Viability of the Loop Quantum Gravity kinematics in Palatini $f(R)$ models

Flavio Bombacigno,\textsuperscript{1,} Simon Boudet,\textsuperscript{1,2} and Giovanni Montani\textsuperscript{1,3}

\textsuperscript{1}Physics Department, “Sapienza” University of Rome, P.le Aldo Moro 5, 00185 (Roma), Italy
\textsuperscript{2}Department of Physics, University of Trento, Via Sommarive 14, 38123 (Trento), Italy
\textsuperscript{3}ENEA, Fusion and Nuclear Safety Department, C. R. Frascati, Via E. Fermi 45, 00044 Frascati (Roma), Italy

We consider special classes of Palatini $f(R)$ theories, featured by additional Loop Quantum Gravity inspired terms, with the aim of testing the feasibility of a loop-like quantization procedure. In particular, by adopting a scalar-tensor representation, we successfully identify a set of modified Ashtekar canonical variables, preserving the $SU(2)$ gauge structure of the standard theory by means of a novel Gauss constraint. We discuss in detail the dynamical role of the additional scalar field affecting these models, stressing its effects on the area operator stemming from such a revised theoretical framework. Especially, we show how a canonical rescaling of the scalar field could absorb the Immirzi parameter in the spectrum, eliminating it from the kinematics. We perform then the analysis in the Einstein frame, and we compare our results with earlier studies in literature, outlining possible differences between metric and Palatini approaches.

I. INTRODUCTION

The most successful approach to the problem of quantizing gravity is, up to now, the so called Loop Quantum Gravity (LQG) theory\textsuperscript{[1,3]}. This formulation, of course, still contains a number of non-trivial and unaddressed questions, as for instance the implementation of the quantum dynamics via the scalar constraint, the construction of a classical limit and the intrinsic ambiguity of the appropriate phase space variable to be adopted, i.e. the ambiguity in the meaning and value of the so-called Immirzi parameter\textsuperscript{[4,11]}. Nonetheless, the great interest for LQG is justified by the possibility to construct a kinematic Hilbert space for the quantum theory, resulting in geometrical operators, like areas and volumes, endowed with discrete spectra\textsuperscript{[12,14]}. By other words, LQG is able to introduce the concept of space discretization, simply starting from a classical Lagrangian for the gravitational field\textsuperscript{[3]}, with the quantum theory just relying on the pre-metric concept of graph. Reasons of this impressive result must be researched in the use of the so-called Ashtekar-Barbero-Immirzi variables\textsuperscript{[15,18]}, which allow to pursue the Hamiltonian formulation of the gravitational field by close analogy with non-Abelian gauge theories. In particular, such variables allow to write down the constraint associated to local spatial rotations in the form of a standard Gauss constraint for the $SU(2)$ group.

In order to achieve this formulation by means of a first order approach, i.e. by considering tetrads and spin connection as independent fields, it is necessary, however, to add to the Einstein-Hilbert Lagrangian new terms, that is Holst and Nieh-Yan contributions\textsuperscript{[19,24]}. In particular, they are such that classical dynamics is always recovered, being the former vanishing on of half-shell (where the equation for the connection holds), and the latter a pure topological term. In both cases, therefore, we deal with a restatement of General Relativity, suitable for loop quantization, which admits Einstein equations as classical limit.

In this respect, the recent interest for $f(R)$ modifications of General Relativity\textsuperscript{[25,26]}, makes very timely questioning about possible LQG extensions of generalized $f(R)$ models, especially via their reformulation in term of scalar-tensor theories\textsuperscript{[27,28]}. A first attempt in this direction was performed in\textsuperscript{[30–32]} (see also\textsuperscript{[33–35]}), where the problem was faced according a second order formulation, i.e. by considering the metric as the only independent field, and authors actually followed in defining Ashtekar-like variables an extended phase-space method\textsuperscript{[3]}. Conclusions of this study suggest that a suitable set of restated variables can be determine to re-obtain proper Gauss and vector constraints, with the non-minimally coupled scalar dynamics affecting the scalar constraint.

Here, we face the same problem, but on a more general framework and adopting the most natural first order formalism, i.e. we deal with Palatini $f(R)$ models\textsuperscript{[37]}, equipped with Holst and Nieh-Yan terms. In particular, we characterize the classical morphology of the resulting theory and we discuss the reformulation in terms of $SU(2)$ variables.

In including Holst and Nieh-Yan terms, we actually have two different choices, consisting in inserting these terms either inside or outside the argument of the function $f$, so leading to four distinct cases. We first analyze the classical dynamics of these four models demonstrating they correspond to only two physically distinct scenarios. In fact, both when the Holst term is included in the argument of $f$ and when the Nieh-Yan term is added into the Lagrangian at the same level of $f$, we see that the classical dynamics is that one of a standard Palatini $f(R)$ model, characterized by a non-dynamical scalar field. Conversely, when the Holst term is added to $f$ and the Nieh-Yan one is plugged inside the function, we deal with a classical scalar-tensor theory and the Immirzi pa-
rameter enters the definition of the parameter $\Omega$, ruling the kinetic term for the dynamical scalar field. Then, we perform the Hamiltonian formulation for this two distinguished cases, and we discuss the resulting morphology in terms of the constraints emerging after the Legendre transformation. In particular, in the case of a Palatini $f(R)$ model and under the assumption of a matter Lagrangian decoupled from connection, we successfully recover the algebraic equation relating the scalar field to the trace of the matter stress energy tensor, leading to a non-dynamical field.

The main merit of this study consists of the determination of suitable generalized Ashtekar-Barbero-Immirzi variables, that for both the cases, i.e. for all original four choices, allows to deal with a Gauss and vector constraint having exactly the same form as in LQG. The different nature of the two considered scenarios and their departure from LQG are then summarized by the morphology of the scalar field only. By other words, we are always able to construct a kinematic Hilbert space for the canonical quantization a la LQG and we are able to study the spectrum of geometrical operators, with special regard on area operator. Here another crucial result from our analysis comes out, since, differently from what assumed in [38], we clarify how the area operator has a clear and unambiguous geometrical nature, being constructed with the real triad of the space. We stress, therefore, how the different link between the real triads and the particular $SU(2)$ variables considered in every specific model, affects the morphology of the area operator as an $SU(2)$ gauge-invariant object. Therefore, also the resulting spectrum is influenced by the considered scenario and in all the cases, a scalar field, non-minimally coupled to gravity emerges. In the case of a standard Palatini $f(R)$ model, such a field is, as above mentioned, non-dynamical and when it is expressed via the trace of the stress energy tensor, a very intriguing coupling takes place among the size of the area associated to a graph and the nature of the matter filling the space. When we deal with a proper scalar tensor theory, instead, the scalar field is truly dynamical and it must be quantized and the nature of the matter filling the space. When

Furthermore, when comparing the metric analysis of [31] to our Palatini formulation, we observe the emergence of a discrepancy in the value of $\Omega$ in the scalar constraints. By other words, starting directly from a metric formulation of $f(R)$ gravity with $SU(2)$ variables provides different dynamical constraints with respect to a first order formulation. In this respect, we outline then the possibility to restore a complete equivalence between these two approaches, by restating our models into the Einstein framework and performing a canonical transformation. We note that similar issues hold also for [36], in which the analysis is actually pursued starting from a first order action, which includes however additional contributions which allow to eliminate torsion from the theory at the very beginning.

The paper is structured as follows. In Sec. III the four models are presented and their effective theories are derived, highlighting equivalences and differences with Palatini $f(R)$ theory in its scalar-tensor formulation. In Sec. III the new modified variables and the correspondent set of constraints are derived, discussing the different dynamical character of the scalar field in each case analysed. Sec. IV deals with the modifications in the spectrum of the area operator and with the Immirzi field role in the new framework. The analysis performed in the conformal Einstein frame and the comparison with earlier studies in literature are contained in Sec. V. In Sec. VI conclusions are drown, while the detailed computations regarding the results presented in Sec. III can be found in the Appendix.

Eventually, notation is established as follows. Spacetime indices are denoted by middle alphabet Greek letters $\mu, \nu, \rho$, spatial ones by letters from the beginning of the Latin alphabet $a, b, c$. Four dimensional internal indices are displayed by capital letters from the middle of the Latin alphabet $I, J, K$, while $i, j, k$ indicate three dimensional internal indices. Spacetime signature is chosen mostly plus, i.e. $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

II. EXTENDED f(R) ACTIONS WITH HOLST AND NIEH-YAN TERMS

The starting point of our analysis is represented by peculiar modifications of Palatini $f(R)$ theories, obtained by implementing Loop Quantum Gravity inspired contributions into the action. For the sake of clarity, then it can be useful to briefly recall the scalar-tensor reformulation of ordinary Palatini $f(R)$ gravity. Let us consider therefore the action

$$S = \frac{1}{2\chi} \int d^4x \sqrt{-g} f(R),$$

(1)
where $\chi = 8\pi G$. The Ricci scalar $\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$ is obtained by the contraction of the Ricci tensor $R_{\mu\nu}$, here considered as a function of the independent connection $\Gamma^\nu_{\mu\rho}$ and related to the Riemann tensor by $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$, with

$$R_{\mu\nu\rho\sigma} = \partial_{\rho} \Gamma^\mu_{\nu\sigma} - \partial_{\sigma} \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho},$$

(2)

The theory can be thus expressed in the Jordan frame by introducing an auxiliary scalar field $\xi$, i.e.

$$S = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [f(\xi) + f'(\xi) (\mathcal{R} - \xi)],$$

(3)

where the variation with respect to $\xi$ leads to the dynamical constraint $\xi = \mathcal{R}$, provided $f'' \neq 0$, which proves the equivalence of (3) with (1). Moreover, the redefinition $\phi = f'(\xi)$ allows us to rewrite (1) in the form of a scalar tensor theory, that is

$$S = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [\phi \mathcal{R} - V(\phi)],$$

(4)

where we defined the potential term $V(\phi) \equiv \phi \xi(\phi) - f(\xi(\phi))$. Now, a procedure can be easily implemented for the peculiar actions considered in this work, namely

$$S_{f(\mathcal{R} - H)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} f(\mathcal{R} - H),$$

(5)

$$S_{f(\mathcal{R} - H)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [f(\mathcal{R}) - H],$$

(6)

$$S_{f(\mathcal{R} + NY)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} f(\mathcal{R} + NY),$$

(7)

$$S_{f(\mathcal{R} + NY)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [f(\mathcal{R}) + NY],$$

(8)

where we introduced, either in the argument of the function $f$ or outside of it, Holst and Nieh-Yan terms, given by, respectively

$$H = \frac{\beta}{2} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma},$$

(9)

$$NY = \frac{\beta}{2} \epsilon^{\mu\nu\rho\sigma} \left( \frac{1}{2} T^\lambda_{\mu\nu} T_{\lambda\rho\sigma} - R_{\mu\nu\rho\sigma} \right),$$

(10)

with $\beta$ the reciprocal of the Immirzi parameter. Torison tensor is displayed by $T^\nu_{\mu\nu} = \Gamma^\nu_{\mu\nu} - \Gamma^\mu_{\nu\nu}$ and, as outlined in [9, 16, 39], it cannot be a priorn neglected in Palatini $f(\mathcal{R})$ generalizations of Holst and Nieh-Yan actions, being its form to be determined dynamically, consistently with a first order formulation. This in turn implies that neither (9) nor (10) are vanishing as in the standard approach, since by virtue of torison the Bianchi identity can be evaded and a possible dynamical coupling with the Nieh-Yan term, stemming from the scalar tensor reformulation, can devoid it of its topological character. In order to see that, let us extend the technique adopted in (3) to models (5)-(8), i.e.

$$S_{f(R-H)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [\phi (\mathcal{R} - H) - V(\phi)],$$

(11)

$$S_{f(R-H)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [\phi \mathcal{R} - H - V(\phi)],$$

(12)

$$S_{f(R+NY)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [\phi (\mathcal{R} + NY) - V(\phi)],$$

(13)

$$S_{f(R+NY)} = \frac{1}{2\chi} \int d^4 x \sqrt{-g} [\phi \mathcal{R} + NY - V(\phi)],$$

(14)

with $\phi$ now defined in terms of the derivative of the function $f(\cdot)$ with respect to its generic argument. Next, if we assume the connection is still metric compatible, then in the presence of torsion it can be written as

$$\Gamma^\nu_{\mu\nu} = C^\nu_{\mu\nu} + K^\nu_{\mu\nu},$$

(15)

where the Christoffel symbol reads

$$C^\nu_{\mu\nu} = \frac{1}{2} g^{\nu\sigma} (\partial_{\rho} g_{\sigma\mu} + \partial_{\sigma} g_{\rho\mu} - \partial_{\rho} g_{\mu\sigma}),$$

(16)

and the independent character of the connection is now encoded in the contortion tensor, given by

$$K^\nu_{\mu\nu} = \frac{1}{2} (T^\nu_{\mu\nu} - T^\nu_{\rho\nu} - T^\nu_{\mu\rho}).$$

(17)

In standard Palatini $f(\mathcal{R})$ gravity [27], it is related to the field $\phi$ by

$$K^\mu_{\rho\mu} = \frac{1}{2} (g_{\rho\nu} \partial_{\mu} \ln \phi - g_{\mu\nu} \partial_{\rho} \ln \phi),$$

(18)

which, taking into account (15), allows us to rewrite the Ricci scalar as

$$\mathcal{R} = R + \frac{3}{2\phi^2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{3\Box \phi}{\phi},$$

(19)

where $R$ denotes the Ricci scalar depending on the metric field only and $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, being $\nabla_\mu$ the covariant derivative defined from connection (16). Therefore, action (4) can be recast, up to boundary terms, as

$$S = \frac{1}{2\chi} \int d^4 x \sqrt{-g} \left[ \phi R + \frac{3}{2\phi} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right].$$

(20)

The analogous of (20) for actions (11)-(14), can be computed by decomposing the torsion tensor in its independent components according to the Lorentz group. These are the trace vector

$$T^\mu_{\mu\nu},$$

(21)

the pseudotrace axial vector

$$S^\mu_{\rho\sigma} T^\nu_{\mu\rho\sigma}$$

(22)
and the antisymmetric tensor $q_{\mu\nu\rho}$ satisfying
\[ \epsilon^{\mu
u\rho\sigma} q_{\nu\rho\sigma} = 0, \quad q^\mu_{\nu\mu} = 0. \] (23)
In terms of these quantities the torsion tensor can be written as
\[ T_{\mu\nu\rho} = \frac{1}{3} (T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu}) - \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} S^\sigma + q_{\mu\nu\rho} \] (24)
and, consequently, the actions can be collectively displayed by
\[ S_b = \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \phi R + \frac{\phi}{24} g^{\mu
u} S_\mu - \frac{g T^{\mu}}{3} T_\mu + \right. \\
\left. -2\phi \nabla_\mu T^{\mu} + b_1 \frac{\phi}{2} \nabla_\mu S_\mu + b_2 \frac{b_2}{3} T^{\mu} S_\mu + \right. \\
\left. \frac{\phi}{2} q_{\mu\nu\rho\sigma} q^{\mu\nu\rho} + b_2 \frac{b_2}{2} \epsilon_{\mu\nu\rho\sigma} q_{\tau\mu\nu} q^{\tau\sigma} - V(\phi) \right], \] (25)
where we used relations
\[ R = R - 2\nabla_\mu T^{\mu} - \frac{2}{3} T_\mu T^\mu + \frac{1}{24} S_\mu S^\mu + \frac{1}{24} q_{\mu\nu\rho\sigma} q^{\mu\nu\rho\sigma}, \] (26)
\[ H = \frac{\beta}{2} \nabla_\mu S_\mu + \beta \frac{T^{\mu}}{3} S_\mu + \frac{\beta}{4} \epsilon^{\mu\nu\rho\sigma} q_{\tau\mu\nu} q^{\tau\sigma}, \] (27)
\[ NY = \frac{\beta}{2} \nabla_\mu S_\mu \] (28)
and we introduced two parameters $b_1$ and $b_2$ which can take values 0 or 1, together with a field $b_\beta$ which is either coincident with $\phi$ or identically equal to 1. The choice of their values allows to select one particular model among the four possible, according table I

| Model | $f(R - H)$ | $f(R)$ | $f(R + NY)$ | $f(R + NY)$ |
|-------|-------------|--------|-------------|-------------|
| $b_1$ | 1           | 0      | 1           | 0           |
| $b_2$ | 1           | 1      | 0           | 0           |
| $b_\beta$ | $\phi$  | 1      | -           | -           |

TABLE I. Values of the auxiliary parameters for each model considered.

The equations of motion for $q_{\mu\nu\rho}$ stemming from (25) are identically solved by $q_{\mu\nu\rho} = 0$, while varying with respect to $T_\mu$ and $S_\mu$ yields, respectively
\[ \phi T_\mu - \frac{3}{2} \partial_\mu \phi - \frac{b_1 b_\beta b_2}{4} S_\mu = 0, \] (29)
\[ \phi S_\mu - 6b_1 \beta \partial_\mu \phi + 4b_2 b_\beta T_\mu = 0, \] (30)
whose solution is given by
\[ T_\mu = \left[ 1 + b_1 b_2 b_\beta^2 / \phi \right] \frac{3}{2\phi} \partial_\mu \phi, \] (31)
\[ S_\mu = \left[ b_1 - 2b_2 b_\beta / \phi \right] \frac{6\beta}{\phi} \partial_\mu \phi. \] (32)
Substituting back (31) and (32) into (25), actions (11)-(14) can be rearranged into the form
\[ S = \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\Omega(\phi)}{\phi} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right], \] (33)
where $\Omega(\phi)$ depends on the particular model addressed according to table II

| Model | $f(R - H)$ | $f(R)$ | $f(R + NY)$ | $f(R + NY)$ |
|-------|-------------|--------|-------------|-------------|
| $\Omega$ | $-3/2$ | $-3/2$ | $\frac{2}{9}(\beta^2 - 1)$ | $-\frac{2}{9}\phi^2$ |

TABLE II. Values of $\Omega$ characterizing the effective theories correspondent to each model.

We see that only $f(R - H)$ and $f(R) + NY$ cases take the scalar tensor parameter $\Omega = -3/2$, corresponding to the effective description (20). By other words, these two alternatives, which are labelled together as case I, are the only models dynamically equivalent to ordinary Palatini $f(R)$ theory for all values of the Immirzi parameter. This implies that the field $\phi$ is not an actual degree of freedom, but it is simply determined by the structural equation as in the standard case [23, 37], i.e.
\[ 2V(\phi) - \phi V'(\phi) = \chi T, \] (34)
where a prime denotes differentiation with respect to the argument and $T$ is trace of the stress energy tensor for some matter contributions, which are assumed do not couple to connection. Case I results also reveal, as one might expect, how the topological character of the Nieh-Yan term be preserved if it is added directly to the Lagrangian. Less trivial, instead, is of course the outcome for the $f(R - H)$ model. In this case, indeed, the vanishing of the Holst term on half shell is to some extent recovered when it is included in the argument of the function $f(\cdot)$.

On the other hand, inserting the Nieh-Yan term in the argument of the function $f(\cdot)$ or featuring the Palatini $f(R)$ theory with an additional Holst contribution (case II), ensures the equivalence to Palatini $f(R)$ gravity only for $\beta = 0$, as it is expected by the coupling of the Immirzi parameter in [27-28]. In case II, therefore, the scalar field acquires in general a dynamical character as described by
\[ (3+2\Omega(\phi))\square \phi + \Omega'(\phi)(\partial \phi)^2 + 2V(\phi) - \phi V'(\phi) = \chi T, \] (35)
and by virtue of (51) and (52) we actually deal with a theory equipped with propagating torsion degrees, by close analogy with [34, 33], where $f(R) - H$ and $f(R) + NY$ models were considered in the presence of a dynamical Immirzi field. In this regard, the additional degree of freedom featuring case II does not make the Palatini extensions here analyzed comparable to metric $f(R)$ gravity, in that corresponding scalar modes are
actually responsible for different gravitational wave polarizations \cite{40,41}. As a result, even though the \( f(R + NY) \) model is formally identical to metric \( f(R) \) gravity for \( \beta = \pm 1 \) (\( \Omega = 0 \)), they are actually endowed with distinct phenomenology. We also note that the case \( f(R) - H \) is singular for \( \beta = \pm i \). However, this value is not truly pathological, since the initial action simply reduces to the \( f(R - H) \) case, provided we fix the Immirzi parameter as \( \beta = \pm i \).

### III. MODIFIED ASHTEKAR VARIABLES

The analysis of Hamiltonian systems corresponding to models \cite{39-43} can be found in appendix \[A\] which we remind the reader for details. Here, we discuss the main results of the procedure, outlining the emergence of a new set of Ashtekar-like variables still suitable for loop quantization.

The gravitational sector of the phase space turns out to be characterized by the set of canonical variables \( \{\pi, \phi; \bar{E}^a_i, K^a_i\} \), where \( \pi \) denotes the conjugate momentum to \( \phi \) and \( \{\bar{E}^a_i, K^a_i\} \) are defined as

\[
K^a_i = \omega^a_{\beta i}, \quad (36)
\]

\[
\bar{E}^a_i = \phi E^a_i, \quad (37)
\]

with \( \omega_{\mu}^{ij} = e^\mu_{\nu} \nabla_\nu e^j_i \) the Riemannian spin connection, and \( E^a_i = \text{det}(e^a_i) e^a_i \) the ordinary densitized triad. The phase space is subject to a set of first class constraints consisting of the rotational constraint

\[
R^a_i \equiv \varepsilon_{ij}^k K^a_j \bar{E}^b_k \approx 0, \quad (38)
\]

the vector constraint

\[
H_a = 2 \bar{E}^b_i D[\tilde{\omega}]{a}^{ik} K^b_j + \pi \partial a \phi \approx 0, \quad (39)
\]

and the scalar constraint

\[H \equiv -\frac{\sqrt{\phi}}{2} \frac{\bar{E}^a_i \bar{E}^b_i}{E} \left(3 R^a_{ib} \tilde{\omega}^b_i + 2 K^a_i K^b_j\right) + \frac{1}{2} \sqrt{\phi} \frac{\phi}{E \phi} \pi^2 + \frac{1}{2} \frac{\phi}{\phi} \partial a \phi + \frac{1}{2} \sqrt{\phi} \frac{V(\phi)}{\phi} \approx 0. \quad (40)\]

where \( \tilde{E} = \text{det}(\bar{E}^a_i), \) \( 3 R^i_{ab} (\tilde{\omega}) = 2 \partial_a \tilde{\omega}_b i + 2 \tilde{\omega}_a^i \tilde{\omega}_b^k \) and the expression of \( \Omega \) selects the particular model considered according to table [III]. In particular, we defined a new type of covariant derivative \( D_{\tilde{\omega}}^{(a)} \), acting on internal spatial indices, by means of the modified spin connection

\[
\tilde{\omega}_a^{ij} = \bar{E}^b_i \left(2 \partial_a \bar{E}^b_j + E^{\nu}d \bar{E}^k_i \partial_a \bar{E}^{\nu k} + \frac{1}{E} \bar{E}^b_i \bar{E}^d_k \partial_b \bar{E}^{\nu d}ight), \quad (41)
\]

which can be expressed in terms of the Riemannian spin connection \( \omega_{\mu}^{ij} \) and the scalar field \( \phi \) as

\[
\tilde{\omega}_a^{ij} = \omega_a^{ij}(E) + \frac{1}{\phi} E^a_i E^{\nu b} \partial_b \phi. \quad (42)
\]

We point out that models belonging to case I are endowed with an additional second class constraint, which can be recast, provided we properly fix the Lagrangian multipliers, in the form \( [44] \), proving the non dynamical character of the field \( \phi \). This result is reinforced by the fact that, in case I, an arbitrary Lagrange multiplier enters the definition of \( \pi \), which can be used for freezing its evolution. In case II, instead, no additional constraints arise and the scalar field and its conjugate momentum are truly dynamical.

Then, performing the canonical transformation

\[
\bar{E}^a_i \rightarrow (36) \bar{E}^a_i = \beta E^a_i, \quad (43)
\]

\[
K^a_i \rightarrow (37) A^a_i = \frac{1}{\beta} K^a_i + \tilde{\Gamma}^a_i, \quad (44)
\]

where \( \tilde{\Gamma}^a_i = -\frac{1}{\beta} \varepsilon^{ijk} \omega_a^j \), a set of modified Ashtekar variables \( \{\beta \bar{E}^a_i, \beta A^a_i\} \) can be obtained, in terms of which the rotational constraint can be combined with the compatibility condition \( D_{\tilde{\omega}}^{(a)} E^a_i = 0 \), satisfied by \( [41] \), yielding the \( SU(2) \) Gauss constraint

\[
G_i = \partial a (\beta \bar{E}^a_i + \varepsilon_{ijk} A^a_j A^a_k) \bar{E}^a_i = 0. \quad (45)
\]

This guarantees that Palatini \( f(R) \)-like models here considered are actually feasible for Loop Quantum Gravity quantization procedure. Especially, by means of the new variables the vector constraint can be rearranged as in standard LQG, along with the additional term associated to the scalar field, i.e.

\[
H_a = (\beta \bar{E}^a_i F_{ab} + \pi \partial a \phi. \quad (46)
\]

where \( F_{ab} = 2 \partial_{[a \beta} A_{b]} + \varepsilon_{ijk} A^a_j A^a_k \). Conversely, the scalar constraint turns out to be modified with respect to the standard case, namely

\[
H = -\frac{\sqrt{\phi}}{2} \frac{\bar{E}^a_i \bar{E}^b_i}{E} \left[\beta^2 \varepsilon^{ijk} F_{ab} + (\beta^2 + 1) 3 R^{ij}(\tilde{\omega}) \right] + \frac{1}{2} \frac{\phi}{\phi} \partial a \phi \phi + \frac{1}{2} \sqrt{\phi} \frac{V(\phi)}{\phi} \approx 0. \quad (47)
\]

where \( (\beta \bar{E} = \text{det}(\bar{E}^a_i)), \) reflecting the difference in the dynamics which exists at a classical level between General Relativity and Palatini \( f(R) \) Gravity.

The preservation of Gauss and vector constraints assure that it is straightforward to extend the usual quantization procedure \[13\] to the new variables \[43-41\]. In particular, we can introduce the fluxes \( \bar{E}^a_i (S) \), obtained smearing \( (\beta \bar{E}^a_i) \) on a surface \( S \), i.e.

\[
\tilde{E}^a_i (S) = \int_S ds \beta \bar{E}^a_i n_a, \quad (48)
\]

where \( n_a \) is the normal vector to the surface, and the holonomies \( h(e)[\beta A^a_i] \), defined as the path ordered exponential of the smearing of \( (\beta A^a_i) \) along a path \( e \), i.e.

\[
h(e)[\beta A^a_i] = \text{Pexp} \left( \int_e (\beta A) \right), \quad (49)
\]
where \((\beta) A = (\beta) A^a_i \tau_i \hat{e}^a\), being \(\hat{e}^a\) the tangent vector to the path \(e\), and \(\tau_i\) the \(SU(2)\) generators. Now, even though they can be promoted to operators acting on spin-network states, just as in the standard case, the quantization of the scalar sector still requires some care, by virtue of the different dynamical character of \(\phi\) in cases I and II. Indeed, since for models belonging to case I the scalar field is not an independent degree of freedom, it has not to be actually quantized, but it can be simply considered a function of matter fields \(\phi(T)\) by means of \([34]\). In case II on the other hand, it embodies a proper gravitational degree of freedom and it can be quantized using the representation adopted in \([42]\), based on the definition of point-like holonomies

\[
U_\lambda(\phi(x)) = e^{i\lambda\phi(x)}, \tag{50}
\]

with \(\lambda \in \mathbb{R}\) and \(x \in \Sigma\), and smeared momenta

\[
P(V) = \int_V d^3x \pi(x), \tag{51}
\]

with \(V\) a 3-dimensional region in \(\Sigma\). In particular, their action on scalar network states defined as

\[
\Phi = \prod_{x \in X} U_\lambda(\phi(x_i)), \tag{52}
\]

where \(X = \{x_i\}\) is a set of points in \(\Sigma\), is given by

\[
\hat{U}_\lambda(\phi(x))\Phi = e^{i\lambda\phi(x)}\Phi, \tag{53}
\]

\[
\hat{P}(V)\Phi = i\hbar \{P(V), \Phi\}. \tag{54}
\]

We remark that in this representation only the holonomy \([50]\) is rigorously defined and there exist no operator \(\hat{\phi}\) associated to the scalar field. However, as outlined in \([34]\), it can be defined by the limit process

\[
\hat{\phi}(x)\Phi = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left[ \hat{U}_\varepsilon(\phi(x)) - \hat{U}_{-\varepsilon}(\phi(x)) \right] \Phi = \phi(x)\Phi, \tag{55}
\]

where \([53]\) is taken into account.

**IV. AREA SPECTRUM AND IMMIRZI AMBIGUITY**

In standard LQG the classical expression for the area of a surface \(S\) is written in terms of densitized triads as

\[
A(S) = \int_S ds \sqrt{E_i^a E_i^b n_a n_b} \tag{56}
\]

and then quantized computing the action of fluxes on spin-network basis states. The area operator turns out to be diagonal in this basis, with the spectrum given by \([2]\)

\[
a = \frac{8\pi\ell_P^2}{\beta} \sum_p \sqrt{j_p(j_p + 1)}, \tag{57}
\]

where \(\ell_P = \sqrt{\hbar G}\) is the Planck length and the sum runs over punctures \(p\) of the surface \(S\) due to edges of the spin-network, colored by spin quantum numbers \(j_p\).

However, in the theories we analyzed, the phase space variable to be quantized is \((\beta) E_i^a\), whereas the physical metric is still associated to the ordinary densitized triad \(E_i^a\). Thus, equation \([55]\) still holds and, in view of its quantization, it has to be rewritten in terms of \((\beta) E_i^a\), namely

\[
A(S) = \frac{1}{\beta} \int_S ds \sqrt{(\beta) E_i^a n_a (\beta) E_i^b n_b} \tag{58}
\]

Now, the square root can be quantized via a regularization procedure as in the standard case, and as long as case I is considered, the scalar field appearing in the last expression should be interpreted as a function of the trace of the stress energy tensor of matter fields \(\phi = \phi(T)\). Then, in this case the spectrum of the area operator is simply given by

\[
a_\phi^{(I)} = \frac{8\pi\ell_P^2}{\beta} \sum_p \frac{1}{\phi(T)} \sqrt{j_p(j_p + 1)}. \tag{59}
\]

When case II is taken into account, instead, the presence in \([58]\) of the dynamical quantity \(\phi\) in the denominator raises the issues about the proper quantization procedure for the scalar field. This problem can be overcome by performing the canonical transformation

\[
\phi' = \frac{1}{\phi}, \quad \pi' = -\phi^2 \pi \tag{60}
\]

which leaves the Gauss and vector constraints invariant \([3]\) but causes the area expression \([58]\) to depend on the scalar field rather than its reciprocal. Then, one can perform the standard regularization procedure for quantizing fluxes and adopt the definition \([59]\) for \(\phi'\), computing the action of both \(\phi'\) and \(E_i(S)\) on a state obtained by the direct product of spin-network and scalar-network states. That results in the modified area spectrum

\[
a_\phi^{(II)} = \frac{8\pi\ell_P^2}{\beta} \sum_p \phi'(p) \sqrt{j_p(j_p + 1)}, \tag{61}
\]

where the only non vanishing contributions are those in which the scalar field is computed in the punctures. This implies that the scalar field contribution to the area operator does not spoil the discrete character of its spectrum, since only a discrete set of its eigenvalues, i.e. \(\phi'(p)\), gives non vanishing contribution, contrary to what argued in \([48, 49]\).

---

2 We consider here only the simple case in which there are no nodes of the graph belonging to the surface nor edges laying on it.

3 Transformation \([60]\) affects, of course, the scalar constraint \([41]\), but in this work we do not address its quantization.
Thus, in both cases a modification in the spectrum of the area operator arises. This is in contrast with the results of \cite{38}, where it is argued that the preservation of Gauss and vector constraints is sufficient to ensure the equivalence of the spectrum with the standard one. Moreover, the dependence of \cite{38} on \(\phi\) (or on \(\phi'\)), can be exploited in order to absorb the Immirzi parameter, eliminating it from the spectrum. Indeed, we can redefine the scalar field and its conjugate momentum as
\begin{align}
^{(\beta)}\phi &= \beta \phi, \\
^{(\beta)}\pi &= \frac{\pi}{\beta},
\end{align}
yielding the new classical expression for the area
\begin{equation}
A(S) = \int_S ds \frac{\sqrt{^{(\beta)}E^a_a (^{(\beta)}E^b_b)}}{^{(\beta)}\phi},
\end{equation}
whose quantization can be addressed as in \cite{38} and \cite{31}, according the case. Therefore, by virtue of \cite{38}, the area spectrum does not contain any dependence on the Immirzi parameter, while the scalar field retains the same meaning as in the cases previously discussed. In this way, the Immirzi parameter still enters the theory as an arbitrary real parameter ruling \cite{43}-\cite{44}, it does not characterize the definition of kinematic quantities like the area\cite{4}. Such an outcome seems to suggest that in Palatini \(f(\mathcal{R})\) extensions of LQG, the Immirzi parameter can be conveniently set to unity in definitions of geometrical objects as in \cite{38}, and its effects on dynamics absorbed in the value of \(\Omega\). We note that in \cite{48}, analogous results are achieved in the context of Conformal-LQG, where an additional conformal transformation is included into the symmetries of the theory. In that case, indeed, it is possible to build an area operator invariant under conformal transformation and independent on the Immirzi parameter, which acquires the role of gauge parameter for the new conformal symmetry. For such a purpose, however, we are forced to consider the conformal rescaled metric as the physical one, in contrast to our assumptions and by analogy with \cite{38}.

\section{V. FORMULATION IN THE EINSTEIN FRAME}

Here we compare the results of the Hamiltonian approach discussed in Sec. \ref{sec:hamiltonian} to analogous studies present in literature \cite{31, 36}, where a different set of canonical variables was obtained, denoted by hatted characters and related to ours by the canonical transformation
\begin{align}
^{\hat{\beta}}E_a^a &= \frac{1}{\phi} E_a^a, \\
^{\hat{\beta}}K_a^i &= \phi K_a^i, \\
^{\hat{\pi}} &= \pi - \frac{1}{\phi} E_a^a K_a^i, \\
^{\hat{\phi}} &= \phi.
\end{align}
These results are to some extent controversial, since performing on \cite{38}-(\ref{eq:40}) such a transformation reproduces the same set of constraints of \cite{31} only if we replace by hand \(\Omega\) with \(\Omega + 3/2\). Furthermore, in \cite{31} starting from a second order analysis of \cite{38}, a LQG formulation of scalar tensor theories was achieved via a symplectic reduction technique \cite{3}. Then, this seems to point out that the implementation of Ashtekar variables could be affected by the peculiar choice of the formalism adopted, when extensions to General Relativity are taken into account, as it occurs for the Jordan frame formulation of Palatini \(f(\mathcal{R})\)-like models considered. In this sense, the fact that in \cite{36} were actually derived results equivalent to \cite{31} according a first order approach, can be traced back to the choice of including additional contributions in the action, featuring a Holst term, with the aim of eliminating torsion from the theory.

Now, we want to show how the phase space structure obtained in \cite{31, 36} could be reproduced by means of the canonical transformation \cite{43}-\cite{44}, provided the analysis of Sec. \ref{sec:hamiltonian} be pursued in the so called Einstein frame, endowed with the conformally rescaled metric \(\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}\). Of course, we note that such an equivalence does not exclude a priori the existence, even at the level of the Jordan frame, of a different, possibly \textit{gauged} canonical transformation (see \cite{12}), able to tackle with this problem.

Let us therefore rewrite in the Einstein frame actions \cite{11}-\cite{14}, that is
\begin{align}
S_{f(R-H)} &= \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \tilde{R} - \phi \tilde{H} - U(\phi) \right],
\end{align}
\begin{align}
S_{f(R)-H} &= \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \tilde{R} - \tilde{H} - U(\phi) \right],
\end{align}
\begin{align}
S_{f(R+NY)} &= \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \tilde{R} + \phi \tilde{NY} - U(\phi) \right],
\end{align}
\begin{align}
S_{f(R)+NY} &= \frac{1}{2\chi} \int d^4x \sqrt{-g} \left[ \tilde{R} + \tilde{NY} - U(\phi) \right],
\end{align}
where the potential is redefined as \(U(\phi) = V(\phi)/\phi^2\) and the Ricci scalar is obtained from
\begin{equation}
\tilde{R} = \tilde{g}_{\mu\nu} \tilde{R}_{\mu\nu},
\end{equation}
with \(\tilde{R}_{\mu\nu}\) denoting the conformally rescaled Ricci tensor, which we assume to depend on the connection \(\tilde{\Gamma}_{\mu\nu}^\rho\). Analogously, we introduced the rescaled Holst and Nieh-Yan terms
\begin{align}
\tilde{H} &= \frac{\beta}{2} \tilde{\epsilon}_{\mu\nu\sigma} \tilde{R}_{\mu\nu\sigma},
\end{align}
\begin{align}
\tilde{NY} &= \frac{\beta}{2} \tilde{\epsilon}_{\mu\nu\sigma} \left( \frac{1}{2} \tilde{g}^{\lambda\tau} \tilde{T}_{\lambda\mu\nu} \tilde{T}_{\rho\sigma} - \tilde{R}_{\mu\nu\sigma} \right),
\end{align}
with \(\tilde{\epsilon}_{\mu\nu\sigma} = \epsilon^{\mu\nu\sigma}/\phi^2\). Now, assuming the compatibility condition for the conformal metric with respect to \(\tilde{\Gamma}_{\mu\nu}^\rho\), namely
\begin{equation}
\nabla_\mu \tilde{g}_{\nu\rho} = 0,
\end{equation}
the connection can be again written as
\[ \tilde{\Gamma}^\rho_{\mu \nu} = \tilde{C}^\rho_{\mu \nu} + \tilde{K}^\rho_{\mu \nu}, \] (75)
where \( \tilde{C}^\rho_{\mu \nu} \) is the Christoffel symbol of the conformal metric and \( \tilde{K}^\rho_{\mu \nu} \) is defined as in (17) but with indices now raised, lowered, and contracted using the conformal metric. Torsion tensor can be therefore displayed as
\[ \tilde{T}_{\mu \nu \rho} = \frac{1}{3} \left( \tilde{T}_\rho \tilde{g}_{\mu \nu} - \tilde{T}_\rho \tilde{g}_{\mu \nu} \right) - \frac{1}{2} \tilde{C}^\rho_{\mu \nu \rho} \tilde{S}^\sigma + \tilde{g}_{\mu \nu \rho}. \] (76)

and the effective action for the four models can be derived, i.e.
\[ S = \frac{1}{2\chi} \int d^4x \sqrt{-g} \left\{ \tilde{R} - \tilde{\Omega}(\phi) \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right\}, \] (77)
where \( \tilde{R} \) is the Ricci scalar depending only on the conformal metric and the function \( \tilde{\Omega} \) takes the values in table III.

| Table III. Values of \( \tilde{\Omega} \) characterizing the effective theories in the Einstein frame. |
|---------------------------------|-----------------|
| Model                          | Case I | Case II |
| \( f(\mathcal{R}) - H \)      | \( f(\mathcal{R}) + NY \) | \( f(\mathcal{R} + NY) \) |
| \( \tilde{\Omega} \)          | 0      | 4\phi^2 + \frac{e^2}{1 + e^2\phi^2} |

which shows how the value \( \Omega = -3/2 \), associated to non dynamical configurations for the field \( \phi \) in the Jordan frame, correspond to \( \tilde{\Omega} = 0 \), in agreement with the representation in the Einstein frame of scalar tensor theories. Therefore, models belonging to case I are dynamically equivalent to Palatini \( f(\mathcal{R}) \) theory for every value of the Immirzi parameter, while in case II the equivalence is correctly recovered only for \( \beta = 0 \).

Then, following the line of appendix A we can perform the Hamiltonian analysis, which reveals a phase space co-ordinated by the set of variables \( \{ \tilde{E}_i^a, K_a^j \} \), where the densitized triad is now defined in terms of the spatial part of the conformal tetrad \( \tilde{e}_\mu^i = \sqrt{\phi} e_\mu^i \), and a rotational constraint coincident for all models with the expression derived in the Jordan frame. Regarding the vector and scalar constraints instead, for case I they read, respectively
\[ H_a = 2\tilde{E}_a^b D^{[\tilde{\omega}]}_b K^i_h \approx 0 \] (79)
\[ H = \frac{\tilde{E}_a^b \tilde{E}_b^i}{2\sqrt{E}} \left( 3 \tilde{R}_{ab}^{ij}(\tilde{\omega}) + 2K^i_a K_h^j \right) + \sqrt{E} U(\phi) \approx 0, \] (80)
where \( \tilde{\omega}^{ij} \) is the spin connection compatible with \( \tilde{e}_\mu^i \). In this case, the conjugated momentum to the scalar field is weakly vanishing, leading to the primary constraint \( \pi \approx 0 \), whose conservation along the dynamics simply amounts to impose the variation of the potential term with respect to \( \phi \) to weakly vanish. If matter is included as in Appendix A this reproduces the structural equation as a secondary constraint, as can be shown using (A.40). In models belonging to case II, instead, the scalar field is dynamical, so that the vector constraints takes the form [39] and the scalar constraint reads
\[ H = \frac{1}{2\sqrt{E}} \left( 3 \tilde{R}_{ab}^{ij}(\tilde{\omega}) + 2K^i_a K_h^j \right) + \frac{1}{2\sqrt{E}} \frac{\phi^2}{\Omega} + \frac{1}{2\sqrt{E}} \tilde{E}_a^b \tilde{E}_b^i \partial_\alpha \phi \partial_\beta \phi + \sqrt{E} U(\phi) \approx 0. \] (81)

If we then perform the canonical transformation (65)-(68), the difference consisting in the shift by \(-3/2 \Omega \) is now compensated taking into account relation (78), and the final expression for the constraints is in agreement with [36], both for \( \Omega \neq -3/2 \) and for \( \Omega = -3/2 \) (\( \tilde{\Omega} = 0 \) and \( \Omega \neq 0 \)). Specifically, in the latter case the primary constraint \( \pi \approx 0 \) becomes
\[ \pi + \frac{1}{\phi} \tilde{E}_a^b \tilde{K}_a^i \approx 0, \] (82)
which reproduces the so called conformal constraint present in [30], and proves the equivalence of first and second order approaches in the Einstein frame.

VI. CONCLUDING REMARKS

In this work, we adopted the point of view that the most natural way to introduce Ashtekar-Barbero-Immirzi variables into modified \( f(\mathcal{R}) \) contexts consists of dealing with a first order (Palatini) formulation of the action. This perspective implies the need to add to the Lagrangian the typical Holst and Nieh-Yan contributions, as discussed for the standard Einstein-Hilbert action in [20]. As shown in Sec. III we have four possible combinations for the Lagrangian, corresponding to include the aforementioned terms either inside the argument of the function \( f \) or simply outside. When the equations of motion are calculated, and the torsion field is expressed via the metric and the scalar field variables, we see that two different physical formulations comes out. In fact, plugging the Holst term in the function \( f \) argument or adding the Nieh-Yan contribute directly into the Lagrangian, simply corresponds to a standard Palatini \( f(\mathcal{R}) \) theory. Conversely, in the opposite case a truly scalar-tensor model is obtained, whose parameter \( \Omega \) turns out to depend on the Immirzi parameter. In these two cases we are able to define variables of the Ashtekar-Barbero-Immirzi type, in terms of which Gauss and vector constraints of LQG
are recovered. The crucial difference between the two cases consists of the non-dynamical role of the scalar mode in the standard Palatini $f(R)$ model, with respect to its non-minimal dynamical coupling with gravity in scalar-tensor models with $Ω \neq -3/2$. Particularly, in the Palatini $f(R)$ model the scalar field can be expressed via the trace of the stress energy tensor, where the matter Lagrangian is necessarily assumed to be connection independent. This difference is also reflected into the morphology of the area operator, obtained starting from its expression in natural geometrical variables, i.e. the natural tetrad fields, and then expressed via the proper $SU(2)$ variables, suitable for loop quantization. As a result, the area spectrum, even if discrete, depends now on the scalar field properties as well. In particular, in the case of a Palatini $f(R)$ theory, we have to deal with the intriguing feature that the geometrical structure of the space depends on the nature of the matter by which it is filled. The most interesting issue, however, comes out for scalar-tensor models, when the scalar field must also be quantized via an holonomy-like representation and the area operators eigenvalues must contains feature of the scalar mode spectrum. This feature is a consequence of the non-minimal coupling of the scalar field to gravity and suggests that geometrical properties of the quantum space are influenced by the particular considered form of the function $f$, i.e. of the potential term $V(ϕ)$. Thus, the form of the Lagrangian one adopted to describe gravity seems to directly influence the space quantum kinematics, differently from the classical scenario, where only the space metric fixes the geometry kinematics, disregarding the Lagrangian form.

We also showed that by a canonical rescaling of the scalar field, it is possible setting the Immirzi parameter equal to one in all the kinematic constraints, while it still affects the scalar constraint morphology. This is not surprising since we are able to directly link the Immirzi parameter to the scalar-tensor parameter $Ω$. However, the $SU(2)$ morphology of the theory and its kinematic properties must not be affected by the $Ω$ parameter, since we can re-absorb its value into the Ashtekar-Barbero-Immirzi variable definition. In turn, the theory dynamics can be instead affected by the value of $Ω$, i.e. of $β$, since the theories are not dynamical equivalent and these parameters can be constrained by experimental observations. In this sense, we can claim that the Immirzi parameter ambiguity is here completely solved, by its link to the physical scalar-tensor parameter and by the independence of the theory kinematics on its specific value. This is, in our opinion, a very relevant result, opening a new perspective for the solution of some of the LQG shortcomings into a revised and extended formulation of the gravitational interaction.

We conclude by stressing a technical issue concerning the possibility to obtain equivalent formulations, when starting from a second order approach as in and according the present Palatini approach. In fact, the scalar constraint appears different in the two analysis and the possibility to restore a complete equivalence implies the choice of a Einstein framework for our formulation, i.e. a conformal rescaling of the tetrad field. This technical evidence suggests a possible physical interpretation for the dynamics in the Einstein framework of a $f(R)$ theory (on the present level it must be considered simply a mathematical tool to make the scalar dynamics minimally coupled), which deserves further investigation.

Appendix A: Derivation of the constrained Hamiltonian system

In this appendix we provide an in-depth analysis of the Hamiltonian formulation of (25), which, by means of tetrad fields $e_μ^I$ and spin connections $ω_μ^I{J}$, can be rewritten, modulo surface terms, as

$$S = \frac{1}{2χ} \int d^3x \ e \left[ ϕR_μ^I{J} + \frac{ϕ}{24} S^μ S_μ - \frac{2}{3} ϕ T^μ T_μ \right.$$

$$\left. + 2T^μ \partial_μ ϕ - b_1 β S^μ \partial_μ ϕ + b_2 b_β \frac{β}{3} T^μ S_μ - V(ϕ) \right], \quad (A1)$$

where $e = \det(e_μ^I)$ and $R_μ^I{J} = 2\partial_μ(ω_ν^I{J}) + 2ω_μ^I{K} ω^K{JV}$ is the strength tensor of the spin connection. Terms in $g_{μνρ}$ have been neglected since they would eventually turn out to yield vanishing contributions as argued further on. The spacetime splitting is achieved via a foliation of the manifold into a family of 3-dimensional hypersurfaces $Σ_t$ defined by the parametric equations $y^μ = y^μ(t, x^a)$, where $t ∈ ℝ$. The submanifold $Σ_t$ is globally defined by a time-like vector $n^μ$ normal to the hypersurfaces, such that $n^μ n_μ = -1$, and an adapted base on $Σ_t$ is then given by $b_a^I := \partial_a y^μ$, satisfying the conditions $g_{μν} n^μ b_a^K = 0$. Defining the deformation vector as $t^μ = \partial_μ y^μ$, it can be decomposed on the basis vectors \{n^μ, b_a^K\} as

$$t^μ = N n^μ + N^a b_a^K, \quad (A2)$$

where $N^a = N b_a^K$, $N$ is the lapse function and $N^a$ the shift vector. The following completeness relation holds

$$h_μν = g_μν + n_μ n_ν, \quad (A3)$$

where $h_μν = h_αβ b_a^K b_b^K$ is the projector on the spatial hypersurfaces and $h_μν$ the 3-metric, related to the triads by $h_αβ = e_α^K e_β^K$. Hence, assuming the time gauge conditions $n^μ = e_0^K$, $e_0^I = 0$ and using $e = N \dot{e}$, with $\dot{e} = \det(e_0^K)$, the action can be rewritten as
\[ S_b = \int dt d^3 x \left[ \phi_a \frac{\partial}{\partial \phi_a} \left( \mathcal{L}_t K^i_a - D_a^{(\omega)} (t \cdot \omega^i) + t \cdot \omega^i K^a_i \right) + \left( -n \cdot T + b_1 \frac{\beta}{4} n \cdot S \right) \mathcal{L}_t \phi + \right. \\
- N_a^a \left[ 2 \phi_a^a D_a^{(\omega)} K^i_b + \left( -n \cdot T + b_1 \frac{\beta}{4} n \cdot S \right) \partial_a \phi \right] + \\
\left. + N \left[ \phi_a^a \frac{\partial}{\partial \phi_a} \left( \frac{3}{2} R_{a b}^{ij} + 2 K^i_b K^j_a \right) + \left( T^a - b_1 \frac{\beta}{4} S^a \right) \partial_a \phi - \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} (n \cdot T)^2 + \\
- b_2 b_1 \frac{\beta}{6} (n \cdot T) (n \cdot S) + \frac{\phi}{48} S^a S_a - \frac{\phi}{3} T^a T_a + b_2 \Phi \frac{\beta}{6} T^a S_a - \frac{1}{2} V(\phi) \right] \right]. \quad (A4) \]

where \( K^i_a \equiv \omega^0_a \), the Lie derivative along the vector field \( t^\mu \) is defined as \( \mathcal{L}_t V_\mu = t^\nu \partial_\nu V_\mu + V_\nu \partial_{\nu t^\mu} \), while \( \cdot \) indicates spacetime indices contractions, namely \( t \cdot \omega^i \equiv t^\mu \omega^i_{\mu} \), \( t \cdot \omega^i \equiv t^\mu \omega^i_{\mu} \), \( n \cdot T \equiv n^\mu T_\mu \) and \( n \cdot S \equiv n^\mu S_\mu \). Moreover, we defined the derivative \( D_a^{(\omega)} \) acting only on spatial internal indices via the spatial components of the spin connection, i.e. \( D_a^{(\omega)} V_\mu^i = \partial_a V_\mu^i + \omega_a^j V_\mu^j \).

Then, the computation of conjugate momenta of \( K^i_a \) and \( \phi \) yields, respectively

\[ \dot{E}_i^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t K_a^i} = \phi \dot{e}_i^a, \quad (A5) \]
\[ \pi \equiv \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t \phi} = \dot{e} \left( -n \cdot T + b_1 \frac{\beta}{4} n \cdot S \right), \quad (A6) \]

while all other momenta vanish, namely

\[ (e) e_i^a : \quad \pi_i^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t e_i^a} = 0; \quad (A7) \]
\[ t \cdot \omega^i : \quad \pi_i^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t (t \cdot \omega^i)} = 0; \quad (A8) \]
\[ \omega^i_a : \quad \pi_i^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t (\omega^i_a)} = 0; \quad (A9) \]
\[ N^a : \quad \pi_a^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t N^a} = 0; \quad (A10) \]
\[ N : \quad \pi = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t N} = 0; \quad (A11) \]
\[ t \cdot \omega^{ik} : \quad \pi_{ik} = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t (t \cdot \omega^{ik})} = 0; \quad (A12) \]
\[ S_a : \quad \pi_a^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t S_a} = 0; \quad (A13) \]
\[ T_a : \quad \pi_a^a = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t T_a} = 0; \quad (A14) \]
\[ n \cdot S : \quad \pi = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t (n \cdot S)} = 0; \quad (A15) \]
\[ n \cdot T : \quad \pi = \frac{\delta S_{\text{tot}}}{\delta \mathcal{L}_t (n \cdot S)} = 0. \quad (A16) \]

Thus, all momenta are non invertible for the correspondent velocities, implying the presence of just as many primary constraints, i.e.

\[ (K) C_i^a \equiv \dot{e}_i^a - \phi \dot{e}_i^a \approx 0; \quad (A17) \]
\[ \pi \equiv \dot{e} \left( n \cdot T - b_1 \frac{\beta}{4} n \cdot S \right) \approx 0; \quad (A18) \]
\[ (e) C_i^a \equiv \pi_i^a \approx 0; \quad (A19) \]
\[ (\omega) C_i^a \equiv \pi_i^a \approx 0; \quad (A20) \]
\[ (\omega) C_i^a \equiv \pi_i^a \approx 0; \quad (A21) \]
\[ (N) C_a \equiv \pi_a^a \approx 0; \quad (A22) \]
\[ (N) C \equiv \pi \approx 0; \quad (A23) \]
\[ (\omega) C_{ik} \equiv \pi_{ik} \approx 0; \quad (A24) \]
\[ (S) C_a \equiv \pi_a^a \approx 0; \quad (A25) \]
\[ (T) C_a \equiv \pi_a^a \approx 0; \quad (A26) \]
\[ (S) C \equiv \pi \approx 0; \quad (A27) \]
\[ (T) C \equiv \pi \approx 0. \quad (A28) \]

A total Hamiltonian can be therefore defined by replacing each non invertible velocity with a Lagrange multiplier, i.e.
\[
H_T = \int d^3x \left\{ -t \cdot \omega^i D^\omega_a (\dot{E}_a^i) + t \cdot \omega^{ik} K_{ai} \dot{E}_k^a + N^a \left( 2 \dot{E}_a^b D^\omega_{[a} K^b_{i]} + \pi \partial_a \phi \right) + \right.
\]
\[\left. - N \sqrt{\frac{\dot{E}_a^b \dot{E}_b^i}{2E}} \left( 3 R^i_{ab} + 2 K^i_{a[b]} \right) - N \sqrt{\frac{E}{\dot{E}_a^b}} \left( T^a - b_1 \frac{\beta}{4} S^a \right) \partial_a \phi - \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} (n \cdot T)^2 \right) \right] \frac{\partial_a \phi}{\phi} - \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} \frac{(n \cdot T)^2}{E} + \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} \frac{(n \cdot T)^2}{E} \right] \right)
\]
(A29)

where \( \dot{E} = \det(\dot{E}_a^i) \) and Lagrangian multipliers are indicated by \( \lambda \) characters. Finally, the phase space is equipped with the following Poisson brackets

\[
\begin{align*}
\{ K^i_{a[} \dot{E}_b^i \} &= \delta^i_j \delta^a_b \delta(x, y); \quad (A30) \\
\{ \phi(x), \pi(y) \} &= \delta(x, y); \quad (A31) \\
\{ e^a_i(x), (\omega^i) \pi^a_i(y) \} &= \delta^a_i \delta^a_b \delta(x, y); \quad (A32) \\
\{ t \cdot \omega^i(x), (\omega^i) \pi^a_i(y) \} &= \delta^a_i \delta^a_b \delta(x, y); \quad (A33) \\
\{ \omega^a_{ij}(x), (\omega^a_{ij}) \pi^b_{kj}(y) \} &= \delta^a_i \delta^a_b \delta(x, y); \quad (A34) \\
\{ N^a(x), (N^a) \pi^b_{ij}(y) \} &= \delta^a_i \delta^a_b \delta(x, y); \quad (A35) \\
\{ N(x), (N) \pi^b_{ij}(y) \} &= \delta^a_i \delta^a_b \delta(x, y); \quad (A36) \\
\{ t \cdot \omega^i(x), (\omega^i) \pi^a_{b[}(y) \right\} &= \delta^a_{[i} \delta^a_{b]} \delta(x, y); \quad (A37) \\
\{ S_a(x), (S) \pi^b_{ij}(y) \} &= \delta^a_{i} \delta^a_{b} \delta(x, y); \quad (A38) \\
\{ T_a(x), (T) \pi^b_{ij}(y) \} &= \delta^a_{i} \delta^a_{b} \delta(x, y); \quad (A39) \\
\{ S(x), (S) \pi^b_{ij}(y) \} &= \delta(x, y); \quad (A40) \\
\{ T(x), (T) \pi^b_{ij}(y) \} &= \delta(x, y). \quad (A41)
\end{align*}
\]

At this stage we have to impose that primary constraints be preserved by the dynamics of the system. This amounts to compute their time evolution, evaluating their Poisson brackets with the total Hamiltonian using (A30)-(A31), and imposing the result to be at least weakly vanishing on the constraint hypersurface. As outlined in the following, this requirement can be accommodated imposing secondary constraints, fixing some of the arbitrary Lagrange multipliers or solving some phase space variables in terms of others.

The time evolution of (A17) and (A19) is proportional to the Lagrange multipliers \((\omega^a) \lambda^a_i \) and \((K^a) \lambda^a_i \), respectively, which can be fixed in order to assure the conservation of these two primary constraints.

The request for the evolution of (A22), (A23), (A21) to be weakly vanishing implies the presence of the vector, scalar and rotational constraints, namely

\[
H_a \equiv \left( 2 \dot{E}_a^b D^\omega_{[a} K^b_{i]} + \pi \partial_a \phi \right) \approx 0, \quad (A42)
\]

\[
H = - \sqrt{\frac{\dot{E}_a^b \dot{E}_b^i}{2E}} \left( 3 R^i_{ab} + 2 K^i_{a[b]} \right) + \right.
\left. - \sqrt{\frac{E}{\dot{E}_a^b}} \left( T^a - b_1 \frac{\beta}{4} S^a \right) \partial_a \phi - \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} (n \cdot T)^2 \right) \right] \frac{\partial_a \phi}{\phi} - \frac{\phi}{48} (n \cdot S)^2 + \frac{\phi}{3} \frac{(n \cdot T)^2}{E} \right] \right)
\]
(A43)

\[
K^a_{[k} \dot{E}_b^a \approx 0. \quad (A44)
\]

Their associated variables, namely \( N^a \), \( N \) and \( t \cdot \omega^{ij} \) have equations of motion proportional to their Lagrange multipliers \((N^a) \lambda^a_i \), \((N) \lambda^a_i \), \((\omega^i) \lambda^a_{ij} \), thus are completely arbitrary and can be considered as Lagrange multipliers themselves.

The time evolution of (A20) yields

\[
(\omega^a) \dot{C}_i = D^a_{[i} \dot{E}_b^a, \quad (A45)
\]

which is strongly vanishing on the whole phase space if the spin connection is fixed to be compatible with \( \dot{E}_a^i \) as in (11).

The Poisson bracket between (A21) and the total Hamiltonian reads

\[
t \cdot \omega^{[i} \dot{E}^{j]}_{[a} - 2 N^a \dot{E}_a^i K^a_{j]} - N^a \dot{E}_a^i K^a_{j]} + \right.
\left. - \frac{\dot{E}_a^b E^{i[b]} \partial_b (N \sqrt{\phi}) - N \sqrt{\phi} D^a_{[i} \dot{E}_a^b \dot{E}^{[b]}_{j]} \approx 0 \right) \quad (A46)
\]

and is weakly vanishing since the third term is proportional to the rotational constraint, which is weakly vanishing, the last term can be dropped once the spin connection is set as in (11) and the remaining terms are proportional to the new Lagrange multipliers and can be made to vanish by definition.

Conservation of (A25) and (A20) is guaranteed solving
$T_a$ and $S_a$ in function of $\phi$ and its spatial derivatives, according to the spatial restriction of expressions (31) and (32). Conservation of (A27) and (A28) is assured fixing the Lagrange multiplier $\lambda$ and imposing in case I $n \cdot S = 0$, or

$$n \cdot S = 4\beta n \cdot T$$  \hspace{1cm} \text{(A47)}

and

$$n \cdot S = \frac{4\beta}{\phi} n \cdot T$$ \hspace{1cm} \text{(A48)}

in the $f(R + NY)$ and $f(R) - H$ cases, respectively. In case I the conservation of (A15) leads to a secondary constraint that takes the form of (33), provided we fix one between $(S)\lambda$ and $(T)\lambda$, and use the unfixed one which enters the definition of $\pi$ for freezing out its dynamics.

In case II instead, the time evolution of (A18) can be set to zero fixing $(S)\lambda$ or $(T)\lambda$, which eventually implies that both of them are no longer arbitrary since the equations of motion for $n \cdot S$ and $n \cdot T$ are proportional to their Lagrange multipliers and relations (A47) and (A48) hold. Thus, in this case there are no arbitrariness degrees of freedom in the definition of $\pi$ that can be used to freeze its dynamics. Moreover, if one of the Lagrange multipliers is used to reproduce the structure equation also in this case, this would imply the non dynamicity of $\phi$, ending up with an inconsistency and forcing to chose another form for the Lagrange multiplier.

Matter is implicitly included into the theory positing its action to depend only on the metric and the matter fields and not on the connection. Assuming that no primary constraints arise in the matter sector, the constraint structure of the theory gets modified by the addition of the terms $\frac{\delta H_{\text{matt}}}{\delta N}$ and $\frac{\delta H_{\text{matt}}}{\delta \phi}$ to the vector and scalar constraints (A42) and (A43), respectively, being $H_{\text{matt}}$ the matter Hamiltonian. In terms of it, from the usual definition of the stress energy tensor of matter in terms of the matter Lagrangian, its trace can be expressed as

$$T = \frac{2\phi^{5/2}}{N\sqrt{E}} \left( \frac{N}{2\phi} \frac{\delta H_{\text{matt}}}{\delta N} + \frac{\delta H_{\text{matt}}}{\delta \phi} \right),$$ \hspace{1cm} \text{(A49)}

a relation useful in order to recover the structural equation in the Einstein frame formulation developed in Sec. V.

Finally, let us notice that, if the terms proportional to $q_{\mu\nu}$ would not have been neglected, then, given the absence of derivatives of $q_{\mu\nu}$, its conjugate momentum would have been weakly vanishing. The additional terms proportional to it via a Lagrange multiplier appearing in the total Hamiltonian would have produced secondary constraints whose solutions would have implied in turn the vanishing of $q_{\mu\nu}$ components, since it does not couple to any derivative of the scalar field $\phi$, contrary to what happens to the other components of torsion.

Lastly, once the secondary constraints are solved the scalar constraint (A43) takes the form (50), while (38) follows from (A44).

[1] F. Cianfrani, O.M. Lecian, M. Lulli and G. Montani, *Canonical Quantum Gravity* (World Scientific Pub Co, Inc., Singapore, 2014)

[2] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2004)

[3] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2007)

[4] C. Rovelli and T. Thiemann, Phys. Rev. D 57, 1009 (1998)

[5] R. Gambini, O. Obregon and J. Pullin, Phys. Rev. D 59, 047505 (1999)

[6] C. H. Chou, R. S. Tung and H. L. Yu, Phys. Rev. D 72, 064016 (2005)

[7] A. Perez and C. Rovelli, Phys. Rev. D 73, 044013 (2006)

[8] S. Mercuri, Phys. Rev. D 73, 084016 (2006)

[9] G. Date, R. K. Kaul and S. Sengupta, Phys. Rev. D 79, 044008 (2009)

[10] S. Mercuri, Phys. Rev. Lett. 103, 081302 (2009)

[11] H. Nicolai, K. Peeters and M. Zamaklar, Class. Quant. Grav. 22, R193 (2005)

[12] C. Rovelli and L. Smolin, Nucl. Phys. B 442, 593 (1995) Erratum: [Nucl. Phys. B 456, 753 (1995)]

[13] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 14, A55 (1997)

[14] J. Lewandowski, Class. Quant. Grav. 14, 71 (1997)

[15] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).

[16] A. Ashtekar, Phys. Rev. D 36, 1587 (1987).

[17] J. F. Barbero G., Phys. Rev. D 51, 5507 (1995)

[18] G. Immirzi, Class. Quant. Grav. 14, L177 (1997)

[19] H. T. Nieh and M. L. Yan, J. Math. Phys. 23, 373 (1982).

[20] S. Holst, Phys. Rev. D 53, 5966 (1996)

[21] C. Soo, Phys. Rev. D 59, 045006 (1999)

[22] H. T. Nieh and C. N. Yang, Int. J. Mod. Phys. A 22, 5237 (2007).

[23] S. Mercuri, Phys. Rev. D 77, 024036 (2008)

[24] K. Banerjee, Class. Quant. Grav. 27, 135012 (2010)

[25] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010)

[26] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. 692, 1 (2017)

[27] P. G. Bergmann, Int. J. Theor. Phys. 1, 25 (1968).

[28] M. Capone and M. L. Ruggiero, Class. Quant. Grav. 27, 125006 (2010)

[29] M. S. Ruf and C. F. Steinwachs, Phys. Rev. D 97, no. 4, 044050 (2018)

[30] X. Zhang and Y. Ma, Phys. Rev. D 84, 064040 (2011)

[31] X. Zhang and Y. Ma, Phys. Rev. D 84, 104045 (2011)

[32] Y. Han, Y. Ma and X. Zhang, Mod. Phys. Lett. A 29, 1450134 (2014)

[33] G. Montani and F. Cianfrani, Phys. Rev. D 80, 084045 (2009)

[34] F. Bombacigno and G. Montani, Phys. Rev. D 97, 124066
[35] F. Bombacigno and G. Montani, Phys. Rev. D 99, no. 6, 064016 (2019)
[36] Z. Zhou, H. Guo, Y. Han and Y. Ma, Phys. Rev. D 87, no. 8, 087502 (2013)
[37] G. J. Olmo, Int. J. Mod. Phys. D 20, 413 (2011)
[38] L. Fatibene, M. Ferraris and M. Francaviglia, Class. Quant. Grav. 27, 185016 (2010)
[39] G. Calcagni and S. Mercuri, Phys. Rev. D 79, 084004 (2009)
[40] D. Liang, Y. Gong, S. Hou and Y. Liu, Phys. Rev. D 95, no. 10, 104034 (2017)
[41] F. Moretti, F. Bombacigno and G. Montani, Phys. Rev. D 100, no. 8, 084014 (2019)
[42] A. Ashtekar, J. Lewandowski and H. Sahlmann, Class. Quant. Grav. 20, L11 (2003)
[43] F. Cianfrani, M. Lulli and G. Montani, Phys. Lett. B 710, 703 (2012)
[44] A. F. Zakharov, A. A. Nucita, F. De Paolis and G. Ingrassia, Phys. Rev. D 74, 107101 (2006)
[45] T. Chiba, T. L. Smith and A. L. Erickcek, Phys. Rev. D 75, 124014 (2007)
[46] H. J. Schmidt, Phys. Rev. D 78, 023512 (2008)
[47] C. P. L. Berry and J. R. Gair, Phys. Rev. D 83, 104022 (2011) Erratum: [Phys. Rev. D 85, 089906 (2012)]
[48] O. J. Veraguth and C. H.-T. Wang, Phys. Rev. D 96, no. 8, 084011 (2017)
[49] C. H.-T. Wang and D. P. F. Rodrigues, Phys. Rev. D 98, no. 12, 124041 (2018)