TRANSVERSE RIEMANN-LORENTZ METRICS WITH TANGENT RADICAL

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Abstract

Consider a smooth manifold with a smooth metric which changes bilinear type from Riemann to Lorentz on a hypersurface Σ with radical tangent to Σ. Two natural bilinear symmetric forms appear there, and we use it to analyze the geometry of Σ. We show the way in which these forms control the smooth extensibility over Σ of the covariant, sectional and Ricci curvatures of the Levi-Civita connection outside Σ.

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1 Preliminaries

Let $M$ be a $m$-dimensional connected manifold ($m > 2$) endowed with a smooth, symmetric $(0,2)$-tensorfield $g$ which fails to have maximal rank on a (non void) subset $\Sigma \subset M$. Thus, at each point $p \in \Sigma$, there exists a nontrivial subspace (the radical) $\text{Rad}_p \subset T_p M$, which is orthogonal to the whole $T_p M$. We say that $(M, g)$ is a singular space (these spaces were analyzed for the first time in [7]). Moreover we say that $(M, g)$ is a transverse singular space if, for any local coordinate system $(x_1, \ldots, x_m)$, the function $\det(g_{ab})_{a,b=1,\ldots,m}$ has non-zero differential at the points of $\Sigma$ (i.e. where $\det(g_{ab})$ vanishes).

This implies at once: (1) the subset $\Sigma$ is a smooth hypersurface in $M$, called the singular hypersurface; (2) at each point $p \in \Sigma$ the radical $\text{Rad}_p$ is one

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dimensional; and (3) the signature of \( g \) changes by +1 or −1 across \( \Sigma \) (see [5] for details). We say that \( g \) has transverse (respectively tangent) radical if \( \text{Rad}_p \cap T_p \Sigma = \{0\} \) (respectively \( \text{Rad}_p \subset T_p \Sigma \)) for all \( p \in \Sigma \). There are several geometric and physical reasons to study such spaces (see the Introduction to [5] for a detailed account) and there are many articles devoted to the case with transverse radical (see [3], [5], [6] and references therein).

In this article we analyze transverse singular spaces with tangent radical, and we focus our attention on the case where the signature changes from Riemann to Lorentz across \( \Sigma \).

In Section 2 we study the local geometry of the singular hypersurface \( \Sigma \), which is degenerate because the radical is assumed to be tangent. When the degeneration of a hypersurface is due to its immersion in an overall semiriemannian space, the Levi-Civita connection remains well-defined at the points of the hypersurface (although it does not induce a connection on it); in that case, and because the hypersurface has a one-dimensional normal vector bundle which is moreover tangent to it, the geometry of the hypersurface (both intrinsic and extrinsic) can be studied using the Levi-Civita connection, looking for some (screen) distribution on the hypersurface complementary to the normal bundle and, because the selected screen is non-canonical, focusing the attention on those properties of the hypersurface which are screen-independent (see e.g. [1]). But when the surrounding space \((M, g)\) is singular at the points of the hypersurface (and independently of whether the radical is transverse or tangent there), the Levi-Civita connection fails to exist at such points; then, the suitable tool to analyze the geometry of the hypersurface seems to be the canonically defined, torsion-free, metric, dual connection on the whole \((M, g)\) (first pointed out in [2]), which, in the case of one-dimensional radical, induces a (conformally defined) symmetric \((0,2)\)-tensorfield \( II \) on the hypersurface. This is what happens with our degenerate hypersurface \( \Sigma \). We point out that, because the normal vector bundle is now two-dimensional, there exists a (local, determined up to a sign) canonical smooth transverse vectorfield \( N \) along \( \Sigma \) which is normal, unitary and \( II \)-isotropic. This vectorfield \( N \) allows us to construct, following the classical scheme, a second fundamental form \( \mathcal{H} \) on \( \Sigma \), which in turn gives rise to a canonical screen distribution \( S \) and also to a canonical vectorfield \( R \) in the radical distribution. Thus, several nice canonical structures arise in this case of tangent radical. All vectorfields tangent to \( \Sigma \) are uniquely decomposable in \( S \)- and \( R \)-components. With that machinery, we analyze a natural family of torsion-free connections on \( \Sigma \), which we call admissible; in
case of $II$-flatness (i.e. when the tensorfield $II$ vanishes), all such connections are metric and have the same covariant curvatures.

In Section 3 we analyze the behaviour of some well-defined semi-Riemannian objects (covariant derivatives, curvatures, ... on $M - \Sigma$), when we approach the singular hypersurface $\Sigma$. We first point out that, by a theorem in [5], the transverse, $II$-isotropic vectorfield $N$ along $\Sigma$ has a canonical (local) extension $N$ to $M$ which is Levi-Civita geodesic outside $\Sigma$, and we use the flow of this extension to (locally) extend to $M$ every vectorfield defined on $\Sigma$. We apply these constructions to analyze whether or not the above mentioned semi-Riemannian objects have good limits on $\Sigma$ and, when this is the case, whether or not these limits only depend on the restriction to $\Sigma$ of the vectorfields we started with. We show how the second fundamental form $H$ and the symmetric $(0,2)$-tensorfield $II$ control these limit properties. We compare our results with the corresponding ones [6] in the case of transverse radical; the conclusion is that the tangent radical case offers a wider variety of results and gives rise to some unavoidable divergences (which are not present when the radical is transverse). In case of $II$-flatness, we show a Gauss-type equation relating the curvature of the admissible connections on $\Sigma$ with the good limit on $\Sigma$ of the Levi-Civita curvature outside $\Sigma$.

It has been argued [11] that singular hypersurfaces with transverse radical could provide a classical picture of the "birth of time" in general relativity. Let us suppose that the radical happens to be tangent on some (closed, with non void interior) region of the singular hypersurface $\Sigma$, and that outside that region the radical recovers its generic behaviour and becomes transverse. In view of the divergences occurring near the hypersurface over the region with tangent radical, it would be interesting to analyze the matching of both regimes (e.g., across some compact submanifold of $\Sigma$) and to look for traces (left by the tangent radical region) near the hypersurface over the transverse radical region. This is beyond the scope of the present article.

1.1 Notations and conventions

Vectorfields on $M$ are denoted by calligraphic letters $A, B, C, \ldots \in \mathfrak{X}(M)$; we use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ to denote vectorfields on $M$ tangent to $\Sigma$. Vectorfields along $\Sigma$ are denoted by capital letters $A, B, C, \ldots \in \mathfrak{X}_\Sigma$; when they are tangent to $\Sigma$, we write $X, Y, Z, \ldots \in \mathfrak{X}(\Sigma)$. Note that $\mathfrak{X}(\Sigma)$ is a $C^\infty(\Sigma)$-submodule of $\mathfrak{X}_\Sigma$. Given $A \in \mathfrak{X}(M)$, we denote $A^\circ \equiv A \big|_{M - \Sigma} \in \mathfrak{X}(M - \Sigma)$ and $A \equiv A \big|_{\Sigma} \in \mathfrak{X}_\Sigma$. In that case, we say that $A$ is an extension of $A$.  

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Let us consider some function $\tau \in C^\infty(M)$ such that $\tau \mid_{\Sigma} = 0$ and $d\tau \mid_{\Sigma} \neq 0$. We say that (locally, around $\Sigma$) $\tau = 0$ is an equation for $\Sigma$. Given $f \in C^\infty(M)$, it holds: $f \mid_{\Sigma} = 0 \iff f = k f \tau$, for some $k_f \in C^\infty(M)$. Denote $\tau^o \equiv \tau \mid_{M-\Sigma}$. When $f \mid_{\Sigma} = 0$, we shall write $\tau^o - 1 f^o \sim 0$ (the function $\tau^o - 1 f^o \in C^\infty(M-\Sigma)$ "has no divergences when approaching $\Sigma"$ or "has a good limit on $\Sigma"$) and we shall say that $\tau^{-1} f$ is well-defined (as an element of $C^\infty(M)$).

Let $(E_1, \ldots, E_m)$ be a (local, $C^\infty$) $\mathfrak{X}(M)$-basis around some point of $\Sigma$. We say that $(E_1, \ldots, E_m)$ is an adapted frame if $E_m \mid_{\Sigma}$ is in the radical distribution and it holds (orthonormality): $g_{ab} = \delta_{ab} (\pm (1 - \delta_{am}) + \delta_{am} \tau)$, for some $\tau \in C^\infty(M)$ ($a, b = 1, \ldots, m$). Thus $\tau \equiv \langle E_m, E_m \rangle = 0$ is an equation for $\Sigma$. The existence of adapted frames around any point $p \in \Sigma$ can be easily proved, e.g. starting with an orthonormal basis $(e_1, \ldots, e_m)$ of $T_p M$ with $e_m \in \text{Rad}_p$, using a local chart of $M$ around $p$ adapted to $\Sigma$, and applying a slight modification of the Gram-Schmidt orthonormalization procedure. If $g$ has transverse (respectively tangent) radical, all $E'_i$s ($i = 1, \ldots, m - 1$) can be chosen to be tangent (respectively, one of the $E'_i$s must be transverse) to $\Sigma$.

We usually write $\langle A, B \rangle$, instead of $g(A, B)$. We denote by $\nabla$ the Levi-Civita connection outside $\Sigma$, and by $R$ its curvature. Connections on $\Sigma$ are denoted by $D, \tilde{D}, \check{D}, \ldots$, and their curvatures by $R_D, R_{\tilde{D}}, R_{\check{D}}, \ldots$.

### 1.2 The Dual Connection

Given a singular space $(M, g)$, there exists a unique torsion-free metric dual connection on $M$, i.e. a unique map $\Box : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ satisfying (for all $f \in C^\infty(M)$ and $A, B, C \in \mathfrak{X}(M)$):

(i) $\Box_f A B = f \Box A B$

(ii) $\Box_A (f B) = A(f) \langle B, \cdot \rangle + f \Box_A B$

(iii) $\Box_A B(C) - \Box_B A(C) = \langle [A, B], C \rangle$ (torsion free)

(iv) $\Box_A B(C) + \Box_B A(C) = A \langle B, C \rangle$ (metric)

Moreover it holds:

(v) $\Box_A B^\alpha(C^\alpha) = \langle \nabla_A B^\alpha, C^\alpha \rangle$ (is compatible with the Levi-Civita connection $\nabla$ on $M-\Sigma$).
The dual connection can be alternatively defined by

\[ 2 \Box A \mathcal{B}(C) := A \langle B, C \rangle + B \langle C, A \rangle - C \langle A, B \rangle + \langle [A, B], C \rangle - \langle [B, C], A \rangle + \langle [C, A], B \rangle. \] (1)

We realize that, \( \forall A, B, C \in \mathcal{X}(M) \), \( \Box A \mathcal{B}(C) \) is \( C^\infty(M) \)-linear in \( A \) and \( C \), thus \( \Box A \mathcal{B} \in \mathcal{X}_\Sigma^* \) is well-defined (we denote \( A \equiv A |_\Sigma \)). This implies that:

1. The dual connection has a good restriction \( \Box : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma) \), which can also be characterized as the unique torsion-free metric dual connection existing on the singular hypersurface \( \Sigma \).
2. Given any vectorfield \( R \in \mathcal{X}_\Sigma \) in the radical distribution, \( \Box A \mathcal{B}(R) \) depends only on \( A \) and \( B \equiv B |_\Sigma \), thus \( \Box A \mathcal{B}(R) \in \mathcal{X}(\Sigma) \) becomes well-defined and we obtain [2] a \( C^\infty(\Sigma) \)-bilinear map \( II_R : \mathcal{X}_\Sigma \times \mathcal{X}_\Sigma \to \mathcal{X}(\Sigma) \), \( (A, B) \mapsto \Box A \mathcal{B}(R) \), which is moreover symmetric because of (iii) above (see [6] for details). In a similar way, given any vectorfield \( N \in \mathcal{X}_\Sigma \) orthogonal to \( \Sigma \), we obtain a \( C^\infty(\Sigma) \)-bilinear, symmetric map \( \mathcal{H}_N : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma) \), \( (X, Y) \mapsto \Box X Y(N) \). We shall come back to these constructions later on.

2 On the Geometry of the singular hypersurface

From now on, we only consider transverse singular spaces \((M, g)\) with tangent radical.

2.1 Fundamental forms

At each point \( p \in \Sigma \), the \( g \)-orthogonal subspace \( T_p^\perp \Sigma \subset T_p M \) is a 2-plane and it holds: \( T_p^\perp \Sigma \cap T_p \Sigma = \text{Rad}_p \). Let \( (\mathcal{E}_1, \ldots, \mathcal{E}_m) \) be an adapted frame around \( p \), thus \( E_m \) is in the radical distribution (we denote \( E_a \equiv E_a |_\Sigma \in \mathcal{X}_\Sigma \), \( a = 1, \ldots, m \), \( \tau \equiv \langle \mathcal{E}_m, \mathcal{E}_m \rangle = 0 \) is an equation for \( \Sigma \) and \( E_1 \) is transverse to \( \Sigma \). Therefore:

\[
\begin{align*}
2II_{E_m}(E_m(p), E_m(p)) &= E_m(p) \langle \mathcal{E}_m, \mathcal{E}_m \rangle = 0 \\
2II_{E_1}(E_1(p), E_m(p)) &= E_1(p) \langle \mathcal{E}_m, \mathcal{E}_m \rangle \neq 0
\end{align*}
\]
It follows that $II_{E_m}$ turns $T_p^\perp \Sigma$ into a Lorentz plane. One of the two $II_{E_m}$-null directions at $p$ is determined by $E_m(p)$. The other one cannot be a $g$-null direction, thus it determines a unique (up to a sign) $g$-unitary vector in $T_p M$ $g$-orthogonal to $\Sigma$. Moving from $p$ to the neighboring points in $\Sigma$, we obtain:

**Proposition 1** Let $(M, g)$ be a transverse singular space with tangent radical and singular hypersurface $\Sigma$. Then there exists locally a canonical smooth vectorfield $N \in \mathfrak{X}_\Sigma$ (determined up to a sign), which is $g$-orthogonal to $\Sigma$, $II_{\text{Rad}}$-isotropic and $g$-unitary, i.e.

$$\langle N, T\Sigma \rangle = 0 \ , \ \ II_{\text{Rad}}(N, N) = 0 \ \text{and} \ \langle N, N \rangle = \pm 1 .$$

If $g$ changes from Riemann to Lorentz, it holds: $\langle N, N \rangle = 1$. We call $N$ the main normal to $\Sigma$.

Over the singular hypersurface $\Sigma$ we have a first (degenerate) fundamental form, namely the restriction $g \mid _\Sigma$. Using now the main normal $N$ to $\Sigma$, we are ready to construct (as in the classical theory) a second fundamental form $\mathcal{H}$ ($\equiv \mathcal{H}_N$, already mentioned in 1.2) on $\Sigma$. We define, for $X, Y \in \mathfrak{X}(\Sigma)$:

$$\mathcal{H}(X, Y) := -\Box_X N(Y) = \Box_X Y(N) ,$$

where the second equality is because of property (iv) of the dual connection. Moreover, property (iii) leads to the conclusion that $\mathcal{H}$ is a symmetric $(0, 2)$-tensor field over $\Sigma$. Note that, as in the classical hypersurface theory, $\mathcal{H}$ is (locally) determined up to a sign.

It turns out that, at each $p \in \Sigma$, the $II_{E_m}$-null direction determined by $E_m(p)$ cannot be a $\mathcal{H}$-null direction, for it holds:

$$2\mathcal{H}(E_m(p), E_m(p)) = -2II_{E_m}(E_m(p), N(p)) = -N(p)(\tau) \neq 0 ;$$

thus we can select a $\mathcal{H}$-unitary vectorfield $R \in \mathfrak{X}(\Sigma)$ in the radical distribution such that $\mathcal{H}(R, R) = \pm 1$. Choosing $N$ such that $\mathcal{H}(R, R) = -1$, we obtain:

**Proposition 2** Let $(M, g)$ be a transverse singular space with tangent radical and singular hypersurface $\Sigma$. Then there exists locally a canonical smooth
vectorfield $R \in \mathfrak{X}(\Sigma)$ (determined up to a sign), which is in the radical distribution and $\mathcal{H}$-unitary, i.e.

$$R(p) \in \text{Rad}_p, \text{ for all } p \in \Sigma, \text{ and } \mathcal{H}(R, R) = -1.$$  

We call $R$ the main radical vectorfield on $\Sigma$.

The main radical vectorfield $R$ induces a canonical $C^\infty(\Sigma)$-bilinear symmetric map

$$II : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma), \ (A, B) \mapsto \square_A B(R)$$

(2)

(thus $II \equiv II_R$, already mentioned in [1.2]), whose restriction to $\mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma)$ yields another symmetric $(0, 2)$-tensorfield $II$ on $\Sigma$. Note that it holds:

$$\begin{cases} 
II(N, N) = 0 \\
II(N, X) = -\mathcal{H}(X, R), \text{ for all } X \in \mathfrak{X}(\Sigma) \\
II(A, R) = 0 \iff A \in \mathfrak{X}(\Sigma)
\end{cases} \quad (3)$$

2.2 The canonical Screen Distribution

A screen distribution is a distribution on the singular hypersurface $\Sigma$ which yields, at each $p \in \Sigma$, a hyperplane of $T_p\Sigma$ transversal to $\text{Rad}_p$. We now define the canonical screen distribution by choosing (at each $p \in \Sigma$)

$$S_p := \{ v \in T_p\Sigma : \mathcal{H}(v, R(p)) = 0 \}, \quad (4)$$

where $R$ is the main radical vectorfield. We shall denote by $S$ either the set $\{ S_p : p \in \Sigma \}$ or the corresponding vector bundle (the screen bundle) $S \to \Sigma$. We denote by $\Gamma(S)$ the $C^\infty(\Sigma)$-module of sections of $S$, which is a submodule of $\mathfrak{X}(\Sigma)$.

From now on, we only consider the case where the signature of $g$ changes from Riemann to Lorentz upon crossing $\Sigma^1$. This means that $g$ is semi-defined on $\Sigma$ and the screen bundle becomes a riemannian vector bundle.

The proof of the following proposition is straightforward.

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1Instead of that hypothesis, we could also make the (weaker) requirement that the metric $g$ is non-degenerate over any slice of the screen distribution. Then the screen bundle would become a semiriemannian vector bundle. All results in this article would remain essentially valid.
Proposition 3  Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical, singular hypersurface \(\Sigma\) and canonical screen distribution \(S\). Then there exist, over each euclidean fibre \(S_p\) of \(S\), two selfadjoint endomorphisms \(\mathcal{H}_p^S\) and \(II_p^S\) such that (for all \(v, w \in S_p\)):

\[
\begin{aligned}
\langle \mathcal{H}_p^S(v), w \rangle &= \mathcal{H}(v, w) \\
\langle II_p^S(v), w \rangle &= II(v, w)
\end{aligned}
\]

these determine vector bundle endomorphisms \(\mathcal{H}^S\) and \(II^S\) of \(S\), called the Weingarten screen maps. The eigenvalues of these screen maps are the principal curvatures, and the corresponding eigenvectors define the principal directions of \(\mathcal{H}^S\) or \(II^S\), respectively.

The natural definition of \(II\)-flatness (respectively, \(\mathcal{H}\)-flatness) of \(\Sigma\) is that \(II^S\) (respectively, \(\mathcal{H}^S\)) is identically zero; or, in other words:

\[II(V, W) = 0\] (respectively, \(\mathcal{H}(V, W) = 0\), for all \(V, W \in \Gamma(S)\).  \hspace{1cm} (5)

Because of (3), \(II\)-flatness is equivalent to \(II(X, Y) = 0\), for all \(X, Y \in \mathfrak{X}(\Sigma)\). And, because of (4), \(\mathcal{H}\)-flatness is equivalent to \(\mathcal{H}(V, X) = 0\), for all \(V \in \Gamma(S)\) and \(X \in \mathfrak{X}(\Sigma)\).

Remark 4  A few words about the definition of \(II\)-flatness. As we mentioned in [1,3], given any transverse singular space \((M, g)\) with singular hypersurface \(\Sigma\), and for any nowhere zero vectorfield \(R \in \mathfrak{X}(\Sigma)\) in the radical distribution, a well-defined \(C^\infty(\Sigma)\)-bilinear, symmetric map \(II : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma), (A, B) \mapsto \Box_A B(R)\) arises. In this general context, \(\Sigma\) was defined \([2]\) to be \(II\)-flat if it holds:

\[II(A, B) = 0, \text{ for all } B \in \mathfrak{X}_\Sigma \Leftrightarrow A \in \mathfrak{X}(\Sigma).\]

Now it is straightforward to see (because \(II(A, R) = 0 \Leftrightarrow A \in \mathfrak{X}(\Sigma)\)) that this requirement is equivalent to the following two conditions:

\[
\left\{
\begin{array}{l}
(i) II(X, Y) = 0, \text{ for all } X, Y \in \mathfrak{X}(\Sigma) \\
(ii) \text{the radical is transverse}
\end{array}
\right.
\]

Because (i) and (ii) are in fact independent conditions, our definition of \(II\)-flatness (which is just condition (i)) turns out to be the natural one for
general case (and coincides with the definition given in [6] for the case with transverse radical)

A vectorfield $A \in \mathfrak{X}_\Sigma$ can be decomposed in normal-, screen- and radical-components, as follows

$$A = \nu(A)N + A^S + \rho(A)R,$$

where $\nu(A) := \langle A, N \rangle$ and $\rho(A) := -\mathcal{H}(A - \nu(A)N, R)$. Thus $\rho \in \mathfrak{X}^*_\Sigma$ is completely determined by the (equally denoted) 1-form $\rho = -\mathcal{H}(., R) \in \mathfrak{X}^*(\Sigma)$. Because $d\rho(V, W) = -\rho([V, W])$, for all $V, W \in \Gamma(S)$, it follows: $d\rho = 0 \Rightarrow$ the distribution $S$ is integrable.

### 2.3 Admissible Connections

We are going to describe some natural connections on $\Sigma$. All these connections arise without any explicit reference to the Levi-Civita connection outside $\Sigma$. We first analyze a canonical screen connection-operator, which yields a natural metric connection on the riemannian screen vector bundle $S \to \Sigma$. Requiring compatibility with that operator and zero torsion leads to a family of connections which we call admissible. These connections are not necessarily metric. It turns out that, if there exists a torsion-free metric connection on $\Sigma$, then: (1) it is necessarily admissible; (2) all admissible connections are metric; and (3) $\Sigma$ is $II$-flat. We finally prove that $II$-flatness is also the necessary and sufficient condition for all admissible connections induce the same covariant curvature tensor. In Section 3, starting with the Levi-Civita connection outside $\Sigma$ (and provided that $\Sigma$ is $II$-flat), we shall induce on $\Sigma$ (like in the classical theory of semiriemannian hypersurfaces) the so called tangential connection, which turns out to be admissible.

Because $g |_S$ does not degenerate, it is natural to introduce the screen connection-operator as the map $D^S : \mathfrak{X}(\Sigma) \times \mathfrak{X}_\Sigma \to \Gamma(S), (X, A) \mapsto D^S_XA$, defined by

$$(D^S_XA)(p) := D^S_{X(p)}A,$$

for all $p \in \Sigma$, where $D^S_{X(p)}A \in S_p \subset T_p \Sigma$ is the unique vector satisfying

$$\Box_{X(p)}A(v) = \langle D^S_{X(p)}A, v \rangle,$$

for all $v \in S_p$. 


The screen connection-operator $D^S$ has the following properties (for all $f \in C^\infty(\Sigma)$, $X \in \mathfrak{X}(\Sigma)$, $A, B \in \mathfrak{X}_\Sigma$ and $V \in \Gamma(S)$):

(i) $H^S(V) = -D^S_X R$

(ii) $H^S(V) = -D^S_X N$

(iii) $D^S_{[X,Y]} A = f D^S_X A$ and $D^S_{X+Y} A = D^S_X A + D^S_Y A$

(iv) $D^S_X(A + B) = D^S_X A + D^S_X B$

(v) $D^S_X( f A) = X(f)A^S + f D^S_X A$; in particular, $D^S_X(f R) = f D^S_X R$

(vi) $\langle D^S_X V, W \rangle + \langle V, D^S_X W \rangle = X \langle V, W \rangle$ (metric)

(vii) $\langle D^S_X A, V \rangle = \Box_X A(V)$ (compatible with $\Box$).

Properties (iii)-(vi) show that $D^S : \mathfrak{X}(\Sigma) \times \Gamma(S) \to \Gamma(S)$ gives a metric connection on the screen riemannian vector bundle $S \to \Sigma$.

However, property (v) shows that the restriction $D^S : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \Gamma(S)$ does not give a connection on $\Sigma$.

When looking for interesting connections $D$ on $\Sigma$, it is natural to impose two requirements: (1) $D$ should satisfy (for all $X, Y \in \mathfrak{X}(\Sigma)$): $(D_X Y)^S = D^S_X Y$, or equivalently: $\langle D_X Y, V \rangle = \Box_X Y(V)$, for all $V \in \Gamma(S)$; and (2) $D$ should be torsion-free. We shall call such connections **admissible**.

The most obvious connection on $\Sigma$ satisfying the first condition is the following one (we denote it by $\tilde{D}$ and we call it the **main connection**)

$$\tilde{D} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma), \quad (X, Y) \mapsto \tilde{D}_X Y := D^S_X Y + X(\rho(Y))R,$$

(6) whose properties are analyzed in the next

**Proposition 5** Let $(M, g)$ be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface $\Sigma$. Then, the main connection $\tilde{D}$ on $\Sigma$:

(a) has torsion $\widetilde{\text{Tor}} = R \otimes d\rho$.

(b) satisfies (for all $X, Y, Z \in \mathfrak{X}(\Sigma)$): $\langle \tilde{D}_X Y, Z \rangle + \langle Y, \tilde{D}_X Z \rangle = X \langle Y, Z \rangle$ (i.e. is metric) if and only if $\Sigma$ is $II$-flat.

**Proof.** In what follows, $S$ is the canonical screen distribution.

(a) Let be $X, Y \in \mathfrak{X}(\Sigma)$. By Property (vii) of $D^S$, one immediately sees that: $(\widetilde{\text{Tor}}(X,Y))^S = 0$. Therefore, $\widetilde{\text{Tor}}(X,Y) = \rho(\widetilde{\text{Tor}}(X,Y))R = d\rho(X,Y)R$.

(b) We have:
\[
\langle \tilde{D}_X Y, Z \rangle = \langle D_X^S Y, Z^S \rangle = \langle D_X^S Y^S, Z^S \rangle + \rho(Y) \langle D_X^S R, Z^S \rangle = \\
= \text{(by property (vii) of } D^S) \langle D_X^S Y^S, Z^S \rangle - \rho(Y) II(X, Z) ;
\]

thus, by property (vi) of \( D^S \), we obtain:

\[
\langle \tilde{D}_X Y, Z \rangle + \langle Y, \tilde{D}_X Z \rangle = X \langle Y, Z \rangle - II(X, \rho(Y)Z + \rho(Z)Y) .
\]

Now \( (\Leftarrow) \) is trivial. Let us prove \( (\Rightarrow) \): If \( \tilde{D} \) is metric, last formula yields:

\[
\rho(X)II(X, X) = 0, \text{ for all } X \in \mathfrak{X}(\Sigma). \text{ Because } T_p\Sigma - S_p \text{ is dense in } T_p\Sigma \text{ (for all } p \in \Sigma), \text{ it follows that } II(X, X) = 0, \text{ for all } X \in \mathfrak{X}(\Sigma). \text{ Being } II \text{ symmetric, this implies that } \Sigma \text{ is } II\text{-flat} \]

Thus, unless \( d\rho = 0 \), the main connection \( \tilde{D} \) is not admissible. However, it is straightforward to check that the connection

\[
\dot{D} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma), (X, Y) \mapsto \dot{D}_X Y := \tilde{D}_X Y - \frac{1}{2}d\rho(X, Y)R \quad (7)
\]

is always admissible (we call it the main admissible connection).

Admissible connections have the following properties:

**Theorem 6** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \( \Sigma \). Then:

(a) Any admissible connection \( D \) on \( \Sigma \) satisfies (for all \( X, Y \in \mathfrak{X}(\Sigma) \))

\[
D_X Y = \dot{D}_X Y + \sigma(X, Y)R ,
\]

where \( \dot{D} \) is the main admissible connection and \( \sigma \) is some symmetric \((0,2)\)-tensorfield on \( \Sigma \).

(b) If there exists a torsion-free metric connection on \( \Sigma \), then: (1) it is admissible; (2) all admissible connections on \( \Sigma \) are metric; and (3) \( \Sigma \) is \( II\)-flat.

(c) If \( \Sigma \) is \( II\)-flat, all admissible connections have the same covariant curvature.
Proof. In what follows, \(X, Y, Z\) are arbitrary in \(\mathfrak{X}(\Sigma)\) and \(V\) is arbitrary in \(\Gamma(S)\), where \(S\) is the canonical screen distribution.

(a) As is well known, any torsion-free connection \(D\) on \(\Sigma\) must satisfy: \(D_X Y = \dot{D}_X Y + \varphi(X, Y)\), where \(\varphi\) is some symmetric \((1, 2)\)-tensorfield on \(\Sigma\). Being both \(D\) and \(\dot{D}\) admissible, it must hold: \(\langle \varphi(X, Y), V \rangle = 0\), and the result follows.

(b) (1) If \(D\) is a torsion-free metric connection on \(\Sigma\), the induced dual connection \(\square D\), defined by: \(\square D_X Y(Q) := \langle D_X Y, Q \rangle\), becomes torsion-free and metric; thus, by uniqueness (see \(\ref{2}\)), it must hold: \(\square D = \square\), and \(D\) becomes admissible. (2) If \(D\) is an admissible connection on \(\Sigma\), it follows from Part (a) that: \(D_X Y = \dot{D}_X Y + (\sigma - \frac{1}{2}d\rho)(X,Y)R\), for some symmetric \((0,2)\)-tensorfield \(\sigma\) on \(\Sigma\). But this implies: \(D\) is metric if and only if \(\dot{D}\) is metric. (3) follows from (2) and from Part (b) of Proposition \(\ref{5}\).

(c) Let \(D\) be any admissible connection on \(\Sigma\) and let us consider its covariant curvature, defined (for all \(X, Y, Z, T \in \mathfrak{X}(\Sigma)\)) by:

\[
\langle R^D(X,Y)Z, T \rangle := \langle D_X(D_Y Z) - D_Y (D_X Z) - D_{[X,Y]} Z, T \rangle .
\]

Using Part (a), we compute:

\[
\langle D_X(D_Y Z), T \rangle = \left\langle D_X(\dot{D}_Y Z + \sigma(Y, Z)R), T \right\rangle = \left\langle \dot{D}_X(\dot{D}_Y Z) + \sigma(Y, Z)\dot{D}_X R, T \right\rangle = \\
= \left\langle \dot{D}_X(\dot{D}_Y Z), T \right\rangle + \sigma(Y, Z) \langle D^S_X R, T \rangle = \left\langle \dot{D}_X(\dot{D}_Y Z), T \right\rangle - \sigma(Y, Z)II(X, T) ,
\]

where last equality is because: \(\langle D^S_X R, T \rangle = \langle D^S_X R, T^S \rangle = \square_X R(T^S) = \square_X R(T) = -II(X, T)\). Therefore we obtain:

\[
\langle R^D(X,Y)Z, T \rangle = \left\langle R^D(X,Y)Z, T \right\rangle - \det \begin{pmatrix}
\sigma(Y, Z) & II(Y, Z) \\
\sigma(X, T) & II(X, T)
\end{pmatrix},
\]

and the result follows

3 Near the singular hypersurface

We analyze in this Section the behaviour of some well-defined Levi-Civita objects (covariant derivatives, curvatures, ... on \(M - \Sigma\)) when we approach
the singular hypersurface \( \Sigma \). Thus we can replace (if necessary) the whole \( M \) by a (small enough) neighborhood of \( \Sigma \) in \( M \).

Typically, we start with vectorfields \( A, B, \ldots \in \mathfrak{X}(M) \), construct some semiriemannian object, say \( \mathcal{O}(A^o, B^o, \ldots) \), on \( M - \Sigma \), and ask under what circumstances this object: (1) has a good limit on \( \Sigma \), i.e. \( \mathcal{O}(A^o, B^o, \ldots) \sim 0 \) (in that case we say that "\( \mathcal{O}(A, B, \ldots) \) is well-defined"); and (2) the restriction \( \mathcal{O}(A, B, \ldots) |_\Sigma \) only depends on \( A|_\Sigma, B|_\Sigma, \ldots \in \mathfrak{X}_\Sigma \) (in that case we say that "\( \mathcal{O}(A, B, \ldots) \) is well-defined").

When dealing with two such objects \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), we usually shall write \( \mathcal{O}_1(A^o, B^o, \ldots) \sim \mathcal{O}_2(A^o, B^o, \ldots) \) to mean \( \mathcal{O}_1(A^o, B^o, \ldots) - \mathcal{O}_2(A^o, B^o, \ldots) \sim 0 \).

### 3.1 Vectorfield extensions

Let \( \nabla \) be the Levi-Civita connection outside \( \Sigma \) and let us consider the main normal \( N \in \mathfrak{X}_\Sigma \), thus \( \langle N, N \rangle = 1 \). Because \( II(N, N) = 0 \), it follows from Theorem 1 in [5] that there exists a unique (we call it canonical) local extension \( \mathcal{N} \in \mathfrak{X}(M) \) of \( N \) which is \( \nabla \)-geodesic outside \( \Sigma \). By continuity, it follows that: \( \langle \mathcal{N}, N \rangle = 1 \).

Given \( A \in \mathfrak{X}_\Sigma \), there exists a unique (we call it canonical) local extension \( \mathcal{A} \in \mathfrak{X}(M) \) such that

\[
[\mathcal{N}, \mathcal{A}] = 0
\]

(\( \mathcal{A} \) is generated from \( A \) by the flow of \( \mathcal{N} \)), and for that extension it holds:

\[
\mathcal{N}^o \langle \mathcal{N}^o, \mathcal{A}^o \rangle = \langle \mathcal{N}^o, \nabla_{\mathcal{N}^o} \mathcal{A}^o \rangle = \langle \mathcal{N}^o, \nabla_{\mathcal{A}^o} \mathcal{N}^o \rangle = 0 , \Rightarrow \mathcal{N} \langle \mathcal{N}, \mathcal{A} \rangle = 0 ;
\]

in particular, if \( X \in \mathfrak{X}(\Sigma) (\Rightarrow \langle N, X \rangle = 0) \), the canonical extension \( \mathcal{X} \in \mathfrak{X}(M) \) satisfies:

\[
\langle \mathcal{N}, \mathcal{X} \rangle = 0 . \tag{8}
\]

Let \( \mathcal{R} \in \mathfrak{X}(M) \) be the canonical extension of the main radical vectorfield \( R \). In what follows, we shall denote \( \tau \equiv \langle \mathcal{R}, \mathcal{R} \rangle \). We obtain from (1):

\[
N(\tau) = 2\Box_R N(R) = -2H(R, R) = 2 ; \tag{9}
\]
thus \( \tau = 0 \) is an equation for \( \Sigma \) and \( \tau^{-1}(\mathcal{N}(\tau) - 2) \) is well-defined (by the way, (9) is still valid if \( \mathcal{N}, \mathcal{R} \) are arbitrary extensions of \( N, R \)).

Let be \( X \in \mathfrak{X}(\Sigma) \) and let \( \mathcal{X} \in \mathfrak{X}(M) \) be its canonical extension. It follows from \( X(\tau) = 0 \) that \( \tau^{-1}\mathcal{X}(\tau) \) is well-defined and it holds:

\[
\mathcal{N}(\mathcal{X}(\tau)) = \mathcal{X}(\mathcal{N}(\tau)) \equiv \mathcal{X}([\tau^{-1}(\mathcal{N}(\tau) - 2)] \tau) , \quad \Rightarrow \quad N(\mathcal{X}(\tau)) = 0 ;
\]

thus \( \tau^{-1}\mathcal{N}(\tau) \) is well-defined and it holds:

\[
(\tau^{-1}\mathcal{N}(\mathcal{X}(\tau))) \tau \equiv \mathcal{N}(\mathcal{X}(\tau)) \equiv \mathcal{N}([\tau^{-1}\mathcal{X}(\tau)] \tau) =
\]

\[
= \mathcal{N}(\tau^{-1}\mathcal{X}(\tau)) + (\tau^{-1}\mathcal{X}(\tau))(2 + [\tau^{-1}(\mathcal{N}(\tau) - 2)] \tau) , \quad \Rightarrow \quad (\tau^{-1}\mathcal{X}(\tau)) |_{\Sigma} = 0 ;
\]

therefore

\[
\tau^{-2}\mathcal{X}(\tau) \quad \text{is well-defined .} \quad (10)
\]

Let be \( X \in \mathfrak{X}(\Sigma) \) and let \( \mathcal{X} \in \mathfrak{X}(M) \) be any extension. A direct computation from (1) leads to (we use (10)):

\[
\mathcal{H}(X, R) := \square_{\mathcal{X}} \mathcal{R}(\mathcal{N}) |_{\Sigma} = -\frac{1}{2}(\mathcal{N}(\mathcal{X}, \mathcal{R})) |_{\Sigma} \equiv -\frac{1}{2}N((\tau^{-1}(\mathcal{X}, \mathcal{R})) \tau) =
\]

\[
= -\frac{1}{2}(\tau^{-1}(\mathcal{X}, \mathcal{R})) |_{\Sigma} N(\tau) = -(\tau^{-1}(\mathcal{X}, \mathcal{R})) |_{\Sigma} ;
\]

therefore, given \( A \in \mathfrak{X}_{\Sigma} \), and for any extension \( \mathcal{A} \in \mathfrak{X}(M) \) of \( A \), it holds:

\[
\rho(A) := -\mathcal{H}(A - \nu(A)N, R) = (\tau^{-1}(\mathcal{A}, \mathcal{R})) |_{\Sigma} . \quad (11)
\]

In what follows, we shall use some fixed \( (C^\infty, \text{local around some point of } \Sigma) \) adapted frame \( (\mathcal{E}_1, \ldots, \mathcal{E}_m = \mathcal{R}) \), with \( E_1 = N, E_2, \ldots, E_{m-1} \in \Gamma(S) \) and \( \mathcal{R} \) the canonical extension of \( R \).

**Remark 7** In this paper, we do not use coordinates. Nevertheless, it is interesting to point out that we can construct, around each point of \( \Sigma \), a coordinate system \( (x^1, \ldots, x^m) \) of \( M \) such that (we call these coordinates
adapted): (1) $\partial x^1 = N$ and $\partial x^m = R$ (the canonical extensions of $N$ and $R$); (2) $x^1 = 0$ is an equation for $\Sigma$; and (3) it holds:

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{\lambda\mu} & x^1 g_\lambda \\ 0 & x^1 g_\lambda & x^1 g_m \end{pmatrix} \quad (a, b = 1, \ldots, m; \lambda, \mu = 2, \ldots, m - 1),$$

for some $g_i \in C^\infty(M)$ ($i = 2, \ldots, m$) with $g_m(0, x^2, \ldots, x^m) = 2$.

We now outline the construction: Around $p \in \Sigma$, we choose coordinates $(x^2, \ldots, x^m)$ in $\Sigma$ such that $\partial x^m = R$. Using the flow of $N$, it is straightforward to construct a coordinate system $(x^1, \ldots, x^m)$ in $M$ such that $\partial x^1 = N$ (thus $g_{11} = 1$) and $\partial x^i$ is the canonical extension of the (equally denoted) vectorfield $\partial x^i \in \mathfrak{X}(\Sigma)$ ($i = 2, \ldots, m$); in particular, $\partial x^m = R$. Obviously, $x^1 = 0$ is an equation for $\Sigma$. It follows from (8): $g_{1i} = 0$ ($i = 2, \ldots, m$). On the other hand, $g_m |_\Sigma = 0$ implies: $g_m = x^1 g_i$, for some $g_i \in C^\infty(M)$ ($i = 2, \ldots, m$). And finally, writing $\tau \equiv g_{mm}$, it follows from (9): $g_m |_\Sigma = 2$.

Note that, in adapted coordinates, a vectorfield $A \equiv \sum f_a \partial x^a \in \mathfrak{X}(M)$ is the canonical extension of $A \equiv \sum f_a |_\Sigma \partial x^a \in \mathfrak{X}_\Sigma$ if and only if all $f_a$'s ($a = 1, \ldots, m$) do not depend on $x^1$.

Calling $\Gamma_{ab} \equiv \square_{\partial x^a} \partial x^b (\partial x^c)$, the first-class Christoffel symbols of $\square$ in that coordinates, and using (1), it is straightforward to see that the components of the second fundamental form $\mathcal{H}$ and of the tensorfield $II$ on $\Sigma$ are given by

$$\begin{cases} \mathcal{H}_{ij} = \Gamma_{1ij} |_\Sigma = -\frac{1}{2} \frac{\partial g_{1i}}{\partial x^j} |_{x^1 = 0} & (i, j = 2, \ldots, m) \\ II_{ij} = \Gamma_{mij} |_\Sigma = -\frac{1}{2} \frac{\partial g_{1m}}{\partial x^j} |_{x^1 = 0} \end{cases}$$

The above construction applies to all local examples of transverse Riemann-Lorentz metrics with tangent radical $\tau$.

### 3.2 Covariant derivatives

Let be $A, B \in \mathfrak{X}(M)$. On $M - \Sigma$ we have:

$$\nabla_A B^o = \sum_{i=1}^{m-1} \square_A B^o (\mathcal{E}_i^o) \mathcal{E}_i^o + \tau^o - 1 \square_A B^o (\mathcal{R}^o) \mathcal{R}^o.$$ 

It thus follows that the vectorfield $\nabla_A B^o$ has a good limit on $\Sigma$ if and only if: $\tau^{-1} \square_A B(\mathcal{R}) \in C^\infty(M)$ or, using (2), if and only if: $II(A, B) = 0$. 

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Once $\nabla_{A'}B^0 \cong 0$, we may define:

$$\nabla_{A}B := \Sigma_{i=1}^{m-1} \Box A B(\mathcal E_i) \mathcal E_i + \tau^{-1} \Box A B(\mathcal R) \mathcal R,$$

which obviously satisfies

$$\langle \nabla_{A}B, C \rangle = \Box A B(C), \quad \text{for all } C \in \mathfrak x(M).$$

It turns out that, in general, the restriction $\nabla_{A}B \mid_{\Sigma}$ depends, not only on the restrictions $A, B \in \mathfrak x_{\Sigma}$, but also on the original vectorfields $A, B$. Indeed, starting with $A' = A + \tau \bar A, B' = B + \tau \bar B$ (for some $\bar A, \bar B \in \mathfrak x(M)$), it is straightforward to see that it holds (we denote $\bar A \equiv \bar A \mid_{\Sigma}, \bar B \equiv \bar B \mid_{\Sigma}$):

$$(\tau^{-1} \langle \nabla_{A'}B', \mathcal R \rangle) \mid_{\Sigma} = (\tau^{-1} \langle \nabla_{A}B, \mathcal R \rangle) \mid_{\Sigma} + A(\tau)(\tau^{-1} \langle \bar B, \mathcal R \rangle) \mid_{\Sigma} + I I (A, \bar B) + I I (\bar A, B),$$

and therefore

$$\nabla_{A'}B' \mid_{\Sigma} = \nabla_{A}B \mid_{\Sigma} + \Sigma_{i=1}^{m-1} \Box A (B' - B)(E_i) E_i +$$

$$+ \{ A(\tau)(\tau^{-1} \langle \bar B, \mathcal R \rangle) \mid_{\Sigma} + I I (A, \bar B) + I I (\bar A, B) \} R. \quad (13)$$

We have the following proposition, whose proof is straightforward using (3) and (4):

**Proposition 8** Let $(M, g)$ be a transverse Riemann-Lorentz space with tangent radical, singular hypersurface $\Sigma$ and canonical screen distribution $S$. Let $\nabla$ be the Levi-Civita connection on $M - \Sigma$.

Let be $A, B \in \mathfrak x(M)$.

(a) It holds: $\nabla_{A'}B^0 \cong 0 \iff I I (A, B) = 0$.

(b) The following two assertions are equivalent: (1) $\nabla_{A'}B^0 \cong 0$, whenever $A, B$ are tangent to $\Sigma$; and (2) $\Sigma$ is $I I$-flat.

Let be $A, B \in \mathfrak x_{\Sigma}$.

(c) $\nabla_{A}B$ is not well-defined, whenever one of the vectorfields $A, B$ is either $N$ or $R$.

(d) The following two assertions are equivalent: (1) $\nabla_{A}B$ is well-defined, whenever $A, B \in \Gamma(S)$; and (2) $\Sigma$ is $I I$-flat. $$
Despite of Proposition R some particular arrangements with covariant derivatives are always ”extension independent”. We mention here three cases:

(a) Let \( A \) be tangent to \( \Sigma \) (i.e. \( A = \mathcal{X} \), with \( X \in \mathfrak{X}(\Sigma) \)). Writing \( \mathcal{X}' = \mathcal{X} + \tau \bar{X}, \mathcal{B}' = \mathcal{B} + \tau \bar{B} \) (for some \( \bar{X}, \bar{B} \in \mathfrak{X}(M) \)), we obtain from (13):

\[
\nabla_{\mathcal{X}' B'} |_{\Sigma} = \nabla_{\mathcal{X} B} |_{\Sigma} + \{ II(X, \bar{B}) + II(\bar{X}, B) \} R ,
\]

from which it follows

\[
\begin{align*}
\{ \langle \nabla_{\mathcal{X}' B'} |_{\Sigma}, C \rangle = \langle \nabla_{\mathcal{X} B} |_{\Sigma}, C \rangle , \forall C \in \mathfrak{X}_\Sigma \\
II(\nabla_{\mathcal{X}' B'} |_{\Sigma}, Z) = II(\nabla_{\mathcal{X} B} |_{\Sigma}, Z) , \forall Z \in \mathfrak{X}(\Sigma) .
\end{align*}
\]

Last equation shows that, if \( \Sigma \) is \( II \)-flat, a natural map

\[
III : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma) , \ (X, Y, Z) \mapsto II(\nabla_{\mathcal{X} Y} |_{\Sigma}, Z) \quad (15)
\]
arises, which turns out to be (straightforward computation) \( C^\infty(\Sigma) \)-trilinear and symmetric in its first two entries. Moreover, it is very easy to see that it holds:

\[
III(\cdot, \cdot, R) = \mathcal{H} \in \mathfrak{T}^2_{Sim}(\Sigma) .
\]

Using (15) and (3) we obtain (for all \( X, Y \in \mathfrak{X}(\Sigma) \) and \( V \in \Gamma(S) \)):

\[
III(X, Y, V) = 0 , \ III(V, R, R) = 0 \quad \text{and} \quad III(R, R, R) = -1 .
\]

We say that \( \Sigma \) is \( III \)-flat if (it is \( II \)-flat and) it holds:

\[
III(V, W, R) = 0 , \ \text{for all} \ V, W \in \Gamma(S) ; \quad (16)
\]
or, because of (3), if \( \nabla_{\mathcal{V} \mathcal{W}} \) is tangent to \( \Sigma \), for all \( \mathcal{V}, \mathcal{W} \in \mathfrak{X}(M) \) tangent to \( S \). It follows from (5) that: \( \Sigma \) is \( III \)-flat if and only if it is \( II \)-flat and \( \mathcal{H} \)-flat.

Remark 9 If \( \Sigma \) is \( II \)-flat but the radical is transverse (remember Remark \( \mathbb{D} \)), a well-defined map \( III : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma) , \ (X, Y, C) \mapsto II(\nabla_{\mathcal{X} Y} |_{\Sigma}, C) \) arises (see \( \mathbb{D} \); observe that the domain of \( III \) in that case is ”larger” than in our case with tangent radical), which is \( C^\infty(\Sigma) \)-trilinear and symmetric in the first two entries, and whose restriction to \( \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \) vanishes. Then \( \Sigma \) was defined \( \mathbb{D} \) to be \( III \)-flat if it holds:

\[
III(X, Y, \text{Rad}) = 0 , \ \text{for all} \ X, Y \in \mathfrak{X}(\Sigma) ;
\]

although natural, this condition is in some sense stronger than (16) \( \blacksquare \)
(b) Let $A$ and $B$ be tangent to $\Sigma$ (i.e. $A = \mathcal{X}, B = \mathcal{Y}$, with $X, Y \in \mathfrak{X}(\Sigma)$). Writing $\mathcal{X}' = \mathcal{X} + \tau \mathcal{X}, \mathcal{Y}' = \mathcal{Y} + \tau \mathcal{Y}$ (for some $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(M)$), we obtain from the symmetry of $\nabla$ and (ii):

$$
(\tau^{-1} \langle R, \nabla \mathcal{X}' \mathcal{Y}' \rangle_{Ant}) |\Sigma = \frac{1}{2} \tau^{-1} \langle R, [\mathcal{X}', \mathcal{Y}'] \rangle |\Sigma = \frac{1}{2} \rho([X, Y]),
$$

(17)

where $\langle R, \nabla \mathcal{X}' \mathcal{Y}' \rangle_{Ant}$ means the antisymmetric part of $\langle R, \nabla \mathcal{X}' \mathcal{Y}' \rangle$ under the permutation of $\mathcal{X}'$ and $\mathcal{Y}'$.

(c) Finally, let be $X \in \mathfrak{X}(\Sigma)$ and let $\mathcal{X}, \mathcal{R}$ be the canonical extensions of $X, R$. Let $\mathcal{X}'$ be any extension of $X$. Writing $\mathcal{X}' = \mathcal{X} + \tau \mathcal{X}$ (for some $\mathcal{X} \in \mathfrak{X}(M)$), we obtain:

$$
\begin{align*}
\nu(\nabla \mathcal{X}' \mathcal{R} |\Sigma) &= \square_X R(N) = \mathcal{H}(X, R) \\
\rho(\nabla \mathcal{X}' \mathcal{R} |\Sigma) &= (\tau^{-1} \langle R, \nabla \mathcal{X}' \mathcal{R} \rangle) |\Sigma = \frac{1}{2} (\tau^{-1} \mathcal{X}'(\tau)) |\Sigma = \frac{1}{2} (\mathcal{X}(\tau)) |\Sigma = \nu(\mathcal{X}) \\
\langle \nabla \mathcal{X}' \mathcal{R} |\Sigma, W \rangle &= \square_X R(W) = -II(X, W), \text{ for all } W \in \Gamma(S)
\end{align*}
$$

(18)

(in the middle line, we have used (ii), (i) and (i) in the first, third and last equality, respectively); in particular: $\nabla \mathcal{R} \mathcal{R} |\Sigma = -N$. 

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3.3 Covariant Curvatures

Given $A, B, C, D \in \mathfrak{X}(M)$, we compute Levi-Civita covariant curvature on $M - \Sigma$:

$$\langle R(A^o, B^o)C^o, D^o \rangle := \Box_{A^o} (\nabla_{B^o} C^o)(D^o) - \Box_{B^o} (\nabla_{A^o} C^o)(D^o) - \Box_{[A^o, B^o]} C^o(D^o) =$$

$$= \sum_{i=1}^{m-1} \{ A^o(\Box_{B^o} C^o(\mathcal{E}_i^o)) - B^o(\Box_{A^o} C^o(\mathcal{E}_i^o)) \} \langle \mathcal{E}_i^o, D^o \rangle +$$

$$+ \sum_{i=1}^{m-1} \{ \Box_{B^o} C^o(\mathcal{E}_i^o) \Box_{A^o} \mathcal{E}_i^o(D^o) - \Box_{A^o} C^o(\mathcal{E}_i^o) \Box_{B^o} \mathcal{E}_i^o(D^o) \} +$$

$$+ A^o(\Box_{B^o} C^o(\mathcal{R}^o)(\tau^{o-1} \langle \mathcal{R}, D \rangle) - B^o(\Box_{A^o} C^o(\mathcal{R}^o)(\tau^{o-1} \langle \mathcal{R}, D \rangle) - \Box_{[A^o, B^o]} C^o(\mathcal{D}^o) +$$

$$+ \tau^{o-1} \{ \Box_{A^o} C^o(\mathcal{R}^o) \Box_{B^o} D^o(\mathcal{R}^o) - \Box_{B^o} C^o(\mathcal{R}^o) \Box_{A^o} D^o(\mathcal{R}^o) \} \equiv$$

$$\equiv \tau^{o-1} \Upsilon(A^o, B^o, C^o, D^o) ,$$

where the (everywhere regular) tensorfield $\Upsilon \in \mathfrak{T}^0_1(M)$ has the same symmetry properties as the covariant curvature and satisfies (as usual, we denote $A \equiv A |_{\Sigma, \ldots \in \mathfrak{X}(\Sigma)}$):

$$\Upsilon(A, B, C, D) = \det \left( \begin{array}{cc} II(A, C) & II(A, D) \\ II(B, C) & II(B, D) \end{array} \right) =$$

If $\Sigma$ is $II$–flat

$$\det \left( \begin{array}{c} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{array} \right) \cdot \det \left( \begin{array}{c} \nu(C) \nu(D) \\ \rho(C) \rho(D) \end{array} \right) .$$

We obtain:

**Theorem 10** Let $(M, g)$ be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface $\Sigma$. Let be $A, B, C, D \in \mathfrak{X}(M)$.

(a) It holds: $\langle R(A^o, B^o)C^o, D^o \rangle \equiv 0 \iff \Upsilon(A, B, C, D) = 0$ .

(b) If we consider the assertions:

(1) $\Upsilon(A, B, C, D) = 0 \iff \det \left( \begin{array}{c} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{array} \right) \cdot \det \left( \begin{array}{c} \nu(C) \nu(D) \\ \rho(C) \rho(D) \end{array} \right) = 0$

(2) $\Sigma$ is $II$–flat

(3) $\Upsilon(A, B, C, D) = 0 \Rightarrow \det \left( \begin{array}{c} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{array} \right) \cdot \det \left( \begin{array}{c} \nu(C) \nu(D) \\ \rho(C) \rho(D) \end{array} \right) = 0$ .
then it holds: \((1) \Leftrightarrow (2) \Rightarrow (3)\).

**Proof.** (a) follows from \(33\). (b) We compute \(\Upsilon(A, B, C, D)\) from \(20\). To prove \((1) \Rightarrow (2)\), we first obtain: \(0 = \Upsilon(V, R, W, N) = II(V, W)\), for all \(V, W \in \Gamma(S)\), and the result follows from \(33\) \(\blacksquare\).

**Remark 11** Theorem \(10\) also says: \(\Sigma\) is \(II\)-flat if and only if \(\langle R(A^0, B^0)C^0, D^0 \rangle \cong 0\), whenever at least three of the vectorfields \(A, B, C, D \in \mathfrak{X}(M)\) are tangent to \(\Sigma\) (as for the sufficient condition: part (a) of the Theorem leads to \(0 = \Upsilon(V, R, W, N) = II(V, W)\), for all \(V, W \in \Gamma(S)\), and the result follows from \(33\)). Concerning other cases: \(\langle R(N^0, V^0)N^0, W^0 \rangle \cong 0\) (always) and \(\langle R(N^0, V^0)N^0, R^0 \rangle \cong 0\) (always), for all \(V, W \in \mathfrak{X}(M)\) tangent to \(\Sigma\). However, even in the \(II\)-flat case, \(\langle R(N^0, V^0)N^0, R^0 \rangle \not\cong 0\).

These results contrast with the corresponding ones in the case of transverse radical, namely \(\langle 12, \text{Theorem 3a} \rangle\): \(\Sigma\) is \(II\)-flat if and only if \(\langle R(A^0, B^0)C^0, D^0 \rangle \cong 0\), for all vectorfields \(A, B, C, D \in \mathfrak{X}(M)\). Thus \(II\)-flatness leads to no divergences in the case with transverse radical \(\blacksquare\).

Once \(\langle R(A^0, B^0)C^0, D^0 \rangle \cong 0\), we may define (by continuity, the only possible definition):

\[\langle R(A, B)C, D \rangle := \tau^{-1}\Upsilon(A, B, C, D)\]

which satisfies (for all \(\tilde{A} \in \mathfrak{X}(M)\)):

\[\langle R(A + \tau\tilde{A}, B)C, D \rangle = \langle R(A, B)C, D \rangle + \Upsilon(\tilde{A}, B, C, D)\]

**Remark 12** One should be careful in writing: \(\langle R(A, B)C, D \rangle = \Box_A(\nabla_B C)(D) - \Box_B(\nabla_A C)(D) - \Box_{[A, B]} C(D)\). The reason is that

\[
\begin{aligned}
\nabla_{B^0} C^0 &\equiv \tau^{0-1} \Box_{B^0} C^0(\mathcal{R}^0) \mathcal{R}^0, \text{ whereas} \\
\Box_{A^0}(\nabla_{B^0} C^0)(D^0) &\equiv -\tau^{0-1} \Box_{B^0} C^0(\mathcal{R}^0)\Box_{A^0}D^0(\mathcal{R}^0);
\end{aligned}
\]

thus it may happen that \(\langle R(A, B)C, D \rangle\) is well-defined, but \(\nabla_B D\) does not exist (e.g. \(\langle R(N^0, V^0)N^0, V^0 \rangle \cong 0\), but it may happen that \(\nabla_{V^0} V^0 \not\cong 0\) \(\blacksquare\).

As it happens with covariant derivatives, \(\langle R(A, B)C, D \rangle\) may sometimes be well-defined. Indeed, starting with \(A' = A + \tau\tilde{A},\ B' = B + \tau\tilde{B} C' = C + \tau\tilde{C} D' = D + \tau\tilde{D}\) (for some \(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathfrak{X}(M)\)), one sees that it holds (we denote \(\tilde{A} \equiv \tilde{A} |_{\Sigma}, \ldots\)):
\[ \langle R(A', B')C', D' \rangle |_\Sigma - \langle R(A, B)C, D \rangle |_\Sigma = \]
\[ = \Upsilon(A, B, C, D) + \Upsilon(A, \bar{B}, C, D) + \Upsilon(A, B, \bar{C}, D) + \Upsilon(A, B, C, \bar{D}) . \]

Last equation and (20) lead to

**Theorem 13** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \(\Sigma\). Let be \(A, B, C, D \in \mathfrak{X}_\Sigma\). Then it holds:

\[ \langle R(A, B)C, D \rangle \text{ is well-defined} \iff \Upsilon(\cdot, B, C, D) = \Upsilon(A, \cdot, C, D) = \Upsilon(A, B, \cdot, D) = \Upsilon(A, B, C, \cdot) = 0 \iff \]

\[ \text{If } \Sigma \text{ is II-flat } \iff \]
\[ \{ \text{either } \det \left( \begin{array}{cc} \nu(A) & \nu(B) \\ \rho(A) & \rho(B) \end{array} \right) = 0 = \det \left( \begin{array}{cc} \nu(C) & \nu(D) \\ \rho(C) & \rho(D) \end{array} \right) \} \]

or one of the above two matrices vanishes

Let be \(p \in \Sigma\) and \(a, b, c, d \in T_p M\). We say that \(\langle R(a, b)c, d \rangle\) is well-defined if there exist (local) extensions \(A, B, C, D \in \mathfrak{X}_\Sigma\) of \(a, b, c, d\) such that \(\langle R(A, B)C, D \rangle\) is well-defined. This definition is independent of the chosen extensions \(A, B, C, D\), as the next Lemma shows:

**Lemma 14** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \(\Sigma\).

(a) Let be \(p \in \Sigma\) and \(a, b, c, d \in T_p M\) such that \(\langle R(a, b)c, d \rangle\) is well-defined. Let \(A, B, C, D \in \mathfrak{X}_\Sigma\) and \(A', B', C', D' \in \mathfrak{X}_\Sigma\) be two sets of (local) extensions of \(a, b, c, d\) such that both \(\langle R(A, B)C, D \rangle\) and \(\langle R(A', B')C', D' \rangle\) are well-defined. Then it holds:

\[ \langle R(A', B')C', D' \rangle (p) = \langle R(A, B)C, D \rangle (p) . \]

(b) Let be \(A, B, C, D \in \mathfrak{X}_\Sigma\) such that \(\langle R(A(p), B(p))C(p), D(p) \rangle\) is well-defined, for all \(p \in \Sigma\). Then \(\langle R(A, B)C, D \rangle\) is well-defined and it holds (for all \(p \in \Sigma\)):

\[ \langle R(A, B)C, D \rangle (p) = \langle R(A(p), B(p))C(p), D(p) \rangle . \]
Proof. (a) Let $A, B, C, D \in \mathfrak{X}(M)$ and $A', B', C', D' \in \mathfrak{X}(M)$ be the canonical extensions of $A, B, C, D$ and $A', B', C', D'$, respectively. Let us consider the function $F := \langle R(A', B')C', D' \rangle - \langle R(A, B)C, D \rangle \in C^\infty(M)$, which satisfies (on $M - \Sigma$)

$$F^o = \langle R(A^o - A^o, B^o)C^o, D^o \rangle + \langle R(A^o, B^o - B^o)C^o, D^o \rangle + \langle (A^o, B^o)(C^o - C^o), D^o \rangle + \langle R(A^o, B^o)C^o, D^o - D^o \rangle.$$ 

Let $\gamma$ be the integral curve of the canonical extension $N$ of $N$ with $\gamma(0) = p$. Because $A' - A$ is the canonical extension of $A' - A$ and $(A' - A)(p) = 0$, it follows that $(A' - A) \circ \gamma = 0$. Analogously, $(B' - B) \circ \gamma = (C' - C) \circ \gamma = (D' - D) \circ \gamma = 0$. Therefore $F^o \circ \gamma = 0$, and we obtain

$$\langle R(A', B')C', D' \rangle (p) - \langle R(A, B)C, D \rangle (p) = \text{(by hypothesis)}$$

$$= \langle R(A', B')C', D' \rangle (p) - \langle R(A, B)C, D \rangle (p) = \lim_{t \to 0}(F^o \circ \gamma)(t) = 0.$$

(b) follows immediately from Theorem 13 because the condition there is tensorial.

Theorem 13 also gives the algebraic (necessary and sufficient) condition on $a, b, c, d$ in order that $\langle R(a, b)C, D \rangle$ is well-defined, namely: $\Upsilon(\cdot, b, c, d) = \Upsilon(a, \cdot, c, d) = \Upsilon(a, b, \cdot, d) = \Upsilon(a, b, c, \cdot) = 0$.

3.4 Sectional Curvatures

We start with two $C^\infty(M)$-linearly independent vectorfields $A, B \in \mathfrak{X}(M)$ with $\text{rank}(g_{A\wedge B}) = 2$ and compute the $\nabla$-sectional curvature on $M - \Sigma$

$$K_{A^o \wedge B^o} := \frac{\langle R(A^o, B^o)A^o, B^o \rangle}{\det(g_{A^o \wedge B^o})}.$$ 

(a) Suppose first that $\text{rank}(g_{A\wedge B}) = \text{(constant)} 2$ (an open condition). In that case, we have:

Proposition 15 Let $(M, g)$ be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface $\Sigma$. Let $A, B \in \mathfrak{X}(M)$ be such that $\text{rank}(g_{A\wedge B}) = 2$.

22
(a) It holds: \( K_{A^o \wedge B^o} \cong 0 \iff \Upsilon(A, B, A, B) = 0 \).

(b) If we consider the assertions:

(1) \( \Upsilon(A, B, A, B) = 0 \iff \det \begin{pmatrix} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{pmatrix} = 0 \)

(2) \( \Sigma \) is II-flat

(3) \( \Upsilon(A, B, A, B) = 0 \Rightarrow \det \begin{pmatrix} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{pmatrix} = 0 \),

then it holds: (1) \( \iff \) (2) \( \Rightarrow \) (3).

Proof. (a) follows from Theorem 10. (b) follows from Theorem 10b.

To prove (1) \( \Rightarrow \) (2), choose \( A \in \Gamma(S) \) and \( B = \nu(B)N + \rho(B)R \), with both \( \nu(B), \rho(B) \in C^\infty(\Sigma) \) nowhere zero.

Remark 16 It follows from Proposition 12 that: \( K_{A^o \wedge V^o} \cong 0 \) (always) and \( K_{\mathcal{N}^o \wedge (V^o + \mathcal{R}^o)} \cong 0 \) (always), for all \( V \in \mathfrak{X}(M) \) nowhere zero and tangent to \( S \).

Moreover, if \( \Sigma \) is II-flat, we obtain: \( K_{A^o \wedge B^o} \cong 0 \), whenever both vector-fields \( A, B \in \mathfrak{X}(M) \) (with \( \text{rank}(g_{A \wedge B}) = 2 \)) are tangent to \( \Sigma \), and \( K_{(\mathcal{N}^o + \mathcal{R}^o)^o \wedge V^o} \cong 0 \), for all \( V \in \mathfrak{X}(M) \) nowhere zero and tangent to \( S \).

Things are again different when the radical is transverse; for then, II-flatness is enough to guarantee (\cite{2}, Theorem 3b) that \( K_{A^o \wedge B^o} \cong 0 \), for all \( A, B \in \mathfrak{X}(M) \) (with \( \text{rank}(g_{A \wedge B}) = 2 \)).

Once \( K_{A^o \wedge B^o} \cong 0 \), we define:

\[
K_{A \wedge B} := \frac{\langle R(A, B)A, B \rangle}{\det(g_{A \wedge B})}.
\]

Now we obtain:

Proposition 17 Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \( \Sigma \). Let \( A, B \in \mathfrak{X}_\Sigma \) be such that \( \text{rank}(g_{A \wedge B}) = 2 \).

(a) It holds:

\( K_{A \wedge B} \) is well-defined \( \iff \) \( \Upsilon(\cdot, B, A, B) = \Upsilon(A, \cdot, A, B) = 0 \iff \)

\[ \text{If } \Sigma \text{ is II-flat } \iff \det \begin{pmatrix} \nu(A) \nu(B) \\ \rho(A) \rho(B) \end{pmatrix} = 0 \ ( \iff \text{dim Span}(A^S, B^S) \geq 1 ) \].
(b) The following three assertions are equivalent:

1. \( K_{A\wedge B} \) is well-defined \( \iff \dim \text{Span}(A^S, B^S) \geq 1 \)

2. \( \Sigma \) is II-flat

3. \( K_{A\wedge B} \) is well-defined \( \iff \dim \text{Span}(A^S, B^S) = 1 \)

**Proof.** (a) is a direct consequence of Theorem 13.

(b) We prove (1) \( \Rightarrow \) (2): if \( \Sigma \) is not II-flat, then \( \exists V \in \Gamma(S) \) such that \( \text{II}(V, V) \neq 0 \), thus (20) leads to \( \Upsilon(N, V, V, R) \neq 0 \) and Theorem 10a shows that

\[
\langle R(N^o, V^o)V^o, R^o \rangle \neq 0 \quad \text{and} \quad \langle R(N^o, V^o)V^o, R^o \rangle = \langle R(N^o, V^o)V^o, R^o \rangle + 2 \langle R(N^o, V^o)R^o, V^o \rangle ,
\]

and the first three terms have good limits on \( \Sigma \) because of Theorem 10a.

Now (2) \( \Rightarrow \) (3) follows from Part (a). Finally, (3) \( \Rightarrow \) (1) is a consequence of Proposition 15

Let be \( p \in \Sigma \) and \( a, b \in T_p M \) such that \( \text{rank}(g_{a\wedge b}) = 2 \). We say that \( K_{a\wedge b} \) is well-defined if there exist (local) extensions \( A, B \in X_{\Sigma} \) of \( a, b \) such that \( K_{A\wedge B} \) is well-defined. This definition is independent of the chosen extensions \( A, B \), as the above Lemma shows. Proposition 17a also gives the algebraic (necessary and sufficient) condition on \( a, b \) in order that \( K_{a\wedge b} \) is well-defined, namely: \( \Upsilon(\cdot, b, a, b) = \Upsilon(a, \cdot, a, b) = 0 \).

(b) Suppose now that \( \text{rank}(g_{A\wedge B}) < 2 \) (\( \iff \text{rank}(g_{A\wedge B}) = 1 \)). It follows:

\[
\det(g_{A\wedge B}) = k \tau , \quad k \in C^\infty(M), \quad \text{with} \quad k|_{\Sigma} \neq 0 \quad \text{(everywhere)}, \quad \Rightarrow \quad \text{(21)}
\]

\[
\Rightarrow \quad (\nu(A)N + A^S) \wedge (\nu(B)N + B^S) = 0
\]

From (20) we compute \( \Upsilon(A, B, A, B) = -\det^2 \left( \begin{array}{cc} \nu(A) & \nu(B) \\ \rho(A) & \rho(B) \end{array} \right) \) and, using (21), we conclude:

\[
\Upsilon(A, B, A, B) = 0 \iff \nu(A) = 0 = \nu(B) \quad \text{(22)}
\]

We finally obtain:

\[
2\text{Such extensions } A, B \text{ will always satisfy, locally around } p, \quad \text{rank}(g_{A\wedge B}) = 2
\]

24
Proposition 18  Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \(\Sigma\). Let \(A, B \in \mathfrak{X}(M)\) be such that \(\text{rank}(g_{A \wedge B}) = 2\), \(\text{rank}(g_{A \wedge B}) = 1\) and \(R \in A \wedge B\).

(a) It holds: \(K_{A \wedge B} \circ \cong 0 \iff \langle R(A, B)A, B \rangle \mid_{\Sigma} = 0 \Rightarrow \nu(A) = 0 = \nu(B)\).

(b) If we consider the assertions:

\begin{enumerate}
\item \(\langle R(A, B)A, B \rangle = 0 \iff \nu(A) = 0 = \nu(B)\) (\(\iff \text{dim Span}(A^S, B^S) = 1\))
\item \(\Sigma\) is III-flat
\item \(\Sigma\) is II-flat
\item \(\langle R(A, B)A, B \rangle\) is well-defined \(\iff \nu(A) = 0 = \nu(B)\),
\end{enumerate}

then it holds: \((1) \iff (2) \Rightarrow (3) \iff (4)\).

**Proof.** (a) follows from \((21)\), Theorem \((10)\) and \((22)\).

(b) When \(\nu(A) = 0 = \nu(B)\), it can be easily proved (as a consequence of Theorem \((13)\)) that:

\(\langle R(A, B)A, B \rangle\) is well-defined \(\iff II(A^S, A^S) = 0 = II(B^S, B^S)\).

Thus \((3) \Rightarrow (4)\) becomes trivial. To prove \((4) \Rightarrow (3)\): if \(\Sigma\) is not II-flat, then \(\exists V \in \Gamma(S)\) such that \(II(V, V) \neq 0\), and \((23)\) shows that \(\langle R(V, R)V, R \rangle\) cannot be well-defined.

We have (trivially): \((1) \Rightarrow (4)\) and \((2) \Rightarrow (3)\).

Because \(R \in A \wedge B\), to prove \((1) \iff (2)\) is equivalent to prove that \(\Sigma\) is III-flat if and only if \(\langle R(V, R)V, R \rangle\) is well-defined and \(= 0\), for all \(V \in \Gamma(S)\). But it holds:

\[\langle R(V^o, R^o)V^o, R^o \rangle = \Box_{V^o}(\nabla_{R^o} V^o)(R^o) - \Box_{R^o}(\nabla_{V^o} V^o)(R^o) - \Box_{[V^o, R^o]} V^o(R^o)\]

As we have already seen, it follows from either (1) or (2) that \(\Sigma\) is II-flat.

On the other hand, \(\nu(\nabla_{R^o} V \mid_{\Sigma}) = \nu(\nabla_{V^o} R \mid_{\Sigma}) = \Box_{V^o} R(N) =: \mathcal{H}(V, R) = 0\) (thus \(\nabla_{R^o} V\) is tangent to \(\Sigma\)). It thus follows that the first and third terms in the righthand side are proportional to \(\tau^o\). And we obtain:
\[
\langle R(V^o, R^o) V^o, R^o \rangle = k^o R^o \quad \Leftrightarrow \quad \Box_{R^o} (\nabla_{V^o} V^o)(R^o) = k^o R^o \quad \Leftrightarrow \\
\forall V \text{ tangent to } S
\]

\[
\Leftrightarrow \quad III\left(\nabla_{V} V \mid_{\Sigma}, R^o\right) = 0 \quad \Leftrightarrow \quad III(V, V, R) = 0 \quad \Leftrightarrow \quad \Sigma \text{ is } III-\text{flat} ;
\]

in the last five implications we have used Proposition 8a, (15), the symmetry of III, the fact that \(III(V, R, R) = 0\) and II-flatness, respectively.

**Remark 19** It follows from Proposition 13 that: \(K_{N^o \wedge R^o} \not\approx 0\) (always).

Moreover, if \(\Sigma\) is II-flat but not III-flat, there exists \(V \in \Gamma(S)\) such that \(\mathcal{H}(V, V) \neq 0\). Let \(V \in \mathfrak{X}(M)\) be any extension of \(V\). Using (19) and II-flatness, we explicitly compute

\[
\langle R(V^o, R^o) V^o, R^o \rangle = k^o R^o - \Box_{V^o} V^o(E^o_1) \Box_{R^o} E^o_1(R^o)
\]

\((k \in C^\infty(M))\). Because \(II(R, N) = 1\), it follows: \(\langle R(V, R) V, R \rangle \mid_{\Sigma} = -\mathcal{H}(V, V) \neq 0\) and, by Theorem 10a, \(K_{V^o \wedge R^o} \not\approx 0\).

This result again contrasts with the corresponding one in the case of transverse radical, where III-flatness guarantees (19), Theorem 3b) that \(K_{A^o \wedge B^o} \approx 0\), for all \(A, B \in \mathfrak{X}(M)\) (with rank\((g_{A \wedge B}) = 1\))

Suppose that \(K_{A^o \wedge B^o} \approx 0\). Because, in this case, both \(\tau^{-1} \langle R(A, B) A, B \rangle\) (by Proposition 15a) and \(\tau^{-1} \det(g_{A \wedge B})\) (by 21) must be well-defined functions and the second one nowhere vanishes, we define:

\[
K_{A \wedge B} := \frac{\tau^{-1} \langle R(A, B) A, B \rangle}{\tau^{-1} \det(g_{A \wedge B})}.
\]

And finally we obtain:

**Proposition 20** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \(\Sigma\). Let \(A, B \in \mathfrak{X}_\Sigma\) be such that \(\text{rank}(g_{A \wedge B}) = 1\) and \(R \in A \wedge B\). Then: \(K_{A \wedge B}\) is not well-defined.
Proof. In order to have \( K_{A^o \wedge B^o} \cong 0 \), we can write (Proposition 18a) \( A = V \in \Gamma(S) \) and \( B = R \). Let us consider extensions \( \mathcal{V}, \mathcal{V}' = \mathcal{V} + \tau f \mathcal{N} \) of \( \mathcal{V} \) and \( \mathcal{R} \) of \( R \) (with \( \mathcal{N}, \mathcal{V}, \mathcal{R} \) canonical and \( f \in C^\infty(M) \)). Then it holds (by Theorem 10a, all terms in the next equality are well-defined):

\[
\langle R(\mathcal{V}', \mathcal{R})\mathcal{V}', \mathcal{R} \rangle = \langle R(\mathcal{V}, \mathcal{R})\mathcal{V}, \mathcal{R} \rangle + 2\tau f \langle R(\mathcal{N}, \mathcal{R})\mathcal{V}, \mathcal{R} \rangle + \tau f^2 \Upsilon(\mathcal{N}, \mathcal{R}, \mathcal{N}, \mathcal{R}) ; \Rightarrow \\
\Rightarrow (\tau^{-1}\langle R(\mathcal{V}', \mathcal{R})\mathcal{V}', \mathcal{R} \rangle) |_\Sigma = (\tau^{-1}\langle R(\mathcal{V}, \mathcal{R})\mathcal{V}, \mathcal{R} \rangle) |_\Sigma + 2f |_\Sigma \langle R(\mathcal{N}, \mathcal{R})\mathcal{V}, \mathcal{R} \rangle |_\Sigma + f^2 |_\Sigma \Upsilon(\mathcal{N}, \mathcal{R}, \mathcal{N}, \mathcal{R}) .
\]

But it follows from (20) that \( \Upsilon(\mathcal{N}, \mathcal{R}, \mathcal{N}, \mathcal{R}) = -1 \); therefore a suitable choice of the function \( f \) leads, again by Proposition 18a, to the result: \( K_{V' \wedge R} |_\Sigma \neq K_{V \wedge R} |_\Sigma \)  

Let be \( p \in \Sigma \) and \( a, b \in T_pM \) such that \( \text{rank}(g_{a \wedge b}) < 2 \) (\( \iff \mathcal{R}(p) \in a \wedge b \), \( \Rightarrow \text{rank}(g_{a \wedge b}) = 1 \)). We say that \( K_{a \wedge b} \) is well-defined if there exist (local) extensions \( A, B \in \mathfrak{X}_\Sigma \) of \( a, b \) with \( \text{rank}(g_{A \wedge B}) < 2 \) and such that \( K_{A \wedge B} \) is well-defined. By Proposition 20, \( K_{a \wedge b} \) cannot be well-defined.

### 3.5 Ricci Curvatures

We start with two vectorfields \( A, B \in \mathfrak{X}(M) \) and compute Ricci-curvature on \( M - \Sigma \):

\[
Ric(A^o, B^o) = \sum_{i=1}^{m-1} \langle R(A^o, E^o_i)B^o, A^o_i \rangle + \tau^{o-1} \langle R(A^o, \mathcal{R}^o)B^o, \mathcal{R}^o \rangle . \quad (24)
\]

Therefore: \( Ric(A^o, B^o) \cong 0 \), whenever all \( m \) terms on the right hand side have good limits on \( \Sigma \). As for the last term, this is equivalent (see 1.1) to say that \( \langle R(A, R)B, \mathcal{R} \rangle |_\Sigma = 0 \).

Now we have:

**Proposition 21** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface \( \Sigma \). Given any extensions \( N' = N + \tau N, V' = V + \tau V, R' = R + \tau R \) of \( N, V \in \Gamma(S), R \) (where \( N, V, R \) are the canonical extensions and \( \mathcal{N}, \mathcal{V}, \mathcal{R} \in \mathfrak{X}(M) \)), it holds:
We apply Theorem \[10\] to all \( m \) terms in \( 21 \) for each case.

(a) The first \( m-1 \) terms have good limits on \( \Sigma \), whereas \( \langle R(\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ) \)
diverges like \( \tau^{o-1} \). Thus \( Ric(\mathcal{N}_\alpha^\circ, \mathcal{N}_\alpha^\circ) \) diverges like \( \tau^{o-2} \).

(b) As in (a), the first \( m-1 \) terms have good limits on \( \Sigma \). Using \( 19 \), we explicitly compute

\[
\langle R(\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ) = k^\circ \tau^o - \Box_{\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ}\mathcal{N}_\alpha^\circ(\mathcal{R}_\alpha^\circ) - \Box_{\mathcal{R}_\alpha^\circ, \mathcal{N}_\alpha^\circ}(\mathcal{R}_\alpha^\circ)(\tau^{o-1} \langle \mathcal{R}_\alpha^\circ, \nabla_{\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ} \rangle) \]
\((k \in C^\infty(M))\), and the result follows from \( 3, 11 \) and \( 18 \). Therefore, the possible divergence of \( Ric(\mathcal{N}_\alpha^\circ, \mathcal{N}_\alpha^\circ) \) when approaching \( \Sigma \) is like \( \tau^{o-1} \).

(c) The first \( m-1 \) terms have good limits on \( \Sigma \) (for \( i = 2, \ldots, m-1 \), because of \( II \)-flatness). Using \( 19 \), we explicitly compute

\[
\langle R(\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ) = k^\circ \tau^o + \Box_{\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ}\mathcal{N}_\alpha^\circ(\mathcal{R}_\alpha^\circ) - \Box_{\mathcal{R}_\alpha^\circ, \mathcal{N}_\alpha^\circ}(\mathcal{R}_\alpha^\circ)(\tau^{o-1} \langle \mathcal{R}_\alpha^\circ, \nabla_{\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ} \rangle) \]
\((k \in C^\infty(M))\), and the result follows from \( 11 \) and \( 14 \). As in (b), the possible divergence of \( Ric(\mathcal{N}_\alpha^\circ, \mathcal{R}_\alpha^\circ) \) when approaching \( \Sigma \) is like \( \tau^{o-1} \).

(d) The first \( m-1 \) terms have good limits on \( \Sigma \). Using \( 19 \), we explicitly compute \( \langle R(\mathcal{R}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{V}_\alpha^\circ, \mathcal{R}_\alpha^\circ) = k^\circ \tau^o \) \((k \in C^\infty(M))\), and the result follows.

(e) The first term diverges like \( \tau^{o-1} \), whereas the next \( m-2 \) terms have good limits on \( \Sigma \). Moreover, \( \langle R(\mathcal{R}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{R}_\alpha^\circ, \mathcal{R}_\alpha^\circ) = \tau^o \langle R(\mathcal{R}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\rangle \). Thus \( Ric(\mathcal{R}_\alpha^\circ, \mathcal{R}_\alpha^\circ) \) diverges like \( \tau^{o-1} \) when approaching \( \Sigma \).

(f) The first \( m-1 \) terms have good limits on \( \Sigma \) (for \( i = 2, \ldots, m-1 \), because of \( II \)-flatness). Using \( 19 \) and \( II \)-flatness, we explicitly compute

\[
\langle R(\mathcal{V}_\alpha^\circ, \mathcal{R}_\alpha^\circ)\mathcal{V}_\alpha^\circ, \mathcal{R}_\alpha^\circ) = k^\circ \tau^o - \Box_{\mathcal{V}_\alpha^\circ, \mathcal{R}_\alpha^\circ}\mathcal{V}_\alpha^\circ(\mathcal{R}_\alpha^\circ) - \Box_{\mathcal{R}_\alpha^\circ, \mathcal{V}_\alpha^\circ}(\mathcal{R}_\alpha^\circ)(\tau^{o-1} \langle \mathcal{J}_\alpha^\circ, \nabla_{\mathcal{V}_\alpha^\circ, \mathcal{R}_\alpha^\circ} \rangle) \]
\((k \in C^\infty(M))\), and the result follows from the fact that \( III \)-flatness is equivalent to \( II \)-flatness plus \( \mathcal{H} \)-flatness. Thus, the possible divergence of \( Ric(\mathcal{V}_\alpha^\circ, \mathcal{V}_\alpha^\circ) \) when approaching \( \Sigma \) is like \( \tau^{o-1} \). \( \blacksquare \)
Remark 22 The results in Proposition 21 sharply contrasts with what happens in the case of transverse radical; for then, $\text{III}$-flatness is equivalent ($[6]$, Theorem 3c) to the fact that $\text{Ric}(A^o,B^o) \cong 0$, for all $A, B \in \mathfrak{X}(M)$. 

Proposition 23 Let $(M, g)$ be a transverse Riemann-Lorentz space with tangent radical and singular hypersurface $\Sigma$. Let be $A, B \in \mathfrak{X}_\Sigma$. Then, even in the $\text{III}$-flat case, $\text{Ric}(A,B)$ is not well-defined.

Proof. We apply Theorem 13 to all $m$ terms in (24) for the cases (d) and (f) in Proposition 21 (the only cases we need to check).

Case (d): $\langle R(R,N)V,N \rangle$ is never well-defined, whereas $\langle R(R,E_\lambda)V,E_\lambda \rangle$ is well-defined if $\Sigma$ is $\text{II}$-flat ($\lambda = 2, \ldots , m-1$) and $\langle R(R,R)V,R \rangle$ is always well-defined (actually, it vanishes). Thus $\text{Ric}(R,V)$ is not well-defined.

Case (f): $\langle R(V,N)W,N \rangle$ is always well-defined and $\langle R(V,E_\lambda)W,E_\lambda \rangle$ is well-defined if $\Sigma$ is $\text{II}$-flat ($\lambda = 2, \ldots , m-1$). However, although $\tau^{\alpha -1} \langle R(V^\alpha,R^\alpha)W^\alpha,R^\alpha \rangle \cong 0$ if $\Sigma$ is $\text{III}$-flat, computation using (19) shows that $(\tau^{\alpha -1} \langle R(V^\alpha,R^\alpha)W^\alpha,R^\alpha \rangle) |_{\Sigma}$ depends on the extensions $V', W'$ of $V, W$. Thus, $\text{Ric}(V,W)$ is not well-defined.

3.6 The tangential connection

When the singular hypersurface $\Sigma$ is $\text{II}$-flat, Proposition 35 induces the following natural connection on $\Sigma$ (we denote it by $\nabla^\Sigma$ and we call it the tangential connection)

$$\nabla^\Sigma : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma), (X,Y) \mapsto \nabla^\Sigma_X Y := \nabla_X Y \mid_{\Sigma} - \nu(\nabla_X Y \mid_{\Sigma}) N,$$

where $X,Y \in \mathfrak{X}(M)$ are the canonical extensions of $X,Y$. Using (12), we obtain:

$$\nabla^\Sigma_X Y = \Sigma^{m -1}_{\lambda=2} \Box_X Y(E_\lambda)E_\lambda + (\tau^{-1} \langle \nabla_X Y, R \rangle) \mid_{\Sigma} R.$$  \hspace{1cm} (25)

As a consequence of (18), the main radical vectorfield $R$ becomes $\nabla^\Sigma$-geodesic.

Equations (14), (16) and (17) lead to:
\[ X(\rho(Y)) = X(\tau^{-1} \langle \mathcal{R}, \mathcal{Y} \rangle) = (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle) \mid_{\Sigma} + (\tau^{-1} \langle \mathcal{R}, \nabla_X \mathcal{Y} \rangle) \mid_{\Sigma} , \Rightarrow \]

\[ \Rightarrow (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle)_{\text{Ant}} \mid_{\Sigma} = \]

\[ = \frac{1}{2} \{ X(\rho(Y)) - Y(\rho(X)) \} - (\tau^{-1} \langle \mathcal{R}, \nabla_X \mathcal{Y} \rangle)_{\text{Ant}} \mid_{\Sigma} = \frac{1}{2} d\rho(X, Y) , \]

from which we obtain (using (25), (6) and (7)):

\[ \nabla^\Sigma_X Y = D^X_X Y + \{ X(\rho(Y)) - (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle) \mid_{\Sigma} \} R = \]

\[ = \tilde{\nabla}_X Y - (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle) \mid_{\Sigma} R = \tilde{\nabla}_X Y - \frac{1}{2} d\rho(X, Y) R - (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle)_{\text{Sim}} \mid_{\Sigma} R = \]

\[ = \tilde{\nabla}_X Y - (\tau^{-1} \langle \nabla_X \mathcal{R}, \mathcal{Y} \rangle)_{\text{Sim}} \mid_{\Sigma} R . \]

It follows from Theorem 6 that \( \nabla^\Sigma \) is an admissible, metric connection and that all admissible connections on \( \Sigma \) have the same covariant curvature as \( \nabla^\Sigma \). We finally compute that covariant curvature:

**Theorem 24** Let \((M, g)\) be a transverse Riemann-Lorentz space with tangent radical and singular, \( II \)-flat hypersurface \( \Sigma \). Let \( R^\Sigma \) be the curvature of the tangential connection \( \nabla^\Sigma \) and let be \( X, Y, Z, T \in \mathfrak{X}(\Sigma) \). Then it holds (Gauss formula):

\[ \langle R^\Sigma(X, Y)Z, T \rangle = \langle R(X, Y)Z, T \rangle - \det \left( \begin{array}{cc} \mathcal{H}(X, Z) & \mathcal{H}(Y, Z) \\ \mathcal{H}(X, T) & \mathcal{H}(Y, T) \end{array} \right) . \]

**Proof** By \( II \)-flatness and Theorem 13, \( \langle R(X, Y)Z, T \rangle \) is well-defined. We compute it using the canonical extensions \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) of \( X, Y, Z \). Taking into account that (by (134)):

\[ \nabla_Y Z \mid_{\Sigma} = \nabla^\Sigma_Y Z + \mathcal{H}(Y, Z)N , \Rightarrow \]

\[ \Rightarrow \Box_X (\nabla_Y Z \mid_{\Sigma}) (T) = \Box_X (\nabla^\Sigma_Y Z) (T) - \mathcal{H}(Y, Z) \mathcal{H}(X, T) = \]

\[ = \langle \nabla^\Sigma_X (\nabla^\Sigma_Y Z), T \rangle - \mathcal{H}(Y, Z) \mathcal{H}(X, T) , \]

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we obtain:

\[ \langle R(X,Y)Z, T \rangle = \square_X (\nabla_Y Z \mid_{\Sigma}) (T) - \square_Y (\nabla_X Z \mid_{\Sigma}) (T) - \square_{[X,Y]} Z (T) = \]

\[ = \langle \nabla_X^\Sigma (\nabla_Y^\Sigma Z), T \rangle - \mathcal{H}(Y,Z)\mathcal{H}(X,T) - \]

\[ - \langle \nabla_Y^\Sigma (\nabla_X^\Sigma Z), T \rangle + \mathcal{H}(X,Z)\mathcal{H}(Y,T) - \langle \nabla_{[X,Y]} Z, T \rangle = \]

\[ = \langle R^\Sigma(X,Y)Z, T \rangle + \text{det} \begin{pmatrix} \mathcal{H}(X,Z) & \mathcal{H}(Y,Z) \\ \mathcal{H}(X,T) & \mathcal{H}(Y,T) \end{pmatrix} \quad \blacksquare \]

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