Low Energy 6-Dimensional $N=2$ Supersymmetric $SU(6)$ Models on $T^2$ Orbifolds

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Abstract

We propose low energy 6-dimensional $N=2$ supersymmetric $SU(6)$ models on $M^4 \times T^2/(Z_2)^3$ and $M^4 \times T^2/(Z_2)^4$, where the orbifold $SU(3)_{\text{C}} \times SU(3)$ model can be embedded on the boundary 4-brane. For the zero modes, the 6-dimensional $N=2$ supersymmetry and the $SU(6)$ gauge symmetry are broken down to the 4-dimensional $N=1$ supersymmetry and the $SU(3)_{\text{C}} \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry by orbifold projections. In order to cancel the anomalies involving at least one $U(1)'$, we add extra exotic particles. We also study the anomaly free conditions and present some anomaly free models. The gauge coupling unification can be achieved at $100 \sim 200$ TeV if the compactification scale for the fifth dimension is $3 \sim 4$ TeV. The proton decay problem can be avoided by putting the quarks and leptons/neutrinos on different 3-branes. And we discuss how to break the $SU(3)_{\text{C}} \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry, solve the $\mu$ problem, and generate the $Z - Z'$ mass hierarchy naturally by using the geometry. The masses of exotic particles can be at the order of 1 TeV after the gauge symmetry breaking. We also forbid the dimension-5 operators for the neutrino masses by $U(1)'$ gauge symmetry, and the realistic left-handed neutrino masses can be obtained via non-renormalizable terms.

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1 Introduction

Grand Unified Theory (GUT) has a two-fold meaning: (1) Gauge coupling unification, i.e., the gauge couplings for the Standard Model (SM) gauge groups $SU(3)_C \times SU(2)_L \times U(1)_Y$ are equal at GUT scale; (2) Fermion unification, i.e., the SM fermions of each generation elegantly fit into the $5 + 10$ representation of $SU(5)$ group, or the spinor 16 representation of $SO(10)$ group if we included the right handed neutrinos. As we know, GUT gives us an simple and elegant understanding of the quantum numbers of quarks and leptons, and the success of gauge coupling unification in the Minimal Supersymmetric Standard Model (MSSM) strongly supports this idea. The radiative electroweak symmetry breaking can also be realized in the supersymmetric GUT due to the large top quark Yukawa coupling. Another impressive success of the supersymmetric GUT is that including the radiative corrections, GUT gives the correct weak mixing angle, as observed in the electroweak (EW) scale experiments. Therefore, the Grand Unified Theory at high energy scale has been widely accepted. However, there are some problems in GUT: the grand unified gauge symmetry breaking mechanism, the doublet-triplet splitting problem, the proton decay problem, and the fermion mass relation problem ($m_e/m_\mu = m_d/m_s$), etc.

About two years ago, a new scenario was proposed to address above questions in GUT [1, 2, 3, 4]. The key point is that the supersymmetric GUT model exists in 5 or higher dimensions and is broken down to the 4-dimensional $N = 1$ supersymmetric Standard-Model-like Model for the zero modes due to the discrete symmetries in the neighborhoods of the branes or on the extra space manifolds, which become non-trivial constraints on the multiplets and gauge generators in GUT [1, 2, 3, 4]. In addition, the discrete symmetry may not act freely on the extra space manifold. When the discrete symmetry does not act freely on the extra space manifold, there exists a brane at each fixed point, line or hypersurface, where only part of the gauge symmetry and supersymmetry might be preserved and the SM fermions can be located [1, 2, 3, 4]. The attractive models have been constructed explicitly, in which the supersymmetric 5-dimensional and 6-dimensional GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ models, where $n$ is the rank of the GUT groups, through the compactification on various orbifolds and manifolds. The GUT gauge symmetry breaking can be achieved, and the doublet-triplet splitting problem, the proton decay problem and the fermion mass relation problem ($m_e/m_\mu = m_d/m_s$) can be solved neatly by the discrete symmetry projections, although the fermion unification might be given up.

In addition to the orbifold GUT models, the low energy partial unification $SU(3)_C \times SU(3)$ model on the space-time $M^4 \times S^1/(Z_2 \times Z_2')$ has been proposed recently, where quarks are put on the 3-brane at the fixed point on which only the Standard Model gauge symmetry is preserved [5, 6, 7]. In this model, the desirable weak mixing angle $\sin^2 \theta_W = 0.2312$ at $m_Z$ scale can be generated naturally and the $U(1)_Y$ charge quantization can be obtained due to the gauge invariant of the Yukawa couplings and anomaly free conditions. We also showed that from the point of view of the gauge coupling runnings, a complete gauge coupling unification might occur
at $10^6$ GeV if the compactification scale of the fifth dimension ($1/R$) is $10^4$ GeV for the supersymmetric model without embedding it into a GUT model [5]. However, there exists the $\mu$ problem. And the left-handed neutrinos can obtain masses via the dimension-5 operators $y_{\nu ij}L_iL_jH_uH_u/M_*$, where $M_*$ is the fundamental scale (or cut-off scale) in the model which is about $10^6$ GeV. Therefore, the masses for the left-handed neutrinos can not be realistic unless $y_{\nu ij}$ is very small, at the order of $10^{-7}$ or less.

On the other hand, the possibility of an extra $U(1)'$ gauge symmetry is well-motivated in superstring constructions [8], grand unified theories [9], and in models of dynamical symmetry breaking [10]. In supersymmetric models, an extra $U(1)'$ can provide an elegant solution to the $\mu$ problem [11, 12], with an effective $\mu$ parameter generated by the vacuum expectation value (VEV) of the Standard Model singlet field $S$ which breaks the $U(1)'$ gauge symmetry. This is somewhat similar to the effective $\mu$ parameter in the Next to Minimal Supersymmetric Standard Model (NMSSM) [13]. However, with a $U(1)'$ the extra discrete symmetries and their associated cosmological problems typically associated with the NMSSM are absent. In superstring-motivated models it is often the case that the electroweak and $U(1)'$ breaking are both driven by the soft supersymmetry breaking parameters, so one typically expects the mass of the $U(1)'$ gauge boson $Z'$ to be of the same order as the electroweak scale [8]. However, there are stringent limits on $Z'$ mass from direct searches during Run I at the Tevatron [14] and from indirect precision tests at the $Z$-pole, at LEP 2, and from weak neutral current experiments [15]. The constraints depend on the particular $U(1)'$ couplings, but in typical models one requires that $M_{Z'} > (500 - 800)$ GeV and the $Z - Z'$ mixing angle $\theta_{Z-Z'}$ to be smaller than a few $\times 10^{-3}$. Thus, how to explain the $Z - Z'$ mass hierarchy is an interesting question. Recently, the supersymmetric model with a secluded $U(1)'$-breaking sector was proposed, where the squark and slepton spectra can mimic those of the MSSM, the electroweak breaking is actually driven by the relatively large $A$ terms, and a large $Z'$ mass can be generated by the VEVs of additional SM singlet fields that are charged under the $U(1)'$ [16]. In this scenario, the superpotential for the Higgs is

$$W = hSH_dH_u + \lambda S_1S_2S_3,$$  \hspace{1cm} (1)

where $S$ and $S_i$ are the Standard Model singlets, $h$ and $\lambda$ are coupling constants. In order to generate the $Z - Z'$ mass hierarchy, one has to choose $h \sim 10\lambda$, which becomes the Yukawa couplings hierarchy.

In this article, we propose the low energy 6-dimensional $N = 2$ supersymmetric $SU(6)$ model on the space-time $M^4 \times T^2/(Z_2)^3$ and $M^4 \times T^2/(Z_2)^4$, where the orbifold $SU(3)_C \times SU(3)$ model can be embedded on the 4-brane at $z = 0$, and the desirable weak mixing angle $\sin^2 \theta_W = 0.2312$ at $m_Z$ scale can be generated naturally and the $U(1)_Y$ charge quantization for the Standard Model fermions can be obtained due to the gauge invariant of the Yukawa couplings and anomaly free conditions. For the zero modes, the 6-dimensional $N = 2$ supersymmetry and the $SU(6)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and the
SU(3)_C × SU(2)_L × U(1)_Y × U(1)′ gauge symmetry by orbifold projections. We present the parity assignment and masses for the bulk gauge fields and Higgs fields on the 4-brane at z = 0, and the number of the 4-dimensional supersymmetry and gauge symmetry on the 3-branes at the fixed points and on the 4-branes on the fixed lines. In order to cancel the anomaly involving at least one U(1)′, we add extra exotic particles which are vector-like under the Standard Model gauge symmetry. We examine the anomaly free conditions and give some anomaly free models. In addition, the gauge coupling unification can be achieved at 100 ∼ 200 TeV if the compactification scale for the fifth dimension (1/R_1) is 3 ∼ 4 TeV. The proton decay problem can be avoided by putting the quarks and leptons/neutrinos on different 3-branes. Moreover, we consider the neutrino masses in detail. We forbid the dimension-5 operators by U(1)′ gauge symmetry which might give very large masses to the left handed neutrinos unless the couplings are at the order of 10^{-8} or less, and the correct active neutrino masses can be obtained via the non-renormalizable terms naturally.

This paper is organized as follows. In Section 2, we review the 6-dimensional N = 2 supersymmetric gauge theory and explain our convention. In Section 3, we consider the low energy 6-dimensional N = 2 supersymmetric SU(6) gauge unification theory on the space-time M^4 × T^2/(Z_2)^3. And in Section 4, we study the low energy 6-dimensional N = 2 supersymmetric SU(6) gauge unification theory on the space-time M^4 × T^2/(Z_2)^4. The discussion and conclusion are given in Section 5.

2 N = 2 6-Dimensional Supersymmetric Gauge Theory and Convention

N = 2 supersymmetric gauge theory in 6-dimension has 16 real supercharges, corresponding to N = 4 supersymmetry in 4-dimension. So, only the vector multiplet can be introduced in the bulk. In terms of 4-dimensional N = 1 supersymmetry language, it contains a vector multiplet V(A_µ, λ_1), and three chiral multiplets Σ_5, Σ_6, and Φ. All of them are in the adjoint representation of the gauge group. In addition, the Σ_5 and Σ_6 chiral multiplets contain the gauge fields A_5 and A_6 in their lowest components, respectively.

In the Wess-Zumino gauge and 4-dimensional N = 1 language, the bulk action is [17]

\[
S = \int d^6x \left\{ \text{Tr} \left[ \frac{1}{4k g^2} \mathcal{W}_a \mathcal{W}_a + \frac{1}{k g^2} \left( \Phi \partial_5 \Sigma_6 - \Phi \partial_6 \Sigma_5 - \frac{1}{\sqrt{2}} \Phi [\Sigma_5, \Sigma_6] \right) \right] \right\}
\]
+H.C.] + \int d^4 \theta \frac{1}{k g^2} \text{Tr} \left[ \sum_{i=5}^{6} \left( (\sqrt{2} \partial_i + \Sigma_i^\dagger) e^{-V} (-\sqrt{2} \partial_i + \Sigma_i) e^V + \partial_i e^{-V} \partial_i e^V \right) + \Phi e^{-V} \Phi e^V \right]. \tag{2}

And the gauge transformation is given by

\begin{align*}
e^V & \rightarrow e^A e^V e^{A^\dagger}, \\
\Sigma_i & \rightarrow e^A (\Sigma_i - \sqrt{2} \partial_i) e^{-A}, \\
\Phi & \rightarrow e^A \Phi e^{-A},
\end{align*}

where \( i = 5, 6 \).

Our convention is the same as that of Ref. [3]. We consider the 6-dimensional space-time which can be factorized into a product of the ordinary 4-dimensional Minkowski space-time \( M^4 \), and the torus \( T^2 \) which is homeomorphic to \( S^1 \times S^1 \). The corresponding coordinates for the space-time are \( x^\mu, (\mu = 0, 1, 2, 3) \), \( y \equiv x^5 \) and \( z \equiv x^6 \). The radii for the circles along the \( y \) direction and \( z \) direction are \( R_1 \) and \( R_2 \), respectively. We also define \( y' \) and \( z' \) by \( y' \equiv y - \pi R_1/2 \) and \( z' \equiv z - \pi R_2/2 \). In addition, we assume that the gauge fields are in the bulk, and the SM fermions and Higgs particles are on the 4-brane on the fixed line (boundary) or on the 3-brane at the fixed point in the extra space orbifold.

In this paper, the orbifold \( T^2/(Z_2)^3 \) are defined by \( T^2 \) moduloing three equivalent classes

\[ y \sim -y, \quad z \sim -z, \quad z' \sim -z'. \tag{6} \]

Precisely speaking, our orbifold \( T^2/(Z_2)^3 \) is \( S^1/Z_2 \times S^1/(Z_2 \times Z_2) \). The fixed points are \( (y = 0, z = 0) \), \( (y = 0, z = \pi R_2/2) \), \( (y = \pi R_1, z = 0) \) and \( (y = \pi R_1, z = \pi R_2/2) \), and the fixed lines are \( y = 0, z = 0, y = \pi R_1 \) and \( z = \pi R_2/2 \). And the extra space orbifold is a rectangle.

In addition, the orbifold \( T^2/(Z_2)^4 \) are obtained by \( T^2 \) moduloing the equivalent classes

\[ y \sim -y, \quad z \sim -z, \quad y' \sim -y', \quad z' \sim -z'. \tag{7} \]

The four fixed points are \( (y = 0, z = 0) \), \( (y = 0, z = \pi R_2/2) \), \( (y = \pi R_1/2, z = 0) \), and \( (y = \pi R_1/2, z = \pi R_2/2) \), and the fixed lines are \( y = 0, z = 0, y = \pi R_1/2 \) and \( z = \pi R_2/2 \).

For a generic bulk field \( \phi(x^\mu, y, z) \), we can define four parity operators \( P^y, P^z, P^{y'} \), and \( P^{z'} \)

\[ \phi(x^\mu, y, z) \rightarrow \phi(x^\mu, -y, z) = P^y \phi(x^\mu, y, z), \tag{8} \]

\[ \phi(x^\mu, y, z) \rightarrow \phi(x^\mu, y, -z) = P^z \phi(x^\mu, y, z), \tag{9} \]

\[ \phi(x^\mu, y, z) \rightarrow \phi(x^\mu, -y, -z) = P^{y'} \phi(x^\mu, y, z), \tag{10} \]

\[ \phi(x^\mu, y, z) \rightarrow \phi(x^\mu, y, -z') = P^{z'} \phi(x^\mu, y, z), \tag{11} \]

\[ \phi(x^\mu, y, z) \rightarrow \phi(x^\mu, -y', -z) = P^{y'} P^z \phi(x^\mu, y, z). \tag{12} \]

\[ \text{(For the model on } T^2/(Z_2)^3 \text{ orbifold, we just neglect the } P^y \text{ parity operator.)} \]
\[
\phi(x^\mu, y', z') \to \phi(x^\mu, -y', z') = P_y^\prime \phi(x^\mu, y', z'),
\]
\[
\phi(x^\mu, y', z') \to \phi(x^\mu, y', -z') = P_{z'} \phi(x^\mu, y', z').
\]

Furthermore, suppose \( G \) is a Lie group and \( H \) is a subgroup of \( G \), we denote the commutant of \( H \) in \( G \) as \( G/H \), i.e.,
\[
G/H \equiv \{ g \in G | gh = hg, \text{ for any } h \in H \}.
\]
And if \( H_1 \) and \( H_2 \) are the subgroups of \( G \), we define
\[
G/\{ H_1 \cup H_2 \} \equiv \{ G/H_1 \} \cap \{ G/H_2 \}.
\]
For simplicity, we define
\[
G/P_y \equiv G/\{ e, P_y \},
\]
\[
G/\{ P_y \cup P_z \} \equiv G/\{ e, P_y \} \cap G/\{ e, P_z \},
\]
similarly for the others.

3. **SU(6) Model on \( M^4 \times T^2/(Z_2)^3 \)**

In this section, we discuss the \( N = 2 \) supersymmetric \( SU(6) \) model on the space-time \( M^4 \times T^2/(Z_2)^3 \). In particular, on the 4-brane at \( z = 0 \), because of the orbifold projections, there exist only the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry and 4-dimensional \( N = 2 \) supersymmetry, where the previous orbifold \( SU(3)_C \times SU(3) \) model in Refs. [5, 6, 7] can be embedded naturally.

3.1 *Orbifold SU(6) Breaking, and Particle Spectrum for Gauge and Higgs Fields*

Because in our model, there exist only the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry and 4-dimensional \( N = 2 \) supersymmetry on the 4-brane at \( z = 0 \), we put one pair of Higgs triplets \( \Psi_u \) and \( \Psi_d \) which transform as \((1,3,c_2)\) and \((1,\bar{3},c_1)\) under the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry on the 4-brane at \( z = 0 \). We require that \( c_1 \neq 0, c_2 \neq 0 \) and \( c_1 + c_2 \neq 0 \). We also add three \( SU(3)_C \times SU(3) \) singlets, \( \Psi_{S_1}, \Psi_{S_2} \) and \( \Psi_{S_3} \) with \( U(1)' \) charges \(-s, -s\) and \( 2s \) respectively on the 4-brane at \( z = 0 \), where \( s \equiv -c_1 - c_2 \). In terms of the 4-dimensional \( N = 1 \) supersymmetry language, the hypermultiplets \( \Psi_u \) and \( \Psi_d \) can be decomposed into two pairs of chiral multiplets \( (\Phi_u, \Phi_u^c) \) and \( (\Phi_d, \Phi_d^c) \), and the hypermultiplet \( \Psi_{S_i} \) can be decomposed into one pair

\[\text{If } c_1 + c_2 = 0, \text{ we can not forbid the brane localized superpotential } \mu H_d H_u. \text{ So, we can not solve the } \mu \text{ problem.}\]
of chiral multiplets \((S_i, S_i^c)\) in which \(i = 1, 2, 3\). Here, the superscript \(c\) means the charge conjugation. To be explicit, we define

\[
\Phi_u = \begin{pmatrix} H_u \\ S_u \end{pmatrix}, \quad \Phi_u^c = \begin{pmatrix} H_u^c \\ S_u^c \end{pmatrix},
\]

\[
\Phi_d = \begin{pmatrix} H_d \\ S_d \end{pmatrix}, \quad \Phi_d^c = \begin{pmatrix} H_d^c \\ S_d^c \end{pmatrix}.
\]

(16)

(17)

Moreover, we introduce extra exotic particles to cancel the anomalies, and put the Standard Model fermions and extra exotic particles on the 3-branes at the orbifold fixed points, which will be explained in detail later.

We choose the following matrix representations for the parity operators \(P^y\), \(P^z\) and \(P^{z'}\), which are expressed in the adjoint representation of \(SU(6)\)

\[
P^y = \text{diag}(+1, +1, +1, -1, -1, +1)
\]

\[
P^z = \text{diag}(+1, +1, +1, -1, -1, -1), \quad P^{z'} = \text{diag}(+1, +1, +1, +1, +1, +1).
\]

(18)

(19)

Thus, under the \(P^y\) and \(P^z\) parities, the \(SU(6)\) gauge generators \(T^A\), where \(A=1, 2, ..., 35\) for \(SU(6)\), are separated into four sets: \(T^a, T^b, T^{\hat{a}}, \) and \(T^{\hat{b}}\) are the gauge generators for the \(SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_I\) gauge symmetry, \(T^a, T^b, T^{\hat{a}}, \) and \(T^{\hat{b}}\) are the other broken gauge generators that belong to \\{\(G/P^y \cap \{\text{coset } G/P^z\}\)\}, \\{\(\text{coset } G/P^y\) \cap \(G/P^z\)\}, and \\{\(\text{coset } G/P^y\) \cap \(\text{coset } G/P^z\)\}, respectively. Therefore, under \(P^y\), \(P^z\) and \(P^{z'}\), the gauge generators transform as

\[
P^y T^a, B (P^y)^{-1} = T^a, B, \quad P^y T^{\hat{a}}, B (P^y)^{-1} = -T^{\hat{a}}, B,
\]

\[
P^z T^A, B (P^z)^{-1} = T^A, B, \quad P^z T^{\hat{A}}, B (P^z)^{-1} = -T^{\hat{A}}, B,
\]

\[
P^{z'} T^A, B (P^{z'})^{-1} = T^A, B.
\]

(20)

(21)

(22)

The generators for \(SU(6)\) are the \(6 \times 6\) traceless Hermitian matrix. According to the \(6 \times 6\) matrix, the decompositions of the \(SU(6)\) generators \(T^{A,B}\) into the \(T^a, b, T^{\hat{a}}, b\), and \(T^{\hat{a}}, b\) are

\[
T^{A,B} = \begin{pmatrix} (T^a, b)_{3 \times 3} & (T^{\hat{a}}, b)_{3 \times 2} & (T^a, b)_{3 \times 1} \\ (T^{\hat{a}}, b)_{2 \times 3} & (T^a, b)_{2 \times 2} & (T^{\hat{a}}, b)_{2 \times 1} \\ (T^a, b)_{1 \times 3} & (T^{\hat{a}}, b)_{1 \times 2} & (T^a, b)_{1 \times 1} \end{pmatrix},
\]

(23)

where \((T^{A,B})_{i,j}\) means a \(i \times j\) matrix.

It is easy to find the corresponding generators in \(SU(6)\) for the generators of the \(SU(3)_C\) and \(SU(2)_L\) gauge symmetry. Here, we write down the explicit generators for \(U(1)_Y\) and \(U(1)_I\)

\[
T_{U(1)_Y} = \text{diag}(0, 0, 0, 1, \frac{1}{2}, \frac{1}{2}, -1),
\]

(24)
\[ T_{U(1)'} = \frac{1}{2\sqrt{3}} \text{diag}(1,1,1,-1,-1,-1) . \]  

Once again, we would like to emphasize that only the $SU(6)/P^z = SU(3)_C \times SU(3) \times U(1)'$ gauge symmetry and 4-dimensional $N = 2$ supersymmetry are preserved on the 4-brane at $z = 0$. Therefore, the previous orbifold $SU(3)_C \times SU(3)$ model can be embedded on the 4-brane at $z = 0$. Similar to the discussions in Refs. [5, 6, 7], the tree level weak mixing angle $\sin^2 \theta_W$ at the $SU(3)$ unification scale is 0.25, which is close to that at weak scale (0.2312). And the correct hypercharges for the Standard Model quarks and leptons can be obtained from the gauge invariant of the Yukawa couplings and four anomaly-free conditions: $[SU(3)_C]^2 U(1)_Y$, $[SU(2)_L]^2 U(1)_Y$, $[U(1)]^3$ and $[\text{Gravity}]^2 U(1)_Y$, since we only introduce the extra exotic particles which are vector-like under the Standard Model gauge symmetry.

For a generic multiplet $\Phi(x^\mu, y)$ which fills a representation of the gauge group $SU(3)$ on the 4-brane at $z = 0$, we can define the parity operator $P_y$

\[ \Phi(x^\mu, y) \rightarrow \Phi(x^\mu, -y) = \eta_\Phi P^\mu \Phi(x^\mu, y)(P^{-1})^{y\mu} , \]  

where $\eta_\Phi = \pm 1$.

The KK mode expansions for the bulk fields and general model building can be found in Ref. [3]. Choosing $\eta_\Phi_u = \eta_\Phi_d = -1$, we obtain the particle spectrum for the vector multiplet and Higgs fields which is given in Table 1. We also present the gauge superfields, the number of the 4-dimensional supersymmetry and gauge groups on the 3-branes at the fixed points or 4-branes on the fixed lines in Table 2. For the zero modes, the 6-dimensional $N = 2$ supersymmetry and the $SU(6)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry.

### 3.2 Anomaly Cancellation and Exotic Particles

The anomaly from the massive KK modes of the Higgs hypermultiplets $\Psi_u$, $\Psi_d$ and $\Psi_S$ on the 4-brane at $z = 0$ can be cancelled by introducing the suitable Chern-Simons terms on the 4-brane at $z = 0$ or the bulk topological term [18, 19] because of the anomaly inflow [20, 21]. The chiral zero modes for $\Sigma^{\hat{a}, \hat{b}}_5$ are just a pair of Higgs doublets with quantum number $(1; 2; 3/2; 0)$ and $(1; 2; -3/2; 0)$ under the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry, then, they do not contribute to anomaly. So, only the chiral zero modes of the Higgs hypermultiplets $\Psi_u$, $\Psi_d$ and $\Psi_S$ will contribute to the localized anomaly, which is split on the 3-branes at $(y = 0, z = 0)$ and $(y = \pi R_1, z = 0)$. In fact, the 4-dimensional anomaly cancellation is sufficient to ensure the consistency of the higher dimensional orbifold theory [18, 19]. In other words, we only need to consider the anomaly localized on the 3-branes at $(y = 0, z = 0)$ and $(y = \pi R_1, z = 0)$ due to the chiral zero modes of the Higgs on the 4-brane at $z = 0$, the Standard Model fermions and exotic particles. In addition, if the anomaly localized on the 3-brane at $(y = 0, z = 0)$ and the anomaly localized on the 3-brane at $(y = \pi R_1, z = 0)$ have the opposite sign and same magnitude, the total anomalies can
be cancelled by introducing the suitable Chern-Simons terms on the 4-brane at $z = 0$
the sum of the localized anomalies on the 3-branes

The anomaly from the Standard Model Higgs and fermions under the Standard Model gauge symmetry cancel each other, and we only introduce the extra exotic particles which are vector-like under the Standard Model gauge symmetry. So, the anomaly will of course involve at least one $U(1)'$.

To generate the $\mu$ term, we introduce a singlet $S$ with $U(1)'$ charge $s$. In order
to cancel the $U(1)'$ anomalies, we will introduce extra exotic particles\[^3\]: $k_3$ copies of $F_i$ and $\tilde{F}_i$ where $i = 1, 2, ..., k_3$, $k'_3$ copies of $F'_i$ and $\tilde{F}'_i$, $k_2$ copies of $X_i$ and $\bar{X}_i$, $k'_2$ copies of $X'_i$ and $\bar{X}'_i$, $k_1$ copies of $Y_i$ and $\bar{Y}_i$, $k'_1$ copies of $Y'_i$ and $\bar{Y}'_i$, $k_0$ copies of $Z_i$ and $\bar{Z}_i$, $k'_0$ copies of $Z'_i$ and $\bar{Z}'_i$. We also include the right handed neutrinos $\bar{\nu}_i$.
The quantum numbers for the Standard Model fermions and extra exotic particles
under the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry are given in Table 3.

And the anomaly free conditions for $[SU(3)_C]^2U(1)'$, $[SU(2)_L]^2U(1)'$, $[U(1)_Y]^2U(1)'$, $U(1)_Y[U(1)']^2$, $[U(1)']^3$ and [Gravity]$^2U(1)'$ respectively are

\[ -3 + 2k_3 - k'_3 = 0, \quad (27) \]

\[ (2k_2 - k'_2 + 1)(c_1 + c_2) + 3(a' + 3b') = 0, \quad (28) \]

\[ \left( \frac{9}{2} - 2k_1 + k'_1 - k_2 + \frac{k'_2}{2} - \frac{2k_3}{3} + \frac{k'_3}{3} \right)(c_1 + c_2) + \frac{3}{2}(a' + 3b') = 0, \quad (29) \]

\[ -3a'^2 + 3b'^2 + 3(a' + c_1)^2 - 6(b' + c_2)^2 + 3(b' + c_1)^2 + 2c_2^2 - 2c_4^2 \]

\[ + k_3((3d_3 + 2s)^2 - d_3^2) + k'_3((d'_3 - s)^2 - d'_3^2) + k_2(d_2^2 - (d_2 + 2s)^2) \]

\[ + k'_2(d'_2^2 - (d'_2 - s)^2) + k_1((d_1 + 2s)^2 - d_1^2) + k'_1((d'_1 - s)^2 - d'_1^2) = 0, \quad (30) \]

\[ 6a'^3 - 3(a' + \delta)^3 - 3(a' + c_1)^3 + 18b'^3 - 9(b' + c_2)^3 - 9(b' + c_1)^3 \]

\[ + k_0(d_0^3 - (d_0 + 2s)^3) + k'_0(d'_0^3 - (d'_0 - s)^3) + k_1(d_1^3 - (d_1 + 2s)^3) \]

\[ + k'_1(d'_1 - (d'_1 - s)^3) + k_2(d_2^3 - (d_2 + 2s)^3) + k'_2(d'_2 - (d'_2 - s)^3) \]

\[ + 3k_3(d_3^3 - (d_3 + 2s)^3) + 3k'_3(d'_3 - (d'_3 - s)^3) + c_1 + c_2^3 + 7s^3 = 0, \]

\[ (-12 + 2k_0 - k'_0 + 2k_1 - k'_1 + 4k_2 - 2k'_2 + 6k_3 - 3k'_3)(c_1 + c_2) + 3(c_2 - \delta) = 0. \quad (32) \]

Using Eqs. (27) and (28), we can reduce the Eq. (29) to

\[ 3 - 2k_1 + k'_1 - 2k_2 + k'_2 = 0, \quad (33) \]

\[^3\text{For simplicity, we do not consider the other exotic particles, for example, the particles transforms as } (3; 1; 2/3; \gamma) \text{ and } (3; 1; -2/3; -(\gamma + 2s)) \text{ or } (3; 1; 2/3; \gamma) \text{ and } (3; 1; -2/3; -(\gamma - s)).\]
and then, reduce Eq. (32) to
\[(2k_0 - k'_0 + 2k_2 - k'_2)(c_1 + c_2) + 3(c_2 - \delta) = 0.\] (34)

The simple solutions of Eqs. (27) and (33) for \(k_i\) and \(k'_i\) are
\[k_1 = 2, \ k'_1 = 1, \ k_2 = 0, \ k'_2 = 0, \ k_3 = 2, \ k'_3 = 1,\] (35)
\[k_1 = 2, \ k'_1 = 0, \ k_2 = 0, \ k'_2 = 1, \ k_3 = 2, \ k'_3 = 1,\] (36)
\[k_1 = 1, \ k'_1 = 1, \ k_2 = 1, \ k'_2 = 0, \ k_3 = 2, \ k'_3 = 1,\] (37)
\[k_1 = 1, \ k'_1 = 0, \ k_2 = 1, \ k'_2 = 1, \ k_3 = 2, \ k'_3 = 1,\] (38)
\[k_1 = 0, \ k'_1 = 1, \ k_2 = 2, \ k'_2 = 0, \ k_3 = 2, \ k'_3 = 1.\] (39)

The \(U(1)'\) charge for the right handed neutrinos is \(-(a' + \delta)\), and we consider the following five scenarios for \(\delta\):

(I) \(\delta = c_2\) and \(a' + c_2 - s = 0\).

(II) \(\delta = c_2 + s\) and \(2a' + 2c_2 + 3s = 0\).

(III) \(\delta = c_2 + 3s\) and \(a' + c_2 + 2s = 0\).

(IV) \(\delta = c_2 - s\) and \(a' + c_2 - 2s = 0\).

(V) \(\delta = c_2 - 2s\) and \(a' + c_2 - 3s = 0\).

It is not difficult to find the irrational solutions to the Eqs. (27-32). Mathematically speaking, the rational \(U(1)'\) charges for all the particles are equivalent to the integer \(U(1)'\) charges for all the particles. To be concrete, we present the sample rational solutions to the Eqs. (27-32) in Tables 4, and 5 for above five scenarios, with the \(k_1, k'_1, k_2, k'_2, k_3\) and \(k'_3\) given in Eqs. (36), and (37), respectively. If \(k_0 = 0\) or \(k'_0 = 0\), we do not have the exotic particles \(Z_i\) and \(\bar{Z}_i\) or \(Z'_i\) and \(\bar{Z}'_i\). Thus, \(d_0\) or \(d'_0\) is not relevant. For simplicity, we write \(d_0 = X\) or \(d'_0 = X\) if \(k_0 = 0\) or \(k'_0 = 0\) in the Tables.

### 3.3 Gauge Coupling Unification

For the gauge coupling unification, the known problem is that the localized gauge kinetic terms on the 3-branes at the fixed points or on the 4-branes at the fixed lines may spoil the desired predictivity of higher dimensional gauge theory. For example,
as shown in the Table 2, the smaller gauge symmetries other than \( SU(6) \) are respected on some 3-branes or 4-branes. This implies that at the fundamental scale, or cutoff scale \( M_* \) in the theory, the \( SU(6) \) gauge coupling relation might not be preserved to a good precision. In this paper, we consider the \( SU(6) \) unification scale as the fundamental scale, or cutoff scale \( M_* \) in the theory. To suppress these uncertainties, one may assume that the theory is strongly coupled at the cutoff scale, and the gauge couplings for the bulk and localized kinetic terms are of the same magnitude at the cutoff scale. If the cutoff scale is much higher than the compactification scale, the uncertainties from the localized terms are then suppressed to a good precision [22]. We are not going to pursue this idea in this paper. Basically it gives a strong constraint. On the other hand, we would like to study by numerical calculations how well the gauge coupling unification can be achieved using the particle contents involved and leave the problem for further study (for example from the point of view of string theory). In fact, at tree-level, there is no localized gauge kinetic term on the orbifold fixed point in the weakly coupled heterotic string theory.

As an example, we discuss the scenario II given in Table 4. Clearly, it’s the KK modes of the Higgs multiplets on the 4-brane at \( z = 0 \) that contribute to the asymmetric radiative corrections to the \( SU(3)_C \) and \( SU(3) \) gauge coupling runnings. One can see from Table 1 that the even modes along the \( z \) direction \( V^{a,b}, V^{\tilde{a},b}, V^{\tilde{a},b}_5, \Sigma^{a,b}_5 \), and \( \Sigma^a_5 \), which are relevant for gauge coupling runnings at the energy scale \( 1/R_1 < \mu < 1/R_2 \), form adjoint representations under the \( SU(3)_C \times SU(3) \) gauge symmetry. For energy scale above \( 1/R_2 \), the spectrum of massive KK modes is actually \( SU(6) \) symmetric in the gauge sector. One can also see from the Table 1 that the Higgs triplets give no asymmetric radiative corrections to the \( SU(2)_L \) and \( U(1)_Y \) gauge coupling runnings. Unlike the story in Ref. [5], the massive KK modes do not help us much on the unification of three gauge couplings. Interestingly, the extra exotic particles can indeed do the job. Notice that there are quite a few exotic particles which are \( SU(2)_L \) singlets and charged under \( U(1)_Y \), the exotic particles do accelerate the \( SU(3) \) asymmetric logarithmic gauge coupling runnings. For the exotic particles with masses around 1.2 TeV, we can have the gauge coupling unification to a good precision (less than 1%) if \( 1/R_1 \simeq 3 \sim 4 \) TeV and \( 1/R_2 \simeq 6/R_1 \) and \( M_* \simeq 50/R_1 \). The gauge coupling (\( \alpha \)) at the unification scale \( M_* \) is around \( 1/55 \), which implies that the gauge coupling \( \alpha'_1 \) for \( U(1)' \) is about \( 1/105 \) at energy scale 1.2 TeV.

The discussions of gauge coupling unification for all five scenarios in Tables 4-5 are similar. In short, the gauge coupling unification can be achieved at the \( 100 \sim 200 \) TeV if the compactification scale for the fifth dimension \( (1/R_1) \) is about \( 3 \sim 4 \) TeV.

### 3.4 \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) Gauge Symmetry Breaking and Particle Masses

In order to avoid the proton decay problem from the non-renormalizable operators, we put the Standard Model quarks, \( Q_i, \bar{U}_i, \bar{D}_i \), and the exotic triplet particles \( F_i, \bar{F}_i, F'_i \) and \( \bar{F}'_i \) on the 3-brane at \( (y = 0, z = 0) \). Moreover, we put the Standard Model
leptons $L_i$, $E_i$ and $\bar{\nu}_i$, $S$, and the rest of the exotic particles, $X_i$, $X'_i$, $X''_i$, $Y_i$, $Y'_i$, $Y''_i$, $Z_i$, $Z'_i$ and $Z''_i$ on the 3-brane at $(y = \pi R_1, z = 0)$. As an example, we consider the scenario II in Table 4 in the last subsection, the proton decay operators are suppressed by the factor $e^{-3(\pi R_1 M_4)^2/4} \sim e^{-18506} \sim 10^{-8142}$ [23].

First, let us discuss the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry breaking and briefly review the results in Ref. [16].

Because on the 4-brane at $z = 0$, there exists the 4-dimensional $N = 2$ supersymmetry, we have to put the superpotential on the 3-brane at $(y = \pi R_1, z = 0)$ where only the 4-dimensional $N = 1$ supersymmetry is preserved. The localized superpotential for the Higgs fields are

$$S = \int d^4 x dy dz \delta(y - \pi R_1) \delta(z) \left[ \int d^2 \theta (h^5 S H d H_u + \lambda^5 S_1 S_2 S_3) + H.C. \right],$$

(40)

where $h^5$ and $\lambda^5$ are 5-dimensional couplings. Similar notations are used for the other 5-dimensional couplings.

In the 4-dimensional effective theory, we obtain the superpotential

$$W = h S H d H_u + \lambda S_1 S_2 S_3,$$

(41)

where $h = h^5/(\pi R_1 M_4)$ and $\lambda = \lambda^5/(\pi R_1 M_4)^{3/2}$, because there is a normalization factor $1/(\pi R_1 M_4)^{1/2}$ for each field on the 4-brane at $z = 0$ [24]. For the scenario II in Table 4 in the last subsection, $(\pi R_1 M_4)^{1/2} \sim 12.5$. Because it is natural to take $h^5 \sim \lambda^5$, we obtain $h \sim 10 \lambda$ which is crucial to generate the $Z - Z'$ mass hierarchy.

The corresponding $F$-term scalar potential is

$$V_F = h^2 \left( |H_d|^2 |H_u|^2 + |S|^2 |H_d|^2 + |S|^2 |H_u|^2 \right)$$

$$+ \lambda^2 \left( |S_1|^2 |S_2|^2 + |S_2|^2 |S_3|^2 + |S_3|^2 |S_1|^2 \right).$$

(42)

The $D$-term scalar potential is

$$V_D = \frac{G^2}{8} \left( |H_u|^2 - |H_d|^2 \right)^2$$

$$+ \frac{1}{2} g_{Z'}^2 \left( s |S|^2 + c_1 |H_d|^2 + c_2 |H_u|^2 - s (|S_1|^2 + |S_2|^2) + 2s |S_3|^2 \right)^2,$$

(43)

where $G^2 = g_1^2 + g_2^2$, $g_1$, $g_2$, and $g_{Z'}$ are the coupling constants for $U(1)_Y$, $SU(2)_L$ and $U(1)'$, respectively.

In addition, we introduce the supersymmetry breaking soft terms

$$V^{(a)}_{soft} = m_H^2 |H_d|^2 + m_{H_u}^2 |H_u|^2 + m_S^2 |S|^2 + \sum_{i=1}^3 m_{S_i}^2 |S_i|^2$$

$$- (A_h h S H_d H_u + A_\lambda \lambda S_1 S_2 S_3 + m_{SS_1}^2 S S_1 + m_{SS_2}^2 S S_2 + H.C.).$$

(44)

To eliminate the runaway directions of the potential that are not bounded from below, we require

$$m_S^2 + m_{S_1}^2 + 2m_{SS_1}^2 > 0, \quad m_S^2 + m_{S_2}^2 + 2m_{SS_2}^2 > 0.$$

(45)
The first condition avoid the runaway direction in which \(< S >= < S_1 >\) with the other VEVs vanishing, for which the quartic and cubic terms in the potential are flat. Similarly for the second condition.

The \(Z - Z'\) mass matrix is

\[
M_{Z-Z'} = \begin{pmatrix} M_Z & M_{ZZ'} \\ M_{Z'Z} & M_{Z'} \end{pmatrix},
\]

(46)

where

\[
M_Z^2 = \frac{G^2}{2} (v_1^2 + v_2^2),
\]

(47)

\[
M_{Z'}^2 = 2g_{Z'}^2 \left( Q_S^2 < S >^2 + Q_{tt'}^2 v_1^2 + Q_{Ht}^2 v_2^2 + \sum_{i=1}^{3} Q_{S_i}^2 < S_i >^2 \right),
\]

(48)

\[
M_{ZZ'}^2 = g_{Z'} G (Q_{Ht} v_1^2 - Q_{Ht} v_2^2),
\]

(49)

where

\[
\langle H_1^0 \rangle \equiv v_1, \langle H_2^0 \rangle \equiv v_2.
\]

(50)

The mass eigenvalues are

\[
M_{Z_1,Z_2}^2 = \frac{1}{2} \left( M_Z^2 + M_{Z'}^2 \mp \sqrt{(M_Z^2 - M_{Z'}^2)^2 + 4M_{ZZ'}^2} \right),
\]

(51)

and the \(Z - Z'\) mixing angle \(\theta_{Z-Z'}\) is given by

\[
\theta_{Z-Z'} = \frac{1}{2} \arctan \left( \frac{2M_{ZZ'}^2}{M_{Z'}^2 - M_Z^2} \right),
\]

(52)

which is constrained to be less than a few times \(10^{-3}\).

It has been shown that this potential can generate 1 TeV scale VEVs for \(S_1\), \(S_2\) and \(S_3\), and 100 ∼ 200 GeV scale VEVs for \(S\), \(H_d\) and \(H_u\) [16]. As an example, we discuss the potential with the \(U(1)'\) charges for the Higgs and the low energy \(U(1)'\) gauge coupling in the scenario II given in Table 4 in the last subsection. With the input \(h = 0.75, \lambda = 0.075, A_h = A_\lambda = 1.0, m_{H_d}^2 = m_{H_u}^2 = m_{S}^2 = -0.010, m_{S_1}^2 = m_{S_2}^2 = 0.031, m_{S_3}^2 = 0.010, m_{S_1}^2 = m_{S_2}^2 = m_{S_3}^2 = -0.010, c_1 = 1/6, c_2 = 5/6, \) and \(s = -1\), we obtain that the VEVs for Higgs fields at the minimum are \(< H_d > = 0.626, < H_u > = 0.631, < S > = 0.949, < S_1 > = 6.45, < S_2 > = 6.44, \) and \(< S_3 > = 6.43. The input parameters with dimensions of mass or mass-squared are chosen in arbitrary units. After finding an acceptable minimum, they are rescaled so that \(\sqrt{v_1^2 + v_2^2} \simeq 174\) GeV. After rescaling, the VEVs for Higgs fields at the minimum are \(< H_d > = 122.5\) GeV, \(< H_u > = 123.5\) GeV, \(< S > = 185.9\) GeV, \(< S_1 > = 1262.8\) GeV, \(< S_2 > = 1260.2\)
GeV, and \( < S_3 >= 1258.5 \) GeV. Thus, we obtain that \( \mu = h < S >= 139.4 \) GeV, \( M_Z = 91.37 \) GeV, \( M_{Z'} = 1513 \) GeV and \( \theta_{Z-Z'} = 1.15 \times 10^{-3} \).

In short, we can solve the \( \mu \) problem and generate \( Z-Z' \) mass hierarchy naturally.

Second, we discuss the Yukawa couplings for the Standard Model quarks and leptons, and the exotic particle masses. We can introduce the following superpotentials localized on the 3-branes at \((y = 0, z = 0)\) and \((y = \pi R_1, z = 0)\)

\[
S = \int d^4x dy dz \delta(y) \delta(z) \int d^2 \theta \left( y_{u_{ij}} Q_i H_u \bar{U}_j + y_{d_{ij}} Q_i H_d \bar{D}_j \\
+ h_{F_1} S_3 F_i \bar{F}_i + h_{F_1} S_1 F'_i \bar{F}'_i + \lambda_{F_1} S_2 F'_i \bar{F}'_i + H.C. \right) \\
+ \int d^4x dy dz \delta(y - \pi R_1) \delta(z) \int d^2 \theta \left( y_{e_{ij}} L_i H_d \bar{E}_j \\
+ h_{X_1} S_3 X_i \bar{X}_i + h_{X_1} S_1 X'_i \bar{X}'_i + \lambda_{X_1} S_2 X'_i \bar{X}'_i \\
+ h_{Y_1} S_3 Y_i \bar{Y}_i + h_{Y_1} S_1 Y'_i \bar{Y}'_i + \lambda_{Y_1} S_2 Y'_i \bar{Y}'_i \\
+ h_{Z_1} S_3 Z_i \bar{Z}_i + h_{Z_1}' S_1 Z'_i \bar{Z}'_i + \lambda_{Z_1}' S_2 Z'_i \bar{Z}'_i + H.C. \right). \tag{53}
\]

In the 4-dimensional effective theory, we have the following superpotential

\[
W = y_{u_{ij}} Q_i H_u \bar{U}_j + y_{d_{ij}} Q_i H_d \bar{D}_j + y_{e_{ij}} L_i H_d \bar{E}_j \\
+ h_{F_1} S_3 F_i \bar{F}_i + h_{F_1} S_1 F'_i \bar{F}'_i + \lambda_{F_1} S_2 F'_i \bar{F}'_i \\
+ h_{X_1} S_3 X_i \bar{X}_i + h_{X_1} S_1 X'_i \bar{X}'_i + \lambda_{X_1} S_2 X'_i \bar{X}'_i \\
+ h_{Y_1} S_3 Y_i \bar{Y}_i + h_{Y_1} S_1 Y'_i \bar{Y}'_i + \lambda_{Y_1} S_2 Y'_i \bar{Y}'_i \\
+ h_{Z_1} S_3 Z_i \bar{Z}_i + h_{Z_1}' S_1 Z'_i \bar{Z}'_i + \lambda_{Z_1}' S_2 Z'_i \bar{Z}'_i. \tag{54}
\]

Therefore, all the exotic particles can obtain the masses around 1 TeV after the gauge symmetry breaking because the VEVs for \( S_i \) are at the order of 1 TeV.

Third, one might notice that there are zero modes for \( S_u^c, S_d^c, \) and \( \Sigma_5^{a,b} \) from Table 1. The zero modes of \( S_u^c \) and \( S_d^c \) can obtain the masses via the following localized superpotential

\[
S = \int d^4x dy dz \delta(y) \delta(z) \left[ \int d^2 \theta (\lambda_{S_1} S_1 S_u^c S_d^c + \lambda_{S_2} S_2 S_u^c S_d^c) + H.C. \right] \\
+ \int d^4x dy dz \delta(y - \pi R_1) \delta(z) \left[ \int d^2 \theta (\lambda_{S_1} S_1 S_u^c S_d^c + \lambda_{S_2} S_2 S_u^c S_d^c) + H.C. \right], \tag{55}
\]

or in the 4-dimensional effective theory, the superpotential is

\[
W = (\lambda_{S_1} + \lambda_{S_1}') S_1 S_u^c S_d^c + (\lambda_{S_2} + \lambda_{S_2}') S_2 S_u^c S_d^c. \tag{56}
\]

Thus, the \( S_u^c \) and \( S_d^c \) can have masses at the order of 1 TeV if \( \lambda_{S_1}, \lambda_{S_1}', \lambda_{S_2} \) and \( \lambda_{S_2}' \) are at the order of 1.

Moreover, \( \Sigma_5^{a,b} \) are just one pair of Higgs doublets with quantum number \((1; 2; 3/2; 0)\) and \((1; 2; -3/2; 0)\) under the \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) gauge
symmetry. The fermionic components of $\Sigma^a_b$ can obtain the gaugino masses via the supersymmetry breaking, for instance, the gaugino masses are proportional to $1/R_1$ in the Scherk-Schwarz mechanism supersymmetry breaking. Meanwhile, the bosonic components of $\Sigma^a_b$ receive the masses of order of $1/(4\pi R_1)$ through the radiative corrections. Because $1/(4\pi R_1)$ is about several hundreds of GeV in our model, the reasonable masses for $\Sigma^a_b$ are around several hundreds of GeV range. Alternatively, $\Sigma^a_b$ can obtain masses at the order of several GeVs via the non-renormalizable localized superpotential,

$$S = \int d^4xdydz \delta(y-\pi R_1)\delta(z) \left[ \int d^2\theta \frac{\lambda_{SS1}^5}{M_*} S S^T_1 tr(\Sigma^a_b \Sigma^a_b) + H.C. \right], \quad (57)$$

or in the 4-dimensional effective theory, the superpotential is

$$W = \frac{\lambda_{SS1}^5}{M_*} S S^T_1 tr(\Sigma^a_b \Sigma^a_b). \quad (58)$$

We also want to point out that $\Sigma^a_b$ can not couple to the Standard Model fermions and exotic particles due to the $U(1)'$ gauge symmetry, so, we have no low bounds on the masses of $\Sigma^a_b$, as long as they are not massless which may cause problem in cosmology.

Fourth, the dimension-5 operators $y_{\nu ij} L_i L_j H_u H_u / M_*$ are forbidden by $U(1)'$ gauge symmetry and we would like to discuss the neutrino masses for all five scenarios with different $\delta$. For scenario (I), we will give a complete discussion, and for the other scenarios, we only give brief descriptions.

(I) $\delta = c_2$ and $a' + c_2 - s = 0$. The neutrinos can obtain masses via the localized superpotential

$$S = \int d^4xdydz \delta(y-\pi R_1)\delta(z) \left[ \int d^2\theta h_{\nu ij}^5 S_3 \bar{\nu}_i \bar{\nu}_j + y_{\nu ij}^5 H_u L_i \bar{\nu}_j + H.C. \right], \quad (59)$$

or in the 4-dimensional effective theory, the superpotential is

$$W = h_{\nu ij} S_3 \bar{\nu}_i \bar{\nu}_j + y_{\nu ij} H_u L_i \bar{\nu}_j. \quad (60)$$

In the basis $\{\nu_1, \nu_2, \nu_3, \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3\}$, the neutrino mass matrix is

$$M = \begin{pmatrix} 0 & M_D^T \\ M_D & M_N \end{pmatrix}, \quad (61)$$

where

$$M_D = \begin{pmatrix} y_{\nu 11} < H^0_u & y_{\nu 12} < H^0_u & y_{\nu 13} < H^0_u \\ y_{\nu 21} < H^0_u & y_{\nu 22} < H^0_u & y_{\nu 23} < H^0_u \\ y_{\nu 31} < H^0_u & y_{\nu 32} < H^0_u & y_{\nu 33} < H^0_u \end{pmatrix}. \quad (62)$$
\[ M_N = \begin{pmatrix} h_{r11} < S_3 > & h_{r12} < S_3 > /2 & h_{r13} < S_3 > /2 \\ h_{r12} < S_3 > /2 & h_{r22} < S_3 > & h_{r23} < S_3 > /2 \\ h_{r13} < S_3 > /2 & h_{r23} < S_3 > /2 & h_{r33} < S_3 > \end{pmatrix}. \] (63)

The left-handed neutrino mass matrix \( M_\nu \) is equivalent to

\[ M_\nu = M_D M_N^{-1} M_D^T. \] (64)

Because \( < S_3 > \) is at the order of 1 TeV in our model, the right-handed neutrino masses are about 1 TeV if \( h_{rij} \) is at the order of 1. Therefore, in order to have the realistic active neutrino masses, we obtain that the neutrino Dirac masses should be at the order of \( 10^{-3} \) GeV, \( \text{i.e.}, \) the Yukawa couplings \( (y_{\nu_{ij}}) \) should be at the order of \( 10^{-5} \sim 10^{-6} \) or less.

(II) \( \delta = c_2 + s \) and \( 2a' + 2c_2 + 3s = 0 \). The neutrinos can obtain masses via the following superpotential in the 4-dimensional effective theory

\[ W = h_{rij} S_1 \bar{\nu}_i \nu_j + h'_{rij} S_2 \bar{\nu}_i \nu_j + y_{\nu_{ij}} \frac{S}{M_s} H_u L_i \bar{\nu}_j. \] (65)

Because in our model, \( < S_1 > \) and \( < S_2 > \) are at the order of 1 TeV, the right-handed neutrino masses can be about 5 TeV if \( h_{rij} \) and \( h'_{rij} \) are about 2.5. So, in order to have the realistic active neutrino masses, we obtain that the couplings \( y_{\nu_{ij}} \) for the non-renormalizable terms should be at the order of or smaller than 0.1 because \( M_s \sim 2 \times 10^5 \) GeV.

(III) \( \delta = c_2 + 3s \) and \( a' + c_2 + 2s = 0 \). The neutrinos can obtain masses via the following superpotential in the 4-dimensional effective theory

\[ W = h_{rij} S_3 \bar{\nu}_i \nu_j + y_{\nu_{ij}} \frac{1}{(\pi R_1 M_s)^{1/2}} \frac{SS_3}{M_s^2} H_u L_i \bar{\nu}_j, \] (66)

where compare to the tree-level Yukawa coupling, we have an extra \( 1/(\pi R_1 M_s)^{1/2} \) factor because \( S_3 \) is on the 4-brane at \( z = 0 \), and then, there is a normalization factor \( 1/(\pi R_1 M_s)^{1/2} \) for it [24]. The right-handed neutrino masses are about 1 TeV if \( h_{rij} \) is at the order of 1. So, in order to have the realistic active neutrino masses, we obtain that the couplings \( y_{\nu_{ij}} \) for the non-renormalizable terms should be at the order of or smaller than 10.

(IV) \( \delta = c_2 - s \) and \( a' + c_2 - 2s = 0 \). The neutrinos can obtain masses via the following superpotential in the 4-dimensional effective theory

\[ W = h_{rij} S_3 \bar{\nu}_i \nu_j + \frac{1}{(\pi R_1 M_s)^{1/2}} (y_{\nu_{ij}} \frac{S_1}{M_s} + y'_{\nu_{ij}} \frac{S_2}{M_s}) H_u L_i \bar{\nu}_j. \] (67)

The right-handed neutrino masses are about 1 TeV if \( h_{rij} \) is at the order of 1. So, in order to have the realistic active neutrino masses, we obtain that the couplings \( y_{\nu_{ij}} \) and \( y'_{\nu_{ij}} \) for the non-renormalizable terms should be at the order of or smaller than \( 10^{-2} \). If there is cancellation in \( (y_{\nu_{ij}} < S_1 > + y'_{\nu_{ij}} < S_2 >) \), the couplings \( y_{\nu_{ij}} \) and \( y'_{\nu_{ij}} \) can be at the order of 0.1.
(V) \( \delta = c_2 - 2s \) and \( \alpha' + c_2 - 3s = 0 \). The neutrinos can obtain masses via the following superpotential in the 4-dimensional effective theory

\[
W = h_{\alpha\beta} S_3 \bar{\nu}_i \nu_j + y_{\alpha\beta} \frac{1}{\pi R_1 M_*} S_1 S_2 H_u L_i \bar{\nu}_j .
\] (68)

The right-handed neutrino masses are about 1 TeV if \( h_{\alpha\beta} \) is at the order of 1. So, in order to have the realistic active neutrino masses, we obtain that the couplings \( y_{\alpha\beta} \) for the non-renormalizable terms should be at the order of or smaller than 10.

In short, the very tiny realistic neutrino masses can be generated naturally in the scenarios (II), (III), (IV) and (V).

4 \( SU(6) \) Model on \( M^4 \times T^2/(Z_2)^4 \)

In this section, we would like to discuss the 6-dimensional \( N = 2 \) supersymmetric \( SU(6) \) model on the space-time \( M^4 \times T^2/(Z_2)^4 \). Here again, on the 4-brane at \( z = 0 \), because of the orbifold projections, there exist only the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry and 4-dimensional \( N = 2 \) supersymmetry, where the previous orbifold \( SU(3)_C \times SU(3) \) model in Refs. [5, 6, 7] can be embedded.

4.1 Orbifold \( SU(6) \) Breaking, and Particle Spectrum for Gauge and Higgs Fields

In our model, there exist only the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry and 4-dimensional \( N = 2 \) supersymmetry on the 4-brane at \( z = 0 \), and the lepton fields forming triplets are on the 3-brane at \( (y = \pi R_1/2, z = 0) \) in which the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry is preserved. Therefore, in order to have the realistic lepton masses, we put one pair of Higgs sextet \( \Psi_u \) and \( \Psi_d \) which transform as \((1, 6, c_2)\) and \((1, 6, c_1)\) under the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry on the 4-brane at \( z = 0 \). We require that \( c_1 \neq 0, c_2 \neq 0 \) and \( c_1 + c_2 \neq 0 \). We also add three \( SU(3)_C \times SU(3) \) singlets, \( \Psi_{S_1}, \Psi_{S_2} \) and \( \Psi_{S_3} \) with \( U(1)' \) charges \(-s, -s \) and \( 2s \) respectively on the 4-brane at \( z = 0 \), where \( s \equiv -c_1 - c_2 \). In terms of the 4-dimensional \( N = 1 \) supersymmetry language, the hypermultiplets \( \Psi_u \) and \( \Psi_d \) can be decomposed into two pairs of chiral multiplets, \( (\Phi_u, \Phi^c) \) and \( (\Phi_d, \Phi^d) \), and the hypermultiplet \( \Psi_{S_i} \) can be decomposed into one pair of chiral multiplets \( (S_i, S^c_i) \) in which \( i = 1, 2, 3 \). Here, the superscript \( c \) means the charge conjugation. To be explicit, we write down the component of \( \Phi_u \) and \( \Phi_d \),

\[
\Phi_u = \begin{pmatrix}
\eta_u^- / \sqrt{2} & \eta_u^0 / \sqrt{2} & H_u^0 / \sqrt{2} \\
\eta_u^- / \sqrt{2} & \eta_u^0 & H_u^+ / \sqrt{2} \\
H_u^0 / \sqrt{2} & H_u^+ / \sqrt{2} & \eta_u^{++}
\end{pmatrix}.
\] (69)

\footnote{If we put one pair of the Higgs triplets on the 4-brane at \( z = 0 \), in order to obtain the realistic lepton masses, we have to put the quarks and leptons on the 3-brane at \( (y = 0, z = 0) \). The discussions for the model building are similar to those in Section 3. However, we might not forbid the non-renormalizable proton decay operators.}
SU(3) on the 4-brane at Model gauge symmetry.

We only introduce the extra exotic particles which are vector-like under the Standard \( \eta \) where \( \eta \) scale. And the correct hypercharges for the Standard Model quarks and leptons can \( \eta \) SU conditions: 

\[
\begin{pmatrix}
\eta^+ / \sqrt{2} & \eta^- / \sqrt{2} \\
H_u^0 / \sqrt{2} & H_d^0 / \sqrt{2} \\
H_u^- / \sqrt{2} & H_d^- / \sqrt{2}
\end{pmatrix}
\]

To simplify the notation, we denote the \( H_u^0 \) and \( H_u^+ \) components of \( \Phi_u \) as the Higgs doublet \( H_u \), and the remaining components ( \( \eta_u^- \), \( \eta_u^0 \) and \( \eta_u^{+++} \)) of \( \Phi_u \) as \( \eta_u \). Similar notation is used for \( \Phi_d \), which decomposes into \( H_d \) and \( \eta_d \).

In addition, we introduce extra exotic particles to cancel the anomalies, and put the Standard Model fermions and extra exotic particles on the 3-branes at the orbifold fixed points, as explained later.

We choose the following matrix representations for the parity operators \( P^\mu \), \( P^{\mu'} \), \( P^\tau \) and \( P^{\tau'} \), which are expressed in the adjoint representation of \( SU(6) \),

\[
P^\mu = \text{diag}(+1, +1, +1, -1, -1, -1) \quad P^{\mu'} = \text{diag}(+1, +1, +1, +1, +1, +1)
\]

\[
P^\tau = \text{diag}(+1, +1, +1, -1, -1, -1) \quad P^{\tau'} = \text{diag}(+1, +1, +1, -1, -1, -1)
\]

Therefore, under \( P^\mu \) and \( P^\tau \) parities, the \( SU(6) \) gauge generators \( T^A \), where \( A=1, 2, \ldots, 35 \) for \( SU(6) \), are separated into four sets: \( T^{a,b}, T^{a,\bar{b}}, T^{\bar{a},b}, \) and \( T^{\bar{a},\bar{b}} \). And under \( P^\mu \), \( P^\tau \), \( P^{\mu'} \) and \( P^{\tau'} \), the gauge generators transform as

\[
P^\mu T^{a,b} (P^\mu)^{-1} = T^{a,b} \quad P^\tau T^{a,\bar{b}} (P^\mu)^{-1} = -T^{a,\bar{b}}
\]

\[
P^\tau T^{a,b} (P^\tau)^{-1} = T^{a,b} \quad P^\mu T^{a,\bar{b}} (P^\tau)^{-1} = -T^{a,\bar{b}}
\]

\[
P^{\mu'} T^{a,b} (P^{\mu'})^{-1} = T^{a,b} \quad P^{\tau'} T^{a,\bar{b}} (P^{\tau'})^{-1} = -T^{a,\bar{b}}
\]

Furthermore, only the \( SU(6)/P^\tau = SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry and 4-dimensional \( N = 2 \) supersymmetry are preserved on the 4-brane at \( z = 0 \). Therefore, the previous orbifold \( SU(3)_C \times SU(3) \) model can be embedded on the 4-brane at \( z = 0 \). Following the discussions in Refs. [5, 6, 7], the tree level weak mixing angle \( \sin^2 \theta_W \) at the \( SU(3) \) unification scale is 0.25, which is close to that at weak scale. And the correct hypercharges for the Standard Model quarks and leptons can be obtained from the gauge invariant of the Yukawa couplings and four anomaly-free conditions: \( [SU(3)_C]^2 U(1)_Y, [SU(2)_L]^2 U(1)_Y, [U(1)_Y]^3 \) and \( [Gravity]^2 U(1)_Y \), because we only introduce the extra exotic particles which are vector-like under the Standard Model gauge symmetry.

For a generic multiplet \( \Phi(x^\mu, y) \) which fills a representation of the gauge group \( SU(3) \) on the 4-brane at \( z = 0 \), we can define two parity operators \( P^\mu \) and \( P^{\mu'} \)

\[
\Phi(x^\mu, y) \rightarrow \Phi(x^\mu, -y) = \eta_{\Phi} P^{\mu + \Phi} \Phi(x^\mu, y)(P^{-1})^{m_\Phi},
\]

\[
\Phi(x^\mu, y') \rightarrow \Phi(x^\mu, -y') = \eta_{\Phi}' P^{\mu + \Phi} \Phi(x^\mu, y')(P^{-1})^{m_\Phi},
\]

where \( \eta_{\Phi} = \pm 1 \) and \( \eta_{\Phi}' = \pm 1 \).
The KK mode expansions for the bulk fields and the general model discussions can be found in Ref. [3]. Choosing \( \eta_{\Phi_u} = \eta_{\Phi_d} = -1 \) and \( \eta_{\Phi_{u'}} = \eta_{\Phi_{d'}} = +1 \), we obtain the particle spectrum for the vector multiplet and Higgs fields which is given in Table 6. We also present the gauge superfields, the number of the 4-dimensional supersymmetry and gauge groups on the 3-brane at the fixed points or on the 4-branes on the fixed lines in Table 7. For the zero modes, the 6-dimensional \( N = 2 \) supersymmetry and the \( SU(6) \) gauge symmetry are broken down to the 4-dimensional \( N = 1 \) supersymmetry and the \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) gauge symmetry.

### 4.2 Anomaly Cancellation and Exotic Particles

The anomaly from the massive KK modes of the Higgs hypermultiplets \( \Psi_u, \Psi_d, \) and \( \Psi_{S_i} \) on the 4-brane at \( z = 0 \) can also be cancelled by introducing the suitable Chern-Simons terms on the 4-brane at \( z = 0 \) or the bulk topological term [18, 19] because of the anomaly inflow [20, 21]. And only the chiral zero modes of the Higgs hypermultiplets \( \Psi_u, \Psi_d, \) and \( \Psi_{S_i} \) will contribute to the localized anomaly, which is split on the 3-branes at \( (y = 0, z = 0) \) and \( (y = \pi R_1/2, z = 0) \).

Because the gauge symmetry on the 3-brane at \( (y = 0, z = 0) \) is \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \), and the gauge symmetry on the 3-brane at \( (y = \pi R_1/2, z = 0) \) is \( SU(3)_C \times SU(3) \times U(1)' \), the particles we put on the 3-brane at \( (y = \pi R_1/2, z = 0) \) must form the complete representations under the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry. Similarly, the 4-dimensional anomaly cancellation is sufficient to ensure the consistency of the higher dimensional orbifold theory [18, 19]. In other words, if the anomaly localized on the 3-brane at \( (y = 0, z = 0) \) and the anomaly localized on the 3-brane at \( (y = \pi R_1/2, z = 0) \) for the gauge group \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) have the opposite sign and same magnitude, the total anomaly can be cancelled by introducing the suitable Chern-Simons terms on the 4-brane at \( z = 0 \) or the bulk topological terms as long as the particles on the 3-brane at \( (y = \pi R_1/2, z = 0) \) form the complete representations under the \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry [18, 19] due to the anomaly inflow [20, 21]. Therefore, the anomaly free condition is that the sum of the anomaly from the chiral zero modes of the Higgs on the 4-brane at \( z = 0 \), the Standard Model fermions and exotic particles on the 3-branes at \( (y = 0, z = 0) \) and \( (y = \pi R_1/2, z = 0) \) for the gauge group \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) should be zero.

Now, we discuss the Standard Model fermions and exotic particles assignment. In order to avoid the proton decay problem from the non-renormalizable operators, we put the Standard Model quarks, \( Q_i, \bar{U}_i \) and \( \bar{D}_i \) on the 3-brane at \( (y = 0, z = 0) \), and put the Standard Model leptons \( L_i \) and \( \bar{E}_i \) which form the triplet \( T_{Li} \), and right handed neutrinos \( \bar{\nu}_i \) on the 3-brane at \( (y = \pi R_1/2, z = 0) \). We also introduce a singlet \( S \) with \( U(1)' \) charge \( s \) on the 3-brane at \( (y = \pi R_1/2, z = 0) \) so that we can generate the \( \mu \) term. Moreover, in order to avoid the anomaly, we introduce the exotic particles. For simplicity, we introduce \( k_3 \) copies of \( F_i \) and \( \bar{F}_i \) where \( i = 1, 2, ..., k_3 \), \( k'_3 \) copies of \( F'_i \) and \( \bar{F}'_i \), \( k'_2 \) copies of \( X'_i \) and \( \bar{X}'_i \), \( k_1 \) copies of \( Y_i \) and \( \bar{Y}_i \), \( k'_1 \) copies of \( Y'_i \) and \( \bar{Y}'_i \) on the 3-brane at \( (y = 0, z = 0) \). And we introduce \( k_2 \) copies of \( T_i \) and \( \bar{T}_i \), \( k_0 \)
copies of \( Z_i \) and \( \bar{Z}_i \), \( k'_i \) copies of \( Z'_i \) and \( \bar{Z}'_i \) on the 3-brane at \( (y = \pi R_1/2, z = 0) \).

The quantum numbers for the Standard Model fermions and extra exotic particles under the \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)' \) and \( SU(3)_C \times SU(3) \times U(1)' \) gauge symmetry are given in Tables 8 and 9, respectively.

Similar to the discussions in the subsection 2.2, the anomaly cancellation conditions are:

\[-3 + 2k_3 - k'_3 = 0 \tag{78}\]

\[(2k_2 - k'_2 + 1)(c_1 + c_2) + 3\left(-\frac{c_1}{2} + 3b'\right) = 0 \tag{79}\]

\[2 - 2k_1 + k'_1 - 4k_2 + k'_2 = 0 \tag{80}\]

\[3b'^2 - 6(b' + c_2)^2 + 3b'(b' + c_1)^2 + c_2^3 - c_1^2 + k_3((d_3 + 2s)^2 - d_3^2) + k'_3((d'_3 - s)^2 - d'_3^2) + k'_2(d'_2 - (d'_2 - s)^2) + k_1((d_1 + 2s)^2 - d_1^2) + k_2((d_1 - s)^2 - d_1^2) = 0 \tag{81}\]

\[-\frac{9}{8}c_1^3 - 3\left(-\frac{c_1}{2} + \delta\right)^3 + 18b'^3 - 9(b' + c_2)^3 - 9(b' + c_1)^3\]
\[+ k_0(d_0^3 - (d_0 + 2s)^3) + k'_0(d'_0^3 - (d'_0 - s)^3) + k_1(d_1^3 - (d_1 + 2s)^3) + k'_1(d'_1 - (d'_1 - s)^3) + 3k_2(d_2^3 - (d_2 + 2s)^3) + 2k'_2(d'_2 - (d'_2 - s)^3) + 3k_3(d_3^3 - (d_3 + 2s)^3) + 3k'_3(d'_3 - (d'_3 - s)^3) + 2c_1^3 + 2c_2^3 + 7s^3 = 0 \tag{82}\]

\[(2k_0 - k'_0 + 2k_2 - k'_2)(c_1 + c_2) + 3(c_2 - \delta) = 0 \tag{83}\]

The simple solutions of Eqs. (78) and (80) for \( k_i \) and \( k'_i \) are:

\[k_1 = 0 \ , \ k'_1 = 2 \ , \ k_2 = 1 \ , \ k'_2 = 0 \ , \ k_3 = 2 \ , \ k'_3 = 1 \tag{84}\]

\[k_1 = 0 \ , \ k'_1 = 1 \ , \ k_2 = 1 \ , \ k'_2 = 1 \ , \ k_3 = 2 \ , \ k'_3 = 1 \tag{85}\]

\[k_1 = 0 \ , \ k'_1 = 0 \ , \ k_2 = 1 \ , \ k'_2 = 2 \ , \ k_3 = 2 \ , \ k'_3 = 1 \tag{86}\]

In particular, for the second solution given by Eq. (85), \( X'_1 \), \( X'_2 \), \( Y'_1 \) and \( \bar{Y}'_1 \) can form one pair of the triplets \( T'_i \) and \( \bar{T}'_i \) with quantum number \( (1; 3; d'_2) \) and \( (1; 3; -(d'_2 - s)) \) respectively under the gauge group \( SU(3)_C \times SU(3) \times U(1)' \) if \( d'_2 = d'_1 \). Consequently, we can put this pair of triplets \( T'_i \) and \( \bar{T}'_i \) on the 3-brane at \( (y = \pi R_1/2, z = 0) \) instead of putting \( X'_1 \), \( X'_2 \), \( Y'_1 \) and \( \bar{Y}'_1 \) on the 3-brane at \( (y = 0, z = 0) \).
The $U(1)'$ charge for the right handed neutrinos are $-(c_1/2 + \delta)$, and we consider the following three scenarios:

(I) $\delta = d_2$ and $-c_1 + 2d_2 - 2s = 0$.

(II) $\delta = d_2 + s$ and $-c_1 + 2d_2 + 3s = 0$.

(III) $\delta = d_2 - s$ and $-c_1 + 2d_2 - 4s = 0$.

It is trivial to find the irrational solutions to the Eqs. (78-83). Mathematically speaking, the rational $U(1)'$ charges for all the particles are equivalent to the integer $U(1)'$ charges for all the particles. As an example, we present the sample rational solutions to the Eqs. (78-83) in Tables 10 and 11 for above three scenarios where the $k_1, k_1', k_2, k_3$ and $k_3'$ are given in Eqs. (84) and (86), respectively. If $k_0 = 0$ or $k_0' = 0$, we do not have the exotic particles $Z_i$ and $\bar{Z}_i$ or $Z_i'$ and $\bar{Z}_i'$. Thus, $d_0$ or $d_0'$ is not relevant. For simplicity, we write $d_0 = X$ or $d_0' = X$ if $k_0 = 0$ or $k_0' = 0$ in the Tables.

### 4.3 Gauge Coupling Unification

As an example, we discuss the gauge coupling unification for the scenario II given in Table 10. Since we have one pair of 6 and 6 Higgs rather than one pair of 3 and 3 Higgs under the $SU(3)$ gauge symmetry on the 4-brane at $z = 0$, the relative running between the $SU(3)_C$ and $SU(3)$ gauge couplings are much faster than the previous example in the subsection 3.3. Fortunately, the exotic fields help to accelerate considerably the relative running between the $SU(2)_L$ and $U(1)_Y$ gauge couplings, and make the unification possible for TeV scale compactification. We find that the gauge coupling unification can be achieved to a good precision (less than 1%) if $1/R_1 \simeq 3 - 4$ TeV with $1/R_2 \simeq 12/R_1$ and $M_* \simeq 28/R_1$. Notice that because of the large Casimir operator for the 6 and 6 representations $(5/2)$, the $SU(3)$ gauge theory is not asymptotically free above the compactification scale of the fifth dimension $(1/R_1)$. The gauge coupling $\alpha$ is around $1/22$ at the unification scale $M_*$ which implies in this scenario $\alpha'_1 \simeq 1/40$ for $U(1)'$ at energy scale 1.2 TeV.

The discussions of gauge coupling unification for all three scenarios in Tables 10 and 11 are similar. In short, the gauge coupling unification can be achieved at the $100 \sim 200$ TeV if the compactification scale for the fifth dimension $(1/R_1)$ is about $3 \sim 4$ TeV.

### 4.4 $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ Gauge Symmetry Breaking and Particle Masses

The discussions for $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry breaking are similar to those in the subsection 3.4 and Ref. [16], so, we do not repeat it here. We would like to emphasize that: (1) In our scenario, the VEVs for $H_1^0, H_2^0$ and $S$ are
at the order of 100 GeV, and the VEVs for $S_1$ are at the order of 1 TeV, then, the $\mu$ problem is solved; (2) We put the Higgs $\Phi_u, \Phi_d$ and $S_1$ on the 4-brane at $z = 0$, and put the singlet $S$ on the 3-brane at $(y = \pi R_1/2, z = 0)$. Thus, we obtain $h \sim 10\lambda$ and generate the $Z - Z'$ mass hierarchy naturally, because for instance, in the scenario II in Table 10, $(\pi R_1 M_1)^{1/2} \sim 9.4$.

As an example, we discuss the potential with the $U(1)'$ charges for the Higgs and the low energy $U(1)'$ gauge coupling in the scenario II given in Table 10 in the last subsection. With the input $h = 0.75$, $\lambda = 0.075$, $A_h = A_\lambda = 1.0$, $m_{H_u}^2 = m_{H_d}^2 = m_S^2 = -0.010, m_{S_1}^2 = m_{S_2}^2 = 0.031, m_{S_3}^2 = -0.010, m_{S_{S_1}}^2 = m_{S_{S_2}}^2 = -0.010, c_1 = -15/29, c_2 = -12/29, and s = 27/29, we obtain that the VEVs for Higgs fields at the minimum are $< H_d >= 0.628, < H_u >= 0.627, < S_1 >= 0.951, < S_1 >= 6.45, < S_2 >= 6.43, and < S_3 >= 6.42$. The input parameters with dimensions of mass or mass-squared are chosen in arbitrary units. After finding an acceptable minimum, they are rescaled so that $\sqrt{v_1^2 + v_2^2} \simeq 174$ GeV. After rescaling, the VEVs for Higgs fields at the minimum are $< H_d >= 123.1$ GeV, $< H_u >= 122.9$ GeV, $< S >= 186.4$ GeV, $< S_1 >= 1264.2$ GeV, $< S_2 >= 1260.7$ GeV, and $< S_3 >= 1258.9$ GeV. Thus, we obtain that $\mu = h < S >= 139.8$ GeV, $M_Z = 91.37$ GeV, $M_{Z'} = 2283$ GeV and $\theta_{Z-Z'} = 0.13 \times 10^{-3}$.

First, we discuss the Yukawa couplings for the Standard Model quarks and leptons, and the masses of exotic particles (include $T'_i$ and $\bar{T}'_i$). The 3-brane localized superpotential is

$$
S = \int d^4xdydz\delta(y)\delta(z) \int d^2\theta \left( y^5_{u_{ij}} Q_i H_u \bar{U}_j + y^5_{d_{ij}} Q_i H_d \bar{D}_j + h^5_{F_i} S_3 F_i \bar{F}_i + h^5_{F_i} S_1 F_i' \bar{F}_i' + \lambda^{5}_{F_i} S_2 F_i F_i' + h^5_{X_i} S_1 X_i' \bar{X}_i' + \lambda^{5}_{X_i} S_2 X_i X_i' + h^5_{Y_i} S_3 Y_i Y_i' + \lambda^{5}_{Y_i} S_2 Y_i Y_i' + H.C. \right) \\
+ \int d^4xdydz(\delta(y - \pi R_1)\delta(z) \int d^2\theta \left( y^5_{e_{ij}} T_i H_L \bar{H}_L + h^5_{T_i} S_3 T_i \bar{T}_i + h^5_{T_i} S_1 T_i' \bar{T}_i' + \lambda^5_{T_i} S_2 T_i T_i' + h^5_{Z_i} S_3 Z_i \bar{Z}_i + h^5_{Z_i} S_1 Z_i' \bar{Z}_i' + \lambda^5_{Z_i} S_2 Z_i Z_i' + H.C. \right). 
$$

(87)

In the 4-dimensional effective theory, we have the following superpotential

$$
W = y^4_{u_{ij}} Q_i H_u \bar{U}_j + y^4_{d_{ij}} Q_i H_d \bar{D}_j + y^4_{e_{ij}} L_i H_d \bar{E}_j + h^4_{F_i} S_3 F_i \bar{F}_i + h^4_{F_i} S_1 F_i' \bar{F}_i' + \lambda^4_{F_i} S_2 F_i F_i' + h^4_{X_i} S_3 X_i \bar{X}_i' + \lambda^4_{X_i} S_2 X_i X_i' + h^4_{Y_i} S_3 Y_i \bar{Y}_i + h^4_{Y_i} S_1 Y_i' \bar{Y}_i' + \lambda^4_{Y_i} S_2 Y_i Y_i' + h^4_{Z_i} S_3 Z_i \bar{Z}_i + h^4_{Z_i} S_1 Z_i' \bar{Z}_i' + \lambda^4_{Z_i} S_2 Z_i Z_i' + H.C. \right) . 
$$

(88)

Therefore, all the exotic particles can obtain masses around 1 TeV after the gauge symmetry breaking because the VEVs for $S_i$ are at the order of 1 TeV.
Second, we would like to discuss the neutrino masses. Because the Higgs sextets \( \Phi_u \) and \( \Phi_d \) can not give the neutrino Dirac masses, we assume that \( T_i \) obtain a small VEV compared to the VEVs of \( H_u \) and \( H_d \). Note that \( T_i \) can not give masses to the leptons and quarks. Because in our models, we just introduce one pair of \( T_i \) and \( \bar{T}_i \), we write \( T \) for \( T_i \) to simplify the notations. Similar to the model in Section 3, the dimension-5 operators \( y_{\nu ij} T_L i T_L i T T^*/M_* \) are forbidden by \( U(1)' \) gauge symmetry.

To be explicit, we will use \( \langle T \rangle_0 \approx 10 \text{ GeV} \) as an example to discuss the neutrino masses for three scenarios with different \( \delta \).

(I) \( \delta = d_2 \) and \( -c_1 + 2d_2 - 2s = 0 \). The neutrinos can obtain masses via the localized superpotential

\[
S = \int d^4x dy dz \delta(y - \pi R_1) \delta(z) \left[ \int d^2 \theta h_{\nu ij} S_3 \bar{\nu}_i \nu_j + y_{\nu ij} T T L_i \bar{\nu}_j + \text{H.C.} \right],
\]

or in 4-dimensional effective theory, the superpotential is

\[
W = h_{\nu ij} S_3 \bar{\nu}_i \nu_j + y_{\nu ij} T T L_i \bar{\nu}_j.
\]

Because in our model, \( \langle S_3 \rangle \) is at the order of 1 TeV, the right-handed neutrino masses are about 1 TeV if \( h_{\nu ij} \) is at the order of 1. Therefore, in order to have the realistic active neutrino masses, we obtain that the neutrino Dirac masses should be at the order of \( 10^{-3} \) GeV, i.e., the Yukawa couplings \( (y_{\nu ij}) \) should be at the order of \( 10^{-4} \sim 10^{-5} \) or less.

(II) \( \delta = d_2 + s \) and \( -c_1 + 2d_2 + 3s = 0 \). The neutrinos can obtain masses via the superpotential in the 4-dimensional effective theory

\[
W = h_{\nu ij} S_1 \bar{\nu}_i \nu_j + h'_{\nu ij} S_2 \bar{\nu}_i \nu_j + y_{\nu ij} S_{M^*} T T L_i \bar{\nu}_j.
\]

In our model, \( \langle S_1 \rangle \) and \( \langle S_2 \rangle \) are at the order of 1 TeV, then, the right-handed neutrino masses are about 2 TeV if \( h_{\nu ij} \) and \( h'_{\nu ij} \) is at the order of 1. In order to have the realistic active neutrino masses, we obtain that the couplings \( y_{\nu ij} \) for the non-renormalizable terms should be at the order of 0.2 or smaller than 0.2 since \( M_* \sim 10^5 \) GeV.

(III) \( \delta = d_2 - s \) and \( -c_1 + 2d_2 - 4s = 0 \). The neutrinos can obtain masses via the superpotential in the 4-dimensional effective theory

\[
W = h_{\nu ij} S_3 \bar{\nu}_i \nu_j + \sqrt{2} (y_{\nu ij} S_{M^*} + y'_{\nu ij} S_{M^*} T T L_i \bar{\nu}_j).
\]

For the scenario II in Table 10 in the last subsection, \( (\pi R_1 M_*/2)^{1/2} \approx 6.63 \). The right-handed neutrino masses are about 2 TeV if \( h_{\nu ij} \) is about 2. And in order to \( T_i^* \) might also obtain the VEV in general. If this is case, we assume that the VEV of \( T_i^* \) is much smaller than that of \( T_i \), similar to the large \( \tan\beta \) scenario for \( H_u \) and \( H_d \) in the Minimal Supersymmetric Standard Model.
have the realistic active neutrino masses, we obtain that the couplings $y_{\nu ij}$ and $y'_{\nu ij}$ for the non-renormalizable terms should be at the order of or smaller than 0.1. If there is cancellation in $(y_{\nu ij} < S_1 > + y'_{\nu ij} < S_2 >)$, the couplings $y_{\nu ij}$ and $y'_{\nu ij}$ can be at the order of 1.

In short, the very small realistic neutrino masses can be generated naturally in scenarios (II) and (III).

5 Discussion and Conclusion

There exist some variants of our models. For instance, for the model in Section 4, we can put one Higgs sextet $\Psi_d$, two singlets $\Psi_{S_1}$ and $\Psi_{S_2}$ on the 4-brane at $z = 0$, and put one Higgs doublet $H_u$, two singlets $S$ and $S_1$ on the 3-brane at $(y = 0, z = 0)$. The anomaly free conditions are the same, and the discussions for the gauge coupling unification, gauge symmetry breaking, $\mu$ problem, $Z - Z'$ mass hierarchy and neutrino masses, etc., are similar.

In addition, we can discuss the models with the general exotic particles. To be explicit, we can add the exotic particles, which transform as $(3; 2; 1/6), (1; 2; -1/2), (3; 1; -2/3), (3; 1; 1/3), (1; 1; 1), (1; 3; 0)$ under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry, and their complex conjugation fields (mirror partners) under the Standard Model gauge symmetry [25, 26]. Similarly, one can calculate the anomaly free conditions and obtain the anomaly free models. In the models with general exotic particles, it is relatively easy to obtain the anomaly free models where the particles have rational $U(1)'$ charges because one has more freedoms [25, 26, 27]. This kind of generalizations is similar and straightforward, so, we do not consider it here.

In this paper, we consider the low energy 6-dimensional $N = 2$ supersymmetric $SU(6)$ gauge unification theory on the space-time $M^4 \times T^2/(Z_2)^3$. First, we discuss the orbifold gauge symmetry breaking, which breaks the $SU(6)$ down to the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry for the zero modes. Then, we present the parity assignment and masses for the bulk gauge fields and Higgs fields on the 4-brane at $z = 0$, and the number of the 4-dimensional supersymmetry and gauge symmetry on the 3-branes at the fixed points and on the 4-branes on the fixed lines. Second, we discuss the anomaly cancellation. In order to cancel the anomalies involving at least one $U(1)'$, we add extra exotic particles which are vector-like under the Standard Model gauge symmetry. And we study the anomaly free conditions and give some anomaly free models. Third, similar to the discussions in Ref. [16], we discuss the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry breaking and solve the $\mu$ problem. The $Z - Z'$ mass hierarchy can be generated naturally because we can have $h \sim 10 \lambda$ if we put the Higgs triplets $\Phi_u, \Phi_d$ and singlet $S_i$ on the 4-brane at $z = 0$ and put the singlet $S$ on the 3-brane at $(y = \pi R_1, z = 0)$. Fourth, we discuss the gauge coupling unification, which shows that the gauge couplings can be unified at $100 \sim 200$ TeV if the compactification scale for the fifth dimension is $3 \sim 4$ TeV. The proton decay problem can be avoided by putting the quarks and leptons/neutrinos on the different 3-branes. Fifth, we discuss the masses for the extra exotic particles, which
can be at the order of 1 TeV after the gauge symmetry breaking, and the masses for the extra zero modes of the chiral superfields from the vector multiplets and Higgs hypermultiplets. In particular, we discuss the neutrino masses in detail. We forbid the dimension-5 operators $y_{\nu ij}L_i L_j H_u H_u / M_*$ by $U(1)'$ gauge symmetry, and the correct active neutrino masses can be obtained via the non-renormalizable terms.

Next, we consider the low energy 6-dimensional $N = 2$ supersymmetric $SU(6)$ gauge unification theory on the space-time $M^4 \times T^2/(Z_2)^4$. First, we discuss the orbifold gauge symmetry breaking, then, we present the parity assignment and masses for the bulk gauge fields and Higgs fields on the 4-brane at $z = 0$, and the number of the 4-dimensional supersymmetry and gauge symmetry on the 3-branes at the fixed points and on the 4-branes on the fixed lines. We would like to point out that, in order to avoid the proton decay problem, we put the quarks on the 3-brane at $(y = 0, z = 0)$ and leptons/neutrinos on the 3-brane at $(y = \pi R_1/2, z = 0)$, which preserve the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ and $SU(3)_C \times SU(3) \times U(1)'$ gauge symmetry, respectively. And in order to generate the correct lepton masses, we add one pair of Higgs sextets on the 4-brane at $z = 0$, instead of one pair of triplets. Second, we discuss the anomaly cancellation by adding the extra exotic particles which are vector-like under the Standard Model gauge symmetry. We also present the anomaly free conditions and some anomaly free models. Third, similar to the discussions in Ref. [16], the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)'$ gauge symmetry can be broken and the $\mu$ problem can be solved. The $Z - Z'$ mass hierarchy can be generated naturally because we can have $h \sim 10\lambda$ if we put the Higgs sextets $\Phi_u$, $\Phi_d$ and singlet $S_i$ on the 4-brane at $z = 0$ and put the singlet $S$ on the 3-brane at $(y = \pi R_1/2, z = 0)$. Fourth, we show that the gauge couplings can be achieved at $100 \sim 200$ TeV if the compactification scale for the fifth dimension is $3 \sim 4$ TeV. Fifth, we discuss the masses for the extra exotic particles, which can be at the order of 1 TeV after the gauge symmetry breaking. Especially, in order to have the correct neutrino masses, we give the small VEVs to the triplet exotic particle $T$ on the 3-brane at $(y = \pi R_1/2, z = 0)$. We forbid the dimension-5 operators $y_{\nu ij} T_{Li} T_{Li} T T / M_*$ by $U(1)'$ gauge symmetry, and the realistic active neutrino masses can be generated via the non-renormalizable terms.

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Table 1: Parity assignment and masses \((n \geq 0, m \geq 0)\) for the vector multiplet and Higgs fields in the \(SU(6)\) model on \(M^4 \times T^2/(Z_2)^3\).

| \((P^y, P^z, P^{z'})\) | Fields | Mass |
|-------------------------|--------|------|
| \((+, +, +)\)           | \(V_{\mu}^{a,b}, \Sigma_5^{\hat{a},\hat{b}}\) | \(\sqrt{n^2/R_1^2 + (2m)^2/R_2^2}\) |
| \((+, +, -)\)           | \(\Sigma_6^{a,b}, \Phi^{\hat{a},\hat{b}}\) | \(\sqrt{n^2/R_1^2 + (2m + 1)^2/R_2^2}\) |
| \((+, -, +)\)           | \(V_{\mu}^{a,b}, \Sigma_5^{\hat{a},\hat{b}}\) | \(\sqrt{n^2/R_1^2 + (2m + 1)^2/R_2^2}\) |
| \((+, -)\)              | \(\Sigma_6^{a,b}, \Phi^{\hat{a},\hat{b}}\) | \(\sqrt{n^2/R_1^2 + (2m + 2)^2/R_2^2}\) |
| \((- , + , + )\)        | \(V_{\mu}^{\hat{a},\hat{b}}, \Sigma_5^{a,b}\) | \(\sqrt{(n + 1)^2/R_1^2 + (2m)^2/R_2^2}\) |
| \((- , +, -)\)          | \(\Sigma_6^{a,b}, \Phi^{\hat{a},\hat{b}}\) | \(\sqrt{(n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}\) |
| \((- , - , +)\)         | \(V_{\mu}^{\hat{a},\hat{b}}, \Sigma_5^{a,b}\) | \(\sqrt{(n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}\) |
| \((- , - , -)\)         | \(\Sigma_6^{a,b}, \Phi^{\hat{a},\hat{b}}\) | \(\sqrt{(n + 1)^2/R_1^2 + (2m + 2)^2/R_2^2}\) |

- \(P^y = +\) \(H_u, S^c_u, H_d, S^c_d, S_i\) \(n/R_1\)
- \(P^y = -\) \(H^c_u, S_u, H^c_d, S_d, S^c_i\) \((n + 1)/R_1\)
Table 2: For the $SU(6)$ model on $M^4 \times T^2/(Z_2)^3$, the gauge superfields, the number of 4-dimensional supersymmetry and gauge symmetry on the 3-brane, which is located at the fixed point $(y = 0, z = 0)$, $(y = 0, z = \pi R_2/2)$, $(y = \pi R_1, z = 0)$, or $(y = \pi R_1, z = \pi R_2/2)$, and on the 4-brane which is located on the fixed line $y = 0, z = 0$, $y = \pi R_1$, or $z = \pi R_2/2$.

| Brane Position | Fields | SUSY | Gauge Symmetry |
|----------------|--------|------|----------------|
| (0, 0)         | $V_{\mu}^{a,b}$, $\Sigma_5^{a,b}$, $\Sigma_6^{a,b}$, $\Phi^{\hat{a},\hat{b}}$ | N=1 | $SU(3) \times SU(2) \times U(1) \times U(1)$ |
| (0, $\pi R_2/2$) | $V_{\mu}^{a,B}$, $\Sigma_5^{a,B}$ | N=1 | $SU(4) \times SU(2) \times U(1)$ |
| ($\pi R_1$, 0) | $V_{\mu}^{a,b}$, $\Sigma_5^{a,b}$, $\Sigma_6^{a,b}$, $\Phi^{\hat{a},\hat{b}}$ | N=1 | $SU(3) \times SU(2) \times U(1) \times U(1)$ |
| ($\pi R_1$, $\pi R_2/2$) | $V_{\mu}^{a,B}$, $\Sigma_5^{a,B}$ | N=1 | $SU(4) \times SU(2) \times U(1)$ |
| $y = 0$        | $V_{\mu}^{a,B}$, $\Sigma_5^{a,B}$, $\Sigma_6^{a,B}$, $\Phi^{\hat{a},\hat{b}}$ | N=2 | $SU(4) \times SU(2) \times U(1)$ |
| $z = 0$        | $V_{\mu}^{A,b}$, $\Sigma_5^{A,b}$, $\Sigma_6^{A,b}$, $\Phi^{A,b}$ | N=2 | $SU(3) \times SU(3) \times U(1)$ |
| $y = \pi R_1$  | $V_{\mu}^{a,B}$, $\Sigma_5^{a,B}$, $\Sigma_6^{a,B}$, $\Phi^{\hat{a},\hat{b}}$ | N=2 | $SU(4) \times SU(2) \times U(1)$ |
| $z = \pi R_2/2$| $V_{\mu}^{A,B}$, $\Sigma_5^{A,B}$ | N=2 | $SU(6)$ |
Table 3: Quantum numbers for the Standard Model fermions ($Q_i$, $\bar{U}_i$, $\bar{D}_i$, $L_i$, $\bar{\nu}_i$, $\bar{E}_i$) and extra exotic particles ($F_i$, $\bar{F}_i$, $F'_i$, $X_i$, $\bar{X}_i$, $X'_i$, $\bar{X}'_i$, $Y_i$, $\bar{Y}_i$, $Y'_i$, $Z_i$, $\bar{Z}_i$, $Z'_i$, $\bar{Z}'_i$) under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^3$.

| Particles | Quantum Numbers | Particles | Quantum Numbers |
|-----------|-----------------|-----------|-----------------|
| $L_i$     | (1; 2; $-1/2$; $a'$) | $Q_i$     | (3; 2; 1/6; $b'$) |
| $\bar{\nu}_i$ | (1; 1; 0; $-(a' + \delta)$) | $\bar{U}_i$ | (3; 1; $-2/3$; $-(b' + c_2)$) |
| $\bar{E}_i$ | (1; 1; $-(a' + c_1)$) | $\bar{D}_i$ | (3; 1; 1/3; $-(b' + c_1)$) |
| $S$     | (1; 1; 0; $s$) | $\bar{F}_i$ | (3; 1; 1/3; $-(d_3 + 2s)$) |
| $F_i$    | (3; 1; $-1/3$; $d_3$) | $\bar{F}'_i$ | (3; 1; 1/3; $-(d'_3 - s)$) |
| $F'_i$   | (3; 1; $-1/3$; $d'_3$) | $X_i$     | (1; 2; $-1/2$; $-(d_2 + 2s)$) |
| $X'_i$   | (1; 2; $1/2$; $d_2$) | $\bar{X}'_i$ | (1; 2; $-1/2$; $-(d'_2 - s)$) |
| $Y_i$    | (1; 1; $-1$; $d_1$) | $\bar{Y}_i$ | (1; 1; 1; $-(d_1 + 2s)$) |
| $Y'_i$   | (1; 1; $-1$; $d'_1$) | $\bar{Y}'_i$ | (1; 1; 1; $-(d'_1 - s)$) |
| $Z_i$    | (1; 1; 0; $d_0$) | $\bar{Z}_i$ | (1; 1; 0; $-(d_0 + 2s)$) |
| $Z'_i$   | (1; 1; 0; $d'_0$) | $\bar{Z}'_i$ | (1; 1; 0; $-(d'_0 - s)$) |

Table 4: Sample: rational $U(1)'$ charges of all the particles for the solution with $k_1 = 2$, $k'_1 = 0$, $k_2 = 0$, $k'_2 = 1$, $k_3 = 2$ and $k'_3 = 1$ in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^3$.

| Scenario | $k_0$ | $k'_0$ | $d_0$ | $d'_0$ | $d_1$ | $d'_1$ | $d_2$ | $d'_2$ | $a'$ | $b'$ | $c_1$ | $c_2$ |
|----------|-------|--------|-------|--------|-------|--------|-------|--------|-----|-----|------|------|
| I        | 1     | 1      | -1    | 1      | 1     | 1      | -1    | -1     | $\frac{15}{8}$ | $\frac{5}{8}$ | $\frac{7}{8}$ | $\frac{1}{2}$ |
| II       | 0     | 2      | X     | -1     | 1     | 1      | -1    | -1     | $\frac{2}{3}$ | $\frac{2}{9}$ | $\frac{1}{6}$ | $\frac{5}{6}$ |
| III      | 0     | 8      | X     | 1      | -1    | -1     | -1    | 1      | -1  | $\frac{1}{3}$ | 1     | -3    |
| IV       | 2     | 0      | 1     | X     | 1     | 1      | -1    | 1      | $\frac{7}{4}$ | $\frac{7}{12}$ | 1     | $\frac{5}{4}$ |
| V        | 4     | 1      | 1     | 1     | -1    | 1      | -1    | -1     | $\frac{11}{8}$ | $\frac{11}{24}$ | $\frac{1}{8}$ | $\frac{1}{4}$ |
Table 5: Sample: rational $U(1)'$ charges of all the particles for the solution with $k_1 = 1$, $k'_1 = 1$, $k_2 = 1$, $k'_2 = 0$, $k_3 = 2$ and $k'_3 = 1$ in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^3$.

| Scenario | $k_0$ | $k'_0$ | $d_0$ | $d'_0$ | $d_1$ | $d_2$ | $d_3$ | $d'_1$ | $d'_3$ | $a'$ | $b'$ | $c_1$ | $c_2$ |
|----------|-------|--------|-------|--------|-------|-------|-------|--------|--------|------|------|------|------|
| I        | 1     | 1      | -1    | -1     | -1    | 1     | 1     | 1      | 1      | 3    | -2/3 | 1    | -2   |
| II       | 0     | 5      | X     | 1      | -1    | 1     | -1    | -1     | 1      | -13/9 | 61/81 | -28/27 | 2/9 |
| III      | 0     | 11     | X     | -1     | -1    | 1     | -1    | -1     | 1      | 2/3  | 13/18 | 5/6  | 7/3  |
| IV       | 1     | 1      | -1    | -1     | 1     | -1    | 1     | -1     | -1     | 3/8  | -11/8 | 5/4  | - |   |
| V        | 2     | 0      | -1    | X      | 1     | 1     | -1    | 1      | 1      | -1/3 | 2/9  | 5/3  | 4/3  |
Table 6: Parity assignment and masses ($n \geq 0, m \geq 0$) for the gauge and Higgs fields in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^4$.

| $(P^y, P^{y'}, P^z, P^{z'})$ | field | mass |
|----------------------------|-------|------|
| $(+, +, +, +)$ | $V^a_b$ | $\sqrt{(2n)^2/R_1^2 + (2m)^2/R_2^2}$ |
| $(+, +, -, +)$ | $V^a_b$ | $\sqrt{(2n)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(-, +, +, +)$ | $V^b_a$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m)^2/R_2^2}$ |
| $(-, +, -, +)$ | $V^b_a$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(-, -, +, +)$ | $\Sigma^a_5$ | $\sqrt{(2n + 2)^2/R_1^2 + (2m)^2/R_2^2}$ |
| $(-, -, -, +)$ | $\Sigma^a_5$ | $\sqrt{(2n + 2)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(+, -, +, +)$ | $\Sigma^a_5$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 2)^2/R_2^2}$ |
| $(+, -, -, +)$ | $\Sigma^a_5$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(+, +, -, -)$ | $\Sigma^a_6$ | $\sqrt{(2n)^2/R_1^2 + (2m + 2)^2/R_2^2}$ |
| $(+, -, -,-)$ | $\Sigma^a_6$ | $\sqrt{(2n)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(-, +, -, -)$ | $\Sigma^a_6$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 2)^2/R_2^2}$ |
| $(-, +, -,-)$ | $\Sigma^a_6$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(-, -, -,-)$ | $\Phi^a_b$ | $\sqrt{(2n + 2)^2/R_1^2 + (2m)^2/R_2^2}$ |
| $(-, -, +,-)$ | $\Phi^a_b$ | $\sqrt{(2n + 2)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(+, -, -,-)$ | $\Phi^a_b$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(+, -, +,-)$ | $\Phi^a_b$ | $\sqrt{(2n + 1)^2/R_1^2 + (2m + 1)^2/R_2^2}$ |
| $(p^y = +, P^{y'} = +)$ | $H_u, H_d, S_i$ | $2n/R_1$ |
| $(p^y = -, P^{y'} = +)$ | $\eta_u, \eta_d$ | $(2n + 1)/R_1$ |
| $(p^y = +, P^{y'} = -)$ | $\eta^c_u, \eta^c_d$ | $(2n + 1)/R_1$ |
| $(p^y = -, P^{y'} = -)$ | $H^c_u, H^c_d, S^c_i$ | $(2n + 2)/R_1$ |
Table 7: For the $SU(6)$ model on $M^4 \times T^2 / (Z_2)^4$, the gauge superfields, the number of 4-dimensional supersymmetry and gauge symmetry on the 3-brane, which is located at the fixed point $(y = 0, z = 0)$, $(y = 0, z = \pi R_2 / 2)$, $(y = \pi R_1 / 2, z = 0)$, or $(y = \pi R_1 / 2, z = \pi R_2 / 2)$, and on the 4-brane which is located on the fixed line $y = 0$, $z = 0, y = \pi R_1 / 2$, or $z = \pi R_2 / 2$.

| Brane Position     | fields                  | SUSY | Gauge Symmetry                              |
|--------------------|-------------------------|------|---------------------------------------------|
| $(0, 0)$           | $V^{a,b}_\mu, \Sigma^{a,b}_5, \Sigma^{a,b}_6, \Phi^{a,b}$ | N=1  | $SU(3) \times SU(2) \times U(1) \times U(1)$ |
| $(0, \pi R_2 / 2)$ | $V^{a,b}_\mu, \Sigma^{a,b}_5$ | N=1  | $SU(4) \times SU(2) \times U(1)$            |
| $(\pi R_1 / 2, 0)$ | $V^{A,b}_\mu, \Sigma^{A,b}_5$ | N=1  | $SU(3) \times SU(3) \times U(1)$            |
| $(\pi R_1 / 2, \pi R_2 / 2)$ | $V^{A,B}_\mu$ | N=1  | $SU(6)$                                    |
| $y = 0$            | $V^{a,b}_\mu, \Sigma^{a,b}_5, \Sigma^{a,b}_6, \Phi^{a,b}$ | N=2  | $SU(4) \times SU(2) \times U(1)$            |
| $z = 0$            | $V^{a,b}_\mu, \Sigma^{a,b}_5, \Sigma^{a,b}_6, \Phi^{a,b}$ | N=2  | $SU(3) \times SU(3) \times U(1)$            |
| $y = \pi R_1 / 2$  | $V^{A,B}_\mu$          | N=2  | $SU(6)$                                    |
| $z = \pi R_2 / 2$  | $V^{A,B}_\mu$          | N=2  | $SU(6)$                                    |

Table 8: Quantum numbers for the Standard Model quarks ($Q_i, \bar{U}_i, \bar{D}_i$) and extra exotic particles ($F_i, \bar{F}_i, F'_i, \bar{F}'_i, X'_i, Y'_i, Y_i, \bar{Y}_i, Y'_i$ and $\bar{Y}'_i$) under the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_Y'$ gauge symmetry in the $SU(6)$ model on $M^4 \times T^2 / (Z_2)^4$. These particles are on the 3-brane at $(y = 0, z = 0)$.

| Particles | Quantum Numbers | Particles | Quantum Numbers |
|-----------|----------------|-----------|----------------|
| $Q_i$     | (3; 2; 1/6; $b'$) | $\bar{D}_i$ | (3; 1; 1/3; $-(b' + c_1)$) |
| $\bar{U}_i$ | (3; 1; $-2/3; -(b' + c_2)$) | $\bar{D}_i$ | (3; 1; 1/3; $-(b' + c_1)$) |
| $F_i$     | (3; 1; $-1/3; d_3$) | $\bar{F}_i$ | (3; 1; 1/3; $-(d_3 + 2s)$) |
| $F'_i$    | (3; 1; $-1/3; d'_3$) | $\bar{F}'_i$ | (3; 1; 1/3; $-(d'_3 + s)$) |
| $X'_i$    | (1; 2; 1/2; $d'_3$) | $\bar{X}'_i$ | (1; 2; $-1/2; -(d'_3 - s)$) |
| $Y_i$     | (1; 1; $-1; d_1$) | $\bar{Y}_i$ | (1; 1; $-1; -(d_1 + 2s)$) |
| $Y'_i$    | (1; 1; $-1; d'_1$) | $\bar{Y}'_i$ | (1; 1; $-1; -(d'_1 - s)$) |

| Particles | Quantum Numbers | Particles | Quantum Numbers |
|-----------|----------------|-----------|----------------|
| $Q_i$     | (3; 2; 1/6; $b'$) | $\bar{D}_i$ | (3; 1; 1/3; $-(b' + c_1)$) |
| $\bar{U}_i$ | (3; 1; $-2/3; -(b' + c_2)$) | $\bar{D}_i$ | (3; 1; 1/3; $-(b' + c_1)$) |
| $F_i$     | (3; 1; $-1/3; d_3$) | $\bar{F}_i$ | (3; 1; 1/3; $-(d_3 + 2s)$) |
| $F'_i$    | (3; 1; $-1/3; d'_3$) | $\bar{F}'_i$ | (3; 1; 1/3; $-(d'_3 + s)$) |
| $X'_i$    | (1; 2; 1/2; $d'_3$) | $\bar{X}'_i$ | (1; 2; $-1/2; -(d'_3 - s)$) |
| $Y_i$     | (1; 1; $-1; d_1$) | $\bar{Y}_i$ | (1; 1; $-1; -(d_1 + 2s)$) |
| $Y'_i$    | (1; 1; $-1; d'_1$) | $\bar{Y}'_i$ | (1; 1; $-1; -(d'_1 - s)$) |
Table 9: Quantum numbers for the Standard Model leptons $T_{Li}$ and right handed neutrinos $\bar{\nu}_i$, and extra exotic particles $T_i, \bar{T}_i, Z_i, Z'_i, \bar{Z}_i, \bar{Z}'_i$ under the $SU(3)_C \times SU(3) \times U(1)'$ gauge symmetry in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^4$. These particles are on the 3-brane at $(y = \pi R_1/2, z = 0)$.

| Particles | Quantum Numbers | Particles | Quantum Numbers |
|-----------|----------------|-----------|----------------|
| $T_{Li}$  | $(1; 3; -c_1/2)$ | $\bar{\nu}_i$ | $(1; 1; -(c_1/2 + \delta))$ |
| $S$       | $(1; 1; 0; s)$  |           |                |
| $T_i$     | $(1; 3; d_2)$   | $\bar{T}_i$ | $(1; 3; -(d_2 + 2s))$ |
| $Z_i$     | $(1; 1; 0; d_0)$ | $\bar{Z}_i$ | $(1; 1; 0; -(d_0 + 2s))$ |
| $Z'_i$    | $(1; 1; 0; d'_0)$ | $\bar{Z}'_i$ | $(1; 1; 0; -(d'_0 - s))$ |

Table 10: Sample: rational $U(1)'$ charges of all the particles for the solution with $k_1 = 0, k'_1 = 2, k_2 = 1, k'_2 = 0, k_3 = 2$ and $k'_3 = 1$ in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^4$.

| Scenario | $k_0$ | $k'_0$ | $d_0$ | $d'_0$ | $d_1$ | $d'_3$ | $b'$ | $c_1$ | $c_2$ | $d_2$ | $d_3$ |
|----------|-------|--------|-------|--------|-------|--------|------|-------|-------|-------|-------|
| I        | 1     | 0      | -1    | X      | -2    | 1      | -1/71| -60/71| 33/71 | -3/71 | 102/71|
| II       | 0     | 1      | X     | -2    | 1     | 1      | 13/58| -15/29| -12/29| -48/49| 63/116|
| III      | 2     | 2      | 1     | 1     | 1     | 2      | -16/2995| -192/2995| 528/2995| 384/2995| 443/2995|

Table 11: Sample: rational $U(1)'$ charges of all the particles for the solution with $k_1 = 0, k'_1 = 0, k_2 = 1, k'_2 = 2, k_3 = 2$ and $k'_3 = 1$ in the $SU(6)$ model on $M^4 \times T^2/(Z_2)^4$.

| Scenario | $k_0$ | $k'_0$ | $d_0$ | $d'_0$ | $d_1$ | $d'_3$ | $b'$ | $c_1$ | $c_2$ | $d_2$ | $d_3$ |
|----------|-------|--------|-------|--------|-------|--------|------|-------|-------|-------|-------|
| I        | 2     | 0      | -1    | X      | 2     | 1      | -7/43| -1/3  | 11/16 | -1/12 | -2    |
| II       | 3     | 2      | -1    | 1      | 1     | 1      | 5/132| 1/2   | 19/32 | 13/43 |       |
| III      | 0     | 2      | X     | 1      | 1     | 1      | 1/45 | 8/15  | 1/10  | -14/15| 2/5   |