DYNAMICAL NONCOMMUTATIVE SPHERES

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Abstract. We introduce a family of noncommutative 4-spheres, such that the instanton projector has its first Chern class trivial: \( ch_1(e) = B\chi + b\xi \). We construct for them a 4-dimensional cycle and calculate explicitly the Chern-Connes paring for the instanton projector.

1. Introduction

The construction of noncommutative spheres based on homological principles was proposed by Connes [3], the basic assumption is that the algebra is generated by the elements of a projector (or unitary matrix over the algebra in the odd case) and its Chern classes in Hochschild homology vanish in all dimensions smaller than the dimension of the manifold.

Connes proved that in dimension 2 only commutative solutions appear. First noncommutative examples of solutions in dimension three and four were constructed in [5] then a systematic analysis of this type of solutions as well as construction of all three-dimensional solutions were given in [3]. All constructed examples of noncommutative three (and four) are of good homological dimension (related to Hochschild or cyclic homology). Moreover, they seem to be (and in some cases certainly are) nice examples of noncommutative spin geometries, as defined by Connes [3].

In this paper we introduce a variation of the noncommutative deformation of a four-sphere. With a subtle generalization of the deformation parameter we shall obtain a family of objects indexed by smooth functions on an interval, a special case of a constant function corresponding to the isospectral deformation [3]. The deformation in question goes beyond the so far considered models for noncommutative spheres like \( SU_q(2) \) and its suspension (see [7]), deformations based on suspensions (and their twists) of Podles spheres ([10], [9]) or the above mentioned isospectral deformations. It rather extends the original ideas of Matsumoto [9] who first considered (in \( C^* \)-algebraic setup)
the three-spheres\(^1\), studied later in \([3]\); in fact he described an entire family of their generalizations (we shall mention them later).

In the paper we present the construction of the deformation, define the instanton projector, differential calculus over the deformed spheres, we construct a four-dimensional cycle, calculate Chern classes of the instanton projector and the corresponding Chern numbers.

The name *dynamical*, which we use for the deformation has been motivated by possible physical applications: although we work here with a deformation of a compact manifold, it is easy to generalize the procedure to construct such deformations of \(\mathbb{R}^4\) or \(M \times \mathbb{R}\). With the natural interpretation of the coordinate as time we obtain *time-dependent noncommutativity*, an idea, which could be motivated, for instance, in string theory from considerations of branes in a non-static \(B\)-field.

## 2. Preliminaries

We shall begin by recalling the main steps of the construction of isospectral deformations, as done in \([3]\). Let \(M\) be a compact manifold, \(\mathcal{A} = C^\infty(M)\) and let the two torus \(T^2\) act on \(\mathcal{A}\).

Since any smooth function (with respect to the action of the torus) could be presented as a doubly infinite norm convergent series of homogeneous elements, where \(f\) is homogeneous of of bidegree \(n_1, n_2\) iff

\[
(u_1, u_2) \triangleright f = (u_1)^{n_1} (u_2)^{n_2} f,
\]

for \(u_1, u_2 \in T^2\), one might introduce a deformed algebra as using a left (or right) twist maps:

\[
\sum_{n_1, n_2} T_{n_1, n_2} = T \mapsto l(T) = \sum_{n_1, n_2} T_{n_1, n_2} \lambda^{n_2 \delta_1}
\]

where \(\lambda\) is a complex number of module 1 and \(\delta_1, \delta_2\) are the generators representations of the (projective) unitary representation of the action of the torus.

Then we have the lemma:

**Lemma 2.1** (\([3]\), Lemma 4). *There exists an associative product on the vector space of smooth functions \(\mathcal{A}\), \((x, y) \mapsto x \ast y\) such that*

\[
l(x)l(y) = l(x \ast y).
\]

\(^1\)The original definition of Matsumoto three-spheres uses different generators, however, in \(C^*\)-algebraic formulations invertible transformations between generators of \([3]\) and \([2]\) could be easily constructed explicitly: these are however only continuous but not smooth.
For the homogeneous elements \((x, y)\) of order \(n_1, n_2\) and \(m_1, m_2\), respectively, it is:

\[x \ast y = \lambda^{n_1 m_2}xy.\]  

(3)

From the algebraic point of view the constructed deformation is a cocycle deformation of the algebra through the twist from the Cartan subalgebra of its symmetry group. This description was developed in [11] and used to demonstrate that twisted isometry of the algebra is the Hopf-algebra isometry of deformed spectral triple, a dual approach to symmetries was suggested in [13], whereas a systematic approach to \(\theta\)-deformations is presented in [5].

2.1. **Dynamical twists.** We shall introduce here a generalization of the above deformation, which we shall study in details in a particular case of the sphere. Our assumptions are as in the situation discussed earlier: we work with smooth functions on a compact oriented manifold, such that its symmetry groups contains a torus and we assume that the (smooth) action of the isometry group is (projectively) lifted to the Hilbert space.

Let \(f\) be a smooth function, and let \(T(f)_{n_1,n_2}\) be its component of the Fourier series with respect to the action of \(T^2:\)

\[T(f) = \sum_{n_1,n_2} T(f)_{n_1,n_2},\]

where the series is norm converging (norms of homogeneous elements, which are the elements of this series, are of rapid decay).

Let \(H\) be a self-adjoint element of the algebra \(C^\infty(M)\) (real smooth function), which is of bidegree \((0,0)\), so it is invariant with respect to the action of the two-torus.

Let us define a map, which shall assign to every element \(f \in C^\infty(M)\) an element of the deformed algebra:

\[T_H(f) = \sum_{n_1,n_2} T(f)_{n_1,n_2} e^{2\pi i n_2 H \delta_1}.\]  

(4)

Let us observe that the series is again infinite norm convergent (we modify each element by multiplication with an operator of norm 1) and since \(H\) commutes with the action of torus the definition well posed (that is the bidegree of an element is stable under multiplication by any function of \(H\)). So we have a lemma:
Lemma 2.2. If \( f, g \) are homogeneous operators of degrees \((n_1, n_2)\) and \((m_1, m_2)\), respectively, then:

\[
T_H(f)T_H(g) = T_H(f \ast g),
\]

where

\[
f \ast g = e^{2\pi i H n_2 m_1} (fg).
\]

Similarly, one may define an opposite deformation:

\[
T^0_H(f) = \sum_{n_1, n_2} e^{2\pi i n_1 H \delta_2} T(f)_{n_1, n_2}.
\]

such that \([T^0_H(x), T_H(y)] = 0\) if \([x, y] = 0\)

Proof. The proof of the lemma follows directly the proof of lemma 4 of \cite{[6]}. \( \Box \)

We shall present now two basic examples of this type of deformation.

Example 2.3 (Three-torus and Heisenberg group algebra). Let \( T^3 \) be a three-torus, consider the natural action of two-torus \( T^2 \subset T^3 \) on \( C^\infty(T^3) \). If we denote the unitary generators of \( C^\infty(T^3) \) by \( U, V, W \), then \( W \) remains the invariant element under the action of \( T^2 \).

If we make the choice of \( H = \theta \) as a constant we obtain a product of a noncommutative torus \( T^2_{\theta} \) with \( S^1 \). However, the simplest nontrivial choice of \( e^{2\pi i H} = W \) gives us the algebra relations:

\[
UV = WVU, \quad [U, W] = [V, W] = 0.
\]

Clearly, the first relation can be generalized to:

\[
UV = f(W)VU.
\]

where \( f(W) \) is a suitable smooth function \( f : S^1 \rightarrow S^1 \), however, in the particular case \( f(W) = W \) the algebra is the group algebra of the discrete 3-dimensional Heisenberg group \cite{[1]}.

We shall study the properties of this algebra, in particular the explicit construction of the \( K \)-cycle and Chern-Connes pairing in a separate paper \cite{[12]}.

Example 2.4 (The 4-sphere). Let us consider a \(*\)-algebra generated by elements \( a, a^*, b, b^* \) and \( t = t^* \) subject to the following set of relations:

\[
\begin{align*}
[a, t] &= 0, & [a^*, t] &= 0, \\
[b, t] &= 0, & [b^*, t] &= 0, \\
[a, a^*] &= 0, & [b, b^*] &= 0, \\
ab &= \lambda(t)ba, & ab^* &= \bar{\lambda}(t)b^*a, \\
a^*b &= \bar{\lambda}(t)a^*b, & a^*b^* &= \lambda(t)b^*a^*.
\end{align*}
\]
where $\lambda(t)$ is a unitary element, $\lambda(t)\bar{\lambda}(t) = 1$, expressed as a function of the central element $t$, so we may assume:

$$\lambda(t) = e^{-i\phi(t)},$$

(11) where $\phi$ is a smooth real function of $-1 \leq t \leq 1$.

Furthermore, we have the restriction:

$$aa^* + bb^* + t^2 = 1,$$

(12) which is the relation defining the (noncommutative) 4-sphere.

One could easily verify that the above set of relations is consistent, for any choice of the function $\phi$, the particular example of $\phi = \theta = \text{const}$ being the isospectral deformation of the sphere.

Passing from algebraic (polynomial) algebra to the algebra of smooth functions one can easily observe that the algebra describes the dynamical deformation of the four-sphere as presented in Lemma 2.2, with $H = \phi(t)$ (the parameter $t$ corresponds to the choice of presentation of $S^4$ as a suspension of $S^3$).

We shall denote this algebra by $S^4_{\lambda}$, let us observe that the center of the algebra in question contains $t$, $aa^*$ and $bb^*$ but could be much bigger depending on the function $\lambda$.

### 3. Instanton bundles over $S^4_{\lambda}$

One of the most appealing feature of the construction of [6] was the existence of the instanton bundle over the deformed algebra. This was shown by the construction of the projector $e$ with vanishing lower Chern classes and $ch_2(e)$ giving rise to a Hochschild cocycle over the algebra.

The projector in our case is unmodified:

$$e = \frac{1}{2} \begin{pmatrix}
1 + t & 0 & a & b \\
0 & 1 + t & -\lambda(t)b^* & a^* \\
a^* & -\bar{\lambda}(t)b & 1 - t & 0 \\
b^* & a & 0 & 1 - t
\end{pmatrix},$$

(13) the only significant distinction for the $\lambda = \text{const}$ case is that no longer all the entries of the projector are the generators of the algebra. Of course, since $\lambda$ is not a constant parameter one may easily verify that the Chern homology elements constructed out of $e$ shall not be the same as in $\lambda = \text{const}$ case. In particular, we have:

$$ch_1(e) = t \otimes x_i \otimes y_i - x_i \otimes t \otimes y_i + x_i \otimes y_i \otimes t,$$

(14)
where:

\[ x_i \otimes y_i = b \otimes b^* - b^* \otimes b + \lambda b^* \otimes \bar{\lambda} b - \bar{\lambda} b \otimes \lambda b^*. \] (15)

It is easy to verify that \( b ch_1(e) \) vanishes, however \( B ch_1(e) \) does not:

\[ B ch_1(e) = 1 \otimes ch_1(e). \] (16)

We shall postpone further discussion of the Chern classes until the last section of the paper, when it shall be clear that although \( ch_1(e) \) does not vanish, its class is trivial.

In fact, using the the natural construction of differential structures on the deformed sphere and the natural trace on the algebra we shall give explicit formula for the volume form, which arises naturally from the Chern class \( e \) and calculate the Chern number of the above projector \( e \).

4. The differential calculus on \( S^4_\lambda \)

Unlike in the \( \lambda = \text{const} \) case we have no clear indication for the construction of differential calculi. We shall look for a guiding principle of the smallest calculi, which, when restricted to commutative subalgebras, remains classical and for \( \lambda = 1 \) gives the correct limit of the differential structures on a four-sphere.

Before we begin let us observe that the commutation relations between algebra generators \( a, a^*, b, b^* \) could be rewritten as

\[ x^i x^j = A_{ij} x^j x^i, \quad 1 \leq i, j \leq 4, \] (17)

where there is no summation in the formula, \( x^i \) denote the generating monomials \( a, a^*, b, b^* \) and the matrix \( A_{ij} \) is \( t \)-dependent, in our case:

\[ A = \begin{pmatrix} 1 & 1 & \lambda(t) & \bar{\lambda}(t) \\ 1 & 1 & \bar{\lambda}(t) & \lambda(t) \\ \lambda(t) & \bar{\lambda}(t) & 1 & 1 \\ \bar{\lambda}(t) & \lambda(t) & 1 & 1 \end{pmatrix}. \] (18)

We make an Ansatz that the bimodule of one forms is generated by \( dx^i \) and a central one-form \( dt \), with quadratic the bimodule commutation rules:

\[ x^i dx^j = A_{ij} dx^j x^i + \frac{1}{2} B_{ij} dt(x^i x^j). \] (19)

We assume as well, that \( t dx^i = dx^i t \). It is easy to see that such relations are consistent with the algebra commutation rules. Further,
if we differentiate (17) and use (19) we obtain the following relation between $B$ and $A$:

$$\frac{1}{2}(B_{ij} - B_{ji}) = \frac{1}{A_{ij}} \dot{A}_{ij},$$

(20)

We shall restrict ourselves only to the antisymmetric solution for $B$, which are explicitly given by the above formula.

Expressing the relations (19) in terms of the generators we have:

$$ada = daa, \quad bdb = dbb,$$

(21)

and

$$a db = \lambda(t) dba + \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt ab,$$

$$a db^* = \bar{\lambda}(t) db^* a - \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt ab^*,$$

$$b da = \bar{\lambda}(t) da b - \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt ba,$$

$$b da^* = \lambda(t) da^* b + \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt ba^*,$$

$$a^* db = \bar{\lambda}(t) db a^* - \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt a^* b,$$

$$b^* da = \lambda(t) da b^* + \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt ab^*,$$

$$a^* db^* = \lambda(t) db^* a^* + \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt a^* b^*,$$

$$b^* da^* = \lambda(t) da^* b^* - \frac{1}{2} \dot{\lambda}(t) \bar{\lambda}(t) dt a^* b^*.$$

(22)

We shall not forget that by differentiating the constraint (12) we have (after using (19)):

$$ada^* + a^* da + b db^* + b^* db + 2tdt = 0,$$

(23)

Note that the left-hand side of (23) side is a central element of the bimodule of one forms and therefore the restriction (23) is compatible with the (22). Now, we are prepared to construct the full differential algebra.

**Proposition 4.1.** Let $\Omega_u(S^4_\lambda)$ be a universal differential algebra, and let $J_1 \subset \Omega^1_u(S^4_\lambda)$ be the kernel of the projection map $\pi : \Omega^1_u(S^4_\lambda) \mapsto \Omega^1(S^4_\lambda)$. Then the differential algebra $\Omega(S^4_\lambda)$ is a $\mathbb{Z}$-graded algebra obtained as a quotient of $\Omega_u(S^4_\lambda)$ by the differential ideal generated by $J^1 + dJ^1$.

Clearly, the subbimodule $J^1$ is in our case defined by relations (19) and (23). Thus by differentiating them we obtain the first set of rules:

$$dx^i dt = -dt dx^i,$$

(24)

$$dt dt = 0.$$

(25)

$$dx^i dx^j = -A_{ij} dx^j dx^i + \frac{1}{2} B_{ij} dt dx^j x^i - \frac{1}{2} B_{ij} dt dx^i x^j$$

(26)
We immediately see that in the differential algebra \( \Omega(S^4_\lambda) \) all generators \( dx^i \) and \( dt \) are nilpotent, and \( da, da^*, \ db, db^* \) are pairwise skew-symmetric:

\[
da da^* = -da^* da, \quad db db^* = -db^* db.
\]

For the remaining relations we have:

\[
(27) \quad da db + \lambda(t) db da = \frac{1}{2} \dot{\lambda}(t) dt db a - \frac{1}{2} \dot{\lambda}(t) \dot{\lambda}(t) dt da b,
\]

\[
= db da^* + \lambda(t) da^* db = \frac{1}{2} \lambda(t) dt da^* b - \frac{1}{2} \dot{\lambda}(t) \dot{\lambda}(t) dt db a^*.
\]

Before we prove more results on the differential algebra we introduced, let us observe interesting relations:

\[
b da da^* = da da^* b - \frac{1}{2} \dot{\lambda}(t) \dot{\lambda}(t) dt (a da^* b + a^* da b) = db db^* b.
\]

where in the last step we used (23).

By differentiating it we obtain:

\[
db da da^* = da da^* db - \frac{1}{2} \dot{\lambda}(t) \dot{\lambda}(t) dt db db^* b.
\]

Similar result can also be proven for \( db^* \):

\[
db^* da da^* = da da^* db^* - \frac{1}{2} \dot{\lambda}(t) \dot{\lambda}(t) dt db db^* b.
\]

and for products of \( da db db^* \). In particular, we can see that:

\[
(28) \quad da da^* db db^* = db da da^* db^* = db^* da da^* db, \\
db da da^* db db^* = da db db^* da^* = da^* db db^* da, \\
db da^* db db^* = db db^* da da.
\]

Next we shall prove that the differential algebra has a finite dimension:

**Lemma 4.2.** The differential algebra \( \Omega(S^4_\lambda) \) has dimension 4, for all \( n > 4 \) we have \( \Omega^n(S^4_\lambda) = 0 \).

*Proof.* Clearly, it is sufficient to show that \( dt da da^* db db^* \) vanishes. Let us consider the relation (23) and multiply it from the left by a two-form \( \frac{1}{2} tda da^* \) and from the right by \( db db^* \).

Using the associativity of the product together with relations (24) and the fact that all one generating one-forms are nilpotent we obtain:

\[
(29) \quad t^2 dt da da^* db db^* = 0.
\]

Similarly, if we multiply (23) from the left by \( dt da a^* \) and by \( db db^* \) from the right we obtain:

\[
(30) \quad aa^* dt da da^* db db^* = 0.
\]
Finally, multiplying it by $b dt da da^*$ from the left and by $db^*$ from the right we get:

\begin{equation}
bb^* dt da da^* db db^* = 0.
\end{equation}

By adding the three identities (29)-(31) and using the constraint (12) we obtain the desired result.

So far we have shown that the maximal degree of forms is 4, it appears however that the structure is exactly as in the "classical" case and we are able to demonstrate that there exist one generating four-form:

**Lemma 4.3.** The bimodule of differential forms of degree 4 is a free bimodule module over the algebra. The generating form $\omega$ can be chosen as:

\begin{equation}
\omega = \frac{1}{4} (t da da^* db db^* - 2a dt da^* db db^* + 2dt da da^* db b^*),
\end{equation}

where the factor $\frac{1}{4}$ was chosen so that it would correspond to the volume form on $S^4$ in the classical limit.

**Proof.** Consider $t \omega$. Using the commutation rules of $dt$ with other one-forms (24) as well as the fact that $t$ is central we might rewrite it conveniently as:

\begin{equation}
t \omega = \frac{1}{4} (t^2 da da^* db db^* + a da^* (2t dt) db db^* + da da^* (2tdt) db b^*) = \ldots
\end{equation}

Next, using (23) and keeping in mind that $dt$ and $dx^i$ are nilpotent we get:

\begin{equation}
\ldots = \frac{1}{4} (t^2 + aa^* + bb^*) da da^* db db^* = \frac{1}{4} da da^* db db^*.
\end{equation}

where we have used first the fact that $a, a^*$ commute with $da, da^*$ (and similar property of $b, b^*$ and their differentials) as well as the defining relation (12).

Similarly one may verify the identities:

\begin{align}
(33) & \quad a \omega = \frac{1}{2} dt da db db^*, \\
(34) & \quad a^* \omega = -\frac{1}{2} dt da^* db db^*, \\
(35) & \quad \omega b = \frac{1}{2} dt da da^* db, \\
(36) & \quad \omega b^* = -\frac{1}{2} dt da da^* db^*.
\end{align}
The form $\omega$ is central, i.e. it commutes with all elements of the algebra. As this result is not evident though it follows from an easy algebraic calculation we shall demonstrate it only for $[b, \omega]$. First, observe that only the first component in the sum (32) might give a nontrivial contribution as the remaining two contain $dt$ and then the nontrivial permutation rules of generators through differentials are homogeneous and will cancel out.

$$[b, \omega] = \frac{1}{4}(bt \, da^* \, db^* - t \, da^* \, db^* b) =$$

$$= \frac{1}{4}t\lambda(t)\bar{\lambda}(t) \left(\bar{\lambda}(t) \, da \, dt (ba^*) - dt \, da^* (ab)\right) \, db^* = \ldots$$

now, if we permute $t$ and use (23) to substitute a nontrivial one-form for $t \, dt$, still using the fact that the one forms are nilpotent:

$$= \ldots \frac{1}{16}\bar{\lambda}(t)\bar{\lambda}(t) \left(-\bar{\lambda}(t) \, da (a \, da^*) (ba^*) + a^* da da^* (ab)\right) \, db^* =$$

$$= \frac{1}{16}\lambda(t)\lambda(t) \left(-da (ada^*) (a^*b) + a^* da da^* (ab)\right) \, db^* = 0.$$

Before we proceed with the construction of the integral of 4-forms, let us observe the properties of a trace on the algebra itself.

**Proposition 4.4.** Let $\int$ be the standard (normalized) integral on $S^4$ and $\eta$ be a linear map on $S^4$, which maps an element of $S^4$ to an element of $C(S^4)$, with the identification of every element with $a, a^*$ to the left of $b, b^*$ with the corresponding function on $S^4$. Then $x \mapsto \int \eta(x)$ is a trace on $S^4$.

Clearly we have a linear map, it remains only to show the cyclicity. First, note that the integral on $S^4$ is nontrivial on functions depending only on $aa^*$ and $bb^*$. Therefore, we might restrict ourselves to such case. Let us take two monomials $p, q$ in $a, a^*, b, b^*$ such that their product is a monomial of $aa^*$ and $bb^*$. Then we shall prove that $\eta(pq) = \eta(qp)$. Let $p = a^{\alpha_p}(a^*)^{\delta_p}b^{\gamma_p}(b^*)^{\delta_p}$ and $q = a^{\alpha_q}(a^*)^{\delta_q}b^{\gamma_q}(b^*)^{\delta_q}$. First, we calculate $pq$ using (14):

$$pq = \lambda(t)^{\gamma_p\beta_q+\delta_p\alpha_q}\bar{\lambda}(t)^{\gamma_q\alpha_p+\delta_q\beta_p}a^{\alpha_p+\alpha_q}(a^*)^{\delta_p+\beta_q}b^{\gamma_p+\gamma_q}(b^*)^{\delta_p+\delta_q},$$

since $\bar{\lambda} = \lambda^{-1}$ we might rewrite the formula as:

$$\eta(pq) = \lambda(t)^{\gamma_p-\delta_p}(\beta_q-\alpha_q)\eta(p)\eta(q).$$

On the other hand, for $qp$ we have:

$$qp = \lambda(t)^{\gamma_q\beta_p+\delta_q\alpha_p}\bar{\lambda}(t)^{\gamma_p\alpha_p+\delta_p\beta_p}a^{\alpha_p+\alpha_q}(a^*)^{\delta_q+\beta_p}b^{\gamma_p+\gamma_q}(b^*)^{\delta_q+\delta_p},$$

which gives:

$$\eta(qp) = \lambda(t)^{\gamma_q-\delta_q}(\beta_p-\alpha_p)\eta(p)\eta(q).$$
Now, it is easy to see that both coefficients are equal, since by our assumption that the product depends only on $aa^*$ and $bb^*$:
\[
\alpha_p + \alpha_q = \beta_p + \beta_q, \quad \gamma_p + \gamma_q = \delta_p + \delta_q,
\]
and thus:
\[
(\gamma_p - \delta_p)(\beta_q - \alpha_q) = (\gamma_q - \delta_q)(\beta_p - \alpha_p).
\]
We now define the integral on 4-forms.

**Proposition 4.5.** There exists a linear functional on $\Omega^4(S^4_\lambda)$ such that $\int (d\rho) = 0$ for every $\rho \in \Omega^3(S^4_\lambda)$ and $\int \omega = \frac{8}{3}\pi^2$.

**Proof.** We begin by defining the integral. Since we know that every four-form $\theta$ could be written as $\theta = x\omega$ we shall set
\[
(37) \quad \int \theta = \int \eta(x).
\]
Note that since $\omega$ is central, $x\omega = \omega x$, we have in effect a linear map $\eta : \Omega^4(S^4_\lambda) \rightarrow \Omega^4(S^4)$. We shall demonstrate that there exists also the extension of map $\eta : \Omega^3(S^4_\lambda) \rightarrow \Omega^3(S^4)$ such that the following diagram is commutative:
\[
\begin{array}{ccc}
\Omega^3(S^4_\lambda) & \xrightarrow{d} & \Omega^4(S^4_\lambda) \\
\eta \downarrow & & \eta \downarrow \\
\Omega^3(S^4) & \xrightarrow{d} & \Omega^4(S^4)
\end{array}
\]
To define the map $\eta$ on three forms we shall use their following presentation as a linear space:

**Observation 4.6.** Every 3-form (over polynomials) could be presented (though not in unique way) as a finite sum of elements of the type:
\[
\rho = t^\alpha p(a,a^*) \chi_i q(b,b^*)
\]
where $\alpha = 0,1$ and $\chi_i$ are forms of the type:
\[
\begin{align*}
da da^* db, & \quad da da^* dt, & \quad da da^* db^*, \\
da db db^*, & \quad da^* db db^*, & \quad dt db db^*, \\
dt da db, & \quad dt da db^*,
\end{align*}
\]
\[
\begin{align*}
& \quad dt da^* db, & \quad dt da^* db^*.
\end{align*}
\]
Of course, these forms are not independent (when we consider them in the bimodule of three-forms). However, it is important that we can map them to $\Omega^3(S^4)$ by setting first $\eta(\chi_i)$, for instance:
\[
\eta(da da^* db) = d\eta(a) d\eta(a^*) d\eta(b),
\]
and then:
\[
\eta(\rho) = \eta(t)^\alpha p(\eta(a), \eta(a^*)) \eta(\chi) q(\eta(b), \eta(b^*)�).
To see that the map is well-defined (as a linear map) let us observe that by using the so ordered product of functions and differentials we see no nontrivial commutation rules. Thus, the characterization of $\Omega^3(S^4_\lambda)$ and $\Omega^3(S^4)$ as a linear space are exactly the same.

Now, using the presentation (4.6) we can easily see that $\eta(d\rho) = d\eta(\rho)$ for every three-form $\rho$. Indeed, the external derivative vanishes on all three-forms $\chi$ and on functions depending only of $a, a^*, t$ and respectively, on $b, b^*, t$ we have standard differentiation:

$$d\eta(p(a, a^*, t)) = \eta(dp(a, a^*, t)),$$

and

$$d\eta(q(b, b^*, t)) = \eta(dq(b, b^*, t)).$$

Since again, we multiply by $a, a^*$ and its differentials from the left and $b, b^*$ and its differentials from the left – we encounter no commutators between $a, a^*$ and their differentials and $b, b^*$ and their differentials. Hence, noncommutativity plays no role in the map $\eta$ and the action of the external derivative.

Using the constructed differential structures and the trace we have:

**Proposition 4.7.** $\Omega^*(S^4_\lambda)$ is a differential graded algebra with a closed graded trace $\int : \Omega^4(S^4_\lambda) \to \mathbb{C}$.

**Proof.** So far we have showed the existence of a closed trace on $\Omega^4(S^3)$. Because of its particular form (4.4) it is evident that $\int x\rho = \int \rho x$ for every four-form $\rho$ and $x \in S^4_\lambda$.

Now, let us take a three-form $\beta$ and a one-form $x dy$:

$$\int (x dy \beta + \beta x dy) = \int (dy \beta x + \beta d(xy) - \beta dx y)$$

$$= \int (dy (\beta x) - y d(\beta x) + \beta d(xy) + d(\beta x)y - (d\beta) xy =$$

$$\int (dy \beta x - [y, d(\beta x)] + d(\beta xy)) = 0.$$

Similarly, we proceed for two-forms. As an immediate corollary we have:

**Corollary 4.8.** Let $\psi$ be a multilinear functional defined as:

$$\psi(a_0, a_1, a_2, a_3, a_4) = \int a_0 da_1 da_2 da_3 da_4.$$

then $\psi$ is a cyclic cocycle.

Having a cyclic cocycle enables us to calculate the Chern-Connes pairing with the instanton projector, which we introduced earlier (13).
4.1. The Chern character. Let us consider the construction of an element of $\Omega^4(S^4_\lambda)$ out of the projector $e$:

$$\text{ch}(e) = -\frac{1}{8\pi^2} \text{Tr}(e \, de \, de \, de),$$

where the trace is over matrix indices of $e$.

We shall use the block form of $e$ and the rules of differential calculi to facilitate the calculations. Let us denote:

$$q = \begin{pmatrix} a & b \\ -\lambda(t)b^* & a^* \end{pmatrix},$$

then we can write $e$ and $de$ as block matrices:

$$e = \frac{1}{2} \begin{pmatrix} t + 1 & q \\ q^* & 1 - t \end{pmatrix},$$

$$de = \frac{1}{2} \begin{pmatrix} dt & dq \\ dq^* & -dt \end{pmatrix},$$

where $1 \pm t$ and $\pm dt$ denote diagonal matrices. Using this fact and that $(dt)^2 = 0$ and $dt$ anticommutes with the rest of the one-forms, we obtain:

$$dedede = \frac{1}{16} \begin{pmatrix} (dq dq^*)^2 & 4dt dq dq^* dq \\ -4dt dq dq^* dq dq^* & (dq dq^*)^2 \end{pmatrix}.$$ (39)

Therefore for the trace of $e \, ed \, de \, de \, de$ we shall have:

$$\ldots = \frac{1}{32} \text{Tr} \{ (1 + t)(dq dq^*)^2 + (1 - t)(dq^* dq)^2 + 4q dt dq dq^* dq dq^* + 4q^* dt dq dq^* dq \},$$

where the trace is now over two-dimensional matrices. As a next step let us calculate $dq dq^*$ and $dq^* dq$:

$$dq dq^* = \begin{pmatrix} da da^* + db db^* & 2db da - \frac{1}{2}\lambda \lambda dt(db a + \lambda da b) \\ 2da^* db^* - \frac{1}{2}\lambda \lambda dt(db^* a^* + \lambda da^* b^*) & da^* da + db^* db + \lambda \lambda dt(dbb^* + db^* b) \end{pmatrix}.$$ (11)

Now, we shall calculate the diagonal part of $(dq dq^*)^2$, the element from the top-left corner, $\{(dq dq^*)^2\}_{11}$, is:

$$\{(dq dq^*)^2\}_{11} = (da da^* + db db^*)^2 + 4db da da^* db^* +$$

$$-\lambda \lambda dt(-a da^* db db^* + da da^* b^* b + da da^* db b^* - a^* da db db^*) = \ldots$$

In the last expression, using (28) we can substitute $-a da^* - a da$ by $b db^* + b^* db + 2t dt$, then, however, we shall encounter at least one element of the type $(dt)^2$, $(db)^2$ or $(db^*)^2$ and therefore it shall vanish. Moreover, using the previously derived rules (28) we see that in the end we obtain:

$$\ldots = 6da da^* db db^*.$$
Quite similarly, for the other diagonal element of \((dq dq^*)^2\) we shall have:

\[
\{(dq dq^*)^2\}_{22} = 4da^* db^* db da + (da^* da + db^* db)^2 = 6da da^* db db^*.
\]

The calculation for the sum of the diagonal elements of \((dq^* dq)^2\) yields (we skip the intermediate technical steps, which are same as in the previous example):

\[
\text{Tr}(dq^* dq)^2 = -12da da^* db db^*.
\]

Coming back to our expression (39) it is easy to demonstrate that 

\[-4\text{Tr}(q dq dq^* dq^* dq^* dq)\] and 

\[4\text{Tr}(q^* dt dq dq^* dq)\]

give the same contributions, which together add up to:

\[24 dt (-a da^* db db^* - da da^* db^* b + da da^* db b^* + a^* da db db^*) .\]

Summing it all together and using again (23) we obtain:

\[\text{ch}(e) = -\frac{1}{2\pi i} \int \text{Tr}(e de de de de) = -\frac{1}{8\pi^2} 24 \int (t da da^* db db^* - 2a dt da^* db db^* + 2dt da da^* db db^*) = -\frac{3}{32\pi^2} 1 = -1.\]

Corollary 4.9. The element \(e de de de de\) gives a nontrivial cohomology class of the complex \(\Omega(S^4)\).

Now, we shall come back to the first Chern form:

\[\text{ch}_1(e) = -\frac{1}{2\pi i} \text{Tr}(e de de) ,\]

which, evidently, does not vanish:

\[\frac{1}{2\pi i} \text{Tr}(e de de) = -\frac{1}{2\pi i} 2\bar{\lambda}(t)\bar{\lambda}(t) dt (b db^* + b^* db),\]

however, it is in the trivial cohomology class. If \(\lambda = e^{-i\phi(t)}\) for a real function \(\phi\) then:

\[\text{ch}_1(e) = \frac{1}{\pi} d (\phi(t)(b db^* + b^* db)).\]

What does it mean? Let us remind that the \(\text{ch}_1(e)\) in the reduced \((b,B)\) double complex was clearly a cycle. Furthermore, one might easily observe that it was depending only on the commutative subalgebra generated by \(t, b\) and \(b^*\), which we shall denote by \(\mathbb{C}[b, b^*, t]\) (we might
equally well describe the algebra as the subalgebra of smooth functions on $S^4$ invariant under the action of $\delta_2$ - and it is the algebra of smooth functions on a three-dimensional closed ball).

Since it is a regular commutative algebra we might use the results relating Hochschild and homology of with the de Rham complex.

**Proposition 4.10.** There exists an element $\chi \in C_1(\mathbb{C}[b,t])$ and $\xi \in C_3(\mathbb{C}[b,t])$ such that:

$$ch_1(e) = B\chi + b\xi.$$  \hspace{1cm} (43)

**Proof.** First, let us observe that since $bch_1(e) = 0$ we might map $ch_1(e)$ to $\Omega^2(\mathbb{C}[t,b])$, the image being exactly the two-form (41). This form is exact, as we have demonstrated explicitly. If we take the one form in $\Omega^1(\mathbb{C}[b,t])$, $\chi_0$, $d\chi_0 = ch_1(e)$, by using the commutative diagram relating Hochschild homology with differential forms (see Proposition 2.3.4, p.69, [8]) we obtain the desired cycle $\chi = \pi^{-1}(\chi_0)$.

Then the Hochschild class of $B\chi$ is the same as this of $ch_1(e)$, so the difference is in the image of $b$, and then by choosing any suitable cycle $\xi$ we get (43).

Therefore, although $ch_1(e)$ does not vanish identically, we still are almost in the same situation. By correcting slightly $ch_2(e)$ we are again able to obtain a Hochschild cycle of dimension 4, which corresponds to the volume form:

$$v = ch_2(e) + B\xi.$$  \hspace{1cm} (44)

Indeed:

$$bv = bch_2(e) + bB\xi = Bch_1(e) - Bb\xi$$

$$= B(ch_1(e) - b\xi) = B(B\chi) = 0.$$  \hspace{1cm} (45)

5. **Conclusions**

The construction presented in this paper extends the notion of noncommutative spheres to objects defined through instanton bundles, whose first Chern class does not vanish but is homologically trivial. Our aim was to demonstrate that such solutions exists, are easily obtained by a slight generalization of the twisted noncommutative spheres. We demonstrated as well the existence of 4-dimensional differential calculus (a 4-dimensional cycle) and calculated explicitly the Chern-Connes pairing.

Of course, it is possible to consider further generalizations going in this direction, for instance one might consider (in the same spirit) the Matsumoto 3-spheres defined through generators as:
\[ [a, a^*] = 0, \quad [b, b^*] = 0, \]
\[ ab = \lambda ba, \quad a^*b = \lambda a^*b, \quad ab^* = \bar{\lambda}^*a, \quad a^*b^* = \lambda^*a^*, \]
and
\[ aa^* + bb^* = 1, \]
where \( \lambda(t) \) is a unitary element from the center of the algebra, \( \lambda(t)\bar{\lambda}(t) = 1 \), for instance:
\[ \lambda = \lambda(bb^*). \]

Similarly as for the four-sphere one may view this algebra as generated by the matrix elements of is generator of \( K_1 \) class:
\[ U = \begin{pmatrix} a & b \\ -\lambda b^* & a^* \end{pmatrix}, \]
Now, it is easy to verify that the Chern character of the generator \( U \) for this algebra is:
\[ ch_{\frac{1}{2}}(U) = b \otimes b^* - b^* \otimes b + \lambda b^* \otimes \bar{\lambda}b - \bar{\lambda}b \otimes \lambda b^*. \]

Again, although this Chern character does not vanish, since it is over a commutative subalgebra we see that the same argument as in the case of 4-sphere applies and it is sufficient to study the image of \( ch_{\frac{1}{2}}(U) \) in the de Rham complex:
\[ \pi(ch_{\frac{1}{2}}(U)) = -\frac{1}{2\pi i}bb^*(\lambda d\bar{\lambda} - \bar{\lambda}d\lambda). \]
If \( \lambda = e^{2\pi i f(bb^*)} \) for some smooth real function \( f \) we get:
\[ \pi(ch_{\frac{1}{2}}(U)) = -2bb^*f'(bb^*)d(bb^*). \]
To proceed further we need to identify the commutative algebra we are working with and it is easy to see that these are functions on a disk. For this reason the above one-form, which is closed is also exact - so again, within the de Rham complex the lower Chern character is of trivial cohomology class.

Although we have concentrated in this paper only on the case of four-dimensional spheres (motivated by the instanton algebra construction of \[3\]) there are numerous examples of other deformation of this type (one of which we already mentioned). Clearly, the procedure might be as well generalized to higher-dimensional spheres.

Their applications to physical theories (allowing, for instance, for a change of commutativity with time) shall be discussed elsewhere \[12\].
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