Approximation of Some Classes of Set-Valued Periodic Functions by Generalized Trigonometric Polynomials

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We generalize some known results on the best, best linear, and best one-sided approximations by trigonometric polynomials from the classes of $2\pi$-periodic functions represented in the form of convolutions to the case of classes of set-valued functions.

1. Introduction

Results of the exact solution of problems of the best, best linear, and best one-sided approximations of the classes of periodic functions represented in the form of convolutions by trigonometric polynomials are well known in the approximation theory. A survey and presentation of the major part of results obtained in this direction and the corresponding references can be found in the works [8–11] and in the monographs [3, 4]. The aim of the present paper is to generalize some of these results to the case of set-valued functions.

The investigations of the problems of approximation of set-valued functions have a relatively short history. For the survey and presentation of the results in this field, see [8–11].

We now briefly describe the structure of the present paper. The required definitions, notation, and facts for numerical sequences of periodic functions are presented in Sec. 2. All necessary definitions and results from the theory of set-valued functions can be found in Sec. 3. In Sec. 4, we pose the problems of approximation of set-valued functions. Some results of approximation of the set-valued periodic functions represented in the form of convolutions by “generalized trigonometric polynomials” are given in Sec. 5.

2. Approximation of the Classes of Numerical Functions

Let $C$ and $L_p$, $1 \leq p \leq \infty$, be spaces of $2\pi$-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with the norms $\| \cdot \|_C$ and $\| \cdot \|_{L_p}$, respectively.

Also let $X$ be $L_p$, $1 \leq p < \infty$, or $C$ and let $H$ be a finite-dimensional subspace of the space $X$. For $f \in X$, we set

$$E(f, H)_X = \inf_{T \in H} \| f - T \|_X.$$  \hspace{1cm} (1)

Let $\mathcal{M} \subset X$ be a class of functions and let

$$E(\mathcal{M}, H)_X = \sup_{f \in \mathcal{M}} E_n(f, H)_X.$$  \hspace{1cm} (2)

Quantities (1) and (2) are called the best approximations of the function $f$ and the class $\mathcal{M}$, respectively, by the subspace $H$ in the metric of the space $X$.

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Further, we set
\[
U(\mathcal{M}, H)_X = \inf_A \sup_{f \in \mathcal{M}} \|f - Af\|_X,
\]
where \(\inf_A\) is taken over all possible linear operators \(A : X \to H\). The quantity \(U(\mathcal{M}, H)_X\) is called the best linear approximation of the class \(\mathcal{M}\) by the subspace \(H\) in the metric of the space \(X\).

As usual, we define the convolution \(K * \varphi\) of functions \(K \in L_1\) (the kernel of the convolution) and \(\varphi \in L_1\) by the equality
\[
K * \varphi(x) = \int_0^{2\pi} K(t) \varphi(x-t) dt.
\]
Let
\[
F_p = \{\varphi \in L_p, 1 \leq p \leq \infty : \|\varphi\|_p \leq 1\}.
\]
By \(K * F_p\) we denote a class of functions of the form
\[
f(x) = K * \varphi(x), \quad \varphi \in F_p.
\]

It is known (see, e.g., [1–3, 5, 6]) that many important classes of numerical periodic functions are classes of the type \(K * F_p\).

By \(H^2_{2n-1}\), \(n = 1, 2, \ldots\), we denote the set of trigonometric polynomials \(T_{n-1}(x)\) of degree at most \(n - 1\), i.e., the set of functions of the form
\[
T_{n-1}(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos kx + b_k \sin kx, \quad a_k, b_k \in \mathbb{R}.
\]

In [1], Nikol’skii established quite general conditions for the kernel \(K\) specifying a class of functions for each of which it is possible to find the quantities
\[
E(K * F_\infty, H^T_{2n-1})_C \quad \text{and} \quad E(K * F_1, H^T_{2n-1})_{L_1}.
\]

Here, we present only the condition \(N^*_n\):

We say that a kernel \(K\) satisfies the condition \(N^*_n\) if there exist a polynomial \(T^* \in H^T_{2n-1}\) and a point \(\theta \in [0, \pi/n]\) such that
\[
(K(x) - T^*(x)) \varphi_n(x - \theta) \geq 0
\]
almost everywhere.

Here and in what follows,
\[
\varphi_n(x) := \text{sgn} \sin nx.
\]

Nikol’skii also proved the following theorem:
Theorem A. If a kernel $K$ satisfies the condition $N_n^*$, then

$$E(K * F \infty, H_{2n-1})_C = U(K * F \infty, H_{2n-1})_C = E_n(K * F_1, H_{2n-1})_{L_1}$$

$$= U_n(K * F_1, H_{2n-1})_{L_1} = E(K, H_{2n-1})_{L_1}$$

$$= \|K - T\|_{L_1} = \|K * \varphi_n\|_C.$$ 

Almost all kernels important for the approximation theory satisfy the condition $N_n^*$ (for examples, see [1, 2, 5, 6]).

As for the best linear approximations of the classes of periodic functions, we present the following theorem, which can be proved by analogy with Theorem 1 in [7]:

Theorem B. Let $p, q \in [1, \infty]$ and $p^{-1} + q^{-1} = 1$. If $K \in L_q$, then, for any $n \in \mathbb{N}$,

$$U(K * F_p, H_{2n-1})_C = E(K, H_{2n-1})_{L_q}.$$ 

3. Definitions and Facts for the Set-Valued Functions

We present necessary definitions and facts for the spaces of nonempty compact subsets of the space $\mathbb{R}^m$ and set-valued functions. For the proofs of the facts presented in what follows, see [12–14]. By $K(\mathbb{R}^m)$ we denote the space of nonempty compact subsets of the space $\mathbb{R}^m$. Further, by $K^c(\mathbb{R}^m)$ we denote the collection of convex elements of the space $K(\mathbb{R}^m)$. We consider set-valued $2\pi$-periodic functions $f: \mathbb{R} \rightarrow K(\mathbb{R}^m)$, i.e., functions $f$ such that $f(x + 2\pi) = f(x)$ for any $x \in \mathbb{R}$.

As usual, we define a linear combination of sets $A, B \subseteq K(\mathbb{R}^m)$ by the equality

$$\lambda A + \mu B = \{\lambda a + \mu b: a \in A, b \in B\}, \quad \lambda, \mu \in \mathbb{R}.$$ 

By $coA$ we denote the convex hull of the set $A \subseteq K(\mathbb{R}^m)$.

If $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$, then $|a|_2 := \sqrt{\sum_{j=1}^{m} a_j^2}$ and $(a, \xi) = \sum_{k=1}^{m} a_k \xi_k$ is the scalar product of elements $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$. For a point $a \in \mathbb{R}^m$ and a set $B \subseteq K(\mathbb{R}^m)$, we define

$$d(a, B) := \inf_{b \in B} |a - b|_2.$$ 

This is the distance from the point $a$ to the set $B$. For the sets $A, B \subseteq K(\mathbb{R}^m)$, we define

$$d(A, B) := \sup_{a \in A} d(a, B).$$ 

This is the distance from the set $A$ to the set $B$. The Hausdorff metric $\delta$ in the space $K(\mathbb{R}^m)$ is defined as follows:

If $A, B \subseteq K(\mathbb{R}^m)$, then

$$\delta(A, B) := \max\{d(A, B), d(B, A)\}.$$ 

Note that $K(\mathbb{R}^m)$ and $K^c(\mathbb{R}^m)$ equipped with the Hausdorff metric are complete metric spaces.
The metric \( \delta (A, B) \) has the following properties:

\[
\delta (\lambda A, \lambda B) = \lambda \delta (A, B) \quad \forall \lambda > 0 \quad \forall A, B \in K(\mathbb{R}^m),
\]

\[
\delta (A + B, C + D) \leq \delta (A, C) + \delta (B, D) \quad \forall A, B, C, D \in K(\mathbb{R}^m),
\]

\[
\delta (\operatorname{co} A, \operatorname{co} B) \leq \delta (A, B) \quad \forall A, B \in K(\mathbb{R}^m),
\]

\[
\delta (\alpha A, \beta A) \leq |\alpha - \beta| \|A\| \quad \forall \alpha, \beta \in \mathbb{R} \quad \forall A \in K^c(\mathbb{R}^m),
\]

where \( \|A\| := \delta (A, \{\theta\}) \) [here and in what follows, \( \theta = (0, \ldots, 0) \) is the null element of the space \( \mathbb{R}^m \)].

The Aumann integral [15] of a set-valued function

\[ f : [0, 2\pi] \rightarrow K(\mathbb{R}^m) \]

is defined as the set of all integrals of integrable selections of the function \( f \):

\[ I(f) = \left\{ \int_0^{2\pi} \phi(x)dx : \phi(x) \in f(x) \text{ almost everywhere; } \phi \text{ is integrable} \right\}. \]

It is known (see, e.g., [16]) that if a function \( f : [0, 2\pi] \rightarrow K(\mathbb{R}^m) \) is measurable and the function \( \|f(\cdot)\| \) is summable (the collection of these functions is denoted by \( L^A_1 \)), then

\[ \int_0^{2\pi} f(x)dx \in K^c(\mathbb{R}^m). \]

We need the following properties of the Aumann integral for the functions \( f, g \in L^A_1 \) (see, e.g., [13, 14]):

\[ \int_0^{2\pi} \operatorname{co}(f(x))dx = \int_0^{2\pi} f(x)dx; \]

\[ \delta \left( \int_0^{2\pi} f(x)dx, \int_0^{2\pi} g(x)dx \right) \leq \int_0^{2\pi} \delta(f(x), g(x))dx. \]

4. Problems of Approximation of Set-Valued Functions

By \( L^A_p, 1 \leq p \leq \infty \), we denote a collection of functions \( f \in L^A_1 \) such that \( \|f(\cdot)\| \in L^A_p \). In \( L^A_p \), we introduce a metric by setting

\[ \delta_{L^A_p} (f, g) := \|\delta(f(\cdot), g(\cdot))\|_{L^A_p}. \]

Let

\[ \Phi_p := \{ f \in L^A_p : \delta_{L^A_p} (f, \{\theta\}) \leq 1 \}. \]
As in Sec. 2, assume that a kernel $K \in L_1$ is given. In what follows, we consider the problems of approximation of the classes $K * \Phi_p$ of set-valued functions that can be represented in the form

$$f(x) = \int_0^{2\pi} K(x-t)g(t)dt, \quad g \in \Phi_p,$$

where the integral is understood in the Aumann sense. In view of the properties of the Aumann integral, the functions from the class $K * \Phi_p$ are convex-valued.

As approximating functions, we use set-valued functions of the form

$$\tau(x) = \int_0^{2\pi} T(x-t)h(t)dt, \quad (3)$$

where $T$ is a polynomial from $H_{2n-1}^T$ and $h$ is a function from $L_1^A$. By $SVH_{2n-1}^T$ we denote the collection of all possible functions of this type.

For a function $f \in K * \Phi_p, \ 1 \leq p \leq \infty$, we set

$$E(f, SVH_{2n-1}^T)_{L_p^A} = \inf_{\tau \in SVH_{2n-1}^T} \delta_{L_p^A}(f, \tau).$$

Let

$$E(K * \Phi_q, SVH_{2n-1}^T)_{L_p^A} = \sup_{f \in K * \Phi_q} E(f, SVH_{2n-1}^T)_{L_p^A}.$$

Parallel with the problem of determination of the quantities $E(K * \Phi_p, SVH_{2n-1}^T)_{L_p^A}$, which can be regarded as a set-valued analog of the problem of the best approximation of functions by trigonometric polynomials, we consider a problem of determination of the quantities

$$U(K * \Phi_p, SVH_{2n-1}^T)_{L_q^A} := \inf_{T \in H_{2n-1}^T} \sup_{f \in K * \Phi_p} \|K * g - T * g\|_{L_q^A}.$$

This problem is a set-valued analog of the problem of the best approximation of the corresponding classes of numerical functions.

We now somewhat extend the collection of approximating functions and, instead of the collection $SVH_{2n-1}^T$, consider a collection $\widetilde{SVH}_{2n-1}$ of set-valued functions of the form

$$\widetilde{\tau}(x) = \tau(x) + B_r(\theta), \quad (4)$$

where

$$\tau \in SVH_{2n-1}^T \quad \text{and} \quad B_r(\theta) = \{z \in \mathbb{R}^m : |z|_{L_2^m} \leq 1\}, \quad r \geq 0.$$

For a function $f \in K * \Phi_p, \ 1 \leq p \leq \infty$, we set

$$E^+(f, SVH_{2n-1}^T)_{L_p^A} = \inf_{\tau \in \widetilde{SVH}_{2n-1} \forall \ f(x) \subset \tau(x)} \delta_{L_p^A}(f, \tau).$$
The problem of determination (estimation) of the quantity

\[ \mathcal{E}^+ \left( \left( K \ast \Phi_p, SV H_{2n-1}^T \right) \right)_{L_p^A} = \sup_{f \in K \ast \Phi_p} \mathcal{E}^+ \left( f, SV H_{2n-1}^T \right)_{L_p^A} \]

is a set-valued analog of the problem of the best one-sided approximation of the classes of numerical functions and, in our opinion, is of significant interest.

5. Results

**Theorem 1.** Let \( n \in \mathbb{N}, \ p, q \in [1, \infty], \) and \( p^{-1} + q^{-1} = 1. \) If \( K \in L_q, \) then

\[ \mathcal{E} \left( K \ast \Phi_p, SV H_{2n-1}^T \right)_{L_q^A} \leq \mathcal{U} \left( K \ast \Phi_p, SV H_{2n-1}^T \right)_{L_q^A} \leq E_n \left( K, H_{2n-1}^T \right)_{L_q}. \]  \hspace{1cm} (5)

If \( K \in L_1, \) then

\[ \mathcal{E} \left( K \ast \Phi_p, SV H_{2n-1}^T \right)_{L_p^A} \leq \mathcal{U} \left( K \ast \Phi_p, SV H_{2n-1}^T \right)_{L_p^A} \leq E_n \left( K, H_{2n-1}^T \right)_{L_1} \]  \hspace{1cm} (6)

for \( p \in [1, \infty]. \)

**Proof.** We first prove that inequality (5) is true. Let \( T^* \in H_{2n-1}^T \) be a polynomial of the best \( L_q \)-approximation for \( K \) and let \( g \in \Phi_p. \) By using the properties of the Hausdorff metric and the integral, we estimate

\[ \delta(K \ast g(x), T^* \ast g(x)) \]

as follows:

\[ \delta(K \ast g(x), T^* \ast g(x)) = \delta \left( \int_0^{2\pi} K(x - t) g(t) dt, \int_0^{2\pi} T^*(x - t) g(t) dt \right) \]

\[ = \delta \left( \int_0^{2\pi} K(x - t) \cos g(t) dt, \int_0^{2\pi} T^*(x - t) \cos g(t) dt \right) \]

\[ \leq \int_0^{2\pi} \delta(K(x - t) \cos g(t), T^*(x - t) \cos g(t)) dt \]

\[ \leq \int_0^{2\pi} |K(x - t) - T^*(x - t)| \delta(\cos g(t), \{0\}) dt. \]

Hence,

\[ \delta(K \ast g(x), T^* \ast g(x)) \leq \int_0^{2\pi} |K(x - t) - T^*(x - t)| \delta(\cos g(t), \{0\}) dt. \]  \hspace{1cm} (7)
By virtue of the Hölder inequality, we get
\[
\|\delta(K * g(\cdot), T^* * g(\cdot))\|_{L\infty} \leq \max_{x \in \mathbb{R}} \left( \int_{0}^{2\pi} |K(x - t) - T^*(x - t)|^q dt \right)^{1/q} \left( \int_{0}^{2\pi} \|\delta (co g(t), \{\theta}\})|^p dt \right)^{1/p}
\]
\[
\leq E_n(K, H^T_{2n-1})_{L_q}.
\]

Relation (5) is true.

Now let \( K \in L_1 \), let \( f = K * g \in K * \Phi_p \), and let \( T^* \in H^T_{2n-1} \) be a polynomial of the best \( L_1 \)-approximation for \( K \). By using inequality (7) and the generalized Minkowski inequality, we obtain
\[
\|\delta(K * g(\cdot), T^* * g(\cdot))\| \leq \left\| \int_{0}^{2\pi} |K(\cdot - t) - T^*(\cdot - t)|\delta (co g(t), \{\theta\}) dt \right\|_{L_p}
\]
\[
\leq \|K - T^*\|_{L_1} \|g(\cdot)\|_{L^p} \leq E(K, H^T_{2n-1})_{L_1}.
\]

Therefore,
\[
\mathcal{E}(K * \Phi_p, SV H^T_{2n-1})_{L^p} \leq \mathcal{U}(K * \Phi_p, SV H^T_{2n-1})_{L^p} \leq E_n(K, H^T_{2n-1})_{L_1}.
\]

Relation (6) is true.

Theorem 1 is proved.

**Theorem 2.** If a kernel \( K(t) \) satisfies the condition \( N^*_n \), then, for \( p = 1 \) or \( p = \infty \), the following relation is true:
\[
\mathcal{E}(K * \Phi_p, SV H^T_{2n-1})_{L^p} = \mathcal{U}(K * \Phi_p, SV H^T_{2n-1})_{L^p} = E \left( K * F_p, H^T_{2n-1} \right)_{L_p} = \|K * \varphi_n\|_{L_\infty},
\]
where \( \varphi_n(x) = \text{sign} \sin nx, \ x \in \mathbb{R} \).

**Proof.** For \( p = 1 \) and \( p = \infty \), the estimate
\[
\mathcal{E}(K * \Phi_p, SV H^T_{2n-1})_{L^p} \leq \mathcal{U}(K * \Phi_p, SV H^T_{2n-1})_{L^p} \leq E \left( K, H^T_{2n-1} \right)_{L_1} = \|K * \varphi_n\|_{L_\infty}
\]
follows from Theorem 1 and Theorem A.

We first obtain the lower bound for \( p = \infty \). We choose an arbitrary \( a \in \mathbb{R}^m \) such that
\[
\delta^h(\{a\}, \{\theta\}) = |a|_\infty^m = 1.
\]
Then

\[ K \ast (\varphi_n(\cdot) \cdot \{a\}) = K \ast \varphi_n(\cdot) \cdot \{a\} \in \Phi_\infty. \]

Assume that \( \tau \in SVH_{2n-1}^T \) has the form (3) and that \( \psi \) is an arbitrary integrable selection from the function \( h \). Then

\[
\|\delta(K \ast \varphi_n(\cdot) \cdot \{a\}, \tau(\cdot))\|_{L_\infty} = \max_{x \in \mathbb{R}} \delta(K \ast \varphi_n(x) \cdot \{a\}, \tau(x)) \\
\geq \max_{x \in \mathbb{R}} d(\tau(x), K \ast \varphi_n(x) \cdot \{a\}) \\
= \max_{x \in \mathbb{R}} d\left(\int_0^{2\pi} T(x - t)\psi(t)dt, K \ast \varphi_n(x) \cdot \{a\}\right) \\
\geq \max_{x \in \mathbb{R}} \left|\int_0^{2\pi} T(x - t)\psi(t)dt - K \ast \varphi_n(x) \cdot \{a\}\right| \\
= \max_{x \in \mathbb{R}} \sup_{\xi \in \mathbb{R}^m, |\xi|_2 = 1} \left|\int_0^{2\pi} T(x - t)\psi(t)dt - K \ast \varphi_n(x) \cdot \{a\}\right| \\
\geq \sup_{\xi \in \mathbb{R}^m, |\xi|_2 = 1} |(\xi, a)| \max_{x \in \mathbb{R}} |K \ast \varphi_n(x)| = \|K \ast \varphi_n\|_{L_\infty}
\]

(the last inequality holds by virtue of the Chebyshev theorem on alternance). Therefore,

\[
\mathcal{E}(K \ast \Phi_\infty, SVH_{2n-1}^T)_{L_\infty} = \sup_{f \in \mathcal{K}^* \Phi_\infty, \tau \in SVH_{2n-1}^T} \inf_{f \in \mathcal{K}^* \Phi_\infty, \tau \in SVH_{2n-1}^T} \|\delta(f(\cdot), \tau(\cdot))\|_{L_\infty} \\
\geq \inf_{\tau \in SVH_{2n-1}^T} \|\delta(K \ast \varphi_n(\cdot) \cdot \{a\}, \tau(\cdot))\|_{L_\infty} \\
\geq \|K \ast \varphi_n\|_{L_\infty}.
\]

The lower bound and, hence, the assertion of the theorem for the case \( p = \infty \) are proved.

We now establish the lower estimate for \( p = 1 \). We choose an arbitrary \( a \in \mathbb{R}^m \) such that \( \delta(\{a\}, \{\theta\}) = |a|_2 = 1 \)
and any function \( g \in F_1 \). It is clear that \( g(\cdot) \cdot \{a\} \in \Phi_1 \) and
\[
f(\cdot) = K \ast g(\cdot) \cdot \{a\} \in K \ast \Phi_1.
\]

Assume that \( \tau \in SVH^T_{2n-1} \) has the form (3) and that \( \psi \) is an arbitrary integrable selection from the function \( h \). Then
\[
\|\delta(f(\cdot), \tau(\cdot))\|_{L_1} = \int_0^{2\pi} \delta(K \ast g(x) \cdot \{a\}, \tau(x)) \, dx
\]
\[
\geq \int_0^{2\pi} d(\tau(x), K \ast g(x) \cdot \{a\}) \, dx
\]
\[
= \int_0^{2\pi} \left( \int_0^{2\pi} T(x-t) h(t) \, dt \right) K \ast g(x) \cdot \{a\} \, dx
\]
\[
\geq \int_0^{2\pi} \int_0^{2\pi} T(x-t) \psi(t) \, dt - K \ast g(x) \cdot \{a\} \bigg|_{L^2}\, dx
\]
\[
= \int_0^{2\pi} \sup_{\xi \in \mathbb{R}^n \mid \xi \mid_2 = 1} \left| \left( \int_0^{2\pi} T(x-t) \psi(t) \, dt \right) - K \ast g(x) \cdot (\xi, a) \right| \, dx
\]
\[
= \int_0^{2\pi} \sup_{\xi \in \mathbb{R}^n \mid \xi \mid_2 = 1} \left| \int_0^{2\pi} T(x-t) (\xi, \psi(t)) \, dt - K \ast g(x) \cdot (\xi, a) \right| \, dx
\]
\[
\geq \sup_{\xi \in \mathbb{R}^n \mid \xi \mid_2 = 1} \int_0^{2\pi} \int_0^{2\pi} T(x-t) (\xi, \psi(t)) \, dt - K \ast g(x) \cdot (\xi, a) \bigg|_{1}\, dx
\]
\[
\geq \sup_{\xi \in \mathbb{R}^n \mid \xi \mid = 1} |(\xi, a)| E(K \ast g, H^T_{2n-1})_{L_1} = E(K \ast g, H^T_{2n-1})_{L_1}.
\]

Hence,
\[
E(K \ast g \cdot \{a\}, SVH^T_{2n-1})_{L^\Delta_1} \geq E(K \ast g, H^T_{2n-1})_{L_1}.
\]

In view of Theorem A, this implies that
\[
E(K \ast \Phi_1, SVH^T_{2n-1})_{L^\Delta_1} \geq E(K \ast F_1, H^T_{2n-1})_{L_1} = \|K \ast \varphi_n\|_{L_\infty}.
\]

Thus, the lower bound is established for \( p = 1 \).

Theorem 2 is proved.
The lower bound established for the case $p = 1$ admits the following generalization:

**Theorem 3.** For any kernel $K \in L_1$, any $p$, $q \in [1, \infty]$, $q \leq p$, and any $n \in \mathbb{N}$,

$$E(K \ast \Phi_p, SVH^{T}_{2n-1})_{L^q} \geq E(K \ast F_p, H^{T}_{2n-1})_{L^q}.$$

**Proof.** We choose an arbitrary $a \in \mathbb{R}^m$ such that $|a|_{\ell^2}^2 = 1$ and any function $g \in F_p$. It is clear that $g(\cdot) \cdot \{a\} \in \Phi_p$ and $f(\cdot) = K \ast g(\cdot) \cdot \{a\} \in K \ast \Phi_p$.

As in the proof of the previous theorem, assume that $	au \in SVH^{T}_{2n-1}$ has the form (3) and that $\psi$ is an arbitrary integrable selection from the function $h$. Then

$$\|\delta(f(\cdot), \tau(\cdot))\|_{L^q} = \left(\int_0^{2\pi} \delta(K \ast g(x) \cdot \{a\}, \tau(x))^q dx\right)^{\frac{1}{q}}$$

$$\geq \left(\int_0^{2\pi} d(\tau(x), K \ast g(x) \cdot \{a\})^q dx\right)^{\frac{1}{q}}$$

$$= \left(\int_0^{2\pi} \left(\int_0^{2\pi} T(x-t)h(t)dt, K \ast g(x) \cdot \{a\}\right)^q dx\right)^{\frac{1}{q}}$$

$$\geq \left(\int_0^{2\pi} \sup_{|a|_{\ell^2} = 1} \left|\left(\xi, \int_0^{2\pi} T(x-t)\psi(t)dt\right) - K \ast g(x) \cdot (\xi, a)\right|^q dx\right)^{\frac{1}{q}}$$

$$\geq \sup_{|a|_{\ell^2} = 1} \left(\int_0^{2\pi} \left|\left(\xi, \psi(t)\right) dt - K \ast g(x) \cdot (\xi, a)\right|^q dx\right)^{\frac{1}{q}}$$

$$\geq \sup_{\xi \in \mathbb{R}^m} \left|\left(\xi, a\right)\right| E(K \ast g, H^{T}_{2n-1})_{L^q} = E(K \ast g, H^{T}_{2n-1})_{L^q}.$$

Hence,

$$E(K \ast g \cdot \{a\}, SVH^{T}_{2n-1})_{L^q} \geq E(K \ast g, H^{T}_{2n-1})_{L^q}$$
and, therefore,
\[ \mathcal{E}(K \ast \Phi_p, SVH_{2n-1}^T)_{L_q^A} \geq E(K \ast F_p, H_{2n-1}^T)_{L_q}. \]

Theorem 3 is proved.

The following theorem is a set-value analog of Theorem B:

**Theorem 4.** Let \( n \in \mathbb{N}, \ p, q \in [1, \infty), \) and \( p^{-1} + q^{-1} = 1. \) If \( K \in L_q, \) then
\[ \mathcal{U}(K \ast \Phi_p, SVH_{2n-1}^T)_{L_{\infty}} = E_n(K, H_{2n-1}^T)_{L_q}. \]

**Proof.** The inequality
\[ \mathcal{U}(K \ast \Phi_p, SVH_{2n-1}^T)_{L_{\infty}} \leq E_n(K, H_{2n-1}^T)_{L_q} \]
follows from Theorem 1. We now prove the opposite inequality.

We choose an arbitrary \( a \in \mathbb{R}^m \) such that \( |a|_{l^2} = 1. \) It is clear that, for any function \( g \in F_p, \) we have
\[ g(\cdot) \cdot \{a\} \in \Phi_p \quad \text{and} \quad f(\cdot) = K \ast g(\cdot) \cdot \{a\} \in K \ast \Phi_p. \]

Thus, for any \( T \in H_{2n-1}^T, \) we obtain
\[
\delta(K \ast g(x)\{a\}, T \ast g(x)\{a\}) = |K \ast g(x) - T \ast g(x)| \cdot |a|_{l^2} = |K \ast g(x) - T \ast g(x)|.
\]

Hence,
\[
\sup_{g \in \Phi_p} \|\delta(K \ast g(x), T \ast g(x))\|_{L_{\infty}} \geq \sup_{g \in F_p} \|\delta(K \ast g(x)\{a\}, T \ast g(x)\{a\})\|_{L_{\infty}} = \sup_{g \in F_p} \|K \ast g(x) - T \ast g(x)\|_{L_{\infty}}
\]
\[
= \sup_{g \in F_p} \max_{x \in \mathbb{R}} \left| \int_{0}^{2\pi} (K(x - t) - T(x - t)g(t)dt \right|
\]
\[
= \max_{x \in \mathbb{R}} \sup_{g \in F_p} \left| \int_{0}^{2\pi} (K(x - t) - T(x - t))g(t)dt \right|
\]
\[
= \max_{x \in \mathbb{R}} \|K(x - \cdot) - T(x - \cdot)\|_{L_q} \geq E(K, H_{2n-1}^T)_{L_q}.
\]

Theorem 4 is proved.
Finally, we present a theorem on the upper bound for the best approximation of a class of set-valued functions by generalized polynomials of the form (4) whose values, for any $x \in \mathbb{R}$, contain the value of the approximating function as a subset.

**Theorem 5.** Let $n \in \mathbb{N}$, $p, q \in [1, \infty]$, and $p^{-1} + q^{-1} = 1$. If $K \in L_q$, then

$$\mathcal{E}^+(K \ast \Phi_p, \text{SV}_{H_{2n-1}^T})_{L_{2n-1}^\infty} \leq 2E_n(K, H_{2n-1}^T)_{L_q}.$$  

**Proof.** Let $T^* \in H_{2n-1}^T$ be the polynomial of the best $L_q$-approximation for $K$. Assume that the function $f = K \ast \Phi_p$ is given. By virtue of Theorem 1, for any $x \in \mathbb{R}$,

$$\delta(K \ast g(x), T^* \ast g(x)) \leq E(K, H_{2n-1}^T)_{L_q} =: e.$$  

We set

$$\tilde{\tau}(x) = T^* \ast g(x) + B_e(\theta).$$  

It is easy to see that

$$K \ast g(x) \subset \tilde{\tau}(x)$$  

and

$$\delta(K \ast g(x), \tilde{\tau}(x)) = \delta(K \ast g(x), T^* \ast g(x) + B_e(\theta))$$

$$\leq \delta(K \ast g(x), T^* \ast g(x)) + \delta(B_e(\theta), \theta)$$

$$\leq 2E(K, H_{2n-1}^T)_{L_q}.$$  

This yields the assertion of the theorem.

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