Thermodynamic Equilibrium of a Wormhole Station

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We study the thermodynamic equilibrium of a station which connects many distinct asymptotic regions of a space-time. An example of the station is a throat of a wormhole connecting two asymptotic regions. The temperatures of the station measured in various asymptotic regions are not necessarily the same. We propose a conjecture relating the temperatures in a coordinate independent manner and present explicit examples which support the conjecture. We also suggest a fundamental question on the identity of mass.

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The zeroth law of thermodynamics states that if each of two thermodynamic systems is in thermal equilibrium with a third one, then they are in thermal equilibrium with each other. In other words, thermal equilibrium between systems is a transitive relation. This character leads to the definition of temperature. In the presence of gravity, the story is not that simple. If a system is in thermodynamic equilibrium, the local temperature at a strong gravity region is higher than that at a low gravity region, which is justified by the redshift of the light. Accordingly, Tolman [1] defined the local temperature as

\[ T(x) = \sqrt{\frac{g_{tt}(\infty)}{g_{tt}(x)}} \beta_+^{-1}, \tag{1} \]

where \( \beta_+^{-1} \) represents the temperature of the system measured by an asymptotically free observer and \( g_{tt}(x) \) is the time-time component of the metric tensor at an event \( x \). Note that the local temperature at the horizon of a black hole diverges because \( g_{tt}(x) \to 0 \).

Meanwhile, for a small statistical system, the temperature is determined from the averaged energy of particles. In the presence of radiation of enough concentration, the temperature can be identified to be

\[ T(x) \equiv \left( \frac{\rho(x)}{\sigma} \right)^{1/4}, \tag{2} \]

where \( \sigma \) and \( \rho(x) \) are the Stefan-Boltzmann constant and the radiation energy density at \( x \), respectively. Given the local temperature \( T(x) \), one can obtain the asymptotic temperature from Eq. (1). Notice that the asymptotic temperature is dependent on the redshift factor between the asymptotic region and the system. Therefore, in the presence of many distinct asymptotic regions observing a given macroscopic system, it is not guaranteed that the measured asymptotic temperatures are the same. There are two reasons for this. First, observers in different asymptotic regions may watch different parts of a given system. Then, the measured local temperature will be different. Second, the redshift factor from the system to the asymptotic regions may be dependent on the choice of the asymptotic region.

A typical example having two distinct asymptotic regions is a wormhole space-time, which is a solution of general relativity with a throat connecting two distinct regions of space-time [2–5]. A throat of a traversable wormhole is made of anisotropic exotic matter which violates energy conditions (null energy condition etc.) [6]. If we do not stick to general relativity, the throat can be composed of an ordinary matter. Various modified gravity theories such as Einstein-Gauss-Bonet theory [7], Lovelock theory [8], \( f(R) \) gravity [9], etc. [10, 11] permit ordinary matter as a source of wormhole solutions. Recently, Susskind and Maldacena proposed ‘ER=EPR’ conjecture that entangled particles are connected by a wormhole [12]. The role of wormholes as a key for a breakthrough in physics is going to be more important.

We consider a four-dimensional space-time in which \( N \)-separated asymptotic regions are connected through a station. The station plays the role of a wormhole connecting the \( N \)-outsides, each of which contains an asymptotically flat region. A wall \( W_n \), which is a border between the station and the outside \( A_n \), is assumed to be hard enough to prevent matter inside from spilling out, where \( n = 1, \cdots, N \). Therefore, the whole system we are interested in is composed of \( N \)-outsides and a station connecting them. A characteristic plot for \( N = 4 \) is given in Fig. 1. A typical example of \( N = 2 \) case is a wormhole space-time in which two asymptotic regions are connected by a station, the wormhole throat. The whole station is filled with self-gravitating matter which is in thermal equilibrium. Its geometry is described by the general theory of relativity without cosmological constant.

We ask whether the temperatures of the station measured in various asymptotic regions are the same or not. Through the wall \( W_n \), the station interacts with \( A_n \) so that an asymptotic observer in \( A_n \) can measure its temperature. We show that, even though the station is in static equilibrium, the measured temperature \( \beta_n^{-1} \) could be different from the temperature \( \beta_m^{-1} \) measured in other outside \( A_m \neq n \). Finding a relation between each asymptotic temperatures is one of the purpose of the present work.
Let us write the coordinate value \( x \) on the wall because of the symmetry. Then, the wormhole station connecting four asymptotic regions.

Consider the geometry around a specific wall \( \mathcal{W}_n \). Let the local temperature must be independent of the position on the wall because of the symmetry. Then, the geometry of the wall will be described by the metric of the form, \( ds^2_{\mathcal{W}_n} = g^{ab}(x^n)dx^a dx^b + r^2(x^n)dt^2 \), \( a,b = 0,1 \). (3)

Let us write the coordinate value \( x^a \) of the wall \( \mathcal{W}_n \) as \( x^n \).

We conjecture for the temperature \( \beta_n^{-1} \) of the station measured in the asymptotic region in \( A_n \)

\[
\beta_n^{-1} \sqrt{-g_n(x^n)} = \mathcal{S},
\]  
where \( g_n(x^n) = \det(g_{ab}) \) and \( \mathcal{S} \) represents a constant with a dimension of temperature which is independent of \( n \). Note that, given a boundary, the left-hand side of Eq. (4) is independent of the coordinate choice. At the present stage, we do not have a general proof for the conjecture in Eq. (4). Rather, we present heuristic arguments which justify the conjecture through a few examples.

**Example 1: Spherically symmetric case.** First, we consider the case that both of the station and the outsides bear the spherical symmetry. For simplicity, we assume that the station consists of a ‘core’ and several ‘branches’ attached to the core. Explicitly, the core may have \( S^3 \) symmetry, which will be broken to \( S^2 \) symmetry at the positions where branches are attached. The core consists of exotic matter and supports the station. The branches consist of radiation and provide physical equilibrium condition between the outsides. Because of the spherical symmetry, \( r(x^n) \) in Eq. (3) can be used as a radial coordinate, i.e. \( x^a \equiv (t,r) \). Without loss of generality, we assume that the core is confined in \( r < r_- \) by using an (imaginary) hard wall. We are not interested in the detailed geometrical structure of the core except that it is in thermal equilibrium with the radiation in branches. A simplest way to go around the difficulty in dealing the details of the core is to set the physical parameters of the branches at \( r = r_- \) to be the same for all the branches and the core to couple with the branches with the same manner. The branch \( B_n \) \( (n = 1, \cdots, N) \) is connected with the outside \( A_n \). Let \( B_n \) be located at \( r_- \leq r \leq r_n \). At \( r = r_n \), another hard wall prevents the radiation from spreading out to \( A_n \).

In the absence of matter in \( A_n \), the geometry will be described by the Schwarzschild metric with a mass parameter \( M_+ \). The geometry of the branch \( B_n \) will be described by a general spherically symmetric metric for \( r_- \leq r \leq r_+ = r_n \) given by

\[
ds^2_n = -\chi^2(r) e^{-2\psi(r)} dt^2 + \frac{dr^2}{\chi^2(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]  
(5)

Now, \( g(r) = -e^{-2\psi} \). The time-symmetric condition for static metric presents \[13\] \( \chi^2(r) = 1 - 2m(r)/r \) where

\[
m(r) \equiv M_- + \int_{r_-}^r (4\pi r^2)\rho(r') dr'.
\]  
(6)

Here \( \rho(r) \) and \( M_- \) are the energy density at \( r \) and a mass parameter related with the region \( r < r_- \), respectively. Explicit value of \( M_- \) should be determined from the physics inside the core. Here, \( M_- \) is set to be independent of \( n \) after assuming that the branches are attached to the core with the same way. In the absence of a black hole, \( m(r) \) satisfies \( m(r) \leq r/2 \) for all \( r \). With a sufficient concentration, the energy and the entropy densities of radiation with temperature \( T(r) \) become

\[
\rho(r) = \sigma T(r)^4, \quad s(r) = \frac{4\sigma}{3} T^{3/4} = \frac{4\sigma^{1/4}}{3} \rho(r)^{3/4},
\]  
(7)

where \( \sigma \) is the Stefan-Boltzmann’s constant. Now, the entropy of the radiation in \( B_n \) will be given by the volume integral over the the branch \[13\]:

\[
S_n = \int_{B_n} s^n n_a d\Sigma = \frac{(4\pi \sigma)^{1/4}}{3} \int_{r_-}^{r_n} \left[ \frac{1}{r^2} \frac{dm(r)}{dr} \right]^{3/4} r^2 dr \chi(r).
\]  
(8)

There is an interesting relation between gravity and thermodynamics, the so-called ‘maximum entropy principle’ (MEP): The maximum entropy state of a system corresponds to its static stable configuration \[13, 14\]. As a result, the variation of the entropy of perfect fluid reproduces the Tolman-Oppenheimer-Volkhoff (TOV) equation for a static star \[15\]. Considering the MEP for a static system, the local variation of the entropy with respect to \( m(r) \) must vanish. This reproduces the TOV equation for the spherically symmetric metric \[13\]. Allowing the mass variation at the boundaries \( r_{\pm} \), the en-
entropy satisfies [16, 17],
\[
\delta S_n = \beta_+ \delta M_n + (\beta_+ - \beta_-) \delta M_- = (\beta_+ - \beta_-) \delta M_+ + \beta_- \delta M_n, \tag{9}
\]
where \(M_n \equiv M_+ - M_-\) is the mass of the radiation in the branch \(B_n\) and by using Eq. (2) and (6),
\[
\beta_\pm \equiv \frac{1}{\chi_\pm} \left( \frac{4 \pi \sigma r^2 \psi_\pm^2}{m^2(r_\pm)} \right)^{1/4} = \frac{1}{\chi_\pm} T_\pm. \tag{10}
\]

Let us assume that \(-\chi_\beta^2\) is the time component of the metric for the outside \(A_1\), i.e. \(\chi_1 = \sqrt{-g_{tt}(r_1)}\). Then, the mass variation through the outer wall of \(B_1\) satisfies, when the heat flow through the inner wall vanishes,
\[
\delta M_1 = \beta_1^{-1} \delta S_1.
\]
This implies that \(\beta_1^{-1}\) is nothing but the asymptotic temperature of the radiation in \(B_1\) when it is measured by an asymptotic observer in \(A_1\). The thermodynamic interaction through the wall at \(r_-\) is given by \(\delta M_n = \beta_\pm \delta S_1\) after setting the heat flow through the outer wall to vanish, \(\delta M_+ = 0\). Note that this relation infers that \(\beta_\pm\) plays the role of a temperature with respect to the heat flow. At the present model, the core is assumed to have an \(S(3)\) symmetry which will be broken to \(S(2)\) of the branches. Because \(r_-\) is identical for all \(n\), the physical parameters at \(r_-\) must be the same for all \(n\).

Then, the local temperature at \(r_\pm\) becomes
\[
T_\pm = \frac{1}{\chi_\pm \beta_\pm}, \quad T_- = \frac{\beta_-^{-1}}{\sqrt{g_{tt}(r_-)}} = \frac{\beta_+^{-1}}{\chi_- e^{-\psi(r_-)}},
\]
where we use \(\psi_\pm \equiv \psi(r_\pm) = 0\). This leads to
\[
e^{-\psi_\pm - \beta_\pm^{-1}} = e^{-\psi_\pm + \beta_\pm^{-1}} \Rightarrow \sqrt{-g_{tt}(r_-)} \beta_\pm^{-1} = \sqrt{-g_{tt}(r_-)} \beta_\pm^{-1}. \tag{11}
\]
Here, we rewrite \(e^{-\psi(r)}\) as a coordinate independent form: \(e^{-\psi(r)} = \sqrt{-g_{tt}(r)}\). The relation in Eq. (11) holds for all \(B_n\). Therefore we find
\[
\beta_\pm^{-1} \sqrt{-g_{tt}(r_-)} = \beta_\pm^{-1} \sqrt{-g_{tt}(r_-)} = \text{constant } \text{for all } n. \tag{12}
\]
Note that the right-hand side is assumed to be the same for all \(n\) because the distribution of the exotic matter is the same for all \(n\). Therefore, this proves the conjecture (4) for the spherically symmetric case. Note that the equivalence is independent of the geometry of the region \(r < r_-\).

In deriving the result in Eq. (12), we use radiation as a matter surrounding the exotic matter. An immediate question is whether this result holds for other kind of matter than the radiation or not.

**Example 2:** Perfect fluids, One may notice that the same result holds even for a general perfect fluid satisfying \(p = w p\). In doing this one may use
\[
\rho = \sigma T^{(w+1)/w}, \quad s = \sigma(w + 1) T^{1/w}. \tag{13}
\]
By using exactly the same method, one can show that Eq. (4) is satisfied.

**Example 3:** Plane geometry. One may also show that the branches may have other geometry than the spherically symmetric one. For example, one can consider the case that the branches \(B_n\) is described by a brane-like metric with a symmetric two-dimensional \((y, z)\)-plane,
\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 (dy^2 + dz^2). \tag{14}
\]
Considering the Einstein equation and the matter in Eq. (13), one can obtain the result in Eq. (4).

**Summarizing,** we studied the thermodynamic equilibrium condition for a space-time in which \(N\)-separated asymptotic regions are connected through a station. A conjecture, \(\beta_n^{-1} \sqrt{-g_n} = \text{constant } (n = 1, \ldots N)\) relating the temperatures of the station measured in the asymptotic regions, was presented, where \(g_n = \det(g_{\mu\nu}(x_n))/\det(h_{ij}(x_n))\). Here \(x_n\) represents the event on the wall \(W_n\) and \(g_{\mu\nu}\) and \(h_{ij}\) are the metric tensor and the induced metric of the wall, respectively. The value \(g_n\) should be the same over all the events on the wall \(W_n\). To support this conjecture, we have illustrated a few examples. The results are important because they are independent of the details of the core geometry made of the exotic matter.

Let us display the metric of an outside, when it is in a vacuum state. Considering the spherically symmetric case, the geometry in the region \(A_n\) is described by the Schwarzschild metric,
\[
ds_n^2 = -g_n \left( 1 - \frac{2 M_{+n}}{r} \right) dt^2 + \frac{dr^2}{1 - 2 M_{+n}/r} + r^2 d\Omega^2, \tag{15}
\]
where \(r > r_+ \equiv r_n\) and \(g_1 = 1\). \(M_{+n} \equiv M_- + M_n\) is the Arnowitt-Deser-Misner (ADM) mass of the station observed in the outside \(A_n\). Here, \(M_n\) is the mass of the branch \(B_n\). \(M_-\) is a mass parameter determined by the property of the core. The value of \(M_-\) must be independent of \(n\) because the core is assumed to play an identical role to the branches. Notice that the ADM masses of the station seen from other outside is not generally identical even though they see the same object. The origin of the difference in current setting is that the masses of the branches are not the same. It is natural to ask whether the mass difference is physically acceptable or not. In other words, is there any physical thermodynamic process between the core and branches to adjust the ADM masses to be the same? Given a static configuration, there is no reason for an additional dynamical evolution in the absence of an external heat flow. In addition, the position of the outer wall of the station \((r_n)\) is chosen by the station constructor which is independent of the global condition. In this sense, if we believe locality, we should accept that the ADM mass may not be identical when the station is seen from observers in distinct asymptotic regions. This raises a fundamental question on the
identity of mass. In particle physics, the mass is treated to be an intrinsic quantity of a particle. On the other hand, in general relativity, the mass especially the ADM mass is a quantity defined by the response of the asymptotic geometry with respect to a local energy. So far, the two definitions have not been contradictory because they present the same value for a point particle or a star if gravitational energy is included. At the present case, the ADM mass is not uniquely defined and therefore its relation to the local energy will be obscured. The problem becomes worse when two outsides are adjoined to form one asymptotic region. If an observer notice both sides of a station, he will not recognize the station as one object but will recognize it as two distinct objects of different mass.

The obscurity above originates from the topological difference of the ‘station’ from an ordinary spacetime having one asymptotic region. For an ordinary star or black hole, matter resides in one asymptotic region. However, wormhole station connects two or more asymptotic regions so that the matter in the station cannot be regarded as located inside any of an asymptotic boundary. Consider two constructors ‘digging’ a wormhole using matter of masses, $M_1$, $M_2$ toward each other. The ADM masses can be defined and give $M_1$, $M_2$ for each. The proper mass parameter corresponding to each will decrease with $r$. If the temperatures on both sides does not meet the equilibrium condition we suggested, the configuration of the throat will not be stable. The mass parameters will be adjusted until the equilibrium is reached. Therefore, our result can be regarded as an equilibrium condition when the construction of station is completed.

The space-time is flat in the asymptotic regions of every $A_n$. However, the asymptotic regions are inequivalent in general because the speed of the proper time for an asymptotic observer, $\frac{\mathrm{d} \tau_n}{\mathrm{d} t} = \sqrt{g_n}$, is dependent on $n$ as shown in Eq. (15). In addition, the measured asymptotic temperature of the station $\beta_n^{-1}$ is generally different from $\beta_m^{-1}$ but is related by $\beta_m^{-1} = \sqrt{g_n/g_m} \beta_n^{-1}$. Note that the temperature ratio is the same as the redshift factor $\sqrt{g_{n,tt}(\infty)/g_{m,tt}(\infty)}$ of the light between the two asymptotic regions in $A_n$ and $A_m$. In $A_n$, one can introduce a new time coordinate $t' = \sqrt{g_n} t$. Renaming $t'$ to $t$ will make the metrics in the asymptotic regions look the same for all $n$ formally. However, this formal coincidence does not imply physical equivalence. Let the asymptotic regions of $A_n$ and $A_n$ are adjoined. Then, we consider a thin and long tube which connects the two throats $W_n$ and $W_n$. The tube may begin at $W_m$, pass through $A_m$ and $A_n$, successively, and end at $W_n$. Now, the two walls interact with each other thermodynamically through the tube. If the transitivity of the thermodynamic equilibrium (the zeroth law of thermodynamics) holds even in this case, the two walls should be in thermal equilibrium. This implies that, between the asymptotic regions of $A_m$ and $A_n$, the redshift corresponding to the temperature ratio should exist to compensate the temperature difference. In other words, between the two asymptotically flat regions, a region of curved space-time exists which produces the redshift factor.

We now consider an opposite situation. Let a wormhole connects two separated regions of an asymptotically flat space-time, where the asymptotic geometry is described by a Minkowski metric. To be consistent with the zeroth law of thermodynamics, the temperatures of the wormhole detected at both sides of its throat should be identical, i.e., $\beta_1^{-1} = \beta_2^{-1}$, where the subscript 1, 2 represents each side. If one constructs the station as a symmetrical shape with respect to the wormhole throat so that $r_1 = r_2$, the solution discussed in this article with $\beta_1 = \beta_2$ will work. On the other hand, the constructor may not want a symmetrical shape. Then, the local temperatures at both sides of the throat do not identical but are related by $\sqrt{-g_m(r_1)} T_1(r_1) = \sqrt{-g_m(r_2)} T_2(r_2)$. The symmetric core solution studied in this work fails to satisfy this boundary condition. In other words, the geometry of the core must be asymmetric. Because there are infinitely many choices of $r_1$ and $r_2$, many asymmetric static states can exist. Which of them is a preferable one is out of the boundary of the present work. To answer the question, exact solutions to the core part are required.

At the present setting, we do not consider the stability issue of the self-gravitating system. Even though the issue does not involved in the definition of the local temperature, it should be examined before the construction of the station especially because the core is made of exotic matter which does not satisfying the energy conditions.

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