On the modified discrete KP equation with self-consistent sources

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The modified discrete KP equation is the Bäcklund transformation for the Hirota’s discrete KP equation or the Hirota-Miwa equation. We construct the modified discrete KP equation with self-consistent sources via source generation procedure and clarify the algebraic structure of the resulting coupled modified discrete KP system by presenting its discrete Gram-type determinant solutions. It is also shown that the commutativity between the source generation procedure and Bäcklund transformation is valid for the discrete KP equation. Finally, we demonstrate that the modified discrete KP equation with self-consistent sources yields the modified differential-difference KP equation with self-consistent sources through a continuum limit. The continuum limit of an explicit solution to the modified discrete KP equation with self-consistent sources also gives the explicit solution for the modified differential-difference KP equation with self-consistent sources.

Keywords: Modified discrete KP equation, Source generalization procedure, Discrete Gram-type determinant, Continuum limit

2000 Mathematics Subject Classification: 37K10, 37K40

1. Introduction

The study of discrete integrable systems has attracted a lot of interest in recent years. The most important and widely studied three-dimensional discrete integrable system is the Hirota-Miwa equation (or discrete KP equation in bilinear form ) [1]- [3]:

\begin{align*}
  a_1(a_2 - a_3)f(k_1 + a_1)f(k_2 + a_2,k_3 + a_3) + a_2(a_3 - a_1)f(k_2 + a_2)f(k_1 + a_1,k_3 + a_3) \\
  + a_3(a_1 - a_2)f(k_3 + a_3)f(k_1 + a_1,k_2 + a_2) = 0
\end{align*}

(1.1)

where \( f = f(k_1,k_2,k_3) \) and \( a_1, a_2, a_3 \) are the difference intervals for discrete independent variables \( k_1, k_2, k_3 \), respectively. We omit the dependence on unshifted variables in the above equation and in the following notations. The Hirota-Miwa equation (1.1) is considered as a master equation which by performing scaling continuum limit gives many well-known semi-discrete and continuous integrable systems.
Under the variable transformations $k_1 \rightarrow a_1 k_1, k_2 \rightarrow a_2 k_2, k_3 \rightarrow a_3 k_3$, the discrete KP equation (1.1) can be written in a compact form by using bilinear operator:

$$[a_1(a_2 - a_3)e^{\frac{1}{2}D_{k_1} - \frac{1}{2}D_{k_2} - \frac{1}{2}D_{k_3}} + a_2(a_3 - a_1)e^{\frac{1}{2}D_{k_2} - \frac{1}{2}D_{k_1} + \frac{1}{2}D_{k_3}} + a_3(a_1 - a_2)e^{\frac{1}{2}D_{k_3} - \frac{1}{2}D_{k_1} - \frac{1}{2}D_{k_2}}]f \cdot f = 0,$$

(1.2)

where the bilinear operator is defined by

$$e^{\delta D_{n}} a(n) \cdot b(n) = a(n + \delta) b(n - \delta).$$

(1.3)

The discrete KP equation (1.1) or (1.2) has the following Bäcklund transformation [1, 4]:

$$[a_1 e^{\frac{1}{2}D_{k_1} - \frac{1}{2}D_{k_2}} - a_2 e^{\frac{1}{2}D_{k_2} - \frac{1}{2}D_{k_1}} + (a_2 - a_3)e^{\frac{1}{2}D_{k_1} + \frac{1}{2}D_{k_2}}]f \cdot g = 0$$

(1.4)

$$[a_1 e^{\frac{1}{2}D_{k_1} - \frac{1}{2}D_{k_3}} - a_3 e^{\frac{1}{2}D_{k_3} - \frac{1}{2}D_{k_1}} + (a_3 - a_1)e^{\frac{1}{2}D_{k_1} + \frac{1}{2}D_{k_3}}]f \cdot g = 0.$$  

(1.5)

Through the variable transformations $k_1 \rightarrow \frac{1}{a_1} k_1, k_2 \rightarrow \frac{1}{a_2} k_2, k_3 \rightarrow \frac{1}{a_3} k_3$, the modified discrete KP equations (1.4-1.5) become

$$a_1 f(k_1 + a_1) g(k_2 + a_2) - a_2 f(k_2 + a_2) g(k_1 + a_1) + (a_2 - a_1) f(k_1 + a_1, k_2 + a_2) g = 0$$

(1.6)

$$a_1 f(k_1 + a_1) g(k_3 + a_3) - a_3 f(k_3 + a_3) g(k_1 + a_1) + (a_3 - a_1) f(k_1 + a_1, k_3 + a_3) g = 0.$$  

(1.7)

Soliton equations with self-consistent sources, describing important physical processes, have been studied by means of various mathematical approaches such as inverse scattering methods [5]-[9], Darboux transformation methods [10]- [13], Hirota’s bilinear method and Wronskian technique [14]- [22]. However, most results have been achieved in continuous case. Comparatively less work has been done in discrete case, especially in the fully discrete case.

In [23], the authors propose a new algebraic method, called the source generalization procedure, to construct and solve the soliton equations with self consistent sources both in continuous and discrete cases. The purpose of this paper is to utilize the source generalization procedure to the modified discrete KP equation (1.6-1.7) to construct the modified discrete KP equation with self-consistent sources and clarify the bilinear structures of the modified discrete KP equations with self-consistent sources. Furthermore, we verify that the commutativity of the source generation procedure and Bäcklund transformation holds for the fully discrete case.

The outline of this paper is as follows. In Sec.2, we construct the modified discrete KP equations with self-consistent sources and give their discrete Gram-type determinant solutions. Sec.3 is devoted to show that the commutativity of the source generation procedure and Bäcklund transformation is valid for the discrete KP equation. We end this paper with a summary and discussion in section 4.

2. The modified discrete KP equation with self-consistent sources

In this section, we construct the modified discrete KP equation with self-consistent sources via source generation procedure. Furthermore, we present the discrete Gram-type determinant solution for the modified discrete KP equation with self-consistent sources and clarify its bilinear structure.
It is given in [24] that the bilinear difference equations (1.6-1.7) have the following discrete Gram-type determinant solution:

\[ f(k_1,k_2,k_3) = |M|, \quad (2.1) \]

\[ g(k_1,k_2,k_3) = \begin{vmatrix} M & \Phi(-1) \\ \Phi(0)^T & 1 \end{vmatrix}. \quad (2.2) \]

where \( M = \det(c_{ij} + m_{ij})_{1 \leq i,j \leq N} \) in which \( c_{ij}(1 \leq i,j \leq N) \) is arbitrary constant and the matrix element \( m_{ij}(1 \leq i,j \leq N) \) is an function of \( k_1,k_2,k_3 \) satisfying the difference equation

\[ \Delta_{k_i}m_{ij} = \phi_i(k_\nu + a_\nu)\bar{\phi}_j(0), \quad (2.3) \]

\[ i,j = 1,2,\cdots,N, \nu = 1,2,3, \]

where unshifted independent variables are suppressed and \( \phi_i \) and \( \bar{\phi}_i \) are arbitrary functions of \( k_1,k_2,k_3 \) and an integer \( s \), satisfying the dispersion relations

\[ \Delta_{-k_\nu}\phi_i(k_1,k_2,k_3,s) = \phi_i(k_1,k_2,k_3,s+1), \quad (2.4) \]

\[ \Delta_{+k_\nu}\bar{\phi}_i(k_1,k_2,k_3,s) = \bar{\phi}_i(k_1,k_2,k_3,s+1), \quad (2.5) \]

where \( \Delta_{-k_\nu}, \Delta_{+k_\nu} \) are defined by

\[
\begin{align*}
\Delta_{-k_\nu}F(k_\nu) & = \frac{F(k_\nu) - F(k_\nu - a_\nu)}{a_\nu}, \\
\Delta_{+k_\nu}F(k_\nu) & = \frac{F(k_\nu + a_\nu) - F(k_\nu)}{a_\nu}, \quad \nu = 0,1,2,3.
\end{align*}
\]

In addition, \( \Phi(s), \bar{\Phi}(s) \) (\( s \) is an integer) are \( N \)th column vectors defined by

\[ \Phi(s) = (\phi_1(s), \phi_2(s), \cdots, \phi_N(s))^T, \quad (2.6) \]

\[ \bar{\Phi}(s) = (\bar{\phi}_1(s), \bar{\phi}_2(s), \cdots, \bar{\phi}_N(s))^T. \quad (2.7) \]

In order to construct the modified fully discrete KP equation with self-consistent sources, we change the discrete Gram-type determinant solutions (2.1-2.2) into the following form:

\[ f(k_1,k_2,k_3) = \det(c_{ij}(k_1) + m_{ij})_{1 \leq i,j \leq N} = |D|, \quad (2.8) \]

\[ g(k_1,k_2,k_3) = \begin{vmatrix} D & \Phi(-1) \\ \Phi(0)^T & 1 \end{vmatrix}, \quad (2.9) \]

where \( N \times N \) matrix \( D = (c_{ij}(k_1) + m_{ij})_{1 \leq i,j \leq N} \) and \( m_{ij}, \Phi(s), \bar{\Phi}(s) \) are given in (2.3-2.7). In addition, \( c_{ij}(k_1) \) satisfies

\[ c_{ij}(k_1) = \begin{cases} c_i(k_1), & 1 \leq i \leq K \leq N \text{ and } j = 1,2,\cdots,K, K \in \mathbb{Z}^+, \\ d_{ij}, & \text{otherwise}, \end{cases} \quad (2.10) \]

with \( c_i(k_1) \) being an arbitrary function of \( k_1 \) and \( K \) being a positive integer. Then we have the following difference formula by employing Eqs. (2.1-2.5):
\[
f(k_1, k_2, k_3) = \begin{vmatrix} M_d & C(k_1) \\ \alpha^T & 1 \end{vmatrix}, \quad g(k_1, k_2, k_3) = \begin{vmatrix} M_d & C(k_1) & \Phi(-1) \\ \alpha^T & 1 & 0 \\ \Phi(0)^T & 0 & 1 \end{vmatrix}, \quad (2.11)
\]

\[
f(k_\mu + a_\mu) = \alpha_\mu \begin{vmatrix} M_d & C(k_1) & \Phi(0_\mu) \\ \alpha^T & 1 & 0 \\ -\Phi(0)^T & 0 & a_\mu^{-1} \end{vmatrix}, \quad f(k_1 + a_1) = a_1 \begin{vmatrix} M_d & C(k_1 + a_1) & \Phi(0_1) \\ \alpha^T & 1 & 0 \\ -\Phi(0)^T & 0 & a_1^{-1} \end{vmatrix}, \quad (2.12)
\]

\[
f(k_2 + a_2, k_3 + a_3) = \frac{(a_2 a_3)^2}{(a_2 - a_3)} \begin{vmatrix} M_d & C(k_1) & \Phi(0_2) & \Phi(0_3) \\ \alpha^T & 1 & 0 & 0 \\ -\Phi(1)^T & 0 & -a_2^2 & -a_3^2 \\ -\Phi(0)^T & 0 & a_2 & a_3 \end{vmatrix}, \quad (2.13)
\]

\[
f(k_1 + a_1, k_\mu + a_\mu) = \frac{(a_1 a_\mu)^2}{(a_1 - a_\mu)} \begin{vmatrix} M_d & C(k_1 + a_1) & \Phi(0_1) & \Phi(0_\mu) \\ \alpha^T & 1 & 0 & 0 \\ -\Phi(1)^T & 0 & -a_1^2 & -a_\mu^2 \\ -\Phi(0)^T & 0 & a_1 & a_\mu \end{vmatrix}, \quad (2.14)
\]

\[
g(k_\mu + a_\mu) = -a_\mu^2 \begin{vmatrix} M_d & C(k_1) & \Phi(-1) & \Phi(0_\mu) \\ \alpha^T & 1 & 0 & 0 \\ \Phi(1)^T & 0 & 0 & a_\mu^2 \\ \Phi(0)^T & 0 & 1 & -a_\mu^{-1} \end{vmatrix}, \quad g(k_1 + a_1) = -a_1^2 \begin{vmatrix} M_d & C(k_1 + a_1) & \Phi(-1) & \Phi(0_1) \\ \alpha^T & 1 & 0 & 0 \\ \Phi(1)^T & 0 & 0 & a_1^{-2} \\ \Phi(0)^T & 0 & 1 & -a_1^{-1} \end{vmatrix}, \quad (2.15)
\]

where \( \mu = 2, 3, C(k_1), \alpha, \Phi(s_v) \) are \( N \)th column vectors given by

\[
C(k_1) = (c_1(k_1), \cdots, c_K(k_1), 0, \cdots, 0)^T,
\]

\[
\alpha = (-1, \cdots, -1, 0, \cdots, 0)^T, \quad \text{number of } -1 = K,
\]

\[
\Phi(s_v) = (\phi_1(k_v + a_v; s), \phi_2(k_v + a_v; s), \cdots, \phi_N(k_v + a_v; s))^T, \quad v = 1, 2, 3,
\]

and \( M_d \) is given by

\[
M_d = \begin{vmatrix}
m_{11} & \cdots & m_{1k} & m_{1.k+1} + d_{1.k+1} & \cdots & m_{1N} + d_{1N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
m_{k1} & \cdots & m_{kk} & m_{k.k+1} + d_{k.k+1} & \cdots & m_{kN} + d_{kN} \\
m_{k+1,1} + d_{k+1,1} & \cdots & m_{k+1,k+1} + d_{k+1,k+1} & \cdots & m_{k+1,N} + d_{k+1,N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
m_{N1} + d_{N1} & \cdots & m_{Nk} + d_{Nk} & m_{N,k+1} + d_{N,k+1} & \cdots & m_{NN} + d_{NN} \\
\end{vmatrix}. \quad (2.16)
\]
According to the above results, \( f, g \) defined in (2.1-2.2) will not satisfy the modified fully discrete KP equation (1.6-1.7) again. In fact, they satisfy the following equations:

\[
a_1 f(k_1 + a_1)g(k_2 + a_2) - a_2 f(k_2 + a_2)g(k_1 + a_1) + (a_2 - a_1) f(k_1 + a_1, k_2 + a_2)g(k_1, k_2, k_3)
\]
\[
= a_2 \sum_{i,j=1}^{K} P_{i,j}(k_1 + a_1)Q_{i,j}(k_2 + a_2),
\]
\[
\text{(2.17)}
\]
\[
a_1 f(k_1 + a_1)g(k_3 + a_3) - a_3 f(k_3 + a_3)g(k_1 + a_1) + (a_3 - a_1) f(k_1 + a_1, k_3 + a_3)g(k_1, k_2, k_3),
\]
\[
= a_3 \sum_{i,j=1}^{K} P_{i,j}(k_1 + a_1)Q_{i,j}(k_1 + a_3),
\]
\[
\text{(2.18)}
\]

where \( P_{i,j}, Q_{i,j} \) for \( r, m = 1, \cdots, K \) are functions of variables \( k_1, k_2, k_3 \) given by

\[
P_{i,j}(k_1, k_2, k_3) = \sqrt{c_i(k_1) - c_i(k_1 - a_1)} D_{ij}(\Phi_j(0))^{T} R_{ij},
\]
\[
\text{(2.19)}
\]
\[
Q_{i,j}(k_1, k_2, k_3) = -\sqrt{\Delta c_i(k_1)} D_{ij} \Phi_i(-1) M_{ij} \Phi_i(-1)^{T},
\]
\[
\text{(2.20)}
\]

in which \( D_{ij} \) is the \((N-1)\)th order matrice obtained by eliminating the \( i \)th row and the \( j \)th column from the matrice \((c_{i,j}(k_2) + m_{i,j}(k_1, k_2, k_3))_{1 \leq i,j \leq N}; \Phi_j(s), \Phi_j(s) \) are \((N-1)\)th column vectors obtained by eliminating the \( j \)th element from \( \Phi(s) \) and \( \Phi(s) \), respectively; \( \Delta c_i(k_1) \) is defined by

\[
\Delta c_i(k_1) = c_i(k_1 + a_1) - c_i(k_1)
\]
\[
\text{(2.21)}
\]

and \( R_{ij}, M_{ij} \) are given by

\[
R_{ij} = (c_i(k_1) + m_{1,j}, \cdots, c_i(k_1) + m_{i-1,j}, c_i(k_1) + m_{i+1,j}, \cdots, c_i(k_1) + m_{i+1,j}),
\]
\[
c_i(k_1) + m_{K,j}, d_{K,l}, d_{K+1,j}, \cdots, d_{N,j} + m_{N,j},
\]
\[
M_{ij} = (c_i(k_1) + m_{1,j}, \cdots, c_i(k_1) + m_{i-1,j}, c_i(k_1) + m_{i+1,j}, \cdots, c_i(k_1) + m_{i+1,j}),
\]
\[
c_i(k_1) + m_{i,j}, d_{i,j}, d_{i+1,j} + m_{i+1,j}, \cdots, d_{i,N} + m_{i+1,j}.
\]

We can also show that functions (2.1-2.2) and new fields (2.19-2.20) for \( r, m = 1, \cdots, K \) satisfy the following bilinear equations:

\[
a_2 f(k_2 + a_2)P_{i,j}(k_3) - a_3 f(k_3 + a_3)P_{i,j}(k_2 + a_2) +
\]
\[
(a_3 - a_2) f(k_2 + a_2, k_3 + a_3)P_{i,j}(k_1, k_2, k_3) = 0,
\]
\[
\text{(2.22)}
\]
\[
a_2 Q_{i,j}(k_2 + a_2)g(k_3 + a_3) - a_3 Q_{i,j}(k_3 + a_3)g(k_2 + a_2) +
\]
\[
(a_3 - a_2) Q_{i,j}(k_2 + a_2, k_3 + a_3)g(k_1, k_2, k_3) = 0.
\]
\[
\text{(2.23)}
\]

So equations (2.17-2.18.22-2.23) constitute the modified discrete KP equation with self-consistent sources. In the following, we will verify that functions (2.1-2.2) and (2.19-2.20) for \( r, m = 1, \cdots, K \) solve bilinear equations (2.17-2.18.22-2.23).

We can compute the following difference formula for \( P_{i,j}, Q_{i,j}, r, m = 1, \cdots, K \) by employing equations (2.3-2.5):
\[ P_{ij}(k_v + a_v) = -\alpha_v^2 \sqrt{c_i(k_v) - c_i(k_v - a_v)} \begin{vmatrix} E_{ij} & \Phi_i(0) & N_{ij} \\ -\Phi_j(1)^T & -a_{v}^{-2} & -\tilde{\Phi_j}(1)^T \\ -\Phi_j(0)^T & a_{v}^{-1} & -\tilde{\Phi_j}(0)^T \end{vmatrix}, \quad (2.24) \]

\[ P_{ij}(k_v + a_v) = -\phi(1)^2 \begin{vmatrix} D_{ij} & R_{ij} & \Phi_i(0_v) \\ -\Phi_j(1)^T & -a_{v}^{-2} \tilde{\Phi_j}(1)^T \\ -\Phi_j(0)^T & a_{v}^{-1} \tilde{\Phi_j}(0)^T \end{vmatrix}, \quad (2.25) \]

\[ Q_{ij}(k_v + a_v) = a_v^2 \begin{vmatrix} D_{ij} & \Psi_i(-1) & \Phi_i(0_v) \\ -\Phi_j(1)^T & -1 & a_{v}^{-1} \\ M_{ij}^{T} & \phi(-1) & \phi(0_v) \end{vmatrix}, \quad (2.26) \]

\[ Q_{ij}(k_v + a_v) = a_v^2 \begin{vmatrix} D_{ij} & \Phi_i(-1) & \Phi_i(0) \\ -\Phi_j(1)^T & 0 & -a_{2}^{-2} \\ M_{ij}^{T} & \Phi_i(-1) & \Phi_i(0) \end{vmatrix}, \quad (2.27) \]

where \( v = 2, 3 \), \( E_{ij} \) is the \((N - 1)\)th order matrix obtained by eliminating the \( i \)th row and the \( j \)th column from the matrix \( (c_{ij}(k_1 + a_1) + m_{ij}(k_2, k_3))_{1 \leq i, j \leq N} \), respectively; \( \Phi_i(s_v) \) is the \((N - 1)\)th column vector obtained by eliminating the \( j \)th element from \( \Phi_i(s_v) \). In addition, \( N_{ij} \) is given by

\[
N_{ij} = (c_1(k_1 + a_1) + m_{i,j}(k_1, k_2, k_3))_{1 \leq i, j \leq N}, 
\]

Substituting equations (2.11-2.15) and (2.24-2.27) into modified discrete KP equation (1.6), we obtain the following determinant identity:

\[
\begin{vmatrix} M_{d} & C(k_1 + a_1) & \Phi(0_1) \\ \frac{a_t}{a_t} & 1 & 0 \\ -\Phi(0)^T & 0 & a_{1}^{-1} \end{vmatrix} = \begin{vmatrix} M_{d} & C(k_1) & \Phi(-1) & \Phi(0) \\ \frac{a_t}{a_t} & 1 & 0 & 0 \\ -\Phi(1)^T & 0 & 0 & -a_{2}^{-2} \\ -\Phi(0)^T & 0 & -1 & a_{2}^{-1} \end{vmatrix} = 0. \quad (2.28)
\]
In order to show that determinant identity (2.28) holds, let us introduce the following \(2(N + 3) \times 2(N + 3)\) determinant which vanishes,

\[
\begin{vmatrix}
M_d & 0 & 0 & 0 & 0 & C(k_1) & \Phi(-1) & \Phi(0_2) \\
\alpha^T & 0 & 0 & 0 & 1 & 0 & 0 \\
-\Phi(1)^T & 0 & 0 & 1 & 0 & 0 & -a_2^{-2} \\
-\Phi(0)^T & 0 & 0 & 0 & 0 & -1 & a_2^{-1} \\
0 & M_d & C(k_1 + a_1) & \Phi(0_1) & 0 & C(k_1) & \Phi(-1) & \Phi(0_2) \\
0 & a^T & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -\Phi(1)^T & 0 & -a_1^{-2} & 1 & 0 & 0 & -a_2^{-2} \\
0 & -\Phi(0)^T & 0 & a_1^{-1} & 0 & 0 & -1 & a_2^{-1} \\
\end{vmatrix} = 0. \tag{2.29}
\]

Applying the Laplace expansion in \((N + 3) \times (N + 3)\) minors to the left-hand side of (2.29), we obtain the determinant identity (2.28). Therefore, functions \(f, g, P_{i,j}, Q_{i,j}\) given in equations (2.1-2.2, 2.19-2.20) are solutions of equation (1.6). In the same way, substitution of equations (2.11-2.15, 2.24-2.27) into equation (1.7) gives the determinant identity:

\[
\begin{align*}
&\begin{vmatrix}
M_d & C(k_1 + a_1) & \Phi(0_1) \\
\alpha^T & 1 & 0 \\
-\Phi(0)^T & 0 & a_1^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & C(k_1) & \Phi(-1) & \Phi(0_3) \\
\alpha^T & 1 & 0 & 0 \\
-\Phi(1)^T & 0 & 0 & -a_3^{-2} \\
-\Phi(0)^T & 0 & -1 & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & C(k_1) & \Phi(0_3) \\
\alpha^T & 1 & 0 \\
-\Phi(0)^T & 0 & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & C(k_1 + a_1) & \Phi(0_1) & \Phi(0_3) \\
\alpha^T & 1 & 0 & 0 \\
-\Phi(1)^T & 0 & -a_1^{-2} & -a_3^{-2} \\
-\Phi(0)^T & 0 & a_1^{-1} & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & \Phi(-1) & \Phi(0_3) \\
\alpha^T & 1 & 0 \\
-\Phi(0)^T & -1 & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & C(k_1 + a_1) & \Phi(0_1) & C(k_1) \\
\alpha^T & 1 & 0 & 1 \\
-\Phi(1)^T & 0 & -a_1^{-2} & 0 \\
-\Phi(0)^T & 0 & a_1^{-1} & 0 \\
\end{vmatrix} = 0. \tag{2.30}
\end{align*}
\]

which can also be proved similarly.

Now we prove \(f, g\) given in (2.1-2.2) and \(P_{i,j}, Q_{i,j}\) for \(r, m = 1, \ldots, K\) given in (2.19-2.20) are solutions to equations (2.22-2.23). Substituting equations (2.11-2.15, 2.19-2.24-2.27) into equations (2.22-2.23), we get the following two identities for the determinant, respectively:

\[
\begin{align*}
&\begin{vmatrix}
M_d & C(k_1) & \Phi(0_2) \\
\alpha^T & 1 & 0 \\
-\Phi(0)^T & 0 & a_2^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
D_{ij} & R_{ij} & \Phi(0_1) \\
-\Phi(1)^T & -\Phi(0)^T & -a_3^{-2} \\
-\Phi(1)^T & -\Phi(0)^T & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
M_d & C(k_1) & \Phi(0_3) \\
\alpha^T & 1 & 0 \\
-\Phi(0)^T & 0 & a_3^{-1} \\
\end{vmatrix} \\
&\begin{vmatrix}
D_{ij} & R_{ij} & \Phi(0_2) \\
-\Phi(1)^T & -\Phi(0)^T & -a_2^{-2} \\
-\Phi(1)^T & -\Phi(0)^T & a_2^{-1} \\
\end{vmatrix}.
\end{align*}
\]
and we obtain the determinant identity (2.31). Let us introduce the following 2 \times 2 determinant which is equal to zero:

\[
\begin{vmatrix}
D_{ij} & \Phi_i(-1) \Phi_j(02) \\
-\Phi_j(0)^T & \Phi_i(02)
\end{vmatrix}
= 0,
\]

and

\[
\begin{vmatrix}
D_{ij} & \Phi_i(-1) \Phi_j(02) \\
-\Phi_j(0)^T & \Phi_i(02)
\end{vmatrix}
= 0.
\]

Now we prove the determinant identity (2.31). Let us introduce the following 2(N + 2) \times 2(N + 2) determinant which is equal to zero:

\[
\begin{vmatrix}
D^R & \Phi_1(02) \Phi_1(03) \\
M_i & \Phi_i(02) \Phi_i(03) \\
-\Phi_1(0)^T & \Phi_1(02) \\
-\Phi_1(0)^T & \Phi_1(02)
\end{vmatrix}
= 0,
\]

where \(D^R\) denotes the \((N - 1) \times N\) matrix obtained by eliminating the \(i\)th row from the matrix \((c_{1,1}(k_1) + m_{ij}(k_1,k_2,k_3))_{1 \leq i,j \leq N},\) and \(M_i\) is given by

\[
M_i = (c_1(k_1) + m_i) \cdots c_i(k_1) + m_i, d_{i,K+1} + m_i, d_{i,K+1}, \cdots, d_{i,N} + m_i)^T.
\]

Applying the Laplace expansion in \((N + 2) \times (N + 2)\) minors to the left-hand side of equation (2.33), we obtain the determinant identity (2.31).

Similarly, the Laplace expansion of the following 2(N + 2) \times 2(N + 2) determinant which is equal to zero in \((N + 2) \times (N + 2)\) minors:

\[
\begin{vmatrix}
D^C & \Phi(-1) 0 0 0 \\
-\Phi_1(0)^T & 0 1 0 0 \\
-\Phi_1(0)^T & -1 0 0 0 \\
0 & 0 & 0 & \Phi(-1) D_j^C \\
0 & 0 & 0 & \Phi_1(0)^T
\end{vmatrix}
= 0,
\]
where \( D^C_j \) denotes the \( N \times (N - 1) \) matrix obtained by eliminating the \( j \)th column from the matrix 
\[
(c_{i,j}(k_1) + m_{ij}(k_1, k_2, k_3))_{1 \leq i, j \leq N},
\]
and \( R_j \) is given by
\[
R_j = (c_1(k_1) + m_{1j}, \ldots, c_K(k_1) + m_{Kj}, k, d_{K+1,j}, \ldots, d_{N,j} + m_{N,j})^T,
\]
yields the determinant identity (2.32).

So functions (2.1-2.2) and (2.19-2.20) for \( r, m = 1, \ldots, K \) are solutions to the modified discrete 
KP equation with self-consistent sources (2.17-2.18, 2.22-2.23).

If we choose the arbitrary function \( c_{ij} \) in (2.10) as a function of \( k_2 \) instead of \( k_1 \), we can produce 
another form of the modified discrete KP equation with self-consistent sources via source generation 
procedure in a similar way:
\[
a_1 f(k_1 + a_1)g(k_2 + a_2) - a_2 f(k_2 + a_2)g(k_1 + a_1) + (a_2 - a_1) f(k_1 + a_1, k_2 + a_2)g(k_1, k_2, k_3)
\]
\[
= a_1 \sum_{i,j=1}^{K} P_{ij}(k_2 + a_2)Q_{ij}(k_1 + a_1),
\]
(2.34)
\[
a_1 f(k_1 + a_1)g(k_1 + a_3) - a_3 f(k_3 + a_3)g(k_1 + a_1) + (a_3 - a_1) f(k_1 + a_1, k_3 + a_3)g(k_1, k_2, k_3)
\]
\[
= 0,
\]
(2.35)
\[
a_1 f(k_1 + a_1)P_{ij}(k_3 + a_3) - a_3 f(k_3 + a_3)P_{ij}(k_1 + a_1) + (a_3 - a_1) f(k_1 + a_1, k_3 + a_3)P_{ij}(k_1, k_2, k_3)
\]
\[
= 0,
\]
(2.36)
\[
a_1 Q_{ij}(k_1 + a_1)g(k_3 + a_3) - a_3 Q_{ij}(k_3 + a_3)g(k_1 + a_1) + (a_3 - a_1) Q_{ij}(k_1 + a_1, k_3 + a_3)g(k_1, k_2, k_3)
\]
\[
= 0.
\]
(2.37)

Equations (2.34-2.37) can be written in a compact way using the bilinear operator (1.3):
\[
[a_1 e^{1/2a_1 D_1} - 1/2a_1 D_2 - 1/2a_1 D_3 - (a_2 - a_1) e^{1/2a_2 D_1} + 1/2a_2 D_2] f \cdot g
\]
\[
= a_1 \sum_{i,j=1}^{K} e^{1/2a_1 D_2} - 1/2a_1 D_3] P_{ij} \cdot Q_{ij},
\]
(2.38)
\[
[a_1 e^{1/2a_1 D_1} - 1/2a_1 D_2 - 1/2a_1 D_3 - (a_3 - a_1) e^{1/2a_3 D_1} + 1/2a_3 D_2] f \cdot g = 0,
\]
(2.39)
\[
[a_1 e^{1/2a_1 D_1} - 1/2a_1 D_3 - 1/2a_2 D_2 - (a_3 - a_1) e^{1/2a_3 D_1} + 1/2a_3 D_3] P_{ij} = 0,
\]
(2.40)
\[
[a_1 e^{1/2a_1 D_1} - 1/2a_3 D_2 - 1/2a_2 D_3 - (a_3 - a_1) e^{1/2a_3 D_1} + 1/2a_3 D_3] Q_{ij} = 0.
\]
(2.41)

We can derive the following discrete Gram-type determinant solution for the modified discrete 
KP equation with self-consistent sources (2.34-2.37) or (2.38-2.41):
\[
f(k_1, k_2, k_3) = \det(\vec{c}_{ij}(k_2) + m_{ij})_{1 \leq i, j \leq N} = |\vec{D}|,
\]
(2.42)
\[
g(k_1, k_2, k_3) = \left[ \begin{array}{c}
D \\
\Phi(0)^T \\
1
\end{array} \right],
\]
(2.43)

where \( m_{ij}(1 \leq i, j \leq N) \), \( \Phi(s) \), \( \Phi(s) \) are given in (2.3-2.7). In addition, \( \vec{c}_{ij}(k_2) \) satisfies
\[
\vec{c}_{ij}(k_2) = \begin{cases} 
\tilde{c}_{ij}(k_2), & 1 \leq i \leq K \leq N \text{ and } j = 1, 2, \cdots, K, K \in \mathbb{Z}^+ \\
d_{ij}, & \text{otherwise,}
\end{cases}
\]

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with \( \bar{c}_i(k_2) (1 \leq i \leq K) \) being an arbitrary function of \( k_2 \), \( \bar{d}_{ij}(1 \leq i, j \leq N) \) being an arbitrary constant and \( K \) being a positive integer, and

\[
P_{i,j}(k_1,k_2,k_3) = \sqrt{c_i(k_2) - c_i(k_2 - a_2)} \left| \begin{array}{c} D_{ij} \\ \Phi_j(0)^T \end{array} \right| R_{ij}, \tag{2.44}
\]

\[
Q_{i,j}(k_1,k_2,k_3) = a_1 \sqrt{\Delta c_i(k_2)} \left| \begin{array}{c} \bar{D}_{ij} \\ \Phi_j(-1)^T \end{array} \right| M_{ij}, \tag{2.45}
\]

where \( \bar{D}_{ij} \) is the \( (N-1) \)th order matrix obtained by eliminating the \( i \)th row and the \( j \)th column from the \( N \times N \) matrix \( \bar{D} \), and

\[
\begin{align*}
R_{ij} &= (\bar{c}_i(k_2) + m_{1j}, \cdots, \bar{c}_{i-1}(k_2) + m_{1i-1j}, \bar{c}_{i+1}(k_2) + m_{1i+1j}, \cdots, \\
\bar{c}_K(k_2) + m_{Kj}, & \bar{d}_{k+1j} + m_{K+1j}, \cdots, \bar{d}_{Nj} + m_{Nj})^T, \\
M_{ij} &= (\bar{c}_i(k_2) + m_{1i}, \bar{c}_i(k_2) + m_{1i-1}, \bar{c}_i(k_2) + m_{1i+1}, \cdots, \\
\bar{c}_i(k_2) + m_{i1}, & \bar{d}_{i+1} + m_{i1}, \cdots, \bar{d}_{iN} + m_{iN})^T.
\end{align*}
\]

### 3. Commutativity of the source generation procedure and Bäcklund transformation

In this section, we show that the commutativity of the source generation procedure and Bäcklund transformation holds for the discrete KP equation. For this purpose, we derive another form of the modified discrete KP with self-consistent sources which is the Bäcklund transformation for the discrete KP equation with self-consistent sources.

We have shown that the Grammian determinants \( f(k_1,k_2,k_3), g(k_1,k_2,k_3), P_{i,j}(k_1,k_2,k_3), Q_{i,j}(k_1,k_2,k_3) \) given in (2.42-2.45) satisfy the modified discrete KP equation with self-consistent sources (2.38-2.41). Now we take

\[
f'(k_1,k_2,k_3) = g(k_1,k_2,k_3), \tag{3.1}
\]

\[
g'_{i,j}(k_1,k_2,k_3) = Q_{i,j}(k_1,k_2,k_3), \tag{3.2}
\]

\[
h_{i,j}(k_1,k_2,k_3) = P_{i,j}(k_1,k_2,k_3), \tag{3.3}
\]

and introduce two new fields

\[
g_{i,j}(k_1,k_2,k_3) = -a_1 a_3 \sqrt{\Delta c_i(k_2)} \left| \begin{array}{c} \bar{D}_{ij}^C \Phi(k_3) \end{array} \right|, \tag{3.4}
\]

\[
h_{i,j}(k_1,k_2,k_3) = -a_3 \sqrt{c_i(k_2) - c_i(k_2 - a_2)} \left| \begin{array}{c} \bar{D}_{ij}^R \Phi(-1) \end{array} \right|, \tag{3.5}
\]

where \( \bar{D}_{ij}^C \) denotes the \( N \times (N-1) \) matrix obtained by eliminating the \( j \)th column from the \( N \times N \) matrix \( \bar{D} \) and \( \bar{D}_{ij}^R \) denotes the \( (N-1) \times N \) matrix obtained by eliminating the \( i \)th row from the \( N \times N \) matrix \( \bar{D} \).
It is not difficult to show that Grammian determinants \( f, g_{i,j}, h_{i,j} \) and \( f', g'_{i,j}, h'_{i,j} \) are two solutions to the discrete KP equation with self-consistent sources derived in [23]:

\[
\begin{align*}
(a_1(a_2-a_3)e^{1/2(-D_1+D_2+D_3)}+a_2(a_3-a_1)e^{1/2(D_1-D_2+D_3)}+a_3(a_1-a_2)e^{1/2(-D_1-D_2+D_3)})f \cdot f &= \sum_{i,j=1}^{K} e^{1/2(D_1-D_2+D_3)}g_{i,j}h_{i,j}, \\
(a_3e^{1/2(D_1-D_3)}-a_1e^{1/2(D_1-D_3)}+(a_1-a_3)e^{1/2(D_1+D_3)})f \cdot h_{i,j} &= 0, \quad j = 1, \cdots, K, \\
(a_3e^{1/2(D_1-D_3)}-a_1e^{1/2(D_1-D_3)}+(a_1-a_3)e^{1/2(D_1+D_3)})g_{i,j}f &= 0, \quad j = 1, \cdots, K.
\end{align*}
\]

Furthermore, we can verify that the Grammian determinants \( f, f', g_{i,j}, g'_{i,j}, h_{i,j}, h'_{i,j} \) satisfy the following six bilinear equations:

\[
\begin{align*}
[a_1e^{1/2D_1-1/2D_2} - a_2e^{1/2D_1-1/2D_3} + (a_2-a_1)e^{1/2D_1+1/2D_3}]f \cdot f' &= \sum_{i,j=1}^{K} e^{1/2D_2-1/2D_3}h_{i,j}g'_{i,j}, \\
[a_1e^{1/2D_1-1/2D_3} - a_3e^{1/2D_1+1/2D_3} + (a_3-a_1)e^{1/2D_1-1/2D_3}]f \cdot f' &= 0, \\
[a_1e^{1/2D_2} - a_3e^{1/2D_3} + (a_3-a_1)e^{1/2D_2+1/2D_3}]g_{i,j}g'_{i,j} &= 0, \quad j = 1, \cdots, K, \\
[a_1e^{1/2D_2} - a_3e^{1/2D_3} + (a_3-a_1)e^{1/2D_2+1/2D_3}]h_{i,j}h'_{i,j} &= 0, \quad j = 1, \cdots, K, \\
e^{1/2D_3}g_{i,j}f' &= (e^{-1/2D_3} - e^{1/2D_3})f \cdot g'_{i,j}, \quad j = 1, \cdots, K, \\
e^{1/2D_3}f \cdot h'_{i,j} &= (e^{-1/2D_3} - e^{1/2D_3})h_{i,j}f', \quad j = 1, \cdots, K.
\end{align*}
\]

Equations (3.9-3.14) constitute another form of the modified discrete KP with self-consistent sources. It is proved in [23] that the discrete KP equation with self-consistent sources (3.6-3.8) possesses the following bilinear Bäcklund transformation:

\[
\begin{align*}
[\gamma e^{1/2D_1-1/2D_2} + z_2e^{1/2D_2-1/2D_3} + \lambda_1z_3e^{1/2D_3+1/2D_2}]f \cdot f' &= \lambda_2 \sum_{i,j=1}^{K} e^{1/2D_2-1/2D_3}h_{i,j}g'_{i,j}, \\
[\beta_1e^{1/2D_1-1/2D_3} - e^{1/2D_3-1/2D_1} + \lambda_1e^{1/2D_2+1/2D_3}]f \cdot f' &= 0, \\
[\beta_1e^{1/2D_2} - e^{1/2D_1} + \lambda_1e^{1/2D_2+1/2D_3}]g_{i,j}g'_{i,j} &= 0, \quad j = 1, \cdots, K, \\
[\beta_1e^{1/2D_2} - e^{1/2D_3} + \lambda_1e^{1/2D_1+1/2D_3}]h_{i,j}h'_{i,j} &= 0, \quad j = 1, \cdots, K, \\
e^{1/2D_3}g_{i,j}f' &= (\beta_2e^{1/2D_3} + \lambda_2e^{1/2D_3})f \cdot g'_{i,j}, \quad j = 1, \cdots, K, \\
e^{1/2D_3}f \cdot h'_{i,j} &= (\beta_2e^{1/2D_3} + \lambda_2e^{1/2D_3})h_{i,j}f', \quad j = 1, \cdots, K,
\end{align*}
\]

where \( \gamma \) is an arbitrary constant and \( \beta_1, \beta_2, \lambda_1, \lambda_2 \) are constants satisfying the relation \( \lambda_1\beta_2a_1 = \lambda_2\beta_1(a_1-a_3) \).

If we take \( \gamma = -a_1, \lambda_1 = \frac{a_3-a_1}{a_1}, \lambda_2 = -1, \beta_1 = \frac{a_1}{a_1}, \beta_2 = 1 \), then equations (3.15-3.20) become (3.9-3.14). Hence the modified discrete KP equation with self-consistent sources (3.9-3.14) is a Bäcklund transformation for the discrete KP equation with self-consistent sources (3.6-3.8).

4. Summary and discussion

In this paper, we have produced two forms of the modified discrete KP equation with self-consistent sources (2.17-2.23) and (2.38-2.41) via the source generation procedure, and given their
discrete Gram-type determinant solutions. We have also constructed another form of the modified discrete KP equation with self-consistent sources (3.9-3.14) which is the Bäcklund transformation for the discrete KP equation with self-consistent sources derived in [23].

We investigate a continuum limit of the modified discrete KP with self-consistent sources (2.34-2.37) with one pair of sources \( P_1(k_1, k_2, k_3), Q_1(k_1, k_2, k_3) \), for simplicity. If we take

\[
\begin{align*}
    a_1 &= \delta, a_2 = \varepsilon, a_3 = 1, D_{k_1} = D_y, D_{k_2} = D_y - \frac{1}{2} \varepsilon D_t, D_{k_3} = D_n, \\
    f(k_1, k_2, k_3) &= F(n, y, t), g(k_1, k_2, k_3) = G(n, y, t), \\
    P_{11}(k_1, k_2, k_3) &= \varepsilon S_{11}(n, y, t), Q_{11}(k_1, k_2, k_3) = \frac{1}{2} \varepsilon H_{11}(n, y, t),
\end{align*}
\]

in (2.38-2.41) with one pair of sources \( P, Q \) and compare \( \varepsilon^2 \delta \) order of the two sides of the equation (2.38) and \( \delta \) order of the two sides of the equations (2.39-2.41), then we obtain the modified differential-difference KP equation with self-consistent sources:

\[
\begin{align*}
    (D_y^2 + D_t)F \cdot G &= H_{11} S_{11}, \\
    D_y e^{\frac{1}{2}D_n}F \cdot G &= (e^{\frac{1}{2}D_n} - e^{-\frac{1}{2}D_n})F \cdot G, \\
    D_y e^{\frac{1}{2}D_n}S_{11} &= (e^{\frac{1}{2}D_n} - e^{-\frac{1}{2}D_n})S_{11}, \\
    D_y e^{\frac{1}{2}D_n}H_{11} \cdot G &= (e^{\frac{1}{2}D_n} - e^{-\frac{1}{2}D_n})H_{11} \cdot G,
\end{align*}
\]

which can be derived by applying source generation procedure to the modified differential-difference KP equation [25]:

\[
\begin{align*}
    (D_y^2 + D_t)F \cdot G &= 0, \\
    D_y e^{\frac{1}{2}D_n}F \cdot G &= (e^{\frac{1}{2}D_n} - e^{-\frac{1}{2}D_n})F \cdot G.
\end{align*}
\]

Now we show that the exact solution to the modified discrete KP with self-consistent sources (2.34-2.37) yields the exact solution to the modified differential-difference KP equation with self-consistent sources (4.4-4.7) through the continuum limit.

We obtain the \( N \)-soliton solution for the modified discrete KP with self-consistent sources (2.34-2.37) by choosing

\[
\begin{align*}
    m_{i,j}(k_1, k_2, k_3) &= \frac{1}{p_i + \bar{p}_j} \left( \frac{1 + \bar{p}_j a_1}{1 - p_i a_1} \right)^{\frac{k_i}{n}} \left( \frac{1 + \bar{p}_j a_2}{1 - p_i a_2} \right)^{\frac{k_j}{n}} \left( \frac{1 + \bar{p}_j a_3}{1 - p_i a_3} \right)^{\frac{k_j}{n}}, 1 \leq i, j \leq N, \\
    \phi_i(k_1, k_2, k_3; s) &= \frac{\bar{p}_j (1 - p_i a_1)}{1 - \bar{p}_j a_1} \left( \frac{1 - p_i a_2}{1 - \bar{p}_j a_2} \right)^{\frac{k_i}{n}} \left( \frac{1 - p_i a_3}{1 - \bar{p}_j a_3} \right)^{\frac{k_i}{n}}, 1 \leq i \leq N, \\
    \tilde{\phi}_j(k_1, k_2, k_3; s) &= \frac{\bar{p}_j (1 + \bar{p}_j a_1) - \frac{k_j}{n}}{1 - \bar{p}_j a_1} \left( \frac{1 + \bar{p}_j a_2}{1 - \bar{p}_j a_2} \right)^{\frac{k_j}{n}} \left( \frac{1 + \bar{p}_j a_3}{1 - \bar{p}_j a_3} \right)^{\frac{k_j}{n}}, 1 \leq j \leq N,
\end{align*}
\]

in equations (2.42-2.45). In equations (4.8-4.10), \( p_i, \bar{p}_i (1 \leq i \leq N) \) are arbitrary constants. For example, if we take \( N = 1, K = 1 \) and

\[
\begin{align*}
    \phi_1(k_1, k_2, k_3; s) &= \frac{\bar{p}_j (1 - p_1 a_1)}{1 - \bar{p}_j a_1} \left( \frac{1 - p_1 a_2}{1 - \bar{p}_j a_2} \right)^{\frac{k_i}{n}} \left( \frac{1 - p_1 a_3}{1 - \bar{p}_j a_3} \right)^{\frac{k_i}{n}}, \\
    \tilde{\phi}_j(k_1, k_2, k_3; s) &= \frac{\bar{p}_j (1 + \bar{p}_j a_1)}{1 + \bar{p}_j a_1} \left( \frac{1 + \bar{p}_j a_2}{1 + \bar{p}_j a_2} \right)^{\frac{k_j}{n}} \left( \frac{1 + \bar{p}_j a_3}{1 + \bar{p}_j a_3} \right)^{\frac{k_j}{n}}, \\
    c_1(k_2) &= \frac{1 + 2a_2 \tilde{p}_1(k_2)}{p_1 + \bar{p}_1},
\end{align*}
\]
where $\beta_1(k_2)$ is an arbitrary function of $k_2$, we obtain the one-soliton solution of the modified discrete KP with self-consistent sources (2.34-2.37):

$$f = \frac{(1 + 2a_2 \beta_1(k_2))^\frac{1}{a_1}}{p_1 + \bar{p}_1} + \frac{1}{p_1 + \bar{p}_1} \left(1 + \bar{p}_1 a_1\right)^\frac{1}{a_1} \left(1 + \bar{p}_1 a_2\right)^\frac{1}{a_2} \left(1 + \bar{p}_1 a_3\right)^\frac{1}{a_3}, \tag{4.14}\$$

$$g = \frac{(1 + 2a_2 \beta_1(k_2))^\frac{1}{a_1}}{p_1 + \bar{p}_1} - \frac{\bar{p}_1}{p_1(p_1 + \bar{p}_1)} \left(1 + \bar{p}_1 a_1\right)^\frac{1}{a_1} \left(1 + \bar{p}_1 a_2\right)^\frac{1}{a_2} \left(1 + \bar{p}_1 a_3\right)^\frac{1}{a_3}, \tag{4.15}\$$

$$P_{11} = \sqrt{c_1(k_2) - c_1(k_2 - a_2)(1 + p_1 a_1)^\frac{1}{a_1}(1 + \bar{p}_1 a_2)^\frac{1}{a_2}(1 + \bar{p}_1 a_3)^\frac{1}{a_3}}, \tag{4.16}\$$

$$Q_{11} = \frac{1}{p_1} \sqrt{c_1(k_2 + a_2) - c_1(k_2)(1 + p_1 a_1)^\frac{1}{a_1}(1 + \bar{p}_1 a_2)^\frac{1}{a_2}(1 + \bar{p}_1 a_3)^\frac{1}{a_3}}. \tag{4.17}\$$

We derive the following variable transformation from equation (4.1):

$$t = -\frac{1}{2} \epsilon k_2, \quad y = k_1 + k_2, \quad n = k_3. \tag{4.18}\$$

Applying equations (4.18) and setting $\beta_1(k_2) = b_1(t), c_1(k_2) = C_1(t)$, we obtain

$$c_1(k_2) = \left(1 + 2a_2 \beta_1(k_2)^\frac{1}{a_1}\right) = \frac{e^{\frac{1}{2} \ln(1 + 2a_2 \beta_1(k_2)^\frac{1}{a_1})}}{p_1 + \bar{p}_1} = \frac{e^{2 \beta_1(k_2) + O(\epsilon)}}{p_1 + \bar{p}_1}, \tag{4.19}\$$

which gives $C_1(t) = \frac{e^{2 \beta_1(t)}}{p_1 + \bar{p}_1}$ in the small limit of $\epsilon$. Furthermore, we can derive the following equations using (4.1-4.3) and (4.18):

$$f = \frac{(1 + 2a_2 \beta_1(k_2))^\frac{1}{a_1}}{p_1 + \bar{p}_1} + \frac{1}{p_1 + \bar{p}_1} \left(1 + \bar{p}_1 a_1\right)^\frac{1}{a_1} \left(1 + \bar{p}_1 a_2\right)^\frac{1}{a_2} \left(1 + \bar{p}_1 a_3\right)^\frac{1}{a_3}
= \frac{e^{2 \beta_1(k_2) + O(\epsilon)}}{p_1 + \bar{p}_1} + \frac{1}{p_1 + \bar{p}_1} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \delta)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \epsilon)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \delta)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \epsilon)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \delta)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \epsilon)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \delta)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \epsilon)} e^{k_3 \ln(1 + p_1)}
= \frac{1}{p_1 + \bar{p}_1} \left(e^{2 \beta_1(t) + O(\epsilon)} + e^{\phi_1 + \bar{p}_1 \gamma} + (\bar{p}_1^2 - p_1^2) + (\ln(1 + \bar{p}_1) + n + O(\delta)) + O(\epsilon^2)\right), \tag{4.20}\$$

$$g = \frac{(1 + 2a_2 \beta_1(k_2))^\frac{1}{a_1}}{p_1 + \bar{p}_1} - \frac{\bar{p}_1}{p_1(p_1 + \bar{p}_1)} \left(1 + \bar{p}_1 a_1\right)^\frac{1}{a_1} \left(1 + \bar{p}_1 a_2\right)^\frac{1}{a_2} \left(1 + \bar{p}_1 a_3\right)^\frac{1}{a_3}
= \frac{1}{p_1 + \bar{p}_1} \left(e^{2 \beta_1(t) + O(\epsilon)} - \frac{\bar{p}_1}{p_1} e^{\phi_1 + \bar{p}_1 \gamma} (\bar{p}_1^2 - p_1^2) + (\ln(1 + \bar{p}_1) + n + O(\delta)) + O(\epsilon^2)\right), \tag{4.22}\$$

$$P_{11} = \sqrt{c_1(k_2) - c_1(k_2 - a_2)(1 + p_1 a_1)^\frac{1}{a_1}(1 + \bar{p}_1 a_2)^\frac{1}{a_2}(1 + \bar{p}_1 a_3)^\frac{1}{a_3}}
= \sqrt{\epsilon \frac{dc_1(k_2)}{dk_2} + O(\epsilon^2)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \delta)} e^{\frac{1}{2} \ln(1 + \bar{p}_1 \epsilon)} e^{k_3 \ln(1 + \bar{p}_1)}
= \sqrt{-\frac{1}{2} \epsilon^2 \frac{dc_1(t)}{dt} + O(\epsilon^4)} e^{\beta_1 + \bar{p}_1 \gamma + (\ln(1 + \bar{p}_1) + n + O(\delta)) + O(\epsilon^2)}
= \epsilon \sqrt{-\frac{1}{2} \frac{dc_1(t)}{dt} + O(\epsilon^2)} e^{\beta_1 + \bar{p}_1 \gamma + (\ln(1 + \bar{p}_1) + n + O(\delta)) + O(\epsilon^2)}, \tag{4.24}\$$

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\( Q_{11} = \frac{1}{p_1} \sqrt{c_1(k_2 + a_2) - c_1(k_2)}(1 - p_1 a_1) e^{\frac{t_1}{p_1} (1 - p_1 a_2)} e^{\frac{t_2}{p_1} (1 - p_1 a_3)} \)

\[
\begin{align*}
Q_{11} & = \sqrt{\frac{dc_1(k_2)}{dk_2}} + O(e^2) e^{-\frac{t_1}{p_1} \ln(1 - p_1 \delta)} e^{-\frac{t_2}{p_1} \ln(1 - p_1 \epsilon)} e^{-k_1 \ln(1 - p_1)} \\
& = \sqrt{-\frac{1}{2} \varepsilon e^{2} \frac{dC_1(t)}{dt} + O(e^4)} e^{p_1 \gamma - p_1^2 \gamma^2 - \ln(1 - p_1)n + O(\delta) + O(e^2)} \\
& = \varepsilon \sqrt{-\frac{1}{2} \frac{dC_1(t)}{dt} + O(e^2)} e^{p_1 \gamma - p_1^2 \gamma^2 - \ln(1 - p_1)n + O(\delta) + O(e^2)}. \quad (4.25)
\end{align*}
\]

From equations (4.3) and (4.24-4.25), we have

\[
\begin{align*}
H_{11} & = \sqrt{-\frac{1}{2} \frac{dC_1(t)}{dt} + O(e^2)} e^{\bar{p}_1 \gamma + \bar{p}_1^2 \gamma^2 + \ln(1 + \bar{p}_1) + O(\delta) + O(e^2)} , \quad (4.26) \\
S_{11} & = \sqrt{-\frac{1}{2} \frac{dC_1(t)}{dt} + O(e^2)} e^{p_1 \gamma - p_1^2 \gamma^2 - \ln(1 - p_1)n + O(\delta) + O(e^2)} . \quad (4.27)
\end{align*}
\]

Through the small limit of \( \delta \) and \( \epsilon \), the functions \( f, g, H_{11}, S_{11} \) given in (4.20-4.22,4.26-4.27) become

\[
\begin{align*}
f & = \frac{e^{2h_1(t)}}{p_1 + \bar{p}_1} (1 + e^{(p_1 + \bar{p}_1)\gamma + (p_1^2 + \bar{p}_1^2)\gamma^2 + \ln(1 + \bar{p}_1) - \ln(1 - p_1)n - 2b_1(t)} , \quad (4.28) \\
g & = \frac{e^{2h_1(t)}}{p_1 + \bar{p}_1} (1 + \frac{\bar{p}_1}{p_1} e^{(p_1 + \bar{p}_1)\gamma + (p_1^2 + \bar{p}_1^2)\gamma^2 + \ln(1 + \bar{p}_1) - \ln(1 - p_1)n - 2b_1(t)} , \quad (4.29) \\
H_{11} & = \sqrt{-\frac{1}{2} \frac{dC_1(t)}{dt}} e^{\bar{p}_1 \gamma + \bar{p}_1^2 \gamma^2 + \ln(1 + \bar{p}_1)n} \sqrt{-\frac{\bar{b}_1(t)}{p_1 + \bar{p}_1} e^{\bar{p}_1 \gamma + \bar{p}_1^2 \gamma^2 + \ln(1 + \bar{p}_1)n + b_1(t)} , \quad (4.30)
\end{align*}
\]

and

\[
\begin{align*}
S_{11} & = \sqrt{-\frac{1}{2} \frac{dC_1(t)}{dt}} e^{p_1 \gamma - p_1^2 \gamma^2 - \ln(1 - p_1)n} \sqrt{-\frac{b_1(t)}{p_1 + \bar{p}_1} e^{p_1 \gamma - p_1^2 \gamma^2 - \ln(1 - p_1)n + b_1(t)} , \quad (4.31)
\end{align*}
\]

respectively. We can verify that the functions \( f, g, H_{11}, S_{11} \) given in (4.28-4.31) solve the modified differential-difference KP equation with self-consistent sources (4.4-4.7).

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