TOPOLOGICAL CLASSIFYING SPACES OF LIE ALGEBRAS
AND THE NATURAL COMPLETION OF CONTRACTIONS

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Abstract

The space $K^n$ of all $n$-dimensional Lie algebras has a natural non-Hausdorff topology $\kappa^n$, which has characteristic limits, called transitions $A \rightarrow B$, between distinct Lie algebras $A$ and $B$. The entity of these transitions are the natural transitive completion of the well known Inönü-Wigner contractions and their partial generalizations by Saletan.

Algebras containing a common ideal of codimension 1 can be characterized by homothetically normalized Jordan normal forms of one generator of their adjoint representation. For such algebras, transitions $A \rightarrow B$ can be described by limit transitions between corresponding normal forms.

The topology $\kappa^n$ is presented in detail for $n \leq 4$. Regarding the orientation of the algebras as vector spaces has a non-trivial effect for the corresponding topological space $K^n$: There exist both, selfdual points and pairs of dual points w.r.t. orientation reflection.

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1 Introduction

The main goal of this paper is to study the topological space of real Lie algebras of a given dimension \( n \leq 4 \).

Extensive studies have been dedicated to generalizations of the classical Lie algebra structure. As an example, think of the famous q-deformations or Santilli's Lie isotopic liftings [34]. However, few work has been dedicated to pursue the theory of deformations and contractions of Lie algebras (or groups) within their category.

Smrz [38] has considered the deformation of Lie algebras outside a fixed subgroup. This kind of deformation is in some sense complementary to a Inönu-Wigner contraction [12], which consists in a parametric linear and isotropic contraction outside a given subgroup of a Lie algebra.

A particularly interesting problem is to find all possible contractions and, more generally, all possible limit transitions between real or complex Lie algebras of fixed dimension \( n \), and to uncover the natural topological structure of the space of all such Lie algebras. It is clear that this requires, as a precondition, to find all isomorphism classes of Lie algebras in the given dimension. Unfortunately, with increasing dimension \( n \) the classification of real and complex Lie algebras becomes rapidly more complicated.

For this goal, the Levi decomposition into a semidirect sum of a radical and a semisimple subalgebra proves to be useful. This way Turkowski has classified real Lie algebras which admit a nontrivial Levi decomposition, up to \( n = 8 \) in [39] and recently for \( n = 9 \) in [41].

In any dimension \( n \), the classification of all nilpotent Lie algebras is an essential step required for a complete classification. For \( n = 7 \), a complete list of all nilpotent, real and complex, Lie algebras has been given by Romdhani [32]; the complex case has been considered first by Ancochea-Bermudez and Goze [1]; complex decomposable algebras have been studied by Charles and Diakite [6]. The variety of structure constants of complex Lie algebras has been examined for \( n = 4, 5, 6 \) by Kirillov and Neretin [13]. Grunewald and O’Halloran [10] have investigated the complex, nilpotent Lie algebras for \( n \leq 6 \).

For \( n = 6 \), all real nilpotent Lie algebras are classified by Morozov [21]; solvable, non nilpotent Lie algebras have been classified by Mubarakzjanov [24]; and solvable real Lie algebras containing nilradicals are classified by Turkovski [40], thus completing the classification of the solvable ones. Both give reference to the early work of Umlauf [42] already classifying the nilpotent complex 6-dimensional Lie algebras. Mubarakzjanov also classified real
Lie algebras up to \( n = 5 \) in [23]. In [22] he treats the case of real \( n = 4 \), giving reference to the early works of Lie [18] for complex algebras with \( n \leq 4 \) and, for the real 3-dimensional case, to Bianchi [3] and later equivalent classifications of Lee [16] and Vranceanu [43].

The 3-dimensional real Lie algebras, are given by the so called Bianchi types, classified independently first by S. Lie [17] and then by L. Bianchi [2]. The original classification of Bianchi revealed 9 inequivalent types of 3-parameter Lie groups \( G_3 \), numbered usually by the Roman numbers I, \ldots, IX. The types of number VI and VII are actually 1-parameter sets of Lie algebras, VI\(_h\) resp. VII\(_h\), with \( h \geq 0 \) all inequivalent. We will refer to the inequivalent 3-dimensional real Lie algebras as the Bianchi types. Our choice of the parameter \( h \) is according to Landau-Lifschitz [15], which agrees for VII\(_h\) with Behr’s choice in [8].

When the isomorphism classes of Lie algebras for a given real (or complex) dimension are known in a given dimension, one can start to compare their algebraic structure systematically and find their algebraic characteristics, i.e. the invariants. So Paterea and Winternitz [27] determined subalgebra structures for real Lie algebras with \( n \leq 4 \). Grigore and Popp [9] developed a general classification of subalgebras of Lie algebras with solvable ideal, and invariants of real Lie algebras have been calculated for \( n \leq 5 \) by Patera, Sharp, Winternitz and Zassenhaus [26].

But the algebraic properties of Lie algebras are also related to the topological structure of the space of all Lie algebras in a given dimension. On the space of all structure constants of real Lie algebras in \( n \) dimensions Segal has introduced (see page 255 in [37]) the subspace topology induced from \( \mathbb{R}^{n^3} \). The space \( K^n \) of all isomorphism classes of real \( n \)-dimensional Lie algebras under general linear isomorphisms \( \text{GL}(n) \) of their generators has a natural weakly separating (i.e. \( T_0 \), not \( T_1 \)) non-Hausdorff topology \( \kappa^n \), induced as the quotient topology from the Segal topology by the equivalence relation given on the structure constants via the action of \( \text{GL}(n) \). This topology has been discovered and described explicitly by Schmidt for \( n \leq 3 \) in [35] and more generally in [36].

As a real vector space, a Lie algebra admits also a natural orientation. By the exponential map, for any Lie algebra there exists an associated Lie group which similarly admits the corresponding orientation as a differentiable manifold.

Note that throughout the following any index or property concerning orientation is set in brackets () iff the corresponding quantity can be considered optionally with or without reference to an orientation.
The present paper is organized by the following sections.

Sec. 2 resumes some well-known facts on Lie algebras and topology needed in the sequel.

Sec. 3 describes the general construction of the topological spaces \((K^n, \kappa^n)\) and \((K^n_{or}, \kappa^n_{or})\), respectively with and without orientation of the Lie algebras as vector spaces.

Sec. 4 shows how those solvable elements of \(K^n\) which contain all the same ideal \(J_{n-1}\) can be characterized against each other by the normalized version (NJNF) of the Jordan normal form (JNF) of a single structure matrix. Correspondingly, an oriented normalized Jordan normal form (ONJNF) for the structure constants of oriented Lie algebras is defined. Hence transitions \(A \rightarrow B\) between Lie algebras can be described by transitions between the corresponding normal forms.

Sec. 5 resumes important general properties (see also Schmidt [36]) of the topology \(\kappa^n_{(or)}\) and shows up further features of orientation duality for arbitrary dimension \(n\). The structure of \(K^n_{or}\), the space of equivalence classes of oriented Lie algebras, as compared to its unoriented counterpart \(K^n\), has also been described in Rainer [29]. A generalization of Schmidt’s notion of atoms is made for arbitrary subsets of \(K^n_{(or)}\). This is applied to the case of the non-selfdual subset \(K^n_{or} \setminus K^n_{SD}\), decomposing it for \(n = 3\) and \(n = 4\) into its connected components \(K^n_{+}\) and \(K^n_{-}\).

Sec. 6 is devoted to the topology of the non oriented \(K^n\) for \(n \leq 4\). The topological structure for \(n \leq 4\) has also been described by Rainer [29]. The \(T_0\) topology \(\kappa^n\) provides for \(n \geq 3\) a rich local structure of \(K^n\), which we describe for \(n \leq 4\).

In Sec. 6.1 the topological structure of \(K^n\) for \(n \leq 3\) is analysed by use of the NJNF. So, using a quite different method, we reproduce the results of Schmidt in [35] and [36].

Sec. 6.2 presents the detailed analysis of the components of \(K^4\), their possible \(\kappa^4\) limits, and transitions between them. Thereby the relation between the different classification schemes of Mubarakzjanov [22], Patera, Winternitz [27] and Petrov [28] is clarified.

Sec. 6.3 determines the topological structure of \(K^4\). Its parametrically connected components are related in a transitive network of \(\kappa^4\) transitions.

Sec. 7 is devoted to the topology of the oriented \(K^4_{or}\) for \(n \leq 4\), which is also described in Rainer [30].

In Sec. 7.1 the topological structure of \(K^4_{or}\) for \(n \leq 3\) is analysed by use of the ONJNF, in correspondence with results listed by Schmidt [36]. We
give the connected components $K^3_\pm$ explicitly.

Using the same method, Sec. 7.2 examines the orientation duality structure of $K^4_\pm$ in detail. In particular, we determine the connected components $K^4_\pm$.

In Sec. 8 we discuss the present results.

2 Preliminaries

In the following we remind shortly some of the notions needed throughout this paper. A (finite-dimensional) Lie algebra is a (finite-dimensional) vector space $V$, equipped with a skew symmetric bilinear product $[\cdot, \cdot]$ called Lie bracket, which maps $(X,Y) \in V \times V$ to $[X,Y] = -[Y,X] \in V$ and satisfies $\sum_{X,Y,Z \in \mathbb{R}} [[X,Y],Z] = 0, \forall X,Y,Z \in V$. The dimension of the Lie algebra is the dimension of the underlying vector space. Here and in the following all Lie algebras and vector spaces are assumed to be finite-dimensional. If the vector space is real resp. complex, we say that the Lie algebra is real resp. complex. If nothing else is specified in the following a Lie algebra or a vector space is assumed to be real.

For a Lie algebra $A$ the descending central series of ideals is defined recursively by

$$C^0 A := A \quad \text{and} \quad C^{i+1} A := [A, C^i A] \subseteq C^i A.$$  

(2.1)

$A$ is called nilpotent, iff there exists a $p \in \mathbb{N}$, such that $C^p A = 0$, i.e. the descending central series of ideals terminates at the zero ideal.

Furthermore for a Lie algebra $A$ the derivative series of ideals is defined recursively by

$$A^{(0)} := A \quad \text{and} \quad A^{(i+1)} := [A^{(i)}, A^{(i)}] \subseteq A^{(i)}.$$  

(2.2)

$A$ is called solvable, iff there exists a finite $q \in \mathbb{N}$, such that $A^{(q)} = 0$, i.e. the derivative series of ideals terminates at the zero ideal.

In the following, we consider real Lie algebras of fixed finite dimension $n \geq 2$ (for $n = 1$ there is only 1 type of Lie algebra, namely the Abelian $A_1$), classified up to equivalence via real $GL(n)$ transformations of their linear generators $\{e_i\}_{i=1, \ldots, n}$, which span an $n$-dimensional real vector space, which may in the following be identified with $\mathbb{R}^n$ or the tangent space $T_x M$ at any point $x$ of an $n$-dimensional smooth real manifold $M$. The Lie bracket $[\ , \ ]$
is given by its action on the generators $e_i$, which is encoded in the structure constants $C_{ij}^k$,

$$[e_i, e_j] = C_{ij}^k e_k.$$  \hfill (2.3)

(The sum convention is always understood implicitly, unless stated otherwise.) The bracket $[,]$ defines a Lie algebra, iff the structure constants satisfy the $n\{\binom{n}{2} + \binom{n}{3}\}$ antisymmetry conditions

$$C_{[ij]}^k = 0,$$  \hfill (2.4)

and the $n \cdot \binom{n}{3}$ quadratic compatibility constraints

$$C_{[ij}^l C_{kl]}^m = 0$$  \hfill (2.5)

with nondegenerate antisymmetric indices $i, j, k$. Here $[,]$ denotes antisymmetrization w.r.t. the indices included.

Note that Eq. (2.5) is satisfied automatically by Eq. (2.4), if the bracket is derived via $[e_i, e_j] \equiv e_i \cdot e_j - e_j \cdot e_i$ from an associative multiplication $e_i \cdot e_j$. In this case Eq. (2.5) is an identity, called Jacobi identity. Otherwise Eq. (2.5) is an axiom, which might be called Jacobi axiom. If there is an (adjoint) matrix representation of the algebra, it is associative and hence satisfies the Jacobi axiom (2.5) trivially, i.e. as identity. We will not assume the existence of any matrix representation nor any associative algebra multiplication, because we want all the data for a Lie algebra to be encoded in the structure constants. Hence we take (2.5) as an axiom.

The space of all sets $\{C_{ij}^k\}$ satisfying the Lie algebra conditions (2.4) and (2.5) can be viewed as a subvariety $W^n \subset \mathbb{R}^{n^3}$ of dimension

$$\dim W^n \leq n^3 - \frac{n^2(n+1)}{2} = \frac{n^2(n-1)}{2}.$$  \hfill (2.6)

For $n = 3$ the structure constants can be written as

$$C_{ij}^k = \varepsilon_{ijl}(n^{lk} + \varepsilon^{lkm}a_m),$$  \hfill (2.7)

where $n^{ij}$ is symmetric and $\varepsilon_{ijk} = \varepsilon^{ijk}$ totally antisymmetric with $\varepsilon_{123} = 1$. With Eq. (2.7) the constraints Eq. (2.5) are equivalent to

$$n^{lm}a_m = 0,$$  \hfill (2.8)

which are 3 independent relations. Actually, Behr has first classified the Lie algebras in $K^3$ according to their possible inequivalent eigenvalues of $n^{lm}$ and values of $a_m$ (see Landau-Lifschitz [15]).
With Eq. (2.8) also Eq. (2.5) is nontrivial for \( n = 3 \). Therefore the inequality in Eq. (2.6) is strict for \( n \geq 3 \).

Throughout the following, we will need the *separation axioms* from topology (for further reference see also Rinow [31]). A given topology on a space \( X \) is *separating* with increasing strength if it satisfies one or more of the following axioms.

**Axiom** \( T_0 \): For each pair of different points there is an open set containing only one of both.

**Axiom** \( T_1 \): Each pair of different points has a pair of open neighbourhoods with their intersection containing none of both points.

**Axiom** \( T_2 \) (Hausdorff): Each pair of different points has a pair of disjoint neighbourhoods.

It holds: \( T_2 \Rightarrow T_1 \Rightarrow T_0 \). If a topology is only \( T_0 \), but not \( T_1 \), we say that it is only *weakly separating* and speak also shortly of the *weak* topology. (The present notion weak should not be confused with another one from functional analysis, which is not meant here. *Separability* of the topological space is defined here by the separation (german: Trennung) axioms \( T_0, T_1 \) or \( T_2 \). This should not be confused with a further notion related to the existence of a countable dense subset.) The separation axioms can equivalently be characterized in terms of sequences and their limits.

**Lemma.** For a topological space \( X \) the following equivalences hold:

a) \( X \) is \( T_0 \) \iff For each pair of points there is a sequence converging only to one of them.

b) \( X \) is \( T_1 \) \iff Each constant sequence has at most one limit.

c) \( X \) is \( T_2 \) (Hausdorff) \iff Each Moore-Smith-sequence has at most one limit.

(As a generalization of an ordinary sequence, a Moore-Smith sequence is a sequence indexed by a (directed) partially ordered set.) \( T_1 \) is equivalent to the requirement that each one-point set is closed.

We define for the following the real *dimension* of a set as the largest number \( k \) such that a subset homeomorphic to \( \mathbb{R}^k \) exists.

### 3 Spaces \( K^n \) and \( K^n_{or} \) of Lie algebras

The space of structure constants \( W^n \) can also be considered as a subvariety of the fibrespace of the tensor bundle \( \wedge^2 T^* M \otimes TM \) over any point of some smooth GL(n)-manifold \( M \). If \( M \) is oriented, the structure group of its
tangent vector bundle $TM$ is reduced from $GL(n)$ to its normal subgroup
\[ \text{GL}^+(n) = \{ A \in \text{GL}(n) : \det A > 0 \}. \] (3.1)

Then $W^n$ gets an additional structure induced from $\wedge^2 T^* M \otimes TM$ by the orientation of $M$.

$GL(n)$ basis transformations induce $GL(n)$ tensor transformations between equivalent structure constants.
\[ C^k_{ij} \sim (A^{-1})^k_h C^h_{fg} A^f_i A^g_j \forall A \in \text{GL}(n), \] (3.2)

where $\sim$ denotes the equivalence relation. This induces the space
\[ K^n = W^n/\text{GL}(n) \] (3.3)
of equivalence classes w.r.t. the nonlinear action of $GL(n)$ on $W^n$. The analogous space for the oriented case is
\[ K^n_{or} = W^n/GL^+(n). \] (3.4)

The $GL(n)$ action on $W^n$ is not free in general. It holds:
\[ \dim W^n > \dim K^n_{(or)} \geq \dim W^n - n^2. \] (3.5)

The first inequality in Eq. (3.5) is a strict one, because the (positive) multiples of the unit matrix in $GL(\text{+})$ gives rise to equivalent points of $K^n_{(or)}$.
Eqs. (2.6) and (3.5) provide only insufficient information on $\dim K^n$. The latter is still unknown for general $n$. (For the analogous complex varieties Neretin [25] has given an upper bound estimate.)

Let $\phi_{(or)} : W^n \to K^n_{(or)}$ be the canonical map for the equivalence relation $\sim$ defined by the action of $GL(\text{+})(n)$ in $W^n$. The natural topology $\kappa^n_{(or)}$ of $K^n_{(or)}$ is given as the quotient topology of the induced subspace topology of $W^n \subset \mathbb{R}^{n^3}$ w.r.t. the $GL(\text{+})(n)$ equivalence relation.

In the oriented case, orientation reversal of the basis yields a natural $Z_2$-action on $K^n_{or}$. This action is not free in general. Hence the fibres of the projection
\[ \pi : K^n_{or} \to K^n = W^n/\text{GL}(n) = K^n_{or}/Z_2 \] (3.6)
can be either $Z_2$ or $E$. In the first case there is a pair of dual points, i.e. points that transform into each other under the $Z_2$-action, in the latter case it is a selfdual point in $K^n_{or}$. The latter therefore decomposes into a selfdual
part $K^n_{SD}$, on which $Z_2$ acts trivially, and 2 conjugate parts $K^n_\pm$. The latter are isomorphic to each other by that reflection in $GL(n)$ that is chosen to define $Z_2$ in Eq. (3.6).

\[
K^n_{or} = K^n_{SD} \oplus K^n_+ \oplus K^n_-, \tag{3.7}
\]

where $\oplus$ denotes the disjoint union of subvarieties.

The projection $\pi$ has the property that its restriction to $K^n_{SD}$ is the identity. Therefore it is useful to make the following

**Definition 1.** A point $A \in K^n$ is called selfdual if $\pi^{-1}(A) \subset K^n_{SD}$ consists of a single point, and non-selfdual if $\pi^{-1}(A)$ consists of a pair of dual points, denoted by $A^R$ and $A^L$ respectively. □

In order to yield a more explicit notion of selfduality, we formulate

**Lemma.** A Lie algebra $A$ is selfdual, $A \in K^n_{SD}$, if and only if there exist two different bases of $\mathbb{R}^n$ possessing different orientation such that all the structure constants $C_{ij}^k$ concerning both bases coincide. □

Obviously a direct sum of a selfdual algebra with any other algebra is selfdual.

Let us mention already here that $K^n_{SD}$ is nonvoid for $n \geq 1$ while $K^n_\pm$ are nonvoid sets only for $n \geq 3$. We will see in Sec. 5 and 7 that the latter are actually nonvoid for $n = 3, 4, 5$ and at least any further odd $n$. In any case $K^n_\pm$ are connected to $K^n_{SD}$. We will see in Sec. 7 that each of $K^n_\pm$ is connected for $n = 3$ and $n = 4$.

Note that for each pair of conjugate Lie algebras $A^R$ and $A^L$ it is a priori completely arbitrary which one is assigned to $K^n_+$ and which one to $K^n_-$. In order to reduce this arbitrariness, in Sec. 4 we will minimize the number of connected components of $K^n_\pm$ to a single component each, thus making $K^n_+$ and $K^n_-$ disconnected to each other. However this requires first a better understanding of the topological structure $K^n_{or}$.

When we do not want to care about effects of orientation, instead of Schmidt’s topological space $(G_n, \tau) \equiv (K^n_{or}, \kappa_{or})$ from [36] we will consider its projection to $(K^n, \kappa^n)$ by Eq. (3.6).

Let us define now the notion of transitions $A \to B$ in $K^n_{or}$.

**Definition 2.** Consider $A, B \in K^n_{or}$, with $A \neq B$. If there is a sequence $\{A_i\}_{i \in \mathbb{N}}$ with $A_i = A$ for all $i \in \mathbb{N}$ which for $i \to \infty$ converges to $B$ in
the topology $\kappa^n_{(or)}$, we say that there is a transition $A \rightarrow B$ in the topology $\kappa^n_{(or)}$. \hfill \Box

Note that this definition makes sense because $K^n$ is a $T_0$ but not a $T_1$ space. A transition is a special kind of limit characteristic for this topology.

**Convention.** We distinguish in notation between a concrete realization of a Lie algebra, $A$, and its equivalence class, $[A]$, where ever this is relevant. In the following, the former will an adjoint representation of the latter, sometimes also called abstract, Lie algebra. However for notational simplicity we prefer to denote a point in $K^n$ by $A$ rather than by $[A]$. If the context does not give the opportunity for confusion, $A$ is implicitly understood as a shorthand for the (abstract) Lie algebra $[A]$. \hfill \Box

In the topology $\kappa^n_{(or)}$, a transition $A \rightarrow B$ occurs if and only if $B \in \text{cl}\{A\}$. For this transition the source $A$ is not closed, and the target $B$ is not open in any subset of $K^n_{(or)}$ containing both of them. In general, a point of $K^n_{(or)}$ will be neither open nor closed. Open points only appear as a source, and not as a target, of transitions. The structure of the rigid Lie algebras, which correspond just to these open points, is examined in Charles [5].

Special kinds of transitions on a certain 2-point set $\{A, B\}$ of Lie algebra isomorphism classes are the contractions of In"on"u-Wigner [12] and their generalization by Saletan [33]. For convenience let us define these here.

Consider a 1-parameter set of matrices $A_t \in \text{GL}(n)$ with $0 < t \leq 1$, having a well defined matrix limit $A_0 := \lim_{t \to 0} A_t$ which is singular, i.e. $\det A_0 = 0$. For given structure constants $C^k_{ij}$ of a Lie algebra $A$ let us define for $0 < t \leq 1$ further structure constants $C^k_{ij}(t) := (A_t^{-1})^k_h C^h_{fg} (A_t)^f_i (A_t)^g_j$, which according to (3.2) all describe the same Lie algebra $A$. If there is a well defined limit $C^k_{ij}(0) := \lim_{t \to 0} C^k_{ij}(t)$ satisfying conditions (2.4) and (2.5) then this limit defines structure constants of a Lie algebra $B$, and the associated limit of Lie algebras $A \rightarrow B$ is called contraction according to Saletan [33] or briefly Saletan contraction. Note that a Saletan contraction $A \rightarrow B$ might yield either $B = A$, then it is called improper, or $B \neq A$, then it is a transition of Lie algebras.

A Saletan contraction is called In"on"u-Wigner contraction if there is a basis $\{e_i\}$ in which

$$A(t) = \begin{pmatrix} E_m & 0 \\ 0 & t \cdot E_{n-m} \end{pmatrix} \quad \forall t \in [0, 1],$$

where $E_k$ denotes the $k$-dimensional unit matrix. This definition closely
follows Conatser [7].

Given the latter decomposition, İnönü and Wigner [12] have shown that the limit \( C^k_{ij}(0) \) exists iff \( e_i, i = 1, \ldots, m \) span a subalgebra \( W \) of \( A \), which then characterizes the contraction.

Saletan [33] gives also a technical criterion for the existence of the limit \( C^k_{ij}(0) \) defining his general contractions.

We only remark here that, while a general Saletan contraction might be nontrivially iterated, the iteration of an İnönü-Wigner contraction is always improper, i.e. no further contraction takes place.

Not every transition \( A \to B \) corresponds to an İnönü-Wigner contraction. We will see some examples of transitions, which are given only by a more general Saletan contraction [33]. However we will find also transitions \( A \to B \), which are not even given by a Saletan contraction.

Transitions \( A \to B \) in the topology \( \kappa^n \) reveal for \( n \geq 3 \) a more complicated structure of the underlying space \( K^n \). In \( K^n \) transitions \( A \to B \) and \( B \to C \) imply a transition \( A \to C \); this means that transitions are transitive. There is a partial order, \( A \geq B : \iff B \in \text{cl}\{A\} \iff A \to B \) (which is also called the specialization order), which gives \( K^n_{(or)} \) the structure of a transitive network of transitions. Since Saletan contractions [33] are not transitive they do not exhaust all kinds of possible \( \kappa^n \) transitions.

Given the topology of \( K^n \), on any 2-point subset \( \{X, Y\} \subset K^n \) we can take the induced topology and consider the set \( T^n := \{\{X, Y\} \subset K^n | X \neq Y\} \) of all 2-point topological subspaces of \( K^n \). Note that a \( T_0 \) topological space, like that of \( K^n \) for \( n \geq 3 \), is in general not determined by the set \( T^n \) of all its induced 2-point topological subspaces. However if the topological space under consideration is finite then \( T^n \) determines already its topology, which is trivially true for \( K^1 \) and \( K^2 \).

4 Normal forms of the structure constants

The structure constants of \( A_n \in [A_n] \in K^n \) are given by the \( n \) matrices \( C_i := (C^k_{ij}), i = 1, \ldots, n \), with rows \( k = 1, \ldots, n \) and columns \( j = 1, \ldots, n \). \( C_i \) is just the matrix of \( \text{ad}_{e_i} \) w.r.t. the basis \( e_1, \ldots, e_n \).

By Eq. (2.4), the column \( j = i \) vanishes identically \( \forall C_i \). Furthermore the diagonals \( (C^j_{ij}), i = 1, \ldots, n \) (no j-summation), determine the rows with \( k = i \), since \( (C^i_{ij}) = (C^i_{ji}), i = 1, \ldots, n \) (no i-summation). Therefore \( A_n \) is described completely by the \( (n - 1) \times (n - 1) \)-matrices \( C_{<i>} := (C^k_{ij}), i = 1, \ldots, n, \) with \( k, j \neq i \) and \( 1 \leq k, j \leq n \).
In the special case where $A_n$ has an ideal $J_{n-1} \in [J_{n-1}] \in K^{n-1}$, we take without restriction $[A_n]/[J_{n-1}] = \text{span}(e_n)$. Then $A_n$ with a given $J_{n-1}$ is described completely by $C_n$ or $C_{<n>}$ only.

**Definition 3.** The normalized JNF (NJNF) of a matrix $C$ is given by the Jordan normal form, abbreviated JNF, of $C$ modulo $\mathbb{R} \setminus \{0\}$, i.e. given by the equivalence class of JNFs, which differ only by a common absolute scale and a common overall sign of their nonzero eigenvalues w.r.t. the eigenvalues of $C$. (The Jordan block structure and the multiplicities are the same for all of them.) \hfill \Box

Thus a normalization convention for the JNF is the division of all nonzero eigenvalues by a fixed element of $\mathbb{R} \setminus \{0\}$. If not stated otherwise, we divide in the following just by the (absolutely) largest eigenvalue in order to represent the NJNF class of the JNF.

Note that the $n^{th}$ row and column of $C_n$ add only an additional eigenvalue 0 (as Jordan block) to the JNF or NJNF of $C_{<n>}$. Since absolute scaling of all eigenvalues of a structure matrix $C_{<n>}$ by $\lambda \in \mathbb{R} \setminus \{0\}$ can be achieved by stretching the basis $\{e_i\}$ homogeneously by $\lambda^{-1}$, it is an equivalence transformation of the algebra. On the other hand it is evident that changing in $C_{<n>}$ the ratio $r$ of any 2 eigenvalues to $r'$, such that $r'$ is not a ratio of any original eigenvalues, changes the equivalence class.

**Theorem.** Consider the set of algebras $A_n$ which have a common (abstract) ideal $J_{n-1}$. Then $A_n^{(1)} \sim A_n^{(2)}$, iff the matrices $C_{<n>}^{(1)}$ and $C_{<n>}^{(2)}$ have the same NJNF.

Proof: $A_n^{(1)} \sim A_n^{(2)}$ iff $\exists M \in \text{GL}(n) : C_{<n>}^{(1)} = (M^{-1})^k C_{<n>}^{(2)} M^f_i M^g_j \sim M^f_i C_{<n>}^{(2)} M^g_j$. By linearity of $[\ , \ ]$ in the second argument, the linearly independent recombinations $\tilde{C}_{i}^{(2)} := M^f_i C_{<n>}^{(2)}$ describe still the same algebra as $C_{i}^{(2)}$. In particular, the (abstract) ideal $J_{n-1}$ is invariant under $M$. Since the algebras have the same ideal $J_{n-1}$, they are characterized by the matrices $C_{<n>}^{(1)}$ resp. $C_{<n>}^{(2)}$. They describe inequivalent algebras, iff $C_{<n>}^{(1)}$ is inequivalent (modulo overall scaling by $M = \lambda E_n$, $\lambda \in \mathbb{R} \setminus \{0\}$) to $C_{<n>}^{(2)}$ and therefore also to $C_{<n>}^{(2)}$. But the equivalence class of any structure matrix $C_{<n>}$ is described by its (real) JNF modulo homogeneous scaling of the eigenvalues with $\lambda \in \mathbb{R} \setminus \{0\}$. \hfill \Box

Already Mubarakzyanov [22] had realized the advantage given by an ideal $J_{n-1}$ of codimension 1. Since then also others, like Magnin [20] within the
nilpotent Lie algebras of dimension \( \leq 7 \), systematically cosidered subclasses of algebras which have a fixed Lie algebra of codimension 1.

In the following, we consider without restriction of generality the ideals \( J_{n-1} \) in the normal form given by the NJNF of the structure constants. The equivalence class \( [A_n] \) of any algebra \( A_n \) with a fixed normal class ideal and additional structure constants from \( C_{\leq n} \) will be characterized in the following by the NJNF of \( C_{\leq n} \) and denoted by

\[
\text{NJNF}(A_n) := \text{NJNF}(C_{\leq n}).
\]

Now we can define the ONJNF of structure matrices \( C_{\leq n} \) of oriented Lie algebras.

**Definition 4.** The ONJNF of the structure matrix \( C_{\leq n} \) of an oriented Lie algebra \( A_n \) is set identical to its NJNF if \( A_n \) is selfdual, and it is given as \( \text{ONJNF}(C_{\leq n}) := \pm \text{NJNF}(C_{\leq n}) \) for \( A_n \in K_n \) respectively. \( \square \)

If \( A_n \) is characterized by an ideal \( J_{n-1} \) in normal form then we set

\[
\text{ONJNF}(A_n) := \text{ONJNF}(C_{\leq n}).
\]

5 General properties of \( \kappa_n \) and \( \kappa_{or} \)

In this section we describe the general topological properties of the topological space \((K^n_{(or)}, \kappa^n_{(or)})\). Let us first remind some general properties from Schmidt [36] (where also more details and proofs can be found).

**Proposition.** \( K^n_{(or)} \) has the following properties w.r.t. \( \kappa^n_{(or)} \):

a) The Abelian algebra \( \{nA_1\} \subset K^n_{(or)} \) is the only closed 1-point set and is contained in any nonempty closed subset of \( K^n_{(or)} \).

b) \( K^n_{(or)} \) is connected and compact.

c) \( K^n_{*(or)} := K^n_{(or)} \setminus \{nA_1\} \) is a compact space, but \( K^n_{*(or)} \) is not a closed subset of \( K^n_{(or)} \).

d) \( K^n_{*(or)} \) is Hausdorff \( (T_2) \) for \( n = 2 \) only.

e) For \( n \geq 2 \) (resp. \( n \geq 3 \)) the separability of \( K^n_{(or)} \) (resp. \( K^n_{*(or)} \) is only weak \( (T_0) \), i.e. for each pair of points there is a sequence converging to only one of them). \( \square \)

d) and e) correspond to the fact that, though \( K^n \) is still an algebraic variety (defined by purely algebraic relations \((2.4), (2.5) \) and \((3.2))\), it can not be expected to be a (topological \( T_1) \) manifold. \( K^n \) is the orbit space of \( W^n \) w.r.t.
the action of the noncompact group $\text{GL}(n)$, which behaves algebraically badly on $W^n$ for $n \geq 2$. So some of the orbits (the elements of $K^n$) are closed in $K^n$, others are not.

Strong separability ($T_1$, i.e. each constant sequence has at most one limit) would imply that there should not exist transitions $A \rightarrow B$ between inequivalent Lie algebra classes $A \not\sim B$, given by a sequence $\{A_i\}$ of Lie algebras of class $A$ converging to a Lie algebra of class $B$. But this is exactly what happens for dimension $n \geq 2$, as will be seen explicitly below. Obviously transitions $A \rightarrow B$ will be transitive, which decisively effects the topology of $K^n$.

Transitions which are impossible in a given dimension $n$ can become possible after Abelian embedding into dimension $n + 1$. Therefore the following lemma holds.

**Lemma 1.** The Abelian embedding $\oplus \mathbb{R}$ of $K_n$ into $K_{n+1}$ is continuous, but for $n \geq 2$ not homeomorphic. □

So we are led to the following

**Definition 5.** The essential dimension of an $n$-dimensional (oriented) Lie algebra $A_n$ is defined as the smallest possible number $n_e \leq n$, such that $A_n = A_{n_e} \oplus \mathbb{R}^{n-n_e}$. The essential-dimensional subset of $K^n$ is defined as $K^{n}_{de} = \{A \in K^n | n_e(A) = n\}$ □

**Lemma 2.** The subsets $\{A \in K^n | n_e(A) \leq m\}$ for any fixed $m \leq n$ need not to be closed. □

This is due to the existence of transitions or limits of structure constants in NJNF such that one or more NJNF eigenvalues degenerate to another one (in Lemma 2 it is the eigenvalue 0), initially distinct from them; in this case the algebraic multiplicity of this eigenvalue increases automatically, but its geometric multiplicity (expressed by its number of Jordan blocks) may remain constant, since an eigenvector of an eigenvalue different from the limit eigenvalue may converge to a principal (not necessarily eigen) vector of the limit eigenvalue (0 for Lemma 2).

A Lie algebra characterized by structure constants $C_{ij}^k$ is called unimodular (on a corresponding Lie group) iff $\text{tr}(C_i) = C_{ik}^k = 0 \forall i$, where the adjoint
representation is generated by the matrices \( C_i \). We denote the subset of all points in \( K^n \) that correspond to unimodular Lie algebras by \( U^n \), and set \( U^n_* = U^n \cap K_*^n \). Since the zero set of a continuous function is always closed, we have

**Lemma 3.** The unimodular subset \( U^n \subset K^n \) is closed and compact. \( \square \)

For \( n \geq 2 \) the structure constants of any Lie algebra admit, as a tensor \( C \), the irreducible decomposition [36]

\[
C^k_{ij} = D^k_{ij} + \delta^k_{[i} v_{j]} \tag{5.1}
\]

in a tracefree part \( D \) with tensor components \( D^k_{ij} \) (the trace free condition for \( D \) can be written as \( \text{tr}(D_i) = D^k_{ik} = 0 \forall i \)), and a vector part, constructed from a vector \( v \) with components \( v_i = C^i_{ij} / (1 - n) \) and the Kronecker symbol of components \( \delta^k_i \) (remind the sum convention over upper and lower indices and the convention to perform an antisymmetric sum over all permutations of the indices included in \( [ \quad ] \)). The Lie algebra is *unimodular* (like any associated connected Lie group), iff it is tracefree, \( v \equiv 0 \), and it is said to be of *pure vector type*, iff \( D \equiv 0 \). In this sense the unimodular and pure vector type are complementary.

The class \( V^{(n)} \) of pure vector type is selfdual for all \( n \geq 2 \). It is the generalization of the unique non-Abelian 2-dimensional algebra \( A_2 \) (see Sec. 6) to arbitrary \( n \). So for each \( n \), there exists exactly one non-Abelian pure vector type Lie algebra, denoted by \( V^{(n)} \) because for \( n = 3 \) it is the Bianchi type \( V \). It has the Abelian ideal \( I^{(n-1)} \) and \( [\text{NJNF}(V^{(n)})]^k_j = \delta^k_j \). For convenience we mention explicitly the nonvanishing commutators of \( V^{(n)} \), for an adapted basis \( \{e_1, \ldots, e_n\} \):

\[
[e_n, e_i] = e_i, \quad i = 1, \ldots, n - 1. \tag{5.2}
\]

The 3-dimensional Heisenberg algebra (= Bianchi type II) is defined in its NJNF by the nonvanishing commutators

\[
[e_3, e_2] = e_1. \tag{5.3}
\]

By Abelian embedding we define the class \( \Pi^{(n)} := \Pi \oplus \mathbb{R}^{n-3} \) for \( n \geq 3 \). Like II, it is unimodular and nilpotent of degree 2. \( \Pi^{(n)} \) is non-selfdual for \( n = 3 \) and selfdual for \( n \geq 4 \). Its nonvanishing structure constants for an adapted basis \( \{e_1, \ldots, e_n\} \) are given by Eq. (5.3) with all indices increased by \( n - 3 \).
In Schmidt [36], an element \( A_n \in K^n_* \) for which its closure in \( K^n \) consists of 2 elements only, \( \text{cl}\{A_n\} = \{A_n, nA_1\} \), was called an atom. Here we will prefer to call equivalently \( A_n \) an atom of \( K^n_* \), iff its closure in \( K^n_* \) is
\[
\text{cl}_{K^n_*}\{A_n\} = \{A_n\}
\]
Let us generalize this:

**Definition 6.** For any subset \( S \subset K^n_{(or)} \), an element \( A \in S \) is called an \( S \)-atom, iff it is closed w.r.t. \( S \), i.e. \( \text{cl}_S\{A\} = \{A\} \).

In the following, we call an \( S \)-atom also synonymously an atom of \( S \) and assume \( S = K^n_* \) if not specified otherwise. Recall from Schmidt [36]

**Theorem 1.** For \( n = 2 \) there is only 1 atom, \( A_2 \equiv V(2) \).

For each \( n \geq 3 \) there exist exactly 2 atoms, the unimodular \( \Pi^{(n)} \) and the pure vector type \( V^{(n)} \).

For \( n \neq 3 \) all atoms are selfdual. If we consider the corresponding atoms of \( K^n_{(or)} \), then only for \( n = 3 \) there is a difference to the nonoriented case. Instead of the unique non-selfdual atom \( \Pi \) in \( K^3 \), there exist 2 non-selfdual atoms, \( \Pi^R \) and \( \Pi^L \), in \( K^3_3 \).

For each \( n \geq 3 \) there is an algebra \( IV^{(n)} \), given by \( [\text{NJNF}(IV^{(n)})]_{ij}^k = \delta_{jk}^i + \delta_{j-2}^k \delta_{i-1}^{n-1} \) w.r.t. to the Abelian ideal \( I^{(n-1)} \). It is selfdual for \( n \geq 4 \) and non-selfdual for \( n = 3 \). For convenience we mention explicitly the nonvanishing commutators of \( IV^{(n)} \), for an adapted basis \( \{e_1, \ldots, e_n\} \) given by
\[
[e_n, e_i] = e_i, \quad i = 1, \ldots, n-2, \quad [e_n, e_{n-1}] = e_{n-2} + e_{n-1}. \tag{5.4}
\]

\( K^n_* \) is generated by infinitesimal deformations of the atoms; this means: \( K^n_* \) itself is the only open subset of \( K^n_* \) which contains all atoms. Since both, \( IV^{(n)} \to \Pi^{(n)} \) and \( IV^{(n)} \to V^{(n)} \), it follows that \( K^n_* \) is connected.

Remark: Connectedness is trivial for \( K^n \), but non-trivial for \( K^n_* \).

To understand better where the exceptionality of \( n = 3 \) w.r.t. to duality comes from, realize that \( n_e(V^{(n)}) = n \) but \( n_e(\Pi^{(n)}) = 3 \) for all \( n \geq 3 \). In particular, \( \Pi^{(n)} \) has essential dimension \( n_e = n \) only for \( n = 3 \); for \( n \geq 4 \) it is decomposable and hence selfdual.

More generally there holds

**Lemma 4.** \( K^n_{NSD} := K^n \setminus K^n_{SD} \) is contained in the subset \( K^n_{de} \) of \( K^n \) for which \( n_e = n \).

To overcome the difference in the essential dimension of the atoms for \( n \geq 4 \), let us search for atoms w.r.t. the subset \( K^n_{de} \) of essential dimension \( n_e = n \) in \( K^n \). We find
Theorem 3. The set $K^n_{de}$ has the following atoms:

a) For $n \geq 2$ exactly 1 pure vector type atom, called $v(n)$.

b) For $n \geq 3$ a nilpotent unimodular atom, called $ii(n)$, located in the subset of algebras with ideal $I^{(n-1)}$.

c) For $n \geq 5$ further \[\left\lfloor \frac{2}{3}(n-4) \right\rfloor \] mixed type atoms, denoted $a_m(n)$, $m = 2 + \left\lceil \frac{n-4}{3} \right\rceil, \ldots, n-3$, all located in the subset of algebras with ideal $I^{(n-1)}$.

(Here $\lfloor x \rfloor$ resp. $\lceil x \rceil$ denotes the largest/smallest integer less/greater or equal than $x$.)

Within the subspace $K^n_{de}|I^{(n-1)} \subset K^n_{de}$ given by $K^n_{de}$-algebras with ideal $I^{(n-1)}$ there are no further $K^n_{de}$-atoms than that of a), b) and c).

$K^n_{de}|I^{(n-1)}$ is connected.

Proof: a) By Theorem 1 the algebra $V^{(n)}$ is an atom of $K^n_+$. Since $K^n_{de} \subset K^n_+$ and $n_e(V^{(n)}) = n$, it follows that $V^{(n)}$ is an atom of $K^n_{de}$. Any algebra with only nonzero components $v_i$ in the vector $v$ of the decomposition (5.1) has a transition or limit to $V^{(n)}$. Hence $v(n) := V^{(n)}$ is the unique (pure) vector type $K^n_{de}$-atom.

b) Some of the algebras with some vanishing component $v_i$ have transitions or limits to an algebra with $v \equiv 0$. Hence we have to search for unimodular $K^n_{de}$-atoms of essential dimension $n_e = n$. Such an atom is the nilpotent algebra $ii(n)$ with NJNF$(ii(n))$ w.r.t. the ideal $I^{(n-1)}$ given for even $n$ as a direct sum of 1 block of NJNF$(ii(4))$ and further blocks of NJNF$(ii(3))$, and for odd $n$ as a direct sum of NJNF$(ii(3))$ blocks only, where

$$NJNF(ii(3)) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$NJNF(ii(4)) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ (5.5)

The algebra $ii(n)$ is essential-dimensional, because any of its subalgebras $ii(3)$ and $ii(4)$ is so; it is an $K^n_{de}$-atom, because its only possible limits necessarily generate a 1 \times 1-block (0) in its NJNF, thus decreasing $n_e$ at least by 1.

c) The mixed atoms can be characterized by their NJNF w.r.t. the ideal $I^{(n-1)}$. Let us set

$$NJNF(a_m(n)) := NJNF(v(m+1)) \oplus NJNF(ii(n-m)).$$
Since for \( m = 2 + \left\lfloor \frac{n-4}{3} \right\rfloor \), \( \ldots, n - 3 \) the geometric multiplicity \( m \) (= the number of Jordan blocks) of the eigenvalue 1 is bigger than that of the eigenvalue 0, any transition yields an additional Jordan block 0 and hence leaves \( K_{de}^n \). So, being essential-dimensional, \( a_m(n) \) is an \( K_{de}^n \)-atom for \( m = 2 + \left\lfloor \frac{n-4}{3} \right\rfloor \), \( \ldots, n - 3 \).

Any algebra of \( K_{de}^n|I^{(n-1)} \) has a combination of transitions and parametric limits leading to at least one of the atoms from a), b) or c), depending on the degeneracy of its eigenvalues. The only nontrivial case, which remains to be checked, are the algebras with their NJNF w.r.t. an ideal \( I^{(n-1)} \) given as \( \text{NJNF}(v(n+1)) \oplus \text{NJNF}(ii(n-m)) \) where \( m = 1, \ldots, 1 + \left\lfloor \frac{n-4}{2} \right\rfloor \) and \( n \geq 4 \). But any of these has a transition to \( ii(n) \).

Let us now consider some algebra in \( K_{de}^n|I^{(n-1)} \) with only nondegenerate nonzero eigenvalues. By continuous deformation of its eigenvalues, such that every deformed algebra remains in \( K_{de}^n|I^{(n-1)} \), and transitions within \( K_{de}^n|I^{(n-1)} \) each of the atoms a), b) and c) can be reached. Since these have just been seen to be the only atoms of \( K_{de}^n|I^{(n-1)} \) it follows that \( K_{de}^n|I^{(n-1)} \) is connected.

\( K_{de}^n \) itself might have further atoms located in \( K_{de}^n \setminus K_{de}^n|I^{(n-1)} \). Since these are difficult to find, in general one cannot see whether \( K_{de}^n \) is connected.

The nonvanishing commutators of \( ii(n) \), \( n \geq 3 \), are given for an adapted basis \( \{e_1, \ldots, e_n\} \) explicitly by

\[
[e_n, e_2] = e_1, \quad [e_n, e_3] = e_2, \quad i = 2, \ldots, n - 1.
\]

\[
[e_n, e_{2i+3}] = e_{2i+2}, \quad i = 1, \ldots, \frac{n-4}{2}.
\] (5.6)

for \( n \) even and by

\[
[e_n, e_{2i}] = e_{2i-1}, \quad i = 1, \ldots, \frac{n-1}{2}.
\] (5.7)

for \( n \) odd.

\( ii(3) \equiv \Pi \) is the Heisenberg algebra. The number of \( \text{NJNF}(ii(3)) \) blocks in its \( \text{NJNF} \) is even for \( n \equiv 0 \mod 4 \) or \( n \equiv 1 \mod 4 \), and it is odd for \( n \equiv 2 \mod 4 \) or \( n \equiv 3 \mod 4 \).

The mixed types \( a_m(n) \), \( n \geq 5 \), have respective algebraic and geometric multiplicities \( m = 2 + \left\lfloor \frac{n-4}{3} \right\rfloor \), \( \ldots, n - 3 \) for the eigenvalue 1.
Their nonvanishing commutators are given w.r.t. an adapted basis \( \{ e_1, \ldots, e_n \} \) as

\[
\begin{align*}
[e_n, e_1] &= e_1, \\
[e_n, e_{m+2}] &= e_{m+1}, \\
[e_n, e_{m+3}] &= e_{m+2}, & i &= 2, \ldots, n-1, \\
[e_n, e_{2i+m+3}] &= e_{2i+m+2}, & i &= 1, \ldots, n - m - 2, \\
[e_n, e_{2i+m}] &= e_{2i+m-1}, & i &= 1, \ldots, n - m - 1/2,
\end{align*}
\]

for \( n - m \) even, and by

\[
\begin{align*}
[e_n, e_1] &= e_1, \\
[e_n, e_{m}] &= e_{m}, \\
[e_n, e_{2i+m}] &= e_{2i+m-1}, & i &= 1, \ldots, n - m - 1/2,
\end{align*}
\]

for \( n - m \) odd.

The reflection \( e_1 \to -e_1 \) leaves \( v(n) \) and any mixed type atom \( a_m(n) \) invariant; hence all these atoms are selfdual. The nilpotent atom \( \text{ii}(n) \) remains as the only possibility for a non-selfdual \( K^n_{de} \)-atom within \( K^n_{de|I^{(n-1)}} \). Therefore, next we want to examine the orientation duality of \( \text{ii}(n) \).

**Theorem 4.** For \( n \geq 3 \) the \( K^n_{de} \)-atom \( \text{ii}(n) \) is non-selfdual only if \( n \equiv 3 \mod 4 \).

\( \text{ii}(n) \) non-selfdual for \( n \equiv 3 \mod 4 \) implies that \( \text{ii}(n) \) is a \( K^n_{de} \)-atom.

**Proof:** A combination of the reflections \( e_n \to -e_n \) and \( e_{2i} \to -e_{2i} \) for \( i = 1, \ldots, \left[ \frac{n-3}{2} \right] \) leaves \( \text{ii}(n) \) invariant. The total number of these reflections is \( \left[ \frac{n+1}{2} \right] \), which is odd for \( n \equiv 1 \mod 4 \) or \( n \equiv 2 \mod 4 \). Furthermore for \( n \) even, \( e_i \to -e_i, i = 1, \ldots, n - 1 \) yields a reflection keeping \( \text{ii}(n) \) invariant. So for all \( n \) but \( n \equiv 3 \mod 4 \) the algebra is selfdual.

Any limit of \( \text{ii}(n) \) is selfdual, because it is a \( K^n_{de} \)-atom and any non-essential-dimensional algebra is decomposable and hence selfdual. Therefore non-selfduality for \( n \equiv 3 \mod 4 \) implies that \( \text{ii}(n) \) is a \( K^n_{de} \)-atom.

For \( n \equiv 3 \mod 4 \) it was impossible to construct a reflection leaving \( \text{ii}(n) \) invariant. But when there is no such reflection the algebra is non-selfdual.

Let us define for \( n \geq 3 \) an algebra \( iv(n) \) given for an adapted basis \( \{ e_1, \ldots, e_n \} \) by the nonvanishing commutators

\[
\begin{align*}
[e_n, e_1] &= e_1, \\
[e_n, e_2] &= e_1 + e_2, \\
[e_n, e_3] &= e_2 + e_3, & i &= 2, \ldots, n-1, \\
[e_n, e_{2i+3}] &= e_{2i+2} + e_{2i+3}, & i &= 1, \ldots, n - 4/2,
\end{align*}
\]

(5.10)
for $n$ even, and by
\[ [e_n, e_{2i}] = e_{2i-1} + e_{2i}, \quad i = 1, \ldots, \frac{n-1}{2}, \] (5.11)
for $n$ odd.

By similar considerations as for ii$(n)$ in Theorem 3 one finds that iv$(n)$ is non-selfdual for only for $n$ odd and selfdual for $n$ even. In any case it has an ideal $I^{(n-1)}$ and for $n$ odd the geometric multiplicity of its eigenvalue 1 of the NJNF w.r.t. $I^{(n-1)}$ can only be increased by yielding at least two $1 \times 1$ blocks of that eigenvalue, hence the resulting algebra of such a transition is selfdual. Apart from limits which increase multiplicity, the only further limits of iv$(n)$ are transitions with the eigenvalue becoming 0, either to ii$(n)$ or some limit thereof. But, according to Theorem 4, for $n \not\equiv 4 \mod 4$, the algebra ii$(n)$ is selfdual. Any limits of ii$(n)$ are selfdual, because it is a $K^n_{de}$-atom and any non-essential-dimensional algebra is decomposable and hence selfdual. Hence non-selfduality of iv$(n)$ for $n$ odd implies that iv$(n)$ is a $K^n_{NSD}$-atom for $n \equiv 1 \mod 4$. Non-selfduality of ii$(n)$ for $n \equiv 3 \mod 4$ implies further that iv$(n)$ is no atom for $n \equiv 3 \mod 4$.

If for $n \equiv 3 \mod 4$ resp. $n$ odd the algebras ii$(n)$ resp. iv$(n)$ are in fact non-selfdual, we get the

**Corollary.** For odd $n \geq 3$ the set $K^n_{NSD}$ has an atom, located within the subspace $K^n_{NSD}|I^{(n-1)}$ of non-selfdual algebras with ideal $I^{(n-1)}$.

For $n \equiv 3 \mod 4$ the atom is nilpotent unimodular, given by ii$(n)$, and for $n \equiv 1 \mod 4$ it is given by iv$(n)$.

The selfduality of the $K^n_{de}$-atoms $a_m(n)$ and $v(n)$ excludes them as candidates for $K^n_{NSD}$-atoms. It remains an open problem to determine at least some $K^n_{NSD}$-atom for arbitrary even $n$, and all $K^n_{NSD}$-atoms for arbitrary $n$. For odd $n$, besides ii$(n)$ or iv$(n)$, there might be further $K^n_{NSD}$-atoms, even within $K^n_{NSD}|I^{(n-1)}$.

However, assume we succeed for some $n$ to determine all $K^n_{NSD}$-atoms and furthermore to show that $K^n_{NSD}$ is connected for that $n$. In Sec. 7 we will actually see that, for $n = 3$ the Heisenberg algebra ii$(3)$ is the only non-selfdual atom, hence $K^n_{NSD}$ is connected, and for $n = 4$, with the topology of $K^4$ obtained in Sec. 6.3 and the non-selfdual algebras of Sec. 7.2.2, the resulting non-selfdual set $K^n_{NSD}$ will be connected, and its explicit structure will reveal the $K^n_{NSD}$-atoms. Let us assume in the following that for a given $n$ the space $K^n_{NSD}$ is connected.
For \( n \equiv 3 \mod 4 \) resp. \( n \equiv 1 \mod 4 \) corresponding to the \( K_{NSD}^n \)-atom \( ii(n) \) resp. \( iv(n) \) there are in any case 2 atoms of \( K_{or,NSD}^n = K^n_+ \oplus K^n_- \), either \( ii(n)^R \) and \( ii(n)^L \), or resp. \( iv(n)^R \) and \( iv(n)^L \).

Similarly, we could pick for arbitrary \( n \) any \( K_{NSD}^n \)-atom \( a \) and will find a corresponding pair of \( K_{or,NSD}^n \)-atoms \( a^R \) and \( a^L \).

At this place, let us make the convention to assign the right atom \( a^R \) to \( K^n_+ \) and the left atom \( a^L \) to \( K^n_- \).

Now consider all other pairs of dual points \( A^R \) and \( A^L \) in \( K^n_{or,NSD} \), which constitute the preimage \( \pi^{-1}(A) \) of a non-selfdual point \( A \in K^n_{NSD} \). For any limit \( A \to B \) or \( C \to A \) in \( K^n \) there exists a corresponding pair of limits \( A^{R/L} \to B' \) or \( C' \to A^{R/L} \) in \( K^n_{or} \), with \( B' \in \pi^{-1}(B) \) resp. \( C' \in \pi^{-1}(C) \). Note however that there are no transitions or limits between conjugate points, neither \( A^R \to A^L \) nor \( A^L \to A^R \), because limits cannot reverse the orientation.

If \( B' \) or \( C' \) is non-selfdual, we demand it, as the limit \( B' = B^{R/L} \) resp. the prelimit \( C' = C^{R/L} \) of \( A^{R/L} \), to be contained in the same component of \( K^n_{or,NSD} \) as \( A^{R/L} \) itself.

Under consideration of the transitivity of transitions in \( K^n_0 \) and use of the assumed connectedness of \( K^n_{or,NSD} \), it follows from assignments for the non-selfdual atoms made above that, all right algebras have to be in \( K^n_+ \) and all left algebras have to be in \( K^n_- \).

If \( K^n_{NSD} \) is connected, this choice is the only one which makes each of \( K^n_+ \) and \( K^n_- \) connected and both disconnected to each other. Therefore it is the canonical assignment in the case of connected \( K^n_{NSD} \). This will be the relevant situation in the following sections.

For \( n \geq 4 \) let us define a selfdual algebra \( A^n_{a,2} \) with Abelian ideal \( I^{(n-1)} \) by \( NJNF(A^n_{a,2}) := [a \cdot NJNF(A_2)] \oplus NJNF(iv(n-1)) \), where \( \oplus \) denotes the direct sum of matrices.

Now it is easy to prove

**Lemma 5.** Within \( K^n \) for \( n \geq 3 \), the subset \( K^n_{SD} \) of selfdual elements in \( K^n_{or} \) has the following properties:

If there exists a non-selfdual algebra, which is the case at least for \( n \) odd, then \( K^n_{SD} \) is not open.

For \( n \) odd \( K^n_{SD} \) is neither open nor closed.

Proof: Assume that there exists a non-selfdual algebra; such an algebra is given by \( iv(n) \) for \( n \) odd. Then there is at least one \( K_{NSD}^n \)-atom. Any
\(K^{n}_{NSD}\)-atom has a selfdual limit. Hence, there exists a selfdual limit from a non-selfdual sequence \(\Rightarrow K^{n}_{(or),NSD}\) not closed \(\Rightarrow K^{n}_{SD}\) not open.

On the other hand, there are also non-selfdual limits from selfdual sequences, like \(VI_{0} \rightarrow II\) for \(n = 3\) and \(A_{n,2}^{a} \rightarrow iv(n)\) with \(a \rightarrow 1\) for odd \(n > 3\) \(\Rightarrow K^{n}_{SD}\) not closed for odd \(n \geq 3\).

Likewise, each of \(K^{n}_{\pm}\) is neither open nor closed for \(n\) odd. Note that \(K^{n}_{SD}\) open would imply \(K^{n}_{SD} = K^{n}\).

In examination of duality of a given algebra, it is useful to remind the obvious

**Lemma 6.** For an algebra \(A \in K^{n}\), following assertions are equivalent:

i) \(A\) is selfdual.

ii) The set \(S(A)\) of all subalgebras of \(A\) is selfdual.

iii) The set \(J(A)\) of all ideals of \(A\) is selfdual.

Note that individual elements of \(S(A)\) and \(J(A)\) taken for themselves can be non-selfdual while \(A\) is selfdual.

Finally we deal with the case of simple Lie algebras.

**Lemma 7.** Simple Lie algebras are not selfdual.

Proof: A simple \(n\)-dimensional Lie algebra \(A_{n}\) can be characterized by a Cartan-Weyl basis. Such a basis consisting of generators \(H_{i}\), \(i = 1, \ldots, l = \text{rank}A_{n}\), which span a maximal Abelian subalgebra (usually called Cartan subalgebra) and \(n - l\) generators \(E_{\alpha}\), each satisfying, for any nonvanishing generator \(H = \alpha^{i}H_{i}\) of the Cartan subalgebra, a root equation \([H, E_{\alpha}] = \alpha E_{\alpha}\) (*) with root \(\alpha = \alpha^{i}\alpha_{i}\). The commutators \([E_{\alpha}, E_{\beta}] = N_{\alpha\beta}E_{\alpha + \beta}\) (**) for \(\alpha + \beta \neq 0\) and \([E_{\alpha}, E_{-\alpha}] = H\) (***) are nonvanishing. From the root equations (*) we see that for any nonvanishing Cartan subalgebra element \(H\) (given by its coroots \(\alpha^{i}\)) the reflection \(H \rightarrow -H\) changes the algebra. Furthermore by (**) and (***)) also none of the reflections \(E_{\alpha} \rightarrow -E_{\alpha}\) keeps the algebra invariant. Since there is no reflection keeping the algebra invariant it can not be selfdual.

For considerations of the topological structure of \(K^{3}\) and \(K^{4}\) in Sec. 6 and 7 respectively, we will define the notion of parametrical connectedness of points in \(K^{n}\) like following:

**Definition 7.** \(X, Y \in K^{n}\) are called parametrically connected iff there exists a continuous curve \(c : [0, 1] \rightarrow K^{n}\) with \(c(0) = X\) and \(c(1) = Y\) such
that, for all \( t_1 \leq t_2 \in [0, 1] \) with \( c(t_1) \neq c(t_2) \), there exist some \( t_0 \in [t_1, t_2] \) such that \( c(t_1) \neq c(t_0) \neq c(t_2) \). Otherwise \( X, Y \in K^n \) are said to be parametrically disconnected. \( \Box \)

Note that, in the topology \( \kappa^n \), arcwise connectedness does not imply parametric connectedness as defined above.

Furthermore, a set \( S \subset K^n \) is called parametrically connected, iff any two points \( X, Y \in S \) are parametrically connected in \( S \).

\( S \subset K^n \) is a parametrically connected component iff \( S \) is parametrically connected but not a proper subset of another parametrically connected set.

6 Topology of \( K^n \) for \( n \leq 4 \)

Sec. 6.1 resumes already existing results for \( n \leq 3 \), Sec. 6.2 describes in detail the components and transitions of \( K^4 \), and Sec. 6.3 gives some overview over the topological structure of \( K^4 \).

6.1 Structure of \( K^n \) for \( n \leq 3 \)

The Lie algebras with \( n \leq 3 \) are well known and listed, e.g. by Patera and Winternitz [27]. \( K^2 \) contains only 2 elements, the Abelian \( 2A_1 \) and \( A_2 \) represented by the algebra with \([e_2, e_1] = e_1\) as only nonvanishing bracket. So \( A_2 \) has the ideal \( J_1 = A_1 \) spanned by \( \{e_1\} \), and is characterized by \( C_{<2>} = (1) \neq 0 \) in contrast to \( 2A_1 \). Note that \( A_2 \equiv V^{(2)} \). Obviously \( \dim K^2_* = 0 \) and the unimodular subset \( U^2_* \subset K^2_* \) is empty.

The elements of \( K^3 \) correspond to the famous Bianchi (or Bianchi-Behr) types. They have been classified independently first by S. Lie [17] and then by L. Bianchi [2]. For their systematic derivation and explanation of their role for cosmological models see e.g. Kramer, Stephani et al. [14]. For convenience of the reader we give for each of the Bianchi types I up to IX an explicit description by the commutators of its generators \( e_1, e_2, e_3 \) according to Landau-Lifschitz [15]:

Types I,II and VIII/IX are given by basic commutators

\[
[e_1, e_2] = n_3 e_3, [e_2, e_3] = n_1 e_1, [e_3, e_1] = n_2 e_2,
\]

with triplets \((n_1, n_2, n_3)\) respectively given by \((0, 0, 0), (1, 0, 0)\) and \((1, 1, \mp 1)\). The 1-parameter families VI\(_h\) / VII\(_h\) with \( h \geq 0 \) are given respectively by

\[
[e_1, e_2] = e_3 + he_2, [e_2, e_3] = 0, [e_1, e_3] = \pm e_2 + he_3,
\]
and especially it is $III = VI_1$. IV resp. V are given by

$$[e_1, e_2] = be_3 + e_2, [e_2, e_3] = 0, [e_1, e_3] = e_3$$

with $b = 1$ resp. $b = 0$.

The 3-dimensional real Lie algebras in the notation of Patera and Winternitz [27] can be characterized by their NJNF, which is simultaneously the normal form (see Eqs. (6.1) up to (6.4) below) of the Bianchi types associated to them like in Table 1.

| 3A_1 | A_1 ⊕ A_2 | A_{3,1} | A_{3,2} | A_{3,3} | A_{3,4} | A_{3,5}^2 | A_{3,6} | A_{3,7}^2 | A_{3,8} | A_{3,9} |
|------|------------|---------|---------|---------|---------|----------|---------|----------|---------|---------|
| I    | III        | II      | IV      | V       | VI_0    | VI_h    | VII_0   | VII_h    | VIII    | IX      |

Table 1: Inequivalent 3-dim. Lie algebras as denoted in [27] (upper row) and corresponding Bianchi types (lower row).

For convenience we explicitly give the nonvanishing commutators for the indecomposable algebras $A_{3,1}$ up to $A_{3,9}$ from [27]:

- $A_{3,1}$ : $[e_2, e_3] = e_1$;
- $A_{3,2}$ : $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$;
- $A_{3,3}$ : $[e_1, e_3] = e_1, [e_2, e_3] = e_2$;
- $A_{3,4}$ : $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$;
- $A_{3,5}^2$ : $[e_1, e_3] = e_1, [e_2, e_3] = ae_2, 0 < |a| < 1$;
- $A_{3,6}^2$ : $[e_1, e_3] = -e_2, [e_2, e_3] = e_1$;
- $A_{3,7}^2$ : $[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2, 0 < a$;
- $A_{3,8}$ : $[e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_3, e_1] = 2e_2$;
- $A_{3,9}$ : $[e_1, e_2] = e_3, [e_2, e_3] = e_2, [e_2, e_3] = e_1$.

In 3 dimensions, all solvable algebras contain the Abelian ideal $J_2 = 2A_1$. Therefore they can be characterized by their NJNF.

$$\text{NJNF}(I) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\text{NJNF}(III) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
\[ \text{NJNF}(\text{II}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{II} = \text{II}^{(3)}, \]
\[ \text{NJNF}(\text{IV}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{IV} = \text{IV}^{(3)}, \]
\[ \text{NJNF}(\text{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{V} = \text{V}^{(3)}. \quad (6.1) \]
\[ \text{NJNF}(\text{VI}_0) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \text{NJNF}(\text{VI}_h) = \begin{pmatrix} 1 \\ a \end{pmatrix}. \quad (6.2) \]

In Eq. (6.2) the range \( 0 < h < \infty, h \neq 1 \) of the parameter according to Landau-Lifschitz [15], denoted here by \( h \), is monotonously homeomorphic to the range \( -1 < a < 1, a \neq 0 \). \( h = 1 \) resp. \( a = 0 \) yields a decomposable algebra, namely \( \text{III} \).

\[ \text{NJNF}(\text{VII}_0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{NJNF}(\text{VII}_h) = \begin{pmatrix} a \\ 1 \\ -1 \\ a \end{pmatrix}. \quad (6.3) \]

In Eq. (6.3) the range \( 0 < h < \infty \) of the parameter according to Landau-Lifschitz [15], denoted here by \( h \), is monotonously homeomorphic to the range \( 0 < a < \infty \).

Note that for a topological characterization of \( K^3 \) it is sufficient to know the relation of the parameters \( a \) and \( h \) in (6.2) and (6.3) at points of qualitative change in the NJNF and to ensure homeomorphisms of the ranges between these critical points. This is precisely the data we have given above. (Though the explicit relation of \( a \) and \( h \) follows from the equivalence transform to normal form, here we do not need to calculate it.) Both \( \text{VI}_h \) and \( \text{VII}_h \) are unimodular for \( h = 0 \) and converge to \( \text{II} \) for \( 0 \leq h < \infty \) and to \( \text{IV} \) for \( h \to \infty \).

The simple algebras \( \text{VIII} = su(1,1) \) and \( \text{IX} = su(2) \) are described respectively by the 3 matrices
\[ C_{<3>} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_{<1>}, -C_{<2>} = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}. \quad (6.4) \]

\( \text{NJNF}(C_{<3>}) = \text{NJNF}(\text{VII}_0) \) for both \( \text{VIII} \) and \( \text{IX} \), but \( \text{NJNF}(C_{<1>}) = -\text{NJNF}(C_{<2>}) \) is equal to \( \text{NJNF}(\text{VII}_0) \) for \( \text{VIII} \) and to \( \text{NJNF}(\text{VI}_0) \) for \( \text{IX} \). Therefore \( \text{IX} \to \text{VII}_0 \), but \( \text{IX} \not\to \text{VI}_0 \), but both
VIII → V10 and VIII → VII0.
Considering all components, their parametrical limits and transitions together, we get the full topological structure of $K^3$, which includes a transitive network of nearest neighbour transitions between different components. The network has been depicted already by Mac Callum [19] and its transitivity was outlined by Schmidt [35]. We have $\dim K^3 = 1$, since its largest parametrically connected components are 1-dimensional.
For the unimodular subvariety $U_3^* \subset K_3^*$ it is $\dim U_3^* = 0$, and \{VIII, IX\} $\subset U_3^*$ is a minimal dense subset of isolated points. Fig. 1 shows the transitive network of transitions in $K_3^*$, with unimodular points encircled.

Fig. 1: Transitive network of transitions in $K_3^*$. 

26
6.2 Components of $K^4$, transitions and parametrical limits

The real 4-dimensional Lie algebras have been classified by Mubarakzyanov [23] and listed by Patera and Winternitz [27]. An early, somehow more coarse classification has been given by Petrov [28]. For the convenience of the reader we explicitly give this classification in terms of nonvanishing basic commutators. In order to avoid confusion with the 3-dimensional Bianchi types we alter the notation of Petrov’s classes [28] from I, . . . , VIII to $\wp_i$, $i = 1, \ldots, 8$. The subclasses $\wp_{1/4}$ are written as $\wp^\alpha/\gamma$ respectively, and $\wp_{2}$ together with $\wp_{3}$ are resumed in a single class $\wp_6^\beta$ in order to correspond to distinct classes of [27]. With this notation Petrov’s classes are characterized like following:

Solvable algebras, without Abelian subgroup $3A_1$:

$\wp_1 : [e_2, e_3] = e_1, [e_1, e_4] = ce_1, [e_2, e_4] = e_2, [e_3, e_4] = (c - 1)e_3, \ c \in \mathbb{R}$;

$\wp_2 : [e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$;

$\wp_3 : [e_2, e_3] = e_1, [e_1, e_4] = qe_1, [e_2, e_4] = e_3, [e_3, e_4] = -e_2 + qe_3, \ q^2 < 4$;

$\wp_4 : [e_2, e_3] = e_2, [e_1, e_4] = e_1$;

$\wp_5 : [e_2, e_3] = e_2, [e_3, e_1] = -e_1, [e_1, e_4] = e_2, [e_2, e_4] = -e_1$;

Solvable algebras, with Abelian subgroup $3A_1$:

$\wp^\alpha_6 : [e_1, e_4] = ae_1 + be_4, [e_2, e_4] = ce_2 + de_4, [e_3, e_4] = ee_3 + fe_4$,

with real tuples $(a, b, c, d, e, f)$ of the form $(0, 0, 0, 0, 0, 0), \ (0, 1, 0, 1, 0, 0), \ (0, 1, 0, 1, 0), \ (1, 1, 0, 0, 0, 0)$ or $(1, 0, c, 0, e, 0)$;

$\wp^\beta_6 : [e_1, e_4] = ke_1 + e_2, [e_2, e_4] = ke_2 + de_3, [e_3, e_4] = ee_3, \ k \in \mathbb{R}, \ d, e \in \{0, 1\}$;

$\wp^\gamma_6 : [e_1, e_4] = ke_1 + e_2, [e_2, e_4] = -e_1 + ke_2, [e_3, e_4] = le_3, \ k, l \in \mathbb{R}$;

Non solvable algebras:

$\wp_7 : [e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_3, e_1] = -2e_2$;

$\wp_8 : [e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2.$
In the following we use the characterization of equivalence classes by their NJNF, according to Sec. 3, in order to find relative positions of the equivalence classes in $K^4$, possible transitions between them and parametrical limits of parametrically connected components of $K^4$.

### 6.2.1 Decomposable Lie algebras

A decomposable 4-dimensional Lie algebra can have the structures $4A_1$, $2A_1 \oplus A_2$, $2A_2$ or $A_1 \oplus A_3$. The first 3 possibilities are unique, since $A_1$ is the unique 1-dim. Lie algebra and $A_2$ is the unique non-Abelian 2-dim. Lie algebra. Note that $2A_2 \equiv \wp_4$ in Petrov’s classification [28]. $A_1 \oplus A_3$ consists of 9 classes, given by $\{A_{3,i}\}_{i=1,...,9}$ listed in Table 1. It is $A_1 \oplus \Pi \equiv \Pi^{(4)}$. $A_1 \oplus \mathrm{VIII}$ and $A_1 \oplus \mathrm{IX}$ are the same as in [28] the $\wp_7$ and $\wp_8$ respectively.

Transitions and limits: Besides the transitions and limits induced by Abelian embedding $\oplus \mathbb{R}$ of transitions in $K^3$, there are further transitions, which prevent the embedding $\oplus \mathbb{R}$ to be a homeomorphism. So for example $\mathcal{V} \oplus \mathbb{R} \rightarrow \Pi \oplus \mathbb{R}$, but $\mathcal{V} \not\rightarrow \Pi$. This demonstrates that, while $\mathcal{V}$ is an atom, $\mathcal{V} \oplus \mathbb{R}$ is not. Furthermore $\mathcal{VI}_0 \oplus \mathbb{R}$ and $\mathcal{VII}_0 \oplus \mathbb{R}$ both go first to $A_{4,1}$ and then to $\Pi \oplus \mathbb{R}$.

$\mathrm{VIII} \oplus \mathbb{R}$ has a limit in the non-decomposable $A_{4,8}$ and, like $\mathrm{IX} \oplus \mathbb{R}$, also in $A_{4,10}$, as described below.

The algebra $2A_2$ in spite of being decomposable is not the limit of any other algebra in $K^4$. It has transitions to $\mathrm{VI}_h \oplus \mathbb{R}$ with $h \geq 0$, to $A_{4,3}$ and to $A_{4,9}^0$.

### 6.2.2 Indecomposable Lie algebras

Coarsely these algebras have already been classified by Petrov [28]. Table 2 relates his classification to that of Patera and Winternitz [27].

| $A_{4,1..4}$ | $A_{4,5}$ | $A_{4,6}$ | $A_{4,7}$ | $A_{4,8/9}$ | $A_{4,10/11}$ | $A_{4,12}$ |
|--------------|-----------|-----------|-----------|-------------|--------------|-----------|
| $\wp_6^g$    | $\wp_6^\alpha$ | $\wp_6^\gamma$ | $\wp_2$   | $\wp_1$     | $\wp_3$     | $\wp_5$   |

Table 2: Classification of Petrov [28] (lower row) and [27] (upper row) of 4-dim. Lie algebras except decomposable ones.

The algebra $\wp_3$, $q = 0$ is the same as $A_{4,10}$. It is the only indecomposable 4-dimensional algebra that corresponds to a maximal isometry group of a 3-dimensional homogeneous Riemannian space (see Sec. 8, 9 below and Bona and Coll [4], Theorem 1).
Either the eigenvalue is

1) 1 eigenvalue with 1 Jordan block:

\[ \lambda = 0 \]

These correspond to \( \varphi_0^\beta \).

ii) 2 eigenvalues with 1 Jordan block each:

\[ \lambda \neq 0 \]

These correspond to \( \varphi_0^\alpha \).

iii) 1 eigenvalue with 2 Jordan blocks:

\[ \lambda = 0 \] and\[ \lambda \neq 0 \]

These correspond to \( \varphi_0^\gamma \).

For convenience we explicitly give the nonvanishing commutators of the indecomposable algebras \( A_{4,1} \) up to \( A_{4,12} \) according to [27]:

\[ A_{4,1} : [e_2, e_4] = e_1, [e_3, e_4] = e_2; \]

\[ A_{4,2} : [e_1, e_4] = ae_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3, a \neq 0; \]

\[ A_{4,3} : [e_1, e_4] = e_1, [e_3, e_4] = e_2; \]

\[ A_{4,4} : [e_1, e_4] = e_1, [e_2, e_4] = e_2 + e_3; \]

\[ A_{4,5}^{a,b} : [e_1, e_4] = e_1, [e_2, e_4] = ae_2, [e_3, e_4] = be_3, -1 \leq a \leq b \leq 1, ab \neq 0; \]

\[ A_{4,6}^{a,b} : [e_1, e_4] = ae_1, [e_2, e_4] = be_2 - e_3, [e_3, e_4] = e_2 + be_3, b \geq 0, a \neq 0; \]

\[ A_{4,7} : [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3, [e_2, e_3] = e_1; \]

\[ A_{4,8} : [e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = -e_3; \]

\[ A_{4,9}^{b} : [e_2, e_3] = e_1, [e_1, e_4] = (1+b)e_1, [e_2, e_4] = e_2, [e_3, e_4] = be_3, -1 < b \leq 1; \]

\[ A_{4,8} \] is the parametrical limit of \( A_{4,9}^{b} \) for \( b \to -1 \); hence by Mubarakzyanov [22] and Petrov [28] \( A_{4,8} \) and \( A_{4,9} \) are subsumed in a single 1-parameter set.

\[ A_{4,10} : [e_2, e_3] = e_1, [e_2, e_4] = -e_3, [e_3, e_4] = e_2; \]

\[ A_{4,11}^{a} : [e_2, e_3] = e_1, [e_1, e_4] = 2ae_1, [e_2, e_4] = ae_2 - e_3, [e_3, e_4] = e_2 + ae_3, 0 < a; \]

\[ A_{4,10} \] is the parametrical limit of \( A_{4,11}^{a} \) for \( a \to 0 \); hence by Mubarakzyanov [22] and Petrov [28] \( A_{4,10} \) and \( A_{4,11} \) are subsumed in a single 1-parameter set.

\[ A_{4,12} : [e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1. \]

The only difference of this classification to that of Mubarakzyanov [22] is that, unlike there, here the endpoints \( A_{4,8} \) and \( A_{4,10} \) are distinguished against the rest of the 1-parameter sets \( A_{4,9} \) and \( A_{4,11} \) respectively.

In the following we reclassify these algebras by their NJNF.

a) Algebras with an Abelian ideal \( J_3 = 3A_1 \equiv I \)

These are the algebras of type \( \varphi_0 \). In the following the cases i) and ii) correspond to \( \varphi_0^\beta \), case iii) to \( \varphi_0^\alpha \) and case iv) to \( \varphi_0^\gamma \).

i) 1 eigenvalue with 1 Jordan block:

Either the eigenvalue is \( \lambda = 0 \) or otherwise it can be normalized to \( \lambda = 1 \).
\[ \text{NJNF}(A_{4,1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{NJNF}(A_{4,4}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \] (6.5)

Transitions: Obviously \( A_{4,4} \rightarrow A_{4,1} \) and, by increasing the geometric multiplicity, \( A_{4,4} \rightarrow IV^{(4)} \equiv A_{4,2}^{1} \) resp. \( A_{4,1} \rightarrow II^{(4)} \equiv II \oplus \mathbb{R} \).

ii) Maximally 2 eigenvalues with together 2 Jordan blocks:
Here JNF(\( A \)) consists of both a \( 1 \times 1 \) and a \( 2 \times 2 \) Jordan block, with eigenvalues \( \lambda_{1} \) and \( \lambda_{2} \) respectively. If \( \lambda_{1} = 0 \), the algebra would become decomposable (\( II \oplus \mathbb{R} \) if \( \lambda_{2} = 0 \), IV \( \oplus \mathbb{R} \) if \( \lambda_{2} \neq 0 \)). Therefore assume \( \lambda_{1} = a \neq 0 \). Either \( \lambda_{2} = 0 \), then \( \lambda_{1} = 1 \) after normalization, or \( \lambda_{2} \neq 0 \), then it can be normalized to \( \lambda_{2} = 1 \). If \( \lambda_{2} = \lambda_{1} = a \), there is only 1 eigenvalue, which can be normalized to \( a = 1 \). Note that \( A_{4,2}^{1} \equiv IV^{(4)} \), which is a case to be considered separately.

\[ \text{NJNF}(A_{4,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{NJNF}(A_{4,2}^{a}) = \begin{pmatrix} a & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (6.6)

Transitions and limits: From \( A_{4,2}^{a} \) with \( a \neq 0, 1 \) to IV \( \oplus \mathbb{R} \) for \( a \rightarrow 0 \), to \( A_{4,4} \) for \( a \rightarrow 1 \), and to \( A_{4,3} \) for \( |a| \rightarrow \infty \). By increasing the geometric multiplicity, to \( A_{4,5}^{1,a} \) for \( 0 < |a| < 1 \), to \( A_{4,5}^{1,-1} = A_{4,5}^{-1,-1} \) for \( a = -1 \) and to \( A_{4,5}^{\frac{1}{a},\frac{1}{a}} \) for \( 1 < |a| < \infty \). Also generally \( A_{4,2}^{a} \rightarrow A_{4,1} \).

From IV \( ^{(4)} \equiv A_{4,2}^{1} \) to II \( ^{(4)} \equiv II \oplus \mathbb{R} \) and \( V \( ^{(4)} \equiv A_{4,5}^{1,1} \), according to the remark at the theorem in Sec. 4.

From \( A_{4,3} \) to \( A_{4,1} \) and, by increasing of geometric multiplicity, to III \( \oplus \mathbb{R} \).

iii) 3 real eigenvalues as Jordan blocks:
Assuming the largest eigenvalue normalized to \( \lambda_{1} = 1 \), there remain \( \lambda_{2} = a \) and \( \lambda_{3} = b \) with \( -1 \leq b \leq a \leq 1 \). If \( a \cdot b = 0 \), the algebra becomes decomposable (\( a = b = 0 \) yields III \( \oplus \mathbb{R} \), for \( a = 1, b = 0 \) it is V \( \oplus \mathbb{R} \), for \( a = 0, b = -1 \) it is VI \( \oplus \mathbb{R} \) and otherwise \( a = 0 \) or \( b = 0 \) yields VI \( \oplus \mathbb{R} \)). Therefore assume \( a \cdot b \neq 0 \). The case \( a = b = 1 \) (single 3-fold degenerate eigenvalue) corresponds to the pure vector type \( A_{4,5}^{1,1} \equiv V \( ^{(4)} \neq V \oplus \mathbb{R} \). In \( A_{4,5}^{1,b} \) and \( A_{4,5}^{a,a} \) there are 2 eigenvalues, one of them 2-fold degenerate.
For the nondegenerate case, \(-1 \leq b < a < 1\). Note that \(A_{4,5}^{a,b} = A_{4,5}^{b,a}\), since permutations are in GL(4).

\[
NJNF(A_{4,5}^{a,b}) = \begin{pmatrix}
1 & a \\
a & b
\end{pmatrix}.
\] (6.7)

Transitions and limits: From \(A_{4,5}^{a,b}\), to \(A_{4,2}^{a} \) for \(b \to a\), to \(A_{4,2}^{b} \) for \(a \to 1\). To \(A_{4,4}\) for \(a \to 1\), to \(IV \oplus \mathbb{R} \) for \(a \to 1\), to \(VII_h \oplus \mathbb{R} \), \(1 < h < \infty\), for \(a \to 0, b \to 0\), to \(A_{4,3}\) for \(a \to 0, b \to 0\), to \(VI_h \oplus \mathbb{R} \), \(0 \leq h < 1\), for \(a \to 0\), and to \(A_{4,2}^{-1}\) for \(b \to -1\) and \(a \to \pm 1\). Also generally \(A_{4,5}^{a,b} \to A_{4,1}\).

Note furthermore that \(A_{4,5}^{a,0} = A_{4,5}^{-a,0}\).

\[
NJNF(A_{4,6}^{a,b}) = \begin{pmatrix}
a & b & 1 \\
b & 1 & -1
\end{pmatrix}.
\] (6.8)

Transitions and limits: For \(a \to 0\), \(A_{4,6}^{a,b} \to VII_h \oplus \mathbb{R}\), with \(0 \leq h < \infty\) corresponding to \(0 \leq b < \infty\). For a fixed ratio \(\frac{a}{b}\) and \(b \to \infty\) there is a limit to \(A_{4,2}^{a}\), if \(a \neq b\), and to \(A_{4,4}\), if \(a = b\). \(A_{4,6}^{a,0} \to A_{4,3}\) for \(b\) finite (esp. \(b = 0\)) and \(|a| \to \infty\), and \(A_{4,6}^{a,b} \to IV \oplus \mathbb{R}\) for \(a\) finite (esp. \(a = 0\)) and \(b \to \infty\). Also generally \(A_{4,6}^{a,b} \to A_{4,1}\).

Note furthermore that \(A_{4,6}^{a,0} = A_{4,6}^{-a,0}\).
b) **Algebras with a nilpotent ideal** $J_3 = A_{3,1} \equiv \Pi$

In the following case i) corresponds to $\varphi_2$, case ii) to $\varphi_1$ and case iii) to $\varphi_3$.

i) 2 eigenvalues with together 2 Jordan blocks:

$$\text{NJNF}(A_{4,7}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (6.9)$$

Transitions: $A_{4,7} \to A_{4,2}^2$ for $J_3 \to \Pi$. Furthermore $A_{4,7} \to A_{4,9}^1$.

ii) 3 real eigenvalues as Jordan blocks:

$$\text{NJNF}(A_{4,9}^b) = \begin{pmatrix} 1 + b & 1 \\ 1 & b \end{pmatrix}, 0 < |b| < 1,$$

$$\text{NJNF}(A_{4,9}^0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{NJNF}(A_{4,9}^1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\text{NJNF}(A_{4,8}) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \quad (6.10)$$

Transitions: From $A_{4,9}^b$ to $A_{4,8}$ for $b \to -1$, to $A_{4,9}^0$ for $b \to 0$, and to $A_{4,7}$ for $b \to 1$. Furthermore, for $J_3 \to \Pi$, to $A_{4,5}^{1+b,b}$ if $0 < b < 1$, and to $A_{4,5}^{1+b,b}$ if $-1 < b < 0$.

For $J_3 \to \Pi$, $A_{4,9}^1 \to A_{4,5}^{1+b}$ and $A_{4,8} \to VI_0 \oplus \mathbb{R}$. $A_{4,9}^0$ goes to $IV \oplus \mathbb{R}$ and further to $V \oplus \mathbb{R}$. Since $VII \oplus \mathbb{R} \to A_{4,8}$, the latter is a limit from a decomposable algebra.

iii) 1 real eigenvalue and 2 complex conjugates:

$$\text{NJNF}(A_{4,11}^a) = \begin{pmatrix} 2a & a & 1 \\ a & 1 & -1 \\ -1 & a \end{pmatrix}, a > 0.$$
\[ \text{NJNF}(A_{4,10}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \]

(6.11)
Transitions: From $A_{4,11}^{a}$ to $A_{4,10}$ for $a \to 0$, to $A_{4,9}^{1}$ for $a \to \infty$ and, for $J_3 \to I$, to $A_{4,6}^{2a,a}$.

For $J_3 \to I$, $A_{4,10} \to VII_0 \oplus \mathbb{R}$. Furthermore both $VIII \oplus \mathbb{R} \to A_{4,10}$ and $IX \oplus \mathbb{R} \to A_{4,10}$.

c) Algebras with a pure vector type ideal $J_3 = A_{3,3} \equiv V$

This case corresponds to type $\wp_5$.

\[
NJNF(A_{4,12}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\] (6.12)

Transitions: $A_{4,12}$ goes to $V \oplus \mathbb{R}$, to $VII_0 \oplus \mathbb{R}$, especially for $J_3 \to I$ to $VII_0 \oplus \mathbb{R}$, and to $A_{4,9}$.

6.3 The topological structure of $K^4$

Since we know all components, their parametrical limits and transitions in $K^4$, we can now put them together, in order to determine the full topological structure of $K^4$. Fig. 2 a), b) and c) show components of $K^4_*$, with $J_3$ equal to I, II and V respectively, as parts of the transitive network of convergence. The dashed lines in Fig. 2 a) indicate the $\kappa^4$ limit lines from lines in Fig. 2 b). $\dim K^4 = 2$, since its largest (parametrically connected) components are 2-dimensional.

For the unimodular subvariety $U_4^* \subset K^4_*$ it is $\dim U_4^* = 1$. The union of $VIII \oplus \mathbb{R}, IX \oplus \mathbb{R}, \{A_{4,5}^{a,-a-1}, -\frac{1}{2} < a < 0\}$ and $\{A_{4,6}^{2b}, 0 < b < \infty\}$ is a dense subset of $U_4^*$ and consists of a minimum number of parametrically connected components, namely 2 isolated points and 2 isolated line segments. In Fig. 2 the unimodular lines are dotted, and the unimodular points encircled.
Fig. 2 a: Transitions and limits at components of $K^4_4$ with ideal $I$. 
Fig. 2 b: Transitions at components of $K_4^4$ with ideal II.

Fig. 2 c: Transitions at components of $K_4^4$ with ideal V.
7 Orientation duality in $K_{or}^n$ for $n \leq 4$

In this section we examine in detail all points in $K_{or}^n$ for $n \leq 4$ under the aspect of orientation duality.

In Sec. 7.1 the topological structure of $K_{or}^n$ for $n \leq 3$ is analysed by use of the (O)NJNF, thus reproducing the results listed by Schmidt [36]. The connected components $K^3_{\pm}$ are determined explicitly.

Using the same method, Sec. 7.2 analyses the orientation duality structure of $K_{or}^4$ in detail. Especially we determine the connected components $K^4_{\pm}$.

7.1 Structure of $K_{or}^n$ for $n \leq 3$

The Lie algebras in $K^n$ for $n \leq 3$ have been classified in Sec. 6.1 using their $n - 1$-dimensional ideals and the NJNF. Their orientation duality has already been listed by Schmidt [36].

$K^2$ contains only 2 elements, the Abelian $2A_1$ and $A_2$ represented by the algebra with $[e_2, e_1] = e_1$ as only nonvanishing bracket. Both are selfdual, because e.g. $e_1 \rightarrow -e_1$ does not change the algebra. So $K^2_{or} = K^2_{SD} = K^2$

The elements of $K^3$ correspond to the familiar Bianchi types. In the following we analyse the orientation duality by looking at the NJNF in $K^3$ and for non-selfduality also considering the ONJNF, defining the elements of $K^3_{\pm}$.

The solvable algebras in $K^3$ contain all the Abelian ideal $J_2 = 2A_1$. In Sec. 6.1 they are classified according to their NJNF. Similarly the solvable algebras in $K^3_{or}$ can be classified according to their ONJNF, which agrees the NJNF in the case of selfduality. So the selfdual algebras in $K^3_{or}$ correspond to the following cases of NJNF w.r.t. the Abelian ideal $J_2$:

$$NJNF(I) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad NJNF(V) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$NJNF(III) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$NJNF(VI_0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad NJNF(VI_h) = \begin{pmatrix} 1 \\ a \end{pmatrix}. \quad (7.1)$$

The algebras I and III are selfdual, since they are decomposable. All algebras in Eq. (7.1) invariant under the reflection $e_1 \rightarrow -e_1$, which guarantees
their selfduality. The parameter range \(0 < h < \infty, h \neq 1\) \((h\) denoting the parameter of Landau-Lifschitz [15]), corresponds monotonously to \(-1 < a < 1, a \neq 0.\) \(h = 1\) resp. \(a = 0\) yields the decomposable \(III\). So \(K_{SD}^3 = \{I, V\} \cup \{VI_h, 0 \leq h < \infty\}\).

The other solvable algebras which are not invariant under any reflection are non-selfdual. According to Sec. 3 and Sec. 4 we choose the reflection \(e_3 \rightarrow -e_3\) to characterize them as algebras in \(K_{\pm}^3\), with their ONJNF respectively given like following:

\[
\text{ONJNF}\{\Pi^{R/L}\} = \pm\text{NJNF}(\Pi) = \pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
\text{ONJNF}\{\IV^{R/L}\} = \pm\text{NJNF}(\IV) = \pm \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

\[
\text{ONJNF}\{\VII_0^{R/L}\} = \pm\text{NJNF}(\VII_0) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
\text{ONJNF}\{\VII_h^{R/L}\} = \pm\text{NJNF}(\VII_h) = \pm \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}. \tag{7.2}
\]

In Eq. (7.2) the parameter range \(0 < h < \infty\) \((h\) denoting the parameter of Landau-Lifschitz [15]) corresponds monotonously to the range \(0 < a < \infty\).

The simple algebras \(VIII = su(1,1)\) and \(IX = su(2)\) are described respectively by the 3 matrices

\[
C_{<3>} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_{<1>} = -C_{<2>} = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.
\]

\(\text{NJNF}(C_{<3>}) = \text{NJNF}(\VII_0)\) for both \(VIII\) and \(IX\), but \(\text{NJNF}(C_{<1>}) = \text{NJNF}(C_{<2>})\) is equal to \(\text{NJNF}(\VII_0)\) for \(VIII\) and to \(\text{NJNF}(\VII_0)\) for \(IX\).

In the Cartan-Weyl basis \(H := -ie_3, E_\pm := e_1 \pm ie_2\) the nonvanishing commutators are given as \([H, E_\pm] = \pm E_\pm\) and, for \(VIII\) or \(IX\) respectively, \([E_+, E_-] = \pm 2H\). Note that the latter are different real sections in the same complex algebra.

According to Lemma 4.5 neither \(VIII\) nor \(IX\) are selfdual. We discriminate the right and left algebra by the reflection \(e_3 \rightarrow -e_3\), defining both pairs \(VIII^{R/L}\) and \(IX^{R/L}\) of points in \(K_{\pm}^3\). So it is

\[
\text{ONJNF}\{C_{<3>}^{R/L}\} = \text{ONJNF}\{\VII_0^{R/L}\}. \tag{7.3}
\]
and $C_{<1>}$ and $C_{<2>}$ interchange under this reflection.

The table below summarizes the duality properties of the Bianchi classes in $K^3$. Note that for any point $A \in K^3 \backslash K_{SD}$ there exists a pair $(A^R, A^L) \in K^+ \oplus K^-$ of points in $K^3 \backslash K_{SD}$ with right/left handed bases respectively.

\[
\begin{array}{cccccccccc}
3A_1 & A_1 \oplus A_2 & A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & A_{3,9} \\
I & III & II & IV & V & VI & VI & VII & VIII & IX \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Table 1: 3-dimensional Lie algebra classes in $K^3$, corresponding Bianchi types and selfduality (yes=1/no=0)

The non-selfdual subset of $K^3_{or}$ has 2 connected 1-dimensional components, $K^3_+$ and $K^3_-$, given respectively by

\[
\text{VIII}^{R/L}/\text{IX}^{R/L} \rightarrow \text{VII}_{0}^{R/L} \leftarrow \text{VII}_{h}^{R/L} \rightarrow \text{IV}^{R/L} \rightarrow \Pi^{R/L}, \quad (7.4)
\]

where $\Pi^{R/L}$ is respectively the atom of $K^3_{\pm}$.

7.2 Structure of $K^4$

In Sec. 6.2 we classified the real 4-dimensional Lie algebras. In this section they are reconsidered under the aspect of orientation duality.

7.2.1 Selfdual Lie algebras

There exist following types of selfdual algebras: a) all decomposable ones, b) indecomposable ones with ideal I, and c) some indecomposable ones with ideal II.

a) Decomposable ones:

All decomposable Lie algebras are selfdual. A decomposable 4-dimensional Lie algebra can have the structures $4A_1$, $2A_1 \oplus A_2$, $2A_2$ or $A_1 \oplus A_3$. The first 3 possibilities are unique, since $A_1$ is the unique 1-dim. Lie algebra and $A_2$ is the unique nonAbelian 2-dim. Lie algebra. $A_1 \oplus A_3$ consists of 9 classes, given by \( \{A_{3,i}\}_{i=1,\ldots,9} \) listed in Table 1. Note that $A_1 \oplus I \equiv I^{(4)}$. 
b) Indecomposable ones with ideal \( J_3 = I \):

Algebras with Abelian ideal \( J_3 = 3A_1 \equiv I \) are selfdual. They are given by the following cases:

i) 1 Jordan block:
These algebras are invariant under a combination of the 3 reflections \( e_i \to -e_i, \ i = 1, \ldots, 3 \).
Either the eigenvalue is \( \lambda = 0 \) or otherwise it can be normalized to \( \lambda = 1 \).

\[
\text{NJNF}(A_{4,1}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
\text{NJNF}(A_{4,4}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

These algebras are a 4-dimensional analogue to II and IV. While the latter are non-selfdual their even dimensional analogues are selfdual. These algebras are the essential dimensional ones, introduced in Sec. 5 and denoted by ii(4) and iv(4). ii(4) is an essential dimensional atom.

ii) 2 Jordan blocks:
All these algebras are all invariant under the reflection \( e_1 \to -e_1 \).

\[
\text{NJNF}(A_{4,3}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{NJNF}(A_{4,2}^a) = \begin{pmatrix} a & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

In the latter case \( a \neq 0 \) and \( A_{4,2}^1 \equiv IV^{(4)} \).

iii) 3 real eigenvalues as Jordan blocks:
All these algebras are all invariant under the reflection \( e_1 \to -e_1 \).
Assuming the largest eigenvalue normalized to \( \lambda_1 = 1 \), there remain \( \lambda_2 = a \) and \( \lambda_3 = b \) with \( -1 \leq b \leq a \leq 1 \). If \( a \cdot b = 0 \), the algebra becomes decomposable \( (a = b = 0 \text{ yields III} \oplus \mathbb{R}, \ a = 1, b = 0 \text{ it is V} \oplus \mathbb{R}, \ a = 0, b = -1 \text{ it is VI}_0 \oplus \mathbb{R} \text{ and otherwise } a = 0 \text{ or } b = 0 \text{ yields VI}_b \oplus \mathbb{R}) \).
Therefore assume \( a \cdot b \neq 0 \). The case \( a = b = 1 \) (single 3-fold degenerate eigenvalue) corresponds to the pure vector type \( A_{4,5}^{1,1} \equiv V^{(4)} \neq V \oplus \mathbb{R} \). In
$A_{4,5}^{1,b}$ and $A_{4,5}^{a,a}$ there are 2 eigenvalues, one of them 2-fold degenerate. For the nondegenerate case, $-1 \leq b < a < 1$. Note that $A_{4,5}^{a,b} = A_{4,5}^{b,a}$, since permutations are in $GL(4)$.

$$NJNF(A_{4,5}^{a,b}) = \begin{pmatrix} 1 & a \\ b & \end{pmatrix}. \quad (7.7)$$

iv) 1 real eigenvalue and 2 complex conjugates:

All these algebras are all invariant under the reflection $e_1 \rightarrow -e_1$.

If $\lambda_{2,3} = r(\cos \theta \pm i \sin \theta)$, by normalization $r \sin \theta = 1$ can be achieved, if $\lambda_2 \neq \lambda_3$ is assured (otherwise the Jordan block becomes diagonal). Set then $r \cos \theta = b$ and $\lambda_1 = a$. Demand $a \neq 0$ to exclude decomposability ($a = 0$ yields VII$_h \oplus \mathbb{R}$ and $b = 0$ then corresponds to $h = 0$) and without restriction $b \geq 0$.

$$NJNF(A_{4,6}^{a,b}) = \begin{pmatrix} a & b & 1 \\ -1 & b & \end{pmatrix}. \quad (7.8)$$

c) Indecomposable ones with ideal $J_3 = II$:

There exist algebras with non-selfdual ideal $II$, which are selfdual.

$$ONJNF(A_{4,8}) = NJNF(A_{4,8}) = \begin{pmatrix} 0 & 1 \\ -1 & \end{pmatrix}. \quad (7.9)$$

This algebra is left invariant by a combination of reflections $e_4 \rightarrow -e_4$, $e_1 \rightarrow -e_1$ and $e_2 \leftrightarrow e_3$.

$$ONJNF(A_{4,10}) = NJNF(A_{4,10}) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (7.10)$$

This algebra is left invariant by a combination of reflections $e_4 \rightarrow -e_4$, $e_1 \rightarrow -e_1$ and $e_2 \rightarrow -e_2$. 

41
7.2.2 Non-selfdual Lie algebras

This kind of algebras exists with a basic ideal $J_3$, given either by the non-selfdual II or by the selfdual V. For all of them we have dual pairs of right and left points in $K^n_{or}$, which transform to each other by $e_4 \rightarrow -e_4$, constituting by Sec. 3 and Sec. 4 the connected components $K^+_4$ respectively.

a) Indecomposable ones with ideal $J_3 = \text{II}$:

The ideal II is non-selfdual. For an algebra $A$ of the kinds listed below the there exists no reflection leaving the set $J(A)$ invariant.

\[
\text{ONJNF}(A^{R/L}_{4,7}) = \pm \text{NJNF}(A_{4,7}) = \pm \begin{pmatrix} 2 & 1 & 1 \\ \end{pmatrix},
\]

\[
\text{ONJNF}(A^{b,R/L}_{4,9}) = \pm \text{NJNF}(A_{4,9}) = \pm \begin{pmatrix} 1 + b & 1 \\ b \\ \end{pmatrix}, 0 < |b| < 1,
\]

\[
\text{ONJNF}(A^{0,R/L}_{4,9}) = \pm \text{NJNF}(A_{4,9}) = \pm \begin{pmatrix} 1 & 1 \\ 0 \\ \end{pmatrix},
\]

\[
\text{ONJNF}(A^{1,R/L}_{4,9}) = \pm \text{NJNF}(A_{4,9}) = \pm \begin{pmatrix} 2 & 1 \\ 1 \\ \end{pmatrix},
\]

\[
\text{ONJNF}(A^{a,R/L}_{4,11}) = \pm \text{NJNF}(A_{4,11}) = \pm \begin{pmatrix} 2a & a & 1 \\ -1 & a \\ \end{pmatrix}, a > 0. \quad (7.11)
\]

b) Indecomposable ones with ideal $J_3 = \text{V}$:

The only case here is given by

\[
\text{ONJNF}(A^{R/L}_{4,12}) = \pm \text{NJNF}(A_{4,12}) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ \end{pmatrix}. \quad (7.12)
\]
Note that besides the selfdual ideal \( V \) there is a second ideal \( VII_0 \) which is not selfdual, causing here the subset of ideals \( S(A_{4,12}) \) to be non-selfdual. Hence \( A_{4,12} \) itself is non-selfdual.

c) The space \( K^4_{NSD} \) and its components \( K^4_\pm \):

Collecting the algebras of the previous subsections a) and b) and recalling transitions and parametrical limits of components in \( K^4_{NSD} \) according to Sec. 6, we find that \( K^4_{NSD} \) is connected, and so is each of \( K^4_\pm \). There are 2 pairs of \( K^4_{or,NSD} \)-atoms, given by \( A_{4,9}^{0,R/L} \) and \( A_{4,9}^{1,R/L} \). In \( n = 4 \) all \( K^4_{NSD} \)-atoms have an ideal II, and hence in the complement of the subspace \( K^4_{NSD}[I] \) of \( K^4_{NSD} \)-algebras with ideal I.

Let us assign e.g. \( A_{4,9}^{0,R/L} \) to \( K^4_+ \) respectively. Then the connectedness of \( K^4_\pm \) and the orientation preservation of limits within \( K^4_{or,NSD} \) imply the assignment \( A^{R/L} \) to \( K^4_\pm \) respectively. Note that with these assignments the component \( K^4_+ \) is given as

\[
\begin{align*}
A_{4,12}^R & \downarrow \quad A_{4,9}^{0,R} \quad A_{4,9}^{0<1,R} \quad A_{4,7}^R \quad A_{4,9}^{1,R} \quad A_{4,11}^{0>1,R} \\
A_{4,9}^{1<b<0,R} & \rightarrow \quad A_{4,9}^{0,R} \quad \leftarrow \quad A_{4,9}^{0<1,R} \quad A_{4,7}^R \quad A_{4,9}^{1,R} \quad \leftarrow \quad A_{4,11}^{0>1,R}
\end{align*}
\]

and the component \( K^4_- \) as

\[
\begin{align*}
A_{4,12}^L & \downarrow \quad A_{4,9}^{0,L} \quad A_{4,9}^{0<1,L} \quad A_{4,7}^L \quad A_{4,9}^{1,L} \quad A_{4,11}^{0>1,L} \\
A_{4,9}^{1<b<0,L} & \rightarrow \quad A_{4,9}^{0,L} \quad \leftarrow \quad A_{4,9}^{0<1,L} \quad A_{4,7}^L \quad A_{4,9}^{1,L} \quad \leftarrow \quad A_{4,11}^{0>1,L}
\end{align*}
\]

So the non-selfdual components of \( K^4_{(or)} \) are 1-dimensional. Note that \( A_{4,1}^{R/L} \equiv \mathrm{ii}(4)^{R/L} \) is the atom of \( K^4_\pm \) respectively.

8 Discussion and outlook

In Sec. 6 we determined Lie algebra transitions in \( K^4 \) as limits induced by the topology \( \kappa^4 \). Any Inönü-Wigner contraction corresponds to a certain transition; explicitly any of the Inönü-Wigner contractions listed in the tables of Huddleston [11] for real 4-dimensional Lie algebras corresponds to a transition in \( K^4 \). Since Inönü-Wigner contractions are only a special case of the more general Saletan contractions, and since even the latter do not
induce all possible transitions in $K^n$ with $n \geq 3$, it should not be surprising that we have obtained transitions, which do not correspond to any Inönü-Wigner contraction, like e.g. transitions $IX \oplus \mathbb{R} \to A_{4,10}$, $VIII \oplus \mathbb{R} \to A_{4,10}$ and transitions from $A_{4,5}^{a,b}$, $A_{4,6}^{a,b}$, $VI_h \oplus \mathbb{R}$ and $VII_h \oplus \mathbb{R}$ to $A_{4,1}$. The transition $IX \oplus \mathbb{R} \to A_{4,10}$ corresponds to a Lie algebra contraction, which was given already in [33] (see Eqs. (35') to (37)) as an example of a Saletan contraction, which can not be obtained as a Inönü-Wigner contraction.

It is also interesting to consider transitions in $K^3$ as obtained in Sec. 5. The Inönü-Wigner contractions for real Lie algebras of dimension $d \leq 3$ are classified already by Conatser [7]. The sequence of transitions $VIII \to VI_{0} \to II \to I$ is generated by an iterated Saletan contraction (see [33], Eqs. (30) and (31)), applied first to the Lie algebra $VIII$, of the 3-dimensional homogenous Lorentz group. On the 4-point subset $\{VIII, VI_{0}, II, I\}$ Saletan contractions are transitive. However this transitivity does not hold for Saletan transitions on general subsets of $K^3$. The sequence of transitions $IX \to VII_{0} \to II \to I$, starting from the Lie algebra $IX$ of the 3-dimensional Euclidean group, can not be obtained by Saletan contractions. The only Saletan contractions starting from $IX$ are in fact given by a Inönü-Wigner contraction $IX \to VII_{0}$ and the trivial contraction $IX \to I$. Though there exists a different Inönü-Wigner contraction corresponding to the transitions $VII_{0} \to II$ there is no Saletan contraction corresponding to $IX \to II$ (for a proof see [33]). This example shows that, on an arbitrary subset of $K^n$ with $n \geq 3$, in general not every transition can be obtained from a Saletan contraction. It implies that, even on a set of points connected by Inönü-Wigner contractions, neither Saletan contractions nor Inönü-Wigner contractions need to be transitive.

Since we consider transitions between different points in $K^n$, improper contractions of an algebra to an equivalent one can not be seen by our method. For $n = 4$ Huddleston [11] identified two types of algebras which admit only trivial and improper contractions. These are precisely the two atoms of $K^4$, namely the unimodular $II^{(4)} \equiv II \oplus \mathbb{R}$ and the pure vector type $V^{(4)} \equiv A_{4,1}^{1,1}$. For arbitrary dimension $n$, the atoms of $K^n$ have been introduced and described first by Schmidt [36].

By now the topological properties of $K^n$ for $n \leq 4$ have been examined. It is natural to demand an investigation for arbitrary dimension $n$. Practically, this is obstructed by the rapidly increasing number of equivalence classes for increasing $n$. A classification for all nilpotent algebras has been done for $n = 6$ by Morozov [21] and for $n = 7$ by Ancochea-Bermudez and Goze [1] in the complex case and by Romdhani [32], who distinguishes
132 components of real indecomposable nilpotent 7-dimensional Lie algebras. For $K^5$ a full classification of all real Lie algebras still distinguishes 40 components (compare Mubarakzjanov [23] and Patera et al. [26]). A determination of all possible transitions would be a rather tidy work. However it is known by [26] that $\dim K^5 = 3$, because the maximal dimension of its components is 3. Unlike for the classification of subalgebra structures of each class in $K^n$ (see Patera, Winternitz [27], and Grigore, Popp [9]), for the determination of all equivalence classes and transitions between them there exists no algorithm at present. However, a systematic exploitation of the NJNF, which has been defined for arbitrary $n$, may contribute some part to further progress.

The NJNF has proven to be a useful tool in characterizing distinct $n$-dimensional Lie algebras with a common ideal $J_{n-1}$ as endomorphisms ade$_n$ of a complementary generator $e_n$ on that ideal, with characteristic Jordan blocks of their eigenvalues normalized by an overall scale. In 4 dimensions, besides decomposable algebras, only cases with ideal I, II, or V appear.

In 4 dimensions, there are no simple algebras. In general for $n \geq 6$ further classes of simple Lie algebras arise, which lead to an additional further sophistication, as compared to $n = 3$.

A combination of the established knowledge on semisimple Lie algebras with the full classification of all Lie algebras would be desirable, but is practically far away, since the dimensionality of the simple Lie algebras increases rapidly with their rank. Note that all simple components belong to the unimodular subset $U^n_n$. In Sec. 7 we found that simple Lie algebras are non-selfdual w.r.t. orientation reflection.

In general, we have neither a formula for $\dim K^n$ nor for $\dim U^n_n$. The $T_0$ topology allows components of different dimensions to converge pointwise to each other, i.e. such that any point of the first component converges to some point of the second component and any point of the second component is the limit of some point of the first.

We have determined the topology of the space $K^n_n$ for $n \leq 4$. The essential difference to $K^n$ is that the single non-selfdual component of the latter is doubled to two components $K^n_+ \text{ and } K^n_-$. For $n = 3$ or 4, the space $K^{n}_{NSD}$ is nonvoid and connected. For $n = 3$ there is a unique $K^3_{NSD}$-atom $\text{ii}(3) = \text{II}$. $K^4_{NSD}$ has two atoms, $A^0_{4,9}$ and $A^1_{4,9}$, and $A^0_{4,9} \text{ and } A^1_{4,9}$ has boundary limits to both of them.

If $K^n_{NSD}$ is connected, the arbitrariness in assigning conjugate pairs of points to $K^n_\pm$ can be reduced to a single decision for one pair only, if we
demand that both of $K^n$ are connected and to each other disconnected.

At present, for general $n \geq 5$ it is not known whether $K_{NSD}$ is connected.

From Eqs. (7.4) and (7.13-14), we see that the non-selfdual subset $K^n_{NSD,(or)}$ of $K^n_{(or)}$ is 1-dimensional for both, $n = 3$ and $n = 4$. We have $\dim K^3_{(or)} = 1$ and $\dim K^n_{(or)} \geq 2$ for dimension $n \geq 4$: In the latter case the contribution of the non-selfdual subset is of dimension less than that of the highest-dimensional component, while in the former case it is of highest dimension. Actually, the topology of the highest-dimensional component of $K^3_{or}$ differs from that of $K^3$ essentially. The question of dimensionality for general $n \geq 5$ remains open, for the non-selfdual subset as well as for $K^n_{(or)}$ itself.

Partial progress has been made by determining a candidate of an atom of the non-selfdual subset in odd dimension $n$. We found an interesting periodicity in the structure of this $K^n_{NSD}$-atom: it is $ii(n)$ for $n = 3 \text{ mod } 4$ and $iv(n)$ for $n = 1 \text{ mod } 4$. However it remains an open problem to determine all atoms of $K^n_{NSD}$ for arbitrary $n \geq 5$. Presently we do not know how an atom for even $n \geq 6$ looks like in general.

With Definition 6 the notion of an atom from Schmidt [36] has been generalized to arbitrary subsets.

Although the present work is on the case of real Lie algebras, we want to make some comments on the analogous complex cases to the pairs of algebras $\text{VII}_h/\text{VI}_h$ ($h \geq 0$), $\text{IX}/\text{VIII}$ of $K^3$ and $\text{A}_{1,6}/\text{A}_{4,5}$, $\text{A}_{4,11}/\text{A}_{4,9}$ and $\text{A}_{4,10}/\text{A}_{4,8}$ of $K^4$. If one considers the analogous 3- or 4-dimensional Lie algebras over the complex basic field the group $\text{GL}(n)$ is now correspondingly the group of nonsingular complex linear transformation. The pairs of complex conjugated eigenvalues associated to the $2 \times 2$ Jordan block of each of the first algebras of the pairs above in the complex remain as 2 Jordan blocks in a corresponding complex JNF. After introducing a similar normalization convention like for the real case, the complex analogues of the NJNF will be the same for members of any pair above. (For $n = 3$ this had already been realized by Bianchi [2]. The complex 4-dimensional case was considered already by Lie [17]).

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