The Threefold Way in Non-Hermitian Random Matrices

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Non-Hermitian random matrices have been utilized in diverse scientific fields such as dissipative and stochastic processes, mesoscopic systems, nuclear physics, and neural networks. However, the only known universal level-spacing statistics are those of the Ginibre ensemble characterized by complex-conjugation symmetry. Here we report our discovery of two distinct universality classes characterized by transposition symmetry. We find that transposition symmetry alters repulsive interactions between two neighboring eigenvalues and deforms their spacing distribution, which is not possible with other symmetries including Ginibre’s complex-conjugation symmetry which can affect only nonlocal correlations. Our results complete the non-Hermitian counterpart of Dyson’s threefold classification of Hermitian random matrices and serve as a basis for characterizing nonintegrability and chaos in open quantum systems with symmetry.

Introduction. Classifying random matrices by symmetries has led to a number of universal phenomena since Dyson’s threefold classification of Hermitian random-matrix ensembles in terms of time-reversal symmetry (TRS, Fig. 1) \cite{1}. Such universality classes emerge in broad scientific fields including nuclear \cite{2} and mesoscopic systems \cite{3, 4}, quantum chaos \cite{5, 6}, and information theory \cite{7}. The most direct manifestation of the universality is the nearest-neighbor spacing distribution, which measures local correlations of eigenvalues. Spacing distributions of nonintegrable systems are known to be described by those of Gaussian random matrices \cite{6, 8, 9}. They belong to the Gaussian unitary ensemble (GUE) for the class without TRS (called class A), the Gaussian orthogonal ensemble (GOE) for the class with TRS whose square is 1 (class AI), and the Gaussian symplectic ensemble (GSE) for the class with TRS whose square is \(-1\) (class AII). Altland and Zirnbauer \cite{10} later extended Dyson’s three classes to ten by introducing particle-hole symmetry (PHS) and chiral symmetry (CS).

While Dyson’s threefold way covers symmetry and universality classes of Hermitian matrices, non-Hermiticity plays a key role in diverse systems such as dissipative systems \cite{12–14}, mesoscopic systems \cite{15}, and neural networks \cite{16}. These systems have been investigated in terms of non-Hermitian random-matrix ensembles introduced by Ginibre (Fig. 1), which are referred to as GinUE, GinOE, and GinSE as non-Hermitian extensions of GUE, GOE, and GSE \cite{11}. These three Gaussian ensembles are defined in terms of complex conjugation (TRS) and have thoroughly been investigated \cite{17–22}. Interestingly, the nearest-neighbor spacing distributions of non-Hermitian systems, which characterize chaotic phases in open quantum systems \cite{13, 23, 24}, belong to the same universality class for the different symmetry classes of GinUE, GinOE, and GinSE \cite{25}. In fact, TRS only creates nonlocal correlations between complex-conjugate pairs of eigenvalues, but does not alter the repulsive interactions between neighboring eigenvalues away from the real axis. The Ginibre distribution is the only hitherto known universal nearest-neighbor spacing distribution \cite{26} even though the three symmetry classes with TRS are considered, in contrast with Dyson’s threefold way where TRS leads to three distinct universality classes. Here, symmetry classification is defined solely by algebraic structures of matrix ensembles, whereas universality classification is defined by those spectral statistics that are independent of detailed structure of matrices.

However, Ginibre’s threefold way (classes A, AI, and AII) is not the only unique extension of Dyson’s threefold way (Fig. 1). Here we report two new universality classes for symmetry classes called AI\(\dagger\) and AII\(\dagger\) that arise from transposition symmetry \cite{27–34}, which is Hermitian conjugate of TRS, i.e., TRS\(\dagger\), and describe a rich variety of experimentally realizable systems such as optical systems with gain and/or loss \cite{35–37}.

We investigate the nearest-neighbor spacing distributions of random matrices which belong to the simplest classes (i.e., classes with a single symmetry) in the non-Hermitian extension of the Altland-Zirnbauer classification.

| Class | Symmetry | Constraint | \(p_{\text{GinUE}}(s)\) ? |
|-------|-----------|------------|-----------------|
| A     | None      | -          | Yes \cite{11}   |
| AI \((\text{D}^{\dagger})\) | TRS, + (PHS\(\dagger\), +) | \(H = H^*\) | Yes \cite{12} |
| AII \((\text{C}^{\dagger})\) | TRS, + (PHS\(\dagger\), +) | \(H = H^T\) | No |
| AI\(\dagger\) | TRS\(\dagger\), + | \(H = \sigma^y H^*\sigma^y\) | Yes \cite{11} |
| AII\(\dagger\) | TRS\(\dagger\), + | \(H = \sigma^y H^T\sigma^y\) | No |
| D     | PHS, +    | \(H = -H^T\) | Yes |
| C     | PHS, +    | \(H = -\sigma^y H^T\sigma^y\) | Yes |
| AIII  | CS (pH)   | \(H = -\sigma^z H^*\sigma^z\) | Yes |
| AIII\(\dagger\) | SLS (CS\(\dagger\)) | \(H = -\sigma^z H\sigma^z\) | Yes |
Dyson’s threefold classification in terms of time-reversal symmetry (TRS) and its non-Hermitian generalizations. Open and filled circles indicate individual eigenvalues and Kramers pairs of eigenvalues, respectively. The repulsive interaction between neighboring eigenvalues is different among Dyson’s three classes. The difference does not appear for Ginibre’s threefold way of non-Hermitian matrices, where TRS affects only the global symmetry of eigenvalues (green arrows). In contrast, transposition symmetry, TRS\(^T\), which is distinct from TRS due to non-Hermiticity, leads to another threefold universality where the neighboring interaction is weaker/stronger for class \(\text{AI}\) compared to \(\text{AII}\)\(^\dagger\).

Non-Hermitian ramification of symmetry classes. Dyson’s threefold classification includes symmetry class \(A\) with no symmetry constraint, class \(\text{AI}\) subject to the constraint \(H = H^* = H^T\), and class \(\text{AII}\) subject to the constraint \(H = \sigma^y H^T \sigma^y = \sigma^y H^T \sigma^y\). Here, without loss of generality, we take a unitary operator \(T\) as the identity (\(\sigma^y\)) in \(H = THT^{-1}\) with \(T^T = +1 (–1)\) for class \(\text{AI}\) (\(\text{AII}\)). Below we consider the same simplification.

Ginibre presented the three non-Hermitian symmetry classes \(A\), \(\text{AI}\), and \(\text{AII}\) by considering complex conjugation symmetry (TRS) \[11\]. Non-Hermitian random matrices in class \(A\) have no symmetry constraint; matrices in class \(\text{AI}\) (\(\text{AII}\)) respect \(H = H^*\) (\(H = \sigma^y H^T \sigma^y\)). In the presence of TRS, eigenvalues are either real or form complex-conjugate pairs \((E_\alpha, E_\alpha^*)\).

We find the non-Hermitian threefold universality distinct from Ginibre’s, noting that complex conjugation and transposition are distinguished for non-Hermitian matrices \[33\]. The three universality classes arise from symmetry classes \(\text{AI}\) and \(\text{AII}\) in addition to class \(A\), where matrices respect transposition symmetry (TRS\(^T\)) and satisfy \(H = H^T = (\sigma^y H^T \sigma^y)\) in class \(\text{AI}\) (\(\text{AII}\)). The TRS\(^T\) imposes constraints on left and right eigenvectors of all the individual complex eigenvalues instead of nonlocal correlations. In class \(\text{AI}\), for example, a right eigenvector and the complex conjugate of the corresponding left eigenvector are proportional to each other.

We can also generalize the Altland-Zirnbauer classification to non-Hermitian 38-fold classification \[33\]. We focus on nine symmetry classes with a single symmetry (e.g., TRS or PHS), as summarized in Table I (see Appendix I for details \[48\]). In class \(D\) (\(C\)), matrices possess PHS and satisfy \(H^T = –H(\sigma^y H^T \sigma^y = –H)\). Class \(\text{AIII}\) has chiral symmetry (CS), which satisfies \(\sigma^y H^T \sigma^y = –H\). Finally, in class \(\text{AII}^\dagger\) (Hermitian conjugate of class \(\text{AIII}\)), sublattice symmetry (SLS) exists and matrices satisfy \(\sigma^y H^T \sigma^y = –H\). Note that these classes create nonlocal correlations between pairs of complex eigenvalues.

Nearest-level-spacing distributions. We now numerically calculate the nearest-neighbor spacing distributions \(p(s)\) in the complex plane away from the symmetric line (the real or imaginary axis) for random matrices in each symmetry class. Here, the nearest neighbor spacing \(s\) for each eigenvalue \(E_\alpha\) is defined as \(s = \min_{\beta(\neq \alpha)} |E_\alpha – E_\beta|\), where \(E_\alpha\) is the unfolded eigenvalue \[25\], and \(p(s)\) is normalized such that \(\int_0^\infty p(s)ds = \int_0^\infty sp(s)ds = 1\).

Figure 2(a) shows \(p(s)\) for Gaussian random matrices. Class \(A\) follows the distributions of GinUE, \(p\text{GinUE}(s) = CP(Cs)\) \[15, 25\], where \(\bar{p}(s) = \lim_{N \to \infty} \prod_{n=1}^{N-1} e_n(s^2)e^{-s^2} \prod_{n=1}^{N-1} \frac{2\pi^{2n+1}}{n!\rho_{v}(s^2)}\) with \(e_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!}\) and \(C = \int_0^\infty ds sp(s) = 1.1429\ldots\) \[25\]. We find similar results for classes \(\text{AI}, \text{AII}, D, C, \text{AIII}, \) and \(\text{AII}^\dagger\) \[48\], indicating that, while symmetries create pairs of eigenvalues, i.e., \((E_\alpha, E_\alpha^*), (E_\alpha, –E_\alpha^*), \) or \((E_\alpha, –E_\alpha^*),\) they do not alter the local correlations between neighboring eigenvalues away from the symmetric line (the real or imaginary axis).

By contrast, \(p(s)\) for classes \(\text{AI}\) and \(\text{AII}\) are distinct from the GinUE in that the peak is higher (lower) and the variance is smaller (larger) for class \(\text{AI}\) (\(\text{AII}\)) than that of the GinUE \[49\]. This is reminiscent of the Hermitian case, where the peak of the nearest-neighbor spacing distribution is higher for class \(\text{AI}\) (GOE) and lower for...
class AII (GSE) than class A (GUE).

We conjecture that GinUE for symmetry classes A, AI, AII, D, C, AIII, and AIII† and the other two distributions for classes AI† and AII† are universal [50]. To quantitatively confirm this, we calculate the cumulants of p(s) on Gaussian and Bernoulli random matrices regarding the symmetry constraints. Figure 2(b) shows that the second cumulants c2 of p(s) as a function of the matrix size for different symmetry classes and matrix ensembles (solid: Gaussian ensembles, dashed: Bernoulli ensembles). Classes A, AI, AII, D, C, AIII, and AIII† have the same cumulants which approach that of pGinUE(s) (thick solid lines), but c2 for classes AI† and AII† approach different values both for the Gaussian and Bernoulli ensembles. Statistics are taken from eigenvalues away from the edges of the spectrum in the shape of a circle and the symmetric line (real or imaginary axis). We average the data over 20000, 40000, 2000, 1000, 300 samples for the matrix size of 100, 500, 1000, 2000, 6000, respectively.

FIG. 2. (a) Nearest-neighbor spacing distributions p(s) of 2000 × 2000 Gaussian random matrices for classes A, AI†, and AII†. While class A obeys the GinUE distribution pGinUE(s), the peak of p(s) is lower for class AI† and higher for class AII† than that of pGinUE(s). (b) Second cumulants c2 of p(s) as a function of the matrix size for different symmetry classes and matrix ensembles (solid: Gaussian ensembles, dashed: Bernoulli ensembles). Classes A, AI, AII, D, C, AIII, and AIII† have the same cumulants which approach that of pGinUE(s) (thick solid lines), but c2 for classes AI† and AII† approach different values both for the Gaussian and Bernoulli ensembles. Statistics are taken from eigenvalues away from the edges of the spectrum in the shape of a circle and the symmetric line (real or imaginary axis). We average the data over 20000, 40000, 2000, 1000, 300 samples for the matrix size of 100, 500, 1000, 2000, 6000, respectively.

The analytic expressions of level-spacing distributions psmall with complex-valued degrees of freedom f for Gaussian-distributed small matrices are obtained as

\[
p_{\text{small}}(s) = \frac{(C_f s)^3}{N_f} K_{f-2}((C_f s)^2),
\]

where \(K_{\nu}(x) = \int_0^\infty dz e^{-x} \cosh z \cos(\nu z)\) is the modified Bessel function, and \(C_f\) and \(N_f\) are some constants. These analytic forms are derived in a unified manner: \(p_{\text{small}}(s)\) is understood by the distribution of

\[s = \sqrt{\sum_{i=1}^f x_i^2},\]

where \(x_i\) are complex Gaussian random variables. All of the three distributions are then obtained from transformations of the so-called \(K\)-distribution with a different shape parameter for each class [53] (see Appendix IV for details [48]). We also make a similar analysis for Hermitian small matrices: \(p_{\text{small}}(s)\) is understood by the distribution of

\[s = \sqrt{\sum_{i=1}^f x_i^2} \text{ with real Gaussian random variables } x_i, \]

which leads to simple transformations of the chi-squared distributions [48]. To our best knowledge, such a unified interpretation of level-spacing distributions for Hermitian matrices from the chi-squared distribution has never been presented.

While the level-repulsion factor for non-Hermitian \(p_{\text{small}}(s \to 0)\) is universally \(\sim s^3\) [54], the entire distribu-
TABLE II. Dyson’s Hermitian threefold way with TRS and our non-Hermitian threefold way with TRS† for Gaussian-distributed small matrices. Although they share the same degrees of freedom  = 2, 3, and 5, the difference between real-valued and complex-valued degrees leads to the different families of level-spacing distributions, i.e., the transformed chi-squared distribution and the K-distribution.

|  | Hermitian (TRS)  | Non-Hermitian (TRS†) |
|---|---|---|
|  |  as real variables |  as complex variables |
|  |  |  |
|  | 2 (A), 3 (A), 5 (AI) | 2 (AI†), 3 (A), 5 (AI†) |
|  |  |  |
|  | chi-squared distribution | K-distribution |
|  | AI > A > AI  | AI† > A > AI† |

The above properties for the peak and variance also hold true in numerical calculations of large matrices as seen from suppression or enhancement of the Ginibre distribution Fig. 2 [55]. As detailed in the next section, such changes due to transposition symmetry account for the distinctive behavior of local spectral statistics and that the randomness ϵ of the model in Eq. (3) is very different from random matrices in class AI or A in Fig. 2. Note that the Hamiltonian is an Ising model with transverse and longitudinal fields \( H = - \sum_{j=1}^{L} (1 + \epsilon_j) \sigma^z_j \sigma^z_{j+1} - \sum_{j=1}^{L} (-1.05 \sigma^x_j + 0.2 \sigma^y_j) \) with \( \epsilon_j \) randomly chosen from \([-0.1, 0.1]\) for each site \( j \) to break unwanted symmetries. Local dissipations \( \hat{\Gamma}_j \) are either (i) dephasing \( \hat{\sigma}^z_j \) or (ii) damping \( \hat{\sigma}^−_j \). This model can be realized using Rydberg atoms [78].

We consider the Liouvillian spectrum of the superoperator \( \mathcal{L} \) in Eq. (3). When we consider \( \mathcal{L} \) as a matrix, the model (i) has a transposition symmetry \( (\hat{\sigma}^x_j, \hat{\sigma}^z_j) \rightarrow (\hat{\sigma}^x_j, \hat{\sigma}^z_j) \) but the model (ii) does not. Both \( \mathcal{L} \) have additional TRS (see Appendix V [48]), but it does not affect the level-spacing distribution away from the real axis, since TRS merely create complex-conjugate pairs of eigenvalues.

Figure 3(b) shows \( p(s) \) for the two models (i) and (ii) after the unfolding procedure of the spectra [25]. We can clearly see that there appear distinct distributions that correspond to the universality classes of the random-matrix ensembles in class AI or A in Fig. 2. Note that the model in Eq. (3) is very different from random matrices in that the matrix is sparse due to the local interactions and that the randomness \( \epsilon_j \) is small. Nevertheless, our results show that local correlations of eigenenergies of nonintegrable dissipative Lindblad systems are well described by non-Hermitian random matrices considering transposition symmetry. We also find the universal results for non-Hermitian many-body systems which belong to class AI† as well as A or AI† (see Appendix IV for the visualization [48]). We can intuitively understand this behavior because \( p_{\text{small}}(s) \) is understood as the normalized distribution of \( s = \sqrt{\sum_{i=1}^{f} s_i^2 / f} \), whose variance becomes smaller for larger \( f \).

We introduce a dissipative one-dimensional spin-1/2 model described by the Lindblad equation \( \frac{d\hat{\rho}}{dt} = \mathcal{L}[\hat{\rho}] \) [77], where

\[
\mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_{j=1}^{L} \gamma \left[ \hat{\Gamma}_j \hat{\rho} \hat{\Gamma}^+_{j} - \frac{1}{2} \left\{ \hat{\Gamma}^+_{j} \hat{\Gamma}_{j}, \hat{\rho} \right\} \right].
\]

The Hamiltonian is an Ising model with transverse and longitudinal fields \( \hat{H} = - \sum_{j=1}^{L} (1 + \epsilon_j) \sigma^z_j \sigma^z_{j+1} \sum_{j=1}^{L} (-1.05 \sigma^x_j + 0.2 \sigma^y_j) \) with \( \epsilon_j \) randomly chosen from \([-0.1, 0.1]\) for each site \( j \) to break unwanted symmetries. Local dissipations \( \hat{\Gamma}_j \) are either (i) dephasing \( \hat{\sigma}^z_j \) or (ii) damping \( \hat{\sigma}^−_j \). This model can be realized using Rydberg atoms [78].
V [48]). These results show that the newly found universality classes play a key role in characterizing nonintegrability of dissipative many-body systems.

**Conclusion.** We have investigated the universality of the nearest-neighbor spacing distributions of non-Hermitian random matrices. We have numerically found new universality classes distinct from the GinUE exclusively for symmetry classes AI† and AII†. The three types of universality for classes A, AI†, AII† defined by TRS† (transposition symmetry) are regarded as a natural generalization of Dyson’s threefold universality for Hermitian matrices. Our results serve as a basis for characterizing nonintegrability and chaos in open quantum systems with symmetry.

Our work paves the way toward understanding universality in non-Hermitian systems and poses many interesting questions. From the fundamental viewpoint, it motivates further study to investigate if new universality appears for other statistics, such as correlation functions of distant eigenvalues and distributions of the edges of the spectrum, as well as to obtain joint probability distributions of eigenvalues for new classes for an arbitrary matrix dimension. As applications, TRS† appears for a wide range of physical systems in addition to dissipative many-body systems reported here, such as systems with gain and loss [36, 37], and classical many-body dynamics for a set of coupled oscillators [79]. It is an interesting future problem to investigate how our new classes lead to universal physical phenomena (such as nonintegrability-integrability transition) in those systems, as Dyson’s classes do in Hermitian systems.

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See Supplemental Material, which includes Refs. [81–88],
for the details on the symmetry classification, the Gaussian/Bernoulli distributions in each class, the numerical results for larger matrix sizes, the analytical results on the small-matrix calculations for non-Hermitian matrices, and the universality of dissipative quantum many-body systems.

The nearest-level-spacing distribution for class AII† is calculated through identification of the two degenerate eigenvalues.

The universality for class A and that for class AI have recently been rigorously proven under certain assumptions on the fourth moment of the random entries in the matrix elements [89].

The minimal matrix is four-by-four for class AII† to take into account the Kramers degeneracy.

One degree is dropped from all the variables because the trace of the matrix does not affect the level spacings.

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In the Hermitian case, the asymptotic behavior of chi-squared distributions leads to the class-dependent level-repulsion factor. However, the K-distribution has the same asymptotic behavior except for the logarithmic correction in class AII†, leading to the class-independent factor \( s^3 \) as noted in Ref. [12] for the non-Hermitian case.

The relation between cumulants higher than the second is not necessarily revealed by small matrices, since we ignore the effect of nearby eigenvalues other than two close ones in calculating \( \hat{p}_{\text{small}}(s) \).

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Other statistics, such as the statistics of the normalized largest eigenvalues of random Hermitian matrices (i.e., the Tracy-Widom distribution) are relevant for other physics such as fluctuations of a growing interface [90]. On the other hand, characterization of chaotic systems requires universal local statistics in the bulk of the spectrum, represented by the level-spacing distributions.

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Supplemental Material for “The Threefold Way in Non-Hermitian Random Matrices”

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I. SYMMETRY CLASSES AND THEIR FUNDAMENTAL PROPERTIES

We formulate symmetries in non-Hermitian systems and describe their fundamental properties according to the classification in Ref. [1]. We also derive explicit forms of the probability distributions of Gaussian ensembles and Bernoulli ensembles for each symmetry class. In the following, a complex eigenvalue is denoted as $E_\alpha$ and the corresponding right (left) eigenvector is denoted as $\phi_\alpha$ ($\chi_\alpha$):

$$H\phi_\alpha = E_\alpha \phi_\alpha, \quad \chi_\alpha^\dagger H = \chi_\alpha^\dagger E_\alpha. \quad (S-1)$$

A. Symmetry for non-Hermitian matrices

Non-Hermiticity alters the nature of symmetry in a fundamental manner. In particular, non-Hermiticity ramifies symmetry [1]. To see this symmetry ramification, let us consider time-reversal symmetry (TRS) as an example. For Hermitian matrices, TRS is defined by

$$T^\dagger H T^{-1} = H, \quad (S-2)$$

where $T$ is a unitary matrix. However, for non-Hermitian matrices, we can define TRS in another way. The crucial observation here is that complex conjugation is equivalent to transposition for Hermitian matrices by definition: $H^* = HT$. As a result, $T^* H T^{-1} = H$ is equivalent to $T^H T^{-1} = H$ for Hermitian $H$. However, since $H^* \neq HT$ for non-Hermitian matrices, we can define another symmetry $C_+$ as

$$C_+^\dagger HT C_+^{-1} = H, \quad (S-3)$$

where $C_+$ is a unitary matrix which is different from $T$. Since the physical TRS is described by Eq. (S-2) [1], we refer to the symmetry in Eq. (S-2) as TRS for non-Hermitian matrices in the main text and the following discussions, and the symmetry in Eq. (S-3) as TRS$^\dagger$.

Such symmetry ramification occurs also for all the other symmetries. For example, let us consider chiral symmetry (CS), which is defined for Hermitian matrices by

$$\Gamma H \Gamma^{-1} = -H, \quad (S-4)$$
where $\Gamma$ is a unitary matrix. Note that CS is equivalent to sublattice symmetry (SLS) for Hermitian matrices, which is defined without Hermitian conjugation (i.e., $\Gamma H \Gamma = -H$). Equation (S-4) can be directly generalized to non-Hermitian matrices, but again, CS can be generalized in a different manner. In fact, for non-Hermitian matrices, Eq. (S-4) is different from sublattice symmetry, defined by

$$SHS^{-1} = -H$$

because $H \neq H^\dagger$. Since the physical CS is described by Eq. (S-4) [1], we refer to the symmetry in Eq. (S-4) as CS, and the symmetry defined by Eq. (S-4) as CS$^\dagger$ or SLS for non-Hermitian matrices.

In a similar manner, particle-hole symmetry (PHS) for Hermitian matrices, satisfying $C H^T C^{-1} = C H^* C^{-1} = -H$, ramifies in the presence of non-Hermiticity. In the main text and the following discussions, we define PHS for non-Hermitian matrices by $C_- H^T C_-^{-1} = -H$ with a unitary matrix $C_-$. On the other hand, we define PHS$^\dagger$ for non-Hermitian matrices by $T_- H^* T_-^{-1} = -H$ with a unitary matrix $T_-$. Non-Hermiticity not only ramifies but also unifies symmetry [2]. To see this symmetry unification, we consider the following antiunitary symmetries:

$$T_+ H^* T_+^{-1} = H, \quad T_- H^* T_-^{-1} = -H.$$  \hspace{1cm} (S-6)

Here $T_+$ denotes TRS, while $T_-$ denotes PHS$^\dagger$. TRS and PHS are clearly disparate from each other for Hermitian matrices. However, when a non-Hermitian matrix $H$ respects TRS, the non-Hermitian matrix $iH$ respects PHS$^\dagger$. Thus, a set of all the non-Hermitian matrices with TRS coincides with a different set of all the non-Hermitian matrices with PHS$^\dagger$; non-Hermiticity unifies TRS and PHS$^\dagger$.

As a result of the symmetry ramification, the 5 classes (AIII, AI, D, AII, and C) for Hermitian matrices with a single symmetry (TRS, PHS, or CS) bifurcate into the 10 classes (AIII, AI, D, AII, C, AIII$^\dagger$, AI$^\dagger$, D$^\dagger$, AII$^\dagger$, and C$^\dagger$). Moreover, as a result of the symmetry unification, classes AI and D$^\dagger$, and classes AII and C$^\dagger$, are equivalent to each other. Adding class A, which has no symmetry, we have in total the 9 classes as listed in Table 1 in the main text for non-Hermitian classes where up to one symmetry is relevant (the other 29 classes have more than one symmetry, such as PHS and TRS). We note that the entire 10 Altland-Zirnbauer symmetry classes for Hermitian matrices [3] ramify into 38 symmetry classes for
non-Hermitian matrices [1]. In the following, we describe basic properties of non-Hermitian matrices for each of the above-mentioned 9 classes.

B. Class A

Matrices in class A are not constrained by any symmetry and thus include most general non-Hermitian matrices. The probability distribution of a Gaussian ensemble is given as

\[
P(H) \propto e^{-\beta \text{Tr}[H^\dagger H]} \propto \exp \left[ -\beta \sum_{i,j} |H_{ij}|^2 \right] \prod_{i,j} dH_{ij} dH_{ij}^*,
\]  
(S-7)

where \( H_{ij} \) is the element in the \( i \)th row and \( j \)th column of the matrix \( H \). On the other hand, for a Bernoulli ensemble, each matrix element is randomly chosen as

\[
H_{ij} = \begin{cases} 
1 + i & \text{with probability } 1/4; \\
1 - i & \text{with probability } 1/4; \\
-1 + i & \text{with probability } 1/4; \\
-1 - i & \text{with probability } 1/4.
\end{cases}
\]  
(S-8)

C. Class AI and class D†

Matrices in class AI respect TRS defined by

\[
\mathcal{T}_+ H^* \mathcal{T}_+^{-1} = H, \quad \mathcal{T}_+ \mathcal{T}_+^* = +1,
\]  
(S-9)

where \( \mathcal{T}_+ \) is a unitary matrix (i.e., \( \mathcal{T}_+ \mathcal{T}_+^\dagger = \mathcal{T}_+^\dagger \mathcal{T}_+ = 1 \)). In the presence of TRS, we have

\[
H (\mathcal{T}_+ \phi_\alpha^*) = \mathcal{T}_+ H^* \phi_\alpha^* = E_\alpha^* (\mathcal{T}_+ \phi_\alpha^*).
\]  
(S-10)

Hence, \( \mathcal{T}_+ \phi_\alpha^* \) is also an eigenvector of \( H \) with its eigenvalue \( E_\alpha^* \), and eigenvalues form \( (E_\alpha, E_\alpha^*) \) pairs in general. Note that the eigenvalue remains real if the corresponding eigenvector satisfies \( \mathcal{T}_+ \phi_\alpha^* \propto \phi_\alpha \). Similarly, matrices in class D† respect the Hermitian conjugate of PHS, denoted as PHS†:

\[
\mathcal{T}_- H^* \mathcal{T}_-^{-1} = -H, \quad \mathcal{T}_- \mathcal{T}_-^* = +1,
\]  
(S-11)

which leads to \( (E_\alpha, -E_\alpha^*) \) pairs in the complex plane. Notably, symmetry classes AI and D† are equivalent to each other [2]. In fact, when a non-Hermitian matrix \( H \) satisfies Eq. (S-9)
and belongs to class AI, another non-Hermitian matrix $iH$ satisfies Eq. (S-11) and belongs to class $D^\dagger$. In particular, the level-spacing distributions are the same for classes AI and $D^\dagger$, provided that the spectrum is rotated by 90 degrees in the complex plane.

For Gaussian ensembles, we can assume that $T_+$ is the identity operator without loss of generality, which leads to

$$H_{ij} = \bar{H}_{ij}^*.$$  \hspace{1cm} (S-12)

Thus, matrices in class AI can be represented as real non-Hermitian matrices. Indeed, the condition (S-9) means that $U^{-1}HU$ is real, where $U = \sqrt{T_+}$ is unitary. Since $P(H)dH = P(U^{-1}HU)d(U^{-1}HU)$ for Gaussian ensembles, we can consider the ensemble of real matrices in Eq. (S-12). Then we can consider the probability distribution of a Gaussian ensemble given as

$$P(H)\,dH \propto \exp\left[-\beta \sum_{i,j} H_{ij}^2\right]\prod_{i,j} dH_{ij}.$$  \hspace{1cm} (S-13)

While a Bernoulli ensemble depends on the explicit form of $T_+$, we here consider $T_+ = 1$. Then each matrix element is randomly chosen as

$$H_{ij} = \begin{cases} 1 & \text{with probability } 1/2; \\ -1 & \text{with probability } 1/2. \end{cases}$$  \hspace{1cm} (S-14)

D. Class AII and class $C^\dagger$

Matrices in class AII respect TRS defined by

$$T_+H^*T_+^{-1} = H, \quad T_+T_+^* = -1.$$  \hspace{1cm} (S-15)

In analogy with class AI, eigenvalues form $(E_\alpha, E_\alpha^*)$ pairs in general. In addition, class AII is equivalent to class $C^\dagger$, whose matrices respect PHS$^\dagger$:

$$T_-H^*T_-^{-1} = -H, \quad T_-T_-^* = -1.$$  \hspace{1cm} (S-16)

In contrast to the Hermitian case and class AII$^\dagger$ described below, only those eigenvalues that lie on the real axis can be two-fold degenerate through formation of Kramers pairs, while generic eigenvalues away from the real axis are not degenerate because they do not form
Kramers pairs. However, Kramers pairs on the real axis do not exist for almost all random matrices, since eigenvalues for generic matrices exhibit level repulsions.

We choose $\mathcal{T}_+$ as the Pauli matrix $\sigma_y$, which again allows us to describe the probability distribution for Gaussian ensembles for general $\mathcal{T}_+$ with $\mathcal{T}_+ \mathcal{T}_+^* = -1$. In this case, we obtain

$$H = I \otimes a + i \sigma_x \otimes b + i \sigma_y \otimes c + i \sigma_z \otimes d,$$

where $a$, $b$, $c$, and $d$ are real non-Hermitian matrices. The probability distribution of a Gaussian ensemble is then given as

$$P(H) \, dH \propto \exp \left[ -\beta \sum_{i,j} \left( a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + d_{ij}^2 \right) \right] \prod_{i,j} da_{ij} db_{ij} dc_{ij} dd_{ij}. \quad (S-18)$$

On the other hand, for a Bernoulli ensemble, each element is randomly chosen as

$$a_{ij}, b_{ij}, c_{ij}, d_{ij} = \begin{cases} 1 \text{ with probability 1/2;} \\ -1 \text{ with probability 1/2.} \end{cases} \quad (S-19)$$

### E. Class AI$^\dagger$

Matrices in class AI$^\dagger$ respect the Hermitian conjugate of TRS, i.e., TRS$^\dagger$, defined by

$$C_+ H^T C_+^{-1} = H, \quad C_+ C_+^* = +1, \quad (S-20)$$

where $C_+$ is a unitary matrix (i.e., $C_+ C_+^\dagger = C_+^\dagger C_+ = 1$). Noting that the transpose of the eigenequation (S-1) gives

$$H^T \chi_\alpha^* = E_n \chi_\alpha^*, \quad (S-21)$$

we have, in the presence of TRS$^\dagger$,

$$H (C_+ \chi_\alpha^*) = C_+ H^T \chi_\alpha^* = E_\alpha (C_+ \chi_\alpha^*). \quad (S-22)$$

Hence, $C_+ \chi_\alpha^*$ is also an eigenvector of $H$ having the same eigenvalue $E_\alpha$ as $\phi_\alpha$. Thus, if there is no degeneracy, the constraint

$$C_+ \chi_\alpha^* \propto \phi_\alpha \quad (S-23)$$

is imposed on the right and left eigenvectors. Importantly, this symmetry constraint is imposed for all the eigenvectors in the entire complex plane in stark contrast to class AI,
which leads to a new universality class of the level-spacing distribution as demonstrated in the main text.

We assume that $C_+$ is the identity (which again allows us to describe the probability distribution for Gaussian ensembles for general $C_+$ with $C_+C_+^* = 1$), which leads to

$$H_{ij} = H_{ji}. \quad (S-24)$$

Thus, matrices in class AI† can be represented as symmetric non-Hermitian matrices. The probability distribution of a Gaussian ensemble is then given as

$$P(H) dH \propto \exp \left[ -\beta \left( \sum_i |H_{ij}|^2 + \sum_{i>j} 2|H_{ij}|^2 \right) \right] \prod_{i \geq j} dH_{ij} dH_{ij}^*. \quad (S-25)$$

On the other hand, for a Bernoulli ensemble, each matrix element is randomly chosen as

$$H_{ij} = \begin{cases} 1 + i & \text{with probability } 1/4; \\ 1 - i & \text{with probability } 1/4; \\ -1 + i & \text{with probability } 1/4; \\ -1 - i & \text{with probability } 1/4 \end{cases} \quad (S-26)$$

under the constraint (S-24).

F. Class AII†

Matrices in class AII† respect TRS† defined by

$$C_+ H^T C_+^{-1} = H, \quad C_+ C_+^* = -1. \quad (S-27)$$

Importantly, there is a non-Hermitian generalization of Kramers degeneracy theorem in class AII† [4, 5]. In fact, from $C_+^T C_+^{-1} = -1$, we have

$$\chi_\alpha^T C_+ \chi_\alpha^* = (\chi_\alpha^T C_+ \chi_\alpha)^T = \chi_\alpha^T C_+^T \chi_\alpha^* = -\chi_\alpha^T C_+ \chi_\alpha^*, \quad (S-28)$$

which leads to $\chi_\alpha^T C_+ \chi_\alpha^* = 0$. Thus, the eigenvectors $\phi_\alpha$ and $C_+ \chi_\alpha^*$ are biorthogonal to each other and linearly independent of each other. This independence implies that all the eigenvectors are at least two-fold degenerate in the presence of TRS† with $C_+ C_+^* = -1$. Since all the eigenvectors are subject to this non-Hermitian extension of the Kramers degeneracy, class AII† is sharply contrasted with class AII in non-Hermitian systems.
We choose $C_+$ as the Pauli matrix $\sigma_y$, which allows us to describe the probability distribution for Gaussian ensembles for general $C_+$ with $C_+ C_+^\dagger = -1$. We then have

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{(S-29)}$$

where non-Hermitian matrices $a$, $b$, $c$, and $d$ satisfy

$$a = d^T, \quad b = -b^T, \quad c = -c^T. \quad \text{(S-30)}$$

The probability distribution of a Gaussian ensemble is then given as

$$P(H) dH \propto \exp \left\{ -2\beta \left[ \sum_i |a_{ii}|^2 + \sum_{i>j} (|a_{ij}|^2 + |b_{ij}|^2 + |c_{ij}|^2 + |d_{ij}|^2) \right] \right\} \prod_i da_{ii} da_{ii}^* \prod_{i>j} da_{ij} db_{ij} dc_{ij} dd_{ij}. \quad \text{(S-31)}$$

On the other hand, for a Bernoulli ensemble, each element is randomly chosen as

$$a_{ij}, b_{ij}, c_{ij}, d_{ij} = \begin{cases} 1 + i & \text{with probability } 1/4; \\ 1 - i & \text{with probability } 1/4; \\ -1 + i & \text{with probability } 1/4; \\ -1 - i & \text{with probability } 1/4 \end{cases} \quad \text{(S-32)}$$

under the constraint (S-30).

**G. Class D**

Matrices in class D respect PHS defined by

$$C_- H^T C_-^{-1} = -H, \quad C_- C_-^* = +1, \quad \text{(S-33)}$$

where $C_-$ is a unitary matrix (i.e., $C_- C_-^\dagger = C_+^\dagger C_- = 1$). In analogy with classes A$^\dagger$ and AII$^\dagger$, we have

$$H \left( C_- \chi_* \right) = -C_- H^T \chi_*^* = -E_\alpha \left( C_- \chi_* \right). \quad \text{(S-34)}$$

Hence, $C_- \chi_*^*$ is also an eigenvector of $H$ with its eigenvalue $-E_\alpha$, and eigenvalues form $(E_\alpha, -E_\alpha)$ pairs in general. Thus, PHS just makes pairs of eigenvalues $(E_\alpha, -E_\alpha)$ and imposes no local constraints on generic eigenvectors away from the zero eigenvalue, which is similar to TRS and PHS$^\dagger$, and in contrast with TRS$^\dagger$. 

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We assume that $C$ is the identity operator, which allows us to describe the probability distribution for Gaussian ensembles for general $C$ with $C C^* = 1$. We then have

$$H_{ij} = -H_{ji}. \quad (S-35)$$

Thus, matrices in class D can be represented as antisymmetric non-Hermitian matrices. The probability distribution of a Gaussian ensemble is then given as

$$P(H) dH \propto \exp \left[ -2\beta \sum_{i>j} |H_{ij}|^2 \right] \prod_{i>j} dH_{ij} dH_{ij}^*. \quad (S-36)$$

On the other hand, for a Bernoulli ensemble, each matrix element is randomly chosen as

$$H_{ij} = \begin{cases} 1 + i & \text{with probability } 1/4; \\ 1 - i & \text{with probability } 1/4; \\ -1 + i & \text{with probability } 1/4; \\ -1 - i & \text{with probability } 1/4 \end{cases} \quad (S-37)$$

subject to the constraint (S-35).

**H. Class C**

Matrices in class C respect PHS defined by

$$C_- H^T C_-^{-1} = -H, \quad C_- C_-^* = -1. \quad (S-38)$$

In analogy with class D, eigenvalues form $(E_\alpha, -E_\alpha)$ pairs in general.

We choose $C_-$ as the Pauli matrix $\sigma_y$, which allows us to describe the probability distribution for Gaussian ensembles for general $C$ with $C_- C_-^* = -1$. We then have Eq. (S-29) with

$$a = -d^T, \quad b = b^T, \quad c = c^T. \quad (S-39)$$

The probability distribution of a Gaussian ensemble is then given as

$$P(H) dH \propto \exp \left\{ -2\beta \left[ \sum_i (|a_{ii}|^2 + |b_{ii}|^2/2 + |c_{ii}|^2/2) + \sum_{i>j} (|a_{ij}|^2 + |b_{ij}|^2 + |c_{ij}|^2 + |d_{ij}|^2) \right] \right\}$$

$$\times \prod_i da_{ii} da_{ii}^* db_{ii} db_{ii}^* dc_{ii} dc_{ii}^* dd_{ii} dd_{ii}^*. \quad (S-40)$$
On the other hand, for a Bernoulli ensemble, each element is randomly chosen as

\[
a_{ij}, b_{ij}, c_{ij}, d_{ij} = \begin{cases} 
1 + i & \text{with probability } 1/4; \\
1 - i & \text{with probability } 1/4; \\
-1 + i & \text{with probability } 1/4; \\
-1 - i & \text{with probability } 1/4 
\end{cases}
\]  

(S-41)

subject to the constraint (S-39).

I. Class AIII

Matrices in class AIII respect CS defined by

\[
\Gamma H^\dagger \Gamma^{-1} = -H, \quad \Gamma^2 = 1,
\]  

(S-42)

where \( \Gamma \) is a unitary matrix (i.e., \( \Gamma \Gamma^\dagger = \Gamma^\dagger \Gamma = 1 \)). In the presence of CS, we have

\[
H (\Gamma \chi_\alpha) = -\Gamma H^\dagger \chi_\alpha = -E^*_\alpha (\Gamma \chi_\alpha).
\]  

(S-43)

Hence, \( \Gamma \chi_\alpha \) is also an eigenvector of \( H \) with its eigenvalue \(-E^*_\alpha\), and eigenvalues form \((E_\alpha, -E^*_\alpha)\) pairs in general. In analogy with the equivalence between TRS and PHS\(^\dagger\) [2], CS is equivalent to pseudo-Hermiticity [6] which is defined by the presence of the unitary matrix \( \eta \) such that

\[
\eta H^\dagger \eta^{-1} = H, \quad \eta^2 = 1.
\]  

(S-44)

This condition implies the presence of \((E_\alpha, E^*_\alpha)\) pairs in the complex plane. In fact, when a non-Hermitian matrix \( H \) satisfies Eq. (S-42) and respects CS, another non-Hermitian matrix \( iH \) satisfies Eq. (S-44) and respects pseudo-Hermiticity. We note that unlike Hermitian systems, CS is distinct from sublattice symmetry (SLS) defined by Eq. (S-48).

We choose \( \Gamma \) to be a Pauli matrix \( \sigma_z \) to obtain a nontrivial result, which leads to Eq. (S-29) with

\[
a = -a^\dagger, \quad b = c^\dagger, \quad c = b^\dagger, \quad d = -d^\dagger.
\]  

(S-45)

We note that \( H \) always reduces to an anti-Hermitian matrix when we take a special choice \( \Gamma = I \), which trivially reduces to class A in Hermitian systems and is not considered here.
The probability distribution of a Gaussian ensemble is then given as

\[
P(H) \, dH \propto \exp \left\{ -\beta \left[ \sum_i (|a_{ii}|^2 + |d_{ii}|^2 + 2|b_{ii}|^2) + 2 \sum_{i>j} (|a_{ij}|^2 + |b_{ij}|^2 + |c_{ij}|^2 + |d_{ij}|^2) \right] \right\}
\times \prod_i da_{ii} dd_{ii} db_{ii}^{*} \prod_{i>j} da_{ij} db_{ij}^{*} dc_{ij}^{*} dd_{ij}^{*}.
\]

On the other hand, for a Bernoulli ensemble, each element is randomly chosen as

\[
a_{ij}, b_{ij}, c_{ij}, d_{ij} = \begin{cases} 
1 + i & \text{with probability } 1/4; \\
1 - i & \text{with probability } 1/4; \\
-1 + i & \text{with probability } 1/4; \\
-1 - i & \text{with probability } 1/4 
\end{cases}
\]

subject to the constraint (S-47).

**J. Class AIII**

Matrices in class AIII respect SLS defined by

\[
SHS^{-1} = -H, \quad S^2 = 1,
\]

where \(S\) is a unitary matrix (i.e., \(SS^\dagger = S^\dagger S = 1\)). In the presence of SLS, we have

\[
H(S\phi_{\alpha}) = -SH\phi_{\alpha} = -E_{\alpha} (S\phi_{\alpha}).
\]

Hence, \(S\phi_{\alpha}\) is also an eigenvector of \(H\) with its eigenvalue \(-E_{\alpha}\), and eigenvalues form \((E_{\alpha}, -E_{\alpha})\) pairs in general.

We choose \(S\) as the Pauli matrix \(\sigma_z\) to obtain a nontrivial result, which leads to Eq. (S-29) with

\[
a = d = 0.
\]

In analogy with class AIII, we do not consider the special choice \(S = I\), which leads to the trivial case \(H = 0\). The probability distribution of a Gaussian ensemble is then given as

\[
P(H) \, dH \propto \exp \left[ -\beta \sum_{i,j} (|b_{ij}|^2 + |c_{ij}|^2) \right] \prod_{i,j} db_{ij} dc_{ij}.
\]
On the other hand, for a Bernoulli ensemble, each element is randomly chosen as

\[ b_{ij}, c_{ij} = \begin{cases} 
  1 + i & \text{with probability } 1/4; \\
  1 - i & \text{with probability } 1/4; \\
  -1 + i & \text{with probability } 1/4; \\
  -1 - i & \text{with probability } 1/4.
\] (S-52)

II. DETAILED NUMERICAL RESULTS FOR RANDOM MATRICES

We here show the detailed numerical results of the nearest-level-spacing distributions \( p(s) \) to confirm the universality. We first show the results of \( p(s) \) for all of the nine classes with a matrix size larger than that discussed in the main text. Figure S-1 shows the results of \( p(s) \) for 6000 \( \times \) 6000 matrices with several different symmetries. We find that classes A, AI, AII, D, C, AIII, and AIII\(^\dagger\) obey the distribution for GinUE, class AI\(^\dagger\) has a lower peak, and class AII\(^\dagger\) has a higher peak than the GinUE. Similar results are obtained for the Bernoulli ensembles.

Next, to confirm our universality more quantitatively, we study up to the fourth cumulants of \( p(s) \) as a function of the size of a matrix for different symmetry classes and matrix distributions (Gaussian or Bernoulli). Note that the second, third, and fourth cumulants are given by

\[
\begin{align*}
 c_2 &= m_2 - m_1^2, \\
 c_3 &= m_3 - 3m_1m_2 + 2m_2^3, \\
 c_4 &= m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4,
\end{align*}
\] (S-53-55)

respectively, where

\[
m_k = \int_0^\infty ds s^k p(s) \quad \text{(S-56)}
\]

is the \( k \)th moment. We also note that we have fixed the scale of \( s \) such that \( m_1 = 1 \). As shown in Fig. S-2, we find the following properties that strengthen our argument on the universality:

- The results for the Gaussian and Bernoulli ensembles in the same symmetry class are almost the same for all the cases. Small deviations for the largest matrices are attributed to the limited number of samples used in our analysis.
FIG. S-1. Nearest-level-spacing distributions $p(s)$ for random matrices whose elements obey Gaussian distributions with their respective symmetry constraints. The distributions $p(s)$ for classes A, AI, D, C, AII, AIII, and AIII† obey the GinUE distribution $p_{\text{GinUE}}(s)$. In contrast, the peak of $p(s)$ is higher for class AI† and lower for class AII† compared with that of the GinUE. The results are obtained from diagonalization of 6000 $\times$ 6000 matrices and the average over 300 ensembles. Statistics are taken from eigenvalues that are away from the edges of the spectrum and the symmetric line (the real axis or the imaginary axis) when it exists.

- For sufficiently large matrix sizes, we clearly find three distinct universality classes even for high-order cumulants: classes A, AI, AII, D, C, AIII, and AIII† have the same cumulants, but classes AI† and AII† have different ones.

- The cumulants $c_{A,2}$, $c_{A,3}$, and $c_{A,4}$ approach values calculated from the exact distribution $p_{\text{GinUE}}(s)$ ($c_{A,2} = 0.0875$, $c_{A,3} = -0.000471$, and $c_{A,4} = -0.00215$). However, the cumulants for classes AI† and AII† approach different values, which indicate that the newly found distributions in AI† and AII† are distinct from that of the GinUE even in the infinite-size limit. From Fig. S-2, we conjecture that $c_{\text{AI}†,2} \approx 0.11$, $c_{\text{AII}†,2} \approx 0.075$, $c_{\text{AI}†,3} \approx 0.004$, $c_{\text{AII}†,3} \approx -0.0025$, and $c_{\text{AI}†,4} \approx -0.003$, $c_{\text{AII}†,4} \approx -0.001$. 
FIG. S-2. Second, third, and fourth cumulants as a function of the matrix size for different symmetry classes and matrix ensembles (solid lines: the Gaussian ensembles, dashed lines: the Bernoulli ensembles). The results for the Gaussian and Bernoulli ensembles of the same symmetry class are almost the same for all the cases. For sufficiently large matrix sizes, we find three distinct universality classes even for high-order cumulants: classes A, AI, AII, D, C, AIII, and AIII† have the same cumulants that approach values calculated from the exact distribution \( p_{\text{GinUE}}(s) \) (thick solid lines), but the cumulants for classes AI† and AII† approach different values. Statistics are taken from eigenvalues that are away from the edges of the spectrum and the symmetric line (the real axis or the imaginary axis) when it exists. We average the data over 20000, 4000, 2000, 1000, 300 samples for the matrix size of 100, 500, 1000, 2000, 6000, respectively.

III. DEGENERATE PERTURBATION THEORY AND LEVEL CORRELATIONS

We analyze two-by-two or four-by-four random matrices for the qualitative understanding of the behavior of larger random matrices on the basis of Refs. [7, 8]. We consider a situation in which elements of a random matrix \( H \) are slightly perturbed in a symmetry-preserving
manner. The eigenvalues of the original matrix are correlated with this perturbation $V$. To simplify the problem, let us assume that two (different) eigenvalues are much closer to each other than to the rest. Then the repulsion of these two eigenvalues can be estimated from diagonalization of $V$ in the subspace spanned by the corresponding eigenvectors. We expect that qualitative behavior of the nearest-level-spacing distributions can be described by this method.

The matrices that we obtain by the above method depend on the symmetry of $H$ (and equivalently $V$). Let us first consider the simplest case, class A, and let $\phi_1 (\chi_1)$ and $\phi_2 (\chi_2)$ be the corresponding right (left) eigenvectors. In this case, the corresponding two-by-two matrix is

$$
\begin{pmatrix}
\chi_1^\dagger V \phi_1 & \chi_1^\dagger V \phi_2 \\
\chi_2^\dagger V \phi_1 & \chi_2^\dagger V \phi_2
\end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

(S-57)

Since $\phi_1$ and $\phi_2$ become independent random vectors for large $H$ and no direct relation between $\phi_\alpha$ and $\chi_\alpha$ exists in general, $a, b, c, d \in \mathbb{C}$ can be regarded as independent random variables. By taking $a, b, c$, and $d$ as Gaussian random variables, we obtain the matrix $H_{\text{small},A}$ in Eq. (1) in the main text [9].

Next, we consider class AI. Whereas time-reversal symmetry imposes some constraints on two eigenvectors that are placed symmetrically around the real axis, it does not on two close eigenvectors that are off the real axis. Thus, the obtained matrix again has the form of Eq. (S-57). Note that $a, b, c$, and $d$ are complex despite time-reversal symmetry of $V$, since the eigenvectors $\phi_1$ and $\phi_2$ spontaneously break time-reversal symmetry. In this sense, the interaction between two eigenvalues that are off the real axis for class AI is also characterized by $H_{\text{small},A}$. The discussion also holds true for classes AII, D, C, AIII, and AIII*: The eigenvectors $\phi_1$ and $\phi_2$ away from the real axis are regarded as two independent random vectors subject to no constraint, and we can consider $H_{\text{small},A}$. In other words, global reflection symmetry of the spectrum in the complex plane does not affect the local statistics away from the real axis, as noted in Ref. [8].

On the other hand, the situation is different for classes with the Hermitian conjugate $C_+$ of time-reversal symmetry. Let us consider class AI*. In this case, we have a condition $C_+ \chi_\alpha^* = \phi_\alpha$ (we have chosen the proportionality factor to be unity). Then, while $\chi_1^\dagger V \phi_1$ and $\chi_2^\dagger V \phi_2$ become some independent complex variables, we obtain the following relation...
for off-diagonal terms:

\[
\chi_1^T V \phi_2 = (\chi_1^T V \phi_2)^T \\
= \phi_2^T V^T \chi_1^* \\
= \chi_2^T C_+^T V^T C_+^{-1} \phi_1 \\
= \chi_2^T C_+^T V^T C_+^{-1} \phi_1 \\
= \chi_2^T V \phi_1,
\]

(S-58)

where we have used \( C_+^T = C_+ \) for class AI\(^\dagger\). Thus, we obtain the symmetric matrix \((b = c)\) in Eq. (S-57), which leads to \( H_{\text{small, AI}} \) in Eq. (1) in the main text.

Finally, we consider class AII\(^\dagger\). Since \( H \) has the Kramers degeneracy, we need to consider the four-by-four matrices spanned by \( \phi_1, \bar{\phi}_1, \phi_2, \) and \( \bar{\phi}_2 \) together with the corresponding left eigenvectors \( \chi_1, \chi_2, \) and \( \bar{\chi}_2 \), where \( \chi_1, \chi_2, \) and \( \bar{\chi}_2 \), where \( \bar{\phi}_\alpha = C_+ \chi_\alpha^* \) and \( \bar{\chi}_\alpha = (\phi_\alpha^T C_+^{-1})^T \) are Kramers conjugate vectors. We have the following relation:

\[
\bar{\chi}_\alpha^T V \bar{\phi}_\beta = \phi_\alpha^T C_+^{-1} V C_+ \chi_\beta^* \\
= (\phi_\alpha^T C_+^{-1} V C_+ \chi_\beta^*)^T \\
= \chi_\beta^T C_+^T V^T (C_+^T)^{-1} \phi_\alpha \\
= \chi_\beta^T C_+^T V^T C_+^{-1} \phi_\alpha \\
= \chi_\beta^T V \phi_\alpha,
\]

(S-59)

where we have used \( C_+^T = -C_+ \) for class AII\(^\dagger\). We also have

\[
\chi_\alpha^T V \bar{\phi}_\beta = (\chi_\alpha^T V C_+ \chi_\beta^*)^T \\
= \chi_\beta^T C_+^T V^T \chi_\alpha^* \\
= -\chi_\beta^T C_+^T V^T \chi_\alpha^* \\
= -\chi_\beta^T V C_+ \chi_\alpha^* \\
= -\chi_\beta^T V \bar{\phi}_\alpha
\]

(S-60)
and

\[ \chi_\alpha^\dagger V \phi_\beta = (\phi_\beta^T C_+^{-1} V \phi_\beta)^T \]
\[ = \phi_\beta^T V^T (C_+^T)^{-1} \phi_\alpha \]
\[ = -\phi_\beta^T V^T C_+^{-1} \phi_\alpha \]
\[ = -\phi_\beta^T C_+^{-1} V \phi_\alpha \]
\[ = -\chi_\beta^\dagger V \phi_\alpha. \] (S-61)

Combining them, we obtain \( H_{\text{small}, \text{AI}}^\dagger \) in Eq. (1) in the main text.

**IV. PROOF OF EQ. (2) IN THE MAIN TEXT AND ITS MEANING**

In this section, we compute the level-spacing probability distributions of \( H_{\text{small}, A} \), \( H_{\text{small}, \text{AI}} \), and \( H_{\text{small}, \text{AI}}^\dagger \). First, we express the level spacing \( s \) in terms of the following stochastic variable:

\[ X_f = z_1^2 + \cdots + z_f^2, \] (S-62)

where \( z_1, \ldots, z_f \) are independently and identically distributed (i.i.d.) complex variables under the Gaussian distribution \( P(z) \propto e^{-|z|^2} \).

We show below the following relation

\[ s \propto |X_f|^{1/2}, \] (S-63)

where \( f = 2, 3, \) and \( 5 \) for classes \( \text{AI}^\dagger, A, \) and \( \text{AI}^{\dagger}\), respectively. After showing (S-63), we derive the probability distribution function of \( |X_f|^2 \), from which the level-spacing distribution follows.

**A. Derivation of Eq. (S-63)**

For classes \( A, \text{AI}^\dagger, \) and \( \text{AI}^{\dagger}\), the set of all the matrices with the corresponding symmetry forms a complex vector space closed under Hermitian conjugation. Therefore, a random \( n \times n \) matrix \( H \) can take the following form:

\[ H = z_0 I + \sum_{j=1}^{f} z_j L_j, \quad z_0, z_1, \ldots, z_f \in \mathbb{C}, \] (S-64)
TABLE S-1. Examples of matrix bases for classes AI†, A, and AII†. Note that $\sigma_{x,y,z}$ are Pauli matrices and that $\gamma_{1,2,3,4,5}$ are the Dirac matrices according to the notation in Ref. [10].

| Class | $f$ | Basis |
|-------|-----|-------|
| AI†   | 2   | $\sigma^x, \sigma^z$ |
| A     | 3   | $\sigma^x, \sigma^y, \sigma^z$ |
| AII†  | 5   | $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ |

where $L_1, \ldots, L_f$ are Hermitian matrices satisfying

$$\text{Tr} L_j = 0, \quad \text{Tr} L_i L_j = n \delta_{ij}. \quad (S-65)$$

We regard the coefficients $z_0, \ldots, z_f$ as random variables rather than matrix entries. The condition (S-65) implies that $z_0, z_1, \ldots, z_n$ are i.i.d. Gaussian variables:

$$P(H) \propto \exp(-\beta \text{Tr}[H^\dagger H]) = \exp[-n\beta(|z_0|^2 + |z_1|^2 + \cdots + |z_f|^2)]. \quad (S-66)$$

Since the term $z_0 I$ does not affect the level spacing, we eliminate $z_0$ from Eq. (S-64) and instead consider the traceless random matrix:

$$\tilde{H} = \sum_{j=1}^f z_j L_j. \quad (S-67)$$

Now we consider $H_{\text{small},A}$, $H_{\text{small},\text{AI}†}$, and $H_{\text{small},\text{AII}†}$ defined in Eq. (1) in the main text. In each of these matrices, the matrix basis $\{L_1, \ldots, L_f\}$ can be taken as listed in Table S-1. The matrices in the basis become spinor matrices characterized by the anticommutation relation $\frac{1}{2}(L_i L_j + L_j L_i) = \delta_{ij} I$. As a consequence, the traceless matrix $\tilde{H}_{\text{small}}$ obtained from $H_{\text{small}}$ satisfies

$$\tilde{H}_{\text{small}}^2 = (z_1^2 + \cdots + z_f^2) I. \quad (S-68)$$

At the same time, $\tilde{H}_{\text{small}}$ has only two distinct eigenvalues $(\lambda, -\lambda)$ of the same multiplicity. Therefore, we also have the following equality:

$$\tilde{H}_{\text{small}}^2 = \lambda^2 I. \quad (S-69)$$

Comparing Eqs. (S-68) and (S-69), the level spacing of $\tilde{H}_{\text{small}}$ (and hence of $H_{\text{small}}$) is found to be described by the following quantity:

$$s = 2|\lambda| = 2\sqrt{n\beta}|X_f|^{1/2}, \quad (S-70)$$
where the factor $\sqrt{n\beta}$ arises from the difference in the scaling between the random variables in (S-66) and those in (S-62).

A similar argument applies to the analysis of Hermitian random matrices if we use real Gaussian variables instead of complex ones. In this case, the stochastic variable $X_f$ is replaced by $X'_f = x_1^2 + \cdots + x_f^2$ with real Gaussian variables $x_1, \ldots, x_f$. Then, $X'_f$ is subject to the chi-squared distribution: $P(X'_f = r) = \chi^2_f(r) \propto r^{f-1}e^{-r}$. Therefore, the level-spacing distribution is the transformed chi-squared distribution,

$$p_{\text{small}}(s) = 2s\chi^2_f(s^2) \propto s^{f-1}e^{-(Cs)^2}, \quad \text{(S-71)}$$

where the level repulsion $O(s^{f-1})$ depends on the degree $f$ of the chi-squared distribution.

Since $p(s)$ for large matrices is well approximated by $p_{\text{small}}$ in the Hermitian case, it is also approximately described by $\chi^2_f(X'_f)$.

In the following, the distribution of $X_f$ for non-Hermitian matrices is shown to be the $K$-distribution. The degree of the level repulsions is confirmed to be independent of $f \geq 2$ except for a logarithmic correction at $f = 2$.

**B. Distribution function of $|X_f|^2$**

The probability density of the squared modulus $|X_f|^2$ is given by

$$P(|X_f|^2 = \rho) = \int \delta(\rho - |z_1^2 + \cdots + z_f^2|^2) \pi^{-f}e^{-\|z\|^2}d^2fz, \quad \text{(S-72)}$$

where $z = (z_1, \ldots, z_f)$ is a complex vector. Let two real vectors $u$ and $v$ be the real and imaginary parts of $z$ ($z = u + iv$), respectively. Then, the variable $X_f$ is expressed in terms of the magnitudes $u, v$ of $u, v$ and the angle $\theta$ between them as

$$X_f = \|u\|^2 - \|v\|^2 + 2iu \cdot v = u^2 - v^2 + 2iuv \cos \theta. \quad \text{(S-73)}$$

Thus,

$$|X_f|^2 = u^4 + v^4 + 2u^2v^2(2\cos^2 \theta - 1). \quad \text{(S-74)}$$

Replacing $(u, v, \theta)$ with $(U, V, q) = (u^2, v^2, \cos \theta)$, we obtain

$$|X_f|^2 = U^2 + V^2 + 2UV(2q^2 - 1). \quad \text{(S-75)}$$
Associated with this change of variables, the measure $d^2fz$ is transformed as:

$$d^2fz = d^fud^fv = \frac{2\pi^{f/2}}{\Gamma(f/2)\Gamma((f-1)/2)}u^{f-1}dv^{f-1}d\theta d\phi$$

$$= \frac{2\pi^{f-1}}{\Gamma(f-1)}u^{f-1}dv^{f-1}d\theta$$

$$= \frac{2\pi^{f-1}}{\Gamma(f-1)}(UV)^{(f-2)/2}dUdV(1-q^2)^{(f-3)/2} dq,$$

(S-76)

where the duplication formula $\Gamma(x)\Gamma(x+1/2) = 2^{1-2x}\pi^{-1/2}\Gamma(2x)$ is used. Therefore, Eq. (S-72) becomes

$$P(|X_f|^2 = \rho) = \frac{1}{\pi \Gamma(f-1)} \int_0^\infty \int_0^\infty dUdV(4UV)^{(f-2)/2}e^{-(U+V)}$$

$$\times \int_{-1}^1 (1-q^2)^{(f-3)/2} dq \delta(\rho - [U^2 + V^2 + 2UV(2q^2 - 1)])$$

(S-77)

We integrate the delta function with respect to $q$. For this purpose, we define

$$q_\rho = \left(\frac{\rho - (U-V)^2}{4UV}\right)^{1/2},$$

(S-78)

which satisfies $0 \leq q_\rho \leq 1$ if and only if $|U-V| \leq \sqrt{\rho} \leq U + V$. Then we obtain

$$\delta(\rho - [U^2 + V^2 + 2UV(2q^2 - 1)]) = \frac{1}{8UVq_\rho}\left[\delta(q-q_\rho) + \delta(q+q_\rho)\right]$$

(S-79)

and

$$P(|X_f|^2 = \rho) = \frac{1}{\pi \Gamma(f-1)} \int_{|U-V|\leq\sqrt{\rho}\leq U+V} dUdV(4UV)^{(f-2)/2}e^{-(U+V)}\frac{(1-q_\rho^2)^{(f-3)/2}}{4UVq_\rho}$$

(S-80)

where we have used

$$\frac{(1-q_\rho^2)^{(f-3)/2}}{q_\rho} = \left(\frac{4UV}{\rho - (U-V)^2}\right)^{1/2} \left(\frac{(U+V)^2 - \rho}{4UV}\right)^{(f-3)/2}.$$

(S-81)

Finally, we change the variables of integration from $(U,V)$ to $(x,y) = (U+V,U-V)$, obtaining

$$P(|X_f|^2 = \rho) = \frac{1}{2\pi \Gamma(f-1)} \int_{\sqrt{\rho}}^\infty dx \int_{-\sqrt{\rho}}^{\sqrt{\rho}} dy e^{-x(x^2-\rho)^{(f-3)/2}/(\rho - y^2)^{1/2}}$$

$$= \frac{1}{2\Gamma(f-1)} \int_{\sqrt{\rho}}^\infty dx e^{-x(x^2-\rho)^{(f-3)/2}}.$$

(S-82)
Here we use the formula:

\[
\int_r^\infty x^a e^{-x} (x^2 - r^2)^{-1/2} = r^{2a} \int_0^\infty dte^{-r \cosh t} (\sinh t)^{2a} = \frac{\Gamma(2\alpha)}{2^{\alpha-1} \Gamma(\alpha)} r^\alpha K_\alpha(r), \quad (S-83)
\]

the proof of which will be provided at the end of this section. Substituting \( \alpha = f/2 - 1 \) and \( r = \sqrt{\rho} \) in (S-83), we obtain the distribution function of \( |X_f|^2 \):

\[
P(|X_f|^2 = \rho) = \frac{2^{-f/2}}{\Gamma(f/2)} \rho^{f/4-1/2} K_{f/2-1}(\sqrt{\rho}).
\quad (S-84)
\]

This probability distribution coincides with the \( K \)-distribution with shape parameters \((f/2, 1)\) \[11, 12\]. In the limit of \( \rho \to +0 \), the right-hand side of (S-84) converges to \( \frac{1}{f-2} \) for \( f > 2 \) and diverges as \( O(\log(1/\rho)) \) at \( f = 2 \).

C. Level-spacing distribution of \( H_{\text{small}} \)

Since the level spacing \( s \) is equal to \( \rho^{1/4} \) multiplied by a scaling factor, the probability distribution of \( s \) reads

\[
p(s) \approx 4s^3 P(|X_f|^2 = \rho)|_{\rho = s^4} = \frac{2^{-f/2}}{\Gamma(f/2)} s^{f+1} K_{f/2-1}(s^2),
\quad (S-85)
\]

where \( \approx \) denotes the equivalence of stochastic variables that differ only by a constant factor. The level repulsion is \( O(s^3) \) for \( f > 2 \) and \( O(s^3 \log(1/s)) \) at \( f = 2 \).

Finally, the distribution function is rescaled to the unit average spacing (i.e., \( \int_0^\infty ds \rho(s) = 1 \)) by multiplying the scaling factor

\[
C_f = \int_0^\infty ds \frac{2^{-f/2}}{\Gamma(f/2)} s^{f+2} K_{f/2-1}(s^2)
\]

\[
= \int_0^\infty \frac{2^{-f/2}}{\Gamma(f/2)} \rho^{f/4-1/4} K_{f/2-1}(\sqrt{\rho}) = \frac{\Gamma(1/4)\Gamma(f/2 + 1/4)}{2\sqrt{2}\Gamma(f/2)}, \quad (S-86)
\]

where the last integration follows from the normalization of the \( K \)-distribution with shape parameters \((f/2 + 1/4, 5/4)\) \[12\].

Thus, we have derived the level-spacing distribution for general \( f \):

\[
p(s) = \frac{2^{-f/2} C_f^{f+2}}{\Gamma(f/2)} s^{f+1} K_{f/2-1}(C_f^2 s^2)
\]

\[
= \frac{1}{N_f} (C_f s)^{f+1} K_{f/2-1}((C_f s)^2), \quad (S-87)
\]
FIG. S-3. Nearest-level-spacing distributions $p(s)$ for two-by-two (for classes A and AI†) and four-by-four (for class AII†) random matrices described by Eq. (1) in the main text. Numerically obtained histograms are well described by analytical predictions in Eq. (S-90). Similarly to Fig. 2 in the main text, $p_{\text{small, AI}^\dagger}(s)$ has a lower peak and $p_{\text{small, AII}^\dagger}(s)$ has a higher peak compared with $p_{\text{small, A}}(s)$. The results are obtained from averages over $10^6$ ensembles.

where we have set $N_f = 2^{f/2-2}\Gamma\left(\frac{f}{2}\right)C_f^{-1}$.

In particular, the Eq. (2) in the main text is obtained by using

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x},$$  \hspace{1cm} (S-88)

$$K_{3/2}(x) = \sqrt{\frac{\pi}{2x}} \left(1 + \frac{1}{x}\right) e^{-x}.$$  \hspace{1cm} (S-89)

Simpler forms of $p(s)$ for specific values of $f$ are:

$$p_{\text{small, A}}(s) = 2C_3^4 s^3 e^{-C_3^2 s^2},$$

$$p_{\text{small, AI}^\dagger}(s) = 2C_2^4 s^3 K_0\left(C_2^2 s^2\right),$$

$$p_{\text{small, AII}^\dagger}(s) = \frac{2C_5^4 s^3}{3} \left(1 + C_5^2 s^2\right) e^{-C_5^2 s^2},$$  \hspace{1cm} (S-90)

where $C_2 = \frac{1}{8\sqrt{2}} \Gamma\left(\frac{1}{4}\right)^2 = 1.16187\ldots$, $C_3 = \frac{3}{4} \sqrt{\pi} = 1.32934\ldots$ and $C_5 = \frac{7}{8} \sqrt{\pi} \simeq 1.5509\ldots$. Here the inversion formula

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}$$  \hspace{1cm} (S-91)

is used to obtain $C_3$ and $C_5$. 

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In Fig. S-3, we visualize Eq. (S-90) and compare it with numerical results, which shows a perfect agreement. Note that, similarly to Fig. 2 in the main text, \( p_{\text{small,AI}}(s) \) has a lower peak and \( p_{\text{small,AII}}(s) \) has a higher peak compared with \( p_{\text{small,A}}(s) \).

D. Proof of Eq. (S-83)

Let us define

\[
f_1(r) = \int_r^{\infty} dx e^{-x}(x^2 - r^2)^{\alpha-1/2}, \tag{S-92}
\]

\[
f_2(r) = r^{-\alpha} f_1(r), \tag{S-93}
\]

\[
f_3(r) = r^{-2\alpha} f_1(r) = \int_0^{\infty} dt e^{-r \cosh t} (\sinh t)^{2\alpha}. \tag{S-94}
\]

First, we derive a differential equation for \( f_3(r) \):

\[
r^2 f_3(r) - r^2 f_3''(r) = r^2 \int_0^{\infty} dt e^{-r \cosh t} (1 - \cosh^2 t)(\sinh t)^{2\alpha}
\]

\[
= r^2 \int_0^{\infty} dt e^{-r \cosh t} (\sinh t)^{2\alpha+2}
\]

\[
= -r \int_0^{\infty} dt \left( \frac{d}{dr} e^{-r \cosh t} \right) (\sinh t)^{2\alpha+1}
\]

\[
= r \int_0^{\infty} dt e^{-r \cosh t} \left( \frac{d}{dr} (\sinh t)^{2\alpha+1} \right)
\]

\[
= (2\alpha + 1) r \int_0^{\infty} dt e^{-r \cosh t} (\cosh t)(\sinh t)^{2\alpha}
\]

\[
= (2\alpha + 1) rf_3'(r). \tag{S-95}
\]

Substituting \( f_3(r) = r^{-\alpha} f_2(r) \) in Eq. (S-95), we obtain a differential equation for \( f_2(r) \):

\[
(r^2 + \alpha^2) f_2(r) + rf_2'(r) + r^2 f_2''(r) = 0, \tag{S-96}
\]

which coincides with the modified Bessel equation. Here \( f_1(r) \) vanishes at \( r \to \infty \), so does \( f_2(r) \), and therefore we can write \( f_2(r) = C K_\alpha(r) \) with a constant \( C \). This constant \( C = \frac{\Gamma(2\alpha)}{2^{\alpha-1}\Gamma(\alpha)} \) can be determined by evaluating \( f_1(0) \) in two ways:

\[
f_1(0) = \int_0^{\infty} dx e^{-x} x^{2\alpha-1} = \Gamma(2\alpha), \tag{S-97}
\]

\[
f_1(0) = \lim_{r \to +0} Cr^\alpha K_\alpha(r) = 2^{\alpha-1}\Gamma(\alpha)C. \tag{S-98}
\]

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V. DETAILS OF DISSIPATIVE QUANTUM MANY-BODY SYSTEMS

A. Lindblad dynamics and symmetry

Let us consider the Lindblad superoperator $L$

$$L[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_{j=1}^{L} \gamma \left[ \hat{\Gamma}_j \hat{\rho} \hat{\Gamma}_j^\dagger - \frac{1}{2} \{ \hat{\Gamma}_j^\dagger \hat{\Gamma}_j, \hat{\rho} \} \right]$$  \hspace{1cm} (S-99)

and its supereigenvalue equation

$$L[\hat{\nu}_\alpha] = \lambda_\alpha \hat{\nu}_\alpha,$$  \hspace{1cm} (S-100)

where $\lambda_\alpha$ is the supereigenvalue for the supereigenstate $\hat{\nu}_\alpha$.

To see the symmetry of Eq. (S-99), it is convenient to use the operator representation of superoperators. That is, we consider the isomorphic mapping

$$\hat{A} |i\rangle \langle j| \rightarrow (\hat{A} \otimes \hat{B}^T) |i\rangle \otimes |j\rangle,$$  \hspace{1cm} (S-101)

where we have doubled the Hilbert space by adding a “copied” space. Then, the Lindblad superopertor can be represented as

$$L \rightarrow \hat{L} = -i(\hat{H} \otimes \mathbb{I} - \mathbb{I} \otimes \hat{H}^T) + \gamma \sum_{j=1}^{L} \left[ \hat{\Gamma}_j \otimes \hat{\Gamma}_j^* - \frac{1}{2} \hat{\Gamma}_j^\dagger \hat{\Gamma}_j \otimes \mathbb{I} - \mathbb{I} \otimes \hat{\Gamma}_j^T \hat{\Gamma}_j^* \right],$$  \hspace{1cm} (S-102)

whose eigenvalues are $\lambda_\alpha$.

For arbitrary $\hat{\Gamma}_j$, $\hat{L}$ has time-reversal symmetry whose square is equal to one. Indeed, for the unitary operation $T_+ = \text{SWAP}$ that exchanges the original and the copied Hilbert spaces (i.e., $T_+ (\hat{A} \otimes \hat{B}) T_+^{-1} = \hat{B} \otimes \hat{A}$), $T_+ \hat{L}^* T_+^{-1} = \hat{L}$ and $T_+ T_+^* = 1$ (see Eq. (S-9)) are satisfied because $\hat{H} = \hat{H}^T$.

On the other hand, for our Ising model with transverse and longitudinal fields in the main text, $\hat{L}$ can also have TRS$^\dagger$ depending on $\hat{\Gamma}_j$. Indeed, since $\hat{H} = \hat{H}^T$ in the conventional Pauli basis, we have

$$\hat{L}^T = -i(\hat{H} \otimes \mathbb{I} - \mathbb{I} \otimes \hat{H}^T) + \gamma \sum_{j=1}^{L} \left[ \hat{\Gamma}_j^T \otimes \hat{\Gamma}_j^\dagger - \frac{1}{2} \hat{\Gamma}_j^\dagger \hat{\Gamma}_j \otimes \mathbb{I} - \mathbb{I} \otimes \hat{\Gamma}_j^T \hat{\Gamma}_j^* \right].$$  \hspace{1cm} (S-103)

Thus, if

$$\hat{\Gamma}_j^T = \hat{\Gamma}_j, \quad (\hat{\Gamma}_j^\dagger \hat{\Gamma}_j)^T = \hat{\Gamma}_j^\dagger \hat{\Gamma}_j,$$  \hspace{1cm} (S-104)

we have the transposition symmetry, $\hat{L}^T = \hat{L}$. This condition is satisfied for (i) dephasing $\hat{\Gamma}_j = \hat{\sigma}_z^j$ but not for (ii) damping $\hat{\Gamma}_j = \hat{\sigma}_-^j$.  

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FIG. S-4. (a) Schematic illustration of a locally interacting system described by the Hamiltonian (S-105). (b) Nearest-level-spacing distributions $p(s)$ for the three non-Hermitian models (i), (ii), and (iii) in Eq. (S-105) after the unfolding procedure of eigenenergies. Comparison with Fig. 2 in the main text reveals that local correlations of eigenenergies of the dissipative systems are well described by non-Hermitian random matrices with their respective symmetry: model (i) obeys class A (blue line), model (ii) obeys class AII (red line, the same data with the middle panel of Fig. 2(a) in the main text), and model (iii) obeys class AIII (purple line, the same data with the right panel of Fig. 2(a) in the main text). We use $h = 0.5, D = 0$ for model (i), $h = 0.5, D = 0.9$ for model (ii), and $h = 0, D = 0.9$ for model (iii). We also use $J = 0.2$ and $L = 13$ for all the models. The results are obtained from averages over 30 ensembles. Statistics are taken from eigenenergies that are away from the edges of the spectrum.

B. Non-Hermitian many-body Hamiltonian

In this section, we show that the level-spacing distributions of non-Hermitian many-body systems exhibit random-matrix universality with the corresponding symmetry. We consider the non-Hermitian one-dimensional spin-1/2 model with a non-Hermitian Ising interaction, transverse and longitudinal fields, and the Dzyaloshinskii-Moriya interaction as follows (Fig S-4(a)):

$$\hat{H}(J, h, D) = \hat{H}_I(J) + \hat{H}_F(h) + \hat{H}_{DM}(D),$$

(S-105)
where $\hat{H}_I(J) = -\sum_{j=1}^{L-1} (1 + iJ\epsilon_j) \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x$, $\hat{H}_F(h) = -h \sum_{j=1}^L (-2.1 \hat{\sigma}_j^x + \hat{\sigma}_j^z)$, and $\hat{H}_{DM}(D) = \sum_{j=1}^{L-1} \vec{D} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+1})$ [13]. Here, $L$ is the size of the system, $\epsilon_j$ is chosen randomly from $[-1, 1]$ for each site $j$ to break unwanted symmetries, $\vec{D} := D \sqrt{2} (\vec{e}_x + \vec{e}_z)$, and the open boundary condition is assumed. Note that non-Hermitian many-body systems can be realized in continuously measured quantum many-body systems [14, 15]. In particular, the non-Hermitian term in Eq. (S-105) arises if we consider the collective dephasing $\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x$ and postselect the null measurement outcome.

The symmetry of this model depends on the parameters $J, h, and D$. For $J \neq 0$, the model belongs to (i) class A for $h \neq 0, D \neq 0$, (ii) class AII* for $h \neq 0, D = 0$ because $\hat{H} = (\hat{H})^T$, and (iii) class AII* for $h = 0, D \neq 0$ and odd $L$ because $\hat{H} = (\prod_{i=1}^{L} \hat{\sigma}_i^y) (\hat{H})^T (\prod_{i=1}^{L} \hat{\sigma}_i^y)^{-1}$ with $(\prod_{i=1}^{L} \hat{\sigma}_i^y) (\prod_{i=1}^{L} \hat{\sigma}_i^y)^* = (-1)^L$ [16].

Figure S-4(b) shows the distributions $p(s)$ for the non-Hermitian models (i), (ii), and (iii) after the unfolding procedure of eigenenergies [8]. We can clearly see that there appear distinct distributions that correspond to the universality classes of the random-matrix ensembles with the same symmetries in Fig. 2 in the main text. Note that the model in Eq. (S-105) is very different from random matrices in that the matrix is sparse due to the local interactions and that the randomness $\epsilon_j$ is small. Nevertheless, our results show that local correlations of eigenenergies of nonintegrable non-Hermitian systems are well described by non-Hermitian random matrices with the corresponding symmetry.

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