On the equal time two-point distribution of the one-dimensional KPZ equation by replica

T Imamura\(^1\), T Sasamoto\(^2,3\) and H Spohn\(^3\)

\(^1\) Research Center for Advanced Science and Technology, The University of Tokyo, Japan
\(^2\) Department of Mathematics and Informatics, Chiba University, Japan
\(^3\) Zentrum Mathematik, Technische Universität München, Germany

E-mail: imamura@jamology.rcast.u-tokyo.ac.jp, sasamoto@math.s.chiba-u.ac.jp, sasamoto@ma.tum.de and spohn@ma.tum.de

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Abstract

In a recent contribution, Dotsenko establishes a Fredholm determinant formula for the two-point distribution of the Kardar–Parisi–Zhang equation in the long time limit and starting from narrow wedge initial conditions. We establish that his expression is identical to the Fredholm determinant resulting from the Airy2 process.

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1. Introduction

The Kardar–Parisi–Zhang (KPZ) equation is a stochastic evolution equation for a height function \(h(x, t)\), which models a surface growing through random deposition [1]. It was noted early on that the \(N\)th exponential moment of the height function can be expressed through the propagator of \(N\) quantum particles interacting with a short range attractive potential [2]. In one spatial dimension, this would be the attractive Lieb–Liniger Hamiltonian. For its propagator we have now rather concise expressions, which allows one to carry out sophisticated replica computations so as to obtain exact distribution functions for the height [3–11]. The various cases are distinguished by the initial conditions (narrow wedge, flat, two-sided Brownian motion in space variable \(x\)), by the location of the spacetime reference points, and by possibly taking a scaling limit (e.g. \(t \to \infty\)).

In the recent contribution [12], Dotsenko develops a novel summation formula, which he uses to study narrow wedge initial conditions. In [13] he jointly considers \(h(0, t), h(0, 2t)\), while in [12] he jointly considers \(h(x_1, t), h(x_2, t)\), both in the scaling limit \(t \to \infty\). It is this latter case which is of interest to us. Dotsenko obtains a Fredholm determinant for the joint distribution. Based on universality, Dotsenko’s expression should agree with the corresponding Fredholm determinant of the Airy2 process, but merely a direct inspection does not suffice to confirm the conjecture.
In our note we close the gap and prove the desired identity. As a by-product, the factorization assumption in [10] can be understood a posteriori as omitting the ‘cross terms’ of Dotsenko’s summation formula.

2. One-point distribution

The notations in [10, 12] differ from each other. To make our contribution self-contained and to fix the notation, we briefly recall the replica computation for the one-point distribution.

In dimensionless form, the KPZ equation reads
\[
\frac{\partial}{\partial t} h = \frac{1}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} + \eta,
\]
where \( h = h(x, t) \) is the height function at location \( x \in \mathbb{R} \) and time \( t \geq 0 \) and \( \eta \) is normalized Gaussian white noise. Through the Cole–Hopf transformation, \( Z = e^\theta \), (2.1) turns into the stochastic heat equation,
\[
\frac{\partial}{\partial t} Z = \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} + \eta Z.
\]

We will consider only the narrow wedge initial condition \( Z(x, 0) = \delta(x) \).

Let us set
\[
h(x, t) = -\frac{1}{2} t + \gamma t \bar{h}(x, t),
\]

\( \gamma = (t/2)^{1/3} \), and introduce the generating function
\[
G_{\gamma, t}(s) = \langle \exp(-e^{\gamma \bar{h}(x, t)-s}) \rangle = \langle \exp(-e^{\theta/24t-Z(x, t)}) \rangle.
\]

Since \( \lim_{t \to \infty} \exp(-e^{-\gamma t}) = \theta(x), \theta \) the step function, \( \theta(x) = 1 \) for \( x > 0 \), \( \theta(x) = 0 \) for \( x < 0 \), it holds
\[
\lim_{t \to \infty} G_{\gamma, t}(s) = \lim_{t \to \infty} \left[ \theta(s - \bar{h}(2\gamma^3 x, t) - x^2) \right] = \lim_{t \to \infty} \text{Prob}[\bar{h}(2\gamma^3 x, t) + x^2 \leq s].
\]

We introduce the moment generating function for \( Z(x, t) \),
\[
G_{\gamma, t}(s) = \sum_{N=0}^{\infty} \frac{(-e^{-s})^N}{N!} \langle Z(x, t)^N \rangle e^{iN/24t}.
\]

The \( N \)th moment, \( \langle Z(x, t)^N \rangle \), can be written using the eigenvalues and eigenfunctions of the \( \delta \)-Bose gas with \( N \) particles,
\[
\langle Z(x, t)^N \rangle = \langle x \rangle e^{-H_N} |0 \rangle = \sum_\gamma \langle x | \Psi_\gamma \rangle \langle \Psi_\gamma | 0 \rangle e^{-E_\gamma}.
\]

where \( |x \rangle = |x_1, \ldots, x_N \rangle \), \( N \)-times, \( H_N \) is the Hamiltonian of the \( \delta \)-Bose gas of \( N \) particles, and \( \gamma \) labels the eigenvalues \( E_\gamma \) and eigenstates \( |\Psi_\gamma \rangle \). Using the usual parametrization of the eigenstates [3, 7], for fixed \( N \) one writes
\[
E_\gamma = \frac{1}{2} \sum_{j=1}^{N} \bar{z_j}^2 = \frac{1}{2} \sum_{a=1}^{M} n_a q_a^2 - \frac{1}{24} \sum_{a=1}^{M} (n_a^3 - n_a),
\]

\[
\langle x | \Psi_\gamma \rangle = (|0 \rangle |\Psi_\gamma \rangle) e^{|0 \rangle |\Psi_\gamma \rangle} e^{-E_\gamma}.
\]

\[
|M|^{-1/2} \prod_{a, \beta=1}^{M} \left( \frac{1}{2} (n_a + n_\beta + i (q_a - q_\beta)) \right)^{M_{a, \beta}}.
\]
\[ \sum = \sum_{M=0}^{N} \frac{1}{M!} \prod_{a=1}^{M} \int_{-\infty}^{\infty} \frac{dq_a}{2\pi} \delta_{\sum_{n=1}^{M} n_a, N} \]  

(2.11)

Hence

\[ G_{t,s}(s) = \sum_{M=0}^{\infty} \left( \frac{-1)^M}{M!} \prod_{a=1}^{M} \int_{-\infty}^{\infty} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} (-1)^{n_a-1} e^{i q_a t/24 + i n_a (q_a - q'_a \gamma)} \right. \]

\[ \times \det \left( \frac{1}{t} \left( (n_a + n\beta) + i(q_a - q'_a) \right) \right) \right) M \]

(2.12)

with the understanding that the term with \( M = 0 \) equals 1.

Using

\[ \frac{1}{t} (n_a + n\beta) + i(q_a - q'_a) = \int_{0}^{\infty} d\omega_a \, e^{-i \frac{1}{t} (n_a + n\beta) + i(q_a - q'_a) \omega_a} \]

(2.13)

and a simple identity for determinants, one has

\[ G_{t,s}(s - x^2/(2r)) = \sum_{M=0}^{\infty} \left( \frac{-1)^M}{M!} \prod_{a=1}^{M} \int_{0}^{\infty} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} (-1)^{n_a-1} \right. \]

\[ \times e^{i q_a t/24 + n_a (q_a - q'_a \gamma)/2} \det \left( e^{-i \frac{1}{t} n_a (q_a + \omega) - i q_a (n_a - \omega) \omega} \right) \]

(2.14)

We rescale \( s \to y s, x \to 2 y t^2 x, \omega_a \to y_a \omega_a, q_a \to q_a / y \). Shifting \( q_a \to q_a + i \lambda \) and noting that \( e^{i (n_a - \omega x)} \) inserted in the determinant does not change its value, we arrive at

\[ G_{t,2y^3 t^2}(y s - x^2) = \det(1 - K_{t,s}), \]

(2.15)

where the kernel of \( K_{t,s} \) is given by

\[ K_{t,s}(\omega, \omega') = \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{i \omega (\gamma - \omega')} \sum_{n=1}^{\infty} (-1)^{n-1} e^{i q_a t/24 - y(\gamma + \frac{1}{2}(\omega + \omega'))} \]

(2.16)

Using

\[ e^{\lambda t^3 / 3} = \int_{-\infty}^{\infty} dy \text{Ai}(y) e^{\lambda y} \]

(2.17)

with \( \lambda = t^3 / 2 = y_t / 2t^3 \) and shifting \( y \to y + q' + \frac{1}{2}(\omega + \omega') \), one obtains

\[ K_{t,s}(\omega, \omega') = 2t^3 \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{i q (\omega - \omega')} \int dy \text{Ai} \left( \frac{2^{2/3}}{t} \left( y + q' + \frac{1}{2}(\omega + \omega') \right) \right) \]

\[ \times \sum_{n=1}^{\infty} (-1)^{n-1} e^{-y \gamma n(x-y)} \]

(2.18)

Summing over \( n \) and using the identity

\[ 2t^3 \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{i q \gamma} \text{Ai} \left( \frac{2^{2/3}}{t} (q^2 + x) \right) = \text{Ai}(x+y) \text{Ai}(x-y), \]

(2.19)

one obtains

\[ K_{t,s}(\omega, \omega') = \int_{-\infty}^{\infty} dy \text{Ai}(\omega + y) \text{Ai}(\omega' + y) \frac{e^{i (y-x)}}{1 + e^{i (y-x)}} \]

(2.20)

In the \( t \to \infty \) limit,

\[ \frac{e^{i (y-x)}}{1 + e^{i (y-x)}} \to \theta(y-s) \]

(2.21)

and the kernel of \( K_{t,s} \) turns into the kernel for the GUE Tracy–Widom distribution.
3. Two-point distribution and factorization assumption

Next we study the joint distribution of \( \tilde{h}(x_1, t), \tilde{h}(x_2, t) \) for \( x_2 - x_1 = O(t^{2/3}) \) in the limit \( t \to \infty \). Following the strategy of section 2, we consider the generating function

\[
G_{t,x_1,x_2}(s_1, s_2) = \exp(-e^{\gamma x_1 - s_1} \tilde{h}(x_1, t) - e^{\gamma x_2 - s_2} \tilde{h}(x_2, t)).
\]

It holds

\[
\lim_{t \to \infty} G_{t,2x_1,2x_2}(s_1, s_2) \gamma_2(s_1 - x_1^2), \gamma_2(s_2 - x_2^2)
\]

\[
= \lim_{t \to \infty} \text{Prob}[\tilde{h}(x_1, t) + x_1^2 \leq s_1, \tilde{h}(x_2, t) + x_2^2 \leq s_2].
\]

By expanding in \( e^{-s_1}, e^{-s_2} \),

\[
G_{t,x_1,x_2}(s_1, s_2) = \sum_{0}^{\infty} \sum_{0}^{\infty} \left( -e^{-s_1} \right)^{L} \left( -e^{-s_2} \right)^{R} \langle Z(x_1, t)^{L} Z(x_2, t)^{R} e^{L+R}|24},
\]

where the term with \( L = 0 = R \) equals 1. Expanding in the eigenstates of the \( \delta \)-Bose gas one arrives at

\[
\langle Z(x_1, t)^{L} Z(x_2, t)^{R} \rangle = \sum_{\psi} |\langle 0| \psi \rangle |^2 e^{-E_{\psi}} \langle x_1, x_2; L, R| \psi \rangle,
\]

where \( |x_1, x_2; L, R \rangle = |x_1, \ldots, x_1, x_2, \ldots, x_2 \rangle \) with \( L \) times the argument \( x_1 \) and \( R \) times the argument \( x_2 \).

By Dotsenko’s summation formula, the last factor in the summation can be written as

\[
\frac{\langle x_1, x_2; L, R| \psi \rangle}{\langle 0| \psi \rangle} \delta_{\sum_{a=1}^{M} n_{a}, L+R} = \left( L + R \right)^{-1} \sum_{m_{0}+l_{0}>0} \prod_{a=1}^{M} \left( \frac{n_{a}}{m_{a}} \right)
\times e^{x_1 m_{a} q_{a} + x_2 m_{a} q_{a} - m_{a} l_{a} x_{2}/2} \delta_{m_{a} + l_{a}, n_{a}} \delta_{\sum_{a=1}^{M} m_{a}, L} \delta_{\sum_{a=1}^{M} l_{a}, R} \times CT,
\]

where \( x = x_2 - x_1 \) and the summation over \( m_{a}, l_{a} \) is for all nonnegative values except for \( m_{a} = 0 = l_{a} \). The factor \( CT \) denotes the cross term which involves a product over all pairs \( \alpha, \beta \), explicitly given by (cf (26) in [12])

\[
CT = \prod_{\alpha \neq \beta} \frac{\Gamma \left[ 1 + \frac{1}{2} (m_{a} + m_{\beta} + l_{a} + l_{\beta}) + i(q_{a} - q_{\beta}) \right] \Gamma \left[ 1 + \frac{1}{2} (-m_{a} + m_{\beta} + l_{a} - l_{\beta}) + i(q_{a} - q_{\beta}) \right]}{\Gamma \left[ 1 + \frac{1}{2} (m_{a} + m_{\beta} + l_{a} + l_{\beta}) + i(q_{a} - q_{\beta}) \right] \Gamma \left[ 1 + \frac{1}{2} (-m_{a} + m_{\beta} + l_{a} - l_{\beta}) + i(q_{a} - q_{\beta}) \right]}.
\]

Substituting in (3.3), we obtain

\[
G_{t,x_1,x_2}(s_1, s_2) = \sum_{M=0}^{\infty} \left( \frac{-1}{M!} \right)^{M} \prod_{a=1}^{M} \int_{-\infty}^{\infty} \frac{dq_{a}}{2 \pi} \sum_{m_{a}+l_{a}>0} (-1)^{m_{a}+l_{a}-1} \left( m_{a} + l_{a} \right) \left( m_{a} \right)
\times \text{det} \left( \begin{array}{c} \frac{1}{2} (m_{a} + l_{a}) + \frac{1}{2} (m_{\beta} + l_{\beta}) + i(q_{a} - q_{\beta}) \\ 0 \end{array} \right)_{a, \beta=1}^{M}
\times e^{(m_{a}+l_{a})/24 - \frac{1}{2} (m_{a}+l_{a}) (m_{\beta}+l_{\beta}) - l_{a} x_{1} + m_{a} x_{1} + \frac{1}{2} (m_{a}+l_{a}) (m_{\beta}+l_{\beta}) - l_{a} x_{1} + m_{a} x_{1} + \frac{1}{2} (m_{a}+l_{a}) (m_{\beta}+l_{\beta}) - l_{a} x_{1} + m_{a} x_{1}} \times CT.
\]
On the other hand, expression (4.8) in [10] after factorization reads, upon replacing \( x \to x/2 \) there to match with our notation\(^4\),

\[
G^\#_{t, x_1, x_2}(s_1, s_2) = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \int_{-\infty}^{\infty} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} \left( -1 \right)^{n_a-1} \det \left( \frac{1}{2n_a + \frac{1}{2}q_{\alpha} + i(q_a - q_{\beta})} \right)_{\alpha, \beta=1}^M \\
\times \prod_{a=1}^{M} e^{\frac{i}{2} \partial_t \partial_z} \left( e^{i \gamma z_{\alpha} - x_1} + e^{i \gamma z_{\beta} - x_2} \right)_{\alpha}^{n_a},
\]

(3.8)

where \( \partial_i = \partial_{x_i}, i = 1, 2 \). Binomially expanding the last factor on the RHS, we note that

\[
e^{-\frac{1}{2} \partial_t \partial_z} \left( e^{i \gamma z_{\alpha} - x_1} + e^{i \gamma z_{\beta} - x_2} \right)_{\alpha}^{n_a} = \sum_{m_a=0}^{n_a} e^{-\frac{1}{2} \partial_t \partial_z} \left( e^{i \gamma z_{\alpha} - x_1} \right)_{\alpha}^{m_a} \epsilon_{\alpha}^{m_a} \left( \gamma z_{\alpha} - x_1 \right) \epsilon_{\alpha}^{m_a} \left( \gamma z_{\alpha} - x_2 \right).
\]

(3.9)

Rewriting the summation over \( n_a \) in (3.8) by setting \( l_a = n_a - m_a \), this expression is the same as (3.7) except for the last factor CT. Hence the factorization assumption in [10] amounts to set the cross term CT = 1.

4. Dotsenko’s formula yields the Airy\(_2\) process

In [10], it is established that under factorization the limit \( t \to \infty \) of the expression in the last section indeed yields the two-point distribution of the Airy\(_2\) process. On the other hand, in [12] Dotsenko argues that, after a suitable analytic continuation, CT \( \to \infty \) as \( t \to \infty \) and then writes a Fredholm determinant expression for the limiting two-point distribution. One expects that Dotsenko’s formula should coincide with the two-point function of the Airy\(_2\) process. In this section, we establish rather straightforwardly the equivalence between expression (51) in [12] and the \( t \to \infty \) limit of (4.19) in [10].

Substituting \( s_1 \to \gamma s_i, x_i \to 2\gamma^2 x_i, i = 1, 2 \) (cf equation (2.31) of [10] and (4.19) in [10]) turns into

\[
G_{t, 2\gamma^2 x_1, 2\gamma^2 x_2}(s_1 - x^2_1, s_2 - x^2_2) = \det(1 - N),
\]

(4.1)

where the kernel of \( N \) is given by

\[
N(z_1, z_2) = \mathbb{1}_{\{z_1, z_2 \geq 0\}} \int_{-\infty}^{\infty} du \ e^{-2\gamma \partial_t \partial_z} e^{-i \gamma z_1 - x_1} \Phi(\gamma t (u - s_1), \gamma t (u - s_2)) \times \text{Ai}(z_1 + u) \text{Ai}(z_2 + u).
\]

(4.2)

Here

\[
\Phi(u, v) = \frac{e^u + e^v}{e^u + e^v + 1} = \sum_{l+m=0}^{\infty} (-1)^{l+m-1} \binom{l+m-1}{m} e^{u+m+iv}.
\]

(4.3)

Therefore

\[
\lim_{t \to \infty} \Phi(\gamma t (u - s_1), \gamma t (u - s_2)) = \theta(u - s_1) + \theta(u - s_2) - \theta(u - s_1) \theta(u - s_2) =: \Phi_\infty(s_1, s_2, u)
\]

(4.4)

and hence

\[
\lim_{t \to \infty} \text{Prob}[\tilde{h}(2\gamma^2 x_1, t) + x_1^2 \leq s_1, \tilde{h}(2\gamma^2 x_2, t) + x_2^2 \leq s_2] = \det(1 - N_\infty),
\]

(4.5)

Note that the definitions of \( s \) in [12] and [10] differ by factor of 2. Also 1/2 in (3.21), (3.23), (3.27) of [10] should instead read 1/4.

\(^4\) Note that the definitions of \( x \) in [12] and [10] differ by factor of 2. Also 1/2 in (3.21), (3.23), (3.27) of [10] should instead read 1/4.
where

\[ N_\infty(z_1, z_2) = \text{\mathcal{F}}_{(z_1, z_2 > 0)} \int_{\infty} \int_{-\infty} du e^{-x_1 u} e^{-(s_1 + t_1 u)(s_1 - t_1 u)} \Phi_\infty(s_1, s_2, u) \Phi(z_1 + u) \Phi(z_2 + u). \]  

(4.6)

The derivative with respect to \( s_1, s_2 \) may seem rather formal, since it is acting on a step function. However, in later sections of [10] there are equivalent expressions for \( N_\infty \), for which such difficulties do not arise (compare with section 5).

Turning to equation (51) in [12], with the understanding that one should replace \( x \to 2^{3/2}x \), \( s \to 2^{3/2}s \) to conform to the notations up to now, the probability in (4.5) is written as a Fredholm determinant with operator kernel

\[ A(\omega_1, \omega_2) = \int_0^\infty dy \int_0^\infty \frac{dq}{2\pi} \Phi(y + q^2 + s_1 + \omega_1 + \omega_2 - i\epsilon y) \Phi(y + q^2 + s_2 + \omega_1 + \omega_2 + i\epsilon y) e^{i\epsilon(y_1 - y_2)} \\
- \int_0^\infty dy \int_0^\infty \frac{dq}{2\pi} (2\pi)^{-3/2} \int d\xi d\eta \Phi(y + q^2 + \omega_1 + \omega_2 - i\epsilon \sqrt{x}) \Phi(y - s_1 + \omega_1 + \omega_2 + i\epsilon \sqrt{x}) e^{i\epsilon(x_1 - x_2)} \\
\times e^{i\epsilon(y_1 - y_2)} \Phi(y - s_1 + i\epsilon y) \Phi(y - s_2 + i\epsilon y) \{ \theta(y - s_1 + i\epsilon y) \Phi(y - s_1 + i\epsilon y) \Phi(y - s_2 + i\epsilon y) \}. \]  

(4.7)

Here, the step functions with complex arguments are to be understood as resulting from a limit in analogy to (2.21). The last term can be rewritten as

\[ (2\pi)^{-3/2} \int d\xi d\eta \Phi(y + q^2 + \omega_1 + \omega_2 - i\epsilon \sqrt{x}) \Phi(y - s_1 + i\epsilon y) \Phi(y - s_2 + i\epsilon y) \]  

(4.8)

\textbf{Proof.} It is easy to see that

\[ e^{-x_1 u_1} e^{-m_1 - l_1} = e^{-m_1 - l_1 - -x_1 u_1} = (2\pi)^{-3/2} \int d\xi d\eta \Phi(y + q^2 + \omega_1 + \omega_2 - i\epsilon \sqrt{x}) \Phi(y - s_1 + i\epsilon y) \Phi(y - s_2 + i\epsilon y). \]  

(4.9)

Multiplying both sides by \((-1)^{l+m-1} e^{m_1 + l_1 + i(s_1 + l_1)q}\), substituting \( s_1 \to \gamma_1 s_1, s_1 \to 2\gamma_1^2 s_1 \) and summing over \( l, m \), we obtain

\[ e^{-x_1 u_1} \Phi(\gamma(s_1 - s_1 + i\epsilon_1 q), \gamma(s_1 - s_2 + i\epsilon_2 q)) \]  

\[ = (2\pi)^{-3/2} \int d\xi d\eta \Phi(y + q^2 + \omega_1 + \omega_2 - i\epsilon \sqrt{x}) \Phi(y - s_1 + i\epsilon y) \Phi(y - s_2 + i\epsilon y). \]  

(4.10)

Taking the limit \( t \to \infty \), we arrive at the desired equality. \( \square \)
Using equation (4.8) one writes
\[
A(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy \, A(y + q^2 + \omega_1 + \omega_2) \, e^{i y (\omega_2 - \omega_1)} \\
\times e^{-x_1 y_2 - x_2 y_1} \Phi_\infty(s_1 - i x_1 q, s_2 - i x_2 q, y) \\
= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta^{3/2}(\omega_2 - \omega_1 - (x_1 y_2 + x_2 y_1)) \, e^{i y (\omega_2 - \omega_1)} \\
\times e^{-x_1 y_2 - x_2 y_1} \Phi_\infty(s_1, s_2, 2\sqrt{3}y). \tag{4.11}
\]

We switch to \( x, s \) as in (4.6) and use (2.19) to obtain
\[
A(\omega_1, \omega_2) = 2^{1/3} \int dy \, A(y + 2^{1/3} \omega_1 - (x_1 \partial_1 + x_2 \partial_2)) A(y + 2^{1/3} \omega_2 + (x_1 \partial_1 + x_2 \partial_2)) \\
\times e^{-x_1 y_2 - x_2 y_1} \Phi_\infty(s_1, s_2, y). \tag{4.12}
\]

Since \( \Phi_\infty \) can be written as a limit of a sum of exponentials (see (4.3) and (4.4)) the derivatives \( \partial_i, i = 1, 2, \) inside the Airy functions can be regarded as numbers. Finally, one can rewrite (4.12) as
\[
A(\omega_1, \omega_2) = 2^{1/3} \int dy \, e^{-x_1 y_2 - x_2 y_1} e^{2^{1/3} (x_1 \partial_1 + x_2 \partial_2)(\partial_{\omega_1} - \partial_{\omega_2})} \\
\times A(y + 2^{1/3} \omega_1) A(y + 2^{1/3} \omega_2) \Phi_\infty(s_1, s_2, y), \tag{4.13}
\]
which is the same as the kernel \( N_\infty \) of (4.6) with the identification \( z_i = 2^{1/3} \omega_i, i = 1, 2. \)

We draw one conclusion from our computation. In both [12] and [10] one has to linearize the factor \( e^{-iml}; \) see (4.9). A threefold Gaussian integration is used in [12], whereas the differential operator of (4.9) is used in [10]. If already in equation (34) of [12] one would use the latter method, then \( N_\infty \) would appear without further efforts.

5. A simplified proof for the Fredholm determinant of \( N \) to determine the Airy\(_2\) process

In [10] it is proved that the Fredholm determinant of \( N \) equals that of the two-point distribution of the Airy\(_2\) process as \( t \to \infty. \) The arguments of [10] are somewhat involved. Here we provide a more straightforward proof.

We begin from an expression \( G_{\gamma_1, \gamma_2}^{\lambda}(x_1 - x_1^2/2t, x_2 - x_2^2/2t) = \text{det}(1 - \tilde N), \) which follows from multiplying out the last factor of (3.8), with the kernel of \( \tilde N \) given by
\[
\tilde N(z, z') = \mathbb{I}_{z, z' > 0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \binom{n}{k} e^{-(x_1 - x_1^2/2t)(k - x_2 - x_2^2/2t)(n-k)} \\
\times e^{-\frac{1}{2} i k(n-k)} \int du \, A(u + z - y) A(u + z' - y) \, e^{i u u} \tag{5.1}
\]
with \( y = (x_1 k + x_2 (n-k))/2\gamma. \) Let us rewrite the integral over \( u \) as
\[
\int du \, dv \delta(u - v) e^{y uu} A(u + z + y) A(v + z' - y) \\
= \int du \, dv \delta(u - v) \, e^{y uu} \\
\times A(u + z + (n-k) x/2\gamma) e^{i u u} A(u + z' + k x/2\gamma), \tag{5.2}
\]
where the integrals over \( u, v \) have both been shifted by \( w = (x_1 k + x_2 (n-k))/2\gamma. \) We note the identity
\[
e^{au} A(u + \lambda) = e^{-tau} e^{-tau^2} e^{-tau} A(u + a) \tag{5.3}
\]
with the Airy operator $H = -(\partial_u)^2 + u$; see appendix B of [10].

**Proof.** Since

$$ (H + \lambda) \Ai(u + \lambda) = 0, \quad (5.4) $$

one arrives at

$$ e^{au} \Ai(u + \lambda) = e^{-\tau(H + \lambda)} e^{au} e^{-\tau(H + \lambda)} \Ai(u + \lambda). \quad (5.5) $$

Using

$$ v_0(\tau) = e^{\tau H} u e^{-\tau H} = u + \tau^2 - 2 \tau \partial_u, $$

$$ e^{a\tau^2} e^{au} e^{-2\tau \partial_u} = e^{a\tau^2 - a^2 \tau} e^{au} e^{-2\tau \partial_u}, \quad (5.6) $$

(5.3) follows from

$$ e^{au} \Ai(u + \lambda) = e^{-\tau(l + a^2)} e^{a\tau^2} e^{-\tau H} e^{au} e^{-2\tau \partial_u} \Ai(u + \lambda). \quad (5.7) \tag*{□} $$

With the help of identity (5.3), the kernel of $\tilde{N}$ can be rewritten as

$$ \tilde{N}(z, z') = \mathbb{1}_{[\zeta, \zeta')] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n-1} \binom{n}{k} \frac{\sin(k-z_1(n-k))}{\sin(n-z_2(n-k))} \times \int_0^\infty du dv |u| e^{-\beta u^2/2} |v| e^{i n \beta u + \gamma \beta v} \Ai(u + z) \Ai(v + z'). \quad (5.8) $$

Using the cyclicity of the determinant, $\det(1 - AB) = \det(1 - BA)$, it holds that $\det(1 - \tilde{N}) = \det(1 - \tilde{L})$ with the kernel of $\tilde{L}$ given by

$$ \tilde{L}(u, v) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n-1} \binom{n}{k} \frac{\sin(k-z_1(n-k))}{\sin(n-z_2(n-k))} |u| e^{-\beta u^2/2} |v| \times e^{i n \beta u + \gamma \beta v} \int_0^\infty \Ai(u + z) \Ai(v + z) e^{-\beta x^2/2} dz. \quad (5.9) $$

From this expression it is easy to see that the Fredholm determinant of $\tilde{L}$ yields that of the Airy$_2$ process for $t \to \infty$; see the arguments in section 2.4 of [10].

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