Abstract

We present a convenient null gauge for the construction of the balanced equations of motion.

This null gauge has the property that the asymptotic structure is intimately related to the interior one; in particular there is a strong connexion between the field equation and the balanced equations of motion.

We present the balanced equations of motion in second order of the acceleration. We solve the required components of the field equation at their respective required orders, $G^2$ and $G^3$.

We indicate how this approach can be extended to higher orders.

1 Introduction

With this new era of gravitational wave observations\cite{1, 2, 3, 4, 5, 6} coming from binary black holes and binary neutron stars, there also comes the need to have at hand the most convenient models of the physical system, to be used at every stage of their dynamics. It has been said that: “it is infeasible to use the NR simulations directly as search templates”\cite{7}. Therefore researchers\cite{7} use “phenomenological waveforms”.

We would like to contribute with models that are useful for the description of the dynamics of coalescence binary systems and their specific relation with the gravitational emitted radiation. Our work concentrates in the construction of equations of motion of compact objects, subjected to the back reaction due to their emission of gravitational radiation.

This is a sequel of references\cite{8, 9} where we have presented the general necessary framework to construct balanced equations of motion for particle in general relativistic theories, and the specific application to the harmonic gauge. In this work we present this framework applied to the case of general relativity in the null gauge.

As in the previous work, we have in mind is a bound binary gravitationally isolated interactive system. When considering an isolated compact object, it induces one to represent it by an asymptotically flat spacetime. Then, in the asymptotic region one can always write the metric as

$$g = \eta_{\text{asy}} + h_{\text{asy}};$$

where $\eta_{\text{asy}}$ is a flat metric associated to inertial frames in the asymptotic region and $h_{\text{asy}}$ the tensor where all the physical information is encoded. But, as we have already noted, there are as many flat metrics $\eta_{\text{asy}}$ as there are BMS\cite{10, 11} proper supertranslation generators. We have explained in\cite{8} the difficulties in finding appropriate rest frames, and how to solve these issues; making use of supertranslation free definition of intrinsic angular momentum\cite{12, 13}. The root of all obstacles is the appearance of gravitational radiation\cite{14, 15, 16}. Because of this when studying the dynamics of compact objects we take into account the back reaction due to gravitational radiation as our starting point.

Whenever necessary we will assign a label $A$ to the particle under consideration, and use $B$ to denote the rest of the system, that is thought to be the other particle, in the binary situation.

We adopt here the viewpoint explained in\cite{8} in which we assume there exists an exact metric $g$ that corresponds to an isolated binary system of compact objects; which it can be decomposed in a form:

$$g = \eta + h_A + h_B + h_{AB};$$

where $\eta$ is a flat metric, $h_A$ is proportional to a parameter $M_A$, that one can think is some kind of measure of the mass of system $A$, similarly $h_B$ is proportional to a parameter $M_B$, and $h_{AB}$ is proportional to both parameters. To study the gravitational radiation emitted by the motion of particle $A$, we model the asymptotic structure of a sub-metric

$$g_A = \eta + h_A;$$

and to describe the rest of the system, we use a sub-metric

$$g_B = \eta + h_B.$$
For more details see article [8], where it is also explained that the appropriate choice of the flat metric $\eta$ should be related to a local notion of center of mass frame.

Although we will be studying the dynamics of system $A$, to simplify the notation we will avoid using a subindex $A$, whenever possible.

We present here a model for a compact object, treated as a particle on an appropriately chosen flat background. The idea one has in mind is to apply this construction to a binary system, so that each of the compact objects will be treated likewise. The flat background metric by construction will share the same asymptotic region as the full metric of the spacetime: so that one of the inertial system at infinity would be related to this flat global metric. The model, for each monopole, will solve the field equations at appropriate orders and by construction, the dynamics will balance the amount of gravitational radiation emitted due to the acceleration of the body.

Although in paper [8] we have presented several delicate issues that one has to consider when constructing a balanced equations of motion, here we present a model that is still simple and so it has the advantage that one should be able to compute without recurring to supercomputers. It is our intention to provide with this model a method for the calculation of observational waveforms, in a wide range of astrophysical parameters. In future works we will apply this model to specific observations.

In section 2 we describe the interior structure of the null gauge model. Section 3 is devoted to the presentation of the main concepts that arise in the neighborhood of future null infinity of an asymptotically flat spacetime. In section 4 we present the null gauge model from the monopole structure of a general asymptotically flat spacetime. Our main result is presented in section 5 where the balanced equations of motion for the null gauge is calculated up to second order in the accelerations. The last section contains final comments on our work.

2 The interior problem in terms of the null gauge

2.1 The notion of a particle over a flat background

Let us consider a massive point particle with mass $M$ describing, in a flat space-time $(M^0, \eta_{ab})$, a curve $C$ which in a Cartesian coordinate system $x^\mu$ reads

$$x^\mu = z^\mu(\tau_0),$$

with $\tau_0$ meaning the proper time of the particle along $C$.

The unit tangent vector to $C$, with respect to the flat background metric is

$$\mathbf{v}^\mu = \frac{dz^\mu}{d\tau_0},$$

that is, $\eta(\mathbf{v}, \mathbf{v}) = 1$. For each point $p = C(\tau_0)$ let $\mathcal{E}_{p}$ denote the future null cone with vertex at $p$. If we call $x^\mu_p$ the Minkowskian coordinates of an arbitrary point on the cone $\mathcal{E}_p$, then we can define the retarded radial distance from the point $p$ by

$$r = v_\mu \left( x^\mu_p - z^\mu(\tau_0) \right).$$

Below we will introduce the four velocity vector $v^a$, proportional to $\mathbf{v}^a$, but normalized with respect to the metric $g_{ab}$.

2.2 The null gauge near a particle over a non-flat background case

Let us now consider the notion of a particle but in the context of a smooth non-flat background.

Given our background metric, one can always construct, in a neighbourhood of the curve $C$ the null surfaces formed by the future directed null geodesic, emanating from points in $C$; that can be labeled by the null function $u$.

Using the null polar coordinate system $(x^0, x^1, x^2, x^3) = (u, r, (\zeta + \overline{\zeta}), \frac{1}{r}(\zeta - \overline{\zeta}))$, in a neighborhood $C$, one can express a null tetrad as:

$$\ell_a = (du)_a,$$

$$\ell^a = \left( \frac{\partial}{\partial r} \right)_a,$$

$$m^a = \xi^j \left( \frac{\partial}{\partial x^j} \right)_a,$$

$$\overline{m}^a = \overline{\xi}^j \left( \frac{\partial}{\partial x^j} \right)_a,$$

$$n^a = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X_j \frac{\partial}{\partial x^j};$$

with $j = 2, 3$, and we use $a, b, \cdots$ as abstract indices.

This null tetrad satisfies

$$g_{ab} \ell^a n^b = 1,$$

and

$$g_{ab} m^a \overline{m}^b = -1,$$

and all other scalar products are zero.

Several useful expressions describing the null gauge we are using can be found in reference [17].

In particular for Minkowski spacetime the components of the null tetrad are given by:

$$U = \frac{V_0}{V_\eta} - \frac{1}{2},$$

$$\xi^2 = \frac{\xi_0^2}{r} \frac{\sqrt{2} P_0 V_\eta}{r},$$

$$\xi^3 = \frac{\xi_3^2}{r} = -i \frac{\sqrt{2} P_0 V_\eta}{r},$$

$$X^3 = 0;$$

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where a dot means \( \partial / \partial u \), \( P_0 = (1 + \zeta \dot{\zeta})/2 \), and \( V_{\eta} \) is given by the following expression

\[
V_{\eta} = \hat{l}^{\alpha} \hat{v}^b \eta_{ab},
\]

where \( (\eta_{\mu\nu}) = \text{diag}(1,-1,-1,-1) \), \( \hat{v}^\mu = \text{v}^\mu(u) \) is the four velocity in Minkowski space-time, which depends only on \( u \) and satisfies the normalization

\[
\hat{v}^a \hat{v}^b \eta_{ab} = 1,
\]

and the null vector \( \hat{l}^a \) is given by

\[
(\hat{l}^a) = \left(1, \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)\right)
\]

\[
= \left(1, \frac{\zeta \dot{\zeta}}{1 + \zeta \dot{\zeta}}, \frac{\zeta - \dot{\zeta}}{1 + \zeta \dot{\zeta}}, \frac{\zeta \dot{\zeta} - 1}{1 + \zeta \dot{\zeta}}\right)
\]

\[
= \left(\sqrt{4\pi} Y_{00}, -\sqrt{\frac{2\pi}{3}} (Y_{11} - Y_{1,-1}), \frac{i}{\sqrt{\frac{2\pi}{3}}} (Y_{11} + Y_{1,-1}), \sqrt{\frac{4\pi}{3}} Y_{1,0}\right).
\]

We distinguish between the abstract indices \( a, b, \cdots \) and the numeric indices \( \mu, \nu, \cdots = 0, 1, 2, 3 \); since in some cases it is convenient to remark the tensorial or vectorial character of the equations, and in others it is convenient to remark how to make the calculations in a particular frame of reference. The relation between these null coordinates and the Cartesian ones \( y^\nu \) is the following

\[
y^\nu = z^\mu(u) + r l^\mu(u, x^2, x^3);
\]

where \( z^\mu(u) \) is a world line with unit tangent vector \( u^\mu \), and the Cartesian components of the vector \( l \) are

\[
l^\mu = \frac{\hat{l}^\mu}{V_{\eta}}.
\]

Let us note that

\[
l^\mu \hat{v}^\nu \eta_{\mu\nu} = 1.
\]

The complex null vectors \( \hat{m} \) and \( \hat{\hat{m}} \), defined as

\[
\hat{m} = \xi_0^\alpha \frac{\partial}{\partial x^\alpha} = \xi_0^a \frac{\partial}{\partial x^a} + \xi_0^a \frac{\partial}{\partial x^a},
\]

are the natural basis for the unit sphere \( r = 1 \). So we will use this as basis vectors in the subspace of the spheres \( u = \text{constant}, r = \text{constant} \).

3 General asymptotically flat spacetimes

3.1 The leading order behaviour of an adapted null tetrad

Let now \( u \) denote a null hypersurface that contains future directed null geodesics that reach future null infinity. Then we can use the same null tetrad prescription, that we used before, but now adapted to this null congruence.

The components \( \xi^t, U \) and \( X^t \) have the asymptotic expansion:

\[
\xi^2 = \frac{\xi_0^2}{r} + O\left(\frac{1}{r^2}\right), \quad \xi^3 = \frac{\xi_0^3}{r} + O\left(\frac{1}{r^2}\right),
\]

with

\[
\xi_0^2 = \sqrt{2} P_0 V, \quad \xi_0^3 = -i \xi_0^2,
\]

where \( V = V(u, \zeta, \dot{\zeta}) \) and the square of \( P_0 = \frac{(1 + \zeta \dot{\zeta})}{2} \) is the conformal factor of the unit sphere;

\[
U = r U_{00} + U_0 + \frac{U_1}{r} + O\left(\frac{1}{r^2}\right),
\]

where

\[
U_{00} = \frac{\dot{V}}{V}, \quad U_0 = -\frac{1}{2} K_V, \quad U_1 = -\frac{\Psi_0 + \bar{\Psi}_0}{2},
\]

where \( K_V \) is the curvature of the 2-metric.

\[
dS^2 = \frac{1}{V^2 P_0^2} d\zeta d\dot{\zeta};
\]

where the regular conformal metric restricted to scri is precisely \( \tilde{g} \). In terms of the edth operator \( \delta_V \) of the sphere \([10]\) the curvature \( K_V \) is given by

\[
K_V = \frac{2}{V} \delta_V \delta_V V - \frac{2}{V^2} \delta_V V \bar{\delta}_V V + V^2.
\]

Finally, the other components of the vector \( n^a \) have the asymptotic form

\[
X^2 = O\left(\frac{1}{r^2}\right), \quad X^3 = O\left(\frac{1}{r^2}\right).
\]

One can see that the previous expressions can be considered as a subset of the present equations; since the first are also expressing an asymptotically flat spacetime.

3.2 The total momentum and flux

Given any section \( S \) at future null infinity, the total momentum of a generic spacetime, in terms of an inertial (Bondi) frame \([11]\), is normally given by

\[
\mathcal{P}^\nu = -\frac{1}{4\pi} \int_S l^\nu (\Psi_2^0 + \sigma_0 \dot{\sigma}_0) dS^2,
\]

where dot means \( \partial / \partial u \); i.e. the partial derivative with respect to the Bondi time \( \dot{u} \); \( \Psi_2^0 \) is a component of the Weyl tensor in the GHP\([12]\) notation and \( \sigma_0 \) is the leading order behavior of the spin coefficient \( r \) in terms of the asymptotic coordinate \( \tilde{r} \) and \( dS^2 \) is the surface element of the unit sphere. So the set of intrinsic Bondi coordinates are \( (\tilde{u}, \tilde{\theta}, \tilde{\phi}) \) or \( (\tilde{u}, \tilde{\zeta}, \tilde{\dot{\zeta}}) \); where \( (\tilde{\zeta}, \tilde{\dot{\zeta}}) \) are complex stereographic coordinates of the sphere; which are related to the standard coordinates by \( \zeta = e^{i\tilde{\phi}} \cot(\tilde{\frac{\tilde{r}}{2}}) \).

The total momentum for the monopole particle can be calculated using the charge integral of the Riemann tensor technique. In this subsection we will use the notation of reference \([12]\).
The total momentum and flux can be calculated in terms of charge integrals; as described in [3][9]. From which it can derive that the time variation of the Bondi momentum, of a generic spacetime, is expressed by

$$\dot{P}^\mu = - \frac{1}{4\pi} \int_S \dot{\tilde{\nu}} \sigma^0_\sigma_0 dS^2 \equiv - F^\mu; \quad (34)$$

that is, $F^\mu$ is the total momentum flux.

In reference [3] we have discussed how to describe the total momentum and flux in terms of a general time coordinate. Let $u =$constant represent a general time coordinate and set of sections such that

$$\frac{\partial \tilde{\nu}}{\partial u} = \tilde{V}(\tilde{u}, \zeta, \tilde{\zeta}) = \tilde{V}(u, \zeta, \tilde{\zeta}) > 0, \quad (35)$$

is the time derivative of the inertial (Bondi) time $\tilde{u}$ with respect to the non-inertial time $u$; so that we can use the Bondi coordinates or the non-inertial coordinates $(u, \zeta, \tilde{\zeta})$. Then, following [3], the general flux expression is

$$\frac{dP^\mu}{du} = - \frac{1}{4\pi} \int_S \dot{\tilde{\nu}} \tilde{\sigma}^0_\sigma_0 dS^2 = - F^\mu; \quad (36)$$

where now $F^\mu$ is the instantaneous momentum flux with respect to the time $u$.

### 4 Monopole geometry from asymptotic structure

#### 4.1 The metric

If we take the general asymptotic form of an adapted null tetrad of an asymptotically flat spacetime [17] and keep only the terms associated with a monopole, then one is left with the line element

$$ds^2 = \left( - \frac{\tilde{V}}{V} r + K_V - 2 \frac{M(u)}{r} \right) du^2 + 2 du dr - \frac{r^2}{\tilde{V}^2} d\zeta d\bar{\zeta}, \quad (37)$$

where $P = P(u, \zeta, \tilde{\zeta}) = V P_0$, and $K_V$ is defined in equation (31). We refer to this as the monopole particle line element and is our model for the sub-metric [3].

This line element of a monopole can be related to what we have called Robinson-Trautman (RT) geometries [19], which are generalizations of Robinson-Trautman spacetimes. Robinson-Trautman [20] spacetimes have been very useful for estimating the total gravitational radiation in the head-on black hole collision [21][22][23]. In reference [21] we have applied these geometries to the description of the total energy radiated in the head-on black hole collision with equal mass; and it was shown that our calculations agree remarkably well with the numerical exact calculations of Anninos et al. [24]. The case of unequal mass black hole collision, was treated numerically in reference [22]; and our technique based on the use of the RT geometries [22] showed again an impressive agreement with the exact calculations.

We assume that the mass parameter $M = \text{constant}$ but to account for a possible time variation of the mass, we also consider the degree of freedom $\mu$; so that instead of $M(u)$ we write $M \mu(u)$.

Then, the components $\xi^0, U$ and $X^i$ are then:

$$\xi^0 = 0, \quad \xi^2 = \frac{\xi_0^2}{r}, \quad \xi^3 = \frac{\xi_0^3}{r} \quad (38)$$

with

$$\xi_0^2 = \sqrt{r} P_0 V, \quad \xi_0^3 = - i \xi_0^2; \quad (39)$$

$$U = r U_00 + U_0 + \frac{U_1}{r}, \quad (40)$$

where

$$U_00 = \frac{\dot{V}}{V}, \quad U_0 = - \frac{1}{2} K_V, \quad U_1 = M \mu, \quad (41)$$

where the curvature $K_V$ of the 2-metric appearing in equation (40), is given by (31), $M$ is a constant and

$$X^0 = 1, \quad X^2 = 0, \quad X^3 = 0; \quad (42)$$

and where $\partial_V$ is the edth operator, in the GHP notation, of the sphere with metric (30).

There is only one Ricci spinor component different from zero, namely

$$\Phi_{22} = 3 M \mu \frac{\tilde{V}}{V} - M \dot{\mu} + \frac{i}{2} \bar{\delta}_V \delta_V K_V. \quad (43)$$

Note the change of sign convention with respect to [19].

It is important to remark here that for a general compact object one would have for $\Phi_{22}$ an asymptotic behavior of the form $\Phi_{22} = \frac{\Phi_{22}}{r^2} + \delta \Phi_{22}(u, r, \zeta, \tilde{\zeta})$ where $\delta \Phi_{22}$ decays to zero faster than $\frac{1}{r}$ for large $r$. Therefore, in general, a condition on $\Phi_{22}$ would not imply direct restrictions in the interior of the spacetime. However, equation (43), tells us that in this monopole geometry a condition on $\Phi_{22}$, which in general would be an asymptotic condition, is a direct restrictions on the motion of the central object, since there is only one single $r$ term dependence. This is a very nice and peculiar property of this gauge.

In this presentation we are going to assume a constant mass particle model; so that from now on we take $\mu = 1$.

#### 4.2 Global quantities of this geometry

Let us recall that the total momentum for RT geometries can be expressed [19] as

$$P^\mu = \frac{1}{4\pi} \int M \frac{P}{V^3} \dot{\nu} dS^2. \quad (44)$$

It is also very interesting to calculate [19] the time variation of the total momentum in these geometries. With respect to the instantaneous inertial (Bondi) time $\tilde{u}$ one has

$$\frac{dP^\mu}{d\tilde{u}} = - \frac{1}{4\pi} \int \left( \frac{\partial \sigma_0 \partial \sigma_0}{d\tilde{u}} + \Phi_{0(B)22} \right) \dot{\nu} dS^2; \quad (45)$$
while the time derivative of the total momentum with respect to the RT time is
\[
\frac{dP^\mu}{du} = -\frac{1}{4\pi} \int \left( \frac{\delta^2 V \delta^2 V}{V} + \Phi^{(0)}_{22} \right) ^\nu dS^2. \tag{46}
\]

It is also convenient to recall the relations between the inertial quantities and the intrinsic ones, namely:
\[
\frac{\partial \sigma_0}{\partial u} = \frac{\delta^2 V}{V} = -\Phi'^0_0, \tag{47}
\]
\[
\Phi^0_{(B)22} = \frac{\Phi^0_{22}}{V^4}, \tag{48}
\]
and
\[
\frac{\partial a}{\partial u} = V. \tag{49}
\]

Note that what it was called \(V\) in the discussion of general asymptotically flat spacetimes, of section 3, becomes just \(V\).

When viewing monopole particles as RT geometries the time parameter \(u\) coincides with the proper time of the particle.

Using the proper time \(u\), one sees that demanding that \(\Phi^0\) has no \(l = 0\) or \(l = 1\) spherical harmonic components, provides the appropriate asymptotic balanced equations of motion; namely
\[
\frac{dP^\mu}{du} = -\frac{1}{4\pi} \int \frac{\delta^2 V \delta^2 V}{V} ^\nu dS^2; \tag{50}
\]
we discuss this in detail below.

In a sense, the last equation is actually an illusion since the Bianchi identities are really identities. The prefer to take the non-trivial equation as that imposed on \(\Phi_{22}\).

5 Monopole dynamics in the null gauge

5.1 Dynamics from the balanced equations of motion approach

We have discussed elsewhere how to construct balanced equations of motion in a general setting applicable to different gauges. Having the flux of momentum as described by the asymptotic balanced equations as given by (50), we set the flux force by
\[
F(u')^\mu = -\nabla F^\mu_V = -\frac{\gamma}{4\pi} \int \frac{\delta^2 V \delta^2 V}{V} ^\nu dS^2 \tag{51}
\]
where we are denoting with \(u'\) the asymptotic time related to the proper time \(\tau\), as explained in [8]. The second equality comes from the following property:
\[
\frac{1}{2V^3} \delta V \delta V K_V = \frac{1}{2V^3} \delta^2 V \delta^2 V - 2V \delta^2 V \delta^2 V \tag{52}
\]
so that the first term on the right hand side does not have \(l = 0\) and \(l = 1\) contributions, in an expansion with spherical harmonics.

There are two natural dynamical times in the interior, for particle \(A\) one is \(\tau\), the proper time with respect to the metric \(g_B\), and the other is the proper time \(\tau_0\), with respect to the metric \(\eta\). Let us denote with \(v\) and \(v\) be the corresponding tangent vectors to the proper times \(\tau_0\) and \(\tau\) respectively. Then the basic differential operators are \(v^a \partial_a v^b\) or \(v^b \nabla_{(B)2} v^a\); where \(\partial_a\) is the covariant derivative associated with the metric \(\eta\).

Let us note that the two velocity vectors are proportional
\[
v^b = \tilde{\eta} v^b. \tag{53}
\]

Notice that
\[
\frac{d\tau}{d\tau_0} = \frac{1}{\tilde{\eta}}. \tag{54}
\]

We can use
\[
1 = g_B(v, v) = \tilde{\eta} v^a \nabla_{(B)2} v^a = \tilde{\eta} \left(1 + h_B(v, v)\right); \tag{55}
\]
which gives \(\tilde{\eta}\) in terms of \(v\) and \(g_B\). Note that one expects \((1 + h_B(v, v)) < 1\).

Expressing the covariant derivative \(\nabla_{(B)2}\) of \(g_B\), in terms of the covariant derivative \(\partial_a\) of \(\eta\), one has
\[
\nabla_{(B)2} v^b = \partial_a v^b + \gamma_{ab} v^c; \tag{56}
\]
and using the relation between the vectors \(v\) and \(v\) one also has
\[
v^a \nabla_{(B)2} v^b = \tilde{\eta} v^a \nabla_{(B)2} v^b = \tilde{\eta} v^a \partial_a v^b + 2\gamma_{ab} v^a v^c
\]
\[
= \tilde{\eta} v^a \partial_a v^b + \gamma_{ab} v^a v^c;
\]
\[
\tau_0 = \tilde{\eta} \int v^a \nabla_{(B)2} v^a d\tau_0 + \frac{u}{\tilde{\eta}} \frac{d\tau}{d\tau_0} v^b. \tag{57}
\]

Then, using the general treatment of the balanced approach [8], one can write at the time \(u'\) the equations of motion it terms of this flux vector as
\[
M \left( \frac{1}{d\tau_0} v^a \partial_a v^b + \frac{1}{d\tau_0} \gamma_{ab} v^a v^c \right.
\]
\[
- \frac{1}{d\tau_0} \frac{d^2 u'}{d\tau_0^2} v^b + \frac{u}{d\tau_0} \frac{d\tau}{d\tau_0} v^b \right) (\tau_0) = F^{(b)}(u', v); \tag{58}
\]
where we note an interior degree of freedom \(u\). The null gauge approach provides with a simple relation between the inner dynamical time \(\tau\) with the asymptotic dynamical time \(u'\); since one has \(u' = \tau\). In other words, in this first version of the model in the null gauge, we neglect a possible degree of freedom, discussed in [8] by taking \(\frac{d\tau}{d\tau_0} = 1\); so that \(\frac{d^2 u'}{d\tau_0^2} = 0\) in the work line. This in turn allows us to express the equations of motion as
\[
M \left( \nabla_{(B)2} v^b + \gamma_{ab} v^a v^c \right.
\]
\[
+ \frac{1}{\tilde{\eta} \frac{d\tau}{d\tau_0}} v^b + \frac{u}{\tilde{\eta}} v^b \right) (\tau_0) = \frac{1}{\tilde{\eta}} F^b_0; \tag{59}
\]
where the momentum flux is calculated in terms of the proper time $\tau_0$.

Thus, equation (59) are the main equations of motion. It is natural to decompose this equation in the direction of $v^a$ and its orthogonal complement. Then, contracting this equation with $\eta_{bd}v^d$ gives

$$M\left(\gamma_a^b \varepsilon v^a v^c \eta_{bd}v^d + \frac{1}{T} \frac{dT}{d\tau_0} + \frac{w}{T}\right) = \frac{1}{T} M F_0^b \eta_{bd}v^d;$$

and it remains the equations of motion

$$Ma^b = MF_\perp^b + \frac{1}{T} F_0^b (\eta_{bd} - v_dv^b);$$

which is orthogonal to $v^a$, where

$$a^a \equiv v^b \partial_b v^a,$$

and $F_\perp^b$ is defined by

$$F_\perp^b = -\gamma_a^b \varepsilon v^a \varepsilon (\eta_{bd} - v_dv^b),$$

which, it should be remarked, only depends on the background geometry $g_B$ and $v$; and let us recall that, in general, the radiation force is given by

$$F^\mu_0 = \frac{1}{4\pi} \int S \varepsilon_0 \varepsilon_0 dS^2;$$

with

$$V = \frac{\partial \bar{u}}{\partial \tau_0},$$

since we are using (19) with $u = \tau_0$.

Equation (60) is, in the general framework, understood as an equation for $w$.

We introduce here a notation that will be useful in the following discussions. Let us define

$$a^a = f^a + f_\perp^a,$$

with $f_\perp^a$ defined by

$$MF_\perp^a = \frac{1}{T} F_0^a.$$

We will also use the notation

$$A \equiv a^\mu l_\mu,$$

and

$$F \equiv F^\mu l_\mu.$$}

Then, defining the scalar

$$f_\perp = f_\perp^a l_a = \frac{1}{M T} F_0^\mu l_\mu,$$

one can write the equations of motion in scalar form by

$$A = F + f_\perp.$$

Note that due to (61) we can also express the equations of motion by

$$A = F_\perp + f_{\lambda\perp};$$

where

$$F_\perp = f_\perp^\mu l_\mu,$$

and

$$f_{\lambda\perp} = \frac{1}{M T} F_0^\mu (\eta_{\mu\nu} - v_\nu v^\nu) l_\mu.$$
One can see from equation (43), that to obtain the asymptotic balance equation, one must demand that the \( l = 0 \) and \( l = 1 \) terms of \( \Phi_{02} \) must vanish; in other words one must satisfy equation

\[
\int \frac{\Phi_{02}}{V^2} \, \bar{\nu} \, dS^2 = 0; \tag{79}
\]

which is our main equation from the asymptotic structure.

Reading the leading order behavior \( \Phi_{02} \) from (43): we can express the complete monopole field equation as:

\[
3M \frac{\dot{V}}{V^2} = -\frac{1}{2V^3} \partial_V \partial_V K_V. \tag{80}
\]

Let us remark that the \( l = 0 \) and \( l = 1 \) terms of the left hand side of (50) constitute minus the time derivative of the total momentum. Then, taking into account property (52), we conclude that it is equivalent to solve equation (79) rather than the asymptotic balance equation (50).

In other words, the \( l = 0 \) and \( l = 1 \) terms of the right hand side of (50) is precisely the radiation flux of total momentum; as corroborated by equation (77). So that in principle we can use the higher order terms of the left hand side of (50) to improve on the left hand side of the equations of motion (53); this is what is done below. This is another subtle advantage of the use of the null gauge.

### 5.4 Properties of the monopole in the null gauge

Let us summarize here what is our approach for the monopole and then remark some convenient properties.

We have taken our definition of the monopole from the leading order behavior of a general asymptotically flat spacetime[17]. The Ricci tensor has only one null component different from zero; namely equation (43).

The asymptotic balance equation, for the total momentum is the requirement of the \( l = 0 \) and \( l = 1 \) terms of equation (50), where it should be remarked that (50) is proportional to (43).

Regarding the properties of this representation of the monopole, let us start by mentioning that in the decomposition (78), the conformal factor \( V_0 \) has the information of the motion of the monopole with respect to the chosen flat background, that in our case is the center of mass frame. Then in general, the evolution of \( V_0 \) will have the information of an accelerated motion of our body \( A \) due to the existence of the rest of the system that we call \( B \). What is the situation if there is no system \( B \)? Then, the interior center of mass should be identified with the asymptotic center of mass, defined in terms of the total angular momentum[12], which in terms of the null gauge, it means that, in the rest frame, \( V = 1 + \) radiation terms. Also, in the case there is no system \( B \), the line element (57) represents the metric of an isolated body; and the complete equation (50) is the Robinson-Trautman equation, that describes a perturbed black hole. We have studied in [25] how these spacetimes decay exponentially fast to the Schwarzschild solution. So, a very convenient property of this monopole description in the null gauge, is that it represents a black hole that satisfies the exact field equations, when isolated.

When it is not isolated, then the nature of equation (50) changes completely, as we will see below, since now the time evolution of \( V_0 \) becomes the driving term of the equation, and we now have an equation for \( V_0 \) as expressed in (78).

The field equation can be expressed as

\[
3M \frac{\dot{V}}{V^2} = -\frac{1}{2V^3} \partial_V \partial_V K_V = -\partial^2 V + \frac{1}{V} \partial^2 V \partial^2 V; \tag{81}
\]

then, in the rest frame, one can replace

\[
3M \frac{\dot{V}}{V^2} = M \frac{(V_0)}{(1 + \dot{V}_0)^3} + 3M \frac{\dot{V}_0}{(1 + \dot{V}_0)^2} - \frac{\dot{V}_0^2}{(1 + \dot{V}_0)^2} \partial^2 V \partial^2 V. \tag{82}
\]

We can in principle solve the coupled system of the balanced equation of motion and the exact equation (81) numerically; where given \( V_0(u_0, \zeta, \bar{\zeta}) \) at an initial time \( u = u_0 \) and \( V_0(u_0, \zeta, \bar{\zeta}) \), then one can integrate in the time domain. Equation (81) is still a parabolic equation with a source for \( V_0 \); so that irrespective of the choice for \( V_0(u_0, \zeta, \bar{\zeta}) \), one expects that all solutions will converge exponentially fast to a solution completely driven by the source acceleration \( V_0(u, \zeta, \bar{\zeta}) \). In a situation in which the radiation effects are bounded, one can study the regime in which \( \dot{V}_0 < 1 \), and so deal with an expansion of the denominators in (82); so that one has

\[
3M \dot{V}_0(1 - 3\dot{V}_0 + 6\dot{V}_0^2 - 10(\dot{V}_0)^3 + \ldots) + 3M \dot{V}_0(1 - 4\dot{V}_0 + 10(\dot{V}_0)^2 + \ldots) \tag{83}
\]

\[
= -\dot{V}_0^2 \partial^2 V + \dot{V}_0^2(1 - \dot{V}_0 \zeta + \ldots) \partial^2 V \partial^2 V.
\]

In this way we will try an approximate solution, that is driven by the source acceleration. The guiding idea is that the \( l = 0 \) term of the right hand side is at least of order \( \mathcal{O}(G^2) \), while the \( l = 1 \) term is at least of order \( \mathcal{O}(G^3) \); therefore, the left hand side is calculated accordingly.

### 5.5 Decomposition of the scalars by their angular behaviour and time derivatives

Given a function \( H(u, \zeta, \bar{\zeta}) \) on the future null cones, one has the natural action of the Lorentz group on the angular coordinates[26, 27], which allows us make the decomposition

\[
\partial_{\zeta_0} \partial_{\zeta_0} H_l = -\frac{l(l+1)}{2} H_l; \tag{84}
\]

for \( l = 0, 1, 2, \ldots \) in terms of the edth operators of the instantaneous rest frame. With this, it is natural then to
define the projection operators \( T \) which have the property \( T(H) = H \).

We use the decomposition for \( V \) in terms of eigenfunctions of the operator \( \partial V \), that is:

\[
V = V_0 \left( 1 + \hat{\gamma} V_1 \right) + V_2 \left( 1 + \hat{\gamma} V_2 \right) + V_3 \left( 1 + \hat{\gamma} V_3 \right) + \ldots \; (85)
\]

where the subindex \( l \) denotes the angular behavior.

In particular, let us note that

\[
\partial V_0 \partial V A = \left( 1 + \hat{\gamma} V_1 \right)^2 \partial V_0 \partial V_3 A = - \left( 1 + \hat{\gamma} V_1 \right)^2 A; \; (86)
\]

that is, we are using:

\[
\partial V_0 \partial V_3 A = - A. \; (87)
\]

In order to study the vacuum equation, it is convenient to make an analysis in terms of the angular behaviour and also in terms of the first few orders of the equation.

In the rest reference frame, one has that the time derivative of \( V \) is

\[
\dot{V} = V_0 \left( 1 + \hat{\gamma} (V_0 + V_1 + V_2 + V_3 + \ldots) \right) \; (88)
\]

And also we will use the formal decomposition of the following expressions

\[
\frac{1}{V^4} = \frac{1}{(1 + \hat{\gamma} V_1)^3},
\]

\[
\frac{1}{V^3} = \frac{1}{(1 + \hat{\gamma} V_1)^2},
\]

\[
\frac{1}{V} = \frac{1}{(1 + \hat{\gamma} V_1)},
\]

Then from the left hand side of (85) we have:

\[
\dot{V} = \frac{\hat{\gamma} (V_0 + V_1 + V_2 + V_3 + \ldots)}{(1 + \hat{\gamma} V_1)} \; (92)
\]

And from the right hand side of (86) we have:

\[
\frac{1}{\partial V_0 \partial V_3} \partial V K_V = \frac{1}{(1 + \hat{\gamma} V_1)} \partial V_0 \partial V_3 K_V
\]

\[
= \frac{1}{(1 + \hat{\gamma} V_1)} \left( - K_{V1} - 3 K_{V2} - 6 K_{V3} + \ldots \right) \; (93)
\]

where we are also using the decomposition of \( K \) in terms of the eigenfunctions of the operator \( \partial V_0 \partial V_3 \).

Let us recall that from (31), that one can also express

\[
K_V = \frac{2(1 + \hat{\gamma} V_1)}{V_0} \partial V_0 \partial V_3 V - \frac{2}{V_0} \partial V_0 \partial V_3 V + V^2, \; (94)
\]

showing the explicit dependence of the edf operators on \( V_0 \).

### 5.6 The first angular terms of the field equation

In this construction the acceleration \( A \), produced by the 'force' \( F \), due to the presence of other systems, provokes the emission of gravitational radiation. In other words, in equation (80) there would be no radiation and therefore no back reaction if the were not external force \( F \); so that it should be consider as the source of the radiation effects.

We study next the way to connect the external force to the radiation degrees of freedom.

In this first version of the null gauge model, we assume that it is only necessary to consider up to the quadrupole structure of the spacetime; in other words, we assume we can build the geometry in terms of \( V_0, V_1 \) and \( V_2 \), assuming that all other \( V_l \)’s with higher \( l \) are negligible. Then expanding (85), in the instantaneous rest frame, in the first few orders one finds

\[
3M \frac{\dot{V}}{V^3} = 3M \left( V_0 + \hat{\gamma} \left[ - 3 \nu_0 (V_0 + V_1 + V_2) \right.ight.
\]

\[
+ \left( \hat{\nu}_0 + \hat{\nu}_1 + \hat{\nu}_2 \right) \right)
\]

\[
+ \hat{\gamma}^2 \left( 6 \hat{\nu}_0 (V_0 + V_1 + V_2) \right.
\]

\[
- 4 (V_0 + \hat{\nu}_1 + \hat{\nu}_2) (V_0 + V_1 + V_2) \right) \ldots
\]

\[
= \frac{1}{2} \left( K_{V1} + 3 K_{V2} + 6 K_{V3} + \ldots \right)
\]

\[
(1 - \hat{\gamma} (V_0 + V_1 + V_2) + \hat{\gamma}^2 (V_0 + V_1 + V_2)^2 + \ldots). \; (95)
\]

In this setting, we will assume that time derivatives of \( V \) increases the order of the term, so that considering terms up to the three order level, the left hand side of expression (85) is

\[
3M \frac{\dot{V}}{V^3} = 3M \left( V_0 + \hat{\gamma} \left[ - 3 \nu_0 (V_0 + V_1 + V_2) \right.ight.
\]

\[
+ \left. \hat{\nu}_0 + \hat{\nu}_1 + \hat{\nu}_2 \right) \right)
\]

\[
+ \hat{\gamma}^2 \left( 6 \hat{\nu}_0 (V_0 + V_1 + V_2) \right.
\]

\[
- 4 (V_0 + \hat{\nu}_1 + \hat{\nu}_2) (V_0 + V_1 + V_2) \right) \ldots
\]

\[
= \frac{1}{2} \left( K_{V1} + 3 K_{V2} + 6 K_{V3} + \ldots \right)
\]

\[
(1 - \hat{\gamma} (V_0 + V_1 + V_2) + \hat{\gamma}^2 (V_0 + V_1 + V_2)^2 + \ldots). \; (96)
\]

In order to calculate the terms that contribute to the \( l = 0 \) and \( l = 1 \) spherical harmonic decomposition of the right hand side of equation (95), we can either use the original expression appearing in the field equation or we can use

\[
\frac{\partial^2 V_0 \partial^2 V}{V}. \; (97)
\]
as appears in [51]. This expression looks simpler to handle since at the instantaneous rest frame one has
\[ \ddot{a}^2 V = \ddot{\gamma}^2 (V_2 + V_3 + \cdots); \]
and since we are neglecting in this first null model higher order angular behavior, we just have
\[ \ddot{a}^2 V = \ddot{\gamma}^2 V_2. \]
The complication is that we have to deal with spin weighted quantities. The other factor is
\[ \frac{1}{V} = (1 - \dot{\gamma}V_1 + (\gamma V_2)^2 - (\dot{\gamma}V_2)^3 + \cdots). \]
Then, one can see that the contribution to the \( l = 0 \) term of [77] comes from
\[ \ddot{\gamma}^2 V_2 \dot{a}^2 V_2; \]
while the first order contribution to the \( l = 1 \) term comes from
\[ -\ddot{\gamma}^2 V_1 \dot{a}^2 V_2 \dot{a}^2 V_2. \]

5.7 Combining the balanced equations of motion with the monopole field equation

In the above equations we have suggested that one can solve the monopole field equation by computing numerically the evolution of \( V_1 \) with high precision. However, it is reasonable to also present an expansion of the monopole field equation in terms of orders, by the choice of a particular sub-gauge. In what follows we present a way to carryout this type of expansion, that we believe will be useful in future calculations.

When analyzing the field equation [80], we use a decomposition of the scalar \( V \), given in [78], which has information of the interior equations of motion, as given by [72] which in turn force the expression, as given by [67]. It is important at this stage to recall a property of expressions as [77] noted in [15]: which says the following: defining
\[ \dot{F}_0 = F_{0 \mu}^m = F_0^0 - F_{0(3)}^0; \]
with \( F_{0(3)}^0 = F_0^0 \dot{1} + F_0^0 \dot{2} + F_0^0 \dot{3} \dot{3} \), then one has that
\[ \frac{1}{2V^3} \dot{a}^2 \dot{V} \dot{a}^2 V_4 = F_0^0 - 3 \dot{F}_{0(3)}^0 + \text{terms with } \dot{Y}_{2,m}; \]
with \( l_2 \geq 2 \), with respect to the angular variables of the inertial system. This is important to notice since we will use equation [72] in the form
\[ A = F + \frac{1}{MV^2} \dot{F}_0. \]
Note that at the instantaneous rest frame one has
\[ \dot{F}_0 = -\frac{1}{MV^2} \dot{F}_0. \]
Replacing \( V_0 \rightarrow A \) in [55], in the instantaneous rest frame, one obtains
\[ 3M \left( (F + \frac{1}{MV^2}(F_0^0 - F_{0(3)}^0))(1 - 3\gamma V_1 + 6(\gamma V_2)^2 + \cdots) + \dot{\gamma}(V_0 + V_1 + V_2) \right) \]
\[ = 3M \left( -\dot{F}_{0(3)}(1 - 3\gamma V_1 + 6(\gamma V_2)^2 + \cdots) + \dot{\gamma}(V_0 + V_1 + V_2) \right) \]
\[ = -\frac{3}{V^2} \dot{F}_{0(3)} \]
\[ = -\frac{1}{2V^2} \dot{a} \dot{V} \dot{a} V \dot{K}_V \]
\[ = -(F_0^0 - 3 \dot{F}_{0(3)}^0) + \text{terms with } \dot{Y}_{2,m}; \]
where we have used that \( F = -\frac{1}{MV^2} F_0^0 - \dot{F}_{0(3)}^0 \) and we have neglected orders higher than the radiation terms. Therefore, up to the third order one has
\[ 3M \left( (F_{0(3)}^0 + \dot{\gamma} \left[ -3(-\dot{F}_{0(3)}^0)(V_0 + V_1 + V_2) \right. \right. \]
\[ + (V_0 + V_1 + V_2)] \left. + \dot{\gamma} 6(-\dot{F}_{0(3)}^0)(V_0 + V_1 + V_2)^2 + \dot{\gamma} \dot{V} \right) \]
\[ = -F_0^0 + 3 \dot{F}_{0(3)}^0 (1 + \frac{1}{V}) + \text{terms with } \dot{Y}_{2,m}; \]

5.8 Choosing an appropriate frame and ansatz

We are going to use both, the notation in terms of vectors, and also in terms of spherical harmonics.

Let us denote with \( v_1^\mu \) the four vector such that
\[ V_1 = v_1^\mu = \frac{v_1^\mu}{V_1}; \]
then, since \( V_1 \) satisfies [54] with \( l = 1 \), we know that \( v_1^\mu \) is orthogonal to \( v_\mu \).

Without loss of generality, we could assume that in the instantaneous rest frame, the acceleration, at zero order, is given by
\[ A = a Y_{10}; \]
where, in order to understand the detail of the notation, it should be remarked that if \( a^\mu \) were the four vector that corresponds to an acceleration \( a^\mu \) in the \( z \) direction, then one would have
\[ a = a^z \sqrt{\frac{4\pi}{3}} Y_{10}. \]
Actually, the source of any acceleration is \( F \); so that it is more convenient to assume that
\[
F = aY_{10} + F^0;
\]
that is
\[
\dot{F}^{(3)} = aY_{10}.
\]

In order to proceed with the calculation it is needed to determine in more detail the relation of the gravitational wave degree of freedom with the mechanical ones. We now take as an ansatz that in this frame, we only need to consider
\[
V_0 = b_0 Y_{00}, \quad V_1 = b_1 Y_{10}, \quad V_2 = b_2 Y_{20}.
\]

With this we can advance in the calculation of (107) to obtain the first orders of the left hand side of the equation.

Note that now we have
\[
3M \left( aY_{10} + \dot{\gamma} \left[ -3aY_{10}(b_0 Y_{00} + b_1 Y_{10} + b_2 Y_{20}) \right. \right.
\]
\[
+ (\dot{V}_0 + \dot{V}_1 + \dot{V}_2)]
\]
\[
+ \dot{\gamma}^2 6aY_{10}(b_0 Y_{00} + b_1 Y_{10} + b_2 Y_{20})^2 + O(\dot{\gamma}^3) \right) = -F_0^0 + 3F_0^{(3)}(1 + \frac{1}{T}) + \text{terms with } Y_{i,m}.
\]

Using the properties of multiplication of spin-weighted spherical harmonics, recalled in appendix A.3 we find that the \( l = 0 \) term of the left hand side of (110) is
\[
3M\dot{\gamma} \left( -\frac{3a b_0}{\sqrt{4\pi}} Y_{00} + \dot{V}_0 + \mathcal{R}_0(\dot{V}_1) \right) + M\dot{\theta}(\dot{\gamma}^2)
\]
while the \( l = 1 \) term of the left hand side of the equation is
\[
3M \left[ aY_{10} + \dot{\gamma} \left( -3 \left( \frac{a b_0}{\sqrt{4\pi}} + \frac{a b_2}{\sqrt{5\pi}} \right) Y_{10} + \mathcal{R}_1(\dot{V}_1) \right) \right.
\]
\[
+ \frac{\dot{\gamma}^2}{\pi} \left[ \frac{1}{4a b_0^2} + \frac{1}{\sqrt{\pi} a b_0 b_2} \right. \right.
\]
\[
\left. + \frac{9}{20} a b_1^2 + \frac{11}{28} a b_2^2 \right] Y_{10} \right] .
\]

Now turning to the right hand side of the equation, let us note first that
\[
\partial^2 V_2 \partial^2 V_2 = 6b_2^2 \left( \frac{1}{\sqrt{4\pi}} Y_{00} - \frac{1}{\sqrt{5\pi}} Y_{20} + \frac{1}{\sqrt{4\pi}} Y_{40} \right);
\]
so that the \( l = 0 \) part of expression (108) is
\[
\dot{\gamma}^2 6b_2^2 \left( \frac{1}{\sqrt{4\pi}} Y_{00} \right);
\]
while the \( l = 1 \) part of (108) becomes
\[
-\dot{\gamma}^3 V_1 \partial^2 V_2 \partial^2 V_2 = -\dot{\gamma}^3 b_1 b_2 \frac{9}{14\pi} Y_{10} + \text{other terms}.
\]

So that
\[
F_0^0 = -\dot{\gamma}^2 b_2^6 \left( \frac{1}{\sqrt{4\pi}} Y_{00} \right),
\]
and
\[
3F_0^{(3)} = -3\dot{\gamma} b_1 b_2 \frac{9}{14\pi} Y_{10}.
\]

We will clarify this below.

Let us study in more detail the term involving \( \dot{V}_1 \). If we use the four dimensional notation, then one would say that
\[
\dot{\gamma} v_i^\mu = -\dot{\alpha}M a^\mu - \alpha M \dot{a}^\mu;
\]
that is, although \( v_i^\mu \) was actually defined in terms of the spacelike part of \( F^\mu \), since the difference between \( a^\mu \) and \( F^\mu \), is order \( \dot{\gamma} \), one will have that expressing \( \dot{\gamma} v_i^\mu \) in terms of \( a^\mu \), instead of \( F^\mu \), will introduce differences of order \( \dot{\gamma} \).

Then we have that
\[
\frac{3}{2} M \ddot{\gamma} \left( -a b_1 + \sqrt{4\pi} (\dot{V}_0 + \mathcal{R}_0(\dot{V}_1)) \right) = \dot{\gamma}^2 b_2^2.
\]

Let us express \( \ddot{\gamma} v_i^\mu \) by
\[
\frac{3}{2} M \ddot{\gamma} \left( -a b_1 + \sqrt{4\pi} (\dot{V}_0 + \mathcal{R}_0(\dot{V}_1)) \right) = \dot{\gamma}^2 b_2^2.
\]

While the source of any acceleration is \( F \), so that it is more convenient to assume that
\[
F = aY_{10} + F^0;
\]
that is
\[
\dot{F}^{(3)} = aY_{10}.
\]
and so, in terms of the scalar, we have

\[ \dot{\bar{V}}_i = -\alpha MA - \alpha MA; \]  

(131)

with

\[ \dot{A} = \dot{a}^a y^a - A^2; \]  

(132)

since \( \dot{\bar{V}}_i = -\alpha MA \).

The \( l = 0 \) term of \( \dot{V}_1 \), in the rest frame, is

\[ \dot{\bar{V}}_0(\dot{A}) = -\alpha M \bar{V}_0(\dot{A}) = -\alpha M (\dot{a}^a - \bar{V}_0(A^2)) \]

(133)

where it should be remarked that \( \dot{a}^a a_{\mu} \leq 0 \).

Then, instead of (107) we have

\[ \dot{\bar{V}} = \dot{a}^a \frac{\dot{y}^a}{\gamma} - A^2; \]  

(134)

with

\[ \dot{A} = \dot{a}^a \frac{\dot{y}^a}{\gamma} - A^2; \]  

(135)

Then, in order to consistently consider the term including the symbol \( \dot{V}_1 \) we should increase the order of the balanced equations of motion; so that we generalize (67) to

\[ \dot{a}^a = f^a + (\alpha \dot{M}^a) + \beta M \dot{f}^a + f^a_\lambda; \]  

(136)

where \( (\alpha \dot{M}^a) \) denotes the time derivative of \( (\alpha M^a) \). This constitutes our second order version of the balanced equations of motion; while (67) was the first order version. Note that now we will have a modification of the interior and asymptotic structure in the null gauge model, terms of orders. From the tight relations between the intermediate quantities we have arrived at the second order balanced equations of motion introduced in [8] to the particular case of the null gauge, that we have described here.

The \( l = 0 \) term of this relation is

\[ 3M \left( \beta - \frac{1}{3} \alpha \right) M \dot{f}^a f^a \]  

(137)

Replacing \( b_1 \) we have

\[ M \left( 4\pi \beta - \frac{1}{3} \alpha \right) M \dot{f}^a f^a + 3\alpha M \alpha^2 + \dot{\gamma} b_0 \sqrt{4\pi} \]  

(138)

Recall that in the frame in which the acceleration is in the z direction \( a = -a^2 \sqrt{\frac{2\pi}{3}} \) and \( f^a f^a = -(a^2)^2 \), so that we could also express it as

\[ M \left( 2\pi M (a^2)^2 \left( \frac{4}{3} - \beta \right) + \dot{\gamma} b_0 \sqrt{4\pi} \right) = \dot{\gamma} b_2^2. \]  

(139)

The \( l = 1 \) component of the equation now is

\[ M \left( a + \dot{\gamma} \left( -3 \frac{ab_0}{\sqrt{4\pi}} + \frac{ab_0}{\sqrt{5\pi}} \right) \right) + \frac{\dot{\gamma}^2}{\sqrt{5}} \left( \frac{1}{4} a b_0^2 + \frac{1}{\sqrt{5}} a b_0 b_2 \right) \]

(140)

and replacing \( b_1 \) we have

\[ \frac{\dot{\gamma}^2}{\sqrt{5}} b_2^2 \left( 1 + \frac{1}{\sqrt{T}} \right). \]

Summarizing, the balanced equations of motion of second order are given by (133) where \( \alpha, \beta \) and \( b_1 \) and \( b_2 \) appearing in \( f^a_\lambda \), satisfy equations (127) (134) and (140). Comparing with the electromagnetic case, one would be tempted to take \( \beta = \alpha \); but we leave the setting of \( \beta \) open for future work.

This is our main result. The tight connexion between the interior and asymptotic structure in the null gauge model for compact objects, has led us to extend the study to the second order balanced equations of motion, as expressed by (134).

We will study how to fix these remaining degrees of freedom by applying these equations to specific observations; which will be carried out in future work.

### 6 Final comments

We have applied the general framework for the construction of balanced equations of motion introduced in [8] to the particular case of the null gauge, that we have described here.

We have presented the balanced equations of motion which are coupled to the field equation of the monopole geometry. We have indicated the way in which system (81) can be solved numerically in an exact form; and we have also presented an expansion of the field equation in terms of orders. From the tight relations between the interior and asymptotic structure in the null gauge model, we have arrived at the second order balanced equations of motion (133); where the intervening quantities \( b_0, \alpha \) and \( b_2 \) are related by the components of the field equation (137) and (140).

After having deduced the form of the balanced equations of motion at second order in the acceleration, the
idea is that if one tackles system [31] by numerical method, one would obtain higher accuracy using the second order version of the equations of motion. Since we do not request low velocities, or weak fields, or condition on the masses, we expect with our models to improve on the range of possible systems that we can study with respect to those covered by the post-Newtonian and the self-force approaches.

When applying these balanced equations of motion to a binary system of comparable masses, one must deal with two dynamical times; one for each particle. So the problem turns into a bound binary retarded dynamical system. When dealing with this type of retarded system, it is common to recur to approximations using some kind of universal dynamical time; as is the case in the post-Newtonian approaches. Instead we intend to apply a method we have developed in [28], based on high order approximations of the trajectory from the force equations.

In the process of relating the equations of motion to the monopole field equation, we have solve the \( l = 0 \) and \( l = 1 \) terms of the field equation at the \( G^2 \) and \( G^3 \) order respectively. This is what is needed to take into account the first effects of back reaction to the equations of motion in the null gauge.

A property of the null gauge model is that one can request increasing precision in the model by demanding other components of the field equations to be satisfied; i.e. \( l = 2, 3, ... \), which of course will require the introduction of new quantities, as \( b_3, b_4, \) etc. At any stage, the neglected \( b_l \) are assumed to be less important, since they represent internal degrees of freedom of the compact object. But there might be occasions in which one would like to resort to this internal structure; as for example the case to build a model of non-black holes compact objects, as are the neutron stars. The degree in which this type of model can be successful for the description of these systems will be a matter of forthcoming work.

The first method that we have suggested for solving the model, is to solve numerically the coupled system of the balanced equation of motion and the exact monopole field equation [31] numerically; which, it should be emphasized, it provides with a global exact solution of the monopole metric, that is, for all \( r \). This, we think, is a very nice property of the null gauge approach to the dynamics of the binary system. So, it seem that the tight relation between the interior and asymptotic structure of the null gauge model, along null directions, of this gauge, will provide more physically interesting results than those from the model based on the harmonic gauge presented in [31]. All these issues will become clear in the application of our models to observations of gravitational waves.

A Appendix

A.1 Coordinate systems and the edth operator

Normally the edth operator is expressed in terms of the complex coordinates \((\zeta, \bar{\zeta})\) or in terms of the spherical coordinates \((\theta, \phi)\). The relation between the two coordinate systems is given by

\[
\zeta = e^{i\phi} \cot(\frac{\theta}{2}).
\]

From this one deduces that the coordinate complex vector is:

\[
\frac{\partial}{\partial \zeta} = -e^{-i\phi} \sin^2(\frac{\theta}{2}) \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin(\theta)} \frac{\partial}{\partial \phi} \right).
\]

In the unit sphere, the complex vectors \( m \) and \( \bar{m} \) are chosen so that \( m \propto \frac{\partial}{\partial \zeta} \) and therefore \( \bar{m} \propto \frac{\partial}{\partial \bar{\zeta}} \). Defining

\[
P_0 = \frac{1}{2} (1 + \zeta \bar{\zeta});
\]

one writes, for the unit sphere,

\[
m = \sqrt{2} P_0 \frac{\partial}{\partial \zeta},
\]

and

\[
\bar{m} = \sqrt{2} P_0 \frac{\partial}{\partial \bar{\zeta}}.
\]

Note that since the scalar product between these vectors is minus one, we have

\[
m_a \equiv -\frac{d\bar{\zeta}}{\sqrt{2}P_0}.
\]

Then, according with our signature conventions one has

\[
-m_a \bar{m}_b - m_b \bar{m}_a \equiv - \frac{d\bar{\zeta} d\zeta}{P_0} = - \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right).
\]

Whenever needed we also use the decomposition of the complex coordinates in terms of

\[
\zeta = \frac{1}{2} (x^2 + ix^3),
\]

and similarly

\[
\bar{\zeta} = \frac{1}{2} (x^2 - ix^3);
\]

so that the coordinate vectors are:

\[
\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial x^2} - i \frac{\partial}{\partial x^3},
\]

and

\[
\frac{\partial}{\partial \bar{\zeta}} = \frac{\partial}{\partial x^2} + i \frac{\partial}{\partial x^3}.
\]

Using now the GHP [18] definition of the edth operator, applied to the unit sphere one can write

\[
\partial f = \sqrt{2} P_0^{1-k} \frac{\partial}{\partial \zeta} (P_0 f).
\]
and
\[ \bar{\partial} f = \sqrt{2} P_0^{l+s} \frac{\partial}{\partial \zeta} (P_0^{-s} f) ; \] (153)

where \( s \) is the spin weight of the quantity \( f \).

Comparing the GHP expressions with equation (3.9) of [29] we see that
\[ \bar{\partial} = \frac{1}{\sqrt{2}} \bar{\partial}_P ; \] (154)

where \( \bar{\partial}_P \) is the NP definition. Instead, their equation (3.8) would need an extra factor \( e^{-i\phi} \), as it can be deduced from [132]; since they use a rotated null tetrad.

A.2 The edth operator as intrinsic objects on the spheres and the spin \( s \) spherical harmonics

In general, for a sphere with metric \( d\zeta d\bar{\zeta}/P^2 \), and \( P = P_0 \), one can define the intrinsic edth operator acting on a function \( f \) of spin weight \( s \) by
\[ \bar{\delta}_V f = \sqrt{2} P^{1-s} \frac{\partial (P^{-s} f)}{\partial \zeta} \] (155)

and similarly
\[ \bar{\delta}_V f = \sqrt{2} P^{1+s} \frac{\partial (P^{-s} f)}{\partial \zeta} . \] (156)

The commutator of these two operators is
\[ (\bar{\delta}_V \bar{\delta}_V - \bar{\delta}_V \bar{\delta}_V) f = s K_V f ; \] (157)

where it is important to notice that the convention that comes from the GHP [18] formalism, differs from the original one suggested by Newman and Penrose in reference [29]. When \( P = P_0 \), equivalently \( V_M = 1 \), it is convenient to refer to the spin \( s \) spherical harmonics [29] \( Y_{lm} \) which have the following properties
\[ \bar{\delta}_s Y_{lm} = \sqrt{\frac{(l-s)(l+s+1)}{2}} \bar{Y}_{l+1}^{m+1} , \] (158)

\[ \bar{\delta}_s Y_{lm} = - \sqrt{\frac{(l+s)(l-s+1)}{2}} \bar{Y}_{l-1}^{m-1} , \] (159)

and
\[ \bar{\delta} s Y_{lm} = - \sqrt{\frac{(l-s)(l+s+1)}{2}} Y_{lm} . \] (160)

This last eigenvalue problem is useful in classifying the functions on the sphere with metric \( d\zeta d\bar{\zeta}/(V_M^2 P_0^2) \) even when \( V_M \neq 1 \).

A.3 Product of spherical harmonics and spin-weighted spherical harmonics

In this section to help the reading we will use the notation \( Y_l^m \equiv Y_{lm} \).

It is convenient to recall that the product of two spherical harmonics can be express, in terms of Clebsch-Gordan coefficients by
\[ Y_{l_1}^{m_1+1}(\theta, \phi) Y_{l_2}^{m_2+2}(\theta, \phi) = \sum_{l_{\text{total}}} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_{\text{total}}+1)}} < l_1l_200|l_{\text{total}}l_{\text{total}}0 > < l_1l_2m_1m_2|l_{\text{total}}(m_1 + m_2) > Y_{l_{\text{total}}}^{m_{\text{total}}}(\theta, \phi) . \] (161)

For the product of two spin-weighted spherical harmonics one has [27]
\[ s_1 Y_{l_1}^{m_1}(\theta, \phi) s_2 Y_{l_2}^{m_2}(\theta, \phi) = \sum_{l} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} < l_1l_2; -s_1, -s_2 | l, -s > < l_1l_2; m_1, m_2 | l, m > Y_l^{m}(\theta, \phi) ; \] (162)

with \( m_1 + m_2, s = s_1 + s_2 \) and \( |l_1 - l_2| \leq l \leq |l_1 + l_2| \), and with a slight change in the notation of the Clebsch-Gordan coefficients.

Then, in particular one has
\[ 2 Y_{l_1}^{0}(\theta, \phi) - 2 Y_{l_1}^{0}(\theta, \phi) = \sum_{l_{\text{total}}} \sqrt{\frac{5^2}{4\pi(2l_{\text{total}}+1)}} < 2, 2, -2, 2 | l, 0 > < 2, 2, 0, 0 | l, 0 > Y_{l_{\text{total}}}^{0}(\theta, \phi) \] (163)
\[ = \frac{5}{\sqrt{4\pi}} \left\{ \frac{\sqrt{5} Y_0^0 - \frac{2}{\sqrt{5}} Y_1^0 + \frac{1}{35} Y_4^0}{\frac{\sqrt{5} Y_0^0 + \frac{1}{7} Y_2^0 + \frac{1}{7\sqrt{4\pi}} Y_2^0} \right\} \]

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