Degrees of Infinite Words, Polynomials, and Atoms
(Extended Version)*†

Jörg Endrullis1, Juhani Karhumäki2, Jan Willem Klop1,3, and Aleksi Saarela2

1 Department of Computer Science
VU University Amsterdam, Amsterdam, the Netherlands
Email: j.endrullis@vu.nl, j.w.klop@vu.nl
2 Department of Mathematics and Statistics & FUNDIM
University of Turku, Turku, Finland
Email: karhumak@utu.fi, amsaar@utu.fi
3 Centrum Wiskunde & Informatica (CWI), Amsterdam, the Netherlands

Abstract
We study finite-state transducers and their power for transforming infinite words. Infinite sequences of symbols are of paramount importance in a wide range of fields, from formal languages to pure mathematics and physics. While finite automata for recognising and transforming languages are well-understood, very little is known about the power of automata to transform infinite words.

The word transformation realised by finite-state transducers gives rise to a complexity comparison of words and thereby induces equivalence classes, called (transducer) degrees, and a partial order on these degrees. The ensuing hierarchy of degrees is analogous to the recursion-theoretic degrees of unsolvability, also known as Turing degrees, where the transformational devices are Turing machines. However, as a complexity measure, Turing machines are too strong: they trivialise the classification problem by identifying all computable words. Finite-state transducers give rise to a much more fine-grained, discriminating hierarchy. In contrast to Turing degrees, hardly anything is known about transducer degrees, in spite of their naturalness.

We use methods from linear algebra and analysis to show that there are infinitely many atoms in the transducer degrees, that is, minimal non-trivial degrees. We also show that there exists an uncomputable degree that has only uncomputable degrees and the trivial bottom degree below itself.

Keywords and phrases finite state transducers, infinite words, infinite sequences, streams, complexity, degrees

1 Introduction

In recent times, computer science, logic and mathematics have extended the focus of interest from finite data types to include infinite data types, of which the paradigm notion is that of infinite sequences of symbols, or words. Infinite words are of paramount importance in a wide range of fields, from formal languages to pure mathematics and physics: they appear in functional programming, formal language theory, in the mathematics of dynamical systems, fractals and number theory, in business (financial data words) and in physics (signal processing).

* This research has been supported by the Academy of Finland under the grant 257857.
† This is an extended version of the papers [11, 12].
An accepted and deep mathematical insight is that together with a class of structures, one has to deal with the ways to transform these structures into each other, such as morphisms in a category. Our objects of interest are words and how they can be transformed into each other via finite-state transducers.

In computer science, infinite words are often referred to as streams. In the real world, we encounter streams in the form of sensor data from continual measurements, streams of financial transactions, and streams of messages in social media. Analysing and processing the large amount of data generated by these applications is one of the major challenges of computer science today, and an active field of research, known as Big Data. When it comes to data sets that are massive in size, even linear algorithms may be too complex for processing the data. For instance, think of an algorithm with linear space complexity (e.g. linear random-access memory) applied to petabytes of input data. This has led to the research field of sublinear algorithms, a rapidly developing area of computer science, that is rooted in the study of Big Data. This field is concerned with the development of algorithms having sublinear-space and/or sublinear-time complexity.

We are interested in the most strict form of sublinear-space complexity: constant-space complexity. Algorithms with constant space-complexity (O(1) space-complexity) are indispensable for programs that are intended to run indefinitely, to continually transform an endless input word into an endless output word. Any algorithm not having constant space-complexity will eventually run out of memory on a real world device (computer) when transforming an infinite word. This motivates the study of constant-space algorithms for the transformation of words. Note that a constant-space algorithm is nothing else than a finite-state automaton.

While finite automata for recognising and transforming languages are well-studied and well-understood, surprisingly, very little is known about the power of finite automata for transforming words. Even for concrete examples of words $w_1$ and $w_2$, there exist no techniques to determine whether $w_1$ can be transformed into $w_2$ by some finite-state transducer. See, e.g., Questions Q4 and Q5. We are interested in understanding the power of finite-state transducers for transforming words.

Such a study can be profitably cast in the form of setting up a hierarchy of degrees,
induced by a transformational device (sometimes called a ‘reduction’). This is a well-known reasoning framework in logic and computer science [16], with many instances, e.g., Wadge degrees [23], Turing degrees [22, 17], r.e. degrees [17], and so on. In our case the hierarchy of degrees is obtained as follows.

The transformation realised by finite-state transducers induces a partial order of degrees of infinite words: for words $v, w \in \Delta^\mathbb{N}$, we write $v \geq w$ if $v$ can be transformed into $w$ by some finite-state transducer. If $v \geq w$, then $v$ can be thought of as at least as complex as $w$. This complexity comparison induces equivalence classes of words, called degrees, and a partial order on these degrees, that we call transducer degrees.

The ensuing hierarchy of degrees is analogous to the recursion-theoretic degrees of unsolvability, also known as Turing degrees, where the transformational devices are Turing machines. The Turing degrees have been widely studied in the 60’s and 70’s. However, as a complexity measure, Turing machines are too strong: they trivialise the classification problem by identifying all computable infinite words. Finite-state transducers (FSTs) give rise to a much more fine-grained hierarchy.

In our view, transducers are the most natural devices for transforming words. Unlike Turing machines, they are not too strong and still very expressive. On the one hand, transducers are ‘weak enough’ to exhibit a rich structure within the computable words. On the other hand, they capture several usual transformations, such as alphabet renaming, insertion and removal of elements, or morphisms as usually studied in theories of infinite sequences [2].

Like the Turing degrees, the transducer degrees have a bottom degree that is less than or equal to all other degrees (pictorial on the right). The bottom degree of the Turing degrees contains all computable words. In contrast, transducer degrees are much more fine-grained. The bottom degree $0$ of the transducer degrees consists only of the ultimately periodic words, that is, words of the form $uvvv\cdots$ for finite words $u, v$.

We present a comparison of some basic properties, as to their validity in the Turing degrees and the transducer degrees. We list a few key results [22, 17] on Turing degrees due to Spector, Kleene, Post, Sacks, Lacombe and Simpson:

(i) There exist $2^{\mathbb{N}^2}$ atom (minimal) degrees.
(ii) Every degree has a minimal cover.
(iii) Every finite set of degrees has a supremum.
(iv) No infinite ascending sequence has a supremum.
(v) There are pairs of degrees without infimum.
(vi) For every degree $\neq 0$ there exists an incomparable degree.
(vii) Every countable partial order can be embedded.
(viii) The recursively enumerable degrees are dense.
(ix) The first-order theory of Turing degrees in the language $\langle \geq, = \rangle$ is recursively isomorphic to that of true second-order arithmetic.

Here the symbols on the right indicate whether the properties also hold for transducer degrees: ✓ if the property also holds for transducer degrees, ✗ if it fails, and ? for questions that are open in transducer degrees.

In previous papers [10, 9, 6, 13], we have discussed several structural properties of the hierarchy of transducer degrees. In this paper, we focus on atom degrees. An atom degree is a minimal non-trivial degree, that is, a degree that is directly above the bottom degree without interpolant degree.
Thus the atom degrees reduce only to 0 or themselves. The following questions are still open for transducer degrees: Q1. Are there \(2^{\aleph_0}\) atoms in the transducer degrees? Q2. Do uncomputable atoms exist in the transducer degrees? Q3. Is the degree of the Thue–Morse word \(T = 011010011010110 \cdots\) an atom?

Figure 2 The partial order of transducer degrees with focus on the properties studied in this paper. Our contribution is indicated using the colour red. Here \(p_k\) is a particular polynomial of order \(k\), see Section 7. The degree of \(\langle p_k \rangle\) is an atom and all other polynomials of order \(k\) can be transduced to \(\langle p_k \rangle\). For \(k \geq 3\), the degree \(\langle n^k \rangle\) is not an atom as shown in Section 6. The definitions of the words Thue–Morse \(T\) and the Mephisto Waltz \(W\) are given in Section 4. The degree of \(C\) is the top degree of the computable words. Finally, the nodes Q1, \ldots, Q7 indicate open problems discussed in Sections 1 and 4.

We show that there are at least \(\aleph_0\) atoms residing in the interesting subclass of words that we call sporadic words, of which the simplest one is 1 1 0 1 0 0 0 1 0 0 0 0 \cdots. (Jacobs [15] called this word ‘rarefied ones’.) Here ‘sporadic’ refers to the fact that the ones are becoming more and more sporadic. In general, they are of the form \((f) = 10^{f(0)} 10^{f(1)} 10^{f(2)} \cdots\), for some \(f : \mathbb{N} \to \mathbb{N}\). This paper studies in particular the case where \(f\) is a polynomial. We consider the ‘atomicity’ of these words depending on the polynomials determining how the ones become ever more sporadic.

In this ‘polynomially sporadic’ subhierarchy of the transducer degrees, we have the following state of affairs:

- \(\langle n \rangle\) is an atom,
- \(\langle n^2 \rangle\) is an atom,
- \(\langle n^3 \rangle\) is not an atom,
- \(\langle an^3 + bn^2 + cn + d \rangle\) is an atom for some \(a, b, c, d > 0\).

This hints at an interesting, rich structure in this hierarchy of sporadic degrees.
Our contribution.

The words \(\langle n \rangle\) and \(\langle n^2 \rangle\) are atoms \([10]\) \([9]\). Surprisingly, we find that this does not hold for \(\langle n^k \rangle\). In particular, we show that the degree of \(\langle n^k \rangle\) is never an atom for \(k \geq 3\) (see Theorem \([23]\)). On the other hand, we prove that for every \(k > 0\), there exists a unique atom among the degrees of words \(\langle p(n) \rangle\) for polynomials \(p(n)\) of order \(k\) (see Theorem \([32]\)). (To avoid confusion between two meanings of degrees, namely degrees of words and degrees of polynomials, we speak of the order of a polynomial.) We moreover show that this atom is the infimum of all degrees of polynomials \(p(n)\) of order \(k\). Figure \([2]\) summarises the state of affairs as in this paper. Finally, we show that there exists an uncomputable word \(U\) that transduces only to uncomputable words or to ultimately periodic words. In particular, this word has no transductions that are computable and not ultimately periodic.

Further related work.

Löwe \([10]\) discussed complexity hierarchies derived from notions of reduction. The paper \([13]\) gives an overview over the subject of transducer degrees and compares them with the well-known Turing degrees \([22]\) \([17]\). Restricting the transducers to output precisely one letter in each step, we arrive at Mealy machines. These give rise to an analogous hierarchy of Mealy degrees that has been studied in \([3]\) \([19]\). The structural properties of this hierarchy are very different from the transducer degrees \([13]\). The paper \([5]\) studies a hierarchy of two-sided infinite sequences arising from the transformation realised by permutation transducers.

2 Preliminaries

Let \(\Sigma\) be an alphabet. The empty word is denote by \(\varepsilon\). Let \(\Sigma^+\) be the set of finite words over \(\Sigma\), and \(\Sigma^* = \Sigma^+ \setminus \{\varepsilon\}\). The set of infinite words over \(\Sigma\) is \(\Sigma^N = \{\sigma : N \rightarrow \Sigma\}\) and we let \(\Sigma^\infty = \Sigma^* \cup \Sigma^N\). Let \(u, w \in \Sigma^\infty\). Then \(u\) is called a prefix of \(w\), denoted \(u \subseteq w\), if \(u = w\) or there exists \(u' \in \Sigma^\infty\) such that \(uu' = w\).

Of particular importance are morphic words \([2]\). For example:

(i) The Thue–Morse word \(T\) arises by starting from the word \(0\), as the limit of repeatedly applying the morphism \(0 \mapsto 01, 1 \mapsto 10\). We abbreviate this by: \(\langle 0 \mid 0 \mapsto 01, 1 \mapsto 10 \rangle\).
   The first iterations are \(0 \mapsto 01 \mapsto 0110 \mapsto \cdots\).
(ii) The period-doubling word \(P = \langle 0 \mid 0 \mapsto 01, 1 \mapsto 00 \rangle\).
(iii) The Mephisto Waltz word \(W = \langle 0 \mid 0 \mapsto 001, 1 \mapsto 110 \rangle\).

Formally, morphic words are defined as follows \([21]\) \([2]\). Let \(s \in \Sigma^+\) be a starting word, \(h : \Sigma \rightarrow \Sigma^*\) a morphism, and \(c : \Sigma \rightarrow \Sigma\) a coding. If the limit \(h^i(s) = \lim_{i \rightarrow \infty} h^i(s)\) exists, then \(c(h^i(s))\) is called a morphic word. Here the limit is taken with respect to the following metric \(d\) on \(\Sigma^\infty\): \(d(u, u) = 0\) and \(d(u, v) = 2^{-n}\) for all \(u, v \in \Sigma^\infty\) with \(u \neq v\), where \(n\) is the length of the longest common prefix of \(u\) and \(v\).

3 Finite-state Transducers

A sequential finite-state transducer (FST) \([2]\) \([20]\), a.k.a. deterministic generalised sequential machine (DGSM), is a finite automaton with input letters and finite output words along the edges. A transducer reads the input word letter by letter, in each step producing an output word and changing its state. Then the output word is the concatenation of all the output words encountered along the edges.
Definition 1. A sequential finite-state transducer $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ consists of a finite input alphabet $\Sigma$, a finite output alphabet $\Gamma$, a finite set of states $Q$, an initial state $q_0 \in Q$, a transition function $\delta : Q \times \Sigma \to Q$, and an output function $\lambda : Q \times \Sigma \to \Gamma^\ast$. Whenever the alphabets $\Sigma$ and $\Gamma$ are clear from the context, we write $A = \langle Q, q_0, \delta, \lambda \rangle$.

An example of an FST is depicted in Figure 3, where we write ‘$a \mid w$’ along the transitions to indicate that the input letter is $a$ and the output word is $w$.

![Figure 3 An FST realising the difference of consecutive bits modulo 2. For example, $T = 01101001 \ldots$ is transformed in $P = 1011101 \ldots$ where the overbar signifies inversion between 0 and 1.](image)

The output given by a transition is allowed to be a word over the output alphabet, and not just a single letter or the empty word $\varepsilon$, although that may also be the case. Thereby finite-state transducers generalize the class of Mealy machines that output precisely one letter in each step.

The transducer in Figure 3 computes the first difference of the input word. For example, it reduces the Thue–Morse sequence $T$ to the inverted period doubling sequence $P$:

\[
\begin{array}{c}
0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ \ldots \\
\rightarrow \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ \ldots
\end{array}
\]

Formally, the transduction of words is defined as follows. We only consider sequential transducers and will simply speak of finite-state transducers henceforth.

Definition 2. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite-state transducer. We homomorphically extend the transition function $\delta$ to $Q \times \Sigma^\ast \to Q$ as follows: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^\ast$ let $\delta(q, \varepsilon) = q$ and $\delta(q, au) = \delta(\delta(q, a), u)$. We extend the output function $\lambda$ to $Q \times \Sigma^\ast \to \Gamma^\ast$ as follows: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^\ast$, let $\lambda(q, \varepsilon) = \varepsilon$ and $\lambda(q, au) = \lambda(q, a) \cdot \lambda(\delta(q, a), u)$.

We note that finite-state transducers can be viewed as productive term rewrite systems [8] and the transduction of infinite words as infinitary rewriting [7].

4 Transducer Degrees

We now explain how FSTs give rise to a hierarchy of degrees of infinite words, called transducer degrees. First, we formally introduce the transducibility relation $\geq$ on words as realised by FSTs.

Definition 3. Let $w \in \Sigma^N$, $u \in \Gamma^N$ for finite alphabets $\Sigma$, $\Gamma$. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a FST. We write $w \geq_A u$ if $u = \lambda(q_0, w)$. We write $w \geq u$, and say that $u$ is a transduct of $w$, if there exists a FST $A$ such that $w \geq_A u$.

Note that the transducibility relation $\geq$ is a pre-order. It thus induces a partial order of ‘degrees’, the equivalence classes with respect to $\geq \cap \leq$. We denote equivalence using $\equiv$. It is not difficult to see that every word over a finite alphabet is equivalent to a word over the alphabet $2 = \{0, 1\}$. Thus every degree contains a representative from $2^N$. For the study of transducer degrees it suffices therefore to consider words over the latter alphabet.
Definition 4. Define the equivalence relation \( \equiv = (\geq \cap \leq) \). The \textit{(transducer) degree} \( w^\equiv \) of an infinite word \( w \) is the equivalence class of \( w \) with respect to \( \equiv \), that is, \( w^\equiv = \{ u \in 2^\mathbb{N} | w \equiv u \} \). We write \( 2^\mathbb{N}/\equiv \) to denote the set of degrees \( \{ w^\equiv | w \in 2^\mathbb{N} \} \).

The \textit{transducer degrees} form the partial order \((2^\mathbb{N}/\equiv, \geq)\) induced by the pre-order \(\geq\) on \(2^\mathbb{N}\), that is, for words \( w, u \in 2^\mathbb{N} \) we have \( w^\equiv \geq u^\equiv \iff w \geq u \).

The \textit{bottom degree} \( 0 \) is the least degree of the hierarchy, that is, the unique degree \( a \in 2^\mathbb{N}/\equiv \) such that \( a \leq b \) for every \( b \in 2^\mathbb{N}/\equiv \); it consists of the ultimately periodic words, that is, words of the form \( u vv \cdots \) for finite words \( u, v \) where \( v \neq \varepsilon \).

Definition 5. An \textit{atom} is a minimal non-bottom degree, that is, a degree \( a \in 2^\mathbb{N}/\equiv \) such that \( 0 < a \) and there exists no \( b \in 2^\mathbb{N}/\equiv \) with \( 0 < b < a \).

Although FSTs are very simple and elegant devices, we hardly understand their power for transforming words \( [13] \). No methods are available to answer simple questions such as:

Consider the period-doubling sequence \( P \) and drop every third element:

\[
P = 0100 0101 0100 0100 0100 \cdots \]
\[
w = 01_0 0_0 1_0 1_0 0_0 0_0 \cdots
\]

Obviously we have \( P \geq w \). Do we have \( w \geq P \)?

Are the degrees of Thue–Morse \( T \) and Mephisto Waltz \( W \) incomparable?

5 Spiralling Words

We now consider \textit{spiralling words} over the alphabet \( 2 = \{0, 1\} \) for which the distance of consecutive 1’s in the word grows to infinity. We additionally require that the sequence of distances between consecutive 1’s is ultimately periodic modulo every natural number. The class of spiralling words permits a characterisation of the ir transducts in terms of weighted products.

For a function \( f : \mathbb{N} \to \mathbb{N} \), we define \( \langle f \rangle \in 2^\mathbb{N} \) by

\[
\langle f \rangle = \prod_{i=0}^{\infty} 10^{f(1)} 10^{f(1)} 10^{f(2)} \cdots.
\]

We write \( \langle f(n) \rangle \) as shorthand for \( \langle n \mapsto f(n) \rangle \).

Example 6. As an example of a transduction between sporadic words, to get a feeling of what finite-state transducers can do on such words, consider \( \langle n^3 \rangle \geq \langle (2n)^3 + (2n + 1)^3 \rangle \).

Here the transducer \( A \) removes the 1 between the appropriate consecutive blocks of 0’s, as in: \( 1010^8 10^{27} 10^{64} 10^{125} \cdots \geq 10^{10^8 (8+27)} 10^{10^{64+125}} \cdots \). It is easy to determine the two-state transducer \( A \) that removes the 1’s at the right places.

Definition 7. A function \( f : \mathbb{N} \to \mathbb{N} \) is called \textit{spiralling} if
\[(i) \lim_{n \to \infty} f(n) = \infty, \text{ and}
(ii) \text{for every } m \geq 1, \text{ the function } n \mapsto f(n) \mod m \text{ is ultimately periodic.}
\]

A word \( \langle f \rangle \) is called \textit{spiralling} whenever \( f \) is spiralling.

---

2 We note that finite state transducers transform infinite words to finite or infinite words. The result of the transformation is finite if the transducer outputs the empty word \( \varepsilon \) for all except a finite number of letters of the input word. We are interested in infinite words only, since the set of finite words would merely entail two spurious extra sub-bottom degrees in the hierarchy of transducer degrees.
For example, \( \langle p(n) \rangle \) is spiralling for every polynomial \( p(n) \) with natural numbers as coefficients\(^3\). Spiralling functions are called 'cyclically ultimately periodic' in the literature \([4]\).

For a tuple \( \vec{\alpha} = (\alpha_0, \ldots, \alpha_m) \), we define

- the length \(|\vec{\alpha}| = m + 1\), and
- its rotation by \( \vec{\alpha}' = (\alpha_1, \ldots, \alpha_m, \alpha_0) \).

Let \( A \) be a set and \( f : \mathbb{N} \rightarrow A \) a function. We write \( S^k(f) \) for the \( k \)-th shift of \( f \) defined by \( S^k(f)(n) = f(n + k) \).

We use 'weights' to represent linear functions.

\[ \textbf{Definition 8.} \text{ A weight } \vec{\alpha} \text{ is a tuple } (a_0, \ldots, a_{k-1}, b) \in \mathbb{Q}^{k+1} \text{ of rational numbers such that } k \in \mathbb{N} \text{ and } a_0, \ldots, a_{k-1} \geq 0. \text{ The weight } \vec{\alpha} \text{ is called}
\]

- \textit{non-constant} if \( a_i \neq 0 \) for some \( i < k \), else \textit{constant},
- \textit{strongly non-constant} if \( a_i, a_j \neq 0 \) for some \( i < j < k \).

Now let us also consider a tuple of tuples. For a tuple \( \vec{\alpha} = (\alpha_0^\top, \ldots, \alpha_m^\top) \) of weights we define \(|\vec{\alpha}| = \sum_{i=0}^{m-1} (|\alpha_i| - 1) \).

\[ \textbf{Definition 9.} \text{ Let } f : \mathbb{N} \rightarrow \mathbb{Q} \text{ be a function. For a weight } \vec{\alpha} = (a_0, \ldots, a_{k-1}, b) \text{ we define}
\]

\[ \vec{\alpha} \cdot f \in \mathbb{Q} \text{ by } \vec{\alpha} \cdot f = a_0 f(0) + a_1 f(1) + \cdots + a_{k-1} f(k-1) + b. \text{ For a tuple of weights}
\]

\[ \vec{\alpha} = (\vec{\alpha}_0, \vec{\alpha}_1, \ldots, \vec{\alpha}_m) \text{, we define the weighted product } \vec{\alpha} \otimes f : \mathbb{N} \rightarrow \mathbb{Q} \text{ by induction on } n:
\]

\[ (\vec{\alpha} \otimes f)(0) = \vec{\alpha}_0 \cdot f, \quad (\vec{\alpha} \otimes f)(n + 1) = (\vec{\alpha} \otimes f)(n) \]

We say that \( \vec{\alpha} \otimes f \) is a \textit{natural weighted product} if \( (\vec{\alpha} \otimes f)(n) \in \mathbb{N} \) for all \( n \in \mathbb{N} \).

We introduce a few operations on weights. We define scalar multiplication \( \circ \) that affects only the last entry of weights (the constant term).

\[ \textbf{Definition 11.} \text{ Let } c \in \mathbb{Q}_{\geq 0}, \vec{\alpha} = (a_0, \ldots, a_{t-1}, b) \text{ a weight, } \vec{\beta} = (\beta_0, \ldots, \beta_{m-1}) \text{ a tuple of weights. We define}
\]

\[ c \vec{\alpha} = (ca_0, \ldots, ca_{t-1}, cb) \quad \vec{\alpha} \circ c = (a_0, \ldots, a_{t-1}, bc) \]

\[ c \vec{\beta} = (c\beta_0, \ldots, c\beta_{m-1}) \quad \vec{\beta} \circ c = (\beta_0 \circ c, \ldots, \beta_{m-1} \circ c) \]

\( ^3 \) The identity function and constants functions are spiralling. Moreover, the class of spiralling functions is closed under addition and multiplication. From this it follows that polynomials with natural numbers as coefficients are spiralling.
The next lemma follows directly from the definitions.

**Lemma 12.** Let $c \in \mathbb{Q}_{\geq 0}$, $\vec{a}$ a tuple of weights, and $f : \mathbb{N} \to \mathbb{Q}$ a function. Then $c(\vec{a} \otimes f) = (c\vec{a}) \otimes f = (\vec{a} \circ c) \otimes (cf)$. □

It is straightforward to define a composition of tuples of weights such that $\vec{\beta} \otimes (\vec{a} \otimes f) = (\vec{\beta} \otimes \vec{a}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$. Note that $\vec{a} \otimes f$ is already defined. For the precise definition of $\vec{\beta} \otimes \vec{a}$, we refer to Appendix A. We will employ the following two properties of composition.

**Lemma 13.** Let $\vec{a}, \vec{\beta}$ be tuples of weights. Then we have that $\vec{\beta} \otimes (\vec{a} \otimes f) = (\vec{\beta} \otimes \vec{a}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$. □

**Lemma 14.** Let $\vec{a}$ be tuple of weights, and $\vec{\beta}$ a tuple of strongly non-constant weights. Then $\vec{\alpha} \otimes \vec{\beta}$ is of the form $\langle \gamma_0, \ldots, \gamma_{k-1} \rangle$ such that for every $i \in \mathbb{N} \cup \{0\}$, the weight $\gamma_i$ is either constant or strongly non-constant. □

We need a few results on weighted products from [6]. The following lemma states that every natural weighted product (see Definition 9) can be realized by a FST.

**Lemma 15 ([6]).** Let $f : \mathbb{N} \to \mathbb{N}$, and $\vec{\alpha}$ a tuple of weights. If $\vec{\alpha} \otimes f$ is a natural weighted product (i.e., $\forall \vec{n} \in \mathbb{N}, (\vec{\alpha} \otimes f)(\vec{n}) \in \mathbb{N}$), then $\langle f \rangle \geq (\vec{\alpha} \otimes f)$. □

For the proof of Theorem 18 below, we use the following auxiliary lemma. The lemma gives a detailed structural analysis, elaborated and explained in [6], of the transductions of a sparring word $f$.

**Lemma 16 ([6]).** Let $f : \mathbb{N} \to \mathbb{N}$ be a sparring function, and let $\sigma \in 2^\mathbb{N}$ be such that $\langle f \rangle \geq \sigma$ and $\sigma \not\in \emptyset$. Then there exist $n_0, m \in \mathbb{N}$, a word $w \in 2^*$, a tuple of weights $\vec{\alpha}$, and tuples of finite words $\vec{p}$ and $\vec{c}$ with $|\vec{\alpha}| = |\vec{p}| = |\vec{c}| = m > 0$ such that $\sigma = w \cdot \prod_{i=0}^{n_0-1} \prod_{j=0}^{m-1} p_i c_i^{p(i,j)}$ where $\varphi(i, j) = (\vec{\alpha} \otimes S^w(f))(\vec{m} + j)$, and

1. $c_i^w \neq p_{i+1} c_i^{w+1}$ for every $j$ with $0 \leq j < m - 1$, and $c_{m-1}^w \neq p_0 c_0^w$, and
2. $c_j \neq \varepsilon$, and $\alpha_j$ is non-constant, for all $j \in \mathbb{N} \cup \{0\}$. □

**Example 17.** We continue Example 10. We have $\vec{\alpha} = (\alpha_0, \alpha_1)$. Accordingly, we have prefixes $p_0, p_1 \in 2^*$ and cycles $c_0, c_1 \in 2^*$. Then the transduct $\sigma$ in Lemma 16 defined by the double product, can be derived as follows:

| $f$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | ... |
|-----|---|---|---|---|----|----|----|----|----|----|-----|
| $\vec{\alpha} \otimes f$ | $\alpha_0$ | $\alpha_1$ | $\alpha_0$ | $\alpha_1$ | $\alpha_0$ | $\alpha_1$ | $\alpha_0$ | $\alpha_1$ | $\alpha_0$ | $\alpha_1$ | ... |
| $\sigma$ | $w \cdot p_0 c_0^{18} \cdot p_1 c_1^{17} \cdot p_0 c_0^{248} \cdot p_1 c_1^{82}$ | ... |

The infinite word $\sigma$ is the infinite concatenation of $w$ followed by alternating $p_0 c_0^{18}$ and $p_1 c_1^{17}$, where the exponents $c_0$ and $c_1$ are the result of applying weights $\alpha_0$ and $\alpha_1$, respectively.

We characterize the transductions of sparring words up to equivalence ($\equiv$).

**Theorem 18 ([6]).** Let $f : \mathbb{N} \to \mathbb{N}$ be sparring, and $\sigma \in 2^\mathbb{N}$. Then $\langle f \rangle \geq \sigma$ if and only if $\sigma \equiv \langle \vec{\alpha} \otimes S^w(f) \rangle$ for some $n_0 \in \mathbb{N}$, and a tuple of weights $\vec{\alpha}$. □

Roughly speaking, polynomials of order $k$ are closed under transduction.

**Proposition 19 ([6]).** Let $p(n)$ be a polynomial of order $k$ with non-negative integer coefficients, and let $\sigma \not\in \emptyset$ with $\langle p(n) \rangle \geq \sigma$. Then $\sigma \geq \langle q(n) \rangle$ for some polynomial $q(n)$ of order $k$ with non-negative integer coefficients.
The Degree of \langle n^k \rangle is Not an Atom for \( k \geq 3 \)

We show that the degree of \( \langle n^k \rangle \) is not an atom for \( k \geq 3 \). For this purpose, we prove a strengthening of Theorem 18, a lemma on weighted products of strongly non-constant weights, and we employ the power mean inequality 14.

**Definition 20.** For \( p \in \mathbb{R} \), the weighted power mean \( M_p(x) \) of \( x = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n_{>0} \) with respect to \( w = \langle w_1, w_2, \ldots, w_n \rangle \in \mathbb{R}^n_{>0} \) with \( \sum_{i=1}^n w_i = 1 \) is

\[
M_{w,0}(x) = \prod_{i=1}^n x_i^{w_i}, \quad M_{w,p}(x) = (\sum_{i=1}^n w_i x_i^p)^{1/p}.
\]

**Proposition 21 (Power mean inequality).** For all \( p, q \in \mathbb{R}, x, \tilde{x} \in \mathbb{R}^n_{>0} : \\
p < q \implies M_{\tilde{w},p}(x) < M_{\tilde{w},q}(x) \\
(p = q \vee x_1 = x_2 = \cdots = x_n) \iff M_{\tilde{w},p}(x) = M_{\tilde{w},q}(x).

Theorem 18 characterises transducts of spiralling sequences only up to equivalence. This makes it difficult to employ the theorem for proving non-transducibility. We improve the characterisation for the case of spiralling transducts as follows.

**Theorem 22.** Let \( f, g : \mathbb{N} \to \mathbb{N} \) be spiralling functions. Then \( \langle g \rangle \geq \langle f \rangle \) if and only if some shift of \( f \) is a weighted product of a shift of \( g \), that is:

\[
S^{n_0}(f) = \tilde{\alpha} \otimes S^{m_0}(g)
\]

for some \( n_0, m_0 \in \mathbb{N} \) and a tuple of weights \( \tilde{\alpha} \).

**Proof.** For the direction \( \langle f \rangle \geq \langle g \rangle \), assume that \( \langle S^{n_0}(f) = \tilde{\alpha} \otimes S^{m_0}(g) \rangle \). Then we have \( \langle g \rangle \equiv \langle S^{m_0}(g) \rangle \geq \langle \tilde{\alpha} \otimes S^{m_0}(g) \rangle = \langle S^{n_0}(f) \rangle \equiv \langle f \rangle \) by invariance under shifts and by Lemma 16.

For the direction \( \langle f \rangle \implies \langle g \rangle \), assume that \( \langle g \rangle \geq \langle f \rangle \). Then by Lemma 15 there exist \( m_0, m \in \mathbb{N}, w \in 2^*, \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\gamma} \) with \( |\tilde{\alpha}| = |\tilde{\beta}| = |\tilde{\gamma}| = m > 0 \) such that:

\[
(f) = w \cdot \prod_{i=0}^\infty \prod_{j=0}^{i-1} p_j \epsilon_{j+1}^{(i,j)}
\]

where \( \varphi(i, j) = (\tilde{\alpha} \otimes S^{m_0}(g))(mj + j) \) such that the conditions [i] and [ii] of Lemma 16 are fulfilled.

Note that, as \( \lim_{n \to \infty} f(n) = \infty \), the distance of ones in the sequence \( \langle g \rangle \) tends to infinity. For every \( j \in \mathbb{N}_{<m} \), the word \( p_j \) occurs infinitely often in \( \langle f \rangle \) by 14, and hence \( p_j \) can contain at most one occurrence of the symbol 1.

By condition [ii], we have for every \( j \in \mathbb{N}_{<m} \) that \( c_j = \epsilon_1 \), and the weight \( \tilde{\alpha}_j \) is not constant. As \( \lim_{n \to \infty} g(n) = \infty \), it follows that \( c_j^2 \) appears infinitely often in \( \langle f \rangle \) by 14. Hence \( c_j \) consists only of 0’s, that is, \( c_j \in \{0\} \) for every \( j \in \mathbb{N}_{<m} \).

By condition [i], we never have \( c_j^{\omega} = p_{j+1} c_{j+1}^{\omega} \) for \( j \in \mathbb{N}_{<m} \) (where addition is modulo \( m \)). As \( c_j^{\omega} = 0^\omega \) and \( p_j 0^\omega = p_{j+1} c_{j+1}^{\omega} \), we obtain that \( p_{j+1} \) must contain a 1. Hence, for every \( k \in \mathbb{N}_{<m} \), the word \( p_k \) contains precisely one 1.

Finally, we apply the following transformations to ensure \( p_j = 1 \) and \( c_j = 0 \) for every \( j \in \mathbb{N}_{<m} : \\
(i) For every \( j \in \mathbb{N}_{<m} \) such that \( c_j = 0^h \) for some \( h > 1 \), we set \( c_j = 0 \) and replace the weight \( \tilde{\alpha}_j \) in \( \tilde{\alpha} \) by \( h \tilde{\alpha}_j \). \\
(ii) For every \( j \in \mathbb{N}_{<m} \) such that \( p_j = 0^h 1^\ell \) for some \( h \geq 1 \) or \( \ell \geq 1 \), we set \( p_j = 1 \) and replace the weight \( \tilde{\alpha}_j \) in \( \tilde{\alpha} \) by \( (\tilde{\alpha}_j + \ell) \) and the weight \( \alpha_j^{-1} \) by \( (\alpha_j^{-1} + h) \). Here, for a weight \( \tilde{\gamma} = (x_0, \ldots, x_{-1}, y) \) and \( z \in \mathbb{Q} \), we write \( \tilde{\gamma} + z \) for the weight \( (x_0, \ldots, x_{-1}, y + z) \).

If \( j = 0 \), we moreover append \( 0^h \) to the word \( w \).
Theorem 23.
Define \( \vec{S}(g) = \vec{S}(g) \). Then we have \( S^{\alpha}(f) = \vec{S}(g) \) for some \( n_0 \in \mathbb{N} \).

\[ \text{Theorem 22 strengthens Theorem 18 in the sense that the characterisation uses equality (\( = \)) and shifts instead of equivalence (\( \equiv \)). But Theorem 22 only characterises spiralling transductions whereas Theorem 18 characterises all transductions. However, next we will employ the gain precision to show that certain spiralling transductions of \( \langle n^k \rangle \) cannot be transduced back to \( \langle n^k \rangle \), and conclude that \( \langle n^k \rangle \) is not an atom for \( k \geq 3 \).} \]

\[ \text{Theorem 23. For } k \geq 3, \text{ the degree of } \langle n^k \rangle \text{ is not an atom.} \]

\[ \text{Proof. Define } f : \mathbb{N} \to \mathbb{N} \text{ by } f(n) = n^k. \text{ We have } \langle f \rangle \geq \langle g \rangle \text{ where } g : \mathbb{N} \to \mathbb{N} \text{ is defined by } g(n) = (2n)^k + (2n + 1)^k; \text{ cf. Example 6.} \text{ Assume that we had } \langle g \rangle \geq \langle f \rangle. \text{ Then, by Theorem 22 we have } S^{\alpha}(f) = \vec{S}(g) \text{ for some } n_0, m_0 \in \mathbb{N} \text{ and a tuple of weights } \vec{a} \text{. Note that } g = \langle \{1, 1, 0\} \rangle \otimes f \text{ and} \]

\[ S^{\alpha}(f) = \vec{a} \otimes S^{\alpha}(\langle \{1, 1, 0\} \rangle \otimes f) = \vec{a} \otimes (\langle \{1, 1, 0\} \rangle \otimes S^{\alpha}(f)) = \vec{b} \otimes S^{\alpha}(f) \]

where \( \vec{b} = \vec{a} \otimes \langle \{1, 1, 0\} \rangle \). By Lemma 14 every weight in \( \vec{b} \) is either constant or strongly non-constant. As \( S^{\alpha}(f) \) is strictly increasing (and hence contains no constant subsequence), each weight in \( \vec{b} \) must be strongly non-constant.

Let \( \vec{b} = \langle b_0, \ldots, b_{r-1} \rangle \). For every \( n \in \mathbb{N} \) we have

\[ S^{\alpha}(f)(f(n)) = (\vec{b} \otimes S^{\alpha}(f))(f(n)) = \vec{a} \cdot S^{\alpha}(f). \]

Then we have

\[ S^{\alpha}(f)(f(n)) = (n_0 + fn)^k = \sum_{i=0}^{k} \binom{k}{i} n_0^i n^{k-i} = n^k + kn_0 n^{k-1} + \ldots + kn_0 n^k. \]

Let \( \vec{b}_0 = (a_0, a_1, \ldots, a_{n-1}, b) \). We define \( c_i = a_i \|\vec{b}\|^k \) and \( d_i = (2m_0 + i) \|\vec{b}\| \). We obtain

\[ \vec{b}_0 \cdot S^{\alpha}(f)(f(n)) = b + \sum_{i=0}^{n-1} a_i (2m_0 + i) \|\vec{b}\| n + i) = b + \sum_{i=0}^{n-1} a_i (\|\vec{b}\|(n + \frac{2m_0 + i}{\|\vec{b}\|}) = b + \sum_{i=0}^{n-1} a_i \|\vec{b}\|(n + d_i)^k = b + \sum_{i=0}^{n-1} c_i (n^k + kd_i n^{k-1} + \ldots + kd_i n^{k-1} + d_i) \].

Recall equation (2). Comparing the coefficients of \( n^k \), \( n^{k-1} \) and \( n \) in (3) and (4) we obtain

\[ \frac{1}{k} = \sum_{i=0}^{n-1} c_i, \] \[ \frac{n_0}{k} = \sum_{i=0}^{n-1} c_i d_i, \] \[ \frac{n_0}{k} = \sum_{i=0}^{n-1} c_i d_i, \] \[ \frac{n_0}{k} = \sum_{i=0}^{n-1} c_i d_i, \]

contradicting the weighted power means inequality (Proposition 21). Clearly all \( d_i \) are distinct, and, as a consequence of \( \vec{b}_0 \) being strongly non-constant, there are at least two \( i \in \mathbb{N} \) for which \( c_i \neq 0 \). Thus our assumption \( \langle g \rangle \geq \langle f \rangle \) is wrong. Hence the degree of \( \langle n^k \rangle \) is not an atom. \]

\[ \square \]
7 Atoms of Every Polynomial Order

The previous section stated that \( \langle n^k \rangle \) is not an atom for \( k \geq 3 \). Now we show that for every \( k \in \mathbb{N} \) there exists a polynomial \( p(n) \) of order \( k \) such that the degree of the word \( \langle p(n) \rangle \) is an atom. Hence there are at least \( \aleph_0 \) atoms in the transducer degrees.

As we have seen in the proof of Theorem 23, whenever \( k \geq 3 \), we have that \( \langle n^k \rangle \geq \langle g(n) \rangle \), but not \( \langle g(n) \rangle \geq \langle n^k \rangle \) for \( g(n) = (2n)^k + (2n + 1)^k \). Thus there exist polynomials \( p(n) \) of order \( k \) for which \( \langle p(n) \rangle \) cannot be transduced to \( \langle n^k \rangle \). However, the key observation underlying the construction in this section is the following: Although we may not be able to reach \( \langle n^k \rangle \) from \( \langle p(n) \rangle \), we can get arbitrarily close (Lemma 25 below). This enables us to employ the concept of \textit{continuity}.

In order to have continuous functions over the space of polynomials to allow limit constructions, we now permit rational coefficients. For \( k \in \mathbb{N} \), let \( \Omega_k \) be the set of polynomials of order \( k \) with non-negative rational coefficients. We also use polynomials in \( \Omega_k \) to denote spiralling sequences. However, we need to give meaning to \( \langle q(n) \rangle \) for the case that the block sizes \( q(n) \) are not natural numbers. For this purpose, we make use of the fact that the degree of a word \( \langle f(n) \rangle \) is invariant under multiplication of the block sizes by a constant, as is easy to see. More precisely, for \( f : \mathbb{N} \rightarrow \mathbb{Q} \), we have \( \langle f(n) \rangle \equiv \langle d \cdot f(n) \rangle \) for every \( d \in \mathbb{N} \) with \( d \geq 1 \). So to give meaning to \( \langle q(n) \rangle \), we multiply the polynomial by the least natural number \( d > 0 \) such that \( d \cdot \langle q(n) \rangle \) is a natural number for every \( n \in \mathbb{N} \).

\textbf{Definition 24.} We call a function \( f : \mathbb{N} \rightarrow \mathbb{Q} \) \textit{naturalisable} if there exists a natural number \( d \geq 1 \) such that for all \( n \in \mathbb{N} \) we have \( \langle d \cdot f(n) \rangle \in \mathbb{N} \).

For naturalisable \( f : \mathbb{N} \rightarrow \mathbb{Q} \) we define \( \langle f \rangle = \langle d \cdot f \rangle \) where \( d \in \mathbb{N} \) is the least number such that \( d \geq 1 \) where for all \( n \in \mathbb{N} \) we have \( \langle d \cdot f(n) \rangle \in \mathbb{N} \). (Note that, for \( f : \mathbb{N} \rightarrow \mathbb{N} \), \( \langle f(n) \rangle \) has been defined in Section 5.)

Observe that every \( q(n) \in \Omega_k \) is naturalisable (multiply by the least common denominator of the coefficients). Also, naturalisable functions are preserved under weighted products.

\textbf{Lemma 25.} Let \( f : \mathbb{N} \rightarrow \mathbb{Q} \) be naturalisable, and \( \vec{\alpha} \) a tuple of weights. Then \( \vec{\alpha} \otimes f \) is naturalisable and \( \langle f \rangle \geq \langle \vec{\alpha} \otimes f \rangle \).

\textbf{Proof.} Let \( \vec{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \) for some \( m \geq 1 \). Let \( c \in \mathbb{N} \) with \( c \geq 1 \) be minimal such that all entries of \( c \vec{\alpha} \) are natural numbers. Let \( d \in \mathbb{N} \) with \( d \geq 1 \) be the least natural number such that \( \forall n \in \mathbb{N} \langle d \cdot f(n) \rangle \in \mathbb{N} \).

Then we obtain \( \langle (dc \vec{\alpha}) \otimes f \rangle(n) \in \mathbb{N} \) for every \( n \in \mathbb{N} \). By the definition of weighted products it follows immediately that \( \langle dc \vec{\alpha} \otimes f \rangle = dc \langle \vec{\alpha} \otimes f \rangle \), and hence \( \vec{\alpha} \otimes f \) is naturalisable. Let \( e \in \mathbb{N} \) with \( e \geq 1 \) be the least natural number such that \( \forall n \in \mathbb{N} \langle e \cdot (\vec{\alpha} \otimes f)(n) \rangle \in \mathbb{N} \).

We have the following transduction

\[
\langle f \rangle = \langle df \rangle \quad \text{by Definition 24}
\]
\[
\geq \langle ((c \vec{\alpha}) \otimes d) \otimes (df) \rangle \quad \text{by Lemma 15}
\]
\[
= \langle (dc \vec{\alpha}) \otimes f \rangle = \langle dc(\vec{\alpha} \otimes f) \rangle \quad \text{by Lemma 12}
\]
\[
\geq \langle (c \frac{e}{dc}, 0) \rangle \otimes \langle dc(\vec{\alpha} \otimes f) \rangle \quad \text{by Lemma 15}
\]
\[
= \langle e(\vec{\alpha} \otimes f) \rangle = \langle \vec{\alpha} \otimes f \rangle \quad \text{by Definition 24}
\]

This concludes the proof. \qed
The following lemma states that every word \( \langle q(n) \rangle \), for a polynomial \( q(n) \in \Omega_k \) of order \( k \), can be transduced arbitrarily close to (but perhaps not equal to) \( \langle n^k \rangle \).

**Lemma 26.** Let \( k \geq 1 \) and let \( q(n) \in \Omega_k \) be a polynomial of order \( k \). For every \( \varepsilon > 0 \) we have \( \langle q(n) \rangle \geq \langle n^k + b_{k-1}n^{k-1} + \ldots + b_1n \rangle \) for some rational coefficients \( 0 \leq b_{k-1}, \ldots, b_1 < \varepsilon \).

**Proof.** Let \( q(n) = a_kn^k + a_{k-1}n^{k-1} + \ldots + a_1n + a_0 \), and let \( \varepsilon > 0 \) be arbitrary. Then for every \( d \in \mathbb{N} \), we have

\[
\langle q(n) \rangle \geq \langle q(dn) \rangle \geq \langle n^k + \frac{a_{k-1}}{a_k}dn^{k-1} + \ldots + \frac{a_1}{a_k}dn^{1} + \frac{a_0}{a_k}dn \rangle
\]

The first transduction selects a subsequence of the blocks. The second transduction is a division of the size of each block (application of Lemma 25 with the weight \( \langle (1/a_k d^k, 0) \rangle \)). The last transduction amounts to removing a constant number of zeros from each block (application of Lemma 25 with the weight \( \langle (1, -a_0/(a_k d^k)) \rangle \)). The last polynomial in the transduction is of the desired form if \( d \in \mathbb{N} \) is chosen large enough. \( \square \)

For polynomials \( p(n) \in \Omega_k \), we want to express weighted products \( \langle \vec{a} \rangle \otimes p \) in terms of matrix products, as follows.

**Definition 27.** For weights \( \vec{a} = (a_0, \ldots, a_{k-1}, b) \) we define a column vector

\[
U(\vec{a}) = (a_0, \ldots, a_{k-1})^T.
\]

**Definition 28.** If \( p(n) = \sum_{i=0}^{k} c_in^i \) is a polynomial of order \( k \), we define a column vector \( V(p(n)) = (c_1, \ldots, c_k)^T \) and a square matrix

\[
M(p(n)) = (V(p(kn + 0)), \ldots, V(p(kn + k - 1))).
\]

We also write \( V(p) \) short for \( V(p(n)) \) and \( M(p) \) for \( M(p(n)) \).

We have omitted the constant term \( c_0 \) from the definition of \( V(p) \). Because for every \( f : \mathbb{N} \rightarrow \mathbb{N} \) and \( c \in \mathbb{N} \) we have \( \langle f(n) \rangle \equiv \langle f(n) + c \rangle \). These words are of the same degree because a FST can add (or remove) a constant number of symbols \( 0 \) to (from) every block of \( 0 \)'s. Similarly, \( b \) was omitted from the definition of \( U(\vec{a}) \).

**Example 29.** Consider the polynomial \( n^3 \):

\[
V(n^3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad M(n^3) = \begin{pmatrix} 0 & 9 & 36 \\ 0 & 27 & 54 \\ 27 & 27 & 27 \end{pmatrix}
\]

where the column vectors of the matrix \( M(n^3) \) are given by \( V((3n)^3), V((3n + 1)^3) \) and \( V((3n + 2)^3) \).

**Lemma 30.** Let \( k \geq 1 \). Let \( \vec{a} = (a_0, \ldots, a_{k-1}, b) \) be a weight and \( p(n) \in \Omega_k \). Then \( M(p) U(\vec{a}) = V(\langle \vec{a} \rangle \otimes p) \).

**Proof.** We calculate

\[
M(p) U(\vec{a}) = \sum_{i=0}^{k-1} a_i V(p(kn + i)) = V\left( \sum_{i=0}^{k-1} a_i p(kn + i) \right)
\]
Lemma 31. By Lemma 30, Theorem 32.

We show that for any countable set $U$ we define
\[ W \in U \sum_{i=0}^{k-1} a_i (kn + i)^k. \]
with $a_0 \ldots, a_{k-1}$ be positive rational numbers, $\vec{a} = (a_0, \ldots, a_{k-1}, 0)$, and
\[ p(n) = (\langle \vec{a} \rangle \otimes n^k)(n) = \sum_{i=0}^{k-1} a_i (kn + i)^k. \]

Then $\langle q(n) \rangle \geq (p(n))$ for all $q(n) \in \Omega_k$. Moreover, the degree $(p(n)) = e$ is an atom. Note that the degree $(p(n)) = e$ is the infimum of all degrees of words $\langle q(n) \rangle$ with $q(n) \in \Omega_k$.

Proof. By Lemma 30, $M(n^k) \in U(\vec{a}) = V(p)$. By Lemma 31, $M(n^k)$ is invertible and we can write $U(\vec{a}) = M(n^k)^{-1}V(p)$. By Lemma 26 for every $\varepsilon > 0$ there exists $q_{\varepsilon} \in \Omega_k$ such that $\langle q(n) \rangle \geq \langle q_{\varepsilon}(n) \rangle$ and
\[ q_{\varepsilon}(n) = n^k + b_{k-1}n^{k-1} + \ldots + b_1 n \]
with $0 \leq b_i \leq \varepsilon$ for every $i \in \{1, \ldots, k-1\}$. We will show that if $\varepsilon$ is small enough, then $\langle q_{\varepsilon}(n) \rangle \geq (p(n))$.

We have $\lim_{\varepsilon \to 0} M(q_{\varepsilon}) = M(n^k)$. As det$(M(n^3)) \neq 0$ and the determinant function is continuous, also det$(M(q_{\varepsilon})) \neq 0$ for all sufficiently small $\varepsilon$. Then $M(q_{\varepsilon})$ is invertible, and we define $U_\varepsilon = M(q_{\varepsilon})^{-1}V(p)$. We would like to have $U_\varepsilon = U(\gamma)$ for some weight $\gamma$. This is not always possible, because some elements of $U_\varepsilon$ might be negative. However, by the continuity of matrix inverse and product,
\[ \lim_{\varepsilon \to 0} U_\varepsilon = \lim_{\varepsilon \to 0} (M(q_{\varepsilon})^{-1}V(p)) = (\lim_{\varepsilon \to 0} M(q_{\varepsilon}))^{-1}V(p) = M(n^k)^{-1}V(p) = U(\vec{a}). \]
Since every element of $U(\vec{a})$ is positive, we can fix a small enough $\varepsilon$ so that every element of $U_\varepsilon$ is positive. Then we have $U_\varepsilon = U(\gamma)$ for some weight $\gamma$.

We have $M(q_{\varepsilon}) U(\gamma) = V(\gamma \otimes q_{\varepsilon})$ by Lemma 30 and $M(q_{\varepsilon}) U(\gamma) = V(p)$ by the definition of $U_\varepsilon$. As a consequence $(\gamma \otimes q_{\varepsilon})(n) = p(n) + c$ for some constant $c$. By Lemma 26, we obtain $\langle q_{\varepsilon}(n) \rangle \geq (p(n))$.

It remains to show that the degree $(p(n)) = e$ is an atom. Assume that $(p(n)) \geq w$ and $w \not\in \emptyset$. By Proposition 19 we have $w \geq \langle q(n) \rangle$ for some $q(n) \in \Omega_k$. As shown above, $\langle q(n) \rangle \geq (p(n))$, thus $w \geq (p(n))$. Hence $(p(n)) = e$ is an atom.

A Hereditary Uncomputable Degree

We show that for any countable set $\mathfrak{D}$ of transducer degrees that does not contain the bottom degree, there exists a degree $3 \not\in \emptyset$ such that $3 \downarrow$ contains no degree from $\mathfrak{D}$. Here $3 \downarrow$ is the cone of 3, that is, the set of degrees below 3:
\[ 3 \downarrow = \{ a | 3 \geq a \}. \]
To this end, we will prove the following theorem.
Theorem 33. Let \( S \subseteq 2^N \) be a countable set of words that contains no ultimately periodic words. Then there exists a word \( w \in 2^N \) that is not ultimately periodic and none of the transducts \( u \) of \( w \), \( w \geq u \), resides in \( S \).

Before proving this theorem, we mention a few corollaries.

Corollary 34. There exists an uncomputable word \( U \in 2^N \) whose finite-state transducts are all uncomputable, ultimately periodic or finite.

Proof. Follows from Theorem 33 with \( S \) the set of computable words that are not ultimately periodic.

Theorem 33 and Corollary 34 have the following immediate implications for the hierarchy of transducer degrees.

Corollary 35. Let \( D \) be a countable set of transducer degrees not containing the bottom degree. Then there exists a degree \( z \neq 0 \) that has no degrees in \( D \) below itself, that is, \( z \uparrow \cap D = \emptyset \).

The following result is somewhat reminiscent of the situation in Turing degrees where there exists a set of incomparable degrees of size continuum.

Corollary 36. Let \( C \) be a countable set of degrees with pairwise almost disjoint cones, that is, for all \( a, b \in C \) with \( a \neq b \), we have \( a \uparrow \cap b \downarrow = \{0\} \). Then \( C \) can be extended to an uncountable set of degrees with pairwise almost disjoint cones.

Proof. Let \( C' \) be a maximal extension of \( C \). If \( C' \) was countable, then by Corollary 35 it could be extended by a disjoint cone: take \( D = \{b : a \in C', a \geq b \} \setminus \{0\} \). This contradicts maximality of \( C' \).

We do not know if ‘uncountable’ can be replaced by continuum in the corollary.

We call a degree uncomputable if it contains an uncomputable word. Note that degrees cannot contain both computable and uncomputable words since the set of computable words is closed under finite-state transduction.

Corollary 37. There exists an uncomputable transducer degree \( U \equiv \) that has only uncomputable degrees and the bottom degree below itself.

Proof. Let \( A = (\Sigma, \Gamma, Q, q_0, \delta, \lambda) \) be a finite-state transducer, and \( w \in \Sigma^* \) a word. Then \( A \) is predetermined by \( w \) if there exists \( u \in \Gamma^N \) such that for every \( w' \in \Sigma^* \) it holds that \( \lambda(q_0, ww') \trianglelefteq u \).

When a transducer is predetermined by \( w \), then it transduces words starting with \( w \) to ultimately periodic words (or finite words).

Lemma 39. Let \( A = (\Sigma, \Gamma, Q, q_0, \delta, \lambda) \) be predetermined by \( w \in \Sigma^* \), and let \( w' \in \Sigma^N \). If the word \( \lambda(q_0, ww') \) is infinite, then it is ultimately periodic.

Proof. Let \( u \in \Gamma^N \) such that
\[
\forall w' \in \Sigma^* \lambda(q_0, ww') \trianglelefteq u \tag{5}
\]
Let \( w' \in \Sigma^N \) such that \( \lambda(q_0, ww') \) is infinite. Note that from (5) it follows that \( \lambda(q_0, ww') = u \). Let \( q = \delta(q_0, w) \). Then \( \lambda(q_0, ww') = \lambda(q_0, w)\lambda(q, w') \). By the infinitary pigeonhole principle, there exists some state \( q' \in Q \) that is visited infinitely often when the automaton reads \( w' \) starting in state \( q \). Consequently there are non-empty words \( w_0, w_1, \ldots \in \Sigma^+ \)
such that \( w' = w_0w_1w_2 \cdots \), and for every \( n \in \mathbb{N} \) we have that \( \delta(q_0, w_0w_1 \cdots w_n) = q' \). As \( \lambda(q_0, w) \) and hence \( \lambda(q, w') \) is infinite, there exists some \( i > 0 \) such that \( \lambda(q', w_i) \neq \varepsilon \).

Define \( v = \lambda(q, w_0w_1 \cdots w_{i-1}w_iw_1w_2 \cdots) \), then it follows that \( v \) is infinite. Moreover \( v \) is ultimately periodic as it is the transduct of an ultimately periodic sequence. We obtain \( \lambda(q_0, w)v = u \) from (33), and thus \( u = \lambda(q_0, ww') \) is ultimately periodic. 

We are now ready to prove Theorem 33.

**Proof of Theorem 33.** Let \( S \subseteq 2^\mathbb{N} \) be a countable set of words that contains no ultimately periodic words. Let \( A \) be the set of all finite-state transducers over the alphabet \( 2 \). Note that \( A \times S \) is countable and let \( (A_0, s_0), (A_1, s_1), \ldots \) be an enumeration of this set. For \( i = 0, 1, 2, \ldots \) we define words \( w_i \in 2^+ \) as follows. Let \( v_i = w_0 \cdots w_{i-1} \). We stipulate that \( v_0 = \varepsilon \). Let \( A_i = (2, 2, Q, q_0, \delta, \lambda) \). If \( A_i \) is predetermined by \( v_i \), then the choice of \( w_i \) is arbitrary; say \( w_i = 0 \). Otherwise, there exist words \( x, y \in \Sigma^+ \) such that neither \( x' \subseteq y' \) nor \( y' \subseteq x' \), where \( x' = \lambda(q_0, v_i x) \) and \( y' = \lambda(q_0, v_i y) \). Then there exists an index \( j < \min(|x'|, |y'|) \) such that \( x'(j) \neq y'(j) \). Define \( w_i = y \) if \( x'(j) = s_i(j) \), and \( w_i = x \), otherwise. This choice guarantees that

\[
\lambda(q_0, w_0 \cdots w_i) \not\subseteq s_i
\]

(6)

Let \( w = w_0w_1w_2 \cdots \). Assume that there exists \( u \in S \) with \( w \geq u \). Then there exist a finite-state transducer \( A \in A \) such that \( w \geq_A u \). However, there is some \( i \in \mathbb{N} \) such that \( (A_i, s_1) = (A, u) \). If \( A_i \) is predetermined by \( w_0w_1 \cdots w_{i-1} \), then \( u \) is ultimately periodic by Lemma 30 and hence \( u \not\in S \). Otherwise property (6) contradicts \( w = w_0w_1w_2 \cdots \geq_A s_i = u \). Thus \( w \) has the required properties. 

**9 Future Work**

We have shown that there are at least \( \aleph_0 \) many atoms in the hierarchy of transducer degrees. They reside in a class of words over \( \{0, 1\} \) in which the distance of ones grows according to a polynomial. In particular, we have proven that, for every \( k \geq 1 \), there exists a polynomial \( p_k(n) \) of order \( k \) such that the degree of the word \( \langle p_k(n) \rangle \) is an atom (see Theorem 52). This atom is the unique atom among, and the infimum of, the degrees of polynomials of order \( k \).

The degrees of \( \langle n \rangle \) and \( \langle n^2 \rangle \) are the unique atoms among the polynomials of order 1 and 2, respectively. Surprisingly, we find that the degree of \( \langle n^k \rangle \) is not an atom whenever \( k \geq 3 \). The degree of \( \langle n^k \rangle \) lies strictly above the degree of \( \langle p_k(n) \rangle \).

Our results hint at an interesting structure of the transducer degrees of words \( \langle p(n) \rangle \) for polynomials \( p(n) \) of order \( k \in \mathbb{N} \). Here, we have only scratched the surface of this structure. Many questions remain open, for example:

- What is the structure of ‘polynomial spiralling’ degrees (depending on \( k \in \mathbb{N} \))? Is the number of degrees finite for every \( k \in \mathbb{N} \)?
- Are there interpolant degrees between the degrees of \( \langle n^k \rangle \) and \( \langle p_k(n) \rangle \)?
- Are there continuum many atoms?
- Is the degree of the Thue–Morse sequence an atom?
References

1. J.-P. Allouche and J. Shallit. The ubiquitous Prouhet–Thue–Morse sequence. In *Sequences and Their Applications: Proceedings of SETA ’98*, pages 1–16. Springer, 1999.

2. J.-P. Allouche and J. Shallit. *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge University Press, 2003.

3. A. Belov. Some algebraic properties of machine poset of infinite words. *ITA*, 42(3):451–466, 2008.

4. J. Berstel, L. Boasson, O. Carton, B. Petazzoni, and J.-É. Pin. Operations preserving regular languages. *Theoretical Computer Science*, 354(3):405–420, 2006.

5. W. Bosma and H. Zantema. Ordering sequences by permutation transducers. *Indagationes Mathematicae*, 28(1):38–54, 2017.

6. J. Endrullis, C. Grabmayer, D. Hendriks, and H. Zantema. The degree of squares is an atom. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of LNCS, pages 109–121. Springer, 2015.

7. J. Endrullis, H. H. Hansen, D. Hendriks, A. Polonsky, and A. Silva. A coinductive framework for infinitary rewriting and equational reasoning. In *Proc. Conf. on Rewriting Techniques and Applications (RTA 2015)*. Schloss Dagstuhl, 2015.

8. J. Endrullis and D. Hendriks. Lazy productivity via termination. *Theoretical Computer Science*, 412(28):3203–3225, 2011.

9. J. Endrullis and D. Hendriks. On Periodically Iterated Morphisms. In *Proc. Joint Meeting of Conference on Computer Science Logic and Symposium on Logic in Computer Science (CSL-LICS 2014)*, pages 39:1–39:10. ACM, 2014.

10. J. Endrullis, D. Hendriks, and J. W. Klop. Degrees of streams. *Journal of Integers*, 11B(A6):1–40, 2011. Proceedings of the Leiden Numeration Conference 2010.

11. J. Endrullis, J. Karhumäki, J.W. Klop, and A. Saarela. Degrees of infinite words, polynomials and atoms. In *Proc. Conf. Developments in Language Theory (DLT 2016)*, LNCS, pages 164–176. Springer, 2016.

12. J. Endrullis, J. Karhumäki, J.W. Klop, and A. Saarela. Degrees of infinite words, polynomials and atoms. *International Journal of Foundations of Computer Science*, 2018.

13. J. Endrullis, J.W. Klop, A. Saarela, and M. Whiteland. Degrees of transducibility. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of LNCS, pages 1–13. Springer, 2015.

14. G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, 1988. Reprint of the 1952 edition.

15. K. Jacobs. *Invitation to mathematics*. Princeton University Press, 1992.

16. B. Löwe. Complexity hierarchies derived from reduction functions. In *Classical and New Paradigms of Computation and their Complexity Hierarchies*, volume 23 of *Trends in Logic*, pages 1–14. Springer, 2004.

17. P. Odifreddi. *Classical Recursion Theory*. Studies in logic and the foundations of mathematics. North-Holland, Amsterdam, 1999.

18. N. Rampersad, J. Shallit, and M. Wang. Avoiding large squares in infinite binary words. *Theoretical Computer Science*, 339(1):19–34, 2005.

19. G. Rayna. Degrees of finite-state transformability. *Information and Control*, 24(2):144–154, 1974.

20. J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2003.

21. A. Salomaa. *Jewels of Formal Language Theory*. Computer Science Press, Rockville, Maryland, 1981.

22. J. R. Shoenfield. *Degrees of Unsolvability*. North-Holland, Elsevier, 1971.

23. R. Van Wesep. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77*, pages 151–170. Springer, 1978.
Appendix

A Weighted Products

We define concatenation and unfolding of tuples of weights.

Definition 40. Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}), \vec{\beta} = (\beta_0, \ldots, \beta_{m-1})$ be a tuple of weights. We define concatenation:

$$\vec{\alpha} ; \vec{\beta} = (\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{m-1}).$$

We define unfolding by induction on $n \in \mathbb{N}$ with $n > 0$:

$$\vec{\alpha}^1 = \vec{\alpha}, \quad \vec{\alpha}^{n+1} = \vec{\alpha} ; \vec{\alpha}^n$$

Unfolding a tuple of weights does not change its semantics.

Lemma 41. Let $f : \mathbb{N} \to \mathbb{Q}$, $\vec{\alpha}$ a tuple of weights and $n \geq 1$. Then $\vec{\alpha} \circ f = \vec{\alpha}^n \circ f$.

Proof. Follows immediately from the cyclic fashion in which the weights in the weighted product are applied.

We will now define the product $\vec{\alpha} \circ \vec{\beta}$ of tuples of weights such that we have $\vec{\alpha} \circ (\vec{\beta} \circ f) = (\vec{\alpha} \circ \vec{\beta}) \circ f$ for every $f : \mathbb{N} \to \mathbb{Q}$. We need one auxiliary definition.

Definition 42. For a weight $\vec{\gamma} = (x_0, \ldots, x_{\ell-1}, y)$ and a tuple of weights $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{\ell-1})$ with $\alpha_i = (a_{i,0}, \ldots, a_{i,m_i}, b_i)$, we define the weight $\vec{\gamma} \cdot \vec{\alpha}$ by

$$\vec{\gamma} \cdot \vec{\alpha} = (x_0 a_{0,0}, \ldots, x_0 a_{0,m_0}, x_1 a_{1,0}, \ldots, x_1 a_{1,m_1}, \ldots, x_{\ell} a_{\ell,0}, \ldots, x_{\ell} a_{\ell,m_\ell}, x_0 b_0 + x_1 b_1 + \cdots + x_{\ell} b_{\ell} + y).$$

We now define the product of tuples of weights.

Definition 43. For tuples of weights $\vec{\alpha}$ and $\vec{\beta}$ with the property $||\vec{\alpha}|| = ||\vec{\beta}||$ we define $\vec{\alpha} \circ \vec{\beta}$ by induction on the tuple length:

$$\vec{\alpha} \circ \vec{\beta} = \langle \alpha_0 \cdot \langle \beta_0, \ldots, \beta_{|\alpha| - 2} \rangle; \langle (\alpha_1, \ldots, \alpha_{|\alpha| - 1}) \circ \langle \beta_1, \ldots, \beta_{|\beta| - 1} \rangle \rangle$$

where $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{|\alpha|-1})$ and $\vec{\beta} = (\beta_0, \ldots, \beta_{|\beta|-1})$. Here we stipulate that $\langle \rangle \circ \langle \rangle = \langle \rangle$.

For $\vec{\alpha}$ and $\vec{\beta}$ such that $||\vec{\alpha}|| \neq ||\vec{\beta}||$ we define $\vec{\alpha} \circ \vec{\beta}$ as follows:

$$\vec{\alpha} \circ \vec{\beta} = (\vec{\alpha} \circ \vec{\beta}) \circ (\vec{\beta} \circ \vec{\alpha})$$

where $c \in \mathbb{N}$ is the least common multiple of $||\vec{\alpha}||$ and $||\vec{\beta}||$.

Example 44. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$

$$\vec{\alpha}_1 = (2, 1, 3) \quad \vec{\alpha}_2 = (1, 1)$$

$$\vec{\beta}_1 = (1, 2, 3, 4) \quad \vec{\beta}_2 = (0, 1, 1)$$
Note that $\vec{\beta}$ is the tuple of weights used in Example 10. We compute $\vec{\alpha} \otimes \vec{\beta}$. We have $|\vec{\alpha}| = 3$ and $|\vec{\beta}| = 2$. Thus, we have to unfold $\vec{\alpha}$ twice and $\vec{\beta}$ three times: $\vec{\alpha}^2 = (\alpha_1, \alpha_2, \vec{\alpha}_1, \alpha_2)$ and $\vec{\beta}^3 = (\beta_1, \beta_2, \vec{\beta}_1, \beta_2, \vec{\beta}_1, \beta_2)$. Then

$$\vec{\alpha} \otimes \vec{\beta} = \vec{\alpha}^2 \otimes \vec{\beta}^3$$

$$= (\alpha_1, \alpha_2, \vec{\alpha}_1, \alpha_2) \otimes (\beta_1, \beta_2, \vec{\beta}_1, \beta_2)$$

$$= (\alpha_1 \cdot \vec{\beta}_1, \vec{\alpha}_2 \cdot \vec{\beta}_1) ; (\alpha_2, \vec{\alpha}_1, \alpha_2) \otimes (\vec{\beta}_1, \beta_2, \vec{\beta}_1, \beta_2)$$

$$= (\alpha_1 \cdot \vec{\beta}_1, \vec{\alpha}_2 \cdot \vec{\beta}_1) ; (\alpha_2 \cdot \vec{\beta}_1, \vec{\alpha}_1, \alpha_2) \otimes (\vec{\beta}_1, \beta_2, \vec{\beta}_1, \beta_2)$$

$$= (\langle 2, 4, 6, 0, 1, 12 \rangle, \langle 1, 2, 3, 5 \rangle, \langle 0, 2, 1, 2, 3, 9 \rangle, \langle 0, 1, 2 \rangle)$$

**Proof of Lemma [14]** Follows directly from the definition of $\vec{\alpha} \otimes \vec{\beta}$. Every weight in $\vec{\alpha} \otimes \vec{\beta}$ is a concatenation of scalar multiplications of weights in $\vec{\beta}$.  

**B Additional Intuition for the Proof of Theorem 32**

Let $k \geq 1$ and $a_0, \ldots, a_{k-1} > 0$. Define

$$p_k(n) = \sum_{i=0}^{k-1} a_i (kn + i)^k .$$

Theorem [32] states that for every polynomial $q(n) \in \Omega_k$, we have $\langle q(n) \rangle \geq \langle p_k(n) \rangle$. Hence the degree of $\langle p(n) \rangle$ is an atom as a consequence of Proposition [19].

We give some more intuition for the proof of Theorem [32], involving a more explicit appeal to the continuity argument. For functions $f_0, \ldots, f_{k-1} : \mathbb{N} \to \mathbb{Q}$, we define the function $\text{zip}(f_0, \ldots, f_{k-1}) : \mathbb{N} \to \mathbb{Q}$ a.k.a. *perfect shuffle* [11, 18], by induction on $n$ as follows

$$\text{zip}(f_0, \ldots, f_{k-1})(0) = f_0(0)$$

$$\text{zip}(f_0, \ldots, f_{k-1})(n + 1) = \text{zip}(f_1, \ldots, f_{k-1}, S^1(f_0))(n)$$

where $S^i(f)$ is the $i$-th shift of $f$, defined by $m \mapsto f(m + i)$.

We have $\langle n^k \rangle \geq \langle p_k \rangle$ by the following transduction:

$$\langle n^k \rangle = \langle \text{zip}(\langle (kn + 0)^k, (kn + 1)^k, \ldots, (kn + k - 1)^k \rangle) \rangle$$

$$\geq \langle a_0 (kn + 0)^k + \cdots + a_{k-1} (kn + k - 1)^k \rangle$$

$$= \langle p_k(n) \rangle$$

Thinking of $n^k$ as an infinite word of natural numbers, then $(kn + 0)^k$, $(kn + 1)^k$, $\ldots$, $(kn + k - 1)^k$ are subsequences of $n^k$. Namely those subsequences picking every $k$-th element starting from element at index $0, 1, \ldots, k - 1$, respectively. Note that the transduction in the second line corresponds to the weighted product $\langle a_0, a_1, \ldots, a_{k-1}, 0 \rangle$, and thus can be realised by a finite state transducer (Lemma [24]).

However, this transduction works only for $\langle n^k \rangle$. It remains to be argued that there exists such a transduction from $\langle q(n) \rangle$ to $\langle p_k(n) \rangle$ for every polynomial $q(n) \in \Omega_k$.

Let us write $\sim_\varepsilon$ for the relation that relates polynomials of the same order whose coefficients differ by at most $\varepsilon > 0$. By Lemma [26] we can get arbitrarily close to $\langle n^k \rangle$. For every $\varepsilon > 0$, there exists $h(n) \in \Omega_k$ such that

$$\langle q(n) \rangle \geq \langle h(n) \rangle \quad \text{and} \quad h(n) \sim_\varepsilon n^k$$
Moreover, for $i \in \mathbb{N}_{<k}$, we have

$$h(kn + i) \sim_{\varepsilon'} (kn + i)^k$$

where $\varepsilon'$ depends on $\varepsilon$ (and $i$). If $\varepsilon$ tends to 0, so will $\varepsilon'$.

The crucial observation is that $(kn + 0)^k, (kn + 1)^k, \ldots, (kn + k - 1)^k$ form a basis of the vector space of polynomials of order $k$ with addition and scalar multiplication. The property of 'being a basis' is continuous. Hence, for small enough $\varepsilon$, and using approximation (7), we conclude that $h(kn + 0), h(kn + 1), \ldots, h(kn + k - 1)$ form a basis as well. Thus, there exist $a'_0, a'_1, \ldots, a'_{k-1} \in \mathbb{Q}$ such that

$$p_k(n) = a'_0 h(kn + 0) + \cdots + a'_{k-1} h(kn + k - 1)$$

We have that

$$\langle h(n) \rangle = \langle \text{zip}(h(kn + 0), h(kn + 1), \ldots, h(kn + k - 1)) \rangle$$

$$\geq \langle a'_0 h(kn + 0) + \cdots + a'_{k-1} h(kn + k - 1) \rangle$$

$$= \langle p_k(n) \rangle$$

However, for this transduction to work, we need to ensure that $a'_0, a'_1, \ldots, a'_{k-1} \geq 0$. Recall that $a_0, a_1, \ldots, a_{k-1} > 0$. Again, by continuity, $a'_i$ approaches $a_i$ as $\varepsilon$ approaches 0. Hence, for small enough $\varepsilon$, we have $a'_i \geq 0$ for every $i \in \mathbb{N}_{<k}$. Thus we have $\langle q(n) \rangle \geq \langle h(n) \rangle \geq \langle p(n) \rangle$. Hence $p(n)$ is an atom.

---

4 The cautious reader will have observed that the basis consists only of $k$ vectors while the vector space has dimension $k + 1$. We tacitly ignore the constant terms of the polynomials since, in the transducer degrees, we have $\{f\} \equiv \{f + c\}$ for every $c \in \mathbb{Q}$. 
