Geodesic equivalence and integrability.

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Abstract
We suggest a construction that, given a trajectorial diffeomorphism between two Hamiltonian systems, produces integrals of them. As the main example we treat geodesic equivalence of metrics. We show that the existence of a non-trivially geodesically equivalent metric leads to Liouville integrability, and present explicit formulae for integrals.

1 Introduction
Integrals of a system are closely related to symmetries. A classical example is Noether’s theorem: if a vector field $X$ on a manifold $M$ preserves a Lagrangian $L : TM \rightarrow \mathbb{R}$, then the function $I_X \overset{def}{=} \frac{d}{dt}(x, \dot{x})X(x)$ is a first integral of the corresponding Lagrangian system.

There are many generalizations of Noether’s theorem, we recall the following two. In the paper [2] it was shown that the existence of a vector field on $T^*M$ which commutes with a Hamiltonian vector field allows one to construct a (multi-valued) integral of the Hamiltonian system. In the paper [11] the result of [2] was generalized to tensor fields. It was shown that if a Hamiltonian flow preserves a tensor field on $T^*M$, then there exists an (also multi-valued) integral of the Hamiltonian system.

In our paper we, following ideas of [11], present a construction which, given a diffeomorphism between two Hamiltonian systems that takes the trajectories and the isoenergy surfaces of the first Hamiltonian system to the trajectories and the isoenergy surfaces of the second one, produces $n$ integrals of the first system, where $n$ is the number of the degrees of freedom of the system.

The construction is applied to geodesically equivalent metrics. Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be smooth metrics on the same manifold $M^n$.

Definition 1. The metrics $g$ and $\bar{g}$ are geodesically equivalent, if they have the same geodesics (considered as unparameterized curves).

This is rather classical material. In 1869 Dini [3] formulated the problem of local classification of geodesically equivalent metrics, and solved it for dimension two. In 1896 Levi-Civita [4] got a local description of geodesically equivalent metrics on manifolds of arbitrary dimension. In the paper [6] a family of (non-trivial) examples of geodesically equivalent metrics on closed manifolds was constructed.
For geodesically equivalent metrics, a trajectorial diffeomorphism Φ is given by Φ(x, ξ) = (x, ∥ξ∥g−1). Here (x, ξ) ∈ TM^n, x is a point of M^n and ξ ∈ TxM^n.

Theorem 1. Let metrics g and ¯g on M^n be geodesically equivalent. Denote by G the linear operator g−1 ¯g = (g^a^b^c^d) = (g^a^b^c^d). Consider the characteristic polynomial det(G − µE) = c_0µ^n + c_1µ^{n−1} + ... + c_n. The coefficients c_1, ..., c_n are smooth functions on the manifold M^n, and c_0 ≡ (−1)n. Then the functions I_k = \left(\frac{det(g)}{det(\bar{g})}\right)^{\frac{k+2}{2n+1}} \bar{g}(S_kξ, ξ), k = 0, ..., n−1, where S_k = \sum_{i=0}^{k} c_iG^{k−i}, are integrals of the geodesic flow of the metric g and pairwise commute.

Remark 1. The integral I_0 = \left(\frac{det(g)}{det(\bar{g})}\right)^{\frac{2n+1}{2}} \bar{g}(ξ, ξ) was obtained by Painlevé, see [4]. The integral I_{n−1} is the energy integral (multiplied by minus two).

The integrals I_1, I_2, ..., I_{n−2} seem to be new, although in each Levi-Civita chart the integrals are linear combinations of Levi-Civita integrals (see Section 3 for definitions). We touch on the connection between the integrals I_0, ..., I_{n−1} and Levi-Civita integrals in Section 5.

Metrics g, ¯g on M^n are strictly non-proportional at a point x ∈ M^n, if the characteristic polynomial \frac{1}{det(g)} det(\bar{g} − tg)|_x has no multiple root.

Corollary 1. Let M^n be a closed real-analytic manifold supplied with two real-analytic metrics g, ¯g such that the metrics g, ¯g are geodesically equivalent and strictly non-proportional at least at one point. Then the fundamental group π_1(M^n) of the manifold M^n contains a commutative subgroup of finite index, and the dimension of the homology group H_1(M^n; Q) is no greater than n.

For dimension two the converse of Theorem 1 is also true, and the condition of Corollary 1 can be weakened.

Corollary 2. Metrics g and ¯g on a surface M^2 are geodesically equivalent, if and only if the function \left(\frac{det(g)}{det(\bar{g})}\right)^{\frac{1}{2}} \bar{g}(ξ, ξ) is an integral of the geodesic flow of the metric g.

Corollary 3. Let metrics g, ¯g on a closed surface of negative Euler characteristic be geodesically equivalent. Then g = C\bar{g}, where C is a constant.

Corollary 4. Let metrics g, ¯g on the torus T^2 be geodesically equivalent. If they are proportional at a point x ∈ T^2, then g = C\bar{g}, where C is a positive constant.

Corollary 5. Let metrics g, ¯g on the sphere S^2 be geodesically equivalent. Then there are three possibilities.
1. The metrics are proportional at exactly two points.
2. The metrics are proportional at exactly four points.
3. The metrics are completely proportional, i.e. \( g = C\bar{g} \), where \( C \) is a positive constant.

In the first case the metrics admit a Killing vector field.

Recall that a vector field on \( M^n \) is Killing (with respect to a metric), if the flow of the field preserves the metric.

**Corollary 6.** Let metrics \( g, \bar{g} \) on a surface \( M^2 \) be geodesically equivalent. If the metrics are proportional at each point of an open non-empty domain \( U \subset M^2 \), then \( g = C\bar{g} \), where \( C \) is a positive constant.

**Corollary 7.** If metrics \( g, \bar{g} \) on a manifold \( M^n \) are geodesically equivalent, and if the metric \( g \) admits a non-trivial Killing vector field, then the metric \( \bar{g} \) also admits a non-trivial Killing vector field.

One of the most famous integrable geodesic flows on closed surfaces is the geodesic flow of the metric on ellipsoid (see [7]). Consider the ellipsoid \( \sum_{i=1}^{n} \left( \frac{x_i^2}{a_i} \right) = 1 \), where \( a_i > 0, \ i = 1, ..., n \).

**Theorem 2.** The restriction of the metric \( \sum_{i=1}^{n} (dx_i)^2 \) to the ellipsoid \( \sum_{i=1}^{n} \left( \frac{x_i^2}{a_i} \right) = 1 \) is geodesically equivalent to the restriction of the metric
\[
\frac{1}{\sum_{i=1}^{n} \left( \frac{x_i^2}{a_i} \right)^2} \left( \sum_{i=1}^{n} \left( \frac{dx_i^2}{a_i} \right) \right)
\]
to the ellipsoid.

The paper is organized as follows. In Section 2 we present the announced construction. Theorem 3 there gives an explicit formula for a one-parameter family of first integrals, if a trajectorial diffeomorphism between two Hamiltonian systems is given.

In Section 3 for use in Sections 4-7 we formulate Levi-Civita and Painlevé results about a local form of geodesically equivalent metrics.

In Section 4 we apply the construction to geodesically equivalent metrics, and prove that the functions \( I_0, ..., I_{n-1} \) from Theorem 2 are integrals of the geodesic flow of the metric \( g \).

In Section 5 we prove that the integrals \( I_0, ..., I_{n-1} \) are in involution.

In Section 6 we prove Corollaries 1, 2, 3, 4, 5, 6, 7.

In Section 7 we prove Theorem 2.

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2 Trajectorial diffeomorphisms and integrals

Let \( v \) and \( \bar{v} \) be Hamiltonian systems on symplectic manifolds \((M, \omega)\) and \((\bar{M}, \bar{\omega})\) with Hamiltonians \( H \) and \( \bar{H} \) respectively. Consider the isoenergy surfaces

\[
Q \overset{\text{def}}{=} \{ x \in M : H(x) = h \}, \quad \bar{Q} \overset{\text{def}}{=} \{ x \in \bar{M} : \bar{H}(x) = \bar{h} \},
\]

where \( h \) and \( \bar{h} \) are regular values of the functions \( H \), \( \bar{H} \) respectively. Let \( U(Q) \subset M \) and \( U(\bar{Q}) \subset \bar{M} \) be neighborhoods of the isoenergy surfaces \( Q \) and \( \bar{Q} \).

**Definition 2.** A diffeomorphism \( \Phi : U(Q) \to U(\bar{Q}) \), \( \Phi(Q) = \bar{Q} \), is said to be trajectorial on \( Q \), if the restriction \( \Phi|_Q \) takes the trajectories of the system \( v \) to the trajectories of the system \( \bar{v} \).

Denote the restriction \( \Phi|_Q \) by \( \phi \). Since \( \phi \) takes the trajectories of \( v \) to the trajectories of \( \bar{v} \), it takes the vector field \( v \) to the vector field that is proportional to \( \bar{v} \). Denote by \( a_1 : Q \to R \) the coefficient of proportionality, i.e. \( \phi_*(v) = a_1 \bar{v} \). Since \( \Phi \) takes \( Q \) to \( \bar{Q} \), it takes the differential \( dH \) to a form that is proportional to \( d\bar{H} \). Denote by \( a_2 : Q \to R \) the coefficient of proportionality, i.e. \( \phi_*dH = a_2 d\bar{H} \). By \( a \) we denote the product \( a_1 a_2 \). We denote the Pfaffian of a skew-symmetric matrix \( X \) by \( \text{Pf}(X) \).

**Theorem 3.** Let a diffeomorphism \( \Phi : U(Q) \to U(\bar{Q}) \), \( \Phi(Q) = \bar{Q} \), be trajectorial on \( Q \). Then for each value of the parameter \( t \) the polynomial

\[
P_{n-1}(t) \overset{\text{def}}{=} \frac{\text{Pf}(\Phi^*\bar{\omega} - t\omega)}{\text{Pf}(\omega)(t - a)}
\]

is an integral of the system \( v \) on \( Q \). In particular, all the coefficients of the polynomial \( P_{n-1}(t) \) are integrals.

**Proof.** Denote by \( \sigma, \bar{\sigma} \) the restrictions of the forms \( \omega, \bar{\omega} \) to \( Q, \bar{Q} \) respectively. Consider the form \( \phi^*\bar{\sigma} \) on \( Q \).

**Lemma 1 (Topalov, [1])**. The flow \( v \) preserves the form \( \phi^*\bar{\sigma} \).

**Proof of Lemma 1**. The Lie derivative \( L_v \) of the form \( \phi^*\bar{\sigma} \) along the vector field \( v \) satisfies

\[
L_v \phi^*\bar{\sigma} = d[\iota_v \phi^*\bar{\sigma}] + \iota_v d[\phi^*\bar{\sigma}] .
\]

On the right side both terms vanish. More precisely, for an arbitrary vector \( u \in \mathcal{T}_xQ \) at an arbitrary point \( x \in Q \) we have

\[
\iota_v \phi^*\bar{\sigma}(u) = \bar{\sigma}(\phi_*(v), \phi_*(u)) = \bar{\sigma}(a_1 \bar{v}, \phi_*(u)) = -a_1 d\bar{H}(\phi_*(u)) = 0.
\]

Since the form \( \bar{\omega} \) is closed, the form \( \bar{\sigma} \) is also closed and \( d[\phi^*\bar{\sigma}] = \phi^*(d\bar{\sigma}) = 0 \), q. e. d.
It is obvious that the kernels of the forms $\sigma$ and $\phi^{*}\bar{\sigma}$ coincide (in the space $T_{x}Q$ at each point $x \in Q$) with the linear span of the vector $v$. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $TQ/\langle v \rangle$. We shall denote the corresponding forms on $TQ/\langle v \rangle$ also by the letters $\sigma, \bar{\sigma}$.

**Lemma 2.** The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^{*}\bar{\sigma})$ on $TQ\langle v \rangle$ is preserved by the flow $v$.

**Proof of Lemma 2.** Since the flow $v$ preserves the Hamiltonian $H$ and the form $\omega$, the flow $v$ preserves the form $\sigma$. Since the flow $v$ preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^{*}\bar{\sigma})$, q. e. d.

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^{*}\bar{\sigma})$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n - 1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow $v$. It is obvious that

$$\delta^{n-1}(t) = (-1)^{n-1}\frac{\text{Pf}(\phi^{*}\bar{\sigma} - t\sigma)}{\text{Pf}(\sigma)}. \quad (2)$$

The last step of the proof is to verify that

$$(t - a)\delta^{n-1} = \frac{\text{Pf}(\Phi^{*}\bar{\omega} - t\omega)}{\text{Pf}(\omega)} \overset{\text{def}}{=} \Delta^n.$$

Take an arbitrary point $x \in Q$. Consider the form $\Phi^{*}\bar{\omega} - a\omega$ on $T_{x}M$. The form $\iota_{v}(\Phi^{*}\bar{\omega} - a\omega)$ equals zero. More precisely, for any vector $u \in T_{x}M$ we have

$$\iota_{v}(\Phi^{*}\bar{\omega} - a\omega) = \bar{\omega}(\Phi_{*}(v), \Phi_{*}(u)) - a\omega(v, u) =$$

$$= \bar{\omega}(a_{1}v, \Phi_{*}(u)) - a\omega(v, u) =$$

$$= -a_{1}dH(\Phi_{*}(u)) + adH =$$

$$= -adH + adH = 0.$$

There exists a vector $A \in T_{x}M$ such that $\omega(A, v) \neq 0$ and the restriction of the form $\iota_{A}(\Phi^{*}\bar{\omega} - a\omega)$ to the space $T_{x}M$ equals zero. More precisely, since the forms $\Phi^{*}\bar{\omega}, \omega$ are skew-symmetric, then the kernel $K_{\Phi^{*}\bar{\omega} - a\omega}$ of the form $\Phi^{*}\bar{\omega} - a\omega$ has an even dimension, and the kernel of the restriction of the form $\Phi^{*}\bar{\omega} - a\omega$ to $T_{x}Q$ has an odd dimension. Thus the intersection $K_{\Phi^{*}\bar{\omega} - a\omega} \cap (T_{x}M \setminus T_{x}Q)$ is not empty. For each vector $A$ from the intersection we obviously have $\omega(A, v) \neq 0$ and $\iota_{A}(\Phi^{*}\bar{\omega} - a\omega) = 0$. Without loss of generality we can assume $\omega(A, v) = 1$.

Consider a basis $(v, e_{1}, ..., e_{2n-2})$ for the space $T_{x}Q$. The set $(A, v, e_{1}, ..., e_{2n-2})$ is a basis for the space $T_{x}M$. In this basis we have

$$\text{det}(\Phi^{*}\bar{\omega} - t\omega) = \text{det}\begin{vmatrix} 0 & a-t & 0 & (\ast) \\ -(a-t) & 0 & 0 & 0 \\ (\ast) & 0 & (\Phi^{*}\bar{\omega} - t\omega)_{(e_{1}, ..., e_{2n-2})} \\ (\ast) & 0 & 0 & 0 \end{vmatrix}$$

$$= (a-t)^2 \text{det}((\Phi^{*}\bar{\omega} - t\omega)_{(e_{1}, ..., e_{2n-2})})$$

$$= (a-t)^2 \text{det}(\phi^{*}\bar{\sigma} - t\sigma),$$

where $(\Phi^{*}\bar{\omega} - t\omega)_{(e_{1}, ..., e_{2n-2})}$ is the matrix of the form $\Phi^{*}\bar{\omega} - t\omega$ in the basis $(e_{1}, ..., e_{2n-2})$. Finally, $\delta^{n-1} = \mathcal{P}^{n-1}$, q. e. d.
3 Levi-Civita theorem

Let \( g \) and \( \tilde{g} \) be smooth metrics on a manifold \( M^n \). Recall that the common eigenvalues of the metrics \( g, \tilde{g} \) at a point \( x \in M \) are roots of the characteristic polynomial \( P_x(t) = \det((G - tE)_{ij}) \), where \( G \) is defined as \( (g^{\alpha \beta} \tilde{g}_{\alpha \beta}) \). Suppose that at every point of an open domain \( D \subset M^n \) the common eigenvalues of the metrics \( g, \tilde{g} \) assume \( m \) distinct values \( \rho^1, \rho^2, \ldots, \rho^m \) (1 \( \leq m \leq n \)) with multiplicities \( k_1, k_2, \ldots, k_m \), respectively.

In the paper \cite{2}, Levi-Civita proved that for every point \( P \in D \) there is an open neighborhood \( U(P) \subset D \) and a coordinate system \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \) (in \( U(P) \)), where \( \bar{x}_i = (x_1^i, \ldots, x_n^i) \), \( 1 \leq i \leq m \), such that the quadratic forms of the metrics \( g \) and \( \tilde{g} \) have the following form:

\[
\begin{align*}
g(\tilde{x}, \tilde{x}) &= \Pi_1(\bar{x})A_1(\bar{x}_1, \bar{x}_1) + \Pi_2(\bar{x})A_2(\bar{x}_2, \bar{x}_2) + \cdots + \Pi_m(\bar{x})A_m(\bar{x}_m, \bar{x}_m), \\
\tilde{g}(\tilde{x}, \tilde{x}) &= \rho^1\Pi_1(\bar{x})A_1(\bar{x}_1, \bar{x}_1) + \rho^2\Pi_2(\bar{x})A_2(\bar{x}_2, \bar{x}_2) + \cdots + \rho^m\Pi_m(\bar{x})A_m(\bar{x}_m, \bar{x}_m),
\end{align*}
\]

where \( A_i(\bar{x}_i, \bar{x}_i) \) are positive-definite quadratic forms in the velocities \( \dot{\bar{x}}_i \) with coefficients depending on \( \bar{x}_i \),

\[
\Pi_i \overset{\text{def}}{=} (\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_i + 1 - \phi_i) \cdots (\phi_m - \phi_i)
\]

and \( \phi_1, \phi_2, \ldots, \phi_m \), \( 0 < \phi_1 < \phi_2 < \ldots < \phi_m \), are smooth functions such that

\[
\phi_i = \begin{cases} \\
\phi_i(\bar{x}_i), & \text{if} \quad k_i = 1 \\
\text{constant}, & \text{else}.
\end{cases}
\]

It is easy to see that the functions \( \rho^i \) as functions of \( \phi_i \) and the function \( \phi_i \) as functions of \( \rho^i \) are given by

\[
\rho^i = \frac{1}{\phi_1 \cdots \phi_m} \frac{1}{\phi_i}
\]

\[
\phi_i = \frac{1}{\rho^i} (\rho^1 \rho^2 \ldots \rho^m)^{\frac{1}{m+1}}
\]

**Definition 3.** Let metrics \( g \) and \( \tilde{g} \) be given by formulae \( \text{(3)} \) and \( \text{(4)} \) in a coordinate chart \( U \). Then we say that the metrics \( g \) and \( \tilde{g} \) have Levi-Civita local form (of type \( m \)), and the coordinate chart \( U \) is a Levi-Civita coordinate chart (with respect to the metrics).

Levi-Civita proved that the metrics \( g \) and \( \tilde{g} \) given by formulae \( \text{(3)} \) and \( \text{(4)} \) are geodesically equivalent. If we replace \( \phi_i \) by \( \phi_i + c \), \( i = 1, \ldots, m \), where \( c \) is a (positive for simplicity) constant, in \( \text{(3)} \) and \( \text{(4)} \), we obtain the following one-parameter family of metrics, geodesically equivalent to \( g \):

\[
g_c(\tilde{x}, \tilde{x}) = \frac{1}{(\phi_1 + c) \cdots (\phi_m + c)} \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c} \Pi_m A_m \right\}.
\]

The next theorem is essentially due to Painlevé, see \cite{4}. 

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\[ \text{(6)} \]
Theorem 4. If the metrics \( g \) and \( g' \) are geodesically equivalent, then the function

\[
I_0 \equiv \left( \frac{\det(g)}{\det(g')} \right)^{\frac{m}{m-1}} g(\dot{x}, \dot{x}),
\]

is an integral of the geodesic flow of the metric \( g \).

Substituting \( g_c \) instead of \( g \) in (7), we obtain the following one-parameter family of integrals

\[
I_c \equiv \left( \frac{\det(g)}{\det(g_c)} \right)^{\frac{m}{m-1}} g_c(\dot{x}, \dot{x}) =
\]

\[
= C[(\phi_1 + c) \cdots (\phi_m + c)] \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c} \Pi_m A_m \right\}
\]

\[
= C\{L_1 c^{m-1} + L_2 c^{m-2} + \cdots + L_m \},
\]

where

\[
L_1 = \Pi_1 A_1 + \cdots + \Pi_m A_m, \text{ which is twice the energy integral,}
\]

\[
L_2 = \sigma_1(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_1(\phi_1, ..., \phi_{m-1})\Pi_m A_m,
\]

\[
L_3 = \sigma_2(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_2(\phi_1, ..., \phi_{m-1})\Pi_m A_m,
\]

\[
\vdots
\]

\[
L_m = (\phi_2 \cdots \phi_m)\Pi_1 A_1 + \cdots + (\phi_1 \cdots \phi_{m-1})\Pi_m A_m,
\]

\( \sigma_k \) denotes the elementary symmetric polynomial of degree \( k \), and

\[
C \equiv [((\phi_1 + c)^{k_1-1} \cdots (\phi_m + c)^{k_m-1})^{\frac{m}{m-1}}
\]

is a constant. Therefore the functions \( L_k, k = 1, ..., m, \) are integrals of the geodesic flows of the metric \( g \). We call these integrals Levi-Civita integrals.

From the results of [8] it follows that Levi-Civita integrals are in involution. More precisely, let \( D = (d_{ij}) \) be an \( m \times m \) matrix. Suppose that for any \( i, j \) the element \( d_{ij} \) depends only on the variables \( \bar{x}_j \). Denote by \( \Delta \) the determinant of the matrix \( D \) and by \( \Delta_{ij} \) the minor of the element \( d_{ij} \). In the paper [8] it was shown that, for arbitrary functions \( A_i(\bar{x}_i, \dot{\bar{x}}_i) \), quadratic in velocities \( \bar{x}_i \), the Lagrangian system with Lagrangian

\[
T_1 = \Delta \left( \frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta_1} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta_2} + \cdots + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta_m} \right)
\]

admits \( (m - 1) \) integrals

\[
T_i = \Delta \left( \frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta_1^2} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta_2^2} + \cdots + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta_m^2} \right),
\]

where \( i = 2, ..., m, \) and if we identify the tangent and cotangent bundles the Lagrangian \( T_1 \) and consider the standard symplectic form on the cotangent bundle, then the integrals are in involution.
If we take $d^i_j = (\phi_j)^{m-i}$, then $\Delta$ and $\Delta^i_j$ are given by

$$\Delta^i_j = (-1)^{m-1} \sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m) \prod_{\alpha > \beta \geq 1, \alpha \neq j, \beta \neq j} (\phi_\alpha - \phi_\beta),$$

$$\Delta = (-1)^m \prod_{\alpha > \beta \geq 1} (\phi_\alpha - \phi_\beta).$$

Therefore,

$$\frac{\Delta \Delta^i_j}{(\Delta^i_j)^2} = \sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m) \Pi_j,$$

so $T_i = -L_i$ and thus the integrals $L_i$ are in involution, q. e. d.

## 4 Geodesic equivalence and corresponding integrals

Let the metrics $g$ and $\bar{g}$ on a manifold $M$ (of dimension $n$) be geodesically equivalent.

Define

$$U^r_g M \overset{\text{def}}{=} \{(x, \xi) \in TM : ||\xi||_g = r\},$$

where $x \in M$, $\xi \in T_x M$ and $||\xi||_g \overset{\text{def}}{=} \sqrt{g(\xi, \xi)} = \sqrt{g_{ij} \xi^i \xi^j}$ is the norm of the vector $\xi$ in the metric $g$.

By the geodesic flow of the metric $g$ we mean the Lagrangian system of differential equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^i}) - \frac{\partial L}{\partial x^i} = 0$$

on $TM$ with Lagrangian $L \overset{\text{def}}{=} \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$. Because of the Legendre transformation, the geodesic flow could be considered as a Hamiltonian system on $TM$ (as a symplectic form we take $\omega_g \overset{\text{def}}{=} d[g_{ij} \xi^i dx^j]$) with the Hamiltonian $H_g \overset{\text{def}}{=} \frac{1}{2} g_{ij} \xi^i \xi^j$.

Since the metrics $g, \bar{g}$ are geodesically equivalent, the mapping $\Phi : TM \to TM$, $\Phi(x, \xi) = \left(x, \xi, \frac{||\xi||_g}{||\xi||_{\bar{g}}} \right)$, takes the trajectories of the geodesic flow of the metric $g$ to the trajectories of the geodesic flow of the metric $\bar{g}$. This mapping is a diffeomorphism (for $r \neq 0$), takes $U^r_g M$ to $U^r_{\bar{g}} M$ and is trajectoryal on $U^r_g M$. Obviously the surfaces $U^r_g, U^r_{\bar{g}}$ are regular isoenergy surfaces \{H_{\bar{g}} = \frac{r}{2}\}, \{H_g = \frac{r}{2}\}.

By Theorem 3 in order to obtain a family of first integrals we have to find the polynomial $\Delta^n(t)$ and divide it by $(t-a)$. In our case $H_g = H_{\bar{g}} \circ \Phi$. Therefore the function $a$ from Theorem 3 equals to $\frac{||\xi||_g}{||\xi||_{\bar{g}}}$. In coordinates we have

$$\omega_g = d[g_{ij} \xi^i dx^j]$$

and

$$\omega_{\bar{g}} = d[\bar{g}_{ij} \xi^i dx^j].$$
Therefore,

$$\Phi^* \omega_{ij} = d \left[ \frac{||\xi||_g}{||\xi||_{\bar{g}}} \bar{g}_{ij} \xi^i d x^j \right] =$$
$$= \frac{\partial}{\partial x^k} \left[ \frac{||\xi||_g}{||\xi||_{\bar{g}}} \bar{g}_{ij} \xi^j \right] d x^k \wedge d x^j -$$
$$- \frac{\partial}{\partial \xi^k} \left[ \frac{||\xi||_g}{||\xi||_{\bar{g}}} \bar{g}_{ij} \xi^j \right] d x^i \wedge d \xi^k.$$

It is easy to see that at a point $\xi \in \mathcal{T}_x M$ the quantities

$$A_{ik} \overset{\text{def}}{=} -\frac{\partial}{\partial \xi^k} \left[ \frac{||\xi||_g}{||\xi||_{\bar{g}}} \bar{g}_{ij} \xi^j \right]$$

form an element of $\mathcal{T}_x M \otimes \mathcal{T}_x M$. Without loss of generality we can assume that in the space $\mathcal{T}_x M$ the metrics $g$ and $\bar{g}$ are given in principal axes. Then

$$A_{ij} \overset{\text{def}}{=} -\rho^i(x) \frac{\partial}{\partial \xi^j} \left( \xi^i \sqrt{\frac{\xi^1 + \cdots + \xi^n}{\rho^1 \xi^1 + \cdots + \rho^n \xi^n}} \right) =$$
$$= \rho^i \delta^j \frac{||\xi||_g}{||\xi||_{\bar{g}}} - \rho^i \xi^i \left( \frac{||\xi||_g}{||\xi||_{\bar{g}}} - \rho^j \frac{||\xi||_g}{||\xi||_{\bar{g}}} \right) =$$
$$= \text{diag}(\mu_1, \ldots, \mu_n) - A \otimes B.$$

Here $\rho^i, i = 1, \ldots, n$ are common eigenvalues (here we allow $\rho^i$ to be equal to $\rho^j$ for some $i, j$) of the metrics $g$ and $\bar{g}$, $\mu_i \overset{\text{def}}{=} -\rho^i \frac{||\xi||_g}{||\xi||_{\bar{g}}}$, $A_i \overset{\text{def}}{=} \rho^i \xi^i$ and

$$B_i \overset{\text{def}}{=} \frac{||\xi||_g}{||\xi||_{\bar{g}}} - \rho^i \frac{||\xi||_g}{||\xi||_{\bar{g}}} =$$

We have

$$\det(\Phi^* \omega_{ij} - t \omega_g) = \det \begin{vmatrix} (* ) & (A_{ij} + t \delta_{ij}) \\ -(A_{ij} + t \delta_{ij}) & 0 \end{vmatrix}$$

$$= \frac{-((A_{ij} + t \delta_{ij})^2)}{\det(A_{ij} + t \delta_{ij})^2}.$$

Therefore,

$$\Delta^n(t) = \det(\text{diag}(t + \mu_1, \ldots, t + \mu_n) - a \otimes b). \quad (8)$$

**Lemma 3.** The following relation holds:

$$\Delta^n(t) = (t + \mu_1) \cdots (t + \mu_n) - (a_1 b_1)(t + \mu_2) \cdots (t + \mu_n) - \ldots$$
$$- (t + \mu_1) \cdots (t + \mu_{n-1})(a_n b_n). \quad (9)$$
The lemma follows from induction considerations.

To divide the polynomial by \((t - a)\) we shall use the Horner scheme. Suppose that \(\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0\) and \(\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0\). Then we have

\[
\begin{align*}
  b_{n-1} &= a_n = 1, \quad (10) \\
  b_{n-2} &= a_{n-1} + a, \quad (11) \\
  & \vdots \\
  b_k &= a_{k+1} + ab_{k+1}, \quad (12) \\
  & \vdots \\
  0 &= a_0 + ab_0. \quad (13)
\end{align*}
\]

It follows from lemma 3 that

\[
a_0 = (\mu_1 \ldots \mu_n) - (A_1B_1)(\mu_2 \ldots \mu_n) - \cdots - (\mu_1 \ldots \mu_{n-1})A_nB_n = (-1)^n \left( \frac{||\xi||_g}{||\xi||_g} \right)^n (\rho^1 \cdots \rho^n) .
\]

Combining with (13) we get

\[
b_0 = - \frac{a_0}{a} = (-1)^{n+1} \left( \frac{||\xi||_g}{||\xi||_g} \right)^{n+1} (\rho^1 \cdots \rho^n). \]

Since \(\frac{1}{2}g_{ij}\xi^i \xi^j\) is an integral of the geodesic flow of the metric \(g\), the function

\[
I_0 \overset{\text{def}}{=} (\rho^1 \cdots \rho^n)^{-\frac{3}{n+1}} \bar{g}(\xi, \xi) \quad (14)
\]

is also an integral of the geodesic flow of the metric \(g\). Using Lemma 3 we have

\[
a_{n-1} = (\mu_1 + \ldots + \mu_n) - (A_1B_1 + \ldots + A_nB_n) = \frac{||\xi||_g}{||\xi||_g} \left\{ (\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) - (\rho^1 + \ldots + \rho^n)(\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) \right\}^{-\frac{3}{n+1}}.
\]

Using (11) we get

\[
b_{n-2} = a_{n-2} + a = \frac{||\xi||_g}{||\xi||_g} \left\{ (\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) - (\rho^1 + \ldots + \rho^n)(\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) \right\}.
\]

Therefore, the function

\[
I_1 \overset{\text{def}}{=} (\rho^1 \cdots \rho^n)^{-\frac{3}{n+1}} \left\{ (\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) - (\rho^1 + \ldots + \rho^n)(\rho^1 \xi^{12} + \ldots + \rho^n \xi^{n2}) \right\}
\]
is an integral. (It is easy to see that \( \frac{||g||^2}{||g||^2} = (\rho^1 \cdots \rho^n) - \frac{||g||^2}{||g||^2} \))

Arguing as above, we see that the functions

\[
I_k \overset{\text{def}}{=} (\rho^1 \cdots \rho^n)^{-\frac{1}{2}} \left\{ (\rho^{1+k} \xi^1 + \cdots + \rho^{n+k} \xi^n)^{-} - (\rho^1 + \cdots + \rho^n)(\rho^{1+k} \xi^1 + \cdots + \rho^{n+k} \xi^n) + \cdots + (-1)^k \sigma_k (\rho^1, \cdots, \rho^n)(\rho^1 \xi^1 + \cdots + \rho^n \xi^n) \right\},
\]

are integrals of the geodesic flow of the metric \( g \), where by \( \sigma_k \) we denote the elementary symmetric polynomial of degree \( k \). It is obvious that \( (-1)^k \sigma_k = c_k \) from Theorem \( \blacksquare \) and therefore \( I_k = \left( \frac{\det(g)}{\det(\bar{g})} \right)^{\frac{k+2}{2}} \bar{g}(S_k \xi, \xi) \). Thus \( I_k, k = 0, ..., n - 1 \), are integrals of the geodesic flow of the metric \( g \), q. e. d.

5 Liouville integrability

The last step of the proof of Theorem \( \blacksquare \) is to verify that the integrals \( I_0, ..., I_{n-1} \) are in involution. We proceed along the following plan. First we show that it is sufficient to prove the involutivity in each Levi-Civita chart. Then we prove that in each Levi-Civita chart the integrals \( I_0, ..., I_{n-1} \) are linear combinations of Levi-Civita integrals, and therefore commute.

Let \( g, \bar{g} \) be metrics on \( M \). A point \( x \in M \) is called stable, if in a neighborhood of \( x \) the number of different eigenvalues of the metrics \( g, \bar{g} \) is independent of the point.

Denote by \( \mathcal{M} \) the set of stable points of \( M \). The set \( \mathcal{M} \) is an open subset of \( M \). Obviously

\[
\mathcal{M} = \bigcup_{1 \leq q \leq n} \mathcal{M}^q,
\]

where \( \mathcal{M}^q \) denotes the set of stable points whose number of distinct common eigenvalues equals \( q \). Points \( x \in M \setminus \mathcal{M} \) are called points of bifurcation.

Lemma 4. The set \( \mathcal{M} \) is everywhere dense in \( M \).

Proof of Lemma \( \blacksquare \). Denote by \( N(x) \) the number of distinct common eigenvalues of the metrics \( g, \bar{g} \) at a point \( x \). Recall that the common eigenvalues of the metrics \( g, \bar{g} \) at a point \( x \in M \) are roots of the characteristic polynomial \( P_x(t) = \det (G - tE)|_x \), where \( G = (g^{\alpha \beta} \bar{g}_{\alpha \beta}) \). In particular, all roots of \( P_x(t) \) are real.

Let us prove that, for a sufficiently small neighborhood of an arbitrary point \( x \in M \), for any \( y \) from the neighborhood the number \( N(x) \) is no greater than \( N(y) \). Take a small \( \epsilon > 0 \) and an arbitrary root \( \rho \) of \( P_x(t) \). Let us prove that for a sufficiently small neighborhood \( U(x) \subset M \), for any \( y \in U(x) \) there is a root \( \rho_y \), \( \rho - \epsilon < \rho_y < \rho + \epsilon \), of the polynomial \( P_y(t) \). If \( \epsilon \) is small, then for a sufficiently small neighborhood \( U(x) \) of the point \( x \), for any \( y \in U(x) \) the numbers \( \rho + \epsilon \) and \( \rho - \epsilon \) are not roots of \( P_y(t) \). Consider the circle \( S_\epsilon \overset{\text{def}}{=} \{ z \in C : ||z - \rho|| = \epsilon \} \) on
the complex plane $C$. Clearly the number of roots (with multiplicities) of the polynomial $P_y$ inside the circle is equal to

$$\frac{1}{2\pi i} \int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)} \, dz.$$ 

Since for any $y \in U(x)$ there are no roots of $P_y$ on the circle $S_\epsilon$, then the function

$$\frac{1}{2\pi i} \int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)} \, dz$$

continuously depends on $y \in U(x)$, and therefore is a constant. Clearly it is positive. Thus for any $y \in U(x)$ there is at least one root of $P_y$ that lies between $\rho + \epsilon$ and $\rho - \epsilon$. Then for any $y$ from a sufficiently small neighborhood of $x$ we have $N(y) \geq N(x)$.

Now let us prove the lemma. Evidently the set $M$ is an open subset of $M$. Then it is sufficient to prove that for any open subset $U \subset M$ there is a stable point $x \in U$. Suppose otherwise, i.e. let all the points of $U$ be points of bifurcation. Take a point $y \in M$ with maximal value of the function $N$ on it.

We have that in a neighborhood $U(y)$ of the point $y$ the function $N$ is constant and equals $N(y)$. Then the point $y$ is a stable point, and we get a contradiction, q. e. d.

Now let the metrics $g, \bar{g}$ be geodesically equivalent. Since the set of points of bifurcation is nowhere dense, it is sufficient to prove the involutivity in each Levi-Civita chart. Let the metrics $g$ and $\bar{g}$ be given by

$$g(\dot{x}, \dot{x}) = \Pi_1(\bar{x})A_1(\bar{x}_1, \dot{x}_1) + \Pi_2(\bar{x})A_2(\bar{x}_2, \dot{x}_2) + \cdots + \Pi_m(\bar{x}_m, \dot{x}_m),$$

$$\bar{g}(\dot{x}, \dot{x}) = \rho^1\Pi_1(\bar{x})A_1(\bar{x}_1, \dot{x}_1) + \rho^2\Pi_2(\bar{x})A_2(\bar{x}_2, \dot{x}_2) + \cdots + \rho^m\Pi_m(\bar{x}_m, \dot{x}_m).$$ (16) (17)

We show that the integrals $I_k$ are linear combinations of the Levi-Civita integrals. We have

$$\bar{G} = \text{diag}(\rho^1, \ldots, \rho^1, \ldots, \rho^m, \ldots, \rho^m),$$

where $\rho^k = \frac{1}{(\sigma^1 \cdots \sigma^m)_{\dot{k}}}$. It is easy to check that

$$S_k = (-1)^k \text{diag}(\sigma^1_{k_1} \ldots, \sigma^1_{k_1}, \ldots, \sigma^m_{k_m} \ldots, \sigma^m_{k_m}),$$

where

$$\sigma^l_k \overset{\text{def}}{=} \sigma_k(\rho^1_{k_1}, \ldots, \rho^1_{k_1}, \rho^l_{k_l}, \ldots, \rho^m_{k_m}).$$ (19) (20)
We have
\[ \sigma_k^l = \frac{1}{(\phi_1 \ldots \phi_m)^k} \sigma_k \left( \frac{1}{\phi_1}, \ldots, \frac{1}{\phi_m} \right) = \frac{1}{(\phi_1 \ldots \phi_m)^k} \sum_{|\alpha|=k} \left( \frac{k_1 - 1}{\alpha_1} \right) \left( \frac{k_2}{\alpha_2} \right) \ldots \left( \frac{k_m}{\alpha_m} \right) \frac{1}{\phi_1^{\alpha_1}} \frac{1}{\phi_2^{\alpha_2}} \ldots \frac{1}{\phi_m^{\alpha_m}}, \]

(21)
(22)

where $|\alpha| \equiv \alpha_1 + \ldots + \alpha_m$ and $\alpha_i \geq 0$. Substituting $\binom{k_1 - 1}{\alpha_1} + \binom{k_1 - 1}{\alpha_i - 1}$ for $\binom{k_1}{\alpha_i}$ (we assume that $\binom{k_1}{\alpha_i} = 1$, $\binom{k_1}{-1} = 0$, $k \geq 0$) for $2 \leq l \leq m$ we obtain
\[ \sigma_k^l = \frac{1}{(\phi_1 \ldots \phi_m)^k} \left( B_k + B_{k-1} \right) + \ldots + B_{k-m+1} \sigma_m \left( \frac{1}{\phi_2}, \ldots, \frac{1}{\phi_m} \right), \]

where
\[ B_k \equiv \sum_{|\alpha|=k} \binom{k_1 - 1}{\alpha_1} \ldots \binom{k_m - 1}{\alpha_m} \frac{1}{\phi_1^{\alpha_1}} \ldots \frac{1}{\phi_m^{\alpha_m}}. \]

(24)

Note that
\[ \left( \frac{\det(g)}{\det(\tilde{g})} \right)^{\frac{k+2}{n+1}} = C_k (\phi_1 \ldots \phi_m)^{k+2}, \]

(25)

where $C_k = \left[ \phi_1^{k_1 - 1} \ldots \phi_m^{k_m - 1} \right]^{\frac{k+2}{n+1}}$. Therefore,
\[ I_k \equiv \left( \frac{\det(g)}{\det(\tilde{g})} \right)^{\frac{k+2}{n+1}} \tilde{g}(S_x \tilde{x}, \tilde{x}) = (-1)^k C_k (\phi_1 \ldots \phi_m)^{k+2} \left\{ \rho^1 \sigma_1^l \Pi A_1 + \ldots + \rho^m \sigma_1^m \Pi A_m \right\} =
\]
\[ = (-1)^k C_k (\phi_1 \ldots \phi_m)^{k+2} \left\{ \frac{1}{\phi_1 \ldots \phi_m} \frac{1}{\phi_1} \left\{ \frac{1}{\phi_1 \ldots \phi_m} \right\} B_k + \ldots + B_{k-m+1} \sigma_m \left( \frac{1}{\phi_2}, \ldots, \frac{1}{\phi_m} \right) \right\} \Pi A_1 + \ldots \right\} =
\]
\[ = (-1)^k C_k \left[ B_k L_m + B_{k-1} L_{m-1} + \ldots + B_{k-m+1} L_1 \right], \]

(26)

where $L_i$ are Levi-Civita integrals.

Finally, since the integrals $I_0, \ldots, I_{n-1}$ are linear combinations of Levi-Civita integrals with constant coefficients, and since Levi-Civita integrals commute, then the integrals $I_0, \ldots, I_{n-1}$ also commute, q. e. d.

**Remark 2.** Let $m$ be the number of distinct common eigenvalues of geodesically equivalent metrics $g, \tilde{g}$ at a point $x$. Then in a neighborhood $U$ of the point $x$ the number of functionally independent almost everywhere Levi-Civita integrals is no less than $m$. Therefore the dimension of the space generated by the differentials $(dI_0, dI_1, \ldots, dI_{n-1})$ no less than $m$ at almost all points of $TU$. 

13
6 Topological obstructions

Corollary follows immediately from the following theorem. Recall that a group $G$ is almost commutative, if there exists a commutative subgroup $P \subset G$ of finite index.

Theorem 5 (Taimanov, [10]). If a real-analytic closed manifold $M^n$ with a real-analytic metric satisfies at least one of the conditions:

a) $\pi_1(M^n)$ is not almost commutative

b) $\dim H_1(M^n; \mathbb{Q}) > \dim M^n$,

then the geodesic flow on $M^n$ is not analytically integrable.

Proof of Corollary 1. If metrics $g, \bar{g}$ are real-analytic and geodesically equivalent, then the integrals $I_0, ..., I_{n-1}$ are also real-analytic. If the metrics are strictly non-proportional at least at one point of $M^n$, then the integrals are functionally independent almost everywhere in a neighborhood of that point. Since the integrals are real-analytic, then they are functionally independent almost everywhere and we can apply Theorem 5, q. e. d.

Proof of Corollary 2. Let metrics $g, \bar{g}$ on a surface $M^2$ be geodesically equivalent. Using Theorem 3 we have that the function

$I_0 = (\det(g))^{\frac{1}{2n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric $g$. In one direction Corollary 2 is proved. In other direction the statement of Corollary 2 can be verified by direct calculation, and it was done in [2].

Proof of Corollaries 3, 4, 5, 6. Let $g$ be a metric on a surface $M^2$. The following lemma is essentially due to [1], see also [5]. For simplicity assume that the surface $M^2$ is oriented, otherwise finitely cover the surface by an oriented one. Consider the complex structure on $M^2$ corresponding to the metric $g$. Let $z$ be a complex coordinate in an open domain $U \subset M^2$. Consider the complex momentum $p$. We shall denote by $\bar{z}$ and $\bar{p}$ the complex conjugation of $z$ and $p$ respectively. In complex variables the Hamiltonian $H : T^*M^2 \to R$ of the geodesic flow of the metric $g$ reads $\lambda \bar{p}$, where $\lambda$ is a real-valued function. Suppose that the real-valued function

$F = A(z)p^2 + B(z)p\bar{p} + \bar{A}(z)\bar{p}^2$

is an integral of the geodesic flow of the metric $g$.

Lemma 5. The form $\frac{1}{A(z)}dzd\bar{z}$ is meromorphic.

Remark 3. If the Hamiltonian and the integral are proportional at each point of $M^2$, i.e. if $F \equiv \alpha(z)H$, where $\alpha : M^2 \to R$, then by definition put $\frac{1}{A(z)}dzd\bar{z}$ equal zero.
Proof of Lemma \[5\]. Since $F$ is an integral of the Hamiltonian system with the Hamiltonian $H$, the Poisson bracket $\{H, F\}$ equals zero. We have

$$\{H, F\} = H_p F_z - H_z F_p + H_{\bar{p}} F_{\bar{z}} - H_{\bar{z}} F_{\bar{p}} = 0$$ (27)

On the right side of (27) each term is a polynomial of third degree in momenta. Then the bracket is also a polynomial of third degree in momenta. In order for a polynomial to equal zero, all coefficients must be zero, in particular the coefficient of $p^3$. Thus $\frac{\partial z}{\partial \lambda}$ equals zero, and $A$ is holomorphic. Then $\frac{1}{A(z)}$ is meromorphic, q. e. d.

Let $g, \bar{g}$ be geodesically equivalent metrics on a closed surface $M^2$ of Euler characteristic $\chi(M^2)$. Then the function $I_0 = \left(\det(g)\det(\bar{g})\right)^{-\frac{1}{n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric $g$, and is quadratic in momenta (if we identify with the help of the metric $g$ the tangent and cotangent bundles of $M^2$). Consider the form $\frac{1}{A(z)}dzdz$ corresponding to the integral $I_0$. Suppose that the form is not identical zero. For a meromorphic 2-form on a closed Riemann surface, the number of poles $P$ minus the number of zeros $Z$ is equal to twice the Euler characteristic. It is easy to see that the form $\frac{1}{A(z)}dzdz$ has no zeros (otherwise the metric $\bar{g}$ has singularities). Then $P = 2\chi(M^2)$, and the Euler characteristic $\chi(M^2)$ can not be negative, q. e. d. Now assume the metrics are proportional at each point of an open subset $U \subset M^2$. Since the form is meromorphic, it must be zero. Thus $\bar{g} = \alpha(z)g$, where $\alpha$ is a function on $M^2$. Let us show that the function $\alpha$ is constant. Actually, $I_0 = 2 \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} H$ (here we identify $T^* M$ and $TM$ with the help of the metric $g$). We have

$$\{H, I_0\} = \{H, 2 \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} H\} = \{H, H\}2 \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} + 2H\left\{\left(\frac{1}{\alpha}\right)^{\frac{1}{2}}, H\right\}.$$ 

Since $\{H, H\}$ equals zero, we have that $\left\{\left(\frac{1}{\alpha}\right)^{\frac{1}{2}}, H\right\}$ equals zero and the function $\alpha$ is constant. This proves Corollaries \[3, 6\].

Remark 4. For non-orientable surfaces the sign of the Euler characteristic coincides with the sign of the Euler characteristic of the oriented covering. Therefore Corollary \[3\] is true also for non-orientable surfaces.

It is easy to see that the form $\frac{1}{A(z)}dzdz$ has poles precisely at points, where the metrics are proportional. If the surface $M^2$ is the torus, then $\chi(M^2) = 0$ and either the metrics $g, \bar{g}$ are proportional at every point, or there are no points of proportionality of the metrics. This proves Corollary \[3\].

The following lemma is essentially due to Kolokol’tzov \[5\]. It completes the proof of Corollary \[3\].

Lemma 6. On the sphere $S^2$ there are the following three possibilities for the form $\frac{1}{A(z)}dzdz$.

1. The form $\frac{1}{A(z)}dzdz$ is identical zero.
2. The form \( \frac{1}{\lambda(z^2)} dzdz \) has exactly two zeros (both zeros are of multiplicity two).

3. The form \( \frac{1}{\lambda(z^2)} dzdz \) has exactly four zeros.

In the second case the metric \( g \) admits a non-trivial Killing vector field.

**Proof of Corollary 7.** Because of Noether’s theorem, if a metric admits a (non-trivial) Killing vector field, then the geodesic flow of the metric admits a (non-trivial) integral, linear in velocities, and vice versa.

Suppose the function

\[
F_1 = \sum_{i=1}^{n} a_i(x) \xi^i
\]

is constant on the trajectories of the geodesic flow of the metric \( \bar{g} \). Then the function

\[
\Phi^*F_1 = \frac{||\xi||_g}{||\xi||_{\bar{g}}} \sum_{i=1}^{n} a_i(x) \xi^i
\]

is constant on the trajectories of the geodesic flow of the metric \( g \). Since the function \( I_0 = \left( \frac{\det(g)}{\det(\bar{g})} \right)^{\frac{1}{n+1}} \bar{g}(\xi, \xi) \) is an integral of the geodesic flow of the metric \( g \), and since the function \( ||\xi||_g = \sqrt{g(\xi, \xi)} \) is also an integral of the geodesic flow of the metric \( g \), then the function

\[
\sqrt{g(\xi, \xi)} \Phi^*F_1 = \left( \frac{\det(g)}{\det(\bar{g})} \right)^{\frac{1}{n+1}} \sum_{i=1}^{n} a_i(x) \xi^i,
\]

linear in velocities, is also an integral of the geodesic flow of the metric \( g \), q. e. d.

### 7 Geodesically equivalent metrics on the ellipsoid.

**Proof of Theorem 2.** We show that in the elliptic coordinate system the restriction of the metrics

\[
ds^2 \overset{\text{def}}{=} \sum_{i=1}^{n} (dx_i)^2 \quad \text{and} \quad dr^2 \overset{\text{def}}{=} \frac{1}{\sum_{i=1}^{n} \left( \frac{x_i}{a_i} \right)^2} \left( \sum_{i=1}^{n} \frac{(dx_i)^2}{a_i} \right)
\]

to the ellipsoid \( \sum_{i=1}^{n} \frac{(x_i)^2}{a_i} = 1 \) have Levi-Civita local form, and therefore are geodesically equivalent. More precisely, consider elliptic coordinates \( \nu^1, ..., \nu^n \). Without loss of generality we can assume that \( a^1 < a^2 < ... < a^n \). Then the relation between the elliptic coordinates \( \bar{\nu} \) and the Cartesian coordinates \( \bar{x} \) is given by

\[
x_i = \sqrt{\prod_{j=1}^{n} (a^i - \nu^j)} \prod_{j=1, j \neq i}^{n} (a^i - a^j).
\]

(28)
Recall that the elliptic coordinates are non-degenerate almost everywhere, and the set 
\[ \{ \nu^1 = 0, a_1 < \nu^2 < a_2, a_2 < \nu^3 < a_3, \ldots, a_{n-1} < \nu^n < a^n \} \]
is the part of the ellipsoid, lying in the quadrant \( \{ x^1 > 0, x^2 > 0, \ldots, x^n > 0 \} \). Since for any \( i \) the symmetry \( x^i \rightarrow -x^i \) takes the ellipsoid to the ellipsoid and preserves the metrics \( ds^2 \) and \( dr^2 \), it is sufficient to check the statement of the theorem only in the quadrant \( \{ x^1 > 0, x^2 > 0, \ldots, x^n > 0 \} \).

In the elliptic coordinates the restriction of the metric \( ds^2 \) to the ellipsoid has the following form
\[
\sum_{i=1}^{n} \Pi_i A_i (d\nu^i)^2,
\]
where \( \Pi_i \) and \( A_i \) are defined as above. The restriction of the metric \( dr^2 \) to the ellipsoid is
\[
(a^1 a^2 \ldots a^n) \sum_{i=1}^{n} \rho^i \Pi_i A_i (d\nu^i)^2,
\]
where \( \rho^i \) are defined as above. We see that the metrics \( ds^2, dr^2 \) have Levi-Civita local form, and therefore are geodesically equivalent, q.e.d.

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