Note on islands in path-length sequences of binary trees

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Abstract

An earlier characterization of topologically ordered (lexicographic) path-length sequences of binary trees is reformulated in terms of an integrality condition on a scaled Kraft sum of certain subsequences (full segments, or islands). The scaled Kraft sum is seen to count the set of ancestors at a certain level of a set of topologically consecutive leaves is a binary tree.

Keywords: binary tree, topological order, lexicographic order, instantaneous codes, prefix-free codes, path-length sequence, Kraft sum, island, full segment

1 Introduction and connections with the theory of islands

In the construction of optimal average-length uniquely decipherable codes, Kraft’s and McMillan’s theorems guarantee that only instantaneous (prefix free) codes need to be considered, and these correspond to binary trees. More precisely, we consider finite rooted binary trees with a "topological" structure specifying for each non-leaf node its left son and right son. Alternatively, this topological structure can be thought of as a labelling of the edges of the
tree in such a way that the two edges from each non-leaf node to its sons are labelled 0 and 1 (for left and right). The 0–1 sequences corresponding to the root-to-leaf paths of the tree then determine an instantaneous code, and all instantaneous codes arise this way. In fact Huffman’s classical algorithm constructs the code by constructing the topological tree. Ordering the set of codewords lexicographically corresponds to a topological left-to-right enumeration of the root-to-leaf paths, and writing down the sequence of lengths produces the lexicographic (topological) length sequence, that in [FS] was shown to fully determine the code.

In [FS] lexicographic length sequences were characterized among all sequences of positive (meaning here non-negative) integers, including 0. This characterization is reformulated in the present note, and a certain integer parameter appearing in the characterization is given a combinatorial meaning. The characterization is established by induction on the number of nodes of an associated (in general not binary) tree.

Consider any finite sequence \( l_1, \ldots, l_n \) of \( n \geq 1 \) positive real numbers. For any non-empty sub-interval \([i, j]\) \( \subseteq [1, n]\) consider the neighborhood maximin parameter \( m[i, j] \) defined by

\[
m[i, j] = \max_{[i, j] \subset [h, k] \subseteq [1, n]} \min(l_h, \ldots, l_k)
\]

where the maximum is taken over all subintervals \([h, k]\) of \([1, n]\) properly containing \([i, j]\), and the maximin is 0 if \([i, j] = [1, n]\). Reformulating a definition in [FS], where the neighborhood maximin parameter was called the largest value near the interval, the interval \([i, j]\) is called a full segment (or island in the terminology of Czédli [C] and [FHRW], where further references are also given) if \( m[i, j] < \min(l_i, \ldots, l_j) \). As two full segments are either disjoint or comparable by inclusion (a fact apparently first noted and used by Gernot Härtel [G]), the set of all full segments is the node set of a rooted tree, called tree of full segments, where \([1, n]\) is the root and for any other full segment \( I \) the father of \( I \) is the smallest full segment containing \( I \) properly. (Similarly, under mild assumptions on the generalized island systems studied in [FHRW], islands are the nodes of a tree of islands.) For any \([i, j] \subseteq [1, n]\) we also write \( K[i, j] \) for the partial Kraft sum \( 2^{-i} + \ldots + 2^{-j} \). For \([i, j] = [1, n]\) this is just the Kraft sum of \( l \).

2. Characterisation of topological length sequences of binary trees

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Using the tree of islands the following variant formulation of the characterization of topologically ordered path-length sequences given in [FS] can be proved:

**Theorem** (variant of Theorem 4 of [FS]) A finite non-empty sequence \( l = (l_1, ..., l_n) \) of positive integers with Kraft sum 1 is the path-length sequence of a topological binary tree if and only if for every full segment \( S = [i, j] \subseteq [1, n] \), the number \( 2^{m(S)}K(S) \) is an integer, where \( m(S) \) and \( K(S) \) denote the neighborhood maximin parameter and the partial Kraft sum of \( S \). In that case, \( 2^{m(S)}K(S) \) is the number of ancestors at level \( m \) below the root of the leaves \( v_i, ..., v_j \) (in the topologically ordered enumeration \( v_1, ..., v_n \) of all leaves).

**Proof.** Suppose \( l \) is the path-length sequence of a binary tree with a left-right topology. For each level \( m \) ancestor of a leaf with index in \( S \), its descendents form a binary tree. If there are \( k \) of these trees, then each term in \( K(S) \) corresponds to exactly one term in the Kraft sum of the path-length sequence of exactly one of the \( k \) binary trees, and it is in value equal to this latter term divided by \( 2^m \).

Conversely, we can proceed by induction on the number of nodes of the tree of full segments of a given sequence. The basis of induction is trivial. Suppose that for the sequence of positive integers \( l \), for every full segment \( S = [i, j] \subseteq [1, n] \) the number \( 2^{m(S)}K(S) \) is an integer. Take a minimal full segment \( S = [i, j] \). Let \( l' \) denote the segment obtained by replacing in \( l \) the -necessarily constant - subsequence \( l_i, ..., l_j \) by a term equal to \( m[i, j] \) repeated \( 2^{m(S)}K(S) \) times. Then the following map \( f \) is a bijection from the set of full segments of \( l \) with \( S \) removed, to the set of all full segments of \( l' \):

\[
f : [h, k] \mapsto [s, t]
\]

where, denoting \( 2^{m(S)}K(S) \) by \( r \),

\[
[s, t] = [h, k] \text{ if } k < i \\
[s, t] = [h - (j - i + 1) + r, k - (j - i + 1) + r] \text{ if } j < h \\
[s, t] = [h, k - (j - i + 1) + r] \text{ otherwise.}
\]

The map preserves partial Kraft sums and neighborhood maximin parameters and it also establishes an isomorphism between the tree of full segments.
of \( l \) with the leaf node \( S \) clipped, and the tree of all full segments of \( l' \). Based on inductive hypothesis, construct the binary tree for \( l' \) then append, to each of the \( r \) leaves starting from the \( i' \)th leaf, uniform binary trees of depth \( l_i - m(S) \).

\[ \square \]

**References**

[C] G. Czédli, The number of rectangular islands by means of distributive lattices. *European J. Combinatorics* 30 (2009), 208-215

[FS] S. Foldes, N.M. Singhi, On instantaneous codes. *J. Combinatorics Inf. Syst. Sci.* 31 (2006) 307-317

[FHRW] S. Foldes, E.K. Horvath, S. Radeleczki, T Waldhauser, A general framework for island systems. To appear in *Acta Sci. Math. (Szeged).* Manuscript on ArXiv, 2013

[H] G. Härtel, personal communication, Tampere University of Technology (2007)