Inverse Problems for Systems of Variable Order Differential Equations with Singularities on Spatial Networks.

V. Yurko

Abstract. Variable order differential equations with non-integrable singularities are considered on spatial networks. Properties of the spectrum are established, and the solution of the inverse spectral problem is obtained.

Key words: spatial networks; differential systems; variable order; inverse problems

AMS Classification: 34A55 34B45 47E05

1. Consider a compact star-type graph $T$ in $\mathbb{R}^n$ with the set of vertices $V = \{v_0, \ldots, v_p\}$ and the set of edges $E = \{e_1, \ldots, e_p\}$, where $v_0, \ldots, v_p$ are the boundary vertices, $v_0$ is the internal vertex, and $e_j = [v_j, v_0]$, $e_1 \cap \ldots \cap e_p = \{v_0\}$. Let $l_j$ be the length of the edge $e_j$. For each edge $e_j \in E$ we introduce the parameter $x_j \in [0, l_j]$ such that $x_j = 0$ corresponds to the boundary vertices $v_1, \ldots, v_p$. A function $Y$ on $T$ may be represented as $Y = \{y_j\}_{j=1}^{\infty}$, where the function $y_j(x_j)$ is defined on the edge $e_j$.

Let $n_j$, $j = 1, p$ be positive integers such that $n_1 \geq n_2 \geq \ldots \geq n_p \geq 2$. Consider the differential equations on $T$:

$$y^{(n_j)}(x_j) + \sum_{\mu=0}^{n_j-2} \left( \frac{\nu_{\mu j}}{x_j^{n_j-\mu}} + q_{\mu j}(x_j) \right) y^{(\mu)}(x_j) = \lambda y_j(x_j), \quad x_j \in (0, l_j), \quad j = 1, p,$$

where $\lambda$ is the spectral parameter, $q_{\mu j}(x_j)$ are complex-valued integrable functions. We call $q_j = \{q_{\mu j}\}_{\mu=0, n_j-2}$ the potential on the edge $e_j$, and we call $q = \{q_j\}_{j=1}^{\infty}$ the potential on the graph $T$. In this paper we study inverse spectral problems for system (1). We provide a procedure for constructing the solution of the inverse problem and prove its uniqueness.

2. Let us construct special fundamental systems of solutions for higher order differential operators with regular singularities. Consider the differential equation

$$\ell y := y^{(n)} + \sum_{j=0}^{n-2} \left( \frac{\nu_j}{x^{n-j}} + q_j(x) \right) y^{(j)} = \lambda y, \quad x > 0$$

on the half-line. Let $\mu_1, \ldots, \mu_n$ be the roots of the characteristic polynomial

$$\Delta(\mu) = \sum_{j=0}^{n} \nu_j \prod_{k=0}^{j-1} (\mu - k), \quad \nu_n = 1, \quad \nu_{n-1} = 0.$$

It is clear that $\mu_1 + \ldots + \mu_n = n(n-2)/2$. For definiteness, we assume that $\mu_k - \mu_j \neq sn$ ($s = 0, \pm 1, \pm 2, \ldots$); $Re \mu_1 < \ldots < Re \mu_n$, $\mu_k \neq 0, 1, 2, \ldots, n-3$ (the other cases require minor modifications). Let $\theta_n = n - 1 - Re(\mu_n - \mu_1)$, $q_{0j}(x) = q_j(x)$ for $x \geq 1$, and $q_{0j}(x) = q_j(x)x^{\min(\theta_n-\mu_j, 0)}$ for $x \leq 1$ and assume that $q_{0j}(x) \in L(0, \infty)$, $j = 0, n-2$.

First of all, we consider the following differential equation without spectral parameter:

$$\ell_0 y := y^{(n)} + \sum_{j=0}^{n-2} \nu_j x^{n-j} y^{(j)} = y.$$

Let $x = r \exp(i \varphi)$, $r > 0$, $\varphi \in (-\pi, \pi]$, $x^\mu = \exp(\mu(\ln r + i \varphi))$ and $\Pi_-$ be the $x$-plane with the cut along the semi-axis $x \leq 0$. Take numbers $c_{j0}$, $j = 1, n$ from the condition

$$\prod_{j=1}^{n} c_{j0} = (\det[\mu_j^{\nu+1} \ell_j, \mu = 1, n])^{-1}.$$
Then the functions
\[
C_j(x) = x^{\mu_j} \sum_{k=0}^{\infty} c_{jk} x^{nk}, \quad c_{jk}(x) = c_{j0} (\prod_{s=1}^{k} \Delta(\mu_j + sn))^{-1}
\]  
(4)
are solutions of (3), and \( \det[C_j^{(\nu-1)}(x)]_{j,\nu=1,n} \equiv 1 \). Moreover, the functions \( C_j(x) \) are analytic in \( \Pi_\nu \). Denote
\[
\varepsilon_k = \exp \left( \frac{2\pi i (k-1)}{n} \right), \quad S_{\nu} = \left\{ x : \arg x \in \left( \frac{\nu \pi}{n}, \frac{(\nu + 1) \pi}{n} \right) \right\},
\]
\[
S_{\nu}^* = \bar{S}_{n-\nu}, \quad S_{\nu}^* = \bar{S}_{n-2\nu + 1} \cup \bar{S}_{n-2\nu + 2}, \quad k = 2, n;
\]
\[
Q_k = \left\{ x : \arg x \in \left[ \min \left( -\pi, (2k + 2) \frac{\pi}{n} \right), \min \left( \pi, (2n - 2k + 2) \frac{\pi}{n} \right) \right] \right\}, \quad k = 1, n.
\]
For \( x \in S_{\nu}^* \) equation (3) has the solutions \( e_k(x), k = 1, n \) of the form
\[
e_k^{(\nu-1)}(x) = \varepsilon_k^{\nu} \exp(\varepsilon_k x) z_{k\nu}(x), \quad \nu = 0, n - 1,
\]
where \( z_{k\nu}(x) \) are solutions of the integral equations
\[
z_{k\nu}(x) = 1 + \frac{1}{n} \int_{x}^{\infty} \left( \sum_{j=1}^{\nu} \varepsilon_j^{\nu+1} \varepsilon_k^{\nu} \exp((\varepsilon_k - \varepsilon_j)(t - x)) \right) \left( \sum_{m=0}^{n-2} \nu_m \varepsilon_k^{n-m-n} z_{km}(t) \right) dt
\]
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
It is easily seen from construction of the functions $e_k(x)$ that $e_1(e^x) = e_{x+1}(x)$. Substituting (5) in this equality and comparing the corresponding coefficients, we obtain (7). After that (8) becomes obvious.

Now we consider the differential equation

$$
\ell_0 y = \lambda y = \rho^ny, \quad x > 0.
$$

(10)

It is evident that if $y(x)$ is a solution of (3), then $y(\rho x)$ satisfies (10). Define $C_j(x, \lambda)$ by

$$
C_j(x, \lambda) = \rho^{-\nu_j}C_j(\rho x) = x^\nu_j \sum_{k=0}^\infty c_{jk}(\rho x)^{n_k}.
$$

The functions $C_j(x, \lambda)$ are entire in $\lambda$, and $\det[C_j^{(\nu-1)}(x, \lambda)]_{j,\nu=1,n} \equiv 1$. From Lemmas 1 and 2 we get the following theorem.

**Theorem 1.** In each sector $S_{k_0} = \{ \rho : \arg \rho \in (k_0\pi, (k_0+1)\pi) \}$ equation (10) has a fundamental system of solutions $B_0 = \{yk(x, \rho)\}_{k=1,n}$ such that $yk(x, \rho) = yk(\rho x)$,

$$
|yk^{(\nu)}(x, \rho)(\rho R_k))^{-\nu}\exp(-\rho R_k x) - 1| \leq \frac{M_0}{|\rho x|}, \quad \rho \in \tilde{S}_{k_0}, \quad |\rho|x \geq 1, \quad \nu = \overline{0, n-1},
$$

(11)

$$
\det[yk^{(\nu)}(x, \rho)]_{k,\nu=1,n} \equiv \rho^{n(n-1)/2}\Omega, \quad \Omega := \det[R_k^{(\nu)}]_{k,\nu=1,n} \neq 0,
$$

(12)

$$
yk(x, \rho) = \sum_{j=1}^n b_{kj}\rho^{\nu_j}C_j(x, \lambda), \quad b_{kj} = \beta_j^0 R_k^{\nu_j}, \quad \beta_j^0 \neq 0,
$$

(13)

where the constant $M_0$ depends only on $\nu_j$.

The functions $yk(x, \rho)$ are analogues of the Hankel functions for the Bessel equation. Denote

$$
C_j^{*}(x, \lambda) = \det[C_k^{(\nu)}(x, \lambda)]_{\nu=0, n-2, k=1, n\setminus n-j+1},
$$

$$
y_j^{*}(x, \rho) = (-1)^{n-j}\left(\rho^{n+1}(n-2)/2\Omega\right)^{-1}\det[yk^{(\nu)}(x, \rho)]_{\nu=0, n-2, k=1, n\setminus j},
$$

$$
F_{ku}(\rho x) = \begin{cases} 
R_k^{\nu}\exp(\rho R_k x), & |\rho|x > 1, \\
(\rho x)^{\mu_1-\nu}, & |\rho|x \leq 1,
\end{cases}
$$

$$
F_k^{*}(\rho x) = \begin{cases} 
\exp(-\rho R_k x), & |\rho|x > 1, \\
(\rho x)^{n-1-\mu_n}, & |\rho|x \leq 1,
\end{cases}
$$

$$
U_k^{0}(x, \rho) = y_k^{(\nu)}(x, \rho)(\rho^{\nu}F_{ku}(\rho x))^{-1}, \quad U_k^{0*}(x, \rho) = y_k^{*}(x, \rho)(F_k^{*}(\rho x))^{-1},
$$

$$
g(x, t, \lambda) = \sum_{j=1}^n (-1)^{n-j}C_j(x, \lambda)C_{n-j+1}(t, \lambda) = \frac{1}{\rho^{n-1}}\sum_{j=1}^n y_j(x, \rho)g_j^{*}(t, \rho).
$$

The function $g(x, t, \lambda)$ is the Green’s function of the Cauchy problem $\ell_0 y - \lambda y = f(x)$, $y^{(\nu)}(0) = 0$, $\nu = 0, n-1$. Using (11)-(13), we obtain

$$
|U_k^{0}(x, \rho)| \leq M_1, \quad |U_k^{0*}(x, \rho)| \leq M_1, \quad x \geq 0, \quad \rho \in S_{k_0},
$$

(14)

$$
|C_k^{(\nu)}(x, \lambda)| \leq M_2|x^{\mu_1-\nu}|,
$$

(15)

$$
\left| \frac{\partial^\nu}{\partial x^\nu}g(x, t, \lambda) \right| \leq M_2 \sum_{j=1}^n |x^{\mu_1-\nu}t^{n-1-\mu_j}|, \quad |\rho x| \leq C_0, \quad t \leq x,
$$

where $M_1$ depends on $\nu_j$, and $M_2$ on $\nu_j$ and $C_0$. 

Now we are going to construct fundamental systems of solutions of equation (2). Denote
\[ J(\rho) = \sum_{m=0}^{n-2} J_m(\rho), \]
\[ J_m(\rho) = |\rho|^{\Re(\mu_1 - \mu_n)} \int_{0}^{1} t^{\theta_{n-m}} |q_m(t)| \, dt + |\rho|^{m-n+1} \int_{|\rho|^{-1}}^{\infty} |q_m(t)| \, dt. \]

\textbf{Lemma 3.} The following estimate holds
\[ J(\rho) \leq \frac{Q}{|\rho|}, \quad |\rho| \geq 1, \quad Q := \sum_{m=0}^{n-2} \int_{0}^{\infty} |q_{0m}(t)| \, dt. \]

Indeed, if \( \theta_m - m \leq 0 \), then \( \Re(\mu_n - \mu_1) \geq n - m - 1 \), and consequently
\[ J_m(\rho) \leq |\rho|^{m-n+1} \left( \int_{0}^{1} t^{\theta_{n-m}} |q_m(t)| \, dt + \int_{|\rho|^{-1}}^{\infty} |q_m(t)| \, dt \right) \leq |\rho|^{m-n+1} \int_{0}^{\infty} |q_{0m}(t)| \, dt. \]

If \( \theta_m - m > 0 \), then
\[ J_m(\rho) \leq |\rho|^{m-n+1} \int_{0}^{\infty} |q_m(t)| \, dt \leq |\rho|^{m-n+1} \int_{0}^{\infty} |q_{0m}(t)| \, dt. \]

Hence \( J(\rho) \leq Q|\rho|^{-1}, \quad |\rho| \geq 1 \), and Lemma 3 is proved.

We now construct the functions \( S_j(x, \lambda), \quad j = \overline{1, n} \) from the system of integral equations
\[ S_j^{(\nu)}(x, \lambda) = C_j^{(\nu)}(x, \lambda) - \int_{0}^{x} \frac{\partial}{\partial x} g(x, t, \lambda) \left( \sum_{m=0}^{n-2} q_m(t)S_j^{(m)}(t, \lambda) \right) \, dt, \quad \nu = \overline{0, n-1}. \] (16)

By (15), system (16) has a unique solution; moreover the functions \( S_j^{(\nu)}(x, \lambda) \) are entire in \( \lambda \) for each \( x > 0 \), the functions \( \{ S_j(x, \lambda) \}_{j=1}^{n} \) form a fundamental system of solutions for equation (2), \( \det[S_j^{(\nu-1)}(x, \lambda)]_{j, \nu=1}^{n} \equiv 1 \), and
\[ S_j^{(\nu)}(x, \lambda) = O(x^{\nu_j-\nu}), \quad \{ S_j(x, \lambda) - C_j(x, \lambda)x^{-\nu_j} = o(x^{\nu_j-\nu}), \quad x \to 0. \] (17)

Let \( S_{k_0, \alpha} = \{ \rho : \rho \in S_{k_0}, \quad |\rho| > \alpha \}, \rho_0 = 2M_1Q + 1. \) For \( k = \overline{1, n}, \quad \rho \in \tilde{S}_{k_0, \rho_0} \) we consider the system of integral equations
\[ U_{kv}(x, \rho) = U_{kv}^{(0)}(x, \rho) + \sum_{m=0}^{n-2} \int_{0}^{\infty} A_{km}^{(0)}(x, t, \rho)U_{km}(t, \rho) \, dt, \quad x \geq 0, \quad \nu = \overline{0, n-1}, \] (18)

where
\[ A_{km}(x, t, \rho) = \frac{q_m(t)F_{km}(\rho t)}{\rho^{n-1-m}F_{km}(\rho x)} \left\{ \begin{array}{ll} - \sum_{j=1}^{k} F_{j\nu}(\rho x)U_{j\nu}^{0}(x, \rho)F_{j}(\rho t)U_{j\nu}^{0,*}(t, \rho), & t \leq x, \\ + \sum_{j=k+1}^{n} F_{j\nu}(\rho x)U_{j\nu}^{0}(x, \rho)F_{j}(\rho t)U_{j\nu}^{0,*}(t, \rho), & t > x. \end{array} \right. \]

Using (14) and Lemma 3, we obtain
\[ \sum_{m=0}^{n-2} \int_{0}^{\infty} |A_{km}(x, t, \rho)| \, dt \leq M_1J(\rho) \leq \frac{M_1Q}{|\rho|}. \]
Consequently, system (18) with $\rho \in \bar{S}_{k_0, p_0}$ has a unique solution, and uniformly in $x \geq 0$,
\begin{equation}
U_{k\nu}(x, \rho) - U_{k\nu}^0(x, \rho) = O(\rho^{-1}), \quad \rho \in \bar{S}_{k_0, p_0}.
\end{equation}

**Theorem 2.** For $x > 0$, $\rho \in \bar{S}_{k_0, p_0}$ there exists an fundamental system of solutions of equation (2), 
$\mathcal{B} = \{Y_k(x, \rho)\}_{k=1, 0}$ of the form
\begin{equation}
Y_k^{(\nu)}(x, \rho) = \rho^\nu F_{k\nu}(\rho x)U_{k\nu}(x, \rho),
\end{equation}
where the functions $U_{k\nu}(x, \rho)$ are solution of (18), and (19) is true.

The function $Y_k^{(\nu)}(x, \rho)$ considered for each $x > 0$, are analytic in $\rho \in \bar{S}_{k_0, p_0}$, continuous in $\rho \in \bar{S}_{k_0, p_0}$ and $\text{det}[Y_k^{(\nu-1)}(x, \rho)]_{k, \nu=1, \infty} = \rho^{n(n-1)/2}\Omega(1 + O(\rho^{-1}))$ as $|\rho| \to \infty$. The functions $Y_k(x, \rho)$ satisfy the equality
\begin{align}
Y_k(x, \rho) &= y_k(x, \rho) - \frac{1}{\rho^{n-1}} \int_0^x \left( \sum_{j=1}^k y_j(x, \rho)y_j^*(t, \rho) \right) \left( \sum_{m=0}^{n-2} q_m(t)Y_k^{(m)}(t, \rho) \right) dt \\
&\quad + \frac{1}{\rho^{n-1}} \int_x^\infty \left( \sum_{j=k+1}^n y_j(x, \rho)y_j^*(t, \rho) \right) \left( \sum_{m=0}^{n-2} q_m(t)Y_k^{(m)}(t, \rho) \right) dt.
\end{align}

Moreover, one has the representation
\begin{equation}
Y_k(x, \rho) = \sum_{j=1}^n b_{kj}(\rho)S_j(x, \lambda),
\end{equation}
where
\begin{equation}
b_{kj}(\rho) = b_{kj}^0(\rho)(1 + O(\rho^{-1})), \quad |\rho| \to \infty, \quad \rho \in \bar{S}_{k_0, p_0}.
\end{equation}

The only part of the theorem that needs proof is the asymptotic formula (21). Let $\rho$ be fixed, $x \leq |\rho|^{-1}$. Then (13) and (20) become
\begin{equation}
\begin{aligned}
U_{k0}^0(x, \rho) &= \sum_{j=1}^n b_{kj}^0(\rho x)^{\mu_j-\mu_1} \tilde{C}_j(x, \lambda), \\
U_{k0}(x, \rho) &= \sum_{j=1}^n b_{kj}(\rho)(\rho)^{-\mu_1}x^{\mu_j-\mu_1} \tilde{S}_j(x, \lambda),
\end{aligned}
\end{equation}
where
\begin{equation}
\tilde{C}_j(x, \lambda) = x^{-\mu_j}C_j(x, \lambda), \quad \tilde{S}_j(x, \lambda) = x^{-\mu_j}S_j(x, \lambda), \quad \tilde{S}_j(0, \lambda) = \tilde{C}_j(0, \lambda) = c_{j0} \neq 0.
\end{equation}
It follows from (22) that
\begin{equation}
U_{k0}(x, \rho) - U_{k0}^0(x, \rho) = \sum_{j=1}^n \left( b_{kj}(\rho)\rho^{-\mu_1} - b_{kj}^0(\rho)^{\mu_j-\mu_1} \right) x^{\mu_j-\mu_1} \tilde{S}_j(x, \lambda)
\end{equation}
\begin{equation}
\quad + \sum_{j=1}^n b_{kj}^0(\rho x)^{\mu_j-\mu_1} (\tilde{S}_j(x, \lambda) - \tilde{C}_j(x, \lambda)).
\end{equation}
Denote
\begin{equation}
\begin{aligned}
\mathcal{F}_{k1}(x, \rho) &= U_{k0}(x, \rho) - U_{k0}^0(x, \rho), \\
\mathcal{F}_{k,s+1}(x, \rho) &= \left( \mathcal{F}_{ks}(x, \rho) - \mathcal{F}_{k1}(0, \rho)\tilde{S}_s(x, \lambda)c_{s0}^{-1} \right) x^{\mu_s-\mu_{s+1}}, \quad s = 1, n - 1.
\end{aligned}
\end{equation}
Lemma 4. The following relations hold
\[(b_{ks}(\rho)\rho^{-\mu_1} - b_{ks}^0\rho^{\mu_2-\mu_1})c_{s0} = F_{ks}(x, \rho), \quad s = 1, n, \quad (25)\]

\[F_{ks}(x, \rho) = ((U_{k0}(x, \rho) - U_{k0}^0(x, \rho)) - \sum_{j=1}^{s-1}(b_{kj}(\rho)\rho^{-\mu_1} - b_{kj}^0\rho^{\mu_2-\mu_1})x^{\mu_j-\mu_1}\hat{S}_j(x, \lambda))x^{\mu_k-\mu_s}, \quad (26)\]

Proof. When \(s = 1\) equality (25) follows from (23) for \(x = 0\), while (26) is obviously true. Suppose now that (25) and (26) have been proved for \(s = 1, \ldots, N - 1\). Then

\[\left((U_{k0}(x, \rho) - U_{k0}^0(x, \rho)) - \sum_{j=1}^{N-1}(b_{kj}(\rho)\rho^{-\mu_1} - b_{kj}^0\rho^{\mu_2-\mu_1})x^{\mu_j-\mu_1}\hat{S}_j(x, \lambda)\right)x^{\mu_k-\mu_N}
\]

\[= \left((U_{k0}(x, \rho) - U_{k0}^0(x, \rho)) - \sum_{j=1}^{N-2}(b_{kj}(\rho)\rho^{-\mu_1} - b_{kj}^0\rho^{\mu_2-\mu_1})x^{\mu_j-\mu_1}\hat{S}_j(x, \lambda)\right)x^{\mu_k-\mu_{N-1}}x^{\mu_{N-1}-\mu_N}
\]

\[-(b_{k, N-1}(\rho)\rho^{-\mu_1} - \hat{b}_{k, N-1}^0\rho^{\mu_{N-1}-\mu_1})\hat{S}_{N-1}(x, \lambda)x^{\mu_{N-1}-\mu_N} = F_{kN}(x, \rho),\]

which gives (26) for \(s = N\). We now write (23) as

\[F_{kN}(x, \rho) = \sum_{j=1}^{n}(b_{kj}(\rho)\rho^{-\mu_1} - b_{kj}^0\rho^{\mu_2-\mu_1})x^{\mu_j-\mu_1}\hat{S}_j(x, \lambda)
\]

\[+ \sum_{j=1}^{n}(b_{kj}^0(\rho x)^{\mu_j-\mu_1}(\hat{S}_j(x, \lambda) - \hat{C}_j(x, \lambda))x^{\mu_k-\mu_N}.
\]

Hence, using (17), we infer \(F_{kN}(0, \rho) = (b_{kN}(\rho)\rho^{-\mu_1} - \hat{b}_{kN}^0\rho^{\mu_{N-1}-\mu_1})c_{N0}\), which gives (25) for \(s = N\). Lemma 4 is proved.

Now we write (18) for \(\nu = 0\) as

\[F_{k1}(x, \rho) = \frac{1}{\rho^{n-1}}(- \int_0^x (\sum_{j=1}^{n}(U_{j0}^0(x, \rho)U_{j0}^{0*,}(t, \rho))(\rho t)^{n-1-\mu_1}V_k(t, \rho) dt
\]

\[+ \int_0^{\infty} \left(\sum_{j=k+1}^{n} U_{j0}^0(x, \rho)U_{j0}^{0*,}(t, \rho))F_{j}^{*}(\rho t)\right)V_k(t, \rho) dt), \quad (27)\]

where

\[V_k(t, \rho) = \sum_{m=0}^{n-2} q_m(t)\rho^m F_{km}(\rho t)U_{km}(t, \rho).
\]

Since for \(t \leq x \leq |\rho|^{-1}\) we have

\[\sum_{j=1}^{n} U_{j0}^0(x, \rho)U_{j0}^{0*,}(t, \rho) = \rho^{\mu_n-\mu_1}x^{-\mu_1}t^{1-n+\mu_n}g(x, t, \lambda),\]

it follows by way of (15) that

\[\left|\sum_{j=1}^{n} U_{j0}^0(x, \rho)U_{j0}^{0*,}(t, \rho)\right| \leq M_3|(\rho x)^{\mu_n-\mu_1}|, \quad 0 \leq t \leq x \leq |\rho|^{-1}. \quad (28)\]
Lemma 5. The following relations hold

\[
\mathcal{F}_{ks}(0, \rho) = \rho^{\mu_s - \mu_1 - n + 1} c_0 \int_0^\infty \left( \sum_{j=k+1}^{n} b_{j,s} F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt, 
\]

(29)

\[
\mathcal{F}_{ks}(x, \rho) = \frac{1}{\rho^{n-1}} \left( - x^{\mu_1 - \mu_*} \int_0^x \left( \sum_{j=k+1}^{n} U_{j,0}^0 (x, \rho) U_{j,s}^0 (t, \rho) \right) (pt)^{n-1-\mu_*} V_k (t, \rho) \, dt 
\]

\[
- \sum_{\ell=1}^{s-1} x^{\mu_* - \mu_*} \int_0^\infty \left( \sum_{j=k+1}^{n} b_{j,\ell}^0 \rho^{\mu_1 - \mu_*} (\hat{S}_\ell (x, \lambda) - \hat{C}_\ell (x, \lambda)) F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt 
\]

\[
+ \int_0^\infty \left( \sum_{j=k+1}^{n} \left( \sum_{\xi=s}^{N} b_{j,\xi}^0 \rho^{\mu_1 - \mu_*} \hat{C}_\xi (x, \lambda) \right) F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt), \quad x \leq |\rho|^{-1}. 
\]

(30)

Proof. For \( s = 1 \), (29) and (30) follow from (27), in view of (22). Suppose now that (29) and (30) have been proved for \( s = 1, \ldots, N \). Then, using (24), we calculate

\[
\mathcal{F}_{k,N+1}(x, \rho) = (\mathcal{F}_{kN}(x, \rho) - \mathcal{F}_{kN}(0, \rho) \hat{S}_N (x, \lambda) e^{-1} N_0) x^{\mu_N - \mu_{N+1}} 
\]

\[
= \frac{1}{\rho^{n-1}} \left( - x^{\mu_1 - \mu_{N+1}} \int_0^x \left( \sum_{j=k+1}^{n} U_{j,0}^0 (x, \rho) U_{j,s}^0 (t, \rho) \right) (pt)^{n-1-\mu_*} V_k (t, \rho) \, dt 
\]

\[
- \sum_{\ell=1}^{N-1} x^{\mu_* - \mu_{N+1}} \int_0^\infty \left( \sum_{j=k+1}^{n} b_{j,\ell}^0 \rho^{\mu_1 - \mu_*} (\hat{S}_\ell (x, \lambda) - \hat{C}_\ell (x, \lambda)) F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt 
\]

\[
+ \sum_{\ell=1}^{N} x^{\mu_* - \mu_{N+1}} \int_0^\infty \left( \sum_{j=k+1}^{n} \left( \sum_{\xi=N}^{N} b_{j,\xi}^0 \rho^{\mu_1 - \mu_*} \hat{C}_\xi (x, \lambda) \right) F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt), \quad x \leq |\rho|^{-1}. 
\]

(31)

We now let \( x \to 0 \) in (30) for \( s = N + 1 \), using (28), to obtain (29) for \( s = N + 1 \). This proves Lemma 3.

It follows from (25) and (29) that

\[
b_{ks}(\rho)^{-\mu_*} - b_{ks}^0 = \frac{1}{\rho^{n-1}} \int_0^\infty \left( \sum_{j=k+1}^{n} b_{j,s} F_j^* (pt) U_{j,s}^0 (t, \rho) \right) V_k (t, \rho) \, dt. 
\]

(31)

Using (31), (14), (19) and Lemma 3, we obtain

\[
b_{ks}(\rho)^{-\mu_*} - b_{ks}^0 = O(J(\rho)) = O(\rho^{-1}), \quad |\rho| \to \infty, \quad \rho \in \tilde{S}_{ks},
\]

i.e. (21) is valid. Theorem 2 is proved.
3. Using constructed fundamental systems of solutions on each edge we can study spectral properties of systems on graphs and solve the inverse spectral problem. Let \( \{ \xi_{kj} \}_{k=1}^{n_j} \) be the roots of the characteristic polynomial

\[
d_j(\xi) = \sum_{\mu=0}^{n_j} \nu_{\mu j} \prod_{k=0}^{\mu-1} (\xi - k), \quad \nu_{n_j,j} := 1, \quad \nu_{n_j-1,j} := 0.
\]

For definiteness, we assume that \( \xi_{kj} - \xi_{snj} \neq sn_j, s \in \mathbb{Z}, \ Re \xi_{lj} < \ldots < Re \xi_{nj,j}, \) \( \xi_{kj} \neq 0, n_j - 3 \) (other cases require minor modifications). We set \( \theta_j := n_j - 1 - Re (\xi_{nj,j} - \xi_{lj}) \), and assume that the functions \( q_{\mu j}^{(\nu)}(x_j), \ \nu = 0, \mu - 1, \) are absolutely continuous, and \( q_{\mu j}^{(\mu)}(x_j)x_j^{\theta_j} \in L(0, l_j) \).

Fix \( j = 1, p \). Let the numbers \( c_{kj 0}, k = 1, n_j \), be such that

\[
\prod_{k=1}^{n_j} c_{kj 0} = \left( \det [\xi_{kj}^{\nu-1}]_{k,\nu=1,n_j} \right)^{-1}.
\]

According to results of the previous section one can construct the fundamental systems of solutions \( \{ S_{kj}(x_j, \lambda) \}_{k=1}^{n_j} \) of Eq. (1) on the edge \( e_j \) such that the functions \( S_{kj}^{(\nu)}(x_j, \lambda), \ \nu = 0, n_j - 1, \) are entire in \( \lambda \), and for each fixed \( \lambda \), and \( x_j \to 0 \), \( S_{kj}(x_j, \lambda) \sim c_{kj 0} x_j^{j \theta_j} \) Consider the linear forms

\[
U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(l_j), \quad j = 1, p, \ \nu = 0, n_j - 1,
\]

where \( \gamma_{j\nu\mu} \) are complex numbers, \( \gamma_{j\nu} := \gamma_{j\nu\mu} \neq 0 \). Denote \( \langle n \rangle := (|n| + n)/2 \), i.e. \( \langle n \rangle = n \) for \( n \geq 0 \), and \( \langle n \rangle = 0 \) for \( n \leq 0 \). Fix \( s = 1, p, k = 1, n_s - 1 \). Let \( \Psi_{sk} = \{ \psi_{skj} \}_{j=1,p} \) be solutions of Eq. (1) on the graph \( T \) under the boundary conditions

\[
\psi_{skj}(x_s, \lambda) \sim c_{sk 0} x_s^{j \theta_j}, \quad x_s \to 0,
\]

\[
\psi_{skj}(x_j, \lambda) = O(x_j^{\xi_{(n_j-k-1)+2,j}}), \quad x_j \to 0, \quad j = 1, p, \ j \neq s,
\]

and the matching conditions at the vertex \( v_0 \):

\[
U_{1\nu}(\psi_{sk 0}) = U_{j\nu}(\psi_{skj}), \quad j = 2, p, \ \nu = 0, k - 1, \ n_j > \nu + 1,
\]

\[
\sum_{j=1, n_j > \nu}^{p} U_{j\nu}(\psi_{skj}) = 0, \quad \nu = k, n_s - 1.
\]

Define additionally \( \psi_{nsn}(x_s, \lambda) := S_{ns}(x_s, \lambda) \). Using the solutions \( \{ S_{\mu j}(x_j, \lambda) \} \), one can write

\[
\psi_{skj}(x_j, \lambda) = \sum_{\mu=1}^{n_s} M_{skj\mu}(\lambda) S_{\mu j}(x_j, \lambda), \quad j = 1, p, \ k = 1, n_s - 1,
\]

where the coefficients \( M_{skj\mu}(\lambda) \) do not depend on \( x_j \). It follows from (36) and the boundary condition (32) for the Weyl-type solutions that

\[
\psi_{skj}(x_s, \lambda) = S_{ks}(x_s, \lambda) + \sum_{\mu=k+1}^{n_s} M_{skj\mu}(\lambda) S_{\mu s}(x_s, \lambda), \quad M_{skj\mu}(\lambda) := M_{skj\mu}(\lambda).
\]

Denote

\[
M_{\nu}(\lambda) = [M_{k\mu}(\lambda)]_{k,\mu=1,n_s}, \quad M_{skj\mu}(\lambda) := \delta_{kj} \quad \text{for} \quad k \geq \nu.
\]

The matrix \( M_{\nu}(\lambda) \) is called the Weyl-type matrix with respect to the boundary vertex \( v_s \). The inverse problem is formulated as follows. Fix \( w = 2, p \).
Inverse problem 1. Given \( \{ M_s(\lambda) \} \), \( s = \overline{1,p} \setminus w \), construct \( q \) on \( T \).

Fix \( s = \overline{1,p} \), \( k = \overline{1,n_s-1} \). Substituting (36) into boundary and matching conditions (32)-(35), we obtain a linear algebraic system with respect to \( M_{skj\mu}(\lambda) \). Solving this system one gets \( M_{skj\mu}(\lambda) = \Delta_{skj\mu}(\lambda)/\Delta_{sk}(\lambda) \), where the functions \( \Delta_{skj\mu}(\lambda) \) and \( \Delta_{sk}(\lambda) \) are entire in \( \lambda \). In particular,

\[
M_{skj\mu}(\lambda) = \Delta_{skj\mu}(\lambda)/\Delta_{sk}(\lambda), \quad k \leq \mu,
\]

where \( \Delta_{sk\mu}(\lambda) := \Delta_{skj\mu}(\lambda) \).

4. In this section we obtain a constructive procedure for the solution of Inverse problem 1 and prove its uniqueness. First we consider auxiliary inverse problems of recovering the differential operator on each each fixed edge. Fix \( s = \overline{1,p} \), and consider the following inverse problem on the edge \( e_s \).

**IP(s).**

Given the matrix \( M_s \), construct the potential \( q_s \) on the edge \( e_s \).

**Theorem 3.** Fix \( s = \overline{1,p} \). The specification of the Weyl-type matrix \( M_s \) uniquely determines the potential \( q_s \) on the edge \( e_s \).

We omit the proof since it is similar to that in [1, Ch.2]. Moreover, using the method of spectral mappings, one can get a constructive procedure for the solution of the inverse problem \( IP(s) \). It can be obtained by the same arguments as for \( n \)-th order differential operators on a finite interval (see [1, Ch.2] for details).

Fix \( j = \overline{1,p} \). Let \( \varphi_{jk}(x_j, \lambda) \), \( k = \overline{1,n_j} \), be solutions of equation (1) on the edge \( e_j \) under the conditions

\[
\varphi_{kj}^{(\nu-1)}(l_j, \lambda) = \delta_{k\nu}, \quad \nu = \overline{1,k}, \quad \varphi_{kj}(x_j, \lambda) = O(x_j^{\xi_{nj}-k+1}), \quad x_j \to 0.
\]

We introduce the matrix \( m_j(\lambda) = [m_{jk\nu}(\lambda)]_{\nu,k=\overline{1,n_j}} \) where \( m_{jk\nu}(\lambda) := \varphi_{jk}^{(\nu-1)}(l_j, \lambda) \). The matrix \( m_j(\lambda) \) is called the Weyl-type matrix with respect to the internal vertex \( v_0 \) and the edge \( e_j \).

**IP[j].**

Given the matrix \( m_j \), construct \( q_j \) on the edge \( e_j \).

This inverse problem is the classical one, since it is the inverse problem of recovering a higher-order differential equation on a finite interval from its Weyl-type matrix. This inverse problem has been solved in [1], where the uniqueness theorem for this inverse problem is proved. Moreover, in [1] an algorithm for the solution of the inverse problem \( IP[j] \) is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

Fix \( j = \overline{1,p} \). Then for each fixed \( s = \overline{1,p} \setminus j \),

\[
m_{jk\nu}(\lambda) = \frac{\psi_{s\nu j}^{(\nu-1)}(l_j, \lambda)}{\psi_{s1 j}(l_j, \lambda)}, \quad \nu = \overline{2,n_j}, \quad 2 \leq k < \nu \leq n_j. \tag{38}
\]

\[
m_{jk\nu}(\lambda) = \frac{\det[\psi_{s\nu j}^{(k)}(l_j, \lambda), \psi_{s\nu j}^{(k-1)}(l_j, \lambda), \ldots, \psi_{s\nu j}^{(1)}(l_j, \lambda), \psi_{s\nu j}^{(\nu-1)}(l_j, \lambda)]_{\nu=\overline{1,k}}}{\det[\psi_{s\nu j}^{(1)}(l_j, \lambda)]_{\nu=\overline{1,k}}}, 2 \leq k < \nu \leq n_j. \tag{39}
\]

Now we are going to obtain a constructive procedure for the solution of Inverse problem 1. For this purpose it is convenient to divide differential equations into \( m \) groups with equal orders. More precisely, let \( \omega_1 > \omega_2 > \ldots > \omega_m > \omega_{m+1} = 1 \), \( n_{p_j-1}+1 = \ldots = n_{p_j} := \omega_j \), \( j = \overline{1,m} \), \( 0 = p_0 < p_1 < \ldots < p_m = p \). Take \( N \) such that \( p_N = w \).

Suppose that we already found the potentials \( q_s \), \( s = \overline{1,p} \setminus p_N \), on the edges \( e_s \), \( s = \overline{1,p} \setminus p_N \). Then we calculate the functions \( S_{kj}(x_j, \lambda) \), \( j = \overline{1,p} \setminus p_N \); here \( k = \overline{1,\omega_i} \) for \( j = \overline{p_i+1, p_{i+1}} \).

Fix \( s = \overline{1,p_1} \) (if \( N > 1 \)), and \( s = \overline{1,p_1-1} \) (if \( N = 1 \)). Our goal now is to construct the Weyl-type matrix \( m_{p_N}(\lambda) \). According to (38)-(39), in order to construct \( m_{p_N}(\lambda) \) we have to calculate the functions

\[
\psi_{skp_N}^{(\nu)}(l_{p_N}, \lambda), \quad k = \overline{1,\omega_N-1}, \quad \nu = \overline{0,\omega_N-1}. \tag{40}
\]
We will find the functions (40) by the following steps.

1) Using (37) we construct the functions

\[ \psi_{sk\lambda}^{(\nu)}(l_s, \lambda), \ k = \overline{1, \omega_N - 1}, \ \nu = 0, \omega_1 - 1, \]  

by the formula

\[ \psi_{sk\lambda}^{(\nu)}(l_s, \lambda) = S_{sk\lambda}^{(\nu)}(l_s, \lambda) + \sum_{\mu = k+1}^{\omega_1} M_{sk\mu}(\lambda) S_{sk\lambda}^{(\nu)}(l_s, \lambda). \]  

2) Consider a part of the matching conditions (34) on \( \Psi_{sk} \). More precisely, let \( \xi = N, m, \ k = \overline{\omega_{\xi+1}, \omega_\xi - 1}, \ l = \xi, m, j = \overline{1, p_l - 1}. \) Then, in particular, (34) yields

\[ U_{p_l \nu}(\psi_{sk\nu}) = U_{j\nu}(\psi_{sk\nu}), \ \nu = \omega_{l+1} - 1, \min(k - 1, \omega_l - 2). \]  

Using (43) we can calculate the functions

\[ \psi_{skj\lambda}^{(\nu)}(l_j, \lambda), \ \xi = N, m, \ k = \overline{\omega_{\xi+1}, \omega_\xi - 1}, \ l = \xi, m, j = \overline{1, p_l - 1}, \ \nu = \omega_{l+1} - 1, \min(k - 1, \omega_l - 2). \]  

3) It follows from (36) and the boundary conditions on \( \Psi_{sk} \) that

\[ \psi_{sk\lambda}^{(\nu)}(l_j, \lambda) = \sum_{\mu = \max(\omega_l - k + 1)}^{\omega_1} M_{sk\mu}(\lambda) C_{sk\mu j}^{(\nu)}(l_j, \lambda), \]  

\[ k = \overline{1, \omega_1 - 1}, \ l = \overline{1, m}, \ j = \overline{p_{l-1} + 1, p_l \setminus s, \ \nu = 0, \omega_l - 1}. \]  

We consider only a part of relations (45). More precisely, let \( \xi = N, m, \ k = \overline{\omega_{\xi+1}, \omega_\xi - 1}, \ l = \overline{1, m}, \ j = \overline{p_{l-1} + 1, p_l}, \ j \neq p_N, \ j \neq s, \ \nu = 0, \min(k - 1, \omega_l - 2). \) Then

\[ \sum_{\mu = \max(\omega_l - k + 1)}^{\omega_1} M_{sk\mu}(\lambda) C_{sk\mu j}^{(\nu)}(l_j, \lambda) = \psi_{sk\lambda}^{(\nu)}(l_j, \lambda), \ \nu = 0, \min(k - 1, \omega_l - 2). \]  

For this choice of parameters, the right-hand side in (46) are known, since the functions (44) are known. Relations (46) form a linear algebraic system \( \sigma_{skj} \) with respect to the coefficients \( M_{sk\mu}(\lambda) \). Solving the system \( \sigma_{skj} \) we find the functions \( M_{sk\mu}(\lambda) \). Substituting them into (45), we calculate the functions

\[ \psi_{sk\lambda}^{(\nu)}(l_j, \lambda), \ k = \overline{1, \omega_N - 1}, \ l = \overline{1, m}, \ j = \overline{p_{l-1} + 1, p_l \setminus p_N}, \ \nu = 0, \omega_l - 1. \]  

Note that for \( j = s \) these functions were found earlier.

4) Let us now use the generalized Kirchhoff’s conditions (35) for \( \Psi_{sk} \). Since the functions (47) are known, one can construct by (35) the functions (40) for \( k = \overline{1, \omega_N - 1}, \ \nu = \overline{1, \omega_N - 1}. \) Thus, the functions (40) are known for \( k = \overline{1, \omega_N - 1}, \ \nu = \overline{0, \omega_N - 1}. \)

Since the functions (40) are known, we construct the Weyl-type matrix \( m_{ps}(\lambda) \) via (38)-(39) for \( j = p_N \). Thus, we have obtained the solution of Inverse problem 1 and proved its uniqueness, i.e. the following assertion holds.

**Theorem 2.** The specification of the Weyl-type matrices \( M_s(\lambda), \ s = \overline{1, p \setminus p_N} \), uniquely determines the potential \( q \) on \( T \). The solution of Inverse problem 1 can be obtained by the following algorithm.

**Algorithm 1.** Given the Weyl-type matrices \( M_s(\lambda), \ s = \overline{1, p \setminus p_N} \),

1) Find \( q_s, \ s = \overline{1, p \setminus p_N} \), by solving the inverse problem IP(s) for each fixed \( s = \overline{1, p \setminus p_N} \).
2) Calculate \( C_{kj}^{(\nu)}(l_j, \lambda), \ j = \overline{1, p \setminus p_N}; \) here \( k = \overline{1, \omega_1}, \ \nu = \overline{0, \omega_1 - 1} \) for \( j = p_{l-1} + 1, p_l \).
3) Fix \( s = \overline{1, p_l} \) (if \( N > 1 \)), and \( s = \overline{1, p_l - 1} \) (if \( N = 1 \)). All calculations below will be made for this fixed \( s \). Construct the functions (41) via (42).
4) Calculate the functions (44) using (43).
5) Find the functions $M_{skj\mu}(\lambda)$, by solving the linear algebraic systems $\sigma_{skj}$.
6) Construct the functions (40) using (35).
7) Calculate the Weyl-type matrix $m_{pN}(\lambda)$ via (38)-(39) for $j = p_N$.
8) Construct the potential $q_{pN}$ by solving the inverse problem $IP[j]$ for $j = p_N$.

We note that inverse spectral problems for Sturm-Liouville operators on spatial networks were studied in [2-7], and for higher order operators in [8].

Acknowledgment. This work was supported by Grant 1.1436.2014K of the Russian Ministry of Education and Science and by Grant 13-01-00134 of Russian Foundation for Basic Research.

REFERENCES

[1] V.A. Yurko. Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.
[2] Belishev M.I. Boundary spectral Inverse Problem on a class of graphs (trees) by the BC method. Inverse Problems 20 (2004), 647-672.
[3] Yurko V.A. Inverse spectral problems for Sturm-Liouville operators on graphs. Inverse Problems, 21, no.3 (2005), 1075-1086.
[4] Brown B.M.; Weikard R. A Borg-Levinson theorem for trees. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005), no.2062, 3231-3243.
[5] Yurko V.A. Inverse problems for Sturm-Liouville operators on bush-type graphs. Inverse Problems, 25, no.10 (2009), 105008, 14pp.
[6] Yurko V.A. An inverse problem for Sturm-Liouville operators on A-graphs. Applied Math. Letters, 23, no.8 (2010), 875-879.
[7] Yurko V.A. Inverse spectral problems for differential operators on arbitrary compact graphs. Journal of Inverse and Ill-Posed Problems, 18, no.3 (2010), 245-261.
[8] Yurko V.A. Inverse spectral problems for arbitrary order differential operators on noncompact trees. Journal of Inverse and Ill-Posed Problems, 20, no.1 (2012), 111-132.

Name: Yurko, Vjacheslav
Place of work: Department of Mathematics, Saratov State University
              Astrakhanskaya 83, Saratov 410012, Russia
E-mail: yurkova@info.sgu.ru