COFINITENESS OF LOCAL COHOMOLOGY MODULES IN THE CLASS OF MODULES IN DIMENSION LESS THAN A FIXED INTEGER

ALIREZA VAHIDI AND MAH DIEH PAPARI-ZAREI

Abstract. Let $n$ be a non-negative integer, $R$ a commutative Noetherian ring with $\dim(R) \leq n + 2$, $a$ an ideal of $R$, and $X$ an arbitrary $R$-module. In this paper, we first prove that $X$ is an $(\text{FD}_{\leq n}, a)$-cofinite $R$-module if $X$ is an $a$-torsion $R$-module such that $\text{Hom}_R \left( \frac{R}{a}, X \right)$ and $\text{Ext}^1_R \left( \frac{R}{a}, X \right)$ are $\text{FD}_{\leq n}$ $R$-modules. Then, we show that $H^i_a(X)$ is an $(\text{FD}_{\leq n}, a)$-cofinite $R$-module and $\{ p \in \text{Ass}_R(H^i_a(X)) : \dim \left( \frac{R}{p} \right) \geq n \}$ is a finite set for all $i$ when $\text{Ext}^i_R \left( \frac{R}{a}, X \right)$ is an $\text{FD}_{\leq n}$ $R$-module for all $i \leq n + 2$. As a consequence, it follows that $\text{Ass}_R(H^i_a(X))$ is a finite set for all $i$ whenever $R$ is a semi-local ring with $\dim(R) \leq 3$ and $X$ is an $\text{FD}_{\leq 1}$ $R$-module. Finally, we observe that the category of $(\text{FD}_{\leq n}, a)$-cofinite $R$-modules forms an Abelian subcategory of the category of $R$-modules.

1. Introduction

We adopt throughout the following notation: let $R$ denote a commutative Noetherian ring with non-zero identity, $a$ and $b$ ideals of $R$, $M$ a finite (i.e., finitely generated) $R$-module, $X$ an arbitrary $R$-module which is not necessarily finite, and $n$ a non-negative integer. We refer the reader to [7, 8, 23] for basic results, notations, and terminology not given in this paper.

Hartshorne, in [14], defined an $a$-torsion $R$-module $X$ to be $a$-cofinite if the $R$-module $\text{Ext}^i_R \left( \frac{R}{a}, X \right)$ is finite for all $i$, and asked the following questions.

**Question 1.1.** Does the category of $a$-cofinite $R$-modules form an Abelian subcategory of the category of $R$-modules?

**Question 1.2.** Is $H^i_a(M)$ an $a$-cofinite $R$-module for all $i$?

The following question is also an important problem in local cohomology [16, Problem 4].

**Question 1.3.** Is $\text{Ass}_R(H^i_a(M))$ a finite set for all $i$?

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There have been many attempts in the literature to study the above questions. Hartshorne in [14] Proposition 7.6 and Corollary 7.7 showed that the answer to these questions is yes if $R$ is a complete regular local ring and $a$ is a prime ideal of $R$ with $\dim \left( \frac{R}{a} \right) \leq 1$. Huneke and Koh in [17] Theorem 4.1 and Delfino in [10] Theorem 3] extended Hartshorne’s result [14] Corollary 7.7 and provided affirmative answers to Questions 1.2 and 1.3 in more general local rings $R$ and one-dimensional ideals $a$. Delfino and Marley in [11] Theorems 1 and 2, Yoshida in [23] Theorem 1.1, Chiriacescu in [9] Theorem 1.4, and Kawasaki in [18] Theorems 1 and 8] showed that the answer to Questions 1.1, 1.3 is yes if $R$ is an arbitrary local ring and $a$ is an arbitrary ideal of $R$ with $\dim \left( \frac{R}{a} \right) \leq 1$. In [24, Corollaries 3.3 and 7.10] and [22, Theorems 2.6 and 2.10], Melkersson provided affirmative answers to these questions for the case that $R$ is an arbitrary ring and either $\dim(R) \leq 2$ or $a$ is an arbitrary ideal of $R$ with $\dim \left( \frac{R}{a} \right) \leq 1$.

Recall that $X$ is said to be an FD$_{< n}$ (or in dimension $< n$) $R$-module if there is a finite submodule $Y$ of $X$ such that $\dim_{R} \left( \frac{Y}{X} \right) < n$ [21]. From [26, Theorem 2.3], the class of FD$_{< n}$ $R$-modules is closed under taking submodules, quotients, and extensions. We say that $X$ is an (FD$_{< n}$,$a$)-cofinite $R$-module if $X$ is an $a$-torsion $R$-module and $\Ext_{R}^{i} \left( \frac{R}{a}, X \right)$ is an FD$_{< n}$ $R$-module for all $i$ [3, Definition 4.1]. Note that $X$ is an $a$-cofinite $R$-module if and only if $X$ is an (FD$_{< 0}$,$a$)-cofinite $R$-module. Thus, as generalizations of Questions 1.1 and 1.3 we have the following questions (see [11, Question 1.4] and [24, Questions 1.5, 1.6, and 1.8]). Here, the set \{ $p \in \Ass_{R}(X) : \dim \left( \frac{X}{p} \right) \geq n$ \} is denoted by $\Ass_{R}(X)_{\geq n}$.

**Question 1.4.** Does the category of (FD$_{< n}$,$a$)-cofinite $R$-modules form an Abelian subcategory of the category of $R$-modules?

**Question 1.5.** Is $H^{i}_{a}(M)$ an (FD$_{< n}$,$a$)-cofinite $R$-module for all $i$?

**Question 1.6.** Is $\Ass_{R}(H^{i}_{a}(M))_{\geq n}$ a finite set for all $i$?

If $R$ is a complete local ring with $\dim \left( \frac{R}{a} \right) \leq n + 1$, then the answer to Questions 1.5 and 1.6 is yes from [11, Theorems 2.5 and 2.10]. In [24, Corollaries 3.3 and 4.5], the first author and Morsali removed the complete local assumption on $R$ and provided affirmative answers to Questions 1.4, 1.6 for the case that $\dim \left( \frac{R}{a} \right) \leq n + 1$, which are generalizations of Melkersson’s results [22, Theorems 2.6 and 2.10]. In this paper, as generalizations of Melkersson’s results [21, Theorems 7.4 and 7.10], we show that the answer to Questions 1.4, 1.6 is also yes if $\dim(R) \leq n + 2$. As a consequence, we provide an affirmative answer to Question 1.3 for the case that $R$ is a semi-local ring with $\dim(R) \leq 3$. This result is a generalization of Marley’s result in [19] where he showed that the answer to Question 1.3 is yes if $R$ is a local ring with $\dim(R) \leq 3$ (see [19, Proposition 1.1 and Corollary 2.5]).

In the main result of Section 2 we observe that if $\dim(R) \leq n + 2$ and $X$ is an $a$-torsion $R$-module such that $\Hom_{R} \left( \frac{R}{a}, X \right)$ and $\Ext_{R}^{i} \left( \frac{R}{a}, X \right)$ are FD$_{< n}$ $R$-modules, then $X$ is an (FD$_{< n}$,$a$)-cofinite $R$-module. Section 3 is devoted to the study of Questions 1.5 and 1.6. We show that $H^{i}_{a}(X)$ is an (FD$_{< n}$,$a$)-cofinite $R$-module and $\Ass_{R}(H^{i}_{a}(X))_{\geq n}$ is a finite set for all $i$ whenever $\dim(R) \leq n + 2$ and $\Ext_{R}^{i} \left( \frac{R}{a}, X \right)$ is an FD$_{< n}$ $R$-module for all $i \leq n + 2$ (e.g., $X$ is an FD$_{< n}$ $R$-module).
It follows that if $R$ is a semi-local ring with $\dim(R) \leq 3$ and $\text{Ext}^i_R \left( \frac{R}{a}, X \right)$ is an $\text{FD}_{<1} R$-module for all $i \leq 3$ (e.g., $X$ is an $\text{FD}_{<1} R$-module), then $H^i_a(X)$ is an $\alpha$-weakly cofinite $R$-module and $\text{Ass}_R(H^i_a(X))$ is a finite set for all $i$. Recall that $X$ is said to be an $\alpha$-weakly cofinite $R$-module if $X$ is an $\alpha$-torsion $R$-module and the set of associated prime ideals of any quotient module of $\text{Ext}^i_R \left( \frac{R}{a}, X \right)$ is finite for all $i$ (see [12, Definition 2.1] and [13, Definition 2.4]). In Section 4, with respect to Question 1.4, we prove that when $\dim(R) \leq n + 2$, the category of $(\text{FD}_{<n}, \alpha)$-cofinite $R$-modules forms an Abelian subcategory of the category of $R$-modules.

2. A CRITERION FOR COFINITENESS

The following two lemmas will be useful in the proof of the main result of this section. Note that when $bX = 0$, $X$ is an $\text{FD}_{<n} R$-module if and only if $X$ is an $\text{FD}_{<n} \frac{R}{b}$-module.

**Lemma 2.1.** Let $t$ be a non-negative integer and let $X$ be an $R$-module such that $bX = 0$ and $\text{Ext}^i_R \left( \frac{R}{a+b}, X \right)$ is an $\text{FD}_{<n} R$-module for all $i \leq t$. Then $\text{Ext}^i_R \left( \frac{R}{a+b}, X \right)$ is an $\text{FD}_{<n} \frac{R}{b}$-module for all $i \leq t$.

**Proof.** We prove this by using induction on $t$. The case $t = 0$ is clear from the isomorphisms

$$
\text{Hom}_R \left( \frac{R}{a+b}, X \right) \cong \left( 0 :_X \frac{a+b}{b} \right) \cong (0 :_X a + b) \cong \text{Hom}_R \left( \frac{R}{a+b}, X \right).
$$

Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\text{Ext}^t_R \left( \frac{R}{a+b}, X \right)$ is an $\text{FD}_{<n} \frac{R}{b}$-module, since $\text{Ext}^i_R \left( \frac{R}{a+b}, X \right)$ is an $\text{FD}_{<n} \frac{R}{b}$-module for all $i \leq t-1$ by the induction hypothesis on $t-1$. From [23, Theorem 11.65], there is a spectral sequence

$$
\text{Ext}^{p,q}_R \left( \frac{R}{a+b}, X \right) \Rightarrow \text{Ext}^{p+q}_R \left( \frac{R}{a+b}, X \right).
$$

Let $r \geq 2$ and set $B^{t,0}_r := \text{Im}(E^{t-r,r-1}_r \to E^{t,0}_r)$. Then $B^{t,0}_r$ is an $\text{FD}_{<n} \frac{R}{b}$-module because $E^{t-r,r-1}_r$ is a subquotient of $E^{t-r,r-1}_r$ that is an $\text{FD}_{<n} \frac{R}{b}$-module by the induction hypothesis and [15, Proposition 3.4]. Thus, from the short exact sequence

$$
0 \rightarrow B^{t,0}_r \rightarrow E^{t,0}_r \rightarrow E^{t,0}_{r+1} \rightarrow 0,
$$

$E^{t,0}_r$ is an $\text{FD}_{<n} \frac{R}{b}$-module whenever $E^{t,0}_{r+1}$ is an $\text{FD}_{<n} \frac{R}{b}$-module. There exists a finite filtration

$$
0 = \phi^{t+1} H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = \text{Ext}^t_R \left( \frac{R}{a+b}, X \right)
$$

such that $E^{t-i}_i \cong \frac{\phi^{t-i} H^t}{\phi^{t-i+1} H^t}$ for all $i, 0 \leq i \leq t$. By assumption, $\text{Ext}^t_R \left( \frac{R}{a+b}, X \right)$ is an $R$-module. Thus, as we noted at the beginning of this section, $\text{Ext}^t_R \left( \frac{R}{a+b}, X \right)$ is an $\text{FD}_{<n} \frac{R}{b}$-module and hence $\phi^t H^t$ is an $\text{FD}_{<n} \frac{R}{b}$-module. Therefore $E^{t,0}_\infty \cong \phi^t H^t$ is an $\text{FD}_{<n} \frac{R}{b}$-module and so $E^{t,0}_{t+2}$ is an $\text{FD}_{<n} \frac{R}{b}$-module, because $E^{t,0}_\infty = E^{t,0}_{t+2}$ as

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$E^t_{j-j-1} = 0 = E^t_{j+j-1}$ for all $j \geq t + 2$. Thus $E^{t,0}_j = \text{Ext}^t_{R_a}(R_{\bar{a}+\bar{b}}, X)$ is an FD$_n$ $\bar{b}$-module.

**Lemma 2.2.** Let $t$ be a non-negative integer and let $X$ be an $R$-module such that $bX = 0$ and $\text{Ext}^t_{R_b}(R_{\bar{a}+\bar{b}}, X)$ is an FD$_n$ $\bar{b}$-module for all $i \leq t$. Then $\text{Ext}^t_{R_a}(R_{\bar{a}}, X)$ is an FD$_n$ $R$-module for all $i \leq t$.

**Proof.** From [23, Theorem 11.65], there is a spectral sequence

$$E^{p,q}_2 := \text{Ext}^p_{\bar{b}}\left(\text{Tor}^R_q\left(\frac{R}{\bar{b}}, \frac{R}{\bar{a}}\right), X\right) \Rightarrow \text{Ext}^{p+q}_{R}(\frac{R}{\bar{a}}, X).$$

Let $0 \leq j \leq i \leq t$. By [15, Proposition 3.4], $E^{i-j,j}_2$ is an FD$_n$ $\bar{b}$-module. Hence $E^{i-j,j}_\infty$ is an FD$_n$ $\bar{b}$-module as $E^{i-j,j}_\infty = E^{i+j,j}_\infty$ and $E^{i-j,j}_i$ is a subquotient of $E^{i,j}_2$. There exists a finite filtration

$$0 = \phi^{i+1}H^i \subseteq \phi^iH^i \subseteq \cdots \subseteq \phi^1H^i \subseteq \phi^0H^i = \text{Ext}_R^i\left(\frac{R}{\bar{a}}, X\right)$$

such that $E^{i-j,j}_i \cong \frac{\phi^{i+j}H^i}{\phi^{i+j+1}H^i}$ for all $j, 0 \leq j \leq i$. Now, from the short exact sequences

$$0 \rightarrow \phi^{i+j+1}H^i \rightarrow \phi^{i+j}H^i \rightarrow E^{i-j,j}_\infty \rightarrow 0,$$

for all $j, 0 \leq j \leq i$, Ext$_R^i\left(\frac{R}{\bar{a}}, X\right)$ is an FD$_n$ $\bar{b}$-module. Therefore Ext$_R^i\left(\frac{R}{\bar{a}}, X\right)$ is an FD$_n$ $R$-module. □

We are now ready to state and prove the main result of this section, which plays an important role in Sections 3 and 4 to study Questions 1.4–1.6.

**Theorem 2.3.** Suppose that dim$(R) \leq n + 2$ and $X$ is an $a$-torsion $R$-module such that Hom$_R\left(\frac{R}{\bar{a}}, X\right)$ and Ext$_R^i\left(\frac{R}{\bar{a}}, X\right)$ are FD$_n$ $R$-modules. Then $X$ is an (FD$_n$, $a$)-cofinite $R$-module.

**Proof.** Assume that $a$ is nilpotent. Then $a^t = 0$ for some integer $t$. By [15, Proposition 3.4], Hom$_R\left(\frac{R}{\bar{a}}, X\right)$ is an FD$_n$ $R$-module and hence $X = (0 :_X a^t)$ is an (FD$_n$, $a$)-cofinite $R$-module. Now, assume that $a$ is not nilpotent. Since $\Gamma_a(R)$ is finite, there is an integer $t$ such that $(0 :_R a^t) = \Gamma_a(R)$. Set $b := (0 :_R a^t)$ and $Y := \frac{X}{(0 :_X a^t)}$. It is easy to see that $bY = 0$, $Y$ is an $(a + b)$-torsion $R$-module, and dim$(\frac{R}{a+b}) \leq n + 1$. Since $(0 :_X a^t)$, Hom$_R\left(\frac{R}{a+b}, X\right)$, and Ext$_R^i\left(\frac{R}{a+b}, X\right)$ are FD$_n$ $R$-modules from [15, Proposition 3.4], Hom$_R\left(\frac{R}{a+b}, Y\right)$ and Ext$_R^i\left(\frac{R}{a+b}, Y\right)$ are FD$_n$ $R$-modules by the short exact sequence

$$0 \rightarrow (0 :_X a^t) \rightarrow X \rightarrow Y \rightarrow 0.$$

Thus, from [21 Corollary 2.3], Ext$_R^i\left(\frac{R}{a+b}, Y\right)$ is an FD$_n$ $R$-module for all $i$. Hence Ext$_R^i\left(\frac{R}{a+b}, Y\right)$ is an FD$_n$ $R$-module for all $i$ by Lemmas 2.1 and 2.2. Therefore $X$ is an (FD$_n$, $a$)-cofinite $R$-module from the above short exact sequence. □

The following corollary is an immediate application of the above theorem.
Corollary 2.4. Suppose that \( \dim(R) \leq n + 2 \) and \( X \) is an arbitrary \( R \)-module such that \( \text{Hom}_R \left( \frac{R}{a}, X \right) \) and \( \text{Ext}^1_R \left( \frac{R}{a}, X \right) \) are \( \text{FD}_{< n} \) \( R \)-modules. Then \( \Gamma_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module.

Proof. By the short exact sequence

\[
0 \longrightarrow \Gamma_a(X) \longrightarrow X \longrightarrow \frac{X}{\Gamma_a(X)} \longrightarrow 0,
\]

\( \text{Hom}_R \left( \frac{R}{a}, \Gamma_a(X) \right) \) and \( \text{Ext}^1_R \left( \frac{R}{a}, \Gamma_a(X) \right) \) are \( \text{FD}_{< n} \) \( R \)-modules. Thus the assertion follows from Theorem 2.3.

By putting \( n = 0 \) in Theorem 2.3 and Corollary 2.4, we have the following results.

Corollary 2.5. Suppose that \( \dim(R) \leq 2 \) and \( X \) is an \( a \)-torsion \( R \)-module such that \( \text{Hom}_R \left( \frac{R}{a}, X \right) \) and \( \text{Ext}^1_R \left( \frac{R}{a}, X \right) \) are finite \( R \)-modules. Then \( X \) is an \( a \)-cofinite \( R \)-module.

Corollary 2.6. Suppose that \( \dim(R) \leq 2 \) and \( X \) is an arbitrary \( R \)-module such that \( \text{Hom}_R \left( \frac{R}{a}, X \right) \) and \( \text{Ext}^1_R \left( \frac{R}{a}, X \right) \) are finite \( R \)-modules. Then \( \Gamma_a(X) \) is an \( a \)-cofinite \( R \)-module.

3. Cofiniteness and associated primes of local cohomology modules

The following is the main result of this section; it shows that the answer to Questions 1.5 and 1.6 is yes if \( \dim(R) \leq n + 2 \).

**Theorem 3.1.** Suppose that \( \dim(R) \leq n + 2 \) and \( X \) is an arbitrary \( R \)-module. Then the following statements are equivalent:

\( i \) \( H^i_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module for all \( i \);

\( ii \) \( \text{Ext}^i_R \left( \frac{R}{a}, X \right) \) is an \( \text{FD}_{< n} \) \( R \)-module for all \( i \);

\( iii \) \( \text{Ext}^i_R \left( \frac{R}{a}, X \right) \) is an \( \text{FD}_{< n} \) \( R \)-module for all \( i \leq n + 2 \).

**Proof.** \( (i) \Rightarrow (ii) \). This follows by Theorem 2.1.

\( (iii) \Rightarrow (i) \). We first show that if \( t \) is a non-negative integer such that \( \text{Ext}^i_R \left( \frac{R}{a}, X \right) \) is an \( \text{FD}_{< n} \) \( R \)-module for all \( i \leq t+1 \), then \( H^i_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module for all \( i \leq t \). We prove this by using induction on \( t \). The case \( t = 0 \) follows from Corollary 2.4. Suppose that \( t > 0 \) and that \( t-1 \) is settled. It is enough to show that \( H^t_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module, because \( H^i_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module for all \( i \leq t - 1 \) from the induction hypothesis on \( t - 1 \). By Theorem 2.3, \( \text{Hom}_R \left( \frac{R}{a}, H^0_a(X) \right) \) and \( \text{Ext}^1_R \left( \frac{R}{a}, H^0_a(X) \right) \) are \( \text{FD}_{< n} \) \( R \)-modules. Therefore \( H^t_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module from Theorem 2.3. This terminates the induction argument. Thus \( H^i_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module for all \( i \neq n+2 \) from Theorem 6.1.2. By Theorem 2.3, \( \text{Hom}_R \left( \frac{R}{a}, H^{n+2}_a(X) \right) \) is an \( \text{FD}_{< n} \) \( R \)-module. Also, from Exercise 7.1.7, \( \text{Supp}_R(H^{n+2}_a(X)) \subseteq \text{Max}(R) \), because each \( R \)-module can be viewed as the direct limit of its finite submodules. Thus \( H^{n+2}_a(X) \) is an \( (\text{FD}_{< n}, a) \)-cofinite \( R \)-module by Lemma 2.1. \( \square \)
Corollary 3.2. Suppose that \( \dim(R) \leq n + 2 \), \( X \) is an arbitrary \( R \)-module, and \( t \) is a non-negative integer such that \( \text{Ext}_R^i \left( \frac{R}{a}, X \right) \) is an \( \text{FD}_{<n} \) \( R \)-module for all \( i \leq t + 1 \) (resp. for all \( i \leq n + 2 \)). Then \( H^i_a(X) \) is an \( \text{(FD}_{<n}, a) \)-cofinite \( R \)-module for all \( i \leq t \) (resp. for all \( i \)). In particular, \( \text{Ass}_R(H^i_a(X)) \geq_n \) is a finite set for all \( i \leq t \) (resp. for all \( i \)).

Proof. The first assertion follows from the proof of Theorem 3.1. The last assertion follows by the first one and [8, Exercise 1.2.28]. 

We have the following corollaries by taking \( n = 0 \) in Theorem 3.1 and Corollary 3.2.

Corollary 3.3 (see [21, Theorem 7.10]). Suppose that \( \dim(R) \leq 2 \) and \( X \) is an arbitrary \( R \)-module. Then the following statements are equivalent:

(i) \( H^i_a(X) \) is an \( a \)-cofinite \( R \)-module for all \( i \);
(ii) \( \text{Ext}_R^i \left( \frac{R}{a}, X \right) \) is a finite \( R \)-module for all \( i \);
(iii) \( \text{Ext}_R^i \left( \frac{R}{a}, X \right) \) is a finite \( R \)-module for all \( i \leq 2 \).

Corollary 3.4. Suppose that \( \dim(R) \leq 2 \) and \( X \) is an arbitrary \( R \)-module such that \( \text{Ext}_R^i \left( \frac{R}{a}, X \right) \) is a finite \( R \)-module for all \( i \leq 2 \). Then \( \text{Ass}_R(H^i_a(X)) \) is a finite set for all \( i \).

If \( R \) is a local ring with \( \dim \left( \frac{R}{a} \right) \leq 2 \), then the answer to Question 1.3 is yes by Bahmanpour–Naghipour’s result [3, Theorem 3.1] (see also [21, Theorem 3.3(c)]). In [24, Corollary 5.6], the first author and Morsali generalized this result to arbitrary semi-local rings. In the next result, by putting \( n = 1 \) in Corollary 3.2 we provide an affirmative answer to Question 1.3 for the case that \( R \) is a semi-local ring with \( \dim(R) \leq 3 \). Note that our result is a generalization of Marley’s result in [19], where he showed that if \( R \) is a local ring with \( \dim(R) \leq 3 \) and \( M \) is a finite \( R \)-module, then \( \text{Ass}_R(H^i_a(M)) \) is a finite set for all \( i \) (see [19, Proposition 1.1 and Corollary 2.5]). Note also that, if \( R \) is a semi-local ring and \( X \) is an \( \text{(FD}_{<1}, a) \)-cofinite \( R \)-module, then \( X \) is an \( a \)-weakly cofinite \( R \)-module by [5, Theorem 3.3].

Corollary 3.5. Suppose that \( R \) is a semi-local ring with \( \dim(R) \leq 3 \), \( X \) is an arbitrary \( R \)-module, and \( t \) is a non-negative integer such that \( \text{Ext}_R^i \left( \frac{R}{a}, X \right) \) is an \( \text{FD}_{<1} \) \( R \)-module for all \( i \leq t + 1 \) (resp. for all \( i \leq 3 \)). Then \( H^i_a(X) \) is an \( a \)-weakly cofinite \( R \)-module for all \( i \leq t \) (resp. for all \( i \)). In particular, \( \text{Ass}_R(H^i_a(X)) \) is a finite set for all \( i \leq t \) (resp. for all \( i \)).

4. Abelianness of the category of cofinite modules

The following theorem is the main result of this section; it shows that the answer to Question 1.4 is also yes if \( \dim(R) \leq n + 2 \).

Theorem 4.1. If \( \dim(R) \leq n+2 \), then the category of \( \text{(FD}_{<n}, a) \)-cofinite \( R \)-modules forms an Abelian subcategory of the category of \( R \)-modules.
Proof. The proof is similar to that of [24, Theorem 3.1]. We bring it here for the sake of completeness. Assume that \( X \) and \( Y \) are \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-modules and \( f : X \rightarrow Y \) is an \( R \)-homomorphism. We show that \( \ker f \), \( \text{im} f \), and \( \text{coker} f \) are \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-modules. From the short exact sequence

\[
0 \longrightarrow \text{im} f \longrightarrow Y \longrightarrow \text{coker} f \longrightarrow 0,
\]

\( \text{Hom}_R \left( R^\alpha, \text{im} f \right) \) is an \( \text{FD}_{<n} R \)-module. Hence \( \text{Hom}_R \left( R^\alpha, \ker f \right) \) and \( \text{Ext}^1_R \left( R^\alpha, \ker f \right) \) are \( \text{FD}_{<n} R \)-modules by the short exact sequence

\[
0 \longrightarrow \ker f \longrightarrow X \longrightarrow \text{im} f \longrightarrow 0.
\]

Therefore \( \ker f \) is an \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-module by Theorem 2.3. Thus \( \text{im} f \) and \( \text{coker} f \) are \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-modules from the above short exact sequences. \( \square \)

As an immediate application of the above theorem, we have the following corollary.

**Corollary 4.2.** Suppose that \( \dim(R) \leq n + 2 \), \( N \) is a finite \( R \)-module, and \( X \) is an \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-module. Then \( \text{Ext}^j_R(N, X) \) and \( \text{Tor}^j_R(N, X) \) are \((\text{FD}_{<n}, \alpha)\)-cofinite \( R \)-modules for all \( j \).

We have the following results by taking \( n = 0 \) in Theorem 4.1 and Corollary 4.2.

**Corollary 4.3** (see [21, Theorem 7.4]). If \( \dim(R) \leq 2 \), then the category of \( \alpha \)-cofinite \( R \)-modules forms an Abelian subcategory of the category of \( R \)-modules.

**Corollary 4.4.** Suppose that \( \dim(R) \leq 2 \), \( N \) is a finite \( R \)-module, and \( X \) is an \( \alpha \)-cofinite \( R \)-module. Then \( \text{Ext}^j_R(N, X) \) and \( \text{Tor}^j_R(N, X) \) are \( \alpha \)-cofinite \( R \)-modules for all \( j \).

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Alireza Vahidi
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697 Tehran, Iran
vahidi.ar@pnu.ac.ir

Mahdieh Papari-Zarei
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697 Tehran, Iran
m.p.zarei@gmail.com

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