Local well-posedness of Euler–Poisson dominated molecular clouds in astrophysics

Chao Liu

Abstract. Motivated by the astrophysical problems of star formations from molecular clouds, we make the first step on the possible mathematical descriptions of molecular clouds. This article first obtain a model describing the physically reasonable molecular clouds by a proposal to remedy the defects of Makino’s star model and removing its exterior regularities (i.e., remove its imposed nonphysically exterior free-falling equations). Since some technical and tricky problems occur, this question is not easy and has been asked by Makino’s original paper [37] and Rendall’s [48] for a long time. We, in this article, establish a reasonable model by Euler–Poisson system for describing the evolution of molecular clouds and obtain the local existence, uniqueness and continuation principle of the classical solution. In the companion article [31], we will develop this description to the long time region, and conclude under certain data (without any symmetry), the developments of the molecular clouds are either global with near-boundary mass accretions (leads to star formations), or blowup at finite time.

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1. Introduction

Understanding the star formation and the evolution of molecular clouds are active and key quests in astrophysics recently (see [28,29,56] for introductions on stellar formation) and numerous studies in astrophysics (see, for example, [13,14,20,21,29,56] and references cited therein), both observational and theoretical, indicate that stars are formed or “born” in a truly vast cold interstellar gas clouds (or the giant molecular clouds\(^1\)). Intuitively, since the molecular clouds are enormous, usually containing up to a million solar masses of gas, the self-gravity is huge and may cause the interstellar gas to condense into new stars. Therefore, mathematically it is can be simplified as the blowup problem or the mass accretion problem of the density of perfect fluids (molecular clouds) with vacuum under Newtonian self-gravity.

Although the evolution of giant molecular clouds is a key ingredient for understanding of the star formation and there are numerous observational, numerical and theoretical works on this topic in astrophysical literature, the main formation mechanics and the origin of their physical conditions still remains uncertain and is an unsolved problem to date due to their extreme complexity (too flexible to capture the main properties) and the outstanding discrepancy between the observations and the theoretical models. In the meanwhile, to the best of our knowledge there is no literature in mathematics to rigorously model and analyze the fully nonlinear evolution of giant molecular clouds (it has been pointed out by Rendall [49, §7.2] that there are no results on Jeans instability available for the fully nonlinear case). We, in this article and the companion [31], attempt to make the first step on the mathematical model and nonlinear analysis of the large, expanding and irregularly-shaped self-gravitational molecular clouds and give possibilities of the star formation (mass accretions), fragmentation of the molecular clouds.

\(^1\)Although interstellar gas clouds and the molecular clouds are not exactly the same, we do not distinguish these two concepts throughout this article since we only focus on the fluid properties and the differences can be ignored. See Appendix E for more detailed meanings.
and other singularities. The complete proofs of the linearized and nonlinear Jeans instability are in progress.

Usually, the molecular cloud is extremely vast and some region of it is more hydrodynamics-dominated (like “fluid”) and other region is more ballistics-dominated (full of free particles, see Appendix E.2 for details). In this article, we aim to (1) use a natural boundary condition, free-fall (or diffuse boundary describing the diffuse clouds in astrophysics), to separate hydrodynamics-dominated domain from the ballistic one, (2) use the Euler–Poisson system to model the hydrodynamics-dominated part of molecular clouds, (3) analyze the local well-posedness and continuation principle. From now on, we only refer to the hydrodynamics-dominated region of clouds when we mention the molecular clouds, and the ballistic region can be viewed as vacuum (density of the fluid vanishes) by neglecting the pressure and density.

1.1. The diffuse boundary problem of molecular clouds. Let us begin with stating the following model of molecular clouds.

(i) Variables: The molecular clouds are characterized by the following variables: We use functions \( \rho : [0, T) \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0} \), \( p : [0, T) \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0} \), and \( \Phi : [0, T) \times \mathbb{R}^3 \to \mathbb{R} \), for some constant \( T > 0 \), to describe the distribution of the mass density, pressure of the fluids and the Newtonian potential of gas clouds, respectively, and denote \( \Omega(t) := \text{supp} \rho(t, \cdot) = \{ x \in \mathbb{R}^3 \mid \rho(t, x) > 0 \} \subset \mathbb{R}^3 \) is the changing volume occupied by the gas at time \( t \). Then the vacuum is identified by \( \rho(t, x) = 0 \) in \( \Omega^c(t) \) and the fluids by \( \rho(t, x) > 0 \) in \( \Omega(t) \). We focus on the isentropic ideal gas throughout this article, that is, the equation of state bridging the pressure \( p \) and the density \( \rho \) of the gas cloud is given by

\[
p = K \rho^\gamma \quad \text{for} \quad x \in \mathbb{R}^3
\]

where \( \gamma > 1 \) and \( K \in \mathbb{R}_{>0} \) are both given constants. The velocity of fluids is only defined on \( \Omega(t) \) and denoted by \( \mathbf{w} := (\mathbf{w}^i) : [0, T) \times \Omega(t) \to \mathbb{R}^3 \) (note that there is no definition in \( \Omega^c(t) \)).

We call \( \Omega(t) \) the hydrodynamic region and \( \Omega^c(t) \) the ballistic region (viewed as vacuum in this model, see Appendix §E.2 for reasons). In ballistic region, rarefied gas dynamics prevails and rarefied gas means there is no mass accretion, which, therefore, does not interest us due to the motivations of star formations. However, in the hydrodynamic region \( \Omega(t) \), the regime of hydrodynamics applies, and in order to clarify the evolution of the hydrodynamic region of clouds, we propose the following Euler–Poisson system with the diffuse boundary to frame out the hydrodynamic region and characterize this system. Later it turns out this system is well-posed and we emphasize again that we are interested in the potion of the hydrodynamic region of molecular clouds.

(ii) Euler–Poisson equations determine the developments of molecular clouds in \( \Omega(t) \) and the Newtonian potential on \( \mathbb{R}^3 \), that is,

\[
\partial_0 \rho + \mathbf{w}^i \partial_i \rho + \rho \partial_i \mathbf{w}^i = 0 \quad \text{in} \quad \Omega(t), \tag{1.2}
\]

\[
\rho \partial_0 \mathbf{w}^k + \rho \mathbf{w}^i \partial_i \mathbf{w}^k + \delta^{ik} \partial_k \rho = -\rho \partial^k \Phi \quad \text{in} \quad \Omega(t), \tag{1.3}
\]

\[
\Delta \Phi = \rho, \quad \text{in} \quad \mathbb{R}^3, \tag{1.4}
\]

for \( t \in [0, T) \). The Newtonian potential \( \Phi \) is given by

\[
\Phi(t, x) = -\frac{1}{4\pi} \int_{\Omega(t)} \frac{\rho(t, y)}{|x - y|} \, d^3y \quad \text{for} \quad (t, x) \in [0, T) \times \mathbb{R}^3, \tag{1.5}
\]

(iii) Initial data are prescribed by

\[
\rho(0, x) = \rho_0(x) \quad \text{for} \quad x \in \mathbb{R}^3 \quad \text{and} \quad \mathbf{w}^k(0, x) = \mathbf{w}^k_0(x) \quad \text{for} \quad x \in \Omega(0). \tag{1.6}
\]
We assume \( \Omega(0) \subset \mathbb{R}^3 \) is a precompact set throughout this article.

\( (iv) \) **Diffuse boundary** is defined by

\[
\lim_{(t,x) \to (\hat{t},\hat{x})} \left( \rho^{-1}(t,x) \partial_i p(t,x) \right) = 0, \tag{1.7}
\]

for any \( (t,x) \in [0,T) \times \Omega(t) \) and \( (\hat{t},\hat{x}) \in [0,T) \times \partial\Omega(t) \) where \( \partial\Omega(t) \) is the boundary of the volume \( \Omega(t) \), that is, the moving interface between the cloud and the exterior vacuum. This boundary condition\(^2\) characterizes the gradually vanishing fluid pressure per mass near the boundary.

\( (v) \) **Classical solution**: We define classical solutions of the diffuse boundary problem \((i)-(iv)\):

**Definition 1.1.** A set of functions \((\rho, \hat{\omega}^i, \Phi, \Omega(t))\) is called a classical solution to above problem \((i)-(iv)\) on \( t \in [0,T) \) for \( T > 0 \) if it solves the system \((i)-(iv)\) and satisfies the following conditions:

1. there is a \( C^1 \)-extension \( \hat{w}^i \) to the boundary, i.e., there exists\(^3\) \( w^i \in C^1([0,T) \times \overline{\Omega(t)}, \mathbb{R}^3) \), such that \( w^i(t,x) = \hat{w}^i(t,x) \) for \( (t,x) \in [0,T) \times \Omega(t) \);
2. \( \rho \in C^1([0,T) \times \mathbb{R}^3, \mathbb{R}) \) and \( \Phi \) is given by (1.5).

**Remark 1.2.** Let us define the sound velocity \( c_s \) by \( c_s^2 := \frac{dp}{d\rho} = K\gamma\rho^{\gamma-1} \). Therefore, the diffuse boundary (1.7) can be expressed as\(^4\)

\[
\lim_{(t,x) \to (\hat{t},\hat{x})} \partial_i c_s^2(t,x) = \lim_{(t,x) \to (\hat{t},\hat{x})} K\gamma\rho^{\gamma-1}(t,x) = 0
\]

for \( (t,x) \in [0,T) \times \Omega(t) \) and \( (\hat{t},\hat{x}) \in [0,T) \times \partial\Omega(t) \).

1.2. **Notation and convention.**

1.2.1. **Vectors and components.** We will use, unless otherwise stated, boldface, e.g., \( \mathbf{x}, \mathbf{y}, \mathbf{w} \) for Latins and the Greeks, e.g., \( \xi, \) to denote vectors and normal font with indices, e.g., \( x^i, y^j, \xi^i \) and \( w^i \), to denote the components of vectors (we also use these components to express the vector if it clear from the context).

1.2.2. **Indices and summation convention.** Throughout this article, unless stated otherwise, we adopt the notation system in general relativity (i.e. Einstein notation, see [55] for more). In specific, we use upper indices to represent components of vectors, lower indices to represent components of covectors, and use lower case Latin letters, e.g. \( i,j,k \), for spatial indices that run from 1 to 3, and lower case Greek letters, e.g. \( \alpha,\beta,\gamma \), for spacetime indices that run from 0 to 3 (we also denote time coordinate \( t \) by \( x^0 := t \)). We use **Einstein summation convention** as well. That is, when an index variable (i.e. dummy index) appears twice, once in an upper superscript and once in a lower subscript, in a single term and is not otherwise defined, it implies summation of that term over all the values of the index. For example, for a summation\(^5\)

\[\sum_{i=1}^{3} x^i \]

\( \rho \) this model simplifies diffuse clouds of astrophysics by ignoring the chemical and radiant complexities (diffuse clouds include only atoms and molecules), see Appendix E.1 for details.

\( w^i \in C^1([0,T) \times \overline{\Omega(t)}, \mathbb{R}^3) \) means \( \hat{w}^i \) and \( \partial_j \hat{w}^i \) have continuous extensions to \( [0,T) \times \overline{\Omega(t)} \).

\( n \) Comparing with the physical vacuum boundary condition of star models (see, for example, [11,12,22,34,35]),

\[ -\infty < \nabla_n(c_s^2) < 0 \quad \text{on} \quad \partial \Omega(t) \],

where \( n \) is the exterior unit normal vector to \( \partial \Omega(t) \), diffuse boundaries do not have such singularities on them, that is why they describe molecular clouds. In fact, the conclusion of the companion paper [31] shows there is a possibility that physical vacuum boundary is formed on the boundary or even in the interior of the clouds.
z = \sum_{i=1}^{3} x^i y_i$, we simply use Einstein notation to denote $z = x^i y_i$. In addition, we raise and lower indices by the Euclidean metric $\delta^{ij}$ and $\delta_{ij}$, that is, for example,

$$W^{ij} := W_{ik}\delta^{ij} \delta_{kj}, \quad W_{ij} := W^{ik}\delta_{0kj} \quad \text{and} \quad W^{ij} := W_{kl}^{i} \delta^{kj}.$$  

1.2.3. **Lagrangian descriptions.** Suppose a field $f : [0, T) \times \Omega(t) \rightarrow V$ (or a property $f$ in Eulerian description) and the flow $\chi : [0, T) \times \Omega(0) \rightarrow \Omega(t) \subset \mathbb{R}^3$ generated by a vector field $w := (w^i)$, such that $\chi(t, \xi) = x \in \Omega(t)$ for every $(t, \xi) \in [0, T) \times \Omega(0)$ where $T > 0$ is a constant, $\Omega(t) \subset \mathbb{R}^3$ is a domain depending on $t$ and $V \subset \mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 1}$, we denote

$$f(t, \xi) := f(t, \chi(t, \xi)) = f(t, x)$$

(1.8)

describing the property $f$ of the parcel labeled by the initial position $\xi$ at time $t$ (the property $f$ along the flow, i.e., in Lagrangian description). Using this notation, we have

$$\partial_t f(t, \xi) = D_t f(t, \xi)$$

where $D_t$ is the material derivative (i.e., $D_t := \partial_t + w^i \partial_i$, see, for instance, [10]). According to the definition of $L^{\infty}$, if $\chi(t, \Omega(0)) = \Omega(t)$, we conclude that

$$\|f(t)\|_{L^{\infty}(\Omega(t))} = \|f(t)\|_{L^{\infty}(\Omega(0))}.$$  

1.2.4. **Sets.** We denote $U$ is strictly contained in $V$ by $U \subset \subset V$, which represents the closure of $U$ is a compact subset of $V$. A set $U$ is called precompact if its closure $\overline{U}$ is compact. We denote the following subset $\Omega_\epsilon$ of $\Omega$ for some constant $\epsilon > 0$ by

$$\Omega_\epsilon := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) < \epsilon \},$$

(1.9)

where $\text{dist}(x, \partial \Omega) := \inf_{y \in \partial \Omega} |x - y|$ and denote $\hat{\Omega}_\epsilon := \Omega \setminus \Omega_\epsilon$. We also use $B(x_0, r)$ to denote a ball centered at $x_0$ with the radius $r$. Therefore, by (1.9), we denote

$$B_\epsilon(x_0, r) := \{ x \in B(x_0, r) \mid \text{dist}(x, \partial B(x_0, r)) < \epsilon \},$$

(1.10)

The complement set of a set $\Omega$ is denoted by $\Omega^c := \mathbb{R}^3 \setminus \Omega$.

1.3. **Preliminary concepts.** In order to state the main theorem concisely, we first introduce a few definitions.

1.3.1. **Decomposition of velocities.** For simplicity of notations, we denote

$$W^i_j(t, x) := w^i_j(t, x) := \begin{cases} \partial_j w^i(t, x) & \text{if } (t, x) \in [0, T) \times \Omega(t) \\ \lim_{(t', x') \to (t, x)} \partial_j w^i(t', x') & \text{if } (t, x) \in [0, T) \times \partial \Omega(t) \end{cases}.$$  

Let us lower the index of $W^i_j$ by $\delta_{ki}$, i.e., $W_{kj}(t, x) := \delta_{ki} W^i_j(t, x)$ and decompose

$$W_{jk}(t, x) = \frac{1}{2} (W_{jk}(t, x) + W_{kj}(t, x)) + \frac{1}{2} (W_{jk}(t, x) - W_{kj}(t, x)) = \Theta_{jk}(t, x) - \Omega_{jk}(t, x)$$

(1.11)

where $\Theta_{jk}$ is the symmetric deformation component ($\Theta_{jk} = \Theta_{kj}$) of $W_{jk}$ and $\Omega_{jk}$ the antisymmetric rotation component ($\Omega_{jk} = -\Omega_{kj}$) defined by

$$\Theta_{jk}(t, x) := \frac{1}{2} (W_{jk}(t, x) + W_{kj}(t, x))$$

(1.12)

$$\Omega_{jk}(t, x) := \frac{1}{2} (W_{kj}(t, x) - W_{jk}(t, x)).$$

(1.13)

Then we have the identities,

$$W_{kj} = \Theta_{jk} + \Omega_{jk} \quad \text{and} \quad W_{jk} = \Theta_{jk} - \Omega_{jk}.$$  

(1.14)
We also denote the divergence of the velocity
\[ \Theta(t, x) := \delta^{jk} \Theta_{jk}(t, x) = \delta^{jk} W_{jk}(t, x) = w_j^i(t, x). \] (1.15)

1.4. Main Theorem. After above concepts, we are in a position to present the main theorem. This theorem states the local existence, uniqueness, continuation principle.

**Theorem 1.3** (Main Theorem). Suppose the initial data \((\rho_0, \omega_0^i)\) of the diffuse boundary problem is given by \((1.6), \Omega(0)\) is a precompact \(C^1\)-domain, \(\rho_0 \in C^0(\mathbb{R}^3, \mathbb{R}_{\geq 0}), \omega_0^i \in C^1(\Omega(0), \mathbb{R}^3)\). If

\[(a)\] \[1 < \gamma \leq \frac{5}{3}, \quad (\rho_0)^{\frac{\gamma-1}{\gamma}} \in H^3(\mathbb{R}^3) \quad \text{and} \quad \omega_0^i \in H^3(\Omega(0), \mathbb{R}^3),\]

or if

\[(b)\] \[1 < \gamma < 2, \quad \rho_0 \in H^3(\mathbb{R}^3), \quad (\rho_0)^{\frac{\gamma-1}{\gamma}} \in H^4(\mathbb{R}^3) \quad \text{and} \quad \omega_0^i \in H^4(\Omega(0), \mathbb{R}^3),\]

then:

1. (Local existence and uniqueness) there is a constant \(T > 0\), such that the diffuse boundary problem \((i)-(iv)\) has a unique classical solution \((\rho, \omega^i, \Phi, \Omega(t))\) defined by Definition 1.1. Moreover, \(\rho^{\frac{\gamma-1}{\gamma}} \in C^{s-2}([0, T) \times \mathbb{R}^3)\) and \(\Phi \in C^2([0, T) \times \mathbb{R}^3)\), and the solution \(\omega^i\) satisfies

\[\omega^i \in C^0([0, T), H^s(\Omega(t))) \cap C^1([0, T), H^{s-1}(\Omega(t)))\]

where \(s = 3, 4\) according \((a)\) and \((b)\) respectively;

2. (Strong continuation principle) if there is a constant \(\epsilon > 0\) such that an estimate holds,

\[\int_0^{T_\epsilon} \left( \|W_{jk}(s)\|_{L^\infty(\Omega(s))} + \|\Theta(s)\|_{L^\infty(\bar{\Omega}(s))} + \|\omega_{jk}(s)\|_{L^\infty(\bar{\Omega}(s))} + \|\nabla \rho^{\frac{\gamma-1}{\gamma}}(s)\|_{L^\infty(\Omega(s))} \right) ds < \infty,\]

for any \(T_\epsilon \in (0, T)\) where \(\Omega(s) := \Omega(s) \setminus \Omega_\epsilon(s)\). Then there is a constant \(T^\star > T > 0\), such that the classical solution \((\rho, \omega^i, \Phi, \Omega(t))\) exists on \(t \in [0, T^\star)\).

1.5. Related work. To our knowledge, the Euler–Poisson system with vacuum was started with Makino [37, 40], which gave the local existence theorem by introducing the Makino’s density and the unphysical concept of tame solutions. Later on, a series of works [33, 38, 39, 41, 47] continued to study the evolution of tame solutions to Euler equation with or without gravity. Note that the blowup results of Euler–Poisson system with vacuum [38, 39, 47] of tame solutions are under the condition of spherical symmetry, but there is no symmetric assumptions for Euler equations in [41]. The advantage of this method is easy to prove blowup, but usually, it does not give more information about the detailed behavior of the finite time blowup solutions. We point out that [38] specified Makino’s conjecture that any tame solution, including nonsymmetric tame solution, will become not tame after a finite time, and this current article can be viewed as a partial answer. Oliynyk [44], Brauer and Karp [6] and the references cited therein have improved the local existence and uniqueness of solutions to Euler–Poisson(–Makino) system with vacuum. Brauer [5], by generalizing the method in [4, 9], gives a strong continuation principle for tame solutions, especially, a more detailed classification of the type of blowups. In our companion article [31], we will develop this description to the long time region, and conclude under certain data (without any symmetry), the developments of the molecular clouds are either global with near-boundary mass accretions (leads to star formations), or blowup at finite time.
1.6. Overview and main ideas. §2 aims to prove the main Theorem 1.3.(1), that is, the local existence and uniqueness theorem of the classical solution to the diffuse boundary problem. The idea of the proof is to connect the classical solution of diffuse boundary problem with the tame solution of the Makino’s initial value problem (see §2.1). Then we introduce the local existence of the tame solution of Makino’s problem in §2.2. The third step is to prove the local existence of the solution to the diffuse boundary problem by Calderón’s extensions of some data of classical solutions and transforming the solution of the diffuse boundary problem to the tame solution of Makino’s problem. The uniqueness of the classical solution is given in §2.5 based on Lemma B.1 (see Appendix B, a Lemma proven by using the relative entropy-entropy flux method given in [35, §2, Step 2]) and Stein Extension Theorem. One ingredient of proving the uniqueness theorem is the preservation of the regularity of the initial $C^1$ boundary and this will be presented in §2.4.

In §3, we introduce the continuation principles of the diffuse boundary problem. These continuation principles allow us to continue the local solutions to a larger time interval provided the suitably bounded spatial derivatives. First we give the weak version (see §3.1) and it will help us extend the classical solutions to several types of singularities and the idea of the proof is to use Makino’s formulation of Euler–Poisson system and energy estimates. However, the standard Gagliardo–Nirenberg–Moser estimates have been replaced by the revised Gagliardo–Nirenberg–Moser estimates on the Lipschitz bounded domain. The strong continuation principle (see §3.2) helps us reduce the types of singularities and remove certain singularities on the boundary. The key tool of the proof for the strong continuation principle is Ferrari–Shirota–Yanagisawa inequality (a bounded domain version of Beale–Kato–Majda estimate). This inequality allows us to control the velocity $\|u^i\|_{W^{1,\infty}}$ “almost only” by the $L^\infty$-norms of expansion and rotation under certain assumptions. In order to use Ferrari–Shirota–Yanagisawa (FSY) inequality, we have to prove a $C^\infty$-approximation lemma C.6 of boundary in Appendix C.2 since FSY inequality requires domain is of class $C^{2,\beta}$, $\beta \in (0,1)$ and $w^i\delta_{ij}n^j = 0$ on $\partial\Omega(t)$ and $n^j$ is the unit outward normal at $x \in \partial\Omega(t)$. To overcome these two requirements and use FSY inequality, our idea is, near the boundary $\partial\Omega(t)$, we select a smooth surface ($C^\infty$-approximation lemma of boundary C.6 ensures this) and decompose the velocity field to two parts around this $C^\infty$ surface, one of these parts is “tangent” to the $C^\infty$ surface on it, then analyze these two parts respectively.

2. Local existence and uniqueness of solutions to diffuse boundary problem

This section contributes to the local existence and uniqueness theorem of the classical solution to the diffuse boundary problem. The idea of the proof is to connect the classical solution of diffuse boundary problem with the tame solution of the Makino’s initial value problem (see §2.1). Then we introduce the local existence of the tame solution of Makino’s problem in §2.2. The third step is to prove the local existence of the solution to the diffuse boundary problem by Calderón’s extensions of some data of classical solutions and transforming the solution of the diffuse boundary problem to the tame solution of Makino’s problem. The uniqueness of the classical solution is given in §2.5 based on Lemma B.1 (see Appendix B, a Lemma proven by using the relative entropy-entropy flux method given in [35, §2, Step 2]) and Stein Extension Theorem. One ingredient of proving the uniqueness theorem is the preservation of the regularity of the initial $C^1$ boundary and this will be presented in §2.4.

2.1. Regular solutions to the initial value problem of Euler–Poisson system. We have given the definition of diffuse boundary (see (1.7)) in the previous section §1.1, and the following proposition claims there are two useful equivalent expressions of this boundary which help us
translate the diffuse boundary problem to a initial value problem and the “Makino’s problem”. The proof of Proposition 2.1 is given in Appendix A for avoiding interrupting the continuity of the statements of this section and please note that we have no any regularity condition on \([0, T) \times \partial \Omega(t)\) in Proposition 2.1.

**Proposition 2.1.** Suppose \(\rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\) and \(w^i \in C^1([0, T) \times \overline{\Omega(t)}, \mathbb{R}^3)\) solve (1.2)–(1.3), \(\Phi\) is given by (1.5), (1.1) holds and \(\gamma > 1\). Then, the following three conditions are equivalent:

1. **Diffuse boundary:**
   \[
   \lim_{(t,x) \to (\bar{t}, \bar{x})} (\rho^{-1}(t,x) \partial_t \rho(t,x)) = 0 \tag{2.1}
   \]
   for any \((t, x) \in [0, T) \times \Omega(t)\) and \((\bar{t}, \bar{x}) \in [0, T) \times \partial \Omega(t)\);

2. **Regularity condition:** \(\rho^{\gamma-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\);

3. **Free-fall boundary:** the velocity of the limiting Lagrangian observer on the boundary satisfies the free-fall equation, i.e., for \((t, \xi) \in [0, T_s) \times \partial \Omega(0)\) where \(T_s \in (0, T)\), \(w^k(t, \xi) := w^k(t, \chi(t, \xi)) = w^k(t, x)\) satisfies
   \[
   \partial_t w^k(t, \xi) = -\partial^k \Phi(t, \xi) \tag{2.2}
   \]

**Remark 2.2.** Note usually (2.2) is not equivalent to the following condition (in fact, this is stronger than (2.2)),

\[
\partial_t w^k(t, x) + w^i(t, x) \partial_i w^k(t, x) = -\partial^k \Phi(t, x).
\]

for \((t, x) \in [0, T) \times \partial \Omega(t)\). Since for the diffuse boundary problem, the (extended) velocity \(w^k\) is only defined in \([0, T) \times \overline{\Omega(t)}\), there are directions \(\nu^\mu \in \mathbb{R}^3\), such that \(\partial_{\nu^\mu} w^k|_{[0, T) \times \partial \Omega(t)}\) is not defined. However, \(\partial_t w^k(t, \xi) = D_t w^k(t, \xi)\) is well defined for \((t, \xi) \in [0, T_s) \times \partial \Omega(0)\) (see Appendix A). In addition, (2.2) can be expressed as \(D_t w^k(t, x) = -\partial^k \Phi(t, x)\).

Next we point out if we introduce a concept of regular solutions to the initial value problem of Euler–Poisson system (i)–(iii), then finding the classical solution of diffuse/free-fall boundary problem (i)–(iv) is equivalent to finding the regular solution of the initial value problem (i)–(iii). Let us first give the definition of the regular solution.

**Definition 2.3.** A set of functions \((\rho, \dot{w}^i, \Phi, \Omega(t))\) is called a regular solution\(^5\) to the initial value problem of Euler–Poisson system (i)–(iii) on \(t \in [0, T)\) for \(T > 0\) if it solves the system (i)–(iii) and satisfies the following conditions:

1. there is a \(C^1\)-extension \(\hat{w}^i\) of \(\dot{w}^i\) to the boundary, i.e., there exists \(\hat{w}^i \in C^1([0, T) \times \overline{\Omega(\bar{t})}, \mathbb{R}^3)\), such that \(\hat{w}^i(t, x) = \dot{w}^i(t, x)\) for \((t, x) \in [0, T) \times \Omega(t)\);

2. \(\rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\) and \(\Phi\) is given by (1.5);

3. \(\rho^{\gamma-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\).

Then, let us state the proposition transforming the diffuse boundary problem to a initial value problem.

**Proposition 2.4.** Finding the classical solution of diffuse/free-fall boundary problem (i)–(iv) is equivalent to finding the regular solution of the initial value problem (i)–(iii).

**Proof.** This proposition can be proved by directly using Proposition 2.1 and Definition 2.3. \(\square\)

\(^5\)Note this regular solution is different with Liu and Yang’s regular solution in [33]. Ours given here does not require the velocity satisfying the free-falling equation on the exterior area of \(\Omega(t)\).
To avoid deviating the main objective of this article (the long time behaviors of the classical solutions) too far, we leave the complete analysis of the local existence of the classical solution of diffuse/pressureless/free-fall boundary problem (i)–(iv) (or equivalently, the regular solution of the initial value problem (i)–(iii)) to another separated paper later, but here only focus on the local existence of a type of “stronger solutions” of the diffuse boundary problem by using Makino’s tame solution (this means the initial data requires additional conditions). We call this “stronger solution” as Makino solution (roughly, it can be regarded as a restriction of Makino’s tame solution on the support of density) and it is defined by the following Definition 2.5 which is a revised regular solution by replacing Definition 2.3.(3) to be the following stronger condition (3*).

**Definition 2.5.** A set of functions \((\rho, \bar{\omega}^i, \Phi, \Omega(t))\) is called a Makino solution to the initial value problem of Euler–Poisson system (i)–(iii) on \(t \in [0, T)\) for \(T > 0\) if it solves the system (i)–(iii) and satisfies the following conditions:

1. there is a \(C^1\)-extension \(w^i\) of \(\bar{w}^i\) to the boundary, i.e., there exists \(w^i \in C^1([0, T) \times \Omega(t), \mathbb{R}^3)\), such that \(w^i(t, x) = \bar{w}^i(t, x)\) for \((t, x) \in [0, T) \times \Omega(t)\);
2. \(\rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\) and \(\Phi\) is given by (1.5);
3. \(\rho^{-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\).

Then immediately, we claim

**Proposition 2.6.** Makino solutions to the initial value problem of Euler–Poisson system (i)–(iii) on \(t \in [0, T)\) for \(T > 0\) are regular solutions to the initial value problem of Euler–Poisson system (i)–(iii) on \(t \in [0, T)\) and furthermore, they are classical solutions of diffuse/free-fall boundary problem (i)–(iv) as well.

**Proof.** Noting \(\partial_t \rho \gamma^{-1} = 2 \rho \frac{\gamma^{-1}}{\gamma} \partial_t \rho \frac{\gamma^{-1}}{\gamma}\), we conclude \(\rho^{\gamma^{-1}} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\) implies \(\rho^{\gamma^{-1}} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\). Then using Definition 2.3 and Proposition 2.4, we complete this proof. \(\Box\)

2.2. Local existence of tame solution to the Makino problem. Before discussing the classical solutions of diffuse boundary problem, we first recall the relevant tame solutions and their local existences given by Makino [38–40]. The Cauchy problem of Euler–Poisson equations with variables defined in \([0, T) \times \mathbb{R}^3\) for some constant \(T > 0\) is given by

\[
\begin{align*}
\partial_t \rho + w^i \partial_i \rho + \rho \partial_t w^i &= 0 & \text{in} & \ [0, T) \times \mathbb{R}^3, \\
\rho \partial_t w^k + \rho w^i \partial_i w^k + \delta^{ik} \partial_k \rho &= - \rho \partial_k \Phi & \text{in} & \ [0, T) \times \mathbb{R}^3, \\
\Delta \Phi &= \rho & \text{in} & \ [0, T) \times \mathbb{R}^3.
\end{align*}
\]  

where \(\rho\) and \(p\) are defined as the same as §1.1.(i), and the velocity \(w := (w^k) : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^3\). The initial data are prescribed by

\[
(\rho, w^k) = (\rho_0, w^k_0), \quad \text{on} \quad \{0\} \times \mathbb{R}^3, 
\]

and \(\Omega(0)\) is precompact.

**Definition 2.7.** A set of functions \((\rho, w^i, \Phi)\) is called a tame solution\(^6\) of Cauchy problem (2.3)–(2.6) on \(t \in [0, T)\) for \(T > 0\), if it solves the system (2.3)–(2.6) and satisfies the following conditions:

1. \((\rho, w^i) \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}^4)\) and \(\Phi\) is given by (1.5);

\(^6\)Also known as “sweet” or “gentle” solutions by Makino, and “Makino” solutions by Brauer [5].
Blowups and long-time developments of irregularly-shaped molecular clouds

(2) \( \rho^{\frac{\gamma}{\gamma-1}} \in C^1([0, T) \times \mathbb{R}^3) \) and \( w^i \) satisfies an extra equation of the free falling,

\[
D_t w^i(t, x) = \partial_i w^i(t, x) + w^k(t, x) \partial_k w^i(t, x) = -\partial_i \Phi(t, x),
\]

for \( (t, x) \in [0, T) \times \Omega^c(t). \)

We refer to, throughout this article, the problem of finding the tame solutions of Euler–Poisson equation (2.3)–(2.6) as solving the Makino problem.

Remark 2.8. The equation of the free falling (2.7) can be interpreted as an equation of motion of a test particle. A test particle is an useful idealized model of an object whose mass is assumed to be negligible, that is, the mass is considered to be insufficient to alter the behavior of the rest of the system. (2.7) expounds that a test particle moves along the integral curves of its velocity field \( w^i(t, x(t)) \) and the acceleration of this test particle is caused only by the Newtonian gravity.

A standard method to obtain the local existence of Euler–Poisson system is to transform this system to a symmetric hyperbolic system, then the theory of symmetric hyperbolic systems can be applied to derive the local existence. However, there is a severe difficulty when vacuum appears. The vanishing of density leads to the degenerated or unbounded coefficients of the symmetric system (detailed explanations of this difficulty and the ideas on how to overcome this difficulty can be found, for example, in [6,32,37]). Let us briefly recall Makino’s ideas [37] in this section. In order to avoid above difficulty, we introduce the Makino’s density

\[
\rho = \left(4K \frac{\gamma}{(\gamma - 1)^2}\right)^{-\frac{1}{\gamma-1}} \alpha^{\frac{2}{\gamma-1}},
\]

that is

\[
\alpha = \frac{2\sqrt{K\gamma}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}}.
\]

Then, we rewrite (2.3)–(2.5), with the help of Makino density \( \alpha \), as a symmetric hyperbolic system with a nonlocal term \( \partial^k \Phi \),

\[
\begin{align*}
\partial_0 \alpha + w^i \partial_i \alpha + \frac{\gamma - 1}{2} \alpha \partial_i w^i &= 0, \\
\partial_0 w^k + w^i \partial_i w^k + \frac{\gamma - 1}{2} \alpha \delta^{ik} \partial_i \alpha &= -\partial^k \Phi,
\end{align*}
\]

with initial data

\[
\begin{align*}
(\alpha, w^i) &= (\alpha_0, w^i_0), & \text{on} & \{0\} \times \mathbb{R}^3,
\end{align*}
\]

Then using above formulation, by constructing contraction mappings and with the help of fixed point theorem, Makino [37] arrived at the following local existence theorem. We omit the detailed proofs which appeared in [37].

Theorem 2.9. (Local Existence of Tame Solution, see [37]) Assume the initial data \( (\rho_0, w^i_0) \in C^1(\mathbb{R}^3, \mathbb{R}^4), \rho_0 \geq 0 \) and has compact support. If

(a) \( 1 < \gamma \leq \frac{5}{3} \), \( (\rho_0)^{\frac{\gamma-1}{2}} \in H^3(\mathbb{R}^3) \) and \( w^i_0 \in H^3(\mathbb{R}^3, \mathbb{R}^3) \),

or if

(b) \( 1 < \gamma < 3 \), \( \rho_0 \in H^3(\mathbb{R}^3), \ (\rho_0)^{\frac{\gamma-1}{2}} \in H^4(\mathbb{R}^3) \) and \( w^i_0 \in H^4(\mathbb{R}^3, \mathbb{R}^3) \),
then there is a constant $T > 0$, such that the Cauchy problem (2.3)–(2.6) has a tame solution $(\rho, w^i) \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ and $\Phi \in C^2([0, T) \times \mathbb{R}^3)$, and the solution $(\rho^{\frac{\gamma-1}{\gamma}}, w^i)$ satisfies
\[ (\rho^{\frac{\gamma-1}{\gamma}}, w^i) \in C^0([0, T), H^s(\mathbb{R}^3)) \cap C^1([0, T), H^{s-1}(\mathbb{R}^3)) \]

where $s = 3, 4$ according (a) and (b) respectively.

**Remark 2.10.** Makino’s tame solutions requires a stronger regularity on the initial data of $\rho_0$, i.e., $\rho_0^{\frac{\gamma-1}{\gamma}} \in H^s$ rather than $\rho_0^{\gamma-1} \in H^s$ (recall Definition 2.3.(3) of regular solutions). There are some improvements of the regularities of solutions with extra exterior velocity constraints in [6,33,44]. However, for simplicity, we currently only use above cited Makino’s local existence theorem 2.9 directly for the followings and leave improvements in near future.

### 2.3. Local existence of classical solutions to the diffuse boundary problem

In this section, we give the local existence theorem of the classical solution to the diffuse boundary problem under certain initial data.

We first prove the following lemma which states the restrictions onto $(t, x) \in [0, T) \times \overline{\Omega(t)}$ of the tame solution $(\rho, w^i)$ of the Makino problem (2.3)–(2.6) is a regular solution to the diffuse boundary problem (1.2)–(1.1).

**Lemma 2.11.** Suppose there is a constant $T > 0$ and $(\rho, w^i, \Phi)$ is a tame solution to the system (2.3)–(2.6). Define $\Omega(t) := \text{supp} \rho(t, \cdot)$ and a function $\hat{w}^i : [0, T) \times \Omega(t) \rightarrow \mathbb{R}^3$ satisfying that
\[ \hat{w}^i(t, x) := w^i(t, x) \]
for $(t, x) \in [0, T) \times \Omega(t)$. Then $(\rho, \hat{w}^i, \Phi, \Omega(t))$ is a Makino solution and further a regular solution of the initial value problem (i)–(iii).

**Proof.** To conclude this Lemma, we only need to compare the Definition 2.5 of Makino solutions and Definition 2.7 of tame solutions. Since $w^i \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ and $\hat{w}^i = w^i$ in $[0, T) \times \Omega(t)$, then there is an extension $\hat{w}^i|_{[0, T) \times \Omega(t)}$ of $\hat{w}^i$ to the boundary $\partial \Omega(t)$ and $w^i|_{[0, T) \times \Omega(t)} \in C^1([0, T) \times \overline{\Omega(t)} \times \mathbb{R}^3)$. Then, we complete the proof. $\square$

Now let us give the local existence theorem of classical solutions to the diffuse boundary problem (i)–(iv) on $t \in [0, T)$ for $T > 0$.

**Theorem 2.12** (Local Existence Theorem). Suppose the initial data $(\rho_0, \hat{w}_0)$ of the diffuse boundary problem (i)–(iv) is given by (1.6), $\Omega(0)$ is precompact and satisfies the Calderón’s uniform cone condition (see Appendix C, Definition C.2), $\rho_0 \in C^1(\mathbb{R}^3, \mathbb{R}_{\geq 0})$, $\hat{w}_0 \in C^1(\Omega(0), \mathbb{R}^3)$. If
\[ (a) \quad 1 < \gamma \leq \frac{5}{3}, \quad (\rho_0)^{\frac{\gamma-1}{\gamma}} \in H^3(\mathbb{R}^3) \quad \text{and} \quad \hat{w}_0^i \in H^3(\Omega(0), \mathbb{R}^3), \]
or if
\[ (b) \quad 1 < \gamma < 2, \quad \rho_0 \in H^3(\mathbb{R}^3), \quad (\rho_0)^{\frac{\gamma-1}{\gamma}} \in H^4(\mathbb{R}^3) \quad \text{and} \quad \hat{w}_0^i \in H^4(\Omega(0), \mathbb{R}^3), \]

then there is a constant $T > 0$, such that the diffuse boundary problem (i)–(iv) has a classical solution $(\rho, \hat{w}^i, \Phi, \Omega(t))$ defined by Definition 1.1. Moreover, $(\rho^\frac{\gamma-1}{\gamma}) \in C^\gamma([0, T) \times \mathbb{R}^3)$ and $\Phi \in C^2([0, T) \times \mathbb{R}^3)$, and the solution $w^i$ satisfies
\[ \hat{w}^i \in C^0([0, T), H^s(\Omega(t))) \cap C^1([0, T), H^{s-1}(\Omega(t))) \]
where $s = 3, 4$ according (a) and (b) respectively.
Proof. We take three steps to prove this theorem:

Step 1: Extend $\tilde{w}_0^i$ from $Ω(0)$ to $\mathbb{R}^3$. Since $Ω(0)$ satisfies the Calderón’s uniform cone condition, by applying the Calderón extension theorem (see Appendix, Theorem D.2), there exists a simple $(s,2)$-extension operator $E : H^s(Ω(0)) → H^s(\mathbb{R}^3)$ (for $s = 3, 4$), such that $w_0^i := E\tilde{w}_0^i = \tilde{w}_0^i$ in $Ω(0)$ for every $\tilde{w}_0^i ∈ H^s(Ω(0))$, and $\|w_0^i\|_{H^s(\mathbb{R}^3)} ≤ K\|\tilde{w}_0^i\|_{H^s(Ω(0))} < ∞$, i.e., the extended data $w_0^i ∈ H^s(\mathbb{R}^3) ⊂ C^4(\mathbb{R}^3)$.

Step 2: Makino’s local existence implies an extended regular solution. Using the extended data $(ρ_0, w_0^i) ∈ C^4(\mathbb{R}^3, \mathbb{R}^4)$ and noting that $(ρ_0, w_0^i)$ satisfies the requirements (a) or (b) in Theorem 2.9, Theorem 2.9 implies there is a constant $T > 0$, such that the Cauchy problem (2.3)–(2.6) has a tame solution $(ρ, w^i) ∈ C^4([0, T) × \mathbb{R}^3, \mathbb{R}^4)$ and $Φ ∈ C^2([0, T) × \mathbb{R}^3)$, which also implies $ρ_0, w^i ∈ C^{s−2}([0, T) × \mathbb{R}^3)$ by the Definition 2.7 of tame solutions.

Step 3: Restraine tame solutions to be classical solutions. Now we have tame solutions $(ρ, w^i, Φ)$ to the Makino’s problem (2.3)–(2.6). by defining $Ω(t) := \text{supp}\ ρ(t, ·)$ and a function $\hat{w}^i : [0, T) × Ω(t) → \mathbb{R}^3$ such that $\hat{w}^i(t, x) := w^i(t, x)$, for $(t, x) ∈ [0, T) × Ω(t)$. We are able to use Lemma 2.11, with the help of Proposition 2.6, to conclude $(ρ, \hat{w}^i, Φ, Ω(t))$ is a regular solution of the initial value problem (i)–(iii) and a classical solutions of diffuse/free-fall boundary problem (i)–(iv) as well.

In the end, noting that for any $H^s(\mathbb{R}^3)$-function $f$, $\|f\|_{H^s(Ω(t))} ≤ \|f\|_{H^s(\mathbb{R}^3)}$ due to the definition of the Calderón extension, we complete the proof. □

Remark 2.13. There may be classical solutions of diffuse boundary problem (or regular solution to the initial value problem) which is not Makino solutions. However, if there is a Makino solution, by Uniqueness Theorem 2.21 later, it is the only solution and there is no other non-Makino classical solution. By weakening the current data, it is possibilities that there is a unique non-Makino solution.

2.4. Preservation of $C^1$-boundary. Although it is possible to weaken the regularity of the boundary by more meticulous considerations, for simplicity and emphasizing the mechanisms of blowups, throughout this article, we assume the initial boundary $∂Ω(0)$ of the support of the fluids is a $C^1$ boundary, and we, in this section, prove $C^1$ regularity of this boundary preserves during the evolution, i.e., $∂Ω(t) ∈ C^1$ for $t$ in the existence interval of the classical solution.

Lemma 2.14. Suppose $Ω_0 ⊂ \mathbb{R}^3$ is a precompact domain satisfying $∂Ω_0 ∈ C^1$, $C^1 ⊃ w := (w^i) : (T_0, T_1) × \mathbb{R}^3 → \mathbb{R}^3$ $(T_0 < 0 < T_1)$ is a vector field. Then

(1) (existence of flow) there exists a bordered subset $[0, T) × [0, T) × \mathcal{D} ⊂ [0, T) × [0, T) × \mathbb{R}^3$ where $T ≤ T_1$ and $\mathcal{D} ⊃ Ω_0$ and a unique time-dependent flow $C^1 ⊃ φ : [0, T) × [0, T) × \mathcal{D} ⊃ (t, t_0, ξ) → x ∈ \mathbb{R}^3$ generalized by the vector field $w$;

(2) (integral curves) for every $ξ ∈ \mathcal{D}$, the curve $φ(0,ξ) : φ(·, 0, ξ) : [0, T) → \mathbb{R}^3$ is the unique maximal integral curve of the vector field $w$ starting at $(0, ξ)$ (i.e., $φ(0,ξ)(0) = ξ$);

(3) (inverse) for every $(t, t_0) ∈ [0, T) × [0, T)$, $φ(t,t_0) := φ(t,t_0, ·) : \mathbb{R}^3 → \mathbb{R}^3$ is a $C^1$ diffeomorphism with inverse $φ(t_0, ·)$;

(4) (diffeomorphism invariance of the boundary) denote $Ω_t := φ(t,0)(Ω_0)$, then $∂Ω_t = φ(t,0)(∂Ω_0)$ and $\mathbb{R}^3 \setminus Ω_t = φ(t,0)(\mathbb{R}^3 \setminus Ω_0)$;

(5) (regularity of evolutional boundary) $∂Ω_t ∈ C^1$;

(6) (foliations) the lateral boundary $[0, T) × ∂Ω_t$ is foliated by integral curves starting from $∂Ω_0$, that is, $[0, T) × ∂Ω_t = \{(t, φ(0,ξ)(t)) \mid t ∈ [0, T), ξ ∈ ∂Ω_0\}$.

Proof. (1), (2) and (3) are directly consequence of the fundamental theorem on flows (see Theorem C.12 and C.13).
(4) is due to the diffeomorphism invariance of the boundary, see Theorem C.11 (see Appendix C).

(5) Since \( \partial \Omega_0 \in C^1 \), by Definition C.1 and Theorem C.4 (see Appendix C), implies that for every point \( \xi_0 \in \partial \Omega_0 \), there is a ball \( B_r(\xi_0) \) and a bijective mapping \( \psi : B_r(\xi_0) \to B \subset \mathbb{R}^3 \) such that (1) \( \psi(B_r(\xi_0) \cap \Omega_0) \subset \mathbb{R}^3_+ \); (2) \( \psi(B_r(\xi_0) \cap \partial \Omega_0) \subset \partial \mathbb{R}^3_+ \); (3) \( \psi \in C^1 \) and \( \psi^{-1} \in C^1 \). It is direct that \( \varphi \in C^1 \) yields \( \varphi^{(t,0)} \in C^1(\mathbb{R}^3, \mathbb{R}^3) \). Then for any point \( x_0 \in \partial \Omega_t \), there is \( \xi_0 \in \partial \Omega_0 \), some neighborhood \( U(x_0) \) of \( x_0 \) and \( r, R > 0 \), such that \( x_0 := \varphi^{(t,0)}(\xi_0) \) and \( B_R(x_0) \subset U(x_0) = \varphi^{(t,0)}(B_r(\xi_0)) \) (since \( \varphi^{(t,0)} \) is a \( C^1 \) diffeomorphism with inverse \( \varphi^{(t,0)} \) by \( (1) \)). Then there is a bijective mapping \( \psi \circ (\varphi^{(t,0)})^{-1} : B_R(x_0) \to B \subset \mathbb{R}^3 \) such that, we claim, (1) \( \psi \circ (\varphi^{(t,0)})^{-1}(B_R(x_0) \cap \Omega_t) \subset \mathbb{R}^3_+ \); (2) \( \psi \circ (\varphi^{(t,0)})^{-1}(B_R(x_0) \cap \partial \Omega_t) \subset \partial \mathbb{R}^3_+ \); (3) \( \psi \circ (\varphi^{(t,0)})^{-1} \in C^1 \) and \( ((\varphi^{(t,0)})^{-1} \circ \psi)^{-1} \in C^1 \). We expound (1) explicitly and then (2) can be similarly proved. The inclusion relations of mappings, with the help of that \( \varphi^{(t,0)} = \varphi^{(t,0)}(\Omega_0) \) and \( \partial \Omega_t = \varphi^{(t,0)}(\partial \Omega_0) \) by \( (4) \), indicate that

\[
(\varphi^{(t,0)})^{-1}(B_R(x_0) \cap \Omega_t) \subset (\varphi^{(t,0)})^{-1}(B_R(x_0)) \cap (\varphi^{(t,0)})^{-1}(\Omega_t) \subset (\varphi^{(t,0)})^{-1}(U(x_0)) \cap \Omega_0 = B_r(\xi_0) \subset \Omega_0.
\]

Then using \( \psi(B_r(\xi_0) \cap \Omega_0) \subset \mathbb{R}^3_+ \), we conclude that \( \psi \circ (\varphi^{(t,0)})^{-1}(B_R(x_0) \cap \Omega_t) \subset \mathbb{R}^3_+ \). This, by Definition C.1 and Theorem C.4 again, concludes that \( \partial \Omega_t \in C^1 \).

(6) Using the previous \( (4) \),

\[
[0, T) \times \partial \Omega_t = \big[ 0, T \big) \times \varphi^{(t,0)}(\partial \Omega_0) = \big\{ (t, \varphi(t, 0, \xi)) \mid t \in [0, T), \xi \in \partial \Omega_0 \big\} = \big\{ (t, \varphi^{(0, \xi)}(t)) \mid t \in \big[ 0, T \big), \xi \in \partial \Omega_0 \big\},
\]

which means all the integral curves \( \varphi^{(0, \xi)}(t) \) (for all \( t \in [0, T) \) and \( \xi \in \partial \Omega_0 \)) form the lateral boundary \( [0, T) \times \partial \Omega_t \).

This lemma gives a geometric preparation for the boundary evolution of the diffuse boundary problem. Under the assumptions of the local existence Theorem 2.12, in order to prove \( \partial \Omega(t) \in C^1 \), we have the velocity field to play the role of above \( C^1 \) vector field, then we have \( \partial \Omega_t \in C^1 \) by using Lemma 2.14 and letting \( \Omega_0 = \Omega(0) \). However, this has not implied \( \partial \Omega(t) \in C^1 \) yet, since, by recalling the definition, \( \Omega(t) := \text{supp} \rho(t, \cdot) = \{ x \in \mathbb{R}^3 \mid \rho(t, x) > 0 \} \). Therefore, to conclude \( \partial \Omega(t) \in C^1 \), we have to verify \( \varphi^{(t,0)}(\Omega(0)) = \Omega_t = \Omega(t) = \text{supp} \rho(t, \cdot) \).

**Theorem 2.15.** Suppose \( (\rho, \dot{w}, \Phi, \Omega(t)) \) is a classical solution to the diffuse boundary problem \( (i) \sim (iv) \) on \( t \in [0, T) \) for \( T > 0 \), if \( \partial \Omega(t) \) is a \( C^1 \) boundary, then \( \partial \Omega(t) \) is of \( C^1 \) for \( t \in [0, T) \) as well.

**Proof.** Firstly, let us derive a flow and \( \Omega_t \). Step 1 and 2 from the proof of Theorem 2.12 imply there is a vector field \( \dot{w}^i \in C^1([T_0, T_1) \times \mathbb{R}^3, \mathbb{R}^3) \) where \( T_0 < 0 < T_1 \) (using these steps to future and past respectively). Then by letting \( \Omega_0 := \Omega(0) \) and Lemma 2.14, we obtain the flow \( \varphi \) described by preceding lemma. Denote the flow \( \varphi \) restricted on the submanifold \( [0, T) \times \{ 0 \} \times \mathcal{D} \) by \( \chi : [0, T) \times \mathcal{D} \ni (t, \xi) \mapsto x \in \mathbb{R}^3 \),

\[
\chi(t, \xi) := \varphi(t, 0, \xi), \quad \chi^i(\xi) := \chi(t, \xi) = \varphi^{(t,0)}(\xi) \quad \text{and} \quad \chi^t(\xi) := \chi(t, \xi) = \varphi^{(0, \xi)}(t).
\]

Then \( \Omega_t = \chi^t(\Omega(0)) \) and \( \partial \Omega_t \in C^1 \) by Lemma 2.14.(5).

Secondly, we prove \( \Omega(t) = \Omega_t = \chi^t(\Omega(0)) \). In other words, we have to verify that the image of the initial support of density is still the support of density, i.e., \( \chi^t(\text{supp} \rho(0, \cdot)) = \text{supp} \rho(t, \cdot) \).

For every \( x \in \chi^t(\Omega(0)) \), there exists a point \( \xi \in \Omega(0) \) which means \( \rho(0, \xi) > 0 \), such that
\( x = \chi^t(\xi) \). Because of the fact \( \chi^t(\xi) = \chi^t(0) \) for every \( \xi \in \Omega(0) \), by denoting the integral curve (characteristic) starting from \( \chi^t(0) = \xi \) by

\[
\chi^t(t) = \chi(t, \xi) = x, \tag{2.13}
\]
we have (by the definition of integral curves)

\[
\frac{d}{dt} \chi^t(t) = w(t, \chi^t(t)). \tag{2.14}
\]
Then (1.2) for the any point on the integral curve \( \chi^t(t) \), with the help of (2.14), yields

\[
\frac{d}{dt} \rho(t, \chi^t(t)) = D_t \rho(t, x) = \partial_0 \rho(t, x) + w^i(t, x) \partial_i \rho(t, x) = - \rho(t, \chi^t(t)) \partial_i w^i(t, \chi^t(t)),
\]
which implies that

\[
\rho(t, x) = \rho(0, \xi) \exp \left( - \int_0^t \partial_i w^i(\tau, \chi^t(\tau)) d\tau \right). \tag{2.15}
\]
This expression (2.15) directly leads to \( \rho(t, x) > 0 \) which means \( x \in \Omega(t) \), and further we derive \( \chi^t(\Omega(0)) \subset \Omega(t) \).

On the other hand, for every \( x \in \Omega(t) \), that is, \( \rho(t, x) > 0 \). Still (2.15) implies \( \rho(0, \xi) > 0 \) where, as above, \( x = \chi^t(\xi) \). This yields \( \xi \in \Omega(0), \ x \in \chi^t(\Omega(0)) \) and, in turn, concludes \( \Omega(t) \subset \chi^t(\Omega(0)) \).

Therefore, \( \Omega(t) = \Omega_t = \chi^t(\Omega(0)) \), which, in turn, implies \( \partial \Omega(t) \in C^1 \) due to \( \partial \Omega_t \in C^1 \), and it is direct that \( T \) can be extended to the existence interval of the solution by using above arguments continuously. Then we complete this proof. \( \square \)

**Remark 2.16.** We point out from the proof of this theorem, the regularity of \( \partial \Omega(t) \) depends on the regularity of initial \( \partial \Omega(0) \) and the regularity of the vector field \( w \). Since \( w \in C^1([0, T) \times \Omega(t), \mathbb{R}^3) \) leads to a \( C^1 \) flow, it seems we can obtain a Lipschitz boundary \( \partial \Omega(t) \) provided \( \partial \Omega(0) \) is Lipschitz. However, due to the implicit function theorem applied by Theorem C.4, the Lipschitz boundary can not be achieved directly.

The next Proposition 2.17 gives the complete boundary \( \partial([0, T] \times \Omega(t)) \) (lateral, top and bottom boundaries) is of Lipschitz. This property enable us to use Stein Extension Theorem D.3 (see Appendix D.2) to extend the solution in \([0, T] \times \Omega(t)\) to the whole \( \mathbb{R}^4 \), which is a crucial step for the uniqueness theorem.

**Proposition 2.17 (Boundary Gluing).** Suppose \( \mathcal{D}_l := [0, T] \times \partial \Omega(t) \), \( \mathcal{D}_t := \{T\} \times \Omega(T) \) and \( \mathcal{D}_b := \{0\} \times \Omega(0) \) are lateral, top and bottom boundaries given by the solution of the diffuse boundary problem, and \( \partial \Omega(0) \) is a \( C^1 \) boundary of \( \Omega(0) \). Then the lateral boundary \( \mathcal{D}_l = [0, T] \times \partial \Omega(t) \in C^1 \) and the complete boundary

\[
\partial([0, T] \times \Omega(t)) = \mathcal{D}_l \cup \mathcal{D}_b \cup \mathcal{D}_t
\]
is a Lipschitz boundary.

We put the proof of this proposition in Appendix B.2 to avoid deviating from the main objective too far.
2.5. Uniqueness of the solution to the diffuse boundary problem. In this section, we prove the uniqueness theorem of the classical solution of the diffuse boundary problem (the proof of this theorem is inspired by the relative entropy-entropy flux method given in [35, §2, Step 2]). This theorem helps us confirm that there is only one classical solution and this one has to be the restriction of the same one, and rule out all other possibilities if the initial data are in the form of Theorem 2.12,(1) and (2). Although the exterior data of velocity \(w^i\) of the corresponding tame solutions can be selected with certain freedoms, the classical solution is unique and independent of selections of the exterior data of velocity of the corresponding tame solutions once the interior data of velocity are fixed. This phenomenon is essentially due to the degenerated sound cones near the boundary (see [5] for details). Before the uniqueness theorem, we first present the following lemma which is a direct result of Lemma B.1 (see Appendix B) and states for the solutions on \(\mathbb{R}^3\) if the portion of data contained in the support of density are the same, then so are the portion of solutions contained in the support of density.

**Lemma 2.18.** Under the assumptions of Lemma B.1, if initial data satisfy

\[
\Omega_1(0) = \Omega_2(0), \quad \rho_1(0, \mathbf{x}) = \rho_2(0, \mathbf{x}) \quad \text{and} \quad w_1^i(0, \mathbf{x}) = w_2^i(0, \mathbf{x})
\]

for \(\mathbf{x} \in \Omega_1(0) = \Omega_2(0)\), then the solutions satisfy

\[
\Omega_1(t) = \Omega_2(t), \quad \rho_1(t, \mathbf{x}) = \rho_2(t, \mathbf{x}) \quad \text{and} \quad w_1^i(t, \mathbf{x}) = w_2^i(t, \mathbf{x}),
\]

for \((t, \mathbf{x}) \in [0, T) \times \Omega_1(t)(= [0, T) \times \Omega_2(t))\).

**Proof.** Lemma B.1 (see Appendix B) implies

\[
\int_{\mathbb{R}^3} \eta^*(t, \mathbf{x})d\mathbf{x} \leq \int_{\mathbb{R}^3} \eta^*(0, \mathbf{x})d\mathbf{x} + C \sup_{0 \leq \tau < T} (\|\nabla_\mathbf{x} w_1^i(\tau, \cdot)\|_{L^\infty} + Z(\tau)) \int_0^t \int_{\mathbb{R}^3} \eta^*(t, \mathbf{x})d\mathbf{x}d\tau \quad (2.16)
\]

where \(\eta^* \geq 0\), \(Z(\tau)\) are defined in Lemma B.1. Note \(\eta^*(t, \mathbf{x}) \equiv 0\) for any \(\mathbf{x} \in (\Omega_1(t) \cup \Omega_2(t))^c\) and use the initial data

\[
\Omega_1(0) = \Omega_2(0), \quad \rho_1(0, \mathbf{x}) = \rho_2(0, \mathbf{x}) \quad \text{and} \quad \dot{w}_1^i(0, \mathbf{x}) = \dot{w}_2^i(0, \mathbf{x})
\]

for \(\mathbf{x} \in \Omega_1(0) = \Omega_2(0)\), then the above domain of integration can be shrunk to \(\Omega_1(t) \cup \Omega_2(t)\) and above inequality (2.16), with the help of \(w_1^i \in W^{1,\infty}([0, T) \times \mathbb{R}^3, \mathbb{R}^3)\), becomes

\[
\int_{\Omega_1(t) \cup \Omega_2(t)} \eta^*(t, \mathbf{x})d\mathbf{x} \leq \int_{\Omega_1(0)} \eta^*(0, \mathbf{x})d\mathbf{x} + C \sup_{0 \leq \tau < T} (\|\nabla_\mathbf{x} w_1^i(\tau, \cdot)\|_{L^\infty} + Z(\tau)) \int_0^t \int_{\Omega_1(t) \cup \Omega_2(t)} \eta^*(t, \mathbf{x})d\mathbf{x}d\tau
\]

\[
\leq C' \int_0^t \int_{\Omega_1(t) \cup \Omega_2(t)} \eta^*(t, \mathbf{x})d\mathbf{x}d\tau.
\]

Then by the Gronwall’s inequality, we arrive at

\[
\int_{\Omega_1(t) \cup \Omega_2(t)} \eta^*(t, \mathbf{x})d\mathbf{x} \equiv 0
\]

for \(t \in [0, T)\). Since \(\eta^* \geq 0\) (see (B.1) in Lemma B.1), then \(\eta^* \equiv 0\) for \((t, \mathbf{x}) \in [0, T) \times (\Omega_1(t) \cup \Omega_2(t))\), by (B.1) in Lemma B.1, we conclude

\[
\rho_1(t, \mathbf{x}) = \rho_2(t, \mathbf{x}) \quad \text{and} \quad \dot{w}_1^i(t, \mathbf{x}) = \dot{w}_2^i(t, \mathbf{x}),
\]

for \((t, \mathbf{x}) \in [0, T) \times (\Omega_1(t) \cup \Omega_2(t))\) and further \(\Omega_1(t) = \Omega_2(t)\) since they are supports of densities \(\rho_1\) and \(\rho_2\), respectively. Then we complete this proof. \(\square\)
Definition 2.19. For a solution \((\rho, \hat{w}, \Phi, \Omega(t))\) to Euler–Poisson system (1.2)–(1.4) on \([0, T) \times \Omega(t)\), suppose \(\rho \in C^1([0, T) \times \mathbb{R}^3)\) and \(\hat{w}^i \in C^1([0, T) \times \Omega(t), \mathbb{R}^3)\), and if there exists an extension function \(w^i \in W^{1,\infty}([0, T) \times \mathbb{R}^3, \mathbb{R}^3)\), such that \(w^i(t, x) = \hat{w}^i(t, x)\) for \((t, x) \in [0, T) \times \Omega(t)\), then we call this solution \((\rho, \hat{w}, \Phi, \Omega(t))\) is extendable.

Note that Local Existence Theorem 2.12 gives the local classical solutions of diffuse boundary problem by the Calderón extension of the data and the symmetric hyperbolic system theory. However, since the classical solution from Local Existence Theorem 2.12 is much stronger than the Definition 1.1 of classical solution (in fact it is a Makino solution i.e. \(\rho^{\frac{\gamma - 1}{\gamma}} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\) and \(\hat{w}^i \in C^0([0, T), H^s(\Omega(t))) \cap C^1([0, T), H^{s-1}(\Omega(t)))\)), it does not mean every classical solutions of diffuse boundary problem comes from this procedure, that is, until now there may be some classical solutions of diffuse boundary problem which are not Makino solution (i.e., \(\rho^{\frac{\gamma - 1}{\gamma}} \notin C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\)) and hence can not be derived by above extension and Makino’s system. The next Uniqueness Theorem 2.20 indicates any solutions (not necessary to be the classical solutions of the diffuse boundary problem) with the same initial data in the hydrodynamic region \(\Omega(t)\) are the same if these solutions are extendable. After giving Uniqueness Theorem 2.20, we point out, in Strong Uniqueness Theorem 2.21, every classical solution defined by Definition 1.1 is extendable and furthermore, with the help of Uniqueness Theorem 2.20, it follows the classical solution exists and unique under the data given in Local Existence Theorem 2.12.

Theorem 2.20. (Uniqueness Theorem) Suppose \(1 < \gamma \leq 2, T > 0\), \((\rho_\ell, \hat{w}_\ell, \Phi_\ell, \Omega_\ell(t))\), for \(\ell = 1, 2\), are two extendable solutions to Euler–Poisson system (1.2)–(1.4) on \([0, T) \times \Omega(t)\), that is they are, respectively, two extendable solutions of the system,

\[
\begin{align*}
\partial_0 \rho_\ell + \hat{w}_\ell^i \partial_i \rho_\ell + \rho_\ell \partial_i \hat{w}_\ell^i &= 0 & \text{in } [0, T) \times \Omega_\ell(t), & \text{(2.17)}
\end{align*}
\]

\[
\begin{align*}
\rho_\ell \partial_0 \hat{w}_\ell^k + \rho_\ell \hat{w}_\ell^i \partial_i \hat{w}_\ell^k + \delta^{ik} \partial_i p_\ell &= - \rho_\ell \partial^k \Phi_\ell, & \text{in } [0, T) \times \Omega_\ell(t), & \text{(2.18)}
\end{align*}
\]

where

\[
\Phi_\ell(t, x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t, y) d^3y}{|x - y|}, \text{ for } (t, x) \in [0, T) \times \mathbb{R}^3;
\]

and the equation of state (1.1) and the diffuse boundary (1.7) hold. If initial data are the same, \(\Omega_1(0) = \Omega_2(0), \rho_1(0, x) = \rho_2(0, x)\) and \(\hat{w}_1^i(0, x) = \hat{w}_2^i(0, x)\) for \(x \in \Omega_1(0) = \Omega_2(0)\), then the solutions are the same as well,

\[
\Omega_1(t) = \Omega_2(t), \rho_1(t, x) = \rho_2(t, x) \text{ and } \hat{w}_1^i(t, x) = \hat{w}_2^i(t, x),
\]

for \((t, x) \in [0, T) \times \Omega_1(t) = [0, T) \times \Omega_2(t)\).

Proof. Firstly, direct calculations (note (2.17)×\(\hat{w}_\ell^i\)+ (2.18) yields the following second equation) lead to that for any \((t, x) \in [0, T) \times \Omega(t)\) the system (2.17)–(2.18) is equivalent to

\[
\begin{align*}
\partial_t \rho_\ell + \partial_i (\rho_\ell \hat{w}_\ell^i) &= 0 & \text{in } [0, T) \times \Omega_\ell(t),
\end{align*}
\]

\[
\begin{align*}
\partial_t (\rho_\ell \hat{w}_\ell^k) + \partial_i (\rho_\ell \hat{w}_\ell^i \hat{w}_\ell^k) + \delta^{ik} \partial_i p_\ell &= - \rho_\ell \partial^k \Phi_\ell, & \text{in } [0, T) \times \Omega_\ell(t).
\end{align*}
\]

Since \((\rho_\ell, \hat{w}_\ell^i, \Phi_\ell, \Omega_\ell(t))\), for \(\ell = 1, 2\), are extendable (using Definition 2.19), there are extensions \(w_\ell^i \in C^1([0, T) \times \Omega(t), \mathbb{R}^3)\)\(\cap W^{1,\infty}([0, T) \times \mathbb{R}^3, \mathbb{R}^3)\) of \(\hat{w}^i\), respectively. Then by direct calculations, we can verify that \(\rho_\ell w_\ell^i \in C^1([0, T) \times \mathbb{R}^3), \rho_\ell \hat{w}_\ell^i w_\ell^k \in C^1([0, T) \times \mathbb{R}^3)\) (note \(\partial_t (\rho_\ell \hat{w}_\ell^i)\big|_{\Omega(t)} = (\partial_t \rho_\ell \hat{w}_\ell^i)\big|_{\Omega(t)} + (\partial_t \rho_\ell \hat{w}_\ell^i)\big|_{\Omega(t)} = 0\) and they both solve the Euler–Poisson system given in Lemma B.1. After verifying them, with the help of Lemma 2.18, we conclude this theorem. \(\square\)
Above Theorem 2.20 claims the extendable solution is unique, the next Theorem 2.21 states with the stronger conditions of the local existence Theorem 2.12, the classical solutions are all extendable, furthermore, under the conditions of the local existence Theorem 2.12, the classical solution is unique.

**Theorem 2.21** (Strong Uniqueness Theorem). Suppose $1 < \gamma \leq 2$, $T > 0$, $(\rho_\ell, \bar{w}_i^\ell, \Phi_\ell, \Omega_\ell(t))$, for $\ell = 1, 2$, are two classical solutions to diffuse boundary problem $(i)$–$(iv)$, defined by Definition 1.1 on $[0, T)$. If the initial domain $\Omega(0)$ is bounded and satisfies $\partial \Omega(0)$ is a $C^1$ boundary, and initial data are the same,

$$\Omega_1(0) = \Omega_2(0), \quad \rho_1(0, x) = \rho_2(0, x) \quad \text{and} \quad \bar{w}_i^1(0, x) = \bar{w}_i^2(0, x)$$

for $x \in \Omega_1(0) = \Omega_2(0)$, then the solutions are the same as well,

$$\Omega_1(t) = \Omega_2(t), \quad \rho_1(t, x) = \rho_2(t, x) \quad \text{and} \quad \bar{w}_i^1(t, x) = \bar{w}_i^2(t, x),$$

for any $T_* \in (0, T)$ and $(t, x) \in [0, T_*] \times \Omega_1(t) (= [0, T_*] \times \Omega_2(t))$.

Furthermore, under the conditions of the local existence Theorem 2.12, and $\partial \Omega(0)$ is a $C^1$ boundary of $\Omega(0)$, then there is a unique classical solution $(\rho, \bar{w}^i, \Phi, \Omega(t))$ in $t \in [0, T_*)$ given by Theorem 2.12 for every $0 < T_* < T$.

**Proof.** Let us first claim every classical solutions to diffuse boundary problem $(i)$–$(iv)$ is extendable. In fact, since $(\rho, \bar{w}^i, \Phi, \Omega(t))$ is a classical solutions to diffuse boundary problem $(i)$–$(iv)$, by Definition 1.1, $\rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})$ and $\bar{w}^i \in C^1([0, T) \times \overline{\Omega(t)}, \mathbb{R}^3)$. Due to the compactness of $\overline{\Omega(t)}$, we obtain $w^i \in W^{1,\infty}([0, T_*] \times \overline{\Omega(t)}, \mathbb{R}^3)$ for any $T_* \in (0, T)$. Since $\partial \Omega(0)$ is a $C^1$ boundary, we apply Proposition 2.17 to conclude $\partial([0, T_*] \times \Omega(t)) = \mathcal{D}_{i} \cup \mathcal{D}_{b} \cup \mathcal{D}_{t} \subset \mathbb{R}^4$ is of Lipschitz. With the help of *Stein Extension Theorem* $D.3$ (see Appendix §D.2) for $(m, p) = (1, \infty)$, there is a Stein extension operator $E : W^{1,\infty}([0, T_*] \times \overline{\Omega(t)}) \to W^{1,\infty}(\mathbb{R}^4)$ satisfying $Ew^i(x^\mu) = \bar{w}^i(x^\mu)$ in $[0, T_*] \times \overline{\Omega(t)}$ and there is a constant $K > 0$ such that $\|Ew^i\|_{W^{1,\infty}(\mathbb{R}^4)} \leq K \|w^i\|_{W^{1,\infty}([0, T_*] \times \overline{\Omega(t)})}$. Therefore, by Definition 2.19, the classical solution is extendable. Further, using Uniqueness Theorem 2.20 completes the proof of the first part.

If assuming conditions of the local existence Theorem 2.12 with $C^1$ boundary $\partial \Omega(0)$ of $\Omega(0)$ for $t \in [0, T)$, then there is at least one Makino solution given by Theorem 2.12 and $\bar{w}^i \in C^0([0, T), H^s(\Omega(t))) \cap C^1([0, T), H^{s-1}(\Omega(t)))$. Using above result, we conclude any two classical solutions are the same, that is, there is a unique classical solution of the system and the unique solution has to be the Makino solution. We complete this proof. \qed

**Remark 2.22.** We point out [5, Proposition 2] proved the uniqueness theorem of the Makino solutions (which allow us obtain a symmetric hyperbolic system) instead of all classical solutions, since [5] presumes two solutions solve Makino’s formulations (make sure the hyperbolic system is not degenerated) of Euler–Poisson system and energy estimates lead to they are equal in the lenslike shape region if data are the same, this can be done only if $\rho^{\frac{1}{2}}(t) \in H^s$ rather than more general condition $\rho^{\gamma-1}(t) \in H^s$ for all classical solutions. Therefore, the uniqueness of [5] implies the solution satisfying $\rho^{\frac{1}{2}}(t) \in H^s$ is unique. In other words, it can not rule out the possibility that there are more than one solutions solve diffuse problem with the same data, one of them is Makino solution satisfying $\rho^{\frac{1}{2}}(t) \in H^s$ and the other satisfying $\rho^{\gamma-1}(t) \in H^s$ but $\rho^{\frac{1}{2}}(t) \notin H^s$. However, above Strong Uniqueness Theorem 2.21 exactly overcomes this difficulty.
3. Continuation principles of diffuse boundary problem

In this section, we introduce the continuation principles of diffuse boundary problem. These continuation principles allow us to continue the local solutions to a larger time interval provided the suitably bounded spatial derivatives. First we give the weak version (see §3.1) and it will help us extend the classical solutions to several types of singularities and the basic idea of the proof is to use Makino’s formulation of Euler–Poisson system and energy estimates. However, the standard Gagliardo–Nirenberg–Moser estimates have been replaced by the revised Gagliardo–Nirenberg–Moser estimates on the Lipschitz bounded domain. The strong continuation principle (see §3.2) helps us reduce the types of interior singularities and remove certain singularities on the boundary. The key tool of the proof for the strong continuation principle is Ferrari–Shirota–Yanagisawa inequality (a bounded domain version of Beale–Kato–Majda estimate). This inequality allows us to control the velocity $\|w^i\|_{W^{1,\infty}}$ “almost only” by expansion and rotation. In order to use Ferrari–Shirota–Yanagisawa (FSY) inequality, we have to prove a $C^\infty$-approximation lemma C.6 of boundary in Appendix C.2 since FSY inequality requires domain is of class $C^{2,\beta}$, $\beta \in (0,1)$ and $w^i \delta_{ij}n^j = 0$ on $\partial\Omega(t)$ and $n^j$ is the unit outward normal at $\mathbf{x} \in \partial\Omega(t)$. To overcome these two requirements and use FSY inequality, our idea is, near the boundary $\partial\Omega(t)$, we select a smooth surface ($C^\infty$-approximation lemma of boundary C.6 ensures this) and decompose the velocity field to two parts around this $C^\infty$ surface, one of these parts is “tangent” to the $C^\infty$ surface on it, then analyze these two parts respectively.

3.1. Weak continuation principle of diffuse boundary problem. Using energy estimates on bounded $\Omega(t)$, we obtain the following weak continuation principle. Note since we will use Makino’s formulations, the derivations of this section work only for Makino solutions (recall Definition 2.5, a special class of regular solutions resulted from Theorem 2.12) instead of the general regular solutions.

**Theorem 3.1** (Weak Continuation Principle of Diffuse Boundary Problem). *Suppose the initial data satisfy requirements of Theorem 2.12, $T > 0$ is some given time. If a classical solution $(\rho, \dot{\mathbf{w}}, \Phi, \Omega(t))$ to the diffuse boundary problem $(i)–(iv)$ exists on $t \in [0,T_\ast)$ for any $T_\ast \in (0,T]$, and satisfy an estimate

$$
\int_0^{T_\ast} \left( \|\nabla \dot{w}^i(s)\|_{L^\infty(\Omega(s))} + \|\nabla \alpha\|_{L^\infty(\Omega(s))} \right) ds < \infty, \tag{3.1}
$$

then there is a constant $T_\ast > T > 0$, such that the classical solution $(\rho, \dot{\mathbf{w}}, \Phi, \Omega(t))$ exists on $t \in [0,T_\ast)$. Moreover, for $t \in [0,T)$, there is an estimate,

$$
\|\mathbf{U}(t)\|_{H^s(\Omega(t))} \leq \|\mathbf{U}(0)\|_{H^s(\Omega(0))} \exp \left[ C \int_0^t \left( \|\nabla \mathbf{U}(s)\|_{L^\infty(\Omega(s))} + C(\|\alpha(s)\|_{L^\infty(\Omega(s))}) \right) ds \right] \tag{3.2}
$$

where $\mathbf{U} := (\alpha, \dot{w}^k)^T$.

*Proof.* Firstly, according to Makino’s formulations (2.8)–(2.9), we reexpress the Euler–Poisson system (1.2)–(1.4) of the diffuse boundary problem in $\Omega(t)$ and rewriting them into the matrix form yields, for $(t, \mathbf{x}) \in [0,T) \times \Omega(t),

$$
A^0 \partial_0 \mathbf{U} + A^i \partial_i \mathbf{U} = \mathbf{F} \tag{3.3}
$$

where

$$
\mathbf{U} := (\alpha, \dot{w}^k)^T, \quad \mathbf{F} := (0, -\partial_j \Phi)^T \tag{3.4}
$$
and
\[ A^0 := \begin{pmatrix} 1 & 0 \\ 0 & \delta_{kj} \end{pmatrix}, \quad A^i = A^i(U) := \begin{pmatrix} \tilde{w}^i & \frac{2-\alpha\delta^i_j}{\gamma-1} \\ \frac{2-\alpha\delta^i_j}{\gamma-1} & \tilde{w}^j \delta_{kj} \end{pmatrix}. \] (3.5)

Secondly, let us derive a local energy estimate on \( \Omega(t) \) by using the local calculus inequalities in Appendix D.3 (as the standard procedure, we, omit the details, first derive the estimate by assuming all the variables are smooth and then using the approximation theorems of Sobolev spaces in [16, §5.3] to conclude the derived inequality holds for Sobolev spaces as well). First acting on both sides of (3.3) by \( A^0\partial^\beta(A^0)^{-1} \) (for \(|\beta| \leq s \) and \( s = 3 \) or 4) yields
\[ A^0 \partial_\beta \mathcal{U}^\beta + A^i \partial_i \mathcal{U}^\beta = \mathcal{G} \] (3.6)
where
\[ \mathcal{U}^\beta := \partial^\beta \mathcal{U} \quad \text{and} \quad \mathcal{G} := -A^0[\partial^\beta, (A^0)^{-1}]\partial^\beta \mathcal{U} + A^0\partial^\beta[(A^0)^{-1} \mathcal{F}]. \]

Let us define the local energy,
\[ \|\partial^\beta \mathcal{U}\|^2_{L^2(\Omega(t))} := \langle \mathcal{U}^\beta, A^0\mathcal{U}^\beta \rangle_t := \int_{\Omega(t)} (\mathcal{U}^\beta)^T A^0 \mathcal{U}^\beta d^3x, \]
and calculate an important boundary term for later use. Note
\[ \tilde{w}^i A^0 - A^i = \begin{pmatrix} 0 & -\frac{2-\alpha\delta^i_j}{\gamma-1} \\ -\frac{2-\alpha\delta^i_j}{\gamma-1} & 0 \end{pmatrix}. \]
Hence, \( (\mathcal{U}^\beta)^T(\tilde{w}^i A^0 - A^i)\mathcal{U}^\beta = -(\gamma - 1)\alpha \partial^\beta w^i \partial^\beta \alpha \). Then since \( w^i \) is the extension of \( \tilde{w}^i \) to the boundary \( \partial \Omega(t) \), let us calculate, by the divergence theorem and \( \alpha|_{\partial \Omega(t)} \equiv 0 \), the boundary term vanishes,
\[ \int_{\Omega(t)} \partial_i \left[ (\mathcal{U}^\beta)^T(\tilde{w}^i A^0 - A^i)\mathcal{U}^\beta \right] d^3x = -(\gamma - 1) \int_{\Omega(t)} \partial_i (\alpha \partial^\beta w^i \partial^\beta \alpha) d^3x = -(\gamma - 1) \int_{\partial \Omega(t)} \alpha \nu_i \partial^\beta w^i \partial^\beta \alpha d^3x = 0 \] (3.7)
where \( \nu_i := \delta_{ij} \nu^j \) and \( \nu^j \) is the unit outward-pointing normal to \( \partial \Omega(t) \).

Then, noting \( \partial_t A^0 \equiv 0 \) and the identity (3.7) on the boundary \( \partial \Omega(t) \), differentiating \( \|\partial^\beta \mathcal{U}\|^2_{L^2(\Omega(t))} \) with respect to \( t \), inserting (3.6) in the followings and using integration by parts and Reynold’s Transport Theorem D.13 lead to, for \( t \in [0, T_*] \),
\[ \partial_t \|\partial^\beta \mathcal{U}\|^2_{L^2(\Omega(t))} = \int_{\Omega(t)} \partial_t \left[ (\mathcal{U}^\beta)^T A^0 \mathcal{U}^\beta \tilde{w}^i \right] d^3x + 2\langle \mathcal{U}^\beta, A^0 \partial_\beta \mathcal{U}^\beta \rangle_t + \langle \mathcal{U}^\beta, (\partial_t A^0) \mathcal{U}^\beta \rangle_t \\
= \int_{\Omega(t)} \partial_t \left[ (\mathcal{U}^\beta)^T(\tilde{w}^i A^0 - A^i)\mathcal{U}^\beta \right] d^3x + \langle \mathcal{U}^\beta, (\partial_t A^i) \mathcal{U}^\beta \rangle_t + 2\langle \mathcal{U}^\beta, \mathcal{G} \rangle_t \\
= \langle \mathcal{U}^\beta, (\partial_t A^i) \mathcal{U}^\beta \rangle_t + 2\langle \mathcal{U}^\beta, \mathcal{G} \rangle_t. \] (3.8)

Standard procedures of hyperbolic system (see, for example, [36, 54], etc.), with the help of the revised Gagliardo–Nirenberg–Moser estimates on the Lipschitz bounded domain, i.e., Proposition D.6, D.8 and D.9 and Corollary D.7 (see Appendix D.3), and taking (3.4) and (3.5) into account, leads to
\[ \begin{align*}
(1) \langle \mathcal{U}^\beta, (\partial_t A^i) \mathcal{U}^\beta \rangle_t & \leq C \|\partial_t A^i\|_{L^\infty(\Omega(t))}\|\mathcal{U}\|_{H^s(\Omega(t))} \leq C \|\nabla \mathcal{U}\|_{L^\infty(\Omega(t))}\|\mathcal{U}\|^2_{H^s(\Omega(t))}, \\
(2) \langle \mathcal{U}^\beta, A^0[\partial^\beta, (A^0)^{-1}]\partial_\beta \mathcal{U} \rangle_t & \leq C \|\mathcal{U}\|_{H^s(\Omega(t))}\|A^i\|_{H^s}\|\partial_\beta \mathcal{U}\|_{L^\infty(\Omega(t))} + \|\partial_t A^i\|_{L^\infty(\Omega(t))}\|\mathcal{U}\|_{H^s(\Omega(t))}.
\end{align*} \] (3.9)
\[
\leq C\|\nabla U\|_{L^\infty(\Omega(t))}\|U\|_{H^{\ast}(\Omega(t))}^2
\] (3.10)
and with the help of [37, Lemma 2] and Proposition D.9,
\[
(3 \langle U^\beta, A^0 \partial^\beta [(A^0)^{-1} F] \rangle_t \leq C(\|\alpha\|_{L^\infty(\Omega(t))})\|U\|_{H^{\ast}(\Omega(t))}^2).
\] (3.11)
where we simply denote \(C(\|\alpha\|_{L^\infty(\Omega(t))}) := C(\|D_\alpha \rho\|_{C^{\alpha_i-1}}(1 + \|\alpha\|_{L^\infty(\Omega(t))}^{\delta-1}))\) to emphasize the constant depending on \(\|\alpha\|_{L^\infty(\Omega(t))}\). Adding up (3.8) for all \(|\alpha| \leq s\), with the help of (3.9)–(3.11), leads to, for \(t \in [0, T_\ast]\),
\[
\partial_t\|U\|_{H^{\ast}(\Omega(t))}^2 \leq C\|\nabla U\|_{L^\infty(\Omega(t))} + C(\|\alpha\|_{L^\infty(\Omega(t))})\|U\|_{H^{\ast}(\Omega(t))}^2.
\]
Then the Gronwall’s inequality implies for all \(t \in [0, T_\ast]\),
\[
\|U(t)\|_{H^{\ast}(\Omega(t))} \leq \|U(0)\|_{H^{\ast}(\Omega(0))} \exp\left[C \int_0^t \|\nabla U(s)\|_{L^\infty(\Omega(s))} + C(\|\alpha(s)\|_{L^\infty(\Omega(s))}) ds\right].
\] (3.12)

Reexpress (1.2) and (2.8) in terms of Lagrangian description (recall §1.2.3), we directly derive
\[
\partial_t \ln \rho(t, \xi) = -\Omega(t, \xi) \quad \text{and} \quad \partial_t \ln \alpha(t, \xi) = -\frac{\gamma - 1}{2} \Omega(t, \xi).
\]
Then integrating them along the integral curves of \(w\) yield
\[
\rho(t, \xi) = \rho(0, \xi) \exp\left(-\int_0^t \Theta(s, \xi) ds\right)
\] (3.13)
and
\[
\alpha(t, \xi) = \alpha(0, \xi) \exp\left(-\frac{\gamma - 1}{2} \int_0^t \Theta(s, \xi) ds\right).
\] (3.14)

Then (3.1), with the help of (1.12) and (1.15), implies
\[
\|\ln \alpha(t)\|_{L^\infty(\Omega(t))} = \|\ln \alpha(t)\|_{L^\infty(\Omega(0))} \leq \|\ln \alpha(0)\|_{L^\infty(\Omega(0))} + \frac{\gamma - 1}{2} \int_0^t \|\Theta(s)\|_{L^\infty(\Omega(t))} ds
\]
\[
\leq \|\ln \alpha(0)\|_{L^\infty(\Omega(0))} + \frac{\gamma - 1}{2} \int_0^t \|\Theta(s)\|_{L^\infty(\Omega(t))} ds
\]
\[
\leq \|\ln \alpha(0)\|_{L^\infty(\Omega(0))} + \frac{\gamma - 1}{2} \int_0^t \|\nabla \tilde{w}^\beta(s)\|_{L^\infty(\Omega(s))} ds < \infty.
\] (3.15)
for \(t \in [0, T_\ast]\). This, by (3.1) and the initial data, in turn, leads to
\[
\|\alpha(t)\|_{L^\infty(\Omega(t))} \leq e^{\|\ln \alpha(t)\|_{L^\infty(\Omega(t))}} \leq C(t) \leq \infty.
\] (3.16)
where
\[
\mathcal{G}(t) := \exp\left(\|\ln \alpha(0)\|_{L^\infty(\Omega(0))} + \frac{\gamma - 1}{2} \int_0^t \|\Theta(s)\|_{L^\infty(\Omega(t))} ds\right).
\] (3.17)
Then using (3.12) and (3.1) again, we conclude for all \(t \in [0, T_\ast]\), there is an estimate \(\|U(t)\|_{H^{\ast}(\Omega(t))} \leq \infty\).

At the end, we prove the local existence can be extended to \(t \in [0, T]\). If, as [36, §2.2], \([0, T_\ast)\) \((T_\ast < T)\) is the maximal interval of the local existence, we can use (3.2) and the local existence Theorem 2.12 to start a solution at time \(T_\ast - \epsilon\) for any small \(\epsilon > 0\) and continue this solution beyond \(T_\ast\) (or in specific, as the proof of theorem 2.12, one can extend the solution to \(\mathbb{R}^3\), then the local existence of standard hyperbolic system with a proceeding time to exceed \(T_\ast\), then restrain the solution to \(\Omega(t)\), then a contradiction follows, which means \(T_\ast = T\). Then continue to begin a solution at time \(T - \epsilon\) for any small \(\epsilon > 0\) and extend it beyond \(T\) to \(T^\ast > T\). Then the proof is completed. \(\square\)
3.2. Strong continuation principle of diffuse boundary problem. Inspired by the ideas of [5], we improve the above continuation principle to decrease the types of singularities. Firstly, Lemma 3.2 (resemble [17, Corollary 1]) is a useful tool for proving the strong version of continuation principle and can be derived directly from the Ferrari–Shirota–Yanagisawa inequality of Theorem D.10 (see Appendix D.4).

Lemma 3.2. Suppose for every \( t \in [0, T) \), \( \Omega(t) \subset \mathbb{R}^3 \) is a bounded, simply connected domain of class \( C^{2, \beta} \), \( \beta \in (0, 1) \). If \( w(\cdot, \cdot) \) is a vector field and \( w^i \delta_{ij} n^j = 0 \) on \( \partial \Omega(t) \) and \( n^j \) is the unit outward normal at \( x \in \partial \Omega(t) \), then for \( t \in [0, T) \),
\[
||w^i(t)||_{W^{1, \infty}(\Omega(t))} \leq C[1 + \ln^{+} ||w^i(t)||_{H^3(\Omega(t))} + ||\Theta(t)||_{L^\infty(\Omega(t))} + ||\Omega_{jk}(t)||_{L^\infty(\Omega(t))} + 1]
\]
where
\[
\ln^{+} a := \begin{cases} 
\ln a, & \text{if } a \geq 1 \\
0, & \text{otherwise}
\end{cases}
\]

Proof. Note the relations between rotation tensor \( \Omega_{ij} \) and the vorticity \( \omega := (\omega_1, \omega_2, \omega_3) = \nabla \times w = (\partial_2 w_3 - \partial_3 w_2, \partial_1 w_3 - \partial_3 w_1, \partial_1 w_2 - \partial_2 w_1) \), i.e.,
\[
(\Omega_{ij}) = \frac{1}{2} \begin{pmatrix}
0 & -\omega_3 & -\omega_2 \\
\omega_3 & 0 & -\omega_1 \\
\omega_2 & \omega_1 & 0
\end{pmatrix},
\]
and \( \theta := \nabla \cdot w = \Theta \). Let \( v := (w, 0) \) (v is defined in Theorem D.10, Appendix D.4), then the requirements of Theorem D.10 is verified and thus we obtain this lemma.

The next theorem gives the continuation principle more meticulous portrayal which will provide a better blowup result when the boundary behavior is more “regular”, see the companion [31].

Theorem 3.3 (Strong Continuation Principle of Diffuse Boundary Problem). Suppose \( \Omega(t) \subset \mathbb{R}^3 \) is a bounded domain, the initial data satisfy requirements of Theorem 2.12, \( T > 0 \) is some given time. If a classical solution \( (\rho, \dot{w}, \Phi, \Omega(t)) \) to the diffuse boundary problem (i)–(iv) exists on \( t \in [0, T_*) \) for any \( T_* \in (0, T] \), and if there is a constant \( \epsilon > 0 \) such that an estimate holds\(^7\)
\[
\int_0^{T_*} \left( ||W_{jk}(s)||_{L^\infty(\tilde{\Omega}_e(s))} + ||\Theta(s)||_{L^\infty(\tilde{\Omega}_e(s))} + ||\Omega_{jk}(s)||_{L^\infty(\tilde{\Omega}_e(s))} + ||\nabla \alpha(s)||_{L^\infty(\tilde{\Omega}_e(s))} \right) ds < \infty.
\]
where \( \tilde{\Omega}_e(s) := \Omega(s) \setminus \Omega_e(s) \). Then there is a constant \( T^* > T > 0 \), such that the classical solution \( (\rho, \dot{w}, \Phi, \Omega(t)) \) exists on \( t \in [0, T^*) \).

Proof. According to the estimate of Theorem 3.1, a direct idea of this proof is to verify that
\[
||W_{jk}(t)||_{L^\infty(\Omega(t))}
\]
is controlled by
\[
||\nabla \alpha(t)||_{L^\infty(\Omega(t))}, \quad ||W_{jk}(t)||_{L^\infty(\Omega_e(t))}, \quad ||\Theta(t)||_{L^\infty(\tilde{\Omega}_e(t))}, \quad \text{and } ||\Omega_{jk}(t)||_{L^\infty(\tilde{\Omega}_e(t))}
\]
for some small \( \epsilon > 0 \). However, in fact, besides (B.9), Lemma 3.2 implies the estimate of \( ||W_{jk}(t)||_{L^\infty(\Omega(t))} \) also slightly depends on the \( H^\gamma \) norm of the unknowns \( ||U||_{H^\gamma(\Omega(t))} \). An approach from [4] (it is widely used, for example, in [5, 17], etc.) indicates Lemma 3.2 along with the energy estimate (3.12) and Gronwall inequality (see Theorem D.11) will help us derive an

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\(^7\)See §1.2.4 and recall the definition (1.9) of \( \Omega_e \).
improved a priori estimate on $\| U \|_{H^s(\Omega(t))}$. Then a continuation argument as the last paragraph of the proof of Theorem 3.1 can be used to conclude this proof.

For later use, let us start with a useful estimate that $\int_0^{T_}\| w^i(t) \|_{L^\infty(\Omega(t))} dt$ is controlled by

$$\int_0^{T_}\| W_{jk}(t) \|_{L^\infty(\Omega(t))} dt, \quad \int_0^{T_}\| \Theta(t) \|_{L^\infty(\hat{\Omega}_c(t))} dt, \quad \text{and} \quad \int_0^{T_}\| \nabla \alpha(t) \|_{L^\infty(\Omega(t))} dt.$$

Step 1: Re-express (2.9) in terms of Lagrangian descriptions (recall §1.2.3),

$$\partial_t u^k(t, \xi) + \gamma - \frac{1}{2} \alpha(t, \xi) \partial_k \alpha(t, \xi) = - \partial_k \Phi(t, \xi).$$

Integrating it along the integral curves of $w$, with the help of [5, Lemma 1], (3.13), (3.15) and notation (1.8) in §1.2.3, yields

$$\| w^k(t) \|_{L^\infty(\Omega(t))}$$

$$\leq \| w^k(0) \|_{L^\infty(\Omega(0))} + \gamma - \frac{1}{2} \int_0^{t} \| \alpha(s) \|_{L^\infty(\Omega(s))} \| \nabla \alpha(s) \|_{L^\infty(\Omega(s))} ds + \int_0^{t} \| \nabla \Phi(s) \|_{L^\infty(\Omega(s))} ds$$

$$\leq \| w^k(0) \|_{L^\infty(\Omega(0))} + \gamma - \frac{1}{2} \int_0^{t} \| \alpha(s) \|_{L^\infty(\Omega(s))} \| \nabla \alpha(s) \|_{L^\infty(\Omega(s))} ds + C \int_0^{t} \| \rho(s) \|_{L^\infty(\Omega(s))} ds$$

$$\leq \| w^k(0) \|_{L^\infty(\Omega(0))} + \gamma - \frac{1}{2} \| \alpha(0) \|_{L^\infty(\Omega(0))} \int_0^{t} \exp \left( \gamma - \frac{1}{2} g(s) \right) \| \nabla \alpha(s) \|_{L^\infty(\Omega(s))} ds$$

$$+ C \| \rho(0) \|_{L^\infty(\Omega(0))} \int_0^{T_} \exp \left( g(s) \right) ds$$

$$\leq \| w^k(0) \|_{L^\infty(\Omega(0))} + \gamma - \frac{1}{2} \| \alpha(0) \|_{L^\infty(\Omega(0))} \exp \left( \gamma - \frac{1}{2} g(T_*) \right) \int_0^{T_} \| \nabla \alpha(s) \|_{L^\infty(\Omega(s))} ds$$

$$+ C \| \rho(0) \|_{L^\infty(\Omega(0))} T_ \exp \left( g(T_*) \right).$$

(3.20)

for $t \in [0, T_*)$ where

$$g(s) := \int_0^{s} \| \Theta(\tau) \|_{L^\infty(\Omega(\tau))} d\tau \leq C \int_0^{T_*} \| W_{jk}(t) \|_{L^\infty(\Omega_c(t))} dt + \int_0^{T_*} \| \Theta(t) \|_{L^\infty(\hat{\Omega}_c(t))} dt.$$

Then, by (3.18), we arrive at

$$\int_0^{T_*} \| w^i(t) \|_{L^\infty(\Omega(t))} dt < \infty.$$

Step 2: Now let us turn to control $\int_0^{T_*} \| W_{jk}(t) \|_{L^\infty(\Omega(t))} dt$. The idea of this proof is under the situation of Theorem 3.1, we try to use Lemma 3.2 to control $\| w^i \|_{W^{1,\infty}}$ in terms of $\| \Theta \|_{L^\infty}$ and $\| \Omega_{jk} \|_{L^\infty}$. However, there are two difficulties obstructing this idea. In order to use Lemma 3.2, the first difficulty is the boundary in this theorem does not have to be $C^{2,\beta}$ as Lemma 3.2 requires; the second one is the orthogonal condition of the boundary limiting velocity $w^i \delta_{ij} n^j = 0$ is not always true in this theorem. It is clear both of these difficulties come from the boundary. To overcome these, our idea is, near the boundary, we select a smooth surface ($C^\infty$-approximation lemma of boundary C.6 ensures this) and decompose the velocity field to two parts around this $C^\infty$ surface, one of these part is “tangent” to the $C^\infty$ surface on it, then analyze these two parts respectively.

First using the $C^\infty$-approximation lemma of boundary (see Lemma C.6 in Appendix §C.2), for every time $t \in [0, T_*)$ and for any small constant $\epsilon > 0$, there is an open domain $D(t)$ with $C^\infty$ boundary, such that $\hat{\Omega}_c(t) \subset D(t) \subset \Omega(t)$. Then by the extension theorems of the outward
unit normal (see, for example, Proposition C.7 or Theorem C.9, see Appendix C.3), since \( D(t) \) is a \( C^\infty \) domain, then there exists \( O(t) \subset \Omega_c(t) \), an open neighborhood of the boundary \( \partial D(t) \), and a unit vector field \( n : O(t) \to \mathbb{R}^3 \) of class \( C^\infty \) in \( O(t) \) with the property that \( n|_{\partial D(t)} \) is the outward unit normal to \( \partial D(t) \), then we further extend \( n \) from \( O(t) \) to \( \overline{O(t)} \) by assuming \( N \) is a smaller neighborhood of \( \partial D(t) \) satisfying \( \partial D(t) \subset N \) and \( \overline{N} \subset O \) and using Proposition C.10 (see Appendix C.3), we denote the extended vector field still by \( n \) and \( \text{supp} \ n \subset O \).

Then, there is a cut-off function \( \phi \in C^\infty_0 \) (see, for example, [30, Proposition 2.25]) such that

\[
\phi(x) = \begin{cases} 
0, & \text{for } x \in O^c \\
1, & \text{for } x \in N \\
(0,1), & \text{for other } x \in \mathbb{R}^3
\end{cases}
\]

Then \( w^i \) can be decomposed as

\[
w^i = (w^i n_j n^j \phi) + (w - (w^j n_j n^i \phi)).
\]

(3.21)

for \( (t,x) \in [0,T_\ast) \times \Omega(t) \). It is clear that \( (w^i - (w^j n_j n^i \phi))|_{O^c} = w^i|_{O^c} \) and \( (w^j n_j n^i \phi)|_{\overline{N}} = (w^j n_j n^i \phi)|_{\overline{N}} \).

We can define the rotation and expansion of vector field \( w^i - (w^j n_j n^i \phi) \) as (1.13) and (1.15), and denote them by \( \hat{\Omega}_{jk} \) and \( \hat{\Theta} \), respectively. Then in order to apply Lemma 3.2, for every \( t \in [0,T_\ast) \) we verify, with the help of Corollary D.7,

\[
\|w^i - (w^j n_j n^i \phi)\|_{H^s(D(t))} \leq \|w^i - (w^j n_j n^i \phi)\|_{H^s(\Omega(t))}
\]

\[
\leq \|w^i\|_{H^s(\Omega(t))} + \|(w^j n_j n^i \phi)\|_{H^s(\Omega(t))} \leq C\|w^i\|_{H^s(\Omega(t))}.
\]

(3.22)

Since \( D(t) \) is a bounded, simply connected, \( C^\infty \) domain, and direct examinations imply \( (w^i - (w^j n_j n^i \phi)n^i n^j \phi)|_{\partial D(t)} = 0 \), Lemma 3.2 yields, for every \( t \in [0,T_\ast) \),

\[
\|w^i - (w^j n_j n^i \phi)\|_{W^{1,\infty}(D(t))} \leq C\left[ (1 + \ln^+ \|w^i - (w^j n_j n^i \phi)\|_{H^s(D(t))}) (\|\hat{\Theta}\|_{L^\infty(D(t))} + \|\hat{\Omega}_{jk}\|_{L^\infty(D(t))}) + 1 \right].
\]

(3.23)

It is direct that

\[
\hat{\Omega}_{jk}|_{\Omega_c(t)} = \Omega_{jk}|_{\Omega_c(t)} \quad \text{and} \quad \hat{\Theta}|_{\Omega_c(t)} = \Theta|_{\Omega_c(t)}.
\]

Furthermore, noting \( \|(w^j n_j n^i \phi)\|_{W^{1,\infty}(\Omega_c(t))} = 0 \), since \( \phi|_{\Omega_c(t)} = 0 \) and

\[
[w^i]\|_{W^{1,\infty}(\Omega_c(t))} \leq \|w^i\|_{W^{1,\infty}(\Omega_c(t))} + \|w^i\|_{W^{1,\infty}(\Omega_c(t))},
\]

and using (3.20) (in Step 1), (3.21), (3.22), (3.23) and \( \hat{\Omega}_c(t) \subset D(t) \subset \Omega(t) \), we estimate

\[
\|W_{ij}(t)\|_{L^\infty(\Omega(t))} \leq \|w^i(t)\|_{W^{1,\infty}(\Omega(t))}
\]

\[
\leq \|w^i\|_{W^{1,\infty}(\Omega(t))} + \|(w^j n_j n^i \phi)\|_{W^{1,\infty}(\Omega_c(t))} + \|w^i - (w^j n_j n^i \phi)\|_{W^{1,\infty}(\Omega_c(t))}
\]

\[
\leq \|w^i\|_{W^{1,\infty}(\Omega_c(t))} + C\left[ (1 + \ln^+ \|w^i - (w^j n_j n^i \phi)\|_{H^s(D(t))}) (\|\hat{\Theta}\|_{L^\infty(D(t))} + \|\hat{\Omega}_{jk}\|_{L^\infty(D(t))}) + 1 \right]
\]

\[
\leq \|w^i\|_{W^{1,\infty}(\Omega_c(t))} + C\left[ (\ln^+(e + C\|w^i\|_{H^s(\Omega(t))})) (\|\Theta\|_{L^\infty(\Omega_c(t))} + \|\hat{\Omega}_{jk}\|_{L^\infty(\Omega_c(t))}) + 1 \right]
\]

\[
\leq \|w^i\|_{W^{1,\infty}(\Omega_c(t))} + C\left[ (\ln^+(e + C\|u\|_{H^s(\Omega(t))})) (\|\Theta\|_{L^\infty(\Omega_c(t))} + \|\hat{\Omega}_{jk}\|_{L^\infty(\Omega_c(t))}) + 1 \right]
\]

\[
(3.24)
\]
for every $t \in [0, T_*)$ where
\[
\|w^j\|_{W^{1,\infty}(\Omega(t))} \leq \|w^i\|_{L^\infty(\Omega(t))} + \|W_{ij}\|_{L^\infty(\Omega(t))}
\]  
(3.25)
and $\|w^j\|_{L^\infty(\Omega(t))}$ can be estimated by (3.20). The last term we have to estimate in above (3.24) is $\|U\|_{H^s(\Omega(t))}$. We use the following procedure (this procedure has been used, for example, in [4, 5, 17]) to derive an estimate of $\|U\|_{H^s(\Omega(t))}$ in terms of (B.9). That is, using (3.2) and (3.24), we arrive at
\[
\|U(t)\|_{H^s(\Omega(t))} \leq \|U(0)\|_{H^s(\Omega(0))} \exp \left[ C \int_0^t \left[ \|\nabla \alpha(s)\|_{L^\infty(\Omega(s))} + \|W_{ij}(s)\|_{L^\infty(\Omega(s))} + C(\|\alpha(s)\|_{L^\infty(\Omega(s))}) \right] ds \right]
\]  
\[
\leq \|U(0)\|_{H^s(\Omega(0))} \exp \left\{ C \int_0^t [b(s) + C(a(s)y(s) + 1)] ds \right\}
\]  
(3.26)
for every $t \in [0, T_*)$ where
\[
y(t) := \ln(C\|U\|_{H^s(\Omega(t))} + e) = \ln(C\|U\|_{H^s(\Omega(t))} + e),
\]
\[
a(t) := \|\Theta\|_{L^\infty(\Omega(t))} + \|\Omega_{jk}\|_{L^\infty(\Omega(t))} + C\|w^i\|_{W^{1,\infty}(\Omega(t))},
\]
and
\[
b(t) := \|\nabla \alpha\|_{L^\infty(\Omega(t))} + \|w^i\|_{W^{1,\infty}(\Omega(t))} + C(\|\alpha(t)\|_{L^\infty(\Omega(t))}).
\]
Then noting
\[
e \leq e \cdot \exp \left\{ C \int_0^t [b(s) + C(a(s)y(s) + 1)] ds \right\}
\]
and calculating $\ln(C \times (3.26)+e)$ yield
\[
y(t) \leq y(0) + C \int_0^t \left[ (b(s) + C) + Ca(s)y(s) \right] ds
\]
\[
\leq y(0) + \int_0^t (C_1 b(s) + C_2) ds + \int_0^t C_3 a(s) y(s) ds
\]
for every $t \in [0, T_*)$. Letting $G(t) := y(0) + \int_0^t C_1 b(s) ds + C_2 t$, a increasing function of $t$, and using generalized Gronwall inequality of integral form (see Theorem D.11 in Appendix §D.5), we arrive at
\[
y(t) \leq G(t) \exp \left( \int_0^t C_3 a(s) ds \right)
\]
for every $t \in [0, T_*)$. From this inequality, we improve a priori estimate of $\|U(t)\|_{H^s(\Omega(t))}$,
\[
\|U(t)\|_{H^s(\Omega(t))} \leq C \left\{ \left[ C\|U(0)\|_{H^s(\Omega(0))} + e\right] \exp \left( C_2 t + C_1 \int_0^t b(s) ds \right) \right\}^{e \exp \left( \int_0^t C_3 a(s) ds \right) - e}
\]  
(3.27)
for every $t \in [0, T_*)$. Using (3.27) and Condition (3.18) guarantee $\|U(t)\|_{H^s(\Omega(t))} < \infty$. With the help of the standard argument for the continuation principle to extend solution, as same as the last paragraph of proof of Theorem 3.1, concludes this Theorem. \[\square\]
APPENDIX A. PRELIMINARY PROPOSITIONS OF DIFFUSE BOUNDARY

The purpose of this Appendix is to prove Proposition 2.1, but before that, let us fix notations and give a simple lemma. Note that we have no any regularity condition on the boundary \([0, T) \times \partial \Omega(t)\) in this Appendix. For any \(\mathbf{x} = (\tilde{x}^\mu)\), a unit vector \(\mathbf{v} = (v^\mu)\) and \(\mathbf{x} = (x^\mu) = (\tilde{x}^\mu + hv^\mu)\) where \(h \in \mathbb{R}\), we denote the one-sided derivatives

\[
\partial_{\mathbf{v}+} f(\tilde{x}^\mu) := \lim_{h \to 0+} \frac{f(\tilde{x}^\mu + hv^\mu) - f(\tilde{x}^\mu)}{h},
\]

and

\[
\partial_{\mathbf{v}-} f(\tilde{x}^\mu) := \lim_{h \to 0-} \frac{f(\tilde{x}^\mu + hv^\mu) - f(\tilde{x}^\mu)}{h}.
\]

We first prove a simple but useful Lemma A.1.

**Lemma A.1.** Suppose \(\rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R})\), \(\gamma > 1\) and \(\Omega(t) = \text{supp} \rho \subset \mathbb{R}^3\). For any points \(x^\mu \in [0, T) \times \Omega(t)\), \(\tilde{x}^\mu \in [0, T) \times \partial \Omega(t)\) and a unit vector \(\mathbf{v} := (v^\mu) \in \mathbb{R}^4\), \(\rho\) satisfies

\[
\lim_{x^\mu \to \tilde{x}^\mu} \partial_{\mathbf{v}} \rho_\gamma^{-1}(x^\mu) = 0. \tag{A.1}
\]

If there is a constant \(\delta_0 > 0\), such that if \(h \in (0, \delta_0)\), then for either \(x^\mu = \tilde{x}^\mu + hv^\mu \in [0, T) \times \Omega(t)\) or \(x^\mu = \tilde{x}^\mu + hv^\mu \in [0, T) \times \Omega^c(t)\), we both have the one-sided derivative

\[
\partial_{\mathbf{v}+} \rho_\gamma^{-1}(\tilde{x}^\mu) := \lim_{h \to 0+} \frac{\rho_\gamma^{-1}(x^\mu) - \rho_\gamma^{-1}(\tilde{x}^\mu)}{h} = 0.
\]

**Proof.** If there is a constant \(\delta_0 > 0\), such that if \(h \in (0, \delta_0)\), then for \(x^\mu = \tilde{x}^\mu + hv^\mu \in [0, T) \times \Omega(t)\), and \(\rho_\gamma^{-1}(x^\mu) = \rho_\gamma^{-1}(\tilde{x}^\mu) = 0\). This yields

\[
\partial_{\mathbf{v}+} \rho_\gamma^{-1}(\tilde{x}^\mu) = \lim_{h \to 0+} \frac{\rho_\gamma^{-1}(x^\mu) - \rho_\gamma^{-1}(\tilde{x}^\mu)}{h} \equiv 0.
\]

If there is a constant \(\delta_0 > 0\), such that if \(h \in (0, \delta_0)\), then \(x^\mu = \tilde{x}^\mu + hv^\mu \in [0, T) \times \Omega(t)\). In this case, we use L’Hospital’s rule and (A.1) to arrive at

\[
\partial_{\mathbf{v}+} \rho_\gamma^{-1}(\tilde{x}^\mu) = \lim_{h \to 0+} \frac{\rho_\gamma^{-1}(x^\mu) - \rho_\gamma^{-1}(\tilde{x}^\mu)}{h} = \lim_{h \to 0+} v^\mu \partial_{\mathbf{v}+} \rho_\gamma^{-1}(x^\mu) = 0.
\]

Then, this completes the proof. \(\square\)

**Lemma A.2.** For \(x^\mu \in [0, T) \times \Omega(t)\) and \(\tilde{x}^\mu \in [0, T) \times \partial \Omega(t)\), if \(w^k(\tilde{x}^\mu) := \lim_{x^\mu \to \tilde{x}^\mu} \hat{w}^k(x^\mu) < \infty\), then

\[
\lim_{x^\mu \to \tilde{x}^\mu} D_t \hat{w}^k(x^\mu) = D_t \lim_{x^\mu \to \tilde{x}^\mu} \hat{w}^k(x^\mu) = D_t w^k(\tilde{x}^\mu). \tag{A.2}
\]

for any \(x^\mu \in [0, T_\ast] \times \Omega(t)\) where \(T_\ast \in (0, T)\) is any constant.

**Proof.** Since \((x^\mu) = (t, x) = (t, \chi(t, \xi)) = (\xi^0, \chi(\xi^\mu))\) and \(\chi\) is a \(C^1\) flow from Lemma 2.14 where we denote \(t = x^0 = \xi^0\), we obtain, by denoting \(\xi^i = \chi(-t, \tilde{x}_j)\) and using the continuity of the flow \(\chi\),

\[
\lim_{\xi^\mu \to \xi^\mu} (t, \chi^i(\xi^\mu)) = \tilde{x}^\mu \quad \text{and} \quad \lim_{x^\mu \to \tilde{x}^\mu} (t, \chi^i(-t, \tilde{x}_j)) = \xi^\mu. \tag{A.3}
\]

Then the limit,

\[
\lim_{\xi^\mu \to \xi^\mu} \partial_0 \hat{w}^k(\xi^0, \chi(\xi^\mu)) = \lim_{\xi^\mu \to \xi^\mu} \partial_0 \hat{w}^k(\xi^\mu) = \lim_{\xi^\mu \to \xi^\mu} D_t \hat{w}^k(\xi^\mu).
\]
If we can prove
\[
\lim_{\xi \to \xi'} \partial_0 \tilde{w}_k^k(\xi) = \partial_0 w^k(\xi') \tag{A.4}
\]
Then it implies (A.2) due to (A.3) and notations in §1.2.3. This is because \( w^k(\tilde{x}_\mu) \equiv \lim_{\xi \to \xi'} \tilde{w}_k^k(x_\mu) \) and (A.3) yield \( \tilde{w}_k^k(\xi) = \lim_{\xi \to \xi'} \tilde{w}_k^k(\xi') \).

Now let us prove (A.4). In order to use Moore–Osgood Theorem D.12 (see Appendix §D.6), we firstly verify, by the definition of derivatives,
\[
\lim_{s \to 0} \frac{\tilde{w}_k^k(\xi^0 + s, \xi^i) - \tilde{w}_k^k(\xi^0, \xi^i)}{s} = \partial_0 \tilde{w}_k^k(\xi^i)
\]
pointwisely for any \( \xi^i \in [0, T) \times \Omega(0) \) and
\[
\lim_{\xi \to \xi'} \frac{\tilde{w}_k^k(\xi^0 + s, \xi^i) - \tilde{w}_k^k(\xi^0, \xi^i)}{s} = \frac{1}{s} \left[ \lim_{\xi \to \xi'} \tilde{w}_k^k(\xi^0 + s, \xi^i) - \lim_{\xi \to \xi'} \tilde{w}_k^k(\xi^0, \xi^i) \right] = \frac{\tilde{w}_k^k(\xi^0 + s, \xi^i) - \tilde{w}_k^k(\xi^0, \xi^i)}{s}
\]
uniformly for any \( s \neq 0 \), since \( \tilde{w}_k^k \in C^1([0, T] \times \Omega(0)) \), then by Heine–Cantor Theorem, \( \tilde{w}_k^k \) is uniformly continuous. This, in turn, yields for every \( \epsilon > 0 \), there is a constant \( \delta > 0 \) such that for any \( s \in \mathbb{R} \setminus \{0\} \) and \( (\xi_0^i + s, \xi^i), (\xi_0^i + s, \xi^i) \in [0, T] \times \Omega(0) \), if \( |(\xi_0^i + s, \xi^i) - (\xi_0^i + s, \xi^i)| < \delta \), then \( \|\tilde{w}_k^k(\xi_0^i + s, \xi^i) - \tilde{w}_k^k(\xi_0^i + s, \xi^i)\| < \epsilon \). Then \( \lim_{\xi \to \xi'} \tilde{w}_k^k(\xi_0^i + s, \xi^i) = \tilde{w}_k^k(\xi_0^i + s, \xi^i) \) uniformly for any \( s \neq 0 \).

Therefore, Moore–Osgood Theorem D.12 (see Appendix §D.6) leads to
\[
\lim_{\xi \to \xi'} \partial_0 \tilde{w}_k^k(\xi) = \lim_{\xi \to \xi'} \lim_{s \to 0} \frac{\tilde{w}_k^k(\xi_0^i + s, \xi^i) - \tilde{w}_k^k(\xi_0^i, \xi^i)}{s} = \lim_{s \to 0} \frac{\tilde{w}_k^k(\xi_0^i + s, \xi^i) - \tilde{w}_k^k(\xi_0^i, \xi^i)}{s} = \partial_0 \tilde{w}_k^k(\xi^i).
\]
This is (A.4) and then we complete the proof.  \( \square \)

**Proof of Proposition 2.1.** Let us first prove (1) \( \iff \) (2).

(2) \( \implies \) (1): That means to prove \( \rho^{\gamma-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}) \) implies (2.1). \( \rho^{\gamma-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}) \) yields \( (\gamma - 1)\rho^{\gamma-1}\partial_\rho = K\gamma\partial_\rho \rho^{\gamma-1} \in C^0([0, T) \times \mathbb{R}^3, \mathbb{R}) \). Note that
\[
\lim_{(t', x') \to (t, x)} \partial_\rho \rho^{\gamma-1}(t', x') = 0,
\]
for \( (t', x') \in [0, T) \times \Omega^\epsilon(t) \), since \( \rho(t', x') = 0 \) for \( (t', x') \in [0, T) \times \Omega^\epsilon(t) \). Therefore, for \( \gamma > 1 \),
\[
\lim_{(t, x) \to (t, x)} \left( \rho^{-1}(t, x) \partial_\rho \rho(t, x) \right) = \lim_{(t, x) \to (t, x)} \frac{K\gamma \partial_\rho \rho^{\gamma-1}(t, x)}{\gamma - 1} = 0,
\]
for any \( (t, x) \in [0, T) \times \Omega(t) \) and \( (t, x) \in [0, T) \times \partial \Omega(t) \).

(1) \( \implies \) (2): First, using \( \rho \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}) \), \( \rho \equiv 0 \) for \( (t, x) \in [0, T) \times \Omega^\epsilon(t) \) and \( \partial_\rho \rho^{\gamma-1} = (\gamma - 1)\rho^{\gamma-2}\partial_\rho \rho \) for \( (t, x) \in [0, T) \times \Omega(t) \) and \( \gamma > 1 \), it is direct to verify \( \rho^{\gamma-1} \in C^1([0, T) \times \Omega(t), \mathbb{R}) \cap C^1([0, T) \times \Omega^\epsilon(t), \mathbb{R}) \cap C^0([0, T) \times \mathbb{R}^3, \mathbb{R}) \). Then, in order to conclude \( \rho^{\gamma-1} \in C^1([0, T) \times \mathbb{R}^3, \mathbb{R}) \), we only need to verify that \( \partial_\rho \rho^{\gamma-1} \) exist on the boundary \( \partial \Omega(t) \) and is continuous.
crossing the boundary, that is, for any \( (\tilde{x}^\mu) := (\tilde{t}, \tilde{x}) \in [0, T) \times \partial \Omega(t), \lim_{(t, x) \to (\tilde{t}, \tilde{x})} \partial_\mu \rho^{-1}(t, x) = \partial_\mu \rho^{-1}(\tilde{t}, \tilde{x}) \) for \( (t, x) \in [0, T) \times \mathbb{R}^3. \)

The equation of continuity (1.2) yields

\[
\partial_t \rho^{-1} = (\gamma - 1) \rho^{-2} \partial_t \rho = (\gamma - 1) \rho^{-2}(-\tilde{w}^i \partial_i \rho - \rho \partial_i \tilde{w}^i) = -\tilde{w}^i \partial_i \rho^{-1} - (\gamma - 1) \partial_i \tilde{w}^i.
\]

Then due to \( w^i \in C^1([0, T) \times \Omega(t), \mathbb{R}^3) \) and (2.1), \( \lim_{x^\mu \to \tilde{x}^\mu} \tilde{w}^i(x^\mu) = w^i(\tilde{x}^\mu) < \infty, \lim_{x^\mu \to \tilde{x}^\mu} \partial_i \rho^{-1} = 0, \lim_{x^\mu \to \tilde{x}^\mu} \rho^{-1} = 0. \) Furthermore,

\[
\lim_{x^\mu \to \tilde{x}^\mu} \partial_\mu \rho^{-1} = -\lim_{x^\mu \to \tilde{x}^\mu} (\tilde{w}^i \partial_i \rho^{-1}) - (\gamma - 1) \lim_{x^\mu \to \tilde{x}^\mu} (\rho^{-1} \partial_i \tilde{w}^i) = 0
\]

for \( \tilde{x}^\mu \in [0, T) \times \partial \Omega(t) \) and \( x^\mu \in [0, T) \times \Omega(t). \) Coupling with (2.1), we have

\[
\lim_{x^\mu \to \tilde{x}^\mu} \partial_\mu \rho^{-1}(x^\mu) = 0.
\]

Let us first prove \( \partial_\nu \rho^{-1}(\tilde{x}^\mu) \) exists for any given unit vector \( \nu := (\nu^\mu) \in \mathbb{R}^4 \) and a point \( (\tilde{x}^\mu) \in [0, T) \times \partial \Omega(t). \)

(a) If there is a constant \( \delta_0 > 0, \) such that if \( |h| \in (0, \delta_0), \) then \( x^\mu = \tilde{x}^\mu + h v^\mu \in [0, T) \times \Omega(t) \) and \( x^\mu = \tilde{x}^\mu - h v^\mu \in [0, T) \times \Omega(t). \) Then in either case, Lemma A.1 yields \( \partial_\nu \rho^{-1}(\tilde{x}^\mu) = 0 \) and \( \partial_{(\nu)\nu} \rho^{-1}(\tilde{x}^\mu) = 0. \) Let us define

\[
\partial_{\nu-} \rho^{-1}(\tilde{x}^\mu) := \lim_{h \to 0-} \frac{\rho^{-1}(x^\mu + h v^\mu) - \rho^{-1}(\tilde{x}^\mu)}{h} = -\partial_{(\nu)\nu} \rho^{-1}(\tilde{x}^\mu) = 0.
\]

(b) For every constant \( \delta_0 > 0, \) there exist \( h_1, h_2 \in (0, \delta_0), \) such that \( x^\mu_1 = \tilde{x}^\mu + h_1 v^\mu \in [0, T) \times \Omega(t) \) and \( x^\mu_2 = \tilde{x}^\mu + h_2 v^\mu \in [0, T) \times \Omega(t). \) Let us denote

\[
\mathcal{H}_1 := \{h_1 \in (0, \delta_0) \mid \tilde{x}^\mu + h_1 v^\mu \in [0, T) \times \Omega(t)\} \neq \emptyset,
\]

\[
\mathcal{H}_2 := \{h_2 \in (0, \delta_0) \mid \tilde{x}^\mu + h_2 v^\mu \in [0, T) \times \Omega(t)\} \neq \emptyset.
\]

Then \( \mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset \) and \( \mathcal{H}_1 \cup \mathcal{H}_2 = (0, \delta_0). \) We need to calculate

\[
\partial_{\nu+} \rho^{-1}(\tilde{x}^\mu) = \lim_{h \to 0+} \frac{\rho^{-1}(x^\mu - h v^\mu) - \rho^{-1}(\tilde{x}^\mu)}{h} = \lim_{h \to 0+} \frac{\rho^{-1}(\tilde{x}^\mu + h v^\mu)}{h}.
\]

for \( h \in (0, \delta_0). \)

(b1) For every \( h \in \mathcal{H}_2, \rho^{-1}(\tilde{x}^\mu + h v^\mu) \equiv 0, \) then

\[
\frac{\rho^{-1}(\tilde{x}^\mu + h v^\mu)}{h} \equiv 0.
\]

(A.5)

(b2) For every \( h \in \mathcal{H}_1, \) there is a constant \( \bar{h} \in \mathcal{H}_2 \) such that \( 0 < \bar{h} < h \) (otherwise, it becomes above case (a)). Let \( \mathcal{H} := \mathcal{H}_2 \cap \{h \mid \bar{h} \leq h < h\} \neq \emptyset, \) then \( h^* := \sup \mathcal{H} \in \mathcal{H} \) (i.e., \( h^* = \max \mathcal{H} \)). This is because \( \rho(\tilde{x}^\mu + \bar{h} v^\mu) \) is continuous in \( h \in [\bar{h}, h), \) the set of preimage of \( \rho(\tilde{x}^\mu + \bar{h} v^\mu) > 0 \) is open. Therefore, the set of preimage of \( \rho(\tilde{x}^\mu + h v^\mu) = 0 \) is closed, which implies \( \mathcal{H} \) is closed and hence \( h^* \in \mathcal{H}. \) According to the definition of the boundary, \( \tilde{x}^\mu + h^* v^\mu \in [0, T) \times \partial \Omega(t). \)

Condition \( \lim_{x^\mu \to \tilde{x}^\mu} \partial_\nu \rho^{-1}(x^\mu) = 0 \) implies for any \( \epsilon > 0, \) there is \( \delta_1 > 0, \) such that for any \( h \in \mathcal{H}_1 \cap (0, \delta_1), \) we derive \( |\partial_\nu \rho^{-1}(\tilde{x}^\mu + h v^\mu)| < \epsilon. \)

With the help of mean value theorem, since \( \rho^{-1}(\tilde{x}^\mu + h v^\mu) \) is continuous in \( \bar{h} \in [h^*, h] \) and differentiable in \( \bar{h} \in (h^*, h) \subset \mathcal{H}_1 \cap (0, \delta_1), \) then there is a \( \hat{h} \in (h^*, h) \subset \mathcal{H}_1 \cap (0, \delta_1), \) such that

\[
0 \leq \frac{\rho^{-1}(\tilde{x}^\mu + h v^\mu)}{h} < \frac{\rho^{-1}(\tilde{x}^\mu + h v^\mu) - \rho^{-1}(\tilde{x}^\mu + h^* v^\mu)}{h - h^*} = \partial_\nu \rho^{-1}(\tilde{x}^\mu + \bar{h} v^\mu) < \epsilon \quad \text{(A.6)}
\]
Then, we claim that for any \( \epsilon > 0 \), there is \( \delta_1 > 0 \), such that \( h \in (0, \delta_1) \),

\[
0 \leq \frac{\rho^{-1}(\dot{x}^\mu + hv^\mu)}{h} < \epsilon \tag{A.7}
\]

This is because for any \( h \in H_2 \cap (0, \delta_1) \), (A.5) ensures (A.7), and for any \( h \in H_1 \cap (0, \delta_1) \), there is \( h^* \) given as above, then using (A.6) yields (A.7), and this, in turn, implies

\[
\partial_t \rho^{-1}(\dot{x}^\mu) = \lim_{h \to 0^+} \frac{\rho^{-1}(\dot{x}^\mu + hv^\mu) - \rho^{-1}(\dot{x}^\mu)}{h} = \lim_{h \to 0^+} \frac{\rho^{-1}(\dot{x}^\mu + hv^\mu)}{h} = 0.
\]

Therefore, in either case, we have \( \partial_t \rho^{-1}(\dot{x}^\mu) \) exist and \( \partial_t \rho^{-1}(\dot{x}^\mu) = 0 \) for every point \((\dot{x}^\mu) \in [0, T) \times \partial \Omega(t) \) and any unit vector \( \mathbf{v} := (v^\mu) \in \mathbb{R}^4 \). Furthermore, this implies for any \((\tilde{\dot{x}}^\mu) := (\tilde{\dot{t}}, \tilde{x}) \in [0, T) \times \partial \Omega(t), \lim_{(t, x) \to (\tilde{t}, \tilde{x})} \partial_t \rho^{-1}(t, x) = \partial_t \rho^{-1}(\tilde{t}, \tilde{x}) = 0 \) for \((t, x) \in [0, T) \times \mathbb{R}^3 \), and we complete the proof of this proposition.

Now let us turn to (1) \( \Leftrightarrow \) (3). Since \( w^i \in C^1([0, T) \times \overline{\Omega(t)} \times \mathbb{R}^3) \), we have

\[
w^i(\dot{x}^\mu) := \lim_{x^\mu \to \dot{x}^\mu} \tilde{w}^i(x^\mu) < \infty \quad \text{and} \quad w^i_{t, i}(\dot{x}^\mu) := \lim_{x^\mu \to \dot{x}^\mu} \partial_t \tilde{w}^i(x^\mu) < \infty \tag{A.8}
\]

for \( \dot{x}^\mu \in [0, T) \times \partial \Omega(t) \) and \( x^\mu \in [0, T) \times \Omega(t) \). Usually, \( w^i_{t, i}(\dot{x}^\mu) \neq \partial_t w^i(\dot{x}^\mu) \) since \( \partial_t w^i(\dot{x}^\mu) \) may not be well defined for all directions.

Then for every \( x^\mu \in [0, T) \times \Omega(t) \), \((\rho, w^i, \Phi)\) satisfies (1.3). Multiplying \( 1/\rho \) on both sides of (1.3) due to \( \rho > 0 \), and taking the limit yields, for \( x^\mu \in [0, T) \times \Omega(t) \) and \( \tilde{x}^\mu \in [0, T) \times \partial \Omega(t) \),

\[
\lim_{x^\mu \to \tilde{x}^\mu} \left[ \partial_t \tilde{w}^k(x^\mu) + \tilde{w}^i(x^\mu) \partial_i \tilde{w}^k(x^\mu) \right] = \lim_{x^\mu \to \tilde{x}^\mu} \partial_t \tilde{w}^k(\tilde{x}^\mu) = - \partial_t \Phi(\tilde{x}^\mu). \tag{A.9}
\]

Note usually we do not see, see Remark 2.2, have \( \lim_{x^\mu \to \tilde{x}^\mu} \partial_t w^k(x^\mu) = \partial_t w^k(\tilde{x}^\mu) \). However by Lemma A.2, we have \( \lim_{x^\mu \to \tilde{x}^\mu} D_t w^k(x^\mu) = D_t w^k(\tilde{x}^\mu) \) for any \( x^\mu \in [0, T_\ast) \times \Omega(t) \) where \( T_\ast \in (0, T) \). Then

\[
\lim_{x^\mu \to \tilde{x}^\mu} \left[ \partial_t \tilde{w}^k(x^\mu) + \tilde{w}^i(x^\mu) \partial_i \tilde{w}^k(x^\mu) \right] = \lim_{x^\mu \to \tilde{x}^\mu} D_t \tilde{w}^k(x^\mu) = D_t w^k(\tilde{x}^\mu) \tag{A.10}
\]

(1) \( \Rightarrow \) (3): With the help of (A.8), (A.10), the diffuse boundary condition (2.1), (A.9) yields

\[
D_t w^k(\tilde{x}^\mu) = - \partial_t \Phi(\tilde{x}^\mu)
\]

for \( \tilde{x}^\mu \in [0, T) \times \partial \Omega(t) \). This concludes (3).

(3) \( \Rightarrow \) (1): (A.9), with the help of (A.8) and (A.10), implies

\[
D_t w^k(\tilde{x}^\mu) + \delta^i_{k} \lim_{x^\mu \to \tilde{x}^\mu} \rho^{-1}(x^\mu) \partial_i p(x^\mu) = - \partial_t \Phi(\tilde{x}^\mu) \tag{A.11}
\]

for \( \tilde{x}^\mu \in [0, T) \times \partial \Omega(t) \) and \( x^\mu \in [0, T) \times \Omega(t) \). Comparing (A.11) with the free falling equation in (3) leads to (1).

\[\square\]

**APPENDIX B. PRELIMINARY LEMMATA OF UNIQUENESS THEOREM 2.20**

**B.1. Relative entropy estimates.** We summarize the following useful lemma from the Step 2 in [35, Page 772–777] where the authors use it to prove the uniqueness theorem for physical vacuum free boundary problem while we will use it to prove the uniqueness Theorem 2.20 of diffuse boundary problem.

**Lemma B.1.** Suppose \( \gamma \in (1, 2], T > 0, \rho_{\ell} \in C^1([0, T) \times \mathbb{R}^3) \) and \( w^i_{\ell} \in C^1([0, T) \times \overline{\Omega(t)}, \mathbb{R}^3) \cap W^{1, \infty}([0, T) \times \mathbb{R}^3, \mathbb{R}^3) \), for \( \ell = 1, 2 \), both solve the system,

\[
\partial_t \rho_{\ell} + \partial_i (\rho_{\ell} w^i_{\ell}) = 0 \quad \text{in} \quad [0, T) \times \mathbb{R}^3,
\]

\[
\partial_t (\rho_{\ell} w^k_{\ell}) + \partial_i (\rho_{\ell} w^i_{\ell} w^k_{\ell}) + \delta^i_k \partial_i p_{\ell} = - \rho_{\ell} \partial^k \Phi_{\ell} \quad \text{in} \quad [0, T) \times \mathbb{R}^3.
\]
\[ \rho_\varepsilon > 0, \quad \text{in } \Omega_\varepsilon(t), \]
\[ \rho_\varepsilon = 0, \quad \text{in } \mathbb{R}^3 \setminus \Omega_\varepsilon(t), \]
where
\[ \Phi_\varepsilon(t, x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x - y|} d^3 y, \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^3, \]
the equation of state (1.1) holds. Define the relative entropy
\[ \eta^\ast := \frac{1}{\gamma - 1} [\rho_\varepsilon^\gamma - \rho_1^\gamma - \gamma \rho_1^{\gamma - 1}(\rho_2 - \rho_1)] + \frac{1}{2} \rho_2 |w_2 - w_1|^2. \]
Then
\[ \eta^\ast(t, x) \geq C(\|\rho_2(\cdot, t)\|_{L^\infty} + \|\rho_2(\cdot, t)\|_{L^\infty})^{\gamma - 2}(\rho_2 - \rho_1)^2 + \frac{1}{2} \rho_2 |w_2 - w_1|^2 \geq 0 \quad \text{(B.1)} \]
and there is an estimate
\[ \int_{\mathbb{R}^3} \eta^\ast(t, x) dx \leq \int_{\mathbb{R}^3} \eta^\ast(0, x) dx + C \sup_{0 \leq \tau < T} (\|\nabla_x w_1(\tau, \cdot)\|_{L^\infty} + Z(\tau)) \int_0^T \int_{\mathbb{R}^3} \eta^\ast(t, x) dx d\tau \]
where
\[ Z(\tau) := \|\rho_2(\cdot, \cdot)\|_{L^\infty} (\|\rho_2(\cdot, \cdot)\|_{L^\infty} + \|\rho_1(\cdot, \cdot)\|_{L^\infty})^{2 - \gamma} (\text{Vol} S(\tau))^\frac{2}{\gamma} \]
and
\[ S(\tau) := \{ x \in \mathbb{R}^3 \mid |\rho_1(\tau, x) - \rho_2(\tau, x)| > 0 \} \]
for \( \tau \in [0, T). \)

\textbf{Proof.} See Step 2 in [35, Page 772–777] for the detailed proof. \( \square \)

\textbf{B.2. Proof of Proposition 2.17 for boundary gluing.} Let us focus on \( \mathcal{D}_l \cup \mathcal{D}_b \) is Lipschitz and \( \mathcal{D}_l \cup \mathcal{D}_t \) can be proven similarly, and then conclude this proposition. The key idea of this proof is to construct a local Lipschitz function near \( \mathcal{D}_l \cap \mathcal{D}_b \) and further the local boundary of \( \mathcal{D}_l \cup \mathcal{D}_b \) near \( \mathcal{D}_l \cap \mathcal{D}_b \) is Lipschitz. We also prove \( \mathcal{D}_l \in C^1 \) and with the help of \( \mathcal{D}_b \in C^1 \), we obtain the complete boundary is Lipschitz.

\textbf{B.2.1. Step 1:} \( \mathcal{D}_l \subset C^1 \) and \( \mathcal{D}_l \) and \( \mathcal{D}_b \) are not tangential to each other near \( \mathcal{D}_l \cap \mathcal{D}_b \). Since \( \partial \Omega(0) \) is a \( C^1 \) boundary of \( \Omega(0) \), by Definition C.1, for every point \( \xi \in \partial \Omega(0) \), there is a rigid map \( \xi \), a neighborhood \( C := B(0, d) \times (-h, h) \subset \mathbb{R}^3 \) where constants \( d, h > 0 \), along with a \( C^1 \) function \( \psi : \mathbb{R}^2 \supset B(0, d) \to (-h, h) \), such that \( \psi(0, 0) = 0 \), \( \nabla \psi(0, 0) = 0 \), and satisfies
\[ \mathcal{C} \cap \xi(\Omega(0)) = \{ (\xi_1, \xi_2, \xi_3) \in \mathcal{C} \mid \xi_3 > \psi(\xi_1, \xi_2) \}; \quad (B.2) \]
\[ \mathcal{C} \cap \xi(\partial \Omega(0)) = \{ (\xi_1, \xi_2, \xi_3) \in \mathcal{C} \mid \xi_3 = \psi(\xi_1, \xi_2) \}. \quad (B.3) \]
Note (B.2) implies if \( \xi_3 > 0 \) and \( (0, 0, \xi_3) \in \mathcal{C} \), a point \( (0, 0, \xi_3) \in \mathcal{C} \cap \xi(\Omega(0)) \) since \( \xi_3 > 0 = \psi(0, 0) \).

By Lemma 2.14 and Theorem 2.15,
\[ \mathcal{D}_l \times \partial \Omega(t) = \{ (t, \varphi(t, 0, \xi)) \mid t \in [0, T), \xi \in \partial \Omega(0) \} \]
where \( \varphi \) is the \( C^1 \) flow generalized by the vector field \( w \) and the inverse of \( x = \varphi(t, 0, \xi) = \varphi(t, 0, \xi) \) is \( \xi = \varphi(0, t, x) = \varphi(0, t, x) \). For simplicity of statements, we extend\(^8\) \( C^1 \) flow \( \varphi \) to \( t \in (-T_1, T) \) \( (T_1 > 0) \) and denote \( \mathcal{D}_l = \{ (t, \varphi(t, 0, \xi)) \mid t \in (-T_1, T), \xi \in \partial \Omega(0) \} \) and the components of \( \xi \) by \( \xi_i := \varphi_i(t, 0, x) \).

\(^8\)This can be done by several ways, for example, use Proposition to extend the vector field \( w^i \) and further extend the flow, or solve the Euler–Poisson system locally to \((-T_1, 0)\).
Let us prove $\mathcal{D}_t \in C^1$. We need to prove for every point $x^\mu \in \mathcal{D}_t$, it locally can be represented by a graph of a $C^1$ function. However, without loss of generality, we only focus on the case $\xi^\mu = (0,0,0,0) \in \mathcal{D}_t$. Because for any case $\xi^\mu = (0, \xi) \in \mathcal{D}_t$, we can use some rigid map $\Sigma$ to move the target point to the origin and further, for any more general case $x^\mu \in \mathcal{D}_t$, since Theorem 2.15 implies $\partial \Omega(t)$ is $C^1$ for every $t \in [0, T]$, we shift $x^\mu \in \mathcal{D}_t$ to $x^3 = 0$ by a rigid map and it turns to the previous case. From now on, we only concentrate on the case $\xi$ is on the origin.

By (B.3), we have
\[
\Psi(t, x) := \varphi^3(0, t, x) - \psi(\varphi^1(0, t, x), \varphi^2(0, t, x)) = 0
\]
for $(t, x) \in \tilde{\mathcal{D}}_t$ to represent the surface $\mathcal{D}_t \subset \tilde{\mathcal{D}}_t$. Therefore, $\Psi \in C^1$. Note $\varphi^{(0,t)} \circ \varphi^{(t,0)} = 1$ ($1$ denotes an identity map), i.e. $\varphi(0, t, \varphi(t, 0, \xi)) = \xi$, then by chain rules, differentiating this identity with respect to the time $t$, we obtain
\[
\partial_t \varphi^j(0, t, x) + w^i \partial x^i = 0. \tag{B.4}
\]

Next, let us calculate the gradient of $\Psi$,
\[
\partial \mu \Psi = \partial \mu \varphi^3(0, t, x) - (\partial \xi^3 \psi) \partial \mu \varphi^1(0, t, x) - (\partial \xi^2 \psi) \partial \mu \varphi^2(0, t, x).
\]
That is, with the help of (B.4),
\[
\partial_t \Psi = - w_i \partial x^i + (\partial \xi^3 \psi) w_i \partial x^i + (\partial \xi^2 \psi) w_i \partial x^i
\]
and
\[
\partial_k \Psi = \frac{\partial \xi^3}{\partial x^k} \partial \xi^3 \psi \partial \xi^1 \psi \partial \xi^2 \psi \partial \xi^k \partial x^k.
\]
Since $\varphi$ is a flow, it is invertible (diffeomorphism), $\det(\frac{\partial \xi^i}{\partial x^k})_{3 \times 3} \neq 0$ and $\frac{\partial \xi^i}{\partial x^k}(t, x) = \delta^i_k$ due to $\varphi(0, 0, \xi) \equiv \xi$. Then a normal vector $n_1$ of the three dimensional hypersurface $\tilde{\mathcal{D}}_t$ at the origin in $\mathbb{R}^4$ is,
\[
n_1 := \partial \mu \Psi(0, 0, 0, 0) = (-w^3, 0, 0, 1) \neq (1, 0, 0, 0) =: n_0 \tag{B.5}
\]
where $n_0$ is a unit normal vector of $\mathcal{D}_b$. This means $\tilde{\mathcal{D}}_t$ and $\mathcal{D}_b$ are not tangential to each other at the origin.

Since $\partial_t \Psi(0, 0, 0, 0) = 1 \neq 0$ as indicated by (B.5), with the help of the implicit function theorem, there is a small neighborhood $C_0$ of the origin and a function $f \in C^1$, such that in this neighborhood, $x^3 = f(x^0, x^1, x^2)$ satisfying $\xi^3 = f(0, \xi^1, \xi^2)$, i.e., $0 = f(0, 0, 0)$. Since Theorem 2.15 implies $\partial \Omega(t)$ is $C^1$ for every $t \in [0, T]$, above arguments hold for every point $x^\mu \in [0, T] \times \partial \Omega(t)$ by shifting every $t > 0$ to 0 (this shift is included in the following rigid map $\Sigma_1$). That is, for every $(t, x) \in [0, T] \times \partial \Omega(t)$, there is a rigid map $\Sigma_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ with $\Sigma_1(t, x) = 0$ and a neighborhood $C_1 := B(0, d_1) \times (h_1, h_1) \subset C_0$, along with a $C^1$ function $\tilde{f} : \mathbb{R}^3 \supset B(0, d_1) \rightarrow (-h_1, h_1)$ ($\tilde{f}$ is a function that $f$ compose a certain rotation such that $\nabla \tilde{f}(0) = 0$, we omit the detailed expression), such that $\tilde{f}(0) = 0, \nabla \tilde{f}(0) = 0$, and Theorem C.11, (B.3) and (B.2) imply
\[
C_1 \cap \Sigma_1([0, T] \times \Omega(t)) = \{x^\mu \in C_1 \mid \Psi(t, x) > 0\} = \{x^\mu \in C_1 \mid x^3 > \tilde{f}(t, x^1, x^2)\};
\]
\[
C_1 \cap \Sigma_1([0, T] \times \partial \Omega(t)) = \{x^\mu \in C_1 \mid \Psi(t, x) = 0\} = \{x^\mu \in C_1 \mid x^3 = \tilde{f}(t, x^1, x^2)\}.
\]
This concludes $\mathcal{D}_t = [0, T] \times \partial \Omega(t) \in C^1$. 
B.2.2. Step 2: Construction of local graph. In order to prove $D_l \cup D_b$ is Lipschitz, we focus on the points on $D_l \cap D_b$ and see if in neighborhoods of these points, $D_l \cup D_b$ can be represented by a local graph of a Lipschitz function. In order to construct a local Lipschitz function (and to avoid the multi-valued function appears), we introduce a rotation in $\mathbb{R}^4$, and will rotate the original coordinate $\{x^\mu\}$ to a new one $\{\hat{x}^\mu\}$, such that $D_l \cup D_b$ is locally graph-represented by a suitable Lipschitz function in $\{\hat{x}^\mu\}$.

$$\mathcal{R}_\vartheta := \begin{pmatrix} \cos \vartheta & 0 & 0 & -\sin \vartheta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin \vartheta & 0 & 0 & \cos \vartheta \end{pmatrix}$$

where $\vartheta \in (0, \frac{\pi}{2})$ is given by

$$\tan \vartheta = w^3 + \sqrt{1 + (w^3)^2} > 0. \quad \text{(B.6)}$$

From this, we also know the inverse of the rotation is $\mathcal{R}_\vartheta^{-1} = \mathcal{R}_{-\vartheta}$.

Then we obtain a coordinate transform $x^\mu = \mathcal{R}_\vartheta \hat{x}^\mu$ for fixed $\vartheta$ defined by (B.6). By reexpressing $\Psi(x^\mu) = 0$, for $x^\mu \in D_l$, in terms of $\hat{x}^\mu$, we obtain $\Psi \circ \mathcal{R}_\vartheta(\hat{x}^\mu) = 0$. We still, with loss of generality, focus on the case $x^\mu = (0, \xi^i) = (0, 0, 0, 0) \in \overline{D_l} \cap \overline{D_b}$. For other general cases if $x^\mu$ is not the origin, we use a rigid map to move it to the origin as before.

By assuming for a point $x^\mu = (0, 0, 0, 0)$, we know $\Psi \circ \mathcal{R}_\vartheta(0, 0, 0, 0) = 0$ and $\partial_{x^\mu}(\Psi \circ \mathcal{R}_\vartheta) = \partial_{\hat{x}^\mu} \Psi$. $(\cos \vartheta, 0, 0, \sin \vartheta) = (-w^3, 0, 0, 1) \cdot (\cos \vartheta, 0, 0, \sin \vartheta) = -w^3 \cos \vartheta + \sin \vartheta = \cos \vartheta \sqrt{1 + (w^3)^2} \neq 0$. Then with the help of the implicit theorem, there is a small neighborhood $0 \in \mathcal{C}_2 := B(0, d_2) \times (-h_2, h_2) \subset \mathbb{R}^3 \times \mathbb{R}$ and in it there is a function $\hat{f} \in C^1(B(0, d_2))$, such that $\hat{x}^0 = \hat{f}(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ satisfying $\hat{f}(0, 0, 0) = 0$ and

$$\begin{align*}
\partial_{\hat{x}^1} \hat{f}\big|_{(0,0,0)} &= -\frac{\partial_{\hat{x}^2}(\Psi \circ \mathcal{R}_\vartheta)}{\partial_{x^\mu}(\Psi \circ \mathcal{R}_\vartheta)} = 0 \quad \text{and} \quad \partial_{\hat{x}^1} \hat{f}\big|_{(0,0,0)} = -\frac{\partial_{\hat{x}^1}(\Psi \circ \mathcal{R}_\vartheta)}{\partial_{x^\mu}(\Psi \circ \mathcal{R}_\vartheta)} = 0, \\
\partial_{\hat{x}^3} \hat{f}\big|_{(0,0,0)} &= -\frac{\partial_{\hat{x}^3}(\Psi \circ \mathcal{R}_\vartheta)}{\partial_{x^\mu}(\Psi \circ \mathcal{R}_\vartheta)} = -\frac{w^3 \sin \vartheta + \cos \vartheta}{-w^3 \cos \vartheta + \sin \vartheta} = \tan \vartheta + \frac{1}{(w^3 - \tan \vartheta) \cos^2 \vartheta} \\
&= \tan \vartheta - \frac{1}{\sqrt{1 + (w^3)^2} \cos^2 \vartheta}. \quad \text{(B.8)}
\end{align*}$$

Then because of $\hat{f} \in C^1(B(0, d_2))$, we obtain $\partial_{\hat{x}^k} \hat{f} \in C^0(B(0, d_2))$, i.e., for any $\epsilon \in (0, \frac{1}{\sqrt{1 + (w^3)^2}})$, there is a constant $\delta > 0$, such that if $|\hat{x}| < \delta$, then $|\partial_{\hat{x}^k} \hat{f}(\hat{x}) - \partial_{\hat{x}^k} \hat{f}(0)| < \epsilon$.

Denote $\mathcal{C}_4 := \mathcal{R}_\vartheta \big( B(0, \delta) \times (-h_2, h_2) \big)$ and define

$$g(x^1, x^2, x^3) := \hat{f}(x^1, x^2, x^3 \cos \vartheta) - x^3 \sin \vartheta$$

\[9\text{In fact, } 2\vartheta \text{ is the angular between } n_1 \text{ and } n_0 \text{ (recall (B.5)),}

$$2\vartheta = \arccos \frac{-w^3}{\sqrt{1 + (w^3)^2}}.$$  

This can be verified by noting $\vartheta \in (0, \frac{\pi}{2})$ and

$$\cos(2\vartheta) = \frac{1 - \tan^2 \vartheta}{1 + \tan^2 \vartheta} = \frac{-w^3}{\sqrt{1 + (w^3)^2}}.$$
Let us now prove
\[ \mathcal{C}_4 \cap \overline{\mathcal{D}_t} \cap \overline{\mathcal{D}_b} = \mathcal{C}_4 \cap \{(0) \times \partial \Omega(0)\} = \{(0, x^1, x^2, x^3) \in \mathcal{C}_4 \mid g(x^1, x^2, x^3) = 0\} \]  \hspace{1cm} (B.9)
and
\[ \mathcal{C}_4 \cap \mathcal{D}_b = \mathcal{C}_4 \cap \{(0) \times \Omega(0)\} = \{(0, x^1, x^2, x^3) \in \mathcal{C}_4 \mid g(x^1, x^2, x^3) < 0\}. \]  \hspace{1cm} (B.10)

First we note for any \( x^\mu \in \mathcal{C}_4 \cap \overline{\mathcal{D}_t} \cap \overline{\mathcal{D}_b} \), since \( x^\mu \in \overline{\mathcal{D}_t} \), we have \( \dot{x}^0 = \hat{f}(\dot{\hat{x}}^1, \dot{\hat{x}}^2, \dot{\hat{x}}^3) = \hat{f}(x^1, x^2, -x^0 \sin \vartheta + x^3 \cos \vartheta) \), and since \( \dot{x}^0 = x^0 \cos \vartheta + x^3 \sin \vartheta \), and \( x^\mu \in \overline{\mathcal{D}_b} \) implies \( x^0 = 0 \), we obtain \( \hat{f}(x^1, x^2, x^3 \cos \vartheta) = x^3 \sin \vartheta \), then \( g(x^1, x^2, x^3) = 0 \), i.e., we complete (B.9).

Let us prove (B.10) now. Firstly, with the help of \( \ddot{x}^3 = -x^0 \sin \vartheta + x^3 \cos \vartheta \) (since \( x^0 = 0 \) for \( x^\mu \in \mathcal{D}_b \)) and the mean value theorem, there is a point \( \mathbf{x}_0 = \lambda_1 \mathbf{x} \) for some constant \( \lambda_1 \in (0, 1) \),
\[ \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) - \dot{x}^3 \tan \vartheta = \dot{x}^1 \partial_{\dot{x}^1} \hat{f}(\mathbf{x}_0) + \dot{x}^2 \partial_{\dot{x}^2} \hat{f}(\mathbf{x}_0) + \dot{x}^3 \partial_{\dot{x}^3} \hat{f}(\mathbf{x}_0) - \dot{x}^3 \tan \vartheta \]
then for \( (0, 0, 0, x^3) \in \mathcal{C}_4 \cap \mathcal{D}_b \) \( (x^3 = \xi^3 > 0) \), using \( \epsilon < \frac{1}{\sqrt{1 + (w^3)^2}} \) and (B.7)-(B.8),
\[ g(0, 0, x^3) = \hat{f}(0, 0, \dot{x}^3) - \dot{x}^3 \tan \vartheta < \dot{x}^3 \partial_{\dot{x}^3} \hat{f}(0) + \dot{x}^3 \epsilon - \dot{x}^3 \tan \vartheta \]
\[ = -\frac{\dot{x}^3}{\sqrt{1 + (w^3)^2}} + \dot{x}^3 \epsilon = -\frac{1}{\sqrt{1 + (w^3)^2}} + \epsilon \cos \vartheta \]
\[ \xi^3 < 0, \]
Next we prove for any \( (0, y^1, y^2, y^3) \in \mathcal{C}_4 \cap \mathcal{D}_b \), \( g(y^1, y^2, y^3) < 0 \). We prove this by contradictions. Assume \( g(y^1, y^2, y^3) \geq 0 \). We focus on \( g(y^1, y^2, y^3) > 0 \) (since \( g(y^1, y^2, y^3) = 0 \) implies \( (0, y^1, y^2, y^3) \in \mathcal{C}_4 \cap \overline{\mathcal{D}_t} \cap \overline{\mathcal{D}_b} \), this leads to a contradiction), then there is a connected open set \( \mathcal{O} \), such that \( (0, 0, 0, x^3) \) and \( (0, y^1, y^2, y^3) \in \mathcal{O} \subset \mathcal{C}_4 \cap \mathcal{D}_b \). Then by the continuity of \( g \) and intermediate value theorem, there is a point \( (0, z^1, z^2, z^3) \in \mathcal{O} \), such that \( g(z^1, z^2, z^3) = 0 \), which implies \( (0, z^1, z^2, z^3) \in \mathcal{C}_4 \cap \overline{\mathcal{D}_t} \cap \overline{\mathcal{D}_b} \), this leads to a contradiction. Therefore, \( g(y^1, y^2, y^3) < 0 \).

Gathering above facts together, we conclude for every point in \( \mathcal{D}_t \cap \mathcal{D}_b \), there is a rigid map \( \mathcal{F}_4 \) (including above rigid movements and rotation \( \mathfrak{R}_\vartheta \)), a small neighborhood \( \mathcal{C}_4 := B(0, \delta) \times (-h_0, h_0) \), along with a function defined by
\[ \mathcal{F}(\dot{x}^1, \dot{x}^2, \dot{x}^3) := \max\{ \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3), \dot{x}^3 \tan \vartheta \} \]
for \( \dot{x}^i \in \overline{B(0, \delta)} \), such that
\[ \mathcal{D}_4 \cap \mathcal{F}_4(\mathcal{D}_t \cap \mathcal{D}_b) = \{ \dot{x}^\mu \in \mathcal{C}_4 \mid \dot{x}^0 = \mathcal{F}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \}; \]
\[ \mathcal{D}_4 \cap \mathcal{F}_4((0, T) \times \Omega(t)) = \{ \dot{x}^\mu \in \mathcal{C}_4 \mid \dot{x}^0 > \mathcal{F}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \}. \]

Firstly, let us first prove (B.13). Since \( \mathcal{C}_4 \cap \mathcal{F}_4(\mathcal{D}_t \cap \mathcal{D}_b) = (\mathcal{C}_4 \cap \mathcal{D}_t \mathcal{D}_b) \cup (\mathcal{C}_4 \cap \mathcal{D}_b \mathcal{F}_4) \). If a point \( x^\mu \in \mathcal{D}_t \cap \mathcal{C}_4 \), then its time component \( t = x^0 \geq 0 \), i.e., \( x^0 = \dot{x}^0 \cos \vartheta - \dot{x}^3 \sin \vartheta \geq 0 \) and it implies \( \dot{x}^0 = \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \geq \dot{x}^3 \tan \vartheta \). In this case, it means (B.12), i.e., \( \dot{x}^0 = \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) = \max\{ \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3), \dot{x}^3 \tan \vartheta \} = \mathcal{F}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \). On the other hand, if a point \( x^\mu \in \mathcal{D}_t \cap \mathcal{C}_4 \), then its time component \( t = x^0 = 0 \), i.e., \( x^0 = \dot{x}^0 \cos \vartheta - \dot{x}^3 \sin \vartheta = 0 \) and it implies \( \dot{x}^0 = \dot{x}^3 \tan \vartheta \), and by (B.10), we arrive at \( g(x^1, x^2, x^3) = \hat{f}(x^1, x^2, x^3 \cos \vartheta) - x^3 \sin \vartheta < 0 \), and by using \( \dot{x}^\mu = \dot{x}^\mu = \mathfrak{R}_\vartheta x^\mu \) and \( x^0 = 0 \), we obtain \( \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) < \dot{x}^3 \tan \vartheta \). This yields \( \dot{x}^0 = \dot{x}^3 \tan \vartheta = \mathcal{F}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \).

Secondly, let us prove (B.14). We first recall \( (0, 0, 0, x^3) \in \mathcal{C}_4 \cap \mathcal{D}_b \) (for \( x^3 > 0 \)) satisfies \( \dot{x}^0 > \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \) due to the fact that \( \dot{x}^\mu = \mathfrak{R}_\vartheta x^\mu \) and \( x^3 \sin \vartheta \geq \hat{f}(0, 0, x^3 \cos \vartheta) \) (see (B.11)). Then, by intermediate value theorem again, we claim for any point \( x^\mu \in (0, T) \times \Omega(t) \), on one hand, \( \dot{x}^0 > \hat{f}(\dot{x}^1, \dot{x}^2, \dot{x}^3) \) for \( \dot{x}^\mu = \mathfrak{R}_\vartheta x^\mu \) (otherwise, proofs by contradictions, there is
Lemma C.3. If boundary and also satisfies the Calderón's uniform cone condition. On the other hand, for any point \( x^0 \in (0, T) \times \Omega(t) \), since \( x^0 = \hat{x}^0 \cos \theta - \hat{x}^3 \sin \theta > 0 \), we obtain \( \hat{x}^0 > \hat{x}^3 \tan \theta \). Therefore, \( \hat{x}^0 > \tilde{\mathfrak{f}}(\hat{x}^1, \hat{x}^2, \hat{x}^3) \).

In the end, let us verify that \( \tilde{\mathfrak{f}} \) is a Lipschitz function. We only focus on the case for two points \( \hat{x}^i_1 \) and \( \hat{x}^i_2 \in B(0, \delta) \), such that \( \tilde{\mathfrak{f}}(\hat{x}^i_1) = \hat{f}(\hat{x}^i_1) \) and \( \tilde{\mathfrak{f}}(\hat{x}^i_2) = \hat{f}(\hat{x}^i_2) = \hat{x}^3_2 \tan \theta \) (i.e., \( \hat{f}(\hat{x}^i_1, \hat{x}^2_1, \hat{x}^3_1) \geq \hat{x}^3_2 \tan \theta \) and \( \hat{f}(\hat{x}^i_2, \hat{x}^2_2, \hat{x}^3_2) \leq \hat{x}^3_2 \tan \theta \)). For other cases, they are either similar to above case or, by using \( \hat{f} \in C^1 \) and \( \hat{x}^3 \tan \theta \in C^1 \), we can obtain the Lipschitz inequality directly. Since \( \hat{f}(\hat{x}^i) - \hat{x}^3 \tan \theta \) is continuous and \( \hat{f}(\hat{x}^i_1) - \hat{x}^3_1 \tan \theta \geq 0 \), \( \hat{f}(\hat{x}^i_2) - \hat{x}^3_2 \tan \theta \leq 0 \), then there exists a point \( \hat{x}^i_0 \in B(0, \delta) \), such that \( \hat{f}(\hat{x}^i_0) - \hat{x}^3_0 \tan \theta = 0 \), then

\[
|\tilde{\mathfrak{f}}(\hat{x}^i_1) - \tilde{\mathfrak{f}}(\hat{x}^i_2)| = |\hat{f}(\hat{x}^i_1) - \hat{x}^3_1 \tan \theta| = |\hat{f}(\hat{x}^i_1) - \hat{f}(\hat{x}^i_0) + \hat{x}^3_0 \tan \theta - \hat{x}^3_2 \tan \theta| \\
\leq |\hat{f}(\hat{x}^i_1) - \hat{f}(\hat{x}^i_0)| + |\hat{x}^3_0 \tan \theta - \hat{x}^3_2 \tan \theta| \leq K(|\hat{x}^i_1 - \hat{x}^i_0| + |\hat{x}^i_2 - \hat{x}^i_0|) = K|\hat{x}^i_1 - \hat{x}^i_2|
\]

where \( K := \max\{|\tan \theta|, \|\nabla \hat{f}\|_{L^\infty}\} \). Then we obtain \( \tilde{\mathfrak{f}} \) is a Lipschitz function. This means \( \mathcal{D}_l \cup \mathcal{D}_b \) is a Lipschitz boundary. Furthermore, the similar proofs arrive at \( (\mathcal{D}_l \cup \mathcal{D}_b) \cup \mathcal{D}_l \) is of Lipschitz as well, then we complete the proof.

**Appendix C. Tools of geometry**

**C.1. Regularity of boundary and the geometric description.** Let us first recall the definition of regularities of the boundary. See, for example, [3].

**Definition C.1.** For a proper, non-empty, open subset \( \Omega \subset \mathbb{R}^n \), we say the boundary \( \partial \Omega \) is of class \( C^k \) (or Lipschitz) locally at a point \( \mathbf{x} \in \partial \Omega \), if it can be expressed as a graph of a \( C^k \) (or Lipschitz) function locally at a neighborhood of \( \mathbf{x} \), that is, at the point \( \mathbf{x} \in \partial \Omega \), there is a composition of a rotation and a translation (i.e., rigid map) \( T : \mathbb{R}^n \to \mathbb{R}^n \) with \( T(\mathbf{x}) = 0 \), and a neighborhood \( \mathcal{C} := B_d(0) \times (-h, h) \subset \mathbb{R}^n \) where constants \( d, h > 0 \), along with a \( C^k \) (or Lipschitz) function \( \phi : \mathbb{R}^{n-1} \supset B_d(0) \to (-h, h) \), such that \( \phi(0) = 0 \), \( \nabla \phi(0) = 0 \), and

\[
\mathcal{C} \cap T(\Omega) = \{ (x', x^n) \in \mathcal{C} | x^n < \phi(x') \}; \\
\mathcal{C} \cap T(\partial \Omega) = \{ (x', x^n) \in \mathcal{C} | x^n = \phi(x') \}.
\]

Furthermore, we say the boundary \( \partial \Omega \) is of class \( C^k \) (or Lipschitz) or \( \Omega \) is a \( C^k \) (or Lipschitz) domain, if \( \partial \Omega \) is of class \( C^k \) (or Lipschitz) locally at every point \( \mathbf{x} \in \partial \Omega \), and we simply denote this by \( \partial \Omega \in C^k \) (or \( \partial \Omega \in \text{Lip} \)).

Another important domain we will mention is the one with the Calderón’s uniform cone condition, this domain has a good extension properties (see §D.1) allowing us to obtain the expected extension.

**Definition C.2 (Calderón’s Uniform Cone Condition (see [2,8])).** A domain \( \Omega \subset \mathbb{R}^n \) satisfies the Calderón’s uniform cone condition if there exists a positive constant \( \epsilon > 0 \), a finite open cover \( \{ U_j \mid j = 1, 2, \ldots, \ell \} \) of \( \partial \Omega \) and a corresponding finitely many sequence \( \{ C_j \} \) of finite cones, such that

1. every point of \( \partial \Omega \) is the center of a sphere of radius \( \epsilon \) entirely contained in one of these sets \( U_j \);
2. every point of \( U_j \cap \Omega \) is the vertex of a translate of \( C_j \) entirely contained in \( \Omega \).

From these definitions, we obtain

**Lemma C.3.** If \( \Omega \) is a bounded domain and \( \partial \Omega \) is a \( C^1 \) boundary, then it is a Lipschitz boundary and also satisfies the Calderón’s uniform cone condition.
Lemma C.6. Suppose Ω is an open set in a smooth n dimensional manifold M, then for any small constant ε > 0, there is an open set D with C∞ boundary, such that Ω \ Ωε ⊂ D ⊂ Ω where Ωε is defined by (1.9) (see §1.2.4).

Proof. By the existence theorem of smooth bump functions (see [30, Proposition 2.25]), there is a smooth bump function f : M → ℝ satisfying f ≡ 1 on Ω \ Ωε, supp f ∈ Ω and 0 ≤ f ≤ 1 on M. By using above Sard’s Theorem C.5, since the set of critical values of the bump function f has measure zero, there always is a small enough regular value δ > 0 of f, such that f−1(δ) forms the boundary of domain D := f−1([δ, 1]) (see Regular Level Set Theorem in [30, Corollary 5.14]) and Ω \ Ωε ⊂ D ⊂ Ω. We have used [30, Proposition 5.47] to conclude D is a regular domain in M.

Proof. For a bounded domain Ω, there is a finite open cover Uj of ∂Ω which can be rigidly moved to $C = B_d(0) \times (-h, h)$ for some constant $d, h > 0$. If ∂Ω is $C^1$, then there is a $C^1$ function $φ$ satisfies the requirements of the definition and it is direct that this function $φ$ is a Lipschitz function on $B_d(0)$, which in turn implies this domain is a Lipschitz domain. Since there is a finite open cover in the form of $C = B_d(0) \times (-h, h)$, we can take $ε$ small enough to ensure Definition C.2.(1) holds. Then it satisfies the Calderón’s uniform cone condition.

Due to the implicit function theorem, the following theorem gives $C^k$ boundary another description in terms of $C^k$ submanifold. We highlight this equivalence only hold for $k \geq 1$ because of the implicit function theorem. Therefore, this excludes the equivalence between Lipschitz boundary and submanifold (see [18, 19]).

Theorem C.4. Let Ω be a nonempty, open, bounded subset of $\mathbb{R}^n$, $n \geq 2$, and assume that $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Then Ω is a $C^k$ domain if and only if for every point $x \in \partial Ω$ there exists an open neighborhood U of x in $\mathbb{R}^n$, $r > 0$, and a $C^k$ diffeomorphism

$$ψ = (ψ_1, \cdots, ψ_n) : U \to B(0, r),$$

for which $ψ(x) = 0$ and which satisfies

$$ψ(Ω \cap U) = B(0, r) \cap \mathbb{R}^n_+,$$
$$ψ(Ω^c \cap U) = B(0, r) \cap \mathbb{R}^n_-, $$
$$ψ(∂Ω \cap U) = B(0, r) \cap ∂\mathbb{R}^n.$$

Proof. See, for example, [3, Theorem 4.6.8] or [18, 19].

C.2. Sard’s Theorem and $C^\infty$-approximation lemma of boundary. For readers’ convenience, we cite the famous Sard’s theorem here, and more information can be found in, for instance, [30]. We will use it to prove a useful lemma of $C^\infty$-approximation of the boundary.

Theorem C.5 (Sard’s Theorem). Suppose M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.

Lemma C.6 ($C^\infty$-approximation of Boundary). Suppose $Ω$ is an open set in a smooth n dimensional manifold $M$, then for any small constant $ε > 0$, there is an open set $D$ with $C^\infty$ boundary, such that $Ω \backslash Ω_ε \subset D \subset Ω$ where $Ω_ε$ is defined by (1.9) (see §1.2.4).

Proof. By the existence theorem of smooth bump functions (see [30, Proposition 2.25]), there is a smooth bump function $f : M \to ℝ$ satisfying $f \equiv 1$ on $Ω \backslash Ω_ε$, supp $f \in Ω$ and $0 \leq f \leq 1$ on $M$. By using above Sard’s Theorem C.5, since the set of critical values of the bump function f has measure zero, there always is a small enough regular value $δ > 0$ of f, such that $f^{-1}(δ)$ forms the boundary of domain $D := f^{-1}([δ, 1])$ (see Regular Level Set Theorem in [30, Corollary 5.14]) and $Ω \backslash Ω_ε \subset D \subset Ω$. We have used [30, Proposition 5.47] to conclude D is a regular domain in M.

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10For simplicity, we have been a little vague here. To be more precise, one should shrink $h$ and $d$ a half and complement finite open sets in this form if necessary to make sure the cone all in Ω.

11The author first learned the idea of this proof from John Lee’s answer in [45].
We have to prove the boundary $\partial \mathcal{D}$ is of class $C^\infty$ next. This is because $f(x) = \delta$ for every regular point, under some local coordinate, $x = (x^1, \cdots, x^{n-1}, x^n) \in \partial \mathcal{D}$ and by letting $C^\infty \ni F(x) := f(x) - \delta$, the implicit function theorem implies there is a small neighborhood around every $x$, such that there is a function $\phi \in C^\infty$, $x^n = \phi(x^1, \cdots, x^{n-1})$. Then by Definition C.1, with suitable rotations and translations, we conclude the boundary is smooth and further complete this proof. \hfill \Box

C.3. Extension of the outward unit normal. The following theorems come from [3, Theorem 5.3.1 and Lemma 4.6.18] which are useful tools used in the proof of strong continuation principle (i.e., Theorem 3.3). We only cite the statements without proofs.

Proposition C.7 (see Lemma 4.6.18 in [3]). Suppose that $\Omega$ is a $C^k$ domain in $\mathbb{R}^m$, $m \geq 2$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$. Then the outward unit normal $n$ is a function of class $C^{k-1}$. That is, there exists an open set $U \subset \mathbb{R}^m$, which contains $\partial \Omega$, along with a function $N : U \to \mathbb{R}^m$ of class $C^{k-1}$ in $U$ with the property that $N|_{\partial \Omega} = n$.

Definition C.8. Given an arbitrary set $\Omega \subset \mathbb{R}^m$, the signed distance (to its boundary $\partial \Omega$) is the function $d : \mathbb{R}^m \to \mathbb{R}$, defined by

$$d(x) := \begin{cases} +\text{dist}(x, \partial \Omega), & \text{if } x \in \Omega \\ -\text{dist}(x, \partial \Omega), & \text{if } x \in \Omega^c \end{cases}$$

Theorem C.9 (Distinguished Extension of the Outward Unit Normal of a Domain). Assume that $\Omega \subset \mathbb{R}^m$, $m \geq 2$, is a domain of class $C^k$, for some $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$. Then, with $d$ denoting the signed distance, there exists $U$ open neighborhood of $\partial \Omega$ and a vector field

$$N = (N^1, \cdots, N^m) : U \to \mathbb{R}^m, \quad N(x) := (\nabla d)(x), \quad \text{for any } x \in U,$$

is a vector-valued function of class $C^{k-1}$ in $U$ which has the following properties:

1. $\|N(x)\| = 1$ for every $x \in U$;
2. $N|_{\partial \Omega} = n$, the outward unit normal to $\Omega$;
3. $\partial_j N_k = \partial_k N_j$ in $U$, for all $j, k \in \{1, \cdots, m\}$;
4. for every $j \in \{1, \cdots, m\}$, the directional derivative $D_N N_j$ vanished in $U$.

The next proposition from [30, Lemma 8.6] extends a smooth vector field from a closed subset to the whole manifold with a compact support.

Proposition C.10 (Global Extension for Vector Fields). Let $M$ be a smooth manifold with or without boundary, and let $\mathcal{N} \subset M$ be a closed subset. Suppose $V$ is a smooth vector field along $\mathcal{N}$. Given any open subset $U$ containing $\mathcal{N}$, there exists a smooth global vector field $\tilde{V}$ on $M$ such that $\tilde{V}|_{\mathcal{N}} = V$ and supp $\tilde{V} \subset U$.

C.4. Flows and fundamental theorems.

Theorem C.11 (Diffeomorphism Invariance of Boundary). Suppose $M$ and $N$ are $C^k$ ($k \geq 1$) manifolds with boundary and $F : M \to N$ is a $C^k$ diffeomorphism. Then $F(\partial M) = \partial N$, and $F$ restricts to a $C^k$ diffeomorphism from IntM to IntN.

We denote $V := (V^1, \cdots, V^n)$ a continuous vector field and use a dot to denote an ordinary derivative with respect to $t$ in this Appendix. Let us see the ordinary differential system,

\begin{align}
\dot{y}^i(t) &= V^i(t, y^1(t), \cdots, y^n(t)), \\
y^i(t_0) &= c^i,
\end{align}
for \( i = 1, \cdots, n \), where \((t_0, c) := (t_0, c^1, \cdots, c^n)\) is an arbitrary point. We first introduce an important Theorem C.12 without proofs, see [30, Theorem D.5, D.6] (or [27, Theorem 1.11], [26, Chapter XIV, §4]) for details.

**Theorem C.12.** Let \( J \subset \mathbb{R} \) be an open interval and \( U \subset \mathbb{R}^n \) be an open subset, and let \( V : J \times U \to \mathbb{R}^n \) be locally Lipschitz continuous and a \( C^k \) vector-valued function for some \( k \geq 1 \).

(1) (Existence) For any \( s_0 \in J \) and \( x_0 \in U \), there exists an open interval \( J_0 \subset J \) containing \( s_0 \) and an open subset \( U_0 \subset U \) containing \( x_0 \), such that for each \( t_0 \in J_0 \) and \( c = (c^1, \cdots, c^n) \in U_0 \), there is a \( C^k \) map \( y : J_0 \to U \) that solves (C.1)–(C.2).

(2) (Uniqueness) Any two differentiable solutions to (C.1)–(C.2) agree on their common domain.

(3) (Regularity) Let \( J_0 \) and \( U_0 \) be as in (1), and define a map \( \theta : J_0 \times J_0 \times U_0 \to U \) by letting \( \theta(t, t_0, c) = y(t) \), where \( y : J_0 \to U \) is the unique solution to (C.1)–(C.2). Then \( \theta \) is of class \( C^k \).

Using above Theorem C.12, we have the fundamental theorem on time-dependent flows. The proof can be found in [30, Theorem 9.48] and [27, IV, §1]

**Theorem C.13** (Fundamental Theorem on Time-Dependent Flows). Let \( M \) be a smooth manifold, let \( I \subset \mathbb{R} \) be an open interval, and let \( V : I \times M \to TM \) be a \( C^k \) \((k \geq 1)\) time-dependent vector field on \( M \). Then there exists an open subset \( E \subset I \times I \times M \) and a \( C^k \) map \( \varphi : E \to M \) called the time-dependent flow of \( V \), with the following properties:

(a) For each \( t_0 \in I \) and \( p \in M \), the set \( E_{(t_0,p)} = \{ t \in I \mid (t, t_0, p) \in E \} \) is an open interval containing \( t_0 \), and the curve \( \varphi_{(t_0,p)} : E_{(t_0,p)} \to M \) defined by \( \varphi_{(t_0,p)}(t) = \varphi(t, t_0, p) \) is the unique maximal integral curve of \( V \) with initial condition \( \varphi_{(t_0,p)}(t_0) = p \).

(b) If \( t_1 \in E_{(t_0,p)} \) and \( q = \varphi_{(t_0,p)}(t_1) \), then \( E_{(t_1,q)} = E_{(t_0,p)} \) and \( \varphi_{(t_1,q)} = \varphi_{(t_0,p)} \).

(c) For each \((t_1, t_0) \in I \times I \), the set \( M_{t_1,t_0} = \{ p \in M \mid (t_1, t_0, p) \in E \} \) is open in \( M \), and the map \( \varphi_{(t_1,t_0)} : M_{t_1,t_0} \to M \) defined by \( \varphi_{(t_1,t_0)}(p) = \varphi(t_1, t_0, p) \) is a diffeomorphism from \( M_{t_1,t_0} \) onto \( M_{t_0,t_1} \) with inverse \( \varphi_{(t_0,t_1)}^{-1} \).

(d) If \( p \in M_{t_1,t_0} \) and \( \varphi_{(t_1,t_0)}(p) \in M_{t_2,t_1} \), then \( p \in M_{t_2,t_0} \) and \( \varphi_{(t_2,t_1)} \circ \varphi_{(t_1,t_0)}(p) = \varphi_{(t_2,t_0)}(p) \).

**Appendix D. Tools of analysis**

**D.1. Calderón extension theorem.** This section states an important tool for the existence theorem. First simple \((m,p)\)-extension operators are introduced for expressing the extension theorem. These definitions and theorems can be found in [2, 8].

**Definition D.1** (Simple \((m,p)\)-extension Operator). Let \( \Omega \) be a domain in \( \mathbb{R}^n \). For given numbers \( m \) and \( p \), a linear operator \( E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n) \) is called a simple \((m,p)\)-extension operator for \( \Omega \) if there is a constant \( K = K(m,p) \) such that for every \( u \in W^{m,p}(\Omega) \), the following conditions hold,

(1) \( Eu(x) = u(x) \) a.e. in \( \Omega \);

(2) \( \| Eu \|_{W^{m,p}(\mathbb{R}^n)} \leq K \| u \|_{W^{m,p}(\Omega)} \).

**Theorem D.2** (Calderón Extension Theorem). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) satisfying the Calderón’s uniform cone condition. Then for any \( m \in \{1, 2, \cdots\} \) and any \( p \) satisfying \( 1 < p < \infty \), there exists a simple \((m,p)\)-extension operator \( E = E(m,p) \) for \( \Omega \).

**Proof.** See [2, Theorem 5.28] or [8] for the construction of this extension and the detailed proof. \( \Box \)
D.2. Stein extension theorem. If the domain $\Omega \subset \mathbb{R}^n$ is more regular, for example, a Lipschitz domain (in fact, see [52], it requires the boundary with a weaker condition, minimal smoothness condition), there is a better and universal extension operator given by E. Stein [52, Chapter V I §3]

**Theorem D.3** (Stein Extension Theorem). Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then there exists a linear operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ satisfying for every $u \in W^{m,p}(\Omega)$,

1. $Eu(x) = u(x)$ a.e. in $\Omega$;
2. there is a constant $K > 0$ such that $\|Eu\|_{W^{m,p}(\mathbb{R}^n)} \leq K\|u\|_{W^{m,p}(\Omega)}$ for all $p \in [1, \infty]$ and $m \in \mathbb{Z}_{\geq 0}$.

**Remark D.4.** For the case $E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$, Evans [16, §5.4] gives another extension theorem provided $\Omega$ is bounded and a $C^1$ domain.

D.3. Gagliardo–Nirenberg–Moser and Moser estimates on bounded domains. The famous Gagliardo–Nirenberg–Moser inequalities are widely used in energy estimates and can be found in many references. Since this article requires these inequalities on a bounded domain, we state them here (see [1, 7, 43]) and derive some useful corollaries which are the bounded domain version of inequalities in [54, §13.3].

In this section, we will always assume $\Omega$ is a *standard domain* in $\mathbb{R}^n$, that means $\Omega$ is $\mathbb{R}^n$, a half space or a *Lipschitz bounded domain* in $\mathbb{R}^n$ (in fact, to serve this paper, we emphasize estimates on the Lipschitz bounded domain). A key condition in the following theorem is given by

$$s_2 \geq 1 \text{ is an integer, } p_2 = 1 \text{ and } s_1 - \frac{1}{p_1} \geq s_2 - \frac{1}{p_2}. \quad (D.1)$$

**Theorem D.5** (General Gagliardo–Nirenberg–Moser Inequalities, see [7]). Assume $\Omega$ is a standard domain in $\mathbb{R}^n$, the real numbers $s, s_1, s_2 \geq 0$, $\theta \in (0, 1)$ and $1 \leq p_1, p_2, p \leq \infty$ satisfy the relations

$$s = \theta s_1 + (1 - \theta)s_2, \text{ and } \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}.$$  

1. If Condition (D.1) fails, then for every $\theta \in (0, 1)$, there exists a constant $C$ depending on $s_1, s_2, p_1, p_2, \theta$ and $\Omega$ such that

$$\|f\|_{W^{s,p}(\Omega)} \leq C\|f\|_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}$$

for all $f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$.

2. If Condition (D.1) holds, there exists some $f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$ such that $f \notin W^{s,p}(\Omega)$ for any $\theta \in (0, 1)$.

**Proof.** See the detailed proof of [7, Theorem 1].

By letting $s = \ell$, $p = 2k/\ell$, $\theta = \ell/k$, $s_1 = k$, $p_1 = 2$, $s_2 = 0$ and $p_2 = \infty$, the following immediate result (D.2) (we also refer to it as Gagliardo–Nirenberg–Moser inequality) derived from Theorem D.5 will play a crucial role in the later estimates,

$$\|f\|_{W^{s,p}(\Omega)} \leq C\|f\|_{H^k(\Omega)}\|f\|^{1-\frac{\ell}{k}}_{L^\infty(\Omega)} \quad (D.2)$$

for all $0 < \ell < k$ and $f \in H^k(\Omega) \cap L^\infty(\Omega)$. The following estimates can be derived in the similar procedures to estimates in [54, §13.3] with minor revisions,
Proposition D.6. If $\Omega$ is a standard domain in $\mathbb{R}^n$ and $|\beta| + |\gamma| = k$, then
\[
\|(D^\beta f)(D^\gamma g)\|_{L^2(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)} + C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}
\]
for all $f, g \in H^k(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $|\beta| = \ell, |\gamma| = m$ and $\ell + m = k$, then using Hölder’s inequality, Gagliardo–Nirenberg–Moser inequality (D.2) and Young’s inequality for conjugate Hölder exponents, in turn, yield
\[
\|(D^\beta f)(D^\gamma g)\|_{L^2(\Omega)} \leq \|D^\beta f\|_{L^\frac{2}{\ell}(\Omega)}\|D^\gamma g\|_{L^\frac{2}{m}(\Omega)} \leq C\|f\|_{H^k(\Omega)}\|f\|_{L^\infty(\Omega)}^{1-\frac{\ell}{k}}\|g\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}^{1-\frac{m}{k}}
\]
\[
= C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)} + C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}.
\]
This completes the proof. \qed

Next, using above Gagliardo–Nirenberg–Moser inequality (D.2) and Proposition D.6 to replace inequality (3.17) and Proposition 3.6 in [54, §13.3] (also see [23,24,36], etc.) respectively, almost the same calculations as Proposition 3.7, 3.9 and Lemma 3.10 in [54, §13.3], we obtain the following three assertions. Omitting the detailed proofs, we only list the results below due to the similarities.

Corollary D.7. If $\Omega$ is a standard domain in $\mathbb{R}^n$ and $|\beta_1| + \cdots + |\beta_\mu| = k$, then
\[
\|f^{(\beta_1)}_1 \cdots f^{(\beta_\mu)}_\mu\|_{L^2(\Omega)} \leq C\sum_\nu \|f^{(\beta_1)}_1\|_{L^\infty(\Omega)} \cdots \|f^{(\beta_\mu)}_\mu\|_{L^\infty(\Omega)} \|f^{(\nu)}_\nu\|_{H^k(\Omega)}
\]
for all $f_\ell \in H^k(\Omega) \cap L^\infty(\Omega)$ ($\ell = 1, \cdots, \mu$).

Proposition D.8 (Moser Estimates I). Suppose $\Omega$ is a standard domain in $\mathbb{R}^n$, we have the estimates
\[
\|fg\|_{H^k(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)} + C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)}
\]
and for $|\alpha| \leq k$,
\[
\|D^\alpha(fg) - f D^\alpha g\|_{L^2(\Omega)} \leq C\|\nabla f\|_{H^{k-1}(\Omega)}\|g\|_{L^\infty(\Omega)} + C\|\nabla f\|_{L^\infty(\Omega)}\|g\|_{H^{k-1}(\Omega)}.
\]

Proposition D.9 (Moser Estimates II). Suppose $\Omega$ is a standard domain in $\mathbb{R}^n$, $F \in C^k(\mathbb{R})$ ($k \in \mathbb{Z}_{\geq 1}$) and $F(0) = 0$. Then, for $u \in H^k(\Omega) \cap L^\infty(\Omega)$ and $u(x) \in \mathcal{G}$ where $\mathcal{G}$ is open and bounded set in $\mathbb{R}$,
\[
\|F(u)\|_{H^k(\Omega)} \leq C_k(\|DF\|_{C^{k-1}(\mathbb{R})}) (1 + \|u\|_{L^\infty(\Omega)}^{k-1})\|u\|_{H^k(\Omega)}.
\]

D.4. Ferrari–Shirota–Yanagisawa estimate. Ferrari–Shirota–Yanagisawa estimate firstly given by Ferrari [17] and Shirota, Yanagisawa [50] respectively in 1993 is a bounded domain version of Beale–Kato–Majda estimate (in $\mathbb{R}^3$, see [4]). We present this inequality without proof and readers may consult the details in [17,50].

Theorem D.10 (Ferrari–Shirota–Yanagisawa Estimate, [17]). Suppose $\Omega \subset \mathbb{R}^3$ is a bounded, simply connected domain of class $C^{2,\beta}$, $\beta \in (0,1)$, and $v := (u, \phi) \in H^3(\Omega)$ is a solution of the system
\[
\nabla \times u - \nabla \phi = \omega \quad \text{in} \quad \Omega,
\]
\[
\nabla \cdot u = \theta \quad \text{in} \quad \Omega,
\]
\[
u \cdot n = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
\phi = 0 \quad \text{on} \quad \partial \Omega.
\]
with \( \psi := (\omega, \theta) \in H^2(\Omega) \) and \( n \) is the unit outward normal at \( x \in \partial\Omega \), then
\[
\|v\|_{W^{1,\infty}(\Omega)} \leq C\left(1 + \ln^+ \frac{\|\psi\|_{H^2(\Omega)}}{\|\psi\|_{L^\infty(\Omega)}}\right)\|\psi\|_{L^\infty(\Omega)}
\]
where
\[
\ln^+ a := \begin{cases} \ln a, & \text{if } a \geq 1 \\ 0, & \text{otherwise} \end{cases}
\]

D.5. **Generalized Gronwall inequality of integral form.** The following form of Gronwall inequality comes from [46, Theorem 1.3.1].

**Theorem D.11** (Generalized Gronwall Inequality of Integral Form). Let \( u \) and \( f \) be continuous and nonnegative functions defined on \( J = [\alpha, \beta] \), and let \( n(t) \) be a continuous, positive and non-decreasing function defined on \( J \); then
\[
u(t) \leq n(t) + \int_{\alpha}^{t} f(s)u(s)ds,
\]
for \( t \in J \), implies that
\[
u(t) \leq n(t)\exp\left(\int_{\alpha}^{t} f(s)ds\right),
\]
for \( t \in J \).

D.6. **Moore–Osgood Theorem.** Moore–Osgood Theorem gives a sufficient condition of interchanging of limits. See [53, Page 139–140] for details.

**Theorem D.12** (Moore–Osgood Theorem). If \( \lim_{x \to p} f(x, y) \) exists pointwise for each \( y \neq q \) and if \( \lim_{y \to q} f(x, y) \) converges uniformly for \( x \neq p \), then the double limit and the iterated limits exist and are equal, i.e.,
\[
\lim_{(x,y) \to (p,q)} f(x, y) = \lim_{y \to q} \lim_{x \to p} f(x, y) = \lim_{x \to p} \lim_{y \to q} f(x, y).
\]

D.7. **Reynold’s transport theorem.** The following Reynold’s transport theorem is a multidimensional version of Leibniz integral rule, which states how to exchange the derivative and the integral if the region of integration is changing with time. The proof can be found in various references of fluid dynamics and calculus (see, for example, [42, Page 578] and [10, §1]).

**Theorem D.13** (Reynold’s Transport Theorem). Suppose a field \( C^1 \ni f : [0, T) \times \Omega(0) \to V \) and the flow \( \chi : [0, T) \times \Omega(0) \to \Omega(t) \subset \mathbb{R}^3 \) generated by a vector field \( w^i \in C^1([0, T) \times \Omega(0), \mathbb{R}^3) \), such that \( \chi(t, \xi) = x \in \Omega(t) \) for every \( (t, \xi) \in [0, T) \times \Omega(0) \) where \( T > 0 \) is a constant, \( \Omega(t) := \chi(t, \Omega(0)) \subset \mathbb{R}^3 \) is a domain depending on \( t \) and \( V \subset \mathbb{R}^n \) for some \( n \in \mathbb{Z}_{\geq 1} \). Then
\[
\frac{d}{dt} \int_{\Omega(t)} f(t, x)d^3x = \int_{\Omega(t)} \left[ \partial_t f(t, x) + \partial_i \left( f(t, x)w^i(t, x) \right) \right]d^3x.
\]

**Appendix E. Astrophysical backgrounds and models**

E.1. **Backgrounds of stellar formation in astrophysics.** Modern astrophysics suggests that there is the interstellar medium instead of the perfect vacuum between stars and galaxies. The interstellar medium, depending on where it is located, can be extremely cold or extremely hot. In the coldest regions of the interstellar medium, the cores of the clouds contain molecular gases (primarily molecular hydrogen gas). Higher temperature and starlight can provide extra energies which may break the molecules apart. Therefore, the molecular gas can survive at
the extremely cold environment and be protected by the dust (in the form of ices, silicates, graphite, metals and organic compounds) from the photons.

In the region where the gas cloud is not quite cold enough for hydrogen molecules to survive, there is a cloud of neutral hydrogen atoms (known as $H \ I$ region in astrophysics). In this article, we will focus on such cold interstellar medium including only molecules, atoms and dusts as the interstellar gas clouds or the molecular clouds (in astrophysics, they are also known as the stellar nursery, etc.).

In addition, to complete the backgrounds, we only point out that there is another interstellar gas of the $H \ II$ regions (a cloud of ionized hydrogen is called an $H \ II$ region) which located near hot stars. The ultraviolet radiation from the stars also ionizes the hydrogen. The very hot stars required to produce H II regions are rare, and only a small fraction of interstellar matter is close enough to such hot stars to be ionized by them. Most of the volume of the interstellar medium is filled with neutral (non-ionized) hydrogen. Ultra-hot interstellar gas comes from supernova.

E.2. Remarks on the mathematical model with diffuse boundary. The molecular clouds, or more generally, the interstellar mediums, are inherently complex in their structures. We prefer a family of clouds which is in a decent “onion-like” structure described in [51, §2]. In this model, the dense cloud is in the core, surrounded by translucent gas, then in turn, surrounded by more diffuse gas which is the most tenuous cloud, fully exposed to starlight. Due to the photo-dissociation, nearly all molecules in the outermost layer of the diffuse gas (the inner part of the diffuse clouds can also be molecular) are quickly destroyed and become the neutral atomic form. There, of course, is no distinct concept of “boundary” in the astrophysical model. We attempt to abstract the crucial properties of the above outermost diffuse part to be the mathematical “boundary”.

E.2.1. Astrophysicists’ ideas. In astronomer Elmegreen’s article [15, §5.2] on the classifications of spiral, he expounded that the pressure boundary for a molecular cloud is usually ignored in discussions of molecular cloud structure and dynamics because the internal or average cloud pressure is much larger than the ambient interstellar pressure when gravity is strong, as in a star-forming cloud, and he pointed out that the flow pattern of the diffuse clouds tends to be more ballistic and is away from the regime of hydrodynamics. In fact, this is because the Knudsen number $K_n$ (see, for example [25], for the theory of classical ballistic transport of rarefied gases), which is defined by the ratio of the molecular mean free path length to a characteristic physical length scale, of the outermost layer of the diffuse clouds is close to or greater then 1, that implies that the mean free path of a molecule is comparable to the diameter of the clouds, and the continuum assumption of fluid mechanics is not a good approximation any more. Consequently, the ballistic transport is observed near the “boundary” of the molecular clouds. That means the pressure exerted around a small parcel can be ignored due to the very large mean free path of the particles on the boundary. The diffuse boundary condition, thus, is a proper abstraction of this situation. In other words, the fictitious “flow” (or test particles) on the boundary are free particles only attracted by the Newtonian gravity without any thermal pressure resisting gravity. We also point out this boundary surely can not be a surface in practice, since the diffuse clouds may extended very far. However, due to the very low pressure as stated by Elmegreen [15, §5.2], we ignore all this part and view it as vacuum in the model.

E.2.2. Mathematical abstractions. In terms of mathematics, as we mentioned in §1.1, we call $\Omega(t)$ the hydrodynamic region and $\Omega^f(t)$ the ballistic region (viewed as vacuum in this model),
since in $\Omega(t)$, the dynamics of the matter is the ideally hydrodynamics-dominated gas and in $\Omega^c(t)$, rarefied ballistics-dominated gas. In ballstic region $\Omega^c(t)$, rarefied gas dynamics prevails and the rarefied gas follows Boltzmann equations (or Vlasov equation, i.e., collisionless Boltzmann equation) usually due to the large enough Knudsen number. However, rarefied gas implies there is no mass accretion (since the density of rarefied gas is very small and can be ignored to be viewed as vacuum), which does not interest us due to the motivations of star formations. By contrast, in the hydrodynamic region $\Omega(t)$, the regime of hydrodynamics applies, and usually, when the Knudsen is very small, the Euler equation governs this system, while the Knudsen is bigger but still smaller than 0.1, the Navier-Stokes equation prevails. However, the temperature is extremely low for the interested molecular clouds (dark cold clouds), therefore, for gas model, the viscosity is very small. By ignoring the extremely small viscosity, the Navier-Stokes equation becomes Euler equation. On the other hand, the decent gravitational model is the Einstein equation of the general relativity. However, due to its extreme complexities, we focus on its Newtonian limit gravity, i.e., the Newtonian gravity. Therefore, in the hydrodynamic region $\Omega(t)$, the Euler–Poisson system applies.

By viewing the ballistic region $\Omega^c(t)$ as vacuum for the hydrodynamic system we are interested in, there is a property characterized by the diffuse boundary near the moving interface between the cloud and the exterior vacuum. The wellposedness shown in this paper implies we have assumed the portion of the hydrodynamic region can be isolated from the clouds and the interaction from the exterior diffuse clouds or the ballistic region can be neglected (due to the interaction is negligible in practice).

E.2.3. About physical vacuum condition. At the end, let us briefly contrast the physical vacuum condition with the diffuse one. The physical vacuum condition implies that the flows near the boundary are still in the regime of the hydrodynamics and the Knudsen number is small up to the boundary and jumps to very large in the exterior region. The models of stars usually do not contain the diffuse area due to the existence of the phase interface.

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**Center for Mathematical Sciences, Huazhong University of Science and Technology, 1037 Luoyu Road, Wuhan, Hubei Province, China; Beijing International Center for Mathematical Research (BICMR), Peking University, No.5 Yiheyuan Road Haidian District, Beijing, China.**

**Email address:** chao.liu.math@foxmail.com