Analysis of forward solvers for electrical impedance tomography in a mammography geometry

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Abstract. We explain how the problem of improving image reconstruction in electrical impedance mammography can be studied by comparing measured data with predictions from the continuum model, ave-gap model, and complete electrode model. The data is measured by the RPI ACT4 system using 60 electrodes arranged in a mammography geometry. Each model's accuracy can be determined by comparing the computed eigenvalues of its Neumann-to-Dirichlet map with those found from experimentally measured N-to-D maps. It will be explained how the different models can be used to reconstruct images of the electrical conductivity and permittivity inside a region bounded by a pair of electrode arrays in a mammography geometry. Reconstructions from experimental data are presented.

1. Introduction
In electrical impedance tomography the goal is to reconstruct electrical properties within a region from voltage and current measurements on the boundary. In reconstruction algorithms it is necessary to have a forward solution that produces accurate voltages from a known set of current patterns (CP) and known admittivity throughout the domain. One can calculate error terms between measured and simulated voltages to get an idea of the quality of the forward solver. Following the work in [4], we look at the eigenvalues of the Neumann-to-Dirichlet map, $R_\gamma$. Since $V^n = R_\gamma T^n$, where $V^n$ and $T^n$ are voltage and current patterns on the electrodes, one can easily understand why the eigenvalues of $R_\gamma$ are known as the characteristic resistances, $\rho_n$. By comparing characteristic resistances between measured and simulated data we have a simple and intuitive way to evaluate the quality of our simulated data.

In our study we focus on a mammography geometry, shown in Figure 1, which can be modeled as a box with electrode arrays on the top and bottom. We calculate the analytic solution for this geometry. Assuming continuous current densities at the top and bottom we can find the exact eigenvalues of $R_\gamma$. In addition, we use the Average-Gap and complete electrode models to calculate the voltages. We describe how one can then find the eigenvalues from these voltages. This analysis is then used to tune a couple of parameters of the forward models, and a reconstruction is shown.
2. Eigenvalue analysis of the data

The problem is modeled by the conductivity equation

\[ \nabla \cdot \gamma \nabla u(x, y, z) = 0, \quad 0 \leq x \leq h_1, 0 \leq y \leq h_2, -\frac{h_3}{2} \leq z \leq \frac{h_3}{2}, \]

\[ \gamma \frac{\partial u}{\partial n} = j^T(x, y), \quad z = \frac{h_3}{2}, \]

\[ \gamma \frac{\partial u}{\partial n} = j^B(x, y), \quad z = -\frac{h_3}{2}, \]

\[ \gamma \frac{\partial u}{\partial n} = 0, \quad x = 0, x = h_1, y = 0, y = h_2, \]

where \( j^T \) and \( j^B \) represent the current densities at the top and bottom of the electrode arrays, respectively. The solution, for \( \gamma = 1 \), can be found to be

\[ u(x, y, z) = b_0 z + \sum_{n, m = 0}^{\infty} \cos(\lambda_n x) \cos(\lambda_m y) \left[ a_{n,m} \cosh(\lambda_n z) + b_{n,m} \sinh(\lambda_n z) \right], \]

where \( \lambda_n = \frac{n\pi}{h_1}, \lambda_m = \frac{m\pi}{h_2} \), and \( \lambda_{n,m} = \sqrt{\lambda_n^2 + \lambda_m^2} \). This represents the solution using the continuum model, i.e. there is no electrode model and we assume that there is a current density applied to the entire top and bottom of the domain.

More accurate voltages can be calculated by properly incorporating electrode effects. We use the Average-Gap model given by

\[ T^n_l = \int_{e_l} \gamma \frac{\partial u}{\partial n} dS, \quad T^n_l = 0, \quad (x, y, z) \notin \bigcup_{l=1}^{L} e_l, \quad V^n_l = \frac{1}{|e_l|} \int_{e_l} u dS, \]

which is discussed in [1], and we use the complete electrode model (CEM), which uses the same boundary conditions (BC) as (3) except the last BC is replaced by \( V^n_l = u + z_l \gamma \frac{\partial u}{\partial n} \), [2]. The term \( z_l \) represents the contact impedance between the electrode and the surface.

Let us say that the matrix \( T \) is defined as the following

\[ T^n_l = \begin{cases} j^T_{p_n,q_n}(x_l, y_l) = M \cos(\lambda_{p_n} x_l) \cos(\lambda_{q_n} y_l) , & \text{if } s_n = 0, \\ j^B_{p_n,q_n}(x_l, y_l) = (-1)^s_n M \cos(\lambda_{p_n} x_l) \cos(\lambda_{q_n} y_l) , & \text{if } s_n = 1. \end{cases} \]

where \( p_n, q_n, \) and \( s_n \) (= 0 or 1) are coefficients of a certain prescribed ordering. Using the continuum model, i.e. combining (2) and (4), we get that \( V^n = R \gamma T^n = \rho_n T^n \), where \( \rho_n := \frac{1}{\gamma k_n} \), where

\[ k_n := \begin{cases} \lambda_{p_n,q_n} \tanh(\lambda_{p_n,q_n} \frac{h_3}{2}), & \text{if } s_n = 0, \\ \lambda_{p_n,q_n} \coth(\lambda_{p_n,q_n} \frac{h_3}{2}), & \text{if } s_n = 1. \end{cases} \]
In the cases of using the Ave-Gap model, CEM, or measured data to calculate the characteristic resistances, the first step is to approximate $R_\gamma$. It should be noted that the vectors $T^n$ are not necessarily eigenvectors of $R_\gamma$. We therefore assume that we have an unknown set of eigenvectors denoted by $t_j$ for $j = 1...L − 1$. Since $T$, if scaled properly, is an orthonormal matrix we can express $t_j$ in terms of $T$ in the following way $t_j = \sum_{k=1}^{L-1} \tau_{j,k} T^k$, where we define $\tau_{j,k} := \langle T^k, t_j \rangle$. Starting with $R_\gamma t_j = \rho_j t_j$ one can find that $[R_\gamma] \tau_j = \rho_j \tau_j$ where we define $[R_\gamma]_{\alpha,k} := \langle T^\alpha, V^k \rangle$. Thus to find $\rho_j$ we simply need to construct $[R_\gamma]$ and calculate its eigenvalues.

3. Reconstruction method: NOSER

In order to make reconstructions we use a Newton one-step error reconstructor (NOSER) method, which was originally described in [3] for a 2D circular problem. The algorithm works the same for $u^n$ from Ave-Gap or CEM. We start by assuming that we have a solution, $v^m$, that satisfies the homogeneous problem, $\gamma = \gamma_0$, for the $m^{th}$ CP. Then by integrating over the domain, comparing solutions, and a step of integration by parts one can find that

$$
\int_\Omega (\gamma - \gamma_0) \nabla u^n \cdot \nabla v^m dP = \int_{\partial \Omega} v^m \gamma \frac{\partial u^n}{\partial n} - u^n \gamma_0 \frac{\partial v^m}{\partial n} ds,
$$

$$
\int_\Omega \delta \gamma \nabla u^n_0 \cdot \nabla v^m dP \approx D_{n,m},
$$

where in the second line we assume $u$ and $\gamma$ are close to constants and higher order terms are ignored. Then we discretize the domain into a coarse set of $N_v$ voxels. This gives us a linear equation to solve $J \delta \gamma = D$, where $J$ is constructed from the Jacobian which is defined to be

$$
J_{n,m,k} := \int_{B_k} \nabla u^n_0 \cdot \nabla v^m dP,
$$

where $B_k$ represents the domain of the $k^{th}$ voxel. Our solution is given by $\gamma_1 = \gamma_0 + \delta \gamma$.

4. Results

The eigenvalue analysis allows us to easily tune parameters for particular geometries or electrode arrays. Since the eigenvalues drop off quickly in our illustrations we multiply them by a term $k_n$ defined in (5). This makes the continuum model, $k_n \rho_n$, be described by a horizontal line. In our illustration we show the tuned results from our mammography tank filled with saline solution of 200 mS/m. We allow for there to be an extra spacing outside of the electrode array to compensate for currents that may travel outside the array, see Figure 1. The best parameters found for this spacing was 1 cm. When using the CEM the best contact impedance was found to be $z = 0.00015/\gamma_0 \Omega m^2$. The characteristic resistances are shown in Figure 2. The best maximum relative errors were 12.1% and 4.7% using the Ave-Gap model and CEM, respectively. The error was defined by $E := \max_n \left[ |\rho_n^{meas} - \rho_n^{sim}| / |\rho_n^{meas}| \right]$. In Figure 3, we show a reconstruction using the tuned CEM voltages of a copper cube of 10 mm sides placed near the lower electrode array at 33 kHz.

5. Conclusion

In this work it is seen that by investigating the eigenvalues of the Neumann-to-Dirichlet map one can tune forward solvers in order to produce good reconstructions. This method provides a simple way to analyze the quality of simulated voltages in comparison to measured voltages, and will be a useful tool in investigating more complex forward models with inevitably more parameters that need to be chosen.
Figure 2. Illustration showing the characteristic resistances using voltages from measurements, the continuum model, the Ave-Gap model, and the CEM. The Ave-Gap and CEM use a spacing of 1 cm around the electrode array, and CEM uses a contact impedance of $z = 0.00015/\gamma_0 \Omega m^2$.

Figure 3. Cross-sections of a reconstruction shown at various heights of a copper cube with 10 mm sides placed within the mammography tank near the lower electrode array using CEM with $z = 0.00015/\gamma_0 \Omega m^2$ and the NOSER algorithm.

References

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