Rational fuzzy attribute logic

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Abstract

We present a logic for reasoning with if-then formulas which involve constants for rational truth degrees from the unit interval. We introduce graded semantic and syntactic entailment of formulas. We prove the logic is complete in Pavelka style and depending on the choice of structure of truth degrees, the logic is a decidable fragment of the Rational Pavelka logic (RPL) or the Rational Product Logic (RΠL). We also present a characterization of the entailment based on least models and study related closure structures.

1 Introduction

In this paper, we are interested in a logic for reasoning with if-then rules describing dependencies between graded attributes. A graded attribute may be seen as a propositional variable which may be assigned a truth degree coming from a scale of degrees which includes intermediate degrees of truth. In this paper, we use particular complete residuated lattices on the real unit interval as the scales of truth degrees. As a consequence, if \( p \) is a propositional variable and \( e(p) \) denotes its truth value under the evaluation \( e \), we admit \( e(p) \in [0,1] \) with the possibility of \( 0 < e(p) < 1 \). The meaning of the degrees assigned to propositional

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variables conforms to the meaning in mathematical fuzzy logics \[14, 15, 29, 32\], i.e., the meaning is comparative: \( e(p) < e(q) \) means that propositional variable \( p \) is (strictly) less true than \( q \) under the evaluation \( e \). Let us stress at this point that the degrees we use are not and shall not be interpreted as degrees of belief or evidence, cf. “the frequentist’s temptation” in [32] and also [34].

Admitting the intermediate degrees of truth may be valuable in situations where attributes represented by propositional variables may not be assigned either of the classic truth degrees 0 (falsity) and 1 (truth). This may occur in situations where human perception and judgment is involved. For instance, if we ask a person whether \$400,000 is a high price for a house in a particular neighborhood, the person may hesitate to describe the price as being (strictly) high or (strictly) not high. Indeed, the person may feel that a sentence “The price of the house sold at \$400,000 is high” is true in a general way, but is not completely true. This fact may be captured by assuming that the attribute “price is high” represented by a propositional variable \( p \) is assigned a truth degree \( e(p) \) which is close to 1 but strictly less than 1 and we may further be interested in reasoning with this fact. Formal treatment of reasoning with such facts is the main subject of mathematical fuzzy logics and is also the main subject of this paper.

In this paper we focus on reasoning with formulas formalizing if-then dependencies between graded attributes. Namely, given an evaluation of propositional variables \( p_1, p_2, \ldots, q_1, q_2, \ldots \), we formalize (the semantics of) rules of the form

\[
\text{if } (p_1 \text{ is true at least to degree } a_1 \text{ and } \cdots \text{ and } p_n \text{ is true at least to degree } a_n),
\text{ then } (q_1 \text{ is true at least to degree } b_1 \text{ and } \cdots \text{ and } q_n \text{ is true at least to degree } b_n).
\]

Moreover, we present a logical system for reasoning with such rules and prove its soundness, completeness, and present its relationship to existing logics. We call
our logic *rational* because we allow only rational degrees from the unit interval to appear in the if-then rules. This is a similar assumption as in the Rational Pavelka logic (RPL) proposed by Hájek [31, 32] which extends Lukasiewicz logic by constants for rational truth degrees and corresponding bookkeeping axioms. In fact, when the standard Lukasiewicz algebra is used as the structure of truth degrees, our logic can be seen as a particular fragment of RPL. Analogously, when the standard Goguen (product) algebra is used as the structure of truth degrees, our logic becomes a fragment of the Rational Product Logic (RΠL) which has been proposed by Esteva, Godo, Hájek, and Navara in [21].

Another common attribute of our logic and RPL (or RΠL) is that we consider degrees of entailment on both the semantic and syntactic levels. That is, for a formula $\varphi$ and a theory $\Sigma$, we consider a degree $||\varphi||_{\Sigma} \in [0, 1]$ to which $\varphi$ is semantically entailed by $\Sigma$ and in general, we may have $0 < ||\varphi||_{\Sigma} < 1$. More importantly, we also consider a degree $|\varphi|_{\Sigma}$ to which $\varphi$ is provable by $\Sigma$ and, again, we may have $0 < |\varphi|_{\Sigma} < 1$. Logics with graded notions of provability were first investigated by Pavelka [42, 43, 44] who was inspired by ideas in the influential paper by Goguen [28]. In his seminal papers, Pavelka proposed very general approach to abstract logics with graded notions of provability and proved completeness of a propositional logic based on the standard Lukasiewicz algebra [36] as the structure of truth degrees. The completeness theorem states that $|\varphi|_{\Sigma} = ||\varphi||_{\Sigma}$, i.e., the degrees of semantic entailment coincide with the degrees of provability. This particular type of completeness of multiple-valued logics has later become known as the Pavelka completeness, cf. [32, Section 9.2].

The contribution of Hájek’s RPL is in simplification of Pavelka’s ideas, including (i) the use of (constants for) only rational truth degrees and thus keeping the language of the logic countable and (ii) considering proofs in the ordinary sense instead of considering them as sequences of weighted formulas as in the original approach [12, 13, 14] by Pavelka. Further details and development of Pavelka-complete logics can be found in [26, 41] and the references therein.
The logic we present is complete in the Pavelka style and depending on the choice of a structure of truth degrees (the standard Lukasiewicz or Goguen algebra), it may be seen as a decidable fragment of RPL or RIIL which uses only formulas in a particular form. Our logic may be seen as logic which falls into the category of fuzzy logics with rational constants for truth degrees [13, 19, 22, 48] and develops our previous results on logic of fuzzy attribute implications which were limited to finite structures of truth degrees [7, 11] or utilized infinitary deduction rules in order to ensure completeness in Pavelka style [8, 37]. In contrast to these previous results, the present approach shows logics with finitary deduction rules and particular structures of truth degrees defined on the real unit interval. In a broader sense, our paper is interested in if-then rules which generalize analogous rules that appear in database systems as functional dependencies [40], logic programming [38], or data analysis [1, 25, 53] as attribute implications or association rules.

Our paper is structured as follows. In Section 2, we survey preliminaries from structures of truth degrees used in this paper (the section may be skipped by readers familiar with residuated lattices). In Section 3, we describe our logic and present its completeness. The proof of completeness together with further notes are presented in Section 4. In Section 5, we present characterization of closure structures related to systems of models of theories.

2 Preliminaries

In our paper, we consider particular linear complete integral commutative residuated lattices [24, 52] as the structures of degrees. The structures are defined as general algebras \( L = \langle L, \land, \lor, \otimes, \to, 0, 1 \rangle \) of type \( (2, 2, 2, 2, 0, 0) \), where

(i) \( \langle L, \land, \lor, 0, 1 \rangle \) is a linear complete lattice [12],

(ii) \( \langle L, \otimes, 1 \rangle \) is a commutative monoid, and

(iii) for all \( a, b, c \in L \), we have \( a \otimes b \leq c \) iff \( a \leq b \to c \).
Property \((iii)\) is called the adjointness of \(\otimes\) (truth function of “fuzzy conjunction”) and \(\to\) (truth function of “fuzzy implication”); note that \(\leq\) denotes the linear order induced by \(\land\), i.e., \(a \leq b\) iff \(a \land b = a\) (equivalently, \(a \lor b = b\)).

In our paper, we use linear complete residuated lattices defined on the real unit interval which are given by left-continuous triangular norms \([36]\) which play central role in the Basic Logic (BL, see \([32]\)) and Monoidal T-norm Logic (MTL, see \([20]\)). That is, \(\langle L, \land, \lor, 0, 1 \rangle\) represents the real unit interval with its natural ordering (\(\land\) and \(\lor\) coincides with operations of minima and maxima, respectively), \(\otimes\) is associative, commutative, neutral with respect to \(1\), and is left-continuous (distributive with respect to general suprema), i.e., it satisfies

\[
\bigvee \{a \otimes b_i; \ i \in I\} = a \otimes \bigvee \{b_i; \ i \in I\} \quad (1)
\]

for all \(a \in [0, 1]\) and \(\{b_i \in [0, 1]; \ i \in I\}\). Moreover, the corresponding (uniquely given) \(\to\) which is adjoint to \(\otimes\) is then given by

\[
a \to b = \bigvee \{c \in [0, 1]; \ a \otimes c \leq b\}. \quad (2)
\]

Two structures which are most relevant for our investigation are the so-called standard Lukasiewicz and Goguen (product) algebras denoted \(L\) and \(\Pi\), respectively, where the multiplications and residua are given by

\[
a \otimes_L b = \max\{0, a + b - 1\}, \quad (3)
\]

\[
a \to_L b = \min\{1, 1 - a + b\}, \quad (4)
\]

and

\[
a \otimes_\Pi b = ab, \quad (5)
\]

\[
a \to_\Pi b = \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{otherwise}. \end{cases} \quad (6)
\]

If \(L\) and \(\Pi\) are clear from the context, we omit the subscripts and write just \(\otimes\) and \(\to\), respectively. Notice that both \(\otimes_L\) and \(\otimes_\Pi\) are continuous functions.
As a consequence,

\[ \bigwedge\{a \otimes b_i; i \in I\} = a \otimes \bigwedge\{b_i; i \in I\} \]  

(7)

for all \( a \in [0, 1] \) and \( \{b_i \in [0, 1]; i \in I\} \). Moreover, \( \rightarrow_L \) is continuous in both its arguments, \( \rightarrow_{\Pi} \) is continuous in the second argument and left-continuous in the first one. As a consequence, the residua in both \( L \) and \( \Pi \) satisfy

\[ \bigwedge\{a \rightarrow b_i; i \in I\} = a \rightarrow \bigwedge\{b_i; i \in I\} \]  

(8)

\[ \bigwedge\{a_i \rightarrow b; i \in I\} = \bigvee\{a_i; i \in I\} \rightarrow b, \]  

(9)

\[ \bigvee\{a_i \rightarrow b_i; i \in I\} = a \rightarrow \bigvee\{b_i; i \in I\} \]  

(10)

and, in addition, \( L \) satisfies

\[ \bigvee\{a_i \rightarrow b_i; i \in I\} = \bigwedge\{a_i; i \in I\} \rightarrow b. \]  

(11)

More details on the structures of degrees can be found in [36].

In order to simplify our considerations about if-then rules between graded attributes, we use \( L \)-fuzzy sets and related notions. Consider a non-empty set \( U \) which acts as a universe of elements. Each map \( A: U \rightarrow L \), where \( L \) is the set of truth degrees in \( L \) is called an \( L \)-fuzzy set [27] (shortly, an \( L \)-set) in \( U \) and the degree \( A(u) \) is interpreted as “the degree to which \( u \) belongs to the \( L \)-set \( A \)”. The collection of all \( L \)-sets in \( U \) is denoted by \( L^U \). Furthermore, \( A \in L^U \) is called finite whenever \( \{u \in U; A(u) > 0\} \) is a finite set; \( A \in L^U \) is called a singleton, written \( \{^a/u\} \), whenever \( A(u) = a \) and \( A(v) = 0 \) for all \( v \in U \setminus \{u\}; A \in L^U \) is called rational whenever \( \{A(u); u \in U\} \subseteq [0, 1]_Q \) where \([0, 1]_Q \) denotes the rational unit interval.

We consider operations with \( L \)-sets which are defined componentwise using the operations in \( L \). Namely, for \( A_i \in L^U \ (i \in I) \), \( A \in L^U \), and \( c \in L \), we define \( \bigcap\{A_i; i \in I\} \) (the intersection of \( A_i \)'s), \( \bigcup\{A_i; i \in I\} \) (the union of \( A_i \)'s), \( c \otimes A \)
(the \(c\)-multiple of \(A\)), and \(c \rightarrow A\) (the \(c\)-shift of \(A\)) by putting

\[
\bigl(\bigcap\{A_i; \ i \in I\}\bigr)(u) = \bigwedge\{A_i(u); \ i \in I\},
\]

\[
\bigl(\bigcup\{A_i; \ i \in I\}\bigr)(u) = \bigvee\{A_i(u); \ i \in I\},
\]

\[
(c \otimes A)(u) = c \otimes A(u),
\]

\[
(c \rightarrow A)(u) = c \rightarrow A(u),
\]

for all \(u \in U\). In addition, if \(|I| = 2\), we use the usual infix notation \(A \cap B\) and \(A \cup B\) to denote \(\text{(12)}\) and \(\text{(13)}\), respectively.

For \(A, B \in L^U\), we consider two basic types of containment relations. Namely, a bivalent containment relation and a graded containment relation. In the first case, we write \(A \subseteq B\) and say that \(A\) is fully contained in \(B\) whenever \(A(u) \leq B(u)\) holds for each \(u \in U\). In the second case, we define a degree \(S(A, B) \in L\) to which \(A\) is a subset of \(B\) by

\[
S(A, B) = \bigwedge\{A(u) \rightarrow B(u); \ u \in U\}.
\]

Using the properties of \(L\), it is easily seen that \(A \subseteq B\) iff \(A(u) \rightarrow B(u) = 1\) holds for all \(u \in U\) which is true iff \(S(A, B) = 1\). See \([5\text{, Theorem 3.12}]\) for further details on properties of graded subsethood.

In the paper, we utilize particular fuzzy closure operators. Recall that an operator \(c : L^U \rightarrow L^U\) is called an \(L\)-closure operator \([4, 47]\) on the set \(U\) whenever the following conditions

\[
A \subseteq c(A),
\]

\[
S(A, B) \leq S(c(A), c(B)),
\]

\[
c(c(A)) = c(A),
\]

are satisfied for all \(A, B \in L^U\), cf. also \([5\text{, Section 7.1}]\). A non-empty system \(S \subseteq L^U\) (of \(L\)-sets in \(U\)) is called directed whenever for any \(A, B \in S\) there is \(C \in S\) such that \(A \subseteq C\) and \(B \subseteq C\).
3 Rational fuzzy attribute logic

In this section, we introduce the rational fuzzy attribute logic (RFAL). We describe its language, formulas, their semantic entailment and a graded notion of provability. We present the Pavelka completeness of our logic which is proved in the next section.

The language of our logic is given by

- a (denumerable) set Var of propositional variables, and
- the set of constants for rational truth degrees, i.e., for each \( a \in [0,1]_Q \) we consider a constant \( a \).

Propositional variables in Var are denoted by \( p,q, \ldots \) Note that the distinction between rational truth degrees in \([0,1]\) and the constants for the degrees is essentially the same as in RPL [31] and is motivated by proper distinction between the syntax and semantics of our logic.

Using symbols for logical connectives \( \land \) (conjunction) and \( \Rightarrow \) (implication), we consider formulas of the following form

\[
((\overline{a_1} \Rightarrow p_1) \land \cdots \land (\overline{a_m} \Rightarrow p_m)) \Rightarrow ((\overline{b_1} \Rightarrow q_1) \land \cdots \land (\overline{b_n} \Rightarrow q_n)), \tag{20}
\]

where \( p_1, \ldots, p_m, q_1, \ldots, q_n \in \text{Var} \) and \( a_1, \ldots, a_m, b_1, \ldots, b_n \) are constants for truth degrees. We call such formulas rational fuzzy attribute implications. Note that according to the standard meaning of \( \land \) and \( \Rightarrow \) which is used in fuzzy logics in the narrow sense, (20) can be understood as a formula expressing the fact that “if \( p_1 \) is true at least to degree \( a_1 \) and \( \cdots \) and \( p_m \) is true at least to degree \( a_n \), then \( q_1 \) is true at least to degree \( b_1 \) and \( \cdots \) and \( q_n \) is true at least to degree \( b_n \)” which corresponds with the intended meaning of the rules we have described in Section 1. Also note that (20) is a well-formed formula in a language of MTL (or a stronger logic) which is extended by constants for truth degrees.

In order to define the semantics of formulas like (20), we use the usual notion of evaluation of propositional variables which uniquely extends to all
formulas, including those in the form of (20). In a more detail, we call any map \( e : \text{Var} \rightarrow [0, 1] \) an *evaluation* (notice that \( e(p) \) may be irrational) and define the degree \( \|\varphi\|_e \) to which a general well-formed formula in our language is true under \( e \):

\[
\|p\|_e = e(p), \tag{21}
\]

\[
\|\varphi \Rightarrow \psi\|_e = \|\varphi\|_e \rightarrow \|\psi\|_e, \tag{22}
\]

\[
\|\varphi \land \psi\|_e = \|\varphi\|_e \land \|\psi\|_e, \tag{23}
\]

\[
\|\exists p\|_e = a, \tag{24}
\]

for all \( p \in \text{Var} \), formulas \( \varphi, \psi \), and \( a \in [0, 1] \) (other connectives may be introduces but we do not need them for our development, cf. [20, 32]). If \( \varphi \) is (20), then \( \|\varphi\|_e \in [0, 1] \) is the *degree to which \( \varphi \) is true under \( e \).* The notion of degrees of semantic entailment of formulas is then defined as follows: An evaluation \( e \) is called a *model* of a set \( \Sigma \) of formulas whenever \( \|\varphi\|_e = 1 \) for all \( \varphi \in \Sigma \). The *degree \( \|\varphi\|_\Sigma \) to which \( \varphi \) is entailed by \( \Sigma \) is defined

\[
\|\varphi\|_\Sigma = \bigwedge \{\|\varphi\|_e; e \text{ is a model of } \Sigma \}. \tag{25}
\]

In this paper, we are primarily interested in syntactic characterization of \( \|\varphi\|_\Sigma \) for \( \varphi \) and all formulas in \( \Sigma \) being of the form (20).

Before we introduce our axiomatization, let us present a concise way of representing formulas like (20) and their entailment. Since both the antecedent and consequent of (20) are of the form of conjunctions of subformulas (and \( \land \) is interpreted by a truth function which is commutative, associative, and idempotent), we may encode the antecedents and consequents by finite rational
\( \text{L}-\text{sets in } \text{Var}. \) Indeed, for (20), we may consider \( A, B \in L^{\text{Var}} \) given by

\[
A(p) = \begin{cases} 
  a_i, & \text{if } p \text{ equals } p_i \text{ for } 1 \leq i \leq m, \\
  0, & \text{otherwise}, 
\end{cases}
\]  

(26)

\[
B(p) = \begin{cases} 
  b_i, & \text{if } p \text{ equals } q_i \text{ for } 1 \leq i \leq n, \\
  0, & \text{otherwise}, 
\end{cases}
\]  

(27)

for all \( p \in \text{Var}. \) Under this notation, (20) may be written as

\[
A \Rightarrow B
\]  

(28)

and the degree \( ||A \Rightarrow B||_e \) to which \( A \Rightarrow B \) is true under \( e \) may be defined as

\[
||A \Rightarrow B||_e = S(A, e) \rightarrow S(B, e),
\]  

(29)

where \( S(A, e) \) and \( S(B, e) \) are subsethood degrees (16). A moment’s reflection shows that if \( \varphi \) denotes (20) and \( A \Rightarrow B \) is the corresponding abbreviation of (20) with \( A \) and \( B \) given by (26) and (27), respectively, then

\[
||\varphi||_e = ||A \Rightarrow B||_e
\]  

(30)

for any evaluation \( e: \text{Var} \rightarrow [0, 1] \). Therefore, in the rest of the paper, we use primarily the abbreviated form (28) of formulas (where both \( A, B \) are finite and rational \( \text{L}-\text{sets in } \text{Var} \)) which simplifies some of our considerations.

\textbf{Remark 1.} The first approach to attribute implications between graded (fuzzy) attributes was introduced by Pollandt [45]. She studied the formulas mainly from the point of view of formal concept analysis [25] of data with fuzzy attributes and she did not present any complete axiomatization of the proposed semantic entailment. Conceptually, the formulas used by Pollandt [45] correspond to (28) and its interpretation as in (29). Later, the approach was generalized by considering truth-stressing hedges [10, 23, 33] as additional parameters of the interpretation of the formulas, see [7] for a survey and [11] for recent
results. The utilization of hedges proved interesting because stronger (and desirable) properties of fuzzy attributes implications (like uniqueness of minimal bases obtained via pseudo-intents \[11, 30\]) result by specific choices of hedges (e.g., by the choice of globalization \[49\] which on linear structures of degrees coincides with Baaż’s \(\Delta\), cf. \[3\]). Even more general concept of a parameterization of fuzzy attribute implications is introduced in \[51\] based on algebras of isotone Galois connections.

From now on, we use the following assumption.

**Assumption 1.** \(L = \langle L, \land, \lor, \otimes, \to, 0, 1 \rangle\) is a complete residuated lattice on the real unit interval and \(\otimes\) and \(\to\) are defined so that for any \(a, b \in [0, 1]_Q\), we have \(a \otimes b \in [0, 1]_Q\) and \(a \to b \in [0, 1]_Q\). In this case, we say that \(L\) is *rationally closed*.

The assumption says basically that \(\otimes\) and \(\to\) applied to rational arguments yield rational results. This is obviously satisfied for both \(L\) and \(\Pi\). As a consequence, if \(A \in L^U\) and \(c\) are rational, then \(c \otimes A\) and \(c \to A\) given by \((14)\) and \((15)\), respectively, are also rational.

Our logic uses a deductive system which consists of a single axiom scheme and two deduction rules:

\[\text{(i) each formula of the form } A \cup B \Rightarrow B \text{ where } A, B \text{ are finite rational } L\text{-sets in } \text{Var is an axiom;}\]

\[\text{(ii) the deduction rules of cut and multiplication:}
\]

\[\text{(Cut) from } A \Rightarrow B \text{ and } B \cup C \Rightarrow D \text{ infer } A \cup C \Rightarrow D,\]

\[\text{ (Mul) from } A \Rightarrow B \text{ infer } c \otimes A \Rightarrow c \otimes B,\]

\[\text{ for all finite rational } L\text{-sets } A, B, C, D \text{ in } \text{Var and } c \in [0, 1]_Q.\]

Observe that \(c \otimes A\) and \(c \otimes B\) are both rational and finite, i.e., (Mul) is a well-defined deduction rule. Analogously, the axioms and (Cut) are well defined because a union of finitely many rational finite \(L\)-sets is a finite rational \(L\)-set.
Remark 2. (a) Our deduction system resembles the famous system by Armstrong [2] which in database systems plays a central role in reasoning about functional dependencies and normalized database schemes [40]. Namely, if we replace $L$-sets by ordinary sets, the axioms together with (Cut) form a system which is equivalent to that of Armstrong. Note that in database systems (Cut) is usually presented under the name pseudotransitivity [35, 40]. Deductive systems for fuzzy attribute implications based on (Mul) and (Cut) are also present in [7, 11] and generalized in [8, 37] by considering infinitary deduction rules to ensure completeness over arbitrary infinite $L$.

(b) Since the axioms and deduction rules we deal with use abbreviations for formulas like (20), it is worth noting that our axioms and deduction rules may also be understood as formulas and deduction rules in the ordinary (narrow) sense. We may prove that in MTL (or a stronger logic) enriched by constants for rational truth degrees and bookkeeping axioms, the axiom schemes are provable, and (Mul) and (Cut) are derived deduction rules. In this remark, let $C$ denote a logic which results from MTL (or a stronger logic) by adding constants for rational truth degrees and bookkeeping axioms which ensure that $a \rightarrow b \iff \pi \Rightarrow b$, $a \otimes b \iff \pi \Theta b$, $a \wedge b \iff \pi \land b$, and $a \vee b \iff \pi \lor b$ are all provable by $C$. We inspect the following cases:

- All axioms are provable by $C$. First, observe that $A \cup B \Rightarrow B$ can equivalently be written as $C \Rightarrow B$ where $C(p) \geq B(p)$ for all $p \in \text{Var}$. Now, fix $p$ and let $C(p) = a$ and $B(p) = b$. Using the bookkeeping axioms together with $b \rightarrow a = 1$, we get

$$\vdash_C b \Rightarrow \pi$$

and as a consequence of the transitivity of implication, we get

$$\vdash_C (\pi \Rightarrow p) \Rightarrow (b \Rightarrow p).$$

Now, we may repeat the idea finitely many times for all $p$ with $C(p) > 0$ and utilize
\[ \vdash_C (\varphi_1 \Rightarrow \psi_1) \Rightarrow ((\varphi_2 \Rightarrow \psi_2) \Rightarrow ((\varphi_1 \land \varphi_2) \Rightarrow (\psi_1 \land \psi_2))) \]

in order to show that \( C \) proves each axiom of our deductive system.

- In case of (Cut), observe that we have

\[ \{ \varphi \Rightarrow \psi, (\psi \land \chi) \Rightarrow \theta \} \vdash_C (\varphi \land \chi) \Rightarrow \theta, \]

where \( \varphi, \psi, \chi, \theta \) are arbitrary formulas. In addition, if a formula of the form [20] is abbreviated by \( B \cup C \Rightarrow D \), then it is equivalent to \( (\psi \land \chi) \Rightarrow \theta \) where \( \psi \Rightarrow \theta \) is abbreviated by \( B \Rightarrow D \) and \( \chi \Rightarrow \theta \) is abbreviated by \( C \Rightarrow D \). Indeed, observe that \( (\pi \Rightarrow \varphi) \land (\tau \Rightarrow \varphi) \) is provably equivalent to \( (\pi \lor \tau) \Rightarrow \varphi \), i.e., \( \overline{a \lor b} \Rightarrow \varphi \) by the bookkeeping axioms. As a consequence, the derived formula \( (\varphi \land \chi) \Rightarrow \theta \) may be abbreviated by \( A \cup C \Rightarrow D \) provided that \( \varphi \Rightarrow \psi \) is abbreviated by \( A \Rightarrow B \). Altogether, the results of (Cut) are derivable in \( C \).

- Finally, for any \( \varphi, \psi \), and \( c \in [0, 1]_Q \), we have

\[ \{ \varphi \Rightarrow \psi \} \vdash_C (\tau \Rightarrow \varphi) \Rightarrow (\tau \Rightarrow \psi). \]

Furthermore, if \( \varphi \Rightarrow \psi \) is of the form [20], then we may use

\[ \vdash_C (\tau \Rightarrow (\varphi_1 \land \cdots \land \varphi_n)) \iff ((\tau \Rightarrow \varphi_1) \land \cdots \land (\tau \Rightarrow \varphi_n)). \]

Thus, using the bookkeeping axioms together with

\[ \vdash_C (\varphi \Rightarrow (\psi \Rightarrow \chi)) \iff ((\varphi \land \psi) \Rightarrow \chi), \]

we conclude that (Mul) is a derivable deduction rule in \( C \).

(c) Note that there are several deductive systems equivalent to the system of our axioms, (Cut), and (Mul). For instance, (Cut) and (Mul) can be reduced to a single deduction rule [11]. Alternatively, one may introduce normalized proofs based on the rule of accumulation analogously as in [9] (cf. also [39]). Other deductive systems may involve the rule of simplification instead of (Cut), see [6] (cf. also [16] and a similar deduction rule proposed by Darwen in [17]).
Considering our deduction system, we introduce the notions of proofs and provability degrees. A proof of \( A \Rightarrow B \) by \( \Sigma \) is any sequence of formulas \( C_1 \Rightarrow D_1, \ldots, C_n \Rightarrow D_n \) such that \( C_n = A, D_n = B \), and for each \( i = 1, \ldots, n \), we have that

(i) \( C_i \Rightarrow D_i \) is an axiom, or

(ii) \( C_i \Rightarrow D_i \in \Sigma \), or

(iii) \( C_i \Rightarrow D_i \) results from some of the formulas in \( \{ C_j \Rightarrow D_j; j < i \} \) by a single application of the deduction rule (Cut) or (Mul).

If there is a proof of \( A \Rightarrow B \) by \( \Sigma \), we denote the fact by \( \Sigma \vdash A \Rightarrow B \) and call \( A \Rightarrow B \) provable by \( \Sigma \). The degree \( |A \Rightarrow B|_\Sigma \) to which \( A \Rightarrow B \) is provable by \( \Sigma \) is defined by

\[
|A \Rightarrow B|_\Sigma = \bigvee \{ c \in [0,1]; \Sigma \vdash c\otimes B \}. \tag{31}
\]

Remark 3. The provability degrees (31) are defined in much the same way as in RPL, i.e., via an ordinary notion of provability and without considering proofs as sequences of weighted formulas as in the original Pavelka approach [42, 43, 44]. Indeed, in RPL [31], the degrees of provability are introduced as

\[
|\varphi|_{\Sigma}^{\text{RPL}} = \bigvee \{ c \in [0,1]; \Sigma \vdash \varphi \Rightarrow \varphi \}. \tag{32}
\]

Also note that if \( L \) is \( \mathbf{L} \), RPL provides a syntactic characterization of the semantic entailment of the formulas considered in our logic. Indeed, for \( \varphi \Rightarrow \psi \) abbreviated by \( A \Rightarrow B \), one can easily see that

\[
\|A \Rightarrow B\|_\Sigma = |\varphi \Rightarrow \psi|_{\Sigma}^{\text{RPL}}. \tag{33}
\]

However, RPL-based proofs of \( \sigma \Rightarrow (\varphi \Rightarrow \psi) \) by \( \Sigma \) may contain formulas which are not of the form (20). In contrast, our deductive system, allows to infer only (abbreviated representations of) formulas like (20). As a consequence, the deduction in our system is simpler than the deduction in RPL and enables
automated deduction as we shall see in Section 4. Analogous remarks can be made for RIIL (but notice that RIIL has an infinitary deduction rule [21]).

The proof of the following assertion is elaborated in Section 4.

**Theorem 2** (Pavelka completeness of RFAL). Let $L$ satisfy Assumption 1 and condition (10). Then, for any $\Sigma$ and $A \Rightarrow B$,

$$|A \Rightarrow B|_\Sigma = ||A \Rightarrow B||_\Sigma.$$  

(34)

In particular, (34) holds for $L$ being $L$ or $\Pi$. □

Using the results on RPL [32], we may get further insight into the previous completeness theorem which is based on our axiomatization. Namely, [32, Lemma 3.3.17] gives that if $\Sigma$ is finite, then each $||A \Rightarrow B||_\Sigma$ and thus $|A \Rightarrow B|_\Sigma$ is rational (for $L$ being $L$). Let us note that the proof of [32, Lemma 3.3.17] is technically quite involved. In Section 4, we show that in case of our logic, which considers only formulas in the special form (20), the argument is considerably easier. The most important consequence of this property is summarized in the following assertion which is also proved in Section 4.

**Theorem 3.** Let $L$ be either of $L$ or $\Pi$. Then, for each finite theory $\Sigma$ and any $A \Rightarrow B$, the degree $|A \Rightarrow B|_\Sigma$ is rational. In addition, we have

$$\Sigma \vdash A \Rightarrow c \otimes B$$

for $c = |A \Rightarrow B|_\Sigma$.  

(35)

Moreover, RFAL based on either of $L$ and $\Pi$ is decidable. □

In particular, for $c = 1$ Theorem 3 yields $\Sigma \vdash A \Rightarrow B$ iff $|A \Rightarrow B|_\Sigma = 1$.

4 Proofs and Notes

In this section, we give proofs of the basic assertions of RFAL presented in the previous section. In the entire section, we assume that Assumption 1 holds and that $L$ satisfies (10).
Lemma 4. For any \( A \Rightarrow B \) and rational \( c \in [0, 1] \), we have

\[
 c \Rightarrow |A \Rightarrow B|_{\Sigma} = |A \Rightarrow c \otimes B|_{\Sigma}. \tag{36}
\]

Proof. We check both inequalities of (36).

In order to prove \( c \Rightarrow |A \Rightarrow B|_{\Sigma} \leq |A \Rightarrow c \otimes B|_{\Sigma} \), observe that by the definition of degrees of provability and using (10), it follows that

\[
 c \Rightarrow |A \Rightarrow B|_{\Sigma} = c \Rightarrow \bigvee \{d \in [0, 1]_Q; \Sigma \vdash A \Rightarrow d \otimes B\}
\]

(37)

\[
 = \bigvee \{c \Rightarrow d; d \in [0, 1]_Q \text{ and } \Sigma \vdash A \Rightarrow d \otimes B\}
\]

(38)

\[
 \leq \bigvee \{c \Rightarrow d; d \in [0, 1]_Q \text{ and } \Sigma \vdash A \Rightarrow (c \Rightarrow d) \otimes c \otimes B\}
\]

(39)

\[
 \leq \bigvee \{e \in [0, 1]_Q; \Sigma \vdash A \Rightarrow e \otimes c \otimes B\}
\]

(40)

because \( \Sigma \vdash A \Rightarrow d \otimes B \) and \( c \Rightarrow d \otimes c \leq d \) yield \( \Sigma \vdash A \Rightarrow (c \Rightarrow d) \otimes c \otimes B \) by projectivity (from \( \Sigma \vdash E \Rightarrow F \cup G \) one may derive that \( \Sigma \vdash E \Rightarrow F \), see [11, Lemma 4.2]), i.e., \( \Sigma \vdash A \Rightarrow e \otimes c \otimes B \) for \( e = c \Rightarrow d \). Hence,

\[
 c \Rightarrow |A \Rightarrow B|_{\Sigma} \leq \bigvee \{e \in [0, 1]_Q; \Sigma \vdash A \Rightarrow e \otimes (c \otimes B)\}
\]

(41)

\[
 = |A \Rightarrow c \otimes B|_{\Sigma}. \tag{42}
\]

Conversely, in order to prove \( c \Rightarrow |A \Rightarrow B|_{\Sigma} \geq |A \Rightarrow c \otimes B|_{\Sigma} \), it suffices to show \( c \otimes |A \Rightarrow B|_{\Sigma} \leq |A \Rightarrow B|_{\Sigma} \), which is indeed the case:

\[
 c \otimes |A \Rightarrow B|_{\Sigma} = c \otimes \bigvee \{d \in [0, 1]_Q; \Sigma \vdash A \Rightarrow d \otimes c \otimes B\}
\]

(43)

\[
 = \bigvee \{c \otimes d; d \in [0, 1]_Q \text{ and } \Sigma \vdash A \Rightarrow (c \otimes d) \otimes B\}
\]

(44)

\[
 \leq \bigvee \{e \in [0, 1]_Q; \Sigma \vdash A \Rightarrow e \otimes B\}
\]

(45)

\[
 = |A \Rightarrow B|_{\Sigma}, \tag{46}
\]

which concludes the proof.

\[
\]

Lemma 5. For any finite index set \( I \) and an \( I \)-indexed set \( \{B_i; i \in I\} \) of finite rational \( \mathbf{L} \)-sets in \( \text{Var} \), we have

\[
 \bigwedge \{|A \Rightarrow B_i|_{\Sigma}; i \in I\} = |A \Rightarrow \bigcup \{B_i; i \in I\}|_{\Sigma} \tag{47}
\]

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for any finite rational \( L \)-set \( A \) in \( \text{Var} \).

**Proof.** Observe that the claim is trivial for \( I = \emptyset \) because

\[
\bigwedge \emptyset = 1 = |A \Rightarrow \bot|_{\Sigma} = |A \Rightarrow \bigcup \emptyset|_{\Sigma} \quad (48)
\]

where \( \bot (p) = 0 \) for all \( p \in \text{Var} \). Therefore, we may only inspect the situation for non-empty finite \( I \). Observe that \( \bigcup \{ B_i; i \in I \} \) is finite and rational, i.e., the formula \( A \Rightarrow \bigcup \{ B_i; i \in I \} \) which appears on the right-hand side of \( (47) \) is well defined. We prove \( (47) \) by checking both inequalities.

By definition of provability degrees and using the complete distributivity of the real unit interval (with its natural ordering, see \[13, 46\]), we have

\[
\bigwedge \{|A \Rightarrow B_i|_{\Sigma}; i \in I\} = \bigwedge \{\bigvee \{c \in [0,1]_Q; \Sigma \vdash c \otimes B_i\}; i \in I\} \quad (49)
\]

\[
= \bigvee \{\bigwedge \{f(i); i \in I\}; f \in \mathcal{F}\}, \quad (50)
\]

where

\[
\mathcal{F} = \prod \{\{c \in [0,1]_Q; \Sigma \vdash A \Rightarrow c \otimes B_i\}; i \in I\} \quad (51)
\]

is the direct product of subsets \( \{c \in [0,1]_Q; \Sigma \vdash A \Rightarrow c \otimes B_i\} \) of \( [0,1]_Q \). That is, \( \mathcal{F} \) is the set of all maps (choice functions) of the form \( f: I \to [0,1]_Q \) such that

\[
\Sigma \vdash A \Rightarrow f(i) \otimes B_i \quad (52)
\]

for all \( i \in I \). Since \( I \) is finite, \( (52) \) yields

\[
\Sigma \vdash A \Rightarrow \bigcup \{f(i) \otimes B_i; i \in I\} \quad (53)
\]

by additivity (from \( \Sigma \vdash E \Rightarrow F \) and \( \Sigma \vdash E \Rightarrow G \) it follows that \( \Sigma \vdash E \Rightarrow F \cup G \), see \[11\] Lemma 4.2)). Furthermore, \( \{f(i); i \in I\} \) has the least element \( \bigwedge \{f(i); i \in I\} \) because \( L \) is linear and \( I \) is non-empty and finite. Therefore, by projectivity, we get

\[
\Sigma \vdash A \Rightarrow \bigcup \{\bigwedge \{f(i); i \in I\} \otimes B_i; i \in I\}. \quad (54)
\]
Hence, using the distributivity of $\otimes$ over general $\bigcup$, it follows that

$$\Sigma \vdash A \Rightarrow c \otimes \bigcup \{B_i; \ i \in I\} \quad (55)$$

for $c = \bigwedge \{f(i); \ i \in I\} \in [0, 1]_Q$. Therefore, (50) may be extended as

$$\bigwedge \{\vert A_i \Rightarrow B_i\vert \Sigma; \ i \in I\} = \bigvee \{\bigwedge \{f(i); \ i \in I\}; \ f \in \mathcal{F}\} \quad (56)$$

$$\leq \bigvee \{c \in [0, 1]_Q; \ \Sigma \vdash A \Rightarrow c \otimes \bigcup \{B_i; \ i \in I\}\} \quad (57)$$

$$= \vert A \Rightarrow \bigcup \{B_i; \ i \in I\}\vert \Sigma, \quad (58)$$

which proves the “$\leq$”-inequality of (47).

Conversely, using projectivity and distributivity of $\otimes$ over general $\bigcup$:

$$\vert A \Rightarrow \bigcup \{B_i; \ i \in I\}\vert \Sigma = \bigvee \{c \in [0, 1]_Q; \ \Sigma \vdash A \Rightarrow c \otimes \bigcup \{B_i; \ i \in I\}\} \quad (59)$$

$$= \bigvee \{c \in [0, 1]_Q; \ \Sigma \vdash A \Rightarrow \bigcup \{c \otimes B_i; \ i \in I\}\} \quad (60)$$

$$\leq \bigvee \{c \in [0, 1]_Q; \ \Sigma \vdash A \Rightarrow c \otimes B_i\} \quad (61)$$

$$= \vert A \Rightarrow B_i\vert \Sigma \quad (62)$$

for arbitrary $i \in I$ which proves the converse inequality of (47).

**Lemma 6.** For any $A \Rightarrow B$, $B \Rightarrow C$, $A \Rightarrow C$, and $\Sigma$,

$$\vert A \Rightarrow B\vert \Sigma \otimes \vert B \Rightarrow C\vert \Sigma \leq \vert A \Rightarrow C\vert \Sigma. \quad (63)$$

**Proof.** Let $\Sigma \vdash A \Rightarrow b \otimes B$ and $\Sigma \vdash B \Rightarrow c \otimes C$ for some $b, c \in [0, 1]_Q$. Using (Mul), we infer $\Sigma \vdash b \otimes B \Rightarrow b \otimes c \otimes C$ from $\Sigma \vdash B \Rightarrow c \otimes C$. Therefore, using (Cut) on $\Sigma \vdash A \Rightarrow b \otimes B$ and $\Sigma \vdash b \otimes B \Rightarrow b \otimes c \otimes C$, we get $\Sigma \vdash A \Rightarrow b \otimes c \otimes C$. As a consequence,

$$\vert A \Rightarrow B\vert \Sigma \otimes \vert B \Rightarrow C\vert \Sigma = \bigvee \{b \otimes c; \ \Sigma \vdash A \Rightarrow b \otimes B \text{ and } \Sigma \vdash B \Rightarrow c \otimes C\} \quad (64)$$

$$\leq \bigvee \{b \otimes c; \ \Sigma \vdash A \Rightarrow b \otimes c \otimes C\} \quad (65)$$

$$= \bigvee \{d \in [0, 1]_Q; \ \Sigma \vdash A \Rightarrow d \otimes C\} \quad (66)$$

$$= \vert A \Rightarrow C\vert \Sigma, \quad (67)$$

which proves (63).
Lemma 7. If $\Sigma \not\vDash A \Rightarrow c \otimes B$ for $c \in [0, 1]_Q$, then $||A \Rightarrow B||_\Sigma \leq c$.

Proof. We assume $\Sigma \not\vDash A \Rightarrow c \otimes B$ and we find an evaluation $e$ which is a model of $\Sigma$ and satisfies $||A \Rightarrow B||_e \leq c$. For every $p \in \text{Var}$, put

$$e(p) = |A \Rightarrow \{1/p\}|_\Sigma.$$  \hfill (68)

Next, we prove an auxiliary claim: For each finite rational $L$-set $C$ in $\text{Var}$, we have $S(C, e) = |A \Rightarrow C|_\Sigma$ where $e$ is given by (68). Using the fact that $C$ may be expressed by a union of finitely many rational singletons in $\text{Var}$ and using (36) together with (47), it follows that

$$S(C, e) = \bigwedge \{C(p) \rightarrow e(p); p \in \text{Var}\}$$  \hfill (69)
$$= \bigwedge \{C(p) \rightarrow |A \Rightarrow \{1/p\}|_\Sigma; p \in \text{Var}\}$$  \hfill (70)
$$= \bigwedge \{|A \Rightarrow C(p) \otimes \{1/p\}|_\Sigma; p \in \text{Var}\}$$  \hfill (71)
$$= \bigwedge \{|A \Rightarrow \{C(p)/p\}|_\Sigma; p \in \text{Var} \text{ and } C(p) > 0\}$$  \hfill (72)
$$= |A \Rightarrow \bigcup \{\{C(p)/p\}; p \in \text{Var} \text{ and } C(p) > 0\}|_\Sigma$$  \hfill (73)
$$= |A \Rightarrow C|_\Sigma.$$  \hfill (74)

Using the claim, we prove that $e$ is a model of $\Sigma$. Indeed, take any $E \Rightarrow F \in \Sigma$ and observe that (63) yields

$$S(E, e) = |A \Rightarrow E|_\Sigma$$  \hfill (76)
$$= |A \Rightarrow E|_\Sigma \otimes 1$$  \hfill (77)
$$= |A \Rightarrow E|_\Sigma \otimes |E \Rightarrow F|_\Sigma$$  \hfill (78)
$$\leq |A \Rightarrow F|_\Sigma$$  \hfill (79)
$$= S(F, e),$$  \hfill (80)

showing $||E \Rightarrow F||_e = 1$ owing to (29). Furthermore, we have

$$S(A, e) = |A \Rightarrow A|_\Sigma = 1,$$  \hfill (81)
because $A \Rightarrow A$ is an axiom. Furthermore, if $d > c$ for $d \in [0,1]_Q$, then our assumption $\Sigma \not\vdash A \Rightarrow c \otimes B$ yields $\Sigma \not\vdash A \Rightarrow d \otimes B$ because otherwise the assumption $\Sigma \not\vdash A \Rightarrow c \otimes B$ would be violated on account of projectivity. Therefore,

$$S(B,e) = |A \Rightarrow B|_{\Sigma} \leq c$$

(82)

As a consequence of (81) and (82), we get

$$||A \Rightarrow B||_{\Sigma} \leq ||A \Rightarrow B||_e$$

(83)

$$= S(A,e) \rightarrow S(B,e)$$

(84)

$$\leq 1 \rightarrow c$$

(85)

$$= c,$$

(86)

which is the desired inequality.

Theorem 2 can be now proved as follows:

**Proof of Theorem 2.** The inequality $|A \Rightarrow B|_{\Sigma} \leq ||A \Rightarrow B||_e$, i.e., Pavelka-style soundness, follows by standard arguments. In order to prove the completeness in Pavelka style, we prove that for each $c \in [0,1]_Q$ such that $c < ||A \Rightarrow B||_e$, we have $\Sigma \vdash A \Rightarrow c \otimes B$ which immediately gives $||A \Rightarrow B||_e \leq |A \Rightarrow B|_{\Sigma}$. But this is a direct consequence of Lemma 7

Since the degrees of semantic entailment of formulas of the form (20) in RPL and RIIL are defined as in our logic, we immediately get the following consequence on the relationship of to these two Pavelka-style complete logics:

**Corollary 8.** The following are consequences of Theorem 2

- If $\mathbf{L}$ is $\mathbf{L}$, then RFAL is a fragment of RPL.
- If $\mathbf{L}$ is $\mathbf{I}$, then RFAL is a fragment of RIIL.

**Remark 4.** (a) Recall that RIIL utilizes an infinitary deduction rule to ensure completeness in Pavelka style. In contrast, our logic for $\mathbf{L}$ being $\mathbf{I}$, has
the usual finitary notion of a proof. In other words, we have shown that RIIL
restricted just to formulas of the form (20) can be axiomatized without infinitary
deduction rules.

(b) RFAL is not Pavelka-style complete with \( L \) being the standard Gödel
algebra (i.e., \( L \) defined on the real unit interval with \( \otimes = \land \)). Observe that for

\[
\Sigma = \{ \{0/p\} \Rightarrow \{a/p\}; a \in [0,0.5)_Q \} \cup \{ \{0.5/p\} \Rightarrow \{1/q\} \},
\]

(87)

we obviously have \( ||\{0/p\} \Rightarrow \{1/q\}||_\Sigma = 1 \) for \( L \) being the standard Gödel algebra
because for each model \( e \) of \( \Sigma \), we have \( e(p) \geq 0.5 \) and thus \( e(q) = 1 \). On the
other hand, we have \( ||\{0/p\} \Rightarrow \{1/q\}||_\Sigma < 1 \). Indeed, each finite \( \Sigma' \subseteq \Sigma \) admits a
rational model \( e \) such that \( e(p) = a < 0.5 \) and \( e(q) = a \). Owing to soundness, we
get \( ||\{0/p\} \Rightarrow \{1/q\}||_{\Sigma'} \leq a < 0.5 \) and thus \( ||\{0/p\} \Rightarrow \{1/q\}||_\Sigma \leq 0.5 < 1 \). Therefore,
in order to ensure completeness in Pavelka style for \( L \) being the standard Gödel
algebra, one has to resort to infinitary deduction rules as in [8, 37].

(c) Note for readers familiar with Pavelka’s abstract logic as it is presented
in [32, Section 9.2]: Our approach uses the traditional understanding of theories
as sets of formulas. Following Pavelka’s ideas, one may extend it to use theories
considered as \( L \)-sets of formulas. This approach is, however, reducible to the
traditional one: For \( \Sigma \) considered as an \( L \)-set which for any \( A \Rightarrow B \) prescribes a
general degree \( \Sigma(A \Rightarrow B) \in [0,1] \) (including irrational degrees), we may consider

\[
\Sigma^* = \{ A \Rightarrow c\otimes B; c \in [0,1)_Q \text{ and } c < \Sigma(A \Rightarrow B) \}
\]

which does the same job, see [32, Section 9.2] for further details on semantic
and syntactic entailment from “fuzzy theories.”

We now turn our attention to the proof of Theorem 3. We utilize a character-
ization of degrees of provability based on constructing least models containing
given evaluations. For each evaluation \( e \) and a theory \( \Sigma \), we consider evaluations
\( e^n_{\Sigma} (n \text{ is a finite ordinal}) \) and \( e^\omega_{\Sigma} (\omega \text{ denotes the least infinite ordinal}) \) as follows:

\[
e^0_{\Sigma} = e, \tag{88}
\]

\[
e^{n+1}_{\Sigma} = e^n_{\Sigma} \cup \bigcup \{ S(A, e^n_{\Sigma}) \otimes B; A \Rightarrow B \in \Sigma \}, \tag{89}
\]

\[
e^\omega_{\Sigma} = \bigcup \{ e^n_{\Sigma}; n < \omega \}, \tag{90}
\]

for all \( n < \omega \). Observe that if \( \Sigma \) is finite and the evaluation \( e \) is finite and rational (recall that \( e \) is in fact an \( L \)-set), then \( e^n_{\Sigma} \) is finite and rational for each \( n < \omega \) provided that \( L \) satisfies Assumption 1. The following assertions show properties of \( e^\omega_{\Sigma} \) given by (90).

**Lemma 9.** Let \( L \) satisfy (10), \( \Sigma \) be a theory and \( e \) be an evaluation. Then, \( e^\omega_{\Sigma} \) given by (90) is the least model of \( \Sigma \) which contains \( e \).

**Proof.** We first show that \( e^\omega_{\Sigma} \) is a model of \( \Sigma \). Directly by (89), for each formula \( A \Rightarrow B \in \Sigma \) and each \( n < \omega \), we have

\[
S(A, e^n_{\Sigma}) \otimes B \subseteq e^{n+1}_{\Sigma}. \tag{91}
\]

Therefore,

\[
\bigcup \{ S(A, e^n_{\Sigma}) \otimes B; n < \omega \} \subseteq \bigcup \{ e^{n+1}_{\Sigma}; n < \omega \} = e^\omega_{\Sigma}. \tag{92}
\]

In addition, using (16), (90), and (10), it follows that

\[
S(A, e^\omega_{\Sigma}) \otimes B = \bigwedge \{ A(p) \Rightarrow e^\omega_{\Sigma}(p); p \in \text{Var} \} \otimes B \tag{93}
\]

\[
= \bigwedge \{ A(p) \Rightarrow \bigvee \{ e^n_{\Sigma}(p); n < \omega \}; p \in \text{Var} \} \otimes B \tag{94}
\]

\[
= \bigwedge \{ \bigvee \{ A(p) \Rightarrow e^n_{\Sigma}(p); n < \omega \}; p \in \text{Var} \} \otimes B. \tag{95}
\]

Let \( \mathcal{F} \) denote the set of all maps of the form \( f: \text{Var} \rightarrow \{ n; n < \omega \} \). Using complete distributivity, the previous equality can be extended as

\[
S(A, e^\omega_{\Sigma}) \otimes B = \bigvee \{ \bigwedge \{ A(p) \Rightarrow e^{f(p)}_{\Sigma}(p); p \in \text{Var} \}; f \in \mathcal{F} \} \otimes B. \tag{96}
\]
In addition, $A$ is a finite $L$-set. Therefore, there are only finitely many $p \in \text{Var}$ such that $A(p) \rightarrow e_\Sigma^f(p) < 1$. As a consequence, for any $f \in \mathcal{F}$, there is $n < \omega$ such that

$$\bigwedge\{A(p) \rightarrow e_\Sigma^f(p); p \in \text{Var}\} \leq \bigwedge\{A(p) \rightarrow e_\Sigma^e(p); p \in \text{Var}\}. \quad (97)$$

Hence, (96) is equivalent to

$$S(A, e_\Sigma^n) \otimes B = \bigvee\{\bigwedge\{A(p) \rightarrow e_\Sigma^e(p); p \in \text{Var}\}; n < \omega\} \otimes B \quad (98)$$

$$= \bigvee\{S(A, e_\Sigma^n); n < \omega\} \otimes B \quad (99)$$

$$= \bigcup\{S(A, e_\Sigma^n) \otimes B; n < \omega\} \quad (100)$$

$$\subseteq e_\Sigma^n \quad (101)$$

using (92). Hence, we have shown $S(A, e_\Sigma^n) \otimes B \subseteq e_\Sigma^n$ for any $A \Rightarrow B \in \Sigma$, i.e.,

$$S(A, e_\Sigma^n) \leq S(B, e_\Sigma^n) \quad (102)$$

for any $A \Rightarrow B \in \Sigma$ and so $e_\Sigma^n$ is a model of $\Sigma$.

Now, consider a model $e'$ of $\Sigma$ such that $e \subseteq e'$, i.e., $e(p) \leq e'(p)$ for all $p \in \text{Var}$. Then, by induction, $e_\Sigma^n \subseteq e'$ gives

$$S(A, e_\Sigma^n) \otimes B \subseteq S(A, e') \otimes B \subseteq e' \quad (103)$$

for all $A \Rightarrow B \in \Sigma$, i.e., (89) immediately yields $e_\Sigma^{n+1} \subseteq e'$. As a consequence, $e_\Sigma^n \subseteq e'$, showing that $e_\Sigma^n$ is indeed the least model of $\Sigma$ containing $e$. \hfill \Box

A description of provability degrees based on least models is now established. Indeed, using the existing result described in [11, Theorem 3.11], it follows that $||A \Rightarrow B||_\Sigma = S(B, e)$ for $e$ being the least model of $\Sigma$ containing $A$. Applying Theorem 2 and Lemma 9, we get the following characterization.

**Corollary 10.** Let $L$ satisfy Assumption [7] and condition (10). Then, for any $\Sigma$ and $A \Rightarrow B$,

$$|A \Rightarrow B|_\Sigma = S(B, A^e_\Sigma). \quad (104)$$

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Proof. Consequence of Theorem 2, Lemma 9, and [11, Theorem 3.11]. □

In case of $L$ and $\Pi$, the observation in Lemma 9 and its consequence in Corollary 10 may further be improved provided that $\Sigma$ is finite in which case one may always take some $e^\omega_\Sigma$ instead of $e^\omega_\Sigma$:

**Lemma 11.** Let $L$ be either of $L$ or $\Pi$. Then, for any finite theory $\Sigma$ and a rational evaluation $e$ there is $n$ such that $e^\omega_\Sigma = e^n_\Sigma$.

**Proof.** We inspect the situation for $L$ and $\Pi$ separately. In the proof, we denote by $O$ the set of all propositional variables which appear in formulas in $\Sigma$. That is, $p \in O$ whenever there is $\mathcal{A} \in \Sigma$ such that $\mathcal{A}(p) > 0$ or $\mathcal{B}(p) > 0$. Analogously, by $K$ we denote the set of non-zero degrees (represented by constants) which appear in formulas in $\Sigma$. That is, $c \in K$ whenever there is $\mathcal{A} \in \Sigma$ such that $\mathcal{A}(p) = c > 0$ or $\mathcal{B}(p) = c > 0$ for some $p \in \text{Var}$.

Note that since $\Sigma$ is finite, then $O$ is finite as well. As a consequence, there are at most finitely many $p \in \text{Var}$ such that $e^\omega_\Sigma(p) > e(p)$. Namely, $e^\omega_\Sigma(p) > e(p)$ implies that $p \in O$ which follows directly by (89). Therefore, in the proof, we tacitly assume that $e(p) = 0$ whenever $p \notin O$.

Let $L$ be $L$ and observe that since $K$ is finite and all degrees in $K$ are rational, then one may consider a finite equidistant Lukasiewicz subchain of $L$ consisting of rational truth degrees which contains all the degrees from $K$ and $\{e(p); p \in \text{Var}\}$. Denote the subchain by $L'$ and its support set by $L'$. Then, inspecting (88) and (89), it follows that each $e^n_\Sigma$ is rational and

$$\{e^n_\Sigma(p); p \in \text{Var}\} \subseteq L'$$

(105)

because each $e^n_\Sigma(p)$ may be expressed by finitely many applications of operations in $L'$. As a consequence, $e^\omega_\Sigma = e^n_\Sigma$ for some $n < \omega$ because only finitely many propositional variables (those in $O$) are assigned finitely many different degrees (those in $L'$), i.e., the chain $e^0_\Sigma \subseteq e^1_\Sigma \subseteq e^2_\Sigma \subseteq \cdots$ consists of only finitely many proper inclusions.
For $L$ being $\Pi$, we proceed by contradiction: We assume there are $p \in \text{Var}$ such that $e_\Sigma^n(p) < e_\Sigma^n(p)$ for all $n < \omega$. Since $\Sigma$ and $O$ are finite, there are pairwise distinct $p_0, \ldots, p_{m-1} \in \text{Var}$ and formulas $A_0 \Rightarrow B_0, \ldots, A_{m-1} \Rightarrow B_{m-1}$ in $\Sigma$ such that for $p_m = p_0$ and for infinitely many $n < \omega$, we have

\begin{align}
  e_\Sigma^{n+1}(p_1) &= (A_0(p_0) \rightarrow e_\Sigma^n(p_0)) \otimes B_0(p_1), \\
  e_\Sigma^{n+2}(p_2) &= (A_1(p_1) \rightarrow e_\Sigma^{n+1}(p_1)) \otimes B_1(p_2), \\
  & \vdots \\
  e_\Sigma^{n+m}(p_m) &= (A_{m-1}(p_{m-1}) \rightarrow e_\Sigma^{n+m-1}(p_{m-1})) \otimes B_{m-1}(p_m),
\end{align}

(106)

and the following conditions are satisfied:

- $e_\Sigma^{n+i}(p_i) > e_\Sigma^{n+i-1}(p_i)$ for all $i = 1, \ldots, m$, and
- $A_i(p_i) > e_\Sigma^{n+i}(p_i)$ for all $i = 0, \ldots, m - 1$.

As a consequence, $e_\Sigma^{n+m}(p_0) = e_\Sigma^{n+m}(p_m) > e_\Sigma^n(p_0)$. Furthermore, directly by the definition of $\rightarrow$ in $\Pi$, we get that

\begin{equation}
  e_\Sigma^{n+i}(p_i) = \frac{e_\Sigma^{n+i-1}(p_{i-1}) \cdot B_{i-1}(p_{i-1})}{A_{i-1}(p_{i-1})}
\end{equation}

(109)

for all $i = 1, \ldots, m$. Therefore, $e_\Sigma^{n+m}(p_0)$ may be expressed as

\begin{equation}
  e_\Sigma^{n+m}(p_0) = e_\Sigma^{n+m}(p_m) = \frac{e_\Sigma^n(p_0) \cdot B_0(p_1) \cdot B_1(p_2) \cdots B_{m-1}(p_m)}{A_0(p_0) \cdot A_1(p_1) \cdots A_{m-1}(p_{m-1})}.
\end{equation}

(110)

That is, we have $e_\Sigma^{n+m}(p_0) = e_\Sigma^n(p_0) \cdot c$ for $c > 1$. Since we have assumed that (106)–(108) hold for infinitely many $n$, then there are $n_1, n_2, \ldots$ such that

\begin{equation}
  e_\Sigma^{n+i}(p_0) \geq e_\Sigma^{n+i}(p_0) = e_\Sigma^n(p_0) \cdot c
\end{equation}

(111)

for all $i = 1, 2, \ldots$ and thus

\begin{equation}
  e_\Sigma^n(p_0) \geq e_\Sigma^n(p_0) \cdot c^{i-1}
\end{equation}

(112)

for all $i = 1, 2, \ldots$ which means that

\begin{equation}
  \lim_{i \to \infty} e_\Sigma^n(p_0) \geq \lim_{i \to \infty} e_\Sigma^{n+1}(p_0) \cdot c^{i-1} = e_\Sigma^n(p_0) \lim_{i \to \infty} c^{i-1} = \infty,
\end{equation}

(113)

which contradicts the fact that $e_\Sigma^n(p_0) < e_\Sigma^n(p_0)$ for all $n < \omega$. \hfill \Box

25
Lemma 12. Let $\Sigma$ be finite. Then, $\Sigma \vdash A \Rightarrow A^n_\Sigma$ for all $n < \omega$.

Proof. The proof goes by induction. Notice that since $\Sigma$ is finite, then $A^n_\Sigma$ is rational and finite for any $n < \omega$, i.e., $A \Rightarrow A^n_\Sigma$ is well defined formula. Suppose that $\Sigma \vdash A \Rightarrow A^n_\Sigma$ and take $E \Rightarrow F \in \Sigma$. Then, using (Mul) for $c = S(E, A^n_\Sigma)$, we have $\Sigma \vdash S(E, A^n_\Sigma) \otimes E \Rightarrow S(E, A^n_\Sigma) \otimes F$. Then, using $S(E, A^n_\Sigma) \otimes E \subseteq A^n_\Sigma$ and (Cut), we get $\Sigma \vdash A \Rightarrow S(E, A^n_\Sigma) \otimes F$. Inspecting (89), we prove $\Sigma \vdash A \Rightarrow A^{n+1}_\Sigma$ by finitely many applications of additivity (from $\Sigma \vdash E \Rightarrow F$ and $\Sigma \vdash E \Rightarrow G$ we derive $\Sigma \vdash E \Rightarrow F \cup G$, see [11, Lemma 4.2]) because $\Sigma$ is finite. 

Proof of Theorem 3. Let $L$ be either of $L$ or $\Pi$ and let $\Sigma$ be finite. Taking into account Corollary 10 and Lemma 11, we get $|A \Rightarrow B|_\Sigma = S(B, A^n_\Sigma) = S(B, A^n_\Sigma)$ (114) for some $n < \omega$. Since both $B$ and $A^n_\Sigma$ are finite and rational, $S(B, A^n_\Sigma)$ is a rational degree which proves that $|A \Rightarrow B|_\Sigma$ is rational. In addition, Lemma 12 yields that $\Sigma \vdash A \Rightarrow A^n_\Sigma$. Hence, using the fact that $S(B, A^n_\Sigma) \otimes B \subseteq A^n_\Sigma$, the projectivity gives $\Sigma \vdash A \Rightarrow c \otimes B$ for $c = S(B, A^n_\Sigma) = |A \Rightarrow B|_\Sigma$.

In addition, it is decidable whether $|A \Rightarrow B|_\Sigma = c$: In finitely many steps, one computes $A^n_\Sigma$ such that $A^n_\Sigma = A^{n+1}_\Sigma$ (which exists owing to Lemma 11) and checks whether $c = S(B, A^n_\Sigma)$. In particular, for $c = 1$, it means $B \subseteq A^n_\Sigma$ if $|A \Rightarrow B|_\Sigma = 1$ if $\Sigma \vdash A \Rightarrow 1 \otimes B$ if $\Sigma \vdash A \Rightarrow B$, i.e., it is decidable whether $A \Rightarrow B$ is provable by $\Sigma$. 

5 Rational $L$-closure Operators

In this section, we present observations on closure structures associated to models of theories consisting of formulas of the form (20). We are motivated by the fact that for any $\Sigma$, we may introduce an operator which maps each evaluation $e$ to $e^n_\Sigma$ defined by (90). Under the assumption of (10), we show that such operators are in fact finitary $L$-closure operators on propositional variables.
Recall from preliminaries the general notion of an \(L\)-closure operator, see Section 2. We call an \(L\)-closure operator \(c: \mathcal{L}U \to \mathcal{L}U\) \textit{finitary} whenever

\[
c(A) = \bigcup \{c(B); \ B \subseteq A \text{ and } B \text{ is finite}\}
\]

for all \(A \in \mathcal{L}U\). For \(A, B \in \mathcal{L}U\), we put

\[
B \subseteq A
\]

whenever \(B\) is rational, finite, and \(B \subseteq A\). Using this notation, we call an \(L\)-closure operator \(c: \mathcal{L}U \to \mathcal{L}U\) \textit{rational} whenever

\[
c(A) = \bigcup \{c(B); \ B \subseteq A\}
\]

for all \(A \in \mathcal{L}U\). By definition, a rational \(L\)-closure operator is finitary. The following assertions show that under the condition (10), finitary \(L\)-closure operators are always rational.

**Lemma 13.** Let \(A\) be a finite \(L\)-set in \(U\) and let \(B\) be a directed system of \(L\)-sets in \(U\). If \(L\) defined on the real unit interval satisfies (10), then

\[
S(A, \bigcup B) = \bigvee \{S(A, B); B \in \mathcal{B}\}.
\]

**Proof.** Note that since \(B\) is directed, it is also non-empty. Let \(\mathcal{F}\) denote the set of all functions from \(\text{Var}\) to \(\mathcal{B}\). Under this notation, using (10) and the complete distributivity, we get

\[
S(A, \bigcup B) = \bigwedge \{A(p) \to (\bigcup B)(p); p \in \text{Var}\}
\]

\[
= \bigwedge \{A(p) \to \bigvee \{B(p); B \in \mathcal{B}\}; p \in \text{Var}\}
\]

\[
= \bigwedge \{\bigvee \{A(p) \to B(p); B \in \mathcal{B}\}; p \in \text{Var}\}
\]

\[
= \bigvee \{\bigwedge \{A(p) \to (f(p))(p); p \in \text{Var}\}; f \in \mathcal{F}\}.
\]

Now, since \(A\) is finite and \(B\) is directed, for each \(f \in \mathcal{F}\) there is \(B \in \mathcal{B}\) such that \(f(p) \subseteq B\) for all \(p \in \text{Var}\) satisfying \(A(p) > 0\) and thus \((f(p))(p) \leq B(p)\) for all
\( p \in \text{Var} \) satisfying \( A(p) > 0 \). Therefore, we have

\[
S(A, \bigcup B) = \bigvee \{ \bigwedge \{ A(p) \rightarrow (f(p))(p); \ p \in \text{Var} \}; \ f \in \mathcal{F} \} \tag{123}
\]

\[
= \bigvee \{ \bigwedge \{ A(p) \rightarrow B(p); \ p \in \text{Var} \}; \ B \in B \} \tag{124}
\]

\[
= \bigvee \{ S(A, B); \ B \in B \}, \tag{125}
\]

which establishes (118). \( \square \)

**Lemma 14.** Let \( A \) be a finite \( L \)-set in \( U \). If \( L \) defined on the real unit interval satisfies (10), then for any \( \varepsilon < 1 \) there is \( B_\varepsilon \subseteq A \) such that \( S(A, B_\varepsilon) > \varepsilon \).

**Proof.** Consider \( L \)-sets \( B_n \) (\( n < \omega \)) with \( B_n \subseteq B_{n+1} \) and \( B_n \subseteq A \) for all \( n < \omega \) such that and \( \bigcup \{ B_n; n < \omega \} = A \). Since \( \{ B_n; n < \omega \} \) is obviously directed, Lemma 13 yields

\[
1 = S(A, \bigcup \{ B_n; n < \omega \}) = \bigvee \{ S(A, B_n); n < \omega \}. \tag{126}
\]

Hence, for every \( \varepsilon < 1 \) there is \( B_\varepsilon \subseteq A \) such that \( S(A, B_\varepsilon) > \varepsilon \) otherwise our observation \( 1 = \bigvee \{ S(A, B_n); n < \omega \} \) would be violated. \( \square \)

**Lemma 15.** Let \( L \) satisfy (10). Then, every finitary \( L \)-closure operator is rational.

**Proof.** Let \( c: L^U \rightarrow L^U \) be an \( L \)-closure operator which is finitary. It suffices to show that for any finite \( L \)-set \( A \) in \( U \), we have \( c(A) = \bigcup \{ c(B); B \subseteq A \} \). Based on the observation in Lemma 14 for each \( n < \omega \) we let \( B_n \in L^U \) such that \( B_n \subseteq A \) and \( S(A, B_n) > 1 - \frac{1}{n} \). Applying the monotony (18) of \( c \), for any \( n < \omega \), we have \( S(A, B_n) \otimes c(A) \subseteq c(B_n) \), i.e.,

\[
c(A) = 1 \otimes c(A) = \bigvee \{ S(A, B_n); n < \omega \} \otimes c(A) \tag{127}
\]

\[
= \bigcup \{ S(A, B_n) \otimes c(A); n < \omega \} \tag{128}
\]

\[
\subseteq \bigcup \{ c(B_n); n < \omega \}. \tag{129}
\]

The converse inclusion holds trivially. As a consequence, \( c \) is rational. \( \square \)
Remark 5. Let us note that the assumption of \( L \) satisfying (10) in Lemma 15 is essential. Indeed, in general there are finitary \( L \)-closure operators which are not rational. For instance, let \( L \) be the standard Gödel algebra, consider \( U = \{ u \} \), and let \( c \) be an irrational number in \([0, 1]\). Put

\[
c(\{a/u\})(u) = \begin{cases} 
a, & \text{for } a < c, \\
1, & \text{otherwise.}
\end{cases}
\]

Obviously, \( c \) satisfies (17) and (19). In order to see that \( c \) satisfies (18), observe that in the non-trivial case for \( \{a/u\} \) and \( \{b/u\} \) with \( a > b \), we have \( S(\{a/u\}, \{b/u\}) = b \). Now, if \( a \geq c \) and \( b < c \), we get

\[
S(c(\{a/u\}), c(\{b/u\})) = 1 \rightarrow b = b = S(\{a/u\}, \{b/u\})
\]

If both \( a \geq c \) and \( b \geq c \), the condition is trivial; the same applies if \( a < c \) and \( b < c \). Altogether, \( c \) is an \( L \)-closure operator. Furthermore,

\[
\bigvee\{c(\{a/u\})(u); a \in [0, 1]_Q \text{ and } a \leq c\} = c < 1 = c(\{c/u\})(u),
\]

i.e., \( c \) is finitary but it is not rational.

Now, for any \( \Sigma \) and evaluation \( e \), we put

\[
c_\Sigma(e) = e_\Sigma^c.
\]

The following assertions characterize operators defined as in (133) for all possible choices of \( \Sigma \).

**Theorem 16.** Let \( L \) satisfy (10). Then, for each \( \Sigma \), \( c_\Sigma : L^{Var} \rightarrow L^{Var} \) defined by (133) is a rational \( L \)-closure operator.

**Proof.** The fact that \( c_\Sigma \) is an \( L \)-closure operator follows by the general result in [11, Theorem 3.9] which holds for any complete residuated lattice \( L \) taken as the structure of degrees. Also, the “\( \supseteq \)”-part of (117) is trivial. Thus, it suffices to check the “\( \subseteq \)”-part of (117). Let \( \mathcal{G} = \{ g \in L^{Var}; g \supseteq e \} \). We proceed by
checking that $\bigcup \{ c_{\Sigma}(g); g \in \mathcal{G} \}$ is a model of $\Sigma$ containing $e$. Observe that $\mathcal{G}$ is directed and so $\{ c_{\Sigma}(g); g \in \mathcal{G} \}$ owing to the monotony (18) of $c$. Take any $A \Rightarrow B \in \Sigma$. Applying Lemma 13 and the fact that for each $g \in \mathcal{G}$, $c_{\Sigma}(g)$ is a model of $\Sigma$ and so $S(A, c_{\Sigma}(g)) \otimes B \subseteq c_{\Sigma}(g)$, it follows that

$$S(A, \bigcup \{ c_{\Sigma}(g); g \in \mathcal{G} \}) \otimes B = \bigvee \{ S(A, c_{\Sigma}(g)); g \in \mathcal{G} \} \otimes B$$

$$= \bigcup \{ S(A, c_{\Sigma}(g)) \otimes B; g \in \mathcal{G} \}$$

$$\subseteq \bigcup \{ c_{\Sigma}(g); g \in \mathcal{G} \},$$

i.e., $\bigcup \{ c_{\Sigma}(g); g \in \mathcal{G} \}$ is indeed a model of $\Sigma$ which obviously contains $e$. □

**Theorem 17.** Let $L$ satisfy [10] and let $c : L^{\text{Var}} \rightarrow L^{\text{Var}}$ be a finitary $L$-closure operator. Then, there is $\Sigma$ such that $c = c_{\Sigma}$.

**Proof.** Let $\top \in L^{\text{Var}}$ such that $\top(p) = 1$ for all $p \in \text{Var}$. We put

$$\Sigma = \{ A \Rightarrow B; A \subseteq \top \text{ and } B \subseteq c(A) \}$$

and prove the claim holds for $\Sigma$.

First, we show that for any evaluation $e$, $c(e)$ is a model of $\Sigma$, i.e., $c(e)$ is closed under $c_{\Sigma}$. Let $A \Rightarrow B \in \Sigma$. Since $B \subseteq c(A)$, (18) and (19) yield

$$S(A, c(e)) \leq S(c(A), c(c(e)))$$

$$= S(c(A), c(e))$$

$$\leq S(B, c(e)),$$

showing that $c(e)$ is a model of $\Sigma$.

Conversely, we show that $c_{\Sigma}(e)$ is a fixed point of $c$. Take any $A \sqsubseteq c_{\Sigma}(e)$. Trivially, $S(A, c_{\Sigma}(e)) = 1$ and thus for any $B \sqsubseteq c(A)$, we have $S(B, c_{\Sigma}(e)) = 1$, i.e., $B \subseteq c_{\Sigma}(e)$ because $A \Rightarrow B \in \Sigma$ and $c_{\Sigma}(e)$ is a model of $\Sigma$. Now, the fact that $B \subseteq c_{\Sigma}(e)$ holds for all $B \sqsubseteq c(A)$ yields that

$$c(A) = \bigcup \{ B; B \subseteq c(A) \} \subseteq c_{\Sigma}(e).$$
Furthermore, the previous inclusion holds for any $A \subseteq c_S(e)$, i.e., utilizing the fact that $c$ is rational which follows by Lemma 15, we have

$$ c(c_S(e)) = \bigcup \{c(A); A \subseteq c_S(e)\} \subseteq c_S(e), \quad (142) $$

proving that $c_S(e)$ is a fixed point of $c$. □

Theorem 16 and Theorem 17 showed that under the assumption (10), the operators on $\mathcal{L}$-sets of propositional variables defined by (133) are exactly all rational $\mathcal{L}$-closure operators on propositional variables. In other words, the systems of fixed points of rational $\mathcal{L}$-closure operators on propositional variables are the systems of models of sets of rational fuzzy attribute implications, cf. [50] for a study of the expressive power of general fuzzy attribute implications parameterized by hedges.

**Conclusion**

Logic for reasoning with graded if-then rules is proposed. The rules can be seen as formulas of the form of implications which contain constants for rational truth degrees. The interpretation of formulas is given by complete residuated lattices defined on the real unit interval. Degrees of semantic entailment and degrees of provability are defined. For complete residuated lattices with residuum which is continuous in the second argument, the logic is Pavelka-style complete which means that degrees of semantic entailment agree with degrees of provability. Characterization of the degrees of provability based on computing least models is established. In case of finite theories and the standard Lukasiewicz or Goguen (product) algebras, the least models may be determined in finitely many steps which shows that the logic based on these structures of degrees is decidable. Structures of models are identified with fixed points of rational $\mathcal{L}$-closure operators. It is shown that the property of being rational is a consequence of the property of being finitary in case of structures of truth degrees with residua continuous in the second argument.
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References

[1] Rakesh Agrawal, Tomasz Imieliński, and Arun Swami, *Mining association rules between sets of items in large databases*, Proceedings of the 1993 ACM SIGMOD International Conference on Management of Data (New York, NY, USA), SIGMOD ’93, ACM, 1993, pp. 207–216.

[2] William Ward Armstrong, *Dependency structures of data base relationships*, Information Processing 74: Proceedings of IFIP Congress (Amsterdam) (J. L. Rosenfeld and H. Freeman, eds.), North Holland, 1974, pp. 580–583.

[3] Mathias Baaz, *Infinite-valued Gödel logics with 0-1 projections and relativizations*, GöDEL ’96, Logical Foundations of Mathematics, Computer Sciences and Physics (Berlin/Heidelberg), Lecture Notes in Logic, vol. 6, Springer-Verlag, 1996, pp. 23–33.

[4] Radim Belohlavek, *Fuzzy closure operators*, Journal of Mathematical Analysis and Applications 262 (2001), no. 2, 473–489.

[5] , *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic Publishers, Norwell, MA, USA, 2002.

[6] Radim Belohlavek, Pablo Cordero, Manuel Enciso, Ángel Mora, and Vilem Vychodil, *An efficient reasoning method for dependencies over similarity and ordinal data*, Modeling Decisions for Artificial Intelligence (Vicenç Torra, Yasuo Narukawa, Beatriz López, and Mateu Villaret, eds.), Lecture Notes in Computer Science, vol. 7647, Springer Berlin Heidelberg, 2012, pp. 408–419.

[7] Radim Belohlavek and Vilem Vychodil, *Attribute implications in a fuzzy setting*, Formal Concept Analysis (Rokia Missaoui and Jürg Schmidt, eds.), Lecture Notes in Computer Science, vol. 3874, Springer Berlin Heidelberg, 2006, pp. 45–60.

[8] , *Fuzzy attribute logic over complete residuated lattices*, Journal of Experimental & Theoretical Artificial Intelligence 18 (2006), no. 4, 471–480.

[9] , *On proofs and rule of multiplication in fuzzy attribute logic*, Foundations of Fuzzy Logic and Soft Computing (P. Melin, O. Castillo, T. Aguilar, L. J. Kacprzyk, and W. Pedrycz, eds.), Lecture Notes in Computer Science, vol. 4529, Springer Berlin Heidelberg, 2007, pp. 471–480.
[10] ______, *Formal concept analysis and linguistic hedges*, International Journal of General Systems 41 (2012), no. 5, 503–532.

[11] ______, *Attribute dependencies for data with grades*, CoRR [abs/1402.2071](2014).

[12] Garrett Birkhoff, *Lattice theory*, 1st ed., American Mathematical Society, Providence, 1940.

[13] Roberto Cignoli, Francesc Esteva, and Lluís Godo, *On Łukasiewicz logic with truth constants*, Theoretical Advances and Applications of Fuzzy Logic and Soft Computing (Oscar Castillo, Patricia Melin, Oscar Montiel Ross, Roberto Sepúlveda Cruz, Witold Pedrycz, and Janusz Kacprzyk, eds.), Advances in Soft Computing, vol. 42, Springer Berlin Heidelberg, 2007, pp. 869–875.

[14] Petr Cintula, Petr Hájek, and Carles Noguera (eds.), *Handbook of Mathematical Fuzzy Logic, Volume 1*, Studies in Logic, Mathematical Logic and Foundations, vol. 37, College Publications, 2011.

[15] Petr Cintula, Petr Hájek, and Carles Noguera (eds.), *Handbook of Mathematical Fuzzy Logic, Volume 2*, Studies in Logic, Mathematical Logic and Foundations, vol. 38, College Publications, 2011.

[16] Pablo Cordero, Ángel Mora, Inmaculada Pérez de Guzmán, and Manuel Enciso, *Non-deterministic ideal operators: An adequate tool for formalization in data bases*, Discrete Applied Mathematics 156 (2008), no. 6, 911–923.

[17] Chris J. Date and Hugh Darwen, *Relational Database Writings 1989–1991*, ch. The Role of Functional Dependence in Query Decomposition, pp. 133–154, Addison-Wesley Publishing Co., Inc., 1992.

[18] Brian A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990.

[19] Francesc Esteva, Joan Gispert, Lluís Godo, and Carles Noguera, *Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results*, Fuzzy Sets and Systems 158 (2007), no. 6, 597–618.

[20] Francesc Esteva and Lluís Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems 124 (2001), no. 3, 271–288.

[21] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara, *Residuated fuzzy logics with an involutive negation*, Archive for Mathematical Logic 39 (2000), no. 2, 103–124.

[22] Francesc Esteva, Lluís Godo, and Carles Noguera, *First-order t-norm based fuzzy logics with truth-constants: Distinguished semantics and completeness properties*, Annals of Pure and Applied Logic 161 (2009), no. 2, 185–202.
[23] Francesc Esteva, Lluís Godo, and Carles Noguera, *A logical approach to fuzzy truth hedges*, Information Sciences 232 (2013), 366–385.

[24] Nikolaos Galatos, Peter Jipsen, Tomacz Kowalski, and Hiroakira Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151*, 1st ed., Elsevier Science, San Diego, USA, 2007.

[25] Bernhard Ganter and Rudolf Wille, *Formal concept analysis: Mathematical foundations*, 1st ed., Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1997.

[26] Giangiacomo Gerla, *Fuzzy Logic. Mathematical Tools for Approximate Reasoning*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[27] Joseph A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications 18 (1967), no. 1, 145–174.

[28] , *The logic of inexact concepts*, Synthese 19 (1979), 325–373.

[29] Siegfried Gottwald, *Mathematical fuzzy logics*, Bulletin of Symbolic Logic 14 (2008), no. 2, 210–239.

[30] Jean-Louis Guigues and Vincent Duquenne, *Familles minimales d’implications informatives resultant d’un tableau de données binaires*, Math. Sci. Humaines 95 (1986), 5–18.

[31] Petr Hájek, *Fuzzy logic and arithmetical hierarchy*, Fuzzy Sets and Systems 73 (1995), no. 3, 359–363.

[32] , *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[33] , *On very true*, Fuzzy Sets and Systems 124 (2001), no. 3, 329–333.

[34] Petr Hájek and Jeff Paris, *A dialogue on fuzzy logic*, Soft Computing 1 (1997), no. 1, 3–5.

[35] Richard Holzer, *Knowledge acquisition under incomplete knowledge using methods from formal concept analysis: Part I*, Fundamenta Informaticae 63 (2004), no. 1, 17–39.

[36] Erich Peter Klement, Radko Mesiar, and Endre Pap, *Triangular Norms*, 1 ed., Springer, 2000.

[37] Tomas Kuhr and Vilem Vychodil, *Fuzzy logic programming reduced to reasoning with attribute implications*, Fuzzy Sets and Systems 262 (2015), 1–20.

[38] John W. Lloyd, *Foundations of Logic Programming*, Springer-Verlag New York, Inc., New York, NY, USA, 1984.

[39] David Maier, *Minimum covers in relational database model*, J. ACM 27 (1980), no. 4, 664–674.
[40] , Theory of Relational Databases, Computer Science Pr, Rockville, MD, USA, 1983.

[41] Vilém Novák, IrinaPerfilieva, and Jiří Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer Academic Publishers, Boston, MA, USA, 1999.

[42] Jan Pavelka, On fuzzy logic I: Many-valued rules of inference, Mathematical Logic Quarterly 25 (1979), no. 3–6, 45–52.

[43] On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi, Mathematical Logic Quarterly 25 (1979), no. 7–12, 119–134.

[44] On fuzzy logic III: Semantical completeness of some many-valued propositional calculi, Mathematical Logic Quarterly 25 (1979), no. 25–29, 447–464.

[45] Silke Pollandt, Fuzzy-Begriffe: Formale Begriffsanalyse unscharfer Daten, Springer, 1997.

[46] George N. Raney, Completely distributive complete lattices, Proc. Amer. Math. Soc. 3 (1952), 677–680.

[47] Ricardo Oscar Rodríguez, Francesc Esteva, Pere García, and Lluís Godo, On implicative closure operators in approximate reasoning, International Journal of Approximate Reasoning 33 (2003), no. 2, 159–184.

[48] Petr Sávický, Roberto Cignoli, Francesc Esteva, Lluís Godo, and Carles Noguera, On product logic with truth-constants, Journal of Logic and Computation 16 (2006), no. 2, 205–225.

[49] Gaisi Takeuti and Satoko Titani, Globalization of intuitionistic set theory, Annals of Pure and Applied Logic 33 (1987), 195–211.

[50] Vilem Vychodil, Fuzzy attribute implications and their expressive power, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 21 (2013), no. 4, 483–496.

[51] Parameterizing the semantics of fuzzy attribute implications by systems of isotone Galois connections, CoRR abs/1410.6960 (2014).

[52] Morgan Ward and Robert P. Dilworth, Residuated lattices, Trans. Amer. Math. Soc. 45 (1939), 335–354.

[53] Mohammed J. Zaki, Mining non-redundant association rules, Data Mining and Knowledge Discovery 9 (2004), 223–248.