RAMSEY NUMBERS OF PARTIAL ORDER GRAPH AND IMPLICATIONS IN RING THEORY

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Abstract. For a partially ordered set \((A, \leq)\), let \(G_A\) be the simple, undirected graph with vertex set \(A\) such that two vertices \(a \neq b \in A\) are adjacent if either \(a \leq b\) or \(b \leq a\). We call \(G_A\) the partial order graph of \(A\). Further, we say that a graph \(G\) is a partial order graph if there exists a partially ordered set \(A\) such that \(G = G_A\). For a class \(C\) of simple, undirected graphs and \(n, m \geq 1\), we define the Ramsey number \(R_C(m, n)\) with respect to \(C\) to be the minimal number of vertices \(r\) such that every induced subgraph of an arbitrary partial order graph consisting of \(r\) vertices contains either a complete \(n\)-clique \(K_n\) or an independent set consisting of \(m\) vertices.

In this paper, we determine the Ramsey number with respect to some classes of partial order graphs. Furthermore, some implications of Ramsey numbers in ring theory are discussed.

1. Introduction

The Ramsey number \(R(m, n)\) gives the solution to the party problem, which asks for the minimum number \(R(m, n)\) of guests that must be invited so that at least \(m\) will know each other or at least \(n\) will not know each other. In the language of graph theory, the Ramsey number is the minimum number of vertices \(v = R(m, n)\) such that all undirected simple graphs of order \(v\) contain a clique of order \(m\) or an independent set of order \(n\). There exists a considerable amount of literature on Ramsey numbers. For example, Greenwood and Gleason [6] showed that \(R(3, 3) = 6\), \(R(3, 4) = 9\) and \(R(3, 5) = 14\); Graver and Yackel [5] proved that \(R(3, 6) = 18\); Kalbfleisch [8] computed that \(R(3, 7) = 23\); McKay and Min [9] showed that \(R(3, 8) = 28\) and Grinstead and Roberts [7] determined that \(R(3, 9) = 36\).

A summary of known results up to 1983 for \(R(m, n)\) is given in Chung and Grinstead [3]. An up-to-date list of the best currently known bounds for generalized Ramsey numbers (multicolor graph numbers), hypergraph Ramsey numbers, and many other types of Ramsey numbers is maintained by Radziszowski [10].

In this paper, we determine the Ramsey number of partial order graphs (also known as Hasse diagrams). For a partially ordered set \((A, \leq)\), let \(G_A\) be the simple, undirected graph with vertex set \(A\) such that two vertices \(a \neq b \in A\) are adjacent if either \(a \leq b\) or \(b \leq a\). We call \(G_A\) the partial order graph of \(A\). Further, we say that a graph \(G\) is a partial order graph if there exists a partially ordered set \(A\) such that \(G = G_A\). For a class \(C\) of simple, undirected graphs and \(n, m \geq 1\), we define the Ramsey number \(R_C(m, n)\) with respect to the class \(C\) to be the minimal number of vertices \(r\) such that every induced subgraph of an arbitrary partial order graph consisting of \(r\) vertices contains either a complete \(n\)-clique \(K_n\) or an independent set consisting of \(m\) vertices. We give some implications of Ramsey numbers of partial order graphs in ring theory. We like to point out that the Cox and Stolee [4] introduced a more general notion of Ramsey number on partially ordered sets and provided bounds for these numbers.

Next, we remind the reader of the graph theoretic definitions that are used in this paper. We say that a graph \(G\) is connected if there is a path between any two distinct vertices of \(G\). For vertices \(x\) and \(y\) of \(G\), we define \(d(x, y)\) to be the length of a shortest path from \(x\) to \(y\) \((d(x, x) = 0\) and \(d(x, y) = \infty\) if there is no such path). The diameter of \(G\) is \(\text{diam}(G) = \sup\{d(x, y)\} |
x and y are vertices of G.} The girth of G, denoted by g(G), is the length of a shortest cycle in G (g(G) = ∞ if G contains no cycles). We denote the complete graph on n vertices or n-clique by K_n, and the complete bipartite graph on m and n vertices by K_{m,n}. The clique number ω(G) of G is the largest positive integer m such that K_m is an induced subgraph of G. The chromatic number of G, χ(G), is the minimum number of colors needed to produce a proper coloring of a G (that is, no two vertices that share an edge have the same color). The domination number of G, γ(G), is the minimum size set S of vertices of G such that each vertex in G \ S is connected to at least one vertex in S by an edge. An independent vertex set of G is a subset of the vertices such that no two vertices in the subset are connected by an edge of G. For a general reference for graph theory we refer to Bollobás’ textbook [2].

In Section 2 we show that the Ramsey number R_{PoG}(n, m) for the class PDG of partial order graphs equals (n - 1)(m - 1) + 1, see Theorem 2.2. In Section 3 we study subclasses of partial order graphs that appear in the context of ring theory. Among other results, we show that for the classes PDG of perfect divisor graphs, DivG of divisibility graphs, InG of inclusion ideal graphs, MatG of matrix graphs and IdemG of idempotents graphs of rings, the respective Ramsey numbers equal to R_{PoG}, see Theorems 3.1, 3.8, 3.12, 3.16 and 3.21 respectively. In Section 4 we present a subclass of partial ordered graphs with respect to which the Ramsey number are non-symmetric. Throughout this paper, Z and Z_n will denote the integers and integers modulo n, respectively. Moreover, for a ring R we assume that 1 ≠ 0 holds, R^* = R \ {0} denotes the set of non-zero elements of R and U(R) denotes the group of units of R.

2. RAMSEY NUMBERS OF PARTIAL ORDER GRAPHS

Definition 2.1. (1) For a partially ordered set (A, ⪯), let G_A be the simple, undirected graph with vertex set A such that two vertices a ≠ b ∈ A are adjacent if either a ≤ b or b ≤ a. We call G_A the partial order graph of A. Further, we say that G is a partial order graph if there exists a partially ordered set A such that G = G_A. By PoG we denote the class of all partial order graphs.

(2) For a class C of simple, undirected graphs and n, m ≥ 1, we set R_C(m, n) to be the minimal number of vertices r such that every induced subgraph of an arbitrary partial order graph consisting of r vertices contains either a complete n-clique K_n or an independent set consisting of m vertices. We call R_C the Ramsey number with respect to the class C.

Theorem 2.2. Let n, m ≥ 1 (n, m need not be distinct). Then for the Ramsey number R_{PoG} with respect to the class PoG of partial order graphs, the following equality holds

\[ R_{PoG}(n, m) = R_{PoG}(m, n) = (n - 1)(m - 1) + 1. \]

Proof. First, we prove that R_{PoG}(n, m) > (n - 1)(m - 1). Let A be a set of cardinality (n-1)(m-1) and A_1, . . . , A_{m-1} an arbitrary partition of A into n - 1 subsets each of cardinality m - 1. Further, for a, b ∈ A, we say a ≤ b if and only if a = b or a ∈ A_i and b ∈ A_j with i < j. Then ≤ is a partial order on A and the partial order graph G_A is a complete (n-1)-partite graph in which each partition has m - 1 independent vertices. It is easily verified that the clique number of G_A is n - 1 and at exactly m - 1 vertices of G_A are independent.

Let G be a partial order graph and H an induced subgraph. We show that if H contains (n - 1)(m - 1) + 1 vertices, then H contains either an n-clique K_n or an independent set of m vertices.

Let G^dir be the directed graph with the same vertex set as G such that (a, b) is an edge if a ≠ b and a ≤ b. Then H^dir (the subgraph of G^dir induced by the vertices of H) contains a directed path of length n if and only if H contains an n-clique K_n.

Note that G^dir does not contain a directed cycle. This allows us to define pos_H(a) to be the maximal length of a directed path in H^dir with endpoint a for a vertex a of H.

It is easily seen, that pos_H(b) ≤ pos_H(a) - 1 for every edge (b, a) in H^dir. In particular, if for two vertices a, b of H, pos_H(a) = pos_H(b), then the two vertices are independent in H.

Moreover, a straightforward argument shows that H contains an n-clique K_n if and only if there exists a vertex a in H with pos_H(a) = n - 1.
Now, assume that $H$ does not contain an $n$-clique $K_n$. This implies that $\text{pos}_H(a) < n - 1$ for all vertices $a$ in $H$. It then follows by the pigeonhole principle that among the $(n-1)(m-1)+1$ vertices in $H$, there are at least $m$ vertices $a$ with $\text{pos}_H(a) = k$ for some $k$, $0 \leq k \leq n-2$. Therefore $H$ contains $m$ independent vertices.

Since $(n-1)(m-1)+1$ is symmetric in $n$ and $m$, it further follows that $R_{\text{PoG}}(m,n) = R_{\text{PDG}}(m,n)$.

3. Subclasses of partial order graphs that appear in ring theory

In this section we discuss subclasses of partial order graphs that appear in the context of ring theory. In particular, we focus on the implications of Theorem 2.2 for certain subclasses of partial order graphs that occur in the context of ring theory. Recall for a class $C$ of graphs, $R_C$ denotes the Ramsey number with respect to $C$, cf. Definition 2.1.

3.1. Perfect divisor graphs.

Definition 3.1. Let $R$ be a commutative ring, $n \in \mathbb{N}_{\geq 2}$ and $S = \{m_1, \ldots, m_n\} \subseteq R^* \setminus U(R)$ be a set of $n$ pairwise coprime non-zero non-units and $m = m_1m_2 \cdots m_n$. (Note that $m = 0$ is possible.)

1. We say $d$ is a perfect divisor of $m$ with respect to $S$ if $d \neq m$ and $d$ is a product of distinct elements of $S$.

2. The perfect divisor graph $\text{PDG}(S)$ of $S$ is defined as the simple, undirected graph $(V,E)$ where $V = \{d \mid d$ perfect-divisor of $m\}$ is the vertex set and for two vertices $a \neq b \in V$, $(a,b) \in E$ if and only if $a \mid b$ or $b \mid a$.

3. By $\text{PDG}$ we denote the class of all perfect divisor graphs.

Lemma 3.2. Let $R$ be a commutative ring, $n \in \mathbb{N}_{\geq 2}$ and $S = \{m_1, \ldots, m_n\} \subseteq R^* \setminus U(R)$ be a set of $n$ pairwise coprime non-zero non-units and $m = m_1m_2 \cdots m_n$. Further, let $V = \{d \mid d$ perfect divisor of $m$ with respect to $S\}$ and define $\leq$ on $V$ such that for all $a, b \in V$, we have $a \leq b$ if and only if $a = b$ or $a \mid b$.

Then $(V, \leq)$ is a partially ordered set of cardinality $|V| = 2^n - 2$ and $\text{PDG}(S)$ is a partial order graph.

Proof. The relation $\leq$ clearly is reflexive and transitive, we prove that it is also antisymmetric. Let $d \in V$ be a perfect divisor of $m$ with respect to $S$. Then $d = \prod_{j \in J} m_j$ for $\emptyset \neq J \subseteq \{1, \ldots, n\}$. We show that for every $1 \leq i \leq m$, $m_i \mid d$ if and only if $i \in J$.

Obviously if $j \in J$, then $m_i \mid d$. Let us assume that $i \in \{1, \ldots, n\} \setminus J$. Then by hypothesis, for $j \in J$ there are elements $a_j$ and $b_j \in R$ such that $a_jm_j + b_jm_i = 1$ holds. Hence

$$1 = \prod_{j \in J} (a_jm_j + b_jm_i) = \left(\prod_{j \in J} a_jm_j\right) + cm_i = ad + cm_i$$

for some $a, c \in R$. Therefore, $d$ and $m_i$ are coprime elements of $R$ which in particular implies that $m_i \nmid d$.

It follows that if $d_1$ and $d_2$ are distinct perfect divisors of $m$ and $d_1 \mid d_2$, then $d_2 \nmid d_1$. Thus $(V, \leq)$ is a partially ordered set.

Moreover, it follows that the elements in $V$ correspond to the non-empty proper subset of $\{1, \ldots, n\}$. Therefore, their number amounts to

$$|V| = |\{\emptyset \neq J \subseteq \{1, \ldots, n\}\}| = \sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2.$$
Theorem 3.3. Let R be a commutative ring, n ∈ \mathbb{N}_{\geq 2} and S = \{m_1, \ldots, m_n\} ⊆ R^\ast \setminus U(R) be a set of n pairwise coprime non-zero non-units, m = m_1m_2 \cdots m_n and PDG(S) the perfect divisor graph of m with respect to S.

Then the following assertions hold:

1. PDG(S) is a connected graph if and only n ≥ 3.
2. If n ≥ 3, then the diameter diam(PDG(S)) = 3.
3. The domination number of PDG(S) is equal 2 if n ≥ 2 and equal 1 if n = 1.
4. If n ≥ 3, then the vertices in \( P_k = \{ \prod_{j \in J} m_j \mid |J| = k \} \) for 1 ≤ k ≤ n − 1 are pairwise not connected by an edge. In particular, PDG(S) is an (n − 1)-partite graph.
5. If a ∈ \( P_k = \{ \prod_{j \in J} m_j \mid |J| = k \} \) for 1 ≤ k ≤ n − 1, then \( \deg(a) = 2^k + 2^{n-k} - 4 \).
6. If n ≥ 3, then for the girth of PDG(S) the following holds

\[
g(PDG(S)) = \begin{cases} 
6 & n = 3 \\
3 & n \geq 4
\end{cases}
\]

7. PDG(S) is planar if and only if n = 3.

Proof. 1: If n = 2, then V consists of 2 vertices \( m_1 \) and \( m_2 \) which are coprime and hence not connected. Assume n ≥ 3 and let \( a = \prod_{j \in J} m_j \) and \( b = \prod_{k \in K} m_k \) be two distinct vertices of PDG(S). Choose \( j \in J \) and \( k \in K \). If \( j = k \), then \( m_j \mid a \) and \( m_j \mid b \) which implies that both \( (m_j,a) \) and \( (m_j,b) \in E \). If \( j \neq k \), then \( m_jm_k \) is a perfect divisor of m with respect to \( S \). A path from a to b is formed by the edges \( (a,m_j), (m_j,m_k), (m_km_k, m_k) \) and \( (m_kb) \). Hence PDG(S) is connected which completes the proof of (1).

2: The path given above is of length 3 which gives an upper bound for the diameter. To prove that the diameter is equal 3, we distinguish two cases, n = 3 and n ≥ 4. For n = 3, the vertices \( a = m_1m_2 \) and \( b = m_3 \) have distance 3, see Figure 1. For n ≥ 4, the vertices \( a = m_1m_2 \) and \( b = m_3m_4 \) have no common neighbor which implies that their distance is at least 3. In both cases it follows that diam(PDG(S)) = 3.

For (3) observe, that every perfect divisor d of m, is either divisible by \( m_1 \) or divides \( m_2m_3 \cdots m_n \). Hence, every vertex of PDG(S) is connected to either one of these two vertices.

4: Let 1 ≤ k ≤ n − 1 and \( J, K \subseteq \{1, \ldots, n\} \) with \( |J| = |K| = k \). Set \( a = \prod_{j \in J} m_j \) and \( b = \prod_{k \in K} m_k \) be two different vertices of PDG(S) which implies \( J \neq K \). In the proof of Lemma 5.2 we have shown that there exists \( j \in J \setminus K \) and \( k \in K \setminus J \) and therefore \( m_j \nmid b \) and \( m_k \nmid a \). In particular, it follows that \( a \nmid b \) and \( b \nmid a \). Hence no two vertices in \( \{ \prod_{j \in J} m_j \mid |J| = k \} \) are not connected by an edge.

5: Let \( a = \prod_{j \in J} m_j \) be perfect divisor of m and set \( k = |J| \). The perfect divisors of m with respect to S which divide a which are connected by an edge to a correspond to the non-empty, proper subsets of J which are are \( \sum_{i=1}^{k-1} \binom{k}{i} = 2^k - 2 \) many. In addition, we need to count the number of perfect divisors of m which are divisible by a. These are exactly the ones of the form \( \prod_{k \in K} m_k \) with \( J \subseteq K \subseteq \{1, \ldots, n\} \) of which there are \( \sum_{i=1}^{n-k-1} \binom{n-k}{i} = 2^{n-k} - 2 \). Hence \( \deg(a) = 2^k + 2^{n-k} - 4 \).

6: For n = 3, we can verify in Figure 1 that there is cycle of length 6 and no shorter cycle. If n ≥ 4, then \( m_1m_2m_3 \) is a perfect divisor and the edges \( (m_1, m_1m_2), (m_1m_2, m_1m_2m_3) \) and \( (m_1m_2m_3, m_1) \) form a cycle of length 3 which is the smallest possible length PDG(S).
Finally, for (7), it is easily verified that \( \text{PDG}(S) \) is planar if \( n = 3 \). If, however, \( n \geq 4 \), then \( \text{PDG}(S) \) contains \( K_{3,3} \) as the minor depicted in Figure 2 and hence is not planar by Wagner’s theorem on planar graphs.

![Figure 2. PDG(S) contains K_{3,3} as a minor for n \geq 4.](image)

Next, we compute the Ramsey number with respect to the class of perfect divisor graphs. Note that \( \text{PDG} \) is a subclass of \( \text{PoG} \) which immediately implies that \( R_{\text{PDG}}(n, m) \leq R_{\text{PoG}}(n, m) \) for all \( n, m \geq 1 \). We use Theorem 3.3 to show that equality holds.

**Theorem 3.4.** Let \( n, m \geq 1 \). Then for the Ramsey number \( R_{\text{PDG}} \) with respect to the class \( \text{PDG} \) of perfect divisor graphs the following holds

\[
R_{\text{PDG}}(n, m) = R_{\text{PoG}}(n, m) = (n - 1)(m - 1) + 1
\]

**Proof.** We set \( w = (n - 1)(m - 1) \) and show that \( R_{\text{PoG}}(n, m) > w = (n - 1)(m - 1) \) by giving an example of perfect divisor graph \( G \) and an induced subgraph \( H \) of \( G \) with \( w \) vertices which is a complete \( (n - 1) \)-partite graph graph on \( w \) vertices in which independent sets are of cardinality at most \( m - 1 \).

Let \( R = \mathbb{Z} \) and let \( S = \{ p_1, p_2, \ldots, p_w \} \) be a set of \( w \) distinct positive prime numbers of \( \mathbb{Z} \). We set \( m = p_1p_2\cdots p_w \) and \( G = \text{PDG}(S) \).

For each \( 1 \leq i \leq n - 1 \), let \( n_i = (i - 1)(m - 1) \) and we set \( a_i = p_1p_2\cdots p_{n_i} \) (where \( a_1 = 1 \)) and

\[
A_i = \{ a_i, a_i+p_{n+1}, \ldots, a_i+p_{n+(m-1)} \}.
\]

Note that \( A_1 = \{ p_1, \ldots, p_{m-1} \} \).

Let \( H \) be the subgraph of \( G \) induced by the vertex set \( A_1 \cup A_2 \cup \cdots \cup A_{n-1} \). By construction, for each \( 1 \leq i \leq n - 1 \), \( |A_i| = m + 1 \) holds and \( A_i \) is contained in the partition \( P_{n+1} \) of \( G \), cf. Theorem 3.1. This implies that each \( A_i \) is an independent vertex set of \( H \) of cardinality \( m - 1 \).

Moreover, since \( G \) is a \( (w - 1) \)-partite graph, it follows that \( H \) is an \((n - 1)\)-partite graph (with partitioning \( A_1 \cup A_2 \cup \cdots \cup A_{n-1} \)). For an example of this construction with \( m = 5 \) and \( n = 4 \) see Example 3.5.

Thus, no more than \( m - 1 \) vertices of \( G \) are independent and a straightforward verification shows that the clique number of \( G \) is at most \( n - 1 \). Thus \( R_{\text{PoG}}(n, m) > w \). Hence by Theorem 3.3, we have \( R_{\text{PDG}}(n, m) = R_{\text{PoG}}(n, m) = w + 1 = (n - 1)(m - 1) + 1 \).

**Example 3.5.** We demonstrate the construction of the previous proof for the example \( R = \mathbb{Z} \) with \( n = 4 \) and \( m = 5 \). That is, we construct a perfect divisor graph which has a complete \( 3 \)-partite \( H \) as subgraph and each of the partitions of \( H \) consist of 4 independent vertices.

Let \( w = (n - 1)(m - 1) = 12 \) and we set \( S = \{ p_1, p_2, \ldots, p_{12} \} \). Next, let \( n_i = (i - 1)(m - 1) \) for \( 1 \leq i \leq 4 \), that is, \( n_1 = 0, n_2 = 4 \) and \( n_3 = 8 \). Then \( a_1 = 1, a_2 = p_1p_2p_3p_4 \) and \( a_3 = p_1p_2\cdots p_8 \).
We set

\[ A_1 = \{a_1p_1, a_1p_2, a_1p_3, a_1p_4\} = \{p_1, p_2, p_3, p_4\} \]
\[ A_2 = \{a_2p_5, a_2p_6, a_2p_7, a_2p_8\} = \{(p_1 \cdots p_4)p_5, (p_1 \cdots p_4)p_6, (p_1 \cdots p_4)p_7, (p_1 \cdots p_4)p_8\} \]
\[ A_3 = \{a_3p_9, a_3p_{10}, a_3p_{11}, a_3p_{12}\} = \{(p_1p_2 \cdots p_8)p_9, (p_1p_2 \cdots p_8)p_{10}, \ldots, (p_1p_2 \cdots p_8)p_{12}\} \]

The subgraph of \(PDG(S)\) induced by \(A_1 \cup A_2 \cup A_3\) is a complete 3-partite graph in which each partition has 4 vertices that are independent, see Figure 3.

![Figure 3. Induced subgraph \(H\) of \(PDG([p_1, \ldots, p_{12}])\)](image)

3.2. The divisibility graph of a commutative ring.

**Definition 3.6.** Let \(R\) be a commutative ring and \(a, b\) be distinct elements of \(R\).

1. If \(a\) is a non-zero non-unit element of \(R\), then we say \(a\) is a proper element of \(R\).
2. If \(a \mid b\) (in \(R\)) and \(b \mid a\) (in \(R\)), then we write \(a \parallel b\).
3. The divisibility graph \(Div(R)\) of \(R\), is the undirected simple graph whose vertex set consists of the proper elements of \(R\) such that two vertices \(a \neq b\) are adjacent if and only if \(a \parallel b\)
or \(b \parallel a\).

The following lemma can be verified by a straight-forward argument.

**Lemma 3.7.** Let \(R\) be a commutative ring and let \(V\) be the set of all proper elements of \(R\) and define \(\leq\) on \(V\) such that for all \(a, b \in V\), we have \(a \leq b\) if and only if \(a = b\) or \(a \parallel b\).

Then \((V, \leq)\) is a partially ordered set and the divisibility graph \(Div(R)\) of \(R\) is a partial order graph.

By Theorem 2.2 it is clear that \(R_{PDG}(n, m) \leq R_{PDG}(n, m)\) holds. However, since the class \(PDG\) of perfect divisor graphs is a subclass of \(DivG\) it follows from Theorem 3.4 that equality holds. We conclude the following theorem.

**Theorem 3.8.** Let \(n, m \geq 1\) be positive integers \((n, m \text{ need not be distinct})\). Then for the Ramsey number \(R_{PDG}\) with respect to the class \(DivG\) of divisibility graphs the following holds

\[ R_{PDG}(n, m) = R_{PDG}(n, m) = (n-1)(m-1) + 1. \]

Moreover, in view of Theorem 3.8 we have the following result.

**Corollary 3.9.** Let \(n, m \geq 1\) be positive integers \((n, m \text{ need not be distinct}), k = (n-1)(m-1)+1, R\) be a commutative ring and \(S\) be a subset of proper elements of \(R\) such that \(|S| \geq k\).

Then one of the following assertions holds:

1. There are \(n\) elements \(a_1, \ldots, a_n \in S\) such that \(a_1 \parallel a_2 \parallel \cdots \parallel a_n\) (in \(R\)).
(2) There are $m$ pairwise distinct elements $b_1, \ldots, b_m \in S$ for all $1 \leq h \neq f \leq m$ either
   \begin{itemize}
     \item $b_h \mid b_f$ or
     \item $b_h \mid b_f$ and $b_f \mid b_h$
   \end{itemize}
holds.

3.3. Inclusion ideal graphs of rings.

Definition 3.10. Let $R$ be a ring.

(1) We call a left (right) ideal $I$ or $R$ non-trivial if $I \neq \{0\}$ and $I \neq R$.
(2) The inclusion ideal graph $\text{In}(R)$ of $R$ is the (simple, undirected) graph whose vertex set is
   the set of non-trivial left ideals of $R$ and two distinct left ideals $I, J$ are adjacent if and only if
   $I \subseteq J$ or $J \subseteq I$ (cf. Akbari et. al [11]).
(3) By $\text{InG}$, we denote the class of all inclusion ideal graphs.

Remark 3.11. The set $V$ of all non-trivial left ideals of a ring $R$ together with the partial order
$\subseteq$ induced by inclusion is a partially ordered set. Hence the inclusion graph $\text{In}(R)$ of a ring $R$ is
a partial order graph.

By Theorem 2.2, it is clear that $\mathcal{R}_{\text{InG}}(n, m) \leq \mathcal{R}_{\text{PoG}}(n, m)$. On the other hand, if $R$ is
commutative, then subgraph of $\text{In}(R)$ induced by the set of principal ideals of $R$ is graph-
isomorphic to $\text{Div}_R(S)$ where $S$ contains a one generator for each ideal in $S$. It follows that
$\mathcal{R}_{\text{PoG}}(n, m) \leq \mathcal{R}_{\text{InG}}(n, m)$.

Hence by Theorems 2.2 and 3.12 we conclude the following theorem.

Theorem 3.12. Let $n, m \geq 1$ be positive integers ($n, m$ need not be distinct). Then for the
Ramsey number $\mathcal{R}_{\text{InG}}$ with respect to the class $\text{InG}$ of inclusion ideal graphs the following holds
$\mathcal{R}_{\text{InG}}(n, m) = \mathcal{R}_{\text{PoG}}(n, m) = (n-1)(m-1) + 1$.

In view of Theorem 3.12, we have the following result:

Corollary 3.13. Let $R$ be a ring, $n, m \geq 1$ be positive integers ($n, m$ need not be distinct) and
$S \subseteq \{ I \mid I \text{ is a non-trivial left ideal of } R \}$ such that $|A| \geq (n-1)(m-1) + 1$.

Then one the following assertions hold:

(1) There are $n$ pairwise distinct elements (non-trivial left ideals) $I_1, \ldots, I_n \in A$ with $I_1 \subset
   I_2 \subset \cdots \subset I_n$.
(2) There are $m$ elements (non-trivial left ideals) $J_1, \ldots, J_m \in A$ such that $J_a \nsubseteq J_b$ for every
   $1 \leq a \neq b \leq m$.

3.4. Matrix graphs over commutative rings.

Definition 3.14. Let $R$ be a commutative ring which is not a field and $j \geq 2$ an integer.

(1) We denote by $R^{j \times j}$ the ring of all $j \times j$ matrices with entries in $R$.
(2) Let $S = \{ A \in R^{j \times j} \mid \det(A) \text{ a proper element of } R \}$ of all $j \times j$ matrices whose determi-
nant is a proper element of $R$, cf. Definition 3.11. We define the matrix graph $\text{MatG}(R)$
of $R$ to be the undirected simple graph with $S$ as its vertex set and two distinct vertices
$A, B \in S$ are adjacent if and only if $\det(A) \parallel \det(B)$ or $\det(A) \parallel \det(B)$.
(3) By $\text{MatG}$ we denote the class of all matrix graphs.

Lemma 3.15. Let $R$ be a commutative ring which is not a field, $j \geq 2$ an integer and
$V = \{ A \in R^{j \times j} \mid \det(A) \text{ is a proper element of } R \}$.

Define $\leq$ on $V$ such that for all $A, B \in V$, we have $A \leq B$ if and only if $A = B$ or $\det(A) \parallel \det(B)$.

Then $(V, \leq)$ is a partially ordered set and the graph $\text{MatG}(R)$ is a partial order graph.

By Theorem 2.2, it is clear that $\mathcal{R}_{\text{MatG}}(n, m) \leq \mathcal{R}_{\text{PoG}}(n, m)$. We prove next that equality holds.

Theorem 3.16. Let $n, m \geq 1$ be positive integers ($n, m$ need not be distinct). Then for the
Ramsey number $\mathcal{R}_{\text{MatG}}$ with respect to the class $\text{MatG}$ of matrix graphs the following holds
$\mathcal{R}_{\text{MatG}}(n, m) = \mathcal{R}_{\text{PoG}}(n, m) = (n-1)(m-1) + 1$. 
Proof. Let $R = \mathbb{Z}$ and $j \geq 2$ and set $w = (n - 1)(m - 1) \geq 1$. Further, let $p_1, p_2, \ldots, p_w$ be distinct positive prime numbers of $\mathbb{Z}$ and choose $X_i \in R^{j \times j}$ with $\det(X_i) = p_i$ for $1 \leq i \leq w$.

We construct a matrix graph $\text{MatG}(R)$ which has a complete $(n - 1)$-partite subgraph $H$ in which each partition has $m - 1$ vertices. The construction is analogous to the one in the proof of Theorem 2.3.

For each $1 \leq i \leq n - 1$, let $n_i = (i - 1)(m - 1)$, $q_i = X_1 X_2 \cdots X_{n_i}$ (hence $q_0 = 1$) and

$$A_i = \{q_i X_{n_i+1}, \ldots, q_i X_{n_i+(m-1)}\}.$$ 

Note that $A_1 = \{X_1, \ldots, X_{m-1}\}$. Since $\det(q_i X_{n_i+j}) = p_1 \ldots p_n p_{n+i}$ it follows that the elements of $A_i$ are pairwise distinct and $|A_i| = m - 1$ for $1 \leq i \leq n - 1$.

Let $S = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and set $G = \text{MatG}(\mathbb{Z})$. Then for each $i$, the vertices in $A_i$ are independent. However, there are edges between all vertices of two distinct sets $A_i$ and $A_j$ with $i \neq j$. Therefore, $G$ is a complete $(n - 1)$-partite graph in which each partition has $m - 1$ vertices that are independent. Thus at most $m - 1$ vertices of $G$ are independent. It is easily verified that the clique number of $G$ is $n - 1$. It follows that $\mathcal{R}_{\text{MatG}}(n, m) > w$ and together with Theorem 2.2 we conclude $\mathcal{R}_{\text{MatG}}(n, m) = \mathcal{R}_{\text{deg}}(n, m) = w + 1 = (n - 1)(m - 1) + 1$. □

Corollary 3.17. Let $R$ be a commutative ring, $j \geq 2$, $n$, $m \geq 1$ be positive integers ($n$, $m$ need not be distinct) and $S \subseteq \{X \in D \mid \det(X) \text{ is a proper element of } R\}$ such that $|S| \geq (n - 1)(m - 1) + 1$.

Then one of the following assertions hold:

1. There are $n$ matrices $X_1, \ldots, X_n \in S$ such that $\det(X_1) \parallel \det(X_2) \parallel \cdots \parallel \det(X_n)$ (in $R$).
2. There are $m$ pairwise distinct matrices $Y_1, \ldots, Y_m \in S$, such that for all $1 \leq h \neq f \leq m$.
   - $\det(Y_h) \parallel \det(Y_f)$ or
   - $\det(Y_h) \parallel \det(Y_f) \text{ and } \det(Y_f) \parallel \det(Y_h)$
   

holds.

3.5. Idempotents graphs of commutative rings.

Definition 3.18. Let $R$ be a commutative ring.

1. We call $a \in R$ idempotent if $a^2 = a$.
2. We define the idempotents graph $\text{Idm}(R)$ of $R$ to be the undirected simple graph with the set of idempotents of $R$ as its vertex set and two distinct vertices $a, b$ are adjacent if and only if $a \mid b$ or $b \mid a$.
3. By $\text{IdemG}$ we denote the class of all idempotents graphs.

First, we show that the divisibility relation is a partial order on the set of idempotent elements of $R$.

Lemma 3.19. Let $R$ be a commutative ring and let $V$ be the set of all idempotent elements of $R$.

We define $\leq$ on $V$ such that for all $a, b \in V$, we have $a \leq b$ if and only if $a \mid b$.

Then $(V, \leq)$ is a partially ordered set and the graph $\text{Idm}(R)$ is a partial order graph.

Proof. Clearly, $\leq$ is reflexive and transitive. Suppose that $a \mid b$ and $b \mid a$ (in $R$), that is, $a = bx$ and $b = ay$ for some $x, y \in R$. Then, since $a$ and $b$ are idempotent, we can conclude that

$$a - ba = (1 - b)a = (1 - b)bx = bx - b^2x = bx - bx = 0$$

and

$$b - ab = (1 - a)b = (1 - a)ay = ay - a^2y = ay - ay = 0$$

and hence $a = ba = ab = b$ which implies that $\leq$ is anti-symmetric. □

By Theorem 2.2 it is clear that $\mathcal{R}_{\text{IdemG}}(n, m) \leq \mathcal{R}_{\text{deg}}(n, m)$. Next, we show that $\mathcal{R}_{\text{IdemG}}(n, m) = \mathcal{R}_{\text{deg}}(n, m)$. We start with the following lemma.

Lemma 3.20. Let $R$ be a commutative ring and $E$ be a set of $w \geq 3$ distinct non-trivial idempotents of $R$ such that $eR$ is a maximal ideal of $R$ for every $e \in E$. Let $x = f_1 f_2 \cdots f_k$ and $y = b_1 b_2 \cdots b_j$ such that $f_1, \ldots, f_k, b_1, \ldots, b_j \in E$ and $2 \leq k, j < w$.

Then
Definition 4.1. \( m \) is \( R \).

Theorem 3.23. Let \( m \) be a subset of idempotent elements of \( R \) and each \( e_i R \) is a maximal ideal of \( R \). \( 1 \leq i \leq w \), by Lemma 3.19 we conclude that \( e_1 R, \ldots, e_w R \) are distinct maximal ideals of \( R \). Since \( k < w \), there exists a maximal ideal \( d R \) for some \( d \in E \) such that \( x = f_1 f_2 \cdots f_k \notin d R \) (note that each \( f_i R \) is a maximal ideal of \( R \)). Thus \( x \neq 0 \).

(ii) We may assume that \( f_i \neq f_j \) for every \( 1 \leq i \leq j \). Hence \( x \in f_1 R \) but \( y \neq f_1 R \) and thus \( x \neq y \). Since all \( f_i \) and \( b_i \) are idempotent elements, multiplicities have no impact which makes the other implication obvious. \( \square \)

Remark 3.22. Induced by \( A \) \( \left| \begin{array}{c} \text{(1) } x \neq 0, \\ \text{(2) } x = y \text{ if and only if } \{f_1, \ldots, f_k\} = \{b_1, \ldots, b_j\}. \end{array} \right. \)

Proof. (i) Since \( e_1, \ldots, e_w \) are distinct non-trivial idempotents of \( R \) and each \( e_i R \) is a maximal ideal of \( R \). \( 1 \leq i \leq w \), by Lemma 3.19 we conclude that \( e_1 R, \ldots, e_w R \) are distinct maximal ideals of \( R \). Since \( k < w \), there exists a maximal ideal \( d R \) for some \( d \in E \) such that \( x = f_1 f_2 \cdots f_k \notin d R \) (note that each \( f_i R \) is a maximal ideal of \( R \)). Thus \( x \neq 0 \).

(ii) We may assume that \( f_i \neq f_j \) for every \( 1 \leq i \leq j \). Hence \( x \in f_1 R \) but \( y \neq f_1 R \) and thus \( x \neq y \). Since all \( f_i \) and \( b_i \) are idempotent elements, multiplicities have no impact which makes the other implication obvious. \( \square \)

Theorem 3.21. Let \( n, m \geq 1 \) be positive integers \((n, m \text{ need not be distinct})\). Then for the Ramsey number \( R_{\text{idem}}(n, m) \) with respect to the class of idempotents graphs the following holds

\[ R_{\text{idem}}(n, m) = R_{\text{DG}}(n, m) = (n - 1)(m - 1) + 1. \]

Proof. We set \( w = (n - 1)(m - 1) \geq 1 \) and show that \( \text{IdemG} \) contains an \((n - 1)-\)partite graph in which each partition consists of \( m - 1 \) independent vertices. For this purpose, set \( R = \prod_{i=1}^w \mathbb{Z}_2 \). It is clear that \( R \) has exactly \( w \) distinct maximal ideals, say \( M_1, \ldots, M_w \), and each \( M_i = p_i R \), \( 1 \leq i \leq w \) for idempotent \( p_i \) of \( R \). We set \( E = \{p_1, p_2, \ldots, p_w\} \). Note that \( |E| = w \) since \( p_1, p_2, \ldots, p_w \) are pairwise distinct.

For each \( 1 \leq i \leq n - 1 \), let \( n_i = (i - 1)(m - 1) \), \( a_i = p_{m-1} \cdots p_{n-1} \) (hence \( a_0 = 1 \)) and \( A_i = \{a_{m-1}, \ldots, a_{n_i(m-1)}\} \). Note that \( A_i = \{p_1, p_2, \ldots, p_{m-1}\} \).

By construction of each \( A_i \) and in light of Lemma 3.20 for each \( 1 \leq i \leq n - 1 \), we have \( |A_i| = m - 1 \) and the vertices of \( A_i \) are independent. Let \( H \) be the subgraph of \( 1 \)-partite graph which is induced by \( A_1 \cup A_2 \cup \cdots \cup A_{n-1} \).

By construction of \( H \) and Lemma 3.20 we conclude that \( H \) is a complete \((n - 1)-\)partite graph in which each partition has \( m - 1 \) vertices that are independent. Thus \( H \) has exactly \( m - 1 \) vertices that are independent. It is easily verified that the clique number of \( H \) is \( n - 1 \). Thus \( R_{\text{idem}}(n, m) \geq w \). Hence by Theorem 2.2 we have \( R_{\text{idem}}(n, m) = R_{\text{DG}}(n, m) = w + 1 = (n - 1)(m - 1) + 1. \)

Remark 3.22. Observe that the ring \( R = \prod_{i=1}^w \mathbb{Z}_2 \) in the proof of Theorem 3.21 is a finite boolean ring. If \( \text{boolG} \) denotes the subclass of \( \text{IdemG} \) consisting of all idempotents graphs of boolean rings.

In view of the proof of Theorem 3.21 we conclude that \( R_{\text{boolG}}(n, m) = R_{\text{IdemG}}(n, m) \). Thus we state this result without a proof.

Theorem 3.23. Let \( n, m \geq 1 \) be positive integers \((n, m \text{ need not be distinct})\).

Then \( R_{\text{boolG}}(n, m) = R_{\text{IdemG}}(n, m) = R_{\text{DG}}(n, m) = (n - 1)(m - 1) + 1. \)

In view of Theorem 3.21 we have the following result.

Corollary 3.24. Let \( n, m \geq 1 \) be positive integers \((n, m \text{ need not be distinct})\), \( k = (n - 1)(m - 1) + 1 \) and \( A \) be a subset of idempotent elements of \( R \) such that \( |A| \geq k \).

Then one of the following assertions holds

1. There are \( n \) pairwise distinct elements (distinct idempotents) \( a_1, \ldots, a_n \in A \) such that \( a_1 | a_2 | \cdots | a_n \) (in \( R \)).

2. There are \( m \) pairwise distinct elements (distinct idempotents) \( b_1, \ldots, b_m \in A \) such that \( b_n \not| b_f \) (in \( R \)) for all \( 1 \leq h \neq f \leq m \).

4. An example class \( C \) of partial order graphs with \( R_C(n, m) \neq R_C(m, n) \)

In this section we present a subclass \( C \) of \( \mathcal{PDG} \) with respect to which the Ramsey numbers \( R_C \) are non-symmetric in \( m \) and \( n \).

Definition 4.1. For \( k \geq 2 \), let \( P_k = \{0, 2k, 3k, \ldots\} = k\mathbb{N}_0 \).

1. For \( a, b \in \mathbb{Z} \) we define \( a \leq b \) if and only if \( a - b \in P_k \).

2. We define the \( k \)-positive cone graph \( \text{ConeG}(k) \) to be the simple, undirected graph with vertex set \( \mathbb{Z} \) such that two vertices \( a, b \) are connected if and only if \( a - b \in P_k \).

3. By \( k\text{-cone} \) we denote the \textit{the class of \( k \)-positive cone graphs}.
Remark 4.2. (1) The relation $a \leq b$ if and only if $a - b \in P_k$ is a partial order on $\mathbb{Z}$ and $\text{ConeG}(k)$ is a partial order graph. In the literature $P_k$ is also known as the positive cone of the partially ordered ring $(\mathbb{Z}, \leq_k)$. 

(2) Two vertices $a, b$ of $\text{ConeG}(k)$ are connected by an edge if and only if $a \equiv b \mod k\mathbb{Z}$.

As the following theorem shows, the Ramsey number $\mathcal{R}_{k\text{-cone}}$ with respect to the class of $k$-positive cone graphs is not always symmetric in $m$ and $n$.

**Theorem 4.3.** Let $k \geq 2$, $n$, $m \geq 1$ be positive integers ($n$, $m$ need not be distinct). Then

(1) If $1 \leq m \leq k + 1$, then

$$\mathcal{R}_{k\text{-cone}}(n, m) = (n - 1)(m - 1) + 1.$$ 

In particular, if $1 \leq n, m \leq k + 1$, then $\mathcal{R}_{k\text{-cone}}(n, m) = \mathcal{R}_{k\text{-cone}}(n, m, n) = (n - 1)(m - 1) + 1$ is symmetric in $n$ and $m$.

(2) If $m \geq k + 1$, then

$$\mathcal{R}_{k\text{-cone}}(n, m) = \mathcal{R}_{k\text{-cone}}(n, k + 1) = (n - 1)k + 1$$

only depends on the first argument $n$. In particular, if $n \neq m$ and either $n \geq k + 1$ or $m \geq k + 1$, then $\mathcal{R}_{k\text{-cone}}(n, m) \neq \mathcal{R}_{k\text{-cone}}(m, n)$.

**Proof.** (1): For $n = 1$ or $m = 1$, the assertion immediately follows, so we assume $n \geq 2$ and $2 \leq m \leq k$. For each $1 \leq i \leq m - 1$, let

$$A_i = \{k + i, 2k + i, \ldots, (n - 1)k + i\}$$

By construction, each $A_i$ contains $n - 1$ distinct elements $a$ with $a - i \in P_k$. Therefore for $a \neq b \in A_i$, either $b - a \in P_k$ or $a - b \in P_k$ and hence each $A_i$ induces a complete subgraph of $\text{ConeG}(k)$ with exactly $n - 1$ vertices. Moreover, since $m - 1 \leq k$, for $a \in A_i$ and $b \in A_j$ with $1 \leq i \neq j \leq m - 1$, then $a \neq b \mod k\mathbb{Z}$ and therefore $a$ and $b$ are not connected by an edge.

Let $H$ be the subgraph of $\text{ConeG}(k)$ which is induced by the vertex set $A_1 \cup \cdots \cup A_{m - 1}$. Then $H$ is a complete $(n - 1)$-partite subgraph in which each independent set of cardinality at most $m - 1$ and hence $\mathcal{R}_{k\text{-cone}}(n, m) \geq w$. It now follows from Theorem 2.2 that $\mathcal{R}_{k\text{-cone}}(n, m) = (n - 1)(m - 1) + 1$.

The symmetry assertion follows immediately from this if, moreover, $1 \leq n \leq k + 1$ holds.

(2): Recall that two vertices $a, b$ of $\text{ConeG}(k)$ are connected by an edge if and only if $a \equiv b \mod k\mathbb{Z}$. Therefore, a maximal independent subset has cardinality $k$ (the number of residue classes mod $k$). Thus if $m \geq k + 1$, then $\text{ConeG}(k)$ cannot contain an independent set with $m$ distinct vertices. Therefore for all $m \geq k + 1$ the equality

$$\mathcal{R}_{k\text{-cone}}(n, m) = \mathcal{R}_{k\text{-cone}}(n, k + 1)$$

The assertion now follows from (1). \hfill \Box

In view of Theorem 4.3 we have the following result.

**Corollary 4.4.** $k \geq 2$ and $n$, $m \geq 1$ be positive integers ($n$, $m$ need not be distinct) and $A$ be a subset of $\mathbb{Z}$. Then

(1) If $2 \leq m \leq k$ and $|A| > (n - 1)(m - 1)$, then there are at least $n$ pairwise distinct elements $a_1, \ldots, a_n \in A$ such that $a_1 \equiv \cdots \equiv a_n \mod k$ or there at least $m$ elements $b_1, \ldots, b_m \in A$ such that $b_i \not\equiv b_j \mod k$ for all $1 \leq i \neq j \leq m$.

(2) If $m > k$ and $|A| > (n - 1)k$, then there are at least $n$ pairwise distinct elements of $A$, say $a_1, \ldots, a_n$ such that $a_1 \equiv \cdots \equiv a_n \mod k$.

**Example 4.5.** The subgraph of $\text{ConeG}(3)$ induced by $\{1, 2, 3, \ldots, 12\}$ consists of $3$ 4-cliques.

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Figure 4. Subgraph of ConeG(3) induced by \{1, 2, 3, \ldots, 12\}

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