Self semi conjugations of Ulam’s Tent-map

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Abstract

We study the self-semiconjugations of the Tent-map $f : x \mapsto 1 - |2x - 1|$ for $x \in [0, 1]$. We prove that each of these semi-conjugations $\xi$ is piecewise linear. For any $n \in \mathbb{N}$ we denote $A_n = f^{-n}(0)$ and describe the maps $\psi : A_n \to [0, 1]$ such that $\psi \circ f = f \circ \psi$. Also we describe all possible restrictions, of self-semiconjugations of the Tent-map onto $A_n$ and prove that for any $\alpha \in A_n \setminus A_{n-1}$ a restriction is completely determined by its value at $\alpha$.

1 Introduction

Motivation

An importance of the notion of topological conjugateness was discovered in the early beginning of the Dynamical systems theory by Henri Poincaré (see [1]). Later Stanislaw Ulam invented (see [2] pp. 401-484, or [3]) the conjugation of continuous interval $[0, 1] \to [0, 1]$ maps

$$f(x) = \begin{cases} 2x, & x < 1/2; \\ 2 - 2x, & x \geq 1/2, \end{cases} \quad (1.1)$$

and $\tilde{g}(x) = 4x(1 - x)$ by the homeomorphism

$$\tilde{\tau}(x) = \sin^2 \left( \frac{\pi x}{2} \right).$$

The elegance of this example made it perhaps the most studied example in the pedagogy of dynamical systems for teaching conjugation. Notice, that due to the form of the graph of $f$ it is often called a **Tent-map**. It is well known that the conjugation of $f$ and $\tilde{g}$ can be illustrated by the claim that the diagram

$$\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\tilde{\tau} & \downarrow & \tilde{\tau} \\
[0, 1] & \xrightarrow{\tilde{g}} & [0, 1]
\end{array}$$

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is commutative. One more important result, which is there in [2], is the way of the construction of the topological conjugacy of Tent-map $f$ and the map

$$g(x) = \begin{cases} 
\gamma_0(x), & \text{if } 0 \leq x < v, \\
\gamma_1(x), & \text{if } v \leq x \leq 1,
\end{cases}$$

(1.2)

for a fixed $v \in (0, 1)$ and continuous monotone functions $\gamma_0, \gamma_1$ such that $\gamma_0(0) = \gamma_1(1) = 0$, $\gamma_1(v) = 1$. Ulam proved in [2] that $f$ and $g$ are topologically conjugated if and only if the integer trajectory $\{g^n(1), n \in \mathbb{Z}\}$ of 1 under $g$ is dense in $[0, 1]$. Moreover, in this case $\tau$ increase and $\tau(f^n(1)) = g^n(1)$ for all $n \in \mathbb{Z}$. One of the simplest maps of the form (1.2) is

$$f_v(x) = \begin{cases} 
x, & 0 \leq x \leq v, \\
\frac{1-x}{1-v}, & v < x \leq 1,
\end{cases}$$

(1.3)

whose graph consists of two line segments extending from $(0, 0)$ to $(v, 1)$ to $(1, 0)$.

The conjugation $h$ of the map $f$ of the form (1.1) and $f_v$ above was treated in [4] and [5]. It is proved in [4] that the derivative of $h$ equals 0 almost everywhere in the sense of Lebesgue’s measure and equals 0 everywhere where it is finite. It is proved in [5] that the length of the graph of $h$ is 2, which is the maximum possible length of monotone $[0, 1] \to [0, 1]$ function. We have studied some properties of this conjugacy in [6] and [7]. We have proved the existence of conjugacy in [6] by Ulam’s method, i.e. proved the density of the integer trajectory $f_v^{-\infty}(1)$ of 1 under $f_v$. We have used the following technical, but important remark in [6].

**Remark 1.1.** [6, Lemma 4]) The complete pre-image $f^{-n}(0)$ of 0 under $f^n$ is

$$A_n = \left\{ \frac{k}{2^n-1}, 0 \leq k \leq 2^n-1 \right\}.$$  

(1.4)

Thus, we considered the sequence $\{h_n, n \geq 1\}$ of piecewise linear functions, such that $h_n(x) = h(x)$ for all $x \in A_n$ and the complete set of the breaking points of $h_n$ is $A_n$. We also have used this sequence in [7], in the proof of the existence and our calculation of the value of the derivative of the conjugacy $h$ at all binary rational points. Then the same problem was solved in [8] for all rational points. Notice, that authors of [4], and [5] use non-explicitly the sequence $\{h_n, n \geq 1\}$ too.

Maps $f_v$ of the form (1.3) are denoted as $T_c$ in [4] and the solution $\varphi$ of the functional equation $\varphi \circ T_{c_1} = T_{c_2} \circ \varphi$ is found as a limit of the sequence $\{\varphi_n, n \geq 0\}$, where $\varphi_0(x) = x$ for all $x \in [0, 1]$ and

$$\varphi_{n+1}(x) = \begin{cases} 
c_2 \varphi_n \left( \frac{x}{c_1} \right) & \text{if } 0 \leq x \leq c_1, \\
(c_2-1) \varphi_n \left( \frac{x-1}{c_1-1} \right) + 1 & \text{if } c_1 < x \leq 1.
\end{cases}$$

(1.5)
Notice, that $\varphi_n = h_{n+1}$ for all $n \geq 0$, if $c_1 = 1/2$ and $c_2 = v$ in (1.3). Indeed, it follows from the commutativity of diagrams

![Diagram](image)

that

$$
\begin{aligned}
    h_{n+1} &= vh_n(2x), \\
    h_{n+1} &= 1 - (1 - v)h_n(2 - 2x),
\end{aligned}
$$

which is the same as (1.5). It is proved in [4, Lemma 3] that for any continuous function $\varphi_0 : [0, 1] \to [0, 1]$ the limit function of (1.5) is the conjugacy, which we call $h$.

Complicatedness of the mentioned properties of $h$ motivate to consider the functional equation

$$
\eta \circ f = f_v \circ \eta
$$

(1.6)

for an unknown continuous $\eta : [0, 1] \to [0, 1]$ (which is not necessary a homeomorphism). It is clear from the commutative diagram

![Diagram](image)

that there is one-to-one correspondence

$$
\begin{aligned}
    \xi &= h^{-1} \circ \eta, \\
    \eta &= h \circ \xi
\end{aligned}
$$

between the solutions $\eta$ of (1.6) and the continuous maps $\xi$ such that

$$
\xi \circ f = f \circ \xi.
$$

(1.7)

Thus, we will concentrate on (1.7) in this article.

**Results**

Our work consists of 3 sections, the first of which is introduction. Section 2 is devoted to the following theorem.
Theorem 1. 1. Let $\xi$ be an arbitrary continuous solution of the functional equation (1.7). Then $\xi$ is one of the following forms:

a. There exists $k \in \mathbb{N}$ such that
$$\xi(x) = \frac{1 - (-1)^{\lfloor kx \rfloor}}{2} + (-1)^{\lfloor kx \rfloor} \{kx\}, \quad (1.8)$$
where $\{\cdot\}$ denotes the function of the fractional part of a number and $\lfloor \cdot \rfloor$ is the integer part.

b. $\xi(x) = x_0$ for all $x$, where either $x_0 = 0$, or $x_0 = 2/3$.

2. For every $k \in \mathbb{N}$ the function (1.8) satisfies (1.7).

We will use the following facts for the proof of Theorem 1.

Lemma 1.2. [9, Theorem 3] If a continuous solution $\xi$ of (1.7) is constant on some interval $[\alpha, \beta] \subseteq [0, 1]$, then $\xi$ is constant on the entire $[0, 1]$.

Lemma 1.3. [9, Theorem 4] If a continuous solution $\xi$ of (1.7) is linear on some interval $[\alpha, \beta] \subseteq [0, 1]$, then $\xi$ is piecewise linear on the entire $[0, 1]$.

Notice, that we call a function linear (piecewise linear) if its graph is a line segment (consists of line segments).

Lemma 1.4. [9, Section 4; Lemmas 10 – 16] Any continuous piecewise linear solution $\xi$ of (1.7) is either constant, or has form (1.8).

Notice that formula (1.8) describes the piecewise linear function $\xi$, whose complete set of breaking points is
$$\left\{ \left( \frac{2t}{k}, 0 \right), 0 \leq t \leq k \right\} \cup \left\{ \left( \frac{2t + 1}{k}, 1 \right), 0 \leq t < k \right\}.$$  

In this case both $\xi \circ f$ and $f \circ \xi$ are piecewise linear functions, whose complete set of breaking points is
$$\left\{ \left( \frac{2t}{2k}, 0 \right), 0 \leq t \leq k \right\} \cup \left\{ \left( \frac{2t + 1}{2k}, 1 \right), 0 \leq t < k \right\}.$$  

Thus, only part 1 of Theorem 1 need to be proved. Lemmas 1.2, 1.3 and 1.4 reduce Theorem 1 to the following fact.

Theorem 2. For any continuous solution of (1.7) there exists an interval $I$, where $\xi$ is linear.
Definition. Say that $\psi : A_n \to [0, 1]$ commutes with the Tent-map $f$, if

$$\psi \circ f = f \circ \psi.$$ \hfill (1.9)

Notice, that $\psi \circ f$ has sense, because $f(A_n) \subset A_n$.

Definition. Call the map $\psi : A_n \to [0, 1]$ Tent-continuable, if there exists a continuous $\xi : [0, 1] \to [0, 1]$ such that $\xi \circ f = f \circ \xi$ and $\xi(x) = \psi(x)$ for all $x \in A_n$.

We will describe in Theorem 3 all the maps $\psi : A_n \to [0, 1]$, which commute with $f$. In Theorem 4 we describe all the Tent-continuable maps.

2 Self semi-conjugation

Suppose that the map $f : [0, 1] \to [0, 1]$ is given by (1.1) and $\xi : [0, 1] \to [0, 1]$ is a continuous solution of (1.7). Denote by $F$ the set of the fixed points of $f$. Clearly, $F = \{0; \frac{2}{3}\}$. Also denote $F_n = f^{-n}(F)$ the complete pre-image of $F$ under $f^n$, i.e.

$$F_n = \{x \in [0, 1] : f^n(x) \in F\}.$$

Denote $B_n = f^{-n}(2/3)$ the complete pre-image of 2/3 under $f^n$. Then $F_n = A_n \cup B_n$ for all $n \geq 1$, where, as above, $A_n = f^{-n}(0)$.

Lemma 2.1. $\xi(F_n) \subseteq F_n$ for any $n \geq 1$.

Proof. If one plug an arbitrary $x \in F$ into (1.7), then it is clear that $\xi(x) \in F$. Moreover, we can rewrite (1.7) as

$$\xi \circ f^n = f^n \circ \xi,$$

whence $\xi(x) \in F_n$, whenever $x \in F_n$. \hfill \qed

We will use the following remark to calculate the explicit expressions for elements of $B_n$ and then for $F_n$.

Remark 2.2. Let

$$x = 0.a_1a_2\ldots a_n\ldots$$

be the binary expression of an arbitrary $x \in [0, 1]$. Then the binary expression of $f(x)$ is

$$f(x) = \begin{cases} 0.a_2a_3\ldots a_n\ldots, & \text{if } a_1 = 0, \\ 0.a_1\overline{a}_3\ldots \overline{a}_n\ldots, & \text{if } a_1 = 1, \end{cases}$$

where $\overline{a}_i = 1 - a_i$. 

Lemma 2.3. For \( n \geq 1 \) the set \( F_n \) is

\[
F_n = \left\{ \frac{1}{2^{n-1}} \cdot (k + \kappa) \right\} \cup \{1\},
\]

where \( 0 \leq k < 2^{n-1} \), and \( \kappa \in \{0; \frac{1}{3}; \frac{2}{3}\} \).

Proof. Notice that the binary form of \( \frac{2}{3} \) is \( 0.(10) \), because

\[
0.(10) = \frac{\frac{1}{2}}{1 - \frac{1}{4}}
\]
as the sum of infinite geometrical series (where \( (10) \) denotes the periodical part of a number).

Now, divide the expression \( \frac{2}{3} = 0.(10) \) by 2 and obtain \( \frac{1}{3} = 0.(01) \). By Remark 2.2 write

\[
B_1 = \{0.0(10); 0.1(01)\}.
\]

It follows from Remark 2.2 and induction on \( n \) that \( B_n \) consists of \( 2^n \) numbers, which have the form

\[
\left\{ \frac{k}{2^n} + p_k, 0 \leq k < 2^n \right\},
\]

where \( p_k \) is an infinite periodical part 01, or 10, starting after the binary digit \( n \) by the following rule: if \( n \)-th digit is 0, then the periodical part is 10 and it is 01 otherwise. In other words,

\[
B_n = \left\{ \frac{2k}{2^n} + \frac{1}{2^n} \cdot \frac{2}{3}, 0 \leq k < 2^{n-1} \right\} \cup \left\{ \frac{2k+1}{2^n} + \frac{1}{2^n} \cdot \frac{1}{3}, 0 \leq k < 2^{n-1} \right\}.
\]

Notice, that \( \frac{2k}{2^n} + \frac{1}{2^n} \cdot \frac{2}{3} = \frac{1}{2^{n-1}} \cdot (k + \frac{1}{3}) \) and \( \frac{2k+1}{2^n} + \frac{1}{2^n} \cdot \frac{1}{3} = \frac{1}{2^{n-1}} \cdot (k + \frac{2}{3}) \). Now lemma follows from Remark 1.1. \( \square \)

2.1 Tangents of secants of \( \xi \) are bounded

By Heine-Cantor theorem the continuity of \( \xi \) on the compact \([0, 1]\) implies its uniform continuity. Thus, for any \( n \geq 1 \) there exists \( m_\xi(n) \) such that

\[
|\xi(a) - \xi(b)| < 2^{-n},
\]

whenever the first \( m_\xi(n) \) binary digits of \( a \) and \( b \) coincide.

Lemma 2.4. For every \( n \in \mathbb{N} \) if the first \( m_\xi(n) + 1 \) binary digits of \( a, b \in [0, 1] \) are equal, then \( |\xi(a) - \xi(b)| < 2^{-n-1} \).

Proof. By Remark 2.2 since the first \( m_\xi(n) + 1 \) binary digits of \( a \) and \( b \) are equal then so are the first \( m_\xi(n) \) binary digits of \( f(a) \) and \( f(b) \). Whence, it follows from (1.7) that

\[
|f(\xi(a)) - f(\xi(b))| < 2^{-n}.
\] (2.1)
Without loss of generality assume

$$\xi(a) > \xi(b) \quad (2.2)$$

and suppose by contradiction that

$$\xi(a) - \xi(b) \geq 2^{-n-1}. \quad (2.3)$$

Notice, that it follow from the construction of $m_\xi(n)$ that

$$\xi(a) - \xi(b) < 2^{-n}. \quad (2.4)$$

Consider two cases.

**Case 1:** Suppose that the first $n$ binary digits of $\xi(a)$ and $\xi(b)$ coincide. Then by (2.2) and (2.3) write the binary form of $\xi(a)$ and $\xi(b)$ as

$$\xi(a) = 0.M \ 1 \ A, \quad \xi(b) = 0.M \ 0 \ B, \quad (2.5)$$

where $A$, $B$ and $M$ are blocks of digits and the length of $M$ is $n$. Indeed,

$$\begin{cases} 
\xi(a) = 0.M \ 0 \ A, \\
\xi(b) = 0.M \ 1 \ B 
\end{cases}$$

contradicts to (2.2). Also the assumption

$$\begin{cases} 
\xi(a) = 0.M \ x \ A, \\
\xi(b) = 0.M \ x \ B 
\end{cases}$$

for some binary digit $x$ contradicts to (2.3). Moreover, it follows from (2.3) that

$$0.A > 0.B$$

in (2.5). For the simplification of reasonings, denote $M'$ the block $M$ without its the first digit and use the line over the name of a block (for example $\overline{A}$, $\overline{M'}$ etc.) for the inversion of all 0-s and 1-s there.

If the first digit of $M$ is 0, then by (2.5) and Remark 2.2 obtain

$$f(\xi(a)) = 0.M' \ 1 \ A, \quad f(\xi(b)) = 0.M' \ 0 \ B,$$

whence

$$f(\xi(a)) - f(\xi(b)) = 2^{-n} + 2^{-n-1} \cdot (0, A - 0, B), \quad (2.6)$$
because $M'$ contains $n - 1$ digits. This contradicts to (2.1).

If the first digit of $M$ equals 1, then

$$
 f(\xi(a)) = 0, M' 0 \overline{A}, \\
 f(\xi(b)) = 0, M' 1 \overline{B}
$$

and

$$
 f(\xi(b)) - f(\xi(a)) = 2^{-n} + 2^{-n-1} \cdot (0, \overline{B} - 0, \overline{A}). 
$$

(2.7)

Notice, that $0, C + 0, \overline{C} = 1$ for every infinite block $C$, whence

$$
 0, \overline{B} - 0, \overline{A} = 0, A - 0, B
$$

and (2.7) implies

$$
 f(\xi(b)) - f(\xi(a)) = 2^{-n} + 2^{-n-1} \cdot (0, A - 0, B),
$$

which contradicts to (2.1).

Consider now an alternative to the case 1, i.e.

**Case 2:** Some of the first $n$ binary digits of $\xi(a)$ and $\xi(b)$ are different. It follows from (2.2) and (2.4) that

$$
 \xi(a) = 0.M 1 0 A, \\
 \xi(b) = 0.M 0 1 B
$$

(2.8)

where 0 and 1 denote blocks of zeros and ones respectively and blocks $A$ and $B$ start from the digits $n + 1$. It follows from (2.3) that

$$
 0.M 0 1 B + 2^{-n-1} \leq 0.M 1 0 A.
$$

Subtract 0. $m$ for both sides of the obtained inequality, where $m$ is the first digit of $M$, multiply the obtained inequality by 2 and get

$$
 0.M' 0 1 B + 2^{-n} \leq 0.M' 1 0 A. 
$$

(2.9)

If the first digit of $M$ is 0, then by (2.8) and Remark 2.2 obtain

$$
 f(\xi(a)) = 0.M' 1 0 A, \\
 f(\xi(b)) = 0.M' 0 1 B.
$$

Now (2.9) implies

$$
 f(\xi(a)) - f(\xi(b)) \geq 2^{-n},
$$

which contradicts (2.1).
If the first digit of \( M \) is 1, then by (2.8) and Remark 2.2 obtain
\[
\begin{align*}
 f(\xi(a)) &= 0.\overline{M'}01A, \\
f(\xi(b)) &= 0.\overline{M'}10B.
\end{align*}
\]
Now,
\[
\begin{align*}
 1 - f(\xi(a)) &= 0.\overline{M'}10A, \\
 1 - f(\xi(b)) &= 0.\overline{M'}01B.
\end{align*}
\]
and again obtain from (2.9) the contradiction with (2.1).

\[\square\]

**Corollary 2.5.** For every \( n, t \in \mathbb{N} \) the equality of \( m\xi(n) + t \) first binary digits of \( a, b \in [0, 1] \) implies \( |\xi(a) - \xi(b)| < 2^{-n-t} \).

**Proof.** This follows from Lemma 2.4 by induction on \( t \).

\[\square\]

### 2.2 Existence of an interval of linearity of \( \xi \)

As it is mentioned in the name of the section, we will prove here Theorem 2. In fact, we will deduce this theorem from Corollary 2.5.

For any \( n \geq 0 \) denote \( \xi_n \) the piecewise linear function, passing through points
\[
\left( \frac{k}{2^n}, \xi \left( \frac{k}{2^n} \right) \right), \quad 0 \leq k \leq 2^n.
\]

**Remark 2.6.** If for an interval \( I \) and some \( n \geq 1 \) the equality \( \xi_k = \xi_{k+1} \) holds for all \( k \geq n \), then \( \xi = \xi_n \) on \( I \).

For any \( n \in \mathbb{N} \) and \( k, 0 \leq k < 2^n \) denote \( I_{nk} \) the interval
\[
I_{nk} = \left( \frac{k}{2^n}, \frac{k+1}{2^n} \right).
\]

Denote by \( t_{nk} \) the tangent of \( \xi_n \) on \( I_{nk} \), i.e.
\[
t_{nk} = 2^n \cdot \left( \xi \left( \frac{k+1}{2^n} \right) - \xi \left( \frac{k}{2^n} \right) \right).
\]

**Remark 2.7.** It follows from Corollary 2.5 that there exists \( t \) such that \( t_{nk} < t \) for all \( n, k \).

**Remark 2.8.** 1. \( T_{nk} = T_{n+1,2k} \cup T_{n+1,2k+1} \) for all \( n \in \mathbb{N} \) and \( k, 0 \leq k < 2^n \).

2. The following statements are equivalent:
   a. \( \xi_n = \xi_{n+1} \) on \( I_{n+1,2k} \);
   b. \( \xi_n = \xi_{n+1} \) on \( I_{n+1,2k+1} \);
   c.
\[
\frac{\xi \left( \frac{k}{2^n} \right) + \xi \left( \frac{k+1}{2^n} \right)}{2} = \xi \left( \frac{2k + 1}{2^{n+1}} \right).
\]
If $\xi$ is not constant, then it follows from continuity of $\xi$ that there exist $n, k$ such that $t_{nk} \neq 0$. We will construct above the sequence of intervals $I = \{I_{pk} : p \geq n\}$ with the following properties:

1. $k_n = k$;
2. $I_{p+1,k_{p+1}} \subseteq I_{pk}$ for all $p$;
3. $|t_{p+1,k_{p+1}}| \geq |t_{pk}|$ and $t_{p+1,k_{p+1}} \cdot t_{pk} > 0$ for all $p$.

This sequence of intervals will be defined inductively. If

$$\frac{\xi\left(k_p\right)}{2} + \frac{\xi\left(k_{p+1}\right)}{2} \neq \xi\left(\frac{2k_p+1}{2^{p+1}}\right),$$

then set $I_{p+1,k_{p+1}}$ that half of $I_{pk}$, where $|t_{p+1,k_{p+1}}| > |t_{kp}|$. Otherwise consider a dichotomy: if $\xi_p = \xi_r$ for all $r$ on $I_{pk}$, then Theorem 2 follows from Remark 2.6. Otherwise find the minimum $r$ such that there exists $s$ with the following properties:

1. $I_{p+r,s} \subset I_{pk}$;
2. $\frac{\xi\left(s\right)}{2^{p+r}} + \frac{\xi\left(s+1\right)}{2^{p+r}} \neq \xi\left(\frac{2s+1}{2^{p+r+1}}\right)$.

In this case denote $p_{p+r} = s$ and find uniquely $k_{p+1}, \ldots, k_{p+r-1}$ such that

$$I_{pk} \supset I_{p+1,k_{p+1}} \supset \ldots \supset I_{p+r-1,k_{p+r-1}} \supset I_{p+r,k_{p+r}} = I_{p+r,s}.$$

This construction can be formalized as follows.

Suppose first that $\xi_n$ increase on $I_n$.

For any $p \geq n$ if

$$\frac{\xi\left(k_p\right)}{2} + \frac{\xi\left(k_{p+1}\right)}{2} < \xi\left(\frac{2k_p+1}{2^{p+1}}\right),$$

then take $k_{p+1} = 2k_p$, i.e. $I_{p+1}$ is the left half of $I_{pk}$. If

$$\frac{\xi\left(k_p\right)}{2} + \frac{\xi\left(k_{p+1}\right)}{2} > \xi\left(\frac{2k_p+1}{2^{p+1}}\right),$$

then take $k_{p+1} = 2k_p + 1$, i.e. $I_{p+1}$ is the right half of $I_{pk}$. If

$$\frac{\xi\left(k_p\right)}{2} + \frac{\xi\left(k_{p+1}\right)}{2} = \xi\left(\frac{2k_p+1}{2^{p+1}}\right),$$

then consider one more dichotomy.

Either the equality

$$\frac{\xi\left(s\right)}{2^{p+r}} + \frac{\xi\left(s+1\right)}{2^{p+r}} = \xi\left(\frac{2s+1}{2^{p+r+1}}\right)$$


holds for all $I_{p+r,s}$ such that $I_{p+r,s} \subset I_{pk_p}$, or there is minimal $r$ such that

$$\frac{\xi \left( \frac{s}{2^p} \right) + \xi \left( \frac{s+1}{2^p} \right)}{2} \neq \xi \left( \frac{2s+1}{2^{p+r+1}} \right)$$

and $I_{p+r,s}$ for some $s$. In the first of these cases notice, that the conditions of Remark 2.6 are satisfied, whence $\xi$ is linear on $I_p$. In the second case there exist numbers $k_{p+1}, \ldots, k_{p+r} = s$, which are uniquely determined by $I_{pk_p}$ and $I_{p+r,s}$, such that

$$I_{pk_p} \supset I_{p+1,k_{p+1}} \supset \ldots \supset I_{p+r-1,k_{p+r-1}} \supset I_{p+r,k_{p+r}} = I_{p+r,s}.$$  

In the case of decrease of $\xi_n$ on $I_{nk}$ the construction is analogous.

**Lemma 2.9.** For any sequence $I = \{I_{pk_p}, p \geq n\}$, which satisfies (2.10), there exists $t$ such that $t_{pk_p} \neq t_{p+1,k_{p+1}}$ implies

$$|t_{p+1,k_{p+1}}| > |t_{pk_p}| + t.$$  

**Proof.** Denote $\alpha_1 = \frac{k}{2^n}$, $\alpha_2 = \frac{k+1}{2^n}$, $\beta_1 = \xi(\alpha_1)$ and $\beta_2 = \xi(\alpha_2)$. In notations above we have that $\alpha_1, \alpha_2 \in A_{n+1}$. Thus, Lemma 2.1 implies that $\beta_1, \beta_2 \in A_{n+1} \cup B_{n+1}$.

By Lemma 2.3 assume that $\beta_1 = \frac{1}{2^n} \cdot (k_1 + \kappa_1)$ and $\beta_2 = \frac{1}{2^n} \cdot (k_2 + \kappa_2)$, where $0 \leq k_1, k_2 \leq 2^n$ and $\kappa_1, \kappa_2 \in \{0, \frac{1}{3}, \frac{2}{3} \}$.

Notice that in this notations we have

$$t_{nk} = 2^n \cdot (\beta_2 - \beta_1) = k_2 - k_1 + \kappa_2 - \kappa_1.$$  

Clearly,

$$\frac{\beta_1 + \beta_2}{2} = \frac{1}{2^n+1} \cdot (k_1 + k_2 + \kappa_1 + \kappa_2).$$  

Consider the case, when $t_{pk_p} > 0$.

If

$$\frac{\beta_1 + \beta_2}{2} < \xi \left( \frac{2k + 1}{2^{p+1}} \right)$$

then it follows from the construction of $I$ that $k_{p+1} = 2k_p$, whence $\xi_{p+1}$ passes on $I_{p+1,k_{p+1}}$ through points $\left( \frac{k_p}{2^n}, \xi \left( \frac{k_p}{2^n} \right) \right)$ and $\left( \frac{2k_{p+1}}{2^{p+1}}, \xi \left( \frac{2k_{p+1}}{2^{p+1}} \right) \right)$.

Denote $\kappa^-(\kappa_1, \kappa_2)$ the minimal $\kappa \in \{1/3, 2/3; 1; 4/3; 5/3 \}$ such that $\kappa_1 + \kappa_2 < \kappa$. Then,

$$t_{p+1,k_p} = 2^{p+1} \left( \xi \left( \frac{2k_p + 1}{2^{p+1}} \right) - \xi \left( \frac{2k_p}{2^{p+1}} \right) \right) =$$

$$= 2^{p+1} \left( \frac{1}{2^{p+1}} \cdot (k_1 + k_2 + \kappa_1 + \kappa_2) - \frac{1}{2^{p+1}} \cdot (k_1 + \kappa_1) \right) >$$

$$> k_2 - k_1 + \kappa^-(\kappa_1, \kappa_2) - 2\kappa_1 =$$

$$= t_{pk_p} + \kappa^-(\kappa_1, \kappa_2) - \kappa_1 - \kappa_2.$$
If
\[
\frac{\beta_1 + \beta_2}{2} > \xi \left( \frac{2k + 1}{2p + 1} \right)
\]
then it follows from the construction of \( \mathcal{I} \) that \( k_{p+1} = 2k_p + 1 \), whence \( \xi_{p+1} \) passes on \( I_{p+1,k_{p+1}} \) through points \( \left( \frac{2k_{p+1}}{2p+1}, \xi \left( \frac{2k_{p+1}}{2p+1} \right) \right) \) and \( \left( \frac{k_{p+1}}{2p}, \xi \left( \frac{k_{p+1}}{2p} \right) \right) \).

Denote \( \kappa^+(\kappa_1, \kappa_2) \) the maximal \( \kappa \in \{ \frac{1}{3}, 0; \frac{2}{3}; 1; \frac{4}{3}; \frac{5}{3} \} \) such that \( \kappa_1 + \kappa_2 > \kappa \). Then,
\[
t_{p+1,k_p} = 2^{p+1} \left( \xi \left( \frac{k_{p+1}}{2p} \right) - \xi \left( \frac{2k_p + 1}{2p+1} \right) \right) =
\]
\[
= 2^{p+1} \left( \frac{1}{2p} \cdot (k_2 + \kappa_2) - \frac{1}{2p+1} \cdot (k_1 + k_2 + \kappa_1 + \kappa_2) \right) >
\]
\[
> k_2 - k_1 + 2\kappa_2 - \kappa^+(\kappa_1, \kappa_2) =
\]
\[
= t_{pk_p} + \kappa_1 + \kappa_2 - \kappa^+(\kappa_1, \kappa_2).
\]

We are left with the case \( t_{p,k_p} < 0 \).

If
\[
\frac{\beta_1 + \beta_2}{2} < \xi \left( \frac{2k + 1}{2p + 1} \right)
\]
then it follows from the construction of \( \mathcal{I} \) that \( k_{p+1} = 2k_p + 1 \), whence \( \xi_{p+1} \) passes on \( I_{p+1,k_{p+1}} \) through points \( \left( \frac{2k_{p+1}}{2p+1}, \xi \left( \frac{2k_{p+1}}{2p+1} \right) \right) \) and \( \left( \frac{k_{p+1}}{2p}, \xi \left( \frac{k_{p+1}}{2p} \right) \right) \). Thus,
\[
t_{p+1,k_p} = 2^{p+1} \left( \xi \left( \frac{k_{p+1}}{2p} \right) - \xi \left( \frac{2k_p + 1}{2p+1} \right) \right) =
\]
\[
= 2^{p+1} \left( \frac{1}{2p} \cdot (k_2 + \kappa_2) - \frac{1}{2p+1} \cdot (k_1 + k_2 + \kappa_1 + \kappa_2) \right) <
\]
\[
< k_2 - k_1 + 2\kappa_2 - \kappa^-(\kappa_1, \kappa_2) =
\]
\[
= t_{pk_p} + \kappa_1 + \kappa_2 - \kappa^-(\kappa_1, \kappa_2).
\]

If
\[
\frac{\beta_1 + \beta_2}{2} > \xi \left( \frac{2k + 1}{2p + 1} \right)
\]
then it follows from the construction of \( \mathcal{I} \) that \( k_{p+1} = 2k_p \), whence \( \xi_{p+1} \) passes on \( I_{p+1,k_{p+1}} \) through points \( \left( \frac{k_p}{2p}, \xi \left( \frac{k_p}{2p} \right) \right) \) and \( \left( \frac{2k_{p+1}}{2p+1}, \xi \left( \frac{2k_{p+1}}{2p+1} \right) \right) \). Thus,
\[
t_{p+1,k_p} = 2^{p+1} \left( \xi \left( \frac{2k_p + 1}{2p+1} \right) - \xi \left( \frac{2k_p}{2p+1} \right) \right) =
\]
\[
= 2^{p+1} \left( \frac{1}{2p+1} \cdot (k_1 + k_2 + \kappa_1 + \kappa_2) - \frac{1}{2p} \cdot (k_1 + \kappa_1) \right) >
\]
\[
> k_2 - k_1 + \kappa^+(\kappa_1, \kappa_2) - 2\kappa_1 =
\]
\[
= t_{pk_k} + \kappa^+(\kappa_1, \kappa_2) - \kappa_1 - \kappa_2.
\]
Now set
\[ t = \min_{\kappa_1, \kappa_2 \in \{0; \frac{1}{6}; \frac{7}{12}; 1\}} \left\{ \kappa^+ (\kappa_1, \kappa_2) - \kappa_1 - \kappa_2; \kappa_1 + \kappa_2 - \kappa^+ (\kappa_1, \kappa_2) \right\} \]
and this finishes the proof. \(\square\)

Now Theorem 2 follows from Lemma 2.9 and Remark 2.7.

3 Piecewise linear approximations of self semi conjugation

Till the end of this section let \( n \geq 1 \) be fixed and \( \psi : A_n \to [0, 1] \) be an arbitrary map, which commutes with \( f \) of the form (1.1). For the simplicity of the further reasonings denote \( \varphi_0(x) = 2x \) and \( \varphi_1(x) = 2 - 2x \), whence \( f \) can be written as

\[
f(x) = \begin{cases} 
\varphi_0(x), & 0 \leq x < 1/2; \\
\varphi_1(x), & 1/2 \leq x \leq 1.
\end{cases}
\]

Notice, that maps \( \varphi_i, i = 0, 1 \) are invertible. The usefulness of this notation can be illustrated by the following fact: if \( \tau \) is the conjugation of the Tent-map \( f \) and the map \( g \) of the form (1.2). Then for any \( n \geq 1 \) and \( i_1, \ldots, i_n \in \{0; 1\} \) the equality

\[
\tau(\varphi_{i_1}^{-1}(\ldots \varphi_{i_m}^{-1}(0) \ldots)) = \gamma_{i_1}^{-1}(\ldots \gamma_{i_n}^{-1}(0) \ldots)
\]

holds. This fact is roved in [6, Theorem 3] for the case \( g = f_v \) of the form (1.3), but only the properties of the map (1.2) are used in the proof. We will need the following technical lemma.

**Lemma 3.1.** 1. The set \( A_n \) from Remark 1.1 can be represented as

\[
A_n = \{ \varphi_{j_n}^{-1}(\ldots (\varphi_{j_1}^{-1}(0)) \ldots), \text{ for all } j_1, \ldots, j_n \in \{0; 1\} \}.
\]

2. For any \( m, t \) such that \( t < m \leq n \) the equality

\[
\varphi_{j_m}^{-1}(\ldots \varphi_{j_{m-t}}(\ldots (\varphi_{j_1}^{-1}(0)) \ldots) \ldots) = \varphi_{j_m-t}^{-1}(\ldots (\varphi_{j_1}^{-1}(0)) \ldots)
\]  \( (3.1) \)

implies that

\[
\dot{j}_1 = \ldots = \dot{j}_t = 0 \quad (3.2)
\]

and

\[
\dot{j}_{t+k} = \dot{j}_k^* \quad (3.3)
\]

for all \( k, 1 \leq k \leq m - t \).
Proof. Part 1 of lemma follows from the definition of $A_n$. To prove Part 2, apply $f^{m-t}$ to both sides of (3.1), whence

$$\varphi_{j_i}^{-1}(\varphi_{j_2}^{-1}(\varphi_{j_1}^{-1}(0))\ldots) = 0$$

and

$$0 \xrightarrow{\varphi_{j_1}} \varphi_{j_1}(0) \xrightarrow{\varphi_{j_2}} \varphi_{j_2}(\varphi_{j_1}(0))\ldots \xrightarrow{\varphi_{j_1}} \varphi_{j_1}(\varphi_{j_1}(0))\ldots = 0$$

is a periodical trajectory of 0 under $f$, which implies (3.2).

Rewrite now (3.1) and (3.2) as

$$\varphi_{j_m}^{-1}(\varphi_{j_{m+1}}^{-1}(0))\ldots = \varphi_{j_{m-t}}^{-1}(\varphi_{j_1}^{-1}(0))\ldots.$$ 

Since $\varphi_0^{-1} : [0, 1] \rightarrow [0, 1/2]$, $\varphi_1^{-1} : [0, 1] \rightarrow [1/2, 1]$ and $\varphi_0^{-1}(1/2) \neq \varphi_1^{-1}(1/2)$, then (3.3) follows.

$\square$

3.1 Maps, which commute with the Tent

Denote 

$$x_0 = \psi(0).$$

Remark 3.2. Plug 0 into (1.9) and obtain that

$$x_0 \in \{0, 2/3\}, \quad (3.4)$$

since $x_0$ appears to be a fixed point of $f$.

Lemma 3.3. For any $m$, $1 \leq m \leq n$ and any $x \in A_m$ there exist $i_1, \ldots, i_m \in \{0, 1\}$ such that

$$\psi(x) = \varphi_{i_m}^{-1}(\varphi_{i_1}^{-1}(x_0))\ldots.$$ 

Proof. Substitute 1 into (1.9) and get that $x_0 = f(\psi(1))$, whence $\psi(1) = \varphi_{i_1}^{-1}(x_0)$ for some $i_1 \in \{0, 1\}$. This proves lemma for $m = 1$.

Assume that for $m = k$ lemma is proved. For any $x \in A_{k+1}$ notice that $f(x) \in A_k$, whence it follows from induction that $\psi(f(x)) = \varphi_{i_k}^{-1}(\varphi_{i_1}^{-1}(x_0))\ldots$ for some $i_1, \ldots, i_k$. Now (1.9) implies $f(\psi(x)) = \varphi_{i_k}^{-1}(\varphi_{i_1}^{-1}(x_0))\ldots$, which means that there exists $i_{k+1}$ such that $\psi(x) = \varphi_{i_{k+1}}^{-1}(\varphi_{i_1}^{-1}(x_0))\ldots.$

By Lemmas 3.1 and 3.3, for any $m \leq n$ and $j_1, \ldots, j_m$ there exist $i_1, \ldots, i_m$ such that

$$\psi(\varphi_{j_m}^{-1}(\varphi_{j_1}^{-1}(0))\ldots) = \varphi_{i_m}^{-1}(\varphi_{i_1}^{-1}(x_0))\ldots. \quad (3.5)$$
For any $m$, $1 \leq m \leq n$ denote by $B_m$ the set of sequences the the length $m$, consisted of 0-s and 1-s. Thus, $\psi$ generates the map $\tilde{\psi} : B_m \to B_m$ such that

$$\tilde{\psi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m) \quad (3.6)$$

whenever (3.5) holds. By arbitrariness of $m$ in (3.6), the map $\tilde{\psi}$ is defined on $B = \bigcup_{i=1}^{n} B_i$.

**Lemma 3.4.** For $m \leq n$ and $t < m$ the equality $\tilde{\psi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)$ implies $\tilde{\psi}(j_1, \ldots, j_{m-t}) = (i_1, \ldots, i_{m-t})$.

**Proof.** Apply $f^t$ to both sides of (3.5), whence lemma follows from (1.9). \hfill \square

**Lemma 3.5.** Denote $i_0 \in \{0; 1\}$ such that $x_0 = \varphi_{i_0}(x_0)$.

If $j_1 = \ldots = j_k = 0$ for some $k \leq m$ in (3.6), then $i_1 = \ldots = i_k = i_0$.

**Proof.** Since $\varphi_{j_1}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots) = \varphi_{j_1}^{-1}(\ldots(\varphi_{j_k+1}^{-1}(0))\ldots)$, then by Lemma 3.3 there exist $i_{k+1}', \ldots, i_m' \in \{0; 1\}$ such that $\psi(\varphi_{i_1}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots)) = \varphi_{i_m}^{-1}(\ldots(\varphi_{i_{k+1}}^{-1}(x_0))\ldots)$, whence

$$\varphi_{i_m}^{-1}(\ldots(\varphi_{i_1}^{-1}(x_0))\ldots) = \varphi_{i_m}'^{-1}(\ldots(\varphi_{i_{k+1}}^{-1}(x_0))\ldots).$$

Now lemma follows from Lemma 3.1. \hfill \square

**Theorem 3.** There is one to one correspondence between maps $\psi : A_n \to [0, 1]$, which commute with $f$, and pairs $(\tilde{\psi}, i_0)$, where $i_0 \in \{0; 1\}$ and $\tilde{\psi}$ is a map $\tilde{\psi} : \bigcup_{i=1}^{n} B_i \to \bigcup_{i=1}^{n} B_i$ with the following properties:

1. For any $m$, $1 \leq m \leq n$, the inclusion $\tilde{\psi}(B_m) \subseteq B_m$ holds.
2. For any $m, t$ such that $t < m \leq n$ the equality $\tilde{\psi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)$ implies $\tilde{\psi}(j_1, \ldots, j_{m-t}) = (i_1, \ldots, i_{m-t})$.
3. If $\tilde{\psi}(j_1, \ldots, j_m) = (i_1, \ldots, i_m)$ and $j_1 = \ldots = j_k = 0$, then $i_1 = \ldots = i_k = i_0$.

**Proof.** If $\tilde{\psi}$ is defied by (3.6) via $\psi$, which commutes with $f$, and $x_0 = \psi(0)$, then theorem follows from Lemmas 3.6, 3.4, and 3.5.

Let $(\tilde{\psi}, i_0)$ be as in theorem. Define $\psi : A_n \to [0, 1]$ as follows. If $i_0 = 0$ then denote $x_0 = 0$, otherwise denote $x_0 = 2/3$. Define $\psi(0) = x_0$. For any $j_1, \ldots, j_n \in \{0, 1\}$ denote $(i_1, \ldots, i_n)$ such that

$$\tilde{\psi}(j_1, \ldots, j_n) = (i_1, \ldots, i_n)$$

and denote

$$x = \varphi_{j_n}^{-1}(\ldots(\varphi_{j_1}^{-1}(0))\ldots). \quad (3.7)$$

Now define

$$\psi(x) = \varphi_{i_m}^{-1}(\ldots(\varphi_{i_1}^{-1}(x))\ldots).$$
Correctness of definition of $\psi$ follows from Lemma 3.1.

Since, by Lemma 3.1, equation (3.7) defines the general form of $x \in A_n$, then it is enough to prove that

$$f(\psi(x)) = \psi(f(x))$$  \hspace{1cm} (3.8)

to conclude that $\psi$ commutes with $f$. Notice that

$$f(\psi(x)) = \varphi_{i_{n-1}}^{-1}(...) \varphi_{i_1}^{-1}(x_0)\ldots.$$  

From another hand,

$$f(x) = \varphi_{j_{n-1}}^{-1}(...) \varphi_{j_1}^{-1}(0)\ldots.$$  

Since $\tilde{\psi}(j_1, \ldots, j_{n-1}) = (i_1, \ldots, i_{n-1})$, then

$$\psi(f(x)) = \varphi_{j_{n-1}}^{-1}(...) \varphi_{j_1}^{-1}(x_0)\ldots$$

and we have (3.8).

\begin{proof}
\end{proof}

**Corollary 3.6.** For any $n \geq 1$ the number of maps $\psi : A_n \to [0, 1]$, which commute with $f$, is

$$\frac{2^{3n-1}}{2^n - 1} \cdot \prod_{k=1}^{n}(2^k - 1).$$

**Proof.** By Theorem \ref{theorem:main} we can calculate the number of maps $\tilde{\psi}$ instead.

Denote by $\mathcal{N}(n)$ the necessary quantity of maps. There are $2^n$ elements of $B_n$. For $\tilde{\psi} : B_n \to B_n$ and for any

$$w = (w_1, \ldots, w_n, w_{n+1}) \neq (0, \ldots, 0, x) \in B_{n+1}$$

we can independently define the extension of $\tilde{\psi}$ as

$$\tilde{\psi}(w) = (\tilde{\psi}(w_1, \ldots, w_n), y),$$

where $w_{n+1}$ and $y$ can be chosen independently, whence we gave 4 ways of extension for each of $2^n - 1$ non-zero words from $B_n$. By Theorem \ref{theorem:main} define

$$\tilde{\psi}(0, \ldots, 0, 0) = (i_0, \ldots, i_0, i_0)$$

and

$$\tilde{\psi}(0, \ldots, 0, 1) = (i_0, \ldots, i_0, z),$$

where $z \in \{0, 1\}$ is arbitrary. Thus, we have $4 \cdot (2^n - 1) \cdot 2$ extensions of $\tilde{\psi}$ from $B_n$ to $B_{n+1}$, whence

$$\mathcal{N}(n + 1) = 8 \cdot (2^n - 1) \cdot \mathcal{N}(n).$$
Calculate $\mathcal{N}(1)$ as follows. $\tilde{\psi}(0) = x_0$, $\tilde{\psi}(1) = z$, whence we can choose $i_0$ and $z$ arbitrary, whence $\mathcal{N}(1) = 4$. We have obtained that

$$\mathcal{N}(n) = 4 \cdot \prod_{k=2}^{n} 8 \cdot (2^{k-1} - 1) = 2^{3n-1} \cdot \prod_{k=1}^{n-1} (2^k - 1).$$

Notice that

$$\frac{2^{3n-1}}{2^n - 1} \cdot \prod_{k=1}^{n} (2^k - 1) = 4$$

for $n = 1$, whence we are done.

\[\square\]

### 3.2 Tent-continuable maps

We will describe in this section all the Tent-continuable maps $\psi : A_n \to [0, 1]$, where, as earlier, $n$ is fixed natural number.

**Lemma 3.7.** Let $\psi_1, \psi_2 : A_n \to [0, 1]$ be Tent-continuable. If $\psi_1(x) = \psi_2(x)$ for all $x \in A_n \setminus A_{n-1}$, then $\psi_1(x) = \psi_2(x)$ for all $x \in A_n$.

**Proof.** The equality (1.9) means that the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & f(x) \\
\downarrow{\psi_i} & & \downarrow{\psi_i} \\
\psi_1(x) & \xrightarrow{f} & f(\psi_1(x))
\end{array}
\]

is commutative for $i = 1, 2$ and arbitrary $x \in A_n \setminus A_{n-1}$.

Since $\psi_1(x) = \psi_2(x)$ for all $x \in A_n \setminus A_{n-1}$, then $\psi_1(x) = \psi_2(x)$ for all $x \in f(A_n \setminus A_{n-1})$. It follows from the definition of $A_n$, that $f(A_k) = A_{k-1}$ for all $k \geq 1$, whence $f(A_n \setminus A_{n-1}) = A_{n-1} \setminus A_{n-2}$. Applying $n - 1$ times the reasonings above obtain that $\psi_1(x) = \psi_2(x)$ for all $x \in A_n$.

It follows from Theorem 1 that either $\psi(x) = 3/2$ for all $x \in A_n$, or $\psi(0) = 0$.

**Lemma 3.8.** If $\psi(0) = 0$, then $\psi(A_n) \subseteq A_n$.

**Proof.** Lemma follows from the definition of $A_n$ and the equality

$$f^n \circ \psi = \psi \circ f^n,$$

which is a corollary of (1.9).

\[\square\]
For any $\alpha \in A_n \setminus A_{n-1}$ and $\beta \in A_n$ denote by $\Xi_{\alpha, \beta}$ the class of all continuous solutions $\xi : [0, 1] \to [0, 1]$ of (1.1), such that $\xi(\alpha) = \beta$. We will need the following technical lemma about the properties of the continuous solutions of (1.7). Denote $\xi(k)$ the map of the form (1.8), where $k \in \mathbb{N}$.

**Lemma 3.9.** For every $\alpha \in A_n \setminus A_{n-1}$ and $\beta \in A_n$ there exists $k_0(\alpha, \beta) \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, either

$$k - k_0(\alpha, \beta) \equiv 0 \mod 2^n,$$

or

$$k + k_0(\alpha, \beta) \equiv 0 \mod 2^n$$

whenever $\xi(k) \in \Xi_{\alpha, \beta}$. Moreover, if $k$ satisfies either (3.9) or (3.10), then $\xi \in \Xi_{\alpha, \beta}$.

**Proof.** By Remark 1.1 there exist $s, p$ such that

$$\alpha = \frac{2s + 1}{2^{n-1}} \quad \text{and} \quad \beta = \frac{p}{2^{n-1}}. \quad (3.11)$$

Since $\xi_k$ is linear on each of the intervals $\left[\frac{t}{k}, \frac{t+1}{k}\right)$, $0 \leq t < k$, denote $t$ such that $\alpha \in \left[\frac{t}{k}, \frac{t+1}{k}\right)$ and consider two cases, whether $t = 2t_0$, or $t = 2t_0 + 1$ for some $t_0 \in \mathbb{N}$.

Suppose that $t = 2t_0$. Then $\xi_k$ increase on $\left[\frac{t}{k}, \frac{t+1}{k}\right)$ with tangent $k$, whence

$$k \cdot \left(\alpha - \frac{2t_0}{k}\right) = \beta. \quad (3.12)$$

Substitute (3.11) and rewrite the last equality as

$$\frac{k(2s + 1)}{2^{n-1}} - 2t_0 = \frac{p}{2^{n-1}}, \quad (3.13)$$

whence

$$k(2s + 1) - p \equiv 0 \mod 2^n. \quad (3.14)$$

Denote $k_0(\alpha, \beta) = k$. Since $2s + 1$ has no common factors with $2^n$, then $2s + 1$ is a generator of the additive group of residuals of $2^n$, whence

$$k_0 = k^* \mod 2^n$$

implies $\xi_{k^*} \in \Xi_{\alpha, \beta}$, whenever $\alpha \in \left[\frac{t}{k^*}, \frac{t+1}{k^*}\right)$ for some even $t$.

Suppose now that $t = 2t_0 + 1$. In this case $\xi(k)$ decrease on $\left[\frac{t}{k}, \frac{t+1}{k}\right)$ with tangent $-k$, whence

$$k \cdot \left(\alpha - \frac{2t_0 + 1}{k}\right) = 1 - \beta. \quad (3.15)$$

Again by (3.11) rewrite this equality

$$\frac{k(2s + 1)}{2^{n-1}} - 2t_0 - 1 = 1 - \frac{p}{2^{n-1}}.$$
whence
\[ k(2s + 1) + p \equiv 0 \mod 2^n. \] (3.15)

This equation (with the same reasonings as (3.14) does) has the unique solution in the additive semigroup of residuals of $2^n$. Notice, that the solution $-k_0(\alpha, \beta)$ is the solution of (3.15), whence

\[ k = -k_0(\alpha, \beta) \mod 2^n \]

and we are done with the first part of lemma.

Resume, that $k_0(\alpha, \beta), 0 \leq k_0(\alpha, \beta) < 2^n$ was constructed as the unique solution of (3.14).

We will now prove the second part of lemma, i.e. if either (3.9), or (3.10) is satisfied, then $\xi(k) \in \Xi_{\alpha, \beta}$. Suppose that (3.9) holds. Then there exists $t_0$ such that (3.13) holds, and (3.13) can be rewritten as (3.12). Since $k \in \mathbb{N}$ and $\alpha, \beta \in [0, 1]$, then $0 \leq \alpha - \frac{2t_0}{k} \leq \frac{1}{k}$ and (3.12) implies that $\xi(k)(\alpha) = \beta$. The case if (3.10) is satisfied, is analogous.

The following theorem directly follows from lemmas 3.7 and 3.9.

**Theorem 4.** 1. For every $x \in A_n \setminus A_{n-1}$ and for every $y \in A_n$ there exists a map $\psi : A_n \to A_n$, which is Tent-continuable and $\psi(x) = y$.

2. Let $\psi_1, \psi_2 : A_n \to A_n$ be Tent-continuable and $\psi_1(x) = \psi_2(x)$ for some $x \in A_n \setminus A_{n-1}$. Then $\psi_1(x) = \psi_2(x)$ for all $x \in A_n$.

**Corollary 3.10.** For every $n \geq 1$ there are $2^{n-1}$ Tent-continuable maps $\psi : A_n \to [0, 1]$.

**Proof.** By item 1 of Theorem 4 for every $x \in A_n \setminus A_{n-1}$ and for every $y \in A_n$ there exist a Tent-continuable $\psi$ such that $\psi(x) = y$. For any $x \in A_n \setminus A_{n-1}$ it follows from Part 2 of Theorem 4 that any $y \in A_n$ defines a Tent-continuable $\psi$ in the unique way.

Thus, take any $x \in A_n \setminus A_{n-1}$ and each of its $2^{n-1}$ images in $A_n$ defines the unique $\psi : A_n \to [0, 1]$, which are Tent-continuable.

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