HARDY-TYPE INEQUALITIES AND PRINCIPLE FREQUENCY OF THE $p$–LAPLACIAN

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ABSTRACT. We prove a sharp $L^p$ weighted Hardy inequality involving boundary distance $\delta$ for any domain $\Omega \subset \mathbb{R}^n$. The inequality may be improved substantially under the additional assumption that $-\log \delta$ is subharmonic. Applications of these inequalities to the principle frequency of the $p$–Laplacian are given.

1. INTRODUCTION

In 1920, Hardy [5] discovered the following famous inequality:

$$\left(1 - \frac{1}{p}\right)^p \int_0^\infty \frac{F(x)^p}{x^p} \, dx \leq \int_0^\infty f(x)^p \, dx,$$

where $F(x) = \int_a^x f(t) \, dt$ and $f \geq 0$. A modern version of Hardy’s inequality for a domain $\Omega \subset \mathbb{R}^n$ may be stated as follows

$$\left(1 - \frac{1}{p}\right)^p \int_\Omega \frac{|f|^p}{\delta^p} \, dx \leq \int_\Omega |\nabla f|^p \, dx, \quad f \in C_0^\infty(\Omega),$$

where $\delta$ denotes the boundary distance of $\Omega$; this holds whenever $-\delta$ is subharmonic on $\Omega$ (e.g., $\Omega$ is convex), and the constant is sharp. The analogue of this inequality for $\mathbb{R}^n$ is

$$\left(\frac{n}{p} - 1\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} \, dx \leq \int_{\mathbb{R}^n} |\nabla f|^p \, dx, \quad f \in C_0^\infty(\mathbb{R}^n),$$

for all $1 \leq p < n$ and the constant is sharp. These inequalities have been generalized and improved in different ways by a large number of authors and many applications have been found; we refer the reader to the book [1] by Balinsky et. al. for the references until 2015 and the article of Kutev and Rangelov [9] for the references up to now.

In this note we will prove the following weighted Hardy inequality:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a domain. Then we have

$$\left(\frac{\alpha + p - n}{p}\right)^p \int_\Omega \frac{|f|^p}{\delta^{\alpha+p}} \, dx \leq \int_\Omega |\nabla f|^p \, dx, \quad f \in C_0^\infty(\Omega),$$

for all $1 \leq p < \infty$ and $\alpha > \max\{n - p, 2 - p\}$. Moreover, the constant $\left(\frac{\alpha+p-n}{p}\right)^p$ is sharp.

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Remark. If \( n = 1 \) and \( \Omega = (0, \infty) \), then we obtain the following one-dimensional weighted Hardy inequality
\[
\left( \frac{\alpha + p - 1}{p} \right)^p \int_0^\infty \frac{|f(x)|^p}{x^{\alpha + \alpha}} \, dx \leq \int_0^\infty \frac{|f'(x)|^p}{x^\alpha} \, dx, \quad f \in C_0^\infty((0, \infty)).
\]
for \( \alpha > 2 - p \), which even holds for \( \alpha > 1 - p \) (cf. [6], Theorem 330).

The case \( \alpha = 0 \) is the most interesting.

Corollary 1.2. Let \( \Omega \subsetneq \mathbb{R}^n \) be a domain with \( n \geq 2 \). Then we have
\[
(1.4) \quad \left( \frac{1}{p} - \frac{n}{p} \right)^p \int_\Omega \frac{|f|^p}{\delta^p} \, dx \leq \int_\Omega |\nabla f|^p \, dx, \quad f \in C_0^\infty(\Omega),
\]
for all \( p > n \), and the constant \( \left( \frac{1}{p} - \frac{n}{p} \right)^p \) is sharp.

Remark. (1) Lewis [10] proved
\[
C_{n,p} \int_\Omega |f|^p / \delta^p \leq \int_\Omega |\nabla f|^p
\]
for \( p > n \), where the constant \( C_{n,p} \) is not explicit.

(2) The number \( 1 - \frac{n}{p} \) has already appeared in Morrey’s inequality:
\[
|f(x) - f(y)| \leq C_{n,p}|x - y|^{1 - \frac{n}{p}} \|
abla f\|_{L^p(B)}, \quad f \in W^{1,p}(B),
\]
where \( B \) is a ball in \( \mathbb{R}^n \) and \( W^{1,p} \) denotes the standard Sobolev space.

Theorem 1.1 may be improved under certain additional conditions.

Theorem 1.3. Let \( \Omega \subsetneq \mathbb{R}^n \) be a domain. If \( -\log \delta \) is subharmonic on \( \Omega \), then we have
\[
(1.5) \quad \left( \frac{\alpha + p - 2}{p} \right)^p \int_\Omega \frac{|f|^p}{\delta^p} \, dx \leq \int_\Omega |\nabla f|^p \, dx, \quad f \in C_0^\infty(\Omega),
\]
for any \( 1 \leq p < \infty \) and \( \alpha > \max\{2 - p, 0\} \). Moreover, the constant \( \left( \frac{\alpha + p - 2}{p} \right)^p \) is sharp.

The condition that \( -\log \delta \) is subharmonic is motivated by function theory of several complex variables. A fundamental result of Oka states that \( -\log \delta \) is plurisubharmonic (hence is subharmonic) for all pseudowconvex domains \( \Omega \subsetneq \mathbb{C}^n \) (cf. [7]). A geometric characterization of bounded pseudoconvex domains with \( C^2 \)-boundary is that the complex Hessian of some/any defining function is semi-positive on every holomorphic tangent space of the domain.

As a popular application of Hardy-type inequalities, we shall investigate the principle frequency (= the first eigenvalue) of the \( p \)–Laplacian \( \Delta_p \), which is defined by
\[
\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u).
\]
Note that the \( p \)–Laplacian is nonlinear except for the case \( p = 2 \), which corresponds to the classical Laplace operator \( \Delta \). Let \( W^{1,p}_0(\Omega) \) denote the closure of \( C_0^\infty(\Omega) \) under the norm of \( W^{1,p}(\Omega) \). We consider the following eigenvalue problem with Dirichlet boundary condition
\[
\Delta_p u + \lambda |u|^{p-2}u = 0.
\]
We say that $\lambda$ is an eigenvalue of $-\Delta_p$ if the above equation has a nontrivial weak solution (eigenfunction) $u_\lambda \in W^{1,p}_0(\Omega)$, i.e.,
\[
\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla f = \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda f, \quad f \in C_0^\infty(\Omega).
\]
For $n \geq 2$ it is known that the first eigenvalue $\lambda_p(\Omega)$ of $-\Delta_p$ can be characterized by
\[
(1.6) \quad \lambda_p(\Omega) = \inf_{f \in C_0^\infty(\Omega) \setminus \{0\}} \left\{ \frac{\int_\Omega |\nabla f|^p dx}{\int_\Omega |f|^p dx} \right\}.
\]
We refer the reader to Lindqvist [11, 12] for an introduction to the $p$-Laplace equation and the $p$-Laplace eigenvalue problem.

Let $R_\Omega$ be the inradius of $\Omega$, i.e., the radius of the largest ball inscribed in $\Omega$.

**Theorem 1.4.**
(1) Let $\Omega \subseteq \mathbb{R}^n$ be a domain with $n \geq 2$. Then we have
\[
(1.7) \quad - \frac{n}{p} \leq R_\Omega \cdot \lambda_p(\Omega)^{\frac{1}{p}} - 1 \leq \frac{C_n \log p}{p}, \quad p > n.
\]
(2) Suppose furthermore that $-\log \delta$ is subharmonic. Then we have
\[
(1.8) \quad - \frac{2}{p} \leq R_\Omega \cdot \lambda_p(\Omega)^{\frac{1}{p}} - 1 \leq \frac{C_n \log p}{p}, \quad p > 2.
\]

**Remark.**
(1) Juutinen et al. [8] obtained the asymptotic formula $R_\Omega \cdot \lambda_p(\Omega)^{\frac{1}{p}} \to 1$ as $p \to \infty$.
(2) Poliquin [13] proved $R_\Omega \cdot \lambda_p(\Omega)^{\frac{1}{p}} \geq C_{n,p}$ for $p > n$, where the constant $C_{n,p}$ is not explicit.

Now we consider the stability problem of the principle frequency about variations of the domain. It is known [12] that if $\Omega_1 \subset \Omega_2 \subset \cdots$ is an exhaustion of $\Omega$ then
\[
(1.9) \quad \lim_{j \to \infty} \lambda_p(\Omega_j) = \lambda_p(\Omega).
\]
It is natural to ask whether (1.9) holds for a decreasing sequence of domains.

**Theorem 1.5.**
(1) Let $\{\Omega_j\}$ be a sequence of proper domains in $\mathbb{R}^n$ such that $\Omega_j \supset \Omega$ and
\[
(1.10) \quad \eta_j := \max_{x \in \partial \Omega} \{\delta_j(x)\} \to 0 \quad (j \to \infty)
\]
where $\delta_j$ denotes the boundary distance of $\Omega_j$. Then (1.9) holds for all $p > n$.
(2) Suppose furthermore that $-\log \delta_j$ is subharmonic. Then (1.9) holds for all $p > 2$.

**Remark.** The condition (1.10) is satisfied for instance, when the Hausdorff distance between $\partial \Omega$ and $\partial \Omega_j$ tends to 0 as $j \to \infty$.

Davis [2, 3] realized that the classical Hardy inequality (1.2) for $p = 2$ can be used to obtain explicit upper bounds on the rate of
\[
|\lambda_2(\Omega) - \lambda_2(\Omega_\varepsilon)|,
\]
where $\Omega_{\varepsilon} = \{ x \in \Omega : \delta(x) > \varepsilon \}$ for $\varepsilon > 0$. The results in Davies [3] were generalized to arbitrary values of $p$ by Fleckinger et. al. [4]. One of the results in [4] states that if $\Omega$ is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ such that
\[ \int_{\Omega} \frac{|f|^p}{\delta^p} dx \leq c_p^p \int_{\Omega} |\nabla f|^p dx, \quad f \in C_0^\infty(\Omega), \]
holds for some $p \geq 2$ and $c_p > 0$, then
\[ \lambda_p(\Omega) \leq \lambda_p(\Omega_{\varepsilon}) \leq \lambda_p(\Omega) + C_\varepsilon^{-\frac{2-p}{2p}} \]
where the constant $C$ depends only on $n, p$ and $\Omega$. This combined with (1.4) and (1.5) gives

**Theorem 1.6.** (1) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Then we have

\[ \lambda_p(\Omega) \leq \lambda_p(\Omega_{\varepsilon}) \leq \lambda_p(\Omega) + C_\varepsilon^{-\frac{2-p}{2p}}, \quad p > n. \]

(2) Suppose furthermore that $-\log \delta$ is subharmonic. Then we have

\[ \lambda_p(\Omega) \leq \lambda_p(\Omega_{\varepsilon}) \leq \lambda_p(\Omega) + C_\varepsilon^{-\frac{2-p}{2p}}, \quad p > 2. \]

2. HARDY-TYPE INEQUALITIES

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For every $\beta \in \mathbb{R}$ we define $\chi_\beta(t) = t^{1-\beta/2}, t > 0$.

**Lemma 2.1.** For $\beta \geq 2$ the inequality

\[ \Delta \chi_\beta(\delta^2) \geq (\beta - 2)(\beta - n)\delta^{-\beta} \]

holds in the sense of distributions in $\Omega$.

**Proof.** For every $a \in \mathbb{R}^n$, the function
\[ u_a(x) := -|x-a|^2 + |x|^2 \]
is harmonic in $\mathbb{R}^n$. Since
\[ \delta(x) = \min_{a \in \partial \Omega} \{|x-a|\}, \]
it follows that
\[ -\delta^2(x) + |x|^2 = \max_{a \in \partial \Omega} \{u_a(x)\} \]
is subharmonic in $\Omega$, i.e.,
\[ -\Delta \delta^2 \geq -2n \]
holds in the sense of distributions in $\Omega$. Thus we have
\[ \Delta \chi_\beta(\delta^2) = \chi''_\beta(\delta^2)|\nabla \delta^2|^2 + \chi'_\beta(\delta^2)\Delta \delta^2 \]
\[ = \beta(\beta - 2)\delta^{-\beta}|\nabla \delta|^2 + (1 - \beta/2)\delta^{-\beta} \Delta \delta^2 \]
\[ \geq (\beta - 2)(\beta - n)\delta^{-\beta} \]
since $|\nabla \delta| = 1$ a.e. on $\Omega$. \qed
Proof of Theorem 1.1. By (2.1) we have
\[(\beta - 2)(\beta - n) \int_\Omega |f|^p / \delta^\beta \leq \int_\Omega \Delta \chi_\beta(\delta^2) \cdot |f|^p = -\int_\Omega \nabla \chi_\beta(\delta^2) \cdot \nabla |f|^p = p(\beta - 2) \int_\Omega \delta^{1-\beta} |f|^{p-1} \nabla \delta \cdot \nabla |f|,\]
so that if \(\beta > 2\) then
\[
\frac{\beta - n}{p} \int_\Omega |f|^p / \delta^\beta \leq \int_\Omega \delta^{1-\beta} |f|^{p-1} \nabla \delta \cdot \nabla |f|
\leq \left[ \int_\Omega |f|^p \nabla \delta \right]^{\frac{p-1}{p}} \left[ \int_\Omega |f|^{p/\delta^\beta-p} \right]^{\frac{1}{p}}
\leq \left[ \int_\Omega |f|^p / \delta^\beta \right]^{\frac{p-1}{p}} \left[ \int_\Omega |\nabla f|^{p/\delta^\beta-p} \right]^{\frac{1}{p}}.
\]

Take \(\beta = p + \alpha\), we immediately obtain (1.3).

To see that the constant \(\left(\frac{\alpha + p - n}{p}\right)^p\) is sharp, we take
\[\Omega = B_2^* := \{ x \in \mathbb{R}^n : 0 < |x| < 2 \}.\]
For every \(0 < \varepsilon < 1\) we set \(\gamma_\varepsilon = \frac{\alpha + p - n}{p} + \varepsilon\). We choose a test function \(f_\varepsilon\) with compact support in \(B_2\) such that \(f_\varepsilon = |x|^{\gamma_\varepsilon}\) on \(B_1^*\). Let \(\sigma_n\) denote the area of the unit sphere in \(\mathbb{R}^n\). By using spherical coordinates, we obtain
\[
\int_{B_2^*} |f_\varepsilon|^p / \delta^{p+\alpha} = \int_{B_1^*} |f_\varepsilon|^p / \delta^{p+\alpha} + O(1)
= \sigma_n \int_0^1 r^{p\gamma_\varepsilon - p - \alpha + n - 1} dr + O(1)
= \sigma_n \int_0^1 r^{-1+p\varepsilon} dr + O(1)
= \sigma_n (p\varepsilon)^{-1} + O(1).
\]
Analogously, we have
\[
\int_{B_2^*} |\nabla f_\varepsilon|^p / \delta^\alpha = \sigma_n \gamma_\varepsilon^p \int_0^1 r^{p(\gamma_\varepsilon - 1) - \alpha + n - 1} dr + O(1)
= \sigma_n \gamma_\varepsilon^p (p\varepsilon)^{-1} + O(1).
\]
Thus
\[
\lim_{\varepsilon \to 0^+} \int_{B_2^*} |\nabla f_\varepsilon|^p / \delta^\alpha / \int_{B_2^*} |f_\varepsilon|^p / \delta^{p+\alpha} = \left(\frac{\alpha + p - n}{p}\right)^p.
\]
Since every $f_\varepsilon$ may be approximated by functions in $C_0^\infty(B_2^*)$ under the norm
\[
\left( \int_{B_2^*} |f|^{p/\delta^{p+\alpha}} \right)^{1/p} + \left( \int_{B_2^*} |\nabla(\cdot)|^p/\delta^\alpha \right)^{1/p},
\]
we are done. □

**Proof of Theorem 1.3.** Since $-\log \delta$ is subharmonic, we have
\[
\Delta \delta^{-\alpha} = \Delta (e^{-\alpha \log \delta}) \geq \frac{\alpha^2}{\delta^{2+\alpha}},
\]
which holds in the sense of distributions in $\Omega$. Hence
\[
\alpha^2 \int_\Omega \frac{|f|}{\delta^{p+\alpha}} \leq \int_\Omega \Delta \delta^{-\alpha} \cdot (|f|/\delta^{p-2}) = -\int_\Omega \nabla \delta^{-\alpha} \cdot \nabla (|f|/\delta^{p-2}) = \alpha p \int_\Omega \frac{|f|^{p-1}}{\delta^{p+\alpha-1}} \nabla \delta \cdot \nabla |f| + \alpha (2-p) \int_\Omega |f|/\delta^{p+\alpha},
\]
so that
\[
\frac{\alpha + p - 2}{p} \int_\Omega |f|^{p/\delta^{p+\alpha}} \leq \int_\Omega \frac{|f|^{p-1}}{\delta^{p+\alpha-1}} \nabla \delta \cdot \nabla |f| \leq \left( \int_\Omega |f|^{p/\delta^{p+\alpha}} \right)^{p-1} \left( \int_\Omega \nabla |f|/\delta^\alpha \right)^{\frac{2}{p}},
\]
which yields (1.5).

To see that the constant $\left( \frac{\alpha + p - 2}{p} \right)^p$ is sharp, we take
\[
\Omega := (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-2} \subset \mathbb{R}^n.
\]
Let $x' = (x_1, x_2)$ and $x'' = (x_3, \cdots, x_n)$. We set
\[
B'_r = \{x' : |x'| < r\} \text{ and } B''_r = \{x'' : |x''| < r\}.
\]
Since $\delta(x) = |x'|$, it follows that $-\log \delta$ is harmonic on $\Omega$. For $0 < \varepsilon < 1$ we set $\gamma_\varepsilon = \frac{\alpha + p - 2}{p} + \varepsilon$.

We choose a function $g_\varepsilon$ with compact support in $B'_2$ such that $g_\varepsilon(x') = |x'|^{\gamma_\varepsilon}$ on $B'_1$. Let $\kappa \in C_0^\infty(B_2'')$ be a nonnegative function satisfying $\kappa = 1$ on $B''_1$. For the test function $f_\varepsilon := g_\varepsilon \cdot \kappa$, we have
\[
\int_\Omega |f_\varepsilon|^{p/\delta^{p+\alpha}} = \int_{B'_r \times \mathbb{R}^{n-2}} |f_\varepsilon|^{p/\delta^{p+\alpha}} + O(1)
\]
\[
= 2\pi \int_{\mathbb{R}^{n-2}} \kappa^p \cdot \int_0^1 r^{p\gamma_\varepsilon - p - \alpha + 1} dr + O(1)
\]
\[
= 2\pi (p\varepsilon)^{-1} \int_{\mathbb{R}^{n-2}} \kappa^p + O(1).
\]
Analogously, we have 
\[
\int_{\Omega} |\nabla f_\varepsilon|^p / \delta^\alpha = \int_{B_1' \times \mathbb{R}^{n-2}} |\nabla f_\varepsilon|^p / \delta^\alpha + O(1)
\]
\[
= 2\pi \gamma_p(p\varepsilon)^{-1} \int_{\mathbb{R}^{n-2}} \kappa^p + O(1).
\]
Thus
\[
\lim_{\varepsilon \to 0^+} \frac{\int_\Omega |\nabla f_\varepsilon|^p / \delta^\alpha}{\int_\Omega |f_\varepsilon|^p / \delta^{p+\alpha}} = \left(\frac{\alpha + p - 2}{p}\right)^p.
\]

3. PRINCIPLE FREQUENCY OF THE \( p \)-LAPLACIAN

**Proof of Theorem 1.4.** Since 
\[
R_\Omega = \max_{z \in \Omega} \{\delta(z)\},
\]
it follows from (1.4) that
\[
\left(1 - \frac{n}{p}\right)^p R_\Omega^{-p} \int_\Omega |f|^p dx \leq \int_\Omega |\nabla f|^p dx, \quad \forall f \in C_0^\infty(\Omega).
\]
This combined with (1.6) gives the first inequality in (1.7).

For the second inequality in (1.7) we first infer from the domain monotonicity property that
\[
\lambda_p(\Omega) \leq \lambda_p(B_1) R_\Omega^{-p}
\]
where \( B_1 \) is the unit ball in \( \mathbb{R}^n \). By using the test function \( f = \chi(|x|) \) where \( \chi \) is a Lipschitz continuous function on \( \mathbb{R} \) with \( \chi|_{[1,\infty)} = 0 \) and spherical coordinates, we have
\[
\lambda_p(B_1) \leq \frac{\int_{B_1} |\nabla f|^p dx}{\int_{B_1} |f|^p dx} \leq \frac{\int_0^1 |\chi'(t)|^p t^{n-1} dt}{\int_0^1 |\chi(t)|^p t^{n-1} dt}.
\]
Take \( \chi(t) = 1 - t \), we obtain
\[
\lambda_p(B_1) \leq \frac{n^{-\frac{1}{p}}}{(p + 1)(p + 2) \cdots (p + n) \cdot \left(p + \frac{1}{p}\right)^n} \leq 1 + \frac{C_n \log p}{p}
\]
for suitable constant \( C_n > 0 \) depending only on \( n \). This combined with (3.1) yields the second inequality in (1.7).

By using Theorem 1.3 instead of Corollary 1.2, we obtain (1.8). \( \square \)

**Proof of Theorem 1.5.** Fix a number \( 0 < \varepsilon < 1 \). Let \( \chi : \mathbb{R} \to [0,1] \) be a cut-off function such that \( \chi|_{(-\infty,1/2]} = 1 \) and \( \chi|_{[1,\infty)} = 0 \). For each \( f \in C_0^\infty(\Omega_j) \setminus \{0\} \) we set
\[
f_{\varepsilon,j} := \chi \left(\frac{\log \delta_j}{\log \varepsilon}\right) f.
\]

\[
\int_{\Omega} |\nabla f_\varepsilon|^p / \delta^\alpha = \int_{B_1' \times \mathbb{R}^{n-2}} |\nabla f_\varepsilon|^p / \delta^\alpha + O(1)
\]
\[
= 2\pi \gamma_p(p\varepsilon)^{-1} \int_{\mathbb{R}^{n-2}} \kappa^p + O(1).
\]
Since \( \text{supp} f_{\varepsilon,j} \subset \{ x \in \Omega_j : \delta_j(x) \geq \varepsilon \} \) and \( \eta_j \to 0 \), we conclude that \( \text{supp} f_{\varepsilon,j} \subset \Omega \) for \( j \geq j_\varepsilon \gg 1 \), so that
\[
\lambda_p(\Omega) \leq \frac{\| \nabla f_{\varepsilon,j} \|_{L^p(\Omega)}}{\| f_{\varepsilon,j} \|_{L^p(\Omega)}} \leq \frac{\| \nabla f \|_{L^p(\Omega_j)} + \| f \nabla \chi(\log \delta_j/\log \varepsilon) \|_{L^p(\Omega)}}{\| f \|_{L^p(\Omega_j)} - (\int_{\delta_j \leq \sqrt{\varepsilon}} |f|^p)^{1/p}}.
\]

Note that
\[
\left(1 - \frac{n}{p}\right)^{-p} \int_{\Omega_j} |\nabla f|^p \geq \int_{\Omega_j} |f|^p/\delta_j^p \geq \int_{\delta_j \leq \sqrt{\varepsilon}} |f|^p/\delta_j^p \geq \varepsilon^{-p/2} \int_{\delta_j \leq \sqrt{\varepsilon}} |f|^p
\]
and
\[
\int_{\Omega_j} |f|^p |\nabla \chi(\log \delta_j/\log \varepsilon)|^p \leq \sup \frac{|\chi'|}{|\log \varepsilon|} \int_{\varepsilon \leq \delta_j \leq \sqrt{\varepsilon}} |f|^p/\delta_j^p \leq \left(1 - \frac{n}{p}\right)^{-p} \sup \frac{|\chi'|}{|\log \varepsilon|} \int_{\Omega_j} |\nabla f|^p.
\]

Thus we have
\[
\lambda_p(\Omega) \leq \frac{\| \nabla f \|_{L^p(\Omega_j)} + \left(1 - \frac{n}{p}\right)^{-1} \sup \frac{|\chi'|}{|\log \varepsilon|} \| \nabla f \|_{L^p(\Omega_j)}}{\| f \|_{L^p(\Omega_j)} - \left(1 - \frac{n}{p}\right)^{-1} \sqrt{\varepsilon} \cdot \| \nabla f \|_{L^p(\Omega_j)}}.
\]

Since we can choose \( f \in C_0^\infty(\Omega_j) \) such that the quotient \( \| \nabla f \|_{L^p(\Omega_j)}/\| f \|_{L^p(\Omega_j)} \) is arbitrarily close to \( \lambda_j(\Omega)^{1/p} \), we obtain
\[
\lambda_p(\Omega) \leq \frac{\lambda_p(\Omega_j)^{1/p} \left(1 + \left(1 - \frac{n}{p}\right)^{-1} \sup \frac{|\chi'|}{|\log \varepsilon|}\right)}{1 - \left(1 - \frac{n}{p}\right)^{-1} \sqrt{\varepsilon} \cdot \lambda_p(\Omega_j)^{1/p}}.
\]

for \( j \geq j_\varepsilon \gg 1 \), that is,
\[
\lambda_p(\Omega_j)^{1/p} \geq \frac{\lambda_p(\Omega)^{1/p}}{1 + \left(1 - \frac{n}{p}\right)^{-1} \sup \frac{|\chi'|}{|\log \varepsilon|} + \left(1 - \frac{n}{p}\right)^{-1} \sqrt{\varepsilon} \cdot \lambda_p(\Omega)^{1/p}}.
\]

On the other hand, the domain monotonicity property implies that
\[
\lambda_p(\Omega_j) \leq \lambda_p(\Omega), \quad \forall j.
\]

Thus we obtain
\[
\lim_{j \to \infty} \lambda_p(\Omega_j) = \lambda_p(\Omega).
\]

The second assertion can be proved by using Theorem 1.3 instead of Corollary 1.2. \(\square\)
REFERENCES

[1] A. A. Balinsky, W. D. Evans and R. T. Lewis, The analysis and geometry of Hardy’s inequality, Springer International Publishing Switzerland 2015.

[2] E. B. Davies, Eigenvalue stability bounds via weighted Sobolev spaces, Math. Z. 214 (1993), 357–371.

[3] E. B. Davies, Sharp boundary estimates for elliptic operators, Math. Proc. Cambridge Philos. Soc. 129 (2000), 165–178.

[4] J. Fleckinger, E. M. Harrell and F. de Thélin, Boundary behavior and estimates for solutions for equations containing the $p$–Laplacian, Electron. J. Diff. Equations 38 (1999), 1–19.

[5] G. H. Hardy, Note on a theorem of Hilbert, Math. Z. 6 (1920), 314–317.

[6] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, 1934.

[7] L. Hörmander, An introduction to complex analysis in several complex variables, Third Edition (Revised), Elsevier, 1990.

[8] P. Juutinen, P. Lindqvist and J. Manfredi, The $\infty$–eigenvalue problem, Arch. Rat. Mech. Anal. 148 (1999), 89–105.

[9] K. Kutev and T. Rangelov, Hardy inequalities with double singular weights, arXiv: 2001.07368v2.

[10] J. L. Lewis, Uniformly fat sets, Trans. Amer. Math. Soc. 308 (1988), 177–196.

[11] P. Lindqvist, Notes on the $p$–Laplace equation, Lectures at University of Jyväskylä, 2006.

[12] P. Lindqvist, A nonlinear eigenvalue problem, In: Topics in Mathematical Analysis (P. Ciatti et. al. eds), Series on Analysis, Applications and Computation-Vol.3, World Scientific Publishing Co. Pte. Ltd. (2008), 175–203.

[13] G. Poliquin, Principal frequency of the $p$-Laplacian and the inradius of Euclidean domains, J. Top. Anal. 7 (2015), 505–511.

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