A note on non-vanishing and applications

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Introduction.
Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n$ defined over the field of complex numbers. By $L$ let us denote the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $X$, by $K_X$ let us denote the canonical divisor of $X$ (a linear equivalence class of the sheaf $\Omega^n_X$ of holomorphic $n$-forms). A (classical) version of adjunction theory concerns studying of an adjoint linear system $|K_X + rL|$, for a suitably chosen positive integer $r$. In particular, a typical problem is to decide whether the adjoint divisor $K_X + rL$ is nef or for which value of $r$ it becomes ample or even very ample. If $K_X + rL$ is nef one may want to find out whether the linear system $|K_X + rL|$ has base points, or for which positive integer $m$ its multiple $|m(K_X + rL)|$ becomes base point free.

Moreover, in the above situation one may release the assumptions concerning $L$ and $X$, and ask all the above questions if $L$ is merely ample and $X$ is possibly singular. Problems concerning adjoint divisors has drawn a lot of attention of algebraic geometers, starting from the classical works of Castelnuovo and Enriques, [CE], who considered adjoint linear systems on surfaces. Among the references on these problems we would like to point out the ones which have inspired us mostly: [BS], [F3], [KMM] and finally the recent paper [K].

The most interesting case concerns situation when the adjoint divisor $K_X + rL$ is nef but not ample. Then, although it can not be said much about the system $|K_X + rL|$ itself, a Kawamata—Shokurov Contraction Theorem asserts that some of its multiple,
$|m(K_X + rL)|$, is base point free for $m \gg 0$ and defines an adjoint contraction morphism $\varphi : X \to Z$ onto a normal projective variety $Z$, with $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$. Understanding of this map seems to be very important for any classification theory of higher dimensional manifolds.

In the present paper we study the situation when $L$ is merely ample and the adjoint contraction morphism has fibers of “small” dimension. This last hypothesis allows us to apply an inductive method which is typical of this theory, which was called “Apollonius method” by Fujita in [F3]. In the present paper we call it horizontal slicing argument (sometime it is called simply slicing but here we need to distinguish it from vertical slicing we also use); it can be briefly summarized as follows.

Consider a general divisor $X'$ from the linear system $|L|$ (a hyperplane section of $X$ if $L$ is very ample) and assume that it is a “good” variety of dimension $n-1$ (i.e. has the same singularities as $X$). By adjunction, $K_{X'} = (K_X + L)|_{X'}$ and by Kodaira-Kawamata-Viehweg Vanishing Theorem, if $r > 1$, the linear system $|m(K_{X'} + (r-1)L)|$ is just the restriction of $|m(K_X + rL)|$, so that the adjoint contraction morphism of $X'$ can be related to the one of $X$. Moreover, fibers of the adjoint morphism of $X'$ will be usually of smaller dimension and an inductive argument can be applied.

The horizontal slicing argument requires therefore the existence of a “good” divisor $X'$ in the linear system $|L|$ (a rung in the language of [F3]). The system, however, for an ample (but not very ample) $L$ may a priori be even empty. To overcome this first difficulty we introduce a local set-up in which the base of the contraction morphism will be affine. We will benefit from this situation also because we will be able to choose effective divisors which are rationally trivial. Then the next point is to ensure that the divisor $X'$ does not contain the whole fiber in question. In other words, we want to ensure that base point locus of $|L|$ ($L$ may be changed by adding a divisor trivial on fibers of $\varphi$) does not contain the fiber. This is what may be called “relative non-vanishing”. (Now we can explain why we use the word “horizontal”: we are used to think about the map $\varphi : X \to Z$ as going vertically, then every divisor from an ample linear system cuts every “vertical” fiber of $\varphi$ of dimension $\geq 1$, so it lies “horizontally”.) To prove it we apply a method of J. Kollár, as in [K]; it is an improved version of the one used by Y. Kawamata in proving the Base Point Free Theorem.

Moreover if the dimension of fibers of $\varphi$ is $\leq r$ we can descend with our induction to the case of adjoint morphism $\varphi$ with 1-dimensional fibers. In such a case a structural theorem for adjoint morphisms of smooth varieties was obtained by Ando: his theorem (2.3) from [A1] asserts, for example, that if $r$ and the dimension of a fiber of $\varphi$ is 1 and the map $\varphi$ is birational then, locally, the map $\varphi$ is a blow-down of a smooth divisor to a smooth codimension-2 subvariety of $Z$. The “relative non-vanishing” turns out to be sufficient to prove the extension of the Ando’s result in the smooth case. Namely we prove a structural theorem for contractions whose fiber is of dimension $r$, see theorem 4.1.

Finally, as it turns out, the Kawamata—Kollár method together with the horizontal slicing allows to extend the non-vanishing to “relative spannedness”. Namely we prove:

**Theorem.** Let $X$ be a normal variety such that $K_X$ is $\mathbb{Q}$-Cartier and $X$ has at worst log terminal singularities; let $L$ be an ample line bundle over $X$. Let $\varphi : X \to Z$ be an adjoint
contraction supported by \( K_X + rL \). Let \( F \) be a fiber of \( \varphi \). Assume moreover that

\[
\begin{align*}
\text{either} & \quad \dim F < r + 1 \quad \text{if} \quad \dim Z < \dim X \\
\text{or} & \quad \dim F \leq r + 1 \quad \text{if} \quad \varphi \text{ is birational}
\end{align*}
\]

Then the evaluation morphism \( \varphi^* \varphi_* L \to L \) is surjective at every point of \( F \).

As a consequence of this one can apply the horizontal slicing (or Apollonius) argument, in the above hypothesis on the dimension of the fibers, on varieties with log terminal singularities.

The paper is organized as follows: In the next section we recall some definitions concerning singularities as well as the definition of the adjoint contraction morphism. Then, in section 2, we introduce the “affine set-up” and we describe the adjoint contraction locally. Section 3 is about non-vanishing and its proof. In Section 4 we prove the extension of Ando’s structure theorem for contractions of smooth varieties. In Section 5 we prove the relative spannedness.

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1. Singularities and adjoint morphism.

In the present section we recall some pertinent definitions and results which will be used in the sequel. Our language is compatible with this of [Ut], or [K], or also [KMM].

Let \( X \) be a projective normal variety of dimension \( n \) defined over the field of complex numbers. Assume that \( D = \sum d_i D_i \) is an effective \( \mathbb{Q} \)-divisor (\( D_i \) are prime Weil divisors) such that \( K_X + D \) is \( \mathbb{Q} \)-Cartier. We say that the pair \((X, D)\) is log canonical (respectively, purely log terminal) if for some (and therefore any) resolution of singularities \( f : Y \to X \) of the pair \((X, D)\) we can write

\[
K_Y = f^*(K_X + D) + \sum e_j E_j
\]

where \( E_j \)'s are irreducible divisors with simple normal crossing and \( e_j > -1 \) (resp. \( e_j \geq -1 \)) (see e.g. [Ut, Ch. 1] for details).

We say that \( X \) has log terminal singularities if the pair \((X, 0)\) is purely log terminal, where \( 0 \) is the “zero” divisor; in particular we mean that \( X \) is considered with no boundary.

Remark. Log terminal singularities of a normal variety \( X \) are rational singularities, therefore they are in particular Cohen-Macaulay, see [KMM, 0.2.17] and [Ke].

Let us recall that a \( \mathbb{Q} \)-Cartier divisor in \( X \) is called nef if it has non-negative intersection with any complete curve in \( X \). Assume that \( K_X \) is not nef (that is, it has negative intersection with a complete curve on \( X \)). Let \( L \) be an ample line bundle (Cartier divisor) on \( X \). If \( X \) has at worst log terminal singularities and for a positive rational number \( r \),
$K_X + rL$ is $\mathbb{Q}$-Cartier and nef (but possibly not ample) then, by the Kawamata—Shokurov Base Point Free Theorem, it is semiample and hence the ring

$$R(X, K_X + rL) := \bigoplus_{m \geq 0} H^0(X, m(K_X + rL))$$

is finitely generated algebra over $\mathbb{C}$. Thus there exists an adjoint model

$$Z := \text{Proj}(R(X, K_X + rL))$$

and a projective surjective morphism $\varphi : X \to Z$ which is given by sections of a high multiple of $K_X + rL$ and such that $\varphi_* O_X = O_Z$ (a contraction with connected fibers). We say that $K_X + rL$ is a good supporting divisor for the morphism $\varphi$; in fact, this divisor is a pull-back of an ample divisor from $Z$. If $\dim Z = \dim X$ then the map $\varphi$ is birational, otherwise we call it of fibre type.

Let us discuss a local (affine) structure of the map $\varphi$: the projective variety $Z$ has a natural affine covering by sets $D_{+}(h) \simeq \text{Spec}(R(X, K_X + rL)(h))$, see [H1, II-2], where $R(X, K_X + rL)(h)$ denotes the subring of elements of degree 0 in the localization of $R(X, K_X + rL)$ with respect to the ideal generated by a homogeneous form $h$. Let $Z_h$ denote such an affine variety, by $X_h$ let us denote its pull-back via $\varphi$ (via base change). Let $\varphi_h$, respectively $L_h$, denote again the restriction of $\varphi$, respectively $L$, to $X_h$. $X_h$ is then projective over $Z_h$. $L_h$ remains clearly $\varphi_h$-ample. $X_h$ is a variety (open subset of $X$), the sections of $m(K_X + rL)$ associated to multiples of $h$ do not vanish on $X_h$ and therefore $m(K_{X_h} + rL_{h})$ is a unit in $\text{Pic}X_h$. Since $m(K_{X_h} + rL_{h})$ is generated for $m$ sufficiently large (not necessary divisible) we can shrink $X_h$ so that actually $K_{X_h} + rL_{h}$ can be assumed to be isomorphic to $O_{X_h}$. The morphism $\varphi_h : X_h \to Z_h$ is given by the evaluation map $H^0(X_h, K_{X_h} + rL) \to (K_{X_h} + rL_h)$. The map $\varphi_h$ is then as follows

$$X_h \ni x \mapsto \text{ideal of sections of } K_{X_h} + rL_{h} \text{ vanishing at } x.$$

Note that this affine definition of $\varphi$ is clearly compatible with restricting $\varphi$ to affine subsets.

Frequently, we will replace $Z_h$ by an affine open subset. Also let us note that, the vanishing of higher direct images $R^i\varphi_* F$ of a sheaf $F$ can be understood as vanishing of $H^i(X_h, F)$. We will also frequently use the existence of effective (Cartier) divisors on $X_h$ (subvarieties of codimension 1 given by functions).
2. Affine set-up.

We want to understand adjoint contraction morphisms (adjoint maps) locally. For this purpose we will "shrink" $X$ so that the target of the map will be affine. This will be our affine set-up. In the present section we review an affine version of adjunction theory and check basic properties of the construction. Most of this seems to be standard but we found no reference for it.

Let, for a while, $X$ denote a scheme (over $\mathbb{C}$). Set \( \Gamma(X) := \text{Spec}(H^0(X, \mathcal{O}_X)) \). The evaluation of global functions yields a morphism \( \varphi : \Gamma(X) \to X \) defined as follows

\[ X_h \ni x \mapsto \text{ideal of global functions vanishing at } x. \]

It is clear that definitions of \( \Gamma(X) \) and \( \varphi \) are functorial. Also, it is not hard to check the following

Claim (2.1). In the above situation the following holds:

- (2.1.1) if $X$ is affine then $\varphi$ is an isomorphism,
- (2.1.2) $(\varphi)_* \mathcal{O}_X = \mathcal{O}_{\Gamma(X)}$,
- (2.1.3) if $X$ is separable over $\mathbb{C}$ then $\varphi$ is separable as well [H1, II.4.6],
- (2.1.4) if $X$ is normal then also $\Gamma(X)$ is normal.

(2.2) Now we are ready to set-up our assumptions:

- (2.2.1) Let $X$ be a normal variety (over $\mathbb{C}$) with at worst log terminal singularities and assume that $K_X$ is $\mathbb{Q}$-Cartier. Assume that $L$ is a line bundle (Cartier divisor) on $X$ and $K_X + rL$ is Cartier for some rational number $r$.
- (2.2.2) Assume that $K_X + rL$ is a unit in $\text{Pic}X$. This is equivalent to say that $K_X + rL$ has a nowhere vanishing section.
- (2.2.3) Set $\varphi := \varphi : X \to \Gamma(X).$ Assume that $Z$ is of finite type over $\mathbb{C}$ and $L$ to be $\varphi$-ample (thus $\varphi$ is projective) and (otherwise stated differently) $r \geq 0$, $r > 0$ if $\dim Z < \dim X$.

Remark (2.3). In the above situation $Z$ is a normal affine variety and the fibers of $\varphi$ are connected. Note also that in this situation $Z := \text{Proj}(R(X, K_X + rL))$ is isomorphic to the affine scheme $\text{Spec}(H^0(X, K_X + rL)) = \Gamma(X)$. Also, the output of the localization procedure described in the preceding section is exactly the same.

If $\varphi : X \to Z$ satisfies all assumptions named in (2.2) then we will call it a local adjoint contraction morphism supported by $K_X + rL$. We merely use this name to distinguish the affine situation we want to deal with.

Lemma 2.4. (Vanishing theorem) Let $\varphi : X \to Z$ be a local contraction morphism supported by $K_X + rL$. Assume that $t$ is an integer and $t > -r$ if $\dim Z < \dim X$ or $t \geq -r$ if $\varphi$ is birational. Then

\[ H^i(X, tL) = 0 \text{ for } i > 0. \]

Proof. Note that $-K_X = rL$ and $L$ is $\varphi$-ample. To prove the vanishing use then the theorem 1.2.5 and the remark 1.2.6 in [KMM], and in the birational case also the theorem 1.2.7 in [KMM].

We will frequently use the following “slicing” arguments.

...
Lemma 2.5. (Vertical slicing) Let $\varphi : X \to Y$ be a local adjoint contraction morphism supported by $K_X + rL$, $X$ is a normal projective variety with at worst log terminal singularities. Assume that $X'' \subset X$ is a non-trivial divisor defined by a global function $h \in H^0(X, K_X + rL) = H^0(X, \mathcal{O}_X)$. Then for a general choice of $h$, singularities of $X''$ are not worse than these of $X$, and any section of $L$ on $X''$ extends to $X$.

**Proof.** The statement on singularities of $X''$ is just a version of Bertini theorem; for the Bertini theorem, as well as for the concept of “general” we refer to [J], pages 66-67. The extension property is given by the vanishing 2.4. Namely, the cokernel of the restriction map $H^0(X, L) \to H^0(X'', L|_{X''})$ is contained in $H^1(X, L \otimes \mathcal{O}(-X'')) = H^1(X, L)$ which vanishes by 2.4.

Lemma 2.6. (Horizontal slicing) Assume the situation from the previous lemma.

(2.6.1) Let $X'$ be a general divisor from the linear system $|L|$. Then the singularities of $X'$ outside of the base point locus of $|L|$ are not worse than those of $X$ and any section of $L$ on $X'$ extends to $X$.

(2.6.2) If $\varphi' := \varphi|_{X'}$, then $K_{X'} + (r - 1)L'$ ($L'$ denoting $L|_{X'}$) is $\varphi'$-trivial and $L'$ is $\varphi'$-ample

(2.6.3) Let $Z' := \Gamma(X')$. If either $r > 1$ for $\varphi$ of fiber type or $r \geq 1$ for $\varphi$ birational then the induced map $Z' \to Z$ is a closed immersion. Therefore the map $\varphi$ restricted to $X'$ has connected fibers.

**Proof.** (2.6.1) is similar to (2.5), (2.6.2) is clear. As for (2.6.3), note that the map $Z' \to Z$ is defined by a homomorphism of rings $H^0(X, K_X + rL) \to H^0(X', K_{X'} + (r - 1)L)$ which is just the restriction morphism with cokernel contained in $H^1(X, K_X + (r - 1)L)$; this vanishes because of (2.4). Then connectedness follows from [H1, III, 11.3].

3. Nonvanishing.

In the present section we prove the following

**Theorem 3.1.** Let $\varphi : X \to Z$ be a local adjoint contraction supported by $K_X + rL$ (see assumptions (2.2)). Let $F$ be a fiber of $\varphi$. Assume moreover that

\[(3.1.1) \quad \text{either } \quad \dim F < r + 1 \quad \text{if } \dim Z < \dim X \quad \text{or } \quad \dim F \leq r + 1 \quad \text{if } \varphi \text{ is birational} \]

Then the base point locus of $L$, $Bs|L| := \text{supp}(\text{coker}(\varphi^* \varphi_* L \to L))$, does not contain any component of $F$. Equivalently, there exists a section of $L$ which does not vanish on any component of $F$.

**Remark** (3.1.1). If $\dim Z = 0$ then the theorem is easy: one writes down Hilbert polynomial $\chi(t) := \chi(X, tL)$ and finds out that it has zeroes at $-\dim F + 1, \ldots, -1$ so that, since $\chi(t) \neq 0$, by Kodaira vanishing $L$ has a non-zero section. Essentially, our point is to extend this argument to our situation.

Actually in the case $\dim Z = 0$ the above method gives more: a theorem of Kobayashi-Ochiai, assuming just normal singularities on $X$, says in fact that, if $r = n + 1$ then $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$, while if $r = n$ then $(\mathbb{Q}, \mathcal{O}_\mathbb{Q}(1))$, where $\mathbb{Q}$ is an hyperquadric in $\mathbb{P}^{n+1}$.  


Moreover if \( r > (n - 1) \), besides the two above pairs, the only pair \((X, L)\) with \(X\) being \(Q\)-Gorenstein is the pair \((\mathbb{P}^2, \mathcal{O}(2))\). For these results see for instance [B-S] or [F4].

**Remark** (3.1.2). The dimension of the fiber \(F\) is bounded from below: \(\dim F \geq r - 1\) and \(\dim F \geq \lfloor r \rfloor\) if \(\varphi\) is birational (where \(\lfloor r \rfloor\) is the integral part of \(r\)); we obtain these bounds in the course of our proof but they are also in [F1] or [F4].

The proof of the theorem is based on the technique developed by Kawamata in proving the base point free theorem; more precisely we proceed as in the recent paper of Kollár; see [K], section 2.1: *Modified base point freeness method*. Our proof differs from Kollár’s only by a special choice of divisors involved:

**Claim** (3.2). We can choose a divisor \(B\) on \(X\) which is a pull back of a divisor on \(Z\) such that \((X, B)\) is log canonical outside \(F\) and \((X, B)\) is not log canonical at a generic point of any component \(F' \subset F\).

The proof of claim is the same as (2.2.1) from [K], or (18.22) from [Ut], we sketch it here: Take some general functions on \(Z\) which vanish on \(\varphi(F)\). Then the pull backs of their divisors contain \(F\). Let \(B\) be their sum. By the general choice of these sections the first part of the claim follows. To see the second part take an irreducible component \(F' \subset F\): blowing up \(F'\) we obtain an exceptional divisor whose discrepancy with the respect to \(K_X + B\) is \(< -1\).

Let \(f : Y \to X\) be a log resolution of \(B\) i.e. \(Y\) is smooth and all relevant divisors are smooth and cross normally. We then write

\[
\begin{align*}
K_Y &= f^*K_X + \sum e_i E_i \quad \text{where, by assumption } e_i > -1 \\
f^*B &= \sum b_i E_i \quad \text{where } b_i \geq 0 \text{ and } \\
f^*(\epsilon L) &= A + \sum p_i E_i \quad \text{if } \dim Z < \dim X, \text{ with } \epsilon > 0, \text{ or } \\
f^*\mathcal{O}_X &= A + \sum p_i E_i \quad \text{if } \varphi \text{ is birational}
\end{align*}
\]

where in the last two formulas \(A\) is assumed to be \(\varphi \circ f\)-ample \(Q\)-divisor and \(0 \leq p_i \ll 1\).

Let \(F' \subset F\) be an irreducible component, define (c.f. [K], (2.1.2))

\[
c := \min \left\{ \frac{e_i + 1 - p_i}{b_i} : F' \subset f(\mathcal{E}_i), b_i > 0 \right\}
\]

By changing the coefficients \(p_i\) a little we can assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by \(E_0\).

**Claim** (3.5) (see [K] Claim 2.1.3). By the choice of \(c\) we have

(i) \(0 < c < 1\),
(ii) \(f(E_0) = F'\),
(iii) if \(cb_i - e_i + p_i < 0\) then \(E_i\) is \(f\)-exceptional,
(iv) if \(cb_i - e_i + p_i \geq 1\) and \(i \neq 0\) then \(F'\) is not contained in \(f(\mathcal{E}_i)\).

Let us note that the \(Q\)-divisor

\[
K_Y + A + \sum (cb_i - e_i + p_i) E_i + f^*(tL)
\]
is numerically equivalent to $f^*((t-r)L)$ in the birational, and to $f^*((t-r+\epsilon)L)$ in the fiber case. We can write

$$
\sum (cb_i - e_i + p_i)E_i = E_0 + H'' - H' + Fr
$$

where $E_0$, $H'$, $H''$ are effective divisor without common irreducible components; $Fr$ is the fractional divisor with rational coefficients between 0 and 1, 0 along $E_0$, defined by $Fr = \sum \{cb_i - e_i + p_i\}E_i$ ($\{a\}$ is the fractional part of the real number $a$). Moreover $H'$ is $f$-exceptional and $F''$ is not contained in $f(H'')$.

Assume that $t$ is an integer such that $t \geq -r$ if $\varphi$ is birational, or such that $t \geq -r + \epsilon$ if $\varphi$ is of fiber type; by construction

$$f^*(tL) - E_0 + H' - H'' - K_Y - Fr$$

is $\varphi \circ f$-ample. This divisor restricted to $E_0$ is numerically equivalent to $H''$ for brevity. By the first vanishing the restriction maps

$$(3.6) \quad H^i(Y, f^*(tL) - E_0 + H' - H'') = 0 \text{ and } H^i(E_0, (f^*(tL) + H' - H'')|_{E_0}) = 0 \text{ for } i > 0.
$$

Let us denote $N(t) := f^*(tL) + H' - H''$ for brevity. By the first vanishing the restriction maps

$$(3.7) \quad H^0(Y, N(t)) \to H^0(E_0, N(t)|_{E_0})$$

are surjective for $t \geq -r$ in the birational and for $t > -r$ in the fiber case.

Assume that then there exists a non-zero section $s$ of $N(t)|_{E_0}$. By surjectivity this extends to a non-zero section of $N(t)$ on $Y$; since $E_0$ is not contained the support of $H''$ we get a section $s \in H^0(Y, f^*(tL) + H')$ which is not identically zero along $E_0$. Moreover $H^0(Y, f^*(tL) + H') = H^0(X, tL)$ since $H'$ is $f$-exceptional, thus $s$ descends to a section of $tL$ which does not vanish along $F' = f(E_0)$ (c.f. [K], (2.1.6)).

From this we have immediately that $H^0(E_0, N(t)) = 0$ for $-r \geq t < 0$ (respectively $-r < t < 0$ in the fiber case) and if $dimF' > 0$.

Let $\chi(t) := \chi(E_0, N(t))$ be the Euler-Poincare characteristic; $\chi$ is a polynomial of degree $\leq d = dimf(E_0) = dimF'$. By (3.6) and the above we have that $\chi(t) = 0$ for $-1 \geq t \geq -r$ (resp. $-1 \geq t > -r$), if $\varphi$ is birational (resp. if $\varphi$ is of fiber type.). Therefore $dimF' \geq |r|$ (respectively $dimF' \geq (r - 1)$). Since $\chi(0) \geq 0$ and $\chi(t) > 0$ for $t \gg 0$ thus since $dimF' \leq r + 1$ (resp. $< r + 1$) it follows that $H^0(E_0, N(1)) \neq 0$.

Therefore by the surjectivity of the map in (3.7) (and the subsequent paragraph) we obtain a section of $L$ in a neighborhood of $F$ which does not vanish along $F'$; this concludes the proof of the theorem.
4. The structure of an adjoint contraction morphism; smooth case.

In the present section we prove a theorem about the local structure of an adjoint contraction morphism.

Let \( \varphi : X \to Z \) be a local adjoint contraction morphism supported by \( K_X + rL, \ r > 0 \). Let \( F \) be a fiber of \( \varphi \). We assume that \( X \) is smooth in a neighbourhood of \( F \). Our result is subsumed in the following

**Theorem 4.1.** If \( \dim F = r \) then \( Z \) is smooth at \( \varphi(F) \) and

(i) if \( \dim Z = \dim X - r \) then \( \varphi \) is a quadric bundle in the neighbourhood of \( F \),

(ii) if \( \dim Z = \dim X - r + 1 \) then \( r \leq \dim X/2 \) and \( F = \mathbb{P}^r \),

(iii) if \( \varphi \) is birational then, in a neighbourhood of \( F \), it is a blow-down of a smooth divisor \( E \subset X \) to a smooth subvariety of \( Z \).

**Remark.** We have just seen, (3.1.2), that every non-trivial fiber of \( \varphi \) is of dimension at least \( r - 1 \) and if \( \dim F = r - 1 \) then \( \dim Z < \dim X \). Fujita proved [F1, (2.12)] that in the latter case \( F \) is irreducible and \( F = \mathbb{P}^{r-1} \), and \( \varphi \) is a projective bundle in a neighbourhood of \( F \). Therefore, by semicontinuity of dimensions of fibers, it follows that if \( \dim F = r \) then either

- \( \dim X - r + 1 \geq \dim Z \geq \dim X - r \), or
- \( \varphi \) is birational and all non-trivial fibers of \( \varphi \) in a neighbourhood of \( F \) are \( r \)-dimensional.

**Remark.** The theorem is a generalization of a result of Ando who proved a version of it for \( r = 1 \), see [A1], Theorem 2.3. We will rely on his result, the only exception is the case (i) which was done by Fujita’s slicing technique in [ABW]. Cases (ii) and (iii) will be reduced by horizontal slicing to the situation of \( r = 1 \) when they coincide and are described in the Ando’s theorem. For \( n = 3 \) the theorem is known; for instance the fact that for \( n = 3 \) and \( r = 2 \) the case (ii) does not occur was proved by Sommese in [S], theorem (3.3), in the case \( L \) is spanned for a slightly more general singularities.

From now on assume that the situation is as in ii) or iii) of (4.1) Although Ando formulated his results for elementary contractions of projective varieties his results hold in our situation, too.

**Proposition 4.2.** (Ando, [A1], 2.3) Assume that \( \varphi : X \to Z \) is a local adjoint contraction supported by \( K_X + L \). Moreover assume that a fiber \( F \) of \( \varphi \) is of dimension 1 and \( \varphi \) is birational. Then locally, in a neighbourhood of \( F \), \( \varphi \) is a blow-down-morphism of a smooth divisor on \( X \) to a smooth codimension 2 subvariety in smooth \( Z \).

**Proof.** The argument is the same as in the course of the proof of theorem 2.3 from [A1], we will only have to ensure that that the fiber \( F \) is irreducible. Take a component \( C_i \) of the fiber \( F \). By 2.2 and 1.5 from [ibid], it is isomorphic to \( \mathbb{P}^1 \) and \( K_X.C_i = -1 = -L.C_i \). By deformation theory, see e.g. the proof of 1.1 from [W3], small deformations of \( C_i \) sweep out a divisor in a neighbourhood of \( F \), call the divisor \( E_i \). Intersection of any two such divisors \( E_i \) and \( E_j \) is of codimension 2 in \( X \) and it has non-empty intersection with \( F \). An intersection \( S \) of general \( n - 2 \) divisors from a good supporting linear system \( m(K_X + L) \), \( m \gg 0 \) will be a smooth surface (\( X \) can be assumed to be smooth); the intersection
$E_i \cap E_j \cap S$ is non-empty. On the other hand, both $E_i \cap S$ and $E_j \cap S$ are contracted to points by a map supported by $L + K_S$, so that $\varphi|_S$ is a contraction of $(-1)$-curves. Therefore $E_i = E_j$ and the fiber $F$ is irreducible.

To conclude the proof we argue like Ando: by [A1], 1.5, we know that the normal bundle to $F$ is $\mathcal{O} \oplus \mathcal{O} \ldots \mathcal{O} \oplus \mathcal{O}(-1)$, so $F$ deforms in a smooth family and finally by Nakano criterium (see [N]) its contraction is a blow-down.

We will need a result about the normalization of components of the fiber $F$.

**Proposition 4.3.** (Fujita [F1], [F2, (2.2)] and Ye, Zhang [YZ, lemma4]) Assume that $X$, $L$, $r$ and $\varphi$ are as at the beginning of the section. Let $F'$ be an irreducible component of a fiber $F$ of $\varphi$.

(4.3.1) If $\text{dim} F' = r - 1$ then the normalization of $F'$ is isomorphic to $\mathbb{P}^{r-1}$ and the pull-back of $L$ is $\mathcal{O}(1)$.

(4.3.2) If $\text{dim} F' = r > \text{dim}(\text{general fiber of } \varphi)$ then the normalization of $F'$ is isomorphic to $\mathbb{P}^r$ and the pull-back of $L$ is $\mathcal{O}(1)$.

Now we can proceed with the proof of (2.1). We can assume $X$ to be smooth, this is preserved by horizontal slicing:

**Lemma 4.4.** Assume that a section $X'$ of $L$ does not contain any component of the fiber $F$. Then $X'$ is smooth on $F$.

**Proof.** Pick up a point $x$ on $F \cap X'$. There exists a rational curve $C$ contained in $F$ such that $C.L = 1$, $C$ contains $x$ and it is not contained in $X'$. This follows from the fact that any two points of $F$ can be joined by such a curve $C$; this in turn follows from the proposition 4.3. Therefore $C.X' = 1$ and thus $X'$ is smooth at $x$.

**Lemma 4.5.** The line bundle $L$ is base point free in a neighbourhood of $F$, that is $B_s|L| := \text{supp}(\text{coker}(\varphi^* \varphi_* L \to L))$ does not meet $F$.

**Proof.** By 4.2 this is true for $r = 1$. For $r > 1$ this follows by induction from a horizontal slicing argument, see 2.6.

**Lemma 4.6.** The fiber $F$ is irreducible.

**Proof.** By 4.2 this is true for $r = 1$. Now assume $r = 2$ and $F$ contains a component of dimension 1. Then by base point freeness we can choose $X' \in |L|$ meeting the 1-dimensional component of $F$ not at the intersection of components. This contradicts 2.6.3 as the map $\varphi|_{X'}$ has a disconnected fiber. Now general case follows by horizontal slicing argument: i.e. by 4.4 and 2.6.

From 4.3, 4.5 and 4.6 it follows now

**Lemma 4.7.** $F$ is isomorphic to $\mathbb{P}^r$ and $L|_F$ is isomorphic to $\mathcal{O}(1)$.

Now we discuss the normal bundle of $F$
Lemma 4.8. The normal bundle of $F$ in $X$ is uniform of the splitting type $(-1, 0, \ldots, 0)$.

Proof. As above we proceed by induction with respect to $r$. For $r = 1$ this follows from Ando’s result. Now assume that the lemma is true for $r - 1$. Consider a divisor $X'$ from $|L|$ let $F' = F \cap X'$. We then have the following exact sequences of vector bundles on any line $C$ in $F'$

$$
0 \rightarrow NF'/X'_C = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \rightarrow NF'/X_C \rightarrow NF/X_C = \mathcal{O}(1) \rightarrow 0
$$

and therefore $NF/X_C = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}$. But, since $L$ is spanned on $F$, the hyperplane $F'$ can be chosen arbitrarily, so that it contains any line $C$.

Lemma 4.9. (Elencwajg, see [W2], Prop. 1.9) The bundle $NF/X$ is either decomposable into a sum of line bundles or, for $r \leq \text{dim}X/2$, it can be isomorphic to $\Omega(1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}$.

Conclusion of the discussion of ii).

We have to prove first that $Z$ is smooth. This is true for $r = 1$ by Ando. For $r > 1$ we argue by induction: take a section $X' \in |L|$. Fibers of $\varphi$ restricted to $X'$ are connected so the morphism associated to $K_{X'} + (r - 1)L$ is again onto $Z$, which, by induction is now smooth.

Lemma 4.10. For $r > n/2$ the case ii) from 4.1 does not occur.

Proof. Because of the lemma 4.9 $NF/X$ is decomposable. Consider a hyperplane $F' \subset F$ which is a specialization of a general fiber. Then $NF'/X$ is a specialization of a trivial bundle, see e.g. [W1]. This however is impossible since from the sequence

$$
0 \rightarrow NF'/F = \mathcal{O}(1) \rightarrow NF'/X \rightarrow NF/X = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \rightarrow 0
$$

we see that the second Chern class of $NF'/X$ can not be 0.

(Note that in [S], theorem 3.3, the case $(n, r) = (3, 2)$ is excluded; in particular we can assume that $\text{dim}F' > 1$).

We finish now the discussion of the case iii).

Let $E$ be the exceptional set of $\varphi$. To conclude, by Nakano criterium [N], we have to prove that $E$ is a smooth divisor in a neighbourhood of $F$, that it has projective bundle structure and $\mathcal{O}_E(E)|_F = \mathcal{O}(-1)$.

First note that a small deformation of $F$ has a decomposable normal bundle. Indeed, take $n - 1 - r$ general very ample divisors on $Z$ and consider the intersection of their pull-back to $X$, call the resulting variety by $X''$. The variety $X''$ is smooth in the neighbourhood of $F$ and the restriction of $\varphi$ contracts a small deformation $F'$ of $F$, being now a divisor in $X''$, to a point. By adjunction we find out that $NF'/X'' = \mathcal{O}(-1)$ so that $NF'/X' = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}$.

But now, because of 4.9 it follows that also $NF/X = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}$. Now we know that $H^1(F, NF/X) = 0$, thus the family of deformations of $F$ inside $X$ is
parametrized by a variety which is smooth at $F$ and is of dimension $n - 1 - r$. Moreover as in the proof of 1.7 in [E] we see that the incidence variety of projective spaces is immersed into $X$, the image of it coincides locally with $E$. Therefore $E$ is smooth in the neighbourhood of $F$ and has a projective bundle structure contracted by $\varphi$. Finally, by adjunction, we see $\mathcal{O}_E(E)|_F = \mathcal{O}(-1)$ which concludes our argument.

A consequence of the theorem 4.1 (iii) is the following generalization of the theorem 1.1 in [ES].

**Corollary 4.11.** Let $\varphi : X \to Y$ be a proper birational morphism between smooth varieties. Put $S = \{y \in Y|\dim \varphi^{-1}(y) \geq 1\}$, $E = \varphi^{-1}(S)$. Assume that $\varphi^{-1}E$ is an irreducible divisor and also that all the fibers of $\varphi|_E$ have the same dimension. Then $S$ is smooth and $\varphi$ is the blowing up of $Y$ at $S$.

**Proof.** Note first that $b_2(X) - b_2(Y) = 1$; for this we use the irreducibility of $E$ as well as the smoothness of $Y$. Let $r$ be the dimension of one fiber of $\varphi^{-1}E$; the corollary follows now immediately from the above theorem taking $L = -E$.

**Remark (4.12).** We can give a better description of the case (ii) of the theorem 4.1. Let us define $E = \varphi^*(L)$ and let $S$ be the singular locus of the sheaf $E$; $S$ is the locus of $r$-dimensional fibers of $\varphi$. Outside of $S$ the sheaf $E$ is locally free of rank $r$. On the other hand let us choose a fiber $F$ of dimension $r$; from 4.1 and 4.5 it follows that, possibly shrinking $Z$ and $X$, we can choose $r + 1$ sections of $L$ spanning it on $X$. This yields a morphism of $X \to P^r_Z = P^r \times Z$ over $Z$. The image of the morphism is a divisor and moreover the morphism in an embedding outside $\varphi^{-1}(S)$. Therefore, by Serre criterion of normality and Zariski Main Theorem this is an embedding. We have the following sequence of sheaves on $P^r_Z$:

$$0 \to \mathcal{O}(1) \otimes \mathcal{O}(-X) \to \mathcal{O}(1) \to \mathcal{O}(1)|_X = L \to 0$$

and pushing it forward we get a presentation of $E$ (note that $X \in |\mathcal{O}(1)|$ so that $\mathcal{O}(1) \otimes \mathcal{O}(-X) = \mathcal{O}$ and the push-forward is right exact):

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z^{r+1} \to E \to 0.$$
Remark (4.13). We have a natural geometric construction which is associated to the situation described in the theorem 4.1.(ii). Let $S \subset Z$ be the set of fibers of dimension $r$; $S$ is the set of singularities of $E$. The set $S$ is of codimension $r + 1$ in $Z$, at least.

Consider a component $\mathcal{H}$ of the Hilbert scheme of $X$ parametrizing deformations of a general fiber of $\varphi$. We claim that $\mathcal{H}$ is smooth. Indeed, this follows because the normal bundle of any deformation of such a fiber is either trivial or $\mathcal{O} \mathbb{P}^{r-1}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(n-2r+1)$. Moreover, we have a birational map $\pi : \mathcal{H} \to Z$ which is biregular outside $\pi^{-1}(S)$, the fibers over $S$ are $\mathbb{P}^r$’s. Thus, by the corollary 4.11, $S$ is smooth and $\mathcal{H}$ is a blow-up of $Z$ along $S$. Considering the universal family over $\mathcal{H}$, we get the following diagram:

\[
\begin{array}{ccc}
Y & \to & \mathcal{H} \\
\downarrow & & \downarrow \\
X & \to & Z
\end{array}
\]

where $Y \to \mathcal{H}$ is a projective bundle and $Y \to X$ a blow-up of $X$ along $\pi^{-1}(S)$.

5. Spannedness.

In the present section we want to extend the non-vanishing argument to prove the following

\textbf{Theorem (5.1).} Let $\varphi : X \to Z$ be a local adjoint contraction supported by $K_X + rL$ (see assumptions (2.2)). Let $F$ be a fiber of $\varphi$. Assume moreover that

\begin{equation}
\text{either } \dim F < r + 1 \quad \text{if } \dim Z < \dim X \\
\text{or } \dim F \leq r + 1 \quad \text{if } \varphi \text{ is birational.}
\end{equation}

Then the evaluation morphism $\varphi^* \varphi_* L \to L$ is surjective at every point of $F$.

The proof will be divided in some lemmata. First of all, by vertical slicing argument and (3.1), we may assume that the base points of the line bundle $L$ are strictly contained in the fiber $F$; we can also assume that $\dim Z \geq 1$, since in the case $\dim Z = 0$ the theorem is trivial (see the remark (3.1.1)). In particular we can assume that the dimension of the base locus of $L$ is less then $(n-1)$.

Now we will estimate singularities of a general divisor from the linear system of $L$.

\textbf{Lemma (5.2).} A general divisor from the linear system $|L|$ is reduced and irreducible, and it has at worst log terminal singularities outside of the base point locus of $L$.

\textbf{Proof.} Let $G$ be a general divisor of $L$; the fact that it has at worst log terminal singularities outside $Bs|L|$ follows from a Bertini theorem. Again by the Bertini theorem (more precisely: theorem 6.3. points 3 and 4 of [J]), and the fact that $Bs|L|$ has codimension at least two, we see that $G$ is generically reduced and irreducible. Note that since we can assume that $\dim Z \geq 1$ and $\dim X \geq 2$, we have that the rational map defined by $|L|$ has image of dimension $\geq 2$. On the other hand $G$ has Cohen Macaulay singularities, since $X$ has log terminal and thus Cohen-Macaulay singularities, therefore $G$ is reduced.
Lemma (5.3). In the situation of the above theorem there exists a divisor \( G \) from \(|L|\) which does not contain any component of the fiber \( F \) and which has at worst log terminal singularities on \( F \).

Proof. We argue similarly as in the proof of (3.1). We can assume that \( Bs|L| \) is not empty because otherwise the lemma is true because of the Bertini theorem.

Let \( G \) be a general divisor from \(|L|\) which does not contain any component of the fiber \( F \). We may assume that points where \( G \) is not log terminal are contained in the base locus of \( L \).

Let \( \Delta \) be a Cartier divisor on \( X \) which is a pull back of a general divisor on \( Z \) passing through \( \varphi(F) \), that is \( \Delta \) contains \( F \); we may assume that \( \Delta \) is log terminal outside \( F \).

As in the proof of the theorem 3.1, let \( f : Y \to X \) be a log resolution (i.e. \( Y \) is smooth and all relevant divisors are smooth and cross normally). As before we write

\[
\begin{align*}
K_Y &= f^*K_X + \sum e'_i E_i \quad \text{where, by assumption } e'_i > -1 \\
f^*G &= \sum b_i E_i \quad \text{with } b_i \geq 0 \\
f^*\Delta &= \sum \delta_i E_i \quad \text{with } \delta_i \geq 0 \text{ and} \\
f^*\epsilon L &= A + \sum p_i E_i \quad \text{if } \dim Z < \dim X \text{ with } \epsilon > 0 \text{ or} \\
f^*O_X &= A + \sum p_i E_i \quad \text{if } \varphi \text{ is birational}
\end{align*}
\]

(5.3)

where in the last two formulas \( A \) is assumed to be ample \( \mathbb{Q} \)-divisor and \( 0 \leq p_i \ll 1 \). Note that \( b_i \geq 0 \) and \( \delta_i \geq 0 \); \( f(E_i) \subset G \) if and only if \( b_i > 0 \), while \( f(E_i) \subset \Delta \) if and only if \( \delta_i > 0 \). Let \( E_1 = G \) be the strict transform of \( G \); in particular \( e'_1 = 0 \) and \( b_1 = 1 \).

By adjunction

\[
K_{\tilde{G}} = (K_Y + \tilde{G})|_{\tilde{G}} = (f^*(K_X + G) + \sum_{i \neq 1} (e'_i - b_i) E_i)|_{\tilde{G}}
\]

and if the divisor \( G \) is not log terminal then, for at least one \( i \neq 1 \), we have \( e'_i - b_i \leq -1 \). Let \( S \) be a component of the locus, \( W \), of non log terminal points of \( G \):

\[
W = \bigcup \{ f(E_i) : e'_i - b_i \leq -1 \text{ with } i \neq 1 \}.
\]

Note that \( W \) is contained in the base locus of \( L \) and therefore in \( F \).

Let \( \Delta = \alpha \tilde{\Delta} \), where \( \alpha \) is a rational number with \( 0 < \alpha < 1 \). Let \( e_i = e'_i - \alpha \delta_i \); if \( \alpha \) is sufficiently small then \( e_i > -1 \) for all \( i \) and, for the \( i \) such that \( e'_i - b_i > -1 \), we have that also \( e_i - b_i > -1 \). On the contrary if \( i \neq 1 \) and \( e'_i - b_i \leq -1 \), then \( \delta_i > 0 \) and therefore \( e_i - b_i < -1 \). This means that the pair \((X, \Delta)\) is purely log terminal and that \( X \setminus W \) is the largest open set such that \((X, \Delta + G)\) is log canonical.

Again, as in the proof of 3.1, we apply Kollár’s argument; let

\[
(5.5) \quad c := \min \left\{ \frac{e_i + 1 - p_i}{b_i} : S \subset f(E_i), b_i > 0 \right\}
\]

By changing the \( p_i \) we can assume that the minimum is achieved for exactly one index (note that this index is not 1). Let us denote the corresponding divisor by \( E_0 \). We have a version of the claim (3.6):
Claim (5.6).
(i) $0 < c < 1$,
(ii) $f(E_0) = S$.
(iii) if $cb_i - e_i + p_i \geq 1$ and $i \neq 0$ then $S \not\subset f(E_i)$.
(iv) if $cb_i - e_i + p_i < 0$ then $e_i > 0$, hence $E_i$ is $f$-exceptional,

Proof. See the proof of the Claim 2.1.3 in [K].

Let us note that the $\mathbb{Q}$-divisor $K_Y + A + \sum (cb_i - e_i + p_i)E_i + f^*(tL)$
is numerically equivalent to $f^*((t-r+c)L)$ in the birational, and to $f^*((t-r+c+\epsilon)L)$
in the fiber case. We can write

$$\sum (cb_i - e_i + p_i)E_i = E_0 + H'' - H' + \sum \{cb_i - e_i + p_i\}E_i$$

where $E_0$, $H'$, $H''$ are effective and without common irreducible components and the fractional divisor has rational coefficients between 0 and 1 and is 0 along $E_0$; moreover $H'$ is $f$-exceptional and $S$ is not contained in $f(H'')$.

Note that since $c < 1$ we can choose $\epsilon$ such that $c + \epsilon < 1$. Therefore, if we assume that $t$ is an integer such that $t \geq r + 1$ (both in the birational or fiber case), then, by construction, the divisor

$$f^*(tL) - E_0 + H' - H'' - K_Y - Fr$$
is $\varphi \circ f$-ample.

Since by the previous theorem we have that $\dim S < \dim F \leq r + 1$ in the birational or, respectively $\dim S < \dim F < r + 1$ in the fiber case we obtain a contradiction arguing in the same way as we proved non-vanishing in the last part of the proof of the previous theorem. Namely we can produce a section of $L$ which does not vanish along $S$ which was supposed in the base locus.

Proof of the theorem 5.1 Note first that if we have a map between noetherian schemes, $X \to Y$, with zero dimensional (not necessarily connected) fibers then any line bundle on $X$ is relatively spanned ([H1], chap III, (3.7)). In particular this says that the theorem is true for $\dim F = 0$ without any hypothesis on $r$. Now we want to apply induction with respect to $\dim F$. Let $X'$ be a general divisor from the linear system $|L|$, $\varphi' := \varphi|_{X'}$ and $L' = L|_{X'}$. By the lemma 5.3 $X'$ does not contain any component of $F$ and it has at worst log terminal singularities while by the lemma 2.6 any section of $L$ on $X'$ extends to $X$. If $r > 1$ or $\geq 1$, respectively in the fiber or birational case, again by lemma 2.6, $\varphi : X' \to Z'$ is a local adjoint contraction supported by $K_{X'} + (r - 1)L$, with fiber $F'$ such that $\dim F' = \dim F - 1$.

Therefore the theorem is proved by induction.
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