Transitions in non-conserving models of Self-Organized Criticality

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Abstract

We investigate a random-neighbours version of the two dimensional non-conserving earthquake model of Olami, Feder and Christensen [Phys. Rev. Lett. 68, 1244 (1992)]. We show both analytically and numerically that criticality can be expected even in the presence of dissipation. As the critical level of conservation, $\alpha_c$, is approached, the cut-off of the avalanche size distribution scales as $\xi \sim (\alpha_c - \alpha)^{-3/2}$. The transition from non-SOC to SOC behaviour is controlled by the average branching ratio $\sigma$ of an avalanche, which can thus be regarded as an order parameter of the system. The relevance of the results are discussed in connection to the nearest-neighbours OFC model (in particular we analyse the relevance of synchronization in the latter).

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In what is now a commonly referenced paper [1], Bak, Tang and Wiesenfeld proposed that extensive systems of many coupled elements naturally evolve toward a dynamical critical state. They named this phenomenon Self–Organized Criticality and tested the idea on a cellular automata, the so called sandpile model. It was shown in that paper, and many others which followed [2], that, independently of the starting, configuration the sandpile model evolves to a stationary state which does not possess any characteristic spatial scale. The amplitude of the response of the system to an external perturbation follows a power law distribution.

From a theoretical point of view the main effort has been directed towards understanding which are the fundamental mechanisms that lead a system to be self–organized critical. It is well known, for instance, that for a particular class of sandpile models a necessary condition is a conserving local dynamics [3]: any amount of dissipation introduces a characteristic length–scale and thus destroys criticality. Nonetheless this does not seem to be a universal requirement, since in some other classes of models, conservation does not appear to be needed. In this respect the earthquake model introduced by Olami, Feder and Christensen [5] is of particular interest. In this model it is possible to directly control the level of conservation of the dynamics, through a parameter $\alpha$. When $\alpha = 1/4$ the system is conserving and it probably belongs to the same universality class as the sandpile model [1]. But, contrary to the latter, there are converging evidences that the OFC model remains critical even when dissipation is introduced ($\alpha < 1/4$) [5–10]. By reducing the value of $\alpha$ one should be able in principle to observe a transition from S.O.C. behaviour to non–S.O.C, although a general agreement on the critical value, $\alpha_c$, does not yet exist. The spectrum of values ranges from the recently proposed $\alpha_c = 0$ [10] to $\alpha_c \simeq 0.18$ [8,9], passing through the originally estimated $\alpha_c \simeq 0.05$ [5,6,12]. It has been suggested that the observed criticality in this model is a consequence of the “imperfect” synchronization in the system [10,11]. The strong correlations so induced would allow an avalanche (i.e. the response to an external perturbation) to be infinitely big, although it keeps on losing “energy”.

In this letter we investigate the random neighbours version of the OFC model [5]. We
find that a sharp transition from non-S.O.C. to S.O.C. behaviour occurs at $\alpha_c \approx 2/9$ in this model. Even more interesting, this transition is correctly described by the branching ratio, which can thus be regarded as an order parameter of the system, in analogy to equilibrium phase transitions. The results obtained for the random neighbours model are interesting in their own right, but their relevance is fully recognised when they are related to the nearest neighbours version. In fact they provide not only a mean–field description of the OFC model, but also properly address the role played by synchronization. For a system where the sole mechanism leading to SOC is synchronization, one would expect the random neighbours version to be non critical, since the latter cannot synchronize.

The OFC model is a a coupled-map lattice model which, despite its simplicity, is thought to capture some of the essential features of earthquake dynamics \[13\]. To each site of a 2–dimensional square lattice is associated a real continuous “energy” $E_i$. The system is driven continuously and uniformly, which means that all the $E_i$ values are simultaneously increased with time at the same rate. Avalanche (or earthquake) dynamics is simulated by assuming that a single site is unable to store more than a finite amount of energy $E_c$. As soon as a site becomes unstable (i.e. $E_i \geq E_c$) an avalanche is triggered: the global driving is stopped and the system evolves according to the following local relaxation rule:

$$
\text{if } E_i \geq E_c \Rightarrow \begin{cases} 
E_i &\rightarrow 0 \\
E_{nn} &\rightarrow E_{nn} + \alpha E_i
\end{cases}
$$

(1)

until all of sites are below $E_c$. In eq.(1) “$nn$” stands for the collection of nearest neighbours to site $i$. The parameter $\alpha \in [0, \frac{1}{4}]$ controls the conservation level of the dynamics ($\alpha = \frac{1}{4}$ corresponds to the conservative case). The random neighbours version we consider differs from the OFC model only in the choice of neighbours: an unstable site distributes an energy $\alpha E_i$ to 4 randomly chosen sites. It is well known that boundary conditions play a crucial role in SOC \[14\]. In accordance with previous studies we use open boundary conditions so that if one of the random neighbours is a boundary site the energy $\alpha E_i$ is simply lost.

The presence of a transition from localised to non–localised behaviour of the avalanches can be understood through the following argument. For notational clarity in the following
we will distinguish between a stable and an unstable site through the superscripts $-$ and $+$. Let $P_+(E^+)$ be the probability that a random site will become active as a consequence of receiving a contribution of magnitude $\alpha E^+$. $P_+(E^+)$ is equal to the probability of a site having a value between $E_c - \alpha E^+$ and $E_c$. The average number of active sites produced by an unstable site with energy $E^+$ is then $4P_+(E^+)$. The branching ratio $\sigma$ is defined as the average number of new active sites created by an unstable site. Averaging over the whole spectrum of possible $P_+(E^+)$ (that is an average over $E^+$) we get for $\sigma$

$$\sigma = 4 \int_{E_c}^{\infty} P_+(E^+)P(E^+)dE^+ \equiv 4P_+$$

(2)

where $P(E)$ is the distribution of the dynamical variable in the system. Obviously the chance to have an infinite avalanche is directly related to the condition $\sigma \geq 1$.

An exact analytic calculation of $P_+$ is, unfortunately, too complicated, since it involves a detailed knowledge of $P(E)$ (see fig.1). In order to continue we have to assume a specific functional form for $P(E)$. For simplicity we approximate the distribution of subcritical $E$-values by a uniform distribution on the interval $[0, E_c]$. We obtain $P_+(E^+) = \alpha E^+/E_c$ and therefore $P_+ = \alpha \langle E^+ \rangle/E_c$ where $\langle E^+ \rangle$ is the average value of a collapsing site. Consequently it follows that

$$\sigma = 4\alpha \frac{\langle E^+ \rangle}{E_c}$$

(3)

It is immediately clear from eq. (3) that $\sigma$ will be greater than 1 even for some values of $\alpha < 1/4$ since $\langle E^+ \rangle > E_c$. The condition $\sigma > 1$ can also be read as a conservation law.

It states that the average contribution $4\alpha \langle E^+ \rangle$ to the avalanche from the collapse of an unstable site must be greater or equal to $E_c$ otherwise the avalanche will die exponentially.

We now estimate $\langle E^+ \rangle$. Consider an active site $i$ and a site $j$ which will be active as a consequence of the action of $i$. We have $E^+_j = E^-_j + \alpha E^+_i$, where the superscripts $+$ and $-$ in the site $j$ distinguish between two successive time steps. Taking the average on both side of this equation we get $\langle E^+_j \rangle = \langle E^-_j \rangle + \alpha \langle E^+_i \rangle$. By assumption $E_c - \alpha E^+_i < E^-_j < E_c$. Thus, if we again approximate the distribution of $E$-values of the subcritical sites by a uniform
distribution on $[0, E_c]$ we will have $\langle E_j^- \rangle = E_c - \frac{1}{2} \alpha \langle E_i^+ \rangle$. Moreover, since $E_j^+$ is a new active site $\langle E_j^+ \rangle = \langle E_i^+ \rangle$. Combining these expressions we obtain

$$\langle E^+ \rangle = \frac{E_c}{1 - \frac{\alpha}{2}} \tag{4}$$

The branching ratio is accordingly given by

$$\sigma = \frac{4 \alpha}{1 - \frac{\alpha}{2}} \tag{5}$$

The condition for infinite avalanches $\sigma \geq 1$ is then

$$\alpha \geq \alpha_c = \frac{2}{9} \simeq 0.222... \tag{6}$$

The above calculation, although very crude in the assumption of a uniform distribution of subcritical $E$-values, predicts two different phases for the random neighbours model: a “low-$\alpha$” phase, where avalanches are essentially smaller than a characteristic size, and a “high-$\alpha$” phase, where on the contrary it is possible to have avalanches of any sizes. Computer simulations indeed confirm this point. Figure 2 shows the avalanche size distribution for increasing values of $\alpha$. The size is measured by counting the total number of toppling in one avalanche. For $\alpha \leq 0.22$ the distribution are very well described by the function

$$P_\alpha(s) \propto s^{-3/2} \exp\left(\frac{s}{\xi}\right) \tag{7}$$

where the cut-off $\xi$, for sufficiently large system, is independent of $L$ ($L$ is linear dimension of the system). Fig. 3 shows the fit of the simulated $P_\alpha(s)$ distributions to the exponential form in Eq. 7. For $\alpha \geq 0.23$ the cut-off in $P_\alpha(s)$ scales with $L$. This is the signature of criticality. In fig.4 we show $\xi$ as a function of $\alpha$. The data fits well the expression $\xi \sim (\alpha_c - \alpha)^{-1.5}$, with $\alpha_c = 0.2255$. Note that $\xi \sim (1/4 - \alpha)^{-2}$ was found in the study of the random neighbour sand pile model [15], i.e. the coherence length is only infinite in this model when conservation is established at $\alpha = 1/4$. Even more exciting is the behaviour of the measured branching ratio. In fig.5 we report the measured branching ratio for different $\alpha$ as a function of $1/L$. One sees that for $\alpha \leq 0.22$ the graphs extrapolate to a value $\lim_{L \to \infty} \sigma$ well below 1. As soon as $\alpha \geq 0.23 \lim_{L \to \infty} \sigma \simeq 1$ in accordance with Eq. 6.
In conclusion, by identifying a transition at a finite conservation level we have shown that a non-conserving system can indeed be critical. The agreement between the analytical calculation and computer simulation is remarkable and somewhat surprising. The assumption of a uniform distribution of subcritical $E$-values is not a very accurate approximation, see Fig.1. In this connection it is interesting to note the observation made by Pietronero, Tartaglia, and Zhang [16]. These authors found that the form of energy distribution depends very much on the energy partition rule (i.e. on how the energy of an unstable site is distributed to its neighbours), whereas the average energy in the system is quite universal. This is a very interesting point, which would deserve further investigations and which might also explain why the uniform approximation for $P(E)$ gives such an accurate estimate of $\alpha_c$. It might be possible in fact that for a particular energy partition rule the approximation becomes almost correct (or even exact). In [16] was shown, for instance, that a random partition makes the peaks of Fig. 1 disappear. We believe this property of universality might be related to the very nature of S.O.C.: measurable quantities should be independent of the details of the model. We have shown that synchronization is not the only mechanism present. Rather, the criticality appears to be related to the dynamics. This of course, does not prevent synchronization to be relevant for lower values of $\alpha$ as has been suggested for the nearest neighbour version of the OFC model [10][11]. Finally we have shown that the branching ratio plays the role of an order parameter in the considered model. We believe that this last point might also be useful for the analysis of other models. The natural step is to use it for the nearest neighbour version the OFC model. Preliminary results are encouraging even though simulations show that the scaling properties of the branching ratio are more complicated than is the case in the random neighbour model.

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REFERENCES

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987); Phys. Rev. A 38, 364 (1988)

[2] see, e.g., L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhu, Phys. Rev. A 39, 6524 (1989); P. Grassberger and S. S. Manna, J. Phys. France 51, 1077 (1990); K. Christensen, H. C. Fogedby, and H. J. Jensen, J. Stat. Phys. 63, 653 (1991).

[3] S.S. Manna, L. B. Kiss, and J. Kertész, J. Stat. Phys. 61, 923 (1990)

[4] D. Dhar, Phys. Rev. Lett. 64, 1613 (1990)

[5] Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. 68, 1244 (1992). K. Christensen and Z. Olami, Phys. Rev. A 46, 1829 (1992); J. Geophys. Res. 97, 8729 (1992). K. Christensen, Z. Olami, and P. Bak, Phys. Rev. Lett. 68, 2417 (1992).

[6] I.M. Jánosi and J. Kertész, Physica A 200, 174 (1993).

[7] J.E.S. Socolar, G. Grinstein, and C. Jayaprakash, Phys. Rev E, 47, 2366 (1993).

[8] P. Grassberger, Phys. Rev. E 49, 2436 (1994).

[9] A. Corral, C. J. Perez, A. Diaz-Guilera, and A. Arenas, Phys. Rev. Lett. 74, 118 (1995)

[10] A.A. Middleton and C. Tang, Phys. Rev. Lett. 74, 742 (1995).

[11] K. Christensen, Ph.D. thesis, University of Aahus, 1992 (unpublished).

[12] To be complete, one should also add to this list $\alpha_c = 0.25$, which is the value suggested by those who just don’t believe a non-conserving system can be critical.

[13] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. 57, 341 (1967).

[14] S. Zapperi, K. B. Lauritsen, and H. E. Stanley, Preprint

[15] A discussion of the random neighbour sandpile model is given by K. Christensen and Z. Olami, Phys. Rev. E 48, 3361 (1993).
Captions

Figure 1.
Energy distribution per site, $P(E)$, for $\alpha = 0.23$ and $L = 400$ (continuous line) ($E_c = 1$). The step function (dashed line) is the approximation used in the calculations. At $E = 0$, $P(E)$ extends up to $\approx 33$.

Figure 2.
Avalanche size distribution for $L = 100, 200$ and $400$ and for (a) $\alpha = 0.20$, (b) $\alpha = 0.21$, (c) $\alpha = 0.22$ and (d) $\alpha = 0.23$.

Figure 3.
Avalanche size distributions for $\alpha < \alpha_c$. The distributions do not scale with system size. The continuous lines are fits of the form $P_\alpha(s) \propto s^{-3/2} \exp(\xi)$

Figure 4.
Cutoff in avalanche size distribution, $\xi$, as a function of $\alpha_c - \alpha$ ($\alpha_c = 0.2255$). The solid line is the interpolation $\xi \sim (0.2255 - \alpha)^{-1.5}$

Figure 5.
Measured branching ratio as a function of $1/L$ ($L = 100, 200, 400$), for different values of $\alpha$. From bottom to top, $\alpha = 0.2, 0.21, 0.22, 0.225, 0.23, 0.24, 0.25$. One can clearly see the abrupt change in behaviour for $\alpha \approx 0.225$. 

[16] L. Pietronero, P. Tartaglia, and Y. C. Zhang, Physica A 173 22 (1991)