Quantum coherence is a fundamental resource that quantum technologies exploit to achieve performance beyond that of classical devices. A necessary prerequisite to achieve this advantage is the ability of measurement devices to detect coherence from the measurement statistics. Based on a recently developed resource theory of quantum operations, here we quantify experimentally the ability of a typical quantum-optical detector, the weak-field homodyne detector, to detect coherence. We derive an improved algorithm for quantum detector tomography and apply it to reconstruct the positive-operator-valued measures (POVMs) of the detector in different configurations. The reconstructed POVMs are then employed to evaluate how well the detector can detect coherence using two computable measures. These results shed new light on the experimental investigation of quantum detectors from a resource theoretic point of view.

Introduction. – Quantum coherence plays an indispensable role in quantum technologies including, for example, quantum computation [1, 2], quantum coding [3] and key distribution [4], quantum metrology [5, 6] and quantum biology [7, 8]. Therefore, the quantitative assessment of quantum coherence as a resource has attracted widespread interest [4, 9, 11–13]. Until recently, most of the research concerned with the assessment of quantum coherence as a resource focused on the coherence in quantum states (see [13] for a review). Following approaches in the resource theory of entanglement [14, 15], the coherence properties of quantum operations have also begun to be examined by their ability to create or increase coherence in quantum states [16–19].

However, to exploit quantum coherence for different applications, it is generally insufficient to only create and manipulate coherence: we also must be able to detect coherence in the sense that its presence makes a difference in measurement statistics [2, 20–22, 24]. To quantify how well a measurement can detect coherence, a theoretical framework in the form of a resource theory on the level of operations has been proposed [2], allowing to address this question rigorously (see [25–31] for related work). Other approaches connecting measurements with quantum coherence and resource theories were recently presented in Refs. [32–34].

Following the methods proposed in Ref. [2], here we quantify experimentally the capability of a quantum-optical detector, the weak-field homodyne detector (WFHD) [35, 36], to detect coherence. In contrast to photon-counting detectors that are sensitive to the intensity or particle behavior of input light fields only [37], the WFHD mixes the input light field with a phase-reference field, the local oscillator (LO), and thus is able to measure the wave-like properties of the input field [38]. The difference between a photon counting detector and the WFHD can be further revealed by the matrix representations of their positive-operator-valued measures (POVMs) in photon-number basis: the former is completely diagonal and hence incoherent, while the latter has off-diagonal matrix elements and provides sensitivity to the coherence of the input states. Moreover, the WFHD can be tuned to interpolate continuously between photon-counting (incoherent) measurements and phase-dependent measurements by adjusting the intensity of the LO [39]. This detector has found important applications in not only state detection [40–42] and state discrimination [43, 44], but also for state preparation [45–49].

In this work we investigate how the capability of the WFHD to detect coherence changes with its configuration, in particular, the LO intensity and the mode overlap between the LO and the input state. We develop an improved quantum detector tomography algorithm building on [1, 51] which is of independent interest to reconstruct experimentally the POVMs of the WFHD under different configurations. Building on the reconstructed POVM we then determine two measures, the diamond measure and the non-stochasticity in detection (NSID) measure [2], to quantify its coherence. Both measures have operational interpretations and for the WFHD in this work can be proven to coincide. These results are a first step towards the experimental investigation and benchmarking of quantum measurements from a resource theoretical point of view, thus providing quantitative tools for assessing improved designs of devices making use of quantum advantages.

Detecting coherence. – To quantify how well a measurement can detect coherence, we use the methods developed in Ref. [2]. In the following, we shortly review the parts relevant for this work but refer to the original paper for details. A quantum state $\rho$ is called incoherent with respect to a fixed orthonormal basis $I = \{ |i\rangle \}$ if $\rho$ is a statistical mixture of ele-
ments of $I$, i.e., $\rho = \sum_{i} c_{i} |i\rangle\langle i|$. All other states are coherent. The total dephasing operation $\Delta$, which is defined by

$$\Delta(\rho) = \sum_{i} |i\rangle\langle i| \rho |i\rangle\langle i|,$$

is a resource destroying map [21], i.e., its output is always incoherent and incoherent states are invariant under its action. Then a POVM $\{\Pi_{n} : \Pi_{n} \geq 0, \sum_{n} \Pi_{n} = 1\}$, where $\mathbb{1}$ is the identity operator, cannot detect coherence if its measurement statistics $\rho_{n} = \text{tr}(\rho \Pi_{n})$ is independent of the coherence within the input state, i.e.

$$\text{tr}(\rho \Pi_{n}) = \text{tr}(\Delta(\rho) \Pi_{n}) \quad \forall \rho, n.$$

This result implies that for an incoherent measurement every $\Pi_{n}$ is diagonal in $I$ [2], while if $\Pi_{n}$ has off-diagonal elements the measurement will be able to detect coherence.

Storing measurement outcomes in the incoherent basis of an auxiliary system, every quantum measurement can be represented by a quantum channel, e.g., a POVM $\{\Pi_{n}\}$ by

$$\Theta(\rho) = \sum_{n} \text{tr}(\rho \Pi_{n}) |n\rangle\langle n|.$$

Treating subselection this way, a general quantum operation $\Phi$ cannot detect coherence iff

$$\Delta \Phi = \Delta \Phi \Delta.$$

The set of these detection-incoherent operations is denoted by $D\mathcal{I}$. This allows us to present two well defined and computable functionals quantifying how well an operation can detect coherence [2]; the diamond measure

$$M_{\diamond}(\Theta) = \min_{\Phi \in D\mathcal{I}} \| \Delta(\Theta - \Phi) \|_{\diamond},$$

and the NSID measure,

$$\hat{M}_{\diamond}(\Theta) = \min_{\Phi \in D\mathcal{I}} \| \Delta(\Theta - \Phi) \|_{1}. $$

It is worthwhile to mention that the NSID measure is directly related to the success probability of simulating $\Theta$ by operations that cannot detect coherence. Furthermore, the diamond measure provides an upper bound on the average coherence that can be prepared remotely when the measurement is applied on one part of the maximally entangled bipartite state (see Sec. V of the Supplemental Material (SM) [52] for details). The coherence of a quantum measurement can be evaluated in two steps: map the measurement to a trace-preserving operation, and then calculate the coherence of the operation using Eq. (5) or Eq. (6). While these measures are generally different, remarkably, for channels with output dimension two (two measurement outcomes) we have been able to prove that the two measures coincide (see Sec. VI of the SM [52] for details).

**Experimental setup.**– The quantum detector we investigate here is a weak-field homodyne detector (WFHD) which is tunable with two parameters. Similar to a standard homodyne detector, the WFHD combines the input state with a coherent optical field $|\alpha_{\text{LO}}\rangle$, the LO. Yet the intensity of the LO $|\alpha_{\text{LO}}|^{2}$ in WFHD is reduced to low photon numbers. Therefore, instead of photodiodes, a photon-counting detector, an avalanche photodiode (APD) is used to detect the interference signal. Since the LO acts as a phase reference, a WFHD is a phase sensitive detector, whose properties have been well studied in [35, 36, 38].

In this work we study the coherence in a WFHD, as shown in Fig. 1, under various configurations. Since an APD is a binary detector, there are two outcomes of the detector: no-click (0) and click (1). We fix the ratio of the beam-splitter as 50:50, and set the average photon number of the LO $|\alpha_{\text{LO}}|^{2}$ to five different values 0.5, 1, 2, 3 and 4. For each LO intensity, the degree of the mode overlap between the LO and the input state is chosen to be $M = 99.87\%$, 85.00\%, 74.99\%. Due to the relatively complex theoretical model of the detector and the experimental imperfections that are difficult to calibrate accurately, we apply quantum detector tomography [53, 54] to characterize experimentally the detector for different configurations. Quantum detector tomography (QDT) allows to reconstruct the POVM of an arbitrary quantum detector from the outcome statistics in response to a set of tomographically complete probe states. In this work we use a set of coherent states $|\alpha\rangle$ as the probe states, which can be generated by modulating the amplitude and phase of the output of a laser. The probe states interfere with the LO at a 50:50 beam-splitter with one output mode coupled into a single mode fiber for APD detection. The other output mode can be used for tracking the relative phase between probe and LO states. More details of the experimental setup can be found in the SM [52].

**Experimental results.**– We adopt a two-step method to
Figure 2. Experimentally reconstructed no-click POVM elements of the weak-field homodyne detector with different LO intensities $|\alpha_{LO}|^2$ and mode overlaps $M$. The reflectivity of the beam-splitter is 0.5 and the quantum efficiency of the APD is 59%. The POVMs are reconstructed up to 70 photons and shown up to 15 photons in the figure. For simplicity, only the diagonal and the first off-diagonal of the POVM elements are shown here. The error bars originate from the fluctuations in the preparation of the probe states used for tomography, for details see the SM [52].

quantify the coherence of the WFHD: first we reconstruct the POVM, which will then allow us to evaluate Eq. (5) or Eq. (6) using numeric methods [2]. In principle, recording the statistics of the measurement outcomes for different probe states, the POVM can be estimated by inverting a set of linear equations given by the Born rule. However, taking into account experimental imperfections and statistical fluctuations, the POVM is usually reconstructed using a constrained convex optimization program. Here, we follow this approach and reconstruct the POVMs using an improved QDT method based on [1, 51]. We truncate the Hilbert space at the photon number of 70 which is sufficient to saturate the detector and reconstruct up to the fifth leading diagonals of the POVM elements. The details are given in the SM [52].

Before moving forward to quantitatively evaluate the coherence of the POVMs with the diamond measure and the NSID measure, we first compare the POVMs associated to the different configurations of the WFHD. The reconstructed POVM elements of the no-click outcome are given in Fig. 2. We only present the diagonals and first off-diagonals to elucidate the difference between different POVMs since they are the most significant ones. The three rows (from top to bottom) correspond to three different degrees of mode overlap between the input states and the LO, $M = 99.87\%$, 85.00\%, 74.99\%, respectively. The five columns (from left to right) represent the five different average photon numbers of the LO used ($|\alpha_{LO}|^2$ from 0.5 to 4). The complete POVMs are given in the SM [52].

The off-diagonal elements of the density matrix determine the coherence of a quantum state [4]. Recalling that the matrices of an incoherent POVM are completely diagonal in the incoherent basis, the off-diagonal part of the POVM elements also plays an important role in quantifying the coherence of a measurement. Indeed the diamond measure is bounded from below by the off-diagonal part of the POVM elements [52]. Therefore, we start the discussion by focusing on the off-diagonal parts of the POVM matrices. For a fixed LO intensity, it can be seen in Fig. 2 that the off-diagonal elements decrease with the reduction of the mode overlap $M$ between the input state and the LO, which suggests a reduced capability to detect the coherence of input states. This result can be understood by dividing the LO into two parts, one that overlaps with the input state and the other with the mode orthogonal to that of the input state [55]. The intensities of the two parts are $M|\alpha_{LO}|^2$ and $(1-M)|\alpha_{LO}|^2$ respectively. Only the first part can interfere with the input state and provide a phase reference, resulting in the off-diagonal elements and the capability to detect coherence. The second part plays the role of background noise that can lead to decoherence of the quantum detectors [5]. Therefore, a decrease in the mode overlap implies not only a reduced phase reference but also an increased background noise. The overall effect is a reduced sensitivity to coherence in the input state. In the limit of no mode overlap at all, the WFHD becomes an intensity detector with the LO acting as background noise, leading to a complete loss of its ability to detect coherence.

For a fixed mode overlap, we should distinguish between two cases. For near perfect mode overlap $M = 99.87\%$, all the LO interferes with the input state. A higher LO intensity implies a better phase reference and therefore larger off-diagonals. With increasing LO intensity, the peak of the off-diagonal elements also shifts to higher photon numbers, which
can be understood by considering that interference between two beams shows maximal visibility when they have the same intensity. In the case of the non-unit mode overlap, again the LO plays a dual role: as phase reference (with the intensity $\mathcal{M}|\alpha_{\text{LO}}|^2$) and as phase-independent background noise (with the intensity $(1 - \mathcal{M})|\alpha_{\text{LO}}|^2$). Both effects increase with the intensity of the LO. The competition between the coherent interference and the incoherent background noise explains why the off-diagonals increase first and then decrease with the increase in the LO intensity for a non-unit mode overlap.

Now, we are ready to move on to the second step, evaluating the exact value of the coherence contained in WFHD. The diamond measure as given in Eq. (5) and the NSID measure in Eq. (6) are calculated. Based on the analysis in [2] and the references therein, the diamond measure can be calculated efficiently using a semidefinite program. The evaluation of the NSID measure is more cumbersome but, as mentioned above, has a clearer operational meaning. In our case, however, the two measures coincide (see the SM [52] for the proof, where we also show that both measures are robust against errors in the reconstructed POVM elements). This allows us to use the efficient semidefinite program to evaluate both measures.

The coherence of the experimentally reconstructed POVMs of the WFHD are shown in Fig. 3 by dots and the error bars originate again from the fluctuations in the intensities and phases of the probe states. For comparison, we also simulated the POVM elements by numerically generating the statistics of the measurement outcomes with the configuration parameters of the WFHD in the experiment, then reconstructing the POVM using the simulated data. The corresponding results are shown by asterisks which are linked by segmented lines for fixed mode overlap to show the tendency on the LO intensity clearly. The experimental results match the simulations well. The remaining discrepancy may originate from the inaccuracies in the parameters such as the mode overlap used in the simulation or from additional imperfections in the experiment not taken into consideration. The measures are presented for three values of the mode overlap $\mathcal{M} = 99.87\%$ (blue), 85.00% (red), 74.99% (magenta) and five different intensities of the LO $|\alpha_{\text{LO}}|^2 = 0.5, 1, 2, 3, 4$ per mode overlap. When the intensity of the LO is zero, the WFHD is degenerated into an APD with additional 50% loss at the BS and the coherence is zero, which is also shown in the figure.

The estimated coherence shows a change similar to that in the off-diagonal elements of the POVM. For a fixed intensity of the LO, it is obvious that the coherence decreases with the reduction in the mode overlap, which agrees with the above analysis based on the POVM elements. For a fixed mode overlap, the relation between the coherence and the LO intensity is less obvious and can be explained with the same arguments we used when discussing the off-diagonals of the POVM elements. When the mode overlap is nearly perfect ($\mathcal{M} = 99.87\%$), increasing LO intensity grants a better phase reference, leading to higher coherence. In case of imperfect mode overlap, the dual role of the LO as both background noise and phase reference makes the connection between coherence and LO intensity more subtle: as we can see, the coherence first increases with increasing LO intensity and then decreases. The value of the coherence of a two-outcome measurement is bounded from above by 1 [2], which we nearly reach with increased LO amplitudes and perfect mode overlap.

Conclusions and outlook.—Detecting coherence, a quantum resource at the core of nonclassical effects such as entanglement, is a necessary prerequisite to its exploitation in quantum technologies [20–22]. It is therefore crucial to have detectors that can measure coherence and moreover, to know their performance precisely. Based on a resource theory, in this work, we experimentally demonstrate a method to quantify the capability of a quantum detector to detect coherence. We develop and apply an improved method of quantum detector tomography to reconstruct the measurement operators (POVMs) of a typical quantum detector, the weak-field homodyne detector (WFHD), with different configurations. The reconstructed POVMs are then used to evaluate the coherence of the detector with two well-defined measures. The results elucidate how the LO intensity and its mode overlap with the input state affect the capability of the WFHD to detect coherence.

This work presents the first rigorous experimental and theoretical analysis of one of the main nonclassical aspects, coherence, of quantum operations and detectors in particular. This may lead to an improved design of devices exploiting quantum effects.

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* These three authors contributed equally
† martin.plenio@uni-ulm.de
‡ lijian.zhang@nju.edu.cn

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[52] See Supplemental Material at [URL will be inserted by publisher] for the experiment details and full data analysis.

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Supplemental Material: Experimental quantification of coherence of a tunable quantum detector

Experiment

In the following, we describe our experiment in more detail, see Fig. 4. The local oscillator (LO) as well as the probe beam were both generated by a Ti:sapphire laser (Coherent Mira HP-900) followed by an APE dual Pulse Select to reduce the repetition rate from 76MHz to 1.9MHz with operating central wavelength at λ₀ = 830 nm. The output laser source from APE dual Pulse Select was coupled into a single mode fibre and filtered by a Semrock interference filter with a bandwidth of ∆λ = 10 nm at the output of the fibre.

The total power of the whole system was controlled by an electrically controlled half-wave plate (HWP) and a Glan-Thompson polariser (GT) with a high dynamical range. In order to make the whole system power stable, we implemented an adaptive power controlling system. We used a beam sampler (Thorlabs BSF10-B) after the GT to track the fluctuation of the total power with a power meter during the data acquisition. The tracked signal was used as feedback to the electrically controlled HWP before the GT to alleviate the change of the total power within a very small range (variance of the power fluctuation is below 10⁻⁴). Thereafter, the output of the adaptive system was coherently separated by a group of birefringent calcite beam displacers (BDs). The state on the upper path is regarded as the probe state, while the other one represents the local oscillator (LO). Both paths were sent to a HWP followed by a GT which allows only the horizontal polarization to transmit so that the magnitudes of the probe state and the LO can be dynamically controlled. The two HWPs placed after the GD are used to adjust the polarization of the probe state and the LO so that they can combine at the second BD. The temporal mode overlap can be changed through the adjustment of the angle of the second BD and therefore, the interference visibility can be controlled dynamically.

The combined beams were interfered at the polarised Mach–Zehnder interferometer (PMZI) which is composed of three wave plates (a HWP sits in the middle of two quarter-wave plates (QWP)). The angles of the two QWPs are set to −45° so that the relative phase between the horizontal polarization (H) and the vertical polarization (V) can be changed by the middle HWP in the form of $e^{2i\theta}$, where $\theta$ is the angle of the middle HWP. A HWP with an angle of 22.5° and a polarized beam splitter (PBS) follow these three wave plates to project the phase modulated state onto the basis $(|H\rangle + |V\rangle)/\sqrt{2}$ (transmission through the PBS) and the basis $(|H\rangle − |V\rangle)/\sqrt{2}$ (reflection by the PBS). The output of one port traveled through a set of precalibrated neutral density filters (NDs) and then coupled into a single mode fibre (with 95% coupling efficiency for probe and 93% for LO respectively) leading to an avalanche photodiode detector (APD) (APD Excelitas with mode number SPCM-AQRH-14) for detection. The quantum efficiency of this type of APD is 59% calibrated with the method of detector tomography. The output of the other port of the PBS was sent to an amplified photodiode detector (Thorlabs PDA36A) so that the relative phase can be fitted with detected interference fringes. Fixing the phase of the LO, this leads to POVM elements that are real.

The data acquisition system is built up by a Field Programmable Gate Array (FPGA) system and a computer which can count the number of clicks occurring during a time span of 1.3s. Since the count rate is high enough, for each probe state, it is sufficient to acquire data during one of these 1.3s system cycles. We used probe states with 41 different amplitudes $|\alpha|^2$, and for each amplitude we implemented 45 phases $\theta$ to cover the $2\pi$ range. To investigate how the detectors coherence depends on the mode overlap and the amplitudes of the LO, we used three degrees of mode overlap $\mathcal{M} = 99.87\%, 85.00\%$, and $74.99\%$ and for each mode overlap five different LO amplitudes $(|\alpha_{LO}|^2 = 0.5, 1, 2, 3$ and $4)$.

Description of the reconstruction algorithm

In this section, we describe the methods underlying the reconstruction of the POVM elements $\Pi_n$ in the main text from experimental data. To this end we have developed an approach, that increases the efficiency and robustness of existing protocols [1] and is expected to find applications in detector tomography beyond the specific application in this work. For completeness, we begin with a review of some results in Ref. [1]. Detector tomography is performed by preparing a set of known probe states $\{\rho_m\}$ incident on a quantum detector and observing the detector outcomes. According to the Born rule, the probability to observe outcome $n = 1,...,N$ is given by

$$p_{n|m} = \text{tr} (\rho_m \Pi_n).$$

(7)

In principle, a direct inversion of this equation allows us to determine $\Pi_n$ but unavoidable errors in the estimation of $p_{n|m}$ and the requirement of non-negativity on the POVM elements and the trace preservation of the POVM make the design of a robust procedure based on convex optimisation indispensable. The probe states must satisfy two requirements: they must be tomographically complete and their accurate preparation experimentally feasible. As in Ref. [1], we use a set of Glauber coherent states $|\alpha\rangle$, since they are easy to produce and form an overcomplete basis in state space. To reconstruct $\Pi_n$, one expands both $|\alpha_m\rangle$ (with $\alpha_m = |\alpha_m| e^{i\theta_m}$) and $\Pi_n$ in the Fock basis and truncates the expansion at the number of photons $d − 1$ that saturate
Figure 4. Experimental setup. The probe states and the local oscillator (LO) are both generated by the Pulse Selector (APE dual Pulse Select) and then pass through a power stable system implemented by a group of half-wave plates (HWPs) and a Glan-Thompson polarizer (GT). Afterwards, they are separated by a beam displacer (BD) in order to control their power separately using the following HWPs and GT. After that the two beams are recombined in the second BD and sent to the weak-field homodyne detector built by a polarized March-Zehnder interferometer, one arm of the Polarized Beam Splitter (PBS) is sent to the APD after the power is lowered by a set of precalibrated neutral density filters (ND), and another arm can be used for tracking the relative phase between probe and LO fitted by the interference fringes.

\[ |\alpha_m \rangle = e^{-|\alpha_m|^2/2} \sum_{j=0}^{d-1} \frac{|\alpha_m|^j}{\sqrt{j!}} e^{ij\theta_m} |j\rangle \]  
(8)

and

\[ \Pi_n = \sum_{j,k=0}^{d-1} \pi_n^{j,k} |j\rangle\langle k| \]  
(9)

Plugging into Eq. (7), we find

\[ p_{n|m} = e^{-|\alpha_m|^2} \sum_{j,k=0}^{d-1} \frac{|\alpha_m|^{j+k}}{\sqrt{j!k!}} e^{i(k-j)\theta_m} \pi_n^{j,k} \]  
(10)

These equations can be written in matrix form and used in a semidefinite program to reconstruct \( \Pi_n \), see Ref. [1] starting with Eq. (6) onward. However, with increasing \( d \), the run time and memory requirements to solve this problem quickly render it infeasible. Therefore, Ref. [1] proposed an algorithm that reconstructs the POVM elements recursively in multiple runs: first the diagonal is reconstructed, then the first off-diagonal and so on. Due to the unavoidable fluctuations of the reference phase off-diagonal of the POVM elements decay rapidly so that only a small number of off-diagonal elements need to be reconstructed. This algorithm allows to reconstruct POVMs for much higher \( d \), but it has two drawbacks: first, the POVM elements must be real and second, the constraints used in the semidefinite program are only necessary but not sufficient for positive semidefinite POVM elements. Application of the methods described in [1] on our experimental data typically led the reconstructed POVM...
elements to violate the semidefiniteness constraint. Therefore, we decided to adopt the reconstruction algorithm in the following way:

Here we take a similar approach and reconstruct only \( t \) leading diagonals but in contrast to Ref. [1] we reconstruct these in one go, ensuring that the POVM elements are indeed positive semidefinite, whilst reducing the calculation time nevertheless. In addition, this approach allows us to avoid errors from approximations necessary in the data preprocessing of the algorithm in Ref. [1] and allows for more freedom in the choice of the probe states.

To bring Eq. (10) into matrix form, we arrange the probabilities \( p_{n|m} \) in a matrix \( P_{n,m} = p_{n|m} \). Next we define \( \tilde{\Pi} \) by concatenating the leading off-diagonals of the POVM elements. The first column of \( \Pi \) contains the relevant part of \( \Pi_1 \), i.e., the diagonal of \( \Pi_1 \), followed by the first off-diagonal above the diagonal, the first off-diagonal below the diagonal, the second off-diagonal above the diagonal and so on, until we reach the number of leading diagonals we intend to reconstruct. The other columns of \( \Pi \) are constructed analogously for the remaining POVM elements. Assuming that the off-diagonals we do not intend to reconstruct are zero (which is a good approximation according to the discussion above as long as we take enough off-diagonals into consideration), one can construct from Eq. (10) a matrix \( F \) such that

\[
P = F\tilde{\Pi}.
\]

To reconstruct positive semidefinite POVM elements, we do not invert the above equation but instead follow the usual approach and solve the following optimization problem

\[
\begin{align*}
\text{minimize:} & \quad \| P - F\tilde{\Pi} \| + g(\tilde{\Pi}) \\
\text{subject to:} & \quad \Pi_n \geq 0, \sum_{n=1}^{N} \Pi_n = 1,
\end{align*}
\]

where \( \| \cdot \| \) is the Frobenius norm and, \( g(\tilde{\Pi}) \) a regularization function needed for numerical stability. We split \( g(\tilde{\Pi}) \) into two parts, \( g(\tilde{\Pi}) = g_1(\tilde{\Pi}) + g_2(\tilde{\Pi}) \), with

\[
g_1(\tilde{\Pi}) = \gamma_1 \sum_{n,i} \sqrt{\sum_j |\pi_n^j+i - \pi_n^{j+1}+1|^2}.
\]

This regularization is physically motivated by the fact that a realistic optical detector is lossy, which leads to a certain smoothness in the POVM representation, see Ref. [1] for a detailed discussion. The second regularization function \( g_2(\tilde{\Pi}) \) is used to remove another source of numerical instability stemming from the truncation of the Fock basis: whenever one reconstructs a set of POVM elements, at least one of them will have diagonal elements that differ significantly from zero but which will be truncated. Therefore, the probabilities in Eq. (10) will be underestimated, which in turn leads to an overestimation of the diagonal elements close to the truncation point during the reconstruction process. Since the condition \( \sum_{n} \Pi_n = 1 \) has to be satisfied and in addition \( \sum_{n} p_{n|m} = 1 \) holds true, we can eliminate one column of \( P \) and the corresponding one of \( \tilde{\Pi} \) without changing the reconstruction as long as we replace the completeness constraint by \( \sum_{n} \Pi_n \leq 1 \). If we drop the POVM element with the highest non-zero truncated elements, this should increase the accuracy of the reconstruction process. In our case, we only register click and no-click events, which means that for a certain intensity, the detector will almost always click. Therefore it is favorable to reconstruct the POVM element corresponding to the no-click event, which is expected to be close to zero above truncation (for high enough \( d \)). This is indeed what we find. The remaining numerical fluctuations that occur from the non-zero elements which were truncated can be stabilized by giving a small bias towards smaller diagonal elements. This is also justified by the discussion above and done by choosing

\[
g_2(\tilde{\Pi}) = \gamma_2 \sum_{n} \sqrt{\sum_j |\pi_n^j|^2}.
\]

To reconstruct a POVM, the free parameters \( \gamma_1, \gamma_2 \) have to be fixed. Of course their choice influences the outcome of the reconstruction process, but as it was the case in Ref. [1], numerical testing showed that there exists a parameter range in which the reconstructed POVM is nearly independent of these parameters.

Reconstructed POVM elements

The reconstruction of the POVM elements was done using the method described in Sec. with \( \gamma_1 = 2, \gamma_2 = 1, d = 70, \) and \( t = 6 \). To compute the measures, we then cut the POVM elements at \( d = 50 \) and used the evaluation method described in
Figure 5. Reconstructed POVM elements of a tunable weak-field homodyne detector. From the top to the bottom, the five rows correspond to five different LO intensities from \(|\alpha_{LO}|^2 = 0.5\) to \(|\alpha_{LO}|^2 = 4\). From left to right, the three different columns represent the different interference visibilities from \(\mathcal{M} = 99.87\%\) to \(\mathcal{M} = 74.99\%\). The yellow and teal bars represent error bars stemming from the sampling method described in the main text.

Ref. [2]. Due to the high repetition rate and the long data acquisition time, statistical fluctuations due to the finite number of click/no-click events are extremely small. The main source of error seems to be the fluctuations of intensities and the relative phases of probe states used for tomography: \(|\alpha_m|\) can have fluctuations of up to 3% and the error in \(\theta_m\) is limited by 0.5%.
To estimate the error emerging from this fluctuations of intensities and relative phases of probe states, we drew 105 additional random samples from the potential input states and reconstructed the POVM elements with the same parameters as above. This leads to the error bars of the POVM elements (and the measures).

In the main text, for clarity, we only showed the diagonals and the first off-diagonals of the reconstructed POVM elements. Here, we present the entire POVM elements in Fig. 5. From the top to the bottom, different rows correspond to different intensities of the LO from \(|\alpha_{1,0}|^2 = 0.5\) to \(|\alpha_{1,0}|^2 = 4\).

It is obvious that the magnitude of the diagonals of the POVM elements decreases with increasing LO intensity, which is a consequence of the increased number of photons arriving at the APD. On the other hand, the magnitude of the off-diagonals increases and higher order off-diagonals appear for higher intensities of the LO. From left to right, the columns represent different interference visibilities ranging from \(M = 99.87\%\) to \(M = 74.99\%\). We observe that the off-diagonals decrease with decreasing mode overlap, especially for the higher order off-diagonals. Therefore, the coherence properties of the POVM will be dominated by the first off-diagonal of the POVMs. For a discussion of the origin of these effects we refer the reader to the main text.

We also use the parameters of our weak-field homodyne detector to simulate its POVM elements theoretically. The fidelity between the simulated POVM elements \(\Pi_{\text{simu}}\) and the experimentally reconstructed POVM elements \(\Pi_{\text{rec}}\) is evaluated with the following equation

\[
F = \text{tr}(\sqrt{\Pi_{\text{rec}}^* \Pi_{\text{simu}}} \sqrt{\Pi_{\text{rec}}^* \Pi_{\text{simu}}})^2 / \text{tr}(\Pi_{\text{rec}}^* \Pi_{\text{rec}}) \text{tr}(\Pi_{\text{simu}}^* \Pi_{\text{simu}}).
\]  

(15)

All of the corresponding fidelities are higher than 90\%. This leads to the good match between the corresponding coherence measures presented in the main article.

**Notes on continuity**

In the reconstruction of the detector POVM small errors are unavoidable due to finite measurement statistics, fluctuations of the parameters of the probe states, and technical imperfections. In order to ensure that this translates into small errors of the diamond and the NSID measure, we show that both measures are (Lipschitz) continuous with respect to their argument.

**Proposition 1.** Let us denote by \(\mathcal{P}_1\) and \(\mathcal{P}_2\) two quantum channels with the same in- and output dimensions. Then we have

\[
|M_\diamond(\mathcal{P}_1) - M_\diamond(\mathcal{P}_2)| \leq \|\mathcal{P}_1 - \mathcal{P}_2\|_o.
\]  

(16)

and

\[
|\tilde{M}_\diamond(\mathcal{P}_1) - \tilde{M}_\diamond(\mathcal{P}_2)| \leq \|\mathcal{P}_1 - \mathcal{P}_2\|_1 \leq \|\mathcal{P}_1 - \mathcal{P}_2\|_o.
\]  

(17)

**Proof.** Since the diamond norm is a norm \([3]\), the triangle inequality holds from which follows directly

\[
\|\Theta\|_o - \|\Phi\|_o \leq \|\Theta - \Phi\|_o.
\]  

(18)

Assuming wlog \(M_\diamond(\mathcal{P}_1) \geq M_\diamond(\mathcal{P}_2)\) and defining \(\Xi \in \mathcal{D}\mathcal{I}\) such that

\[
M_\diamond(\mathcal{P}_2) = \min_{\Phi \in \mathcal{D}\mathcal{I}} \|\Delta \mathcal{P}_2 - \Delta \Phi\|_o = \|\Delta \mathcal{P}_2 - \Delta \Xi\|_o,
\]  

(19)

we therefore have

\[
M_\diamond(\mathcal{P}_1) - M_\diamond(\mathcal{P}_2)
\]

\[
= \min_{\Phi \in \mathcal{D}\mathcal{I}} \|\Delta \mathcal{P}_1 - \Delta \Phi\|_o - \|\Delta \mathcal{P}_2 - \Delta \Xi\|_o
\]

\[
\leq \|\Delta \mathcal{P}_1 - \Delta \Xi\|_o - \|\Delta \mathcal{P}_2 - \Delta \Xi\|_o
\]

\[
\leq \|\Delta \mathcal{P}_1 - \Delta \Xi - \Delta \mathcal{P}_2 + \Delta \Xi\|_o
\]

\[
= \|\Delta \mathcal{P}_1 - \Delta \mathcal{P}_2\|_o
\]

\[
\leq \|\Delta\|_o \|\mathcal{P}_1 - \mathcal{P}_2\|_o
\]

\[
= \|\mathcal{P}_1 - \mathcal{P}_2\|_o.
\]  

(20)

Using that

\[
\tilde{M}_\diamond(\mathcal{P}_2) = \min_{\Phi \in \mathcal{D}\mathcal{I}} \|\Delta \mathcal{P}_2 - \Delta \Phi\|_1,
\]  

(21)
we can prove analogously that
\[ |\tilde{M}_\varnothing(\mathcal{P}_1) - \tilde{M}_\varnothing(\mathcal{P}_2)| \leq \|\mathcal{P}_1 - \mathcal{P}_2\|_1 \leq \|\mathcal{P}_1 - \mathcal{P}_2\|_\infty, \]
(22)
where the last inequality follows directly from the definitions of the diamond and trace norm.

This implies that the two measures are (Lipschitz) continuous with respect to their arguments. Assuming that the \(\mathcal{P}_i\) are channels identified with POVMs with two elements \(\{P^n_i\}_{n=1,2}\), we further find
\[
\|\mathcal{P}_1 - \mathcal{P}_2\|_1 = \max_{\psi} \left\| \sum_{n=1,2} \text{tr} \left[ (P^n_1 - P^n_2) |\psi\rangle \langle n| \right] \right\|_1 \\
= \max_{\psi} \sum_{n=1,2} |\text{tr} \left[ (P^n_1 - P^n_2) |\psi\rangle \langle n| \right]| \\
= \max_{\psi} \left| \text{tr} \left[ (P^1_1 - P^2_1) |\psi\rangle \langle \psi| \right] \right| \\
+ |\text{tr} \left[ (\mathbb{1} - P^1_1 + P^2_1) |\psi\rangle \langle \psi| \right]| \\
= 2 \max_{\psi} |\text{tr} \left[ (P^1_1 - P^2_1) |\psi\rangle \langle \psi| \right]| \\
= 2 \sigma_{\text{max}}(P^1_1 - P^2_1) \\
= 2 \|P^1_1 - P^2_2\|_\infty,
\]
(23)
where \(\sigma_{\text{max}}(A)\) denotes the largest singular value of \(A\), which is equal to its spectral norm denoted by \(\|A\|_\infty\). The spectral norm \(\|A\|_\infty\) is upper bounded by the trace norm \(\|A\|_1\), which is the sum of the singular values of \(A\). Therefore we find in this case
\[
|\tilde{M}_\varnothing(\mathcal{P}_1) - \tilde{M}_\varnothing(\mathcal{P}_2)| \leq 2 \|P^1_1 - P^2_2\|_\infty \\
\leq 2 \|P^1_1 - P^2_2\|_1.
\]
(24)

In addition, the diamond norm can be bounded by the trace norm.

A general bipartite state \(|\psi\rangle\) (on the system and the auxiliary system) can be written in the form of its Schmidt decomposition,
\[
|\psi\rangle = \sum_{i=1}^d \lambda_i |\phi_i\rangle \otimes |\xi_i\rangle,
\]
(25)
with \(\lambda_i \geq 0\) and \(\sum_i \lambda_i^2 = 1\). Together with the fact that the trace norm of linear operators \(x\) and \(y\) obeys
\[
\|x \otimes y\|_1 = \|x\|_1 \|y\|_1,
\]
(26)
which follows directly from the singular value decomposition, we find
\[
\|\mathcal{P}_1 - \mathcal{P}_2\|_\infty = \max_{\psi} \left\| \left( (\mathcal{P}_1 - \mathcal{P}_2) \otimes \mathbb{1} \right) |\psi\rangle \langle \psi| \right\|_1 \\
= \max_{\lambda_i, |\phi_i\rangle, |\xi_i\rangle} \left\| \left( (\mathcal{P}_1 - \mathcal{P}_2) \otimes \mathbb{1} \right) \sum_{i,j} \lambda_i \lambda_j |\phi_i\rangle \langle \phi_j| \otimes |\xi_i\rangle \langle \xi_j| \right\|_1 \\
\leq \max_{\lambda_i, |\phi_i\rangle, |\xi_i\rangle} \sum_{i,j} \lambda_i \lambda_j \left\| \left( (\mathcal{P}_1 - \mathcal{P}_2) \otimes \mathbb{1} \right) |\phi_i\rangle \langle \phi_j| \otimes |\xi_i\rangle \langle \xi_j| \right\|_1 \\
= \max_{\lambda_i, |\phi_i\rangle, |\xi_i\rangle} \sum_{i,j} \lambda_i \lambda_j \left\| \left( (\mathcal{P}_1 - \mathcal{P}_2) \otimes \mathbb{1} \right) |\phi_i\rangle \langle \phi_j| \right\|_1 \\
\leq \max_{\lambda_i} \sum_{i,j} \lambda_i \lambda_j \max_{\|x\|_1 \leq 1} \left\| \left( \mathcal{P}_1 - \mathcal{P}_2 \right) x \right\|_1 \\
= d \left\| (\mathcal{P}_1 - \mathcal{P}_2) \right\|_1,
\]
(27)
where \(\max_{\lambda_i, |\phi_i\rangle, |\xi_i\rangle}\) denotes a maximization over the sets \(\{\lambda_i\}, \{|\phi_i\rangle\},\) and \(\{|\xi_i\rangle\}\) appearing in the Schmidt decomposition of \(|\psi\rangle\). Using the above results, for dichotomous POVMs, this allows us to bound
\[
|\tilde{M}_\varnothing(\mathcal{P}_1) - \tilde{M}_\varnothing(\mathcal{P}_2)| \leq 2d \|P^1_1 - P^2_2\|_1.
\]
(28)
A bound on the diamond measure of POVMs

In this section, we derive a bound on the diamond measure of a POVM in terms of the off-diagonal part of the POVM elements. The dual form of the semidefinite program for the diamond measure given in Ref. [2] can be simplified. The optimization problem there is given by

\[\begin{align*}
\text{Primal problem} & \quad \text{minimize: } 2 \| \text{tr}_B(Z) \|_\infty \\
& \quad \text{subject to: } Z \geq J(\Delta \Theta) - W, \\
& \quad [I - \Delta]W = 0, \\
& \quad \text{tr}_B(W) = 1_A, \\
& \quad Z \geq 0, \\
& \quad W \geq 0, \\
\end{align*}\]

\[\begin{align*}
\text{Dual problem} & \quad \text{maximize: } 2 (\text{tr}(J(\Delta \Theta)X) - \text{tr}(Y_2)) \\
& \quad \text{subject to: } X \leq 1_B \otimes \rho : \rho \geq 0, \text{tr}(\rho) = 1, \\
& \quad [I - \Delta]Y_1 - X + 1_B \otimes Y_2 \geq 0, \\
& \quad X \geq 0, \\
& \quad Y_1 = Y_1^\dagger, \\
& \quad Y_2 = Y_2^\dagger. \\
\end{align*}\]

Since \([I - \Delta]Y_1\) has only zeros on the diagonal, the constraint

\[I - \Delta]Y_1 - X + 1_B \otimes Y_2 \geq 0 \tag{30}\]

enforces that each diagonal element of \(X\) is smaller than the corresponding diagonal element of \(1_B \otimes Y_2\), which is also sufficient, since we can choose \(Y_1 = X - 1_B \otimes Y_2\). This could also be derived by first noting that the primal problem is the same if we replace \(W\) by \(\Delta W\) and remove the condition \([I - \Delta] W = 0\).

Therefore, we can simplify the optimization problem to

\[\begin{align*}
\text{Primal problem} & \quad \text{minimize: } 2 \| \text{tr}_B(Z) \|_\infty \\
& \quad \text{subject to: } Z \geq J(\Delta \Theta) - \Delta W, \\
& \quad \text{tr}_B(\Delta W) = 1_A, \\
& \quad Z \geq 0, \\
& \quad \Delta W \geq 0, \\
\end{align*}\]

\[\begin{align*}
\text{Dual problem} & \quad \text{maximize: } 2 (\text{tr}(J(\Delta \Theta)X) - \text{tr}(Y_2)) \\
& \quad \text{subject to: } \Delta [1_B \otimes Y_2] - X \geq 0, \\
& \quad X \geq 0, \\
& \quad Y_2 = Y_2^\dagger. \\
\end{align*}\]

It is now easy to see that the optimal point of \(Y_2\), i.e., \(Y_2^\ast\), can be chosen to be diagonal with \(\langle j | Y_2^\ast | j \rangle = \max_i \langle i | X^\ast | i, j \rangle\).

This allows for an interesting side remark: assume we had \(X^\ast = 1_B \otimes \rho^\ast\). With the above reasoning and using \(\text{tr}_B(J(\Delta \Theta)) = 1_A\) for all Choi states corresponding to trace preserving operations, this would imply that the optimal value is given by

\[2 (\text{tr}(J(\Delta \Theta)X^\ast) - \text{tr}(Y_2^\ast)) = 2 (\text{tr}(J(\Delta \Theta)1_B \otimes \rho^\ast) - \text{tr}(\rho^\ast)) = 2 (\text{tr} [\text{tr}_B(J(\Delta \Theta)) \rho^\ast] - \text{tr}(\rho^\ast)) = 0, \tag{32}\]

independent of \(\Theta\). Therefore, for \(\Theta \notin D\mathcal{L}\), \(X^\ast\) cannot be of this form. More general, and for the same reason, \(X^\ast \neq 1_B \otimes A\).

Coming back to our main purpose, assume we have a POVM with elements \(\{P_n\}_{n=1}^N\), acting on a Hilbert space with dimension \(d\), which we expand in the incoherent basis as

\[P_{ij} = \sum_{n=1}^N |P_{n,ij}|e^{i\phi_{n,ij}} |i\rangle |j\rangle. \tag{33}\]

We define

\[X = \frac{1}{2d} \left[ \frac{1}{d} \sum_{n,i,j} e^{-i\phi_{n,ij}} |n\rangle \langle n| \otimes |i\rangle \langle j| + 1 \right], \tag{34}\]
where \( n \) runs from 1 to \( N \). Note that \( X \) is hermitian, because the POVM elements are. Since \( (\Delta \otimes \mathbb{1})X = X \), we know from Lem. 14 in Ref. [2] that the (normalized) eigenvectors of \( X \) are given by separable states of the form \(|b\rangle \otimes |\phi\rangle\), where we expand \(|\phi\rangle = \sum_i x_i e^{i\xi_i} |i\rangle\) with \( x_i \geq 0 \). From

\[
(b) \otimes (\phi) X (b) \otimes (\phi) = \frac{1}{2d} \left[ \frac{1}{d} \sum_{i,j} e^{-i\phi_{ij}} x_i x_j e^{i(\xi_i - x_i)} + 1 \right]
\]

and

\[
\sum_{i,j} e^{-i\phi_{ij}} x_i x_j e^{i(\xi_j - x_j)} \geq - \sum_{i,j} x_i x_j \geq -d,
\]

\[
\sum_{i,j} e^{-i\phi_{ij}} x_i x_j e^{i(\xi_i - x_i)} \leq \sum_{i,j} x_i x_j \leq d
\]

follows then that the eigenvalues of \( X \) are between 0 and \( 1/d \). This implies \( 0 \leq X \leq \mathbb{1} \otimes \frac{1}{2^T} \), which means that \( X \) together with \( Y_2 \) diagonal and \(|j\rangle Y_2 |j\rangle = \max_{i,j} \langle i,j | X | i,j \rangle \) is a feasible point of the dual problem. From

\[
\text{tr} \left[ J(\Delta \mathcal{P})X \right] = \text{tr} \left[ \left( \sum_{n,i,j} |P_{n,i,j}\rangle \langle n| \otimes |i\rangle \langle j| \right) \left( \frac{1}{2d} \left[ \frac{1}{d} \sum_{n,i,j} e^{-i\phi_{n,i,j}} |n\rangle \langle i| \otimes |j\rangle \langle n| + \mathbb{1} \right] \right) \right]
\]

\[
= \frac{1}{2d} \left[ \frac{1}{d} \sum_{n,i,j} |P_{n,i,j}| e^{i\phi_{n,i,j}} + \sum_{n,i} |P_{n,i}| e^{i\phi_{n,i}} \right]
\]

\[
= \frac{1}{2d} \left[ \frac{1}{d} \sum_{n,i,j} |P_{n,i,j}| + \text{tr} \left[ \sum_n P_n \right] \right]
\]

\[
= \frac{1}{2d} \left[ \frac{1}{d} \sum_{n,i,j} |P_{n,i,j}| + 1 + d \right]
\]

and

\[
\text{tr}[Y_2] = \sum_j \max_i \langle i,j | X | i,j \rangle = \sum_j \max_n \frac{1}{2d} \left[ \frac{1}{d} e^{-i\phi_{n,j}} + 1 \right] = \frac{1}{2d} (1 + d)
\]

follows that

\[
M_\alpha (\mathcal{P}) \geq 2 \left( \text{tr} \left[ J(\Delta \mathcal{P})X \right] - \text{tr}[Y_2] \right) = \frac{1}{d^2} \sum_{n,i \neq j} |P_{n,i,j}|.
\]

This matches our intuition that a POVM can detect coherence well if it has large off-diagonals. If we do our measurement on one half of a bipartite state

\[
|\lambda\rangle = \frac{1}{\sqrt{d}} \sum_b |bb\rangle,
\]

upon outcome \( n \) which appears with probability \( p_n \), the state of the remaining half is given by

\[
\rho_n = \frac{1}{d p_n} \sum_{i,j} P_{n,i,j} |i\rangle \langle j|,
\]

which has \( l_1 \) coherence [4]

\[
C_{l_1}(\rho_n) = \frac{1}{d p_n} \sum_{i \neq j} |P_{n,i,j}|.
\]
Therefore, the average coherence obtained is bounded from above by the ability of the POVM to detect coherence, i.e.,
\[
\frac{1}{d} \sum_{n} p_n C_1(\rho_n) = \frac{1}{d^2} \sum_{i \neq j} |P_{n,j}| \leq M_{\omega}\left(\bar{P}\right).
\] (43)

This result should be compared to the results presented in Ref. [5], where it was shown that the nonclassicality of a single mode state which has been prepared by a measurement on the other half of a maximally entangled two mode state depends directly on the nonclassicality of that measurement. This can be seen as localization of coherence, since we transform entanglement into local coherence.

**Equivalence of NSID and diamond measure for output dimension two**

In this section we prove that the NSID and the diamond measure are equal on channels with output dimension two. To do this, we need the following Lemma.

**Lemma 2.** For every hermiticity preserving linear map $\mathcal{X}$ with output dimension two that is either trace preserving or has only outputs with trace zero, there exists an orthonormal basis $B$ such that
\[
\Delta \mathcal{X} = \Delta \mathcal{X} \Delta_B,
\] (44)
where $\Delta_B$ denotes total dephasing in $B$. For higher dimensional output, this is not true in general.

**Proof.** Let $B = \{ |\phi_j\rangle \}$ and remember that every hermiticity preserving linear map $\mathcal{X}$ can be expanded as
\[
\mathcal{X}(x) = \sum_{n} \lambda_n K_n x K_n^\dagger
\] (45)
with $\lambda_n = \pm 1$. From this follows
\[
\begin{align*}
\Delta \mathcal{X} &= \Delta \mathcal{X} \Delta_B \\
\Leftrightarrow \sum_{n,i} \lambda_n |i\rangle\langle i| K_n |k\rangle\langle l| K_n^\dagger |i\rangle\langle i| \\
&= \sum_{n,i,j} \lambda_n |i\rangle\langle i| K_n |\phi_j\rangle \langle \phi_j| K_n^\dagger |i\rangle\langle i| \quad \forall k, l \\
\Leftrightarrow \sum_{n} \lambda_n \langle l| K_n^\dagger |i\rangle\langle i| K_n |k\rangle \\
&= \sum_{n,j} \lambda_n \langle l| K_n^\dagger |i\rangle\langle i| K_n |\phi_j\rangle \langle \phi_j| K_n^\dagger |k\rangle \quad \forall k, l, i \\
\Leftrightarrow \mathcal{X}^\dagger (|i\rangle\langle i|) &= \Delta_B \mathcal{X}^\dagger (|i\rangle\langle i|) \quad \forall i,
\end{align*}
\] (46)
where $\mathcal{X}^\dagger$ is the dual of $\mathcal{X}$ with respect to the Hilbert-Schmidt inner product. Defining
\[
x_i := \mathcal{X}^\dagger (|i\rangle\langle i|),
\] (47)
this in turn is equivalent to the statement that all $x_i$ are diagonal in a common orthonormal basis. Assuming that $\mathcal{X}$ has outputs of dimension two, this holds true if $x_0$ and $x_1$ commute, i.e.,
\[
\begin{align*}
0 &= x_0 x_1 - x_1 x_0 \\
&= \mathcal{X}^\dagger (|0\rangle\langle 0|) \mathcal{X}^\dagger (|1\rangle\langle 1|) - \mathcal{X}^\dagger (|1\rangle\langle 1|) \mathcal{X}^\dagger (|0\rangle\langle 0|) \\
&= [\mathcal{X}^\dagger (\mathbb{I}) - \mathcal{X}^\dagger (|1\rangle\langle 1|)] [\mathcal{X}^\dagger (\mathbb{I}) - \mathcal{X}^\dagger (|0\rangle\langle 0|)] \\
&- \mathcal{X}^\dagger (|1\rangle\langle 1|) \mathcal{X}^\dagger (|0\rangle\langle 0|) \\
&= [\mathcal{X}^\dagger (\mathbb{I})]^2 - \mathcal{X}^\dagger (\mathbb{I}) \mathcal{X}^\dagger (|0\rangle\langle 0|) - \mathcal{X}^\dagger (|1\rangle\langle 1|) \mathcal{X}^\dagger (\mathbb{I}).
\end{align*}
\] (48)

In case $\mathcal{X}$ is trace preserving, $\mathcal{X}^\dagger (\mathbb{I}) = \mathbb{I}$ and we find indeed
\[
\begin{align*}
[\mathcal{X}^\dagger (\mathbb{I})]^2 - \mathcal{X}^\dagger (\mathbb{I}) \mathcal{X}^\dagger (|0\rangle\langle 0|) - \mathcal{X}^\dagger (|1\rangle\langle 1|) \mathcal{X}^\dagger (\mathbb{I}) \\
&= \mathbb{I} - \mathcal{X}^\dagger (|0\rangle\langle 0|) - \mathcal{X}^\dagger (|1\rangle\langle 1|) = 0.
\end{align*}
\] (49)
If $\mathcal{X}$ has only output with trace zero, $\mathcal{X}^\dagger(\mathbb{I}) = 0$, which also implies that Eq. 48 holds true. In case the output is higher dimensional, starting from Eq. (46), it is easy to find counterexamples.

This allows us to prove the promised statement.

**Proposition 3.** On operations with output dimension two, the diamond measure and the NSID measure are equal.

**Proof.** Let us first remember that the diamond measure is given by

$$M_\diamond(\Theta) = \min_{\Phi \in DI} \|\Delta \Theta - \Delta \Phi\|_\diamond$$

and the NSID measure by

$$\tilde{M}_\diamond(\Theta) = \min_{\Phi \in DI} \|\Delta \Theta - \Delta \Phi\|_1$$

Since both $\Theta$ and $\Phi$ are trace preserving operations, the output of $\Theta - \Phi$ has always trace zero. Therefore, according to Lem. 2, there exists an orthonormal basis $B$ such that $\Delta (\Theta - \Phi) = \Delta (\Theta - \Phi) \Delta_B$. From this follows that for every $\Phi$,

$$\|\Delta \Theta - \Delta \Phi\|_\diamond = \max_\rho \|\Delta (\Theta - \Phi) \otimes \mathbb{I}\rho\|_1$$

From Lem. 14 in [2], we know that $\sigma = \Delta_B \otimes \mathbb{I}\rho$ has separable eigenvectors, and, more concretely, with $B = \{|c_i\rangle\}$,

$$|\sigma| = \sum_{i,j} q_{ij} p_i |c_i\rangle\langle c_i| \otimes |\phi_j\rangle\langle \phi_j|.$$  

Using convexity, this allows us to deduce

$$\max_{\sigma = \Delta_B \otimes \mathbb{I}\rho} \|\Delta (\Theta - \Phi) \otimes \mathbb{I}\sigma\|_1$$

Putting everything together, we therefore proved

$$\|\Delta \Theta - \Delta \Phi\|_\diamond \leq \max_i \|\Delta (\Theta - \Phi) |c_i\rangle\langle c_i|\|_1$$

$$\leq \max_\rho \|\Delta (\Theta - \Phi) \rho\|_1$$

$$= \|\Delta \Theta - \Delta \Phi\|_1.$$  

(55)
The reverse inequality follows directly from the definition of the diamond norm, which implies

$$\|\Delta \Theta - \Delta \Phi\|_\diamond = \|\Delta \Theta - \Delta \Phi\|_1.$$  \hspace{1cm} (56)

Since this holds for every $\Phi$, the proof is completed.

* These three authors contributed equally
† martin.plenio@uni-ulm.de
‡ lijian.zhang@nju.edu.cn

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