Nearly-Perfect Non-Magnetic Invisibility Cloaking: Analytic Solutions and Parametric Studies

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Coordinate-transformation approaches to invisibility cloaking rely on the design of an anisotropic, spatially inhomogeneous “transformation medium” capable of suitably re-routing the energy flux around the region to conceal without causing any scattering in the exterior region. It is well known that the inherently magnetic properties of such medium limit the high-frequency scaling of practical “metamaterial” implementations based on subwavelength inclusions (e.g., split-ring resonators). Thus, for the optical range, non-magnetic implementations, based on approximate reductions of the constitutive parameters, have been proposed.

In this paper, we present an alternative approach to non-magnetic coordinate-transformation cloaking, based on the mapping from a nearly-transparent, anisotropic and spatially inhomogeneous virtual domain. We show that, unlike its counterparts in the literature, our approach is amenable to exact analytic treatment, and that its overall performance is comparable to that of a non-ideal (lossy, dispersive, parameter-truncated) implementation of standard (magnetic) cloaking.

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I. INTRODUCTION AND BACKGROUND

Invisibility of objects to an interrogating (electromagnetic, acoustic, elastic, quantum) wave illumination is a fascinating research topic of long-standing interest, with a wealth of intriguing theoretical and application-oriented implications. For instance, in electromagnetics (EM) engineering, “invisible” sources, scatterers and antennas have been investigated for several decades (see, e.g., Refs. 1,2,3,4,5,6 for a sparse sampling). However, during the last few years, interest in this topic has gained renewed momentum, under the suggestive association with the “cloaking” concept, mainly motivated by the rapid advances in the engineering of metamaterials (EM “wormhole”) with precisely controllable constitutive (e.g., anisotropy, spatial inhomogeneity, dispersion) properties. Among the most prominent approaches to (passive) invisibility cloaking, it is worth recalling those based on scattering cancellation8,9, coordinate transformations10,11,12,13 (experimentally demonstrated at microwave7 and visible14 frequencies), anomalous localized resonances15, inverse design of scattering optical elements16, and transmission-line networks17. The reader is referred to Ref. 18 (and references therein) for a recent comparative review of these various approaches.

In particular, the coordinate-transformation (also referred to as “transformation-optics” or “transformation EM”) approach10,11,12,13, directly related to our investigation, relies on the formal invariance of Maxwell’s equations under coordinate transformations, which allows a preliminary design in an auxiliary curved-coordinate space containing a “hole,” and its subsequent translation into a conventionally flat, Cartesian space filled by an anisotropic and spatially inhomogeneous “transformation medium” that suitably bends the ray trajectories so as to re-route the energy flux around the concealment region. The reader is referred to Refs. 19 and 20 (and references therein) for a recent collection and review of applications and theoretical aspects. Among the most interesting new twists and extensions of the basic cloaking idea above, it is worth mentioning the concepts of EM “wormhole”21, anti-cloak22,23, carpet cloak24,25,26,27, cloak at a distance28, and open cloak29, as well as the extensions to acoustic30,31, elastic32, and quantum33 waves.

In view of the considerable complexity of the arising transformation media, EM modeling of such structures typically relies on heavily numerical (finite-element) simulations34. Nevertheless, at least for canonical (e.g., cylindrical35,36 and spherical17) geometries, analytic full-wave approaches, based on suitable mappings of standard Fourier-Bessel or Mie expansions, have been developed. Remarkably, via these approaches, it was possible to prove analytically that in the ideal case (implying a lossless, anisotropic, spatially inhomogeneous transformation medium, with extreme values of the constitutive parameters ranging from zero to infinity) the cloaking would be perfect, i.e., without any transmission into the concealment region and any external scattering, and may be, in principle, attained at any frequency – not necessarily in the asymptotic high-frequency regime that the intuitive ray-bending picture would suggest. However, the inherent limitations arising from the unavoidable losses35,37, dispersion38,39, perturbations40 and simplifications/reductions41 in the constitutive parameters were also pointed out.

The first experimental verification of coordinate-transformation cloaking was achieved at microwave (X-band) frequencies7, where the involved scales allowed the metamaterial fabrication via low-loss metallic split-ring-resonator inclusions. High-frequency scaling of this technological solution seems to be within reach for the low-THz region42, but is limited by satura-
tion effects. Thus, for the visible range, an alternative route has been followed, based on the use of non-magnetic materials, which has led to the experimental demonstration of an optical cloak. Non-magnetic approaches to coordinate-transformation cloaking are based on approximate reductions of the constitutive parameters that preserve the ray trajectories inside the cloak shell, at the expense of destroying the perfect impedance matching with the background medium, which can only be partially restored by suitable tweaking of the extra parameters available in quadratic or higher-order coordinate transformations. The above parameter reductions also prevent application of the exact analytic approaches in Refs. [37,38] and thus their analytic modeling is limited to asymptotic (semiclassical) approximations.

In this paper, we propose a different approach to non-magnetic coordinate-transformation cloaking which, acknowledging the inherent limitations of realistic metamaterial implementations, instead of applying the coordinate transformation to a perfectly transparent virtual domain (as in standard transformation optics), considers a weakly non-transparent anisotropic and spatially inhomogeneous domain. With reference to a two-dimensional (cylindrical) scenario, we show that, via a judicious choice of the constitutive parameters and the coordinate transformation, it is possible to achieve a non-magnetic transformation-medium without resorting to approximate reductions, thereby maintaining the applicability of the computationally-effective and insight-providing exact analytic modeling (via generalization of the results in Refs. [32,33]). Moreover, via suitable parameter optimization at a given frequency, the inherently non-zero scattering can be minimized so that, when the unavoidable non-idealities (losses, dispersion, parameter truncations) of metamaterial implementations are taken into account, the overall performance becomes comparable to that of a standard (magnetic) cloak within broad parametric ranges.

Accordingly, the rest of the paper is laid out as follows. In Sec. II we outline the problem formulation and the proposed strategy. In Sec. III we derive the analytic solutions. In Sec. IV we present some representative results from our parametric studies, and compare them with those achievable via standard (magnetic) cloaking. Finally, in Sec. V we provide some concluding remarks and hints for future research.

II. PROBLEM FORMULATION AND PROPOSED STRATEGY

A. Virtual Space: Nearly-Transparent Domain

In the two-dimensional (2-D) scenario of interest, we start considering an auxiliary virtual space \((x', y', z)\) featuring a circular cylindrical domain (infinite along \(z\)) of radius \(R_2\), made of an anisotropic and radially inhomogeneous medium immersed in vacuum. We choose for the relative permittivity and permeability tensors \(\varepsilon_\prime\) and \(\mu_\prime\), respectively) a rather general parametric form featuring several degrees of freedom and yet amenable to analytic solution of the relevant Helmholtz equation; for the transverse-magnetic (TM) polarization (magnetic field parallel to the cylinder) of interest here, their relevant components (in the associated cylindrical coordinates \(r', \phi, z\)) are given by

\[
\varepsilon_\prime(r') = \varepsilon_\prime(\phi) = \left(\frac{r'}{R_2}\right)^{-\gamma} \exp \left[ -\alpha \left( \frac{r'}{R_2} - 1 \right) \right], \\
\mu_\prime(r') = P(r') \left( \frac{r'}{R_2} \right)^{\gamma} \exp \left[ \alpha \left( \frac{r'}{R_2} - 1 \right) \right],
\]

where \(\gamma > 0\), \(\alpha > 0\), and

\[
P(r') = 1 - p + \frac{R_2}{r'}, \quad 0 \leq p \leq 1.
\]

Note that the constitutive parameters in (1) are always positive, and locally matched with vacuum at the interface \(r' = R_2\), whereas they may exhibit singular behaviors at \(r' = 0\),

\[
\lim_{r' \to 0} \varepsilon_\prime(r') = \infty, \quad \lim_{r' \to 0} \mu_\prime(r') = \begin{cases} 
\infty, & 0 < \gamma < 1, \\
\exp(-\alpha p), & \gamma = 1, \\
0, & \gamma > 1.
\end{cases}
\]

We highlight that, unlike the typical transformation-optics framework, our virtual domain is not perfectly transparent. Nevertheless, the constitutive relationships in (1) contain three adjustable parameters (\(\alpha, \gamma, p\)) that can be optimized (see Sec. IV) for achieving a nearly-transparent response at a given frequency. Figure
shows an example of constitutive parameters optimized for an electrical size $R_2 = 3\lambda_0$ (with $\lambda_0$ denoting the vacuum wavelength). The log-log scale utilized highlights the algebraic singular behavior of the permeability for $r' \to 0$, but is still not able to capture the extremely slow (in view of the chosen parameters) divergence of the permittivities. For the same configuration, Fig. 2(a) illustrates the ray tracing, obtained as in Ref. 13. As a first indication of the nearly-transparent behavior of this optimized parametric configuration, one can observe the practically straight ray trajectories, with only a slight bending for those passing nearby the cylinder center.

B. Coordinate Transformation

Following the transformation-optics approach\textsuperscript{10,11,12,13}, in order to achieve invisibility cloaking, we construct a cylindrical coordinate transformation, from the virtual space $(x',y',z)$ to the actual physical space $(x,y,z)$ of interest,

$$r = g(r'), \quad r' \leq R_2,$$

which compresses the cylindrical nearly-transparent region $r' \leq R_2$ into a concentric annulus $R_1 \leq r \leq R_2$, i.e., satisfies the boundary conditions

$$g(0) = R_1, \quad g(R_2) = R_2.$$  

As previously mentioned, the non-flat metric underlying the transformation in (4) and (5) creates a “hole” of radius $R_1$, i.e., a region of space effectively impermeable by the EM fields where an object can be concealed. Invoking the formal invariance of Maxwell’s equations under coordinate transformations, the above behavior can be equivalently obtained in a globally flat space by filling up the transformed region $R_1 \leq r \leq R_2$ with an anisotropic, spatially inhomogeneous “transformation medium,” whose relative permittivity and permeability tensors ($\varepsilon$ and $\mu$, respectively) are given by\textsuperscript{10,11,12,13}

$$\left\{ \begin{array}{l} \varepsilon(x,y) = J \cdot \left( \begin{array}{c} \varepsilon_x(r') \\ \varepsilon_y(r') \\ \varepsilon_z(r') \\ \varepsilon_r(r') \\ \varepsilon_\phi(r') \\ \varepsilon_\theta(r') \end{array} \right) J^T \left[ \det (J) \right]^{-1}, \end{array} \right.$$  

where $J = \partial(x,y,z)/\partial(x',y',z)$ is the Jacobian matrix of the transformation in (4), the superfix $T$ indicates transposition, and the symbol $\det(\cdot)$ denotes the determinant. For the TM polarization of interest here, the relevant (cylindrical) components can be directly found from (4) as\textsuperscript{13}

$$\varepsilon_r(r') = \varepsilon_r(r') g(r') \frac{r'}{r},$$

$$\varepsilon_\phi(r) = \varepsilon_\phi(r') \frac{r}{g(r')} \frac{r'}{r'},$$

$$\mu_z(r) = \mu_z(r') \frac{g(r')}{\dot{g}(r')},$$

where the overdot \(\dot{\cdot}\) denotes differentiation with respect to the argument, and $r' = f(r)$ [with $f(r) = g^{-1}(r')$ denoting the inverse mapping]. In standard approaches to non-magnetic cloaking\textsuperscript{43,44,45,46,47}, based on perfectly-transparent virtual domains (i.e., $\varepsilon_r = \varepsilon_\phi = \mu_z = 1$), reliance is made on the dependence of the ray trajectories on the products $\varepsilon_r \mu_z$ and $\varepsilon_\phi \mu_z$ in order to find a set of reduced parameters featuring $\mu_z = 1$. As anticipated, this destroys the perfect transparency of the cloak, and prevents application of the exact analytic framework in Refs. 35 and 36. In our approach, instead, we capitalize on the extra degrees of freedom available in order to enforce $\mu_z(r) = 1$ in (7c), which yields the following first-order differential equation

$$\dot{g}(r') - r' \mu_z(r') \frac{g(r')}{\dot{g}(r')} = 0.$$  

A similar approach was used in Ref. 49 in order to obtain a spatially homogeneous axial permeability (or permit-
tivity, for the transverse-electric polarization). However, relying on a vacuum (i.e., \(\varepsilon' = \mu' = 1\)) virtual domain, such approach did not provide enough degrees of freedom. As a consequence, once the cloaking boundary conditions in (3) were enforced, the value of the transformed axial permeability was inevitably dictated by the shape factor \(R_1/R_2\), and (most important) always larger than the vacuum value (i.e., \(\mu_2 > 1\), for the case of interest here). In our approach, this limitation is overcome via the ex-vacuum value (i.e., the function \(\varepsilon' = \mu' = 1\)) virtual domain permeability \(\mu_2'(r)\) [cf. Eqs. (1) and (2)]. In particular, thanks to the functional form in (1b), the differential equation in (8) admits a closed-form analytic solution as

\[
g(r') = \sqrt{R_1^2 + R_2^2 h \left( R_1^2; \alpha, \gamma, p \right)},
\]

where

\[
h \left( \frac{r'}{R_1^2}; \alpha, \gamma, p \right) = 2 \exp(-\alpha) \times \left[ \left( 1 - \frac{p}{\gamma + 2} \right) \left( \frac{r'}{R_2} \right)^{\gamma + 2} M \left( \alpha \gamma + 3, \frac{\gamma + 2}{\gamma + 2} \right) \right],
\]

with \(M(\cdot, \cdot, \cdot)\) denoting the confluent hypergeometric function [Eq. (13.1.2) in Ref. 50]. Note that, in light of the assumed constraints (\(\alpha > 0, \gamma > 0, 0 \leq p \leq 1\)), the function \(h(r'/R_2; \alpha, \gamma, p)\) in (10) is always positive, which ensures that the mapping \(g(r')\) in (9) is always real. Moreover, in (9) and (10), the arising integration constant has been exploited to enforce the first boundary condition in (5); the second boundary condition, instead, yields

\[
h(1; \alpha, \gamma, p) = 1 - \left( \frac{R_1}{R_2} \right)^2,
\]

which can be satisfied by properly tweaking the parameters \(\alpha, \gamma, \) and \(p\). In particular, in view of the linear dependence involved, Eq. (11) may be straightforwardly solved with respect to \(p\). However, this seemingly simplest approach is not necessarily the most effective in a broader perspective of achieving a nearly-transparent response. In fact, our parametric studies (see Sec. IV A below) indicate that it is generally more convenient to satisfy the constraint in (11) by fixing \(\alpha\) (via numerical solution), and exploit the parameters \(\gamma\) and \(p\) for minimizing the scattering response.

It should be noted that, since \(\mu_2'(r)\) in (11) and \(g(r')\) in (9) are always positive, the constraint in (8) also implies \(g(r') > 0\), and hence the invertibility of the coordinate mapping. However, the inverse mapping \(f(r) = g^{-1}(r')\) cannot generally be calculated analytically. Nevertheless, as can be observed from the example in Fig. 2(b), the mapping behavior is fairly regular, and therefore its numerical inversion does not pose any problem. Figure 2(c) shows the ray trajectories obtained by transforming [via the mapping in Fig. 2(b)] those in the virtual space [cf. Fig. 2(a)], from which the ray bending around the interior region \((r < R_1)\) to conceal (typical of coordinate-transformation-based cloaking) is fairly evident.

C. Real Space: Non-Magnetic Transformation Medium

By substituting (11) and \(r' = f(r)\) in (7), and taking into account (5), we readily obtain the explicit expressions of the constitutive parameters of the desired non-magnetic transformation medium in the real space,

\[
\varepsilon_r(r) = \frac{R_2^2 \left[ f(r) \right]^{1-\gamma}}{rf(r)} \exp \left\{ -\alpha \left[ \frac{f(r)}{R_2} - 1 \right] \right\}, \tag{12a}
\]

\[
\varepsilon_\phi(r) = \frac{R_2^2 \gamma f(r) \left[ f(r) \right]^{1-\gamma}}{\left[ f(r) \right]^{1-\gamma}} \exp \left\{ -\alpha \left[ \frac{f(r)}{R_2} - 1 \right] \right\}, \tag{12b}
\]

\[
\mu_z(r) = 1. \tag{12c}
\]

Note that, since \(f(r)\) and \(f'(r)\) are always positive in the transformed region \(R_1 < r < R_2\), both permittivity components in (12a) and (12b) are likewise positive, thereby yielding a double positive medium. Moreover, like the virtual-space medium in (11), our non-magnetic transformation medium in (12) is locally matched with vacuum at \(r = R_2\). In view of the singular behavior exhibited by the virtual-domain medium at \(r' = 0\) [cf. (3)], it is interesting to investigate the behavior of the transformation medium at the image point \(r = R_1\). Recalling that \(M(a, b, 0) = 1\) [cf. Eq. (13.1.2) in Ref. 50], we obtain from (11), in the limit \(r' \to 0\),

\[
h \left( \frac{r'}{R_2^2}; \alpha, \gamma, p \right) \sim 2 \exp(-\alpha) \left( \frac{\chi}{\gamma - s + 2} \right) \left( \frac{r'}{R_2} \right)^{\gamma - s + 2}, \tag{13}
\]

where

\[
\chi = \left\{ \begin{array}{ll}
1, & p = 0, \\
p, & p \neq 0,
\end{array} \right. \quad s = \left\{ \begin{array}{ll}
0, & p = 0, \\
1, & p \neq 0.
\end{array} \right. \tag{14}
\]

By substituting (13) into (9), and approximating the square root via its first-order Taylor expansion, we then obtain

\[
g(r') \sim R_1 \sqrt{1 + \frac{2 \exp(-\alpha) \chi R_2^{s-\gamma}(r')^{\gamma-s+2}}{\left(2 \gamma - s + 2\right)R_2^2}} \sim R_1 + \frac{\exp(-\alpha) \chi R_2^{s-\gamma}(r')^{\gamma-s+2}}{\left(2 \gamma - s + 2\right)R_1}, \tag{15a}
\]

and hence, via differentiation,

\[
\dot{g}(r') \sim \exp(-\alpha) \chi R_1^{-1} R_2^{s-\gamma}(r')^{\gamma-s+1}. \tag{15b}
\]
Recalling that \( f(r) = 1/\hat{g}(r') \) and \( r' = f(r) \), and substituting into (12a) and (12b), we then obtain

\[
\varepsilon_r(r) \sim \frac{\chi R^3}{R^1} [f(r)]^{2-s}, \quad (16a)
\]
\[
\varepsilon_\phi(r) \sim \text{exp}(2\alpha R^2 R^2 - s) [f(r)]^{s-2g-2}, \quad (16b)
\]

from which, recalling that \( 0 \leq s \leq 1 \) and \( \gamma > 0 \), it finally follows

\[
\varepsilon_r(R_1) = 0, \quad \lim_{r \to R_1} \varepsilon_\phi(r) = \infty. \quad (17)
\]

Thus, as in standard (magnetic) cloaking, at the inner interface \( r = R_1 \), the permittivities exhibit either zero or infinite values. Figure 3 shows the transformation-medium constitutive parameters obtained from the (optimized) virtual-domain medium in Fig. 1 via the coordinate mapping in Fig. 2(b), and corresponding to the ray trajectories in Fig. 2(c).

It is worth pointing out that, in principle, different choices of the parameters and functional forms of the virtual-domain medium (11) are possible, yielding finite variation ranges of the resulting transformation media. Our choice here was only aimed at facilitating direct comparison with standard (magnetic) cloaks. Nevertheless, the effects of unavoidable parameter truncations are investigated in our parametric studies (see Sec. IV.B below).

III. ANALYTIC SOLUTIONS

The EM response of the cloaking configuration of interest can be computed analytically by generalizing the Fourier-Bessel-type approach in Refs. 35 and 36. In what follows, we address such generalization for the case of TM-polarized plane-wave illumination with unit-amplitude \( z \)-directed magnetic field and time-harmonic \( \text{exp}(\pm i\omega t) \) dependence, by suitable coordinate mapping [via the inverse of (1)] of the virtual-space solution.

A. Virtual Space

It is expedient to expand the incident plane wave \( H^i_z \) (assumed to impinge from the positive \( z' \) direction) and the scattered field \( H_z^s \) into Fourier-Bessel series,

\[
H_z^i(r', \phi) = \text{exp}(ik_0 z') = \sum_{m=\pm \infty} i^m J_m(k_0 r') \text{exp}(im\phi), \quad (18a)
\]
\[
H_z^s(r', \phi) = \sum_{m=\pm \infty} c_m H_m^{(1)}(k_0 r') \text{exp}(im\phi), \quad (18b)
\]

where \( k_0 = \omega/c = 2\pi/\lambda_0 \) denotes the vacuum wavenumber (with \( c \) denoting the speed of light in vacuum), \( J_m(\cdot) \) and \( H_m^{(1)}(\cdot) \) are the \( m \)-th order Bessel and Hankel functions of first kind, respectively (cf. Sec. 9.1 in Ref. 50), and \( c_m \) are unknown coefficients. The magnetic field \( H_z^s \) transmitted inside the cylindrical domain \( r' < R_2 \), ruled by the Helmholtz equation

\[
\frac{1}{r' \mu_z} \frac{\partial}{\partial r'} \left( r' \frac{\partial}{\partial r'} \right) H_z^s(r', \phi) + \left[ \frac{1}{r'^2 \varepsilon_z} \frac{\partial^2}{\partial \phi^2} + k_0^2 \right] H_z^s(r', \phi) = 0, \quad (19)
\]

can be likewise expanded into this type of series

\[
H_z^s(r', \phi) = \sum_{m=\pm \infty} \left[ a_m \Psi_m^{(1)}(k_0 r') + b_m \Psi_m^{(2)}(k_0 r') \right] \times \text{exp}(im\phi), \quad r' \leq R_2. \quad (20)
\]

In (20), \( a_m \) and \( b_m \) are unknown coefficients, while \( \Psi_m^{(1,2)}(\cdot) \) denote independent solutions of the radial Helmholtz equations

\[
\left\{ \frac{d^2}{dr'^2} + \left[ \frac{1}{r'^2} \frac{1}{\varepsilon_z} \frac{d}{dr'} \right] \frac{d}{dr'} \Psi_m(k_0 r') + \left[ k_0^2 \mu_z(r') \varepsilon_z(r') - \frac{m^2}{r'^2 \varepsilon_z(r')} \right] \Psi_m(k_0 r') = 0. \quad (21)
\]

It can be shown (see Appendix A for details) that, for the constitutive parameters in (1), these solutions can be expressed in closed form as

\[
\Psi_m^{(1,2)}(k_0 r') = \left( \frac{r'}{R_2} \right)^{\text{imp}} \exp \left[ -\left( \alpha + \xi R_2 \right) r'/2R_2 \right] \times \left\{ \begin{array}{l}
M(\zeta_m; \nu_m + 1, \xi r') \\
U(\zeta_m; \nu_m + 1, \xi r')
\end{array} \right\}, \quad (22)
\]
where (the already defined) $M(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are confluent hypergeometric functions [cf. Eqs. (13.1.2) and (13.1.3), respectively, in Ref. 50], and

\[
\begin{align*}
\nu_m &= \sqrt{\gamma^2 + 4m^2}, \\
\xi &= R_2^{-1} \sqrt{\alpha^2 - 4(1 - p)k_0^2 R_2^2}, \quad 0 \leq \arg(\xi) < \pi, \quad (23b) \\
\zeta_m &= \left(\frac{\nu_m + 1}{\nu_m} + \frac{\alpha}{2} - \frac{2\nu_m k_0^2 R_2^2}{\nu_m + 1} + \frac{\alpha(\gamma + 1)}{2} - 2\nu_m k_0^2 R_2^2 \right)^{1/2}.
\end{align*}
\]

Recalling that the wavefunctions $\Phi_m^{(2)}$ exhibit a singular behavior at $r' = 0$ (see Appendix 13), the field-finiteness condition yields $b_m = 0$ in (20). The remaining unknown expansion coefficients ($\alpha_m$ and $c_m$) can be computed by enforcing the continuity of the magnetic and electric field tangential components at the interface $r = R_2$, viz.,

\[
H_1^e(R_2, \phi) = H_1^e(R_2, \phi) + H_2^e(R_2, \phi), \quad (24a)
\]

\[
E_\phi^e(R_2, \phi) = E_\phi^e(R_2, \phi) + E_\phi^e(R_2, \phi), \quad (24b)
\]

where the tangential electric fields readily follow from the curl Maxwell equation

\[
E_\phi(r', \phi) = \frac{1}{i\omega \varepsilon_0 c_0} \frac{\partial H_x(r', \phi)}{\partial r'}.
\]

By substituting the Fourier expansions (18) and (20) into (24) [with (25)], we obtain a doubly countable infinity of linear equations,

\[
a_m \Phi_1^{(1)}(k_0 R_2) + c_m H_1^{(1)}(k_0 R_2), \quad (26a)
\]

\[
a_m \Phi_1^{(1)}(k_0 R_2) = i^m J_m(k_0 R_2) + c_m H_1^{(1)}(k_0 R_2), \quad (26b)
\]

which can be solved in a straightforward fashion, yielding

\[
a_m = -\frac{2^{m+1}}{\pi k_0 R_2 W(m, \nu_m)}, \quad (27a)
\]

\[
c_m = W(m, \nu_m) J_m(k_0 R_2) - F(k_0 R_2) G(k_0 R_2).
\]

B. Real Space

We can now address the solution in the real space $(r, \phi, z)$, by generalizing the approach in Refs. 33 and 36. First, we note that, since the coordinate transformation in (11) is restricted within the cloak shell $R_1 < r < R_2$, the fields in the (vacuum) exterior region $r > R_2$ admit the same expressions as in (18) (with $r$ substituting $r'$). For the same reason, the field transmitted into the (vacuum) concealment region $r < R_1$, can be expanded in terms of a standard Fourier-Bessel series,

\[
\Phi_m^{(2)}(r, \phi) = \sum_{m=-\infty}^{\infty} d_m J_m(k_0 r) \exp(\im \phi), \quad r \leq R_1,
\]

where $d_m$ are unknown coefficients, and the field-finiteness condition has been enforced. Following Refs. 33 and 36, the field transmitted into the cloak shell $R_1 \leq r < R_2$ is obtained by (inverse) coordinate mapping [via (1)] of the virtual-space solution (24), viz.,

\[
\Phi_m^{(2)}(r, \phi) = \sum_{m=-\infty}^{\infty} \left[ a_m \psi_1^{(1)}(k_0 r) + b_m \psi_1^{(2)}(k_0 r) \right] \exp(\im \phi), \quad R_1 \leq r \leq R_2,
\]

where $a_m$ and $b_m$ are unknown expansion coefficients, and

\[
\psi_1^{(1,2)}(k_0 r) = \psi_1^{(1,2)}(k_0 f(r)).
\]

Once again, the four sets of unknown expansion coefficients ($a_m, b_m, c_m, d_m$) can be computed by enforcing the continuity of the tangential fields, this time at the interfaces $r = R_1$ and $r = R_2$. While the continuity at the outer interface ($r = R_2$) does not pose any particular problem, much more involved is dealing with the inner interface ($r = R_1$), in view of the singular behavior exhibited by the wavefunctions $\psi_m^{(2)}$ in (30) [cf. Eq. (13.1.3) in Ref. 50 and Appendix 13] and the azimuthal permittivity $\varepsilon_\phi$ in (12) [cf. 17]. In order to circumvent this problem, as in Refs. 33 and 36, we therefore follow a limiting approach, slightly shifting the inner cloak boundary to $r = R_1 + \Delta$, where $\Delta$ denotes a small quantity (which eventually we let tend to zero). Accordingly, we obtain

\[
\begin{align*}
&d_m J_m(k_0(R_1 + \Delta)) = a_m \psi_1^{(1)}(k_0(R_1 + \Delta)) + b_m \psi_1^{(2)}(k_0(R_1 + \Delta)), \quad (31a) \\
&d_m \dot{J}_m(k_0(R_1 + \Delta)) = a_m \dot{\psi}_1^{(1)}(k_0(R_1 + \Delta)) + b_m \dot{\psi}_1^{(2)}(k_0(R_1 + \Delta)), \quad (31b) \\
&a_m \psi_1^{(1)}(k_0 R_2) + b_m \psi_1^{(2)}(k_0 R_2) = i^m J_m(k_0 R_2) + c_m H_1^{(1)}(k_0 R_2), \quad (31c) \\
&a_m \psi_1^{(1)}(k_0 R_2) + b_m \psi_1^{(2)}(k_0 R_2) = i^m \dot{J}_m(k_0 R_2) + c_m \dot{H}_1^{(1)}(k_0 R_2). \quad (31d)
\end{align*}
\]
FIG. 4: (Color online) Curves in the \((\gamma, p)\) plane yielding \(\alpha = 0\) roots of \((31)\), for the parameters in Fig. 1 and shape-factor values \(R_1/R_2 = 1/4, 1/3, 1/2\) (black-solid, red-dashed, blue-dotted, respectively). The regions below the curves are associated with positive \(\alpha\) values, and thus represent the admissible \((\gamma, p)\) search spaces in the scattering-width minimization problem.

It can be shown (see Appendix B for details) that, in the limit \(\Delta \to 0\), Eqs. (31a) and (31b) yield

\[
\begin{align*}
    b_m & \sim a_m \Omega_1 \left( \frac{\Delta}{R_2} \right) \frac{\nu_m}{\nu_m + 2}, \quad (32a) \\
    d_m & \sim a_m \Omega_2 \left( \frac{2}{R_2} \right) \frac{\gamma_m + \nu_m}{\gamma_m + \nu_m + 2}, \quad (32b)
\end{align*}
\]

where \(\Omega_{1,2}\) are irrelevant constants. Recalling that \(0 \leq s \leq 1, \gamma > 0\) and \(\nu_m > 0\), it is readily realized that the expansion coefficients \(b_m\) and \(d_m\) in (32) vanish as \(\Delta \to 0\). As a consequence, recalling (30) and that \(f(R_2) = R_2\), Eqs. (31c) and (31d) are readily recognized to become identical to (26a) and (26b), respectively, and therefore the remaining unknown coefficients \(a_m\) and \(c_m\) are still given by (24).

To sum up, the above analytic solutions resemble those obtained for the standard (magnetic) cloak \(\gamma, p\) in the exact suppression of the field transmitted into the concealment region (i.e., \(d_m = 0\)). The expectable differences show up in the non-zero scattering coefficients \(c_m\), which are directly inherited from the nearly-transparent medium \(\Omega\) in the virtual space.

In what follows, we show that, via judicious optimization at a given frequency, the scattering response can be effectively reduced so that the overall performance of the proposed nonmagnetic cloak becomes comparable to that of a standard (magnetic) cloak when the unavoidable non-idealities (parameter truncations, losses, dispersion) are taken into account.

IV. REPRESENTATIVE RESULTS

We now move on to the presentation of the salient results from an extensive series of parametric studies, starting with the virtual-domain parameter optimization, and continuing with the performance comparison between the proposed approach and standard (magnetic) cloaking.

A. Virtual Space: Optimization for Near-Transparency

As previously mentioned, the inherently non-zero scattering response of our proposed non-magnetic cloak is directly inherited by the non-transparent virtual domain \(\Omega\). In our study, such response is compactly parameterized in terms of the total scattering cross-sectional width per unit length

\[
Q_s = \frac{4}{k_0} \sum_{m=-\infty}^{\infty} |c_m|^2, \quad (33)
\]

where the scattering coefficients \(c_m\) are given by (27b). Our approach is based on the minimization of the scattering width in (33), at a given frequency, by acting on the free parameters \((\alpha, \gamma, p)\) available in the virtual-domain medium constitutive parameters \(\Omega\). Clearly, different observables (e.g., more directly tied to the near-field or angular distributions) may be considered, giving rise to different optimized configurations. It is also worth recalling that the above parameters are actually constrained via the cloak condition in \(\Omega\). In our numerical studies, we found it most effective to satisfy this constraint by fixing \(\alpha\) via numerical solution of \(\Omega\), and exploit the parameters \(\gamma\) and \(p\) for minimizing \(Q_s\). Figure 4 shows,
for representative values of the shape factor $R_1/R_2$, the $(\gamma,p)$ parametric ranges for which Eq. (11) admits a solution $\alpha > 0$ (consistent with the model assumptions), which constitute the search spaces in our minimization problem. In view of the reduced number of parameters involved and the computationally-effective analytic modeling, such minimization can be readily pursued via exhaustive parameter scanning. For a fixed frequency and shape factor, Fig. 5 shows the scattering width in Eq. (33) as a function of the two free parameters $\gamma$ and $p$, within the above defined admissible ranges, from which a rather broad minimum is identified. The corresponding parameter configuration yields the constitutive relationships shown in Fig. 1 and the associated (practically straight) ray trajectories in Fig. 2(a).

B. Real Space: Comparison with Standard (Magnetic) Cloak

Referring to the non-magnetic ideal (lossless, non-truncated) constitutive parameters in Fig. 3 and associated ray trajectories in Fig. 2(c) [derived from the above optimized virtual-domain via the mapping in Fig. 2(b)], Fig. 7 shows the real part of the magnetic field map (for plane-wave excitation) computed via the expansions in Eq. (18) and (20) [with (27)], from which the near-transparency is evident.

Although the present prototype study was not focused on practical applications, we did explore the effects of the unavoidable non-idealities, namely, the parameter truncations and material losses. As for the standard (magnetic) cloak, suitable truncation of the permittivities in Eq. (12) at the inner interface $r = R_1$ is necessary in view of their singular behavior [cf. (17)]. As in Refs. 35 and 36, in our parametric studies, we truncated the cloak shell at an interface $r = R_1 + \Delta$, considering the thin annulus $R_1 \leq r < R_1 + \Delta$ as part of the concealment region. Figure 8 shows, for an empty (vacuum) concealment region, the scattering width as a function of the truncation parameter $\Delta/R_1$, for various values of the loss-tangent.
ranging from zero to $10^{-2}$. For comparison, we also studied the response of a standard (magnetic) cloak,

\begin{align}
\varepsilon_r^{(ref)}(r) &= \frac{r - R_1}{r}, \\
\varepsilon_\phi^{(ref)}(r) &= \frac{r}{r - R_1}, \\
\mu_z^{(ref)}(r) &= \left(\frac{R_2}{R_2 - R_1}\right)^2 \frac{r}{r - R_1}.
\end{align}

likewise truncated so as to guarantee comparable variation ranges of the permittivities. As one can observe, in spite of its non-magnetic character, the proposed cloak turns out to outperform the standard one (in terms of smaller scattering width) over a wide parametric range extending up to moderate values of the truncation parameter and losses.

It is worth pointing out that parameter truncations allow field penetration through the cloak shell, and thus, unlike in the ideal case, the overall response generally depends on the possible presence of objects in the concealment region. We therefore studied some representative scenarios, with the concealment region entirely filled up by a dielectric (with relative permittivity $\varepsilon_{obj} = 4$ and 16) or a perfect electric conductor (PEC), still amenable to analytic solution via straightforward generalization of (29). Figures 9, 10, 11 show the corresponding responses, where, in order to better visualize the cloaking effect, the scattering width has been normalized with respect to the reference value $Q_s^{(obj)}$ exhibited by the object in vacuum.

values of the truncation parameter $\Delta/R_1$ and no losses) tend to disappear when moderate values of the truncation parameter and loss-tangent are considered. In particular, for $\tan \delta = 10^{-2}$, the proposed non-magnetic cloak is still capable of reducing the scattering width of over an order of magnitude with respect to the uncloaked case, slightly outperforming a comparably-truncated standard cloak. As an example, Fig. 12 compares the responses pertaining to a PEC cylinder free-standing in vacuum and cloaked with the proposed and standard approaches, for a truncation parameter $\Delta/R_1 = 10^{-2}$ and loss tangent $\tan \delta = 10^{-2}$, in terms of the bistatic scattering width

\begin{equation}
\sigma_s(\phi) = \lim_{r \to \infty} 2\pi r \frac{|H_z^2(r, \phi)|^2}{|H_z^1(r, \phi)|^2} = \frac{4}{k_0} \left| \sum_{m=-\infty}^{\infty} (-i)^m c_m \exp(i m \phi) \right|^2,
\end{equation}

FIG. 9: (Color online) As in Fig. 8 but with the concealment region entirely filled by a dielectric object with relative permittivity $\varepsilon_{obj} = 4$. In order to better visualize the cloaking effect, the scattering width is normalized with respect to reference value $Q_s^{(obj)}$ exhibited by the object in vacuum.

FIG. 10: (Color online) As in Fig. 9 but for an object with $\varepsilon_{obj} = 16$.

FIG. 11: (Color online) As in Fig. 9 but for a PEC object.
V. CONCLUSIONS AND OUTLOOK

In this paper, we have presented an alternative approach to non-magnetic coordinate-transformation-based invisibility cloaking. Unlike other approaches in the literature, the proposed strategy does not rely on approximate parameter reductions but rather on the design, via parametric optimization, of a nearly-transparent anisotropic and spatially inhomogeneous virtual domain, and is amenable to exact analytic treatment.

After derivation of the relevant analytic solutions, we have presented a body of representative parametric studies. Our results indicate that, when the unavoidable non-idealities (parameter truncations, losses, dispersion) of typical metamaterial implementations are taken into account, the overall performance attainable is comparable to that of a standard (magnetic) cloak.

The idea underlying the proposed strategy is rather general and, besides the cloaking scenarios, it may open up interesting perspectives for other transformation-optics applications, such as hyperlensing\cite{53,54}. In this framework, it should also be emphasized that the class of constitutive relationships in \cite{1} represents only one example of nearly-transparent media amenable to analytic solutions, and exploration of further classes is certainly worth of interest and currently being pursued. In particular, configurations featuring a larger number of parameters may be utilized, together with more sophisticated optimization strategies, in order to achieve broadband or multi-band responses, and/or to enforce constraints in the variation ranges of the constitutive parameters of the transformation medium. Of particular interest are also the “masking” application scenarios\cite{53,54}, where one is interested in changing the scattering signature of an object (e.g., making it appear larger, smaller, or of different shape), and thus the desired scattering response is inherently non-zero. In such scenarios, the virtual-domain medium may offer extra degrees of freedom exploitable for the design of the desired scattering signature.

APPENDIX A: PERTAINING TO EQ. (22)

Particularizing the Helmholtz equation in \cite{21} to the constitutive parameters in \cite{11}, we obtain

\[
\frac{d^2}{dr^2} + \left( \frac{1 + \gamma}{r'} + \alpha \right) \frac{1}{R_2} \frac{d}{dr'} \Psi_m(k_0 r') + \left[ k_0^2 (1-p) + \frac{p k_0^2 R_2}{r'} - \left( \frac{m}{r'} \right)^2 \right] \Psi_m(k_0 r') = 0, \quad (A1)
\]

which, letting \( \tau = \xi r' \) [with \( \xi \) defined in \cite{23,25}], becomes

\[
\frac{d^2}{d\tau^2} + \left( \frac{\alpha}{\xi R_2} + \frac{1 + \gamma}{\tau} \right) \frac{1}{\xi^2} \frac{d}{d\tau} \Psi_m \left( \frac{k_0 \tau}{\xi} \right) + \left[ (1-p) k_0^2 + \frac{p k_0^2 R_2}{\xi^2} - \frac{m^2}{\tau^2} \right] \Psi_m \left( \frac{k_0 \tau}{\xi} \right) = 0. \quad (A2)
\]

and, via the mapping

\[
\Psi_m \left( \frac{k_0 \tau}{\xi} \right) = \tau^{\frac{m}{2}} \exp \left[ -\frac{(\xi R_2 + \alpha) \tau}{2k R_2} \right] \Psi_m (\tau), \quad (A3)
\]
with \( \nu_m \) and \( \zeta_m \) defined in (23a) and (23c), respectively. Equation (13.1.1) is readily recognized to be the Kummer equation [cf. Eq. (13.1.1) in Ref. 50], and admits two independent solutions in terms of confluent hypergeometric functions [cf. Eqs. (13.1.2), (13.1.3) in Ref. 50], from which the final solutions in (22) follow straightforwardly via inverse mapping. Note that, as a possible alternative route, the mapping

\[
\Psi_m(t) = \tau^{-\frac{1}{2}} \exp \left( -\frac{\alpha \tau}{2 \xi R_2} \right) \tilde{\Psi}_m(\tau) \tag{A4}
\]

would lead to the Whittaker equation [cf. Eq. (13.1.31) in Ref. 50].

**APPENDIX B: PERTAINING TO THE APPROXIMATIONS IN (32)**

We start recalling that, for \( r' \to 0 \), the wavefunctions \( \Psi_m^{(1)} \) in (22) are regular, since \( \nu_m \geq \gamma \) [cf. (23a)] and \( M(\zeta_m, \nu_m + 1, 0) = 1 \) [cf. Eq. (13.1.2) in Ref. 50], whereas \( \Psi_m^{(2)} \) are singular, since [cf. Eq. (13.1.3) in Ref. 50]

\[
U(\zeta_m, \nu_m + 1, \xi r') \sim \frac{\pi (\xi r')^{-\nu_m}}{\sin[\pi (\nu_m + 1)] \Gamma(\zeta_m) \Gamma(1 - \nu_m)}, \quad r' \to 0, \tag{B1}
\]

where \( \Gamma(\cdot) \) is the Gamma function [cf. Eq. (6.1.1) in Ref. 50]. We then define

\[
\Delta' = f(R_1 + \Delta), \tag{B2}
\]

where, in view of (13a),

\[
\Delta = g(\Delta') - R_1 \sim \exp(-\alpha) \chi R_2^{\gamma+2} (\Delta')^{\gamma-2}. \tag{B3}
\]

Thus, combining (22), (B1) and (B2), we obtain

\[
\Psi_m^{(1,2)}(k_0 \Delta') \sim s_{1,2} \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma}, \tag{B4a}
\]

\[
\tilde{\Psi}_m^{(1,2)}(k_0 \Delta') \sim s_{1,2} \left( \frac{\pm \nu_m - \gamma}{2} \right) \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma - 2}, \tag{B4b}
\]

where \( s_{1,2} \) are irrelevant constants. Then, by recalling (30) and that

\[
\tilde{\Psi}_m^{(1,2)}(k_0 r') = \frac{\Psi_m^{(1,2)}(k_0 r')}{g(r')}, \tag{B5}
\]

and substituting into (31), we obtain

\[
d_m J_m(k_0 R_1) \sim a_m s_1 \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma} + b_m s_2 \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma - 2}, \tag{B6a}
\]

\[
d_m J_m(k_0 R_1) \sim \left( \frac{\Delta'}{R_2} \right)^{\gamma+1} \left[ a_m \eta_1 \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma - 2} + b_m \eta_2 \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma - 2} \right], \tag{B6b}
\]

where \( \eta_{1,2} \) are other irrelevant constants. Equations (B6) can be readily solved with respect to \( b_m \) and \( d_m \), yielding

\[
b_m \sim a_m \left( \frac{\Delta'}{R_2} \right)^{\gamma} \times \left[ \frac{\eta_1 J_m(k_0 R_1) \left( \frac{\Delta'}{R_2} \right)^{\gamma}}{\xi_1 J_m(k_0 R_1) - \eta_2 J_m(k_0 R_1) \left( \frac{\Delta'}{R_2} \right)^{\gamma}} \right], \tag{B7a}
\]

\[
d_m \sim \frac{a_m (\eta_1 \xi_1 - \eta_2 \xi_2) \left( \frac{\Delta'}{R_2} \right)^{\pm \nu_m - \gamma}}{\xi_1 J_m(k_0 R_1) - \eta_2 J_m(k_0 R_1) \left( \frac{\Delta'}{R_2} \right)^{\gamma}}, \tag{B7b}
\]

from which, neglecting the higher-order terms in \( (\Delta'/R_2) \) and recalling (B3), the approximations in (32) follow straightforwardly.

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