Current understanding of the pseudospin symmetry in atomic nuclei

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Abstract. We use the relativistic mean field framework to analyse the reliability of the explanation of the pseudospin symmetry (PSS) that has been accepted, quite generally, by the scientific community, in the last decade. We make a comparative analysis of the mechanisms responsible for the breaking of the spin and pseudospin symmetries that shows the different nature of these symmetries. We propose an explanation of the PSS, also valid in the non-relativistic limit, in which the effect of the deviation of the single-particle central potential from a harmonic oscillator on the breaking of the degeneracy of pseudospin doublets is partially compensated by the effect of the spin-orbit interaction.

1. Introduction

1.1. The pseudospin symmetry

When one observes the single-particle (SP) energy spectrum of the shell model, one can distinguish couples of states with close energy values. In figure 1, they are represented for neutrons with thicker lines. By analogy with the spin-orbit formalism, each pair of these states is defined as a pseudospin doublet (PSD). Thus, about forty years ago an attempt has been made to introduce the pseudospin (PS) and the pseudo-orbital angular momentum $l$ as new dynamical variables, instead of those used in the conventional shell model, to classify the SP energy levels in nuclei \cite{1, 2}. The main purpose of this classification was to get a small magnitude of the “new” effective pseudospin-orbit force. The quantum number $\tilde{l}$ is defined as

$$\tilde{l} = (2j - l) = \begin{cases} l + 1, & \text{for } j = l + 1/2; \\ l - 1, & \text{for } j = l - 1/2. \end{cases} \quad (1)$$

If we consider the two states “a” and “b” of a PSD, they can be labelled by the quantum numbers $n_a$, $l_a$, $j_a = l_a + 1/2$ and $n_b = n_a - 1$, $l_b = l_a + 2$, $j_b = l_a + 3/2$, where $n$, $l$, and $j$ are the SP radial, orbital and total angular momentum quantum numbers, respectively. Thus, $\tilde{l}_a = \tilde{l}_b = l_a + 1$. If the pseudospin symmetry (PSS) were exact, the two states $a$ and $b$ would be degenerate. Similarly, we shall consider a spin doublet (SD) shows spin symmetry (SS) when the two respective partners have the same energy.
**Figure 1.** Qualitative evolution of the neutron energy levels from the shell model taking for the central potential, successively, a harmonic oscillator and a Woods-Saxon potential and no spin-orbit interaction (we use standard notation). The NL-SH column indicates qualitative results of the Dirac-Hartree approximation with the NL-SH force [31]. Notice that, in the relativistic case, the single-particle energy does not increase linearly with \( N \) for a harmonic oscillator potential.
In the last decade, it has been established that the PSS has its origin in the relativistic symmetry of the Dirac equation [3]-[15], claiming that is not possible to explain this symmetry, properly, in the non-relativistic framework. Thus, in the recent years, many authors have studied also the PSS using the relativistic formalism either based on a Dirac equation with appropriate potentials fitted to reproduce bulk properties of finite nuclei [13]-[15] or through the Hartree [16]-[23] and Hartree-Fock [24]-[26] relativistic approximations.

In spite of the claimed important progress made in understanding the PSS during the last decade, there are still essential points of controversy, mainly related to the effect of the PSS breaking term and the sufficient conditions for the PSS in finite nuclei, between the different groups working in this field [18]-[25]. To give a concrete example of the doubts that some physicists have on the explanation of the PSS based on relativistic grounds, we reproduce here a few lines from the paper [28] by J. Piekarewicz “notice that while the qualitative nature of the pseudospin symmetry emerges naturally in the relativistic approach, it is not without subtleties. For example, in the limit of exact pseudospin symmetry the nucleons spectrum is unbound. Thus, a realistic description of nuclei necessarily requires the breaking of the symmetry. Moreover, as nucleons become bound, the pseudospin-orbit potential becomes singular. As a result, great care is required in dealing with such a potential and useful insights into this subtle symmetry will undoubtedly continue to emerge.”

The aim of this work is to analyse, in the framework of the Dirac Hartree approximation (DHA), the explanation of the PSS given in the last decade. To this end, we study the effect of the PSS breaking term and the reliability of the sufficient conditions proposed to insure an approximate PSS in finite nuclei, making a comparative analysis with the corresponding situation in relation with the SS.

1.2. Dirac Hartree framework and basic equations for the spin and pseudospin symmetries

In the DHA (the tensor contribution of the vector mesons being ignored), the SP states are obtained from a Dirac equation that can be written as

\[ \left[ -i\alpha \cdot \nabla + \beta(M + \Sigma) + \Sigma_0 \right] \psi(\vec{r}) = E \psi(\vec{r}), \]  

(2)

where, \( E = M + \epsilon \) is the relativistic energy, \( M \) is the nucleon mass, \( \Sigma \) is the scalar self-energy coming from the scalar \( \sigma \) meson and \( \Sigma_0 \) is the time component of the vector self-energy coming from the vector \( \omega \) and \( \rho \) mesons and the Coulomb field [29, 30]. For spherical nuclei (to which we restrict ourselves in this work), the nucleon Dirac spinor \( \psi(\vec{r}) \) can be written, in standard notation, as

\[ \psi(\vec{r}) = \frac{1}{r} \begin{pmatrix} iG(r)y_{\ell j}(\theta, \phi) \\ F(r)y_{\ell j}(\theta, \phi) \end{pmatrix}, \]  

(3)

where \( \frac{G(\vec{r})}{r} \) and \( \frac{F(\vec{r})}{r} \) represent the radial parts of its big and small components, respectively. As noticed in [3], \( \tilde{l} = l \pm 1 \) appears in the small component of the spinor.

By substituting eq. (3) into eq. (2), one can obtain the following Dirac equation for the \( G \) and \( F \) components:

\[ \frac{d}{dr} G(r) = -\frac{\kappa}{r} G(r) + W F(r), \]  

(4)

\[ \frac{d}{dr} F(r) = V G(r) + \frac{\kappa}{r} F(r). \]

In this equation,

\[ \kappa = (2j + 1)(l - j) = j(j + 1) - \tilde{l}(\tilde{l} + 1) + 1/4, \]  

(5)
whereas

\[ V \equiv \Sigma_S + \Sigma_0 - \epsilon, \quad W \equiv 2M + \Sigma_S - \Sigma_0 + \epsilon \]  

are energy dependent potentials.

From the Dirac equation (4), one can get the two following equivalent second order differential equations for the \( G \) and \( F \) components of the nucleon Dirac spinor:

\[ -G'' + \left[ \frac{W'}{W} \left( \frac{G'}{G} + \frac{\kappa}{r} \right) + \frac{l(l+1)}{r^2} + V \right] G = 0, \]  

(7)

\[ -F'' + \left[ \frac{V'}{V} \left( \frac{F'}{F} - \frac{\kappa}{r} \right) + \frac{\tilde{l}(\tilde{l}+1)}{r^2} + VW \right] F = 0, \]  

(8)

where the prime stands for the radial derivative. The following relations

\[ l(l+1) = \kappa(\kappa+1) \]  

(9)

and

\[ \tilde{l}(\tilde{l}+1) = \kappa(\kappa-1) \]  

(10)

hold, in accordance with eq. (5), and the quantity

\[ VW = 2MV + 2\epsilon\Sigma_0 + (\Sigma_S^2 - \Sigma_0^2) - \epsilon^2 \]  

(11)

represents an effective state-dependent central potential (multiplied by \( 2M \)). The quantity \( \Sigma_S + \Sigma_0 \) from \( VW \) in eqs. (7) and (8) is the essential part of the central potential, whereas the quantity \( \Sigma_S - \Sigma_0 \) from \( W \), through the term entering \( W'\kappa \), determines the spin-orbit (SO) potential in eq. (7).

### 1.3. Breaking mechanisms for spin and pseudospin symmetries

The solutions of eq. (7) with the same number of nodes (\( n \)) of \( G \) and the same value of \( l \) form a SD, whereas the solutions of eq. (8) with the same number of nodes (\( \tilde{n} \)) of \( F \) and the same value of \( \tilde{l} \) form a PSD. As \( \kappa \) depends on \( j \), the \( \kappa \) terms entering equations (7) and (8), to which we shall refer to as the \( G - \kappa \) and \( F - \kappa \) terms, respectively, are, formally, responsible for the splitting of the SDs and the PSDs, i.e., they break the SS and the PSS. Actually, in this latter case, the splitting of the PSDs can be avoided by an appropriate choice of the self-energies \( \Sigma_S \) and \( \Sigma_0 \) [22, 23].

From eqs. (7) and (8), it is evident that the \( G - \kappa \) and \( F - \kappa \) terms are also responsible for the breaking of the relations \( G_a \propto G_b \) and \( F_{a'} \propto F_{b'} \) for the two partner of the SDs (\( a, b \)) and PSDs (\( a', b' \)), respectively. However, if \( W \) is constant \( W'W = 0 \) (there is no SO interaction) and the SS is restored with \( G_a \propto G_b \) (we shall denote this particular form of SS as SS*). Similarly, if \( V \) is constant \( V'V = 0 \) and the PSS is restored with \( F_{a'} \propto F_{b'} \) (we shall denote this particular form of PSS as PSS*).

Although eqs. (7) and (8) look very similar, there is an essential difference between them. Indeed, \( V(r) \) becomes zero at some point (\( r_0 \)) in the nuclear surface, so that, for \( r \to r_0 \), \( V'/V \propto [(r-r_0)r]^{-1} \) and the \( F - \kappa \) term, \( \frac{\kappa}{r^2}F \), in equation (8) is singular at \( r = r_0 \).
At present, the PSS is considered slightly broken in nuclei due to some of the following hypotheses:
1) The magnitude of $\Sigma_S + \Sigma_0$ is small [3]-[7],
2) The $F - \kappa$ term is small [8]-[11], or
3) The different contributions to the energy in equation (8) partially compensate each other [14], [18]-[26].

In Sect. 2-5, we analyse the hypotheses 1) and 2), showing that they present very serious problems to justify the approximate PSS observed in heavy nuclei, and we compare the spin and pseudospin symmetries to shed new light for explaining the PSS. In Sect. 6, we give a simple explanation of the PSS, which is valid for both the relativistic and non-relativistic approximations. Finally, the conclusions are summarized in Sect. 7.

2. Pseudospin symmetry and the magnitude of $\Sigma_S + \Sigma_0$
In what follows, we shall designate by “a” and “b” the two states of a SD or PSD with $\kappa < 0$ and $\kappa > 0$, respectively. As shown in ref. [4], in the limit $\Sigma_S + \Sigma_0 = 0$, two PS partners, $a$ and $b$, have the same energy and $F_a$ and $F_b$ are identical up to a phase (from eq. (8) one can immediately see that $F_a \propto F_b$), i.e., we have exact PSS* (in references [3]-[11] only the PSS* has been considered). But the condition $\Sigma_S + \Sigma_0 = 0$ does not allow bound states (except for models with unrealistic values of $\Sigma_S - \Sigma_0$).

In the inner part of real nuclei, $\Sigma_S + \Sigma_0 \approx -60$ MeV and can be considered small (in relation to the nucleon mass $M$ or $|\Sigma_S - \Sigma_0|$) but it is not zero. Then, in references [3]-[7], it is claimed that approximate PSS*, i.e., $\epsilon_a \simeq \epsilon_b$ and $F_a \approx F_b$, can be expected on the ground of the smallness of $\Sigma_S + \Sigma_0$. However, one can see that this cannot be the main reason for the observed approximate PSS* in nuclei considering, for example, the solution of a Dirac equation for the single-nucleon states with appropriate values of $\Sigma_S + \Sigma_0$ but taking $\Sigma_S - \Sigma_0 = 0$ (i.e., the SO interaction is neglected). Under these conditions, it is clear that the PSDs (mainly those with $\tilde{n} < 3$) do not satisfy the property of quasi-degeneracy imposed by the PSS, in spite that $|\Sigma_S + \Sigma_0|$ is realistic (see, for example, figure 1 or any other shell model scheme obtained in the non-relativistic framework without SO interaction). This argument and the apparent effect of the potential $\Sigma_S - \Sigma_0$ in figure 1 also suggest that the SO interaction plays an essential role in the achievement of the quasi-degeneracy of the PSDs in the nuclear shell model.

To understand better the effect of the SO interaction, we have presented in table 1 the SP energies of the neutron PSDs of the $^{208}Pb$ nucleus with the DHA (as explained in the table caption) for two intensities of the SO potential: for the realistic one (column $\epsilon$) and for other choice with a stronger strength than the physical one (column $\epsilon^*$). These results show that the condition of quasi-degeneracy of the PSDs, $\epsilon_a \simeq \epsilon_b$, for the PSS (or PSS*) is strongly improved as the magnitude of the potential $\Sigma_S - \Sigma_0$ is considerably increased with respect to the realistic values. For the realistic case, it can be seen that, as the $\tilde{n}$ number of a PSD increases, the relation $\epsilon_a \simeq \epsilon_b$ and, consequently, the PSS are satisfied better, but PSDs satisfying $\epsilon_a \simeq \epsilon_b$ with a small value of $\tilde{n}$ can be found as well [18]-[21]. Moreover, in ref. [20], we have also found that the relation $F_a \approx F_b$ (and, consequently, the PSS*) improves, without exception, as $\tilde{n}$ increases. We have mainly attributed this fact to a more important role of the term $F''$ in eq. (11), though $F_a$ and $F_b$ always differ from each other considerably near the singularity point $r_0$ (see figures 2 and 3).

However, we have verified that, if we multiply the central potential $\Sigma_S + \Sigma_0$ in the Dirac eq. (5) by a reduction factor $(RF < 1)$, neither the PSS nor the PSS* are, necessarily, improved [21]. This conclusion can be extracted from figure 4, where we show the energies of the neutron PSD of the $^{40}Ca$ nucleus as a function of $RF$. This result is not an exception and all PSDs that become degenerate ($\epsilon_a = \epsilon_b$) for a given magnitude of the quantity $\Sigma_S + \Sigma_0$ split if this
This result was unexpected by those authors that have been explained the approximate PSS (or PSS* or PSS) obtained increasing the spin-orbit interaction by replacement of the self-consistent values of $\Sigma_S - \Sigma_0$ by $2 \times (\Sigma_S - \Sigma_0)$ in $W$ and making the transformation $r \rightarrow 1.4 \times r$ to get a single-particle spectrum with a density of levels similar to the self-consistent one ($\epsilon$). Energies are given in MeV.

| PSD | $\hat{n}$ | state | $\ell$ | $\kappa$ | $\varepsilon(\kappa)$ | $\varepsilon$ | $\varepsilon^*$ | $\varepsilon^{(\text{exp})}$ |
|-----|---------|-------|-----|-----|-------------|-------------|-------------|----------------|
| 1   | 2       | $2s_{1/2}$ (a) | 1   | -1  | 0.03        | -41.42      | -38.29      | -               |
|     |         | $1d_{3/2}$ (b) | 1   | 2   | -0.46       | -44.67      | -40.43      | -               |
| 2   | 2       | $2p_{3/2}$ (a) | 2   | -2  | -0.31       | -30.91      | -27.61      | -               |
|     |         | $1f_{5/2}$ (b) | 2   | 3   | 0.13        | -35.16      | -29.54      | -               |
| 3   | 2       | $2d_{5/2}$ (a) | 3   | -3  | -1.12       | -20.62      | -18.18      | -               |
|     |         | $1g_{7/2}$ (b) | 3   | 4   | 0.60        | -24.75      | -18.94      | -               |
| 4   | 3       | $3p_{1/2}$ (a) | 1   | -1  | -0.30       | -17.96      | -14.68      | -               |
|     |         | $2d_{5/2}$ (b) | 1   | 2   | 0.59        | -18.87      | -14.67      | -               |
| 5   | 2       | $2f_{7/2}$ (a) | 4   | -4  | -2.50       | -10.80      | -10.52      | -10.1          |
|     |         | $1h_{9/2}$ (b) | 4   | 5   | 1.66        | -13.88      | -9.90       | -10.8          |
| 6   | 3       | $3p_{3/2}$ (a) | 2   | -2  | -0.85       | -7.85       | -7.17       | -8.3           |
|     |         | $2f_{5/2}$ (b) | 2   | 3   | 1.23        | -8.68       | -7.14       | -7.9           |

magnitude changes, in particular, when it decreases (it is just what happens with the PSDs 4 and 6 in the column $\epsilon^*$ of table 1). Thus, the relation $\epsilon_a \simeq \epsilon_b$ can get worse as $|\Sigma_S + \Sigma_0|$ decreases. Furthermore, $F_a$ becomes very different from $F_b$ as the magnitude of $\Sigma_S + \Sigma_0$ decreases when $\epsilon_a$ or $\epsilon_b$ is close to the continuum. This is due to the fact that both states, except in very particular cases, approach the continuum, as functions of the magnitude of $\Sigma_S + \Sigma_0$, differently. Then, when a state of a PSD becomes unbound, its $F(r)$ component spreads all over the space, whereas its partner remains bound with a localized wave function.

We have also realized that, when $\Sigma_S + \Sigma_0$ is a Coulomb potential and $|\Sigma_S + \Sigma_0| \rightarrow 0$, $\epsilon_a$ and $\epsilon_b$ do not approach each other faster than the energies of other couples of states (not forming a PSD) do, $F_a$ and $F_b$ remaining also different in this limit. The different result found firstly in [3] was due to an error made by the author in the interpretation of the principal quantum number [25]. Thus, the exact PSS obtained for $\Sigma_S + \Sigma_0 = 0$ does not ensure, in general, that for $|\Sigma_S + \Sigma_0|$ small we could expect approximate PSS. These arguments indicate that neither the PSS nor the PSS* can be based on the smallness of $|\Sigma_S + \Sigma_0|$.

3. Spin symmetry and the magnitude of $\Sigma_S - \Sigma_0$

In the limit $\Sigma_S - \Sigma_0 = 0$, two states $a$ and $b$ of a SD have the same energy $\epsilon_a = \epsilon_b$ (i.e., the nucleus exhibits exact SS) and, moreover, it is evident from eq. (7) that $G_a \propto G_b$ (i.e., the nucleus satisfies the condition for exact SS*). In real nuclei, $|\Sigma_S - \Sigma_0|$ is large, therefore, in this case, no good approximate SS or SS* has been expected [6]. In fact, nuclei have large SO splittings. However, as suggested by the almost perturbative character of the $G - \kappa$ term predicted in [20] and confirmed in [21], the $G$ functions of the spin partners are very similar\(^1\)

\(^1\) This result was unexpected by those authors that have been explained the approximate PSS (or PSS*) on the base of the smallness of $|\Sigma_S + \Sigma_0|$, by analogy to the large SO splittings due to the large magnitude of $|\Sigma_S - \Sigma_0|$ (see, for example, ref. [6]).
Figure 2. The small components $F(r)$ of the Dirac spinor for the two neutron states of the $(2s_{1/2}, 1d_{3/2})$ PSD of the $^{208}$Pb nucleus with the NL-SH set [31] (these results are taken from ref. [21]).

Figure 3. The same as figure 2 but for the $(3s_{1/2}, 2d_{3/2})$ PSD.
Figure 4. The single-particle energies $\epsilon_a$ and $\epsilon_b$ of the two states $a = 2s_{1/2}$ and $b = 1d_{3/2}$ of the neutron PSD of the $^{40}$Ca nucleus and the difference $\epsilon_a - \epsilon_b$ as a function of the reduction factor $RF$ of $\Sigma_S + \Sigma_0$ in the potential $V$ for the NL-SH set [31] (these results are taken from ref. [21]). (actually, much more similar than the small components $F$ of the PS partners [21, 6], as can be seen in figures 5 and 6). Moreover, if $|\Sigma_S - \Sigma_0|$ decreases, the SO splittings decrease and the $G$ functions of a SD become more similar, i.e., both the SS and the SS* are progressively improved because, the SS and the SS* are, actually, identical. These results indicate that the relation between the SS or SS* and $|\Sigma_S - \Sigma_0|$ is very different to the relation between the PSS or PSS* and $|\Sigma_S + \Sigma_0|$, due to the different behaviour of the $G - \kappa$ and $F - \kappa$ terms in equations (7) and (8), respectively.

4. Pseudospin symmetry and the $F - \kappa$ term

As we have explained above, the $F - \kappa$ term breaks the PSS* but not, necessarily, the PSS. The PSS has been considered slightly broken in nuclei by several authors [8]-[11] because the $F - \kappa$ term is small. More precisely, the condition required is that the magnitude of the $F - \kappa$ term should be much smaller than that of the pseudocentrifugal barrier (PCB) $\times F$, i.e.:

$$\left| \frac{V'_{\kappa}}{V} \right| \ll \frac{\hat{l}(\hat{l} + 1)}{r^2}.$$  

(12)

In this case, one can expect that the energy difference between two PS partners, which have the same value of $\hat{l}$, be much smaller than the difference of energies between two states with different values of $\hat{l}$ that, consequently, belong to different PSDs. The condition given by eq. (12) has been considered less restrictive than the relation $\Sigma_S \approx -\Sigma_0$ (or $\Sigma_S + \Sigma_0 \approx C_{ps} = constant$).

As $V(r_0) = 0$, the $F - \kappa$ term is large around $r_0$ and, consequently, $F_a$ and $F_b$ differ mostly around $r_0$ (see figures 2 and 3). It is remarkable that the $F - \kappa$ term together with the one originated from the PCB control the behaviour of the $F$ functions in the nuclear surface, both terms being essential. In fact, the relation between $\hat{l}$ and $\kappa$ given by eq. (10) must be exactly fulfilled to get physical $F(r)$ functions (if this relation is slightly violated, the continuous solutions
Figure 5. The big components $G(r)$ of the Dirac spinor for the two neutron states of the $(2f_{7/2}, 2f_{5/2})$ SD of the $^{208}$Pb nucleus with the NL-SH set [31] (these results are taken from ref. [21]).

Figure 6. The same as figure 4 but for the $(3p_{3/2}, 3p_{1/2})$ SD.
of eq. (8), $F(r)$, with a continuos derivative, become zero at the singularity point $r_0$ [20]-[22]). Then, the $F - \kappa$ term should not be considered much smaller than the PCB, in contrast to the conclusions established in references [8]-[11].

However, the $F - \kappa$ term is roughly an odd function around $r_0$. This allows a small PS splitting being $F_a$ and $F_b$ quite different. In fact, the property $\epsilon_a = \epsilon_b$, within a realistic DHA, needs the condition $F_a \neq F_b$, since, if $F_a(r) = F_b(r)$, the corresponding PS splitting would be due just to the contribution of the $F - \kappa$ term, which is, necessary, different from zero (the exact PSS* is forbidden in this case). Then, we have quasi-degenerate PSDs, not because the $F - \kappa$ term is small, but due to the compensation of its contribution by those coming from the other terms entering eq. (8) different from the $F - \kappa$ term, the precise balance of this compensation depending on the $\Sigma_S$ and $\Sigma_0$ self-energies [19]-[21]. We can say the PSS is an accidental symmetry for these reasons.

Table 1 shows that the contribution of the $F - \kappa$ term to the splittings of the PSDs is not very large, except for the 5th PSD. However, it is surprising that this contribution, except for the 1st PSD, has a sign opposite to that of the splitting itself. This is due to the effect that the $F - \kappa$ term produces on $F$. It is large enough to compensate the contribution of the $F - \kappa$ term to $\epsilon$ and, furthermore, to generate the PSDs splittings [19, 21, 24].

5. Spin symmetry and the $G - \kappa$ term
The $G - \kappa$ term breaks the SS and, of course, the SS*. Since $W'/W \approx (\Sigma_S' - \Sigma_0')/2M$, the $G - \kappa$ term can be considered as a “small” relativistic term. In fact, the relation $G_a \approx G_b$ is very well satisfied in all cases, improving also as the number of nodes of $G_{a,b}$ increases (see figures 5 and 6) and, though the SO splittings are large, the $G - \kappa$ term behaves almost in a perturbative way. We can conclude that the $G - \kappa$ term, which breaks the condition $G_a = G_b$ (and, consequently, the SS*) and is considered large, produces a smaller effect in nuclei than the $F - \kappa$ term, which breaks the condition $F_a = F_b$ (and, consequently, the PSS*) and has been considered small by some groups [8]-[11].

6. Pseudospin symmetry as a result of compensation of different effects
6.1. Original concept of the pseudospin symmetry
We have seen that the conditions $\epsilon_a \simeq \epsilon_b$ and $F_a \approx F_b$, for the two states of a PSD, are not strictly dependent on each other. We can use this fact as an argument to recover the original concept of the PSS that grounds on the degeneracy of the PSDs, no matter whether $F_a$ and $F_b$ are proportional each other or not. That is why we have proposed in refs. [22, 23] an explanation of the PSS in real nuclei starting from a situation of exact degeneracy of the PSDs, being $F_a$ and $F_b$ not proportional to each other. In the starting model, $\Sigma_S - \Sigma_0 = 0$ (i.e., there is no SO interaction), whereas $\Sigma_S + \Sigma_0$ is a harmonic oscillator potential. This model, defined by the Hamiltonian $H_0$, produces degenerate SDs with $G_a \propto G_b$ and degenerate PSDs but with $F_a$ not proportional to $F_b$ [15]. Then, although a more realistic quantity $\Sigma_S + \Sigma_0$ breaks the degeneracy of the PSDs, the choice of an appropriate potential $\Sigma_S - \Sigma_0 \neq 0$ partially restores the quasidegeneracy of the PSDs. Thus, we have a continuous (almost perturbative) way that allows to connect the model satisfying exact degeneracy of the PSDs with realistic models of nuclei fulfilling approximate PSS.

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2 This transition is made in two steps: a) the change of the oscillator harmonic potential to a more realistic one and b) the inclusion of the SO potential. Both steps can be realized in an almost perturbative way (we have realized above that this is the case for the SO potential).
6.2. Pseudospin symmetry breaking term

Since the consideration of the $F - \kappa$ term as the PSS breaking term requires the two conditions $\epsilon_a \simeq \epsilon_b$ and $F_a \approx F_b$ to be directly dependent on each other, our present form of understanding the PSS indicates that the $F - \kappa$ term is not the appropriate choice for the PSS breaking term in this model. Let us see how we can define an appropriate breaking term in accordance with the present interpretation of the PSS. Let us write the SP Hamiltonian corresponding to eq. (2) as $H = H_0 + H_1$, where $H_0$ exhibits exact degeneracy of PSDs and $H_1$ represents the corresponding breaking term.

If the Dirac equation corresponding to $H$ is given by eq. (4), and we write $W = W_0 + W_1$, $V = V_0 + V_1$ and $\epsilon = \epsilon_0 + \epsilon_1$, where $W_0$, $V_0$ and $\epsilon_0$ represent, respectively, the potentials entering eq. (4) and the SP energy corresponding to $H_0$, we have $W_0 = 2M + \epsilon_0$, $V_0 = \Omega - \epsilon_0$, $\Omega$ being an appropriate harmonic oscillator potential for a given nucleus; $W_1 = \Sigma_S - \Sigma_0 + \epsilon_1$ and $V_1 = \Sigma_S + \Sigma_0 - \Omega - \epsilon_1$.

From the Dirac eq. (4) or from eq. (8), we get the following equivalent equation for the small component $F$:

$$-F'' + \left[ \frac{V_0' + V_1'}{V_0 + V_1} \left( \frac{F''}{F} - \frac{\kappa}{r} \right) + \frac{\tilde{I}(\tilde{I} + 1)}{r^2} + (V_0 + V_1)(W_0 + W_1) \right] F = 0,$$

(13)

where the $F - \kappa$ term breaks, explicitly, the relation $F_a \propto F_b$, even if $V_1 = W_1 = 0$. However, in this case, we have $\epsilon_0 = \epsilon_0$ [15] (indicating that the $F - \kappa$ term is not the appropriate PSS breaking term). The fact that the $F - \kappa$ term is not a perturbative term facilitates this equality $^3$.

The self-consistency effects due to the $F - \kappa$ term, which breaks the proportionality between $F_a$ and $F_b$, allow to compensate, exactly, the contribution of the $F - \kappa$ term to the pseudospin-orbit splittings. It is important to remind that this situation of exact degeneracy of the PSDs is compatible with bound nuclei, on the contrary to what happens with the exact PSS considered in refs. [3]-[7]. When the potentials $V_1$ and $W_1$ entering eq. (13) are considered, their effects are almost perturbative and the real nuclei with broken PSS are recovered.

6.3. Condition $F_a \approx F_b$ for the two pseudospin partners

The model can explain the relation $\epsilon_a \simeq \epsilon_b$ for the pseudospin partners, but still the question remains, how can we explain the relation $F_a \approx F_b$ found also in relativistic calculations for PSDs with $\tilde{n} \geq 3$? The similarity between $F_a$ and $F_b$ for the two states of a PSD in the inner region of the nuclei can be explained, independently from $\epsilon_a \simeq \epsilon_b$, because the effects of the $F - \kappa$ term, responsible for the lack of proportionality between $F_a$ and $F_b$, are mainly important in the nuclear surface due to the singularity of the $F - \kappa$ term at $r_0$. Furthermore, since the role of the terms containing $F'$ and, mainly, $F''$ in eqs. (8) or (13) increases with the number of nodes $\tilde{n}$ of $F$ [21], $F_a$ and $F_b$ become more similar as $\tilde{n}$ increases and, generally, the same happens with $\epsilon_a$ and $\epsilon_b$ (although the relation $F_a \approx F_b$ cannot justify the small splitting of PSDs found in calculations [19, 21]).

The improvement of the relation $\epsilon_a \simeq \epsilon_b$ with $\tilde{n}$ can be understood taking into account that: (A) the splitting of the PSDs produced by the deviation of $\Sigma_S + \Sigma_0$ from a harmonic oscillator potential (i.e. by $V_1$) decreases with $\tilde{n}$ inside a major shell (see figure 1), and (B) the SO interaction in realistic relativistic models, which partially compensates the effect of $V_1$, is not strong enough to approach the two states of the PSDs with $\tilde{n} \leq 2$ what would be necessary to reach a relation $\epsilon_a \simeq \epsilon_b$ so good as for the PSDs with $\tilde{n} \geq 3$. The results of table 1 show that when the strength of the SO interaction is increased (column $\epsilon^*$), the PSS is improved, mainly for the PSDs with $\tilde{n} = 2$ (which, for $\Sigma_S - \Sigma_0 = 0$, show sizable PS orbit splittings).

$^3$ Notice that our present form of understanding the PSS is based on the fact that the $F - \kappa$ term is large rather than small. The case considered here supplies also an example of the independence between the two relations $\epsilon_{a0} = \epsilon_{b0}$ and $F_a \propto F_b$. 
6.4. Pseudospin symmetry in the non-relativistic framework

It is important to notice that the explanation for the PSS given in this section is not based on the relativistic character of the model used. It can be formulated in similar terms for the non-relativistic models, starting from the non-relativistic SP harmonic oscillator potential without SO interaction. This model produces also degenerate SDs and PSDs. When a more realistic SP potential is used, the degeneracy of the PSDs is broken, but it is partially restored, as in the relativistic case, if an adequate SO interaction is added by hand. Thus, the PSS can be explained in a simple and similar way in the relativistic and non-relativistic approximations, although the SO interaction, which plays an important role in this explanation, is more naturally incorporated in the relativistic models. It is important that the explanation of the PSS given in the relativistic framework be valid also in the the non-relativistic case, since, for realistic models, the quantities $V_1$ and $W_1$ cannot be neglected in the non-relativistic limit.

7. Conclusions

Our conclusions can be summarized in the following points:

1.- The PSS in nuclei is not due to the smallness (or flatness) of the central potential $\Sigma_S + \Sigma_0$ or to a small $F - \kappa$ term, but, rather, it is related to the partial compensation of the contributions of different terms entering eq. (8). This compensation is possible because the $F - \kappa$ term, which diverges in the nuclear surface, produces effects by any means perturbative, in particular, in the $F$ wave functions of the Dirac spinors. In this compensation, the SO interaction plays an important role, since, partially, restores the PSS, which is broken due to the deviation of the real $\Sigma_S + \Sigma_0$ from a harmonic oscillator potential. It is important to remark, that the mechanisms allowing the compensation, which is crucial to explain the PSS, do not have an exclusively relativistic character, except in what it concerns the SO interaction (if one considers that this interaction has a particular relativistic character).

2.- The $F - \kappa$ term, although "large", is the appropriate breaking term for the PSS* but not for the PSS (as implicitly indicates conclusion 1).

3.- Though the $F - \kappa$ term is not small, the PSS and the PSS* are improved as $\bar{n}$ increases due, in particular, to an increasing role of the $F^{\prime\prime}$ term in eq. (8) with the increase of $\bar{n}$. Thus, the PSS and the PSS* (with essential differences between $F_a$ and $F_b$ in the nuclear surface) show up themselves more often in the PSDs of heavy nuclei.

4.- The PSS is favoured because the $F - \kappa$ term is roughly an odd function around its singularity point $r_0$, whereas the relation $F_a \approx F_b$ is favoured by the flatness of $\Sigma_S + \Sigma_0$ inside the nucleus.

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Notice that the $F - \kappa$ term cannot be considered either as a genuine relativistic term because it cannot be neglected, in eqs. (8) or (13) for the function $F$, as the nucleon mass $M$ becomes very large.
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