Critical exponents of domain walls in the two-dimensional Potts model

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Abstract
We address the geometrical critical behavior of the two-dimensional Q-state Potts model in terms of the spin clusters (i.e. connected domains where the spin takes a constant value). These clusters are different from the usual Fortuin–Kasteleyn clusters, and are separated by domain walls that can cross and branch. We develop a transfer matrix technique enabling the formulation and numerical study of spin clusters even when $Q$ is not an integer. We further identify geometrically the crossing events which give rise to conformal correlation functions. This leads to an infinite series of fundamental critical exponents $h_{\ell_1,\ell_2}$, valid for $0 \leq Q \leq 4$, that describe the insertion of $\ell_1$ thin and $\ell_2$ thick domain walls.

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(Some figures in this article are in colour only in the electronic version)
frustrating since those are the very clusters that one would observe in an actual experiment on a magnetic alloy in the Potts universality class. The case of the Ising model $Q = 2$ is an exception to this rule [4, 5], but this is due to its ‘coincidental’ equivalence to the $O(n)$ vector model with $n = 1$. Indeed, defining the Ising spins on the triangular lattice, the corresponding $O(n)$ model is described by self and mutually avoiding loops on the hexagonal lattice, and such loops are readily treated by CFT and SLE techniques.

For general $Q$, the salient feature of Potts spin clusters is that the domain walls (DWs) separating different clusters undergo branchings and crossings (see figure 1). These phenomena are, however, absent for $Q = 2$ with the above choice of lattice. It is precisely these branchings and crossings that make the application of exact techniques—such as CG mappings or direct Bethe ansatz diagonalization—very difficult, if not impossible. The belief that spin clusters are indeed conformally invariant for other values of $Q \neq 2$ in the critical regime $0 \leq Q \leq 4$ has even been challenged at times, but seems however well established at present [6, 7].

Some progress has been accomplished in the $Q = 3$ case [5, 8] by speculating that the spin clusters in the critical Potts model would be equivalent to FK clusters in the tricritical Potts model [8, 9]. This equivalence has however not been proven, and is moreover restricted so far to the simplest geometrical questions [10]. The equivalence can also be understood as a relationship with the dilute $O(n)$ model [11].

The Potts model has been used recently to build a new class of 2D quantum lattice models that exhibit topological order [12]. Both FK clusters and DWs between spin clusters are important in these models.

Apart from issues of branching and crossing, another major hurdle in the study of Potts spin clusters has come from the lack of a formulation that can be conveniently extended to $Q$ a real variable. In the case of FK clusters, this formulation led naturally to the introduction of powerful algebraic tools via the Temperley–Lieb (TL) algebra, and to the equivalence with the six-vertex model—the eventual key to the exact solution of the problem [1]. Factors of $Q$ then appear naturally through a parameter in the TL algebra, or—via a geometrical construction—as complex vertex weights in the six-vertex formulation. Also for spin clusters can $Q$ be promoted to an arbitrary variable: the weight of a set of spin clusters is simply the chromatic polynomial of the graph dual to the DWs. From the point of view of the TL algebra, the DWs are composite (spin-1) objects, hence more complicated than the FK loops. Recent work on the related Birman–Wenzl–Murakami (BWM) algebra [13, 14] suggests that this formulation
might be amenable to the standard algebraic and Bethe ansatz techniques, although such a lofty goal has not been achieved so far.

We report in this communication major progress toward the understanding of Potts spin clusters. Our results are of two kinds. On the one hand, we develop a transfer matrix technique which allows the formulation and numerical study of the spin clusters for all real $Q$. On the other hand, we identify the geometrical events that give rise to conformal correlations, and provide exact (albeit numerically determined) expressions for an infinite family of critical exponents, similar to the familiar ‘$L$-legs’ exponents [1, 3] for FK loops. Surprisingly, we find that the geometrical properties of spin clusters encompass all integer indices $(r, s)$ in the Kac table $h_{r,s}$. An analytical derivation of our results appears for now beyond reach, in part because the algebraic properties of our transfer matrix are still ill understood. We do however provide some exact results based on an approach which does not involve a CG mapping, but rather the use of a massless scattering description.

2. DW expansion

The $Q$-state Potts model is defined by the partition function

$$ Z = \sum_{\sigma} \prod_{(ij) \in E} \exp(K \delta_{\sigma_i, \sigma_j}), $$

where $K$ is the coupling between spins $\sigma_i = 1, 2, \ldots, Q$ along the edges $E$ of some lattice $\mathcal{L}$. We use the square lattice in the computations below (since its transfer matrix construction is slightly easier) and the triangular lattice in the figures (since it is more familiar to readers acquainted with the SLE literature)—but our continuum limit results are of course independent of this choice. The Kronecker delta function $\delta_{\sigma_i, \sigma_j}$ equals 1 if $\sigma_i = \sigma_j$, and 0 otherwise.

The DW expansion of (1) involves all possible configurations of DWs that can be drawn on the dual of $\mathcal{L}$ (see figure 2). A DW configuration is given by a graph $G$ (not necessarily connected). The faces of $G$ are the spin clusters. Since we do not specify the color of each of these clusters, a DW configuration has to be weighted by the chromatic polynomial $\chi_G(Q)$ of the dual graph $\hat{G}$. Initially $\chi_G(Q)$ is defined as the number of colorings of the vertices of the graph $\hat{G}$, using colors $\{1, 2, \ldots, Q\}$, with the constraint that neighboring vertices have different colors. This is indeed a polynomial in $Q$ for any $G$, and so can be evaluated for any real $Q$ (but $\chi_G(Q)$ is integer only when $Q$ is integer). For example, the chromatic polynomial...
Figure 3. The two different types of DWs: a thin DW corresponds to the interface between two clusters of different colors (a), while for a thick DW the two clusters have the same color. An illustration for the $Q = 3$ Potts model is given (left) as well as a schematic picture for non-integer $Q$ (right).

of the graph $\hat{G}$ in figure 2 is $Q(Q-1)^2(Q-2)^2$. The partition function (1) can thus be written as a sum over all possible DW configurations:

$$Z = e^{NK} \sum_G (e^{-K})^{\text{length}(G)} X_G(Q),$$

where $N$ is the number of spins and $\text{length}(G)$ denotes the total length of the DWs.

3. The bulk DW exponents

The fundamental geometric object we consider is a connected part of a DW that separates two clusters. One can ask how the probability, that a certain number of such DWs connect a small neighborhood $A$ to another small neighborhood $B$, decays when the distance $x$ between $A$ and $B$ increases. Each DW separates two spin clusters which connect $A$ and $B$. There are in fact two types of such DWs, depending on the relative coloring of the two clusters that are separated. If the two clusters have different colors (see figure 3(a)), they can touch, whereas if the two clusters have the same color (see figure 3(b)), they cannot touch (otherwise they would not be distinct). Because of this property, we define a thin (respectively a thick) DW as a DW that separates two distinct clusters having different colors (respectively the same color).

We can now state the central claim of this communication. Consider the 2D Potts model for any real $Q$ in the critical regime $0 \leq Q \leq 4$. Then the probability $P$ that the two regions $A$ and $B$, with separation $x \gg 1$, are connected by $\ell_1$ thin DW and $\ell_2$ thick DW
decays algebraically in the plane with some exponent $h(Q, \ell_1, \ell_2)$, namely $P \propto x^{-4h(Q, \ell_1, \ell_2)}$.

Equivalently, on a long cylinder of size $L \times \ell$ with $\ell \gg L$, and $A$ and $B$ identified with the opposite ends of the cylinder, the decay is exponential: $P \propto e^{-4\pi(\ell/L)h(Q, \ell_1, \ell_2)}$. Below we check this assertion numerically, and we observe that the numerical values of the exponents match the formula

$$h(Q, \ell_1, \ell_2) = h_{\ell_1 - \ell_2, 2\ell_1}, \quad (3)$$

where we have used the Kac parametrization of CFT:

$$h_{r,s} = \frac{(r - s\kappa/4)^2 - (1 - \kappa/4)^2}{\kappa} \quad (4)$$

and $2 \leq \kappa \leq 4$ parametrizes $Q = 4 \left(\cos \frac{\kappa\pi}{4}\right)^2 \in [0, 4]$.

### 4. Transfer matrix formulation

The DW expansion (2) may appear unwieldy and difficult to study in a Monte Carlo simulation for non-integer $Q$. However, it can be tackled in a transfer matrix formalism that is no more complicated than the one [15] routinely used in the study of the FK clusters.

Consider a strip of the square lattice of width $L$ spins (boundary conditions will be detailed later). The basis states on which the transfer matrix $T$ acts contain one color label $c_i$ per spin. By definition, one has $c_i = c_j$ if and only if $\sigma_i = \sigma_j$ (i.e. the two spins on sites $i$ and $j$ have the same color). The color labels $c_i$ contain less information than the spin colors $\sigma_i$ themselves. For instance, any configuration in which the first and third spins have the same color, no matter which one, and no other spins have identical colors, is represented by

$$c_1 c_1 c_1, \quad c_1 c_2 c_1, \quad c_1 c_1 c_2, \quad c_1 c_2 c_2, \quad c_1 c_2 c_3 \quad (5)$$

Note that the last state would carry zero weight for $Q = 2$, but apart from that the number of basis states for any given $L$ will be finite and independent of $Q$.

We can write $T$ as a product of elementary transfer matrices, each represented symbolically as a rhombus surrounding a single lattice edge. This edge links spins (shown as solid circles) on diametrically opposite sites of the rhombus. On an $L = 4$ square lattice with periodic boundary conditions, this reads $T = \bigotimes_i \bigotimes_j$.

A rhombus $\Diamond$ corresponds to a vertical edge, and acts on a basis state $s$ as follows. If exactly $Q_i$ distinct color labels $\{c_k\}$ are used in $s$, then either the new color label $c'_i$ of the spin $\sigma_i$ can be unchanged, $c'_i = c_i$ (with weight $e^K$), or be any one of the other labels already in use $c'_i = c_k$ (each with weight 1), or be a new one $c'_i \notin \{c_k\}$ (with weight $Q - Q_i$). Note that this latter weight is in general non-integer, and is responsible for the correct computation of the chromatic polynomial $\chi_G(Q)$. A rhombus $\blacktriangleleft$ adding a horizontal edge between vertices $i$ and $i + 1$ corresponds simply to a diagonal matrix, with a weight $e^K$ if $c_i = c_{i+1}$, and 1 otherwise.
With these rules at hand, one can write the (periodic) $L = 3$ transfer matrix for arbitrary $Q$ in the basis (5) as an instructive example: $T = h_1 \cdot h_2 \cdot h_3 \cdot v_1 \cdot v_2 \cdot v_3$ with

$$v_1 = \begin{pmatrix} e^K & 0 & 0 & 1 & 0 \\ 0 & e^K & 1 & 0 & 1 \\ 0 & 1 & e^K & 0 & 1 \\ Q - 1 & 0 & 0 & e^K + Q - 2 & 0 \\ 0 & Q - 2 & Q - 2 & 0 & e^K + Q - 3 \end{pmatrix}$$

(6)

and $h_1 = \text{diag}(e^K, 1, e^K, 1, 1)$. The remaining matrices can be obtained from those given by cyclic permutations of the color labels. The reader can now check that $T$ gives the same free energy as the FK transfer matrix, even when $Q$ is non-integer.

However, to obtain the desired two-point correlation functions of DW, the basis states (5) need to be endowed with some additional information about the connectivity of the spin clusters. We need to know whether two spins having the same label $c_i$ also belong to the same cluster. Thus the states we use in the final transfer matrix have the form

$$c_1 \ c_2 \ c_1 \ c_1 \ c_5 \ c_1 \ c_2 \ c_1 \ c_1 \ c_5$$

(7)

In the left state, the spins on vertices 3 and 4 are in the same cluster, but not in the same cluster as spin 1. In the right state, the spins 1, 3 and 4 are all in the same cluster. These two states are different. In the transfer matrix evolution, when two neighbor vertices correspond to the same color, the corresponding clusters are joined up.

The final transfer matrix thus keeps enough information, both about the mutual coloring of the sites and about the connectivity of the clusters, to give the correct Boltzmann weights to the different configurations, even for non-integer $Q$ and to follow the evolution of the boundary of a particular set of clusters. These boundaries are precisely the DWs (see figure 3).

5. Numerical results

We have numerically diagonalized the transfer matrix in the DW representation for periodic strips of width up to $L = 11$ spins. We verified that the leading eigenvalue in the ground state sector coincides with that of the FK transfer matrix, including for non-integer $Q$. As to the excitations, we explored systematically all possible coloring combinations for up to four marked spin clusters, for a variety of values of the parameter $\kappa$. Finite-size approximations of the critical exponents $h$ were extracted from the leading eigenvalue in each sector, using standard CFT results [1], and fitting for both the universal corrections in $L^{-2}$ and the non-universal $L^{-4}$ term. Final results (see table 1) for the exponents were obtained by extrapolating those approximants to the $L \to \infty$ limit, fitting them to first- and second-order polynomials in $L^{-1}$, and gradually excluding data points corresponding to the smallest $L$. Error bars were obtained by carefully comparing the consistency of the various extrapolations. Note that (3) is only valid for $(\ell_1, \ell_2) = (0, 1)$ if the spin cluster is forbidden to wrap around the periodic direction. Without that restriction we obtain $h_{0,1/2}$.

6. Analytical results

While it is far from obvious to derive (3) by CG methods, minor progress can be achieved using a rather different set of ideas. Indeed, we can learn about the dynamics of DW in the critical theory by using known information about the low-temperature ($K > K_c$) phase of
the Potts model. Albeit non-integrable on the lattice, the corresponding deformation by the operator $\Phi_{21}$ is integrable in the continuum [16]. It can be described using a basic set of kinks $K_{ab}$ separating two vacua, i.e. ordered regions where the dominant value of the spin is $a$, respectively $b$. These kinks scatter with a known $S$-matrix related to the BWM algebra [14]. Importantly, the dynamics conserves the number of kinks: the process $K_{ab}K_{bc} \rightarrow K_{ac}$ is forbidden (as in any elastic relativistic scattering theory), although kinks do appear as bound states in kink–kink processes. Many properties of these kinks can be calculated using integrability techniques. When the mass $m \rightarrow 0$ (i.e. $K \rightarrow K_c$), the $S$-matrix provides a ‘massless scattering’ description [17] of some of the degrees of freedom of the critical theory itself. It is not entirely clear what a kink, which is well defined for ‘massless scattering’ description [17] of some of the degrees of freedom of the critical theory itself. Indeed, the fact that the $S$-matrix satisfies relations from the BWM algebra allows us to re-express it in terms of the $a(2)$ or Bullough–Dodd $S$-matrix [18], for which the thermodynamic Bethe ansatz was studied in [19]. This describes the dynamics of the field theory as well,

$$S = \frac{1}{8\pi} \int d^2x \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + g(2 e^{-\frac{i}{\beta} \Phi} + e^{i\sqrt{2}\beta \Phi}) \right]$$  \hspace{2cm} (8)

with $\beta^2 = \frac{\xi}{\pi}$. Giving each kink the fugacity $Q - 1$ produces the correct central charge $c = 1 - \frac{3(\xi - 4)}{4\xi}$, where each kink has a $U(1)$ charge equal to 0, ±1. The scaling of the sector with charge $j$ produces a gap $\Delta_j = \frac{j^2}{2\pi^2}$, so the leading dimension is $\Delta_j - (1 - c)/24 = h_{j/2,0}$. This agrees with $h_{-\ell_2,0}$ with $\ell_2 = j/2$, so there are two kinks per thick DW.

### 7. Fractal dimensions

The dimension of a spin cluster is $d = \min(2, 2 - 2h_{0,1/2}) = \min(2, \frac{(8\pi \gamma)(5 + 3\pi)}{32\pi \gamma})$, in agreement with [4, 5]. The boundary of a cluster has dimension $d_b = 2 - 2h_{-1,0} = 1 + \frac{\xi}{2\pi}$. According to (3), the set of points where a thin (respectively thick) DW has minimal width (one respectively two lattice spacings) has dimension $d_1 = 2 - 2h_{2,4} = \frac{1}{2\pi}(4 - k)(5\kappa - 4)$ (respectively $d_2 = \min(0, 2 - 2h_{-2,0}) = 0$). The fact that $d_1 \geq 0$ and $d_2 = 0$ validates the epithets ‘thin’ and ‘thick’ for the scaling limit. For $Q = 4$ (or $\kappa = 4$), we have $d_1 = d_2 = 0$, so thin and thick DWs become indistinguishable. Indeed one has $h_{\ell_1,-\ell_2,2\ell_1} = (\ell_1 + \ell_2)^2/4$ in that case.

| $p = \frac{1}{c+3}$ | (2, 0) | (0, 2) | (3, 0) | (2, 1) | (0, 3) |
|---------------------|--------|--------|--------|--------|--------|
| 2                   | 2.01(1)| 5.99(2)| 2.97(4)| 8.94(2)|        |
| 3                   | 4.01(1)| 7.99(2)| 6.02(2)| 8.04(2)| 11.98(3)|
| 4                   | 5.93(2)| 10.01(2)| 8.89(5)| 10.93(5)| 15.05(4)|
| 5                   | 7.77(4)| 12.09(8)| 11.6(1)| 13.8(1)| 18.2(2) |
| Exact               | $2(p-1)$| $2(p+1)$| $(p-1)$| $(p-1)$| $(p+1)$|
For the Ising model $Q = 2$ (or $\kappa = 3$), the absence of branchings means that one thick DW equals two thin DWs. Indeed $h_{\ell_1 - \ell_2} = (4\ell^2 - 1)/48$ with $\ell = \ell_1 + 2\ell_2$ in that case, and this agrees with the exponent $h_{\ell/2,0}$ for $\ell$ loop strands [1, 3] in the dilute O(1) model.

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