Three dimensional Eddington–inspired Born–Infeld gravity: solutions

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Three dimensional Eddington-inspired Born–Infeld gravity is studied with the goal of finding new solutions. Beginning with cosmology, we obtain analytical and numerical solutions for the scale factor $a(t)$, in spatially flat ($k = 0$) and spatially curved ($k = \pm 1$) Friedmann-Roberston-Walker universes with (i) pressureless dust ($P = 0$) and (ii) perfect fluid ($P = \frac{\rho}{3}$), as matter sources. When the theory parameter $\kappa > 0$, our cosmological solutions are generically singular (except for the open universe, with a specific condition). On the other hand, for $\kappa < 0$ we do find non-singular cosmologies.

We then move on towards finding static, circularly symmetric line elements with matter obeying (i) $p = 0$ and (ii) $p = \frac{\rho}{3}$. For $p = 0$, the solution found is nonsingular for $\kappa < 0$ with the matter–stress–energy representing inhomogeneous dust. For $p = \frac{\rho}{3}$ we obtain nonsingular solutions, for all $\kappa$ and discuss some interesting characteristics of these solutions. Finally, we look at the rather simple $p = -\rho$ case where the solutions are either de Sitter or anti-de Sitter or flat spacetime.

I. INTRODUCTION

Theories of gravity different from General Relativity (GR) have been actively pursued by many, for a variety of reasons. One such reason relates to the possibility of avoiding the singularity problem in GR involving the occurrence of a big bang in cosmology or black holes in astrophysics. In a classical metric theory of gravity one is aware that these singularities are inevitable, as proved through the Hawking–Penrose theorems [1], under very general and physical assumptions. However, it is quite possible that in an alternate theory one might obtain non–singular geometries (for example non-singular Friedman–Robertson–Walker type cosmologies) as solutions—a feature which does not exist in the FRW cosmology based on GR. It may be noted that the removal/resolution of a singularity is also expected to be a basic feature of a quantum theory of gravity.

In this article, we look into one such alternate theory. Historically, its origin goes back to Eddington who showed us how de Sitter gravity could be obtained using an action where the Einstein–Hilbert term, $\sqrt{-g}R$ ($R$ is the Ricci scalar), is replaced by $\sqrt{-\det(R_{ij})}$ [2]. Eddington’s formulation allowed the choice of the connection (instead of the metric) as the basic variable—therefore, it is essentially an affine formulation. However, coupling of matter remained a problem in this formulation.

We are also aware of Born–Infeld electrodynamics [3], which was introduced in order to get rid of the infinity in the field at the location of the charge/current. A gravity theory in the metric formulation inspired by Born-Infeld electrodynamics was suggested by Deser and Gibbons [4]. Later Vollick [6] worked on the various aspects of the Deser-Gibbons proposal in a Palatini formulation and also introduced a non-trivial way of coupling matter in such theories. More recently, Banados and Ferreira [7] have come up with a formulation wherein the matter coupling is different and simpler from that introduced in Vollick’s work [6]. We will focus in the theory proposed in [7] and call it as Eddington-inspired Born–Infeld (EiBI) gravity, for obvious reasons. Note that the EiBI theory has the feature that it reduces to GR, in vacuum.

It may be noted that the theory we consider falls within the class of bimetric theories of gravity (also called bi-gravity). The current bimetric theories have their origin in the seminal work of Isham, Salam and Strathdee [8]. Numerous papers on varied aspects of such bimetric theories have appeared in the last few years. The central feature here is the existence of a physical metric which couples to matter and another auxiliary metric which is not used for matter couplings. One needs to solve for both metrics through the field equations.

Let us now briefly recall Eddington–inspired Born–Infeld gravity. Since we deal with three spacetime dimensions in this article, we prefer to write down the action and ensuing field equations in three dimensional spacetime. The action for the theory developed in [7], is given as:

$$S_{BI}(g, \Gamma, \Psi) = \frac{2}{\kappa} \int d^3x \left[ \sqrt{|g_{ij} + \kappa R_{ij}|} - \lambda \sqrt{-g} \right] + S_M(g, \Psi)$$

(1)
where $\Lambda = \frac{\kappa \rho}{a}$. Variation w.r.t $\Gamma(q)$, leads to the metric compatibility condition for the $q$ metric. Therefore, the $q_{ij}$ is a valid Riemannian metric and is defined through the relation,

$$q_{ij} = g_{ij} + \kappa R_{ij}(q)$$

(2)

Variation w.r.t $g_{ij}$ gives

$$\sqrt{-q} q^{ij} = \lambda \sqrt{-g} g^{ij} - \kappa \sqrt{-g} T^{ij}$$

(3)

In order to obtain solutions, we need to assume a $g_{ij}$ and a $q_{ij}$ with unknown functions, as well as a matter stress energy ($T^{ij}$). Thereafter, we write down the field equations and obtain solutions using some additional assumptions about the metric functions and the stress energy.

Quite some work on various fronts have been carried out on various aspects of this theory in the last couple of years. Astrophysical aspects have been discussed in the references in [9] while cosmology in those cited in [10]. Other topics such as a domain wall brane has been analysed in [11]. More recently, generic features of paradigms on matter-gravity couplings have been discussed in [12]. However, in [13] a major problem related to surface singularities has been noticed which has put the theory on shaky ground insofar as stellar physics is concerned.

Our work here is reasonably modest. In the two subsequent sections, we discuss cosmological and circularly symmetric solutions in three spacetime dimensions, successively. In the final section, we briefly summarize and conclude. Some of our solutions are analytical and simple. They also maintain some of the generic features noted in the original work of Banados and Ferreira [7].

II. COSMOLOGY

Let us assume a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) line element in 2+1 dimensions, given as:

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right]$$

(4)

where $k = +1, 0, -1$ for closed, flat and open universe respectively. The energy–momentum tensor is taken to be that of a fluid with $T^{ij} = (P + \rho) u^i u^j + P g^{ij}$. The conservation of energy–momentum leads to

$$\dot{\rho} = -2 \frac{\dot{a}}{a} (P + \rho)$$

(5)

which implies $\rho \propto \frac{1}{a^2}$ (for $P = 0$) and $\rho \propto \frac{1}{a^4}$ (for $P = \frac{\rho}{2}$). Further, we assume the auxiliary line element to be of the form

$$ds_q^2 = -U(t) dt^2 + a^2(t) V(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right]$$

(6)

Using the auxiliary metric ($q^{ij}$), physical metric ($g^{ij}$) and stress-energy tensor ($T^{ij}$) in the field equation obtained by varying w.r.t. $g_{ij}$, we get two equations, given by,

$$U = \frac{D}{1 + \kappa \rho_T}$$

(7a)

$$V = \frac{D}{1 - \kappa P_T}$$

(7b)

where, $D = (1 + \kappa \rho_T)(1 - \kappa P_T)^2$, $\rho_T = \rho + \Lambda$, and $P_T = P - \Lambda$. We define two quantities $G_2(\rho, \Lambda)$ and $F_2(\rho, \Lambda)$, given as,

$$G_2(\rho, \Lambda) = \frac{1}{\kappa} \left( 1 + U - \frac{2U}{V} \right) - \frac{2k}{a^2} \frac{U}{V}$$

(8a)

$$F_2(\rho, \Lambda) = 1 - \frac{\kappa (P_T + \rho_T)(1 - w - \kappa P_T - kw \rho_T)}{(1 + \kappa \rho_T)(1 - \kappa P_T)}$$

(8b)
where, in Eq. (10), we have also assumed $P = w \rho$. Combining Eq. (2) and the results from Eqs. (8), we obtain the Friedmann equation given by,

$$H^2 = \frac{G_2}{2F_2^2}$$  \(9\)

Let us now examine two special cases: (i) pressureless dust ($w = 0$) filled flat universe ($k = 0$) with the cosmological constant, $\Lambda = 0$ and (ii) radiation dominated ($w = \frac{1}{3}$) flat universe with $\Lambda = 0$.

For the case (i), using the equation of state (Eq. (5)) and the Friedmann equation (Eq. (9)), we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\bar{\rho}(1 + \bar{\rho})}{\kappa}$$  \(10\)

where, $\bar{\rho} = \kappa \rho$. Note that from the conservation law $pa^2$ is a constant. Using this in Eq. (10), we obtain the solution for the scale factor $a(t)$ for $\kappa > 0$ as,

$$a(t) = \sqrt{C_1 \left(\frac{t^2}{\kappa} - 1\right)}$$  \(11\)

where the constant $C_1 = |\bar{\rho}| a^2 > 0$. The solution clearly demonstrates that for $\kappa > 0$ the EiBI theory cannot avoid the initial singularity ($a(t)$ can become zero at a finite $t$ and hence we have infinite curvature). However, for $\kappa < 0$, the Eq. (10) becomes,

$$\left|\frac{\dot{a}}{a}\right|^2 = \frac{4|\rho|(1 - |\rho|)}{|\kappa|}$$  \(12\)

From Eq. (12), we note that there exists a maximum density ($\rho_B$) or, equivalently, a minimum value for the scale factor ($a_B$). The solution for the scale factor $a(t)$ for $\kappa < 0$ is given by,

$$a(t) = \sqrt{C_2 \left(\frac{t^2}{|\kappa|} + 1\right)}$$  \(13\)

where $C_2$ is a constant. It is easy to see that the scale factor is never zero and thus, there is no curvature singularity. Both these solutions are plotted in the top row of Fig. 1.

For case (ii), (i.e., radiation dominated universe), the Friedmann equation becomes:

$$H^2 = \frac{2}{\kappa} \left[2\bar{\rho} - 1 + \left(1 - \frac{\bar{\rho}}{2}\right)^2 (1 + \bar{\rho}) - \frac{\kappa k}{a^2} (2 - \bar{\rho})\right] \frac{(1 + \bar{\rho})(2 - \bar{\rho})^2}{(4 - \bar{\rho} + 4\bar{\rho}^2)^2}$$  \(14\)

For the flat universe ($k = 0$), the last term in the square-bracket of the right hand side of the Eq. (14) does not contribute and the equation becomes,

$$H^2 = \frac{2}{\kappa} \left[2\bar{\rho} - 1 + \left(1 - \frac{\bar{\rho}}{2}\right)^2 (1 + \bar{\rho})\right] \frac{(1 + \bar{\rho})(2 - \bar{\rho})^2}{(4 - \bar{\rho} + 4\bar{\rho}^2)^2}$$  \(15\)

In this case, for $\kappa > 0$, $H^2 \geq 0$ for an arbitrary, but physically justifiable value of $\bar{\rho}$ (i.e. for $\bar{\rho} \geq 0$). Thus, here also, a curvature singularity appears in the solution. Using the equation of state $\rho a^2 = C_3$, where $C_3$ is a constant, we can solve numerically the Eq. (15) for the time evolution of the scale-factor $a(t)$, which is shown in Fig. 4. However, for $\kappa < 0$, the Eq. (15) is rewritten as,

$$H^2 = \frac{2}{|\kappa|} \left[1 + 2|\bar{\rho}| + \left(1 + \frac{|\bar{\rho}|}{2}\right)^2 (|\bar{\rho}| - 1)\right] \frac{(1 - |\bar{\rho}|)(2 + |\bar{\rho}|)^2}{(4 + |\bar{\rho}| + 4|\bar{\rho}|^2)^2}$$  \(16\)

The presence of the factor $(1 - |\bar{\rho}|)$ in the Eq. (16) leads to $H^2$ being negative when $|\bar{\rho}| > 1$. Therefore, in this case, for $\kappa < 0$, there exists a maximum density or, equivalently, a minimum, non-zero value for the scale factor.

Let us now define two dimensionless variables $\Omega = \frac{\rho}{\rho_m} (0 \leq \Omega \leq 1)$, $z = \frac{a}{a_0}$. Using these we recast the equations as:

$$|\kappa H^2(\Omega)| = 2 \left[1 + 2\Omega + \left(1 + \frac{\Omega}{2}\right)^2 (\Omega - 1)\right] \frac{(1 - \Omega)(2 + \Omega)^2}{(4 + \Omega + 4\Omega^2)^2}$$  \(17a\)
\[\dot{a}^2 = \frac{2z^2}{|\kappa|} \left[ 1 + \frac{2}{z^2} + \left( 1 + \frac{1}{2z^2} \right)^2 \left( \frac{1}{z^2} - 1 \right) \right] \frac{(1 - \frac{1}{z^2}) (2 + \frac{1}{z^2})^2}{(4 + \frac{1}{z^2} + \frac{4}{z^2})^2} \] (17b)

Numerically solving the above equation, we plot and note the time evolution of the scale factor using the Eqs. (17).

In the bottom row of Fig. 1 we note a non-singular scale factor for \(\kappa < 0\) – a result showing the existence of a bounce, similar to that obtained in 3 + 1 dimensions. For \(\kappa > 0\), the solution appears to be singular. The above solutions

\[
a(t) = \sqrt{\left( 1 - \frac{k}{\rho_0} \right) C_1 \left( \frac{t^2}{|\kappa|} - 1 \right)}, \quad \kappa > 0
\] (18a)

\[
a(t) = \sqrt{\left( 1 - \frac{k}{\rho_0} \right) C_2 \left( \frac{t^2}{|\kappa|} + 1 \right)}, \quad \kappa < 0
\] (18b)

where \(\rho_0\) is a constant (\(\rho a^2 = \rho_0\) for \(P = 0\)). For a \(k = +1\) solution there is a lower bound on \(\rho_0\) (\(\rho_0 > 1\)) whereas, for \(k = -1\), \(\rho_0\) is an arbitrary positive, real constant. When \(k = 0\), we recover the earlier results (Eq. 11 and Eq. 13).

It is easy to see that there is nothing new in the \(P = 0, k \neq 0\) solutions.

However, in the radiation dominated \((P = \frac{2}{3})\) case we do get some interesting results, though the main conclusion regarding the appearance of a singularity or otherwise, is almost the same. The introduction of spatial curvature results in an additional term as shown in the square-bracket of the R.H.S. of Eq. (14) and this leads to all the differences. For a \textit{closed universe} \((k = +1)\), instead of a \textit{bounce} we get an \textit{oscillation} of the universe for \(\kappa < 0\). For \(\kappa > 0\) there is an additional feature implying a maximum value of the scale-factor, along with the singularity. These are shown in Fig 2. In an \textit{open universe} \((k = -1)\), we do not see any characteristic novelties for \(\kappa < 0\), but for \(\kappa > 0\) along with the singularity, we also get, under certain circumstances, a \textit{non-singular loitering} phase of the early universe. This is shown through the plots in Fig. 3.
FIG. 2: Plot of $H^2$ and the time evolution of the scale-factor $a(t)$ for closed universe ($k = +1$). $\rho_1$ is a constant, $(\rho^3 a^3 = \rho_1)$.

The main differences, in the context of cosmology, between the results in 2 + 1 dimensions and those in the 3 + 1 dimensional version of EiBI theory are the following. In 2 + 1 dimensions, for $\kappa > 0$, the cosmological solutions are singular for matter satisfying $P = 0$ and $P = \rho / 2$. In contrast, the 3 + 1 dimensional cosmological solutions are non-singular for both $P = 0$ and $P = \rho / 2$. Further, in 2 + 1, for $P = 0$, we find a bounce solution for $\kappa < 0$, whereas in 3 + 1, as obtained by I. Cho et al. [10] (see Fig.2 there), the early universe is de Sitter spacetime and $H^2$ is constant at early times. In [10], an analytical expression for the scale factor $a(t)$ in the 3 + 1 dimensional $P = 0$ case, has indeed been obtained but the expression involves non-invertible functions. In 2 + 1, with $P = 0$, we find analytical scale factors which involve simple functions. The loitering phase of the early universe for $\kappa > 0$ in 3 + 1 dimension is absent in 2 + 1, except for a specific situation with an open universe ($k = -1$). For a radiation dominated closed universe ($k = +1$), we obtain a big crunch solution for $\kappa > 0$ (see Fig. 2), whereas, in 3 + 1 dimensions, the solution under a similar situation has a non-singular loitering phase. Thus, the lower dimensional toy model cosmologies have expected similarities and differences when compared with their higher dimensional counterparts.

III. CIRCULARLY SYMMETRIC, STATIC SOLUTIONS

Let us now turn to a completely different class of line elements –i.e. those which are circularly symmetric and static. We consider a simple ansatz for the physical line element $g_{ij}$,

$$ds^2 = -f^2(l)dt^2 + dl^2 + r^2(l)d\theta^2$$

(19)

where $r(l)$, $f(l)$ are non-negative functions and $l$ extends from minus infinity to plus infinity. $r(l)$ represents the radius of a circle at each value of $l$. $f^2(l)$ is the so-called redshift function. The energy-momentum tensor is assumed as, $T^{ij}=$diag.$\left(\rho f^2(l), p_1, p_2, p_3\right)$. Let us further assume the auxiliary line element to be of the form

$$ds_q^2 = -h^2(l)dt^2 + u^2(l)\left[dl^2 + r^2(l)d\theta^2\right]$$

(20)

where $u(l)$ and $h(l)$ are non-negative functions of $l$. The field equation obtained from $g_{ij}$ variation yields,

$$\rho = \frac{1}{\kappa}\left(\frac{f}{h} u^2 - 1\right) - \Lambda ; \quad p_1 = p_2 = p = \Lambda + \frac{1}{\kappa}\left(1 - \frac{h}{f}\right)$$

(21)
The other field equation obtained from $\Gamma$ variation yield the following equations,

$$\begin{align*}
1 - \frac{f^2}{h^2} &= -\frac{\kappa}{u^2} \left( \frac{h''}{h} + \frac{h'}{h} \frac{r'}{r} \right) \\
u^2 - 1 &= -\kappa \left[ \frac{h''}{h} - \frac{h'}{h} \frac{u'}{u} + \frac{r''}{r} + \frac{r'}{r} \frac{u'}{u} + \frac{u''}{u} - \left( \frac{u'}{u} \right)^2 \right] \\
u^2 - 1 &= -\kappa \left[ \frac{h'}{h} \frac{r'}{r} + \frac{h'}{h} \frac{u'}{u} + \frac{r''}{r} + \frac{r'}{r} \frac{u'}{u} + \frac{u''}{u} - \left( \frac{u'}{u} \right)^2 \right]
\end{align*}$$

Consistency of the last two equations (since the L. H. S. of both are the same) leads to the simple relation

$$h' = \pm C_1 u^2 r$$

where $C_1$ is a constant. Further, the conservation law $\nabla_j T^{ij} = 0$ implies

$$p' + \frac{f'}{f} (\rho + p) = 0$$

Using the expressions for $\rho$ and $p$ given earlier, we arrive at

$$hh' = f f'u^2$$

which, using the expression $h' = \pm C_1 u^2 r$ becomes

$$ff' = \pm C_1 rh$$

We can now look for possible solutions of the above equations.
A. \( C_1 = 0, \ p = 0 \)

The first of these involves assuming \( C_1 = 0 \) which, from the equations imply \( p = 0 \). Equivalently, \( f = h = 1 \), i.e. \( f \) and \( h \) are both constants. With \( f = h = 1 \), Eqn (22) is vacuous and the remaining single equation for \( u^2 - 1 \) is

\[
u^2 - 1 = -\kappa \left[ \frac{r''}{r} + \frac{r' u'}{ru} + \frac{u''}{u} - \left( \frac{u'}{u} \right)^2 \right]
\]

(29)

To solve this equation we need another condition. With \( h = f = 1 \), we note that \( p = p_1 = p_2 = \Lambda \) and hence there is no scope of assuming an equation of state. In other words the solution we are looking for is produced by \textit{inhomogeneous dust} in the presence of a \( \Lambda \). To find one such solution, we assume a relationship between \( u \) and \( r \). Let us take

\[
\frac{r'}{r} + \frac{u'}{u} = D
\]

(30)

where \( D \) is a constant. Thus, we can transform the second order differential equation (Eq. 29) into an easily solvable first order ordinary differential equation for \( u \). Other choices for the R. H. S. of Eq. 30 (may not be a constant) could give other mathematical solutions for which we need to verify whether the energy-density distributions (\( \rho_T \)) are physically acceptable.

We now demonstrate one solution assuming the condition in Eq. (30) and \( D = 1/\sqrt{|\kappa|} \). Let us also assume \( \kappa < 0 \). The solution we obtain is specified by the \( r(l) \) and \( u(l) \) given below:

\[
\begin{align*}
r^2 &= r_0^2 \cosh \frac{2l}{\sqrt{|\kappa|}}; \\
u &= \frac{e^{\sqrt{|\kappa|} l}}{\sqrt{\cosh \frac{2l}{\sqrt{|\kappa|}}}}
\end{align*}
\]

(31)

The energy density is given as

\[
\rho = -\frac{1}{|\kappa|} \tanh \frac{2l}{\sqrt{|\kappa|}} - \Lambda
\]

(32)

and is positive as long as \( \Lambda < 0 \) and \( \frac{1}{\sqrt{|\kappa|}} < |\Lambda| \). The corresponding Ricci scalar for the physical \( g \) metric is,

\[
R = -\frac{2 + 2 \text{sech}^2 \left( \frac{2l}{\sqrt{|\kappa|}} \right)}{|\kappa|}
\]

(33)

which is clearly non-singular and has asymptotically constant negative curvature. The \textit{Kretschmann scalar} is

\[
K = R^{ijkl} R_{ijkl} = \frac{4}{|\kappa|^2} \left[ 2 - \tanh^2 \left( \frac{2l}{\sqrt{|\kappa|}} \right) \right]^2
\]

(34)

and is non-singular as well. The physical line element, for this case can be written as,

\[
ds^2 = -dt^2 + dl^2 + r_0^2 \cosh \left( \frac{2l}{\sqrt{|\kappa|}} \right) d\theta^2
\]

(35)

One can also rewrite it as,

\[
ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\theta^2
\]

(36)

where,

\[
b(r) = r + \frac{1}{|\kappa|} \left( \frac{r_0^4}{r} - r^3 \right)
\]

(37)
and \( r = r_0 \sqrt{\cosh \left( \frac{2l}{\sqrt{|\kappa|}} \right)} \). Thus, \( r \geq r_0 \) and a metric singularity occurs in the transformed metric function only at \( r = r_0 \). We plot the transformed metric function \( 1/(1 - b(r)/r) \) and have shown it in Fig. 4. Note that this solution is not asymptotically flat (i.e. \( b(r) \) does not tend to zero as \( r \to \infty \)). However, there is a minimum radius \( r = r_0 \) and the geometry is symmetric as \( l \rightarrow -l \). Also \( b(r = r_0) = r_0 \). Thus, the features are similar to that of a Lorentzian wormhole [14] though the geometry is not asymptotically flat (in fact it is asymptotically anti-de Sitter).

In Fig. 4 we have shown the \( g \) and \( q \) metric functions, the Ricci scalar \( (R) \), and the energy-density \( (\rho_T = \rho + \Lambda) \) for the solution quoted above. We note that the energy-density is asymmetric as \( l \rightarrow -l \), even though the physical metric \( g \) is symmetric in \( l \). This happens because the \( q \) metric is asymmetric and the field equation which contains the energy–momentum tensor, has contributions from the \( g \) and \( q \) metrics. The asymmetry is also evident from the expression for \( \rho \) which clearly depends only on \( u^2 \).

If \( \kappa > 0 \), assuming the same relation (Eq. 30) between \( r \) and \( u \), we get a singular solution. In this case, \( l \) becomes restricted and at the boundary of \( l \), both the Ricci scalar and the energy density \( (\rho) \) diverge.

\[ B. \quad C_1 \neq 0, \quad p = \frac{\rho}{2} \]

In this case, the equations we need to solve are Eqns. (21), (22), (28) or (24), (25) and (28) which are six equations in the six variables \( \rho, p, f, h, r \) and \( u \). However, we now show that Eqns. (22) and (24) (or (23)) are not independent equations. To this end, we define

\[ \eta = \frac{r'}{r} ; \quad \xi = \frac{u'}{u} ; \quad \chi = \xi + \eta \quad (38) \]

Using these definitions and the other equations, one can show that Eqn. (22) reduces to

\[ \chi = \pm \left( \frac{f^2}{2 \kappa C_1 r h} - \frac{h}{2 C_1 r \kappa} \right) \quad (39) \]

Further, one may rewrite Eqn. (24) using the new variables defined above. \( \chi' \) as obtained from the reduced version of (22) matches with the \( \chi' \) as obtained from (24). Hence these two equations are not independent. We exploit this feature of the system of equations to impose an equation of state restriction. This is chosen to be \( p = \frac{\rho}{2} \). Using the above equation of state, it is straightforward to write down all the unknown functions as functions of \( \rho \).
FIG. 5: (a)Plot of physical metric function $g_{\theta\theta}$, auxiliary metric functions $q_{ll}$ and $q_{\theta\theta}$ and the Ricci scalar ($R$) for $p = \Lambda$. (b)Plot of energy density ($\rho_T = \rho + \Lambda$). The value of the parameters ($r_0, \kappa$) are shown on the frame of each plot.
These are given below as,

\[ u^2 = \frac{1}{2} (1 + \bar{\rho}) (2 - \bar{\rho}) \]  
\[ f = C_4 \left( \frac{\kappa}{\rho} \right)^{\frac{1}{2}} \]  
\[ h = \frac{1}{2} (2 - \bar{\rho}) C_4 \left( \frac{\kappa}{\rho} \right)^{\frac{1}{2}} \]  
\[ r = \pm \frac{2 C_4}{3 C_1} \left( \frac{\kappa}{\rho} \right)^{\frac{1}{2}} \frac{\bar{\rho}'}{\bar{\rho}'(2 - \bar{\rho})} \]  

where, we have assumed \( \Lambda = 0 \), defined \( \bar{\rho} = \kappa \rho \) and introduced a non-zero positive constant \( C_4 \). Further, we have, from the field equations, the following equation for \( \bar{\rho} \),

\[ \bar{\rho}'' + \frac{1}{6} \frac{\bar{\rho}^2 (8 \bar{\rho}^2 + \bar{\rho} - 16)}{\bar{\rho}(1 + \bar{\rho})(2 - \bar{\rho})} + \frac{3}{8 \kappa} \bar{\rho}^2 (4 - \bar{\rho}) = 0 \]  

Due to the presence of the factor \((1 + \bar{\rho})(2 - \bar{\rho})\) in the denominator of the second term in the L.H.S, the Eqn. (44) can be solved only for \( \bar{\rho}_{\text{max}} < 2 \) for \( \kappa > 0 \) and \( \bar{\rho}_{\text{max}} < 1 \) for \( \kappa < 0 \). Though the range of \( \rho \) is restricted but \( \rho_{\text{max}} \) can approach its limiting values and be as close as required. One can obtain qualitative analytical information about the nature of the solutions by using approximations. For instance, when \( \kappa > 0 \), we can show, using a Taylor series about \( \bar{\rho} \to 2 \), that the solution will behave as \(-\frac{1}{2} \bar{\rho}^2 + \text{const.}\). On the other hand, when \( \bar{\rho} \to 0 \), the solution behaves generically as \( \frac{1}{(a + b)^2} \) \((a, b \text{ being constants})\). The equation for \( \bar{\rho} \) can be easily solved numerically and the solutions for \( \kappa > 0 \) and \( \kappa < 0 \) are shown in Figs. [14]. Note that the plot for \( \bar{\rho} \) as a function of \( l \) matches qualitatively with the approximate solutions found in the neighborhoods of \( \bar{\rho} = 2 \) and \( \bar{\rho} = 0 \).

We also note that the Ricci scalar is regular everywhere though the \( r(l) \) does become zero at \( l = 0 \). We will now try to see analytically, why the solution is regular everywhere. The expression for the Ricci scalar as a function of \( \bar{\rho} \) turns out to be

\[ R = -2 \left[ \frac{\bar{\rho}'''}{\bar{\rho}'} + \frac{\bar{\rho}''}{(2 - \bar{\rho})} \left( 3 \frac{3}{2 - \bar{\rho}} - \frac{14}{5 \bar{\rho}} \right) + \bar{\rho}^2 \left( \frac{4}{\bar{\rho}^2} + \frac{2}{(2 - \bar{\rho})^2} - \frac{3}{\bar{\rho}(2 - \bar{\rho})} \right) \right] \]  

From the differential equation for \( \bar{\rho} \) we can show, by taking another derivative w.r.t. \( l \), that \( \frac{\overline{\rho}'''}{\overline{\rho}'} \) is finite everywhere including the location of the maximum density at \( l = 0 \) where \( \bar{\rho}' = 0 \) and \( \bar{\rho}'' < 0 \). The remaining terms are all finite everywhere and thus the Ricci scalar is finite. One can also check that the Kretschmann scalar

\[ R_{ijkl} R^{ijkl} = 4 \left[ \left( \frac{f'''}{f} \right)^2 + \left( \frac{f'}{r f} \right)^2 + \left( \frac{r''}{r} \right)^2 \right] \]  

is finite everywhere and therefore \( R_{ijkl} R^{ijkl} \) is also finite. Thus at the location of a maximum density the geometry remains nonsingular though the radius \( r(l) \) becomes zero. This implies the vanishing of the circumference of the circle at \( t = \text{constant} \) and \( l = 0 \). The vanishing of \( r(l) \) at \( l = 0 \) would have implied a singularity (as in cosmology, where \( a(t) = 0 \) implies a singularity) in GR. However, in EIBI theory it does not imply a curvature singularity essentially due to presence of a non-zero \( \kappa \). In the cosmological context, such regular solutions with vanishing volume spatial hypersurfaces have been noted earlier in the literature [15]. The form of \( f^2(l) \) as shown in Figs. [14] indicates the absence of any horizon (\( f(l) \) is never zero) in the geometry. The energy density and the pressure are both finite and positive definite everywhere as is clear from the graphs in Figs. [14]. We also note that the spacetime is asymptotically flat and tends to one of the vacuum \((p = 0, \rho = 0, \Lambda = 0)\) solutions, \( r = \text{const.}, \) \( u^2 = 1 \) and \( h^2 = f^2 = (cl + d)^2 \) \((c \text{ and } d \text{ being constants})\), at infinity.

C. \( C_1 \neq 0, \rho = -\rho \)

In this simple case we solve a similar set of five equations, assuming the equation of state \( p = -\rho \), with \( \Lambda = 0 \). From the Eqn. (26), we have \( p = -\rho_0 \) \((\rho_0 \text{ is a positive constant})\). Using this in Eqn. (21), we get \( u^2 = (1 + \rho_0)^2 \)
FIG. 6: Plot of all metric functions, energy density and Ricci scalar for $p = \rho/2$, using $\kappa = 1, C_1 = C_4 = 1, \rho(0) = 1.9$ and $\rho'(0) = 0$.

and $\frac{r^2}{C_1^2} = \frac{1}{(1 + \tilde{\rho}_0)^2}$. Further, using these results in the remaining equations, we finally obtain the second order linear homogeneous differential equation for $h$, which is

$$h'' + \frac{\tilde{\rho}_0(\tilde{\rho}_0 + 2)}{2\kappa}h = 0$$  \hspace{1cm} (47)

Solving the Eqn. (47) and using other relations, we get trivial but non-singular solutions. For $\kappa > 0$, the solutions for $h(l), f(l)$ and $r(l)$ are

$$h = \left\{ \sin \left( \sqrt{\frac{\tilde{\rho}_0(\tilde{\rho}_0 + 2)}{2\kappa}}l \right), \cos \left( \sqrt{\frac{\tilde{\rho}_0(\tilde{\rho}_0 + 2)}{2\kappa}}l \right) \right\} = f(1 + \tilde{\rho}_0)$$

$$r = \frac{\sqrt{\tilde{\rho}_0(\tilde{\rho}_0 + 2)/2\kappa}}{C_1(1 + \tilde{\rho}_0)^2} \left\{ \cos \left( \sqrt{\frac{\tilde{\rho}_0(\tilde{\rho}_0 + 2)}{2\kappa}}l \right), \sin \left( \sqrt{\frac{\tilde{\rho}_0(\tilde{\rho}_0 + 2)}{2\kappa}}l \right) \right\}$$
FIG. 7: Plot of all metric functions, energy density and Ricci scalar for $p = \rho/2$, using $\kappa = -1, C_1 = C_4 = 1, \rho(0) = 0.9$ and $\rho'(0) = 0$.

The resulting physical line element is de-Sitter spacetime. Similarly, for $\kappa < 0$, we get de-Sitter spacetime (if $|\bar{\rho}_0| < 2$ but $|\bar{\rho}_0| \neq 1$). For $\kappa < 0$ and $|\bar{\rho}_0| > 2$, we get,

$$h = \exp \left( \pm \sqrt{\frac{|\bar{\rho}_0|(|\bar{\rho}_0| - 2)}{2|\kappa|}} \right) = f(|\bar{\rho}_0| - 1)$$

$$r = \frac{\sqrt{|\bar{\rho}_0|(|\bar{\rho}_0| - 2/|\kappa|)}}{C_1(1 - |\bar{\rho}_0|)^2} \exp \left( \pm \sqrt{\frac{|\bar{\rho}_0|(|\bar{\rho}_0| - 2)}{2|\kappa|}} \right)$$

This solution represents anti-de Sitter spacetime. For $\kappa < 0$ and $|\bar{\rho}_0| = 2$, we find,

$$h = Al + B = f, (A, B \text{ being constants}),$$

$$r^2 = \frac{A^2}{C_1^2}, \text{(constant)}$$

which is just flat spacetime.
IV. CONCLUSIONS

The main purpose behind this article was to find simple solutions in three dimensional EiBI gravity. We have focused here on two types of solutions – cosmological and static, circularly symmetric. We summarise our results pointwise below.

• We have found analytical, spatially flat cosmologies for pressureless dust \((P = 0)\) and numerical solutions for \(P = \frac{\kappa}{2}\). We have also explored the cases when spatial curvature is present. The cosmological solutions presented here seem to point to a generic feature that singularities are not present if \(\kappa < 0\) whereas singularities do arise if \(\kappa > 0\).

• In the circularly symmetric, static case we have found an analytical solution for inhomogeneous dust \((p = 0)\). We have also obtained static, circularly symmetric, numerical solutions for the equation of state \(p = \frac{\mu}{\kappa}\) and have briefly analysed the \(p = -\rho\) case. In the class of circularly symmetric, static solutions found we note that they are nonsingular for all \(\kappa\), except in the \(p = 0\) case for \(\kappa > 0\). The \(p = \frac{\mu}{\kappa}\) solution exhibits the curious feature of being a regular (non-singular) solution though the radius of the circle \(r(l)\) vanishes at \(l = 0\), which usually would imply a singularity. This feature is exclusively due to the structure of the modified theory (EiBI).

We are aware that in 2 + 1 General Relativity extensive work has been done on cosmological solutions (see for example [4]). It may be possible to use some of these ideas in 2 + 1 EiBI gravity too. Further, our results for the circularly symmetric, static case can easily be generalised to equations of state such as \(p = w\rho\) or the well-known polytropic one.

In conclusion, we raise a few relevant questions. For instance, one may ask– are there black hole solutions in the 2 + 1 dimensional EiBI theory? The answer is surely hidden in the field equations and is worth exploring. In the context of the cosmological solutions, it is important to know about the behaviour of fluctuations about a given background solution. A study of such fluctuations can surely be carried out using the exact analytical solution found here (for a spatially flat FRW line element with \(P = 0\)) as the background. For the more general cases (say \(P = \frac{\kappa}{2}\)) the numerically obtained spacetimes may be used as backgrounds to investigate fluctuations, numerically.

Finally, it is possible that our work on solutions in this toy three dimensional theory may provide viable pointers towards finding new analytical or numerical geometries in the real four dimensional world.

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