CONVOLVED FIBONACCI NUMBERS AND THEIR APPLICATIONS

TAEKYUN KIM, DMITRY V. DOLGY, DAE SAN KIM, AND JONG JIN SEO

Abstract. In this paper, we present a new approach to the convolved Fibonacci numbers arising from the generating function of them and give some new and explicit identities for the convolved Fibonacci numbers.

1. Introduction

As is well known, the Fibonacci numbers are given by the numbers in the following integer sequence:

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\]

The sequence \(F_n\) of Fibonacci numbers is defined by the recurrence relation as follows:
\begin{equation}
F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad (n \geq 2), \quad (\text{see}\ [1–8]).
\end{equation}

The sequence can be extended to negative index \(n\) arising from the rearranged recurrence relation
\begin{equation}
F_{n-2} = F_n - F_{n-1}, \quad (\text{see}\ [1–13])
\end{equation}
which yields the sequence of “negafibonacci” numbers satisfying
\begin{equation}
F_{-n} = (-1)^{n+1} F_n, \quad (\text{see}\ [11, 12]).
\end{equation}

It is well known that the generating function of Fibonacci numbers is given by
\begin{equation}
\frac{1}{1 - t - t^2} = \sum_{n=0}^{\infty} F_n t^n, \quad (\text{see}\ [3–6]).
\end{equation}

The convolved Fibonacci numbers \(p_n(x), \ (n \geq 0)\), are defined by the generating function
\begin{equation}
\left(\frac{1}{1 - t - t^2}\right)^x = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}, \quad (x \in \mathbb{R}).
\end{equation}

2010 Mathematics Subject Classification. 05A19, 11B83, 34A30.

Key words and phrases. Fibonacci numbers, convolved Fibonacci numbers, linear differential equation.
From (1.4) and (1.5), we note that

\[
\frac{p_n(1)}{n!} = F_n, \quad (n \geq 0).
\]

In this paper, we present a new approach to the convolved Fibonacci numbers arising from the generating function of them and give some new and explicit identities for the convolved Fibonacci numbers.

2. CONVOLVED FIBONACCI NUMBERS AND THEIR APPLICATIONS

From (2.45), we note that

\[
\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = \left( \frac{1}{1-t-t^2} \right)^x = \left( \frac{1}{1-t-t^2} \right) \left( \frac{1}{1-t-t^2} \right)^{x-1}
\]

\[
= \left( \sum_{l=0}^{\infty} p_l(1) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} p_m(x-1) \frac{t^m}{m!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} p_l(1) p_{n-l}(x-1) \right) \frac{t^n}{n!}.
\]

By comparing the coefficients on both sides of (2.1), we obtain the following proposition.

**Proposition 1.** For \( n \geq 0, x \in \mathbb{R}, \) we have

\[
p_n(x) = \sum_{l=0}^{n} \binom{n}{l} p_l(1) p_{n-l} (x-1).
\]

Let us take \( x = r \in \mathbb{N}. \) Then, by Proposition 1 we get

\[
p_n(r) = \sum_{l_1=0}^{n} \binom{n}{l_1} p_{l_1}(1) p_{n-l_1}(r-1)
\]

\[
= \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \binom{n-l_1}{l_2} p_{l_1}(1) p_{l_2}(1) p_{n-l_1-l_2}(r-2)
\]

\[
= \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \sum_{l_3=0}^{n-l_1-l_2} \binom{n-l_1}{l_2} \binom{n-l_1-l_2}{l_3} p_{l_1}(1) p_{l_2}(1) p_{l_3}(1) p_{n-l_1-l_2-l_3}(r-3)
\]

\[
\vdots
\]

\[
= \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_{r-1}=0}^{n-l_2-l_{r-2}} \binom{n-l_1}{l_2} \binom{n-l_1-l_2}{l_3} \cdots
\]
\[ \times \left( n - l_1 - l_2 - \cdots - l_{r-2} \right) \]
\[ \times \left( \prod_{k=1}^{r-1} p_{l_k} (1) \right) p_{n-l_1-l_2-\cdots-l_{r-1}} (1). \]

Therefore, by (2.2), we obtain the following corollary.

**Corollary 2.** For \( r \in \mathbb{N} \) and \( n \geq 0 \), we have

\[ p_n (r) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_{r-1}=0}^{n-l_1-\cdots-l_{r-2}} \left( \begin{array}{c} n \\ l_1 \end{array} \right) \left( \begin{array}{c} n - l_1 \\ l_2 \end{array} \right) \cdots \]
\[ \times \left( \begin{array}{c} n - l_1 - l_2 - \cdots - l_{r-2} \\ l_{r-1} \end{array} \right) \left( \prod_{k=1}^{r-1} p_{l_k} (1) \right) p_{n-l_1-l_2-\cdots-l_{r-1}} (1). \]

We observe that

(2.3) \[ \frac{1}{1 - t - t^2} = \left( \frac{1}{1 - t - t^2} \right)^r \]
\[ = \left( \sum_{l=0}^{\infty} p_l (r) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} p_m (x-r) \frac{t^m}{m!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p_l (r) p_{n-l} (x-r) \right) \frac{t^n}{n!}. \]

Therefore, by (2.3) and (2.2), we obtain the following theorem.

**Theorem 3.** For \( n \geq 0, r \in \mathbb{N} \), we have

\[ p_n (x) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p_l (r) p_{n-l} (x-r) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p_{n-l} (r) p_l (x-r). \]

Let us take \( x = r + 1 \) in Theorem 3. Then, we have

(2.4) \[ p_n (r + 1) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p_{n-l} (r) p_l (1) \]
\[ = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p_{n-l} (r) l! p_l (1) \]
\[ = \sum_{l=0}^{n} (n)_l p_{n-l} (r) F_l, \]
where \( (x)_n = x (x-1) \cdots (x-n+1) \), \( (n)_0 = 1 \).
Corollary 4. For $r \in \mathbb{N}$, $n \geq 0$, we have

$$p_n (r + 1) = \sum_{l=0}^{n} (n)_l \cdot p_{n-l} (r) \cdot F_l.$$ 

Taking $r = 1$ in Corollary 4, we have

$$p_n (2) = \sum_{l=0}^{n} (n)_l \cdot p_{n-l} (1) \cdot F_l$$

$$= \sum_{l=0}^{n} (n)_l \cdot (n-l)! \cdot \frac{p_{n-l} (1) \cdot F_l}{(n-l)!}$$

$$= n! \sum_{l=0}^{n} \binom{n}{l} \binom{n-l}{l}^{-1} \cdot F_{n-l} F_l$$

$$= n! \sum_{l=0}^{n} F_l F_{n-l}.$$ 

Thus, by (2.5), we get

$$\frac{p_n (2)}{n!} = \sum_{l=0}^{n} F_l F_{n-l}, \quad (n \geq 0).$$ 

From (2.6), we can also derive the following equation.

$$p_n (3) = \sum_{l_1=0}^{n} (n)_{l_1} \cdot p_{n-l_1} (2) \cdot F_{l_1}$$

$$= \sum_{l_1=0}^{n} (n)_{l_1} \cdot (n-l_1)! \cdot \frac{p_{n-l_1} (2) \cdot F_{l_1}}{(n-l_1)!}$$

$$= n! \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} F_{l_1} F_{l_2} F_{n-l_1-l_2}.$$ 

Thus, by (2.7), we get

$$\frac{p_n (3)}{n!} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} F_{l_1} F_{l_2} F_{n-l_1-l_2}.$$ 

For $r = 3$ in Corollary 4, we have

$$p_n (4) = \sum_{l_1=0}^{n} (n)_{l_1} \cdot p_{n-l_1} (3) \cdot F_{l_1}$$

$$= n! \sum_{l_1=0}^{n} \frac{p_{n-l_1} (3)}{(n-l_1)!} \cdot F_{l_1}.$$
\[ n! \sum_{l_1=0}^{n-l_1} \sum_{l_2=0}^{n-l_1-l_2} \sum_{l_3=0}^{n-l_1-l_2-l_3} F_{l_1} F_{l_2} F_{l_3} F_{n-l_1-l_2-l_3}. \]

From (2.9), we note that
\[ \frac{p_n (4)}{n!} = n! \sum_{l_1=0}^{n-l_1} \sum_{l_2=0}^{n-l_1-l_2} \sum_{l_3=0}^{n-l_1-l_2-l_3} F_{l_1} F_{l_2} F_{l_3} F_{n-l_1-l_2-l_3}. \]

Continuing this process, we have
\[ \frac{p_n (r + 1)}{n!} = n! \sum_{l_1=0}^{n-l_1} \sum_{l_2=0}^{n-l_1-l_2} \cdots \sum_{l_r=0}^{n-l_1-l_2-\cdots-l_{r-1}} F_{l_1} F_{l_2} \cdots F_{l_r} F_{n-l_1-l_2-\cdots-l_r}, \]
where \( r \in \mathbb{N} \).

**Theorem 5.** For \( r \in \mathbb{N} \) and \( n \geq 0 \), we have
\[ \frac{p_n (r + 1)}{n!} = n! \sum_{l_1=0}^{n-l_1} \sum_{l_2=0}^{n-l_1-l_2} \cdots \sum_{l_r=0}^{n-l_1-l_2-\cdots-l_{r-1}} F_{l_1} F_{l_2} \cdots F_{l_r} F_{n-l_1-l_2-\cdots-l_r}. \]

Let
\[ F(t, x) = (1 - t - t^2)^{-x} = e^{-x \log(1-t-t^2)}. \]

Then, by (2.12), we get
\[ F^{(1)}(t, x) = \frac{dF}{dt}(t, x) = x (1 + 2t) (1 - t - t^2)^{-x-1} = x (1 + 2t) F(t, x + 1), \]
\[ F^{(2)}(t, x) = \frac{dF^{(1)}}{dt}(t, x) = 2x F(t, x + 1) + \langle x \rangle_2 (1 + 2t)^2 F(t, x + 2), \]
where \( \langle x \rangle_n = x (x + 1) \cdots (x + n - 1), (n \geq 1), \langle x \rangle_0 = 1 \).

From (2.14), we note that
\[ F^{(3)}(t, x) = \frac{dF^{(2)}}{dt}(t, x) = 6 \langle x \rangle_2 (1 + 2t) F(t, x + 2) + \langle x \rangle_3 (1 + 2t)^3 F(t, x + 3). \]
\[ F^{(4)}(t, x) = \frac{dF^{(3)}}{dt}(t, x) = 12 \langle x \rangle_2 F(t, x + 2) + 12 \langle x \rangle_3 (1 + 2t)^2 F(t, x + 3) \]
\[ + \langle x \rangle^4_4 (1 + 2t)^4 F(t, x + 4), \]

(2.17)
\[ F^{(5)}(t, x) = \frac{dF^{(4)}}{dt}(t, x) \]
\[ = 60 \langle x \rangle^3_3 (1 + 2t) F(t, x + 3) + 20 \langle x \rangle^3_4 (1 + 2t)^3 F(t, x + 4) \]
\[ + \langle x \rangle^5_5 (1 + 2t)^5 F(t, x + 5) \]

and

(2.18)
\[ F^{(6)}(t, x) = \frac{dF^{(5)}}{dt}(t, x) \]
\[ = 120 \langle x \rangle^3_3 F(t, x + 3) + 180 \langle x \rangle^3_4 (1 + 2t)^2 F(t, x + 4) \]
\[ + 30 \langle x \rangle^5_5 (1 + 2t)^4 F(t, x + 5) + \langle x \rangle^6_6 (1 + 2t)^6 F(t, x + 6). \]

Thus, we are led to put

(2.19)
\[ F^{(N)}(t, x) = \left( \frac{d}{dt} \right)^N F(t, x) \]
\[ = \sum_{i=0}^{\left\lfloor \frac{N}{2} \right\rfloor} a_i(N) \langle x \rangle^{N-i}_N (1 + 2t)^{N-2i} F(t, x + N - i) \]

where \( N \in \mathbb{N} \).

Taking the derivatives of (2.19) with respect to \( t \), we have

(2.20)
\[ F^{(N+1)}(t, x) = \sum_{i=0}^{\left\lfloor \frac{N}{2} \right\rfloor} a_i(N) \langle x \rangle^{N-i}_N (1 + 2t)^{N-2i} F^{(1)}(t, x + N - i) \]
\[ + \sum_{i=0}^{\left\lfloor \frac{N}{2} \right\rfloor} a_i(N) \langle x \rangle^{N-i}_N 2(N - 2i)(1 + 2t)^{N-2i-1} F(t, x + N - i) \]
\[ = \sum_{i=0}^{\left\lfloor \frac{N}{2} \right\rfloor} 2(N - 2i) a_i(N) \langle x \rangle^{N-i}_N (1 + 2t)^{N-2i-1} F(t, x + N - i) \]
\[ + \sum_{i=0}^{\left\lfloor \frac{N}{2} \right\rfloor + 1} a_i(N) \langle x \rangle^{N-i+1}_N (1 + 2t)^{N-2i+1} F(t, x + N - i + 1) \]
\[ = \sum_{i=1}^{\left\lfloor \frac{N}{2} \right\rfloor + 1} 2(N - 2i + 2) a_{i-1}(N) \langle x \rangle^{N-i+1}_N \]
\begin{equation}
\left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) + \sum_{i=0}^{\lfloor N/2 \rfloor} a_i (N) \langle x \rangle_{N-i+1} \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) \right) \cdot
\end{equation}

On the other hand, by replacing \( N \) by \( N + 1 \) in (2.19), we get
\begin{equation}
F^{(N+1)}(t, x) = \sum_{i=0}^{\lfloor N/2 \rfloor} a_i (N + 1) \langle x \rangle_{N-i+1} \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) .
\end{equation}

**Case 1.** Let \( N \) be an even number. Then we have
\begin{equation}
\sum_{i=1}^{\lfloor N/2 \rfloor} 2 \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) + \sum_{i=0}^{\lfloor N/2 \rfloor} a_i (N) \langle x \rangle_{N-i+1} \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1)
\end{equation}

Comparing the coefficients on both sides of (2.22), we get
\begin{align}
\sum_{i=0}^{\lfloor N/2 \rfloor} a_i (N) & = a_i (N), \\
2 \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) & = 2a_i (N) + a_i (N) , \\
1 \leq i \leq \frac{N}{2}.
\end{align}

**Case 2.** Let \( N \) be an odd number. Then, we have
\begin{equation}
\sum_{i=1}^{\lfloor (N+1)/2 \rfloor} 2 \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1) + \sum_{i=0}^{\lfloor (N+1)/2 \rfloor} a_i (N) \langle x \rangle_{N-i+1} \left(1 + 2t\right)^{N-2i+1} F(t, x + N - i + 1)
\end{equation}

Comparing the coefficients on both sides of (2.25), we have
\begin{align}
a_0 (N) & = a_0 (N), \\
a_N/2 (N + 1) & = 2a_{N-1} (N),
\end{align}
and

(2.27) \[ a_i (N + 1) = 2 (N - 2i + 2) a_{i-1} (N) + a_i (N), \quad (1 \leq i \leq \frac{N - 1}{2}) \]

In addition, we have the following “initial conditions”:

(2.28) \[ F^{(0)} (t, x) = F (t, x) = a_0 (0) F (t, x) \]

Thus, by (2.28), we get \( a_0 (0) = 1 \).

From (2.13) and (2.19), we note that

(2.29) \[ F^{(1)} (t, x) = a_0 (1) x (1 + 2t) F (t, x + 1) = x (1 + 2t) F (t, x + 1) \]

Thus, by (2.29), we see that \( a_0 (1) = 1 \).

By (2.14) and (2.19), we easily get

(2.30) \[ F^{(2)} (t, x) = \sum_{i=0}^{1} a_i (2) \langle x \rangle_{2-i} (1 + 2t)^{2-2i} F (t, x + 2 - i) \]

Thus, by comparing the coefficients on both sides of (2.30), we get

(2.31) \[ a_0 (2) = 1, \quad \text{and} \quad a_1 (2) = 2. \]

In (2.19), it is not difficult to show that

(2.32) \[ a_{\frac{N+1}{2}} (N) = 0, \quad \text{for all positive integers } N. \]

From (2.32), we note that

(2.33) \[ a_1 (1) = a_2 (3) = a_3 (5) = a_4 (7) = \cdots = 0. \]

By (2.32), we get

(2.34) \[ F^{(N)} (t, x) = \sum_{i=0}^{\left[ \frac{N+1}{2} \right]} a_i (N) \langle x \rangle_{N-i} (1 + 2t)^{N-2i} F (t, x + N - i), \]

where

(2.35) \[ a_0 (N + 1) = a_0 (N), \quad a_{\frac{N+1}{2}} (N) = 0, \quad \text{for all positive integers } N, \]

and

(2.36) \[ a_i (N + 1) = 2 (N - 2i + 2) a_{i-1} (N) + a_i (N), \quad (1 \leq i \leq \left[ \frac{N + 1}{2} \right]). \]

From (2.35), we note that

(2.37) \[ a_0 (N + 1) = a_0 (N) = a_0 (N - 1) = \cdots = a_0 (1) = 1. \]
For \( i = 1, 2, 3 \) in (2.36), we have

\[
(2.38) \quad a_1 (N + 1) = 2 \sum_{k=0}^{N-1} (N - k) a_0 (N - k),
\]

\[
(2.39) \quad a_2 (N + 1) = 2 \sum_{k=0}^{N-3} (N - 2 - k) a_1 (N - k),
\]

and

\[
(2.40) \quad a_3 (N + 1) = 2 \sum_{k=0}^{N-5} (N - 4 - k) a_2 (N - k).
\]

Thus, we can deduce that, for \( 1 \leq i \leq \left[ \frac{N+1}{2} \right] \),

\[
(2.41) \quad a_i (N + 1) = 2 \sum_{k=0}^{N-2i+1} (N - k - 2i + 2) a_{i-1} (N - k)
\]

\[
= 2 \sum_{k=0}^{N+2-2i} k a_{i-1} (k + 2i - 2).
\]

Now, we give explicit expressions for \( a_i (N + 1) \).

From (2.37), (2.38), (2.39) and (2.40), we have

\[
(2.42) \quad a_1 (N + 1) = 2 \sum_{k_1=1}^{N} k_1 a_0 (k_1) = 2 \sum_{k_1=1}^{N} k_1
\]

\[
(2.43) \quad a_2 (N + 1) = 2 \sum_{k_2=1}^{N-2} k_2 a_1 (k_2 + 2) = 2^2 \sum_{k_2=1}^{N-2} \sum_{k_1=1}^{k_2+1} k_2 k_1,
\]

\[
(2.44) \quad a_3 (N + 1) = 2 \sum_{k_3=1}^{N-4} k_3 a_2 (k_3 + 4) = 2^3 \sum_{k_3=1}^{N-4} \sum_{k_2=1}^{k_3+1} \sum_{k_1=1}^{k_2+1} k_3 k_2 k_1
\]

and

\[
(2.45) \quad a_4 (N + 1) = 2^4 \sum_{k_4=1}^{N-6} \sum_{k_3=1}^{k_4+1} \sum_{k_2=1}^{k_3+1} \sum_{k_1=1}^{k_2+1} k_4 k_3 k_2 k_1.
\]

Continuing this process, we have

\[
(2.46) \quad a_i (N + 1) = 2^i \sum_{k_{i-1}=1}^{N-2i+1} \sum_{k_{i-2}=1}^{k_{i-1}+1} \cdots \sum_{k_1=1}^{k_{i-2}+1} \left( \prod_{l=1}^{i} k_l \right), \quad \left( 1 \leq i \leq \left[ \frac{N+1}{2} \right] \right).
\]

Therefore, by (2.46), we obtain the following theorem.
Theorem 6. For $N = 0, 1, 2, \ldots$, the family of differential equations

$$F^{(N)}(t, x) = \left(\frac{d}{dt}\right)^N F(t, x)$$

$$= \left(\sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i(N) \langle x \rangle_{N-i} (1 + 2t)^{N-2i} (1 - t - t^2)^{-N+i}\right) F(t, x)$$

have a solution

$$F(t, x) = (1 - t - t^2)^{-x}$$

where

$$a_0(N) = 1, \quad a_{\frac{N+1}{2}}(N) = 0, \quad \text{for all positive integers } N,$$

and

$$a_i(N) = 2^i \sum_{k_i=1}^{N-2i+1} \sum_{k_{i-1}=1}^{k_i+1} \cdots \sum_{k_1=1}^{k_2+1} \left(\prod_{l=1}^{i} k_l\right), \quad \left(1 \leq i \leq \left[\frac{N}{2}\right]\right).$$

From (1.4), we note that

$$1 = \sum_{k=0}^{\infty} F_k t^k (1 - t - t^2) = \sum_{k=0}^{\infty} F_k t^k - \sum_{k=1}^{\infty} F_{k-1} t^k - \sum_{k=2}^{\infty} F_{k-2} t^k. \quad (2.47)$$

Comparing the coefficients on the both sides of (2.47), we have

$$F_0 = 1, \quad F_1 - F_0 = 0 \iff F_1 = F_0 = 1, \quad (2.48)$$

and

$$F_k - F_{k-1} - F_{k-2} = 0 \quad \text{if } k \geq 2. \quad (2.49)$$

By (1.4), we easily get

$$F(t, x) = (1 - t - t^2)^{-x} = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!}, \quad (2.50)$$

and

$$F^{(N)}(t, x) = \left(\frac{d}{dt}\right)^N F(t, x) = \sum_{k=0}^{\infty} p_{k+N}(x) \frac{t^k}{k!}. \quad (2.51)$$

On the other hand, by Theorem 6, we get

$$F^{(N)}(t, x) = \sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i(N) \langle x \rangle_{N-i} (1 + 2t)^{N-2i} F(t, x + N - i) \quad (2.52)$$
\[
\sum_{i=0}^{[N+1]} a_i (N) \langle x \rangle_{N-i} (1 + 2t)^{N-2i} \sum_{m=0}^{\infty} p_m (x + N - i) \frac{t^m}{m!}
\]

\[
= \sum_{i=0}^{[N+1]} a_i (N) \langle x \rangle_{N-i} \sum_{l=0}^{\infty} (N - 2i)_l 2^l l! \sum_{m=0}^{\infty} p_m (x + N - i) \frac{t^m}{m!}
\]

\[
= \sum_{i=0}^{[N+1]} a_i (N) \langle x \rangle_{N-i} \sum_{k=0}^{\infty} \left( \sum_{l=0}^{k} \binom{k}{l} (N - 2i)_l 2^l p_{k-l} (x + N - i) \right) \frac{t^k}{k!}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{[N+1]} \sum_{l=0}^{k} \binom{k}{l} (N - 2i)_l 2^l a_i (N) \langle x \rangle_{N-i} p_{k-l} (x + N - i) \right) \frac{t^k}{k!}
\]

Therefore, by comparing the coefficients on both sides of (2.51) and (2.52), we obtain the following theorem.

**Theorem 7.** For \( k, N = 0, 1, 2, \ldots \), we have

\[
p_{k+N} (x) = \sum_{i=0}^{[N+1]} \sum_{l=0}^{k} \binom{k}{l} (N - 2i)_l 2^l a_i (N) \langle x \rangle_{N-i} p_{k-l} (x + N - i),
\]

where

\[
a_0 (N) = 1, \quad a_{N+1} (N) = 0, \quad \text{for all positive integers } N,
\]

\[
a_i (N) = 2^i \sum_{k_1=1}^{N-2i+1} \sum_{k_{i-1}=1}^{k_1+1} \cdots \sum_{k_1=1}^{k_{i-1}+1} \prod_{l=1}^{i} k_l, \quad \left( 1 \leq i \leq \left[ \frac{N}{2} \right] \right).
\]

When \( k = 0 \) in Theorem 7, we have the following corollary.

**Corollary 8.** For \( N = 0, 1, 2, \ldots \), we have

\[
p_N (x) = \sum_{i=0}^{[N+1]} a_i (N) \langle x \rangle_{N-i}.
\]

Let us take \( x = 1 \) in Corollary 8. Then, we easily get

\[
p_N (1) = \sum_{i=0}^{[N+1]} a_i (N) (N - i)! = N! + \sum_{i=1}^{[N+1]} a_i (N) (N - i)!
\]

Thus, by (2.53), we get

\[
p_N (1) = 1 + \frac{[N+1]}{N!} \sum_{i=1}^{[N+1]} a_i (N) (N - i)!
\]

\[
\frac{p_N (1)}{N!} = \frac{1}{N!} \sum_{i=1}^{[N+1]} a_i (N) (N - i)!
\]
Therefore, by (1.6) and (2.54), we obtain the following corollary.

**Corollary 9.** For \( N = 0, 1, 2, \ldots \), we have

\[
F_N - 1 = \frac{1}{N!} \left( \sum_{i=1}^{[N/2]} \sum_{k_1=1}^{N+1-2i} \sum_{k_1=1}^{k_2+1} \cdots \sum_{k_1=1}^{k_{N-1}+1} 2^i \left( \prod_{j=1}^{\text{i}} k_j \right) (N - i)! \right).
\]

**Remark.** Recently, several authors have studied special polynomials and sequences arising from the generating functions (see [1–16]).

**Acknowledgements.** This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund.

**References**

1. K. T. Atanassov, *An extremal problem related to Fibonacci numbers*, Adv. Stud. Contemp. Math. (Kyungshang) 7 (2003), no. 1, 87–92.
2. , *A relation between the prime and the Fibonacci numbers*, Adv. Stud. Contemp. Math. (Kyungshang) 6 (2003), no. 1, 53–56.
3. G. E. Bergum and V. E. Hoggatt, Jr., *Numerator polynomial coefficient array for the convolved Fibonacci sequence*, Fibonacci Quart. 14 (1976), no. 1, 43–48. MR 0392804
4. , *Limits of quotients for the convolved Fibonacci sequence and related sequences*, Fibonacci Quart. 15 (1977), no. 2, 113–116. MR 0441845
5. G.-S. Choi, S.-G. Hwang, I.-P. Kim, and B. L. Shader, (±1)-invariant sequences and truncated Fibonacci sequences, Linear Algebra Appl. 395 (2005), 303–312. MR 2112892
6. H. W. Corley, *The convolved Fibonacci equation*, Fibonacci Quart. 27 (1989), no. 3, 283–284. MR 1002075
7. Y. He and W. Zhang, *A convolution formula for Bernoulli polynomials*, Ars Combin. 108 (2013), 97–104. MR 3060257
8. D. Kang, J. Jeong, S.-J. Lee, and S.-H. Rim, *A note on the Bernoulli polynomials arising from a non-linear differential equation*, Proc. Jangjeon Math. Soc. 16 (2013), no. 1, 37–43. MR 3059283
9. H. S. Kim, J. Neugers, and K. S. So, *Generalized Fibonacci sequences in groupoids*, Adv. Difference Equ. (2013), 2013:26, 10. MR 3022836
10. T. Kim and D. S. Kim, *A note on nonlinear Changhee differential equations*, Russ. J. Math. Phys. 23 (2016), no. 1, 88–92. MR 3482767
11. A. K. Kwasniewski, *Fibonacci-triad sequences*, Adv. Stud. Contemp. Math. (Kyungshang) 9 (2004), no. 2, 109–118. MR 2090114
12. P. Moree, *Convoluted convolved Fibonacci numbers*, J. Integer Seq. **7** (2004), no. 2, Article 04.2.2, 14. MR 2084694

13. J.-W. Park, *On the q-analogue of λ-Daehee polynomials*, J. Comput. Anal. Appl. **19** (2015), no. 6, 966–974. MR 3309750

14. A. G. Shannon and C. K. Cook, *Generalized Fibonacci-Feinberg sequences*, Adv. Stud. Contemp. Math. (Kyungshang) **21** (2011), no. 2, 171–179. MR 2815707

15. X. Ye and Z. Zhang, *On some formulas of the reciprocal sum and the alternating sum for generalized Fibonacci numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **10** (2005), no. 2, 143–148. MR 2130540

16. Z. Zhang and G. Xi, *Some computational formulas of mixed-convoluted sum for Fibonacci and Lucas sequences*, Adv. Stud. Contemp. Math. (Kyungshang) **14** (2007), no. 2, 283–292. MR 2316993

Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

E-mail address: tkkim@kw.ac.kr

Institute of Mathematics and Computer Science, Far Eastern Federal University, 690950 Vladivostok, Russia

E-mail address: dvdolgy@gmail.com

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

E-mail address: dskim@sogang.ac.kr

Department of Applied Mathematics, Pukyong National University, Busan, Republic of Korea

E-mail address: seo2011@pknu.ac.kr