Combinatorial solutions to the Hamiltonian constraint in (2+1)-dimensional Ashtekar gravity

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Abstract

Dirac’s quantization of the (2+1)-dimensional analog of Ashtekar’s approach to quantum gravity is investigated. After providing a diffeomorphism-invariant regularization of the Hamiltonian constraint, we find a set of solutions to this Hamiltonian constraint which is a generalization of the solution discovered by Jacobson and Smolin. These solutions are given by particular linear combinations of the spin-network states. While the classical counterparts of these solutions have degenerate metric, due to a ‘quantum effect’ the area operator has nonvanishing action on these states. We also discuss how to extend our results to (3+1)-dimensions.

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1 Introduction

About a decade ago Ashtekar discovered the new canonical variables which describe the canonical formulation of general relativity [1]. These new variables simplify the form of the Hamiltonian constraint compared to the ADM formalism [2]. Thus we expect that, using these new variables, we can solve the Wheeler-Dewitt equation [3], namely the quantum version of the Hamiltonian constraint equation, whose solution has not yet been found in the conventional metric formulation. Moreover, because Ashtekar’s new variables can be regarded as an $SL(2, \mathbb{C})$-connection and its conjugate momentum, we can embed the phase space of general relativity into that of an $SL(2, \mathbb{C})$ gauge theory [1]. These virtues of the new variables have led many people to vigorous studies [4] on the nonperturbative formulation of quantum general gravity in terms of the new variables, namely, Ashtekar’s formalism.

While much progress has been made on Ashtekar’s formalism, we have not yet reached the complete formulation of nonperturbative quantum general relativity because we have not yet overcome several problems. We list a few of these problems:

i) The problem of constructing the physical Hilbert space. We can separate this problem into two parts. One is that of finding all the solutions to the constraint equations which involves solving the Wheeler-Dewitt equation. The other is that of constructing inner product in the space of the solutions to all the constraints; the problem of imposing the ‘reality condition’ is considered to be contained in this problem.

ii) The issue of constraint closure under the commutator algebra. This is intimately related to the problem of choosing an appropriate operator ordering and that of inventing a regularization which is physically relevant.

Under such situation it would be useful to study some toy models which give some lessons on the technical and conceptual problems in the full (3+1)-dimensional theory. (2+1)-dimensional general relativity is considered as one of such toy models [5].

It has been shown by Achucarro and Townsend [6] and Witten [7] that the first order form of (2+1)-dimensional Einstein gravity is equivalent to the Chern-Simons gauge theory with a non-compact gauge group $G$, where $G$ is $SO(3,1)$, $ISO(2,1)$, and $SO(2,2)$ when the cosmological constant is positive, zero, and negative respectively. Witten further showed that the
phase space of the canonical formulation of this (2+1)-dimensional Einstein gravity is
described by the moduli space of flat $G$-connections on a 2 dimensional space $\Sigma$ modulo
gauge transformations\cite{7}, which is finite dimensional. We will refer to this canonical
formalism as Witten’s formalism. Because Witten’s formalism has a form analogous to
that of Ashtekar’s formalism, it has also been investigated as a toy model of Ashtekar’s
formalism\cite{8}. However, there is an essential difference between these two formulations.
While the constraints in Witten’s formalism is at most first order in conjugate momenta,
those in Ashtekar’s formalism involves the Hamiltonian constraint which is second order
in conjugate momenta. The reason that this difference is essential is the following. If
all the constraints are at most first order in momenta, Dirac’s quantization\cite{9} and the
reduced phase space (RPS) quantization\cite{10}, namely the quantization on the space of
all the classical solutions to the constraints, yield almost the same results. This is not
the case when there exist constraints which are more than first order in momenta; in
this case the Hilbert space of Dirac’s quantization is expected to be larger than that of
the RPS quantization\cite{11}. In (3+1)-dimensions we cannot in practice carry out the RPS
quantization, because this quantization requires the full knowledge of the solutions to
Einstein equations.

We therefore suspect that, while Witten’s formalism may be useful in obtaining concep-
tual intuition, the techniques developed there cannot be applied to Ashtekar’s formalism.
Thus we would be glad if there exists
any formalism in (2+1)-dimensions which is more similar to Ashtekar’s formalism.

Such a formulation indeed exists. It was found by Bengtsson\cite{12} in the context of
reformulating the constraint algebra of quantum gravity from the Yang-Mills fields. The
relevant gauge group is $SO(2,1)$, or $SL(2,R)$, which is isomorphic to the local Lorentz
group in (2+1)-dimensions. In contrast to Witten’s formalism, this ‘(2+1)-dimensional
Ashtekar formalism’ seems to be investigated scarcely, except in a few works\cite{13}\cite{14}.
From the reasoning mentioned above, however, studying this formalism is expected to
yield many useful intuitions on the quantum gravity in (3+1)-dimensions, both in the
technical and the conceptual aspects.

In this paper we investigate the quantization of this (2+1)-dimensional Ashtekar for-

\footnote{In this paper we consider the cosmological constant to be zero, except in §2.}
malism. We will adopt Dirac’s quantization procedure, in the representation where the spin connection is diagonalized. We use spin network states translated in terms of the spinor representation. A spin network state[15] is a generalization of the Wilson loop to the ‘graphs’, i.e. the set of curves embedded in the spatial hypersurface Σ. It is known[16] that the spin network states form an orthogonal basis in the space $L^2(A/G)$ of gauge-invariant square-integrable functionals of connections, at least when the gauge group G is compact. Returning to (2+1)-dimensional Ashtekar formalism, the Hamiltonian constraint operator needs to be regularized because it involves two functional derivatives at a point. We introduce a regularization which preserves the covariance under the diffeomorphisms. This regularization is a sort of point-splitting regularization. The difference from the conventional ones is that we use the curvilinear coordinate frame in which the curves in the ‘graph’ serve as coordinate curves. A merit of this regularization is that we can get rid of ‘acceleration terms’[14] which manifestly violates the diffeomorphism covariance. After the regularization we work out the action of the Hamiltonian constraint on the ‘basic configurations’, each of which consists of one or two spinor parallel propagators. Then the evaluation of the action of the Hamiltonian constraint on the spin network states reduces to the problem of the combinatorics. This simplifies the attempts to look for solutions to the Hamiltonian constraint equation. We will construct in this paper ‘combinatorial solutions’, in each of which the action of the Hamiltonian constraint on the ingredients cancels in an elementary algebraic way. The set of these solutions is a generalization of the solution found by Jacobson and Smolin[17] to spin network states. Each solution is labelled by a set $\{\alpha_i\}$ of smooth loops each of which is equipped with the spin-$\frac{1}{2}$ representation of $SL(2,\mathbb{R})$. We will see that classical counterparts of these solutions are solutions to the classical Hamiltonian constraint having degenerate metric. It is shown, however, these solutions can have nonzero area due to a sort of quantum effect. In this paper we do not use the essential merit of (2+1)-gravity, i.e. its topological nature. Most of the results obtained in this paper can therefore be extended to (3+1)-dimensions. In particular, the combinatorial solutions remain to be the solutions to (3+1)-dimensional Hamiltonian constraint if we appropriately modify the intertwining operators.

The organization of this paper is as follows. After briefly reviewing (2+1)-dimensional
Ashtekar formalism and the spin network states in §2, we provide a diffeomorphism-
covariant regularization in §3. §4 is the main part of this paper. There we provide a set of
combinatorial solutions as well as the solutions which have been known up to now. In §5
we investigate the action of a few operators which measure e.g. the area and the length, on
the combinatorial solutions. Finally in §6, after briefly summarizing the obtained results,
we make an attempt to extend these results to (3+1)-dimensions. In this paper we use
graphical representations frequently. In Appendix we list the action of the Hamiltonian
constraint on the basic configurations in terms of the graphical representation.

Here we give the convention for the indices and the signatures of the metric used in
this paper: i) $\mu, \nu, \rho, \cdots$ denote 2+1 dimensional spacetime indices and the metric $g_{\mu\nu}$ has
the signature $(-,+,+)$; ii) $i, j, k, \cdots$ are used for spatial indices; iii) $a, b, c, \cdots$ represent
indices of the $SO(2,1)$ representation of the local Lorentz group, with the metric $\eta_{ab} =
\text{diag}(-,+,+)$; iv) $A, B, C, \cdots$ are indices of the $SL(2,\mathbb{R})$ spinor representation; v) $\epsilon_{abc}$
stands for the totally antisymmetric pseudo-tensor with $\epsilon_{012} = -\epsilon^{012} = 1$; vi) $\tilde{\epsilon}^{ij}$ ($\epsilon_{ij}$)
denotes the totally antisymmetric tensor density of weight $+1$ ($-1$) with $\tilde{\epsilon}^{12} = \epsilon_{12} = 1$;
and vii) $\epsilon^{AB}$ and $\epsilon_{AB}$ are the rank-2 antisymmetric tensors with $\epsilon^{12} = \epsilon_{12} = 1$.

\section{Preliminaries}

In this section we provide two backgrounds which are necessary for reading this paper,
namely (2+1)-dimensional Ashtekar formalism and the spin-network states.

\subsection{Ashtekar’s formalism in (2+1)-dimensions}

(2+1)-dimensional analog of Ashtekar’s approach to quantum general relativity was first
discovered by Bengtsson\cite{12} in the context of the Poisson algebra of the constraints con-
structed from the Yang-Mills fields. Here we will briefly look at this ‘(2+1)-dimensional
Ashtekar formalism’ from the viewpoint of the Lagrangian formalism.

In the first order formalism of (2+1)-dimensional general relativity, we use as inde-
pendent variables the triad one-form $e^a = e^a_{\mu} dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}_{\mu} dx^\mu$.
Its action is given by the Einstein-Palatini action possibly with a cosmological constant.
\[ I_{EP} = \int_M \epsilon_{abc} e^a \wedge [d\omega^{bc} + \omega^b_d \wedge \omega^{dc} - \frac{1}{3} \Lambda e^b \wedge e^c]. \tag{2.1} \]

Let us assume \( M \approx R \times \Sigma \) and construct a canonical formalism. For simplicity we restrict ourselves to the case when the spatial hypersurface \( \Sigma \) is a compact, oriented two dimensional manifold without boundary. If we set \( \omega^a \equiv \frac{1}{2} \epsilon^{a}_{bc} \omega^{bc} \) and make a naive \((2+1)\)-decomposition: \( x^0 = t \), we obtain the canonical formulation a la Witten:\[ I_W = (I_{EP})|_{M \approx R \times \Sigma} = \int dt \int_\Sigma d^2x (-2\tilde{\epsilon}^{ij} e_{ia} \dot{\omega}_j^a + e_t^a \Psi_a + \omega_{ta} \mathcal{G}^a). \tag{2.2} \]

The first order constraints obtained from the variations of the Lagrange multipliers \( e_t^a \) and \( \omega_{ta} \) are:

\[ \Psi^a = \tilde{\epsilon}^{ij} (F_{ij}^a - \Lambda \epsilon^a_{bc} e^b_{ij} e^c_j), \quad F_{ij}^a \equiv \partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^a_{bc} \omega_i^b \omega_j^c, \]
\[ \mathcal{G}^a = 2\tilde{\epsilon}^{ij} (\partial_t e_{ia}^j + \epsilon^a_{bc} \omega_i^b \dot{e}_{ij}). \tag{2.3} \]

It is well known that the physical phase space of this formalism is isomorphic to the moduli space of flat \( G \)-connections on \( \Sigma \) modulo gauge transformations. The gauge group \( G \) is \( SO(3,1), ISO(2,1) \) and \( SO(2,2) \) when the cosmological constant \( \Lambda \) is positive, zero, and negative respectively.

In order to obtain \((2+1)\)-dimensional Ashtekar formalism, we first remember the ADM decomposition\[2\], which is written in the first order formalism by:

\[ e_t^a e_{ja} = h_{ij}, \quad e_t^a e_{ia} = N^j h_{ij} \equiv N_i, \quad e_t^a e_{ta} = -N^2 + N^i N_i, \]

where \( h_{ij} \) is the induced metric on \( \Sigma \), \( N^i \) and \( N \) are the celebrated shift vector and the lapse function respectively. A way to satisfy these equations is to set \[2\]:

\[ e_t^a = N^i e_i^a + N \tilde{n}^a, \]
\[ (\tilde{N} \equiv (\det(h_{ij}))^{-1/2} N, \quad \tilde{n}^a \equiv \frac{1}{2} \epsilon^a_{bc} \epsilon^{kl}_{ri} e^b_k e^c_l). \tag{2.4} \]

By introducing the ‘momentum variable’ \( \tilde{E}_a^i \equiv \tilde{\epsilon}^{ij} e_{ja} \) and a new Lagrange multiplier \( \eta^a \equiv -(\omega_t^a - N^j \omega_j^a) \), we find the action of \((2+1)\)-dimensional analog of Ashtekar’s formalism:

\[ I_A = \int dt \left[ \int_\Sigma d^2x 2\tilde{E}_a^i \dot{\omega}_i^a - H \right]. \tag{2.5} \]

\[ ^2\text{The upper(lower) tilde stands for the density weight +1 (-1)} \]
The Hamiltonian $H$ is a linear combination of the first order constraints which we call the Hamiltonian constraint, the diffeomorphism constraint, and the Gauss law constraint respectively:

$$H = \mathcal{H}(\mathcal{N}) + \mathcal{D}_i(N^i) + \mathcal{G}_a(\eta^a),$$

$$\begin{align*} 
&\mathcal{H}(\mathcal{N}) \equiv -\int_\Sigma d^2x N^i \left[ \epsilon_{abc} \tilde{E}^i_{bc} F_{ij}^a - \Lambda \epsilon_{ijk} \epsilon^{ij} \tilde{E}^a_{ij} \tilde{E}_b^k \right] \\
&\mathcal{D}_i(N^i) \equiv 2 \int_\Sigma d^2x N^i (\tilde{E}_a^i F_{ij}^a - \omega^i_b D_j \tilde{E}^a_j) = 2 \int_\Sigma d^2x \tilde{E}_a^i \mathcal{L}_\mathcal{N} \omega^a_i \\
&\mathcal{G}_a(\eta^a) \equiv 2 \int_\Sigma d^2x \eta^a D_j \tilde{E}_a^j,
\end{align*}$$

(2.6)

where $\mathcal{L}_\mathcal{N}$ and $D_j$ denote respectively the Lie derivative w.r.t. $N^i$ and the covariant derivative defined by the spin connection $\omega^a_i$.

From the action (2.3) we can read off the basic Poisson brackets:

$$\{\omega^a_i(x), \tilde{E}_b^j(y)\} = \frac{1}{2} \delta^a_b \delta^j_i \delta^2(x, y).$$

(2.7)

Under the Poisson bracket, the diffeomorphism and the Gauss law constraints respectively generate (infinitesimal) spatial diffeomorphisms and the local Lorentz transformations:

$$\{(\omega^a_i, \tilde{E}_b^j), \mathcal{D}_i(N^i)\} = (\mathcal{L}_\mathcal{N} \omega^a_i, \mathcal{L}_\mathcal{N} \tilde{E}_a^i)$$

$$\{(\omega^a_i, \tilde{E}_b^j), \mathcal{G}_a(\eta^a)\} = (-D_i \eta^a, -\epsilon_{abc} \tilde{E}^i_{bc} \tilde{E}^j).$$

(2.8)

The Hamiltonian constraint generates the ‘bubble-time evolutions’:

$$\{\omega^a_i, \mathcal{H}(\mathcal{N})\} = -\mathcal{N}(\epsilon^a_{bc}) \tilde{N}^{jb} F_{ij}^c - 2\Lambda \epsilon_{ijk} \epsilon^{ij} \tilde{E}^a_{ij} \tilde{E}_b^k$$

$$\{\tilde{E}_a^i, \mathcal{H}(\mathcal{N})\} = D_j (\mathcal{N} \epsilon_{abc} \tilde{E}^i_{bc} \tilde{E}^j).$$

(2.9)

Using these equations, we can compute the Poisson brackets between the constraints:

$$\begin{align*} 
\{\mathcal{D}_i(M^i), \mathcal{D}_j(N^j)\} &= -\mathcal{D}_i (\mathcal{L}_\mathcal{N} M^i), \\
\{\mathcal{G}_a(\eta^a), \mathcal{D}_j(N^j)\} &= -\mathcal{G}_a (\mathcal{L}_\mathcal{N} \eta^a), \\
\{\mathcal{H}(\mathcal{N}), \mathcal{D}_j(N^j)\} &= -\mathcal{H} (\mathcal{L}_\mathcal{N} \mathcal{N}), \\
\{\mathcal{G}_a(\eta^a), \mathcal{G}_b(\eta^a)\} &= \mathcal{G}_a (\epsilon^{abc} \eta^c), \\
\{\mathcal{H}(\mathcal{N}), \mathcal{G}_a(\eta^a)\} &= 0, \\
\{\mathcal{H}(\mathcal{N}), \mathcal{H}(\mathcal{M})\} &= \mathcal{D}_i (K^i) + \mathcal{G}_a(\omega^a_i K^i), \quad K^i \equiv \tilde{E}^i_{bc} (\mathcal{N} \partial_j \mathcal{M} - \mathcal{M} \partial_j \mathcal{N}).
\end{align*}$$

(2.10)

Here we will make some remarks. The Witten constraint equations:

$$\Psi^a \approx \mathcal{G}^a \approx 0$$
and the Ashtekar constraint equations:

$$H = -\tilde{n}^a \Psi_a \approx 0, \quad V_i = -\epsilon^a_i \Psi_a \approx 0, \quad G^a \approx 0 \quad (2.11)$$

are equivalent if $\epsilon_i^a$ ($i = 1, 2$) and $\tilde{n}^a$ form a non-degenerate frame, namely, if the space-time metric $g_{\mu\nu} = \epsilon^a_\mu \epsilon^a_{\nu}$ is non-degenerate. If $g_{\mu\nu}$ is degenerate, however, there exist solutions to the Ashtekar constraint equations which are not subject to the Witten constraint equations. The phase space of (2+1)-Ashtekar formalism is thus expected to contain that of Witten’s formalism as a subspace \[13\].

### 2.2 Spin network states

In this paper we will make an attempt to quantize (2+1)-dimensional gravity on the space $\mathcal{A}$ of spin connections $\omega_i^a$, which can be regarded as $SL(2, \mathbb{R})$-connections. As will be seen in the next section, the Gauss law constraint tells us that the wavefunctions be the gauge invariant functionals of the spin connection. We are thus interested in the functions on the quotient space $\mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the group of $SL(2, \mathbb{R})$ gauge transformations.

It is well known that the Wilson loops yield the gauge invariant information on the connection \[18\]. A Wilson loop is defined if a loop $\gamma : [0, 1] \to \Sigma$ ($\gamma(0) = \gamma(1)$) and a representation $\rho$ of $SL(2, \mathbb{R})$ is given:

$$W(\gamma, \rho) \equiv \text{Tr} \rho (h_{\gamma}[0, 1]),$$

where

$$h_{\alpha}[0, 1] \equiv \mathcal{P} \exp \left\{ \int_0^1 ds \dot{\alpha}^i(s) A_i(\alpha(s)) \right\} \quad (2.13)$$

is the parallel propagator of the $SL(2, \mathbb{R})$ connection $A_i \equiv \omega_i^a J_a$ ($J_a$: the generators of $SL(2, \mathbb{R})$) along a curve $\alpha : [0, 1] \to \Sigma$, where $\alpha(0)$ does not in general coincide with $\alpha(1)$. Under the gauge transformation

$$A_i(x) \to g(x) A_i(x) g^{-1}(x) - \partial_i g(x) g^{-1}(x) \quad (g(x) \in SL(2, \mathbb{R})),$$

the parallel propagator $h_{\alpha}[0, 1]$ transforms as:

$$h_{\alpha}[0, 1] \to g(\alpha(0)) h_{\alpha}[0, 1] g^{-1}(\alpha(1)). \quad (2.14)$$

Gauge invariance of the Wilson loop \[2.12\] is an immediate consequence of this.
Spin network states are the generalized version of this Wilson loop [15][16]. A spin network state is in one-to-one correspondence with a ‘colored graph’. A colored graph is specified by a set of ‘colored edges’, the edges $e$ (the segments of the curves in $\Sigma$) each of which being labelled by a representation $\rho_e$ of $SL(2, \mathbb{R})$, and a set of ‘colored vertices’, the vertices $v$ each of which being equipped with an intertwining operator $i_v$ [3]. The representations $\rho_e$, of course, must satisfy some conditions at each vertex in order to ensure the gauge invariance.

The $SL(2, \mathbb{R})$ group has finite dimensional representations and infinite dimensional ones. We will henceforth restrict our attention to the former. It is a well-known fact that the $(l + 1)$-dimensional irreducible representation, i.e. the spin-$l \over 2$ representation, is expressed as the symmetrized tensor product of $l$ copies of the fundamental representation (the spinor representation). Thus we can make our analysis by using the spinor representation only.

The parallel propagator in the spinor representation along a curve $\alpha$ is

$$h_\alpha[0, 1]^B_A = \left[ \mathcal{P} \exp \int_0^1 ds \dot{\alpha}^i(s) \omega_i^a(\alpha(s)) \lambda_a \right]^B_A,$$

(2.15)

where $\lambda_a$ are spin-$l \over 2$ generators subject to $\lambda_a \lambda_b = \frac{1}{4} \eta_{ab} + \frac{1}{2} \epsilon_{abc} \lambda^c$. This spinor propagator is subject to the following identity:

$$\epsilon_{AD} \epsilon^{BC} h_\alpha[0, 1]^D_C = (h_\alpha[0, 1]^{-1})^B_A = h_{\alpha^{-1}}[0, 1]^B_A,$$

(2.16)

where $\alpha^{-1}$: $\alpha^{-1}(s) = \alpha(1-s)$ is the curve obtained by reversing the orientation of $\alpha$. This identity is useful because it enables us to choose a relevant orientation of the propagators according to a particular problem.

In the spinor representation, there are three invariant tensors (tensors which are invariant under the $SL(2, \mathbb{R})$ transformations) $\epsilon^A_B \equiv \delta^A_B$, $\epsilon^{AB}$ and $\epsilon_{AB}$. Because the intertwining operators are constructed from the invariant tensors, the total rank of an intertwining operator is even. Thus the sum of the numbers of the spinor propagators ending and starting at a vertex is necessarily even. Using identity (2.16) and the equation $\epsilon_{AC} \epsilon^{BC} = \delta^B_A$, we can equalize at each vertex the number of in-coming propagators with that of out-going propagators.

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3The intertwining operator $i_v$ is an operator which extracts the trivial representation from the direct product of the representations labelling the edges which have an end at $v$. 

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propagators. As a consequence we can regard the intertwining operators to be a sum of the products of $\delta^B_A$. The product of the propagators along two successive curves $\alpha, \beta$ ($\alpha(1) = \beta(0)$) is the propagator along the composite curve $\alpha \circ \beta$:

$$h_\alpha[0,1]_A^C h_\beta[0,1]^B_C = h_{\alpha \circ \beta}[0,1]^B_A.$$

We can thus consider the spin network states to be (the linear combinations of) products of the Wilson loops in the spinor representation along the loops each of which is composed of the edges in the graph.

In addition to eq.(2.16), there are two useful identities in the spinor representation; the two-spinor identity

$$\delta^B_A \delta^D_C - \delta^D_A \delta^B_C = \epsilon_{AC} \epsilon^{BD} \quad \text{(or } \phi^A \epsilon^{BC} + \phi^B \epsilon^{CA} + \phi^C \epsilon^{AB} = 0 \text{)} \quad (2.17)$$

and the Fiertz identity

$$(\lambda^a)_A^B (\lambda_a)_C^D = \frac{1}{2} (\delta^D_A \delta^B_C - \frac{1}{2} \delta^B_A \delta^D_C). \quad (2.18)$$

The two-spinor identity tells us that the antisymmetrized tensor product of the two identical spinor propagators gives the trivial representation

$$h_\alpha[0,1]_A^B h_\alpha[0,1]^D_C = \frac{1}{2} \epsilon_{AC} \epsilon^{BD} \quad (2.19)$$

and that, at an intersection $\alpha(s_0) = \beta(t_0)$ of two curves $\alpha$ and $\beta$, the following identity holds:

$$h_\alpha[0,1]_A^B h_\beta[0,1]^D_C = (h_\alpha[0,s_0] h_\beta[t_0,1])_A^D \epsilon_{EC} (h_\beta^{-1}[0,1-t_0] h_\alpha[s_0,1])_E^F \epsilon^{FD}. \quad (2.20)$$

Owing to eq.(2.19) we do not have to take account of antisymmetrizing the identical propagators.

For the purpose of the calculation, it is convenient to introduce the graphical representation. We will denote a spinor propagator $h_\alpha[0,1]_A^B$ by an arrow from $\alpha(0)$ to $\alpha(1)$ with its tail (tip) being associated with the spinor index $A$ ($B$). Identities (2.16), (2.19) and (2.20) are then expressed as follows:

\[\text{The indices enclosed by } [\text{ and } () \text{ are supposed to be antisymmetrized and symmetrized respectively.}\]
\[ \epsilon_{AD} \epsilon^{BC} \left[ \begin{array}{c} D \\ C \end{array} \right] = \left[ \begin{array}{c} A \\ B \end{array} \right], \quad \text{(2.16)} \]

\[ \left[ \begin{array}{c} B \\ D \\ A \\ C \end{array} \right] - \left[ \begin{array}{c} B \\ D \\ A \\ C \end{array} \right] = \epsilon_{AC} \epsilon^{BD}, \quad \text{(2.19)} \]

\[ \left[ \begin{array}{c} C \\ D \\ A \\ B \end{array} \right] - \left[ \begin{array}{c} C \\ D \\ A \\ B \end{array} \right] = \epsilon_{EC} \epsilon^{FD}, \quad \text{(2.20)} \]

3 Dirac quantization

Now that we have provided the necessary backgrounds, let us try to quantize (2+1)-dimensional gravity in Ashtekar’s form. In this paper we will adopt the quantization procedure proposed by Dirac\[9\]. In Dirac’s quantization procedure, we first quantize the phase space by promoting the canonical variables \( (\omega^a_i, \tilde{E}^a_i) \) to the operators \( (\hat{\omega}^a_i, \hat{\tilde{E}}^a_i) \) which act on an appropriately chosen Hilbert space, and by replacing \( i \)-times the basic Poisson bracket \( (2.7) \) with the quantum commutation relation:

\[ [\hat{\omega}^a_i(x), \hat{\tilde{E}}^j_b(y)] = i \frac{\delta^a_i}{2} \delta^j_b \delta(x, y). \]  

We will use as a pre-constrained Hilbert space the space \( L^2(\mu, A) \) of the square-integrable functionals of \( \omega^a_i \) w.r.t a certain measure \( \mu \) on \( A \). \( \hat{\omega}^a_i \) and \( \hat{\tilde{E}}^a_i \) then act on the wavefunctions by multiplication and functional differentiation, respectively:

\[ \hat{\omega}^a_i(x) \cdot \Psi(\omega) = \omega^a_i(x) \cdot \Psi(\omega), \quad \hat{\tilde{E}}^i_a(x) \cdot \Psi(\omega) = -i \frac{\delta^i}{2 \delta \omega^a_i(x)} \Psi(\omega). \]  

(3.2)
Next we impose the constraint equations as the operator equations which restrict the
wavefunctions allowed in the theory\footnote{There is the issue on the choice of operator ordering which is intimately related to the problem on
the closure of the constraint algebra \cite{19,20}. We will not discuss on these issues and simply choose the
ordering in which the momentum operators $\hat{E}_a$ are placed to the right of the spin connections $\hat{\omega}_a^i$.}. The Gauss law and the diffeomorphism constraints:

$$\hat{G}_a(\eta^a)\Psi(\omega) = i \int_{\Sigma} d^2 x D_i \eta^a(x) \frac{\delta}{\delta \omega_i^a(x)} \Psi(\omega)$$

$$\hat{D}_i(N_i)\Psi(\omega) = -i \int_{\Sigma} d^2 x L N_i \omega_i^a(x) \frac{\delta}{\delta \omega_i^a(x)} \Psi(\omega)$$

(3.3)

have simple geometrical interpretations. The former requires the wavefunctionals to be
invariant under the small gauge transformations, and the latter tells us that the wave-
functionals be invariant under the spatial diffeomorphisms connected to the identity.

Because the spin network states are shown to span the dense subset of the space of
the gauge-invariant functionals of $SL(2,\mathbb{R})$ connections\cite{11}, to work in the set of these
states (or its completion) automatically satisfies the Gauss law constraint. Solving the
diffeomorphism constraint, on the other hand, requires a special consideration and we
will postpone this procedure until the next section. Here we only mention that the
exponentiated diffeomorphism constraint gives the diffeomorphism operator:

$$\hat{U}(\phi)\omega_i^a(x) = \frac{\partial}{\partial x^j} \omega_i^a(\phi^{-1}(x)),$$

(3.4)

whose action on a parallel propagator can be cast into the action on the curve along which
the propagator is evaluated:

$$\hat{U}(\phi)h_\alpha[0,1] = h_{\phi^{-1}\alpha}[0,1].$$

(3.5)

Unlike the previous two constraints, to define the Hamiltonian constraint operator

$$\hat{H}_{(N)} = \frac{1}{4} \int_{\sigma} d^2 x N(x) \epsilon_{abc} F_{ij}^a \frac{\delta}{\delta \omega_{ib}(x)} \frac{\delta}{\delta \omega_{jc}(x)},$$

we have to prescribe some regularization because it involves two functional derivatives
at an identical point. The problem is that whether we can find a regularization which
preserves the closure of the constraints under the commutator algebra. In particular, it
is important whether there exists a regularization which is invariant under the diffeomor-
phisms. In $(2+1)$-dimensions we can regularize the Hamiltonian constraint at least in
a diffeomorphism-invariant manner if we restrict the types of graphs on which the spin
network states are defined.
3.1 A diffeomorphism-covariant regularization

Among the regularizations of the Hamiltonian constraint, the most familiar one is the point-splitting regularization \[ \hat{H}(\tilde{\mathcal{N}}) = \lim_{\epsilon \to 0} \hat{H}^\epsilon(\tilde{\mathcal{N}}), \] (3.6)

\[ \hat{H}^\epsilon(\tilde{\mathcal{N}}) = \int_{\Sigma} d^2x \int_{\Sigma} d^2y \tilde{\mathcal{N}}(x) \tilde{f}_\epsilon(x,y) \xi^{ij} \text{Tr}(h_{yx}[0,1] \tilde{B}(x) \lambda^b h_{xy}[0,1] \lambda^c) \frac{\delta}{\delta \omega_{ib}(x)} \frac{\delta}{\delta \omega_{jc}(y)}, \]

where \( h_{xy}[0,1] \) is the parallel propagator along a curve from \( x \) to \( y \) which shrinks to \( x \) in the limit \( y \to x \), \( \tilde{B} \equiv \tilde{B}^a \lambda_a \equiv \frac{1}{2} \tilde{\epsilon}^{ij} F_{ij}^a \lambda^a \) is the magnetic field of the spin connection and the regulator \( \tilde{f}_\epsilon(x,y) \) is subject to the condition

\[ \tilde{f}_\epsilon(x,y) \xrightarrow{\epsilon \to 0} \delta^2(x,y). \]

In order to define a particular regulator, we usually have to introduce some extra background structures such as a fiducial metric or a fixed frame. These background structures do not behave covariantly under the action of the diffeomorphism constraint. This is the source of the diffeomorphism non-invariance of the regularized Hamiltonian constraint.

It is obvious that the action of \( \hat{H}(\tilde{\mathcal{N}}) \) on the parallel propagators is nonvanishing only on the curves along which the propagators are evaluated. Eq.(3.5) tells us that these curves are subject to the action of the diffeomorphism constraint. We therefore expect that if we introduce the curvilinear coordinate frame in which these curves play the role of coordinate curves (figure 1), then we can define a regularization of the Hamiltonian constraint which preserves the diffeomorphism covariance. In this subsection we explicitly carry out such a regularization procedure.

To simplify the analysis we restrict the types of the vertices to those at which only two smooth curves with linear independent tangent vectors intersect. We therefore do not consider vertices at which more than three independent curves intersect (figure 2(a), (b)), cusps (figure 2(c)), or vertices at which two independent curves with coincident tangent vectors intersect (figure 2(d)). We consider only configurations depicted in figure 3.

Because the Hamiltonian constraint involves two functional derivatives, it is convenient to separate its action as follows:

\[ \hat{H}(\tilde{\mathcal{N}}) = \hat{H}_1(\tilde{\mathcal{N}}) + \hat{H}_2(\tilde{\mathcal{N}}), \] (3.7)
Figure 1: A curvilinear coordinate frame used in our regularization.

Figure 2: Configurations not considered in this paper

Figure 3: The configurations considered in this paper: (a) two-point vertex; (b) three-point vertex; and (c) four-point vertex
where $\hat{\mathcal{H}}_1(\mathcal{N})$ and $\hat{\mathcal{H}}_2(\mathcal{N})$ stand for, respectively, the action of $\hat{\mathcal{H}}(\mathcal{N})$ on the single loops and that on the pairs of loops. This separation simplifies considerably the evaluation of the action of the Hamiltonian constraint on the spin-network states.

Now we demonstrate a few examples of the calculation.

First we consider the action of the regularized Hamiltonian \((3.6)\) on a single smooth loop $\alpha$:

$$\hat{\mathcal{H}}^c(\mathcal{N})h_\alpha[0,1]_A^B = \int \int d^2x \int d^2y_N(x)\bar{f}_c(x,y)\epsilon_{ijj}\text{Tr}(h_{y_{[0,1]}B}(x)\lambda^i h_{xy}[0,1]\lambda^c)$$

$$\times \int ds\delta^2(x,\alpha(s))\dot{\alpha}^i(s) \int dt\delta^2(y,\alpha(t))\dot{\alpha}^j(t)$$

$$\times \{\theta(t-s)h_\alpha[0,s]\lambda^j h_\alpha[s,1] + \theta(s-t)h_\alpha[0,t]\lambda^j h_\alpha[t,s]h_\alpha[1,1]\}_{A}^B$$

$$= \int \int dsdt\epsilon_{ijj}\dot{\alpha}^i(s)\dot{\alpha}^j(t)\bar{f}_c(\alpha(s),\alpha(t))\mathcal{N}(\alpha(s))\text{Tr}(h_{st}\tilde{B}(\alpha(s))\lambda^i h_{st}1\lambda^c)$$

$$\times \{\theta(t-s)h_\alpha[0,s]\lambda^j h_\alpha[s,1] + \theta(s-t)h_\alpha[0,t]\lambda^j h_\alpha[t,s]h_\alpha[1,1]\}_{A}^B$$

where we have abbreviated $h_{\alpha(s)\beta(t)}[0,1]$ to $h_{st}$. Because $\alpha$ serves as a coordinate curve, $\dot{\alpha}(s) \propto \dot{\alpha}(t)$ holds even when $t$ does not coincide with $s$ (but is sufficiently close to $s$). We find that the result vanishes due to the factor $\epsilon_{ijj}\dot{\alpha}^i(s)\dot{\alpha}^j(t)$. Hence our regularization does not suffer from ‘acceleration terms’ \([14]\) which manifestly violates diffeomorphism covariance.

Next we calculate the action on a single loop with a kink, which we will denote by $\alpha_1 \circ \beta_2$ \([15]\):

$$\hat{\mathcal{H}}^c(\mathcal{N})(h_\alpha[0,s_0]h_\beta[t_0,1])_A^B = \hat{\mathcal{H}}^c(\mathcal{N})(h_\alpha[0,s_0]h_\beta[t_0,1])_A^B$$

$$= \int_0^{s_0} ds \int_{t_0}^{1} dt\epsilon_{ijj}\dot{\alpha}^i(s)\dot{\beta}^j(t)\bar{f}_c(\alpha(s),\beta(t))$$

$$\times \{\mathcal{N}(\alpha(s))\text{Tr}(h_{st}[0,1]B(\alpha(s))\lambda^j h_{st}1\lambda^c)h_\alpha[0,s]\lambda^j h_\alpha[s,s_0]h_\beta[t_0,t]\lambda^j h_\beta[t,1]$$

$$- \mathcal{N}(\beta(t))\text{Tr}(h_{st}[0,1]B(\beta(t))\lambda^j h_{st}[0,1]\lambda^c)h_\alpha[0,s]\lambda^j h_\alpha[s,s_0]h_\beta[t_0,t]\lambda^j h_\beta[t,1]\}_{A}^B$$

$$\equiv \int_0^{s_0} ds \int_{t_0}^{1} dt\epsilon_{ijj}\dot{\alpha}^i(s)\dot{\beta}^j(t)\bar{f}_c(\alpha(s),\beta(t)) \times I_1,$$ \hspace{1cm} (3.8)

We first simplify the integrand. By using $h_{\alpha(s)\beta(t)}[0,1]_A^B = \delta_A^B + O(\epsilon)$ and similar approximations, we find

$$I_1 = \left\{ \frac{\epsilon_{abc}}{4} \mathcal{N}(x_0)\tilde{B}_a(x_0) (h_\alpha[0,s_0]\lambda^j h^c_\lambda - \lambda^c h_\lambda h_\beta[t_0,1]) + O(\epsilon) \right\}_A^B$$

\(^6\)In this paper we usually assume that the curves $\alpha$ and $\beta$ intersect with each other at $\alpha(s_0) = \beta(t_0) = x_0$. $\alpha_1 \subset \alpha$ is the curve from $\alpha(0)$ to $\alpha(s_0)$ and $\beta_2 \subset \beta$ is the curve from $\beta(t_0)$ to $\beta(1)$.
\[ \hat{\mathcal{H}}_1(\mathcal{N})(h_\alpha[0, s_0]h_\beta[t_0, 1])^B_A = -\frac{1}{2} \left\{ \mathcal{N}(x_0)h_\alpha[0, s_0]\hat{B}(x_0)h_\beta[t_0, 1] + O(\epsilon) \right\}^B_A. \]

The \(O(\epsilon)\) part also transforms covariantly under the gauge transformations. Now we consider the part involving the regulator:

\[ \int_0^{\delta_0} ds \int_{t_0}^1 dt \epsilon_{ij} \hat{\omega}^i(s) \hat{\omega}^j(t) \hat{f}_s(\alpha(s), \beta(t)). \]

In the conventional point splitting regularization we fix a background metric to define \(\hat{f}_s\). In our regularization, however, we first exploit the transformation property of the \(\delta\)-function:

\[ |\det(\partial_t\phi^i(x))|\delta^n(\phi(x), \phi(y)) = \delta^n(x, y), \]

and fix the explicit form of the regulator after we transfer into the curvilinear coordinate system in which the loop parameters \(s, t\) play the role of the coordinate. For example, we set

\[ \epsilon_{ij} \hat{\omega}^i(s) \hat{\omega}^j(t) \hat{f}_s(\alpha(s), \beta(t)) = \frac{\sigma(\alpha, \beta)}{4\epsilon^2} \theta(\epsilon - |s - s_0|)\theta(\epsilon - |t - t_0|), \]

where \(\sigma(\alpha, \beta)\) stands for the signature which takes the value \(+1(-1)\) if \((\hat{\omega}(s_0), \hat{\omega}(t_0))\) forms a right-(left-)handed frame. Putting these ingredients into together and taking the limit \(\epsilon \to 0\), we finally find:

\[ \hat{\mathcal{H}}_1(\mathcal{N})(h_\alpha[0, s_0]h_\beta[t_0, 1])^B_A = -\frac{1}{8} \sigma(\alpha, \beta) \mathcal{N}(x_0)(h_\alpha[0, s_0]\hat{B}(x_0)h_\beta[t_0, 1])^B_A \]

\begin{equation}
(3.10) \end{equation}

where \(\hat{\Delta}\) is the area derivative which acts on the functionals of graphs \([\gamma]\):

\[ \hat{\Delta}(\gamma, x_0)\Psi[\alpha, \cdots] \equiv \lim_{s \to 0} \frac{\Psi[\alpha \circ \gamma_{x_0}, \cdots] - \Psi[\alpha, \cdots]}{s(\gamma_{x_0})}. \]

To derive the last equality of eq.\((3.10)\), we have used the result in ref.\([21]\).

The third example is a pair of smooth loops intersecting at a vertex. The first example shows that the action of \(\hat{\mathcal{H}}_1(\mathcal{N})\) is zero. Thus we have only to compute the action of \(\hat{\mathcal{H}}_2(\mathcal{N})\):

\[ \hat{\mathcal{H}}_2(\mathcal{N}) \left( h_\alpha[0, 1] h_\beta[0, 1] \right)^B_C \]

\[ = \int_0^1 ds \int_0^1 dt \epsilon_{ij} \hat{\omega}^i(s) \hat{\omega}^j(t) \hat{f}_s(\alpha(s), \beta(t)) \mathcal{N}(x_0) \text{Tr}(\hat{B}(x_0)\lambda^b \lambda^c) \]

\[ \times \left\{ (h_\alpha[0, s] \lambda_b h_\alpha[s, 1])^B_A (h_\beta[0, t] \lambda_c h_\beta[t, 1])^C_D - (b \leftrightarrow c) + O(\epsilon) \right\}. \]

\(^7\text{We regard } \alpha \text{ as a part of a graph. } \gamma_{x_0} \text{ is an infinitesimal loop with a basepoint } x_0. \text{ The ‘coordinate area’ } s(\gamma) \text{ of a loop } \gamma \text{ is defined by: } s(\gamma) \equiv \oint_{\gamma_{x_0}} f(\gamma).\)
We can reduce the integrand by using the Fiertz identity (2.18) and eq.(3.9). The final result is:

\[
\hat{H}_2(N) \left( h_\alpha[0, 1]_A B h_\beta[0, 1]_C D \right)
\]

\[
= \frac{\sigma(\alpha, \beta)}{4} N(x_0) \left\{ (h_\alpha[0, s_0] h_\beta[t_0, 1])_A D (h_\beta[0, t_0] B^2(x_0) h_\alpha[s_0, 1])_C B \right\}
\]

\[
= \frac{\sigma(\alpha, \beta)}{4} N(x_0) \left( \bar{\Delta}(\beta_1 \circ \alpha_2, x_0) - \bar{\Delta}(\alpha_1 \circ \beta_2, x_0) \right)
\times \left\{ (h_\alpha[0, s_0] h_\beta[t_0, 1])_A D (h_\beta[0, t_0] h_\alpha[s_0, 1])_C B \right\}.
\] (3.11)

The action on the rest of the configurations can be calculated similarly. We list the action of \(\hat{H}(N)\) on all the basic configurations in Appendix. There we make use of the graphical representation.

Since we have at hand the action of the Hamiltonian constraint on the basic configurations, it is not difficult to calculate its action on general spin network states. The idea is the following. Appendix tells us that the action of \(\hat{H}(N)\) has nonvanishing contributions only at the vertices. Because the action of \(\hat{H}(N)\) is local, we can separate its action on the individual vertices, i.e.

\[
\hat{H}(N)\Psi^{\text{spinnet}}(\omega) = \sum_{v \in V} \left( \hat{H}(N)\Psi^{\text{spinnet}}(\omega) \right) |_v,
\] (3.12)

where \(V\) denotes the set of all vertices involved in the graph on which the spin network state \(\Psi^{\text{spinnet}}\) is defined.

If we use the separation (3.7), we can easily evaluate the action on each vertex \(v\). First we add up the contributions from all the kinks and thus obtain the action of \(\hat{H}_1(N)\) on the vertex \(v\). We then compute the sum of the action of \(\hat{H}_2(N)\) on all the pairs of parallel propagators. The total sum of these contributions yields the action of \(\hat{H}(N)\) at the vertex \(v\). Symbolically we can write:

\[
\left( \hat{H}(N)\Psi^{\text{spinnet}}(\omega) \right) |_v = \sum_{k \in K_v} \left( \text{the action of } \hat{H}_1(N) \text{ on a kink } k \right) + \sum_{p \in P_v} \left( \text{the action of } \hat{H}_2(N) \text{ on a pair } p \text{ of propagators} \right),
\] (3.13)

where \(K_v\) (\(P_v\)) is the set of all the kinks (all the pairs of parallel propagators) at the vertex \(v\). Using eqs.(3.12), (3.13) and the equations in Appendix, we can therefore reduce...
the problem of evaluating the action of the Hamiltonian constraint on the spin network states to that of combinatorics. This applies to more general operators which involve the product of a finite number of momenta $E_a$. Thus, by virtue of this property of the spin network, the problems of gauge theories or those of quantum gravity may be formulated in a graphical and combinatorial manner which is somewhat similar to that of a perturbation theory using Feynmann diagrams.

3.2 Covariance under the diffeomorphisms

In the last subsection we have obtained the action of the regularized Hamiltonian constraint on the spin network states. The expression does not explicitly depend on the background structure used to define the regulator (3.9), and so it is expected to be covariant under the spatial diffeomorphisms. We will demonstrate explicitly that this is indeed the case.

As an illustration we consider the action on a kink:

$$\hat{U}(\phi)\hat{\mathcal{H}}(\mathcal{N})\hat{U}(\phi^{-1}) \cdot (h_{\alpha}[0,s_0]h_{\beta}[t_0,1])^B_A.$$  

Using eq.(3.5) and the following equation: $\hat{U}(\phi)\tilde{B}(x) = \det(\partial_i\phi^{-1}(x)^j)\tilde{B}(\phi^{-1}(x))$ we can easily compute the above expression:

$$\hat{U}(\phi)\hat{\mathcal{H}}(\mathcal{N})\hat{U}(\phi^{-1}) \cdot (h_{\alpha}[0,s_0]h_{\beta}[t_0,1])^B_A,$$

$$= -\frac{\sigma(\phi \cdot \alpha, \phi \cdot \beta)}{8} \mathcal{N}(\phi(x_0))\hat{U}(\phi) \cdot (h_{\phi,\alpha}[0,s_0]\tilde{B}(\phi(x_0))h_{\phi,\beta}[t_0,1])^B_A$$

$$= -\frac{\sigma(\phi \cdot \alpha, \phi \cdot \beta)}{8} \mathcal{N}(\phi(x_0))\left(h_{\alpha}[0,s_0] \left(\det(\partial_i\phi^{-1}(x)^j)|_{x=\phi(x_0)}\tilde{B}(x_0)\right)h_{\beta}[t_0,1]\right)^B_A$$

$$= -\frac{\sigma(\phi \cdot \alpha, \phi \cdot \beta)}{8} \left(\phi_\#\mathcal{N}(x_0)(h_{\alpha}[0,s_0]B(x_0)h_{\beta}[t_0,1])^A\right)^B,$$

where $(\phi_\#\mathcal{N})(x) \equiv \det^{-1}(\partial_i\phi^j(x))\mathcal{N}(\phi(x))$. This is the desired result. Though we do not demonstrate explicitly, we can derive this result also using the expression involving the area derivative.

The proof is essentially the same also in the general case. The action of $\hat{\mathcal{H}}(\mathcal{N})$ on the spin network states can be always expressed by the composite operation of: i) the topological manipulation in which the curves through the vertices are cut and rejoined with appropriate relative weights; and ii) the action of the area derivative multiplied by the value of ‘lapse’ $\mathcal{N}(x_v)$ at each
vertex. It is obvious that, when conjugated by a diffeomorphism, operation i) does not pick anything while the change in operation ii) amounts to replacing $\mathcal{N}$ by $\phi_+ \mathcal{N}$. Thus we have proved the diffeomorphism covariance of our regularized Hamiltonian constraint:

$$\hat{U}(\phi) \hat{\mathcal{H}}(\mathcal{N}) \hat{U}(\phi^{-1}) = \hat{\mathcal{H}}(\phi_+ \mathcal{N}).$$  (3.14)

Finally we should mention that, if we ‘differentiate’ this equation, we obtain the quantum commutator version of the Poisson algebra (2.10) between the Hamiltonian constraint and the diffeomorphism constraint.

## 4 Solutions to the Hamiltonian constraint

In this section we provide solutions to the Hamiltonian constraint. After briefly reviewing the solutions which have been found so far, we construct the ‘combinatorial solutions’ which are the generalization of the solution found by Jacobson and Smolin to the spin network states.

### 4.1 Topological solutions

As we have seen in the last section, the nonvanishing action of the Hamiltonian constraint $\hat{\mathcal{H}}(\mathcal{N})$ on spin network states necessarily involves the area derivative $\tilde{\Delta}(\alpha, x_v)$ at the vertex $v$. Hence the action of $\hat{\mathcal{H}}(\mathcal{N})$ vanishes if the action of the area derivative is zero everywhere, which means that the spin connection is flat:

$$\tilde{B}(x)^a = \frac{1}{2} \epsilon^{ij} F_{ij}^a(x) = 0.$$  (4.1)

We therefore find the following ‘distributional’ solution:

$$\Psi^{\text{top}}(\omega) \equiv \psi(\omega) \prod_{a,x} \delta(\tilde{B}^a(x)), $$  (4.2)

where $\psi(\omega)$ is an arbitrary (square-integrable) gauge-invariant function of $\omega^a$. This is the very solution of the constraints in Witten’s formalism of (2+1)-gravity [4] with a vanishing cosmological constant:

$$\hat{\Psi}^a(x) \Psi^{\text{top}}(\omega) = \hat{\mathcal{G}}^a(x) \Psi^{\text{top}}(x) = 0.$$
Because this kind of solution has a support only on the flat connections, \( \psi(\omega) \) amounts to the function on the moduli space of flat \( SL(2, \mathbb{R}) \) connections on \( \Sigma \) modulo gauge transformations. In terms of spin network states, this solution depends only on the homotopy classes of the ‘colored graphs’. As a corollary, it follows that this solution is invariant under the diffeomorphisms and so this topological solution is the solution to all the constraint. We can thus consider the Hilbert space of Witten’s formalism as a subspace of the Hilbert space of (2+1)-dimensional Ashtekar formalism.

### 4.2 Trivial solutions

One of the important results obtained in the last section is that the nonvanishing contributions to the action of \( \hat{\mathcal{H}}(N) \) are only from the vertices, i.e. the points where the analyticity of the curves breaks down. From this result we realize that, if we consider the spin network states consisting only of the Wilson loops along the smooth loops \( \{\alpha_i\} \) \((i = 1, \cdots, I)\) without any intersection:

\[
\Psi_{\{(\alpha_i, \rho_i)\}}(\omega) = \prod_{i=1}^{I} W(\alpha_i, \rho_i),
\]

then these states solve the Hamiltonian constraint. These states are related with the solutions to the Hamiltonian constraint which have been found in the loop representation [22].

### 4.3 Combinatorial solutions

If we look at Appendix carefully, we find that the linear combination:

\[
h_{\alpha}[0, 1]_A^B h_{\beta}[0, 1]_C^D - 2(h_{\alpha}[0, s_0] h_{\beta}[t_0, 1])_A^D (h_{\beta}[0, t_0] h_{\alpha}[s_0, 1])_C^B
\]

has a vanishing action of \( \hat{\mathcal{H}}(N) \). This is the solution given by Jacobson and Smolin [17]. We therefore expect that, even if the graph has vertices, some appropriate linear combinations of the spin network states defined on a graph may solve the Hamiltonian constraint. Here we will find such ‘combinatorial solutions’ which are considered to be generical within the configuration of the graphs considered in this paper.

We have seen in eq.(3.12) that \( \hat{\mathcal{H}}(N) \) acts on each vertices independently. In order to be a solution to the Hamiltonian constraint equation, the spin network state must solve this constraint equation \textit{at every vertex}. We can construct a solution by looking for
the intertwining operators which give the vanishing action of $\hat{\mathcal{H}}(\mathcal{N})$ at each vertex, and by gluing these solutions at adjacent (but separate) vertices using an adequate parallel propagator as a glue. Thus we have only to concentrate on one vertex, say, at $x_0$. As in Appendix we introduce the ‘rescaled Hamiltonian’:

$$\hat{\mathcal{H}}_{x_0}^\prime \equiv -\sigma(\alpha, \beta) \frac{8}{N(x_0)} \left( \text{the action of } \hat{\mathcal{H}}(\mathcal{N}) \text{ at } x_0 \right). \quad (4.5)$$

In searching for the combinatorial solutions, the following identity proves to be useful:

$$\begin{align*}
\left(\begin{array}{ccc}
A & D & F \\
C & E & B
\end{array}\right) - \left(\begin{array}{ccc}
A & D & F \\
C & E & B
\end{array}\right)
&= \left(\begin{array}{ccc}
\Delta & D & F \\
C & E & B + \Delta
\end{array}\right) - \left(\begin{array}{ccc}
\Delta & D & F \\
C & E & B - \Delta
\end{array}\right). \quad (I)
\end{align*}$$

This identity is proved as follows. We can write down the difference between the l.h.s and the r.h.s of identity (I) as:

$$(\text{l.h.s}) - (\text{r.h.s}) = \tilde{B}_{GIK}^{HJL}(\epsilon_G^{L} \epsilon_J^{L} - \epsilon_G^{J} \epsilon_J^{L}) + \tilde{B}_I^{H} (\epsilon_I^{L} \epsilon_J^{H} - \epsilon_I^{H} \epsilon_J^{L}) - \tilde{B}_I^{J} (\epsilon_I^{L} \epsilon_K^{J} - \epsilon_I^{J} \epsilon_K^{L}).$$

Using the two-spinor identity (2.17), we can reduce $\tilde{B}_{GIK}^{HJL}$ to zero:

$$\tilde{B}_{GIK}^{HJL} = \tilde{B}_I^{H} \epsilon_G^{L} \epsilon_J^{JL} - \tilde{B}_I^{J} \epsilon_K^{L} \epsilon_J^{JL} + \tilde{B}_I^{H} \epsilon_K^{L} \epsilon_J^{JL} - \tilde{B}_I^{J} \epsilon_K^{L} \epsilon_J^{JL} = \epsilon_{IK} \tilde{B}_I^{H} \epsilon_J^{JL} - \epsilon_J^{JL} \tilde{B}_I^{H} \epsilon_{IK} = 0.$$
Now we are in the place where we demonstrate explicitly there exist a set of solutions to $\mathcal{H}'_{x_0}$ which are the extensions of the solution found by Jacobson and Smolin to (four-valent) spin network states. These solutions are given by the following expression,

$$
\prod_{i=1}^{m} (h_\alpha[0, s_0]_{A_i} E_i h_\alpha[s_0, 1]_{F_i})^{B_i} \prod_{j=1}^{n} (h_\beta[0, t_0]_{C_j} h_\beta[t_0, 1]_{H_j})^{D_j} \times \mathcal{I}^r(m, n)_{E_i \cdots E_m; G_1 \cdots G_n}, \quad (4.6)
$$

where $\mathcal{I}^r(m, n)$ is the relevant intertwining operator:

$$
\begin{align*}
\mathcal{I}^r(m, n)_{E_i \cdots E_m; H_1 \cdots H_n} &\equiv \min(m, n) \left( \prod_{i=1}^{r} \frac{-2}{m + n - 2i + 1} \right) \mathcal{I}^r(m, n)_{E_i \cdots E_m; H_1 \cdots H_n}, \\
\mathcal{I}^r(m, n)_{E_i \cdots E_m; G_1 \cdots G_n} &\equiv \sum_{1 \leq k_1 < \cdots < k_r \leq m} \sum_{1 \leq l_1 < \cdots < l_r \leq n} \prod_{i=1}^{r} \left( \epsilon_{E_{k_i}} F_{k_i} \epsilon_{G_{l_i}} F_{l_i} \right) \prod_{k' = 1}^{m} \prod_{l' = 1}^{n} \epsilon_{E_{k'}} \epsilon_{G_{l'}}. \quad (4.7)
\end{align*}
$$

$P_r$ in the above expression is the group of the permutations of $r$ numbers $(l_1, \cdots, l_r)$.

(Proof): To simplify the calculation we first define the following two sets of functionals:

$$
\begin{align*}
C^r(m, n)_{A_1 \cdots A_m; C_1 \cdots C_n} &\equiv \prod_{i=1}^{m} (h_\alpha[0, s_0]_{A_i} E_i h_\alpha[s_0, 1]_{F_i})^{B_i} \prod_{j=1}^{n} (h_\beta[0, t_0]_{C_j} h_\beta[t_0, 1]_{H_j})^{D_j} \\
&\times \mathcal{I}^r(m, n)_{E_i \cdots E_m; H_1 \cdots H_n}, \quad (4.8)
\end{align*}
$$

$$
\begin{align*}
B^r(m, n)_{A_1 \cdots A_m; C_1 \cdots C_n} &\equiv \prod_{i=1}^{m} (h_\alpha[0, s_0]_{A_i} E_i h_\alpha[s_0, 1]_{F_i})^{B_i} \prod_{j=1}^{n} (h_\beta[0, t_0]_{C_j} h_\beta[t_0, 1]_{H_j})^{D_j} \\
&\times \sum_{\sigma \in P_r} \left( \prod_{i=1}^{r} \epsilon_{E_{k_i}} F_{k_i} \epsilon_{G_{l_i}} F_{l_i} \right) \prod_{k' = 1}^{m} \prod_{l' = 1}^{n} \epsilon_{E_{k'}} \epsilon_{G_{l'}}. \\
&\times \left( \sum_{1 \leq k_i < \cdots < k_r \leq m} \sum_{1 \leq l_i < \cdots < l_r \leq n} \prod_{i=1}^{r} \epsilon_{E_{k_i}} F_{k_i} \epsilon_{G_{l_i}} F_{l_i} \right) \prod_{k' = 1}^{m} \prod_{l' = 1}^{n} \epsilon_{E_{k'}} \epsilon_{G_{l'}}.
\end{align*}
$$

We should notice that $B^0(m, n) = B^{\min(m, n)+1}(m, n) = 0$. Intuitively, the $C^r(n, m)$-functional is obtained as follows: Prepare $m$ propagators $h_\alpha$ along $\alpha$ and $n$ propagators $h_\beta$ along $\beta$; then choose $r$ pairs of $h_\alpha$ and $h_\beta$; cut and reglue each pair in the orientation preserving fashion; and finally sum up the results obtained from all the choices of $r$ pairs. The $B^r(m, n)$-functional is the result of the action of $(\mathcal{H}'_{x_0})_1$ on the $C^r(m, n)$-functional.
\[ C^r(m,n) = \sum_{1 \leq k_1 < \cdots < k_r \leq m} \sum_{1 \leq l_1 < \cdots < l_r \leq n} \sum_{\sigma \in P_r} f \sum_{i=1}^r \]

\[ B^r(m,n) = \sum_{1 \leq k_1 < \cdots < k_r \leq m} \sum_{1 \leq l_1 < \cdots < l_r \leq n} \sum_{\sigma \in P_r} f \sum_{i=1}^r \]

Figure 4: Graphical representations of \( C^r(m,n) \) and \( B^r(m,n) \)

We will henceforth omit the ‘external’ spinor indices each of which is associated with the tip or the tail of a parallel propagator \( h_\alpha[0,1] \) or \( h_\beta[0,1] \), because they only serve as the labels of the propagators and do not play any essential role. Graphical representations of \( C^r(m,n) \) and \( B^r(m,n) \) are provided in figure 4. There, we distinct the propagators by numbering them.

Now the action of \( \hat{\mathcal{H}}_{x_0} \) on \( C^r(m,n) \) is computed as:

\[ \hat{\mathcal{H}}_{x_0} C^r(m,n) = (m + n - 2r + 1)B^r(m,n) + 2B^{r+1}(m,n). \]  

(4.9)

We will only give a sketch of the proof of the above equation. The contributions to the action of \( \hat{\mathcal{H}}_{x_0} \) on \( C^r(m,n) \) is classified into four types (see Appendix): i) contributions of type (A.2); ii) contributions of type (A.3); iii) contributions of type (A.4) and those of type (A.5); and iv) is the same as iii) except that \( \alpha \) being replaced by \( \beta \).
We can easily show that the class i)-contributions and the class ii)-contributions sum up to yield $B^r(m,n)$ and $2B^{r+1}(m,n)$ respectively.

It is somewhat difficult to evaluate class iii)- and class iv)-contributions. First we see that, in the sum over the contributions belonging to each of the two classes, the terms like the second term in the r.h.s of eq.(A.4) and those like the first term in the r.h.s of eq.(A.5) cancel out. Then we can apply identity (I) to the surviving contributions, the sum of these terms turns out to be proportional to $B^r(m,n)$. A careful consideration on permutations and combinations tells us that the class iii)- and the class iv)- contributions respectively add up to yield $(m-r)B^r(m,n)$ and $(n-r)B^r(m,n)$ respectively. Thus the total sum of all the contributions to $\hat{H}'_{x_0}C^r(m,n)$ is:

$$B^r(m,n)+2B^{r+1}(m,n)+(m-r+n-r)B^r(m,n)=(m+n-2r+1)B^r(m,n)+2B^{r+1}(m,n).$$

This is nothing but the r.h.s of (4.9).

Now we have derived eq.(4.9), it is an elementary exercise of the linear algebra to show the following equation holds:

$$\hat{H}'_{x_0}\left(\sum_{r=0}^{\min(m,n)} \left(\prod_{i=1}^{r} \frac{-2}{m+n-2i+1}\right) C^r(m,n)\right) = 0. \quad (4.10)$$

Separating the expression in the large parenthesis into the product of parallel propagators along $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ and the intertwining operator, we obtain eq.(4.6).

Let us now consider whether there are any combinatorial solutions other than eq.(4.6). The key point in finding these combinatorial solutions is whether there exist more than two configurations, or graphs, such that the action of $\hat{H}'_{x_0}$ on them cancel with each other. At the four point vertex, we have such configurations like the two terms in eq.(4.4). It is obvious from eqs.(A.2)(A.3) we cannot find any combinatorial solutions from two point vertices.

Similarly at three point vertices, no nontrivial configuration seems to exist on which the total action of $\hat{H}'_{x_0}$ vanishes. Hence we conjecture that

(Conjecture1): We cannot construct any nontrivial combinatorial solutions which involve two or three point vertices.

Thus it is sufficient to consider only four point vertices. If we closely look at the proof of eq.(4.6), it seems to be essential that, in a configuration, the parallel propagators
along the curves $\alpha_1 \circ \beta_2$ and those along $\beta_1 \circ \alpha_1$ appear in pairs. This indicates that, in a configuration which is a part of a combinatorial solution, the number of the parallel propagators along each smooth curve is conserved at each vertex.

(Conjecture 2): In any nontrivial combinatorial solutions, the configurations at each vertex are constructed by cutting and rejoining the parallel propagators along the smooth curves through that vertex.

If we allow only configurations appearing in conjecture 2, it is almost evident that eq.(4.6) provides the combinatorial solutions at a vertex $x_0$ which are general under the assumption made in §2. The reasoning is as follows. Let us consider to cancel the action of $\hat{H}'_{x_0}$ on a configuration which appear in the sum in $C^r(m,n)$. To cancel the contributions like the second term in eq.(A.4) and the first term in eq.(A.5), we have to sum up all the configurations in $C^r(m,n)$. The problem is thus reduced to that of finding linear combinations of $C^r(m,n) (r = 1, \cdots, \min(m,n))$ which yield zero eigenvalue of $\hat{H}'_{x_0}$. We can find immediately the unique solution, which is identical to (4.6). Thus we conclude

(Conjecture): Eq.(4.6) yields the combinatorial solutions which are general on the restricted graphs in which at most two independent curves may intersect at each vertices.

To prove these conjectures rigorously, of course, we need to ensure that any ‘miraculous’ cancellations can never occur in general except those in eq.(4.6). If these conjectures turn out to be true, we can extend them to more general cases in which any types of vertices (except those shown in figure 2(c), 2(d)) are considered. Because the essence is in the cancellation between the actions of $\hat{H}'_{x_0}$ on the two terms in eq.(4.4), we conjecture that the combinatorial solutions consist only of the configurations in which the number of the propagators along each smooth curve is conserved at every vertex. A corollary of this generalized conjecture is that we cannot construct a combinatorial solution from the trivalent graphs. Thus to consider the space of the solutions to all the constraints, we would be forced to consider $n$-valent graphs with $n \geq 4$. While we have not yet found the proof, we expect that these conjectures are indeed true.

Finally we provide the procedure to construct a combinatorial solution which is also gauge invariant:

1) Choose a set of smooth loops $\alpha_i : [0,1] \to \Sigma (i = 1, 2, \cdots, N; \alpha_i(0) = \alpha_i(1))$ in the 2 dimensional space $\Sigma$. Each loop in this set can intersect with itself or with the others,
provided that more than two loops do not intersect at a point.

2) Equip each loop \( \alpha_i \) with an \( l_i \)-th power of spinor Wilson loops: \((h_{\alpha_i}[0,1]^{A_i})^{l_i}\).

3) At each point where two loops, e.g. \( \alpha_i \) and \( \alpha_j \) intersect, cut the spinor Wilson loops and rejoin them by using as a glue the intertwining operator \( I^c(l_i, l_j) \) in eq.(4.7).

We should note that eq.(4.6) remains to be the solution even if we permute the external indices. Moreover, from eq.(2.13), we see that the antisymmetrized tensor product of two spinor propagators yields the invariant spinor. As a result the antisymmetrization reduces the number of the spinor propagators by two. We can therefore replace procedure 2) by:

2') Equip each loop \( \alpha_i \) with a Wilson loop in the spin-\( l_i \) representation, which is given by the symmetrized trace of \( l_i \) spinor propagators:

\[ h_{\alpha_i}[0,1]^{\{A_1 \cdots A_i\}}. \]

We will denote the combinatorial solution constructed by procedures 1), 2') and 3) by \( \Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega) \). If we fix a set of loops \( \{\alpha_i\} \) \((i = 1, \cdots, N)\), we can construct \((\mathbb{Z}_+)^N\) combinatorial solutions where \( \mathbb{Z}_+ \) is the set of non-negative integers. The trivial solutions described in the last subsection are contained in the combinatorial solutions \( \Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega) \) as particular configurations in which all the loops \( \alpha_i \) do not intersect.

### 4.4 Solutions to all the constraints

In the previous subsections we have found the solutions to both the Gauss law and the Hamiltonian constraints. To construct the physical Hilbert space which is spanned by physical wavefunctions, however, the diffeomorphism constraint remains to be solved. Because the diffeomorphism constraint equation tells us that the physical states are invariant under the diffeomorphisms

\[ \hat{U}(\phi) \cdot \Psi_{\text{phys.}}(\omega) = \Psi_{\text{phys.}}(\omega), \quad \phi \in \text{Diff}(\Sigma), \]

a prescription to construct such states from the combinatorial solutions is to average the action of the diffeomorphism operators on these states:

\[
\Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega) = \left(Vol(\text{Diff}(\Sigma))\right)^{-1} \int_{\text{Diff}(\Sigma)} [D\phi] \hat{U}(\phi) \cdot \Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega), \quad \phi \in \text{Diff}(\Sigma),
\]

\[
\Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega) = \left(Vol(\text{Diff}(\Sigma))\right)^{-1} \int_{\text{Diff}(\Sigma)} [D\phi] \Psi_{\{( \phi \cdot \alpha_i, l_i) \}}^{\text{combi.}}(\omega), \quad \phi \in \text{Diff}(\Sigma),
\]  

\[ (4.11) \]
where $[D\phi]$ is an ‘invariant measure on the space $\text{Diff}(\Sigma)$ of the diffeomorphisms on $\Sigma$. This average is formal in the sense that it is indefinite because it involves the ratio of two divergent expressions, namely, $\text{Vol}(\text{Diff}(\Sigma))$ and $\int_{\text{Diff}(\Sigma)}[D\phi]$. An attempt to make this formal average be mathematically rigorous can be seen in ref.\cite{23}. A rough outline of the strategy used there is the following. Because there is an infinite ‘isotropy group’, i.e. the group of diffeomorphisms which leave invariant the set $\{((\alpha_i, l_i))\}$ of colored loops, we can replace the formal average over the entire group $\text{Diff}(\Sigma)$ of diffeomorphisms by a discrete sum on the orbit $\{((\alpha_i, l_i))\}$ of $\{((\alpha_i, l_i))\}$ under $\text{Diff}(\Sigma)$:

$$\Psi_{\{(\alpha_i, l_i)\}}^{\text{phys.}}(\omega) = \sum_{\{(\alpha_i', l_i')\} \in \{((\alpha_i, l_i))\}} \Psi_{\{(\alpha_i', l_i')\}}^{\text{combi.}}(\omega). \quad (4.12)$$

This is well-defined as a distributional wavefunction which belongs to the dual $\Phi'$ of the space $\Phi \equiv Cyl^\infty(A/G)$ of the smooth cylindrical functions.\footnote{A cylindrical function is the function which is the projective limit of the functions on the space of connections in a ‘lattice gauge theory’ defined on the graphs embedded in $\Sigma$. For details of the Cylindrical function and the projective limit, see e.g. ref.\cite{24}.} Hence, from the combinatorial solutions $\Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}$, we can construct the physical wavefunctions $\Psi_{\{(\alpha_i, l_i)\}}^{\text{phys.}}$ which depend only on the differential topology of the set $\{((\alpha_i, l_i))\}$ of smooth colored loops embedded in $\Sigma$.

5 Physical properties of the combinatorial solutions

We have constructed infinitely many solutions $\Psi_{\{(\alpha_i, l_i)\}}^{\text{combi.}}(\omega)$ to the Hamiltonian constraint for each fixed set $\{\alpha_i\}$ of smooth loops. Let us now try to give physical interpretations to these solutions. For this purpose we will consider the action of a few types of operators on these solutions.

First we consider the ‘normal vector operator’

$$\hat{n}^a(\eta_a) \equiv \int_{\Sigma} d^2x \eta_a(x) \hat{n}^a(x) = \int_{\Sigma} d^2x \eta_a \frac{1}{2} \epsilon^{abc} \epsilon_{ij} \hat{E}^i_b(x) \hat{E}^j_c(x) \quad (5.1)$$

which is not gauge invariant. This operator needs to be regularized because it involves two functional derivatives at a point. We adopt the same regularization as that of the Hamiltonian constraint:

$$\hat{n}^a(\eta_a) = \lim_{\epsilon \to 0} (\hat{n}^a(\eta_a))^\epsilon,$$
\[(\tilde{n}^a(\eta))^\epsilon \equiv -\frac{1}{2} \int_\Sigma d^2x \int_\Sigma d^2y \tilde{f}_i(x, y) \epsilon_{ij} \]
\[\times \ \text{Tr}(h_{yx}[0, 1]|\eta(x)\lambda^b h_{xy}[0, 1]|\lambda^c) \frac{\delta}{\delta \omega^a_i(x)} \frac{\delta}{\delta \omega^c_j(y)}, \tag{5.2}\]

where we have set \(\eta \equiv \eta^a \lambda_a\). This is the same expression as that of the regularized Hamiltonian constraint with \(\eta(x)\) being substituted for \(8 \tilde{N}(x) \tilde{B}(x)\). The action of \(\hat{n}^a(\eta_a)\) can therefore be calculated in the same way as that of computing the action of \(\hat{H}(\tilde{N})\). When the combinatorial solutions were derived, the action of \(\hat{B}\), or the area derivative \(\hat{\Delta}\), did not play any essential roles. Thus each combinatorial solution gives zero eigenvalue to the operator which is obtained by replacing \(\tilde{N}(x) \tilde{B}^a(x)\) by a certain functional \(F^a(\omega, x)\) of the spin connection. The normal vector operator is one such operator. Classically, vanishing of the ‘normal vector’ \(\tilde{n}^a\) indicates that the induced metric \(h_{ij}\) on \(\Sigma\) is degenerate. Hence we can consider that the combinatorial solutions correspond to the classical solutions with degenerate metric which do not belong to the solutions in Witten’s formalism.

In practice, the ‘normal vector’ operator is not so useful because it is not invariant under gauge transformations. Thus we have to consider some operators which carry the gauge invariant information on \(\tilde{n}^a\). A useful candidate is provided by the operators one of which measures the area of some region \(D\) in \(\Sigma\). Classical description of the area is given by:
\[S(D) = \int_D d^2x \sqrt{h(x)} = \int_D d^2x (\tilde{n}^a a_{\tilde{n}^a})^{1/2}.\]

In order to define its quantum version we will mimic the prescription for defining the operator version of the volume or area in (3+1)-dimensional gravity\[25\]. Dividing the region \(D\) into small regions \(D_I\) \((I = 1, 2, \cdots)\) whose coordinate areas are supposed to be of order \(\delta^2\), we define the area operator as follows:
\[\hat{S}(D) = \lim_{\delta \to 0} \sum_I \hat{S}_I \]
\[\hat{S}_I^2 \equiv -\hat{S}_I^a \hat{S}_I^b \eta_{ab} \]
\[\hat{S}_I^a \equiv \int_{D_I} d^2x \tilde{n}^a(x). \tag{5.3}\]

Because the last expression involves two functional derivatives at a point, we further have to regularize this expression:
\[\hat{S}_I^a = \lim_{\epsilon \to 0} (\hat{S}_I^a)^\epsilon, \]

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This is nothing but eq.\((5.2)\) with the smearing function \(\eta_a\) being the characteristic function:

\[
\eta_a'(x) = \begin{cases} 
\delta_a, & \text{if } x \in D_I \\
0, & \text{if } x \in \Sigma \setminus D_I.
\end{cases}
\]

As a consequence, the action of the single \(\hat{S}^a_t\)-operator on the combinatorial solutions vanishes

\[
\hat{S}^a_t \cdot \Psi_{\{\alpha_i, l_i\}}^{\text{combi.}}(\omega) = 0.
\]

From this result we can say that the naive ‘squared infinitesimal area operator’ which is defined by

\[
(\hat{S}^a_t)^2_{\text{naive}} \equiv -\lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} (\hat{S}^a_t)^{\epsilon}(\hat{S}^a_{Ia})^{\epsilon'} = -\lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} (\hat{S}^a_t)^{\epsilon}(\hat{S}^a_{Ia})^{\epsilon'}
\]

has zero eigenvalues on the combinatorial solutions

\[
(\hat{S}^a_t)^2_{\text{naive}} \cdot \Psi_{\{\alpha_i, l_i\}}^{\text{combi.}}(\omega) = 0.
\]

A more careful consideration shows, however, that the result depends on the detail of the limit \(\epsilon, \epsilon' \to 0\). A candidate for the regularized squared infinitesimal area operator which we feel more natural than eq.\((5.5)\) in the field theoretical viewpoint is:

\[
(\hat{S}^a_t)^2_{\text{natural}} \equiv -\lim_{\epsilon=\epsilon' \to 0} (\hat{S}^a_t)^{\epsilon}(\hat{S}^a_{Ia})^{\epsilon'} = \lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} (\hat{S}^a_t)^{\epsilon}(\hat{S}^a_{Ia})^{\epsilon'}
\]

\[
\times (\eta^{ac}\eta^{bd} + O(\epsilon, \epsilon')) \frac{\delta}{\delta \omega^a_i(x)} \frac{\delta}{\delta \omega^b_j(x')} \frac{\delta}{\delta \omega^c_j(y)} \frac{\delta}{\delta \omega^d_l(y')}. \tag{5.7}
\]

Owing to eq.\((5.6)\), when we are interested in the action on the combinatorial solutions, we have only to consider the difference:

\[
\delta(\hat{S}^a_t)^2 \equiv (\hat{S}^a_t)^2_{\text{natural}} - (\hat{S}^a_t)^2_{\text{naive}}
\]

\[
\quad = \{ \lim_{\epsilon=\epsilon' \to 0} \lim_{\epsilon' \to 0} \left( \frac{1}{2} \left( \lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} + \lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \right) \right) (\hat{S}^a_t)^{\epsilon}(\hat{S}^a_{Ia})^{\epsilon'} \}. \tag{5.8}
\]

\[
\lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \text{denotes the limit in which we first take } \epsilon' \text{ to zero and then } \epsilon \text{ to zero.}
\]
Let us now consider the action of this difference $\delta(\hat{S}_I)^2$ on the spin network states. In our regularization introduced in §3, due to the antisymmetric factor $\epsilon_{ij}$, the nonvanishing contributions arise only when two functional derivatives, ‘hands’, from $(\hat{S}_I^a)^c$ act on the parallel propagators along independent curves. The same discussion holds also for $(\hat{S}_I^a)^{c'}$. Thus we see that not more than two hands act on one parallel propagator along a smooth curve (or segment). When two hands act on a smooth curve (or segment), one hand comes from $(\hat{S}_I^a)^c$ and the other from $(\hat{S}_I^a)^{c'}$. Next we consider the dependence on the limit taken. In this case we are interested only in the action of a pair of hands, one of which is from $(\hat{S}_I^a)^c$ and the other from $(\hat{S}_I^a)^{c'}$, on one propagator. We immediately see that the action of such pair on the propagator which traverses the intersection along a smooth curve is independent of the way of taking the limit. Therefore, the only case depending on how to take the limit turns out to be the case in which such a pair of hands act on the parallel propagator along a segment which starts from (or terminates at) the intersection considered. The relation among the limitation procedures is:

$$\left\{ \lim_{\epsilon = \epsilon' \to 0} - \frac{1}{2} \left( \lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} + \lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} \right) \right\} \left[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{hand-configuration.png}
\end{array} \right] = 0 ,$$

where a black small circle and a white one represent a hand from $\delta(\hat{S}_I^a)^c$ and one from $\delta(\hat{S}_{Ia})^{c'}$ respectively.

Using this relation, we conclude that the nonvanishing contributions to $\delta(\hat{S}_I)^2$ are obtained from only three configurations which are depicted in figure 5, where the action of $\delta(\hat{S}_I)^2$ on these configurations is also given. We should notice that, as a consequence of the above argument, the action of $\delta(\hat{S}_I)^2$ turns out to be divided into two parts:

$$\delta(\hat{S}_I)^2 = \delta(\hat{S}_I)^2|_1 + \delta(\hat{S}_I)^2|_2 ,$$

where $\delta(\hat{S}_I)^2|_1$ and $\delta(\hat{S}_I)^2|_2$ denote respectively the action on the single propagators and that on the pairs of propagators.

\footnote{Of course we assume that the intersection point under consideration resides in the region $D_I$.}
Now we have the basic actions in figure 5, we can calculate the action of \((\hat{S}_I)^2\) on the combinatorial solutions. We will only give its action on the simplest solution eq.(4.4):

\[
(\hat{S}_I)^2_{\text{natural}} (h_\alpha \otimes h_\beta - 2h_{\alpha_1\beta_2} \otimes h_{\beta_1\alpha_2}) = -2^{-8} (h_\alpha \otimes h_\beta - 2h_{\alpha_1\beta_2} \otimes h_{\beta_1\alpha_2}),
\]

where we have simplified the notation. From this result we expect that the combinatorial solutions \(\Psi^{\text{combi}}\) have nonzero eigenvalues of \((\hat{S}_I)^2_{\text{natural}}\). We can regard this as a sort of quantum effect because the classical counterpart of \(\Psi^{\text{combi}}\) has degenerate metric and so its area factor vanishes.

We can also construct the operator which measures the length \(L(\eta)\) of a curve \(\eta\)
embedded in $\Sigma$. The classical expression of $L(\eta)$ is

$$L(\eta) = \int_\eta ds \sqrt{h_{ij}(s)\dot{\eta}^i(s)\dot{\eta}^j(s)},$$

(5.11)

where $h_{ij} = \epsilon_{ijk}\epsilon_{jkl}\tilde{E}_a^k\tilde{E}_b^l\eta^{ab}$ is the induced metric on $\Sigma$. We can define its quantum operator version in the way which is identical to that of defining the ‘area’ operators in (3+1)-dimensional gravity[25]. We divide the curve $\eta$ into small segments $\eta_I$ with length of order $\delta$ and define $\hat{L}(\eta_I)$ as

$$\hat{L}(\eta_I) = \lim_{\delta \to 0} \sum_I \hat{L}(\eta_I)$$

$$\left(\hat{L}(\eta_I)\right)^2 = -\frac{1}{2} \int_{\eta_I} d\sigma \dot{\eta}_i^i(\sigma) \epsilon_{ijk} \int_{\eta_I} d\tau \dot{\eta}_j^j(\tau) \epsilon_{jkl}$$

$$\times \text{Tr} (\lambda^a h_{\sigma\tau}[0, 1] \lambda^b h_{\tau\sigma}[0, 1] \frac{\delta}{\delta \omega^a_k(\eta_I(\sigma))} \frac{\delta}{\delta \omega^b_l(\eta_I(\tau))}).$$

(5.12)

Because this operator involves two functional derivatives, we can separate its action as:

$$\left(\hat{L}(\eta_I)\right)^2 = \left|\left(\hat{L}(\eta_I)\right)^2\right|_1 + \left|\left(\hat{L}(\eta_I)\right)^2\right|_2,$$

(5.13)

where the first term in the r.h.s is the action on the single propagators and the second one denotes the action on the pairs of propagators. In principle we can calculate the action even when the curve $\eta$ passes through the intersections. For simplicity, however, we will only consider the case where the segment $\eta_I$ intersects with a curve $\alpha$ at the point which is away from the intersections. The basic actions in this case is given by

$$\left|\left(\hat{L}(\eta_I)\right)^2\right|_1 = -\frac{3}{16} h_{\alpha[0, 1]}^B A$$

$$\left|\left(\hat{L}(\eta_I)\right)^2\right|_2 \left( h_{\alpha[0, 1]}^B A h_{\alpha[0, 1]}^D C \right) = -\frac{1}{4} \left( h_{\alpha[0, 1]}^B A h_{\alpha[0, 1]}^D C - \frac{1}{2} h_{\alpha[0, 1]}^B A h_{\alpha[0, 1]}^D C \right).$$

(5.14)

Eq.(5.14) tells us that, while each tensor product of $l$ spinor propagators

$$h_{\alpha[0, 1]}^{B_1\cdots B_l} A_1\cdots A_l \equiv \prod_{i=1}^l h_{\alpha[0, 1]}^{B_i} A_i$$

are not eigenvectors unless $l = 1$ or 0, the symmetrized tensor products yield the eigenvectors. Using eq.(5.13), we can easily calculate their eigenvalues:

$$\left|\left(\hat{L}(\eta_I)\right)^2 h_{\alpha[0, 1]}^{(B_1\cdots B_l)} A_1\cdots A_l \right| = p \left( -\frac{3}{16} \right) h_{\alpha[0, 1]}^{(B_1\cdots B_l)} A_1\cdots A_l$$

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\[
-\frac{1}{4} \sum_{1 \leq i < j \leq l} \left( h_{\alpha}[0, 1]^{(B_1 \cdots B_j \cdots B_i)} A_1 \cdots A_i \cdots A_j \cdots A_l - \frac{1}{2} h_{\alpha}[0, 1]^{(B_1 \cdots B_i)} \right)
= \left( -\frac{3p}{16} - \frac{1}{8} \cdot \frac{p(p-1)}{2} \right) h_{\alpha}[0, 1]^{(B_1 \cdots B_i)}
= -\frac{p(p+2)}{16} h_{\alpha}[0, 1]^{(B_1 \cdots B_i)}. \tag{5.15}
\]

The result is problematic because the eigenvalue is negative while classically \((L(\eta_l))^2\) takes positive values\(^{11}\). If we ignore the minus sign by taking the ‘absolute value’ of this operator, however, we see that the combinatorial solution \(\Psi_{\text{combi}}^{\{\alpha_i, l_i\}}\) becomes the eigenvector of the length operator \(\hat{L}(\eta)\) if \(\eta\) does not pass through the intersections. The eigenvalue in this case is given by
\[
\frac{1}{4} \sum_{i} n(i) \sqrt{l_i(l_i + 1)}, \tag{5.16}
\]
where the loop \(\alpha_i\) is supposed to intersect with \(\eta\) \(n(i)\)-times. We should mention that this result is almost identical to that of the ‘area operator’ in (3+1)-dimensions. This is probably because the 2-dimensional surface and the 1-dimensional curve are respectively the dual of the loop in 3- and 2-dimensions.

6 Discussion: an extension to (3+1)-dimensional quantum gravity

In this paper we have investigated the Dirac quantization of (2+1)-dimensional analog of Ashtekar’s approach to quantum gravity using the spin network states. We have regularized the Hamiltonian constraint by introducing the (local) curvilinear coordinate frame in which the loop parameters play the role of coordinates. Then we have found a set of combinatorial solutions
\[
\{\Psi_{\text{combi}}^{\{\alpha_i, l_i\}}\}
\]
to the Hamiltonian constraint. Each solution is labelled by a set \(\{\alpha_i\}\) of smooth curves which may intersect with one another or with themselves and which is equipped with a

\(^{11}\) This situation arises also in (3+1)-dimensions, where the ‘squared-infinitesimal area’ operator which is naively defined from the classical expression has negative eigenvalues on the spin network states. This issue might tell us that we must consider more ‘singular’ configurations in which loops are non-analytic, i. e. have kinks or intersections, almost everywhere.
non-negative number \( l \). These combinatorial solutions correspond to the classical solutions which do not belong to the solution space of Witten’s formalism and in which the induced metric on \( \Sigma \) is degenerate. We can construct the operator which measures the area of a 2-dimensional region. The result of its action depends on the way of regularization and the action on the combinatorial solutions in a more natural regularization yields nonvanishing result. This can be considered as a quantum effect.

As a by-product we have established a prescription for calculating the action of an operator \( \hat{O}^{(n)} \) which involves an \( n \)-th order product of the momentum \( \hat{E}^i_a \). We first work out all the actions on the basic configurations which consist of at most \( n \) parallel propagators in the spinor representation. Next we divide the action of the operator as:

\[
\hat{O}^{(n)} = \sum_{k=1}^{n} \hat{O}^{(n)}|_k,
\]

where \( \hat{O}^{(n)}|_k \) denotes the action of \( \hat{O}^{(n)} \) on the sets each of which is composed of \( k \) propagators. The problem of calculating the action of \( \hat{O}^{(n)} \) thus reduces to that of combinatorics. In particular, the graphical representation introduced in this paper is expected to provide a powerful tool for calculation.

In this paper we did not exploit the merit of (2+1)-dimensions, except the possibility of the naive regularization. Thus we anticipate that most of the results obtained in this paper can be extended to (3+1)-dimensions.

An essential difference between these two cases is that we cannot naively apply to (3+1)-dimensions the regularization of the Hamiltonian constraint adopted in this paper. Roughly speaking, many attempts to regularize the Hamiltonian constraint in (3+1)-dimensions have been made in two directions. One is the point splitting regularization \[22\] \[14\] \[20\] and the other is the regularization based on the extended loops \[26\]. The former needs to be accompanied with a multiplicative renormalization and the latter is an attempt to elaborate the ‘flux-tube regularization’ introduced in ref.\[17\]. If we naively use the point-splitting regularization to (3+1)-dimensions, \( \xi_{ij}\dot{\alpha}^i(s)\dot{\beta}^j(t) \) and \( \tilde{B}^a \equiv \frac{1}{2}\tilde{\epsilon}^{ij}F_{ij}^a \) which appeared in (2+1)-dimensions is replaced, respectively, by \[20\] \( \xi_{abc}\dot{\alpha}^b(s)\dot{\beta}^c(t) \) and \( \tilde{B}^{ia} \equiv \frac{1}{2}\tilde{\epsilon}^{abc}F_{bc}^i \). Naively we cannot regularize the Hamiltonian constraint in a way which respects both diffeomorphism covariance and invariance under the reparametrization of

\[12\] We have used here the conventional notation in (3+1)-dimensions.
the curves. Nevertheless we can consider that, if a modification is made, the combinatorial solutions \( \Psi_{\{\alpha_i, l_i\}}^{\text{combi}} \) obtained in this paper still yield the solutions of the renormalized Hamiltonian constraint \( \mathcal{H}^{\text{ren}} \) in (3+1)-dimensions. If we make the replacement mentioned above, the basic action of \( \mathcal{H}^{\text{ren}} \) is represented in the same way as that shown in Appendix, up to constant factors which may depend on the angle \( \theta \) between two curves measured in some fixed background metric. First we should note that, in the configurations considered here, the ‘acceleration term’, if any, cannot be distinguished from a diffeomorphism\(^{13}\). We can thus neglect this term.

In obtaining the combinatorial solutions, the overall factor is irrelevant. We suppose that the ratio of the factor in (A.3) to that in (A.2) be \( Z(\theta) \) and that the ratio of the factor in (A.4) to that in (A.2) be \( Y(\theta) \). The discussion analogous to that leading to eq.(4.6) shows that, if we use the intertwining operator:

\[
I_{(Y,Z)}^r(m,n) \equiv \sum_{r=0}^{\min(m,n)} \left( \prod_{i=0}^{r} \frac{-Z(\theta)}{(m+n-2i)Y(\theta)+1} \right) I^r(m,n)
\]

instead of \( I^o(m,n) \) in eq.(4.7), then \( \Psi_{\{\alpha_i, l_i\}}^{\text{combi} \ast} \) thus obtained becomes the combinatorial solution to the renormalized Hamiltonian constraint

\[
\mathcal{H}^{\text{ren}} \Psi = 0.
\]

In (3+1)-dimensions, however, we also have to consider ‘essentially 3 dimensional intersections’ at which at least three curves with non-degenerate tangent vectors intersect. These intersections may yield the combinatorial solutions which is physically more interesting. While the calculation may become more complicated than in (2+1)-dimensions, we believe that no essential difficulty arises in carrying out this calculus.

We can easily see that the combinatorial solutions \( \Psi_{\{\alpha_i, l_i\}}^{\text{combi} \ast} \) in (3+1)-dimensions also correspond to classical solutions whose metric is degenerate. Hence naively the action of the volume operator on these solutions is expected to vanish. As we have seen in §5, however, the action of an operator is regularization-dependent. In particular, it is suspicious whether there exists a prescription in which both the Hamiltonian constraint and the volume operator are regularized consistently. Thus it is no wonder that the combinatorial solutions should have non-vanishing action of the volume operator.

\(^{13}\) In other words, the acceleration terms disappear if we adopt the regularization similar to that used in this paper.
In this paper we have not exploited the essential virtue of (2+1)-gravity, namely, its topological nature. The analysis made here gave only results which is somewhat formal and abstract. To make further study we need to acquire an intimate acquaintance with the classical solutions in a fixed topology. By comparing Dirac’s quantization method in a fixed topology to the reduced phase space quantization which is based on the space of classical solutions, we may have some conceptually important lessons on the quantum gravity in (3+1)-dimensions. While we expect that Dirac’s quantization yields physical states corresponding to quantum fluctuations from the classical solutions\cite{11}, we have not yet found such solutions. These solutions as well as the solutions in (3+1)-dimensions which is the quantum counterpart of the classical solutions with non-degenerate metric may play some important roles. While we have not mentioned here, the closure of the constraints under the commutation relations is essential to the question whether the Dirac quantization can be carried out consistently. We have shown in (2+1)-dimensions that the algebra is closed, up to the commutator of two Hamiltonian constraints. This remaining commutator is the most interesting in studying quantum gravity and its detailed investigation is longed for.

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A Appendix

In this appendix we list the action of the Hamiltonian constraint on the basic configurations. Here the graphical notation is used. We will denote the parallel propagators along the curves $\alpha$ and $\beta$ by an upward vertical arrow and a rightward horizontal arrow respectively. For example, we write the tensor product of the propagators along $\alpha$ and $\beta$ as follows $^{14}$:

$$h_\alpha[0,1]_A^B h_\beta[0,1]_C^D = \begin{array}{c}
    B \\
    C \\
    A \\
\end{array} .$$

(A.0)

It is convenient to separate $\hat{\mathcal{H}}(\mathcal{N})$ as in eq.(3.7):

$$\hat{\mathcal{H}}(\mathcal{N}) = \hat{\mathcal{H}}(\mathcal{N})_1 + \hat{\mathcal{H}}(\mathcal{N})_2.$$  

Because we can show that the action on the smooth lines without intersection vanishes:

$$\begin{align*}
\hat{\mathcal{H}}_1(\mathcal{N}) & = \begin{array}{c}
    B \\
    A \\
\end{array} = \begin{array}{c}
    B \\
    D \\
    A \\
    C \\
\end{array} = 0, \\
\hat{\mathcal{H}}_2(\mathcal{N}) & = \begin{array}{c}
    B \\
    D \\
    A \\
    C \\
\end{array} = 0,
\end{align*}$$

(A.1)

we have only to concentrate on the action at the vertex. In the present case the only vertex is at $x_0$. In order to simplify the notation of the equations, we introduce the following ‘rescaled Hamiltonian’:

$$(\hat{\mathcal{H}}_{x_0}^I)_I \equiv -\sigma(\alpha,\beta) \frac{8}{N(x_0)} \left( \text{the action of } \hat{\mathcal{H}}_I(\mathcal{N}) \text{ at } x_0 \right), \quad (I = 1, 2).$$

$^{14}$We will assume that $\alpha$ and $\beta$ intersect once at $x_0 = \alpha(s_0) = \beta(t_0)$.  

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In the following we provide some of the basic actions of the Hamiltonian constraint in the graphical notation. The dot in the diagram denotes the part on which the area derivative $\tilde{\Delta}$ acts, or the location into which the magnetic field $\tilde{B}$ is inserted. Using equations (A.1-8) and identities (2.16) (2.19) (2.20), we can write out all the basic action, namely all the action on a single propagator and a pair of propagators, of the Hamiltonian constraint $\hat{H}(N)$.

\[
(\hat{H}_{x_0}^I)_1 \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
, \quad (A.2)
\]

\[
(\hat{H}_{x_0}^I)_2 \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
= 2 \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
 - \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
, \quad (A.3)
\]

\[
(\hat{H}_{x_0}^I)_2 \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
 - \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
C \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
, \quad (A.4)
\]

\[
(\hat{H}_{x_0}^I)_2 \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
 - \begin{array}{c}
\begin{array}{c}
A \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
B \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
D \\
\end{array}
\end{array}
, \quad (A.5)
\]

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\[(\hat{\mathcal{H}}'_{x_0})_2 \begin{pmatrix} R \\ A \\ C \end{pmatrix} = \begin{pmatrix} D \\ B \end{pmatrix} - \begin{pmatrix} D \\ B \end{pmatrix}, \]  
(A.6)

\[(\hat{\mathcal{H}}'_{x_0})_2 \begin{pmatrix} \Delta \\ B \\ D \\ C \end{pmatrix} = \begin{pmatrix} \Delta \\ B \\ D \\ C \end{pmatrix} - \begin{pmatrix} \Delta \\ B \\ D \\ C \end{pmatrix}, \]  
(A.7)

\[(\hat{\mathcal{H}}'_{x_0})_2 \begin{pmatrix} C \\ B \\ D \end{pmatrix} = (\hat{\mathcal{H}}'_{x_0})_2 \begin{pmatrix} B \\ D \\ A \\ C \end{pmatrix} = 0. \]  
(A.8)

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