IN THE NAME OF GOD

CHARGED ROTATING BLACK BRANES IN VARIOUS DIMENSIONS

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IN

PHYSICS

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SEPTEMBER 2004
To:

my parents,

my wife,

and...

Farin
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Abstract

Charged Rotating Black Branes in Various Dimensions

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In this thesis, two different aspects of asymptotically charged rotating black branes in various dimensions are studied. In the first part, the thermodynamics of these spacetimes is investigated, while in the second part the no hair theorem for these spacetimes in four dimensions is considered. In part I, first, the Euclidean actions of a d-dimensional charged rotating black brane are computed through the use of the counterterms renormalization method both in the canonical and the grand-canonical ensemble, and it is shown that the logarithmic divergencies associated to the Weyl anomalies and matter field vanish. Second, a Smarr-type formula for the mass as a function of the entropy, the angular momenta and the electric charge is obtained, which shows that these quantities satisfy the first law of thermodynamics. Third, by using the conserved quantities and the Euclidean actions, the thermodynamics potentials of the system in terms of the temperature, the angular velocities and the electric potential are obtained both in the canonical and the grand-canonical ensemble. Fourth, a stability analysis in these two ensembles is performed, which shows that the system is thermally stable. This is in commensurable with the fact that there is no Hawking-Page phase transition for black object with zero curvature horizon. Finally, the logarithmic correction of the entropy due to the thermal fluctuation around the equilibrium is calculated.

In part II, the cosmological defects are studied, and it is shown that the Abelian Higgs field equations in the background of a four-dimensional rotating charged black string have vortex solutions. These solutions, which have axial symmetry, show that the rotating black string can support the Abelian Higgs field as hair. It is found that in the case of rotating black string, there exists an electric field coupled to the Higgs scalar field. This electric field is due to an electric charge per unit length, which increases as the rotation parameter becomes larger. Also it is found that the vortex thickness decreases as the rotation parameter grows up. Finally the self-gravity of the Abelian Higgs field is investigated, and it is shown that the effect of the vortex is to induce a deficit angle in the metric under consideration which decreases as the rotation parameter increases.
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B.1 The manifold, its boundaries and the normal vectors to these boundaries.
Chapter 1

Introduction

Black holes are truly unique objects: For theoretical physicists they pose various interesting problems, which may offer a way for solving the difficult problems of quantum gravity. Currently about 20 stellar binaries are known in our galaxy which are believed to contain black holes of some solar masses, whereas supermassive black holes provide the only explanation for the processes observed in the centers of active galaxies [1]. The gravitational wave detectors GEO 600 [2], VIRGO [3] and LIGO [4], are used to directly observe processes involving black holes, including collisions of black holes, in our cosmic neighborhood of about 25 Mpc.

The theoretical aspect of black hole physics is the essential aim of this thesis. Many works have been done on spherical black holes which are asymptotically flat, de Sitter or anti-de Sitter. In recent years the thermodynamics and no-hair theorem of static solutions of Einstein equation such as Schwarzschild, Reissner-Nordstrom and ... black holes have been considered. But there exist only a few works on these aspects for stationary black holes with less symmetries. This is due to the difficulties which will appear in considering less symmetric black holes. In this thesis we want to consider the thermodynamics (part I) and no-hair theorem (part II) of these kind of black holes. Rotating charged black brane is an example of black holes with less symmetries which
1.1 Thermodynamics of Black Holes (Part I)

The connection between heat and mechanical energy was one of the most interesting discoveries of thermodynamics. This realization provided the first clues about how phenomena at the macroscopic level must arise from the statistical properties of the mechanics of microscopic objects. The failure of the classical statistical mechanics, was the first indication of quantum mechanical effects. The study of systems at the macroscopic scale yields insight into the more fundamental theories of nature and has allowed physicists to take the first steps to understand these theories.

Although quantum mechanics seems to be quite sufficient for most applications, the theory suffers from serious problems in its foundation, especially when one attempts to develop a quantum theory of gravitation. As an aid in understanding quantum gravity, one needs a system in which both the quantum and the classical behavior exist. The black hole is such a system. One hopes to gain insight into the nature of quantum gravity by studying the thermodynamics of black holes. The black hole is an object that is considered of classical mechanics and quantum mechanics: of the macroscopic and the microscopic point of view.

The quantities of particular interest in gravitational thermodynamics are the physical entropy $S$ and the temperature $\beta^{-1}$, where these quantities are respectively proportional to the area and surface gravity of the event horizon(s) [5, 6, 7]. Other black hole properties, such as energy, angular momentum and conserved charges can also be given a thermodynamic interpretation. In finding the thermodynamic quantities, one should use the quasilocal definitions for the thermodynamic variables. By quasilocal, we mean that the quantity is constructed from information that exists on the boundary of a gravitating system alone. Just as the Gauss’ law, such quasilocal quantities will yield
information about the spacetime contained within the system boundary. Two advantages of using such a quasilocal method are the following: first, one is able to effectively separate the gravitating system from the rest of the universe; second, the formalism does not depend on the particular asymptotic behavior of the system, so one can accommodate a wide class of spacetimes with the same formalism.

Brown and York [8] developed a method for calculating energy and other charges contained within a specified surface surrounding a gravitating system. Although their quasilocal energy depends only on quantities defined on the boundary of the gravitating system, it yields information about the total gravitational energy contained within the boundary. Also they showed that this quasilocal energy is the thermodynamic internal energy [9]. The quasilocal energy and momentum are obtained from the gravitational action via a Hamilton-Jacobi analysis. Although the analysis of Brown and York was restricted to general relativity in four dimensions, the thermodynamic variables were obtained from the action rather than from the field equations.

In general, the problem of calculating gravitational thermodynamic quantities remains a lively subject of interest. Because they typically diverge for both asymptotically flat, asymptotically anti-de Sitter (AdS) and de Sitter (dS) spacetimes, a common approach toward evaluating them has been to carry out all computations relative to some other spacetime that is regarded as the ground state for the class of spacetimes of interest. This is done by taking the original action $I_G = I_M + I_{\partial M}$ for gravity coupled to matter fields and subtracting from it a reference action $I_0$, which is a functional of the induced metric on the boundary $\partial M$. Conserved and/or thermodynamic quantities are then computed relative to reference spacetime, which can then be taken to (spatial) infinity if desired. This approach has been widely successful in providing a description of gravitational thermodynamics in regions of both finite and infinite spatial extent [8, 10]. Unfortunately it suffers from several drawbacks. The choice of reference spacetime is not always unique [11], nor is
it always possible to embed a boundary with a given induced metric into the reference background. Indeed, for Kerr spacetimes, this latter problem forms a serious obstruction towards calculating the subtraction energy, and calculations have only been performed in the slow-rotating regime [12]. Currently, one of the useful theories which helps us obtain the thermodynamic quantities is the ‘AdS/conformal field theory (CFT)’ correspondence which can be used to compute the conserved quantities of asymptotically anti-de Sitter (AAdS) spacetimes.

This principle posits a relationship between supergravity or string theory in bulk AdS spacetimes and conformal field theories on the boundary. It offers the possibility that a full quantum theory of gravity could be described by a well understood CFT theory.

Since quantum field theories in general contain counterterms, it is natural from the AdS/CFT viewpoint to append a boundary term $I_{ct}$ to the action that depends on the intrinsic geometry of the (timelike) boundary at large spatial distances. This requirement, along with general covariance, implies that these terms be functional of curvature invariants of the induced metric and have no dependence on the extrinsic curvature of the boundary. An algorithmic procedure [13] exists for constructing $I_{ct}$ for asymptotically AdS spacetimes, and so its determination is unique. The addition of $I_{ct}$ will not affect the bulk equations of motion, thereby eliminating the need to embed the given geometry in a reference spacetime.

Although, there is no proof of this conjecture, it furnishes a means for calculating the action and thermodynamic quantities intrinsically without reliance on any reference spacetime [14, 15, 16]. A dictionary is emerging that translates between different quantities in the bulk gravity theory and its counterparts on the boundary, including the partition functions and correlation functions. Another interesting application of the AdS/CFT correspondence is the interpretation of Hawking-Page phase transition between thermal AdS and AAdS black hole as the confinement-deconfinement phases of the Yang-Mills
(dual gauge) theory defined on the AdS boundary [17].

The AdS/CFT correspondence has been also extended to the case of asymptotically de Sitter spacetimes [18, 19]. Although the (A)dS/CFT correspondence applies for the case of special infinite boundary, it was also employed to the computation of the conserved and thermodynamic quantities in the case of a finite boundary [20]. This conjecture has also been applied for the case of black objects with constant negative or zero curvature horizons [21, 22].

Due to the AdS/CFT correspondence, AAdS black holes continue to attract a great deal of attention. For AAdS spacetimes, the presence of a negative cosmological constant makes it possible to have a large variety of black holes/branes, whose event horizons are hypersurfaces with positive, negative or zero scalar curvature [23]. Static and rotating uncharged solutions of Einstein Relativity with negative cosmological constant and with planar symmetry (planar, cylindrical and toroidal topology) may be found in Ref. [24]. Unlike the zero cosmological constant planar case, these solutions include the presence of black strings (or cylindrical black holes) in the cylindrical model, and of toroidal black holes and black membranes in the toroidal and planar models, respectively. The extension to include the Maxwell field has been done in Ref. [25]. This metric was the AAdS rotating type solutions of Einstein’s equation. The authors in Ref. [25] found the static and rotating pure electrically charged black holes that are the electric counterparts of the cylindrical, toroidal and planar black holes which is found in [24]. The metric with electric charge and zero angular momentum was also discussed in Ref. [26].

The generalization of this AAdS charged rotating solution of Einstein-Maxwell equations to the higher dimensions has been done in Ref. [27]. Many authors have considered thermodynamics and stability conditions of these black holes [21, 28].
1.2 The No-Hair Theorem (Part II)

The conjecture that after the gravitational collapse of the matter field, the resultant black hole is characterized by at most its electromagnetic charge, mass, and angular momentum is known as the classical ‘no-hair’ theorem and was first proposed by Ruffini and Wheeler [29]. Nowadays we are faced with the discovery of black hole solutions in many theories in which Einstein’s equation is coupled with some self interacting matter fields, and therefore this conjecture needs more investigation.

In certain special cases the conjecture has been verified. For example, a scalar field minimally coupled to gravity in asymptotically flat spacetimes cannot provide hair for the black hole [30]. But this conjecture cannot extend to all forms of matter fields. It is known that some long range Yang-Mills quantum hair could be painted on the black holes [31]. Explicit calculations have also been carried out which verify the existence of a long range Nielsen-Olesen vortex solution as stable hair for a Schwarzschild black hole in four dimensions [32]. For the extreme black hole of Einstein-Maxwell gravity, it has been shown that flux expulsion occurs for thick strings (thick with respect to the radius of horizon), while flux penetration occurs for thin strings [33, 34, 35, 36]. Of course, one may note that these situations fall outside the scope of classical no-hair theorem due to the non-trivial topology of the string configuration.

Recently, some effort has been made to extend these ideas to the case of (anti-)de Sitter spacetimes. While a scalar field minimally coupled to gravity in asymptotically de Sitter spacetimes cannot provide hair for the black hole [37], it has been shown that in asymptotically AdS spacetimes there is a hairy black hole solution [38]. Also, in Ref. [31], it is shown that there exists a solution to the $SU(2)$ Einstein-Yang-Mills equations which describes a stable Yang-Mills hairy black hole that is asymptotically AdS. In addition the idea of Nielsen-Olesen vortices has been extended to the case of asymptotically (anti-)de Sitter spacetimes.
spacetimes [39]. The investigation of Nielsen-Olesen vortices in the background of charged black holes was done in Refs. [33, 36]. More recently the stability of the Abelian Higgs field in AdS-Schwarzschild and Kerr-AdS backgrounds has been investigated and it has been shown that these asymptotically AdS black holes can support an Abelian Higgs field as hair [40, 41]. In the case of stationary black hole solutions, the explicit calculations which can investigate the existence of a long range Nielsen-Olesen vortex solution as a hair is escorted with much more difficulties due to the rotation parameter [40]. In this thesis we study the Abelian Higgs hair in a four dimensional rotating charged black string that is a stationary model with cylindrical symmetry. Various features of this kind of solutions of Einstein-Maxwell equation have been considered [21]. Here, we want to investigate the influence of rotation on the vortex solution of Einstein-Maxwell-Higgs equation. Since an analytic solution to the Abelian Higgs field equation appears to be intractable, we confirm by numerical calculations that the rotating charged black string can be dressed by Abelian Higgs field as hair.

1.3 Overview

In chapter 2, we present the counterterm renormalization method in order to obtain the finite action, the divergence free stress energy-momentum tensor, and the conserved quantities of asymptotically AdS spacetimes. In chapter 3, the classical law of mechanics of black holes is posed. The metric of a four dimensional black string, which was constructed by Lemos [25], is introduced in chapter 4. In chapter 5, we study the thermodynamics of an \((n+1)\)-dimensional charged rotating black brane. The conserved charges are obtained and also the stability of the black brane is discussed. The discussion of no-hair theorem and the theory of Abelian Higgs hair for black holes is presented in chapter 6, while the existence of Abelian Higgs hair for charged rotating black string is studied in chapter 7. Finally, in chapter 8, we summarize the results.
Part I

Thermodynamics of Black Holes
and AdS/CFT Correspondence
Chapter 2

The Counterterm

Renormalization

The counterterm renormalization which is based on the AdS/CFT correspondence, was first proposed by Maldacena, and is known as the Maldacena’s conjecture [42]. In this chapter, we first discuss the AdS/CFT correspondence briefly, and then introduce the counterterm renormalization method. However, we take more attention on the gravitational aspect of the theory while the conformal field theory aspect of the correspondence would not be covered as much.

The starting point of some exciting developments over the last few years, AdS/CFT correspondence, was a conjecture about the duality of classical supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, $U(N)$ super Yang-Mills theory in the large $N$ limit. Moreover, Witten [43] suggested that via this identification any field theory action on $(n + 1)$-dimensional anti-de Sitter spacetime gives rise to an effective action of a field theory on the $n$-dimensional horizon of AdS spacetime. Most importantly, this field theory on the AdS horizon should be a conformal field theory, because the AdS symmetries act as conformal symmetries on the AdS horizon.
The general correspondence formula is [43]

\[ \int_{\Psi_0} D\Psi e^{-S_{\text{AdS}}[\Psi]} = \langle \exp \int d^nx \mathcal{O}(x) \Psi_0(x) \rangle, \]

(2.1)

where the functional integral on the left hand side is over all the fields \( \Psi \) whose asymptotic boundary values are \( \Psi_0 \), and \( \mathcal{O} \) denotes the conformal operators of the boundary conformal field theory. In the classical limit, the correspondence formula can be written as [43, 44]

\[ S_{\text{AdS}}[\Psi_0] = W_{\text{CFT}}[\Psi_0], \]

(2.2)

where \( S_{\text{AdS}} \) is the classical on-shell action of an AdS field theory, expressed in terms of the field boundary values \( \Psi_0 \), and \( W_{\text{CFT}} \) is the CFT effective action. However, one should expect \( S_{\text{AdS}} \) to be divergent as it stands, because of the divergence of the AdS metric on the AdS horizon (see Appendix A). Thus, in order to extract the physically relevant information, the on-shell action has to be renormalized by adding counterterms, which cancel the infinities. After defining the renormalized, finite action by

\[ S_{\text{AdS,fin}} = S_{\text{AdS}} - S_{\text{div}}, \]

(2.3)

where \( S_{\text{div}} \) stands for the local counterterms and \( S_{\text{AdS,fin}} \) is the CFT effective action, then the meaningful correspondence formula is

\[ S_{\text{AdS,fin}} = W_{\text{CFT}}. \]

(2.4)

Given a field theory action on AdS spacetime and a suitable regularization method, it is straightforward to calculate the renormalized on-shell action \( S_{\text{AdS,fin}} \). On the other hand, the CFT effective action \( W_{\text{CFT}}[\Psi_0] \) contains all the information about the conformal field theory living on the AdS horizon. Moreover, any field theory on AdS spacetime, which includes gravity, has a corresponding counterpart CFT, whose action might not even be known.

Before closing this section, an important aspects related to the correspondence formula shall be discussed briefly. Why can one be sure that \( W_{\text{CFT}} \) is the
generating functional of a conformal field theory? The AdS symmetries act as conformal symmetries on the AdS horizon. Thus, by virtue of the invariance of the AdS action under AdS symmetries, $W_{CFT}$ is invariant under conformal transformations, as long as the counterterms do not break these symmetries. This case, which is the generic one, is given when all the counterterms are covariant. Hence, the CFT correlation functions will obey all the restrictions imposed by conformal invariance. A brief discussion of AdS symmetries and conformal symmetries will be given in Appendix A.

The famous exception appears when $S_{div}$ contains at least one non-covariant term, which inevitably breaks some of the AdS symmetries. Therefore, the conformal symmetry of the CFT effective action will be broken too. The interesting and encouraging fact about this is that the breaking of conformal invariance is a strong signature of the quantum character of the CFT and tells about the anomalies in the algebra of quantum conformal operators. The most notable example is the Weyl anomaly, which is discussed in section(2.2.1).

### 2.1 The AdS/CFT Correspondence

In a generally covariant theory, it is unnatural to assign a local energy-momentum density to the gravitational field. For instance, candidate expressions depending only on the metric and its first derivatives will always vanish at a given point in locally coordinates. Instead, one can consider the so-called ‘quasilocal stress tensor’, defined locally on the boundary of a given spacetime region. Consider the gravitational action as a functional of the boundary metric $\gamma_{\mu\nu}$.

The quasilocal stress tensor associated with a spacetime region has been defined by Brown and York to be [8]:

$$T^{ij} = 2 \frac{\delta S_{grav}}{\sqrt{-\gamma} \delta \gamma_{ij}}, \quad (2.5)$$

where $S_{grav}$ is the gravitational action. The resulting stress tensor typically diverges as the boundary is taken to infinity. However, one is always free to add
a boundary term to the action without disturbing the bulk equations of motion. To obtain a finite stress tensor, Brown and York propose a subtraction derived by embedding a boundary with the same intrinsic metric $\gamma_{\mu \nu}$ in some reference spacetime. Unfortunately this prescription suffers from several drawbacks. The choice of reference spacetime is not always unique [11], nor is it always possible to embed a boundary with a given induced metric into the reference background. Indeed, for Kerr spacetimes this latter problem forms a serious obstruction toward calculating the subtraction energy, and calculations have only been performed in the slow-rotating regime [12]. Therefore, the method of subtraction of Brown-York is generally not well defined.

For asymptotically anti-de Sitter (AdS) spacetimes, the AdS/CFT correspondence was an attractive resolution to this difficulty. In the gravitational aspects, it says there is an equivalence between a gravitational theory in a $(n + 1)$-dimensional anti-de Sitter spacetime and a conformal theory in a $n$-dimensional spacetime which can in some sense be viewed as the boundary of the higher dimensional spacetime. According to this correspondence, Eq. (2.5) can be interpreted as the expectation value of the stress tensor in the CFT:

$$\langle T^{ij} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{eff}}{\delta \gamma_{ij}}.$$ (2.6)

The divergences which appear as the boundary goes to infinity, are simply the standard ultraviolet divergences of quantum field theory, and may be removed by adding local counterterms to the action. These counterterms depend only on the intrinsic geometry of the boundary and are defined once and for all. By using this method, the ambiguous prescription involving embedding the boundary in a reference spacetime is removed.
2.1.1 Counterterm Method

We write the standard action for the gravitational field in vacuum, in a unit system with $G = 1$, as:

\[
S = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left( R + \frac{n(n-1)}{\ell^2} \right) + \frac{1}{8\pi} \int_{\partial B} d^n x \sqrt{-\gamma} \Theta - \frac{1}{8\pi} \int_{\Sigma} d^n x \sqrt{h} K, \tag{2.7}
\]

where $\Theta$ and $K$ are the trace of the extrinsic curvature of the boundaries with induced metric $\gamma_{ij}$ and $h_{ij}$ (Appendix B). The first integral is the Einstein-Hilbert volume term. The second term represents an integral over the three-timelike-boundary $\partial B$ and the third term is an integral over the three-spacelike-boundary $\Sigma$. These two surface terms are called the Gibbons-Hawking boundary terms. The necessity of the boundary terms can be seen as follows. The variation of the volume term in action (2.7) with respect to $g^{\mu\nu}$ is given by

\[
\delta S = \delta S_{M_{n+1}} + \delta S_{M_n}, \tag{2.8}
\]

where

\[
\delta S_{M_{n+1}} = \frac{1}{8\pi} \int_{M_{n+1}} d^{n+1} x \sqrt{-g} \delta g^{\varsigma\zeta} \left[ -\frac{1}{2} g_{\varsigma\zeta} \left( R + \frac{n(n-1)}{\ell^2} \right) + R_{\varsigma\zeta} \right] \tag{2.9}
\]

and

\[
\delta S_{M_n} = \frac{1}{8\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} n_\mu \left[ \partial^\mu (g_{\xi\nu} \delta g^{\xi\nu}) - D_\nu (\delta g^{\mu\nu}) \right]. \tag{2.10}
\]

Here $\hat{g}_{ij}$ (i.e. $\gamma_{ij}$ or $h_{ij}$) is the metric on $M_n$ (i.e. $\Sigma$ or $\partial B$) induced from $g_{\mu\nu}$ and $n_\mu$ is the unit vector normal to $M_n$ (Appendix B). Now we choose the metric in the following form:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} \hat{g}_{ij} dx^i dx^j. \tag{2.11}
\]

---

\[1\] This fixes our conventions for the Riemann curvature to be $R_{\mu\nu\lambda}^\sigma = -2\partial_{[\mu} \Gamma_{\nu]\lambda}^{\sigma} + 2\Gamma_{\mu\rho}^\alpha \Gamma_{\nu}\rho^\alpha$, where the antisymmetrization is defined with strength one, i.e. $[\mu\nu] = \frac{1}{2} (\mu\nu - \nu\mu)$. Also $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$. With these conventions spheres have a positive scalar curvature. The cosmological constant is written as $\Lambda = -n(n-1)/2\ell^2$; in this notation pure $AdS_{n+1}$ has radius $\ell$. 

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Then \( n_\mu \) and its covariant derivatives are given by,

\[
\begin{align*}
n_\mu &= \left( \frac{2\rho}{l}, 0, ..., 0 \right), \quad \nabla_\rho \rho^\mu = \nabla_\rho n^i = \nabla_i n^\rho = 0, \quad \nabla_i n^j = \frac{\rho}{l} \hat{g}_{ik} \hat{g}^{ij},
\end{align*}
\]

(2.12)

where the prime denotes the derivative with respect to \( \rho \). In the coordinate choice (2.11), the surface terms \( \delta S_{M_n} \) in (2.10) have the following form

\[
\delta S_{M_n} = \lim_{\rho \to 0} \frac{1}{8\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} \left[ \partial^\rho (\hat{g}_{ij} \delta \hat{g}^{ij}) \right].
\]

(2.13)

Note that the terms containing \( \delta \hat{g}^{\rho\rho} \) or \( \delta \hat{g}^{\rho i} \) vanish. The variant \( \delta S_{M_n} \) contains the derivative of \( \delta \hat{g}^{ij} \) with respect to \( \rho \), which makes the variational principle ill-defined. In order to have a well defined variation principle on the boundary, the variation of the action, after using the partial integration, should be written as

\[
\delta S_{M_n} = \lim_{\rho \to 0} \frac{1}{8\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} \delta \hat{g}^{ij} \{ \cdots \}.
\]

(2.14)

If the variation of the action on the boundary contains \( (\delta \hat{g}^{ij})' \), however, we cannot partially integrate it with respect to \( \rho \) on the boundary to rewrite the variation in the form of (2.14) since \( \rho \) is the coordinate expressing the direction perpendicular to the boundary. Therefore the ‘extremization’ of the action is ambiguous. Such a problem was well studied in [45] for the Einstein gravity. It is easy to show that the boundary term which can remove the terms containing \( (\delta \hat{g}^{ij})' \) is

\[2\]

\[\text{(2.15)}\]

then the variation over the metric \( \hat{g}^{ij} \) gives

\[
\delta S_{GH}^{(\hat{g}^{ij})} = -\frac{1}{4\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} \left[ \delta \hat{g}^{ij} \left\{ -\hat{g}_{ik} \hat{g}_{jl} (\hat{g}^{kl})' - \frac{1}{2} \hat{g}_{ij} \hat{g}_{kl} (\hat{g}^{kl})' \right\} + \hat{g}_{ij} (\delta \hat{g}^{ij})' \right].
\]

(2.16)

On the other side, the surface terms in the variation of the bulk Einstein action in (2.13), have the form

\[
\delta S_{Einstein}^{(\hat{g}^{ij})} = \lim_{\rho \to 0} \frac{1}{8\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} \left[ \hat{g}_{ij} (\delta \hat{g}^{ij})' + \hat{g}_{ij} (\delta \hat{g}^{ij})' \right].
\]

(2.17)

Then the terms containing \( (\delta \hat{g}^{ij})' \) in (2.15) and (2.16) are canceled with each other [46].
\[ S_{GH}^{b} = -\frac{1}{4\pi} \int_{M_n} d^n x \sqrt{-\hat{g}} \nabla_{\mu} n^\mu. \quad (2.18) \]

Variation of the action (2.7) with respect to the metrics \( \gamma_{\mu\nu} \) and \( h_{\mu\nu} \) of the boundary \( ^3B \) and \( \Sigma \) gives [8]:

\[ \int_{^3B} d^3 x \Pi^{ij} \delta \gamma_{ij} + \int_{t_i}^{t_f} d^3 x P^{ij} \delta h_{ij}. \quad (2.19) \]

Here, \( \Pi^{ij} \) denotes the gravitational momentum conjugate to \( \gamma_{ij} \), as defined with respect to the three boundary \( ^3B \), while \( P^{ij} \) denotes the gravitational momentum conjugate to \( h_{ij} \), as defined with respect to the spacelike hypersurfaces \( t_i \) and \( t_f \) given as

\[ \Pi^{ij} = -\frac{1}{16\pi} \sqrt{-\gamma} (\Theta \gamma^{ij} - \Theta^{ij}), \quad (2.20) \]

\[ P^{ij} = \frac{1}{16\pi} \sqrt{h} (Kh^{ij} - K^{ij}), \quad (2.21) \]

where \( \Theta \) and \( K \) are trace of the extrinsic curvatures \( \Theta^{ij} \) and \( K^{ij} \) of three-boundaries \( ^3B \) and \( \Sigma \) (see Appendix B). In the AdS frame work, it is natural to contribute only the time like boundary term \( ^3B \) in action (2.7) and \( \Pi^{ij} \) in Eq. (2.20).

The energy-momentum tensor for boundary \( ^3B \) can obtain as:

\[ T^{ij} = -\frac{1}{8\pi} (\Theta \gamma^{ij} - \Theta^{ij}). \quad (2.22) \]

Concrete computations show that in most spacetimes both the action integral (2.7) and the energy-momentum tensor (2.22) diverge as the boundary \( ^3B \) goes to infinity. We therefore think of these as the unrenormalized quantities.

The divergences must be cancelled in order to achieve physically meaningful expressions. Thus, one may add a counterterm

\[ \tilde{S} = \frac{1}{8\pi} \int d^n x \sqrt{-\gamma} \tilde{\mathcal{L}}, \quad (2.23) \]
to the action, along with the corresponding counterterm energy-momentum tensor:

\[ \tilde{T}_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta \tilde{S}}{\delta \gamma_{ab}}. \]  

(2.24)

The counterterms, by definition, contain the divergent part of the corresponding unrenormalized quantities, but finite terms may depend on the details of the renormalization.

The counterterm should be a function of the boundary geometry only. Moreover, suppose the counterterm is an analytical function of the boundary geometry, and expand it as a power series in the metric and its derivatives. Dimensional analysis shows that in AdS\(_{n+1}\) only terms of order \(m < n/2\) contribute to the divergent part of the action. (By terms of order \(m\), we mean terms containing \(2m\) derivatives.) Therefore one may truncate the series at this order and obtain a finite polynomial [15]. This agrees with the expectations from the interpretation of the divergences in terms of a dual boundary theory that obeys the usual axioms of quantum field theory, including locality.

### 2.1.2 The Counterterm Generating Algorithm

The structure of divergences is tightly constrained by the Gauss-Codacci equations (B.2)-(B.4). Using Eq. (2.22), the Gauss-Codacci equations for timelike hypersurface boundary \(3B\) can be written as [47]:

\[ G_{ab} = G_{ab}(\gamma) + n^\mu \nabla_\mu T_{ab} - \frac{1}{2} \gamma_{ab} \left( \frac{T^2}{n-1} - T_{cd}T^{cd} \right) + \frac{1}{n-1} T_{ab} T, \]  

(2.25)

\[ G_{a\mu} n^\mu = -\nabla_b T_{ba}, \]  

(2.26)

\[ G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} \left( \frac{T^2}{n-1} - T_{cd}T^{cd} - R(\gamma) \right), \]  

(2.27)

where \(n^\mu\) is an outward pointing unit vector normal to the boundary \(3B\). We will always consider solutions of the bulk equations of motion. So the following
equations
\[ G_{ab} = \frac{1}{2} n(n-1) \ell^2 \gamma_{ab}, \]
\[ G_{a\mu} n^\mu = 0, \]
\[ G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} n(n-1) \ell^2, \]
(2.28)
determine the left hand side of the Gauss-Codacci equations.

In principle, one could solve the Gauss-Codacci equations (2.25)-(2.27) for the unrenormalized energy-momentum tensor \( T_{ab} \), and then identify its divergent part with \( -\tilde{T}_{ab} \). However, this strategy is rather complicated due to the presence of the normal derivatives in (2.25). The appearance of these normal derivatives expresses the intuitive fact that, to determine the solution throughout, both the boundary values and their derivatives are needed. However, the counterterm should be determined independently of data that is extrinsic to the boundary, such as the normal derivative.

Explicit computations show that one can find a coordinate system so that the divergent part of the normal derivatives expressed in terms of the intrinsic boundary data. We implement this observation covariantly, as follows. We impose the constraint equation (2.27):
\[ \frac{1}{n-1} \tilde{T}^2 - \tilde{T}_{ab} \tilde{T}^{ab} = \frac{n(n-1)}{\ell^2} + R, \]
(2.29)
and further insist that the counterterm energy-momentum tensor must derive from a counterterm action, which is itself intrinsic to the boundary:
\[ \tilde{T}^{ab} = \frac{2}{\sqrt{-\gamma}} \delta \delta^{ab} \int d^n x \sqrt{-\gamma} \tilde{\mathcal{L}}. \]
(2.30)
As we will show, the conditions (2.29) and (2.30) fully determine the counterterm. The form of (2.30) ensures that the counterterm energy-momentum is conserved, which in turn implies (2.26). It is important to stress that the remaining Gauss-Codacci equations (2.25) are also satisfied: they can be viewed as expressions for the normal derivatives specified implicitly in our construction. We note that the normal derivatives which are determined, do not in general vanish.
We are now prepared to describe an algorithm that determines the counterterm as an expansion in the parameter $\ell$. The leading order term scale as $\ell^{-1}$ and terms at a given order $\ell^{2m-1}$ with $m \geq 0$ are denoted by $\tilde{T}_{ab}^{(m)}$ and $\tilde{L}^{(m)}$. The starting point is to note that the curvature term in (2.29) can be neglected to the leading order in $\ell$, so that the metric is the only tensor characterizing the boundary geometry to the leading order, and therefore $\tilde{T}_{ab}^{(0)}$ is proportional to the metric. Explicit computation will be given in Subsec. (2.1.4).

Higher order counterterms are now given by induction. Assuming that $\tilde{T}_{ab}$ is known up to and including order $m-1$, the following three steps determine $\tilde{T}_{ab}^{(m)}$:

step 1: Insert the known terms in (2.29); the resulting expression is a linear equation with the trace $\tilde{T}^{(m)}$ as the only unknown.

step 2: With the trace $\tilde{T}^{(m)}$ in hand, integrate (2.30) and find $\tilde{L}^{(m)}$. This step is purely algebraic, as discussed in the following subsection.

step 3: Finally, take the functional derivative of $\tilde{L}^{(m)}$ with respect to $\gamma_{ab}$, and so find the full tensor $\tilde{T}_{ab}^{(m)}$ from (2.30).

The fact that $\tilde{T}_{ab}^{(0)}$ is proportional to the metric $\gamma_{ab}$ is crucial to make step 1 possible. We stress that higher orders of $\tilde{T}_{ab}$ in general will depend also on other tensor structures.

### 2.1.3 Some Comments on Weyl Rescaling

Under Weyl rescaling we have

$$g'_{ij} = [\lambda(x)]^2 g_{ij}(x)$$  \hspace{1cm} (2.31)

where $\lambda(x)$ is an arbitrary function. If the action $S$ is invariant under any such Weyl rescalings, the definition of energy momentum tensor

$$\delta S = -\frac{1}{2} \int_{\Omega} d^n x \sqrt{g(x)} T_{ij}(x) \delta g^{ij}(x),$$  \hspace{1cm} (2.32)
implies the tracelessness of the energy momentum tensor,

\[ T^i_i = 0. \]  

(2.33)

The inverse statement is true as well. Moreover, Eq. (A.13) implies that, if an action is invariant under Weyl rescaling, it is also conformally invariant. However the inverse is not true, because, while \( \lambda \) is arbitrary in Eq. (2.31), it is not so in Eq. (A.13).

The integration in step 2 in previous section is interesting and deserves comment. It is related to the behavior of the various terms under the local Weyl variations which transform the metric as:

\[ \delta_W \gamma_{ab} = \sigma \gamma_{ab}, \]  

(2.34)

where \( \sigma \) is an arbitrary function. Consider the counterterm action at the \( m \)th order and note that dimensional analysis gives the behavior under a global Weyl rescaling. The result of a local Weyl variation can therefore be written in the form [48]:

\[ \delta_W \int d^n x \sqrt{-\gamma} \mathcal{L}(m) = \int d^n x \sqrt{-\gamma} \sigma \left( \frac{n-2m}{2} \mathcal{L}^{(m)} + \nabla_a X^a(m) \right), \]  

(2.35)

where \( X^a(m) \) is some unspecified expression (involving \( 2m + 1 \) derivatives). However, it follows from (2.30) that:

\[ \delta_W \int d^n x \sqrt{-\gamma} \tilde{\mathcal{L}}^{(m)} = \frac{1}{2} \int d^n x \sqrt{-\gamma} \sigma \tilde{T}^{(m)}, \]  

(2.36)

and therefore

\[ (n-2m) \tilde{\mathcal{L}}^{(m)} = \tilde{T}^{(m)}, \]  

(2.37)

The practical significance of this identity is that it renders the integration in step 2 almost trivial. We also note that [48]:

\[ \delta_W \int d^n x \sqrt{-\gamma} \tilde{T}^{(m)} = \frac{n-2m}{2} \int d^n x \sqrt{-\gamma} \sigma \tilde{T}^{(m)}, \]  

(2.38)

and therefore \( \sqrt{-\gamma} \tilde{T}^{(m)} \) transforms as a conformal density with Weyl weight \( \frac{1}{2}(n-2m) \), up to a total derivative [49, 50]. This constrains the form of the counterterms.
In even dimensions it is clear that (2.37) prevents $\tilde{T}^{(n/2)}$ from being obtained as the variation of any local action. This is the origin of trace anomalies. For an even $n$, the trace $\tilde{T}^{(n/2)}$ is, therefore, identified with the trace anomaly of the dual boundary theory. This result for the anomaly agrees with that of [14], as may be verified by looking at the explicit expressions given below.

2.1.4 Explicit Computations of Counterterms

At this point, we evaluate the first few orders of counterterms explicitly.

Expanding $\tilde{T} = \gamma^{ab} \tilde{T}_{ab}$ as a power series of $\ell$:

$$\tilde{T} = \tilde{T}^{(0)} + \tilde{T}^{(1)} + \tilde{T}^{(2)} + \ldots$$

$$= \frac{a_0}{\ell} + a_1 \ell + a_2 \ell^3 + a_3 \ell^5 + \ldots,$$

(2.39)

In leading order one may neglect $R$ in Eq. (2.29) and uses the fact that $\gamma^{ab} \gamma_{ab} = n$ to obtain

$$\frac{1}{n-1} (\gamma^{ab} \tilde{T}^{(0)}_{ab}) (\gamma_{cd} \tilde{T}^{(0)}_{cd}) - \tilde{T}^{(0)}_{ab} \tilde{T}^{(0)}_{ab} = \frac{n(n-1)}{\ell^2}$$

$$\rightarrow \tilde{T}^{(0)}_{ab} \tilde{T}^{(0)}_{ab} = \frac{n(n-1)^2}{\ell^2}$$

$$\rightarrow \tilde{T}^{(0)}_{ab} = -\frac{(n-1)}{\ell} \gamma_{ab}.$$  

(2.40)

Thus,

$$\tilde{T}^{(0)} = -\frac{n(n-1)}{\ell},$$

(2.41)

and by using Eq. (2.37) one obtains:

$$\tilde{L}^{(0)} = -\frac{n-1}{\ell}.$$  

(2.42)

Up to the first order by inserting Eq. (2.41) in Eq. (2.39) and using Eq. (2.29) one obtains:

$$\tilde{T}^2 = \tilde{T}^{(0)^2} + 2a_1 \ell \tilde{T}^{(0)} = \frac{n^2(n-1)^2}{\ell^2} + Rn(n - 1)$$

$$\rightarrow a_1 = -\frac{R}{2} \rightarrow \tilde{T}^{(1)} = -\frac{\ell}{2} R.$$  

(2.43)

Now Eq. (2.37) gives:

$$\tilde{L}^{(1)} = -\frac{\ell}{2(n - 2)} R.$$  

(2.44)
and the variation (2.30) yields:

\[
\tilde{T}^{(1)}_{ab} = \frac{\ell}{n-2} \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right). 
\] (2.45)

By using this algorithm, one can generate the next few order in expansion as:

\[
\tilde{T} = -\frac{n(n-1)}{\ell} - \frac{\ell}{2} R - \frac{\ell^3}{2(n-2)^2} \left( R_{ab} R^{ab} - \frac{n}{4(n-1)} R^2 \right)
+ \frac{\ell^5}{(n-2)^3(n-4)} \left\{ \frac{3n+2}{4(n-1)} RR_{ab} R^{ab} - \frac{n(n+2)}{16(n-1)^2} R^3 
- 2 R^{ab} R_{acbd} R^{cd} + \frac{n-2}{2(n-1)} R^{ab} \nabla_a \nabla_b R - R^{ab} \Box R_{ab} + \frac{1}{2(n-1)} R R \right\} + \cdots ,
\] (2.46)

\[
\tilde{L} = -\frac{n-1}{\ell} - \frac{\ell}{2(n-2)} R - \frac{\ell^3}{2(n-2)^2(n-4)} \left( R_{ab} R^{ab} - \frac{n}{4(n-1)} R^2 \right)
+ \frac{\ell^5}{(n-2)^3(n-4)(n-6)} \left\{ \frac{3n+2}{4(n-1)} RR_{ab} R^{ab} - \frac{n(n+2)}{16(n-1)^2} R^3 
- 2 R^{ab} R_{acbd} R^{cd} + \frac{n-2}{2(n-1)} R^{ab} \nabla_a \nabla_b R - R^{ab} \Box R_{ab} + \frac{1}{2(n-1)} R R \right\} + \cdots .
\] (2.47)

The energy-momentum tensor can be obtained through the variation of Eq. (2.47):

\[
\tilde{T}_{ab} = -\frac{n-1}{\ell} \gamma_{ab} + \frac{\ell}{n-2} \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right)
+ \frac{\ell^3}{(n-2)^2(n-4)} \left\{ - \frac{1}{2} \gamma_{ab} \left( R_{cd} R^{cd} - \frac{n}{4(n-1)} R^2 \right) - \frac{n}{2(n-1)} RR_{ab} 
+ 2 R^{cd} R_{acbd} - \frac{n-2}{2(n-1)} \nabla_a \nabla_b R + \Box R_{ab} - \frac{1}{2(n-1)} \gamma_{ab} \Box R \right\} + \cdots ,
\] (2.48)

where \(\Box = \nabla_a \nabla^a\) and \(R, R_{abcd}\), and \(R_{ab}\) are the Ricci scalar, Riemann and Ricci tensors of the boundary metric \(\gamma_{ab}\). The most laborious step is to find the full energy momentum-tensor from the counterterm. Accordingly, we have resisted carrying out this computation to the fourth order [48]. At last the divergent free or finite energy-momentum tensor can be obtained from \(\tilde{T}_{(finite)ab} = \tilde{T}_{ab} + \tilde{T}_{ab},\)
therefore

\[ T_{(\text{finite})}^{ab} = \frac{1}{8\pi} \left\{ (\Theta^{ab} - \Theta \gamma^{ab}) - \frac{n-1}{\ell} \gamma^{ab} + \frac{\ell}{n-2} (R^{ab} - \frac{1}{2} R \gamma^{ab}) \ight. \\
\left. + \frac{\ell^3 \Upsilon(n-5)}{(n-4)(n-2)^2} \left[ -\frac{1}{2} \gamma^{ab} (R^{cd} R_{cd} - \frac{n}{4(n-1)} R^2) - \frac{n}{(2n-2)} R R^{ab} \ight. \\
\left. + 2 R_{cd} R^{abcd} - \frac{n-2}{2(n-1)} \nabla^a \nabla^b R + \nabla^2 R^{ab} - \frac{1}{2(n-1)} \gamma^{ab} \nabla^2 R \right] + \ldots \right\}, \]

(2.49)

where \( \Upsilon(x) \) is the step function which is equal to one for \( x \geq 0 \) and zero otherwise.

### 2.2 The Other Divergent Terms and It’s Counterterms

Additional divergent terms may be created due to a matter field on manifold \( M \), and Weyl anomalies. First, we will discuss briefly the Weyl anomaly in the dual conformal field theory and then reproduce the logarithmic divergence for the matter field which is related to anomalies in the dual conformal field theories and the Weyl anomalies.

#### 2.2.1 Weyl Anomalies

As mentioned before the breaking of AdS symmetry and consequently breaking of conformal invariance is a signature of anomalies in the algebra of quantum conformal operators. The conformally invariance is a consequence of the action invariability under the Weyl rescalings.

Previous section dealt with the counterterms which were required to regularize the gravity action. These were calculated for an arbitrary boundary metric without any need to linearize around a given background. The calculation of the Weyl anomalies can be found in Refs. [14] and [51]. Let \( W \) denote
the effective action of a quantum field theory, defined by

\[ W = - \ln \int \mathcal{D} \phi e^{-S[\phi]}. \]  

(2.50)

Defining an infinitesimal Weyl rescaling by \( \delta g_{ij}(x) = 2\lambda(x)g_{ij}(x) \), the Weyl anomaly is given by

\[ \langle T^i_i(x) \rangle = A[g_{ij}(x)] = \frac{1}{\sqrt{g(x)}} \frac{\delta W}{\delta \lambda(x)}. \]  

(2.51)

**Scale Invariance and Its Breaking by Non-Covariant Counterterms:**

It is natural to extend the AdS/CFT correspondence on a pure AdS spacetime to the arbitrary Einstein spaces \( \Omega \) with negative cosmological constants [43]. Such a space \( \Omega \) possesses a horizon manifold \( \partial \Omega \), on which it determines a conformal structure. For simplicity, \( \partial \Omega \) is assumed to have the topology of a \( n \)-sphere in the sequel. Then, in generalization of Eq. (A.5), there is a set of coordinates on \( \Omega \) for which the metric takes the form [52]

\[ ds^2 = l^2 x_0^2 \left[ (dx_0)^2 + \hat{g}_{ij}(x, x_0)dx^i dx^j \right], \]  

(2.52)

where \( \hat{g}_{ij}(x, 0) = \hat{g}_{ij}(x) \) is the horizon metric. For the following consideration it is useful to use the dimensionless variable \( \rho = x_0^2/l^2 \) and write the metric as

\[ ds^2 = \frac{l^2}{4\rho^2} (d\rho)^2 + \frac{1}{\rho} \hat{g}_{ij}(x, \rho)dx^i dx^j. \]  

(2.53)

Besides any coordinate symmetries, the metric (2.53) is invariant under

\[ \rho \to \sigma \rho \quad \text{and} \quad \hat{g}_{ij} \to \sigma \hat{g}_{ij}, \]  

(2.54)

which constitutes a global rescaling of the horizon \( \partial \Omega \).

Obviously, the gravity action on the manifold \( \Omega \) is invariant under the rescaling (2.54). Hence, the AdS/CFT correspondence implies the scale invariance of the CFT effective action, if it were not for non-covariant divergent terms, which have to be cancelled by non-covariant counterterms. Such divergent terms have the form

\[ S_{\text{div}} = \ln \epsilon \int_{\partial \Omega} d^n x \sqrt{\hat{g}} \mathcal{L}_c, \]  

(2.55)
which is known as ‘logarithmic divergence’. Here, \( \int_{\partial \Omega} d^n x \sqrt{g} \mathcal{L}_c \) is itself scale
invariant, and the cut-off boundary is characterized by \( \rho = \epsilon \). Hence, for the
scale transformation (2.54) with \( \sigma = 1 + 2 \lambda \) one obtains

\[
\delta S_{\text{div}} = 2 \lambda \int_{\partial \Omega} d^n x \sqrt{\hat{g}} \mathcal{L}_c,
\]

(2.56)

which, because of \( S = S_{\text{fin}} + S_{\text{div}} \) and \( \delta S = 0 \), implies

\[
\frac{\partial S_{\text{fin}}}{\partial \lambda} = -2 \int_{\partial \Omega} d^n x \sqrt{\hat{g}} \mathcal{L}_c.
\]

(2.57)

Equation (2.57) shows that the scale invariance of the CFT effective action is
broken due to the necessity to renormalized with a non-covariant counterterm.
The right hand side of equation (2.57) is the integrated Weyl anomaly. Within
the AdS/CFT correspondence, \( S_{\text{fin}} \) is identified with the effective action \( W \)
of the boundary CFT. Hence, comparing Eqs. (2.57) and (2.51) yields the
anomaly

\[
\mathcal{A} = -2 \mathcal{L}_c + D_i J^i,
\]

(2.58)

where \( J^i \) is continuous, but otherwise arbitrary.

### 2.2.2 The Logarithmic Divergencies

The action of gravity in the presence of an electromagnetic field can be written
as

\[
S_{\text{bulk}} = -\frac{1}{16 \pi} \int_M d^{n+1} x \sqrt{-g} \left( R + \frac{n(n - 1)}{\ell^2} - \frac{1}{4} F^2 \right).
\]

(2.59)

In the neighborhood of a boundary \( \partial M \), we will assume that the metric can
be expressed in the form

\[
ds^2 = \frac{\ell^2}{x^2} dx^2 + \frac{1}{x^2} \tilde{\gamma}_{ij} dx^i dx^j,
\]

(2.60)

with the induced hypersurface metric \( \gamma_{ij} = \tilde{\gamma}_{ij}/\epsilon^2 \) where \( \epsilon \ll 1 \). The nondegenerate metric \( \tilde{\gamma}_{ij} \) admitting the expansion

\[
\tilde{\gamma}_{ij} = \gamma_{ij}^0 + x^2 \gamma_{ij}^2 + x^4 \gamma_{ij}^4 + x^6 \gamma_{ij}^6 \ldots.
\]

(2.61)
Here $\tilde{\gamma}^0_{ij}$ is the leading order of the metric $\tilde{\gamma}_{ij}$. If $M$ is an Einstein manifold with negative cosmological constant, then according to [52, 53] such an expansion always exists. For solutions of gauged supergravity theories with matter fields, demanding that this expansion is well-defined as $x \to 0$, will impose conditions on the matter fields induced on $\partial M$. The equations of motion derived from the action (2.59) are [54]

$$R_{mn} = -\frac{n}{\ell^2}g_{mn} + \frac{1}{2}F_{mq_1...q_{p-1}}F^n_{q_1...q_{p-1}} - \frac{1}{4(n-1)}F^2g_{mn};$$

$$R = -\frac{n(n+1)}{\ell^2} + \frac{(n-3)}{4(n-1)}F^2.$$  \hspace{1cm} (2.62)

Using Eq. (2.62), it is easy to show that the on-shell bulk action can be written as:

$$S_{\text{bulk}} = \frac{1}{8\pi} \int_M d^{n+1}x \sqrt{-g} \left( \frac{n}{\ell^2} + \frac{1}{4(n-1)}F^2 \right). \hspace{1cm} (2.63)$$

Let us assume that in the vicinity of the conformal boundary the field can be expanded as a power series in $x$ as

$$F_{\mu\nu} = F^0_{\mu\nu} + xdx_{[\mu}A^1_{\nu]} + x^2F^2_{\mu\nu} + x^2dx_{[\mu}A^2_{\nu]} + x^3F^3_{\mu\nu}...,$$  \hspace{1cm} (2.64)

where $G^i_\mu$ is a tensor dependent only on $x^i$ and $F^0_{\mu\nu}$ is the leading order of $F_{\mu\nu}$ or, in other word, is the induced field on the boundary. In order to write down the explicit form of the action, we have to behave as follow [54]:

1. **Step 1.** Expand out the $\sqrt{-g}$ and $F^2$ as leading order of $\tilde{\gamma}^0_{ij}$ and $F^0_{\mu\nu}$.

2. **Step 2.** Classify the expansion of $\sqrt{-g}$ in two part, one without field ($F = 0$) depending only on $\tilde{\gamma}^0_{ij}$, $\sqrt{-g^{(1)}}$, and the other, $\sqrt{-g^{(2)}}$, which depend only on $F^0$. By substituting the explicit form for the metric and the field strength as functions of $\tilde{\gamma}^0$ and $F^0_{\mu\nu}$ in Eq. (2.63), we find that one encounters with two kinds of divergent terms. The first type of these divergent terms depend only on $\tilde{\gamma}^0$ and its curvature invariants, given as:

$$S^{(1)} = -\frac{(n-1)}{8\pi\ell e^n} \int d^n x \sqrt{-\tilde{\gamma}^0} - \frac{(n-4)(n-1)\ell}{16\pi(n^2-4)e^{n-2}} \int d^n x \sqrt{-\tilde{\gamma}^0} R^0 + ...$$  \hspace{1cm} (2.65)
which can be removed (at least for \( n \) odd) by the counterterms given in Sec.(2.1.4). The second type of divergent terms in the bulk action is

\[
S^{(2)}_{\text{bulk}} = \frac{\ell}{8\pi} \int \frac{d^{n+1}x}{x^{n+1}} \sqrt{-\tilde{\gamma}} \{ x^4 \frac{(n+8)}{32(n-1)} (F_{\mu\nu}^0)^2 + \ldots \},
\]

(2.66)

and in the surface action is

\[
S_{\text{surf}} = -\frac{1}{8\pi} \int d^n x \sqrt{-\tilde{\gamma}} \{ \frac{(n-4)}{32(n-1)} \ell \epsilon^{n-4} (F_{\mu\nu}^0)^2 + \ldots \}.
\]

(2.67)

Hear \( R^0, R^0_{ij} \) and \( R^0_{ijkl} \) are the Ricci scalar, Ricci tensor and Riemann tensor respectively of the metric \( \tilde{\gamma}_{ij} \). There are no divergences for \( n < 4 \). In \( n = 4 \) there is a logarithmic divergence due to the Weyl anomaly term [14]

\[
S_{\log} = -\frac{\ell^3}{64\pi} \ln \epsilon \int d^4 x \sqrt{-\tilde{\gamma}} \left[ (R^0_{ij})^2 - \frac{1}{3} (R^0)^2 \right],
\]

(2.68)

and an additional logarithmic divergence in the action given by

\[
S_{\log}^{\text{em}} = \frac{\ell}{64\pi} \ln \epsilon \int d^4 x \sqrt{-\tilde{\gamma}} (F_{\mu\nu}^0)^2.
\]

(2.69)

In \( n > 4 \) the \( F_{\mu\nu}^0 \) will cause a power law divergence in the action [54]

\[
S_{\text{div}} = -\frac{\ell}{256\pi \epsilon^{n-4}} \frac{(n-8)}{(n-4)} \int d^n x \sqrt{-\tilde{\gamma}} (F_{\mu\nu}^0)^2,
\]

(2.70)

which can be removed by a counterterm of the form

\[
S_{\text{ct}} = \frac{\ell}{256\pi} \int d^n x \sqrt{-\gamma} \frac{(n-8)}{(n-4)} (F_{\mu\nu})^2.
\]

(2.71)

In \( n = 6 \) as well as the logarithmic divergence associated with the Weyl anomaly of the dual theory, which is given by [54]

\[
S_{\log} = \frac{\ell^3}{8\pi} \ln \epsilon \int d^6 x \sqrt{-\tilde{\gamma}} \left\{ \frac{3}{50} (R^0)^3 + R^{(0)ij} R^{(0)kl} R^{(0)}_{ijkl} - \frac{1}{2} R^0 R^{(0)ij} R^{(0)}_{ij} \\
+ \frac{1}{5} R^{(0)ij} D^0_i D^0_j R^0 - \frac{1}{2} R^{(0)ij} \Box^0 R^0_{ij} + \frac{1}{20} R^0 \Box^0 R^0 \right\},
\]

(2.72)

as was found in [14], there is an anomaly of the form

\[
S_{\log}^{\text{em}} = \frac{\ell^3}{8\pi} \ln \epsilon \int d^6 x \sqrt{-\tilde{\gamma}} \left\{ \frac{1}{16} R^0 (F_{2}^0)^2 - \frac{1}{8} R^{(0)ij} (F_{2}^0)^i j (F_{2}^0)^{ij} \\
+ \frac{1}{64} (F_{2}^0)^{ij} \left[ D^0_{ij} D^{(0)jk} F^0_{k}\ - D^0_{i} D^{(0)kj} F^0_{kj} \right] \right\},
\]

(2.73)

due to field anomaly.
2.3 Conserved Charges and the AdS/CFT Correspondence

One of the important applications of AdS/CFT correspondence is the computation of conserved charges for the spacetimes. In this section, we give the formalism of calculating the conserved charges of a spacetime through the use of Brown and York formalism and AdS/CFT correspondence.

The start point for this subject is to write the spacetime metric as the usual ADM decomposition \[55\]:

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt), \quad (2.74) \]

where \( N \) is the laps function and \( V^i \) is the shift vector (see Appendix B).

Throughout this analysis, it is assumed that the hypersurface foliation \( \Sigma \) is orthogonal to \( 3B \), meaning that on the boundary \( 3B \) the hyper-surface normal \( u^\mu \) and the three-boundary normal \( n^\mu \) satisfy \((u.n)|_{3B} = 0 \) (see Appendix B).

In the canonical formalism, the boundary \( B \) is specified as a fixed surface in \( \Sigma \). The Hamiltonian must evolve the system in a manner consistent with the presence of this boundary, and cannot generate transformations that map the canonical variables across \( B \). This means that the component of the shift vector normal to the boundary must be restricted to vanish, \( V^n_i |_B = 0 \). From a spacetime point of view, this is the condition that the two-boundary evolves into a three-surface that contains the unit normal \( u^\mu \) to the hypersurfaces \( \Sigma \). Therefore, \( u^\mu \) and \( n^\mu \) are orthogonal on \( 3B \). The metric on \( 3B \) can be decomposed as

\[ \gamma_{ij}dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt), \quad (2.75) \]

where \( x^a \ (a = 1, 2) \) and \( \sigma_{ab} \) are coordinates and induced metric on \( B \) respectively. The extrinsic curvature of \( B \) as a surface embedded in \( \Sigma \) is denoted by \( k_{ab} \). These tensors can be viewed as spacetime tensors \( \sigma_{\mu\nu} \) and \( k_{\mu\nu} \), or as tensors on \( \Sigma \) or \( 3B \) by using indices \( i, j, k, l \). Also, \( \sigma^\nu_\nu \) is the projection tensor onto \( B \).
Now we define proper energy surface density $\varepsilon$ which is the projection of stress tensor normal to the space like surface $B$, while proper momentum density $j_a$ and spatial stress $s^{ab}$ are the normal–tangential and tangential–tangential projections of the stress tensor respectively. On base of these definitions the variation of gravitational action on the three-boundary $^3B$ can be written as

$$\delta S_{\text{cl}} = \int d^3x \sqrt{\sigma} (-\varepsilon \delta N + j_a \delta V^a + \frac{1}{2} N s^{ab} \delta \sigma_{ab}) \delta \gamma_{ij}. \quad (2.76)$$

Then on the two-surface $B$, we have

$$\varepsilon \equiv -\frac{1}{\sqrt{\sigma}} \frac{\delta S_{\text{cl}}}{\delta N} = u_i u_j T^{ij}, \quad (2.77)$$

$$j_a \equiv \frac{1}{\sqrt{\sigma}} \frac{\delta S_{\text{cl}}}{\delta V^a} = -\sigma_{ai} u_j T^{ij}, \quad (2.78)$$

$$s^{ab} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{cl}}}{\delta \sigma_{ab}} = \sigma^a_i \sigma^a_j T^{ij}, \quad (2.79)$$

where the second equality in Eqs. (2.77)-(2.79) follow from definition (2.5) for $T^{ij}$ and the relationships

$$\frac{\partial \gamma_{ij}}{\partial N} = -2 \frac{u_i u_j}{N}, \quad (2.80)$$

$$\frac{\partial \gamma_{ij}}{\partial V^a} = -2 \frac{\sigma a_{(i} u_{j)}}{N}, \quad (2.81)$$

$$\frac{\partial \gamma_{ij}}{\partial \sigma_{ab}} = \sigma^a_i \sigma^a_j. \quad (2.82)$$

The total quasilocal energy of the system is

$$E = \int_B d^2x \sqrt{\sigma} \varepsilon, \quad (2.83)$$

which can be meaningfully associated with the thermodynamic energy of the system [56].

When there is a Killing vector field $\xi$ on the boundary $^3B$, an associated conserved charge is defined by [8]

$$Q(\xi) = -\int_B d^2x \sqrt{\sigma} (u_i T^{ij} \xi_j). \quad (2.84)$$
From Eqs. (2.77)-(2.79) one can see that $-u_i T^{ij} = (\varepsilon u^i + j^i)$. Hence we have

$$Q(\xi) = -\int_B d^2x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i. \quad (2.85)$$

For boundaries with time like Killing vector ($\xi = \partial/\partial t$) and rotational Killing vector fields ($\varsigma = \partial/\partial \phi$) we obtain

$$M = \int_B d^2x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i, \quad (2.86)$$

$$J = \int_B d^2x \sqrt{\sigma} j^i \varsigma_i, \quad (2.87)$$

provided the surface $B$ contains the orbits of $\varsigma$. These quantities are respectively the conserved mass and angular momentum of the system enclosed by the boundary.

By computing $\varepsilon$ and $j_a$, one can find the conserved mass and angular momentum. But for infinite spacetimes, the surface stress energy-momentum tensor diverges. To solve this problem, one may use the AdS/CFT correspondence conjecture and obtain the finite surface stress energy-momentum tensor as obtained in the Eq. (2.49). Examples of this kind of computations will be given in chapters 4 and 5.
Chapter 3

Thermodynamics of Black Holes

The defining feature of a black hole is its future event horizon or shortly its event horizon. This one way membrane separates events which are inside the horizon from those that are outside the horizon; the events inside the event horizon are never within the causal past of the ones outside. This feature can be seen in figure (3.1), which depicts the formation of a black hole by the spherical collapse of a star. The conformal transformation that brings points at infinity to a finite distance preserves the light-cone structure so that light rays travelling radially inwards or outwards travel on lines inclined by 45°; these rays are called null. In Fig. (3.1), the points at infinity include timelike and spacelike infinities (i⁺, i⁻ and i⁰), and null infinities (I⁺ and I⁰). Each point represents a sphere. Notice that the event horizon in figure (3.1) is a null surface, so the events within it are never in the past light-cone of events outside of the horizon. However, charged black holes have two inner and outer horizons. In this case, one may encounter with an extreme black hole with a degenerate horizon, if one choose the black hole’s parameters properly. The presence of an event horizon causes some unusual effects on quantum fields existing in the black hole spacetime. By acting as a one-way membrane, the event horizon can trap one of the ‘virtual’ particle-pairs produced by quantum processes. The escaping particle (which is no longer virtual) appears to have
been radiated from the black hole. This process is shown in figure (3.2).

The radiation, known as Hawking radiation, has a thermal spectrum (if one neglects scattering of the gravitational field) with a temperature proportional to the surface gravity of the event horizon. Although this picture is drawn from semi-classical quantum field theory, it should be qualitatively correct since the gravitational field at the event horizon of a black hole need not be very strong.

Another strange property of black holes was realized by Wheeler: one can forever hide information from the outside world by dropping it into a black hole. Indeed, this property seems to pose a problem with the second law of

\[1\text{Recently, S. W. Hawking in '17th International Conference on General Relativity and Gravitation in Dublin' said that: black holes, the mysterious massive vortexes formed from collapsed stars, do not destroy everything they consume but instead eventually fire out matter and energy "in a mangled form." Also he told "There is no baby universe branching off, as I once thought. The information remains firmly in our universe." Then he said "I'm sorry to disappoint science fiction fans, but if information is preserved, there is no possibility of using black holes to travel to other universes," he said. "If you jump into a} \]
thermodynamics since a black hole will quickly return to a very simple state even if an object with a large amount of entropy is dropped into it. Bekenstein [57, 58] speculating that the black hole itself is a thermodynamic system with the area of the event horizon representing the entropy. Since the area always increases when matter of positive energy is added to a black hole, the problem with the second law of thermodynamics can be resolved. The fact that the black hole is also surrounded by quantum fields with a thermal spectrum seems to support Bekenstein’s speculation. Various attempts for understanding the entropy of black holes in terms of the number of quantum states contained within or near the event horizon have been made (two recent reviews are given in references [59, 60]). Therefore, black holes are interesting systems to study, if only theoretically, as their classical and semi-classical properties may hint at the nature of a quantum theory of gravity. Black holes can be treated as a thermodynamic system whose properties must be reproduced in the statistics.

black hole, your mass energy will be returned to our universe, but in a mangled form, which contains the information about what you were like, but in an unrecognizable state.” He added: “information is not lost in the formation and evaporation of black holes. The way the information gets out seems to be that a true event horizon never forms, just an apparent horizon.” By these statements, in spite of the old idea about black holes, the information does not hide in black holes and black holes never fully evaporate. ‘The information is preserved’.
of the quantum fields. However, the thermodynamic properties of black holes must first be understood.

### 3.1 Classical Black Hole Thermodynamics

In this section, using reference [62], we will give a brief review of the null hypersurfaces and Killing horizons. Also the laws of classical black hole mechanics will be described.

#### 3.1.1 The Null Hypersurfaces and Killing Horizons

Let $S(x)$ be a smooth function of the spacetime coordinates $x^\mu$ and consider a family of hypersurfaces $S = \text{const}$. The vector fields normal to the hypersurface are

$$ l = \tilde{f}(x) \frac{\partial}{\partial x^\mu}, $$

where $\tilde{f}$ is an arbitrary non-zero function. If $l^2 = 0$ for a particular hypersurface $\mathcal{N}$ in the family, then $\mathcal{N}$ is said to be a null hypersurface. For example the surface $r = 2M$ is a null hypersurface for Schwarzschild spacetime. In the null hypersurfaces, $l$ is itself a tangent vector.

**Definition 1** A null hypersurface $\mathcal{N}$ is a Killing horizon of a Killing vector field $\xi$ if, on $\mathcal{N}$, $\xi$ is normal to $\mathcal{N}$.

Let $l$ be normal to $\mathcal{N}$ such that $l \cdot Dl^\mu = 0$ (affine parameterization). Then, since, on $\mathcal{N}$,

$$ \xi = fl $$

for some function $f$, it follows that

$$ \xi \cdot D\xi^\mu = \xi \cdot D(fl^\mu) = \frac{\xi \cdot Df}{f} fl^\mu = \kappa \xi^\mu, \quad \text{on } \mathcal{N} $$

33
where $\kappa = \xi \cdot \partial \ln |f|$ is called the surface gravity. In this way, the surface gravity can be obtained from

$$\kappa^2 = -\frac{1}{2} (D^\mu \xi^\nu) (D_\mu \xi_\nu) |_N.$$  \hfill (3.4)

**Proposition 1** $\kappa$ is constant on orbits of $\xi$.

**Proof.** Let $t$ be tangent to $N$. Then, since (3.4) is valid everywhere on $N$

$$t \cdot \partial \kappa^2 = -(D^\mu \xi^\nu) t^\rho D_\rho D_\mu \xi_\nu |_N$$  \hfill (3.5)

but for Killing vector $\xi^\nu$ we have

$$D_\rho D_\mu \xi^\nu = R^\rho_{\mu \rho \sigma} \xi^\sigma$$  \hfill (3.6)

where $R^\rho_{\mu \rho \sigma}$ is the Riemann tensor. Hence Eq. (3.5) can be written as

$$t \cdot \partial \kappa^2 = -(D^\mu \xi^\nu) t^\rho R^\rho_{\nu \mu \rho \sigma} \xi_\sigma.$$  \hfill (3.7)

Now, $\xi$ is tangent to $N$ (in addition to being normal to it). Choosing $t = \xi$ we have

$$\xi \cdot \partial \kappa^2 = -(D^\mu \xi^\nu) R^\rho_{\nu \mu \rho \sigma} \xi^\rho \xi^\sigma$$

$$= 0 \quad \text{(since } R^\rho_{\nu \mu \rho \sigma} = -R^\rho_{\nu \rho \sigma \mu})$$  \hfill (3.8)

so $\kappa$ is constant on orbits of $\xi$. ■

For surface gravity, also there is an interpretation as follow:

Surface gravity of the event horizon can be thought as the force required to hold a unit test mass on the event horizon in place by an observer who is far from the black hole.

A bifurcate Killing horizon is a pair of null surfaces, $N_A$ and $N_B$, which intersect on a spacelike 2-surface, $C$ (called the ‘bifurcation surface’), such that $N_A$ and $N_B$ are each Killing horizons with respect to the same Killing
field $\xi^a$. It follows that $\xi^a$ must vanish on $C$; conversely, if a Killing field, $\xi^a$, vanishes on a two-dimensional spacelike surface, $C$, then $C$ will be the bifurcation surface of a bifurcate Killing horizon associated with $\xi^a$ (see [47] for further discussion).

**Proposition 2** If $N$ is a bifurcate Killing horizon of $\xi$, with bifurcation 2-surface, $C$, then $\kappa^2$ is constant on $N$.

**Proof.** $\kappa^2$ is constant on each orbit of $\xi$. The value of this constant is the value of $\kappa^2$ at the limit point of the orbit on $C$, so $\kappa^2$ is constant on $N$ if it is constant on $C$. But we saw previously that

$$t \cdot \partial \kappa^2 = - (D^\mu \xi^\nu) t^\rho R_{\nu\mu\rho} \xi^\sigma \big|_N$$

$$= 0 \quad \text{on} \quad C \quad (\text{since} \quad \xi_\sigma \big|_C = 0). \quad (3.9)$$

Since $t$ can be any tangent to $C$, $\kappa^2$ is constant on $C$, and hence on $N$. ■

### 3.1.2 Conserved Charges

Let $V$ be a volume of spacetime on a spacelike hypersurface $\Sigma$, with boundary $B$. To every Killing vector field $\xi$, we can associate the Komar integral [61]

$$Q_\xi(V) = \frac{c}{16\pi} \int_B dS_{\mu\nu} D^\mu \xi^\nu \quad (3.10)$$

for some constant $c$. Using Gauss’ law

$$Q_\xi(V) = \frac{c}{8\pi} \int_V dS_{\mu\nu} D\mu D^\nu \xi^\nu, \quad (3.11)$$

and the fact that

$$D\nu D\mu \xi^\nu = R_{\mu\nu} \xi^\nu, \quad (3.12)$$

we obtain

$$Q_\xi(V) = \frac{c}{8\pi} \int_V dS_{\mu} R^\mu_{\nu} \xi^\nu$$

$$= \frac{c}{8\pi} \int_V dS_{\mu} (T^\mu_{\nu} \xi^\nu - \frac{1}{2} T \xi^\nu) \quad \text{(by Einstein’s equation)}$$

$$= \frac{1}{8\pi} \int_V dS_{\mu} J^\mu(\xi), \quad (3.13)$$
where $J^\mu(\xi)$ defined as:

$$J^\mu(\xi) = c(T^\mu_\nu \xi^\nu - \frac{1}{2} T \xi^\mu).$$  

(3.14)

One can verify that the current $J^\mu(\xi)$ is conserved. To prove this, using the fact that $D_\mu T^{\mu\nu} = 0$. We obtain

$$D_\mu J^\mu(\xi) = c \left( T^{\mu\nu} D_\mu \xi^\nu - \frac{1}{2} T D_\mu \xi^\mu \right) - \frac{c}{2} \xi \cdot \partial T.$$  

(3.15)

Since for Killing vector $D_\mu \xi^\nu = D_\mu \xi^\mu = 0$, then

$$D_\mu J^\mu(\xi) = \frac{c}{2} \xi \cdot \partial T.$$  

(3.16)

Now using Einstein’s equation, one can show that $D_\mu J^\mu(\xi) \propto \xi^\mu \partial_\mu R$ which is zero.

Since $J^\mu(\xi)$ is a ‘conserved current’, the charge $Q_\xi(V)$ is time-independent provided $J^\mu(\xi)$ vanishes on $B$.

If one chooses time-translation Killing vector field $\xi = k$, then one can obtain quasi-local energy in the volume $V$ by

$$E(V) = -\frac{1}{8\pi} \oint_B dS_{\mu\nu} D_\mu k^\nu.$$  

(3.17)

For Schwarzschild spacetime one can obtain $E(V) = M$ for any $V$ with $B$ outside the horizon.

In axisymmetric spacetimes, by choosing $\xi = m = \partial/\partial \phi$ and $c = 1$, we can calculate the angular momentum as

$$J(V) = \frac{1}{16\pi} \oint_B dS D^\mu m_\nu.$$  

(3.18)

It is worthwhile to mention that the Komar integral formalism is valid only for asymptotically flat spacetimes. For asymptotically AdS or dS spacetimes, one should use the formalism which was given in chapter 2.

### 3.1.3 The Laws of Black Hole Mechanics

Previously, we showed that $\kappa^2$ is constant on a bifurcate Killing horizon. The proof fails if we have only part of a Killing horizon, without the bifurcation...
2-sphere, as happens in gravitational collapse. However in this case, if one accepts some conditions which will be discussed, then we have the following laws.

**Zeroth Law:**

The surface gravity $\kappa$ is constant on the event horizon $H$ if $T_{\mu\nu}$ obeys the dominant energy condition. This resembles the zeroth law of thermodynamics, which says that the temperature is constant in thermodynamic equilibrium.

**Smarr’s Formula:**

Let $\Sigma$ be a spacelike hypersurface in a stationary exterior black hole spacetime with an inner boundary, $H$, on the event horizon and another boundary at $i_0$ (see figure(3.1)). The surface $H$ is a 2-sphere that can be considered as the ‘boundary’ of the black hole.

Applying Gauss’ law to the Komar integral for $J$ we have

$$J = \frac{1}{8\pi} \int_\Sigma ds_\mu D_\nu D^\mu m^\nu + \frac{1}{16\pi} \oint_H ds_\mu D^\mu m^\nu$$

$$= \frac{1}{8\pi} \int_\Sigma ds_\mu R^\mu_\nu m^\nu + J_H.$$  \hspace{1cm} (3.19)

Using Einstein equation, one obtains:

$$J = \int_\Sigma ds_\mu (T^\mu_\nu m^\nu - \frac{1}{2} T m^\mu) + J_H.$$  \hspace{1cm} (3.20)

In the absence of matter other than electromagnetic field, we have

$$T_{\mu\nu} = T_{\mu\nu}(F) = \frac{1}{4\pi} (F_{\mu\lambda} F^{\lambda}_\nu - \frac{1}{4} F^{\rho\lambda} F_{\rho\lambda} g_{\mu\nu}),$$  \hspace{1cm} (3.21)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (3.22)

Since $g^{\mu\nu} T_{\mu\nu}(F) = T(F) = 0$, we have

$$J = \int_\Sigma ds_\mu T^\mu_\nu(F) m^\nu + J_H.$$  \hspace{1cm} (3.23)
for an isolated black hole (i.e. $T_{\mu\nu} = T_{\mu\nu}(F)$).

Now apply Gauss’ law to the Komar integral for the total energy (= mass).

$$M = -\frac{1}{4\pi} \int_\Sigma ds_\mu R^\mu_\nu k^\nu - \frac{1}{8\pi} \oint_H ds_\mu D^\mu k^\nu. \quad (3.24)$$

By inserting

$$\xi = k + \Omega_H m, \quad (3.25)$$

we have

$$M = \int_\Sigma ds_\mu (-2T^\mu_\nu k^\nu + Tk^\mu) - \frac{1}{8\pi} \oint_H ds_\mu (D^\mu \xi^\nu - \Omega_H D^\mu m^\nu) \quad (3.26)$$

since $\Omega_H$ is constant on $H$, for $T_{\mu\nu} = T_{\mu\nu}(F)$ we have

$$M = -2 \int_\Sigma ds_\mu T^\mu_\nu(F) k^\nu + 2\Omega_H J_H - \frac{1}{8\pi} \oint_H ds_\mu D^\mu \xi^\nu. \quad (3.27)$$

Using (3.23) for an isolated black hole,

$$M = -2 \int_\Sigma ds_\mu T^\mu_\nu(F) \xi^\nu + 2\Omega_H J - \frac{1}{8\pi} \oint_H ds_\mu D^\mu \xi^\nu. \quad (3.28)$$

The first term can be written as:

$$-2 \int_\Sigma ds_\mu T^\mu_\nu(F) \xi^\nu = \Phi_H Q, \quad (3.29)$$

and for the third term we have [62]

$$-\frac{1}{8\pi} \oint_H ds_\mu D^\mu \xi^\nu = \frac{\kappa A}{4\pi}, \quad (3.30)$$

where $A$ is the ‘area of the horizon’. Also $\Phi_H$ and $Q$ are the co-rotating electric potential on the horizon and electrical charge, which are defined by

$$\Phi = \xi^\nu A_\nu, \quad (3.31)$$

$$Q = \int_\Sigma ds_\mu D_\nu F^\mu_\nu = \oint_{\partial\Sigma} F^{\mu_1} ds_i. \quad (3.32)$$

Hence

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q. \quad (3.33)$$

This is Smarr’s formula for the mass of charged rotating black holes.
**First Law:**

If a stationary black hole of mass $M$, charge $Q$ and angular momentum $J$, with event horizon of area $A$, surface gravity $\kappa$, electric surface potential $\Phi_H$ and angular velocity $\Omega_H$, is perturbed such that it settles down to another black hole with mass $M + \delta M$ charge $Q + \delta Q$ and angular momentum $J + \delta J$, then the conservation of energy requires that

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J + \Phi_H \delta Q.$$ (3.34)

This has the same form as the first law of thermodynamics, and since $\kappa$ is the analog of temperature, the area plays the role of entropy. This statement of the first law uses the fact that the event horizon of a stationary black hole must be a Killing horizon.

To prove Eq. (3.34), for $Q = 0$, we use the Smarr’s formula for mass, Eq. (3.33), then by Euler’s theorem for homogeneous function $M(A, J)$, we have

$$A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J} = \frac{1}{2} M = \frac{\kappa}{8\pi} A + \Omega_H J.$$ (3.35)

At the first step of this calculation, we use the fact that $A$ and $J$ both have dimension of $M^2$ so the function $M(A, J)$ must be homogeneous of degree 1/2, and at the second step we use Eq. (3.33) Therefore

$$A \left( \frac{\partial M}{\partial A} - \frac{\kappa}{8\pi} \right) + J \left( \frac{\partial M}{\partial J} - \Omega_H \right) = 0.$$ (3.36)

But $A$ and $J$ are free parameters so,

$$\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi}, \quad \frac{\partial M}{\partial J} = \Omega_H.$$ (3.37)

In the case $Q \neq 0$, we can generalize this equation and write

$$\frac{\partial M}{\partial Q} = \Phi_H.$$ (3.38)
The Second Law (Hawking’s Area Theorem):

The analogy of area and entropy is confirmed by the second law of black hole mechanics [63]. This is a statement about non stationary processes in a spacetime containing black hole, including collisions and fusions of black holes. Two assumptions can be made:

1. The time evolution of the system must be under sufficient control. This is implemented by requiring that the spacetime is strongly asymptotically predictable (cosmic censorship hypothesis). In physical formulation this explains that the complete gravitational collapse of a body always results in a black hole rather than a naked singularity; i.e., all singularities of gravitational collapse are ‘hidden’ within horizon, and can not be ‘seen’ by distant observers.

2. The matter, represented by the stress energy tensor should behave ‘reasonable’. This is done by imposing the ‘weak energy condition’ on the stress energy tensor. For more detailed about energy conditions, we refer to [47].

Under these assumptions the second law states that the total area of all event horizons is non-decreasing,

\[ \delta A \geq 0. \]  

(3.39)

This is striking analogue of the entropy law of thermodynamics. For more detail we refer to [62].

The Third Law:

Here several versions exist, and the status of this law does not seem to be fully understood. We only touch upon this and refer to [47] for a more detailed account. One version of the law states that the extreme limit cannot be reached in finite time in any physical process. Here the obvious problem is to define
what a physical process is and to bring such non-stationary processes under sufficient control. Another version, which does not refer to non-stationary properties, states that black holes of vanishing temperature (surface gravity) have vanishing entropy. This is in obvious contradiction to the fact that the area of an extreme black hole can be non-vanishing. There are however subtleties at the quantum level, and these have been used as arguments in favor of the second version of the third law. We will return to this when discussing quantum aspects of black holes.

3.2 Quantum Aspects of Black Holes and Black Hole Thermodynamics

The laws of black hole mechanics have been known for quite some time, but were mostly considered as a curious formal analogy. The most obvious reason for not believing in a thermodynamic content is that a classical black hole is just black: It cannot radiate and therefore one should assign temperature zero to it, so that the interpretation of the surface gravity as temperature has no physical content.

This changes dramatically when taking into account quantum effects. One can analyze black holes in the context of quantum field theory in curved backgrounds, where matter is described by quantum field theory while gravity enters as a classical background, see for example [64]. In this framework it was discovered that black holes can emit Hawking radiation [7]. The spectrum is (almost) Planckian with a temperature, the so-called Hawking temperature, which is indeed proportional to the surface gravity,

\[ T_H = \frac{\kappa}{2\pi}. \]  

(3.40)

This motivates to take the analogy of area and entropy seriously. Since the Hawking temperature fixes the factor of proportionality between temperature
and surface gravity, one finds the Bekenstein-Hawking area law,

\[ S = \frac{A}{4}. \]  

(3.41)

Before the discovery of Hawking radiation, Bekenstein had already given an independent argument in favour of assigning entropy on black holes [5, 65]. He pointed out that in a spacetime containing a black hole one could adiabatically transport matter into it. This reduces the entropy in the observable world and thus violates the second law of thermodynamics. He, therefore, proposed to assign entropy to black holes, such that a generalized second law is valid, which states that the sum of thermodynamic entropy and black hole entropy is non-decreasing. With the discovery of Hawking radiation one can give an additional argument in favour of this generalization: By Hawking radiation a black hole looses mass and shrinks. This is not in contradiction with the second law of black hole mechanics, because one can show that the weak energy condition is violated in the near horizon region if the effect of quantum fields is taken into account. Bekenstein’s generalized second law claims that the loss in black hole entropy is always (at least) compensated by the thermodynamic entropy of the Hawking radiation, so that the total entropy is non-decreasing.

One example of unusual thermodynamic behavior of black holes is provided by the mass dependence of the temperature of uncharged black holes. For the Schwarzschild black hole one finds \( \kappa = (4M)^{-1} \), which shows that the specific heat is negative: The black hole heats up while loosing mass. This behavior is unusual, but nevertheless not unexpected because gravity is a purely attractive force. The fact that uncharged black holes seem to fully decay into Hawking radiation leads to the information or unitarity problem of quantum gravity, see for example [66]. Charged black holes behave differently, since the Hawking temperature vanishes in the extreme limit, and therefore extreme black holes are stable against decay by thermic radiation.

We already mentioned that one version of the third law states that extreme black holes have vanishing entropy. This statement depends on subtleties of the quantum mechanical treatment of such objects [67, 68]: The entropy can be
computed in semiclassical quantum gravity, i.e. by quantizing gravity around a black hole configuration. One can use either the Euclidean path integral formulation or a Minkowskian canonical framework. The result for the entropy depends on whether the extreme limit is taken before or after quantization: If one quantizes around extreme black holes the entropy vanishes. But if one quantizes around general charged black hole configurations, one finds an entropy that is non-vanishing when taking the extreme limit. The second option seems to be more natural and it is the one supported by string theory.

The identification of the area with entropy leads to several questions. Standard thermodynamics provides a macroscopic effective description of systems in terms of macroscopic observables like temperature and entropy. At the fundamental microscopic level, systems are described by statistical mechanics in terms of microstates which encode, say, the positions and momenta of all particles that constitute the system. At this level one can define the microscopic or statistical entropy as the quantity which characterizes the degeneracy of microstates in a given macrostate, where the macrostate is characterized by specifying the macroscopic observables. Assuming ergodic behavior the macroscopic and microscopic entropy agree. One should therefore address the question whether there exists a fundamental, microscopic level of description of black holes, where one can identify microstates and count how many of them lead to the same macrostate. The macrostate of a black hole is characterized by its mass, charge and angular momentum. Denoting the number of microstates leading to the same mass \( M \), charge \( Q \) and angular momentum \( J \) by \( N(M, Q, J) \), the statistical or microscopic black hole entropy is defined by

\[
S_{\text{micro}} = \log N(M, Q, J). \tag{3.42}
\]

If the Bekenstein-Hawking entropy is the analogue of thermodynamic entropy and if stationary black holes are the analogue of thermodynamic equilibrium states, then the Bekenstein-Hawking entropy must coincide with the micro-
scopic entropy,
\[ S = S_{\text{micro}}. \]  

One of the astonishing properties of the Bekenstein-Hawking entropy is its simple and universal behavior: the entropy is just proportional to the area. The fact that the entropy is proportional to the area and not to the volume has led to the speculation that quantum gravity is in some sense non-local and admits a holographic representation on boundaries of spacetime [68, 69].

### 3.2.1 The Hawking Temperature

As we mentioned in the previous sections, we can assign temperature to a black hole. In order to calculate this temperature we start with the example of the Schwarzschild metric.

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (3.44)

This represents the gravitational field that a black hole would settle down to, if it were non rotating. In the usual \( r \) and \( t \) coordinates there is an apparent singularity at the Schwarzschild radius \( r = 2M \). However, this is just caused by a bad choice of coordinates. One can choose other coordinates in which the metric is regular there. If one puts \( t = i\tau \) one gets a positive definite metric. We shall refer to such positive definite metrics as Euclidean even though they may be curved. In the Euclidean-Schwarzschild metric there is again an apparent singularity at \( r = 2M \). However, one can define a new radial coordinate \( x \) to be \( 4M(1 - 2Mr^{-1})^{1/2} \). Then we have

\[ ds^2 = x^2\left(\frac{d\tau}{4M}\right)^2 + \left(\frac{r^2}{4M^2}\right)^2dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (3.45)

where \( \kappa = (4M)^{-1} \). The metric in the \( x - \tau \) plane then becomes like the origin of polar coordinates if one identifies the coordinate with period \( 8\pi M \). Similarly other Euclidean black hole metrics will have apparent singularities on their horizons which can be removed by identifying the imaginary time coordinate with period \( \beta = 2\pi/\kappa \).
What is the significance of having imaginary time identified with some period $\beta$? To see this, consider the amplitude to go from some field configuration $\phi_1$ on the surface $t_1$ to a configuration $\phi_2$ on the surface $t_2$. This will be given by the matrix element of $\exp[iH(t_2 - t_1)]$. However, one can also represent this amplitude as a path integral over all fields between $t_1$ and $t_2$ which agree with the given fields $\phi_1$ and $\phi_2$ on the two surfaces as Fig.(3.4).

\[
\langle \phi_2, t_2 \mid \phi_1, t_1 \rangle = \langle \phi_2 \mid \exp(-iH(t_2 - t_1)) \mid \phi_1 \rangle = \int D[\phi] \exp(iS[\phi])
\]

One now chooses the time separation $(t_2 - t_1)$ to be pure imaginary and equal
to $\beta$. One also puts the initial field $\phi_1$ equal to the final field $\phi_2$ and sums over a complete basis of states $\phi_n$. On the left, one has the expectation value of $e^{-\beta H}$ summed over all states. This is just the thermodynamic partition function $Z$ at the temperature $T = \beta^{-1}$. On the right hand of the equation (3.46) one has a path integral. One puts $\phi_1 = \phi_2$ and

$$t_2 - t_1 = -i\beta, \quad \phi_1 = \phi_2$$

$$Z = \sum <\phi_n | \exp(-\beta H) | \phi_n>$$

$$= \int D[\phi] \exp(-i\tilde{I}[\phi])$$

(3.47) sums over all field configurations $\phi_n$. This means that effectively one is doing the path integral over all fields $\phi$ on a spacetime that is identified periodically in the imaginary time direction with period $\beta$. Thus the partition function for the field $\phi$ at temperature $T$ is given by a path integral over all fields on a Euclidean spacetime. This spacetime is periodic in the imaginary time direction with period $\beta = T^{-1}$. If one does the path integral in flat spacetime identified with period in the imaginary time direction, one gets the usual result for the partition function of black body radiation. However, as we have just seen, the Euclidean- Schwarzschild solution is also periodic in imaginary time with period $2\pi/\kappa$. This means that fields on the Schwarzschild background will behave as if they were in a thermal state with temperature $2\pi/\kappa$ [70].
Chapter 4

Rotating Charged Black Strings in 4-Dimensions

The theory of gravitational collapse and the theory of black holes are two distinct but linked subjects. From the work of Oppenheimer and Snyder [71] and Penrose’s theorem [72] we know that if general relativity is correct, then realistic, slightly non-spherical, complete collapse leads to the formation of a black hole and a singularity. There are also studies hinting that the introduction of a cosmological constant ($\Lambda$) does not alter this picture [73]. Collapse of cylindrical systems and other idealized models was used by Thorne to mimic prolate collapse [74]. This study led to the formulation of the hoop conjecture which states that horizons form when and only when a mass gets compacted into a region whose circumference in every direction is less than its Schwarzschild circumference, $4\pi M$. Thus, cylindrical collapsing matter will not form a black hole. However, the hoop conjecture was given for spacetimes with zero cosmological constant. In the presence of a negative cosmological constant one can expect the occurrence of major changes. Indeed, we show in this chapter that there are black hole solutions with cylindrical symmetry if a negative cosmological constant is present (a fact that does not happen for zero cosmological constant). These cylindrical black holes are also called black strings. We study
the charged rotating black string and show that apart from spacetime being asymptotically anti-de Sitter in the radial direction (and not asymptotically flat) the black string solution has many similarities with the Kerr-Newman black hole. A 4-D metric, \( g_{\mu\nu} \) \((\mu\nu = 0, 1, 2, 3)\), with one Killing vector can be written (in a particular instance) as,

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{mn} dx^m dx^n + \frac{r^2}{\ell^2} dz^2,
\]

(4.1)

where \( g_{mn} \) and \( \phi \) are metric functions, \( m, n = 0, 1, 2 \) and \( z \) is the Killing coordinate. Equation (4.1) is invariant under \( z \to -z \). A cylindrical symmetric metric can then be taken from (4.1) by imposing that the azimuthal coordinate, \( \varphi \), also yields a Killing direction.

We show that the theory has black holes similar to the Kerr-Newman black holes, with a polynomial timelike singularity hidden behind the event and Cauchy horizons. When the charge is zero, the rotating solution does not resemble so much the Kerr solution, the singularity is spacelike hidden behind a single event horizon. In addition, in the non-rotating uncharged case, apart from the topology and asymptotics, the solution is identical to the Schwarzschild solution.

Cylindrical symmetry, as emphasized by Thorne [74], is an idealized situation. It is possible that the Universe we live in, contains an infinite cosmic string. It is also possible, however less likely, that the Universe is crossed by an infinite black string. Yet, one can always argue that close enough to a loop string, spacetime resembles the spacetime of an infinite cosmic string. In the same way, one could argue that close enough to a toroidal finite black hole, spacetime resembles the spacetime of the infinite black string.
4.1 Equations and Solutions

We consider Einstein-Hilbert action in four dimensions with a cosmological term in the presence of an electromagnetic field. The total action is

\[ S_G = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} (R - 2\Lambda - F_{\mu\nu} F^{\mu\nu}), \quad (4.2) \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( A_\mu \) is the vector potential. We study solutions of the Einstein-Maxwell equations with cylindrical symmetry. By this, we mean spacetimes admitting three kinds of topology as [75]

(i) \( R \times S^1 \), the standard cylindrically symmetric model.

(ii) \( S^1 \times S^1 \) the flat torus \( T^2 \) model.

(iii) \( R^2 \).

We will focus upon (i) and (ii). We then choose a cylindrical coordinate system \( (x^0, x^1, x^2, x^3) = (t, r, \varphi, z) \) with \( -\infty < t < +\infty, 0 \leq r < +\infty, -\infty < z < +\infty \) and \( 0 \leq \varphi < 2\pi \). In the toroidal model (ii) the range of the coordinate \( z \) is \( 0 \leq \alpha z < 2\pi \). The electromagnetic four potential is given by \( A_\mu = -h(r)\delta_\mu^0 \), where \( h(r) \) is yet unknown function of the radial coordinate \( r \). Solving the Einstein-Maxwell equations yielded by (4.2) for a static cylindrically symmetric spacetime we find,

\[ ds^2 = -f(r)dt^2 - \frac{dr^2}{f(r)} + r^2 d\varphi^2 + \frac{r^2}{\ell^2} dz^2, \quad (4.3) \]

where

\[ f(r) = \frac{r^2}{\ell^2} - \frac{b\ell}{r} + \frac{\lambda^2 \ell^2}{r^2}, \quad (4.4) \]

and

\[ h(r) = \frac{\lambda \ell}{r} + \text{const.} \quad (4.5) \]

where \( b \) and \( \lambda \) are integration constants. It is easy to show that \( \lambda/2 \) is the linear charge density of the \( z \)-line, and \( b/4 \) is the mass per unit length of the \( z \)-line as we will see in the next section. Depending on the relative values of \( b \) and \( \lambda \), the metric (4.3) can represent a static black string. In this case there
is a black string that is not simply connected, i.e., closed curves encircling the horizon cannot be shrunk to a point.

There is also a stationary solution that follows from equations (4.2) given by

\[
\begin{align*}
\frac{ds^2}{2} &= -\Xi^2 \left[ f(r) - \frac{a^2 r^2}{\Xi^2 \ell^4} \right] dt^2 - \frac{\Xi a \ell}{r} \left( b - \frac{\lambda^2 \ell}{r} \right) 2d\varphi dt \\
&\quad + \left[ \Xi^2 r^2 - a^2 f(r) \right] d\varphi^2 + \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2} dz^2,
\end{align*}
\]

(4.6)

where

\[
A_\mu = -h(r)(\Xi \delta_\mu^0 + a \delta_\mu^2) \quad \text{and} \quad \Xi^2 = 1 + \frac{a^2}{\ell^2},
\]

(4.7)

where \( a \) is constant, \( h(r) = \lambda \ell / r \), and the coordinates have the same range as in the static case. Solution (4.6) can represent a stationary black string. If one compactifies the \( z \) coordinate \( (0 \leq \alpha z < 2\pi) \) one has a closed black string. In this case, one can also put the coordinate \( z \) to rotate. However, this simply represents a bad choice of coordinates. One can always find principal directions in which spacetime rotates only along one of these coordinates (\( \varphi \), say) as in (4.6).

For an observer at radial infinity, the standard cylindrical spacetime model (with \( R \times S^1 \) topology) given by the metric (4.6) extends uniformly over the infinite \( z \)-line. Thus one expects that, as \( r \to \infty \), the total energy as well as the total charge is infinite. The quantities that can be interpreted physically are the mass and charge densities, i.e., mass and charge per unit length of the string. In fact we have already found above the finite and well defined line charge density (of the \( z \)-line) as an integration constant in Einstein-Maxwell equations. For the close black string (the flat torus model with \( S^1 \times S^1 \) topology) the total energy and total charge are well defined quantities. In order to properly define such quantities we use the Hamiltonian formalism and the prescription of Brown and York [8].

There is a suitable canonical form for the metric (4.6) as follow:

\[
\frac{ds^2}{2} = -N^0 dt^2 + R^2 (N^r dt + d\varphi)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2} dz^2,
\]

(4.8)
where
\[ N^{02} = f(r) \frac{r^2}{R^2}, \quad N^\varphi = -\Xi a \left( \frac{b \ell}{r} - \frac{\lambda^2 \ell^2}{r^2} \right), \quad R^2 = \Xi^2 r^2 - a^2 f(r). \] (4.9)

In metric (4.8) \( N^0 \) and \( N^\varphi \) are respectively the lapse and shift functions. Now we see that the metric given in equation (4.8) admits the two Killing vectors needed in order to define mass and angular momentum: a timelike Killing vector \( \xi = (\partial/\partial t) \) and a spacelike (axial) Killing vector \( \varsigma = (\partial/\partial \varphi) \). The total mass as well as the total angular momentum per unit length of this metric can be obtained from Eqs. (2.86) and (2.87), that was expressed in chapter (2), as follow [25, 21]:
\[ M = \frac{1}{8} b(3\Xi^2 - 1), \quad J = \frac{3}{8} b a \Xi. \] (4.10)

Also the electric charge per unit length \( Q \), are found by calculating the flux of the electromagnetic field at infinity. So
\[ Q = \Xi \lambda/2. \] (4.11)

Let us recall that spacetime of (4.6) is pure anti-de Sitter metric, if we choose \( b = \lambda = 0 \). In this case
\[ ds^2 = -\frac{r^2}{\ell^2} dt^2 + \frac{\ell^2 dr^2}{r^2} + r^2 d\varphi^2 + \frac{r^2}{\ell^2} dz^2. \] (4.12)
This is also the background reference spacetime, since metric (4.6) reduces to (4.12) if the black hole is not present.

One can derive \( b \) and \( \lambda \) from equations (4.10), and rewrite the metric (4.6) as
\[ ds^2 = -\left( \frac{r^2}{\ell^2} - \frac{2(M + \Omega) \ell}{r} + \frac{4Q^2 \ell^2}{r^2} \right) dt^2 - \frac{16J \ell}{3r} \left( 1 - \frac{2Q \ell}{(M + \Omega) r} \right) dtd\varphi \]
\[ + \left[ r^2 + \frac{4(M - \Omega) \ell^3}{r} \left( 1 - \frac{2}{(M + \Omega) r} \right) \right] d\varphi^2 \]
\[ + \frac{dr^2}{r^2 - \frac{2(3\Omega - M) \ell^2}{r}} + \frac{3\Omega - M}{\Omega + M} \frac{4Q^2 \ell^2}{r^2} + \frac{r^2}{\ell^2} dz^2, \] (4.13)
where
\[ \Omega = \sqrt{M^2 - \frac{8J^2}{9\ell^2 \alpha}}. \] (4.14)

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4.2 Causal Structure of the Charged Rotating Black String Spacetime

In order to study the metric and its causal structure it is useful to define the parameter \( \alpha \) (with units of angular momentum per unit mass), that can show the effect of rotation (called rotation parameter) as:

\[
\frac{\alpha^2}{\ell^2} \equiv 1 - \frac{\Omega}{M} \quad (4.15)
\]

such that

\[
1 + \frac{\Omega}{M} = 2(1 - \frac{\alpha^2}{2\ell^2}) \quad , \quad 3 \frac{\Omega}{M} - 1 = 2(1 - \frac{3\alpha^2}{2\ell^2}). \quad (4.16)
\]

The relation between \( J \) and \( \alpha \) is given by

\[
J = \frac{3}{2} \alpha M \sqrt{1 - \frac{\alpha^2}{2\ell^2}}. \quad (4.17)
\]

The range of \( \alpha \) is \( 0 \leq \alpha/\ell \leq 1 \). With these definitions the metric (4.13) assumes the form

\[
ds^2 = \left( \frac{r^2}{\ell^2} - \frac{4M(1 - \frac{\alpha^2}{2\ell^2})}{r} + \frac{4Q^2\ell^2}{r^2} \right) dt^2
\]

\[- \frac{4\alpha M\ell}{r} \sqrt{1 - \frac{\alpha^2}{2\ell^2}} \left( 1 - \frac{Q^2\ell}{M(1 - \frac{\alpha^2}{2\ell^2})r} \right) 2dd\varphi
\]

\[+ \left( \frac{r^2}{\ell^2} - \frac{4M(1 - \frac{3\alpha^2}{2\ell^2})}{r} + \frac{4Q^2\ell^2}{r^2} \left( 1 - \frac{3\alpha^2}{2\ell^2} \right) \right)^{-1} dr^2
\]

\[+ \left[ r^2 + \frac{4M\alpha^2\ell}{r} \left( 1 - \frac{Q^2\ell}{(1 - \frac{\alpha^2}{2\ell^2})Mr} \right) \right] d\varphi^2 + \frac{r^2}{\ell^2} dz^2. \quad (4.18)
\]

It is worthwhile to mention that, there is a relation between the new parameter \( \alpha \) and the old parameter \( a \) as:

\[
\alpha^2 = a^2(1 + \frac{3a^2}{2\ell^2})^{-1}. \quad (4.19)
\]

Hence the old parameter \( a \), also may be called ‘rotation parameter’. In order to compare metric (4.18) with the well-known Kerr-Newman metric, we write
explicitly here the Kerr-Newman metric on the equatorial plane

\[
\begin{align*}
\text{ds}^2 &= -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dt^2 - \frac{2ma}{r}(1 - \frac{e^2}{2mr})2dtd\varphi \\
&\quad + \left[r^2 + a(1 + \frac{2m}{r} - \frac{e^2}{r^2})\right]d\varphi^2 + r^2d\theta^2 \\
&\quad + (1 - \frac{2m}{r} + \frac{a^2 + e^2}{r^2})^{-1}dr^2,
\end{align*}
\]

(4.20)

where \((m, a, e)\) are the mass, specific angular momentum and charge parameters of Kerr-Newman spacetimes, respectively. We can now see that the metric for a rotating cylindrical symmetric asymptotically anti-de Sitter spacetime, given in (4.18), has many similarities with the metric on the equatorial plane for the Kerr-Newman metric in (4.20).

Metric (4.18) has a singularity at \(r = 0\). The Kretschmann scalar or scalar of Riemann tensor, \(K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\) is

\[
K = \frac{24}{\ell^4} \left(1 + \frac{b^2\ell^6}{2r^6}\right) - \frac{48\lambda^2\ell^3}{r^7} \left(b - \frac{7\lambda^2\ell}{6r}\right),
\]

(4.21)

where \(b\) and \(\lambda\) can be picked up from (4.6) and (4.18). Thus \(K\) diverges at \(r = 0\). The solution has totally different character depending on whether \(r > 0\) or \(r < 0\). The important black hole solution exists for \(r > 0\) or \((M > 0)\) which we consider.

To analyze the causal structure and follow the procedure of Boyer and Lindquist [76] and Carter [77] we put metric (4.18) in the form,

\[
\begin{align*}
\text{ds}^2 &= -f(r)(\Xi dt - ad\varphi)^2 + r^2\left(\Xi d\varphi - \frac{a}{\ell^2}dt\right)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2}dz^2.
\end{align*}
\]

(4.22)

There are horizons whenever

\[
f(r) = 0,
\]

(4.23)

i.e., at the roots of \(f(r)\). One knows that the non-extremal situations in the Kerr-Newman metric are given by \(0 \leq \frac{a^2}{m^2} \leq 1 - \frac{e^2}{m^2}\). Here, to have horizons one needs either one of the two conditions:

\[
0 \leq \frac{\alpha^2}{\ell^2} \leq \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}\frac{\alpha^2}{\ell^2})^3},
\]

(4.24)
or
\[ \frac{2}{3} < \frac{\alpha^2}{\ell^2} \leq 1. \]  \hspace{1cm} (4.25)

Thus there are five distinct cases depending on the value of the charge and angular momentum:

(I) \( 0 \leq \frac{\alpha^2}{\ell^2} \leq 2/3 - (128/81)Q^6[M^4(1 - \alpha^2/2\ell^2)^3]^{-1} \), which yields the black hole solution with event and Cauchy horizons.

(II) \( \frac{\alpha^2}{\ell^2} = 2/3 - (128/81)Q^6[M^4(1 - \alpha^2/2\ell^2)^3]^{-1} \), which corresponds to the extreme case, where the two horizons merge.

(III) \( 2/3 - (128/81)Q^6[M^4(1 - \alpha^2/2\ell^2)^3]^{-1} < \frac{\alpha^2}{\ell^2} < \frac{2}{3} \), corresponding to naked singularity solutions.

(IV) \( \frac{\alpha^2}{\ell^2} = 2/3 \), which gives a null singularity.

(V) \( 2/3 < \frac{\alpha^2}{\ell^2} < 1 \), which gives a black hole solution with one horizon.

The most interesting solutions are given in items (I) and (II). Solutions (IV) and (V) do not have partners in the Kerr-Newman family. In figure (4.1), we show the black hole and naked singularity regions, and the extremal black hole line dividing those two regions. Now we analyze each item in turn.

4.2.1 (I) Black Hole With Two Horizons

This is the charged-rotating black string spacetime. As we will see this is indeed very similar to the Kerr-Newman black hole. The structure has event and Cauchy horizons, timelike singularities, and closed timelike curves.

Now, following Boyer and Lindquist, we choose a new angular coordinate which straightens out the helicoidal null geodesics that pile up around the event horizon. A good choice is

\[ \varphi = \Xi \varphi - \frac{a}{\ell^2} t. \]  \hspace{1cm} (4.26)
Figure 4.1: The five regions and lines which yield solutions of different natures are shown.

In this case the metric reads,

$$ds^2 = -f(r) \left( \xi dt - \frac{a}{\xi} d\varphi \right)^2 + r^2 d\varphi^2 + \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2} dz^2.$$  \hspace{1cm} (4.27)

The horizons are given at the zeros of the lapse function, i.e. when $f(r) = 0$. We find that $f(r)$ has two roots $r_+$ and $r_-$ given as

$$r_{\pm} = b^4 \frac{s \pm \sqrt{2s^2 - 4q^2 - s}}{2\alpha}$$  \hspace{1cm} (4.28)

where,

$$s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left( \frac{4q^2}{3} \right)} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left( \frac{4q^2}{3} \right)} \right)^{\frac{1}{3}},$$  \hspace{1cm} (4.29)

$$q^2 = \frac{\lambda^2}{b^4}.$$  \hspace{1cm} (4.30)

Now we introduce a Kruskal coordinate patch around $r_+$ and $r_-$. The first patch constructed around $r_+$ is valid for $r_- < r < \infty$.

In the region $r_- < r \leq r_+$ the null Kruskal coordinates $U$ and $V$ are given
by,

\[
U = \left( \frac{r_+ - r}{\ell b^2} \right)^{\frac{1}{2}} \left( \frac{r - r_-}{\ell b^2} \right)^{\frac{r_+^2 - r_-^2}{2r_+^2}} F(r) \exp \left( -\frac{A r_+ - r_- + t}{B 2r_+^2} \right) \]

\[
V = \left( \frac{r_+ - r}{\ell b^2} \right)^{\frac{1}{2}} \left( \frac{r - r_-}{\ell b^2} \right)^{\frac{r_+^2 - r_-^2}{2r_+^2}} F(r) \exp \left( A r_+ - r_- + t \right) \frac{1}{B 2r_+^2} \]

(4.31)

For \( r_+ \leq r < \infty \) we put,

\[
U = -\left( \frac{r - r_+}{\ell b^2} \right)^{\frac{1}{2}} \left( \frac{r - r_-}{\ell b^2} \right)^{\frac{r_+^2 - r_-^2}{2r_+^2}} F(r) \exp \left( -\frac{A r_+ - r_- + t}{B 2r_+^2} \right) \]

\[
V = \left( \frac{r - r_+}{\ell b^2} \right)^{\frac{1}{2}} \left( \frac{r - r_-}{\ell b^2} \right)^{\frac{r_+^2 - r_-^2}{2r_+^2}} F(r) \exp \left( A r_+ - r_- + t \right) \frac{1}{B 2r_+^2} \]

(4.32)

The following definitions have been introduced in order to facilitate the notation,

\[
A \equiv (r_+^2 + r_-^2)^2 + 2 (r_+ + r_-)^4, \quad (4.33)
\]

\[
B \equiv \ell \left[ (r_+ + r_-)^2 + 2r_-^2 \right], \quad (4.34)
\]

\[
C \equiv \ell \left[ (r_+ + r_-)^2 + 2r_+^2 \right], \quad (4.35)
\]

\[
D \equiv \ell \left( r_+ + r_- \right)^3, \quad (4.36)
\]

\[
E \equiv \ell \frac{(r_+^2 + r_-^2)^2 + 2 (r_+ + r_-)^2 (r_+^2 + r_-^2) + 2r_+ r_-}{\sqrt{(r_+ + r_-)^2 + 2 (r_+^2 + r_-^2)}}, \quad (4.37)
\]

and finally,

\[
F(r) \equiv \frac{1}{\ell^2 b^2} \left[ r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-) \right]^{-\frac{r_+^2 - r_-^2}{2r_+^2}} \exp \left( \frac{E r_+ - r_-}{B 2r_+^2} \arctan \frac{2r_+ (r_+ + r_-)}{\sqrt{(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)}} \right). \quad (4.38)
\]

In this first coordinate patch, \( r_- < r \leq \infty \), the metric can be written as,

\[
ds^2 = -\ell^2 b^2 \frac{(r - r_-)^{1+\frac{r_+^2 - r_-^2}{2r_+^2}}}{k^2 r^2} G_+(r) dU dV
\]

\[
+ \frac{\alpha \ell}{\sqrt{1 - \frac{\alpha^2}{2r^2}}} \frac{b^2 (r - r_-)^{1+\frac{r_+^2 - r_-^2}{2r_+^2}}}{k^2 r^2} G_+(r) (V dU - U dV) d\varphi
\]

\[
+ \left( r^2 - f(r) \frac{\alpha^2}{1 - \frac{\alpha^2}{2r^2}} \right) d\varphi^2 + \frac{r^2}{\ell^2} d\zeta^2, \quad (4.39)
\]

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where,

\[
G_+(r) \equiv \frac{r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-)}{F^2(r)},
\]

and

\[
k_+ = A \frac{r_+ - r_-}{B} \frac{r_{+}^2}{2r_{+}^2}.
\]

We see that the metric given in (4.39) is regular in this patch, and in particular is regular at \( r_+ \). It is however singular at \( r_- \). To have a metric non-singular at \( r_- \) one has to define new Kruskal coordinates for the patch \( 0 < r < r_+ \).

For \( 0 < r \leq r_- \) and \( r_- \leq r < r_+ \) we have

\[
U = - \left( \frac{r_+ - r}{\ell b^2} \right)^{-\frac{B}{2r^2 - 2}} \left( \frac{r_+ - r}{\ell b^2} \right) H(r) \exp \left( \frac{A r_+ - r_-}{C} \frac{t}{2r^2 - 2} \right),
\]

\[
V = \left( \frac{r_+ - r}{\ell b^2} \right)^{-\frac{B}{2r^2 - 2}} \left( \frac{r_+ - r}{\ell b^2} \right) H(r) \exp \left( - \frac{A r_+ - r_-}{C} \frac{t}{2r^2 - 2} \right),
\]

and

\[
U = \left( \frac{r_+ - r}{\ell b^2} \right)^{-\frac{B}{2r^2 - 2}} \left( \frac{r_+ - r}{\ell b^2} \right) H(r) \exp \left( \frac{A r_+ - r_-}{C} \frac{t}{2r^2 - 2} \right),
\]

\[
V = \left( \frac{r_+ - r}{\ell b^2} \right)^{-\frac{B}{2r^2 - 2}} \left( \frac{r_+ - r}{\ell b^2} \right) H(r) \exp \left( - \frac{A r_+ - r_-}{C} \frac{t}{2r^2 - 2} \right),
\]

where,

\[
H(r) = \left( \frac{1}{\ell^2 b^2} \left[ r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-) \right] \right)^{-\frac{B}{2r^2 - 2}} \\
\exp \left( - \frac{E r_+ - r_-}{C} \arctan \frac{2r + (r_+ + r_-)}{\sqrt{(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)}} \right)
\]

The metric for this second patch can be written as

\[
ds^2 = - \ell^2 b^2 \left( \frac{r_+ - r}{\ell b^2} \right)^{\frac{1 + B}{2r^2 - 2}} G_-(r) dU dV \\
- \frac{\alpha \ell}{b^2} \left( \frac{r_+ - r}{\ell b^2} \right)^{\frac{1 + B}{2r^2 - 2}} \frac{r_{+}^2}{k_{-} r^2} G_-(r) (V dU - U dV) d\varphi \\
+ \left( r^2 - f(r) \frac{\alpha^2}{1 - \frac{\alpha^2}{2\ell^2}} \right) d\varphi^2 + \frac{r^2}{\ell^2} d\zeta^2,
\]

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where,
\[ G_-(r) \equiv \frac{r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-)}{H^2(r)} \]  \hspace{1cm} (4.46)

and
\[ k_- = \frac{A r_+ - r_-}{B 2 r_-^2}. \]  \hspace{1cm} (4.47)

The metric is regular at \( r_- \) and is singular at \( r = 0 \). To construct the Penrose diagram we have to define the Penrose coordinates, \( \psi, \xi \) by the usual arctangent functions of \( U \) and \( V \),
\[ U = \tan \frac{1}{2} (\psi - \xi) \quad \text{and} \quad V = \tan \frac{1}{2} (\psi + \xi) \]  \hspace{1cm} (4.48)

From (4.48), (4.31) and (4.32) we see that: (i) the line \( r = \infty \) is mapped into two symmetrical curved timelike lines, and (ii) the line \( r = r_+ \) is mapped into two mutual perpendicular straight lines at 45\(^0\). From (4.42) and (4.43) we see that: (i) \( r = 0 \) is mapped into a curved timelike line and (ii) \( r = r_- \) is mapped into two mutual perpendicular straight null lines at 45\(^0\). One has to join these two different patches (see [78, 79]) and then repeat them over in the vertical. The result is the Penrose diagram shown in figure (4.2). The lines \( r = 0 \) and \( r = \infty \) are drawn as vertical lines, although in the coordinates \( \psi \) and \( \xi \) they should be curved outwards, bulged. It is always possible to change coordinates so that the lines are indeed vertical.

4.2.2 (II) Extreme Case:

The extreme case is given when \( Q \) is connected to \( M \) and \( \alpha \) through the relation,
\[ Q^6 = \frac{27}{64} M^2 \left( 1 - \frac{3}{2} \frac{\alpha^2}{\ell^2} \right) \left( 1 - \frac{1}{2} \frac{\alpha^2}{\ell^2} \right)^3, \]  \hspace{1cm} (4.49)

which can also be put in the form \( \alpha^2/\ell^2 = 2/3 - (128/81)Q^6[M^4(1-\alpha^2/2\ell^2)^3]^{-1} \) as above. In figure (4.1) we have drawn the line which gives the values of \( Q \) and \( \alpha \) (in suitable \( M \) units) compatible with this case. The event and Cauchy horizons join together in one single horizon \( r_+ \) given by
\[ r_+ = \frac{4Q^2\ell}{3M(1 - \frac{\alpha^2}{2\ell^2})} \]  \hspace{1cm} (4.50)
Figure 4.2: The Penrose diagram representing the nonextreme charged rotating black string.

The function \( f(r) \) is now,

\[
f(r) = \frac{(r - r_+)^2(r^2 + 2r_+ + 3r_+^2)}{r^2 \ell^2} \tag{4.51}
\]

so the metric (4.22) turns to

\[
ds^2 = -\frac{(r - r_+)^2(r^2 + 2r_+ + 3r_+^2)}{r^2 \ell^2} (\Xi dt - ad\varphi)^2 + \frac{r^2 \ell^2 dr^2}{(r - r_+)^2(r^2 + 2r_+ + 3r_+^2)} + r^2(\Xi dt - ad\varphi)^2 + \frac{r^2}{\ell^2} dz^2. \tag{4.52}
\]

There are no Kruskal coordinates. To draw the Penrose diagram we resort first to the double null coordinates \( u \) and \( v \),

\[
u = \frac{1}{\ell} (\Xi t - r_*) \quad \text{and} \quad v = \frac{1}{\ell} (\Xi t + r_*) \tag{4.53}
\]

where \( r_* \) is the tortoise coordinate given by

\[
r_* = \frac{2\ell}{9r_+} \ln \left( \frac{r - r_+}{\ell b_+^4} \right) - \frac{\ell}{6(r - r_+)} - \frac{\ell}{9r_+} \ln \left[ \frac{(r^2 + 2r_+ + 3r_+^2)^2}{\ell^2 b_+^4} \right] + \frac{7\ell}{18\sqrt{2}r_+} \arctan \frac{r + r_+}{\sqrt{2}r_+}. \tag{4.54}
\]
Defining the new angular coordinate as before \( \varphi = \Xi \varphi - (a/\ell^2) t \), the metric (4.52) is now
\[
d s^2 = -\frac{(r - r_+)^2(r^2 + 2r_+ r + 3r_+^2) \, dt^2}{\ell^2 r^2} + \frac{\ell^2 r^2 \, dr^2}{(r - r_+)^2(r^2 + 2r_+ r + 3r_+^2)}
+ \frac{a f(r)}{\Xi^2} 2 dt d\varphi + (r^2 - f(r) \frac{a^2}{\Xi^2}) d\varphi^2 + \frac{r^2}{\ell^2} dz^2.
\] (4.55)

Now defining the Penrose coordinates [79, 80] \( \psi \) and \( \xi \) via the relations
\[
u = \tan \frac{1}{2}(\psi - \xi) \quad \text{and} \quad v = \tan \frac{1}{2}(\psi + \xi),
\] (4.56)
one can write the metric (4.55) as:
\[
d s^2 = -\frac{(r - r_+)^2(r^2 + 2r_+ r + 3r_+^2)}{r^2} \frac{d\psi^2 - d\xi^2}{(\cos \psi + \cos \xi)^2}
+ \frac{a f(r)}{\Xi^2} 2 dt d\varphi + (r^2 - f(r) \frac{a^2}{\Xi^2}) d\varphi^2 + \frac{r^2}{\ell^2} dz^2,
\] (4.57)
where \( t \) is given implicitly in terms of \( \psi \) and \( \xi \). From the defining Eqs. (4.53) and (4.56) we have
\[
\frac{\sin \xi}{\cos \psi + \cos \xi} = \frac{r_*}{\ell}
\] (4.58)

Then, one can draw the Penrose diagram (see figure (4.3)). The lines \( r = r_+ \) are given by the equation \( \psi = \pm \xi + n\pi \) with \( n \) any integer, and therefore are lines at 45°. The lines \( r = 0 \) and \( r = \infty \) are timelike lines given by an equation of the form \( \sin \xi/(\cos \psi + \cos \xi) = \text{const.} \), where the constant is easily found from \( r_* \). These are not straight vertical lines. However by a further coordinate transformation it is possible to straighten them out as it is shown in Fig. (4.3).

The metric (4.57) is regular at \( r = r_+ \), because the zeros of the denominator and numerator cancel each other.

### 4.2.3 (III) Naked Singularity

In this case there are no roots for \( f(r) \) as defined in (4.4). Therefore there are no horizons. The singularity is timelike and naked. Infinity is also timelike. There is an infinite redshift surface if the following inequality is satisfied
\[
Q^6 \leq \frac{27}{64} (1 - \frac{1}{2} \frac{\alpha^2}{\ell^2})^4 M^4.
\] (4.59)
Figure 4.3: *The Penrose diagram for the extremal charged rotating black string.*

The Penrose diagram is sketched in Fig. (4.4).

In this chapter, first, we made the black string metric in 4–dimensions and then the properties of this metric studied. At the next chapter we generalize this metric to higher dimensions and its thermodynamic properties will be studied. Also; In chapter 7, The ‘no-hair’ theorem for this metric will be discussed.
Figure 4.4: The Penrose diagram for the charged rotating naked singularity.
Chapter 5

Thermodynamics of

$(n + 1)$-Dimensional Charged Rotating Black Branes

The theory of AdS/CFT correspondence was given in Chapter 2. There, we studied all divergent terms and obtained the counterterms to finite the Gravitational action. In Chapter 3, we studied the theory of thermodynamics of black holes and in chapter 4, we gave the metric of a 4-dimensional charged rotating black string that was obtained by Lemos [25]. In this chapter, we study the thermodynamics of the $(n + 1)$-dimensional charged rotating black brane introduced by Awad [27], and consider its stability.

1We published the paper related to this subject in Phys. Rev. D (see Ref. [81])
5.1 The Action and Thermodynamic Quantities of Asymptotically AdS (AAdS) Charged Rotating Black Brane

The gravitational action for Einstein-Maxwell theory in \((n+1)\) dimensions for AAdS spacetimes is

\[
I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} (R - 2\Lambda - F^{\mu\nu}F_{\mu\nu}) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n\mathbf{x} \sqrt{-\gamma}\Theta(\gamma), \tag{5.1}
\]

where \(F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\) is the electromagnetic tensor field and \(A^{\mu}\) is the vector potential. The first term is the Einstein-Hilbert volume term with negative cosmological constant \(\Lambda = -n(n-1)/2\ell^2\) and the second term is the Gibbons Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold \(\mathcal{M}\) has metric \(g_{\mu\nu}\) and covariant derivative \(\nabla_\mu\). \(\Theta\) is the trace of the extrinsic curvature \(\Theta^{\mu\nu}\) of any boundary(ies) \(\partial\mathcal{M}\) (that is \(\Sigma\)) of the manifold \(\mathcal{M}\), with induced metric(s) \(\gamma_{i,j}\) (see Appendix B).

The counterterm for asymptotically AdS spacetimes up to seven dimensions as was obtained in chapter 2 is

\[
I_{ct} = \frac{1}{8\pi} \int_{\partial\mathcal{M}_\infty} d^n\mathbf{x} \sqrt{-\gamma}\{\frac{n-1}{\ell} - \frac{\ell\Upsilon(n-3)}{2(n-2)}\frac{R}{\ell^2} - \frac{\ell^3\Upsilon(n-5)}{2(n-4)(n-2)^2}\left(R_{ab}R^{ab} - \frac{n}{4(n-1)}R^2\right) + \ldots\}, \tag{5.2}
\]

where \(R, R_{abcd}\), and \(R_{ab}\) are the Ricci scalar, Riemann and Ricci tensors of the boundary metric \(\gamma_{ab}\).

The metric of \((n+1)\)-dimensional AAdS charged rotating black brane with \(k\) rotation parameters is [27]

\[
ds^2 = -f(r) \left(\Xi dt - \sum_{i=1}^{k} a_id\phi_i\right)^2 + \frac{r^2}{\ell^2} \sum_{i=1}^{k} \left(a_idt - \Xi\ell^2 d\phi_i\right)^2
+ \frac{dr^2}{f(r)} - \frac{r^2}{\ell^2} \sum_{i<j}^{k} (a_id\phi_j - a_jd\phi_i)^2 + r^2 d\Omega^2, \tag{5.3}
\]
where $\Xi = \sqrt{1 + \sum_i a_i^2 / \ell^2}$ and $d\Omega^2$ is the Euclidean metric on the $(n - 1 - k)$-dimensional submanifold with volume $V_{n-1}$. The maximum number of rotation parameters in $(n + 1)$ dimensions is $[(n + 1)/2]$, where $[x]$ denotes the integer part of $x$. In Eq. (5.3) $f(r)$ is

$$f(r) = \frac{r^2}{\ell^2} - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}}, \quad (5.4)$$

and the gauge potential is given by

$$A_\mu = -\sqrt{n - 1 \over 2n - 4} q \left( \Xi \delta^0_\mu - \delta^i_\mu a_i \right), \quad \text{(no sum on } i). \quad (5.5)$$

In 4—dimensions with $n = 3$, simply, one can see that this metric exchange to the previous metric of Eq. (4.12). The Einstein equation for this spacetime can be written as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{n(n - 1)}{2\ell^2} g_{\mu\nu} = 8\pi T_{(\text{em})\mu\nu}, \quad (5.6)$$

where $R$ is the Ricci scalar. The stress energy-momentum tensor $T_{(\text{em})\mu\nu}$ is

$$T_{(\text{em})\mu\nu} = \frac{1}{8\pi}(2F^\lambda_\mu F_{\lambda\nu} - \frac{1}{2} F^\sigma_\lambda F_{\sigma\lambda} g_{\mu\nu}). \quad (5.7)$$

Of course, as expressed in chapter 2, there are also logarithmic divergences due to the Weyl anomaly and matter field. To obtain the total action, we first calculate the logarithmic divergences due to the Weyl anomaly and matter field given in Eqs. (2.68), (2.72), (2.69), and (2.73). The leading metric $\tilde{\gamma}_{ij}^0$ in Eq.(2.60) can be obtained as

$$\tilde{\gamma}_{ij}^0 dx^i dx^j = -\frac{1}{\ell^2} dt^2 + d\phi^2 + d\Omega^2. \quad (5.8)$$

Therefore the curvature scalar $R^0(\tilde{\gamma}^0)$ and Ricci tensor $R_{ij}^0(\tilde{\gamma}^0)$ are zero. Also it is easy to show that $F_{ij}^0$ in Eqs. (2.69) and (2.73) vanishes. Thus, all the logarithmic divergences for the $(n + 1)$-dimensional charged rotating black brane are zero. It is also a matter of calculation to show that the counterterm action due to the electromagnetic field in Eq. (2.71) is zero. Thus, the total renormalized action is

$$I = I_G + I_{ct}. \quad (5.9)$$
In order to obtain the Einstein-Maxwell equations by the variation of the volume integral with respect to the fields, one should impose the boundary condition \( \delta A_\mu = 0 \) on \( \partial \mathcal{M} \). Thus the action (5.9) is appropriate to study the grand-canonical ensemble with fixed electric potential \([82]\). To study the canonical ensemble with fixed electric charge one should impose the boundary condition \( \delta (n^a F_{ab}) = 0 \), and therefore the total action is \([6]\)

\[
\tilde{I} = I - \frac{1}{4\pi} \int_{\partial \mathcal{M}_\infty} d^nx \sqrt{-\gamma} n_a F^{ab} A_b. \tag{5.10}
\]

The divergence free stress-energy tensor for \( n \leq 6 \) was given by Eq. (2.49).

As in the case of rotating black hole solutions of Einstein’s gravity, the above metric given by Eqs. (5.3)-(5.5) has two types of Killing and event horizons. It was proved that a stationary black hole event horizon should be a Killing horizon in the four-dimensional Einstein gravity \([6]\). The Killing vector of this \((n + 1)\)-dimensional metric is

\[
\chi = \partial_t + \sum_{i=1}^k \Omega_i \partial_{\phi_i}, \tag{5.11}
\]

which is the null generator of the event horizon. The metric of Eqs. (5.3)-(5.5) has two inner and outer event horizons located at \( r_- \) and \( r_+ \), if the metric parameters \( m \) and \( q \) are chosen to be suitable \([27]\). The two horizons \( r_- \) and \( r_+ \) are the real roots of \( f(r) = 0 \). For later use in the thermodynamics of the black brane, it is better to present an expression for the critical value of the charge and mass in term of the radius of the event horizon \( r_+ \). It is easy to show that the metric has two inner and outer horizons provided the charge parameter, \( q \) is less than \( q_{\text{crit}} \) given as

\[
q_{\text{crit}} = \sqrt{\frac{n}{n-2}} \frac{r_+^{n-1}}{\ell}, \tag{5.12}
\]

or \( m \) is greater than \( m_{\text{crit}} \), that is

\[
m_{\text{crit}} = \left( \frac{n-1}{n-2} \right) \frac{2r^n}{\ell^2}. \tag{5.13}
\]

In the case that \( q = q_{\text{crit}} \) or \( m = m_{\text{crit}} \), we will have an extreme black brane.
As mentioned in chapter 2 and 4, mass and angular momentum are conserved charges which may be calculated for black holes (string). By calculating the finite or divergence free stress energy-momentum tensor and using Eqs. (2.86) and (2.87), we obtain the total mass as well as the total angular momentum as:

\[ M = \frac{V_{n-1}(2\pi)^k}{16\pi} m \left[ n \Xi^2 - 1 \right], \]  

\[ J_i = \frac{V_{n-1}(2\pi)^k}{16\pi} n \Xi m a_i, \]  

The Hawking temperature and the angular velocities of the \( r_+ \) can be calculated as expressed in chapter 3. Hence we have

\[ T = \frac{1}{\beta_+} = \left( \frac{4\pi \Xi}{f'(r_+)} \right)^{-1}, \]  

\[ = \frac{nr_+^{(2n-2)} - (n-2)q^2 \ell^2}{4\pi \ell^2 \Xi r_+^{(2n-3)}}, \]  

\[ \Omega_j = \frac{a_j}{\Xi \ell^2}, \]  

where \( \beta_+ \) is the inverse Hawking temperature. These quantities also were obtained by Awad in Ref. [27]. Equation (5.12) shows that the temperature \( T \) in Eq. (5.16) is positive for the allowed values of the metric parameters and vanishes for the extremal solution.

By using Eqs. (5.1), (5.2), (5.9), and (5.10), the Euclidean actions in the grand-canonical and the canonical ensemble can be calculated as

\[ I = -\frac{\beta_+ V_{n-1}(2\pi)^k}{16\pi} \frac{r_+^{(2n-2)} + q^2 \ell^2}{r_+^{(n-2)} \ell^2}, \]  

\[ \tilde{I} = -\frac{\beta_+ V_{n-1}(2\pi)^k}{16\pi} \frac{r_+^{(2n-2)} - (2n-3)q^2 \ell^2}{r_+^{(n-2)} \ell^2}. \]

The electric charge \( Q \), can be found by calculating the flux of the electromagnetic field at infinity, yielding

\[ Q = \frac{\Xi V_{n-1}(2\pi)^k}{4\pi} \sqrt{\frac{(n-1)(n-2)}{2}} q. \]
The electric potential \( \Phi \), measured at infinity with respect to the horizon, is defined by [82]

\[
\Phi = A_\mu \chi_\mu |_{r \to \infty} - A_\mu \chi_\mu |_{r=r_+},
\]

where \( \chi \) is the null generators of the event horizon given by Eq. (5.11). One obtains

\[
\Phi = \sqrt{\frac{(n-1)}{2(n-2)}} \frac{q}{\Xi r_+^{(n-2)}}. \tag{5.21}
\]

The area law of the entropy is universal as was seen in Eq. (3.41), and applies to all kinds of black holes/branes [83]. At first, we should calculate the area of event horizon \( A \). This can be done by calculating \( \int \sqrt{\sigma} d^{n-1}x \). We obtain

\[
A = V_{n-1} \Xi r_+^{(n-1)}, \tag{5.22}
\]

and therefore

\[
S = \frac{\Xi V_{n-1} r_+^{(n-1)}}{4}. \tag{5.23}
\]

For \( n = 3 \), these quantities given in Eqs. (5.14)-(5.23) reduce to those calculated in Ref. [21].

### 5.2 Thermodynamics of black brane

#### 5.2.1 Generalization of Smarr Formula

At first we obtain the mass as a function of the extensive quantities \( S, J, \) and \( Q \). Using the expression for the entropy, the mass, the angular momenta, and the charge given in Eqs. (5.14), (5.15), (5.20), (5.23), and the fact that \( f(r_+) = 0 \), one can obtain a Smarr-type formula as

\[
M(S, J, Q) = \frac{(nZ - 1) \sqrt{\sum_i J_i^2}}{n \ell \sqrt{Z(Z - 1)}}, \tag{5.24}
\]

where \( Z = \Xi^2 \) is the positive real root of the following equation:

\[
(Z - 1)^{(n-1)} - \frac{Z}{16S^2} \left\{ \frac{4\pi(n-1)(n-2)\ell SJ}{n[(n-1)(n-2)S^2 + 2\pi^2 Q^2 \ell^2]} \right\}^{(2n-2)} = 0. \tag{5.25}
\]
One may then regard the parameters $S$, $J$, and $Q$ as a complete set of extensive parameters for the mass $M(S, J, Q)$ and define the intensive parameters conjugate to $S$, $J$ and $Q$. These quantities are the temperature $T$, the angular velocities $\Omega_i$, and the electric potential $\Phi$,

$$T = \left( \frac{\partial M}{\partial S} \right)_{J, Q}, \quad \Omega_i = \left( \frac{\partial M}{\partial J_i} \right)_{S, Q}, \quad \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S, J}. \quad (5.26)$$

It is a matter of straightforward calculation to show that the intensive quantities calculated by Eq. (5.26) coincide with Eqs. (5.16), (5.17), and (5.21) found in Sec. (5.1). Thus, the thermodynamic quantities calculated in Sec. (5.1) satisfy the first law of thermodynamics which was expressed in Eq. (3.34) as:

$$dM = TdS + \sum_{i=1}^k \Omega_i dJ_i + \Phi dQ. \quad (5.27)$$

5.2.2 Thermodynamic potentials

Now we obtain the thermodynamic potential in the grand-canonical and canonical ensembles. Using the definition of the Gibbs potential $G(T, \Omega, \Phi) = I/\beta$, we obtain

$$G = \frac{-V_{n-1}}{16\pi} \left( \frac{2}{n^2(n-1)(1 - \sum_i \ell^2\Omega_i^2)} \right)^{n/2} \left( \gamma^2 + n^2(n-2)\Phi^2 \right) (\gamma \ell)^{(n-2)}, \quad (5.28)$$

where

$$\gamma = \sqrt{2n - 2T^2\pi \ell} + \sqrt{2(n-1)^2T^2 \ell^2 + n(n-2)^2\Phi^2}. \quad (5.29)$$

Using the expressions (5.16), (5.17), and (5.21) for the inverse Hawking temperature, the angular velocities and the electric potential, one obtains

$$G(T, \Omega, \Phi) = M - TS - \sum_i \Omega_i J_i - \Phi Q, \quad (5.30)$$

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which means that \( G(T, \Omega, \Phi) \) is, indeed, the Legendre transformation of the
\( M(S, J_i, Q) \) with respect to \( S, J_i, \) and \( Q \). It is a matter of straightforward
calculation to show that the extensive quantities

\[
J_i = -\left( \frac{\partial G}{\partial \Omega_i} \right)_{T, \Phi}, \quad Q = -\left( \frac{\partial G}{\partial \Phi} \right)_{T, \Omega}, \quad S = -\left( \frac{\partial G}{\partial T} \right)_{\Omega, \Phi},
\]

(5.31)

turn out to coincide precisely with the expressions (5.15), (5.20), and (5.23).

For the canonical ensemble, the Helmholtz free energy \( F(T, J, Q) \) is defined
as

\[
F(T, J, Q) = \tilde{I} + \sum_i \Omega_i J_i,
\]

(5.32)

where \( \tilde{I} \) is given by Eq. (5.19). One can verify that the conjugate quantities

\[
\Omega_i = \left( \frac{\partial F}{\partial J_i} \right)_{T, Q}, \quad \Phi = \left( \frac{\partial F}{\partial Q} \right)_{T, J}, \quad S = -\left( \frac{\partial F}{\partial T} \right)_{J, Q},
\]

(5.33)

agree with expressions (5.17), (5.21), and (5.23). Also it is worthwhile to
mention that \( F(T, J, Q) \) is the Legendre transformation of the \( M(S, J_i, Q) \)
with respect to \( S \), i.e.

\[
F(T, J, Q) = M - TS.
\]

(5.34)

5.2.3 Stability in the canonical and the grand-canonical
ensemble

The stability of a thermodynamic system with respect to the small variations
of the thermodynamic coordinates, is usually performed by analyzing the be-
behavior of the entropy \( S(M, J, Q) \) around the equilibrium. The local stability
in any ensemble requires that \( S(M, J, Q) \) be a convex function of their ex-
tensive variables or its Legendre transformation must be a concave function
of their intensive variables. Thus, the local stability can in principle be car-
rried out by finding the determinant of the Hessian matrix of \( S \) with respect
to its extensive variables $X_i$, $H_{X_iX_j}^S = [\partial^2 S/\partial X_i \partial X_j]$, or the determinant of the Hessian of the Gibbs function with respect to its intensive variables $Y_i$, $H_{Y_iY_j}^G = [\partial^2 G/\partial Y_i \partial Y_j]$ [84, 82]. Also, one can perform the stability analysis through the use of the Hessian matrix of the mass with respect to its extensive parameters [85]. In our case the entropy $S$ is a function of the mass, angular momenta, and the charge. In the canonical ensemble, the charge and the angular momenta are fixed parameters, and therefore the positivity of the thermal capacity $C_{J,Q} = T(\partial S/\partial T)_{J,Q}$ is sufficient to assure the local stability. The thermal capacity $C_{J,Q}$, at constant charge and angular momenta is

$$C_{J,Q} = \frac{\Xi V}{4} r^{(n-1)} [(nm^{(2n-2)} - (n - 2)q^2 \ell^2)]^{(2n-2) + q^2 \ell^2}$$

$$\times [(n - 2)\Xi^2 + 1][(n - 2)q^4 \ell^4[(3n - 6)\Xi^2 - (n - 3)] - 2q^2 \ell^2 r^{(2n-2)}[(3n - 6)\Xi^2 - n^2 + 3] + nr^{(4n-4)}[(n + 2)\Xi^2 - (n + 1)]]^{-1}. \quad (5.35)$$

Figure (5.1) shows the behavior of the heat capacity as a function of the charge parameter. It shows that $C_{J,Q}$ is positive in various dimensions and goes to zero as $q$ approaches its critical value (extreme black brane). Thus, the $(n+1)$-dimensional AAdS charged rotating black brane is locally stable in the canonical ensemble.

In the grand-canonical ensemble, we find it more convenient to work with the Gibbs potential $G(T, \Omega_i, \Phi)$. Here the thermodynamic variables are the temperature, the angular velocities, and the electric potential. After some algebraic manipulation, we obtain

$$|H_{G_{T,\Omega_i,\Phi}}| = \left( \frac{n-3}{n} \right) \left[ \frac{nm^{\gamma} k}{16\pi} \right] \left[ \frac{(n - 2)\Xi^2 + 1}{r^{\gamma}_\Omega + (\frac{a^2}{2}) ml^2} \right] (\Xi)^{2k+2}\Xi^2 r^{3n-4}. \quad (5.36)$$

As one can see from Eq. (5.36), $|H_{G_{T,\Omega_i,\Phi}}|$ is positive for all the phase space, and therefore the $(n+1)$-dimensional AAdS charged rotating black brane is locally stable in the grand-canonical ensemble. As mentioned in the previous paragraph, one can also perform the local stability analysis through the use
of the determinant of the Hessian matrix of $M$ with respect to $S$, $J$ and $Q$, which has the same result, since $|H_{S,J,Q}^M| = |H_{T,\Omega,\Phi}^G|^{-1}$.

5.2.4 Logarithmic correction to the Bekenstein-Hawking entropy

In recent years, there are several works in literature suggesting that for a large class of black holes, the area law of the entropy receives additive logarithmic corrections due to thermal fluctuation of the object around its equilibrium [86]. Typically, the corrected formula has the form

$$S = S_0 - K \ln(S_0) + ..., \quad (5.37)$$

where $S_0$ is the standard Bekenstein-Hawking term and $K$ is a number. In Ref. [87] an expression has been found for the leading-order correction of a generic thermodynamic system in terms of the heat capacity $C$ as [87]

$$S = S_0 - K \ln(CT^2). \quad (5.38)$$
Equation (5.38) has been considered by many authors for Schwarzschild-AdS, Reissner-Nordstrom-AdS, BTZ, and slowly Kerr-AdS spacetimes [88]. Thus, it is worthwhile to investigate its application for the charged rotating black brane considered in this chapter. Using Eqs. (5.16), (5.23), and (5.35) with \(q = 0\), one obtains

\[
S = S_0 - \frac{n + 1}{2(n - 1)} \ln(S_0) - \Gamma_n(\Xi),
\]

(5.39)

where \(\Gamma_n(\Xi)\) is a positive constant depending on \(\Xi\) and \(n\). Equation (5.39) shows that the correction of the entropy is proportional to the logarithm of the area of the horizon. For small values of \(q\) the logarithmic correction in Eq. (5.38) can be expanded in terms of the power of \(S_0\) as

\[
S = S_0 - \frac{n + 1}{2(n - 1)} \ln(S_0) - \Gamma_n(\Xi) + \frac{\ell^2 \Xi^2 [(n - 4) \Xi^2 + 2]}{16[(n + 2) \Xi^2 - n - 1]} \frac{q^2}{S_0^2} + \ldots
\]

(5.40)

Again the leading term is a logarithmic term of the area.
Part II

HAIRY BLACK HOLE
Chapter 6

The Theory of Abelian Higgs

Hair for Black Holes

Black hole ‘hair’ is defined to be any field(s) associated with a stationary black hole configuration which can be detected by asymptotic observers but which cannot be identified with the electromagnetic or gravitational degrees of freedom. Back in the heyday of black hole physics a number of results were proven [89, 90, 91] which seemed to imply that black holes ‘have no-hair’. These results implied that given certain assumptions the only information about a black hole which an observer far from the hole can determine experimentally is summarized by the electric charge, magnetic charge, angular momentum and mass of the hole. Such uniqueness results are referred to as ‘no-hair’ theorems. These important results would seem to imply that a black hole horizon can support only these limited gauge charges; for a long time people thought that other matter fields simply could not be associated with a black hole. Thus, for example, lepton or baryon number were not good quantum numbers for black holes, despite being defined for a neutron star. However, this idea was to some extent discredited when various authors [92], using numerical techniques, discovered black hole solutions of the Einstein- Yang-Mills equations that support Yang-Mills fields which can be detected by asymptotic observers [93]; one therefore says that these black holes are ‘coloured’. Once the solu-
tions of [93] were discovered, it wasn’t long before other people were finding similar solutions in Einstein-non-Abelian gauge systems [94].

However, these exotic solutions do not violate the original no-hair results since all such solutions are known to be unstable (see e.g. [95]). Since the original no-hair theorems assumed a stationary picture they simply do not apply to coloured holes. On the other hand, coloured holes do still exist and so they are said to ‘evade’ the usual no-hair results. These results teach us that we have to tread carefully when we start talking about black hole hair. There are other amusing tricks which allow one to evade no-hair theorems. We will stick with our definition of hair as any property which can be measured by asymptotic observers. Furthermore, we shall follow [32] and use the term ‘dressing’ for the question of whether or not fields actually live on the horizon.

With all of this in mind, we want to analyze the extent to which hair is present in situations where we allow the topology of some field configurations to be non-trivial; in particular, an interesting question is whether or not topological defects, such as domain walls, strings, or textures [96], can act as ‘hair’ for a black hole. In [32] evidence was presented that a Nielsen-Olesen \((U(1))\) vortex can act as ‘long’ hair for a Schwarzschild black hole.

Recently it has been shown that these ideas can be extended to the case of anti-de Sitter (AdS) and de Sitter (dS) spacetimes. For asymptotically AdS spacetimes, it has been shown that conformally coupled scalar field can be painted as hair [97]. Another asymptotically AdS hairy black hole solution has been investigated in Ref. [98]. Also it was shown that there exist a solution to the \(SU(2)\) Einstein-Yang-Mills equations which describes a stable Yang-Mills hairy black hole, that is asymptotically AdS [31]. Although the idea of Nielson-Olesen vortices has been first introduced in flat spacetimes [99], but recently it has been extended to (A)dS spacetimes [100, 101]. The existence of long range Nielson-Olesen vortex as hair for asymptotically AdS black holes has been investigated in Refs. [39, 41] for Schwarzschild-AdS black hole and charged black string. The explicit calculations which can investigate
the existence of a long range Nielson-Olesen vortex solution as a stable hair for
a stationary black hole solution is escorted with much more difficulties due to
the rotation parameter [40]. In the next chapter, we study the Abelian Higgs
hair for a four dimensional rotating charged black string that is a stationary
model for Einstein-Maxwell equation with cylindrical symmetry.

6.1 Abelian Higgs Field as a Source of the
Topological Defects

On a cold day, ice forms quickly on the surface of a pond. But it does not
grow as a smooth, featureless covering. Instead, the water begins to freeze in
many places independently, and the growing plates of ice join up in random
fashion, leaving zig–zag boundaries between them. These irregular margins are
an example of what physicists call ‘topological defects’ – defects because they
are places where the crystal structure of the ice is disrupted, and topological
because an accurate description of them involves ideas of symmetry embodied
in topology, the branch of mathematics that focuses on the study of continuous
surfaces.

Current theories of particle physics likewise predict that a variety of topo-
logical defects would almost certainly have formed during the early evolution
of the universe. Just as water turns to ice (a phase transition) when the tem-
perature drops, so the interactions between elementary particles run through
distinct phases as the typical energy of those particles falls with the expansion
of the universe. When conditions favor the appearance of a new phase, it gen-
erally crops up in many places at the same time, and when separate regions
of the new phase run into each other, topological defects are the result. The
detection of such structures in the modern universe would provide precious
information on events in the earliest instants after the big bang.

A central concept of particle physics theories attempting to unify all the
fundamental interactions is the concept of symmetry breaking. As the universe
expanded and cooled, first the gravitational interaction, and subsequently all other known forces would have begun adopting their own identities. In the context of the standard hot big bang theory the spontaneous breaking of fundamental symmetries is realized as a phase transition in the early universe. Such phase transitions have several exciting cosmological consequences and thus provide an important link between particle physics and cosmology.

There are several symmetries which are expected to break down in the course of time. In each of these transitions the spacetime gets ‘oriented’ by the presence of a hypothetical force field called the ‘Higgs field’, named for Peter Higgs. This field orientation signals the transition from a state of higher symmetry to a final state where the system under consideration obeys a smaller group of symmetry rules. As a simple analogy we may consider the transition from liquid water to ice; the formation of the crystal structure ice (where water molecules are arranged in a well defined lattice), breaks the symmetry possessed when the system was in the higher temperature liquid phase, when every direction in the system was equivalent. In the same way, it is precisely the orientation in the Higgs field which breaks the highly symmetric state between particles and forces. Kibble [102] first saw the possibility of defect formation when he realized that in a cooling universe phase transitions proceed by the formation of uncorrelated domains that subsequently coalesce, leaving behind relics in the form of defects. In the expanding universe, widely separated regions in space have not had enough time to ‘communicate’ amongst themselves and are therefore not correlated, due to a lack of causal contact. It is therefore natural to suppose that different regions ended up having arbitrary orientations of the Higgs field and that, when they merged together, it was hard for domains with very different preferred directions to adjust themselves and fit smoothly. In the interfaces of these domains, defects form.

Different models for the Higgs field lead to the formation of a whole variety of topological defects, with very different characteristics and dimensions. Some of the proposed theories have symmetry breaking patterns leading to the
formation of ‘domain walls’ (mirror reflection discrete symmetry): incredibly thin planar surfaces trapping enormous concentrations of mass–energy which separate domains of conflicting field orientations, similar to two–dimensional sheet–like structures found in ferromagnets. Within other theories, cosmological fields get distributed in such a way that the old (symmetric) phase gets confined into a finite region of space surrounded completely by the new (non–symmetric) phase. This situation leads to the generation of defects with linear geometry called ‘cosmic strings’. Theoretical reasons suggest these strings (vortex lines) do not have any loose ends in order that the two phases not get mixed up. This leaves infinite strings and closed loops as the only possible alternatives for these defects to manifest themselves in the early universe. ‘Magnetic monopole’ is another possible topological defect. Cosmic strings bounded by monopoles is yet another possibility in grand unified theories (GUT) phase transitions.

Cosmic strings are without any doubt the topological defect most thoroughly studied, both in cosmology and solid–state physics (vortices).

6.2 Structure formation from defects

In this section we will provide just a quick description of the remarkable cosmological features of cosmic strings. Many of the proposed observational tests for the existence of cosmic strings are based on their gravitational interactions. In fact, the gravitational field around a straight static string is very unusual [103]. A simple computations indicates that space is flat outside of an infinite straight cosmic string and therefore test particles in its vicinity should not feel any gravitational attraction.

In fact, a full general relativistic analysis confirms this and test particles in the space around the string feel no Newtonian attraction; however there exists something unusual, a sort of wedge missing from the space surrounding the string and called the ‘deficit angle’, usually noted ∆, that makes the topology
of space around the string that of a cone (Fig. (6.1)). To see this, consider the metric of a source with energy–momentum tensor [103], [104]

\[ T_{\mu \nu} = \delta(x) \delta(y) \text{diag}(\mu, 0, 0, T) . \]  

(6.1)

In the case with \( T = \mu \) (a rather simple equation of state) this is the effective energy–momentum tensor of an unperturbed string with string tension \( \mu \) as seen from distances much larger than the thickness of the string (a Goto–Nambu string).

The gravitational field around the cosmic string [neglecting terms of order \((G\mu)^2\)] is found by solving the linearized Einstein equations with the above \( T_{\mu \nu} \). One gets

\[ h_{00} = h_{33} = 4G(\mu - T) \ln(r/r_0), \]  

(6.2)

\[ h_{11} = h_{22} = 4G(\mu + T) \ln(r/r_0), \]  

(6.3)

where \( h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu} \) is the metric perturbation, the radial distance from the string is \( r = (x^2 + y^2)^{1/2} \), and \( r_0 \) is a constant of integration.

For an ideal, straight, unperturbed string, the tension and mass per unit length are \( T = \mu = \mu_0 \) and one gets

\[ h_{00} = h_{33} = 0, \quad h_{11} = h_{22} = 8G\mu_0 \ln(r/r_0). \]  

(6.4)

By a coordinate transformation one can bring this metric to a locally flat form

\[ ds^2 = dt^2 - dz^2 - dr^2 - (1 - 8G\mu_0)r^2 d\phi^2, \]  

(6.5)

which describes a conical and flat (Euclidean) space with a wedge of angular size \( \Delta = 8\pi G\mu_0 \) (the deficit angle) removed from the plane and with the two faces of the wedge identified.

### 6.3 The Abelian Higgs Vortex

The Lagrangian of Einstein gravity in the presence of electromagnetic and abelian Higgs field is

\[ \mathcal{L} = \mathcal{R} - 2\Lambda - F_{\mu \nu}F^{\mu \nu} + \mathcal{L}_{\text{Higgs}} . \]  

(6.6)
Figure 6.1: Cosmic strings affect surrounding spacetime by removing a small angular wedge which is called deficit angle $\Delta$, ($\Delta \approx 10^{-5}$ radian), creating a conelike geometry.
where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, with electromagnetic field $A_{\mu}$. The $L_{\text{Higgs}}$ is the Lagrangian of Higgs field which defines as follow

$$L_{\text{Higgs}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} |D_{\mu} \Phi|^2 - \xi \left( |\Phi|^2 - \eta^2 \right)^2. \quad (6.7)$$

The matter content of the abelian Higgs system consists of the complex Higgs field, $\Phi$, and a $U(1)$ gauge field $B_{\nu}$ with strength, $F_{\mu\nu} = \partial_{[\mu}B_{\nu]}$. Both the Higgs scalar and the gauge field become massive in the broken symmetry phase. The gauge covariant derivative is $D_{\mu} = \nabla_{\mu} + ieB_{\mu}$, where $\nabla_{\mu}$ is the spacetime covariant derivative. The parameter $\eta$ is the symmetry breaking energy scale and $\xi$ is the Higgs coupling. These can be related to the Higgs mass by $m_{\text{Higgs}} = 2\eta\sqrt{\xi}$. There is another relevant mass scale, i.e., that of the vector field in the broken phase, $m_{\text{vector}} = \sqrt{2}e\eta$. On length scales smaller than $m_{\text{vector}}^{-1}$, $m_{\text{Higgs}}^{-1}$, the vector and Higgs field behave as essentially massless. It is also convenient to define the Bogomolnyi parameter $\beta = 2\xi/e^2 = m_{\text{Higgs}}^2/m_{\text{vector}}^2$. This Lagrangian is invariant under the action of the Abelian group $G = U(1)$. This is the Abelian Higgs model [105]. The fields in $L_H$ will be treated as ‘test field’, i.e., their energy momentum tensor is supposed to yield a negligible contribution to the source of the gravitational field. Notice that we have two different gauge fields, $F$ and $F$, and each is treated in a different manner. It is only $F$ that couples to the Higgs scalar field and is therefore subject to spontaneous symmetry breaking. The other gauge field, $F$, can be thought of as the free massless Maxwell field which apart from modifying the background geometry, its dynamic will be of little concern to us here. For detailed study about the Higgs mechanism, we refer to [96], [106] and [107]. In this part, we use units in which $8\pi G = c = 1$.

The action (6.7) is invariant under the following transformations,

$$\Phi \rightarrow \Phi e^{i\Lambda(x)}, \quad B_{\mu} \rightarrow B_{\mu} - \nabla_{\mu}\Lambda(x), \quad (6.8)$$

which is spontaneously broken in the ground state, $\Phi = \eta e^{i\Lambda_0}$. Besides this ground state, another solution, the vortex, is present when the phase of $\Phi(x)$
is a non-single valued quantity. To better describe this, define the real fields
$X, P_\mu, \omega(x)$, by

$$\Phi(x^\mu) = \eta X(x^\mu) e^{i\omega(x^\mu)}, \quad (6.9)$$

$$B_\mu(x^\mu) = \frac{1}{e} [P_\mu(x^\mu) - \nabla_\mu \omega(x^\mu)]. \quad (6.10)$$

The flux of Higgs gauge field $B_\mu$ is quantized by identifying the vortex line. The flux is given by

$$\Phi_H = \int \mathcal{F}_{\mu\nu} d\sigma^{\mu\nu} = \oint B_\mu(x^\mu) dx^\mu. \quad (6.11)$$

Using the Eq. (6.10), then the integration (6.11) without any $P$ field is

$$\Phi_H = -\frac{1}{e} \oint \nabla_\mu \omega(x) dx^\mu. \quad (6.12)$$

The line integral over the gradient of phase $\Phi$ does not necessarily vanish. The only requirement on the phase is that $\Phi$ is single valued, i.e., $\oint d\omega = 2\pi N$ ($N$ is an integer). In this case a vortex is present. The integer $N$ is called the winding number of the vortex. If $N \neq 0$, and if the spatial topology is trivial, then, by continuity, the integration loop must encircle a point of unbroken symmetry ($X = 0$), namely, the vortex core. Therefore the flux of Higgs gauge field can be obtained as:

$$\Phi_H = -\frac{2\pi N}{e}. \quad (6.13)$$

Thus, the flux of vortex lines is quantized, and $-2\pi/e$ being the quantum.

The field equations that follow by varying $X$ in the action (6.7) and employing a suitable choice of gauge, are

$$\nabla_\mu \nabla^\mu X - XP_\mu P^\mu - 4\xi \eta^2 X(X^2 - 1) = 0, \quad (6.14)$$

while by varying $B_\mu$ one finds

$$\nabla_\mu \tilde{F}^{\mu\nu} - 4\pi e^2 \eta^2 P_\nu X^2 = 0. \quad (6.15)$$

By varying the action with respect to $g_{\mu\nu}$, one obtains

$$G_{\mu\nu} - \frac{3}{f^2} g_{\mu\nu} = (T_{\mu\nu}^{em} + T_{\mu\nu}^{Higgs}). \quad (6.16)$$
where $\tilde{F}^{\mu\nu} = \nabla^\mu P^\nu - \nabla^\nu P^\mu$ is the field strength of the corresponding gauge field $P^\mu$, and $T^\text{em}_{\mu\nu}$ and $T^\text{Higgs}_{\mu\nu}$ are the stress energy tensors of the electromagnetic and Higgs fields given by

$$T^\text{em}_{\mu\nu} = 2 F^\sigma_{\mu} F^\sigma_{\nu} - \frac{1}{2} F^2 g_{\mu\nu}$$

$$T^\text{Higgs}_{\mu\nu} = \eta^2 \nabla_\mu X \nabla_\nu X + \eta^2 X^2 P_\mu P_\nu + \frac{1}{4\pi e^2} F^\mu_\sigma F^\nu_\sigma + g_{\mu\nu} L_H. \quad (6.18)$$

The field $\omega$ is not dynamical. Some of the static black holes may be dressed by the vortices of Nielsen-Olesen type [99] appear as cylindrically symmetric solutions,

$$\Phi = X(r_c)e^{iN_\phi}, \quad P_\phi = NP(r_c), \quad (6.19)$$

where $r_c$ is the cylinder radial coordinate, and all other components of $P_\mu$ are zero.

The Abelian Higgs hair for rotating charged black string will be studied in the next chapter.
Chapter 7

Abelian Higgs Hair for Charged Rotating black string

The metric of the charged rotating black string which had been considered in Chapter 4, can be rewritten as:

\[ ds^2 = -\Gamma (\Xi dt - ad\phi)^2 + \frac{r^2}{\ell^4} (adt - \Xi \ell^2 d\phi)^2 + \frac{dr^2}{\Gamma} + \frac{r^2}{\ell^2} dz^2, \]  

(7.1)

where \( \Gamma = f(r) \) and the other parameters were defined in chapter 4 (see Eqs.(4.4) and (4.7).

We seek a cylindrically symmetric solution for the Higgs field equations (6.14) and (6.15) in the background of a charged rotating black string. Thus, we assume that the fields \( X \) and \( P_\mu \) are functions of \( r \). As mention in the previous chapter, for the case of vanishing rotation parameter (static case), one can choose the Nielsen-Olesen type gauge field as \( P_\mu (r) = (0, 0, Np(r), 0) \). Indeed, the field equations (6.14) and (6.15) reduces to two equations for the two unknown functions \( X(r) \) and \( P(r) \). Here, for stationary case (non vanishing rotation parameter), the field equations (6.14) and (6.15) reduces to three equations and therefore one may use the following gauge choice

\[ P_\mu (r) = (S(r), 0, NP(r), 0). \]  

(7.2)

\[ ^1 \text{The paper related to this subject has been submitted to Can. J. Phys. (see Ref. [108])} \]
The field equations (6.14) and (6.15) reduce to

\[ r^2 \Gamma X'' + \left( 4 \ell^{-2} r^3 - b \ell \right) X' - 4 r^2 X \left( X^2 - 1 \right) - X \left( a S + \Xi N P \right)^2 \]
\[ - r^2 \Gamma^{-1} X \left( \Xi S + Na \ell^{-2} P \right)^2 = 0, \]  
(7.3)

\[ N \left( r^3 \ell^2 \Xi^2 - r^5 a^2 \ell^{-2} \right) P'' + N \left( 2 r^4 + b \ell^3 r \Xi^2 - 2 \lambda^2 \ell^4 \Xi^2 \right) P' \]
\[ + \ell^2 a \Xi \left( \lambda^2 \ell^2 - b \ell r \right) \left( r S'' - S' + \alpha r X^2 \Gamma^{-1} S \right) - a \Xi \lambda^2 \ell^4 S' \]
\[ + N \alpha r^3 X^2 \left( \ell^2 \Xi^2 - r^2 a^2 \ell^{-2} \Gamma^{-1} \right) P = 0, \]  
(7.4)

\[ r^3 \left( \Xi^2 r^2 - \Gamma a^2 \right) S'' + \left[ 2 r^4 - a^2 (b \ell r - 2 \lambda^2 \ell^2) \right] S' + \alpha^2 r^3 \Gamma^{-1} \left( \Xi^2 r^2 - \Gamma a^2 \right) \times \]
\[ X S - N \Xi a \left( r P'' - P' + \alpha r X^2 \Gamma^{-1} P \right) \left( \lambda^2 \ell^2 - b \ell r \right) - \lambda^2 \ell^2 P' \] = 0,  
(7.5)

where \( \alpha = 4 \pi c^2 / \xi \) and the prime denotes a derivative with respect to \( r \). It is worthwhile to mention that even in the pure flat or (anti-)de Sitter spacetimes no exact analytic solutions are known for equations (6.14) and (6.15) coupled with (6.16). Even in situation of no electromagnetic charge or any horizon. Thus we should try to solve these coupled differential equations approximately. In the first order approximation, we solve Eqs. (6.14) and (6.15) in the background of charged rotating black string and then we will carry out numerical calculations to solve Eq. (6.16). For asymptotically AdS spacetimes, it had showed that the Abelian Higgs equations of motion in the background of charged black string spacetime (static case) have vortex solution [41]. Here we want to investigate the influence of rotation parameter on the vortex solutions.

### 7.1 Numerical solutions

We pay attention now to the numerical solutions of Eqs. (7.3)-(7.5) outside the black string horizon. First, we must take appropriate boundary conditions. Since at a large distance from the horizon the metric (7.1) reduces to AdS spacetime, we demand that our solutions go to the solutions of the vortex equations in AdS spacetime given in [100]. This requires that we demand \((X \to 1, P \to 0)\) as \( r \) goes to infinity and \((X = 0, P = 1)\) on the horizon. For consistency with the non-rotating case [41], we take \( S = 0 \) on the horizon. Also
one may note that the electric field, $\tilde{F}_{tr}$ which is proportional to $S'$ should be zero as $r$ goes to infinity. We employ a grid of points $r_i$ with division $dr$, where $r_i$ goes from $r_H$ to some large value of $r$ ($r_\infty$) which is much greater than $r_H$. We rewrite Eqs. (7.3)-(7.5) in a finite deference language and use the successive over relaxation method [109] to calculate the numerical solutions of $X(r)$, $P(r)$ and $S(r)$ for different values of the rotation parameter and winding number. The numerical results of calculations are shown in Figs. (7.1)-(7.9). In these figures, first, we investigate the influence of rotation parameter on the solutions of field equations (7.3)-(7.5). We carry out all the calculation for $\ell = 1$, $\lambda = 0.2$ and $b = 0.3$ for which the radius of horizon is $r_H = 0.6173$. Figures (7.1) and (7.2) show the behavior of the electric, $E_{\text{Higgs}} = \tilde{F}_{tr}$, and the magnetic, $H_{\text{Higgs}} = \tilde{F}_{\phi r}$, fields associated with the field $P_{\mu}$ respectively for different values of the rotation parameters. We use the subscript ‘Higgs’ for these electromagnetic field to emphasis that they are coupled with the Higgs scalar field $\Phi$. As one can see in Fig. (7.1), $E_{\text{Higgs}}$ is zero for $a = 0$, and becomes larger as $a$ increases. The magnetic field $H_{\text{Higgs}}$ is plotted in Fig. (7.2) for different values of angular momentum. This figure shows that the variation of $H_{\text{Higgs}}$ with respect to the rotation parameter is very slow. Overall, these figures show that the vortex thickness decreases as the rotation parameter increases. As we mentioned, in the case of non vanishing rotation parameter we encounter with the electric field $E_{\text{Higgs}}$. One may compute the source of this electric field through the use of Gauss law numerically. It is notable that the computation of this electric type charge for $E_{\text{Higgs}}$ shows that it increases as $a$ becomes larger. This is analogous to the rotating solutions of Einstein-Maxwell equation discussed in the context of cosmic string theory for which the electric charge of the string is proportional to the rotation parameter of the string [110]. The effect of rotation on field $X(r)$ also is shown in Figs. (7.3).

Next, we investigate the influence of the winding number $N$ on the solutions of field equations (7.3)-(7.5) for the case of rotating charged black string. The
results for $a = 0.5$ are shown in Figs. (7.4)-(7.6) for different values of $N$. As in the case of asymptotically flat, dS, and AdS spacetimes considered in Refs. [32, 41, 39, 100, 101], increasing the winding number yields a greater vortex thickness.

The effects of charge per unit length, $\lambda$, or mass per unit length, $b$, parameters on the solutions of field equations (7.3)-(7.5) for the case of rotating charged black string are shown in Figs. (7.7)-(7.9). As one can see from Fig. (7.7), the vortex thickness decreases slowly by increasing the charge per unit length and from Fig. (7.8), the dependence of $H_{\text{Higgs}}$ on $\lambda$ is very small (almost negligible), but $E_{\text{Higgs}}$ will become stronger by increasing the charge per unit length (see Fig. (7.9)).

### 7.2 Asymptotic Behavior of the Solutions of Einstein-Maxwell-Higgs Equation

In previous section we found the solutions of Higgs field equation in the background of charged rotating black string. Here, we want to solve the coupled Einstein-Maxwell-Higgs differential equation (6.14)-(6.16). This is a formidable problem even for flat or AdS spacetimes, and no exact solutions have been found for these spacetimes yet. Indeed, besides the electromagnetic stress energy tensor, the energy-momentum tensor of the Higgs field is also a source for Einstein equation (6.16). However, some physical results can be obtained by making some approximations. First, we assume that the thickness of the skin covering the black string is much smaller than all the other relevant length scales. Second, we assume that the gravitational effects of the Higgs field are weak enough so that the linearized Einstein-Abelian Higgs differential equations are applicable. We choose $g_{\mu\nu} \simeq g_{\mu\nu}^{(0)} + \varepsilon g_{\mu\nu}^{(1)}$, where $g_{\mu\nu}^{(0)}$ is the rotating charged black string metric in the absence of the Higgs field and $g_{\mu\nu}^{(1)}$ is the first order correction to the metric. Employing the two assumptions concerning the thickness of the vortex and its weak gravitational field, the first
order approximation to Einstein equation (6.16) can be written as:

\[ G_{\mu\nu}^{(1)} - \frac{3}{\ell^2} g_{\mu\nu}^{(1)} = T_{\mu\nu}^{(0)} , \]

(7.6)

where \( G_{\mu\nu}^{(1)} \) is the first order correction to the Einstein tensor due to \( g_{\mu\nu}^{(1)} \) and \( T_{\mu\nu}^{(0)} \) is the energy-momentum tensor of the Higgs field in the rotating charged black string background metric with components:

\[
\begin{align*}
T_t^{(0)}(r) &= \{-\Gamma_1 r^4 \left(r^2 \Xi^2 - \alpha a^2 \right) S^2 - \alpha r^2 \ell^4 X^2 - N^2 \Gamma (\Xi^2 \ell^4 \Gamma - a^2 r^2) P^2 \\
&- \alpha N^2 X^2 \ell^4 \Gamma P^2 - \alpha a^2 \left(\Gamma \ell^2 - r^2 \right) \left(N^2 P^2 - S^2 \ell^2\right) X^2 - \alpha r^2 \ell^4 X^2 S^2 \\
&- 2 \alpha r^2 \ell^4 \Gamma \left(X^2 - 1 \right)^2 \}/2r^2 \ell^4 \Gamma ,
\end{align*}
\]

\[
\begin{align*}
T_\phi^{(0)}(r) &= \{ \Gamma_1 r^4 \left(-\Gamma a^2 + r^2 \Xi^2 \right) S^2 - \alpha r^2 \ell^4 \Gamma^2 X^2 + N^2 \Gamma (\Xi^2 \ell^4 \Gamma - a^2 r^2) P^2 \\
&+ \alpha \ell^4 X^2 \left(N^2 \Gamma P^2 + r^2 S^2\right) + \alpha X^2 a^2 \left(\Gamma \ell^2 - r^2 \right) \left(N^2 P^2 - \ell^2 S^2\right) \}
&- 2 \alpha \Gamma r^2 \ell^4 \left(X^2 - 1 \right)^2 \}/2r^2 \ell^4 \Gamma ,
\end{align*}
\]

\[
\begin{align*}
T_\phi^{(0)}(r) &= \{ 2 N^2 \ell^2 \Gamma a \Xi \left(\Gamma \ell^2 - r^2 \right) P^2 + 2 \Gamma \ell^4 N \left(\Gamma a^2 - r^2 \Xi^2 \right) P^2 S' - r^2 \ell^2 PS \\
&+ 2 \ell^2 N X^2 \alpha \left(P a \left(\Gamma \ell^2 - r^2 \right) \left(aS + \Xi NP\right)\right) \}/2r^2 \ell^4 \Gamma ,
\end{align*}
\]

\[
\begin{align*}
T_r^{(0)}(r) &= \{ \Gamma_1 r^4 \left(\Gamma a^2 - \Xi^2 \ell^2 \right) S^2 + \alpha r^2 \ell^4 \Gamma^2 X^2 + N^2 \Gamma \left(\Xi^2 \ell^4 \Gamma - a^2 r^2 \right) P^2 \\
&+ 2 a \Xi \ell^2 N \Gamma \left(\Gamma \ell^2 - r^2 \right) P^2 S' - 2 \alpha r^2 \ell^4 \Gamma \left(X^2 - 1 \right)^2 - \alpha X^2 a^2 \times \\
&\left(\Gamma \ell^2 - r^2 \right) \left(N^2 \Gamma P^2 + \ell^2 S^2\right) - 2 \alpha \ell^2 X^2 Na \Xi S P \left(\Gamma \ell^2 - r^2 \right) \\
&- \alpha \ell^4 X^2 \left(N^2 \Gamma P^2 - \ell^2 S^2\right) \}/2r^2 \ell^4 \Gamma ,
\end{align*}
\]

\[
\begin{align*}
T_z^{(0)}(r) &= \{ \Gamma_1 r^4 \left(-\Gamma a^2 + r^2 \Xi^2 \right) S^2 - \alpha r^2 \ell^4 \Gamma^2 X^2 - N^2 \Gamma \left(\Xi^2 \ell^2 \Gamma - a^2 r^2 \right) P^2 \\
&- 2 a \ell^2 \Xi N \Gamma \left(\Gamma \ell^2 - r^2 \right) P^2 S' - 2 \alpha r^2 \ell^4 \Gamma \left(X^2 - 1 \right)^2 - \alpha X^2 a^2 \times \\
&\left(\Gamma \ell^2 - r^2 \right) \left(N^2 \Gamma P^2 + \ell^2 S^2\right) - 2 \alpha \ell^2 X^2 Na \Xi S P \left(\Gamma \ell^2 - r^2 \right) \\
&- \alpha \ell^4 X^2 \left(N^2 \Gamma P^2 - \ell^2 S^2\right) \}/2r^2 \ell^4 \Gamma ,
\end{align*}
\]

(7.7)

where \( X, P, \) and \( S \) are the solutions of the Abelian Higgs system. The behavior of \( T_{\mu\nu}(r) \) is shown in Fig. (7.10). As we mentioned in the last section, the vortex thickness decreases as the rotation parameter increases. This fact is more clear in Fig. (7.11) which shows \( T_t^{(0)} \) for various values of \( a \).

For convenience, we use the following form of the metric which has cylin-
spherical symmetry

\[ ds^2 = -\tilde{A}(r)dt^2 + \tilde{B}(r)dr^2 + \tilde{C}(r)d\tau d\phi + \tilde{D}(r)d\phi^2 + \tilde{E}(r)dz^2. \]  

(7.8)

In order to solve numerically Eq. (7.6), it is better to write the metric function \( \tilde{A}(r) \) to \( \tilde{E}(r) \) as

\[
\begin{align*}
\tilde{A}(r) &= A_0(r)[1 + \varepsilon A(r)], \\
\tilde{B}(r) &= B_0(r)[1 + \varepsilon B(r)], \\
\tilde{C}(r) &= C_0(r)[1 + \varepsilon C(r)], \\
\tilde{D}(r) &= D_0(r)[1 + \varepsilon D(r)], \\
\tilde{E}(r) &= E_0(r)[1 + \varepsilon E(r)],
\end{align*}
\]

(7.9)

where \( A_0(r) = \Gamma \Xi^2 - r^2a^2\ell^{-4} \), \( B_0(r) = \Gamma^{-1} \), \( C_0 = 2a\Xi(\Gamma - r^2\ell^{-2}) \), \( D_0(r) = r^2\Xi^2 - \Gamma a^2 \), \( E_0(r) = r^2\ell^{-2} \), yielding the metric of the stationary rotating charged black string in four dimensions. The Einstein equations (7.6) in terms of the functions \( A(r) \) to \( E(r) \) are given in the Appendix C. Here we want to obtain the behavior of these functions for large values of the coordinate \( r \).

As one can see from Fig. (7.10), the components of the energy-momentum tensor rapidly go to zero outside the skin, so the situation is the same as what happened in the static black string spacetime considered in [41]. One can solve the linearized Einstein equation for large values of \( r \) numerically. The results which are displayed in Fig. (7.12) show that \( A(r) = B(r) = E(r) = 0 \), and \( 2C(r) = D(r) = 2 \). Hence the metric (7.8) can be written as

\[ ds^2 = -A_0(r)dt^2 + B_0(r)dr^2 + (1 + \varepsilon)C_0(r)d\tau d\phi + (1 + 2\varepsilon)D_0(r)d\phi^2 + E_0(r)dz^2. \]

(7.10)

It is worthwhile to mention that the metric (7.10) is the first order solution in \( \varepsilon \) of the Einstein-Maxwell-Higgs equations far from the thin string. Of course, one may note that the metric (7.10) is the first order approximation of the following metric

\[ ds^2 = -\Gamma(\Xi dt - a\gamma d\phi)^2 + \frac{r^2}{\ell^4} (adt - \Xi\ell^2\gamma d\phi)^2 + \frac{dr^2}{\Gamma} + \frac{r^2}{\ell^2} dz^2. \]

(7.11)
Figure 7.1: $E_{Higgs} \times 10^3$ versus $r$ for $N = 1$, $a = 0$ (touch the horizontal axis), 0.25 (dotted), 0.5 (solid) and 0.7 (bold).

which is the exact solution of Einstein-Maxwell gravity. In Eq. (7.11), $\gamma$ is defined as $\gamma = 1 + \varepsilon$. The above metric describes a stationary rotating charged black string with a deficit angle $\Delta = 2\pi \varepsilon$. The size of deficit angle $\Delta$ is proportional to $2\pi \int r T_{t}^{(0)} dr$ [32]. Numerical computation shows that the absolute values of this integral decreases as the rotation parameter increases, which can also been from Fig. (7.11). So, using a physical Lagrangian based model, we have established that the presence of the Higgs field induces a deficit angle in the rotating charged black string metric which decreases as the rotation parameter increases.
Figure 7.2: $H_{Higgs}$ versus $r$ for $N = 1$, $a = 0$ (solid) and 0.7 (bold).

Figure 7.3: $X(r)$ versus $r$ for various $a$ (all curves touch each other).

Figure 7.4: $E_{Higgs} \times 10^3$ versus $r$ for $N = 1$ (dotted), 3 (solid), and 5 (bold).
Figure 7.5: $H_{Higgs}$ versus $r$ for $N = 1$ (dotted), 3 (solid), and 5 (bold).

Figure 7.6: $X(r)$ versus $r$ for $N = 1$ (dotted), 3 (solid), and 5 (bold).
Figure 7.7: $X_{\text{Higgs}}$ versus $r$ for fix $r_+ = 0.6173$, $N = 1$, $a = 0.25$ and $\lambda = 0.2$ (dotted), $0.4$ (solid) and $0.5$ (bold).

Figure 7.8: $H_{\text{Higgs}}$ versus $r$ for fix $r_+ = 0.6173$, $N = 1$, $a = 0.25$ and various $\lambda$ (the dependence on $\lambda$ is very small).
Figure 7.9: $E_{Higgs} \times 10^3$ versus $r$ for fix $r_+ = 0.6173$, $N = 1$, $a = 0.25$ and $\lambda = 0.2$ (dotted), 0.4 (solid) and 0.5 (bold).
Figure 7.10: $T_l^{t(0)} = T_z^{z(0)}$ (dotted), $T_\phi^{\phi(0)}$ (solid), $T_\varphi^{\varphi(0)}$ (bold) and $T_r^{r(0)}$ (thick-bold), versus $r$ for $N = 1, a = 0.5$.

Figure 7.11: $|T_t^{t(0)}|$ versus $r$ for $a = 0$ (solid) and $a = 0.7$ (bold).
Figure 7.12: $A(r)$, $B(r)$ and $E(r)$ touch the horizontal axis, $C(r)$ (solid) and $D(r)$ (bold).
Chapter 8

Conclusion

In this thesis, two aspects of asymptotically charged rotating black branes in various dimensions have been studied. In part I, the thermodynamics of these spacetimes has been investigated, while in the second part the no-hair theorem for the four-dimensional case of these spacetimes has been considered. In chapter 1 of this thesis we give a brief review of the conjecture of AdS/CFT correspondence, which presents an equivalence between a gravitational theory in an \((n+1)\)-dimensional AAdS spacetime and a conformal field theory on the \(n\)-dimensional boundary of the bulk. This conjecture furnished a means for calculating the action and thermodynamic quantities intrinsically without reliance on any reference spacetime. Chapter 2 devoted to a brief review of the classical laws of thermodynamics of black holes in Einstein gravity, and the analogy between thermodynamic of black holes and thermodynamics in thermal physics. Also, the quantum aspect of black holes was discussed, and it was described that the laws of thermodynamics of black holes is not only a mathematical analogy, but have also physical interpretation. In chapter 3, we focus on the asymptotically charged rotating black string with zero curvature horizon, and the maximal analytical extension of this solution was studied. In chapter 4, the thermodynamics of asymptotically charged rotating black branes in various dimensions, which first was introduced by Awad [27], was studied. We calculated the conserved quantities and the Euclidean actions of
charged rotating black branes both in canonical and grand-canonical ensemble through the use of counterterms renormalization procedure. Also we obtained the charge and electric potential of the black brane in an arbitrary dimension. We found that the logarithmic divergencies associated to the Weyl anomalies and matter field are zero. We obtained a Smarr-type formula for the mass as a function of the extensive parameters $S$, $J$ and $Q$, calculated the temperature, the angular velocity, and the electric potential, and showed that these quantities satisfy the first law of thermodynamics. Using the conserved quantities and the Euclidean actions, the thermodynamics potentials of the system in the canonical and grand-canonical ensemble were calculated. We found that the Helmholtz free energy, $F(T, J, Q)$, is a Legendre transformation of the mass with respect to $S$ and the Gibbs potential is a Legendre transformation of the mass with respect to $S$, $J$ and $Q$ in the grand-canonical ensemble.

Also, we studied the phase behavior of the charged rotating black branes in $(n + 1)$ dimensions and showed that there is no Hawking-Page phase transition in spite of the angular momentum of the branes. Indeed, we calculated the heat capacity and the determinant of the Hessian matrix of the Gibbs potential with respect to $S$, $J$ and $Q$ of the black brane and found that they are positive for all the phase space, which means that the brane is stable for all the allowed values of the metric parameters. This analysis has also be done through the use of the determinant of the Hessian matrix of $M(S, J, Q)$ with respect its extensive variables and we got the same phase behavior. This phase behavior is in commensurable with the fact that there is no Hawking-Page transition for black object whose horizon is diffeomorphic to $\mathbb{R}^p$ and therefore the system is always in the high temperature phase [17].

Finally, we obtained the logarithmic correction of the entropy due to the thermal fluctuation around the thermal equilibrium. For the case of uncharged rotation black brane, we found that only a term which is proportional to $\ln(\text{area})$ will appear. But we found that for the charged rotating black brane, the correction contains other powers of the ‘area’ including the logarithmic
In part II of this thesis, we investigated the no-hair theorem for the four-dimensional AAdS charged rotating black string. In chapter 6, we studied the cosmological defects and the theory of the Abelian Higgs field for an arbitrary spacetime. We obtained the field equations for Einstein-Maxwell-Higgs system in the background of a stationary rotating charged black string. Since there is no analytic solutions for Einstein-Maxwell-Higgs system, even for the flat spacetimes, we attempted to solve them numerically. We obtained the numerical solutions for various values of rotation parameter and found that for a fixed horizon radius, by increasing the rotation parameter the vortex thickness decrease very slowly. Also the numerical solutions for various values of winding number and charge per unit length \( \lambda \), were obtained. These solutions shows that the vortex thickness increases as the winding number increases and is unchanged for various \( \lambda \) with constant \( r_+ \).

The main difference between the case of Abelian Higgs field in the background of static black string considered in [41], and this work is that the time component of the gauge field coupled to the Higgs scalar field is not zero for non zero rotation parameter. Indeed, we found that for the case of rotating black string, there exist an electric field coupled to the Higgs scalar field. This electric field increases as the rotation parameter becomes larger. Numerical calculations show that the electric charge which creates this electric field grows up as the rotation parameter increases. This is analogous to the results that Dias and Lemos have found recently for the magnetic rotating string [110]. They showed that the charge per unit length of a rotating string in the Einstein-Maxwell gravity increases as the rotation parameter becomes larger, and we found that the electric charge of the field \( F'_{\mu\nu} \) coupled to the Higgs field has the same feature.

Also, the effect of a thin vortex on pure AdS spacetime was studied. By including the self-gravity of a thin vortex in the rotating charged black string background in the first order approximation, we found out that the effect of a
thin vortex on the stationary charged rotating black string is to create a deficit angle in the metric, as in the case of pure AdS [19], Schwarzschild-AdS [20], Kerr-AdS, and Reissner-Nordström-AdS [40] spacetimes. We found that the deficit angle decreases as the rotation parameter increases.
Appendix A

The Symmetries of Anti-de Sitter Spacetimes and The Conformal Field Theory

Information on the geometrical properties of AdS spaces can be found in most advanced text books on general relativity (see for example Ref. [111]). Field theories on AdS spaces have been considered first in Refs. [112, 113].

On the other hand, conformal field theories in dimensions $n > 2$ have enjoyed a growing attention after great successes of the case $n = 2$. Earlier studies can be found in [114], and more recent relevant references are [115].

Here, we give a brief review of AdS space which focuses on two points. First, an AdS space is explicitly constructed. Secondly, the AdS symmetries are found, and the symmetry algebra is represented in a form which reveals its isomorphism to the conformal algebra.

The review of the basics of CFTs first recalls the definition of conformal transformations and the explicit expressions for conformal transformations of Euclidean space. Secondly, the expressions for the symmetry algebra operators acting on quasi-primary conformal fields are given. Quasi-primary conformal fields are important, because they form the basic field content of any CFT.
A.1 The Geometry of AdS$_{n+1}$

As is well known, an AdS space is a maximally symmetric space which can be represented as a hyperboloid embedded into a higher dimensional Minkowski space. The following considerations apply to the AdS spaces with Euclidean signature, but many results notably those regarding the symmetry algebra, can be straightforwardly carried over to the general case. Let the dimensions of the AdS and the embedding Minkowski spaces be $n+1$ and $n+2$, respectively, and let the embedding be defined by

$$y^A y^B \eta_{AB} = -\ell^2, \quad y^{-1} > 0,$$

(A.1)

where $\ell$ is the ‘radius’ of the hyperboloid, and $A, B = -1, 0, \ldots, n$. The Minkowski metric tensor is given by

$$\eta_{-1-1} = -1, \quad \eta_{\mu\nu} = \delta_{\mu\nu}, \quad \text{and} \quad \eta_{-1\mu} = 0$$

(A.2)

(with $\mu = 0, 1, \ldots n$). The metric

$$ds^2 = dy^A dy^B \eta_{AB}$$

(A.3)

readily represents the AdS metric, if one takes the coordinates $y^\mu$ as AdS coordinates and define $y^{-1}$ via equation (A.1).

It is useful to introduce a new set of AdS coordinates by

$$x^0 = \frac{\ell^2}{y^0 + y^{-1}} \quad \text{and} \quad x^i = \frac{x^0 y^i}{\ell}.$$  

(A.4)

The domain of the new variables is given by $0 < x^0 < \infty$, $x^i \in \mathbb{R}(i = 1, \ldots, n)$. More over the AdS metric (A.3) take the form

$$ds^2 = \frac{\ell^2}{(x^0)^2} \delta_{\mu\nu} dx^\mu dx^\nu.$$  

(A.5)

Obviously, AdS$_{n+1}$ is an open space, i.e. it does not possess a boundary. However, it is useful to consider the boundary of the coordinate patch, namely the conformally compactified Euclidean space given by $x^0 = 0$ plus the single point $x_0 = \infty$, as a pseudo boundary, which will be called the AdS horizon.
For completeness the expressions for the affine connections, the curvature tensor, Ricci tensor and curvature scalar are provided here:

\[
\Gamma^\mu_{\nu\lambda} = \frac{1}{x_0} \left( \delta^\mu_0 \delta_{\nu\lambda} - \delta^\mu_\lambda \delta_{0\nu} - \delta^\mu_\nu \delta_{0\lambda} \right),
\]
(A.6)

\[
R^\mu_{\nu\lambda\rho} = \frac{1}{x_0^2} \left( \delta^\mu_\rho \delta_{\nu\lambda} - \delta^\mu_\lambda \delta_{\nu\rho} \right),
\]
(A.7)

\[
R_{\mu\nu} = -\frac{n}{x_0^2} \delta_{\mu\nu},
\]
(A.8)

\[
R = -n(n+1)\ell^{-2}.
\]
(A.9)

### A.1.1 The Symmetry Group:

The definition (A.1) of anti–de Sitter space is invariant under transformations of the embedding Minkowski space of the form \((y')^A = R^A_B y^B\), where the \((n + 2) \times (n + 2)\) matrix \(R\) satisfies \(R^T \eta R = \eta\) and \(R^{-1} > 0\). The group of such matrices consists of two subsets, one being the Lie group \(SO(n + 1, 1)\), whereas the other can be represented by \(I \times SO(n + 1, 1)\) with an inversion \(I\). In the first symmetry, One can introduce a conformal basis of \(SO(n + 1, 1)\) and show that the conformal algebra and \(SO(n + 1, 1)\) algebra are isomorphic [116].

In the second set of symmetries, one encounters the inversion \(I\), whose action on the coordinates \(y^A\) of the embedding Minkowski space can be defined by the matrix

\[
I^A_B = \delta^A_B - 2\delta^A_0 \delta_B^0.
\]
(A.10)

Obviously,

\[
II = 1,
\]
(A.11)

which must hold in every representation. Using Eq. (A.4), the transformation induced on the \(x^\mu\) coordinates is found to be

\[
x'^\mu = l^2 \frac{x^\mu}{x^2}.
\]
(A.12)
A.2 Basics of Conformal Field Theory in $n$-Dimensions

Let $g_{ij}$ be the metric tensor of some $n$–dimensional manifold with respect to some coordinates $x^i$. A transformation $x \rightarrow x'(x)$, under which the metric tensor changes as

$$g'_{ij}(x) = [\lambda(x)]^2 g_{ij}(x)$$  \hspace{1cm} (A.13)

is called a conformal transformation. Conformal transformations are a generalization of ordinary symmetry transformations, in which every symmetry transformation satisfies Eq. (A.13) with $\lambda(x) = 1$.

Given such a transformation $x \rightarrow x'$, one can define the transformation matrix

$$R^\mu_\nu(x) = \lambda(x) \frac{\partial x'^\mu}{\partial x^\nu}$$  \hspace{1cm} (A.14)

which satisfies

$$g'_{\mu\nu}(x') R^\mu_\nu(x) R^\rho_\lambda(x) = g'_{\nu\lambda}(x)$$  \hspace{1cm} (A.15)

Obviously, the conformal transformations form a group.

A.2.1 Conformal Symmetry

A very important case, of conformal symmetry which has been studied, is the case of conformal symmetry on a flat manifold, which will be assumed to be Euclidean here. Let $g_{ij} = \delta_{ij}$. Then, for $n > 2$, the transformations satisfying equation (A.13) are [117]:

\[ \]
dilatations: \( x'^i = cx^i \)
translations: \( x'^i = x^i + b^i \)
rotations: \( x'^i = R^i_j x^j \)
inversion: \( x'^i = \frac{x^i}{x^0} \),

as well as any combination of the above. It is assumed that \( c > 0 \) and \( R \in O(n) \).

It has been shown by a direct calculations that the infinitesimal versions of the transformations (A.16) are identical with the symmetry transformations of \( AdS_{n+1} \) restricted to its horizon, \( x_0 = 0 \). Hence, the same is true for the finite transformations connected to the identity. Moreover, the inversion is identical to the AdS inversion formula (A.12) restricted to \( x_0 = 0 \). Thus, the AdS symmetry transformations directly correspond to conformal transformations of the AdS horizon.
Appendix B

Boundaries of the Spacetime

Let spacetime be an \((n + 1)\)-dimensional Lorentzian manifold equipped with a metric \(g_{\mu\nu}\). We consider a region of this manifold, \(\mathcal{M}\), and we define various tensors on the boundary. We require that the region \(\mathcal{M}\) have the topology of the direct product of a spacelike hypersurface, \(\Sigma\), with a real (timelike) interval. This requirement allows me to foliate the manifold into leaves \(\Sigma_t\) of constant foliation parameter \(t\). The vector \(u^\mu\) is the future-directed timelike vector to \(\Sigma_t\). The boundary of the region \(\mathcal{M}\) is the union of ‘initial’ and ‘final’ hypersurfaces, \(\Sigma_i\) and \(\Sigma_f\), and the timelike hypersurface \(\partial B\). This timelike hypersurface can also be foliated into the spacelike quasilocal surfaces \(B_t\). The (spacelike) outwards normal vector to the boundary element \(\partial B\) is \(n^\mu\) while the bi-normal to the quasilocal surface \(B\) is \(n^{\mu\nu} = 2u^{[\mu}n^{\nu]}\). For simplicity, we enforce the condition \(g_{\mu\nu}u^\mu n^\nu = 0\) and \(g_{\mu\nu}n^\mu n^\nu = 1\). In Fig (B.1) you can see these definitions.

B.1 The Timelike Boundary

Now we examine the geometry on the timelike boundary \(\partial B\). The two fundamental forms on \(\partial B\) are the induced metric \(\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu\), and the extrinsic curvature \(\Theta_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}\). The restriction of the induced metric to the boundary \(\partial B\) can be viewed as the physical metric on the \(n\)-dimensional manifold
Figure B.1: The manifold, its boundaries and the normal vectors to these boundaries.

$3 B$. Alternately, one can view the operator $\gamma^{\mu \nu}$ as a projection operator that will take a vector on the tangent space of $\mathcal{M}$ to a vector on the tangent space of $3 B$. The derivative operator compatible with the metric $\gamma_{\mu \nu}$ is $D$.

The second fundamental form can be thought of as the failure of the vectors $n^\mu$ to coincide when parallel-transported along the boundary $3 B$. The definition of extrinsic curvature yields

$$
\Theta_{\mu \nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu \nu}
$$

$$
= -\frac{1}{2} \mathcal{L}_n (g_{\mu \nu} - n_{\mu} n_{\nu})
$$

$$
= -(\nabla_{\mu} n_{\nu} - n^{\lambda} n_{\mu} \nabla_{\lambda} n_{\nu})
$$

$$
= -\gamma^{\alpha}_{\mu} \nabla_{\alpha} n_{\nu},
$$

which is the difference between the normal vector and the parallel transport (along $3 B$) of a nearby normal. In the third line we have used the hypersurface orthogonality of the unit vector, the definition of the Lie derivative, and the compatibility of the metric with the derivative operator $\nabla$. From this viewpoint, the second fundamental form describes how curved the surface $3 B$ is.
B.2 The Spacelike Boundary

On the spacelike boundary $\Sigma$, the induced metric is $h_{ab} = g_{ab} + u_a u_b$ while the extrinsic curvature of $\Sigma$ embedded in $\mathcal{M}$ is $K_{ab} = -\frac{1}{2} \mathcal{L}_u h_{ab} = -h^c{}_a \nabla_c u_b$. Here, the operator $h^c{}_a$ is a projection operator onto the tangent space of $\Sigma$.

Define the lapse function as the normalization of the unit normal $u^a$ relative to the vector $t^a$: $N = -t^a u_a = (u^a \nabla_a t)^{-1}$. The shift vector $V^a = h^a{}_b t^b$ is the projection of the vector $t^a$ onto the surface $\Sigma$. Then the vector $t^a$ can be decomposed into a portion normal to $\Sigma$ and a portion tangent to $\Sigma$: $t^a = Nu^a + V^a$.

Let $\mathcal{D}$ be the derivative operator compatible with the metric $h_{ab}$. It is straightforward to show that $\mathcal{D}_a T^{b...c}_{d...e} = h^b{}_f ... h^j{}_e h^k{}_a \nabla_k T^{f...g}_{i...j}$. By the Gauss-Codacci relations which are covariant expressions of the bulk Einstein tensor $G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$ in terms of the boundary Einstein tensor $G_{ab}(h)$ (which only depends on the induced metric $h_{ab}$) and the extrinsic curvature $K_{ab}$, we can write

$$R_{abcd}[h] = h^b{}_a h^c{}_b h^d{}_e R_{jklm}[g] + 2K_{d[a} K_{b]c}$$

(B.2)

and

$$2\mathcal{D}_b K^b{}_a = n^c h^d{}_a R_{cd}[g].$$

(B.3)

An immediate consequence of (B.2) is

$$2u^a u^b G_{ab}[g] = h^{ac} h^{bd} R_{abcd}[g]$$

(B.4)

$$= R[h] + K^2 - K^{ab} K_{ab}.$$ 

B.3 The Quasilocal Surface

Because $u^a$ and $n^e$ are taken to be orthogonal, the quasilocal surface $B$ can be viewed either as the boundary of $\Sigma$ or as a leaf in the foliation of $^3B$. The
induced metric on $B$ is

$$\sigma_{ab} = g_{ab} + u_a u_b - n_a n_b. \quad (B.5)$$

The extrinsic curvature of $B$ embedded in $\Sigma$ is $k_{ab} = \sigma^c D_c n_b$. A straightforward analysis yields the following relationship between the various extrinsic curvatures [8]:

$$\Theta_{ab} = k_{ab} + u_a u_b n_c u^d \nabla_d u^c + 2\sigma^c (u^b) n^d K_{cd} \quad (B.6)$$

Thus, the projection of $\Theta_{ab}$ onto $B$ is the extrinsic curvature $k_{ab}$. The quantities $\Theta_{ab}$ and $K_{ab}$ are related by $\sigma^a c ^b \Theta_{ab} = -\sigma^a c ^b K_{ab}$. Finally, the trace of (B.6) yields

$$\Theta = k - n_c a^c \quad (B.7)$$

where $\Theta$ and $k$ are the traces of the extrinsic curvatures $\Theta_{ab}$ and $k_{ab}$ respectively.

Variations of the induced metric, $\gamma_{ab}$, on the timelike boundary $^3B$ can be decomposed into pieces that are normal-normal, normal-tangential, and tangential-tangential to the quasilocal surface $B$:

$$\delta \gamma_{ab} = \sigma_a^c \sigma_b^d \delta \sigma_{cd} - 2 \frac{2}{N} u_a u_b \delta N - 2 \frac{2}{N} u_{(a} \sigma_{b)c} \delta N^c. \quad (B.8)$$

The variation of the metric on the quasilocal surface can also be decomposed into a variation of the square-root of the determinant, $\sqrt{\sigma}$, plus a variation of the conformally invariant part of the metric $\zeta_{ab}$:

$$\delta \sigma_{ab} = \frac{2}{n - 2} \left( \frac{\sigma_{ab}}{\sqrt{\sigma}} \right) \delta \sqrt{\sigma} + \left( \sqrt{\sigma} \right)^{\frac{2}{n-2}} \delta \zeta_{ab}, \quad (B.9)$$

where $\zeta_{ab} = (\sqrt{\sigma})^{-2/(n-2)} \sigma_{ab}$. The second term in equation (B.9) represents changes in the ‘shape’ of the quasilocal surface that preserve the determinant whereas the changes in the determinant (given by the first term) reflect a change in the ‘size’ of the quasilocal surface while maintaining the same shape.
Appendix C

The Einstein Equations in Terms of the Functions $A, \ldots, E$

The Einstein equation mentioned in Sec. (7.2) in terms of the metric functions $A(r)$ to $E(r)$ are

$$
\Xi^2 F_1 (A + D - 2 C - 4 \lambda \ell^4 a^2 r^2 (r^2 a^2) D + 4 \lambda^2 \ell^8 \Xi^2 \Gamma^2 \times \\
(\Gamma a^2 - r^2 \Xi^2) A - r^4 \ell^4 \Gamma^2 \{4 \lambda \ell^4 \Xi^2 \Gamma (b \ell^3 - 4 \lambda^3 a^2)\} B' - \Xi^2 a^2 \ell^8 r \Gamma^2 \times \\
\{2 b^2 r^2 + \lambda^2 (4 \lambda^2 \ell^2 - 6 b \ell r)\} A' - 12 \ell^4 r^6 \Gamma^2 B - \Xi^2 r^2 \ell^2 \Gamma^2 F_2 (A' + D' - 2 C') \\
+ \ell^2 \Gamma^2 r^4 \{4 \lambda \ell^4 \Gamma \Xi^2 + (4 \lambda - b \ell^3) (a^2 - \ell^2)\} (E' + D') + 2 \Gamma^3 r^6 \ell^6 (E'' + D'') \\
+ \Gamma^2 \ell^4 a^2 \{2 \Xi^2 \ell^4 [b^2 r + \lambda^2 (2 \lambda \ell^2 r^{-1} - 3 b \ell)] + r^3 (4 \lambda^2 \ell^2 - 3 b \ell r)\} D' \\
- 2 \Xi^2 a^2 \lambda^2 \ell^6 r \Gamma^2 (\Gamma r \ell^2 - r^3 - b \ell^3) (D'' - C'') = 4 r^6 \ell^6 \Gamma^2 \mathcal{T}_t^{(0)}(0),
$$

$$
8 \Gamma^2 (\Gamma \ell^2 - r^2) \lambda^2 \Xi^6 a^2 D + 8 \lambda^2 \Xi^6 \ell^6 a^2 \{\ell^2 r^2 - (\Gamma \ell^2 - r^2) a^2\} C \\
+ a^2 \Xi^3 F_3 (A + D - 2 C) + r^4 \ell^4 a \Xi^2 (4 \lambda \ell^4 \Gamma - 4 r^3 + b \ell^3) (B' - E' - 2 D') \\
- r^2 \ell^2 a \Xi \Gamma F_5 (r, b, \lambda) (A' - D') - 2 r^2 \ell^2 \Xi a \Gamma F_4 (C' - D') + 2 \ell^4 \Gamma^2 \Xi a \times \\
\{\ell^5 \ell^2 (\Xi^2 - 1) \Gamma (\lambda^2 \ell - b r) + \Xi^2 r^6 (r^2 - \Gamma \ell^2)\} (D'' - C'') = 4 r^6 \ell^6 \Gamma^2 \mathcal{T}_t^{(0)},
$$
\[ F_6 (A + D - 2C) + 8 a^2 \lambda^2 \ell^6 \Gamma^2 \{ \ell^2 \Xi^2 - r^2 (-1 + \Xi^2) \} (C - A) + 4r^2 \ell^6 \times \lambda \Gamma^2 (\Xi^2 \ell^2 + a^2) A + r^2 \Gamma \ell^{-4} \Xi^2 F_7 (A' + D' - 2C') - 4 \Xi^2 a^2 \ell^6 r \Gamma^2 \times \{ b^2 r \ell^2 + \lambda^2 (2 \Gamma r \ell^2 - b \ell^3 - 2r^3) \} C' + \ell^4 r^2 \Gamma^2 F_8 A' + r^4 \ell^4 \Gamma^2 (4r^3 - b \ell^3) E' + r^4 \ell^4 \Gamma^2 \{ 4r a^2 \Gamma - \Xi^2 (4r^3 - b \ell^3) \} (B' - E') + 2r^6 \ell^6 \Gamma^3 (E'' + A'') - 2 \Xi^2 a^2 \lambda^2 \ell^6 r \Gamma^2 (\Gamma \ell^2 r - r^3 - b \ell^3) (A'' - C'') - 12 \ell^4 r^6 \Gamma^2 B = 4r^6 \ell^6 \Gamma^2 T_\varphi^{(0)}; \]

\[ \Xi^2 F_9 (A + D - 2C) - 8 r^2 \lambda^2 \Xi^2 \ell^6 \Gamma C + 4r^2 \lambda^2 \ell^4 \Gamma (\Xi^2 \ell^2 - a^2) D + r^4 \ell^4 \Gamma (4r^3 - b \ell^3) E' + r^4 \ell^4 \Gamma \{ 4r a^2 \Gamma - \Xi^2 (4r^3 - b \ell^3) \} D' + r^4 \Gamma \{ 4r \ell^4 \Xi^2 \Gamma - a^2 (4r^3 - b \ell^3) \} A' + \Xi^2 a^2 b \ell^3 r^2 \Gamma (\Gamma \ell^2 - r^2) \times (A' + D' - 2C') - 12 r^6 \ell^2 \Gamma B = 4r^6 \ell^4 \Gamma T_\varphi^{(0)}; \]

\[ \Xi^2 F_{10} (A + D - 2C) - 16r^2 \lambda^2 \ell^6 \Gamma^2 (\Xi^2 \ell^2 - a^2) D + (32 \lambda^2 \Xi^2 \ell^8 r^2 \Gamma^2) C - 4 \ell^4 r^4 \Gamma^2 (4r^3 - b \ell^3) B' + 4 \ell^4 r^2 \Gamma^2 \{ 8r^3 \ell^2 \Gamma + (4\lambda^2 \ell^2 - 3b \ell r) \left( a^2 - 2 \ell^2 \right) \} D' + 4 \ell^2 r^2 \Gamma \Xi^2 F_{11} (A' + D' - 2C') + \{ 8r^4 + (3 \Xi^2 - 2) b \ell^2 r - 4 \lambda^2 \ell^4 \Xi^2 \} \times 4 \ell^4 r^3 \Gamma^2 A' - 8 \Xi^2 a^2 \lambda^2 \ell^6 r \Gamma^2 (r \ell^2 \Gamma - r^3 - b \ell^3) (A'' + D'' - 2C'') + 8 r^6 \ell^6 \Gamma^3 \times (A'' + D'') - 48 \ell^4 r^6 \Gamma^2 B = 16 r^6 \ell^6 \Gamma^2 T_\varphi^{(0)}; \]
where $T_{\mu}^{(0)}$ is the energy momentum tensor of the Higgs field given in Eqs. (7.7) and the functions $F_i$'s are:

\begin{align*}
F_1 &= \Xi^4(32 \Gamma^3 \ell \ell r^4 - \Gamma^2 \ell \ell^2 \ell^6 - 32 \Gamma \ell^3 \ell^2 r^4 b + 8 \Gamma^2 \ell^7 r^3 b \\
&\quad+ 16 r^10 - 16 \Gamma^4 \ell^6 r^2 - 8 r^7 b \ell^3 + 8 \Gamma \ell^5 r^5 b) + \Xi^2(2 \Gamma^2 \ell^4 r^2 + 12 \Gamma^4 \ell^2 r^2 \\
&\quad- 28 \Gamma^3 \ell^6 r^4 - 12 \Gamma^2 \ell^4 r^6 - 60 \Gamma \ell^2 r^8 - 32 r^10 + 16 r^7 b \ell^3 - 12 \Gamma \ell^5 r^5 b \\
&\quad+ 4 \Gamma^3 \ell^6 r b - 8 \Gamma^2 \ell^7 r^3 b - 2 r^10 b^2 \ell^6 + 4 \Gamma^4 \ell^8 r^2 + r^4 b^2 \ell^6 - 8 r^7 b \ell^3 \\
&\quad+ 4 \Gamma^3 \ell^6 r b + 16 r^10 + 4 \Gamma \ell^5 r^5 b + 12 \Gamma^2 \ell^4 r^6 - 28 \Gamma \ell^2 r^8 - 4 \Gamma^3 \ell^6 r^4),
\end{align*}

\begin{align*}
F_2 &= \Xi^4(4r^7 - 4 \Gamma^3 \ell \ell^6 r - b \ell^3 r^4 - b \ell^7 \Gamma^2 + 2 b \ell^5 \Gamma r^2 + 12 \Gamma^2 \ell^4 r^3 - 12 \Gamma \ell^2 r^5) \\
&\quad+ \Xi^2(2 r^4 b \ell^3 - 8 r^7 - b \ell^7 \Gamma^2 + 16 r^5 \Gamma \ell^2 - 8 r^3 \Gamma^2 \ell^4 - r^2 b \ell^3 \Gamma) - r^4 b \ell^3 \\
&\quad+ 4 r^7 + 2 \Gamma^2 \ell^7 b - \Gamma \ell^5 r^2 b + 4 \Gamma^3 \ell^6 r - 4 \Gamma^2 \ell^4 r^3 - 4 \Gamma \ell^2 r^5,
\end{align*}

\begin{align*}
F_3 &= \Xi^2(32 \Gamma\ell^2 r^8 - 32 \Gamma^3 \ell \ell^6 r^4 + 8 r^7 \ell b^3 - r^4 b^2 \ell^6 - 8 \Gamma^2 \ell^7 r^3 b - 8 \Gamma \ell^5 r^5 b \\
&\quad+ 8 \Gamma^3 \ell^6 r b + \Gamma^2 \ell^4 r^2 - 16 r^10 + 16 \Gamma^4 \ell^8 r^2 + 8 \Gamma \ell^3 r^5 b + 16 r^10 \\
&\quad+ 32 \Gamma^3 \ell^6 r^4 + 8 \Gamma^2 \ell^7 r^3 b - 8 \Gamma^3 \ell^9 r b - 16 \Gamma^4 \ell^8 r^2 - 32 \Gamma \ell^2 r^8 \\
&\quad+ r^4 b \ell^6 - 8 r^7 b \ell^3 - b^2 \ell^{10} \Gamma^2,
\end{align*}

\begin{align*}
F_4 &= \Xi^4(4r^7 - b \ell^7 \Gamma^2 - b \ell^3 r^4 + 12 \Gamma^2 \ell^4 r^3 - 12 \Gamma \ell^2 r^5 - 4 \Gamma^3 \ell^6 r + 2 b \ell^5 \Gamma r^2) \\
&\quad+ \Xi^2(8 r^5 \Gamma \ell^2 - b \ell^7 \Gamma^2 + 4 r^7 - 4 r^3 \Gamma^2 \ell^4 + r^4 b \ell^3) \\
&\quad+ 2 \Gamma^2 \ell^7 b + 4 \Gamma^3 \ell^6 r - 4 \Gamma^2 \ell^4 r^3,
\end{align*}

\begin{align*}
F_5 &= \Xi^4(4 \Gamma^3 \ell^6 r - 4 r^7 + b \ell^7 \Gamma^2 - 2 b \ell^5 \Gamma r^2 - 12 \Gamma^2 \ell^4 r^3 + 12 \Gamma \ell^2 r^5 + b \ell^3 r^4) \\
&\quad+ \Xi^2(2 r^2 b \ell^5 \Gamma - b \ell^7 \Gamma^2 + 4 r^7 + 12 r^3 \ell^4 \ell - 12 r^5 \Gamma \ell^2 - 4 \Gamma^3 \ell^6 r - r^4 b \ell^3) \\
&\quad- \Gamma \ell^5 r^2 b - 4 \Gamma^2 \ell^4 r^3 + 4 \Gamma \ell^2 r^5,
\end{align*}

\begin{align*}
F_6 &= \Xi^6(b^2 \ell^{10} \Gamma^2 + 16 \Gamma^4 \ell^8 r^2 - 32 r^4 \Gamma^3 \ell^6 + 32 r^8 \Gamma \ell^2 + 8 r^7 b \ell^3 + 8 \Gamma^3 \ell^9 r b \\
&\quad- b^2 \ell^9 r^4 - 8 r^5 \Gamma \ell^5 b - 16 r^10 - 8 r^3 \Gamma^2 \ell^7 b) + \Xi^4(30 r^4 \Gamma^3 \ell^6 \\
&\quad+ 4 b \ell^3 r^7 + 6 b \ell^7 \Gamma^2 r - 8 b \ell^3 \Gamma^3 r - 18 r^8 \Gamma \ell^2 + 6 b \ell^5 \Gamma r^5 - 4 r^6 \ell^2 \ell^4 + 8 r^{10} \\
&\quad- b^2 \ell^{10} \Gamma^2 + 16 \Gamma^4 \ell^8 r^2) + 4 \Xi^2(r^6 \Gamma^2 \ell^4 - 6 r^4 \Gamma^3 \ell^6 - r^5 \Gamma \ell^5 b + 2 \Gamma^3 \ell^9 r b \\
&\quad+ r^8 \Gamma \ell^2 + 4 \Gamma^4 \ell^8 r^2 + \Gamma^2 \ell^{10} b^2 / 4 + 4(\Gamma^2 \ell^4 r^6 - \Gamma^3 \ell^6 r^4 - \Gamma^2 \ell^2 r^3 b),
\end{align*}

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\[ \begin{align*}
F_7 &= \Xi^4(11 r b \ell^5 \lambda^4 - 10 r^2 b^2 \ell^4 \lambda^2 - 4 \lambda^6 \ell^6 + 3 r^3 b^3 \ell^3) + \Xi^2(3 r^6 b^2 - 7 r^5 b \ell \lambda^2 \\
&\quad - 6 r^3 b^3 \ell^3 - 22 r b \ell^5 \lambda^4 + 8 \lambda^6 \ell^6 + 4 r^4 \lambda^4 \ell^2 + 20 r^2 b^2 \ell^4 \lambda^2) - 3 r^6 b^2 \\
&\quad + 7 r^5 b \ell \lambda^2 - 4 r^4 \lambda^4 \ell^2 + 11 r b \ell^5 \lambda^4 + 3 r^3 b^3 \ell^3 - 4 \lambda^6 \ell^6 - 10 r^2 b^2 \ell^4 \lambda^2, \\
F_8 &= 4 \Gamma \ell^2 \r^3 + 2 \ell^3 \Xi^2 b + 8 r^3 \Xi^2 \Gamma \ell^2 - 4 \ell^3 \Xi^1 b r^2 + 8 \Xi^4 r^5 + 4 \ell^3 \Xi^4 b \Gamma \\
&\quad - 4 \ell^6 \Xi^2 b \Gamma + 8 \Xi^4 \Gamma^2 \ell^4 r - 16 \Xi^4 \Gamma \ell^2 r^3 - 8 \Xi^2 \Gamma^2 \ell^4 r + 4 r^5 - r^2 b \ell^3, \\
F_9 &= \Xi^2(12 r^4 \Gamma^2 \ell^4 - 24 r^6 \Gamma \ell^2 - 4 \ell^7 b \Gamma^2 r - \ell^8 b \Gamma + 4 r^3 b \ell^5 \Gamma + 12 r^8) \\
&\quad 16 r^4 \Gamma^2 \ell^4 + 28 r^6 \Gamma \ell^2 + b^2 \ell^8 \Gamma - 8 r^3 \Gamma \ell^5 b + 4 b \ell^7 \Gamma^2 r - 12 r^8, \\
F_{10} &= 24 \Gamma^3 \ell^9 r b - 16 \Gamma^2 \ell^7 r^3 b + \Xi^2(76 \Gamma^3 \ell^6 r^4 - 40 \Gamma^4 \ell^8 r^2 + 20 \Gamma^2 \ell^7 r^3 b \\
&\quad - 48 \Gamma^2 \ell^4 r^6 + 28 \Gamma \ell^2 r^8 - 16 r^{10} - 24 \Gamma^3 \ell^9 r b - 2 \Gamma^2 \ell^{10} b^2 - 4 \Gamma \ell^5 r^5 b \\
&\quad - r^4 b^2 \ell^6 + 8 r^7 b \ell^3) + 40 \Gamma^4 \ell^8 r^2 - 72 \Gamma^3 \ell^6 r^4 + 44 \Gamma^2 \ell^4 r^6 + 16 r^{10} \\
&\quad + 4 \ell \ell^5 r^5 b - 28 \Gamma \ell^2 r^8 + r^4 b^2 \ell^6 - 8 r^7 b \ell^3 + 2 b^2 \ell^{10} \Gamma^2, \\
F_{11} &= \Xi^2(4 r^7 - r^4 b \ell^3 - 28 r^3 \Gamma^2 \ell^4 - 5 r^2 b \ell^2 \Gamma + 6 b \ell^7 \Gamma^2 + 8 r^5 \Gamma \ell^2 + 16 \Gamma^3 \ell^6 r) \\
&\quad + r^4 b \ell^3 - 8 \Gamma \ell^2 r^5 + 5 \ell^5 r^2 b - 4 r^7 - 6 \Gamma^2 \ell^7 b + 28 \Gamma^2 \ell^4 r^3 - 16 \Gamma^3 \ell^6 r.
\end{align*}\]
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