Super-exponential query complexity reduction via noise-resistant quantum search

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Abstract

In the SEARCH WITH ADVICE problem, a single entry of interest within a database of \( N \) entries is to be found assuming that an ordering of the entries, from that with the highest probability of being the entry of interest (as determined by a so-called advice distribution) to that with the lowest, is provided. We present a quantum algorithm that, in the presence of significant levels of quantum noise, solves SEARCH WITH ADVICE for a power law advice distribution with average-case query complexity \( O(1) \) as \( N \to \infty \). Since as we also show the best classical algorithms for this problem exhibit average-case query complexity of order no better than \( \log(N) \), our quantum algorithm provides a super-exponential reduction in query complexity.

Keywords: quantum database search, query complexity, Grover algorithm, quantum noise

1 Introduction

Well-known theoretical results have established that quantum computation offers the promise of dramatic reductions in the computational complexity of algorithms for data decryption (Shor (1994)) and database search (Grover (1996, 1997)), among other critical applications. Yet one of the key challenges for practical quantum computation remains the sensitivity of quantum computations to errors caused by ambient quantum noise. One of the most commonly suggested ways of dealing with this problem is by means of fault-tolerant quantum error correction (see for example Chapter 10 in Nielsen and Chuang (2000)). However, this approach can be challenging to implement as it is very expensive with respect to computational resources, and large circuit sizes are in general required to apply it. Rather than expending great effort to explicitly correct errors caused by quantum noise, the approach we adopt in this paper is instead to accept their presence in the context of quantum computations and to attempt

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to reach a viable solution anyway. Under this as a guiding principle, we analyze the computational complexity of quantum algorithms for database search.

Assuming the absence of quantum noise, the renowned Grover quantum search algorithm (Grover (1996,1997)), already cited above, finds a single item of interest (called the “marked element”) in a database of \( N \) items with just \( O(\sqrt{N}) \) database query calls as \( N \to \infty \) in the worst case, as compared with a worst-case query complexity of \( \Omega(N) \) for the best classical algorithms. In this article, however, we analyze the performance of quantum database search algorithms in the presence of quantum noise. We study a generalization, called SEARCH WITH ADVICE, of the standard, unstructured search problem addressed by Grover. In SEARCH WITH ADVICE (see §2 below), a so-called “advice” probability distribution giving the probability of each item’s actually being the marked element, is presupposed, and a “hint”, in the form of an ordering of the \( N \) items from highest advice probability to lowest, is actually given as input to the search algorithm intended to be used to solve the problem (so that SEARCH WITH ADVICE specializes to the standard search problem under a uniform advice distribution).

In this article, the average-case query complexity of an algorithm is defined as the expected number of queries to an oracle function required by it (see also §2). Under a direct generalization of depolarizing channel noise, with respect to which noise in the form of a corresponding quantum channel occurs with probability \( p = p(N) \), the quantum algorithm we present here (as our Algorithm 2 in §3 below) solves the SEARCH WITH ADVICE problem for advice distribution following a power law with exponent \( \frac{1}{N} - 2 \) with average-case query complexity of just \( O(1) \) as \( N \to \infty \), given a noise level as large as \( p = p(N) = \frac{1}{\log(N)} \) (see our own Corollary 1 in §4 below). This means that the expected query complexity is in fact bounded by a fixed constant (not depending on \( N \)) for all positive integers \( N \) no matter how large, and this contrasts with a minimal average-case query complexity no better than order \( \log(N) \) (that is, \( \Omega(\log(N)) \) as \( N \to \infty \)) for the best classical algorithm applied to the same problem (once again see Corollary 1 below). Hence the quantum search algorithm we introduce in this paper achieves a dramatic super-exponential reduction in computational complexity over the best possible classical algorithm for this search problem, even in the presence of quantum noise levels declining quite slowly as \( N \to \infty \). Quantum supremacy (Preskill (2013)) refers to an empirical demonstration that a quantum processor can perform some computational task faster than any classical computer. Offering such a dramatic quantum computational speed-up over corresponding classical algorithms as well as being as robust as it is with respect to quantum noise, our quantum algorithm for solving SEARCH WITH ADVICE appears, for example, to possess significant potential as a means to establishing practical quantum supremacy.

Our database search algorithm here is a combination of a quantum database search algorithm in Montanaro (2010), which solves the SEARCH WITH ADVICE problem without addressing the possibility of quantum noise, and that in Vrana et al. (2014), which solves the standard, unstructured database search
problem in the presence of quantum noise. In fact, by comparison, the algorithm in Vrana et al. (2014) (see in particular their Algorithm 3 in their §2.4) exhibits a worst-case query complexity no better than order $\frac{N}{\log(N)}$ (that is, a query complexity of $\Omega\left(\frac{N}{\log(N)}\right)$) for the standard database search problem (with no advice distribution) in the context of the same noise model as here with noise probability once again $p = p(N) = \frac{1}{\log(N)}$. This is not nearly as dramatic a computational speed-up relative to the classical case as we are able to achieve here in the average-case complexity setting for SEARCH WITH ADVICE. Rather similarly to the results in Vrana et al. (2014) and once again for the search problem without advice distribution, the authors of Cohn et al. (2016) have deduced that the maximum depolarizing channel noise probability possible to achieve any computational advantage over the fastest classical algorithms is $p = p(N) = \Omega\left(\frac{\log(\sqrt{N})}{\sqrt{N}}\right)$ as $N \to \infty$ (see § III.B.1 in Cohn et al. (2016)), which is a much smaller level of quantum noise than we allow to gain a much greater quantum speed-up in the average case for power law distributions under SEARCH WITH ADVICE. Moreover, in Corollary 2 in §4, we also establish for the same problem that the amount of quantum noise can even be increased to order $p = p(N) = \left(\frac{1}{\log(N)}\right)^q$ for any $q$, $0 < q < 1$, while still solving the problem with query complexity of order no larger than $(\log(N))^{1-q}$. Hence an at least polynomial speedup with respect to the best classical algorithms is still achieved in this case.

The paper is organized as follows. In the next section we formally state the SEARCH WITH ADVICE problem and introduce some necessary background concepts and notation. In §3 we present our noise-resistant quantum search algorithm for solving this problem. Our results on the query complexity of this quantum algorithm and corresponding classical algorithms for solving the same problem are stated and proved in §4.

2 The Search with Advice Problem

We now state the SEARCH WITH ADVICE problem.

**Problem:** SEARCH WITH ADVICE

**Input:** An oracle function $f : \{1, \ldots, N\} \to \{0, 1\}$ on $N = 2^q$ elements, for any positive integer $q$, that takes the value 1 on precisely one element $n_{me} \in \{1, \ldots, N\}$, and an “advice” probability distribution $\mu = (\mu_n), n = 1, \ldots, N$, where $\mu_n$ is an assessed probability that in fact $n = n_{me}$.

**Output:** The unique element $n_{me}$, called the “marked element”, for which $f(n_{me}) = 1$.

In this paper our interest is in analyzing and comparing algorithms — both classical and quantum — that solve the SEARCH WITH ADVICE problem.
above with minimal query complexity. In the context of SEARCH WITH ADVICE, the query complexity (Montanaro (2010)) of a classical or quantum algorithm is the number of queries to the oracle function $f$ in the statement of the problem that is required to identify the marked element. Consider any quantum or classical algorithm $A$, which, given access to the oracle $f$, is designed to solve SEARCH WITH ADVICE by identifying the marked element. Indeed we call $A$ a valid algorithm if it outputs the marked element with certainty. Let $D$ denote the class of valid deterministic classical algorithms.

We intend to investigate the average-case query complexity (that is, here, the expected query complexity) of efficient quantum and classical algorithms for solving SEARCH WITH ADVICE. So, let $T_A(\mu)$ denote the expectation of the number of queries to $f$ used by $A$, where this expectation is taken over the distribution $\mu$ and (potentially) the internal randomness of $A$. That is, this average-case query complexity $T_A(\mu)$ is defined by

$$T_A(\mu) = \sum_{n=1}^{N} \mu_n T_A(n),$$

where, in turn, $T_A(n)$ is defined as the expectation of the number of queries to $f$ used by $A$ to identify $n$ as the marked element, if in fact $n$ is the marked element. We also define a key corresponding quantity of interest, the classical (that is, classical algorithm) average-case query complexity of $\mu$, as

$$D(\mu) = \inf_{A \in D} T_A(\mu).$$

We assume in this paper that the advice distribution $\mu$ is non-increasing, so that $\mu_{n_1} \geq \mu_{n_2}$ whenever $n_1 \leq n_2$. With this assumption, the optimal classical algorithm to find $n_{mc}$ is clearly to query $f(1)$ through $f(N)$ in turn, so the classical average-case query complexity is easily seen in this case to be

$$\sum_{n=1}^{N} \mu_n n = D(\mu) = \min_{A \in D} T_A(\mu).$$

3 Noise-resistant geometric quantum search algorithm

The idea behind the quantum search algorithm we present in this section is to combine an algorithm in Vrana et al. (2014), which is designed to be robust against quantum noise, with one in Montanaro (2010), which achieves super-exponential expected computational advantage over classical algorithms in the absence of quantum noise. Algorithm 2 in this paper achieves super-exponential expected computational advantage over the optimal classical algorithm in the presence of significant levels of quantum noise. The algorithms in Montanaro (2010) and Vrana et al. (2014) are both ultimately based on the original quantum search algorithm of Grover (Grover (1996, 1997)), so in that sense, our
own quantum search algorithm here for SEARCH WITH ADVICE, which we present in this section, is as well.

To describe our algorithm, for any positive integer $N_1, 1 \leq N_1 \leq N$, let $C^{d_1}$ be a quantum state space of dimension

$$d_1 = d_1(N_1) = 2^{\min(\lceil \log_2(N_1) \rceil + 1, q)}, \tag{4}$$

where we recall that by definition $N = 2^q$ for some positive integer $q$. Note that this definition implies that $N_1 \leq d_1 = d_1(N_1) \leq 2N_1$. Define the action of the quantum oracle operator on a corresponding computational basis, which we enumerate as $|n\rangle$, $n = 1, \ldots, d_1$, by the unitary map

$$O_f : C^{d_1} \to C^{d_1}, \quad |n\rangle \mapsto (-1)^{f(n)}|n\rangle. \tag{5}$$

Note of course that $O_f(|n\rangle) = |n\rangle$ only at $n = n_{me}$ (and of course we still assume there is exactly one marked element in the entire set $\{1, \ldots, N\}$). Let

$$|\psi\rangle = \frac{1}{\sqrt{d_1}} \sum_{n=1}^{d_1} |n\rangle \tag{6}$$

be the equal superposition state, which can be generated using the Hadamard transform (see Nielsen and Chuang (2000), Chapter 6), and also define a corresponding unitary operator via

$$U(\psi) = I - 2|\psi\rangle\langle\psi|, \tag{7}$$

where $I$ is of course the identity operator. In addition, we require a suitable model of quantum noise, and we consider a generalization (as in Vrana et al. (2014)) of the depolarizing channel. So let $T$ be any arbitrary, given quantum channel (quantum operation) acting on density operators $\rho$ on the state space $C^{d_1}$, and define a quantum noise model, parametrized by a probability value $p \in [0, 1]$, via

$$N_p(\rho) = (1 - p)\rho + pT(\rho). \tag{8}$$

When $T = \frac{I}{d_1}$, (8) is of course the standard $d_1$-dimensional depolarizing channel (see Nielsen and Chuang (2000)).

We now state Subroutine 1, which will be called by Algorithm 1 below and hence by our main quantum search algorithm (Algorithm 2) for SEARCH WITH ADVICE. Subroutine 1 is in essence the Grover iteration step appearing in the standard version of the Grover algorithm but with quantum noise present (see also Vrana et al. (2014) or for, for the version of the Grover iteration step without noise, Chapter 6 in Nielsen and Chuang (2000)).

**Subroutine 1:** Grover iteration with quantum noise

**Input:** The oracle function $f$ from the statement of the SEARCH WITH ADVICE problem above; a positive integer $N_1$ with $1 \leq N_1 \leq N$; a density operator $\rho : C^{d_1} \to C^{d_1}$ where $d_1 = d_1(N_1) = 2^{\min(\lceil \log_2(N_1) \rceil + 1, q)}$; a desired
number of iterations $M$. Moreover, assume that the equal superposition state $|\psi\rangle$ as in (6) has been prepared and that quantum noise is modeled as in (8) above for some $p \in [0,1]$.

**Output:** The density operator resulting from application of $M$ Grover iterations with noise to $\rho$.

```math
count := 1;
Groverstep := \rho;
while count \leq M do
    Groverstep := U(|\psi\rangle)(\mathcal{O}_f(N_p(Groverstep))\mathcal{O}_f^\dagger)|\psi\rangle;
    count := count + 1;
end while
return Groverstep;
```

Exploiting the basic Grover iteration above, we now state a version of the Grover quantum search algorithm (Algorithm 1 below) which will be invoked within our noise-resistant geometric quantum search algorithm (Algorithm 2) below. Algorithm 1 here has itself appeared as Algorithm 1 in §2.4 of Vrana et al. (2014). In order to state it, define, for any real numbers $\epsilon, c > 0$ and all nonnegative integers $i$, the value

$$
\alpha_i(\epsilon, c) = \frac{1}{\sqrt{1 + \frac{i}{c \log(1/\epsilon)}}}.
$$

(9)

**Algorithm 1:** Noise-resistant Grover search

**Input:** The function $f$ from the statement of the SEARCH WITH ADVICE problem above; two integers $n_1, n_2 \in \{1, ..., N\}$, indicating where a search of some subset of consecutive numbers from among the set $1, ..., N$ is to begin and end, respectively, inclusive of the two numbers $n_1, n_2$; two adjustable parameters $\epsilon, c > 0$. Assume as well that quantum noise (corresponding to some $p \in [0,1]$) affects Algorithm 1 through its presence in Subroutine 1, and define $N_1 = n_2 - n_1 + 1$.

**Output:** The marked element $n_{me}$ if found; otherwise 0.

for $i = 0, 1, 2, \ldots$ do

1. Prepare the equal superposition state $|\psi\rangle = \frac{1}{d_1^{1/2}} \sum_{n=1}^{d_1} |n\rangle$ on a quantum register $C^{d_1}$, where $d_1 = d_1(N_1)$ is as in (4);

2. Let $\rho = \frac{1}{d_1} \sum_{n=1}^{d_1} |n\rangle\langle n|$, and apply Grover iteration with quantum noise (Subroutine 1) above with inputs $f$, $N_1$, $\rho$, $|\psi\rangle$, and $M = \left[\alpha_i(\epsilon, c) \frac{\pi}{4} \sqrt{d_1}\right]$;

3. Measure $\min(\log_2(N_1) + 1, q)$ qubits in the standard basis, and check the result using one oracle invocation;

end for

if $n_{me}$ found then

return $n_{me}$

else
The geometric quantum search algorithm of Montanaro (2010) (Algorithm 1 in §2.1 there), on which Algorithm 2 below is in part based, does not incorporate the possibility of quantum noise as is done here in Subroutine 1 and Algorithm 1 as above. Our own noise-resistant geometric quantum search algorithm (Algorithm 2) for SEARCH WITH ADVICE, which does incorporate quantum noise, in essence merges the algorithms of both Montanaro (2010) and Vrana et al. (2014). Informally Algorithm 2 consists in partitioning the input into successive blocks which increase in size geometrically (hence the algorithm’s name) and performing Algorithm 1 on each of these blocks. We are now in a position to state it.

Algorithm 2: Noise-resistant geometric quantum search

Input: The oracle function \( f : \{1, \ldots, N\} \to \{0, 1\} \) from the SEARCH WITH ADVICE problem as above; Advice distribution \( \mu = (\mu_n) \) (though all that is actually needed to implement the algorithm is just the ordering of the numbers \( 1, \ldots, N \) as \( n_1, \ldots, n_N \), where the \( n_i \in \{1, \ldots, N\} \) for all \( i = 1, \ldots, N \), the \( n_i \) are all distinct, and \( \mu_{n_i} \geq \mu_{n_{i+1}}, i = 1, \ldots, N - 1 \)); chosen real values \( \epsilon, c > 0 \). Assume as well that quantum noise is present in Algorithm 2 through its appearance in Subroutine 1 (for some \( p \in [0, 1] \)).

Output: The marked element \( n_{\text{me}} \).

\[
\begin{align*}
\text{start} &:= 1; \\
\text{end} &:= 1; \\
\text{step} &:= 0; \\
\text{while} \ \text{start} \leq N \text{ do} & \text{ if} \ \text{end} - \text{start} \geq 100 \text{ then} \\
& \text{ Perform Noise-resistant Grover search (Algorithm 1) with } \epsilon, c > 0 \text{ to identify } n_{\text{me}} \text{ or its absence in the subset } \{\text{start}, \ldots, \text{end}\} \text{ (where } n_1 = \text{start} \text{ and } n_2 = \text{end} \text{ in the notation of Algorithm 1);} \\
& \text{ else } \\
& \text{ Perform classical (non-quantum) search (as in §2) to identify } n_{\text{me}} \text{ or its absence within the set } \{\text{start}, \ldots, \text{end}\}; \\
& \text{ end if} \\
& \text{ if } n_{\text{me}} \text{ was found then} \\
& \text{ return } n_{\text{me}}; \\
& \text{ else} \\
& \text{ step} := \text{step} + 1; \\
& \text{ start} := \text{end} + 1; \\
& \text{ end} := \min(\text{start} + \lceil \epsilon\text{step} \rceil - 1, N); \\
& \text{ end if} \\
& \text{ end while}
\end{align*}
\]
4 Query Complexity Results

Now for any valid quantum or classical algorithm $A$ as at the beginning of §2, recall, for any advice measure $\mu$, the average-case query complexity values $T_A(\mu)$ and $T_A(n)$, $n = 1, \ldots, N$, from (1). We denote by $A_{QS}$ our Algorithm 2 in §3, which is a valid algorithm, and we can consider the associated quantities $T_{A_{QS}}(\mu)$ and $T_{A_{QS}}(n)$, $n = 1, \ldots, N$. Also, we can write $A_{QS} = A_{QS}(p)$ to make the ambient quantum noise level $p \in [0, 1]$ (as in Subroutine 1) explicit.

The following Theorem 1 is a direct consequence of Theorem 3 in §2.4 of Vrana et al. (2014), along with the basic observation (see §1.2 in Montanaro (2010) or Motwani and Raghavan (1995), Exercise 1.3 in §1.2 there) that, given a (classical or quantum) search algorithm $A$ that uses $k$ query calls and outputs the marked element $n_{me}$ with probability $s$, there is a classical algorithm $A_1$ that takes $A$ and $s$ as inputs and outputs the marked element with certainty, doing so using an expected number of queries of at most $k + 1 + s$. In the statement of Theorem 1 below we analyze the quantum algorithm $A$ given as Algorithm 1 in §3 above, and, as the observation just mentioned is a standard one from the theory of randomized algorithms, we will for any value probability value $s$ simply identify the algorithm $A$ given in §3 as Algorithm 1 with the algorithm $A_1 = A_1(A, s)$ and can use the same notation to refer to both of them.

**Theorem 1** (Vrana et al. (2014)). With respect to the above Algorithm 1 (for which in this theorem statement we use the notation $A$), let $N_1$ be any integer with $100 \leq N_1 \leq N$, and also let $d_1 = d_1(N_1)$ be as in (4). Suppose as well that $\epsilon \in (0, \frac{1}{2})$ is given. Furthermore, assume that $p = p(N)$, which may depend on $N$, is any value $p \in [0, 1]$ which describes the ambient quantum noise level via $N_0(p) = (1 - p)\rho + pT(\rho)$, where in turn $\rho$ is any density operator acting on the state space $\mathbb{C}^{d_1}$ and $T$ is a given quantum channel acting on the corresponding quantum register when executing Algorithm 1. Then, Algorithm 1 with $c = 10$ and $\epsilon$ as given (or, more technically, the algorithm $A_1 = A_1(A, 1 - \epsilon)$ as discussed in the previous paragraph) finds the marked element with certainty (if it is present within the subset of $\{1, \ldots, N\}$ of size $N_1$ being searched by Algorithm 1) after an expected number of not more than

$$\left(\frac{100}{1 - \epsilon}\right) \left(1.02 + d_1p + \sqrt{d_1}\right) \log \left(\frac{1}{\epsilon}\right)$$

oracle queries, where $d_1 \leq 2N_1$.

For an arbitrary advice distribution, Theorem 1 enables us to now state and prove Lemma 1. Lemma 1 — our first new result — is a bound on the expected query complexity of Algorithm 2 in §3.

**Lemma 1.** Let $N$ be any positive integer, and assume that $\epsilon \in (0, \frac{1}{2}]$ and $p \in [0, 1]$, where $p = p(N)$ may depend on $N$. Then the expected number of queries used by Algorithm 2 in §3 for any given advice distribution $\mu = (\mu_n)$ is
upper-bounded by

\[ T_{AQS}(p)(\mu) \leq e^2 \sum_{n=1}^{N} \mu_n G(n, p, \epsilon), \quad (11) \]

where

\[ G(r, p, \epsilon) = \frac{200}{1 - \epsilon} (1.04 + rp + \sqrt{r}) \log\left(\frac{1}{\epsilon}\right) \]

for \( r \in [0, \infty), p \in [0, 1], \epsilon \in (0, \frac{1}{2}] \).

**Proof.** In the \( m \)th repetition of the loop, the (at most) \([e^m]\) elements contained in the range

\[ R_m = \{1 + \sum_{i=0}^{m-1} [e^i], \ldots, \min([e^m] + \sum_{i=0}^{m-1} [e^i], N)\} \quad (12) \]

will be searched. By Theorem 1 with \( c = 10 \), the noisy Grover search step in this iteration uses at most an expected

\[ \left(\frac{100}{1 - \epsilon}\right) (1.02 + 2[e^m]p + \sqrt{2[e^m]}) \log\left(\frac{1}{\epsilon}\right) + 1 \]

number of queries. So, an expected total of at most

\[ \sum_{i=0}^{m} \left(\frac{100}{1 - \epsilon}\right) (1.02 + 2[e^m]p + \sqrt{2[e^m]}) \log\left(\frac{1}{\epsilon}\right) + 1 \leq \sum_{i=0}^{m} G(e^i, p, \epsilon) \quad (13) \]

queries will be used by Algorithm 2 to search for the marked element up to and including the \( m \)th repetition of the loop. But it is clear from (12) that, for any \( n \in R_m \), \( m \leq \log_e(n) + 1 \). The average-case query complexity is therefore upper-bounded by

\[ \sum_{n=1}^{N} \mu_n \left( \sum_{i=0}^{[\log_e(n)]+1} G(e^i, p, \epsilon) \right), \quad (14) \]

and, estimating the inner sum above by an integral, we obtain an upper bound of

\[ \sum_{n=1}^{N} \mu_n \left( \int_{0}^{[\log_e(n)]+2} G(e^s, p, \epsilon) ds \right) \leq \sum_{n=1}^{N} \mu_n \left( e^2 G(n, p, \epsilon) \right), \quad (15) \]

completing the proof.

We use Lemma 1 to prove Theorem 2, which we now state. Theorem 2 asserts that, for advice distributions defined by certain power law distributions, Algorithm 2 as presented in the previous section achieves greater-than-exponential speed-ups in average-case query complexity for SEARCH WITH ADVICE, relative to the best possible classical algorithms for this problem. Theorem 2 and its Corollaries 1 and 2, which also follow below, are the main results of the paper.
Theorem 2. For any positive integer $N \geq 100$ and any $\delta \in (0, \frac{1}{4}]$, define an advice distribution $\mu = \mu_\delta$ on $\{1, \ldots, N\}$ by taking $\mu_{x,n} = \alpha(\delta-2)n^{(\delta-2)}$ for each $n \in \{1, \ldots, N\}$, where $\frac{1}{\alpha(\delta-2)} = \sum_{n=1}^{N} n^{(\delta-2)}$. Then, $D(\mu_\delta) \geq \frac{3N^{\delta-3}}{88}$, but, for any $p \in [0,1]$, $T_{AQ_\delta}(p)(\mu_\delta) \leq 400e^2 \left( c_1 + \frac{2p(N^\delta-1)}{\delta} \right)$, for some constants $c_1, c_2 > 0$, where the level $p = p(N)$ of quantum noise as in (8) may depend on $N$. Furthermore, we note that we can take $c_1 = 9$ and $c_2 = 1.04$.

Proof. For any desired $r$, $-2 < r < -1$, define a probability measure on $\{1, \ldots, N\}$ via $\mu_n = \alpha_r n^r$, $n = 1, \ldots, N$. Since $\mu$ is prescribed to be a probability measure, we have that $\frac{1}{\alpha_r} = \sum_{n=1}^{N} n^r$. This sum can be estimated by an integral, giving

$$\int_1^N x^r dx \leq \frac{1}{\alpha_r} \leq 1 + \int_1^N x^r dx. \quad (16)$$

This implies that

$$\frac{N^{r+1} - 1}{r+1} \leq \frac{1}{\alpha_r} \leq \frac{N^{r+1} - 1}{r+1} + 1. \quad (17)$$

We have, for $r = \delta - 2$,

$$D(\mu) = \alpha_r \sum_{n=1}^{N} n^{r+1} \geq \alpha_r \int_1^N x^{r+1} dx$$

$$\geq \frac{(N^{r+2} - 1)(r+1)}{(N^{r+1} + r)(r+2)} = \frac{(N^\delta - 1)(\delta - 1)}{(N^{\delta-1} + \delta - 2)/(\delta)}. \quad (18)$$

Since $|\delta - 1| > \frac{3}{4}$ and $|N^{\delta-1} + \delta - 2| \leq 2$ for $\delta \in (0, \frac{1}{4}]$, we obtain the lower bound on $D(\mu)$ in the statement of the theorem. Again for $-2 < r < -1$ and applying Lemma 1 with any $\epsilon \in (0, \frac{1}{2}]$,

$$T_{AQ_\delta}(p)(\mu) \leq e^2 \sum_{n=1}^{N} \mu_n G(n, p, \epsilon)$$

$$= e^2 \alpha_r \sum_{n=1}^{N} n^r G(n, p, \epsilon)$$

$$\leq e^2 \alpha_r \left( G(1, p, \epsilon) + \int_1^N x^r G(x, p, \epsilon) dx \right)$$

$$\leq 200(1 - \epsilon)^{-1} e^2 \log(\epsilon^{-1}) \left( \frac{r+1}{N^{r+1} - 1} \right)$$

$$\times \left( 2.04 + p + \frac{1.04N^{r+1} - 1.04}{r+1} + \frac{pN^{r+2} - p}{r+2} + \frac{N^{\delta} - 1}{r + \frac{\delta}{2}} \right)$$

$$\leq 200(1 - \epsilon)^{-1} e^2 \log(\epsilon^{-1}) \left( 1.04 + 1.04 \left( 7.04 + \frac{p(N^\delta - 1)}{\delta} \right) \right), \quad (19)$$

having chosen $r = \delta - 2$. Taking $\epsilon = \frac{1}{2}$ now leads to the stated result. \qed
Corollary 1 which we now present shows that, for quantum noise levels as large as \( p = p(N) = \frac{1}{\log(N)} \) and advice distribution following a power law, the best possible classical algorithm for SEARCH WITH ADVICE in this case has expected query complexity growing no more slowly than a rate of order \( \log(N) \) as \( N \to \infty \), whereas the query complexity of our geometric quantum search algorithm, Algorithm 2, is in fact bounded by a fixed constant (not depending on \( N \)) for all positive integers \( N \).

**Corollary 1.** For any value \( \delta \) with \( 0 < \delta \leq \frac{1}{4} \), define an advice distribution \( \mu_\delta \) on \( \{1, ..., N\} \) by taking \( \mu_{\delta,n} = \alpha_{(\delta-2)} n^{(\delta-2)} \) for each \( n \in \{1, ..., N\} \) where \( \frac{1}{\alpha_{(\delta-2)}} = \sum_{n=1}^{N} n^{(\delta-2)} \). Suppose that the ambient quantum noise level as in (8) is described by \( p = p(N) = \frac{1}{\log(N)} \). Then,

\[
D(\mu_\delta) \geq \Omega(\log(N)), \quad \text{but} \quad T_{A_{QS}(p(N))}(\mu_\delta) \leq O(1). \tag{20}
\]

In fact, for all \( N \geq 100, \)

\[
D(\mu_\delta) \geq c_3 \log(N), \quad \text{and} \quad T_{A_{QS}(p(N))}(\mu_\delta) \leq c_4, \tag{21}
\]

for constants \( c_3, c_4 > 0 \), where we can take \( c_3 = \frac{3}{8} \) and \( c_4 = 4440e^2 < 32810 \).

**Proof.** The first derivative of the function \( g(\delta) = N^{\delta}, \delta \in \mathbb{R} \), is the function \( g'(\delta) = \log(N)N^{\delta} \). Hence, by the Mean Value Theorem, \( \frac{N^{\delta_1} - 1}{\delta_1} = \log(N)N^{\delta_1} \), where \( 0 \leq \delta_1 \leq \delta \leq \frac{1}{N} \). But, for such \( \delta_1 \), \( 1 \leq N^{\delta_1} \leq N^{1/N} \leq 2 \). So, taking \( p = p(N) = \frac{1}{\log(N)} \) in the statement of Theorem 2 establishes the result. \( \square \)

In Corollary 2, we extend Corollary 1 by allowing increased levels of quantum noise, at the expense of obtaining less dramatic reductions in query complexity by means of quantum search.

**Corollary 2.** For any value \( \delta \) with \( 0 < \delta \leq \frac{1}{4} \), define an advice distribution \( \mu_\delta \) on \( \{1, ..., N\} \) by taking \( \mu_{\delta,n} = \alpha_{(\delta-2)} n^{(\delta-2)} \) for each \( n \in \{1, ..., N\} \) where \( \frac{1}{\alpha_{(\delta-2)}} = \sum_{n=1}^{N} n^{(\delta-2)} \). Suppose that the ambient quantum noise level as in (8) is described by \( p = p(N) = \frac{1}{(\log(N))^q} \), for some \( q, 0 < q \leq 1 \). Then,

\[
D(\mu_\delta) \geq \Omega(\log(N)), \quad \text{but} \quad T_{A_{QS}(p(N))}(\mu_\delta) \leq O((\log(N))^{1-q}). \tag{22}
\]

In fact, for all \( N \geq 100, \)

\[
D(\mu_\delta) \geq c_3 \log(N), \quad \text{and} \quad T_{A_{QS}(p(N))}(\mu_\delta) \leq c_4(\log(N))^{1-q}, \tag{23}
\]

for constants \( c_3, c_4 > 0 \), where we can take \( c_3 = \frac{3}{8} \) and \( c_4 = 4440e^2 < 32810 \).

**Proof.** The proof is the same as that for Corollary 1 except that we now take \( p = p(N) = \frac{1}{(\log(N))^q} \). \( \square \)
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