A three-dimensional $q$-Lie algebra of $SU_q(2)$ is realized in terms of first- and second-order differential operators. Starting from the $q$-Lie algebra one has constructed a left-covariant differential calculus on the quantum group. The proposed construction is inverse to the standard Woronowicz approach; the left-invariant vector fields are introduced as initial objects whereas the differential 1-forms are defined in a dual manner.
1 Introduction

Researches of non-commutative geometry of the quantum groups [1-5] led to a series of papers concerned with non-equivalent differential calculi on the quantum groups and quantum spaces. In particular, bicovariant $4D_\pm$ and left-covariant $3D$ differential calculi on the quantum group $SU_q(2)$ had been considered [3-6]. For a two-parameter deformed linear quantum group $GL_{p,q}(2)$ the corresponding covariant differential calculus on the quantum group and quantum plane was proposed as well [7 - 9]. As a rule, the standard Woronowicz approach is successfully used to construct differential calculi on quantum groups. In the framework of this approach the differential 1-forms are treated as initial basic elements whereas the vector fields are defined as dual ones. As a consequence the geometrical content of the vector fields and their connection with usual derivatives on the quantum group remain hidden.

In this paper an explicit representation of the $q$-Lie algebra of left-invariant vector fields on the quantum group $SU_q(2)$ is proposed in terms of first- and second-order differential operators. Using the representation one has constructed the covariant $3D$ differential calculus on the quantum group. In some sense, the construction is inverse to the standard Woronowicz approach. In Section 2 a left-covariant differential calculus on the quantum group $U_q(2)$ is presented. The main commutation relations are differed from ones considered in [7]. Here we introduce covariant tensor notations which are very convenient in constructing various covariant geometrical objects with a definite $U(1)$ charge. Section 3 deals with the left-invariant vector fields on the quantum group $SU_q(2)$. Defining relations for an exterior differential algebra complete the differential calculus. In Conclusion we determine the connection between the $SU_q(2)$ left-covariant $3D$ $q$-Lie algebra and the Drinfeld-Jimbo quantum enveloping algebra. Main formulas and conventions used throughout the paper are contained in Appendix.

2 Differential calculus on the quantum group $U_q(2)$

Let $x^i, y_i$ ($i = 1, 2$) be generators (coordinates) of the function algebra $C^2_{q,q^{-1}}$ on the quantum hermitean vector space $U^2_q$ with an involution $*: x^i = y_i$. It is...
convenient to parametrize the matrix $T^i_j \in U_q(2)$ by the coordinates $x, y$

$$T^i_j = \begin{pmatrix} y^1 & x^1 \\ y^2 & x^2 \end{pmatrix} = (y^i x^j). \tag{2.1}$$

We use the $R$-matrix formulation of quantum groups following Faddeev, Reshetikhin and Takhtajan [4]. The main commutation relation for the quantum group generators $T^i_j$ has a standard form

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}. \tag{2.2}$$

The parametrization (2.1) was introduced in the harmonic formalism [13] applied to extended supersymmetric theories and supergravities. If one imposes the unimodularity constraint

$$D \equiv \det_q T^i_j = x_i y^i = 1, \tag{2.3}$$

then the variables $(x, y)$ will be just the quantum harmonic functions $(u^\pm)$ on the coset $S^2_q \sim SU(2)/U(1)$ with corresponding $U(1)$ charges $\pm 1$:

$$x^i \equiv u^{+i}, \quad y^i \equiv u^{-i}. \tag{2.4}$$

For convenience we shall not use the notations $u^\pm$ throughout the paper keeping in mind that all geometrical objects (like coordinates, derivatives, differentials etc.) have definite $U(1)$ charges. All further consideration respects the global covariance under the action of the group $U(1)$.

Let us define the derivatives $\partial_i \equiv \frac{\partial}{\partial x^i}, \quad \bar{\partial}^j \equiv \frac{\partial}{\partial y^j}$ on the quantum group by the formulas

$$\partial_i x^j = \gamma^j_i, \quad \bar{\partial}^j y_j = \delta^j_i, \tag{2.5}$$

where $\gamma^j_i, \delta^j_i$ are quantum analogues to classical Kronecker symbol. The commutation relations between the coordinates and derivatives are uniquely defined (up to the symmetry connected with the exchange $q \to \frac{1}{q}$):

$$R_{12} (\partial_T)_1 (\partial_T)_2 = (\partial_T)_2 (\partial_T)_1 R_{21}, \quad (\partial_T)^i_j \equiv \begin{pmatrix} \bar{\partial}_1 & \bar{\partial}_2 \\ \partial_1 & \partial_2 \end{pmatrix}, \tag{2.6}$$

$$\partial_i x^k = \gamma^k_i + qY^m_{ni} x^m \partial_n, \quad \bar{\partial}^j y_j = \delta^j_i + q y^m \bar{\partial}_n \hat{R}^m_{nj},$$

$$\partial_i y_j = q \hat{R}^{-1}_{ji} y_k \partial_k, \quad \bar{\partial}^j x^j = \frac{1}{q} \hat{R}^{ij}_{kl} x^k \bar{\partial}^j. \tag{2.7}$$
All these relations are consistent with the quantum group structure. The definitions (2.7) do not differ from ones considered in ref. [7] on principal. Our choice is conditioned by a requirement of manifest $U_q(2)$ and $U(1)$ covariance. We consider all objects with upper (lower) indices to be transformed under the quantum group co-action $\Delta$ like classical co-(contra-)variant tensors. For instance, a second rank tensor $N_{ij}$ will be transformed as follows

$$(N'_{ij}) = (T^\dagger)^k_i T^j_l N^l_k. \quad (2.8)$$

Hereafter the signs $\otimes$ of tensor product are omitted for simplification in writing formulas. It should be noted that an alternative way of introducing manifest tensor notations is adoption of the left $\bar{\partial}$ and right $\overrightarrow{\partial}$ derivatives [10, 11].

Let us construct the left-invariant differential operators $\partial_a \ (a = 1, 2, 3, 4)$ on the $U_q(2)$ space:

$$\partial_1 = x_k \partial^k, \quad \partial_2 = y_k \bar{\partial}^k, \quad \partial_3 = x_k \bar{\partial}^k, \quad \partial_4 = y_k \partial^k \quad (2.9)$$

The differential operators $\partial_a$ form a generalized $q$-Lie algebra of left-invariant vector fields. It is easy to check that the operators $\partial_3, \partial_4$ maintain the unimodularity condition $s \equiv D - 1 = 0$, i.e.

$$\partial_{3,4}(sf(x, y)) = 0. \quad (2.10)$$

Here, $f(x, y)$ is an arbitrary function. At the same time it is impossible to construct a third linearly independent left-invariant first-order differential operator obeying the condition (2.10). Consequently, one cannot realize a $q$-Lie algebra on the $SU_q(2)$ in terms of first-order differential operators.

Now we define an exterior differential $\hat{d}$ by following relations

$$\hat{d} \equiv d + \bar{d}, \quad \hat{d}^2 = d^2 = \bar{d}^2 = 0$$

$$d = dx^i \gamma^{-1k}_i \partial_k, \quad \overrightarrow{\partial} = dy_i \bar{\partial}^i$$

$$d(fg) = df \cdot g + f \cdot dg, \quad \bar{d}(fg) = \bar{d}f \cdot g + f \cdot \bar{d}g$$

$$d\bar{d} + \bar{d}d = 0. \quad (2.11)$$

Here a standard Leibnitz rule for the exterior differential is used. Commutation relations for the basic differential 1-forms, coordinates and derivatives can be easily derived in a similar way as in ref. [8] taking into account full consistency with the complex structure provided by the $*$-involution. The final relations are collected in
Appendix. The left-invariant Cartan 1-forms $\omega^a(a = 1, 2, 3, 4)$ generate a basis in the space of differential forms

$$
\begin{align*}
\omega^1 &= \frac{1}{q} y_i dx^i, & \omega^2 &= q^2 x_i dy^i, \\
\omega^3 &= -y_i dy^i, & \omega^4 &= q x_i dx^i.
\end{align*}
$$

The exterior differential defined by relations (2.11) can be rewritten as follows

$$
\hat{d} = \frac{1}{D} \omega^a \partial_a.
$$

Commutation relations for the basic differential 1-forms $\omega^a$ and the Cartan-Maurer equation are derived straightforwardly using definitions (2.11) (see Appendix). Analysis of the differential calculus on $U_q(2)$ implies that one cannot directly reduce it to the differential calculus on the quantum group $SU_q(2)$.

3 Left-invariant vector fields of $SU_q(2)$ and differential calculus

To define the left-invariant vector fields on the quantum group $SU_q(2)$ we shall use the $U_q(2)$-covariant differential calculus. Let us introduce the next notations for the left-invariant 1-order differential operators in correspondence with notations of the classical harmonic approach [13]

$$
D^{++} \equiv x_i \bar{\partial}^i, \quad D^{--} \equiv -y_i \partial^i.
$$

The action of the operators $D^{++}, D^{--}$ on the coordinates $(x, y)$ have simple properties:

$$
\begin{align*}
D^{++} x^i &= 0, & D^{--} x^i &= y^i, \\
D^{++} y_i &= x_i, & D^{--} y_i &= 0.
\end{align*}
$$

A Leibnitz rule for these operators is simplified when acting on the functions with definite $U(1)$ charges

$$
D^{\pm \pm} (f^{(m)} g^{(n)}) = (D^{\pm \pm} f^{(m)}) g^{(n)} + q^{-m} f^{(m)} (D^{\pm \pm} g^{(n)}).
$$

As it is mentioned above, one cannot construct a $q$-analogue for the classical $U(1)$ generator $D^0$ in terms of first-order differential operators. Nevertheless, the interesting feature of non-commutative geometry is that the quantum $U(1)$ generator
$D^0$ does exist. The operator is realized as a left-invariant second-order differential operator

$$D^0 \equiv -x_i \partial^i - q^2 y_i \bar{\partial}^i + (1 - q^2)x_i y_k \bar{\partial}^k \partial^i.$$ (3.4)

It is not hard to check that the operator $D^0$ has eigenfunctions $f^{(n)}$ with eigenvalues corresponding to $q$-generalized $U(1)$ charge $(n)$:

$$D^0 f^{(n)} = \{n\}_q f^{(n)},$$

$$\{n\}_q \equiv \frac{1 - q^{-2n}}{1 - q^{-2}}.$$ (3.5)

Due to that property (3.5) the algebra of functions $f^{(n)}$ with a definite $U(1)$ charge is just the function algebra on a quantum sphere $S^2_q = SU_q(2)/U(1)$. The Leibnitz rule for the operator $D^0$ has the next form

$$D^0(f^{(m)} g^{(n)}) = (D^0 f^{(m)}) g^{(n)} + q^{-2m} f^{(m)} D^0 g^{(n)}.$$ (3.6)

By direct checking one can verify that the operators $D^{\pm \pm 0}$ satisfy a generalized $q$-Lie algebra of $SU_q(2)$ [8]

$$[D^0, D^{++}] = (2)_q D^{++},$$

$$[D^0, D^{--}] = (-2)_q D^{--},$$

$$[D^{++}, D^{--}]_q^2 = D^0,$$ (3.7)

here, $[A, B]_q^s \equiv AB - q^s BA$.

It should be noted that the algebra (3.7) is valid irrespective of whether one imposes the constraint $D = 1$. An important property of the operators $D^{\pm \pm 0}$ is conservation of the unimodularity constraint

$$D^{\pm \pm 0}(D f(x, y)) \approx 0.$$ (3.8)

The last relation allows to construct the differential calculus on the $SU_q(2)$ in a consistent manner. Note, that the braiding matrix corresponding to the $q$-Lie algebra (3.7) is unitary and the generalized Jacobi identity is available

$$[D^0, [D^{++}, D^{--}]_q^2] + [D^{++}, [D^{--}, D^0]_q^4]_q^{-4} + q^2 [D^{--}, [D^0, D^{++}]_q^4]_q^{-2} \equiv 0.$$ (3.9)
It is easy to check another relation which is similar to Jacobi identity

\[ [D^0, [D^{++}, D^{--}]] + [D^{++}, [D^{--}, D^0]] q^{-4}] q^6 + q^6[D^{--}, [D^0, D^{++}]] q^{-4}] q^{-6} \equiv 0, \]  

(3.10)

Using properties of the differential operators \( D^{\pm\pm} \) we can define a covariant algebra of left-invariant vector fields \( \nabla^{\pm\pm} \) on the quantum group \( SU_q(2) \) in axiomatic way and then introduce the basic differential Cartan 1-forms as dual objects. Let us define the left-invariant vector fields \( \nabla^{\pm\pm} \) by the same relations (3.2, 3.3, 3.5, 3.6) that the operators \( D^{\pm\pm} \) obey with only exchanging \( D^{\pm\pm} \rightarrow \nabla^{\pm\pm} \). The basic left-invariant differential 1-forms \( \omega^{\pm\pm} \) are then defined as dual objects to vector fields \( \nabla^{\pm\pm} \)

\[ \omega^{++}(\nabla^{--}) = 1, \quad \omega^{--}(\nabla^{++}) = q, \quad \omega^0(\nabla^0) = 1. \]  

(3.11)

An exterior differential on \( SU_q(2) \) is defined in a standard manner with a usual Leibnitz rule

\[ \delta \equiv \omega^{++}\nabla^{--} + \omega^{--}\nabla^{++} + \omega^0\nabla^0, \]
\[ \delta(fg) = \delta f \cdot g + f \cdot \delta g, \]
\[ \delta^2 = 0, \]  

(3.12)

where the \( f, g \) – are arbitrary functions on the quantum group \( SU_q(2) \). Using these formulas it is not difficult to obtain all commutation relations for the differential 1-forms \( \omega^{\pm\pm} \) and corresponding Cartan-Maurer equations

\[ (\omega^\alpha)^2 = 0, \quad \alpha = (++, --, 0), \quad \omega^{++}\omega^{--} = -2q^2 \omega^{--}\omega^{++}, \]
\[ \omega^{\pm\pm} f^{(m)} = q^m f^{(m)} \omega^{\pm\pm}, \quad \omega^{++}\omega^0 = -q^2 \omega^0\omega^{++}, \]
\[ \omega^0 f^{(m)} = q^{2m} f^{m} \omega^0, \quad \omega^{--}\omega^0 = -q^4 \omega^0\omega^{--}, \]
\[ \delta \omega^{++} = (-2) q^{2} \omega^{++} \omega^0, \]
\[ \delta \omega^{--} = 2 q^{2} \omega^{--} \omega^0, \]
\[ \delta \omega^0 = \omega^{++} \omega^{--}. \]  

(3.13)

All these relations are consistent with the unimodularity condition \( D = 1 \). An exterior algebra of the differential forms is defined straightforwardly. The final construction of the covariant differential calculus on the quantum group \( SU_q(2) \) presented here agrees with one considered in Woronowicz approach [3, 8].
4 Conclusion

Now we shall determine the explicit connection between the generalized $q$-Lie algebra (3.7) and the quantum enveloping Drinfeld-Jimbo algebra $U_q(su(2))$. For this purpose one considers the differential operators $\mu, \nu$ on the quantum group $U_q(2)$:

$$\mu = 1 + (q^2 - 1)y_i \partial^i, \quad \nu = 1 + (1 - \frac{1}{q^2})x_i \partial^i.$$  \hfill (4.1)

These operators have simple commutation relations with the operators $D^{\pm \pm 0}$. For instance, we have the following formulae containing the operator $\mu$

$$\mu D^{--} = q^2 D^{--} \mu, \quad \mu D^{++} = \frac{1}{q^2} D^{++} \mu,$$

$$\mu D^0 = D^0 \mu, \quad \mu \nu = \nu \mu. \hfill (4.2)$$

Equations for the $\nu$ operator have a similar form.

Let us define new operators $D^{++}, D^{--}, D^0$ multiplying the $D^{\pm \pm 0}$ by corresponding factors

$$D^{++} = \mu^{-\frac{1}{2}} D^{++}, \quad D^{--} = \nu^{-\frac{1}{2}} D^{--},$$

$$D^0 = \frac{1}{q} \mu \nu D^0 \equiv [\partial^0]_q. \hfill (4.3)$$

It is easy to verify that the operators $D^{\pm \pm 0}$ form just the Drinfeld-Jimbo quantum enveloping algebra $U_q(su(2))$

$$[\partial^0, D^{++}] = 2D^{++}, \quad [\partial^0, D^{--}] = -2D^{--},$$

$$[D^{++}, D^{--}] = [\partial^0]_q. \hfill (4.4)$$

The operator $D^0$ counts the $q$-generalized $U(1)$ charge when acting on the functions with a definite $U(1)$ charge ($m$)

$$D^0 f^{(m)} = [m]_q f^{(m)}. \hfill (4.5)$$

Observe that the way of constructing the Drinfeld-Jimbo algebra from the $q$-Lie algebra (3.7) is by no means unique.

We confine our consideration to a simple case of the quantum group $SU_q(2)$. However, one should expect that similar representation of the $q$-Lie algebra in terms of differential operators hold for other quantum groups. The $q$-Lie algebra (3.7) has a natural geometrical origin in our approach. It turns out to be closely connected with
a gauge covariant differential algebra of $SU_q(2)$ in constructing the non-standard Leibnitz rule. These questions will be considered in a separate paper elsewhere.

**Acknowledgments**

Author would like to thank B. Zupnik, A. Isaev and Ch. Devchand for useful discussions and interest to work.

**A Appendix**

We use the next notations for the invariant $SU_q(2)$ tensors

\[
\varepsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad \gamma^i_j = \begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix}, \\
\varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{q} & 0 \end{pmatrix}, \quad \gamma^{-1j}_i = \begin{pmatrix} \frac{1}{q} & 0 \\ 0 & q \end{pmatrix}
\]

(A.1)

\[
\varepsilon_{ik}\varepsilon^{jk} = \varepsilon_{ki}\varepsilon^{kj} = \delta^j_i, \quad \varepsilon_{ik}\varepsilon^{kj} = -\gamma^{-1j}_i, \quad \varepsilon_{ki}\varepsilon^{jk} = -\gamma^j_i
\]

The invariant metric $\varepsilon_{ij}$ is used to raise and lower the $SU_q(2)$ indices as follows

\[
A_i = \varepsilon_{ij} A^j, \quad A^i = A_j \varepsilon^{ji}
\]

(A.2)

The $R$-matrix and auxiliary matrices $X, Y$ are defined as in [7]

\[
\hat{R}_{kl}^{ij} = \hat{R}_{kl}^{ji} = \delta_i^k \delta_j^l (1 + (q - 1)\delta^{ij}) + (q - \frac{1}{q})\delta_i^k \delta_j^l \theta(i - j), \quad X_{s}^{ri} = \hat{R}_{ij}^{rs} q^{2(r-s)} = \hat{R}_{ij}^{rs} q^{2(s-r)}, \quad Y_{s}^{ri} = (\hat{R}^{-1})_{ij}^{rs} q^{2(s-r)} = (\hat{R}^{-1})_{ij}^{rs} q^{2(r-s)}, \quad Y_{it}^{jk} R_{km} = \delta_i^t \delta_j^m.
\]

(A.3)

The main commutation relations in a case of the quantum group $U_q(2)$ have the following form

\[
R_{12} dT_1 dT_2 = -dT_2 dT_1 R_{12}^{-1}, \\
dx_{1} x_2 = q R_{21} x_2 d x_1, \quad \partial_j dx^i = q X_{ij}^k dx^k \partial_k, \\\ndy_1 y_2 = \frac{1}{q} R_{12}^{-1} y_2 dy_1, \quad \bar{\partial}^i dx^j = \frac{1}{q} X_{ij}^k dx^k \bar{\partial}^j, \quad \bar{\partial}^i dy^j = \frac{1}{q} X_{ij}^k dy^k \bar{\partial}^j, \quad \partial_i dy_j = q (\hat{R}^{-1})_{ji}^k dy_k \partial_l.
\]

(A.4)
The Cartan-Maurer equations for the quantum group $U_q(2)$ can be written as follows

$$
d\omega_1 = \frac{1}{D} (\omega_3 \omega_4 + \omega_2 \omega_1), \quad d\omega_2 = -\frac{1}{D} (q^2 \omega_3 \omega_3 + \omega_1 \omega_2),
$$

$$
d\omega_3 = -\frac{1 + q^2}{q^4 D} \omega_3 \omega_2, \quad d\omega_4 = \frac{1 + q^2}{D} \omega_3 \omega_1.
$$

(A.5)
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