On the completely faithfulness of the $p$-free quotient modules of dual Selmer groups

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Abstract

In this paper, we consider the question of completely faithfulness of the $p$-free quotient module of the dual Selmer groups of elliptic curves defined over a noncommutative $p$-adic Lie extension. Our question will refine previous questions on the completely faithfulness of dual Selmer groups. We will also formulate and study an analogous question for the dual Selmer groups of Hida deformations. We then give positive answer to our question in certain cases. We will also establish a control theorem between the two questions in certain cases.

Keywords and Phrases: Completely faithful modules, Selmer groups, elliptic curves, Hida deformations.

Mathematics Subject Classification 2010: 11F80, 11G05, 11R23, 11R34, 16S34.

1 Introduction

Let $p$ be an odd prime. We are interested in a certain class of modules defined over the Iwasawa algebra

$$Z_p[G] = \lim_{\leftarrow} Z_p[G/U]$$

of a noncommutative compact pro-$p$ torsionfree Lie group $G$. The modules belonging to this class are called completely faithful modules (see Section 2 for the precise definition). The motivation of studying these modules arises from noncommutative Iwasawa theory, where one asks whether one can find a global annihilator for the dual Selmer group of an elliptic curve defined over a noncommutative $p$-adic Lie extension. Such a global annihilator, if it exists, will give insight to the noncommutative $p$-adic $L$-function. Unfortunately, such an approach is not feasible in general, and this was first pointed out by Hachimori and Venjakob in [11]. There, by building on a previous work of Venjakob [24], they constructed examples of completely faithful dual Selmer groups over a certain class of $Z_p \rtimes Z_p$-extensions. We should mention that the question of whether the dual Selmer group is completely faithful was also raised in [6]. Since then, many authors, including the present author, have studied completely faithful modules over Iwasawa algebra of other compact $p$-adic Lie groups and constructed examples of completely faithful dual Selmer groups over $p$-adic Lie extensions with Galois groups realized by such $p$-adic Lie groups (see...
In this paper, we will consider a refinement of the question of completely faithfulness which we will now describe.

To do this, we need to first introduce some notation and terminology. Let $F$ be an number field and $E$ an elliptic curve defined over $F$. Let $F_{\infty}$ be a $p$-adic Lie extension of $F$ with the property that the Galois group $G = \text{Gal}(F_{\infty}/F)$ is a noncommutative compact pro-$p$ Lie group with no $p$-torsion and that $F_{\infty}$ contains the cyclotomic $\mathbb{Z}_p$-extension $F^{\text{cyc}}$ of $F$. Write $H = \text{Gal}(F_{\infty}/F^{\text{cyc}})$. A common phenomenon in all the examples of completely faithful dual Selmer groups constructed mentioned in the previous paragraph is that they are all finitely generated over $\mathbb{Z}_p[H]$. In fact, the question raised in [6] was also posed under the condition when the dual Selmer group in question is finitely generated over $\mathbb{Z}_p[H]$. It is then natural to ask what we can say if the dual Selmer group is not finitely generated over $\mathbb{Z}_p[H]$.

Now write $X(E/F_{\infty})$ for the dual Selmer group of $E$ defined over $F_{\infty}/F$. Denote by $X(E/F_{\infty})(p)$ the $p$-primary submodule of $X(E/F_{\infty})$ and write $X_f(E/F_{\infty}) = X(E/F_{\infty})/X(E/F_{\infty})(p)$. As we will see in Lemma 2.2(d), a necessary condition for a $\mathbb{Z}_p[G]$-module $M$ to be completely faithful, its $p$-primary submodule must be pseudo-null. As is well-known, there are cases of $X(E/F_{\infty})(p)$ being not pseudo-null. In fact, building on this fact, the author has produced examples of dual Selmer groups that are not completely faithful in his previous paper on the subject (see [17]). Therefore, one may cut off the $p$-primary submodule of $X(E/F_{\infty})$, and ask the natural question of whether $X_f(E/F_{\infty})$ is completely faithful. This will be the main theme of the paper. In fact, throughout the paper, we say that Property CF holds for $X_f(E/F_{\infty})$ whenever $X_f(E/F_{\infty})$ is either pseudo-null or completely faithful, and we will be interested in the question of the validity of Property CF. We shall show that this property is invariant under isogeny (see Proposition 3.3). We will then show how the results of [17, 24] can be applied to provide strong theoretical evidence to our question (see Theorem 3.6). As a corollary, we show that the central torsion submodule of $X_f(E/F_{\infty})$ is trivial (see Corollary 3.7).

We will also propose and study an analogous question on Property CF for the dual Selmer group attached to a Hida deformation (see Section 4). Again, building on the results of [17, 24], we shall establish the validity of Property CF in certain cases (see Theorem 4.3), and apply them to show that the central torsion submodule is trivial (see Corollary 4.4). We will then establish a control theorem between the Property CF for an elliptic curve and its Hida deformation for a specific class of $p$-adic Lie extensions (see Theorem 5.7). We will also establish a control theorem for certain extensions of $p$-adic Lie extensions (see Theorem 5.11). It may be worthwhile to mention that our control theorems depend crucially on the results of a preprint of Csige [8], where he proves a slight generalization of a previous result of Ardakov [1].

We end the introductory section mentioning that the discussion in the paper can be applied to consider Selmer groups of $p$-ordinary modular forms, or even more general $p$-adic representations. In fact, one can also consider Selmer groups for $p$-adic representations defined over coefficient rings $\mathbb{Z}_p[X_1, X_2, ..., X_n]$ and its various specialization. It may also be of interest to consider the question of completely faithfulness for the Galois group of the maximal abelian $p$-extension (unramified outside a finite set of primes) of a strongly admissible totally real $p$-adic Lie extension. However, in this paper, we will concentrate mainly
on elliptic curves and Hida deformations. Finally, we mention that the result of Csige [8] concerning on the $K_0$-invariance of completely faithful modules implies that the class of modules in an appropriate $K_0$-group (see [8] for details) are partitioned into two sets corresponding to whether they come from a completely faithful modules or not. This seems to suggest that understanding the complete faithfulness of the $p$-free part of dual Selmer groups might give insight towards understanding the characteristic element (in the sense of [5]) attached to such modules, and in particularly, the $p$-free part of the dual Selmer groups (provided they are completely faithful). We do not attempt to study this topic here, but hope to return to it in a subsequent paper. We should remark that, on the contrary, the characteristic elements of the $p$-primary part of the dual Selmer groups are relatively well understood (see [2]).

2 Algebraic Preliminaries

In this section, we establish some more or less standard algebraic preliminaries and notation which are necessary for us in order to prepare for the discussion and the proofs of our results. Throughout the paper, we will always work with left modules over a ring. Let $\Lambda$ be a (not necessarily commutative) Noetherian ring which has no zero divisors. It is well-known that $\Lambda$ admits a skew field of fractions $K(\Lambda)$ which is flat over $\Lambda$ (see [9, Chapters 6 and 10] or [15, Chapter 4, §9 and §10]). For a finitely generated $\Lambda$-module $M$, we then define its $\Lambda$-rank to be

$$\text{rank}_\Lambda M = \dim_{K(\Lambda)} K(\Lambda) \otimes_\Lambda M.$$  

We will say that $M$ is a torsion-module if $\text{rank}_\Lambda M = 0$.

For a nonzero $\Lambda$-module $M$, the global annihilator ideal of $M$ is defined to be

$$\text{Ann}_\Lambda(M) = \{ \lambda \in \Lambda : \lambda m = 0 \text{ for all } m \in M \}$$

which can be easily seen to be a two-sided ideal of $\Lambda$. We will say that $M$ is a faithful $\Lambda$-module if $\text{Ann}_\Lambda(M) = 0$.

Now if $x \in \Lambda$, we denote $M[x]$ to be the set consisting of elements of $M$ annihilated by $x$. If $x$ is not central, $M[x]$ is at most an additive subgroup of $M$. However, if we assume further that $x\Lambda = \Lambda x$, then it is straightforward to verify that $M[x]$ is a $\Lambda$-submodule of $M$. Similarly, one can also verify easily that

$$xM = \{xm : m \in M\}$$

is a $\Lambda$-submodule of $M$, and that $M/xM$ is a $\Lambda/x\Lambda$-module.

Suppose now that $\Lambda$ is a Auslander regular ring (see [23, Definition 3.3]) with no zero divisors. We say that a finitely generated torsion $\Lambda$-module $M$ is pseudo-null if $\text{Ext}^1_\Lambda(M, \Lambda) = 0$. Denote by $\mathcal{M}$ the category of all finitely generated torsion $\Lambda$-modules and by $\mathcal{C}$ the full subcategory of all pseudo-null modules in $\mathcal{M}$. We write $q : \mathcal{M} \to \mathcal{M}/\mathcal{C}$ for the quotient functor. For a finitely generated torsion $\Lambda$-module $M$, we say that $M$ is completely faithful if $\text{Ann}_\Lambda(N) = 0$ for any $N \in \mathcal{M}$ such that $q(N)$ is isomorphic to a non-zero subquotient of $q(M)$. 3
Lemma 2.1. Suppose that  
\[ \alpha : M \longrightarrow N \]
is a \( \Lambda \)-homomorphism of finitely generated torsion \( \Lambda \)-modules with the property that \( \ker \alpha \) and \( \coker \alpha \) are pseudo-null \( \Lambda \)-modules, then \( M \) is a completely faithful \( \Lambda \)-module (resp., pseudo-null \( \Lambda \)-module) if and only if \( N \) is a completely faithful \( \Lambda \)-module (resp., pseudo-null \( \Lambda \)-module).

Proof. This is straightforward from the definition. \( \square \)

The class of Auslander regular rings with no zero divisors that we will consider in this paper comes from the Iwasawa algebras of certain compact \( p \)-adic Lie groups, first discovered by Venjakob, which we now recall. From now on, \( p \) will denote a fixed odd prime. Let \( G \) be a compact pro-\( p \) \( p \)-adic Lie group without \( p \)-torsion. A theorem of Venjakob asserts that \( \mathbb{Z}_p \llbracket G \rrbracket \) is an Auslander regular ring (see [23 Theorems 3.26]). Furthermore, the ring \( \mathbb{Z}_p \llbracket G \rrbracket \) has no zero divisors (cf. [18]), and therefore, as seen above, there is a well-defined notion of \( \mathbb{Z}_p \llbracket G \rrbracket \)-rank, torsion \( \mathbb{Z}_p \llbracket G \rrbracket \)-module, pseudo-null \( \mathbb{Z}_p \llbracket G \rrbracket \)-module and completely faithful \( \mathbb{Z}_p \llbracket G \rrbracket \)-module.

For a given finitely generated \( \mathbb{Z}_p \llbracket G \rrbracket \)-module \( M \), we denote \( M(p) \) to be the \( \mathbb{Z}_p \llbracket G \rrbracket \)-submodule of \( M \) consisting of elements of \( M \) which are annihilated by some power of \( p \). Since the ring \( \mathbb{Z}_p \llbracket G \rrbracket \) is Noetherian, the module \( M(p) \) is finitely generated over \( \mathbb{Z}_p \llbracket G \rrbracket \). Therefore, it follows that one can find an integer \( r \geq 0 \) such that \( p^r \) annihilates \( M(p) \). Following [14] Formula (33), we define

\[ \mu_G(M) = \sum_{i \geq 0} \text{rank}_{\mathbb{Z}_p \llbracket G \rrbracket} \left( p^i M(p)/p^{i+1} \right). \]

(For another alternative, but equivalent, definition, see [23 Definition 3.32].) By the above discussion, the sum on the right is a finite one. Also, it is clear from the definition that \( \mu_G(M) = \mu_G(M(p)) \). We will say that a finitely generated \( \mathbb{Z}_p \llbracket G \rrbracket \)-module \( M \) is \( p \)-primary if \( M = M(p) \).

We now record certain properties of the \( \mu_G \)-invariant which we will use frequently in the paper.

Lemma 2.2. Let \( G \) be a compact pro-\( p \) \( p \)-adic Lie group without \( p \)-torsion and \( M \) a finitely generated \( \mathbb{Z}_p \llbracket G \rrbracket \)-module. Then we have the following statements.

(a) The \( \mu_G \)-invariant is additive on short exact sequences of torsion \( \mathbb{Z}_p \llbracket G \rrbracket \)-modules.

(b) Suppose that \( G \) has a closed normal subgroup \( H \) such that \( G/H \cong \mathbb{Z}_p \). If \( M \) is a \( \mathbb{Z}_p \llbracket G \rrbracket \)-module which is finitely generated over \( \mathbb{Z}_p \llbracket H \rrbracket \), then we have \( \mu_G(M) = 0 \).

(c) We have \( \mu_G(M) = 0 \) if and only if \( M(p) \) is pseudo-null over \( \mathbb{Z}_p \llbracket G \rrbracket \). In particular, if \( M \) is a \( p \)-primary \( \mathbb{Z}_p \llbracket G \rrbracket \)-module, then \( \mu_G(M) = 0 \) if and only if \( M \) is pseudo-null over \( \mathbb{Z}_p \llbracket G \rrbracket \).

(d) Suppose that \( M \) is a completely faithful \( \mathbb{Z}_p \llbracket G \rrbracket \)-module. Then \( M(p) \) is pseudo-null over \( \mathbb{Z}_p \llbracket G \rrbracket \) and \( \mu_G(M) = 0 \).
Proof. Statement (a) follows from [23, Corollary 3.37] (see also [14, Proposition 1.8]). Statement (b) follows from [14, Lemma 2.7]. Statement (c) is shown in [23, Remark 3.33]. We now show statement (d).

Since $M(p)$ is a $\mathbb{Z}_p[[G]]$-submodule of $M$ which is annihilated by some power of $p$ and $M$ is completely faithful, this forces $M(p)$ to be a pseudo-null $\mathbb{Z}_p[[G]]$-module. By statement (c), this in turn implies that $\mu_G(M) = \mu_G(M(p)) = 0$.

We also record the following well-known and important result of Venjakob (cf. [24, Example 2.3 and Proposition 5.4]) which we require in the discussion of the paper.

**Theorem 2.3** (Venjakob). Suppose that $H$ is a closed normal subgroup of $G$ with $G/H \cong \mathbb{Z}_p$. Let $M$ be a compact $\mathbb{Z}_p[G]$-module which is finitely generated over $\mathbb{Z}_p[H]$. Then $M$ is a pseudo-null $\mathbb{Z}_p[G]$-module if and only if $M$ is a torsion $\mathbb{Z}_p[H]$-module.

Finally, we record another useful lemma whose easy proof is left to the reader as an exercise.

**Lemma 2.4.** Let $H$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $N$ be a closed normal subgroup of $H$ such that $N \cong \mathbb{Z}_p$ and such that $H/N$ is also a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $M$ be a finitely generated $\mathbb{Z}_p[H]$-module. Then $H_i(N, M)$ is finitely generated over $\mathbb{Z}_p[H/N]$ for each $i$ and $H_i(N, M) = 0$ for $i \geq 2$. Furthermore, we have an equality

$$\text{rank}_{\mathbb{Z}_p[H]} M = \text{rank}_{\mathbb{Z}_p[H/N]} M_N - \text{rank}_{\mathbb{Z}_p[H/N]} H_1(N, M).$$

### 3 A question on completely faithfulness

In this section, we introduce the Selmer group of an elliptic curve. Fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Therefore, an algebraic (possibly infinite) extension of $\mathbb{Q}$ will mean an subfield of $\overline{\mathbb{Q}}$. A finite extension $F$ of $\mathbb{Q}$ will be called a number field. Let $E$ be an elliptic curve which is defined over a number field $F$. Suppose that for every prime $v$ of $F$ above $p$, our elliptic curve $E$ has either good ordinary reduction or multiplicative reduction at $v$.

Let $v$ be a prime of $F$. For every finite extension $L$ of $F$, we set

$$J_v(E/L) = \bigoplus_{w | v} H^1(L_w, E)_{p^\infty},$$

where $w$ runs over the (finite) set of primes of $L$ above $v$. If $L$ is an infinite algebraic extension of $F$, we define

$$J_v(E/L) = \lim_{\rightarrow} J_v(E/L),$$

where the direct limit is taken over all finite extensions $L$ of $F$ contained in $L$. For any algebraic (possibly infinite) extension $L$ of $F$, the Selmer group of $E$ over $L$ is defined to be

$$S(E/L) = \ker \left( H^1(L, E_{p^\infty}) \to \bigoplus_v J_v(E/L) \right),$$

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where $v$ runs through all the primes of $F$.

We will be interested in studying the Selmer groups defined over a certain class of infinite algebraic extensions which we define now. An extension $F_\infty$ of $F$ is said to be an admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F_\infty/F)$ is a compact $p$-adic Lie group, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension $F^{\text{cyc}}$ of $F$ and (iii) $F_\infty$ is unramified outside a finite set of primes of $F$. In the event that $\text{Gal}(F_\infty/F)$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion, we will say $F_\infty$ is a strongly admissible $p$-adic Lie extension of $F$.

Let $S$ be any finite set of primes of $F$ which contains the primes above $p$, the infinite primes, the primes at which $E$ has bad reduction and the primes that are ramified in $F_\infty/F$. Denote by $F_S$ the maximal algebraic extension of $F$ unramified outside $S$. For each algebraic (possibly infinite) extension $\mathcal{L}$ of $F$ contained in $F_S$, we write $G_S(\mathcal{L}) = \text{Gal}(F_S/\mathcal{L})$. The following alternative equivalent description of the Selmer group of $E$ over $F_\infty$

$$S(E/F_\infty) = \ker \left( H^1(G_S(F_\infty), E_{p^{\infty}}) \xrightarrow{\lambda_{E/F_\infty}} \bigoplus_{v \in S} J_v(E/F_\infty) \right)$$

is well-known (for instance, see [7, Lemma 2.2]). We will denote by $X(E/F_\infty)$ the Pontryagin dual of $S(E/F_\infty)$. We will also write $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F^{\text{cyc}})$ and $\Gamma = \text{Gal}(F^{\text{cyc}}/F)$.

As mentioned in the Introduction, the question on whether $X(E/F_\infty)$ is completely faithful when $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[[H]]$ with positive $\mathbb{Z}_p[[H]]$-rank has been studied by many in [3, 6, 11, 17]. A natural question will be to ask what if $X(E/F_\infty)$ is not finitely generated over $\mathbb{Z}_p[[H]]$. In this case, the present author has constructed examples of dual Selmer groups which are faithful but not completely faithful (see [17]). However, as observed loc. cit., the examples of dual Selmer groups with such a property have positive $\mu$-invariant. Furthermore, in view of Lemma 2.2 (4), it would seem that to be able to talk about completely faithfulness, one need to cut out off the $p$-primary submodules. We will therefore introduce the following property which will be the main theme of the paper.

**Property CF.** For every noncommutative strongly admissible $p$-adic Lie extension $F_\infty$ of $F$, the $p$-free quotient module $X(E/F_\infty)/X(E/F_\infty)(p)$ is either a pseudo-null $\mathbb{Z}_p[[G]]$-module or a completely faithful $\mathbb{Z}_p[[G]]$-module.

For the remainder of the paper, we will write $X_f(E/F_\infty) = X(E/F_\infty)/X(E/F_\infty)(p)$. In this paper, we will be interested in the question on whether Property CF holds for $X_f(E/F_\infty)$. We first show that this question covers the original question of Coates et al.

**Lemma 3.1.** Let $E$ be an elliptic curve over $F$ which has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$. Let $F_\infty$ be a noncommutative strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[[H]]$. Then the following statements are equivalent.

(a) $X(E/F_\infty)$ is completely faithfully over $\mathbb{Z}_p[G]$.

(b) $X_f(E/F_\infty)$ is completely faithfully over $\mathbb{Z}_p[[G]]$.  


Proof. Since \( X(E/F_\infty) \) is finitely generated over \( \mathbb{Z}_p[[H]] \), so is \( X(E/F_\infty)(p) \). It then follows from an application of Lemma \([22] \) (b) and (c) that \( X(E/F_\infty)(p) \) is pseudo-null over \( \mathbb{Z}_p[G] \). The equivalence of the statements now follows from Lemma \([21] \). \( \square \)

Remark 3.2. In fact, if \( E \) has good ordinary reduction at all the primes above \( p \) and \( F \) is not totally real, then we have the following observation, namely, if \( X(E/F_\infty) \) is finitely generated over \( \mathbb{Z}_p[[H]] \), then \( X(E/F_\infty)(p) = 0 \) and \( X(E/F_\infty) = X_f(E/F_\infty) \) (for instance, see \([11\) Page 456]).

We now show that Property CF is invariant under isogeny.

Proposition 3.3. Let \( E_1 \) and \( E_2 \) be two elliptic curves over \( F \) with either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \) which are isogenous to each other. Let \( F_\infty \) be a noncommutative strongly admissible noncommutative p-adic Lie extension of \( F \) with \( G = \text{Gal}(F_\infty/F) \). Assume that \( X(E_1/F_\infty) \) is a torsion \( \mathbb{Z}_p[G] \)-module. Then \( X(E_2/F_\infty) \) is a torsion \( \mathbb{Z}_p[G] \)-module. Furthermore, Property CF holds for \( X(E_1/F_\infty) \) if and only if Property CF holds for \( X(E_2/F_\infty) \).

Proof. Let \( \varphi : E_2 \rightarrow E_1 \) be an isogeny defined over \( F \). Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X(E_1/F_\infty)(p) & \longrightarrow & X(E_1/F_\infty) & \longrightarrow & X_f(E_1/F_\infty) & \longrightarrow & 0 \\
& & f & & g & & h & & \\
0 & \longrightarrow & X(E_2/F_\infty)(p) & \longrightarrow & X(E_2/F_\infty) & \longrightarrow & X_f(E_2/F_\infty) & \longrightarrow & 0
\end{array}
\]

with exact rows, where the vertical maps are induced by \( \varphi \). It is not difficult to show that \( g \) has kernel and cokernel that are killed by \( p^n \) for some large enough \( n \) (for instance, see the proof of \([11\) Theorem 5.1] or \([14\) Theorem 3.1]). Thus, it follows that \( X(E_2/F_\infty) \) is also a torsion \( \mathbb{Z}_p[G] \)-module.

Now, clearly, \( f \) has kernel and cokernel that are \( p \)-primary \( \mathbb{Z}_p[G] \)-modules. Hence we conclude that \( h \) must also have kernel and cokernel that are \( p \)-primary \( \mathbb{Z}_p[G] \)-modules. Since \( X_f(E_1/F_\infty) \) has no \( p \)-torsion by definition, we have \( \ker h = 0 \). This in turn implies that we have \( \ker f = \ker g \) and the following short exact sequence

\[
0 \longrightarrow \text{coker } f \longrightarrow \text{coker } g \longrightarrow \ker h \longrightarrow 0.
\]

On the other hand, it follows from the exact sequence of torsion \( \mathbb{Z}_p[G] \)-modules

\[
0 \longrightarrow \ker g \longrightarrow X(E_1/F_\infty) \longrightarrow X(E_2/F_\infty) \longrightarrow \text{coker } g \longrightarrow 0
\]

and Lemma \([22\) (a)] that

\[
\mu_G(\ker g) - \mu_G(\text{coker } g) = \mu_G(X(E_1/F_\infty)) - \mu_G(X(E_2/F_\infty)).
\]

Similarly, we have

\[
\mu_G(\ker f) - \mu_G(\text{coker } f) = \mu_G(X(E_1/F_\infty)(p)) - \mu_G(X(E_2/F_\infty)(p)).
\]
Since $\mu_G(M) = \mu_G(M(p))$, the above two equalities combine to give

$$\mu_G(\ker g) - \mu_G(\coker g) = \mu_G(\ker f) - \mu_G(\coker f).$$

As observed above, we already have $\ker f = \ker g$. Therefore, it follows that $\mu_G(\coker f) = \mu_G(\coker g)$ and this in turn implies that $\mu_G(\coker h) = 0$. Since $\coker h$ is $p$-primary, it follows from Lemma 2.2(d) that $\coker h$ is a pseudo-null $\mathbb{Z}_p[G]$-module. In conclusion, we have shown that the map

$$h : X_f(E_1/F_\infty) \to X_f(E_2/F_\infty)$$

is injective with a pseudo-null cokernel. The final assertion of the proposition now follows from an application of Lemma 2.1. □

To facilitate further discussion, we recall the following conjecture on the structure of $X(E/F_\infty)$ which was first raised in [5].

$\mathfrak{M}_H(G)$-Conjecture. For every admissible $p$-adic Lie extension $F_\infty$ of $F$, $X_f(E/F_\infty)$ is a finitely generated $\mathbb{Z}_p[H]$-module.

We remark that if one assumes the $\mathfrak{M}_H(G)$-Conjecture, and takes Theorem 2.3 into account, then Conjecture CF is equivalent to saying that if $X_f(E/F_\infty)$ is either a torsion $\mathbb{Z}_p[H]$-module or a completely faithful $\mathbb{Z}_p[G]$-module. The following lemma is not used in the paper but we have decided to include it, as we believe that it is an interesting observation. In particular, the lemma says that in order $X(E/F_\infty)$ to be completely faithful $\mathbb{Z}_p[G]$-module (and assuming that the $\mathfrak{M}_H(G)$-conjecture holds), it is necessary that it is finitely generated over $\mathbb{Z}_p[H]$.

Lemma 3.4. Let $E$ be an elliptic curve over $F$ which has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$. Let $F_\infty$ be a noncommutative strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that $X(E/F_\infty)$ satisfies the $\mathfrak{M}_H(G)$-Conjecture and that $X(E/F_\infty)$ is a completely faithful $\mathbb{Z}_p[G]$-module. Then $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$.

Proof. Since $X(E/F_\infty)$ is completely faithful, we may apply Lemma 2.2(d) to conclude that $\mu_G(X(E/F_\infty)) = 0$. By the assumption that $X(E/F_\infty)$ satisfies the $\mathfrak{M}_H(G)$-Conjecture, it follows from [16, Theorem 3.1] that $\mu_f(X(E/F_\infty^{\text{cyc}})) = 0$ and from [7, Proposition 2.5] that $X(E/F_\infty^{\text{cyc}})$ is torsion over $\mathbb{Z}_p[\Gamma]$. By the structure theory of finitely generated $\mathbb{Z}_p[\Gamma]$-modules, this in turn implies that $X(E/F_\infty^{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$. By virtue of [7, Theorem 2.1], we then have that $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$ as required. □

For the remainder of the section, we will show that the results in [17, 24] can be applied to give strong evidence to our conjecture in certain cases. We shall recall the result that we require. Before that, we introduce the following terminology for convenience. We say that $G$ satisfies (NH) if the group $G$ contains two closed normal subgroups $N$ and $H$ with the following two properties:

(i) $N \subseteq H$, $G/H \cong \mathbb{Z}_p$ and $G/N$ is a non-abelian group isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$.
There is a finite family of closed normal subgroups \( N_i \) (\( 0 \leq i \leq r \)) of \( G \) such that \( 1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N \) and such that \( N_i/N_{i-1} \cong \mathbb{Z}_p \) for \( 1 \leq i \leq r \).

**Theorem 3.5** (Lim, Venjakob). Suppose that \( G \) satisfies (NH). Let \( M \) be a \( \mathbb{Z}_p[G] \)-module which is finitely generated over \( \mathbb{Z}_p[H] \) and has positive \( \mathbb{Z}_p[H] \)-rank. Then \( M \) is a completely faithful \( \mathbb{Z}_p[G] \)-module.

**Proof.** When \( r = 0 \), this is precisely [24, Corollary 4.3]. For \( r > 0 \), one proceeds by an inductive argument. We refer readers to [17, Theorem 3.3] for the details. \( \square \)

We can now prove the following theorem which validates Property CF when the strongly admissible extension has a Galois group of the form (NH).

**Theorem 3.6.** Let \( E \) be an elliptic curve over \( F \) which has either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \). Let \( F_\infty \) be a strongly admissible \( p \)-adic Lie extension of \( F \) with \( G = \text{Gal}(F_\infty/F) \). Suppose that the following statements hold.

(i) The \( \mathfrak{M}_H(G) \) conjecture holds for \( X(E/F_\infty) \).

(ii) \( G \) satisfies (NH).

Then Property CF holds for \( X_f(E/F_\infty) \).

**Proof.** If \( X_f(E/F_\infty) \) is a torsion \( \mathbb{Z}_p[H] \)-module, then we are done by an application of Theorem 2.3. In the case that \( X_f(E/F_\infty) \) is a finitely generated \( \mathbb{Z}_p[H] \)-module with positive \( \mathbb{Z}_p[H] \)-rank, it will follow from Theorem 3.5 that \( X_f(E/F_\infty) \) is a completely faithful \( \mathbb{Z}_p[G] \)-module. \( \square \)

At this point of writing, we feel that there is currently insufficient evidence (even taking the preceding theorem into account) for a positive answer to the question of Property CF being satisfied in general. Hence it may be premature to make a conjecture on Property CF, and therefore, we have refrain from doing so.

We end the section mentioning an interesting corollary of Theorem 3.6. For every prime \( v \) of \( F \) above \( p \), fix a prime \( w \) of \( F_\infty \) above \( v \). We denote \( G_w \) (resp., \( I_w \)) to be the decomposition group (resp. the inertia group) of \( w \) in \( G \). We will write \( f_{\infty,w} \) for the residue field of \( F_{\infty,w} \), and \( \tilde{E}_v \) for the reduction of \( E \) mod \( v \). We can now state the corollary (compare with [19, Theorem 6.5]).

**Corollary 3.7.** Let \( E \) be an elliptic curve over \( F \) which has good ordinary reduction at every prime of \( F \) above \( p \). Let \( F_\infty \) be a noncommutative strongly admissible \( p \)-adic Lie extension of \( F \) which is contained in \( M_\infty \). Suppose that the \( \mathfrak{M}_H(G) \) conjecture holds for \( X(E/F_\infty) \). Suppose further that the following statements hold.

(i) For every prime \( v \) of \( F \) above \( p \), either the decomposition group of \( \text{Gal}(F_\infty/F) \) at \( v \) has dimension \( \geq 3 \), or \( \dim G_v = \dim I_v = 2 \) and \( \tilde{E}_v(f_{\infty,w}) \) is finite.
(ii) For every prime \( v \) of \( F \) in \( S \) but not above \( p \), the decomposition group of \( \text{Gal}(F_\infty/F) \) at \( v \) has dimension \( \geq 2 \).

Then the \( \mathbb{Z}_p[[C]] \)-torsion submodule of \( X_f(E/F_\infty) \) is zero, where \( C \) is the centre of \( G = \text{Gal}(F_\infty/F) \).

In the event that \( X(E/F_\infty) \) is finitely generated over \( \mathbb{Z}_p[H] \), we have that the \( \mathbb{Z}_p[[C]] \)-torsion submodule of \( X(E/F_\infty) \) is zero.

Proof. It follows from Theorem 3.6 that \( X_f(E/F_\infty) \) is either pseudo-null or completely faithful. On the other hand, by [10] Theorem 3.2, \( X(E/F_\infty) \) has no nonzero pseudo-null \( \mathbb{Z}_p[[G]] \)-submodule. By [22] Lemma 4.2, this in turn implies that \( X_f(E/F_\infty) \) has no nonzero pseudo-null \( \mathbb{Z}_p[[G]] \)-submodule. Therefore, if \( X_f(E/F_\infty) \) is pseudo-null, we must have \( X_f(E/F_\infty) = 0 \) which in turn implies the same for its \( \mathbb{Z}_p[[C]] \)-torsion submodule. Now suppose that \( X_f(E/F_\infty) \) is completely faithful over \( \mathbb{Z}_p[G] \). Since \( X_f(E/F_\infty) \) is a Noetherian \( \mathbb{Z}_p[G] \)-module, so is its \( \mathbb{Z}_p[[C]] \)-torsion submodule. It then follows that the \( \mathbb{Z}_p[[C]] \)-torsion submodule of \( X_f(E/F_\infty) \) has a global annihilator, and therefore, must be pseudo-null over \( \mathbb{Z}_p[G] \) by the completely faithfulness of \( X_f(E/F_\infty) \). But since \( X_f(E/F_\infty) \) has no nonzero pseudo-null \( \mathbb{Z}_p[[G]] \)-submodule, this implies that the \( \mathbb{Z}_p[[C]] \)-torsion submodule of \( X_f(E/F_\infty) \) is zero. The second assertion can be proved similarly. \( \square \)

4 Analogue of Property CF for Hida deformations

In this section, we discuss an analogue of Property CF for Selmer groups of Hida deformations. As before, \( p \) will denote an odd prime. Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with ordinary reduction at \( p \) and assume that \( E[p] \) is an absolutely irreducible \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-representation. By Hida theory (for instance, see [12] [13]), there exists a commutative complete \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-local domain \( R \) which is flat over the power series ring \( \mathbb{Z}_p[[X]] \) in one variable, and a free \( R \)-module \( T \) of rank 2 with \( T/P \cong T_pE \) for some prime ideal \( P \) of \( R \). We will assume that \( R = \mathbb{Z}_p[[X]] \) in all our discussion. Furthermore, the prime ideal \( P \) can be expressed as the ideal generated by \( X - a \) for some \( a \in p\mathbb{Z}_p \). For more detailed description of fundamental and important arithmetic properties of the Hida deformations, we refer readers to [7] [12] [13] [20]. We will just mention two properties of \( T \) which we require to define an appropriate Selmer group of the Hida deformation. The first is that \( T \) is unramified outside the set \( S \), where \( S \) is any finite set of primes of \( F \) which contains the primes above \( p \), the infinite primes, the primes at which \( E \) has bad reduction and the primes that are ramified in \( F_\infty/F \). The second property we will mention is that there exists an \( R \)-submodule \( T^+ \) of \( T \) which is invariant under the action of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) and such that both \( T^+ \) and \( T/T^+ \) are free \( R \)-modules of rank one.

Set \( A = T \otimes_R \text{Hom}_{cts}(R, \mathbb{Q}_p/\mathbb{Z}_p) \) and \( A^+ = T^+ \otimes_R \text{Hom}_{cts}(R, \mathbb{Q}_p/\mathbb{Z}_p) \). We note that one has \( E_{p^\infty} = A[P] \). Following [12] Section 4 or [20] Section 6, we define the Selmer group of the Hida deformation over an admissible \( p \)-adic Lie extension \( F_\infty \) of \( \mathbb{Q} \) by

\[
S(A/F_\infty) = \text{ker} \left( H^1(G_S(F_\infty), A) \rightarrow \bigoplus_{v \in S} J_v(A, F_\infty) \right),
\]

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where

\[ J_v(A, F_\infty) = \begin{cases} 
\prod_{w|v} H^1(F_\infty, w, A/A^+), & \text{if } v \text{ divides } p, \\
\prod_{w|v} H^1(F_\infty, w, A), & \text{if } v \text{ does not divide } p.
\end{cases} \]

We will denote by \( X(A/F_\infty) \) the Pontryagin dual of this Selmer group. We will consider this dual Selmer group as a (compact) Gal(\( F_\infty/F \))-module for some finite extension \( F \) of \( \mathbb{Q} \) in \( F_\infty \), where \( F_\infty \) is a strongly admissible \( p \)-adic Lie extension of \( F \).

For a \( R/[G] \)-module \( M \), we denote by \( M(R) \) the \( R \)-torsion submodule of \( M \). It is an easy exercise to verify that this is a \( R/[G] \)-submodule of \( M \).

We now record the following lemma.

**Lemma 4.1.** Suppose that \( M \) is a completely faithful \( R/[G] \)-module. Then \( M(R) \) is pseudo-null over \( R/[G] \).

**Proof.** Since \( M \) is finitely generated over \( R/[G] \), we can find \( x \in R \) such that \( M(R) = M[x] \). Since \( M \) is completely faithful over \( R/[G] \), this in turn implies that \( M(R) \) is pseudo-null over \( R/[G] \). \( \square \)

Therefore, the above result shows that in considering completely faithfulness for a \( R/[G] \)-module, we must first cut off its \( R \)-torsion submodule. In view of this, the analogue of Property CF for Hida deformations is as follows.

**Property CF (for Hida deformations).** For every noncommutative strongly admissible \( p \)-adic Lie extension \( F_\infty \) of \( F \), \( X(A/F_\infty)/X(A/F_\infty)(R) \) is either a pseudo-null \( R/[G] \)-module or a completely faithful \( R/[G] \)-module.

It would be of interest to prove a control theorem connecting the Property CF in both cases. At this point of writing, we are only able to establish such a relation for a specific class of strongly admissible \( p \)-adic Lie extensions (see Theorem 5.7). We now record the following lemma which is the analogue of Lemma 3.1 and has a similar proof.

**Lemma 4.2.** Let \( F_\infty \) be a noncommutative strongly admissible \( p \)-adic Lie extension of \( F \) with \( G = \text{Gal}(F_\infty/F) \). Suppose that \( X(A/F_\infty) \) is finitely generated over \( R[H] \). Then the following statements are equivalent.

(a) \( X(A/F_\infty) \) is completely faithfully over \( R[H] \).

(b) \( X(A/F_\infty)/X(A/F_\infty)(R) \) is completely faithfully over \( R[H] \).

As in the case of elliptic curves, we recall the \( \mathfrak{M}_H(G) \)-conjecture for Hida deformations which was first stated in [7].

**\( \mathfrak{M}_H(G) \)-Conjecture (for Hida deformations).** For every admissible \( p \)-adic Lie extension \( F_\infty \) of \( F \), \( X(A/F_\infty)/X(A/F_\infty)(R) \) is a finitely generated \( R[H] \)-module.

A parallel argument to that in Theorem 3.6 will establish the following theorem which provides evidence to our conjecture for the Hida deformations situation.

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Theorem 4.3. Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that the following statements hold.

(i) The $\mathfrak{M}_H(G)$ conjecture holds for $X(A/F_\infty)$.

(ii) $G$ satisfies (NH).

Then Property CF holds for $X(A/F_\infty)/X(A/F_\infty)(R)$.

We can also establish the following analogue of Corollary 3.7. Before that, we need to introduce some notation. For every prime $v$ of $F$ above $p$, denote $G_v$ (resp., $I_v$) to be the decomposition group (resp. the inertia group) of some prime of $F_\infty$ above $v$ in $G$. Recall that the decomposition group $G_p$ at $p$ acts on $T/F^+T$ via an unramified character $\eta$ (for instance, see [20, Section 1]). Denote by $Fr_p$ the Frobenius at $p$.

Corollary 4.4. Retain the assumptions of the previous theorem. Suppose further that the following statements hold.

(i) For every prime $v$ of $F$ above $p$, either the decomposition group of $\text{Gal}(F_\infty/F)$ at $v$ has dimension $\geq 3$, or $\dim G_v = \dim I_v = 2$ and $\eta(Fr_p)$ is not of finite order.

(ii) For every prime $v$ of $F$ in $S$ but not above $p$, the decomposition group of $\text{Gal}(F_\infty/F)$ at $v$ has dimension $\geq 2$.

Then the $R[[C]]$-torsion submodule of $X(A/F_\infty)/X(A/F_\infty)(R)$ is zero, where $C$ is the centre of $G$.

Proof. By [20, Theorems 6.10 and 6.12], $X(A/F_\infty)$ has no nonzero pseudo-null $R[[G]]$-submodules. By a similar argument to that in [22, Lemma 4.2], one can deduce from this that $X(A/F_\infty)/X(A/F_\infty)(R)$ has no nonzero pseudo-null $R[[G]]$-submodules. The remainder of the proof proceeds as in Corollary [37].

5 Control theorems for Property CF

In this section, we will prove control theorems for Property CF for a specific subclass of strongly admissible $p$-adic Lie extensions. Readers may compare our results with those in [17], where the author established control theorems for faithful dual Selmer groups only but for a larger class of strongly admissible $p$-adic Lie extensions than those considered in this section. We first prove the control theorem which concerns extensions of strongly admissible $p$-adic Lie extensions. We will assume that $p \geq 5$ throughout the section.

Theorem 5.1. Let $p \geq 5$. Let $E$ be an elliptic curve over $F$ with either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$. Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that the following statements hold.

(i) The $\mathfrak{M}_H(G)$ conjecture holds for $X(E/F_\infty)$. 

Proof. 

...
(ii) \( G = H \times \Gamma \), where \( H \cong N \times \mathbb{Z}_p^r \times H_0 \). Here \( r \geq 0 \), \( H_0 \) is a torsion-free compact \( p \)-adic analytic group, whose Lie algebra is split semisimple over \( \mathbb{Q}_p \) and \( N \cong \mathbb{Z}_p \).

(iii) \( X(E/F_\infty) \) has no nonzero pseudo-null \( \mathbb{Z}_p[G] \)-submodules.

(iv) Set \( L_\infty := F_\infty^N \). For every prime \( v \) of \( F \) above \( p \), the decomposition group of \( \text{Gal}(L_\infty/F) \) at \( v \) has dimension \( \geq 2 \).

(v) Property CF holds for \( X_f(E/L_\infty) \).

Then Property CF holds for \( X_f(E/F_\infty) \).

Remark 5.2. Note that condition (iii) is satisfied in many cases (for instance, see [3] Corollary 2.8), [10] Theorem 3.2], [11] Theorem 2.6 and [19] Theorem 5.1).

Before proving Theorem 5.1, we will need to first establish the following algebraic result.

Proposition 5.3. Let \( p \geq 5 \). Given that \( G = H \times \Gamma \), where \( H \cong N \times \mathbb{Z}_p^r \times H_0 \). Here \( r \geq 0 \), \( H_0 \) is a torsion-free compact \( p \)-adic analytic group, whose Lie algebra is split semisimple over \( \mathbb{Q}_p \) and \( N \cong \mathbb{Z}_p \). Let \( M \) be \( \mathbb{Z}_p[G] \)-module which is finitely generated over \( \mathbb{Z}_p[H] \), has no nonzero pseudo-null \( \mathbb{Z}_p[G] \)-submodules and has the property that \( H_1(N,M) = 0 \). If \( M_N \) is a pseudo-null \( \mathbb{Z}_p[G/N] \)-module (resp. completely faithful \( \mathbb{Z}_p[G/N] \)-module), then \( M \) is a pseudo-null \( \mathbb{Z}_p[G/N] \)-module (resp. completely faithful \( \mathbb{Z}_p[G/N] \)-module).

The crucial ingredient to the proof of Proposition 5.3 is the following theorem of Csige [8] Theorem 2.4.1 which generalizes a previous result of Ardakov [11] Theorem 1.3].

Theorem 5.4 (Ardakov, Csige). Let \( p \geq 5 \). Given that \( G \cong \mathbb{Z}_p^r \times H_0 \times \Gamma \). Here \( r \geq 0 \), \( H_0 \) is a torsion-free compact \( p \)-adic analytic group, whose Lie algebra is split semisimple over \( \mathbb{Q}_p \). Let \( M \) be \( \mathbb{Z}_p[G] \)-module which has no nonzero pseudo-null \( \mathbb{Z}_p[G] \)-submodules. Write \( Z = \mathbb{Z}_p^r \times \Gamma \). Then \( M \) is a completely faithful \( \mathbb{Z}_p[G] \)-module if and only if the \( \mathbb{Z}_p[Z] \)-torsion submodule of \( M \) is trivial.

We are now set to prove Proposition 5.3.

Proof of Proposition 5.3. We first consider the case when \( M_N \) is a pseudo-null \( \mathbb{Z}_p[G/N] \)-module. Since \( M \) is finitely generated over \( \mathbb{Z}_p[H] \), we have that \( M_N \) is finitely generated over \( \mathbb{Z}_p[H/N] \). By Theorem 2.3 this implies that \( M_N \) is a finitely generated torsion \( \mathbb{Z}_p[H/N] \)-module. By Lemma 2.3 we have

\[
0 = \text{rank}_{\mathbb{Z}_p[H/N]} M_N = \text{rank}_{\mathbb{Z}_p[H]} M + \text{rank}_{\mathbb{Z}_p[H/N]} H_1(N,M) = \text{rank}_{\mathbb{Z}_p[H]} M,
\]

where the last equality follows from the fact that \( H_1(N,M) = 0 \). Hence we have that \( M \) is a finitely generated torsion \( \mathbb{Z}_p[H] \)-module, or equivalently, a pseudo-null \( \mathbb{Z}_p[G] \)-module.

Now we consider the case when \( M_N \) is a completely faithful \( \mathbb{Z}_p[G/N] \)-module. Set \( Z = N \times \mathbb{Z}_p^r \times \Gamma \). Let \( W \) be the \( \mathbb{Z}_p[Z] \)-torsion submodule of \( M \). By Theorem 5.4 it suffices to show that \( W = 0 \). Since \( Z \) is a central subgroup of \( G \) and \( M \) is finitely generated over \( \mathbb{Z}_p[G] \), we have \( W = M[z] \) for some \( z \in \mathbb{Z}_p[Z] \).
It follows from this that $M/W = zM$ is a $\mathbb{Z}_p[G]$-submodule of $M$. Since $H_1(N, M) = 0$ by hypothesis and $H_1(N, -)$ is left exact, we have $H_1(N, M/W) = 0$. Similarly, we also have $H_1(N, W) = 0$ which is equivalent to saying that $W[\gamma_N - 1] = 0$, where $\gamma_N$ is a topological generator of $N$. Note that the augmentation kernel $I_N$ of $\mathbb{Z}_p[G] \rightarrow \mathbb{Z}_p[G/N]$ is generated by $\gamma_N - 1$. Now if $z \in I_N$, then we may write $z = z'(\gamma_N - 1)^m$ for some $z' \notin I_N$ and integer $m$ (since $\gamma_N - 1$ is central in $\mathbb{Z}_p[G]$). By virtue of $W[\gamma_N - 1] = 0$, one checks easily that $W = M[z] = M[z']$. Therefore, we may assume that the $z$ we choose initially does not lies in $I_N$, or in other words, the image of $z$ in $\mathbb{Z}_p[G/N]$, which we denote by $\bar{z}$, is nonzero. Now, applying $N$-invariant to the short exact sequence

$$0 \rightarrow W \rightarrow M \rightarrow M/W \rightarrow 0,$$

we obtain a short exact sequence

$$0 \rightarrow W_N \rightarrow M_N \rightarrow (M/W)_N \rightarrow 0.$$  

(Here we make use of the above observation that $H_1(N, M/W) = 0$.) Clearly, $\bar{z}$ annihilates $W_N$ and lies in $\mathbb{Z}_p[Z/N]$. In particular, this implies that $W_N$ is contained in $\mathbb{Z}_p[Z/N]$-torsion submodule of $M_N$. But since $M_N$ is a completely faithful $\mathbb{Z}_p[G/N]$-module, it follows from a similar argument to that in Lemma 1.1 that the $\mathbb{Z}_p[Z/N]$-torsion submodule of $M_N$ is a pseudo-null $\mathbb{Z}_p[G/N]$-module, and this in turn implies that $W_N$ is a pseudo-null $\mathbb{Z}_p[G/N]$-module. On the other hand, since $M$ is finitely generated over $\mathbb{Z}_p[H]$, so is $W$. We may now apply a similar argument to that in the previous paragraph to conclude that $W$ is a pseudo-null $\mathbb{Z}_p[G]$-module. Since $M$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodule, this forces $W = 0$. The proof of Proposition 5.3 is now complete. □

Remark 5.5. We emphasis that we do not assume nor require that $M_N$ has no nonzero pseudo-null $\mathbb{Z}_p[G/N]$-submodules. In fact, in our application, $M_N$ may have nonzero pseudo-null $\mathbb{Z}_p[G/N]$-submodules (see the proof of Theorem 5.1).

Remark 5.6. We thought that it may be worthwhile to mention the following. In the proof of Proposition 5.3 when trying to show that $W_N$ is torsion over $\mathbb{Z}_p[Z/N]$, one may like to make use of Lemma 2.4 to obtain the following

$$\text{rank}_{\mathbb{Z}_p[Z/N]} W_N = \text{rank}_{\mathbb{Z}_p[Z]} W + \text{rank}_{\mathbb{Z}_p[Z/N]} H_1(N, W)$$

and try to argue from there. However, to be able to apply Lemma 2.4 meaningfully, we require $W$ to be finitely generated $\mathbb{Z}_p[Z]$. But this needs not be true a priori (for instance, see [10, Theorem 6.5]).

We can now prove our control theorem.

Proof of Theorem 5.1. It suffices to verify that all the hypothesis of Proposition 5.3 are satisfied (taking $M = X_f(E/F_\infty)$). Assumption (i) guarantees us that $X_f(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$. It follows from assumption (iii) and [22, Lemma 4.2] that $X_f(E/F_\infty)$ has no nonzero pseudo-null $\mathbb{Z}_p[G]$-submodules. By the argument in [17, Remark 6.2], we have $H_1(N, X(E/F_\infty)) = 0$. Since $X_f(E/F_\infty) =$
$p^n X(E/F_\infty)$ for a large enough $m$, we also have $H_1(N, X_f(E/F_\infty)) = 0$ by the left exactness of $H_1(N, -)$. Therefore, it remains to show that $X_f(E/F_\infty)_N$ is completely faithful over $\mathbb{Z}_p[G/N]$. By the argument of [7] Lemma 2.4, one can show that the map 

$$\alpha : X(E/F_\infty)_N \rightarrow X(E/L_\infty)$$

has kernel which is finitely generated over $\mathbb{Z}_p[H/N]$, and cokernel which is finitely generated over $\mathbb{Z}_p$. We shall now show that $\text{ker } \alpha$ is a finitely generated torsion $\mathbb{Z}_p[H/N]$-module, and therefore, is pseudo-null over $\mathbb{Z}_p[G/N]$. Consider the following commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & S(E/L_\infty) & \rightarrow & H^1(G_S(L_\infty), E_{p_\infty}) & \rightarrow & \bigoplus_{v \in S} J_v(E/F_\infty) & \rightarrow & 0 \\
& & \downarrow^a & & \downarrow^b & & \downarrow^{c=\oplus c_v} & & \\
0 & \rightarrow & S(E/F_\infty)^N & \rightarrow & H^1(G_S(F_\infty), E_{p_\infty}) & \rightarrow & \bigoplus_{v \in S} J_v(E/F_\infty)^N & \rightarrow & 0
\end{array}
$$

with exact rows. Let $v \in S$ such that the decomposition group of $\text{Gal}(L_\infty/F)$ at $v$ has dimension 1. Then $v$ does not divide $p$ by assumption (iv). Let $w$ be a prime of $F_\infty$ above $v$. By abuse of notation, we will also denote by $w$ the prime of $L_\infty$ below $w$. Therefore, we have $L_{\infty,w} = F_{\infty,w}^{\text{cyc}} = F_{v_\infty}^{\text{ur},p}$, where $F_{v_\infty}^{\text{ur},p}$ is the maximal unramified pro-$p$ extension of $F_v$. Since $F_{\infty}$ is obtain from $L_\infty$ by adjoining a $\mathbb{Z}_p$-extension of $F$, it follows that $F_{\infty}/L_\infty$ is also unramified outside $p$. Since $L_{\infty,w}$ is the maximal unramified pro-$p$ extension of $F_v$, we have $L_{\infty,w} = F_{\infty,w}$, or in other words, we have $N_w = 0$, where $N_w$ is the decomposition group of $N = \text{Gal}(F_\infty/L_\infty)$ at $w$. In particular, we have $\text{ker } c_v = 0$ when the decomposition group of $\text{Gal}(L_\infty/F)$ at $v$ has dimension 1. It remains to consider $\text{ker } c_v$ for those $v$ such that the decomposition group of $\text{Gal}(L_\infty/F)$ at $v$ has dimension $\geq 2$. For each of these $v$, one can apply a similar argument in the spirit of the proof of [20] Lemma 8.7 to show that $\text{ker } c_v$ is a cofinitely generated torsion $\mathbb{Z}_p[H/N]$-module. In conclusion, we have that $\text{ker } c$, and hence $\text{coker } a$, is a cofinitely generated torsion $\mathbb{Z}_p[H/N]$-module. But $\text{coker } a$ is precisely the Pontryagin dual of $\text{ker } \alpha$. Hence we have shown what we want.

Now consider the following commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & X(E/F_\infty)(p)_N & \rightarrow & X(E/F_\infty)_N & \rightarrow & X_f(E/F_\infty)_N & \rightarrow & 0 \\
& & \downarrow^{\alpha'} & & \downarrow^{\alpha} & & \downarrow^{\alpha''} & & \\
0 & \rightarrow & X(E/L_\infty)(p) & \rightarrow & X(E/L_\infty) & \rightarrow & X_f(E/L_\infty) & \rightarrow & 0
\end{array}
$$

with exact rows (Here the top exact row follows from the above mentioned fact that $H_1(N, X_f(E/F_\infty)) = 0$). Since $\text{coker } a$ is pseudo-null over $\mathbb{Z}_p[G/N]$, so is $\text{coker } \alpha''$. We shall now show that $\text{ker } \alpha''$ is pseudo-null over $\mathbb{Z}_p[G/N]$. It follows from assumption (i) that $X_f(E/F_\infty)_N$ is finitely generated over $\mathbb{Z}_p[H/N]$ and this in turn implies that $\text{ker } \alpha''$ is finitely generated over $\mathbb{Z}_p[H/N]$. Combining this with the above observation that $\text{coker } a$ is finitely generated over $\mathbb{Z}_p$, we have that $\text{coker } \alpha'$ is finitely generated over $\mathbb{Z}_p[H/N]$. Since $\text{coker } \alpha'$ is $p$-primary, it follows from Lemma 2.2 (b) and (c) that $\text{coker } \alpha'$ is pseudo-null
over \( \mathbb{Z}_p[G/N] \). Combining this with the above observation that \( \ker \alpha \) is also pseudo-null over \( \mathbb{Z}_p[G/N] \), we have that \( \ker \alpha'' \) is pseudo-null over \( \mathbb{Z}_p[G/N] \). In conclusion, we have shown that the map

\[
\alpha'': X_f(E/F_\infty)_N \rightarrow X_f(E/L_\infty)
\]

has kernel and cokernel that are pseudo-null over \( \mathbb{Z}_p[G/N] \). Since \( X_f(E/L_\infty) \) is completely faithful over \( \mathbb{Z}_p[G/N] \) by hypothesis, so is \( X_f(E/F_\infty)_N \) by Lemma 2.1. This is what we want to show and the proof of Theorem 5.1 is now complete.

The next control theorem is in the direction of a Hida deformation. Recall that \( A \) is the \( R \)-cofree Galois module attached to the Hida deformation as defined at the end of Section 4, where \( A \) and has the property that \( X \) is a finitely generated \( R \)-module with the property that \( \ker \alpha'' \) is also pseudo-null over \( \mathbb{Z}_p[G/N] \). As before, we denote by \( X(A/F_\infty) \) the dual Selmer group of the Hida deformation.

**Theorem 5.7.** Let \( p \geq 5 \). Let \( F_\infty \) be a strongly admissible \( p \)-adic Lie extension of \( F \) with Galois group \( G \). Suppose that the following statements hold.

(i) The \( 2\mathfrak{M}_H(G) \) conjecture holds for \( X(A/F_\infty) \).

(ii) \( G = H \times \Gamma \), where \( H \cong \mathbb{Z}_p^r \times H_0 \). Here \( r \geq 0 \), and \( H_0 \) is a torsion-free compact \( p \)-adic analytic group, whose Lie algebra is split semisimple over \( \mathbb{Q}_p \).

(iii) \( X(A/F_\infty) \) has no nonzero pseudo-null \( R[G] \)-submodules.

(iv) For every \( v \in S \), the decomposition group of \( \text{Gal}(F_\infty/F) \) at \( v \) has dimension \( \geq 2 \).

(v) Property CF holds for \( X_f(E/F_\infty) \).

Then Property CF holds for \( X(A/F_\infty)/X(A/F_\infty)(R) \).

**Remark 5.8.** Condition (iii) is known to be satisfied in many cases (see [20] Theorems 6.10 and 6.12).

We record a preliminary lemma. Recall that a polynomial \( X^n + c_{n-1}X^{n-1} + \cdots + c_0 \) in \( \mathbb{Z}_p[X] \) is said to be a Weierstrass polynomial if \( p \) divides \( c_i \) for every \( 0 \leq i \leq n-1 \).

**Lemma 5.9.** Let \( M \) be a finitely generated \( R[G] \)-module with the property that \( M = M(R) \). If \( P \) is a prime ideal of \( R \) generated by \( X - a \) for some \( a \in \mathbb{Z}_p \) with the property that \( M[X - a] = 0 \). Then \( M/PM \) is a finitely generated \( p \)-primary \( \mathbb{Z}_p[G] \)-module.

**Proof.** By the finite generation of \( M \) over \( R[G] \), we can find a Weierstrass polynomial \( f(X) \) such that \( f(X)M = 0 \). Now write \( f(X) = (X - a)^m g(X) \), where \( g(a) \neq 0 \). Since \( M[X - a] = 0 \), one easily checks that \( g(X)M = 0 \). Clearly, \( g(a) \) is nonzero and annihilates \( M/PM \). Since \( a \in \mathbb{Z}_p \) and \( g(X) \) is also a Weierstrass polynomial, we have \( g(a) \in \mathbb{Z}_p \). This shows that \( M/PM \) is \( p \)-primary.

We can now give the proof of Theorem 5.7.
Proof of Theorem 5.7: The proof is essentially similar to that in Theorem 5.1, where we choose an identification $R[G] \cong \mathbb{Z}_p[N \times G]$ such that $M/P M = M/N$ for every $R[G]$-module $M$. Here $N \cong \mathbb{Z}_p$. The only thing which perhaps requires additional attention is to establish the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & X(A/F_\infty)(R)/P \\
\alpha' \downarrow & & \alpha \downarrow \\
0 & \to & X(E/F_\infty)(p)
\end{array}
\]

with exact rows. By the preceding lemma, $X(A/F_\infty)(R)/P$ is $p$-primary (noting that $X(A/F_\infty)(R)\cdot (X - a) = H_1(N, X(A/F_\infty)R)) = 0$ by a similar argument to that in Theorem 5.1). This in turn implies that $X(A/F_\infty)(R)/P$ is sent into $X(E/F_\infty)(p)$ under $\alpha$, and thus inducing the required maps $\alpha'$ and $\alpha''$. The remainder of the proof proceeds as in Theorem 5.1.

The following is an analogue of Theorem 5.1 for Hida deformations which has a parallel proof that we will omit.

**Theorem 5.10.** Let $p \geq 5$. Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that the following statements hold.

(i) The $\mathfrak{M}_H(G)$ conjecture holds for $X(A/F_\infty)$.

(ii) $G = H \times \Gamma$, where $H \cong N \times \mathbb{Z}_p^r \times H_0$. Here $r \geq 0$, $H_0$ is a torsion-free compact $p$-adic analytic group, whose Lie algebra is split semisimple over $\mathbb{Q}_p$ and $N \cong \mathbb{Z}_p$.

(iii) $X(A/F_\infty)$ has no nonzero pseudo-null $R[G]$-submodules.

(iv) Set $L_\infty := F_\infty^N$. For every prime $v$ of $F$ above $p$, the decomposition group of $\text{Gal}(L_\infty/F)$ at $v$ has dimension $\geq 2$.

(v) Property CF holds for $X(A/L_\infty)/X(A/L_\infty)(R)$.

Then Property CF holds for $X(A/F_\infty)/X(A/F_\infty)(R)$.

6 Examples

In this section, we will give some examples to illustrate the results in this paper.

(a) Let $E$ be the elliptic curve 11a2 of Cremona’s table which is given by

\[y^2 + y = x^3 - x.\]

Take $p = 5$, $F = \mathbb{Q}(\mu_5)$ and $L_\infty = \mathbb{Q}(\mu_{5\infty}, 11^{5-\infty})$. Let $F_\infty$ be a strongly admissible 5-adic Lie extension of $F$ that contains $L_\infty$ and that the group $N = \text{Gal}(F_\infty/L_\infty)$ satisfies the conditions in Theorem 3.6. As observed in [11, 17], $X(E/F_\infty)$, and hence $X_f(E/F_\infty)$, has positive $\mathbb{Z}_5[H]$-rank. It is not difficult to
show that if \( E' \) is either \( 11a1 \) or \( 11a3 \), then \( X_f(E'/F_\infty) \) also has positive \( \mathbb{Z}_5[H] \)-rank. Theorem 5.6 then tells us that \( X_f(E/F_\infty) \) is completely faithful over \( \mathbb{Z}_5[G] \), and in particular satisfies Conjecture CF, for \( E = 11a1, 11a2 \) and \( 11a3 \).

Furthermore, if \( F_\infty \) satisfies the conditions in Corollary 5.7 then we have that the \( \mathbb{Z}_5[C] \)-torsion submodule of \( X_f(E/F_\infty) \) is zero, where \( C \) is the center of \( G = \text{Gal}(F_\infty/F) \). Some interesting examples of strongly admissible 5-adic extensions \( F_\infty \) of \( F \) which satisfies the conditions of Theorem 5.6 and Corollary 5.7 that one can take are

\[
M_\infty(\mu_{5^\infty}, 2^{5^{-\infty}}, 11^{5^{-\infty}}), \quad M_\infty(\mu_{5^\infty}, 3^{5^{-\infty}}, 11^{5^{-\infty}}), \quad M_\infty(\mu_{5^\infty}, 2^{5^{-\infty}}, 3^{5^{-\infty}}, 11^{5^{-\infty}}),
\]

where \( M_\infty \) is any \( \mathbb{Z}_5 \)-extension of \( F \) disjoint from \( F^{\text{cyc}} \) for \( 1 \leq r \leq 2 \).

(b) Let \( p = 5 \). Let \( E \) be the elliptic curve \( 21a4 \) of Cremona’s tables given by

\[
y^2 + xy = x^3 + x.
\]

Let \( p = 5 \) and \( F = \mathbb{Q}(\mu_5) \). As discussed in [3 Section 7], \( X(E/F) \) is finite as suggested by its \( p \)-adic \( L \)-function which in turn implies that \( X(E/F^{\text{cyc}}) \) has trivial \( \mu_5 \)-invariant and \( \lambda \)-invariant. We will assume this latter property throughout our discussion here. Let \( A \) be an elliptic curve \( 66c1 \) of Cremona’s tables given by

\[
A : y^2 + xy = x^3 - 45x - 81.
\]

Set \( L_\infty = F(A_{5^\infty}) \). Since \( A \) has no complex multiplication and \( A(\mathbb{Q})_{\text{tor}} \cong \mathbb{Z}/2 \times \mathbb{Z}/5 \), it follows that \( L_\infty \) is a noncommutative pro-\( p \) extension of \( F \). Since \( p \geq 5 \), \( L_\infty \) is in fact a strongly admissible \( p \)-adic Lie extension of \( F \).

Let \( P_0, P_1, P_2 \) be defined as in [3 Definition 1.2]. Then in this situation, we have \( P_0 = \{2, 3, 11\} \). Since \( E \) has split multiplicative reduction at \( 3 \) and \( 3 \) does not split in \( F/\mathbb{Q} \), we have \( P_1 = \{3\} \). We shall now show that \( P_2 \) is empty. As observed in [3 Section 7], \( 2 \) does not lie in \( P_2 \). Now \( 11 \) splits completely over \( F/\mathbb{Q} \), and therefore, for every prime \( u \) of \( F \) above \( 11 \), we have \( F_u = \mathbb{Q}_{11} \). Since \( E \) has good reduction at \( 5 \), we have an injection \( E(\mathbb{Q}_{11})[5] \hookrightarrow E(\mathbb{F}_{11}) \) (see [21 Chap. VII., Proposition 3.1(b)]).

By inspection, we have \( |E(\mathbb{F}_{11})| = 8 \). This in turn implies that \( E(\mathbb{Q}_{11})[5] = 0 \) and so all the primes of \( F \) above \( 11 \) do not lie in \( P_2 \). (Alternatively, to see that \( E(\mathbb{Q}_{11})[5] = 0 \), one can also prove by contradiction. Suppose that \( E(\mathbb{Q}_{11})[5] \neq 0 \). Then since \( |E(\mathbb{Q})| = 4 \), it follows from the injections \( E(\mathbb{Q}) \hookrightarrow E(\mathbb{F}_{11}) \) and \( E(\mathbb{Q}_{11})[5] \hookrightarrow E(\mathbb{F}_{11}) \) that 20 divides \( |E(\mathbb{F}_{11})| \). But by the Hasse bound (cf. [21 Chap. V., Theorem 1.1]), we have

\[
|E(\mathbb{F}_{11})| \leq 11 + 1 + 2\sqrt{11} = 18.633... < 20
\]

and this gives the required contradiction.) Now we may apply [3 Corollary 6.3] to conclude that \( X(E/L_\infty) \) is completely faithfully over \( \mathbb{Z}_5[\text{Gal}(L_\infty/F)] \) (note that \( X(E/L_\infty) \) is finitely generated over \( \mathbb{Z}_5[\text{Gal}(L_\infty/F^{\text{cyc}})] \)).

Now let \( F_\infty \) be the compositum of \( L_\infty \) and \( M_\infty \), where \( M_\infty \) is any \( \mathbb{Z}_5 \)-extension of \( F \) disjoint from \( F^{\text{cyc}} \) for \( 1 \leq r \leq 2 \). All the hypothesis in Theorem 5.1 can be verified easily. We will just make the
remark that assumption (iv) follows from the fact that \( A \) has good ordinary reduction at 5 and \cite{4} Lemma 2.8(ii). Therefore, we have that \( X(E/F_\infty) \) is a completely faithful \( \mathbb{Z}_5[[\text{Gal}(F_\infty/F)]] \)-module by an iterative application of Theorem \ref{5.1}. One can then apply Proposition \ref{5.3} repeatedly to show that \( X_f(E'/F_\infty) \) is also completely faithful over \( \mathbb{Z}_5[[G]] \) for \( E' = 21a1, 21a2, 21a3, 21a5, 21a6 \).

We finally mention that one can also combine the main result of Csige with a similar argument in \cite{3} Section 6] to obtain the completeness faithfulness for \( X(E/F_\infty) \) directly. It would be of interest to have an example that is not covered by the discussion in \cite{3} Section 6] (i.e., the \( \mathbb{Z}_p[[H]] \)-rank of \( X_f(E/F_\infty) \) is \( \geq 2 \)) but which can be tackled by Theorem \ref{5.6}. Unfortunately, at present, the author does not have one.

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