The Influence of an External Chromomagnetic Field on Color Superconductivity

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Abstract

We study the competition of quark-antiquark and diquark condensates under the influence of an external chromomagnetic field modelling the gluon condensate and in dependence on the chemical potential and temperature. As our results indicate, an external chromomagnetic field might produce remarkable qualitative changes in the picture of the color superconducting phase formation.

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I. INTRODUCTION

Low energy (large distance) effects in QCD can only be studied by approximate (nonperturbative) methods in the framework of various effective models or in terms of lattice calculations. At present time, one of the most popular QCD-like effective theories is the well-known Nambu–Jona-Lasinio (NJL) model \[1\], which is a relativistic quantum field theory with four-fermion interactions. The physics of light mesons (see e.g. \[2\] and references therein), diquarks \[3,4\] and meson-baryon interactions \[3,7\] based on dynamical chiral symmetry breaking can be effectively described by NJL chiral quark models. Moreover, NJL models are widely used in nuclear physics and astrophysics (neutron stars) for the investigation of quark matter \[8\], to construct alternative models of electroweak interactions \[9\] and in cosmological applications \[10\]. Moreover, its (2+1)-dimensional analogue serves as a satisfactory microscopic theory for several effects in the physics of high-temperature superconductors \[11\].

The NJL model displays the same symmetries as QCD. So it can be successfully used for simulating some of the QCD vacuum properties under the influence of external conditions such as temperature $T$ and chemical potential $\mu$ \[12\]. The role of such considerations significantly increases especially in the cases, where numerical lattice calculations are not admissible in QCD, i.e. at nonzero density and in the presence of external electromagnetic fields \[13,14\]. Recently, it was shown in the framework of a (2+1)-dimensional NJL model that an arbitrary small external magnetic field induces the spontaneous chiral symmetry breaking (CSB) even under conditions, when the interaction between fermions is arbitrary weak \[15\]. Later it was shown that this phenomenon (called magnetic catalysis effect) has a rather universal character and gets its explanation on the basis of the dimensional reduction mechanism \[16\]. (The recent reviews \[17\] consider the modern status of the magnetic catalysis effect and its applications in different branches of physics.)

As an effective theory for low energy QCD, the NJL model does not contain any dynamical gluon fields. Such nonperturbative feature of the real QCD vacuum, as the nonzero gluon condensate $< F_{\mu\nu}^a F^{a\mu\nu} > \equiv < FF >$ can, however, be mimicked in the framework of NJL models with the help of external chromomagnetic fields. In particular, for a QCD-motivated NJL model with gluon condensate (i.e. in the presence of an external chromomagnetic field) and finite temperature, it was shown that a weak gluon condensate
plays a stabilizing role for the behavior of the constituent quark mass, the quark condensate, meson masses and coupling constants for varying temperature \[18\]. Then, in a series of papers, devoted to the NJL model with gluon condensate, it was shown that an external chromomagnetic field, similar to the ordinary magnetic field, serves as a catalyzing factor in the fermion mass generation and dynamical breaking of chiral symmetry as well \[19\]. The basis for this phenomenon is the effective reduction of the space dimensionality in the presence of external chromomagnetic fields \[20\].

There exists the exciting idea proposed more than twenty years ago \[21\]-\[23\] that at high baryon densities a colored diquark condensate \(< qq >\) might appear. In analogy with ordinary superconductivity, this effect was called color superconductivity (CSC). In the framework of the one-gluon exchange approximation in QCD the colored Cooper pair formation is predicted self-consistently at extremely high values of the chemical potential \(\mu \gtrsim 10^8\) MeV \[24\]. Unfortunately, such baryon densities are not observable in nature and not accessible in experiments (the typical densities inside the neutron stars or in the future heavy ion experiments correspond to \(\mu \sim 500\) MeV). The possibility for the existency of the CSC phase in the region of moderate densities was proved quite recently (see e.g. the papers \[25\]-\[28\] as well as the review article \[29\] and references therein). In these papers it was shown on the basis of different effective theories for low energy QCD (instanton model, NJL model etc) that the diquark condensate \(< qq >\) can appear already at a rather moderate baryon density (\(\mu \sim 400\) MeV), which can possibly be detected in the future experiments on ion-ion collisions. Since quark Cooper pairing occurs in the color anti-triplet channel, the nonzero value of \(< qq >\) means that, apart from the electromagnetic \(U(1)\) symmetry, the color \(SU_c(3)\) should be spontaneously broken down inside the CSC phase as well. In the framework of NJL models the CSC phase formation has generally be considered as a dynamical competition between diquark \(< qq >\) and usual quark-antiquark condensation \(< \bar{q}q >\). However, the real QCD vacuum is characterized in addition by the appearence of a gluon condensate \(< FF >\) as well, which might change the generally accepted conditions for the CSC observation. In particular, one would expect that, similarly to the case of quark-antiquark condensation, the process of diquark condensation might be induced by external chromomagnetic fields. For a \((2+1)\)-dimensional quark model, this was recently demonstrated in \[30\]. There, a \(SU(2)_L \times SU(2)_R\)
The chirally symmetric (2+1)-dimensional NJL model with three colored quarks of two flavors was considered at zero $T$, $\mu$. It was shown that in this case for arbitrary fixed values of coupling constants there exists a critical value of the external chromomagnetic field at which a CSC second order phase transition is induced in the system. Since the two-flavored QCD$_3$ and the considered NJL model are not in the same universality class of theories (QCD$_3$ with $N_f = 2$ has a higher flavor symmetry $SU(4)$), the obtained results are intrinsic to real QCD$_4$ rather than to QCD$_3$. Indeed, our recent investigations on the basis of a (3+1)-dimensional NJL model [31] and $\mu = 0$ show that some types of sufficiently strong external chromomagnetic fields may catalyze the diquark condensation.

As argued above, CSC might occur inside neutron stars and possibly become observable in ion-ion collisions, i.e. at nonzero baryon densities. Taking into account the fact that at finite chemical potential the magnetic generation of dynamical CSB qualitatively differs from the $\mu = 0$ case [14], one might expect analogous effects for CSC, too. By this reason, the investigation of the chromomagnetic generation of CSC under the influence of a finite chemical potential (finite particle density) is a very interesting and actual physical problem.

The aim of the present paper is to study the influence of external conditions such as chemical potential, temperature and especially of the gluon condensate (as modelled by external color gauge fields) on the phase structure of quark matter with particular emphasize of its CSC phase. To this end, we shall extend our earlier analysis of the chromomagnetic generation of CSC at $\mu = 0$ [30], [31] to the case of an (3+1)-dimensional NJL type model with finite chromomagnetic field, temperature and chemical potential presenting a generalization of the free field model of [28].

The paper is organized as follows. In Sections II and III the extended NJL model under consideration is presented, and its effective potential ($\equiv$ thermodynamic potential) at nonzero external chromomagnetic field, chemical potential and temperature is obtained in the one-loop approximation. This quantity contains all the necessary informations about the quark and diquark condensates of the theory. In following Section IV the phase struc-
ture of the model is discussed on the basis of numerical investigations of the global minimum point of the effective potential. As our main result, it is shown that the external chromomagnetic field can induce the transition to the CSC phase and diquark condensation. Thereby, the characteristics of the CSC phase can significantly change in dependence on the strength of the chromomagnetic field. Finally, section V contains a summary and discussion of the results. Few details of the effective potential calculation are relegated to an Appendix.

II. THE MODEL

Let us first give several (very approximative) arguments motivating the chosen structure of our QCD-motivated extended NJL model introduced below. For this aim, consider two-flavor QCD with nonzero chemical potential and color group $SU_c(N_c)$ and decompose the gluon field $A^a_\mu(x)$ into a condensate background (“external”) field $A^a_\mu(x)$ and the quantum fluctuation $a^a_\mu(x)$ around it, i.e. $A^a_\mu(x) = A^a_\mu(x) + a^a_\mu(x)$. By integrating in the generating functional of QCD over the quantum field $a^a_\mu(x)$ and further “approximating” the nonperturbative gluon propagator by a $\delta$–function, one arrives at an effective local chiral four-quark interaction of the NJL type describing low energy hadron physics in the presence of a gluon condensate. Finally, by performing a Fierz transformation of the interaction term, one obtains a four-fermionic model with $(\bar{q}q)$– and $(qq)$–interactions and an external condensate field $A^a_\mu(x)$ of the color group $SU_c(N_c)$ given by the following Lagrangian

$$L = \bar{q} \gamma^\nu (i \partial_\nu + g A^a_\mu(x) \frac{\lambda^a}{2}) q + \frac{G_1}{2N_c} \left[ (\bar{q}q)^2 + (\bar{q}i\gamma^5 \vec{\tau} q)^2 \right] + \frac{G_2}{N_c} \left[ i \bar{q} c \varepsilon (i \lambda^{ab}_{as}) \gamma^5 q c \right] [i \bar{q} c \varepsilon (i \lambda^{ab}_{as}) \gamma^5 q c] \right].$$

It is necessary to note that in order to obtain realistic estimates for masses of vector/axial-vector mesons and diquarks in extended NJL–type of models [3], we have to allow for independent coupling constants $G_1, G_2$, rather than

\footnote{The most general four-fermion interaction would include additional vector and axial-vector $$(\bar{q}q)$$ as well as pseudo-scalar, vector and axial-vector-like $(qq)$ -interactions. For our goal of studying the effect of chromomagnetic catalysis for the competition of quark and diquark condensates, the interaction structure of (1) is, however, sufficiently general.}
to consider them related by a Fierz transformation of a current-current interaction via gluon exchange. Clearly, such a procedure does not spoil chiral symmetry.

In (1) $g$ denotes the gluon coupling constant, $\mu$ is the quark chemical potential, $q_c = C\bar{q}^t$, $\bar{q}_c = q^t C$ are charge-conjugated spinors, and $C = i\gamma^2\gamma^0$ is the charge conjugation matrix ($t$ denotes the transposition operation). In what follows we assume $N_c = 3$ and replace the antisymmetric color matrices $\lambda_{as}$ (with factor $i$) by the antisymmetric $\epsilon^b$ operator. Moreover, summation over repeated color indices $a = 1, \ldots, 8$; $b = 1, 2, 3$ and Lorentz indices $\nu = 0, 1, 2, 3$ is implied. The quark field $q \equiv q_{\alpha i}$ is a flavor doublet and color triplet as well as a four-component Dirac spinor, where $i = 1, 2$; $\alpha = 1, 2, 3$. (Latin and Greek indices refer to flavor and color indices, respectively; spinor indices are omitted.) Furthermore, we use the notations $\lambda^a/2$ for the generators of the color $SU_c(3)$ group appearing in the covariant derivative as well as $\bar{\tau} \equiv (\tau^1, \tau^2, \tau^3)$ for Pauli matrices in the flavor space; $(\epsilon)_{ik} \equiv \epsilon_{ik}$, $(\epsilon^b)^{\alpha\beta} \equiv \epsilon^{\alpha\beta b}$ are totally antisymmetric tensors in the flavor and color spaces, respectively. Clearly, the Lagrangian (1) is invariant under the chiral $SU(2)_L \times SU(2)_R$ and color $SU_c(3)$ groups.

Next, let us for a moment suppose that in (1) $A^a_\mu(x)$ is an arbitrary classical gauge field of the color group $SU_c(3)$. (The following investigations do not require the explicit inclusion of the gauge field part of the Lagrangian). The detailed structure of $A^a_\mu(x)$ corresponding to a constant chromomagnetic gluon condensate will be given below.

The linearized version of the model (1) with auxiliary bosonic fields has the following form

$$
\tilde{L} = \bar{q}[\gamma^\nu(i\partial_\nu + gA_\nu^a(x)\lambda^a/2) + \mu\gamma^0]q - \bar{q}(\sigma + i\gamma^5\bar{\tau}\pi)q - \frac{3}{2G_1}(\sigma^2 + \pi^2) - \frac{3}{G_2}\Delta^b \Delta^b \Delta^b - \Delta^b[i\bar{q}^t C\varepsilon^b\gamma^5 q] - \Delta^b[i\bar{q}\varepsilon^b\gamma^5 C\bar{q}].
$$

The Lagrangians (1) and (2) are equivalent, as can be seen by using the equations of motion for bosonic fields, from which it follows that

$$
\Delta^b \sim iq^t C\varepsilon^b\gamma^5 q, \quad \sigma \sim \bar{q}q, \quad \bar{\pi} \sim i\bar{q}\gamma^5\bar{\tau}q.
$$

Clearly, the $\sigma$ and $\bar{\pi}$ fields are color singlets. Besides, the (bosonic) diquark field $\Delta^b$ is a color antitriplet and a (isoscalar) singlet under the chiral $SU(2)_L \times SU(2)_R$ group. Note further that the $\sigma$, $\Delta^b$, are scalars, but the $\bar{\pi}$
are pseudo-scalar fields. Hence, if $\sigma \neq 0$, then chiral symmetry of the model is spontaneously broken, whereas $\Delta^b \neq 0$ indicates the dynamical breaking of both the color and electromagnetic symmetries of the theory.

In the one-loop approximation, the effective action for the boson fields which is invariant under the chiral (flavor) as well as color and Lorentz groups is expressed through the path integral over quark fields:

$$
\exp(iS_{\text{eff}}(\sigma, \vec{\pi}, \Delta^b, \Delta^{*b}, A^a_{\mu}) = N' \int [d\vec{q}][dq] \exp(i \int \tilde{L} d^4x),
$$

where

$$
S_{\text{eff}}(\sigma, \vec{\pi}, \Delta^b, \Delta^{*b}, A^a_{\mu}) = -N_c \int d^4x \left[ \frac{\sigma^2 + \vec{\pi}^2}{2G_1} + \frac{\Delta^b \Delta^{*b}}{G_2} \right] + \tilde{S},
$$

(4)

$N'$ is a normalization constant. The quark contribution to the partition function is then given by:

$$
Z_q = \exp(i\tilde{S}) = N' \int [d\vec{q}][dq] \exp(i \int [\bar{q}Dq + \bar{q}\mathcal{M}\bar{q} + q^t\bar{M}q]d^4x). \tag{5}
$$

In (5) we have used the following notations

$$
\mathcal{D} = D + \gamma^\mu gA^a_\mu(x)\frac{\lambda^a}{2}; \quad D = i\gamma^\mu \partial_\mu - \sigma - i\gamma^5\vec{\pi}\vec{\gamma} + \mu\gamma^0,
$$

$$
\bar{\mathcal{M}} = -i\Delta^{*b}C\varepsilon^b\gamma^5, \quad \mathcal{M} = -i\Delta^b\varepsilon^b\gamma^5C, \tag{6}
$$

where $D$ is the Dirac operator in the coordinate, spinor and flavor spaces, whereas $\mathcal{D}$, $\mathcal{M}$ and $\bar{\mathcal{M}}$ are in addition operators in the color space, too. Let us next assume that in the ground state of our model $\langle \Delta^1 \rangle = \langle \Delta^2 \rangle = \langle \pi \rangle = 0$ and $\langle \sigma \rangle, \langle \Delta^3 \rangle \neq 0$. Obviously, the residual symmetry group of such a vacuum is $SU_c(2)$ whose generators are the first three generators of initial $SU_c(3)$. Now suppose that in this frame the constant external chromomagnetic field, simulating the presence of a gluon condensate $\langle \mathcal{F}\mathcal{F} \rangle = 2H^2$, has the following form $H^a = (H^1, H^2, H^3, 0, \ldots, 0)$. Furthermore, due to the residual $SU_c(2)$ invariance of the vacuum, one can put $H^1 = H^2 = 0$ and $H^3 \equiv H$.

Some remarks about the structure of the external chromomagnetic fields $A^a_\nu(x)$ used in (4) are needed. From this moment on, we assume $A^a_\nu(x)$ in

\[3\]If $\langle \pi \rangle \neq 0$ then one would have spontaneous breaking of parity. For strong interactions parity is, however, a conserved quantum number, justifying the assumption $\langle \pi \rangle = 0$. \]
such a form that the only nonvanishing components of the corresponding field strength tensor $F_{\mu\nu}^a$ are $F_{12}^3 = -F_{21}^3 = H = \text{const}$. The above homogeneous chromomagnetic field can be generated by the following vector-potential

$$A_\nu^3(x) = (0, 0, Hx^1, 0); \quad A_\nu^a(x) = 0 \quad (a \neq 3),$$

which defines the well known Matinyan–Savvidy model of the gluon condensate in QCD \[32\].

In QCD the physical vacuum may be interpreted as a region splitted into an infinite number of domains with macroscopic extension \[33\]. Inside each such domain there can be excited a homogeneous background chromomagnetic field, which generates a nonzero gluon condensate $\langle FF \rangle \neq 0$. (Averaging over all domains results in a zero background chromomagnetic field, hence color as well as Lorentz symmetries are not broken.)

In order to find nonvanishing condensates $\langle \sigma \rangle$ and $\langle \Delta^3 \rangle$, we should calculate the effective potential whose global minimum point provides us with these quantities. Suppose that (apart from the external vector-potential $A_\mu^a(x)$ \[7\]) all boson fields in $S_{\text{eff}}$ \[41\] do not depend on space-time. In this case, by definition, $S_{\text{eff}} = -V_{\text{eff}} \int d^4x$, where

$$V_{\text{eff}} = \frac{3(\sigma^2 + \bar{\pi}^2)}{2G_1} + \frac{3\Delta^b \Delta^{*b}}{G_2} + \tilde{V}; \quad \tilde{V} = -\frac{\tilde{S}}{v}, \quad v = \int d^4x.$$  \hspace{1cm} (8)

Due to our assumption on the vacuum structure, we put $\Delta^{1,2} \equiv 0$ as well as $\bar{\pi} = 0$. Then, taking into account the form of the vector-potential \[7\], one can easily see that the functional integral for $\tilde{S}$ in \[5\] is factorized

$$Z_q = \exp(i\tilde{S}(\sigma, \Delta)) = N' \int [dq_3][dq_3] \exp(i \int \bar{q}_3 \tilde{D} q_3 d^4x) \times$$

$$\times \int [d\bar{Q}][dQ] \exp(i \int \bar{Q} \tilde{D} Q + \bar{Q} M \tilde{Q}^t + Q^t M Q] d^4x),$$

where $\Delta \equiv \Delta^3$, $q_3$ is the quark field of color 3 and $Q \equiv (q_1, q_2)^t$ is the doublet, composed from quark fields of the colors 1,2. Moreover, $\tilde{D} = D|_{\bar{\pi} = 0}$ ($D$ is presented in \[6\]) and

$$\tilde{D} = \bar{D} + \gamma^\mu g A^{3}_{\mu}(x) \frac{\sigma_3}{2}; \quad \tilde{M} = -i \Delta^* C \varepsilon \tilde{\epsilon} \gamma^5, \quad M = -i \Delta \varepsilon \tilde{\epsilon} \gamma^5 C.$$  \hspace{1cm} (11)

4Strictly speaking, our following calculations refer to some given macroscopic domain. The obtained results turn out to depend on color and rotational (Lorentz) invariant quantities only, and are independent on the concrete domain.
In (11) $\sigma_3$, $\tilde{\epsilon}$ are matrices in the two-dimensional color subspace, corresponding to the $SU_c(2)$ group:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Clearly, the integration over $q_3$ in (9) yields $\det \tilde{D}$.

Defining $\Psi^t = (Q^t, \bar{Q})$ and introducing the matrix-valued operator

$$Z = \begin{pmatrix} 2 \bar{M} & -\tilde{D}^t \\ \tilde{D} & 2M \end{pmatrix},$$

the gaussian integral over $\bar{Q}$ and $Q$ in (10) can be rewritten in compact matrix notation and be evaluated as

$$\int [d\Psi] e^{i \int \Psi^t Z \Psi dx} = \det^{1/2} Z.$$  (12)

Then, by using in (12) the general formula

$$\det \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} = \det[-\bar{B}B + \bar{B}A\bar{B}^{-1}A] = \det[\bar{A}A - \bar{A}B\bar{A}^{-1}B],$$

one obtains the result:

$$\exp(i \tilde{S}(\sigma, \Delta)) = N' \det(\tilde{D}) \det^{1/2}[4M\bar{M} + M\tilde{D}^t M^{-1}\tilde{D}] =
= N' \det[(i\tilde{\partial} - \sigma + \mu\gamma^0)] 
\cdot \det^{1/2} \left[ 4|\Delta|^2 + (-i\tilde{\partial} - \sigma + \mu\gamma^0 - g\tilde{A}^3\sigma_3/2)(i\tilde{\partial} - \sigma + \mu\gamma^0 + g\tilde{A}^3\sigma_3/2) \right].$$  (13)

Recall that the operator under the first det-symbol in (13) acts only in the flavor, coordinate and spinor spaces, whereas the operator under the second det-symbol acts in the two-dimensional color subspace, too.

III. THE GENERAL CASE $\mu \neq 0, T \neq 0, H \neq 0$

A. The effective potential

First of all, let us calculate the effective action from (13) at zero temperature $T$. It is convenient to rewrite the second determinant in (13) in the form
\[
\det[4M \tilde{M} + M \tilde{D}^t(M)^{-1} \tilde{D}] = \\
= \det \left[ 4|\Delta|^2 + \mu^2 - p_0^2 + \sigma^2 - (\gamma \tilde{\nabla})^2 - 2\mu \gamma^0(\sigma + \gamma \tilde{\nabla}) \right]
\]  
(14)

where the \(p^0\)-momentum space representation and \(\gamma \tilde{\nabla} = \gamma_k(i \partial_k + gA_k^3\sigma_3/2)\), \((k = 1, 2, 3)\) have been used. Similarly to quantum electrodynamics, it is easily seen that the operator \(H \equiv \gamma^0(\sigma + \gamma \tilde{\nabla})\) is the Hamiltonian for quarks with color indices \(\alpha = 1, 2\) and flavor \(i = 1, 2\) in the background vector-potential \([7]\). Its eigenvalues are \(\pm \varepsilon_{\{n\}}\), where \(\varepsilon_{\{n\}} = \sqrt{\sigma^2 + p_0^2 + gH(n + 1/2) - gH \zeta/2}\), and corresponding eigenstates are denoted by \(\Phi_{\{n\}}^{\pm}_{p_2i\alpha}\). The set of quark quantum numbers in the background field are defined as follows: \(\{n\} \equiv \{n = 0, 1, 2, \ldots ; -\infty < p_3 < +\infty; \zeta = \pm 1\}\), \(i, \alpha = 1, 2, -\infty < p_2 < \infty\). Each of the eigenvalues \(\pm \varepsilon_{\{n\}}\) for \(H\) is evidently four-fold degenerate with respect to flavor and color quantum numbers \(i, \alpha\). It is also degenerate with respect to the quantum number \(p_2\), which quasi-classically characterizes the center of orbit motion for a charged particle in an external magnetic field. Since

\[
H H \Phi_{\{n\}}^{\pm}_{p_2i\alpha} = [\sigma^2 - (\gamma \tilde{\nabla})^2] \Phi_{\{n\}}^{\pm}_{p_2i\alpha} = \varepsilon_{\{n\}}^2 \Phi_{\{n\}}^{\pm}_{p_2i\alpha},
\]

one can easily conclude that in the basis \(\Phi_{\{n\}}^{\pm}_{p_2i\alpha}\) the operator in the determinant \((14)\) is diagonal. Moreover, its diagonal matrix elements are equal to \(4|\Delta|^2 - p_0^2 + (\mu \pm \varepsilon_{\{n\}})^2\). Upon multiplying these quantities, one can find the determinant from \((14)\). In a similar way, it is possible to calculate the first determinant from \((13)\). Hence, taking into account the relation \(\text{Det}O = \exp(\text{Tr} \ln O)\), and following the standard procedure (see, e.g., \([34]\)), the following expression for \(\tilde{V}\) is obtained from \((13)-(14)\) (omitting an infinite \(\sigma\)- and \(\Delta\)-independent constant):

\[
\tilde{V} = - \frac{\bar{S}}{v} = i N_f \int \frac{dp_0}{2\pi} \left\{ \sum_{\{p\}_0, \pm} \ln \left( (E_p \pm \mu)^2 - p_0^2 \right) + \right. \\
\left. + A \sum_{\{n\}, \pm} \ln \left( 4|\Delta|^2 - p_0^2 + (\varepsilon_{\{n\}} \pm \mu)^2 \right) \right\},
\]

(15)

where \(\{p\}_0\) denotes the set of quark quantum numbers for vanishing background field \(\{\{p\}_0\} \equiv \{-\infty < p_1, p_2, p_3 < +\infty \}\)), and \(E_p = \sqrt{p^2 + \sigma^2}\). The factor \(N_f\) in front of the integral in \((15)\) is the result of summation over flavor indices \(i = 1, \ldots, N_f\), whereas the degeneracy factor \(A \equiv gH/(8\pi^2)\) is due to the integration over the momentum \(p_2\) and summation over the color indices \(\alpha = 1, 2\). Moreover, \(\Sigma_{\{p\}_0} \equiv \int d^3p/(2\pi)^3; \Sigma_{\{n\}} \equiv \int dp_0 \Sigma_{n, \zeta}\).
In the case of finite temperature $T = 1/\beta > 0$ the corresponding expression for $\tilde{V}_T$ can be obtained from (13) by means of the following replacements:

$$\int \frac{dp_0}{2\pi} (\cdots) \rightarrow iT \sum_l (\cdots); \quad p_0 \rightarrow i\omega_l \equiv 2\pi iT(l + 1/2); \quad l = 0, \pm 1, \pm 2, \ldots,$$

where $\omega_l$ is the Matsubara frequency. Hence,

$$\tilde{V}_T = -N_f T \sum_{l=-\infty}^{l=\infty} \left\{ \sum_{\{p\},0,\pm} \ln \left( (E_p \pm \mu)^2 + \omega_l^2 \right) + A \sum_{\{n\},\pm} \ln \left( 4|\Delta|^2 + \omega_l^2 + (\varepsilon_{\{n\}} \pm \mu)^2 \right) \right\}. \quad (16)$$

In order to transform (16), let us first perform the summation over the Matsubara frequencies. It is evident that

$$\sum_l \ln(\omega_l^2 + \Omega^2) = \sum_l \Omega^2 \int_{1/\beta^2}^{\omega_l^2 + a^2} \frac{1}{\omega_l^2 + a^2} + \sum_l \ln \left( \omega_l^2 + \frac{1}{\beta^2} \right), \quad (17)$$

where $\Omega$ stands, according to (16), for $\sqrt{(E_p \pm \mu)^2}$ or $\sqrt{4|\Delta|^2 + (\varepsilon_{\{n\}} \pm \mu)^2}$, i.e. $\Omega \geq 0$. Note that we can neglect the contribution from the last term in (17), since it does not depend on $\sigma$ and $\Delta$. The first term in (17) can be presented in the following form (see Appendix):

$$\sum_l \Omega^2 \int_{1/\beta^2}^{\omega_l^2 + a^2} \frac{1}{\omega_l^2 + a^2} = 2 \ln \text{ch}(\Omega/2) + \text{const}$$

$$= \Omega \beta + 2 \ln(1 + e^{-\Omega\beta}) + \text{const}. \quad (18)$$

Performing the summation over Matsubara frequencies in the second term in (16), and taking into account the degeneracy of the quark spectrum $\varepsilon_n$ in the chromomagnetic field with respect to combination of quantum numbers $n$ and $\zeta$, we can use the following expression for the energy spectrum: $\varepsilon_n = \sqrt{gHn + p_3^2 + \sigma^2}$, where $n = 0, 1, 2, \ldots$ is the Landau quantum number, and $-\infty < p_3 < \infty$. Then, summing over the spin quantum number $\zeta = \pm 1$, we have to account for the fact that for the ground state with $n = 0$ only one spin projection $\zeta = -1$ is possible. Hence, a factor $\alpha_n = 2 - \delta_{n0}$ should be included in the final expression. As for the summation over Matsubara frequencies in the first term in (16), it is necessary to take into account the
fact that the function (18) is even with respect to the variable $\Omega$. Finally, we thus arrive at the following result for the thermodynamic potential

$$V_{H\mu T}(\sigma, \Delta) = N_c \left( \frac{\sigma^2}{2G_1} + \frac{|\Delta|^2}{G_2} \right) - 2N_f \int \frac{d^3 p}{(2\pi)^3} (N_c - 2) \left\{ E_p + T \ln\left[ (1 + e^{-\beta(E_p-\mu)})(1 + e^{-\beta(E_p+\mu)}) \right] \right\} - N_f A \sum_{n=0}^{\infty} dp_3 \alpha_n \left\{ \sqrt{(\varepsilon_n - \mu)^2 + 4|\Delta|^2} + \sqrt{(\varepsilon_n + \mu)^2 + 4|\Delta|^2} \right. + 2T \ln\left[ (1 + e^{-\beta\sqrt{(\varepsilon_n-\mu)^2+4|\Delta|^2}})(1 + e^{-\beta\sqrt{(\varepsilon_n+\mu)^2+4|\Delta|^2}}) \right] \}.$$ 

For convenience, expressions are again written in terms of $N_f$ and $N_c$ even though in the following we will be concerned only with $N_f = 2$ and $N_c = 3$.

### B. Regularization

First of all, let us subtract from (19) an infinite constant in order that the effective potential obeys the constraint $V_{H\mu T}(0, 0) = 0$. After this subtraction the effective potential still remains UV divergent. This divergence could evidently be removed by introducing a simple momentum cutoff $|p| < \Lambda$. Instead of doing this, we find it convenient to use another regularization procedure. To this end, let us recall that all UV divergent contributions to the subtracted potential $V_{H\mu T}(\sigma, \Delta) - V_{H\mu T}(0, 0)$ are proportional to powers of meson and/or diquark fields $\sigma, \Delta$. So, one can insert some momentum-dependent form factors in front of composite $\sigma$–and $\Delta$–fields in order to regularize the UV behaviour of integrals and sums.  

It is clear by now that we are going to study the effects of an external chromomagnetic condensate field in the framework of the NJL-type model

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5 A suitable physical motivation of such form factors follows in the framework of nonlocal NJL type models based on the one-gluon-exchange approximation to QCD with nontrivial gluon propagator. In particular, in ref. [35] it was shown that the arising exponential form factors for composite mesons, as obtained from the solution of the (nonlocal) Bethe-Salpeter(BS)-equation, make the quark loop expansion including meson (diquark) vertices convergent. In this case, there is no need for introducing a sharp momentum cutoff as in the local NJL model. Hence, the introduction of smoothing form factors in our expressions of an approximate local NJL model may be interpreted as a regularization procedure taking some effects of the originally non-local current-current interaction afterwards into account.
(1), which in addition to two independent coupling constants $G_1, G_2$ includes regularizing meson (diquark) form factors. Of course, it would be a very hard task to study the competition of DSB and CSC for arbitrary values of coupling constants $G_1, G_2$ and any form factors. Thus, in order to restrict this arbitrariness and to be able to compare our results (at least roughly) with other approaches, we find it convenient to investigate the phase structures of the model (1) at $H = 0$ and $H \neq 0$ only for some fixed values of $G_1, G_2$ and some simple expressions for meson/diquark form factors (for simplicity, meson/diquark form factors are chosen to be equal). We expect that qualitatively the obtained (integrated) results do not depend significantly on the chosen regularization procedures, including the momentum cutoff one.

Let us choose the form factors

$$\phi = \frac{\Lambda^4}{(\Lambda^2 + \vec{p}^2)^2}, \quad \phi_n = \frac{\Lambda^4}{(\Lambda^2 + p_3^2 + gHn)^2},$$

which have to be included in the energy spectra by a corresponding multiplication of the $\sigma-, \Delta-$ fields:

$$E_p^r = \sqrt{\vec{p}^2 + \phi^2 \sigma^2}, \quad \varepsilon_n^r = \sqrt{gHn + p_3^2 + \phi_n^2 \sigma^2}, \quad |\Delta^2| \rightarrow \phi_n^2 |\Delta^2|. \quad (21)$$

Note that with the choice of simple form factors (20) our expression for the thermodynamic potential at $H = 0$ formally coincides with the corresponding expression of Ref. [27] obtained for an NJL type model with instanton-induced four-fermion interactions. In particular, by a suitable choice of coupling constants $G_1, G_2$, we will later “normalize” our phase portraits for $H = 0$ to the curves of this paper in order to illustrate the influence of a nonvanishing chromomagnetic field. \footnote{The application of the smooth meson form factors (20) leads in a natural way to a suppression of higher Landau levels which is of particular use here. Hence, this regularization scheme is particularly suitable for the manifestation of the (chromo)magnetic catalysis effect of dynamical symmetry breaking. Indeed, the (chromo)magnetic catalysis effect and the underlying mechanism of dimensional reduction are closely related to the infrared dominance of the lowest Landau level with $n = 0$ [17,20].}

\footnote{It is necessary to underline that in our case the meson/diquark form factors (20) mimick solutions of the BS-equation for some non-local four-fermion interaction arising from the one-gluon exchange approach to QCD. Contrary to this, the instanton-like form factor used in [27] has another physical nature. It appears as quark zero mode wave function in the presence of instantons [25].}
As a result, instead of (19) we shall deal with the following regularized potential $V_r(\sigma, \Delta)$:

$$V_r(\sigma, \Delta) = V_0 - 2N_f(N_c - 2) \int_{-\infty}^{\infty} V_1 \frac{d^3 p}{(2\pi)^3} - \frac{gHN_f}{8\pi^2} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \alpha_n V_2 dp_3, \quad (22)$$

where $V_0 = N_c \left( \frac{\sigma^2}{2G_1} + \frac{|\Delta|^2}{G_2} \right)$,

$$V_1 = E^r_p + T \ln \left[ \left( 1 + e^{-\beta(E^r_p + \mu)} \right) \left( 1 + e^{-\beta(E^r_p - \mu)} \right) \right],$$

$$V_2 = \sqrt{(\varepsilon^r_n - \mu)^2 + 4|\Delta|^2\phi^2_n} + \sqrt{(\varepsilon^r_n + \mu)^2 + 4|\Delta|^2\phi^2_n} + 2T \ln \left[ \left( 1 + e^{-\beta \sqrt{(\varepsilon^r_n - \mu)^2 + 4|\Delta|^2\phi^2_n}} \right) \left( 1 + e^{-\beta \sqrt{(\varepsilon^r_n + \mu)^2 + 4|\Delta|^2\phi^2_n}} \right) \right], \quad (23)$$

and $E^r_p, \varepsilon^r_n$ are given in (21). Despite the $\Lambda$-modification, the expression (22) contains yet UV-divergent integrals. However, as it was pointed out from the very beginning, we shall numerically study the subtracted effective potential, i.e. the quantity $V_r(\sigma, \Delta) - V_r(0, 0)$, which has no divergences.

In the next section the dependency of the global minimum point of the regularized potential (22) on the external parameters $H, \mu, T$ will be investigated.

**IV. NUMERICAL DISCUSSIONS**

In the previous section we have chosen the form factors as in (20) in order to roughly normalize our numerical calculations at $H \neq 0$ on the results obtained at $H = 0$ in [27]. Comparing the effective potential (22) at $gH = 0$ with the corresponding one from ref. [27] (denoting their respective diquark field and coupling constants by a tilde), we see that these quantities coincide if $2\Delta = \tilde{\Delta}$, $G_1 = 2N_c\tilde{G}_1$ and $G_2 = N_c\tilde{G}_2$. Using further their value $\Lambda = 0.8$ GeV as well as their numerical ratio of coupling constants, we get in our case the following relations

$$G_1 = 2N_c\frac{6.47}{\Lambda^2}, \quad G_2 = \frac{3}{8}G_1. \quad (24)$$

Now, let us perform the numerical investigation of the global minimum point (GMP) of the potential (22) for form factors and values of coupling constants as given by (20) and (24), respectively. In this case, the GMP of the function
$V_r(\sigma, \Delta)$ at $T = \mu = H = 0$ is at the point $\sigma = 0.4$ GeV, $\Delta = 0$ in correspondence with phenomenological results and [27]. Note, further that in the case of zero temperature we have

$$V_1|_{T=0} = E'_p + (\mu - E'_p) \theta(\mu - E'_p),$$

$$V_2|_{T=0} = \sqrt{(\varepsilon^n_r - \mu)^2 + 4|\Delta|^2\phi^2_n} + \sqrt{(\varepsilon^n_r + \mu)^2 + 4|\Delta|^2\phi^2_n}.$$

In order to study the phase structure of the model using numerical methods, the summation over $n$ in (22) is limited by a maximum value $n_{\text{max}} = (2.5\Lambda)^2/gH$, where other terms of the series can be neglected due to their smallness.

First of all, it should be remarked that, as in paper [27] at $gH = 0$, a mixed phase of the model was not found for $H \neq 0$, i.e. for a wide range of parameters $\mu, H, T$ we did not find a global minimum point of the potential (22), at which $\sigma \neq 0, \Delta \neq 0$. The results of our numerical investigations of the GMP of $V_r(\sigma, \Delta)$ are graphically represented in the set of figures.

In Fig. 1, one can see the phase portrait of the model in terms of $\mu, gH$ at $T = 0$. In this case, the plane of parameters $\mu, gH$ is divided into three regions I, II, III. The totally symmetric phase of the model is in region I, since for the points inside I the GMP of the potential lies at $\sigma = 0, \Delta = 0$. In region II we have a phase with broken chiral symmetry, corresponding to the GMP of the potential at $\sigma \neq 0, \Delta = 0$. Finally, the color superconducting (CSC) phase with the GMP of the potential at $\sigma = 0, \Delta \neq 0$, corresponds to the points from region III of this figure. The boundary between regions II and III is given by the critical curve $\mu_{\text{cr1}}(gH)$ describing a first order phase transition. The boundary between regions I and III is the critical curve $\mu_{\text{cr2}}(gH)$ of a second order phase transition. (The curves in the figure are obtained by interpolation in the most simple manner, i.e. by a second order polynomial.) Earlier, in the papers [14] the model (1) at $G_2 = 0$ and in the presence of an external magnetic field was considered. As shown there, for small values of the magnetic field the critical curves $\mu_c(H)$ as well as different thermodynamical and dynamical parameters of the system have an oscillating behaviour. In order to make more accurate interpolations and to become sure whether analogous oscillations appear in the present case, too, (i.e. for the critical curves from Fig.1, or not), one should make an enormous amount of numerical calculations, which proved to be rather difficult to accomplish. Due to this, we can make only a conjecture of an oscillating behavior of the
functions $\mu_{cr}(gH)$ judging from the positions of the points we have really calculated. Note further that in the region of low chromomagnetic fields $gH < 0.2 \text{ GeV}^2$ we have extrapolated the critical curves to the known point at $gH = 0$.

In Fig. 2, the phase portrait of the model in terms of $\mu, gH$ at $T = 0.15 \text{ GeV}$ is presented. The corresponding remarks and explanations are the same as in the case of Fig. 1.

The phase diagrams in the $(T, \mu)$ plane for $gH = 0$ and $gH = 0.4 \text{ GeV}^2$ are schematically represented in Fig. 3. The phase diagram in the $(T, gH)$ plane for $\mu = 0.4 \text{ GeV}$ is represented in Fig. 4. It is necessary to point out that at $gH = 0$ the numerical results of Figs. 1–4 coincide with those obtained in [27]. Moreover, it should be emphasized that in all the above mentioned figures, the second order phase transition takes place at the boundary of the region I. At the boundary between regions II and III a first order phase transition takes place.

Let us for a moment fix the values of the chemical potential and temperature at varying values of $gH$. In this case in the Figs. 1, 2 one will have a straight line along the $gH$ axis. In particular, if $\mu = 0.4 \text{ GeV}$ and $T = 0$, this line originates at $gH = 0$ in the CSC phase III. At some (critical) value $(gH)_c \approx 0.1 \text{ GeV}^2$ it crosses the line $\mu_{cr1}(gH)$ and then, at yet greater values of $gH$, it passes through the phase II (see Fig. 1). Accordingly, at $gH < (gH)_c$ the GMP of the effective potential lies in the point $(\sigma = 0, \Delta \neq 0)$, where $\Delta$ is equal to the diquark condensate in the true stable vacuum, whereas at $gH > (gH)_c$ the point $(0, \Delta \neq 0)$ ceases to be a GMP. In this case it is only a local minimum point, so $\Delta \neq 0$ corresponds to a metastable ground state of the system (for the stable ground state at $gH > (gH)_c$ the GMP is of the form $\sigma \neq 0,\Delta = 0$). Thus, the value $(gH)_c$ is a so-called evaporation point for the diquark condensate.

In Fig. 5 the diquark condensate $\Delta$ is shown as a function of $gH$ for three values of the temperature at $\mu = 0.4 \text{ GeV}$. Thick curves correspond to a stable diquark condensate (the point $(0, \Delta \neq 0)$ is the global minimum of the effective potential), and thin curves correspond to a quasistable diquark condensate (this point is a local minimum). Here we should note that recent investigations yield the following value of the QCD gluon condensate at $T = \mu = 0$: $gH \approx 0.6 \text{ GeV}^2$ [30]. So, assuming that this value might
be extrapolated to $\mu = 0.4$ GeV, $T = 0$, one would conclude from Figs. 1, 5 that the CSC phase does not exist for such a large value of the gluon condensate. In contrast, if $gH = 0$, then there is a CSC phase at $\mu = 0.4$ GeV, $T = 0$, since the order parameter (diquark condensate) takes in this case the nonzero value $\Delta \approx 0.05$ GeV. If one takes $T = 0.1$ GeV and $\mu = 0.4$ GeV, then at sufficiently small values of $gH$ there is a symmetric phase of the theory, where both $< \bar{q}q >$ and $< qq >$ condensates are zero (see Fig. 5). However, at the point $gH \approx 0.1$ GeV$^2$ there is a second order phase transition to the CSC phase (here only $< qq > \neq 0$), and at the point $gH \approx 0.2$ GeV$^2$ the external field destroys the CSC in favour of chiral phase, where $< \bar{q}q > \neq 0$ and $< qq > = 0$.

In Fig. 6, the diquark condensate $\Delta$ is shown as a function of the chromomagnetic field $gH$ for two values of the temperature and for $\mu = 0.8$ GeV. In contrast to the previous Fig. 5, the diquark condensate at $\mu = 0.8$ GeV is stable at least for all values of $gH \in (0, 0.9)$GeV$^2$. First, let us look at the bottom curve of this figure, corresponding to $T = 0.15$ GeV. One can easily see that there is a critical value $gH_c \approx 0.45$ GeV$^2$ such, that at $gH < gH_c$ one has $\Delta = 0$. At $gH > gH_c$ we then get a stable nonzero diquark condensate. Hence, the external chromomagnetic field induces the color superconducting second order phase transition for some values of the temperature, including the value $T = 0.15$ GeV.

Considering the upper curve of Fig. 6, we come to the following important conclusion. At $\mu = 0.8$ GeV, $T = 0$, $gH = 0$, the GMP of the effective potential corresponds to the CSC phase with a stable diquark condensate $\Delta \approx 0.08$ GeV. However, assuming that the value of the gluon condensate $gH \approx 0.6$ GeV$^2$ would hold also for the above nonvanishing chemical potential, one would get a value of the diquark condensate $\Delta \approx 0.44$ GeV, which is significantly larger in magnitude, than at $gH = 0$.

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8In the paper [37] it was shown in the framework of a quark-meson model that at ordinary nuclear density $\rho_0$ the gluon condensate decreases by no more than six percent, compared with its value at zero density. At densities $3\rho_0$ the value of $< FF >$ decreases by fifteen percent. This means that for values of the chemical potential $\mu < 1$ GeV the gluon condensate is a slowly decreasing function vs $\mu$, i.e. it is approximately the same as at $\mu = 0$.

9Recently, a similar prediction, namely that nonperturbative gluon fluctuations might be strong enough to destroy the CSC, was done in [38], but in a rather qualitative form.
As a general conclusion, we see that taking into account an external chromomagnetic field at least in the form as considered in the model above, might, in principle, lead to remarkable qualitative and quantitative changes in the picture of the diquark condensate formation, obtained in the framework of NJL models at $H = 0$. Clearly, a detailed quantitative discussion would, however, require to have a more detailed information on the gluon condensate as function of the temperature and chemical potential. A further interesting generalization could be to extend this kind of approach to inhomogeneous background field configurations [39]. This concerns, in particular, the nonabelian condensate fields of the Stochastic Vacuum Model of QCD realizing Wilson’s area law of confinement [40].

V. SUMMARY AND CONCLUSIONS

In the present paper the influence of different physical factors on the phase structure of the two-flavor NJL model (1) with two independent structures of four-quark interactions has been considered. This model is adequate for the description of the low energy physics of two-flavor QCD both in $q\bar{q}$- and $qq$ channels. In the papers [26]-[28] it was shown that in QCD-motivated type of models (1) with $H = 0$ the new color superconducting (CSC) phase can exist for moderate values of the chemical potential (baryon density). As generalization of the “free field” NJL model of ref. [28], we have, in particular, taken into account such nonperturbative feature of the real QCD vacuum as the nonzero gluon chromomagnetic condensate $< F^{a}_{\mu\nu} F^{a}_{\mu\nu}> \equiv 2H^2$, which in the framework of a NJL model can be simulated as an external chromomagnetic field. The modern estimates give the following fixed value $gH_{phys} \approx 0.6 \text{ GeV}^2$ [30] for the gluon condensate in QCD at $\mu, T = 0$. Despite of this fact, we considered it, however, useful to treat $H$ as a free external parameter of the model.

Since in the CSC phase the original $SU_c(3)$ symmetry of the theory is spontaneously broken down to $SU_c(2)$, five color gauge bosons acquire masses. The corresponding external fields are expelled from the CSC phase (Meissner effect). However, the other three “color isospin” bosons stay massless, in accordance with the residual $SU_c(2)$ symmetry of the vacuum. Clearly, the corresponding external fields may then penetrate into the CSC phase. It is just the influence of these types of external chromomagnetic fields on the formation of CSC which was studied in our previous paper [30] in the frame-
work of a (2+1)-dimensional NJL model for vanishing chemical potential and temperature. There it was shown that chromomagnetic fields may induce the CSC phase transition. In the present paper, the chromomagnetic generation of CSC has been studied in the framework of a (3+1)-dimensional NJL model for finite chemical potential and temperature. The vector-potential of the external chromomagnetic field was chosen to be of the Matinyan-Savvidy form \( (7) \) and lies in the algebra of the residual \( SU_c(2) \) group, too.

The coupling constants \( G_1, G_2 \) of our model (1) are considered as free independent parameters. In our numerical estimates we found it, however, convenient to use the values \( (24) \) in order to "normalize" our calculations at \( H \neq 0 \) on the known results at \( H = 0 \) \[27\]. However, we hope that our qualitative conclusions remain also valid for values of \( G_1, G_2 \) in some neighbourhood of \( (24) \). The results of numerical investigations of the effective potential (22) are presented in a set of Figs 1–6, where phase diagrams for the extended NJL model (1) in terms of \( \mu, T, H \) as well as the behaviour of diquark condensates versus \( gH \) are shown.

The main conclusion of our investigations is that the inclusion of an external chromomagnetic field can significantly change the phase portrait, obtained at \( H = 0 \). Indeed, at \( H = T = 0 \) the values of the chemical potential corresponding to the CSC phase approximately lie in the interval \( 0.3 \text{ GeV} < \mu < 1.2 \text{ GeV} \) (see Fig. 1). If the external chromomagnetic field of the type \( (7) \) is switched on at \( \mu = 0.4 \text{ GeV} \), then at \( gH_c \approx 0.1 \text{ GeV}^2 \) there is a transition of the system from CSC to a phase, where only chiral symmetry is broken down. Thus, at \( gH \approx 0.6 \text{ GeV}^2, \mu = 0.4 \text{ GeV} \) and \( T = 0 \) the CSC can not be observed at all. However, if \( T = 0 \) and the chemical potential is fixed at \( \mu = 0.8 \text{ GeV} \), then for all values \( 0 \leq gH \leq 0.9 \text{ GeV}^2 \) one can observe the CSC phase in which the diquark condensate \( \Delta(H) \) is nonzero. It is worth remarking that in this case the function \( \Delta(H) \) is a monotonically increasing one (see Fig. 6) and the value of the diquark condensate at \( gH_{phys} \) comes out to be significantly greater in magnitude than at vanishing \( H \), i.e. \( \Delta(H_{phys}) \approx 5\Delta(0) \) (see Fig. 6). \[10\]

\[10\] We roughly suppose throughout the present paper that at \( (\mu, T) \neq 0 \) the real gluon condensate is the same as at \( (\mu, T) = 0 \). However, using a given \( \mu, T \)-dependency of the gluon condensate, it would be possible to extract physical informations about the CSC phase using our phase diagrams on Figs. 1-4 and graphics of \( \Delta(H) \) functions of Figs. 5, 6.
Finally, one should note that at $\mu \neq 0, T \neq 0$ the external chromomagnetic field can induce the CSC phase transition. For example, at $T = 0.15$ GeV and $\mu = 0.8$ GeV there is a symmetric phase of the theory in which $\sigma = \Delta = 0$ (both chiral and diquark condensates are zero) for all $0 \leq gH \leq 0.45$ GeV$^2$ (see Fig. 2). In the point $gH \approx 0.45$ GeV$^2$ a phase transition of the second order from the symmetric to the CSC phase is induced by the external chromomagnetic field. Notice that the CSC induction by some types of external chromomagnetic fields was observed in the framework of a NJL model at zero $\mu, T$ (see [30,31]). The present analysis shows that this effect takes place at some nonzero values of $\mu, T$, too.

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APPENDIX:

Let us sketch the calculation of the integral in eq. (18) of the text. Evidently, it can be rewritten as a contour integral

$$\sum_{l} \int_{1/\beta^2}^{\Omega^2} da^2 \frac{1}{\omega_l^2 + a^2} = - \int_{1/\beta^2}^{\Omega^2} da^2 \int_{C_0} \frac{d\omega}{2\pi i} \frac{1}{\omega^2 + a^2} \frac{\beta}{2} \tan \frac{\beta \omega}{2},$$  \hspace{1cm} (A1)

where $\tan(\beta \omega/2)$ has poles inside the integration path $C_0$ (see Fig. 7):

$$\frac{\beta \omega}{2} = \pm \frac{\pi}{2} (2l + 1), \hspace{1cm} l = 0, 1, 2, \ldots$$  \hspace{1cm} (A2)

The integral over $C_0$ is equal to the integral along the contour $C$ (Fig. 8), and hence

$$- \int_{C_0} \frac{d\omega}{2\pi i} \int_{1/\beta^2}^{\Omega^2} da^2 \frac{\beta}{2} \tan \frac{\beta \omega}{2}$$

$$= - \int_{1/\beta^2}^{\Omega^2} da^2 \int_{C} \frac{d\omega}{2\pi i} \frac{\beta}{2} \left( \frac{1}{\omega - ia} - \frac{1}{\omega + ia} \right) \frac{1}{2ia} \tan \frac{\beta \omega}{2}$$
\[ \beta \int_{1/\beta}^{\Omega} \text{d}a \text{th} \frac{\beta a}{2} = 2 \ln \text{ch} \frac{\beta a}{2} \bigg|_{a=1/\beta}^{a=\Omega} = 2 \ln \text{ch} \frac{\Omega/\beta}{2} + \text{const}. \quad (A3) \]
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Figure captions

Fig. 1. Phase portrait of the model in terms of variables \((\mu, gH)\) at \(T = 0\). Regions I, II, III describe the totally symmetric phase \((\sigma = 0, \Delta = 0)\), the phase with broken chiral symmetry \((\sigma \neq 0, \Delta = 0)\) and the color superconducting phase \((\sigma = 0, \Delta \neq 0)\), respectively.

Fig. 2. Phase portrait of the model in terms of variables \((\mu, gH)\) at \(T = 0.15\) GeV.

Fig. 3. Phase portrait of the model in terms of variables \((T, \mu)\) at \(gH = 0\) and \(gH = 0.4\) GeV².

Fig. 4. Phase portrait of the model in terms of variables \((gH, T)\) at \(\mu = 0.4\) GeV.

Fig. 5. Diquark condensate as a function of an external chromomagnetic field for \(\mu = 0.4\) GeV and three values of the temperature, \(T = 0\), \(T = 0.1\) and \(T = 0.15\) GeV.

Fig. 6. Diquark condensate as a function of an external chromomagnetic field for \(\mu = 0.8\) GeV and two different values of the temperature, \(T = 0\) and \(T = 0.15\) GeV.

Fig. 7. Integration path \(C_0\) used in eq. (A1) of Appendix.

Fig. 8. Integration path \(C\) used in eq. (A3) of Appendix.
FIG. 4.

FIG. 5.
FIG. 6.

\[ T = 0 \]

\[ T = 0.15 \text{ GeV} \]

\[ \Delta, \text{ GeV} \]

\[ gH, \text{ GeV}^2 \]

FIG. 7.

FIG. 8.