On the orthogonal democratic systems in the $L^p$ spaces*

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Abstract

The concept of bidemocratic pair for a Banach space was introduced in [4]. We construct a family of orthonormal systems $\mathcal{F}_l$, $l \in (0, \infty)$ of functions defined on $[-1, 1]$ such that the pair $(\mathcal{F}_l, \mathcal{F}_l)$ is bidemocratic for $L^p[-1, 1]$ and for $L^{p'}[-1, 1]$ if $l \in (0, \frac{p}{2(p-2)})$, where $p > 2$ and $p' = \frac{p}{p-1}$. The system $\mathcal{F}_l$ is not democratic for $L^{p'}[-1, 1]$ when $l \in (\frac{p}{2(p-2)}, \frac{p}{p-2})$. When $l > \frac{p}{2(p-2)}$ the pair $(\mathcal{F}_l, \mathcal{F}_l)$ is not bidemocratic neither for $L^p[-1, 1]$ nor for $L^{p'}[-1, 1]$.

1 Introduction

Greedy algorithms have been studied extensively during last two decades. S.V. Konyagin and V.N. Temlyakov [7] gave a characterization of greedy bases: a basis is greedy if and only if it is unconditional and democratic. An infinite system $X = \{x_k\}_{k=1}^\infty$ in a Banach space $\mathbb{B}$ will be called a democratic system for $\mathbb{B}$ if there exists a constant $D > 1$ such that, for any two finite sets of indices $P$ and $Q$ with the same cardinality $|P| = |Q|$, we have

$$\left\| \sum_{k \in P} \frac{x_k}{\|x_k\|} \right\| \leq D \left\| \sum_{k \in Q} \frac{x_k}{\|x_k\|} \right\|.$$  (1.1)

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A pair of systems \( X = \{ x_k \}_{k=1}^{\infty} \subset \mathcal{B} \), \( X^* = \{ x_k^* \}_{k=1}^{\infty} \subset \mathcal{B}^* \) is called biorthogonal if \( x_k^*(x_m) = \delta_{km} \), where \( \delta_{km} \) is the Kronecker symbol. In [1] bidemocratic bases have been studied. Following [1] we put

\[
\varphi_X(n) = \sup_{|P| \leq n} \left\| \sum_{k \in P} x_k \right\|, \quad \varphi_{X^*}^*(n) = \sup_{|P| \leq n} \left\| \sum_{k \in P} \| x_k \| x_k^* \right\|_{\mathcal{B}^*},
\]

and will say that a pair of biorthogonal systems \((X, X^*)\) is bidemocratic for \( \mathcal{B} \) if there exists \( C > 0 \) such that for any \( n \in \mathbb{N} \)

\[
\varphi_X(n) \varphi_{X^*}^*(n) \leq Cn. \tag{1.2}
\]

Modifying the definition given in [1] we say that \( \varphi_X(n) \) is the fundamental function and \( \varphi_{X^*}^*(n) \) is the dual fundamental function of the pair of biorthogonal systems \((X, X^*)\). It is proved in [1] that a bidemocratic basis is a democratic basis. The above definition of bidemocratic system is given for minimal systems which are not necessarily bases. Further we will check that if a pair of biorthogonal systems \((X, X^*)\) is bidemocratic for \( \mathcal{B} \) then the system \( X \) is democratic in \( \mathcal{B} \). It is clear that if a system is democratic for \( \mathcal{B} \) then any its infinite subsystem is also democratic. Using the concept of bidemocratic pair we find conditions for which the inverse assertion is also true. This idea was used in [1] (see also [5]) to give a complete characterization of weight functions \( \omega \) for which the higher rank Haar wavelets are bidemocratic systems for \( L^p(\mathbb{R}, \omega), 1 < p < \infty \).

One of the main purposes of the article [1] was the study of the duality properties of the greedy algorithms. For example, if \( \mathcal{B} \) is a reflexive Banach space, the pair of biorthogonal systems \((X, X^*)\) is bidemocratic for \( \mathcal{B} \) and \( \| x_j \|_\mathcal{B} \| x_j^* \|_{\mathcal{B}^*} = \theta, j \in \mathbb{N} \) for some \( \theta \geq 1 \) then the pair of biorthogonal systems \((X^*, X)\) is bidemocratic for \( \mathcal{B}^* \). Of course, we came to the same conclusion if \( \varphi_X \simeq \varphi_{X^*}^* \) and \( \varphi_{X^*}^* \simeq \varphi_{X^*}^* \). We say that \( \varphi \) and \( \psi \) are equivalent, \( \varphi \simeq \psi \) if \( \varphi \) and \( \psi \) defined on \( \mathbb{N} \) with values in \( \mathbb{R}^+ = \{ t \in \mathbb{R} : t \geq 0 \} \) and for some \( 0 < C_1 < C_2 \) we have that \( C_1 \varphi(n) \leq \psi(n) \leq C_2 \varphi(n), n \in \mathbb{N} \).

We construct a family of orthonormal systems such that they are bidemocratic for \( L^p \) but for a subset of parameters they are not democratic for the dual space \( L^{p'} \), for another set of parameters those systems are democratic for \( L^p \) but not bidemocratic for \( L^p \). Finally, for another set of parameters they are bidemocratic for \( L^{p} \).

The characteristic function of a set \( E \) is denoted by \( I_E \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( E \subseteq \mathbb{R}, |E| > 0 \) be a measurable set then we write \( \phi \in L^p(E), 1 \leq p < \infty \).
if $\phi : E \to \mathbb{R}$ is measurable on $E$ and the norm is defined by

$$\|\phi\|_{L^p(E)} := \left( \int_E |\phi(t)|^p dt \right)^{\frac{1}{p}} < +\infty.$$  

2 Democratic systems

Let $N_j \subset \mathbb{N}, 1 \leq j \leq \nu$ be such that $N_j \cap N_i = \emptyset$ if $i \neq j$, card $N_j = \infty, 1 \leq j \leq \nu$ and $\bigcup_{j=1}^{\nu} N_j = \mathbb{N}$. For a given pair of biorthogonal systems $(X, X^*)$ consider the biorthogonal pairs $(X_j, X_j^*)$, where $X_j = \{x_k\}_{k \in N_j}, X_j^* = \{x_k^*\}_{k \in N_j}, 1 \leq j \leq \nu$.

Proposition 2.1. Let $(X_j, X_j^*), 1 \leq j \leq \nu$ be pairs of biorthogonal systems defined as above. If pairs $(X_j, X_j^*), 1 \leq j \leq \nu$ are bidemocratic for $B$ and for any $1 \leq i < j \leq \nu$ the functions $\varphi_{X_i}(\cdot), \varphi_{X_j}(\cdot)$ are equivalent then the pair of biorthogonal systems $(X, X^*)$ is bidemocratic for $B$. Moreover, $\varphi_X(\cdot)$ and $\varphi_{X_j}(\cdot)$ are equivalent for any $1 \leq j \leq \nu$.

Proof. Let $\tilde{\varphi}_X(n) = \max_{1 \leq j \leq \nu} \varphi_{X_j}(n), n \in \mathbb{N}$.

We have that for any $n \in \mathbb{N}$

$$\frac{1}{\nu} \varphi_X(n) \leq \tilde{\varphi}_X(n) \leq \varphi_X(n).$$

The right hand inequality is obvious. On the other hand

$$\varphi_X(n) \leq \sum_{j=1}^{\nu} \varphi_{X_j}(n) \leq \nu \tilde{\varphi}_X(n).$$

Let $P \subset \mathbb{N}$ be a finite set. We have that

$$\varphi_X(|P|) \varphi_{X^*}(|P|) \leq \nu \tilde{\varphi}_X(|P|) \sum_{j=1}^{\nu} \varphi_{X_j^*}(|P|)$$

$$\leq \nu \sum_{j=1}^{\nu} C_j \varphi_{X_j}(|P|) \varphi_{X_j^*}(|P|) \leq \nu C \sum_{j=1}^{\nu} C_j |P|,$$

where $C = \max_{1 \leq j \leq \nu} C_j$. 

\hfill \Box
The condition (1.2) yields

**Remark 2.1.** If a pair of biorthogonal systems \((X, X^*)\) is bidemocratic for \(\mathbb{B}\) then there exists \(C > 0\) such that for any \(k \in \mathbb{N}\)

\[\|x_k\|_\mathbb{B} \cdot \|x_k^*\|_{\mathbb{B}^*} \leq C.\] (2.1)

It is proved in [1] that a bidemocratic basis is a democratic basis. The proof given in [1] for bases also works for the proof of the following

**Proposition 2.2.** Let \((X, X^*)\) be a pair of biorthogonal systems bidemocratic for \(\mathbb{B}\). Then the system \(X\) is democratic for \(\mathbb{B}\).

We are going to construct a family of orthonormal systems in order to clarify some duality properties of orthonormal systems if it is democratic for the \(L^p, 1 < p < \infty\) spaces.

Let \(\chi\) be an orthonormal system of functions defined on \([-1, 1]\) as follows:

For any \(n \in \mathbb{N}\) we divide the interval \((-2^{-n+1}, -2^{-n}]\) into \(2^n\) equal intervals \(\Delta^n_j, 1 \leq j \leq 2^n\) such that \(\Delta^n_i \cap \Delta^n_j = \emptyset\) if \(i \neq j\).

Set \(\chi^n_j(x) = 2^n I_{\Delta^n_j}(x) : 1 \leq j \leq 2^n, n \in \mathbb{N}\), where \(I_E(\cdot)\) is the characteristic function of the set \(E \subset [-1, 1]\). It is clear that the system \(\chi = \{\chi^n_j(x) : 1 \leq j \leq 2^n, n \in \mathbb{N}\}\) is an orthonormal system of functions on \([-1, 1]\).

We put \(k_0 = 0\) and \(k_n = k_{n-1} + 2^n = 2(2^n - 1), n \in \mathbb{N}\). In our construction we use the Rademacher system \(\{r_k(t)\}_{k=1}^\infty\), which is an orthonormal system of functions defined on \([0, 1]\) (see [2], [3]). Let

\[f_j^{(n,l)}(x) = \begin{cases} \sqrt{1 - 2^{-\frac{p}{2}}} \chi^n_j(x), & \text{if } x \in [-1, 0); \\ 2^{-\frac{p}{2}} \varphi_{k_{n-1}+j}(x), & \text{if } x \in [0, 1], \end{cases}\]

\(1 \leq j \leq 2^n, n \in \mathbb{N}\) and \(l \in (0, \infty)\).

Let \(f_k^l(x) = f_j^{(n,l)}(x)\) if \(k = k_{n-1} + j\) and \(1 \leq j \leq 2^n\). For any fixed \(l \in (0, \infty)\) the system \(\mathfrak{F}_l = \{f_k^l(x)\}_{k=1}^\infty\) is an orthonormal system of functions defined on \([-1, 1]\).

**Proposition 2.3.** For any \(l \in (0, \infty)\) the system \(\mathfrak{F}_l\) is a democratic system for \(L^p[-1, 1], 2 \leq p < \infty\), and \(\varphi_{\mathfrak{F}_l}(m) \asymp m^{\frac{1}{p}}\).

**Proof.** The proposition is obviously true if \(p = 2\). Thus we only will consider the case \(p > 2\). We have that

\[|c_{n,l}|^p := \left\| f_j^{(n,l)} \right\|_{L^p[-1,1]}^p = 2^{n(p-2)}(1 - 2^{-\frac{p}{2}})^\frac{p}{2} + 2^{-\frac{p}{2}}.\] (2.2)
Let $|b_{k,l}|^p := |c_{n,l}|^p$ if $k = k_{n-1} + j$ and $1 \leq j \leq 2^n$.

We prove that there exists $0 < C_1(l, p) < C_2(l, p)$ such that for any finite set $A \subset \mathbb{N}, |A| = m$

$$C_1(l, p)m^\frac{1}{p} \leq \left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l \right\|_{L^p(-1,1)} \leq C_2(l, p)m^\frac{1}{p}. \quad (2.3)$$

Let $n_0 = [l] + 1$, where $[l] \in \mathbb{N}$ and $[l] \leq l < [l] + 1$. Thus $2^{-\frac{n_0}{2}} \leq 2^{-\frac{n}{2}}$ if $n \geq n_0$. We have that

$$2^{-\frac{n_0}{2}} \cdot 2^{-n(p-2)}(1 - 2^{-\frac{n}{2}})^{-\frac{p}{2}} \leq 2^{-\frac{n}{2}}2^{2-p}2^{\frac{p}{2}} = 2^{2-p}$$

when $n \geq n_0$. Then for any $A \subset [n_0, \infty) \cap \mathbb{N}, |A| = m$ we have that

$$\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l \right\|_{L^p(-1,0]}^p \geq m \frac{1}{1 + 2^{2-p}} > \frac{m}{2}.$$ 

The supports of functions on $[-1,0)$ are not empty and do not coincide. Thus changing the constant we easily get the left side inequality in (2.3) for the general case.

Let $m > 2^{n_0}, m \in \mathbb{N}$ and $\nu \in \mathbb{N}$ be such that $2^{\nu-1} < m \leq 2^\nu$. By the Khintchine inequality (see [2]) it follows that

$$\int_{[0,1]} \left| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l(x) \right|^p dx \leq D_p \left( \sum_{n=1}^\nu |c_{n,l}|^{-2\frac{p}{2}2^n} \right)^{\frac{p}{2}}.$$ 

Clearly

$$|c_{n,l}|^p \geq \frac{1}{2}2^{n(p-2)} \quad \text{for } n \geq n_0. \quad (2.4)$$

Thus it follows that

$$\sum_{n=n_0}^\nu |c_{n,l}|^{-2\frac{p}{2}2^n} \leq 2^\frac{p}{2} \sum_{n=n_0}^\nu 2^{-2n(p-2)} 2^{n(1-\frac{p}{2})}.$$ 

Let $\kappa = \frac{4}{p} - 1 - \frac{1}{4}$. If $\kappa > 0$ we write

$$\sum_{n=n_0}^\nu |c_{n,l}|^{-2\frac{p}{2}2^n} \leq 2^\frac{2}{\kappa} \sum_{n=0}^{2\kappa n} 2^{\frac{1}{2\kappa - 1}2^{\kappa(n+1)}} \leq 2^{2(\kappa + \frac{1}{4})} \frac{m^\kappa}{2^{\kappa - 1}}.$$
Thus it follows
\[
\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l \right\|_{L^p[-1,1]}^p \leq m + D_p(C'_p + C_pm^\kappa)^{\frac{p}{2}}
\]
\[= m(1 + D_p(C'_p m^{\frac{-2}{p}} + C_pm^{\kappa-\frac{2}{p}})^{\frac{p}{2}}).
\]

Whence we obtain the right hand inequality in (2.3) because \(\kappa - \frac{2}{p} < 0\). If \(\kappa \leq 0\) then the proof is obvious.

**Proposition 2.4.** The system \(\mathfrak{F}_l\) is a democratic system for \(L^r[-1,1], 1 \leq r < 2, l \in (0, \frac{r}{2(2-r)}]\) and \(\varphi_{\mathfrak{F}_l}(m) \asymp m^\frac{1}{r} \).

**Proof.** We have that
\[
|\hat{c}_{n,l}|^r := \left\| f_j^{(n,l)} \right\|_{L^r[-1,1]}^r = 2^{n(r-2)}(1 - 2^{-\frac{n}{r}}) \frac{2}{2} + 2^{-\frac{n}{r}}. \tag{2.5}
\]

As above we put \(|\hat{b}_{k,l}|^r := |\hat{c}_{n,l}|^r\) if \(k = k_{n-1} + j\) and \(1 \leq j \leq 2^n\).

If \(l \in (0, \frac{r}{2(2-r)}]\) then \(2^{n(r-2)} \geq 2^{-\frac{n}{r}}\) and it follows that
\[
\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l \right\|_{L^r[-1,1]}^r \geq \frac{m}{2} (1 - 2^{-\frac{1}{r}})^{\frac{r}{2}}.
\]

On the other hand we have that there exists \(n_1 \in \mathbb{N}\) such that
\[
|\hat{c}_{n,l}|^r \geq \frac{1}{2} 2^{n(r-2)} \quad \text{for } n \geq n_1.
\]

Let \(A_n = [k_{n-1} + 1, k_n] \cap A, n \in \mathbb{N}\) and \(\Omega_A = \{n \in \mathbb{N} : A_n \neq \emptyset\}\).

Let \(m > 2^{n_1}, m \in \mathbb{N}\) and \(\nu \in \mathbb{N}\) be such that \(2^{\nu-1} < m \leq 2^\nu\) then by the Khintchine inequality it follows that
\[
\int_{[0,1]} \left| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^l(x) \right|^r dx \leq D_r \left( \sum_{n \in \Omega_A} |\hat{c}_{n,l}|^{-2} 2^{-\frac{n}{r}} |A_n| \right)^{\frac{r}{2}}
\]
\[
\leq 2D_r \left( \sum_{n \in \Omega_A} 2^{2n(2-r)/r} 2^{-\frac{n}{r}} |A_n| \right)^{\frac{r}{2}} \leq 2D_r m^{\frac{r}{2}}
\]

if \(l \leq \frac{r}{2(2-r)}\). Afterwards we proceed as in the proof of Proposition 2.3 and easily finish the proof.
Proposition 2.5. The system $\mathfrak{F}_l$ is not a democratic system for $L^r[-1,1]$, $1 \leq r < 2$ if $l \in \left(\frac{r}{2(r-2)}, \frac{r}{2-r}\right)$.

Proof. If $l > \frac{r}{2(2-r)}$ then $2^{n(r-2)} < 2^{-\frac{nr}{2}}$. Thus for any $n \in \mathbb{N}$

$$2^{-\frac{nr}{2}} \leq |\hat{c}_{n,l}|^r \leq 2 \cdot 2^{-\frac{nr}{2}}$$

(2.6)

where $\hat{c}_{n,l}$ is defined by (2.5). Let $B_n = [k_{n-1} + 1, k_n] \cap \mathbb{N}, n \in \mathbb{N}$. Then it follows that

$$\left\| \sum_{k \in B_n} \frac{1}{|b_{k,l}|} f_k \right\|_{L^r[-1,1]}^r = \left\| \frac{1}{|\hat{c}_{n,l}|} \sum_{k \in B_n} f_k \right\|_{L^r[-1,1]}^r \geq \frac{1}{2} 2^{\frac{nr^2}{2}} 2^{2n(r-2)} (1 - 2^{-\frac{nr}{2}})^{\frac{r}{2}}$$

$$+ \frac{1}{2} \left\| \sum_{j=k_{n-1}+1}^{k_n} r_j(\cdot) \right\|_{L^r[0,1]}^r \geq \frac{1}{2} 2^n \omega + (1 - 2^{-\frac{nr}{2}})^{\frac{r}{2}} + D_r 2^{\frac{nr}{2}},$$

where $\omega = \frac{r}{2} + r - 1$ and $D_r > 0$. Observe that

$$\frac{r}{2} < \omega < 1 \text{ if } l \in \left(\frac{r}{2(2-r)}, \frac{r}{2-r}\right).$$

Afterwards we consider $B^*_n = [k_{n^2-1} + 1, k_{n^2-1} + 2^n] \cap \mathbb{N}, n \in \mathbb{N}$. In this case we have

$$\left\| \sum_{k \in B^*_n} \frac{1}{|b_{k,l}|} f_k \right\|_{L^r[-1,1]}^r = \left\| \frac{1}{|\hat{c}_{n^2,l}|} \sum_{k \in B^*_n} f_k \right\|_{L^r[-1,1]}^r \leq \frac{1}{2} 2^{\frac{n^2r^2}{2}} 2^{n^2(r-2)} 2^n (1 - 2^{-\frac{n^2r}{2}})^{\frac{r}{2}}$$

$$+ \left\| \sum_{j=k_{n^2-1}+1}^{k_{n^2}+2^n} r_j(\cdot) \right\|_{L^r[0,1]}^r \leq \frac{1}{2} 2^n \omega + (1 - 2^{-\frac{n^2r}{2}})^{\frac{r}{2}} + D_r 2^{\frac{n^2r}{2}},$$

where $D_r > 0$ and $\omega = \frac{r}{2} + r - 2 = \omega - 1 < 0$ if $l \in \left(\frac{r}{2(2-r)}, \frac{r}{2-r}\right)$. The inequality $\frac{r}{2} < \omega$ yields that the system $\mathfrak{F}_l$ is not a democratic system. \qed

Proposition 2.6. The system $\mathfrak{F}_l$ is a democratic system for $L^r[-1,1]$, $1 \leq r < 2$, if $l \in \left[\frac{r}{2-r}, \infty\right)$. Moreover, $\varphi_{\mathfrak{F}_l}(n) \asymp n^{\frac{r}{2}}$.

Proof. If $l \in \left[\frac{r}{2-r}, \infty\right)$ we have that the inequalities (2.6) hold. Hence, for any $A \subset \mathbb{N}, |A| = m$

$$\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k \right\|_{L^r[-1,1]}^r \geq \frac{1}{2} \left\| \sum_{n \in \Omega_{A_k}} \sum_{k \in A_n} r_k \right\|_{L^r[0,1]}^r \geq C_r m^{\frac{r}{2}},$$

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where the last inequality follows by the Khintchine inequalities and $C_r > 0$. Let $\nu \in \mathbb{N}$ be such that $2^{\nu-1} \leq m < 2^\nu$. Then

$$\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^r \right\|_{L^r[-1,0]} \leq \sum_{n \in \Omega_A} \sum_{k \in A_n} 2^{n(r-2+\frac{1}{r})},$$

$$\leq \sum_{n \in \Omega_A} 2^{n(\frac{\nu}{2}-1)}|A_n| \leq \sum_{k=1}^{\nu} 2^k \leq \frac{2^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}} - 1} m^{\frac{\nu}{2}}.$$

By Khintchine’s inequalities we obtain

$$\left\| \sum_{k \in A} \frac{1}{|b_{k,l}|} f_k^r \right\|_{L^r[0,1]} \leq \left\| \sum_{n \in \Omega_A} \sum_{k \in A_n} r_k \right\|_{L^r[0,1]} \leq D_r m^{\frac{\nu}{2}}.$$

By resuming the propositions proved above we easily obtain the following theorem.

**Theorem 2.1.** Let $p > 2$ and $p' = \frac{p}{p-1}$. Then the pair $(\mathcal{F}_l, \mathcal{H}_l)$ is bidemocratic for $L^p[-1,1]$ and for $L^{p'}[-1,1]$ if $l \in (0, \frac{p}{2(p-2)}]$. The system $\mathcal{F}_l$ is not democratic for $L^{p'}[-1,1]$ when $l \in (\frac{p}{2(p-2)}, \frac{p}{p-2})$.

When $l > \frac{p}{2(p-2)}$, the pair $(\mathcal{F}_l, \mathcal{H}_l)$ is not bidemocratic neither for $L^p[-1,1]$ nor for $L^{p'}[-1,1]$.

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