly on \(y_2, y_3\) and \(y_4\). Precisely, we have \(x^2 = (x_1 = ... = x_k = 0, x_{k+1} = ... = x_{(n+k)/2} = 1, x_{(n+k)/2+1} = ... = x_n = 2)\). For this location profile, the optimal candidate pair is among \((y_3, y_4), (y_2, y_3)\) and \((y_2, y_4)\) with the minimum social cost \(OPT(x^2) = \min\{(n - k)/2, k\}\). Since \(k \leq n/3\), we have \(OPT(x^2) = k\) which indicates the unique optimal candidate pair is \((y_3, y_4)\). Since the mechanism is strategy-proof, the action profile would be \(a^k\). The expected social cost achieved by the mechanism is at least \(p_{3,4}(a^k) \cdot k + (1 - p_{3,4}(a^k)) \cdot (n - k)/2\). We define such election as \(f_2\) and have

\[
dist(f, f_2) \geq \frac{p_{3,4}(a^k) \cdot k + (1 - p_{3,4}(a^k)) \cdot \frac{n-k}{2}}{k}
\]

\[
= \frac{3k - n}{2k} \cdot p_{3,4}(a^k) + \frac{n-k}{2k}. \tag{2}
\]

Next, we show that \(p_{3,4}(a^0)\) actually equals to \(p_{3,4}(a^k)\). To prove this, we consider a location profile where \(k\) voters are located at the midpoint of the first two candidates. To be precise, \(x^3 = (x_1 = ... = x_k = -\sigma/2 + 1, x_{k+1} = ... = x_{(n+k)/2} = 1, x_{(n+k)/2+1} = ... = x_n = 2)\). For any \(0 \leq t \leq k - 1\), we compare the mechanisms given action profiles \(a^t\) and \(a^{t+1}\). Note that voter \(t\) has the same cost when either \(y_1\) or \(y_2\) is elected. When the action profile is \(a^t\), voter \(t\)'s cost is \((1 - p_{3,4}(a^t)) \cdot (\sigma/2 - 1) + p_{3,4}(a^t) \cdot \sigma/2\). Similarly, when the action profile is \(a^{t+1}\), voter \(t\)'s cost is \((1 - p_{3,4}(a^{t+1})) \cdot (\sigma/2 - 1) + p_{3,4}(a^{t+1}) \cdot \sigma/2\). Voter \(t\) should be indifferent in voting either \(y_1\) or \(y_2\) since the mechanism is strategy-proof. Therefore, we have \(p_{3,4}(a^t) = p_{3,4}(a^{t+1})\). Since it holds for any \(0 \leq t \leq k - 1\), we can get \(p_{3,4}(a^0) = p_{3,4}(a^k)\).

Combine the two lower bounds in the inequality \((1)\) and \((2)\), we have

\[
Dist(f) \geq \max \left\{ \left( \frac{2k(\sigma - 1)}{n-k} - 1 \right) p_{3,4}(a^0) + 1, \frac{3k - n}{2k} \cdot p_{3,4}(a^0) + \frac{n-k}{2k} \right\}. \tag{3}
\]
Then, we remove $p_{3,4}(a^0)$ from the lower bound. The first function is increasing in $p_{3,4}(a^0)$ while the second is decreasing in $p_{3,4}(a^0)$. Thus, the max of the two functions achieves the minimum value when they are equal. It can be shown that when

$$p_{3,4}(a^0) = \frac{n-3k}{2k} / \left( \frac{2k(\sigma-1)}{n-k} + \frac{n-5k}{2k} \right),$$

the two functions are equal. We plug it into the inequality and get

$$\text{Dist}(f) \geq \frac{\sigma-2}{\frac{2(n-1)}{n-k} + \frac{n-5k}{2k}} \quad (4)$$

We finish the proof in two different cases depending on the relation between parameters $\sigma$ and $n$. We know that $k$ is chosen from $[\frac{n}{2\sigma-1}, \frac{n}{3}]$ and $k$ should be an integer.

- Case 1, $n \geq 2\sqrt{\sigma-1} + 1$. Set $k = \lfloor \frac{n}{2\sqrt{\sigma-1}+1} \rfloor$ and we have $\frac{1}{2} \cdot \frac{n}{2\sqrt{\sigma-1}+1} \leq k \leq \frac{n}{2\sqrt{\sigma-1}+1}$. Then we get

$$\frac{2k}{n-k} \leq \frac{2n}{n-2\sqrt{\sigma-1}+1} - \frac{n}{2\sqrt{\sigma-1}+1} = 2 = \frac{1}{\sqrt{\sigma-1}}$$

$$\frac{n}{2k} \leq \frac{n}{\frac{n}{2\sqrt{\sigma-1}+1}} = 2\sqrt{\sigma-1} + 1$$

We plug the two parts into the inequality and we know $\sigma \geq 3$. Then we get

$$\text{Dist}(f) \geq \frac{\sigma-2}{\frac{\sigma-1}{\sqrt{\sigma-1}} + 2\sqrt{\sigma-1} + \frac{1}{2} - 2} = \frac{\sigma-2}{3\sqrt{\sigma-1} - \frac{3}{2}}$$

which is larger than $\frac{\sqrt{\sigma}}{3}$ for $\sigma \geq 3$.

- Case 2, $n < 2\sqrt{\sigma-1} + 1$. We set $k = 1$ and get

$$\text{Dist}(f) \geq \frac{\sigma-2}{\frac{\sigma}{n-1} + \frac{n-1}{n-2} - 2} \geq \frac{\sigma-2}{\frac{2(\sigma-1)}{n-1} + \frac{2(\sigma-1)}{n-1} - 2}$$

The numerator is at least $(\sigma-1)/2$ when $\sigma \geq 3$. Then,

$$\text{Dist}(f) \geq \frac{1}{2}(\sigma-1) \geq \frac{n-1}{8} \geq \frac{n}{12}$$

The last inequality holds as long as $n \geq 3$.

To sum up, the distortion is at least $\frac{1}{12} \min\{n, \sqrt{\sigma}\}$.

We can reuse the same instance and extend the proof when the mechanism is restricted to be deterministic.

**Theorem 2.** For two-winner election on the real line and $m = 4$, the distortion of any strategy-proof deterministic SCV mechanism cannot be smaller than $\frac{1}{3} \min\{n, \sqrt{\sigma}\}$. 
Proof. We consider the Instance \[\text{II}\]. There are six kinds of candidate pair. We divide them into two kinds of deterministic mechanism: the one is \((y_3, y_4)\), the other contains \((y_1, y_2), (y_1, y_3), (y_1, y_4), (y_2, y_3)\) and \((y_2, y_4)\). We define the same series of action profiles as in the proof of Theorem \(\text{I}\). For \(0 \leq t \leq k\), let \(a^t\) denote \((a_1 = \ldots = a_{k-t} = y_1, a_{k-t+1} = \ldots = a_k = y_2, a_{k+1} = \ldots = a_{(n+k)/2} = y_3, a_{(n+k)/2+1} = \ldots = a_n = y_4)\). The parameter \(k\) is also chosen from the interval \(\left[\frac{n}{2\sigma-1}, \frac{n}{\sigma}+1\right]\).

Next, we consider the location profile where \(k\) voters are located at the middle point of first two candidates. Precisely, we have \(x^1 = (x_1 = \ldots = x_k = -\sigma/2 + 1, x_{k+1} = \ldots = x_{(n+k)/2} = 1, x_{(n+k)/2+1} = \ldots = x_n = 2)\). The social cost achieved by the mechanism is \(k(\sigma/2 - 1)\). Under such conditions, since \(k \geq \frac{n}{2\sigma-1}\), the optimal candidate pair is \((y_1, y_3)\) or \((y_1, y_4)\).

The optimal social cost is \(\text{OPT}(x^1) = (n - k)/2\). Therefore, we have

\[
\text{Dist}(f) \geq \frac{k(\sigma - 1)}{n - k} = \frac{2k(\sigma - 1)}{n - k} \tag{5}
\]

If the mechanism is the second kind for action profiles \(a^0\) and \(a^k\), we consider the location profile where voters are located exactly at \(y_1, y_3\) and \(y_4\). Precisely, we have \(x^2 = (x_1 = \ldots = x_k = -\sigma + 2, x_{k+1} = \ldots = x_{(n+k)/2} = 1, x_{(n+k)/2+1} = \ldots = x_n = 2)\). The social cost achieved by the mechanism is at least \((n - k)/2\) implies that the deterministic mechanism outputs \((y_2, y_3)\) or \((y_2, y_4)\). However, due to \(k \leq n/3\), the optimal candidate pair is \((y_3, y_4)\) and \(\text{OPT}(x^2) = k\). Consequently, we have

\[
\text{Dist}(f) \geq \frac{n-k}{k} = \frac{n-k}{2k} \tag{6}
\]

We have either inequality \(\text{[5]}\) or inequality \(\text{[6]}\) holds. Thus, combining the two lower bounds, we get

\[
\text{Dist}(f) \geq \min\left\{ \frac{2k(\sigma - 1)}{n - k}, \frac{n - k}{2k} \right\}
\]

At last, we finish the proof in two different cases depending on the relation between \(\sigma\) and \(n\). We know that \(k\) should be an integer in \(\left[\frac{n}{2\sigma-1}, \frac{n}{\sigma}+1\right]\).

- Case 1, \(n \geq 2\sqrt{\sigma - 1} + 1\). Set \(k = \left\lfloor \frac{n}{2\sqrt{\sigma - 1}+1} \right\rfloor\) and we have \(k \geq \frac{\sqrt{n}}{\sigma} \frac{n}{2\sqrt{\sigma - 1}+1}\). Then we get

\[
\text{Dist}(f) \geq \frac{2k(\sigma - 1)}{n - k} \geq \frac{\sigma - 1}{2\sqrt{\sigma - 1} + \frac{n}{\sigma}}
\]

which is larger than \(\frac{\sigma}{3}\) for \(\sigma \geq 3\).
Two-Winner Election Using Favorite-Candidate Voting Rule

– Case 2, \( n < 2\sqrt{\sigma} - 1 + 1 \). We set \( k = 1 \) and get

\[
\text{Dist}(f) \geq \frac{n - k}{2k} = \frac{n - 1}{2} \geq \frac{n}{3}
\]

The last inequality holds as long as \( n \geq 3 \).

To sum up, the distortion is at least \( \frac{1}{3} \min\{n, \sqrt{\sigma}\} \).

For \( m > 4 \), we can easily nest Instance \( \text{I} \) into an instance with \( m \) candidates. The candidates are located at \( y_1 = -\sigma + (m - 2), y_2 = 0, y_3 = 1, ..., y_m = m - 2 \). Suppose \( n \) voters only vote for \( y_1, y_2, y_3 \) and \( y_4 \), so the case reduces to Instance \( \text{I} \). We define \( \sigma' = \sigma - m + 4 \) among the four candidates, that is, the lower bound of distortion is at least \( \Omega(\min\{n, \sqrt{\sigma - m + 4}\}) \) in the subset. It means that for any \( m \) candidates, there exists a case where no strategy-proof mechanism can achieve a distortion smaller than \( \Omega(\min\{n, \sqrt{\sigma - m + 4}\}) \).

**Corollary 1.** For two-winner election on the real line and \( m > 4 \), no strategy-proof scv mechanism can achieve a distortion smaller than \( \Omega(\min\{n, \sqrt{\sigma - m + 4}\}) \).

When there are exact 3 candidates, electing two candidates is equivalent to eliminating the least popular candidate. Chen, Li, and Wang (2020) propose Power-Proportional mechanism with distortion at most \( \frac{7}{3} \). We prove that this is also the lower bound of distortion. Consequently, we get the tight bound for the line metric when \( m = 3 \).

**Instance 2** Three candidates are located at \( y_1 = -2, y_2 = 0, y_3 = 2 \).

**Theorem 3.** For two-winner election on the real line and \( m = 3 \), the distortion of any strategy-proof randomized scv mechanism cannot be smaller than \( \frac{7}{3} \).

**Proof.** We consider the Instance \( \text{II} \). Three voters participate in the game. Consider the situation where each candidate receives a vote. So the action profile is \( \mathbf{a} = (a_1 = y_1, a_2 = y_2, a_3 = y_3) \). Based on the probability distribution, there are three different cases.

– Case 1, \( p_{1,2}(\mathbf{a}) \leq \frac{1}{3} \). Consider the location profile \( \mathbf{x}^1 = (x_1 = -2, x_2 = 0, x_3 = 1) \), we have \( OPT(\mathbf{x}^1) = 1 \) where the optimal candidate pair is \( (y_1, y_2) \). The social cost achieved by the mechanism is \( p_{1,2}(\mathbf{a}) \cdot 1 + p_{1,3}(\mathbf{a}) \cdot 3 + p_{2,3}(\mathbf{a}) \cdot 3 \). We have

\[
dist(f, \Gamma) \geq \frac{p_{1,2}(\mathbf{a}) \cdot 1 + p_{1,3}(\mathbf{a}) \cdot 3 + p_{2,3}(\mathbf{a}) \cdot 3}{1} = p_{1,2}(\mathbf{a}) + (1 - p_{1,2}(\mathbf{a})) \cdot 3 \geq \frac{7}{3}
\]

– Case 2, \( p_{2,3}(\mathbf{a}) \leq \frac{1}{3} \). By symmetry, the case is the same as Case 1. We can also get \( dist(f, \Gamma) \geq \frac{7}{3} \).
– Case 3, $p_{1,3}(a) \leq \frac{1}{3}$. Consider the location profile $x^2 = (x_1 = -2, x_2 = -1, x_3 = 2)$, we have $OPT(x^2) = 1$ where the optimal candidate pair is $(y_1, y_3)$. The social cost achieved by the mechanism is $p_{1,2}(a) \cdot 3 + p_{1,3}(a) \cdot 1 + p_{2,3}(a) \cdot 3$. We have

$$dist(f, \Gamma) \geq \frac{p_{1,2}(a) \cdot 3 + p_{1,3}(a) \cdot 1 + p_{1,2}(a) \cdot 3}{1} = p_{1,3}(a) + (1 - p_{1,3}(a)) \cdot 3 \geq \frac{7}{3}$$

In all three cases, the distortion is at least $\frac{7}{3}$.

For two-facility game, Procaccia and Tennenholtz [2009] suggest Two-Extremes mechanism. We assume that $y_1 < y_2 < \ldots < y_m$ on the line. The mechanism outputs the rightmost and leftmost candidates receiving votes. Here we give a tight distortion on this classic deterministic mechanism in our setting.

**Mechanism 1 (Two-Extremes)** Given an election $\Gamma = (\Omega, M, a)$, there is a subset $C \subset M$ receiving votes. First, consider $|C| = 1$ and the only candidate which receives votes is $y_i$. If $y_i \neq y_1$, the deterministic mechanism $f$ outputs $(y_1, y_i)$. If $y_i = y_1$, the mechanism outputs $(y_1, y_2)$. Then, consider $|C| > 1$, a deterministic mechanism $f$ outputs two winners $f(a)$ which is the leftmost and rightmost in $C$, that is

$$f(a) = (\min_{i \in C} i, \max_{j \in C} j)$$

**Theorem 4.** Two-Extremes mechanism is a strategy-proof deterministic scv mechanism and has a distortion at most $2n - 3$. The ratio is tight.

**Proof.** First, we show that Two-Extremes mechanism is strategy-proof. Given actions of voters other than $k$, suppose the leftmost and the rightmost candidates are $y_i$ and $y_j$. Consider the case that voter $k$’s location is between $y_i$ and $y_j$, then she has no incentive to misreport another candidate that is beyond $y_i$ and $y_j$, which only makes the outcome worse. If she votes for another candidate between $y_i$ and $y_j$, the outcome would not change. In this situation, voter $k$ has no incentive to misreport. Then, we consider the case that voter $k$’s location is outside $y_i$ and $y_j$. Voter $k$’s favorite candidate is determined to be elected as long as she reports truthfully. Hence, Two-Extremes mechanism is strategy-proof.

Next, we show that Two-Extremes mechanism has a distortion at most $2n - 3$. If all voters vote for the same candidate, Two-Extremes mechanism would achieve the lowest social cost. Then, we consider the case where at least two candidates receive votes.

We assume that the leftmost and rightmost candidate who collect positive number of votes are $y_k$ and $y_l$, the optimal candidate pair is $(y_k^*, y_l^*) (k^* < l^*)$. Obviously, we have $k \leq k^* < l^* \leq l$. 

We know that the distortion $\text{Dist}$ voter 1 and voter 2 are locate exactly at the same time to bound the distortion. In fact, it means that all voters except voter 1 is located exactly at $y_k$ and voter 2 votes for $y_l$. The optimal social cost is

$$SC((y_k, y_l), x) = \sum_{i \in N \setminus \{1,2\}} \min\{d(x_i, y_k), d(x_i, y_l)\} + d(x_1, y_k) + d(x_2, y_l)$$

For any voter $i$, consider the cost when the candidate pair is $(y_k, y_l)$ and $(y_k^*, y_l^*)$. We have

$$\min\{d(x_i, y_k), d(x_i, y_l)\} \leq \min\{d(x_i, y_k), d(x_i, y_l^*) + d(y_l^*, y_l)\} \leq \min\{d(x_i, y_k), d(x_i, y_l^*)\} + \max\{d(y_k^*, y_k), d(y_l^*, y_l)\}$$

So the social cost achieved by the mechanism is upper bounded by

$$SC((y_k, y_l), x) = \sum_{i \in N \setminus \{1,2\}} \min\{d(x_i, y_k), d(x_i, y_l)\} + d(x_1, y_k) + d(x_2, y_l) \leq \sum_{i \in N \setminus \{1,2\}} \min\{d(x_i, y_k^*), d(x_i, y_l^*)\} + (n-2) \max\{d(y_k^*, y_k), d(y_l^*, y_l)\} + d(x_1, y_k) + d(x_2, y_l)$$

We know that the distortion $\text{Dist}(f) \geq 1$. Consequently, we can eliminate $\sum_{i \in N \setminus \{1,2\}} \min\{d(x_i, y_k), d(x_i, y_l)\}$ from the top and bottom of the fraction at the same time to bound the distortion. In fact, it means that all voters except voter 1 and voter 2 are locate exactly at $y_k^*$ and $y_l^*$. We get

$$\text{Dist}(f) \leq \frac{(n-2) \max\{d(y_k^*, y_k), d(y_l^*, y_l)\}}{d(x_1, y_k) + d(x_2, y_l)} + \frac{d(x_1, y_k) + d(x_2, y_l)}{d(x_1, y_k^*) + d(x_2, y_l^*)}$$

Next, we consider the location of voter 1 and voter 2. Take voter 1 for example, note that $y_k$ is always the nearest candidate of voter 1. The possible location $x_1$ is either $x_1 \leq y_k$ or $x_1 \in (y_k, (y_k + y_k^*)/2]$. When voter 1 is located at the left side of $y_k$, we have $d(x_1, y_k) = d(x_1, y_k) + d(y_k, y_k^*)$. So we also can eliminate $d(x_1, y_k)$ from the top and bottom of the fraction at the same time to achieve the upper bound. In fact, it means that voter 1 is located exactly at $y_k$. When voter 1 is located between $y_k$ and the midpoint of $(y_k, y_k^*)$, as voter 1 moves from left to right, $d(x_1, y_k)$ is increasing and $d(x_1, y_k^*)$ is decreasing. To sum up, the worst case is that voter 1 is located exactly at the midpoint of $(y_k, y_k^*)$. The
same goes for voter 2. We have

$$\text{Dist}(f) \leq \frac{(n-2)}{2}(d(y_k, y_l) + d(y_l, y_l))$$

$$\leq \frac{(n-2)}{2} \max\{d(y_k, y_k), d(y_l, y_l)\} + 1$$

$$= 2n - 3$$

The worst case is as follows. We use Instance 2 where $y_1 = -2, y_2 = 0, y_3 = 2$. There are $n$ voters participating in the game. The voters are located at the midpoint of $(y_1, y_2), y_2$ and $y_3$. Precisely, the location profile is $x = (x_1 = -1, x_2 = \ldots = x_{n-1} = 0, x_n = 2)$. The optimal candidate pair is $(y_2, y_3)$ and $OPT(x) = 1$. The social cost achieved by Two-Extremes mechanism is $2(n-2) + 1 = 2n - 3$.

### 4 Multi-Dimensional Metric Space

This section focuses on the multi-dimensional Euclidean space with at least three dimensions. In one-dimensional space, only the midpoint of two candidates has the same distance to them. However, there is a subspace where the points have the same distance to the two candidates in multi-dimensional space. Besides, we can construct the instance with symmetrical points in multi-dimensional space.

We consider the anonymity. If we only use the number of votes obtained by each candidate when designing a mechanism, instead of paying attention to the votes of each voter, we say that the mechanism is anonymous.

#### 4.1 Randomized Mechanism

When $m$ candidates are located in at least $(m-1)$-dimensional space, we construct an instance which is more amenable to show constraints on anonymous strategy-proof mechanisms. We provide a necessary condition for any strategy-proof mechanism electing $w$ candidate where $w < m-1$. With this piece of machinery, we first exhibit a lower bound of the distortion for any strategy-proof mechanism to elect two winners. We then show the bound is asymptotically tight by introducing “Pair-Independent” mechanism. At last, we demonstrate the unique anonymous strategy-proof mechanism for single-winner election with finite distortion is Random Dictator.

First, we introduce a well-defined instance in $(m-1)$-dimensional space. Intuitively, $m-1$ candidates are located at vertices of a regular $(m-1)$-hedron. Besides, the last candidate is farther and has the same distance to them.

**Instance 3** There are $m$ candidates located in $(m-1)$-dimensional space. For candidate $y_i (i < m)$, the coordinate in $i$-th dimension is 1, and the coordinates are all zeros in other dimensions. For candidate $y_m$, the coordinates all equal to $r$. Formally, we have $y_1 = (1, 0, \ldots, 0), \ldots, y_{m-1} = (0, 0, \ldots, 1)$, and $y_m = (r, r, \ldots, r)$ where $r > 2$. 

Lemma 1. For any anonymous strategy-proof $w$-winner ($w < m - 1$) election mechanism applied on the Instance 3, the probability of any committee of candidates being elected only depends on the number of votes obtained by the $w$ candidates, no matter what the vote distribution is over any other candidates.

Proof. Set $L = \{1, \ldots, l_w\}$ is used to represent a subset of $\{1, \ldots, m\}$ with size $w$. For any $a_{-k}$, we denote the action profile where voter $k$ chooses $y_j$ by $a_1 = (a_k = y_j, a_{-k})$ and another action profile where voter $k$ chooses $y_j$ by $a_2 = (a_k = y_j, a_{-k})$. We also extend the definition of function $p$ by allowing probability assignment on a $w$-winner profile.

First we show that for any $L$ not including elements $i$ and $j$, $p_L(a_1) = p_L(a_2)$.

Based on whether $i$ or $j$ equals to $m$, there are two cases.

1. Case 1, $i = m$ or $j = m$. W.l.o.g., we assume $j = m$. For any set $L$ with size $w$ and $r/2 \leq \alpha_1 \leq \alpha_2 \leq (r + 1)/2$, we define a subspace $U_L(\alpha_1, \alpha_2)$ as

$$\{ (t_1, t_2, \ldots, t_{m-1}) \in R^{m-1} | t_i = \frac{r + 1}{2}, t_1 = \frac{(m-2)r}{2} - (w-1)\alpha_1 - (m-w-2)\alpha_2, t_2 = \ldots = t_{m-1} = \alpha_1, t_h = \alpha_2, \forall h \notin L \cup \{i\} \}$$

Then $\forall x \in U_L(\alpha_1, \alpha_2)$, we first consider the distances from $x$ to $y_i$ and to $y_j$. Actually, $d(x, y_i)$ and $d(x, y_j)$ are equal and can be written as

$$\left( \left( \frac{r - 1}{2} \right)^2 + t_i + (w-1)\alpha_1^2 + (m-w-2)\alpha_2^2 \right)^{\frac{1}{2}}$$

For simplicity, we express the distances between $x$ and the other candidates in terms of $d(x, y_i)$.

$$d(x, y_h) = (d(x, y_i)^2 + r + 1 - 2\alpha_2)^{\frac{1}{2}}, \forall h \notin L \cup \{i\}$$

$$d(x, y_j) = \ldots = d(x, y_{l_w}) = (d(x, y_i)^2 + r + 1 - 2\alpha_1)^{\frac{1}{2}}$$

$$d(x, y_j) = (d(x, y_i)^2 + r + 1 - 2t_i)^{\frac{1}{2}}$$

For $r/2 \leq \alpha_1 \leq \alpha_2 \leq (r + 1)/2$ and $h \notin L \cup \{i\}$, it is easy to check that $d(x, y_i) = d(x, y_j) \leq d(x, y_h) \leq d(x, y_{l_w}) = \ldots = d(x, y_{l_w}) \leq d(x, y_i)$. For candidates $y_i$ and $y_j$, $U_L(\alpha_1, \alpha_2)$ is a subspace in which voter $k$ has the same shortest distance from her to candidate $i$ and to candidate $j$. There are three possibilities in the outcome: the distance from voter $k$ to the nearest candidate could be $d(x, y_i)$, $d(x, y_j)$ and $d(x, y_h)$ for $h \notin L \cup \{i\}$.

For convenience, we define $\sum_{L' \ni \{i\}, j \notin L'} p_{L'}(a) - p_L(a)$ as $e_1(a)$ and define $1 - \sum_{L' \ni \{i\}, j \notin L'} p_{L'}(a)$ as $e_2(a)$. Voter $k$’s cost by voting $y_i$ would be

$$p_L(a_1) \cdot d(x, y_i) + e_1(a_1) \cdot d(x, y_h) + e_2(a_1) \cdot d(x, y_j)$$

Since voter $k$ has the same cost when voting $y_i$ and $y_k$, we have $(p_L(a_1) - p_L(a_2)) \cdot d(x, y_j) + (e_1(a_1) - e_1(a_2)) \cdot d(x, y_h) = (e_2(a_2) - e_2(a_1)) \cdot d(x, y_j)$.
We square both sides of the equation. Then the right-hand side has no square root term while the left side has one. The coefficient is $2(p_L(a^1) - p_L(a^2))(e_1(a^1) - e_2(a^2))$. Note that the equality above always holds when $r/2 \leq \alpha_1 \leq \alpha_2 \leq (r + 1)/2$. So we have $p_L(a^1) = p_L(a^2), e_1(a^1) = e_1(a^2)$ and $e_1(a^1) = e_2(a^2)$. Therefore, we conclude that if $i = m$ or $j = m$, for any subset $L$ not including elements $i$ and $j$, $p_L(a^1) = p_L(a^2)$.

- Case 2, $i \neq m$ and $j \neq m$. The proof is similar to Case 1. We can also get $p_{L^2}(a^1) = p_{L^2}(a^2)$. Based on whether the subset containing $m$, we construct two different subspace. There are also three possibilities in the outcome and we conclude that the corresponding probabilities are equal.

First, we consider the subset containing $m$. For any set $L_1 = \{l_1, l_2, ..., l_w\}$ with size $w$ and $m \in L_1$, w.l.o.g., we suppose $l_1 = m$, and for $0 \leq \alpha_1 \leq \alpha_2 \leq 1/2$, we define a subspace $U_{L_1}(\alpha_1, \alpha_2)$ as

$$
\{(t_1, t_2, ..., t_{m-1}) \in R^{m-1} | t_i = \frac{1}{2}, t_j = \frac{1}{2} \\
t_{t_2} = ... = t_{t_w} = \alpha_1, t_h = \alpha_2, \forall h \notin \{i, j\} \cup L_1 \}
$$

For $h \notin L_1 \cup \{i, j\}$, we have $d(x, y_i) = d(x, y_j) \leq d(x, y_h) \leq d(x, y_{t_1}) = ... = d(x, y_{t_w}) \leq d(x, y_{t_1})$ under such condition. We consider the three distances $d(x, y_i), d(x, y_j), d(x, y_h)$ and their corresponding possibilities. Same as Case 1, we can get $p_{L_1}(a^1) = p_{L_1}(a^2)$.

Second, we consider the subset not containing $m$. For any set $L_2 = \{l_1, l_2, ..., l_w\}$ with size $w$ and $m \notin L_2$, and for $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 1/2$, we define a subspace $U_{L_2}(\alpha_1, \alpha_2, \alpha_3)$ as

$$
\{(t_1, t_2, ..., t_{m-1}) \in R^{m-1} | t_i = \frac{1}{2}, t_j = \frac{1}{2}, t_1 = \alpha_1, \\
t_{t_2} = ... = t_{t_w} = \alpha_1, t_h = \alpha_2, \forall h \notin \{i, j\} \cup L_2 \}
$$

For $h \notin L_2 \cup \{i, j\}$, we have $d(x, y_i) = d(x, y_j) \leq d(x, y_h) \leq d(x, y_{t_w}) = ... = d(x, y_{t_w}) \leq d(x, y_{t_1}) \leq d(x, y_{t_1})$ under such condition. Let $L_1 = \{m, l_2, ..., l_w\}$.

We consider the three distances $d(x, y_i), d(x, y_j), d(x, y_h)$ and their corresponding possibilities. Similarly, we have $p_{L_1}(a^1) + p_{L_2}(a^1) = p_{L_1}(a^2) + p_{L_2}(a^2)$. We know that $p_{L_1}(a^1) = p_{L_1}(a^2)$. So we can get $p_{L_2}(a^1) = p_{L_2}(a^2)$.

To sum up, no matter what candidates $i$ and $j$ are and what subset $L$ is, we have $p_L(a^1) = p_L(a^2)$. In addition, note that $a^1$ and $a^2$ are two action profiles that only have differences on the number of votes obtained by voter $i$ and $j$. Besides, the gap is only one vote. Furthermore, it implies no matter we change the number of votes obtained by voter $i$ and $j$, as long as the rest of votes remains the same, the probability $p_L(a)$ remains indifferent. Because we can arbitrarily elect $i, j \notin L$, so we only need to fix the votes of the elements in $L$, the probability $p_L(a)$ is always indifferent. And it means the probability $p_L(a)$ only can be related with the number of votes obtained by $y_1, ..., y_{t_w}$.

For a strategy-proof mechanism, the probability of a candidate pair only depends on the number of votes that the two candidates receive. We call it
independent mechanism. For the sake of simplicity, we use \( q_{i,j}(n_i, n_j) \) to represent the probability that candidate pair \((y_i, y_j)\) is elected where \( n_i \) and \( n_j \) are the number of votes for candidates \( y_i \) and \( y_j \) in the action profile \( a \). That is, there exists an independent mechanism \( q \) such that \( p_{i,j}(a) = q_{i,j}(n_i, n_j) \). Now we use Lemma \( \text{I} \) to show a lower bound of distortion when electing two winners.

**Theorem 5.** For two-winner election in multi-dimensional Euclidean space and \( m > 3 \), no randomized strategy-proof scv mechanism can achieve a distortion smaller than \( \sigma/6 \).

**Proof.** We consider Instance \( \text{I} \). Then, \( \forall i, j \in [m - 1] \), \( d(y_i, y_j) = d_{\text{min}} \) and \( d(y_i, y_m) = d_{\text{max}} \). Given any strategy-proof mechanism, we can construct a randomized strategy-proof anonymous mechanism by using uniform permutation of voters and the distortion will not be worse.

First, we consider a location profile where voters are located exactly at \( y_1 \) and \( y_2 \). Precisely, we have \( x^1 = (x_1 = x_2 = \ldots = x_k = y_1, x_{k+1} = \ldots = x_n = y_2) \). The optimal candidate pair is \((y_1, y_2)\) and \( \text{OPT}(x^1) = 0 \). It is easy to see that any mechanism with finite distortion would elect \((y_1, y_2)\) deterministically. So we have \( q_{1,3}(k, 0) = 0 \) for any \( 1 \leq k < n \) and \( q_{3,4}(0, 0) = 0 \). In addition, using the same argument, we can derive the following claim

\[
q_{i,j}(0, k) = 0, \forall i, j \leq m, 0 \leq k < n
\]  

(7)

Next, we consider a location profile where voters are located exactly at \( y_1, y_2 \) and \( y_3 \). These three candidates receive approximately equal number of votes. We set \( n_1, n_2, n_3 \in \{\lfloor n/3 \rfloor, \lceil n/3 \rceil\} \) such that \( n_1 + n_2 + n_3 = n \). In particular, we set \( x^2 = (x_1 = \ldots = x_{n_1} = y_1, x_{n_1+1} = \ldots = x_{n_1+n_2} = y_2, x_{n_1+n_2+1} = \ldots = x_n = y_3) \). According to Eq. 7, there are only three possibilities for how the mechanism elects the candidate pair. We have

\[
q_{1,2}(n_1, n_2) + q_{1,3}(n_1, n_3) + q_{2,3}(n_2, n_3) = 1.
\]

W.l.o.g., we assume \( q_{1,2}(n_1, n_2) \geq \frac{1}{3} \).

Third, we consider a location profile where voters are located exactly at \( y_1, y_2 \) and \( y_m \). We consider \( x^3 = (x_1 = \ldots = x_{n_1} = y_1, x_{n_1+1} = \ldots = x_{n_1+n_2} = y_2, x_{n_1+n_2+1} = \ldots = x_n = y_m) \). The optimal candidate pair is \((y_1, y_m)\) or \((y_2, y_m)\) with the social cost \( \text{OPT}(x^3) = \min\{n_1, n_2\} \cdot d_{\text{min}} \leq \lfloor n/3 \rfloor \cdot d_{\text{min}} \). The expected social cost of the mechanism is lower bounded by

\[
\frac{\lfloor n/3 \rfloor}{\lceil n/3 \rceil} \left( q_{1,2}(n_1, n_2) \cdot d_{\text{max}} + (1 - q_{1,2}(n_1, n_2)) \cdot d_{\text{min}} \right) \geq \frac{\lfloor n/3 \rfloor}{\lceil n/3 \rceil} \cdot d_{\text{max}}/3.
\]

Therefore, we have \( \text{Dist}(f) \geq \frac{\lfloor n/3 \rfloor}{\lceil n/3 \rceil} \cdot d_{\text{max}}/3 \geq \frac{\sigma}{6} \).

Next, we demonstrate the randomized strategy-proof mechanisms that achieve the upper bound only with the \( \sigma \) parameter. We first show that any monotone mechanism satisfying Lemma \( \text{I} \) is strategy-proof.
Definition 1. (Monotone Mechanism) We say an independent mechanism is monotone if the probability of a committee being elected weakly increases when the number of votes received by each member weakly increases. In particular, for two-winner election, \(\forall (y_i, y_j) \in M^2\), it holds that
\[
q_{i,j}(n_i + 1, n_j) \geq q_{i,j}(n_i, n_j)
\]

Theorem 6. Any monotone mechanism satisfying Lemma \(7\) for two-winner election is a strategy-proof mechanism.

Proof. The mechanism that satisfies Lemma \(7\) is denoted by function \(q\). Consider a location profile \(x\) and an arbitrary voter \(i\). Suppose \(y_{k^*}\) is the nearest candidate to the voter \(i\). We consider voter \(i\)'s cost when reporting some other candidate \(y_{k'}\) and prove that it is more than her cost when reporting \(y_{k^*}\). We use \(n_j\) to denote the number of voters who vote for \(y_j\) when voter \(i\) reports \(y_{k^*}\). For example, when voter \(i\) reports \(y_{k'}\), the number of voters who vote for \(y_{k'}\) would be \(n_{k'} + 1\). The outcome of the mechanism has three categories depending on whether \(y_{k^*}\) or \(y_{k'}\) is elected.

When \(y_{k^*}\) is elected, the cost is \(\sum_l q_{k^*, l}(n_{k^*} - 1, n_l) d(x_i, y_{k^*})\). Compared to reporting truthfully, the cost in this category reduces by
\[
\sum_l [-q_{k^*, l}(n_{k^*} - 1, n_l) + q_{k^*, l}(n_{k^*}, n_l)] d(y_{k^*}, x_i) \tag{8}
\]

When \(y_{k^*}\) is not elected but \(y_{k'}\) is elected, the cost is \(\sum_{l \neq k'} q_{k', l}(n_{k'} + 1, n_l) \min\{d(x_i, y_{k'}), d(x_i, y_l)\}\). Since the mechanism is monotone, the probability of candidate pair \((y_{k'}, y_l)\) increases compared to reporting truthfully. Note that \(\min\{d(x_i, y_{k'}), d(x_i, y_l)\} \geq d(x_i, y_{k^*})\), the cost in this category increases by at least
\[
\sum_{l \neq k^*} [q_{k^*, l}(n_{k^*} + 1, n_l) - q_{k^*, l}(n_{k^*}, n_l)] d(y_{k^*}, x_i) \tag{9}
\]

When neither candidate \(y_{k^*}\) nor \(y_{k'}\) is elected, the cost is
\[
\sum_{(j,l): \{j,l\} \cap \{k^*, k'\} = \emptyset} q_{j,l}(n_j, n_l) \min\{d(x_i, y_j), d(x_i, y_l)\}
\]
The cost in this category does not change.

To figure out how voter \(i\)'s cost changes, we compare the cost of increasing in the second category and decreasing in the first category. Notice that the probability that one of \(y_{k'}\) and \(y_{k^*}\) is elected does not change. Thus we have
\[
\sum_l q_{k^*, l}(n_{k^*} - 1, n_l) + \sum_{l \neq k^*} q_{k^*, l}(n_{k^*} + 1, n_l) = \sum_l q_{k^*, l}(n_{k^*} - 1, n_l) + \sum_{l \neq k^*} q_{k^*, l}(n_{k^*}, n_l).
\]
It implies that
\[
\sum_l q_{k^*, l}(n_{k^*} - 1, n_l) < \sum_{l \neq k^*} q_{k^*, l}(n_{k^*} + 1, n_l) - q_{k^*, l}(n_{k^*}, n_l)
\]
Thus we have the amount of increase in Eq. (9) is at least the amount of decrease
in Eq. (8). So
cost\left(f(a_i = y_k^*, a_{-i}), x_i\right) \geq cost(f(a_i = y_k^*, a_{-i}), x_i) and the proof
completes.

Now we are ready to present a strategy-proof randomized mechanism which
ensures the distortion is only related to the parameter $\sigma$.

**Mechanism 2 (Pair-Independent)** Given an election $\Gamma = (\Omega, M, a)$, when all
voters vote the same candidate $y_i$, for all candidate pairs containing $y_i$, the
winning probability is $\frac{1}{m - 1}$. When more than one candidate gets votes, for every
candidate pair $(y_i, y_j) \in M^2$, $n_i$ and $n_j$ are the number of votes for candidates
$y_i$ and $y_j$ in the action profile $a$, the winning probability is

$$q_{i,j}(n_i, n_j) = \frac{n_i}{n - n_j} + \frac{n_j}{n - n_i} - \frac{n_i + n_j}{n}$$

It is easy to see that the sum of the probability is always 1. When only one can-
didate gets votes, the mechanism guarantees this candidate elected with prob-
ability 1. When more than one candidate receive votes, the mechanism never
elects a candidate who receives zero vote and it assigns a higher probability for
a pair of candidates when they receive more votes. Thus, the mechanism is well
defined.

**Theorem 7.** Pair-Independent is a randomized strategy-proof scv mechanism
and has a distortion at most $1 + 6\sigma$.

Proof. First, we show the mechanism is monotone. If there are more than one
candidate receiving votes, consider the votes of candidates $y_i$ and $y_j$,

$$q_{i,j}(n_i + 1, n_j) - q_{i,j}(n_i, n_j) = \left(\frac{1}{n - n_j} - \frac{1}{n}\right) + \left(\frac{n_j}{n - n_i} - \frac{n_j}{n - n_i - 1}\right) \geq 0$$

If candidate $y_i$ receives all the votes, we have

$$q_{i,j}(n, 0) - q_{i,j}(n - 1, 0) = \frac{1}{m - 1} > 0$$

The Pair-Independent is strategy-proof by Theorem 6. Next, we show the dis-
tortion of the mechanism. Given a location profile $x$ and suppose the optimal
candidate pair is $(y_{k^*}, y_{l^*})$. Given a randomized scv mechanism, we examine the
following ratio

$$\frac{\sum_{(y_k, y_l) \in M^2} q_{k,l}(n_k, n_l) SC((y_k, y_l), x)}{SC((y_{k^*}, y_{l^*}), x)}.$$
For each candidate pair \((y_k, y_l) \neq (y_k^*, y_l^*)\), the social cost \(SC((y_k, y_l), x)\) is upper bounded by

\[
\sum_{i \in N} \min\{d(x_i, y_k), d(x_i, y_l)\} \\
\leq \sum_{i : a_i = y_k} d(x_i, y_k) + \sum_{i : a_i = y_l} d(x_i, y_l) \\
+ \sum_{i : a_i \notin \{y_k, y_l\}} \min\{d(x_i, y_k), d(x_i, y_l)\} \\
\leq SC((y_k^*, y_l^*), x) + (n - n_k - n_l)d_{max}
\]

Then the social cost \(SC(f(a), x)\) is upper bounded by

\[
SC((y_k^*, y_l^*), x) + \sum_{(k, l) \neq (k^*, l^*)} q_{k,l}(n_k, n_l)(n - n_k - n_l)d_{max}
\]

The optimal social cost is lower bounded by

\[
SC((y_k^*, y_l^*), x) \\
= \frac{1}{2} \sum_{i \in N} 2 \min\{d(x_i, y_k^*), d(x_i, y_l^*)\} \\
\geq \frac{1}{2} \sum_{i \in N} (\min\{d(x_i, y_k^*), d(x_i, y_l^*)\} + d(x_i, a_i)) \\
\geq \frac{1}{2} \sum_{i \in N} \min\{d(a_i, y_k^*), d(a_i, y_l^*)\} \\
\geq \frac{1}{2}(n - n_k^* - n_l^*)d_{min}
\]

Combining the upper bound of the social cost achieved by mechanism and the lower bound of optimal social cost, the ratio is upper bounded by

\[
1 + \frac{2\sigma}{n - n_k^* - n_l^*} \sum_{k < l} q_{k,l}(n_k, n_l)(n - n_k - n_l)
\]

We plug the probability function into the summation and get

\[
\sum_{k < l} q_{k,l}(n_k, n_l)(n - n_k - n_l) \\
= \sum_{k < l} \left( \frac{n_k}{n - n_l} + \frac{n_l}{n - n_k} - \frac{n_k + n_l}{n} \right) (n - n_k - n_l) \\
= \sum_{k < l} \left( \frac{(n_k + n_l)^2}{n} - \frac{n_k^2}{n - n_l} - \frac{n_l^2}{n - n_k} \right) \\
= \left( \sum_{k \in [m]} n_k \right)^2 n - \sum_{k \in [m]} \left( \frac{m - 2}{n} n_k^2 - \sum_{l \neq k} n_l \right)
\]
For any \( k \in [m], l \neq k \), we have \( \frac{n_k^2}{n} - \frac{n_l^2}{n} \leq 0 \). So it holds that \( \frac{(m-2)n^2}{n} - \sum_{l \neq k} \frac{n_l^2}{n-m} \leq 0 \). We only keep two terms containing \( n_{k^*} \) and \( n_{l^*} \), and get

\[
\sum_{k<l} q_{k,l}(n_k, n_l)(n - n_k - n_l) \leq n - \frac{n_{k^*}^2}{n-n_{l^*}} - \frac{n_{l^*}^2}{n-n_{k^*}}
\]

Consider the function \( g(x) = \frac{x^2}{x+b} \), and \( g''(x) = \frac{2b^2}{(x+b)^3} \geq 0 \). So \( g(x) \) is a convex function. We have \( g(n_{k^*}) + g(n_{l^*}) \geq 2g\left(\frac{n_{k^*}+n_{l^*}}{2}\right) \). Define \( n^* = \frac{n_{k^*}+n_{l^*}}{2} \), we have

\[
\text{Dist}(f) \leq 1 + \frac{2\sigma}{n - n_{k^*} - n_{l^*}} \left( n - \frac{n_{k^*}^2}{n-n_{l^*}} - \frac{n_{l^*}^2}{n-n_{k^*}} \right)
\]

\[
\leq 1 + \frac{2\sigma}{n - 2n^*} \cdot \left( n - \frac{2n^* \cdot n^*}{n-n^*} \right)
\]

\[
= 1 + 2\sigma \cdot \frac{n + n^*}{n - n^*}
\]

\[
\leq 1 + 6\sigma
\]

Finally, we also find a desirable conclusion for single-winner election by Lemma \[1\]

**Mechanism 3 (Random Dictator)** Given an election \( \Gamma = (\Omega, M, a) \), \( n_i \) is the number of votes for candidates \( y_i \) in the action profile \( a \), for every candidate \( y_i \in M \), the winning probability is \( n_i/n \).

**Theorem 8.** The only anonymous strategy-proof scv mechanism with finite distortion is Random Dictator for single-winner election.

**Proof.** Using Lemma \[1\] the mechanism can be denoted by function \( q_k \) for any \( y_k \). We elect \( y_i \) arbitrarily, and \( \forall k \neq i, n_k \in [n-1] \), we have \( q_i(1) + q_k(n_k) = q_k(n_k+1) + q_i(0) \). Therefore, we know \( q_k(n_k+1) - q_k(n_k) = q_i(1) - q_i(0) \) is a constant. It means that \( \forall k \neq i \), the function \( q_k \) is a linear function. Besides, all functions have the same coefficient. Because the sum of probability is always 1, \( q_i \) must be a linear function and also has the same coefficient. Suppose \( q_k(n_k) = c \cdot n_k + b_k (b_k \geq 0) \). Consider the instance \[3\] and \( n \) voters are located exactly at \( y_m \). Precisely, we have \( \mathbf{x} = (x_1 = ... = x_n = y_m) \). For this location profile, we have \( \text{OPT}(\mathbf{x}) = 0 \). The expected social cost is

\[
\sum_{k \neq m} q_k(n_k) \cdot n \cdot d_{max}
\]

\[
= \sum_{k \neq m} (c \cdot n_k + b_k) \cdot n \cdot d_{max}
\]

\[
= n \cdot d_{max} \sum_{k \neq m} b_k
\]
If the mechanism has finite distortion, we must have \( \forall k \neq i, b_k = 0 \). According to \( y_i \) can be elected arbitrarily, we also get \( b_i = 0 \). Since the sum of the probability is 1, we have that the coefficient \( c \) is \( 1/n \). To sum up, the only anonymous strategy-proof mechanism with finite distortion for single-winner election is Random Dictator.

### 4.2 Deterministic Mechanism

We move on to discussing deterministic mechanisms in multi-dimensional Euclidean space with at least three dimensions. We introduce a special case of Instance 3.

**Instance 4** Four candidates are located at \( y_1 = (1, 0, 0), y_2 = (0, 1, 0), y_3 = (0, 0, 1), y_4 = (r, r, r) \).

We mainly consider the anonymous mechanisms that only use the number of votes. According to Lemma 1, we demonstrate that no deterministic strategy-proof mechanism can achieve a finite distortion.

**Theorem 9.** For two-winner election in multi-dimensional Euclidean space and \( m > 3 \), no anonymous deterministic strategy-proof scv mechanism can achieve a finite distortion.

**Proof.** For a deterministic mechanism, it outputs a candidate pair \((y_i, y_j)\) with probability 1. According to Lemma 1, it means that \( q_{i,j}(n_i, n_j) \) should be 0 or 1. There are only four voters, and we consider a location profile \( x^4 = (x_1 = x_2 = x_3 = y_1, x_4 = y_2) \). So the optimal candidate pair is \((y_1, y_2)\) and \( OPT(x^4) = 0 \). It is easy to see that any mechanism with finite distortion would elect \((y_1, y_2)\) deterministically, that is, \( q_{1,2}(3, 1) = 0 \). Similarly, we have \( \forall (y_i, y_j) \in M^2, q_{i,j}(3, 1) = q_{i,j}(2, 2) = q_{1,2}(1, 3) = 1 \). Due to the sum of probability is 1, we get \( \forall (y_i, y_j) \in M^2, q_{i,j}(0, 1) = q_{i,j}(1, 0) = q_{i,j}(2, 0) = q_{i,j}(0, 2) = 0 \). Next, we consider a location profile \( x^6 = (x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4) \). There are two cases depending on whether the mechanism elects \( y_4 \).

- **Case 1**, the mechanism does not elect \( y_4 \). W.l.o.g., we assume that it outputs \((y_1, y_2)\). Then, we have \( q_{1,2}(1, 1) = 1 \) and \( \forall (y_i, y_j) \in M^2 \setminus \{y_i, y_j\}, q_{i,j}(1, 1) = 0 \). Next, we consider the following three location profiles \( x^7 = (x_1 = x_2 = y_2, x_3 = y_3, x_4 = y_4) \), \( x^4 = (x_1 = x_1, x_2 = x_3 = y_2, x_4 = y_4) \) and \( x^5 = (x_1 = y_1, x_2 = x_3 = y_2, x_4 = y_3) \). We have the corresponding three equations \( q_{2,4}(2, 1) + q_{2,4}(2, 1) = 1, q_{1,2}(1, 2) + q_{2,4}(2, 1) = 1 \) and \( q_{1,2}(1, 2) + q_{2,4}(2, 1) = 1 \). So we get \( 2(q_{1,2}(1, 2) + q_{2,4}(2, 1) + q_{2,4}(2, 1)) = 3 \) which contradicts the condition where \( q_{i,j}(n_i, n_j) \) should be 0 or 1.
- **Case 2**, the mechanism elects \( y_4 \). W.l.o.g., we assume that it outputs \((y_1, y_4)\). Then, we have \( q_{1,4}(1, 1) = 1 \). It is similar to Case 1. We consider the location profiles \( x^6 = (x_1 = y_1, x_2 = x_3 = y_3, x_4 = y_4) \), \( x^7 = (x_1 = y_1, x_2 = y_2, x_3 = x_4 = y_4) \) and \( x^8 = (x_1 = y_2, x_2 = x_3 = x_3 = x_4 = y_4) \). We have \( 2(q_{1,4}(1, 2) + q_{2,4}(1, 2) + q_{3,4}(1, 2)) = 3 \). It also contradicts the same condition.
Therefore, no anonymous deterministic strategy-proof scv mechanism can achieve a finite distortion.

If the anonymity is not required, the mechanism can use the full information contained in the profile, including which voter votes for which candidate. In such case, Lemma \[\text{4}\] is not feasible and we design the strategy-proof deterministic mechanism called “Sequential Dictator”. Informally, we arrange the voters in a sequence by their serial number, and the mechanism always elects the first two candidates.

**Mechanism 4 (Sequential Dictator)** Given an election \(\Gamma = (\Omega, M, a)\), when all the voters vote the same candidate \(y_i\), if \(y_i \neq y_1\), the deterministic mechanism outputs \((y_1, y_j)\). If \(y_i = y_1\), the mechanism outputs \((y_1, y_2)\). When more than one candidate receive votes, the deterministic mechanism outputs the first two different candidates in the sequence \([a_1, a_2, \ldots, a_n]\), that is

\[
f(a) = (a_1, a_i), i = \min\{j | a_j \neq a_1, a_j \in M\}
\]

**Theorem 10.** Sequential Dictator is a deterministic strategy-proof scv mechanism and has a distortion at most \(2(n - 2)\sigma + 1\). The ratio is tight.

**Proof.** First, we show that Sequential Dictator mechanism is strategy-proof. Given a action profile \(a\), we can generate the corresponding sequence by the serial number. For voter \(i\), if the two winners have been already elected by the action of former voters, it is no matter what voter \(i\) casts. Thus, voter \(i\) has no reason to misreport. If the candidate pair has not been determined, what vote \(i\) submits becomes the winner. It is obvious that the voter reports her favorite choice. Hence, Sequential Dictator mechanism is strategy-proof. Next, we show Sequential Dictator mechanism has a distortion at most \(2(n - 2)\sigma + 1\).

Part of proof is the same as Theorem 7. Given a location profile \(x\) and suppose the optimal candidate pair is \((y_{k\ast}, y_{l\ast})\). The deterministic mechanism outputs \((y_k, y_l)\) which is different from the optimal one. The social cost achieved by the mechanism is upper bounded by

\[
SC((y_{k\ast}, y_{l\ast}), x) + (n - n_k - n_l)d_{max}
\]

The optimal social cost \(SC((y_{k\ast}, y_{l\ast}), x)\) is lower bounded by

\[
\frac{1}{2}(n - n_{k\ast} - n_{l\ast})d_{min}
\]

Consequently, the distortion is upper bounded by

\[
1 + 2\sigma \frac{n - n_k - n_l}{n - n_{k\ast} - n_{l\ast}}
\]

We decrease \((n_k + n_l)\) and increase \((n_{k\ast} + n_{l\ast})\) as much as possible. We set \(n_k + n_l = 2\) and \(n_{k\ast} + n_{l\ast} = n - 1\). Therefore, the distortion of Sequential Dictator is at most \(2(n - 2)\sigma + 1\).
The worst case is as follows. We use Instance 4 again. The first voter is located at \(y_1\) and the second is located at the midpoint of \((y_1, y_2)\). The others are located at \(y_4\). Precisely, the location profile is \(x = (x_1 = y_1, x_2 = (1/2, 1/2, 0), x_3 = \ldots = x_n = y_4)\). The second voter reports \(y_2\). Thus, Sequential Dictator outputs \((y_1, y_2)\). And the social cost achieved by the mechanism is \((n-2)d_{\text{max}} + 1/2*d_{\text{min}}\). Whereas, the optimal candidate pair could be \((y_1, y_4)\) and \(\text{OPT}(x) = 1/2*d_{\text{min}}\). The ratio is \(2(n - 2)\sigma + 1\).

5 Conclusion

In this paper, we are concerned with the strategy-proof scv mechanisms for two-winner election in Euclidean space. We show the distortion in both one-dimensional and multi-dimensional space. Remarkably, we propose the “Pair-Independent” mechanism which achieves asymptotically tight bound in multi-dimensional space. Many problems remain open. Can we have sufficient and necessary conditions like Lemma 1 for strategy-proof mechanisms in low dimension space? Are mechanisms using location information strictly more potent than mechanisms using single-candidate voting rule?
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