Minimal Cuntz-Krieger Dilations and Representations of Cuntz-Krieger Algebras

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Abstract

Given a contractive tuple of Hilbert space operators satisfying certain $A$-relations we show that there exists a unique minimal dilation to generators of Cuntz-Krieger algebras or its extension by compact operators. This Cuntz-Krieger dilation can be obtained from the classical minimal isometric dilation as a certain maximal $A$-relation piece. We define a maximal piece more generally for a finite set of polynomials in $n$ noncommuting variables. We classify all representations of Cuntz-Krieger algebras $\mathcal{O}_A$ obtained from dilations of commuting tuples satisfying $A$-relations. The universal properties of the minimal Cuntz-Krieger dilation and the WOT-closed algebra generated by it is studied in terms of invariant subspaces.

KEY WORDS: dilation, commuting tuples, complete positivity, Cuntz algebras, Cuntz-Krieger algebras

MATHEMATICS SUBJECT CLASSIFICATION: 47A20, 47A13, 46L05
1. Introduction

Cuntz-Krieger algebras were introduced by J. Cuntz and W. Krieger in [CK] as examples of simple purely infinite $C^*$-algebras not stably isomorphic to Cuntz algebras. Let $A = (a_{ij})_{n \times n}$ be a square $0-1$-matrix i.e. $a_{ij} \in \{0, 1\}$ and each row and column has at least one non-zero entry. The Cuntz-Krieger algebra $\mathcal{O}_A$ is defined as follows:

**Definition 1.** $\mathcal{O}_A$ is the universal $C^*$-algebra generated by $n$ partial isometries $s_1, \cdots, s_n$ with orthogonal ranges satisfying

$$s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^*$$

$$I = \sum_{i=1}^{n} s_i s_i^*.$$  

We denote the tuple $(s_1, \cdots, s_n)$ by $\mathfrak{s}$.

Notice that $s_i s_j = a_{ij} s_i s_j$ for all $s_i$ and $s_j$ in $\mathcal{O}_A$. In this paper we study dilations related to these algebras. Equations (1.1) are called Cuntz-Krieger relations. An $n$-tuple of bounded operators $T = (T_1, \cdots, T_n)$ on a Hilbert space is said to be a contractive $n$-tuple if $T_1 T_1^* + \cdots + T_n T_n^* \leq I$. Such tuples are also called row contractions as the condition is equivalent to saying that the operator $(T_1, \cdots, T_n)$ from $H \oplus \cdots \oplus H$ ($n$-times) to $H$ is a contraction. We will only consider contractive tuples. For such tuples Davis [D], Bunce [Bu], Frazho [Fr] and more extensively Popescu ([Po1-6], [AP]) constructed dilations consisting of isometries with orthogonal ranges. Under natural minimality conditions this dilation is unique up to unitary equivalence. We refer to it as the minimal isometric dilation or the standard (noncommuting) dilation $\hat{T}$ on the full Fock space. The standard noncommuting dilation has similar characterizations as classical dilations of single operators notably the minimal normal extension of subnormal operators. The universal role of the unilateral shift for single operators is played by the tuple of creation operators on the full Fock space.

However, tuples are more complex than single operators and one may impose symmetry conditions on the tuple and study dilations within this restricted class of tuples. For instance if all operators in the tuple commute i.e. $T$ is a commuting tuple, Arveson [Ar3] showed that there is a unique minimal commuting dilation with similar properties. Crucial in his approach is the tuple of creation operators on symmetric Fock space playing the role of the shift for single operators. In [BBD] the relation between the minimal commuting and the standard dilation has been investigated. It was shown that for every contractive tuple there is a maximal subspace on which it forms a commuting tuple and that the minimal commuting dilation is precisely this maximal commuting piece of the standard noncommuting dilation.

In this article we consider the following class of tuples and investigate dilations within this class and their connection with the commuting and noncommuting standard dilation.

**Definition 2.** Let $A = (a_{ij})_{n \times n}$ be a $0$-$1$-matrix. A contractive $n$-tuple $T$ is an $A$-relation tuple or is said to satisfy $A$-relations if $T_i T_j = a_{ij} T_i T_j$ for $1 \leq i, j \leq n$. 


Given such a tuple $T$ there is a unique minimal dilation to partial isometries satisfying $A$-relations i.e. generators of a Cuntz-Krieger algebras if $T$ is unital or an extension of it by compact operators if $T$ is contractive. We call this the minimal Cuntz-Krieger dilation of $T$. It will be denoted by $\tilde{T}$ and acts on $\tilde{H}$.

For an arbitrary tuple we define a maximal $A$-relation piece and compare the maximal $A$-relation piece of the standard isometric dilation with the minimal Cuntz-Krieger dilation. As for commuting tuples both turn out to be the same. We also prove similar results for the maximal commuting $A$-relation piece.

We begin in section 2 by defining the maximal piece of a tuple of operators with respect to a finite set of polynomials in $n$-noncommuting variables. Similarly to results by Arias and Popescu [AP] there is a canonical homomorphism between the WOT-closed (non-selfadjoint) algebra generated by them and the WOT-closed algebra generated by the original tuple modulo a two-sided ideal. The maximal $A$-relation piece of a tuple of $n$-isometries with orthogonal ranges is a special case of this. In fact the maximal commuting piece and maximal $q$-commuting piece (c.f. [BBD], [De], [AP]) can all be treated using this approach.

In section 3 we show that the maximal $A$-relation piece of the standard dilation of an $n$-tuple satisfying $A$-relations is the minimal Cuntz-Krieger dilation. The section begins with two different constructions of minimal Cuntz-Krieger dilations, one using positive definite kernels the other using a modification of Popescu’s Poisson transform.

In section 4 we study the minimal Cuntz-Krieger dilation of commuting $A$-relation tuples from a representation theoretical point of view. If such a tuple is also unital then it determines a unique representation of the Cuntz-Krieger algebra $O_A$. Generalizing results from [BBD] for $O_n$ we are able to show that these representations are determined by the GNS-representations of analogues of Cuntz states.

Based on ideas of Bunce, Popescu [Po5] showed that the minimal isometric dilation can be characterized by a universal property of the $C^*$-algebra generated by it. In section 5 we first point out that minimal Cuntz-Krieger dilations can be characterized in a similar way. We study the structure of the WOT-closed algebra generated by the operators constituting the minimal Cuntz-Krieger dilation and describe this algebra by making use of its invariant and wandering subspaces. We use techniques of Davidson, Kribs and Pitts ([DPS], [DP2]) to understand the structure of ‘free semigroup algebras’ i.e. WOT-closed algebras generated by a finite number of isometries with orthogonal ranges. In this section many proofs are only sketched or omitted.

All Hilbert spaces in this paper are complex and separable. We denote the full Fock space over $\mathcal{L}$ by $\Gamma(\mathcal{L})$ which is defined as

$$\Gamma(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \ldots .$$

Let the vacuum vector $1 \oplus 0 \oplus 0 \oplus \ldots$ be denoted by $\omega. \mathbb{C}^n$ is the $n$-dimensional complex Euclidean space with standard orthonormal basis $\{e_1, \ldots, e_n\}$. The left creation operator $L_i$ on $\Gamma(\mathbb{C}^n)$ is defined by

$$L_i x = e_i \otimes x,$$
where $1 \leq i \leq n$ and $x \in \Gamma(C^n)$. The $L_i$’s are clearly isometries with orthogonal ranges. We denote the tuple $(L_1, \cdots, L_n)$ by $L$. Also $\sum L_i L_i^* = I - P_0$ where $P_0$ is the projection onto the vacuum space.

Let $\Lambda$ be the set $\{1, 2, \cdots, n\}$ and $\Lambda^m$ the $m$-fold cartesian product of $\Lambda$ for $m \in \mathbb{N}$. For an operator tuple $(T_1, \cdots, T_n)$ on a Hilbert space $\mathcal{H}$ and for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) = \alpha_1 \alpha_2 \cdots \alpha_m$ in $\Lambda^m$, the operator $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$ will be denoted by $T^\alpha$. Let $\Lambda$ denote $\cup_{m=0}^\infty \Lambda^m$, where $\Lambda^0$ is $\{0\}$ and $\Lambda^0$ is the identity operator. We may think of elements in $\Lambda$ as words with concatenation as product written $\alpha \beta$. $\Lambda$ is the free semigroup with $n$ generators. Given a 0-1-matrix $A$ as above we can define $\Lambda^m_A = \{\alpha_1 \alpha_2 \cdots \alpha_m : a_{\alpha_i, \alpha_{i+1}} = 1$ for $i = 1, \ldots, n-1\}$ and the subsemigroup $\Lambda_A = \cup_{m=0}^\infty \Lambda^m_A$. $o(\alpha)$ and $t(\alpha)$ denote the first and last letter (i.e. index) of $\alpha$.

**Definition 3.** Let $\mathcal{H}$ and $\mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ is a closed subspace of $\mathcal{L}$ and $T$, $R$ be $n$-tuples of operators on $\mathcal{H}$, $\mathcal{L}$ respectively. Then $R$ is a *dilation* of $T$ or $T$ a *piece* of $R$ if

$$R_i^* h = T_i^* h$$

for all $h \in \mathcal{H}$, $1 \leq i \leq n$. A dilation is said to be *minimal dilation* if

$$\overline{\text{span}}\{R^\alpha h : \alpha \in \Lambda, h \in \mathcal{H}\} = \mathcal{L}.$$

(1) A dilation $R$ of $T$ is said to be *isometric* if $R$ consists of isometries with orthogonal ranges.

(2) When $T$ satisfies $A$-relations, a dilation $R$ of $T$ is said to be a *Cuntz-Krieger dilation* if $R$ consists of partial isometries with orthogonal ranges satisfying $A$-relations and

$$R_i^* R_i = I - \sum_{j=1}^n (1 - a_{ij}) R_j R_j^* = P_0 + \sum_{j=1}^n a_{ij} R_j R_j^*,$$

where $P_0 = 1 - \sum_{j=1}^n R_j R_j^*$.

Thus, like for isometric dilations, $C^*(R) = \overline{\text{span}}\{R^\alpha (R^\beta)^* : \alpha, \beta \in \Lambda\}$ for any Cuntz-Krieger dilation $R$ of $T$. Moreover, since

$$R_\alpha R_\beta = \delta_{\alpha, \beta} R_{t(\alpha)} R_{t(\beta)} = \delta_{\alpha, \beta} \left(I - \sum_{j=1}^n (1 - a_{t(\alpha), j}) R_j R_j^*\right)$$

whenever $\alpha \neq 0$, it follows that if $R_1$ and $R_2$ are any two minimal Cuntz-Krieger dilations then $\sum R_\alpha^* h_i \mapsto \sum R_\alpha^* h_i$ extends to a unitary equivalence.

Given any dilation $R$ of $T$ all $R_i^*$ leave $\mathcal{H}$ invariant and if $p, q$ are polynomials in $n$-noncommuting variables then

$$T_\alpha^*(T_\beta^*)^* = P_H R_\alpha (R_\beta)^*|_{\mathcal{H}} \text{ and } p(T)(q(T))^* = P_H p(R)(q(R))^*|_{\mathcal{H}}.$$

It follows that if $R$ is a Cuntz-Krieger dilation, then there is a unique completely positive map $\rho : C^*(R) \to C^*(T)$ mapping $R^\alpha (R_\beta)^*$ to $T_\alpha^*(T_\beta^*)^*$. 


Finally let us recall a concept needed later. For an \( n \)-tuple \( R \) of bounded operators on \( \mathcal{L} \), a subspace \( \mathcal{K} \) of \( \mathcal{L} \) is said to be \textit{wandering} for the tuple if \( R^\alpha \mathcal{K} \) are pairwise orthogonal for all \( \alpha \in \tilde{\Lambda} \).

2. Maximal A-relation Piece and A-Fock Space

We begin with an \( n \)-tuple of bounded operators \( \overrightarrow{R} \) on a Hilbert space \( \mathcal{L} \) and a finite set of polynomials \( \{p_\xi\}_{\xi \in \mathcal{I}} \) in \( n \)-noncommuting variables with finite index set \( \mathcal{I} \). Consider

\[
\mathcal{C}(R) = \{ \mathcal{M} : R^*_i \mathcal{M} \subseteq \mathcal{M} \text{ and } (p_\xi (R))^* h = 0, \forall h \in \mathcal{M}, 1 \leq i \leq n, \xi \in \mathcal{I} \}.
\]

\( \mathcal{C}(R) \) consists of all co-invariant subspaces of \( R \) such that the compressions form a tuple \( R^p = (R_1^p, \ldots, R_n^p) \) satisfying \( p_\xi (R^p) = 0 \) for all \( \xi \in \mathcal{I} \). It is a complete lattice, in the sense that arbitrary intersections and closed spans of arbitrary unions of such spaces are again in this collection. Its maximal element is denoted by \( \mathcal{L}^p(R) \) (or by \( \mathcal{L}^p \) when the tuple under consideration is clear). Since \( (p_\xi (R))^* (R^\alpha)^* \mathcal{M} = 0 \) for all \( \mathcal{M} \in \mathcal{C}(R) \), \( \alpha \in \tilde{\Lambda} \) and \( \xi \in \mathcal{I} \) we have \( \mathcal{L}^p(R) \subseteq \bigcap_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} \ker (p_\xi (R))^* (R^\alpha)^* \). On the other hand this intersection lies in \( \mathcal{C}(R) \) hence

\[
\mathcal{L}^p(R) = \bigcap_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} \ker (p_\xi (R))^* (R^\alpha)^* = \left[ \bigvee_{\alpha \in \tilde{\Lambda}, \xi \in \mathcal{I}} R^\alpha p_\xi (R) (\mathcal{L}) \right]^\perp.
\]

Therefore we have:

\textbf{Lemma 4.} Let \( \overrightarrow{R} \) be an \( n \)-tuple of operators on a Hilbert space \( \mathcal{L} \) and \( \mathcal{K} = \text{span} \{ R^\alpha p_\xi (R) R^\beta h : h \in \mathcal{L}, \xi \in \mathcal{I} \text{ and } \alpha, \beta \in \tilde{\Lambda} \} \). Then \( \mathcal{L}^p(R) = \mathcal{K}^\perp = \{ h \in \mathcal{L} : (R^\alpha p_\xi (R) R^\beta)^* h = 0, \forall \xi \in \mathcal{I} \text{ and } \alpha, \beta \in \tilde{\Lambda} \} \).

\( \mathcal{L}^p(R) \) can also be thought of as follows. Let \( \mathcal{R} \) be the (non self-adjoint) WOT-closed algebra generated by \( \overrightarrow{R} \). Then \( \mathcal{L}^p(R) = (\mathcal{J} \mathcal{L})^\perp \), where

\[
\mathcal{J} = \text{span} \{ R^\alpha p_\xi (R) R^\beta : \alpha, \beta \in \tilde{\Lambda}, \xi \in \mathcal{I} \} \subseteq \mathcal{R}
\]

is the WOT-closed ideal generated by \( \{ p_\xi (R) : \xi \in \mathcal{I} \} \).

\textbf{Definition 5.} The \textit{maximal piece} of \( \overrightarrow{R} \) with respect to \( \{ p_\xi \}_{\xi \in \mathcal{I}} \) is defined as the piece obtained by compressing \( \overrightarrow{R} \) to the maximal element \( \mathcal{L}^p(R) \) of \( \mathcal{C}(R) \) denoted by \( \overrightarrow{R}^p = (R_1^p, \ldots, R_n^p) \). The maximal piece is said to be \textit{trivial} if the space \( \mathcal{L}^p(R) \) is the zero space.

Let \( \mathcal{R}^p \) be the WOT-closed algebra generated by \( \overrightarrow{R}^p \). By co-invariance the map

\[
\Psi_{\mathcal{L}^p(R)} : \mathcal{R} \rightarrow \mathcal{R}^p, \quad X \mapsto P_{\mathcal{L}^p(R)} X = P_{\mathcal{L}^p(R)} X P_{\mathcal{L}^p(R)}
\]

is a WOT-continuous homomorphism of \( \mathcal{R} \) whose kernel is a WOT-closed ideal of \( \mathcal{R} \). Since \( \Psi_{\mathcal{L}^p}(p_\xi (R)) = p_\xi (R^p) = 0 \) for all \( \xi \in \mathcal{I} \) we certainly have \( p_\xi (R) \in \ker \Psi_{\mathcal{L}^p} \) i.e. \( \mathcal{J} \subseteq \ker \Psi_{\mathcal{L}^p} \). Thus there is a canonical surjective and contractive homomorphism

\[
\Psi : \mathcal{R} / \mathcal{J} \rightarrow \mathcal{R}^p.
\]
When the polynomials are \( p_{(l,m)} = z_l z_m - a_{lm} z_l z_m, (l, m) \in \{1, \ldots, n\} \times \{1, \ldots, n\} = I \)
we call \( L^\alpha(R) \) the **maximal A-relation subspace** and the corresponding piece the **maximal A-relation piece** \( R^A \). The maximal A-relation subspace is explicitly given by

\[
L_A(R) = \{ R^\alpha(R_i R_j - a_{ij} R_i R_j) h : h \in L, \alpha \in \Lambda, i, j = 1, \ldots, n \}.
\]

When the noncommuting polynomials are \( q_{(l,m)} = z_l z_m - z_m z_l \) then we obtain the maximal commuting subspace

\[
L^\alpha(R) = \{ R^\alpha(R_i R_j - R_j R_i) h : h \in L, \alpha \in \Lambda, i, j = 1, \ldots, n \}.
\]

studied in [BBD].

**Lemma 6.** Let \( R \) and \( T \) be two \( n \)-tuples of bounded operators on \( M \) and \( H \) respectively.

1. The maximal A-relation piece of \( (R_1 \oplus T_1, \ldots, R_n \oplus T_n) \) is \( (R_1^A \oplus T_1^A, \ldots, R_n^A \oplus T_n^A) \) acting on \( M_A \oplus H_A \) and the maximal A-relation piece of \( (R_1 \otimes I, \ldots, R_n \otimes I) \) acting on \( M \otimes H \) is \( (R_1^A \otimes I, \ldots, R_n^A \otimes I) \) on \( M_A \otimes H \).

2. Suppose \( H \subseteq M \) and \( R \) is a dilation of \( T \) then \( R^A \) is a dilation of \( T^A \) with \( H_A(T) = M_A(R) \cap H \).

**Proof:** Follows from Lemma 4 (compare with [BBD] for part (2)).

Cuntz-Krieger relations are naturally related to A-Fock space, a variant of the usual Fock space.

**Definition 7.** For a given \( A = (a_{ij})_{n \times n} \) as above, the **A-Fock space** is defined as the maximal A-relation subspace \( (\Gamma(C^n))_A(L) \) with respect to the left creation operators. It is denoted by \( \Gamma_A \). We also define the \( n \)-tuple \( S = (S_1, \ldots, S_n) \), where the \( S_i \)'s are the compressions of left creation operators \( L_i \) onto \( \Gamma_A \).

The A-Fock space has a very concrete description justifying the terminology.

**Proposition 8.** \( \Gamma_A = \overline{\text{span}} \{ e^\alpha : \alpha \in \Lambda_A \} \) and \( L^A = S \).

**Proof:** Let \( \alpha \in \Lambda^m \) be such that there exist \( 1 \leq k \leq m - 1 \) for which \( a_{\alpha_k \alpha_{k+1}} = 0 \). Denoting \( \alpha_k, \alpha_{k+1} \) by \( s, t \), it is clear that

\[
e^\alpha \in \overline{\text{span}} \{ L^\gamma (L_s L_t - a_{st} L_s L_t) h : h \in \Gamma(C^n), \gamma \in \Lambda \},
\]

which implies that such \( e^\alpha \) are orthogonal to \( \Gamma(C^n)_A(L) \), whereas if for all \( 1 \leq k \leq m - 1 \), \( a_{\alpha_k \alpha_{k+1}} = 1 \) then for all \( 1 \leq i, j \leq m - 1 \), \( \beta \in \Lambda, h \in \Gamma(C^n) \)

\[
\langle e^\alpha, L^\beta (L_i L_j - a_{ij} L_i L_j) h \rangle = 0,
\]

and thus such \( e^\alpha \in \Gamma_A \). By taking completions the Proposition follows. \( \square \)

Similar Fock spaces were also studied by Muhly [Mu], Solel and others.

Now suppose that \( \alpha = \alpha_1 \cdots \alpha_m \in \Lambda_A^m \) and \( m > 0 \), then

\[
S_i e^\alpha = P_{\Gamma_A} L_i e^\alpha = \begin{cases} 
  e_i & \text{if } |\alpha| = 0 \\
  a_{\alpha_1} e_i \otimes e^\alpha & \text{if } |\alpha| \geq 1
\end{cases}
\]
Clearly all mutually orthogonal. Similarly one observes that the other
k
onal ranges is an
as for
Proposition 9 (2.3)
⟨
Equation (2.1) follows from
Fix a matrix
A
s ∈ \mathcal{K} \mathcal{A}(\mathcal{V}) by \mathcal{P}. Any \ k_{A} in \mathcal{K}_{A}(\mathcal{V}) can be written as \ k_{A} = \bigoplus_{p=1}^{n} \mathcal{V}_{p}k_{p} \oplus \mathcal{K}_{0} for some \ k_{p} \in \mathcal{K}, 1 \leq p \leq n and some \ k_{0} \in \mathcal{I} = \mathcal{I} \mathcal{A}(\mathcal{V}) \mathcal{K}. (Any \ k \in \mathcal{K} can be written in this form.) Clearly \ k_{0} is in \mathcal{K}_{A}(\mathcal{V}) using Lemma 4, since the ranges of the \mathcal{V}_{q}’s and \mathcal{I} = \mathcal{I} \mathcal{A}(\mathcal{V}) \mathcal{K} \mathcal{F} are all mutually orthogonal. Similarly one observes that the other \mathcal{V}_{q}’s also belong to \mathcal{K}_{A}(\mathcal{V}) as for \ k \in \mathcal{K}, \alpha \in \Lambda

\langle k_{p}, V_{\alpha} (\mathcal{V}_{j} \mathcal{V}_{j} - a_{ij} \mathcal{V}_{j})k \rangle = \langle V_{p}k_{p}, V_{p}^{\alpha} (\mathcal{V}_{j} \mathcal{V}_{j} - a_{ij} \mathcal{V}_{j})k \rangle

where again we use that the ranges of the \mathcal{V}_{q}’s and \mathcal{I} = \mathcal{I} \mathcal{A}(\mathcal{V}) \mathcal{K} \mathcal{F} are all mutually orthogonal. Next we show that

(2.1) \quad PV_{i}k_{0} = V_{i}k_{0},

and

(2.2) \quad PV_{i}V_{p}k_{p} = a_{ip}V_{p}k_{p}.

Equation (2.1) follows from \langle V_{i}k_{0}, V_{\beta} (\mathcal{V}_{s} \mathcal{V}_{t} - a_{st} \mathcal{V}_{s} \mathcal{V}_{t})k \rangle = 0, for all \beta \in \Lambda, 1 \leq s, t \leq n, \ k \in \mathcal{K} (since \ k_{0} is orthogonal to the range of \mathcal{V}_{t}, 1 \leq t \leq n). When \ a_{ip} = 0, we have \mathcal{P}V_{i}V_{p}k_{p} = \mathcal{P}(V_{i}V_{p} - a_{ip}V_{p}k_{p})k_{p} = 0 = a_{ip}V_{p}k_{p}. So it is enough to show for \ a_{ip} = 1 that \mathcal{V}_{i}V_{p}k_{p} \in \mathcal{K}_{A}(\mathcal{V}) \mathcal{F}. When \ |\alpha| > 1 or \ |\alpha| = 0, it is easy to see that for \ 1 \leq s, t \leq n, \ k \in \mathcal{K}

(2.3) \quad \langle V_{i}V_{p}k_{p}, V_{\alpha} (\mathcal{V}_{s} \mathcal{V}_{t} - a_{st} \mathcal{V}_{s} \mathcal{V}_{t})k \rangle = 0

as the \mathcal{V}_{i}’s are isometries with orthogonal ranges and \ k_{p} \in \mathcal{K}_{A}(\mathcal{V}). When \ |\alpha| = 1,

\langle V_{i}V_{p}k_{p}, V_{i} (V_{p}V_{i} - a_{pt}V_{p}V_{i})k \rangle = \langle V_{p}k_{p}, (V_{p}V_{i} - a_{pt}V_{p}V_{i})k \rangle

= \langle \oplus_{q=1}^{n} V_{q}k_{q} \oplus k_{0}, (V_{p}V_{i} - a_{pt}V_{p}V_{i})k \rangle = 0.
Clearly equation (2.3) holds in all other cases when $|\alpha| = 1$. So equation (2.2) is true in general and we have

\[
V_i^A(V^A)_i^*V_i^A k_A = PV_i^*PV_i k_A
= PV_i^*P(\bigoplus_p V_i V_p k_p \oplus V_i k_0)
= PV_i^*(\bigoplus_{p=1}^n a_{ip} V_i V_p k_p \oplus V_i k_0)
= \bigoplus_{p=1}^n a_{ip} PV_i V_p k_p \oplus PV_i k_0
= PV_i k_A = V_i^A k_A.
\]

Thus the $V_i^A$'s are partial isometries. Now the assertion of the Proposition that for $1 \leq i \neq j \leq n$, the range of $V_i^A$ is orthogonal to the range of $V_j^A$ can be proved in the following way:

\[
(V_j^A)^*V_j^A k_A = V_j^* PV_j k_A
= V_j^* PV_i (\bigoplus_{p=1}^n V_p k_p \oplus k_0)
= V_j^* (\bigoplus_{p=1}^n a_{ip} V_i V_p k_p \oplus V_i k_0) = 0.
\]

Alternatively this follows since $V^A$ is contractive. \hfill \Box

**Corollary 10.** The following holds for $S$:

1. $I - \sum_{i=1}^n S_i S_i^* = P_0$ where $P_0$ is the projection onto the vacuum space.
2. The $S_i$'s are partial isometries with orthogonal ranges.
3. $S_i^* S_i = I - \sum_{j=1}^n (1 - a_{ij}) S_j S_j^* = P_0 + \sum_{j=1}^n a_{ij} S_j S_j^*$

**Proof:**

1. $I - \sum S_i S_i^* = P_{\Gamma_A}(I - \sum L_i L_i^*) P_{\Gamma_A} = P_0$.
2. Follows from Proposition 9 or can be checked directly from the relations before Proposition 9.
3. Suppose $e^\alpha \in \Gamma_A$, and when $|\alpha| > 0$ let $\alpha = \alpha_1 \cdots \alpha_m$. Then

\[
[I - \sum_{j} (1 - a_{ij}) S_j S_j^*] e^\alpha = \begin{cases} 
\omega & \text{if } |\alpha| = 0 \\
 a_{i\alpha_1} e^\alpha & \text{if } |\alpha| \geq 1.
\end{cases}
\]

\hfill \Box

**3. Minimal Cuntz-Krieger Dilations and Standard Noncommuting Dilations**

The main purpose of this section is to prove the following result.

**Theorem 11.** Let $\overline{T}$ be a contractive $A$-relation tuple on $\mathcal{H}$. Then there exists a minimal Cuntz-Krieger dilation $\tilde{T}$ on $\mathcal{H}$ unique up to unitary equivalence. $\tilde{T}$ is unital iff $T$ is unital.
Uniqueness has already been pointed out and the last part follows since $\sum_{|\alpha|=k} \tilde{T}^\alpha (\tilde{T}^\alpha)^*$ form a decreasing sequence of projections converging weakly to a limit projection $P$. If $\mathcal{H}$ is unital then $P_{\mathcal{H}} \leq P$ and so $P = 1$ by minimality.

We will give two proofs of the existence. The first is direct and uses positive definite kernels. It gives an explicit construction of the dilation Hilbert space and the dilated tuple. The second construction is an adaptation of Popescu’s Poisson transform method which uses completely positive maps. Though elegant it is less direct. Then we show that the minimal Cuntz-Krieger dilation can also be obtained as the maximal $A$-relation piece of the standard dilation.

**First proof via positive definite Kernels**

Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a contractive $n$-tuple on a Hilbert space $\mathcal{H}$ satisfying $A$-relations. Assume we already found the minimal Cuntz-Krieger dilation $\tilde{T}$ on the Hilbert space $\tilde{\mathcal{H}}$. Then

$$K((\alpha, u), (\beta, v)) := \langle \tilde{T}^\alpha u, \tilde{T}^\beta v \rangle$$

clearly defines a positive definite kernel on the set

$$X = \tilde{\Lambda}_A \times \mathcal{H}.$$  

By minimality $\tilde{\mathcal{H}} = \text{span}\{\tilde{T}^\alpha u : u \in \mathcal{H}, \alpha \in \tilde{\Lambda}_A\}$ which is precisely the kernel Hilbert space. Moreover $\tilde{T}_i$ corresponds to the map $(\alpha, u) \mapsto (i\alpha, u)$.

Using co-invariance of $\mathcal{H}$ under $\tilde{T}$ and the relation $\tilde{T}_i^* \tilde{T}_i = I - \sum_j (1 - a_{ij}) \tilde{T}_j^* \tilde{T}_j$ we find that

$$K((\alpha, u), (\beta, v)) = \begin{cases} 
\langle u, v \rangle & \text{if } \alpha = \beta = 0 \\
\langle u, [I - \sum_k (1 - a_{(\alpha)k}) T_k T_k^*] v \rangle & \text{if } \alpha = \beta \neq 0 \\
\langle u, \tilde{T}^\gamma v \rangle & \text{if } \beta = \alpha \gamma \\
\langle u, (\tilde{T}^\gamma)^* v \rangle & \text{if } \alpha = \beta \gamma \\
0 & \text{otherwise.} 
\end{cases}$$

and this kernel depends only on $\mathcal{T}$. We will show directly by induction that the kernel $K$ thus defined is always positive definite.

To simplify the calculations we assume that $\mathcal{T}$ is unital i.e. $\sum_i T_i T_i^* = I$. There is no loss in doing so since positivity for the kernel defined by the $(n+1)$-tuple $(T_1, \ldots, T_n, (I - \sum_i T_i T_i^*)^{1/2})$ implies positivity of $K$. Under this assumption $T_i Q_i = T_i$, where $Q_i = I - \sum_k (1 - a_{ik}) T_k T_k^* = \sum_{j=1}^n a_{ij} T_j T_j^*$ and $K((\alpha, u), (\alpha, v)) = \langle u, Q_{t(\alpha)} v \rangle$, whenever $\alpha \neq 0$.

Now let $A^{(m)}$ denote operator matrices with entries in $B(\mathcal{H})$ indexed by $\alpha, \beta \in \tilde{\Lambda}$, where $|\alpha|, |\beta| \leq m$ and define $K^{(m)} = (K_{\alpha,\beta}^{(m)})$ by

$$K_{\alpha,\beta}^{(m)} := \begin{cases} 
I & \text{if } \alpha = \beta = 0 \\
Q_{t(\alpha)} & \text{if } \alpha = \beta \neq 0 \\
\tilde{T}^\gamma & \text{if } \beta = \alpha \gamma \\
(\tilde{T}^\gamma)^* & \text{if } \alpha = \beta \gamma \\
0 & \text{otherwise.} 
\end{cases}$$
i.e. $K^{(m)}$ is a compression of $K$. Clearly it suffices to show that all $K^{(m)}$ are positive. For $m = 1$ this follows from the equation

$$\begin{bmatrix}
I & T_1 & T_2 & \cdots & T_n \\
T_1^* & Q_1 & 0 & \cdots & 0 \\
T_2^* & 0 & Q_2 & 0 & \cdots \\
& \vdots & & \ddots & \vdots \\
T_n^* & 0 & \cdots & 0 & Q_n
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & Q_1 & 0 & \cdots \\
0 & 0 & 0 & Q_2 & 0 & \cdots \\
\vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & 0 & Q_n
\end{bmatrix}$$

If $m > 1$ define matrices $L_1, L_2, \ldots, L_{m-1}$ by

$$L_{k;\alpha,\beta} := \begin{cases}
T_i & \text{if } \beta = \alpha i, |\beta| = k \\
I & \text{if } \alpha = \beta \text{ and } |\beta| \geq k \\
0 & \text{otherwise}.
\end{cases}$$

Finally let

$$Q^{(m)}_{\alpha,\beta} := \begin{cases}
Q_{t(\alpha)} & \text{if } \alpha = \beta \text{ and } |\alpha| = m \\
0 & \text{otherwise}.
\end{cases}$$

Then it is not hard to check that

$$K^{(m)} = L_1L_2 \cdots L_{m-1}Q^{(m)}L^*_m \cdots L^*_2 L^*_1$$

which shows that $K$ is positive. By Kolmogorov's Theorem there exists a Hilbert space $\tilde{H}$ and an injective map $\lambda : X \to \tilde{H}$ such that $\overline{\text{span}}\{\lambda(\alpha, u) : 1 \leq i \leq n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \tilde{H}$ and

$$K((\alpha, u), (\beta, v)) = \langle \lambda(\alpha, u), \lambda(\beta, v) \rangle.$$ 

It remains to show that $\tilde{T} = (\tilde{T}_1, \cdots, \tilde{T}_n)$ consisting of maps $\tilde{T}_i : \tilde{H} \to \tilde{H}$ defined by

$$\tilde{T}_i \lambda(\alpha, u) = \lambda(i \alpha, u),$$

constitute a tuple $\tilde{T}$ which is the minimal Cuntz-Krieger dilation of $T$. First note that $\tilde{T}_i$ is a well-defined contraction. Indeed, thinking of $K$ as a block matrix we have

$$K_{i\alpha,i\beta} = \begin{cases}
Q_i & \text{if } \alpha = \beta = 0 \\
a_{i\alpha(\alpha)}a_{i\alpha(\beta)}K_{\alpha,\beta} & \text{otherwise},
\end{cases}$$
where we define \( a_{i\alpha(\alpha)} = 1 \) for all \( i \) if \( \alpha = 0 \). So if \( F \subseteq X \) is a finite subset then

\[
\left\| \sum_{(\alpha,u) \in F} \lambda(i\alpha,u) \right\|^2 = \left\| \sum_{(\alpha,u) \in F} a_{i\alpha(\alpha)} \lambda(i\alpha,u) \right\|^2
\]

\[
= \sum_{\alpha=\beta=0} \langle u, Q_i v \rangle + \sum_{\alpha \neq 0 \text{ or } \beta \neq 0} a_{i\alpha(\alpha)} a_{i\alpha(\beta)} \langle u, K_{\alpha,\beta} v \rangle
\]

\[
\leq \sum_{\alpha=\beta=0} \langle u, v \rangle + \sum_{\alpha \neq 0 \text{ or } \beta \neq 0} \langle u, K_{\alpha,\beta} v \rangle
\]

\[
= \left\| \sum_{(\alpha,u) \in F} \lambda(\alpha,u) \right\|^2.
\]

For \( i \neq j \)

\[
\langle \bar{T}_i \lambda(\alpha,u), \bar{T}_j \lambda(\beta,v) \rangle = \langle \lambda(i\alpha,u), \lambda(j\beta,v) \rangle
\]

\[
= K((i\alpha,u),(j\beta,v)) = 0
\]

since neither \( i\alpha = j\beta\gamma \) nor \( j\beta = i\alpha\gamma \) is possible. As required for dilations we have \( \bar{T}_i^* \lambda(0,u) = \lambda(0,T_i^*u) \) as may be seen as follows.

\[
\langle \bar{T}_i^* \lambda(0,u), \lambda(\beta,v) \rangle = \langle \lambda(0,u), \bar{T}_i \lambda(\beta,v) \rangle
\]

\[
= \langle \lambda(0,u), \lambda(i\beta,v) \rangle
\]

\[
= K((0,u),(i\beta,v))
\]

\[
= K((0,T_i^*u),(\beta,v))
\]

\[
= \langle \lambda(0,T_i^*u), \lambda(\beta,v) \rangle.
\]

Next we show that \( \bar{T}_i \) is a partial isometry by evaluating \( \bar{T}_i^* \bar{T}_i \lambda(\alpha,u) \). By definition of \( K \) we have

\[
\langle \lambda(\alpha,u), \bar{T}_i^* \bar{T}_i \lambda(\beta,v) \rangle = \begin{cases} a_{i\alpha(\alpha)} a_i(\beta,v) \langle \lambda(\alpha,u), \lambda(\beta,v) \rangle & \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \\ \langle u, Q_i v \rangle & \text{if } \alpha = \beta = 0. \end{cases}
\]

Thus \( \bar{T}_i^* \bar{T}_i \lambda(\beta,u) = a_{i\alpha(\beta)} \lambda(\beta,u) \) if \( \beta \neq 0 \) and

\[
\langle \lambda(0,u), \bar{T}_i^* \bar{T}_i \lambda(0,v) \rangle = \langle u, Q_i v \rangle
\]

\[
= \langle u, \sum_k a_{ik} T_k^* T_k v \rangle
\]

\[
= \langle \lambda(0,u), \sum_k a_{ik} \lambda(k,T_k^* v) \rangle.
\]

Since \( \langle \lambda(\alpha,u), \lambda(k,T_k^* v) \rangle = \delta_{k\alpha(\alpha)} \langle u, (T_i^\alpha)^* v \rangle \) for \( \alpha \neq 0 \) we have

\[
\langle \lambda(\alpha,u), \bar{T}_i^* \bar{T}_i \lambda(0,v) \rangle = a_{i\alpha(\alpha)} \langle \lambda(\alpha,u), \lambda(0,v) \rangle
\]

\[
= \langle \lambda(\alpha,u), \sum_k a_{ik} \lambda(k,T_k^* v) \rangle.
\]
and therefore
\[ \tilde{T}_i^* \tilde{T}_i \lambda(\alpha, u) = \begin{cases} \sum_k a_{ik} \lambda(k, T_k^* u) & \text{if } \alpha = 0 \\ a_{i\alpha(\alpha)} \lambda(\alpha, u) & \text{otherwise}. \end{cases} \]

It follows that \( \tilde{T}_i^* \tilde{T}_i = \tilde{T}_i \), i.e. the \( \tilde{T}_i \) are partial isometries. Finally minimality holds as
\[ \overline{\text{span}}\{\tilde{T}_i^\alpha \lambda(0, u) : \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \overline{\text{span}}\{\lambda(\alpha, u) : \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \tilde{\mathcal{H}}. \]

**SECOND PROOF USING POPESCU’S METHOD**

Recall that \( L, S \) denote the \( n \)-tuples of creation operators on \( \Gamma(\mathbb{C}^n) \), \( \Gamma_A \) respectively.

**DEFINITION 12.** For a contractive tuple \( T = (T_1, \cdots, T_n) \) on a Hilbert space \( \mathcal{H} \) the operator \( \Delta_T = (I - \sum_{i=1}^n T_i T_i^*)^{\frac{1}{2}} \) is called defect operator of \( T \). If \( \sum_{\alpha \in \Lambda^n} \tilde{T}^\alpha(T^\alpha)^* \) converges to zero in the strong operator topology as \( m \) tends to infinity then this tuple is said to be pure.

Now let \( T \) be a pure tuple on \( \mathcal{H} \) satisfying \( A \)-relations. Similarly as in [Po1]
\[ (3.1) \quad K h = \sum_{\alpha \in \Lambda_A} e^\alpha \otimes \Delta_T(T^\alpha)^* h \]
defines an isometry \( K : \mathcal{H} \rightarrow \Gamma_A \otimes \overline{\Delta_T(\mathcal{H})} \) such that \( \tilde{T}^\alpha = K^* (S^\alpha \otimes I) K \). Moreover for all \( \alpha \in \tilde{\Lambda}, S_i^* \otimes I \) leaves the range of \( K \) invariant and
\[ \overline{\text{span}}\{(S_i \otimes I)K h : i = 1, \ldots, n, h \in \mathcal{H}\} = \Gamma_A \otimes \overline{\Delta_T(\mathcal{H})} \]
since \( (I - \sum S_i S_i^*) \otimes I)K h = \omega \otimes \Delta_T h \) and \( \overline{\text{span}}\{S^\alpha \omega : \alpha \in \tilde{\Lambda}_A\} = \Gamma_A \), the tuple \( (S_1 \otimes I, \ldots, S_n \otimes I) \) is the minimal Cuntz-Krieger dilation of \( T \). Note that for \( \alpha, \beta \in \tilde{\Lambda}_A \) and \( P_0 = I - \sum_{i,j} S_i S_j^* \) we have
\[ K^* [S^\alpha P_0(S^\beta)^* \otimes I] K h = K^* [S^\alpha P_0(S^\beta)^* \otimes I] \left( \sum_{\gamma} S^\gamma \omega \otimes \Delta_T(T^\gamma)^* h \right) \]
\[ = K^* \left( \sum_{\gamma} S^\alpha P_0(S^\beta)^* S^\gamma \omega \otimes \Delta_T(T^\gamma)^* h \right) \]
\[ = K^* \left( S^\alpha \omega \otimes \Delta_T(T^\beta)^* h \right) = \tilde{T}^\alpha \Delta_T^2(T^\beta)^* h. \]

As in [Po1] starting with a contractive tuple \( T \) on a Hilbert space \( \mathcal{H} \), the tuple \( rT = (rT_1, \cdots, rT_n) \) is pure for \( 0 < r < 1 \). By (3.1) there is an isometry \( K_r : \mathcal{H} \rightarrow \Gamma_A \otimes \overline{\Delta_r(\mathcal{H})} \) defined by
\[ (3.3) \quad K_r h = \sum_{\alpha} e^\alpha \otimes \Delta_r((rT^\alpha)^* h), \]
where \( \Delta_r = (I - r^2 \sum T_i T_i^*)^{\frac{1}{2}} \). From this we obtain a unital completely positive map \( \psi_r : C^*(\mathbb{S}) \rightarrow B(\mathcal{H}) \) defined by \( \psi_r(X) = K_r^* (X \otimes I) K_r, X \in C^*(\mathbb{S}) \). As the family of maps \( \psi_r \), where \( 0 < r < 1 \), is uniformly bounded, \( \psi_r \) converges pointwise. Taking the
limit as \( r \) increases to 1, we get a unique unital completely positive map \( \theta \) from \( C^*(S) \) to \( B(\mathcal{H}) \) satisfying

\[
\theta(S^\alpha(S^\beta)^*) = T^\alpha(T^\beta)^* \quad \text{for } \alpha, \beta \in \tilde{\Lambda}_A.
\]

Once we have this map we can use a minimal Stinespring dilation \( \pi_1 : C^*(S) \to B(\tilde{\mathcal{H}}) \) of \( \theta \) such that

\[
\theta(X) = P_{\mathcal{H}} \pi_1(X) |_{\mathcal{H}} \quad \forall X \in C^*(S)
\]

and \( \text{span}\{\pi_1(X)h : X \in C^*(S), h \in \mathcal{H}\} = \mathcal{H} \). The tuple \( \tilde{T} = (\tilde{T}_1, \cdots, \tilde{T}_n) \) where \( \tilde{T}_i = \pi_1(S_i) \), is the minimal Cuntz-Krieger dilation of \( T \) which is unique up to unitary equivalence. \( \tilde{T} \) consists of partial isometries with orthogonal ranges satisfying \( A \)-relations.

We remark that if \( R \) is a tuple consisting of partial isometries orthogonal ranges satisfying the condition

\[
R_i^* R_i = I - \sum_{j=1}^n (1 - a_{ij}) R_j R_j^*
\]

(e.g. a Cuntz-Krieger dilation) then the completely positive map \( \Theta \) in (3.4) mapping \( S^\alpha(S^\beta)^* \) to \( R^\alpha(R^\beta)^* \) is *-homomorphisms because the \( S_i \)'s and \( R_i \)'s have orthogonal ranges and for \( 1 \leq i \leq n \)

\[
\Theta(S_i^* S_i) = \Theta(I - \sum_{j} (1 - a_{ij}) S_j S_j^*) = I - \sum_{j} (1 - a_{ij}) R_j R_j^*
\]

\[
= R_i^* R_i = \Theta(S_i^*) \Theta(S_i).
\]

**The algebra generated by a Cuntz-Krieger dilation**

The tuple obtained from the above constructions satisfies Cuntz-Krieger relations, that is:

\[
(3.5) \quad \tilde{T}_i^* \tilde{T}_i = I - \sum_{j} (1 - a_{ij}) \tilde{T}_j \tilde{T}_j^*.
\]

We consider the \( C^* \)-algebra generated by such tuples.

First consider the \( C^* \)-algebra generated by left creation operators on \( A \)-Fock space. Since for any \( \alpha, \beta \in \tilde{\Lambda}_A \) the rank one operator \( \eta \mapsto (S^\beta \omega, \eta) S^\alpha \omega \) on \( \Gamma_A \) can be written as

\[
S^\alpha(I - \sum S_i S_i^*) (S^\beta)^* = \sum P_0(S^\beta)^* \quad \text{and they span the subalgebra of compact operators in } C^*(S),
\]

we conclude that \( C^*(S) \) also contains all compact operators. For \( \tilde{T} = \pi_1(S) \) the Hilbert space \( \tilde{\mathcal{H}} \) can be decomposed as \( \tilde{\mathcal{H}} = \tilde{\mathcal{H}}_C \oplus \tilde{\mathcal{H}}_N \), where

\[
\tilde{\mathcal{H}}_C := \text{span}\{\pi_1(X)h : h \in \mathcal{H}, X \in C^*(\tilde{T}) \text{ and compact}\}
\]

is bi-invariant with respect to the \( \tilde{T}_i \)’s, that is, invariant with respect to \( \tilde{T}_i \) and \( \tilde{T}_i^* \) for all \( i \). Thus \( \pi_1 \) can be decomposed as \( \pi_{1C} \oplus \pi_{1N} \), where \( \pi_{1C}(X) = P_{\tilde{\mathcal{H}}_C} \pi_1(X) P_{\tilde{\mathcal{H}}_C} \) and \( \pi_{1N}(X) = P_{\tilde{\mathcal{H}}_N} \pi_1(X) P_{\tilde{\mathcal{H}}_N} \). As \( \pi_{1N} \) annihilates compacts, \( \pi_{1N}(I - \sum S_i S_i^*) = \pi_{1N}(P_0) = 0 \), hence \( (\pi_{1N}(S_1), \cdots, \pi_{1N}(S_n)) \) satisfy Cuntz-Krieger relations (in particular \( A \)-relations) and generate a Cuntz-Krieger algebra.
Let $K$ be the range of $\pi_1(P_0)$ then $\pi_{1C}(S^\alpha)k \mapsto e^\alpha \otimes k$ extends to a unitary equivalence between $\pi_{1C}(S)$ and $S \otimes I$ on $\Gamma_A \otimes K$ so that $K$ is a wandering subspace for $\tilde{T}$ generating $\tilde{\mathcal{H}}_C$.

The isometry in Stinespring’s Theorem is of the form $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that $V_1$ maps $\mathcal{H}$ to $\Gamma_A \otimes K$ and $V_2$ maps $\mathcal{H}$ to $\tilde{\mathcal{H}}_N$. Now for $\alpha, \beta \in \tilde{\Lambda}$

$$T^\alpha \Delta^2_T(T^\beta)^*h = \theta(S^\alpha(I - \sum_i S_i^*S_i^\beta)(S^\beta)^*)(h)$$

$$= V_1^*[S^\alpha P_0(S^\beta)^\ast \otimes I]V_1(h) + V_2^*[\pi_1N(S^\alpha P_0(S^\beta)^\ast)]V_2(h)$$

$$= V_1^*[S^\alpha P_0(S^\beta)^\ast \otimes I]V_1(h)$$

as $\pi_{1N}$ annihilate compacts. Comparison with identity (3.2) shows that $V_1$ may be taken to be $K$. Hence $K := \Delta_T(\mathcal{H})$.

In fact given just a tuple $R$ verifying

$$R_i^*R_i = I - \sum_{j=1}^n (1 - a_{ij})R_jR_j^*$$

it is clear that we will always obtain a decomposition of this type as the minimal Cuntz-Krieger dilation of such a tuple is the tuple itself. Such a decomposition is called Wold decomposition.

Using arguments similar to Theorem 1.3 in [Po1] we conclude that

$$\tilde{\mathcal{H}}_N = \bigcap_{m=0}^\infty \text{span}\{\tilde{T}^\alpha h : h \in \mathcal{H}, |\alpha| = m\}.$$ 

**Corollary 13.** Suppose $\tilde{T}$ is the minimal isometric dilation of a contractive tuple $T$ satisfying $A$-relations.

1. $\text{rank } (I - \sum_i \tilde{T}_i^*\tilde{T}_i) = \text{rank } (I - \sum_i T_i^*T_i)$.
2. $\lim_{k \to \infty} \sum_{|\alpha| = k} \tilde{T}^\alpha(\tilde{T}^\alpha)^* = P_{\tilde{\mathcal{H}}_N}$.

**Proof:** Clear. \(\square\)

Now we would like to see how the minimal Cuntz-Krieger dilation and minimal isometric dilation are related. The following is an analogue of Theorem 13 in [BBD] for the maximal $A$-relation piece.

**Theorem 14.** Let $T$ be a contractive $n$-tuple on a Hilbert space $\mathcal{H}$ satisfying $A$-relations. Then the maximal $A$-relation piece of the standard noncommuting dilation $\tilde{T}$ of $T$ is a realization of the minimal Cuntz-Krieger dilation $\tilde{T}$ of $T$.

**Proof:** Let $\theta : C^\ast(S) \to B(\mathcal{H})$ be the unital completely positive map as in equation (3.4), $\pi_1$ the corresponding minimal Stinespring dilation and $\tilde{T}_i = \pi_1(S_i)$ as before. Since the standard tuple $S$ on $\Gamma_A$ is also a contractive tuple, there is a unique completely
positive map \( \varphi \) from the \( C^\ast \)-algebra \( C^\ast (L) \) generated by the left creation operators to \( C^\ast (S) \), satisfying

\[
\varphi(L^\alpha (L^\beta)^*) = S^\alpha (S^\beta)^* \quad \text{for } \alpha, \beta \in \tilde{\Lambda}.
\]

Thus \( \psi \) as defined before on \( C^\ast (L) \) satisfies \( \psi = \theta \circ \varphi \). Let the minimal Stinespring dilation of \( \pi_1 \circ \varphi \) be the \( * \)-homomorphism \( \pi : C^\ast (L) \to B(\mathcal{H}_1) \) for some Hilbert space \( \mathcal{H}_1 = \text{span}\{\pi(X)h : X \in C^\ast (L), h \in \tilde{\mathcal{H}}\} \) such that

\[
\pi_1 \circ \varphi(X) = P_{\tilde{\mathcal{H}}} \pi(X)|_{\tilde{\mathcal{H}}} \quad \forall X \in C^\ast (L).
\]

In the following commuting diagram

\[
\begin{array}{ccc}
C^\ast (L) & \xrightarrow{\varphi} & C^\ast (S) \\
\pi & \uparrow & \theta \\
B(\mathcal{H}) & \xrightarrow{\pi_1} & B(\mathcal{H})
\end{array}
\]

all horizontal arrows are unital completely positive maps, down arrows are compressions and diagonal arrows are minimal Stinespring dilations. Let \( \hat{L}_i = \pi(L_i) \) and \( \hat{L} = (\hat{L}_1, \cdots, \hat{L}_n) = \hat{T} \). We will first show that \( \hat{T} \) is the maximal A-relation piece of \( \hat{L} \) and then show that \( \hat{L} \) is the standard noncommuting dilation of \( \hat{T} \).

To this end we use the presentation of the minimal isometric dilation \( \hat{L} \) which was given by Popescu [Po1]. In the one variable case it was given by Schäffer c.f. also [BBD]. Define \( D : \mathcal{H} \oplus \cdots \oplus \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) by

\[
D^2 = [\delta_{ij} I - \tilde{T}_i^* \tilde{T}_j]_{n \times n} = [\delta_{ij} (I - \tilde{T}_i^* \tilde{T}_i)]_{n \times n}
\]

as used by Popescu. Note that \( D^2 \) is a projection as all \( \tilde{T}_i \)'s are partial isometries hence \( D^2 = D \). Let \( \mathcal{D} \) denote the range of \( D \). We identify \( \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) with \( \mathbb{C}^n \otimes \mathcal{H} \) and hence \((h_1, \cdots, h_n)\) with \( \sum_{i=1}^n e_i \otimes h_i \) and \( \mathbb{C} \omega \otimes \mathcal{D} \) with \( \mathcal{D} \).

\[
D(h_1, \cdots, h_n) = D(\sum_{i=1}^n e_i \otimes h_i) = \sum_{i=1}^n e_i \otimes (I - \tilde{T}_i^* \tilde{T}_i)h_i.
\]

For \( h \in \mathcal{H}, d_\alpha \in \mathcal{D}, 1 \leq i \leq n, \) the standard noncommuting dilation \( \hat{L} = (\hat{L}_1, \cdots, \hat{L}_n) \) is given by

\[
\hat{L}_i (h \oplus \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha) = \tilde{T}_i h \oplus D(e_i \otimes h) \oplus e_i \otimes \left( \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha \right)
\]
on the dilation space $\mathcal{H}_1 = \tilde{\mathcal{H}} \oplus (\Gamma(C^n) \otimes \mathcal{D})$. As $\tilde{T}$ satisfies $A$-relations and $\hat{L}_i^*$ leaves $\tilde{\mathcal{H}}$ invariant, $\tilde{\mathcal{H}} \subseteq (\mathcal{H}_1)_A(\tilde{L})$. To show the reverse inclusion suppose that there exists a non-zero $z \in \tilde{\mathcal{H}}^\perp \cap (\mathcal{H}_1)_A(\tilde{L})$. $z$ can be written as $0 \oplus \sum \omega \otimes z_\alpha$ such that $z_\alpha \in \mathcal{D}$. Since $\langle \omega \otimes z_\alpha, (\tilde{L}_i^*)^* z \rangle = \langle e^\alpha \otimes z_\alpha, z \rangle$ and $(\tilde{L}_i^*)^* z \in (\mathcal{H}_1)_A(\tilde{L})$, we can assume $\|z_0\| = 1$ without loss of generality. Also $z_0 = D(h_1, \ldots, h_n)$ for some $h_i \in \tilde{\mathcal{H}}$ as projections have closed ranges. Now consider

$$\sum_{i,j=1}^n (\hat{L}_i \hat{L}_j - a_{ij} \hat{L}_i \hat{L}_j) \hat{T}_j^* h_i = \sum_{i,j=1}^n (\check{T}_i \check{T}_j - a_{ij} \check{T}_i \check{T}_j) \check{T}_j^* h_i + \sum_{i=1}^n D(e_i \otimes \sum_{j=1}^n (1 - a_{ij}) \check{T}_j \check{T}_j^* h_i)$$

$$+ \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \check{T}_j^* h_i)$$

$$= 0 + \sum_{i=1}^n D[e_i \otimes (I - \check{T}_i \check{T}_i^*) h_i] + x$$

$$= D^2(h_1, \ldots, h_n) + x = \tilde{z}_0 + x.$$

where $x = \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \check{T}_j^* h_i)$. Thus $\langle z, \tilde{z}_0 + x \rangle = 0$ by Lemma 4. Moreover

$$\|x\|^2 = \| \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \check{T}_j^* h_i) \|^2$$

$$= \sum_{i=1}^n \sum_{j=1}^n (1 - a_{ij}) D(e_j \otimes \check{T}_j^* h_i, \sum_{j'=1}^n (1 - a_{ij'}) e_{j'} \otimes \check{T}_{j'}^* h_i)$$

$$= \sum_{i=1}^n \sum_{j=1}^n (1 - a_{ij})(I - \check{T}_j \check{T}_j^*) \check{T}_j^* h_i, \sum_{j=1}^n (1 - a_{ij}) \check{T}_j^* h_i)$$

$$= \sum_{i=1}^n \sum_{j=1}^n (1 - a_{ij})(\check{T}_j^* - \check{T}_j^*) h_i, \sum_{j=1}^n (1 - a_{ij}) \check{T}_j^* h_i) = 0,$$

i.e. $x = 0$ and therefore $\|\tilde{z}_0\|^2 = \langle z, \tilde{z}_0 \rangle = 0$ which is a contradiction. Hence $z = 0$ which implies $\mathcal{H} = (\mathcal{H}_1)_A(\tilde{L})$.

To show finally that $\hat{L}$ is standard note that

$$\mathcal{H}_1 = \operatorname{span} \{ \hat{L}_i^\alpha x : \alpha \in \tilde{\Lambda}_A, \ x \in \tilde{\mathcal{H}} \}$$

and

$$\tilde{\mathcal{H}} = \operatorname{span} \{ \tilde{T}_i^\alpha z : \alpha \in \tilde{\Lambda}_A, \ z \in \mathcal{H} \},$$
moreover \( P_{\hat{H}} \hat{L}^\alpha = \hat{T}^\alpha \) and \( P_{\hat{H}} \hat{L}^\alpha = T^\alpha \) for \( \alpha \in \tilde{\Lambda}_A \) by assumption. Hence
\[
\mathcal{H}_1 = \text{span}\{ \hat{L}^\alpha x : \alpha \in \tilde{\Lambda}_A, x \in \mathcal{H} \}
\]
\[
= \text{span}\{ \hat{L}^\alpha \hat{T}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A, z \in \mathcal{H} \}
\]
\[
= \text{span}\{ \hat{L}^\alpha P_{\hat{H}} \hat{L}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A, z \in \mathcal{H} \}
\]
\[
\subseteq \text{span}\{ \hat{L}^\alpha \hat{L}^\beta z : \alpha, \beta \in \tilde{\Lambda}_A, z \in \mathcal{H} \} = \mathcal{H}_1.
\]

In the same way one can show that similar results holds even for \( q \)-commuting tuples (c.f. [BBD]). To keep the presentation simpler we have worked with the above special case. The following example illustrates the forgoing results.

Example 15. For \( \mathcal{H} = \mathbb{C}^2 \), let \( T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then one observes that \( T \) satisfies \( A \)-relations for the matrix \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( T_1 T_1^* + T_2 T_2^* = I \) and the \( T_i \)'s are partial isometries. Further \( D \) used in the above Theorem turns out to be
\[
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let us denote the two basis vectors in the range of \( D \) corresponding to the entries 1 appearing in \( D \) by \( f_1 \) and \( f_2 \) such that
\[
D(e_1 \otimes (a_1, a_2) + e_2 \otimes (b_1, b_2)) = a_1 f_1 + b_2 f_2
\]
for all \( a_1, a_2, b_1, b_2 \in \mathbb{C} \). The dilation space for the minimal isometric dilation \( \hat{V} = (\hat{V}_1, \hat{V}_2) \) of \( \check{T} \) is \( \mathcal{H} \oplus \Gamma(\mathbb{C}^n) \otimes \mathcal{D} \) where \( \mathcal{D} \) is the range of \( D \).
\[
\hat{V}_1 \hat{V}_1(a_1, a_2) = (0, 0) + \omega \otimes (a_2, 0) + e_1 \otimes (a_1, 0),
\]
\[
\hat{V}_2 \hat{V}_2(a_1, a_2) = (0, 0) + \omega \otimes (0, a_1) + e_2 \otimes (0, a_2).
\]
As \( a_1, a_2 \) are arbitrary using the above equations together with the description of the maximal \( A \)-relation piece from Lemma 4 we get \( \mathcal{H} = \mathcal{H} \).

4. Representations of Cuntz-Krieger Algebras

In general a Cuntz-Krieger algebra \( \mathcal{O}_A \) admits many inequivalent representations. When \( \check{T} = (T_1, \ldots, T_n) \) is a tuple on the Hilbert space \( \mathcal{H} \) satisfying \( A \)-relations and \( \sum_{i=1}^n T_i T_i^* = I \), the minimal Cuntz-Krieger dilation \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_n) \) is such that \( C^*(\hat{T}) \) is a Cuntz-Krieger algebra. If \( \check{T} \) is moreover commuting then the unital completely positive map \( \rho_\check{T} : \mathcal{O}_A \to C^*(\hat{T}) \) given by \( \rho_\check{T}(s_i) = \hat{T}_i \) is a representation of \( \mathcal{O}_A \). We will classify such representations here.
For a tuple $R = (R_1, \cdots, R_n)$ on a Hilbert space $\mathcal{K}$, we use the concept of *maximal commuting piece* and the space $\mathcal{K}^c(R)$ as defined before Lemma 6 and section 2 of [BBD]. We refer to $\mathcal{K}^c(R)$ as *maximal commuting subspace*.

**Definition 16.** (1) A commuting tuple $T = (T_1, \cdots, T_n)$ is called **spherical unitary** if $\sum T_i T_i^* = I$ and all $T_i$’s are normal.

(2) A representation $\rho$ of $O_A$ on $B(K)$ for some Hilbert space $K$, is said to be **spherical** if $R_i = \rho(s_i), 1 \leq i \leq n$ and $\mathcal{K} = \{ R^\alpha k : k \in \mathcal{K}^c(R) \text{ and } \alpha \in \tilde{A} \}$.

If $T$ is a spherical unitary then by Fuglede’s Theorem $C^*(T)$ is a commutative $C^*$-algebra i.e. $T_i^*$ and $T_j$ also commute for all $i$ and $j$.

**Definition 17.** The **maximal commuting $A$-subspace** of a $n$-tuple of isometries $V$ with orthogonal ranges is defined as the intersection of its maximal commuting subspace and maximal $A$-relation subspace. The $n$-tuple obtained by compressing each $V_i$ to the maximal commuting $A$-subspace is called **maximal commuting $A$-piece**.

**Remark 18.** Making use of Lemma 4, it follows that

$\mathcal{K}_A \cap \mathcal{K}^c = (\mathcal{K}_A)^c = (\mathcal{K}^c)_A$

i.e. the maximal commuting $A$-subspace of a $n$-tuple is in fact the maximal commuting subspace of the maximal $A$-relation piece or the maximal $A$-relation subspace of the maximal commuting piece.

Let $P_0 = 1$ on $\mathcal{C}$ and $P_m$ be the projection $\frac{1}{m!} \sum_{\sigma \in S_m} U^m_\sigma$ acting on $(\mathcal{C}^n)^\otimes^m$ where $U^m_\sigma(y_1 \otimes \cdots \otimes y_m) = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(m)}$, $y_i \in \mathcal{C}^n$. We denote $\oplus_{m=0}^\infty P_m$ by $P^\infty$. Given $A$-relations define $\Lambda^m_{m_A} = \{ \alpha \in \tilde{A} : \text{ either } |\alpha| = m > 1 \text{ and } a_{\alpha_i} = 1 \text{ for } 1 \leq i \neq j \leq m, \text{ or } |\alpha| \leq 1 \} \subseteq \Lambda^m_{m_A}$.

**Definition 19.** The subspace of $\Gamma_A$ defined by

$\text{span}\{ P^\infty e^\alpha : \alpha \in \Lambda^m_{m_A} \}$

is called **commuting $A$-Fock space** and denoted by $\Gamma_{sA}$.

To see that $\Gamma_{sA}$ is the maximal commuting $A$-subspace of $\mathcal{L}$ we first note that the maximal commuting $A$-subspace of $\mathcal{L}$ is the intersection of symmetric Fock space $\Gamma_s(\mathcal{C}^n)$ (c.f. [BBD]) and the maximal $A$-relation subspace of $\mathcal{L}$. Also

$\Gamma_s(\mathcal{C}^n) = \text{span}\{ P^\infty e^\alpha : \alpha \in \tilde{A} \}$.

Suppose $\alpha \in \Lambda^m$ and for all $1 \leq k \neq l \leq m, a_{\alpha_k} = 1$ then for $h \in \Gamma(\mathcal{C}^n)$ and all $i, j$

$\langle P^\infty e^\alpha, L^j_i (L_i L_j - a_{ij} L_i L_j) h \rangle = 0$.

So, from the definition it is clear that

$\Gamma_{sA} \subseteq \Gamma_s(\mathcal{C}^n) \cap \Gamma_A$. 
Let $\hat{P}$ denote the projection onto $\Gamma_s(\mathbb{C}^n) \cap \Gamma_A$ and let $z \in \Gamma_s(\mathbb{C}^n) \cap \Gamma_A$ be arbitrary. Suppose $\alpha \in \Lambda^m$ is such that $\langle e^\alpha, z \rangle$ is not equal to 0. As $z \in \Gamma_A$, it follows that $\alpha \in \Lambda_A^m$. Further for any $\sigma \in S_m$

$$\langle U^m_\sigma e^\alpha, z \rangle = \langle U^m_\sigma e^\alpha, \hat{P}z \rangle = \langle \hat{P}U^m_\sigma e^\alpha, z \rangle = \langle \hat{P}e^\alpha, z \rangle = \langle e^\alpha, z \rangle.$$ 

Thus $\langle U^m_\sigma e^\alpha, z \rangle$ is not equal to 0. This implies that $\alpha \in \Lambda_{sA}^m$ and hence $z \in \Gamma_{sA}$. We conclude that $\Gamma_{sA}$ is the maximal commuting $A$-subspace of $L$.

Also we would like to remark that the Fermionic Fock space (c.f. [De]) $\Gamma_s(\mathbb{C}^n)$ is the intersection of the maximal $q$-commuting subspace (defined in [De]) and the maximal $A$-relation subspace with respect to the following $q = (q_{ij})_{n \times n}$ and $A = (a_{ij})_{n \times n}$:

$$q_{ij} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise}. \end{cases}$$

It is easy to see this using arguments similar to that we use to show that $\Gamma_{sA}$ is the maximal commuting $A$-subspace with respect to $L$. In other words, the Fermionic Fock space $\Gamma_s(\mathbb{C}^n)$ is the maximal piece for the set of polynomials:

$$p_{ij}(z) = z_j z_i - q_{ij} z_i z_j \quad \text{and} \quad p_{2ij}(z) = z_i z_j - a_{ij} z_i z_j \forall 1 \leq i, j \leq n.$$ 

Notice that $L^*_i$ leaves $\Gamma_{sA}$ invariant as $S^*_i$ leaves $\Gamma_{sA}$ invariant. Let the compression of $L_i$ onto $\Gamma_{sA}$ be denoted by $W_i$, i.e. $W$ is the maximal commuting $A$-relation piece of $L$.

Suppose $\alpha \in \Lambda_{sA}^m$, and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \cdots, \alpha_m)$ where $m = |\alpha|$. The operator $W_i$ turns out to be

$$W_i P^s e^\alpha = \begin{cases} e_i & \text{if } |\alpha| = 0 \\ \frac{1}{m} P^s e_i \otimes e^\alpha & \text{if } a_{\alpha j} a_{\alpha i} = 1, \forall 1 \leq j \leq m \text{ otherwise}. \end{cases}$$

Form this it follows that $W = (W_1, \cdots, W_n)$ is the maximal commuting piece satisfying $A$-relations of $L$. Let us denote the maximal commuting piece of $L$ on $\Gamma(\mathbb{C}^n)$ by $C = (C_1, \cdots, C_n)$. (These are just the creation operators on symmetric Fock space.) Then for $\alpha \in \Lambda_{sA}^m, \alpha = (\alpha_1, \cdots, \alpha_m), m > 1$ the commutators verify

$$[W_i, W^*_i] P^s e^\alpha = \begin{cases} \left[ C_i, C^*_i \right] P^s e^\alpha & \text{if } a_{\alpha j} a_{\alpha i} = 1, \forall 1 \leq j \leq m \text{ or if } \alpha = 0 \\ \frac{1}{m} P^s e^\alpha & \text{if } \alpha_j = i \text{ for some } 1 \leq j \leq m \text{ and } a_{ii} = 0 \text{ otherwise}. \end{cases}$$

It is known that $[C_i, C^*_i]$ is compact for all $i$ (c.f. [Ar3, 5.3]), so the $[W_i, W^*_i]$’s are compact. Clearly, the vacuum vector is contained in $\Gamma_{sA}$ and $I - \sum W_i W^*_i$ is the projection on to the vacuum space. It contains all the rank one operators of the type $\mu \to (W^\alpha \omega, \mu) W^\beta \omega$ on $\Gamma_{sA}$ as those can be written as $W^\alpha (I - \sum W_i W^*_i) (W^\beta)^*$. As these rank one operators span the subalgebra of compact operators, we conclude that $C^*(W)$ contains the subalgebra of all compacts $\mathcal{K}$ as an ideal. Since the image of $W$ in $C^*(W)/\mathcal{K}$ is a spherical unitary
it follows from Fuglede’s Theorem that also \([W_i, W_j^*]\) where \(i \neq j\) must be compact and we also conclude that
\[
C^*(W) = \text{span}\{W^\alpha(W^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}.
\]

For a commuting pure tuple \(T\) satisfying \(A\)-relations, with easy computations it can be seen that the range of the isometry \(K_r : \mathcal{H} \to \Gamma_A \otimes \Delta_T(\mathcal{H}), 1 \leq r \leq 1\), defined in equation (3.4) is contained in \(\Gamma_A \otimes \Delta_T(\mathcal{H})\) and we obtain a unital completely positive map \(\phi : C^*(W) \to B(\mathcal{H})\) defined as strong operator topology limit of \(K_r^*(\otimes I)K_r\) as \(r\) increases to 1. Let \(\pi_0 : C^*(W) \to B(\mathcal{H}_0)\) be the minimal Stinespring dilation of \(\phi\) for some Hilbert space \(\mathcal{H}_0\) and \(\tilde{W}_i = \pi_0(W_i)\) where \(\mathcal{H}_0 = \text{span}\{W^\alpha h : \alpha \in \tilde{\Lambda}, h \in \mathcal{H}\}\).

**Definition 20.** The above defined tuple \(\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_n)\) is said to be the **standard commuting \(A\)-dilation of \(T\)**.

**Remark 21.** It follows from Theorem 15 in [BBD] that for spherical unitaries \(T\) satisfying \(A\)-relation the maximal commuting piece of the standard noncommuting dilation is \(\tilde{T}\). As \(T\) satisfies \(A\)-relations, it is clear that \(\tilde{T}\) is also the maximal commuting \(A\)-piece.

So far for a commuting contractive tuple satisfying \(A\)-relations we have four types of standard minimal dilations: the isometric dilation, the Cuntz-Krieger dilation (or \(A\)-dilation), the commuting dilation and the commuting \(A\)-dilation. These are obtained by considering Stinespring dilations of suitable completely positive maps on \(C^*(L), C^*(S), C^*(C)\), and \(C^*(W)\) respectively. The last dilation is in a certain sense the intersection of the previous two. The next Lemma, which is a generalization of Theorem 13 in [BBD] makes this statement rigorous. This will be crucial for classifying certain types of representations of Cuntz-Krieger algebras.

**Lemma 22.** The maximal commuting piece of the minimal Cuntz-Krieger dilation of a commuting tuple \(T\) satisfying \(A\)-relations is the standard commuting \(A\)-dilation.

**Proof:** Let the unital completely positive map \(\phi : C^*(W) \to B(\mathcal{H})\), \(\pi_0\) and \(\mathcal{H}_0\) be as above. We denote the operator \(\pi_0(W_i)\) by \(W_i\) and denote the \(n\)-tuple \((W_1, \ldots, W_n)\) by \(\tilde{W}\). As \(\tilde{W}\) is a contractive tuple satisfying \(A\)-relation, there is a unital completely positive map \(\eta : C^*(S) \to C^*(W)\) such that \(\eta(S^\alpha(S^\beta)^*) = W^\alpha(W^\beta)^*\). The completely positive map \(\theta\) as in equation (3.4) is equal to \(\phi \circ \eta\). Let \(\tilde{\pi}_1\) be the minimal Stinespring dilation of \(\pi_0 \circ \eta\) and \(V_i = \tilde{\pi}_1(S_i)\). We have the following commuting diagram.

\[
\begin{array}{ccc}
C^*(S) & \xrightarrow{\eta} & C^*(W) \\
\downarrow & & \downarrow \phi \\
\mathcal{B}(\mathcal{H}_1) & \xrightarrow{\tilde{\pi}_1} & \mathcal{B}(\mathcal{H}) \\
\downarrow & & \downarrow \\
\mathcal{B}(\mathcal{H}_0) & \xrightarrow{\pi_0} & \mathcal{B}(\mathcal{H})
\end{array}
\]

As before horizontal arrows are completely positive maps, diagonal arrows are \(*\)-homomorphism and down arrows are compressions.
Since $C^*(W)$ contains all compact operators, $\mathcal{H}_0$ can again be decomposed as $\mathcal{H}_{0C} \oplus \mathcal{H}_{0N}$ where $\mathcal{H}_{0C} = \text{span}\{\pi_0(X)h : h \in \mathcal{H}, X \in C^*(W), X \text{ compact}\}$ and $\mathcal{H}_{0N} = \mathcal{H}_0 \ominus \mathcal{H}_{0C}$. Correspondingly,

$$\pi_0(X) = \begin{pmatrix} \pi_{0C}(X) & \pi_{0N}(X) \end{pmatrix},$$

where $\pi_{0C}(X)$ and $\pi_{0N}(X)$ are compressions of $\pi_0(X)$ to $\mathcal{H}_{0C}$ and $\mathcal{H}_{0N}$ respectively. Furthermore $\mathcal{H}_{0C} = \Gamma_{sA} \otimes \Delta_T(\mathcal{H})$ and $\pi_{0C}(X) = X \otimes I$. Let $E_i = \pi_{0N}(W_i)$ and $E = (E_1, \cdots, E_n)$. As $[W_i, W_i^*]$ and $I - \sum W_iW_i^*$ are compacts, clearly $E$ consists of pairwise commuting normal operators i.e. $E$ is a spherical unitary satisfying A-relations.

From the properties of Popescu’s Poisson transform and $\Gamma_{sA}$, it follows that $(W_1 \otimes I, \cdots, W_n \otimes I)$ is the maximal commuting A-piece of its standard noncommuting dilation $(L_1 \otimes I, \cdots, L_n \otimes I)$. Also from Remark 21 we conclude that $E$ is the maximal commuting A-piece of its standard noncommuting dilation. Using Remark 18 and Theorem 14 we observe that each of them is the maximal commuting piece of their minimal Cuntz-Krieger dilation. Hence by Lemma 6, $\overline{W}$ is the maximal commuting piece of $\overline{V}$. From this using arguments similar to Theorem 13 in [BBD] it can be shown that $\overline{W}$ is the minimal Cuntz-Krieger dilation of $\overline{W}$. Hence the Lemma follows. □

If a commuting contractive tuple $T$ also satisfies A-relations for $A = (a_{ij})_{n \times n}$, then without loss of generality we may assume $A$ to be symmetric, i.e., $A = A^t$. In this case $A$ is the adjacency matrix of a (nondirected) graph $G$ with vertex set $\{1, 2, \cdots, n\}$ and set of edges $E = \{(i, j) : a_{ij} = 1, 1 \leq i < j \leq n\}$. A vertex $i$ is said to be a zero vertex if $a_{ii} = 0$. Let us associate to this graph a subset $M$ of $\{(z_1, \cdots, z_n) : \sum_{i=1}^n |z_i|^2 = 1\}$ defined as the set of elements satisfying A-relations, that is

$$M = \{(z_1, \cdots, z_n) : \sum_{i=1}^n |z_i|^2 = 1, z_iz_j = a_{ij}z_iz_j, 1 \leq i, j \leq n\}.$$

The set $M$ can be described in the following way: For a zero vertex $i$, the corresponding $z_i$ of any element of $M$ will always taken to be zero. For any element $(z_1, \cdots, z_n)$ of $M$, some elements $z_{i_1}, \cdots, z_{i_k}$ for different $1 \leq i_k \leq n$ can be simultaneously choosen to be non-zero if and only if $i_1, \cdots, i_k$ are nonzero vertices and form vertices of an induced subgraph of $G$ which is also complete.

Let $C^*_n$ be the $C^*$-algebra of continuous complex valued functions on $M$. Consider the tuple $z = (z_1, \cdots, z_n)$ of co-ordinate functions $z_i$ in $C^*_n$. To any spherical unitary $R = (R_1, \cdots, R_n)$ satisfying A-relations there corresponds a unique representation of $C^*_n$ mapping $z_i$ to $R_i$. As for any commuting tuple $T$ satisfying A-relations with $\sum T_iT_i^* = I$, the standard commuting dilation $\overline{T} = (\hat{T}_1, \cdots, \hat{T}_n)$ is a spherical unitary (c.f. section 3 of [BBD]), we have a representation $\eta_{\overline{T}}$ of $C^*_n$ such that $\eta_{\overline{T}}(z_i) = \hat{T}_i$. From Theorem 14, it is easy to see that if $D$ and $E$ are two commuting $n$-tuples of operators satisfying the same A-relations (on not necessarily the same Hilbert space), the corresponding representations $\rho_D$ and $\rho_E$ of $O_A$ are unitarily equivalent if and only if the representations $\eta_D$ and $\eta_E$ of $C^*_n$ are unitarily equivalent.
Any \( z = (z_1, \ldots, z_n) \in M \) satisfying \( A \)-relations as operator tuple on \( \mathbb{C} \) and is a spherical unitary. We can obtain a one dimensional representation \( \eta_z \) of \( C^M_n \) which maps \( f \) to \( f(z) \). Let \((V^n_z, \cdots, V^0_z)\) and \((S^n_z, \cdots, S^0_z)\) be the standard noncommuting dilation and the minimal Cuntz-Krieger dilation respectively of this operator tuple \( z = (z_1, \cdots, z_n) \). The dilation space of the standard noncommuting dilation is

\[ \mathcal{H}^z = \mathbb{C} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}^n_z), \]

where \( \mathbb{C}^n_z \) is the \((n - 1)\)-dimensional subspace of \( \mathbb{C}^n \) orthogonal to \((\overline{z}_1, \ldots, \overline{z}_n)\) and

\[ V^n_i(h \oplus \sum_{\alpha} e^\alpha \otimes d_{\alpha}) = a_i \oplus D(e_i \otimes h) \oplus e_i \otimes (\sum_{\alpha} e^\alpha \otimes d_{\alpha}). \]

Using the minimal Cuntz-Krieger dilation \( S^z \) we get a representation \( \vartheta : \mathcal{O}_A \to C^*(S^z) \) mapping \( s_i \) to \( S^z_i \). This is the GNS representation of the Cuntz-Krieger state

\[ S^\alpha(z^\beta)^* \to z^\alpha(z^\beta). \]

which exists by (3.4). We call such states Cuntz-Krieger states.

**Theorem 23.** Any spherical representation of \( \mathcal{O}_A \) (on a separable Hilbert space) can be written as direct integral of GNS representations of Cuntz-Krieger states.

**Proof:** An arbitrary representation of \( C^M_n \) is a countable direct sum of multiplicity free representations. Also any multiplicity free representation of \( C^M_n \) can be seen as a map which sends \( g \in C^M_n \) to an operator which acts as multiplication by \( g \) on \( L^2(M, \mu) \) for some finite Borel measure \( \mu \) on \( M \) and the associated representation \( \vartheta \) of \( \mathcal{O}_A \) can be expressed as direct integral of representations \( \vartheta_z \) with respect to the measure \( \mu \) acting on \( \mathcal{H}^z \mu(dz) \). Thus the Theorem follows. \( \square \)

**Example 24.** Let \( A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \). Then any commuting contractive \( A \)-relation tuple also satisfies \( A' \)-relations, where \( A' \) is the symmetric matrix

\[ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Furthermore the set of vertices of the corresponding graph is \( \{1, 2, 3, 4\} \), the set of edges is \( E = \{(1, 2), (1, 3)\} \) and 3 is a zero vertex. Hence \( M = [(\mathbb{C}^2 \times \{0\})] \cup (\{0\}^3 \times \mathbb{C}) \cap \partial B_n \) where \( \partial B_n = \{(z_1, \cdots, z_n) : \sum_{i=1}^n |z_i|^2 = 1\} \).

**Corollary 25.** Any representation of \( \mathcal{O}_A \) can be decomposed as \( \pi_s \oplus \pi_t \) where \( \pi_s \) is spherical representation and \( (\pi_t(s_1), \cdots, \pi_t(s_n)) \) has trivial maximal commuting piece.

**Proof:** Similar to proof of Theorem 19 in [BBD]. \( \square \)

It also follows that for irreducible representations of \( \mathcal{O}_A \), the maximal commuting piece of \( (\pi(s_1), \cdots, \pi(s_n)) \) is either one dimensional or trivial.
5. Universal Properties and WOT-closed Algebras Related to Minimal Cuntz-Krieger Dilation

Assume \( \tilde{T} \) to be the minimal Cuntz-Krieger dilation of a contractive tuple \( T \) satisfying \( A \)-relations. Define \( C^*(\tilde{T}) \) to be the unital \( C^* \)-algebra generated by \( \tilde{T} \). Clearly the linear map from \( C^*(\tilde{T}) \) to \( B(\mathcal{H}) \) such that \( \tilde{T}^\alpha (\tilde{T}^\beta)^* \mapsto P_\mathcal{H} \tilde{T}^\alpha (\tilde{T}^\beta)^* \) is a unital completely positive map. We will investigate some universal properties of minimal Cuntz-Krieger dilations using methods employed by Popescu [Po5] for minimal isometric dilations. Proposition 26 is a nonspatial characterization of the minimal Cuntz-Krieger dilation and Theorem 27 describes functoriality and commutant lifting in this setting. The proofs are omitted as they are similar to those appearing in Section 2 of [Po5].

**Proposition 26.** Suppose \( \tilde{T} \) is the minimal Cuntz-Krieger dilation of a contractive tuple \( T \) on a Hilbert space \( \mathcal{H} \) satisfying \( A \)-relations with respect to some matrix \( A \).

1. Consider a unital \( C^* \)-algebra \( C^*(d) \) generated by the entries of the tuple \( d = (d_1, \ldots, d_n) \) where the entries satisfy
   \[ d_i^* d_i = I - \sum_{j=1}^n (1 - a_{ij}) d_j d_j^* \]
   Assume that \( d \) also satisfies \( d_i^* d_j = 0 \) for \( 1 \leq i \neq j \leq n \). Let there be a completely positive map \( \rho : C^*(d) \to B(\mathcal{H}) \) such that \( \rho(d^\alpha (d^\beta)^*) = \tilde{T}^\alpha (\tilde{T}^\beta)^* \). Then there is an \( * \)-homomorphism form \( C^*(d) \) to \( C^*(\tilde{T}) \) such that \( d_i \mapsto \tilde{T}_i \) for all \( 1 \leq i \leq n \).

2. Suppose \( \pi : C^*(\tilde{T}) \to B(\mathcal{K}) \) is a \(*\)-homomorphism and \( \theta : C^*(\tilde{T}) \to C^*(T) \) the completely positive map obtained by restricting the compression map (to \( B(\mathcal{H}) \)) of \( B(\mathcal{K}) \) to \( C^*(\tilde{T}) \). Assume the minimal Stinespring dilation of \( \pi \circ \theta \) to be \( \tilde{\pi} \).
   i.e. \( \pi \circ \theta (X) = P_\mathcal{K} \tilde{\pi}(X)|_\mathcal{K} \). Then \( (\tilde{\pi}(\tilde{T}_1), \ldots, \tilde{\pi}(\tilde{T}_n)) \) is the minimal Cuntz-Krieger dilation of \( (\pi(T_1), \ldots, \pi(T_n)) \).

**Theorem 27.** Let \( T \) be a contractive \( n \)-tuple on \( \mathcal{H} \) satisfying \( A \)-relations and \( \tilde{T} \) be its minimal Cuntz-Krieger dilation.

1. Suppose \( \pi_1 \) and \( \pi_2 \) are two \(*\)-homomorphism from \( C^*(T) \) to \( B(\mathcal{K}_1) \) and \( B(\mathcal{K}_2) \) respectively, for some Hilbert spaces \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \). Let \( \theta \) be as defined in the previous Proposition. If \( X \) is an operator such that \( X \pi_1(Y) = \pi_2(Y) X \) for all \( Y \in C^*(T) \), and \( \pi_1 \) and \( \pi_2 \) are the minimal Stinespring dilations of \( \pi_1 \circ \theta \) and \( \pi_2 \circ \theta \) respectively then there exists an operator \( \tilde{X} \) such that \( \tilde{X} \pi_1 = \tilde{\pi}_2 \tilde{X} \) and \( \tilde{X} P_{\mathcal{K}_1} = P_{\mathcal{K}_2} \tilde{X} \).

2. If \( X \in C^*(T) \) then there exists a unique \( \tilde{X} \in C^*(\tilde{T}) \cap \{ P_\mathcal{H} \} \) such that \( P_\mathcal{H} \tilde{X} = X \). Also the map \( X \mapsto \tilde{X} \) is a \(*\)-isomorphism.

Many of the results and arguments for minimal Cuntz-Krieger dilation in the following part of this section are similar to those of [DKS] and [DP2] for standard noncommuting
dilation. Using equation (3.4) we observe that
\[
\tilde{T}_j^* \tilde{T}_i^* \tilde{T}_i \tilde{T}_j = \tilde{T}_j^* [I - \sum_k (1 - a_{ik}) \tilde{T}_k^* \tilde{T}_k] \tilde{T}_j = a_{ij} \tilde{T}_j^* \tilde{T}_i^* \tilde{T}_j = a_{ij} \tilde{T}_j^* \tilde{T}_j.
\]
From this follows that for \( \alpha = (\alpha_1, \ldots, \alpha_m) \)
\[
(\tilde{T}^\alpha) (\tilde{T}^\alpha)^* = a_{\alpha_1 \alpha_2} \cdots a_{\alpha_{m-1} \alpha_m} \tilde{T}_{\alpha_m}^* \tilde{T}_{\alpha_m} = a_{\alpha_1 \alpha_2} \cdots a_{\alpha_{m-1} \alpha_m} [I - \sum_k (1 - a_{\alpha_m k}) \tilde{T}_k^* \tilde{T}_k].
\]
and
\[
\tilde{T}^\alpha (\tilde{T}^\alpha)^* \tilde{T}^\alpha = \tilde{T}^\alpha
\]
so that each \( \tilde{T}^\alpha \) is a partial isometry. Let \( \hat{\mathcal{H}} \) and \( \tilde{\mathcal{H}} \) be the dilation spaces associated with standard noncommuting dilation \( \tilde{T} \) and \( \hat{T} \) respectively as before and let us denote \( \hat{\mathcal{H}} \ominus \mathcal{H} \) by \( \mathcal{E} \). We know that \( \hat{T}_i \) and \( \tilde{T}_i \) leaves \( \mathcal{E} \) and \( \hat{\mathcal{H}} \ominus \mathcal{H} \) respectively invariant. Let \( \Phi : B(\mathcal{H}_1) \to B(\mathcal{H}_1) \) be the completely positive map defined by
\[
\Phi(X) = \sum_{i=1}^n \hat{T}_i P_{\hat{\mathcal{H}}} X P_{\hat{\mathcal{H}}} \hat{T}_i^*.
\]
Thus, \( \Phi(P_\mathcal{E}) \leq \Phi(I) \). Also let \( Q_i := P_\mathcal{E} \tilde{T}_i P_\mathcal{E} \). Then for \( h \in \mathcal{E} \)
\[
\langle h, \sum_{|\alpha|=m} Q^\alpha (Q^\alpha)^* h \rangle = \langle h, \sum_{|\alpha|=m} \tilde{T}^\alpha P_\mathcal{E} (\tilde{T}^\alpha)^* h \rangle = \langle h, \sum_{|\alpha|=m} \hat{T}^\alpha P_{\hat{\mathcal{H}}} P_\mathcal{E} P_{\hat{\mathcal{H}}} (\hat{T}^\alpha)^* h \rangle = \langle h, \Phi^m (P_\mathcal{E}) h \rangle \leq \langle h, \Phi^m (I) h \rangle.
\]
But as \( \lim_{m \to \infty} \langle h, \Phi^m (I) h \rangle = 0 \) we have \( \lim_{m \to \infty} \langle h, \sum_{|\alpha|=m} Q^\alpha (Q^\alpha)^* h \rangle = 0 \) which implies that \( Q \) is pure. In the above computation we used \( \hat{T}_i^* \) invariance of \( \hat{\mathcal{H}} \) for \( 1 \leq i \leq n \). Here we are interested in understanding the structure of WOT-closed algebra generated by the minimal Cuntz-Krieger dilation \( \tilde{T} \) of some contractive tuple \( T = (T_1, \cdots, T_n) \) satisfying \( A \)-relations where \( T_i \in B(\mathcal{H}) \). Let \( \mathcal{A} \) denote the WOT-closed algebra generated by all \( \tilde{T}_i, 1 \leq i \leq n \).

**Lemma 28.**

1. If \( \mathcal{A} \) has no wandering vector then every non-trivial invariant subspace with respect to \( \mathcal{A} \) is also reducing.
2. \( \mathcal{K} := \mathcal{E} \ominus \sum_{i=1}^n \tilde{T}_i \mathcal{E} \) is a wandering subspace for \( \tilde{T} \).

**Proof:** Let there be no wandering vector for \( \mathcal{A} \) and let if possible \( \mathcal{N} \) be a non-trivial invariant subspace for \( \mathcal{A} \). If \( \sum_{i=1}^n \tilde{T}_i \mathcal{N} \) is not equal to \( \mathcal{N} \) then \( \mathcal{N} \ominus \sum_{i=1}^n \tilde{T}_i \mathcal{N} \) would be
wandering as seen using orthogonality of the ranges of the \( \tilde{T}_i \)'s, equation (5.1) and the following: For \( n_1, n_2 \in \mathcal{N} \ominus \sum_{i=1}^{n} \tilde{T}_i \mathcal{N} \)

\[
\langle \tilde{T}_i \tilde{T}_i \cdots \tilde{T}_{\alpha m} n_1, n_2 \rangle = \langle a_{\alpha_1} \tilde{T}_1 \cdots \tilde{T}_{\alpha m} n_1, n_2 \rangle = 0.
\]

But this is not possible by our assumption. So

\[
(5.2) \quad \mathcal{N} = \sum_{i=1}^{n} \tilde{T}_i \mathcal{N}.
\]

Now let \( h \in \mathcal{N} \) be arbitrary. From the above equation it follows that one can write \( h \) as \( \sum_{i,j=1}^{n} T_i T_j n_{ij} \) for some \( n_{ij} \in \mathcal{N} \). From this and equation (5.1) it is clear that \( \tilde{T}_k h \in \mathcal{N} \) for all \( 1 \leq k \leq n \). So \( \mathcal{N} \) is reducing for \( A \). Hence (1) follows.

\( \mathcal{E} \) is also an invariant subspace for \( A \). The nontrivial case is when \( \mathcal{E} \) is non-zero. \( \mathcal{E} \neq \sum_{i=1}^{n} \tilde{T}_i \mathcal{E} \) as otherwise \( \mathcal{E} \) would be reducing which is not possible as \( H \) spans \( \tilde{H} \). It follows from above that \( K \) is a wandering subspace of \( A \).

So we can write \( \tilde{H} = H \oplus H' \oplus (\Gamma_A \otimes K) \) for some Hilbert space \( H' \). So \( \sum_{\alpha \in \Lambda} \tilde{T}_i \mathcal{K} = \Gamma_A \otimes K \) and this is left invariant by all \( \tilde{T}_i \). Also \( \tilde{T}_i P_{\Gamma_A \otimes K} \) is \( S_i \otimes I \) for \( 1 \leq i \leq n \).

Let us denote by \( \mathcal{B} \) the WOT-closed algebra generated by \( T_1, \cdots, T_n \). In order to get reducing subspaces for \( \mathcal{A} \) it's sufficient to demand for \( \mathcal{B}^\ast \)-invariant subspace as seen in the next Lemma. (\( \mathcal{A}[\mathcal{L}] \) denotes the closed linear span of \( \mathcal{A} \mathcal{L} \).)

**Lemma 29.** Let \( \mathcal{L} \) be a \( \mathcal{B}^\ast \)-invariant subspace of \( H \). Then \( \mathcal{A}[\mathcal{L}] \) reduces \( A \). If \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are orthogonal \( \mathcal{B}^\ast \)-invariant subspace of \( H \) then \( \mathcal{A}[\mathcal{L}_1] \) and \( \mathcal{A}[\mathcal{L}_2] \) are also mutually orthogonal.

**Proof:** \( \tilde{T}_i^\ast \) leaves \( \mathcal{L} \) invariant as \( \tilde{T}_i^\ast \) and \( T_i^\ast \) leaves \( H \) and \( \mathcal{L} \) respectively invariant. Thus

\[
\mathcal{A}[\mathcal{L}] = \text{span} \{ \tilde{T}_i^\ast h : \alpha \in \Lambda, h \in \mathcal{L} \}.
\]

Now for any \( x \in \mathcal{L} \) and \( \alpha = (\alpha_1, \cdots, \alpha_m) \), using equation (5.1)

\[
\tilde{T}_i^\ast \tilde{T}_i^\alpha x = \begin{cases} 
[I - \sum_{k} (1 - a_{ik}) \tilde{T}_k \tilde{T}_i^k] x & \text{if } \alpha_1 = i, |\alpha| = 1 \\
\alpha_1 \alpha_2 \tilde{T}_2 \cdots \tilde{T}_{\alpha m} x & \text{if } \alpha_1 = i, |\alpha| > 1 \\
0 & \text{if } \alpha_1 \neq i \\
\tilde{T}_i^\ast x & \text{if } |\alpha| = 0
\end{cases}
\]

As \( \mathcal{L} \) is invariant for \( \mathcal{A}^\ast \), \( \tilde{T}_i^\ast \tilde{T}_i^\alpha x \in \mathcal{A}[\mathcal{L}] \) and hence reduce \( A \).

Further when \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are orthogonal \( \mathcal{B}^\ast \)-invariant subspaces, to establish that \( \mathcal{A}[\mathcal{L}_1] \) is \( \mathcal{A}[\mathcal{L}_2] \) are orthogonal it is sufficient to check if \( |\alpha| \leq |\beta| \) and \( l_1 \in \mathcal{L}_1, l_2 \in \mathcal{L}_2 \), then

\[
\langle \tilde{T}_i^\alpha l_1, \tilde{T}_j^\beta l_2 \rangle = 0.
\]

This is checked easily for all cases by orthogonality of ranges of different \( \tilde{T}_i \)'s, \( \mathcal{B}^\ast \)-invariance of \( \mathcal{L}_i \) and the equation (5.1) except \( \alpha = (\alpha_1, \cdots, \alpha_m) = \beta \). In this case

\[
\langle \tilde{T}_i^\alpha \tilde{T}_i^\alpha l_1, l_2 \rangle = \langle a_{\alpha_1} \alpha_2 \cdots a_{\alpha_{m-1}} \alpha_m [I - \sum_k (1 - a_{\alpha m k}) \tilde{T}_k \tilde{T}_k^\alpha] l_1, l_2 \rangle
\]

\[
= a_{\alpha_1} \alpha_2 \cdots a_{\alpha_{m-1}} \alpha_m \{ \langle l_1, l_2 \rangle - \sum_k (1 - a_{\alpha m k}) \langle \tilde{T}_k^\alpha l_1, \tilde{T}_k^\alpha l_2 \rangle \} = 0,
\]

So

\[
\mathcal{A}[\mathcal{L}_1] \perp \mathcal{A}[\mathcal{L}_2].
\]
hence the Lemma follows.

Recall that \( \mathcal{H}_N \) denotes the summand on which the compact operators in \( C^*(\tilde{T}) \) act trivially. Let \( \mathcal{H} := \mathcal{H}_N \cap \mathcal{H} \). In the next Proposition we assume \( \mathcal{H} \) to be finite dimensional.

**Proposition 30.** Let \( \tilde{T} \) be a contractive tuple satisfying \( \Lambda \)-relations of operators on a finite dimensional Hilbert space \( \mathcal{H} \).

1. Let \( \mathcal{K} \) be a reducing subspace of \( \mathcal{H}_N \) with respect to \( \mathcal{A} \) and let \( h \in \mathcal{H} \) such that \( P_{\mathcal{K}h} \) is non-zero. Then there exists \( k \in \mathcal{A}^*[h] \cap \mathcal{H}_N \) such that \( P_{\mathcal{K}k} \) is non-zero.
2. Any non-zero subspace of \( \mathcal{H}_N \) which is co-invariant with respect to \( T_i, 1 \leq i \leq n \) has a non-trivial intersection with \( \mathcal{H}_N \).
3. \( \mathcal{H}_N = \mathcal{A}[\mathcal{H}_N] \). When \( \sum T_i^* T_i = I \) and \( \mathcal{B} = B(\mathcal{H}) \) every co-invariant subspace of \( \mathcal{H} \) with respect to all \( T_i \)'s contains \( \mathcal{H} \).

**Proof:** Proof is similar to the proof of Lemma 4.1, Corollary 4.2 and Corollary 4.3 in [DPS]. It uses above two Lemmas, pureness of \( Q \), Wold decomposition of \( \tilde{T} \) and compactness of the unit ball of finite dimensional \( \mathcal{T} \) Hilbert space \( \mathcal{H} \). (2) and (3) are corollaries of (1).

Now we consider the tuple \( R \) consisting of right creation operators on \( \Gamma(\mathbb{C}^n) \) given by \( R_i x = x \otimes e_i \). One can easily notice using methods similar to proof of Lemma 4 that for polynomials \( p_{l,m} = z_l z_m - a_{ml} z_l z_m, (l,m) \in \{1, \cdots, n\} \times \{1, \cdots, n\} = \mathcal{I} \), we get \( (\Gamma(\mathbb{C}^n))^p(R) = \Gamma_A \). Let \( X_i \) denote the compression of \( R_i \) to \( \Gamma_A \), i.e. \( X_i = P_{\Gamma_A} R_i |_{\Gamma_A} \). Suppose \( e^\alpha \in \Gamma_A \), and when \( |\alpha| > 0 \) let \( \alpha = (\alpha_1, \cdots, \alpha_m) \). Then

\[
X_i e^\alpha = P_{\Gamma_A} R_i e^\alpha = \begin{cases} e_i & \text{if } |\alpha| = 0 \\ a_{\alpha m} e^\alpha \otimes e_i & \text{if } |\alpha| \geq 1 \end{cases}
\]

Moreover from Proposition 9 it follows that \( X \) consists of isometries with orthogonal ranges satisfying \( \Lambda' \)-relations, where \( \Lambda' \) is the transpose of \( \Lambda \). Let \( S \) and \( \mathcal{X} \) denote the WOT-closed algebras generated by \( S_1, \cdots, S_n \) and \( X_1, \cdots, X_n \) respectively. Now we shall analyze the structure of these WOT-closed algebras. Let \( Q_k \) denote the projection onto \( \text{span}\{e^\alpha : \alpha \in \tilde{\Lambda}_A, |\alpha| = k\} \).

**Proposition 31.**

1. \( S \) coincides with the commutant of \( \mathcal{X} \) in \( B(\Gamma_A) \), that is \( S = \mathcal{X}' \). Also \( \mathcal{X} = S' \) and hence \( S \) and \( \mathcal{X} \) are double commutants of themselves.
2. \( S \) and \( \mathcal{X} \) are inverse closed and the only normal elements in \( S \) and \( \mathcal{X} \) are scalars.

**Proof:** Any element in \( S \) can be written as a formal sum \( \sum b_a \mathcal{X}^a \), where \( b_a \in \mathbb{C} \) are given by \( X \omega = \sum b_a e^a \). Let for \( \beta = (\beta_1, \cdots, \beta_m), \beta' \) denote \((\beta_m, \cdots, \beta_1)\).

\[
\mathcal{X}^{\alpha} X^{\beta} e^{\gamma} = \begin{cases} a_{\alpha |\alpha| \gamma} a_{\gamma |\beta| \beta} e^\alpha \otimes e^\gamma \otimes e^\beta & \text{if } |\gamma| > 0 \\ a_{\alpha |\beta| \alpha} e^\alpha \otimes e^\beta & \text{if } |\gamma| = 0 \end{cases} = X^{\beta'} \mathcal{X}^{\alpha} e^{\gamma}.
\]
So, $S \subseteq X'$. The converse is similar to the proof of Theorem 1.2 in [DP2] after noticing that $X_i Q_k = Q_{k+1} X_i$ and considering the Cesáro sums

$$p_k(L) = \sum_{|\alpha| < k} \left( 1 - \frac{|\alpha|}{k} \right) d_{\alpha} S^\alpha$$

for $L \omega = \sum_{\alpha \in \lambda_A} d_{\alpha} e^\alpha$.

$S$ and $X'$ are inverse closed as this is the case for any algebra which is a commutant. This and (2) can be proved by taking the same approach as that of the proof of Corollary 1.4, 1.5 in [DP2]. □

**Proposition 32.** Any element $A \in \mathcal{L}$ leaves the range of $X^\alpha (X^\alpha)^*$ invariant.

**Proof:** Note that one can argue as we did for $S^\alpha$ and show that $X^\alpha$ are partial isometries. Further as $\mathcal{L} = X' \quad X^\alpha (X^\alpha)^* A X^\alpha (X^\alpha)^* = X^\alpha (X^\alpha)^* X^\alpha A (X^\alpha)^* = X^\alpha A (X^\alpha)^* = A X^\alpha (X^\alpha)^*$, the proposition follows. □

In these algebras the wandering subspace description is much simpler than the general case as can be seen from the next result.

**Proposition 33.**

1. If $N$ is an invariant subspace of $\mathcal{L}$ then $M = N \ominus \sum_{i=1}^n S_i N$ is a wandering subspace and $\mathcal{L}[M] = N$.
2. A subspace is cyclic and invariant with respect to $\mathcal{L}$ if and only if it is the range of some element in $X$.

**Proof:** Follows from the Wold decomposition using methods similar to the proof of Theorem 2.1 in [DP2]. □

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