Probing tripartite entanglement and coherence dynamics in pure and mixed independent classical environments

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Abstract
The type of environments to which a quantum system is exposed has a significant impact on the quantum system’s entanglement and coherence protection. In this regard, we investigate the time evolution of tripartite entanglement and coherence in GHZ-like state when subject to independent classical environments. In particular, we focus on local environments with the same and mixed disorders, resulting in various Gaussian noisy conditions, namely pure power-law noise, pure fractional Gaussian noise, power-law noise maximized, and fractional Gaussian noise maximized configurations. We show that the environments with mixed disorders are more detrimental than those having single kind of disorder for entanglement and coherence preservation using time-dependent quantum negativity, entanglement witnesses, purity, and decoherence metrics. Besides, there is no ultimate solution for avoiding the negative consequences of fractional Gaussian noise-assisted classical environments in both pure and mixed noise conditions. Not only the noise, but also the number of qubits driven by a certain noise, has been discovered to strongly influence the amount, nature of the decay, and preservation intervals. We also show that the GHZ-like states can be modelled in classical channels driven by pure power-law noise for extended quantum correlations, coherence, and quantum information preservation. In addition, we compare the entanglement measurement efficiency between entanglement witness and negativity measures.

Keywords Entanglement · Coherence · Independent classical fields · Pure and mixed Gaussian noises · GHZ-like state

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1 Introduction

Information processing and computing based on quantum mechanics are the focus of quantum information processing [1–7]. In this context, quantum computers have emerged as forefront advanced devices that are much superior in operation and most certain in practical applications with high speed and accuracy to their classical equivalents [8–10]. Various quantum phenomena, that control the tasks to be performed are the key concerns and working concepts of quantum computers [11–15]. The phenomena along with other principles of quantum computing are just as essential to consider as quantum computing devices are [16–19]. Among many other non-local phenomena, entanglement and coherence are a few of the most important for establishing efficient quantum mechanical operations [20–22]. The successful deployment of these protocols needs the useful transfer and preservation of such non-local correlations and coherence during realistic quantum operations. Quantum correlations dynamics of different quantum systems for this purpose are studied and took precedence over the research problems in quantum information sciences [23,24].

Entanglement, a non-local phenomenon, has been identified as a critical component in differentiating between local and non-local correlations [25,26]. As a result, entanglement has received a lot of attention and is being studied for novel non-local operations like quantum communications [27], quantum cryptography [28], quantum dense coding [29,30], quantum teleportation [31], quantum secure direct communication [32,33], and quantum key distribution [34,35]. Maintainable entanglement is required for any quantum computation that cannot be efficiently performed on a classical computer [36,37]. Besides, quantum coherence is a prerequisite for entanglement and other types of quantum correlations, and it can be wielded to preserve entanglement in a quantum system. In this work, because of as excellent resources, we will focus on investigating quantum entanglement and coherence in a tripartite state.

Quantum computing is incredibly difficult to design and programme. It is hampered by errors such as noise, malfunctions, quantum decoherence, and quantum disentanglement [38–40]. This becomes critical to their functioning and breaks down before any non-trivial program can complete. This is due to the limitation because a quantum system cannot be kept isolated from the effects of the linked environments [41–43]. The interaction of quantum systems with their surroundings, which results in decoherence and entanglement degradation, is becoming one of the major obstruction for the successful employment of quantum information processing protocols [44–46]. This weakening or disappearance of the entanglement is known as disentanglement and is caused by decoherence. This environmental defect, where the initial state entanglement of the system is constrained, is due to environmental noises. It is therefore very important to optimize and characterize the evolution of the quantum systems in presence of such fatal interfering noises in particular, for practical utilization of quantum information processing. Environmental noise can be understood into two different categories, which are classical and quantum interaction pictures. The classical description is more important than the quantum equivalent since it allows for a greater number of degrees of freedom to explore the time evolution of quantum systems [47–53]. In this context, we will be studying noises of Gaussian nature in pure and mixed forms over the time evolution of tripartite entangled and coherent state.
The time evolution of entanglement, coherence, and their protection has been studied extensively and has remained the major focus in research aspects for various types of quantum systems. In addition, for both bipartite and tripartite quantum systems, different non-Gaussian noises such as random telegraph, static, and coloured noises are studied in detail [47,48,54–56]. The findings show that these noises have dephasing effects that cause non-local correlations, coherence, and quantum information to be lost in a non-monotonic fashion, rendering the quantum process inefficient. In this regard, the multi-qubit systems have been shown with more cryptographic behaviour, and more superior in carrying information than the single-qubit and two-qubit systems [57–60].

In this paper, we investigate the dynamics of entanglement and coherence when a system of three non-interacting qubits is prepared as a maximally entangled Greenberger–Horne–Zeilinger ($\mathcal{X}_{\text{GHZ}}$) state and exposed to classical environments. Under various conditions, we are particularly interested in environments operating in independent mode and governed by Gaussian processes. The reason for investigating this type of configuration is that various types of disorders manifest themselves in the physical world. The most common are those defined by their auto-correlation functions or spectrum and having a Gaussian nature. Besides, such classical environments may experience similar or dissimilar disorders. We provide a detailed analysis of environments with similar disorders causing pure noises and dissimilar disorders causing non-identical mixed noises. The basic idea is to establish a foundation for protecting entanglement and coherence in such differing environments. We focus on the independent coupling of the three non-interacting qubits with three individual environments. This independent system–environment coupling is further extended to be described noisy conditions such as, pure power-law (PL), pure fractional Gaussian (FG), power-law noise maximized (PLM), and fractional Gaussian noise maximized (FGM) configuration. The first two cases are related with classical environments with only pure noise. The latter cases are related to mixed noise situations where the effects of a Gaussian noise are maximized relative to other. Different measures, such as quantum negativity, entanglement witness, purity, and decoherence will evaluate the dynamics of the tripartite quantum correlation and coherence in the presence of the current pure and mixed noisy situations.

The paper is organized as: In Sect. 2, the estimators used to measure tripartite entanglement, purity, and environmental decoherence are illustrated. Section 3 presents the physical model that accounts for the system–environment interaction, and the application of the pure and mixed noisy configurations. Section 4 deals with the results obtained for our physical model, as well as the discussions that followed. Section 5 represents the conclusive comments based on the investigation carried out.

2 Tripartite entanglement and coherence measures

This section describes the quantifiers used to evaluate tripartite entanglement and coherence.
2.1 Quantum negativity

The metric of quantum entanglement known as negativity is simple to compute. It is a separability criterion derived from the PPT criterion. It has been demonstrated to be an entanglement monotone and therefore an appropriate entanglement measure. Negativity for a system of three qubits is computed by taking the geometric mean of the bipartite negativities of any possible system bipartition, and is written as [61,62]:

\[ N_3(t) = \sqrt[3]{N_{1|23}N_{2|13}N_{3|12}}, \]  

(1)

where the negativity for the subsystem 1 and joint system 23 is defined as \( N_{1|23} = \sum_i |\lambda_i(\rho_{T1})| - 1 \). After performing the partial transpose with respect to subsystem 1, \( \lambda_i(\rho_{T1}) \) is the \( i^{th} \) eigenvalue of the density matrix of the three-qubit state. If the density matrix \( \rho_{T1} \) has at least one negative eigenvalue, then the corresponding negativity for the bipartition of the system is written as \( \rho_{T1} = 2\max\{0, -\lambda_{min}\} \) where \( \lambda_{min} \) is the smallest eigenvalue [63].

2.2 Entanglement witness

Entanglement witnesses, which are Hermitian operators with at least one negative eigenvalue, facilitate distinguishing between entangled and separable quantum systems [64–66]. Entanglement witnesses exist as a result of functional analysis’s of Hahn–Banach theorem, which provides a necessary and sufficient condition for detecting entanglement [67]. Mathematically, entanglement witness is written as [68]:

\[ E(t) = -\text{Tr}[\mathbb{W}_o\rho_{123}(t)], \]  

(2)

where \( \mathbb{W}_o = \frac{1}{2}I - \rho_o \) with \( \rho_o \) being the initial density matrix defined as \( \rho_o = |\psi\rangle\langle\psi| \) and \( \rho_{123}(t) \) is the time evolved density matrix of the state \( \psi \). For \( E(t) = 0 \), the state will be separable. Positive equation (2) results indicate the presence of experimentally detectable tripartite entanglement in the system. However, as described in [69], the negative outcomes of Eq. (2) do not guarantee that the state is completely separable. Here, the entanglement witness will be used to distinguish tripartite entangled states of the GHZ-class from separable ones.

2.3 Purity

One of the most basic quantifiable measures of quantum coherence is purity. Within Gaussian noise-driven classical fields, it will estimate the degree of mixedness and loss of coherence of the coherent state in time. For a time evolved state \( \rho_{123}(t) \), purity is given by [55]:

\[ P(t) = \text{Tr}[\rho_{123}(t)]^2. \]  

(3)

For \( n \)-dimensional quantum system, the purity criteria ranges as \( \frac{1}{n} \leq P(t) \leq 1 \). The state is completely pure and coherent at \( P(t) = 1 \), while becomes completely mixed and decoherent at the lower bound value \( \frac{1}{n} \).
Fig. 1 The schematic diagram of three non-interacting qubits $Q_1$, $Q_2$ and $Q_3$ coupled with independent classical environments $E_1$, $E_2$ and $E_3$ represented by square like boxes is shown. The model with single-coloured regions represents the presence of pure Gaussian noise (left) while the configuration with two coloured regions shows the presence of mixed Gaussian noises (right), i.e. power-law noise maximized configuration and fractional Gaussian noise maximized configuration. The black wavy lines represent the dynamics of the system with reduced amplitudes means the detrimental effects of the noise in the corresponding subspaces of the qubits. The red wavy lines indicate the action of the noises with the connecting lines among the qubits reflect the non-local correlations between the subsystems and showing that the three qubits are prepared as a single composite state. The identical size and shape of the qubits show that they are assembled with equal energy splitting.

2.4 Decoherence

Decoherence occurs when quantum system’s wave functions become entangled with their coupled environments. As a result, instead of being a single coherent quantum superposition, the system behaves like a classical statistical ensemble of its constituents. Decoherence will be a valid measure to compute coherence loss in the time evolved state of the system because the system–environment interaction is described classically here. The von-Neumann entropy approach can be used to estimate the decoherence effects for the time evolved density matrix $\rho_{123}(t)$ as [55,69]:

$$D(t) = -\text{Tr}[\rho_{123}(t)\ln\rho_{123}(t)].$$  \hspace{1cm} (4)

For entangled and coherent quantum states, $D(t) = 0$, whereas any other value of this measure will indicate the corresponding amount of coherence loss.

3 The model

Our physical model consists of three identical non-interacting qubits with equivalent energy splitting $\epsilon_n$ that are exposed to external independent classical random fields. We assume these fields to be characterized by Gaussian statistics, which further are presented in pure and mixed noisy configurations.
We investigate the adverse effects of pure power-law (PL) and fractional Gaussian (FG) noise in the first case. In the mixed input noise cases, we investigate power-law noise maximized (PLM) and fractional Gaussian noise maximized (FGM) configurations. In the PLM arrangement, two qubits are linked with PL noise and one with FG noise. Two qubits are paired with FG noise and one is paired with PL noise in FGM situation. The main purpose of this study is to differentiate between local environments involving similar and dissimilar disorders or noises. The stochastic Hamiltonian governs the current physical configuration, which can be stated as [55]:

$$H_{123}(t) = H_1(t) \otimes I_2 \otimes I_3 + I_1 \otimes H_2(t) \otimes I_3 + I_1 \otimes I_2 \otimes H_3(t), \quad \text{(5)}$$

where $H_n(t)$ is the single qubit Hamiltonian and is defined by $H_n(t) = \varepsilon_n I_n + \lambda \Omega_n(t) \sigma^x_n$ with $n \in \{1, 2, 3\}$. Here, $\Omega_n(t)$ is the stochastic parameter randomly flipping between $\pm 1$ while $\lambda$ is the coupling constant. $I_n$ and $\sigma^x_n$ are the identity and Pauli matrices acting on the subspaces of the qubits. The time evolution of the system can be obtained by [54]:

$$\rho_{123}(t) = U_{123}(t) \rho_{GHZ}(0) U_{123}(t)^\dagger, \quad \text{(6)}$$

where $U_{123}(t)$ is the time unitary operator and is defined as $U_{123}(t) = \exp[-\int_{t_0}^t H_{123}(s) ds]$ with $\hbar = 1$. Here, $\rho_{GHZ}(0)$ is the initial density matrix of the system and is given as [70]:

$$\rho_{GHZ}(0) = \frac{I_{(8 \times 8)}(1 - r)}{8} + r \chi_{GHZ}\chi_{GHZ}^\dagger, \quad \text{(7)}$$

where $r = 1$ refers to the pure GHZ-class state’s initial density matrix. $\chi_{GHZ}$ is the three-qubit maximally entangled Greenberger–Horne–Zeilinger state and is defined as $\chi_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

### 3.1 Impacts of noises

In this section, we examine the FG and PL noises in detail.

#### 3.1.1 Pure power-law noise

In classical environments, noises are caused by defects. Thermal fluctuations defects cause PL noise to be produced in this case. The low frequency PL noise spectrum expands as $\frac{1}{f^\alpha}$, where $f$ is the cyclic frequency and $\alpha$ is a real number. Thermal fluctuations and resistance, voltages across diodes and in vacuum tubes, and nearly in every solid-state device, frequency variations in harmonic fluctuations, and voltages in most superconducting devices are all sources of this noise [71–75]. In order to include the stochastic process in the case of classical noises, the $\beta$-function must be specified, which introduces the noise phase to the system and reads as [62,63]:

$$\beta_n(t) = \int_0^t \int_0^t dz dz' k(z - z'). \quad \text{(8)}$$
The auto-correlation function of the noise, which relate the noise phase to the phase of the system in the local fields are written as:

\[ J_{PL}(t - t', \chi, \Gamma, \alpha) = \frac{\alpha - 1(\chi \Gamma)}{2(\chi |t - t'| + 1)^2}. \] (9)

Now, upon assuming the dimensionless noisy quantities \( g = \frac{\chi}{\Gamma} \) and \( \tau = \Gamma t \) and inserting the auto-correlation function from Eq. (9) into Eq. (8), the corresponding \( \beta \)-function is obtained as:

\[ \beta_{PL}(t) = 1 + \frac{1}{g} \left[ \frac{g \tau (\alpha - 2) + (1 + g \tau)^{2 - \alpha} - 1}{\alpha - 2} \right]. \] (10)

### 3.1.2 Pure fractional Gaussian noise

In addition, we take into account a second vulnerability caused by the discrete Brownian motion of medium particles, which results in FG noise [61,76,77]. On traffic controls, electrical measurements, and meteorological data, the FG noise has been extensively studied [78–80]. Multi-scale patterns, Tsallis permutation entropy, empirical mode decomposition, creating self-similar network traffic, and other hydrological issues have all been modelled using FG noise [81–85]. In the current case, the auto-correlation function of the FG noise is written as:

\[ J_{FG}(t - t') = \frac{1}{2}(|t|^{2H} + |t'|^{2H} - |t - t'|^{2H}), \] (11)

by substituting Eq. (11) into Eq. (8), we get:

\[ \beta_{FG} = \frac{\tau^{2H+2}}{2H + 2}, \] (12)

where \( H \) is known as the Hurst exponent and ranges as \( 0 < H < 1 \) [86].

In the current cases, the phase of the system is described by \( \phi_n(t) = n \lambda \Omega_n(t) \). The dephasing effects of the noise are determined by averaging the time evolved density matrices over the corresponding noise phases as \( \langle \exp[n \lambda \Omega_n(t)]\rangle = \langle \exp[-\frac{1}{2}n^2 \beta_{X'}(t)]\rangle \) where \( X' \in \{FG, PL\} \). We get the final density matrix of the system within independent classical fields driven by the detrimental effects of the pure Gaussian noise case as:

\[ \rho_p(\tau) = \left\langle \left\langle \left( U_{123}(t) \rho_o U_{123}(t)^\dagger \right)_{\theta_1} \right\rangle_{\theta_2} \right\rangle_{\theta_3}. \] (13)

where \( \theta \) corresponds to the phase of either pure PL or FG noise.
3.1.3 Power-law noise maximized and fractional Gaussian noise maximized configurations

In case of the PLM and FGM configurations, two qubits are influenced by the first mentioned noise and the third qubit by the later noise. The basic idea is to represent the mixed noises arising from non-identical defects in classical independent environments. This can be traced to the model represented in [87] for bipartite and independent environments with non-Gaussian noises, namely static and random telegraph noise. The final density matrix for the system of three qubits when coupled to independent classical fields is given by [88]:

\[ \rho_{\text{PLM}}(\tau) = \langle \langle \langle U_{123}(t) \rho_0 U_{123}(t)^\dagger \rangle_\phi \rangle_\varphi \rangle_\psi, \]  
(14)

\[ \rho_{\text{FGM}}(\tau) = \langle \langle \langle U_{123}(t) \rho_0 U_{123}(t)^\dagger \rangle_\phi \rangle_\varphi \rangle_\psi, \]  
(15)

where \( \phi \) and \( \varphi \) are the corresponding noises phases of the PL and FG noise.

4 Results and discussion

In this section, the analytical and numerical results obtained for the quantum negativity, entanglement witness, purity, and decoherence measures for the dynamics of the three non-interacting qubits when coupled independently to external random fields generating pure and mixed classical noises are presented.

4.1 Pure power-law and fractional Gaussian noise

The analytical results obtained for the dynamics of quantum negativity, entanglement, witness, purity, and decoherence when coupled to classical independent environments with pure PL and FG noise are presented here. Using Eq. (13), the final density matrix ensemble for the current pure noise case obtained is X-shaped, as shown in “Appendix”. This resembles the time evolved density ensemble state obtained for non-local-system environment coupling driven by random telegraph noise given in [89]. All the diagonal elements are non-vanishing in the final density matrix to represent the three qubits to be in the chosen basis. Using Eqs. (1), (2, 3) and (4) for the final density matrix of the three-qubit state, the analytical expressions for the entanglement and coherence measurement are followed as (also shown in Figs. 2 and 3):

\[ N_p^3(t) = \left( -1 + X_1 + \frac{1}{2}(X_2 + X_3X_4) \right)^n \]  
(16)

\[ E_p(t) = -\frac{1}{4}(1 + 3\eta_1). \]  
(17)
Fig. 2 Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{GHZ}(0)$ when subjected to independent classical fields with pure PL noise when $g = 10^{-2}$ and $\alpha = 2.1$

\[ P_p(t) = -\frac{1}{4}(1 + 3\eta_2), \]  
\[ D_p(t) = -\frac{3}{4}M \log \left[ \frac{1}{4}M \right] - \frac{1}{4}N \log \left[ \frac{1}{4}N \right] , \]  

where $\eta_i = \exp[-n\beta_{\chi \gamma}]$ with $n \in \{4, 8\}$.

Figure 2 analyses the time evolution of the entanglement and coherence initially encoded in the tripartite entangled state when subjected to independent environments characterized by pure PL noise. Not all the off-diagonal elements are vanishing, thus, represents the existence of the coherence between the qubits. Under the current noisy effects, negativity, witness, and purity were found decreasing functions of entanglement and coherence. In contrast, the decoherence measure remained an increasing function of coherence decay. The dynamical outlook appraises the dominant character of the PL noise to degrade the initial encoded entanglement and coherence in the system. We noticed that for the utmost low values of $g$, quantum correlations, as well as coherence, can be successfully preserved in the three-qubit systems, however, for finite lengthy intervals. The random behaviour of the classical environments is completely suppressed upon the superposition of the noise phase over the joint phase of the system and environments. Because of this, the decay outlook turns out to be completely monotonic rather than showing any entanglement revivals. The backflow mechanism
of information between the system and environment is thus vanished, resulting in permanent information loss. In contrast, temporary information losses have been observed under static, dynamic, and coloured noise for different quantum systems [47,48,55]. The coherence decay rate for the three-qubit system result into the disentanglement of the three qubits. This suggests that the decoherence between the system and the environment caused the system to be entanglement free. Besides, all the measure have produced the same qualitative time evolution and provide consistency in results.

Figure 3 evaluates dynamics of the entanglement and coherence initially encoded in a three-qubit state and subjected to pure FG noise in classical independent environments. The decay outlook in the current case differs from that observed under pure PL noise. The pure FG noise seems more detrimental relatively, causing the three-qubit state to lose coherence and entanglement in a very short time as compared to the PL noise case. The unusual character of the $H$ has been noticed to robust the entanglement and coherence which contradicts most of the noise parameters properties [47,48,55,61]. However, because of the dominant destructive power of the current noise phase, even this robustness cannot prevent the state from becoming decoherent and disentangled. All other dynamical aspects match with those given in Fig. 2. The current qualitative dynamics resembles those obtained for bipartite entangled and mixture states under FG noise given in [63].
4.2 Power-law noise maximized configuration

The dynamics of negativity, entanglement witness, purity, and decoherence are evaluated for the PLM where two qubits are individually coupled with PL noise and one with FG noise. The final density matrix ensemble for the current mixed noise case has an X-shape, as shown in “Appendix”, and looks similar to the pure noise case. However, compared to the previous noise case, the density matrix elements and the phases differ. The structure of the current density matrix is like that obtained in [89] for non-local system–environment coupling under random telegraph noise. In the final density matrix, all the diagonal elements are non-vanishing, indicating that the three qubits are in the chosen entangled basis. Besides, some non-vanishing elements of the final density matrix represent that the system is still coherent under the mixed noise situation. All the operations are done over the final density matrix given in Eq. (14). Using Eqs. (1), (2), (3) and (4), the analytical results for the quantum negativity, entanglement witness, purity and decoherence are followed as (also shown in Fig. 4):

\[
N_{\text{PLM}}^3 = \left(-1 + \sum_{i=1}^{3} \frac{1}{2} W_i + \sum_{j=1}^{4} \frac{1}{4} X_j\right)^n \quad (20)
\]

\[
E_{\text{PLM}}(t) = \frac{1}{4} \left(-1 + e^{-4\beta_{\text{PL}}} + 2e^{-2(\beta_{\text{m}ix})}\right), \quad (21)
\]

\[
P_{\text{PLM}}(t) = \frac{1}{4} \left(1 + e^{-8\beta_{\text{PL}}} + 2e^{-4(\beta_{\text{m}ix})}\right), \quad (22)
\]

\[
D_{\text{PLM}}(t) = -\frac{1}{2} \mathcal{P} \log \left[\frac{1}{4}\mathcal{P}\right] - \frac{1}{2} \mathcal{Q} \log \left[\frac{1}{4}\mathcal{Q}\right], \quad (23)
\]

where \(\beta_{\text{m}ix}\) represents the local-mixed noise phases of the PL and FG noise and is defined as \(\beta_{\text{m}ix} = \beta_{(\mathcal{F}_G + \mathcal{P}_{\mathcal{L}})}\).

Figure 4 shows that under the current mixed noisy effects, negativity, witness and purity are found decreasing functions of entanglement, while decoherence remained an increasing function of coherence decay. This shows the detrimental effects of the mixed noise phases on the tripartite entanglement and coherence dynamics. The decoherence and purity decay rates hit the saturation levels faster than the witness. It infers the coherence loss to be quicker than the entanglement. It means incoherency between the system and environment gives rise to the disentanglement between the three qubits. The rise in decoherence allows the qubits to be entangled partially with the surrounding environments that result in quantum information loss and separability. Besides, there is considerable difference between the entanglement dynamical map shown by negativity and entanglement witness. Unlike, entanglement witness, the negativity provides a clear picture whether the state is entangled or separable. According to the negativity, the state becomes separable after a very short time. One can note that the entanglement time shown by the negativity is nearly equivalent to the time shown by entanglement witness in the positive regime. This means, in the negative regimes of the entanglement witness, the state becomes separable. All the measures display monotonic decay rather than showing any rebirths showing similarity with the decay
Fig. 4 Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields for the case of power-law noise maximized configuration, when $H = 0.1$, $\alpha = 3$ and $g = 1$

carried by the Ornstein–Uhlenbeck noise [62,63]. This exponential decay apprises that the information lost by the system is not recovered back from the coupled classical channels and is permanently lost. The overall qualitative behavioural decay shown by the measures is likely the same and infer that, after a finite interaction time, the state becomes completely disentangled and decoherent. The slopes decay faster as the value of $g$ increases, indicating the noxious nature of the parameter. We also find the current negative consequences encountered under the current mixed noisy situation completely different from that of the non-Gaussian noises, such as of random and static nature investigated in [47,48,61,90,91].

4.3 Fractional Gaussian noise maximized configuration

Here, we analyse the dynamics of negativity, entanglement witness, purity, and decoherence for three non-interacting maximally entangled qubits. Here, the independent system–environment coupling is assumed to be characterized by FGM configuration, where two qubits are driven by FG and the third one by PL noise. The final density matrix state for the current mixed noisy situation has a similar shape and characteristics as the PLM, but with different noise phases. Employing Eqs. (1), (2), (3) and (4) for the final density matrix of the three-qubit state given in Eq. (15), the analytical
results for the negativity, witness, purity and decoherence are followed as (also shown in Fig. 5):

\[
N_{\text{FGM}}^3(t) = \left( -1 + Y_1 + \sum_{i=2}^{4} \frac{1}{2} Y_i + \sum_{j=1}^{4} \frac{1}{4} Z_j \right)^n,
\]

(24)

\[
E_{\text{FGM}}(t) = \frac{1}{4} \left( -1 + e^{-4\beta_{FG}} + 2e^{-2(\beta_{\text{mix}})} \right),
\]

(25)

\[
P_{\text{FGM}}(t) = \frac{1}{4} \left( 1 + e^{-8\beta_{FG}} + 2e^{-4(\beta_{\text{mix}})} \right),
\]

(26)

\[
D_{\text{FGM}}(t) = -\frac{1}{2} R \log \left[ \frac{1}{4} \right] - \frac{1}{2} S \log \left[ \frac{1}{4} S \right].
\]

(27)

Figure 5 evaluates the dynamics of tripartite non-local correlation and coherence within independent classical channels driven by FGM configuration. In the current case, the quantitative and qualitative dynamical behaviour of the entanglement and coherence seems to be increasingly different from that observed under PLM configuration in Fig. 4. For the increasing values of the Hurst index \((H)\), the entanglement and coherence preservation becomes more robust initially, which completely contradicts the

\[\text{Fig. 5} \quad \text{Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of } \tau \text{ for the three non-interacting qubits, initially prepared in the states } \rho_{\text{GHZ}}(0) \text{ when subjected to independent classical fields for the case of fractional Gaussian noise maximized configuration, when } H = 0.9, \alpha = 3 \text{ and } g = 0.1.\]
previous results for the noise parameter $g$ of the PL noise. The dynamics of entanglement and coherence are not only affected by the coupled classical environments but are also significantly altered by the number of qubits driven by a specific noise and the values of the corresponding noise parameters. The behaviour of the noise phases over the qubit subspaces caused the initially encoded non-local correlation and coherence to degrade. Under the presence of mixed noisy phases, negativity, witness, and purity, all are found to be decreasing functions of the entanglement and coherence, respectively. In contrast, the decoherence measure is found to be the increasing function of coherence decay. However, the rates of entanglement decay and coherence decay differ slightly in time. The coherence seems to be a necessary prerequisite for the entanglement preservation. As can be seen, the disentanglement rate directly increases with the increasing speed of the decoherence. The three measures are monotonous functions in time, with no evidence of the entanglement of sudden death and birth phenomena. This resembles the bipartite and tripartite entanglement decay observed in [62,63,90] due to the Ornstein–Uhlenbeck noise. However, the quantitative aspects are completely different. This kind of monotonic decay results in the off-limitation of the backflow of information from the environment to the tripartite states after it has been lost. After a finite interaction time, the measures show that the three qubits become separable, mixed, and decoherent. Thus, all the measures show the same qualitative dynamical behaviour, suggesting good agreement among them.

4.4 Detailed time evolution analysis

In this section, we provide a detailed analysis including comparative dynamics, memory properties, and detrimental effects of the pure PL and FG noise along with the mixed Gaussian noise cases, PLM and FGM configurations. The focus of the section is to estimate the degrading effects explicitly against different values of the corresponding noise parameters in the given pure and mixed noisy configurations.

4.4.1 The case of comparing the noisy effects arising from similar defects in independent classical environments

When coupled to independent classical fluctuating fields generating pure PL noise (red-lined slopes) and FG (blue-lined slopes), the time evolution of negativity, entanglement witness, purity, and decoherence for the tripartite $X_{GHZ}$ state is described. The current section relates to the ability of the two pure Gaussian noises arising from similar defects in classical environments to degrade entanglement and coherence in terms of the noise parameters $g$ and $H$.

Figure 6 analyses the dynamics of time-dependent negativity, entanglement witness, purity, and decoherence for tripartite states within independent classical fields driven by pure FG and PL noise. The difference between the resultant dynamics of the entanglement and coherence is very much distinguishable under the two different Gaussian noises. We noticed that the entanglement and coherence remain much short-lived under the FG noise. Besides, we found the phase of the PL noise readily exploitable to get long-ranged quantum correlations and coherence protection. The
spectrum of the FG noise seems much narrower than that of the PL noise. Following this, one can note that the variation in the preservation time of the FG noise is negligible compared to that shown by the opponent noise. The statement can be verified by comparing the results given in Figs. 3 and 6. Other qualitative dynamical properties of the three qubits are in good agreement with those defined in Figs. 3 and 4. However, with PL noise, the preservation interval differs significantly. The current long interval preservation of the non-local correlation and coherence under PL noise is completely attributable to the utmost low values of the noise parameter $g$. Thus, making it easier for the quantum practitioners to design long-range entanglement, coherence, and memory properties for the successful deployment of the quantum mechanical protocols. In comparison, the preservation effects under FG noise are much lesser than those obtained under Ornstein–Uhlenbeck noise [62,63], but much longer preserved entanglement and coherence have been obtained under the current PL noise. Besides, all the measures have produced the same qualitative dynamical behaviour of the entanglement and coherence and are in good agreement with each other. The pure FG noisy effects in the current environments can be traced back to that for bipartite maximally entangled and mixture states given on [63]. Most notably, by comparing the results of negativity and entanglement witness, one can easily deduce that the negative areas of the graphs strongly show separability of the GHZ state.
4.5 The case of comparing the noisy effects arising from dissimilar defects in independent classical environments

Time evolution of the negativity, entanglement, purity, and coherence in tripartite $\mathcal{X}_{\text{GHZ}}$ state, when coupled to independent classical fluctuating fields with unlike defects generating mixed noises, namely PLM and FGM configurations, is addressed. The current section focuses on the potential of the two mixed noisy configurations to degrade entanglement and coherence. In the first case, we investigate the detrimental effects of the mixed-noise configuration when two qubits are driven by independent classical fields with PL noise and the third by FG noise (as shown by the red-lined slopes). In the later noisy configuration, two qubits are under the influence of FG noise and one under the PL noise (as shown by the blue-lined slopes).

In Figure 7, time evolution of the negativity, entanglement witness, purity, and decoherence under the destructive effects of two different mixed-noisy configurations is reported. The current results show that both noisy schemes, the tripartite state becomes separable and decoherent after a finite interaction time. The adverse negative effects caused by these two mixed noisy schemes differ significantly. As shown, when two of the qubits were coupled to FG noise, the initially encoded entanglement and coherence suffered a greater loss. This suggests that the FG noise phase is more disruptive than the PL noise phase, which produced smaller disentanglement and decoherence effects. As a result, the nature of decay can be deduced to largely depend on the type of

![Fig. 7 Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields under power-law noise maximized configuration (red-lined slopes) and fractional Gaussian noise maximized configuration (blue-lined slopes) when $H = 10^{-1}, \alpha = 2.1$ (non-dashed slopes) and $H = 0.8, \alpha = 10$ (dashed slopes) with $g = 10^{-4}$]
noise involved. In Figs. 6 and 7, the net difference between the preservation effects is significantly large. Entanglement and coherence are preserved for a longer time under pure PL noise, but not for long enough under both mixed noisy configurations. The main reason for modelling long entanglement and coherence preservation is the easily exploitable noise phase of the PL noise. However, it is difficult to avoid mixed noisy effects, especially when FG noise is involved, resulting in reduced memory effects. The dominant degrading effects resulting in faster entanglement and coherence decay are easily deducible under FG noise in both pure and mixed noisy configurations, as shown in Figs. 3 and 7. In comparison, the current results under both mixed noisy configurations differ completely from those defined in [61] for joint detrimental effects of static and random telegraph noise. In addition, the qualitative dynamical behaviour of the negativity, entanglement, purity, and decoherence coincides with one another and with previous findings. The decay observed is fully monotonic, according to the measures, and no entanglement sudden death and birth revivals have been observed. We showed that, in addition to the noisy character of the parameter $g$, $\alpha$ has the dephasing nature too. Entanglement and coherence decay increase in direct proportion to the value of $\alpha$. As $H$ increases, the entanglement and coherence become more robust at first, but the total preservation duration remains unaffected. The purity and coherence decay rates have been observed to be faster than the disentanglement rates of the three qubits, which are, however, directly related. Excluding the decay rates, all the three measures showed consistent results, indicating good agreement among them. The current detrimental effects resemble the decay caused by the Ornstein–Uhlenbeck noise [62,63,90] but strongly disagree with the adverse effects because of non-Gaussian noises given in [47,48,55,61].

### 4.5.1 Power-law noise maximized configuration: entanglement and coherence dynamics under the wide range of the noise parameter $g$

The time evolution of negativity, entanglement witness, purity, and coherence for three non-interacting qubits coupled independently to external fields generating mixed Gaussian noises is presented. Here, we assume the two of the independent fields have PL noise and one has FG noise, with one qubit coupled to each. In this context, we focus on the ability of the current mixed noise case to destroy the entanglement and coherence because of different noise parameter values.

Figure 8 shows the time evolution of negativity, entanglement witness, purity, and decoherence against the mixed Gaussian noises originated from the independent coupling of the three qubits to classical fields. We found that the current mixed noisy configuration has different adverse effects than the case of the maximized effects of FG noise. The spectrum of the PL noise is not as discrete as that of the FG noise, encompassing a variety of decay. Here, the deterioration of entanglement and coherence increases directly with the increasing values of $g$. The shifting of the blue-lined slopes follows this towards the red end for the increasing values of $g$. Furthermore, $g$ controls not only the decay rate but also the decay levels. As seen, the decay accumulates smaller values for smaller values of the parameter (as shown through blue and green slopes), while the decay levels rise for higher values of $g$ (as shown through red and black slopes). In agreement with the previous results, the nature of the decay
encountered is completely monotonous functions in time and no entanglement sudden death and birth revivals were observed which resembles the dephasing effects in tripartite entanglement and coherence caused by Ornstein–Uhlenbeck noise [62,88]. This explains the quantum information’s irreversible decay because of the present detrimental effects generated by the unlike defects. However, under the non-Gaussian noises, the overall dynamical behaviour of the current tripartite state becomes increasingly different, as shown in [47,48,54,56,90,92]. The FG noisy effects over the third qubit are reduced in the PLM configuration, showing tiny evidence of presence. As a result, the nature of the noise and its application to the number of qubits play a pivotal role in determining the amount and nature of the decay as well as the memory properties preservation intervals. In the current mixed noise case, even for smaller values of the noise parameters, the disentanglement and decoherence effects are unavoidable. However, the decay at the smaller values of $g$ is relatively much smaller than that at the higher values. All the measures produced the same qualitative results, ensuring higher consistency and validity in the results. The disentanglement rate shown by the entanglement witness occurs later in the qualitative scales than the coherence decay rate shown by the purity and decoherence measure. This shows that the generation of decoherence causes the disentanglement between the system and the environment and that it increases in direct relation to the rate of coherence decay.

Fig. 8 Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields under power-law noise maximized configuration when $g = 10^{-2}$ (blue), $10^{-1}$ (green), 1 (red) and 10 black with $\alpha = 3$ and $H = 10^{-1}$
4.5.2 Fractional Gaussian noise maximized configuration: entanglement and coherence dynamics under the discrete nature of the parameter $H$.

In this section, we investigate the time evolution of negativity, entanglement witness, purity, and coherence for the $\lambda_{\text{GHZ}}$ state under the influence of FGM configuration. In the current case, the first two qubits are driven by FG noise and one by PL noise. Here, the preservation of the non-local correlation and coherence against the different values of the noise parameter $H$ is primarily investigated.

Figure 9 evaluates the dynamical behaviour of tripartite entanglement and coherence when subjected to independent environments with joint effects of FG and PL noise. Unlike the PL noise, the width between the slopes for different values of $H$ is much smaller in comparison, which is because of the discrete nature of the FG noise with a narrow spectrum. For increasing values of $H$, the non-local correlation and coherence remained more robust initially, contrary to the characteristic behaviour of $g$. As with increasing choices of $H$, the slopes shift from blue towards the black-end, indicating later decay. The noxious effects caused by the FG noise, on the other hand, are inevitable, and the state becomes completely disentangled and decoherent after a very short interaction period. The increased effects of the FG noise over the two qubits suppress the PL noisy effects over the third qubit to the point that they do not even show up. It should be noted, however, that the PL noise parameters $g$ and $\alpha$ are kept minimal. The nature of the decay was unaffected by the increasing values of the noise parameters, remaining monotonic and free of entanglement sudden death, and birth.

![Figure 9](image_url)

**Fig. 9** Dynamics of the negativity (a), entanglement witness (b), purity (c) and decoherence (d) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields under fractional Gaussian maximized configuration when $H = 10^{-2}$ (blue), $0.2$ (green), $0.5$ (red) and $0.9$ black with $\alpha = 3$ and $g = 10^{-3}$.
revivals. This agrees with the previous results obtained for Ornstein–Uhlenbeck noise [62,63,88], however, long-lived correlations and coherence with strong entanglement rebirths have been observed under static, random telegraph, and coloured noises comparatively [47,48,54,56,90,92]. In addition, the current noisy effects resemble those obtained by quantum negativity and discord under the FG noise, for example given in [63]. Besides, the observed disentanglement rate is slower than the purity and coherence decay rates, which is consistent with previous findings. The measures have shown similar qualitative dynamical behaviour, implying close connections between them.

4.6 Equivalence between negativity and entanglement witness

Detecting genuine tripartite and multipartite entanglement was one of the early problems in quantum information processing. Many important metrics for bipartite quantum correlation measurement, including as concurrence, entanglement witnesses, quantum negativity, quantum discord, and other related measurements, were already in use. Many of these approaches were upgraded to detect multipartite entanglement and were completely trusted, with the exception of the entanglement witness. This is due to the metric’s ambiguous character, which allows it to provide both positive and negative numerical results. The issue was that this operator’s negative output was still deemed to infer or suggest entanglement, as previously suggested in [93]. This section delves deeper into the issue, focusing on the credibility of the entanglement witness. This is done by comparing it to quantum negativity, a well-known quantum entanglement metric.

Figures 10 and 11 assess the entanglement witness’s compatibility with quantum negativity. The findings are obtained for pure PL, FG, PLM, and FGM noisy configurations with the identical noise parameter values in each case. When the negativity and witness findings are compared, it’s obvious to see that both metrics have comparable entanglement preservation intervals. This points to the validity of our findings as well as high reliability of the measure. We found that the positive results of the entanglement witness indicate that the state is in the entanglement regime. At the same time, the negativity and entanglement witness slopes approach 0. This is the negativity measure’s final saturation level, while the entanglement witness’s slopes approach negative regimes, indicating the state’s separability. As a result, we find that when coupled to classical fields driven by Gaussian noise, the negative regimes of the entanglement witness significantly support the separability of the maximally entangled three-qubit GHZ-like state. The entanglement witness in this case successfully quantified entanglement and produced consistent results, which is similar to quantum negativity.

5 Conclusion

In a brief description, the dynamics of tripartite entanglement and coherence are investigated for a system of three non-interacting qubits prepared as maximally entangled GHZ-like states coupled with independent classical environments. Four different
Fig. 10  Upper panel: Dynamics of the negativity (a) and entanglement witness (b) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields under pure power-law noise with $g = 1$ and $\alpha = 2.1$. Lower panel: Same as the upper panel but for the pure fractional Gaussian noise when $H = 5 \times 10^{-3}$.

Fig. 11  Upper panel: Dynamics of the negativity (a) and entanglement witness (b) as a function of $\tau$ for the three non-interacting qubits, initially prepared in the states $\rho_{\text{GHZ}}(0)$ when subjected to independent classical fields under power-law noise maximized configuration with $g = 1$, $\alpha = 5$ and $H = 10^{-1}$. Lower panel: Same as the upper panel but for the pure fractional Gaussian noise maximized configuration.
noisy models are used to describe the external fields: pure power-law, pure fractional Gaussian, power-law noise maximized, and fractional Gaussian noise maximized configurations. Taking ensemble averages over the stochastic process in both the pure and mixed independent environmental noisy configurations, we computed the final density matrices for the three qubits. Finally, we investigated the entanglement and coherence in the three non-interacting maximally entangled qubits using estimators such as quantum negativity, entanglement witness, purity, and decoherence measures.

Our findings reveal that in pure and mixed noisy configurations, the noxious effects on the dynamics of the tripartite non-local correlation and coherence are fundamentally different. In the mixed noisy situation, the PL noise was more destructive than in the pure noisy case. As demonstrated in Fig. 6, the associated phase in pure PL noise can be effectively exploited to induce entanglement and coherence preservation over a long period of time. FG noise, on the other hand, is equally detrimental in both pure and mixed noisy contexts, causing entanglement and coherence to disappear in a very short duration. It’s also worth mentioning that the dephasing effects of noise introduced to a single qubit are completely suppressed in the mixed noisy arrangement, offering only an insignificant evidence of presence. As a result, it’s been proven that the type of noise given to the corresponding number of qubits affects both the decay rate and the amount of decay. Besides this, the decay amounts are smaller when two qubits are coupled with PL noise and greater when two qubits are coupled with FG noise, as shown in Fig. 7. In addition, both pure and mixed noisy effects within independent classical fields are unavoidable, and with either a long or short preservation period, the state eventually becomes separable. Moreover, we found the tripartite and bipartite quantum correlations dynamics under the mixed non-Gaussian noisy configuration given in [61, 90] completely different than the current Gaussian mixed noise cases. The basic difference between the two is the presence of the entanglement revivals, due to which, unlike the current case, the state remained inseparable for longer intervals. The decay was monotonic, and there was no sign of entanglement sudden death and birth phenomenon, which contradicts the findings in [47, 48, 55, 56, 61, 90–93] for various non-Gaussian noises, however, coincides with those given in [94, 95]. This signifies that entanglement and coherence, as well as quantum information degradation, are irreversible once lost under the current Gaussian noises, as also observed under Ornstein–Uhlenbeck noise in [62, 63, 88].

In contrast to the characteristics of most of the noisy parameters described in [55, 61, 62, 63, 90, 91], entanglement, coherence, and memory features were initially more robust for the upper bound of the $H$ in the case of noise parameters. The preservation intervals, on the other hand, were almost unchanged across the whole discrete range of the $H$. Compared to the broad range of $g$ of the PL noise, the variation in decay induced by the FG noise is nearly trivial over the entire range of the parameter $H$. This characteristic of the FG noise is fully attributable to the discrete spectrum of the noise. In the case of PL noise, the parameters $g$ and $\alpha$ play opposing roles in the dynamics of non-local correlation and coherence, with both decreasing as this parameter is increased. Aside from that, due to the broad spectrum of the power-law noise, a variety of decay has been seen against various values of $g$. Most significantly, much longer and greater entanglement and coherence retention may be achieved for
low values of $g$, far longer than those obtained under Gaussian and non-Gaussian
noises investigated in [47,48,55,56,62,88,90,92,93].

The measures utilized displayed comparable qualitative dynamical behaviour, imply-
ing that they were in strict agreement and ensuring consistency and validity in the
results. Quantum negativity and entanglement witness successfully captured entangle-
ment in the case of entanglement estimation. Entanglement witness, unlike quantum
negativity, produces mostly negative values, indicating that the current state is sep-
arable. The purity and decoherence measures, on the other hand, deteriorated at a
faster rate than the entanglement witness in quantitative analysis. This means that the
amount of decoherence between the system and its environment affects the rate of
disentanglement proportionally.

Finally, we show that in any situation, the current Gaussian noisy effects within classi-
cal independent fields are unavoidable. We find that fractional Gaussian noise is more
detrimental than power-law noise for tripartite entanglement, coherence, and memory
characteristics. The pure power-law noise-assisted local channels are detected with the
least detrimental effects, allowing the survival of non-local correlation and coherence
for longer intervals, notably for the extremely low values of the parameters $g$ and $\alpha$.
We found that no such exploitation is conceivable in the case of fractional Gaussian
noise that can mimic long enough preservation effects.

**Data availability** The authors confirm that the data supporting the findings of this study are available within
the article and its supplementary materials.

**Appendix**

In this section, we give the details of the final density matrices obtained for the time
evolution of the three qubits initially prepared in the state $\rho_{\text{GHZ}}(0)$ under the effects of
pure and mixed noisy configurations. Using Eq. (13), we get the final density matrix
under pure Gaussian noise case as:

\[
\rho_p(t) = \frac{1}{8} \begin{bmatrix}
1 + 3X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 + 3X_1 \\
0 & 1 - X_1 & 0 & 0 & 0 & 0 & 1 - X_1 & 0 \\
0 & 0 & 1 - X_1 & 0 & 0 & 1 - X_1 & 0 & 0 \\
0 & 0 & 0 & 1 - X_1 & 1 - X_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - X_1 & 1 - X_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - X_1 & 0 & 0 & 1 - X_1 & 0 \\
0 & 0 & 1 - X_1 & 0 & 0 & 0 & 1 - X_1 & 0 \\
1 + 3X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 + 3X_1
\end{bmatrix}
\]
Next, using Eq. (15), we obtained the final density matrix ensemble state as:

$$\rho_{XYZ}(t) = \frac{1}{8} \begin{bmatrix}
\mathcal{H}_1 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_1 \\
0 & \mathcal{H}_2 & 0 & 0 & 0 & 0 & \mathcal{H}_2 \\
0 & 0 & \mathcal{H}_3 & 0 & 0 & \mathcal{H}_3 & 0 \\
0 & 0 & 0 & \mathcal{H}_3 & \mathcal{H}_3 & 0 & 0 \\
0 & 0 & 0 & \mathcal{H}_3 & \mathcal{H}_3 & 0 & 0 \\
0 & \mathcal{H}_2 & 0 & 0 & 0 & 0 & \mathcal{H}_2 \\
\mathcal{H}_1 & 0 & 0 & 0 & 0 & 0 & \mathcal{H}_1
\end{bmatrix}$$

(29)

where $$\rho_{XYZ}(t) \in \{\rho_{PLM}(t), \rho_{FGM}(t)\}$$, $$\mathcal{X}_i = \exp[-n\beta_{AB}(t)]$$ with $$\beta_{AB}(t) \in \beta_{PL}(t), \beta_{FG}(t)$$ and $$\mathcal{Y} = \exp[-n\beta_{mix}(t)]$$ with $$\beta_{mix}(t) = \beta_{PL}(t) + \beta_{FG}(t)$$, $$\mathcal{H}_1 = 1 + \mathcal{X}_2 + 2\mathcal{Y}, \mathcal{H}_2 = 1 + \mathcal{X}_2 - 2\mathcal{Y}, \mathcal{H}_3 = 1 - \mathcal{X}_2$$.

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