TRACIAL INVARIANTS, CLASSIFICATION AND II$_1$ FACTOR REPRESENTATIONS OF POPA ALGEBRAS

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Abstract. Using various finite dimensional approximation properties, four convex subsets of the tracial space of a unital C$^*$-algebra are defined. One subset is characterized by Connes’ hypertrace condition. Another is characterized by hyperfiniteness of GNS representations. The other two sets are more mysterious but are shown to be intimately related to Elliott’s classification program.

Applications of these tracial invariants include:
1. An analogue of Szegő’s Limit Theorem for arbitrary self-adjoint operators.
2. A McDuff factor embeds into $R^\omega$ if and only if it contains a weakly dense operator system which is injective.
3. There exists a simple, quasidiagonal, real rank zero C$^*$-algebra with non-hyperfinite II$_1$ factor representations and which is not tracially AF. This answers negatively questions of Sorin Popa and, respectively, Huaxin Lin.
4. If $A$ is any one of the standard examples of a stably finite, non-quasidiagonal C$^*$-algebra and $B$ is a C$^*$-algebra with Lance’s WEP and at least one tracial state then there is no unital $\ast$-homomorphism $A \to B$. In particular, many stably finite, exact C$^*$-algebras can’t be embed into a stably finite, nuclear C$^*$-algebra.

Dedicated to my big, beautiful family.

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1. Introduction

One of von Neumann’s main motivations for initiating the study of operator algebras was to provide a framework for studying unitary representations of locally compact groups. Hence it is no surprise that representation theory of \( C^* \)-algebras has, historically, received a lot of attention. Over the last couple of decades, however, representation theory has become less fashionable as exciting new fields developed such as Connes’ noncommutative geometry, Jones’ theory of subfactors, Elliott’s classification program and, most recently, Voiculescu’s theory of free probability.

In this paper we revisit representation theory but stick to the finite (i.e. tracial) case. There are two general questions which we study: Approximation properties of traces on \( C^* \)-algebras and \( II_1 \) factor representations of Popa algebras. While we believe these topics to be of independent interest, perhaps the most surprising part of this work is that combining these two aspects of representation theory leads to a variety of new results which provide a common thread between several important problems in operator algebras. Our results are most strongly connected to questions in the classification program and free probability but there are also relations with single operator theory (more specifically, a classical theorem of Szegő on spectral distributions of self-adjoint operators – see Section 10), the \( C^* \)-algebraic structure of the hyperfinite \( II_1 \) factor and even a possible connection with the Novikov conjecture (though this is only speculation at the moment).

Since we deal with some virtually disjoint subfields of operator algebras, we feel it is worthwhile to paint a broad picture of the topics covered before proving any results.

1.1. Approximating Traces on \( C^* \)-algebras

We will study four subsets of the tracial space of a unital, separable \( C^* \)-algebra. These subsets are defined via certain finite dimensional approximation properties. Though these approximation properties may seem artificial at first glance, it turns out that they are actually very natural. Indeed, the definition of quasidiagonality immediately leads to a norm approximation property for certain traces, while the deep fact that the double dual of a nuclear \( C^* \)-algebra is injective implies that every trace on a nuclear \( C^* \)-algebra enjoys a strong 2-norm approximation property.

One of the objectives of this paper is to point out how approximation properties of traces carry information about (tracial) representation theory of \( C^* \)-algebras. This has already been witnessed in the work of several authors (e.g. [17], [18], [31], [33], [6], [43], [4]). These papers are primarily concerned with Connes’ notion of a hypertrace. For example, in [6] Bekka defines a unitary representation of a locally compact group to be amenable if the \( C^* \)-algebra generated by the representation admits a hypertrace. (We recommend looking at [4] for a very nice introduction to the interactions between representation theory and hypertraces.)

It is a simple consequence of Voiculescu’s Theorem that any trace which satisfies one of the four approximation properties defined in these notes will extend to a hypertrace in the sense of Connes. A wonderful result of Eberhard Kirchberg provides a converse: If a trace extends to a hypertrace then it is ‘liftable’ (see [33, Definition 3.1, Proposition 3.2]). It is a simple exercise to show that liftable traces enjoy a weak 2-norm approximation property and thus hypertraces are characterized by a certain finite dimensional approximation property.

The three other approximation properties studied in this paper are natural variations on the approximation property enjoyed by hypertraces. One of these also turns out to be intimately related to representation theory: A trace is ‘uniformly weakly approximately finite dimensional’ (see Definition 3.1 below) if and only if the associated GNS representation gives a hyperfinite von Neumann algebra (see Theorem 3.8). The remaining two approximation properties seem harder to understand, but it is natural to expect that they are also related
to representation theory in a way which is not yet understood. A better understanding of these other two sets of traces is closely related to questions about quasidiagonal C*-algebras.

The tracial invariants introduced here have a number of applications. For example, in Section 10 we will see that approximation properties of traces on type I C*-algebras together with Voiculescu’s Theorem easily yield an analogue of a classical theorem of Szegö. We also observe that they provide natural obstructions to the existence of (unital) *-homomorphisms between certain classes of operator algebras (cf. Corollary 7.2). They also clarify results of J. Rosenberg and S. Wassermann on the existence of stably finite, non-quasidiagonal C*-algebras (see the examples at the end of Section 3 and the middle of section 7).

1.2. II$_1$ Factor Representations of Popa Algebras

Another goal of these notes is to study II$_1$ factor representations of Popa algebras.

Definition 1.1. A simple, separable, unital C*-algebra, $A$, is called a Popa algebra if for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a nonzero finite dimensional C*-subalgebra $B \subset A$ with unit $e$ such that $\|ex - xe\| \leq \varepsilon$ for all $x \in \mathcal{F}$ and $e\mathcal{F}e \subset B$ (i.e. for each $x \in \mathcal{F}$ there exists $b \in B$ such that $\|exe - b\| \leq \varepsilon$).

Popa algebras are always quasidiagonal (QD). Using his local quantization technique in the C*-algebra setting Popa nearly provides a converse in [43]: Every simple, unital, quasidiagonal C*-algebra with 'sufficiently many projections' (e.g. real rank zero) is a Popa algebra. Thus the class of Popa algebras is much larger than one might first guess.

For some time there was speculation that quasidiagonality may be closely related to nuclearity. For example, in [43, pg. 157] Popa asked whether every Popa algebra with unique trace is necessarily nuclear. (Counterexamples were first constructed by Dadarlat in [19].) More generally, he asked in [43, Remark 3.4.2] whether the hyperfinite II$_1$ factor $R$ was the only II$_1$ factor which could arise from a GNS representation of a Popa algebra. Support for a positive answer was provided by the following consequence of Connes’ celebrated classification theorem:

Theorem. Let $M$ be a (separable) II$_1$ factor. Then $M \cong R$ if and only if for every finite subset $\mathcal{F} \subset M$ and $\varepsilon > 0$ there exists a nonzero finite dimensional C*-subalgebra $B \subset M$ with unit $e$ such that $\|ex - xe\| \leq \varepsilon$ for all $x \in \mathcal{F}$ and $e\mathcal{F}e \subset B$ (i.e. for each $x \in \mathcal{F}$ there exists $b \in B$ such that $\|exe - b\| \leq \varepsilon$).

Since Popa algebras are simple, the finite dimensional approximation property from Definition 1.1 obviously passes to II$_1$ factor representations (or any other representation). Since the 2-norm version of this approximation property characterizes $R$ it was natural to expect that the II$_1$ factor representation theory of Popa algebras would be trivial. We will see, however, that the representation theory of Popa algebras is actually very rich. For example, we will construct a Popa algebra $A$ with the property that for every (separable) II$_1$ factor $M$ there exists a tracial state $\tau$ on $A$ such that $\pi_\tau(A)'' \cong M \bar{\otimes} R$ (see Theorem 8.4). On the other hand we will see that if $A$ is a locally reflexive (e.g. exact) Popa algebra with unique trace $\tau$ then $\pi_\tau(A)'' \cong R$ thus giving a positive answer to Popa’s question in this case (see Corollary 8.2).

1.3. Applications
As mentioned above, our study of finite representation theory leads to a number of results which are closely related to some important open problems.

Elliott’s Classification Program
There are three new results which those in the classification program may find of interest. First, we will show that if Elliott’s conjecture holds for simple, nuclear, QD C*-algebras then
every trace on every nuclear, QD (not necessarily simple!) C*-algebra satisfies the strongest kind of approximation property (cf. Proposition 4.2). Hence we get a new necessary condition for Elliott’s conjecture to hold. Second, we will point out why this necessary condition may also eventually become part of a sufficient condition for classification (see Theorem 4.3).

The third result of relevance to the classification program is inspired by Huaxin Lin’s class of tracially AF algebras. Roughly speaking a tracially AF algebra is a Popa algebra with the additional property that the finite dimensional algebra $B$ from Definition 1.1 can always be taken ‘large in trace’ (see [36] for the precise definition).

Thanks to Huaxin Lin’s remarkable classification result for tracially AF algebras (cf. [35]) it is now of fundamental importance to understand when a Popa algebra is a tracially AF algebra. Indeed, Elliott’s classification conjecture predicts that every nuclear Popa algebra with real rank zero, unperforated K-theory and Riesz decomposition property must be tracially AF (see Proposition 2.4.2). Thus, by Lin’s classification theorem, verifying this case of Elliott’s conjecture is equivalent to the following question (modulo a UCT assumption):

Does every nuclear Popa algebra with real rank zero, unperforated K-theory and having the Riesz decomposition property also have tracial topological rank zero, in the sense of Huaxin Lin [37]?

Using the tracial invariants alluded to above and representation theory of Popa algebras we will show that it is possible to construct an exact Popa algebra with virtually every nice property (including all those above) but which is not of tracial topological rank zero (cf. Corollary 5.3). This example is a bit surprising as some experts did not expect that the question above would have much to do with nuclearity. (See, for example, [36, page 694] where it was “tempting to conjecture that every quasidiagonal, simple C*-algebra of real rank zero, stable rank one and with weakly unperforated $K_0$ is tracially AF.”) Indeed, the question above can be regarded as a finite analogue of the question of whether every nuclear, simple, infinite C*-algebra is purely infinite and hence our example in Corollary 5.3 should be regarded as a finite analogue of M. Rørdam’s recent construction of a (non-nuclear) simple, infinite but not purely infinite C*-algebra (see [47]).

**Free Probability**

There are two new results arising from this work which are relevant to free probability. One is related to Connes’ embedding problem and the other is related to the semicontinuity question for Voiculescu’s free entropy dimension.

As mentioned above, we will show that every McDuff factor contains a weakly dense Popa algebra. It seems quite likely that many other II$_1$ factors contain weakly dense Popa algebras as well and hence it is natural to ask how free entropy dimension behaves on Popa algebras. For example, is it true that for Popa algebras the free entropy dimension with respect to any trace and any set of generators is always $\leq 1$? The point is that Popa algebras do not arise from any sort of (reduced) free product construction (since such free products are essentially never QD) and have abstract properties which are quite similar to properties which characterize the hyperfinite II$_1$ factor. Thus it is important to determine how free entropy dimension behaves on these algebras.

Regarding Connes’ embedding problem (i.e. the question of whether or not microstates always exist) we obtain the following result (see Theorem 9.1): A II$_1$ factor $M$ has microstates if and only if there exists a weakly dense operator system $X \subset R \hat{\otimes} M$ which is injective. Hence we see that the difference between embedding into the ultrapower of $R$ and actually being isomorphic to $R$ is rather delicate. Our own feeling is that this difference is too delicate to expect that every II$_1$ factor has microstates, but we have not yet been able to construct a counterexample.
Szegö’s Limit Theorem

Inspired by ideas of Arveson (see also Bédos’ work in this direction; [5]) we observe in Section 10 that the techniques and ideas of this paper easily yield a very general existence theorem, analogous to the classical limit theorem of Szegö, for arbitrary self-adjoint operators on a separable Hilbert space. In particular, Theorem 10.1 is a vast generalization of [3, Theorem 4.5] in the sense that all of the hypotheses regarding the larger C*-algebra are unnecessary.

Quasidiagonality and the Hyperfinite II$_1$ Factor

It is an open question whether or not $R$ is a quasidiagonal (QD) C*-algebra. While many experts believe that $R$ is not QD, there is little concrete evidence to support a negative answer. On the other hand, a positive answer would imply, for example, that every simple, unital, nuclear, stably finite C*-algebra is QD (something predicted by Elliott’s conjecture) and that $C^*_r(\Gamma)$ is QD for every discrete amenable group $\Gamma$ (a conjecture of J. Rosenberg).

Hence, aside from being a natural and basic problem, we feel this to be an important question as well. In the appendix we will reformulate this problem in terms of approximation properties of traces on (separable) C*-algebras. Thus it is our hope that future work on approximation properties of traces will shed light on this problem.

The paper is organized as follows. In section 2 we collect a number of preliminary results which we will need. The main result of this section shows how to construct Popa algebras with specified GNS representations. In section 3 we define various subsets of traces and study their properties. This section also contains a number of examples. In section 4 we study the case that $A$ is a nuclear C*-algebra. It is shown that Elliott’s conjecture predicts that every trace on every nuclear, quasidiagonal C*-algebra (simple or not) has the strongest kind of approximation property. In section 5 we consider the exact case and show how to construct exact Popa algebras which are not tracially AF. Section 6 treats the locally reflexive case while Section 7 treats the WEP case. In section 8 we discuss II$_1$ factor representations of Popa algebras and observe that even very nice Popa algebras can have non-hyperfinite II$_1$ factor representations. Section 9 observes that the techniques of this paper easily yield new characterizations of McDuff factors which are embeddable into the ultrapower of the hyperfinite II$_1$ factor. One consequence of this result is that a number of well known II$_1$ factors (e.g. $R\bar{\otimes}L(\Gamma)$ for any i.c.c., residually finite, discrete group) contain weakly dense injective subspaces, but, of course, are not themselves injective. Section 10 contains an analogue of Szegö’s Limit Theorem. Section 11 just contains a number of questions which arise naturally from this work. Finally, we have an appendix which discusses the question of whether or not the hyperfinite II$_1$ factor is a quasidiagonal C*-algebra.

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2. Preliminaries

In this section we present a number of results (many of which are well known, but restated in particularly convenient ways) which serve as the backbone for the rest of the paper. The main new result, Theorem 2.5.1, states that residually finite dimensional C*-algebras always admit approximately trace preserving embeddings into Popa algebras.
2.1. Notation. Before stating any results we wish to introduce some notation which will be used throughout this paper.

We first remark that, unless otherwise noted or obviously false, all $C^*$-algebras in this paper are assumed to be unital and separable. Similarly, all von Neumann algebras will be assumed to have separable preduals (with the exception of $R^\omega$, which is well known to be non-separable).

For a Hilbert space $H$, we will let $B(H)$ and $\mathcal{K}(H)$ denote the bounded and, respectively, compact operators on $H$. We let $\| \cdot \|$ denote the operator norm on $B(H)$ while $\| \cdot \|_{HS}$ and $<\cdot, \cdot>_{HS}$ will denote the Hilbert-Schmidt norm and, respectively, inner product on the Hilbert-Schmidt operators on $H$.

When $A$ is a $C^*$-algebra with state $\eta$ we will denote the associated GNS Hilbert space, representation and von Neumann algebra by $L^2(A, \eta)$, $\pi_\eta : A \to B(L^2(A, \eta))$ and $\pi_\eta(A)''$, respectively.

The symbols $\circ$, $\otimes$ and $\overline{\otimes}$ will denote the algebraic, minimal and W*-tensor products, respectively.

If $A$ is a $C^*$-algebra we will let $A^{\text{op}}$ denote the opposite algebra (i.e. $A^{\text{op}} = A$ as involutive normed linear spaces, but multiplication in $A^{\text{op}}$ is defined by $a \circ b = ba$; the latter multiplication being the given multiplication in $A$). $A^{**}$ will denote the enveloping von Neumann algebra of $A$ (i.e. the Banach space double dual of $A$). Contrary to our standing assumption that von Neumann algebras should have separable preduals, $A^{**}$ often has a non-separable predual.

If $\tau \in T(A)$ is a tracial state then there is a canonical antilinear isometry $J : L^2(A, \tau) \to L^2(A, \tau)$ defined by $J(\hat{a}) = a^*$. One defines a $*$-homomorphism $\pi^{\text{op}}_\tau : A^{\text{op}} \to B(L^2(A, \tau))$ by $\pi^{\text{op}}_\tau(a) = J \pi_\tau(a^*) J$. Since $J \pi_\tau(A)J \subset \pi_\tau(A)'$ one then gets an algebraic homomorphism $\pi_\tau \otimes \pi^{\text{op}}_\tau : A \otimes A^{\text{op}} \to B(L^2(A, \tau))$ defined on elementary tensors by $\pi_\tau \otimes \pi^{\text{op}}_\tau(a \otimes b) = \pi_\tau(a) \pi^{\text{op}}_\tau(b)$.

Completely positive maps (cf. [38]) will play an important role in these notes. We will use the abbreviations c.p. and u.c.p. for ‘completely positive’ and ‘unital completely positive’, respectively.

The hyperfinite $\Pi_1$ factor will appear many times and will always be denoted by $R$. We will use $tr_n$ to denote the unique tracial state on the $n \times n$ matrices. For a von Neumann algebra, $M$, with faithful, normal, tracial state $\tau$, we will let $\| \cdot \|_{2, \tau}$ be the associated 2-norm (i.e. $\|x\|_{2, \tau} = \tau(x^*x)^{1/2}$). If $M$ has a unique trace (i.e. is a factor) then we will drop the dependence on $\tau$ and simply write $\| \cdot \|_2$.

Finally, we will need the ultrapower of the hyperfinite $\Pi_1$ factor. That is, given a free ultrafilter $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$ one defines an ideal $I_\omega \subset l^\infty(R) = \{(x_n) \in \Pi_{n \in \mathbb{N}}R : \sup_{n \in \mathbb{N}}\|x_n\| < \infty\}$ by $I_\omega = \{(x_n) \in l^\infty(R) : \lim_{n \to \omega}\|x_n\|_2 = 0\}$. Then the ultrapower of $R$ with respect to $\omega$ is defined to be the ($C^*$-algebraic) quotient: $R^\omega = l^\infty(R)/I_\omega$. $R^\omega$ is a $\Pi_1$ factor with trace $\tau_\omega((x_n) + I_\omega) = \lim_{n \to \omega}\tau_R(x_n)$.

2.2. Voiculescu’s Theorem. We will need the following version of Voiculescu’s Theorem.

Theorem 2.2.1. Let $A \subset B(H)$ be in general position (i.e. $A \cap \mathcal{K}(H) = \{0\}$). If $\phi : A \to M_n(\mathbb{C})$ is a u.c.p. map then there exist isometries $V_k : \mathbb{C}^n \to H$ such that $\|\phi(a) - V_k^*aV_k\| \to 0$, for all $a \in A$, as $k \to \infty$. Moreover if $P_k = V_k^*V_k$ then one has the following estimates on commutators:

1. $\limsup \|P_k a - a P_k\| \leq \max\{\|\phi(aa^*) - \phi(a)\phi(a^*)\|^{1/2}, \|\phi(a^*a) - \phi(a^*)\phi(a)\|^{1/2}\}$;
2. $\limsup \frac{\|P_k a - a P_k\|_{HS}}{\|P_k\|_{HS}} \leq \left(\text{tr}_n(\phi(aa^*) - \phi(a)\phi(a^*)) + \text{tr}_n(\phi(a^*a) - \phi(a^*)\phi(a))\right)^{1/2}$. 

Proof. This result is really contained in the standard proof of Voiculescu’s Theorem (cf. [1] or [21]). However, the commutator estimates, which are the important part for us, become especially transparent if we use the usual versions of Voiculescu’s Theorem and Stinespring’s Theorem.

So, let $\pi : A \to B(K)$ be the Stinespring dilation of $\phi$ with isometry $V : C^n \to K$ such that $\phi(a) = V^*\pi(a)V$. Define the projection $P = VV^*$ and note that we have the identity $P\pi(a) - \pi(a)P = Pa(1-P) - (1-P)aP$. Note also that $Pa(1-P)$ and $(1-P)aP$ have orthogonal domains and ranges and, in particular, are perpendicular in the Hilbert space of Hilbert-Schmidt operators. Using this one can verify the identities

$$\|P\pi(a) - \pi(a)P\| = \max\{\|\phi(aa^*) - \phi(a)\phi(a^*)\|^{{1/2}}, \|\phi(a^*a) - \phi(a^*)\phi(a)\|^{{1/2}}\},$$

and

$$\frac{\|P\pi(a) - \pi(a)P\|_H}{\|P\|_H} = \left(\text{tr}_n(\phi(aa^*) - \phi(a)\phi(a^*)) + \text{tr}_n(\phi(a^*a) - \phi(a^*)\phi(a))\right)^{{1/2}}.$$ 

Note that the same estimates hold for the representation $\iota \oplus \pi : A \to B(H \oplus K)$, where $\iota : A \hookrightarrow B(H)$ is the given inclusion. Since, $\iota$ is approximately unitarily equivalent to $\iota \oplus \pi$, it is clear how to complete the proof.

In one place, we will need the following technical version of Voiculescu’s Theorem. A proof can be found in [13] or [14].

**Proposition 2.2.2.** Let $A \subset B(H)$ be in general position and $\Phi : A \to B(K)$ be a u.c.p. map which is a faithful $*$-homomorphism modulo the compacts (i.e. composing with the quotient map to the Calkin algebra yields a faithful $*$-monomorphism $A \hookrightarrow Q(K)$). Then there exists a sequence of unitaries $U_n : K \to H$ such that for every $a \in A$ we have

$$\limsup \|a - U_n\Phi(a)U_n^*\| \leq 2\max\{\|\Phi(aa^*) - \Phi(a)\Phi(a^*)\|^{{1/2}}, \|\Phi(a^*a) - \Phi(a^*)\Phi(a)\|^{{1/2}}\}.$$ 

2.3. Elliott’s Intertwining Argument. Perhaps the single most important argument in the classification program is due to George Elliott. Though usually done in the setting of C*-algebras we will need Elliott’s approximate intertwining argument in the setting of von Neumann algebras. While not the most general possible form, the following version is more than sufficient for our purposes. The set-up is as follows.

Assume that $M \subset B(L^2(M, \tau))$ and $N \subset B(L^2(N, \gamma))$ are von Neumann algebras acting standardly, with faithful, normal tracial states $\tau$ and, respectively, $\gamma$. Let $X_1 \subset X_2 \subset \ldots \subset M$ and $Y_1 \subset Y_2 \subset \ldots \subset N$ be (not necessarily unital) $C^*$-subalgebras such that $\cup X_i$ is weakly dense in $M$ and $\cup Y_i$ is weakly dense in $N$. Further assume that we have c.p. maps $\alpha_n : Y_n \to X_n$, $\beta_n : X_n \to Y_{n+1}$, which are contractive both with respect to the operator norms and the 2-norms coming from $\tau$ and $\gamma$, and finite subsets $\Lambda_i \subset X_i$ and $\Omega_i \subset Y_i$ with the following properties:

1. $\Lambda_i \subset \Lambda_{i+1}$, $\Omega_i \subset \Omega_{i+1}$, for all $i \in \mathbb{N}$, and the linear spans of $\cup \Lambda_i$ and $\cup \Omega_i$ are norm dense in $\cup X_i$ and $\cup Y_i$, respectively, and hence weakly dense in $M$ and $N$, respectively.

To simplify things, we will also assume that $x_1, x_2 \in \Lambda_i \Rightarrow x_1x_2 \in \Lambda_{i+1}$ and, similarly, that $\Omega_{i+1}$ contains the product of any pair of elements from $\Omega_i$.

2. $\alpha_i(\Omega_i) \subset \Lambda_i$ and $\beta_i(\Lambda_i) \subset \Omega_{i+1}$ for all $i \in \mathbb{N}$.

3. Both $\{\alpha_i\}$ and $\{\beta_i\}$ are weakly asymptotically multiplicative. That is, $\|\alpha_i(y_1y_2) - \alpha_i(y_1)\alpha_i(y_2)\|_{1,\tau} \to 0$, as $i \to \infty$, for all $y_1, y_2 \in \cup Y_i$ and similarly for $\{\beta_i\}$.

**Theorem 2.3.1.** (Elliott’s Intertwining) In the setting described above, if it happens that $\|x - \alpha_{n+1} \circ \beta_n(x)\|_{1,\tau} < 1/2^n$ and $\|y - \beta_n \circ \alpha_n(y)\|_{1,\gamma} < 1/2^n$ for all $x \in \Lambda_n$, $y \in \Omega_n$ and all $n \in \mathbb{N}$, then $M \cong N.$
Proof. As this argument is well known, we will be a bit sketchy. Two facts which we will need are: i) every norm bounded sequence which is Cauchy in 2-norm converges (in 2-norm) and ii) on norm bounded subsets, the 2-norm topology is the same as the strong operator topology (since our von Neumann algebras are acting standardly on the $L^2$ spaces coming from their traces).

Since the $\alpha_n$’s and $\beta_n$’s are 2-norm contractive, one first checks that for each $y \in \bigcup \Omega_i$, the sequence $\{\alpha_n(y)\}$ is Cauchy in 2-norm (and similarly for each $x \in \bigcup \Lambda_i$). Hence this is also true on the linear spans of $\bigcup \Lambda_i$ and $\bigcup \Omega_i$.

Since the $\alpha_n$’s and $\beta_n$’s are norm contractive, it follows that for each $y$ in the linear span of $\bigcup \Omega_i$, the sequence $\{\alpha_n(y)\}$ is convergent in $M$ (and similarly for all $x \in \text{span}(\bigcup \Lambda_i)$). Hence we can define linear maps $\Phi : \text{span}(\bigcup \Lambda_i) \to N$ and $\Psi : \text{span}(\bigcup \Omega_i) \to M$ by $\Phi(x) = \lim \beta_n(x)$ and $\Psi(y) = \lim \alpha_n(y)$. Note that $\Phi$ and $\Psi$ are contractive with respect to both the operator norms and the 2-norms. This implies that $\Phi$ and $\Psi$ can be (uniquely) extended to the norm closures of $\text{span}(\bigcup \Lambda_i)$ and $\text{span}(\bigcup \Omega_i)$ (which are weakly dense C*-algebras, by condition (1) above) and, moreover, that these extensions are 2-norm contractive as well. Now one uses Kaplansky’s density theorem (and the fact that our extensions are still norm and 2-norm contractive) to extend beyond these weakly dense C*-subalgebras to all of $M$ and $N$. (i.e. For each $x \in M$ we take a norm bounded sequence $\{x_n\}$ from the norm closure of $\text{span}(\bigcup \Lambda_i)$ which converges to $x$ in 2-norm. The image of $\{x_n\}$ in $N$ is then a norm bounded sequence which is Cauchy in 2-norm and hence we map $x$ to the (2-norm) limit of this sequence.)

To save notation, we will also let $\Phi : M \to N$ and $\Psi : N \to M$ denote the maps constructed in the previous paragraph. Note that these maps are 2-norm contractive and linear. They are also *-preserving since the strong and strong-* topologies agree on bounded subsets of a tracial von Neumann algebra (since $\|x\|_{2,\tau} = \|x^*\|_{2,\tau}$). It is also easy to check that they are mutual inverses on the spans of $\bigcup \Lambda_i$ and $\bigcup \Omega_i$. By 2-norm contractivity it follows that they are mutual inverses on all of $M$ and $N$. Hence we only have to observe that both $\Phi$ and $\Psi$ are multiplicative on $M$ and $N$. Since multiplication is continuous, on bounded sets, in the 2-norm, it is not hard to check that $\Phi$ and, respectively, $\Psi$ are multiplicative on the norm closures of the linear spans of $\bigcup \Lambda_i$ and, respectively, $\bigcup \Omega_i$. Finally, another application of Kaplansky’s density theorem and a standard interpolation argument allow one to deduce multiplicativity on all of $M$ and $N$.

2.4. Consequences of Elliott’s Conjecture. We remind the reader that all algebras are assumed separable and unital.

We state here the special case of Elliott’s Conjecture which will be relevant for us. We then deduce a few statements which are predicted by this conjecture. We are indebted to M. Rørdam and H. Lin for some helpful discussions regarding these issues.

For a stably finite C*-algebra $A$, the Elliott invariant is the triple $(K_0(A), K_1(A), T(A))$, where $T(A)$ is the set of tracial states on $A$, together with the natural pairing $P_A : K_0(A) \times T(A) \to \mathbb{R}$. Given two algebras $A$ and $B$, we say that their Elliott invariants are isomorphic if $K_1(A) \cong K_1(B)$ and there exist a scaled, ordered group isomorphism $\Phi : K_0(A) \to K_0(B)$ and an affine homeomorphism $T : T(A) \to T(B)$ such that $P_A(x, \tau) = P_B(\Phi(x), T(\tau))$, for all $(x, \tau) \in K_0(A) \times T(A)$.

(Special case of) Elliott’s Conjecture: If two simple, stably finite, nuclear C*-algebras have isomorphic Elliott invariants (as described above) then they are isomorphic.

Proposition 2.4.1. Elliott’s conjecture predicts that if $A$ is a stably finite, simple, nuclear C*-algebra then for any UHF algebra, $U$, $A \otimes U$ is an inductive limit of subhomogeneous algebras (i.e. ASH).
Proof. By [46, Theorem 5.2 (b)] it follows that the pairing $P_{A \otimes U} : K_0(A \otimes U) \times T(A \otimes U) \to \mathbb{R}$ is weakly unperforated (which means that if $P_{A \otimes U}(x, \tau) > 0$ for all $\tau$ then $x > 0$). Note that in order to apply Rørdam’s results from [46] we have to rely on U. Haagerup’s result that quasitraces on (unital) exact $C^*$-algebras are traces (cf. [25]). Once we know that the invariant of $A \otimes U$ is weakly unperforated, we are done since Elliott showed how to construct a simple ASH algebra with arbitrary weakly unperforated invariant (see, for example, the appendix of [24]). Indeed, if Elliott’s conjecture holds, we can find an ASH algebra whose invariant is isomorphic to the invariant of $A \otimes U$ and hence $A \otimes U$ is isomorphic to an ASH algebra. \qed

Note that Elliott’s conjecture also predicts that every simple, stably finite, nuclear $C^*$-algebra is QD, since quasidiagonality passes to subalgebras and ASH algebras are QD.

**Proposition 2.4.2.** Elliott’s conjecture predicts that if $A$ is simple, stably finite, nuclear, real rank zero (cf. [12]), has weakly unperforated invariant (see the proof of the previous proposition for this definition) and $K_0(A)$ has the Riesz interpolation property (cf. [21, Section IV.6]) then $A$ is tracially AF in the sense of [36].

Proof. We first remark that in the real rank zero case the tracial simplex is no longer relevant and hence Elliott’s invariant reduces to $K$-theory alone. Indeed, if both $A$ and $B$ are $C^*$-algebras of real rank zero and $\Phi : K_0(A) \to K_0(B)$ is a scaled, ordered group isomorphism such that $\Phi([1_A]) = [1_B]$ then $\Phi$ induces an affine homeomorphism $T(A) \to T(B)$ since we may (affinely, homeomorphically) identify $T(A)$ (resp. $T(B)$) with the states in $\text{Hom}(K_0(A), \mathbb{R})$ (resp. $\text{Hom}(K_0(B), \mathbb{R})$). To see that this is true, we first note that the obvious map $T(A) \to \text{Hom}(K_0(A), \mathbb{R})$ is affine and injective since $A$ has real rank zero. It is also onto the states in $\text{Hom}(K_0(A), \mathbb{R})$ since every state on $K_0(A)$ comes from a trace on $A$ when $A$ is unital and exact (cf. [26]). Finally, it is easy to check (again using real rank zero) that a sequence of traces $\tau_n \in T(A)$ converges to $\tau \in T(A)$ in the weak-* topology if and only if their images in $\text{Hom}(K_0(A), \mathbb{R})$ converge in the topology of pointwise convergence and hence our identification is also a homeomorphism.

Finally, in [23] it is shown how to construct simple AH algebras with real rank zero and with arbitrary unperforated $K$-theory and Riesz interpolation property. As observed by Lin, [36, Proposition 2.6], the Elliott-Gong construction always yields tracially AF algebras and hence we can find a simple, nuclear, tracially AF algebra with the same $K$-theory as $A$. Hence, if Elliott’s conjecture holds, $A$ is isomorphic to a tracially AF algebra. \qed

### 2.5. Approximately Trace Preserving Embeddings into Popa Algebras.

We now present the main technical result of this paper. The informed reader will note that every aspect of this result can be traced back to the classification program. Indeed, we will adapt the inductive limit techniques of Dadarlat [19] to construct new Popa algebras and use Elliott’s intertwining argument to understand their GNS representations.

In the following theorem $\mathcal{C}$ will denote some collection of $C^*$-algebras which is closed under increasing unions (i.e. inductive limits with injective connecting maps) and tensoring with finite dimensional matrix algebras. A $C^*$-algebra $E$ is called residually finite dimensional if $E$ has a separating family of finite dimensional representations (i.e. for every $0 \neq x \in E$ there exists a $\ast$-homomorphism $\pi : A \to M_n(\mathbb{C})$ such that $\pi(x) \neq 0$).

**Theorem 2.5.1.** Let $E \in \mathcal{C}$ be a residually finite dimensional $C^*$-algebra. Then there exists a Popa algebra $A \in \mathcal{C}$ such that for every $\varepsilon > 0$ we can find a $\ast$-monomorphism $\rho : E \hookrightarrow A$ with the property that for each trace $\tau \in T(E)$, there exists a trace $\gamma \in T(A)$ such that

1. $|\gamma \circ \rho(x) - \tau(x)| < \varepsilon \|x\|$ for all $x \in E$ and,
2. $\pi_\gamma(A)'' \cong R \otimes \pi_\gamma(E)''$.

The proof of this result becomes much more transparent once the main idea is understood. Hence we think it is worthwhile to give the main idea first and leave the details to the end.

So suppose that $E$ is a residually finite dimensional C*-algebra and $\tau \in T(E)$. Let $\mathcal{U}$ be some UHF algebra. Then the canonical, unital inclusion $E \hookrightarrow E \otimes \mathcal{U}$ is honestly trace preserving (in fact, yields an isomorphism of tracial spaces) and the weak closure in any GNS representation is obviously of the form $R \otimes \pi_\tau(E)''$. The problem, of course, is that $E \otimes \mathcal{U}$ is not a Popa algebra. So the idea is that we will use an inductive limit construction to get a sequence

$$E \to E \otimes M_{k(1)}(\mathbb{C}) \to E \otimes M_{k(1)}(\mathbb{C}) \otimes M_{k(2)}(\mathbb{C}) \to \cdots,$$

such that the limit is a Popa algebra, but the connecting maps above will be chosen so that when one applies a trace it will (approximately) look like the sequence which yields $E \otimes \mathcal{U}$.

We now describe the basic construction which will be needed to get our Popa algebras. Let $\pi : E \to M_k(\mathbb{C})$ be a representation, $\tau \in T(E)$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ very large and consider the map $\rho : E \to E \otimes M_n(\mathbb{C})$ given by

$$x \mapsto 1_E \otimes \text{diag}(0_{n-k}, \pi(x)) + x \otimes \text{diag}(1_{n-k}, 0_k),$$

where $\text{diag}(0_{n-k}, \pi(x))$ is the block diagonal element in $M_n(\mathbb{C})$ whose first $n-k$ entries down the diagonal are zero and the bottom block is given by $\pi(x)$, while $\text{diag}(1_{n-k}, 0_k) \in M_n(\mathbb{C})$ has $n-k$ 1’s down the diagonal followed by $k$ zeros. The key remarks about this choice of connecting map are:

1. If $n^{-\alpha} > 1 - \epsilon$ then $|\tau \otimes \text{tr}_n(\rho(x)) - \tau(x)| < 2\epsilon \|x\|$ for all $x \in E$. That is, in trace the connecting map $\rho$ is almost the same as the map $x \mapsto x \otimes 1_{M_n}$ (which would be the natural connecting maps to use if we were trying to construct $E \otimes \mathcal{U}$).
2. If $I \subset E$ is an ideal and there exists an element $x \in I$ such that $\pi(x) \neq 0$ then the ideal generated by $\rho(I)$ is all of $E \otimes M_n(\mathbb{C})$. This follows from the definition of $\rho$ and the simplicity of $M_n(\mathbb{C})$. It is this fact that will allow us to deduce simplicity in our inductive limits.
3. There exists a finite dimensional C*-algebra $B \subset E \otimes M_n(\mathbb{C})$ with unit $e$ such that $e\rho(x) - \rho(x)e = 0$ and $e\rho(x)e \in B$, for all $x \in E$. (Let $B = \text{diag}(0, \ldots, 0, \pi(E)) \subset M_n(\mathbb{C})$.) This remark will immediately imply that our inductive limits satisfy the finite dimensional approximation property which defines Popa algebras.
4. The representation $\pi \otimes \text{id} : E \otimes M_n(\mathbb{C}) \to M_k \otimes M_n(\mathbb{C})$ is again a finite dimensional representation and hence this whole procedure can be reapplied to the algebra $E \otimes M_n(\mathbb{C})$ (thus yielding an inductive system).

We now enter the gory details. So let $E$ be a residually finite dimensional C*-algebra and $\pi_i : E \to M_{k(i)}(\mathbb{C})$ be a separating sequence of representations. In fact, we will assume that for every $x \in E$, $\|x\| = \lim_{i} \|\pi_i(x)\|$ (taking direct sums, it is not hard to see that every residually finite dimensional C*-algebra has such a sequence). Note that for every $n \in \mathbb{N}$, $\pi_i \otimes \text{id} : E \otimes M_n(\mathbb{C}) \to M_{k(i)} \otimes M_n(\mathbb{C})$ is a separating sequence of the same type.

Now choose natural numbers $1 = n(0) \leq n(1) \leq n(2) \leq \ldots$ such that

$$\frac{n(0)n(1) \cdots n(j - 1)k(j)}{n(j)} < 2^{-j},$$

for all $j \in \mathbb{N}$. One then defines algebras $E = E_0, E_1 = E_0 \otimes M_{n(1)}, E_2 = E_1 \otimes M_{n(2)}, E_3 = E_2 \otimes M_{n(3)}, \ldots$ and inclusions $\rho_i : E_i \hookrightarrow E_{i+1}$ as in the basic construction described above where the inclusion $\rho_i$ uses the finite dimensional representation $\pi_{i+1} \otimes \text{id} \otimes \cdots \otimes \text{id} : E \otimes M_{n(1)} \otimes \cdots \otimes M_{n(i)} \to M_{k(i+1)} \otimes M_{n(1)} \otimes \cdots \otimes M_{n(i)}$ in the lower right hand corner. Letting $\Phi_{j,i} : E_i \to E_j$, $i \leq j$, be defined by $\Phi_{j,i} = \rho_{j-1} \circ \cdots \circ \rho_i$ we get an inductive system $\{E_i, \Phi_{j,i}\}$.
The are some projections in the above inductive system which we will need. Let $P_i \in E_i$ be the projection
\[ P_i = 1_{E_{i-1}} \otimes \text{diag}(1_{n(i)-n(1)-\cdots-n(i-1)k(i)}, 0_{n(i)-n(i)k(i)}). \]
Note that $P_{i+1}$ commutes with all of $\rho_i(E_i)$ (and, in particular, with $\rho_i(P_i)$). Note also that if we write $E_{i+1} = E_i \otimes M_{n(i+1)}$, then
\[ P_{i+1} \rho_i(P_i) = P_i \otimes \text{diag}(1_{n(i+1)-n(1)-\cdots-n(i)k(i+1)}, 0_{n(i)-n(i)k(i+1)}). \]

Letting $A$ be the inductive limit of the inductive system above, we only have to show that $A$ is the Popa algebra we are after.

**Proof of Theorem 2.5.1:** We keep all the notation above. We leave it to the reader to verify that $A$ is a Popa algebra as this follows from our remarks above and the construction of $A$. (That $A$ is unital and satisfies the right finite dimensional approximation property is obvious while simplicity follows from the remark that any ideal in $A$ must eventually intersect some $E_i$ (cf. [21, Lemma III.4.1]).) Note also that $A$ was constructed as an inductive limit of matrices over $E$ and hence belongs to the class $\mathcal{C}$ when $E$ does.

Now observe that given a trace $\tau \in T(E)$ we can define traces $\tau_j \in T(E_j)$ by $\tau_j = \tau \otimes tr_{n(1)} \otimes \cdots \otimes tr_{n(j)}$. Then the embedding $\rho_j : E_j \to E_{j+1}$ almost intertwines $\tau_j$ and $\tau_{j+1}$. More precisely, a straightforward (but rather unpleasant) calculation shows that for $i < j$,
\[ \tau_j(\Phi_{j,i}(x)) = \frac{j-1}{\prod_{s=1}^{j-1} n(s+1) - n(1)n(2) \cdots n(s)k(s+1)} \tau_i(x) + \lambda_{i,j} \eta_{i,j}(x), \]
where $\lambda_{i,j} = 1 - \prod_{s=1}^{j-1} (1 - \frac{n(1)n(2) \cdots n(s)k(s+1)}{n(s+1)})$ and $\eta_{i,j}$ is some tracial state on $E_i$. Hence we get the estimate
\[ |\tau_j(\Phi_{j,i}(x)) - \tau_i(x)| \leq 2\lambda_{i,j} \|x\|, \]
for all $x \in E_i$. But, it can be shown by induction that $\prod_{s=1}^{j-1} (1 - \frac{n(1)n(2) \cdots n(s)k(s+1)}{n(s+1)}) \geq 1 - 2^{-s-1} \geq 1 - 2^{-i} + 2^{-j} \geq 1 - 2^{-i}$, for all $i < j \in \mathbb{N}$. Hence we get that
\[ |\lambda_{i,j}| = |1 - \prod_{s=1}^{j-1} (1 - \frac{n(1)n(2) \cdots n(s)k(s+1)}{n(s+1)})| \leq 2^{-i}. \]

We have almost established part (1) in Theorem 2.5.1. For each $i \in \mathbb{N}$, extend $\tau_i$ to a state on $A$ (after identifying $E_i$ with its image in $A$). It is clear that if we take any weak-$*$ cluster point, $\gamma$, of this sequence then we will get a trace on $A$. Moreover, by the estimates above, we have that for each $x \in E_i$,
\[ |\gamma(x) - \tau_i(x)| \leq 2^{-i} \|x\|. \]
Since we always have $\tau$-preserving embeddings of $E$ into $E_i$, it should be clear how to construct the embedding $\rho$ in the statement of the theorem.

Our last task is to prove that $\pi_\gamma(A)'' \cong \pi_\tau(E)'' \otimes R$. To do this, we will need to study the projections $P_j \in E_j$ defined above. The idea is that we will use the $P_j$’s to construct different projections $Q^{(i)} \in \pi_\gamma(A)''$ with the following properties:

1. $Q^{(i)} = P_i Q^{(i+1)} = Q^{(i+1)} P_i$ and hence $Q^{(i)} \leq Q^{(i+1)}$ for all $i \in \mathbb{N}$.
2. $Q^{(i+1)} \in \pi_\gamma(E_i)'$, for all $i \in \mathbb{N}$ (where we have identified $E_i$ with its image in $A$).
3. \( \gamma(Q^{(i)}) \geq 1 - 2^{-i} \).

4. For each \( i \in \mathbb{N} \), \( \frac{2Q^{(i+1)}x}{\gamma(Q^{(i+1)})} = \tau_i(x) \), for all \( x \in E_i \).

5. The natural inclusion of the weak closure of \( Q^{(i)} \pi_{\gamma}(E_{i-1})Q^{(i)} \) into the weak closure of \( Q^{(i+1)} \pi_{\gamma}(E_i)Q^{(i+1)} \) (which is a natural inclusion by (1) above) is isomorphic to the (non-unital) inclusion \( \pi_{\gamma-1}(E_{i-1})'' \hookrightarrow \pi_{\gamma}(E_i)' \approx \pi_{\gamma-1}(E_{i-1})'' \otimes M_{n(i)} \) given by

\[
x \mapsto x \otimes \text{diag}(1_{n(i)-n(1)n(2)\ldots n(i-1)k(i)}, 0_{n(1)n(2)\ldots n(i-1)k(i)})\text{.}
\]

We claim that the construction of such \( Q^{(i)} \)'s will complete the proof. Indeed, if we can do this then one uses part (5) and Elliott’s approximate intertwining argument to compare the (non-unital) inclusions \( Q^{(1)} \pi_{\gamma}(E_0) \subset Q^{(2)} \pi_{\gamma}(E_1) \subset \ldots \) to the natural (unital) inclusions \( E_0 \subset E_0 \otimes M_{n(1)} \subset E_0 \otimes M_{n(1)} \otimes M_{n(2)} \subset \ldots \). Part (3) ensures that the former sequence recaptures \( \pi_{\gamma}(A)'' \) while the latter sequence gives \( E_0 \otimes \mathcal{U} \) in the limit, where \( \mathcal{U} \) is a UHF algebra, and hence the weak closure will be as desired and the proof will be complete.

The construction of the \( Q^{(i)} \)'s is fairly simple. For each \( i \in \mathbb{N} \) we define projections \( Q^{(i)}_n = \pi_{\gamma}(P_iP_{i+1} \ldots P_{i+n}) \in \pi_{\gamma}(A)' \). Since the \( Q^{(i)}_n \)'s are decreasing (as \( n \to \infty \)), there exists a strong operator topology limit. Define \( Q^{(i)} = \text{sot} - \lim_{n \to \infty} Q^{(i)}_n \). Then \( Q^{(i)} \) is a projection and it is straightforward to verify conditions (1) and (2) above. (Recall that \( P_j \) commutes with \( \Phi_{j,i}(E_i) \) whenever \( i < j \).) Thus we are left to verify the last three conditions.

Proof of (3). It suffices to show that \( \gamma(Q^{(i)}_n) \geq 1 - 2^i - 2^{-i-n} \) for all \( n \). But we may identify \( Q^{(i)}_n \) with a projection in \( E_{i+n} \) and so using the first part of the proof of this theorem (and using the identification) we get \( |\gamma(Q^{(i)}_n) - \pi_{\gamma}(Q^{(i)}_n)| < 2^{-i-n} \). However, it follows from the construction of \( Q^{(i)}_n \) that

\[
\tau_{i+n}(Q^{(i)}_n) = \prod_{s=1}^{n} \text{tr}_{n(i+s)}(\text{diag}(1_{n(i+s)-n(1)n(2)\ldots n(i+s-1)k(i+s)}, 0_{n(1)n(2)\ldots n(i+s-1)k(i+s)}))\text{.}
\]

Thus, by the calculations given in the first part of the proof of this theorem, we see that \( \tau_{i+n}(Q^{(i)}_n) \geq 1 - 2^{-i} \) and hence \( \gamma(Q^{(i)}_n) \geq 1 - 2^i - 2^{-i-n} \).

Proof of (4). We must show that if \( x \in E_{i-1} \) then \( \tau_{i-1}(x) \gamma(Q^{(i)}) = \lim_{n \to \infty} \gamma(Q^{(i)}_n)x \). In order to show this, it suffices to prove that

\[
\tau_{i-1}(x) = \frac{\tau_{i+n}(Q^{(i)}_n)x}{\tau_{i+n}(Q^{(i)}_n)}\text{,}
\]

for all \( n \in \mathbb{N} \). This last equality, however, is evident from the construction.

Proof of (5). It suffices to show (essentially due to the uniqueness, up to unitary equivalence, of GNS representations together with part (4)) that there exists a surjective \( * \)-homomorphism \( \eta : E_{i-1} \otimes M_{n(i)} \to Q^{(i+1)} \pi_{\gamma}(E_i) \) such that for every \( x \in E_{i-1} \),

\[
\eta(x \otimes \text{diag}(1_{n(i)-n(1)n(2)\ldots n(i-1)k(i)}, 0_{n(1)n(2)\ldots n(i-1)k(i)})) = Q^{(i)} \pi_{\gamma}(x) = Q^{(i+1)} P_i \pi_{\gamma}(x)\text{.}
\]

But since

\[
P_i \rho_{i-1}(x) = x \otimes \text{diag}(1_{n(i)-n(1)n(2)\ldots n(i-1)k(i)}, 0_{n(1)n(2)\ldots n(i-1)k(i)}) \in E_i = E_{i-1} \otimes M_{n(i)}\text{,}
\]

we get the desired homomorphism by identifying \( E_{i-1} \otimes M_{n(i)} \) with it’s image in \( A \), passing to the GNS construction \( \pi_{\gamma} \) and then cutting down by \( Q^{(i+1)} \). \( \square \)

As mentioned above, the construction used in this section is a simple adaptation of the Bratteli systems introduced in [19]. Chris Phillips is currently writing a manuscript [40] which contains a detailed exposition of (a generalization of) these inductive systems. He also proves a number of results concerning the UCT [48] which we will find use for later.
The two UCT results which are needed in this work are stated below. See section 2 of [40] for a detailed explanation of what is meant by the UCT and the proofs of the following results.

**Proposition 2.5.2** (Two out of Three Principle). If $0 \to I \to E \to B \to 0$ is a semi-split extension (i.e. there exists a contractive c.p. splitting $B \to E$) and any two of $I$, $E$ and $B$ satisfy the UCT then so does the third.

**Proposition 2.5.3.** Let $\phi_n : A_n \to A_{n+1}$ be injective *-homomorphisms and let $A$ denote the inductive limit of this sequence. Assume further that for each $n$, there exists a contractive c.p. map $\psi_n : A_{n+1} \to A_n$ such that $\psi_n \circ \phi_n = \text{id}_{A_n}$ for all $n$. If each $A_n$ satisfies the UCT then so does $A$.

**Remark 2.5.4.** Note that the inductive limit construction used in Theorem 2.5.1 does have the one-sided c.p. inverses required in Proposition 2.5.3 and hence the Popa algebra constructed will satisfy the UCT whenever the original residually finite dimensional algebra does. To see that such c.p. inverses exist, we let $\tau : E \to M_k(\mathbb{C})$ be a *-homomorphism and $\rho : E \to E \otimes M_n(\mathbb{C})$ be as in the basic construction. The desired map $E \otimes M_n(\mathbb{C}) \to E \cong E \otimes e_{1,1}$ is just given by compressing to the $(1,1)$ corner.

2.6. **Miscellaneous.** Here we collect a few facts for future reference. The first two are well known and the second two are just a bit of trickery. We begin with a simple adaptation of the fact that if $M$ is a von Neumann algebra with faithful, normal tracial state $\tau$ and $1_M \in N \subset M$ is a sub-von Neumann algebra then there always exists a $\tau$-preserving (hence faithful) conditional expectation $M \to N$ (cf. [30, Exercise 8.7.28]).

**Lemma 2.6.1.** Let $A$ be a $C^*$-algebra, $\tau \in T(A)$ be a tracial state and $1_A \in B \subset A$ be a finite dimensional subalgebra. Then there exists a conditional expectation $\Phi_B : A \to B$ such that $\tau \circ \Phi_B = \tau$.

**Proof.** Assume first that $\tau|_B$ is faithful. Write $B \cong M_{n(1)}(\mathbb{C}) \oplus \cdots \oplus M_{n(k)}(\mathbb{C})$, and let $\{e_{ij}^{(1)}\}_{1 \leq i,j \leq n(1)} \cup \cdots \cup \{e_{ij}^{(k)}\}_{1 \leq i,j \leq n(k)}$ be a system of matrix units for $B$. Then the desired conditional expectation is given by

$$\Phi_B(x) = \sum_{s=1}^k \sum_{i,j=1}^{n(s)} \frac{\tau(x e_{ij}^{(s)})}{\tau(e_{ij}^{(s)})} e_{ij}^{(s)}.$$

When $\tau|_B$ is not faithful, the formula above no longer makes sense. However, one can decompose $B$ as the direct sum of two finite dimensional algebras, $B_0 \oplus B_f$, where $\tau|_{B_0} = 0$ and $\tau|_{B_f}$ is faithful. Letting $e_0$ (resp. $e_f$) be the unit of $B_0$ (resp. $B_f$) we get a $\tau$-preserving conditional expectation by mapping each $a \in A$ to $E_{B_0}(e_0 a e_0) + E_{B_f}(e_f a e_f)$, where $E_{B_0} : e_0 A e_0 \to B_0$ is any conditional expectation (which exist by finite dimensionality) and $E_{B_f} : e_f A e_f \to B_f$ is a $\tau|_{e_f A e_f}$ preserving conditional expectation as in the first part of the proof.

**Lemma 2.6.2.** If $B$ is a finite dimensional $C^*$-algebra with tracial state $\tau$ then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and a unital *-homomorphism $\rho : B \to M_n(\mathbb{C})$ such that $|\tau(x) - tr_n \circ \rho(x)| < \varepsilon \|x\|$ for every $x \in B$.

**Proof.** If $\tau$ is a rational convex combination of extreme traces then one can find an honestly trace preserving embedding by inflating the summands of $B$ according to the rational numbers appearing in the convex combination. The general case then follows by approximation. \qed
We remind the reader of the following theorem of Voiculescu (cf. [49, Theorem 1]): A separable, unital C*-algebra A is quasidiagonal (QD) if and only if there exists a sequence of u.c.p. maps \( \varphi_n : A \to M_{k(n)}(\mathbb{C}) \) which are asymptotically multiplicatively (i.e. \( \| \varphi_n(ab) - \varphi_n(a)\varphi_n(b) \| \to 0 \) for all \( a, b \in A \)) and asymptotically isometric (i.e. \( \|a\| = \lim \|\varphi_n(a)\| \), for all \( a \in A \)).

**Lemma 2.6.3.** Assume that A is a QD C*-algebra and \( \psi_n : A \to M_{l(n)}(\mathbb{C}) \) is an asymptotically multiplicative (but not necessarily asymptotically isometric) sequence of u.c.p. maps. Then there exists a sequence of u.c.p. maps \( \Phi_n : A \to M_{l(n)}(\mathbb{C}) \) which are asymptotically multiplicative, asymptotically isometric and such that \( |tr_{l(n)}(\Phi_n(a)) - tr_{l(n)}(\psi_n(a))| \to 0 \) for all \( a \in A \).

**Proof.** Let \( \varphi_n : A \to M_{k(n)}(\mathbb{C}) \) be an asymptotically multiplicative, asymptotically isometric sequence of u.c.p. maps. Choose integers \( s(n) \) such that \( \frac{k(n)}{s(n)} \to 1 \) (i.e. such that \( k(n) \to 0 \)). Then one defines \( \Phi_n : A \to M_{s(n)l(n) + k(n)}(\mathbb{C}) \) to be the block diagonal map with one summand equal to \( \varphi_n \) and \( s(n) \) summands equal to \( \psi_n \).

Note that the previous lemma can also be formulated in terms of \(*\)-homomorphisms when A is a residually finite dimensional C*-algebra.

Though we will try to keep everything unital, non-unital maps are sometimes unavoidable. The next lemma keeps everything running smoothly.

**Lemma 2.6.4.** Let \( \phi_n : A \to M_{k(n)}(\mathbb{C}) \) be contractive c.p. maps (though we still assume A is unital) which are asymptotically multiplicative with respect to the 2-norms on \( M_{k(n)}(\mathbb{C}) \). If \( tr_{k(n)}(\phi_n(1_A)) \to 1 \) then there exist u.c.p. maps \( \psi_n : A \to M_{k(n)}(\mathbb{C}) \) which are also asymptotically multiplicative (with respect to 2-norms) and such that \( |tr_{k(n)}(\psi_n(a)) - tr_{k(n)}(\phi_n(a))| \to 0 \) as \( n \to \infty \), with convergence being uniform on the unit ball of A.

**Proof.** By [16, Lemma 2.2] we can find u.c.p. maps \( \psi_n : A \to M_{k(n)}(\mathbb{C}) \) such that \( \phi_n(a) = c_n\psi_n(a)c_n \) for all \( n \) and \( a \in A \), where \( c_n = \phi_n(1_A)^{1/2} \). Since \( tr_{k(n)}(\phi_n(1_A)) \to 1 \) it follows that \( \|c_n - 1_{M_{k(n)}}\|_2 \to 0 \). Using this remark, the Cauchy-Schwartz inequality and the general inequality \( \|x - c_nxc_n\|_2 \leq 2\|x\|\|c_n - 1_{M_{k(n)}}\|_2 \) (for all \( x \in M_{k(n)}(\mathbb{C}) \)) it is straightforward to verify that \( \psi_n \) have the properties asserted in the statement of the lemma.

### 3. Approximately Finite Dimensional Traces

We remind the reader that all C*-algebras (resp. von Neumann algebras) in these notes are assumed to be separable and unital (resp. have separable preduals). We also remind the reader that all notation not defined below should have been explained in Section 2.1.

We now define the tracial invariants mentioned in the introduction. We will be concerned with those traces that can be approximated by almost multiplicative maps to matrix algebras. We can require these maps to be almost multiplicative either in the operator norm or the 2-norm and we can require either weak-* or norm convergence in the dual space. Hence we get four types of approximation. Kirchberg has studied similar sets of traces in his work on Connes’ embedding problem (see [31, Lemma 4.5]) and residually finite groups with Kazhdan’s property T (see [33, Definition 3.1]). Indeed, it is not hard to see that our definition of \( T(A)_{w,AFD} \) below agrees with Kirchberg’s notion of liftable traces from [33]. The main theorem concerning the set \( T(A)_{w,AFD} \) (Theorem 3.6 below) will thus depend heavily on [33, Proposition 3.2].

**Definition 3.1.** Let \( A \) be a C*-algebra and \( T(A) \) denote the (possibly empty) set of tracial states on \( A \). We will say that a trace \( \tau \in T(A) \) is *approximately finite dimensional* if there
exists a sequence of u.c.p. maps $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ and $\tau(a) = \lim_{n \to \infty} tr_{k(n)} \circ \phi_n(a)$, for all $a, b \in A$. If we have norm convergence in the dual space $A^*$ (i.e. $\|\tau - tr_{k(n)} \circ \phi_n\|_{A^*} \to 0$) then $\tau$ will be called uniformly approximately finite dimensional.

A trace $\tau \in T(A)$ is called weakly approximately finite dimensional if there exists a sequence of u.c.p. maps $\phi_n : A \to M_{k(n)}(\mathbb{C})$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \to 0$ and $\tau(a) = \lim_{n \to \infty} tr_{k(n)} \circ \phi_n(a)$, for all $a, b \in A$. Similarly, $\tau$ will be called uniformly weakly approximately finite dimensional if $tr_{k(n)} \circ \phi_n \to \tau$ in the norm topology on $A^*$.

We then put

$$T(A)_{AFD} = \{\tau \in T(A) : \tau \text{ is approximately finite dimensional}\},$$

$$T(A)_{w-AFD} = \{\tau \in T(A) : \tau \text{ is weakly approximately finite dimensional}\},$$

$$UT(A)_{AFD} = \{\tau \in T(A) : \tau \text{ is uniformly approximately finite dimensional}\},$$

$$UT(A)_{w-AFD} = \{\tau \in T(A) : \tau \text{ is uniformly weakly approximately finite dimensional}\}.$$

Evidently we have the following inclusions of the sets defined above.

$$T(A) \supset T(A)_{w-AFD} \supset T(A)_{AFD},$$

$$UT(A)_{w-AFD} \supset UT(A)_{AFD}.$$ 

We will see in Section 6 that if $A$ is locally reflexive (e.g. nuclear or exact) then $T(A) \supset T(A)_{w-AFD} = UT(A)_{w-AFD} \supset T(A)_{AFD} = UT(A)_{AFD}$. 

Though they may seem unnatural at first, the definitions above are inspired by the theories of quasidiagonal and nuclear C*-algebras.

Recall that a C*-algebra $A$ is quasidiagonal (QD) if there exists a faithful representation $\pi : A \to B(H)$ such that one can find an increasing sequence of finite rank projections $P_1 \leq P_2 \leq \ldots$ with the property that $\|\pi(a)P_n - P_n\pi(a)\| \to 0$ for all $a \in A$ and $P_n \to 1_H$ in the strong operator topology. (This is not the right definition for non-separable algebras.)

**Example 3.2.** (cf. [50, 2.4], [13, Proposition 6.1]) If $A$ is QD then $T(A)_{AFD} \neq \emptyset$ (in the non-unital case it can happen that $T(A) = \emptyset$; e.g. the suspension of a Cuntz algebra). Indeed if $\pi : A \to B(H)$ and $P_1 \leq P_2 \leq \ldots$ are as in the definition of quasidiagonality then one defines u.c.p. maps by

$$\phi_n(a) = P_n\pi(a)P_n.$$ 

The asymptotic commutativity of $P_n$ ensures that $\phi_n$ are asymptotically multiplicative in norm. Note also that $P_nB(H)P_n \cong M_{\text{rank}(P_n)}(\mathbb{C})$. Finally, a straightforward calculation shows that any weak-* cluster point of the sequence of states $\{tr_{\text{rank}(P_n)} \circ \phi_n\}$ is necessarily a tracial state and hence $T(A)_{AFD} \neq \emptyset$.

From the previous example we see that the approximately finitely dimensional traces form a very natural subset of the tracial space of a QD C*-algebra. Will we see later that $UT(A)_{AFD}$ is also fairly natural (at least in the classification program). A much deeper fact is that the sets $T(A)_{w-AFD}$ and $UT(A)_{w-AFD}$ are large for an important class of C*-algebras (see also Corollary 7.1).

**Proposition 3.3.** *If $A$ is a nuclear C*-algebra then $T(A) = T(A)_{w-AFD} = UT(A)_{w-AFD}$.***
Proof. The proof is a simple consequence of the following deep fact: If $A$ is nuclear and $\tau \in T(A)$ then $\pi, (A)$ is hyperfinite (cf. [15], [17], [41], [42]). Given this result, the equation above follows easily from Lemmas 2.6.1 and 2.6.2. \hfill \Box

**Proposition 3.4.** The sets $T(A)_{\text{AFD}}$, $T(A)_{\text{w-AFD}}$, $UT(A)_{\text{AFD}}$ and $UT(A)_{\text{w-AFD}}$ are all convex. The sets $T(A)_{\text{AFD}}$ and $T(A)_{\text{w-AFD}}$ are closed in the weak-* topology while $UT(A)_{\text{w-AFD}}$ and $UT(A)_{\text{AFD}}$ are closed in norm (i.e. the norm on $A^*$) and thus, by the Hahn-Banach theorem, closed in the weak topology coming from $A^{**}$.

Proof. It is not hard to verify that the first assertion follows from Lemma 2.6.2. Since $T(A)_{\text{AFD}}$ and $T(A)_{\text{w-AFD}}$ (resp. $UT(A)_{\text{w-AFD}}$ and $UT(A)_{\text{AFD}}$) are defined via weak-* convergence (resp. norm convergence), it is also easy to see that these sets are closed in this topology. \hfill \Box

We will soon see that if $\Gamma$ is a non-amenable, residually finite, discrete group then the canonical trace on $C^*(\Gamma)$ (which gives the left regular representation in the GNS construction) is always a weak-* limit of uniformly approximately finite dimensional traces, but is not itself uniformly weakly approximately finite dimensional. Thus, the sets $UT(A)_{\text{w-AFD}}$ and $UT(A)_{\text{AFD}}$ need not be weak-* closed in general.

In [33, Definition 3.1] Kirchberg introduced the notion of a liftable trace. His definition is as follows: $\tau \in T(A)$ is liftable if there exists a completely positively liftable $\ast$-homomorphism $\pi : A \to R_\omega$ such that $\tau_\omega \circ \pi = \tau$. (Here, $R_\omega$ is as defined in Section 2.1 and ‘completely positively liftable’ means that there exists a completely positive lifting (from $A \to l^\infty(R)$ for $\pi$.) To the experts it is immediate that this notion is the same as weakly approximately finite dimensional, as defined above, and it is a good exercise for those not familiar with these notions. In any case, the next proposition is now a direct consequence of [33, Lemma 3.4].

**Proposition 3.5.** $T(A)_{\text{w-AFD}}$ is a face in $T(A)$. We will soon see that $UT(A)_{\text{w-AFD}}$ is also a face, but we do not know whether the sets $T(A)_{\text{AFD}}$ and $UT(A)_{\text{AFD}}$ are always faces.

The hardest part in the proof of the following theorem is the implication (7) $\implies$ (8). As previously indicated, this was already done by Kirchberg in [33, Proposition 3.2]. His proof amounts to a very clever (and technically difficult) reduction to [17, Theorem 1.2.2]. Indeed, as will become clear, most of the content of this theorem can be traced back to the fundamental work of Alain Connes [17].

**Theorem 3.6.** Let $A \subset B(H)$ be in general position (i.e. $A \cap K(H) = \{0\}$) and $\tau \in T(A)$. Then the following are equivalent:

1. $\tau \in T(A)_{\text{w-AFD}}$.
2. There exist finite rank projections $P_n \in B(H)$ (not necessarily increasing) such that
   \[ \frac{\|aP_n - P_n a\|_{HS}}{\|P_n\|_{HS}} \to 0 \quad \text{and} \quad \tau(a) = \lim_{n \to \infty} \frac{< aP_n, P_n >_{HS}}{< P_n, P_n >_{HS}}, \]
   for all $a \in A$.
3. For all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,
   \[ |\tau(\sum_{i=1}^n a_i b_i^*)| \leq \| \sum_{i=1}^n a_i \otimes b_i^*\|_{A \otimes A^{op}}. \]
4. For all \( a_1, \ldots, a_n, b_1, \ldots, b_n \in A \),
\[
\| \pi_\tau \circ \pi_\tau^{op} \left( \sum_{i=1}^{n} a_i \otimes b_i \right) \| \leq \| \sum_{i=1}^{n} a_i \otimes b_i \|_{A \otimes A^{op}}.
\]

5. \( \pi_\tau \circ \pi_\tau^{op} \) extends to a representation of \( A \otimes A^{op} \).

6. There exists a u.c.p. map \( \Phi : B(H) \to \pi_\tau(A)'' \) such that \( \Phi(a) = \pi_\tau(a) \), for all \( a \in A \).

7. \( \tau \) extends to a hypertrace (i.e. there exists a state \( \varphi \) on \( B(H) \) such that \( \varphi|_A = \tau \) and \( A \subset B(H) \) \( \varphi = \{ T \in B(H) : \varphi(ST) = \varphi(TS) \}, \) for all \( S \in B(H) \}).

8. There exists a \( \ast \)-monomorphism \( \Phi : \pi_\tau(A)'' \to R^\omega \) such that \( \tau'' = \tau_\omega \circ \Phi \) and \( \Phi \circ \pi_\tau : A \to R^\omega \) can be lifted to a u.c.p. map \( A \to \ell^\infty(R) \), where \( \tau'' \) is the vector trace on \( \pi_\tau(A)'' \) induced by \( \tau \).

**Proof.** (1) \( \implies \) (2). This follows from Voiculescu’s Theorem.

(2) \( \implies \) (3) and (3) \( \implies \) (4) are essentially the same as (6) \( \implies \) (5) and (5) \( \implies \) (4), respectively, from [17, Theorem 5.1].

(4) \( \implies \) (5) is immediate.

(5) \( \implies \) (6) is a consequence of [31, Observation 3.0] but we remind the reader of Kirchberg’s elegant proof. Since \( A \otimes A^{op} \subset B(H) \otimes A^{op} \) we can extend \( \pi_\tau \circ \pi_\tau^{op} : A \otimes A^{op} \to B(L^2(A, \tau)) \) to a completely positive map \( \Phi : B(H) \otimes A^{op} \to B(L^2(A, \tau)) \). Since \( \Phi|_{A \otimes A^{op}} \) is a homomorphism it follows that \( A \otimes A^{op} \) (and, in particular, \( 1 \otimes A^{op} \)) is in the multiplicative domain of \( \Phi \). Hence, for every \( T \in B(H) \), it follows that \( \Phi(T \otimes 1) \in \Phi(1 \otimes A^{op})' = \pi_\tau^{op}(A^{op})' = \pi_\tau(A)'' \).

(6) \( \implies \) (7). Since \( A \) is contained in the multiplicative domain of \( \Phi \), it is easy to verify that \( \varphi(T) = \langle \Phi(T) \eta_\tau, \eta_\tau \rangle \) gives the desired hypertrace (here, \( \eta_\tau = 1 \in L^2(A, \tau) \) is the canonical trace vector).

(7) \( \implies \) (8). This follows from (iii) \( \implies \) (ii) in [33, Proposition 3.2]. Indeed Kirchberg’s result states that, assuming (7), we can find a u.c.p. liftable \( \ast \)-homomorphism \( \Psi : A \to R^\omega \) such that \( \tau = \tau_\omega \circ \Psi \). Hence the weak closure of \( \Psi(A) \) (inside \( R^\omega \)) will be canonically isomorphic to \( \pi_\tau(A)'' \) and the desired map \( \Phi \) is just this identification.

Finally, (8) \( \implies \) (1) is immediate to the reader who has verified that liftable traces and weakly approximately finite dimensional traces are the same thing.

**Remark 3.7.** Note that the proofs of (5) \( \implies \) (6) and (6) \( \implies \) (7) never used our assumption that \( A \) is in general position. Hence every weakly approximately finite dimensional trace extends to a hypertrace in any faithful representation of \( A \).

The space \( UT(A)_{w-AFD} \) also admits a number of nice characterizations. We thank Yasuyuki Kawahigashi, Sergei Neshveyev and Narutaka Ozawa for discussions in Oberwolfach which slightly shortened our original proof of (1) \( \implies \) (2). Ozawa also added (6) to the list below and showed us the elegant proof.

**Theorem 3.8.** For a trace \( \tau \in T(A) \), the following are equivalent:

1. \( \tau \in UT(A)_{w-AFD} \).

2. There exists u.c.p. maps \( \psi_n : A^{**} \to M_{k(n)}(\mathbb{C}) \) such that for each free ultrafilter \( \omega \in \beta \mathbb{N} \setminus \mathbb{N} \) we have
\[
\lim_{n \to \omega} \| \psi_n(xy) - \psi_n(x)\psi_n(y) \|_2 = 0 \quad \text{and} \quad \lim_{n \to \omega} tr_{k(n)} \circ \psi_n(x) = \tau^{**}(x),
\]
for all \( x, y \in A^{**} \), where \( \tau^{**} \) is the normal trace on \( A^{**} \) induced by \( \tau \).

3. There exist u.c.p. maps \( \psi_n : \pi_\tau(A)^{**} \to M_{k(n)}(\mathbb{C}) \) such that for each free ultrafilter \( \omega \in \beta \mathbb{N} \setminus \mathbb{N} \) we have
\[
\lim_{n \to \omega} \| \psi_n(xy) - \psi_n(x)\psi_n(y) \|_2 = 0 \quad \text{and} \quad \lim_{n \to \omega} tr_{k(n)} \circ \psi_n(x) = \tau''(x),
\]
for all $x,y \in \pi_{\tau}(A)^{\prime\prime}$, where $\tau^{\prime\prime}$ is the normal trace on $\pi_{\tau}(A)^{\prime\prime}$ induced by $\tau$.

4. There exists a u.c.p. liftable, normal $*$-monomorphism $\sigma : \pi_{\tau}(A)^{\prime\prime} \rightarrow R^{\omega}$.

5. $\pi_{\tau}(A)^{\prime\prime}$ is hyperfinite.

6. $\pi_{\tau} : A \rightarrow \pi_{\tau}(A)^{\prime\prime}$ is weakly nuclear (i.e. there exists u.c.p. maps $\phi_{n} : A \rightarrow M_{k(n)}(\mathbb{C})$, $\psi_{n} : M_{k(n)}(\mathbb{C}) \rightarrow \pi_{\tau}(A)^{\prime\prime}$ such that $\psi_{n} \circ \phi_{n}(a) \rightarrow \pi_{\tau}(a)$ in the $\sigma$-weak topology for every $a \in A$).

Proof. $(1) \implies (2)$. Let $\phi_{n} : A \rightarrow M_{k(n)}(\mathbb{C})$ be 2-norm asymptotically multiplicative u.c.p. maps such that $tr_{k(n)} \circ \phi_{n} \rightarrow \tau$ in the norm on $A^{\ast}$. Let $\phi_{n}^{\ast} : A^{\ast} \rightarrow M_{k(n)}(\mathbb{C})$ be the canonical extensions to the double dual. Since $\|tr_{k(n)} \circ \phi_{n} - \tau\|_{A^{\ast}} \rightarrow 0$, it follows that $tr_{k(n)} \circ \phi_{n}^{\ast}(x) \rightarrow \tau^{\ast\ast}(x)$ for every $x \in A^{\ast\ast}$. Hence we can construct a u.c.p. map $\Phi : A^{\ast\ast} \rightarrow R^{\omega}$ such that $i)$ $\tau_{\omega} \circ \Phi = \tau^{\ast\ast}$, $ii)$ $\Phi|_{A}$ is a $*$-homomorphism and $iii)$ $\Phi$ has a u.c.p. lifting $A^{\ast\ast} \rightarrow l^{\infty}(R)$. If we knew that $\Phi$ was a homomorphism then it would follow that $\lim_{n \rightarrow \infty} \|\phi_{n}^{\ast}(xy) - \phi_{n}^{\ast}(x)\phi_{n}^{\ast}(y)\|_{2} = 0$ for all $x,y \in A^{\ast\ast}$ and hence this is what we will show.

First note that $\Phi$ is normal: if $\{x_{\lambda}\} \subset A^{\ast\ast}$ is a norm bounded, increasing net of self adjoint elements with strong operator topology limit $x$ then $\{\Phi(x_{\lambda})\}$ is increasing up to $\Phi(x)$ (in the strong operator topology - i.e. 2-norm) since $\Phi(x_{\lambda}) \leq \Phi(x)$ and $\tau_{\omega} \circ (\Phi(x_{\lambda})) = \tau^{\ast\ast}(x_{\lambda}) \rightarrow \tau^{\ast\ast}(x) = \tau_{\omega} \circ \Phi(x)$. It follows that $\Phi$ is continuous from the $\sigma$-weak topology on $A^{\ast\ast}$ to the $\sigma$-weak topology on $R^{\omega}$ (i.e. w.r.t. the weak-$*$ topologies coming from the preduals). Letting $\Psi : A^{\ast\ast} \rightarrow R^{\omega}$ be the (weak-$*$ continuous) $*$-homomorphism which extends $\Phi|_{A}$ (and which exists by universality of $A^{\ast\ast}$) it follows that $\Phi = \Psi$ since they are continuous and agree on $A$. Hence $\Phi$ is also multiplicative.

$(2) \implies (3)$. Assuming $(2)$, we can use the maps $\psi_{n}$ to construct a u.c.p. liftable, $*$-homomorphism $\sigma : A^{\ast\ast} \rightarrow R^{\omega}$ such that $\tau_{\omega} \circ \sigma = \tau^{\ast\ast}$. It follows that $\sigma(A^{\ast\ast}) \cong \pi_{\tau}(A)^{\prime\prime}$ and, hence, $A^{\ast\ast} \cong ker(\sigma) \oplus \pi_{\tau}(A)^{\prime\prime}$. Restricting the maps $\psi_{n}$ to this non-unital copy of $\pi_{\tau}(A)^{\prime\prime}$ gives c.p. maps with the desired properties. Then applying Lemma 2.6.4 we can replace these nonunital maps with unital ones and we get $(3)$.

$(3) \implies (4)$ is immediate.

$(4) \implies (5)$. Since $l^{\infty}(R)$ is injective and there exists a (trace preserving) conditional expectation $R^{\omega} \rightarrow \sigma(\pi_{\tau}(A)^{\prime\prime})$, it is not hard to see that $\pi_{\tau}(A)^{\prime\prime}$ must be injective, assuming $(4)$.

$(5) \implies (1)$ is contained in the proof of Proposition 3.3 and hence $(1) - (5)$ are equivalent.

$(5) \implies (6)$ is trivial and hence we are left to prove $(6) \implies (5)$. So assume that $\pi_{\tau} : A \rightarrow \pi_{\tau}(A)^{\prime\prime}$ is weakly nuclear and $\phi_{n} : A \rightarrow M_{k(n)}(\mathbb{C})$, $\psi_{n} : M_{k(n)}(\mathbb{C}) \rightarrow \pi_{\tau}(A)^{\prime\prime}$ are u.c.p. maps whose composition converges to $\pi_{\tau}$ in the point-$\sigma$-weak topology. Using these maps it is not hard to see that the canonical homomorphism $A \otimes \pi_{\tau}(A)^{\prime} \rightarrow B(L^{2}(A,\tau))$, $a \otimes x \mapsto \pi_{\tau}(a)x$, is continuous with respect to the minimal tensor product norm. (Use the fact that the natural map on the maximal tensor product approximately factorizes through $M_{k(n)} \otimes_{max} \pi_{\tau}(A)^{\prime} = M_{k(n)} \otimes \pi_{\tau}(A)^{\prime}$ and hence factors through the minimal tensor product.) As in the proof of $(5) \implies (6)$ from the last theorem, it follows that there exists a conditional expectation $B(L^{2}(A,\tau)) \rightarrow \pi_{\tau}(A)^{\prime}$ and hence $\pi_{\tau}(A)^{\prime}$ is injective. This implies that $\pi_{\tau}(A)^{\prime\prime}$ is also injective and the proof is complete.

Note that part $(3)$ in the previous theorem could be used as an abstract (i.e. representation free) definition of quasidiagonality, analogous to Voiculescu’s abstract characterization [49, Theorem 1], in the setting of tracial von Neumann algebras. The equivalence of $(3)$ and $(5)$ would then say that quasidiagonality is equivalent to hyperfiniteness. (Compare with the $C^{\ast}$-case where Dadarlat has constructed non-nuclear Popa algebras [19].)

**Corollary 3.9.** UT(A)_{w-afd} is always a face in T(A).
Proof. If $0 < s < 1$, $\tau, \gamma \in T(A)$ and $s\tau + (1-s)\gamma \in UT(A)_{w-AFD}$ then we can find (non-unital) normal embeddings of $\pi_\tau(A)''$ and $\pi_\gamma(A)''$ into the weak closure of the GNS representation of $A$ with respect to $s\tau + (1-s)\gamma$ (cf. [39, Proposition 3.3.5]). As the latter algebra is injective, it follows that both $\pi_\tau(A)''$ and $\pi_\gamma(A)''$ are injective (hence hyperfinite). Thus, both $\tau$ and $\gamma$ belong to $UT(A)_{w-AFD}$. □

The set $T(A)_{AFD}$ seems a bit hard to get a handle on. This is largely due, at least in this author’s opinion, to our present lack of understanding of the class of QD C*-algebras. However, if one knows a C*-algebra to be QD then the set $T(A)_{AFD}$ is precisely the set of traces which can be encoded in the definition of quasidiagonality.

Proposition 3.10. Let $A \subset B(H)$ be in general position. If $A$ is QD then there exists an increasing sequence of finite rank projections $P_1 \leq P_2 \leq \ldots$, converging strongly to the identity, which asymptotically commutes (in norm) with every element in $A$ and such that for each $\tau \in T(A)_{AFD}$ there exists a subsequence $\{n_k\}$ such that

$$\frac{<aP_{n_k}, P_{n_k}>_{HS}}{<P_{n_k}, P_{n_k}>_{HS}} \rightarrow \tau(a), \text{ as } k \rightarrow \infty,$$

for all $a \in A$.

Proof. We claim that it suffices to prove that for every finite set $\mathcal{F} \subset A$, finite dimensional subspace $X \subset H$, $\varepsilon > 0$ and trace $\tau \in T(A)_{AFD}$ there exists a finite rank projection $P \in B(H)$ such that

1. $\|[a, P]\| < \varepsilon$ for all $a \in \mathcal{F}$.
2. $P(x) = x$ for all $x \in X$.
3. $\left|\frac{<aP, P>_{HS}}{<P, P>_{HS}} - \tau(a)\right| < \varepsilon$ for all $a \in \mathcal{F}$.

Assume for the moment that we were able to prove this local version. Then, if $\{a_n\} \subset A$ is a sequence which is dense in the unit ball of $A$ and $\{\tau_j\}$ is any sequence of traces in $T(A)_{AFD}$ we could apply the above local approximation property to construct a sequence $P_1 \leq P_2 \leq \ldots$ which was converging strongly to the identity, asymptotically commuting in norm with $A$ and such that

$$\left|\frac{<a_iP_n, P_n>_{HS}}{<P_n, P_n>_{HS}} - \tau_n(a_i)\right| < 1/n$$

for all $n \in \mathbb{N}$ and $1 \leq i \leq n$. Since $A$ is separable, the weak-* topology on $T(A)$ is metrizable and hence we can always find a sequence of traces $\{\tau_j\} \subset T(A)_{AFD}$ such that there exists a weak-* dense subset $Y \subset T(A)_{AFD}$ with the property that every element of $Y$ appears infinitely many times in the sequence $\{\tau_j\}$. The sequence of projections associated with such a sequence of traces will have all the properties asserted in the statement of the proposition. Hence it suffices to prove the local statement in the first paragraph of the proof.

The required local statement is now a consequence of Voiculescu’s Theorem (version 2.2.2) and a little trickery. Let $\tau \in T(A)_{AFD}$ be arbitrary. Since $A$ is QD, by Lemma 2.6.3 we can find a sequence of u.c.p. maps $\phi_n : A \rightarrow M_{k(n)}(\mathbb{C})$ which are asymptotically multiplicative, asymptotically isometric and such that $tr_{k(n)} \circ \phi_n \rightarrow \tau$ in the weak-* topology. If a finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ are given then, by passing to a subsequence if necessary, we may assume that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| < \varepsilon$ and $\|tr_{k(n)} \circ \phi_n(a) - \tau(a)\| < \varepsilon$ for all $a, b \in \mathcal{F}$. Letting $K = \oplus_n \mathbb{C}^{k(n)}$ and $\Phi = \oplus_n \phi_n : A \rightarrow B(K)$ we have that $\Phi$ is a faithful $*$-homomorphism modulo the weak topology. Hence we can find a unitary operator $U : K \rightarrow H$ such that $U\Phi(a)U^*$ is nearly equal (in norm) to $a$, for all $a \in \mathcal{F}$. Hence, if $Q_s \in B(K)$ is the orthogonal projection onto $\oplus_s \mathbb{C}^{k(n)}$ we have that $\|UQ_sU^*, a\|$ is small for all $s \in \mathbb{N}$ and for all $a \in A$. Moreover, compressing $a \in \mathcal{F}$ to the range of $UQ_sU^*$ will almost recover the trace
τ (for all s). Finally, if \( X \subset H \) is any finite dimensional subspace then \( X \) will almost be contained in the range of \( UQ_s U^* \) for sufficiently large \( s \) and hence a tiny (norm) perturbation of a sufficiently large \( UQ_s U^* \) will actually be the identity on \( X \) (and still almost commute with \( \mathcal{F} \) and almost recover the trace \( \tau \)).

As mentioned above, the set \( \text{UT}(A)_{\text{AFD}} \) is the one most relevant to the classification program. It also seems to be the most difficult one to understand in general. However, for certain C*-algebras it is easily seen to be all of \( \text{T}(A) \).

**Definition 3.11.** We will call a C*-algebra \( A \) *tracially type I* if for each finite subset \( \mathcal{F} \subset A \) and \( \varepsilon > 0 \) there exists a type I subalgebra \( B \subset A \) with unit \( e \) such that \( \| [x, e] \| < \varepsilon \) for all \( x \in \mathcal{F} \), \( e \mathcal{F} e \subset \varepsilon B \) and \( \tau(e) > 1 - \varepsilon \) for every \( \tau \in \text{T}(A) \).

One could also adopt Lin’s notions used to define tracial topological rank. However, for our purposes the simpler definition above is sufficient. It is not hard to verify that if \( A \) is a C*-algebra with finite tracial topological rank (either [37, Definition 3.1] or [37, Definition 3.4]) then \( A \) is tracially type I in the sense described above. Moreover, any ASH algebra (with or without slow dimension growth) is evidently tracially type I.

**Proposition 3.12.** If \( A \) is tracially type I then \( \text{T}(A) = \text{UT}(A)_{\text{AFD}} \).

**Proof.** Let \( A \) be tracially type I and \( \tau \in \text{T}(A) \) be arbitrary. If \( \mathcal{F} \subset A \) is arbitrary and \( \varepsilon > 0 \) is given then we must produce a u.c.p. map \( \phi : A \to M_n(\mathbb{C}) \) such that \( \phi \) is \( \varepsilon \)-multiplicative on \( \mathcal{F} \) and \( |\tau(x) - tr_n \circ \phi(x)| < \varepsilon \) for all \( x \) in the unit ball of \( A \).

Choose \( \delta > 0 \) very small and apply the definition of tracially type I to \( \mathcal{F} \) and \( \delta \) to get a type I subalgebra \( B \subset A \) with unit \( e \). Note that for every positive \( x \) in the unit ball of \( A \) we have \( |\tau(x) - \tau(\mathcal{F} e)\mathcal{F} e| < \delta \). Also, the map \( x \mapsto \mathcal{F} e \) is almost multiplicative on \( \mathcal{F} \).

If \( \pi_\tau : A \to B(L^2(A, \tau)) \) is the GNS representation, then the weak closure of \( \pi_\tau(B) \) is a type I von Neumann algebra with faithful, normal trace given by \( \tau(\cdot)/\tau(e) \).

**Case 1:** Assume that the weak closure of \( \pi_\tau(B) \) is isomorphic to

\[
\prod_{i=1}^k M_n(i)(\mathbb{C}) \otimes L^\infty(X_i, \mu_i)
\]

for some probability spaces \( (X_i, \mu_i) \). Then the weak closure is a (non-separable) AF algebra. In particular, we can find a finite dimensional subalgebra, \( C \), of the weak closure of \( \pi_\tau(e\mathcal{F} e) \) which almost contains (up to \( \delta \)) the set \( \pi_\tau(e\mathcal{F} e) \) *in norm*. We then get the desired map \( \phi \) by composing \( x \mapsto \pi_\tau(e\mathcal{F} e) \) with a \( \tau(\cdot)/\tau(e) \) preserving conditional expectation from the weak closure of \( \pi_\tau(e\mathcal{F} e) \) to \( C \) (actually, one must also apply Lemma 2.6.2 to get from \( C \) to a matrix algebra).

**Case 2:** The only other possibility is that the weak closure of \( \pi_\tau(B) \) is isomorphic to

\[
\prod_{i=1}^\infty M_n(i)(\mathbb{C}) \otimes L^\infty(X_i, \mu_i),
\]

which is no longer an AF algebra. However, we are saved by the fact that \( \tau(\cdot)/\tau(e) \) is a normal trace and hence we have

\[
1 = \sum_{i=1}^\infty \frac{\tau(e_i)}{\tau(e)},
\]

where \( e_i \) denotes the unit of \( M_n(i)(\mathbb{C}) \otimes L^\infty(X_i, \mu_i) \). Putting \( E_n = e_1 + \ldots + e_n \) we then get that \( \tau(E_n)/\tau(e) \to 1 \). Moreover, since \( \pi_\tau(e\mathcal{F} e) \) is almost contained in \( \pi_\tau(B) \) it follows that the map \( x \mapsto E_n \pi_\tau(e\mathcal{F} e) E_n \) is still almost multiplicative (in norm) on \( \mathcal{F} \). Taking \( n \) large enough, we now complete the proof as in case 1.

\[\square\]
As usual, groups provide some interesting and instructive examples. As we are sticking to unital algebras, we will only consider the discrete case.

**Example 3.13.** (Compare with [4, Corollary 2.11] .) If $\Gamma$ is a discrete group then there is a sort of "all or nothing" principle for the weakly approximately finite dimensional traces on the reduced group $C^*\text{-algebra} C^*_r(\Gamma)$. More precisely, we have $T(C^*_r(\Gamma)) = UT(C^*_r(\Gamma))_{w\text{-AFD}}$ if and only if $\Gamma$ is amenable and $T(C^*_r(\Gamma))_{w\text{-AFD}} = \emptyset$ if $\Gamma$ is not amenable. To see that this is the case, we first recall that if $\Gamma$ is amenable then $C^*_r(\Gamma)$ is nuclear and hence every trace is in $UT(C^*_r(\Gamma))_{w\text{-AFD}}$. On the other hand, if there exists a trace $\alpha \in T(C^*_r(\Gamma))_{w\text{-AFD}}$ then (in the left regular representation) $\alpha$ extends to a hypertrace $\varphi$ on $B(l^2(\Gamma))$. A simple calculation then shows that $\varphi$ defines a left invariant mean on $l^\infty(\Gamma) \subset B(l^2(\Gamma))$ and hence $\Gamma$ is amenable. (See [4, Proposition 2.12] for analogues of this to certain (twisted) crossed product $C^*$-algebras.)

The previous example clarifies an observation of J. Rosenberg: If $C^*_r(\Gamma)$ is QD then $\Gamma$ is amenable. Since $T(A)_{AFD} \neq \emptyset$ for every QD $C^*$-algebra the above example shows that it takes much less than quasidiagonality to imply amenability for $C^*_r(\Gamma)$.

There are two natural traces on the full group $C^*$-algebra, $C^*(\Gamma)$. Namely the one coming from the trivial one dimensional representation (which is clearly in $UT(C^*(\Gamma))_{AFD}$) and the one which gives the left regular representation after applying the GNS construction. This latter trace will be denoted by $\tau$, but in the following examples it is important to remember that we are no longer talking about the reduced group $C^*$-algebra.

**Example 3.14.** If $\Gamma$ is a residually finite discrete group then the canonical trace $\tau$ is always in $T(C^*(\Gamma))_{AFD}$. To see this, we let $\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots$ be a descending sequence of normal subgroups each of which has finite index in $\Gamma$ and such that their intersection is the neutral element. Let $\phi_n : C^*(\Gamma) \to B(l^2(\Gamma/\Gamma_n))$ be the unitary representation induced by the left regular representation of $\Gamma/\Gamma_n$. Since $l^2(\Gamma/\Gamma_n)$ is a finite dimensional Hilbert space it follows that $\tau \in T(C^*(\Gamma))_{AFD}$. Also note that $\tau$ is uniformly weakly approximately finite dimensional if and only if $\Gamma$ is amenable, by Theorem 3.8. It follows that the sets $UT(A)_{w\text{-AFD}}$ and $UT(A)_{AFD}$ need not be closed in the weak-$*$ topology, in general.

**Example 3.15.** If $\Gamma$ is a discrete group with Kazhdan’s Property T then $T(C^*(\Gamma))_{AFD} = T(C^*(\Gamma))_{w\text{-AFD}}$ and, moreover, $\tau \in T(C^*(\Gamma))_{w\text{-AFD}}$ if and only if $\gamma$ is residually finite. Both of these claims follow from [33, Proposition 2.3]. The basic idea is that Property T implies that any u.c.p. map which is weakly almost multiplicative can be approximated (in a probabilistic sense) by an honest homomorphism.

**Example 3.16.** Let $F_\infty$ be the free group on (countably) infinitely many generators. Then $T(C^*(F_\infty))_{w\text{-AFD}} = T(C^*(F_\infty))_{AFD} = [T(C^*(F_\infty))_{FD}]^\perp$, where $[T(C^*(F_\infty))_{FD}]^\perp$ denotes the weak-$*$ closure of the set of traces on $C^*(F_\infty)$ whose GNS representations are finite dimensional. This observation appears in Kirchberg’s work on Connes’ embedding problem for $\Pi_1$ factors (see [31, Lemma 4.5]). It is also shown in [31] that Connes’ embedding problem is equivalent to verifying the equation $T(C^*(F_\infty))_{w\text{-AFD}} = T(C^*(F_\infty))$. Indeed, what Kirchberg observes in [31] is that a trace $\tau$ on $C^*(F_\infty)$ is weakly approximately finite dimensional if and only if there exists a $\tau$-preserving embedding $\pi_\tau(C^*(F_\infty))'' \hookrightarrow R_\omega$. (See also [28].) From this we see that, in general, $T(A)_{w\text{-AFD}} \neq UT(A)_{w\text{-AFD}}$ and $T(A)_{AFD} \neq UT(A)_{AFD}$. (Since the latter sets give hyperfinite GNS representations, by Theorem 3.8, while there are plenty of examples of non-hyperfinite, $R_\omega$-embeddable von-Neumann algebras.) Compare with the locally reflexive case; Theorem 6.1.
In this section we discuss the connection between approximately finite dimensional traces and Elliott’s conjecture that simple, separable, nuclear $C^*$-algebras are classified by their K-theoretic and tracial data. Our first goal is to show that a necessary condition for Elliott’s conjecture to hold is that every unital, nuclear, QD $C^*$-algebra $A$ satisfies the equation $T(A) = UT(A)_{\text{AFD}}$ (even the non-simple ones!). Then, at the end of the section, we will point to a few references which lead us to believe that knowing an equation like $T(A) = UT(A)_{\text{AFD}}$ may eventually be part of the sufficiency as well.

Our first lemma states that when dealing with many classes of $C^*$-algebras, it often suffices to restrict attention to Popa algebras.

**Lemma 4.1.** If $\mathfrak{C}$ is a collection of $C^*$-algebras which contains $\mathbb{C}$ and is closed under i) increasing unions (i.e. inductive limits with injective connecting maps), ii) quasidiagonal, semi-split extensions (i.e. if $0 \to I \to E \to B \to 0$ is a semi-split (cf. [9]), short exact sequence, $I$ contains an approximate unit of projections which is quasicentral in $E$ and both $I, B \in \mathfrak{C}$ then $E \in \mathfrak{C}$) and iii) tensoring with finite dimensional matrix algebras then the following are equivalent:

1. $T(A) = T(A)_{\text{AFD}}$ (resp. $T(A) = UT(A)_{\text{AFD}}$) for every QD $A \in \mathfrak{C}$.
2. $T(A) = T(A)_{\text{AFD}}$ (resp. $T(A) = UT(A)_{\text{AFD}}$) for every residually finite dimensional $A \in \mathfrak{C}$.
3. $T(A) = T(A)_{\text{AFD}}$ (resp. $T(A) = UT(A)_{\text{AFD}}$) for every Popa algebra $A \in \mathfrak{C}$.

If, moreover, the class $\mathfrak{C}$ is closed under tensor products with (non-unital) abelian algebras (it actually suffices to know $A \in \mathfrak{C} \implies A \otimes C_0((0,1]) \in \mathfrak{C}$) then the following are equivalent:

4. $T(A) = T(A)_{\text{w-AFD}}$ (resp. $T(A) = UT(A)_{\text{w-AFD}}$) for every $A \in \mathfrak{C}$.
5. $T(A) = T(A)_{\text{w-AFD}}$ (resp. $T(A) = UT(A)_{\text{w-AFD}}$) for every QD $A \in \mathfrak{C}$.
6. $T(A) = T(A)_{\text{w-AFD}}$ (resp. $T(A) = UT(A)_{\text{w-AFD}}$) for every residually finite dimensional $A \in \mathfrak{C}$.
7. $T(A) = T(A)_{\text{w-AFD}}$ (resp. $T(A) = UT(A)_{\text{w-AFD}}$) for every Popa algebra $A \in \mathfrak{C}$.

**Proof.** We first prove the equivalence of (1) - (3) and then indicate the changes necessary to prove the second part. The proofs are the same whether dealing with $T(A)_{\text{AFD}}$ or $UT(A)_{\text{AFD}}$ and hence we just treat the uniformly approximately finite dimensional case.

(1) $\implies$ (3) is immediate. (3) $\implies$ (2) follows from Theorem 2.5.1.

(2) $\implies$ (1). Let $A \in \mathfrak{C}$, $\tau \in T(A)$ and $\mathfrak{F} \subset A$ be an arbitrary finite set. Since $A$ is QD we can find a sequence of u.c.p. maps $\varphi_n : A \to M_{k(n)}(\mathbb{C})$ which are asymptotically multiplicative and asymptotically isometric (i.e. $\|a\| = \lim \|\varphi_n(a)\|$, for all $a \in A$). Passing to a subsequence, if necessary, we may assume that $\varphi_1$ (and all the other $\varphi_n$’s) is as close to multiplicative on $\mathfrak{F}$ as we like. Put $\Phi = \oplus_n \varphi_n : A \to \Pi_n M_{k(n)}(\mathbb{C})$, and let $E$ be the $C^*$-algebra generated by $\Phi(A)$. Note that $\Phi : A \to E$ is as close to multiplicative on $\mathfrak{F}$ as we like, by construction.

Now observe that we have a semi-split, quasidiagonal, short exact sequence:

$$0 \to \oplus M_{k(n)}(\mathbb{C}) \to E + \oplus M_{k(n)}(\mathbb{C}) \to A \to 0.$$  

Since $\mathfrak{C}$ is closed under all of the operations used, it follows that $E + \oplus M_{k(n)}(\mathbb{C}) \in \mathfrak{C}$ and it is clear that $E + \oplus M_{k(n)}(\mathbb{C})$ is residually finite dimensional. Hence every trace on $E + \oplus M_{k(n)}(\mathbb{C})$ is uniformly approximately finite dimensional by (2). In particular, the trace $\tau \in T(A)$ defines a trace on $E + \oplus M_{k(n)}(\mathbb{C})$ which is uniformly approximately finite dimensional. Since the splitting $\Phi : A \to E \subset E + \oplus M_{k(n)}(\mathbb{C})$ is almost multiplicative, it follows that we can construct a u.c.p. map on $A$ (by composing maps on $E$ with $\Phi$) which is
almost multiplicative on $\mathcal{F}$ and which approximately recaptures the trace $\tau$. This completes the proof of $\mathcal{F} \implies \mathcal{E}$.

For the equivalence of (4) - (7), we really only need to show the implication $(5) \implies (4)$ as the arguments above go through without change for the other implications. To prove this we will need the following fact which follows from Theorems 3.6 (part (8)) and 3.8 (part (5)): If $\pi : B \to A$ is a surjective $*$-homomorphism and $\phi : A \to B$ is a c.p. splitting (i.e. $\pi \circ \phi = id_A$) then for each $\tau \in T(A)$ we have that $\tau \in T(A)_{w,\text{AFD}}$ (resp. $\tau \in UT(A)_{w,\text{AFD}}$) if and only if $\tau \circ \pi \in T(B)_{w,\text{AFD}}$ (resp. $\tau \circ \pi \in UT(B)_{w,\text{AFD}}$). (Thanks to Lemma 2.6.4 it does not matter whether or not the splitting $A \to B$ is unital. Note also that the existence of a c.p. splitting is not necessary in the case of uniformly weakly approximately finite dimensional traces.)

With this observation in hand, $(5) \implies (4)$ becomes very simple. Indeed, let $B$ be the unitization of the cone over $A$ (i.e. the unitization of $C_0((0, 1]) \otimes A$). Then $B$ is QD (cf. [49]) and belongs to $\mathcal{E}$. Moreover, there is a natural surjective $*$-homomorphism $B \to A \oplus C \to A$. A (non-unital) c.p. splitting for this quotient map is given by $a \mapsto e \otimes a$, where $e \in C_0((0, 1])$ is any non-negative function such that $e(1) = 1$. Hence if every trace on $B$ is (uniformly) weakly approximately finite dimensional then every trace on $A$ enjoys the same property.

The assumptions on the class $\mathcal{E}$ may seem unusual, but note that any one of the following classes of C*-algebras is closed under the operations needed in the lemma above: nuclear C*-algebras, exact C*-algebras (cf. [32, Section 7]), real rank zero C*-algebras (cf. [12, 2.10, 3.1, 3.14], it is easy to prove that if an extension is semi-split and quasidiagonal then every projection in the quotient lifts to a projection in the middle algebra) - though these algebras are not closed under tensoring with $C_0((0, 1])$.

**Proposition 4.2.** If Elliott’s conjecture holds for all nuclear, simple, QD C*-algebras with stable rank one and unperforated K-theory then $T(A) = UT(A)_{\text{AFD}}$ for every nuclear, QD C*-algebra $A$.

**Proof.** We will apply the previous lemma to the set $\mathcal{E}$ of all nuclear C*-algebras. We remark that extensions of nuclear C*-algebras are again nuclear by [15, Corollary 3.3].

So assume that Elliott’s Conjecture holds for all nuclear, simple, QD C*-algebras with stable rank one and unperforated K-theory and let $A$ be QD and nuclear. By the previous lemma, we may assume that $A$ is simple. Let $B$ be a UHF algebra and note that every trace on $A$ extends (in fact, uniquely) to a trace on $A \otimes B$. Hence it suffices to show that $T(A \otimes B) = UT(A \otimes B)_{\text{AFD}}$. However, by [45] and [46], $A \otimes B$ has stable rank one and unperforated K-theory. Thus $A \otimes B$ is classifiable. But then, just as in the proof of Proposition 2.4.1, it follows that $A \otimes B$ is an ASH algebra. Hence, by Proposition 3.12, we see that $T(A \otimes B) = UT(A \otimes B)_{\text{AFD}}$.

The point of the above result is that verifying the equation $T(A) = UT(A)_{\text{AFD}}$ for every nuclear, unital, QD C*-algebra $A$ is a necessary condition for Elliott’s conjecture to hold. However, we also believe that knowing the equation $T(A) = UT(A)_{\text{AFD}}$ for a particular $A$ may someday be part of the sufficient conditions for classification. In [35] Huaxin Lin proved that certain tracially AF algebras are classifiable. One of the key technical tools he needed in the proof (see [35, Lemmas 2.7 and 2.10]) was a good understanding of the approximation properties of traces on tracially AF algebras. Hence we would suggest that it is natural to try to classify algebras which satisfy some form of tracial approximation property (something like the equation $T(A) = UT(A)_{\text{AFD}}$) as a replacement for, say, assuming tracially AF or ASH. Indeed, the second part of [43, Theorem 3.3] shows that a simple, QD C*-algebra with real rank zero and satisfying the equation $T(A) = UT(A)_{\text{AFD}}$ has a finite dimensional
approximation property which is vaguely similar to being tracially AF (even without the assumption of nuclearity). For example, we have the following result.

**Theorem 4.3.** Let $A$ be a locally reflexive Popa algebra with real rank zero and unique trace $\tau$. Then for every finite set $F \subset A$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$, subalgebras $Q_1, \ldots, Q_m \subset A$ each of which is isomorphic to $M_n(\mathbb{C})$ and with units $e_1, \ldots, e_m$ such that $\tau(e_1) = \tau(e_2) = \cdots = \tau(e_m)$, $\|[e_i, x]\| < \varepsilon$ for $1 \leq i \leq m$ and all $x \in F$, $e_i F e_i \in Q_i$ for $1 \leq i \leq m$ and, finally,

$$\| \frac{1}{m \tau(e_1)} \sum_{k=1}^{m} e_k - 1_A \|_{\tau, 1} < \varepsilon,$$

where $\|x\|_{\tau, 1} = \tau(|x|)$.

**Proof.** By [43, Theorem 3.3] it suffices to show that $\tau \in UT(A)_{AFD}$. However, we will see later that the assumption of local reflexivity implies that $T(A)_{AFD} = UT(A)_{AFD}$ (cf. Theorem 6.1) and hence we are done. \hfill \Box

Note that if one could somehow adapt Popa’s techniques to further arrange that $\tau(e_1) = 1/m$ then the approximation property above would look very similar to a tracially AF algebra.

As mentioned in the proof above, $T(A)_{AFD} = UT(A)_{AFD}$ for every nuclear $C^*$-algebra and hence deciding whether $T(A) = UT(A)_{AFD}$ is equivalent to deciding if $T(A) = T(A)_{AFD}$ for every nuclear, QD $C^*$-algebra. If the hyperfinite $II_1$ factor is a QD $C^*$-algebra then this question has an affirmative answer (see the appendix).

5. The Exact Case

The previous section attempts to argue that deciding whether or not $T(A) = UT(A)_{AFD}$ for every nuclear, QD $C^*$-algebra is a natural and important open question. In this section we will show that this equation fails in the category of exact $C^*$-algebras. In fact, it is possible to construct very nice exact $C^*$-algebras for which $T(A) \neq T(A)_{w-AFD}$. Such algebras are not tracially type I (in particular, do not have finite tracial topological rank in the sense of [37]) by Proposition 3.12.

We won’t actually need condition (3) in the next proposition. However, we feel that it may be of independent interest. We wish to thank G. Pisier for providing the first proof of $(3) \implies (4)$, though Ozawa later observed that exactness was an unnecessary assumption (see Theorem 3.8).

**Proposition 5.1.** Let $A$ be a unital, exact $C^*$-algebra and $\tau \in T(A)$. Then the following are equivalent:

1. $\tau \in T(A)_{w-AFD}$.
2. $\tau \in UT(A)_{w-AFD}$.
3. The GNS representation is a nuclear map into $\pi_{\tau}(A)^\prime\prime$. That is, there exist u.c.p. maps $\phi_n : A \to M_{k(n)}(\mathbb{C})$ and $\psi_n : M_{k(n)}(\mathbb{C}) \to \pi_{\tau}(A)^\prime\prime$ such that $\|\pi_{\tau}(a) - \psi_n \circ \phi_n(a)\| \to 0$ for all $a \in A$.
4. $\pi_{\tau}(A)^\prime\prime$ is hyperfinite.

**Proof.** We will see in the next section that (1) and (2) are equivalent since Kirchberg has shown that exactness implies local reflexivity. Hence by Theorem 3.8 we see that (1), (2) and (4) are equivalent. Since nuclearity obviously implies weak nuclearity we also get the implication $(3) \implies (4)$ from Theorem 3.8. Hence we are only left to prove $(1) \implies (3)$.

We first recall one of Kirchberg’s characterizations of exactness: $A$ is exact if and only if every u.c.p. map $A \to D$, where $D$ is an injective $C^*$-algebra, is nuclear. From this fact, we see that it suffices to show that the GNS representation $\pi_{\tau} : A \to \pi_{\tau}(A)^\prime\prime$ factors
through an injective C*-algebra. More precisely, it suffices to show that there exist u.c.p. maps \( \Phi : A \to l^\infty (R) \) and \( \Psi : l^\infty (R) \to \pi_r (A)'' \) such that \( \pi_r = \Psi \circ \Phi \), where \( R \) denotes the hyperfinite \( \text{II}_1 \) factor. Indeed, if we can do this, then the map \( \Phi \) is nuclear, by Kirchberg’s characterization of exactness, and hence \( \pi_r = \Psi \circ \Phi \) is also nuclear.

So assume that \( \tau \in T(A)_{w\text{-AFD}} \). Then there exists a u.c.p. map \( \Phi : A \to l^\infty (R) \) such that \( \sigma \circ \Phi \) is a \( * \)-homomorphism with \( \tau = \tau_w \circ \sigma \circ \Phi \), where \( \sigma : l^\infty (R) \to R^w \) is the quotient mapping and \( \tau_w \) is the unique trace on the ultrapower of the hyperfinite \( \text{II}_1 \) factor. Since the weak closure of \( \sigma \circ \Phi (A) \) is canonically isomorphic to \( \pi_r (A)'' \), and there is a conditional expectation \( E : R^w \to \pi_r (A)'' \), we get the desired map by defining \( \Psi = E \circ \sigma \).

We now come to the main result of this section.

**Theorem 5.2.** There exists an exact, Popa algebra, \( A \), with real rank zero, stable rank one, UCT, Blackadar’s fundamental comparison property (i.e. if \( p, q \in A \) are projections such that \( \tau (q) < \tau (p) \) for all \( \tau \in T(A) \) then \( q \) is equivalent to a subprojection of \( p \)), unperforated K-theory, Riesz interpolation property and which is approximately divisible and an increasing union of residually finite dimensional subalgebras such that \( T(A) \neq T(A)_{w\text{-AFD}} \).

**Proof.** We first claim that it suffices to construct a C*-algebra \( C \) which is residually finite dimensional, exact, real rank zero, satisfies the UCT and such that \( T(C) \neq T(C)_{w\text{-AFD}} \). Indeed, if we can find such a \( C \) then by applying Theorem 2.5.1 to the class of all exact, real rank zero C*-algebras which satisfy the UCT we can find an exact Popa algebra with real rank zero, UCT and such that \( T(A) \neq T(A)_{w\text{-AFD}} \). (See Remark 2.5.4 for the UCT assertion and [12] for a proof that matrices over a real rank zero algebra also have real rank zero.)

Then replacing \( A \) with \( A \otimes U \), where \( U \) is some UHF algebra, will be the desired example since this operation preserves Popa’s property, exactness, real rank zero, UCT and picks up stable rank one, Riesz interpolation (cf. [10, Corollary 3.15]), Blackadar’s fundamental comparison property and hence unperforated K-theory (cf. [45], [46]). Moreover, it is clear that this example will be an inductive limit of residually finite dimensional subalgebras and be approximately divisible in the sense of [10].

The construction of the desired residually finite dimensional C*-algebra is a consequence of another one of Kirchberg’s characterizations of exactness [32, Theorem 1.3]: A separable C*-algebra \( A \) is exact if and only if there exists a subalgebra \( B \) of the CAR algebra, \( M_{2^\infty} \), and an AF ideal \( J \subset B \) such that \( A \cong B/J \). We remark if \( A \) is exact and \( 0 \to J \to B \to A \to 0 \) is the short exact sequence given by Kirchberg’s theorem, then this sequence is automatically semi-split (i.e. there exists a c.p. splitting \( A \to B \)) since \( B \) is exact and \( J \) is nuclear (cf. the bottom of page 41 in [32]).

Since \( C^*_r (\mathbb{F}_2) \) has a unique trace, it follows from [46, Theorem 7.2] that \( C^*_r (\mathbb{F}_2) \otimes M_{2^\infty} \) is exact and has real rank zero. By Kirchberg’s characterization, we can find an exact, QD C*-algebra \( B \) with an AF ideal \( J \subset B \) such that \( B/J \cong C^*_r (\mathbb{F}_2) \otimes M_{2^\infty} \). Moreover, the short exact sequence \( 0 \to J \to B \to C^*_r (\mathbb{F}_2) \otimes M_{2^\infty} \to 0 \) is semi-split.

Since \( J \) is AF, it follows from [12, Theorem 3.14 and Corollary 3.16] that \( B \) also has real rank zero. Note that from part (4) of Proposition 5.1 it follows that \( T(B) \neq T(B)_{w\text{-AFD}} \). Finally, since \( C^*_r (\mathbb{F}_2) \) satisfies the UCT, it follows from the ‘two out of three principle’ (cf. Proposition 2.5.2) that \( B \) also satisfies the UCT. \( B \) is almost the desired algebra; we only have to replace \( B \) with something residually finite dimensional (\( B \) is only QD).

From the proof of (2) \( \Rightarrow \) (1) in Lemma 4.1 we can use \( B \) to construct a residually finite dimensional C*-algebra \( C \) such that \( C \) is exact, real rank zero, satisfies the UCT and such that \( T(C) \neq T(C)_{w\text{-AFD}} \).
Corollary 5.3. There exists an exact, Popa algebra, $A$, with real rank zero, stable rank one, UCT, unperforated $K$-theory, Riesz interpolation and Blackadar’s fundamental comparison property which is approximately divisible and an increasing union of residually finite dimensional subalgebras but such that $A$ is not tracially AF.

Proof. Since $T(A) = UT(A)_{AFD}$ for every tracially type I algebra (see Proposition 3.12), the example in the previous theorem can’t be tracially AF. \hfill \Box

It may seem unusual to mention in the previous two results that $A$ is an inductive limit of residually finite dimensional subalgebras. Our main reason for pointing out this fact is that some of Lin’s recent structural work on the class of tracially AF $C^*$-algebras relies heavily on a theorem of Blackadar and Kirchberg stating that every simple, nuclear, QD $C^*$-algebra is an inductive limit of such subalgebras. In fact, for some of Lin’s structural work, this is the only place that nuclearity is used (i.e. his results hold more generally if one replaces the assumption of nuclearity by the assumption of an inductive limit decomposition by residually finite dimensional subalgebras).

In [4] Bédos asked whether or not every separable, unital hypertracial $C^*$-algebra is nuclear (see [4, Section 3] in the language of the present paper, a $C^*$-algebra is hypertracial if every quotient has at least one weakly approximately finite dimensional trace). It is easy to see that every simple, unital, QD $C^*$-algebra is hypertracial and hence Dadarlat’s examples of non-nuclear Popa algebras provide counterexamples to this question [19]. Theorem 5.2 above provides further examples. Indeed, for every non-hyperfinite II$_1$ factor $M$ which contains a weakly dense exact $C^*$-subalgebra the proof of Theorem 5.2 shows that we can construct an exact Popa algebra with stable rank one, Blackadar’s comparison property (hence unperforated $K$-theory), Riesz property, approximate divisibility and which is an increasing union of residually finite dimensional subalgebras but which is not nuclear since it will have $M\otimes R$ as the weak closure of some GNS representation.

6. The Locally Reflexive Case

The main result of this section is:

Theorem 6.1. If $A$ is locally reflexive then $T(A)_{w-AFD} = UT(A)_{w-AFD}$ and $T(A)_{AFD} = UT(A)_{AFD}$.

Proof. We only give the proof of $T(A)_{AFD} = UT(A)_{AFD}$ as it will be clear that essentially the same proof gives the other equality.

So let $\tau \in T(A)_{AFD}$ be arbitrary. Evidently it suffices to prove that if $\mathcal{F} \subset A$ is an arbitrary finite set and $\varepsilon > 0$ then there exists a u.c.p. map $\varphi : A \to B$, where $B$ is a finite dimensional $C^*$-algebra, such that $\|\varphi(xy) - \varphi(x)\varphi(y)\| < \varepsilon$, for all $x, y \in \mathcal{F}$ and such that there exists a trace $\gamma$ on $B$ such that $\|\tau - \gamma \circ \varphi\|_{A^*} < \varepsilon$.

In order to do this, we will show that for each finite dimensional operator system $X \subset A^{**}$ containing both the set $\mathcal{F}$ and $\{ab : a, b \in \mathcal{F}\}$, there exists a sequence of normal, u.c.p. maps $\psi_n : A^{**} \to M_{s(n)}(\mathbb{C})$ such that $tr_{s(n)} \circ \psi_n(x) \to \tau^{**}(x)$, for all $x \in X$ and each $\psi_n$ is $\varepsilon$-multiplicative on $\mathcal{F}$. If we are able to do this then one can construct a net of normal, u.c.p. maps $\varphi_\lambda : A^{**} \to M_{k(\lambda)}(\mathbb{C})$ with the property that $tr_{k(\lambda)} \circ \varphi_\lambda \in A^*$ for all $\lambda$, each $\varphi_\lambda$ is $\varepsilon$-multiplicative on $\mathcal{F}$ and (here is the key) $tr_{k(\lambda)} \circ \varphi_\lambda \to \tau^{**}$ in the weak topology coming from $A^{**}$. Hence, by the Hahn-Banach theorem, $\tau^{**}$ belongs to the norm closure of the convex hull of $\{tr_{k(\lambda)} \circ \varphi_\lambda \} \subset A^*$. Then one would be able to choose a finite set $\lambda_1, \ldots, \lambda_p$ and positive real numbers $\theta_1, \ldots, \theta_p$ such that $\sum \theta_i = 1$ and $\|\tau^{**} - \sum \theta_i tr_{k(\lambda_i)} \circ \varphi_{\lambda_i}\|_{A^*} < \varepsilon$. Finally one defines $B = M_{k(\lambda_1)}(\mathbb{C}) \oplus \cdots \oplus M_{k(\lambda_p)}(\mathbb{C})$, $\varphi = \varphi_{\lambda_1} \oplus \cdots \oplus \varphi_{\lambda_p}$ and $\gamma = \sum \theta_i tr_{k(\lambda_i)}$. 


As we have arranged that \( \varphi_\lambda \) is \( \varepsilon \)-multiplicative on \( \mathfrak{F} \) for every \( \lambda \), it is clear that \( \varphi \) will also be close to multiplicative on \( \mathfrak{F} \).

So let \( X \subset A^{**} \) be any finite dimensional operator system containing the sets \( \mathfrak{F} \) and \( \{ab : a, b \in \mathfrak{F}\} \). Since \( \tau \in T(A)_{AFD} \) we can find a sequence of u.c.p. maps \( \varphi_m : A \to M_{k(m)}(\mathbb{C}) \) which are asymptotically multiplicative (in norm) and which recapture \( \tau \) (as a weak-* limit) after composing with the traces on \( M_{k(m)}(\mathbb{C}) \). Note that by passing to a subsequence, if necessary, we may further assume that each \( \varphi_m \) is as close to multiplicative as one likes on the set \( \mathfrak{F} \). Since \( A \) is locally reflexive, we can find a net of u.c.p. maps \( \alpha_i : X \to A \) such that \( \alpha_i(x) \to x \) in the weak-* topology (coming from \( A^* \)) for all \( x \in X \). Another standard Hahn-Banach argument allows us to extract a subnet \( \beta_i : X \to A \) from the convex hull of the \( \alpha_i \)'s such that \( \beta_i(x) \to x \) in the weak-* topology, for all \( x \in X \), and \( \|a - \beta_i(a)\| \to 0 \) for all \( a \in \mathfrak{F} \cup \{ab : a, b \in \mathfrak{F}\} \). Again, passing to a subnet, we may assume that \( \|\beta_i(a) - a\| < \varepsilon \) for all \( a \in \mathfrak{F} \cup \{ab : a, b \in \mathfrak{F}\} \) and for all \( t \). In particular, this implies that \( \varphi_m \circ \beta_t \) is nearly multiplicative on \( \mathfrak{F} \) for all \( t, m \).

We almost have the desired maps \( \psi_n \). Since \( X \) is finite dimensional we can choose a linear basis \( \{x_1, \ldots, x_q\} \). For each \( n \in \mathbb{N} \) first choose \( t(n) \) such that \( |\tau^{**}(x_i) - \tau(\beta_{t(n)}(x_i))| < 1/n \) for \( 1 \leq i \leq q \). Then choose \( m_n \) such that \( |\tau(\beta_{t(n)}(x_i)) - tr_{k(m_n)} \circ \varphi_{m_n}(\beta_{t(n)}(x_i))| < 1/n \) for \( 1 \leq i \leq q \). Then defining \( \tilde{\psi}_n = \varphi_{m_n} \circ \beta_{t(n)} : X \to M_{k(m_n)}(\mathbb{C}) \) we have that \( tr_{k(m_n)} \circ \tilde{\psi}_n(x) \to \tau^{**}(x) \) for all \( x \in X \).

By Arveson's Extension Theorem, we may assume that each \( \tilde{\psi}_n \) is actually defined on all of \( A^{**} \). The only problem is that we can’t be sure that the Arveson extensions are normal on \( A^{**} \). However a tiny perturbation of the \( \tilde{\psi}_n \) will yield normal maps \( \psi_n \) with all the right properties. (Use the fact that c.p. maps to matrix algebras are nothing but positive linear functionals on matrices over the given algebra and that the set of normal linear functionals on a von Neumann algebra is dense in the dual space.)

The following corollary generalizes [31, Theorem 7.5] and the discrete case of [31, Proposition 7.1].

**Corollary 6.2.** Let \( \Gamma \) be a discrete group such that every finitely generated subgroup has Kirchberg’s factorization property (cf. [33]; for example, if \( \Gamma \) is an increasing union of residually finite subgroups). Then \( \Gamma \) is amenable if and only if \( C^*(\Gamma) \) is locally reflexive.

**Proof.** According to [33, Lemma 4.1] the factorization property for a discrete group \( \Gamma \) is equivalent to knowing that the canonical trace on \( C^*(\Gamma) \) (which gives the left regular representation after applying GNS) is weakly approximately finite dimensional. (This is essentially a special case of the equivalence of (1) and (5) in Theorem 3.6.) Thus our assumptions imply that the canonical trace on \( C^*(\Gamma) \) is weakly approximately finite dimensional (cf. [31, Lemma 7.3, part (v)]).

Now the proof is simple. The ‘only if’ part is immediate since \( C^*(\Gamma) = C^*_r(\Gamma) \) is nuclear whenever \( \Gamma \) is amenable. The opposite direction follows from Theorem 6.1 since injectivity of the von Neumann algebra generated by a discrete group in it’s left regular representation implies amenability of the group.

### 7. The WEP Case

Here we observe that if \( A \) has the WEP of E.C. Lance then \( T(A) = T(A)_{w-AFD} \). As an application it follows that none of the known examples of non-quasidiagonal \( C^* \)-algebras can be embed into a nuclear, stably finite \( C^* \)-algebra (even though many are exact and stably finite). This should be contrasted with Kirchberg’s embedding theorem which states that every exact \( C^* \)-algebra embeds into the Cuntz algebra \( \mathcal{O}_2 \).
Recall that a C*-algebra $A$ is said to have the weak expectation property (WEP) if for every faithful, nondegenerate representation $\pi : A \to B(H)$ there exists a u.c.p. map $\Phi : B(H) \to \pi(A)'$ such that $\Phi(\pi(a)) = \pi(a)$, for all $a \in A$. This is a large class of C*-algebras (including every injective C*-algebra and every nuclear C*-algebra). In fact, [31, Corollary 3.5] states that every (separable) C*-algebra is contained in a (separable) simple C*-algebra with the WEP.

Kirchberg studied this class of algebras in [31] and proved the following result: $A$ has the WEP if and only if there is a unique C*-norm on $A \otimes C^*(F_\infty)$, where $C^*(F_\infty)$ denotes the full group C*-algebra of the free group on infinitely many generators.

**Proposition 7.1.** If $A$ has the WEP then $T(A) = T(A)_{\text{w-AFD}}$.

**Proof.** Let $\tau \in T(A)$ and $\pi_{\tau}$ be the associated GNS representation. Let $\rho : C^*(F_\infty) \to \pi_{\tau}(A)'$ be any *-homomorphism with weakly dense range. By Kirchberg’s tensorial characterization of the WEP we get a *-homomorphism $\pi_{\tau} \otimes \rho : A \otimes C^*(F_\infty) \to C^*(\pi_{\tau}(A), \pi_{\tau}(A)')$. As in the proof of (5) $\implies$ (6) from Theorem 3.6 it follows that $\tau$ is weakly approximately finite dimensional. \qed

Our next three corollaries are related to some observations of Bedøs (see, for example, [4, Corollary 2.11]).

**Corollary 7.2.** Let $A$ be such that $T(A)_{\text{w-AFD}} = \emptyset$ and $B$ be a C*-algebra with the WEP and at least one tracial state. Then there is no unital *-homomorphism $A \to B$.

We already observed that the set of weakly approximately finite dimensional traces is empty when $A = C^*_r(\Gamma)$ for a non-amenable, discrete group $\Gamma$ and hence the corollary above covers these examples. These are the standard examples of stably finite, non-QD C*-algebras. However, S. Wassermann has produced other examples of stably finite, non-QD C*-algebras in [51], [52] (though, to the best of our knowledge, it is not known if any of these are exact). A careful inspection of Wassermann’s proofs shows that he proves much more than non-quasidiagonality; he actually shows that his examples admit no weakly approximately finite dimensional traces and hence are also covered by the corollary above. At the moment the only other known examples of stably finite, non-QD C*-algebras are things which contain one of the examples described above – e.g. many reduced free products with respect to tracial states contain reduced free group C*-algebras. (Caution: There are plenty of stably finite C*-algebras for which it is not known if they are QD, but the examples above are the only ones which are known to not be QD.)

**Corollary 7.3.** None of the known examples (i.e. the examples described above) of stably finite, non-QD C*-algebras can be embed into a finite, hyperfinite von Neumann algebra.

Since (unital) stably finite, exact C*-algebras always have at least one tracial state we have:

**Corollary 7.4.** None of the known examples of stably finite, non-QD C*-algebras can be embed into a stably finite, nuclear C*-algebra.

The corollary above is well known to the free probability community in the case of $C^*_r(F_n)$ (or many other free product groups). Indeed, if $C^*_r(F_n) \subset B$ and $B$ has a trace $\tau$ then the restriction of $\tau$ to $C^*_r(F_n)$ is the canonical trace (by uniqueness) and hence the weak closure of $C^*_r(F_n)$ in the GNS representation of $B$ with respect to $\tau$ will not be hyperfinite and hence the weak closure of $B$ can’t be hyperfinite.

Note, of course, that not only are embeddings impossible, but there are no non-zero homomorphisms from any of the standard examples of stably finite, non-QD C*-algebras to a finite, hyperfinite von-Neumann algebra (or stably finite, nuclear C*-algebra).
The author’s interest in embedding questions of the above type stems from a question of Blackadar and Kirchberg which asks whether every nuclear, stably finite C*-algebra is QD (cf. [11]). One consequence of the results above is that it is impossible to give counterexamples to this question by embedding one of the known non-QD C*-algebras into a nuclear, stably finite C*-algebra. Hence constructing a counterexample (if one exists) will require completely new ideas.

Our final result of this section generalizes [6, Theorem 6.8]. If \( \pi : G \to B(H) \) is a unitary representation of a locally compact group \( G \) then Bekka defines the representation to be amenable if there exists a state \( \varphi \) on \( B(H) \) such that \( \varphi(\pi(g)T\pi(g^{-1})) = \varphi(T) \) for all \( T \in B(H) \) and for all \( g \in G \). If \( A = C^*(\{\pi(g) : g \in G\}) = \text{span}\{\pi(g) : g \in G\}^{-} \) (norm closure, since \( G \) is a group) and \( A \) falls in the centralizer of some state on \( B(H) \) then clearly the representation is amenable. However, the converse is also true. Namely, if \( \pi \) is amenable and \( \varphi \) is a state such that \( \varphi(\pi(g)T\pi(g^{-1})) = \varphi(T) \) then we get

\[
\varphi(\pi(g)T) = \varphi(\pi(g^{-1})[\pi(g)T]\pi(g)) = \varphi(T\pi(g))
\]

for all \( T \in B(H) \). Since \( A \) is the norm closure of the span of \( \pi(G) \) it follows that \( A \) also falls in the centralizer of \( \varphi \). In summary: A representation \( \pi : G \to B(H) \) is amenable in the sense of Bekka if and only if \( T(A)_{w-AFD} \neq \emptyset \), where \( A = C^*(\{\pi(g) : g \in G\}) \). In light of Proposition 7.1, the following corollary in now immediate.

**Corollary 7.5.** Let \( \pi : G \to B(H) \) be a strongly continuous, unitary representation of a locally compact group \( G \) and \( A = C^*(\{\pi(g) : g \in G\}) \). If \( A \) has the WEP then \( \pi \) is an amenable representation if and only if \( A \) has a tracial state.

### 8. \( \text{II}_1 \) Factor Representations of Popa Algebras

The main motivation for Popa’s work in [43] was to try to understand the relationship between quasidiagonality and nuclearity. Indeed, in [43] Popa asked whether every Popa algebra with unique trace was nuclear. More generally, he also asked whether \( R \) was the only \( \text{II}_1 \) factor that could be realized as the weak closure of the GNS representation of a Popa algebra. There was serious speculation that both questions should be true. However counterexamples to Popa’s first question were constructed by Dadarlat in [19]. Interestingly enough, though, in [19] Dadarlat constructs nonnuclear tracially AF algebras and hence all of their \( \text{II}_1 \) factor representations are hyperfinite (even though the algebras are not nuclear).

Further support for Popa’s second question was also provided in [43, Remark 3.4.2] where Popa proved that if a factorial trace is in UT(A)\(_{\text{AFD}}\) then it gives the hyperfinite \( \text{II}_1 \) factor (compare with Theorem 3.8). However, the results of the previous sections also answer Popa’s second question.

**Proposition 8.1.** There exists an exact, Popa algebra, \( A \), with real rank zero, stable rank one, UCT, unperforated K-theory, Riesz interpolation and Blackadar’s fundamental comparison property which is approximately divisible and an increasing union of residually finite dimensional subalgebras but such that \( A \) has non-hyperfinite \( \text{II}_1 \) factor representations.

**Proof.** Let \( A \) be the example constructed in Theorem 5.2. Then any extreme point of \( T(A) \) which does not belong to \( T(A)_{w-AFD} \) will give a non-hyperfinite \( \text{II}_1 \) factor representation. \( \square \)

Note that if \( A \) is any QD, locally reflexive C*-algebra then \( A \) has at least one hyperfinite \( \text{II}_1 \) (or finite dimensional) factor representation. (Use the fact that \( T(A)_{w-AFD} \) is a face in \( T(A) \), hence contains an extreme point of \( T(A) \) and, finally, Theorem 6.1.) In particular, the following result shows that Popa’s \( \text{II}_1 \) factor representation question has an affirmative answer in the locally reflexive, unique trace case.
Corollary 8.2. If $A$ is a locally reflexive, QD $C^*$-algebra with unique trace $\tau$ then either $\pi_\tau(A)^{\prime\prime} \cong R$ or $\pi_\tau(A)^{\prime\prime}$ is a finite dimensional matrix algebra.

Remark 8.3. One of the deficiencies of Theorem 2.5.1 is that it will usually produce a tracial space with an infinite number of extreme points. Since we know that the unique trace case always produces $R$ (for infinite dimensional, locally reflexive Popa algebras), a natural question then becomes: Is there an exact Popa algebra such that $T(A)$ has a finite number of extreme points, some of which do not give hyperfinite GNS representations?

It is also natural to wonder whether local reflexivity is really needed in the corollary above. That is, if a Popa algebra has unique trace then must one get $R$ in the GNS representation?

Dropping the assumption of exactness, it is now easy to construct a sort of universal Popa algebra which realizes every McDuff factor as a GNS representation. (Recall that a $\text{II}_1$ factor is called McDuff if it is isomorphic to something of the form $M \bar{\otimes} R$.)

Theorem 8.4. There exists a Popa algebra $A$ with the property that for each McDuff factor, $M$, there exists a trace $\tau_M \in T(A)$ such that $\pi_\tau(A)^{\prime\prime} \cong M$.

Proof. Since every $\text{II}_1$ factor arises as the weak closure of a GNS representation of $C^*(\mathbb{F}_\infty)$, and $C^*(\mathbb{F}_\infty)$ is residually finite dimensional, this theorem follows from Theorem 2.5.1.

Corollary 8.5. If $M \subset B(L^2(M))$ is a McDuff factor then there exists a weakly dense $C^*$-subalgebra $A \subset M$ such that:

1. For each finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional subalgebra $B \subset A$ with unit $e$ such that $\|E_B(exe) - (x - e^+xe^+)\|_2 < \varepsilon \|e\|_2$, where $E_B : eMe \to B$ is a trace preserving conditional expectation.

2. There exist finite rank projections, $P_1 \leq P_2 \leq \ldots$, such that $\|[P_n, a]\| \to 0$ for all $a \in A$.

3. There exists a state $\phi$ on $B(L^2(M))$ such that $A \subset B(L^2(M)) : \phi(TS) = \phi(ST), S \in B(L^2(M))$.

4. There exists a sequence of $\text{II}_1$ factors, $R_n \subset B(L^2(M))$, such that $R_n \cong R$ for all $n$ and for each $a \in A$ we can find $x_n \in R_n$ such that $\|a - x_n\| \to 0$.

5. There exists a sequence of normal, u.c.p. maps $\varphi_n : M \to M_{k(n)}(\mathbb{C})$ such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 \to 0$ for all $a, b \in A$.

6. There exists a completely positively liftable u.c.p. map $\Phi : M \to R^\omega$ such that $\Phi|_A$ is a $*$-monomorphism.

Proof. Let $A$ be the universal Popa algebra from the previous theorem. Evidently $A$ satisfies (1) above since it satisfies the stronger norm approximation property. This also immediately gives (5). Since $A$ is QD we get (2) and the fact that $T(A)_{\text{AFD}}$ is not empty. Hence, from Theorem 3.6, we get (3). Since $A$ also embeds into $R$, we get (4) from Voiculescu’s Theorem. Finally, note that (6) follows from (5).

When we first discovered part (6) in the corollary above, we thought that it would be useful in showing that every $\text{II}_1$ factor embeds into $R^\omega$. However, if the map $\Phi$ is normal then $M \cong R$.

Another curious consequence of this work is that McDuff factors which are generated by exact $C^*$-algebras always have ‘norm microstates’ on a dense subalgebra.

Theorem 8.6. If $M \subset B(L^2(M))$ is McDuff and contains a weakly dense, exact $C^*$-subalgebra $A \subset M$ and finite dimensional matrix subalgebras $M_n \subset B(L^2(M))$ such that for each $a \in A$ there exists a sequence $a_n \in M_n$ such that $\|a - a_n\| \to 0$. (Hence, for every noncommutative polynomial $P$ in $k$ variables and finite set $\{a^{(i)}\}_{i=1}^k \subset A$ we have $\|P(a^{(1)}, \ldots, a^{(k)}) - P(a^{(1)}_n, \ldots, a^{(k)}_n)\| \to 0$ as $n \to \infty$.)
Proof. Since every exact C*-algebra is the quotient of an exact, residually finite dimensional C*-algebra (cf. [13, Corollary 5.3]), it follows from Theorem 2.5.1 that $M$ contains a weakly dense, exact Popa algebra. In particular, $M$ contains a weakly dense, exact, QD C*-algebra. Since $M$ is a factor, it can’t contain any nonzero compact operators and hence the result now follows from [20] (see also [14] for the general case).

Note that the preceding theorem covers many group von Neumann algebras (e.g. $\Gamma = G_1 \times G_2$ where $G_1$ is discrete, amenable and i.c.c. while $G_2$ is discrete, exact and i.c.c.). We are not, however, claiming that this result implies $R^\omega$ embeddability for such group von Neumann algebras. Indeed, it is not at all clear that the existence of norm microstates implies the existence of ‘weak’ microstates (in the sense of Voiculescu) since there does not appear to be any way of understanding how the traces behave on the norm approximations.

9. CONNES’ EMBEDDING PROBLEM

In this section we show that the techniques of this paper easily yield a new characterization of those McDuff factors which are embeddable into $R^\omega$. An open problem of Connes asks whether or not every (separable) $\Pi_1$ factor embeds into $R^\omega$ and it is easy to see that this is the case if and only if every McDuff factor embeds into $R^\omega$. Embeddable $\Pi_1$ factors already admit a number of characterizations (see [31], [27]) but the results of this section show that the difference between embeddability and hyperfiniteness is quite delicate (at least for McDuff factors).

Theorem 9.1. If $M \subset B(L^2(M))$ is a McDuff factor with trace $\tau_M$ then the following are equivalent:

1. $M$ is embeddable into $R^\omega$.
2. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and a sequence of normal, u.c.p. maps $\varphi_n : M \rightarrow M_{k(n)}(\mathbb{C})$ such that
   (a) $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 \rightarrow 0$ and
   (b) $|\tau_{k(n)} \circ \varphi_n(a) - \tau_M(a)| \rightarrow 0$ for all $a, b \in A$.
3. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and finite rank projections $\{P_n\}$ such that
   (a) $\frac{\|P_n\|_{HS}}{\|P_n\|_{HS}} \rightarrow 0$ and
   (b) $\langle P_n, P_n \rangle_{HS} \rightarrow \tau_M(a)$ for all $a \in A$.
4. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and a state $\phi$ on $B(L^2(M))$ such that
   (a) $A \subset B(L^2(M))_\phi$ and
   (b) $\phi|_A = \tau_M$.
5. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and an embedding $\Phi : M \hookrightarrow R^\omega$ such that $\Phi|_A$ is completely positively liftable.
6. There exists a weakly dense $C^*$-subalgebra $A \subset M$ such that $C^*(A, JAJ) \cong A \otimes A^{op}$.
7. $M$ has a weak expectation relative to a weakly dense $C^*$-subalgebra. (i.e. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and a u.c.p. map $\Phi : B(L^2(M)) \rightarrow M$ such that $\Phi|_A = id_A$.
8. There exists a weakly dense operator system $X \subset M$ such that $X$ is injective.

Proof. (1) $\Rightarrow$ (2). Choose a trace $\gamma \in T(C^*(F_\infty))$ such that the weak closure of the GNS representation is isomorphic to $M$. Since $M$ embeds into $R^\omega$ it follows that $\gamma$ is in the weak-* closure of the set $T(C^*(F_\infty))_{FD}$ (traces whose GNS representations are finite dimensional). This follows easily from the remark that unitaries in $R^\omega$ always lift to unitaries in $l^\infty(R)$ (see [31, Lemma 4.5]).
From Theorem 2.5.1 it follows that we can find a Popa algebra $A$ with a trace $\tau \in T(A)_{\text{AFD}}$ such that the weak closure of the GNS representation of $A$ with respect to $\tau$ is isomorphic to $M \cong M \hat{\otimes} R$, since $M$ is McDuff. (2) now follows from the definition of $T(A)_{\text{AFD}}$, Arveson’s Extension Theorem and the remark that u.c.p. maps from a von Neumann algebra to a matrix algebra can always be approximated by normal, u.c.p. maps.

Using Theorem 3.6 it is easy to see that statements (2) - (7) are equivalent. Since, (5) obviously implies (1), it follows that (1) - (7) are equivalent.

$$(8) \implies (1)$$ is a consequence of [31, Theorem 1.4] together with the equivalence of (vi) and (iii) in [31, Proposition 1.3] (recall that $B(H)$ has the WEP).

$$(7) \implies (8).$$ If $A \subset M$ is weakly dense and $\Phi : B(L^2(M)) \to M$ is a weak expectation relative to $A$ then [7, Theorem 2.1] ensures that we can find an idempotent u.c.p. map $\Psi : B(L^2(M)) \to M$ such that $\Psi(a) = a$ for all $a \in A$. The desired injective operator system is then $X = \Psi(B(L^2(M)))$ (since $\Psi \circ \Psi = \Psi$).

In [8] Blackadar proved the existence of a non-injective factor which has a weak expectation relative to a dense subalgebra. Other examples are exhibited by Kirchberg in the remark after [31, Corollary 3.5]. However, in both of these papers it is far from clear if finite examples can be constructed using their techniques. Thus part (7) of Theorem 9.1 seems to give the first examples of non-hyperfinite II$_1$ factors with weak expectations and shows that in fact many well known II$_1$ factors have weak expectations.

In order to illustrate just how delicate the results above are, we remind the reader of the various characterizations of the hyperfinite II$_1$ factor (most of which are due to Alain Connes).

**Theorem 9.2.** If $M \subset B(L^2(M))$ is a II$_1$ factor with trace $\tau_M$ then the following are equivalent:

1. $M \cong R$.
2. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and a sequence of normal, u.c.p. maps $\varphi_n : M \to M_{k(n)}(\mathbb{C})$ such that
   
   (a) $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 \to 0$ for all $a, b \in A$ and
   (b) $|\tau_{k(n)} \circ \varphi_n(x) - \tau_M(x)| \to 0$ for all $x \in M$.
3. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and finite rank projections $\{P_n\}$ such that
   
   (a) $\|P_{n,a}\|_{HS} \to 0$ for every $a \in A$ and
   (b) $\langle P_{n,a}P_{n,b} \rangle_{HS} \to \tau_M(x)$ for all $x \in M$.
4. There exists a weakly dense $C^*$-subalgebra $A \subset M$ and a state $\phi$ on $B(L^2(M))$ such that
   
   (a) $A \subset B(L^2(M))_{\phi}$ and
   (b) $\phi|_M = \tau_M$.
5. There exists a completely positively liftable embedding $\Phi : M \hookrightarrow R^\omega$.
6. $C^*(M, JM) \cong M \otimes M^{op}$.
7. There exists a u.c.p. map $\Phi : B(L^2(M)) \to M$ such that $\Phi|_M = id_M$ (i.e. $M$ is injective).

**Proof.** (1) $\implies$ (2) is obvious. (2) $\implies$ (5) is contained in the argument in the proof of (1) $\implies$ (2) from Theorem 3.8. Since (5) is equivalent to hyperfiniteness for general finite von Neumann algebras, we see that (1), (2) and (5) are equivalent.

The equivalence of (1), (6) and (7) is due to Connes (cf. [17, Theorem 5.1]). This paper also contains his adaptation of Day’s trick to deduce (3) from injectivity (7). Hence we are left to prove (3) $\implies$ (4) and (4) $\implies$ (1).
(3) $\implies$ (4) is well known: simply take a cluster point of the states

$$T \mapsto < TP_n, P_n >_{HS}$$

on $B(L^2(M))$. Finally, (4) $\implies$ (1) also follows from Connes’ work since the density of $A$ in $M$, together with the fact that the hypertrace takes the correct value on all of $M$, implies that actually $M$ is contained in the centralizer of $\varphi$ and hence is hyperfinite (cf. proof of [18, Theorem 5.1]).

10. **Szegő’s Limit Theorem for Self Adjoint Operators**

Here we observe that Proposition 3.12 together with Voiculescu’s Theorem yield a general form of a classical theorem of Szegő. We recommend [2], [3] and [5] for nice discussions of this problem, its relation to numerical approximation of spectra and overviews of the vast body of previous work on this problem. (We are particularly fond of the introduction in [5].)

Let $T \in B(H)$ be a self adjoint operator, $A = C^*(T, I_H)$ be the $C^*$-algebra generated by $T$ and the identity operator and $\varphi \in S(A) = T(A)$ be an arbitrary (tracial) state. Let $\mu_{\varphi}$ be the spectral distribution of $T$ with respect to $\varphi$; i.e. the unique probability measure on $\mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} f(x) d\mu_{\varphi}(x) = \varphi(f(T)),$$

for all $f \in C_0(\mathbb{R})$.

If $\{P_n\}$ is any sequence of finite rank projections then we will regard the compressions $P_n TP_n$ as self adjoint operators on the finite dimensional Hilbert space $P_n(H)$. We will let $\mu_n$ denote the spectral distribution of $P_n TP_n$ with respect to the unique trace on the matrix algebra in which $P_n TP_n$ sits.

**Theorem 10.1.** Let $T \in B(H)$ be a self adjoint operator and $A = C^*(T, I_H)$. Then, there exists a sequence of finite rank projections $P_1 \leq P_2 \leq P_3 \ldots \in B(H \otimes_2 H)$ such that $P_n \to I_{H \otimes_2 H}$ in the strong operator topology and for every $\varphi \in T(A)$ there exists a subsequence $\{n_k\}$ such that

$$\mu_{n_k} \to \mu_{\varphi}$$

in the weak-* topology, where $\mu_{n_k}$ denotes the spectral distribution of $P_{n_k} T \otimes I_H P_{n_k}$ as above.

**Proof.** Since abelian $C^*$-algebras are type I, every state (i.e. trace) is uniformly approximately finite dimensional. Hence, since $A \otimes I_H$ contains no non-zero compact operators, we can apply Proposition 3.10 to find a sequence of projections $P_1 \leq P_2 \leq P_3 \ldots \in B(H \otimes_2 H)$ which asymptotically commute with $T \otimes I_H$ in norm and recapture the entire state space of $A$. The theorem now follows from [5, Theorem 6].

Note that if $A \cap K(H) = \{0\}$ then the filtration can be found on $H$ (i.e. one does not have to change the representation in this case). It should also be remarked that numerical analysts will likely not find the above theorem to be of any use as the construction of the projections $P_n$ requires knowledge of the spectral distribution.

11. **Questions**

The following questions seem natural, in light of the present work.

1. Is every $\text{II}_1$ factor representation of a Popa algebra McDuff? While this seems unlikely, the inductive limit constructions used in the classification program tend to produce McDuff factors.
2. Is there an example such that $T(A)_{w-AFD} \neq T(A)_{AFD}$ (or $UT(A)_{w-AFD} \neq UT(A)_{AFD}$)? The only obvious obstruction is related to quasidiagonality since the existence of a faithful trace in $T(A)_{AFD}$ implies quasidiagonality. However, as we saw in Section 7, $T(A)_{w-AFD} = \emptyset$ for any of the standard examples of stably finite, non-QD $C^*$-algebras. For example, one can construct a $C^*$-algebra which has the WEP and a faithful trace but which is not QD? The most natural candidate seems to be the hyperfinite $II_1$ factor.

3. Can a free group factor or a $II_1$ factor with property T contain a weakly dense, QD $C^*$-subalgebra? How about a Popa algebra? Our constructions always give McDuff factors (so we have some place to hide the Popa algebra). Note that amenability can't be an obstruction since $L(G_1 \times G_2) \cong L(G_1) \bar{\otimes} L(G_2)$ and hence many non-amenable groups give McDuff factors and hence contain dense QD subalgebras by Theorem 8.4.

4. Can one give estimates of the free entropy dimension of a finite set of elements (not necessarily generators) in a Popa algebra which is independent of the particular trace? This is related to the semicontinuity problem for free entropy dimension.

5. Can one prove a classification theorem for simple, nuclear, real rank zero $C^*$-algebras which satisfy the UCT and such that $T(A) = T(A)_{AFD} = UT(A)_{AFD}$, by Theorem 6.1)? In [43, Theorem 3.3] Popa proves that such algebras have an internal finite dimensional approximation property which should be of use. Presumably the role of nuclearity needs to be clarified as Popa never assumes nuclearity in [43].

6. Is every (nuclear) Popa algebra with real rank zero, unperforated K-theory, Riesz decomposition property and unique trace (or, perhaps, finitely many extreme traces) necessarily tracially AF?

7. Can an infinite, simple, discrete group with Kazdan’s property T be embed into the unitary group of an $R^\omega$-embeddable McDuff factor? (Compare with [44] where it is shown that no such embedding exists into the unitary group of $L(F_n) \otimes R$ or, more generally, $L(\Gamma)$ for any a-T-amenable discrete group $\Gamma$.)

8. Let $\Gamma$ be a discrete group such that the group von Neumann algebra of $\Gamma$ contains a weakly dense operator system which is injective. Does it follow that $\Gamma$ is an exact group? Perhaps just uniformly embeddable into Hilbert space? Since these notions pass to subgroups, this would imply that every residually finite group (and every other group which embeds into the unitary group of $R^\omega$) is exact (or uniformly embeddable). It would also show that non-embeddable groups, whose existence has been asserted by Gromov, give counterexamples to Connes’ embedding problem.

12. Appendix: Is $R$ Quasidiagonal?

Here we discuss a basic problem in operator algebra theory about which very little is known. Namely, whether or not the hyperfinite $II_1$ factor is QD. One has to be careful as we are now thinking of $R$ as a $C^*$-algebra and as such it is no longer separable (in norm). This has led to some confusion as the ‘classical’ definition of quasidiagonality is not the correct notion for non-separable $C^*$-algebras. Voiculescu gives the correct local definition in the non-separable case in [50]. (See also the appendix to [13] for a detailed treatment of the non-separable case.)

We are not aware of a reference for the following fact, however it is known to a number of experts. We thank George Elliott for showing us a very nice proof. We have only modified the last few lines of Elliott’s argument so that we can deduce a slightly stronger statement.

**Lemma 12.1.** Let $R$ act on $L^2(R)$ via the GNS construction. There is no sequence of nonzero, finite rank projections $P_1, P_2, \ldots$ such that $|[x, P_n]| \to 0$ for all $x \in R$. 

Corollary 12.3. The proof goes by contradiction. So let $P_1, P_2, \ldots$ be finite rank projections such that $\| [x, P_n] \| \to 0$ for all $x \in R$. Put $K = \oplus_{n \in \mathbb{N}} L^2(R) = L^2(R) \otimes \ell^2(\mathbb{N})$ and $P = \oplus_{n \in \mathbb{N}} P_n$. Then $(x \otimes 1)P - P(x \otimes 1)$ is a compact operator for every $x \in R$. Hence, down in the Calkin algebra $P$ will land in the commutant of $R \otimes 1$. But then by a theorem of Johnson and Parrott (see the remarks after [29, Lemma 3.3]) it follows that each $T_{i,j} \in R'$ and $P - T$ is compact on $K$. In particular, this implies that $\| P_n - T_{n,n} \| \to 0$. Thus $\| T_{n,n} \| \to 1$ and down in the Calkin algebra the norm of $T_{n,n}$ is tending to zero. However this is a contradiction since the commutant of $R$ is a II$_1$ factor (isomorphic to $R$) and hence a simple C$^*$-algebra. Thus the mapping to the Calkin algebra is isometric.

Note that the proof above never used the fact that the $P_n$'s are projections and hence also holds for sequences of finite rank operators whose norms are tending to one. However, since we can always construct a quasicentral net of finite rank operators for $R$ we are left to conclude that the lemma above has more to do with sequences versus nets (i.e. separable versus non-separable Hilbert spaces) than it does with quasidiagonality.

The only other obvious strategy for proving nonquasidiagonality of $R$ is to embed a non-QD C$^*$-algebra into it. However we already observed in Section 7 that none of the known examples of non-QD C$^*$-algebras can be embed into $R$.

The tracial invariants studied in this paper lead to another approach for proving that $R$ is not QD. They also show that an affirmative answer to this question would have some remarkable consequences.

Proposition 12.2. $R$ is QD if and only if for every (separable) C$^*$-algebra $A$ we have $T(A)_{\text{AFD}} \supset UT(A)_{w-\text{AFD}}$.

Proof. We begin with the necessity. Let $A$ be arbitrary. It suffices to show that the extreme points of $UT(A)_{w-\text{AFD}}$ belong to $T(A)_{\text{AFD}}$. However, every extreme point of $UT(A)_{w-\text{AFD}}$ is also an extreme point of $T(A)$, since $UT(A)_{w-\text{AFD}}$ is a face, and hence gives $R$ in the GNS representation. Thus, if we assume that $R$ is QD then it's unique trace must belong to $T(R)_{\text{AFD}}$ and this completes the proof.

For the sufficiency, we first point out that $R$ is QD if and only if all of it’s separable C$^*$-subalgebras are QD. So let $A \subset R$ be an arbitrary separable, unital subalgebra. Let $\tau \in T(A)$ be the restriction of the unique trace on $R$ to $A$. Clearly $\tau$ is faithful and belongs to $UT(A)_{w-\text{AFD}}$. Hence it also belongs to $T(A)_{\text{AFD}}$. This implies that $A$ is QD by Voiculescu’s abstract characterization of quasidiagonality (cf. [49, Theorem 1]).

Applying the proposition above and Theorem 6.1 we immediately get the following corollary.

Corollary 12.3. If $R$ is QD then for every locally reflexive C$^*$-algebra $A$ we have $T(A)_{w-\text{AFD}} = UT(A)_{w-\text{AFD}} = T(A)_{\text{AFD}} = UT(A)_{\text{AFD}}$.

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