ASYMPTOTICALLY HYPERBOLIC METRICS ON THE UNIT BALL WITH HORIZONS

YUGUANG SHI\(^1\) AND LUEN-FAI TAM\(^2\)

Abstract. In this paper, we construct a family of asymptotically hyperbolic manifolds with horizons and with scalar curvature equal to \(-6\). The manifolds we constructed can be arbitrary close to anti-de Sitter-Schwarzschild manifolds at infinity. Hence, the mass of our manifolds can be very large or very small. The main arguments we used in this paper is gluing methods which was used in [12].

1. Introduction

In the past few years, there are many works on the construction of asymptotically flat (AF) and scalar flat manifolds which contain minimal spheres. See [4], [12], [8], and [14] for examples, and for existence of many blackholes, please see [5]. From the point of view of general relativity, these are examples of globally regular and asymptotically flat initial data for the Einstein vacuum equations containing a trapped surfaces. According to [13], if the topology of an AF manifold is nontrivial, then this manifold always contains an outer most minimal sphere. The examples in [4], [8], [12], [14], have the interesting property that the manifolds in those examples are all diffeomorphic to \(\mathbb{R}^3\).

Another natural class of manifolds that are of interest in general relativity consists of asymptotically hyperbolic (AH) manifolds (see Definition 1.1). Such manifolds arise when considering solutions to the Einstein fields equations with a negative cosmological constant, or when considering “hyperboloidal hypersurfaces” in space-times which are asymptotically flat in isotropic directions. Therefore, it seems to be interesting to find 3-dimensional AH manifolds with \(R = -6\) with trivial topology which contain horizons. In the asymptotically hyperbolic context, horizons refer not only to boundaries of domains which are minimal but also to boundaries satisfying \(H = \pm 2\). Here \(H\) is the mean curvature of the boundaries with respect to the outward unit normal vectors. More precisely, we are interested in following:

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To find an AH manifold which is diffeomorphic to an open 3-ball in \( \mathbb{R}^3 \) with scalar curvature \( R = -6 \) which contains spheres with \( H = 0 \) or \( \pm 2 \).

In the AF context, in \([12]\), Miao constructs an AF and scalar flat manifold with topology \( \mathbb{R}^3 \) and containing a horizon (see also \([4]\)). The main arguments in \([12]\) is to glue \( S^3 \) with Schwarzschild manifold and then conformally deform the metric to a scalar flat AF metric so that it still contains a minimal sphere. We will use similar methods to study our problem. More precisely, we will glue the anti-de Sitter-Schwarzschild space (see Section 1 for details) with part of the unit ball and obtain a complete metric with scalar curvature \( R \geq -6 \), which contains topological spheres with \( H = 0 \) or \( \pm 2 \), so that the metric is conformal to the hyperbolic metric on the unit ball in \( \mathbb{R}^3 \). Moreover, outside a compact set, the manifold is part of the anti-de Sitter-Schwarzschild space. Then we will deform the metric to obtain an AH with \( R = -6 \) which contains spheres with mean curvature \( 0, \pm 2 \). We can show that the mass of our manifolds (in the sense of \([16]\)) can be close enough to that of the anti-de Sitter-Schwarzschild space provided that the perturbation is small enough. Hence the mass can be very large or very small.

The outline of the paper is as follows. In Section 1 we discuss some basic facts of anti-de Sitter-Schwarzschild space. Most of them are well known, but we cannot find the details in literatures. In Section 2, we will construct AH metric on the unit ball in \( \mathbb{R}^3 \) which contains horizons. The metrics are rotationally symmetric and are conformal to the hyperbolic metric, with scalar curvature \( R \geq -6 \) so that \( R = -6 \) near infinity. In Section 3, we will do the deformation to obtain new AH metrics on the ball with scalar curvature equal to \( -6 \) which contain horizons. We also discuss the mass of these AH metrics in this section.

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### 2. The anti-de Sitter-Schwarzschild metric

In order to construct metrics which are asymptotically hyperbolic (AH) and contains horizons, we will make use of the anti-de Sitter-Schwarzschild metric. Therefore in this section, we will discuss this metric in details. Let us first recall the definition of asymptotically hyperbolic manifolds and its mass. We will use the definitions in \([16]\), see also \([6]\). We are only interested in the case that the manifold has dimension three.
Definition 2.1. A complete noncompact Riemannian manifold \((X^3, g)\) is said to be asymptotically hyperbolic if there is a compact manifold \((\overline{X}, \overline{g})\) with boundary \(\partial X\) and a smooth function \(t\) on \(\overline{X}\) such that the following are true:

(i) \(X = \overline{X} \setminus \partial X\).
(ii) \(t = 0\) on \(\partial X\), and \(t > 0\) on \(X\).
(iii) \(\overline{g} = t^2 g\) extends to be \(C^3\) up to the boundary.
(iv) \(|dt|_{\overline{g}} = 1\) at \(\partial X\).
(v) Each component \(\Sigma\) of \(\partial X\) is the standard two sphere \((S^2, g_0)\) and there is a collar neighborhood of \(\Sigma\) where

\[ g = \sinh^{-2} t(dt^2 + g_t) \]

with

\[ g_t = g_0 + \frac{t^3}{3} h + O(t^4) \]

where \(h\) is a \(C^2\) symmetric two tensor on \(S^2\).

With the above notation, let \((X, g)\) be an AH manifold with scalar curvature \(R \geq -6\), then the mass of an end of \(X\) corresponding to a boundary component \(\Sigma\) of \(\partial X\) is defined as

\[ M = \frac{1}{16\pi} \left[ \left( \int_{S^2} \text{trace}_{g_0}(h) dV_{g_0} \right)^2 - \left( \int_{S^2} \text{trace}_{g_0}(h)(x) x dV_{g_0} \right)^2 \right]^{\frac{1}{2}} \]

where \(x\) is the standard coordinates of a point on \(S^2\) in \(\mathbb{R}^3\). This is well-defined by [10].

Next we want to describe the anti-de Sitter-Schwarzschild metric which is obtained by gluing two copies of manifolds with boundary with metric

\[ ds^2 = \frac{dr^2}{1 + r^2 - \frac{M}{r}} + r^2 d\sigma^2 \]

with \(M > 0\) defined on \((a(M), \infty) \times SS^2\) where \(a(M) > 0\) is the unique root of \(1 + r^2 - \frac{M}{r} = 0\) and \(d\sigma^2\) is the standard metric on the standard sphere \(SS^2\). The construction and some properties of the metric are well-known. But for the sake of completeness and for reference later, we will give details of the metric and its properties.

Let

\[ h(r) = \int_r^\infty \frac{1}{\sqrt{t(t + t^3 - M)}} dt \]

for \(r > a(M)\). Let \(\rho(r)\) be the function defined by the relation

\[ e^{\rho} = \frac{1 + e^{-h}}{1 - e^{-h}} \]
that is
\[ \rho(r) = \log \left( \coth \frac{h(r)}{2} \right). \]

Then \( \rho : (a(M), \infty) \to (\rho(M), \infty) \) is a smooth increasing function in \( r \) with \( \rho(M) = \rho(a(M)) > 0 \). Let \( \phi > 0 \) be the smooth function in \( \rho \) on \( (\rho(M), \infty) \) by

\[ \phi^4(\rho) = \frac{r^2(\rho)}{\sinh^2 \rho}. \]

We have

\[ ds^2 = \phi^4(d\rho^2 + \sinh^2 \rho d\sigma^2). \]

Here \( d\rho^2 + \sinh^2 \rho d\sigma^2 \) is the standard metric on the hyperbolic space \( \mathbb{H}^3 \). Observe that \( \phi \) is continuous up to \( \rho(M) \) and is positive at \( \rho(M) \).

Also,

\[ \phi_\rho = -\frac{1}{2} \phi^{-1}(\sinh h + rh_r \cosh h) \frac{\sinh h}{h_r} \]

and

\[ \phi_{\rho\rho} = -\frac{1}{4} \phi^{-3} \left[ (\sinh h + rh_r \cosh h) \frac{\sinh h}{h_r} \right]^2 + \frac{1}{2} \phi^{-1} \left[ 3 \sinh h \cosh h - \frac{\sinh^2 h h_{rr}}{h_r^2} + r(h^2 h + \cosh^2 h)h_r \right] \frac{\sinh h}{h_r} \]

From these, it is easy to see that \( \phi \) as a function of \( \rho \) is \( C^2 \) up to \( \rho(M) \).

The scalar curvature of \( ds^2 \) is:

\[ R = \phi^{-5}(-6\phi - 8\Delta_{\mathbb{H}^3} \phi) \]

\[ = \phi^{-5}(-6\phi - 8(\phi_{\rho\rho} + 2 \coth \rho \phi_\rho)) \]

\[ = 2 \left[ h_r^{-2} r^{-4} + r^{-2} + 2h_r r^{-3} r^{-3} \right] \]

\[ = -6. \]

Let us use the ball model for \( \mathbb{H}^3 \). Let

\[ b(M) = \frac{e^{\rho(M)} - 1}{e^{\rho(M)} + 1}. \]

Then

\[ ds^2 = \frac{4\psi^4(\tau)}{(1 - \tau^2)^2}(d\tau^2 + \tau^2 d\sigma^2), \]
on the annulus $b(M) \leq |x| < 1$ in $\mathbb{R}^3$, where $\tau = |x|$ and 

$$\psi(\tau) = \phi \left( \log \left( \frac{1 + \tau}{1 - \tau} \right) \right).$$

Now we use the transformation $x \rightarrow b^2(M)x/|x|^2$ to transform the annulus $b(M) \leq |x| < 1$ to $b^2(M) < |x| \leq b(M)$, such that $|x| = 1$ is mapped to $|x| = b^2(M)$. Pull back the metric $ds^2$ to $b^2(M) < |x| < b(M)$, we extend the metric $ds^2$ to $b^2 < |x| < 1$ and is still denoted by $ds^2$ such that the metric is of the form

$$ds^2 = f^4(\tau)ds^2_{\mathbb{H}^3},$$

where

$$f(\tau) = \left\{ \begin{array}{ll}
\psi(\tau) & \text{for } b \leq |x| < 1; \\
\psi(b^2(\tau)) \left( \frac{1 - \tau^2}{\tau^2 - b^4(M)} \right)^{\frac{3}{2}} & \text{for } b^2 < |x| \leq b.
\end{array} \right.$$
Here $B(\rho_0(M))$ is the geodesic ball with center at $\rho = 0$. The metric $g_{\text{AdS-Sch,}M}$ is called the anti-de Sitter-Schwarzschild metric (with mass $M$).

With the above notation, we have:

**Proposition 2.2.** For each $M > 0$, the anti-de Sitter-Schwarzschild metric $g_{\text{AdS-Sch,}M} = \phi_M^4(d\rho^2 + \sinh^2 \rho d\sigma^2)$ is complete and is defined on $\mathbb{H}^3 \setminus B(\rho_0(M))$. Moreover:

(i) $\phi > 1$, $\lim_{\rho \to \infty} \phi = 1$ and $\lim_{\rho \to \rho_0(M)} \phi = \infty$.

(ii) The manifold is asymptotically hyperbolic with constant scalar curvature -6.

(iii) Denote $\phi_M$ by $\phi$, then $\phi_\rho < 0$ and

$$\text{(2.9)} \quad (\sinh^2 \rho \cdot \phi_\rho)_\rho = \frac{3}{4} \sinh^2 \rho \cdot \phi(\phi^4 - 1).$$

(iv) There exist unique $\rho_2 > \rho_1 > \rho'_2 > \rho_0(M)$ such that the level surface of $\rho = \rho_1$ is minimal, the mean curvature of $\rho = \rho_2$ is 2 and $\rho = \rho'_2$ is -2 with respect to the unit normal in the direction $\frac{\partial}{\partial \rho}$.

**Proof.** (i) The results are immediate from the definition of $\phi_M$.

(ii) This follows from (2.5) and [16].

(iii) $\phi$ satisfies (2.9) because the scalar curvature is -6. From the equation, we have $\sinh^2 \rho \phi_\rho$ is strictly increasing. Suppose $\phi_\rho \geq 0$ for some $\rho^*$, then $\phi_\rho > 0$ for all $\rho > \rho^*$. Since $\phi > 1$, this contradicts (i).

(iv) Denote $\phi_M$ simply by $\phi$. The mean curvature of the level surface $\rho =$constant for $\rho > \rho(M)$ is

$$H = \frac{1}{\phi^2} \left( 2 \cosh \rho \frac{\sinh \rho}{\sinh \rho} + \frac{4}{\phi} \phi_\rho \right)$$

$$= \frac{1}{\phi^2} \left( 2 \cosh h - 2\phi^{-2}(\sinh h + \rho h_c \cosh h) \frac{\sinh h}{h_c} \right)$$

$$= -2\phi^{-4} \sinh^2 h h_c^{-1}$$

$$= -2r^{-2} h_c^{-1}$$

$$= 2 \left( 1 + r^{-2} - Mr^{-3} \right)^{\frac{1}{2}}.$$ 

From this the results follow. 

Next we will discuss the behaviors of the metrics $g_{\text{AdS-Sch,}M}$ as $M$ changes. Before we do this, we need the following lemma which may be well-known:
Lemma 2.3. Let \((N, g)\) be a complete noncompact Riemannian manifold. Suppose \(u_1 \geq 1\) and \(u_2 \geq 1\) are such that

\[
\Delta u_1 + \frac{3}{4} u_1 (1 - u_1^4) = \Delta u_2 + \frac{3}{4} u_2 (1 - u_2^4)
\]

on \(N \setminus B(\rho^*)\) where \(\Delta\) is Laplacian of \(N\) and \(B(\rho^*)\) is the geodesic ball of radius \(\rho^*\) with center at a fixed point. Suppose \(u_1 \geq u_2\) at \(\partial B(\rho^*)\) and suppose \(\lim_{x \to \infty} (u_1(x) - u_2(x)) = 0\), then \(u_1 \geq u_2\) in \(N \setminus B(\rho^*)\). If in addition that \(N = \mathbb{H}^3\), then

\[
|u_1 - u_2|(x) \leq C \left(\sup_{\partial B(\rho^*)} |u_1 - u_2|\right) e^{-3\rho(x)}
\]

outside \(B(\rho^*)\) where \(\rho\) is the distance function from a fixed point and \(C\) is a constant depending only on \(\rho^*\). In case \(\rho^* = 0\), then \(u_1 = u_2\).

Proof. Let us prove the last statement and the first assertion can be proved similarly. Let \(\eta = u_1 - u_2\), then

\[
\Delta \eta = 3\eta + \frac{3}{4} \eta(-5 + G)
\]

where \(G = u_1^4 + u_1^3 u_2 + u_1^2 u_2^2 + u_1 u_2^3 + u_2^4 \geq 5\). Let \(\xi(\rho) = e^{-2\rho} \sinh^{-1} \rho\), then it is easy to check that

\[
\Delta \xi = 3\xi.
\]

Let

\[
A = \sup_{\partial B(\rho^*)} \frac{|u_1 - u_2|}{\xi(\rho^*)}.
\]

Then by maximum principle, we can conclude that

\[
\eta \leq A \xi
\]

outside \(B(\rho^*)\). Similarly, one can prove that \(-\eta \leq A \xi\). From this the second part of the lemma is proved. \(\square\)

Now we are ready to discuss the behaviors of the metrics \(g_{AdS-Sch,M}\). Let \(a(M), \rho(M)\) and \(b(M)\) be as before. The metric \(g_{AdS-Sch,M}\) is of the form \(\phi_\tau^4 ds_{\mathbb{H}^3}^2\) which is defined and is complete on \(\mathbb{H}^3\).\(M < \tau < 1\) in the ball model of \(\mathbb{H}^3\).

Proposition 2.4. With the above notation, we have the following:

(i) \(a(M), \rho(M), b(M)\) are continuous monotonic increasing functions of \(M\).
(ii)

\[
\lim_{M \to 0} \frac{a(M)}{M} = 1; \quad \lim_{M \to 0} \rho(M) = \lim_{M \to 0} b(M) = 0.
\]

(2.12)

\[
\lim_{M \to \infty} \frac{a}{M^3} = 1; \quad \lim_{M \to \infty} \rho(M) = \infty; \quad \lim_{M \to \infty} b = 1.
\]

(2.13)

(iii) \( M_1 > M_2 > 0 \) if and only if \( \varphi_{M_1} > \varphi_{M_2} \) on \( b^2(M_1) < \tau < 1 \).

(iv) For each fixed \( 0 < \tau < 1 \), \( \varphi_M(\tau) \) is a continuous function of \( M \) whenever it is defined and \( \lim_{M \to 0} \varphi_M(\tau) = 1 \).

**Proof.** (i) It is easy to see that \( M_1 > M_2 \) implies \( a(M_1) > a(M_2) \) and \( a(M) \) is continuous in \( M \). Next we want to prove that \( h(a(M_1)) < h(a(M_2)) \). Given \( M > 0 \), let us denote \( a = a(M) \) for simplicity. Then \( t + t^3 - M = (t - a)(t^2 + at + a^2) \) by direct computation. Hence

\[
h(a(M)) = \int_a^\infty \frac{1}{\sqrt{t(t - a)(t^2 + at + a^2)}} dt
\]

(2.14)

\[
= \int_0^\infty \frac{1}{\sqrt{t(t + a)(t^2 + 3at + 1 + 3a^2)}} dt.
\]

From this it is easy to see that \( h(a(M_1)) < h(a(M_2)) \) if \( M_1 > M_2 \). Hence \( \rho(M_1) > \rho(M_2) \) and \( b(M_1) > b(M_2) \). From (2.14) it is easy to see that \( h(a(M)) \) is a continuous function of \( a(M) \) and hence is continuous in \( M \). So \( \rho(M) \) and \( b(M) \) are continuous in \( M \). This proves (i).

(ii) Given \( M > 0 \) denote \( a(M) \) simply by \( a \). Then \( a + a^3 - M = 0 \) and so \( a < M \) and \( a \to 0 \) as \( M \to 0 \).

\[
M > a = M - a^3 > M - M^3.
\]

From this, we can conclude that \( a/M \to 1 \) as \( M \to 0 \). By (2.14), we have \( h(a(M)) \to \infty \) if \( M \to 0 \) (and so \( a(M) \to 0 \)). Hence (2.12) is true.

It is easy to see that if \( M \to \infty \) then \( a = a(M) \to \infty \) and \( a^3 < M \). On the other hand, let \( 1 > \delta > 0 \), then \( a^3 \geq M - \delta a^3 \) provided \( M \) is large enough. From this, (2.13) follows.

(iii) Suppose \( M_1 > M_2 \), then \( b^2(M_1) > b^2(M_2) \). Hence \( \phi_{M_2} \) is bounded on \( \tau = b^2(M_1) \) and \( \phi_{M_1} = \infty \) at \( \tau = b^2(M_1) \). Since both \( \phi_{M_1} > 1 \) and \( \phi_{M_2} > 1 \) satisfies the equation: \( \Delta u + \frac{3}{2} u(1 - u^4) = 0 \) outside the geodesic ball in \( \mathbb{H}^3 \) corresponding to \( |x| < b^2(M_1) \) in the ball model, and since \( \phi_{M_1} \phi_{M_2} \to 1 \) as \( \tau \to 1 \), (iii) follows from Lemma 2.3.

(iv) Let \( \tau \) be fixed. For any \( M_0 \) such that \( \tau > b^2(M_0) \) then \( \tau > b^2(M) \) provided \( M \) is close enough to \( M_0 \). By the construction of \( \phi_M \), it is sufficient to prove that case that \( \tau > b(M_0) \). By the construction, it is
sufficient to prove the following: if \( \rho > 0 \) is fixed such that \( \rho > \rho(M_0) \),
then \( \phi_M(\rho) \to \phi_{M_0}(\rho) \) as \( M \to M_0 \). Now by (2.1),
\[
\phi_M^4(\rho) = \frac{r^2}{\sinh^2 \rho},
\]
where \( r \) and \( \rho \) is related by
\[
\sinh h(r) = \frac{1}{\sinh \rho}
\]
with
\[
h(r) = \int_r^\infty \frac{1}{\sqrt{t(t+t^3-M)}} \, dt.
\]
From these it is easy to see the result follows.

To prove the second assertion in (iv), it is sufficient to prove that for fixed \( \rho \), \( r \sinh h(r) \to 1 \) as \( M \to 0 \), where \( r \) and \( h(r) \) are given by
\[
\sinh h(r) = \frac{1}{\sinh \rho}
\]
with
\[
h(r) = \int_r^\infty \frac{dt}{\sqrt{t(t+t^3-M)}}.
\]
Then as \( M \to 0 \), \( r \to r_0 \) such that \( \sinh h(r_0) = 1/ \sinh \rho \) and
\[
h(r_0) = \int_{r_0}^\infty \frac{dt}{\sqrt{t(t+t^3)}}.
\]
Hence \( \sinh h(r_0) = 1/r_0 \) and so \( r_0 = \sinh \rho \). From (2.1), the result follows.

As an application, we have the following uniqueness result:

**Corollary 2.5.** Suppose \( g = \phi^4 ds_{H^3}^2 \) is a conformal metric defined on \( H^3 \setminus B_\rho \) for some \( \rho > 0 \) such that the scalar curvature is \(-6\). Suppose \( \lim_{x \to \infty} \phi(x) = 1 \), \( \phi > 1 \) and \( \phi = \text{constant on } \partial B_\rho \). Then \( g = g_{\text{AdS-Sch},M} \) on \( H^3 \setminus B_\rho \) for some \( M > 0 \).

**Proof.** The corollary follows from Lemma 2.3, Propositions 2.2 and 2.4.

### 3. Conformal AH metrics on the unit ball

In this section, we will construct asymptotically hyperbolic (AH) metrics on the unit ball in \( \mathbb{R}^3 \) which contains horizons and which is conformal to the hyperbolic metric. Moreover, the scalar curvature \( R \) satisfies \( R \geq -6 \) and the manifold is a part of the anti-de Sitter-Schwarzschild near infinity.

Let \( M > 0 \) and let \( g_{\text{AdS-Sch},M} = \phi_M^4 ds_{H^3}^2 \) be the anti-de Sitter-Schwarzschild metric defined in Proposition 2.2. Let \( \rho_2 > \rho_1 > \rho'_2 > \rho'_1 > \rho_3 > \cdots \).
\[ \rho_0(M) > 0 \] be as in the proposition. First we want to construct a \( C^{2,1} \) metric with the properties mentioned above such that it is anti-de Sitter-Schwarzschild outside \( B(\tau_2) \) for some \( \rho'_2 > \tau_2 > \rho_0(M) \), where \( B(\tau_2) \) is the geodesic ball with center at \( \rho = 0 \) of the hyperbolic space with metric of the form \( d\rho^2 + \sinh^2 \rho d\sigma^2 \). Let us denote \( \phi_M \) simply by \( \phi \). Note that if \( f^4ds_{\mathbb{H}^3}^2 \) is a conformal metric such that \( f \) depends only on \( \rho \), then the scalar curvature is given by

\[
R = f^{-5}(-6f - \Delta_{\mathbb{H}^3} f) = f^{-5}[-6f - 8(f_{\rho\rho} + 2 \coth \rho \cdot f_\rho)].
\]

**Lemma 3.1.** With the above notations, there exist \( \rho_0(M) < \tau_1 < \tau_2 < \rho'_2 \) and a \( C^{2,1} \) function \( \psi(\rho) \) on \([0, \infty)\) such that \( \psi(\rho) > 1 \), \( \psi(\rho) = \text{constant} \) on \([0, \tau_1] \) and such that \( \psi(\rho) = \phi(\rho) \) on \([\tau_2, \infty)\). Moreover, the metric \( \psi^4ds_{\mathbb{H}^3}^2 = \psi^4(d\rho^2 + \sinh^2 \rho d\sigma^2) \) has scalar curvature \( R > -6 \) on \( B(\tau_2) \).

**Proof.** For any \( \rho_0(M) < \tau_1 < \tau_2 < \rho'_2 \), let \( \xi(\rho) = (\rho - \tau_1)^2(a\rho + b) \) where \( a \) and \( b \) are chosen so that

\[
\begin{cases}
\xi(\tau_1) = 0 \\
\xi_\rho(\tau_1) = 0;
\end{cases}
\]

\[
\begin{cases}
\xi(\tau_2) = A = \sinh^2 \tau_2 \cdot \phi_\rho(\tau_2); \\
\xi_\rho(\tau_2) = B = \frac{3}{4} \sinh^2 \tau_2 \cdot \phi(\tau_2)(\phi^4(\tau_2) - 1).
\end{cases}
\]

Then

\[
a = (\tau_2 - \tau_1)^{-2} \left[B - 2A(\tau_2 - \tau_1)^{-1}\right]
\]

and

\[
b = (\tau_2 - \tau_1)^{-2} \left[A - \tau_2 B + 2A\tau_2 (\tau_2 - \tau_1)^{-1}\right].
\]

Since \( A < 0 \) and \( B > 0 \), we have \( a > 0 \) and \( b < 0 \). Since \( \xi(\tau_2) = A < 0 \), so \( a\tau_2 + b < 0 \). Since \( a > 0 \), we have \( a\rho + b < 0 \) for all \( \rho < \tau_2 \). In particular,

\[
\xi \leq 0
\]

on \([\tau_1, \tau_2] \).

Define \( \psi \) as follows

\[
\psi(\rho) = \begin{cases}
\phi(\tau_2) - \int_{\tau_2}^{\rho} \frac{\xi(t)}{\sinh^2 t} dt, & 0 \leq \rho < \tau_1; \\
\phi(\tau_2) - \int_{\rho}^{\tau_2} \frac{\xi(t)}{\sinh^2 t} dt, & \tau_1 \leq \rho \leq \tau_2; \\
\phi(\rho), & \tau_1 < \rho < \infty.
\end{cases}
\]

Since in \([\tau_1, \tau_2] \), \( \sinh^2 \rho \cdot \psi_\rho(\rho) = \xi(\rho) \), and since the metric \( \phi^4ds_{\mathbb{H}^3}^2 \) has constant curvature \(-6\), by Proposition 2.1 and the definition of \( \xi \), one can see that \( \psi \) is \( C^{2,1} \).

We want to compute the scalar curvature of \( \psi^4ds_{\mathbb{H}^3}^2 \). Since \( \xi < 0 \), \( \psi > 1 \), then the scalar curvature on \( B(\tau_1) \) is larger than \(-6 \) because
ψ > 1 and is constant there. Outside $B(\tau_2)$, $\psi = \phi$ and the scalar curvature is -6. In $[\tau_1, \tau_2]$,

$$\xi_\rho(\rho) = 3a\rho^2 + 2(b - 2a\tau_1)\rho + (a\tau_1^2 - 2b\tau_1).$$

Hence for $\rho \in [\tau_1, \tau_2]$,

$$\xi_\rho(\tau_2) - \xi_\rho(\rho) = 3a(\tau_2^2 - \rho^2) + 2(b - 2a\tau_1)(\tau_2 - \rho) = a(\tau_2 - \rho) \left[ 3(\tau_2 + \rho) + \frac{2b}{a} - 4\tau_1 \right] \geq a(\tau_2 - \rho) \frac{2A}{B - 2A(\tau_2 - \tau_1)^{-1}} = a(\tau_2 - \rho) \frac{B(\tau_2 - \tau_1)^2}{B(\tau_2 - \tau_1) - 2A} \geq 0$$

and is positive if $\rho < \tau_2$, because $a > 0$, $B > 0$ and $A < 0$.

So in $B(\tau_2) \setminus B(\tau_1)$, we have

$$[\sinh^2 \rho \psi_\rho(\rho)]_\rho = \xi_\rho(\rho) < \xi_\rho(\tau_2) = \frac{3}{4} \phi(\tau_2)(\phi^4(\tau_2) - 1) \sinh^2 \tau_2 \leq \frac{3}{4} \phi(\rho)(\phi^4(\rho) - 1) \sinh^2 \rho \leq \frac{3}{4} \psi(\rho)(\psi^4(\rho) - 1) \sinh^2 \rho$$

and the scalar curvature is larger than -6 by (3.1), provided that $\phi(\rho)(\phi^4(\rho) - 1) \sinh^2 \rho$ is decreasing on $[\tau_1, \tau_2]$. Here we have used the fact that $\psi \geq \phi > 1$ on $[\tau_1, \tau_2]$.

Now

$$[\log (\phi(\rho)(\phi^4(\rho) - 1) \sinh^2 \rho)]_\rho = \frac{2 \cosh \rho}{\sinh \rho} + \frac{5\phi^4 - 1}{\phi^4 - 1} (\log \phi)_\rho = \frac{2 \cosh \rho}{\sinh \rho} + (\log \phi)_\rho + \frac{4\phi^4}{\phi^4 - 1} (\log \phi)_\rho.$$

Since $\phi \to \infty$ as $\rho \to \rho_0(M)_+$ by Proposition 2.2 there exists $\rho_0(M) < \tau_2 < \rho_2$ such that

(3.4) $$[\log (\phi(\rho)(\phi^4(\rho) - 1) \sinh^2 \rho)]_\rho < 0$$

at $\tau_2$. Hence one can choose $\alpha < \tau_1 < \tau_2 < \rho_0$ such that (3.4) is true in $[\tau_1, \tau_2]$. This completes the proof of the lemma. □
Next we want to modify $\psi$ in the lemma so that it is smooth. Using the same notation as in Lemma 3.1. Let

$$f = \Delta_{\mathbb{H}^3}\psi + \frac{3}{4}\psi(1 - \psi^4).$$

Then $f$ is Lipschitz and $f = f(\rho) < 0$ on $[0, \tau_2)$ and $f = 0$ on $[\tau_2, \infty)$. For any $\epsilon > 0$ let $0 \leq \chi_\epsilon \leq 1$ be a cutoff function on $[0, \infty)$ such that $\chi_\epsilon = 1$ on $[0, \tau_2 - \epsilon)$ and $\chi_\epsilon = 0$ on $[\tau_2 - \frac{1}{2}\epsilon, \infty)$. Define $f_\epsilon = f\chi_\epsilon$. Then $f_\epsilon \geq f$ and $f_\epsilon - f \leq C(\epsilon)$ where $C(\epsilon)$ is a function of $\epsilon$ with $\lim_{\epsilon \to 0} C(\epsilon) = 0$. Note that $f_\epsilon$ is smooth and $f_\epsilon = f = 0$ on $[\tau_2, \infty)$.

We want to prove the following:

**Theorem 3.2.** For any $\tau_2 > \epsilon > 0$, there is a unique $\phi_\epsilon$ which depends only on $\rho$ such that

(i)  
(3.5) \[ \Delta_{\mathbb{H}^3}\phi_\epsilon + \frac{3}{4}\phi_\epsilon(1 - \phi_\epsilon^4) = f_\epsilon, \]

and hence the scalar curvature of the metric $g_\epsilon = \phi_\epsilon^4 ds_{\mathbb{H}^3}^2$ is not less than -6 in $B(\tau_2)$ and is -6 outside $B(\tau_2)$.

(ii)  
$\psi > \phi_\epsilon > 1$

and $\lim_{\rho \to \infty} \phi_\epsilon = 1$

(iii) $\psi(x) - \phi_\epsilon(x) \leq C(\epsilon)e^{-3\rho(x)}$ in $\mathbb{H}^3 \setminus B(\tau_2)$, where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

(iv) $g_\epsilon^4 ds_{\mathbb{H}^3}^2 = g_{\text{AdS-Sch,M}_\epsilon}$ for some $M_\epsilon > 0$ on $\mathbb{H}^3 \setminus B(\tau_2)$. In particular, $g_\epsilon$ is AH. Moreover, if $\epsilon > 0$ is small enough, then $\tau_2 < \rho_{2,\epsilon}^\epsilon$ where $\rho = \rho_{2,\epsilon}^\epsilon$ is the surface with constant mean curvature $-2$ in the metric $g_{\text{AdS-Sch,M}_\epsilon}$.

(v) Let $M_\epsilon$ be as in (iv), then $M - M_\epsilon \leq C(\epsilon)$, where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

**Proof.** The existence part follows from [1]. In fact, let $\psi$ as in Lemma 3.1. By the definitions of $f$ and $f_\epsilon$, we have

$$\Delta_{\mathbb{H}^3}\psi + \frac{3}{4}\psi(1 - \psi^4) \leq f_\epsilon$$

and $\psi > 1$, $\psi \in C^{2,1}_{\text{loc}}(M)$. Since $f_\epsilon \leq 0$, we have

$$\Delta_{\mathbb{H}^3}\psi_0 + \frac{3}{4}\psi_0(1 - \psi_0^4) = 0 \geq f_\epsilon$$
where \( \psi_0 = 1 \) is the constant function. By [15], for any integer \( k \geq 1 \), we can find a unique solution \( \psi_k \)

\[
\begin{aligned}
\Delta_{\mathbb{H}^3} \psi_k + \frac{3}{4} \psi_k (1 - \psi_k^4) &= f_\epsilon, \quad \text{in } B(k); \\
\psi_k|_{\partial B(k)} &= \psi|_{\partial B(k)},
\end{aligned}
\] (3.6)

with \( 1 \leq \psi_k \leq \psi \). Hence one can choose a subsequence of \( \psi_k \) which converges uniformly on compact subsets of \( \mathbb{H}^3 \) together with its derivatives to a solution \( \phi_\epsilon \) of (3.5). Moreover, \( 1 \leq \phi_\epsilon \leq \psi \) by the strong maximum principle. Moreover, since \( f_\epsilon \) is a function of \( \rho \), \( \phi_\epsilon \) is also a function of \( \rho \) by Lemma 2.3. This proves (i) and (ii).

For \( k > \tau_2 \), let \( \eta = \psi_k - \psi \). Then

\[
\Delta_{\mathbb{H}^3} \eta + \frac{3}{4} \eta (1 - G) = f_\epsilon - f
\]

where \( G = \psi^4 + \psi^3 \psi_k + \cdots + \psi_k^4 \) > 5. Multiply both sides by \( \eta \) and integrating by parts, we get

\[
\int_{B(k)} |\nabla \eta|^2 + \frac{3}{4} \int_{B(k)} G \eta^2 - \frac{3}{4} \int_{B(k)} |\eta|^2 = - \int_{B(k)} (f_\epsilon - f) \eta = - \int_{B(\rho_2)} (f_\epsilon - f) \eta.
\]

Hence there exists a function \( C(\epsilon) \) such that

\[
\int_{B(k)} |\eta|^2 \leq C(\epsilon)
\]

here and below, \( C(\epsilon) \) denotes a function of \( \epsilon \) such that \( \lim_{\epsilon \to 0} C(\epsilon) = 0 \).

Hence we have

\[
\int_M |\phi_\epsilon - \psi|^2 \leq C(\epsilon).
\]

By mean value inequality [10], we conclude that for any \( \rho > \tau_2 \)

\[
\sup_{B(\rho) \setminus B(\tau_2)} |\phi_\epsilon - \psi| \leq C(\epsilon).
\]

Since both \( \psi \) and \( \phi_\epsilon \) satisfy

\[
\Delta_{\mathbb{H}^3} u + \frac{3}{4} (1 - u^4) = 0
\]
on \( \mathbb{H}^3 \setminus B(\tau_2) \), by Lemma 2.3, we conclude that

\[
\sup_{\mathbb{H}^3 \setminus B(\tau_2)} |\psi(x) - \phi_\epsilon(x)| \leq C(\epsilon) e^{-3\rho(x)}.
\]

This proves (iii).

The first part of (iv) follows from Corollary 2.5. (v) follows from (ii), (iii) and Proposition 2.4. The second part of (iv) follows from (v) and Proposition 2.4.

\[\square\]
4. AH METRICS WITH $R = -6$ ON THE UNIT BALL WITH HORIZONS

Using the metrics constructed in §2, we will construct AH metrics on the unit ball with $R = -6$ which contains a minimal sphere and spheres with constant mean curvature $\pm 2$. More precisely, we have:

**Theorem 4.1.** Let $D$ be the unit ball in $\mathbb{R}^3$. For any $M > 0$ and $\delta > 0$, there is a smooth complete metric $g$ on $D$ with constant scalar curvature $-6$ such that the following are true:

(i) $(D, g)$ is asymptotically hyperbolic with mass $M_g$ satisfying $|M_g - M| < \delta$.

(ii) There exist surfaces $S_1$, $S_2$, and $S_3$ which are topological spheres with constant mean curvature $-2, 0, 2$ respectively such that $S_1$ is in the interior of $S_2$ and $S_2$ is in the interior of $S_3$.

(iii) Outside a compact set the metric $g$ is conformal to the standard hyperbolic metric of $D$.

**Proof.** Let $M > 0$ and $\epsilon > 0$ be given, let $g_1 = g_\epsilon = \phi_\epsilon^4 ds^2_{\mathbb{H}^3}$ be the metric constructed in Theorem 3.2. The scalar curvature $R_1$ of $g_1$ is $-6$ outside the geodesic ball $B(\tau_2)$ with respect to $ds^2_{\mathbb{H}^3}$, and $R_1 \geq -6$. Let $0 \leq \xi \leq 1$ be a smooth function which is positive on $B(\tau_2)$, zero outside $B(\tau_2)$. For $\delta > 0$, let $f_\delta = -6 - \delta \xi$. Then $f_\delta < R_1$ in $B(\tau_2)$ and $f_\delta = R_1$ outside $B(\tau_1)$. By the result of Lohkamp [11, Theorem 1], there is a metric $g_2$ such that the scalar curvature $R_2$ of $g_2$ satisfies $f_\delta - \delta \leq R_2 \leq f_\delta$ on $B(\tau_2 + \delta)$. Moreover, $g_2 = g_1$ outside $B(\tau_2 + \delta)$ and $g_2$ can be chosen to be close to the metric $g_1$ in the $C^0$-topology. In particular, if $\delta > 0$ is small enough, then the first eigenvalue of the Laplacian operator of $(D, g_2)$ is bounded below by a constant $C_1 > 0$ independent of $\delta$. Since $0 \geq R_2 + 6 \geq -2\delta$, by [9], if $\delta$ is small enough then there is a positive solution $v$ of

$$
\Delta_{g_2} v - \frac{1}{8}(R_2 + 6)v = 0.
$$

We want to conformally deform $g_2$ to an AH metric with constant scalar curvature -6. To do this, for any $k > 0$, consider the following boundary value problem

$$
\begin{cases}
\Delta_{g_2} u_k - \frac{1}{8}R_2 u_k - \frac{3}{4}u_k^5 = 0, & \text{in } \tilde{B}(k); \\
u_k|_{\partial \tilde{B}(k)} = 1,
\end{cases}
$$

where $\tilde{B}(k)$ is the geodesic ball with respect to $g_2$ of radius $k$ with center at the origin of $D$. By rescaling $v$ in $\tilde{B}(k)$ we may assume that $v > 1$ in $\tilde{B}(k)$. Then we have

$$
\Delta_{g_2} v - \frac{1}{8}R_2 v - \frac{3}{4}v^5 \leq 0
$$

in $\hat{B}(k)$ and the constant function $v_1 = 1$ satisfies

$$\Delta_{g_2}v_1 - \frac{1}{8}R_2v_1 - \frac{3}{4}v_1^5 \geq 0.$$  

Here we have used the fact that $R_2 \leq -6$. By [15] as in the proof of Theorem 3.2, (4.1) has a solution $u_k \geq 1$. Suppose $u_k$ attains maximum at a point $x_0 \in B(k)$, then we have

$$\frac{1}{8}R_2u_k + \frac{3}{4}u_k^5 = \Delta_{g_2}u_k \leq 0$$

at $x_0$. Hence

$$\max_{B(k)} u_k \leq \max_{D} \left(-\frac{1}{6}R_2\right)^\frac{1}{4}.$$  

In particular, $u_k$ are uniformly bounded. By taking a subsequence if it is necessary, we see that there is a smooth function $u \geq 1$ on $D$ satisfying:

$$\Delta_{g_2}u - \frac{1}{8}R_2u - \frac{3}{4}u^5 = 0.$$  

We claim that $\lim_{x \to \infty} u(x) = 1$. Let $g = u^4g_2$, then the scalar curvature of $g$ is $-6$. Moreover, outside $B(\tau_2 + \delta)$, $g = u^4g_1 = u^4\phi^4ds^2_{\mathbb{H}^3}$. $u\phi_\epsilon$ satisfies:

$$\Delta_{\mathbb{H}^3}(u\phi_\epsilon) + \frac{3}{4}u\phi_\epsilon[1 - (u\phi_\epsilon)^4] = 0.$$  

Use the functions in defining the anti-de Sitter-Schwarzschild metric in Proposition 2.2 as comparison functions we conclude that $u\phi_\epsilon \to 1$ as $x \to \infty$. This proves the claim.

We want to prove that $g$ is an AH metrics with mass $M_g$ such that $|M_g - M| = C(\delta)$ with $C(\delta) \to 0$ as $\delta \to 0$ and that $g$ has surfaces $S_1, S_2, S_3$ as in the theorem. Note that

$$|u - 1| \leq C(\delta)$$

on $B(\tau_2 + \delta)$ as in the proof of Theorem 3.1. Here $C(\delta) \to 0$ as $\delta \to 0$. By this and Lemma 1.3, and together with the standard theory of elliptic partial differential equations, we see that

$$(4.2) \quad \|u\phi_\epsilon - \phi_\epsilon\|_{C^2,0(\Omega)} \leq C(\Omega, \delta).$$

Here $C(\Omega, \delta)$ is a positive constant which depend only on $\Omega$ and $\delta$, and $C(\Omega, \delta) \to 0$ as $\delta \to 0$. The results will be consequences of Lemma 2.3 and the following Lemmas 4.2, 4.3, and 4.4. □
Before we state and prove the lemmas, let us consider the metric:

\[ g = u^4 ds_{H^3}^2 \]
\[ = \frac{4u^4}{(1 - |x|^2)^2} (dr^2 + r^2 d\sigma^2) \]
\[ = \frac{4u^4}{(1 - |x|^2)^2} e^{2t} (dt^2 + d\sigma^2) \]
\[ = \frac{4u^4}{(1 - |x|^2)^2} g_0 \]  

(4.3)

where \( r = |x|, r = e^t \), and \( d\sigma^2 = h_{\alpha\beta} d\sigma_\alpha d\sigma_\beta \) is the standard metric on \( SS^2 \) and \( g_0 \) is the Euclidean metric. Suppose \( g \) has constant scalar curvature \(-6\) and \( u \to 1 \) as \( |x| \to 1 \), then by [2], \( u \) is smooth as a function of \( x \) up to \( |x| = 1 \).

**Lemma 4.2.** Assume that \(|u - 1| \leq C e^{-3d_\beta(x,0)}\). Then \( g \) is AH.

**Proof.** Let \( \rho = \frac{1-|x|^2}{2u^2} \). Then \( \rho \) is smooth up to \(|x| = 1\) and \(|\nabla_0 \rho| = 1 \) at \(|x| = 1\), where \( \nabla_0 \) is the Euclidean gradient. As in [3, Lemma 5.3] (see also p.102 in [7]), let \( \theta \) be the solution of the equation, with \( \theta = 1 \) at \( r = 1 \):

\[ \rho|\nabla_0 \theta|^2 + 2\theta \langle \nabla_0 \theta, \nabla_0 \rho \rangle = \theta^4 \rho + \theta^2 a \]  

(4.4)

with \( \theta = 1 \) at \( t = 0 \), where \( a \rho = 1 - |\nabla_0 \rho|^2 \) is a smooth function, and \( \nabla_0 \) is with respect to the Euclidean metric \( g_0 \). Let \( f \) be such that \( \sinh f = \theta \rho \). Then

\[ g = \sinh^{-2} f (df^2 + g_f) \]

where \( df^2 + g_f = \theta^2 g_0 = \hat{g} \), and \( g_f \) is the restriction of \( \hat{g} \) on the level surface \( f = \text{constant} \). Near \( r = 1 \), i.e., \( t = 0 \), the level surface is a graph of a function \( t = t(\sigma) \) where \( \sigma \in SS^2 \). Since \( f(t(\sigma_1, \sigma_2), \sigma_1, \sigma_2) = c \), we have

\[ f_t t_{\sigma_\alpha} + f_{\sigma_\alpha} = 0. \]  

(4.5)

Since \(|\hat{\nabla} f|^2 = 1\), here \( \hat{\nabla} \) is with respect to the metric \( \hat{g} \), we have

\[ \theta^{-2} e^{-2t} (f_t^2 + h^{\alpha\beta} f_{\sigma_\alpha} f_{\sigma_\beta}) = 1. \]  

(4.6)
In local coordinates \((\sigma_1, \sigma_2)\), the metric \(\gamma = g_f\) on the level surface is:

\[
\gamma_{\alpha\beta} = \gamma\left(\frac{\partial}{\partial \sigma_\alpha}, \frac{\partial}{\partial \sigma_\beta}\right) \\
= \theta^2 e^{2t} \left( t_{\sigma_\alpha} t_{\sigma_\beta} + h_{\alpha\beta} \right) \\
= \theta^2 e^{2t} \left( \frac{f_{\sigma_\alpha} f_{\sigma_\beta}}{f_t^2} + h_{\alpha\beta} \right) \\
= \frac{f_{\sigma_\alpha} f_{\sigma_\beta}}{1 - h^{\xi\zeta} f_{\sigma_\xi} f_{\sigma_\zeta}} + (\theta^2 e^{2t} - 1) h_{\alpha\beta} + h_{\alpha\beta}.
\]

We want to prove that the last term is \(h_{\alpha\beta} + O(t^3)\) near \(t = 0\).

Since \(|u - 1| = O(e^{-3d(x,0)})\), \(|u - 1| = O(t^3)\), at \(t = 0\), and \(u\) is smooth up to the boundary, we have \(u_t = u_{tt} = 0\) at \(t = 0\). Hence at \(t = 0\),

\[
(4.8) \quad \rho_t = -1, \rho_{tt} = -2, \rho_{ttt} = -4, \rho_{tttt} = 8(-1 + u_{ttt}), \rho_{t\sigma_\alpha} = \rho_{tt\sigma_\alpha} = \rho_{ttt\sigma_\alpha} = 0
\]

Now

\[
|\nabla_0 \rho|^2 = e^{-2t} (\rho_t^2 + h^{\alpha\beta} \rho_{\sigma_\alpha} \rho_{\sigma_\beta}) = e^{-2t} A
\]

we have at \(t = 0\),

\[
A = 1, \quad A_t = 4, \quad A_{tt} = 16, A_{ttt} = 64 - 16u_{ttt}
\]

Hence at \(t = 0\),

\[
(4.9) \quad (1 - |\nabla_0 \rho|^2)_t = -e^{-2t}[2A + A_t] = -2,
\]

\[
(1 - |\nabla_0 \rho|^2)_{tt} = -e^{-2t}[4A - 4A_t + A_{tt}] = -4
\]

\[
(1 - |\nabla_0 \rho|^2)_{ttt} = -e^{-2t}[-8A + 12A_t - 6A_{tt} + A_{ttt}] = 8 - 16u_{ttt}.
\]

Now \(a \rho = 1 - |\nabla_0 \rho|^2\), at \(t = 0\), we have

\[
(4.10) \quad a = 2, \quad a_t = 0, \quad a_{tt} = -\frac{16}{3} u_{ttt}
\]

By \((4.4)\),

\[
(4.11) \quad \rho(\theta_t^2 + h^{\alpha\beta} \theta_{\sigma_\alpha} \theta_{\sigma_\beta}) + 2\theta(\theta_t \rho_t + h^{\alpha\beta} \theta_{\sigma_\alpha} \rho_{\sigma_\beta}) = e^{2t}(\theta^4 \rho + \theta^2 a)
\]

Note that at \(t = 0\), \(\theta = 1, \theta_{\sigma_\alpha} = 0\). Hence at \(t = 0\), \(\theta_t = -1\). Now

\[
(4.12) \quad \rho_t(\theta_t^2 + h^{\alpha\beta} \theta_{\sigma_\alpha} \theta_{\sigma_\beta}) + \rho(\theta_t^2 + h^{\alpha\beta} \theta_{\sigma_\alpha} \theta_{\sigma_\beta})_t \\
+ 2\theta_t(\theta_t \rho_t + h^{\alpha\beta} \theta_{\sigma_\alpha} \rho_{\sigma_\beta}) + 2\theta(\theta_t \rho_t + \theta_t \rho_t + [h^{\alpha\beta} \theta_{\sigma_\alpha} \rho_{\sigma_\beta}]_t) \\
= e^{2t}(\theta^4 \rho + \theta^2 a) + e^{2t}(\theta^4 t \rho + \theta^4 \rho_t + 2\theta t a + \theta a_t)
\]
So at $t = 0$, $\theta_{tt} = 1$. Here we have used the fact that $\theta_{\sigma \alpha} = \rho_{\sigma \alpha} = 0$ at $t = 0$.

(4.13)

$$\rho_{tt}(\theta^2 + h^{\alpha \beta} \theta_{\sigma \alpha} \theta_{\sigma \beta}) + 2\rho_t(2\theta_t \rho_{tt} + [h^{\alpha \beta} \theta_{\sigma \alpha} \theta_{\sigma \beta}]_t) + \rho(\theta^2 + h^{\alpha \beta} \theta_{\sigma \alpha} \theta_{\sigma \beta})_{tt}$$

$$+ 2\theta_t(\theta_t \rho_{tt} + h^{\alpha \beta} \theta_{\sigma \alpha} \rho_{\sigma \beta}) + 4\theta_t(\theta_t \rho_t + \theta_t \rho_{tt} + [h^{\alpha \beta} \theta_{\sigma \alpha} \rho_{\sigma \beta}]_t)$$

$$+ 2\theta(\theta_{tt} \rho_t + 2\theta_{tt} \rho_{tt} + \theta_t \rho_{tt} + [h^{\alpha \beta} \theta_{\sigma \alpha} \rho_{\sigma \beta}]_t)$$

$$= e^{2t} \{ 4(\theta^4 + \theta^2 a) + 4[(\theta^4)_{tt} \rho + \theta^4 \rho_t + 2\theta \theta a + \theta^2 a_t]$$

$$+ (\theta^4)_{ttt} \rho + 8\theta^3 \theta_t \rho_t + \theta^4 \rho_{tt} + (2\theta^2_i + 2\theta \theta a + 2\theta^2 a_t \alpha + \theta^2 a_{tt}) \}$$

Hence at $t = 0$, $\theta_{tt} = -1 + \frac{1}{8} u_{tt}$.

(4.14)

$$\theta^2 e^{2t - 1}_{tt} = 2\theta_{tt} e^{2t} + 2e^{2t} \theta^2 = 0$$

$$\theta^2 e^{2t - 1}_{tt} = e^{2t} (2\theta_{tt} + 2\theta_t^2 + 4\theta^2 + 8\theta \theta_t) = 0$$

$$\theta^2 e^{2t - 1}_{tt} = 2e^{2t} (2\theta_{tt} + 2\theta_t^2 + 4\theta^2 + 8\theta \theta_t)$$

$$+ e^{2t} (2\theta_{tt} + 6\theta_t \theta_t + 8\theta \theta_t + 8\theta^2 + 8\theta \theta_t) = 2\theta_{tt} + 2$$

Since sinh $f = \theta \rho$,

$$\cosh ff = \theta_t \rho + \theta \rho_t = -1$$

at $t = 0$. Hence $f_t = -1$ at $t = 0$ and so $f_{\sigma \alpha} = 0$ at $t = 0$.

$$\cosh ff + \sinh ff = \theta_{tt} + 2\theta_{tt} \rho_t + \theta_{tt} \rho_{tt} = -4.$$ 

So $f_t = -4$ at $t = 0$ and $f_{tt \sigma \alpha} = 0$ at $t = 0$. From these and (4.7), we conclude that

$$\gamma_{\alpha \beta} = h_{\alpha \beta} + O(t^3),$$

near $t = 0$. Hence $g$ is AH because $u$ is smooth at $|x| = 1.$ \hfill $\Box$

To get an expression for the mass, let us compute $f_{tt}$, we have

(4.15)

$$\cosh ff_{tt} + 3 \sinh ff f_{tt} + \cosh ff_t^3 = \theta_{tt} + 3\theta_{tt} \rho_t + 3\theta_t \rho_{tt} + \theta \rho_{tt}$$

$$= -3 + 6 - 4$$

$$= -1.$$ 

So $f_{tt} = 0$ at $t = 0$. Hence if

$$\gamma_{\alpha \beta} = h_{\alpha \beta} + \frac{f_t^3}{3} \eta_{\alpha \beta} + O(f^4)$$

then

$$\gamma_{\alpha \beta} = h_{\alpha \beta} - \frac{t^3}{3} \eta_{\alpha \beta} + O(t^4).$$
Hence
\[ \eta_{\alpha\beta} = -\frac{1}{2} (\gamma_{\alpha\beta})_{tt} \]
\[ = -\frac{1}{2} (\theta^2 e^{2t} - 1)_{tt} h_{\alpha\beta} \]
\[ = - (\theta_{tt} + 1) h_{\alpha\beta} \]
\[ = -\frac{1}{8} u_{tt} h_{\alpha\beta} \tag{4.16} \]
evaluated at \( t = 0 \).

**Lemma 4.3.** \( tr_h \gamma = -\frac{1}{4} u_{tt} \). Hence suppose \( g_1 = u_1^4 ds_{H^3}^2 \) and \( g_2 = u_2^4 ds_{H^3}^2 \) be two metrics defined outside some compact set of \( H^3 \) such that \( u_1 \) and \( u_2 \) are smooth up to \( \partial B(1) \) if we use ball model for \( H^3 \). Moreover, assume that \( |u_1(x) - 1| + |u_2(x) - 1| \leq C e^{-3d(x,0)} \). Suppose \( u_1 \) and \( u_2 \) are close in the sense that \( |u_1 - u_2| \leq \epsilon e^{-3d(x,0)} \). Then there is an absolute constant \( C_1 \) such that \( |M_1 - M_2| \leq C_1 \epsilon \), here \( M_1 \), \( M_2 \) is the mass of \( g_1 \), \( g_2 \) respectively.

**Proof.** To prove the second part, with the same notation as in Lemma 4.2, we have \( |(u_1)_{tt} - (u_2)_{tt}| \leq C_2 \epsilon \) at \( t = 0 \) for some absolute constant \( C_2 \). The result follows from first part and the definition of mass. \( \square \)

Since \( g_2 \) in the proof Theorem 4.1 is anti-de Sitter-Schwarzchild outside \( B(\tau_2 + \delta) \), we may assume \( g_2 = \phi^4 (dr^2 + r^2 d\sigma^2) \) for some \( r \geq \delta > 0 \), where \( d\sigma^2 = h_{\alpha\beta} d\sigma_\alpha d\sigma_\beta \) is the standard metric on \( S S^2 \). Without loss of generality, we may assume \( (D_{1-\delta} \setminus D_\delta, g_0) \) containing the compact surface with mean curvature \( \pm 2 \) and 0, here and in the sequel, the mean curvature is always with respect to the outward unit normal vector, and for simplicity, we denote \( D_{1-\delta} \setminus D_\delta \) by \( N \).

Obviously, it is enough to show

**Lemma 4.4.** Let \( g_2 = \phi^4 (dr^2 + r^2 d\sigma^2) \) be as above, which is a Riemannian metric on \( N \). Then there is an \( \epsilon > 0 \) such that for any \( \tilde{\phi} \) with \( \| \phi - \tilde{\phi} \|_{C^{2,\alpha}(N)} \leq \epsilon \), there are compact surfaces in \( (N, g) \) with mean curvature equal to \( \pm 2 \) and 0, here \( g = \tilde{\phi}^4 (dr^2 + r^2 d\sigma^2) \).

**Proof.** We will use implicit function theorem. Let us discuss the case that \( H = 0 \) and \( H = \pm 2 \) together, we adopt the coordinates \( (r, \sigma) \) on \( N \), here \( \sigma \in S S^2 \). Consider the Banach spaces \( B_1 = C^{2,\alpha}(N) \), \( B_2 = C^{2,\alpha}(S S^2) \) and \( B_3 = C^\alpha(S S^2) \) and let \( U \) be the open set in \( B_1 \times B_2 \) consisting of \( (\Phi, v) \) such that \( \Phi > 0 \) and \( \delta < v < 1 - \delta \). For \( (\Phi, v) \in U \), let define \( H(\Phi, v) \) to be the mean curvature of the surface given by \( (\sigma, v(\sigma)) \) in \( (N, \Phi^4 (dr^2 + \sigma^2 d\sigma^2)) \). Then \( H : U \rightarrow B_3 \).
We want to compute the differential at $\Phi = \phi = \phi(r)$ and $v = c=$constant with $\delta < c < 1 - \delta$. Let $\nabla = \nabla_{SS^2}$. Consider a surface given by $(\sigma, v(\sigma))$ and let $f(r, \sigma) = v(\sigma) - r$. Then the surface is given by the level surface $f = 0$. Then

$$
\begin{align*}
\nabla f &= \phi^{-4}(\frac{\partial}{\partial r} + r^{-2}\nabla v); \\
\Delta f &= -\frac{1}{\phi^2 r^2}(\phi^2 r^2)_r + \phi^{-4}r^{-2}\tilde{\Delta}v; \\
|\nabla f|^2 &= \phi^{-4}(1 + r^{-2}|\tilde{\nabla}v|^2) = \phi^{-4}v^2
\end{align*}
$$

(4.17)

where $\nabla$, $\Delta$ are the gradient and Laplacian with respect to $g_2$, $\tilde{\Delta}$ is the Laplacian on $SS^2$ and $\psi = (1 + r^{-2}|\tilde{\nabla}v|^2)^{\frac{1}{2}}$. Therefore the mean curvature of the level surface $f = 0$ is

$$
H(\phi, v) = \text{div} \left( \frac{\nabla f}{|\nabla f|} \right)
$$

$$
= \frac{\Delta f}{|\nabla f|} + \langle \nabla(\frac{1}{|\nabla f|}), \nabla f \rangle
$$

(4.18)

$$
= -\phi^{-4}r^{-2}(\phi^2 r^2)_r + \phi^{-2}\psi^{-1}r^{-2}\tilde{\Delta}v - \phi^{-4}(\phi^2 \psi^{-1})_r + \phi^{-2}r^{-2}(\tilde{\nabla}(\psi^{-1}), \tilde{\nabla}v)
$$

$$
= -\phi^{-4} \left[ 2\psi^{-1}(\phi^2)_r + 2\psi^{-1}r^{-1}\phi^2 + \phi^2(\psi^{-1})_r \right] + \phi^2\psi^{-1}r^{-2}\tilde{\Delta}v + \phi^{-2}r^{-2}(\tilde{\nabla}(\psi^{-1}), \tilde{\nabla}v)
$$

evaluated at $r = v$. By the expression of $H$, it is easy to see that $H$ is $C^1$.

We want to compute $\frac{d}{dt}H(\phi, c + t\eta)|_{t=0}$, where $c$ is a constant. It is easy to see that

$$
\frac{d}{dt}\psi(c + t\eta, \sigma)|_{t=0} = \frac{d}{dt}\frac{\partial}{\partial r}(\psi(c + t\eta, \sigma))|_{t=0} = 0.
$$

(4.19)

Since $c$ is a constant, we have

$$
\frac{d}{dt} \left[ \phi^2\psi^{-1}r^{-2}\tilde{\Delta}v + \phi^{-2}r^{-2}\langle \tilde{\nabla}(\psi^{-1}), \tilde{\nabla}v \rangle \right] |_{t=0} = \phi^{-2}c^{-2}\tilde{\Delta}\eta
$$

(4.20)

Hence

$$
\frac{d}{dt}H(\phi, c + t\eta)|_{t=0} = 4\phi^{-5}\phi_r \left[ 4\phi\phi_r + 2c^{-1}\phi^2 \right] \eta
$$

$$
- \phi^{-4} \left[ 4\phi^2 + 4\phi\phi_{rr} - 2c^{-2}\phi^2 + 4c^{-1}\phi\phi_r \right] \eta
$$

$$
+ \phi^{-2}c^{-2}\tilde{\Delta}\eta.
$$

(4.21)

Since the metric $g_2$ has constant scalar curvature -6 on $N$ and the surface $r = c$ has constant mean curvature $H$,

$$
H = \phi^{-2}(\frac{2}{r} + 4\phi^{-1}\phi_r) = \phi^{-2}(2c^{-1} + 4\phi^{-1}\phi_r)
$$
and so
\[ \phi_r = \frac{1}{4} \phi^3 H - \frac{1}{2} \phi c^{-1}. \]

Also
\[ 8 \Delta_0 \phi = 6 \phi^5, \]
where \( \Delta_0 \) is the Euclidean Laplacian. Hence
\[ \phi_{rr} = -\frac{2}{r} \phi_r + \frac{3}{4} \phi^5 = -2c^{-1} \left( \frac{1}{4} \phi^3 H - \frac{1}{2} \phi c^{-1} \right) + \frac{3}{4} \phi^5. \]

Therefore
\[ (4.22) \]
\[ \frac{d}{dt} H(\phi, c + t\eta)|_{t=0} \]
\[ = \eta \phi^{-4} \left[ 4 \phi^3 H \phi_r - 4 \phi^2 - 4 \phi \left( -2c^{-1} \phi_r + \frac{3}{4} \phi^5 \right) + 2c^{-2} \phi^2 - 4c^{-1} \phi \phi_r \right] + \phi^{-2} c^{-2} \Delta \eta. \]
\[ = \eta \phi^{-4} \left[ 4 \phi^3 H \phi_r - 4 \phi^2 + 4 \phi c^{-1} \phi_r + 2c^{-2} \phi^2 - 3 \phi^6 \right] + \phi^{-2} c^{-2} \Delta \eta \]
\[ = \eta \phi^{-4} \left[ 4 \phi^3 H \phi_r - 4 \phi^2 + 4 \phi c^{-1} \phi_r + 2c^{-2} \phi^2 - 3 \phi^6 \right] + \phi^{-2} c^{-2} \Delta \eta \]
\[ = \eta \phi^{-4} \left[ 4 \phi^3 H \phi_r - 4 \phi^2 + \phi^4 c^{-1} H - 3 \phi^6 \right] + \phi^{-2} c^{-2} \Delta \eta \]
\[ = \eta \phi^{-4} \left[ \phi^6 H^2 - 2 \phi^4 c^{-1} H - \frac{1}{4} \phi^6 H^2 + \phi^4 c^{-1} H - \phi^2 c^{-2} + \phi^4 c^{-1} H - 3 \phi^6 \right] + \phi^{-2} c^{-2} \Delta \eta \]
\[ = \eta \phi^{-4} \left[ \frac{3}{4} \phi^6 H^2 - \phi^2 c^{-2} - 3 \phi^6 \right] + \phi^{-2} c^{-2} \Delta \eta \]
\[ = \phi^{-2} c^{-2} \Delta \eta - \Theta \eta \]
where \( \Theta \geq \phi^{-2} c^{-2} \) is a positive function if \(|H| \leq 2.\)

Let \( c \) be such that \( H(\phi, c) = 2 \), then \( \delta < c < 1 - \delta \). \( \frac{d}{dt} H(\phi, c)|_{t=0}: C^{2,\alpha}(SS^2) \to C^{0}(SS^2) \) is bijective, thus, by implicit function theorem, we see that there is \( \epsilon > 0 \) so that for any \( \|\phi - \tilde{\phi}\|_{C^{2,\alpha}(N)} \leq \epsilon \), there is a smooth function \( v \) on \( SS^2 \) with \( \|v - c\|_{C^{2,\alpha}(SS^2)} \leq \epsilon \) such that \( H(\tilde{\phi}, v) = 2 \). Thus, there is a compact surface in \( (N, g) \) with the mean
curvature equal to 2. by the same arguments, one may show there are compact surfaces with mean curvature $-2$ and 0. \[\square\]

Together with (4.2), we see that there exists surfaces $S_1$, $S_2$, $S_3$ in the manifolds that we constructed. Note that the surfaces are diffeomorphic to $SS^2$ and are close to the constant mean curvature surfaces in the metric $g_2$. Thus, we finish to prove Theorem 3.1.

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Key Laboratory of Pure and Applied Mathematics, School of Mathematics Science, Peking University, Beijing, 100871, P.R. China.
E-mail address: ygshi@math.pku.edu.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.
E-mail address: lftam@math.cuhk.edu.hk