KdV and Almost Conservation Laws

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Abstract. In this article we illustrate a new method to extend local well-posedness results for dispersive equations to global ones. The main ingredient of this method is the definition of a family of what we call almost conservation laws. In particular we analyze the Korteweg-de Vries initial value problem and we illustrate in general terms how the “algorithm” that we use to formally generate almost conservation laws can be used to recover the infinitely many conserved integrals that make the KdV an integrable system.

1. Introduction

This short survey paper is concerned with a new method to prove global well-posedness results for dispersive equations below energy spaces, namely $H^1$ for the Schrödinger equation and $L^2$ for the KdV equation.

Even though I am the single writer of this article, all the new statements that I will make below have been proved together with my collaborators J. Colliander, M. Keel, H. Takaoka and T. Tao. What started as a simple lunch at Stanford two years ago, evolved into a very fruitful collaboration in mathematics and a pleasant friendship. For whatever the reader appreciates in what follows, we all take the credit, for the mistakes, inaccuracies and the typos, I am the only one to blame!

Before starting with the story that I am set to tell, I should warn the reader that because this article is a written version of the talk that I gave at the conference on Harmonic Analysis in Mt. Holyoke College, I will not present the complete proofs of the statements, but rather the main ideas involved in them. The interested reader can check the references that I will list for a detailed proof of all the claims made. I also apologize in advance for not citing all the work that has been published in the context of well-posedness for dispersive equations. Here I will limit the bibliography to those publications that are in direct contact with the methods and the findings that I am about to describe.

We end this section with some notations. Throughout the paper we use $C$ to denote various constants. If $C$ depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C\|u_0\|_2$ will depend on $\|u_0\|_2$. We use $A \lesssim B$
to denote an estimate of the form $A \leq CB$, where $C$ is an absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a + \varepsilon$ and $a - \varepsilon$, for some $0 < \varepsilon \ll 1$.

We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R})$ norm. For a fixed interval of time $[0, T]$ and a Banach space of functions $X$, we denote with $C([0, T], X)$ the space of the continuous maps from $[0, T]$ to $X$.

We define the spatial Fourier transform of $f(x)$ by
$$
\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx
$$
and the spacetime Fourier transform of $u(t, x)$ by
$$
\mathcal{F}(u)(\tau, \xi) := \hat{u}(\tau, \xi) := \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x\xi + t\tau)} u(t, x) \, dt \, dx.
$$
Note that the derivative $\partial_x$ is conjugated to multiplication by $i\xi$ by the Fourier transform. We shall also define $D_x$ to be the operator conjugate to multiplication by $\langle \xi \rangle := 1 + |\xi|$. We can then define the Sobolev norms $H^s$ by
$$
\|f\|_{H^s} := \|D_x^s f\|_2 = \|\langle \xi \rangle^s \hat{f}\|_{L^2_x}.
$$

2. Well-posedness and conservation laws

We consider the initial value problem (IVP) given by
\begin{equation}
\label{eq:2.1}
\begin{cases}
\partial_t u + P(D) u + N(u) = 0 \\
u(x, 0) = u_0(x),
\end{cases}
\end{equation}
where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ or $\mathbb{T}^n$, $P(D)$ is a differential operator with constant coefficients and $N(u)$ is the nonlinear part of the equation. For the moment we do not assume any special structure either for $P(D)$ or $N(u)$, we only assume that in terms of derivatives $P(D)$ is of at least one order higher than $N(u)$, in other words we assume that the first equation in (2.1) is semilinear. The function $u_0$ is called the initial profile and in general we assume that $u_0 \in H^s$.

We will use the following definition for well-posedness:

**Definition 2.1.** The IVP (2.1) is locally well-posed (l.w.p.) in $H^s$ if for any $u_0 \in H^s$ there exists $T = T(\|u_0\|_{H^s})$ and a unique solution $u \in C([0, T], H^s)$ for (2.1). Moreover the map that associates to each initial data its evolution is continuous.

We say that the IVP is globally well-posed (g.w.p.) in $H^s$ if for any $T > 0$ the definition above is satisfied.

The question of l.w.p is certainly the first one that one investigates. After a positive result, then one tries to extend the local result to a global one.

To convince the reader that proving well-posedness for a small interval of time is simpler than proving it for any fixed interval of large size, we briefly recall the contraction method. We first use the Duhamel principle to write (2.1) as the integral equation:
\begin{equation}
\label{eq:2.2}
u(t, x) = W(t)u_0 + \int_0^t W(t - t') N(u(t')) \, dt',
\end{equation}
where $W(t)u_0(x)$ is the solution of the linear problem
\begin{equation}
\label{eq:2.3}
\begin{cases}
\partial_t v + P(D) v = 0 \\
v(x, 0) = u_0(x).
\end{cases}
\end{equation}
If one is willing to reduce the size of the interval of existence of the solution then one can replace (2.2) with

\[(2.4) \quad u(t, x) = \psi(t/\delta)W(t)u_0 + \psi(t/\delta) \int_0^t W(t - t')N(u(t')) \, dt',\]

where \(\psi(t)\) is a smooth cut-off function for the interval \([-2, 2]\). We can still claim that \(u\) solves (2.3) in \([0, \delta]\) if and only if \(u\) solves (2.4) in the same interval. Now, consider the operator

\[(2.5) \quad Lv(t, x) = \psi(t/\delta)W(t)u_0 + \psi(t/\delta) \int_0^t W(t - t')N(v(t')) \, dt',\]

and assume that we are able to prove that there exists a Banach space \(X^s\) and \(s_0 \in \mathbb{R}\), such that for any \(s \geq s_0\) we have \(X^s \subset C(\mathbb{R}, H^s)\) and

\[(2.6) \quad \|\psi(t/\delta)W(t)u_0\|_{X^s} \leq C_0\|u_0\|_{H^s},\]

\[(2.7) \quad \left\| \psi(t/\delta) \int_0^t W(t - t')N(v(t')) \, dt' \right\|_{X^s} \leq C_1 \delta^\alpha \|u\|_{X^s} F\|u\|_{X^{s_0}},\]

\[(2.8) \quad \left\| \psi(t/\delta) \int_0^t W(t - t')[N(v_1(t')) - N(v_2(t'))] \, dt' \right\|_{X^s} \leq C_1 \delta^\alpha \max\{\tilde{F}(\|v_1\|_{X^{s_0}}), \tilde{F}(\|v_2\|_{X^{s_0}})\} \|v_1 - v_2\|_{X^s},\]

where \(\alpha > 0\), and \(F, \tilde{F}: \mathbb{R} \to \mathbb{R}^+\) are functions bounded on bounded sets. If we set \(a = 2C_0\|u_0\|_{H^{s_0}}\) and we take \(\delta^\alpha = 1/4(C_1 \max\{F(a), \tilde{F}(a)\})^{-1}\), then the operator \(L\) defined in (2.5) maps the ball \(B_a\) in \(X^s\) centered at the origin and radius \(a\) into itself and is a contraction. Hence a unique fixed point exists and this is the unique solution for (2.1). Using a combination of (2.6) and (2.8) one also obtains, for free, the continuity with respect to the initial data.

Arguably, this method has been used to prove the best results on local well-posedness for a variety of dispersive equations (see [2, 3, 16, 17, and 8], just to name a few).

We assume now for simplicity that \(N(u)\), the nonlinear part of the equation, is polynomial and that again (2.6), (2.7) and (2.8) still hold. Then the method we just described gives well-posedness in \(H^s\), \(s \geq s_0\) in an interval of time \([0, T]\) such that

\[(2.9) \quad T = C\|u_0\|_{H^{s_0}}^{-\beta},\]

for some \(\beta > 0\). We discuss now how to extend this short time result to a long time one.

Accordingly to (2.9), if we are willing to restrict our result to data small in \(H^s\), then we can enlarge the time of existence. But this is not our goal here. We are looking in fact for a long time well-posedness for any initial data in \(H^s\).

The first attempt that one can try is to iterate the short time result. Again by looking at (2.3), it is clear that the obstacle in doing so will be the growth of \(H^{s_0}\) norm of the solution \(u(t)\) of (2.1). It is at this stage that uniform bounds for the Sobolev norms of the solution \(u\) are needed and the conservation laws are the first source for such bounds.

The existence of useful conservation laws depends on the structure of the equation in (2.1). So to continue our general exposition in this first section we do not
write explicitly any conservation laws involving the solution \( u \), but instead we assume a consequence of them, whenever they are available, that is we assume that there exists \( s_* \in \mathbb{R} \) such that
\[
\|u(t)\|_{H^{s_*}} \leq C^*,
\]
where \( C^* \) does not depend on \( t \). If now \( s_* \geq s_0 \), then by (2.9) and (2.10) we can take \( T^* = C(C^*)^{-\beta} \) and iterate the local well-posedness result presented above. In the rest of the paper we will refer to this as \textit{the method of conservation laws}.

We consider now two special examples of the IVP (2.1). We start with the cubic defocusing Schrödinger equation in \( \mathbb{R}^2 \):
\[
\begin{cases}
  i\partial_t u + \Delta u - |u|^2 u = 0, \\
  u(x,0) = u_0(x).
\end{cases}
\]
There are two conservation laws for this problem: the Hamiltonian
\[
\int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2(x,t) + \frac{1}{4} |u|^4(x,t) \, dx = C_1,
\]
and the \( L^2 \) norm
\[
\int_{\mathbb{R}^2} |u|^2(x,t) \, dx = C_0.
\]
Using the Gagliardo-Nirenberg inequality, (2.12) and (2.13), one obtains (2.10) for \( s_* = 1 \). On the other hand one can prove that if \( s > 0 \) then the IVP (2.11) is well-posed in \( H^s \) for an interval of time \([0,T]\), where \( T \lesssim \|u_0\|_{H^s}^\beta \), for some \( \beta > 0 \), (see [7] and [2]). Then by the method of conservation laws presented above one obtains global well-posedness for \( s \geq 1 \). So this method leaves the gap \( s \in (0,1) \) open because the l.w.p., in the sense defined here, is barely missed at \( s = 0 \), where the next conservation laws (2.13) could have been used!

Next we pass to the KdV initial value problem
\[
\begin{cases}
  \partial_t u + \partial_x u + \frac{1}{3} \partial_x^2 u^2 = 0, \\
  u(x,0) = u_0(x),
\end{cases}
\]
where \( x \in \mathbb{R} \) or \( T \). The KdV equation is special, in fact it enjoys infinitely many conserved integrals. Here we recall only the first four of them (notice that here \( u \) is a real function!):
\[
\begin{align*}
(2.15) \quad &\int u(x,t) \, dx = C_0 \\
(2.16) \quad &\int u^2(x,t) \, dx = C_1 \\
(2.17) \quad &\int \partial_x u^2 + \frac{2}{3} u^3 \, dx = C_2 \\
(2.18) \quad &\int (\partial_x^2 u)^2 - \frac{5}{3} u(\partial_x u)^2 + \frac{5}{9} u^4 = C_3.
\end{align*}
\]
Bourgain proved local well-posedness in \( L^2 \), hence by (2.14) and the method of conservation laws, global well-posedness in \( L^2 \) [2]. Later Kenig, Ponce and Vega showed that the IVP (2.14) on the line is locally well-posed in \( H^s, s > -3/4 \), leaving the gap \( s \in (-3/4,0) \) open for global well-posedness. Similarly they proved \(^1\) and this is sharp!
that KdV on the circle is locally well-posed in $H^s$, $s \geq -1/2$, leaving here the gap $s \in [-1/2,0)$.

Similar results to the ones presented here for the IVP \((2.11)\) and \((2.14)\) are available, with the obvious changes, also for the modified KdV equations \([16]\), the 1D Schrödinger equation with derivative nonlinearity \([19]\), the KP-II equation \([6]\) and the Zakharov system \([1]\).

**Remark 2.2.** The method of conservation laws has two types of limitations. In general they only provide bounds for the $H^1$ norm (coming from the Hamiltonian), and the $L^2$ norm. Hence when good local results are available for rough data these uniform bounds are not enough to cover all the possible indices, and gaps are left as we showed above. The second limitation is that in higher dimensions well-posedness results are available only for relatively smooth data (in general in $H^{s_n}$, where $n$ is the dimension and $s_n > n/2$). Then again uniform bounds in $H^1$ and $L^2$ are not enough (at least not yet!) to control these higher Sobolev norms.

### 3. The method of Bourgain

The method that we are about to describe is used to prove global well-posedness for rough initial data in low dimensions. It partially solves the first limitation of the method of conservation laws discussed in Remark 2.2. This method was first introduced by Bourgain \([4]\) who considered the cubic, defocusing NLS on $\mathbb{R}^2$, (but soon the reader will appreciate its generality).

As recalled above, for this IVP, the method of conservation laws leaves the gap $(0,1)$ between l.w.p and g.w.p.. So assume that $u_0 \in H^s$ and $s < 1$. We split $u_0 = \phi_0 + \psi_0$, such that
\[
(3.1) \quad \hat{\phi}_0(\xi) = \chi_{|\xi| \leq N} \hat{u}_0(\xi) \quad \hat{\psi}_0(\xi) = \chi_{|\xi| > N} \hat{u}_0(\xi),
\]
that is we decompose $u_0$ into low and high frequency parts. One can immediately observe that the low frequency part $\phi_0$ is smoother, but has a large norm:
\[
\|\phi_0\|_{H^1} \lesssim N,
\]
while the high frequency part $\psi_0$ clearly does not improve its smoothness, but its lower order norms are small:
\[
\|\psi_0\|_{H^s} \lesssim N^{\sigma-s}, \quad \text{for} \quad \sigma \leq s.
\]

Then we evolve these two initial data. We call $u^0$ the evolution of the low frequency part $\phi_0$ under the equation in \((2.11)\). We call $v^0$ the evolution of the high frequency part $\psi_0$ under the difference equation
\[
i \partial_t v^0 + \Delta v^0 = |v^0 + u^0|^2(v^0 + u^0) - |u^0|^2 u^0.
\]
We can rewrite $v^0(t,x) = S(t)\psi_0(x) + w^0(t,x)$, where $e^{it\Delta} \psi_0(x)$ is the solution of the associated linear problem
\[
(3.2) \quad \begin{cases}
i \partial_t v + \Delta v = 0, \\
v(x,0) = \psi_0(x).
\end{cases}
\]
and
\[
w^0(t,x) = \int_0^t S(t-t')(|v^0 + u^0|^2(v^0 + u^0) - |u^0|^2 u^0) dt'.
\]

\[\text{So far this has only been proved in low dimensions.}\]
is the nonlinear part. Clearly \( u^0(t) + v^0(t) = u(t) \), where \( u \) is the solution of (2.11). There are two key parts in Bourgain’s argument. The first is that there exists \( \delta = \delta(\|\phi_0\|_{H^1}) > 0 \) such that both \( u^0(t) \) and \( v^0(t) \) are defined for \( t \in [0, \delta] \). The second, more surprising, is that \( \|w_0\|_{H^1} \lesssim 1/N^\alpha \), for some \( \alpha = \alpha(s) > 0 \).

This is now the right set up for iteration. At this point we know that the unique solution \( u(x,t) = u^0(x,t) = v^0(x,t) \) lives for all times in \([0, \delta]\). To proceed from \( \delta \) to \( 2\delta \) we start a new IVP at time \( \delta \), by assigning the new initial data

\[
\phi_1 = u^0(\delta) + w^0(\delta) \\
\psi_1 = e^{i\delta \Delta} \psi_0
\]

and we repeat the argument above. An iteration like this would work on any finite interval \([0, T]\), as long as the total error is at most comparable with the size of \( \|\phi_0\|_{H^1} \), the quantity that defines \( \delta \), that is

\[
\sum_{i=1}^{M} \|w^i\|_{H^1} \sim \|\phi_0\|_{H^1} \sim N,
\]

where \( M \sim \delta^{-1}T \). By simple calculations on the explicit formula for \( \delta \) and \( \alpha(s) \) that we do not report here, one obtains the following result \([4]\)

**Theorem 3.1 (Bourgain).** The Shrödinger IVP (2.11) in \( \mathbb{R}^2 \) is globally well-posed in \( H^s \) for \( s > 3/5 \).

Using this method several authors extended global well-posedness results for variety of equations, see for example \([13]\) for the KdV equation, \([14]\) for the modified KdV, \([18]\) for wave equations, \([22]\) and \([23]\) for the KP-II equation and \([21]\) for the Schrödinger equations with derivative nonlinearity.

### 4. The almost conservation laws: a first attempt

We restrict the description of this method to the KdV initial value problem (2.14). We remark at the end on the applications to other equations.

To help the reader in understanding this method we decided to reproduce in a coherent way the evolution of thoughts that guided us to our recent findings. We start by proving the conservation of the \( L^2 \)-norm for the solution \( u \) of (2.14) by integration by parts. We refer to this proof as a proof in physical space in contrast with another one that we will give later and that will be performed in frequency space. If we multiply the equation in (2.14) by \( u \) we obtain

\[
\frac{1}{2} \partial_t u^2 = -\partial_x (u \partial_x^2 u) + \partial_x \left( \frac{1}{2} |\partial_x u|^2 \right) - \frac{1}{3} \partial_x u^3
\]

and integration over the line, or in the periodic case, over the circle, we obtain the desired identity

\[
\frac{d}{dt} \|u\|_{L^2}^2 = 0.
\]

\(^4\)The reader should appreciate the remarkable fact that the nonlinear part of the evolution of the high frequency of the initial data is smoother than the data itself and small in the energy norm, hence it can be treated as an error!
This type of proof does not involve any analysis of the interaction of frequencies, which we believe is the key to understand the evolution not just of the of the $L^2$, but also of the $H^s$ norms, for any $s \in \mathbb{R}$.

We recall that in Section 3 we observed that the method of conservation laws cannot establish global results for $(2.14)$ on the line, when the initial data $u_0 \in H^s$ for $s \in (-3/4, 0)$. So we assume that $s < 0$. There are no conservation laws, that we are aware of, for the $H^s$ norm, when $s$ is negative, hence some new idea has to be considered. We borrow from Bourgain [4] the splitting process into low and high frequency, but this time the splitting is done in a smooth way and on the solution $u$ itself, not the initial data. This argument has been successfully used by Keel and Tao for the 1D wave map problem [13]. So we consider the multiplier

$$
(4.2) \quad \hat{I}u(\xi) = m(\xi)\hat{u}(\xi), \quad m(\xi) = \begin{cases} 
1, & |\xi| < N,
N^{-s}|\xi|^s, & |\xi| \geq 10N
\end{cases}
$$

where $m$ is smooth and monotone and $N$ is a large number to be fixed later. The operator $I$ (barely) maps $H^s(\mathbb{R}) \mapsto L^2(\mathbb{R})$. Observe that on low frequencies \{ $\xi : |\xi| < N$ \}. $I$ is the identity operator. Note also that $I$ commutes with differential operators. We now want to repeat the argument presented above to prove the conservation of the $L^2$ norm, but this time for $\|Iu(t)\|_{L^2}$. Using the Fundamental Theorem of Calculus, the equation, and integration by parts, we have

$$
(4.3) \quad \|Iu(t)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \int_0^t \frac{d}{dt}(Iu(\tau), Iu(\tau))d\tau,
$$

$$
= \|Iu(0)\|_{L^2}^2 + 2\int_0^t (I\hat{u}(\tau), Iu(\tau))d\tau,
$$

$$
= \|Iu(0)\|_{L^2}^2 + 2\int_0^t (I(-u_{xxx} - \frac{1}{2}\partial_x[u^2])(\tau), Iu(\tau))d\tau
$$

$$
= \|Iu(0)\|_{L^2}^2 + \int_0^t (I(-\partial_x[u^2]), Iu)d\tau,
$$

where $(\cdot, \cdot)$ is the scalar product in $L^2$. The error that could make $\|Iu(t)\|_{L^2}$ too large in the future is

$$
(4.4) \quad R(t) = \int_0^t (I(-\partial_x[u^2]), Iu)d\tau.
$$

The idea is to use local well-posedness estimates to show that locally in time $R(t)$ is small. To do so we first have to recall the precise local well-posedness result of Kenig, Ponce and Vega [17]. We define the space $X^{s,b}$, $s, b \in \mathbb{R}$ as the closure of the Schwartz’s functions with respect to the norm

$$
\|f\|_{X^{s,b}} = \left(\int_{\mathbb{R}^2} |\hat{f}(\xi, \tau)(1 + |\xi|)^{2s}(1 + |\tau - \xi^3|)^{2b}|^2 d\xi d\tau\right)^{1/2}.
$$

Observe that for $b > 1/2$, it follows that $X^{s,b} \subset C([0,T], H^s)$. Kenig, Ponce and Vega proved the following bilinear estimate [17]

**THEOREM 4.1** (Kenig-Ponce-Vega). For $s > -3/4$ and $b > 1/2$, there exists $b' < b$ such that

$$
(4.5) \quad \|\partial_x(uv)\|_{X^{s,b'-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b'}}.
$$

Moreover if $s < -3/4$, there is no $b$ and $b'$ such that $(4.5)$ is true.
This bilinear estimate is essential to obtain an estimate like (4.7) and hence to use a fixed point theorem. The local well-posedness result can be summarized in the following theorem. Assume that \( \psi(t) \) is a cut-off function relative to the interval \([-2,2]\).

**Theorem 4.2 (Kenig-Ponce-Vega).** For any \( u_0 \in H^s, s > -3/4 \) there exist \( T = C(\|u_0\|_{H^s})^{-\alpha} \) and a unique solution \( u \) for (2.14) such that \( u \) exists for all \( t \in [-T,T] \) and in particular

\[
\|\psi(\cdot/T)u\|_{X^{s,\beta}} \leq C\|u_0\|_{H^s}.
\]

A modification of this theorem can be proved when we introduce the multiplier operator \( I \). In fact we have

**Theorem 4.3.** For any \( u_0 \in H^s, s > -3/2 \) there exist \( T = C(\|u_0\|_{L^2})^{-\alpha} \) and a unique solution \( u \) for (2.14) such that \( u \) exists for all \( t \in [-T,T] \) and in particular

\[
(4.6) \quad \|\psi(\cdot/T)u\|_{X^{0,\beta}} \leq C\|u_0\|_{L^2}.
\]

Now let's go back to the estimate of the error \( R(t) \). Using Plancherel

\[
|R(t)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x I(u^2)(\xi,\tau)| (1 + |\xi|)^s (1 + |\tau - \xi^3|)^{\beta - 1}
\times \left| \chi_{1}(\xi,\tau) (1 + |\xi|)^s (1 + |\tau - \xi^3|)^{-\beta + 1} \right| d\xi d\tau,
\]

where \( \chi_{1} \) is the characteristic function of \([0,t]\). Then by the Cauchy-Schwarz inequality we have

\[
(4.7) \quad |R(t)| \leq \|\partial_x I(u^2)\|_{X^{0,1-\nu}} \|Iu\|_{X^{0,1-\nu'}}.
\]

Hence if we could prove a bilinear inequality like

\[
(4.8) \quad \|\partial_x I(u^2)\|_{X^{0,1-\nu'}} \leq N^{-\beta} \|Iu\|_{X^{0,\beta}}^2,
\]

for some \( \beta > 0 \), then we would be done because the factor \( N^{-\beta} \) would make the error small. But unfortunately, even though (4.8) looks a lot like (4.7), it is false due to the interaction of very low frequencies \((|\xi| < < N)\) with very large frequencies \((|\xi| >> N)\). But not everything is lost, in fact we can introduce for free a suitable cancellation\(^4\) by rewriting (14.3) as

\[
(4.9) \quad R(t) = \int_0^t \int \partial_x \left\{ (I(u))^2 - I(u^2) \right\} Iu \ dx d\tau,
\]

and we replace (4.7) with

\[
(4.10) \quad |R(t)| \leq \|\partial_x \{ (I(u))^2 - I(u^2) \}\|_{X^{0,1-\nu'}} \|Iu\|_{X^{0,1-\nu}}.
\]

Now the following desired proposition is true (see (11))

**Proposition 4.4. (Extra smoothing)** The bilinear estimate

\[
(4.11) \quad \|\partial_x \{ (I(u)(v) - I(u)v)\|_{X^{0,-1/2-}} \leq C N^{-\frac{3}{4}} \|Iu\|_{X^{0,1/2+}} \|Iv\|_{X^{0,1/2+}}.
\]

holds.

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\(^4\)This will be explained in more details in Theorem 4.5.

\(^5\)This is not obvious at first sight, for more explanation one should consult (11).

\(^6\)This cancellation is recognisable once one writes the expression in Fourier transform and uses the mean value theorem, see (11) for a precise calculation.
Combining (4.3) with (4.10) and (4.11), we obtain the almost conservation law
\[ \|Iu(t)\|_{L^2}^2 \leq \|Iu(0)\|_{L^2}^2 + C N^{-\frac{3}{10}} \|u\|_{X_{0,1/2+}}^3. \]

In proving the following theorem we describe in detail how one obtains a global result by an iteration based on (4.12).

**Theorem 4.5.** The initial value problem (2.14) is globally well posed in $H^s$ for all $s$ such that $s > -\frac{3}{10}$.

**Proof.** The proof is taken from [11]. Global well-posedness of (2.14) will follow if we show well-posedness on $[0,T]$ for arbitrary $T > 0$. We renormalize things a bit via scaling. If $u$ solves (2.14) then $u_\lambda(x,t) = (\frac{1}{\lambda})^2 u(\frac{x}{\lambda}, \frac{t}{\lambda^3})$ solves (2.14) with initial data
\[ u_{0,\lambda}(x,t) = \left(\frac{1}{\lambda}\right)^2 u_0 \left(\frac{x}{\lambda}\right). \]

Note that $u$ exists on $[0,T]$ if and only if $u_\lambda$ exists on $[0,\lambda^3 T]$. A calculation shows that
\[ \|Iu_{0,\lambda}\|_{L^2} \leq C \lambda^{-\frac{3}{10}} N^{-\frac{s}{2}} \|u_0\|_{H^s}. \]

Here $N = N(T)$ will be selected later but we choose $\lambda = \lambda(N)$ right now by requiring
\[ C \lambda^{-\frac{3}{10}} N^{-\frac{s}{2}} \|u_0\|_{H^s} \sim 1 \implies \lambda \sim N^{-\frac{2}{3+s}}. \]

We now drop the $\lambda$ subscript on $u_0$ by assuming that
\[ \|Iu_0\|_{L^2} = \epsilon_0 \ll 1, \]
and our goal is to construct the solution of (2.14) on the time interval $[0, \lambda^3 T]$.

The local well-posedness result of Theorem 4.3 shows we can construct the solution for $t \in [0,1]$ if we choose $\epsilon_0$ small enough. Using (4.4) and (4.16), the almost $L^2$ conservation property (4.12) we obtain
\[ \|Iu(t)\|_{L^2}^2 \leq \epsilon_0^2 + N^{-\frac{3}{10}}. \]

We can iterate this process $N^{\frac{3}{10}}$ times before doubling $\|Iu(t)\|_{L^2}$. Therefore, we advance the solution by taking $N^{\frac{3}{10}}$ time steps of size $O(1)$. We now restrict $s$ by demanding that
\[ N^{\frac{3}{10}} \gtrsim \lambda^3 T = N^{-\frac{3+s}{2}} T \]
is ensured for large enough $N$, so $s > -\frac{3}{10}$.

5. The almost conservation laws: the final version

The cancellation that we introduced in (4.10), and that can be seen explicitly in frequency space by taking Fourier transforms, led us to try to understand more deeply the interaction of frequencies during the evolution of the solution $u(x,t)$ of (2.14). For this purpose we propose here a proof in frequency space of the $L^2$ conservation law for the solution of (2.14). By the Plancherel theorem we have
\[ \|u(t)\|_{L^2}^2 = \int \hat{u}(\xi) \overline{\hat{u}}(\xi) d\xi = \int \hat{u}(\xi) \overline{\hat{\Delta}(\xi)} d\xi = \int_{\xi_1+\xi_2=0} \hat{u}(\xi_1) \overline{\hat{u}(\xi_2)} \xi_1 d\xi_1 \xi_2, \]

where $\Delta$ is the dispersion relation for the KdV equation.

---

\[ \text{We refer to these types of estimate as almost conservation laws because of the presence of the decaying factor } N^{-\beta}. \]
since \( u \) is \( \mathbb{R} \)-valued. Therefore, by substituting in the equation we obtain
\[
\partial_t \int \hat{u}(\xi) \overline{\bar{u}}(\xi) d\xi = 2 \int_{\xi_1 + \xi_2 = 0} \hat{u}(\xi_1) \overline{\bar{u}}(\xi_2) d\xi_1 d\xi_2
\]
\[
= 2 \int_{\xi_1 + \xi_2 = 0} \left[ -(i\xi_1)^3 \hat{u}(\xi_1) - \frac{1}{2} (i\xi_1) \overline{\bar{u}}^2(\xi_1) \right] \hat{u}(\xi_2) d\xi_1 d\xi_2.
\]
Now we symmetrize the first term and we expand the convolution to get
\[
\frac{d}{dt} \int u^2(x) dx = \partial_t \int \hat{u}(\xi) \overline{\bar{u}}(\xi) d\xi = -\int_{\xi_1 + \xi_2 = 0} i(\xi_1^3 + \xi_2^3) \hat{u}(\xi_1) \overline{\bar{u}}(\xi_2) d\xi_1 d\xi_2
\]
\[
- \int_{\xi_1 + \xi_2 + \xi_3 = 0} i(\xi_1 + \xi_2) \hat{u}(\xi_1) \overline{\bar{u}}(\xi_2) \overline{\bar{u}}(\xi_3) d\xi_1 d\xi_2 d\xi_3.
\]
The first term is clearly zero. Upon writing \( \xi_1 + \xi_2 = -\xi_3 \) and symmetrizing, the second term vanishes too. This \textit{symmetrization/cancellation} describes the non linear interaction of the frequencies of the solution \( u \) for the KdV equation in (2.14). We stress here once more that we think this is an important mechanism to understand in order to keep track of the various pieces of \( \hat{u} \) once we perform a frequency localization like we did by introducing the multiplier operator \( I \).

It is time now to introduce some notation that will make the rest of our presentation less cumbersome. We start with the following definitions:

**Definition 5.1.** A \( k \)-multiplier \( m \) is a function \( m : \mathbb{R}^k \rightarrow \mathbb{C} \). A \( k \)-multiplier is symmetric if \( m(\xi) = m(\sigma(\xi)) \) for all \( \sigma \in S_k \). The symmetrization of a \( k \)-multiplier is
\[
[m]_{\text{sym}}(\xi) = \frac{1}{n!} \sum_{\sigma \in S_k} m(\sigma(\xi)).
\]
A \( k \)-multiplier generates the \( k \)-linear functional via the integration
\[
\Lambda_k(m) = \int_{A_k} m(\xi_1, \ldots, \xi_k) \hat{u}(\xi_1) \ldots \hat{u}(\xi_k),
\]
where \( A_k = \{ (\xi_1, \ldots, \xi_k) / \xi_1 + \cdots + \xi_k = 0 \} \).

We immediately observe that we can rewrite \( \|Iu\|_{L^2} \) using the \( \Lambda \) notation above. In fact
\[
\|Iu(t)\|_{L^2}^2 = \int_{A_2} m(\xi_1 m(\xi_2) \hat{u}(\xi_1, t) \hat{u}(\xi_2, t) = \Lambda_2(m(\xi_1) m(\xi_2)).
\]
It is then clear the purpose of next proposition:

**Proposition 5.2.** Suppose \( u \) satisfies the KdV equation, and \( m \) is a symmetric \( k \)-multiplier and
\[
\Lambda_k(m) = \int_{A_k} m(\xi_1, \ldots, \xi_k) \hat{u}(\xi_1) \ldots \hat{u}(\xi_k),
\]
is the \( k \)-linear functional generated by \( m \). Then
\[
\frac{d}{dt} \Lambda_k(m) = \Lambda_k(m \alpha_k) - i \frac{k}{2} \Lambda_{k+1} (\tilde{m}(\xi_1, \ldots, \xi_{k+1})),
\]
where
\[
\alpha_k = i(\xi_1^3 + \cdots + \xi_k^3)
\]
and
\[ \tilde{m}(\xi_1, \ldots, \xi_{k+1}) = m(\xi_1, \ldots, \xi_{k-1}, [\xi_k + \xi_{k+1}]) (\xi_k + \xi_{k+1}). \]

We now describe the general principle behind the almost conservation laws.

Let \( m \) be an \( \mathbb{R} \)-valued even 1-multiplier. Define again the multiplier operator \( I \) via
\[ \tilde{I}(\xi) = m(\xi) f(\xi). \]

For convenience of notation we rename \( E_i^j(t) = \|u(t)\|^i_{\|H_i^2}, \) and
\[ E_i^2(t) = \|Iu(t)\|^i_{\|I^2} = \Lambda_2(m(\xi_1)m(\xi_2)). \]

Our goal now is to define a hierarchy of modified energies \( E_i^j(t), i = 2, 3, \ldots \) for the solution of the IVP \([2.14]\) such that, when \( m \) is like in \([1.2]\),
\[ \|u(t)\|^i_{\|H^i} \lesssim E_i^j(t) \lesssim E_i^{j+1}(t), \]
\[ (E_i^{j+1}(b) - E_i^{j+1}(a)) \ll (E_i^j(b) - E_i^j(a)), \]
for any fixed interval \([a, b]\. In other words we want to find better generations of energies that are comparable to the original norm \( \|u(t)\|^j_{\|H^i} \), but which increments decrease as the generations evolve.

We now present an algorithm that formally\(^8\) provides improved generations of energies. Using Proposition \([5.2]\) we calculate
\[ \frac{d}{dt} E_i^j(t) = \Lambda_2(m(\xi_1)m(\xi_2)\sigma_2 - i\Lambda_3(m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)). \]

We should point out that for \( m \) as in \([1.2]\)
\[ R(t) = \int_0^t \Lambda_3(-i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}) \, ds \]
where \( R(t) \) is the error defined in Section \([4]\). We proved in Proposition \([4.4]\) that even though \( R(t) \) is a threelinear expression coming from a bilinear expression such as \( E_i^j(t) \), the symmetrization\(^9\) allows us to obtain a decay in \( N \) which is in fact what gives \([5.3]\). So our goal is to push this idea further in the following way: we first denote
\[ M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}. \]

Then we define the third generation of modified energy as
\[ E_i^3(t) = E_i^2(t) + \Lambda_3(\sigma_3), \]
where \( \sigma_3 \) is a multiplier that will be chosen later. Now again by Proposition \([5.2]\) we have
\[ \frac{d}{dt} E_i^3(t) = \Lambda_3(M_3) + \Lambda_3(\sigma_3\sigma_3) + \Lambda_4(M_4), \]
where
\[ M_4(\xi_1, \ldots, \xi_4) = \sigma_3(\xi_3 + \xi_4). \]

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\(^8\)This is the same operator introduced in Section \([1.4]\) when we take \( m \) as in \([1.2]\).

\(^9\)At this stage not all the mathematical quantities that we write are proven to make sense, we will worry about this later, for the moment we just want to give the original flow of ideas that brought us to rigorous and useful results!

\(^{10}\)This is what we called then cancellation.
We choose \( \sigma_3 \) to cancel the \( \Lambda_3 \) terms, that is
\[
\sigma_3 = -\frac{M_3}{\alpha_3}.
\]
Because \( \alpha_3 = \sum_{i=1}^{3} \xi_i^3 \), we expect that
\[
|M_4| = \left| M_4 \frac{(\xi_3 + \xi_4)}{\alpha_3} \right| \ll |M_4|,
\]
and hence (5.4). Certainly at this point our expectation is a pure leap of faith because anybody could argue that when \( \sum_{i=1}^{3} \xi_i^3 = 0 \), the left hand side of the expression in (5.7) would become infinity unless a miraculous cancellation occurs in the numerator. What really amazed us was that indeed such a miracle happens! The “miracle” is a combination of the type of frequency cancellation that we observed in the proof of the \( L^2 \) conservation law via the frequency method, with several applications of the Mean Value Theorem that we can perform since we are assuming that the multiplier \( m \) is smooth, see [12] for details.

The process we described above may be iterated to formally generate a sequence of modified energies \( \{E_j(t)\}_{j=2}^{\infty} \) with the property that
\[
\frac{d}{dt} E_j(t) = \Lambda_{j+1}(M_{j+1}).
\]
The hard part of the argument is to present a rigorous proof for the statement
\[
|M_{j+1}| \ll |M_j|
\]
in an appropriate sense!

Before we proceed to a less formal, but more technical discussion on the algorithm above, we want to convince the reader that in principle our method could be used to recover all the conservation laws that the KdV equation enjoys. We didn’t set to the onerous task of checking this in detail, but we can show at least an example that is not trivial, see also [12], the paper where this computation first appeared.

We first specify the multiplier \( m(\xi) = i\xi \). Then
\[
E^2(t) = \| \partial_x u \|_L^2 = \Lambda_2((i\xi_1)(i\xi_2)).
\]
Next we define \( E^3(t) = E^2(t) + \Lambda_3(\sigma_3) \), and we use Proposition 5.3 to see that
\[
\partial_t E^3(t) = \Lambda_3(i(\xi_1 + \xi_2)i\xi_3(\xi_1 + \xi_2)) + \Lambda_3(\sigma_3\alpha_3) + \Lambda_4(M_4),
\]
where \( M_4 \) is explicitly obtained from \( \sigma_3 \). Noting that \( i(\xi_1 + \xi_2)i\xi_3(\xi_1 + \xi_2) = -\xi_3^3 \) on the set \( \xi_1 + \xi_2 + \xi_3 = 0 \), we know that
\[
\partial_t E^3(t) = \Lambda_3(-\frac{1}{3}\alpha_3) + \Lambda_3(\sigma_3\alpha_3) + \Lambda_4(M_4).
\]
The choice of \( \sigma_3 = \frac{1}{3} \) results in a cancellation of the \( \Lambda_3 \) terms and
\[
M_4 = [(\xi_1 + \xi_2)]_{sym} = \xi_1 + \xi_2 + \xi_3 + \xi_4,
\]
so \( M_4 = 0 \). Therefore, \( E^3(t) = \Lambda_2((i\xi_1)(i\xi_2)) + \Lambda_3(\frac{1}{3}) \) is an exactly conserved quantity. The modified energy construction applied to the Dirichlet energy \( \| \partial_x u \|_L^2 \) led us to the Hamiltonian for KdV described in (2.17).

\[\text{11 The reader should observe that } \sum_{i=1}^{3} \xi_i^3 = 0 \text{ is the relationship that defines the resonance set of three wave interaction!}\]
to higher order derivatives in $L^2$ we expect that will similarly lead to the higher conservation laws of KdV.

Assume now that the initial data $u_0$ of our IVP is in $H^s$, $s \in (-3/4, 0)$. Let $m$ be the multiplier defined in (4.2).

Using multilinear type estimate one can show that

\begin{equation}
\|u(t)\|_{H^s} \lesssim E_1^2(t) \lesssim E_1^2(t).
\end{equation}

But the heart of the matter is the following proposition

**Proposition 5.3.** For fixed $T > 0$

\begin{equation}
E_1^4(T) - E_1^4(0) = \int_0^T \Lambda_5(M_5(\tau))d\tau \leq C_T N^{-3+\epsilon}\|Iu\|_{X^{0,1/2+}}^5.
\end{equation}

For a complete proof see [12].

At this point probably the reader would like to ask the following question: Why did we stop at $E_1^4$? The obvious answer that we can give is that we stopped because the decay of the increment of this modified energy, given by (5.10), is enough to obtain the best possible result:

**Theorem 5.4.** The IVP (2.14) is globally well-posed in $H^s$ for $s > -3/4$.

But there is a much deeper reason why we didn’t pursue the estimates of the increment of the energies $E_k^4$, for $k > 4$. The formal expression for the increment of these energies becomes more and more complex. Nice algebraic properties like (5.11) and (5.12) below are no longer available! Also it seems to us that the reason why we didn’t need to estimate the increment for all the modified energies is that $-3/4$ is larger than the scaling index, which, in this case, is $-3/2$.

The proof of Theorem 5.4 is similar to the proof of Theorem 4.5 if one uses (5.9) and replaces (4.12) with (5.10), see [12] for details.

To give an idea of the type of miracle that makes (5.8) analytically correct we consider $M_4$, defined in (5.6). The complete estimate of $M_4$ is very involved, so we will restrict ourselves to some special cases. The computations that follow are directly taken from [12]. We recall the following arithmetic facts that may be easily verified:

\begin{equation}
\xi_1 + \xi_2 + \xi_3 = 0 \implies \alpha_3 = \xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3,
\end{equation}

\begin{equation}
\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 \implies \alpha_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4).
\end{equation}

Recall that,

\begin{equation}
M_4(\xi_1, \xi_2, \xi_3, \xi_4) = c[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{\text{sym}},
\end{equation}

where $\sigma_3 = -\frac{M_3}{\alpha_3}$ and

\begin{equation}
M_3(x_1, x_2, x_3) = -i[m(x_1)m(x_2 + x_3)(x_2 + x_3)]_{\text{sym}}
\end{equation}

\begin{equation}
= -\frac{i}{3}[m^2(x_1)x_1 + m^2(x_2)x_2 + m^2(x_3)x_3],
\end{equation}

\begin{footnote}{12}The scaling index is the the Sobolev index $s_c$ such that the rescaled initial data $u_{0, \lambda}$ defined in (1.13) has the property that $\|u_{0, \lambda}\|_{H^{s_c}}$ is independent of $\lambda$.\end{footnote}
and by (5.11) \( \alpha_3(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 = 3x_1x_2x_3 \). We shall ignore the irrelevant constant in (5.13). Therefore,

\[
M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{1}{2} \left[ \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3 + \xi_4)(\xi_3 + \xi_4)}{3\xi_1\xi_2} \right]_{\text{sym}}
\]

Using the identity (5.12) and lots of symmetrizations and clever tricks like in [12], one can reexpress \( M_4 \) as

\[
M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{1}{36} \frac{1}{\xi_1\xi_2\xi_3\xi_4} \times \left\{ \xi_1\xi_2\xi_3[m^2(\xi_1) + m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] + \xi_1\xi_2\xi_4[m^2(\xi_1) + m^2(\xi_2) + m^2(\xi_4) - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] + \xi_1\xi_3\xi_4[m^2(\xi_1) + m^2(\xi_3) + m^2(\xi_4) - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] + \xi_2\xi_3\xi_4[m^2(\xi_2) + m^2(\xi_3) + m^2(\xi_4) - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] \right\}. 
\]

Assume now that \( m \) is like in (4.2) and that \( \xi_i = 0 \) for \( i = 1, \ldots, 4 \). Then obviously \( M_4 = 0 \). To make things more interesting let’s now assume that only \( \xi_1 = 0 \). Then the numerator of \( M_4 \) takes the form of

\[
\xi_2\xi_3\xi_4[m^2(\xi_2) + m^2(\xi_3) + m^2(\xi_4) - m^2(\xi_2) - m^2(\xi_3) - m^2(\xi_4)]
\]

which is once again zero.

We end this section and the article with some general remarks. Using the arguments presented in this section we are able to completely fill the gap between local well-posedness and global well-posedness also for the periodic KdV and the continuous and periodic mKdV. The periodic KdV problem is more difficult because the scaling argument used in the proof of Theorem 5.4 changes the period of the rescaled solution, hence all the estimates have to be independent of the rescaling parameter \( \lambda \) up to a factor \( \lambda^{5/4} \). To approach the mKdV problem we use the Miura transformation that relates solutions of the KdV to solution of the mKdV equation in an explicit way. For details the reader should see [12].

The method of almost conservation laws that we presented here is very general. We used it to obtain similar sharp results for the 1D Schrödinger equation with derivative nonlinearity [8] [9], and to obtain partial results for the IVP (2.11) [10], that improve Bourgain’s results in [4].

We believe that given a dispersive equation, the method we developed gives an analytic tool to study the nonlinear interactions of parts of the solution of the equation carried by different frequencies. We are now entering the domain of the weak turbulence theory!

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