Model Dependence of Baryon Decay Enhancement by Cosmic Strings

C.J. Fewster∗ and B.S. Kay†

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge,
Silver Street, Cambridge CB3 9EW, U.K.

July, 1992

Abstract
Cosmic strings arising from GUTs can catalyse baryon decay processes with strong interaction cross sections. We examine the mechanism by which the cross section is enhanced and find that it depends strongly on the details of the distribution of gauge fields within the string core. We propose a calculational scheme for estimating wavefunction amplification factors and also a physical understanding of the nature of the enhancement process.

∗E-mail address: cjf10@uk.ac.cam.phx
†Address from 1 October 1992: Department of Mathematics, University of York, Heslington, York YO1 5DD, U.K.
1 Introduction

Grand unified theories (GUTs) predict a rich variety of topologically stable “defects” – domain walls, monopoles, cosmic strings – whose localized concentrations of unbroken gauge fields and Higgs condensate would be expected to catalyse baryon decay. In particular, there would be a non-zero amplitude for quarks which penetrate the core of such a defect to decay into leptons, as quarks and leptons appear in the same multiplet in GUTs. The typical size of such defects (monopole radius, or cosmic string radius) will be of the order of the GUT length scale ($10^{-30}$ cm). Naively, one might expect the cross section for monopole catalysed baryon decay processes to be of the order of the corresponding area ($10^{-60}$ cm$^2$) and that for decay processes mediated by a string, the cross section per unit length of string would be of the order of the GUT length. In the case of monopoles however, it has long been understood, since the work of Callan and Rubakov [1, 2], that cross sections for such processes will be greatly enhanced and actually be of the order of the QCD area ($10^{-30}$ cm$^2$). The essential point is that there is a mechanism (due to the long range external magnetic field of the monopole) for the amplification of the quark wavefunction at the monopole core and thereby an increased probability of penetration. Since quark masses and energies will be of the order of the QCD scale which is typically 15 orders of magnitude smaller than the inverse radius of the monopole, one is thus interested in the low energy scattering of quarks on monopole targets. (We shall, however, treat energies sufficiently high that quarks may be treated as free.)

In the case of cosmic strings, mechanisms for a similar enhancement of baryon catalysis have often been discussed [3, 4, 5]. Alford and Wilczek [3] showed that a GUT string can carry fractional flux in units of $2\pi/q$, where $q$ is the charge of a quark or lepton in the theory. Such a particle thus has a significant low energy elastic cross section for scattering from the string due to the presence of the topologically non-trivial, but pure gauge external field configuration, as shown long ago by Aharonov and Bohm [7]. It was also noted in [3] that the fermion wavefunction is amplified at the string. Physically, this means that the presence of the external gauge fields has increased the probability of fermions penetrating the region of unbroken symmetry, thereby enhancing the catalysis rate.

In this paper, we follow the computational scheme for catalysis cross sections originally proposed in [5] which breaks the calculation into two steps. In the first, the decay cross section is computed using free fermions (i.e. ignoring the external gauge fields) to give a GUT scale cross section, known as the geometric cross section. This step clearly depends strongly on the decay mechanism – whether baryon decay is mediated by interactions with internal X and Y gauge fields or by interaction with a scalar condensate in the string core. This is partly determined by the model of the string used. In the second step, one computes the degree to which the geometric cross section is enhanced by considering the scattering of fermions off the string. The prescription used is derived from first
order perturbation theory and gives the decay cross section as

\[ \frac{d\sigma}{d\Omega} = A^4 \left. \frac{d\sigma}{d\Omega} \right|_{\text{geom}} \]  

(1)

where \((d\sigma/d\Omega)_{\text{geom}}\) is the geometric cross section and \(A\) is the wavefunction amplification factor

\[ A = \frac{|\psi(a)|}{|\psi_{\text{free}}(a)|} \]  

(2)

i.e. the ratio of the magnitude of the spinor in the magnetic field to the free spinor evaluated at \(a\) the core radius of the string (at infinity, both spinors are normalised by scattering boundary conditions).

It was shown in [5] that the total cross section can be enhanced up to the QCD scale. The second step also depends on the internal model of the string (gauge fields or scalar condensate) and also on the net flux carried by the string; however, it was concluded in [5] that the distribution of gauge fields within the core does not affect the amplification factor. This claim was made on the basis of a consideration of two simple models of flux distribution: the case of a flux ring at the core radius and the case of uniformly distributed flux within the core.

Here, we re-examine this claim by considering more general models of the flux distribution. We shall concentrate exclusively on the case where the dominant low energy scattering of fermions off the string is due to interactions with gauge fields and where the decay process itself is mediated by gauge fields, although our general methodology could easily be extended to cover interactions with scalar fields. Under certain assumptions, we demonstrate that the results of [5] are indeed independent of the details of the flux distribution. However, when the flux is allowed e.g. to change sign in the string interior, we find that the amplification factors can be strongly dependent on the details of the flux distribution. It might be objected that such a field configuration is unphysical, and certainly a single gauge field whose flux changed sign in the core would probably be unstable. However, the situation we envisage is where two or more gauge fields may be represented by a single effective \(U(1)\) gauge field. For example, these fields might be the \(X\) and \(Y\) gauge fields and the electromagnetic field. If the separate gauge fields have different ranges (as they do in this example) and if they are of opposite, but constant sign (and there is no \textit{a priori} reason to prevent this) then the effective gauge field could certainly change sign in the core without prejudicing the stability of the string. Thus our results may be important for computations with realistic string models.

In order to treat these problems, we develop and extend a calculational scheme (the scattering length formalism) which was originally developed [8, 9] (see also [13]) to study the general problem of the large scale behaviour of small objects. A remarkable property of Dirac operators coupled to external \(U(1)\) gauge fields allows us to calculate the relevant parameters (the scattering lengths introduced
below) analytically for arbitrary flux distributions and by developing the analogue to the low energy expansion of potential scattering theory [12] we are thus able to provide a simple means of estimating the amplification factors for baryon decay. Our formalism leads to a physical interpretation of the enhancement process. We also comment on the validity of the thin wire approximation to the flux distribution used in [3, 4] and relate our results to our other work on the large scale effects of small objects [8, 9].

Our principal assumptions are as follows:

1. Quarks are treated (as in [4, 5]) as free Dirac particles with energies above the confinement scale. (On the GUT scale, this still corresponds to low energies.)

2. Decay processes are mediated by interactions with gauge fields in the core.

3. The cosmic string is assumed to be an infinitely long straight cylindrical string along the $z$-axis of radius $a$.

4. All fields are cylindrically symmetric about the $z$-axis.

5. The quark wavefunction is $z$-translationally invariant. (Due to the low energies of incoming quarks, we expect that including the $z$-dependence of the quark wavefunction will not significantly alter the physics.)

Our conventions are as follows: the metric has signature $+ − − −$, the incoming quark has charge $−e$ and the electromagnetic vector potential is defined by $A_\mu = (\phi, −A)$ with $\nabla \wedge A = B$, where $B$ is the magnetic field. $(r, \theta, z)$ are cylindrical polar coordinates about the $z$-axis.

2 The Dirac Equation with a Flux Tube

We consider the minimally coupled Dirac equation $(i\gamma^\mu(\partial_\mu − i e A_\mu) − \tilde{m})\psi = 0$ in the $\gamma$-matrix representation

$$\begin{align*}
\gamma^0 &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{align*}$$

For a flux tube of magnetic flux $B = B(r)\hat{z}$ with $B(r)$ vanishing for $r > a$, a simple Stokes’ theorem argument gives

$$A = \frac{\alpha(r)}{er} \hat{\theta}$$

\[\footnote{We refer to $A_\mu$ in terms appropriate to an electromagnetic field; however it should of course be thought of as the effective gauge field.}\]
where $\alpha(r)$ is defined by

$$\alpha(r) = e \int_0^r B(r')r' dr'.$$

We define $\Phi = \alpha(a) \equiv \text{(total magnetic flux)}/(2\pi/e)$. In addition, we define $\nu = \Phi - [\Phi]$, where $[\Phi]$ is the greatest integer strictly less than $\Phi$. We will only be interested in the case of non-integer flux.

Diagonalising $J_z$, the angular momentum operator about the $z$-axis, given by

$$J_z = -i\partial_\theta + \frac{1}{2} \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right)$$

with eigenvalues $n + \frac{1}{2}$ ($n \in \mathbb{Z}$) using the ansatz

$$\psi_4 = e^{-i\omega t} \exp \left\{ i \left[ n + \frac{1}{2} - \frac{1}{2} \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right) \right] \theta \right\} \left( \begin{array}{c} F_n^+(r) \\ G_n^+(r) \\ F_n^-(r) \\ G_n^-(r) \end{array} \right)$$

we find that the Dirac equation separates into two 2-spinor equations for $F_n^+, G_n^+$ and $F_n^-, G_n^-$ with radial equations

$$\begin{align*}
-i(\omega + m)G_n + \left( \frac{d}{dr} - \frac{n + \alpha(r)}{r} \right) F_n &= 0 \\
-i(\omega - m)F_n + \left( \frac{d}{dr} + \frac{n + 1 + \alpha(r)}{r} \right) G_n &= 0
\end{align*}$$

where $m = \tilde{m}$ for $F_n^+, G_n^+$ and $m = -\tilde{m}$ for $F_n^-, G_n^-$. In the rest frame of the particle, $\uparrow$ and $\downarrow$ correspond to spin aligned and anti-aligned with $\hat{z}$. We now drop the arrows and proceed to treat only one 2-spinor in each angular momentum sector:

$$\psi_n = \left( \begin{array}{c} F_n(r) \\ G_n(r) \end{array} \right).$$

It will turn out that the amplification factor has the same order of magnitude whichever 2-spinor is chosen.

The equations (5) decouple to give

$$\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + V^+_n + m^2 - \omega^2 \\
-\frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + V^-_n + m^2 - \omega^2
\end{array} \right\} F_n &= 0 \\
\left\{ \begin{array}{l}
-\frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + V^+_n + m^2 - \omega^2 \\
-\frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} + V^-_n + m^2 - \omega^2
\end{array} \right\} G_n &= 0
\end{align*}$$

where the effective potentials are given by

$$V^+_n = \left( \frac{n + \alpha(r)}{r} \right)^2 + \frac{\alpha'(r)}{r}$$

$$V^-_n = \left( \frac{n + 1 + \alpha(r)}{r} \right)^2 - \frac{\alpha'(r)}{r}. (10)$$
In the subsequent discussion we shall see that an understanding of the mechanism for enhancement depends crucially on a consideration of the details of the effective potentials \( V_n^\pm \) (i.e. on the details of the flux distribution) inside the cosmic string, and that one loses essential insight if one passes to the thin wire approximation, or if one takes non-smooth models for the flux distribution.

Equations (7) and (8) are not independent, but are coupled via the first order equations (5). It is easy to see that the coupling ensures that if one component is regular at the origin, then so is the other. Solving the system at momentum \( k = \sqrt{\omega^2 - \tilde{m}^2} \) with regular boundary conditions at \( r = 0 \), we find that outside the string core, where \( \alpha(r) \equiv \Phi \), \( F_{n,k}(r) \) and \( G_{n,k}(r) \) may be written as

\[
F_{n,k}(r) \propto \cos \theta_n J_{|n+\Phi|}(kr) - \sin \theta_n J_{|-n+\Phi|}(kr) \tag{11}
\]

\[
G_{n,k}(r) \propto \cos \varphi_n J_{|n+\Phi+1|}(kr) - \sin \varphi_n J_{|-n+\Phi+1|}(kr) \tag{12}
\]

where \( \theta_n \) and \( \varphi_n \) must be determined by matching the external solution to the solution inside the string core. The coupling between equations (5) implies that \( \tan \theta_n = -\tan \varphi_n \) in all sectors except \( n = -1 - \lfloor \Phi \rfloor \), where we require \( \tan \theta_n = -\cot \varphi_n \).

If we were to employ the “thin wire approximation”, in which the flux is concentrated in a flux tube of infinitesimal radius, the external solutions (11),(12) would hold down to \( r = 0 \). We could then apply the criterion of local square integrability (with measure \( rdr \)) at \( r = 0 \) to fix \( \theta_n = \varphi_n = 0 \) (for all \( k \)) in all sectors other than \( n = -1 - \lfloor \Phi \rfloor \), which we refer to as the critical sector. In the critical sector any choice of \( \theta_n \) and \( \varphi_n \) consistent with the coupling \( \tan \theta_n = -\cot \varphi_n \) leads to a locally square integrable wavefunction. We therefore see that in the thin wire approximation, all wavefunctions are regular and vanishing at the origin except in the critical sector where there is a 1-parameter family of possible boundary conditions, for each of which at least one of the 2-spinor components must be irregular at \( r = 0 \).

Mathematically, this corresponds to the fact that the Hamiltonians \( H_n^{\uparrow \downarrow} \) derived from (5) are essentially self-adjoint\(^2\) on \( C^\infty \)-spinors compactly supported away from \( r = 0 \) in \( L^2((0, \infty)^2, rdr) \) in all sectors other than the critical sector where \( H_n^{\uparrow \downarrow} \) have deficiency indices \( \langle 1, 1 \rangle \). This entails that there is a 1-parameter family of self-adjoint extensions labelled by elements of \( U(1) \), each of which corresponds to a different choice of boundary condition in the critical sector. The self-adjointness of the Hamiltonian is necessary to ensure a unitary time evolution, so it is only by choosing a particular self-adjoint extension that we can specify a well-defined global dynamics for the system (see also [8, 9]).

Following the procedure for computing amplification factors introduced in \( E \) (see equation (2)), we now impose scattering boundary conditions on the wavefunction in each sector and then compare the magnitude of the wavefunction

\(^2\)An operator \( A \) on some domain is **essentially self-adjoint** if its operator closure \( \overline{A} \) is self-adjoint i.e. \( \overline{A} \) and its adjoint \( \overline{A}^* \) have the same domain on which they act in the same way.
at \( r = a \) against that of the free wavefunction, which is of order 1. The scattering boundary conditions are derived in Appendix A and give the normalised spinor as:

\[
\psi_n = \left( \begin{array}{c}
\left[ 1 - (-i)^{2|n+\Phi|} \tan \theta_n \right]^{-1} (-i)^{|n+\Phi|} J_{n+\Phi}(kr) \\
\left[ 1 - (-i)^{2|n+\Phi+1|} \tan \varphi_n \right]^{-1} \Lambda(-i)^{|n+\Phi+1|} J_{n+\Phi+1}(kr) \\
\left[ 1 - (-i)^{-2|n+\Phi|} \cot \theta_n \right]^{-1} (-i)^{-|n+\Phi|} J_{-|n+\Phi|}(kr) \\
\left[ 1 - (-i)^{-2|n+\Phi+1|} \cot \varphi_n \right]^{-1} \Lambda(-i)^{-|n+\Phi+1|} J_{-|n+\Phi+1|}(kr)
\end{array} \right)
\]

(13)

where \( \Lambda = -k/(\omega + m) \) which is of order 1 at the energies of interest (where \( k \sim \tilde{m} \)), for either choice \( m = \pm \tilde{m} \).

### 3 Scattering Length Formalism

In order to determine the wavefunction amplification, it now suffices to specify \( \theta_n \) and \( \varphi_n \). This is accomplished by performing the analogue of the low energy expansion in potential scattering theory. If we denote the logarithmic derivative \( F'_{n,k}/F_{n,k}|_{r=a} \) by \( D_n \), we may expand \( D_n^+ \) in powers of \((ka)^2\): \( D_n^+ = D_n^{(0)+} + (ka)^2 D_n^{(1)+} + O(ka)^4 \). Note that \( ka \), the product of fermion momentum and string radius is of order \( 10^{-15} \). The matching between internal and external solutions

\[
cot \theta_n = \frac{D_n J_{-|n+\Phi|}(kr) - J'_{-|n+\Phi|}(kr)}{D_n J_{|n+\Phi|}(kr) - J'_{|n+\Phi|}(kr)}
\]

(14)

(where the prime denotes differentiation with respect to \( r \)) may be expanded to give

\[
cot \theta_n = \left( \frac{k R_n^+}{2} \right)^{-2|n+\Phi|} \frac{\Gamma(1 + |n + \Phi|)}{\Gamma(1 - |n + \Phi|)} \left\{ 1 - \frac{|n + \Phi|}{2} (kr_n^+)^{2} + O(k)^4 \right\}
\]

(15)

where we call \( R_n^+ \) the scattering length of the effective potential \( V_n^+ \) given by

\[
R_n^+ = a \left[ \frac{a D_n^{(0)+} - |n + \Phi|}{a D_n^{(0)+} + |n + \Phi|} \right]^{1/(2|n+\Phi|)}
\]

(16)

and \( r_n^+ \), which generalises the effective range of potential scattering, is given by

\[
(r_n^+)^2 = \frac{a^2}{2\pi|n + \Phi|^2} \left[ \frac{(a/R_n^+)^{-2|n+\Phi|}}{1 - |n + \Phi|} - 2 + \frac{(a/R_n^+)^{2|n+\Phi|}}{1 + |n + \Phi|} \right] + 2a D_n^{(1)+} \left( \left( \frac{a}{R_n^+} \right)^{|n+\Phi|} - \left( \frac{a}{R_n^+} \right)^{-|n+\Phi|} \right)^2.
\]

(17)
Our parameter $\theta_n$ is related to the phase shift $\delta_n$ relative to the Aharonov-Bohm scattering (defined by $F_{n,k} \propto \cos \delta_n J_{|n+\Phi|}(kr) - \sin \delta_n N_{|n+\Phi|}(kr)$) by

$$\cot \delta_n = \frac{\cos(|n + \Phi| \pi) - \cot \theta_n}{\sin(|n + \Phi| \pi)}.$$  

(18)

Note that there is another definition of scattering length (which is the one used in [8, 9]): equation (7) may be solved exactly at $\omega^2 = m^2$ for $r > a$ (where $V_n(r) = (n + \Phi)^2/r^2$) and takes the simple form

$$F_n(r) \propto \left( \frac{r}{R_n^+} \right)^{|n+\Phi|} - \left( \frac{r}{R_n^-} \right)^{-|n+\Phi|}$$  

(19)

which we can use to define $R_n^+$. Note that (to ensure reality of $F_n(r)$ up to a phase) $(R_n^+)^{2|n+\Phi|}$ must be real (although possibly negative) and so the allowed values of $R_n^+$ lie on a contour in the complex plane. We make this definition of $R_n^+$ to ensure that our ‘scattering length’ really has dimensions of length; it will turn out that this is a natural parametrisation to use. $R_n^+$ may be expressed in terms of $F_n(r)$ by means of a fitting formula [9]

$$R_n^+ = a \left\{ 1 - \frac{2|n + \Phi|}{a} \left| \frac{r^{n+\Phi} F_n}{(r^{n+\Phi} F_n')^{r=a}} \right| \right\}^{1/(2|n+\Phi|)}$$  

(20)

Substituting $D_{n}^{+(0)} = F'_n(a)/F_n(a)$ in (20) reduces it to (16) and we see that our two definitions of scattering length agree.

Similar considerations for $\cot \varphi_n$ (when the logarithmic derivative $D_{n}^- = G'_n/G_n|_{r=a}$ is expanded as $D_{n}^- = D_{n}^{+(0)} + (ka)^2D_{n}^{-(1)} + O(ka)^4$) give

$$\cot \varphi_n = \left( \frac{kR_n^-}{2} \right)^{-2|n+\Phi+1|} \frac{\Gamma(1 + |n + \Phi + 1|)}{\Gamma(1 - |n + \Phi + 1|)} \left\{ 1 - \frac{|n + \Phi + 1|\pi}{2}(kr_n^-)^2 + O(k)^4 \right\}$$  

(21)

where $R_n^-$ is determined from the zero energy external solution $G_n(r)$ by the fitting formula

$$R_n^- = a \left\{ 1 - \frac{2|n + \Phi + 1|}{a} \left| \frac{r^{n+\Phi+1} G_n}{(r^{n+\Phi+1} G_n')^{r=a}} \right| \right\}^{1/(2|n+\Phi+1|)}$$  

(22)

and $r_n^-$ is given by

$$\left( r_n^- \right)^2 = \frac{a^2}{2\pi|n + \Phi + 1|^2} \left[ \frac{(a/R_n^-)^{-2|n+\Phi+1|}}{1 - |n + \Phi + 1|} - 2 + \frac{(a/R_n^-)^{2|n+\Phi+1|}}{1 + |n + \Phi + 1|} \right] + 2aD_{n}^{-(1)} \begin{pmatrix} \left( \frac{a}{R_n^-} \right)^{n+\Phi+1} - \left( \frac{a}{R_n^-} \right)^{-|n+\Phi+1|} \end{pmatrix}^2.$$  

(23)
In this case, the zero energy external solution is

\[ G_n(r) \propto \left( \frac{r}{R_n^+} \right)^{|n+\Phi+1|} - \left( \frac{r}{R_n^-} \right)^{-|n+\Phi+1|} \tag{24} \]

The remarkable feature of equations (7) and (8) to which we referred in the introduction is that they may be solved analytically at zero kinetic energy (\(\omega^2 = \tilde{m}^2\)). This is because they factorise at zero kinetic energy (see Appendix B) and may be viewed as a consequence of an abstract supersymmetry possessed by Dirac operators coupled to external magnetic fields (see [11]). In Appendix B, we derive these solutions and place bounds on their corresponding scattering lengths sector by sector. We summarise our results in Table 1. In each sector, one of the two scattering lengths \(R_n^\pm\) is either zero or infinite and so the corresponding low energy expansion ((15) or (21)) breaks down. One could derive the form of the expansion in these special cases; however, the other low energy expansion remains well-defined and so, by using the formulae connecting \(\theta_n\) and \(\varphi_n\), we can always determine both as functions of \(k\) at low energies.

In sectors where the bounds derived in Appendix B allow a range of possible scattering lengths, the precise details of the flux distribution fix a particular choice via the fitting formulae. The bounds derived in Appendix B can be shown to be ‘best possible’ and so, in sectors where our bounds permit, \(R_n^\pm\) can be made arbitrarily large for arbitrarily small values of \(a\). This persists in the limit as \(a \to 0\) contradicting the results of [14].

Large scattering lengths occur only when the corresponding effective potential exhibits a potential well, as may be seen by the following argument. If the effective potential in, say, equation (7) is everywhere non-negative, then a convexity argument applied to the differential equation (see [9]) shows that the zero energy solution \(F_n(r) \neq 0\) with regular boundary conditions at the origin must satisfy \(F_n'(a)/F_n(a) \geq 0\). Inserting this in the appropriate fitting formula (20) we find that the corresponding scattering length satisfies \(|R_n^+| < a\). Thus to generate large scattering lengths it is necessary for the effective potential to exhibit a well. Note that if \(\Phi > 0\), it is necessary that the magnetic field \(B(r)\) change sign within the core of the string for \(R_n^+\) to be large in the critical sector; for the existence of a well in \(V_n^+\) implies that \(\alpha'(r)/r = B(r)\) must be negative in some interval in \((0,a)\). Similarly, for \(\Phi < 0\), \(R_n^-\) is large only if \(B(r)\) changes sign. We have already observed that such configurations are not necessarily unstable when an effective gauge potential is considered.

Of particular interest in Table 1 are the cases \(-[\Phi] \leq n \leq -1\) for \(\Phi > 1\) and \(0 \leq -2-[\Phi]\) for \(\Phi < -1\). The infinite scattering lengths in these sectors are due to the presence of bound states of zero kinetic energy located at \(\omega = -\epsilon(\Phi)\tilde{m}\) (where \(\epsilon(x) = \pm 1\) as \(x\) is greater than or less than 0). In accordance with a theorem of Aharonov and Casher [10], there are precisely \([\Phi]\) such states for \(\Phi > 0\) and \(-[\Phi] - 1\) such if \(\Phi < 0\). Although one of the scattering lengths in the critical
sector \( n = -1 - [\Phi] \) is infinite, this is not a bound state, as the wavefunction at zero kinetic energy fails to be square integrable at infinity. (Recall: \([\Phi]\) is the greatest integer \textit{strictly} less than \(\Phi\).)

4 Calculation of Amplification Factors

The information in Table 1 and expansions (15) and (21) enables us to gain some insight into the form of the wavefunction amplification at low energies (in comparison with the GUT scale), as we now have some control over the leading order behaviour of \(\cot \theta_n\) and \(\cot \varphi_n\). We can use this to construct simple order of magnitude arguments which are sufficient to demonstrate the range of possible behaviour. In particular, we will assume that \(aD^{(1)}_{n\pm}\) have magnitude of order 1 or smaller – we will not consider the additional behaviour occurring if \(D^{(1)}_{n\pm}\) are finely tuned so as to produce cancellations in (17) and (23). We also ignore the effect of higher terms in the low energy expansions (15) and (21). These assumptions amount to the approximation

\[
(kr_n^+)^2 \approx \begin{cases} 
O \left[ (ka)^2 (a/R_n^+)^{2|n+\Phi|} \right] & R_n^+ \ll a \\
O \left[ (ka)^2 \right] & R_n^+ \sim a \\
O \left[ (ka)^2 (R_n^+/a)^{2|n+\Phi|} \right] & R_n^+ \gg a 
\end{cases} 
\tag{25}
\]

This allows us to conclude that \(\cot \theta\) is well-approximated by the first term in (14) provided that

\[
a(ka)^{1/|n+\Phi|} \ll |R_n^+| \ll a(ka)^{-1/|n+\Phi|} \tag{26}
\]

and so \(\cot \theta_n = O((kR_n^+)^{-2|n+\Phi|})\). In the language of [8], we say that \(R_n^+\) is ‘believable’ at scale \(k^{-1}\). For very small scattering lengths \(R_n^+ \ll a(ka)^{1/|n+\Phi|}\), we have \(\cot \theta_n \gg (ka)^{-2(1+|n+\Phi|)}\) and for very large scattering lengths \(R_n^+ \gg a(ka)^{-1/|n+\Phi|}\), we have \(\cot \theta \sim (ka)^{2(1-|n+\Phi|)}\). Note in particular that QCD scale scattering lengths \(R_n^+ = O(k^{-1})\) are classified as ‘very large’ unless \(|n+\Phi| < 1\) when they fall within the range (26). Similar conclusions may be derived for \(\cot \varphi\).

We treat the case \(\Phi > 0\) and distinguish two cases: the critical sector \(n = -1 - [\Phi]\) and other sectors with \(n \neq -1 - [\Phi]\). In the critical sector, we find (inserting the coupling relation \(\tan \theta_n = -\cot \varphi_n\) in (13))

\[
\psi_n = \left[ 1 - (-i)^{2(1-\nu)} \tan \theta_n \right]^{-1} \begin{pmatrix} (-i)^{1-\nu} J_{1-\nu}(kr) \\ \Lambda(-i)^{-\nu} J_{-\nu}(kr) \end{pmatrix} + \left[ 1 - (-i)^{2(\nu-1)} \cot \theta_n \right]^{-1} \begin{pmatrix} (-i)^{\nu-1} J_{\nu-1}(kr) \\ \Lambda(-i)^{\nu} J_{\nu}(kr) \end{pmatrix} \tag{27}
\]

and so we see that the wavefunction is always amplified at \(r = a\), regardless of the value of \(\theta_n\). However, the degree to which it is amplified is determined by \(\theta_n\) and
can vary between \((ka)^{-\nu}\) and \((ka)^{\nu-1}\), depending on the details of the internal flux distribution. As \(|n + \Phi| < 1\) in this sector, we find that we can approximate the expression \((15)\) for \(\cot \theta\) as

\[
\text{values of GUT scale scattering lengths) the amplification is of order (} \phantom{i} \text{ka} \phantom{a} \text{kR} \phantom{a} \text{for which} \phantom{a} \text{in the lower component of the spinor. For scattering lengths on the QCD scale,} \text{the amplification is of order (} \phantom{i} \text{ka} \phantom{a} \text{kR} \phantom{a} \text{for which} \phantom{a} \text{in the lower component) and (} \phantom{i} \text{ka} \phantom{a} \text{ν} \phantom{a} \text{for which} \phantom{a} \text{in the upper component).}
\]

For scattering lengths on the QCD scale, for which \(kR_n^+ \sim 1\), the amplification is given by the larger of \((ka)^{-\nu}\) (lower component) and \((ka)^{\nu-1}\) (upper component), whilst for \(R_n^+\) in excess of the QCD scale, the amplification is of order \((ka)^{\nu-1}\), occurring in the upper component. Thus the amplification factor is strongly dependent on the details of the flux distribution.

In sectors other than \(n = -1 - [\Phi]\), we find

\[
\psi_n = \left[1 - (-i)^{2|n+\Phi|} \tan \theta_n\right]^{-1} \left(\frac{(-i)^{|n+\Phi|} J_{|n+\Phi|}(kr)}{\Lambda(-i)^{|n+\Phi+1|} J_{|n+\Phi+1|}(kr)}\right) \\
+ \left[1 - (-i)^{2|n+\Phi|} \cot \theta_n\right]^{-1} \left(\frac{(-i)^{-|n+\Phi|} J_{-|n+\Phi|}(kr)}{\Lambda(-i)^{-|n+\Phi+1|} J_{-|n+\Phi+1|}(kr)}\right),
\]

(28)

We treat the case \(-[\Phi] \leq n \leq -1\) first which arises only for \(\Phi > 1\) and corresponds to the Aharonov-Casher states. Here, we have \(0 < R_n^+ < a\). For cases in which \(R_n^+\) is of the order of \(a\), we see that \(\cot \theta_n = O((ka)^{-2|n+\Phi|})\) as \((kr_n^+)^2 \ll 1\) and so

\[
\psi_n \sim \left(\frac{O((ka)^{|n+\Phi|})}{O((ka)^{2|n+\Phi|-|n+\Phi+1|})}\right) = \left(\frac{O((ka)^{|n+\Phi|})}{O((ka)^{n+\Phi-1})}\right)
\]

(29)

and we therefore find amplification only for \(n = -[\Phi]\), where the amplification factor is \((ka)^{\nu-1}\) in the lower component. However, if \(R_n^+\) is ‘very small’ so that \((kr_n^+)^2\) is no longer negligible, \(\cot \theta_n \gg (ka)^{-2(1+|n+\Phi|)}\) and it is easy to see that there is no amplification even in the sector \(n = -[\Phi]\).

We now treat the case \(n \leq -2 - [\Phi]\). In these sectors it is easy to see, using the same methods as above, that ‘very small’ scattering lengths or scattering lengths within a few orders of magnitude of the core radius \(a\) give no amplification. However, if \(R_n^+\) is of the order of the QCD scale or larger, \(\cot \theta_n = O((ka)^{2(1-|n+\Phi|)})\), giving

\[
\psi_n \sim \left(\frac{O((ka)^{|n+\Phi|-2})}{O((ka)^{2|n+\Phi|-2-|n+\Phi+1|})}\right)
\]

(30)

and so there is amplification only in the sector \(n = -2 - [\Phi]\), with amplification factor \((ka)^{-\nu}\) in the upper component. The possibility of amplification from this sector does not appear to have been noted before. To conclude the analysis for \(\Phi > 0\), it remains to consider \(n \geq 0\). In these sectors, \(R_n^+\) is forced to be zero and we must therefore consider \(R_n^-\) to give \(\cot \varphi_n\) and then use \(\cot \theta_n = -\cot \varphi_n\).
As before, we find no amplification if $R_n^−$ is ‘very small’ or within a few orders of magnitude of $a$. In the case where $R_n^−$ is QCD scale or larger, we find $\cot \varphi_n = O((ka)^2(1−|n+\Phi+1|))$ and so

$$\psi_n \sim \begin{cases} O((ka)^2(1−|n+\Phi+1|)) & \text{if } |n+\Phi+1| ≤ 1−|\Phi| \leq \frac{1}{2}(2−|n+\Phi+1|) \\ O((ka)^{n+\Phi+1}) & \text{if } |n+\Phi+1| > \frac{1}{2}(2−|n+\Phi+1|) \end{cases}.$$ (31)

Thus there is no amplification for $n ≥ 0$ unless $[\Phi] = 0$ in which case, there is amplification of order $(ka)^{\nu−1}$ in the lower component in sector $n = 0$ only. Note that in this case, there are no Aharonov-Casher states, and that $n = 0$ is adjacent to the critical sector.

We can derive the analogous results for $\Phi < 0$ by sending $\Phi \to −\Phi, \nu \to 1−\nu, n \to −1−n$ and $R^+ \to R^-$. We find that the critical sector $n = −1−[\Phi]$ always exhibits amplification: for values of $R_n^−$ within a few orders of magnitude of $a$ or smaller, amplification occurs in the upper component with factor $(ka)^{\nu−1}$, whilst for $R_n^−$ in excess of the QCD scale, the lower component is amplified by factor $(ka)^{−\nu}$. In the Aharonov-Casher states $0 ≤ n ≤ −2−[\Phi]$, we find that only $n = −2−[\Phi]$ contributes with amplification $(ka)^{−\nu}$ in the upper component unless $R_n^− ≪ a(kα)^{1/[n+\Phi+1]}$ when there is no amplification. The sector $n = −[\Phi]$ allows amplification $(ka)^{\nu−1}$ of the lower component only for large $R_n^−$ and can be amplified in $n = −1$ only for $[\Phi] = −1$, when the upper component is amplified by $(ka)^{−\nu}$ only for large $R_n^+$. No other sectors contribute.

In conclusion, and subject always to the provisos stated before equation (25) there are thus at most three sectors which can contribute to wavefunction amplification: the critical sector $n = −1−[\Phi]$ (in which there is always amplification) and the two adjacent sectors. We have also seen that, of the sectors adjacent to the critical sector, only the Aharonov-Casher sector (when present) amplifies generically and requires an anomalously small scattering length to be suppressed, while non-Aharonov-Casher sectors require anomalously large scattering lengths in order to contribute. We summarise our results in Figure 1, where we graph $p(\Phi)$, which determines the overall amplification factor of equation (2) as $A = (ka)^{−\nu(\Phi)}$. Figure 1(a) shows the maximum (solid line) and minimum (dotted line) possible amplification for each flux. In the most general case, when we make no assumptions about the form of the flux distribution, we can say no more than this without explicitly computing the relevant scattering lengths.

However, if we know that the flux distribution is single-signed within the core, then if $\Phi > 0$ we have $V_n^+ ≥ 0$ for all $n$ and so our earlier arguments show that $0 < R_n^+ < a$; conversely, if $\Phi < 0$, we know that $0 < R_n^- < a$. In either case, (and provided $|\Phi| > 1$) the relevant scattering lengths in the contributing sectors are bounded between 0 and $a$. (In the case $|\Phi| < 1$ (in which there are no Aharonov-Casher states), there is a possible contribution in the $n = 0$ (n = −1) sector for $\Phi > 0$ ($\Phi < 0$) which is due to $R_0^−$ ($R_{−1}^+$) and therefore unaffected by these bounds.) Thus for a sufficiently ‘nice’ subclass of single-signed flux
distributions, the scattering lengths of interest will all be within a few orders of magnitude of \( a \) (i.e. of the GUT scale) and so \( p(\Phi) \) follows the graph shown in Fig. 1(b) as only the critical sector and (if \( |\Phi| > 1 \)) the adjacent Aharonov-Casher sector contribute. The solid lines indicate the range of \( \Phi \) for which the critical sector provides the dominant contribution to the amplification, whilst the dotted portions indicate the ranges where the adjacent Aharonov-Casher state gives the dominant amplification. This situation holds for many simple models of flux distribution (in particular for those examined in [5]) and, as we have indicated, for all sufficiently ‘nice’ single-signed flux distributions. The results of [5] correspond to (and agree with) our results in this case and so our discussion has demonstrated the extent to which those results can be considered generic.

We note that for \( |\Phi| > 1/2 \) Fig. 1(b) follows the maximum amplification plot and so the effect of abnormally large or small scattering lengths could change the amplification factor only by suppressing it, as would occur, for example, if the scattering length in the critical sector was large (QCD scale or larger), or the scattering length in the adjacent Aharonov-Casher sector was much smaller than \( a \). When \( |\Phi| < \frac{1}{2} \), however, it is possible to increase the amplification factor considerably by tuning the scattering length in the critical sector to be large (a similar effect occurs in this case if the scattering length in \( n = 0 \) (\( \Phi > 0 \)) or \( n = -1 \) (\( \Phi < 0 \)) is tuned to be large). The special status of \( |\Phi| < \frac{1}{2} \) in Fig. 1(b) is due to the absence of an Aharonov-Casher state in this case, which provides the dominant amplification when \( |\Phi| > 1 \) and the fractional part of \( |\Phi| \) is less than \( \frac{1}{2} \).

Finally, Fig. 1(c) shows the results which would be obtained using the thin-wire approximation, on the assumption that the scattering length in the critical sector (which is the only free parameter) is of the GUT scale. The inadequacy of this approximation is seen by its disimilarity to Fig. 1(b) and the importance of the adjacent Aharonov-Casher state becomes clear. It is important to note that the thin-wire does not support Aharonov-Casher states as they fail to be normalisable.

5 Conclusion

We first consider the relation of our current results to our other work [8, 9] on the large scale effects of small objects. In [8] we point out that in many physical situations, a small object may be replaced by a point-like or line-like idealisation and that if the dynamics of the idealised system admits more than one consistent choice of boundary condition (in our case, the Hamiltonian fails to be essentially self-adjoint on a suitable domain), this is often a signal that the large scale behaviour may be sensitive to the details of the internal structure of our original small object. Furthermore, in such cases, the large scale dynamics of the true system is well-approximated by the idealisation with an appropriate
choice of boundary condition (here a self-adjoint extension) and that therefore
the parameter(s) labelling the choice of boundary condition (in our case, the
scattering lengths) parametrise the possible large scale behaviour. This is the
content of the “principle of sensitivity” enunciated in [8].

In the case at hand for the specific purpose of computing wavefunction ampli-
fication factors the thin wire approximation fails to be a good idealisation of
the true system because there are contributions from non-critical sectors, gener-
ically from the adjacent Aharonov-Casher sector. We note however, that the
Aharonov-Casher states appear for a quite special and deep reason: an index
theorem related to the abstract supersymmetry of the Dirac operator, and so
confirms our general philosophy in [8] that when the principle of sensitivity fails
to apply, it fails for ‘interesting reasons’.

However, we see that the scattering length formalism developed in [8, 9] is
still applicable and that the amplification factor in the critical sector is strongly
sensitive to the details of the internal flux distribution. Moreover, if one considers
the elastic scattering cross section rather than wavefunction amplification factors,
it is found that the main deviation from the pure Aharonov-Bohm cross section
at low energies (large scales) occurs in the critical sector and is parametrised by
the scattering length there. Also, one can show [9] that if a sequence of Dirac Hamiltonians describing flux tubes of steadily decreasing radius has a limit (in a
suitable sense of convergent dynamics, technically strong resolvent convergence)
which is self-adjoint (i.e. a well-defined limiting dynamics) then the limit must
be a self-adjoint extension of the idealised thin-wire approximation. Thus the
principle of sensitivity seems to apply as far as scattering cross sections are con-
cerned.

We also note that if one modifies the Hilbert space or the domain on which the
Hamiltonian is defined in an appropriate way, it is possible to arrange that the
sectors \( n = -2\Phi \), \( n = -1\Phi \) and \( n = \Phi \) are precisely the sectors in which
the Hamiltonian fails to be essentially self-adjoint and that therefore it might be
that our results can be reconciled with a discussion of self-adjoint extensions after
all. This may be done in a variety of ways; for example by taking the Hilbe-
t space to be the Sobolev space given by the completion of the space of smooth
spinors compactly supported away from the flux line in the norm defined by
\( \langle \phi | \psi \rangle = \langle \phi | H^2 \psi \rangle_{L^2} \), where \( H \) is the thin wire Hamiltonian. Alternatively,
one can keep the original Hilbert space, whilst restricting the domain of the
Hamiltonian to be the range of the massless thin wire Hamiltonian acting on
smooth spinors compactly supported away from the flux line. We hope to return
to the significance of these modified versions of the thin wire approximation
elsewhere.

We conclude with various remarks. Firstly, the above arguments have estab-
lished that the amplification factors for baryon decay enhancement calculations
can depend substantially on the internal distribution of the magnetic flux. In
particular, we note that the case $|\Phi| < \frac{1}{2}$ in which we have seen that amplification can be increased includes two of the most physically interesting cases in the GUT model of [3], where scattering of the $d$ quark is modelled by $\Phi = -\frac{1}{4}$, and the electron as $\Phi = \frac{1}{4}$. For these values of $\Phi$, the difference between $(ka)^{n'}$ and $(ka)^{n-1}$ amounts to 8 orders of magnitude. It is therefore of some importance that the amplification factor be correctly computed, taking into account the details of the model. The scattering length formalism presented here provides a convenient calculational scheme.

Secondly, we turn to the physical interpretation of the process of wavefunction amplification. From above, it is clear that for a particular component to be amplified it is necessary that its corresponding (zero energy) scattering length be large (QCD scale or above). (That it is not sufficient may be seen by considering the Aharonov-Casher states). We also saw above that large scattering lengths occur only when the effective potential exhibits a well of negative potential. This makes it reasonable to suggest that the physical cause of wavefunction amplification (and therefore of baryon decay enhancement) is a resonance phenomenon caused by the spin-flux interaction: incoming quarks may tunnel into the well and be delayed, perhaps being reflected by the walls of the well before tunnelling out. Quarks are therefore present in the core of the string much longer than would naively be expected and therefore decay processes occur with increased probability.

This interpretation of the enhancement process as a resonance phenomenon depends on an examination of the details of the effective potentials (9) and (10) – in particular the presence or absence of wells. Thus our interpretation did not emerge clearly from previous work on this subject, where the restricted range of particular models treated did not display all of the possible qualitative features discussed above. We have seen in particular that the thin wire approximation (and therefore an approach based solely on self-adjoint extensions on the usual domain) is inadequate for this problem, as we have found possible contributions to enhancement not only from the critical sector $n = -1 - [\Phi]$ (which provides the only contribution in the thin-wire approximation) but from the two sectors adjacent to this sector. In particular the adjacent Aharonov-Casher sector provides the dominant enhancement for certain ranges of $\Phi$ (provided the associated scattering length is of the order of $a$, which is the case e.g. for the simple flux distributions models of [5]). This relation with the Aharonov-Casher state has not been noted before.

Finally, we note that although we find one more contributing sector than [3], the amplification arising from the new sector is at most of the order of that from the other two sectors. This is in accord with the unitarity bounds established in [3].

**Acknowledgments:** We thank Gary Gibbons, Mark Hindmarsh and Lloyd Alty for useful conversations and B. Thaller for making a copy of reference [11] available to us in advance of publication. C.J.F. thanks Churchill College, Cambridge
for the award of a Gateway Studentship. B.S.K. thanks SERC for the award of an
Advanced Fellowship and the Schweizerischer Nationalfonds for partial support.
We both thank the Institute for Theoretical Physics at the University of Berne,
Switzerland for hospitality as this work was completed.

\section{Scattering Normalisation}

In this Appendix, we derive the scattering boundary conditions required above.
The scattering theory is determined by a relation of form

\[ \alpha_n (-i)^{n+\Phi} J_{|n+\Phi|}(kr) + \beta_n (-i)^{-|n+\Phi|} J_{-|n+\Phi|}(kr) \xrightarrow{r \to \infty} (-i)^n J_n(kr) + \frac{f_n e^{ikr}}{\sqrt{r}} \]  

(32)

where the \( f_n \) are the scattering amplitudes and the integer order Bessel func-
tions arise from the expansion of the incoming plane wave. All Bessel functions
may be replaced by their asymptotic forms \( J_{\mu}(x) \sim \sqrt{2/(\pi x)} \cos(x - (\mu + \frac{1}{2})\pi/2) \). The scattering
normalisation is then determined by requiring the coefficients of \( e^{-ikr} \) to
match and gives \( \alpha_n + \beta_n = 1 \). This leads to the normalised spinor (13)
given in the text.

\section{Bounds on Scattering Lengths}

We derive here the range of allowed \( R_n^+ \) sector by sector. The scattering
lengths \( R_n^- \) may be derived from these by \( R_n^- = R_{-1-n}^+(\Phi) \). Equation (7) at zero
kinetic energy (\( \omega^2 = \tilde{m}^2 \)) is

\[ \left\{ \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - \left( \frac{n + \alpha(r)}{r} \right)^2 + \frac{\alpha'(r)}{r} \right\} \psi = 0. \]  

(33)

This factorises as

\[ \left( \frac{d}{dr} + \frac{n + 1 + \alpha(r)}{r} \right) \left( \frac{d}{dr} - \frac{n + \alpha(r)}{r} \right) \psi = 0 \]  

(34)

and so may be solved exactly to give two independent solutions

\[ \psi^{(1)}(r) = r^n \exp \left\{ \int_0^r \frac{\alpha(r')}{r'} dr' \right\} \]  

(35)

\[ \psi^{(2)}(r) = r^n \exp \left\{ \int_0^r \frac{\alpha(r')}{r'} dr' \right\} \int_0^r r'^{-1-2n} \exp \left\{ -2 \int_0^{r'} \frac{\alpha(r'')}{r''} dr'' \right\} dr'. \]  

(36)

The solution \( F_n(r) \) to (7) at zero kinetic energy is the solution with regular
boundary conditions at \( r = 0 \). For \( n \geq 0 \) this is clearly \( \psi^{(1)}(r) \) whilst for \( n < 0 \),
\( \psi^{(2)} \) is the appropriate solution.
Case (i): $n \geq 0$ For $r > a$, $\psi^{(1)}(r) \propto r^{n+\Phi}$. Comparing with (19) or using the fitting formula (20), we see that $R_n^+ = 0$ if $n \geq -[\Phi]$ (matching to $r^{n+\Phi}$) or $R_n^+ = \infty$ otherwise (matching to $r^{-[n+\Phi]}$).

Case (ii): $n < 0$ In general, $\psi^{(2)}(r)$ matches to a non-trivial linear combination of $r^{\pm[n+\Phi]}$ so the situation is more complex. The logarithmic derivative is given by

$$
\frac{F'_n}{F_n} \bigg|_{r=a} = \frac{n + \Phi}{a} + \frac{f(a)}{af(a)}
$$

(37)

where $f(r)$ is defined by

$$
f(r) = r^{-1-2n} \exp \left\{ -2 \int_0^r \frac{\alpha(r')}{r'} dr' \right\}
$$

(38)

and $\bar{f}(r)$ by

$$
\bar{f}(r) = \frac{1}{r} \int_0^r f(r') dr'
$$

(39)

Clearly, $f(r)$ and $\bar{f}(r)$ are positive and non-vanishing except at $r = 0$. We may rewrite the fitting formula (20) as

$$
(R_n^+/a)^{2n+\Phi} = 1 - \frac{2|n + \Phi|}{a} \left[ \frac{|n + \Phi|}{a} + \frac{F'_n}{F_n} \bigg|_{r=a} \right]^{-1}.
$$

(40)

Thus if $n \geq -[\Phi]$, we obtain

$$
R_n^+ = a \left[ 1 - \left( 1 + \frac{1}{2|n + \Phi|} \frac{f(a)}{\bar{f}(a)} \right)^{-1} \right]^{1/(2|n+\Phi|)}
$$

(41)

from which we can conclude the bound $0 < R_n^+ < a$, given our observations about $f(r)$ and $\bar{f}(r)$. On the other hand, if $n < -[\Phi]$ we find

$$
R_n^+ = a \left[ 1 - 2|n + \Phi| \frac{\bar{f}(a)}{f(a)} \right]^{1/(2|n+\Phi|)}
$$

(42)

yielding the bound $-\infty < (R_n^+/a)^{2|n+\Phi|} < a^{2|n+\Phi|}$. Note that $R_n^+$ must be finite as a consequence of the non-vanishing of $f(a)$.

Furthermore, one can show that the above bounds are best possible in the sense that, for given $n$ and any radius $a$, there exist magnetic flux distributions supported within radius $a$ with any scattering length in the above allowed ranges. This may be proven by observing that a potential $V(r)$ takes the form (9) with $\alpha(r)$ smooth and obeying $\alpha(r) = \lambda r^2 + O(r^4)$ as $r \to 0$ and $\alpha(r) = \Phi$ for $r \geq a$, if and only if the equation

$$
\left( -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + V \right) u = 0
$$

(43)
has a smooth solution \( u(r) \) which is non-vanishing in \((0, \infty)\) and obeys \( u(r) = r^n(1 + \lambda r^2 + O(r^4)) \) as \( r \to 0 \) and \( u(r) \propto r^{n+\Phi} \) for \( r \geq a \), whereupon we may identify

\[
\alpha(r) = r \frac{u'}{u} - n. \tag{44}
\]

Full details will appear in [9].

We remark that in [14] it is mistakenly concluded that in the limit as \( a \to 0 \), the upper spinor component is always either \( J_{|\Phi|} \) or \( J_{-|\Phi|} \) and that (translated into our language) \( R_n^+ \to 0 \) if \( \Phi > 0 \) in the critical sector. However, our arguments above and in [8] hold for arbitrarily small \( a \) and thus the range of scattering lengths allowed in the limit as \( a \to 0 \) is simply the appropriate limit of the range for finite \( a \). Thus in the critical sector for \( \Phi > 0 \), any scattering length in the range \( -\infty < (R_n^+)^{2|\Phi|} \leq 0 \) is allowed as the limit of scattering lengths of a sequence of flux tubes of decreasing radius. In [9] we also prove rigorous statements about the convergence of the associated sequence of Hamiltonians.

References

[1] C.G. Callan, Phys. Rev. D 25 (1982) 2058

[2] V. Rubakov, Pis’ma Zh. Eksp. Teor. Fiz. 33 (1981) 658; JETP Lett. 33 (1981) 644

[3] M.G. Alford and F. Wilczek, Phys. Rev. Lett. 62 (1989) 1071

[4] M.G. Alford, J. March-Russell and F. Wilczek, Nucl. Phys. B328 (1989) 140

[5] W.B. Perkins, L. Perivolaropoulos, A.-C. Davis, R.H. Brandenberger and A. Matheson, Nucl. Phys. B353 (1991) 237

[6] Ph. de Sousa Gerbert, Phys. Rev. D 40 (1989) 1346

[7] Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485

[8] B.S. Kay and C.J. Fewster, When Can A Small Object Have A Big Effect At Large Scales? in preparation

[9] C.J. Fewster and B.S. Kay, Low Energy Quantum Dynamics in Magnetic Fields of Small Support in preparation

[10] Y. Aharonov and A. Casher, Phys. Rev. A 19 (1979) 2461

[11] B. Thaller, The Dirac Equation Springer-Verlag in press
[12] R.G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill 1966)

[13] B.S. Kay and U.M. Studer, Comm. Math. Phys. **139** (1991) 103

[14] C.R. Hagen, Phys. Rev. Lett. **64** (1990) 503

[15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol II: Fourier Analysis, Self-Adjointness* (Academic Press 1975)
\[
\begin{array}{|c|c|c|}
\hline
n \geq 0 & R^+_n = 0 & -\infty < (R^-_n)^{2|n+\Phi+1|} < a^{2|n+\Phi+1|} \\
[-\lfloor \Phi \rfloor \leq n \leq -1] & 0 < R^+_n < a & R^-_n = \infty \\
n = -1 - [\Phi] & -\infty < (R^+_n)^{2|n+\Phi|} < a^{2|n+\Phi|} & R^-_n = \infty \\
n \leq -2 - [\Phi] & -\infty < (R^+_n)^{2|n+\Phi|} < a^{2|n+\Phi|} & R^-_n = 0 \\
\hline
\end{array}
\]

**Table 1(a):** Allowed scattering lengths for \( \Phi > 0 \).

\[
\begin{array}{|c|c|c|}
\hline
n \geq -[\Phi] & R^+_n = 0 & -\infty < (R^-_n)^{2|n+\Phi+1|} < a^{2|n+\Phi+1|} \\
n = -1 - [\Phi] & R^+_n = \infty & -\infty < (R^-_n)^{2|n+\Phi+1|} < a^{2|n+\Phi+1|} \\
0 \leq n \leq -2 - [\Phi] & R^+_n = \infty & 0 < R^-_n < a \\
n \leq -1 & -\infty < (R^+_n)^{2|n+\Phi|} < a^{2|n+\Phi|} & R^-_n = 0 \\
\hline
\end{array}
\]

**Table 1(b):** Allowed scattering lengths for \( \Phi < 0 \).
Fig. 1(a) Maximum and minimum amplification factors.

Fig. 1(b) Amplification assuming GUT scale scattering lengths.

Fig. 1(c) Amplification from the critical sector with GUT scale scattering lengths.