Structural stability for the Forchheimer equations interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^3$

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Abstract

The structural stability for the Forchheimer fluid interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^3$ was studied. We assumed that the nonlinear fluid was governed by the Forchheimer equations in $\Omega_1$, while in $\Omega_2$, we supposed that the flow satisfies the Darcy equations. With the aid of some useful a priori bounds, we were able to demonstrate the continuous dependence results for the Forchheimer coefficient $\lambda$.

MSC: 35B40; 35Q30; 76D05

Keywords: Structural stability; Forchheimer equations; Darcy equations; Interface boundary condition

1 Introduction

Many papers in the literature studied the structural stability for the partial differential equations. They obtained the results of continuous dependence or convergence on the equations. Unlike the traditional stability study, they focused on the changes of the coefficients of the equations. This is to say, the structural stability mainly focuses on changes in the model itself, while the traditional stability focuses on the initial data. For a review of the nature of the structural stability, one could see the monograph of Ames and Straughan [3]. In continuum mechanics problems, it is important to obtain the continuous dependence result on the model itself. This problem is discussed for several different partial differential equations by Hirsch and Smale [8]. We usually want to know if a small change in the constructive coefficient in the equations themselves will lead to drastic changes in the solutions. If the answer is no, we can do further studies. It is very important for us to study the structural stability for the model.

There are many models that have been studied in a porous medium. Nield and Beijan [14] and Straughan [27, 28] discussed these models in their books. The authors of [2, 16, 17] studied these models in an unbounded domain and obtained some Saint-Venant-type results. They mainly focused on the studies of the Brinkman, Darcy, and Forchheimer equations in porous media.
Recently, some authors began to study the structural stability for equations in porous media. They obtained some continuous dependence results. For a review of these papers, one could see Payne and Straughan \cite{19–22}, Scott \cite{23}, Scott and Straughan \cite{24}, Straughan \cite{26}, Ames and Payne \cite{1}, Celebi, Kalantarov and Ugurlu \cite{4, 5}, Franchi and Straughan \cite{6}, Harfash \cite{7}, Kaloni and Guo \cite{9}, Li and Lin \cite{10}, Lin and Payne \cite{11, 12}, Payne, Song and Straughan \cite{18}, and Straughan and Hutter \cite{30}. The Brinkman, Forchheimer, and Darcy equations are widely studied in these papers. They consider only one fluid in the domain. In reality, there typically exists more than one fluid in a domain. It is interesting to study two fluids interfacing with each other in one domain.

In \cite{21}, Payne and Straughan established the structural stability result for the Brinkman–Darcy interfacing equations. They studied the continuous dependence result for the interface boundary coefficient $\alpha$. We change the Brinkman equations to the Forchheimer equations. However, if we use the same method as in \cite{21}, we cannot obtain a similar result. Since the equations do not contain the term $\Delta u$, it is difficult to deal with the nonlinear term $|u_i|u_i$. Recently, in \cite{13} and \cite{25}, the authors studied the structural stability for the Forchheimer–Darcy interfacing problems in a bounded domain. In order to obtain their results, the authors obtained the results $\sup_{[0,\tau]} \|T\|_\infty \leq T_M$ and $\sup_{[0,\tau]} \|S\|_\infty \leq S_M$ for the temperatures $T$ and $S$ using the method proposed by Payne, Rodrigues, and Straughan in \cite{15}. In the present paper, the equations for the temperatures are not the same as in \cite{13} and \cite{25}. We cannot get the same results by using the method proposed in \cite{15}. We must seek a new method to get the results. How to get the maximum estimates and the related bounds for $T$ and $S$ is the biggest innovation of this paper. In our opinion, it is of great significance to study the structural stability for the Forchheimer–Darcy interfacing fluids.

The purpose of this paper is to study the manner in which a solution to a flow in a fluid which borders a porous medium depends on a coefficient in the Forchheimer equation. Thus, let an appropriate part of the plane $z = x_3 = 0$ denote the boundary between a porous medium occupying a bounded region $\Omega_2$ in $\mathbb{R}^3$ and a nonlinear fluid occupying a bounded region $\Omega_1$ in $\mathbb{R}^3$, and the governing equations be Forchheimer equations. We denote the interface by $L$, and further denote the remaining parts of the boundaries of $\Omega_1$ and $\Omega_2$ by $\Gamma_1$ and $\Gamma_2$. We also denote $\partial \Omega_1 = \Gamma_1 \cup L$ and $\partial \Omega_2 = \Gamma_2 \cup L$.

We are interested in the solution of the following initial-boundary value problem. The governing equations for Forchheimer flow are (see \cite{29})

$$\begin{align*}
\frac{\partial u_i}{\partial t} &= -\lambda_i |u_i| u_i - p_j + g_i T, \\
\frac{\partial u_i}{\partial x_i} &= 0, \\
\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \kappa \Delta T + Q,
\end{align*}$$

(1.1)

where $u_i$, $p$, and $T$ are the velocity, pressure, and temperature, $\kappa$ is the thermal diffusivity. Here $g_i(x)$ are gravity vector functions, and $Q(x, t)$ is a prescribed heat source. We assume $g_i$ satisfy $|g| \leq G_1$. Here also $\Delta$ is the Laplace operator.

Equations (1.1) hold in the region $\Omega_1 \times [0, \tau]$, where $\Omega_1$ is a bounded, simply connected, and star-shaped domain with boundary $\partial \Omega_1$ in $\mathbb{R}^3$, and $\tau$ is a given number satisfying $0 \leq \tau < \infty$. 
The Darcy equations governing the flow are (see [27])

\[
\begin{align*}
\rho_i &= -q_i + g_i S, \\
\frac{\partial v_i}{\partial x_i} &= 0, \\
\frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} &= \kappa \Delta S + Q_i,
\end{align*}
\]

(1.2)

where \(v_i\), \(q_i\), and \(S\) are the velocity, pressure, and temperature, while \(Q_i(x, t)\) is a prescribed heat source.

Equations (1.2) hold in the region \(\Omega_2 \times [0, \tau]\), where \(\Omega_2\) is a bounded, simply connected, and star-shaped domain with boundary \(\partial \Omega_2\) in \(\mathbb{R}^3\), and \(\tau\) is a given number satisfying \(0 \leq \tau < \infty\).

We impose the boundary and initial conditions as follows:

\[
\begin{align*}
u_i &\big|_{\partial \Omega_1} = 0, \\T &\big|_{\partial \Omega_1} = T_0(x), \quad (x, t) \in \Omega_1 \times [0, \tau], \\
v_i &\big|_{\partial \Omega_2} = 0, \\S &\big|_{\partial \Omega_2} = S_0(x), \quad (x, t) \in \Omega_2 \times [0, \tau].
\end{align*}
\]

(1.3)

We assume further that

\[
\begin{align*}
u_i(x, 0) &= f_i(x), \\T(x, 0) &= T_0(x), \quad x \in \Omega_1, \\
S(x, 0) &= S_0(x), \quad x \in \Omega_2.
\end{align*}
\]

(1.4)

Finally, the interfacing conditions are taken from [21] as

\[
\begin{align*}
u_3 &\big|_{\Gamma_1} = v_3, \\T &\big|_{\Gamma_1} = S, \\T_3 &\big|_{\Gamma_1} = S_3, \\
q &\big|_{\Gamma_1} = p
\end{align*}
\]

(1.5)

on \(L \times \{ t > 0 \}\).

In the next section, we will derive some a priori bounds which will be used in deriving our main results. In Sect. 3, the convergence results for the Forchheimer coefficient are obtained.

In this present paper, the comma is used to indicate differentiation, and the differentiation with respect to the direction \(x_k\) is denoted as ",\(k\)"; thus \(u_j\) denotes \(\frac{\partial u}{\partial x_j}\). Hence, \(u_{i,j} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}\).

2 A priori bounds

We now begin to derive a priori bounds for both \(T\) and \(S\).

First, we introduce the function \(H\), which takes the same boundary values as \(T\):

\[
\begin{align*}
\Delta H &= H_t, \quad (x, t) \in \Omega_1 \times [0, \tau], \\
H(x, 0) &= T_0(x), \quad x \in \Omega_1, \\
H(x, t) &= T_{1i}(x, t), \quad (x, t) \in \Gamma_1 \times [0, \tau], \\
H_{ij}(x, t) &= T_{1ij}(x, t), \quad (x, t) \in \Gamma_1 \times [0, \tau].
\end{align*}
\]

(2.1)
Next, we introduce the function $I$, which takes the same boundary values as $S$:

$$\begin{align*}
\Delta I &= I, \\ I(x,0) &= S_0(x), \\ I(x,t) &= S_U(x,t), \\ I_t(x,t) &= S_{U,t}(x,t),
\end{align*}$$

(2.2)

On $\Omega \times \{t > 0\}$, we let

$$\begin{align*}
H &= I, \\ H_j &= I_j, \\ H_\eta &= I_\eta.
\end{align*}$$

(2.3)

If we let

$$F = \begin{cases} 
H, & (x,t) \in \Omega_1 \times [0,\tau], \\
I, & (x,t) \in \Omega_2 \times [0,\tau],
\end{cases}$$

(2.4)

we get

$$\begin{align*}
\Delta F &= F, \\
F(x,t) &= \begin{cases} 
T_U(x,t), & (x,t) \in \Gamma_1 \times [0,\tau], \\
S_U(x,t), & (x,t) \in \Gamma_2 \times [0,\tau],
\end{cases} \\
F(x,0) &= \begin{cases} 
T_0(x), & x \in \Omega_1, \\
S_0(x), & x \in \Omega_2.
\end{cases}
\end{align*}$$

(2.5)

If we let

$$F_M = \max \left\{ \sup_{\Omega_1} T_0, \sup_{\Omega_2} S_0, \sup_{\Gamma_1 \times [0,\tau]} T_U, \sup_{\Gamma_2 \times [0,\tau]} S_U \right\},$$

(2.6)

we know by maximum principle that $|F| \leq F_M$.

The following lemmas will be used in deriving our main result.

**Lemma 1** For the temperatures $T$ and $S$, we have the following estimates:

$$\begin{align*}
\int_{\Omega_1} T^2 \, dx + \int_{\Omega_2} S^2 \, dx &+ \kappa \int_0^\tau \int_{\Omega_1} T_j T_{j\eta} \, dx \, d\eta + \kappa \int_0^\tau \int_{\Omega_2} S_j S_{j\eta} \, dx \, d\eta \\
&\leq 4 \int_0^\tau \int_{\Omega_1} T^2 \, dx \, d\eta + 4 \int_0^\tau \int_{\Omega_2} S^2 \, dx \, d\eta + \frac{4F_M^2}{\kappa} \int_0^\tau \int_{\Omega_1} u_j u_{j\eta} \, dx \, d\eta \\
&\quad + \frac{4F_M^2}{\kappa} \int_0^\tau \int_{\Omega_2} v_j v_{j\eta} \, dx \, d\eta + 4 \int_{\Omega_1} H^2 \, dx + 2 \int_{\Omega_1} H^2 \, dx \, d\eta \\
&\quad + 2\kappa \int_0^\tau \int_{\Omega_1} H_{j\eta} H_{j\eta} \, dx \, d\eta + 2 \int_0^\tau \int_{\Omega_1} H_{j\eta} H_{j\eta} \, dx \, d\eta + 4 \int_0^\tau \int_{\Omega_1} Q^2 \, dx \, d\eta
\end{align*}$$
\[ +4 \int_{\Omega_1} I^2 \, dx + 2 \int_0^t \int_{\Omega_1} I^2 \, dx \, d\eta + 2\kappa \int_0^t \int_{\Omega_1} I_j I_j \, dx \, d\eta \]
\[ + 2 \int_0^t \int_{\Omega_2} I_{\eta} I_{\eta} \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} Q_i^2 \, dx \, d\eta, \] (2.7)

**Proof** Multiplying (1.1) by \(2(T-H)\) and integrating over \(\Omega_1 \times [0,t]\), we find

\[ 2 \int_0^t \int_{\Omega_1} u_i T T_j \, dx \, d\eta - 2 \int_0^t \int_{\Omega_1} u_i H T_j \, dx \, d\eta \]
\[ = 2\kappa \int_0^t \int_{\Omega_1} (T-H) \Delta T \, dx \, d\eta + 2 \int_0^t \int_{\Omega_1} (T-H) Q \, dx \, d\eta \]
\[ - 2 \int_0^t \int_{\Omega_1} (T-H) T_j \, dx \, d\eta. \] (2.8)

For the first function on the left-hand side of (2.8), using the divergence theorem and equations (1.4), (2.1), we find

\[ 2 \int_0^t \int_{\Omega_1} u_i T T_j \, dx \, d\eta = \int_0^t \int_{\Omega_1} u_i (T^2) \, dx \, d\eta = \int_0^t \int_{\Omega_1} T^2 \, u_3 h_3^{(1)} \, dS \, d\eta \]
\[ = - \int_0^t \int_{\Omega_1} S^3 \, v_3 h_3^{(2)} \, dS \, d\eta. \] (2.9)

For the second function on the left-hand side of (2.8), we have

\[ 2 \int_0^t \int_{\Omega_1} u_i H T_j \, dx \, d\eta \leq \frac{2F_M^2}{\kappa} \int_0^t \int_{\Omega_1} u_i u_i \, dx \, d\eta + \frac{\kappa}{2} \int_0^t \int_{\Omega_1} T_j T_j \, dx \, d\eta. \] (2.10)

For the first function on the right-hand side of (2.8), using the divergence theorem and equations (1.3), (1.5), and (2.1), we get

\[ 2 \kappa \int_0^t \int_{\Omega_1} (T-H) \Delta T \, dx \, d\eta \]
\[ = 2\kappa \int_0^t \int_{\Omega_1} (T-3h_3^{(1)}) \, dS \, d\eta - 2\kappa \int_0^t \int_{\Omega_1} (H T_3 h_3^{(1)}) \, dS \, d\eta \]
\[ - 2\kappa \int_0^t \int_{\Omega_1} T_j T_j \, dx \, d\eta + 2\kappa \int_0^t \int_{\Omega_1} H_j T_j \, dx \, d\eta \]
\[ \leq -2\kappa \int_0^t \int_{\Omega_1} \Delta h_3 h_3^{(2)} \, dS \, d\eta + 2\kappa \int_0^t \int_{\Omega_1} I S_3 h_3^{(2)} \, dS \, d\eta \]
\[ - \kappa \int_0^t \int_{\Omega_1} T_j T_j \, dx \, d\eta + \kappa \int_0^t \int_{\Omega_1} H_j H_j \, dx \, d\eta. \] (2.11)

For the second function on the right-hand side of (2.8), we get

\[ 2 \int_0^t \int_{\Omega_1} (T-H) Q \, dx \, d\eta \]
\[ \leq 2 \int_0^t \int_{\Omega_1} Q^2 \, dx \, d\eta + \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta + \int_0^t \int_{\Omega_1} H^2 \, dx \, d\eta. \] (2.12)
For the third function on the right-hand side of (2.8), using equations (1.4) and (2.1), we find

\[-2 \int_0^t \int_{\Omega_1} (T - H)T_{,\eta} \, dx \, d\eta \]
\[= 2 \int_0^t \int_{\Omega_1} (T - H)_{,\eta} T \, dx \, d\eta - 2 \int_{\Omega_1} (T - H)T \, dx \]
\[\leq - \int_{\Omega_1} T^2 \, dx - \int_0^t T_{,\eta}^2 \, dx + 2 \int_{\Omega_1} HT \, dx - 2 \int_0^t \int_{\Omega_1} T_{,\eta} T \, dx \, d\eta \]
\[\leq - \int_{\Omega_1} T^2 \, dx - \frac{1}{2} \int_{\Omega_1} T^2 \, dx + 2 \int_{\Omega_1} H^2 \, dx + \int_0^t \int_{\Omega_1} T_{,\eta} H_{,\eta} \, dx \, d\eta \]
\[+ \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta. \quad (2.13)\]

Combining (2.8)–(2.13), we obtain

\[\int_{\Omega_1} T^2 \, dx + \kappa \int_0^t \int_{\Omega_1} T_{,\eta} T_{,\eta} \, dx \, d\eta + 4\kappa \int_0^t \int_L S_3 L_3^{(2)} \, dS \, d\eta \]
\[- 4\kappa \int_0^t \int_L S_3 L_3^{(2)} \, dS \, d\eta \]
\[\leq 2 \int_0^t \int_L S_2 v_3 L_3^{(2)} \, dx \, d\eta + 4 \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta + \frac{4F_M^2 \kappa}{\kappa} \int_0^t \int_{\Omega_1} u_i u_i \, dx \, d\eta \]
\[+ 4 \int_0^t \int_{\Omega_1} Q_i^2 \, dx \, d\eta + 4 \int_0^t \int_{\Omega_1} H^2 \, dx \, d\eta + 2 \int_0^t \int_{\Omega_1} H_{,\eta} H_{,\eta} \, dx \, d\eta \]
\[+ 2\kappa \int_0^t \int_{\Omega_1} H_{,\eta} H_{,\eta} \, dx \, d\eta + \int_0^t H_{,\eta} H_{,\eta} \, dx \, d\eta. \quad (2.14)\]

Similarly, we get

\[\int_{\Omega_2} S^2 \, dx + \kappa \int_0^t \int_{\Omega_2} S_{3,\eta} S_{3,\eta} \, dx \, d\eta - 4\kappa \int_0^t \int_L S_3 L_3^{(2)} \, dS \, d\eta \]
\[+ 4\kappa \int_0^t \int_L S_3 L_3^{(2)} \, dS \, d\eta \]
\[\leq -2 \int_0^t \int_L S_2 v_3 L_3^{(2)} \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} S^2 \, dx \, d\eta + \frac{4F_M^2 \kappa}{\kappa} \int_0^t \int_{\Omega_2} v_i v_i \, dx \, d\eta \]
\[+ 4 \int_0^t \int_{\Omega_2} Q_i^2 \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} I_i^2 \, dx \, d\eta + 2 \int_0^t \int_{\Omega_2} I_i I_i \, dx \, d\eta \]
\[+ 2\kappa \int_0^t \int_{\Omega_2} I_i I_i \, dx \, d\eta + \int_0^t I_i I_i \, dx \, d\eta. \quad (2.15)\]

Combining (2.14) and (2.15), we can get the desired result (2.7). \(\square\)

Lemma 2 If

\[F_1(t) = \int_{\Omega_1} H_{,\eta} H_{,\eta} \, dx + \int_{\Omega_2} I_i I_i \, dx,\]
\[ D_1(t) = \left( \int_{\Omega_1} T_{0 j} T_{0 j} \, dx + \int_{\Omega_2} S_{0 j} S_{0 j} \, dx \right) \]
\[ + \left( \frac{4}{d} + \frac{8}{m} \right) \left( \int_0^t \int_{\Gamma_1} |\nabla \mathcal{H}|^2 \, dS \, d\eta + \int_0^t \int_{\Gamma_2} |\nabla \mathcal{I}|^2 \, dS \, d\eta \right) \]
\[ + \frac{d^2}{4m} \left( \int_0^t \int_{\Gamma_1} (T_{1 j} T_{1 j})^2 \, dS \, d\eta + \int_0^t \int_{\Gamma_2} (S_{1 j} S_{1 j})^2 \, dS \, d\eta \right), \]

we have

\[ F_1(t) \leq D_1(t) + \frac{4}{d^2} \int_0^t \int_0^t D_1(\eta) e^{\frac{4}{d^2} (t - \eta)} \, d\eta \, d\eta = m_1(t), \quad (2.16) \]

where \( m \) and \( d \) are positive constants to be defined later.

**Proof** Start with the identity

\[ 2 \int_{\Omega_1} x_i \mathcal{H} \Delta \mathcal{H} \, dx = 2 \int_{\Omega_1} x_i \mathcal{H} \mathcal{H} \, dx. \quad (2.17) \]

For the first function on the left-hand side of (2.17), using the divergence theorem and equations (2.1), (2.3), we get

\[ 2 \int_{\Omega_1} x_i \mathcal{H} \Delta \mathcal{H} \, dx = 2 \int_{\Gamma_1} x_i \mathcal{H} \mathcal{H} \, dS + 2 \int_{\Gamma_2} x_i \mathcal{H} \mathcal{H} (1) \mathcal{S} \, dS \]
\[ - 2 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx - 2 \int_{\Omega_1} x_i \mathcal{H} \mathcal{H} \, dx \]
\[ = 2 \int_{\Gamma_1} x_i \mathcal{H} \mathcal{H} \, dS - 2 \int_{\Gamma_2} x_i \mathcal{H} \mathcal{H} \, dS \]
\[ - 2 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx - 2 \int_{\Omega_1} x_i \mathcal{H} \mathcal{H} \, dx. \quad (2.18) \]

For the fourth function on the right-hand side of (2.18), using the divergence theorem and equations (2.1), (2.3), we get

\[ -2 \int_{\Omega_1} x_i \mathcal{H} \mathcal{H} \, dx = 3 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx - \int_{\Gamma_1} x_i \mathcal{H} \mathcal{H} n_i \, dS - \int_{\Gamma_2} \mathcal{H} \mathcal{H} x_3 n_3 (1) \, dS \]
\[ = 3 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx - \int_{\Gamma_1} x_i \mathcal{H} \mathcal{H} n_i \, dS + \int_{\Gamma_2} l_i l_j x_3 n_3 (2) \, dS. \quad (2.19) \]

For the first function on the right-hand side of (2.17), we get

\[ 2 \int_{\Omega_1} x_i \mathcal{H} \mathcal{H} \, dx \leq 2 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx + \frac{1}{2} \int_{\Omega_1} x_i x_i \mathcal{H} \mathcal{H} \, dx \]
\[ \leq 2 \int_{\Omega_1} \mathcal{H} \mathcal{H} \, dx + \frac{d^2}{2} \int_{\Omega_1} \mathcal{E} \mathcal{E} \, dx, \quad (2.20) \]

where \( d^2 = \max_{\Omega} x_i x_i. \)
Combining (2.17)–(2.20), we obtain
\[
2 \int_{\Gamma_1} x_i H_j H_j n_j \, ds - \int_{\Gamma_1} x_i H_j H_j n_i \, ds + \int_L I_j L_j x_3 n_3^{(2)} \, ds \\
\leq \int_{\Omega_1} H_j H_j \, dx + \frac{d^2}{2} \int_{\Omega_1} H_j H_j \, dx + 2 \int_L I_j L_j x_3 n_3^{(2)} \, ds.
\] (2.21)

Similarly, we get
\[
2 \int_{\Gamma_2} x_i L_j L_j n_j \, ds - \int_{\Gamma_2} x_i L_j L_j n_i \, ds - \int_L I_j L_j x_3 n_3^{(2)} \, ds \\
\leq \int_{\Omega_2} I_j I_j \, dx + \frac{d^2}{2} \int_{\Omega_2} I_j I_j \, dx - 2 \int_L I_j L_j x_3 n_3^{(2)} \, ds.
\] (2.22)

Combining (2.21) and (2.22), we obtain
\[
2 \int_{\Gamma_1} x_i H_j H_j n_j \, ds + 2 \int_{\Gamma_2} x_i L_j L_j n_j \, ds - \int_{\Gamma_1} x_i H_j H_j n_i \, ds - \int_{\Gamma_2} x_i L_j L_j n_i \, ds \\
\leq \int_{\Omega_1} H_j H_j \, dx + \int_{\Omega_2} I_j I_j \, dx + \frac{d^2}{2} \int_{\Omega_1} H_j H_j \, dx + \frac{d^2}{2} \int_{\Omega_2} I_j I_j \, dx.
\] (2.23)

Since
\[
H_j = \frac{\partial H}{\partial n} n_i + s_i \nabla H, \quad I_j = \frac{\partial I}{\partial n} n_i + s_i \nabla I,
\] (2.24)

where \( n \) and \( s \) are the normal and tangential vectors to \( \partial \Omega \), respectively, and \( \nabla H \) and \( \nabla I \) are the tangential derivatives, we have
\[
\int_{\Gamma_1} x_i n_i \left( \frac{\partial H}{\partial n} \right)^2 \, ds + \int_{\Gamma_2} x_i n_i \left( \frac{\partial I}{\partial n} \right)^2 \, ds \\
\leq \int_{\Gamma_1} x_i n_i |\nabla H|^2 \, ds - 2 \int_{\Gamma_2} x_i n_i |\nabla I|^2 \, ds + \int_{\Gamma_2} x_i n_i |\nabla I|^2 \, ds \\
- 2 \int_{\Gamma_2} x_i s_i \nabla H \frac{\partial H}{\partial n} \, ds + \int_{\Omega_1} H_j H_j \, dx + \int_{\Omega_2} I_j I_j \, dx \\
+ \frac{d^2}{2} \int_{\Omega_1} H_j H_j \, dx + \frac{d^2}{2} \int_{\Omega_2} I_j I_j \, dx.
\] (2.25)

We know \( \Omega \) is star-shaped with respect to the region and, setting \( m = \min_{\Omega} x_i n_i > 0 \), we then obtain
\[
m \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds + m \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds \\
\leq \left( d + \frac{2d^2}{m} \right) \int_{\Gamma_1} |\nabla H|^2 \, ds + \frac{m}{2} \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds \\
+ \left( d + \frac{2d^2}{m} \right) \int_{\Gamma_2} |\nabla I|^2 \, ds + \frac{m}{2} \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds + \int_{\Omega_1} H_j H_j \, dx \\
+ \int_{\Omega_2} I_j I_j \, dx + \frac{d^2}{2} \int_{\Omega_1} H_j H_j \, dx + \frac{d^2}{2} \int_{\Omega_2} I_j I_j \, dx.
\] (2.26)
Multiplying (2.1) by $2H$, and integrating over $\Omega$, we find

$$
2 \int_{\Omega} H, H, dx = 2 \int_{\Omega} H, \Delta H, dx = 2 \int_{\Gamma_1} T_{\Omega,1} \frac{\partial H}{\partial n} dS + 2 \int_{L} H_{\Omega,1} \frac{\partial H}{\partial n}^{(1)} dS - 2 \int_{\Omega} H, H, dx
$$

$$
\leq \frac{d^2}{2m} \int_{\Gamma_1} (T_{\Omega,1})^2 dS + \frac{2m}{d^2} \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 dS - 2 \int_{L} I, I, H_{\Omega,1}^{(2)} dS
$$

$$
- \frac{d}{dt} \int_{\Omega} H, H, dx.
$$

(2.27)

Similarly, we get

$$
2 \int_{\Omega} I, I, dx \leq \frac{d^2}{2m} \int_{\Gamma_2} (S_{\Omega,2})^2 dS + \frac{2m}{d^2} \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 dS
$$

$$
+ 2 \int_{L} I, I, I_{\Omega,2}^{(2)} dS - \frac{d}{dt} \int_{\Omega} I, I, dx.
$$

(2.28)

Combining (2.26)–(2.28), we obtain

$$
\frac{d}{dt} \int_{\Omega} H, H, dx + \frac{d}{dt} \int_{\Omega} I, I, dx
$$

$$
\leq \frac{4}{d^2} \left( \int_{\Omega} H, H, dx + \int_{\Omega} I, I, dx \right) + \left( \frac{4}{d} + \frac{8}{m} \right) \left( \int_{\Gamma_1} |\nabla H|^2 dS + \int_{\Gamma_2} |\nabla I|^2 dS \right)
$$

$$
+ \frac{d^2}{4m} \left( \int_{\Gamma_1} (T_{\Omega,1})^2 dS + \int_{\Gamma_2} (S_{\Omega,2})^2 dS \right).
$$

(2.29)

Therefore, integrating (2.29) yields

$$
F_1(t) \leq D_1(t) + \frac{4}{d^2} \int_0^t F_1(\eta) d\eta.
$$

(2.30)

Gronwall inequality now implies (2.16).

Lemma 3 For the functions $H$ and $I$, we have the following estimates:

$$
\int_0^t \int_{\Omega} H, H, dx d\eta + \int_0^t \int_{\Omega} I, I, dx d\eta \leq m_3(t),
$$

(2.31)

where

$$
m_3(t) = \frac{d^2 m(t)}{2m} + \frac{1}{2} (\int_{\Omega} H, H, dx + \int_{\Omega} I, I, dx) + \frac{m}{d^2} \left( \int_0^t \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 dS d\eta \right)
$$

$$
+ \int_{\Gamma_1} (T_{\Omega,1})^2 dS + \int_{\Gamma_2} (S_{\Omega,2})^2 dS d\eta).
$$
Proof. Multiplying (2.1) by $2H_x$ and integrating over $\Omega_1$, we find

$$2 \int_{\Omega_1} H_xH_x \, dx = 2 \int_{\Omega_1} H_x\Delta H \, dx$$

$$= 2 \int_{\Gamma_1} T_{11} \frac{\partial H}{\partial n} \, ds + 2 \int_L H_xH_3 \nu^{(3)}_3 \, ds - 2 \int_{\Omega_1} H_xH_x \, dx$$

$$\leq \frac{d^2}{m} \int_{\Gamma_1} (T_{11})^2 \, ds + \frac{m}{d^2} \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds$$

$$- 2 \int_L I_4I_3\nu^{(2)}_3 \, ds - \frac{d}{dt} \int_{\Omega_1} H_xH_x \, dx. \quad (2.32)$$

Similarly, we get

$$2 \int_{\Omega_2} I_4I_4 \, dx$$

$$\leq \frac{d^2}{m} \int_{\Gamma_2} (S_{11})^2 \, ds + \frac{m}{d^2} \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds + 2 \int_L I_4I_4\nu^{(2)}_4 \, ds - \frac{d}{dt} \int_{\Omega_2} I_4I_4 \, dx. \quad (2.33)$$

Combining (2.26), (2.32), and (2.33), we obtain

$$\left( \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds + \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds \right) + \frac{d^2}{m} \left( \int_{\Omega_1} H_xH_x \, dx + \int_{\Omega_2} I_4I_4 \, dx \right)$$

$$\leq \frac{4}{m} \left( \int_{\Omega_1} H_xH_x \, dx + \int_{\Omega_2} I_4I_4 \, dx \right) + \frac{d^4}{m^2} \left( \int_{\Gamma_1} (T_{11})^2 \, ds + \int_{\Gamma_2} (S_{11})^2 \, ds \right)$$

$$+ \left( \frac{4d}{m} + \frac{8d^2}{m^2} \right) \left( \int_{\Gamma_1} \left| \nabla_xH \right|^2 \, ds + \int_{\Gamma_2} \left| \nabla_xI \right|^2 \, ds \right). \quad (2.34)$$

Therefore, integrating (2.34) yields

$$\left( \int_0^t \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds \, d\eta + \int_0^t \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds \, d\eta \right)$$

$$\leq \frac{4}{m} \left( \int_0^t m_1(\eta) \, d\eta + \left( \frac{4d}{m} + \frac{8d^2}{m^2} \right) \left( \int_0^t \int_{\Gamma_1} \left| \nabla_xH \right|^2 \, ds \, d\eta + \int_0^t \int_{\Gamma_2} \left| \nabla_xI \right|^2 \, ds \, d\eta \right) \right. \right.$$

$$\left. + \frac{d^4}{m^2} \left( \int_0^t \int_{\Gamma_1} (T_{11})^2 \, ds \, d\eta + \int_0^t \int_{\Gamma_2} (S_{11})^2 \, ds \, d\eta \right) \right)$$

$$+ \frac{d^2}{m} \left( \int_{\Omega_1} T_{00}I_4 \, dx + \int_{\Omega_2} S_{00}I_4 \, dx \right) = m_2(t). \quad (2.35)$$

Combining (2.32) and (2.33), we obtain

$$\int_{\Omega_1} H_xH_x \, dx + \int_{\Omega_2} I_4I_4 \, dx + \frac{1}{2} \left( \frac{d}{dt} \int_{\Omega_1} H_xH_x \, dx + \frac{d}{dt} \int_{\Omega_2} I_4I_4 \, dx \right)$$

$$\leq \frac{d^2}{2m} \left( \int_{\Gamma_1} (T_{11})^2 \, ds + \int_{\Gamma_2} (S_{11})^2 \, ds \right)$$

$$+ \frac{m}{2d^2} \left( \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, ds + \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, ds \right). \quad (2.36)$$

Therefore, integrating (2.36) yields the desired result (2.31). \qed
Lemma 4 For the functions \( H \) and \( I \), we have the following estimates:

\[
\int_{\Omega_1} H^2 \, dx + \int_{\Omega_2} I^2 \, dx \leq m_4(t),
\]

(2.37)

with \( m_4(t) = \int_{\Omega_1} T_0^2 \, dx + \int_{\Omega_2} S_0^2 \, dx + \int_0^t \left( \int_{\Gamma_1} T_0^2 \, d\Sigma + \int_{\Gamma_2} S_0^2 \, d\Sigma \right) \, d\eta + \int_0^t \left( \int_{\Omega_1} \partial H \cdot \nu \, dS + \int_{\Omega_2} \partial I \cdot \nu \, dS \right) \, d\eta + m_2(t). \)

Proof Multiplying (2.1) by \( 2H \) and integrating over \( \Omega_1 \), we find

\[
\frac{d}{dt} \int_{\Omega_1} H^2 \, dx = 2 \int_{\Omega_1} HH_x \, dx = 2 \int_{\Omega_1} H \Delta H \, dx
\]

\[
= 2 \int_{\Gamma_1} H \frac{\partial H}{\partial n} \, dS + 2 \int_{\Gamma_2} HH_3 n_3^{(1)} \, dS - 2 \int_{\Omega_1} H_x H_2 \, dx
\]

\[
\leq \int_{\Gamma_1} T_0^2 \, dS + \int_{\Gamma_2} S_0^2 \, dS + \left( \frac{\partial H}{\partial n} \right)^2 \, dS - 2 \int_{\Omega_1} H_3 n_3^{(2)} \, dS.
\]

(2.38)

Similarly, we get

\[
\frac{d}{dt} \int_{\Omega_2} I^2 \, dx \leq \int_{\Gamma_2} S_0^2 \, dS + \int_{\Gamma_1} \left( \frac{\partial I}{\partial n} \right)^2 \, dS + 2 \int_{\Omega_1} H_3 n_3^{(2)} \, dS.
\]

(2.39)

Combining (2.38) and (2.39), we obtain

\[
\frac{d}{dt} \int_{\Omega_1} H^2 \, dx + \frac{d}{dt} \int_{\Omega_2} I^2 \, dx
\]

\[
\leq \int_{\Gamma_1} T_0^2 \, dS + \int_{\Gamma_2} S_0^2 \, dS + \int_{\Gamma_1} \left( \frac{\partial H}{\partial n} \right)^2 \, dS + \int_{\Gamma_2} \left( \frac{\partial I}{\partial n} \right)^2 \, dS.
\]

(2.40)

Therefore, integrating (2.40) yields the desired result (2.37).

\[\Box\]

Lemma 5 For the temperatures \( T \) and \( S \), we have the following estimates:

\[
\int_{\Omega_1} T^2 \, dx + \int_{\Omega_2} S^2 \, dx + \kappa \int_0^t \int_{\Omega_1} T_x T_x \, dx \, d\eta + \kappa \int_0^t \int_{\Omega_2} S_x S_x \, dx \, d\eta
\]

\[
\leq 4 \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} S^2 \, dx \, d\eta + \frac{4F_M^2}{\kappa} \int_0^t \int_{\Omega_2} u_i u_i \, dx \, d\eta
\]

\[
+ \frac{4F_M^2}{\kappa} \int_0^t \int_{\Omega_2} v_i v_i \, dx \, d\eta + 2 \int_0^t m_4(\eta) \, d\eta
\]

\[
+ 2\kappa \int_0^t m_1(\eta) \, d\eta + 2m_3(t) + 4 \int_0^t \int_{\Omega_1} Q^2 \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} Q^2 \, dx \, d\eta.
\]

(2.41)

Proof A combination of (2.7), (2.16), (2.31), and (2.37) leads to the desired result (2.41).

\[\Box\]

Lemma 6 For the solutions \((u_i, T)\) and \((v_i, S)\) of equations (1.1) and (1.2), if we let

\[
F_3(t) = \int_{\Omega_1} T^2 \, dx + \int_{\Omega_2} S^2 \, dx + \int_{\Omega_2} u_i u_i \, dx, \quad m_5 = \max \{4 + G_1^2 + \frac{4F_M^2}{\kappa} G_1^2 + 2 + \frac{4F_M^2}{\kappa} \}, \quad D_3(t) = (1 + \frac{4F_3^2}{\kappa} G_2^2) + \frac{4F_3^2}{\kappa} G_2^2 + 2 + \frac{4F_3^2}{\kappa} (1 + D_2(t) = (1 + \frac{4F_3^2}{\kappa} G_2^2) f_{ij} \, dx + m_4(t) + 2 \int_0^t m_4(\eta) \, d\eta + 2\kappa \int_0^t m_1(\eta) \, d\eta + 2m_3(t) + 4 \int_0^t \int_{\Omega_1} Q^2 \, dx \, d\eta +
\]

\[
+ 2\kappa \int_0^t m_1(\eta) \, d\eta + 2m_3(t) + 4 \int_0^t \int_{\Omega_1} Q^2 \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} Q^2 \, dx \, d\eta.
\]
\[ 4 \int_\Omega Q^2 \, dx \, d\eta, \] we get

\[ F_2(t) \leq D_2(t) + m_5 e^{m_5 t} \int_0^t D_2(\eta) e^{-m_5 \eta} \, d\eta = m_6(t), \] (2.42)

\[ \int_0^t \int_\Omega |u|^3 \, dx \, d\eta \leq \frac{G_1^2 + 1 + |G_1^2 - 1|}{4\lambda} m_6(t) + \frac{1}{2\lambda} \int_\Omega f_\delta \, dx = \frac{m_7(t)}{\lambda}, \] (2.43)

\[ \int_0^t \int_\Omega T_j T_j \, dx \, d\eta + \int_0^t \int_\Omega S_i S_i \, dx \, d\eta \leq \frac{1}{\kappa} D_3(t) + \frac{m_5}{\kappa} \int_0^t m_6(\eta) \, d\eta = m_8(t), \] (2.44)

and

\[ \int_0^t \int_\Omega v_i v_i \, dx \, d\eta \leq \frac{G_1^2 + 1 + |G_1^2 - 1|}{2} \int_0^t m_6(\eta) \, d\eta + \int_\Omega f_\delta \, dx = m_9(t), \] (2.45)

where \( m_7(t) = \frac{G_1^2 + 1 + |G_1^2 - 1|}{4} m_6(t) + \frac{1}{2} \int_\Omega f_\delta \, dx. \)

Proof. Multiplying \((1.1)\) by \(2u_i\) and integrating over \(\Omega_1\), we find

\[ \frac{d}{dt} \int_\Omega u_i u_i \, dx = 2 \int_\Omega u_i u_i \, dx + 2 \int_\Omega p_i u_i \, dx \]

\[ = -2\lambda \int_\Omega |u| |u| u_i \, dx - 2 \int_\Omega p_i u_i \, dx + 2 \int_\Omega g_i T u_i \, dx. \] (2.46)

For the second function on the right-hand side of \((2.46)\), using the divergence theorem and equations \((1.3), (1.5)\), we get

\[ -2 \int_\Omega p_i u_i \, dx = -2 \int_\Omega p u_3 u_3^{(1)} \, dS = 2 \int_\Omega q v_i u_3^{(2)} \, dS = 2 \int_\Omega q_i v_i \, dx. \] (2.47)

If we insert \((1.2)\) and \((2.47)\) into \((2.46)\), we get

\[ \frac{d}{dt} \int_\Omega u_i u_i \, dx + 2\lambda \int_\Omega |u| |u| u_i \, dx \]

\[ \leq 2 \int_\Omega q_i v_i \, dx + 2 \int_\Omega g_i T u_i \, dx \]

\[ \leq 2 \int_\Omega (g_i S - v_i) v_i \, dx + 2 \int_\Omega g_i g_i T^2 \, dx + \int_\Omega u_i u_i \, dx \]

\[ \leq 2 \left[ \int_\Omega g_i S^2 \, dx + G_1 \int_\Omega T^2 \, dx + \int_\Omega u_i u_i \, dx \right] \]

\[ \leq 2 \left[ \frac{G_1^2}{2} \int_\Omega S^2 \, dx + G_1^2 \int_\Omega T^2 \, dx + \int_\Omega u_i u_i \, dx \right]. \] (2.48)

Therefore, integrating \((2.48)\) yields

\[ \int_\Omega u_i u_i \, dx \leq G_1^2 \int_0^t \int_\Omega T^2 \, dx \, d\eta + \frac{1}{2} G_1^2 \int_0^t \int_\Omega S^2 \, dx \, d\eta \]

\[ + \int_0^t \int_\Omega u_i u_i \, dx \, d\eta + \int_\Omega f_\delta \, dx. \] (2.49)
Similarly, we get
\[
\int_0^t \int_{\Omega_2} v_i v_i \, dx \, d\eta \leq G^2_1 \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta + G^2_1 \int_0^t \int_{\Omega_2} S^2 \, dx \, d\eta \\
+ \int_0^t \int_{\Omega_1} u_i u_i \, dx \, d\eta + \int_{\Omega_1} f_i f_i \, dx.
\] (2.50)

Combining (2.41), (2.49), and (2.50), we obtain
\[
\int_{\Omega_1} T^2 \, dx + \int_{\Omega_2} S^2 \, dx + \int_{\Omega_1} u_i u_i \, dx + \kappa \int_0^t \int_{\Omega_1} T_i T_i \, dx \, d\eta + \kappa \int_0^t \int_{\Omega_2} S_i S_i \, dx \, d\eta \\
\leq \left(4 + G^2_1 + \frac{4}{\kappa} F^2 M G^2_1\right) \int_0^t \int_{\Omega_1} T^2 \, dx \, d\eta + \left(4 + \frac{1}{2} G^2_1 + \frac{4}{\kappa} F^2 M G^2_1\right) \int_0^t \int_{\Omega_2} S^2 \, dx \, d\eta \\
+ \left(2 + \frac{8}{\kappa} F^2 M\right) \int_0^t \int_{\Omega_1} u_i u_i \, dx \, d\eta + \left(2 + \frac{4}{\kappa} F^2 M\right) \int_0^t \int_{\Omega_1} f_i f_i \, dx + m_4(t) \\
+ 2 \int_0^t m_4(\eta) \, d\eta + 2\kappa \int_0^t m_1(\eta) \, d\eta + 2m_3(t) + 4 \int_0^t \int_{\Omega_1} Q^2 \, dx \, d\eta \\
+ 4 \int_0^t \int_{\Omega_2} Q^2 \, dx \, d\eta.
\] (2.51)

We can get
\[
F_2(t) \leq D_2(t) + m_5 \int_0^t F_2(\eta) \, d\eta.
\] (2.52)

Gronwall inequality now implies the desired result (2.42). \(\square\)

Similarly, we can also get the desired result (2.43).

Combining (2.51) and (2.42), we obtain the desired result (2.44).

Combining (2.50) and (2.42), we obtain the desired result (2.45).

**Lemma 7** For the temperatures \(T\) and \(S\), we have the following estimates:
\[
\max \left\{ \sup_{[0, \tau]} \| T \|_{\infty}, \sup_{[0, \tau]} \| S \|_{\infty} \right\} \leq e^{2r} \max \left\{ \sup_{[0, \tau]} \| Q \|_{\infty}, \sup_{[0, \tau]} \| Q_i \|_{\infty}, F_M \right\} = N_M.
\] (2.53)

**Proof** Multiplying (1.1)3 by \(2r(T^{2r-1} - H^{2r-1})\) and integrating over \(\Omega_1 \times [0, t]\), (where \(r > 2\)), we find
\[
2r \int_0^t \int_{\Omega_1} u_i T^{2r-1} T_i \, dx \, d\eta - 2r \int_0^t \int_{\Omega_1} u_i H^{2r-1} T_i \, dx \, d\eta \\
- 2r \int_0^t \int_{\Omega_1} (T^{2r-1} - H^{2r-1}) Q \, dx \, d\eta \\
= 2r \kappa \int_0^t \int_{\Omega_1} (T^{2r-1} - H^{2r-1}) \Delta T \, dx \, d\eta \\
- 2r \int_0^t \int_{\Omega_1} (T^{2r-1} - H^{2r-1}) T_i \, dx \, d\eta.
\] (2.54)
For the first function on the right-hand side of (2.54), using the divergence theorem and equations (1.3), (1.5), we get

\[
2\kappa \int_0^t \int_{\Omega_1} (T^{2r-1} - H^{2r-1}) \Delta T \, dx \, d\eta
\]
\[
= 2\kappa \int_0^t \int_L T^{2r-1} \, T_3 \, \eta^3 \, dS \, d\eta - 2\kappa \int_0^t \int_{\Omega_1} H^{2r-1} \, T_3 \, \eta^3 \, dS \, d\eta
\]
\[
- \frac{2\kappa(2r-1)}{r} \int_0^t \int_{\Omega_1} (T^r)^j (T^r)^j \, dx \, d\eta + 2\kappa(2r-1) \int_0^t \int_{\Omega_1} H^{2r-2} H_j T_j \, dx \, d\eta
\]
\[
\leq -2\kappa \int_0^t \int_L S^{2r-1} S_3 \eta^2 \, dS \, d\eta + 2\kappa \int_0^t \int_{\Omega_1} T^{2r-1} S_3 \eta^3 \, dS \, d\eta
\]
\[
+ \kappa(2r-1) F^{2r-2}_M \int_0^t m_1(\eta) \, d\eta + \kappa(2r-1) F^{2r-2}_M m_8(t). \quad (2.55)
\]

For the second function on the left-hand side of (2.54), using the divergence theorem and equations (1.4), (2.1), we find

\[
2 \left| \int_0^t \int_{\Omega_1} u_i T^{2r-1} \, dx \, d\eta \right| = \int_0^t \int_{\Omega_1} u_i (T^{2r})^j \, dx \, d\eta = \int_0^t \int_{\Omega_1} T^{2r} u_3 \eta^3 \, dS \, d\eta
\]
\[
= - \int_0^t \int_L S^{2r} v_3 \eta^2 \, dS \, d\eta. \quad (2.56)
\]

For the second function on the left-hand side of (2.54), we get

\[
2 \left| \int_0^t \int_{\Omega_1} u_i H^{2r-1} T_j \, dx \, d\eta \right|
\]
\[
\leq 2 F^{2r-1}_M \int_0^t \int_{\Omega_1} u_i T_j \, dx \, d\eta
\]
\[
\leq F^{2r-1}_M \int_0^t \int_{\Omega_1} u_i u_i \, dx \, d\eta + F^{2r-1}_M \int_0^t \int_{\Omega_1} T_i T_j \, dx \, d\eta
\]
\[
\leq F^{2r-1}_M \int_0^t m_6(\eta) \, d\eta + F^{2r-1}_M m_8(t). \quad (2.57)
\]

For the third function on the left-hand side of (2.54), using Young inequality, we get

\[
2 \left| \int_0^t \int_{\Omega_1} (T^{2r-1} - H^{2r-1}) Q \, dx \, d\eta \right| \leq 2 \int_0^t \int_{\Omega_1} Q^{2r} \, dx \, d\eta
\]
\[
+ (2r-1) \int_0^t \int_{\Omega_1} T^{2r} \, dx \, d\eta
\]
\[
+ (2r-1) \int_0^t \int_{\Omega_1} H^{2r} \, dx \, d\eta. \quad (2.58)
\]
Similarly, we get

\[
\int_{\Omega_2} S^2 dx + 2 \int_{\Omega_2} S^{2r-1} S_3 n_3^2 dS d\eta - 4r \int_{\Omega_2} \int_{\Omega_2} S^{2r-1} S_3 n_3^2 dS d\eta
\]
\[+ 4r \int_{\Omega_2} \int_{\Omega_2} I^{2r-1} S_3 n_3^2 dS d\eta\]
\[\leq (4r - 2) \int_{\Omega_2} \int_{\Omega_2} S^2 dx d\eta + (4r - 2) \int_{\Omega_2} \int_{\Omega_2} I^2 dx d\eta + 4 \int_{\Omega_2} \int_{\Omega_2} Q^2 dx d\eta\]
\[+ (4r - 2) \int_{\Omega_2} \int_{\Omega_2} I^{2r-2} m_3(t) + 2r(2r - 1) I^{2r-2} m_3(t)\]
\[+ 2r(2r - 1) F_M^{2r-2} m_3(t) + 2r(2r - 1) F_M^{2r-2} m_3(t)\]
\[+ \left[ 2 F_M^{2r-1} + 2r(2r - 1) F_M^{2r-2} \right] m_8(t) + 2 F_M^{2r-1} m_8(t) . \]  
(2.61)
Combining (2.60) and (2.61), we get

\[
\int_{\Omega_1} T^{2r} \, dx + \int_{\Omega_2} S^{2r} \, dx \\
\leq (4r - 2) \int_0^t \int_{\Omega_1} T^{2r} \, dx \, d\eta + 4 \int_0^t \int_{\Omega_1} Q^{2r} \, dx \, d\eta \\
+ (4r - 2) 2 \pi r \int_{\Omega_1} H^{2r} \, dx + (4r - 2) \int_0^t \int_{\Omega_1} H^{2r} \, dx \, d\eta \\
+ (4r - 2) \int_0^t \int_{\Omega_2} S^{2r} \, dx \, d\eta + 4 \int_0^t \int_{\Omega_2} Q^{2r} \, dx \, d\eta + (4r - 2) 2 \pi r \int_{\Omega_2} I^{2r} \, dx \\
+ (4r - 2) \int_0^t I^{2r} \, dx \, d\eta + 4 r (2r - 1) F^{2r-2}_M \int_0^t m_1(\eta) \, d\eta \\
+ 4r(2r - 1) F^{2r-2}_M m_3(t) + 2F^{2r-1}_M m_9(t) \\
+ [2F^{2r-1}_M + 4r(2r - 1)F^{2r-2}_M] \int_0^t m_6(\eta) \, d\eta \\
+ [4F^{2r-1}_M + 4r(2r - 1)F^{2r-2}_M] m_8(t). \tag{2.62}
\]

Letting

\[
F_3(t) = \int_{\Omega_1} T^{2r} \, dx + \int_{\Omega_2} S^{2r} \, dx,
\]

\[
D_3(t) = 4 \int_0^t \int_{\Omega_1} Q^{2r} \, dx \, d\eta + (4r - 2) 2 \pi r \int_{\Omega_1} H^{2r} \, dx + (4r - 2) \int_0^t \int_{\Omega_1} H^{2r} \, dx \, d\eta \\
+ 4 \int_0^t \int_{\Omega_2} Q^{2r} \, dx \, d\eta + (4r - 2) 2 \pi r \int_{\Omega_2} I^{2r} \, dx + (4r - 2) \int_0^t \int_{\Omega_2} I^{2r} \, dx \, d\eta \\
+ 4r(2r - 1) F^{2r-2}_M \int_0^t m_1(\eta) \, d\eta + 4r(2r - 1) F^{2r-2}_M m_3(t) + 2F^{2r-1}_M m_9(t) \\
+ [2F^{2r-1}_M + 4r(2r - 1)F^{2r-2}_M] \int_0^t m_6(\eta) \, d\eta + [4F^{2r-1}_M + 4r(2r - 1)F^{2r-2}_M] m_8(t),
\]

we get

\[
F_3(t) \leq D_3(t) + (4r - 2) \int_0^t F_3(\eta) \, d\eta. \tag{2.63}
\]

Gronwall inequality now implies

\[
\int_0^t F_3(\eta) \, d\eta \leq \int_0^t D_3(\eta) e^{(4r-2)(t-\eta)} \, d\eta \leq e^{(4r-2)t} \int_0^t D_3(\eta) \, d\eta. \tag{2.64}
\]

Raising to the power of \(\frac{1}{r}\) both sides of (2.64), we have

\[
\left[ \int_0^t F_3(\eta) \, d\eta \right]^\frac{1}{r} \leq \left[ e^{(4r-2)t} \right]^\frac{1}{r} \left[ \int_0^t D_3(\eta) \, d\eta \right]^\frac{1}{r}. \tag{2.65}
\]
From the definition of $F_3(t)$, we have

$$\max \left\{ \left( \int_0^t \int_{\Omega} T^2 r \, dx \, d\eta \right)^{\frac{1}{2}}, \left( \int_0^t \int_{\Omega} S^2 r \, dx \, d\eta \right)^{\frac{1}{2}} \right\} \leq \left[ \int_0^t F_3(\eta) \, d\eta \right]^{\frac{1}{2}} \leq \left[ e^{(4r-2)t} \int_0^t D_3(\eta) \, d\eta \right]^{\frac{1}{2}}. \quad (2.66)$$

Using the facts

$$\lim_{r \to \infty} \left( \int_0^t \int_{\Omega} T^2 r \, dx \, d\eta \right)^{\frac{1}{2}} = \sup_{[0,\tau]} \| T \|_{\infty},$$

$$\lim_{r \to \infty} \left( \int_0^t \int_{\Omega} S^2 r \, dx \, d\eta \right)^{\frac{1}{2}} = \sup_{[0,\tau]} \| S \|_{\infty},$$

and the equality

$$\lim_{n \to \infty} \left( a_1^n + a_2^n + \cdots + a_p^n \right)^{\frac{1}{n}} = \max \{ a_1^n, a_2^n, a_3^n, \ldots, a_p^n \},$$

with $a_1, a_2, \ldots, a_p$ all nonnegative constants, we can get the desired result (2.53). \hfill \Box

### 3 Continuous dependence results for the Forchheimer coefficient $\lambda$

In this section, we will discuss the continuous dependence on the Forchheimer coefficient $\lambda$. Let $(u_i, T, p)$ and $(v_i, S, q)$ be the solutions of (1.1)–(1.5) with $\lambda = \lambda_1$. Similarly, we set $(u_i^*, T^*, p^*)$ and $(v_i^*, S^*, q^*)$ to be the solutions of (1.1)–(1.5) with $\lambda = \lambda_2$.

We define $\omega_i = u_i - u_i^*, \theta = T - T^*, \pi = p - p^*, \hat{\lambda} = \lambda_1 - \lambda_2$, and $\omega_m = v_i - v_i^*, \theta_m = S - S^*, \pi_m = q - q^*$.

We find that $(\omega_i, \theta, \pi)$ satisfy the following equations:

\[
\begin{align*}
\frac{\partial \omega_i}{\partial t} &= -(\lambda_1 |u_i| u_i - \lambda_2 |u^*_i| u^*_i) - \pi_j + g_i \theta, \\
\frac{\partial \omega_i}{\partial x_j} &= 0, \\
\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} + \omega_i \frac{\partial T^*}{\partial x_i} &= \kappa \Delta \theta, \quad (3.1)
\end{align*}
\]

and $(\omega_m^m, \theta_m^m, \pi_m^m)$ satisfy

\[
\begin{align*}
\omega_m^m &= -\pi_m^m + g_m \theta, \\
\frac{\partial \omega_m^m}{\partial x_i} &= 0, \\
\frac{\partial \theta_m^m}{\partial t} + v_i \frac{\partial \theta_m^m}{\partial x_i} + \omega_m^m \frac{\partial S^*}{\partial x_i} &= \kappa \Delta \theta_m^m. \quad (3.2)
\end{align*}
\]

The boundary conditions are

\[
\begin{align*}
\omega_i &= 0, \quad \theta = 0, \quad (x, t) \in \Gamma_1 \times [0, \tau], \\
\omega_m^m n_i &= 0, \quad \theta_m^m = 0, \quad (x, t) \in \Gamma_2 \times [0, \tau], \quad (3.3)
\end{align*}
\]
and additionally the initial conditions are given at $t = 0$, i.e.,

$$
\begin{align*}
\omega_1(x, 0) &= 0, \quad \theta(x, 0) = 0, \quad x \in \Omega_1, \\
\theta^m(x, 0) &= 0, \quad x \in \Omega_2.
\end{align*}
$$

(3.4)

The conditions on interface $L$ are

$$
\begin{align*}
\omega_3 &= \omega_3^m, \quad \theta = \theta^m, \quad \theta_3 = \theta_3^m, \\
\pi &= \pi^m.
\end{align*}
$$

(3.5)

**Theorem** Let $(u_i, T, p)$ and $(v_i, S, q)$ be the classical solutions to the initial-boundary value problem (1.1)–(1.5) with $\lambda = \hat{\lambda}_1$, while $(u_i^*, T^*, p^*)$ and $(v_i^*, S^*, q^*)$ are the classical solutions to the initial-boundary value problem (1.1)–(1.5) with $\lambda = \hat{\lambda}_2$. We define $(\omega_i, \theta, \pi)$ and $(\omega_i^m, \theta^m, \pi^m)$ to be the differences of these two solutions, respectively. Then the solutions $(u_i, T, p)$ and $(v_i, S, q)$ converge to the solutions $(u_i^*, T^*, p^*)$ and $(v_i^*, S^*, q^*)$ as the Forchheimer coefficient $\hat{\lambda}$ tends to 0. The differences of solutions satisfy

$$
\int_{\Omega_1} \theta^2 \, dx + \int_{\Omega_2} (\theta^m)^2 \, dx + \frac{N_2^2}{2\kappa} \int_{\Omega_1} \omega_i \omega_i \, dx \leq \hat{\lambda}^2 m_{11}(t),
$$

(3.6)

where $m_{10} = \max\{\frac{N_2^2}{\kappa}, \frac{N_2^2 G_1^2}{2\kappa}\}$, $m_{11}(t) = \frac{N_2^2}{2\kappa \lambda_2} m_7(t) + \frac{N_2^2 G_1^2}{2\kappa \lambda_2} e^{m_{10} t} \int_0^t m_7(\eta) e^{-m_{10} \eta} \, d\eta$.

Moreover, the differences of velocities satisfy the following estimates:

$$
\int_0^t \int_{\Omega_2} \omega_i^m \omega_i^m \, dx \, d\eta \leq \hat{\lambda}^2 \left( \frac{m_7(t)}{\lambda_1 \lambda_2} + m_{12} \int_0^t m_{11}(\eta) \, d\eta \right),
$$

(3.7)

where $m_{12} = \max\{G_1^2, \frac{2\kappa}{N_2}\}$.

**Proof** Multiplying (3.1) by $2\omega_i$ and integrating over $\Omega_i$, we see

$$
\frac{d}{dt} \int_{\Omega_i} \omega_i \omega_i \, dx = -2\hat{\lambda} \int_{\Omega_i} |u_i| u_i \omega_i \, dx - 2\lambda_3 \int_{\Omega_i} (|u_i| u_i - |u_i^*| u_i^*) \omega_i \, dx - 2 \int_{\Omega_i} \pi_j \omega_i \, dx \\
+ 2 \int_{\Omega_i} g_i \theta \omega_i \, dx.
$$

(3.8)

For the third function on the right-hand side of (3.8), using the divergence theorem and Eqs. (3.3), (3.5), we get

$$
-2 \int_{\Omega_1} \pi_j \omega_i \, dx = -2 \int_L \pi_j \omega_i n_j^{(3)} (s) \, dS = 2 \int_L \pi^m \omega_i n_j^{(2)} (s) \, dS = 2 \int_{\Omega_2} \pi_j \omega_i^m \, dx.
$$

(3.9)
For the second function on the right-hand side of (3.8), we have

\[
2(\text{Re}u_i - \text{Re}u_i^*)\omega_i
\]

\[
= 2\left( |u|^3 + |u^*|^3 \right) - 2u_i u_i^* (|u| + |u^*|)
\]

\[
= (|u| + |u^*|) \left( |u|^2 + |u^*|^2 - 2|u||u^*| \right) + (|u|^3 + |u^*|^3 - 2u_i u_i^*)
\]

\[
= (|u| + |u^*|) \left( |u|^2 + |u^*|^2 + \omega_i \omega_i \right)
\]

\[
\geq |u|\omega_i \omega_i. \quad (3.10)
\]

For the first function on the right-hand side of (3.8), we have

\[
-2\hat{\lambda} \int_{\Theta_1} |u| u_i \omega_i \, dx \leq \frac{\hat{\lambda}^2}{\lambda_2} \int_{\Theta_1} |u|^3 \, dx + \lambda_2 \int_{\Theta_1} |u| \omega_i \omega_i \, dx. \quad (3.11)
\]

Combining (3.8)–(3.11), we have

\[
\frac{d}{dt} \int_{\Theta_1} \omega_i \omega_i \, dx
\]

\[
\leq \frac{\hat{\lambda}^2}{\lambda_2} \int_{\Theta_1} |u|^3 \, dx + 2 \int_{\Theta_1} \pi_{m}^{\omega_i} \omega_{i}^{m} \, dx + 2 \int_{\Theta_1} g_i \omega_i \, dx
\]

\[
\leq \frac{\hat{\lambda}^2}{\lambda_2} \int_{\Theta_1} |u|^3 \, dx + 2 \int_{\Theta_1} (g_i \theta_i^m - \omega_{i}^{m}) \omega_{i}^{m} \, dx + \int_{\Theta_1} \omega_i \omega_i \, dx + G_i^2 \int_{\Theta_1} \theta^2 \, dx
\]

\[
\leq \frac{\hat{\lambda}^2}{\lambda_2} \int_{\Theta_1} |u|^3 \, dx - \int_{\Theta_1} \omega_{i}^{m} \omega_{i}^{m} \, dx + \int_{\Theta_1} \omega_i \omega_i \, dx
\]

\[
+ G_i^2 \int_{\Theta_1} \theta^2 \, dx + G_i^2 \int_{\Theta_1} (\theta_i^m)^2 \, dx. \quad (3.12)
\]

In order to estimate \( \int_{\Theta_1} \theta \theta \, dx + \int_{\Theta_2} \theta^m \theta^m \, dx \), we multiply (3.1) by \( 2 \theta \) and get

\[
\frac{d}{dt} \int_{\Theta_1} \theta \, dx = 2 \int_{\Theta_1} \theta \, dx
\]

\[
= 2 \int_{\Theta_1} \theta \left( \kappa \Delta \theta - u_i \theta_j - \omega_i T_i^j \right) \, dx
\]

\[
= 2\kappa \int_{\Theta_1} \theta \Delta \theta \, dx - 2 \int_{\Theta_1} \theta u_i \theta_j \, dx - 2 \int_{\Theta_1} \theta \omega_i T_i^j \, dx. \quad (3.13)
\]

For the first function on the right-hand side of (3.13), using the divergence theorem and Eqs. (3.3), (3.5), we get

\[
2\kappa \int_{\Theta_1} \theta \Delta \theta \, dx = 2 \int_{L} \kappa \theta \theta_3^{(1)} \, dS - 2\kappa \int_{\Theta_1} \theta \theta_j \, dx
\]

\[
\leq -2 \int_{L} \theta^m \kappa \theta_3^{(2)} \, dS - 2\kappa \int_{\Theta_1} \theta \theta_j \, dx. \quad (3.14)
\]
For the second function on the right-hand side of (3.13), using the divergence theorem and Eqs. (1.3), (3.5), we get
\[
-2 \int_{\Omega_1} \theta u_i \theta_j \, dx = - \int_{\Omega_1} u_i (\theta_j)^2 \, dx = - \int_L \theta_j n_3^{(1)} \, dS
= \int_L v_3 (\theta^m)^2 n_3^{(2)} \, dS = 2 \int_{\Omega_2} \theta^m v_j \theta_j \, dx.
\] (3.15)

For the third function on the right-hand side of (3.13), using the divergence theorem and Eqs. (1.5), (3.3), and (3.5), we get
\[
-2 \int_{\Omega_1} \theta \omega_i T^*_i \, dx = -2 \int_L \theta \omega_i T^*_i \, dx + 2 \int_{\Omega_1} \theta_i \omega_i T^* \, dx
= 2 \int_L \theta^m \omega_i^m S^* n_3^{(2)} \, dS + 2 \int_{\Omega_1} \theta_i \omega_i T^* \, dx.
\] (3.16)

Combining (3.13)–(3.16), we get
\[
\frac{d}{dt} \left( \int_{\Omega_1} \theta^2 \, dx + \int_{\Omega_2} (\theta^m)^2 \, dx \right) \leq -2 \kappa \int_{\Omega_1} \theta_j \theta_j \, dx + 2 \int_{\Omega_1} \theta_j \omega_i T^* \, dx - 2 \int_L \theta^m \kappa \theta^m n_3^{(2)} \, dS
+ 2 \int_L \theta^m \omega_i^m S^* n_3^{(2)} \, dS + 2 \int_{\Omega_2} \theta_i \omega_i T^* \, dx.
\] (3.17)

Similarly, we multiply (3.2) by $2 \theta^m$, we have
\[
\frac{d}{dt} \int_{\Omega_2} (\theta^m)^2 \, dx \leq \frac{N_M^2}{2\kappa} \int_{\Omega_2} \omega_i \omega_i^m \, dx + 2 \int_{\Omega_2} \theta^m \kappa \theta^m n_3^{(2)} \, dS
- 2 \int_L \theta^m \omega_i^m S^* n_3^{(2)} \, dS - 2 \int_{\Omega_2} \theta_i \omega_i T^* \, dx.
\] (3.18)

Combining (3.17) and (3.18), we have
\[
\frac{d}{dt} \left( \int_{\Omega_1} \theta^2 \, dx + \int_{\Omega_2} (\theta^m)^2 \, dx \right) \leq \frac{N_M^2}{2\kappa} \int_{\Omega_1} \omega_i \omega_i \, dx + \frac{N_M^2}{2\kappa} \int_{\Omega_2} \omega_i \omega_i^m \, dx.
\] (3.19)

Combining (3.12) and (3.19), we have
\[
\frac{d}{dt} \left( \int_{\Omega_1} \theta^2 \, dx + \int_{\Omega_2} (\theta^m)^2 \, dx \right) \leq \frac{\lambda^2}{\lambda_2}\frac{N_M^2}{2\kappa} \int_{\Omega_1} |u|^3 \, dx + \frac{N_M^2}{\kappa} \int_{\Omega_1} \omega_i \omega_i \, dx + \frac{N_M^2 G_1^2}{2\kappa} \int_{\Omega_1} \theta^2 \, dx
+ \frac{N_M^2 G_1^2}{2\kappa} \int_{\Omega_1} (\theta^m)^2 \, dx.
\] (3.20)
If we let $F_4(t) = \int_{\Omega_2} \theta^2 \, dx + \int_{\Omega_2} (\theta^m)^2 \, dx + \frac{N_M^2}{2 \kappa} \int_{\Omega_1} \omega_i \omega_i \, dx$, $m_{10} = \max\{N_M^2, N_M^2 G_1^2\}$. Therefore, integrating (3.20) yields

$$F_4(t) \leq \hat{\lambda}^2 \frac{N_M^2}{2 \kappa \lambda_1 \lambda_2} m_7(t) + m_{10} \int_0^t F_4(\eta) \, d\eta.$$  

(3.21)

Gronwall inequality implies

$$F_4(t) \leq \hat{\lambda}^2 \frac{N_M^2}{2 \kappa \lambda_1 \lambda_2} m_7(t) + \hat{\lambda}^2 \frac{N_M^2 m_{10}}{2 \kappa \lambda_1 \lambda_2} e^{m_{10} t} \int_0^t m_7(\eta) e^{-m_{10} \eta} \, d\eta = \hat{\lambda}^2 m_{11}(t),$$  

(3.22)

where $m_{11}(t) = \frac{N_M^2}{2 \kappa \lambda_1 \lambda_2} m_7(t) + \frac{N_M^2 m_{10}}{2 \kappa \lambda_1 \lambda_2} e^{m_{10} t} \int_0^t m_7(\eta) e^{-m_{10} \eta} \, d\eta$.

Inserting (3.22) into (3.12), we have

$$\int_0^t \int_{\Omega_2} \omega_i^m \omega_i^m \, dx \, d\eta \leq \hat{\lambda}^2 \left( \frac{m_7(t)}{\lambda_1 \lambda_2} + m_{12} \int_0^t m_{11}(\eta) \, d\eta \right),$$  

(3.23)

where $m_{12} = \max\{G_1^2, \frac{N_M}{N_M^2}\}$. □

Acknowledgements
The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions.

Funding
The work was supported National Natural Science Foundation of China (Grant No. 61907010), Natural Science Foundation in Higher Education of Guangdong, China (Grant Nos. 2018KZDXM048; 2019KZDXM036; 2019KZDXM042; 2020ZDKZ3051), the General Project of Science Research of Guangzhou (Grant No. 201707010126), and the science foundation of Huashang College Guangdong University of Finance & Economics (Grant No. 2019HSD528).

Availability of data and materials
This paper focuses on theoretical analysis, not involving experiments and data.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 November 2020 Accepted: 20 April 2021 Published online: 29 April 2021

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