Confinement-Deconfinement Phase Transition at Nonzero Chemical Potential

D. Toublan\textsuperscript{1} and Ariel R. Zhitnitsky\textsuperscript{2}

\textsuperscript{1}Physics Department, University of Illinois at Urbana-Champaign, Urbana, IL 61801
\textsuperscript{2}Department of Physics and Astronomy, University of British Columbia, Vancouver, BC V6T 1Z1, Canada

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We present arguments suggesting that large size overlapping instantons are the driving mechanism of the confinement-deconfinement phase transition at nonzero chemical potential $\mu$. The arguments are based on the picture that instantons at very large chemical potential in the weak coupling regime are localized configurations with finite size $\rho \sim \mu^{-1}$. At the same time, the same instantons at smaller chemical potential in the strong coupling regime are well represented by the so-called instanton-quarks with fractional topological charge $1/N_c$. We estimate the critical chemical potential $\mu_c(T)$ where this phase transition takes place as a function of temperature in the domain where our approach is justified. In this picture, the long standing problem of the “accidental” coincidence of the chiral and deconfinement phase transitions at nonzero temperature (observed in lattice simulations) is naturally resolved. We also derive results at nonzero isospin chemical potential $\mu_I$ where direct lattice calculations are possible, and our predictions can be explicitly tested.

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\textit{Introduction.} — Color confinement, spontaneous breaking of chiral symmetry, the $U(1)$ problem and the $\theta$ dependence are some of the most interesting questions in QCD. Unfortunately, progress in the understanding of these problems has been extremely slow. At the end of the 1970’s A. M. Polyakov \cite{1} demonstrated charge confinement in QED. This was the first example where nontrivial dynamics was shown to be a key ingredient for confinement: The instantons (the monopoles in 3d) play a crucial role in the dynamics of confinement in QED. Soon afterwards instantons in four dimensional QCD were discovered \cite{2}. However, their role in QCD remains unclear due to the divergence of the instanton density for large size instantons.

Approximately at the same time instanton dynamics was developed in two dimensional, classically conformal, asymptotically free models (which may have some analogies with QCD). Namely, using an exact accounting and resummation of the $n$-instanton solutions in 2d $CP^{N_c-1}$ models, the original problem of a statistical instanton ensemble was mapped unto a 2d-Coulomb Gas (CG) system of pseudo-particles with fractional topological charges $\sim 1/N_c$ (the so-called instanton-quarks) \cite{3}. The instanton-quarks do not exist separately as individual objects. Rather, they appear in the system all together as a set of $\sim N_c$ instanton-quarks so that the total topological charge of each configuration is always an integer. This means that a charge for an individual instanton-quark cannot be created and measured. Instead, only the total topological charge for the whole configuration is forced to be integer and has a physical meaning. This picture leads to the elegant explanation of confinement and other important properties of the 2d $CP^{N_c-1}$ models \cite{4}. Unfortunately, despite some attempts \cite{4}, there is no demonstration that a similar picture occurs in 4d gauge theories, where the instanton-quarks would become the relevant quasiparticles. Nevertheless, there remains a strong suspicion that this picture, which assumes that instanton-quarks with fractional topological charges $\sim 1/N_c$ become the relevant degrees of freedom in the confined phase, may be correct in QCD$_4$.

On the phenomenological side, the development of the instanton liquid model (ILM) \cite{5,6} has encountered successes (chiral symmetry breaking, resolution of the $U(1)$ problem, etc) and failures (confinement could not be described by well separated and localized lumps with integer topological charges). Therefore, it is fair to say that at present, the widely accepted viewpoint is that the ILM can explain many experimental data (such as hadron masses, widths, correlation functions, decay couplings, etc), with one, but crucial exception: confinement. There are many arguments against the ILM approach, see e.g. \cite{7}, there are many arguments supporting it \cite{8}.

In this letter we present new arguments supporting the idea that the instanton-quarks are the relevant quasiparticles in the strong coupling regime. In this case, many problems formulated in \cite{7} are naturally resolved as both phenomena, confinement and chiral symmetry breaking are originated from the same vacuum configurations, instantons, which may have arbitrary scales: the finite sized localized lumps, as well as set of $N_c$ fractionally charged $1/N_c$ correlated objects with arbitrary large separations.

More importantly, we make some very specific predictions which can be tested with traditional Monte Carlo techniques, by studying QCD at nonzero isospin chemical potential \cite{8}. We start by reviewing recent work for QCD at large $\mu$ in the deconfined phase \cite{9}, where the instanton calculations are under complete theoretical control, since the instantons are well-localized objects with a typical size $\rho \sim 1/\mu$. We then discuss the dual rep-
representation of the low-energy effective chiral Lagrangian in the regime of small chemical potential where confinement takes place. We shall argue that the corresponding dual representation corresponds to a statistical system of interacting pseudo-particles with fractional $1/N_c$ topological charges which can be identified with instanton-quarks \( \frac{1}{2} \) suspected long ago \( [3, 4] \). Based on these observations we conjecture that the transition from the description in terms of well localized instantons with finite size at large \( \mu \) to the description in terms of the instanton quarks with fractional $1/N_c$ topological charges precisely corresponds to the deconfinement-confinement phase transition. In what follows we explicitly calculate the critical chemical potential \( \mu_c \) where this phase transition occurs. Our conjecture can be explicitly and readily tested in numerical simulations due to the absence of the sign problem at arbitrary value of the isospin chemical potential. If our conjecture turns out to be correct, it would be an explicit demonstration of the link between confinement and instantons.

**Instantons at large \( \mu \).**— At low energy and large chemical potential, the \( \eta' \) is light and described by the Lagrangian derived in \( [3] \):

\[
L_\phi = f^2(\mu)(\partial \phi \partial \phi) - u^2(\partial \phi \partial \phi) - V_{\text{inst}}(\phi),
\]

where the \( \phi \) decay constant, \( f^2(\mu_B) = \mu_B^2/8\pi^2 \) and \( f^2(\mu) = 3\mu^2/16\pi^2 \), and its velocity, \( u^2 = 1/3 \). We define baryon and isospin chemical potentials as \( \mu_{B,I} = (\mu_u \pm \mu_d)/2 \). The nonperturbative potential \( V_{\text{inst}} \sim \cos(\phi - \theta) \) is due to instantons, which are suppressed at large chemical potential.

The instanton-induced effective four-fermion interaction for 2 flavors, \( u, d \), is given by \( [12, 13] \):

\[
L_{\text{inst}} = \int d\rho n(\rho) \left\{ (\bar{u}_R \gamma^\mu u_L) (\bar{d}_R \gamma^\mu d_L) + \frac{3}{32} \left[ (\bar{u}_R \gamma^\mu \gamma^\mu u_L)(\bar{d}_R \gamma^\mu \gamma^\mu d_L) - \frac{3}{4} (\bar{u}_R \sigma_{\mu\nu} \gamma^\mu u_L)(\bar{d}_R \sigma_{\mu\nu} \gamma^\mu d_L) \right] \right\} + \text{H.c.}
\]

We study this problem at nonzero temperature and chemical potential for \( T \ll \mu \), and we use the standard formula for the instanton density at two-loop order \( [6] \):

\[
n(\rho) = C_N (\beta_1(\rho))^{2N_c} \rho^{-5} \exp[\beta_{II}(\rho)]
\]

\[
\exp[-(N_f/2\pi^2 + 1/3 (2N_c + N_f)\rho^2)],
\]

where

\[
C_N = 0.466e^{-1.679N_c/2N_f}/(N_c - 1)!(N_c - 2)!,
\]

\[
\beta_{I}(\rho) = -b \log(\rho\Lambda), \quad \beta_{II}(\rho) = \beta_I(\rho) + \frac{b'}{2b} \log \left( \frac{2\beta_I(\rho)}{b} \right),
\]

\[
b = \frac{11}{3} N_c - \frac{2}{3} N_f, \quad b' = \frac{34}{3} N_c^2 - \frac{13}{3} N_f N_c + \frac{N_f}{N_c}.
\]

By taking the average of Eq. (2) over the state with nonzero vacuum expectation value for the condensate, one finds

\[
V_{\text{inst}}(\phi) = -\int d\rho n(\rho) \left\{ \frac{4}{3} \rho^2 \phi^2 \right\} 12 |X(\mu)|^2 \cos(\phi - \theta)
\]

\[
= -a(\mu, T) \mu^2 \Delta^2 \cos(\phi - \theta),
\]

where \( |X(\mu_B)| = 3\mu_B^2 \Delta / \sqrt{\beta_I(\rho)} \) and \( |X(\mu_I)| = 3\sqrt{3}\mu_I^2 \Delta / \sqrt{\beta_I(\rho)} \), and \( \Delta \) is the gap \( [3, 11] \). Therefore the mass of the \( \phi \) field is given by

\[
m = \sqrt{a(\mu, T) \mu \Delta / 2 \beta_I(\rho)}.
\]

The approach presented above is valid as long as the \( \phi \) field is lighter than \( \sim 2\Delta \), the mass of the other mesons in the system \( [3] \), that is if

\[
a(\mu, T) \lesssim 8f^2(\mu)/\mu^2.
\]

This is exactly the vicinity where the Debye screening scale and the inverse gap become of the same order of magnitude \( [3] \), and therefore, where the instanton expansion breaks down.

For reasons which will be clear soon, we want to represent the Sine-Gordon (SG) partition function \( [10, 11] \) in the equivalent dual Coulomb Gas (CG) representation \( [3] \):

\[
Z = \sum_{M_{\pm} = 0}^\infty \frac{(\lambda/2)^M}{M_+! M_-!} \int d^4x_1 \ldots d^4x_M \ e^{-i\theta \sum_{a=0}^M Q_a} \cdot e^{-\frac{\lambda}{2} \sum_{a>b=0}^M Q_a Q_b G(x_a - x_b)},
\]

\[
G(x_a - x_b) = \frac{1}{4\pi^2 (x_a - x_b)^2}, \quad \lambda = \frac{a(\mu, T) \mu^2 \Delta^2}{n}.
\]

Physical interpretation of the dual CG representation \( [7] \):

a) Since \( Q_{\text{net}} \equiv \sum_{a} Q_a \) is the total charge and it appears in the action multiplied by the parameter \( \theta \), one concludes that \( Q_{\text{net}} \) is the total topological charge of a given configuration.

b) Each charge \( Q_a \) in a given configuration should be identified with an integer topological charge well localized at the point \( x_a \). This, by definition, corresponds to a small instanton positioned at \( x_a \).

c) While the starting low-energy effective Lagrangian contains only a colorless field \( \phi \) we have ended up with a representation of the partition function in which objects carrying color (the instantons) can be studied.

d) In particular, \( II \) and \( I \) interactions (at very large distances) are exactly the same up to a sign, order \( g^0 \), and are Coulomb-like. This is in contrast with semiclassical expressions when \( II \) interaction is zero and \( I \) interaction is order \( 1/g^2 \).

e) The very complicated picture of the bare \( II \) and \( I \) interactions becomes very simple for dressed
instantons/anti-instantons when all integrations over all possible sizes, color orientations and interactions with background fields are properly accounted for.

f) As expected, the ensemble of small \( \rho \sim 1/\mu \) instantons cannot produce confinement. This is in accordance with the fact that there is no confinement at large \( \mu \).

**Instantons at small \( \mu \)**[14]. — We want to repeat the same procedure that led to the CG representation in the confined phase at small \( \mu \) to see if any traces from the instantons can be recovered. We start from the chiral Lagrangian and keep only the diagonal elements of the chiral matrix \( U = \exp\{i\text{diag}(\phi_1, \ldots, \phi_{N_f})\} \) which are relevant in the description of the ground state. Singlet combination is defined as \( \phi = \text{Tr} \ U \). The effective Lagrangian for the field is

\[
L_{\eta} = f^2(\partial_\mu \phi)^2 + E \cos \left( \frac{\phi - \theta}{N_c} \right) + \sum_{a=1}^{N_f} m_a \cos \phi_a \quad (8)
\]

A Sine-Gordon structure for the singlet combination corresponds to the following behavior of the \((2k)^{th}\) derivative of the vacuum energy in pure gluodynamics[14].

\[
\frac{\partial^{2k} E_{\text{vac}}(\theta)}{\partial \theta^{2k}} \bigg|_{\theta=0} \sim \int \prod_{i=1}^{2k} dx_i (Q(x_1) \ldots Q(x_{2k})) \sim (\frac{i}{N_c})^{2k},
\]

where \( Q = \frac{g^2}{2\pi^2} G_{\mu\nu} G_{\mu\nu} \) is the topological density. The same structure was also advocated in [12] from a different perspective. As in [5], the Sine-Gordon effective field theory [5] can be represented in terms of a classical statistical ensemble (CG representation) given by [1] with the replacements \( \lambda \to E, \ u \to 1 \). The fundamental difference in comparison with the previous case [7] is that while the total charge is integer, the individual charges are fractional \( \pm 1/N_c \). This is a direct consequence of the \( \theta/N_c \) dependence in the underlying chiral Lagrangian [5] before integrating out \( \phi \) fields.

**Physical Interpretation of the CG representation**[1].

a) As before, one can identify \( Q_{\text{net}} = \sum_a Q_a \) with the total topological charge of the given configuration.

b) Due to the \( 2\pi \) periodicity of the theory, only configurations which contain an integer topological number contribute to the partition function. Therefore, the number of particles for each given configuration \( Q_i \) with charges \( \sim 1/N_c \) must be proportional to \( N_c \).

c) Therefore, the number of integrations over \( dx_i \) in CS integrals exactly equals \( 4N_c k \), where \( k \) is integer. This number \( 4N_c k \) exactly corresponds to the number of zero modes in the \( k \)-instanton background. This is basis for the conjecture [10] that at low energies (large distances) the fractionally charged species, \( Q_i = \pm 1/N_c \) are the instanton-quarks suspected long ago [3].

d) For the gauge group, \( G \) the number of integrations would be equal to \( 4kC_2(G) \) where \( C_2(G) \) is the quadratic Casimir of the gauge group (\( \theta \) dependence in physical observables comes in the combination \( \frac{\theta}{2\pi} \)). This number \( 4kC_2(G) \) exactly corresponds to the number of zero modes in the \( k \)-instanton background for gauge group \( G \).

e) The CG representation corresponding to eq. [5] describes the confinement phase of the theory. One immediate objection: it has long been known that instantons can explain most low energy QCD phenomenology [2] with the exception confinement; and we claim that confinement arises in this picture: how can this be consistent? We note that quark confinement cannot be described in the dilute gas approximation, when the instantons and anti-instantons are well separated and maintain their individual properties (sizes, positions, orientations), as at large \( \mu \). However, in strongly coupled theories the instantons and anti-instantons lose their individual properties (instantons will “melt”) their sizes become very large and they overlap. The relevant description is that of instanton-quarks which can be far away from each other, but still strongly correlated.

**Conjecture.** — We thus conjecture that the confinement-deconfinement phase transition takes place at precisely the value where the dilute instanton calculation breaks down: At low \( \mu \) color is confined (because of the instanton-quarks), whereas at large \( \mu \) color is not confined (because of dilute instantons). The value of the critical chemical potential as a function of temperature, \( \mu_c(T) \) is given by saturating the inequality [6].

Few remarks are in order. We can calculate the temperature dependence of \( \mu_c(T) \) only at relatively low \( T \) where our calculations are justified. We also note that the critical chemical potentials given below are not sensitive to the specific assumptions made in the derivation of eq. [5] as the numerical estimates below are based on approximating critical values from large \( \mu \). We expect that our numerical results for \( \mu_c(T) \) are not very sensitive to many unavoidable uncertainties due to the large power of \( (\Lambda_{QCD}/\mu)^b \) entering the instanton density.

**Results.** — The critical chemical potential as a function of temperature is implicitly given by \( a(\mu_c(T), T) = 8f^2(\mu_c(T))/\mu_c(T)^2 \). We can calculate \( a(\mu_c(T), T) \) from [10]. We are however limited to temperatures where Cooper pairing takes place, i.e. for \( T \lesssim 0.567 \Delta [10] \). We have determined the critical chemical potential in different cases at nonzero baryon or isospin chemical potential. We find that the values of the critical chemical potentials at \( T = 0 \) are given by (we use \( m_s \approx 150 \text{ MeV} \) which is numerically close to 0.75\( \Delta \) for \( N_f = 3 \)):

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1 If we used 1-loop \( \beta \) function, we would get larger coefficients in the table. It is simple reflection of the fact that we are very close to the critical values where the interaction is essential. While numerical predictions are not robust, the general picture of the transition is not sensitive to the details and remains the same.
As an example, we explicitly show the results as a function of temperature for $N_c=3$ at nonzero $\mu_1$ in FIG. 1, where direct lattice calculation are possible. We notice that with our conventions the transition from the normal phase to pion condensation happens at $\mu_1 = m_\pi/2$.

**FIG. 1:** Critical isospin chemical potential for the confinement-deconfinement phase transition as a function of temperature (solid curve). The dashed curve represents the largest temperatures that can be reached in our approach, given by $0.567\Delta$ (see text for more details).

**Conclusion.** — In this article we have conjectured that there is a confinement-deconfinement phase transition at nonzero chemical potential and small temperature that is driven by instantons (which transformed from well-localized objects to strongly overlapped configurations). Furthermore we make a quantitative prediction for the critical value of the chemical potential where this transition takes place: $\mu_c \sim 3\Lambda_{\text{QCD}}$ at $T = 0$. This prediction can be readily tested on the lattice at nonzero isospin chemical potential. Our conjecture corresponds to the statement that the confinement-deconfinement transition and the topological charge density distribution (instanton-quark to instanton transition) must experience sharp changes exactly at the same critical value $\mu_c(T)$. There are well-established lattice methods which allow to measure the topological charge density distribution, see e.g. [5–17]. Independently, there are well-established lattice methods which allow to introduce isospin chemical potential into the system, see e.g. [8]. We claim that the topological charge density distribution measured as a function of $\mu_1$ will experience sharp changes at the same critical value $\mu_1 = \mu_c(T)$ where the phase transition (or rapid crossover) occurs. If our conjecture is correct, the phase diagram of QCD at nonzero temperature and isospin chemical potential should be given by FIG. 2.

Finally, at the intuitive level there seems to be a close relation between our conjecture about instanton quarks and the “periodic instanton” analysis [17, 18]. Indeed, in these papers it has been shown that the large size instantons and monopoles are intimately connected and instantons have the internal structure resembling the instanton-quarks. Unfortunately, one should not expect to be able to account for large instantons using semiclassical technique to bring this intuitive correspondence onto the quantitative level. However, this analogy may help us to understand the relation between picture advocated by ’t Hooft and Mandelstam [21] and confinement due to the instanton-quarks, as conjectured in the present paper. The key point of the ’t Hooft - Mandelstam approach is the assumption that dynamical monopoles exist and Bose condense. If our conjecture is correct, then one can argue (on the basis of semiclassical analysis [12], see also [10]) that the instanton-quarks carry the magnetic charges and are responsible for confinement. In this case both pictures could be the two sides of the same coin.

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**FIG. 2:** Phase diagram of QCD at nonzero temperature and isospin chemical potential. First and second order phase transitions are depicted by solid and dashed curves, respectively. Chiral symmetry is broken everywhere except in the QGP phase. The confined phases are shaded. Confinement of pure glue is expected at very low $T$ and very large $\mu_1$ [8].

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