GENERALIZED SPIN REPRESENTATIONS.
PART 2: CARTAN–BOTT PERIODICITY FOR THE SPLIT REAL $E_n$ SERIES

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Abstract. In this article we analyze the quotients of the maximal compact subalgebras of the split real Kac–Moody algebras of the $E_n$ series resulting from the generalized spin representation introduced in [HKL13]. It turns out that these quotients satisfy a Cartan–Bott periodicity.

Our findings are also meaningful in the finite-dimensional cases of $A_2 \oplus A_1$, $A_4$, $E_6$, $E_7$, $E_8$, where it turns out that the generalized spin representation is injective. Consequently the observed Cartan–Bott periodicity provides a structural explanation for the seemingly sporadic isomorphism types of the maximal compact Lie subalgebras of the split real Lie algebras of types $E_6$, $E_7$, $E_8$.

1. Introduction

In this article we continue the investigation of the generalized spin representations introduced in the first part [HKL13]. We focus on the $E_n$ series and use the original description of the generalized spin representation from [DKN06], [DBHP06], [HKL13] via Clifford algebras.

The $E_n$ series is traditionally only defined for $n \in \{6,7,8\}$. However, using the Bourbaki style labeling shown in Figure 1, it naturally extends to arbitrary $n \in \mathbb{N}$. Using this description, one has $E_1 = A_1$, $E_2 = A_1 \oplus A_1$, $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$ (see Figure 2).

![Figure 1. The Dynkin diagram of type $E_n$](image)

An elementary combinatorial counting argument using binomial coefficients allows us to determine lower bounds for the $\mathbb{R}$-dimension of the images of the generalized spin representation. These images have to be compact, whence reductive by [HKL13] Theorem 4.11] and even semisimple, if the diagram be irreducible, thus providing an upper bound for the $\mathbb{R}$-dimension via the maximal compact Lie subalgebras of the Clifford algebras. As it turns out, the lower and the upper bounds coincide, providing the following Cartan–Bott periodicity.

**Theorem A** (Cartan–Bott periodicity of the $E_n$ series). Let $n \in \mathbb{N}$ with $n \geq 4$, let $\mathfrak{k}$ be the maximal compact Lie subalgebra of the split real Kac–Moody Lie algebra of type $E_n$, let $C = C(\mathbb{R}^n, q)$ be the Clifford algebra with respect to the standard positive definite quadratic form $q$ and let $\rho : \mathfrak{k} \to C$ be the standard generalized spin representation.

Then $\text{im}(\rho)$ is isomorphic to...
Along the way we arrive at a structural explanation for the isomorphism types of the maximal compact Lie subalgebras of the semisimple split real Lie algebras of types $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$, $E_6$, $E_7$, $E_8$.

**Theorem B.** The maximal compact Lie subalgebras of the semisimple split real Lie algebras of types $A_2 \oplus A_1$, $A_4$, $D_5$, $E_6$, $E_7$, $E_8$ are isomorphic to $u(2)$, $sp(2) \cong so(5)$, $sp(2) \oplus sp(2) \cong so(5) \oplus so(5)$, $sp(4)$, $su(8)$, $so(16)$, respectively.

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2. Cartan–Bott periodicity of Clifford algebras

Let \( \mathbb{N} = \{1, 2, 3, \ldots\} \) be the set of natural numbers, and let \( \mathbb{R}, \mathbb{C}, \) resp. \( \mathbb{H} \) denote the reals, complex numbers resp. quaternions. For \( n \in \mathbb{N} \) and a division ring \( \mathbb{D} \), denote by \( M(n, \mathbb{D}) \) the \( \mathbb{D} \)-algebra of \( n \times n \) matrices over \( \mathbb{D} \).

Let \( V \) be an \( \mathbb{R} \)-vector space and \( q: V \to \mathbb{R} \) a quadratic form with associated bilinear form \( b \). Then the Clifford algebra \( C(V, q) \) is defined as \( C(V, q) := T(V)/(vw + wv - b(v, w)) \) where \( T(V) \) is the tensor algebra of \( V \); cf. [KY05, Section 4.3], [LM89, Chapter 1, §1].

Let \( V = \mathbb{R}^n \) with standard basis vectors \( v_i \), let \( q = x_1^2 + \cdots + x_n^2 \). Then in \( C(V, q) \) we have \( v_i^2 = 1 \) and \( v_i v_j = -v_j v_i \).

**Proposition 2.1** (Cartan–Bott periodicity). For \( n \geq 2 \), the Clifford algebra \( C(\mathbb{R}^n, q) \) is isomorphic to the following algebra:

\[
(0) \, \mathbb{R} \otimes_\mathbb{R} M(2^\infty, \mathbb{R}), \text{ if } n \equiv 0 \pmod{8}, \\
(1) \, (\mathbb{R} \oplus \mathbb{R}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 1 \pmod{8}, \\
(2) \, M(2, \mathbb{R}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 2 \pmod{8}, \\
(3) \, M(2, \mathbb{C}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 3 \pmod{8}, \\
(4) \, M(2, \mathbb{H}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 4 \pmod{8}, \\
(5) \, (M(2, \mathbb{H}) \oplus M(2, \mathbb{H})) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 5 \pmod{8}, \\
(6) \, M(4, \mathbb{H}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 6 \pmod{8}, \\
(7) \, M(8, \mathbb{C}) \otimes_\mathbb{R} M(2^{n-1}, \mathbb{R}), \text{ if } n \equiv 7 \pmod{8}.
\]

**Proof.** See e.g. [KY05] Proposition 4.4.1 + Table 4.4.1. \( \square \)

Since \( C(V, q) \) is an associative algebra, it becomes a Lie algebra by setting \( [A, B] := AB - BA \). With this in mind, Proposition 2.1 implies the following:

**Corollary 2.2.** For \( n \geq 2 \), the maximal semisimple compact Lie subalgebra of the Clifford algebra \( C(\mathbb{R}^n, q) \) is isomorphic to the following Lie algebra:

\[
(0) \, \mathfrak{so}(2^\infty), \text{ if } n \equiv 0 \pmod{8}, \\
(1) \, \mathfrak{so}(2^{n-1}) \oplus \mathfrak{so}(2^{n-1}), \text{ if } n \equiv 1 \pmod{8}, \\
(2) \, \mathfrak{so}(2^{n-1}), \text{ if } n \equiv 2 \pmod{8}, \\
(3) \, \mathfrak{su}(2^{n-1}), \text{ if } n \equiv 3 \pmod{8}, \\
(4) \, \mathfrak{sp}(2^{n-1}), \text{ if } n \equiv 4 \pmod{8}, \\
(5) \, \mathfrak{sp}(2^{n-2}) \oplus \mathfrak{sp}(2^{n-2}), \text{ if } n \equiv 5 \pmod{8}, \\
(6) \, \mathfrak{sp}(2^{n-2}), \text{ if } n \equiv 6 \pmod{8}, \\
(7) \, \mathfrak{su}(2^{n-2}), \text{ if } n \equiv 7 \pmod{8}.
\]

3. A lower bound on the dimension of a subalgebra

**Definition 3.1.** For \( n \geq 3 \) let \( \mathfrak{m} \) be the Lie subalgebra of \( C(\mathbb{R}^n, q) \) generated by \( v_1 v_2 v_3 \) and by \( v_i v_{i+1} \), \( 1 \leq i < n \).

**Lemma 3.2.** Let \( n \geq 3 \). Then \( \mathfrak{m} \) contains all products of the form \( v_{j_1} v_{j_2} \cdots v_{j_k} \) for \( 2 \leq k \leq n \) and \( k \equiv 2, 3 \pmod{4} \) with pairwise distinct \( j_i \in \{1, \ldots, n\} \), with the possible exception of \( v_1 v_2 \cdots v_n \). The exception can only happen if \( n \equiv 3 \pmod{4} \).

**Proof.** It is well-known that all products \( v_{j_1} v_{j_2}, j_1 \neq j_2 \), are contained in \( \mathfrak{m} \): Indeed, \( A^2 \mathbb{R}^n \cong \mathfrak{so}(n) \) (cf., e.g., [LM89, Proposition 6.1]) is generated as a Lie algebra by the \( v_i v_{i+1}, 1 \leq i < n \) (cf., e.g., [Ber89 Theorem 1.31], [HKL13 Theorem 2.1]).
Moreover, for pairwise distinct $j_1, 1 \leq t \leq k + 1$, one has

$$[v_{j_1} v_{j_2}, v_{j_2} v_{j_3} \cdots v_{j_{k+1}}] = 2v_{j_1} v_{j_3} \cdots v_{j_k}.$$ 

Since re-ordering of the factors simply yields scalar multiples, this shows inductively that, as long as $k + 1 \leq n$, once an arbitrary factor of the form $v_{j_1} v_{j_2} \cdots v_{j_k}$ is contained in the Lie subalgebra, all factors of that form are contained in the Lie subalgebra. This statement is also true in the situation $k = n$, because in that case all factors of that form are scalar multiples of one another.

We finally prove the claim by induction over $k$. For $k = 2$ and $k = 3$, this is obvious. Suppose the claim holds for $k \equiv 3 \pmod{4}$, then the next value for $k$ to consider is $k + 3 \equiv 2 \pmod{4}$. By induction hypothesis $v_4 v_5 \cdots v_{k+3} \in m$ and

$$0 \neq [v_1 v_2 v_3, v_4 v_5 \cdots v_{k+3}] = 2v_1 v_2 v_4 \cdots v_{k+3}.$$ 

If on the other hand the claim holds for $k \equiv 2 \pmod{4}$, then the next value for $k$ to consider is $k + 1 \equiv 3 \pmod{4}$. If $k + 2 \leq n$, then by induction hypothesis $v_3 v_4 \cdots v_{k+2} \in m$ and

$$0 \neq [v_1 v_2 v_3, v_4 v_5 \cdots v_{k+2}] = 2v_1 v_2 v_4 \cdots v_{k+2}.$$ 

That is, the presence of all elements of the form $v_{j_1} v_{j_2} \cdots v_{j_k}$ with pairwise distinct $j_i \in \{1, \ldots, n\}$ inductively allows us to construct all elements of the form $v_{j_1} v_{j_2} \cdots v_{j_k}$ for $k \equiv 2, 3 \pmod{4}$ with pairwise distinct $j_i \in \{1, \ldots, n\}$ for all $k \leq n$, with the possible exception of the situation $k = n \equiv 3 \pmod{4}$, as the element $v_{k+2}$ does not exist in that case.

**Remark 3.3.** It will turn out later, as a consequence of the proof of Theorem [A] based on dimension arguments, that the above elements in fact generate $m$ as an $\mathbb{R}$-vector space and that for $n \equiv 3 \pmod{4}$ the element $v_1 v_2 \cdots v_n$ indeed is not contained in $m$, unless of course $n = 3$.

**Definition 3.4.** For $k \in \{0, 1, 2, 3\}$, let

$$\delta_k : \mathbb{N} \to \mathbb{N} : n \mapsto \sum_{i=k}^{n} \binom{n}{i}.$$ 

**Remark 3.5.** Let $n \in \mathbb{N}$ and let $M$ be a set of size $n$. Then the number of subsets of $M$ of size $k \pmod{4}$ is precisely $\delta_k(n)$. Therefore

$$\delta_0(n) + \delta_1(n) + \delta_2(n) + \delta_3(n) = 2^n.$$ 

**Consequence 3.6.** Let $n \geq 3$. Then

$$\dim m \geq \begin{cases} \delta_2(n) + \delta_3(n) & \text{if } n \not\equiv 3 \pmod{4}, \\ \delta_2(n) + \delta_3(n) - 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

### 4. Combinatorics of Binomial Coefficients

We now turn the lower bound from Consequence 3.6 into a numerically explicit bound by deriving a closed formula in $n$ for the functions $\delta_k$.

**Proposition 4.1.** Let $n \in \mathbb{N}$ and $k \in \{0, 1, 2, 3\}$.

1. If $n \equiv 0 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} & \text{for } k \in \{1, 3\}, \\ 2^{n-2} + (-1)^\frac{n}{2} 2^{\frac{n}{2}-1} & \text{for } k \in \{0, 2\}. \end{cases}$$

2. If $n \equiv 1 \pmod{4}$, then

$$\delta_k(n) = \begin{cases} 2^{n-2} + (-1)^\frac{n+1}{2} 2^{\frac{n+1}{2}} & \text{for } k \in \{0, 1\}, \\ 2^{n-2} - (-1)^\frac{n+2}{2} 2^{\frac{n+2}{2}} & \text{for } k \in \{2, 3\}. \end{cases}$$
(2) If $n \equiv 2 \pmod{4}$, then
$$\delta_k(n) = \begin{cases} 
2^{n-2} & \text{for } k \in \{0, 2\}, \\
2^{n-2} + (-1)^{n-2+1} \frac{n-1}{4} 2^{\frac{n}{2}-1} & \text{for } k \in \{1, 3\}.
\end{cases}$$

(3) If $n \equiv 3 \pmod{4}$, then
$$\delta_k(n) = \begin{cases} 
2^{n-2} - (-1)^{n-2} \frac{n-1}{4} 2^{\frac{n}{2}} & \text{for } k \in \{0, 3\}, \\
2^{n-2} + (-1)^{n-2} \frac{n-1}{4} 2^{\frac{n}{2}} & \text{for } k \in \{1, 2\}.
\end{cases}$$

Proof. Note first that the claimed identities hold for $n \in \{1, 2\}$. The pairing $S \mapsto S \triangle \{n\}$, where $\triangle$ denotes symmetric difference, provides a bijection between the set of subsets of $M$ of even order with the set of subsets of $M$ of odd order. Combined with Remark 3.5 we conclude

(1)
$$\delta_0(n) + \delta_2(n) = \delta_1(n) + \delta_3(n) = 2^{n-1}.$$  

Moreover, the pairing $S \mapsto M \setminus S$ provides a bijection

(i) between the set of subsets of $M$ of order $1 \pmod{4}$ and the set of subsets of $M$ of order $3 \pmod{4}$, if $n \equiv 0 \pmod{4}$,

(ii) between the set of subsets of $M$ of order $0 \pmod{4}$ and the set of subsets of $M$ of order $1 \pmod{4}$ and between the set of subsets of $M$ of order $2 \pmod{4}$ and the set of subsets of $M$ of order $3 \pmod{4}$, if $n \equiv 1 \pmod{4}$,

(iii) between the set of subsets of $M$ of order $0 \pmod{4}$ and the set of subsets of $M$ of order $2 \pmod{4}$, if $n \equiv 2 \pmod{4}$,

(iv) between the set of subsets of $M$ of order $0 \pmod{4}$ and the set of subsets of $M$ of order $1 \pmod{4}$ and the set of subsets of $M$ of order $2 \pmod{4}$, if $n \equiv 3 \pmod{4}$.

Hence
$$\delta_1(n) = \delta_3(n) \quad \text{for } n \equiv 0 \pmod{4},$$
$$\delta_0(n) = \delta_1(n) \quad \text{and} \quad \delta_2(n) = \delta_3(n) \quad \text{for } n \equiv 1 \pmod{4},$$
$$\delta_0(n) = \delta_2(n) \quad \text{for } n \equiv 2 \pmod{4},$$
$$\delta_0(n) = \delta_3(n) \quad \text{and} \quad \delta_1(n) = \delta_2(n) \quad \text{for } n \equiv 3 \pmod{4}.$$  

Together with Equation 1 this already yields the claim for (a), case $k \in \{1, 3\}$ and for (c), case $k \in \{0, 2\}$.

We will now prove case $k = 0$ of (b), (d) by induction, which by the above observations implies all claims made in (b), (d). Let $M$ be a set of order $n + 2$ and let $a, b \in M$ be distinct elements so that $M = M' \cup \{a, b\}$ for a set $M'$ of cardinality $n$. A subset $S \subset M$ of cardinality $0 \pmod{4}$ satisfies exactly one of the following:

(i) $S \subset M'$ has cardinality $0 \pmod{4}$,

(ii) $S \setminus \{a\} \subset M'$ has cardinality $3 \pmod{4}$,

(iii) $S \setminus \{b\} \subset M'$ has cardinality $3 \pmod{4}$,

(iv) $S \setminus \{a, b\} \subset M'$ has cardinality $2 \pmod{4}$.

Hence for $n \equiv 1 \pmod{4}$ resp. $n + 2 \equiv 3 \pmod{4}$ we have
$$\delta_0(n + 2) = \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n)$$
$$= 2^{n-1} + 2\left(2^{n-2} - (-1)^{n-2} \frac{n-1}{4} 2^{\frac{n}{2}-1}\right)$$
$$= 2^n - (-1)^{n-2} \frac{n-1}{4} 2^{\frac{n}{2}}$$
$$= 2^{n+2} - (-1)^{(n+2)-1} \frac{2^{(n+2)-3}}{4} 2^{\frac{(n+2)-3}{2}}.$$
and similarly for \( n \equiv 3 \pmod{4} \) resp. \( n + 2 \equiv 1 \pmod{4} \) we have
\[
\delta_0(n + 2) = \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n) \\
= 2^{n-1} + 2 \left( 2^{n-2} - (-1)^{\frac{n-1}{2} + \frac{n-1}{2}} \right) \\
= 2^n - (-1)^{\frac{n+3}{2} + \frac{n+3}{2}} \\
= 2^{(n+2)-1} - (-1)^{\frac{n+3}{4} + 2\cdot\frac{n+2}{2}}.
\]
Next we prove case \( k = 0 \) of (a) using (c) as an induction hypothesis and afterwards case \( k = 1 \) of (c) using (a) as an induction hypothesis. By the above observations this implies all claims made in (a) and (c).

In order to establish case \( k = 0 \) of (a) we use the exact same combinatorial induction step as above and arrive again at
\[
\delta_0(n + 2) = \delta_0(n) + \delta_2(n) + 2\delta_3(n) = 2^{n-1} + 2\delta_3(n) \\
= 2^{n-1} + 2 \left( 2^{n-2} - (-1)^{\frac{n-1}{2} + \frac{n-1}{2}} \right) \\
= 2^n - (-1)^{\frac{n+3}{2} + \frac{n+3}{2}} \\
= 2^{(n+2)-1} - (-1)^{\frac{n+3}{4} + 2\cdot\frac{n+2}{2}}
\]
as claimed.

In order to establish case \( k = 1 \) of (c) we use the same combinatorial induction step as above but need to observe that if \( S \subset M \) is a subset of cardinality \( 1 \pmod{4} \), then \( S \setminus \{a, b\} \) may have cardinality \( 1 \pmod{4}, 3 \pmod{4} \) or, in two different ways, \( 0 \pmod{4} \). Therefore
\[
\delta_1(n + 2) = 2\delta_0(n) + \delta_1(n) + \delta_3(n) = 2\delta_0(n) + 2^n-1 \\
= 2 \left( 2^{n-2} - (-1)^{\frac{n-1}{2} + \frac{n-1}{2}} \right) + 2^{n-1} \\
= 2^n + (-1)^{\frac{n+3}{2} + \frac{n+3}{2}} \\
= 2^{(n+2)-1} + (-1)^{\frac{n+3}{4} + \frac{1}{2} + \frac{n+2}{2}}.
\]

Combining this with Consequence 3.6 yields the following:

**Consequence 4.2.** Let \( n \in \mathbb{N} \) and \( n \geq 2 \).

1. If \( n \equiv 0 \pmod{8} \), then
   \[
   \dim m \geq \delta_2(n) + \delta_3(n) = 2^{n-2} - 2^{\frac{n-2}{2}} + 2^{n-2} = 2^{\frac{n-2}{2}} (2^{\frac{n-2}{2}} - 1) \\
   = \dim_G(\mathfrak{so}(2^{\frac{n-2}{2}})).
   \]

2. If \( n \equiv 1 \pmod{8} \), then
   \[
   \dim m \geq \delta_2(n) + \delta_3(n) = 2 \left( 2^{n-2} - 2^{\frac{n-1}{2}} \right) = 2^{\frac{n-1}{2}} (2^{\frac{n-1}{2}} - 1) \\
   = \dim_G(\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})).
   \]

3. If \( n \equiv 2 \pmod{8} \), then
   \[
   \dim m \geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-2} - 2^{\frac{n-1}{2}} = 2^{\frac{n-2}{2}} (2^{\frac{n-2}{2}} - 1) \\
   = \dim_G(\mathfrak{so}(2^{\frac{n-2}{2}})).
   \]
(3) If $n \equiv 3 \pmod{8}$, then
\[
\dim m + 1 \geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-1} + 2^{n-2} - 2^{n-1} = 2^{n-1}
\]
\[
= \dim_{\mathbb{R}}(su(2^{n-1})).
\]

(4) If $n \equiv 4 \pmod{8}$, then
\[
\dim m \geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-1} + 2^{n-2} = 2^{n-2} (2^{n-1} + 1)
\]
\[
= \dim_{\mathbb{R}}(sp(2^{n-2})).
\]

(5) If $n \equiv 5 \pmod{8}$, then
\[
\dim m \geq \delta_2(n) + \delta_3(n) = 2\left(2^{n-2} + 2^{n-1}\right) = 2^{n-1} (2^{n-1} + 1)
\]
\[
= \dim_{\mathbb{R}}(sp(2^{n-2})).
\]

(6) If $n \equiv 6 \pmod{8}$, then
\[
\dim m \geq \delta_2(n) + \delta_3(n) = 2^{n-2} + 2^{n-2} + 2^{n-1} = 2^{n-2} (2^{n-1} + 1)
\]
\[
= \dim_{\mathbb{R}}(sp(2^{n-2})).
\]

(7) If $n \equiv 7 \pmod{8}$, then
\[
\dim m + 1 \geq \delta_2(n) + \delta_3(n) = 2^{n-2} - 2^{n-1} + 2^{n-2} + 2^{n-1} = 2^{n-1}
\]
\[
= \dim_{\mathbb{R}}(su(2^{n-1})).
\]

5. Generalized spin representations of the split real $E_n$ series and the resulting quotients

The example of a generalized spin representation of the maximal compact subalgebra of the split real Kac–Moody Lie algebra of type $E_10$ described in [DKN06], [DBHP06], [HKL13] generalizes directly to the whole $E_n$ series as follows.

Let $n \in \mathbb{N}$, let $\mathfrak{g}$ be the split real Kac–Moody Lie algebra of type $E_n$, let $\mathfrak{k}$ be its maximal compact subalgebra, and let $X_i, 1 \leq i \leq n$, be the Berman generators of $\mathfrak{k}$ (cf. [Ber89] Theorem 1.31), [HKL13] Theorem 2.1] enumerated in Bourbaki style as shown in Figure 1. i.e., $X_1, X_2, X_3, \ldots, X_n$ belong to the $A_{n-1}$ subdiagram, generating $\mathfrak{so}(n)$, and $X_2$ to the additional node. As in Section 2, let $q$ be the standard positive definite quadratic form on $\mathbb{R}^n$ and let $\mathcal{C} = \mathcal{C}(\mathbb{R}^n, q)$ be the corresponding Clifford algebra, considered as a Lie algebra.

**Proposition 5.1.** Let $n \geq 3$. The assignment
\[
\begin{align*}
X_1 &\mapsto v_1 v_2, \\
X_2 &\mapsto v_1 v_2 v_3, \\
X_j &\mapsto v_{j-1} v_j \text{ for } 3 \leq j \leq n
\end{align*}
\]
defines a Lie algebra homomorphism $\rho$ from $\mathfrak{k}$ to the Lie subalgebra $\mathfrak{m}$ of $\mathcal{C}$ generated by $v_1 v_2 v_3$ and by $v_i v_{i+1}, 1 \leq i < n$, called the standard generalized spin representation of $\mathfrak{k}$.

**Proof.** The proof is based on the criterion established in [HKL13] Remark 4.5] and is exactly the same as in the $E_{10}$ case discussed in [HKL13] Example 4.1].

**Proof of Theorem 4.1.** By [HKL13] Theorem 4.11] and since $E_n$ is simply laced and connected for $n \geq 4$, the image $\mathfrak{m}$ of $\rho$ is semisimple and compact. By Lemma 4.2 and Consequence 4.2, $\dim_{\mathbb{R}}(\mathfrak{m})$ is at least as large as the dimension of the semisimple maximal compact Lie subalgebra of $\mathcal{C}$ as given in Corollary 2.2. The claim follows.
Proof of Theorem [8] Let \( g \) be a semisimple split real Lie algebra of type \( E_4 = A_4, E_5 = D_5, E_6, E_7 \) or \( E_8 \) and \( g = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n} \) its Iwasawa decomposition. Since \( \dim_{\mathbb{R}}(\mathfrak{t}) = \dim_{\mathbb{R}}(\mathfrak{n}) \), from the combinatorics of the respective root system we conclude that the maximal compact Lie subalgebra \( \mathfrak{k} \) has dimension

\[
10 = \frac{4 \cdot 5}{2} = \frac{2^\frac{3}{2} \cdot (2^\frac{3}{2} + 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5)) \quad \text{if } n = 4,
\]

\[
20 = 2 \cdot 10 = \dim_{\mathbb{R}}(\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5) \oplus \mathfrak{so}(5)) \quad \text{if } n = 5,
\]

\[
36 = 4 \cdot 9 = \frac{2^2(2^2 + 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{sp}(2^2)) \quad \text{if } n = 6,
\]

\[
63 = 2^6 - 1 = \dim_{\mathbb{R}}(\mathfrak{su}(8)) \quad \text{if } n = 7,
\]

\[
120 = \frac{16 \cdot 15}{2} = \frac{2^\frac{5}{2} \cdot (2^\frac{5}{2} - 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{so}(16)) \quad \text{if } n = 8.
\]

For \( n \geq 4 \) we may now apply Theorem [8] and deduce that the standard generalized spin representation \( \rho \) has to be injective in these cases.

This leaves the case \( E_3 = A_2 \oplus A_1 \). Since this diagram is not irreducible, [HKL13, Theorem 4.11] only implies that \( \text{im}(\rho) = m \) is compact but not that it is semisimple (and indeed, it is not). However, \( n = 3 \) is also an exceptional case for Lemma [3.2]. Taking that into consideration, it follows that \( \dim_{\mathbb{R}}(m) \geq 2^2 \) (\( 1, v_1v_2, v_2v_3, v_1v_2v_3 \) is a basis of \( m \)). On the other hand, the Clifford algebra \( \mathcal{C} \) is isomorphic to \( M(2, \mathbb{C}) \), hence \( \mathfrak{t} \cong \mathfrak{u}(2) \), and this has dimension 4. Thus \( \rho \) is also injective when \( n = 3 \). The claim follows. \( \square \)

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