On the Order Optimality of Large-scale Underwater Networks

Won-Yong Shin, Member, IEEE, Daniel E. Lucani, Member, IEEE,
Muriel Médard, Fellow, IEEE, Milica Stojanovic, Fellow, IEEE,
and Vahid Tarokh, Fellow, IEEE

Abstract

Capacity scaling laws are analyzed in an underwater acoustic network with $n$ regularly located nodes on a square, in which both bandwidth and received signal power can be limited significantly. A narrow-band model is assumed where the carrier frequency is allowed to scale as a function of $n$. In the network, we characterize an attenuation parameter that depends on the frequency scaling as well as the transmission distance. Cut-set upper bounds on the throughput scaling are then derived in both extended and dense networks having unit node density and unit area, respectively. It is first analyzed that under extended networks, the upper bound is inversely proportional to the attenuation parameter, thus resulting in a highly power-limited network. Interestingly, it is seen that the upper bound for extended networks is intrinsically related to the attenuation parameter but not the spreading factor. On the other hand, in dense networks, we show that there exists either a bandwidth or power limitation, or both, according to the path-loss attenuation regimes, thus yielding the upper bound that has three fundamentally different operating regimes. Furthermore, we describe an achievable scheme based on the simple nearest-neighbor multi-hop (MH) transmission. We show that under extended networks, the MH scheme is order-optimal for all the operating regimes. An achievability result is also presented in dense networks, where the operating regimes that guarantee the order optimality are identified. It thus turns out that frequency scaling is instrumental towards achieving the order optimality in the regimes. Finally, these scaling results are extended to a random network realization. As a result, vital information for fundamental limits of a variety of underwater network scenarios is provided by showing capacity scaling laws.

Index Terms

Achievability, attenuation parameter, bandwidth, capacity scaling law, carrier frequency, cut-set upper bound, dense network, extended network, multi-hop (MH), operating regime, path-loss attenuation regime, power-limited, underwater acoustic network.
I. INTRODUCTION

Gupta and Kumar’s pioneering work [1] characterized the connection between the number $n$ of nodes and the sum throughput in a large-scale wireless radio network. They showed that the total throughput scales as $\Theta(\sqrt{n/\log n})$ when a multi-hop (MH) routing strategy is used for $n$ source–destination (S–D) pairs randomly distributed in a unit area. MH schemes are then further developed and analyzed in [3]–[9], while their throughput per S–D pair scales far slower than $\Theta(1)$. Recent results [10], [11] have shown that an almost linear throughput in the radio network, i.e. $\Theta(n^{1-\epsilon})$ for an arbitrarily small $\epsilon > 0$, which is the best we can hope for, is achievable by using a hierarchical cooperation (HC) strategy. Besides the schemes in [10], [11], there have been other studies to improve the throughput of wireless radio networks up to a linear scaling in a variety of network scenarios by using novel techniques such as networks with node mobility [12], interference alignment [13], and infrastructure support [14].

Together with the studies in terrestrial radio networks, the interest in study of underwater networks has been growing, due to recent advances in acoustic communication technology [15]–[18]. In underwater acoustic communication systems, both bandwidth and received signal power are severely limited owing to the exponential (rather than polynomial) path-loss attenuation with long propagation distance and the frequency-dependent attenuation. This is a main feature that distinguishes underwater systems from wireless radio links. Hence, the system throughput is affected by not only the transmission distance but also the useful bandwidth. Based on these characteristics, network coding schemes [17], [19], [20] have been presented for underwater acoustic channels, while network coding showed better performance than MH routing in terms of reducing transmission power. MH networking has further been investigated in other simple but realistic network conditions that take into account the practical issues of coding and delay [21], [22].

One natural question is what are the fundamental capabilities of underwater networks in supporting a multiplicity of nodes that wish to communicate concurrently with each other, i.e., multiple S–D pairs, over an acoustic channel. To answer this question, the throughput scaling for underwater networks was first studied [23], where $n$ nodes were arbitrarily located in a planar disk of unit area, as in [1], and the carrier frequency was set to a constant independent of $n$. That work showed an upper bound on the throughput of each node, based on the physical model [1], which scales as $n^{-1/\alpha} e^{-W_0(\Theta(n^{-1/\alpha}))}$, where $\alpha$ corresponds to the spreading factor of the underwater channel and $W_0$ represents the branch zero of the Lambert $W$ function [24]. Since the spreading factor typically has values in the range $1 \leq \alpha \leq 2$ [23], the throughput per node decreases almost as $O(n^{-1/\alpha})$ for large enough $n$, which is considerably faster than the $\Theta(\sqrt{n})$ scaling characterized for wireless radio settings [1].

In this paper, capacity scaling laws for underwater networks are analyzed in two fundamental but different networks: extended networks [4], [5], [10], [25], [26] of unit node density and dense networks [1], [6], [10] of unit area. Unlike the work in [23], the information-theoretic notion of network capacity is adopted in terms of characterizing the model for successful transmission. Especially, we are interested in the case where the carrier frequency scales as a certain function of $n$ in a narrow-band model. Such an assumption leads to a significant change in the scaling behavior owing to the attenuation characteristics. Recently, the optimal capacity scaling of wireless radio networks has been studied in [28], [29] according to operating regimes that are determined by the relationship between the carrier frequency and the number $n$ of nodes. The frequency scaling scenario of our study essentially follows the same arguments as those

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1We use the following notation: i) $f(x) = O(g(x))$ means that there exist constants $C$ and $c$ such that $f(x) \leq C g(x)$ for all $x > c$. ii) $f(x) = o(g(x))$ means that $\lim_{x \to \infty} f(x)/g(x) = 0$. iii) $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$. iv) $f(x) = \omega(g(x))$ if $g(x) = o(f(x))$. v) $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$ [2].

2Note that the HC scheme deals with a subtle issue around quantization, which is not our main concern in this work.

3The Lambert $W$ function is defined to be the inverse of the function $z = W(z)e^{W(z)}$ and the branch satisfying $W(z) \geq -1$ is denoted by $W_0(z)$.

4Since the two networks represent both extreme network realizations, a realistic one would be in-between. In wireless radio networks, the work in [27] generalized the results of [10] to the case where the network area can scale polynomially with the number $n$ of nodes. In underwater networks, we leave this issue for further study.
We aim to study both information-theoretic upper bounds and achievable rate scalings while allowing the frequency scaling with \( n \). To the best of our knowledge, such an attempt has never been done before in underwater networks.

We explicitly characterize an attenuation parameter that depends on the transmission distance and also on the carrier frequency, and then identify fundamental operating regimes depending on the parameter. For networks with \( n \) regularly distributed nodes, we derive upper bounds on the total throughput scaling using the cut-set bound. In extended networks, our upper bound is based on the characteristics of power-limited regimes shown in [10]. We show that the upper bound is inversely proportional to the attenuation parameter. This leads to a highly power-limited network for all the operating regimes, where power consumption is important in determining performance. Interestingly, it is seen that contrary to the case of wireless radio networks, our upper bound heavily depends on the attenuation parameter but not on the spreading factor (corresponding to the path-loss exponent in wireless networks). On the other hand, in dense networks, our upper bound basically follows the arguments, similarly as in [27]: there exists either a bandwidth or power limitation, or both, according to the operating regimes (i.e., path-loss attenuation regimes). Specifically, the network is bandwidth-limited as the path-loss attenuation is small. This is because network performance on the total throughput is roughly linear in the bandwidth. However, at the medium attenuation regime, the network is both bandwidth- and power-limited since the amount of available bandwidth and received signal power affects the performance. Finally, the network becomes power-limited as the attenuation parameter increases exponentially with respect to more than \( \sqrt{n} \), i.e., at the high attenuation regime. Hence, our results indicate that the upper bound for dense networks has three fundamentally different operating regimes according to the attenuation parameter. In addition, to show constructively our achievability result for extended networks, we describe the conventional nearest-neighbor MH transmission [1], and analyze its achievable throughput. We show that under extended networks, the achievable rate scaling based on the MH routing exactly matches the upper bound on the capacity scaling for all the operating regimes. An achievability is also presented in dense networks by utilizing the existing MH routing scheme with a slight modification—we identify the operating regimes such that the optimal capacity scaling is guaranteed. Therefore, this key result indicates that frequency scaling is instrumental towards achieving the order optimality in the regimes. Furthermore, a random network scenario is discussed in this work. We show that under extended random networks, the conventional MH-based achievable scheme is not order-optimal for any operating regimes.

The rest of this paper is organized as follows. Section II describes our system and channel models. In Section III, the cut-set upper bounds on the throughput are derived. In Section IV, the achievable throughput scalings are analyzed. These results are extended to the random network case in Section V. Finally, Section VI summarizes the paper with some concluding remarks.

Throughout this paper the superscript \( H \), \( \lfloor \cdot \rfloor_{ki} \), and \( \| \cdot \|_2 \) denote the conjugate transpose, the \((k,i)\)-th element, and the largest singular value, respectively, of a matrix. \( \mathbf{I}_n \) is the identity matrix of size \( n \times n \), \( \text{tr}(\cdot) \) is the trace, \( \det(\cdot) \) is the determinant, and \( |\mathcal{X}| \) is the cardinality of the set \( \mathcal{X} \). \( \mathbb{C} \) is the field of complex numbers and \( E[\cdot] \) is the expectation. Unless otherwise stated, all logarithms are assumed to be to the base 2.

## II. SYSTEM AND CHANNEL MODELS

We consider a two-dimensional underwater network that consists of \( n \) nodes on a square such that two neighboring nodes are 1 and \( 1/\sqrt{n} \) units of distance apart from each other in extended and dense networks, respectively, i.e., a regular network [25], [26]. We randomly pick a matching of S–D pairs, so that each node is the destination of exactly one source. Each node has an average transmit power constraint \( P \) (constant), and we assume that the channel state information (CSI) is available at all receivers, but not at the transmitters. It is assumed that each node transmits at a rate \( T(n)/n \), where \( T(n) \) denotes the total throughput of the network.

Now let us turn to channel modeling. We assume frequency-flat channel of bandwidth \( W \) Hz around carrier frequency \( f \), which satisfies \( f \gg W \), i.e., narrow-band model. This is a highly simplified model,
but nonetheless one that suffices to demonstrate the fundamental mechanisms that govern capacity scaling. Assuming that all the nodes have perfectly directional transmissions, we also disregard multipath propagation, and simply focus on a line-of-sight channel between each pair of nodes used in [10], [11], [27]. An underwater acoustic channel is characterized by an attenuation that depends on both the distance \( r_{ki} \) between nodes \( i \) and \( k \) \((i, k \in \{1, \cdots, n\})\) and the signal frequency \( f \), and is given by

\[
A(r_{ki}, f) = c_0 r_{ki}^\alpha a(f)^r_{ki}
\]

for some constant \( c_0 > 0 \) independent of \( n \), where \( \alpha \) is the spreading factor and \( a(f) > 1 \) is the absorption coefficient [16]. For analytical tractability, we assume that the spreading factor \( \alpha \) does not change throughout the network, i.e., that it is the same from short to long range transmissions, as in wireless radio networks [1], [4], [10]. The spreading factor describes the geometry of propagation and is typically \( 1 \leq \alpha \leq 2 \)—its commonly used values are \( \alpha = 1 \) for cylindrical spreading, \( \alpha = 2 \) for spherical spreading, and \( \alpha = 1.5 \) for the so-called practical spreading. Note that existing models of wireless networks typically correspond to the case for which \( a(f) = 1 \) (or a positive constant independent of \( n \)) and \( \alpha > 2 \).

A common empirical model gives \( a(f) \) in dB/km for \( f \) in kHz as [16], [30]:

\[
10 \log a(f) = a_0 + a_1 f^2 + a_2 \frac{f^2}{b_1 + f^2} + a_3 \frac{f^2}{b_2 + f^2},
\]

where \( \{a_0, \cdots, a_3, b_1, b_2\} \) are some positive constants independent of \( n \). As stated earlier, we will allow the carrier frequency \( f \) to scale with the number \( n \) of nodes. As a consequence, a wider range of both \( f \) and \( n \) is covered, similarly as in [27]–[29]. In particular, we consider the case where the frequency scales at arbitrarily increasing rates relative to \( n \), which enables us to really capture the dependence on the frequency in performance.\(^5\) The absorption \( a(f) \) is then an increasing function of \( f \) such that

\[
a(f) = \Theta \left( e^{c_1 f^2} \right)
\]

with respect to \( f \) for some constant \( c_1 > 0 \) independent of \( n \).

The noise \( n_i \) at node \( i \in \{1, \cdots, n\} \) in an acoustic channel can be modeled through four basic sources: turbulence, shipping, waves, and thermal noise [16]. We assume that \( n_i \) is the circularly symmetric complex additive colored Gaussian noise with zero mean and power spectral density (PSD) \( N(f) \), and thus the noise is frequency-dependent. The overall PSD of four sources decays linearly on the logarithmic scale in the frequency region 100 Hz – 100 kHz, which is the operating region used by the majority of acoustic systems, and thus is approximately given by [16], [31]

\[
\log N(f) = a_4 - a_5 \log f
\]

for some positive constants \( a_4 \) and \( a_5 \) independent of \( n \).\(^6\) This means that \( N(f) = O(1) \) since

\[
N(f) = \Theta \left( \frac{1}{f a_5} \right)
\]

in terms of \( f \) increasing with \( n \). From (3) and (5), we may then have the following relationship between the absorption \( a(f) \) and the noise PSD \( N(f) \):

\[
N(f) = \Theta \left( \frac{1}{(\log a(f))^{a_5/2}} \right).
\]

From the narrow-band assumption, the received signal \( y_k \) at node \( k \in \{1, \cdots, n\} \) at a given time instance is given by

\[
y_k = \sum_{i \in I} h_{ki} x_i + n_k,
\]

\(^5\) The counterpart of \( \alpha \) in wireless radio channels is the path-loss exponent.

\(^6\) Otherwise, the attenuation parameter \( a(f) \) scales as \( \Theta(1) \) from (2), which is not a matter of interest in this work.

\(^7\) Note that in our operating frequencies, \( a_5 = 1.8 \) is commonly used for the above approximation [16].
where
\[ h_{ki} = \frac{e^{j\theta_{ki}}}{\sqrt{A(r_{ki}, f)}} \] (8)
represents the complex channel between nodes \( i \) and \( k \), \( x_i \in \mathbb{C} \) is the signal transmitted by node \( i \), and \( I \subset \{1, \cdots, n\} \) is the set of simultaneously transmitting nodes. The random phases \( \theta_{ki} \) are uniformly distributed over \([0, 2\pi]\) and independent for different \( i, k, \) and time. We thus assume a narrow-band time-varying channel, whose gain changes to a new independent value for every symbol. Note that this random phase model is based on a far-field assumption [10], [11], [27], [33] which is valid if the wavelength is sufficiently smaller than the minimum node separation.

Based on the above channel characteristics, operating regimes of the network are identified according to the following physical parameters: the absorption \( a(f) \) and the noise PSD \( N(f) \) which are exploited here by choosing the frequency \( f \) based on the number \( n \) of nodes. In other words, if the relationship between \( f \) and \( n \) is specified, then \( a(f) \) and \( N(f) \) can be given by a certain scaling function of \( n \) from (3) and (5), respectively.

III. CUT-SET UPPER BOUND

To access the fundamental limits of an underwater network, new cut-set upper bounds on the total throughput scaling are analyzed from an information-theoretic perspective [33]. Consider a given cut \( L \) dividing the network area into two equal halves, as in [10], [27] (see Figs. 1 and 2 for extended and dense networks, respectively). Under the cut \( L \), source nodes are on the left, while all nodes on the right are destinations. In this case, we have an \( \Theta(n) \times \Theta(n) \) multiple-input multiple-output (MIMO) channel between the two sets of nodes separated by the cut.

A. Extended Networks

In this subsection, an upper bound based on the power transfer argument [10] is established for extended networks, where the information flow for a given cut \( L \) is proportional to the total received signal power from source nodes. Note, however, that the present problem is not equivalent to the conventional extended network framework [10] due to quite different channel characteristics, and the main result is shown here in a somewhat different way by providing a simpler derivation than that of [10].

As illustrated in Fig. 1 let \( S_L \) and \( D_L \) denote the sets of sources and destinations, respectively, for the cut \( L \) in an extended network. We then take into account an approach based on the amount of power transferred across the network according to different operating regimes, i.e., path-loss attenuation regimes. As pointed out in [10], the information transfer from \( S_L \) to \( D_L \) is highly power-limited since all the nodes in the set \( D_L \) are ill-connected to the left-half network in terms of power. This implies that the information transfer is bounded by the total received power transfer, rather than the cardinality of the set \( D_L \). For the cut \( L \), the total throughput \( T(n) \) for sources on the left is bounded by the (ergodic) capacity of the MIMO channel between \( S_L \) and \( D_L \) under time-varying channel assumption, and thus is given by

\[ T(n) \leq \max_{Q_L \succeq 0} E \left[ \log \det \left( I_{n/2} + \frac{1}{N(f)} H_L Q_L H_L^H \right) \right], \] (9)

where \( H_L \) is the matrix with entries \([H_L]_{ki} = h_{ki}\) for \( i \in S_L, k \in D_L \), and \( Q_L \in \mathbb{C}^{\Theta(n) \times \Theta(n)} \) is the positive semi-definite input signal covariance matrix whose \( k \)-th diagonal element satisfies \([Q_L]_{kk} \leq P\) for \( k \in S_L \).

The relationship (9) will be further specified in Theorem 1. Before that, we first apply the techniques of [26], [34] to obtain the total power transfer of the set \( D_L \). These techniques involve the design of the

*In [32], instead of simply taking the far-field assumption, the physical limit of wireless radio networks has been studied under certain conditions on scattering elements. Further investigation is also required to see whether this assumption is valid for underwater networks of unit node density in the limit of large number \( n \) of nodes.
optimal input signal covariance matrix $Q_L$ in terms of maximizing the upper bound (9) on the capacity. If the matrix $H_L$ has independent entries, each $h_{ki}$ of which is a proper complex random variable [35], and has the same distribution as $-h_{ki}$ for $i \in S_L, k \in D_L$, then the optimal $Q_L$ is diagonal, i.e., the maximum in (9) is attained with $[Q_L]_{kk} = P$ for $k \in S_L$, where $Q_L$ is the diagonal matrix. We start from the following lemma.

Lemma 1: Each element $h_{ki}$ of the channel matrix $H_L$ is a proper complex random variable, where $i \in S_L, k \in D_L$.

The proof of this lemma is presented in Appendix A. It is readily proved that $h_{ki}$ has the same distribution as $-h_{ki}$ for all $i$ and $k$ since the random phases $\theta_{ki}$ are uniformly distributed over $[0, 2\pi)$. Thus, using the result of Lemma 1, we obtain the following result.

Lemma 2: The optimal input signal covariance matrix $Q_L$ that maximizes the upper bound (9) is unique and is given by the diagonal $Q_L$ with entries $[Q_L]_{kk} = P$ for $k \in S_L$.

We refer to Section III of [34] for the detailed proof. From Lemma 2, the expression (9) is then rewritten as

$$T(n) \leq E \left[ \log \det \left( I_{n/2} + \frac{1}{N(f)}H_LQ_LH_L^H \right) \right]$$

$$= E \left[ \log \det \left( I_{n/2} + \frac{P}{N(f)}H_LH_L^H \right) \right]$$

$$\leq E \left[ \sum_{k \in D_L} \log \left( 1 + \frac{P}{N(f)} \sum_{i \in S_L} |h_{ki}|^2 \right) \right]$$

$$= \sum_{k \in D_L} \log \left( 1 + \frac{P}{N(f)} \sum_{i \in S_L} \frac{1}{A(r_{ki}, f)} \right)$$

$$\leq \sum_{k \in D_L} \sum_{i \in S_L} \frac{P}{A(r_{ki}, f)N(f)}$$

(10)

where the second inequality is obtained by applying generalized Hadamard’s inequality [36] as in [10], [26]. The last two steps come from (1) and the fact that $\log(1 + x) \leq x$ for any $x$, which is only tight as $x$ is small. Note that the right-hand side (RHS) of (10) represents the total amount of received signal-to-noise ratio (SNR) from the set $S_L$ of sources to the set $D_L$ of destinations for a given cut $L$. To further compute (10), we define the following parameter

$$P_L^{(k)} = \frac{P}{c_0} \sum_{i \in S_L} r_{ki}^{-\alpha} a(f)^{-r_{ki}}$$

(11)

for some constant $c_0 > 0$ independent of $n$, which corresponds to the total power received from the signal sent by all the sources $i \in S_L$ at node $k$ on the right (see (1) and (3)). For convenience, we now index the node positions such that the source and destination nodes under the cut $L$ are located at positions $(-i_x + 1, i_y)$ and $(k_x, k_y)$, respectively, for $i_x, k_x = 1, \cdots, \sqrt{n}/2$ and $i_y, k_y = 1, \cdots, \sqrt{n}$. The scaling result of $P_L^{(k)}$ defined in (11) can then be derived as follows.

Lemma 3: In an extended network, the term $P_L^{(k)}$ in (11) is given by

$$P_L^{(k)} = O \left( k_x^{-\alpha} a(f)^{-k_x} \right),$$

where $k_x$ represents the horizontal coordinate of node $k \in D_L$ for $k_x = 1, \cdots, \sqrt{n}/2$.

The proof of this lemma is presented in Appendix B. We are now ready to show the cut-set upper bound in extended networks.
Theorem 1: For an underwater regular network of unit node density, the total throughput $T(n)$ is upper-bounded by

$$T(n) \leq \frac{c_2 \sqrt{n}}{a(f)N(f)},$$

where $c_2 > 0$ is some constant independent of $n$.

Proof: From (1) and (10)–(12), we obtain the following upper bound on the total throughput $T(n)$:

$$T(n) \leq \frac{1}{N(f)} \sum_{k \in D_L} P_L^{(k)}$$

$$\leq \frac{1}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \sum_{k_y=1}^{\sqrt{n}} \frac{1}{k_x^{2a-1}a(f)^{k_x}}$$

$$\leq c_3 P \frac{\sqrt{n}}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \frac{1}{a(f)^{k_x}}$$

$$\leq \frac{c_3 P \sqrt{n}}{N(f)} \frac{1}{a(f)^{\alpha} - 1}$$

$$\leq \frac{c_4 P \sqrt{n}}{a(f)N(f)},$$

where $c_3$ and $c_4$ are some positive constants independent of $n$, which is equal to (13). This completes the proof of the theorem.

We remark that this upper bound is expressed as a function of the absorption $a(f)$ and the noise PSD $N(f)$, whereas an upper bound for wireless radio networks depends only on the constant value $\alpha$ [10].

In addition, using (3), (5), and (6) in (13) results in two other expressions on the total throughput $T(n) = O(\sqrt{n}(\log a(f))^{a_5/2})$ and $T(n) = O(\sqrt{n}f^{a_5})$ for some positive constants $c_1$ and $a_5$ shown in (3) and (4), respectively. Hence, from (14), it is seen that the upper bound is inversely proportional to the attenuation parameter $a(f)$ and decays fast with increasing $a(f)$, thereby leading to a highly power-limited network irrespective of the parameter $a(f)$.

B. Dense Networks

In a dense network, it is necessary to narrow down the class of S–D pairs according to their Euclidean distance to obtain a tight upper bound. In this subsection, we derive a new upper bound based on hybrid approaches that consider either the sum of the capacities of the multiple-input single-output (MISO) channel between transmitters and each receiver or the amount of power transferred across the network according to operating regimes, similarly as in [27].

For the cut $L$, the total throughput $T(n)$ for sources on the left half is bounded by the capacity of the MIMO channel between $S_L$ and $D_L$, corresponding to the sets of sources and destinations, respectively, and thus is given by (9). In the extended network framework, upper bounding the capacity by the total
received SNR yields a tight bound due to poor power connections for all the operating regimes. In a dense network, however, we may have arbitrarily high received SNR for nodes in the set $D_L$ that are located close to the cut, or even for all the nodes, depending on the path-loss attenuation regimes, and thus need the separation between destination nodes that are well- and ill-connected to the left-half network in terms of power. More precisely, the set $D_L$ of destinations is partitioned into two groups $D_{L,1}$ and $D_{L,2}$ according to their location, as illustrated in Fig. 2. Then, since Lemmas 1 and 2 also hold for the dense network, by applying generalized Hadamard’s inequality [36], we have

$$T(n) \leq \max_{Q_L \geq 0} E \left[ \log \det \left( I_n/2 + \frac{1}{N(f)} H_L Q_L H_L^H \right) \right]$$

$$\leq E \left[ \log \det \left( I_n/2 + \frac{P}{N(f)} H_L H_L^H \right) \right]$$

$$\leq E \left[ \log \det \left( I_{|D_{L,1}|} + \frac{P}{N(f)} H_{L,1} H_{L,1}^H \right) \right]$$

$$+ E \left[ \log \det \left( I_{|D_{L,2}|} + \frac{P}{N(f)} H_{L,2} H_{L,2}^H \right) \right].$$

(15)

where $H_{L,l}$ is the matrix with entries $[H_{L,l}]_{k,i} = h_{ki}$ for $i \in S_L$, $k \in D_{L,l}$, and $l = 1, 2$. Note that the first and second terms in the RHS of (15) represent the information transfer from $S_L$ to $D_{L,1}$ and from $S_L$ to $D_{L,2}$, respectively. Here, $D_{L,1}$ denotes the set of destinations located on the rectangular slab of width $x_L/\sqrt{n}$ immediately to the right of the centerline (cut), where $x_L \in \{0, 1, \ldots, \sqrt{n}/2\}$. The set $D_{L,2}$ is given by $D_L \setminus D_{L,1}$. It then follows that $|D_{L,1}| = x_L \sqrt{n}$ and $|D_{L,2}| = (\sqrt{n}/2 - x_L) \sqrt{n}$.

Let $T_i(n)$ denote the $l$-th term in the RHS of (15) for $l \in \{1, 2\}$. It is then reasonable to bound $T_1(n)$ by the cardinality of the set $D_{L,1}$ rather than the total received SNR. In contrast, using the power transfer argument for the term $T_2(n)$, as in the extended network case, will lead to a tight upper bound. It is thus important to set the parameter $x_L$ according to the attenuation parameter $a(f)$, based on the selection rule for $x_L$ [27], so that only $D_{L,1}$ contains the destination nodes with high received SNR. To be specific, we need to decide whether the SNR received by a destination $k \in D_L$ from the set $S_L$ of sources, denoted by $P_L^{(k)}/N(f)$, is larger than one, because degrees-of-freedom (DoFs) (also known as capacity pre-log factor) of the MISO channel are limited to one. If destination node $k$ has the total received SNR greater than one, i.e., $P_L^{(k)} = \omega(N(f))$, then it belongs to $D_{L,1}$. Otherwise, it follows that $k \in D_{L,2}$.

For analytical tractability, suppose that

$$a(f) = \Theta \left( (1 + \epsilon_0)^{n^\beta} \right) \quad \text{for} \quad \beta \in [0, \infty),$$

(16)

where $\epsilon_0 > 0$ is an arbitrarily small constant, which represents all the operating regimes with varying $\beta$. As before, let us index the node positions such that the source and destination nodes are located at positions $(i_x/\sqrt{n}, i_y/\sqrt{n})$ and $(k_x/\sqrt{n}, k_y/\sqrt{n})$, respectively, for $i_x, k_x = 1, \ldots, \sqrt{n}/2$ and $i_y, k_y = 1, \ldots, \sqrt{n}$. We then obtain the following scaling results for $P_L^{(k)}$ as shown below.

**Lemma 4:** In a dense network, the term $P_L^{(k)}$ in (11) is given by

$$P_L^{(k)} = \begin{cases} 
O(n) & \text{if } 1 \leq \alpha < 2 \text{ and } k_x = o(n^{1/2 - \beta + \epsilon}) \\
O(n \log n) & \text{if } \alpha = 2 \text{ and } k_x = o(n^{1/2 - \beta + \epsilon}) \\
O(n^{\alpha/2} (1 + \epsilon_0)^{n^{\beta - 1/2}} \max \left\{ 1, n^{1/2 - \beta} \right\}) & \text{if } k_x = \Omega \left( n^{1/2 - \beta + \epsilon} \right)
\end{cases}$$

(17)

and

$$P_L^{(k)} = \begin{cases} 
\Omega \left( n^{\alpha/2 - 2/\epsilon} \max \left\{ 1, n^{1/2 - 2/\epsilon} \right\} \right) & \text{if } k_x = o(n^{1/2 - \beta + \epsilon}) \\
\Omega \left( \frac{1}{(1 + \epsilon_0)^{n^{\beta - 1/2}}} \max \left\{ 1, n^{1/2 - \beta} \right\} \right) & \text{if } k_x = \Omega \left( n^{1/2 - \beta + \epsilon} \right)
\end{cases}$$

(18)
for arbitrarily small positive constants $\epsilon$ and $\epsilon_0$, where $x_k/\sqrt{n}$ is the horizontal coordinate of node $k \in D_{L,2}$.

The proof of this lemma is presented in Appendix C. Although the upper and lower bounds for $P_L^{(k)}$ are not identical to each other, showing these scaling results is sufficient to make a decision on $x_L$ according to the operating regimes. It is seen from Lemma 4 that when $x_k = o\left(n^{1/2-\beta+\epsilon}\right)$, $P_L^{(k)}$ does not depend on the parameter $\beta$ (or $a(f)$), while for $x_k = \Omega\left(n^{1/2-\beta+\epsilon}\right)$, node $k \in D_{L,2}$ gets ill-connected to the left half in terms of power since $P_L^{(k)}$ decreases exponentially with $n$. More specifically, when $x_k = o\left(n^{1/2-\beta+\epsilon}\right)$, it follows that $P_L^{(k)} = \omega(n^{\alpha\beta})$ from (18), resulting in $P_L^{(k)} = \omega(N(f))$ due to $N(f) = O(1)$. In contrast, under the condition $x_k = \Omega\left(n^{1/2-\beta+\epsilon}\right)$, it is observed from (17) that $P_L^{(k)}$ is exponentially decaying as a function of $n$, thus leading to $P_L^{(k)} = o(N(f))$. As a consequence, using the result of Lemma 4 three different regimes are identified and the selected $x_L$ is specified accordingly:

$$x_L = \begin{cases} \sqrt{n/2} & \text{if } \beta = 0 \\ n^{1/2-\beta+\epsilon} & \text{if } 0 < \beta \leq 1/2 \\ 0 & \text{if } \beta > 1/2 \end{cases}$$

for an arbitrarily small $\epsilon > 0$. It is now possible to show the proposed cut-set upper bound in dense networks.

**Theorem 2:** Consider an underwater regular network of unit area. Then, the upper bound on the total throughput $T(n)$ is given by

$$T(n) = \begin{cases} O(n \log n) & \text{if } \beta = 0 \\ O\left(n^{1-\beta+\epsilon} \log n\right) & \text{if } 0 < \beta \leq 1/2 \\ O\left(n^{(1+\alpha+\beta\alpha_5)/2} \log n\right) & \text{if } \beta > 1/2, \end{cases}$$

where $\epsilon$ and $\epsilon_0$ are arbitrarily small positive constants, and $\alpha_5$ and $\beta$ are defined in (4) and (16), respectively.

**Proof:** We first compute the first term $T_1(n)$ in (15), focusing on the case for $0 \leq \beta \leq 1/2$ since otherwise $T_1(n) = 0$. Since the nodes in the set $D_{L,1}$ have good power connections to the left-half network and the information transfer to $D_{L,1}$ is limited in bandwidth (but not power), the term $T_1(n)$ is upper-bounded by the sum of the capacities of the MISO channels. More specifically, by generalized Hadamard’s inequality [36], $T_1(n)$ can be easily bounded by

$$T_1(n) \leq \sum_{k \in D_{L,1}} \log \left(1 + \frac{P}{N(f)} \sum_{i \in S_L} \frac{1}{A(r_{ki}, f)} \right) \leq x_L \sqrt{n} \log \left(1 + \frac{P \sqrt{n}^{\alpha/2+1}}{a(f) \sqrt{n} N(f)} \right) \leq x_L \sqrt{n} \log \left(1 + \frac{P \sqrt{n}^{\alpha/2+1}}{N(f)} \right) \leq c_5 x_L \sqrt{n} \log n$$

for some constant $c_5 > 0$ independent of $n$, where the last two steps are obtained from the fact that $0 < a(f) \leq 1$ and $N(f)$ tends to decrease polynomially with $n$ from the relation in (6). The upper bound for the second term $T_2(n)$ in (15) is now derived under the condition $\beta \in (0, \infty)$. Similarly as in the steps of (10), we have

$$T_2(n) \leq \sum_{k \in D_{L,2}} \log \left(1 + \frac{P}{N(f)} \sum_{i \in S_L} \frac{1}{A(r_{ki}, f)} \right) \leq \sum_{k \in D_{L,2}} \sum_{i \in S_L} \frac{P}{A(r_{ki}, f) N(f)} = \frac{1}{N(f)} \sum_{k \in D_{L,2}} P_L^{(k)},$$

(22)
which corresponds to the sum of the total received SNR from the left-half network to the destination set $D_{L,2}$. Hence, combining the two bounds (21) and (22) along with the choices for $x_L$ specified in (19), we obtain the following upper bound on the total throughput $T(n)$:

$$
T(n) \leq \begin{cases} 
    c_5 n \log n + \frac{1}{N(f)} \sum_{k \in D_{L,2}} P_L^{(k)} & \text{if } \beta = 0 \\
    \frac{1}{N(f)} \sum_{k \in D_L} P_L^{(k)} & \text{if } 0 < \beta \leq 1/2 \\
    \frac{c_5 n \log n}{c_5 n^{1-\beta+\epsilon} \log n + \frac{1}{N(f)} \sum_{k \in D_L} P_L^{(k)}} & \text{if } \beta > 1/2 
\end{cases}
$$

for an arbitrarily small $\epsilon > 0$ and some constant $c_6 > 0$ independent of $n$, where the equality comes from (17). The second inequality holds due to the fact that the term $\frac{1}{N(f)} \sum_{k \in D_{L,2}} P_L^{(k)}$ tends to decay exponentially with $n$ under the conditions $0 < \beta \leq 1/2$ and $x_L \leq k_x \leq \sqrt{n}/2$, and thus the total is dominated by the information transfer $T_1(n)$ in (21). Now let us focus on the last line of (23), which corresponds to the total amount of SNR received by all nodes for the condition $\beta > 1/2$. For $\beta > 1/2$, using the two relationships (6) and (16) follows that

$$
\frac{n^{1+\alpha/2}}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \frac{1}{(1 + \epsilon_0)^{\beta-1/2}} \leq \frac{n^{1+\alpha/2}}{N(f)} \frac{1}{(1 + \epsilon_0)^{\beta-1/2}} - 1 
\leq \frac{c_7 n^{1+\alpha+\beta\alpha_0/2}}{(1 + \epsilon_0)^{\beta-1/2}}
$$

for some constant $c_7 > 0$ independent of $n$, where the second inequality holds due to $(1 + \epsilon_0)^{\beta-1/2} = \omega(1)$ under the condition. This coincides with the result shown in (20), which completes the proof.

Note that the upper bound [10] for wireless radio networks of unit area is given by $O(n \log n)$, which is the same as the case with $\beta = 0$ (or equivalently $\alpha(f) = \Theta(1)$) in the dense underwater network. Now let us discuss the fundamental limits of the network according to three different operating regimes shown in (20).

**Remark 1:** The upper bound on the total capacity scaling is illustrated in Fig. 3 versus the parameter $\beta$ (logarithmic terms are omitted for convenience). We first address the regime $\beta = 0$ (i.e., low path-loss attenuation regime), in which the upper bound on $T(n)$ is active with $x_L = \sqrt{n}/2$, or equivalently $D_{L,1} = D_L$, while $T_2(n) = 0$. In this case, the total throughput of the network is limited by the DoFs of the $\Theta(n) \times \Theta(n)$ MIMO channel between $S_L$ and $D_L$, and is roughly linear in the bandwidth, thus resulting in a bandwidth-limited network. In particular, our interest is in the operating regimes for which the network becomes power-limited as $\beta > 0$. In the second regime $0 < \beta \leq 1/2$ (i.e., medium path-loss attenuation regime), as pointed out in the proof of Theorem 2, the upper bound on $T(n)$ is dominated by the information transfer from $S_L$ to $D_{L,1}$, that is, the term $T_1(n)$ in (21) contributes more than $T_2(n)$ in (22). The total throughput is thus limited by the DoFs of the MIMO channel between $S_L$ and $D_{L,1}$, since more available bandwidth leads to an increment in $T_1(n)$. As a consequence, in this regime, the network is both bandwidth- and power-limited. In the third regime $\beta > 1/2$ (i.e., high path-loss attenuation regime), the upper bound (22) is active with $x_L = 0$, or equivalently $D_{L,2} = D_L$, while $T_1(n) = 0$. The information transfer to $D_L$ is thus totally limited by the sum of the total received SNR from the left-half network to
the destination nodes in $D_L$. In other words, in the third regime, the network is limited in power, but not in bandwidth.

Note that the upper bound on $T(n)$ decays polynomially with increasing $\beta$ in the regime $0 < \beta \leq 1/2$, while it drops off exponentially when $\beta > 1/2$. In addition, two other expressions on the total throughput $T(n)$ are summarized as follows.

**Remark 2:** From (6) and (16), the upper bound and the corresponding operating regimes can also be presented below in terms of the attenuation parameter $a(f)$:

$$T(n) = \begin{cases} O(n \log n) & \text{if } a(f) = \Theta(1) \\
O\left(\frac{n^{1+\epsilon} \log a(f)}{\log a(f)}\right) & \text{if } a(f) = \omega(1) \text{ and } a(f) = O\left((1 + \epsilon_0)^{\sqrt{n}}\right) \\
O\left(\frac{n^{(1+\alpha)/2} \log a(f)}{a(f)^{1/2}}\right)^{\alpha_5/2} & \text{if } a(f) = \omega\left((1 + \epsilon_0)^{\sqrt{n}}\right).
\end{cases}$$

Note that as $a(f) = \omega\left((1 + \epsilon_0)^{\sqrt{n}}\right)$, we also obtain

$$T(n) = O\left(\frac{n^{(1+\alpha)/2}}{a(f)^{1/2} \sqrt{n} N(f)}\right),$$

which is expressed as a function of the spreading factor $\alpha$ as well as the absorption $a(f)$ and the noise PSD $N(f)$. Using (3) and (5) further yields the following expression

$$T(n) = \begin{cases} O(n \log n) & \text{if } f = \Theta(1) \\
O\left(\frac{n^{1+\epsilon} \log n}{f^2}\right) & \text{if } f = \omega(1) \text{ and } f = O\left(n^{1/4}\right) \\
O\left(\frac{n^{(1+\alpha)/2} \log n}{e^{1/f^2} \sqrt{n}}\right) & \text{if } f = \omega\left(n^{1/4}\right),
\end{cases}$$

which represents the upper bound for the operating regimes identified by frequency scaling.

### IV. Achievability Result

In this section, to show the order optimality in underwater networks, we analyze the achievable throughput scaling for both extended and dense networks with the existing transmission scheme, commonly used in wireless radio networks. Under an extended regular network, the conventional MH transmission is introduced and its optimal achievability result is shown. Under a dense regular network, we examine the operating regimes for which the achievable throughput matches the upper bound shown in Section III-B.

#### A. Extended Networks

The nearest-neighbor MH routing protocol [1] will be briefly described to show the order optimality. The basic procedure of the MH protocol under our extended regular network is as follows:

- Divide the network into square routing cells, each of which has unit area.
- Draw a line connecting a S–D pair. A source transmits a packet to its destination using the nodes in the adjacent cells passing through the line.
- Use the full transmit power at each node, i.e., the transmit power $P$.

The achievable rate of MH is now shown by quantifying the amount of interference.

**Lemma 5:** Consider an extended regular network that uses the nearest-neighbor MH protocol. Then, the total interference power $P_I$ from other simultaneously transmitting nodes, corresponding to the set $I \subset \{1, \ldots, n\}$, is upper-bounded by $\Theta(1/a(f))$, where $a(f)$ denotes the absorption coefficient greater than 1.
Proof: There are $8k$ interfering routing cells, each of which includes one node, in the $k$-th layer $l_k$ of the network as illustrated in Fig. 4. Then from (1), (7), and (8), the total interference power $P_I$ at each node from simultaneously transmitting nodes is upper-bounded by

$$P_I = \sum_{k=1}^{\infty} (8k) \frac{P}{c_0 k^\alpha a(f)^k}$$

$$= \frac{8P}{c_0} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha-1} a(f)^k}$$

$$\leq \frac{8P}{c_0} \sum_{k=1}^{\infty} \frac{1}{a(f)^k}$$

$$\leq \frac{c_8}{a(f)},$$

where $c_0$ and $c_8$ are some positive constants independent of $n$, which completes the proof. \hfill \blacksquare

Note that the received signal power no longer decays polynomially but rather exponentially with propagation distance in our network. This implies that the absorption term $a(f)$ in (1) will play an important role in determining the performance. It is also seen that the upper bound on $P_I$ does not depend on the spreading factor $\alpha$. Using Lemma 5, it is now possible to simply obtain a lower bound on the capacity scaling in the network, and hence the following result presents the achievable rates under the MH protocol.

**Theorem 3:** In an underwater regular network of unit node density,

$$T(n) = \Omega \left( \frac{n^{1/2}}{a(f) N(f)} \right)$$

(24)

is achievable.

Proof: Suppose that the nearest-neighbor MH protocol is used. To get a lower bound on the capacity scaling, the signal-to-interference-and-noise ratio (SINR) seen by receiver $i \in \{1, \ldots, n\}$ is computed as a function of the absorption $a(f)$ and the PSD $N(f)$ of noise $n_i$. Since the Gaussian is the worst additive noise [37], [38], assuming it lower-bounds the throughput. Hence, by assuming full CSI at the receiver, from (1), (7), and (8), the achievable throughput per S–D pair is lower-bounded by

$$\log(1 + \text{SINR}) \geq \log \left( 1 + \frac{P/(c_0 a(f))}{N(f) + c_8/a(f)} \right)$$

$$\geq \log \left( 1 + \frac{c_9 P}{a(f) N(f)} \right),$$

for some positive constants $c_0$, $c_8$, and $c_9$ independent of $n$, where the second inequality is obtained from the relationship (6) between $a(f)$ and $N(f)$, resulting in $N(f) = \Omega(1/a(f))$. Due to the fact that $\log(1 + x) = (\log e)x + O(x^2)$ for small $x > 0$, the rate of

$$\Omega \left( \frac{1}{a(f) N(f)} \right)$$

is thus provided for each S–D pair. Since the number of hops per S–D pair is given by $O(\sqrt{n})$, there exist $\Omega(\sqrt{n})$ source nodes that can be active simultaneously, and therefore the total throughput is finally given by (24), which completes the proof of the theorem. \hfill \blacksquare

Now it is examined how the upper bound shown in Section [II-A] is close to the achievable throughput scaling.
Remark 3: Based on Theorems 1 and 3, it is easy to see that the achievable rate and the upper bound are of exactly the same order. MH is therefore order-optimal in regular networks with unit node density for all the attenuation regimes.

We also remark that applying the hierarchical cooperation strategy [10] may not be helpful to improve the achievable throughput due to long-range MIMO transmissions, which severely degrade performance in highly power-limited networks. To be specific, at the top level of the hierarchy, the transmissions between two clusters having distance $O(\sqrt{n})$ become a bottleneck, and thus cause a significant throughput degradation. It is further seen that even with the random phase model, which may enable us to obtain enough DoF gain, the benefit of randomness cannot be exploited because of the power limitation.

B. Dense Networks

From the converse result in Section III-B, it is seen that in dense networks, there exists either a bandwidth or power limitation, or both, according to the path-loss attenuation regimes. Based on the earlier studies [1], [10], [27], [39] for wireless radio networks, it follows that using MH routing is preferred at power-limited regimes, while the HC strategy may have a better performance at bandwidth-limited regimes. Thus, existing schemes need to be used carefully, depending on operating regimes.

In this subsection, the nearest-neighbor MH routing in [1] is described with a slight modification. The basic procedure of the MH protocol under our dense regular network is similar to the extended network case, and is briefly described as follows:

- Divide the network into $n$ square routing cells, each of which has the same area.
- Draw a line connecting an S–D pair.
- At each node, use the transmit power of $P_{\text{min}} \left\{ 1, \frac{a(f) \alpha}{n^{\alpha/2}} \right\}$.

The scheme operates with the full power when $a(f) = \Omega\left(\frac{n^{\alpha/2}}{f^{\alpha/2}}\right)$, the transmit power $Pa(f)^{1/\sqrt{n}}N(f)/n^{\alpha/2}$, which scales slower than $\Theta(1)$, is sufficient so that the received SNR at each node is bounded by 1 (note that having a higher power is unnecessary in terms of throughput improvement).

The amount of interference is now quantified to show the achievable throughput based on MH.

**Lemma 6:** Consider a dense regular network that uses the nearest-neighbor MH protocol. Then, the total interference power $P_I$ from other simultaneously transmitting nodes, corresponding to the set $I \subset \{1, \cdots, n\}$, is bounded by

$$P_I = \begin{cases} 
O\left(\frac{\max\{n^{(1/2-\beta)(2-\alpha)}, \log n\}}{n^{\alpha_s n^{\beta/2}}}ight) & \text{if } 0 \leq \beta < 1/2 \\
O\left(n^{1-\beta a_5/2}\right) & \text{if } \beta = 1/2 \\
O\left(n^{\alpha/2}(1+\epsilon_0)n^{\beta-1/2}\right) & \text{if } \beta > 1/2 
\end{cases}$$

(25)

for an arbitrarily small $\epsilon_0 > 0$, where $a_5$ and $\beta$ are defined in (4) and (16), respectively.

The proof of this lemma is presented in Appendix D. From (6) and (16), we note that when $\beta = 1/2$, it follows that $P_I = O(N(f))$, i.e., $P_I$ is upper-bounded by the PSD $N(f)$ of noise. Using Lemma 6, a lower bound on the capacity scaling can be derived, and hence the following result shows the achievable rates under the MH protocol in a dense network.

9In wireless radio networks of unit node density, the hierarchical cooperation provides a near-optimal throughput scaling for the operating regimes $2 < \alpha < 3$, where $\alpha$ denotes the path-loss exponent that is greater than 2 [10]. Note that the analysis in [10] is valid under the assumption that $\alpha$ is kept at the same value on all levels of hierarchy.
**Theorem 4:** In an underwater regular network of unit area,

\[
T(n) = \begin{cases} 
\Omega \left( \frac{\sqrt{n}}{\max \{n^{1/2 - \beta}(2 - \alpha), \log n \}} \right) & \text{if } 0 \leq \beta < 1/2 \\
\Omega \left( \frac{\sqrt{n}}{n^{1/2 - \beta}(2 - \alpha)} \right) & \text{if } \beta = 1/2 \\
\Omega \left( \frac{n^{1+\alpha+\beta a_5}/2}{(1+\epsilon_0)n^{\beta-1/2}} \right) & \text{if } \beta > 1/2 
\end{cases}
\]  

(26)

is achievable.

**Proof:** Suppose that the nearest-neighbor MH protocol described above is used. Then, from (1), the received signal power \( P_r \) from the desired transmitter is given by

\[
P_r = \frac{P \min \left\{ 1, \frac{a(f)^{1/\sqrt{n}} N(f)}{n^{\alpha/2}} \right\}}{c_0 a(f)^{1/\sqrt{n}}},
\]

which can be rewritten as

\[
\frac{P n^{\alpha/2}}{c_0 (1 + \epsilon_0) n^{\beta-1/2}} \min \left\{ 1, \frac{(1 + \epsilon_0)^{n^{\beta-1/2}}}{n^{\alpha+\beta a_5}/2} \right\}
\]

\[
= \begin{cases} 
\frac{P n^{\alpha/2}}{c_0 n^{\beta-1/2} n^{\alpha+\beta a_5}/2} & \text{if } 0 \leq \beta \leq 1/2 \\
\frac{P n^{\alpha+\beta a_5}/2}{c_0 (1+\epsilon_0)n^{\beta-1/2}} & \text{if } \beta > 1/2
\end{cases}
\]  

(27)

with respect to the parameter \( \beta \) using (6) and (16). A lower bound on the throughput is now obtained using (25) and (27). By assuming the worst case noise, which lower-bounds the transmission rate, and full CSI at the receiver, the achievable throughput per S–D pair is then lower-bounded by

\[
\log(1 + \text{SINR}) = \log \left( 1 + \frac{P_r}{N(f) + P_I} \right)
\]

\[
= \begin{cases} 
\Omega \left( \log \left( 1 + \frac{1}{\max \{n^{1/2 - \beta}(2 - \alpha), \log n \}} \right) \right) & \text{if } 0 \leq \beta < 1/2 \\
\Omega(1) & \text{if } \beta = 1/2 \\
\Omega \left( \log \left( 1 + \frac{n^{\alpha+\beta a_5}/2}{(1+\epsilon_0)n^{\beta-1/2}} \right) \right) & \text{if } \beta > 1/2
\end{cases}
\]

\[
= \begin{cases} 
\Omega \left( \frac{1}{\max \{n^{1/2 - \beta}(2 - \alpha), \log n \}} \right) & \text{if } 0 \leq \beta < 1/2 \\
\Omega(1) & \text{if } \beta = 1/2 \\
\Omega \left( \frac{n^{\alpha+\beta a_5}/2}{(1+\epsilon_0)n^{\beta-1/2}} \right) & \text{if } \beta > 1/2,
\end{cases}
\]

where the second equality holds since \( N(f) = \Theta(n^{-\beta a_5}/2) \) and thus \( P_I = \Theta(P_r) = \Theta(N(f)) \) for \( \beta < 1/2, P_I = \Theta(P_r) = \Theta(N(f)) \) for \( \beta = 1/2, \) and \( P_I = o(N(f)) \) for \( \beta > 1/2. \) The last equality comes from the fact that \( \log(1 + x) = (\log e)x + O(x^2) \) for small \( x > 0. \) Since there are \( \Omega(\sqrt{n}) \) S–D pairs that can be active simultaneously in the network, the total throughput is finally given by (26), which completes the proof. \( \Box \)

Note that the achievable throughput [1] for wireless radio networks of unit area using MH routing is given by \( \Omega(\sqrt{n}) \), which is the same as the case for which \( \beta = 1/2 \) (or equivalently \( a(f) = \Theta ( (1 + \epsilon_0)^{\sqrt{n}}) \)) in a dense underwater network. The lower bound on the total throughput \( T(n) \) is also shown in Fig. 5 according to the parameter \( \beta. \) From this result, an interesting observation follows. To be specific, in the regime \( 0 \leq \beta \leq 1/2, \) the lower bound on \( T(n) \) grows linearly with increasing \( \beta, \) because the total interference power \( P_I \) in (25) tends to decrease as \( \beta \) increases. In this regime, note that \( P_I = \Omega(P_r). \)
Meanwhile, when $\beta > 1/2$, the lower bound reduces rapidly due to the exponential path-loss attenuation in terms of increasing $\beta$.

In addition, similarly as in Section III-B two other expressions on the achievability result are now summarized as in the following.

Remark 4: From (6) and (16), the lower bound on the throughput $T(n)$ and the corresponding operating regimes can also be presented below in terms of the attenuation parameter $a(f)$:

$$T(n) = \begin{cases} 
\Omega \left( \frac{\sqrt{n}}{\max\{a(f)^{\beta}(1+\epsilon_f)\}} \right) & \text{if } a(f) = \Omega(1) \text{ and } a(f) = o \left( (1+\epsilon_0)^{\beta} \right) \\
\Omega \left( \frac{n^{(1+\beta)/2}a(f)^{\beta}}{a(f)^{1/\sqrt{n}}} \right) & \text{if } a(f) = \Theta \left( (1+\epsilon_0)^{\beta} \right) \\
\Omega \left( \frac{n^{(1+\beta)/2}a(f)^{\beta}}{a(f)^{1/\sqrt{n}}} \right) & \text{if } a(f) = \omega \left( (1+\epsilon_0)^{\beta} \right). 
\end{cases}$$

Furthermore, using (3) and (5) follows that

$$T(n) = \begin{cases} 
\Omega \left( \frac{\sqrt{n}}{\max\{a(f)^{\beta}(1+\epsilon_f)\}} \right) & \text{if } f = \Omega(1) \text{ and } f = o \left( n^{1/4} \right) \\
\Omega \left( \frac{n^{(1+\beta)/2}a(f)^{\beta}}{a(f)^{1/\sqrt{n}}} \right) & \text{if } f = \Theta \left( n^{1/4} \right) \\
\Omega \left( \frac{n^{(1+\beta)/2}a(f)^{\beta}}{a(f)^{1/\sqrt{n}}} \right) & \text{if } f = \omega \left( n^{1/4} \right),
\end{cases}$$

which represents the lower bound for the operating regimes obtained from the relationship between the frequency $f$ and the number $n$ of nodes.

Now let us turn to examining how the upper bound shown in Section III-B is close to the achievable throughput scaling.

Remark 5: Based on Theorems 2 and 4, it is seen that if $\beta \geq 1/2$, then the achievable rate of the MH protocol is close to the upper bound up to $n^\epsilon$ for an arbitrarily small $\epsilon > 0$ (note that the two bounds are of exactly the same order especially for $\beta > 1/2$). The condition $\beta \geq 1/2$ corresponds to the high path-loss attenuation regime, and is equivalent to $a(f) = \Omega \left( (1+\epsilon_0)^{\beta} \right)$ or $f = \Omega \left( n^{1/4} \right)$. Therefore, the MH is order-optimal in regular networks of unit area under the aforementioned operating regimes, whereas in extended networks, using MH routing results in the order optimality for all the operating regimes.

Finally, we remark that applying the HC strategy [10] does not guarantee the order optimality in the regime $0 \leq \beta < 1/2$ (i.e., low and medium path-loss attenuation regimes). The primary reason is specified under each operating regime: for the condition $\beta = 0$, following the steps similar to those of Lemma 6 it follows that $P_t = \omega \left( P_t \right)$ at all levels of the hierarchy, thereby resulting in SINR = $o(1)$ for each transmission (the details are not shown in this paper). It is thus not possible to achieve a linear throughput scaling. Now let us focus on the case where $0 < \beta < 1/2$. At the top level of the hierarchy, the transmissions between two clusters having distance $O(1)$ becomes a bottleneck because of the exponential path-loss attenuation with propagation distance. Hence, the achievable throughput of the HC decays exponentially with respect to $n$, which is significantly lower than that in (26).

V. EXTENSION TO RANDOM NETWORKS

In this section, we would like to mention a random network configuration, where $n$ S–D pairs are uniformly and independently distributed on a square of unit node density (i.e., an extended random network).

We first discuss an upper bound for extended random networks. A precise upper bound can be obtained using the binning argument of [10] (refer to Appendix V in [10] for the details). Consider the same cut $L$, which divides the network area into two halves, as in the regular network case. For analytical convenience, we can artificially assume the empty zone $E_L$, in which there are no nodes in the network, consisting of
a rectangular slab of width \(0 < \bar{c} < \frac{1}{\sqrt{2}\log n}\), independent of \(n\), immediately to the right of the centerline (cut), as done in [27] (see Fig. 5). Let us state the following lemma.

**Lemma 7:** Assume a two dimensional extended network where \(n\) nodes are uniformly distributed. When the network area is divided into \(n\) squares of unit area, there are fewer than \(\log n\) nodes in each square with high probability.

Since the result in Lemma 7 depends on the node distribution but not the channel characteristics, the proof essentially follows that presented in [4]. By Lemma 7 we now take into account the network transformation resulting in a regular network with at most \(\log n\) and \(2\log n\) nodes, on the left and right, respectively, at each square vertex except for the empty zone (see Fig. 5). Then, the nodes in each square are moved together onto one vertex of the corresponding square. More specifically, under the cut \(L\), the node displacement is performed in the sense of decreasing the Euclidean distance between source node \(i \in S_L\) and the corresponding destination \(k \in D_L\), as shown in Fig. 5 which will provide an upper bound on \(P_L^{(k)}\) in (11). It is obviously seen that the amount of power transfer under the transformed regular network is greater than that under another regular network with at most \(\log n\) nodes at each vertex, located at integer lattice positions in a square region of area \(n\). Hence, the upper bound for random networks is boosted by at least a logarithmic factor of \(n\) compared to that of regular networks discussed in Section III.

Now we turn our attention to showing an achievable throughput for extended random networks. In this case, the nearest-neighbor MH protocol [1] can also be utilized since our network is highly power-limited. Then, the area of each routing cell needs to scale with \(2\log n\) to guarantee at least one node in a cell [1], [6]. Each routing cell operates based on 9-time division multiple access to avoid causing large interference to its neighboring cells [1], [6]. For the MH routing, since per-hop distance is given by \(\Theta(\sqrt{\log n})\), the received signal power from the intended transmitter and the SINR seen by any receiver are expressed as

\[
\frac{c_{10}P}{(\log n)^{\alpha/2}a(f)^{\delta \sqrt{\log n}}}
\]

and

\[
\Omega\left(\frac{1}{(\log n)^{\alpha/2}a(f)^{\delta \sqrt{\log n}} N(f)}\right),
\]

respectively, for some constants \(c_{10} > 0\) and \(\delta \geq \sqrt{2}\) independent of \(n\). Since the number of hops per S–D pair is given by \(O(\sqrt{n/\log n})\), there exist \(O(\sqrt{n/\log n})\) simultaneously active sources, and thus the total achievable throughput \(T(n)\) is finally given by

\[
T(n) = \Omega\left(\frac{n^{1/2}}{(\log n)^{(\alpha+1)/2}a(f)^{\delta \sqrt{\log n}} N(f)}\right)
\]

for some constant \(\delta \geq \sqrt{2}\) independent of \(n\) (note that this relies on the fact that \(\log(1 + x)\) can be approximated by \(x\) for small \(x > 0\)). Hence, using the MH protocol results in at least a polynomial decrease in the throughput compared to the regular network case shown in Section IV. This comes from the fact that the received signal power tends to be mainly limited due to exponential attenuation with transmission distance \(\Theta(\sqrt{\log n})\). Note that in underwater networks, randomness on the node distribution causes a huge performance degradation on the throughput scaling. Therefore, we may conclude that the
existing MH scheme does not satisfy the order optimality under extended random networks regardless of the attenuation parameter \(a(f)\).

VI. CONCLUSION

The attenuation parameter and the capacity scaling laws have been characterized in a narrow-band channel of underwater acoustic networks. Provided that the frequency \(f\) scales relative to the number \(n\) of nodes, the information-theoretic upper bounds and the achievable throughputs were obtained as functions of the attenuation parameter \(a(f)\) in regular networks. In extended networks, based on the power transfer argument, the upper bound was shown to decrease in inverse proportion to \(a(f)\). In dense networks, the upper bound was derived characterizing three different operating regimes, in which there exists either a bandwidth or power limitation, or both. In addition, to show the achievability result, the nearest-neighbor MH protocol was introduced with a simple modification, and its throughput scaling was analyzed. We proved that the MH protocol is order-optimal in all operating regimes of extended networks and in power-limited regimes (i.e., the case where the frequency \(f\) scales faster than or as \(n^{1/4}\)) of dense networks. Therefore, it turned out that there exists a right frequency scaling that makes our scaling results for underwater acoustic networks to break free from scaling limitations related to the channel characteristics that were described in [23]. Our scaling results were also extended to the random network scenario, where it was shown that the conventional MH scheme does not satisfy the order optimality for all the operating regimes.

APPENDIX

A. Proof of Lemma 7

The following definition is used to simply provide the proof.

**Definition 1 [35]:** A complex random variable \(Y\) is said to be proper if \(\tilde{\Sigma}_Y = 0\), where \(\tilde{\Sigma}_Y\), called the pseudo-covariance, is given by \(E[(Y - E[Y])^2]\).

Since the \((k, i)\)-th element of the channel matrix \(H_L\) is given by (8), it follows that

\[
E \left[ (h_{ki} - E[h_{ki}])^2 \right] = \frac{1}{A(r_{ki}, f)} E \left[ (e^{j\theta_{ki}} - E[e^{j\theta_{ki}}])^2 \right].
\]

From the fact that

\[
E \left[ e^{j\theta_{ki}} \right] = E \left[ \cos(\theta_{ki}) + j \sin(\theta_{ki}) \right] = 0
\]
due to uniformly distributed \(\theta_{ki}\) over \([0, 2\pi]\), we thus have

\[
E \left[ (h_{ki} - E[h_{ki}])^2 \right] = \frac{1}{A(r_{ki}, f)} E \left[ e^{j2\theta_{ki}} \right] = \frac{1}{A(r_{ki}, f)} E \left[ \cos(2\theta_{ki}) + j \sin(2\theta_{ki}) \right] = 0,
\]

which complete the proof, because (28) holds for all \(i \in S_L\) and \(k \in D_L\).

B. Proof of Lemma 3

An upper bound on \(P_L^{(k)}\) can be found by using the node-indexing and layering techniques similar to those shown in Section VI of [39]. As illustrated in Fig. 6, layers are introduced, where the \(i\)-th layer \(l^i\) of the network represents the ring with width 1 drawn based on a destination node \(k \in D_L\), whose coordinate is given by \((k_x, k_y)\), where \(i \in \{1, \cdots, \sqrt{n}\}\). More precisely, the ring is enclosed by the circumferences
of two circles, each of which has radius \(k_x + i\) and \(k_x + i - 1\), respectively, at its same center (see Fig. 6). Then from (11), the term \(P_L^{(k)}\) is given by

\[
P_L^{(k)} = \frac{P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0 \sum_{i_x=1}^{\sqrt{n}/2} \sum_{i_y=1}^{\sqrt{n}} (i_x + k_x - 1)^2 + (i_y - k_y)^2} a(f)^{\alpha/2} a((i_x + k_x - 1)^2 + (i_y - k_y)^2/a)^{\alpha/2}.
\]

It is further assumed that all the nodes in each layer are moved onto the innermost boundary of the corresponding ring, which provides an upper bound for \(P_L^{(k)}\). From the fact that there exist \(\Theta(k_x + i)\) nodes in the layer \(l_i'\), since the area of \(l_i'\) is given by \(\pi(2k_x + 2i - 1)\), \(P_L^{(k)}\) is then upper-bounded by

\[
P_L^{(k)} \leq \frac{P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0 \sum_{i_x=1}^{\sqrt{n}/2} \sum_{i_y=1}^{\sqrt{n}} (i_x + k_x - 1)^2 + (i_y - k_y)^2} \sum_{i'=k_x}^{i+x} \frac{1}{i^{\alpha} a(f)^{i'}}
\]

\[
\leq \frac{2c_{11} P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0 k_x^{\alpha-1}} \sum_{i'=k_x}^{i+x} \frac{1}{i^{\alpha} a(f)^{i'}} + \int_{k_x}^{\infty} \frac{1}{i^{\alpha} a(f)^{i'}} dx
\]

\[
\leq \frac{c_{12} P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{k_x^{\alpha-1} a(f)^{k_x}}
\]

for some positive constants \(c_0\), \(c_{11}\), and \(c_{12}\) independent of \(n\), where the fourth inequality holds since \(a(f) > 1\), which finally yields (12). This completes the proof.

C. Proof of Lemma 4

Upper and lower bounds on \(P_L^{(k)}\) in a dense network are derived by basically following the same node indexing and layering techniques as those in Appendix B. We refer to Fig. 6 for the detailed description (note that the destination nodes are, however, located at positions \((k_x, k_y)\) in dense networks). Similarly to the extended network case, from (11), the term \(P_L^{(k)}\) is then given by

\[
P_L^{(k)} = \frac{P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0 \sum_{i_x=1}^{\sqrt{n}/2} \sum_{i_y=1}^{\sqrt{n}} (i_x + k_x - 1)^2 + (i_y - k_y)^2} n^{\alpha/2} a((i_x + k_x - 1)^2 + (i_y - k_y)^2/a)^{\alpha/2}.
\]

First, focus on how to obtain an upper bound for \(P_L^{(k)}\). Assuming that all the nodes in each layer are moved onto the innermost boundary of the corresponding ring, we then have

\[
P_L^{(k)} \leq \frac{P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0} \sum_{i'=k_x}^{i+x} \frac{c_{11}(i' + 1)}{i^{\alpha} a(f)^{i'}}
\]

\[
\leq \frac{2c_{11} P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{c_0} \sum_{i'=k_x}^{i+x} \frac{1}{i^{\alpha-1} a(f)^{i'}}
\]

\[
= \frac{c_{12} P_{n^{\alpha/2}} \Delta k \sqrt{n/2} \sqrt{n}}{i^{\alpha-1}}
\]

\[
(29)
\]

for some positive constants \(c_0\), \(c_{11}\), and \(c_{12}\) independent of \(n\) and an arbitrarily small \(\epsilon_0 > 0\), where the equality comes from the relationship (16) between \(a(f)\) and \(\beta\). We first consider the case where \(k_x = o(n^{1/2-\beta+\epsilon})\) for an arbitrarily small \(\epsilon > 0\). Under this condition, from the fact that the term \(i^{\alpha-1}\)
in the RHS of (29) is dominant in terms of upper-bounding $P_L^{(k)}$ for $i' = k_x, \cdots, \sqrt{n}$, (29) is further bounded by

$$P_L^{(k)} \leq c_{12} P n^{\alpha/2} \sum_{i' = k_x}^{\sqrt{n}} \frac{1}{i'^{\alpha - 1}}$$

$$\leq c_{12} P n^{\alpha/2} \left( \frac{1}{i'^{\alpha - 1} \sum_{k_x}^{\sqrt{n}}} + \int_{k_x}^{\sqrt{n}} \frac{1}{x^\alpha - 1} dx \right),$$

which yields $P_L^{(k)} = O \left( n^{\alpha/2} (\sqrt{n})^{2-\alpha} \right) = O(n)$ for $1 \leq \alpha < 2$ and $P_L^{(k)} = O \left( n \log n \right)$ for $\alpha = 2$. When $k_x = \Omega(n^{1/2-\beta+\epsilon})$, the upper bound (29) for $P_L^{(k)}$ is dominated by the term $(1 + \epsilon_0)^{n^{\beta-1/2}}$, and thus is given by

$$P_L^{(k)} \leq c_{13} P n^{\alpha/2} \max \left\{ 1, n^{1/2-\beta} \right\}$$

for some constant $c_{13} > 0$ independent of $n$, which is the last result in (17).

Next, let us turn to deriving a lower bound for $P_L^{(k)}$. Since each layer has at least one node that is onto the innermost boundary of the corresponding ring, the lower bound similarly follows

$$P_L^{(k)} \geq \frac{P n^{\alpha/2}}{c_0} \sum_{i' = k_x}^{k_x + \sqrt{n}/2 - 1} \frac{1}{i'^{\alpha} a(f)i'^{\sqrt{n}}}$$

$$= c_{12} P n^{\alpha/2} \sum_{i' = k_x}^{k_x + \sqrt{n}/2 - 1} \frac{1}{i'^{\alpha} (1 + \epsilon_0)^{i'^{\beta-1/2}}}.$$ (31)

For the condition $k_x = o(n^{1/2-\beta+\epsilon})$, (31) is represented as

$$P_L^{(k)} \geq c_{12} P n^{\alpha/2} \sum_{i' = k_x}^{k_x + \sqrt{n}/2 - 1} \frac{1}{i'^{\alpha} (1 + \epsilon_0)^{i'^{\beta-1/2}}}$$

$$\geq c_{12} P n^{\alpha/2} \sum_{i' = k_x}^{2k_x - 1} \frac{1}{i'^{\alpha} (1 + \epsilon_0)^{i'^{\beta-1/2}}}$$

$$\geq \frac{c_{12} P n^{\alpha/2}}{n^\epsilon} \sum_{i' = k_x}^{2k_x - 1} \frac{1}{i'^{\alpha \epsilon}}$$

$$\geq \frac{c_{14} P n^{\alpha/2}}{n^\epsilon k_x^{\alpha - 1}}.$$
for an arbitrarily small $\epsilon' > 0$ and some constant $c_{14} > 0$ independent of $n$, where the third inequality holds due to $(1 + \epsilon_0)k_x n^{\beta - 1/2} = O(n^{\epsilon'})$. On the other hand, similarly as in the steps of (30), the condition $k_x = \Omega(n^{1/2 - \beta + \epsilon})$ yields the following lower bound for $P_L^{(k)}$:

$$P_L^{(k)} \geq c_{12} n^{\beta - 1/2} + \int_{k_x + 1}^{k_x + \sqrt{n}/2 - 1} \frac{1}{(1 + \epsilon_0)^x} dx$$

$$\geq c_{15} \max \left\{ 1, \frac{n^{1/2 - \beta}}{(1 + \epsilon_0)^{n^{\beta - 1/2}}} \right\}$$

some constant $c_{15} > 0$ independent of $n$, which finally complete the proof of the lemma.

### D. Proof of Lemma [6]

The layering technique illustrated in Fig. 4 is applied as in the extended network case. From (11), the total interference power $P_I$ at each node from simultaneously transmitting nodes is then upper-bounded by

$$P_I = \sum_{k=1}^{\infty} (8k) c_0(k/\sqrt{n})^\alpha a(f)^{k/\sqrt{n}}$$

$$= 8P n^{\alpha/2} \min \left\{ 1, \frac{a(f)^{1/\sqrt{n}N(f)}}{n^{\alpha/2}} \right\} \sqrt{n} \sum_{k=1}^{\infty} k^{\alpha-1} a(f)^{k/\sqrt{n}}. \quad (32)$$

Using (6) and (16), the upper bound (32) on $P_I$ can be expressed as

$$P_I \leq c_{16} P n^{\alpha/2} \min \left\{ 1, \frac{(1 + \epsilon_0)n^{\beta - 1/2}}{n^{(\alpha + \beta)a}} \right\} \sqrt{n} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}}$$

$$\leq \begin{cases} 
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } 0 \leq \beta < 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} \frac{1}{(1 + \epsilon_0)^{k^n}} & \text{if } \beta = 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } \beta > 1/2 
\end{cases}$$

$$\leq \begin{cases} 
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } 0 \leq \beta < 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} \frac{1}{(1 + \epsilon_0)^{k^n - 1}} & \text{if } \beta = 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } \beta > 1/2 
\end{cases}$$

$$\leq \begin{cases} 
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } 0 \leq \beta < 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} \frac{1}{(1 + \epsilon_0)^{k^n - 1}} & \text{if } \beta = 1/2 \\
\frac{c_{16} P \sqrt{n}}{n^{(\alpha + \beta)a/2}} \sum_{k=1}^{\infty} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta - 1/2}} & \text{if } \beta > 1/2 
\end{cases}$$
for some positive constants $c_{16}$ and $c_{17}$ independent of $n$. Based on the argument in Appendix C when $0 ≤ β < 1/2$, it follows that

\[
\sum_{k=1}^{\sqrt{n}} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta-1/2}} \leq \sum_{k=1}^{\sqrt{n}^{1/2-\beta}} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta-1/2}} + \sum_{k=n^{1/2-\beta}}^{\sqrt{n}} k^{\alpha-1} (1 + \epsilon_0)^{kn^{\beta-1/2}}
\]

\[
\leq \left(1 + \int_{1}^{\sqrt{n}^{1/2-\beta}} \frac{1}{x^{\alpha-1}} dx\right) + \frac{1}{n^{(1/2-\beta)(\alpha-1)}} \left(1 + \epsilon_0 + \int_{n^{1/2-\beta}}^{\sqrt{n}} (1 + \epsilon_0)^{xn^{\beta-1/2}} dx\right)
\]

\[
\leq 2 \int_{1}^{\sqrt{n}^{1/2-\beta}} \frac{1}{x^{\alpha-1}} dx + \frac{2}{n^{(1/2-\beta)(\alpha-1)}} \int_{1}^{n^{1/2-\beta}} (1 + \epsilon_0)^{x} dx
\]

\[
\leq \begin{cases} 
4n^{(1/2-\beta)(2-\alpha)} & \text{if } 1 ≤ \alpha < 2 \\
\log n & \text{if } \alpha = 2,
\end{cases}
\]

which results in $P_1 = O\left(\max\left\{\frac{n^{(1/2-\beta)(2-\alpha)}}{n^{\beta\alpha/2}},\log n\right\}\right)$. This completes the proof of the lemma.

REFERENCES

[1] P. Gupta and P. R. Kumar, “The capacity of wireless networks,” IEEE Trans. Inf. Theory, vol. 46, pp. 388–404, Mar. 2000.
[2] D. E. Knuth, “Big Omicron and big Omega and big Theta,” ACM SIGACT News, vol. 8, pp. 18–24, Apr.-June 1976.
[3] P. Gupta and P. R. Kumar, “Towards an information theory of large networks: an achievable rate region,” IEEE Trans. Inf. Theory, vol. 49, pp. 1877–1894, Aug. 2003.
[4] M. Franceschetti, O. Dousse, D. N. C. Tse, and P. Thiran, “Closing the gap in the capacity of wireless networks via percolation theory,” IEEE Trans. Inf. Theory, vol. 53, pp. 1009–1018, Mar. 2007.
[5] F. Xue, L.-L. Xie, and P. R. Kumar, “The transport capacity of wireless networks over fading channels,” IEEE Trans. Inf. Theory, vol. 51, pp. 834–847, Mar. 2005.
[6] A. El Gamal, J. Mammen, B. Prabhakar, and D. Shah, “Optimal throughput-delay scaling in wireless networks-Part I: The fluid model,” IEEE Trans. Inf. Theory, vol. 52, pp. 2568–2592, June 2006.
[7] A. El Gamal and J. Mammen, “Optimal hopping in ad hoc wireless networks,” in Proc. IEEE INFOCOM, Barcelona, Spain, Apr. 2006, pp. 1–10.
[8] Y. Nebat, R. L. Cruz, and S. Bhardwaj, “The capacity of wireless networks in nonergodic random fading,” IEEE Trans. Inf. Theory, vol. 55, pp. 2478–2493, June 2009.
[9] W.-Y. Shin, S.-Y. Chung, and Y. H. Lee, “Improved power-delay trade-off in wireless networks using opportunistic routing,” IEEE Trans. Inf. Theory, under revision for possible publication, available at [http://arxiv.org/abs/0907.2455](http://arxiv.org/abs/0907.2455).
[10] A. Özgür, O. Lévéque, and D. N. C. Tse, “Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks,” IEEE Trans. Inf. Theory, vol. 53, pp. 3549–3572, Oct. 2007.
[11] U. Niesen, P. Gupta, and D. Shah, “On capacity scaling in arbitrary wireless networks,” IEEE Trans. Inf. Theory, vol. 55, pp. 3959–3982, Sept. 2009.
[12] M. Grossglauser and D. N. C. Tse, “Mobility increases the capacity of ad hoc wireless networks,” IEEE/ACM Trans. Networking, vol. 10, pp. 477–486, Aug. 2002.
[13] V. R. Cadambe and S. A. Jafar, “Interference alignment and degrees of freedom of the $K$ user interference channel,” IEEE Trans. Inf. Theory, vol. 54, pp. 3425–3441, Aug. 2008.
[14] A. Zemlianov and G. de Veciana, “Capacity of ad hoc wireless networks with infrastructure support,” IEEE J. Select. Areas Commun., vol. 23, pp. 657–667, Mar. 2005.
[15] J. Partan, J. Kurose, and B. N. Levine, “A survey of practical issues in underwater networks,” in Proc. Int. Workshop on Underwater Networks (WUWNet), Los Angeles, CA, Sept. 2006.
[16] M. Stojanovic, “On the relationship between capacity and distance in an underwater acoustic communication channel,” ACM SIGMOBILE Mobile Computing and Communications Review (MC2R), vol. 11, pp. 34–43, Oct. 2007.
[17] D. E. Lucani, M. Méard, and M. Stojanovic, “Underwater acoustic networks: channel models and network coding based lower bound to transmission power for multicast,” IEEE J. Select. Areas Commun., vol. 26, pp. 1708–1719, Dec. 2008.
[18] D. E. Lucani, M. Stojanovic, and M. Méard, “On the relationship between transmission power and capacity of an underwater acoustic communication channel,” in Proc. OCEANS’08, Kobe, Japan, Apr. 2008, pp. 1–6.
[19] Z. Guo, P. Xie, J. H. Cui, and B. Wang, “On applying network coding to underwater sensor networks,” in Proc. Int. Workshop on Underwater Networks (WUWNet), Los Angeles, CA, Sept. 2006, pp. 109–112.

[20] D. E. Lucani, M. Méard, and M. Stojanovic, “Network coding schemes for underwater networks: the benefits of implicit acknowledgment,” in Proc. Int. Workshop on Underwater Networks (WUWNet), Montreal, Canada, Sept. 2007, pp. 25–32.

[21] C. Carbonelli and U. Mitra, “Cooperative multihop communication for underwater acoustic networks,” in Proc. Int. Workshop on Underwater Networks (WUWNet), Los Angeles, CA, Sept. 2006, pp. 97–100.

[22] W. Zhang, M. Stojanovic, and U. Mitra, “Analysis of a linear multihop underwater acoustic network,” IEEE J. Oceanic Eng., vol. 35, pp. 961–970, Oct. 2010.

[23] D. E. Lucani, M. Méard, and M. Stojanovic, “Capacity scaling laws for underwater networks,” in Proc. Asilomar Conf. on Signals, Systems and Computers, Pacific Grove, CA, Oct. 2008, pp. 2125–2129.

[24] F. Chapeau-Blondeau and A. Monir, “Numerical evaluation of the Lambert W function and application to generalization of generalized Gaussian noise with exponent 1/2,” IEEE Trans. Signal Process., vol. 50, pp. 2160–2165, Sept. 2002.

[25] L.-L. Xie and P. R. Kumar, “A network information theory for wireless communication: scaling laws and optimal operation,” IEEE Trans. Inf. Theory, vol. 50, pp. 748–767, May 2004.

[26] A. Jovicic, P. Viswanath, and S. R. Kulkarni, “Upper bounds to transport capacity of wireless networks,” IEEE Trans. Inf. Theory, vol. 50, pp. 2555–2565, Nov. 2004.

[27] A. Özgür, R. Johari, D. N. C. Tse, and O. Lévêque, “Information-theoretic operating regimes of large wireless networks,” IEEE Trans. Inf. Theory, vol. 56, pp. 427–437, Jan. 2010.

[28] S.-H. Lee and S.-Y. Chung, “Capacity scaling of wireless ad hoc networks: effect of finite wavelength,” IEEE Trans. Inf. Theory, submitted for publication, available at [http://arxiv.org/abs/1002.1337](http://arxiv.org/abs/1002.1337).

[29] A. Özgür, O. Lévêque, and D. N. C. Tse, “Linear capacity scaling in wireless networks: beyond physical limits?” in Proc. Inf. Theory and Applications Workshop (ITA), San Diego, CA, Jan./Feb. 2010, pp. 1–10.

[30] L. Berkovskikh and Y. Lysanov, Fundamentals of Ocean Acoustics. New York: Springer, 1982.

[31] R. Cover, Underwater Acoustic Systems. New York: Wiley, 1989.

[32] M. Franceschetti, M. D. Migliore, and P. Minero, “The capacity of wireless networks: information-theoretic and physical limits,” IEEE Trans. Inf. Theory, vol. 55, pp. 3413–3424, Aug. 2009.

[33] T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: Wiley, 1991.

[34] V. Veeravalli, Y. Liang, and A. M. Sayeed, “Correlated MIMO wireless channels: capacity, optimal signaling, and asymptotics,” IEEE Trans. Inf. Theory, vol. 51, pp. 2058–2072, June 2005.

[35] F. D. Neeser and J. L. Massey, “Proper complex random processes with applications to information theory,” IEEE Trans. Inf. Theory, vol. 39, pp. 1293–1302, July 1993.

[36] F. Constantinescu and G. Scharf, “Generalized Gram-Hadamard inequality,” Journal of Inequalities and Applications, vol. 2, pp. 381–386, 1998.

[37] M. Méard, “The effect upon channel capacity in wireless communications of perfect and imperfect knowledge of the channel,” IEEE Trans. Inf. Theory, vol. 46, pp. 3072–3081, Nov. 2001.

[38] W.-Y. Shin, S.-W. Jeon, N. Devroye, M. H. Vu, S.-Y. Chung, Y. H. Lee, and V. Tarokh, “Improved capacity scaling in wireless networks with infrastructure,” IEEE Trans. Inf. Theory, under revision for possible publication, available at [http://arxiv.org/abs/0811.0726](http://arxiv.org/abs/0811.0726).

[39] R. Meester and R. Roy, Continuum Percolation. Cambridge, U.K.: Cambridge Univ. Press, 1996.
Fig. 1. The cut $L$ in a two-dimensional extended regular network. $S_L$ and $D_L$ represent the sets of source and destination nodes, respectively.

Fig. 2. The cut $L$ in a two-dimensional dense regular network. $S_L$ and $D_L$ represent the sets of source and destination nodes, respectively, where $D_L$ is partitioned into two groups $D_{L,1}$ and $D_{L,2}$.
Fig. 3. Upper (solid) and lower (dashed) bounds on the capacity scaling $T(n)$.

Fig. 4. Grouping of interference routing cells in extended networks. The first layer $l_1$ represents the outer 8 shaded cells.
Fig. 5. The node displacement to square vertices, indicated by arrows. The empty zone $E_L$ with width constant $\bar{c}$ is assumed for simplicity.

Fig. 6. Grouping of source nodes in extended networks. There exist $\Theta(k_x)$ nodes in the first layer $l'_1$. This figure indicates the case where one destination is located at the position $(k_x, k_y)$. The source nodes are regularly placed at spacing 1 on the left half of the cut $L$. 