Exact ground state of one- and two-dimensional frustrated quantum spin systems.

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We outline the recent results on the ground state for a class of one- and two-dimensional frustrated quantum spin models with competing ferro-(F)- and antiferromagnetic (AF) interactions. Frustrated spin systems are known to have many interesting properties due to large quantum fluctuations. As a result of these fluctuations usual mean-field approach gives quite crude (if not false) description of these systems. Therefore, exactly solvable models are very instructive at investigations of such systems. The exact ground state wave function of proposed models has a structure of the valence-bond state (VBS) type. One of the 1D model describes the transition line between the F and AF phase. The exact singlet ground state on this line has a double-spiral ordering. Using different approximation methods we study the magnetization curve in the AF phase. The second considered set of the 1D and 2D models has an exact non-degenerate ground state with exponentially decaying spin correlations. We also proposed the 1D and 2D electronic models with exact ground state represented in terms of singlet bond functions which are the generalization of the RVB functions including ionic states.

I. INTRODUCTION

There is currently much interest in quantum spin systems that exhibit frustration. This has been stimulated in particular by the study of the magnetic properties of the cuprates which become high-Tc superconductors when doped. Frustrated spin systems are known to have many interesting properties that are quite unlike those of conventional magnetic systems. The simplest model of such kind is the Heisenberg spin chain with nearest- and next-nearest-neighbor interactions $J_1$ and $J_2$. This model is well studied for $J_1$, $J_2 > 0$. In particular, it has been found that at $J_2 = 0.24J_1$ the transition from the gapless state to the dimerized one takes place. The point $J_2 = J_1/2$ corresponds to the well-known Majumdar-Ghosh model for which the exact ground state consists of dimerized singlets and there is a gap in the spectrum of excited states.

Less studied is frustrated spin models with competing interactions of ferro- and antiferromagnetic types. The physical interest for these models is connected with the study of the real compounds containing $CuO$ chains with edge-sharing $CuO_4$ units, like $La_2Cu_8O_{24}$ and $La_2CuO_2$ and $Cu_2Y_2Cu_3O_{10}$. These compounds $Cu-O-Cu$ bond angle is nearly 90° and the nearest-neighbor $Cu-Cu$ spin interaction is ferromagnetic according to Goodenough-Kanamori-Anderson rule while the next-nearest-neighbor interaction is antiferromagnetic. The magnetic properties of these models are very different from those for models with pure antiferromagnetic interactions. Their ground state can be either ferromagnetic or singlet depending on the relation between the exchange integrals. One of the most interesting problems related to these models is the character of the transition between ferromagnetic (F) and antiferromagnetic (AF) phases.

Of a special importance are models for which it is possible to construct an exact ground state. Last years a considerable progress has been achieved by using so-called Matrix-Product (MP) form of the ground state wave function. The ground state wave function in the MP method is represented by Trace of a product of matrices describing single-site states. The MP ground state has a structure of the type where each neighboring pair of spins has valence bond and, in fact, the MP form is a convenient representation of the valence bond states. Its origin can be traced to the S=1 quantum spin chain with bilinear and biquadratic interactions. At present, various 1D spin models with the exact MP ground state are found.

In this paper we present a set of 1D and 2D spin-1/2 models with competing F and AF interactions for which the singlet ground state wave function can be found exactly. This function has a special form expressed in terms of auxiliary Bose operators. This form of the wave function is similar to the MP one but with infinite matrices. For special values of model parameters it can be reduced to the standard MP form.

One of models is 1D quantum spin model describing the F-AF transition point. Spin correlations in the singlet ground state show giant spiral magnetic structure with period equals to the system size. On the antiferromagnetic side of this point the ground states can be either gapless with incommensurate spiral ground state or gapped with exponential decay of correlators. There are regions in the AF phase where the magnetization as a function of magnetic field has jumps.

The second considered model is the special case of the spin ladder with exchange integrals depending on one parameter. The exact ground state of this model is non-degenerated singlet with exponentially decaying spin correlations,
and there is an energy gap.

It will be shown that proposed form of exact wave function can be generalized to higher dimensions and two dimensional frustrated spin model with exact ground state can be constructed.

It is known that the exact ground state of some 1D and 2D quantum spin models can be represented in RVB form. The RVB function in the Fermi representation consists of homopolar configurations of electron pair. We generalize this two-particle function to include ionic states and denote it as “singlet bond” (SB) function. It is natural to try to find electronic models with exact ground state formed by SB functions in the same manner as for known spin models. In the paper some models of interacting electrons are presented. The Hamiltonians of these models include the correlated hopping of electrons and spin interactions. One of the 1D models describes the transition point between the phases with and without an off-diagonal long-range order.

The paper is organized as follows. In Sec.2 we consider the frustrated spin chain at F-AF transition point and describe the exact singlet ground state wave function as well as details of the spin correlation function calculations. We discuss the phase diagram of this model and its magnetic properties in the AF phase. In Sec.3 the special spin ladder will be considered. Two-dimensional frustrated spin model with the exact ground state is considered in Sec.4. Sec.5 is devoted to the construction of the electronic models with SB type of wave function. The results of paper is summarized in Sec.6.

II. ZIGZAG SPIN MODEL.

A. Zigzag spin model at F-AF transition point.

Let us consider $s = \frac{1}{2}$ spin chain with nearest- and next-nearest neighbor interactions given by the Hamiltonian

$$
H = -\sum_{i=1}^{M} (S_{2i-1} \cdot S_{2i} - \frac{1}{4}) + J_{23} \sum_{i=1}^{M} (S_{2i} \cdot S_{2i+1} - \frac{1}{4}) + J_{13} \sum_{i=1}^{N} (S_i \cdot S_{i+2} - \frac{1}{4}),
$$

(1)

with periodic boundary conditions and even number of spins $N = 2M$. This model is equivalent to a ladder model with diagonal coupling, so called a ‘zigzag’ spin ladder.

If $J_{23} < 1$, then the ground state of (1) is ferromagnetic (singlet) at $\delta < 0$ ($\delta > 0$), where $\delta = J_{23} + \frac{2J_{13}}{2J_{13}}$ (Fig.1). The equation $\delta = 0$ defines the line of transition points from the ferromagnetic to the singlet state, when energies of these states are zero. The model (1) along this line is given by the Hamiltonian depending on the parameter $x$ ($x > -1/2$):

$$
H = -\sum_{i=1}^{M} (S_{2i-1} \cdot S_{2i} - \frac{1}{4}) + \frac{2x}{2x + 1} \sum_{i=1}^{M} (S_{2i} \cdot S_{2i+1} - \frac{1}{4}) - x \sum_{i=1}^{N} (S_i \cdot S_{i+2} - \frac{1}{4}),
$$

(2)

with periodic boundary conditions.
We note that the Hamiltonian (2) has a symmetry: its spectrum coincides with the spectrum of \( \tilde{H}(x) \) obtained by the following transformation

\[
\tilde{H}(x) = -\frac{2x}{2x+1} H(-x - \frac{1}{2}), \quad -\frac{1}{2} < x < 0
\]

This transformation permutes the factors at the first and the second terms in the Hamiltonian (2). Thus, due to the symmetry it is sufficient to consider the range \( x \geq -\frac{1}{2} \).

First, we will show that the ground state energy of (2) is zero. Let us represent the Hamiltonian (2) as a sum of Hamiltonians \( h_n \) of cells containing three sites

\[
H = \sum_{i=1}^{M} (h_{2i-1} + h_{2i}) \tag{3}
\]

where

\[
h_{2i-1} = -\frac{1}{2}(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i} - \frac{1}{4}) + \frac{x}{2x+1}(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} - \frac{1}{4}) - x(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i+1} - \frac{1}{4}),
\]

\[
h_{2i} = -\frac{1}{2}(\mathbf{S}_{2i+1} \cdot \mathbf{S}_{2i+2} - \frac{1}{4}) + \frac{x}{2x+1}(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} - \frac{1}{4}) - x(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+2} - \frac{1}{4})
\]

Eigenvalues of each \( h_n \) are

\[
\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \frac{4x^2 + 2x + 1}{4x + 2} > 0
\]

We will present a singlet wave function which is the exact one of each \( h_n \) with zero energy and, therefore, it is the exact ground state wave function of (2). This function has a form

\[
\Psi_0 = P_0 \Psi, \quad \Psi = \langle 0_b | g_1 \otimes g_2 \otimes g_3 \otimes \ldots \otimes g_N | 0_b \rangle \tag{4}
\]

where

\[
g_i = b^+ | \uparrow \rangle_i + | \downarrow \rangle_i, \quad g_i' = (b^+ b - 2x) | \uparrow \rangle_i + b | \downarrow \rangle_i \tag{5}
\]

Here we introduced one auxiliary Bose-particle \( b^+ \) (the Bose operators \( b^+ \) and \( b \) do not act on spin states \( | \uparrow \rangle_i \) and \( | \downarrow \rangle_i \)) and the Bose vacuum \( | 0_b \rangle \). Therefore, the direct product \( g_1 \otimes g_2' \otimes \ldots \otimes g_N' \) is the superposition of all possible spin configurations multiplied on the corresponding Bose operators, like \( b^+ b b^+ \ldots | \uparrow \downarrow \downarrow \uparrow \ldots \). \( P_0 \) is a projector onto the singlet state. This operator can be written as \( 2 \)

\[
P_0 = \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \int_0^{\pi} \sin \gamma d\gamma \, e^{i\alpha S^z} e^{i\gamma S^z} e^{i\beta S^z}, \tag{6}
\]

where \( S^z(\alpha) \) are components of the total spin operator.

The form of wave function (4) resembles the MP form, but with an infinite matrix which is represented by Bose operators. Therefore, we have to pick out the \( \langle 0_b | \ldots | 0_b \rangle \) element of the matrix product instead of the usual trace in the MP formalism \( \boxed{[8]} \), because the trace is undefined in this case. The function \( \Psi \) contains components with all possible values of spin \( S (0 \leq S \leq N/2) \) and, in fact, a fraction of the singlet is exponentially small at large \( N \). This component is filtered out by the operator \( P_0 \).

One can easily check that each cell Hamiltonian \( h_{2i-1} \) and \( h_{2i} \) for \( i = 1, \ldots (M - 1) \) gives zero when acting on the corresponding part in wave function \( \Psi - g_{2i-1} \otimes g_{2i+1}' \otimes g_{2i+2}' \) and \( g_{2i}' \otimes g_{2i+1} \otimes g_{2i+2}' \) (one should take care of Bose commutation relations). Since each \( h_i \) is a non-negatively defined operator, then \( \Psi \) is the exact ground state wave function of an open chain:

\[
H_{\text{open}} = \sum_{i=1}^{M-1} (h_{2i-1} + h_{2i}) \tag{7}
\]

As mentioned above, the function \( \Psi \) contains components of all possible values of total spin \( S \), and, therefore, the ground state of the open chain is multiply degenerate. However it can be proven \( \boxed{[8]} \) that for the cyclic chain \( \boxed{[8]} \) only
singlet and ferromagnetic components of $\Psi$ have zero energy. Therefore, for cyclic chain (3) $\Psi_0$ is the singlet ground state wave function degenerate with the ferromagnetic state.

In particular case, $x = -1/4$, when $J_{12} = J_{23} = -1$ and $J_{13} = \frac{1}{4}$, another form of the exact singlet ground state wave function has been found in [15]. It reads

$$\Psi = \sum_{[i,j][k,l][m,n]...} |i,j\rangle |k,l\rangle |m,n\rangle...,$$  \hspace{1cm} (8)

where $[i,j]$ denotes the singlet pair and the summation is made for any combination of spin sites under the condition that $i < j, k < l, m < n$...

The following general statements relevant to the Hamiltonian (2) can be proved:

1). The ground states of open chain described by (7) in the sector with fixed total spin $S$ are non-degenerate and their energies are zero.

2). For cyclic chains the ground state in the $S = 0$ sector is non-degenerate. The ground state energies for $0 < S < M$ are non-zero.

3). The singlet ground state wave function for open and cyclic chains coincide with each other.

B. Spin correlations in the ground state.

For the sake of simplicity we show the calculation of the spin correlation function in the symmetric case $x = -1/4$, when the Hamiltonian (2) takes the form

$$H = - \frac{1}{4} \sum_{i=1}^{N} (S_i \cdot S_{i+1} - \frac{1}{4}) + \frac{1}{4} \sum_{i=1}^{N} (S_i \cdot S_{i+2} - \frac{1}{4}),$$  \hspace{1cm} (9)

Since in this case there is one spin in elementary cell, the singlet ground state wave function $\Psi_0$ can be written in a more simple and symmetric form:

$$\Psi_0 = P_0 \Psi, \quad \Psi = \langle 0_b | g_1 \otimes g_2 \otimes \cdots \otimes g_N | 0_b \rangle,$$  \hspace{1cm} (10)

where

$$g_i = b^+ |\uparrow\rangle_i + b |\downarrow\rangle_i,$$  \hspace{1cm} (11)

One can check that wave functions (10) is the singlet ground state wave function with zero energy of Hamiltonian (9). Therefore, the equivalence of wave functions (4) and (10) follows from the non-degeneracy of ground state in the $S = 0$ sector (though functions $\Psi$ in Eqs. (10) and (4) are different).

Now we calculate the norm and correlation function of the wave function $\Psi_0$ (10). The norm of the singlet wave function $\Psi_0$ is

$$\langle \Psi_0 | \Psi_0 \rangle = \langle \Psi | P_0 | \Psi \rangle,$$  \hspace{1cm} (12)

It is easy to check that the function $\Psi$ has $S^z = 0$. Then the projector $P_0$ in Eq. (10) takes the form

$$P_0 = \frac{1}{2} \int_{0}^{\pi} \sin \gamma d\gamma e^{izS^z} e^{iz' S^z},$$  \hspace{1cm} (13)

where $z = \tan \frac{\pi}{8}$, $z' = \sin \frac{\pi}{8} \cos \frac{\pi}{8}$ and $S^{+(-)}$ are the operators of the total spin.

Therefore, the norm takes the form

$$\langle \Psi_0 | \Psi_0 \rangle = \frac{1}{2} \int_{0}^{\pi} \sin \gamma d\gamma \langle 0_b, 0_b \rangle \prod_{i=1}^{N} (g_i^+ e^{izS^z} e^{iz' S^z} g_i) | 0_a, 0_b \rangle,$$

where

$$g_i^+ e^{izS^z} e^{iz' S^z} g_i = (a^+ \langle \uparrow | + a \langle \downarrow | ) e^{izS^z} e^{iz' S^z} (b^+ |\uparrow\rangle_i + b |\downarrow\rangle_i)$$

$$= a^+ b^+ + (1 - z'z)ab + izab^+ + iz'a^+b,$$

and $a^+$ and $a$ are the Bose operators. Thus the norm can be rewritten as
\[ \langle \Psi_0 | \Psi_0 \rangle = \frac{1}{2} \int_0^\pi \sin \gamma d\gamma \langle 0 | G^N | 0 \rangle, \]  
(14)

where \(|0\rangle = |0_a, 0_b\rangle\) is the Bose vacuum of \(a^+\) and \(b^+\) particles and

\[ G = u(a^+b^+ + ab) + iv(ab^+ + a^+b), \]

where \(u = \cos \frac{\gamma}{2}, v = \sin \frac{\gamma}{2} \).

Let us introduce the auxiliary function \(P(\xi)\):

\[ P(\xi) = \langle 0 | e^{\xi G} | 0 \rangle, \]  
(15)

then

\[ \langle 0 | G^N | 0 \rangle = \left. \frac{d^NP}{d\xi^N} \right|_{\xi=0} \]

The function \(P(\xi)\) can be easily found:

\[ P(\xi) = \frac{1}{\sqrt{1 - u^2 \sin^2 \xi}} \]  
(16)

Integrating Eq.(16) over \(\gamma\), we obtain

\[ \langle \Psi_0 | \Psi_0 \rangle = \frac{1}{2} \int_0^\pi \sin \gamma d\gamma \left. \frac{d^NP}{d\xi^N} \right|_{\xi=0} = \left. \frac{d^NP}{d\xi^N} \left( \frac{1}{\cos^2(\frac{\xi}{2})} \right) \right|_{\xi=0} \]  
(17)

Thus, finally, we arrive at

\[ \langle \Psi_0 | \Psi_0 \rangle = 2 \left. \frac{d^{N+1}}{d\xi^{N+1}} \left( \tan \frac{\xi}{2} \right) \right|_{\xi=0} = \frac{4(2^{N+2} - 1)}{N + 2} |B_N + 2| \]  
(18)

Here \(B_N\) are the Bernoulli numbers.

To calculate the spin correlators we need to introduce operators

\[ G_z = g_i^+ e^{izS_i^z} e^{iz'S_i^z} 2S_i^z g_i = u(a^+b^+ - ab) + iv(ab^+ - a^+b), \]
\[ G_+ = g_i^+ e^{izS_i^z} e^{iz'S_i^z} S_i^+ g_i = ua^+b + iuvab, \]
\[ G_- = g_i^+ e^{izS_i^z} e^{iz'S_i^z} S_i^- g_i = uab^+ + iuvab \]

Then, the correlator \(\langle S_1 \cdot S_{l+1} \rangle\) will be defined by

\[ \langle \Psi_0 | S_1 \cdot S_{l+1} | \Psi_0 \rangle = \frac{1}{2} \int_0^\pi \sin \gamma d\gamma \langle 0 | \frac{1}{4} G_z G_k G_z G^{N_l-2} + \frac{1}{2} G_+ G_l G_- G^{N_l-2} | 0 \rangle \]  
(19)

(since \(\langle 0 | G_- \ldots | 0 \rangle = 0\)).

The expectation values in Eq.(19) can be represented as

\[ \langle 0 | G_z G_l G_z G^{N_l-2} | 0 \rangle = \left. \frac{\partial^l}{\partial \xi^l} \frac{\partial^{N-l-2}}{\partial \xi^{N-l-2}} \langle 0 | G_z e^{\xi G} G_z e^{\xi G} | 0 \rangle \right|_{\xi=0}, \]
\[ \langle 0 | G_+ G_l G_- G^{N_l-2} | 0 \rangle = \left. \frac{\partial^l}{\partial \xi^l} \frac{\partial^{N-l-2}}{\partial \xi^{N-l-2}} \langle 0 | G_+ e^{\xi G} G_- e^{\xi G} | 0 \rangle \right|_{\xi=0} \]  
(20)

After a procedure similar to that for the norm and the integration over \(\gamma\), we obtain

\[ \langle \Psi_0 | S_1 \cdot S_{l+1} | \Psi_0 \rangle = \left. \frac{\partial^l}{\partial \xi^l} \frac{\partial^{N-l-2}}{\partial \xi^{N-l-2}} \left( \frac{3 \cos(\xi - \zeta)}{8 \cos^4(\frac{\xi + \zeta}{2})} \right) \right|_{\xi=0} \]  
(21)

It can be shown that in the thermodynamic limit, Eqs.(18) and (21) result in
$\langle \mathbf{S}_i \cdot \mathbf{S}_{i+1} \rangle = \frac{1}{4} \cos \left( \frac{2\pi l}{N} \right)$ \hspace{1cm} (22)

So, we reproduce the result obtained in Refs.\[15,11\] that in the thermodynamic limit a giant spiral spin structure is realized, with the period of the spiral equal to the system size.

For the general case of model (2) the calculation of the singlet ground state correlation functions can be performed by the similar way. The final result in the thermodynamic limit is\[14,11\]:

$\langle \mathbf{S}_n \mathbf{S}_{n+2l} \rangle = \frac{1}{4} \cos \left( \frac{4\pi l}{N} \right)$,

$\langle \mathbf{S}_n \mathbf{S}_{n+2l-1} \rangle = \frac{1}{4} \cos \left( \frac{2(2l-1)\pi}{N} + (-1)^n \Delta \varphi \right)$ \hspace{1cm} (23)

The latter equations mean that the long-range double-spiral order exists in the singlet ground state of Hamiltonian (2). The pitch angle of each spiral is $\frac{2\pi}{N}$ and there is a small shift angle $\Delta \varphi = \frac{2\pi(4x+1)}{N}$ between them. This shift angle reflects the fact that the unit cell contains two sites unless $x = -\frac{1}{4}$.

We note that for special values of model parameter $x = \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, ...$ Eqs. (23) are not valid and spin correlations decay exponentially. These cases will be considered in Sec.III (see Eqs. (42)).

C. Phase diagram of ‘zigzag’ model

So far we considered the model (1) at the transition line from ferromagnetic to antiferromagnetic state. Now we discuss the phase diagram of this model. The exact ground state in the AF phase is generally unknown. But it is interesting to note that the ground state on the line $J_{13} = -1/2$ is the product of singlets on ladder diagonals (2,3), (4,5), ... as in the point $J_{13} = J_{23} = -1/2$ on the transition line. The spectrum of (1) on the transition line is gapless. There are some regions on the plane ($J_{13}$, $J_{23}$) which were studied by different approximation methods.

First let us consider behaviour of the system in the transition region near the ‘symmetric’ point $x = -\frac{1}{4}$, where

$$J_{12} = J_{23} = -1, \quad J_{13} = \frac{1}{4} + \delta$$ \hspace{1cm} (24)
It would be mentioned that several copper oxide compounds are described by this model. These compounds contain CuO chains and Cu – O – Cu angle $\theta$ is near $90^\circ$. In this case the usual antiferromagnetic super-exchange of two nearest-neighbor magnetic Cu ions is suppressed and the exchange integral $J_{12}$ is negative. On the other hand the next-nearest-neighbor interaction $J_{13}$ between Cu ions does not depend on $\theta$ and is antiferromagnetic. The estimation of the ratio $J_{13}/J_{12}$ for $La_6Cu_8Cu_2O_{18}$ and $Li_2CuO_2$ gives $\delta = 0.11$ and $\delta = 0.37$ respectively.

Though the model (24) is not exactly solvable in this case, its properties for $\delta \ll 1$ can be studied. The consideration is based on the classical approximation. In this approximation the ground state spin structure is the spiral with the period $\sim \delta^{-1/2}$ and the ground state energy $E = -2N\delta^2$. Using this approach the regular expansion in powers of the small parameter $\delta$ can be developed (22). The second-order quantum corrections coincides with the classical energy. The calculation of higher orders of the perturbation theory in $\delta$ leads to infrared-divergent integrals and it is necessary to sum them in all orders to obtain the contributions proportional to $\delta^{5/2}$, $\delta^3$ etc. In particular, the ground state energy calculated up to terms $\sim \delta^{5/2}$ is

$$E_0 = -4N\delta^2 + 4.14N\delta^{5/2}$$

The excitation spectrum is gapless and has a sound like behaviour. Corresponding sound velocity is $v = 4\delta^{3/2}$.

The study of the dependence of the ground state energy $E(m)$ on magnetization $m = S/S_z$ at $\delta \ll 1$ shows that $\delta^2E/\delta m^2 < 0$ in a finite interval of values of $m$. This implies the thermodynamic instability of the uniform state against the phase separation. This instability arises due to existence of multimagnon bound states. Energy of $n$-magnon bound state ($n \gg 1$) is $\varepsilon_n = -nb, b = |\varepsilon_1| + \epsilon_b$ where $\varepsilon_1$ is the energy of one-magnon state and $\epsilon_b$ is binding energy per magnon. For the model (24) $\varepsilon_1 = -8\delta^2$ and $\epsilon_b \sim \delta^{5/2}$. As a result the function $E(m)$ has the form:

$$E(m) = E_0 + 2\pi\delta^{3/2}m^2, \quad m < m_c = \frac{2}{\pi}\sqrt{\delta}$$

$$E(m) = -\frac{b}{2}(1-2m), \quad m_c < m < \frac{1}{2} \quad (25)$$

At $m > m_c$ the system is in a two-phase state consisting of the ferromagnetic ($m = 1/2$) phase and the phase with $m = m_c$. According to (23) the magnetization as a function of magnetic field $h$ has a jump from $m = m_c$ to $m = 1/2$ (metamagnetic transition) at $h = h_c = b$. These values $m_c$ and $h_c$ are close to those obtained in (22) by extrapolation of finite cluster calculations. We stress that the jump of the magnetization exists if there are multimagnon bound states leading to the linear dependence of $E(m)$ for some region of $m$.

As for properties of the model far from the transition point $\delta = 0$, now-days there is no clear understanding, but we believe that the spectrum remains gapless. For the case $\delta \gg 1$ this fact is predicted in (22). As for the jump in $m(h)$ it is unknown if it remains at $\delta \simeq 1$. Numerical calculations show that, at least, there is an abrupt increase of magnetization at some critical value of magnetic field.

This perturbation theory can be generalized for the vicinity of any point on the transition line. It gives the similar properties of the system in the region $J_{23} < 0$, but it diverges at $J_{23} > 0$.

At $J_{13} = 0$ and $J_{23} > 0$ the model (1) reduces to the alternating Heisenberg chain studied in (22). The lowest excitation is the triplet and there is the gap. At $J_{23} = 0$ and $J_{13} > 0$ the model (1) reduces to the spin ladder with antiferromagnetic interactions along legs and the ferromagnetic interactions on rungs. It is evident that there is a gap at $J_{13} < 1$ (in this case the model is equivalent to the spin $S = 1$ Heisenberg chain). It was shown in (22) that the gap exists at $J_{13} \gg 1$.

The region $J_{23} > 0$, $J_{13} < 0$ was studied by different methods in (22). The exact diagonalization of finite chains shows the gap $\Delta$ in the excitation spectrum which is closed on the transition line. The presence of singlet-triplet gap is also confirmed by calculations with use of variational wave function in MP form (22) when all matrix elements are considered as variational parameters. The MP variational function gives very good accuracy (22), besides it gives exact ground state results for special points on the transition line and for the line $J_{13} = -\frac{1}{2}$ (dimer state).

Another important property of the model with $J_{23} > 0$, $J_{13} < 0$ is the existence of the multimagnon bound states for definite region of the values $J_{23}$ and $J_{13}$. This region is shown on Fig.1. On the boundary of this region the multimagnon bound states disappear. As it was noted before the existence of the bound states leads to the linear regions on the dependence $E(m)$ and to the metamagnetic transition. The dependence $E(m)$ and $m(h)$ was found using MP variational function. The calculations show that out of the bound state region the function $m(h)$ is typical for the gapped antiferromagnet, i.e. it is monotonically increasing function (Fig.3). In other words, $m(h) = 0$ for $h < \Delta$ and $m(h) = 1/2$ for $h > |\varepsilon_1|$. In the bound state region the magnetization curve has the jump. It is interesting that inside the stripe $-\frac{1}{2} < J_{13} < \frac{1}{2}$ this jump takes place immediately from zero to the maximal value $m = 1$ (Fig.3) at $h_c = |\varepsilon(0)|$ ($\varepsilon(0)$ is a ground state energy per site). Such behaviour of $m(h)$ is explained by the fact that inside this stripe $\Delta > |\varepsilon(0)|$ and the transition from the singlet state to the ferromagnetic one comes to pass through
FIG. 3. Dependence of magnetization $m$ on magnetic field $h$ for the case: (a) outside bound state region; (b) in the bound state region, but outside the stripe $-\frac{1}{2} < J_{13} < -\frac{1}{6}$; (c) inside the stripe $-\frac{1}{2} < J_{13} < -\frac{1}{6}$.

states with intermediate values of spin. In the bound state region outside the stripe the magnetization curve has a form as shown on Fig.(3). In this case the critical field $h_c = \Delta$.

Thus, summarizing all above, we expect that the phase diagram of the model (1) has the form shown in Fig.(2).

III. SPIN LADDER MODEL

Let us consider more general spin ladder model, for which ‘zigzag’ model (2) is a particular case. So, we consider the cyclic ladder model containing $N = 2M$ spins $s = \frac{1}{2}$ (Fig.4). The proposed form of wave function (4) can be generalized for a ladder model as follows:

$$\Psi_0 = P_0 \Psi, \quad \Psi = \langle 0 | g_1 \otimes g_2 \otimes \ldots \otimes g_M | 0 \rangle,$$

where each $g_i$ corresponds to the $i$th rung of the ladder:

$$g_i = -b^+(2x - b^+b) |\uparrow\uparrow\rangle_i + b |\downarrow\downarrow\rangle_i + (b^+b - x) |\uparrow\downarrow + \downarrow\uparrow\rangle_i + y |\uparrow\downarrow - \downarrow\uparrow\rangle_i$$

where $x$ and $y$ are two parameters of the model. It is obvious that the ladder wave function (26)-(27) reduces to the original function (4) at $y = x$.

Now we will construct the Hamiltonian for which $\Psi_{\text{ladder}}$ is the exact ground state wave function. This Hamiltonian describes two-leg $s = \frac{1}{2}$ ladder with periodic boundary conditions (Fig.4) and can be represented in a form

$$H = \sum_{i=1}^{N} h_{i,i+1},$$

where $h_{i,i+1}$ describes the interaction between neighboring rungs. The spin space of two neighboring rungs consists of six multiplets: two singlet, three triplet and one quintet. At the same time, one can check that the product $g_i \otimes g_{i+1}$ contains only three of the six multiplets of each pair of neighboring rungs: one singlet, one triplet and one quintet. The specific form of the singlet and triplet components present in the product $g_i \otimes g_{i+1}$ depends on parameters $x$.
and $y$. The Hamiltonian $h_{i,i+1}$ can be written as the sum of the projectors onto the three missing multiplets with arbitrary positive coefficients $\lambda_1$, $\lambda_2$, $\lambda_3$:

$$h_{i,i+1} = \sum_{k=1}^{3} \lambda_k P_k^{i,i+1},$$  \hspace{1cm} (29)$$

where $P_k^{i,i+1}$ is the projector onto the missing multiplets in the corresponding cell Hamiltonian.

The wave function (26) is an exact wave function of the ground state of the Hamiltonian $h_{i,i+1}$ with zero energy, because

$$h_{i,i+1}|\Psi\rangle = 0, \quad i = 1,\ldots,N-1$$  \hspace{1cm} (30)$$

and $\lambda_1$, $\lambda_2$, $\lambda_3$ are the excitation energies of the corresponding multiplets.

So, $\Psi$ is the exact ground state wave function with zero energy for the total Hamiltonian of an open ladder

$$H_{\text{open}} = \sum_{i=1}^{M-1} h_{i,i+1}$$  \hspace{1cm} (31)$$

$$H_{\text{open}}|\Psi\rangle = 0$$  \hspace{1cm} (32)$$

Since the function $\Psi$ contains components with all possible values of total spin $S$ ($0 \leq S \leq M$), then the ground state of open ladder is multiple degenerate. But for a cyclic ladder only singlet and ferromagnetic components of $\Psi$ have zero energy. Therefore, for a cyclic ladder $\Psi_0$ is a singlet ground state wave function degenerated with ferromagnetic state. Besides, all the statements given for ‘zigzag’ model (25) are valid for ladder model (26).

Since the specific form of the existing and missing multiplets in the wave function (26) on each two nearest neighbor spin pairs depends on the parameters $x$ and $y$, the projectors in (29) also depend on $x$ and $y$. Each projector can be written in the form

$$P_k^{1,2} = J_{12}^{(k)}(S_1 \cdot S_2 + S_3 \cdot S_4) + J_{13}^{(k)}(S_1 \cdot S_3 + S_2 \cdot S_4) + J_{14}^{(k)}S_1 \cdot S_4 + J_{23}^{(k)}S_2 \cdot S_3$$

$$+ J_1^{(k)}(S_1 \cdot S_2)(S_3 \cdot S_4) + J_2^{(k)}(S_1 \cdot S_3)(S_2 \cdot S_4) + J_3^{(k)}(S_1 \cdot S_4)(S_2 \cdot S_3) + C^{(k)}$$  \hspace{1cm} (33)$$

and this representation is unique for a fixed value of the parameters $x$ and $y$.

Substituting the above expressions for the projectors into Eq. (29), we obtain the general form of the Hamiltonians $h_{i,i+1}$. Inasmuch as the Hamiltonians $h_{i,i+1}$ have the same form for any $i$, it suffices here to give the expression for $h_{1,2}$:

$$h_{1,2} = J_{12}(A_{12} + A_{34}) + J_{13}(A_{13} + A_{24}) + J_{14}A_{14} + J_{23}A_{23}$$

$$+ J_1A_{12}A_{34} + J_2A_{13}A_{24} + J_3A_{14}A_{23}$$  \hspace{1cm} (34)$$

where

$$A_{ij} = S_i \cdot S_j - \frac{1}{4}$$

and all exchange integrals depend on the model parameters and the spectrum of excited states $J_i = J_i(x, y, \lambda_1, \lambda_2, \lambda_3)$ as follows:

$$J_{12} = -\frac{\lambda_2}{2} + \frac{\lambda_3}{2} \frac{4y^2 - 1}{4y^2 + 1}, \hspace{1cm} J_{23} = -\frac{\lambda_2}{2} + \frac{\lambda_3}{2} \frac{(2y - 1)^2}{4y^2 + 1},$$

$$J_{13} = -\frac{\lambda_2}{2} - \frac{\lambda_3}{2} \frac{4y^2 - 1}{4y^2 + 1}, \hspace{1cm} J_{14} = -\frac{\lambda_2}{2} - \frac{\lambda_3}{2} \frac{(2y + 1)^2}{4y^2 + 1},$$

$$J_1 = 2J_{12} - \lambda_1 \frac{y^4 - x^2(x + 1)^2}{3y^4 + x^2(x + 1)^2}, \hspace{1cm} J_2 = 2J_{13} + 2\lambda_1 \frac{y^4 + y^2x(x + 1)}{3y^4 + x^2(x + 1)^2},$$

$$J_3 = J_{14} + J_{23} + 2\lambda_1 \frac{y^4 - y^2x(x + 1)}{3y^4 + x^2(x + 1)^2}$$  \hspace{1cm} (35)$$
(one should keep in mind that only positive $\lambda_i$ can be substituted to these expressions). The model (35) has evident symmetry: the change of sign of $y$ is equivalent to renumbering of sites $1 \leftrightarrow 2$, $3 \leftrightarrow 4 \ldots$. Therefore, we will consider only the case $y > 0$.

In general, the Hamiltonian $h_{i,i+1}$ contains all the terms presented in (33), but we can simplify it by setting, for example, $J_2 = J_3 = 0$ and solving equations (33) for $\lambda_1, \lambda_2, \lambda_3$. All $\lambda_k$ turn out to be positive in this case for any $x$ and $y$ except two lines $y = 0$ and $x = -1/2$, where ground state is multiple degenerated. The Hamiltonian $h_{i,i+1}$ in this case takes the form

$$h_{1,2} = J_{12}(A_{12} + A_{34}) + J_{13}(A_{13} + A_{24}) + J_{14}A_{14} + J_{23}A_{23} + J_1A_{12}A_{34}$$

(36)

$$J_{12} = -x(x+1) + y^2 - \frac{1}{2}, \quad J_{14} = x(x+1) - y^2 - y,$$

$$J_{13} = -x(x+1) - y^2, \quad J_{23} = x(x+1) - y^2 + y, \quad J_1 = \frac{(x(x+1) - y^2)^2}{y^2} - 1$$

The calculation of the norm of (29) and the singlet ground state correlation functions can be performed in the similar way to the corresponding calculations for the case $y = x = -1/4$. Therefore, we give here the final result for spin correlation functions at $N \rightarrow \infty$

$$\langle S_n \cdot S_{n+2l} \rangle = \frac{1}{4} \cos \left( \frac{4\pi l}{N} \right),$$

$$\langle S_{2n-1} \cdot S_{2n+2l} \rangle = \frac{1}{4} \cos \left( \frac{4\pi l}{N} - \Delta \varphi \right)$$

(37)

These equations mean that the spiral on each leg with pitch angle $\frac{4\pi}{N}$ is formed and the shift angle between spirals on the upper and the lower legs is $\Delta \varphi = \frac{8\pi y}{N}$. At $y = 0$, when spins on each rung form a local triplet, the shift angle vanishes and the spirals on both legs become coherent (we note that the shift angles in Eqs. (23) and (37) are defined in a different way).

Thus, there is just one full rotation of the spin over the length of the ladder, independent of the size of the system and for fixed $l \ll N$ at $N \rightarrow \infty$ two spins on the ladder are parallel.

We emphasize that the spin correlation function $\langle S_i \cdot S_j \rangle$ does not depend on the choice of $\lambda_k$ for a fixed parameters $x, y$, because the ground state wave function of the three-parameter set of Hamiltonians (32) is the same.

### A. Special cases

There are special values of the parameter $x$ for which Eqs. (33) are not valid. For cases of integer or half-integer $x = j$, which correspond to the special cases of the model (35), in Eq. (27) one can easily recognize the Maleev’s boson representation of spin $S = j$ operators:

$$S^+ = b^+(2j - b^+b), \quad S^- = b, \quad S^z = b^+b - j$$

(38)

Generally, the wave functions (3) and (24) resemble MP form but with an infinite matrices represented by the Bose operators. However, in accordance to Maleev’s representation in the special cases the infinite matrices formed by the Bose operators $b^+$ and $b$ can be broken off to the size $n = 2j + 1$ and wave function (26) is reduced to the usual MP form

$$\Psi_0 = Tr \left( g_1 \otimes g_2 \otimes \cdots \otimes g_N \right),$$

(39)

where $g = -xT + yS$ is the $n \times n$ matrix describing states of spin pair on corresponding rung of the ladder. Singlet state matrix is

$$S = I \langle s \rangle$$

(40)

where $I$ is unit matrix and $|s\rangle$ is the singlet state. Triplet state matrix $T$ is expressed by Clebsch-Gordan coefficients $C_{m_1,m_2} = \langle \langle 1,m_1 \rangle (j,m_2) \| (j,m_1 + m_2) \rangle$ as follows:
FIG. 5. Stripe spin structure on the ladder model.

\[
T = \frac{1}{C_{0,j}} \begin{pmatrix}
C_{0,j} |0\rangle & C_{1,j-1} |1\rangle & 0 & 0 & 0 \\
C_{-1,j} |\!\!-1\rangle & C_{0,j-1} |0\rangle & . & 0 & 0 \\
0 & . & . & C_{1,j} |1\rangle & 0 \\
0 & 0 & . & . & C_{-1,j+1} |\!\!-1\rangle \\
0 & 0 & 0 & C_{0,j} |1\rangle & 0
\end{pmatrix},
\]

where $|\sigma\rangle$ is the triplet state with $S^z = \sigma$.

Exact calculation of the correlators in the thermodynamic limit is performed using standard transfer matrix technique and results in

\[
\langle S_1 S_2 \rangle = \frac{x(x+1)-3y^2}{4\omega},
\]

\[
\langle S_i S_{i+2l} \rangle = \frac{x(x+1)(4y^2-1)}{4\omega^2} \left( \frac{\omega - 1}{\omega} \right)^{l-1},
\]

\[
\langle S_{2i+1} S_{2i+2l+2} \rangle = -\frac{x(x+1)(2y-1)^2}{4\omega^2} \left( \frac{\omega - 1}{\omega} \right)^{l-1},
\]

\[
\langle S_{2i+2} S_{2i+2l+1} \rangle = -\frac{x(x+1)(2y+1)^2}{4\omega^2} \left( \frac{\omega - 1}{\omega} \right)^{l-1},
\]

where

\[
\omega = x(x+1) + y^2
\]

In the particular case of zero singlet weight, $y = 0$, when spins on each rung form a local triplet, correlation functions (42) coincide with those obtained in [34, 29]. According to Eqs. (42) the spin correlations have an exponential decay and the correlation length $r_c$ is

\[
r_c = 2\ln \left| \frac{\omega}{\omega - 1} \right|
\]

In particular, for special points of ‘zigzag’ model (2) with $J_{23} = \frac{2x}{2x+1}$ and $J_{13} = -x$ the correlation length $r_c = -2\ln \left( 1 - \frac{1}{x(2x+1)} \right)$.

The correlation length $r_c$ diverges when $x \to \infty$ or $y \to \infty$. In these cases the singlet ground state has collinear or stripe spin structure, i.e. spin-spin correlations are ferromagnetic along legs and antiferromagnetic between them (Fig. 5), with a magnetic order $m$:

\[
\langle S_i S_{i+2l} \rangle \simeq -\langle S_i S_{i+1+2l} \rangle \simeq m^2, \quad m \simeq \frac{xy}{x^2 + y^2}
\]

When $y = x$ (the ‘zigzag’ model) the magnetic order is equal to the classical value $m = 1/2$.

We note that the wave function (26) show double-spiral ordering for all values of $x$ and $y$ excluding the special lines $x = j$. The crossover between spiral and stripe states occurs in the exponentially small (at $N \to \infty$) vicinity of the special lines.
B. Spectrum of the model

Generally, the excitation spectrum of model (28, 30) cannot be calculated exactly. It is clear that this spectrum is gapless because, for example, the one-magnon energy is $\sim N^{-4}$ at $N \to \infty$. Moreover, the lowest singlet excitation is gapless as well. For model (28, 30), lying on the special lines this fact can be established from the following consideration. As mentioned above, the crossover between spiral and stripe states occurs in the exponentially small vicinity of the special lines. It means that two wave functions $\Psi_0(j)$ and $\Psi_0(j+\delta)$ corresponding to points $(x = j, y)$ and $(x = j + \delta, y)$ respectively are almost orthogonal at $\delta \sim e^{-N}$. Hence, we can consider a spiral wave function $\Psi_0(j+\delta)$ as a variational function of the excited singlet state at special point $x = j$. The energy of this excited state at special point $x = j$ is

$$E_{ex} = \langle \Psi_0(j) | H(j) | \Psi_0(j+\delta) \rangle$$

$$\sim \left( \Psi_0(j+\delta) | H(j+\delta) - \delta \frac{dH(j+\delta)}{dx} | \Psi_0(j+\delta) \right) \sim \text{const} \cdot \delta \sim e^{-N}$$

Thus, on the special lines the stripe ground state of considered model is asymptotically degenerated with an excited spiral singlet state at the thermodynamic limit. It is not clear if the degeneracy is exponentially large or not. This consideration is valid for any integer or half-integer $x = j$, but it is not valid for the parameters $x \neq j$, which are out of special lines. We performed numerical diagonalization of finite ladders for various parameters $x$ and $y$ and found that the exponential degeneracy possibly takes place for all parameters $x$ and $y$, but we can not confirm it strictly.

It is interesting to note that singlet wave function (26) can be also represented in a special recurrent form (36)

$$\Psi_0 = P_0 \Psi_M,$$

$$\Psi_M = (s_1^+ + \nu_1 s_2^+ + \nu_2 s_3^+ + \ldots + \nu_N s_N^+) \ldots$$

$$(s_{2n-1}^+ + \nu_1 s_{2n}^+ + \nu_2 s_{2n+1}^+ + \ldots + \nu_N s_{2n+1}^+) \ldots$$

(44)

where $s_i^+$ is the $s = \frac{1}{2}$ raising operator. Eq. (45) contains $M$ operator multipliers and the vacuum state $|\downarrow\downarrow\ldots\downarrow\rangle$ is the state with all spins pointing down. The function $\Psi_M$ is the eigenfunction of $S_z$ with $S_z = 0$ but it is not the eigenfunction of $S^2$. $P_0$ is a projector onto the singlet state. Two parameters $\nu_1$ and $\nu_2$ in wave function (43) is connected with parameters $x$ and $y$ in (27) as

$$\nu_1 = \frac{1+x-y}{1+x+y}, \quad \nu_2 = \frac{1}{1+x+y}$$

The norm of the wave function (14) and expectation values can be also calculated with use of recurrent technique developed in (21, 24). Certainly, it gives the same expressions (17, 12) for spin correlation function.

C. Antiferromagnetic ladder model

Now we should consider in particular the special case $x = 1/2$ (14). In this case the product $g_i \otimes g_{i+1}$ contains only one singlet and one triplet and does not contain quintet. Therefore, in this case the cell Hamiltonian can be written in the form (11)

$$H = \sum_i h_{i,i+1}, \quad h_{i,i+1} = \sum_{k=1}^4 \lambda_k P_k^{i,i+1},$$

(46)

where $P_k^{i,i+1}$ is a projector onto the quintet state. If all $\lambda_k > 0$, the ferromagnetic state has positive energy $E = M \lambda_4$ and wave function $\Psi_0$ (26) is now non-degenerate singlet ground state wave function for the Hamiltonian (10), while for Hamiltonian (28, 29) $\Psi_0$ is also exact ground state but degenerate with ferromagnetic state. Using a freedom in choice of $\lambda_k$ we can exclude all four-spin interactions in $h_{i,i+1}$

$$h_{i,i+1} = J_{12} (S_{2i-1} \cdot S_{2i} + S_{2i+1} \cdot S_{2i+2}) + J_{13} (S_{2i-1} \cdot S_{2i+1} + S_{2i} \cdot S_{2i+2}) + J_{14} S_{2i-1} \cdot S_{2i+2} + J_{23} S_{2i} \cdot S_{2i+1} + C$$

(47)
FIG. 6. Singlet-triplet gap of the model (48) as a function of the parameter $y$. The circles denote the results of the extrapolation of an exact finite-chain calculations. The solid line represents the dependence $\Delta(y)$ given by Eq.(50).

and all exchange integrals $J_{ij}$ depend on one model parameter $y < 3/2$ (this inequality is necessary to satisfy $\lambda_k > 0$)

$$J_{12} = \frac{3}{2}(4y^2 - 1), \quad J_{14} = -y(3 + 2y)(2y - 1)^2;$$
$$J_{13} = -2y^2(4y^2 - 1), \quad J_{23} = y(3 - 2y)(2y + 1)^2, \quad C = 9y^2 + \frac{3}{4} \quad (48)$$

It follows from Eq.(42) that the ground state has ultrashort-range correlations with $r_c \sim 1$. For example, $r_c(y = 0) = 2\log \frac{3}{2}$, which coincides with correlation length of AKLT model. But at $y = \frac{1}{2}$ all correlations are zero except $\langle S_{2i} \cdot S_{2i+1} \rangle = -\frac{3}{4}$. It implies that at $y = \frac{1}{2}$ model (46-48) has a dimer ground state.

The Hamiltonian (46-48) of the cyclic ladder has a singlet-triplet gap $\Delta$ for finite $N$. It is evident that for $y = \frac{1}{2}$ the gap exists for $N \to \infty$ and $\Delta(\frac{1}{2}) = 4$. The existence of the finite gap at the thermodynamic limit in the range $0 < y < \frac{3}{2}$ follows from the continuity of the function $\Delta(y)$. It is also clear that $\Delta(y)$ at $N \to \infty$ vanishes at the boundary points $y = 0$ and $y = \frac{3}{2}$ when the ground state is degenerate and there are low-lying spin wave excitations.

Unfortunately, the method of the exact calculation of $\Delta(y)$ in the thermodynamic limit is unknown. For the approximate calculation $\Delta(y)$ we use the trial function of the triplet state in the form

$$\Psi_t = \sum_n s_n^+ e^{ikn} \Psi_0, \quad (49)$$

The trial function $\Psi_t$ gives $\Delta(y)$ which at $N \to \infty$ has minima at $k = \pi$ and $k = 0$ for $0 < y < \frac{1}{2}$ and $\frac{1}{2} < y < \frac{3}{2}$, respectively

$$\Delta(y) = \frac{32y^2(4y^2 + 1)}{4y^2 + 3}, \quad 0 < y < \frac{1}{2}$$
$$\Delta(y) = \frac{128y^2}{(4y^2 + 1)(4y^2 + 3)}, \quad \frac{1}{2} < y < \frac{3}{2} \quad (50)$$

The dependence of $\Delta(y)$ given by Eq.(50) is shown on Fig.6 together with the results of extrapolations of exact finite-chain calculations. Both dependences agree very well for $y \leq \frac{1}{2}$. However, $\Delta(y)$ given by Eq.(50) is not zero at $y = \frac{3}{2}$, while numerical calculations fit the dependence $\Delta(y) \sim \sqrt{\frac{3}{2} - y}$ at $y \to \frac{3}{2}$.

We note that the trial function of the type (49) gives the value 0.7407 for the singlet-triplet gap in the AKLT model. This estimate is close to the value 0.7143 obtained by another approach in [4].

For other special cases one can also construct Hamiltonians for which $\Psi_0$ is non-degenerate singlet ground state wave function. But in these cases one have to introduce more distant interactions.
IV. VALENCE-BOND-STATE MODELS

A. One-dimensional model

We studied previously a one-parameter ladder model (46-48) with non-degenerate singlet ground state. The exact ground state wave function of the cyclic ladder was written in the MP form (39). Now we write the wave function $\Psi_0$ in a form more suitable for subsequent generalization to other types of lattices.

We consider a ladder of $N = 2M$ spins 1/2. The wave function of this system is described by the $N$th-rank spinor

$$\Psi = \Psi_{\lambda\mu\nu...\tau},$$

(51)

where the indices $\lambda, \mu, \nu,..., \tau = 1, 2$ correspond to different projections of the spin 1/2.

We partition the system into pairs of spins located on rungs of the ladder. The wave function can then be written as the product of $M$ second-rank spinors

$$\Psi = \Psi_{\lambda\mu}^{(1)}\Psi_{\nu\rho}^{(2)}...\Psi_{\sigma\tau}^{(M)}.$$  

(52)

We then form a scalar from Eq. (52), simplifying the latter with respect to index pairs:

$$\Psi_s = \Psi_{\lambda}^{\nu(1)}\Psi_{\nu}^{\kappa(2)}...\Psi_{\sigma}^{\lambda(M)}.$$  

(53)

Here subscripts correspond to the covariant components of the spinor, which are related to the contravariant components (superscripts) through the metric spinor

$$g_{\lambda\mu} = g^{\lambda\mu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(54)

$$\Psi_{\lambda} = g_{\lambda\mu}\Psi_{\mu}, \quad \Psi^{\lambda} = g^{\mu\lambda}\Psi_{\mu}.$$  

The scalar function (53) can thus be written in the form

$$\Psi_s = \Psi_{\lambda}^{\mu(1)}g_{\mu\nu}\Psi_{\nu}^{\rho(2)}g_{\rho\kappa}...\Psi_{\tau}^{\lambda(M)}g_{\tau\lambda}.$$  

(55)

The scalar function $\Psi_s$ obviously describes the singlet state.

The second-rank spinor describing the pair of spins 1/2 can be written in the form

$$\Psi_{\lambda\mu} = c_t\Psi_{t\mu}^t + c_s\Psi_{s\mu}^s,$$  

(56)

where $\Psi_{t\mu}^t$ and $\Psi_{s\mu}^s$ are symmetric and antisymmetric second-rank spinors, respectively, and $c_t$ and $c_s$ are arbitrary constants. We know that the symmetric second-rank spinor describes a system with spin $S = 1$, so that the pair of spins 1/2 in this case forms a triplet. If $\Psi_{t\mu}^t$ is an antisymmetric second-rank spinor reducible to a scalar multiplied by $g_{t\mu}$, the spin pair exists in the singlet state. Consequently, the ratio of the constants $c_t$ and $c_s$ determines the relative weights of the triplet and singlet components on the pair of spins $s = 1/2$ and is a parameter of the model. In particular, for $c_s = 0$ the wave function (56) contains only a triplet component, and for $c_t = 0$ it contains only a singlet component.

We note that the wave function (53) has MP form (39) with the matrices $g_i$ representing a mixed second-rank tensor:

$$g_i = \Psi_{\lambda}^{\nu(i)} = c_t\frac{1}{2}|\uparrow\downarrow + \downarrow\uparrow\rangle_i + c_s\frac{1}{2}|\downarrow\downarrow\rangle_i - |\uparrow\uparrow\rangle_i,$$  

(57)

where $I$ is the unit matrix.

We now choose a Hamiltonian $H$ for which the wave function (55) is an exact ground-state wave function. To do so, we consider the part of the system (cell) consisting of two nearest neighbor spin pairs. In the wave function (55) the factor corresponding to the two spin pairs is a second-rank spinor:

$$\Psi_{\lambda}^{\mu(1)}g_{\mu\nu}\Psi_{\nu}^{\rho(2)}g_{\rho\kappa}...\Psi_{\tau}^{\lambda(M)}.$$  

The general form of the Hamiltonian (46) can be reduced to a more simple form (47) with $c_s/c_t = 2y$. The Hamilto

Thus, the singlet ground state wave function of the model (46) can be also written in a spinor form (55).
B. Two-dimensional model

Now we consider an \( N \times N \)-site square lattice with cyclic boundary conditions. We replace each site of the lattice by a square (Fig. 7) with spins \( s = 1/2 \) at its corners, making the total number of spins equal to \( 4N^2 \). To avoid misunderstanding, however, from now on we continue to refer to these squares as sites. The wave function of the system is described by the product of fourth-rank spinors

\[
\Psi = \prod_n \Psi_{\lambda \mu \nu \rho}(n).
\]  

(59)

By analogy with (55), from Eq. (59) we form the scalar

\[
\Psi = \prod_n \Psi_{\lambda \mu \nu \rho}(n) g_{\lambda \mu \nu \rho}(n+a+b).
\]  

(60)

where \( a \) and \( b \) are unit vectors in the \( x \) and \( y \) directions.

The singlet wave function (60) is conveniently identified graphically with a square lattice, each site corresponding to a fourth-rank spinor \( \Psi_{\lambda \mu \nu \rho} \) (whose form is identical for all sites), and each segment linking sites corresponds to a metric spinor \( g_{\lambda \mu} \) (Fig. 8).

To completely define the wave function (60), it is necessary to know the form of the site spinor \( \Psi_{\lambda \mu \nu \rho} \). The specific form of the fourth-rank spinor \( \Psi_{\lambda \mu \nu \rho} \) [and, hence, the wave function (60)] describing the system of four spins \( s = 1/2 \) is governed by 14 quantities, which are parameters of the model.
We now choose a Hamiltonian $H$ for which the wave function $\Psi$ is an exact ground state wave function. As in the one-dimensional case, we seek the required Hamiltonian in the form of a sum of cell Hamiltonians acting in the space of two nearest neighbor spin quartets:

$$H = \sum_n H_{n,n+a} + \sum_n H_{n,n+b}. \quad \text{(61)}$$

The first term in Eq. (61) is the sum of the cell Hamiltonians in the horizontal direction, and the second term is the same for the vertical. The cell Hamiltonians along each direction have the same form, but the “horizontal” and “vertical” Hamiltonians differ in general. In the following discussion, therefore, we consider only the Hamiltonians $H_{1,2}$ and $H_{1,3}$ (Fig. 9), which describe interactions of “sites” in the $x$ and $y$ directions, respectively.

For the wave function $\Psi$ to be an exact eigenfunction of the Hamiltonian $H$, it is sufficient that the sixth-rank spinors

$$\Psi^{\lambda_1\mu_1\nu_1\rho_1}(1)\Psi^{\lambda_2\mu_2\nu_2\rho_2}(2)\Psi^{\nu_1,\lambda_2},$$

$$\Psi^{\lambda_1\mu_1\nu_1\rho_1}(1)\Psi^{\lambda_3\mu_3\nu_3\rho_3}(3)\Psi^{\rho_1,\mu_3}, \quad \text{(62)}$$

be eigenfunctions of the corresponding cell Hamiltonians $H_{1,2}$ and $H_{1,3}$.

In general, when the site spinor $\Psi^{\lambda\mu\nu\rho}$ is not symmetric with respect to any indices, the possible states of two quartets of spins $s = 1/2$ consist of 70 multiplets. A wave function represented by a sixth-rank spinor contains only 20 of them. Accordingly, the cell Hamiltonians $H_{1,2}$ and $H_{1,3}$ can be represented by the sum of projectors onto the 50 missing multiplets:

$$H_{1,2} = \sum_{k=1}^{50} \lambda_k P_k^{1,2}, \quad H_{1,3} = \sum_{k=1}^{50} \mu_k P_k^{1,3}, \quad \text{(63)}$$

where the positive constants $\lambda_k$ and $\mu_k$ are the excitation energies of $H_{1,2}$ and $H_{1,3}$, and the specific form of the projectors depends on 14 model parameters.

Inasmuch as

$$H_{n,n+a}|\Psi_s\rangle = 0, \quad H_{n,n+b}|\Psi_s\rangle = 0, \quad \text{(64)}$$

for the total Hamiltonian $H$ we have the expression

$$H|\Psi_s\rangle = 0. \quad \text{(65)}$$

Consequently, $\Psi_s$ is the ground-state wave function of the total Hamiltonian $H$, because it is a sum of nonnegative definite cell Hamiltonians. Also, it can be rigorously proved that the ground state of $H$ is nondegenerate.

As mentioned above, the specific form of the projectors depends on 14 model parameters, and in general the cell Hamiltonians (63), expressed in terms of scalar products of the type $s_i \cdot s_j$, $(s_i \cdot s_j)(s_k \cdot s_l)$, etc., have an extremely cumbersome form. We therefore consider a few special cases.
FIG. 10. Pattern of independent singlet pairs (double lines).

When the site spinor $\Psi^{\lambda\mu\nu\rho}$ is a symmetric fourth-rank spinor $Q^{\lambda\mu\nu\rho}$ (corresponding to the two-dimensional AKLT model[^4]), only the quintet component out of the six multiplets on each spin quartet is present in the wave function [^4]. The sixth-rank spinors [^12] are symmetric with respect to two triplets of indices and, hence, contain four multiplets with $S = 0, 1, 2, 3$ formed from two quintets. Consequently, the cell Hamiltonian ($H_{1,2}$ and $H_{1,3}$ coincide in this case) has the form

$$H_{1,2} = \sum_{k=1}^{66} \lambda_k P_k^{1/2}. \quad (66)$$

If we set $\lambda_k = 1$ ($k = 1, 66$), we can write Eq. (66) in the form

$$H_{1,2} = P_4(S_1 + S_2) + [1 - P_2(S_1)P_2(S_2)], \quad (67)$$

where $S_i$ is the total spin of the quartet of spins $s = 1/2$ on the $i$th site, $S_i = s_1(i) + s_2(i) + s_3(i) + s_4(i)$, and $P_l(S)$ is the projector onto the state with spin $S = l$.

If the four spins $s = 1/2$ at each site are replaced by a single spin $S = 2$ and if the wave function (60) is treated as a wave function describing a system of $N^2$ spins $S = 2$, the second term in the Hamiltonian (67) vanishes, and we arrive at the Hamiltonian of the two-dimensional AKLT model:

$$H_{1,2} = P_4(S_1 + S_2) = \frac{1}{28} S_1 \cdot S_2 + \frac{1}{40} (S_1 \cdot S_2)^2 + \frac{1}{180} (S_1 \cdot S_2)^3 + \frac{1}{2520} (S_1 \cdot S_2)^4. \quad (68)$$

Another interesting special case is encountered when the system decomposes into independent one-dimensional chains. This happens if the site spinor $\Psi^{\lambda\mu\nu\rho}$ reduces to a product of two second-rank spinors, each describing two spins 1/2. For example,

$$\Psi^{\lambda\mu\nu\rho}(s_1, s_2, s_3, s_4) = \varphi^{\lambda\nu}(s_1, s_3)\varphi^{\mu\rho}(s_2, s_4). \quad (69)$$

In this case the Hamiltonians $H_{1,2}$ and $H_{1,3}$ contain interactions of four rather than eight spins 1/2 and have the form [^4].

The simplest case is when the site spinor $\Psi^{\lambda\mu\nu\rho}$ is a product of four first-rank spinors:

$$\Psi^{\lambda\mu\nu\rho}(s_1, s_2, s_3, s_4) = \varphi^{\lambda}(s_1)\varphi^{\mu}(s_2)\varphi^{\nu}(s_3)\varphi^{\rho}(s_4). \quad (70)$$

Now the system decomposes into independent singlet pairs (Fig.Fig.10), and the total Hamiltonian of the system has the form

$$H = \sum_{i,j} \left( s_i \cdot s_j + \frac{3}{4} \right), \quad (71)$$

where $s_i$ and $s_j$ are the spins forming the singlet pairs.
C. Spin correlation functions in the ground state

We now look at the problem of calculating the norm and the correlation function of the model described by the wave function \( \Psi_{\lambda \mu \nu \rho}(n) \). The expression for the norm of the wave function \( G = \langle \Psi_s | \Psi_s \rangle \) has the form

\[
G = \prod_n \left\langle \Psi_{\lambda \mu \nu \rho}^a(n) \right| \Psi_{\lambda \mu \nu \rho}^b(n) \rangle g_{\alpha a \lambda \mu \nu \rho} g_{\beta b \lambda \mu \nu \rho},
\]

where \( R_{\alpha \beta} \) is a \( 4 \times 4 \times 4 \times 4 \) matrix.

According to the selection rules for the projection of the total spin \( S_z \), only 70 of the 256 elements in the expression \( \langle \Psi_{\lambda \mu \nu \rho}^a(n) | \Psi_{\lambda \mu \nu \rho}^b(n) \rangle \) are non-vanishing. Consequently, the matrix \( R \) also contains at most 70 elements. If we regard the elements of \( R \) as Boltzmann vertex weights, the problem of calculating the norm reduces to the classical 70-vertex model. Since the exact solution for the 70-vertex model is unknown, numerical methods must be used to calculate the norm and the expectation values.

To calculate the above-indicated expected values, we carry out Monte Carlo calculations on 20 \times 20-site lattices. As mentioned, the ground-state wave function of the model depends on 14 parameters and, of course, cannot possibly be analyzed completely. We confine the numerical calculations to the case in which the spinor \( \Psi_{\lambda \mu \nu \rho} \) depends on one parameter \( \alpha \):

\[
\Psi_{\lambda \mu \nu \rho} = \cos \alpha \cdot Q_{\lambda \mu \nu \rho} + \sin \alpha \cdot \left( A_{\lambda \mu \nu \rho} - Q_{\lambda \mu \nu \rho} \right),
\]

where \( \alpha \in [-\pi/2; \pi/2] \), the spinor \( Q_{\lambda \mu \nu \rho} \) is symmetric with respect to all indices, and

\[
A_{\lambda \mu \nu \rho} = \varphi^\lambda(s_1) \varphi^\mu(s_2) \varphi^\nu(s_3) \varphi^\rho(s_4).
\]

In this case we have a one-parameter model with two well-known limiting cases. One corresponds to \( \alpha = \pi/4 \), for which \( \Psi_{\lambda \mu \nu \rho} = A_{\lambda \mu \nu \rho} \), and the system decomposes into independent singlet pairs (Fig. 10); the other limiting case corresponds to \( \alpha = 0 \) (our model reduces to the two-dimensional AKLT model in this case, the spins at each site forming a quintet).

In the given model there are four spins \( s = 1/2 \) at each site, and the enumeration of each spin is determined by the order number of the lattice site to which it belongs and by its own number at this site. The spin correlation function therefore has the form

\[
f_{ij}(r) = \langle s_i(n) \cdot s_j(n + r) \rangle.
\]

In determining the spin structure of the ground state, however, it is more practical to consider the more straightforward quantity \( F(r) \):

\[
F(r) = \sum_{i,j=1}^4 \langle s_i(n) \cdot s_j(n + r) \rangle = \langle S(n) \cdot S(n + r) \rangle.
\]

The function \( F(r) \) is left unchanged by a change of sign of \( \alpha \). We note, however, that only the total correlation function, and not \( f_{ij}(r) \), possesses symmetry under a change of sign of \( \alpha \). This assertion is evident, for example, in Fig. 13, which shows the dependence of \( f_{31}(r) \) on \( \alpha \) as an illustration.

The correlation function decays exponentially as \( r \) increases, differing from the one-dimensional model in that the pre-exponential factor also depends on \( r \). Figure 14 shows the dependence of the correlation length \( r_c \) on the parameter \( \alpha \). The correlation length is a maximum at the point \( \alpha = 0 \) (two-dimensional AKLT model), decreases as \( \alpha \) increases, and at \( \alpha = \pi/4 \), when the system decomposes into independent singlet pairs (Fig. 10), it is equal to zero. With a further increase in \( \alpha \) the correlation length increases and attains a second maximum at \( \alpha = \pi/2 \). Like the correlation function \( F(r) \), the function \( r_c(\alpha) \) is symmetric with respect to \( \alpha \). It is evident from Fig. 12 that the parameter \( \alpha \) has two ranges corresponding to states with different symmetries. In the range \(|\alpha| < \pi/4\) the correlation function \( F(r) \) exhibits antiferromagnetic behavior:

\[
F(r) \propto (-1)^{r_x + r_y} e^{-|r|/r_c},
\]

\[
18
\]
whereas the spins at one site are coupled ferromagnetically, $\langle s_i(n) \cdot s_j(n) \rangle > 0$. On the other hand, in the range $\pi/4 < |\alpha| < \pi/2$ the correlation function $F(r)$ is always negative:

$$F(r) \propto -e^{-|r|r_c}$$

and all the correlation functions at one site are also negative.

These ranges have two end points in common, $\alpha = \pm \pi/4$, where $r_c = 0$. Whereas $\alpha = \pi/4$ corresponds to the trivial partition of the system into independent singlet pairs, the case $\alpha = -\pi/4$ is more interesting. In this case one can calculate all spin correlations exactly. The correlations of spins located on neighboring ‘sites’ of lattice at $\alpha = -\pi/4$ are antiferromagnetic, while all other correlators are zero.

The Hamiltonian for model (73) has a very cumbersome form and for the cases $\alpha = -\pi/4$ and $\alpha = \pi/2$ was given in [31]. Our results suggest that the spin correlation functions decay exponentially with a correlation length $\sim 1$ for an arbitrary parameter $\alpha$. We also assume that the decay of the correlation function is of the exponential type for the 14-parameter model as well, i.e., for any choice of site spinor $\psi^{\lambda\nu\rho}$. This assumption is supported in special cases: 1) the partition of the system into one-dimensional chains with exactly known exponentially decaying correlation functions; 2) the two-dimensional AKLT model, for which the exponential character of the decay of the correlation function has been rigorously proved [32]. Further evidence of the stated assumption lies in the numerical results obtained for various values of the parameter in the one-parameter model.
D. Generalization of the model to other types of lattices

The wave function \( \psi^{(81)} \), \( \psi^{(55)} \), \( \psi^{(60)} \) can be generalized to any type of lattice. The general principle of wave function construction for a system of spins 1/2 entails the following:

1) Each bond on a given lattice has associated with it two indices running through the values 1 and 2, one at each end of the bond.

2) Each bond has associated with it a metric spinor \( g_{\lambda \mu} \) with the indices of the ends of this bond.

3) Each site of the lattice (a site being interpreted here, of course, in the same sense as in Sec.IVB) with \( m \) outgoing bonds has associated with it an \( m \)-th-rank spinor with the indices of the bonds adjacent to the site.

4) The wave function is the product of all spinors at sites of the lattice and all metric spinors.

It is obvious that each index in the formulated wave function is encountered twice, so that the wave function is scalar and, hence, singlet.

The wave function so constructed describes a system in which each lattice site contains as many spins \( s = 1/2 \) as the number of bonds emanating from it.

To completely define the wave function, it is necessary to determine the specific form of all site spinors. The coefficients that determine their form are then parameters of the model.

The Hamiltonian of such a model is the sum of the cell Hamiltonians acting in the spin space of the subsystem formed by the spins at two mutually coupled sites:

\[
H = \sum_{\langle ij \rangle} H_{ij};
\]

Each cell Hamiltonian is the sum of the projectors with arbitrary positive coefficients onto all multiplets possible in the corresponding two-site subsystem except those present in the constructed wave function:

\[
H_{i,j} = \sum_k \lambda_k P_{k}^{ij}.
\]

Then \( H_{i,j} \psi_s = 0 \) and, accordingly, \( H \psi_s = 0 \).

Consequently, \( \psi_s \) is an exact ground-state wave function.

We note that any two lattice sites can be joined by two, three, or more bonds, because this does not contradict the principle of construction of the wave function. Moreover, the general principle of construction of the wave function is valid not only for translationally symmetric lattices, but for any graph in general. As an example, let us consider the system shown in Fig. 13. The wave function of this system has the form

\[
\psi_s = \psi^{(1)} \psi^{(2)} \psi^{(3)} \psi^{(4)} g_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} g_{\mu_1 \mu_2 \mu_3 \mu_4} g_{\nu_1 \nu_2 \nu_3 \nu_4} g_{\rho_1 \rho_2 \rho_3 \rho_4} g_{\tau_1 \tau_2 \tau_3 \tau_4};
\]

and describes a system containing ten spins 1/2.

If the given lattice has dangling bonds (as occurs for systems with open boundary conditions), the resulting wave function represents a spinor of rank equal to the number of loose ends. The ground state of this kind of system is therefore \( 2^l \)-fold degenerate, where \( l \) is the number of loose ends. For an open one-dimensional chain, for example, the ground state corresponds to four functions — one singlet and three triplet components. For higher-dimensional lattices this degeneracy depends on the size of the lattice and increases exponentially as its boundaries grow.
V. ELECTRONIC MODELS

In recent years there has been increasing interest in studying models where at least the ground state can be found exactly. The most popular method for the construction of exact ground state is so-called optimal ground state approach (OGS). In the OGS method the ground state of the system is simultaneously the ground state of each local interaction. In this Section we propose new 1D and 2D models of interacting electrons with the exact ground state which are different from those constructed in the OGS method. The ground state wave function of our models is expressed in terms of the two-particle ‘singlet bond’ (SB) function located on sites \( i \) and \( j \) of the lattice:

\[
\langle i, j \rangle = c_{i\uparrow}^+ c_{j\downarrow}^+ - c_{i\downarrow}^+ c_{j\uparrow}^+ + x \left( c_{i\uparrow}^+ c_{i\downarrow}^+ + c_{i\downarrow}^+ c_{j\uparrow}^+ \right) |0\rangle, \tag{82}
\]

where \( c_{i\sigma} \) are the Fermi operators and \( x \) is an arbitrary coefficient. The SB function is the generalization of the RVB function including ionic states. The presence of the ionic states is very important from the physical point of view because, as a rule, the bond functions contain definite amount of the ionic states as well.

It is known a set of 1D and 2D quantum spin models the exact ground state of which can be represented in the RVB form. It is natural to try to find electronic models with exact ground state at half-filling formed by \( x \) electron states. The ground state of our models is expressed in terms of the two-particle 'singlet bond' state which are different from those constructed in the OGS method. The ground state wave function of our models includes ionic states. The presence of the ionic states is very important from the physical point of view because, as a rule, the bond functions contain definite amount of the ionic states as well.

In general case, the local Hamiltonian \( h_{i,j} \) of the lattice:

\[
H = \sum_{i=1}^{N} h_{i} \tag{85}
\]

The basis of three-site local Hamiltonians \( h_{i} \) consists of 64 states, while only eight of them are present in \( \Psi_{1} \) and \( \Psi_{2} \). These 8 states are

\[
[1, i + 1] \varphi_{i+2}, \quad \varphi_{1}[i + 1, i + 2], \quad \varphi_{2}[i + 1, i + 2], \tag{86}
\]

where \( \varphi_{i} \) is one of four possible electronic states of \( i \)-th site: \( |0\rangle_{i}, |\uparrow\rangle_{i}, |\downarrow\rangle_{i}, |\uparrow\downarrow\rangle_{i}, \)

The local Hamiltonian \( h_{i} \) for which all the functions \( \varphi_{i} \) are the exact ground state wave functions can be written as the sum of the projectors onto other 56 states \( |\chi_{k}\rangle \)

\[
h_{i} = \sum_{k} \lambda_{k} |\chi_{k}\rangle \langle \chi_{k}|, \tag{87}
\]

where \( \lambda_{k} \) are arbitrary positive coefficients. This means that the wave functions \( \Psi_{1} \) and \( \Psi_{2} \) are the ground states of each local Hamiltonian with zero energy. Hence, \( \Psi_{1} \) and \( \Psi_{2} \) are the 'optimal' ground state wave functions of the total Hamiltonian \( H \) with zero energy, similarly to the models in OGS. In general case, the local Hamiltonian \( h_{i} \) is many-parametrical and depends on parameters \( \lambda_{k} \) and \( x \). We consider one of the simplest forms of \( h_{i} \) including the correlated hopping of electrons of different types and spin interactions between nearest- and next-nearest neighbor sites:

\[
h_{i} = 2 - x (t_{i,i+1} + t_{i+1,i+2}) \]
\[
+ \left( x^{2} - (1 + x^{2}) (1 - n_{i+1})^{2} \right) T_{i,i+2} \]
\[
+ \frac{8}{3} x^{2} (S_{i} \cdot S_{i+1} + S_{i+1} \cdot S_{i+2} + S_{i} \cdot S_{i+2}), \tag{88}
\]
FIG. 14. The lattice of the Shastry – Sutherland model

where

\[ T_{i,j} = \sum_{\sigma} (c_{i,\sigma}^+ c_{j,\sigma} + c_{j,\sigma}^+ c_{i,\sigma})(1 - n_{i,-\sigma} - n_{j,-\sigma}), \]

\[ t_{i,j} = \sum_{\sigma} (c_{i,\sigma}^+ c_{j,\sigma} + c_{j,\sigma}^+ c_{i,\sigma})(n_{i,-\sigma} - n_{j,-\sigma})^2 \]  \tag{89}

and \( s_i \) is the \( SU(2) \) spin operator.

Each local Hamiltonian \( h_i \) is a non-negatively defined operator at \( |x| \leq 1 \). The following statements related to the Hamiltonian (88) are valid:

1. The functions (83) and (84) are the only two ground state wave functions of the Hamiltonian (88) at \( N_e = N \) (\( N_e \) is the total number of electrons). They are not orthogonal, but their overlap is \( \sim e^{-N} \) at \( N \gg 1 \).

2. The ground state energy \( E_0(N_e/N) \) is a symmetrical function with respect to the point \( N_e/N = 1 \) and has a global minimum \( E_0 = 0 \) at \( N_e/N = 1 \).

3. The translational symmetry of (88) is spontaneously broken in the ground state leading to the dimerization:

\[ \langle |t_{i,i+1} - t_{i+1,i+2}| \rangle = 2 \]

The excited states of the model can not be calculated exactly but we expect that there has to be a gap, because the ground state is formed by the ultrashort-range SB functions. If it is the case, the function \( E_0(N_e/N) \) has a cusp at \( N_e/N = 1 \).

Actually, this model is the fermion version of the Majumdar – Ghosh spin model. Moreover, it reduces to the Majumdar – Ghosh model at \( x = 0 \) and in the subspace with \( n_i = 1 \).

For \( x = 1 \) the Hamiltonian (88) simplifies and takes the form:

\[ H = -2 \sum_j t_{j,j+1} - \sum_j e^{i\pi n_{j+1}} T_{j,j+2} \]  \tag{90}

The 2D model.

We can easily construct the 2D electronic model with the exact ground state which is analogous to the Shastry – Sutherland model (Fig. 14). The Hamiltonian of this model is:

\[ H = \sum_{\{i,j,k\}} h_{i,j} + h_{i,k} + h_{j,k}^d, \]  \tag{91}

where the sum is over all triangles \( \{i,j,k\} \), one of which is shown on Fig.14. So, each diagonal line belongs to the two different triangles. The local Hamiltonians \( h_{j,k}^d \) acting on the diagonal of the triangle \( \{i,j,k\} \), and \( h_{i,j}, h_{i,k} \) have the form (for the sake of simplicity we put \( x = 1 \))

\[ h_{j,k}^d = -2 t_{j,k} + 4 \]

\[ h_{i,j} = -t_{i,j} - e^{i\pi n_k} T_{i,j} \]

\[ h_{i,k} = -t_{i,k} - e^{i\pi n_j} T_{i,k} \]  \tag{92}

It is easy to check that

\[ h_{j,k}^d \ |\varphi_i \ [j,k]\rangle = (h_{i,j} + h_{i,k}) \ |\varphi_i \ [j,k]\rangle = 0 \]
All other states of the local Hamiltonian \( h_{i,j} + h_{i,k} + h_{j,k}^d \) have higher energies. Therefore, the ground state wave function in the half-filling case is the product of the SB functions located on the diagonals shown by dashed lines on Fig.3. This model has non-degenerate singlet ground state with ultrashort-range correlations.

**The ladder model.**

Let us now consider electronic models with a more complicated ground state including different configurations of short-range SB functions. The form of these ground states is similar to that for spin models proposed in \(^{13}\) and generalized in \(^{13}\). In the 1D case our model describes two-leg ladder model (Fig.4). Its ground state is a superposition of the SB functions where each pair of nearest neighbor rungs of the ladder is connected by one SB.

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The wave function of this ground state can be written in a form (93):

\[
\Psi_s = \psi^\lambda h(1) g_{\mu
u} \psi^{\rho\nu}(2) g_{\rho k} \ldots \psi^\sigma_t(N) g_{\tau \lambda}
\]  

The functions \( \psi^\lambda_h(i) \) describes \( i \)-th rung of the ladder

\[
\psi^\lambda_h(i) = c_1 \varphi^\lambda_{2i-1} \varphi^\mu_{2i} + c_2 \varphi^\lambda_{2i} \varphi^\mu_{2i-1} \tag{94}
\]

with

\[
\varphi^\lambda_k = \begin{pmatrix}
|\uparrow\rangle_k \\
|\downarrow\rangle_k \\
|2\rangle_k \\
|0\rangle_k
\end{pmatrix}, \quad g_{\lambda\mu} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & x \\
0 & 0 & x & 0
\end{pmatrix} \tag{95}
\]

It is easy to see that

\[ g_{\lambda\mu} \varphi^\lambda_i \varphi^\mu_j = [i, j] \]

Therefore, the function \( \Psi_s \) is a singlet wave function depending on two parameters \( x \) and \( c_1/c_2 \). Actually, this form of \( \Psi_s \) is equivalent to the MP form with 4 \times 4 matrices \( A_{\lambda\mu}(i) = g_{\lambda\mu} \psi^{\mu\nu}(i) \). Moreover, at \( x = 0 \) and \( c_1/c_2 = -1 \) the function \( \Psi_s \) reduces to the wave function of the well-known AKLT spin-1 model.

In order to find the Hamiltonian for which the wave function (93) is the exact ground state wave function, it is necessary to consider what states are present on the two nearest rungs in the \( \Psi_s \). It turns out that there are only 16 states from the total 256 ones in the product \( \psi^\lambda_h(i) g_{\mu\nu} \psi^{\rho\nu}(i+1) \). The local Hamiltonian \( h_i \) acting on two nearest rungs \( i \) and \( i + 1 \) can be written in the form of (87) with the projectors onto the 240 missing states. The total Hamiltonian is the sum of local ones (85). The explicit form of this Hamiltonian is very cumbersome and, therefore, it is not given here.

The correlation functions in the ground state (13) can be calculated exactly in the same manner as it was done for spin models. It can be shown that all of correlations exponentially decay in the ground state. We expect also that this model has a gap.

This method of construction of the exact ground state can be generalized also to 2D and 3D lattices, as it was done in Sec.IV. Following \(^{13}\), one can rigorously prove that the ground state of these models is always a non-degenerate singlet.

**1D models with the giant spiral order.**

There is one more spin-\( 1/2 \) model with an exact ground state of the RVB type (13). This is the model (8) describing the F-AF transition point. The exact singlet ground state can be expressed by the combinations of the RVB functions \( (i,j) \) distributed uniformly over the lattice points (8). The analog of the wave function (8) in the SB terms is:

\[
\Psi_0 = \sum_{i<j \ldots} (-1)^P [i,j][k,l][m,n] \ldots, \tag{96}
\]

where \( P = (i,j,k,l,\ldots) \) is the permutation of numbers \( (1,2,\ldots,N) \). It is interesting to note that the singlet wave function (96) can be also written in the MP form but with an infinite size matrices (8).

In order to find the Hamiltonian for which the wave function (96) is the exact ground state wave function, let us consider what states are present on the two nearest sites in the \( \Psi_0 \). It turns out that there are only 9 states from the total 16 ones in (86). They are

\[
|\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle, \quad |\uparrow\downarrow + \downarrow\uparrow\rangle, \quad |20 - 02\rangle, \\
|\uparrow\downarrow - \downarrow\uparrow + x|20 + 02\rangle, \quad |\uparrow 0 - 0 \uparrow\rangle, \\
|\uparrow 2 - 2 \uparrow\rangle, \quad |\downarrow 0 - 0 \downarrow\rangle, \quad |\downarrow 2 - 2 \downarrow\rangle
\]  

(97)
The local Hamiltonian $h_{i,i+1}$ can be written as the sum of the projectors onto the 7 missing states \((87)\). At \(|x| > 1\) the most simple form of this total Hamiltonian is:

$$H = \sum_{i=1}^{N} \left( T_{i,i+1} - \frac{2}{x} t_{i,i+1} - 4 S_i \cdot S_{i+1} + \frac{4}{x^2} \eta_i \cdot \eta_{i+1} + \frac{4 x^2 - 3}{x^2} \eta_i^z \eta_{i+1}^z + 1 \right)$$ \hspace{1cm} (98)

We use here \(\eta\) operators:

$$\eta_i^+ = c_{i,\uparrow}^+ c_{i,\uparrow}^-, \quad \eta_i^- = c_{i,\uparrow} c_{i,\downarrow}, \quad \eta_i^z = \frac{1 - n_i}{2},$$

which form another \(SU(2)\) algebra\(^4\), and \(\eta_1 \cdot \eta_2\) is a scalar product of pseudo-spins \(\eta_1\) and \(\eta_2\). We note that this Hamiltonian commutes with \(S^2\), but does not commute with \(\eta^2\). It can be proved\(^2\) that only three multiplets are the ground states of \((98)\): the singlet state \((96)\), the trivial ferromagnetic state \(S = N/2\) and the state with \(S = N/2 - 1\). The norm and the correlators of the electronic model \((98)\) in the singlet ground state are exactly calculated\(^[17]\). For example, the norm of \((94)\) is:

$$\langle \Psi_0 | \Psi_0 \rangle = \left. \frac{d^N}{d\xi^N} \left( 2 \frac{1 + \cosh(x\xi)}{\cos^2(\xi/2)} \right) \right|_{\xi=0}$$

The correlators at \(N \gg 1\) are

$$\langle \eta_i^+ \eta_{i+l}^- \rangle = O \left( \frac{1}{N^2} \right), \quad \langle \eta_i^- \eta_{i+l}^+ \rangle = O \left( \frac{1}{N^2} \right),$$

$$\langle c_{i,\sigma}^+ c_{i+l,\sigma} \rangle = O \left( \frac{1}{N} \right), \quad \langle S_i S_{i+l} \rangle = \frac{1}{4} \cos \left( 2\pi l / N \right)$$ \hspace{1cm} (99)

The correlator \(\langle \eta_i^+ \eta_{i+l}^- \rangle\) which determines the off-diagonal long-range order (ODLRO)\(^4\) vanishes in the thermodynamic limit. At the same time the spin-spin correlations have a spiral form, and the period of the spiral equals to the system size as in the spin model \((8)\).

Another electronic model can be obtained by making the canonical transformation \(c_{i,\uparrow}^+ \rightarrow c_{i,\uparrow}^+\) and \(c_{i,\downarrow}^+ \rightarrow c_{i,\downarrow}^+\). As a result of this transformation, the SB function \((82)\) becomes:

\[
\{i,j\} = c_{i,\uparrow} c_{i,\uparrow} c_{j,\downarrow} c_{j,\downarrow} - x (c_{i,\uparrow} c_{j,\downarrow} + c_{i,\downarrow} c_{j,\uparrow}) |0\rangle,
\]

and the wave function \((99)\) changes to

$$\Psi_0 = \sum_{i<j} \{i,j\} \{k,l\} \{m,n\} \ldots$$ \hspace{1cm} (101)

The function \((101)\) for \(|x| > 1\) is the exact ground state wave function of the transformed Hamiltonian:

$$H = \sum_{i=1}^{N} \left( -T_{i,i+1} - \frac{2}{x} t'_{i,i+1} - 4 \eta_i \cdot \eta_{i+1} + \frac{4}{x^2} S_i \cdot S_{i+1} + \frac{4 x^2 - 3}{x^2} S_i^z S_{i+1}^z + 1 \right),$$ \hspace{1cm} (102)

where

\[
t'_{i,i+1} = \sum_{\sigma} \sigma (c_{i,\sigma}^+ c_{i+1,\sigma}^+ c_{i,\sigma} c_{i+1,\sigma}) (n_{i,-\sigma} - n_{i+1,-\sigma})^2
\]

This Hamiltonian commutes with \(\eta^2\) but does not commute with \(S^2\). Therefore, the eigenfunctions of the Hamiltonian \((102)\) can be described by quantum numbers \(\eta\) and \(\eta^z\). For the cyclic model the states with three different values of \(\eta\) have zero energy\(^4\) [as it was for the model \((8)\)]. They include one state with \(\eta = 0\) \((101)\), all states with \(\eta = N/2:\]
\[ \Psi_{N/2, \eta^z} = (\eta^+)^{N/2 - \eta^z} |0\rangle, \]

and the states with \( \eta = N/2 - 1 \). Therefore, for the case of one electron per site (\( \eta^z = 0 \)), the ground state of the model (102) is three-fold degenerate.

The correlation functions in the ground states with \( \eta = N/2 \) and \( \eta = N/2 - 1 \) for the half-filling case coincide with each other and at \( N \gg 1 \) they are:

\[ \langle c_{i,\sigma}^+ c_{i+l,\sigma} \rangle = O \left( \frac{1}{N} \right), \quad \langle S_i S_{i+l} \rangle = O \left( \frac{1}{N^2} \right), \]
\[ \langle \eta_i^+ \eta_{i+l}^+ \rangle = O \left( \frac{1}{N} \right), \quad \langle \eta_i^- \eta_{i+l}^- \rangle = \frac{1}{4} + O \left( \frac{1}{N} \right) \]

The existence of the ODLRO immediately follows from the latter equations. The correlation functions in the ground state (101) have similar forms as in Eqs.(99):

\[ \langle c_{i,\sigma}^+ c_{i+l,\sigma} \rangle = O \left( \frac{1}{N} \right), \quad \langle S_i S_{i+l} \rangle = O \left( \frac{1}{N^2} \right), \]
\[ \langle \eta_i^- \eta_{i+l}^+ \rangle = 2 \langle \eta_i^- \eta_{i+l}^- \rangle = \frac{1}{6} \cos \left( \frac{2\pi l}{N} \right) \]

The giant spiral ordering in the last equation implies the existence of the ODLRO and, therefore, the superconductivity in the ground state.

Similarly to the original spin model (9), the last two electronic models (98), (102) describe the transition points on the phase diagram between the phases with and without a long-range order (ferromagnetic for the model (98) and off-diagonal for the model (102)). Therefore, we suggest the formation of the ground state with giant spiral order (ferromagnetic or off-diagonal) as a probable scenario of the subsequent destruction of the ferromagnetism and superconductivity.

VI. CONCLUSION

We have considered the class of the 1D and 2D spin and electronic models with the exact ground states.

The one of these models is the spin-1/2 ladder with competing interactions of the ferro- and antiferromagnetic types at the F-AF transition line. The exact singlet ground state wave function on this line is found in the special form expressed in terms of auxiliary Bose-operators. The spin correlators in the singlet state show double-spiral ordering with the period of spirals equal to the system size.

In general case the proposed form of the wave function corresponds to the MP form but with matrices of infinite size. However, for special values of parameters of the model it can be reduced to the standard MP form. In particular, we consider spin-1/2 ladder with nondegenerate antiferromagnetic ground state for which the ground state wave function is the MP one with 2 \times 2 matrices. This model has some properties of 1D AKLT model and reduces to it in definite limiting case.

The ground state wave function of the spin ladders can be represented in an alternative form as a product of second-rank spinors associated with the lattice sites and the metric spinors corresponding to bonds between nearest neighbor sites. Two-dimensional spin-1/2 model is constructed with exact ground state wave function of this type. The ground state of this model is a nondegenerate singlet with exponentially decay of spin correlators. We believe the model has a gap in the spectrum of excitations.

We propose new models of interacting electrons with the exact ground state formed by the singlet bond functions in the same manner as for some spin models. In particular, we considered the models describing the boundary points on the phase diagram between the phases with and without long-range order (ferromagnetic or off-diagonal).

In conclusion we note that the construction of considered models is based on following property. Their Hamiltonians are the sums of the Hamiltonians that are local and non-commuting with each other. In the same time the ground state wave function of the total Hamiltonian is the ground state one for each for them. It is clear that these models are rather special. Nevertheless, the study of them is useful for understanding properties of the real frustrated spin systems and strongly correlated electronic models.

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