Pattern formation of quantum Kelvin-Helmholtz instability in binary superfluids

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We study theoretically nonlinear dynamics induced by shear-flow instability in segregated two-component Bose-Einstein condensates in terms of the Weber number, defined by extending the past theory on the Kelvin-Helmholtz instability in classical fluids. Numerical simulations of the Gross-Pitaevskii equations demonstrate that dynamics of pattern formation is well characterized by the Weber number $W_e$, clarifying the microscopic aspects unique to the quantum fluid system. For $W_e \lesssim 1$, the Kelvin-Helmholtz instability induces flutter-finger patterns of the interface and quantized vortices are generated at the tip of the fingers. The associated nonlinear dynamics exhibits a universal behavior with respect to $W_e$. When $W_e \gtrsim 1$ in which the interface thickness is larger than the wavelength of the interface mode, the nonlinear dynamics is effectively initiated by the counter-superflow instability. In a strongly segregated regime and a large relative velocity, the instability causes transient zipper pattern formation instead of generating vortices due to the lack of enough circulation to form a quantized vortex per a finger. While, in a weakly segregating regime and a small relative velocity, the instability leads to sealed skin pattern in the overlapping region, in which the frictional relaxation of the superflow cannot be explained only by the homogeneous counter-superflow instability. We discuss the details of the linear and nonlinear characteristics of this dynamical crossover from small to large Weber numbers, where microscopic properties of the interface become important for the large Weber number.

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I. INTRODUCTION

Hydrodynamic instability in superfluids is one of the important topics in a research field of quantum fluids, being deeply related with a generation mechanism of quantum turbulence \cite{1}. The Kelvin-Helmholtz instability (KHI), one of the fundamental instabilities in classical hydrodynamics, occurs when two phase-separated fluid components undergo a shear flow beyond the critical relative velocity \cite{2,3}. The KHI in quantum fluids, referred to as quantum KHI (QKHI), has been studied in superfluid helium \cite{4,5}, atomic Bose-Einstein condensates (BECs) \cite{7,10}, and nuclear superfluids in a neutron stars \cite{11}. A cold atomic BEC is a versatile system to study the hydrodynamic instability and the associated nonlinear dynamics, because ideal configurations suitable to study the relevant problems can be prepared in a well controlled manner; for example, a flat interface between different superfluids can be prepared by using binary BECs with tunable interatomic interactions \cite{12,13}. The interface dynamics, the hydrodynamic instabilities and the nonlinear dynamics in immiscible two-component BECs have been studied in some papers \cite{16,31}. Even for the miscible case, the binary BECs exhibit the countersuperflow instability (CSI) \cite{22}, which results in a train of solitons in a one-dimensional (1D) case or the complicated turbulent structure in 2D or 3D systems \cite{33,57}.

The linear stability analysis of stationary flowing states in immiscible binary superfluids can be explored in the hydrodynamic model based on the low-energy effective action of a quantized interface excitation, i.e., ripplon. In the previous study \cite{7}, the QKHI of a thin interface of strongly-segregated binary BECs has been studied. The nonlinear stage of the evolution has shown that, just above the critical relative velocity, the initial flat interface between the two condensates deforms into sawtooth waves and generates singly quantized vortices on the peaks and troughs of the waves. The subsequent work addresses the stability analysis and resulting nonlinear dynamics with increasing the interface thickness to the miscible limit, revealing the crossover behavior from the KHI to the CSI \cite{8}.

In this work, we study theoretically the nonlinear evolution in immiscible two-component BECs with a shear flow in a wide range of system parameters. The characteristics of the nonlinear dynamics is summarized in the phase diagram parametrized by the relative velocity and the intercomponent coupling strength. We find that the comparison relation of the two important scales, namely, the wavelength of the unstable interface excitations and the thickness of the interface, determines the boundary of different regimes of nonlinear evolution of the QKHI. We introduce the Weber number, a dimensionless quantity given by the ratio of the inertial force to the surface tension and extended to the segregated superfluids, to characterize the dynamics of the QKHI. This number is related to the ratio of the above-mentioned two length scales, separating the dynamical behavior between the universal macroscopic regime and the microscopic one. For relatively small Weber number less than unity, the interface wave evolves to elongated flutter-finger patterns, as seen in the classical fluid dynamics. The fingers are disintegrated through the creation of quantized vortices at each tip of the fingers, supported by the fact that the
vorticity along the interface with respect to one wavelength of the unstable wave is larger than a single quantum circulation \( \kappa = \hbar/m \). When the Weber number is typically larger than unity, the microscopic aspect of the interface structure becomes important in the nonlinear dynamics. We clarify the detailed characteristics of the pattern forming dynamics from not only the simulations of the GP equations but also the linear stability analysis based on the Bogoliubov-de Genne (BdG) equations. For the strongly segregated regime, a small amplitude interface wave forms a zipper pattern and does not emit the quantized vortices at the tip of the wave, since the vorticity per one wavelength is not enough to evolve a single vortex. However, the nonlinear dynamics causes multistep collapses of the interface, leading eventually to the turbulent state. When the intercomponent coupling strength is decreased to the miscibility limit, the interface instability exhibits a crossover from the KHI dynamics to the CSI-like behavior, as discussed in Ref. \[8\]. The analysis reveals the mechanism of frictional relaxation of the shear flow by forming a “sealskin” pattern through sheared CSI at the inhomogeneous overlapping region.

This paper is organized as follows. In Sec. II, we introduce the formulation and the setup of the problem to study the QKHI in phase separated two-component BECs. Several characteristic length scales are introduced in order to classify different regimes of the nonlinear dynamics. In Sec. III, we introduce the Weber number and construct the phase diagram Fig. 2 of the nonlinear dynamics. After that, we show the simulation results of the nonlinear dynamics associated with the shear-flow instability in Sec. IV and V for the small and large Weber number, respectively. Section VI is devoted to conclusion and discussion.

II. FORMULATION

We first give a brief introduction of the QKHI in phase-separated two-component BECs. The details are found in Refs. \[1\] \[4\] \[5\]. Also we introduce the several length scales of the problem; especially, the wavelength of the unstable interface mode and the interface thickness play an important role to understand the nonlinear dynamics.

A. Equations of motion

We consider two-component BECs in a homogeneous space without an external potential. In the mean-field theory at low temperatures, the two-component BECs are described by the condensate wave functions \( \Psi_j(r,t) = \sqrt{n_j(r,t)} e^{i\theta_j(r,t)} \) with the particle densities \( n_j \) and the phases \( \theta_j \), obeying the coupled GP equations

\[
\begin{align*}
\hbar \frac{\partial \Psi_j}{\partial t} &= \left( -\frac{\hbar^2}{2m_j} \nabla^2_j + \mu_j + g_j |\Psi_j|^2 + g_{jj'} |\Psi_{j'}|^2 \right) \Psi_j \quad (j, j' = 1, 2, j \neq j'). \\
\end{align*}
\]

Here, \( m_j \) is the atomic mass and \( g_1, g_2, \) and \( g_{12} \) are the coupling constants in the nonlinear terms which are related to the s-wave scattering lengths \( a_1, a_2, \) and \( a_{12} \), respectively, as \( g_j = 4\pi \hbar^2 a_j/m_1 \) and \( g_{12} = 2\pi \hbar^2 a_{12}(m_1 + m_2)/m_1 m_2 \). Throughout this work, we consider immiscible BECs under the condition \( g_{12} > \sqrt{g_1 g_2} \)[33][40]. The immiscible ground state for the binary condensates with an equal particle number consists of the configuration in which one component occupies a half of the space and the other does the rest; the stable interface is formed between them. We assume that the first and second components are located located in \( y \leq 0 \) and \( y \geq 0 \), respectively, and the interface between them is located near \( y \approx 0 \) plane.

The QKHI can be studied by making the linear stability analysis around the stationary state which has the straight interface at \( y = 0 \) and the shear flow velocities \( v_j = V_j x \) with \( V_1 = V_{12}/2 \) and \( V_2 = -V_{12}/2 \), the relative velocity being determined as \( V_{12} = |v_1 - v_2| \) along the \( x \)-axis. Substituting the form \( \Psi_j(r) = \phi_j(y) e^{im_j V_j x/\hbar} \), the profile \( \phi_j(y) \) can be calculated by solving the time-independent GP equations

\[
\left( -\frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial y^2} - \mu_j + \frac{m_j V_j^2}{2} + g_j |\phi_j|^2 + g_{jj'} |\phi_{j'}|^2 \right) \phi_j = 0.
\]

Far from the interface, the bulk density for \( j \)-th component is simply given by the constant \( n_{oj} = |\mu_j - m_j V_j^2/2|/g_j \), which is used as a boundary condition for the solution of Eq. (2).

B. The KH theory

There is one-to-one correspondence between classical hydrodynamics and the present system when one introduce the scalar velocity potential \( \Phi_j = (\hbar/m_j) \theta_j \), the velocity field being given by \( v_j = \nabla \Phi_j \). From the GP equation (1), the equation of motion of \( \Phi_j \) is written as

\[
\frac{\partial \Phi_j}{\partial t} + \frac{v_j^2}{2} - \frac{\mu_j}{m_j} + P_j + Q_j = 0,
\]

where \( P_j = g_j n_j/m_j \) and \( Q_j = -\hbar^2 (\nabla^2 \sqrt{\mu_j})/(2m_j^2 \sqrt{\mu_j}) \) represent the pressure function and the quantum pressure, respectively. When the quantum pressure term is neglected, we have a problem similar to the classical hydrodynamics of the KHI. The detail of the analysis has been described in Refs. \[7\][8]. Note that, in the standard problem in the classical hydrodynamics, the interface between two species of fluids is stabilized by the gravitational potential, which is absent in our system.
We suppose that a position of the time-dependent curved interface can be described by the displacement field \( y = \eta(x, z, t) \) and neglect the \( z \)-dependence by assuming the uniformity along the \( z \)-axis. A small-amplitude interface wave is represented by the localized small fluctuation of the velocity potential \( \delta \Phi_j = \Phi_j - V_j x \) and the small displacement \( \eta \) with the form

\[
\delta \Phi_j = A_j e^{-(1/2)k^2} \cos(kx - \omega t),
\]

\[
\eta = B \sin(kx - \omega t),
\]

where \( k \) and \( \omega \) represent the wavenumber and the frequency of the interface wave, respectively. The dispersion relation of the interface wave is written as [7 8 41]

\[
\omega = \frac{(\rho_1 V_1 + \rho_2 V_2)k}{\rho_1 + \rho_2} \pm \frac{1}{\sqrt{\rho_1 + \rho_2}} \sqrt{\alpha k^3 - \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} V_R^2 k^2}.
\]

Here, \( k > 0 \) and \( \rho_j = m_n \rho_0 j \) is the bulk mass density. The parameter \( \alpha \) stands for the surface tension of the interface, corresponding to the excess energy due to the presence of the interface [42] and being determined later as a function of \( \gamma_{12} \).

When the inside of the square root in Eq. (6) becomes negative, the imaginary part \( \text{Im}(\omega) \) appears and the shear flow states are dynamically unstable. From Eq. (6), the instability occurs for the nonzero relative velocity \( V_R > 0 \). The imaginary part \( \text{Im}(\omega) \) appears in a range \( 0 < k < k_+ \) with

\[
k_+ = \frac{\rho_1 \rho_2}{\alpha(\rho_1 + \rho_2)} V_R^2.
\]

The wave number of the most unstable (fastest growing) mode of the QKHII corresponds to

\[
k_0 = \frac{2}{3} k_+ = \frac{2\rho_1 \rho_2}{3\alpha(\rho_1 + \rho_2)} V_R^2,
\]

which means that \( \text{Im}(\omega) \) takes a maximum at \( k = k_0 \).

In the following, we confine ourselves to the situation without the center-of-mass velocity of the two components, corresponding to the vanishing first term of the right hand side of Eq. (6). The plus-minus sign represents the conjugate modes propagating to the opposite directions; we shall take only the plus sign below. If the center-of-mass velocity is alive, there is another critical velocity associated with the Landau instability given by the condition \( \omega < 0 \). Although this instability is significant when the system is subject to an energy dissipation as in the system of the superfluid helium [43], we will not consider this instability in the following by supposing the cold atom system which is almost isolated from a surrounded environment.

C. Characteristic length scales

Before the numerical simulations of the real time dynamics, it is instructive to understand the aspect of the dynamical instability and expected nonlinear dynamics by comparing the several characteristic length scales in our problem. In the following, we confine ourselves to the parameters as \( m_1 = m_2 = m \) and \( g_1 = g_2 = g \). We also assume the condition of the chemical potential as \( \mu_1 - m V_1^2/2 = \mu_2 - m V_2^2/2 \equiv \mu \). Then, the number density in the bulk region is \( n_1 = n_2 = \mu/g = n_0 \) and the mass density is written as \( \rho_j = mn_0 = m\mu/g \equiv \rho \).

The first characteristic length scale is the healing length

\[
\xi = \frac{\hbar}{\sqrt{2m\mu}},
\]

which determines the scale with which the amplitude of the wave function of the one component heals from zero to the bulk when the other component is absent. The healing length comes from a purely quantum origin, which is a balance between quantum pressure term and the nonlinear coupling constant, providing a length scale not found in classical hydrodynamics. In this work, all lengths are scaled by \( \xi \). By scaling the coordinate and the wave function as \( r \to \xi r \) and \( \phi_j \to \sqrt{\xi_0} \phi_j \), the stationary GP equation (2) has a single parameter \( g_{12}/g \).

In Fig. 1(b), we show the value of \( \gamma_{12}/g - 1 > 0 \), where the numerical solution of \( \phi_j(y) \) obtained from Eq. (2) is used to calculate Eq. (11). The approximate analytic formula of \( \alpha \) without an external potential has been obtained as

\[
\alpha_{\text{weak}} \simeq 2R_0 \xi \sqrt{\frac{g_{12}}{g} - 1}
\]

In Fig. 1(b), we show the value of \( \alpha \) as a function of \( \Delta \equiv g_{12}/g - 1 > 0 \), where the numerical solution of \( \phi_j(y) \) obtained from Eq. (2) is used to calculate Eq. (11). The approximate analytic formula of \( \alpha \) without an external potential has been obtained as

\[
\alpha_{\text{weak}} \simeq 2R_0 \xi \sqrt{\frac{g_{12}}{g} - 1}
\]
in the weakly segregating limit \( g_{12}/g \approx 1 \) \cite{44} and

\[
\alpha_{\text{weak}} \simeq 4 P_0 \xi \left[ \frac{2 \sqrt{2}}{3} - 0.514 \left( \frac{g_{12}}{g} \right)^{-1/4} - 0.110 \left( \frac{g_{12}}{g} \right)^{-3/4} - 0.134 \left( \frac{g_{12}}{g} \right)^{-5/4} \right], \tag{13}
\]

in the strongly segregating limit \( g_{12}/g \gg 1 \) \cite{42,43}. Here, \( P_0 = \mu^2/2g \) represents the equilibrium pressure. We confirm that the two analytical formula can describe well the numerical result for the corresponding limits.

The surface tension density \( \sigma(y) \) in Eq. (11) is localized around the position of the interface, as shown in Fig. 1(d). From the distribution of \( \sigma(y) \), we can obtain the third length scale

\[
l_d = \sqrt{\frac{\int dy [y^2 \sigma(y)]}{\alpha}}, \tag{14}
\]

which represents the thickness of the interface. Figure 1(c) shows \( l_d \) as a function of \( \Delta \). In the weakly segregating limit \( (\Delta \ll 1) \), we have \( l_d \propto \xi/\sqrt{\Delta} \), consistent with the analytical evaluations in Refs. \cite{40,42,44}, where the total density \( n_1 + n_2 \) is almost uniform. With increasing \( \Delta \) in the strongly segregating regime \( (\Delta \gg 1) \), \( l_d \) takes a minimum \( l_d \approx 0.67 \xi \) around \( \Delta \approx 10 \) and approaches slowly to the value \( \approx 0.8 \xi \), which is obtained in the limit \( \Delta \to \infty \). The latter behavior is due to the fact that, after the interface becomes thinnest, the condensate domains are repelled further to get rid of the overlapping region completely, which leads to the imperceptible increase of \( l_d \). At \( \Delta \to \infty \) the total density at the interface becomes zero and the profile is given by the dark-soliton solution \cite{42}.

### III. Phase diagram based on Weber number

#### A. Weber number

In the classical hydrodynamics, when discussing the interface dynamics of the phase separated fluid, the Weber number

\[
We = \frac{\rho V^2 L}{\alpha}, \tag{15}
\]

is a useful dimensionless quantity to characterize the nonlinear dynamics \cite{45,46}. The Weber number includes the characteristic length \( L \), which is taken as the wavelength of the initial perturbation in the classical case.

In our simulations below, since the dynamical instability is caused by the random noise, \( L \) is naturally given by \( \lambda_0 \). Then, the Weber number is simply given by \( We = 6 \pi (= \text{const.}) \) from Eq. (10), which is not suitable to classify the dynamics of our problem. Instead of using \( \lambda_0 \), we here take the interface thickness as the length scale \( L = l_d \); the Weber number in our problem is written as

\[
We = \frac{6 \pi l_d}{\lambda_0}. \tag{16}
\]

This definition is generally applicable to any systems in terms of the thickness of the interface between two separated fluids.

The thickness \( l_d \) depends on the system parameters through the internal structure of the interface described by the “microscopic theory” beyond the hydrodynamic theory of KHI. In our case, the structure and thus the thickness are uniquely determined by the dimensionless...
parameter $\Delta$. In this sense, the definition of the Weber number as Eq. (16) enables us to access more microscopic behaviors of the instability beyond the KH theory. In fact, according to the conditions under which the KH theory holds, the hydrodynamic treatment breaks down when $\lambda_0$ is similar or less than $l_d$, namely $We > 1$. We shall demonstrate that the microscopic behavior described by quantum fluid dynamics becomes prominent typically for $We > 1$.

B. phase diagram

Here, we summarize the prospect of the nonlinear dynamics in terms of Weber number by comparing the simulations results in the past works [7, 8, 29]. We show the phase diagram of the dynamics in the $\Delta-V_R^2$ plane in Fig. 2. Here, $V_R$ is scaled by the characteristic velocity $V \equiv \xi \mu/\hbar = \sqrt{\mu/(2m)}$. Some contours of the typical values of $We$ are also shown. Eq. (16) means $V_R^2 \propto \alpha l/d$ for a given $We$, the contour lines of $We$ have a behavior $V_R^0 \propto \Delta$ in the weakly-segregating limit with $\alpha_{\text{weak}} \propto \Delta^{1/2}$ and $l_d \propto \Delta^{-1/2}$.

The (red) solid curve $k_{\text{d}}l_d = 1$ [We = 4 from Eq. (16)] gives roughly the boundary of two characteristic nonlinear dynamics. For $k_{\text{d}}l_d \lesssim 1$ or $We \lesssim 1$ (the lower-right region of the diagram) the interface is thin and the linear stability is well described by the KH theory. On the other region with $k_{\text{d}}l_d \gtrsim 1$ or $We \gtrsim 1$, the thickness of the interface is larger than the wavelength of the excitation so that the resulting instability cannot be described by the KH theory in which the internal structure of the interface is neglected. We also depict two other (blue dashed and green dotted) curves that classify the different dynamical regimes associated with the characteristic pattern formation, which are derived in the following sections.

There are some studies showing the simulation results of the nonlinear dynamics of the KHI. Takeuchi et al. [7] considered the KHI in a strongly segregated condensates with $\Delta = 9$ and $V_R = 0.98 \times \sqrt{2}V$, where the authors also introduced an external potential to sustain the stable interface. The simulation results demonstrated that the initially growing sinusoidal wave deformed into a sawtooth wave. The vorticity increased on the edges of the sawtooth waves, developing as a quantized vortex and being released into each bulk. The subsequent paper by Suzuki et al. [8] showed that the dynamics for $V_R = \sqrt{2} \times 10^{-1}V$ and two different values of $\Delta$, namely $\Delta = 10^{-2}$ and $\Delta = 10^{-5}$, which aimed to discuss the dependence of the interface thickness of the nonlinear dynamics. There, the dynamics exhibits a crossover-like behavior between the KHI and the CSI; in the latter the instability of the density wave arises in the overlapping region of the two components. Finally, Kovyakov et al. [29] showed the dynamics for $\Delta = 10^{-2}$ and $V_R = V/3$ [27]. They observed that, after the periodic interface wave is excited, the vorticity accumulated on the mode with the largest wavelength in the system to make a vortex bundle in the later stage, which is similar to the role-up pattern seen in the classical KHI. We have also observed similar role-up patterns in the later stage of the instability for different parameter regimes in our simulations. This behavior is conventionally explained by the fact that a bundle of quantized vortices can be regarded as a coarse-graining vortex in classical fluids. This scenario would be universal in the later stage of quantum KHI when the system can form a large vortex bundle without external potentials and we thus focus on the nonlinear dynamics in the early stage before forming the roll-up patterns.
IV. UNIVERSAL MACROSCOPIC REGIME 
\( (We \lesssim 1) \)

In this section and the next one, we demonstrate the nonlinear dynamics by numerically solving the time-dependent GP equation \( (1) \) to corroborate the phase diagram of Fig. 2. We study the dynamics in the 2D system by assuming the uniformity along the z-axis and do not consider a contribution of an external trap. Here, we show the dynamics of the interface for \( We \lesssim 1 \), namely, the lower right region of Fig. 2 (the large \( \Delta \) and the small \( V_R \)). In this regime, referred to as the universal macroscopic regime hereafter, the interface thickness \( \ell_d \) is smaller than the wavelength \( \lambda_0 \) of the KH theory; the nonlinear dynamics has a similarity with the KH-induced dynamics in classical hydrodynamics.

The numerical calculations are done under the following procedures. We first solve Eq. (2) through the imaginary time propagation to obtain the stationary solution \( \phi_j(y) \). Setting the initial wave function as \( \Psi_j(x, y, t = 0) = \phi_j(y) e^{imV_j x/k} \) with \( V_j = -(1)^{j-1}V_R/2 \), we then solve Eq. (1) to see the time developments of \( \Psi_j(x, y, t) \) for given values of \( V_R \) and \( \Delta \), where the simulations are done in a 2D \( x-y \) system with the size \([-L_{x,y}, +L_{x,y}] \). The periodic boundary condition is given for the \( x \)-direction along which the condensates initially have uniform counterflow, while the Neumann boundary condition is given at \( y = \pm L_y \). The system size \( (L_x, L_y) \) is prepared properly for each parameter set enough to omit the influence of numerical boundaries. To initiate the dynamical instability, we give a small random noise (of the order \( 10^{-5} \)) to the initial wave functions.

A. Flutter-finger pattern

The dynamics can be visualized directly through the profile of the condensate density. We show the density difference \( \Delta n = (n_1 - n_2)/n_0 \) below, in which the bulk region of the first (second) component corresponds to \( \Delta n \simeq 1 \) \((-1)\), while the interface is distributed around \( \Delta n = 0 \). Figure 3(a) shows a typical dynamics of the condensate density for \( \Delta = 1.0 \) and \( V_R/V = \sqrt{0.45} \approx 0.67 \). In this parameter, the instability begins to grow after \( \tilde{t} \simeq (gn_0/k) t \approx 800 \), where the amplitude of the sinusoidal interface wave is monotonically increased due to the exponential growth with \( \text{Im}(\omega) \neq 0 \) described by Eq. (6). The wavelength given by the analytical prediction of Eq. (10) is \( \lambda_0 \approx 64.9\xi \), reasonable agreement with the numerical result \( \sim 60\xi \) in the panel at \( \tilde{t} = 875 \) Fig. 3(a). Then, the sinusoidal wave develops finger patterns. Although the formation of such a finger pattern has been seen in the simulations of the classical fluid dynamics \( [45, 46] \), the subsequent nonlinear evolution exhibits a quite different behavior from that of the classical one. The fingers are elongated gradually in the oblique direction and eventually disintegrate into bubble-like domains of the condensates.

For \( We \lesssim 1 \), the eigenmode of the unstable excitation is localized on the interface; the associated density modulation occurs periodically along the interface \( [8] \). This means that the form of the excitation is sinusoidal in the early stage of the instability. After the amplitude of the interface wave becomes large to form a finger pattern, the fingers are pushed by the counterflowing other components, like a grass fluttered in the wind. Thus, the fingers of the component 1 (2) grow along the upper left (lower right) direction in the nonlinear stage of the evolution. We call this stripe pattern as the “flutter-finger” pattern.

Figure 3(b) shows the corresponding phase profile \( \theta_1 = \text{arg}(\Psi_1) \) of the first component. Each bubble-like domain contains a quantized vortex, as seen in the panel at \( \tilde{t} = 1500 \) of Fig. 3(b), forming a coreless vortex \([48, 19]\) with the vortex core filled by the density of the other component. Also, one can see that the branch cuts are located near the tips of the fingers of the \( \Psi_2 \)-domain as precursors of the vortices. The emission of the quantized vortices from the finger pattern is a distinguishable feature from the classical problem.

B. Surface vorticity

The appearance of the finger patterns involves the characteristic vorticity distribution, which eventually develops to the quantized vortices. To show this, we introduce the mass current velocity defined as

\[
v = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2}
\]

(17)

and the associated vorticity

\[
\omega = \nabla \times v,
\]

(18)

which provide a useful description of vortices in the two-component system \([7, 50]\). In the 2D calculation, we concern only with the z-component as \( \omega = \omega(x, y) e_z \). Figure 3(c) shows the distribution of the vorticity \( \omega(x, y) \). Initially, the vorticity is distributed uniformly along the interface, which forms a linear vortex sheet. The circulation per a unit length along the sheet, denoted as \( \rho_T \), can be easily calculated according to the Stokes theorem as

\[
\rho_T = \int dS \cdot \omega = \int d\ell \cdot v = v_1 - v_2 = V_R.
\]

(19)

Here, the area of the integral is taken as a rectangular enclosing the vortex sheet and having a unit horizontal width and a vertical width sufficiently larger than the interface thickness. As shown in the Appendix A, these vorticity distribution can be understood according to the analogy to the electrostatic problem. Far from the interface, \( v \) is coincident with \( v_j \) appearing in Eq. (1), and if the time and spatial scale is slow, \( v \) satisfies the Laplace equation. Then, the vorticity distribution \( \rho_T \) on
FIG. 3. Snapshots of the unstable dynamics for \( V_R/V = \sqrt{0.45} \) and \( \Delta = 1.0 \), where the corresponding Weber number is \( We = 0.291 \). In (a), the density difference \( \Delta n = (n_1 - n_2)/n_0 \) is depicted, in which the red (blue) region corresponds to the area where the density \( n_1 \) \( (n_2) \) is located. In (b), the profile of the phase \( \theta_1 \) of the first component is shown only in the region \( \Delta n > 0 \); there is only noisy phase fluctuations in the region \( \Delta n < 0 \) since the amplitude \( n_1 \) is almost zero there. A vortex is located at the end point of a branch cut (jump from \( \theta_1 = -\pi \) to \( \pi \)). The distributions of the vorticity \( \omega \) in the \( x-y \) plane is shown in (c). The spatial region of the plot is \(-150 \xi \leq x, y \leq 150 \xi \). The time is represented by the dimensionless value \( gn_0t/\hbar \equiv \tilde{t} \).

the sheet has a one-to-one correspondence with that of an electronic charge on a flat-plate conductor, being obtained by solving the Laplace equations with the suitable boundary condition.

This electrostatics analogy is approximately applicable to our dynamic situation, in which the vorticity is accumulated at the tip of each finger to form quantized vortices (see the panel at \( \tilde{t} = 1250 \)) during the slow growth of the finger pattern. The electrostatics predicts that a surface charge density becomes larger on a sharp end of a charged object than that on the other region [51]; see also the discussion in Appendix A. The accumulation of the vorticity is thus enhanced as the fingers grow and, when the local accumulated vorticity becomes comparable to the single quantum circulation \( \kappa = h/m \), the vortices are eventually emitted from the tip and the fingers undergo self-collapse. After the vortices are emitted, the vorticity on the sheet is reduced even to negative values locally according to the conservation of the vorticity.

The vorticity charged per a single interface wave is also an important quantities to understand the pattern formation. To make a vortex from a single finger, the vorticity per a half of the unstable wavelength should contains the vorticity above the quantum circulation \( \kappa = h/m \). Since an interface per a unit wavelength possesses a vorticity \( \rho \Gamma \lambda_0 \) and two fingers can grow from a single wave, the quantity \( \lambda_0 \rho \Gamma/(2\kappa) \equiv \nu \kappa \) determines whether an elongated finger possesses vorticity enough to make a quantized vortex. Using the relation of Eq. (19), we also plot the curve \( \nu \kappa = 1 \) in the diagram of Fig. 2, whose behavior is almost coincident with the curve \( We \simeq 1 \). This is because the relation \( \nu \kappa = 1 \) can be written as \( (V_R/V)^2 = 9\alpha^2/4 \), which has a similar dependence of \( We \) with respect to \( \Delta \) in the weakly segregating limit \( \alpha_{weak} \propto \Delta^{1/2} \). For \( We < 1 \), in the right side of this curve, the vortex sheet in one finger contains the vorticity enough to generate a single quantized vortex. Then, the event of the vortex generation takes place in the first
We focus on the length of the fingers seen in the simulations. Here, the finger length $\Lambda$ is defined as the difference between the amplitude at the highest top and that at the lowest bottom of the interface wave, where the interface position is identified by zeros of the density difference $\Delta n$. We extract the maximum length $\Lambda_{\text{max}}$ at the moment when the first self-collapse of the fingers takes place. The timing of the first self-collapse is related to the time evolution of the total length of the interface because the evolution changes qualitatively after the vortex nucleation. The interface length is roughly proportional to the time derivative of the inter-component interaction energy $E_{12} = (1 + \Delta) \int dr |\Psi_1|^2 |\Psi_2|^2$, since the two components overlap only in the interface layer. The time derivative $\dot{E}_{12} = dE_{12}/dt$, shown in Fig. 4(a) as a typical example, indicates the first exponential growth of the interface wave at $t \approx 700$ and the subsequent nonlinear elongation of the fingers for $700 \lesssim t \lesssim 1400$. The growth of the fingers is suppressed by the creation of the quantized vortices; a signal of the vortex creation can be seen as an occurrence of a rapid chaotic oscillation of $\dot{E}_{12}$ and we take the value $\Lambda_{\text{max}}$ at this moment.

Figure 4(b) shows the average $\overline{\Lambda}_{\text{max}}$ taken from the five simulations with a random initial noise. We find that the length of the growing fingers can be elongated further with increasing $\Delta$ for a fixed $V_R$. For smaller $V_R$, the maximum length of the finger patterns increases rapidly with the increase in $\Delta$, where the measured finger length exhibits large error bars. The growth time of the instability becomes extremely long in the limit of $\Delta \gg 1$ and $V_R \ll 1$, the numerical demonstration being difficult there. As shown in Fig. 4(c), the results are explained more clearly by plotting $\Lambda_{\text{max}}/l_d$ with respect to the Weber number, where the data are well described by the $W^{-1}$-behavior. Thus, Fig. 4(c) explains not only the fact that the finger length decreases with $W_e$, but also that the Weber number can give a common index to characterize the nonlinear dynamics of the KHI for different parameter values. For example, the qualitative behavior of the dynamics is common between our Fig. 4 and Fig. 2(a) in Ref. [3], which are done with the different system parameters but nearly equal $W_e$. The behavior $\Lambda_{\text{max}}/l_d \propto W^{-1}$ implies $\Lambda_{\text{max}} \propto \lambda_0$ from Eq. (16), which means that the maximum amplitude of the interface wave is as large as the its wavelength.

V. MICROSCOPIC REGIME : $W_e \gtrsim 1$

Next, we consider the dynamics of the interface by increasing $W_e$ much larger than unity. This regime, referred to as the microscopic regime, corresponds to the condition that the interface thickness $l_d$ is larger than the wavelength $\lambda_0$ of the KH theory. We find that, even though $W_e$ is common, the nonlinear dynamics in this regime are qualitatively different for strongly and weakly segregated cases in the microscopic regime; we discuss separately these situations in the following.

The KH theory is not applicable directly to this regime, since it is based on the assumption of the thin interface thickness compared with the other length scales. Thus, we analyze the BdG equation numerically to see the interface instability against a shear flow more microscopically. We linearize the time-dependent GP equation (1) around
the stationary solutions \( \phi_j^0 \) as

\[
\Psi_j = \left[ \phi_j^0(y) + u_j(y) e^{i k x - i \omega t} - v_j(y) e^{-i k x + i \omega t} \right] e^{i m V_j x / h}
\]

to obtain the BdG equation

\[
\hat{H} u = \hbar \omega u,
\]

\[
\hat{H} = \begin{pmatrix}
\hat{h}_1^+ & -g(\phi_0^\dagger)^2 & g_{12}\phi_0^\dagger \phi_0^\dagger & -g_{12}\phi_0^\dagger \phi_2^\dagger \\
-g(\phi_0)^2 & \hat{h}_1 & g_{12}\phi_1^\dagger \phi_0^\dagger & -g_{12}\phi_1^\dagger \phi_2^\dagger \\
g_{12}\phi_1^\dagger \phi_0^\dagger & -g_{12}\phi_1^\dagger \phi_2^\dagger & \hat{h}_2 & g(\phi_2^\dagger)^2 \\
g_{12}\phi_1^\dagger \phi_2^\dagger & -g_{12}\phi_2^\dagger \phi_2^\dagger & g(\phi_2)^2 & -\hat{h}_2
\end{pmatrix},
\]

where \( u = (u_1, v_1, u_2, v_2)^T \) and

\[
\hat{h}_j^\pm = \frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial y^2} \right] \left( k \pm \frac{m V_j}{\hbar} \right)^2 - g m - \frac{m V_j^2}{2} + 2g|\phi_j^0|^2 + g_{12}|\phi_j|^2
\]

\[
[j = 1(2) \text{ for } j = 2(1)]
\]

We numerically diagonalize the discretized BdG hamiltonian \( \hat{H} \) to calculate the eigenfrequency \( \omega \) for a given value of \( k \), the wave number of the plane wave excitation along the translationally invariant \( x \)-axis. The interface mode is described by the eigenmodes \([u_j(y), v_j(y)]\) localized around the interface in the \( y \)-direction. When the frequency has a non-zero imaginary part \( \text{Im}[\omega] \neq 0 \), the system is dynamically unstable.

In the previous study [8], the BdG spectrum was compared between the miscible condensates with external gradient potential and immiscible ones without potential in connection with CSI. For comparison we use the dispersion relation of the counterflowing miscible condensates in a homogenous system, referred to as homogeneous CSI; the dispersion is given by [32,33]

\[
(h\omega)^2 = \varepsilon_0(\varepsilon_0 + 2gn) + \varepsilon_R^2
\]

\[
\pm 2\sqrt{\varepsilon_0\varepsilon_R^2(\varepsilon_0 + 2gn) + (g_{12}n)^2\varepsilon_0}
\]

A. Zipper pattern formation

1. Development of the density and the phase

First, we show the simulation results of the GP equations in the strongly segregating regime. The typical numerical results are shown in Fig. 5, where we increase \( V_R \) with fixed \( \Delta = 1 \) from the parameters of Fig. 3. For \( V_R/V = 2 \) in Fig. 5(a), the initially growing interface wave forms a saw-tooth shape. Then, the saw-tooth pattern transforms to a transient zipper pattern, where each cusp is torn off from the lump and just slides to merge with the next lump, instead of emitting the vortices at the tip of the interface wave. After that, the dynamics exhibits a recurrence of the saw-tooth and zipper patterns alternatively. However, the periodic pattern is disturbed in the long-time nonlinear evolution, eventually evolving a large-scale turbulent structure. A further increase in \( V_R \) results in the growth of the saw-tooth pattern with a shorter wavelength, as shown in Fig. 5(b) depicting the dynamics for \( V_R/V = \sqrt{20} \). The transient zipper pattern is again resulted from the sliding motion of the lumps, and also there appears a density filament in the bulk region [the panel at \( \bar{t} = 62.5 \) in Fig. 5(b)]. After the zipper pattern and the filaments appear, the interface deforms furthermore with larger length scales [the panel at \( \bar{t} = 75 \) in Fig. 5(b)] and evolves the turbulent structure.

The absence of the vortex emission during the first growth of the instability is due to the fact that \( \nu_c \) in these parameters is less than unity; a half of the wavelength of the interface does not contain enough vorticity to generate a single quantized vortex. Figure 5(c) shows the evolution of \( \theta_1 = \arg(\psi_1) \) corresponding to Fig. 5(b). It is clear that the number of density lumps is incommensurate with that of the branch cuts that will evolve to quantized vortices; see Fig. 5(b) for comparison. This is consistent with \( \nu_c < 1 \) and thus each density lump does not emit a vortex but forms the zipper pattern. In this sense, the occurrence of the zipper pattern is a characteristic phenomenon of strongly segregated condensates in the microscopic regime.

Note that, in the case of Fig. 5(b), the Mach number in the bulk region \( Ma = V_j/\sqrt{\mu/m} = V_R/(2\sqrt{2}V) \) is more than unity. Thus, the bulk flow is supersonic so that the density filament can be identified as an appearance of a shock wave. Usually, a Mach cone structure is generated by an obstacle forced to move through a superfluid with supersonic velocity [52,55]. Since there is no forced obstacle in our case, it does not lead to the clear formation of a Mach cone.

2. The results of the BdG analysis

The spectrum of the BdG equation is useful to understand the numerical observation. We show in Fig. 6 the imaginary part \( \text{Im}[\omega] \) of the eigenfrequency as a function
of $k$ for $\Delta = 1$ and several values of $V_R$. Since there is not an external potential, the imaginary part always appear for $V_R \neq 0$ in the range $0 < k < k_{\text{max}}$. For small values of $V_R \lesssim 1$, an example being shown in Fig. 6(a) for $V_R/V = 1$, there is a single branch associated with the dynamical instability, which is consistent with Eq. (6) of the KH theory.

However, the spectrum is deviated from Eq. (6) with increasing $V_R \gtrsim 1$. In Fig. 6(b), we see that $\text{Im}[\omega]$ of the BdG result for $V_R/V = 2$ lies deeply inside the dispersion curve of Eq. (6). We confirm that the peak of $\text{Im}[\omega]$ of the BdG result determines the wavelength $\lambda_0^{\text{BdG}}$ of the growing wave; for example, we have $k_0^{\text{BdG}} \xi = 0.592$ ($\lambda_0^{\text{BdG}}/\xi = 10.6$) for $V_R/V = 2$, reasonably agreement with the results of Fig. 6(a). It is thus suggested that the behavior of Fig. 6(a) is deviated from the KH theory. Above a certain value of $V_R$, a main branch is shifted to the higher-$k$ region and there appear many sub-branches to form complicated spectral form as shown in Fig. 6(c) for $V_R/V = \sqrt{20}$. The main peak appears at $k_0^{\text{BdG}} \xi = 1.939$ for $V_R/V = \sqrt{20}$; the corresponding wavelength $\lambda_0^{\text{BdG}}/\xi = 3.239$ is again reasonably agreement with the spatial period $\approx 2.9$ of the pattern at $\tilde{t} = 42.5$ seen in Fig. 6(b). Although the initial growth rate of the unstable dynamics is determined by the main peak in the BdG spectrum, the subsequent evolution is affected by the excitations of the unstable sub-branch distributed in a wider range of the wave number, which leads to the multistep growth of the instability and complicated nonlinear dynamics.

Note that the main unstable branch of $\text{Im}[\omega]$ resembles the imaginary part of the CSI dispersion Eq. (21). To this end, we use $g_{12} = 2g$ and $n = n_0(0.5 - \delta)$ at the center of the interface $(y = 0)$, where $\delta$ takes a finite value $0 < \delta < 0.5$ ($\delta \approx 0$ for $\Delta \ll 1$ and $\delta = 0.5$ for $\Delta \to \infty$) extracted from the numerical solution [see the right panel of Fig. 4(a)]. This implies that the instability in the microscopic regime $We > 1$ can be described by the CSI even for the strongly segregating regime. In the theory of the CSI, for $2 \sqrt{2(1 - g_{12}/g)} < V_R/V < 2 \sqrt{2(1 + g_{12}/g)}$ the unstable region is broadly distributed in the wave-number space $(k_\parallel, k_\perp)$, while for

FIG. 5. Typical dynamics of the density difference $\Delta n$ in the case of strongly phase-separated condensate with $\Delta = 1.0$ and the relative velocity (a) $V_R/V = 2$ ($We = 2.58$) and (b) $V_R/V = \sqrt{20}$ ($We = 12.92$). The red (blue) region corresponds to the area where the density $n_1$ ($n_2$) is located. The profile of the phase $\theta_1$ in the region with $\Delta n > 0$, corresponding to (b), is shown in the panels (c). The spatial region of the plot is $-25\xi \leq x, y \leq 25\xi$. 

(a) $\Delta = 1$ $V_R/V = 2$ $We = 2.58$

(b) $\Delta = 1$ $V_R/V = \sqrt{20}$ $We = 12.92$

(c) $\Delta = 1$ $V_R/V = \sqrt{20}$ $We = 12.92$
the CSI in Ref. [34], where the green dotted curve represents Eq. (24) in the theory of CSI crossover [8], where the authors found the continuous change of the \( V_{R} \)-dependence of the unstable range of the wavenumber \( 0 < k < k_{+} \) from the KHI with \( k_{+} \propto V_{R} \), which has been confirmed by the BdG analysis. However, we show here that the theory of CSI is not simply applicable due to a difference in the mechanism of the frictional relaxation of the relative motion between our system and the CSI in uniform systems.

1. Development of the density and the phase

Figure 7 shows the interface dynamics for several values of \( \Delta \) toward the miscible limit \( \Delta \to 0 \), where the corresponding Weber number changes from \( \mathcal{O}(1) \) to \( \mathcal{O}(10^{2}) \); the initial states of the simulations are shown in Fig. 1(a). For \( \Delta = 0.1 \), Fig. 7(a) shows that the nonlinear grows of interface wave leads to the finger patterns and the fingers collapse to emit the quantized vortices at the stage of the shorter finger length than that shown in Fig. 5(a). This can be seen from the result of Fig. 4, where the elongation of the finger pattern is suppressed with decreasing \( \Delta \). Thus, we can say that the instability in this case is fairly in the macroscopic regime similar to that in Sec. IV.

A further decrease in \( \Delta \) qualitatively changes the nonlinear dynamics. In Fig. 7(b) for \( \Delta = 10^{-2} \) with \( We \approx 20 \), the crossover between the obliquely striped density pattern \( S \) and the flutter-finger pattern, where the density modulation develops an array of multiple dipole-like structures consisting of pairs of density dips of each component along the overlapping region, as seen in the panels \( t = 750 \) and 815 of Fig. 7(b). Here, the density dip of the one-component is filled with the density of the other component. These dipoles correspond to the pair of quantized vortices with the same circulation in each component. The array of the dipoles is subsequently collapsed by emitting the density wave into the bulk region, irregular density patterns being eventually appeared. Figure 7(c) for \( \Delta = 10^{-3} \), satisfying \( We \approx 200 \), clearly shows that the central overlapping region undergoes a characteristic modulation that leads to the obliquely striped density patterns \( S \). These density stripes are subsequently collapsed from the central region \( (t = 750) \) and causes the turbulent structures \( (t = 875) \).

Note that the finger pattern of the \( \Psi_{1} \)- (\( \Psi_{2} \))-component in the universal macroscopic regime [Fig. 7(a)] grows to an upper left (lower right) direction, while the orientation of the stripe modulation in this case [Fig. 7(c)] is in the upper right (lower left) direction. The structure in Fig. 7(c) looks effectively to apply ‘friction’ to the bulk flow, whereas the flutter-finger pattern seems to be parrying the flow as grass flutters in the wind. In fact, the CSI causes the frictional relaxation against the relative motion \( \Psi_{1} \)- \( \Psi_{2} \), which occurs more effectively when the two components overlap each other more and more. This is just like the function of ski skins attached to the bottom of nordic skis, designed to let the ski slide forward on snow but not backward by resembling sealskin. Here, the directions of the stripe and the bulk flow correspond
to those of hairs on the sealskin and the motion of the ski, respectively. As seen in the profile of $\theta_1 = \arg \psi_1$ of Fig. 7(d), in the interface layer in one component has a flow so as to penetrate into the other component and against its counterflow (see the panel $\tilde{t} = 650$). To distinguish the pattern formation in the early stage from the flutter-finger pattern we call this stripe pattern as the “sealskin pattern” in the diagram of Fig. 2.

2. *The results of the BdG analysis*

For the CSI in a homogeneous system, the dynamical instability induces formation of a vortex–anti-vortex pair (in 2D) or a vortex ring (in 3D), which cause frictional relaxation of the countersuperflow due to the phase slip through a dissociation (expansion) of the vortex pair (vortex ring) [33, 34]. In contrast, the friction in our case is caused by the penetration of the bulk flow along the oblique density stripe, like the sealskin. This qualitative difference can be clarified through the microscopic
analysis based on the BdG equation.

![Image of a diagram](image-url)

**FIG. 8.** (a) A semi-log plot of $\text{Im}[\omega]$ of the dispersion relations for $V_R/V = \sqrt{3}/2$ and $\Delta = 0.001$. The numerical results obtained by the BdG equation are plotted by red points. The blue dashed curve represents the dispersion relation Eq. (6) of the KHI. Also, we plot $\text{Im}[\omega]$ obtained from the dispersion relation Eq. (24) with $k_{\perp} = 0$ by the green dashed-dotted curve and $k_{\perp} = k_{\perp}^{\text{fit}}$ extracted from the eigenfunction in (b) by the grey dashed-dotted curve, where the fitting function $u_{\text{fit}} = |u_j|^{\epsilon^{\text{fit}}_{u} n(y-y_0)}$ with the parameters $k_{\perp}^{\text{fit}}$ and $y_{0}^{\text{fit}}$ is employed; we here get $k_{\perp} = 0.155$. In the panels (b), we plot the eigenfunctions $(u_1, v_1, u_2, v_2)$ corresponding to the largest value of $\text{Im}[\omega]$ in (a).

Figure 8(a) shows the imaginary part of the BdG excitation spectrum as a function of $k$ for the parameters corresponding to Fig. 7(c). The spectrum exhibits not only a main single branch but also an anomalously large number of unstable branches inside the main branch. From Fig. 8(a), the maximum of $\text{Im}[\omega]$ occurs at $k_\perp = 0.218$, which determines the wave length of the initially growing mode. However, the corresponding eigenfunction along the $y$-direction also has a certain finite wave number, denoted as $k_y$, as shown in Fig. 8(b). From the distribution of the real and imaginary parts of the eigenfunctions, the sign of $k_y$ is negative. Also, note that the eigenfunctions satisfy the antisymmetric relation $\text{Re}[u_1(y)] = \text{Im}[v_2(-y)]$ and $\text{Im}[u_1(y)] = \text{Re}[v_2(-y)]$ in the upper panel of (b) and the similar relation in the lower panel. Thus, the norm of the excitations for each component $N_j \equiv \int dy[|u_j|^2 - |v_j|^2]$ takes opposite sign. Since the Bogoliubov theory predicts that the current induced by the excitation is given by $\hbar k_y N_j$ in the vertical direction, the eigenmode represents the counterpropagating excitations perpendicular to the interface. We find that $N_1 < 0$ and $N_2 > 0$ from Fig. 8(b) and the unstable mode results in the positive (negative) current for $\psi_1 (\psi_2)$-component. These properties realize the excitation mode that induces the encroaching flow into the interface layer.

In Fig. 8(a), we compare the BdG results with the dispersion relations of the KH theory and the CSI in a homogeneous system. The dispersion relation Eq. (6) of the KHI is only coincident with the BdG result asymptotically at $k \rightarrow 0$, which implies the breakdown of the KH theory. The appearance of the instability band may be explained as a signature of the CSI [33, 34], where the CSI appears in some range of $k_\parallel$ as well as the wave number component $k_{\perp}$ perpendicular to the direction of the counterflow. When we plot $\text{Im}[\omega]$ of Eq. (24) with $k_{\perp} = 0$, the unstable range of $k$ agrees with the BdG results, although their magnitudes are quite different. The coincidence of the unstable range has been seen in Ref. [8]. Since the most unstable mode has a finite wave number in the $y$-direction as seen in Fig. 8(b), we also plot Eq. (24) with $k_{\perp} = 0.155$ extracted from the numerical fitting of the eigenfunction in Fig. 8(b). Then, the unstable range becomes narrow, which cannot reproduce the numerical result of the BdG result, and also the magnitude still overestimates the BdG result. Although the local approximation of the homogeneous CSI makes us expect the excitation modes having a momentum antiparallel to the initial condensate velocity leading the frictional relaxation, we also find that the excitation has nontrivially the momentum perpendicular to the interface to realize the obliquely encroaching flow. By using the BdG result, in fact, the oblique direction of the encroaching flow is explained by the BdG result as $\theta = \tan^{-1}(k_\parallel/k_\perp) = \tan^{-1}1.406 = 0.3\pi$, reasonably agreement with the GP result [panel $t = 650$ of Fig. 7(c)]. These facts imply that the shear flow instability in the weakly segregating regime is qualitatively different from the homogeneous CSI; the difference could come from the fact that the density profile has a spatial gradient by the external potential in the former and Ref. [8] but not in the latter. To distinguish the homogeneous CSI with the CSI that causes sealskin pattern with a shear flow, we call the latter as a sheared CSI.

**VI. CONCLUSION AND DISCUSSION**

In this work, we have studied detailed nonlinear dynamics of an interface in segregating two-component BECs with a shear flow by varying the intercomponent coupling strength and the relative velocity of the two
components. The nonlinear dynamics induced by the KHI is characterized by the Weber number $We$, adopted to the segregated binary superfluids with the interface thickness $l_d$. The main result is summarized in the phase diagram of Fig. 2. For $We \lesssim 1$, the dynamics is characterized by a universal macroscopic behavior, which is relevant to the KHI in classical fluid dynamics. The dynamics induced by the KHI exhibit the formation of the flutter-finger pattern and its subsequent collapse by emitting the coreless quantized vortices at the tips of the fingers. These dynamical properties are characterized by the single parameter $We$: we find that the growing finger length divided by $l_d$ can be scaled as $We^{-1}$. For $We \gtrsim 1$, however, the nonlinear dynamics is caused by the microscopic mechanism beyond the conventional KHI and cannot be classified only by $We$. For $\Delta > 1$, a strongly segregating regime, the small amplitude interface wave forms a transient zipper pattern. Since the vorticity per a single wavelength is not enough to evolve into the vortex, the transition to the vortex turbulence configuration needs multiple steps of the instability growth. In the weakly segregating regime $\Delta \ll 1$ with a large overlapping region, the instability gives rise to the frictional relaxation by forming a so-called sealskin pattern. We suggest the underlying mechanism of the sealskin pattern formation as the sheared CSI, which is qualitatively different from the homogeneous CSI.

Finally, let us discuss the crossover of the dynamical regime between the zipper and sealskin formation. As discussed above, the mechanism behind the sealskin pattern formation is partly the CSI around the overlapping region. Then, it is natural to compare the length scale of the overlapping region with the perturbation length. Since the overlapping region vanishes when the interface thickness becomes thin like $\sim \xi$, the overlapping length can be defined as $l_{\text{overlap}} = l_d - \xi$. The perturbation length can be estimated by the wave number that gives the maximum of $\text{Im}[\omega]$, which is approximately given by the dispersion relation of the homogeneous CSI. Since the characteristic wave number of the CSI is given by $k_{\text{CSI}} = mV_R/h$ for a large $V_R$ (see Appendix A), the curve $k_{\text{CSI}} l_{\text{overlap}} \sim 1$ could gives a rough boundary between the zipper and sealskin region in the phase diagram of Fig. 2. The curve has a similar $\Delta$-dependence with $We$ for weakly segregating limit but exhibits a divergent behavior around $\Delta \sim 1$, where $l_d \sim \xi$. We confirm through the GP simulations that the period and the amplitude of the sealskin pattern are decreased when the parameters are changed toward this curve from the $\Delta = 0.001$ and $V_R/V = \sqrt{0.45}$ (the parameters of Fig. 2(c)) and the zipper patterns begin to appear around the parameters on the boundary curve.

Since it is difficult to treat the extreme parameter values with $We \ll 1$ or $We \gg 1$ in numerical simulations, it should be noted that Fig. 2 represents the dynamical phase diagram of the crossover transition between the universal macroscopic regime and the microscopic regime. In outlook, more details in the latter regime are remained to be studied. For example, we have to consider the finite size effect of the CSI to account the full behavior of the sheared CSI, seen in Figs. 6 and 8. Also, when the bulk velocity enters a supersonic regime, it is interesting to clarify the relation of the shock wave formation to the interface instability. Furthermore, it is necessary to consider the simulations under the realistic experimental setup. These issues are merit for further studies and will be reported elsewhere.

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### Appendix A: Analogy with electrostatics in strongly-segregated condensates with a shear flow

In this Appendix, we describe an effective description of the strongly segregated condensates with a shear flow in Sec. IV A by introducing the analogy with the electrostatics. This analogy is useful to understand qualitatively the vorticity distribution when the vortex sheet at the interface is deformed to the finger pattern.

In our setup, the first (second) component exists in the $y < 0$ ($y > 0$) region and the flat interface exists at $y = 0$ as shown in Fig. 9(a). The velocity of the each component is written as $v_1 = (V_R/2, 0, 0)$ for $y < 0$ and $v_2 = (-V_R/2, 0, 0)$ for $y > 0$. The mass current velocity $\mathbf{E}$ is characterized by the Weber number $We$ and the relative velocity $V_R$.
\[ \mathbf{v} = \frac{(\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2)}{(\rho_1 + \rho_2)} \] is given as
\[ \mathbf{v} = \begin{cases} (+V_R/2, 0, 0) & \text{for } y < 0 \\ (-V_R/2, 0, 0) & \text{for } y > 0 \end{cases} \quad \text{(A1)} \]

If the density is stationary, we can apply approximately the incompressible condition \( \nabla \cdot \mathbf{v} = 0 \), thereby defining the stream function \( \psi \) satisfying
\[ v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}. \quad \text{(A2)} \]

Then, the vorticity \( \omega = \nabla \times \mathbf{v} = (0, 0, \omega) \) can be expressed by the stream function as
\[ \omega = -\nabla^2 \psi. \quad \text{(A3)} \]

Since \( \omega = 0 \) in the region far from the interface, the stream function there obeys Poisson equation
\[ \nabla^2 \psi = 0. \quad \text{(A4)} \]

Since \( \mathbf{v} \) is given by Eq. (A1) far from the interface, the solution of Eq. (A4) is written as
\[ \psi = \begin{cases} \frac{V_R}{2} y & \text{for } y < 0 \\ -\frac{V_R}{2} y & \text{for } y > 0 \end{cases} \quad \text{(A5)} \]

where we set \( \psi = 0 \) at \( y = 0 \).

According to the electrostatic analogy, the stream function \( \psi \) and the vorticity \( \omega \) are related with the electrostatic potential and the charge density, respectively. The analog electric field \( \mathbf{E} \) is given by
\[ \mathbf{E} = -\nabla \psi = \begin{cases} (0, -V_R/2, 0) & \text{for } y < 0 \\ (0, +V_R/2, 0) & \text{for } y > 0 \end{cases} \quad \text{(A6)} \]

Thus, the situation is related with the electric field created by the uniformly distributed positive charge density \( \rho_e \) along the \( y = 0 \) line, as shown in Fig. 9(b). By using the Gauss’s law, we get \( E = \rho_e/\epsilon_0 \), where the dielectric constant is taken to be unity. Since the circulation along the sheet per unit length has the correspondence \( \rho_v \leftrightarrow \rho_e \) we obtain the relation
\[ \rho_v = V_R, \quad \text{(A7)} \]

which is equivalent to Eq. (19).

The electrical flux lines exhibit a similar behavior with the branch cuts of the phase of the condensate wave function, as seen in Fig. 3. When the interface is deformed to the finger pattern, the vorticity, namely the positive charge, should be accumulated around the tips of the fingers, as seen in Fig. 3(c). This is because the electric field far from the interface is fixed by Eq. (A6), which gives the boundary condition to determine the charge distribution along the winding interface. As seen in the finger formation of the KHI dynamics, the strong deformation of the plane provides a cancelation of the electric field inside the domain of the fingers. Then, the charge density is more concentrated around the tips of the fingers. This charge distribution is actually observed in Fig. 3(b) and (c).

We confirm that the relation Eq. (A7) holds exactly by using the numerical solution of Eq. (2). In particular, for \( \Delta = 2 \) we can use the exact solution of Eq. (2) [13] to confirm this relation. The density profile of the strongly-segregating BEC with \( \Delta = 2 \) is given by
\[ \begin{aligned} |\phi_1| &= \frac{\sqrt{n_0}}{2} \left[ 1 - \tanh \left( \frac{y}{\sqrt{2}} \right) \right], \\
|\phi_2| &= \frac{\sqrt{n_0}}{2} \left[ 1 + \tanh \left( \frac{y}{\sqrt{2}} \right) \right] \quad \text{(A8)} \end{aligned} \]

According to the definition of \( \mathbf{v} \), we have
\[ v_x(y) = -\frac{V_R \tanh(y/\sqrt{2})}{1 + \tanh^2(y/\sqrt{2})} \quad \text{(A9)} \]

and the vorticity is
\[ \omega(x, y) = \frac{V_R}{\sqrt{2}} \sech^2(\sqrt{2}y). \quad \text{(A10)} \]

Thus, the linear density of the vorticity is given by
\[ \rho_v = \int dy \omega(x, y) = V_R, \quad \text{(A11)} \]

which is consistent with the above discussion.

**Appendix B: The dispersion relation of the miscible binary BECs with counter-superflow**

We here describe briefly the derivation of the dispersion relation of the miscible two-component BECs with counterflow and show the dynamical instability known as the CSI [32-34]. The dispersion relation can be derived from the BdG analysis for a system of a uniform binary BECs with counter-superflow [8].

We consider a small excitation \( \delta \Psi_j \) above a uniform state with a velocity \( v_j \) as
\[ \Psi_j = (\sqrt{n_j} + \delta \Psi_j) e^{-i j \omega t / \hbar + im_j v_j t / \hbar}, \quad \text{(B1)} \]

where \( \mu_j = g_j n_j + g_{jj} n_j^2 + m_j v_j^2 / 2 \) and \( j, \bar{j} = 1, 2 \) \( (j \neq \bar{j}) \). Although the miscibility condition \( g_1 g_2 > g_{12}^2 \) is generally supposed when the uniform solution \( \sqrt{n_j} \) is employed, the dispersion relation is irrelevant to such a condition, which we do not assume here; the uniform solution itself is of course unstable for \( g_1 g_2 < g_{12}^2 \).

Substituting Eq. (B1) into Eq. (1) and taking the first order of \( \delta \Psi_j \), we obtain \( (j \neq \bar{j}) \)
\[ i \hbar \frac{\partial \delta \Psi_j}{\partial t} = \left[ -\frac{\hbar^2}{2m_j} \left( \nabla + i m_j v_j / \hbar \right)^2 - \mu_j + 2 g_{j\bar{j}} n_{\bar{j}} \\
+ g_{j\bar{j}} n_{\bar{j}} \right] \delta \Psi_j + g_{j\bar{j}} n_{\bar{j}} \delta \Psi_{\bar{j}}^* \\
+ g_{j\bar{j}} \sqrt{n_j n_{\bar{j}}} \left( \delta \Psi_j + \delta \Psi_{\bar{j}} \right). \quad \text{(B2)} \]
We expand the small excitation by the plane wave as
\[ \delta \Psi_j = U_j \mathbf{K} e^{i \mathbf{K} \cdot \mathbf{r} - i \omega t} - V_j e^{i \mathbf{K} \cdot \mathbf{r} + i \omega t} \] (B3)

with the wave vector \( \mathbf{K} \) and the complex frequency \( \omega \), and substitute it into Eq. (B2), which yield (\( j \neq j' \))

\[
\begin{align*}
\hbar^2 & \left( K^2 + \frac{2m}{\hbar} \mathbf{K} \cdot \mathbf{v}_j \right) + g_j n_j \right] U_j \mathbf{K} - g_j n_j V_j \mathbf{K} + g_j \sqrt{n_j n_j} \left( U_j \mathbf{K} - V_j \mathbf{K} \right) = \hbar \omega U_j \mathbf{K}, \\
\hbar^2 & \left( K^2 - \frac{2m}{\hbar} \mathbf{K} \cdot \mathbf{v}_j \right) + g_j n_j \right] V_j \mathbf{K} - g_j n_j U_j \mathbf{K} - g_j \sqrt{n_j n_j} \left( U_j \mathbf{K} - V_j \mathbf{K} \right) = -\hbar \omega V_j \mathbf{K}.
\end{align*}
\] (B4a, b)

Diagonalizing the eigenvalue equation (B4), we obtain the Bogoliubov excitation spectrum. Although the forms of the eigenvalues are generally complicated, the simplified form can be obtained by assuming \( m_1 = m_2 = m, g_1 = g_2 = g \), and \( n_1 = n_2 = n \). Then, the eigenvalue of Eq. (B4) becomes

\[ \hbar \omega = \frac{\hbar}{2} \left( \mathbf{v}_1 + \mathbf{v}_2 \right) \cdot \mathbf{K} \pm \sqrt{\varepsilon_0^2 + \varepsilon_r^2 - 2 \varepsilon_0 \varepsilon_r g n + 2 \varepsilon_0 \varepsilon_r^2 g n + \varepsilon_r^4 g_1^2 n^2} \] (B5)

with \( \varepsilon_0 = \hbar^2 K^2 / (2m) \), \( \varepsilon_r = \hbar k \eta V_R / 2 \), and the relative velocity \( V_R = \mathbf{v}_1 - \mathbf{v}_2 \). Here, the wave number \( \mathbf{K} \) is decomposed to the components of the parallel and perpendicular directions as \( \mathbf{K} = (k_{||}, k_{\perp}) \). The first term is neglected by assuming the situation of the vanishing center-of-mass velocity, namely \( \mathbf{v}_1 + \mathbf{v}_2 = 0 \).

\[ (k_{||}, k_{\perp}) \] plane for several values of \( V_R \), representing the wave-number region associated with the dynamical instability. The unstable region appears inside the semicircle in the positive \((k_{||}, k_{\perp})\) plane. The imaginary part of \( \hbar \omega \) is finite when the expression in the larger square root in Eq. (B5) with the negative sign becomes negative. The condition of the CSI is thus given by

\[ \varepsilon_r^2 - 2(g + g_{12}) n \varepsilon_0 < \varepsilon_0^2 < \varepsilon_r^2 - 2(g - g_{12}) n \varepsilon_0 \] (B6)

At the miscible-immiscible boundary \( g = g_{12} \), the right inequality of Eq. (B6) reduces to \( (k_{||} - m V_R / 2 \hbar)^2 + k_{\perp}^2 < (m V_R / 2 \hbar)^2 \). As a result, the wave number of the unstable modes are characterized by \( k_{||} \lesssim m V_R / \hbar \) and \( k_{\perp} \lesssim m V_R / 2h \). With increasing \( V_R \), the unstable modes are distributed in the higher-\( K \) region with a crescent shape, where the inner boundary of the crescent is determined by the left inequality of Eq. (B6).

![Figure 10](image)

**FIG. 10.** The imaginary part of the eigenvalue of Eq. (B5) in the \( \mathbf{K} \)-space for \( g_{12} = g \), \( n = n_0 / 2 \), and several values of \( V_R \): (a) \( V_R = \sqrt{0.4} \), (b) \( V_R = \sqrt{5} \), (c) \( V_R = \sqrt{10} \), and (d) \( V_R = \sqrt{20} \). The plot range is determined as \( 0 < k_{||} \xi < V_R / (2V) \) and \( 0 < k_{\perp} \xi < V_R / (4V) \). The eigenvalue is scaled by \( \mu = g n_0 \).

Figure 10 shows the imaginary part of Eq. (B5) in the table.
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