Mathematical modeling the wildfire propagation in a Randers space

Hengameh R. Dehkordi
Center of Mathematics, Computation and Cognition,
Federal University of ABC,
Santo André, Brazil

MSC 2010: 53B40; 53B50; 83C57; 83C80
Keywords: Finsler geometry; Randers metric; Huygens’ Principle; wavefronts; wave rays; causal structure; analogue gravity.

Abstract

The devastating effects of wildfires on the wildlife and their impact on human lives and properties are undeniable. This shows the importance of studying the spread of the wildfire, predicting its behavior and presenting more reliable models for its propagation. Here, by using the validity of the Huygens’ envelope principle for wavefronts in Finsler spaces, we present some models for the propagation of the wildfire in an $n$-dimensional smooth manifold under the presence of wind. In the models, trajectories of fire particles are tracked and the equations that give the wavefront at each time are provided. Furthermore, we determine the paths and points of great importance in the process of wildfire management, called strategic paths and points. Finally, we consider two examples of spreading the wildfire in some agricultural land or woodland, for the sake of illustration.

1 Introduction

Every year, wildfires cause significant damage to the wildlife, jungles, grasslands, agricultural lands, and natural resources; and threaten infrastructures, properties, and human lives [12, 28]. Sometimes, the renovation of destroyed regions and the recovery of damaged faunas are impossible. Indeed, wildfires have strongly negative ecological effects and every year engulf millions hectares of rainforests [13]. Global warming and carbon dioxide released into the atmosphere due to the wildfires are other issues that can not be ignored [34].

Providing a more accurate and reliable model for spreading the wildfire in time plays an important role in the wildfire management strategies and without any doubt reduces both the financial and life losses due to wildfire. Finsler geometry,
which is a classical branch of differential geometry started by P. Finsler in 1918, is a strong tool to model some real phenomena in anisotropic or inhomogeneous media \[15, 6, 9, 16, 17, 35, 25\]. On the propagation of waves and tracking the wave rays, several authors have already applied the Finsler metric \[14, 18, 24\]. As a particular case, the propagation of the wildfire waves in dimension 2 was studied in \[24\], in which the author showed that, for a wildfire spreading, the wavefronts and wave rays are respectively the geometric spheres and geodesics of the corresponding Finsler metric. Very recently, the validity of Huygens’ principle was verified for wavefronts in Finsler spaces of any dimension \(n\) \[11\].

The Huygens’ principle is frequently used for modeling the growth of the wildfire in dimension 2. A literature review shows that several authors used some elliptical template fields, such as double ellipse, lemniskata, oval shape, and tear shape, as spherical wavefronts in the process of applying the Huygens’ principle \[1, 8, 19, 24, 29\]. However, the spherical wavefronts used in this process have some deviations from the spherical wavefront in reality \[24\]. Consequently, sometimes, the model presented based on these spherical wavefronts is neither accurate enough nor reliable. The reason why these template fields are not so suitable is that when one of the above mentioned templates is used, in fact, it is assumed that the space is of zero curvature which is far from the reality. The more the curvature differs from zero, the less the model, based on these template fields, approximates the propagation \[21\]. By the way, the Finsler geometry, not only, provides the model for the propagation in any dimension \(n\), but also, takes the curvature of the space into account. In particular, if there exist analytical solutions for the system of Finsler geodesic equations (wave rays), the model almost coincides with the reality. Even in the case that one has to apply some numerical methods to solve the system of Finsler geodesic equations (wave rays), the model is still reliable enough. In fact, since the Finsler metric takes the curvature of the space into account, it makes Finsler geometric spheres have negligible deviation from the spherical wavefronts created by the wildfire. By the way, the Finsler geometric spheres are those used in the process of the Huygens’ principle which provide us the wavefronts and finally the model of the propagation.

The use of simulators, such as Phoenix, IGNITE, Bushfire, FireMaster, FARSITE and Prometheus is another technique widely used by the researchers to predict the behavior of the fire \[22, 27, 33\]. To the best of our knowledge, FARSITE and Prometheus are considered to be the best ones among these simulators \[32\]. The challenge we confront dealing with simulators is that they produce errors during the process. Therefore, regarding to simulators, new problems appear which are reducing the errors and time of computations \[2, 20\].

In \[11\], it was shown that for a given propagation of waves, if one finds a Finsler metric \(F\) in such a way that one of the preimages of the Finsler distance function coincides with the wavefront at some time \(t\), then the Finsler distance function provides a model for that propagation. In fact, the Finsler geodesics and the distance function are applied to modeling the propagation of the waves. In our work, by using the results of \[11\], we suggest two different strategies to predict the progress of fire waves in the case of Randers metrics. In other words, in order to provide the model of a given propagation, one may use the Finsler geometric spheres, and then apply...
the Huygens’ principle, or solve the system of geodesic equations, which are indeed paths of fire particles. By the way, by the spread of wildfire in a Randers space, it means a fire spreading across a smooth manifold $M$ under the presence of some wind $W$. The wind here is a smooth vector field $W$ such that $|W| < 1$.

Through three main results, that is Theorems \[3.2\] and \[3.7\] we provide the equations of the wave rays and wavefronts at each time $\tau$. Moreover, the equations of strategic paths and points are presented. Let us explain that by strategic paths we mean the paths along which the fire engulfs more regions or it reaches to some special zone that has some priority (protecting fauna, houses, etc.) and should be protected from the fire. In other words, firefighters and equipment should be located along the strategic paths in order to control the fire or prevent it from progressing toward some special direction or area. In fact, having strategic paths increases the chance of success in wildfire management. After finding the model of the propagation and strategic paths, by using some information on the behavior of a wildfire, one determines some points where the firefighters and equipment should be located to attack the fire. Such points, that are along strategic paths, are called strategic points. It should be pointed out that, firstly, the time is so important when it comes to responding to an emergency incident and, secondly, our sources, forces, and equipment are limited against wildfires. Therefore, finding the strategic points, which leads to save the time and expense, is vital to wildfire management, especially in wildfires of the big scale and magnitude that are happening every year around the world, such as the United States, Australia, and Brazil. Hence, it is fair to claim that the study of the strategic points and paths is of great importance in the firefighting process and really demands more attention. Whereas, to the best of our knowledge, there is no study related to such paths or points.

\subsection*{1.1 Hypotheses, methodology, remarks and Outline of the paper}

Throughout this work, it is assumed that a wildfire is sweeping across some space $M$ which is a smooth manifold of dimension $n$ and some fuel has been distributed smoothly through $M$. A mild wind $W$, that is a smooth vector field, is blowing in $M$. Although the wind might or not be space-dependent, it must be time-independent at intervals of time. Also, it is assumed that the fire is stopped before it creates singularities or cut loci. To be closer to the reality, the focus of our work is on the dimension 3 (for instance $M$ can be any open subset of $\mathbb{R}^3$), however our results are valid for any dimension $n$. It is assumed that $M$ contains some flat field of fuel $D$, that is $D$ is a 2-dimensional subspace of $M$ such that its slope is negligible. In fact, if one walks around in $D$, he would feel some going up and down but negligible. For instance, $D$ might be the bed of an agricultural land, a forest, and so forth.

On the methodology, we show that the model of the propagation is merely determined by considering some translated ellipsoid template frames. This way, it is enough to determine the equation of such ellipsoids from the experimental or laboratory data. Afterwards, equations of wavefronts and wave rays associated with the propagation are determined, and finally the model is presented. Following this strategy, we investigate the spread of the wildfire across some space under the presence
of some wind which might be a constant, Killing or smooth vector field.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2. The new results and certain discussions on the spread of fire waves, in a Randers space, are presented in Section 3. Section 4 provides some examples in which certain wildfire is spreading in some agricultural land, woodland or forest under the influence of different winds. In Section 5, concluding remarks and some idea for future works are provided.

2 preliminaries

Here, for the sake of self-contained paper, we first provide a brief review of the Finsler geometry, and then recall the Huygens’ envelope principle.

2.1 Geometric Preliminaries

Some definitions and results from the Finsler geometry that are used to establish our main results are stated here. For the details and further information see [4, 31].

Given a real finite-dimensional vector space $\mathcal{V}$, a non-negative function $F : \mathcal{V} \rightarrow [0, \infty)$ is called a Minkowski norm if it satisfies the following properties:

(i) $F$ is smooth on $\mathcal{V}\setminus\{0\}$,

(ii) $F$ is positive homogeneous of degree 1, that is $F(\lambda V) = \lambda F(V)$ for every $\lambda > 0$,

(iii) for each $V_1 \in \mathcal{V}\setminus\{0\}$, the fundamental tensor $g_{V_1}$ which is the symmetric bilinear form defined as

$$g_{V_1}(V_2, V_3) := \frac{1}{2} \left( \frac{\partial^2}{\partial t \partial s} F^2(V_1 + tV_2 + sV_3) \right)_{s=t=0}$$

is positive definite on $\mathcal{V}$.

Let $M$ be a smooth manifold, $p = (x_1, ..., x_n) \in M$ a point and $T_pM$ the space tangent to $M$ at $p$. Assume that $V = (v_1, ..., v_n) \in T_pM$ is a vector according to the canonical basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ for $T_pM$ and $TM$ the tangent bundle, i.e. the collection of all vectors tangent to $M$ or in other words

$$TM = \bigcup_{p \in M} \{(p, V) : V \in T_pM\}.$$ 

A function $F : TM \rightarrow [0, \infty)$ is called a Finsler metric if it enjoys the following properties:

(i) $F$ is smooth on $TM\setminus\{0\}$,

(ii) for each $p \in M$, $F_p := F|T_pM$ is a Minkowski norm on $T_pM$. 

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In fact, a Minkowski metric is a metric that depends on the direction and a Finsler metric is a metric that depends on the foot point and also the direction of the vector. A Riemannian metric on $M$ is a smooth function $h : T_pM \times T_pM \to \mathbb{R}$. The smoothness condition on $h$ refers to the fact that the function $p \in M \to h_p(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \in \mathbb{R}$ must be smooth. Now, consider a Riemannian metric $r$ on $M$, $r : TM \times TM \to [0, \infty)$, and a 1-form $\beta : TM \to [0, \infty)$ such that $r(\beta^*, \beta^*) < 1$, where $\beta^*$ stands for the dual vector of $\beta$. Considering $\alpha(\cdot) = \sqrt{r(\cdot, \cdot)}$, then $F = \alpha + \beta$ is a special case of the Finsler metric which is called Randers metric. It is well known that given any Randers metric $F$ on $M$, there is a Riemannian metric $h$ on $M$ and some vector field $W$ with $|W| < 1$ related to this Randers metric $F$. Due to Zermelo, the problem which associates the Randers metric $F$ with the Riemannian metric $h$ is called the Zermelo’s navigation problem. For a literature review on the Zermelo’s problem of navigation see [26]. We recall the following result from [5], that together with the given explanation below clarify the relation between the Randers metric $F$ and Riemannian metric $h$ associated with it.

**Proposition 2.1.** [5] A strongly convex Finsler metric $F$ is of Randers type if and only if it solves the Zermelo navigation problem on a Riemannian manifold $(M, h)$, under the influence of some wind $W$ which satisfies $h(W, W) < 1$.

In other words, given a smooth manifold $M$ and a Finsler metric $F$ on it, $F$ is some Randers metric on $M$ if and only if there exist some Riemannian metric $h$ and some smooth vector field $W$ satisfying $h(W, W) < 1$ on $M$ associated with $F$. We call the pair $(h; W)$ as the zermelo’s data associated with Randers metric $F$. Given $M$, $h$ and the smooth vector field $W$ such that $h(W, W) < 1$, then the Randers metric $F$ is as

$$F(V) = \alpha(V) + \beta(V) = \frac{\sqrt{h^2(W, V) + \lambda h(V, V)}}{\lambda} - \frac{h(W, V)}{\lambda}, \quad (2.1)$$

where $\lambda = 1 - h(W, W)$. As some interesting and useful relation, according to Proposition 2.1 and Eq. (2.1), one says

$$F(V) = 1 \text{ if and only if } h(V - W, V - W) = 1 \quad (2.2)$$

(see the details in section 1.1.2 of [5]). Later, we use this relation in the proof of our results.

Given a Minkowski space $(\mathcal{V}, F)$, the *indicatrix* $\mathcal{I}_F$ is the set

$$\mathcal{I}_F = \{ V \in \mathcal{V} : F(V) = 1 \}$$

and by the *indicatrix of radius* $\tau$ we mean the set

$$\mathcal{I}_F^\tau = \{ V \in \mathcal{V} : F(V) = \tau \}.$$

Indeed, the indicatrix $\mathcal{I}_F$ ($\mathcal{I}_F^\tau$) is a hypersurface in $\mathcal{V}$ consisting of the collection of all end points of the vectors of length 1 ($\tau$) with the initial point of each vector.
at origin. Similarly, given some Riemannian metric \( h \) on \( \mathcal{V} \), by the Riemannian indicatrix, \( \mathcal{I}_h \), and the Riemannian indicatrix of radius \( \tau \), \( \mathcal{I}_h^\tau \), we mean the sets
\[
\mathcal{I}_h = \{ V \in \mathcal{V} : |V| = 1 \} \quad \text{and} \quad \mathcal{I}_h^\tau = \{ V \in \mathcal{V} : |V| = \tau \},
\]
where \(|.| = \sqrt{h(\cdot, \cdot)}\). Later, we will see in this work that given the equation of a Finsler indicatrix, one finds the Finsler metric associated with this indicatrix.

Given some Finsler space \((M, F)\) and a piece-wise smooth curve \( \gamma : [0, 1] \rightarrow M \), the length of \( \gamma \) is defined as \( L[\gamma] := \int_0^1 F(\gamma'(t))dt \). Similar to the Riemannian space, the distance from a point \( p \in M \) to another point \( q \in M \) in the Finsler space \((M, F)\) is
\[
d(p, q) := \inf_{\gamma} \int_0^1 F(\gamma'(t))dt, \tag{2.3}
\]
where the infimum is taken over all piece-wise smooth curves \( \gamma : [0, 1] \rightarrow M \) joining \( p \) to \( q \). A smooth curve in a Finsler manifold is called a geodesic (shortly \( F \)-geodesic) if it is locally the shortest time path connecting any two nearby points on this curve.

Given a compact subset \( A \subseteq M \), we define the Finsler distance function \( \rho : M \rightarrow \mathbb{R} \) with \( \rho(p) = d(A, p) \). It can be proved that \( \rho \) is locally Lipschitz continuous \([31]\) and therefore it is differentiable almost everywhere.

Given some smooth vector field \( W \) on \( M \), the flow of \( W \) is the smooth map \( \varphi : (-\epsilon, \epsilon) \times M \rightarrow M \) such that for all \( p \in M \), \( \varphi^p(t) := \varphi(t, p) \) is an integral curve of \( W \), that is \( \frac{d\varphi^p}{dt}(t)|_{t=0} = W(p) \) \([23]\). The flow has some interesting properties as follows:

- \( \varphi(t, \varphi(s, p)) = \varphi(t + s, p) \)
- \( \varphi(0, p) = p \), i.e., \( \varphi(0, \cdot) \) is the identity of \( M \).

Let \( \varphi_t := \varphi(t, \cdot) \). Then, \( \varphi_t : M \rightarrow M \) is a diffeomorphism. A vector field \( W \) on a Riemannian space is called Killing if and only if its flow is an isometry of \((M, h)\).

One can also say that \( W \) is Killing if and only if \( L_W h = 0 \), where \( L \) is the Lie derivative.

**Lemma 2.2.** \([30]\) Assume \((M, h)\) is a Riemannian manifold. Given a unitary Riemannian geodesic (\( h \)-geodesic) \( \gamma_h : (-\epsilon, \epsilon) \rightarrow M \) and a Killing vector field \( W \), the unitary \( F \)-geodesics are \( \varphi(t, \gamma_h(t)) \), where \( \varphi : (-\epsilon, \epsilon) \times U \rightarrow M \) is the flow of \( W \) through \( \gamma_h(t) \).

Given some Finsler space \((M, F)\) and a submanifold \( A \subseteq M \), we say that a vector \( V \) is orthogonal to \( A \) with respect to \( F \), and write \( V \perp_\mathcal{F} A \), if for every vector \( U \) tangent to \( A \) one has \( g_F(V, U) = 0 \). Similarly, given the Riemannian manifold \((M, h)\), a vector \( V \) is orthogonal to \( A \) with respect to \( h \), i.e. \( V \perp_\mathcal{F} A \), if for every vector \( U \) tangent to \( A \) one has \( h(V, U) = 0 \). The following corollary states the relation between \( F \)-orthogonality and \( h \)-orthogonality.

**Corollary 2.3.** \([10]\) Given a Randers manifold \((M, F)\), assume that \((h; W)\) is the Zermelo’s data associated with it. Then, for any two non-zero vectors \( U \) and \( V \) tangent to \( M \), \( U \perp_\mathcal{F} V \) if and only if \( h(U, \frac{V}{F(V)} - W) = 0 \).
2.2 Huygens’ principle

Let \( P \) be a source which emits waves. Given any time \( t > 0 \), we consider the collection of all the points to which the wave reaches at time \( t \). This collection is called the \textit{wavefront} at time \( t \). In the case that \( P \) is a single point, the wavefront is called the \textit{spherical wavefront}. Given a wavefront, assume that each point on this wavefront acts as a point source that emits spherical wavefronts. At any time later, a surface tangent to each spherical wavefront (at a single point) is called the \textit{envelope}. For a given wavefront \( B \), by the \textit{wave ray} it means the shortest time path connecting any point of \( B \) to the wavefront at any time later. We recall the Huygens’ Theorem as follows:

\textbf{Theorem 2.4.} Let \( \phi_p(t) \) be the wavefront of the point \( p \) after time \( t \). For every point \( q \) of this wavefront, consider the wavefront after time \( s \), i.e. \( \phi_q(s) \). Then, the wavefront of point \( p \) after time \( s + t \), \( \phi_p(s + t) \), will be the envelope of wavefronts \( \phi_q(s) \), for \( q \in \phi_p(t) \).

We recall the following result from [11] which is used throughout this work several times. It is assumed that, in a Finsler space \((M, F)\), a wildfire is spreading and sweeping some area \( U \subset M \) in the interval of time from \( t = s > 0 \) to \( t = r \). It is also assumed that \( U \) is a smooth manifold and \( d \) is the Finsler distance function.

\textbf{Theorem 2.5.} Let \( \rho : M \to \mathbb{R} \) with \( \rho(p) = d(A, p) \) where \( A \) is a compact subset of \( M \) and \( \rho(U) = [s, r] \), where \( 0 < s < r \). Suppose that \( \rho^{-1}(s) \) is the wavefront at time \( 0 \) and there is no cut loci in \( \rho^{-1}(s, r) \). Then, for each \( t \in [s, r] \), \( \rho^{-1}(t) \) is the wavefront at time \( t - s \) and the Huygens’ principle is satisfied by the wavefronts

\[ \{ \rho^{-1}(t) \}_{t \in [s, r]} . \]

Furthermore, the track of each fire particle is a geodesic of \( F \) and also it is orthogonal to each wavefront \( \rho^{-1}(t) \) at time \( t - s \).

This theorem says that once one finds the Finsler metric associated with a wildfire spreading in a smooth manifold \( M \), by using the distance function \( \rho \), the model for the propagation is provided. As a result from Theorem 2.5, one has the following corollary.

\textbf{Corollary 2.6.} Let \( \rho : M \to \mathbb{R} \) with \( \rho(p) = d(A, p) \) where \( A \) is a compact subset of \( M \) and \( \rho(U) = [0, r] \). Suppose that there is no cut loci in \( \rho^{-1}(0, r) \). Then, for each \( t \in [0, r] \), \( \rho^{-1}(t) \) is the wavefront at time \( t \) and the Huygens’ principle is satisfied by the wavefronts

\[ \{ \rho^{-1}(t) \}_{t \in [0, r]} . \]

Furthermore, the track of each fire particle is a geodesic of \( F \) and also it is orthogonal to each wavefront \( \rho^{-1}(t) \) at time \( t \).

\textbf{Proof.} It is enough to take the limit of the function \( \rho \) to extend the results of Theorem 2.5 to the desired case. \( \square \)
3 Modeling the propagation by using the Finsler geometry

We first recall some of the hypotheses stated in Introduction section. It is assumed that a wildfire is spreading throughout a 3-dimensional smooth manifold $M$. The space $M$ contains some field of fuel $D$, such as some field of grass, woods, etc., such that the fuel has been distributed smoothly through $M$. Some mild wind $W$ is blowing across $M$. By a mild wind it means given any point $p \in M$ the origin of the tangent space at $p$ remains inside the indicatrix, i.e. $0_p \in I_p$. We consider three different states for the wind: some constant vector, Killing vector field, and smooth vector filed.

Here, it is shown that for modeling a propagation one first needs to find the equation of some ellipsoid - as several authors have used elliptic fields in the case of dimension 2 [1]. From this ellipsoid, that is actually our indicatrix, we calculate the Riemannian metric, then the wave rays, and finally we provide the model.

Before presenting the main results, we state and prove Lemma 3.1 which says that the Finsler indicatrix of radius $\tau$, i.e. $I^\tau_F$, is the translation of the Riemannian indicatrix of radius $\tau$, i.e. $I^\tau_h$, by the vector $\tau W$.

**Lemma 3.1.** Given a smooth manifold $M$ and a Randers metric $F$ on it, let $(h;W)$ be the Zermelo’s data associated with it. Assume that $I^\tau_F$ and $I^\tau_h$ are the Finsler and Riemannian indicatrices of radius $\tau$, respectively. Then, $I^\tau_F = I^\tau_h + \tau W$.

**Proof.** One has $V \in I^\tau_h$ if and only if $|V| = \tau$ if and only if $|\frac{V}{\tau}| = 1$ if and only if $\frac{V}{\tau} \in I^1_h$. In other words, $I^\tau_h = \tau I^1_h$. Similarly, one shows $I^\tau_F = \tau I^\tau_F$. Furthermore, according to the relation $2.2$, $F(V) = 1$ if and only if $|V - W| = 1$, or in other words $V \in I^\tau_F$ if and only if $V - W \in I^\tau_h$ which leads to $I^\tau_F = I^\tau_h + W$. Finally, from these relations we have

\[ I^\tau_F = \tau I^\tau_F = \tau(I^\tau_h + W) = \tau I^\tau_h + \tau W = I^\tau_h + \tau W, \]

which completes the proof. 

It is worth mentioning that given a Randers indicatrix, one can find the Randers metric associated with it. In fact, given the equation of Randers indicatrix $I^\tau_F$, by using Lemma 3.1 one finds the Riemannian indicatrix $I^\tau_h$, next, the quadratic equation $Q_h$ associated with $I^\tau_h$, and then the Riemannian metric which is $h = \frac{1}{2} \text{Hess} Q_h$. Afterwards by using Eq. (2.1), one finds the Randers metric $F$.

3.1 Constant wind

Here, it is assumed that the fuel has distributed uniformly and smoothly through $M$ with the same height at all the points belonging to $D$. For instance, $M$ is some cylinder whose base is $D$. Also the temperature and moisture are constant at all points of $M$. The constant wind $W$ is blowing across $M$. Our first objective is finding the wavefronts and wave rays of the propagation, and then the strategic paths and points.
3.1.1 Wavefronts and Wave rays

Theorem 3.2. Assume that a wildfire is spreading in some space \( M \), where \( M \) is some smooth 3-dimensional manifold. Some constant wind \( W = (0, W_2, W_3) \) is blowing across \( M \) and \( A \) is the wavefront at time 0. Then:

(i) Given any point \( p \) in \( M \), the spherical wavefront of some radius \( \tau \) and center of \( p \) coincides with the translation of the rotated ellipsoid, given by the Eq. (3.3), by vector \( \tau W \).

(ii) Given any point \( p \) in \( A \), the wave rays emanating from \( p \) are straight lines given by \( \gamma_p(t) = p + tV \), where \( V \) is any vector such that \( |V - W| = 1 \) and \( V - W \) is orthogonal to \( A \) with respect to \( h \). Here, \( h \) is the Riemannian metric in the Zermelo’s data \((h; W)\) associated with \( F \).

(iii) The wavefront at time \( \tau \) is the set

\[
\{ p + \tau V : p \in A, |V - W| = 1, V - W \perp h A \}.
\]

Proof. (i) : First of all, observe that since we have the same conditions in all points of \( M \), the spherical wavefronts of some radius \( \tau \) centered at different points of \( M \) are the same geometric objects, up to a translation. Consequently, the Finsler metric \( F \) associated with the spherical spheres does not depend on the point, but on the direction. In other words, \( F \) is a Minkowski-Finsler type. As a consequence, the Finsler geodesics are straight lines and the space is of zero curvature. Therefore, the spherical wavefront of radius \( \tau \) centered at point \( p \in T_p M \) coincides with the indicatrix of radius \( \tau \) centered at 0, i.e. \( I_{F,\tau} \). Let us find the equation of \( I_{F,\tau} \). Assume \( h \) is the metric in the Zermelo data \((h; W)\) whose Zermelo’s solution is \( F \). By Lemma 3.1, \( I_h \) is a translation of \( I_F \) by \(-W\). Therefore, let’s find \( I_h \), in order to find \( I_{F,\tau} \). Assume that \( Q_h \) is the quadratic equation of \( I_h \), that is

\[
Q_h(u, v, w) = [u, v, w]^t h [u, v, w] = 1,
\]

where \( h \) is the matrix of the metric \( h \) and \([\cdots]^t\) is the transpose of \([\cdots] \). We show through the following lemma that \( Q_h \) is a rotated ellipsoid. Observe that \( h \) is a symmetric and positive-definite matrix, since it is the matrix of the metric \( h \). Also, the fact that \( Q_h \) at each point has the same equation implies that \( h \) has constant components.

Lemma 3.3. Let \( Q_h(u, v, w) \) be a quadratic equation such that

\[
Q_h(u, v, w) = [u, v, w]^t h [u, v, w] = 1,
\]

where \( h \) is a symmetric and positive-definite matrix with constant components. Then, \( Q_h \) is the equation of a rotated ellipsoid.

Proof. Since \( h \) is a symmetric and positive-definite matrix, by Spectral theorem, there exists some orthogonal matrix \( P \) (i.e. \( P P^T = P^T P = I \)) such that \( h = P^T D P \), where \( D = diag(\lambda_1, \lambda_2, \lambda_3) \) is a diagonal matrix. Therefore, one can write
\[
1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}^T P^T \mathcal{D} P \begin{bmatrix} u \\ v \\ w \end{bmatrix} = (P \begin{bmatrix} u \\ v \\ w \end{bmatrix})^T \mathcal{D} P \begin{bmatrix} u \\ v \\ w \end{bmatrix}\\
= \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}^T \mathcal{D} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \lambda_1 u'^2 + \lambda_2 v'^2 + \lambda_3 w'^2,
\]

where we put \( P \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \) and the last equality is because \( \mathcal{D} \) is a diagonal matrix.

Furthermore, since \( h \) is positive definite, the elements on the principle diagonal of \( \mathcal{D} \) are positive real numbers and we can write

\[
1 = \lambda_1 u'^2 + \lambda_2 v'^2 + \lambda_3 w'^2 = \frac{u'}{a}^2 + \frac{v'}{b}^2 + \frac{w'}{c}^2,
\]

(3.1)

where \( a = \frac{1}{\sqrt{\lambda_1}}, b = \frac{1}{\sqrt{\lambda_2}}, c = \frac{1}{\sqrt{\lambda_3}} \). Eq. (3.1) is the equation of an ellipsoid in the vector space \( T_pM \) with basis \( \mathcal{B} = \{ P \frac{\partial}{\partial x}, P \frac{\partial}{\partial y}, P \frac{\partial}{\partial z} \} \), which is the rotation of the canonical basis. Therefore, it is deduced that \( Q_h \) is some rotated ellipsoid in the system with basis of \( \{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \} \).

From Lemma 3.3, the quadratic equation of spherical wavefronts with respect to the Riemannian metric \( h \), i.e. \( Q_h(u,v,w) \), is some rotated ellipsoid given by Eq. (3.1). Let us figure out what is the angle of the rotation. Observe that, according to the facts that the heat goes toward above, i.e. \( z \)-axis, the fuel has distributed uniformly through \( M \), and \( W \) is in the \( yz \)-plan, it is deduced that \( Q_h \) is rotated around \( x \)-axis. That is the matrix of rotation is

\[
P = R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}.
\]

(3.2)

Here, we used a right-handed coordinate system and a right-handed rotation through an angle \( \alpha \) around \( x \)-axis. To summarize we have:

\[
Q_h(u,v,w) = \left( \frac{u}{a} \right)^2 + \left( \frac{v \cos \alpha - w \sin \alpha}{b} \right)^2 + \left( \frac{v \sin \alpha + w \cos \alpha}{c} \right)^2 = 1,
\]

(3.3)

were \( a, b, c \) and \( \alpha \) are constant numbers and will be determined from the experimental data. Finally, the Riemannian indicatrix is

\[
\mathcal{I}_h = \{(u,v,w) : Q_h(u,v,w) = 1\}
\]

and therefore, by Lemma 3.1 one obtains

\[
\mathcal{I}_F^\tau = \{(u,v,w) \in \mathbb{R}^3 : Q_h\left( \frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau} \right) = 1 \} + \tau W.
\]

(3.4)
Eq. (3.4), that is the equation of the spherical wavefront at time $\tau$ and center of $p$, is the translation of the rotated ellipsoid given by Eq. (3.3) and so the proof of item (i) is complete. As it was mentioned in Introduction section, the constants in the equation of the indicatrix are determined by using the experimental data.

(ii): Given some point $p \in A$, by Theorem 2.5 and since geodesics of Minkowski spaces are straight lines, the wave rays emanating from the point $p$ and velocity vector $V$ are unit speed straight lines $\gamma_F(t) = p + tV$, provided that vectors $V$ are orthogonal to $A$ with respect to $F$. Therefore, by Corollary 2.3 and the fact that $F(V) = 1$ if and only if $|V - W| = 1$, one can only verify that $V - W$ is orthogonal to $A$ with respect to $h$ and $|V - W| = 1$.

(iii): Observe that by Theorem 2.5 the wavefront at time $\tau$ is $p^{-1}(\tau)$, where $p$ is the Finsler distance function. Assume that $\gamma_F$ is a unit speed geodesic (wave ray) that minimizes the distance from $A$ to $q \in p^{-1}(\tau)$. That is it emanates from some point $p \in A$ and reaches to $q$ at time $\tau$, or in other words,

$$\tau = d(A, q) = d(p, q) = d(\gamma_F(0), \gamma_F(\tau)).$$

Note that, by item (ii), $\gamma_F$ is the straight line $\gamma_F(t) = p + tV$ with $|V - W| = 1$ and $V - W$ being orthogonal to $A$ with respect to $h$. It concludes that

$$p^{-1}(\tau) = \{q = \gamma_F(\tau) = p + \tau V : p \in A, |V - W| = 1, V - W \perp_h A\}.$$ 

Observe that there may not be any relations between the angle $\alpha$ and angles that the wind, $W$, makes with the axes. We only can say that the stronger the wind, the closer the angle $\alpha$ to the angle that the wind makes with the $z$-axis. In fact, since the heat goes towards above, the indicatrix is an ellipsoid with the major axis along the $z$-axis, before the influence of the wind. Then, the wind makes it rotate and translate toward the direction of $W$. Consequently, for the stronger wind we have the smaller deviation between the angle of rotation and the direction of $W$.

Corollary 3.4. Assume that a wildfire is spreading throughout a 3-dimensional Randers space $(M, F)$ such that $W = 0$ and $A$ is the wavefront at time 0. Then:

(i) the spherical wavefront at time $\tau$ is the ellipsoid $$(\frac{a}{a})^2 + (\frac{b}{b})^2 + (\frac{c}{c})^2 = 1$$ where $a = b \leq c$.

(ii) Given any point $p$ on $A$, the wave rays emanating from $p$ are straight lines given by $\gamma_F(t) = p + tV$, where $V$ is some vector such that $|V| = 1$ and $V$ is orthogonal to $A$ with respect to $h$.

(iii) The wavefront at time $\tau$ is the set

$$\{p + \tau V : p \in A, |V| = 1, V \perp_h A\}.$$
Proof. One can follow the proof of Theorem 3.2 considering the fact that \( W = 0 \). This point should be noticed that since the fuel has distributed uniformly, the fire should spread in all the horizontal directions with the same velocity. However, since the heat goes toward above, that is \( z \)-axis, the fire grows faster toward the vertical direction. Therefore, the spherical wavefronts are ellipsoids (indeed ellipsoids of revolution) with the major axis along with \( z \)-axis which means \( a = b \leq c \) and \( \alpha = 0 \).

Corollary 3.5. If \( A \) is a point, that is considered as the origin, then wave rays are straight lines given by \( \gamma_F(t) = tv \), where \( |V - W| = 1 \), and the wavefront at time \( \tau \) is the translation of the rotated ellipsoid \( Q_h(u, v, w) = 1 \) by the vector \( \tau W \), where \( Q_h(u, v, w) = 1 \) is given by Eq. (3.3).

Proof. Considering the fact that the wavefront at time zero is the origin, from Theorem 3.2 nothing is left to be proved.

3.1.2 Strategic paths and points in the case of the constant wind

Here, we determine the equations of strategic paths for two different situations. The situations are as follows. First, all the points of \( M \) have the same priority from the fire fighting point of view. Therefore, the strategic path is the path along which the fire engulfs more area. Second, some special area or point has higher priority and the objective is protecting it against the wildfire. Hence, for a given point, the strategic path is the path through which the fire particles reach to this point at some time \( \tau \). Or, for a given area \( B \), the strategic path is the path through which the fire meets \( \partial B \) for the first time at time \( \tau \), where \( \partial B \) is the frontier of \( B \). By the way, depending on the case, we may have more than one strategic path.

Lemma 3.6. Assume that a wildfire is spreading in the space \( M \), where \( M \) is a smooth 3-dimensional manifold. Some constant wind \( W = (0, W_2, W_3) \) is blowing across \( M \) and \( A \) is the wavefront at time 0. Then:

(i) In the case that all the points of \( M \) have the same priority, the strategic path is \( \gamma_p(t) = p + tV \) provided that \( p \in A \) and the Euclidean norm of \( V \), i.e. \( v_1^2 + v_2^2 + v_3^2 \), is the maximum among all the vectors \( V \) satisfying \( |V - W| = 1 \) and \( V - W \perp h A \).

(ii) Given any point \( q \), the strategic path that reaches to \( q \) is \( \gamma_p(t) = q + (t - \tau)V \) where \( |V - W| = 1 \), \( V - W \perp h A \), and \( \tau \) is the time when the wavefront meets \( q \).

(iii) Given any area \( B \), provided that \( \partial B \) is some smooth curve, the strategic path which reaches to it is \( \gamma_p(t) = q + (t - \tau)V \) such that \( |V - W| = 1 \), \( V - W \perp h A \), and \( \tau \) is the time when the wavefront meets \( \partial B \) for the first time (at point \( q \)).

Proof. (i): First, since the strategic path is some wave ray of the fire that emanates from \( A \), by the item (ii) of Theorem 3.2 its equation must be \( \gamma_p(t) = p + tV \) such
that $p \in A, |V - W| = 1$, and $V - W \perp A$. Next, the fact that the strategic path is the wave ray through which the fire engulfs more region implies that the Euclidean velocity of such a wave ray is the maximum. In other words, $v_1^2 + v_2^2 + v_3^2$ is the maximum among all such vectors $V$.

(ii): For a given point $q$, assuming that $\tau$ is the time when the wavefront meets $q$ for the first time, there exists a unique wave ray that emanates from some point belonging to $A$ and reaches to $q$ at time $\tau$ (up to a reparameterization). Because the wave rays are integral curves of the gradient of the distance function \[^{[11]}\]. Hence, from item (ii) of Theorem \[^{[3.2]}\] the path of this wave ray must be a unit speed straight line that emanates form $A$, $F$-orthogonally, and reaches to the wavefront at time $\tau$. Consequently, it is not difficult to show that $\gamma_F(t) = q + (t - \tau)V$ where $|V - W| = 1$, and $V - W \perp h A$.

(iii): As the first step one has to find the wavefront that meets $\partial B$ for the first time. As the next step, one finds the intersection of this wavefront, i.e. $\rho^{-1}(\tau)$, and $\partial B$, i.e. $q \in \rho^{-1}(\tau) \cap \partial B$. Once one finds the point $q$, the rest of the proof is similar to that of item (ii).

\[\square\]

3.2 Wind as Some Smooth Vector Field

Here, through Theorem \[^{[3.7]}\] we first study the case that the wind is a special case of a smooth vector field, that is the Killing vector field. The positive aspect of this kind of vector field is that there exists some direct relation between the wave rays of propagation and geodesics of the Riemannian metric $h$. Therefore, in order to find the wave rays, one just needs to solve the system of equations of $h$-geodesics. To see the system of equations of geodesics of a Riemannian metric, see Chapter 6 of \[^{[23]}\]. In Theorem \[^{[3.9]}\] we study the propagation for the case that the wind is some smooth vector field on $M$, not necessarily Killing.

Theorem 3.7. Assume that a wildfire is spreading in the space $M$, where $M$ is a 3-dimensional smooth manifold. The wind $W$, which is a Killing vector filed, is blowing across $M$ and $A$ is the wavefront at time 0. Then:

(i) Given any point $p$ in $A$, the wave rays emanating from $p$ are $\gamma_p(t) := \varphi(t, \gamma_h(t))$, where $\varphi$ is the flow of $W$ and $\gamma_h$ is the $h$-geodesic such that $\gamma_h(0) = p$, $|\gamma_h'(t)| = 1$, and $d\varphi_p \gamma_h'(0) \perp h A$.

(ii) The spherical wavefront at time $\tau$ and the center of some point $p \in M$ is the set

$\{\varphi(\tau, \gamma_h(\tau)) : \gamma_h \text{ is the unit speed } h\text{-geodesic that } \gamma_h(0) = p\}$.

(iii) The wavefront at time $\tau$ is the set

$\{\varphi(\tau, \gamma_h(\tau)) : \gamma_h \text{ is an } h\text{-geodesic that } d\varphi_p \gamma_h'(0) \perp h A, |\gamma_h'(t)| = 1\}$.
Proof. Before proceeding with the proofs of items (i), (ii), and (iii), let’s see what the indicatrix and Riemannian metric \( h \) are. Given \( p \in M \), one is faced with Theorem 3.2 on \( T_pM \) with constant wind \( W(p) \) on it. Although, since the direction of \( W \) depends on \( p \), one cannot assume that \( W \) is always in \( yz \)-plane. It implies that at each point \( p \), \( h(p) \) is \( h(p) = (P^TDP)(p) \) with \( P(p) \in SO(3) \). That is \( P(p) = R_z(\gamma(p))R_y(\beta(p))R_x(\alpha(p)) \), where

\[
R_x(\alpha(p)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha(p) & -\sin \alpha(p) \\ 0 & \sin \alpha(p) & \cos \alpha(p) \end{pmatrix}, \quad R_y(\beta(p)) = \begin{pmatrix} \cos \beta(p) & 0 & \sin \beta(p) \\ 0 & 1 & 0 \\ -\sin \beta(p) & 0 & \cos \beta(p) \end{pmatrix},
\]

\[
R_z(\gamma(p)) = \begin{pmatrix} \cos \gamma(p) & -\sin \gamma(p) & 0 \\ \sin \gamma(p) & \cos \gamma(p) & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Therefore, the indicatrix (as a subset of \( T_pM \)) is the translation of the rotated ellipsoid, given by

\[
Q_h(p; (u, v, w)) = \left( \frac{u'}{a(p)} \right)^2 + \left( \frac{v'}{b(p)} \right)^2 + \left( \frac{w'}{c(p)} \right)^2 = 1,
\]

by the vector \( W(p) \), where

\[
u' = u \cos \beta(p) \cos \gamma(p) + v (\sin \alpha(p) \sin \beta(p) \cos \gamma(p) - \cos \alpha(p) \sin \gamma(p))
\]
\[
+ w (\cos \alpha(p) \sin \beta(p) \cos \gamma(p) + \sin \alpha(p) \sin \gamma(p)),
\]

\[
v' = u \cos \beta(p) \sin \gamma(p) + v (\cos \alpha(p) \cos \gamma(p) + \sin \alpha(p) \sin \beta(p) \sin \gamma(p))
\]
\[
+ w (\cos \alpha(p) \sin \beta(p) \sin \gamma(p) - \sin \alpha(p) \cos \gamma(p)),
\]

\[
w' = -u \sin \beta(p) + v \sin \alpha(p) \cos \beta(p) + w \cos \alpha(p) \cos \beta(p).
\]

Observe that \( a, b, c, \alpha, \beta, \) and \( \gamma \) are smooth functions which may not be constant, unlike Theorem 3.2. In other words, \( Q_h(\cdot, \cdot, \cdot) = 1 \) is a quadratic equation which depends on two components, that are the point \( p \) and vector \((u, v, w) \in T_pM\), such that at each point \( p \in M \) it gives some rotated ellipsoid in \( T_pM \). Similar to the proof of Theorem 3.2 one finds the metric \( h(p) \) on \( M \). Indeed, the matrix of \( h \) is a symmetric matrix whose components are smooth functions of \( p \) and \( h(p) \) is a positive-definite symmetric matrix with constant components, at each point \( p \).

(i): First, by Theorem 2.5, the wave rays are unitary \( F \)-geodesics which are \( F \)-orthogonal to \( A \). To find unitary \( F \)-geodesics, by Lemma 2.2 it is enough to find the unitary \( h \)-geodesics. Furthermore, for some \( F \)-geodesic \( \gamma_F(t) \) and its corresponding \( h \)-geodesic \( \gamma_h(t) \) one has \( \gamma_F(0) = \varphi(0, \gamma_h(0)) = \gamma_h(0) \) that means both \( \gamma_F \) and \( \gamma_h \) have the same initial point. Hence, given \( p \in A \), to find the wave rays emanating from \( p \), it suffices to search for the unitary \( h \)-geodesics \( \gamma_h(t) \) emanating from \( p \) provided that \( d\varphi_{p} \gamma_h'(0) \perp_h A \). The last relation satisfies the condition that \( \gamma_F(t) \)
must be $F$-orthogonal to $A$. Because by Corollary \ref{cor:orthogonal}, $\gamma_{F}'(0) \perp_{F} A$ if and only if $\gamma_{p}'(0) - W \perp_{h} A$. Furthermore, from the chain rule we have

\[ \gamma_{F}'(0) = d\varphi(0, \gamma_{h}(0)) + d\varphi(0, \gamma_{h}(0))\gamma_{h}'(0) = W(\gamma_{h}(0)) + d\varphi_{p}\gamma_{h}'(0). \]

\[ (ii) : \] Observe that, by Theorem \ref{thm:wavefront}, the spherical wavefront of radius $\tau$ and center $p$ is the set $\rho^{-1}(\tau)$, where $\rho(.) := d(p,.) : M \to \mathbb{R}$, that is the Finsler distance function. Given $q \in \rho^{-1}(\tau)$, assume that $\gamma_{p}(t)$ is a unitary $F$-geodesic that minimizes the distance from $p$ to $q$, or in other words

\[ d(p, q) = d(\gamma_{p}(0), \gamma_{p}(\tau)) = \tau. \]

Therefore,

\[ \rho^{-1}(\tau) = \{ \gamma_{p}(\tau) : \gamma_{p}(t) \text{ is the } F \text{-geodesic that } d(p, \gamma_{p}(\tau)) = \tau, F(\gamma_{p}'(\tau)) = 1, \gamma_{p}(0) = p \}. \]

Now, by item $(i)$, $\gamma_{p}(t) = \varphi(t, \gamma_{h}(t))$ where $\gamma_{h}(t)$ is a unit speed $h$-geodesic such that $\gamma_{h}(0) = p$, concluding the proof of item $(ii)$.

\[ (iii) : \] The wavefront at time $\tau$, by Theorem \ref{thm:wavefront} is the set $\rho^{-1}(\tau)$, where $\rho(.) = d(A,.) : M \to \mathbb{R}$. In other words,

\[ \rho^{-1}(\tau) = \{ q : d(A, q) = \tau \}. \]

Suppose $\gamma_{p}(t)$ is the equation of some unit speed $F$-geodesic that minimizes the distance from $A$ to $q \in \rho^{-1}(\tau)$. That is $\gamma_{p}(0) \in A$, $q = \gamma_{p}(\tau)$, and $d(A, q) = d(\gamma_{p}(0), \gamma_{p}(\tau)) = \tau$. By item $(i)$, $\gamma_{p}(t) = \varphi(t, \gamma_{h}(t))$ where $\gamma_{h}(t)$ is a unitary $h$-geodesic such that $d\varphi_{p}\gamma_{h}'(0) \perp_{h} A$. Therefore, to sum up, as it was shown in the proof of item $(ii)$, one has

\[ \rho^{-1}(\tau) = \{ \varphi(\tau, \gamma_{h}(\tau)) : |\gamma_{h}'| = 1, d\varphi_{p}\gamma_{h}'(0) \perp_{h} A, \gamma_{h}(\tau) \text{ is an } h \text{-geodesic} \}, \]

which concludes the proof.

\[ \square \]

### 3.2.1 Strategic paths and points in the case of the Killing vector field

Through Lemma \ref{lem:Killing} below, we verify the strategic paths and points in the case of the wind being a Killing vector field.

**Lemma 3.8.** Assume that some wildfire is spreading in the space $M$, where $M$ is a 3-dimensional smooth manifold. The wind $W$, that is a Killing vector field, is blowing throughout $M$ and $A$ is the wavefront at time 0. Then:

(i) In the case that all the points of $M$ have the same priority, given any time $\tau$, the strategic path until time $\tau$ is $\gamma_{F}(t) := \varphi(t, \gamma_{h}(t))$, where $\varphi$ is the flow of $W$ and $\gamma_{h}$ is the $h$-geodesic such that $|\gamma_{h}'(t)| = 1$ and $d\varphi_{p}\gamma_{h}'(0) \perp_{h} A$, provided that $|\gamma_{h}(\tau) - \gamma_{h}(0)|$ is maximum among all such $h$-geodesics.
(ii) Given any point \( q \) belonging to the wavefront at time \( \tau \), the strategic path which reaches to \( q \) is the curve \( \gamma_F(t) := \varphi(t, \gamma_h(t)) \), where \( \varphi \) is the flow of \( W \) and \( \gamma_h \) is the \( h \)-geodesic such that \( |\gamma_h'(t)| = 1 \), \( d\varphi_p \gamma_h'(0) \perp h A \) and \( \varphi(\tau, \gamma_h(\tau)) = q \).

(iii) Given any area \( B \), provided that \( \partial B \) is some smooth curve, the strategic path which reaches to it is \( \gamma_F(t) := \varphi(t, \gamma_h(t)) \) where \( \varphi \) is the flow of \( W \) and \( \gamma_h \) is the unitary \( h \)-geodesic such that \( d\varphi_p \gamma_h'(0) \perp h A \) and \( \varphi(\tau, \gamma_h(\tau)) = q \). Here \( \tau \) is the time when the wavefront meets \( \partial B \) for the first time at \( q \).

Proof. To prove item (i), each strategic path must be an \( F \)-geodesic. Therefore, by item (i) of Theorem 3.7, the strategic path must be \( \varphi(t, \gamma_h(t)) \), where \( \varphi \) is the flow of \( W \) and \( \gamma_h \) is the unitary \( h \)-geodesic such that \( d\varphi_p \gamma_h'(0) \perp h A \). However, as the wave rays might be some curves, one can not claim that there exits a unique strategic path that remains valid for any time \( t \). Indeed, a wave ray might be a strategic path just before some time \( \tau \) and after this time one has to choose another wave ray as the strategic path. Since the strategic path is the path through which the fire engulfs more area, the Euclidean length of the strategic path is the maximum. That is \( |\gamma_h(\tau) - \gamma_h(0)| \) must be maximum among all wave rays joining \( A \) to the wavefront at time \( \tau \), closing the proof of item (i).

To prove item (ii), similar to item (i), the strategic path is \( \varphi(t, \gamma_h(t)) \), where \( \gamma_h \) is the unitary \( h \)-geodesic such that \( d\varphi_p \gamma_h'(0) \perp h A \). However, among all such wave rays, we have to search for the wave ray which passes through \( q \) at time \( \tau \). In other words, the wave ray that satisfies the condition \( \varphi(\tau, \gamma_h(\tau)) = q \) is the one we need.

To prove item (iii), the same proof as that of item (iii) of Lemma 3.6 can be made and therefore we avoid repeating it.

In the next theorem, we present the model for the propagation of the wildfire in some space \( M \) under the presence of the wind \( W \) that is a smooth vector field, not necessarily constant or \( \text{Killing} \).

**Theorem 3.9.** Assume that some wildfire is spreading in a smooth manifold \( M \) of dimension 3 across which the wind \( W \), that is a smooth vector field, is blowing. Suppose \( A \) is the wavefront at time 0. Then:

(i) The wave rays are the unit speed \( F \)-geodesics that are \( F \)-orthogonal to \( A \), where \( F \) is the Finsler metric whose indicatrix at each point \( p \in M \) is the translation of the rotated ellipsoid, given by Eq. (3.5), by the vector \( W(p) \).

(ii) Given some point \( p \in M \), the spherical wavefront of radius \( \tau \) and center of \( p \) is:

\[ \{ \gamma_p(\tau) : \gamma_p(\tau) \text{ is the unit speed } F \text{-geodesic that } \gamma_p(0) = p \} \]

(iii) The wavefront at time \( \tau \) is the set:

\[ \{ \gamma_p(\tau) : \gamma_p(\tau) \text{ is the unit speed } F \text{-geodesic that } \gamma_p'(0) \perp F A \} \].
Proof. To prove item (i), it should be noted that once one has the Finsler metric \( F \) on \( M \), by Theorem 2.5, the wave rays are unitary \( F \)-geodesics that are \( F \)-orthogonal to \( A \). To find the metric \( F \), we apply Eq. (2.1) which demands finding the Riemannian metric \( h \). To have \( h \), one has to find the equation of Riemannian/Finsler indicatrix, as it was explained after Lemma 3.1. By following the same argument as that of Theorem 3.7, it can be shown that at each point \( p \in M \) the Finsler indicatrix is the translation of the rotated ellipsoid, given by Eq. (3.5), by the vector \( W(p) \). It completes the proof of item (i).

The proofs of items (ii) and (iii) are similar to those of items (ii) and (iii) of Theorem 3.7 respectively, in which one does not involve \( \gamma_h(t) \) and \( \varphi(.,.) \) in the arguments.

3.2.2 Strategic paths and points in the case of the wind being a smooth vector field

Here, we verify the strategic paths and points in the case that wind is not necessarily a Killing vector field.

Corollary 3.10. Assume a wildfire is spreading in some space \( M \), where \( M \) is a smooth manifold of dimension 3. The wind \( W \), that is a smooth vector filed, is blowing throughout \( M \) and \( A \) is the wavefront at time 0. Then:

(i) If all the points of \( M \) have the same priorities, given any time \( \tau \), the strategic path till time \( \tau \) is the unitary \( F \)-geodesic \( \gamma_F(t) \) that is \( F \)-orthogonal to \( A \), provided that \( |\gamma_F(\tau) - \gamma_F(0)| \) is maximum among all such \( F \)-geodesics.

(ii) Given any point \( q \) belonging to the wavefront at time \( \tau \), the strategic path which reaches to \( q \) is the unitary \( F \)-geodesic \( \gamma_F(t) \) that is \( F \)-orthogonal to \( A \) and \( \gamma_F(\tau) = q \).

(iii) Given any area \( B \), provided that \( \partial B \) is some smooth curve, the strategic path that intersects \( \partial B \) for the first time at time \( \tau \) is the unitary \( F \)-geodesic \( \gamma_F(t) \) which is \( F \)-orthogonal to \( A \) and \( \gamma_F(\tau) = q \), were \( q \) is the intersection of \( \partial B \) and the wavefront at time \( \tau \).

Proof. One applies Theorem 3.9 and follows a proof similar to that of Lemma 3.8 in which the flow of \( W \) and Riemannian geodesic \( h \) do not get involved.

To summarize the results of this section, for a wildfire spreading in some space, there are two ways to find the model of the propagation. Depending on the conditions and wind, one uses one of the methods below:

(i) Using the Huygens’ principle. It means, given the wavefront at some time \( t_0 \geq 0 \), by finding the geometric Finsler spheres of some radius \( r > 0 \) centered at different points belonging to this wavefront, the wavefront at time \( t_0 + r \) will be the surface which is tangent to all of these spheres. This method is more applicable when we want to use some computer programming and software for modeling the propagation.
(ii) Using geodesics. The unitary $F$-geodesics that start $F$-orthogonally from some wavefront are the wave rays and all of them reach to the same wavefront at the same time. It means given some wavefront $A$, after time $\tau$, the wavefront will be the set
\[ \{ \gamma_F(\tau) : \gamma_F \text{ is } F\text{-geodesic}, \ F(\gamma_F'(0)) = 1, \ \text{and } \gamma_F'(0) \perp F A \}. \]

Observe that one can find the $F$-geodesics without even calculating the metric $F$. In the proofs of Theorems 3.2 and 3.7, the formulas of $F$-geodesics are provided. In Theorem 3.9, by formulas and relations provided in Section 2.3 of [7], one obtains the system of equations of the $F$-geodesics from the system of equations of the $h$-geodesics. Therefore, given a propagation, to provide the model, it suffices to find the metric $h$, and then the system of equations of the geodesics associated with it, and finally the wavefronts. This way, the calculations and computations associated with finding $F$-geodesics would be reduced reasonably. Once one finds the $F$-geodesics, it will be easy to find the spherical wavefronts and wavefronts. By the way, in the case that one does not want to get involved with the orthogonality in the Finslerian sense, it would be recommended to find spherical wavefronts, and then apply the Huygens’ principle or apply Corollary 2.3 above. If the system of equations of the geodesics has no analytic solutions, one can find numerical solutions to approximate the wavefronts and wave rays. In the next section, some examples are given to clarify and illustrate the methods.

4 Examples

Here, we provide two examples in which we assume that a wildfire is spreading across some agricultural land or woodland $M \subset \mathbb{R}^3$, provided that $M$ is a smooth manifold. It is assumed that the fuel has been distributed smoothly across $M$ and some wind $W$ is blowing through $M$. We denote the floor of $M$ with $D$. In Example 4.1, the wind is constant and in Example 4.2, it is a Killing vector field. In both examples, the fire starts from $A \subset M$, which is certain point, smooth curve or cylinder. The objective in these examples is finding the wavefronts, wave rays and strategic paths.

Example 4.1. Assume a wildfire is spreading in some agricultural land that can be seen as a smooth manifold $M$ of dimension 3. Here, $D$ is the floor of $M$ with $D$. In Example 4.1, the wind is constant and in Example 4.2 it is a Killing vector field. In both examples, the fire starts from $A \subset M$, which is certain point, smooth curve or cylinder. The objective in these examples is finding the wavefronts, wave rays and strategic paths.

Case 1. $A$ is some point in $D$.

Case 2. $A$ is the path of the closed curve
\[ C(s) = \left( \frac{1}{4} \cos s(\cos s + 6), \frac{4}{13} \sin s(-\sin s + 3), 0 \right), s \in [0, 2\pi]. \] (4.1)

Case 3. $A$ is the image of the surface
\[ S(s_1, s_2) = \left( \frac{1}{4} \cos s(\cos s + 6), \frac{4}{13} \sin s(-\sin s + 3), s_2 \right), s_1 \in [0, 2\pi], \ s_2 \in [0, 2]. \] (4.2)
Before verifying the cases, let’s see what the Riemannian metric and indicatrix are. Since the wind is a constant vector field, by Theorem 3.2, the indicatrix is the translation of the rotated ellipsoid, given by Eq. (3.3), by the vector $W$. Assume that, from the experimental data, we are given the constants in Eq. (3.3) as follows:

\[ a = \frac{1}{2}, \ b = 1, \ c = 2, \ \alpha = \frac{\pi}{6}. \]

Therefore, the equation of the spherical wavefront is Eq. (4.3) below:

\[ 64u^2 + 13v^2 + 7w^2 - (26/3 + \sqrt{3})v + (2\sqrt{3} - 7/3)w - 6\sqrt{3}vw = \frac{\sqrt{3}}{3} + \frac{635}{36}. \tag{4.3} \]

Hence, the matrix of the Riemannian metric is

\[ h = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{13}{16} & -\frac{3\sqrt{3}}{16} \\ 0 & -\frac{3\sqrt{3}}{16} & \frac{7}{16} \end{pmatrix}. \]

In the sequel, we find the wavefronts, wave rays and strategic paths for each of the cases listed above, separately.

**Case 1**

We consider the point $A$ as origin of the coordinate system. By Theorem 3.2 the wavefront at time $\tau$ is the translation of

\[ 4\left(\frac{u}{\tau}\right)^2 + \frac{13}{16}\left(\frac{v}{\tau}\right)^2 + \frac{7}{16}\left(\frac{w}{\tau}\right)^2 - \frac{3\sqrt{3}}{8}\left(\frac{v}{\tau}\right)\left(\frac{w}{\tau}\right) = 1, \]

by the vector $\tau W$. As it can be seen, in this case, applying the spherical wavefronts is an easier method to find the model for the propagation.

To find the wave rays, by item (ii) of Theorem 3.2 the set of the wave rays is

\[ \{\gamma_F(t) = tV : |V - W| = 1\}, \]

where $V = (v_1, v_2, v_3)$ and the equality $|V - W|^2 = 1$ is equivalent to

\[ 64v_1^2 + 13v_2^2 + 7v_3^2 - (26/3 + \sqrt{3})v_2 + (2\sqrt{3} - 7/3)v_3 - 6\sqrt{3}v_2v_3 = \frac{\sqrt{3}}{3} + \frac{635}{36}. \tag{4.4} \]

In order to track the strategic path which reaches to the wavefront at time $\tau$, it suffices to find the point $q = (u, v, w)$ for which $\sqrt{u^2 + v^2 + w^2}$ is maximum among all the other points belonging to this wavefront. Afterwards, the strategic path will be $\gamma_F(t) = t\frac{q}{\tau}$. The Fig. 1 depicts the wavefront at time 1, in $D$ and $M$, together with the strategic path.

**Case 2**
To find the wavefront at time $\tau$, one may use the spherical wavefront given by Eq. (4.3) centered at different points belonging to $C([0,\pi])$, and then apply the Huygens’ principle.

If one wants to find the wave ray that emanates from some point $p = C(s_p)$, it is as $\gamma_p(t) = p + tV$, where $|V - W| = 1$ and $\langle V - W, C'(s) \rangle = 0$ where

$$C'(s) = (-\frac{1}{4} \sin 2s - \frac{6}{4} \sin s, -\frac{4}{13} \sin 2s + \frac{12}{13} \cos s, 0)$$

and $|V-W|$ is in the direction of the propagation. To sum up, to have $\gamma_p(t) = p + tV$, one has to find $V = (v_1, v_2, v_3)$ by solving the following system of equations at point $p$

$$\begin{cases} 
\langle V - W, C'(s_p) \rangle = 0 \\
|V - W|^2 = 1,
\end{cases} \quad (4.5)$$

where $|V - W|^2 = 1$ is equivalent to Eq. (4.4) and the first equation is equivalent to

$$-v_1(\sin 2s_p + 6 \sin s_p) + \left(\frac{13}{16} (v_2 - \frac{1}{3}) - \frac{3\sqrt{3}}{16} (v_3 - \frac{1}{6})\right) \frac{1}{13} (-\sin 2s_p + 3 \cos s_p) = 0.$$
In the case that all the points of the wavefront have the same priority, to find
the strategic path that reaches to the wavefront at time $\tau$ one has to find the wave
ray $\gamma_p(t) = p + tV$ for which $v_1^2 + v_2^2 + v_3^2$ is maximum. Fig. 2 depicts some of
the wavefronts from time 1 to 10 together with the strategic path (in purple color)
together with the path (in black color) through which the fire is progressing slower.

![Image](image.png)

Figure 2: The wavefront from time 1 to 10 and strategic path (in purple color) for Case 2

**Case 3**

In order to find the wavefront at time $\tau$, one way is finding the spherical wave-
fronts centered at points $p \in S([0,2\pi] \times [0,2])$, and then the hypersurface which is
tangent to all of these spherical wavefronts.

To discover the wave ray that emanates from some point $p = S(s_1,p, s_2,p)$, i.e.
$\gamma_p(t) = p + tV$, one has to detect the vectors $V$ such that at point $p$:

$$
\begin{align*}
\langle V - W, \frac{\partial s_1}{\partial s_2} \rangle &= 0, \\
\langle V - W, \frac{\partial s_2}{\partial s_2} \rangle &= 0, \\
|V - W| &= 1,
\end{align*}
$$

provided that $V - W$ is in the direction of the propagation. The first equation in
the system \[4.6\] is equivalent to
\[-\frac{2}{5}v_1 \sin s_1(2 \cos s_1 + 1) + \frac{1}{128} \cos s_1(2 \sin s_1 + 1)(13(v_2 - \frac{1}{3}) + 3\sqrt{3}(v_3 - \frac{1}{6})) = 0,
\]
and the second equation is equivalent to
\[\frac{3\sqrt{3}}{7}(v_2 - \frac{1}{3}) = v_3 - \frac{1}{6}.
\]
Once one has the wave rays, the set \(\{\gamma_p(\tau) = p + \tau V : p \in S([0, \pi] \times [0, 2])\}\) is the wavefront at time \(\tau\).

If all the points of \(M\) have the same priority, the strategic path that reaches to the wavefront at time \(\tau\) is the wave ray \(\gamma_p(t) = p + tV\) for which \(v_1^2 + v_2^2 + v_3^2\) is maximum.

**Example 4.2.** Assume that a wildfire is spreading in some agricultural land which can be seen as a smooth manifold \(M\) of dimension 3 and some wind \(W\) is blowing across \(M\). Here, \(D\) is the floor of \(M\) and \(W(p) = k(y, 0, 0)\), where \(p = (x, y, z) \in M\) and \(k\) is some small enough constant number so that the origin of the tangent space remains inside the indicatrix. Similar to Example 4.1, we consider three different cases for the wavefront at time 0, i.e. \(A\). In other words, \(A\) is a point or some curve given by Eq. (4.1) or surface given by Eq. (4.2). As the first step, one has to focus on finding the indicatrix. Since the wind is some smooth vector field, by Theorems 3.7 and 3.7, the indicatrix must be the translation of the rotated ellipsoid, given by Eq. (3.5), by the vector \(W\). Keeping in mind Eq. (3.5), assume that from the experimental data we are given \(a = 1, b = \frac{1}{2}, c = 2, \alpha = \gamma = 0, \text{and} \beta = y\). Therefore, the indicatrix is the translation of the following ellipsoid by \(W\):

\[Q_h(u, v, w) = (u \cos y + w \sin y)^2 + 4v^2 + \left(\frac{-u \sin y + w \cos y}{2}\right)^2 = 1. \quad (4.7)
\]

From Eq. (4.7), the Riemannian metric is \(h = P^TDP\), where \(D = \text{diag}(1, \frac{1}{2}, 2)\) and \(P(x, y, z) = R_y(y)\). From the fact that \(\frac{\partial}{\partial y}(P_{ki}P_{kj}) = 0\), it is deduced that \(W\) is a Killing vector field. It is easy to see that the flow of \(W\) is as \(\varphi(t, p) = kt(y, 0, 0) + p\), where \(p = (x, y, z)\). We apply Theorem 3.7 to deal with the propagation and obtain the wavefronts, wave rays, and strategic paths for each case.

**Case 1**

The wave rays emanating from some point \(p\) and the unitary velocity vector \(V\) are
\[\gamma_p(t) = kt(y(t), 0, 0) + (x(t), y(t), z(t)),\]
where \(\gamma_p(t) = (x(t), y(t), z(t))\) is the solution of the following system
\[
\begin{align*}
x'' + y'z' &= 0, \\
y'' &= 0, \\
z'' - x'y' &= 0,
\end{align*}
\]
with the initial conditions $\gamma_h(0) = p = (x, y, z)$, $\gamma'_h(0) = V - W$ and $|V - W| = 1$. After some calculations, if $v_2 \neq 0$, we have

$$
\begin{aligned}
&x(t) = x - \frac{v_1}{v_2} + t \frac{v_3}{v_2} \cos v_2 + t \frac{v_1 - ky}{v_2} \sin v_2,
&y(t) = tv_2 + y,
&z(t) = z - \frac{v_1 - ky}{v_2} - t \frac{v_1 - ky}{v_2} \cos v_2 + t \frac{v_3}{v_2} \sin v_2.
\end{aligned}
$$

(4.9)

and if $v_2 = 0$,

$$
\begin{aligned}
&x(t) = (v_1 - ky)t + x,
&y(t) = y,
&z(t) = v_3t + z,
\end{aligned}
$$

(4.10)

are component of $\gamma_h(t)$, provided that $|V - W| = 1$. One can easily verify that $|V - W|^2 = 1$ is equivalent to

$$
4(v_1 - ky)^2 + \frac{13}{16} v_2^2 - \frac{3\sqrt{3}}{8} v_2 v_3 + \frac{7}{16} v_3^2 = 1.
$$

(4.11)

Once one has the curves $\gamma_h(t) = (x(t), y(t), z(t))$, which are satisfied in the systems 4.9 or 4.10 the wavefront at time $\tau$ is the set

$$
\{k\tau(y(\tau), 0, 0) + (x(\tau), y(\tau), z(\tau))\}.
$$

The strategic path for the situation that all the points belonging to the space have the same priority is the wave ray $\gamma_F(t) = kt(y, 0, 0) + \gamma_h(t)$ provided that $|k\tau(y, 0, 0) + \gamma_h(\tau) - p|$ is maximum among all the curves $\gamma_h(t)$. If one wants to find the wave ray that reaches to some point $q$ belonging to the wavefront at time $\tau$, it is $\gamma_F(t) = kt(y, 0, 0) + \gamma_h(t)$ such that $\gamma_h(\tau) = q - k\tau(y, 0, 0)$.

**Case 2**

For Case 2, the wave ray which emanates from some point $p = C(s_p)$ is $\gamma_F(t) = kt(y(t), 0, 0) + (x(t), y(t), z(t))$ such that $\gamma_h(t) = (x(t), y(t), z(t))$ is a solution of the systems 4.9 or 4.10 and it also satisfies $|\gamma'_h(0)| = 1$, $\gamma_h(0) = p$, and $\gamma'_h(0) \perp C'(s)$ in the direction of the propagation.

The wavefront at time $\tau$ is the set

$$
\{k\tau(y(\tau), 0, 0) + (x(\tau), y(\tau), z(\tau))\}.
$$

The calculations related to the strategic paths, and also the wave rays and wavefront in Case 3 are similar to the previous case and also Case 3 of Example 4.1, and therefore we avoid redoing them.
5 Conclusion and Final Remarks

In this work, we presented a model for waves propagation of the wildfire in some smooth manifold $M$ under the presence of the wind $W$ which was assumed to be a smooth vector field. Three different cases for the wind were considered; the wind as some constant vector, Killing vector field, and smooth vector field. These three cases were separately studied through Theorems 3.2, 3.7, and 3.9. This way, the equations of the wavefronts and wave rays were provided. Moreover, the concepts of strategic paths and points were introduced and their equations were determined for different situations. Finally two examples were given to illustrate the results.

In this work, it was assumed that the wind $W$ is time-independent. However, one can apply our results for the case that the wind is some time-dependent vector field $W(t,p)$, $t \in [0,s]$, assuming that $\{t_1,\ldots,t_n\}$ is a partition of $[0,s]$ such that at each interval $[t_i,t_{i+1}] \subset [0,s]$, the wind is the time-independent vector field $W_i(p)$. Therefore, at each interval $[t_i,t_{i+1}]$, we are faced with a new propagation and have to apply one of the Theorems 3.2, 3.7, or 3.9 to modeling it. Generally, there is no relations between the metrics $F_i$ and $h_i$ related to one interval with those of another one.

By the way, in the case that we also need the model of the propagation of the wildfires in $D$, it suffices to consider the intersection of the wavefronts with $D$. In other words, to give the model of the propagation on the ground, it suffices to find the model of the propagation in $M$, and then intersect it with $D$.

For further works in the future, one may work on the cases that the wildfire creates singularities or cut loci. Besides this, it is interesting to see how the wildfire spreads when the fuel has not been distributed smoothly through $M$. Moreover, studying the propagation of the wildfire on the mountain or some slope will be welcomed as it is closer to the reality. In this regard, one problem can be finding the relations between the propagation of wildfires on two domains under the same conditions but different slopes. By the way, as some more problems, one may study the model associated with a strong wind, that is when the origin of tangent space does not locate inside the indicatrix. Finally, one can consider a spread of the wildfire and compare the models presented by some simulators and the results of this work.

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