ON THE SIGNATURE OF CERTAIN INTERSECTION FORMS

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ABSTRACT. We prove a conjecture of Zuber on the signature of intersection forms associated with affine algebras of type A.

\section{Introduction}

Let \( N \geq 2 \) be a positive integer and \( \lambda' = (\lambda'_1, ..., \lambda'_{N-1}), 0 < \lambda'_i < 1, i = 1, 2, ..., N - 1 \) with \( \sum_{0<i<N} \lambda'_i < 1 \). Define

\[
p_i(\lambda') := \lambda'_i + \cdots + \lambda'_{N-1} - \frac{1}{N} \sum_{0<j<N} j\lambda'_j, i = 1, 2, ..., N - 1
\]

and

\[
p_N(\lambda') := -\frac{1}{N} \sum_{0<j<N} j\lambda'_j.
\]

Define

\[
q_i(\lambda') := -Np_i(\lambda') + \frac{N + 1}{2} - i
\]

and

\[
g_i(\lambda') := (-1)^i \prod_{r=1}^{N} 2\cos(\pi(p_r(\lambda') - \frac{i}{N}))
\]

for \( i = 1, 2, ..., N \).

If \( S \) is a finite sequence of real numbers, we define \( b_+(S) \) (resp. \( b_-(S), b_0(S) \)) to be the number of positive (resp. negative, zero) elements in \( S \). Let \( a(S) := b_+(S) - b_-(S) \) and denote by \( Q_{\lambda'}, G_{\lambda'} \) the following two sets:

\[
Q_{\lambda'} := \{\cos(\pi q_1(\lambda')), ..., \cos(\pi q_N(\lambda'))\},
\]

\[
G_{\lambda'} := \{g_1(\lambda'), ..., g_N(\lambda')\}.
\]

Notice that since \( \cos(\pi q_i(\lambda')) > 0 \) iff \( q_i(\lambda') \in ]2p - \frac{1}{2}, 2p + \frac{1}{2}[ \) for some integer \( p \), \( b_+(Q_{\lambda'}) \) is much easier to calculate than \( b_+(G_{\lambda'}) \) and the same is true for \( b_- \)'s. The main theorem in this paper is the following:

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Theorem 1. Let $\lambda' = (\lambda'_1, ..., \lambda'_{N-1})$ be as above. Then:

$$b_+(Q_{\lambda'}) = b_+(G_{\lambda'}), b_-(Q_{\lambda'}) = b_-(G_{\lambda'}), b_0(Q_{\lambda'}) = b_0(G_{\lambda'})$$

This theorem implies Zuber’s conjecture about the signature of intersection forms associated with affine algebras of type A (cf.[Z1]) which is the motivation of this paper. Note that $b_0(Q_{\lambda'}) = b_0(G_{\lambda'})$ is already noticed in a slightly different form in [Z1].

Zuber’s conjecture appeared as Conjecture 2.5 of [Z1]. It is based on the mysterious connections between integrable models with two supersymmetries (N=2) in two dimensions (cf.[CV]) and the class of graphs constructed in [Z1] (see also [Z2]). In the special case when the graphs are regular (cf. §2.3), the conjecture can be proved (cf. Page 14 of [GZV]) by combining the results of [GZV] and [S]. In fact, in [GZV], a connection between regular graphs and singularity theory is established, and combine with [S] which is based on mixed Hodge structures, gives a rather indirect proof of Zuber’s conjecture in the special case when the graphs are regular. However, for other graphs in [Z1] (see also [X]), the connection with singularity theory, if any, is not clear at all. We also failed to prove Zuber’s conjecture by using the connection with [CV] as mentioned in [Z1].

We came to the realization that a statement as in theorem 1 may be true by first observing lemma 1 (cf. §2.1) which was already noticed in a slightly different form in [Z1]. We then checked that theorem 1 is true explicitly in the case when $N = 3, 4$ and some other cases which motivated us to give a general proof.

The idea of the proof of theorem 1 is as follows. When $\lambda'$ changes, $b_+(G_{\lambda'})(\text{resp. } b_-(G_{\lambda'}))$ may change only if some of $g_i(\lambda')$’s become 0 or change its sign, i.e., $g_i(\lambda')$ intersect the hyperplanes on which $g_i(\lambda') = 0$. By lemma 1 of §2.1, these hyperplanes are the same as the hyperplanes on which some $q_j(\lambda')$’s lie in $\mathbb{Z} + \frac{1}{2}$. Consider the domain $D := \{\lambda' = (\lambda'_1, ..., \lambda'_{N-1})|0 < \lambda'_i < 1, \sum_{0<i<N} \lambda'_i < 1\}$. $D$ is separated by the above hyperplanes into disjoint open regions. In each open region, the set of numbers compared in theorem 1 should be completely determined. In §2.1, we determine these numbers in a given open region and find that they miraculously satisfy theorem 1. In §2.2, we show that theorem 1 also holds for any $\lambda' \in D$ which is on the boundary of the open region: this follows from §2.1 and lemma 1. In §2.3, after introducing Zuber’s conjecture, we show how theorem 1 implies that the conjecture is true. In §3, we present our conclusions and questions.

2. The Proof

We shall use the notations of §1. Recall

$$D = \{\lambda' = (\lambda'_1, ..., \lambda'_{N-1})|\lambda'_1 > 0, \sum_{0<i<N} \lambda'_i < 1\}.$$ For $\lambda' \in D$, recall

$$p_i(\lambda') = \lambda'_1 + ... + \lambda'_{N-1} - \frac{1}{N} \sum_{0<j<N} j\lambda'_j$$

$$= \frac{1}{N}[-\lambda'_1 - 2\lambda'_2 - ... - (i-1)\lambda'_{i-1}) + (N-i)\lambda'_i + ... + \lambda'_{N-1}]$$
for \( i = 1, 2, \ldots, N - 1 \). We have:

\[
p_i(\lambda') > \frac{1}{N} \left[ (-\lambda'_1 - 2\lambda'_2 - \ldots - (i-1)\lambda'_{i-1}) \right]
\]

\[
> -\frac{1}{N} (i-1) \sum_{0<j<N} \lambda'_j > -\frac{1}{N} (i-1)
\]

, and

\[
p_i(\lambda') < \frac{1}{N} \left[ (N-i)\lambda'_i + \ldots + \lambda'_{N-1} \right]
\]

\[
< \frac{1}{N} (N-i) \sum_{0<j<N} \lambda'_j < \frac{1}{N} (N-i)
\]

Similarly one can show \( \frac{1}{N} (1-N) < p_N(\lambda') < 0 \). So we have:

\[
\frac{1}{N} (1-i) < p_i(\lambda') < \frac{1}{N} (N-i)
\]

, \( i = 1, 2, \ldots, N \). If for some \( i \), \( q_i(\lambda') = -Np_i(\lambda') + \frac{1}{2} (N+1) - i = j + \frac{1}{2} \) with \( j \in \mathbb{Z} \), then

\[
i - \frac{1}{2} N < i + j < i + \frac{1}{2} N - 1
\]

Define \( 0 < r_{\lambda'}(i) < N+1 \) to be the unique integer such that \( \frac{1}{N} (r_{\lambda'}(i) + j + i) \in \mathbb{Z} \). In fact, if \( j + i < 0, r_{\lambda'}(i) = -(j+i) \), if \( 0 \leq j + i < N, r_{\lambda'}(i) = N - (j+i) \), and if \( N \leq (j+i) < \frac{3}{2} N, r_{\lambda'}(i) = 2N - (j+i) \). It follows that

\[
g_{r_{\lambda'}(i)}(\lambda') = 0
\]

We have the following lemma which, in a slightly different form also appeared on Page 17 of [Z1].

**Lemma 1.** For any \( \lambda' \in D \), the map \( i \to r_{\lambda'}(i) \) defined above from \( \{i| q_i(\lambda') \in \mathbb{Z} + \frac{1}{2}\} \) to \( \{r| g_r(\lambda') = 0\} \) is a one to one and onto map. Moreover \( q_i(\lambda') \in \mathbb{Z} + \frac{1}{2} \) iff \( g_{r_{\lambda'}(i)}(\lambda') = 0 \) and \( r_{\lambda'}(i) \) depends only on \( i \) and \( q_i(\lambda') \).

**Proof.** By using definitions we have

\[
g_r(\lambda') = (-1)^r \prod_{i=1}^{N} 2\sin \left( \frac{\pi}{N} (q_i(\lambda') + i + r - \frac{1}{2}) \right)
\]

If \( g_r(\lambda') = 0 \), then there exists \( i := i_{\lambda'}(r) \) such that

\[
\frac{1}{N} (q_i(\lambda') + i + r - \frac{1}{2}) \in \mathbb{Z}
\]
Let \( j \in \mathbb{Z} \) with \( q_i(\lambda')+i+r-\frac{1}{2} = j+i+r \), then \( q_i(\lambda') = -Np_i(\lambda')+\frac{1}{2}(N+1)-i \in \mathbb{Z}+\frac{1}{2} \). Using the fact that \( 0 < |p_a-p_b| < 1 \) for any \( 1 \leq a \neq b \leq N \), it is easy to see that such an \( i := i_{\lambda'}(r) \) is also unique. It is then easy to check that the map \( i \to r_{\lambda'}(i) \) and \( r \to i_{\lambda'}(r) \) are inverse to each other. The rest of the lemma follows from the definitions of \( r_{\lambda'}(i) \).

Q.E.D.

Let \( \beta_i := \frac{1}{2}N - i + \gamma_i \) with \( \gamma_i \in \mathbb{Z}, i = 1, 2, ..., N \). Let \( D_\gamma := \{ \lambda' \in D | \beta_i < Np_i(\lambda') < \beta_i + 1, i = 1, 2, ..., N \} \). \( D_\gamma \) will be called open regions. It is clear that if \( \gamma \neq \gamma' \), then \( D_\gamma \cap D_{\gamma'} = \emptyset \). Notice that \( q_i(\lambda') \in \mathbb{Z} + \frac{1}{2} \) iff \( \lambda' \) lies on the boundary of some \( D_\gamma \), and by lemma 1, \( g_{r_{\lambda'}(i)}(\lambda') = 0 \) iff \( \lambda' \) lies on the boundary of some \( D_\gamma \).

Suppose \( D_\gamma \neq \emptyset \) and \( \lambda' \in D_\gamma \). Then we have:

1. If \( i < j \), then \( \beta_i \geq \beta_j \) which follows from the fact that \( i < j \), then \( p_i(\lambda') > p_j(\lambda') \) and \( \beta_i - \beta_j \in \mathbb{Z} \);
2. \( \beta_1 - \beta_N \leq N \) which follows from \( p_1(\lambda') - p_N(\lambda') < 1 \).

By (1) we can assume that

\[
\beta_1 = \ldots = \beta_{i_1} > \beta_{i_1+1} = \ldots = \beta_{i_2} > \ldots > \beta_{i_{t-1}+1} = \ldots = \beta_{i_t}
\]

where \( 1 \leq i_1 < i_2 < \ldots < i_t = N \). We determine the sign of \( g_r(\lambda') \) for a fixed \( 1 \leq r \leq N \). Since \( 0 < p_1(\lambda') - p_N(\lambda') < 1 \), there is at most one \( k + \frac{1}{2} \) with \( k \in \mathbb{Z} \) such that

\[
p_N(\lambda') - \frac{r}{N} < k + \frac{1}{2} < p_1(\lambda') - \frac{r}{N}
\]

Also notice that if

\[
\frac{\beta_i - r}{N} < k + \frac{1}{2} < \frac{\beta_i - r + 1}{N}
\]

then we have

\[
\gamma_i - i - \frac{N}{2} < r < \gamma_i - i - \frac{N}{2} + 1
\]

which is impossible since \( r, \gamma_i \) are integers. So if there is a \( k \in \mathbb{Z} \) such that

\[
p_N(\lambda') - \frac{r}{N} < k + \frac{1}{2} < p_1(\lambda') - \frac{r}{N}
\]

then there is a unique integer, denoted by \( 1 \leq f(r) \leq t - 1 \), such that:

\[
\frac{\beta_{i_{f(r)+1}} + 1 - r}{N} \leq k + \frac{1}{2} \leq \frac{\beta_{i_{f(r)}} - r}{N}
\]

and the sign of \( g_r(\lambda') \) is:

\[
(-1)^r(-1)^{kk'}(-1)^{(k-1)(N-k')} = (-1)^{r+Nk-i_{f(r)}}
\]

where \( k' := \{ \beta_j : \beta_j < \beta_{f(r)} \} = N - i_{f(r)} \). If there is no \( k_1 \in \mathbb{Z} \) such that

\[
p_N(\lambda') - \frac{r}{N} < k_1 + \frac{1}{2} < p_1(\lambda') - \frac{r}{N}
\]
then there is a $k \in \mathbb{Z}$ such that:
\[
k - \frac{1}{2} \leq \frac{\beta_N - r}{N} < \frac{\beta_1 + 1 - r}{N} \leq k + \frac{1}{2}
\]
and the sign of $g_r(\lambda')$ is:
\[
(1)^{r + kN}
\]
. We define $f(r) = t$ in this case. Let $s := r + kN$, then the signs of the set $\{g_r(\lambda')\}$ with $1 \leq f(r) \leq t - 1$ are given by
\[
(1)^{s - i_{f(r)}}
\]
with $\gamma_{i_{f(r)+1}} + 1 - i_{f(r)+1} \leq s \leq \gamma_{i_{f(r)} - i_{f(r)}}$, and the sign of the set $\{g_r(\lambda')\}$ with $f(r) = t$ is given by
\[
(1)^{s - i_t}
\]
with $\gamma_{i_1} + 1 - i_1 - N \leq s \leq \gamma_{i_t} - i_t$. Now we determine the sign of $\cos(\pi q_i(\lambda'))$.
Recall $\beta_i = \frac{N}{2} - i + \gamma_i$, $q_i(\lambda') = -Np_i(\lambda') + \frac{N+1}{2} - i$ and $\beta_i < Np_i(\lambda') < \beta_i + 1$, we have
\[
-\gamma_i - \frac{1}{2} < q_i(\lambda') < -\gamma_i + \frac{1}{2}
\]
. So $\cos(\pi q_i(\lambda')) > 0$ (resp. $\cos(\pi q_i(\lambda')) < 0$) iff $\gamma_i \in 2\mathbb{Z}$ (resp. $\gamma_i \in 2\mathbb{Z} + 1$). Recall from the introduction we have that for a finite sequence $S$ of real numbers $a(S) = b_+(S) - b_-(S)$. To save some writing for any integer $x$ we define $\{x\} := \frac{1 - (-1)^x}{2}$.

Then the $a$ of the following sequence $\{\cos(\pi q_i(\lambda'))i_{u-1} + 1 \leq i \leq i_u\}$ is
\[
(1)^{\gamma_{i_u}} \{i_u - i_{u-1}\}
\]
and the $a$ of the following sequence $\{(-1)^{s - i_u}, \gamma_{i_{u+1}} + 1 - i_{u+1} \leq s \leq \gamma_{i_u} - i_u\}$ is
\[
(1)^{\gamma_{i_u}} \{\gamma_{i_u} - \gamma_{i_{u+1}} + i_{u+1} - i_u\}
\]
, where we define $i_{u-1} = 0$ if $u = 1$, and $\gamma_{i_{u+1}} - i_{u+1} = \gamma_{i_1} - i_1 - N$ if $u = t$. It follows that $a(G_{\lambda'}) - a(Q_{\lambda'})$ is given by
\[
\sum_{u=1}^{t} (-1)^{\gamma_{i_u}} \{\gamma_{i_u} - \gamma_{i_{u+1}} + i_{u+1} - i_u\} - \{i_u - i_{u-1}\}
\]
\[
= (-1)^{\gamma_{i_1}} \{\gamma_{i_1} - \gamma_{i_2} + i_2 - i_1\} - \{i_1\} +
(1)^{\gamma_{i_2}} \{\gamma_{i_2} - \gamma_{i_3} + i_3 - i_2\} - \{i_2 - i_1\} +
\vdots + (-1)^{\gamma_{i_{t-1}}} \{\gamma_{i_{t-1}} - \gamma_{i_t} + i_t - i_{t-1}\} - \{i_{t-1} - i_{t-2}\} +
(1)^{\gamma_{i_t}} \{\gamma_{i_t} - \gamma_{i_1} + i_1\} - \{i_t - i_{t-1}\}
\]

. By using
\[
\{\gamma_{i_u} - \gamma_{i_{u+1}} + i_{u+1} - i_u\} = \{\gamma_{i_u} - \gamma_{i_{u+1}}\} + (-1)^{\gamma_{i_u} - \gamma_{i_{u+1}}} \{i_{u+1} - i_u\}
\]
which follows easily from the definition of \( \{ \cdot \} \), we see that \( \pm \{ i_{u+1} - i_u \} \) terms cancelled each other in the above summation and the remaining terms are:

\[
(-1)^{\gamma_{i1}} \{ \gamma_{i1} - \gamma_{i2} \} + (-1)^{\gamma_{i2}} \{ \gamma_{i2} - \gamma_{i3} \} + \ldots + (-1)^{\gamma_{i_l}} \{ \gamma_{i_l} - \gamma_{i1} \}
\]

which is also 0 since \( \{ x \} = \frac{1 - (1)^x}{2} \). So we have shown that

\[
a(G_{\lambda'}) - a(Q_{\lambda'}) = 0
\]

i.e.,

\[
b_+(G_{\lambda'}) - b_-(G_{\lambda'}) = b_+(Q_{\lambda'}) - b_-(Q_{\lambda'})
\]

Since

\[
b_+(G_{\lambda'}) + b_-(G_{\lambda'}) = N = b_+(Q_{\lambda'}) + b_-(Q_{\lambda'})
\]

it follows that theorem 1 is true for \( \lambda' \in D_\gamma \).

2.2. The boundary case. Assume \( \lambda' \in D \) and \( \lambda' \) is on the boundary of some \( D_\gamma \).

Assume \( \{ q_i(\lambda') \} \} \) \( q_i(\lambda') \in \mathbb{Z} + \frac{1}{2} \} = \{ q_{k_1}(\lambda'), \ldots, q_{k_s}(\lambda'), s \geq 1 \} \). Let \( k_i \to r_\lambda(k_i) \) be as in lemma 1. We can choose a small neighborhood \( W \) of \( \lambda' \) such that for any \( \mu \in W \) and \( l \neq k_i, i = 1, \ldots, s \) (resp. \( m \neq r(k_i), i = 1, \ldots, s \) ), \( \cos(\pi q_i(\mu)) \) (resp. \( g_m(\mu) \)) has the same sign as \( \cos(\pi q_i(\lambda')) \) (resp. \( g_m(\lambda') \)) since \( \cos(\pi q_i(\lambda')) \) (resp. \( g_m(\lambda') \)) is not zero. Let \( \mu_1 \in D_\gamma \cap W \). We compare \( b_+(G_{\lambda';\mu_1}) \) (resp. \( b_+(Q_{\lambda';\mu_1}) \)) with \( b_+(G_{\mu_1}) \) (resp. \( b_+(Q_{\mu_1}) \)). Since \( b_+(G_{\mu_1}) = b_+(Q_{\mu_1}) \) by §2.1, to prove \( b_+(G_{\lambda'}) = b_+(Q_{\lambda'}) \) we just have to show that if \( \cos(\pi q_i(\mu_1)) > 0 \) for some \( k_i, 1 \leq i \leq s \), then \( g_{r_\lambda(k_i)}(\mu_1) > 0 \) and vice versa. Let us consider a small line segment with end points \( \mu_1, \mu_2 \) which passes from \( D_\gamma \) to its neighbor \( D_\gamma' \), intersects the hyperplane \( q_{k_i}(\mu) = q_{k_i}(\lambda') \) at \( \mu_0 \), and does not intersect any other hyperplanes. Then we have \( \cos(\pi q_{k_i}(\mu_0)) = 0 \), so by lemma 1, \( g_{r_\lambda(k_i)}(\mu_0) = 0 \). Again by lemma 1, \( r_{\mu_0}(k_i) \) depends only on \( k_i \) and \( q_{k_i}(\mu_0) = q_{k_i}(\lambda') \), so \( r_{\mu_0}(k_i) = r_{\lambda}(k_i) \). As \( \mu \) goes from \( \mu_1 \) to \( \mu_2 \) on the above line segment, \( \cos(\pi q_{k_i}(\mu)) \), \( g_{r_\lambda(k_i)}(\mu) \) change their signs while the signs of all other \( \cos(\pi q_{k_i}(\mu)) \), \( g_{j}(\mu) \)'s do not change. By §2.1, \( b_+(Q_{\mu_1}) = b_+(Q_{\mu_1}) \), \( l = 1, 2 \), it follows that if \( \cos(\pi q_{k_i}(\mu_1)) > 0 \) for some \( k_i, 1 \leq i \leq s \), then \( g_{r_\lambda(k_i)}(\mu_1) > 0 \) and vice versa. So we have proved that \( b_+(G_{\lambda'}) = b_+(Q_{\lambda'}) \), and since \( b_0(G_{\lambda'}) = b_0(Q_{\lambda'}) \) by lemma 1, and both \( G_{\lambda'} \) and \( Q_{\lambda'} \) have \( N \) elements, theorem 1 is proved for \( \lambda' \in D \) which lies on the boundary of some \( D_\gamma \).

By §2.1, §2.2, theorem 1 is proved.

2.3. Zuber’s conjecture. To describe Zuber’s conjecture, we have to introduce some notations from [Z1] to which the reader is referred for more details.

Let \( \Lambda_1, \ldots, \Lambda_{N-1} \) be the fundamental weights of \( SL(N) \). Let \( k \in \mathbb{N} \). Recall that the set of integrable weights of the affine algebra \( SL(N) \) at level \( k \) is the following subset of the weight lattice of \( SL(N) \):

\[
P_{++}^{(h)} = \{ \lambda = \lambda_1 \Lambda_1 + \ldots + \lambda_{N-1} \Lambda_{N-1} | \lambda_i \in \mathbb{N}, \lambda_1 + \ldots + \lambda_{N-1} < h \}
\]
where \( h = k + N \). This set admits a \( \mathbb{Z}_N \) automorphism generated by

\[
\sigma : \lambda = (\lambda_1, \lambda_2, ..., \lambda_{N-1}) \to \sigma(\lambda) = (h - \sum_{j=1}^{N-1} \lambda_j, \lambda_1, ..., \lambda_{N-2})
\]

. We then introduce the weights \( e_i \) of the standard \( N \)-dimensional representation of \( SL(N) \)

\[
e_1 = \Lambda_1, e_i = \Lambda_i - \Lambda_{i-1}, i = 2, ..., N - 1, e_N = -\lambda_{N-1}
\]

endowed with the scalar product \((e_i, e_j) = \delta_{ij} - \frac{1}{N}\). We shall be concerned with type II class of graphs introduced in section 1 of \([Z1]\). These graphs generalize the classical \( A, D, E \) Dynkin diagrams which may be regarded as related to the \( SL(2) \) algebra. The axioms on these graphs are given in §1.2 of \([Z1]\) as follows:

1. A set \( \nu \) of \( |\nu| = n \) vertices is given. These vertices are denoted by Latin letters \( a, b, ..., \). There exists an involution \( a \to \bar{a} \) and the set \( \nu \) admits a \( \mathbb{Z}_n \) grading denoted by \( \tau(a) \) such that \( \tau(\bar{a}) = -\tau(a) \mod N \);

2. A set of \( N - 1 \) commuting \( n \times n \) matrices \( G_{p}, p = 1, 2, ..., N - 1 \) is given. Their matrix elements are assumed to be non-negative integers, so they may be regarded as adjacency matrices of \( N - 1 \) graphs \( g_p \). \( g_1 \) is also assumed to be connected;

3. The edges of the graphs \( g_p \) are compatible with the grading \( \tau \) in the sense that \( (G_{p})_{ab} = 0 \) if \( \tau(b) \neq \tau(a) + p \mod N \);

4. The matrices are transposed of one another \( G_{p}^t = G_{N-p} \) and \( (G_{p})_{ab} = (G_{p})_{ba} \);

5. As a consequence of (2) and (4), the matrices \( G_p \) are commuting normal matrices and may thus be simultaneously diagonalized in a common orthonormal basis. This basis, denoted by \( \psi^{(\lambda,i)} \), is assumed to be labelled by the weights \( \lambda \) of \( SL(N) \), that are restricted to \( P_{++}^{(h)} \), for some integer \( h > N \), in a way that the eigenvalues \( \gamma_{p}^{(\lambda,i)} \) have the form \( \gamma_{p}^{(\lambda,i)} = \chi_p(M(\lambda)) \), where \( \chi_p \) is the ordinary character for the \( p \)-th fundamental representation of the group \( SU(N) \), and \( M(\lambda) \) denotes the diagonal matrix \( M(\lambda) = \text{diag}(\epsilon_j(\lambda))_{j=1,..,N} \). Here \( \epsilon_j(\lambda) := \exp(-\frac{2\pi i}{h}(e_j, \lambda)) \), and \( i \) in \( (\lambda, i) \) is an index integer, \( 1 \leq i \leq m_{\lambda} \) with \( m_{\lambda} \) being the multiplicity of eigenvalue \( \gamma_{p}^{(\lambda,i)} \). The set of \( (\lambda, i) \)’s will be denoted by \( \text{Exp} \).

There exists a special class of solutions known for all \( N \) and \( h > N \), namely the fusion graphs of the affine algebra \( SL(N) \) at level \( k = h - N \). The vertices are the integrable weights described above, i.e., \( \nu = P_{++}^{(h)} \). The matrices \( G_{p} \) are the Verlinde matrices, which describe the fusion by the \( p \)-th fundamental representation. The fusion rules are given on Page 288 of \([K]\). Their diagonalization is known, thanks to the Verlinde formula (cf. Page 288 of \([K]\)), and the eigenvalues are the \( \gamma_{p}^{(\lambda,i)} \), where \( \lambda \) takes all the values in \( = P_{++}^{(h)} \). We will call these graphs as regular graphs in this paper. In the case of \( N = 2 \), these regular graphs reduce to the \( A_{h-1} \) Dynkin diagrams.

More solutions are known (cf.[Z1]). In [X] (in particular Th.3.10 and (5) of Th.3.8), infinite series of such graphs are constructed from the maximal conformal
inclusions of the form $SU(N) \subset G$ with $G$ being a simple and simply connected compact Lie group.

Given graphs of the previous type, let $V$ be a complex vector space with a basis $\alpha_a$ labelled by the vertices of the set $\nu$. A bilinear form $g$ is defined by:

$$g_{ab} = \langle \alpha_a, \alpha_b \rangle = 2\delta_{ab} + G_{ab}$$

in terms of the matrix $G_{ab} = \sum_{p=1}^{N-1} (G_p)_{ab}$. $g$ will be called the intersection form. This is the intersection form in the title of this paper.

The eigenvalue of the matrix $(g_{ab})$ with eigenvector $\psi_p^{(\lambda, i)}$ is (cf. (34) of [Z1]):

$$g^{(\lambda)} = \prod_{i=1}^{N} (1 + \exp(-\frac{2\pi i}{h}(e_i, \lambda)))$$

. For $(\lambda, i) \in \text{Exp}$ define real numbers which depend only on $\lambda$ (cf. (46) of [Z1]) by:

$$q^{(R)}_{\lambda} := \frac{1}{h} \sum_{j=1}^{N-1} j(\lambda_j - 1) + \frac{(N - h)(N - 1)}{2h}$$

. We can now state Zuber’s conjecture on the signature of $g$ (cf. Conjecture 2.5 of [Z1]):

**Zuber’s Conjecture.** The signature of the bilinear form $g$ for class II graphs is $(x+, y-, z0)$ where $x$ is the number of $q^{(R)}_{\lambda}$ which fall in an interval $]2p - \frac{1}{2}, 2p + \frac{1}{2}[$ for some $p \in \mathbb{Z}$ ($p$ may depend on $q^{(R)}_{\lambda}$), $y$ is the number of those in an interval $]2p' + \frac{1}{2}, 2p' + \frac{3}{2}[$ for some $p' \in \mathbb{Z}$ ($p'$ may depend on $q^{(R)}_{\lambda}$), and $t = n - r - s$ is the number of those $q^{(R)}_{\lambda}$ which are half-integers.

We now prove this conjecture.

Let us first notice a simple consequence of the axioms on the graphs. It follows from Prop.1.2 of [Z1] that $\text{Exp}$ is invariant under the action of $\sigma$. In fact, if $\sum_{a} \psi_{a}^{(\lambda, i)} a$ is an eigenvector of $G_{p}$ with eigenvalue $\gamma^{(\lambda)}_{p}$, then Prop.1.2 of [Z1] implies that $\sum_{a} \psi_{a}^{(\lambda, i)} \exp(\frac{2\pi i \sigma(a)}{N}) a$ is an eigenvector of $G_{p}$ with eigenvalue $\gamma^{(\lambda)}_{\sigma(a)}$. Since $a \rightarrow \exp(\frac{2\pi i \sigma(a)}{N}) a$ is an invertible map, it follows that the multiplicity of eigenvalue $\gamma^{(\lambda)}_{\sigma(a)}$ is the same as that of eigenvalue $\gamma^{(\lambda)}_{p}$. We can therefore define $\sigma(\lambda, i) = (\sigma(\lambda), i)$. It follows that $\text{Exp}$ can be written as a disjoint union of the orbits under the action of $\sigma$. To prove Zuber’s conjecture, we just have to show it is true on each orbit.

Let $(\lambda, i) \in \text{Exp}$ and let $d$ be the smallest positive integer such that $\sigma^{d}(\lambda) = \lambda$. Then $d | N$ and let $N = dd'$. Let $G'_{\lambda} := \{g^{(\sigma^{i}(\mu))}, i = 1, 2, ..., d\}$, $Q'_{\lambda} := \{\cos(\pi q^{(R)}_{\sigma^{i}(\mu)}), i = 1, 2, ..., d\}$, we need to show

$$b_{+}(G'_{\lambda}) = b_{+}(Q'_{\lambda}), b_{0}(G'_{\lambda}) = b_{0}(Q'_{\lambda}), b_{-}(G'_{\lambda}) = b_{-}(Q'_{\lambda})$$.
Note that \( b_0(G'_\lambda) = b_0(Q'_\lambda) \) was already noticed on page 17 of [Z1]. Let \( \lambda' = (\frac{\lambda_1}{h}, ..., \frac{\lambda_{N-1}}{h}) \), then \( p_i(\lambda') = \frac{\epsilon_i(\lambda)}{h}, i = 1, 2, ..., N \). To use theorem 1, we make use of the following identities which follow from the definitions:

\[
q_{\sigma^{-j}(\lambda)}^{(R)} = q_j(\lambda'),
\]

and

\[
g^{(\sigma'(\lambda))} = \prod_{l=1}^{N} (1 + \exp(\frac{2\pi ij}{N} \epsilon_l(\lambda)))
\]

\[
= \prod_{l=1}^{N} 2\cos(\pi p_l(\lambda') - \frac{j}{N})) \times \exp(\pi ij - \sum_{l=1}^{N} \frac{1}{h}(\lambda, \epsilon_l))
\]

\[
= (-1)^j \prod_{l=1}^{N} 2\cos(\pi p_l(\lambda') - \frac{j}{N}))
\]

\[
= g_j(\lambda')
\]

Now it is clear that the size of \( G_{\lambda'} \) (resp. \( Q_{\lambda'} \)) is \( d_1 \) times the size of \( G'_\lambda \) (resp. \( Q'_\lambda \)) and we have:

\[
b_+(G_{\lambda'}) = d_1 b_+(G'_\lambda), b_0(G_{\lambda'}) = d_1 b_0(G'_\lambda), b_-(G_{\lambda'}) = d_1 b_-(G'_\lambda)
\]

and

\[
b_+(Q_{\lambda'}) = d_1 b_+(Q'_\lambda), b_0(Q_{\lambda'}) = d_1 b_0(Q'_\lambda), b_-(Q_{\lambda'}) = d_1 b_-(Q'_\lambda)
\]

By theorem 1, we have proved:

\[
b_+(G'_\lambda) = b_+(Q'_\lambda), b_0(G'_\lambda) = b_0(Q'_\lambda), b_-(G'_\lambda) = b_-(Q'_\lambda).
\]

Let us summarize the result in the following:

**Corollary 1.** Zuber’s Conjecture as stated above is true.

§5. Conclusions and questions

In this paper we proved Zuber’s conjecture on the signature of certain intersection forms by using theorem 1.

Our results imply that the infinite series of graphs which are constructed in [X] by using subfactors associated with conformal inclusions satisfy Zuber’s conjecture. This lends further support to the idea that these graphs may be associated with the integrable models in [CV] which is the basis of Zuber’s conjecture. Such a relation is not very clear and should be very interesting.
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