Revisiting Li-Yau type inequalities on Riemannian manifolds

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Abstract

Inspired Yau’s work (Comm. Anal. Geom., 1994), in this short note we provide a new version of Li-Yau gradient estimate for the linear heat equation, which generalizes some known results and gives new gradient estimates. Also we explain the different known results as different cases here.

Keywords: Li-Yau inequality, Gradient estimates, Heat equation.

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1 Introduction

In their well known work, Li and Yau [9] proved an upper bound on the gradient estimates of positive solutions to the heat equation, which is so-called Li-Yau inequality. It gives parabolic Harnack inequality, which provides a comparison between heat at two different points in space and at different times. It also gives good bounds of the associated heat kernel, Green functions and lower bound of Dirichlet or Neumann eigenvalue. Since then it becomes a powerful tool in heat kernel analysis, PDE, entropy theory, differential geometry etc. It also plays an important role in the Perelman’s solution to the Poincaré conjecture. More precisely, it claims that, for an $n$-dimensional compact Riemannian manifold with Ricci curvature bounded below by $-K (K \geq 0)$, if $u$ is a positive solution to the heat equation $\partial_t u = \Delta u$, then for all $\alpha > 1$,

$$|\nabla \log u|^2 - \alpha (\log u)_t \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{2(\alpha - 1)}. \quad (1.1)$$

If we take the Gaussian kernel $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ on $\mathbb{R}^n$, in this case the inequality (1.1) is equality for $K = 0, \alpha = 1$. There are a series work on improving...
this inequality for negative curvature and for small time and large time, see [7, 13, 8, 5, 1, 17] and the references therein. We briefly recall them as follows:

B. Davies in [4] improved the estimate to

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2K}{4(\alpha - 1)}$$  \hspace{1cm} (1.2)

holds for any $\alpha > 1, t > 0$.

In [7], R. S. Hamilton proved a new gradient estimate for the heat equation, the new viewpoint is that we can see constant $\alpha$ in [9] as function of time $t$:

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq e^{2\alpha Kt} \frac{nK}{2t}.$$  \hspace{1cm} (1.3)

Bakry-Qian in [3] improved the above inequality to the following: for all $t > 0$,

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \sqrt{nK} \sqrt{\frac{|\nabla u|^2}{u^2}} + \frac{nK}{4} + \frac{n}{2t},$$  \hspace{1cm} (1.4)

and

$$|\nabla f|^2 - \left(1 + \frac{2}{3}Kt\right) f_t \leq \frac{nK}{2t} \left(1 + \frac{1}{3}Kt\right).$$  \hspace{1cm} (1.5)

Li-Xu in [8] re-found (1.5) and further got the following gradient estimate:

$$|\nabla f|^2 - \left(1 + \frac{\sinh(Kt) \cosh(Kt) - Kt}{\sin^3(Kt)}\right) f_t \leq \frac{nK}{2t} (\coth(Kt) + 1).$$  \hspace{1cm} (1.6)

Meanwhile, Bakry and Ledoux in [2] found a logarithmic Sobolev form of the Li-Yau parabolic inequality by semigroup method, which generalizes (1.1). See also [5] and [6] in this direction. The author himself in [11] provides a general form of gradient estimate which generalizes (1.5) and (1.6), see also [5, 17].

In the very recent, Bakry, Bolley and Gentil in [1] have obtained the following refined global Li-Yau inequality:

$$\frac{|\nabla u|^2}{u^2} < \frac{n}{2} \Phi_t \left(\frac{4}{nK} \Delta u \right),$$  \hspace{1cm} (1.7)

where

$$\Phi_t(x) = \begin{cases} \frac{-K}{2} \left( x - 2 + 2\sqrt{1 - x \coth(-Kt\sqrt{1 - x})} \right), & x \leq 1 \\ \frac{-K}{2} \left( x - 2 + 2\sqrt{x - 1} \cot(-Kt\sqrt{x - 1}) \right), & 1 \leq x < 1 + \frac{x^2}{K^2t} \end{cases}.$$  

This bound works for both negative and positive curvature. They show estimate (1.7) is stronger than (1.1), (1.2), (1.3), (1.4) and (1.6), see section 5 in [1].

Inspired Yau’s work [14], see also [10], we give a new version of gradient estimate for the linear heat equation, if the parameter functions satisfy certain nonlinear condition, see Theorem 2.1. We further analysis the nonlinear condition, and divide it into two cases (see Theorem 2.3). Then we show the known results (1.1), (1.5) and (1.6) are particular ones of the respective cases (see Corollary 2.5 and Theorem 2.10), also we refind the Hamilton type gradient estimate which refines (1.3), see Corollary 2.7 and Corollary 2.8. Evenmore, we can derive new estimate refining (1.2) by combining the two cases, see Remark 2.6.
2 Main results

Theorem 2.1. Let \( M \) be a compact manifold without boundary, \( \text{Ricci}(M) \geq -K \). Let \( u \) be any positive solution of the heat equation

\[
 u_t = \Delta u
\]  
(2.1)
on \( M \). For \( c(t) \in C^1(0, \infty) \) and \( \alpha \in C^1[0, \infty) \) be two arbitrary positive functions, if the conditions below are satisfied:

(A) \( c(0^+) := \lim_{t \downarrow 0} c(t) = +\infty \);

(B) (nonlinear condition) the following quadratic inequality holds for all \( x \geq 0, t > 0 \),

\[
 -\frac{2(1 - \alpha(t))^2}{n\alpha^2(t)} x^2 + \left(2K + \frac{4(1 - \alpha)c(t)}{n\alpha^2} - \frac{\alpha'}{\alpha}\right) x - \frac{2c^2(t)}{n\alpha^2} - c'(t) + \frac{\alpha'(t)}{\alpha(t)}c(t) \leq 0.
\]

Then we have for all \( t > 0 \),

\[
 |\nabla \log u|^2 - \alpha(t)(\log u)_t - c(t) \leq 0.
\]  
(2.2)

Proof. Without loss of any generalization, we can assume \( \alpha \geq \varepsilon > 0 \), otherwise \( u_\varepsilon := u + \varepsilon \), obviously \( u_\varepsilon \) is also a solution to the heat equation (2.1), then letting \( \varepsilon \to 0 \) gives the desired result. For arbitrary positive function \( \alpha \in C^1[0, \infty) \), denote

\[
 F = |\nabla \log u|^2 - \alpha(t)(\log u)_t.
\]

Direct computation gives

\[
 \partial_t F = 2\nabla \log u \cdot \nabla \frac{u_t}{u} - \alpha'(t)(\log u)_t - \alpha(t)(\log u)_{tt}
 = 2\nabla \log u \cdot \nabla \Delta \frac{u}{u} - \alpha'\Delta \frac{u}{u} - \alpha(t)(\log u)_{tt}
 = 2\nabla \log u \cdot (\Delta \log u + |\nabla \log u|^2) - \alpha'\Delta \log u - \alpha'\nabla \log u \cdot \nabla (\log u) - \alpha(t)(\log u)_{tt},
\]

\[
 \Delta F = \Delta |\nabla \log u|^2 - \alpha \Delta (\log u)_t,
\]

and

\[
 \nabla \log u \cdot \nabla F = \nabla \log u \cdot \nabla |\nabla \log u|^2 - \alpha \nabla \log u \cdot \nabla (\log u)_t
 = \nabla \log u \cdot \nabla |\nabla \log u|^2 - \frac{1}{2} \alpha \partial_t (|\nabla \log u|^2).
\]

This yields

\[
 (\partial_t - \Delta) F = - (\Delta |\nabla \log u|^2 - 2 \nabla \log u \cdot \nabla \Delta \log u) + 2\nabla \log u \cdot \nabla |\nabla \log u|^2 - \alpha'\Delta \log u
 - \alpha'\nabla \log u \cdot \nabla (\log u)_t
 = - \left( |\nabla \log u|^2 - 2 \nabla \log u \cdot \nabla \Delta \log u \right) + 2\nabla \log u \cdot \nabla |\nabla \log u|^2 - \alpha'\Delta \log u
 - \alpha'\nabla \log u \cdot \nabla (\log u)_t
 = 2\nabla \log u \cdot \nabla F' - 2 (|\text{Hess} \log u|^2 + \text{Ricci}(\nabla \log u, \nabla \log u))
 - \alpha'(t) \left( \Delta \log u + |\nabla \log u|^2 \right) + c(t),
\]

(2.3)
where the last equality follows from the Bochner-Weitzenböck formula.

For the functional \( \Psi := F - c(t) \), we have obviously \( \Psi(0+) \leq 0 \). Assuming that:

if \( \Psi = F - c(t) \leq 0 \) for \( t \leq t_0 \) and \( \Psi(x_0, t_0) = F(x_0, t_0) - c(t_0) = 0 \) for some \( x_0 \in M \),

by the maximum principle we would have at the point \((x_0, t_0)\),

\[
\frac{d}{dt} \Psi \geq 0, \quad \Delta \Psi \leq 0, \quad \nabla \Psi = 0.
\]

Noticing that at \((x_0, t_0)\),

\[
\Delta \log u = -\frac{F}{\alpha} + \frac{1 - \alpha}{\alpha} |\nabla \log u|^2 = -\frac{c}{\alpha} + \frac{1 - \alpha}{\alpha} |\nabla \log u|^2,
\]

substituting into (2.3), it follows at \((x_0, t_0)\)

\[
0 \leq (\partial_t - \Delta) \Psi = 2 \nabla \log u \cdot \nabla \Psi - 2 \left( |\text{Hess} \log u|^2 + \text{Ricci}(\nabla \log u, \nabla \log u) \right) - \alpha'(t) \left( \Delta \log u + |\nabla \log u|^2 \right) - c'
\]

\[
\leq -\frac{2}{n} (\Delta \log u)^2 + 2K |\nabla \log u|^2 - \alpha'(t) \left( \Delta \log u + |\nabla \log u|^2 \right) - c'
\]

\[
= -\frac{2}{n} \left( \frac{1 - \alpha}{\alpha} |\nabla \log u|^2 - \frac{c}{\alpha} \right)^2 + 2K |\nabla \log u|^2 - \alpha' \left( \frac{|\nabla \log u|^2}{\alpha} - \frac{c}{\alpha} \right) - c'
\]

\[
= -\frac{2}{n} \left( \frac{\alpha - 1}{\alpha} \right)^2 |\nabla \log u|^4 + \left( 2K + \frac{4(1 - \alpha)c}{n\alpha^2} - \frac{\alpha'}{\alpha} \right) |\nabla \log u|^2
\]

\[- \frac{2c^2}{n\alpha^2} - c' + \frac{c'}{\alpha}.\]

Combining with nonlinear condition (B), we have at \((x_0, t_0)\),

\[
\partial_t \Psi = 0, \quad \Delta \Psi = 0, \quad \nabla \Psi = 0.
\]

By the strong maximum principle, we see that \( \Psi(x_0, t) \leq 0, \forall t \in (t_0, t_0 + \delta) \) for some \( \delta > 0 \), thus we have

\[
\Psi = |\nabla \log u|^2 - \alpha(\log u)_t - c(t) \leq 0, \forall t > 0.
\]

We complete the proof. \( \square \)

**Proposition 2.2.** For the quadratic function \( f(x) = ax^2 + bx + c \), where \( a, b, c \) are some constants and \( a < 0 \). \( f(x) \leq 0 \) holds for all \( x \geq 0 \) if and only if the following two cases satisfy any one of them: **Case 1:** \( b \leq 0 \) and \( c \leq 0 \); **Case 2:** \( b^2 - 4ac \leq 0 \).

**Proof.** The proof is elementary. \( \square \)

Combining with Theorem 2.1 and Proposition 2.2, we have the following theorem.

**Theorem 2.3.** Let \( M \) be a compact manifold without boundary, \( \text{Ricci}(M) \geq -K \). Let \( u \) be any positive solution of the heat equation \( u_t = \Delta u \) on \( M \). For \( c(t) \in C^1(0, \infty) \), and \( \alpha \in C^1[0, \infty) \) be an arbitrary positive function, if the conditions below are satisfied:


(a). \( c(0+) := \lim_{t \downarrow 0} c(t) = +\infty \);

(b). Suppose the following two cases satisfy any one of them:

**Case 1:** \( \forall t > 0, (1 - \alpha)c(t) \leq \frac{n\alpha(\alpha' - 2K\alpha)}{4} \) and \( \left( \frac{\alpha}{c} \right)' \leq \frac{2}{n\alpha} \). \hfill (2.4)

**Case 2:** \( \forall t > 0, (1 - \alpha)^2 c'(t) \geq (1 - \alpha)(2K - \alpha')c(t) + \frac{n(2K\alpha - \alpha')^2}{8} \). \hfill (2.5)

Then we have for all \( t > 0 \),

\[
|\nabla \log u|^2 - \alpha(t)(\log u)_t - c(t) \leq 0.
\] \hfill (2.6)

**Proof.** The proof follows from the combination of Theorem 2.1 and Proposition 2.2. By the assumption of \( c(0+) = +\infty \), we have Condition A holds in Theorem 2.1. To verify the nonlinear condition B in Theorem 2.1, by Proposition 2.2, we divide it into two cases.

1. **Case 1 in Proposition 2.2**

\[
\begin{align*}
2K + 4(1 - \alpha)c(t) - \frac{\alpha'}{\alpha} - \frac{2c^2(t)}{n\alpha^2} + c'(t) + \frac{\alpha'(t)}{\alpha(t)}c(t) & \leq 0, \\
\left( 2K + \frac{4(1 - \alpha)c(t)}{n\alpha^2} - \frac{\alpha'}{\alpha} \right)^2 - 8(1 - \alpha)^2 - \frac{8c^2(t)}{n\alpha^2} + c'(t) - \frac{\alpha'(t)}{\alpha(t)}c(t) & \leq 0,
\end{align*}
\]

We can rewrite them as (2.4).

2. **Case 2 in Proposition 2.2**

\[
\left( 2K + \frac{4(1 - \alpha)c(t)}{n\alpha^2} - \frac{\alpha'}{\alpha} \right)^2 - \frac{8(1 - \alpha)^2}{n\alpha^2} + c'(t) - \frac{\alpha'(t)}{\alpha(t)}c(t) \leq 0,
\]

which can be rewritten as (2.5).

By Theorem 2.1, the proof is completed. \( \square \)

**Remark 2.4.** The conclusion of Theorem 2.3 holds for all \( t \leq T \) (\( T > 0 \)) if the second assumption is replaced by (2.4) holds for \( t \in (0, t_0] \) and (2.5) holds for \( t \in (t_0, T] \) for some \( 0 < t_0 < T \). This is because condition B in Theorem 2.1 still holds in this case.

### 2.1 Case 1 in Theorem 2.3

**Corollary 2.5.** Let \( M \) be a compact manifold without boundary, \( \text{Ricci}(M) \geq -K(K \geq 0) \). Let \( u \) be any positive solution of the heat equation \( u_t = \Delta u \) on \( M \) and let \( \alpha > 1 \) be a positive constant. We have for all \( t > 0 \),

\[
|\nabla \log u|^2 - \alpha(t)(\log u)_t \leq \frac{n\alpha^2}{2} \max \left\{ \frac{1}{t}, \frac{K}{\alpha - 1} \right\}
\] \hfill (2.7)
Proof. We verify Case 1 in Theorem 2.3, i.e. (2.4). By \((\frac{1}{c})' \leq \frac{2}{n\alpha t}\) and \(c(0+) = \infty\), we can obtain \(\frac{1}{c(t)} \leq \frac{2}{n\alpha K}\). Meanwhile \(\frac{c(t)}{t} \leq 2(\alpha - 1)\), hence we can choose \(c(t) = \frac{n\alpha^2}{2} \max \left\{ \frac{1}{t}, \frac{K}{\alpha - 1} \right\}\) to verify (2.4). Although the function \(c(t) = \frac{n\alpha^2}{2} \max \left\{ \frac{1}{t}, \frac{K}{\alpha - 1} \right\}\) is not \(C^1\) in time \(t\), we can find some smooth enough functions \(c_\varepsilon(t)\) satisfying (2.4) and \(c_\varepsilon(t) \to c(t)\) as \(\varepsilon \to 0\), thus we complete the proof.

Remarks 2.6.

(1). Combining with Remark 2.4, we can strengthen the above result to

\[
|\nabla \log u|^2 - \alpha (\log u)_t \leq \left\{ \frac{n\alpha^2}{2t}, \frac{K\alpha^2}{4(\alpha - 1)} \right\} \left( 1 + e^{-2\left(\frac{K}{\alpha - 1}t - 1\right)} \right),
\]

\(t \leq \frac{\alpha - 1}{K}\).

(2.8)

(2). (2.7) and (2.8) improve the Li-Yau inequality (1.1) and Davies’s result (1.2) respectively. Also (2.8) improves Corollary 1.2, Corollary 1.4 in [16].

Proof of (2.8). By Corollary 2.5, we can choose \(c(t) = \frac{n\alpha^2}{2t}\), for \(0 < t \leq t_0 := \frac{\alpha - 1}{K}\.

For \(t > t_0\), we only need \(c(t)\) satisfies (2.5) with \(c(t_0) = \frac{nK\alpha^2}{2(\alpha - 1)}\) and \(\alpha > 1\) is a constant. Solving it with equality gives

\[
c(t) = \frac{K\alpha^2}{4(\alpha - 1)} \left( 1 + e^{-2\left(\frac{K}{\alpha - 1}t - 1\right)} \right),\]

\(t \geq t_0 = \frac{\alpha - 1}{K}\).

It’s easy to see the right hand side in (2.8) is \(C^1\) in time \(t\). Hence the desired result immediately follows by Remark 2.4.

Corollary 2.7. Let \(M\) be a compact manifold without boundary, \(\text{Ricci}(M) \geq -K(K \geq 0)\). Let \(u\) be any positive solution of the heat equation \(u_t = \Delta u\) on \(M\). We have for all \(t > 0\) and \(\theta \geq 0\)

\[
|\nabla \log u|^2 - e^{2\theta Kt}(\log u)_t \leq \frac{nKe^{4\theta Kt}}{e^{2\theta Kt} - 1} \max \left\{ \frac{1 - \theta}{2}, \theta \right\}.
\]

(2.9)

where \(\max \left\{ \frac{1 - \theta}{2}, \theta \right\} = \theta\) if \(\theta \geq \frac{1}{2}\); \(= \frac{1 - \theta}{2}\) if \(\theta \in [0, \frac{1}{2}]\).

Proof. Let’s take \(\alpha = e^{2\theta Kt}\). It follows from (2.4)

\[
\begin{cases}
  c(t) &\geq \frac{nKe^{4\theta Kt}}{e^{2\theta Kt} - 1} \cdot \frac{1 - \theta}{2} \\
  c(t) &\geq \frac{n\alpha(t)}{2\int_{t_0}^{t} \alpha(s) ds} = \frac{\theta nKe^{4\theta Kt}}{e^{2\theta Kt} - 1}.
\end{cases}
\]

Hence we can take

\[
c(t) = \frac{nKe^{4\theta Kt}}{e^{2\theta Kt} - 1} \max \left\{ \frac{1 - \theta}{2}, \theta \right\}.
\]

We can verify \(c(t)\) satisfies (2.4) and \(c(0+) = +\infty\). By Theorem 2.3 we complete the proof.
If we take $\theta = 1$ in Corollary 2.7, we re-find the refined Hamilton type gradient estimate (see Theorem 2.1 and Theorem 3.1 in [12])

$$|\nabla \log u|^2 - e^{2Kt}(\log u)_t \leq \frac{nKe^{4Kt}}{e^{2Kt} - 1}.$$ 

In [12], the local estimate is also obtained, hence (2.1) holds in the setting of complete Riemannian manifold. The corresponding Harnack inequality and heat kernel estimate are derived.

In the positive curvature case, we have

**Corollary 2.8.** Let $M$ be a compact manifold without boundary, $\text{Ricci}(M) \geq -K (K < 0)$. Let $u$ be any positive solution of the heat equation $u_t = \Delta u$ on $M$. We have for all $t > 0$ and $\theta \in (0, \frac{1}{3}]$

$$|\nabla \log u|^2 - e^{2\theta Kt}(\log u)_t \leq \frac{nK\theta e^{4\theta Kt}}{e^{2\theta Kt} - 1}.$$  \hspace{1cm} (2.10)

**Proof.** For $\theta \in (0, \frac{1}{3}]$, we take $\alpha = e^{2K\theta t}$. It follows from (2.4)

$$\begin{aligned}
c(t) & \leq \frac{nK\theta e^{4\theta Kt}}{e^{2\theta Kt} - 1} \cdot \frac{1-\theta}{2} \\
\alpha(t) & \geq \frac{\theta nKe^{4Kt}}{e^{2\theta Kt} - 1}.
\end{aligned}$$

Hence we can take

$$c(t) = \frac{nK\theta e^{4\theta Kt}}{e^{2\theta Kt} - 1}.$$ 

We can verify $c(t)$ satisfies (2.4) and $\lim_{t \downarrow 0} c(t) = +\infty$. By Theorem 2.3 we complete the proof. \hfill \Box

**Remark 2.9.** If we take $\theta = 1/3$ in Corollary 2.8, we re-find Theorem 2.1 in [6], they also get the estimate of the associated heat kernel and the lower bounds for the eigenvalues of Laplacian. We can derive the following new Harnack inequality from (2.10): for $0 \leq s < t$ and $x, y \in M$,

$$u(s, x) \leq \left( 1 - e^{2\theta Kt} \right)^{\frac{\theta}{2}} e^{-\frac{\theta K}{2} - \frac{\theta^2}{2} - \frac{2\theta K}{1 - e^{2\theta K}} u(t, y)}, \forall 0 < \theta \leq \frac{1}{3}.$$ 

### 2.2 Case 2 in Theorem 2.3

Take $\alpha$ to be certain expression, we find the following

**Theorem 2.10.** Let $M$ be a compact manifold without boundary, $\text{Ricci}(M) \geq -K (K \geq 0)$. Let $u$ be any positive solution of the heat equation $u_t = \Delta u$ on $M. For a given $C^1$ positive function $a(t) : (0, \infty) \rightarrow (0, \infty)$, we always suppose $a(t)$ satisfies the following assumptions:

(A1). For all $t > 0$, $a(t) > 0$ and $\lim_{t \rightarrow 0} a(t) = 0$, $\lim_{t \rightarrow 0} \frac{a(t)}{a'(t)} = 0$.

(A2). For any $L > 0$, $\frac{a^2}{a}$ is continuous and integrable on the interval $[0, L]$. 

Then we have for all $t > 0$,

$$|\nabla \log u|^2 - \alpha(t)(\log u)_t \leq c(t),$$

where

$$\alpha(t) = \frac{2K}{a(t)} \int_0^t a(s)ds + 1, \quad c(t) = \frac{nK^2}{2a(t)} \int_0^t a(s)ds + \frac{n}{8a(t)} \int_0^t a^2(s) ds. \quad (2.12)$$

**Proof.** We only need to verify Assumption (a) and **Case 2** of (b) in Theorem 2.3 i.e. (2.5). For such function $a$, we take $\alpha(t) = \frac{2K}{a(t)} \int_0^t a(s)ds + 1$, direct computation gives: $c(t)$ defined in (2.12) satisfies (2.5) and $c(0+) = +\infty$.

**Remarks 2.11.**

(1). If we take $a(t) = t^2$ and $a(t) = \sinh^2(Kt)$, we can get the estimates (1.5) and (1.6) respectively. Different choice of $a(t)$ gives various differential Harnack inequalities, e.g. see [11].

(2). Compared with Theorem 1.1 in [11], we drop the assumption of $a' > 0$. In fact, going through carefully the proof of Theorem 1.1 in [11], we can also drop the assumption of $a' > 0$.

(3). We mention here that: For Theorem 1.1 and Theorem 1.2 in [8], it does need the assumption of nonpositive Ricci curvature ($K \geq 0$) for the local differential Harnack inequality, since in their proof it is necessary to assume $\alpha > 1$, see Line-4 of Page 4468. For the compact manifolds with convex boundary, the global differential Harnack inequalities (2.11) works both for the negative curvature and positive curvature for time $t$ satisfying $\alpha(t) > 0$, since the proof of Theorem 1.1 in [11] works in this case. Indeed (2.8) in [11] gives $(\Delta - \partial_t)(aF) \geq -2\nabla(aF) \cdot \nabla f$ and we consider $aF$ instead of $F$ after (2.8).

Combining Corollary 2.5, Remarks 2.6, Corollary 2.7, Corollary 2.8 and Theorem 2.10, we see that Theorem 2.3 or more generally Theorem 2.1 provides a general form of differential Harnack inequality, which unifies and partial improves the classical Li-Yau inequality (1.1), Davies’s estimate (1.2), Hamilton differential Harnack inequality (1.3), Li-Xu’s result (1.5), (1.6), Baudoin-Garofalo [5], Qian [11] etc.

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