Generalized twisted modules associated to general automorphisms of a vertex operator algebra

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Abstract
We introduce a notion of strongly $\mathbb{C}\times\mathbb{Z}$-graded, or equivalently, $\mathbb{C}/\mathbb{Z}$-graded generalized $g$-twisted $V$-module associated to an automorphism $g$, not necessarily of finite order, of a vertex operator algebra. We also introduce a notion of strongly $\mathbb{C}$-graded generalized $g$-twisted $V$-module if $V$ admits an additional $\mathbb{C}$-grading compatible with $g$. Let $V = \bigoplus_{n \in \mathbb{Z}} V(n)$ be a vertex operator algebra such that $V(0) = \mathbb{C}1$ and $V(n) = 0$ for $n < 0$ and let $u$ be an element of $V$ of weight 1 such that $L(1)u = 0$. Then the exponential of $2\pi\sqrt{-1} \operatorname{Res}_x Y(u, x)$ is an automorphism $g_u$ of $V$. In this case, a strongly $\mathbb{C}$-graded generalized $g_u$-twisted $V$-module is constructed from a strongly $\mathbb{C}$-graded generalized $V$-module with a compatible action of $g_u$ by modifying the vertex operator map for the generalized $V$-module using the exponential of the negative-power part of the vertex operator $Y(u, x)$. In particular, we give examples of such generalized twisted modules associated to the exponentials of some screening operators on certain vertex operator algebras related to the triplet $W$-algebras. An important feature is that we have to work with generalized (twisted) $V$-modules which are doubly graded by the group $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}$ and by generalized eigenspaces (not just eigenspaces) for $L(0)$, and the twisted vertex operators in general involve the logarithm of the formal variable.

1 Introduction
The present paper is the first in a series of papers developing systematically a general theory of twisted modules for vertex operator algebras. This the-
ory is in fact equivalent to the study of orbifold theories in conformal field theory. Though twisted modules associated to finite-order automorphisms of a vertex operator algebra have been introduced and studied (see below for more discussions and references), curiously, twisted modules associated to infinite-order automorphisms of a vertex operator algebras have not been previously formulated and studied mathematically. To develop our general theory of twisted representations of vertex operator algebras, it is necessary to first find the correct category of twisted modules associated to both finite- and infinite-order automorphisms of a vertex operator algebra, and to study and construct such twisted modules. In this paper, we introduce a notion of strongly $\mathbb{C}^\times$-, or equivalently, $\mathbb{C}/\mathbb{Z}$-graded generalized $g$-twisted module associated to an automorphism of a vertex operator algebra and more generally, a notion of strongly $\mathbb{C}$-graded generalized $g$-twisted $V$-module when $V$ has an additional $\mathbb{C}$-grading compatible with $g$. We also construct examples. The category of such generalized twisted modules will be the category of twisted modules that we shall study in this series of papers.

Twisted modules for vertex operator algebras were introduced in the construction of the moonshine module vertex operator algebra $V^\natural$ by I. Frenkel, J. Lepowsky and A. Meurman [FLM1], [FLM2] and [FLM3]. Given a vertex operator algebra $V$ and an automorphism $g$ of $V$ of finite order, the notion of $g$-twisted $V$-module is a natural but very subtle generalization of the notion of $V$-module such that the action of $g$ is incorporated. The main axiom in the definition of the notion of $g$-twisted $V$-module is the twisted Jacobi identity of the type proved in [Le1], [FLM2], [Le2] and [FLM3]. In fact, $V^\natural$ was the first example of an orbifold theory, before orbifold theories were studied in physics.

Orbifold theories are important examples of conformal field theories obtained from known conformal field theories and their automorphisms (see [DHVW1], [DHVW2], [H], [DFMS], [HV], [NSV], [M], [DGH], [DVV], [DGM], as well as, for example [KS], [FKS], [Ba1], [Ba2], [BHS], [dBHO], [HO], [GHHO], [Ba3] and [HH]). Mathematically, the study of orbifold theories is equivalent to the theory of twisted representations of vertex operator algebras, in the sense that any result or conjecture on orbifold conformal field theories can be formulated precisely as a result or conjecture in the theory of twisted representations of vertex operator algebras.

There have been a number of papers constructing and studying twisted modules associated to finite-order automorphisms of vertex operator algebras (see for example, [Le1], [FLM2], [Le2], [FLM3], [D], [DL], [DonLM1],
[DonLM2], [Li], [BDM], [DoyLM1], [DoyLM2], [BHL], and the references in these papers, especially in the last three papers). However, as far as the author knows, the precise notion of $g$-twisted $V$-module for an infinite-order automorphism $g$ has not been previously formulated, and such twisted modules have not been previously constructed and studied.

The main difficulty in formulating such a general notion of $g$-twisted $V$-module is that the right-hand side

$$\frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \delta \left( \eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) Y^g(Y(g^ju, x_0)v, x_2)$$

of the twisted Jacobi identity (see [Le1], [FLM2], [Le2] and [FLM3]; cf. [DL]) and the form of the expression

$$(x_2 + x_0)^N Y^g(Y(u, x_0)v, x_2)$$

needed in the weak associativity relation (see [Li]) cannot be generalized to the general case. In the present paper, we first reformulate the notion of $g$-twisted $V$-module in the case of finite order $g$ using a duality property (incorporating both commutativity and associativity) formulated in terms of complex variables and then generalize this equivalent definition to the general case. The precise generalization involves more subtle issues which fortunately have been treated in the logarithmic tensor product theory developed by Lepowsky, Zhang and the author in [HLZ1] and [HLZ2]. First, in the general case, we have to consider a twisted generalization of the notion of generalized $V$-module in the sense of [HLZ1] and [HLZ2] instead of the notion of $V$-module. Second, the twisted vertex operators in general involve the logarithm of the formal variable. Third, we have to consider an additional grading by $\mathbb{C}^\times$ (equivalently by $\mathbb{C}/\mathbb{Z}$) or by $\mathbb{C}$ on such a twisted generalization to introduce a notion of strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized $g$-twisted $V$-module in the spirit of the the notion of strongly $\hat{A}$-graded generalized $V$-module in [HLZ1] and [HLZ2] for a suitable abelian group $\hat{A}$. It is important to note that in the general case, all these three new ingredients are necessary. In particular, many of the general results obtained in [HLZ1] and [HLZ2] are needed for the study of strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized $g$-twisted $V$-modules introduced and constructed in the present paper.

Even when $g$ is a finite-order automorphism of $V$, how to explicitly construct a nontrivial $g$-twisted $V$-module is still an open problem in general.
The problem of constructing strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized $g$-twisted $V$-module for a general automorphism of $V$ is certainly more difficult. However, when we consider certain special types of automorphisms of $V$, we might still be able to construct such generalized twisted $V$-modules associated to these automorphisms even though the general problem is not solved; indeed, in the finite-order case, the construction of certain twisted modules for lattice vertex operator algebras was given by Frenkel, Lepowsky and Meurman as one of the many hard steps in the construction of the moonshine module in the work [FLM3] and has been used to solve many other significant problems.

Given $u \in V_{(1)}$ such that $L(1)u = 0$, we show that $g_u = e^{2\pi \sqrt{-1} \text{Res}_x Y(u,x)}$ is an automorphism of $V$. In general, its order might not be finite. In the present paper, generalizing a construction by Li [Li] in the case that $\text{Res}_x Y(u,x)$ acts on $V$ semisimply and has eigenvalues belonging to $\frac{1}{k} \mathbb{Z}$, we construct a strongly $\mathbb{C}$-graded generalized $g_u$-twisted $V$-module for $u \in V_{(1)}$ from a strongly $\mathbb{C}$-graded generalized $V$-module with an action of $g_u$ such that the $\mathbb{C}$-grading is given by the generalized eigenspaces of the action of $\text{Res}_x Y(u,x)$. We modify the vertex operator map of the generalized $V$-module using the exponentials of the negative-power part of the vertex operator $Y(u,x)$. We study such exponentials for $u$ not necessarily of weight 1 and apply the result to the case of $u \in V_{(1)}$ later in the construction. Our results show the importance of studying exponentials of vertex operators.

As explicit examples, we apply this construction to certain vertex operator algebras constructed from certain lattice vertex operator algebras and automorphisms obtained by exponentiating suitable screening operators. The fixed-point subalgebras of these vertex operator algebras under the group generated by these automorphisms are in fact the triplet $\mathcal{W}$-algebras which in recent years have attracted a lot of attention from physicists and mathematicians. These algebras were introduced first by Kausch [Ka1] and have been studied extensively by Flohr [FL1] [FL2], Gaberdiel-Kausch [GK1] [GK2], Kausch [Ka2], Fuchs-Hwang-Semikhatov-Tipunin [FHST], Abe [A], Feigin-Gaınudtinov-Semikhatov-Tipunin [FGST1] [FGST2] [FGST3], Carqueville-Flohr [CF], Flohr-Gaberdiel [FG], Fuchs [Fu], Adamović-Milas [AM1] [AM2], Adamović-Milas [AM3], Flohr-Grabow-Koehn [FKK], Flohr-Knuth [FK], Gaberdiel-Runkel [GR1] [GR2], Gaınudtinov-Tipunin [GT], Pearce-Rasmussen-Ruelle [PRR1], [PRR2], Nagatomo-Tsuchiya [NT] and Rasmussen [R]. Our results show, in particular, that in this special case, the intertwining operators for suitable fixed-point subalgebras given by Adamović and Milas in Theorem 9.1 in [AM3] are in fact twisted vertex operators. This connection between suitable
interwining operators and twisted vertex operators shows that the triplet logarithmic conformal field theories are closely related to orbifold theories. Note that up to now, the triplet $W$-algebras are the main known examples of vertex operator algebras for logarithmic conformal field theories. We expect that our theory will provide an orbifold-theoretic approach to the triplet logarithmic conformal field theories and will give more interesting examples of logarithmic conformal field theories.

There are certainly many examples of vertex operator algebras which have automorphisms of infinite orders. For example, the automorphism groups of the vertex operator algebras for the Wess-Zumino-Novikov-Witten models contain the automorphism groups of the finite-dimensional Lie algebras that one starts with. There are also spectral flow automorphisms of the vertex operator algebras for the Wess-Zumino-Novikov-Witten models and of superconformal vertex operator algebras. Many of these automorphisms are of infinite orders and do not act semisimply. In the case that these automorphisms are the exponentials of weight 1 elements of the vertex operator algebras, strongly $\mathbb{C}$-graded generalized twisted $V$-modules associated to these automorphisms are constructed by applying the results of the present paper.

The present paper is organized as follows: In Section 2, we reformulate the notion of $g$-twisted $V$-module in the case of finite-order $g$ using complex variables and duality properties. Then we introduce the notions of strongly $\mathbb{C}/\mathbb{Z}$- and $\mathbb{C}$-graded generalized $g$-twisted $V$-module in the general case in Section 3. In Section 4, we study the exponentials of the negative-power parts of vertex operators. The construction of strongly $\mathbb{C}$-graded generalized $g_{u}$-twisted $V$-modules for $u \in V_{(1)}$ is given in Section 5. The explicit examples related to the triplet $W$-algebras are also given in this section.

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2 Twisted modules for finite-order automorphisms of a vertex operator algebra

The notion of $g$-twisted module associated to a finite order automorphism $g$ of $V$ is formulated by collecting the properties of twisted vertex operators obtained in [Le1], [FLM2], [Le2] and [FLM3]; in particular, it uses a twisted Jacobi identity (see (2.1) below) as the main axiom. See [FLM3], [D] and [DL]. For an infinite-order automorphism $g$ of $V$, we cannot write down a generalization of the twisted Jacobi identity in the case of finite-order automorphisms. The twisted Jacobi identity can also be replaced by the weak commutativity and (twisted) weak associativity in the case of finite-order automorphisms (see (2.3) below). But the side of the weak associativity containing the iterate of the twisted vertex operator map cannot be generalized to the case of an (necessarily infinite-order) automorphism which acts on the vertex operator algebra or the twisted module to be defined nonsemisimply. These difficulties force us to use the complex variable approach. To motivate our notion of $g$-twisted module for a general automorphism $g$ of $V$, we reformulate the definition of $g$-twisted module in terms of complex variables in this section.

Let $(V,Y,1,\omega)$ be a vertex operator algebra and $g$ an automorphism of $V$ of order $k \in \mathbb{Z}_+$. Then $V = \bigsqcup_{j \in \mathbb{Z}/k\mathbb{Z}} V^j$ where $V^j = \{v \in V \mid gv = \eta^j v\}$ for $j \in \mathbb{Z}/k\mathbb{Z}$, where $\eta = e^{2\pi \sqrt{-1}/k}$. We first recall the notion of $g$-twisted $V$-module.

**Definition 2.1.** A $g$-twisted $V$-module $W$ is a $\mathbb{C}$-graded vector space $W = \bigsqcup_{n \in \mathbb{C}} W_{(n)}$ equipped with a linear map

$$Y^g : V \otimes W \rightarrow W[[x^{1/k}, x^{-1/k}]]$$

$$v \otimes w \mapsto Y^g(v, x)w = \sum_{n \in \frac{1}{k}\mathbb{Z}} Y^g_n(v)wx^{-n-1}$$

satisfying the following conditions:

1. The *grading-restriction condition*: For each $n \in \mathbb{C}$, $\dim W_{(n)} < \infty$ and $W_{(n+l/k)} = 0$ for all sufficiently negative integers $l$.

2. The *formal monodromy condition*: For $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$, $Y^g(v, x) = \sum_{n \in j/k+z} Y^g_n(v)x^{-n-1}$. 

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3. The lower-truncation condition: For \( v \in V \) and \( w \in W \), \( Y^g_n(v)w = 0 \) for \( n \) sufficiently large.

4. The identity property: \( Y^g(1, x) = 1 \).

5. The twisted Jacobi identity: For \( u, v \in V \) and \( w \in W \),

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y^g(u, x_1)Y^g(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y^g(v, x_2)Y^g(u, x_1)
\]

\[
= \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \delta \left( \eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) Y^g(Y(g^j u, x_0)v, x_2). \quad (2.1)
\]

6. The \( L^g(0) \)-grading condition: For \( n \in \mathbb{Z} \), let \( L^g(n) = Y^g_{n+1}(\omega) \), that is, \( Y^g(\omega, x) = \sum_{n \in \mathbb{Z}} L^g(n)x^{-n-2} \). Then \( L^g(0)w = mw \) for \( w \in W_m \).

7. The \( L^g(-1) \)-derivative condition: For \( v \in V \),

\[
\frac{d}{dx} Y^g(v, x) = Y^g(L(-1)v, x).
\]

We denote a \( g \)-twisted \( V \)-module defined above by \((W, Y^g)\) (or briefly, by \( W \)). Note that in the definition above, we use \( Y^g_n(u) \) instead of \( u_n^g \) for \( n \in \frac{1}{k}\mathbb{Z} \) to denote the components of the vertex operator \( Y^g(u, x) \). In the present paper, we shall always use such notations to denote the components of a vertex operator. For example, for our vertex operator algebra \((V, Y, 1, \omega)\), we shall use \( Y_n(u) \) instead of \( u_n \) for \( n \in \mathbb{Z} \) to denote the components of the vertex operator \( Y(u, x) \). Similarly, for a \( V \)-module \((W, Y_W)\), we shall use \( (Y_W)_n(u) \) instead of \( u_n \) for \( n \in \mathbb{Z} \) to denote the components of the vertex operator \( Y_W(u, x) \).

This definition of \( g \)-twisted \( V \)-module is a natural generalization of the definition of \( V \)-module given in [Bo], [FLM3] and [FHL]. The main axiom is the twisted Jacobi identity. The following result is proved by Li [Li]:

**Proposition 2.2.** The twisted Jacobi identity in Definition 2.1 can be replaced by the following two properties:
1. The weak commutativity or locality: For \( u, v \in V \), there exists \( N \in \mathbb{Z}_+ \) such that
\[
(x_1 - x_2)^N Y^g(u, x_1) Y^g(v, x_2) = (x_2 - x_1)^N Y^g(v, x_2) Y^g(u, x_1). \tag{2.2}
\]

2. The weak associativity: For \( u \in V^j \) and \( w \in W \), there exists \( M \in \mathbb{Z}_+ \) such that
\[
(x_2 + x_0)^{j/k + M} Y^g(Y(u, x_0)v, x_2) = (x_0 + x_2)^{j/k + M} Y^g(u, x_0 + x_2) Y^g(v, x_2)
\]
for \( v \in V \), where \( \tilde{j} \in j \) satisfies \( 0 \leq \tilde{j} < k \).

One advantage of the definition, the weak commutativity and the weak associativity above is that they use only formal variables so that one can discuss the multivaluedness of the correlation functions algebraically. But as we mentioned above, this formulation of \( g \)-twisted \( V \)-module for \( g \) of finite order cannot be generalized to the case that \( g \) is of infinite order. So we first reformulate this definition using complex variables and a duality property.

We need a complex variable formulation of vertex operator maps and some notations first. For any \( z \in \mathbb{C}^\times \), we shall use \( \log z \) to denote \( \log|z| + \sqrt{-1} \arg z \) where \( 0 \leq \arg z < 2\pi \). In general, we shall use \( l_k(z) \) to denote \( \log z + 2\sqrt{-1}\pi \) for \( k \in \mathbb{Z} \). Let \( W_1 \), \( W_2 \) and \( W_3 \) be \( \mathbb{C} \)-graded vector spaces. The \( \mathbb{C} \)-gradings on \( W_1 \) and \( W_2 \) induce a \( \mathbb{C} \)-grading on \( W_1 \otimes W_2 \). For a formal variable \( x^{1/k} \), we define its degree to be \(-1/k\). Then the grading on \( W_3 \) and the degree of \( x \) together give \( W_3[[x^{1/k}, x^{-1/k}]] \) a \( \mathbb{C} \)-graded vector space structure. Let
\[
X : W_1 \otimes W_2 \rightarrow W_3[[x^{1/k}, x^{-1/k}]]
\]
\[
w_1 \otimes w_2 \mapsto X(w_1, x)w_2 = \sum_{n \in \mathbb{Z}} X_n(w_1)w_2x^{-n-1}
\]
is a linear map preserving the gradings. For any \( w_1 \in W_1 \) and \( w_2 \in W_2 \), we define \( X^p(w_1, z)w_2 \) for \( p \in \mathbb{Z}/k\mathbb{Z} \) to be the elements \( X(w_1, x)w_2 \big|_{x^n = e^{n\tilde{p}(z)}} \) of \( \mathbb{W}_3 = \prod_{n \in \mathbb{C}} (W_3)_{(n)} \) where for \( p \in \mathbb{Z}/k\mathbb{Z} \), \( \tilde{p} \) is the integer in \( p \) satisfying \( 0 \leq \tilde{p} < k \). When \( p = 0 \), for simplicity, we shall denote \( X^0(w_1, z)w_2 \) simply as \( X(w_1, z)w_2 \). For \( z \in \mathbb{C}^\times \), we have the linear maps
\[
X^p(\cdot, z) : W_1 \otimes W_2 \rightarrow \mathbb{W}_3,
\]
\[
w_1 \otimes w_2 \mapsto X^p(w_1, z)w_2 \tag{2.4}
\]
and we shall denote $X^0(\cdot, z)\cdot$ simply as $X(\cdot, z)\cdot$. For any $w'_3 \in W'_3$,

$$\langle w'_3, X^p(w_1, z)w_2 \rangle$$

are branches of a multivalued analytic function defined on $\mathbb{C}^\times$. We can view this multivalued analytic function on $\mathbb{C}^\times$ as a single-valued analytic function defined on a $k$-fold covering space of $\mathbb{C}^\times$. For $p \in \mathbb{Z}/k\mathbb{Z}$, we have a map

$$X^p : \mathbb{C}^\times \to \text{Hom}(W_1 \otimes W_2, \overline{W}_3)$$
$$z \mapsto X^p(\cdot, z)\cdot.$$

We shall call $X^p$ the $p$-th analytic branch of the map $X$.

In particular, for the twisted vertex operator map $Y^g$, we have its $p$-th analytic branch $Y^{gp}$. In terms of these analytic branches, we have:

**Proposition 2.3.** The formal monodromy condition in Definition 2.1 can be replaced by the following condition: For $p \in \mathbb{Z}/k\mathbb{Z}$, $z \in \mathbb{C}^\times$ and $v \in V$,

$$Y^{gp+1}(gv, z) = Y^{gp}(v, z). \quad (2.5)$$

*Proof.* Let $W$ be a $g$-twisted $V$-module satisfying Definition 2.1. Then for $v \in V^j$, we have $gv = \eta^j v \in V^j$. It is easy to see that this fact and the formal monodromy condition give

$$Y^{gp+1}(gv, z) = Y^{gp}(v, z)$$

for $p \in \mathbb{Z}/k\mathbb{Z}$, $z \in \mathbb{C}^\times$ and $v \in V^j$.

Conversely, assume that $W$ satisfies the modification of Definition 2.1 by replacing the formal monodromy condition with the equivariance condition. Then it is easy to see that the fact $gv = \eta^j v \in V^j$ for $v \in V^j$ and the equivariance condition give

$$\sum_{n \in \mathbb{Z}/k\mathbb{Z}} Y^g_n(v) \eta^j e^{(-n-1)lp+1(z)} = \sum_{n \in \mathbb{Z}/k\mathbb{Z}} Y^g_n(v) e^{\frac{2\pi(n+1)\sqrt{-1}}{k}e^{(-n-1)lp+1(z)}}$$

for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$. So we have

$$\sum_{n \in \mathbb{Z}/k\mathbb{Z}} Y^g_n(v) (\eta^j - e^{\frac{2\pi(n+1)\sqrt{-1}}{k}e^{(n+1)lp(z)}}) e^{(n+1)lp(z)} = 0,$$
which implies that
\[ Y_n^g(v)(\eta^j - e^{\frac{2\pi(n+1)i}{k}}) = 0 \]
for \( n \in \frac{1}{k}\mathbb{Z} \). So we have either \( Y_n^g(v) = 0 \) or \( \eta^j = e^{\frac{2\pi(n+1)i}{k}} \), that is, either \( Y_n^g(v) = 0 \) or \( n \in j/k + \mathbb{Z} \). So we obtain \( Y^g(v, x) = \sum_{n \in j/k + \mathbb{Z}} Y_n^g(v)x^{-n-1} \).

We shall call (2.5) in Proposition 2.3 the equivariance property.

For \( n \in \mathbb{C} \), let \( \pi_n : W \rightarrow W(n) \) be the projection. We have:

**Theorem 2.4.** The twisted Jacobi identity in Definition 2.1 can be replaced by the following property: For any \( u, v \in V, w \in W \) and \( w' \in W' \), there exists a multivalued analytic function of the form

\[ f(z_1, z_2) = \sum_{r,s=N_1}^{N_2} a_{rs}z_1^{r/k}z_2^{s/k}(z_1 - z_2)^{-N} \]

for \( N_1, N_2 \in \mathbb{Z} \) and \( N \in \mathbb{Z}_+ \), such that the series

\[ \langle w', Y^{gp}(u, z_1)Y^{gp}(v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{gp}(u, z_1)\pi_nY^{gp}(v, z_2)w \rangle, \quad (2.6) \]

\[ \langle w', Y^{gp}(v, z_2)Y^{gp}(u, z_1)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{gp}(v, z_2)\pi_nY^{gp}(u, z_1)w \rangle, \quad (2.7) \]

\[ \langle w', Y^{gp}(Y(u, z_1 - z_2)v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{gp}(\pi_nY(u, z_1 - z_2)v, z_2)w \rangle \]

(2.8)

are absolutely convergent in the regions \(|z_1| > |z_2| > 0, |z_2| > |z_1| > 0, |z_2| > |z_1 - z_2| > 0\), respectively, to the branch

\[ \sum_{r,s=N_1}^{N_2} a_{rs}e^{(r/k)p(z_1)}e^{(s/k)p(z_2)}(z_1 - z_2)^{-N} \]

(2.9)

of \( f(z_1, z_2) \).

**Proof.** Assume that \( W \) is a \( g \)-twisted \( V \)-module satisfying Definition 2.1. By Proposition 2.2, there exists \( N \in \mathbb{Z}_+ \) such that (2.2) holds.

For \( w \in W \) and \( w' \in W' \), we know that

\[ \langle w', Y^g(u, x_1)Y^g(v, x_2)w \rangle \]

(2.10)
involves finitely many positive powers of $x_1^{1/k}$ and finitely many negative powers of $x_2^{1/k}$ and
\[ \langle w', Y^g(v, x_2)Y^g(u, x_1)w \rangle \tag{2.11} \]
involves finitely many negative powers of $x_1^{1/k}$ and finitely many positive powers of $x_2^{1/k}$. Since $(x_1 - x_2)^N$ is a polynomial in $x_1^{1/k}$ and $x_2^{1/k}$, the same statements are also true for
\[ (x_1 - x_2)^N \langle w', Y^g(u, x_1)Y^g(v, x_2)w \rangle \]
and
\[ (x_1 - x_2)^N \langle w', Y^g(v, x_2)Y^g(u, x_1)w \rangle, \]
respectively. By (2.2), they are actually equal. Thus they are both equal to
\[ \sum_{r,s=N_1}^{N_2} a_{rs}x_1^{r/k}x_2^{s/k} \in \mathbb{C}[x_1^{1/k}, x_1^{-1/k}, x_2^{1/k}, x_2^{-1/k}]. \]
Since (2.10) involves finitely many positive powers of $x_1^{1/k}$ and finitely many negative powers of $x_2^{1/k}$, it must be equal to the expansion of
\[ \sum_{r,s=N_1}^{N_2} a_{rs}x_1^{r/k}x_2^{s/k} (x_1 - x_2)^{-N} \]
as a series of the same form. Thus the expansion of (2.9) in the region $|z_1| > |z_2| > 0$ is equal to (2.6). The same argument shows that the expansion of (2.9) in the region $|z_2| > |z_1| > 0$ is equal to (2.7).

For $j \in \mathbb{Z}/k\mathbb{Z}$, let $\tilde{j} \in j$ satisfying $0 \leq \tilde{j} < k$. For $u \in V^j$, by Proposition 2.2 there exists $M \in \mathbb{Z}_+$ such that (2.3) holds.

For $w \in W$ and $w' \in W'$, we know that (2.10) involves finitely many positive powers of $x_1^{1/k}$ and finitely many negative powers of $x_2^{1/k}$. So
\[ \langle w', Y^g(u, x_0 + x_2)Y^g(v, x_2)w \rangle \tag{2.12} \]
involves finitely many positive powers of $x_0^{1/k}$ and finitely many negative powers of $x_2^{1/k}$. We also know that
\[ \langle w', Y^g(Y(u, x_0)v, x_2)w \rangle \tag{2.13} \]
involves finitely many negative powers of $x_0^{1/k}$ and finitely many positive powers of $x_2^{1/k}$. Since $(x_0 + x_2)^{j/k + M}$ involves finitely many positive powers of $x_0^{1/k}$ and no negative powers of $x_2^{1/k}$ and $(x_2 + x_0)^{j/k + M}$ involves finitely many positive powers of $x_2^{1/k}$ and no negative powers of $x_2^{1/k}$, the same statements above for (2.12) and (2.13) are also true for

$$(x_0 + x_2)^{j/k + M} \langle w', Y^g(u, x_0 + x_2)Y^g(v, x_2)w \rangle$$

and

$$(x_2 + x_0)^{j/k + M} \langle w', Y^g(Y(u, x_0)v, x_2)w \rangle,$$

respectively. Since we have proved that (2.10) is equal to

$$N_2 \sum_{r,s=N_1} a_{rs} x_1^{r/k} x_2^{s/k} (x_1 - x_2)^{-N},$$

we obtain

$$(x_0 + x_2)^{j/k + M} \langle w', Y^g(u, x_0 + x_2)Y^g(v, x_2)w \rangle = \sum_{r,s=N_1} a_{rs} (x_0 + x_2)^{j/k + M} x_0^{s/k} x_2^{r/k} w.$$

By (2.3), we also have

$$(x_2 + x_0)^{j/k + M} \langle w', Y^g(Y(u, x_0)v, x_2)w \rangle = \sum_{r,s=N_1} a_{rs} (x_2 + x_0)^{j/k + M} x_2^{s/k} x_0^{r/k} w.$$

Thus the expansion of (2.3) in the region $|z_2| > |z_1 - z_2| > 0$ is equal to (2.8).

Conversely, assume that the property in the theorem holds. Let

$$f_j(x_1, x_2, x_0) = \sum_{r,s=N_1} a_{rs} \eta^{-j} x_1^{r/k} x_2^{s/k} x_0^{-N}$$

for $j \in \mathbb{Z}/k\mathbb{Z}$. Then this property gives

$$\langle w', Y^g(u, x_1)Y^g(v, x_2)w \rangle = f_0(x_1, x_2, x_1 - x_2),$$

$$\langle w', Y^g(v, x_2)Y^g(u, x_1)w \rangle = f_0(x_1, x_2, -x_2 + x_1),$$

$$\langle w', Y^g(Y(u, z_1 - z_2)v, z_2)w \rangle = f_0(x_2 + x_0, x_2, x_0)$$

for $j \in \mathbb{Z}/k\mathbb{Z}$. Then this property gives

$$\langle w', Y^g(u, x_1)Y^g(v, x_2)w \rangle = f_0(x_1, x_2, x_1 - x_2),$$

$$\langle w', Y^g(v, x_2)Y^g(u, x_1)w \rangle = f_0(x_1, x_2, -x_2 + x_1),$$

$$\langle w', Y^g(Y(u, z_1 - z_2)v, z_2)w \rangle = f_0(x_2 + x_0, x_2, x_0)$$

for $j \in \mathbb{Z}/k\mathbb{Z}$. Then this property gives
for \( u, v \in V, w \in W \) and \( w' \in W' \). From (2.14), (2.16) and the equivariance property, we have

\[
\langle w', Y^g(Y(g^j u, z_1 - z_2)v, z_2)w \rangle = f_j(x_2 + x_0, x_2, x_0) \tag{2.17}
\]

for \( u, v \in V, w \in W \) and \( w' \in W' \).

We have the formal variable identity

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \delta \left( \eta_j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right).
\]

Using this identity, the properties of the delta-function and the explicit form of \( f_0(x_1, x_2, x_0) \), we can prove

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) f_0(x_1, x_2, x_1 - x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) f_0(x_1, x_2, -x_2 + x_1)
\]

\[
= \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \delta \left( \eta_j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) f_j(x_2 + x_0, x_2, x_0). \tag{2.18}
\]

From (2.14), (2.16) and (2.18), we obtain

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle w', Y^g(u, x_1)Y^g(v, x_2)w \rangle
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w', Y^g(v, x_2)Y^g(u, x_1)w \rangle
\]

\[
= \frac{1}{k} \sum_{j \in \mathbb{Z}/k\mathbb{Z}} x_2^{-1} \delta \left( \eta_j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) \langle w', Y^g(Y(g^j u, x_0)v, x_2)w \rangle
\]

for \( u, v \in V, w \in W \) and \( w' \in W' \). Since \( w \in W \) and \( w' \in W' \) are arbitrary, we obtain the twisted Jacobi identity (2.11) for \( u, v \in V \).

We shall call the property in Theorem 2.4 the \textit{duality property}.

\textbf{Remark 2.5.} For simplicity, we give only the definition of twisted module for a vertex operator algebra, not for a vertex operator superalgebra. But it is trivial to generalize it to that of twisted module for a vertex operator superalgebra.
3 Definitions of strongly $\mathbb{C}/\mathbb{Z}$- and $\mathbb{C}$-graded generalized $g$-twisted module

We give a definition of strongly $\mathbb{C}/\mathbb{Z}$-graded generalized $g$-twisted $V$-module for an automorphism $g$ of $V$ of possibly infinite order in this section. In the case that $V$ admits an additional $\mathbb{C}$-grading compatible with $g$, we also give a definition of strongly $\mathbb{C}$-graded generalized $g$-twisted $V$-module. In the case that $g$ is of finite order, $g$ always acts on $V$ semisimply so that we have $V = \bigsqcup_{j \in \mathbb{Z}/k\mathbb{Z}} V^j$. When the order of $g$ is infinite, it does not have to act on $V$ semisimply. In particular, we have to allow not only nonintegral powers of $x$ in the twisted vertex operators, but also integral powers of $\log x$ in these operators. As we have mentioned in the introduction, the left-hand side of the twisted Jacobi identity cannot be generalized to the case that the order of $g$ is infinite. Also we cannot straightforwardly generalize the formal monodromy condition to the general case. On the other hand, the equivariance property and the duality property in Proposition 2.3 and Theorem 2.4, respectively, can be generalized easily. Our definition will use the generalizations of these axioms.

First we need to further generalize our consideration of analytic branches of the map $X$ in the preceding section. For formal variables $x$ and $\log x$, we define their degrees to be $-1$ and $0$. Let $W_1$, $W_2$ and $W_3$ be $\mathbb{C}$-graded vector spaces. Then the grading on $W_3$ and the degrees of $x$ and $\log x$ together give $W_3\{x\} [\log x]$ a $\mathbb{C}$-grading. Let

$$X : W_1 \otimes W_2 \to W_3\{x\} [\log x]$$

$$w_1 \otimes w_2 \mapsto X(w_1, x)w_2 = \sum_{n \in \mathbb{C}} \sum_{k=1}^{K} X_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k$$

is a linear map preserving the gradings. For any $w_1 \in W_1$ and $w_2 \in W_2$, we define $X^p(w_1, z)w_2$ for $p \in \mathbb{Z}$ to be the elements $X(w_1, x)w_2|_{x^n = e^{u_p(z)}, \log x = l_p(z)}$ of $W_3$. When $p = 0$, we shall denote $X^0(w_1, z)w_2$ simply as $X(w_1, z)w_2$. For $z \in \mathbb{C}^\times$, we have the linear maps

$$X^p(\cdot, z) : W_1 \otimes W_2 \to \overline{W_3},$$

$$w_1 \otimes w_2 \mapsto X^p(w_1, z)w_2$$

(3.1)

and we shall denote $X^0(\cdot, z)$ simply as $X(\cdot, z)$.. For any $w'_3 \in W'_3$, we have

$$\langle w'_3, X^p(w_1, z)w_2 \rangle$$
are branches of a multivalued analytic function defined on \( \mathbb{C}^\times \). We can view this multivalued analytic function on \( \mathbb{C}^\times \) as a single-valued analytic function defined on a covering space of \( \mathbb{C}^\times \). For \( p \in \mathbb{Z} \), we have a map

\[
X^p : \mathbb{C}^\times \to \text{Hom}(W_1 \otimes W_2, \overline{W}_3)
\]

\[
z \mapsto X^p(\cdot, z)
\]

We shall call \( X^p \) the \( p \)-th analytic branch of the map \( X \).

We also need a \( \mathbb{C}^\times \)-, or equivalently, \( \mathbb{C}/\mathbb{Z} \)-grading structure on \( V \) given by an automorphism \( g \) of \( V \). Since \( g \) preserve \( V(n) \) for \( n \in \mathbb{Z} \) and \( V(n) \) for \( n \in \mathbb{Z} \) are all finite dimensional, we have

\[
V(n) = \coprod_{a \in \mathbb{C}^\times} V^{[a]}_{(n)}
\]

for \( n \in \mathbb{Z} \), where for \( n \in \mathbb{Z} \) and \( a \in \mathbb{C}^\times \), if \( a \) is an eigenvalue for the operator \( g \) on \( V(n) \), \( V^{[a]}_{(n)} \) is the generalized eigenspace of \( V(n) \) for \( g \) with the eigenvalue \( a \), and if \( a \) is not an eigenvalue for \( g \) on \( V(n) \), \( V^{[a]}_{(n)} = 0 \). Since the multiplicative abelian group \( \mathbb{C}^\times \) is isomorphic to the additive abelian group \( \mathbb{C}/\mathbb{Z} \) through the isomorphism \( a \mapsto \frac{1}{2\pi \sqrt{-1}} \log a + \mathbb{Z} \), we can use \( \mathbb{C}/\mathbb{Z} \) instead of \( \mathbb{C}^\times \) so that

\[
V(n) = \coprod_{a \in \mathbb{C}/\mathbb{Z}} V^{[a]}_{(n)}
\]

and \( V \) is doubly graded by \( \mathbb{C} \) and \( \mathbb{C}/\mathbb{Z} \) as

\[
V = \coprod_{n \in \mathbb{Z}, a \in \mathbb{C}/\mathbb{Z}} V^{[a]}_{(n)}.
\]

Let \( V^{[\alpha]} = \coprod_{n \in \mathbb{Z}} V^{[\alpha]}_{(n)} \) for \( \alpha \in \mathbb{C}/\mathbb{Z} \). Then \( V = \coprod_{a \in \mathbb{C}/\mathbb{Z}} V^{[a]} \) and for \( v \in V^{[\alpha]} \),

\[
qv = e^{2\pi \sqrt{-1}\alpha} v.
\]

It is easy to see that the vertex operator algebra \( V \) together with this \( \mathbb{C}/\mathbb{Z} \)-grading is a strongly \( \mathbb{C}/\mathbb{Z} \)-graded vertex operator algebra in the sense of [HLZ2].

If, in addition to the \( \mathbb{C} \)-grading given by the eigenvalues of \( L(0) \), \( V \) has another \( \mathbb{C} \)-grading \( V = \coprod_{a \in \mathbb{C}} V^{[a]} \) (not just a \( \mathbb{C}/\mathbb{Z} \)-grading as we have given above) which is compatible with \( g \) (that is, for \( v \in V^{[\alpha]} \), \( gv = e^{2\pi \sqrt{-1}\alpha} v \)), then \( V = \coprod_{n,a \in \mathbb{C}} V^{[a]}_{(n)} \) equipped with this \( \mathbb{C} \)-grading has a strongly \( \mathbb{C} \)-graded vertex operator algebra structure.
For a strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded) vertex operator algebra, a natural module category is the category of strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded, respectively) generalized $V$-modules. Moreover, if the $\mathbb{C}/\mathbb{Z}$-grading (or $\mathbb{C}$-grading) is given by (or compatible with) $g$, it is natural to consider strongly $\mathbb{C}/\mathbb{Z}$-graded (or $\mathbb{C}$-graded) generalized $V$-modules with actions of $g$ such that the $\mathbb{C}^\times$-gradings (or $\mathbb{C}$-gradings) are given by (or compatible with) generalized eigenspaces for $g$ in the obvious sense. The category of the generalizations of $g$-twisted modules should be a generalization of the category of such strongly $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-graded generalized $V$-modules. In particular, there should also be an action of $g$ and also a $\mathbb{C}/\mathbb{Z}$-grading (or $\mathbb{C}$-grading) given by (or compatible with) generalized eigenspaces for the action of $g$. Here is our generalization of $g$-twisted $V$-module for finite-order $g$ to the general case:

**Definition 3.1.** Let $(V, Y, 1, \omega)$ be a vertex operator algebra and let $g$ be an automorphism of $V$. A strongly $\mathbb{C}/\mathbb{Z}$-graded generalized $g$-twisted $V$-module is a $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$-graded vector space $W = \bigoplus_{n, \alpha \in \mathbb{C}/\mathbb{Z}} W^{[\alpha]}_{[n]}$ equipped with a linear map

$$Y^g : V \otimes W \to W \{x\} [\log x]$$

$$v \otimes w \mapsto Y^g(v, x)w$$

and an action of $g$ satisfying the following conditions:

1. The grading-restriction condition: For each $n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}$, $\dim W^{[\alpha]}_{[n]} < \infty$ and $W^{[\alpha]}_{[n+l]} = 0$ for all sufficiently negative real number $l$.

2. The equivariance property: For $p \in \mathbb{Z}, z \in \mathbb{C}^\times, v \in V$ and $w \in W$, $Y^{g:p+1}(gv, z)w = Y^{g:p}(v, z)w$, where for $p \in \mathbb{Z}, Y^{g:p}$ is the $p$-th analytic branch of $Y^g$.

3. The identity property: For $w \in W$, $Y^g(1, x)w = w$.

4. The duality property: Let $W' = \bigoplus_{n, \alpha \in \mathbb{C}/\mathbb{Z}} (W^{[\alpha]}_{[n]})^*$ and, for $n \in \mathbb{C}$, $\pi_n : W \to W_{[n]}$ be the projection. For any $u, v \in V, w \in W$ and $w' \in W'$, there exists a multivalued analytic function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^N a_{ijkl} z_1^m z_2^n (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$
for $N \in \mathbb{N}$, $m_1, \ldots, m_N, n_1, \ldots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series

$$
\langle w', Y^{g,p}(u, z_1)Y^{g,p}(v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g,p}(u, z_1)\pi_n Y^{g,p}(v, z_2)w \rangle, \tag{3.2}
$$

$$
\langle w', Y^{g,p}(v, z_2)Y^{g,p}(u, z_1)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g,p}(v, z_2)\pi_n Y^{g,p}(u, z_1)w \rangle, \tag{3.3}
$$

$$
\langle w', Y^{g,p}(Y(u, z_1 - z_2)v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g,p}(\pi_n Y(u, z_1 - z_2)v, z_2)w \rangle \tag{3.4}
$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to the branch

$$
\sum_{i,j,k,l=0}^N a_{ijkl} e^{m_{i}l_p(z_1)} e^{n_{j}l_p(z_2)} l_p(z_1)^{k} l_p(z_2)^{l}(z_1 - z_2)^{-t}
$$

of $f(z_1, z_2)$.

5. The $L^g(0)$- and $g$-grading conditions: Let $L^g(n)$ for $n \in \mathbb{Z}$ be the operator on $W$ given by $Y^g(\omega, x) = \sum_{n \in \mathbb{Z}} L^g(n)^{-n-2}$. Then for $n \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W^{[\alpha]}$, there exists $K, \Lambda \in \mathbb{Z}_+$ such that $(L^g(0) - n)Kw = (g - e^{2\pi i \alpha})^\Lambda w = 0$.

6. The $L(-1)$-derivative condition: For $v \in V$,

$$
\frac{d}{dx} Y^g(v, x) = Y^g(L(-1)v, x).
$$

If $V$ has a strongly $\mathbb{C}$-graded vertex operator algebra structure compatible with $g$, then a strongly $\mathbb{C}$-graded generalized $g$-twisted $V$-module is a $\mathbb{C} \times \mathbb{C}$-graded vector space $W = \prod_{n, \alpha \in \mathbb{C}} W^{[\alpha]}$ equipped with a linear map

$$
Y^g : V \otimes W \rightarrow W \{x\}[\log x]
$$

$$
v \otimes w \mapsto Y^g(v, x)w
$$

and an action of $g$, satisfying the same axioms above except that $\mathbb{C}/\mathbb{Z}$ is replaced by $\mathbb{C}$ and the $g$-grading condition is replaced by the following grading-compatibility condition: For $\alpha, \beta \in \mathbb{C}$, $v \in V^{[\alpha]}$ and $w \in W^{[\beta]}$, $Y^g(v, x)w \in W^{[\alpha + \beta]} \{x\}[\log x]$. 17
Remark 3.2. As in the preceding section, for simplicity, we give only the
definition of such generalized twisted module for a vertex operator algebra,
not for a vertex operator superalgebra. It is also trivial in this general case to
generalize the definition above to that of twisted module for a vertex operator
superalgebra.

4 Exponentiating integrals of negative parts
of vertex operators

In this section we shall exponentiate integrals of negative parts of vertex
operators and prove formulas needed in the next section. We need first
several commutator formulas.

The first is the commutator formula for vertex operators which is the
special case of the Jacobi identity when we take the coefficients of \( x_0^{-1} \) and
expand the formal delta-function on the right-hand side explicitly:

**Proposition 4.1.** Let \((V, Y, 1, \omega)\) be a vertex operator algebra and \((W, Y_W)\)
a weak \(V\)-module. The for \(u, v \in V\), we have the commutator formula

\[
[Y_W(u, x_1), Y_W(v, x_2)] = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \binom{m}{k} x_1^{-m-1} x_2^{-k} Y_W(Y_k(u)v, x_2). \tag{4.1}
\]

Let \(u \in V\). In this section, we do not require that the conformal weight of
\(u\) be 1. Let \(Y_W^{\leq -2}(u, x)\) be the part of \(Y_W(u, x)\) consisting all the monomials
in \(x\) with powers of \(x\) less than or equal to \(-2\), that is,

\[
Y_W^{\leq -2}(u, x) = \sum_{m \in \mathbb{Z}_+} Y_m(u)x^{-m-1}.
\]

For any vector space \(W\) and

\[
f(x) = \sum_{n \in \mathbb{Z} \setminus \{-1\}} a_n x^n \in W[[x]] + x^{-2}W[[x^{-1}]],
\]

let

\[
\int_0^x f(y) = \sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{a_n}{n+1} x^{n+1}.
\]
Then we obtain a linear map (integration from 0 to $x$)

$$\int_0^x : W[[x]] + x^{-2}W[[x^{-1}]] \to xW[[x]] + x^{-1}W[[x^{-1}]].$$

We have:

**Proposition 4.2.** For $u, v \in V$,

$$\left[ \left( \int_0^{x_1} Y^\leq_{-2}(u, y) \right), Y_W(v, x_2) \right] = Y_W \left( \int_0^{-x_1-x_2} Y^\leq_{-2}(u, y)v, x_2 \right) + \log \left( 1 + \frac{x_2}{x_1} \right) Y_W(0)(u)v, x_2),$$

(4.2)

where $\int_0^{-x_1-x_2} Y^\leq_{-2}(u, y)v$ is defined by expanding the series in powers of $-x_1 - x_2$ as a series in positive powers of $x_2$ and in powers of $x_1$.

**Proof.** From (4.1), we obtain

$$[Y^\leq_{-2}(u, x_1), Y_W(v, x_2)] = \sum_{m \in \mathbb{Z}_+} \sum_{k \in \mathbb{N}} \binom{m}{k} x_1^{-m-1} x_2^{-k} Y_W(0)(u)v, x_2),$$

(4.3)

Then

$$\left[ \int_0^{-x_1} Y^\leq_{-2}(u, y), Y_W(v, x_2) \right]$$

$$= \sum_{k, l \in \mathbb{N}, k+l \neq 0} \binom{k+l}{k} \frac{(-1)^{k+l+1}}{k+l} x_1^{-k-l} x_2^l Y_W(0)(u)v, x_2),$$

$$= \sum_{k \in \mathbb{Z}_+} \frac{(-1)^k}{-k} \sum_{l \in \mathbb{N}} \binom{-k}{l} x_1^{-k-l} x_2^l Y_W(0)(u)v, x_2),$$

$$+ \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l+1}}{l} x_1^{-l} x_2^l Y_W(0)(u)v, x_2).$$
\[
\sum_{k \in \mathbb{Z}_+} \frac{(-1)^k}{k} (x_1 + x_2)^{-k} Y_W(Y_k(u)v, x_2) + \log \left(1 + \frac{x_2}{x_1}\right) Y_W(Y_0(u)v, x_2)
\]

\[
= Y_W \left( \int_0^{-x_1 - x_2} Y_{\leq -2}(u, y)v, x_2 \right) + \log \left(1 + \frac{x_2}{x_1}\right) Y_W(Y_0(u)v, x_2).
\]

We also have:

**Proposition 4.3.** For \( u \in V \),

\[
\left[ L(-1), \int_0^{-x} Y_{\leq -2}(u, y) \right] = -\frac{d}{dx} \int_0^x Y_{\leq -2}(u, y) - Y_0(u)x^{-1}. \quad (4.4)
\]

**Proof.** By the \( L(-1) \)-derivative property, we have

\[
[L(-1), Y_W(u, y)] = \frac{d}{dy} Y_W(u, x),
\]

which gives

\[
[L(-1), Y_{\leq -2}^W(u, y)] = \frac{d}{dy} Y_{\leq -2}^W(u, y) - Y_0(u)y^{-2}. \quad (4.5)
\]

Note that the integration \( \int_0^x : W[[x]] + x^{-2}W[[x^{-1}]] \to xW[[x]] + x^{-1}W[[x^{-1}]] \)
defined above is invertible and its inverse is in fact the restriction of \( \frac{d}{dx} \) to
the space \( xW[[x]] + x^{-1}W[[x^{-1}]] \). Integrating both sides of (4.5) with respect
to \( y \) and then substituting \( -x \) for \( y \), we obtain

\[
\left[ L(-1), \int_0^{-x} Y_{\leq -2}^W(u, y) \right] = Y_{\leq -2}^W(u, -x) - Y_0(u)x^{-1}
\]

\[
= -\frac{d}{dx} \int_0^{-x} Y_{\leq -2}^W(u, y) - Y_0(u)x^{-1}.
\]

Let

\[
\tilde{\Delta}_W^{(u)}(x) = \exp \left( \left( \int_0^{-x} Y_{\leq -2}^W(u, y) \right) + Y_0(u) \log x \right).
\]

Then we see that \( \Delta_W^{(u)}(x) \in (\text{End})[[x^{-1}]][\log x] \).
Proposition 4.4. For $u, v \in V$,
\[
\tilde{\Delta}_W^{(u)}(x) Y(v, x_2) = Y(\tilde{\Delta}_W^{(u)}(x + x_2)v, x_2) \tilde{\Delta}_W^{(u)}(x),
\]
(4.6)
\[
[L(-1), \tilde{\Delta}_W^{(u)}(x)] = -\frac{d}{dx} \tilde{\Delta}_W^{(u)}(x).
\]
(4.7)

Proof. By (4.2) and (4.4), we obtain (4.6) and (4.7), respectively. 

5 Generalized twisted $V$-modules associated to weight 1 elements of $V$

In this section, we construct strongly $C$-graded generalized twisted $V$-modules associated to $u \in V(1)$ under the assumption that $V(0) = C1$ and $V(n) = 0$ for $n < 0$ and that $L(1)u = 0$.

For such an element $u$, we have an operator $Y_0(u) = \text{Res}_x Y(u, x)$ on $V$. Since the conformal weight of $Y_0(u)$ is 0, it preserves the finite-dimensional homogeneous subspaces $V(n)$ for $n \in \mathbb{C}$. In particular, for any $n \in \mathbb{C}$, $V(n) = \bigoplus_{\alpha \in \mathbb{C}} V_{\alpha}^{(n)}$ where $V_{\alpha}^{(n)}$ is the generalized eigenspaces for $Y_0(u)$ with eigenvalue $\alpha$ if $\alpha$ is an eigenvalue of $Y_0(u)$ and is 0 for other $\alpha \in \mathbb{C}$. Thus $V$ has an additional $\mathbb{C}$-grading and, equipped with this $\mathbb{C}$-grading, is a strongly $\mathbb{C}$-graded vertex operator algebra.

Given a strongly $\mathbb{C}$-graded generalized $V$-module, we would like to construct a strongly $\mathbb{C}$-graded generalized twisted $V$-module by modifying the vertex operator map for the module using the operators $\tilde{\Delta}_V^{(u)}(x)$ constructed in the preceding section. But in general $\tilde{\Delta}_V^{(u)}(x)v \in V[[x^{-1}]][[\log x]]$ for $v \in V$. On the other hand, in our definition of strongly $C^\times$-graded generalized twisted module, the image of the vertex operator map must be in $V\{x\}[\log x]$. So we cannot use $\tilde{\Delta}_V^{(u)}(x)$ directly. Instead, we first need to construct a series $\Delta_V^{(u)}(x)$ by modifying $\tilde{\Delta}_V^{(u)}(x)$ and discuss under what assumptions, $\Delta_V^{(u)}(x)v$ is in $V\{x\}[\log x]$ for $v \in V$.

Let $(Y_0(u))_{ss}$ be the semisimple part of $Y_0(u)$, that is, for any generalized eigenvector $v \in V$ for $Y_0(u)$ with eigenvalue $\lambda$, $(Y_0(u))_{ss}v = \lambda v$. Then $Y_0(u) = (Y_0(u))_{ss} + (Y_0(u) - (Y_0(u))_{ss})$ and $(Y_0(u) - (Y_0(u))_{ss})N_u = 0$. The commutator formula (4.1) for vertex operators gives
\[
[Y_0(u), Y(v, x_2)] = Y(Y_0(u)v, x_2)
\]
(5.1)
for any $v \in V$.

The commutator formula (5.1) gives

\[
[(Y_0(u))_{ss}, Y(v, x_2)] = Y((Y_0(u))_{ss}v, x_2), \quad (5.2)
\]

\[
[(Y_0(u) - (Y_0(u))_{ss}), Y(v, x_2)] = Y((Y_0(u) - (Y_0(u))_{ss})v, x_2). \quad (5.3)
\]

The statement that (5.2) is a consequence of (5.1) is a special case of a very
general fact in linear algebra. Here we give a direct proof in this special case.

Let $v, w \in V$ be generalized eigenvectors for $Y_0(u)$ with eigenvalues $\lambda_1$ and
$\lambda_2$, respectively. Then there exist $N_1, N_2 \in \mathbb{Z}_+$ such that $(Y_0(u) - \lambda_1)^{N_1}v = (Y_0(u) - \lambda_2)^{N_2}w = 0$. From (5.1), we obtain

\[
(Y_0(u) - \lambda_1 - \lambda_2)Y(v, x_2)w
= Y((Y_0(u) - \lambda_1)v, x_2)w + Y(v, x_2)(Y_0(u) - \lambda_2)w.
\]

Using this formula repeatedly, we obtain

\[
(Y_0(u) - \lambda_1 - \lambda_2)^{N_1+N_2}Y(v, x_2)w
= \sum_{i=0}^{N_1+N_2} \binom{N_1+N_2}{i} Y((Y_0(u) - \lambda_1)^i v, x_2)(Y_0(u) - \lambda_2)^{N_1+N_2-i}w
= 0.
\]

This shows that $Y(v, x_2)w$ is a formal series whose coefficients are generalized
eigenvectors for $Y_0(u)$ with eigenvalue $\lambda_1 + \lambda_2$. Thus

\[
(Y_0(u))_{ss}Y(v, x_2)w = (\lambda_1 + \lambda_2)Y(v, x_2)w
= Y((Y_0(u))_{ss}v, x_2)w + Y(v, x_2)(Y_0(u))_{ss}w,
\]

which is equivalent to

\[
[(Y_0(u))_{ss}, Y(v, x_2)]w = Y((Y_0(u))_{ss}v, x_2)w.
\]

Since $v$ and $w$ are arbitrary generalized eigenvectors for $Y_0(u)$, we obtain (5.2). The formula (5.3) follows immediately from (5.1) and (5.2).

For any generalized eigenvector $v \in V$ for $Y_0(u)$ with eigenvalue $\lambda$, we have $e^{tY_0(u)}v = e^{t(Y_0(u))_{ss}}e^{tY_0(u) - t(Y_0(u))_{ss}}v = e^{t\lambda}e^{tY_0(u) - t(Y_0(u))_{ss}}v$. Since $V$ is spanned by such generalized eigenvector $v \in V$ for $Y_0(u)$, we have a well-defined operator $e^{tY_0(u)}$ on $V$ for every $t \in \mathbb{C}$. 22
Proposition 5.1. For $t \in \mathbb{C}$, the operator $e^{tY_0(u)}$ is an automorphism of $V$.

Proof. Since $Y_0(u)$ preserve the grading of $V$, $e^{tY_0(u)}$ also preserve the grading of $V$. From (5.2) and (5.3), we obtain

$$e^{tY_0(u)}Y(v, x) = Y(e^{tY_0(u)}v, x)e^{tY_0(u)}.$$

The first half of the creation property says that $Y(u, x)1 \in V[[x]]$. In particular, $Y_0(u)1 = 0$. So $e^{tY_0(u)}1 = 1$. Since $u \in V(1)$ and $V(n) = 0$ for $n < 0$, $L(n)u = 0$ for $n \geq 2$. We also have $L(1)u = 0$. So $u$ is a lowest weight vector of weight 1 and $Y(u, x)$ is a primary field. Thus we have

$$[L(-2), Y(u, x)] = x^{-1} \frac{d}{dx} Y(u, x) - z^{-2}Y(u, x).$$

Taking the coefficients of $x^{-1}$ of both sides of this formula, we obtain

$$[L(-2), Y_0(u)] = 0.$$

Thus $Y_0(u)\omega = L(-2)Y_0(u)1 = 0$ which implies $e^{tY_0(u)}\omega = \omega$. \hfill \blacksquare

By the skew-symmetry for $V$, we have

$$Y(u, x)u = e^{xL(-1)}Y(u, -x)u.$$

Taking Res$_x$ of both sides, we have

$$Y_0(u)u = \sum_{m \in \mathbb{N}} \frac{(-1)^{n-1}}{m!} (L(-1))^m Y_m(u)u.$$

Since wt $Y_m(u)u = 1 - m - 1 + 1 = 1 - m < 0$ and $V(n) = 0$ when $n < 0$, $Y_m(u)u = 0$ for $m > 1$. Since $V(0) = \mathbb{C}1$, $Y_1(u)u$ is proportional to 1 since wt $Y_1(u)u = 1 - 1 - 1 = 0$. So $(L(-1))^m Y_m(u)u = 0$ for $m \geq 0$. Thus we obtain $Y_0(u)u = -Y_0(u)u$ which implies $Y_0(u)u = 0$. By the component form of the commutator formula (5.1), we obtain

$$[Y_0(u), Y_n(u)] = Y_n((Y_0(u)u)) = 0$$

for $n \in \mathbb{Z}$. Then we can define

$$\Delta^{(u)}_Y(x) = x^{Y_0(u)}e^{\int_0^x Y^{\leq -2}(u, y)\log x e^{\int_0^x Y^{\leq -2}(u, y)}},$$

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where \((Y_0(u))_{ss}\) is the semisimple part of \(Y_0(u)\) and, on an eigenvector of \((Y_0(u))_{ss}\) (that is, a generalized eigenvector of \(Y_0(u)\)), \(x^{(Y_0(u))_{ss}}\) is defined to be \(x^a\) if the eigenvalue of the eigenvector is \(a\).

**Theorem 5.2.** For any \(v \in V\), there exist \(m_1, \ldots, m_l \in \mathbb{R}\) such that

\[
\Delta^{(u)}_V(x)v \in x^{m_1} V[x^{-1}][\log x] + \cdots + x^{m_l} V[x^{-1}][\log x]
\]

and the series \(\Delta^{(u)}_V(x)\) satisfies

\[
\Delta^{(u)}_V(x) Y(v, x_2) = Y(\Delta^{(u)}_V(x + x_2)v, x_2) \Delta^{(u)}_V(x) \quad (5.4)
\]

and

\[
[L(-1), \Delta^{(u)}_V(x)] = -\frac{d}{dx} \Delta^{(u)}_V(x). \quad (5.5)
\]

**Proof.** For any \(v \in V\),

\[
e^{\int_{0}^{-x} y^{\leq -2(u,y)}v} = \sum_{k \in \mathbb{N}} \frac{1}{k!} \left( \int_{0}^{-x} Y^{\leq -2(u,y)} \right)^k v
\]

\[
= \sum_{k \in \mathbb{N}} \frac{1}{k!} \left( \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{-n} Y_n(u)x^{-n} \right)^k v
\]

\[
= \sum_{k \in \mathbb{N}} \frac{1}{k!} \sum_{n_1, \ldots, n_k \in \mathbb{Z}_+, n_1 + \cdots + n_k = l} \frac{(-1)^l}{(-n_1) \cdots (-n_k)} Y_{n_1}(u) \cdots Y_{n_k}(u) v x^{-l}.
\]

(5.6)

Since

\[
\text{wt } Y_{n_1}(u) \cdots Y_{n_k}(u) = (1 - n_1 - 1) + \cdots + (1 - n_k - 1)
\]

\[
= -n_1 - \cdots - n_k
\]

\[
= -l,
\]

we see that \(Y_{n_1}(u) \cdots Y_{n_k}(u) w = 0\) when \(l\) is sufficiently large. Thus the right-hand side of (5.6) is an element of \(V[x^{-1}]\) and our first conclusion

\[
\Delta^{(u)}_V(x)v \in x^{m_1} V[x^{-1}][\log x] + \cdots + x^{m_l} V[x^{-1}][\log x]
\]

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follows immediately.

From (4.2), we obtain

\[ e^{\int_0^x Y \leq -2(u,y)v} Y(v, x_2)e^{-\int_0^x Y \leq -2(u,y)v} = Y \left( e^{\int_0^x Y \leq -2(u,y)v} v, x_2 \right) \]

\[ = Y \left( e^{\int_0^x Y \leq -2(u,y)v} v, x_2 \right) \cdot \frac{d}{dx} e^{\int_0^x Y \leq -2(u,y)v} = \frac{d}{dx} e^{\int_0^x Y \leq -2(u,y)v} \frac{d}{dx} \left( e^{\int_0^x Y \leq -2(u,y)v} v, x_2 \right). \]  (5.7)

On the other hand, by (5.2) and (5.3), we have

\[ x^0(v)Y(v, x_2) = x^0(v)Y^0(v, x_2) = x^0(v)Y^0(v, x_2) \leq -2(u, v) \]

\[ = x^0(v)Y^0(v, x_2) \leq -2(u, v) \leq -2(u, v) \leq -2(u, v). \]  (5.8)

Using (5.7) and (5.8), we obtain (5.4).

Finally, notice that both \([L(-1), \cdot]\) and \(-\frac{d}{dx}\) are derivatives on the space \(\mathbb{C}[\int_0^x Y \leq -2(u, v)]\) of power series in \(\int_0^x Y \leq -2(u, v)\). So from (4.4), we obtain

\[ [L(-1), \int_0^x Y \leq -2(u, v)] + \frac{d}{dx} \int_0^x Y \leq -2(u, v) \int_0^x Y \leq -2(u, v) \]

\[ = \left[ L(-1), \int_0^x Y \leq -2(u, v) \right] e^{\int_0^x Y \leq -2(u, v)} \]

\[ + \left( \frac{d}{dx} \int_0^x Y \leq -2(u, v) \right) e^{\int_0^x Y \leq -2(u, v)} \]

\[ = -Y^0(v) x^{-1} e^{\int_0^x Y \leq -2(u, v)}. \]  (5.9)

The equality (5.9) is equivalent to (5.5).

Let \(W = \bigoplus_{n, \alpha \in \mathbb{C}} W^{[\alpha]}_{(n)}\) be a strongly \(\mathbb{C}\)-graded generalized \(V\)-module (see [HLZ1] and [HLZ2]) with the action \(e^{2\pi \sqrt{-1} Y(v)_{0}(u)}\) of the automorphism \(e^{2\pi \sqrt{-1} Y(v)_{0}(u)}\) of \(V\) such that the \(\mathbb{C}\)-grading is given by the generalized eigenspaces of \((Y(v)_{0}(u))\) and thus is compatible with the generalized eigenspaces of the action \(e^{2\pi \sqrt{-1} Y(v)_{0}(u)}\) of \(e^{2\pi \sqrt{-1} Y(v)_{0}(u)}\). For example, when \(W\) is a \(V\)-module, \(W\) has a natural \(\mathbb{C}\)-grading given by the generalized
eigenspaces for \((Y_W)_0(u)\) and together with this grading, \(W\) is such a generalized \(V\)-module. We define a linear map

\[
Y^{(u)}_W : V \otimes W \to W\{x\}\{\log x\}
\]

by

\[
Y^{(u)}_W(v, x)w = Y_W(\Delta^{(u)}_V(x)v, x).
\]

First, we have:

**Lemma 5.3.** The vertex operator \(Y^{(u)}_W(\omega, x)\) is in \(W[[x, \frac{1}{x}]]\) and if we write

\[
Y^{(u)}_W(\omega, x) = \sum_{n \in \mathbb{Z}} L^{(u)}_W(n) x^{-n-2},
\]

then

\[
L^{(u)}_W(0) = L_W(0) + (Y_W)_0(u) - \frac{1}{2} \mu,
\]

where \(\mu \in \mathbb{C}\) is given by

\[
(Y_W)_1(u) = \mu \mathbf{1}
\]

(note that since \((Y_W)_1(u) \in V_0 = \mathbb{C} \mathbf{1}\), it must be proportional to \(\mathbf{1}\)). In particular, for \(n \in \mathbb{C}\) and \(\alpha\) an eigenvalue of \((Y_W)_0(u)\), \(W^{[\alpha]}_{(n-\alpha+\frac{1}{2}\mu)}\) is the generalized eigenspace for both \(L^{(u)}_W(0)\) and \((Y_W)_0(u)\) with the eigenvalues \(n\) and \(\alpha\), respectively.

**Proof.** By \((4.1)\), we have

\[
[Y(\omega, x_1), Y(u, x_2)] = \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(\omega, x_0 u, x_2).
\]

Taking the coefficients of \(x_1^0\) in both sides and noticing that from our assumption, \(L(n)u = 0\) for \(n > 0\), we obtain

\[
[L(-2), Y(u, x_2)] = -x_2^{-1} Y(L(-1)u, x_2) + x_2^{-2} Y(L(0)u, x_2)
\]

\[
= -x_2^{-1} \frac{d}{dx_2} Y(u, x_2) + x_2^{-2} Y(u, x_2).
\]

Taking the coefficients of \(x_2^{-m-1}\) in both sides, we obtain

\[
[L(-2), Y_m(u)] = m Y_{m-2}(u)
\]

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for $m \in \mathbb{Z}$. Then we have

$$Y_m(u)\omega = Y_m(u)L(-2)1$$

$$= L(-2)Y_m(u)1 - mY_{m-2}(u)1$$

for $m \in \mathbb{Z}$. Since $Y_m(u)1 = 0$ for $m \geq 0$ and $Y_{-1}(u)1 = u$, we obtain

$$Y_m(u)\omega = 0$$

for $m \in \mathbb{N}$ but $m \neq 1$ and $Y_1(u)\omega = -u$.

In particular, $Y_0(u)\omega = (Y_0(u))_{ss}\omega$. We also know that $Y_m(u)u = 0$ for $m > 1$ and $Y_1(u)u = \mu 1$. So

$$e^{r_0 Y_{\leq -2}(u, y)}\omega = \omega + ux^{-1} - \frac{1}{2}\mu 1 x^{-2}$$

and

$$\Delta Y_{(u)}(x)\omega = x^{(Y_0(u))_{ss}}(\omega + ux^{-1} - \frac{1}{2}\mu 1 x^{-2})$$

$$= \omega + ux^{-1} - \frac{1}{2}\mu 1 x^{-2}.$$ 

Thus

$$Y_W^{(u)}(\omega, x) = Y_W(\omega, x) + Y(u, x)x^{-1} - \frac{1}{2}\mu x^{-2} \in W[[x, x^{-1}]]$$

and $L_W^{(u)}(0) = L_W(0) + (Y_W)_0(u) - \frac{1}{2}\mu$.

The second conclusion follows immediately.

We have the following consequence:

**Corollary 5.4.** The space $W$ is also doubly graded by the generalized eigenspaces for $L_W(0)$ and $(Y_W)_0(u)$, that is, $W$ has a $\mathbb{C} \times \mathbb{C}$-grading $W = \bigsqcup_{n, \alpha \in \mathbb{C}} W_{[n]}^{[\alpha]}$ where for $n, \alpha \in \mathbb{C}$,

$$W_{[n]}^{[\alpha]} = W_{(n-\alpha + \frac{1}{2}\mu)},$$

that is, $w \in W_{[n]}^{[\alpha]}$ for $n, \alpha \in \mathbb{C}$ if and only if $w$ is a generalized eigenvector for $L_W(0)$ with the eigenvalue $n$ and a generalized eigenvector for $(Y_W)_0(u)$ with the eigenvalue $\alpha$. 

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Proof. By Lemma 5.3, every element of $W$ is a finite sum of vectors which are generalized eigenvectors for both $I_{W}(0)$ and $(Y_{W})_{0}(u)$. 

The following theorem is the main result of the present paper:

**Theorem 5.5.** The pair $(W, Y_{W}^{(u)})$, where $W$ is equipped with the $\mathbb{C} \times \mathbb{C}$-grading given in Corollary 5.4, is a strongly $\mathbb{C}$-graded generalized $e^{2\pi \sqrt{-1}Y_{0}(u)}$-twisted $V$-module.

**Proof.** We first prove that $W$ satisfies the grading-restriction condition. By Lemma 5.3, $L_{W}(0) = I_{W}^{(u)}(0) - (Y_{W})_{0}(u) + \frac{1}{2} \mu$. Since the generalized $V$-module $W$ satisfies the grading-restriction condition, $\dim W_{(n)} < \infty$ for $n, \alpha \in \mathbb{C}$ and for any fixed $n, \alpha \in \mathbb{C}$, $W_{(n+l)}^{[\alpha]} = 0$ when $l$ is a sufficiently negative integer. So for $n, \alpha \in \mathbb{C}$, $W_{(n+l)}^{[\alpha]} = W_{(n-l-\alpha+\frac{1}{2} \mu)}^{[\alpha]}$ are finite dimensional and for any fixed $n, \alpha \in \mathbb{C}$, $W_{(n+l)}^{[\alpha]} = W_{(n+l-\alpha+\frac{1}{2} \mu)}^{[\alpha]} = 0$ when $l$ is a sufficiently negative integer.

From the first part of the creation property, $Y(u, x)1 \in V[[x]]$ for vertex operator algebra, we have $Y_{n}(u)1 = 0$ for $n \geq 0$. In particular, $(Y_{0}(u))_{ss}1 = (Y_{0}(u) - (Y_{0}(u))_{ss})1 = 0$. We obtain

$$\Delta_{V}^{(u)}(x)1 = x^{(Y_{0}(u))_{ss}} e^{-Y_{0}(u)(Y_{0}(u))_{ss}} \log x \ e^{-Y_{0}(u)(Y_{0}(u))_{ss}} \log x \ e^{-Y_{0}(u)(Y_{0}(u))_{ss}} 1$$

$$= 1.$$

For $w \in W$,

$$Y_{W}^{(u)}(1, x)w = Y_{W}(\Delta_{V}^{(u)}(x)1, x)w = Y_{W}(1, x)w = w,$$

proving the identity property.

Let $v \in V$ be a generalized eigenvector for $Y_{0}(u)$ with eigenvalue $\lambda$. Then for $v \in V$ and $z \in \mathbb{C}^{\times}$,

$$\Delta_{V}^{(u)}(x) e^{2\pi i Y_{0}(u)} v \bigg|_{x^{n} = e^{u l_{p}(z)}, \log x = l_{p}(z)} = x^{\lambda} e^{2\pi i \langle Y_{0}(u) - (Y_{0}(u))_{ss} \rangle \log x} e^{2\pi i \langle Y_{0}(u) - (Y_{0}(u))_{ss} \rangle \log x} v \bigg|_{x^{n} = e^{u l_{p}(z)}, \log x = l_{p}(z)}$$

$$= x^{\lambda} e^{2\pi i \langle Y_{0}(u) - (Y_{0}(u))_{ss} \rangle \log x} v \bigg|_{x^{n} = e^{u l_{p+1}(z)}, \log x = l_{p+1}(z)}$$

$$= \Delta_{V}^{(u)}(x) v \bigg|_{x^{n} = e^{u l_{p+1}(z)}, \log x = l_{p+1}(z)}.$$
Thus

\begin{align*}
(Y^{(u)}_W e^{2\pi i Y_0(u)} p (e^{2\pi i Y_0(u)} v, z) w
& = Y_W (\Delta^{(u)}_V (x) e^{2\pi i Y_0(u)} v, x) w |_{x^n = e^{n \mu (z)}, \log x = l_p (z)} \\
& = Y_W (\Delta^{(u)}_V (x) v, x) w |_{x^n = e^{n \mu (z)}, \log x = l_{p+1} (z)} \\
& = (Y^{(u)}_W e^{2\pi i Y_0(u)} p+1 (v, z) w
\end{align*}

for \( v \in V \) and \( w \in W \), that is, the equivariance property holds.

By Corollary 5.4, the \( L^{(u)} (0) \)-grading condition and the grading-compatibility condition are satisfied.

By the \( L^{(-1)} \)-derivative property for the vertex operator map \( Y_W \) and \[5.5\], we have

\[
\frac{d}{dx} Y^{(u)}_W (v, x) = \frac{d}{dx} Y_W (\Delta^{(u)}_V (x)v, x)
\]

\[
= Y_W (L(-1) \Delta^{(u)}_V (x)v, x) + Y_W \left( \left( \frac{d}{dx} \Delta^{(u)}_V (x)v \right), x \right)
\]

\[
= Y_W (L(-1) \Delta^{(u)}_V (x)v, x) + Y_W (-[L(-1), \Delta^{(u)}_V (x)]v, x)
\]

\[
= Y_W (\Delta^{(u)}_V (x) L(-1)v, x)
\]

\[
= Y^{(u)}_W (L(-1)v, x),
\]

for \( v \in V \), proving the \( L^{(-1)} \)-derivative property.

Finally we prove the duality property. By Corollary 5.4, \( W_{[n]}^{[\alpha]} = W_{[n-\alpha+\frac{1}{2} \mu]}^{[\alpha]} \).

So the graded dual of \( W \) with respect to the grading \( W = \bigcup_{n, \alpha \in \mathbb{C}} W_{[n]}^{[\alpha]} \) and the graded dual with respect the new \( \mathbb{C} \times \mathbb{C} \)-grading \( W = \bigcup_{n, \alpha \in \mathbb{C}} W_{[\alpha]}^{[n]} \) in Corollary 5.4 are equal as vector spaces, though their gradings are different. We shall use \( W' \) to denote the underlying vector space of these two graded duals. It will be clear from the context which grading we will be using.

For \( v_1, v_2 \in V, w \in W \) and \( w' \in W' \), we know that there exist \( m_1, \ldots, m_r, n_1, \ldots, n_s \in \mathbb{R} \) such that

\[
\Delta^{(u)}_V (x_1) e^{2\pi i Y_0(u)} v_1 \in x_1^{m_1} V [x_1^{-1}] [\log x_1] + \cdots + x_1^{m_r} V [x_1^{-1}] [\log x_1]
\]

and

\[
\Delta^{(u)}_V (x_2) e^{2\pi i Y_0(u)} v_2 \in x_2^{n_1} V [x_2^{-1}] [\log x_2] + \cdots + x_2^{n_s} V [x_2^{-1}] [\log x_2].
\]

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Thus, using the rationality, commutativity and associativity properties for the $V$-module $W$ and the fact that in the region $|z_2| > |z_1 - z_2| > 0$

$$\Delta_V^{(u)}(x_2 + x_0)v_1 \bigg|_{x_0^n = (z_1 - z_2)^n, \ x_2^n = e^{-n \bar{\rho}(x_2)}, \ \log x_2 = \bar{\rho}(z_2)}$$

$$= \Delta_V^{(u)}(x_1)v_1 \bigg|_{x_1^n = e^{n \bar{\rho}(z_1)}, \ \log x_1 = \bar{\rho}(z_1)},$$

we see that there exists $a_{ijkl} \in \mathbb{C}$ for $i, j, k, l = 0, \ldots, N$, $\alpha_i, \beta_j \in \mathbb{C}$ for $i, j = 0, \ldots, N$ and $t \in \mathbb{N}$ such that the series

$$\langle w', (Y_W^{(u)})^{e^{2\pi i \bar{\rho}(u)}}p(v_1, z_1)(Y_W^{(u)})^{e^{2\pi i \bar{\rho}(u)}}p(v_2, z_2)w \rangle$$

$$= \langle w', Y_W(\Delta_V^{(u)}(x_1)v_1, x_1) \cdot Y_W(\Delta_V^{(u)}(x_2)v_2, x_2)w \rangle \bigg|_{x_1^n = e^{n \bar{\rho}(z_1)}, \ \log x_1 = \bar{\rho}(z_1), \ x_2^n = e^{n \bar{\rho}(z_2)}, \ \log x_2 = \bar{\rho}(z_2)},$$

$$\langle w', (Y_W^{(u)})^{e^{2\pi i \bar{\rho}(u)}}p(v_2, z_2)(Y_W^{(u)})^{e^{2\pi i \bar{\rho}(u)}}p(v_1, z_1)w \rangle$$

$$= \langle w', Y_W(\Delta_V^{(u)}(x_2)v_2, x_2) \cdot Y_W(\Delta_V^{(u)}(x_1)v_1, x_1)w \rangle \bigg|_{x_1^n = e^{n \bar{\rho}(z_1)}, \ \log x_1 = \bar{\rho}(z_1), \ x_2^n = e^{n \bar{\rho}(z_2)}, \ \log x_2 = \bar{\rho}(z_2)},$$

$$\langle w', Y_W(Y(\Delta_V^{(u)}(x_2 + x_0)v_1, x_0) \cdot \Delta_V^{(u)}(x_2)v_2, x_2)w \rangle \bigg|_{x_0^n = (z_1 - z_2)^n, \ x_2^n = e^{n \bar{\rho}(z_2)}, \ \log x_2 = \bar{\rho}(z_2)}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to the branch

$$\sum_{i,j,k,l=0}^{N} a_{ijkl} e^{m_i \bar{\rho}(z_1)} e^{n_j \bar{\rho}(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of the multivalued analytic function

$$\sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^m z_2^n (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}.$$
\[ \Delta_{V}^{(u)}(x_2, v_2) \bigg|_{x_2^n = (z_1 - z_2)^n, x_2^p = e^{{\alpha}(z_2)}, \log x_2 = \alpha(z_2)} = \langle w', Y_{W}(\Delta_{V}^{(u)}(x_2))Y(v_1, x_0)v_2, x_2)w \rangle \bigg|_{x_2^n = (z_1 - z_2)^n, x_2^p = e^{{\alpha}(z_2)}, \log x_2 = \alpha(z_2)} = \langle w', (Y_{W}^{(u)})^{2\pi i\gamma}(w)Y(v_1, z_1 - z_2)v_2, z_2)w \rangle. \]

Thus the duality property is proved. ■

**Remark 5.6.** In the case that Res\()_{x}Y(u, x)\) acts on \(V\) semisimply and has eigenvalues belonging to \(1/\mathbb{Z}\), the theorem above reduces to the construction by Li in [Li].

We have the following immediate consequence:

**Corollary 5.7.** The correspondence given by \((W, Y_{W}) \rightarrow (W, Y_{W}^{(u)})\) is a functor from the category of strongly \(\mathbb{C}\)-graded generalized \(V\)-modules to the category of strongly \(\mathbb{C}\)-graded generalized \(e^{2\pi \sqrt{-1} Y_{0}(u)}\)-twisted \(V\)-modules. ■

We now apply the theorem above to give some explicit examples. Let \(p\) and \(q\) be a pair of coprime positive integers, \(L\) the one-dimensional positive-definite even lattice generated by \(\gamma\) with the bilinear form \(\langle \cdot, \cdot \rangle\) given by \(\langle \gamma, \gamma \rangle = 2pq\), and \(\tilde{L}\) its dual lattice. Let \(V_{L}\) be the vertex algebra associated to \(L\) and \(V_{\tilde{L}}\) the generalized vertex algebra associated to \(\tilde{L}\) (see [DL]). The element

\[ \omega = \frac{1}{pq} \gamma(-1)^2 1 + \frac{p - q}{2pq} \gamma(-2) 1 \in V_{L} \subset V_{\tilde{L}} \]

is a conformal element. If instead of the usual conformal element \(\frac{1}{pq} \gamma(-1)^2 1\), we take \(\omega\) to be the conformal element for \(V_{L}\) and \(V_{\tilde{L}}\), we obtain a vertex operator algebra (still denoted \(V_{L}\)) and a generalized vertex operator algebra (still denoted \(V_{\tilde{L}}\), respectively. Note that in the grading given by the conformal element \(\omega\), the weights of \(e^{\gamma/q}\) and \(e^{-\gamma/p}\) are 1. Consider the \(V_{L}\)-modules \(V_{L-\frac{2}{p}}\) and \(V_{L+\frac{2}{q}}\) which are both graded by \(\mathbb{Z}\). Then we have vertex-operator-algebraic extensions \(V_{0} = V_{L} \oplus V_{L-\frac{2}{p}}\), \(V_{0} = V_{L} \oplus V_{L+\frac{2}{q}}\) and \(V(p, q) = V_{L} \oplus V_{L-\frac{2}{p}} \oplus V_{L+\frac{2}{q}}\) of \(V_{L}\) for which the vertex operators are given by

\[
\begin{align*}
Y_{V_{0}}(u_1 + v_1, x)(u_2 + v_2) &= Y_{V_{L}}(u_1 + v_1, x)u_2 + Y_{V_{L}}(u_1, x)v_2, \\
Y_{V_{0}}^\circ(u_1 + w_1, x)(u_2 + w_2) &= Y_{V_{L}}(u_1 + w_1, x)u_2 + Y_{V_{L}}(u_1, x)w_2,
\end{align*}
\]

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\[ Y_{V(p,q)}(u_1 + v_1 + w_1, x)(u_2 + v_2 + w_2) = Y_{V_L}(u_1 + v_1 + w_1, x)u_2 + Y_{V_L}(u_1, x)(v_2 + w_2) \]

for \( u_1, u_2 \in V_L \) and \( v_1, v_2 \in V_{L-z_p} \) and \( w_1, w_2 \in V_{L+z_q} \). Note that \( V_0 \) and \( V_0^o \) are vertex operator subalgebras of \( V(p,q) \) and that \( e^{-\gamma/p} \in V_0 \) and \( e^{\gamma/q} \in V_0^o \). Also note that \( (V_0)_{(0)} = (V_0^o)_{(0)} = (V(p,q))_{(0)} = C \mathbf{1} \) and \( (V_0)_{(n)} = (V_0^o)_{(n)} = (V(p,q))_{(n)} = 0 \) for \( n < 0 \).

Let \( Q = \text{Res}_x Y_{V(p,q)}(e^{\gamma/q}, x) \) and \( \tilde{Q} = \text{Res}_x Y_{V(p,q)}(e^{-\gamma/p}, x) \). These operators are called screening operators. Then \( e^{2\pi\sqrt{-1}Q} \) and \( e^{2\pi\sqrt{-1}\tilde{Q}} \) are automorphisms of \( V_0 \) and \( V_0^o \), respectively, and are both automorphisms of \( V(p,q) \). In fact, the triplet vertex operator algebra \( W(p,q) \) (see [FGST2] and [AM3]) is a vertex operator subalgebra of the fixed-point subalgebra of \( V(p,q) \) under the group generated by \( e^{2\pi\sqrt{-1}Q} \) and \( e^{2\pi\sqrt{-1}\tilde{Q}} \).

Let \( W = \bigoplus_{a,\alpha \in \mathbb{C}} W^{[a]}_{(\alpha)} \) be a strongly \( \mathbb{C} \)-graded generalized \( V_0 \)-, \( V_0^o \)- or \( V(p,q) \)-module with an action \( e^{2\pi\sqrt{-1}(Y_W)_{(0)}(e^{\gamma/q})} \) of \( e^{2\pi\sqrt{-1}Q} \) or \( e^{2\pi\sqrt{-1}\tilde{Q}} \) of \( e^{2\pi\sqrt{-1}Q} \) or \( e^{2\pi\sqrt{-1}\tilde{Q}} \) or either of them, respectively, such that the \( \mathbb{C} \)-grading is given by the generalized eigenspaces of \( (Y_W)_{(0)}(e^{\gamma/q}) \) or \( (Y_W)_{(0)}(e^{-\gamma/p}) \). For example, we can take \( W \) to be \( V_0 \), \( V_0^o \) or \( V(p,q) \) themselves. In [AM3] (Theorem 9.1), Adamović and Milas in [AM3] proved that \( Y_W^{(e^{-\gamma/p})} \) and \( Y_W^{(e^{\gamma/q})} \) are intertwining operators of suitable types. Applying Theorem 5.5 we obtain immediately:

**Theorem 5.8.** For a strongly \( \mathbb{C} \)-graded generalized \( V_0 \)-module (or \( V_0^o \)- or \( V(p,q) \)-module) \( (W,Y_W) \), the pair \( (W,Y_W^{(e^{\gamma/q})}) \) (or the pair \( (W,Y_W^{(e^{-\gamma/p})}) \) or the pairs \( (W,Y_W^{(e^{\gamma/q})}) \) and \( (W,Y_W^{(e^{-\gamma/p})}) \)) is a strongly \( \mathbb{C} \)-graded generalized \( e^{2\pi\sqrt{-1}Q} \)-twisted \( V_0 \)-modules (or is a strongly \( \mathbb{C} \)-graded generalized \( e^{2\pi\sqrt{-1}Q} \)-twisted \( V_0^o \)-module or are strongly \( \mathbb{C} \)-graded generalized \( e^{2\pi\sqrt{-1}Q} \)- and \( e^{2\pi\sqrt{-1}\tilde{Q}} \)-twisted \( V(p,q) \)-modules, respectively.)

**References**

[A] T. Abe, A \( \mathbb{Z}_2 \)-orbifold model of the symplectic fermionic vertex operator superalgebra, *Math. Z.* 255 (2007), 755–792.

[AM1] D. Adamović and A. Milas, Logarithmic intertwining operators and \( \mathcal{W}(2,2p-1) \)-algebras, *J. Math. Phys.* 48, 073503 (2007).
[AM2] D. Adamović and A. Milas, On the triplet vertex algebra $W(p)$, 
Adv. in Math. 217 (2008), 2664-2699.

[AM3] D. Adamović and A. Milas, Lattice construction of logarithmic modules for certain vertex operator algebras, to appear; 
arXiv:0902.3417

[Ba1] P. Bantay, Algebraic aspects of orbifold models, Int. J. Mod. Phys. A9 (1994), 1443–1456.

[Ba2] P. Bantay, Characters and modular properties of permutation orbifolds, Phys. Lett. B419 (1998), 175–178.

[Ba3] P. Bantay, Permutation orbifolds and their applications, in: Vertex Operator Algebras in Mathematics and Physics, Proc. workshop, Fields Institute for Research in Mathematical Sciences, 2000, ed. by S. Berman, Y. Billig, Y.-Z. Huang and J. Lepowsky, Fields Institute Communications, Vol. 39, Amer. Math. Soc., 2003, 13–23.

[BDM] K. Barron, C. Dong and G. Mason, Twisted sectors for tensor products vertex operator algebras associated to permutation groups, Commun. Math. Phys., 227 (2002), 349–384.

[BHL] K. Barron, Y.-Z. Huang and J. Lepowsky, An equivalence of two constructions of permutation-twisted modules for lattice vertex operator algebras, Jour. Pure Appl. Alg. 210 (2007), 797–826.

[Bo] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986), 3068–3071.

[BHS] L. Borisov, M. Halpern and C. Schweigert, Systematic approach to cyclic orbifolds, Internat. J. Modern Phy. A 13 (1998), no. 1, 125–168.

[CF] N. Carqueville and M. Flohr, Nonmeromorphic operator product expansion and $C_2$-cofiniteness for a family of $W$-algebras, J.Phys. A39 (2006), 951–966.

[dBHO] J. de Boer, M. Halpern and N. Obers, The operator algebra and twisted KZ equations of WZW orbifolds, J. High Energy Phys. 10 (2001), no. 11.
| Reference | Authors | Title |
|-----------|---------|-------|
| [DVVV]    | R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde | The operator algebra of orbifold models, *Comm. Math. Phys.* **123** (1989), 485–526. |
| [DFMS]    | L. Dixon, D. Friedan, E. Martinec and S. Shenker | The conformal field theory of orbifolds, *Nucl. Phys.* **B282** (1987), 13–73. |
| [DGH]     | L. Dixon, P. Ginsparg and J. Harvey | Beauty and the beast: Superconformal conformal symmetry in a Monster module, *Comm. Math. Phys.* **119** (1989), 221-241. |
| [DHVW1]   | L. Dixon, J. Harvey, C. Vafa and E. Witten | Strings on orbifolds, *Nucl. Phys.* **B261** (1985), 678–686. |
| [DHVW2]   | L. Dixon, J. Harvey, C. Vafa and E. Witten | Strings on orbifolds, II, *Nucl. Phys.* **B274** (1986), 285–314. |
| [DGM]     | L. Dolan, P. Goddard and P. Montague | Conformal field theory of twisted vertex operators, *Nucl. Phys.* **B338** (1990), 529–601. |
| [D]       | C. Dong | Twisted modules for vertex algebras associated with even lattice, *J. of Algebra* **165** (1994), 91–112. |
| [DL]      | C. Dong and J. Lepowsky | The algebraic structure of relative twisted vertex operators, *J. Pure Appl. Algebra* **110** (1996), 259–295. |
| [DonLM1]  | C. Dong, H. Li and G. Mason | Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571-600. |
| [DonLM2]  | C. Dong, H. Li and G. Mason | Modular invariance of trace functions in orbifold theory and generalized moonshine, *Commun. Math. Phys.*, **214** (2000), 1–56. |
| [DoyLM1]  | B. Doyon, J. Lepowsky and A. Milas | Twisted modules for vertex operator algebras and Bernoulli polynomials, *Int. Math. Res. Not.* **44** (2003), 2391–2408. |
| [DoyLM2]  | B. Doyon, J. Lepowsky and A. Milas | Twisted vertex operators and Bernoulli polynomials, *Commun. in Contemporary Math.* **8** (2006), 247–307. |

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[FGST1] B. L. Feigin, A. M. Gaǐnutdinov, A. M. Semikhatov, and I. Yu Tipunin, I, The Kazhdan-Lusztig correspondence for the representation category of the triplet $W$-algebra in logarithmic conformal field theories (Russian), *Teoret. Mat. Fiz.* **148** (2006), no. 3, 398–427.

[FGST2] B. L. Feigin, A. M. Gaǐnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Logarithmic extensions of minimal models: characters and modular transformations, *Nucl. Phys. B* **757** (2006), 303–343.

[FGST3] B. L. Feigin, A. M. Gaǐnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center, *Comm. Math. Phys.* **265** (2006), 47–93.

[Fl1] M. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, *Int. J. Mod. Phys.* **A11** (1996), 4147–4172.

[Fl2] M. Flohr, On fusion rules in logarithmic conformal field theories, *Int. J. Mod. Phys.* **A12** (1996), 1943–1958.

[FG] M. Flohr and M. R. Gaberdiel, Logarithmic torus amplitudes, *J. Phys. A* **39** (2006), 1955–1968.

[FK] M. Flohr and H. Knuth, On Verlinde-Like formulas in $c_{p,1}$ logarithmic conformal field theories, to appear; [arXiv:0705.0545](http://arxiv.org/abs/0705.0545)

[FGK] M. Flohr, C. Grabow and M. Koehn, Fermionic Expressions for the characters of $c(p, 1)$ logarithmic conformal field theories, *Nucl. Phys. B* **768** (2007), 263–276.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **104**, 1993.

[FLM1] I. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function $J$ as character, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3256–3260.
[FLM2] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in: *Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego*, ed. by S.-T. Yau, World Scientific, Singapore, 1987, 150–188.

[FLM3] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.*, Vol. 134, Academic Press, 1988.

[Fu] J. Fuchs, On nonsemisimple fusion rules and tensor categories, in: *Lie algebras, vertex operator algebras and their applications, Proceedings of a conference in honor of James Lepowsky and Robert Wilson, 2005*, ed. Y.-Z. Huang and K. Misra, Contemporary Mathematics, Vol. 442, Amer. Math. Soc., Providence, 2007.

[FHST] J. Fuchs, S. Hwang, A.M. Semikhatov and I. Yu. Tipunin, Non-semisimple Fusion Algebras and the Verlinde Formula, *Comm. Math. Phys.* 247 (2004), no. 3, 713–742.

[FKS] J. Fuchs, A. Klemm and M. Schmidt, Orbifolds by cyclic permutations in Gepner type superstrings and in the corresponding Calabi-Yau manifolds, *Ann. Phys.* 214 (1992), 221–257.

[GK1] M. R. Gaberdiel and H. G. Kausch, Indecomposable fusion products, *Nucl. Phys.* B477 (1996), 298–318.

[GK2] M. R. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, *Phys. Lett.* B386 (1996), 131–137.

[GR1] M. R. Gaberdiel and I. Runkel, The logarithmic triplet theory with boundary, *J. Phys.* A39 (2006), 14745-14780.

[GR2] M. R. Gaberdiel and I. Runkel, From boundary to bulk in logarithmic CFT, *J. Phys.* A41 (2008), 075402.

[GT] A. M. Gaǐnutdinov and I. Yu. Tipunin, Radford, Drinfeld, and Cardy boundary states in $(1,p)$ logarithmic conformal field models, *J. Phys.* A42 (2009), 315207.

[GHHO] O. Ganor, M. Halpern, C. Helfgott and N. Obers, The outerautomorphic WZW orbifolds on $\mathfrak{so}(2n)$, including five triality orbifolds on $\mathfrak{so}(8)$, *J. High Energy Phys.* 12 (2002), no. 19.
[HH] M. Halpern and C. Helfgott, The general twisted open WZW string, *Internat. J. Modern Phys.* A20 (2005), 923–992.

[HO] M. Halpern and N. Obers, Two large examples in orbifold theory: abelian orbifolds and the charge conjugation orbifold on $\mathfrak{su}(n)$, *Internat. J. Modern Phys.* A17 (2002), 3897–3961.

[HV] S. Hamidi and C. Vafa, Interactions on orbifolds, *Nucl. Phys* B279 (1987), 465–513.

[H] J. Harvey, Twisting the heterotic string, in: *Unified String Theories, Proc. 1985 Inst. for Theoretical Physics Workshop*, Ed. by M. Green and D. Gross, World Scientific, Singapore, 1086, 704–718.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, A logarithmic generalization of tensor product theory for modules for a vertex operator algebra, *Internat. J. Math.* 17 (2006), 975–1012.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, to appear; [arXiv:0710.2687](http://arxiv.org/abs/0710.2687).

[Ka1] H. G. Kausch, Extended conformal algebras generated by multiplet of primary fields, *Phys. Lett.* 259 B (1991), 448–455.

[Ka2] H. G. Kausch, Symplectic fermions, *Nucl. Phys.* B 583 (2000), 513–541.

[KS] A. Klemm and M.G. Schmidt, Orbifolds by cyclic permutations of tensor product conformal field theories, *Phys. Lett.* B245 (1990), 53–58.

[Le1] J. Lepowsky, Calculus of twisted vertex operators, *Proc. Nat. Acad. Sci. USA* 82 (1985), 8295–8299.

[Le2] J. Lepowsky, Perspectives on vertex operators and the Monster, in: Proc. 1987 Symposium on the Mathematical Heritage of Hermann Weyl, Duke Univ., *Proc. Symp. Pure. Math., American Math. Soc.* 48 (1988), 181–197.
[Li] H. Li, Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, in: Moonshine, the Monster, and related topics Mount Holyoke, 1994, ed. C. Dong and G. Mason, Contemporary Math., Vol. 193, Amer. Math. Soc., Providence, 1996, 203–236.

[M] G. Moore, Atkin-Lehner symmetry, Nucl. Phys. B293 (1987), 139–188.

[NT] K. Nagatomo and A. Tsuchiya, The Triplet Vertex operator algebra $W(p)$ and the restricted quantum group at root of unity, to appear; arXiv:0902.4607.

[NSV] K. S. Narain, M. H. Sarmadi and C. Vafa, Asymmetric orbifolds, Nucl. Phys. B288 (1987), 551–577.

[PRR1] P. A. Pearce, J. Rasmussen and P. Ruelle, Integrable boundary conditions and $W$-extended fusion in the logarithmic minimal models $LM(1, p)$, J. Phys. A41 (2008) 295201.

[PRR2] P. A. Pearce, J. Rasmussen and P. Ruelle, Grothendieck ring and Verlinde formula for the $W$-extended logarithmic minimal model $WL\!M(1, p)$, to appear; arXiv:0907.0134.

[R] J. Rasmussen, Fusion matrices, generalized Verlinde formulas, and partition functions in $WL\!M(1, p)$, to appear; arXiv:0908.2014.

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