Wild high-dimensional Cantor fences in $\mathbb{R}^n$, Part I

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Abstract

Let $\mathcal{C}$ be the Cantor set. For each $n \geq 3$ we construct an embedding $A : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^n$ such that $A(\mathcal{C} \times \{s\})$, for $s \in \mathcal{C}$, are pairwise ambiently incomparable everywhere wild Cantor sets (generalized Antoine’s necklaces). This serves as a base for another new result proved in this paper: for each $n \geq 3$ and any non-empty perfect compact set $X$ which is embeddable in $\mathbb{R}^{n-1}$, we describe an embedding $A : X \times \mathcal{C} \to \mathbb{R}^n$ such that each $A(X \times \{s\})$, $s \in \mathcal{C}$, contains the corresponding $A(\mathcal{C} \times \{s\})$, and is “nice” on the complement $A(X \times \{s\}) - A(\mathcal{C} \times \{s\})$; in particular, the images $A(X \times \{s\})$, for $s \in \mathcal{C}$, are ambiently incomparable pairwise disjoint copies of $X$. This generalizes and strengthens theorems of J.R. Stallings (1960), R.B. Sher (1968), and B.L. Brechner–J.C. Mayer (1988).

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1 Introduction and Statements of Main Results

Theory of wild embeddings comes back to papers of L. Antoine, P.S. Urysohn, J.W. Alexander. Let us refer to the reviews [18], [19], [23], [21] and the books [45], [25] which contain hundreds of references on the subject.

It is known that for $n \geq 3$ there exist uncountably many inequivalent embeddings of the Cantor set in $\mathbb{R}^n$ [47], [57, Cor. 5.2]; this together with “feathers method” (see Statement 2.10) implies that each non-empty perfect compact subset of $S^{n-1}$ can be embedded in $\mathbb{R}^n$ in uncountably many inequivalent ways, compare [48, p. 114], [57, Thm. 7.1], [16, Thm. 2, 3, and Remark on p. 320].

In this paper, we study disjoint wild embeddings. We would like to embed uncountably many copies of a compactum simultaneously, so that they are mutually exclusive. The assumption of being wild is essential. Concentric spheres of arbitrary radii form a family of cardinality continuum; in contrast to this, by results of R.H. Bing, it is impossible to place an uncountable collection of pairwise disjoint wild closed surfaces in $\mathbb{R}^3$ (scheme of proof is given in [10]; the full proof needs results of [11] and [13]; see also a short sketch of Bing’s idea in [18, Thm. 3.6.1]). For higher dimensions, Bing’s non-embedding result has at the moment only partial generalizations, see [17, Thm. 1, 2], [19, Thm. 10.5], [23, p. 383, Thm. 3C.2].

J.R. Stallings constructed a family of continuum cardinality of pairwise disjoint wild 2-disks in $\mathbb{R}^3$ [51]. J. Martin showed that all except countably many disks in such a collection must be locally tame except on their boundaries, hence must lie on 2-spheres [38] (this result was predicted in [13] where important facts needed for Martin’s arguments are proved).

R.B. Sher modified Stallings’ construction so that no two disks of the family are ambiently homeomorphic, that is, no self-homeomorphism of $\mathbb{R}^3$ can map one disk onto another [46].

J.L. Bryant noticed that the Stallings’ construction can be extended to the case of $(n - 1)$-disks in $\mathbb{R}^n$, $n \geq 4$ by taking direct product with $I^{n-3}$ [17, p. 479]. Our construction is more complicated, and it provides disjoint embeddings with additional property of pairwise incomparability, not only for disks, but for arbitrary perfect compacta. Our method unifies and extends ideas of [51], [46], [16].
Theorem 1.1. For each \( n \geq 3 \) and any non-empty perfect compact set \( X \) which is embeddable in \( \mathbb{R}^{n-1} \), there exists an embedding \( A : X \times C \to \mathbb{R}^n \) such that the images \( A(X \times \{s\}) \), for \( s \in C \), are non-locally-hyperplanar and pairwise ambiently incomparable copies of \( X \) in \( \mathbb{R}^n \).

Further:
(i) \( A \) extends to an isotopy \( X \times I \to \mathbb{R}^n \);
(ii) if, in addition, \( X \) is a polyhedron, then each \( A(X \times \{s\}) \), \( s \in C \), is wild.

The notion of local hyperplanarity is introduced in Definition 2.2, see below.

Remark 1.2. Let us make some remarks concerning possible analogues and generalizations of Theorem 1.1.
1) There is no evident analogue for countable, compact sets: any countable, compact set can be straightened by an ambient isotopy \([35, 1.1, 1.2]\), see also \([37, \text{Thm. I.4.2}]\).
2) By \([30, \text{Thm. 1.8.10, 4.1.5}]\), each \( n \)-dimensional compact set in \( \mathbb{R}^n \) contains an open ball; hence Theorem 1.1 does not extend to the case of \( n \)-dimensional compacta \( X \).
3) There is no evident analogue of Theorem 1.1 for \( n = 2 \). In \( \mathbb{R}^2 \) each zero-dimensional compactum is tame \([5, 75, \text{p. 87–89}]\, [37, \text{Cor. II.3.2, Cor. II.3.3}]\, [41, \text{Chap. 13}]\). E.D. Tymchatyn and R.B. Walker showed that for each embedding \( F : I \times C \to \mathbb{R}^2 \) there exists a homeomorphism \( h : \mathbb{R}^2 \cong \mathbb{R}^2 \) such that \( h \circ F(I \times C) = I \times C \) \([52]\).

Our proof of Theorem 1.1 is based on the following two theorems which are interesting in themselves:

Theorem 1.3. Let \( n \geq 3 \). There exists an embedding \( A : C \times C \to \mathbb{R}^n \) such that for each \( s \in C \) the corresponding Cantor set \( A(C \times \{s\}) =: \mathcal{A}_s \) is everywhere wild; moreover, for \( s \neq t \), the sets \( \mathcal{A}_s \) and \( \mathcal{A}_t \) are ambiently incomparable. In fact, each \( \mathcal{A}_s \) is an Antoine’s necklace for \( n = 3 \) and a generalized Antoine’s necklace for \( n \geq 4 \).

Theorem 1.4. For any non-empty perfect compact set \( X \subset \mathbb{R}^n \), where \( n \geq 2 \), there exists a homeomorphism \( F : \mathbb{R}^n \to \mathbb{R}^n \) such that \( F(X) \subset I^n \) and \( F(X) \cap \partial I^n = \{0\}^{n-1} \times \mathcal{C} \), where \( \mathcal{C} \subset I \) is the middle-thirds Cantor set.
Before we finish the introductory part, let us mention some other results in this field.

R.B. Sher (assuming the Continuum Hypothesis) constructed an uncountable collection of pairwise disjoint arcs in $\mathbb{R}^3$ which contain a representative (under self-homeomorphism of $\mathbb{R}^3$) of each arc locally tame modulo a compact 0-dimensional set [48, Thm. 3].

E.R. Apodaca embedded $I \times C$ in $\mathbb{R}^3$ so that $I \times \{s\}, s \in C$, are pairwise inequivalently embedded wild arcs, each locally tame except at one point. Also, identifying $I \times C$ with $I \times \{0\} \times C \subset I \times I \times C$, he extended his embedding to an embedding $I \times I \times C \rightarrow \mathbb{R}^3$ [7].

R.J. Daverman constructed examples of $k$-cells $B$ in $\mathbb{R}^n$ ($n \geq 4$, $3 \leq k \leq n$) such that each 2-disk $D$ in $B$ (in $\partial B$ in case $k = n$) has non-simply connected complement in $\mathbb{R}^n$ and hence is wildly embedded in $\mathbb{R}^n$ [22]; this implies also a disjoint embedding result.

Under certain restrictions, if a space $Y$ contains uncountably many pairwise disjoint copies of a compactum $X$, then $Y$ contains a copy of $X \times C$ [28], [8]. It would be interesting to specify this result so that the copies $X \times \{s\}$ run over the prescribed set of embedding types (up to ambient homeomorphism of $Y$).

In a forthcoming paper, instead of Antoine’s necklaces we will exploit Bing-Whitehead Cantor sets which have simply-connected complements but are nevertheless wild [33, Theorem]; we will construct uncountable families of pairwise disjoint wild $k$-disks in $\mathbb{R}^n$ such that the complement of each $k$-disk is simply connected; the result generalizes [33, Cor. 1] and is announced in [31, Thm. 1].

2 Preliminaries

A compactum is called perfect if it has no isolated points.

By $C$ we denote the usual middle-thirds Cantor set on $I = [0,1]$. Any topological space homeomorphic to $C$ is called a Cantor set. (These are exactly non-empty metric zero-dimensional perfect compacta, see [30, Problem 1.3.F, p. 29], [41, Thm. 12.8].)

For a topological manifold-with-boundary $M$, denote by $\bar{M}$ and $\partial M$ the interior and the boundary of $M$, correspondingly. In particular, $\bar{I} = (0,1)$.

For a subset $A$ of a topological space $X$, its closure is denoted by $\bar{A}$. 

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Definition 2.1. A zero-dimensional compact set $K \subset \mathbb{R}^n$ is called 
**tame** if there exists a homeomorphism $h$ of $\mathbb{R}^n$ onto itself such that $h(K)$ is a subset of the $Ox_1$-axis of $\mathbb{R}^n$; and it is called **wild** otherwise.

A zero-dimensional compact set $K \subset \mathbb{R}^n$ is **locally tame at a point** $x \in K$ if there is a neighborhood $U$ of $x$ in $\mathbb{R}^n$ such that $K \cap \overline{U}$ is a tame (zero-dimensional) compactum, and **locally wild at $x$** otherwise.

A zero-dimensional compact set $K \subset \mathbb{R}^n$ is **everywhere wild** if it is not locally tame at each of its points.

A Cantor set $K$ in $\mathbb{R}^n$ is tame iff for some homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$, we have $h(K) = \{0\}^{n-1} \times C$ (refer e.g. to [15, Prop. 6.1.17]).

Each countable, compact subset of $\mathbb{R}^n$ is tame [35, 1.1, 1.2], [37, Thm. I.4.2].

We introduce the following

**Definition 2.2.** Call a subset $X \subset \mathbb{R}^n$ **locally hyperplanar at a point** $x \in X$ if there exist a neighborhood $U$ of $x$ in $\mathbb{R}^n$ and an embedding $h : U \to \mathbb{R}^n$ such that $h(X \cap U)$ is contained in the standard hyperplane $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$.

A zero-dimensional compact subset of a hyperplane $K \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ is tame in $\mathbb{R}^n$, see [59, Thm. 2] (using Klee flattening theorem [45, Thm. 2.5.1]) or [42, Theorem 3], [39, Cor. 1]. Hence, in the next statement, (a) implies (b); the converse implication is evident.

**Statement 2.3.** Let $n \geq 3$. For any zero-dimensional compact set $K \subset \mathbb{R}^n$ and a point $x \in K$ the following conditions are equivalent:

(a) $K$ is locally wild at $x$;

(b) $K$ is not locally hyperplanar at $x$.

L. Antoine in [3] sketched and in [5, 78, p. 91–92] explicitly constructed a Cantor set in $\mathbb{R}^3$ which is now widely known as Antoine’s necklace. Antoine proved that this Cantor set is wild [5, Part 2, Chap. III]. Using “feelers” he showed that each zero-dimensional compactum in $\mathbb{R}^n$, $n \geq 2$, can be included in a Jordan arc [5, 72, p. 82–84] (this result was announced by M. Denjoy in [26, 27]); thus Antoine described a first example of a wild arc in $\mathbb{R}^3$ [5, 83, p. 97] (see also [11, Thm. 18.8]). In $\mathbb{R}^n$, for each $n \geq 3$, many other examples of wild Cantor sets are now known. We will use generalized Antoine’s Necklaces as a base for our proofs (see Section 4.4).

We would like to mention that besides widely known Antoine’s necklace, there is an essentially different construction of a wild Cantor set in $\mathbb{R}^3$ given
by P.S. Urysohn [53, n.40, p. 121–122], see also [54, p. 330–332]. According to P.S. Alexandroff [54, p. 488], P.S. Urysohn wrote this paper in 1922–23. Urysohn’s set has infinite genus, in contrast to that of Antoine. Urysohn writes with regret [54, p. 332, footnote] that Antoine’s thesis came to his knowledge only after he had almost finished his mémoire.

**Definition 2.4.** [36, Def. 1], [37, Def. I.3.2] A set $K$ in an $n$-dimensional topological manifold $M$ is called *cellularly separated* (in $M$) if for each open neighborhood $V$ of $K$ there exists a family $\{u_\alpha\}$ of open subsets of $M$ such that $K \subset \bigcup_\alpha u_\alpha \subset \bigcup_\alpha \overline{u}_\alpha \subset V$; each $u_\alpha$ is homeomorphic to $\mathbb{R}^n$; each $\overline{u}_\alpha$ is a topological $n$-cell; and $\overline{u}_\alpha \cap \overline{u}_\beta = \emptyset$ for $\alpha \neq \beta$.

The next proposition was proved by L.V. Keldysh [36, Thm. 1], see also [37, Thm. I.4.2] (for $n = 3$, the equivalency $(a) \iff (c)$ follows also from [12, Thm. 1.1]; for arbitrary $n$, see also a later paper [42, Thm. 1]).

**Statement 2.5.** For each $n$ and each zero-dimensional compact set $K \subset I^n \subset \mathbb{R}^n$ the following conditions are equivalent:

- (a) $K$ is cellularly separated in $I^n$;
- (b) there exists an isotopy $F_t$ of $\mathbb{R}^n$ onto itself such that $F_0 = \text{id}$; $F_t = \text{id}$ on $\mathbb{R}^n - I^n$ for each $t \in I$; and $F_1(K) \subset O_{x_1}$;
- (c) $K$ is tame in $\mathbb{R}^n$.

The next Proposition follows from the proof of [12, Thm. 8] (for $n = 3$ see also [12, Thm. 6.1]):

**Statement 2.6.** If a Cantor set $K \subset \mathbb{R}^n$ is a union of countably many tame compact sets $K_1, K_2, \ldots$, then $K$ is tame.

**Definition 2.7.** A subset $P \subset \mathbb{R}^n$ is called a *polyhedron* if it is a union of a finite collection of simplices. A compactum $X \subset \mathbb{R}^n$ homeomorphic to a polyhedron is called *tame* if there exists a homeomorphism $h$ of $\mathbb{R}^n$ onto itself such that $h(X)$ is a polyhedron in $\mathbb{R}^n$; and called *wild* otherwise.

**Remark 2.8.** Any orientation-preserving self-homeomorphism of $\mathbb{R}^n$ is isotopic to the identity map. This follows from the Alexander theorem [37, Thm. V.3.1] together with the Annulus Theorem whose history is presented in [21]. This means that in Definitions 2.1 and 2.7 one can replace “a homeomorphism of $\mathbb{R}^n$” by “an isotopy of $\mathbb{R}^n$”. (For zero-dimensional compacta refer also to Statement 2.5)
Remark 2.9. M.A. Shtan’ko presented a general definition of tameness for embedded compacta and studied its properties [49], [50]. A brief but detailed overviews analyzing its relations with other possible definitions can be found in [40] and [21].

As we said, Antoine constructed the first wild arc in \( \mathbb{R}^3 \). In [4], he announced and in [6] described in detail a 2-sphere which contains the Antoine’s necklace (the description can also be found in [2] and in [11, Thm. 18.7]). Antoine mentions that a homeomorphism between this 2-sphere and a standard 2-sphere in \( \mathbb{R}^3 \) can be extended over their interior domains, but cannot be extended over their exterior domains, in contrast to the case of simple closed curves in plane.\(^1\) This homeomorphism cannot be extended over neighborhoods of the 2-spheres.\(^2\) A better-known example of a wild 2-sphere in \( \mathbb{R}^3 \) is the horned sphere of J.W. Alexander [1].

The feelers idea is very fruitful and widely used. As said, it comes back to [4], [5], [6], [2], [34], [14], [43, proof of Theorem 3, Corollary 4], [59, Thm. 1] and it can be stated as follows:

Statement 2.10. For \( n \geq 3 \), any Cantor set \( K \) which is a cellularly separated subset of \( \partial I^n \) and for any embedding \( f : K \to \mathbb{R}^n \), there exists an embedding \( F : I^n \to \mathbb{R}^n \) such that \( F|_K = f \), and \( F \) is piecewise linear on \( I^n - K \).

Definition 2.11. Two subsets \( X, Y \subset \mathbb{R}^n \) are ambiently homeomorphic (or equivalently embedded) if there exists a homeomorphism \( h \) of \( \mathbb{R}^n \) onto itself such that \( h(X) = Y \).

Definition 2.12. Let \( X, Y \subset \mathbb{R}^n \). We say that \( X \) can be ambiently embedded in \( Y \) if there exists a homeomorphism \( h \) of \( \mathbb{R}^n \) onto itself such that \( h(X) \subset Y \). Two subsets \( X, Y \subset \mathbb{R}^n \) are called ambiently comparable if at least one of them can be ambiently embedded into the other.

Remark 2.13. Using Statements 4.4 and 4.6 and their higher-dimensional analogues (Subsection 4.4), one can easily construct for each \( n \geq 3 \) two wild

\(^1\)“It en résulte qu’il existe dans \( E_3 \) une surface homéomorphe à une sphère, la correspondance s’étendant aux intérieurs de ces surfaces, mais pas aux extérieurs... On sait que, au contraire, la correspondance entre deux courbes de Jordan planes peut s’étendre à la totalité de leurs plans.” [4] p. 285

\(^2\)“...qu’il existe, dans l’espace à 3 dimensions, des surfaces homéomorphes (même simplement connexes), telles qu’aucune correspondance entre ces surfaces ne peut être étendue à leurs voisinages.” [6] p. 282–283
Cantor sets $X, Y \subset \mathbb{R}^n$ such that $X$ ambienly embeds in $Y$, and $Y$ ambiently embeds in $X$, but $X$ and $Y$ are inequivalently embedded.

**Remark 2.14.** Suppose that $X, Y \subset \mathbb{R}^n$ and $X$ ambienly embeds in $Y$. If $X$ is not locally hyperplanar at a point $x$, then $Y$ is not locally hyperplanar at the point $h(x)$.

### 3 Reduction of Theorem 1.1 to Theorems 1.3 and 1.4

By Theorem 1.4 we may assume that $X \subset I^{n-1}$ and $X \cap \partial I^{n-1} = \{0\}^{n-2} \times C$, where $C$ is the middle-thirds Cantor set on $I$. Hence $\{0\}^{n-2} \times C \times C \subset X \times C \subset I^{n-1} \times I$. Take an embedding

$$\{0\}^{n-2} \times C \times C \cong C \times C \subseteq \mathbb{R}^n,$$

where the first identification is simply $(0, \ldots, 0, x, y) \mapsto (x, y)$, and $A$ is the embedding given by Theorem 1.3. Extend it to an embedding $\tilde{A} : I^n \to \mathbb{R}^n$ which is piecewise linear on $I^n - \{0\}^{n-2} \times C \times C$ (Statement 2.10). Define the embedding $A$ as the restriction of $\tilde{A}$ on $X \times C$. Let us show that $A$ has the desired properties.

Fix any $s \in C$. Evidently $A(X \times \{s\}) \supset A(C \times \{s\})$, hence the set $A(X \times \{s\})$ is not locally hyperplanar at any point $x \in A(\{0\}^{n-2} \times C \times \{s\})$ (see Statement 2.3 and Remark 2.14). Since $A : I^n \to \mathbb{R}^n$ is piecewise linear on $I^n - \{0\}^{n-2} \times C \times C$, its restriction on $I^{n-1} \times \{s\}$ is piecewise linear on $I^{n-1} \times \{s\} - \{0\}^{n-2} \times C \times \{s\}$. By [45, Thm. 1.7.2], the set $A(I^{n-1} \times \{s\})$ (and therefore $A(X \times \{s\})$ is locally hyperplanar at each point $x \in A(I^{n-1} \times \{s\} - \{0\}^{n-2} \times C \times \{s\})$. By Remark 2.14 the sets $A(X \times \{s\})$ for $s \in C$ are pairwise ambiently incomparable.

To prove (i), note that $\tilde{A}|_{X \times I}$ is the required isotopy.

It remains to prove (ii). Now $X$ is a polyhedron with dim $X \geq 1$. Suppose that $A(X \times \{s\})$ is tame for some $s \in C$. Hence there exists a polyhedron $P$ in $\mathbb{R}^n$ such that: $P$ contains a Cantor set $K$ which is everywhere wild as a subset of $\mathbb{R}^n$, dim $P \geq 1$, and $P$ is embeddable in $\mathbb{R}^{n-1}$. Represent $P$ as a finite union of simplices $P = \bigcup_{i=1}^{q} \sigma_i$. (We do not assume that the simplices $\sigma_i$ are pairwise disjoint.) For each $i = 1, \ldots, q$ we have dim $\sigma_i \leq n - 1$; by
Thm. 2] the zero-dimensional compact set $K \cap \sigma_i$ is tame in $\mathbb{R}^p$. By Statement 2.6 $K$ is tame, a contradiction. Theorem 1.1 is proved.

4 Proof of Theorem 1.3

The idea is to construct “the Cantor set of Antoine’s necklaces $\cup A_s$” and the embedding $A$ simultaneously. We proceed in a countable number of steps. In our construction of $\cup A_s$, we alternate steps of two types: ramification and inserting simple chains. The resulting set $\cup A_s$ can be considered as a generalization of ramified Antoine’s Necklaces, see [29, p. 383], [22], [55].

4.1 On Antoine’s Necklaces

In 1920-21, L. Antoine constructed and investigated [3], [5, p. 91–105] his famous wild Cantor set now called Antoine’s Necklace.

Let us recall his construction.

Definition 4.1. Let $\Pi$ be a plane in $\mathbb{R}^3$. Let $D \subset \Pi$ be a disk of radius $r$ with center $Q$, and $\ell \subset \Pi$ a straight line such that $d(Q, \ell) = R > r$. A standard solid torus is the solid torus of revolution $T$ generated by revolving $D$ in $\mathbb{R}^3$ about $\ell$. The central circle of $T$ is the circle generated by rotating the point $Q$. The center of $T$ is the center of its central circle.

Antoine takes a standard solid torus $T$ in $\mathbb{R}^3$; assume that the ratio $\frac{r}{R}$ is small enough (where $r$ and $R$ are numbers described in Definition 4.1). There exists an integer $k$ (sufficiently large in comparison with $\frac{r}{R}$) such that a simple chain of cyclically linked $k$ tori, each geometrically similar to $T$, can be placed inside $T$, so that their centers lie on the central circle of $T$ and form a regular convex $k$-gon. Then, he applies a similarity transformation to place a chain of $k$ tori in the interior of each torus of the previous level, and so on. Since diameters of the tori tend to zero, the limit set is a Cantor set. Antoine showed [5, Part II, Chap. III] that this Cantor set is wild. Other proofs can be found e.g. in [24, Prop. 9.5], [41, Section 18], [37, §IV.4].

To generalize this construction, one may allow the tori to be non-standard solid tori, that is, not necessary tori of revolution; also, one may vary the number of tori on each stage, see [47]. For us, it will be convenient to restrict ourselves by the following definition.
**Definition 4.2.** A simple chain in a standard solid torus $T \subset \mathbb{R}^3$ is a finite family $T_1, \ldots, T_q$, $q \geq 3$, of pairwise disjoint congruent standard solid tori such that

1) $T_1 \cup \ldots \cup T_q \subset \mathring{T}$;
2) centers of $T_1, \ldots, T_q$ are subsequent vertices of a regular convex $q$-gon inscribed in the central circle of $T$;
3) $T_i$ and $T_j$ are linked for $|i-j| \equiv 1 \mod q$, and are not linked otherwise;
4) for each $i$, the central circle of $T_i$ is zero-homotopic in $T$.

That is, the chain $T_1, \ldots, T_q$ looks like a usual “necklace” which winds once around the hole of $T$; no one of the tori $T_i$ embraces the hole of $T$.

An Antoine’s Necklace is a Cantor set in $\mathbb{R}^3$ which can obtained as an intersection $A = \bigcap_{i \geq 0} M_i$, where each $M_i$ is the union of a finite number of pairwise disjoint standard solid tori such that:

A1) $M_0 \subset \mathbb{R}^3$ is a standard solid torus;
A2) $M_{i+1} \subset M_i$ for each $i$;
A3) for each $i$ and each component $T$ of $M_i$, the intersection $M_{i+1} \cap T$ is a union of solid tori, which form a simple chain in $T$.

The sequence $\{M_i\}$ is called a canonical defining sequence for $A$.

(Since $\dim A = 0$, conditions A1)–A3) imply that diameters of components of $M_i$ tend to zero as $i \to \infty$. Conversely, if a sequence $\{M_i\}$ satisfies A1)–A3) and diameters of components of $M_i$ tend to zero as $i \to \infty$, then the intersection $\bigcap_{i \geq 0} M_i$ is a Cantor set.)

Each Antoine’s necklace is everywhere wild; see [5, 79–82, p. 93–96].

**Remark 4.3.** Many authors describe an Antoine’s necklace as obtained by first taking a standard solid torus, then placing a simple chain of four solid tori in its interior, then again placing a simple chain of four tori in each of these four tori, and so on. A simple chain is often defined “by a picture”. This is not the original Antoine’s construction. It is true that the diameters of the tori can be made arbitrary small in case we use four non-standard tori at each stage (in fact, it suffices to take two tori [9]), but this is not evident since the tori may become “more and more curved” on each stage. For simplicity of exposition, we prefer to use “a large number” of standard tori, as Antoine did. (For an interesting related result, see [58].)

R.B. Sher classified Antoine’s necklaces up to ambient homeomorphism (in fact, he considered more general Antoine-type sets than those described in Definition 12):
Statement 4.4. [47, Thm. 2] Let $A$, $B$ be two Antoine’s necklaces in $\mathbb{R}^3$ with canonical defining sequences $\{M_i\}$, $\{N_i\}$. Then $A$ and $B$ are ambiently homeomorphic if and only if there exists a homeomorphism $h$ of $\mathbb{R}^3$ onto itself such that $h(M_i) = N_i$ for each $i \in \mathbb{N}$.

Remark 4.5. Sher’s Theorem implies that ambiently homeomorphic necklaces have the same number of components on each level $i$. This statement is weaker than Sher’s Theorem itself (see an example in [32, Theorem 2]), and its proof can be obtained directly; see [57, Theorem 4.6].

Statement 4.6. Let $A, B \subset \mathbb{R}^3$ be two Antoine’s necklaces with canonical defining sequences $\{M_i\}$ and $\{N_i\}$ correspondingly. Then, $A$ can be ambiently embedded in $B$ if and only if there exist an integer $k \geq 0$, a component $T$ of $N_k$, and a self-homeomorphism $h$ of $\mathbb{R}^3$ such that $h(M_i) = T \cap N_i + k$ for each $i \in \mathbb{N}$.

Proof. Let $h$ be a self-homeomorphism of $\mathbb{R}^3$ onto itself such that $h(A) \subset B$. Theorem 4.6 of [57] implies that for some component $T$ of $N_k$, $h$ maps $A$ homeomorphically onto $T \cap B$; now it remains to apply Sher’s Theorem. The converse statement is evident.

4.2 Notation and Tools needed for the proof of Theorem 1.3

1) The usual “middle-thirds” Cantor set $C$ in $[0, 1] \subset \mathbb{R}^1$ is defined as follows.

Take $K_0 = I = [0, 1]$. Put $K_1 = \Delta_0 \cup \Delta_1$, where $\Delta_0 = [0, \frac{1}{3}]$ and $\Delta_1 = [\frac{2}{3}, 1]$. Suppose that we have constructed a family $S(I; N)$ of $2^N$ segments $\Delta_{i_1 i_2 \ldots i_N}$, where $(i_1, \ldots, i_N) \in \{0, 1\}^N$; put $K_N = \cup \{J \mid J \in S(I; N)\}$. Divide each $\Delta_{i_1 i_2 \ldots i_N} \in S(I; N)$ into 3 equal parts; define $\Delta_{i_1 i_2 \ldots i_N 0}$ and $\Delta_{i_1 i_2 \ldots i_N 1}$ to be its first and third part, correspondingly (going from left to right). We thus obtain the family $S(I; N + 1)$ of $2^{N+1}$ segments $\Delta_{i_1 i_2 \ldots i_N i_{N+1}}$, where $(i_1, \ldots, i_{N+1}) \in \{0, 1\}^{N+1}$. Define $K_{N+1} = \cup \{J \mid J \in S(I; N + 1)\}$, and so on. Finally, put $C = \cap_{N=0}^{\infty} K_N$.

2) For any segment $L = [a, b]$ and any given integer $N \geq 0$, define the family of $2^N$ segments on $L$ by the formula

$$S(L; N) = \{H_L(J) \mid J \in S(I; N)\}$$

where $H_L : I \to L$ is the homeomorphism defined by $t \mapsto a(1-t) + bt$, $t \in I$.  

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3) Let \( \{L_i = [a_i, b_i]\} \) be a finite family of pairwise disjoint subsegments of \([0, 1]\). We say that this system is \textit{regularly indexed} or \textit{regularly ordered} if \( i < j \) implies \( b_i < a_j \).

4) For each non-negative integer \( s \), each array \((a_{i_1}, a_{i_1i_2}, \ldots, a_{i_1i_2\ldots i_s})\) of positive integers, and each array

\[
(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_s}) \in \{1, \ldots, 4^{a_{i_1}}\} \times \{1, \ldots, 4^{a_{i_1i_2}}\} \times \ldots \times \{1, \ldots, 4^{a_{i_1i_2\ldots i_s}}\}
\]

defs a segment \( J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_s}) \) inductively as follows.

For \( s = 0 \), the unique segment is \( J = I = [0, 1] \). Suppose that we have defined all segments \( J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_{s-1}}) \) for each \( q \leq s \). Then, to obtain \( J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_s}) \), consider the segment \( J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_{s-1}}) \). Enumerate \( 4^{a_{i_1i_2\ldots i_s}} \) segments of the family \( S(J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_{s-1}}), 2a_{i_1i_2\ldots i_s}) \) (see formula (11)) in regular order. Define \( J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1i_2\ldots i_s}) \) to be the \( k_{i_1i_2\ldots i_s} \)-th segment of this family.

5) Let \( G \) be a family of subsets of \( \mathbb{R}^n \). The number of its elements is denoted by \#(\( G \)); the mesh of the family is \( \|G\| = \sup \{\text{diam } A \mid A \in G\} \). By \( \widetilde{G} \) we denote the union \( \bigcup \{g \mid g \in G\} = \{x : x \in g \text{ for some } g \in G\} \). For two families \( G, G' \) of subsets of \( \mathbb{R}^n \), the inequality \( G \leq G' \) means that \( G \) is a refinement of \( G' \).

Next proposition follows from [41, Theorem 12.7]:

**Statement 4.7.** Suppose that for each integer \( k \geq 0 \) we have families \( G_k \) and \( G'_k \) of subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) correspondingly such that

(a) both \( G_k \) and \( G'_k \) are finite families of pairwise disjoint compacta;
(b) \( G_{k+1} \subseteq G_k \) and \( G'_{k+1} \subseteq G'_k \);
(c) \#(\( G_k \)) = \#(\( G'_k \));
(d) \( \|G_k\| \to 0 \) and \( \|G'_k\| \to 0 \) as \( k \to \infty \).

Moreover, suppose that for each \( k \geq 0 \) we are given a bijection \( f_k : G_k \to G'_k \) such that the triple of conditions \( g_k \in G_k, g_{k+1} \in G_{k+1}, g_{k+1} \subset g_k \) implies \( f_{k+1}(g_{k+1}) \subset f_k(g_k) \).

Then each point \( P \in \bigcap_{k \geq 0} \widetilde{G}_k \) has a unique representation of the form

\[
\{P\} = \bigcap_{k \geq 0} g_{k,P}, \text{ where } g_{k,P} \in G_k \text{ for each } k \geq 0; \text{ the formula } \{f(P)\} = \bigcap_{k \geq 0} f_k(g_{k,P}) \text{ provides a well-defined map } f \text{ of } \bigcap_{k \geq 0} \widetilde{G}_k \text{ to } \bigcap_{k \geq 0} \widetilde{G}'_k; \text{ and } f \text{ is a homeomorphism onto.}
\]

Below we will need the 2-fold Ramification procedure.
Definition 4.8. Suppose we are given a standard solid torus \( T \) in \( \mathbb{R}^3 \); \( T \) is generated by revolving a disk \( D \) about an axis \( \ell \) (Definition 4.1). Choose two disjoint subdisks \( D_1, D_2 \) of \( \mathring{D} \) and take the standard solid tori \( T_1, T_2 \) obtained by revolving \( D_1, D_2 \) about \( \ell \). We call \( T_1 \cup T_2 \) a 2-fold ramification of \( T \).

The following fact is elementary (the number of links is even for simplicity; we have in mind that the central circle of every second link lie in the same plane which contains the central circle of \( T \), and it is perpendicular to planes containing central circles of other links):

Statement 4.9. Let \( T \) be a solid torus of revolution. For each \( \varepsilon > 0 \) there exists an integer \( N \) such that for each \( k \geq N \) one can construct in \( T \) a simple chain of \( 2k \) congruent standard solid tori whose diameters do not exceed \( \varepsilon \).

4.3 Proof of Theorem 1.3 for \( n = 3 \).

The desired embedding \( A \) will be constructed using Statement 4.7. We will inductively define families \( G_k \) and \( G_k' \) of subsets of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) correspondingly, together with bijections \( f_k \), so that \( \bigcap_k G_k = \mathcal{C} \times \mathcal{C} \), and \( \bigcap_k G_k' \) is a Cantor set of pairwise ambiently incomparable Antoine's necklaces. For even and for odd values of \( k \) the constructions are different; we use therefore indices of two types: the first ones are usual; and the second ones are in square brackets. The whole array of indices is written in angle brackets.

In the description, we will use a special sequence \( a_0, a_1, a_{00}, a_{01}, a_{10}, a_{11}, \ldots \). We assume that its members are pairwise distinct positive integers. We also assume that it increases sufficiently fast; exact meaning of this will be clarified below.

Let us describe several initial steps for clarity.

Step 0. The family \( G_0 \) consists of one element \( K = I \times I \). The family \( G_0' \) contains one element \( T \) which is a standard solid torus in \( \mathbb{R}^3 \). The bijection \( f_0 : G_0 \rightarrow G_0' \) is the obvious one.

Step 1. The family \( G_1 \) contains 2 elements: \( K \langle [0] \rangle = I \times \Delta_0 \) and \( K \langle [1] \rangle = I \times \Delta_1 \). The family \( G_1' \) contains 2 elements \( T \langle [0] \rangle \) and \( T \langle [1] \rangle \) which are obtained from \( T \) by the 2-ramification procedure. Define the bijection \( f_1 : G_1 \rightarrow G_1' \) by \( K \langle [i] \rangle \mapsto T \langle [i] \rangle \), \( i = 0, 1 \).

Step 2. Let \( G_2 \) contain elements of 2 types.
Type 1: \( K \langle [0]; k_0 \rangle = J \langle k_0 \rangle \times \Delta_0 \), where for \( k_0 = 1, \ldots, 4^{a_0} \) the segments
$J(k_0)$ run over the family $S(I; 2a_0)$, see the formula (11) above.

Type 2: $K([1]; k_1) = J(k_1) \times \Delta_1$, where for $k_1 = 1, \ldots, 4^{a_1}$ the segments $J(k_1)$ run over the family $S(I; 2a_1)$.

Also, elements of $G'_2$ are of two types.

Type 1: solid tori $T([0]; k_0)$, $k_0 = 1, \ldots, 4^{a_0}$, which form a simple chain in $T([0])$.

Type 2: solid tori $T([1]; k_1)$, $k_1 = 1, \ldots, 4^{a_1}$, which form a simple chain in $T([1])$.

We can assume that $a_0$ and $a_1$ are sufficiently large, so that $\|G'_2\| < \frac{1}{2}$ (see Statement 19).

The bijection $f_2 : G_2 \to G'_2$ is defined by $K([i]; k_i) \mapsto T([i]; k_i)$ for each $i = 0, 1$; $k_i = 1, \ldots, 4^{a_i}$.

Let us now describe the general rule.

Step 2$s+1$. On Step 2$s$ we constructed the family $G_{2s}$. Its elements are of the form

$$K([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s}) = J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1\ldots i_s}) \times \Delta_{i_1\ldots i_s},$$

where each $i_j \in \{0, 1\}$, each $k_{i_1\ldots i_j} \in \{1, \ldots, 4^{a_{i_1\ldots i_j}}\}$, and the segments $J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1\ldots i_s})$ are defined as in Section 12.

Define the family $G_{2s+1}$. Its elements are

$$K([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s}, [i_{s+1}]) = J(k_{i_1}, k_{i_1i_2}, \ldots, k_{i_1\ldots i_s}, [i_{s+1}]) \times \Delta_{i_1\ldots i_{s+1}},$$

where each $i_j \in \{0, 1\}$ and each $k_{i_1\ldots i_j} \in \{1, \ldots, 4^{a_{i_1\ldots i_j}}\}$.

To obtain the family $G'_{2s+1}$, apply the 2-ramification procedure to each $T([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s})$; denote the two resulting tori by

$$T([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s}, [0])$$

and

$$T([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s}, [1]).$$

The family $G'_{2s+1}$ consists of all solid tori

$$T([i_1], k_{i_1}, [i_2], k_{i_1i_2}, \ldots, [i_s], k_{i_1\ldots i_s}, [i_{s+1}]),$$

where indices run over the same domains as for $G_{2s+1}$. Note that the inequality $\|G'_{2s}\| < \frac{1}{2^n}$ implies $\|G'_{2s+1}\| < \frac{1}{2^n}$. 

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The bijection \( f_{2s+1} : G_{2s+1} \to G'_{2s+1} \) maps

\[
K([i_1], k_{i_1}, [i_2], \ldots, [i_s], k_{i_{11}} \ldots i_{s+1}) \mapsto T([i_1], [k_{i_1}], [i_2], \ldots, [i_s], k_{i_{11}} \ldots i_{s+1}).
\]

**Step 2s+2.** The family \( G_{2s+2} \) consists of the elements

\[
K([i_1], k_{i_1}, [i_2], k_{i_{12}}, \ldots, [i_{s+1}], k_{i_{1(s+1)}}) = J(k_{i_1}, k_{i_{12}}, \ldots, k_{i_{1(s+1)}}) \times \Delta_{1, s+1},
\]

where each \( i_j \in \{0, 1\} \), and each \( k_{i_1 \ldots i_j} \in \{1, \ldots, 4^{a_{i_1 \ldots i_j}}\} \).

Let us describe the family \( G'_{2s+2} \). For any fixed values of \([i_1], k_{i_1}, [i_2], k_{i_{12}}, \ldots, [i_{s+1}], k_{i_{1(s+1)}}\), where each \( i_j \in \{0, 1\} \) and each \( k_{i_1 \ldots i_j} \in \{1, \ldots, 4^{a_{i_1 \ldots i_j}}\} \), take a simple chain of \( 4^{a_{i_1 \ldots i_{s+1}}} \) standard solid tori in \( T([i_1], k_{i_1}, [i_2], k_{i_{12}}, \ldots, [i_{s+1}]) \). Denote these tori by \( T([i_1], k_{i_1}, [i_2], k_{i_{12}}, \ldots, [i_{s+1}], k_{i_{1(s+1)}}) \), where \( k_{i_{11}} \ldots i_{s+1} \in \{1, \ldots, 4^{a_{i_1 \ldots i_{s+1}}}\} \). All tori obtained in this way form the family \( G'_{2s+2} \).

We can assume that the numbers \( a_{i_1 \ldots i_{s+1}} \) are sufficiently large, so that

\[
\|G'_{2s+2}\| < \frac{1}{2s+2} \quad \text{(see Statement 4.9)}.
\]

The bijection \( f_{2s+2} : G_{2s+2} \to G'_{2s+2} \) is defined by

\[
K([i_1], k_{i_1}, [i_2], \ldots, [i_{s+1}], k_{i_{1(s+1)}}) \mapsto T([i_1], [k_{i_1}], [i_2], \ldots, [i_{s+1}], k_{i_{1(s+1)}}).
\]

The process continues ad infinitum.

By construction, \( \|G'_{k}\| \to 0 \) as \( k \to \infty \). Hence \( K' = \bigcap_k \widetilde{G}_k \) is a Cantor set. The intersection \( \bigcap_k \widetilde{G}_k \) is exactly \( C \times C \).

Statement 4.7 provides the homeomorphism \( C \times C \cong K' \subset \mathbb{R}^3 \); thus we obtain the desired embedding \( A : C \times C \to \mathbb{R}^3 \).

In fact, each \( s \in C \) is the intersection of a (unique) system

\[
\Delta_{i_1} \supset \Delta_{i_1 i_2} \supset \ldots
\]

where each \( i_j \in \{0, 1\} \). The image \( A(C \times \{s\}) \) is easily seen to be the intersection of the corresponding decreasing sequence of sets

\[
T([i_1]) \supset \bigcup_{k_{i_1} = 1, \ldots, 4^{a_{i_1}}} T([i_1], k_{i_1}) \supset
\]

\[
\supset \bigcup_{k_{i_1} = 1, \ldots, 4^{a_{i_1}}} T([i_1], k_{i_1}, [i_2]) \supset \bigcup_{k_{i_1} = 1, \ldots, 4^{a_{i_1}}, k_{i_1 i_2} = 1, \ldots, 4^{a_{i_1 i_2}}} T([i_1], k_{i_1}, [i_2], k_{i_1 i_2}) \supset \ldots
\]

that is, \( A(C \times \{s\}) =: A_s \) is an Antoine’s necklace in \( \mathbb{R}^3 \).

Recall that the numbers \( a_{i_1 \ldots i_k} \) are pairwise different. By Statement 4.6, the necklaces \( A_s \) are pairwise ambiently incomparable.
4.4 Proof of Theorem 1.3 for \( n \geq 4 \) (sketch).

This case has minor difference with the case of \( n = 3 \). Higher-dimensional generalizations of Antoine’s construction were suggested by A.A. Ivanov [34], [35] and by W.A. Blankinship [14]; they both used \( n \)-dimensional solid tori \( D^2 \times (S^1)^{n-2} \) in \( \mathbb{R}^n \) and showed that the complements of these “generalized Antoine’s necklaces” have non-trivial fundamental groups. Analogous to Antoine’s construction, Blankinship takes the same number \( k \) of smaller tori inside each torus on each level, on each stage they are placed likewise on the first stage. A broader class of sets can be obtained by varying numbers of tori constructed on each level in each component. (Ivanov’s scheme allows this.) W.T. Eaton gives a brief and very clear description of such generalized Antoine-type sets [29, p. 380–381] and proves that they are wild [29, p. 381–383]. (Suchlike sets were described and studied also in [44].) Each of the sets described by Eaton is everywhere wild; further, for these sets we have a weaker but satisfactory analogue of Statement 4.6, see [57, Theorems 4.6, 5.1].

Having said all this, we see that a detailed proof for \( n \geq 4 \) is a hand-by-hand repetition of the case \( n = 3 \). It is therefore omitted.

5 Proof of Theorem 1.4

Let us introduce some additional notation. An \( n \)-cube neighborhood \( N \) of a point \( p \in \mathbb{R}^n \) is a set of the form \( N = [a_1, b_1] \times \ldots \times [a_n, b_n] \) such that each \( b_i - a_i > 0 \) and \( p \in \overline{N} \).

For an \( n \)-cube neighborhood \( N \) of both points \( p, p' \) define a homeomorphism \( H_{N,p,p'} : N \cong N \) as follows. Every point in \( N \), other than \( p \), lies on a unique line segment \( pq \) where \( q \in \partial N \). The homeomorphism \( H_{N,p,p'} \) takes \( p \) to \( p' \); and it maps the line segment \( pq \) onto \( p'q \) linearly. Note that \( H_{N,p,p'} \) is the identity on \( \partial N \). Extending \( H_{N,p,p'} \) by the identity map, we get a homeomorphism \( \tilde{H}_{N,p,p'} : \mathbb{R}^n \cong \mathbb{R}^n \).

For \( n \geq 2 \), let \( \pi : \mathbb{R}^n \to \mathbb{R}^{n-1} \) and \( \lambda : \mathbb{R}^n \to \mathbb{R} \) be the projection maps given by \( \pi(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}) \) and \( \lambda(x_1, \ldots, x_{n-1}, x_n) = x_n \).

Proof of the next Lemma is straightforward.

Lemma 5.1. In \( \mathbb{R}^n, n \geq 2 \), let \( N \) be an \( n \)-cube neighborhood of two points, \( p \) and \( q \) with \( \lambda(p) > \lambda(q) \). We suppose that \( x \in N \) and \( \lambda(x) \geq \lambda(p) \). Then \( \lambda(H_{N,p,q}(x)) \geq \lambda(q) \); and equality holds if and only if \( x = p \).
Lemma 5.2. Let $X \subset \mathbb{R}^n$ be a non-empty compact set, $n \geq 2$. Then there exists a homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$ such that $h(X) \subset I^n$, and $h(X) \cap \partial I^n$ is a single point in $I^{n-1} \times \{0\}$.

Proof. Since $X$ is bounded, a linear homeomorphism $L : \mathbb{R}^n \cong \mathbb{R}^n$ takes $X$ into $\tilde{I}^n$. Let $\delta = \inf_{p \in L(X)} \lambda(L(X))$. Recall that $L(X)$ is compact; hence $\delta > 0$, and for some point $p \in L(X)$ we have $\lambda(p) = \delta$. Let $N = I^{n-1} \times [-1, 1]$. Take $p' \in I^{n-1} \times \{0\}$. By Lemma 5.1, $\tilde{H}_{N,p,p'} \circ L$ is the desired homeomorphism.

Statement 5.3. Let $X \subset I^n$ be a non-empty perfect set, $n \geq 2$. Suppose that $X \cap \partial I^n = \{p_1, \ldots, p_k\} \subset \tilde{I}^{n-1} \times \{0\}$. For each $i$, let $N_i$ an $n$-cube neighborhood of $p_i$ in $\mathbb{R}^n$ such that $\pi(N_i) \subset \pi(I^n)$, and $N_i \cap N_j = \emptyset$ if $i \neq j$. Then there exists a homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$ such that

1) outside $\bigcup_{i=1}^k N_i$, $h$ is the identity map;
2) $h(X) \subset I^n$;
3) $h(X) \cap \partial I^n = \{p_1, q_1, p_2, q_2, \ldots, p_k, q_k\}$, where $q_i \in \pi(\tilde{N}_i) \times \{0\}$ and $p_i \neq q_i$ for each $i$.

Proof. Let is fix an $i$ and construct the homeomorphism in $N_i$. For simplicity, we omit the index $i$ and write $N = N_i$ and $p = p_i$. Since $X$ is perfect, there exists a point $r \in X \cap \tilde{N} \cap \tilde{I}^n$. Consider two cases.

Case 1. A point $r$ can be chosen so that $\pi(r) \neq \pi(p)$. Let $q = (\pi(r), 0)$. Let $M \subset N$ be an $n$-cube neighborhood of both $r$ and $q$ which does not contain $p$. The set $X \cap M$ is compact and non-empty, hence $\delta = \inf_{x \in X \cap M} \lambda(x)$ is positive, and for some $s \in X \cap M$ we have $\lambda(s) = \delta$.

Subcase 1.1. Suppose that this point $s$ can be chosen in $X \cap \tilde{M}$. Then, $\tilde{H}_{M,s,q}$ is the desired homeomorphism.

Subcase 1.2. Suppose that such a point $s$ cannot be chosen in $X \cap \tilde{M}$. Take any point $s' \in M$ with $\lambda(s') = \delta$. We come to Subcase 1.1 replacing $X$ by $\tilde{H}_{M,r,s'}(X)$ (the number $\delta$ may have changed).

Case 2. For each $r \in X \cap \tilde{N} \cap \tilde{I}^n$, we have $\pi(r) = \pi(p)$. Let $M$ be an $n$-cube neighborhood of $r$ such that $M \subset \tilde{N} \cap \tilde{I}^n$. Take any point $r' \in M$ with $\pi(r') \neq \pi(p)$. Replace $X$ by $\tilde{H}_{M,r,r'}(X)$; Case 1 now applies.

Statement 5.4. Let $X \subset \mathbb{R}^n$ be a non-empty perfect compact set, $n \geq 2$. There exists a homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$ such that

1) $h(X) \subset I^n$;
2) $h(X) \cap \partial I^n \subset \hat{I}^{n-1} \times \{0\}$;
3) $h(X) \cap \partial I^n$ is a Cantor set cellularly separated as a subset of $\hat{I}^{n-1} \times \{0\}$.

Proof. The homeomorphism $h$ is constructed as a limit of a sequence of homeomorphisms of $\mathbb{R}^n$. This sequence is defined inductively.

By Lemma 5.2 we may assume that $X \subset I^n$ and $X \cap \partial I^n$ is a single point in $\hat{I}^{n-1} \times \{0\}$; call it $p$. Let $N$ be an $n$-cube neighborhood of $p$ such that $\pi(N) \subset \hat{I}^{n-1}$. Apply Statement 5.3 to $X$ and $N$; we obtain a homeomorphism $h_0 : \mathbb{R}^n \cong \mathbb{R}^n$ such that $h_0 = \text{id}$ outside $N$; $h_0(X) \subset I^n$; and $h_0(X) \cap \partial I^n = \{p_0, p_1\} \subset \pi(N) \times \{0\}$, where $p_1 \neq p_0 = p$.

Let $N_0, N_1$ be disjoint $n$-cube neighborhoods of points $p_0, p_1$ correspondingly with $N_i \subset \hat{N}$. Apply Statement 5.3 to $h_0(X)$, $N_0, N_1$; we obtain a homeomorphism $h_1 : \mathbb{R}^n \cong \mathbb{R}^n$ with $h_1 = \text{id}$ outside $N_0 \cup N_1$; $h_1(h_0(X)) \subset I^n$; and $h_1(h_0(X)) \cap \partial I^n = \{p_{00}, p_{10}, p_{10}, p_{11}\}$, where $p_{00} = p_0 \neq p_{01} \in \pi(N_0) \times \{0\}$, $p_{10} = p_1 \neq p_{11} \in \pi(N_1) \times \{0\}$.

For each $(i_1, i_2) \in \{0, 1\}^2$, let $N_{i_1i_2}$ be an $n$-cube neighborhood of $p_{i_1i_2}$ such that $N_{i_1i_0} \cup N_{i_1i_1} \subset \hat{N}_{i_1}$, and $N_{i_1i_0} \cap N_{i_1i_1} = \emptyset$. Apply Statement 5.3 to $h_1(h_0(X))$ and four neighborhoods $N_{i_1i_2}$; get a homeomorphism $h_2$.

Continuing in this way, we obtain for each $k \geq 1$ and each $(i_1, \ldots, i_k) \in \{0, 1\}^k$ a homeomorphism $h_k : \mathbb{R}^n \cong \mathbb{R}^n$, and a family of $2^k$ pairwise disjoint $n$-cube neighborhoods $N_{i_1i_k}$ of points $p_{i_1\ldots i_k}$, such that

1. $h_k = \text{id}$ outside $\bigcup_{(i_1, \ldots, i_k) \in \{0, 1\}^k} N_{i_1\ldots i_k}$;
2. $N_{i_1\ldots i_k} \cup N_{i_1\ldots i_{k-1}} \subset \hat{N}_{i_1\ldots i_k}$;
3. $h_{k-1} \circ h_{k-2} \circ \ldots \circ h_0(X) \subset I^n$;
4. $h_{k-1} \circ h_{k-2} \circ \ldots \circ h_0(X) \cap \partial I^n = \{p_{i_1\ldots i_k} \mid (i_1, \ldots, i_k) \in \{0, 1\}^k\}$;
5. $p_{i_1\ldots i_k} \in \pi(N_{i_1\ldots i_k}) \times \{0\}$.

Note that for each $k$ we can choose the $n$-cubes $N_{i_1\ldots i_k}$ sufficiently small; how small depends on the homeomorphism $h_{k-1} \circ h_{k-2} \circ \ldots \circ h_0$; thus the sequence $\{h_k \circ h_{k-1} \circ \ldots \circ h_0\}$ can be constructed so that it converges to a homeomorphism $h : \mathbb{R}^n \cong \mathbb{R}^n$ [20, Thm. 5.1]. We have

$$h(X) \cap \partial I^n = \bigcap_{k \geq 1} \bigcup_{(i_1, \ldots, i_k) \in \{0, 1\}^k} N_{i_1\ldots i_k} \subset \hat{I}^{n-1} \times \{0\},$$
this is a Cantor set cellularly separated in $\hat{I}^{n-1} \times \{0\}$.

**Proof of Theorem 1.4.** We may assume that $X$ satisfies the conclusion of Statement [5.4]. That is, $X \subset I^n$ and $K = X \cap \partial I^n$ is a cellularly separated Cantor set in $I^{n-1} \times \{0\}$. There exists a homeomorphism $G : \partial I^n \cong \partial I^n$ such that $G(K) = \{0\}^{n-1} \times C$ (see Statement [2.5]). Extend $G$ to a homeomorphism $F : \mathbb{R}^n \cong \mathbb{R}^n$; we get the desired map.

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