PROOF OF SOME LITTLEWOOD IDENTITIES CONJECTURED BY LEE, RAINS AND WARNAAAR

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Abstract. We prove a novel pair of Littlewood identities for Schur functions, recently conjectured by Lee, Rains and Warnaar in the Macdonald case, in which the sum is over partitions with empty 2-core. As a byproduct we obtain a new Littlewood identity in the spirit of Littlewood’s original formulae.

1. Introduction

The classical Littlewood identities are the following three summation formulae for Schur functions:

\begin{align}
\sum_{\lambda} s_\lambda(x) &= \prod_{i \geq 1} \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j}, \\
\sum_{\lambda \text{ even}} s_\lambda(x) &= \prod_{i \geq 1} \frac{1}{1 - x_i^2} \prod_{i < j} \frac{1}{1 - x_i x_j}, \\
\sum_{\lambda' \text{ even}} s_\lambda(x) &= \prod_{i < j} \frac{1}{1 - x_i x_j},
\end{align}

where \( x = (x_1, x_2, x_3, \ldots) \) is a countable alphabet. Here and throughout the rest of the paper “\( \lambda \) even” means the partition \( \lambda \) has only even parts and \( \lambda' \) denotes the conjugate of \( \lambda \). These identities were first written down together by Littlewood [16, p. 238], however (1.1a) was already known to Schur [27]. They have since afforded many far-reaching generalisations and have found applications in areas such as combinatorics, representation theory and elliptic hypergeometric series. In particular there are many generalisations of (1.1) at the Schur level [3, 7, 10, 11, 12, 13, 21, 22, 28]. Also see [25] for comprehensive references to the literature.

The purpose of this note is to prove the Schur function case of a pair of Littlewood identities for Macdonald polynomials recently conjectured by Lee, Rains and Warnaar [15, Conjecture 9.5]. To state these we need some notation. Denote the multiset of hook lengths of a partition \( \lambda \) by \( \mathcal{H}_\lambda \). We refine this by writing \( \mathcal{H}_\lambda^{e/o} \) for the submultiset of even/odd hook lengths. The standard infinite \( q \)-shifted factorial is given by \( (a; q)_\infty := \prod_{i \geq 0} (1 - aq^i) \)

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and we define a statistic

\[ \varsigma(\lambda) := \sum_{(i,j) \in \lambda} (-1)^{\lambda_i + \lambda_j - i - j + 1} (\lambda_i - i), \]

in terms of the Young diagram of \( \lambda \); see Subsection 2.1 below. Finally, let \( \hat{\Lambda}_F \) denote the completion of the ring of symmetric functions over the field \( F \) with respect to the natural grading by degree.

**Theorem 1.1.** As identities in \( \hat{\Lambda}_Q(q) \) at the alphabet \( x = (x_1, x_2, x_3, \ldots) \) we have that

\[ \sum_{\lambda \atop 2\text{-core}(\lambda) = 0} q^{\varsigma(\lambda)} \frac{\prod_{h \in \mathcal{H}_\lambda} (1 - q^h)}{\prod_{h \in \mathcal{H}_\lambda} (1 - q^{h_i})} s_\lambda(x) = \prod_{i \geq 1} \left( \frac{(qx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \right) \prod_{i < j} \frac{1}{1 - x_i x_j}, \]

and

\[ \sum_{\lambda \atop 2\text{-core}(\lambda) = 0} q^{\varsigma(\lambda)} \frac{\prod_{h \in \mathcal{H}_\lambda} (1 - q^h)}{\prod_{h \in \mathcal{H}_\lambda} (1 - q^{h_i})} \bar{s}_\lambda(x) = \prod_{i \geq 1} \left( \frac{(qx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \right) \prod_{i < j} \frac{1}{1 - x_i x_j}. \]

The condition \( 2\text{-core}(\lambda) = 0 \) generalises both the even row and even column conditions of (1.1b) and (1.1c). Indeed, by Lemma 2.2 we have that \( \varsigma(\lambda) = 0 \) if and only if \( \lambda \) is even. Thus when setting \( q = 0 \) (1.3) and (1.4) collapse to (1.1b) and (1.1c) respectively. In this sense these identities are in the spirit of Kawanaka’s identity [13, Theorem 1.1]

\[ \sum_\lambda \prod_{h \in \mathcal{H}_\lambda} \frac{1 + q^h}{1 - q^h} s_\lambda(x) = \prod_{i \geq 1} \left( \frac{-qx_i}{x_i} \right)_\infty \prod_{i < j} \frac{1}{1 - x_i x_j}, \]

since this reduces to (1.1a) when \( q = 0 \). Unlike Kawanaka’s identity one can make sense of the \( q \to 1 \) limit of (1.3) and (1.4). In either case we obtain the following Littlewood-type identity.

**Corollary 1.2.** As an identity in \( \hat{\Lambda}_Q \) at the alphabet \( x = (x_1, x_2, x_3, \ldots) \),

\[ \sum_{\lambda \atop 2\text{-core}(\lambda) = 0} \frac{\prod_{h \in \mathcal{H}_\lambda} h^{\lambda}}{\prod_{h \in \mathcal{H}_\lambda} h^{\lambda_i}} s_\lambda(x) = \prod_{i \geq 1} \frac{1}{(1 - x_i^2)^{1/2}} \prod_{i < j} \frac{1}{1 - x_i x_j}. \]

The outline of the paper is as follows. In the next section we give preliminaries regarding partitions, Schur functions and Koornwinder polynomials. In Section 3 we prove a pair of vanishing integrals for Schur functions again conjectured by Lee, Rains and Warnaar in the Macdonald case [15, Conjecture 9.2]. Then, in Section 4, we follow the techniques of [25] to prove the bounded analogues of Theorem 1.1 conjectured in [15, Conjecture 9.4]. The theorem then follows by taking an appropriate limit. We conclude with a derivation of Corollary 1.2.

2. Partitions and \((BC_n)\)-Symmetric Functions

2.1. Partitions. A partition \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) is a weakly decreasing sequence of nonnegative integers such that finitely many \( \lambda_i \) are nonzero. The sum of the entries is denoted \( |\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots \) and if \( |\lambda| = n \) we say \( \lambda \) is a partition of \( n \). Nonzero entries are called parts, and the number
of parts is called the length, denoted \( l(\lambda) \). We denote by \( \mathcal{P} \) the set of all partitions and by \( \mathcal{P}_n \) the set of all partitions with length at most \( n \). In particular \( \mathcal{P}_0 = \{\emptyset\} \) where \( \emptyset \) denotes the unique partition of zero. If \( \lambda \in \mathcal{P}_n \) we write \( \lambda + \delta \) for the partition \( (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n) \). The number \( m_i(\lambda) \) of occurrences of an integer \( i \) as a part of \( \lambda \) is called the multiplicity. Sometimes we express a partition in terms of its multiplicities as \( \lambda = (m_1 \cdot 1, m_2 \cdot 2, m_3 \cdot 3, \ldots) \).

In particular \( \mathcal{P}_0 = \{0\} \) where 0 denotes the unique partition of zero. If \( \lambda \in \mathcal{P}_n \) we write \( \lambda + \delta \) for the partition \( (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n) \).

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We denote by \( \mu \subset \lambda \) if the partition \( \mu \) is contained in \( \lambda \), i.e. if \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \). If \( \lambda \subseteq (m^n) \) for some nonnegative integers \( m, n \), then we write \( (m^n) - \lambda \) for the complement of \( \lambda \) inside \( (m^n) \), that is, \( (m^n) - \lambda := (m - \lambda_n, m - \lambda_{n-1}, \ldots, m - \lambda_1) \). A partition is identified with its Young diagram, which is the left-justified array of squares with \( \lambda_i \) squares in row \( i \) with \( i \) increasing downward. For example

is the Young diagram of \((6, 4, 3, 1)\). The conjugate of a partition, written \( \lambda' \), is obtained by reflecting the Young diagram in the main diagonal, so that \((6, 4, 3, 1)' = (4, 3, 3, 2, 1, 1)\). The arm and leg lengths of a square \( s = (i, j) \in \lambda \) are given by

\[
a(s) := \lambda_i - j \quad \text{and} \quad l(s) := \lambda'_j - i,
\]

which are the number of boxes strictly to the right and below \( s \) respectively. The hook length is the sum of these including \( s \) itself, so that \( h(s) := a(s) + l(s) + 1 \). Using the same example as above, in the Young diagram

we have labelled the square \( s = (2, 2) \) so that \( a(s) = 2, l(s) = 1 \) and \( h(s) = 4 \).

As in the introduction we denote the multiset of hook lengths of \( \lambda \) by \( H_\lambda \). This is further refined as \( H_\lambda^e \) and \( H_\lambda^o \), the multisets of hook lengths which are even or odd, respectively. In terms of these we define the hook polynomials

\[
H_\lambda(q) := \prod_{h \in H_\lambda} (1 - q^h)
\]

\[
H_\lambda^e/o(q) := \prod_{h \in H_\lambda^e/o} (1 - q^h),
\]

which are invariant under conjugation of \( \lambda \). For \( z \in \mathbb{C} \) we also need the content polynomials

\[
C_\lambda(z; q) := \prod_{(i,j) \in \lambda} (1 - z q^{j-i})
\]

\[
C_\lambda^e/o(z; q) := \prod_{(i,j) \in \lambda, \text{even/odd}} (1 - z q^{j-i}).
\]
In this paper we will frequently encounter partitions with empty 2-core, written 2-core(λ) = 0. One definition of such partitions is that their diagrams may be tiled by dominoes. Our running example of (6, 4, 3, 1) has empty 2-core since it admits the tiling

which is clearly not unique. We will now give some conditions which are equivalent to λ having empty 2-core which all easily follow by induction on |λ|. The reader interested in more general statements involving Littlewood’s decomposition of a partition into its r-core and r-quotient for all r ≥ 2 may consult, for example, [17] or [19, p. 12–15].

**Lemma 2.1.** For λ ∈ P_{2n} the following are equivalent:

1. 2-core(λ) = 0.
2. |H^o_λ| = |H^e_λ| = n.
3. The set \{λ_1 + 2n − 1, λ_2 + 2n − 2, . . . , λ_{2n−1} + 1, λ_{2n}\}
   contains n even and n odd integers.

2.2. **Auxiliary results.** Here we prove some properties of the statistic ς(λ) (1.2). Firstly, as we have already used in the introduction, we have the following characterisation of the vanishing of ς(λ).

**Lemma 2.2.** Let 2-core(λ) = 0. Then ς(λ) ≥ 0 with ς(λ) = 0 if and only if λ is even.

**Proof.** If λ is even then ς(λ) = 0 since the number of even and odd hook lengths in each row is equal. Assume that λ is not even. Then λ has an even number of odd parts. Let λ_{i_1}, λ_{i_2} be the final two odd rows of λ. Since 2-core(λ) is empty these must be separated by an even number of even rows (possibly zero). Ignoring the rows above, the contribution to ς(λ) below and including row λ_{i_1} may be computed as

\[ \lambda_{i_1} - \lambda_{i_2} + i_2 - i_1 + 2 \sum_{j=i_1+1}^{i_2} (-1)^{i_1+j-1} (\lambda_j - j). \]

Since the numbers λ_j − j are strictly decreasing this sum is positive. The next nonzero contribution to ς(λ) will come from the pair of odd rows above in the same fashion. Thus repeating the above shows that if λ has empty 2-core and contains at least two odd rows then ς(λ) > 0. □

Note that ς((2, 1, 1, 1)) = 0, so that ς(λ) may vanish for partitions with nonempty 2-core.

**Lemma 2.3.** For λ ∈ P_{2n} there holds

\[ ς(λ) = \sum_{(i,j) ∈ λ + δ} (-1)^{λ_i - i - j + 1} (λ_i - i) - \sum_{1 ≤ i < j ≤ 2n} (-1)^{λ_i - λ_j + j - i} (λ_i - i). \]
Moreover, if $2\text{-core}(\lambda) = 0$, then

\begin{equation}
\zeta(\lambda') = \frac{|\lambda|}{2} - n^2 - n + \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_j - j).
\end{equation}

Proof. We interpret the definition of $\zeta(\lambda)$ as a sum over the Young diagram of $\lambda$ where each square has weight $(-1)^{\lambda_i + \lambda_j - i - j + 1} (\lambda_i - i)$. In the Young diagram of $\lambda + \delta$ place the integer $(-1)^{\lambda_i - i - j + 1} (\lambda_i - i)$ in box $(i, j)$. Summing over $i, j$ gives the first sum on the right of (2.1). To identify the second sum, we remove the columns with index $\lambda_j + 2n - j + 1$ for $2 \leq j \leq 2n$ whose entries are $(-1)^{\lambda_i - \lambda_j + j - i} (\lambda_i - i)$. The remaining diagram is that of $\lambda$ with entries $(-1)^{\lambda_i + \lambda_j - i - j + 1} (\lambda_i - i)$, which shows the first identity.

The proof of the second identity is similar. Note that by (1.2), $\zeta(\lambda')$ may be written as

$$\zeta(\lambda') = \sum_{(i, j) \in \lambda} (-1)^{\lambda_i + \lambda_j - i - j + 1} (\lambda_j' - j).$$

We thus fill the diagram of $\lambda + \delta$ with integers $(-1)^{\lambda_i - i - j + 1} (2n - j)$, so that removing the same columns as before now gives

$$\zeta(\lambda') = \sum_{(i, j) \in \lambda + \delta} (-1)^{\lambda_i - i - j + 1} (2n - j) - \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (j - \lambda_j - 1).$$

A simple calculation shows that for $2\text{-core}(\lambda) = 0$,

$$\sum_{(i, j) \in \lambda + \delta} (-1)^{\lambda_i - i - j + 1} (2n - j) + \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j - i - j + 1} (2n - j) = \frac{|\lambda|}{2} - n^2 - n,$

completing the proof. \qed

2.3. Schur functions. For completeness we give a definition of the Schur functions in terms of the classical ratio of alternants. For $\lambda \in \mathcal{P}_n$ the Schur function is defined as

$$s_\lambda(x_1, \ldots, x_n) := \det_{1 \leq i, j \leq n} \left( x_i^{\lambda_j + n - j} - x_j^{\lambda_i + n - j} \right),$$

and $s_\lambda(x_1, \ldots, x_n) := 0$ for $l(\lambda) > n$. The set of the $s_\lambda(x_1, \ldots, x_n)$ indexed over $\mathcal{P}_n$ forms a $\mathbb{Z}$-basis for the ring of symmetric functions in $n$ variables, denoted $\Lambda_n$. We also use the Schur functions in countably many variables $x = (x_1, x_2, x_3, \ldots)$, such as in Theorem [19] which may be defined by the Jacobi–Trudi determinant [19, p. 41]. The set of such $s_\lambda(x)$ when indexed over all partitions $\lambda$ form a $\mathbb{Z}$-basis for the ring of symmetric functions $\Lambda$. We also require the ring $\Lambda$ which is the completion of $\Lambda$ with respect to the natural grading by degree [23, p. 66].

Several of the results we need below are best stated in terms of Macdonald polynomials, which are a $q, t$-analogue of the Schur functions [19, §VI]. We simply note that the Macdonald polynomials $P_\lambda(x; q, t)$ are a basis for $\Lambda_{\mathbb{Q}(q, t)}$ and reduce to the Schur functions when $q = t$, i.e., $P_\lambda(x; q, q) = s_\lambda(x)$. 

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2.4. Koornwinder polynomials and integrals. The Koornwinder polynomials are a family of BC\(_n\)-symmetric functions depending on six parameters first introduced by Koornwinder [14] as a multivariate analogue of the Askey–Wilson polynomials [1]. Here we write \( x = (x_1, \ldots, x_n) \), \( x^\pm = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) \) and for a single-variable function \( g(x_i) \) we set
\[
\begin{align*}
g(x_i^+) &:= g(x_i)g(x_i^{-1}) \\
g(x_i^-) &:= g(x_ix_j)g(x_i^{-1}x_j)g(x_ix_j^{-1})g(x_i^{-1}x_j^{-1}).
\end{align*}
\]
Below the function will be one of \( g(x_i) = (x_i; q)_\infty \) or \( g(x_i) = (1 - x_i) \). Also for the infinite \( q \)-shifted factorial we adopt the usual multiplicative notation
\[
(a_1, \ldots, a_n; q)_\infty := (a_1; q)_\infty \cdots (a_n; q)_\infty.
\]
Let \( W := \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \) be the group of signed permutations on \( n \) letters. A Laurent polynomial \( f(x) \in \mathbb{C}[x^\pm] \) is called BC\(_n\)-symmetric if it is invariant under the natural action of \( W \) on the \( n \) variables where the reflections act by \( x_i \mapsto 1/x_i \). For \( \lambda \in \mathcal{P}_n \) define the orbit-sum indexed by \( \lambda \) as
\[
m^\text{BC}_\lambda(x) := \sum_\alpha x^\alpha,
\]
where the sum is over all elements \( \alpha \) of the \( W \)-orbit of \( \lambda \), the reflections act on sequences by \( \alpha_i \mapsto -\alpha_i \), and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). The orbit-sums form a basis for the ring \( \Lambda^\text{BC}_n \) of BC\(_n\)-symmetric functions. For \( q, t, t_0, t_1, t_2, t_3 \in \mathbb{C} \) with \( |q|, |t|, |t_0|, |t_1|, |t_2|, |t_3| < 1 \), define the Koornwinder density by
\[
\Delta(x; q, t; t_0, t_1, t_2, t_3) := \prod_{i=1}^n \frac{(x_i^{\pm 2}; q)_\infty}{\prod_{r=0}^3 (t^r x_i^{\pm}; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(x_i^\pm x_j^\pm; q)_\infty}{(t x_i^\pm x_j^\pm; q)_\infty}.
\]
This further allows one to define an inner product on \( \Lambda^\text{BC}_n \) by
\[
\langle f, g \rangle_{q, t, t_0, t_1, t_2, t_3}^{(n)} := \int_{\mathbb{T}^n} f(x)g(x^{-1})\Delta(x; q, t; t_0, t_1, t_2, t_3) \; dT(x),
\]
where \( \mathbb{T}^n \) is the standard \( n \)-torus and the measure \( T(x) \) is given by
\[
dT(x) := \frac{1}{2^n n!(2\pi i)^n} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.
\]
The Koornwinder polynomials are defined to be the unique BC\(_n\)-symmetric functions satisfying
\[
K_\lambda = m^\text{BC}_\lambda + \sum_{\mu \leq \lambda} c_\lambda \mu m^\text{BC}_\mu,
\]
where \( c_\lambda \mu \in \mathbb{C}(q, t, t_0, t_1, t_2, t_3) \), and for which
\[
\langle K_\lambda, K_\mu \rangle_{q, t, t_0, t_1, t_2, t_3}^{(n)} = 0 \quad \text{if } \lambda \neq \mu.
\]
Note that \( \mu \leq \lambda \) denotes the extension of the usual dominance order to all partitions \( \lambda, \mu \in \mathcal{P}_n \). \( \mu \leq \lambda \) if and only if \( \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \) for all \( i \geq 1 \). The Koornwinder polynomials satisfy many nice properties such
as the quadratic norm evaluation and evaluation symmetry [4, 26]. The key identity we need is [25, Equation (2.6.9)] (see also [23, Corollary 7.2.1])

\[
\lim_{m \to \infty} (x_1 \ldots x_n)^m K_{(m^n)}(x; q, t; t_0, t_1, t_2, t_3) = P_\lambda(x; q, t) \prod_{i=1}^n \frac{(t_0 x_i, 1 x_i, t_2 x_i, t_3 x_i; q) \infty}{(x_i^2; q) \infty} \prod_{1 \leq i < j \leq n} \frac{(t x_i x_j; q) \infty}{(x_i x_j^2; q) \infty}.
\]

We will only use this for \( \lambda = 0 \), in which case \( P_0(x; q, t) = 1 \).

For a basis \( \{ f_\lambda \} \) of \( \Lambda_n^{BC} \) we write \( [f_\lambda]g \) for the coefficient of \( f_\lambda \) in the expansion \( g = \sum c_\lambda f_\lambda \) where the \( c_\lambda \) lie in some coefficient ring. The virtual Koornwinder integral of a \( BC_n \)-symmetric function \( f \) is defined as

\[
I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) := [K_0(x; q, t; t_0, t_1, t_2, t_3)] f.
\]

This is extended to allow for symmetric function arguments via the homomorphism \( \Lambda_{2n} \to \Lambda_n^{BC} \) for which \( f(x_1, \ldots, x_{2n}) \to f(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) \). Of course since \( K_0 = 1 \) the orthogonality of the Koornwinder polynomials allows us to express this as

\[
I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) = \frac{\langle f, 1 \rangle^{(n)}_{q, t, t_0, t_1, t_2, t_3}}{\langle 1, 1 \rangle^{(n)}_{q, t, t_0, t_1, t_2, t_3}}.
\]

Note that the denominator has the explicit evaluation

\[
\langle 1, 1 \rangle^{(n)}_{q, t, t_0, t_1, t_2, t_3} = \prod_{i=1}^n \frac{(t, t_0 t_1 t_2 t_3 t_i^{n+i-2}; q) \infty}{(q, t_i; q) \infty \prod_{0 \leq r < s \leq 3} (t_r t_s t_i^{3-r-s}; q) \infty},
\]

which is Gustafson’s generalised Askey–Wilson integral [9]. The virtual Koornwinder integral can be evaluated for many choices of the argument \( f \), see [15, 23, 24, 25]. In particular, the vanishing integrals of the next section may be expressed in terms of virtual Koornwinder integrals. We need one final identity involving virtual Koornwinder integrals. To state this conveniently, let

\[
f^{(n)}_\lambda(q, t; t_0, t_1, t_2, t_3) := [P_\lambda(x; q, t)](x_1 \cdots x_n)^m K_{(m^n)}(x; q, t; t_0, t_1, t_2, t_3).
\]

**Proposition 2.4** ([25, Proposition 4.9]). For nonnegative integers \( n, m \) and \( \lambda \subseteq (2m)^n \),

\[
f^{(n)}_\lambda(q, t; t_0, t_1, t_2, t_3) = (-1)^{\lambda|} I_K^{(m)}(P_\lambda(t, q); t, q; t_0, t_1, t_2, t_3).
\]

3. **Vanishing Integrals**

In this section we evaluate a pair of vanishing integrals for Schur functions conjectured by Lee, Rains and Warnaar in the Macdonald case [15, Conjecture 9.2].

For \( a, b, q \in \mathbb{C} \) with \(|a|, |b|, |q| < 1 \) we define

\[
I^{(n)}_\lambda(a, b; q) := \frac{1}{Z_n(a, b; q)} \int_{\mathbb{T}^n} s_\lambda(x_1^\pm, \ldots, x_n^\pm) \prod_{i=1}^n \frac{(x_i^{\pm 2}; q) \infty}{(ax_i^{\pm 2}, bx_i^{\pm 2}, q^2) \infty} \times \prod_{1 \leq i < j \leq n} (1 - x_i^\pm x_j^\pm) \, dT(x),
\]

where \( \{x_i^\pm \} \) parametrise the \( (2m)^n \) roots of unity.
where \( \lambda \) is a partition with length at most \( 2n \) and the normalising factor is given by

\[
Z_n(a, b; q) := \prod_{i=1}^{n} \frac{(x_i^{\pm 2}; q)_\infty}{(ax_i^{\pm 2}, bx_i^{\pm 2}; q^2)_\infty} \prod_{1 \leq i < j \leq n} (1 - x_i^\pm x_j^\pm) \, dT(x)
\]

\[
= \prod_{i=1}^{n} \frac{(abq^{n+i-2}; q)_\infty}{(q^i, -aq^{i-1}, -bq^{i-1}; q)_\infty(abq^{2i-2}; q^2)_\infty}.
\]

Note that in terms of virtual Koornwinder integrals this is

\[
I^{(n)}_\lambda(a, b; q) = I^{(n)}_\Lambda(s_\lambda; q, a^{1/2}, -a^{1/2}; b^{1/2}, -b^{1/2}).
\]

Lee, Rains andWarnaar prove the following properties of the above integral.

**Proposition 3.1** ([15] Proposition 9.3). For \( a, b, q \in \mathbb{C} \) with \( |a|, |b|, |q| < 1 \) and \( \lambda \) a partition of length at most \( 2n \) the integral \( I^{(n)}_\lambda(a, b; q) \) vanishes unless \( 2\text{-core}(\lambda) = 0 \). Furthermore

\[
(3.1a) \quad I^{(n)}_\lambda(q, q; q) = \prod_{i=1}^{n} \frac{(1 - q^{2i-1})^{2n-2i+1}}{(1 - q^{2i})^{2n-2i}}
\]

\[
\times \prod_{1 \leq i, j \leq 2n} \text{Pf} \left( q^{(\lambda_i - \lambda_j + j-i)/2} \right).
\]

and

\[
(3.1b) \quad I^{(n)}_\lambda(1, q^2; q) = \frac{1}{2^{n-1}(1 + q^n)} \prod_{i=1}^{n} \frac{(1 - q^{2i-1})^{2n-2i+1}}{(1 - q^{2i})^{2n-2i}}
\]

\[
\times \prod_{1 \leq i, j \leq 2n} \text{Pf} \left( \frac{1 + q^{\lambda_i - \lambda_j + j-i}}{1 - q^{\lambda_i - \lambda_j + j-i}} \right).
\]

Lee, Rains and Warnaar also give a conjectural Macdonald polynomials analogue of this proposition [15] Conjecture 9.2. There the generalisations of (3.1) are explicit products. Our next proposition gives the evaluation of the Pfaffians in the previous proposition, verifying the conjecture of Lee, Rains and Warnaar for \( q = t \).

**Proposition 3.2.** For \( \lambda \) with length at most \( 2n \) and \( 2\text{-core}(\lambda) = 0 \),

\[
(3.2) \quad I^{(n)}_\lambda(q, q; q) = q^{\nu(\lambda)} \frac{C^{(n)}_\lambda(q^2n; q)H^{(q)}_\lambda(q)}{C^{(n)}_\lambda(q^{2n}; q)H^{(q)}_\lambda(q)}
\]

and

\[
(3.3) \quad I^{(n)}_\lambda(1, q^2; q) = q^{\nu(\lambda)} \frac{1 + q^{2(\lambda_\lambda) - 2n}}{1 + q^n} \frac{C^{(n)}_\lambda(q^2n; q)H^{(q)}_\lambda(q)}{C^{(n)}_\lambda(q^{2n}; q)H^{(q)}_\lambda(q)}.
\]

Proof. Since the structure of the Pfaffians is similar, we focus on the second identity, and evaluate (3.1b).

Fix a partition \( \lambda \in \mathcal{P}_{2n} \) with empty 2-core. Define the set \( J \subseteq \{1, \ldots, 2n\} \) as the collection of integers \( j \) for which column \( j \) has a nonzero entry in the first row, and set \( I := \{1, \ldots, 2n\} \setminus J \). Since \( 2\text{-core}(\lambda) = 0 \) it follows that \( |I| = |J| = n \). The elements of \( I \) and \( J \) are labeled by \( i_k \) and \( j_k \) respectively,
where $1 \leq k \leq n$ and ordered naturally. With this established we define the $n \times n$ matrix $M$ with entries $M_{k, \ell}$ by

$$M_{k, \ell} := \frac{1 + q^{\lambda_k - \lambda_\ell + j - i}}{1 - q^{\lambda_k - \lambda_\ell + j - i}}.$$ 

The Pfaffian in (3.1b) may be expressed in terms of the determinant of $M$. Indeed, by pushing the rows with indices in $J$ to the right we see that

$$\text{Pf}_{1 \leq i, j \leq 2n} \left( \frac{1 + q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\lambda_i - \lambda_j + j - i}} \chi(\lambda_i - \lambda_j + j - i \text{ odd}) \right)$$

$$= (-1)^{\binom{n}{2}} + \sum_{j \in J} \text{Pf} \left( \begin{array}{cc} 0 & M^t \\ -M & 0 \end{array} \right)$$

$$= (-1)^{\sum_{j \in J} \det M}.$$

The determinant may be evaluated simply by applying the following generalisation of Cauchy’s double alternant which may be found in [5, Example 3.1; $a = 0$]:

$$\det_{1 \leq i, j \leq n} \left( bx_i + cy_j \right) = (b - c)^{n-1} \left( b \prod_{i=1}^{n} x_i + \left(-1\right)^{n-1} c \prod_{i=1}^{n} y_i \right)$$

$$\times \prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j).$$

We apply this with $(b, c, x_k, y_k) \mapsto (-1, 1, q^{\lambda_k - \lambda_\ell}, -q^{\lambda_k - \lambda_\ell})$ for $1 \leq k, \ell \leq n$. After some elementary manipulations the evaluation may now be expressed as

$$I^{(n)}_\lambda(1, q^2; q)$$

$$= \frac{\prod_{i \in I} q^{\lambda_i - i}}{1 + q^{H}} \prod_{j \in J} q^{\lambda_j - j} \prod_{i=1}^{n} \left(1 - q^{2i-1}\right)^{2n-2i+1}$$

$$\times \prod_{1 \leq i < j \leq 2n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{q^{\lambda_j - j}} \prod_{1 \leq i < j \leq 2n} \frac{q^{\lambda_j - j}}{1 - q^{\lambda_i - \lambda_j + j - i}}.$$ 

The terms of the form $1 - q^x$ can be simplified thanks to the identity [19, p. 10–11]

$$C^s_\lambda(2n; q) H^s_\lambda(q) = \prod_{s \in \lambda} \frac{1 - q^{n+c(s)}}{1 - q^{h(s)}} \prod_{i=1}^{n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\lambda_i - \lambda_j + j - i}},$$

where $l(\lambda) \leq n$. Restricting all products to even/odd exponents implies that

$$\frac{C^s_\lambda(2n; q) H^s_\lambda(q)}{C^s_\lambda(2n; q) H^s_\lambda(q)}$$

$$= \prod_{1 \leq i < j \leq 2n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\lambda_i - \lambda_j + j - i}} \prod_{1 \leq i < j \leq 2n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\lambda_i - \lambda_j + j - i}}$$

$$\times \prod_{i=1}^{n} \frac{1 - q^{2i-1}(2n-2i+1)}{1 - q^{2i}(2n-2i)}. $$
It remains to show that the powers of $q$ agree in the prefactor. Since
\[
\prod_{i \in I} q^{\lambda_i - i} + \prod_{j \in J} q^{\lambda_j - j} = \prod_{\lambda_i - i \text{ even}}^{2n} q^{\lambda_i - i} + \prod_{\lambda_i - i \text{ odd}}^{2n} q^{\lambda_i - i},
\]
this may be reduced to the pair of identities
\[
\varsigma(\lambda) = 2n \sum_{\lambda_i - i \text{ even}}^{\lambda_i - i = 1} (\lambda_i - i) + \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_j - j),
\]
and
\[
n + 2\varsigma(\lambda') - 2\varsigma(\lambda) = \sum_{\lambda_i - i \text{ odd}}^{\lambda_i - i = 1} (\lambda_i - i) - \sum_{\lambda_i - i \text{ even}}^{\lambda_i - i = 1} (\lambda_i - i).
\]
In the first of these write
\[
2n \sum_{\lambda_i - i \text{ even}}^{\lambda_i - i = 1} (\lambda_i - i) = \sum_{(i, j) \in \lambda + \delta} (-1)^{\lambda_i - j + i} (\lambda_i - i) + \sum_{\lambda_i - i = 1}^{2n} (\lambda_i - i)
\]

\[
= \varsigma(\lambda) + \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_j - j) + \sum_{\lambda_i - i = 1}^{2n} (\lambda_i - i),
\]
where in the second equality we have applied (2.1) from Lemma 2.3. Since
\[
\sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_j - j) + \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_i - i) + \sum_{\lambda_i - i = 1}^{2n} (\lambda_i - i)
\]
\[
= \sum_{i, j = 1}^{2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_i - i)
\]
\[
= 0,
\]
the first identity follows. For the second identity, a similar rewriting, now using (2.2) of Lemma 2.3 shows us that
\[
\sum_{\lambda_i - i \text{ odd}}^{\lambda_i - i = 1} (\lambda_i - i) - \sum_{\lambda_i - i \text{ even}}^{\lambda_i - i = 1} (\lambda_i - i)
\]
\[
= -2 \sum_{(i, j) \in \lambda + \delta} (-1)^{\lambda_i - j + i} (\lambda_i - i) - \sum_{\lambda_i - i = 1}^{2n} (\lambda_i - i)
\]
\[
= -2\varsigma(\lambda) - |\lambda| + 2n^2 + n - 2 \sum_{1 \leq i < j \leq 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_i - i)
\]
\[
= n + 2\varsigma(\lambda') - 2\varsigma(\lambda).
\]
This finishes the evaluation of (3.1b). The evaluation of (3.1a) is almost identical except one directly applies (2.2) of Lemma 2.3 to compute the exponent of $q$ in the prefactor. \qed
4. Bounded Littlewood identities

Here we use the integral evaluations of the previous section to prove a bounded analogue of Theorem 1.1. This is followed by proofs of the theorem and of Corollary 1.2.

4.1. A bounded analogue of Theorem 1.1

Bounded Littlewood identities are generalisations of ordinary Littlewood identities in which the largest part of the indexing partition has an upper bound, say \( m \), such that sending \( m \) to infinity recovers an ordinary (unbounded) Littlewood identity. The first example of such an identity was discovered by Macdonald [18, §1.5] where he used a bounded analogue of (1.1a) to prove the MacMahon and Bender–Knuth conjectures on plane partitions [2, 20]. Bounded analogues of the remaining two classical identities (1.1b) and (1.1c) were obtained by Désarménien, Proctor and Stembridge [7, 22, 28] and Okada [21] respectively. A host of other bounded identities for Hall–Littlewood and Macdonald polynomials may be found in [25] and references therein. For further discussion of the history of bounded Littlewood identities see [10]. We now state the bounded analogue of Theorem 1.1.

**Theorem 4.1.** For nonnegative integers \( m \) and \( n \),

\[
\sum_{\lambda \text{ 2-core} = 0} q^{\varepsilon(\lambda)} \frac{C^e_\lambda(q^{-2m}; q)H^e_\lambda(q)}{C^o_\lambda(q^{-2m}; q)H^o_\lambda(q)} s_\lambda(x) = (x_1 \cdots x_n)^m K_{(m^n)}(x; q, q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}),
\]

and

\[
\sum_{\lambda \text{ 2-core} = 0} q^{2\varepsilon(\lambda') - \varepsilon(\lambda)} + q^{m + \varepsilon(\lambda)} \frac{C^e_\lambda(q^{-2m}; q)H^o_\lambda(q)}{C^o_\lambda(q^{-2m}; q)H^o_\lambda(q)} s_\lambda(x)
\]

\[
= (x_1 \cdots x_n)^m K_{(m^n)}(x; q, q; 1, -1, q, -q).
\]

These identities are indeed bounded since \( C^e_\lambda(q^{-2m}; q) \) vanishes if \( \lambda_1 > 2m \). Since, by [15, Lemma 4.1], the Koornwinder polynomials on the right reduce to classical group characters for \( q = 0 \), one recovers the previously mentioned Désarménien–Proctor–Stembridge and Okada identities respectively in this case. The Koornwinder polynomials for \( q = t \) on the right-hand side may alternatively be expressed as a ratio of determinants of Askey–Wilson polynomials [1]; see, e.g., [6, Definition 4.1]. This, however, does not seem to shed light on a more explicit expression for the evaluation of these sums. In particular, the specialisations of \( K_{(m^n)} \) above are not contained in [15, Lemma 4.1].

The following argument is sketched in [15, §9], but we give the details in the Schur case. Assuming the Macdonald polynomial version of the vanishing integrals [15, Conjecture 9.2], the same argument gives the conjectural Littlewood identities.

**Proof of Theorem 4.1.** The goal is to find an expression for the coefficient of \( s_\lambda(x) \) in the Schur expansion of the right-hand side. By Proposition 2.4
In the case of (4.1) this yields
\[ I^{(m)}_\lambda(x; q, q, t_0, t_1, t_2, t_3) = (-1)^{|\lambda|} I^{(m)}_K(s_\lambda(x); q, q, t_0, t_1, t_2, t_3). \]
If we specialise \((t_0, t_1, t_2, t_3) = (q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2})\) then this reduces to
\[ I^{(m)}_\lambda(x; q, q; q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}) = (-1)^{|\lambda|} I^{(m)}_\lambda(q, q, q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}) = 0. \]
The integral on the right is \((3.2)\), as desired, and vanishes unless \(2\text{-core}(\lambda) = 0\). In this case the sign disappears since \(|\lambda|\) is even and we obtain
\[ (-1)^{|\lambda|} I^{(m)}_\lambda(q, q, q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}) = q^{\langle \lambda \rangle} \frac{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}. \]
By \([13]\) Lemma 2.3 we may alternatively express this as
\[ q^{\langle \lambda \rangle} \frac{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)} = q^{\langle \lambda \rangle} \frac{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}. \]
This establishes \((4.1)\). For \((4.2)\) the same procedure applies with the substitution \((t_0, t_1, t_2, t_3) = (1, -1, q, -q)\) and by using the integral \((3.3)\). \(\square\)

4.2. **Proof of Theorem 1.1** With the bounded identities established we may take the \(m \to \infty\) limit of both identities to obtain their unbounded counterparts. For the Koornwinder side we use \((2.3)\) with \((\lambda, q, t) = (0, q, q)\) and \((t_0, t_1, t_2, t_3) = (q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2})\) or \((t_0, t_1, t_2, t_3) = (1, -1, q, -q)\).

In the case of \((4.1)\) this yields
\[ \lim_{m \to \infty} (x_1 \cdots x_m)^m K_{(m)}(x; q, q; q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}) \]
\[ = \prod_{i=1}^n \left( \frac{q^{1/2} x_i, -q^{1/2} x_i, q^{1/2} x_i, -q^{1/2} x_i, q^{1/2} x_i, -q^{1/2} x_i, q_\infty}{(x_i^2; q_\infty)} \right) \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \]
\[ = \prod_{i=1}^n \left( \frac{q x_i^2, q^2}{x_i^2; q^2} \right) \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}, \]
where we have used
\[ (a, -a; q)_\infty = (a^2; q^2)_\infty. \]
For the limit of the summand we use it in conjugate form \((4.3)\) so that
\[ \lim_{m \to \infty} q^{\langle \lambda \rangle} \frac{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)}{C^e_\lambda(q^{2m}; q) H^0_\lambda(q)} = q^{\langle \lambda \rangle} \frac{H^0_\lambda(q)}{H^0_\lambda(q)}. \]
Thus we have proved \((4.3)\). As before the same procedure yields \((4.4)\).

4.3. **Proof of Corollary 1.2** In order to obtain Corollary 1.2 we take \(q \to 1\) in either \((4.3)\) or \((4.4)\). Let \((a; q)_n := \prod_{k=0}^{n-1} (1 - a q^k)\). Then we may take the limit of the product-side of \((4.3)\) by using
\[ \lim_{q \to 1} \frac{(q x_i^2, q^2)_\infty}{(x_i^2; q^2)_\infty} = \lim_{q \to 1} \frac{\sum_{n=0}^\infty (q; q^n) x_i^{2n}}{\sum_{n=0}^\infty (q; q^n) x_i^{2n}} = \frac{\sum_{n=0}^\infty 1 \cdot 3 \cdots (2n - 1) x_i^{2n}}{2 \cdot 4 \cdots 2n x_i^{2n}} \]
\[ = \frac{1}{(1 - x_i^2)^{1/2}}. \]
where in the first line we have applied the \( q \)-binomial theorem [8 Equation (1.3.2)]:
\[
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.
\]
The \( q \to 1 \) limit of the product-side of (1.4) gives the same result. The limit of either sum follows from the characterisation of partitions with empty 2-core in Lemma 2.1, namely that \( |H^e_\lambda| = |H^o_\lambda| \).

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References

[1] R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalise Jacobi polynomials, Mem. Amer. Math. Soc. 54 (1985), iv+55.
[2] E. A. Bender and D. E. Knuth, Enumeration of plane partitions, J. Combin. Theory Ser. A 13 (1972), 40–54.
[3] D. M. Bressoud, Elementary proofs of identities for Schur functions and plane partitions, Ramanujan J. 4 (2000), 69–80.
[4] J. F. van Diejen, Self-dual Koornwinder–Macdonald polynomials, Invent. math. 126 (1996), 319–339.
[5] W. Chu, Generalizations of the Cauchy determinant, Publ. Math. Debrecen 58 (2001), 353–365.
[6] S. Corteel, O. Mandalstham and L. Williams, Combinatorics of the two-species ASEP and Koornwinder moments, Adv. Math. 321 (2017), 160–204.
[7] J. Désarménien, La démonstration des identités de Gordon et MacMahon et de deux identités nouvelles, Sém Lothar. Combin. 15 (1986), Art. B15a, 11pp.
[8] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd ed., Encyclopedia of Mathematics and its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.
[9] R. A. Gustafson, A generalization of Selberg’s beta integral, Bull. Amer. Math. Soc. (N.S.) 22 (1990), 97–105.
[10] J. Huh, J. S. Kim, C. Krattenthaler and S. Okada, Bounded Littlewood identities for cylindric Schur functions, arXiv:2301.13117.
[11] M. Ishikawa and M. Wakayama, Applications of minor-summation formula II. Pfaffians and Schur polynomials, J. Combin. Theory Ser. A 88 (1999), 136–157.
[12] F. Jouhet and J. Zeng, Some new identities for Schur functions, Adv. Appl. Math. 27 (2001), 493–509.
[13] N. Kawakita, A q-series identity involving Schur functions and related topics, Osaka J. Math. 36 (1999), 157–176.
[14] T. H. Koornwinder, Askey–Wilson polynomials for root systems of type BC, in Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Tampa, FL, 1991), Contemp. Math. 138, Amer Math. Soc., Providence, RI, 1992, 189–204.
[15] C.-h. Lee, E. M. Rains and S. O. Warnaar, An elliptic hypergeometric function approach to branching rules, SIGMA 16 (2020), paper 142, 52 pp.
[16] D. E. Littlewood, The Theory of Group Characters and Matrix Representations of Groups, Oxford University Press, New York, 1940.
[17] D. E. Littlewood, Modular representations of symmetric groups, Proc. Roy. Soc. London. Ser. A 209 (1951), 333–353.
[18] I. G. Macdonald, Symmetric functions and Hall polynomials, The Clarendon Press, Oxford University Press, New York, 1979.
[19] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., The Clarendon Press, Oxford University Press, New York, 1995.
[20] P. A. MacMahon, Partitions of numbers whose graphs possess symmetry, Trans. Cambridge Phil. Soc. 17 (1899), 149–170.
[21] S. Okada, Applications of minor summation formulas to rectangular-shaped representations of classical groups, J. Algebra 205 (1998), 337–367.
[22] R. A. Proctor, New symmetric plane partition identities from invariant theory work of De Concini and Procesi, European J. Combin. 11 (1990), 289–300.
[23] E. M. Rains, $BC_n$-symmetric polynomials, Transform. Groups 10 (2005), 63–132.
[24] E. M. Rains and M. J. Vazirani, Quadratic transformations of Macdonald and Koornwinder polynomials Transform. Groups 12 (2007), 725–759.
[25] E. M. Rains and S. O. Warnaar, Bounded Littlewood identities, Mem. Amer. Math. Soc. 270 (2021), vii+115 pp.
[26] S. Sahi, Nonsymmetric Koornwinder polynomials and duality, Ann. Math. 150 (1999), 267–282.
[27] I. Schur, Aufgabe 569, Arch. Math. Phys. 27 (3) (1918), 163; Ges. Abhandlungen, Vol. 3, p. 456.
[28] J. R. Stembridge, Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities, Trans. Amer. Math. Soc. 319 (1990), 469–498.

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