A Cubic Whitney and Further Developments in Geometric Discretisation

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Abstract
Geometric discretisation draws analogies between discrete objects and operations on a complex with continuum ones on a manifold. We generalise the theory to the cubic case and incorporate metric, by adding volume factors to our discrete Hodge star and then by modifying our inner product which leads to the same result.

1 Introduction
Geometric discretisation (GD) [1, 2] appears quite complicated, at first, using unfamiliar maps and objects but nothing could be further from the truth. Simply put we can translate from continuum objects to discrete ones, and back again in such a way that not only can we make the discrete ones as close as we want to the continuum ones, but we have discrete operators which satisfy the same identities as their continuum counterparts. We have $d^2 = 0$, Stokes’ theorem, the Leibniz rule and more; meaning that the discrete theory mimics the continuum one to a remarkable extent. Here we are interested in the theory itself, as opposed to applications of which there are many. In particular, we introduce a cubic Whitney map [3] and metric, needed for the discrete Hodge star which is after all not purely a topological object.

The basic structure of the theory, using the Whitney and de Rham map, to translate from the discrete the the continuum and back again, is the same as considered by Dodziuk [4]. GD uses a subdivided space though, in order to have a discrete Hodge [1]; Hiptmair [5] considers the discrete Hodge star using finite elements which map also to dual spaces and satisfy $\delta = \ast d\ast$, where he refers to cubic work done by Nedelec [6] which I have only recently come across. My focus has been on application to lattice field theory [7], though the relationship with finite elements has also been of interest.

GD deals with operators, as well as functions, in such a way that the identities and rules which they obey hold. Notably we have

- Stokes’ Theorem (Gauss’ Law in electromagnetism)

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• The Leibniz rule (The product law for differential forms)
• A discrete Hodge star \( \star \) where \( \star^2 = I \) and \( \delta = \star d \star \).
• \( d^2 = 0 \)
• A skew symmetric wedge

and DO NOT have associativity of the wedge. The presence of a Hodge star with the associated properties is the most significant feature of the system.

There are two aspects to our work here. First, we introduce a cubic Whitney map and then tackle convergence. We find that by the addition of volume factors to the Hodge star we can demonstrate convergence. This is not entirely satisfactory since the inner product is intimately related to \( \star \); we cannot alter one without the other. That being the case, we look at how a natural inner product leads to the introduction of exactly the volume factors which we had anticipated above.

We have also been considering the implications of this theory to lattice field theory, where the lack of a discrete Hodge star has been a problem [8] but have been aware that related work has been ongoing from engineering [5, 6, 9]. The possibility of fruitful cross-fertilisation is very much on the cards and something which we have always been interested in developing further.

In short we begin with a review of GD before going on to our new work where:

1. We develop a cubic theory.
2. We demonstrate converge using a heuristic involving “natural volume factors”.
3. We show that convergence, using a new inner product which retains the relationship of the Hodge star to the inner product, leads to precisely the same factors as we used for the heuristic.

We finish with a brief discussion of our current and future work.

2 Geometric discretisation

Geometric discretisation (GD) [1] is a discretisation scheme based on a correspondence between discrete objects and operations on a complex, \( K \), [10, 11] with continuum ones on a manifold, \( M \), [12] which captures topology [2].

The de Rham, \( A^K \), and Whitney map, \( W^K \), play a central role since they allow us to move from continuum to discrete and back again [3] whilst maintaining topology\(^1\). For the moment, we note that this provides a very clean

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\(^1\)Both these operators commute with \( d \), so if \( f \) is exact or closed in the continuum, it is also in the discrete.
structure and allows us, for example, to induce a natural discrete wedge from the continuum one
\[ x \wedge^K y = A^K(W^K(x) \wedge^K W^K(y)), \]
which inherits skewsymmetry and the Leibniz rule; though not associativity.

The other key idea is that of a subdivided space \( B \), containing both the original, \( K \), and dual, \( L \), spaces, which allows us to introduce a discrete Hodge star operator \( \star \). This has the property that \( \star \star = 1 \) and \( \delta = \star d \star \), with appropriate signs, which is of interest \[8\]. For example, we are able to capture chirality in the Dirac-Kähler formalism as a result \[14\].

In summary we have the following in GD, with the noticeable exception of associativity for the wedge:

- An exterior derivative which maps from \( p \)-cochains to \((p + 1)\)-cochains with \( d^2 = 0 \).
- A wedge product, with which we can take the product of a \( p \)-cochain and a \( q \)-cochain to get a \((p + q)\)-cochain. This has the properties that
  - Skewsymmetry: \( x^p \wedge y^q = (-1)^{pq} y^q \wedge x^p \).
  - Leibniz Rule: \( d(x^p \wedge y^q) = dx^p \wedge y^q + (-1)^p x^p \wedge (dy^q) \).
- The Hodge Star: This duality map associates an \((n - p)\)-cochain in \( L \) to each \( p \)-cochain in \( K \), capturing \( \star \star = 1 \) and \( \delta = \star d \star \) with appropriate signs.

For associativity we get
\[ (x^p \wedge^K y^q) \wedge^K z^r = \left( \frac{p + q + 1}{r + p + 1} \right) x^p \wedge^K (y^q \wedge^K z^r). \tag{1} \]

Given a triangulation\(^2\) of our space we can translate from differential geometry to our discrete structure. We start by looking at how forms are projected onto the triangulation and back before moving on to various operations.

### 2.1 de Rham map

\( p \)-cochains, \( C^p(K) \), on the triangulation are the discrete analogies of \( p \)-forms, \( \Omega^p(M) \), on the manifold. The \( p \)-forms can be projected onto the \( p \)-cochains using the de Rham map, \( A^K \), which involves integrating them over the associated \( p \)-chains. In other words we have \( A^K : \Omega^p(M) \to C^p(K) \) defined as
\[ < A^K \omega^p, \sigma^p > = \int_{\sigma^p} \omega^p, \]
which has the property that \( dA^K = A^K d^K \) (Stokes’ Theorem).

\(^2\) We do not worry about how to construct a triangulation of the space we are working with though we do know that they exist for the cases we are interested in since Radó\[13\] proved this for compact spaces. In fact we know any differentiable manifold can be triangulated\[14\].

3
Figure 1: $d$, $A^K$ and $W^K$ provide a commutative diagram if we restrict ourselves to the image of the Whitney map (a.k.a. Whitney elements), a finite dimensional space of functions.

Figure 2: The standard triangle.

If we take the standard triangle $[0, 1, 2]$, shown in Fig.2, we can see that what happens explicitly. Since this is in 2D we have $1$, $dx$, $dy$ and $dxdy$ as our possible differential forms, all with possible function coefficients $f$. The possible cochains are $[0]$, $[1]$, $[2]$, $[0, 1]$, $[0, 2]$, $[1, 2]$ and $[0, 1, 2]$. 1 is a 0-form and so is mapped onto the vertices. This means that $f$ is mapped to $f([0]) + f([1]) + f([2])$. Similarly $fdx$ is mapped to $(\int_{[0, 1]} f\,dx)[0, 1] + (\int_{[0, 2]} f\,dx)[0, 2] + (\int_{[1, 2]} f\,dx)[1, 2]$.

2.2 Whitney map

The Whitney map is the complimentary operation, from $p$-cochains to $p$-forms. In order to introduce this we need the barycentric coordinates, $\mu_i$’s. Given an $n$-dimensional complex we have $\mu_0, \ldots, \mu_n$ defined on each $n$-simplex which have the property that:

- $\mu_i([v_j]) = \delta_{ij}$.
- $\sum_i \mu_i(x) = 1$ for all $x$ inside the triangle.
- $\mu_i = 0$ outside the triangle.
In our standard triangle, with vertices with coordinates \((0,0), (1,0)\) and \((0,1)\), we can define

\[
\begin{align*}
\mu_0 &= 1 - x - y \\
\mu_1 &= x \\
\mu_2 &= y
\end{align*}
\]

which satisfy the conditions necessary.

We can then define \(W^K\) as

\[
W^K[v_0, \ldots, v_p] = \sum_i p!(-1)^i \mu_i d\mu_0 \wedge \cdots \hat{d}\mu_i \cdots \wedge d\mu_i
\]

where \(\hat{d}\mu_i\) denotes that \(d\mu_i\) is emitted. With this we have that \(dW = Wd\) \(^{1}\) \(^{15}\) and \(A^KW^K = I\). This leads to a commutative diagram of sorts, Fig. 1 since we can starting from a form, map it to the triangulation and then act on it with \(d\) or act on it with \(d\) before taking our approximation; either way we get the same result. It is a “commutative diagram” since \(W^KA^K\) is not equal to the identity, which means that not all routes are equivalent.

### 2.3 Wedge

Having established the basic mechanic of using \(W^K\) and \(A^K\), to map to and fro between our spaces, we can induce a discrete wedge product \(\wedge^K\) on the discrete side:

\[
x^p \wedge^K y^q = A^K(W^K(x^p) \wedge W^K(y^q)).
\]

This is both distributive and anti-symmetric but not associative\(^3\), as can be seen from Eqn.1.

### 2.4 Hodge star

The Hodge star is the jewel in the GD\(^4\) crown. There are various problems with having a discrete star satisfying both \(\star\star = I\) and \(\delta = \star d\star\) type behaviour as discussed by Rabin \(^8\). These can be resolved by working in a subdivided space and saying that \(\star\) maps from the original triangulation to a dual space. The advantage of this can be seen from the Figures 3-5.

First from Fig. 3 we see that with a dual space we have a trivial identification of original and dual objects and so capture \(\star\star = I\); while in Fig. 4 we see that without a dual space, we do not return to where we began and so do not have \(\star\star = I\). Finally, in Fig. 5 we see that without the use of a subdivided space we do not get \(\delta = \star d\star\) but end up linking vertices to vertices “two units away”.

Aside from associativity of the wedge, we lack a Whitney map on the dual space. Partially motivated by this, we develop a cubic version of the theory next, whose dual space is also cubic.

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\(^{3}\)It is an anti-symmetrisation like \(^{16}\) and \(^{17}\)

\(^{4}\)This was developed by Adams \(^1\)
Figure 3: A dual space leads to $\star \star = I$ type behaviour.

Figure 4: Without a dual space $\star$ doesn’t know which vertex to map back to and so end up mapping to all of them.

Figure 5: Without a subdivided space we do not get $= \star\delta\star$ type behaviour. Initially $\star$ maps the vertex to the edge, before $\partial$ maps to its edges. Finally the second $\star$ maps these vertices to edges which results in a line which is twice as long as it would have been if it initial vertex had been acted on with $d$. The use of a subdivided space halves the lengths and thereby fixes the problem. Of course multiply both sides by $\star$ leads to $d = \star\delta\star$. 
3 Cubic theory

We begin by introducing our notation before looking at the operators $d$ and $\partial$ in this language and moving on to the de Rham and Whitney maps, which we show to have the desired properties.

Note that the dual to a cube is also a cube which means that we also have a Whitney map from the dual space which was not the case with simplices. We unfortunately were unable to develop a generic method to determine Whitney maps, which is the tricky part to generalise, for arbitrary cells.

3.1 Cubic Notation

We describe the various vertices, edges and faces, taken from Fig. 6 in our new notation below:

| Vertex     | 0,0     |
|------------|---------|
| Vertex 1   | 0,1     |
| Vertex 2   | 1,0     |
| Vertex 3   | 1,1     |
| Edge from 0 to 1 | 0, y |
| Edge from 0 to 2 | x, 0 |
| Edge from 1 to 3 | x, 1 |
| Edge from 2 to 3 | 1, y |
| Face “[0,1,2,3]” | x, y |

In n-dimensions, our general object, for the unit hypercube, is thus $[a_0, \ldots, a_n]$, where $a_i$ can be 0, 1 or $x_i$. 

Figure 6: Binary notation applied to a square, where vertex number expressed in binary leads to the $x$ and $y$ coordinates of that vertex. This makes it very easy to know which combinations of vertices are edges since they must differ by a power of two. In 3D for example we can tell that the edge [2,3] is connected to [6,7].
3.2 Cubic Operations

We define $D = d_{cube}$ as follows:

$$D[a_0, \ldots, a_n] = \sum_i s_i p_i [a_0, \ldots, a_n],$$  \hfill (2)

where $s_i = 0$ if $a_i = x_i$, $s_i = +1$ if $a_i = 1$ and $s_i = -1$ if $a_i = 0$. $p_i$ equals $(-1)^n$, where $n$ is the number of $a_j$'s equal to $x_j$ for $j < i$.

Next we introduce the boundary operator:

$$\partial : [a_0, \ldots, a_n] = \sum_i p_i ([a_0, \ldots, 1, \ldots, a_n] - [a_0, \ldots, 0, \ldots, a_n]),$$

where we have replaced the $i$th slot with a 0 and a 1, the second one having opposite the orientation. $p_i$ is the same as for $D$. In other words

$$\partial [x] = [1] - [0]$$

and

$$\partial [x, y] = -[0, y] + [1, y] + [x, 0] - [x, 1]$$

which is what we expect the boundary operation to be. To test it more thoroughly we apply it to $[x, y, z]$:

$$\partial [x, y, z] = -[0, y, z] + [1, y, z] + [x, 0, z] - [x, 1, z] - [x, y, 0] + [x, y, 1]$$

$$\partial^2 [x, y, z] = [0, 0, z] - [0, 1, z] + [0, y, 1] - [0, y, 0] + [1, 1, z] - [1, 0, z] + [1, y, 0] - [1, y, 1] + [1, 0, z] - [0, 0, z] + [x, 0, 0] - [x, 0, 1] - [1, 1, z] + [0, 1, z] + [x, 1, 1] - [x, 1, 0] - [1, y, 0] + [0, y, 0] + [x, 1, 0] - [x, 0, 0] + [1, y, 1] - [0, y, 1] - [x, 1, 1] + [x, 0, 1] = 0.$$

The boundary map is clearly taking us to faces of the object it is acting on but from this we can see that the orientation of the various factors is being dealt with properly too.

We want to show that $\partial^2 = 0$ in general. We can see that $\partial$ maps an $m$-cell to two oppositely oriented pieces. We say that the first $\partial$ sets $a_k$ to either 0 or 1 and the second the same to $a_l$. We denote this as

$$\partial \sigma^m = \sigma_0^{m-1} - \sigma_1^{m-1}$$

in short hand, where we mean sum over $i$ where $a_i = 0$ by $\sigma_0^{m-1}$. Then

$$\partial^2 \sigma^m = \sigma_{00}^{m-2} - \sigma_{01}^{m-2} - \sigma_{10}^{m-2} + \sigma_{11}^{m-2}.$$ 

In fact we have two cases to consider depending on whether $k < l$ or $k > l$. The idea is that if $k < l$ then we get the same sign factor whether the first $\partial$ removes $k$ or the second. This is not the case with $l$ since the $p_i$ factor changes. Thus we get two oppositely oriented versions of the same term, which cancel. The same argument applies if $k > l$. 

8
3.3 de Rham and Whitney

We next introduce $A^K$ and $W^K$ in the cubic framework, showing they satisfy the desired properties

- $A^K W^K = I$
- $dW^K = W^K D$
- $DA^K = A^K d$

The Whitney map is defined as

$$W^K[a_0, \ldots, a_n] = W^K[a_0] W^K[a_1] \ldots W^K[a_n], \tag{3}$$

where, for edges $[0, x_i]$ of length $h_j$, we have

$$W^K[0] = \frac{h_j - x_i}{h_j}, \tag{4}$$
$$W^K[1] = \frac{x_i}{h_j}, \tag{5}$$
$$W^K[x_i] = \frac{dx_i}{h_j}. \tag{6}$$

From Eqn.3 the general case consists of products of terms of the form

$W^K[a_i]$. $W^K[a_i]$ maps to $\frac{h_j - x_i}{h_j} = 1$ where $a_i = x_i = 0$, to $\frac{x_i}{h_j} = \frac{h_j}{h_j} = 1$ where $a_i = x_i = h_j$ and since $W^K[x_i] = \frac{dx_i}{h_j}$, we get

$$\int_0^{h_j} \frac{dx_i}{h_j} = 1,$$

when $a_i = x_i$. Thus $A^K W^K = I$ in general.

From Eqn.2 and Eqn.3 we see that

$$W^K D[a_0, \ldots, a_n] = \sum_i s_i p_i W^K[a_0] \ldots W^K[a_n]$$

and

$$dW^K[a_0, \ldots, a_n] = \sum_i W^K[a_0] \ldots dW^K[a_i] \ldots W^K[a_n] \tag{7}$$
$$= \sum_i s_i p_i dW^K[a_i] W^K[a_0] \ldots W^K[a_i] \ldots W^K[a_n]. \tag{8}$$

Thus

$$dW^K = W^K D.$$

Finally, from Stokes' theorem we see that $A^K d$ acting on a $p$-form

$$A^K d\omega^p = \sum_j \int_{\sigma_j^{p+1}} d\omega^p[\sigma_j^{p+1}] = \sum_j \int_{\partial \sigma_j^{p+1}} \omega^p[\sigma_j^{p+1}] = \sum_{i,j} \int_{\Omega_{i,j}} \omega^p[\sigma_j^{p+1}],$$
is the same as
\[
\sum_{i,j} I_{i,j} \int_{\sigma_i^p} \omega^p[\sigma_j^{p+1}] = \sum_i d \int_{\sigma_i^p} \omega^p[\sigma_i^p] = dA^K \omega^p.
\]

For this to hold in the cubic case we need $\partial$ and $D$ to be compatible, as they are in the simplicial case via the incidence matrix.

Looking at $\partial$ we can see that the sign, of incidence matrix elements, is determined by whether we introduce a 0 or a 1. This is also the case in $D$. We get a (-1) contribution if we introduce or remove a 0. The remaining factor is the $p_i$ one which also occurs in both $\partial$ and $D$. So the incidence matrix associated with $\partial$ induces a $D$ which is the same as the one we have that been using. Since $D$ is compatible $A^K D = dA^K$ follows.

Note that since $A^K W^K = I$ we also know that
\[
D^2 = (A^K W^K) D^2 = d^2 (A^K W^K) = 0.
\]

Thus we have

1. Introduced a cubic notation.
2. Defined appropriate boundary and coboundary operator.
3. Developed a cubic Whitney map.
4. Shown that these satisfy the desired properties.

Note that if we map $dx$ onto the standard triangle then due to the diagonal edge, $[1, 2]$, we introduce a $dy$ component when mapping back using $W^K$. This does not happen in the cubic.

Next we use this formalism to incorporate metric into geometric discretisation and show that image of the discrete Hodge star converges to the continuum one.

4 Incorporating metric via heuristic

We have treated GD primarily as a topological theory but $\ast$ has metric dependence which must be considered also. Since we can only compare two similar objects, we map continuum ones to the triangulation and back again, using $W^K A^K$, and note the difference. So when looking at functions, or forms, we consider
\[
f - W^K A^K f,
\]
and its dependence on the size of the discrete cells used.

In 1D, for example, we can see that via the de Rham map, $f(x)$ goes to $f(0)(0) + f(1)(1)$. $W$ then maps this to $f(0) + (f(1) - f(0)x$: a piecewise linear approximation. For $f dx$ we get $\int_{[01]} f dx$ on the complex which $W$ maps to
$(\int_{[01]} f dx) d\mu_0$. Since $\mu_0$ goes from 0 to 1 as $x$ goes from 0 to $L$, the length of $[01]$, we get

$$W^K A^K (f dx) = W \int_{[01]} f d\mu_0 = \frac{\int_{[01]} f dx}{L}$$

or the average value of $f$ along the edge $[01]$.

What we find is that the approximation made involves taking the average of the continuum object for forms and piecewise linear approximations for functions. The de Rham map takes the integral over a triangle, say, while the Whitney map results in dividing this by the volume so

$$f d^p x \to (\int_\sigma f d^p x) \sigma^p \to \frac{\int_\sigma f d^p x}{Vol \sigma^p} = f_{\text{average}} d^p x.$$

We can analyse $\star$ using this picture. If we start with a $p$-form, we integrate it over a $p$-simplex. This leads to a $p$-cochain in our discrete structure. We can act on this with $\star$, which maps it to an $(n-p)$-cochain which we can be mapped back to continuum space with Whitney(if we had a dual Whitney map that is, which we do for the cubic case). The problem then is that the Whitney map divides by the “wrong” volume; we no longer have an average:

$$f d^p x \to (\int_\sigma f d^p x) \sigma^p \to \frac{\int_\sigma f d^p x}{Vol \sigma^{n-p}} d^p x \neq f_{\text{average}} d^p x.$$

The idea then is to introduce volume factors to $\star$ which fixes this. We say that

$$\star_{\text{NEW}} \sigma^p \to \frac{Vol \sigma^{n-p}}{Vol \sigma^p} \star \sigma^p.$$ 

Using this we have anticipated the problem with $W$ so that now, when we divide by the “wrong” volume factor, it simply cancels with what we have, leaving the correct term to get the average. This also has the property that when we look at $\star\star$ the factors cancel out, as they should.

Once we have this, we can see that $\delta = \star d\star$ and the Laplacian should also converge. For the first case we want $W^K \star d \star A^K$ to converge to the continuum case. We can rewrite this as $W^K \star A^K W^K d \star A^K$ since $A^K W^K = I$, but this is $(W^K \star A^K) d(W^K \star A^K)$ since $d W^K = W^K d$. So if we have $W^K \star A^K$ converging to $\star$, which we do with the addition of the volume factors, we have the desired result. The Laplacian follows once we have $\delta$ converging since we can move $d$ through $A^K$ and $W^K$. Note that we only have these result for the cubic case since we do not have $\star$ acting on the dual space otherwise. Dodziuk [4] has previously shown convergence of functions, forms and various operators though his system was did not involve a subdivided space.

5 Incorporating metric via inner product

The previous modification is nice, since with it we have our discrete $\star$ converging to the continuum one BUT it is against the spirit of GD. By adding an $ad$ hoc
term we break the relationship between the inner product and $\star$. We desire our discrete theory to be as close as we can to the continuum one. We look at the inner product and what we can do with that next.

5.1 The inner product is the star

Any linear map on a vector space can be expressed in terms of the dual basis list elements. If we have any $p$-form $\lambda$ we can wedge it to an $(n-p)$-form $\mu$ to get an $n$-form $f d\sigma = \lambda \wedge \mu$. This is a linear map so $f$ is uniquely determined. Since it is linear we can also express it as $<\star \lambda, \mu> = f$ where $\star \lambda$ belongs to the dual space. This $\star$ is the hodge star and we can see how closely it is related to $<,>$. We cannot change one without changing the other.

We had in our original formulation that

$$(e^i, e^j) = \delta_{ij}$$

which means that the inner product of an edge with itself is 1 whilst we’d expect from the continuum case to get its length. This can be achieved by using a new inner product

$$(e^i, e^j) = \int W^B(e^i) \wedge \star W^B(e^j)$$

as used by Dodziuk [4]. We want to investigate the effect this has on $\star$. We have introduced metric information into our system and ideally this should filter through the system leaving all the properties which we are happy with whilst sorting out the metric dependence of $\star$.

5.1.1 Determining $\star$

Our inner product is given by

$$< e^i_B, e^j_B > = \int W^B(e^i_B) \wedge \star W^B(e^j_B),$$

(9)

Note that $e^i_B$ and $e^j_B$ both need to be of the same dimension since we need an $n$-form on the RHS for it to be nonzero. We then define $\star_K$ as follows:

$$< \star_K e^l_K, e^m_L > = \int W^B(\epsilon^l_K) \wedge W^B(\epsilon^m_L),$$

(10)

where using the fact that $e^l_K$ and $e^m_L$ can be expressed in terms of elements of $B$.

Now we write

$$\star_K e^l_K = h^l_m e^m_L.$$  (11)

We can using (9) and (10) determine $h^l_m$ which determines our matrix for the hodge star operator.
Firstly we define two more matrices $A$ and $B$ which allow us to move from basis elements of $K$ and $L$ to $B$ respectively. So

\begin{align*}
e^l_K &= A^l_i e^i_B, \\ e^m_L &= B^m_j e^j_B.
\end{align*}

(12)

and (13) can be rewritten as

\begin{align*}
< \star K e^l_K, e^m_L > &= < h^l_B, e^m_B >, \\
&= h^l_B B^m_j < e^j_B, e^j_B >, \\
&= h^l_B B^m_j \int W(e^j_B) \wedge \star W(e^j_B). \\
I^{ij} &= \int W(e^i_B) \wedge \star W(e^j_B). \\
< \star K e^l_K, e^m_L > &= h^l_B B^m_j I^{ij} B^j_m, \\
&= h^l_B X^{om}.
\end{align*}

(14)

(15)

(16)

(17)

Also using (10) we have that

\begin{align*}
< \star K e^l_K, e^m_L > &= \int W(e^l_K) \wedge W(e^m_L), \\
&= \int W(A^l_i e^i_B) \wedge W(B^m_j e^j_B), \\
&= A^l_i B^m_j \int W(e^j_B) \wedge W(e^i_B). \\
J^{ij} &= \int W(e^i_B) \wedge W(e^j_B). \\
< \star K e^l_K, e^m_L > &= A^l_i J^{ij} B^m_j, \\
&= S^{lm}.
\end{align*}

(20)

(21)

(22)

(23)

(24)

(25)

Since (10) = (25) we know that

\begin{equation}
h^l_B X^{om} = S^{lm},
\end{equation}

which is just a matrix equation. If we determine $X^{-1}$ we can right multiply by this to get $h$ which is $\star K$. We can get $\star L$ using a similar calculation.

5.2 Cubic case

Given an inner product, the Hodge star operator can be determined, as we have just seen. We consider here the cubic case and show that volume factors emerge.

\footnote{We have investigated the simplicial case computationally but the nature of the volume factors is unclear due to its more complicated form. We resulted with $\star \star$ diagonal but not equal to the identity; this could be normalised but the metric dependence, which was the objective, was not clarified.}

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\end{equation}

which is just a matrix equation. If we determine $X^{-1}$ we can right multiply by this to get $h$ which is $\star K$. We can get $\star L$ using a similar calculation.

5.2 Cubic case

Given an inner product, the Hodge star operator can be determined, as we have just seen. We consider here the cubic case and show that volume factors emerge.

\footnote{We have investigated the simplicial case computationally but the nature of the volume factors is unclear due to its more complicated form. We resulted with $\star \star$ diagonal but not equal to the identity; this could be normalised but the metric dependence, which was the objective, was not clarified.}

Since $X^{-1}$ we know that

\begin{equation}
h^l_B X^{om} = S^{lm},
\end{equation}

which is just a matrix equation. If we determine $X^{-1}$ we can right multiply by this to get $h$ which is $\star K$. We can get $\star L$ using a similar calculation.
in precisely the manner we expected from the heuristic. We take a simple one square system which is subdivided as our reference system.

We begin by determining the Whitney map of the various cells which we have. For example

\[
W^B[0] = \frac{1}{a}(a - x)(a - y) \quad A \\
= \frac{1}{a}(x)(a - y) \quad B \\
= \frac{1}{a}(x)(y) \quad C \\
= \frac{1}{a}(a - x)(y) \quad D \\
W^B[7] = -\frac{1}{a}(a - x)y \quad C \\
= -\frac{1}{a}(x)y \quad D \\
W^B[0123] = \frac{1}{a^2}dx \wedge dy \quad A, B, C, D
\]

Once we have this we can look at \(\langle,\rangle_K\) where

\[
\langle x^p_K, y^p_K \rangle_K = \int W^B(Bx^p_K) \wedge *W^B(By^p_K).
\]

If we let the \(e^i\) be the basis list elements of \(K\) then we can define

\[
I^{ij} = \langle e^i, e^j \rangle_K.
\]

Note that

- I is diagonal otherwise \(e^i\) and \(e^j\) are in different squares of \(B\) and so their images under \(W^B\) (though not under \(W^K\)) do not overlap.

- The various integrals that appear are the same since we are always integrating either \(x^2\) from 0 to \(a\) or something which can be expressed as this with a change of variables.

- The \(dx\) and \(dy\) integrals are independent and both of the form mentioned above.
The are only two possible cases which lead to the same result:

\[ \int_0^a x^2 \, dx = \left. \frac{x^3}{3} \right|_0^a = \frac{a^3}{3} \]

\[ \int_0^a (a-x)^2 \, dx = \int_a^0 y^2 (-dy) = \frac{a^3}{3} \]

So any of the integrals which we have are equal to \( \frac{a^3}{3} \).

In the case of vertices we have two such integrals leading to \( \frac{k^4}{3} \) factors with a symmetry term \( F \) to specify how often a term occurs; this is 4 for vertices since each vertex occurs in 4 squares, 2 for edges and 1 for faces. For edges we only have one integral, since we don’t get \( x^2y^2 \) terms but \( x^2dy \) ones instead, and for faces we get no such terms since the integral is simply \( dx dy \). \( B \) also introduces a factor, as in the simplicial case; though here we get \( 2^p \) instead of \( p! \) since the volumes involved are different. Finally \( W^K \) has a \( \frac{1}{a^2} \) term associated with it so for vertices we get

\[ < [0], [0] >_K = \frac{1 \times 4a^6}{9a^4}, \]

for edges

\[ < [01], [01] >_K = \frac{4 \times 2a^4}{3a^4} \]

and for faces

\[ < [0123], [0123] >_K = \frac{16 \times 1a^2}{a^4}. \]

For \( L \) we similarly get

\[ < [0123], [0123] >_L = \frac{1 \times 4a^2}{9} \]

\[ < [01], [01] >_L = \frac{4 \times 2}{3} \]

\[ < [0], [0] >_L = \frac{16 \times 1}{a^2} \]

We only mention the results for particular vertices, edges and faces but they are all the same.

To determine \( \star_K \) we use

\[ < \star_K x^p_K, y^{n-p}_L >_L = \int W^B (Bx^p_K) \wedge W^B (By^{n-p}_L). \]

We have the LHS except \( \star_K \) so we need the RHS next. Note that since both \( <, >_K \) and \( <, >_L \) are diagonal, we only get contributions when \( y = \hat{x} \).

---

6 This is related to maintaining \( A^K W^K = I \).

7 The \( F \) factors are all the same since we are interested in complexes without boundary in which case every vertex has \( F = 4 \), every edge has \( F = 2 \) and every face has \( F = 1 \).
We have two $B$ factors on the right hand side which leads to a factor of 4. If we are dealing with a vertex then we wedge it with a face. $B$ of the vertex gives 1, while $B$ of the face gives 4. For edges we get $2 \times 2$ since we wedge two edges together and for a face we wedge it with a vertex leading to a $4 \times 1 = 4$ factor. We look at the various cases next.

Performing the RHS integral for the vertices, if we have $x = [0]$ then $y = [\hat{0}]$, we get:

$$= 4 \int_A \frac{1}{a^4} (a - x)(a - y) dx \wedge dy$$

$$= 4 \frac{1}{a^4} \frac{a^2}{2} \frac{a^2}{2}$$

$$= 1.$$ 

For the edges we get:

$$= 4 \int_A W^B[01] \wedge W^B[03]$$

$$= 4 \int_A \frac{1}{a^4} \frac{a^2}{2} \frac{a^2}{2}$$

$$= 1.$$ 

And for the the faces:

$$= \int W^B[0123] \wedge W^B[0]$$

$$= 4 \int_A \frac{1}{a^4} \frac{a^2}{2} \frac{a^2}{2}$$

$$= 1.$$ 

The dual calculations are the same apart from sign factors which arise when the $W^K(x)$ and $W^K(y)$ terms are flipped. So for the edges case we get a ($-1$) sign.

Determining $\star_K$ is now trivial since both sides of the equation are diagonal. The $i$th diagonal element of $\star_K$ is simply the $i$th diagonal element of the RHS(which are all just $\pm 1$) divided by the $i$th diagonal element of $<, >_L$ which only depends on whether dealing with vertices, edges or faces. So on 0-cochains have

$$\star_K[0] = \frac{a^2}{16}[\hat{0}]$$

on edges we have

$$\star_K[01] = \frac{3}{8}[\hat{01}]$$

and for faces we have

$$\star_K[0123] = \frac{9}{4a^2}[\hat{0123}].$$
We can get $\star_L$ in the same way:

\[
\star_L[0] = \frac{9}{4a^2}[0] \\
\star_L[01] = -\frac{3}{8}[01] \\
\star_L[0123] = \frac{a^2}{16}[0123]
\]

With this we get

\[
\star_K \star_L = \pm \frac{9}{64} I
\]

which is diagonal and proportional to the identity.

We can then introduce normalisation factors on the RHS — as is done in the
simplicial theory — by simply adding a factor of $\frac{8}{3}$ to the definition of $\star_K/\star_L$:

\[
< \star_K f^p_L, y_{n-p} >_L = \frac{8}{3} \int W^B(B x^p_K) \wedge W^B(B y_{L}^{n-p}).
\]

Using this we have $\star_K \star_L = \pm I$ as required.

Here whilst we get $\star \star = I$ type behaviour, it can be improved. $\star_K$ acting on a vertex introduces a factor of $\frac{1}{8}$, after the above normalisation has been made. When $\star_L$ acts on this it multiplies it by 6 so we get the desired $\star \star = I$. Since does not occur in the continuum we should normalise at this level instead which can be done by using appropriate $p$ dependent factors in Eq. 27.

We can now determine convergence results since we can see how the length
scale $a$ appears in the various operators. In short they appear as you would expect them to using the heuristic which means that that $\delta$ and the Laplacian converge to the continuum case.

Firstly we look explicitly at what happens for $\delta = \star d \star$ in 2D. In the continuum case we know that:

1. $\star d \star f = \star df \, dx \, dy = 0$
2. $\star d \star f \, dx = \star d(f \, dy) = \star \frac{\partial f}{\partial x} \, dx \, dy = \frac{\partial f}{\partial x}$
3. $\star d \star f \, dx \, dy = \star df = \star (\frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy) = \frac{\partial f}{\partial x} \, dy - \frac{\partial f}{\partial y} \, dx$.

We can now look at what happens in the cubic case and compare $W^K \star_L d_L \star_K A^K$ with the above.

Note that $\partial = \star d \star$, with some sign factor which means that $\star d \star$ applied to $[xy]$ gives us its boundary, upto sign. For example, if we take the edge $[0, y]$ we map $-f_i$ onto it, where $f_i$ is $f \, dx \, dy$ integrated over the face $i$. The adjacent face will also contribute to this edge with a $+f_j$ term. Face $i$ is $a$ away in the $x$ direction from face $j$ and so we effectively get an $f(x + a) - f(x)$ term which after Whitney will give

\[
\frac{f(x + a) - f(x)}{a} \, dy.
\]
In this way we can see how we agree with the continuum result. The 1-form case follows similarly with the boundary now vertices and the 0-form case is trivial.

As we mentioned before, the volume factors which arise are precisely those which we required from our heuristic (Sec 4) which can be generalised to higher dimensions. First we note that this is the case in 2D since $\star$ applied to vertices leads to $a^2$ factor which is the volume of the cell mapped to divided by that of the cell acted on. For edges we get a factor independent of $a$ which again agrees with this picture as does the $\frac{a}{2}$ factor for faces.

In general we have $a^{3D-2p}/a^{2D}$ for $p$-cochains. In two dimensions this means that we have $a^{6-2p}/a^4$ so $a^2$ for vertices, 1 for edges and $a^{-2}$ for faces. Similarly in three dimensions we have $a^{9-2p}/a^6$ or $a^3$ for vertices, $a$ for edges, $a^{-1}$ for faces and $a^{-3}$ for cubes. In general the heuristic expression is the volume of the space mapper to, $a^{D-p}$, divided by the volume of the object being acted on, $a^p$. Since

$$\frac{a^{3D-2p}}{a^{2D}} = a^{D-2p} = \frac{a^{D-p}}{a^p},$$

the two agree.

We had already seen that the heuristic suggested a modification that could be made to $\star^K$ so that it, along with the coderivative and Laplacian, would converge to the continuum result. This problem with this was that it broke the relationship between the inner product and $\star$. In order to retain this we made a natural modification to the discrete inner product to see what effect this had on $\star^K$ and found that this resulted in adding precisely the factors which we wanted from the heuristic. As a result, we can use the various convergence results which we had before, except that now the relationship of $\star^K$ and the inner product is preserved.

6 Conclusions

We have thus introduced a cubic Whitney map which we have used to demonstrate convergence of our discrete functions and operators to their continuum counterparts. This involved the introduction of some modifications for the Hodge star, either via a heuristic, which was unsatisfactory due to the relationship of the inner product and $\star$, or via a new inner product (the one in fact used by Dodziuk).
With metric and convergence results, applications become a possibility. The work, for example, of Bossavit [9], Hiptmair [5] and Nedelec [6] are naturally of interest though within a different context, namely finite element. GD also involves complexes, with finite dimensional subspaces within them, but does not use variational arguments for existence theorems for example.

Our current focus is on application to lattice field theory where following Rabin [8] we have shown that within the Dirac-Kähler formalism [18], where Nielsen-Ninomya [19] is not applicable as shown by Becher [20], we have chirality (∗ in this picture) whilst avoiding degeneracy.

Work has also been done extending the mathematical structures which GD deals and should soon follow.

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