Formulating Beurling LASSO for Source Separation via Proximal Gradient Iteration

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Beurling LASSO generalizes the LASSO problem to finite Radon measures regularized via their total variation. Despite its theoretical appeal, this space is hard to parametrize, which poses an algorithmic challenge. We propose a formulation of continuous convolutional source separation with Beurling LASSO that avoids the explicit computation of the measures and instead employs the duality transform of the proximal mapping.

1 Introduction

LASSO is a finite-dimensional least-squares problem that is regularized via the 1-norm [1]. In the basis pursuit denoising formulation [cf. 2, Section 3.1], it can be written as:

$$v = \arg \min_{v} \frac{1}{2} \|Av - b\|_2^2 + \alpha \|v\|_1, \quad \alpha > 0,$$  

(1)

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Algorithmically, it is usually solved via proximal gradient methods [cf. 3, Section 4.2], often called ISTA in that setting, or an accelerated version called FISTA [4].

A shortcoming of (1) is that $v$ is discrete. Therefore, while it is possible to represent convolutions in this formulation such as via convolutional sparse coding [5], those are necessarily limited to a predetermined grid. An approach to off-the-grid convolutions was proposed by Ekanadham et al. [6] as continuous basis pursuit; it still uses a grid but interpolates between the points.

Beurling LASSO avoids the discretization of the solution altogether by operating on measures, and it is not limited to convolutions. However, it is hard to represent such measures parametrically, which poses a problem in practical applications. For the purposes of source

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separation, however, we do not need explicit access to the full solution; instead, we present an approach to obtain the separated sources via proximal gradient iteration.

## 2 Beurling LASSO

**Beurling LASSO (BLASSO)** [7; 8; 9; cf. 10] is a variation of basis pursuit denoising where the solution is a finite Radon measure. Formulations of basis pursuit denoising in infinite-dimensional vector spaces are often called Tikhonov-regularized problems [cf. 11, Chapter 4]. We first state:

$$v = \arg \min_{v \in \mathcal{X}} \frac{1}{2} \|Av - b\|^2_{\mathcal{H}} + \alpha \|v\|_{\mathcal{X}}, \quad \alpha > 0,$$

where $\mathcal{X}$ is a real Banach space and $\mathcal{H}$ is a real Hilbert space, $A: \mathcal{X} \to \mathcal{H}$ is a continuous linear operator, and $b \in \mathcal{H}$.

Banach spaces can be very general; for instance, $C_0(\mathbb{R})$ is a Banach space with the norm

$$\|x\|_{\infty} = \max_{\omega \in \mathbb{R}} |x(\omega)|.$$

Via the Riesz representation theorem [cf. 12, 6.19], it follows that its dual space $\mathcal{M}(\mathbb{R})$ is that of finite regular signed Borel measures, also called finite Radon measures [cf. 10, Section 2]. It becomes a Banach space when equipped with the norm of total variation (TV):

$$\|\nu\|_{\text{TV}} = \|\nu\|(\mathbb{R}) = \sup_{x \in C_0(\mathbb{R})} \left\{ \int x \, d\nu : \|x\|_{\infty} \leq 1 \right\}, \quad \nu \in \mathcal{M}(\mathbb{R}),$$

which is the dual norm of $\|\cdot\|_{\infty}$.

Since $C_0(\mathbb{R}) \supset \mathcal{S}(\mathbb{R})$ (where $\mathcal{S}(\mathbb{R})$ is the Schwartz space over $\mathbb{R}$ [cf. 13, 7.3]), it follows that $\mathcal{M}(\mathbb{R}) = C_0(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, and therefore, any finite Radon measure can also be regarded as a distribution.

**Example 2.1:** [cf. 10, Sections 1–3] Let $X \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra over $\mathbb{R}$.

- The Dirac measure is defined as:

$$\delta(X) = \begin{cases} 1, & \text{if } 0 \in X, \\ 0, & \text{otherwise}, \end{cases}$$

and it can be identified with the Dirac $\delta$-distribution.

- The Dirac measure can be translated, and linear combinations of translated Dirac measures can be summed. In fact, for any series $(c_j) \in \ell_1(\mathbb{Z})$, we can define:

$$\nu_c(X) = \sum_{j \in \mathbb{Z}} c_j \chi_{j \in X},$$
such that \( \| \nu_c \|_{TV} = \| (c_j) \|_{\ell_1} \). Here, \( \chi_{j \in X} = 1 \) if \( j \in X \) and 0 otherwise.

- For any function \( f \in L_1(\mathbb{R}) \), we can define:
  \[
  \nu_f(X) = \int_X f \, d\lambda,
  \]
  where \( \lambda \) is the Lebesgue measure. Then \( \| \nu_f \|_{TV} = \| f \|_{L_1} \), which may seem surprising since the TV-norm on functions has a different definition and can also be used as a regularizer [cf. 14, Section 8.1]. However, when considering
  \[
  F(t) = \int_{-\infty}^{t} f(\omega) \, d\omega \quad \text{such that} \quad \nu_f((a,b]) = F(b) - F(a),
  \]
  it follows that \( \| F \|_{TV} = \| \nu_f \|_{TV} \) [cf. 15, Theorem 3.29].

With these identifications, we can conclude that both \( \ell_1(\mathbb{Z}) \subset \mathcal{M}(\mathbb{R}) \) and \( L_1(\mathbb{R}) \subset \mathcal{M}(\mathbb{R}) \).

We now state the BLASSO problem as:

\[
\min_{\nu \in \mathcal{M}(\mathbb{R})} \left[ \frac{1}{2} \| A\nu - b \|_2^2 + \alpha \| \nu \|_{TV} \right], \quad \alpha > 0. \tag{2}
\]

The immediate difficulty is that while the space \( \mathcal{M}(\mathbb{R}) \) is versatile, it is also hard to parametrize. It can be shown that under certain assumptions, the TV-norm induces sparsity such that the solution \( \nu \) is a finite linear combination of translated Dirac measures [16, Section 4.1]. A solver which makes explicit use of this representation is the sliding Frank-Wolfe algorithm [17] which employs a non-convex solver (such as BFGS) in order to refine the shifts and amplitudes.

3 The Dual Problem

Another approach to solving the problem (2) is to transform it in such a way that the “problematic” space \( \mathcal{M}(\mathbb{R}) \) does not need to be explicitly handled anymore. For this, we need some elements of convex analysis:

**Definition 3.1:** [cf. 13, 4.2; 18, Sections 2.3, 2.4] Let \( \mathcal{X} \) be a real topological vector space vector space and \( x : \mathcal{X} \to \mathbb{R} \).

- The topological vector space \( \mathcal{X}^* \) is defined such that either \( \mathcal{X}^* = \mathcal{X}' \) or \( (\mathcal{X}^*)_\prime = \mathcal{X} \), where \( \mathcal{X}' \) is the topological dual space of \( \mathcal{X} \), and \( (\mathcal{X}^*)_\prime \) is the topological dual space of \( \mathcal{X}^* \). For \( \omega \in \mathcal{X} \) and \( \omega^* \in \mathcal{X}^* \), we note the dual pairing \( \langle \omega^*, \omega \rangle \) such that either \( \langle \omega^*, \omega \rangle = \omega^*(\omega) \) or \( \langle \omega^*, \omega \rangle = \omega(\omega^*) \). For reflexive spaces, this distinction does not matter, and for real Hilbert spaces, \( \langle \cdot, \cdot \rangle \) coincides with the inner product up to isomorphism. It always holds that \( \langle \omega, \omega^* \rangle = \langle \omega^*, \omega \rangle \).
- The function \( x^* : \mathcal{X}^* \to \mathbb{R} \) as given by
  \[
  x^*(\omega^*) = \sup_{\omega \in \mathcal{X}} \{ \langle \omega^*, \omega \rangle - x(\omega) \}.
  \]
is the convex conjugate of $x$. Similarly, the convex biconjugate $x^{**}: \mathcal{X} \to \mathbb{R}$ is given by $x^{**} = (x^*)^*$, exploiting the symmetry of the dual pairing.

- The subdifferential of $x$ is given by:

$$\partial x(\omega) = \{ \omega^* \in \mathcal{X}^* : \langle \omega^*, \tilde{\omega} - \omega \rangle \leq x(\tilde{\omega}) - x(\omega) \text{ for all } \tilde{\omega} \in \mathcal{X} \}. \quad \diamondsuit$$

**Lemma 3.2:** [cf. 18, Theorems 2.3.1, 2.4.2] Let $\mathcal{X}$ be a real topological vector space and $x: \mathcal{X} \to \mathbb{R}$ be a function with convex conjugate $x^*: \mathcal{X}^* \to \mathbb{R}$. Then:

(i) For all $\omega \in \mathcal{X}$ and $\omega^* \in \mathcal{X}^*$, we have the Fenchel-Young inequality which states that $\langle \omega^*, \omega \rangle \leq x(\omega) + x^*(\omega^*)$.

(ii) We have $\omega^* \in \partial x(\omega)$ iff $\langle \omega^*, \omega \rangle = x(\omega) + x^*(\omega^*)$.

(iii) For all $\omega \in \mathcal{X}$, it holds $x^{**}(\omega) \leq x(\omega)$.

**Proof:** Part (i) is shown via:

$$x(\omega) + x^*(\omega^*) = x(\omega) + \sup_{\omega^*} \{ \langle \omega^*, \tilde{\omega} \rangle - x(\tilde{\omega}) \} \geq x(\omega) + \langle \omega^*, \omega \rangle - x(\omega) = \langle \omega^*, \omega \rangle.$$

Conversely, $\omega^* \in \partial x(\omega)$, by the definition of the subdifferential, holds if and only if:

$$\langle \omega^*, \tilde{\omega} - \omega \rangle \leq x(\tilde{\omega}) - x(\omega) \text{ for all } \tilde{\omega} \in \mathcal{X},$$

and therefore equivalently:

$$x(\omega) + x^*(\omega^*) = x(\omega) + \sup_{\omega^*} \{ \langle \omega^*, \tilde{\omega} \rangle - x(\tilde{\omega}) \} \leq \langle \omega^*, \omega \rangle.$$

In combination, this gives us (ii). Using (i) again, we find:

$$x^{**}(\omega) = \sup_{\omega^*} \{ \langle \omega^*, \omega \rangle - x^*(\omega^*) \} \leq \sup_{\omega^*} \{ x(\omega) + x^*(\omega^*) - x^*(\omega^*) \} = x(\omega),$$

yielding (iii). \quad \Box

**Theorem 3.3 (Duality):** [cf. 18, Theorem 2.7.1] Let $\mathcal{X}, \mathcal{Y}$ be real topological vector spaces. Let $\Psi: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, and assume the product topology on $\mathcal{X} \times \mathcal{Y}$. Then, for $\gamma \in \mathcal{Y}$, we have weak duality:

$$\inf_{\tilde{\omega}} \Psi(\tilde{\omega}, \gamma) \geq \sup_{\tilde{\gamma}^*} \{ \langle \tilde{\gamma}^*, \gamma \rangle - \Psi^*(0, \tilde{\gamma}^*) \}.$$

If $\mathcal{X}$ is locally convex and there exists $\omega \in \mathcal{X}$ such that $\Psi(\omega, \gamma) = \min_{\tilde{\omega}} \Psi(\tilde{\omega}, \gamma)$, then for any $(\omega^*, \gamma^*) \in \partial \Psi(\omega, \gamma)$, we have strong duality:

$$\Psi(\omega, \gamma) = \langle \gamma^*, \gamma \rangle - \Psi^*(0, \gamma^*) = \max_{\tilde{\gamma}^*} \{ \langle \tilde{\gamma}^*, \gamma \rangle - \Psi^*(0, \tilde{\gamma}^*) \}, \quad (3)$$

and $\omega^* = 0$. \quad \diamondsuit
Weak duality follows via Lemma 3.2.iii.

We set \( h(\gamma) = \inf_{\tilde{\omega}} \Psi(\tilde{\omega}, \gamma). \) The convex conjugate can be determined as:

\[
h^*(\gamma^*) = \sup_{\tilde{\gamma}} [\langle \gamma^*, \tilde{\gamma} \rangle - \inf_{\tilde{\omega}} \Psi(\tilde{\omega}, \tilde{\gamma})] = \sup_{\tilde{\omega}, \tilde{\gamma}} [(0, \tilde{\omega}) + \langle \gamma^*, \tilde{\gamma} \rangle - \Psi(\tilde{\omega}, \tilde{\gamma})] = \Psi^*(0, \gamma^*),
\]

and the biconjugate is:

\[
h^{**}(\gamma) = \sup_{\tilde{\gamma}^*} [\langle \gamma^*, \tilde{\gamma} \rangle - h^*(\tilde{\gamma}^*)] = \sup_{\tilde{\gamma}^*} [\langle \gamma^*, \gamma \rangle - \Psi^*(0, \gamma^*)].
\]

Weak duality follows via Lemma 3.2.iii.

If \((\omega^*, \gamma^*) \in \partial \Psi(\omega, \gamma), \) then, by definition:

\[
\langle \omega^*, \tilde{\omega} - \omega \rangle + \langle \gamma^*, \gamma - \tilde{\gamma} \rangle \leq \Psi(\tilde{\omega}, \tilde{\gamma}) - \Psi(\omega, \gamma) \quad \text{for all} \quad \tilde{\omega} \in \mathcal{X}, \quad \tilde{\gamma} \in \mathcal{Y}.
\]

If \( h(\gamma) = \Psi(\omega, \gamma) \) for some \( \omega \in \mathcal{X}, \) then \( \Psi(\tilde{\omega}, \gamma) \geq \Psi(\omega, \gamma) \) for all \( \tilde{\omega} \in \mathcal{X}, \) and it follows via the Hahn-Banach theorem [cf. 13, 3.6] that \( \omega^* = 0. \) Thus, \((0, \gamma^*) \in \partial \Psi(\omega, \gamma) \) and also \( \gamma^* \in \partial h(\gamma). \) From Lemma 3.2.iii, it then follows:

\[
h(\gamma) = \langle \gamma^*, \gamma \rangle - h^*(\gamma^*) \leq h^{**}(\gamma).
\]

With Lemma 3.2.iii, this yields \( h(\gamma) \leq h^{**}(\gamma) \leq h(\gamma) \) and thus \( h(\gamma) = h^{**}(\gamma) \) with \( \gamma^* \) as a maximizer of the supremum, giving \( \gamma^* \).

**Corollary 3.4 (Fenchel-Rockafellar):** [cf. 18, Corollary 2.8.5] Let \( \mathcal{X}, \mathcal{Y} \) be real topological vector spaces and \( f: \mathcal{X} \to \mathbb{R}, g: \mathcal{Y} \to \mathbb{R} \). Assume that \( A: \mathcal{X} \to \mathcal{Y} \) is a continuous linear operator such that \( A^*: \mathcal{Y}^* \to \mathcal{X}^* \) is its adjoint and that \( \gamma \in \mathcal{Y} \) is a fixed. If \( \mathcal{X} \) is locally convex, there exists \( \omega \in \mathcal{X} \) with \( f(\omega) + g(A\omega - \gamma) = \min_{\omega} [f(\tilde{\omega}) + g(A\tilde{\omega} - \gamma)] \) (solving the primal problem), and \( \omega^* \in \partial f(\omega), \gamma^* \in \partial g(A\omega - \gamma), \) then we have strong duality with:

\[
f(\omega) + g(A\omega - \gamma) = \langle \gamma^*, \gamma \rangle - f^*(A^*\gamma^*) - g^*(\gamma^*) = \max_{\tilde{\gamma}} [\langle \tilde{\gamma}^*, \gamma \rangle - f^*(A^*\tilde{\gamma}^*) - g^*(-\tilde{\gamma}^*)],
\]

where the maximum is called the dual problem (right-hand side). Also, \(-\gamma^* \in \partial g(A\omega - \gamma). \)

**Proof:** We set \( \Psi(\omega, \gamma) = f(\omega) + g(A\omega - \gamma). \) Then:

\[
\Psi^*(0, \gamma^*) = \sup_{\omega, \gamma} [\langle \gamma^*, \gamma \rangle - f(\omega) - g(A\omega - \gamma)]
\]

\[
= \sup_{\omega, \gamma} [\langle \gamma^*, A\omega - \gamma \rangle - f(\omega) - g(\gamma)]
\]

\[
= \sup_{\omega, \gamma} [\langle A^*\gamma^*, \omega \rangle - \langle \gamma^*, \gamma \rangle - f(\omega) - g(\gamma)]
\]

\[
= f^*(A^*\gamma^*) + g^*(-\gamma^*).
\]

If \( \omega^* \in \partial f(\omega) \) and \( \gamma^* \in \partial g(A\omega - \gamma), \) then, for all \( \tilde{\omega} \in \mathcal{X} \) and \( \tilde{\gamma} \in \mathcal{Y}: \)

\[
\langle \omega^*, \tilde{\omega} - \omega \rangle + \langle \gamma^*, A\omega - \tilde{\gamma} - A\omega + \gamma \rangle \leq f(\tilde{\omega}) - f(\omega) + g(A\tilde{\omega} - \tilde{\gamma}) - g(A\omega - \gamma),
\]
and so \((\omega^* + A^*\gamma^*, -\gamma^*) \in \partial\Psi(\omega, \gamma)\). We can now apply Theorem 3.3 to obtain strong duality, and it follows that \(\omega^* + A^*\gamma^* = 0\). Therefore:

\[
\langle -\gamma^*, \hat{\gamma} - \gamma \rangle \leq g(A\omega - \hat{\gamma}) - g(A\omega - \gamma),
\]

and thus \(-\gamma^* \in \partial g(A\omega - \gamma)\). □

### 3.1 Duality on Beurling LASSO

For \(2\), we can choose:

\[
f(\nu) = \alpha \|\nu\|_{TV}, \quad g(\gamma) = \frac{1}{2}\|\gamma\|^2_{\mathcal{H}}, \quad \alpha > 0, \quad \gamma = A\nu - b,
\]

in order to apply Corollary 3.4. First we have to show that the minimum is attained. For the primal problem, this was done by Bredies and Pikkarainen [8, Proposition 3.1] via the direct method. We now reenact the proof with some detail added in. We begin with some well-known statements from functional and convex analysis:

**Lemma 3.5:** [cf. 19, Proposition VI.1.3] Let \(\mathcal{X}, \mathcal{Y}\) be topological vector spaces and let \(B: \mathcal{Y} \to \mathcal{X}\) be a continuous linear operator. Then its adjoint \(B^*: \mathcal{X}' \to \mathcal{Y}'\) is weak*-weak*-continuous. □

**Proof:** By the definition of the adjoint, we have, for any \(\omega^* \in \mathcal{X}'\) and \(\gamma \in \mathcal{Y}\):

\[
\langle B\gamma, \omega^* \rangle = \langle \gamma, B^*\omega^* \rangle.
\]

Now, as \(\omega^*\) converges in the weak* topology over \(\mathcal{X}'\), then \(B\gamma, \omega^*\) converges in \(\mathbb{R}\), so \(B^*\omega^*\) converges in the weak* topology over \(\mathcal{Y}'\). □

**Lemma 3.6:** [cf. 20, Section 6.8; 19, Proposition VI.1.4] Let \(\mathcal{H}\) be a real Hilbert space and let \(\mathcal{X}\) be a real Banach space. Let \(A: \mathcal{X}' \to \mathcal{H}\) and \(B: \mathcal{H} \to \mathcal{X}'\) be continuous linear operators such that \(B^* = A\). For any \(\omega^* \in \mathcal{X}'\) and \(\gamma \in \mathcal{H}\), it holds that \(\omega^*(B\gamma) = A^*\gamma(\omega^*)\). □

**Proof:** Since \(\mathcal{H}\) is a real Hilbert space, we have for any \(\gamma \in \mathcal{H}\) that \(\gamma = \langle \gamma, \cdot \rangle \in \mathcal{H}^*\), and therefore, with \(\omega^* \in \mathcal{X}\):

\[
\langle B\gamma, \omega^* \rangle = \langle \gamma, B^*\omega^* \rangle = \langle \gamma, A\omega^* \rangle = \langle A^*\gamma, \omega^* \rangle.
\]

**Lemma 3.7:** [cf. 20, Lemma 6.22] Let \(\mathcal{X}\) be a real Banach space and let \(\mathcal{X}'\) be its dual space with the norm \(\|\cdot\|_{\mathcal{X}'}\). Then \(\|\cdot\|_{\mathcal{X}'}\) is weak* lower semicontinuous. □
With Lemma 3.5, it follows that assume that there exists a continuous linear operator $A$ and a constant $M > 0$ such that $A$ is weak*-weak*-continuous and bounded. Since $f$ and $g$ are both composed of norms, this means that (2) is coercive and weak* lower semicontinuous. Then, for any $\omega^* \in V$, the set $\{\omega^* \in V : \omega^* \in U\}$ is weak* compact. According to Lemma 3.8, the infimum is attained. Then, for any $\omega^* \in V$, we have:

$$\liminf_{\omega^* \to \omega^* \text{ weak*}} \|\omega^*\|_\nu = \sup_{V \ni \omega^*} \inf_{\omega^* \in V} \|\omega^*\|_\nu$$

$$= \sup_{V \ni \omega^*} \inf_{\omega^* \in V} \langle \omega^*, \tilde{\omega} \rangle$$

$$\geq \sup_{V \ni \omega^*} \inf_{\omega^* \in V} \langle \omega^*, \tilde{\omega} \rangle$$

$$= \sup_{\|\omega\|_1 = 1} \inf_{\omega^* \in V} \langle \omega^*, \tilde{\omega} \rangle$$

$$= \sup_{\|\omega\|_1 = 1} \langle \omega^*, \tilde{\omega} \rangle$$

$$= \|\omega^*\|_\nu.$$

\[ \square \]

**Lemma 3.8:** [cf. 20 Theorem 2.43] Let $\mathcal{X}$ be a dual real Banach space. Let $f : \mathcal{X} \to \mathbb{R}$ be a weak* lower semicontinuous function. If $U \subset \mathcal{X}$ is a weak* compact set, then $f$ attains its minimum on $U$.

**Proof:** We show this by contradiction. Assume that the infimum $a := \inf_{\omega^* \in U} f(\omega^*)$ is not attained. Then, for any $\omega^* \in U$, since $f$ is weak* lower semicontinuous, there exists a weak* open neighborhood $V_{\omega^*} \subset \mathcal{X}$ with $\omega^* \in V_{\omega^*}$ such that $f(\tilde{\omega}^*) \geq (a + f(\omega^*))/2$ for all $\omega^* \in V_{\omega^*}$. Then $\cup_{\omega^* \in U} V_{\omega^*}$ is an open cover of the compact set $U$, so there exist $\omega^*_1, \ldots, \omega^*_m \in U$ such that $U = \cup_{k=1}^m V_{\omega^*_k}$. Therefore, there is a $k \in \{1, \ldots, m\}$ such that $\inf_{\omega^* \in V_{\omega^*_k}} f(\omega^*) \leq a$. However, since $f(\tilde{\omega}^*) \geq (a + f(\omega^*_k))/2 > a$ for all $\omega^* \in V_{\omega^*_k}$, this is impossible.

**Corollary 3.9:** Let $\mathcal{X}$ be a dual real Banach space. Let $f : \mathcal{X} \to \mathbb{R}$ be a weak* lower semicontinuous and coercive function. Then $f$ attains its minimum on $\mathcal{X}$.

**Proof:** Since $f$ is coercive, for any constant $C > 0$, there exists a value $M > 0$ such that if $f(\omega^*) \leq C$ with $\omega^* \in \mathcal{X}$, then $\|\omega^*\|_\mathcal{X} \leq M$. According to the Banach-Alaoglu theorem [cf. 13 3.15; 19 Theorem V.3.1], the set

$$U = \{\omega^* \in \mathcal{X} : \|\omega^*\|_\mathcal{X} \leq M\}$$

is weak* compact. According to Lemma 3.8, $f$ thus attains its minimum on $U \subset \mathcal{X}$.

Assume that there exists a continuous linear operator $B : \mathcal{H} \to C_0(\mathbb{R})$ such that $A = B^*$. With Lemma 3.5 it follows that $A$ is weak*-weak*-continuous, and it is also bounded. Since $f$ and $g$ are both composed of norms, this means that (2) is coercive and weak* lower semicontinuous in $\nu$. Via Corollary 3.9 the minimum is attained.

For any $\gamma \in \mathcal{H}$, we set $\gamma^* = \langle \gamma, \cdot \rangle$ (in the sense of the inner product), so we have:

$$\langle \gamma^*, \tilde{\gamma} - \gamma \rangle = \langle \gamma, \tilde{\gamma} - \gamma \rangle \leq 2\|\tilde{\gamma}\|^2_\mathcal{H} - \frac{1}{2}\|\gamma\|^2_\mathcal{H}$$

for all $\tilde{\gamma} \in \mathcal{H}$. 7
Thus, $\gamma^* \in \partial g(\gamma)$; in fact, $\partial g(\gamma) = \{\gamma^*\}$ since if $\hat{\gamma}^* = \langle \hat{\gamma}, \cdot \rangle$ with $\hat{\gamma} \neq \gamma$, we then have:

$$0 < \frac{1}{2} \|\hat{\gamma} - \gamma\|_H^2 = -\frac{1}{2} \|\hat{\gamma}\|_H^2 + \langle \hat{\gamma}, \hat{\gamma} - \gamma \rangle + \frac{1}{2} \|\gamma\|_H^2,$$

so $\hat{\gamma}^* \notin \partial g(\gamma)$.

Considering $f$, we know [cf. 12, 6.12] that for any $\nu \in M(\mathbb{R})$, there exists a Borel-measurable function $u: \mathbb{R} \to \{-1,1\}$ from which we can construct a linear functional $\nu^* \in M'(\mathbb{R})$ with $\nu^*(\nu) = \int u \, d\nu = \|\nu\|_{TV}$. Then:

$$\langle \nu^*, \tilde{\nu} - \nu \rangle = \int u \, d(\tilde{\nu} - \nu) = \int u \, d\tilde{\nu} - \int u \, d\nu \leq \|\tilde{\nu}\|_{TV} - \|\nu\|_{TV} \quad \text{for all} \quad \tilde{\nu} \in M(\mathbb{R}),$$

and therefore $\alpha \nu^* \in \partial f(\nu)$. We can now apply Corollary 3.4 in order to obtain strong duality.

Even though generally $\partial f(\nu) \nsubseteq C_0(\mathbb{R})$, we can identify $A^* \gamma^* = B \gamma^*$ according to Lemma 3.6, and therefore it is sufficient to regard $f^*: C_0(\mathbb{R}) \to \mathbb{R}$ in order to interpret the result of Corollary 3.4. We compute:

$$f^*(\nu^*) = \sup_{\nu \in M(\mathbb{R})} \left[ \langle \nu^*, \nu \rangle - f(\nu) \right]$$

$$= \sup_{\nu \in M(\mathbb{R})} \left[ \langle \nu^*, \nu \rangle - \alpha \|\nu\|_{TV} \right]$$

$$= \begin{cases} 0, & \text{for } \|\nu^*\|_{\infty} \leq \alpha, \\ \infty, & \text{otherwise} \end{cases}$$

$$= \mathbb{1}_{\|\cdot\|_{\infty} \leq \alpha}(\nu^*),$$

where $\mathbb{1}_{\|\cdot\|_{\infty} \leq \alpha}$ is the indicator function, since, according to the Hahn-Banach theorem, if $\nu^* \neq 0$, then there exists $\nu \in M(\mathbb{R})$ such that $\langle \nu^*, \nu \rangle = \|\nu^*\|_{\infty}\|\nu\|_{TV}$ becomes arbitrarily large. For the conjugate of $g$, we have:

$$g^*(\gamma^*) = \sup_{\gamma \in H} \left[ \langle \gamma^*, \gamma \rangle - \frac{1}{2} \|\gamma\|_H^2 \right]$$

$$= \sup_{\gamma \in H} \left[ \frac{1}{2} \|\gamma^*\|_H^2 - \frac{1}{2} \|\gamma^* - \gamma\|_H^2 \right]$$

$$= \frac{1}{2} \|\gamma^*\|_H^2.$$

We can thus formulate the dual problem as:

$$\max_{r \in H} \left\{ \langle r, b \rangle - \frac{1}{2} \|r\|_H^2 : \|A^* r\|_{\infty} \leq \alpha \right\},$$

(4)

where we have $r = b - A\nu$ due to $\partial g(A\nu - b) = \{b - A\nu\}$. In other words, the solution of the dual problem is nothing but the residual of the primal problem. In some applications like denoising, it could potentially be sufficient to know $A\nu$ while avoiding stating $\nu$ directly. Also, the benefit of solving $A\nu = b - r$ rather than $2$ is that it is only a linear equation
and no longer an optimization problem. This property is exploited by Catala et al. [9] in a semidefinite relaxation approach.

While the objective of the dual problem (4) is linear and quadratic, its constraint still involves the global absolute maximum of a function $A^*r \in C_0(\mathbb{R})$. However, if $\mathcal{H}$ is discrete, then knowledge about the structure of $A^*$ can be used to predict a neighborhood of the maximum. Catala et al. [9, Algorithm 1] again propose a Frank-Wolfe-type algorithm with BFGS.

### 4 Application to Source Separation

Conceptually speaking, continuous LASSO is always a hard problem, and even Beurling LASSO cannot eliminate the difficulty. However, it gives a powerful framework in order to analyze the problem in other ways. In source separation, we can avoid parametrizing the Radon measure $\nu$ by using an intermediate representation instead.

As an illustrative example, let us consider two patterns $y_1, y_2$, where $y_1$ is the upper half of an ellipse and $y_2$ is triangular-shaped. Giving a mixture spectrum, the task is to separate the contributions of the individual patterns.

In Figure 1 the different stages of representation are displayed. The left plot is the complete mixture spectrogram with the contributions of both patterns. In the middle column, these contributions are separated. The plots in the right column are linear combinations of shifted Dirac measures (indicated as arrows). Convolving the spectra in the right column with the respective patterns gives the spectra in the middle column.

![Figure 1: Separation of the contributions of two different patterns in a spectrum](image)

To formalize this process, we have to extend our framework. While the spectra in the right column of Figure 1 can be understood as a Radon measure each, the space $\mathcal{M}(\mathbb{R})$ only accounts for one measure, not multiple ones. Thus, to operate with multiple patterns, we have to consider $\mathcal{M}(\mathbb{R})^n$ which is then the dual space of $C_0(\mathbb{R})^n$, where $n$ is the number of patterns ($n = 2$ in the figure). When we equip the latter with the norm

$$\|z\|_\infty = \max_{i=1,\ldots,n} \|z_i\|_\infty, \quad z = (z_1, \ldots, z_n) \in C_0(\mathbb{R})^n$$

(which is compatible with the product topology), it gives the dual norm:

$$\|\nu\|_{TV} = \sum_{i=1}^n \|\nu_i\|_{TV}, \quad \nu = (\nu_1, \ldots, \nu_n) \in \mathcal{M}(\mathbb{R})^n,$$
complying with Bredies and Pikkarainen [8]. For the Hilbert space $\mathcal{H}^n$, we use:
\[
\|x\|_{\mathcal{H}^n}^2 = \sum_{i=1}^{n} \|x_i\|_{\mathcal{H}}^2, \quad x = (x_1, \ldots, x_n) \in \mathcal{H}^n.
\]

Following Figure 1, the operator $A: M(\mathbb{R})^n \to \mathcal{H}^n$ now convolves the measure $\nu \in M(\mathbb{R})^n$ component-wise with the patterns $y_1, \ldots, y_n \in C_0(\mathbb{R})$:
\[
A\nu = \begin{pmatrix} A_1 \nu_1 \\ \vdots \\ A_n \nu_n \end{pmatrix} = \begin{pmatrix} \nu_1 \ast y_1 \\ \vdots \\ \nu_n \ast y_n \end{pmatrix} =: x,
\]
where the convolution is defined via:
\[
(\nu \ast y)(\omega) = \int y(\omega - s) \, d\nu(s).
\]

The space $\mathcal{H}$ and the patterns $y_1, \ldots, y_n$ have to be chosen such that the pre-adjoint operator is well-defined, that is, there exists a continuous linear operator $B: \mathcal{H}^n \to C_0(\mathbb{R})^n$ such that $A = B^*$.

The operator $C: \mathcal{H}^n \to \mathcal{H}$ sums the components of the individual patterns:
\[
Cx = \sum_{k=1}^{n} x_k,
\]
and it is obviously linear and continuous. Combined, we formulate the primal problem as:
\[
\min_{x,\nu} \left\{ \frac{1}{2} \|Cx - b\|^2_{\mathcal{H}} + \alpha \|\nu\|_{TV} : A\nu = x \right\}.
\]  
(5)

so we have:
\[
f(x) = \min_{\nu} \left[ \alpha \|\nu\|_{TV} + \iota_0(A\nu - x) \right], \quad \alpha > 0,
\]
with:
\[
f^*(x^*) = \sup_x \left[ \langle x^*, x \rangle - f(x) \right]
\]
\[
= \sup_{\nu,x} \left[ \langle x^*, x \rangle - \alpha \|\nu\|_{TV} - \iota_0(A\nu - x) \right]
\]
\[
= \sup_{\nu,x} \left[ \langle x^*, A\nu - x \rangle - \alpha \|\nu\|_{TV} - \iota_0(x) \right]
\]
\[
= \sup_{\nu} \left[ \langle A^* x^*, \nu \rangle - \alpha \|\nu\|_{TV} \right]
\]
\[
= \iota_{\|\cdot\| \leq \alpha}(A^* x^*).
\]

It would now be straight-forward to apply Corollary 3.4 again, but it would still only give the residual, not expose $x$ directly. However, unlike the original problem (2), the new problem
(5) is now one where the solution $x \in \mathcal{H}^n$ lies in a Hilbert space and only the constraint is problematic.

Just like normal LASSO is often solved via the proximal mapping [cf. 3 Section 1.1], we can formulate the proximal mapping for (5) and apply Corollary 3.4 on it:

$$
\text{prox}_f(x) = \arg \min_{\tilde{x} \in \mathcal{H}^n} \left[ \frac{1}{2} \| \tilde{x} - x \|_{\mathcal{H}^n}^2 + f(\tilde{x}) \right]
$$

$$
= x - \arg \max_{x^* \in \mathcal{H}^n} \left[ \langle x^*, x \rangle - \frac{1}{2} \| x^* \|_{\mathcal{H}^n}^2 - f(x^*) \right]
$$

$$
= x - \arg \max_{x^* \in \mathcal{H}^n} \left[ \frac{1}{2} \| x \|_{\mathcal{H}^n}^2 - \| x^* - x \|_{\mathcal{H}^n}^2 - f(x^*) \right]
$$

$$
= x - \text{prox}_{f^*}(x),
$$

where we set:

$$
g(x) = \frac{1}{2} \| x \|_{\mathcal{H}^n}^2, \quad \text{so} \quad g^*(x^*) = \frac{1}{2} \| x^* \|_{\mathcal{H}^n}^2.
$$

This result is also known as Moreau decomposition [cf. 3 Section 2.5]. When substituting $\tilde{x} = Ax$, the primal problem here is formally equivalent to (2), so an optimal $\nu$ exists, and therefore also an optimal $\tilde{x}$. The proximal gradient iteration [cf. 3 Section 4.2] for (5) is then:

$$
x^{i+1} = \text{prox}_{\lambda f}(x^i - \lambda (Cx^i - b))
$$

$$
= x^i - \lambda (Cx^i - b) - \text{prox}_{\lambda f^*}(x^i - \lambda (Cx^i - b))
$$

$$
= x^i - \lambda (Cx^i - b) - \arg \max_{x^* \in \mathcal{H}^n} \left\{ \frac{1}{2} \| x^* - x^i + \lambda (Cx^i - b) \|_{\mathcal{H}^n}^2 : \| A^* x^* \|_{\infty} \leq \lambda \alpha \right\},
$$

with $\lambda > 0$.

So far, we have not specified the choice of the Hilbert space $\mathcal{H}$. With $\nu_i \in \mathcal{M}(\mathbb{R})$, $x_i \in \mathcal{H}$, and $i = 1, \ldots, n$, we have:

$$
(A_i \nu_i, x_i) = \int (\nu_i \ast y_i) (\omega) x_i(\omega) \, d\omega = \int \int y_i(\omega - s) \, d\nu_i(s) \, x_i(\omega) \, d\omega.
$$

For the pre-adjoint operator to exist, we need to be able to swap the integrals. If $\mathcal{H} = L_2(\mathbb{R})$, then this is well-defined for $y_i \in C_0(\mathbb{R}) \cap L_2(\mathbb{R})$: As can be shown by applying a version of the convolution theorem [cf. 21 Theorem 2.5.9.a] in combination with Riemann-Lebesgue lemma [cf. 13 7.5, 21 Theorem 1.4.1.c], the function given by $A^* x_i(s) = \int y_i(\omega - s) x_i(\omega) \, d\omega$ then lies in $C_0(\mathbb{R})$ as well.

For computations, it is practical to choose a discrete Hilbert space such as $\mathcal{H} = \ell_2(\mathbb{Z})$. In this case, we need to ensure sufficient decay of the patterns even when they are sampled. A possible choice is $y_i \in C_0(\mathbb{R}) \cap W(\mathbb{R})$ (where $W(\mathbb{R})$ is the Wiener space [cf. 22 Definition 6.1.1]), yielding $A^* x_i \in C_0(\mathbb{R}) \cap L_2(\mathbb{R})$. Note that discretizing $x_i \in \mathcal{H}$ does not restrict the space for $\nu_i$; however, if the grid is too coarse, then some features of $y_i$ may disappear between the sampling points.
5 Conclusion

With (5), we have given an explicit proximal gradient iteration in order to separate the convolutional contributions of given patterns from a mixture. Implicitly, it solves the continuous problem (2), but by avoiding representing the measures directly, the computation can be carried out in a discrete Hilbert space.

Even though the linear operator $A$ is a convolution in our example, the formulation is not limited to convolutions as long as the pre-adjoint operator can be stated. However, the caveat is that the bounds of $A^*x^*$ give constraints over a continuous function. How to incorporate those in a practical solution algorithm is yet to be determined.

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