Abstract: In the Economic Nonlinear Model Predictive (ENMPC) context, closed-loop stability relates to the existence of a storage function satisfying a dissipation inequality. Finding the storage function is in general— for nonlinear dynamics and cost— challenging, and has attracted attentions recently. Q-Learning is a well-known Reinforcement Learning (RL) techniques that attempts to capture action-value functions based on the state-input transitions and stage cost of the system. In this paper, we present the use of the Q-Learning approach to obtain the storage function and verify the dissipativity for discrete-time systems subject to state-input constraints. We show that undiscounted Q-learning is able to capture the storage function for dissipative problems when the parameterization is rich enough. The efficiency of the proposed method will be illustrated in the different case studies.

Keywords: Economic Nonlinear Model Predictive Control, Reinforcement Learning, Q-learning, Dissipativity, Storage Function

1. INTRODUCTION

Economic Nonlinear Model Predictive Control (ENMPC) optimizes an economic performance rather than penalizing deviations from a desired steady-state reference (Rawlings and Amrit (2009)). Dissipativity is a fundamental concept in ENMPC, when the objective stage cost is not necessarily positive definite (Diehl et al. (2010)). In contrast to tracking costs, an optimal economic policy may not lead to the closed-loop stability of the optimal steady-state point (Amrit et al. (2011)). The dissipativity property allows one to equalize the ENMPC with a tracking MPC which has a well-established stability conditions. Unfortunately, it is not trivial to prove that a given problem is dissipative (Pirkelmann et al. (2019)). In order to prove it, one has to find a storage function that satisfies the dissipation inequality. Scherer and Weiland (2015) used Linear Matrix Inequality (LMI) techniques to compute the storage function for linear systems with a quadratic stage cost. A similar method is used by Koch et al. (2020) to verify dissipativity properties based on noisy data for linear systems. Furthermore, the Sum-of-Squares (SOS) method is used by Pirkelmann et al. (2019) for polynomials dynamics and stage cost. Zanon et al. (2016) showed how tracking MPC schemes can approximate generic economic NMPC locally and provided conditions for the local existence of a storage function using a Semi-Definite Program (SDP). Grune and Müller (2016) showed that strict dissipativity with the addition of a suitable controllability property leads the turnpike property in optimal control. Roughly speaking, the turnpike property means that open-loop optimal trajectories spend most of the time in a vicinity of the steady-state solution.

Reinforcement Learning (RL) offers tools for tackling Markov Decision Processes (MDP) without having an accurate knowledge of the probability distribution underlying the state transition (Sutton and Barto (2018); Bertsekas (2019)). RL seeks to optimize the parameters underlying a given policy in view of minimizing the expected sum of a given stage cost. Q-learning attempts to find the optimal policy by capturing the optimal action-value function underlying the MDP. Model Predictive Control (MPC) scheme used as a value function approximator, provides a formal framework to analyse the stability and feasibility of the closed-loop system (Rawlings et al. (2017)). Gros and Zanon (2020, 2019) showed that adjusting the MPC model, cost and constraints allows the MPC to capture the optimal policy for the system even if using an inaccurate model of the system. RL is proposed and recent research have focused on MPC-based approximation for RL (see e.g. Koller et al. (2018); Mannucci et al. (2017); Gros and Zanon (2020); Bahari Kordabad et al. (2021b,a); Nejatbakhsh Esfahani et al. (2021)).

In this paper, we propose to use undiscounted Q-learning as a method to compute the storage function and verify the dissipativity conditions. We assume that the given problem is dissipative and replace the ENMPC by its equivalent tracking MPC-scheme (positive stage cost) with an additional storage function. We parametrized the storage function, stage cost and terminal cost in the tracking MPC-scheme, and use it as a function approximator in Q-learning based on the economic stage cost. Then Q-learning adjusts the parameters by capturing the optimal action-value function. Using different examples we show that this method can be used for more general, nonlinear...
dynamics and cost to deliver the storage function with high accuracy.

The rest of the paper is structured as follows. Section 2 provides the ENMPC formulation and the dissipativity conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.

2. ECONOMIC NONLINEAR MPC

In this section we detail the ENMPC formulation and the dissipativity condition. Consider the following discrete time, constrained dynamic system:

\[ s_{k+1} = f(s_k, a_k), \quad h(s_k, a_k) \leq 0, \]

(1)

where \( k = 0, 1, \ldots \) is the physical time index, \( s_k \in X \subseteq \mathbb{R}^n \) is the state, \( a_k \in U \subseteq \mathbb{R}^m \) is the input. Vector field \( f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) expresses the state transition. Function \( h \) is a mixed input-state constraint. The set of feasible state-input pairs is defined as follows:

\[ \mathcal{Z} := \{ (s, a) \in X \times U \mid f(s, a) \in X, h(s, a) \leq 0 \} \]

(2)

The selected economic stage cost, denoted by \( L : \mathcal{Z} \rightarrow \mathbb{R} \), is not necessarily positive definite. The following standard optimality conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.

2. ECONOMIC NONLINEAR MPC

In this section we detail the ENMPC formulation and the dissipativity condition. Consider the following discrete time, constrained dynamic system:

\[ s_{k+1} = f(s_k, a_k), \quad h(s_k, a_k) \leq 0, \]

(1)

where \( k = 0, 1, \ldots \) is the physical time index, \( s_k \in X \subseteq \mathbb{R}^n \) is the state, \( a_k \in U \subseteq \mathbb{R}^m \) is the input. Vector field \( f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) expresses the state transition. Function \( h \) is a mixed input-state constraint. The set of feasible state-input pairs is defined as follows:

\[ \mathcal{Z} := \{ (s, a) \in X \times U \mid f(s, a) \in X, h(s, a) \leq 0 \} \]

(2)

The selected economic stage cost, denoted by \( L : \mathcal{Z} \rightarrow \mathbb{R} \), is not necessarily positive definite. The following standard optimality conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.

2. ECONOMIC NONLINEAR MPC

In this section we detail the ENMPC formulation and the dissipativity condition. Consider the following discrete time, constrained dynamic system:

\[ s_{k+1} = f(s_k, a_k), \quad h(s_k, a_k) \leq 0, \]

(1)

where \( k = 0, 1, \ldots \) is the physical time index, \( s_k \in X \subseteq \mathbb{R}^n \) is the state, \( a_k \in U \subseteq \mathbb{R}^m \) is the input. Vector field \( f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) expresses the state transition. Function \( h \) is a mixed input-state constraint. The set of feasible state-input pairs is defined as follows:

\[ \mathcal{Z} := \{ (s, a) \in X \times U \mid f(s, a) \in X, h(s, a) \leq 0 \} \]

(2)

The selected economic stage cost, denoted by \( L : \mathcal{Z} \rightarrow \mathbb{R} \), is not necessarily positive definite. The following standard optimality conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.

2. ECONOMIC NONLINEAR MPC

In this section we detail the ENMPC formulation and the dissipativity condition. Consider the following discrete time, constrained dynamic system:

\[ s_{k+1} = f(s_k, a_k), \quad h(s_k, a_k) \leq 0, \]

(1)

where \( k = 0, 1, \ldots \) is the physical time index, \( s_k \in X \subseteq \mathbb{R}^n \) is the state, \( a_k \in U \subseteq \mathbb{R}^m \) is the input. Vector field \( f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) expresses the state transition. Function \( h \) is a mixed input-state constraint. The set of feasible state-input pairs is defined as follows:

\[ \mathcal{Z} := \{ (s, a) \in X \times U \mid f(s, a) \in X, h(s, a) \leq 0 \} \]

(2)

The selected economic stage cost, denoted by \( L : \mathcal{Z} \rightarrow \mathbb{R} \), is not necessarily positive definite. The following standard optimality conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.

2. ECONOMIC NONLINEAR MPC

In this section we detail the ENMPC formulation and the dissipativity condition. Consider the following discrete time, constrained dynamic system:

\[ s_{k+1} = f(s_k, a_k), \quad h(s_k, a_k) \leq 0, \]

(1)

where \( k = 0, 1, \ldots \) is the physical time index, \( s_k \in X \subseteq \mathbb{R}^n \) is the state, \( a_k \in U \subseteq \mathbb{R}^m \) is the input. Vector field \( f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n \) expresses the state transition. Function \( h \) is a mixed input-state constraint. The set of feasible state-input pairs is defined as follows:

\[ \mathcal{Z} := \{ (s, a) \in X \times U \mid f(s, a) \in X, h(s, a) \leq 0 \} \]

(2)

The selected economic stage cost, denoted by \( L : \mathcal{Z} \rightarrow \mathbb{R} \), is not necessarily positive definite. The following standard optimality conditions and explains how one can parameterize the ENMPC-scheme. Section 3 details the Q-learning approach and constrained learning steps and discusses efficiency of the proposed method. Section 4 illustrates the simulations results for the different case studies. Section 5 delivers a conclusion.
functions. We force \( \tilde{\rho} \) to be lower bounded by a \( K_\infty \) function in the Q-learning steps or by construction, but we do not impose any restriction on \( \tilde{\rho} \) or the storage function \( \lambda_\theta \). If the parameterization is rich enough, Q-learning will be able to capture the optimal action-value function under some mild assumptions shown in Zanon and Gros (2020). As a result, the stage cost, terminal cost and storage function can be adjusted to deliver the optimal action-value and optimal policy. As a result:

\[
\rho(\|s-s_c\|) \leq \tilde{\rho}(s,a) \quad (14)
\]

holds for some \( \rho(.) \in K_\infty \), and it implies the dissipativity inequality (7).

Next section details the conditions on the stage cost \( \tilde{\rho} \) and discusses how \( \theta \) should be updated using undiscounted Q-learning to capture the storage function, optimal policy and optimal action-value function while respecting the possible constraints on the parameters \( \theta \).

3. EVALUATION OF THE STORAGE FUNCTION USING Q-LEARNING

Q-learning is a classical model-free RL algorithm that tries to capture the optimal action-value function via tuning the parameter vector \( \theta \). The MPC-based action-value function can be extracted as follows (see Gros and Zanon (2019)):

\[
Q_\theta(s,a) = \min_{u,x} \quad (12a) \\
\quad s.t. \quad (12b),(12c) \quad (15a)
\]

The parameterized deterministic policy can be obtained as:

\[
\pi_\theta(s) = \tilde{u}^*_\theta(s,\theta), \quad (16)
\]

where \( \tilde{u}^*_\theta(s,\theta) \) is the first element of \( \tilde{u}^* \), which is the input solution of the MPC scheme (12). It is shown by Gros and Zanon (2019) that the value function \( V_\theta \) in (12), the action-value function \( Q_\theta \) in (15) and the policy \( \pi_\theta \) in (16) satisfy the fundamental Bellman equations. For an undiscounted problem, basic Q-learning uses the following update rule for the parameters \( \theta \) at state \( s_k \) (see e.g. Sutton and Barto (2018)):

\[
\delta_k = \ell(s_k,a_k) + V_\theta(s_{k+1}) - Q_\theta(s_k,a_k) \quad (17a) \\
\theta_{k+1} = \theta_k + \alpha \delta_k \nabla_\theta Q_\theta(s_k,a_k) \quad (17b)
\]

where the scalar \( \alpha > 0 \) is the learning step-size, \( \delta_k \) is labelled the Temporal-Difference (TD) error and the input \( a_k \) is selected according to the corresponding parametric policy \( \pi_\theta(s_k) \).

Since the constraints in the MPC (12) are independent of \( \theta \), the sensitivity \( \nabla_\theta Q_\theta(s) \) required in Eq.(17b) only depends on the cost and it is given by (see Gros and Zanon (2019)):

\[
\nabla_\theta Q_\theta(s,a) = \nabla_\theta \Psi_\theta(u^*,x^*) \quad (18)
\]

where \( \Psi_\theta \) gathers the cost for (12a) and \( x^*,u^* \) is the solution of (15) for a given \( s,a \) pair.

In the following we made an assumption in parameterization that allows us to evaluate optimal action-value and the storage function.

**Assumption 2.** We assume that the parametrization is rich enough. I.e. there exists \( \theta^* \) such that:

\[
\pi_\theta(s) = \pi^*(s), \quad V_\theta(s) = V^*(s), \quad (19a)
\]

\[
Q_\theta(s,a) = Q^*(s,a) \quad (19b)
\]

**Lemma 1.** If the assumption 2 holds, then \( \theta^* \) is a fixed point of (17).

**Proof.** From (6) and (19), we have:

\[
\delta = \ell(s,a) + V_\theta(f(s,a)) - Q_\theta(s,a) \\
= \ell(s,a) + V^*(f(s,a)) - Q^*(s,a) = 0
\]

Let us define \( \Psi_\theta^N(s) \) and \( \Lambda_\theta^N(s,a) \) as follows:

\[
\Psi_\theta^N(s) := \min_{u,x} T_\theta(x_N) + \sum_{k=0}^{N-1} \tilde{\theta}(x_k, u_k) \quad (20a)
\]

\[
\Lambda_\theta^N(s,a) := \min_{u,x} T_\theta(x_N) + \sum_{k=0}^{N-1} \tilde{\theta}(x_k, u_k) \quad (20b)
\]

One can observe that:

\[
\Lambda_\theta^N(s,a) = \tilde{\theta}(s,a) + \Psi_\theta^{N-1}(f(s,a)) \quad (21)
\]

Before providing the main theorem that expresses how Q-learning can capture a valid storage function we make a basic stability assumption.

**Assumption 3.** For all \( s \in X \), there exists an input \( a \) (such that \( (s,a) \in Z \)) satisfying:

\[
T_\theta(f(s,a)) - T_\theta(s) \leq -\tilde{\rho}(s,a), \quad h(s,a) \leq 0 \quad (22)
\]

Assumption 3 is a basic assumption in the MPC context that allows one to discuss about the stability. If there exists a terminal constraint in form \( x_f \in \mathbb{X}_f \subseteq \mathbb{X} \) in the MPC (12), then the assumption 3 will be limited to be satisfied for all \( s \in \mathbb{X}_f \). To satisfy that in the Q-learning context, one needs to provide a generic function approximator for the terminal cost \( T_\theta \). The following Lemma on the monotonocity property of the MPC value function is useful in the building of the theorem.

**Lemma 2.** Under the assumptions 1 and 3, we have:

\[
\Psi_\theta^{N+1}(s) \leq \Psi_\theta^N(s), \quad \forall N \geq 0 \quad (23)
\]

**Proof.** See Rawlings et al. (2017).

**Theorem 1.** Assumption (1)-(3), for a dissipative problem with rich enough parameterization, the undiscounted Q-learning yields a valid storage function which satisfies (7) if there exists a \( \mathbb{X}_\infty \) functions \( \alpha_\ell(.) \) satisfying

\[
\alpha_\ell(\|s-s_c\|) \leq \tilde{\rho}(s,a), \quad \forall (s,a) \in Z \quad (24)
\]

**Proof.** Using (19b) and (15), we have:

\[
Q^*(s,a) = Q_\theta^*(s,a) = -\lambda_\theta^*(s) + \Lambda_\theta^*(s,a) \\
= -\lambda_\theta^*(s) + \tilde{\theta}(s,a) + \Psi_\theta^{N-1}(f(s,a)) \quad (25)
\]

Moreover, from (6),(19a) and (12), we have:

\[
Q^*(s,a) = \ell(s,a) + V^*(f(s,a)) = \ell(s,a) + V_\theta(f(s,a)) \\
= \ell(s,a) - \lambda_\theta^*(f(s,a)) + \Psi_\theta^{N-1}(f(s,a)) \\
\leq \ell(s,a) - \lambda_\theta^*(f(s,a)) + \Psi_\theta^{N-1}(f(s,a)) \quad (26)
\]
where we used (23) in the last inequality. From (25) and (26), we have:
\[
\lambda_{\theta}(f(s, a)) - \lambda_{\theta}(s) \leq -\dot{\lambda}_{\theta}(s, a) + \ell(s, a) \\
\leq -\alpha_1(\|s - s_e\|) + \ell(s, a)
\]

In order to satisfy (27), we generalize it to hold for all parameters set \( \theta \), i.e.
\[
\alpha_1(\|s - s_e\|) \leq \dot{\lambda}_{\theta}(s, a), \quad \forall (s, a) \in \mathbb{Z}
\]  

(27)

Under assumption 3, if the inequality (27) holds, MPC-scheme (12) is stable (Rawlings et al. (2017)), i.e. a dissipative problem has an equivalent tracking MPC which is stable. Then the policies converge to the cost-free steady-state point \( \ell(s_e, a_e) = 0 \) as time goes to infinity. Under assumption 1 and 2, the undiscounted Q-learning (17) converges to its fixed-point for a dissipative problem (Tsitiklis (1994)). From Theorem 1, the inequality (27) yields the dissipativity inequality (7). Unfortunately, Q-learning steps delivered by (17) do not necessarily respect the requirement of the inequality (27) for stage cost \( \ell_{\theta} \). We discuss briefly here two solutions to satisfy (27) and each solution will be extended in future works.

1) Parametrize the stage cost \( \ell_{\theta} \) such that be zero at steady-state and strictly positive otherwise using a positive output function approximator. E.g., for a quadratic approximator, one can use the following function approximator:
\[
\ell_{\theta}(s, a) = (s - s_e)^T (a - a_e)(M(\theta) + \epsilon I)(s - s_e) + (a - a_e)^T \theta
\]
where \( M \) is an arbitrary parameters matrix and \( \epsilon \) is a small positive constant. In order to provide a more generic positive stage cost, one can use a positive activation function in the output layer of a neural network approximator.

2) Using SOS method for the polynomial approximator of the stage cost \( \ell_{\theta} \). In this case, the positivity of the stage cost can be represented as the following linear matrix inequality:
\[
0 \prec R(\theta_k), \quad \forall k \geq 0
\]

(29)

where
\[
\dot{\lambda}(s, a) = m^T (s - s_e, a - a_e) R(\theta) m(s - s_e, a - a_e)
\]
and where \( m \) is a vector of monomials of \( s - s_e \) and \( a - a_e \) without bias (constant) value to preserve to be zero at the steady-state point. In this case, the update rule (17) can be seen as minimizing a quadratic cost subject to some constraints in the parameters taking the form of a Semidefinite program (SDP):
\[
\min_{\Delta \theta} \frac{1}{2} \| \Delta \theta \|^2 - \alpha \delta \nabla^2 Q_0(s_k, a_k) \Delta \theta
\]

s.t. \( 0 \prec R(\Delta \theta + \theta_k) \)

(31a)

(31b)

where \( \Delta \theta := \theta_{k+1} - \theta_k \). One can observe that \( \theta_k \) will satisfy (29). Note that using SOS method in this method has less complexity and more accuracy in the storage function rather than solving (7) directly using SOS method. Indeed, to solve dissipativity (7) using SOS one needs to approximate the exact stage cost \( \ell \) and dynamics by polynomials and the obtained storage function is based of these approximations and may have a bias with the exact storage function (Pirkelmann et al. (2019)). However, to solve (31), one only needs to provide a SOS MPC stage cost \( \ell_{\theta} \). Using a generic approximator for the storage function and terminal cost, Q-learning will be able to find the best polynomial stage cost and capture the optimal action-value function. Indeed, the storage function is used as a generic approximator, and \( \ell \) and dynamics do not need to be approximated by the polynomial.

In practice, if the parametrization is not rich (assumption 2 does not hold), \( \delta \) will not converge to zero necessarily. In this case, Q-learning will find the optimal parameters among the provided functions family. Then we will check the dissipativity inequality (7) for the optimal parameters. If it holds, then the problem is dissipative, otherwise the algorithm is inconclusive, i.e. the parametrization is not rich enough or the problem is non-dissipative. The proposed approach is summarized in the Algorithm 1.

Algorithm 1: Q-learning to evaluate storage function and verify the dissipativity

\begin{itemize}
  \item \textbf{Input:} Parameterize \( \lambda_{\theta}, T_{\theta}, \) and \( \ell_{\theta} \)
  \item Initialize \( \theta_0 \)
  \item \textbf{while} \( \|\delta\| > \epsilon \) \textbf{do}
  \item \quad Update \( \theta \) from (31) if \( \ell_{\theta} \) is SOS (or from (17) if \( \ell_{\theta} \) is positive definite by construction)
  \item \quad Compute rotated storage function \( \ell_{\theta} \) from learned storage function in (13)
  \item \quad \textbf{if} Converge \textbf{then}
  \item \quad \quad System is dissipative and \( \lambda_{\theta} \) is a valid storage function
  \item \quad \quad Inconclusive
  \item \quad \textbf{else}
  \item \quad \quad Inconclusive
  \item \textbf{end}
\end{itemize}

4. SIMULATION

In this section, we provide four numerical examples in order to illustrate the efficiency of the proposed method.

4.1 LQR case

First, we look at a trivial Linear dynamics. Quadratic stage cost, regulator (LQR) problem with non-positive stage cost. LQR is a well-known problem, because the exact optimal policy, value function and stage cost are obtainable using other techniques, e.g., Riccati equation. Consider the following linear dynamics with a quadratic economic stage cost:

\[
s_{k+1} = 2s_k - a_k, \quad L(s, a) = s^2 + a^2 + 4sa \geq 0
\]

(32)

The optimal steady state is \( (s_o, a_o) = (0, 0) \). Using Riccati equation, the optimal value function and policy read as:

\[
V^*(s) = 11.7446s^2, \quad \pi^*(s) = 1.6861s
\]

(33)

The storage function \( \lambda_{\theta} \), terminal cost \( T_{\theta} \) and stage cost approximations \( \ell_{\theta} \) are selected as follows:
where $\theta = \{\theta_1, \ldots, \theta_6\}$ is the set of parameters adjusted by Q-learning. The RL steps are restricted to making $T_\theta$ positive definite. We initialize $\theta_0 = [0.1, 1, 1, 0, 0, 0]^\top$. Fig. 1 shows the convergence of the parameters resulting from Q-learning during 50 episodes. As can be seen in Fig. 2, after 50 iterations Q-learning is able to capture the optimal value and optimal policy functions. The learned storage function is $\lambda_\theta(s) = \theta_1 s^2$, $T_\theta(s) = \theta_2 s^2$ (34a)

\[ \hat{\ell}_\theta(s) = \begin{bmatrix} s \\ \theta \end{bmatrix}^\top \begin{bmatrix} \theta_3 & \theta_4 \\ \theta_5 & \theta_6 \end{bmatrix} \begin{bmatrix} s \\ \theta \end{bmatrix} \] (34b)

for $0 < \rho = 2.3286$.

4.2 Non-dissipative dynamics

We provide next an example of non-dissipative dynamics and stage cost. Consider the following dynamics:

\[ s_{k+1} = a_k, \quad L(s, a) = -2 s^2 + a^2 + sa + s^2 a^2 \] (36)

the optimal steady-state is $s_c = a_c = 0$. In the Appendix, it is shown that there is no storage function for this example, i.e., (36) is non-dissipative. A quadratic stage cost, terminal cost, and storage function with adjustable parameters are used for the simulation. Fig. 3 shows the learned rotated stage cost after convergence. It can be seen that Q-learning does not manage to learn a positive definite rotated stage cost.

4.3 Non-polynomial case

Consider the following dynamics with a non-polynomial economic stage cost:

\[ s_{k+1} = a_k, \quad L(s, a) = -\ln(5s^{0.34} - a), \] (37)

and

\[ X = [0, 10], \quad U = [0.01, 10] \] (38)

This model is a benchmark optimal investment problem where $s$ denotes the investment in a company and the term $5s^{0.34}$ is the return from this investment after one period. Then $5s^{0.34} - a$ is the amount of money that can be used for consumption in the current time period. Then the objective is to maximize the sum of logarithmic utility function. The optimal steady-state point is $s_c = a_c = 2.2344$. It is shown in Pirkelmann et al. (2019) that a storage function in the form

\[ \lambda_\theta(s) = \theta(s - s_c), \] (39)

is valid for this problem and the Sum-of-Squares (SOS) approach delivers an approximated storage function ($\theta = 0.23$) using an order-3 Taylor approximation for the stage cost. The analytical solution can be obtained for $\theta$ (see Pirkelmann et al. (2019)). We use the approximated storage function obtained from SOS as an initial guess for $\lambda_\theta$ and apply the proposed learning method. The following stage cost is used in the simulation:

\[ \hat{\ell}_\theta(s, a) = L(s, a) = L(s_c, a_c) + \theta(s - s_c) - \theta(a - a_c), \] (40)

and we use a long horizon $N = 100$ without terminal cost. Fig. 4 shows the update of the parameter $\theta$ over the 100 episodes. As can be seen, the learning-based storage function converges to the analytical solution, while the parameter $\theta$ resulting from SOS method has a bias. Note that this example is simple and the bias issue is not significant, but for a more complex problem in practice the SOS method may lead a storage function that has more bias with the true storage function. Fig. 5 illustrates the rotated stage cost from the learned storage function.

5. CONCLUSION

This paper presented the use of undiscounted Q-learning to evaluate the storage function and verify the dissipativity in the ENMPC-scheme for general discrete-time dynamics and cost subject to input-state constraint. We
parametrized the equivalent tracking MPC-scheme and delivered it to the RL as a function approximator. We showed that if the parameterization is rich enough, Q-learning will be able to deliver a valid storage function for a dissipative problem. The efficiency of the method was illustrated in the different case studies. Combining this method with SOS, applying for the noisy data and stochastic systems can be considered in future research.

REFERENCES

Amrit, R., Rawlings, J.B., and Angeli, D. (2011). Economic optimization using model predictive control with a terminal cost. Annual Reviews in Control, 35(2), 178–186.

Bahari Kordabad, A., Cai, W., and Gros, S. (2021a). MPC-based reinforcement learning for economic problems with application to battery storage. arXiv preprint arXiv:2104.02411.

Bahari Kordabad, A., Nejatbakhsh Esfahani, H., Lekkas, A.M., and Gros, S. (2021b). Reinforcement learning based on scenario-tree MPC for ASVs. arXiv e-prints, arXiv:2103.

Bertsekas, D.P. (2019). Reinforcement learning and optimal control. Athena Scientific Belmont, MA.

Diehl, M., Amrit, R., and Rawlings, J.B. (2010). A lyapunov function for economic optimizing model predictive control. IEEE Transactions on Automatic Control, 56(3), 703–707.

Gros, S. and Zanon, M. (2019). Data-driven economic nmpc using reinforcement learning. IEEE Transactions on Automatic Control, 65(2), 636–648.

Gros, S. and Zanon, M. (2020). Reinforcement learning for mixed-integer problems based on mpc. arXiv preprint arXiv:2004.01430.

Grüne, L. and Müller, M.A. (2016). On the relation between strict dissipativity and turnpike properties. Systems & Control Letters, 90, 45–53.

Koch, A., Berberich, J., and Allgöwer, F. (2020). Provably robust verification of dissipativity properties from data. arXiv preprint arXiv:2006.05974.

Koller, T., van Kampen, E.J., de Visser, C., and Chu, Q. (2017). Safe exploration algorithms for reinforcement learning controllers. IEEE transactions on neural networks and learning systems, 29(4), 1069–1081.

Nejatbakhsh Esfahani, H., Bahari Kordabad, A., and Gros, S. (2021). Safe reinforcement learning based on MPC/MHE for unmodeled and partially observable dynamics. arXiv e-prints, arXiv:2103.

Pirkelmann, S., Angeli, D., and Grüne, L. (2019). Approximate computation of storage functions for discrete-time systems using sum-of-squares techniques. IFAC-PapersOnLine, 52(16), 508–513.

Rawlings, J.B. and Amrit, R. (2009). Optimizing process economic performance using model predictive control. In Nonlinear model predictive control, 119–138. Springer.

Rawlings, J.B., Mayne, D.Q., and Diehl, M. (2017). Model predictive control: theory, computation, and design, volume 2. Nob Hill Publishing Madison, WI.

Scherer, C. and Weiland, S. (2015). Linear matrix inequalities in control. Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, 3(2).

Sutton, R.S. and Barto, A.G. (2018). Reinforcement learning: An introduction. MIT press.

Tsitsiklis, J.N. (1994). Asynchronous stochastic approximation and q-learning. Machine learning, 16(3), 185–202.

Zanon, M. and Gros, S. (2020). Safe reinforcement learning using robust mpc. IEEE Transactions on Automatic Control.

Zanon, M., Gros, S., and Diehl, M. (2016). A tracking mpc formulation that is locally equivalent to economic mpc. Journal of Process Control, 45, 30–42.

APPENDIX. NON-DISSIPATIVITY OF THE EXAMPLE 4.2

Let us assume that the system-stage cost described in (36) is dissipative. Then there exists a storage function $\lambda(s)$ and $0 < \rho$ that satisfy the following inequality:

$$\lambda(a) - \lambda(s) \leq -2s^2 + a^2 + sa + s^2a^2 - \rho s^2,$$  \hspace{0.5cm} (A.1)

and $\lambda(0) = 0$. For $s = 0$ we have:

$$\lambda(a) \leq a^2 \Rightarrow \lambda(s) \leq s^2,$$ \hspace{0.5cm} (A.2)

where we changed the variable name $a \to s$ in the last inequality. For $a = 0$, (A.1) reads as:

$$-\lambda(s) \leq -2s^2 - \rho s^2 \Rightarrow 2s^2 \leq \lambda(s),$$ \hspace{0.5cm} (A.3)

the contradiction from (A.2) and (A.3) shows that the storage function does not exist and the system is non-dissipative.