Twist Symmetry and Gauge Invariance

M. Chaichian and A. Tureanu

High Energy Physics Division, Department of Physical Sciences, University of Helsinki
and
Helsinki Institute of Physics,
P.O. Box 64, FIN-00014 Helsinki, Finland

Abstract

By applying properly the concept of twist symmetry to the gauge invariant theories, we arrive at the conclusion that previously proposed in the literature noncommutative gauge theories, with the use of $\star$-product, are the correct ones, which possess the twisted Poincaré symmetry. At the same time, a recent approach to twisted gauge transformations is in contradiction with the very concept of gauge fields arising as a consequence of local internal symmetry. Detailed explanations of this fact as well as the origin of the discrepancy between the two approaches are presented.
1 Introduction

The study of noncommutative quantum field theories (NC QFT) with Heisenberg-like commutation relations [1] (for a review, see [2]) has got a new impetus after it was realized that, although they violate Lorentz invariance, they are however subject to a Lorentz-invariant interpretation due to their twisted Poincaré symmetry [3].

The gauge invariance of noncommutative field theories has been investigated for a long time, since the low-energy limit obtained from string theory in the presence of a constant background field is a noncommutative gauge theory, related to a commutative one by the Seiberg-Witten map [4]. Noncommutative gauge theories have been studied also in their own right, without the use of the Seiberg-Witten map, and this study was initiated in [4], by building NC QED. It was also understood that the use of the $\star$-product imposes rather strict constraints on the noncommutative gauge symmetry, among which was the fact that only NC gauge $U(n)$ groups close (and not $SU(n)$). Moreover, there is a no-go theorem [5] stating that only certain representations of the gauge group are allowed (fundamental, antifundamental and adjoint) (see also [6]) and the matter fields can be charged under at most two gauge groups. Using these features of the noncommutative gauge theories a noncommutative version of the Standard Model was built [7, 8], with the gauge group $U_\star(3) \times U_\star(2) \times U_\star(1)$, which solved the problem of electric charge quantization in NC QED arrived at in [4].

Another noncommutative version of the Standard Model was built using the Seiberg-Witten map, where the gauge invariance is defined with respect to the infinitesimal local transformations of NC $su(3) \times su(2) \times u(1)$ (by the use of the Seiberg-Witten map, one can close these noncommutative gauge algebras [9] and some others as well [10], however not the corresponding gauge groups).

In this Letter we argue that the noncommutative gauge theories constructed with the use of $\star$-product (i.e. $\star$-action of the gauge algebra generators on the fields) remain the only correct ones, possessing also twisted Poincaré symmetry. We show that a recent approach to twisted gauge transformations [11, 12], apparently allowing any gauge group to close, just as in the commutative case, is in contradiction with the very idea of introducing gauge fields
when symmetry under local transformation is required.

2 Gauge transformations in NC field theory

In all these approaches to noncommutative gauge theories [4]-[10] and in the further developments based on them, the essential aspect was that the gauge transformations of the fields, whether infinitesimal or finite, were considered as \( \star \)-gauge transformations. For example, in the case of the gauge \( U_\star(n) \) group, an arbitrary element of the group will be

\[
U(x) = \exp_\star(i\alpha^a(x)T_a)
\]

where \( T_a, a = 1, ..., n^2 \) are the generators of the \( U(n) \) group, with the algebra \( [T_a, T_b] = if_{abc}T_c \), \( \alpha^a(x), a = 1, ..., n^2 \) are the gauge parameters and the \( \star \)-exponential means

\[
\exp_\star(i\alpha^a(x)T_a) = 1 + i\alpha^a(x)T_a + \frac{i^2}{2!}\alpha^a(x) \star \alpha^b(x)T_aT_b + ...
\]

Under the transformations of the \( U_\star(n) \) gauge group, the matter fields can be in the fundamental representation:

\[
\psi(x) \rightarrow \psi'(x) = U(x) \star \psi(x)
\]

or antifundamental representation:

\[
\chi(x) \rightarrow \chi'(x) = \chi(x) \star U^{-1}(x)
\]

\( U^{-1}(x) \) is the \( \star \)-inverse of \( U(x) \), while the gauge fields are, as they should, in the adjoint representation:

\[
A_\mu(x) \rightarrow A'_\mu(x) = U(x) \star A_\mu(x) \star U^{-1}(x) + iU(x) \star \partial_\mu U^{-1}(x),
\]

where the matrix form of the gauge fields \( A_\mu(x) = T_aA^a_\mu(x) \) is used. The covariant derivative, defined as usual

\[
D_\mu = \partial_\mu - iA_\mu(x),
\]

but acting with \( \star \):

\[
D_\mu \psi(x) = \partial_\mu \psi(x) - iA_\mu(x) \star \psi(x),
\]
transforms appropriately, like the original fields
\[ D_\mu \psi(x) \rightarrow D'_\mu \psi'(x) = U(x) \ast (D_\mu \psi(x)) , \] (8)
while the field strength tensor \( F_{\mu\nu} \)
\[ F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i [A_\mu(x), A_\nu(x)] \ast , \] (9)
can be easily shown to transform as
\[ F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = U(x) \ast F_{\mu\nu}(x) \ast U^{-1}(x) , \] (10)
such that the action of a noncommutative theory with fermionic matter fields:
\[ S = \int d^4x \left[ i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} \text{Tr}(F^\mu_\nu \ast F_{\mu\nu}) \right] \] (11)
is invariant under the noncommutative gauge group \( U_\ast(n) \). We emphasize that the action (11) is also twisted-Poincaré invariant.

3 The concept of gauge invariance

Recently, there have been attempts to approach gauge invariance of noncommutative theories by using the mechanism of the twist [11, 12], as explained in Section 4. We argue here that the very approach of gauge transformation by the twist is in contradiction with the gauge principle itself [13]. For this purpose, we shall briefly review the introduction of the gauge field in the usual commutative theory, following the classical paper of Utiyama [14] (for a pedagogical presentation, see [15]).

Let us consider a Lagrangean density \( \mathcal{L}(\Phi_i(x), \partial_\mu \Phi_i(x)) \), where \( \Phi_i(x) \) are the fields, and a Lie group of internal global transformations, \( G \). Under the infinitesimal transformations of the corresponding algebra \( G \), the fields and their derivatives transform as
\[ \Phi_i(x) \rightarrow \Phi'_i(x) = \Phi_i(x) + \delta \Phi_i(x), \quad \delta \Phi_i(x) = iT^a_{ij} \alpha_a \Phi_j(x), \] (12)
\[ \partial_\mu \Phi_i(x) \rightarrow \partial_\mu \Phi'_i(x) = \partial_\mu \Phi_i(x) + \delta(\partial_\mu \Phi_i(x)), \quad \delta(\partial_\mu \Phi_i(x)) = iT^a_{ij} \alpha_a \partial_\mu \Phi_j(x), \] (13)
where \( T^a_{ij} \) are the generators of the group in component form and \( \alpha_a \) are the global parameters of the group. In terms of group representations, [12] and [13] show that, if a field \( \Phi_i(x) \) is
a representation of the global Lie algebra $\mathcal{G}$, then its first derivative with respect to space time $\partial_\mu \Phi_i(x)$ (and actually its derivatives of any order) is also a representation of $\mathcal{G}$.

The invariance of the Lagrangean under the transformations of the algebra $\mathcal{G}$ is expressed by the condition:

$$\frac{\partial L}{\partial \Phi_i} \delta \Phi_i + \frac{\partial L}{\partial (\partial_\mu \Phi_i)} \delta(\partial_\mu \Phi_i) = 0 , \quad (14)$$

which, upon taking into account $(12)$ and $(13)$, becomes:

$$\frac{\partial L}{\partial \Phi_i} T^a_{ij} \alpha_a \Phi_j + \frac{\partial L}{\partial (\partial_\mu \Phi_i)} T^a_{ij} \alpha_a \partial_\mu \Phi_i = 0 . \quad (15)$$

If now we make the transformations local by taking the infinitesimal parameters dependent on coordinates, $\alpha_a(x)$, the transformation of the fields will be of the same form as $(12)$

$$\delta_\alpha \Phi_i(x) = i T^a_{ij} \alpha_a(x) \Phi_j(x) , \quad (16)$$

however, the variation of the derivatives, taking into account that $\delta$ and $\partial_\mu$ act in different spaces and therefore they commute, will read:

$$\delta_\alpha (\partial_\mu \Phi_i(x)) = i T^a_{ij} \alpha_a(x) \partial_\mu \Phi_j(x) + i T^a_{ij} \Phi_j(x) \partial_\mu \alpha_a(x) . \quad (17)$$

In other words, if the transformation is local (gauge), the derivatives $\partial_\mu \Phi_i(x)$ are not representations of the gauge algebra. This is the essential point of gauge transformations. Consequently, the variation of the Lagrangean density will be nonzero

$$\delta L = \frac{\partial L}{\partial (\partial_\mu \Phi_i)} i T^a_{ij} \Phi_i(x) \partial_\mu \alpha_a(x) \neq 0 . \quad (18)$$

Therefore, in order to achieve the invariance under the gauge transformations, compensating (gauge) fields need to be introduced into the Lagrangean, whose transformations would annihilate $(18)$. The gauge fields, transforming as

$$\delta_\alpha A^a_\mu(x) = f_{abc} \alpha_b(x) A^c_\mu(x) + \partial_\mu \alpha^a(x) \quad (19)$$

enter the Lagrangean in the combination

$$D_\mu \Phi_i(x) = \partial_\mu \Phi_i(x) - i T^a_{ij} \Phi_j(x) A^a_\mu(x) , \quad (20)$$
which is the covariant derivative, transforming like the original field (and therefore being a representation of the gauge algebra), i.e.

$$\delta_\alpha(D_\mu \Phi_i(x)) = iT_{ij}^a \alpha_a(x)(D_\mu \Phi_j(x)). \quad (21)$$

The purpose of this review was to show that the usual derivatives of the fields are not representations of the gauge algebra and this is essential for the introduction of the gauge fields.

4 Twist approach to noncommutative gauge transformations

Twisted Poincaré symmetry of noncommutative QFT is important because of the Lorentz-invariant interpretation (in the sense of one-particle wave-functions) which it provides for a theory which effectively breaks Lorentz symmetry [3]. This interpretation is exclusively due to the content of the representation theory in the noncommutative case, which is the same as in the commutative case. Essential for twisting the coproduct of the Lorentz generators with the Abelian twist

$$\mathcal{F} = \exp \left( \frac{i}{2} \theta^{\mu \nu} P_\mu \otimes P_\nu \right) \quad (22)$$

is the fact that, once a field is a representation of the Lorentz generators $M_{\mu \nu}$, any derivative of any order of the field is still a representation of $M_{\mu \nu}$.

The idea of the works [11, 12] was that, since the gauge generators, defined as

$$\alpha(x) = \alpha^a(x)T_a \quad (23)$$

do not commute with the generators of the Poincaré algebra (in particular, with the momentum generator, $P_\mu$), one could extend the Poincaré algebra by semidirect product with the gauge generators and apply the twist (22) also to the coproduct of the gauge generators

$$\Delta_0(\delta_\alpha(x)) = \delta_\alpha(x) \otimes 1 + 1 \otimes \delta_\alpha(x) \rightarrow \Delta_\ell(\delta_\alpha(x)) = \mathcal{F} \Delta_0(\delta_\alpha(x))\mathcal{F}^{-1}. \quad (24)$$
We recall that twisting the coproduct of the generators of the Poincaré algebra requires a consistent deformation of the product of fields:

\[
m \circ (\phi \otimes \psi) = \phi \psi \rightarrow m \circ (\phi \otimes \psi) = m \circ \mathcal{F}^{-1}(\phi \otimes \psi) \equiv \phi \star \psi .
\]  

(25)

Since the procedure of the twist states clearly that, for consistency, \(\star\)-products appear only in the algebra of the representations of the Poincaré algebra (i.e., the usual product of fields is replaced by \(\star\)-product, as in (25)), while the generators act on the fields as usual, it was natural to take as infinitesimal gauge transformation of the individual fields the usual form (without \(\star\)-product):

\[
\delta_\alpha \Phi(x) = i \alpha(x) \Phi(x), \]

\[
\delta_\alpha \Phi^\dagger(x) = -i \Phi^\dagger(x) \alpha(x) .
\]  

(26)

However, the variation of a term of the Lagrangean written as a \(\star\)-product of fields under the gauge transformation, reads:

\[
\delta_\alpha(\Phi^\dagger(x) \star \Phi(x)) = m \circ \Delta_0(\alpha(x))(\Phi^\dagger(x) \otimes \Phi(x))
\]

\[
= m \circ \mathcal{F}^{-1} \mathcal{F} \Delta_0(\alpha(x)) \mathcal{F}^{-1}(\Phi^\dagger(x) \otimes \Phi(x))
\]

\[
= m \circ \Delta_0(\alpha(x)) \mathcal{F}^{-1}(\Phi^\dagger(x) \otimes \Phi(x)) .
\]  

(27)

It is then claimed [11, 12] that the result of the above action is

\[
\delta_\alpha(\Phi^\dagger(x) \star \Phi(x)) = i \alpha^a(x)[-\Phi^\dagger(x)T_a \star \Phi(x) + \Phi^\dagger(x) \star (T_a \Phi(x))] .
\]  

(28)

This claim is based on the fact that it is considered that the derivatives of any order of the field \(\Phi(x)\) are in the same representation of the gauge algebra as the field itself, i.e.:

\[
\delta_\alpha((-i)^n P_{\mu_1}...P_{\mu_n} \Phi(x)) = \delta_\alpha(\partial_{\mu_1}...\partial_{\mu_n} \Phi(x)) = \alpha(x)(\partial_{\mu_1}...\partial_{\mu_n} \Phi(x)) ,
\]  

(29)

because only in this case we have, from (24),

\[
\delta_\alpha(\Phi^\dagger \star \Phi) = m \circ \Delta_0(\alpha) \mathcal{F}^{-1}(\Phi^\dagger(x) \otimes \Phi(x))
\]

\[
= (\delta_\alpha \Phi^\dagger) \Phi + \Phi^\dagger(\delta_\alpha \Phi)
\]

\[
+ \sum_{n=1}^\infty \frac{(-i)^n}{n!} \theta^{\mu_1 \nu_1}...\theta^{\mu_n \nu_n}[(\delta_\alpha P_{\mu_1}...P_{\mu_n} \Phi^\dagger)(P_{\nu_1}...P_{\nu_n} \Phi) + (P_{\mu_1}...P_{\mu_n} \Phi^\dagger)(\delta_\alpha P_{\nu_1}...P_{\nu_n} \Phi)]
\]  

7
\[
= (\delta_\alpha \Phi^\dagger)\Phi(x) + \Phi^\dagger(\delta_\alpha \Phi)
\]
\[
+ \sum_{n=1}^{\infty} \frac{i^n}{n!} \theta^{\mu_1 \nu_1} ... \theta^{\mu_n \nu_n} [-i\alpha^a (\partial_{\mu_1} ... \partial_{\mu_n} \Phi^\dagger T_a) (\partial_{\nu_1} ... \partial_{\nu_n} \Phi) + (\partial_{\mu_1} ... \partial_{\mu_n} \Phi^\dagger) (i\alpha^a \partial_{\nu_1} ... \partial_{\nu_n} (T_a \Phi)]
\]
\[
= i\alpha^a [-\Phi^\dagger(T_a) \ast \Phi(x) + \Phi^\dagger(x) \ast (T_a \Phi(x))] .
\] (30)

It is obvious that, were (30) correct, it would immediately follow that any algebra which closes in the commutative case, would trivially close in this case as well, since the gauge parameters \(\alpha^a(x)\) are not affected by the \(\ast\)-product.

It is easy to see that, using (29), one obtains as well

\[
\delta_\alpha (\Phi^\dagger \ast \partial_\mu \Phi) = i\alpha^a [-\Phi^\dagger(T_a) \ast \partial_\mu \Phi(x) + \Phi^\dagger(x) \ast (T_a \partial_\mu \Phi(x))] ,
\] (31)

which shows that, upon gauge transforming the kinetic terms of the Lagrangeans, the latter remain invariant, without any need for the introduction of the gauge fields.

We can now see that the essence of the gauge invariance is in contradiction with the essence of the twist approach to gauge transformations advocated in [11, 12]: the twist requires that the fields and their derivatives of any order be representations of the gauge generator, as in (29), contrary to (17), the crucial point of the gauge invariance machinery.

Actually, (29) and (30) are correct only if the parameters \(\alpha^a\) do not depend on coordinates, i.e. for global internal transformations, which would also explain (31), but in this case the whole twist approach is redundant.

We can therefore conclude that the approach of [11, 12] is indeed leading to global internal transformations of \(\ast\)-products of fields and that gauge transformations apriorically cannot be implemented by twisting the gauge generators with the twist element (22), because they do not satisfy the condition (29).

5 Conclusions

We have shown that gauge transformations of the action of NC QFT [11] cannot be introduced by twisting with (22) the coproduct of the usual gauge generators. Such a procedure, to be consistent, would require that if a field is transformed in a representation of the gauge algebra, then its derivatives of any order also transform according to the representations of
the gauge algebra, what is in obvious contradiction with the very concept of gauge transformations.

This leaves us with the only option of formulating noncommutative gauge theories via the $\star$-product, as initiated in [4] or by using the Seiberg-Witten map. The latter theories have the twisted Poincaré symmetry built in.

A full understanding of the concept of twist symmetry is important. In this respect, we would like to mention that an improper use of the twist in the quantization of noncommutative field theories may also erroneously lead to a violation of the spin-statistics relation and the Pauli exclusion principle, as it has been clarified in [16]. The proper use of the concept of twist, as outlined in the present paper, will be important also in constructing a noncommutative version of the gravitational theory [17].

Acknowledgements

We are grateful to Claus Montonen, Kazuhiko Nishijima and Peter Prešnajder for illuminating discussions.

References

[1] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 032, hep-th/9908142

[2] R. J. Szabo, *Phys. Rept.* **378** (2003) 207, hep-th/0109162

[3] M. Chaichian, P. Kulish, K. Nishijima and A. Tureanu, *Phys. Lett.* **B 604** (2004) 98, hep-th/0408069

[4] M. Hayakawa, hep-th/9912167, *Phys. Lett.* **B 478** (2000) 394, hep-th/9912094

[5] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari and A. Tureanu, *Phys. Lett.* **B 526** (2002) 132, hep-th/0107037

[6] S. Terashima, *Phys. Lett.* **B 482** (2000) 276, hep-th/0002119
[7] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari and A. Tureanu, \textit{Eur. Phys. J. C} \textbf{29} (2003) 413, hep-th/0107055.

[8] M. Chaichian, A. Kobakhidze and A. Tureanu, \textit{Spontaneous Reduction of Noncommutative Gauge Symmetry and Model Building}, \textit{Eur. Phys. J. C}, in print, hep-th/0408065.

[9] B. Jurčo, S. Schraml, P. Schupp and J. Wess, \textit{Eur. Phys. J. C} \textbf{17} (2000) 521, hep-th/0006246.

[10] L. Bonora, M. Schnabl, M. M. Sheikh-Jabbari and A. Tomasiello, \textit{Nucl. Phys. B} \textbf{589} (2000) 461, hep-th/0006091.

[11] D.V. Vassilevich, \textit{Twist to close}, hep-th/0602185.

[12] P. Aschieri, M. Dimitrievic, F. Meyer, S. Schraml and J. Wess, \textit{Twisted Gauge Theories}, hep-th/0603024.

[13] C. N. Yang and R. L. Mills, \textit{Phys. Rev.} \textbf{96} (1954) 191.

[14] R. Utiyama, \textit{Phys. Rev.} \textbf{101} (1956) 1597.

[15] M. Chaichian and N. F. Nelipa, \textit{Introduction to Gauge Field Theories}, Springer Verlag (1984).

[16] A. Tureanu, \textit{Twist and Spin-Statistics Relation in Noncommutative Quantum Field Theory}, hep-th/0603219.

[17] M. Chaichian and A. Tureanu, in preparation.