Bandit Algorithms for Prophet Inequality and Pandora’s Box

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Abstract

The Prophet Inequality and Pandora’s Box problems are fundamental stochastic problem with applications in Mechanism Design, Online Algorithms, Stochastic Optimization, Optimal Stopping, and Operations Research. A usual assumption in these works is that the probability distributions of the $n$ underlying random variables are given as input to the algorithm. Since in practice these distributions need to be learned, we initiate the study of such stochastic problems in the Multi-Armed Bandits model.

In the Multi-Armed Bandits model we interact with $n$ unknown distributions over $T$ rounds: in round $t$ we play a policy $x(t)$ and receive a partial (bandit) feedback on the performance of $x(t)$. The goal is to minimize the regret, which is the difference over $T$ rounds in the total value of the optimal algorithm that knows the distributions vs. the total value of our algorithm that learns the distributions from the partial feedback. Our main results give near-optimal $\tilde{O}(\text{poly}(n)\sqrt{T})$ total regret algorithms for both Prophet Inequality and Pandora’s Box.

Our proofs proceed by maintaining confidence intervals on the unknown indices of the optimal policy. The exploration-exploitation tradeoff prevents us from directly refining these confidence intervals, so the main technique is to design a regret upper bound that is learnable while playing low-regret Bandit policies.
1 Introduction

The field of Stochastic Optimization deals with optimization problems under uncertain inputs and has had tremendous success since the work of Bellman in 1957 [Bel57]. A standard model is that the inputs are random variables that are drawn from known probability distributions. The goal is to design a policy (an adaptive algorithm) to optimize the expected objective function. Examples include Prophet Inequality [HKS07, CHMS10, KW12, Rub16], Pandora’s Box [KWW16, Sin18b, GKS19], Stochastic Probing [CIK+09, GN13, AN16, GNS17], and Optimal Auction Design [Har22, Rou16]. Most prior works assume that the underlying distributions are known to the algorithm and the challenge is in computing an (approximately) optimal policy. In many applications, however, the distributions are unknown and need to be learned while already making decisions.

In this paper we consider Stochastic Optimization problems in an Online Learning model with partial “bandit” feedback. In particular, we will study in this model the fundamental Prophet Inequality and Pandora’s Box problems that have applications in areas such as Mechanism Design, Online Algorithms, Microeconomics, Operations Research, and Optimal Stopping.

Formally, we study stochastic learning problems that are played over $T$ rounds: in each round $t \in [T]$ we play some policy $x(t)$ and receive a partial feedback on its performance for the underlying unknown-but-fixed distributions. The goal is to minimize the regret, which is the difference over $T$ rounds in the expected total value of the optimal algorithm that knows the underlying distributions and the total value of our algorithm that learns distributions from partial feedback.

The Bandits model has been extensively studied in “single-stage” stochastic learning problems; see, e.g., books [LS20, Sli19, BC12] and related work in Section 1.4. Although both Prophet Inequality and Pandora’s Box admit polytime optimal policies when the distributions are given, these are multi-stage (adaptive) policies where we have limited understanding for unknown distributions.

Can we design low regret Bandit algorithms for multi-stage stochastic problems?

Our main result is an affirmative answer to this question for both the Prophet Inequality and Pandora’s Box problems: we design near-optimal $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret algorithms\(^1\).

1.1 Prophet Inequality under Bandit Feedback

Consider the classical Optimal Stopping problem of Prophet Inequality [KS77, KS78, SC84] where we are given distributions $D_1, \ldots, D_n$ of $n$ independent random variables. The outcomes $X_i \sim D_i$ for $i \in [n]$ are revealed one-by-one and we have to immediately select/discard $X_i$ with the goal of maximizing the selected random variable in expectation. The optimal policy for this problem is given by a simple (reverse) dynamic program: always select $X_n$ on reaching it and select $X_i$ for $i < n$ if its value is more than the expected value of this optimal policy on $X_{i+1}, \ldots, X_n$. Thus, the optimal policy with expected value $\text{Opt}$ can be thought of as a fixed-threshold policy where we select $X_i$ iff $X_i > \tau_i$ for $\tau_i$ being the expected value of this policy after $i$. How to design this optimal policy for unknown distributions? (See Remark 1.2 on the “hindsight optimum” benchmark.)

As a motivating example, consider a scenario where you want to sell a perishable item (e.g., cheese) in the market each day for the entire year. For simplicity, assume that there are 8 buyers, one arriving in each hour between 9 am to 5 pm. Your goal is to set price thresholds for each hour to maximize the total value. If the buyer value distributions are known, this can be modeled as a Prophet Inequality problem with $n = 8$ distributions. However, for unknown value distributions

\(^1\)We use $\tilde{O}(\cdot)$ to hide $\text{polylog}(nT)$ factors in $O(\cdot)$. 

this becomes the following repeated game with a fixed arrival order where on each day you play some price thresholds and obtain a value along with feedback.

**Online Learning Prophet Inequality.** In this problem the distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of Prophet Inequality are unknown to the algorithm in the beginning. We make the standard normalization assumption that each $\mathcal{D}_i$ is supported on $[0, 1]$. Now we play a $T$ rounds repeated game$^2$: in round $t \in [T]$ we play a policy $x^{(t)}$, which is a set of $n$ thresholds, and receives as reward its value on freshly drawn independent random variables $X_1^{(t)} \sim \mathcal{D}_1, \ldots, X_n^{(t)} \sim \mathcal{D}_n$, i.e., the reward is $X_{\text{Alg}(t)}^{(t)}$ where $\text{Alg}(t) \in [n]$ is the smallest index with $X_i^{(t)} > x_i^{(t)}$. The goal is to minimize the total regret:

$T \cdot \text{Opt} - E \left[ \sum_{t=1}^T X_{\text{Alg}(t)}^{(t)} \right]$. Since per-round reward is bounded by 1, the goal is to get $o(T)$ regret. Moreover, standard examples show that every algorithm incurs $\Omega(\sqrt{T})$ regret; see Section 5.

An important question is what amount of feedback the algorithm receives after a round. One might consider a full-feedback setting, where after each round $t$ the algorithm gets to know the entire sample $X_1^{(t)}, \ldots, X_n^{(t)}$ as feedback, which could be used to update beliefs regarding the distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$. Here it’s easy to design an $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret algorithm using existing techniques. This is because after discretization, we may assume that the there are only $T$ candidate thresholds for each $X_i$, so there are only $T^n$ candidate policies. Now the classical multiplicative weights algorithm [AHK12] implies that the regret is $O(\sqrt{T \log(\# \text{policies})}) = \tilde{O}(\text{poly}(n)\sqrt{T})$. Although this naïve algorithm is not polytime, a recent work of [GHTZ21] can be interpreted as giving a polytime $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret algorithm under full-feedback$^3$. These results, however, do not extend to bandit feedback, where the algorithm does not see the entire sample.

**Bandit Feedback.** In many applications, it is unreasonable to assume that the algorithm gets the entire sample $X_1^{(t)}, \ldots, X_n^{(t)}$. For instance, in the above scenario of selling a perishable item, we may only see the winning bid (e.g., if you don’t run the shop and delegate someone else to sell the item at the given price thresholds). There are several reasonable partial feedback models, namely:

(a) We see $X_1^{(t)}, \ldots, X_{\text{Alg}(t)}^{(t)}$ but not $X_{\text{Alg}(t)+1}^{(t)}, \ldots, X_n^{(t)}$, meaning that we do not observe the sequence after it has been stopped.

(b) We see the index $\text{Alg}(t)$ and the value $X_{\text{Alg}(t)}$ that we select but no other $X_i$.

(c) We only see the value of $X_{\text{Alg}(t)}$ that we select and not even the index $\text{Alg}(t)$.

What is the least amount of feedback needed to obtain $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret?

Our first main result is that even with the most restrictive feedback (c), it is possible to obtain $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret. Thus, the same bounds also hold under (a) and (b). Note that these bounds are almost optimal because standard examples show that even with full feedback every algorithm incurs $\Omega(\sqrt{T})$ regret (see Section 5).

**Theorem 1.1.** There is a polytime algorithm with $O(n^3\sqrt{T \log T})$ regret for the Bandit Prophet Inequality problem where we only receive the selected value as the feedback.

We remark that it’s possible to improve the $n^3$ factor in Theorem 1.1, but we do not optimize it to keep the presentation clean.

One might wonder whether $\tilde{O}(\text{poly}(n)\sqrt{T})$ regret holds even for adversarial online learning, i.e., where $X_1^{(t)}, \ldots, X_n^{(t)}$ are chosen by an adversary in each round $t$ and we compete against the

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$^2$We will always assume $T \geq n$ since otherwise getting an $O(\text{poly}(n))$ regret algorithm is trivial.

$^3$Their results are in the PAC model for “strongly monotone” stochastic problems that imply $\tilde{O}(\sqrt{nT})$ regret under full-feedback using the standard doubling-trick.
optimal fixed-threshold policy in hindsight. In Section 5 we prove that this is impossible since every online learning algorithm incurs $\Omega(T)$ regret for adversarial inputs, even under full-feedback.

**Remark 1.2 (Hindsight Optimum).** There is a lot of work on Prophet Inequality (with Samples) where the benchmark is the expected hindsight optimum $\mathbb{E} [\max X_i]$; see Section 1.4. However, we will be interested in the more realistic benchmark of the optimal policy, or in other words the optimal solution to the underlying MDP, which is standard in stochastic optimization. Firstly, comparing to the hindsight optimum does not make any sense for most stochastic problems, including Pandora’s Box. Secondly, it gives us a much more fine-grained picture than comparing to the offline optimum [RWW20]. This might give the impression that there is nothing to be learned about the distributions for Prophet Inequality and sublinear regrets are impossible. However, this is incorrect as Theorem 1.1 obtains sublinear regret bounds w.r.t. the optimal policy.

### 1.2 Pandora’s Box under Bandit Feedback

The Pandora’s Box problem was introduced by Weitzman in 1979 [Wei79], motivated by Economic search applications. In the classical setting, we are given distributions $D_1, \ldots, D_n$ of $n$ independent random variables. The outcome $X_i \sim D_i$ for $i \in [n]$ can be obtained by the algorithm by paying a known inspection cost $c_i$. The goal is to find a policy to adaptively inspect a subset $S \subseteq [n]$ of the random variables to maximize utility: $\mathbb{E} [\max_{i \in S} X_i - \sum_{i \in S} c_i]$. Note that unlike the Prophet Inequality, we may now inspect the random variables in any order by paying a cost and we don’t have to immediately accept/reject $X_i$. Even though this problem has an exponential state space, Weitzman [Wei79] showed a simple optimal policy where we inspect in a fixed order (based on “indices”) along with a stopping rule. We study this problem in the Online Learning model where the distributions $D_i$ supported on $[0,1]$ are unknown-but-fixed. We will assume that the deterministic costs $c_i \in [0,1]$ are known to the algorithm.

Formally, in Online Learning for Pandora’s Box we play a $T$ rounds repeated game where in round $t \in [T]$ we play a policy $x^{(t)}$, which is an order of inspection along with a stopping rule. As reward, we receive our utility (value minus total inspection cost) on freshly drawn independent random variables $X_1^{(t)} \sim D_1, \ldots, X_n^{(t)} \sim D_n$. The goal is to minimize the total regret, which is the difference over $T$ rounds in the expected utility of the optimal algorithm that knows the underlying distributions and the total utility of our algorithm.

In the full-feedback setting the algorithm receives the entire sample $X_1^{(t)}, \ldots, X_n^{(t)}$ as feedback in each round. Here, it’s again easy to design an $O(\text{poly}(n) \sqrt{T})$ regret polytime algorithm relying on the results in [GHTZ21, FL20]. But these results do not extend to partial feedback.

There are again multiple ways of defining partial feedback. For example, we could see the values of all $X_i$ for $i \in S$, meaning that we get to see the values of the inspected random variables. Indeed, our results again apply to the most restrictive form of partial feedback: We only see the total utility of a policy and not even the indices of inspected random variables or any of their values.

**Theorem 1.3.** There is a polytime algorithm with $O(n^{5.5} \sqrt{T \log T})$ regret for the Bandit Pandora’s Box problem where we only receive utility (selected value minus total cost) as the feedback.

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4If the costs $c_i$ are unknown but fixed then the problem trivially reduces to the case of known costs. This is because we could simply open each box once without keeping the prize inside and receive as feedback the cost $c_i$. 
Again, standard examples show that every algorithm incurs \( \Omega(\sqrt{nT}) \) regret even with full feedback; see Section 5. Furthermore, we prove in Section 5 that Theorem 1.3 cannot hold for adversarial online learning where \( X_1^{(t)}, \ldots, X_n^{(t)} \) are chosen by an adversary: every online learning algorithm incurs \( \Omega(T) \) regret for adversarial inputs, even under full-feedback.

### 1.3 High-Level Techniques

Let’s consider the general Prophet Inequality problem or the subproblem of Pandora’s Box where the optimal order is given. In both cases, a policy is described by \( n \) thresholds \( \tau_1, \ldots, \tau_n \in [0,1] \), defining when to stop inspecting. It would be tempting to apply standard multi-armed bandit algorithms to maximize the expected reward over \( [0,1]^n \). However, such approaches are bound to fail because the expected reward is not even continuous\(^5\), let alone convex or Lipschitz. Discretizing the action space and applying a bandit algorithm only leads to \( \Omega(T^{2/3}) \) regret. Another reasonable approach is to try to learn the distributions \( \mathcal{D}_i \). However, recall that we only get feedback regarding the overall reward of a policy and do not see which \( X_i \) is selected. It is possible to obtain samples from each \( X_i \) by considering policies that ignore all other boxes; however, such algorithms that use separate exploration and exploitation also have \( \Omega(T^{2/3}) \) regret.

Our algorithms combine exploration and exploitation. We maintain confidence intervals \([\ell_i, u_i]\) for \( i \in [n] \) satisfying w.h.p. that the optimal thresholds \( \tau^*_i \in [\ell_i, u_i] \). The crucial difference from UCB algorithms [ACBF02] is that our confidence intervals apply to the price thresholds, meaning that they apply to the space of all policies, whereas in a UCB-style algorithm one has confidence intervals on the distributions. Interestingly, our algorithms do not actually try to learn the underlying distributions, apart from a short initialization phase where we obtain an estimate \( \hat{F}_i \) of the CDF of \( X_i \).

Afterwards, the confidence intervals are refined. This is the main challenge since we need to keep the regret during refinement bounded. Unlike UCB algorithms, we do not obtain unbiased samples from the distributions and cannot just play the “upper confidences” during refinement.

Our refinement works in \( O(\log T) \) phases. In each phase, we start with confidence intervals \([\ell_i, u_i]\) that satisfy: (i) \( \tau^*_i \in [\ell_i, u_i] \) and (ii) playing any thresholds within the confidence intervals incur at most some \( \epsilon \) regret. During the phase, we refine the confidence interval to \([\ell'_i, u'_i]\) while only playing thresholds within our original confidence intervals, so that we don’t incur much regret. We will show that the new confidence intervals satisfy that \( \tau^*_i \in [\ell'_i, u'_i] \) and that playing any thresholds within \([\ell'_1, u'_1], \ldots, [\ell'_n, u'_n]\) incur at most \( \frac{\epsilon}{2} \) regret. Thus, the regret bound goes down by a constant factor in each phase. Next, we describe this refinement.

**Bounding Function to Refine for** \( n = 2 \). To illustrate the idea behind a refinement phase, let’s discuss the case of \( n = 2 \); see Section 2 for more technical details. In this case, there is only one confidence interval \([\ell, u]\) that we have to refine. Our idea is to define a “bounding function” \( \delta(\cdot) \) such that the expected regret in a single round when using threshold \( \tau \in [\ell, u] \) is bounded by \( |\delta(\tau)| \). Ideally, we would like to choose the optimal threshold \( \tau^* \) for which \( \delta(\tau^*) = 0 \). However, this requires the knowledge of \( \delta \), which we don’t have since the distributions are unknown. Instead, we compute an estimate \( \hat{\delta} \) of \( \delta \) and construct the new confidence interval \([\ell', u']\) to include all \( \tau \) for which \( |\hat{\delta}(\tau)| \) is small. The main technical difficulty is to obtain \( \hat{\delta} \) while only playing low-regret policies. We achieve this by choosing \( \delta \) such that \( \delta \) can be obtained by using only the estimates \( \hat{F}_i \) of the CDF and the

\(^5\) For example, consider the Prophet Inequality instance in which \( X_1 \) is a distribution that returns \( \frac{1}{3} \) w.p. \( \frac{1}{3} \) and \( \frac{2}{3} \) otherwise, while \( X_2 \) is a distribution that always returns \( \frac{2}{3} \). The reward of this example is a piece-wise constant function: When \( \tau < \frac{1}{3} \) or \( \tau \geq \frac{2}{3} \), the expected reward is \( \frac{1}{3} \). When \( \frac{1}{3} \leq \tau < \frac{2}{3} \), the expected reward is \( \frac{2}{3} \).
empirical average rewards when choosing the boundaries of the confidence interval as thresholds. Note that we do not make any statements about the width of the confidence interval; we only ensure that the regret is bounded when choosing any threshold inside the confidence interval.

**Prophet Inequality for General** \( n \). In the case of general \( n \), each refinement phase updates the confidence intervals from the last random variable \( X_n \) to the first one \( X_1 \). To refine confidence interval \([\ell_i, u_i]\), we use our algorithm for the \( n = 2 \) case as a subroutine, i.e., we play \( \ell_i \) and \( u_i \) sufficiently many times keeping the other thresholds fixed. However, there are several challenges in this approach. The first important one is that the probability of reaching \( X_i \) will change depending on which thresholds are applied before it. We deal with this issue by always using thresholds from our confidence intervals that maximize the probability of reaching \( X_i \). Another important challenge while refining \([\ell_i, u_i]\) is that the current choice of thresholds for \( X_{i+1}, \ldots, X_n \) is not optimal, so we maybe learning a threshold different from \( \tau_i^* \). We handle this issue by choosing the other thresholds in a way that they only improve from phase to phase. We then leave some space in the confidence intervals to accommodate for the improvements in later phases.

**Pandora’s Box for General** \( n \). We still maintain confidence and refine them using ideas similar to Prophet Inequality for general \( n \). The main additional challenge arising in Pandora’s box is that the inspection ordering is not fixed. The optimal order is given by ordering the random variables by decreasing thresholds. However, there might be multiple orders consistent with our confidence intervals. Therefore, we keep a set \( S \) of constraints corresponding to a directed acyclic graph on the variables, where an edge from \( X_i \) to \( X_j \) means that \( X_i \) comes before \( X_j \) in the optimal order. We update this set by consider pairwise swaps. Then, during refinement of confidence interval \([\ell_i, u_i]\), we choose an inspection order satisfying these constraints while (approximately) maximizing a difference of products objective.

### 1.4 Further Related Work

There is a long line of work on both Prophet Inequality (PI) and Pandora’s Box (PB), so we only discuss the most relevant papers. For more references, see [Luc17, Sin18a]. Both PI and PB are classical single-item selection problems, but were popularized in TCS in [HKST09] and [KWW16], respectively, due to their applications in mechanism design. Extensions of these problems to combinatorial settings have been studied in [CHMS10, KW12, FGL15, FSZ16, Rub16, RS17, EFST20] and in [KWW16, Sin18b, GKS19, GJSS18, FTW+21], respectively. Although the optimal policy for PI with known distributions is a simple dynamic program, designing optimal policies for free-order or in combinatorial PI settings is challenging. Some recent works designing approximately-optimal policies are [ANSS19, PPSW21, SS21, LLP+21, BDL22].

Starting with Azar, Kleinberg, and Weinberg [AKW14], there is a lot of work on PI-with-Samples where the distributions are unknown but the algorithm has sample access to it [CDFS19, RWW20, GHTZ21, CDF+22]. These works, however, compete against the benchmark of expected hindsight optimum, so lose at least a multiplicative factor of \( 1/2 \) due the classical single-item PI and do not admit sublinear regret algorithms.

The field of Online Learning under both full- and bandit-feedback is well-established; see books [CBL06, BC12, Haz16, Sli19, LS20]. Most of the initial works focused on obtaining sublinear regret for single-stage problems (e.g., choosing the reward maximizing arm). The last decade has seen progress on learning multi-stage policies for tabular MDPs under bandit feedback; see [LS20, Chapter 38]. However, these algorithms have a regret that is polynomial in the state space, so they do not apply to PI and PB that have large MDPs.
Finally, there is some work at the intersection of Online Learning and Prophet Inequality/Pandora’s Box [EHLM19, ACG+22, GT22]. These models are significantly different from ours, so do not apply to our problems. The closest one is [GT22], where the authors consider Pandora’s Box under partial feedback (akin to model (a)), but for adversarial inputs (i.e., no underlying distributions). They obtain $O(1)$-competitive algorithms and leave open whether sublinear regrets are possible [Ger22]. Our lower bounds in Section 5.2 resolve this question by showing that sublinear regrets are impossible for adversarial inputs (even under full feedback), and one has to lose a multiplicative factor in the approximation.

2 Prophet Inequality and Pandora’s Box for $n = 2$

In this section we give $O(\sqrt{T}\log T)$ regret algorithms for both Bandit Prophet Inequality and Bandit Pandora’s Box problems with $n = 2$ distributions. We discuss this special case before since it’s already non-trivial and showcases one of our main ideas of designing a regret bounding function that is learnable while playing low-regret Bandit policies.

Our algorithms run in $O(\log T)$ phases, where the number of rounds doubles each phase. Starting with an initial confidence interval containing the optimal threshold $\tau^*$, the goal of each phase is to refine this interval such that the one-round regret drops by a constant factor for the next phase. In Section 2.1 we discuss each phase’s algorithm for Prophet Inequality with $n = 2$. In Section 2.2 we give a generic doubling framework that combines all phases to prove total regret bounds. Finally, in Section 2.3 we extend these ideas to Pandora’s Box with $n = 2$.

2.1 Prophet Inequality via an Interval-Shrinking Algorithm

We first introduce the setting of the Bandit Prophet Inequality Problem with two distributions. Let $D_1, D_2$ denote the two unknown distributions over $[0, 1]$ with cdfs $F_1, F_2$ and densities $f_1, f_2$. Consider a $T$ rounds game where in each round $t$ we play a threshold $\tau^{(t)} \in [0, 1]$ and receive as feedback the following reward:

- Independently draw $X_1^{(t)}$ from $D_1$. If $X_1^{(t)} \geq \tau^{(t)}$, return $X_1^{(t)}$ as the reward.
- Otherwise, independently draw $X_2^{(t)}$ from $D_2$ and return it as the reward.

The only feedback we receive is the reward, and not even which random variable gets selected.

If the distributions are known then the optimal policy is to play $\tau^* := \mathbb{E}[X_2]$ in each round. For $\tau \in [0, 1]$, let $R(\tau)$ be the expected reward of playing one round with threshold $\tau$, i.e.,

$$R(\tau) := F_1(\tau) \cdot \mathbb{E}[X_2] + \int_\tau^1 x \cdot f_1(x)dx = 1 + F_1(\tau)(\mathbb{E}[X_2] - \tau) - \int_\tau^1 F_1(x)dx,$$

where the second equality uses integration by parts. The total regret is $T \cdot R(\tau^*) - \sum_{t=1}^T R(\tau^{(t)})$.

Initialization. For the initialization, we get $\Theta(\sqrt{T}\log T)$ samples from both $D_1$ and $D_2$ by playing $\tau = 0$ and $\tau = 1$, respectively. This incurs $\Theta(\sqrt{T}\log T)$ regret since each round incurs at most 1 regret. The following simple lemma uses the samples to obtain initial distribution estimates.

Lemma 2.1. After getting $\Theta(\sqrt{T}\log T)$ samples from $D_1$ and $D_2$, we can:

- Calculate $\hat{F}_1(x)$ such that w.h.p.\footnote{In this paper, we say a statement holds “with high probability” when it holds with probability $1 - \frac{1}{T}$ for any} $|\hat{F}_1(x) - F_1(x)| \leq T^{-\frac{1}{4}}$ for all $x \in [0, 1]$ simultaneously.
• Calculate \( \ell \) and \( u \) such that \( u - \ell \leq T^{-\frac{1}{4}} \) and w.h.p. \( \mathbb{E}[X_2] \in [\ell, u] \).

Proof. The first statement follows the DKW inequality (Theorem A.3). After taking \( N = C \cdot \sqrt{T} \log T \) samples, the probability that \( \exists x \) s.t. \( |\hat{F}_1(x) - F_1(x)| > \varepsilon = T^{-\frac{1}{4}} \) is at most \( 2 \exp(-2N\varepsilon^2) = 2T^{-2C} \). So, the first statement holds w.h.p. when \( C \) is sufficiently large.

The second statement follows the Hoeffding’s Inequality (Theorem A.1). After taking \( N = C \cdot \sqrt{T} \log T \) samples, let \( \mu \) be the average reward. Let \( \ell = \mu - \varepsilon \) and \( u = \mu + \varepsilon \) for \( \varepsilon = \frac{1}{2}T^{-\frac{1}{4}} \).

Then, \( u - \ell \leq T^{-\frac{1}{4}} \) by definition. Since the reward of each sample in inside \([0, 1]\), by Hoeffding’s Inequality the probability that \( |\mu - \mathbb{E}[X_2]| > \varepsilon \) is bounded by \( 2 \exp(-2N\varepsilon^2) = 2T^{-2C} \). So, the second statement holds w.h.p. when \( C \) is sufficiently large. \( \square \)

Next we discuss our core algorithm.

Interval-Shrinking Algorithm. Starting with an initial confidence interval containing \( \tau^* = \mathbb{E}[X_2] \), our Interval-Shrinking algorithm (Algorithm 1) runs for \( \Theta(\frac{\log T}{\varepsilon^2}) \) rounds and outputs a refined confidence interval. In the following lemma, we will show that this refined interval still contains \( \tau^* \) and that the regret of playing any \( \tau \) inside this refined interval is bounded by \( O(\varepsilon) \).

|Algorithm 1: Interval-Shrinking Algorithm for Prophet Inequality|
|---|
|**Input:** Interval \([\ell, u]\), approximate cdf \( \hat{F}_1(x) \), and accuracy \( \varepsilon \). |
|1 Run \( \Theta(\frac{\log T}{\varepsilon^2}) \) rounds with \( \tau = \ell \). Let \( \hat{R}_\ell \) be the average reward. |
|2 Run \( \Theta(\frac{\log T}{\varepsilon^2}) \) rounds with \( \tau = u \). Let \( \hat{R}_u \) be the average reward. |
|3 For \( \tau \in [\ell, u] \), define \( \hat{\Delta}(\tau) := \hat{F}_1(u)(\tau - u) - \hat{F}_1(\ell)(\tau - \ell) + \int_\ell^u \hat{F}_1(x)dx \). |
|4 For \( \tau \in [\ell, u] \), define \( \hat{\delta}(\tau) := \hat{\Delta}(\tau) - (\hat{R}_u - \hat{R}_\ell) \). |
|5 Let \( \ell' := \min\{\tau \in [\ell, u] \text{ s.t. } \hat{\delta}(\tau) \geq -5\varepsilon\} \) and let \( u' := \max\{\tau \in [\ell, u] \text{ s.t. } \hat{\delta}(\tau) \leq 5\varepsilon\} \). |
|**Output:** \([\ell', u']\)|

Lemma 2.2. Suppose we are given:

• Initial interval \([\ell, u]\) of length \( u - \ell \leq T^{-\frac{1}{4}} \) and satisfying \( \tau^* \in [\ell, u] \).

• Distribution estimate \( \hat{F}_1(x) \) satisfying \( |F_1(x) - \hat{F}_1(x)| \leq T^{-\frac{1}{4}} \) for all \( x \in [0, 1] \) simultaneously.

Then, for \( \varepsilon > T^{-\frac{1}{2}} \) Algorithm 1 runs thresholds inside \([\ell, u]\) for \( \Theta(\frac{\log T}{\varepsilon^2}) \) rounds and outputs w.h.p. a sub-interval \([\ell', u']\) \( \subseteq [\ell, u] \) satisfying:

1. \( \tau^* \in [\ell', u'] \).
2. For every \( \tau \in [\ell', u'] \) the expected one-round regret of playing \( \tau \) is at most \( 10\varepsilon \).

Proof Overview of Lemma 2.2. The main idea is to define a bounding function

\[
\delta(\tau) := (F_1(u) - F_1(\ell)) \cdot (\tau - \tau^*).
\]

As we show in Claim 2.3 below, this function satisfies \( R(\tau^*) - R(\tau) \leq |\delta(\tau)| \) for all \( \tau \in [\ell, u] \), i.e., \( |\delta(\tau)| \) is an upper bound on the one-round regret when choosing \( \tau \) instead of \( \tau^* \). So, ideally, we would like to choose \( \tau \) that minimizes \( |\delta(\tau)| \). However, we do not know \( \delta(\tau) \). Therefore, we derive

fixed constant \( c \). One should notice that if we combine \( \text{poly}(T) \) number of “w.h.p.” statements with the union bound, the combined statement still holds with high probability.
an estimate \( \hat{\delta}(\tau) \) for all \( \tau \in [\ell, u] \) and discard \( \tau \) for which \( |\hat{\delta}(\tau)| \) is too large because these cannot be the minimizers.

In order to estimate \( \delta(\tau) \), we rewrite it in a different way as a sum of terms that can be estimated well. First, consider the difference in expected rewards when choosing thresholds \( u \) and \( \ell \), i.e.,

\[
R(u) - R(\ell) = (F_1(u) - F_1(\ell))\tau^* - \int_{\ell}^{u} x f_1(x) dx = F_1(u) \cdot (\tau^* - u) - F_1(\ell) \cdot (\tau^* - \ell) + \int_{\ell}^{u} F_1(x) dx,
\]

where we used integration by parts. Adding this with \( \delta(\tau) \) gives \( \delta(\tau) + (R(u) - R(\ell)) \) equals

\[
F_1(u)(\tau - u) - F_1(\ell)(\tau - \ell) + \int_{\ell}^{u} F_1(x) dx =: \Delta(\tau),
\]

which gives an alternate way of expressing \( \delta(\tau) = \Delta(\tau) - (R(u) - R(\ell)) \). (Another way of understanding the definition of \( \Delta(\tau) \) is that it represents the difference of playing thresholds \( u \) and \( \ell \), assuming that \( E[X_2] = \tau \).) So, we define the estimate

\[
\hat{\delta}(\tau) := \hat{\Delta}(\tau) - (\hat{R}_u - \hat{R}_\ell),
\]

where \( \Delta \) uses the estimate \( \hat{F}_1 \) instead of \( F_1 \) in (2) and to estimate \( \hat{R}_u \) and \( \hat{R}_\ell \) we use empirical averages obtained in the current phase. The advantage is that besides the coarse knowledge of \( \hat{F}_1 \) we assumed to be given, we only need to choose thresholds from within our current confidence interval to obtain \( \hat{\delta} \).

Claim 2.4 will show that \( |\hat{\delta}(\tau)| \) gives an upper bound on one-round regret with threshold \( \tau \).

Completing the Proof of Lemma 2.2. Now we complete the missing details. We first prove that \( |\hat{\delta}(\tau)| \) gives an upper bound on one-round regret with threshold \( \tau \).

Claim 2.3. If \( \tau, \tau^* \in [\ell, u] \), then \( R(\tau^*) - R(\tau) \leq |\hat{\delta}(\tau)| \).

Proof. Consider \( R(\tau^*) - R(\tau) \). The two settings are different only when \( X_1 \) is between \( \tau^* \) and \( \tau \), and the difference of the reward is bounded by \( |\tau^* - \tau| \).

\[
|F_1(\tau^*) - F_1(\tau)| \leq |\tau^* - \tau| \cdot |F_1(u) - F_1(\ell)| = |\delta(\tau)|,
\]

where the second inequality uses \( \tau^*, \tau \in [\ell, u] \) implies \( |F_1(\tau) - F_1(\tau^*)| \leq |F_1(u) - F_1(\ell)| \).

Next, we prove that \( \hat{\delta}(\tau) \) is a good estimate of \( \delta(\tau) \).

Claim 2.4. In Algorithm 1, if the conditions in Lemma 2.2 hold then w.h.p. \( |\hat{\delta}(\tau) - \delta(\tau)| \leq 5 \cdot \epsilon \) for all \( \tau \in [\ell, u] \) simultaneously.

Proof. Recall that \( \delta(\tau) = \Delta(\tau) - (R(u) - R(\ell)) \). We first bound the error \( |\hat{\Delta}(\tau) - \Delta(\tau)| \). Notice,

\[
|\hat{\Delta}(\tau) - \Delta(\tau)| \leq |F_1(u) - \hat{F}_1(u)| \cdot |\tau - u| + |F_1(\ell) - \hat{F}_1(\ell)| \cdot |\tau - \ell| + \int_{\ell}^{u} |F_1(x) - \hat{F}_1(x)| dx.
\]

The main observation is that all three terms on the right-hand-side can be bounded by \( T^{-\frac{1}{2}} \) since \( |F_1(x) - \hat{F}_1(x)| \leq T^{-\frac{1}{4}} \) and \( u - \ell \leq T^{-\frac{1}{4}} \). Hence, \( |\hat{\Delta}(\tau) - \Delta(\tau)| \leq 3T^{-\frac{1}{2}} \leq 3 \epsilon \).

Next, we bound the errors for \( \hat{R}_\ell - R(\ell) \) and for \( \hat{R}_u - R(u) \). For \( |\hat{R}_\ell - R(\ell)| \), notice that \( \hat{R}_\ell \) is an estimate of \( R(\ell) \) with \( N = C \cdot \log \frac{T}{\epsilon^2} \) samples. Since the reward of each sample is in \([0, 1]\), by Hoeffding’s Inequality (Theorem A.1) the probability that \( |\hat{R}_\ell - R(\ell)| > \epsilon \) is bounded by \( 2 \exp(-2N\epsilon^2) = 2T^{-2C} \). Then, \( |\hat{R}_\ell - R(\ell)| \leq \epsilon \) holds w.h.p. when \( C \) is sufficiently large. The error bound for \( |\hat{R}_u - R(u)| \) is identical.
Finally, summing the three errors completes the proof of Claim 2.4.

Now, we are ready to prove Lemma 2.2.

Proof of Lemma 2.2. We will assume that $|\hat{\delta}(\tau) - \delta(\tau)| \leq 5\epsilon$, which is true w.h.p. by Claim 2.4.

Observe that $\hat{\delta}(\tau)$ is a monotone increasing function because $\hat{\delta}'(\tau) = \hat{\Delta}'(\tau) = \hat{F}_1(u) - \hat{F}_1(\ell) \geq 0$. Therefore, according to the definition of $\ell'$ and $u'$, we have $[\ell', u'] = \{ \tau \in [\ell, u] : |\hat{\delta}(\tau)| \leq 5\epsilon \}$. Now, we can use this property to prove the two statements of this lemma separately.

For Statement 1, notice that $\delta(\tau^*) = 0$. Claim 2.4 gives $|\hat{\delta}(\tau^*)| \leq 5\epsilon$. Then, since $\tau^* \in [\ell, u]$ and $|\hat{\delta}(\tau^*)| \leq 5\epsilon$, we must have $\tau^* \in [\ell', u']$ as $[\ell', u'] = \{ \tau \in [\ell, u] : |\hat{\delta}(\tau)| \leq 5\epsilon \}$.

Next, we prove Statement 2. By Claim 2.3, it suffices to bound $|\delta(\tau)|$ for all $\tau \in [\ell', u']$. By Claim 2.4, we have w.h.p. for all $\tau \in [\ell', u']$ that $|\delta(\tau)| \leq |\hat{\delta}(\tau)| + 5\epsilon \leq 10\epsilon$, where the last inequality uses the definition of $\ell'$ and $u'$.

2.2 Doubling Framework for Low-Regret Algorithms

In this section we show how to run Algorithm 1 for multiple phases with a doubling trick to get $O(\sqrt{T} \log T)$ regret. Instead of directly proving the regret bound for Prophet Inequality with $n = 2$, we first give a general doubling framework that will later be useful for Prophet Inequality and Pandora’s Box problems with $n$ random variables:

Lemma 2.5. Consider an online learning problem with size $n$. Assume the one-round regret for every possible action is bounded by 1. Suppose there exists an action set-updating algorithm $\text{Alg}$ satisfying: Given accuracy $\epsilon$ and action set $A$, algorithm $\text{Alg}$ runs $\Theta(n^\alpha \log T)$ rounds in $A$ and outputs $A' \subseteq A$ satisfying the following with high probability:

- The optimal action in $A$ belongs to $A'$.
- For $a \in A'$, the one-round regret of playing $a$ is bounded by $\epsilon$.

Then, Algorithm 2 gives an $O(n^{\alpha/2} \sqrt{T} \log T)$ regret algorithm.

Algorithm 2: General Doubling Algorithm

Input: Time horizon $T$, problem size $n$, action space $A$, Algorithm $\text{Alg}$ and parameter $\alpha$.

1. Let $i = 1$, $\epsilon_1 = 1$, $A_1 = A$.
2. while $\epsilon_i > \Theta(n^{\alpha/2} \log T)$ do
   3. Call $\text{Alg}$ with input $\epsilon_i$ and $A_i$, and get output $A_{i+1}$.
   4. $\epsilon_{i+1} \leftarrow \frac{\epsilon_i}{2}$
   5. $i \leftarrow i + 1$
3. Run $a \in A_i$ for the remaining rounds.

The proof of the lemma uses simple counting; see Appendix B.

Based on Lemma 2.5, we can immediately give the Bandit Prophet Inequality regret bound.

Theorem 2.6. There exists an algorithm that achieves $O(\sqrt{T} \cdot \log T)$ regret w.h.p. for Bandit Prophet Inequality problem with two distributions.

Proof. The initialization runs $O(\sqrt{T} \log T)$ rounds, so the regret is $O(\sqrt{T} \log T)$. For the following interval shrinking procedure, Algorithm 1 matches the algorithm $\text{Alg}$ described in Lemma 2.5 with $\alpha = 0$. Therefore, applying Lemma 2.5 completes the proof.
2.3 Extending to Pandora’s Box with a Fixed Order

In order to extend the approach to Pandora’s Box, in this section we consider a simplified problem with a fixed box order. There are two boxes taking values in \([0, 1]\) from unknown distributions \(D_1, D_2\) with cdfs \(F_1, F_2\) and densities \(f_1, f_2\). The boxes have known costs \(c_1, c_2 \in [0, 1]\). We assume that we always pay \(c_1\) to observe \(X_1\) (i.e., \(E[X_1] > c_1\)), and then decide whether to observe \(X_2\) by paying \(c_2\). Indeed, it might be better to open the second box before the first box or not to open any box. We make these simplifying assumptions in this section to make the presentation cleaner. Generally, determining an approximately optimal order will be one of the main technical challenges that we will need to handle for general \(n\) in Section 4.

Formally, consider a \(T\) rounds game where in each round \(t\) we play a threshold \(\tau^{(t)} \in [0, 1]\) and receive as feedback the following utility:

- Independently draw \(X_1^{(t)}\) from \(D_1\). If \(X_1^{(t)} \geq \tau^{(t)}\), we stop and receive \(X_1^{(t)} - c_1\) as the utility.
- Otherwise, we pay \(c_2\) to see \(X_2^{(t)}\) drawn independently from \(D_2\), and receive \(\max\{X_1, X_2\} - (c_1 + c_2)\) as utility.

The only feedback we receive is the utility, and not even which random variable gets selected.

To see the optimal policy, define a gain function \(g(v) := E[\max\{0, X_2 - v\} - c_2]\) to represent the expected additional utility from opening \(X_2\) assuming we already have \(X_1 = v\), i.e.,

\[
g(v) = -c_2 + \int_v^1 (x - v)f_2(x)dx = -c_2 + (1 - v) - \int_v^1 F_2(x)dx. \quad (3)
\]

The optimal threshold \(\tau^*\) is now the solution to \(g(\tau^*) = 0\), i.e., \(E[\max\{X_2 - \tau^*, 0\}] = c_2\). Since our algorithm does not know \(F_2(x)\) but only an approximate distribution \(\hat{F}_2(x)\), we get an estimate \(\hat{g}(v)\) of \(g(v)\) by replacing \(F_2(x)\) with \(\hat{F}_2(x)\) in (3).

For \(\tau \in [0, 1]\), let reward function \(R(\tau)\) denote the expected reward of playing \(\tau\). With the definition of gain function \(g(v)\) and linearity of expectation, we can write

\[
R(\tau) := -c_1 + E[X_1] + \int_0^\tau f_1(x)g(x)dx.
\]

The total regret of our algorithm is now defined as \(T \cdot R(\tau^*) - \sum_{t=1}^T R(\tau^{(t)})\).

**Interval-Shrinking Algorithm.** Starting with an initial confidence interval \([\ell, u]\) containing \(\tau^*\), we again design an Interval-Shrinking algorithm (Algorithm 3) that runs for \(\Theta(\frac{\log T}{\epsilon})\) rounds and outputs a refined confidence interval \([\ell', u']\). We will show that this refined interval still contains \(\tau^*\) and that the regret of playing any \(\tau\) inside this refined interval is bounded by \(O(\epsilon)\). Now we give the algorithm and the theorem.

**Lemma 2.7.** Suppose we are given:

- Initial interval \([\ell, u]\) satisfying \(\tau^* \in [\ell, u]\), gain function \(|g(\tau)| \leq T^{-\frac{1}{4}}\), and bounding function \(|\delta(\tau)| \leq 16\epsilon\) where \(\delta\) is defined in (4).
- CDF estimate \(\hat{F}_1(x)\) which is constructed via \(\Theta(\frac{\log T}{\epsilon})\) new i.i.d. samples of \(X_1\).
- CDF estimate \(\hat{F}_2(x)\) which is constructed via \(\Theta(\frac{\log T}{\epsilon})\) new i.i.d. samples of \(X_2\).

Then, for \(\epsilon > T^{-\frac{1}{4}}\), Algorithm 3 runs thresholds inside \([\ell, u]\) for \(\Theta(m \log T)\) rounds and outputs w.h.p. a sub-interval \([\ell', u'] \subseteq [\ell, u]\) satisfying:

1. \(\tau^* \in [\ell', u']\).
2. Simultaneously for every \(\tau \in [\ell', u']\), we have \(|\delta(\tau)| \leq 8\epsilon\).
\textbf{Algorithm 3:} Interval-Shrinking Algorithm for Pandora’s Box

\begin{tabular}{l}
\textbf{Input:} \text{Interval } [\ell, u], \text{ length } m, \text{ and CDF estimates } \hat{F}_1(x), \hat{F}_2(x). \\
1 Run $\Theta\left(\frac{\log T}{\epsilon^2}\right)$ rounds with $\tau = \ell$. Let $\check{R}_\ell$ be the average reward. \\
2 Run $\Theta\left(\frac{\log T}{\epsilon^2}\right)$ rounds with $\tau = u$. Let $\check{R}_u$ be the average reward. \\
3 For $\tau \in [\ell, u]$, define $\hat{\Delta}(\tau) := (\hat{g}(u) - \hat{g}(\tau))\hat{F}_1(u) - (\hat{g}(\ell) - \hat{g}(\tau))\hat{F}_1(\ell) - \int_\ell^u \hat{g}'(x)\hat{F}(x)dx$. \\
4 For $\tau \in [\ell, u]$, define $\hat{\delta}(\tau) := \hat{\Delta}(\tau) - (\check{R}_u - \check{R}_\ell)$. \\
5 Let $\ell' = \min\{\tau \in [\ell, u] \mid \hat{\delta}(\tau) \geq -4\epsilon\}$ and let $u' = \max\{\tau \in [\ell, u] \mid \hat{\delta}(\tau) \leq 4\epsilon\}$.
\end{tabular}

\textbf{Output:} $[\ell', u']$

3. Simultaneously for every $\tau \in [\ell', u']$, the expected one-round regret of playing $\tau$ is at most $8\epsilon$.

To understand the main idea of the proof, let’s compare the expected reward of choosing the optimal threshold $\tau^*$ and an arbitrary threshold $\tau \in [\ell, u]$. The difference is given by

$$R(\tau^*) - R(\tau) = \int_0^{\tau^*} f_1(x)g(x)dx - \int_0^\tau f_1(x)g(x)dx = \int_\tau^{\tau^*} f_1(x)g(x)dx.$$ 

Note that $g$ is non-increasing since $g'(x) = F_2(x) - 1 \leq 0$. So, using $\tau^*, \tau \in [\ell, u]$ imply $|F_1(\tau^*) - F_1(u)| \leq |F_1(\ell) - F_1(u)|$, we get $R(\tau^*) - R(\tau) \leq |(F_1(\ell) - F_1(u)) \cdot g(\tau)|$. This motivates defining the bounding function

$$\delta(\tau) := (F_1(u) - F_1(\ell)) \cdot (g(\tau^*) - g(\tau)) = - (F_1(u) - F_1(\ell)) \cdot g(\tau), \quad (4)$$

and we get the following upper bound on the one-round regret when choosing $\tau$ instead of $\tau^*$.

\textbf{Claim 2.8.} If $\tau, \tau^* \in [\ell, u]$ then $R(\tau^*) - R(\tau) \leq |\delta(\tau)|$.

In order to define an estimate $\hat{\delta}(\tau)$ that can be computed using the available information, again consider the rewards when playing thresholds $u$ and $\ell$. The difference is given by

$$R(u) - R(\ell) = \int_\ell^u f_1(x)g(x)dx = F_1(u)g(u) - F_1(\ell)g(\ell) - \int_\ell^u F_1(x)g'(x)dx$$

$$= F_1(u)g(u) - F_1(\ell)g(\ell) - \int_\ell^u F_1(x) \cdot (F_2(x) - 1)dx.$$ 

Adding this equation with the definition of $\delta(\tau)$ gives $\delta(\tau) + R(u) - R(\ell)$ equals

$$F_1(u) \cdot (g(u) - g(\tau)) - F_1(\ell) \cdot (g(\ell) - g(\tau)) - \int_\ell^u F_1(x) \cdot (F_2(x) - 1)dx =: \Delta(\tau), \quad (5)$$

which gives us an alternate way to express $\delta(\tau) = \Delta(\tau) - (R(u) - R(\ell))$. So, we define the estimate

$$\hat{\delta}(\tau) := \hat{\Delta}(\tau) - (\hat{R}_u - \hat{R}_\ell),$$

where $\hat{\Delta}$ uses the estimates $\hat{F}_1$ and $\hat{g}$ instead of $F_1$ and $g$ in (5), and to estimate $(\hat{R}_u - \hat{R}_\ell)$ we use empirical averages obtained in the current phase. We have the following claim on the accuracy of $\hat{\delta}$ in Appendix B.2, which is similar to Claim 2.4.

\textbf{Claim 2.9.} In Algorithm 3, if the conditions in Lemma 2.7 hold, then w.h.p. $|\hat{\delta}(\tau) - \delta(\tau)| \leq 4\epsilon$ simultaneously for all $\tau \in [\ell, u]$. 


The proof of Claim 2.9 is different from Claim 2.4: After the initialization, it’s not possible to give an initial confidence interval of length at most $T^{-\frac{1}{4}}$. So, we cannot prove an $O(T^{-\frac{1}{4}})$ accuracy for $\Delta(\tau)$. Instead, we use the fact that $\text{Var}\Delta(\tau) \leq O(\epsilon)$ to give an $O(\epsilon)$ accuracy bound using Bernstein inequality (Theorem A.2) for a single $\tau$. To extend the bound to the whole interval, we discretize and apply a union bound. To avoid the dependency from the previous phases when discretizing, in each phase we use new samples to construct $\hat{F}_1$ and $\hat{F}_2$. This is the reason that we introduce sample sets in Algorithm 3.

Now the proof of Lemma 2.7 is similar to the proof of Lemma 2.2 via Claims 2.8 and 2.9.

Finally, we state the main theorem for Pandora’s Box problem with two boxes in a fixed order.

**Theorem 2.10.** For Bandit Pandora’s Box learning problem with two boxes in a fixed order, there exists an algorithm that achieves $O(\sqrt{T\log T})$ total regret.

The proof of Theorem 2.10 is similar to Theorem 2.6: We first show that $\Theta(\sqrt{T\log T})$ initial samples are sufficient to meet the conditions in Lemma 2.7. Combining this with Lemma 2.5 proves the theorem. See Appendix B.2 for details.

## 3 Prophet Inequality for General $n$

In the Bandit Prophet Inequality problem, there are $n$ unknown independent distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ taking values in $[0, 1]$ with cdfs $F_1, \ldots, F_n$ and densities $f_1, \ldots, f_n$. Consider a $T$ rounds game where round $t$ we play thresholds $\tau^{(t)} = (\tau_1^{(t)}, \tau_2^{(t)}, \ldots, \tau_{n-1}^{(t)}, \tau_n^{(t)} = 0)$ and receive as feedback the following reward:

- Starting with $i = 1$, independently draw $X_i^{(t)}$ from $\mathcal{D}_i$. If $X_i^{(t)} \geq \tau_i^{(t)}$ then $i$ is accepted and return $X_i^{(t)}$ as the reward.
- Otherwise, updates $i \leftarrow i + 1$ and go back to the previous step.

The only feedback is the reward, and we do not see the index $i$ of the selected random variable. Since we have $\tau_n^{(t)} = 0$, the algorithm will always select. In the following, we omit $\tau_n^{(t)}$ and only use $\tau^{(t)} := (\tau_1^{(t)}, \tau_2^{(t)}, \ldots, \tau_{n-1}^{(t)})$ to represent a threshold setting.

Let $\text{Opt}_i$ represent the optimal expected reward if only running on distributions $\mathcal{D}_i, \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n$. Then, the optimal $i$-th threshold setting is exactly $\text{Opt}_{i+1}$. We can calculate $\{\text{Opt}_{i+1}\}$ as follows:

- Let $\text{Opt}_n = E[X_n]$
- For $i = n - 1 \rightarrow 1$: Let $\text{Opt}_i = R(1, 1, \ldots, 1, \text{Opt}_{i+1}, \text{Opt}_{i+2}, \ldots, \text{Opt}_n)$, where the function $R(\tau)$ represents the expected one-round reward under thresholds $\tau = (\tau_1, \ldots, \tau_{n-1})$.

The total regret is defined

$$T \cdot \text{Opt}_1 - \sum_{t=1}^{T} R(\tau^{(t)}).$$

**High-Level Approach.** Following the doubling framework from Algorithm 2, we only need to design an initialization algorithm and a constraint-updating algorithm. For the initialization, we get $O(\text{poly}(n)\sqrt{T\log T})$ i.i.d. samples for each $X_i$ by playing thresholds $(1, 1, \ldots, \tau_{i-1} = 1, \tau_i = 0, 0, \ldots, 0)$. Besides, we run $O(\text{poly}(n)\sqrt{T\log T})$ samples to get the initial confidence intervals with small length. For the constraint-updating algorithm, we reuse the idea from the $n = 2$ case where we shrink confidence intervals by testing $X_i$ with thresholds $\ell_i$ or $u_i$. However, there are two major new challenges while testing $X_i$. 

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The first challenge while testing $X_i$ is that we may stop early, and not get sufficiently many samples for $X_i$. Although the probability of reaching $X_i$ could be very small, this also means that we will not reach $X_i$ frequently. To avoid this problem, for $j < i$, we use the upper confidence bounds as thresholds since they maximize the probability of reaching $X_i$. In particular, it is at least as high as in the optimal policy. Therefore, we will be able to show that the probability term cancels in calculation, so the total loss from $X_i$ can still be bounded.

The second challenge is that when we are testing $X_i$, we need to also set thresholds $\tau_j$ for $j > i$. The problem is that the optimal choice for $\tau_j$ depends on $\tau_i$ for $j > i$. To cope with this problem, in our algorithm we use the lower confidence bounds as thresholds for $j > i$ as they only improve from phase to phase. Formally, let $\text{Alg}_i$ denote the expected reward if only running on distributions $\mathcal{D}_i, \ldots, \mathcal{D}_n$ with lower confidence bounds as the thresholds, i.e.,

$$\text{Alg}_i := R(1, \ldots, \tau_i = \ell_i, \tau_{i+1} = \ell_{i+1}, \ldots, \tau_{n-1} = \ell_{n-1}).$$

Now, under our threshold setting, we can only hope to learn $\text{Alg}_{i+1}$, while the optimal threshold is $\text{Opt}_{i+1}$. So, our key idea is to first get a new confidence interval for $\text{Alg}_{i+1}$, and then obtain new upper confidence bound for $\text{Opt}_{i+1}$ by bounding the difference between $\text{Opt}_{i+1}$ and $\text{Alg}_{i+1}$.

### 3.1 Interval-Shrinking Algorithm for General $n$

In this section, we give the interval shrinking algorithm, and provide the regret analysis to show that we can get a new group of confidence intervals that achieves $O(\epsilon)$ regret after $\tilde{O}(\text{poly}(n)\epsilon^2)$ rounds. We first give the algorithm and the corresponding lemma.

**Algorithm 4:** Interval shrinking Algorithm for general $n$

**Input:** Intervals $[\ell_1, u_1], \ldots, [\ell_{n-1}, u_{n-1}]$, CDF estimates $\hat{F}_1(x), \ldots, \hat{F}_n(x)$, and $\epsilon$.

1. For $i \in [n-1]$, define $P_i := \prod_{j \in [i-1]} \hat{F}_j(u_j)$
2. for $i = n-1 \rightarrow 1$ do
   3. Run $\Theta(\frac{\log T}{\epsilon^2})$ rounds with thresholds setting $(u_1, \ldots, u_i-1, \ell_i, \ell_{i+1}', \ldots, \ell_{n-1}')$ and $(u_1, \ldots, u_i-1, u_i, \ell_{i+1}', \ldots, \ell_{n-1}')$. Let $\hat{D}_i$ be the difference of the average rewards.
   4. For $\tau \in [\ell_i, u_i]$, define $\hat{\Delta}_i(\tau) := \hat{P}_i(\hat{F}_i(u_i)-(\tau - u_i) - \hat{P}_i(\ell_i)(\tau - \ell_i)) + \int_{\ell_i}^{u_i} F_i(x)dx$.
   5. For $\tau \in [\ell_i, u_i]$, define $\hat{\delta}_i(\tau) := \hat{\Delta}_i(\tau) - \hat{D}_i$.
   6. Let $\ell_i' = \min\{\tau \in [\ell_i, u_i] \text{ s.t. } \hat{\delta}_i(\tau) \geq -\epsilon\}$.
   7. Let $u_i' = \max\{\tau \in [\ell_i, u_i] \text{ s.t. } \hat{\delta}_i(\tau) \leq (2n-2i-1)\epsilon\}$.

**Output:** $[\ell_2', u_2'], \ldots, [\ell_n', u_n']$

**Lemma 3.1.** Suppose we are given:

- Distribution estimates $\hat{F}_i(x)$ for $i \in [n-1]$ satisfying $|\prod_{i \in S} \hat{F}_i(x) - \prod_{i \in S} F_i(x)| \leq T^{-1/4}$ for all $x \in [0, 1]$ and $S \subseteq [n]$.
- Initial intervals $[\ell_i, u_i]$ for $i \in [n-1]$ of length $u_i - \ell_i \leq T^{-1/4}$ that satisfy $\text{Opt}_{i+1} \in [\ell_i, u_i]$ and $\text{Alg}_{i+1} \in [\ell_i, u_i]$.

Then, for $\epsilon > 12T^{-1}$ Algorithm 4 runs $\Theta(\frac{n\log T}{\epsilon^2})$ rounds such that in each round the threshold $\tau$ satisfies $\tau_i \in [\ell_i, u_i]$ for all $i \in [n-1]$. Moreover, w.h.p. the following statements hold:

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7We don’t explicitly update the upper confidence bound using this method, but this shows our intuition.
(i) \( \text{Opt}_{i+1} \in [\ell'_i, u'_i] \) for all \( i \in [n-1] \).
(ii) Let \( \text{Alg}_i' := R(1, \ldots, 1, \ell'_i, \ldots, \ell'_{n-1}) \) for \( i \in [n-1] \). Then \( \text{Alg}_{i+1}' \in [\ell'_i, u'_i] \).
(iii) For every threshold setting \( \tau = (\tau_1, \ldots, \tau_{n-1}) \) where \( \tau_i \in [\ell'_i, u'_i] \), the expected one-round regret of playing \( \tau \) is at most \( 2n^2 \epsilon \).

3.1.1 Generalizing Notation and Claims from \( n = 2 \)

We introduce some notation to prove Lemma 3.1. First, we define a single-dimensional function \( R_i(\tau) \) to generalize reward function \( R(\tau) \) from the \( n = 2 \) case in Section 2.1. Ideally, \( R_i(\tau) \) should represent the reward of playing \( \tau_i = \tau \), but thresholds \( \tau_j \) for \( j > i \) also affect its expected reward. So, to match the setting in Algorithm 4, we set thresholds \( \tau_{i+1}, \ldots, \tau_{n-1} \) to be the updated lower bounds, i.e., define

\[
R_i(\tau) := R(1, \ldots, 1, \tau_i = \tau, \ell'_{i+1}, \ldots, \ell'_{n-1}).
\]

Next, we introduce \( P_i \), representing the maximum probability of observing \( X_i \) when we have confidence intervals \( \{[\ell_i, u_i]\} \), i.e.,

\[
P_i := \prod_{j=1}^{i-1} F_j(u_j).
\]

Replacing \( F_j \) with \( \hat{F}_j \) in this equation defines estimate \( \hat{P}_i \).

Notice that \( P_i \) also equals the probability of reaching \( X_i \) when we play thresholds \( \tau_j = u_j \) for all \( j < i \) in Algorithm 4. So, the loss of playing a sub-optimal threshold \( \tau_i \) will be \( P_i \cdot (R_i(\text{Alg}_{i+1}') - R_i(\tau)) \) because \( P_i \) is the probability of reaching \( X_i \) and \( \text{Alg}_{i+1}' \) is the optimal threshold when \( \tau_j = \ell'_j \) for all \( j > i \). We define the generalized bounding function:

\[
\delta_i(\tau) := P_i \cdot (F_i(u_i) - F_i(\ell_i)) \cdot (\tau - \text{Alg}_{i+1}').
\]

We will show in Claim 3.2 below that \( |\delta_i(\tau)| \) upper bounds \( P_i \cdot (R_i(\text{Alg}_{i+1}') - R_i(\tau)) \) for all \( \tau \in [\ell_i, u_i] \). Since we don’t know \( \delta_i(\tau) \), we will estimate it by writing in a different way.

Consider the difference in expected rewards between \( \tau_i = u_i \) and \( \tau_i = \ell_i \) when the other thresholds are set to \( \tau_j = u_j \) for \( j < i \) and \( \tau_j = \ell'_j \) for \( j > i \). The difference between these two settings only comes from \( \tau_i \), so the expected difference is

\[
P_i \cdot (R_i(u_i) - R_i(\ell_i)) = P_i \cdot (F_i(u_i) - F_i(\ell_i)) \int_{\ell_i}^{u_i} x f_i(x) dx
\]

Adding this with \( \delta_i(\tau) \) implies \( \delta_i(\tau) + P_i \cdot (R_i(u_i) - R_i(\ell_i)) \) equals

\[
P_i \cdot (F_i(u_i)(\tau - u_i) - F_i(\ell_i)(\tau - \ell_i) + \int_{\ell_i}^{u_i} F_i(x) dx) =: \Delta_i(\tau), \tag{6}
\]

which gives another way of writing \( \delta_i(\tau) = \Delta_i(\tau) - P_i \cdot (R_i(u_i) - R_i(\ell_i)) \). Since \( \hat{D}_i \) from Algorithm 4 is the difference between average rewards of taking samples with \( \tau_i = u_i \) and \( \tau_i = \ell_i \), it is an unbiased estimator of \( P_i \cdot (R_i(u_i) - R_i(\ell_i)) \). So, we define estimate

\[
\hat{\delta}_i(\tau) := \hat{\Delta}_i(\tau) - \hat{D}_i,
\]

where \( \hat{\Delta}_i(\tau) \) is obtained by replacing \( F_i \) with \( \hat{F}_i \) and \( P_i \) with \( \hat{P}_i \) in (6).
Similar to Claim 2.3 and Claim 2.4, we introduce the following claims for Algorithm 4.

Claim 3.2. For \( i \in [n-1] \), if \( \text{Alg}_{i+1}^c \in [\ell_i, u_i] \) and \( \tau \in [\ell_i, u_i] \), then \( P_i \cdot (R_i(\text{Alg}_{i+1}^c) - R_i(\tau)) \leq |\delta_i(\tau)|. \)

Proof. We only need to prove that \( R_i(\text{Alg}_{i+1}^c) - R_i(\tau) \leq |\delta_i(\tau)|/P_i = (F_i(u_i) - F_i(\ell_i)) \cdot (\tau - \text{Alg}_{i+1}^c). \) Now the proof is identical to Claim 2.3 by replacing function \( R(\cdot) \) with \( R_i(\cdot) \).

Claim 3.3. In Algorithm 4, if the conditions in Lemma 3.1 hold, then w.h.p. \( |\hat{\delta}_i(\tau) - \delta_i(\tau)| \leq \epsilon \) simultaneously for all \( \tau \in [\ell_i, u_i] \).

Proof. There are two terms in \( \delta_i(\tau) = \Delta_i(\tau) - P_i \cdot (R_i(u_i) - R_i(\ell_i)) \). We prove that the error of each term is bounded by \( \frac{\epsilon}{2} \) with high probability, which will complete the proof by a union bound.

We first bound \( |\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \). There are three terms in \( \frac{\Delta_i(\tau)}{P_i} = F_i(u_i)(\tau - u_i) - F_i(\ell_i)(\tau - \ell_i) + \int_{\ell_i}^{u_i} F_i(x)dx. \) Since the conditions in Lemma 3.1 guarantee that \( |\hat{F}_i(x) - F_i(x)| \leq T^{-1/4} \) and \( u_i - \ell_i \leq T^{-1/4} \), the error in each term is at most \( \frac{1}{\sqrt{T}} \) and the total error \( \left| \frac{\Delta_i(\tau)}{P_i} - \hat{\Delta}_i(\tau) \right| \leq \frac{3}{\sqrt{T}}. \)

For \( P_i \), the preconditions in Lemma 3.1 guarantee that \( |\hat{P}_i - P_i| \leq T^{-1/4} \). Moreover, observe that \( u_i - \ell_i \leq T^{-1/4} \) implies that \( \frac{|\Delta_i(\tau)|}{P_i} \leq 3T^{-1/4}. \) So,

\[
|\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \leq \left| \hat{P}_i \left( \frac{\hat{\Delta}_i(\tau)}{P_i} - \frac{\Delta_i(\tau)}{P_i} \right) \right| + \left| (\hat{P}_i - P_i) \frac{\Delta_i(\tau)}{P_i} \right| \leq \frac{6}{\sqrt{T}} \leq \frac{\epsilon}{2},
\]

where the last inequality uses \( \epsilon > 12T^{-\frac{1}{2}}. \)

Thus, \( |\hat{D}_i - P_i \cdot (R_i(u_i) - R_i(\ell_i))| \leq \frac{\epsilon}{2} \) holds w.h.p. for sufficiently large \( C. \)

3.1.2 Proof of Lemma 3.1

We need some other properties of Algorithm 4 to prove Lemma 3.1. The next claim shows that the expected reward of playing lower confidence bounds increases phase to phase.

Claim 3.4. Assume the conditions in Lemma 3.1 and the bound in Claim 3.3 hold. Then, for \( i \in [n] \), we have \( \text{Alg}_{i+1}^c \geq \text{Alg}_i \).

Proof. We prove by induction for \( i \) going from \( n \) to \( 1 \). The base case \( i = n \) holds because \( \text{Alg}_n^c = \text{Alg}_n = \text{Opt}_n = \mathbb{E}[X_n] \) by definition.

For the induction step, assume that \( \text{Alg}_{i+1}^c \geq \text{Alg}_{i+1} \) by induction hypothesis. Observe that

\[
R(1, \ldots, 1, \ell_i, \ell_{i+1}, \ldots, \ell_{n-1}) = \mathbb{E}[X_i \cdot 1_{X_i > \ell_i}] + \mathbb{P}[X_i \leq \ell_i] R(1, \ldots, 1, 1, \ell_{i+1}, \ldots, \ell_{n-1})
\leq \mathbb{E}[X_i \cdot 1_{X_i > \ell_i}] + \mathbb{P}[X_i \leq \ell_i] R(1, \ldots, 1, 1, \ell_{i+1}', \ldots, \ell_{n-1}')
= R(1, \ldots, \ell_i, \ell_{i+1}', \ldots, \ell_{n-1}'),
\]

where the last inequality follows from the induction hypothesis.
where the inequality uses induction hypothesis as $R(1, \ldots, 1, 1, \ell_{i+1}, \ldots, \ell_{n-1}) = \text{Alg}_{i+1} \leq \text{Alg}'_{i+1} = R(1, \ldots, 1, 1, \ell'_{i+1}, \ldots, \ell'_{n-1})$.

Next, we have $R_i(\ell_i) \leq R_i(\ell'_i)$, i.e.,

$$R(1, \ldots, 1, \ell_{i}, \ell'_{i+1}, \ldots, \ell'_{n-1}) \leq R(1, \ldots, 1, \ell_{i}, \ell'_{i+1}, \ldots, \ell'_{n-1}). \quad (8)$$

To prove this, we first observe that if $\ell_i = \ell'_i$, then the inequality is an equality. Otherwise, there must be $\hat{\delta}_i(\ell'_i) = -\epsilon$. Next, combining the definition of $\delta_i(\tau)$ and Claim 3.3, we have $\hat{\delta}_i(\text{Alg}'_{i+1}) \geq \delta_i(\text{Alg}_{i+1}) - \delta_i(\text{Alg}'_{i+1}) \geq -\epsilon$. Since $\hat{\delta}_i(\tau) = P_i \cdot (\hat{F}_i(u_i) - \hat{F}_i(\ell_i)) \geq 0$ means $\hat{\delta}_i(\tau)$ is increasing, there must be $\text{Alg}'_{i+1} \geq \ell'_i$.

Now consider function $R_i(\tau)$. Recall that $R_i(\tau) = R(1, \ldots, 1, \tau_i = \tau', \ell'_{i+1}, \ldots, \ell'_{n-1})$. Therefore,

$$R_i(\tau) = \Pr[X_i \leq \tau] \cdot \text{Alg}'_{i+1} + \mathbb{E}[X_i \cdot 1_{X_i > \tau}] = F_i(\tau) \cdot \text{Alg}'_{i+1} + \int_{\tau}^{1} f_i(x) \, dx,$$

which means $R_i'(\tau) = f_i(\tau)(\text{Alg}'_{i+1} - \tau)$, showing that $R_i(\tau)$ is a unimodular function and reaches its maximum when $\tau = \text{Alg}'_{i+1}$. Hence, (8) holds because $\ell_{i+1} \leq \ell'_i \leq \text{Alg}'_{i+1}$.

Combining (7) and (8) proves the claim. \qed

Next, we prove that $\text{Alg}'_{i+1} \in [\ell_i, u_i]$, which is crucial for us to use Claim 3.2.

**Claim 3.5.** Assume that the preconditions in Lemma 3.1 and the bound in Claim 3.3 hold, then $\text{Alg}'_{i+1} \in [\ell_i, u_i]$ for all $i \in [n-1]$.

**Proof.** Claim 3.4 shows that $\text{Alg}_{i+1} \leq \text{Alg}'_{i+1}$. On the other hand, $\text{Alg}'_{i+1} \leq \text{Opt}_{i+1}$ holds because $\text{Opt}_{i+1}$ is the maximum achievable reward. Then, Claim 3.5 holds because $\text{Opt}_{i+1}, \text{Alg}_{i+1} \in [\ell_i, u_i]$ by the preconditions in Lemma 3.1. \qed

Finally, we show that $\text{Alg}'_i$ cannot be much smaller than $\text{Opt}_i$.

**Claim 3.6.** Assume that the preconditions in Lemma 3.1 and the bound in Claim 3.3 hold, then $\text{Opt}_i - \text{Alg}'_i \leq \frac{2(n-i-1)\epsilon}{P_i}$ for all $i \in [n-1]$.

**Proof.** We prove by induction for $i$ going from $n$ to 1. The base case $i = n$ holds because $\text{Opt}_n = \text{Alg}_n = \mathbb{E}[X_n]$.

For the induction step, we assume that $\text{Opt}_{i+1} - \text{Alg}'_{i+1} \leq \frac{2(n-i-1)\epsilon}{P_{i+1}}$ and would like to show that $\text{Opt}_i - \text{Alg}'_i \leq \frac{2(n-i-1)\epsilon}{P_i}$. We first have

$$R(1, \ldots, 1, \text{Opt}_{i+1}, \text{Opt}_{i+2}, \ldots, \text{Opt}_n) = \mathbb{E}[X_i \cdot 1_{X_i > \text{Opt}_{i+1}}] + \Pr[X_i \leq \text{Opt}_{i+1}] \text{Opt}_{i+1}$$

$$\leq \mathbb{E}[X_i \cdot 1_{X_i > \text{Opt}_{i+1}}] + \Pr[X_i \leq \text{Opt}_{i+1}] \left(\text{Alg}'_{i+1} + \frac{2(n-i-1)\epsilon}{P_{i+1}}\right)$$

$$\leq \mathbb{E}[X_i \cdot 1_{X_i > \text{Opt}_{i+1}}] + \Pr[X_i \leq \text{Opt}_{i+1}] \text{Alg}'_{i+1} + \frac{2(n-i-1)\epsilon}{P_i}$$

$$= R(1, \ldots, 1, \text{Opt}_{i+1}, \ell'_{i+1}, \ldots, \ell'_{n-1}) + \frac{2(n-i-1)\epsilon}{P_i}, \quad (9)$$

where we use the induction hypothesis in the second line, and the fact that $\Pr[X_i \leq \text{Opt}_{i+1}] \leq \Pr[X_i \leq u_i] = \frac{P_{i+1}}{P_i}$ in the third line.
Next, since $\text{Alg}_{i+1}'$ is the optimal threshold, we have

$$R(1, \ldots , 1, \text{Opt}_{i+1}, \ell'_{i+2}, \ldots , \ell'_{n-1}) \leq R(1, \ldots , 1, \text{Alg}_{i+1}', \ell'_{i+2}, \ldots , \ell'_{n-1}).$$ (10)

Finally,

$$|\delta'(\ell'_i)| = |\delta'(\ell'_i)| + |\hat{\delta}(\ell'_i) - \delta'(\ell'_i)| \leq \epsilon + \epsilon = 2\epsilon,$$

where the bound of $|\hat{\delta}(\ell'_i) - \delta'(\ell'_i)|$ is from Claim 3.3. Combining this with Claim 3.2, we have $R_i(\text{Alg}_{i+1}') - R_i(\ell'_i) \leq \frac{|\delta'(\ell'_i)|}{P_i} \leq \frac{2\epsilon}{P_i}$, which is exactly

$$R(1, \ldots , 1, \text{Alg}_{i+1}', \ell'_{i+1}, \ldots , \ell'_{n-1}) \leq R(1, \ldots , 1, \ell'_i, \ell'_{i+1}, \ldots , \ell'_{n-1}) + \frac{2\epsilon}{P_i}.$$

Summing this with (9) and (10) completes the induction step.

Finally, we can prove Lemma 3.1.

**Proof of Lemma 3.1.** In this proof, we assume Claim 3.3 always holds. Then the whole proof should succeed with high probability.

We prove the three statements separately:

**Statement (i).** For the upper bound, Claim 3.6 shows that $\text{Opt}_i - \text{Alg}_i' \leq \frac{2(n-i)\epsilon}{P_i}$. Therefore, $\delta_i(\text{Opt}_{i+1}) \leq P_i \cdot F_i(u_i) \cdot \frac{2(n-i-1)\epsilon}{P_{i+1}} = 2(n-i-1)\epsilon$. Combining this with Claim 3.3, we have $\hat{\delta}_i(\text{Opt}_{i+1}) \leq 2(n-i-1)\epsilon + \epsilon = (2n-2i-1)\epsilon$. Then $\text{Opt}_{i+1} \leq u_i', \text{ because } \text{Opt}_{i+1} \in [\ell_i, u_i]$, $u_i' = \max\{\tau : \tau \in [\ell_i, u_i] \land \hat{\delta}_i(\tau) \leq (2n-2i-1)\epsilon\}$ and the monotonicity of $\hat{\delta}_i$ function.

For the lower bound, at least we have $\text{Opt}_{i+1} \geq \text{Alg}_{i+1}'$. Therefore, $\hat{\delta}_i(\text{Opt}_{i+1}) \geq 0$, so $\delta_i(\text{Opt}_{i+1}) \geq -\epsilon$. Then $\text{Opt}_{i+1} \geq \ell'_i$, because $\text{Opt}_{i+1} \in [\ell_i, u_i]$, $\ell'_i = \max\{\tau : \tau \in [\ell_i, u_i] \land \hat{\delta}_i(\tau) \geq -\epsilon\}$ and the monotonicity of $\hat{\delta}_i$ function. Combining the two bounds proves Statement (i).

**Statement (ii).** The proof idea is the same as Statement (i). Notice that $\delta_i(\text{Alg}_{i+1}') = 0$. Then, according to Claim 3.3, $|\hat{\delta}_i(\text{Alg}_{i+1}')| \leq \epsilon$. So Statement (ii) hold because $\text{Alg}_{i+1}' \in [\ell_i, u_i]$, which is from Claim 3.5, and $[\ell_i, u_i] \supseteq \{\tau \in [\ell_i, u_i] : |\hat{\delta}_i(\tau)| \leq \epsilon\}$.

**Statement (iii).** We prove the following stronger statement by induction on $i$: If $\tau_j \in [\ell'_j, u'_j]$ for all $j \in \{i, \ldots , n\}$, then

$$\text{Alg}_i' - R(1, \ldots , 1, \tau_i, \ldots , \tau_{n-1}) \leq \frac{(n-i+1)^2\epsilon}{P_i}.$$ When the statement above holds, taking $i = 1$ gives $R(\tau_1, \ldots , \tau_{n-1}) \geq \text{Alg}_1' - n^2\epsilon$. Furthermore, Claim 3.6 shows that $\text{Alg}_1' \geq \text{Opt}_1 - 2(n-1)\epsilon$. Combining these two inequalities proves Statement (iii).

It remains to prove the induction statement. The base case $i = n$ holds trivially.

For the induction step, we will assume that the statement holds for $i + 1$ and we have to show it also holds for $i$. By induction hypothesis,

$$R(1, \ldots , 1, \tau_i, \ldots , \tau_{n-1}) = \mathbb{E}[X_i \cdot 1_{X_i \geq \tau_i}] + \mathbb{P}[X_i < \tau_i] R(1, \ldots , 1, \tau_{i+1}, \ldots , \tau_{n-1})$$

$$\geq \mathbb{E}[X_i \cdot 1_{X_i \geq \tau_i}] + \mathbb{P}[X_i < \tau_i] \left( R(1, \ldots , 1, \ell'_{i+1}, \ldots , \ell'_{n-1}) - \frac{(n-i)^2\epsilon}{P_{i+1}} \right)$$

$$\geq \mathbb{E}[X_i \cdot 1_{X_i \geq \tau_i}] + \mathbb{P}[X_i < \tau_i] (R(1, \ldots , 1, \ell'_{i+1}, \ldots , \ell'_{n-1}) - \frac{(n-i)^2\epsilon}{P_i})$$

$$= R(1, \ldots , 1, \tau_i, \ell'_{i+1}, \ldots , \ell'_{n-1}) - \frac{(n-i)^2\epsilon}{P_i}. $$

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Furthermore, \(|\delta_i(\tau_i)| \leq |\delta_i(\tau_i)| + \epsilon\) by Claim 3.3 and \(|\delta_i(\tau)| \leq (2n - 2i - 1)\epsilon\) by the definitions of \(\ell_i\) and \(u_i\), which means \(|\delta_i(\tau)| \leq 2(n - i)\epsilon\). So, Claim 3.2 implies
\[
R(1, \ldots, 1, \text{Alg}_{i+1}'m, \ell_{i+1}', \ldots, \ell_{n-1}') - R(1, \ldots, 1, \tau_i, \ell_{i+1}', \ldots, \ell_{n-1}') \leq \frac{2(n - i)\epsilon}{P_i}.
\]
Finally, using \(R(1, \ldots, 1, \text{Alg}_{i+1}'m, \ell_{i+1}', \ldots, \ell_{n-1}') \geq \text{Alg}_{i}'\), we get
\[
R(1, \ldots, 1, \tau_i, \ldots, \tau_{n-1}) \geq \text{Alg}_{i}' - \frac{2(n - i)\epsilon}{P_i} - \frac{(n - i)^2\epsilon}{P_i} \geq \text{Alg}_{i}' - \frac{(n - i + 1)^2\epsilon}{P_i}. \quad \square
\]

### 3.2 Initialization and Putting Everything Together

Now, we can give the initialization algorithm. The main goal of the initialization is to satisfy the conditions listed in Lemma 3.1. Starting from the second call of Algorithm 4, the confidence interval length constraint and the distribution estimates constraints hold from the initialization, and the constraints \(\text{Opt}_{i+1}, \text{Alg}_{i+1} \in [\ell_i, u_i]\) are guaranteed by Statements (i) and (ii) in Lemma 3.1. Then, we can apply Lemma 2.5 to bound the total regret.

We first give the initialization algorithm:

**Algorithm 5: Initialization**

1. **Input:** Time horizon \(T\), problem size \(n\).
2. **for** \(i = 1 \rightarrow n\) **do**
   1. Run \(\Theta(n^2\sqrt{T} \log T)\) free samples for \(X_i\) to estimate \(\hat{F}_i(x)\).
3. **for** \(i = n - 1 \rightarrow 1\) **do**
   1. Run \(\Theta(n^2\sqrt{T} \log T)\) samples under the threshold setting \((1, \ldots, 1, \tau_{i+1} = \ell_{i+1}', \ldots, \tau_{n-1} = \ell_{n-1}')\). Let \(\mu_i\) be the average reward.
   2. Let \(\ell_i = \mu_i - \frac{T^{-1/4}}{10n}\), \(u_i = \mu_i + (2n - 2i - 1) \cdot \frac{T^{-1/4}}{10n}\).

**Output:** \([\ell_1, u_1], \ldots, [\ell_{n-1}, u_{n-1}]\).

**Lemma 3.7.** Algorithm 5 runs \(\Theta(n^3\sqrt{T} \log T)\) rounds. The output satisfies w.h.p. all constraints listed in Lemma 3.1.

**Proof.** For the accuracy bound of \(\hat{F}_i(x)\), we first show that \(|\hat{F}_i(x) - F_i(x)| \leq \frac{T^{-1/4}}{2n}\) w.h.p. after running \(N = C \cdot n^2\sqrt{T} \log T\) samples. With DKW inequality (Theorem A.3), we have
\[
\Pr \left[ |\hat{F}_i(x) - F_i(x)| > \epsilon = \frac{T^{-1/4}}{2n} \right] \leq 2 \exp(-2N\epsilon^2) = 2T^{-C/4}.
\]
So the bound holds with high probability. By the union bound, w.h.p. \(|\hat{F}_i(x) - F_i(x)| \leq \frac{T^{-1/4}}{2n}\) holds for every \(i \in [n]\). Then, for the accuracy of \(\prod_{i \in S} F_i(x)\), we have \((1 - \frac{T^{-1/4}}{2n})^n - 1\) \(\prod_{i \in S} F_i(x) \leq \prod_{i \in S} F_i(x) \leq (1 + \frac{T^{-1/4}}{2n})^n - 1\). For the lower bound, we have \((1 + \frac{T^{-1/4}}{2n})^n - 1 \geq 1 - T^{-\frac{3}{2}}\) \(-T^{-\frac{1}{2}}\). For the upper bound, we have \((1 + \frac{T^{-1/4}}{2n})^n - 1 \leq \exp(\frac{T^{-1/4}}{2n} \cdot n) - 1 \leq 1 + 2 \cdot \frac{T^{-1/4}}{2n} - 1 = T^{-\frac{1}{2}}\). Combining two bounds finishes the proof.
For the confidence interval, the constraints \( u_i - \ell_i \leq T^{-1/4} \) hold by definition. Then, we only remain to show \( \text{Opt}_{i+1} \in [\ell_i, u_i] \) and \( \text{Alg}_{i+1} \in [\ell_i, u_i] \).

We start from proving \( \text{Alg}_{i+1} \in [\ell_i, u_i] \). Notice that \( \mu_i \) is an estimate of \( \text{Alg}_{i+1} \) with \( N = C \cdot n^2 \sqrt{T} \log T \) samples. With Hoeffding’s Inequality (Theorem A.1), we have

\[
\Pr \left[ |\mu_i - \text{Alg}_{i+1}| > \varepsilon = \frac{T^{-1/4}}{10n} \right] < 2 \exp(-2N \varepsilon^2) = 2T^{-\sqrt{1/50}}.
\]

Notice that \( \ell_i = \mu_i - \frac{T^{-1/4}}{10n} \) and \( u_i \geq \mu_i + \frac{T^{-1/4}}{10n} \). Then, by the union bound, \( \text{Alg}_{i+1} \in [\ell_i, u_i] \) holds for all \( i \) w.h.p. when \( C \) is sufficiently large.

For \( \text{Opt}_{i+1} \), we prove the statement by doing induction with the assumption that \( |\text{Alg}_{i+1} - \mu_i| \leq \frac{T^{-1/4}}{10n} \) for all \( i \). The base case is \( i = n \), the statement simply holds because \( \text{Alg}_n = \text{Opt}_n \). Next, we consider \( i \), with the condition that \( \text{Opt}_{j+1} \in [\ell_i, u_j] \) for all \( j > i \). For the lower bound, since we know that \( \text{Alg}_{i+1} \geq \ell_i \), there must be \( \text{Opt}_{i+1} \geq \ell_i \), because \( \text{Opt}_{i+1} \geq \text{Alg}_{i+1} \). For the upper bound, we first bound the difference between \( \text{Alg}_{i+1} \) and \( \text{Opt}_{i+1} \). Consider the setting \( (1, \ldots, 1, \tau_{i+1} = \ell_{i+1}, \ldots, \tau_{n-1} = \ell_{n-1}) \) and \( (1, \ldots, 1, \tau_{i+1} = \text{Opt}_{i+2}, \ldots, \tau_{n-1} = \text{Opt}_n) \). The first setting incurs an extra loss only when its behavior is different from the second setting. Assume the two settings behave differently when meeting a threshold \( \tau_j \). Notice that this extra loss is bounded by \( |\ell_j - \text{Opt}_{j+1}| \). Since \( \text{Opt}_{j+1} \in [\ell_i, u_j] \) for all \( j > i \), this difference is upper bounded by \( \max_{j > i} u_j - \ell_j = u_{i+1} - \ell_{i+1} = (2n - 2i - 2) \cdot \frac{T^{-1/4}}{10n} \). Therefore,

\[
\text{Opt}_{i+1} \leq \text{Alg}_{i+1} + (2n - 2i - 2) \cdot \frac{T^{-1/4}}{10n} \leq \mu_i + (2n - 2i - 1) \cdot \frac{T^{-1/4}}{10n} = u_i.
\]

Combining the lower bound and the upper bound proves \( \text{Opt}_{i+1} \in [\ell_i, u_i] \). Finally, notice that the assumption \( |\text{Alg}_{i+1} - \mu_i| \leq \frac{T^{-1/4}}{10n} \) holds w.h.p., so w.h.p. \( \text{Opt}_{i+1} \in [\ell_i, u_i] \) holds for all \( i \).

Now we are ready to prove the main theorem.

**Theorem 1.1.** There is a polytime algorithm with \( O(n^3 \sqrt{T} \log T) \) regret for the Bandit Prophet Inequality problem where we only receive the selected value as the feedback.

**Proof.** For the initialization, Algorithm 5 runs \( \Theta(n^3 \sqrt{T} \log T) \) rounds, so the total regret from the initialization is \( O(n^3 \sqrt{T} \log T) \).

For the main algorithm, we run Algorithm 2 with Algorithm 4 being the required \( \text{Alg} \). This is feasible because the requirements in Lemma 3.1 are guaranteed by the initialization and Lemma 3.1 itself. Besides, Lemma 3.1 implies that Algorithm 4 upper-bound the one-round regret by \( \epsilon \) after \( \Theta(n^3 \log T) \) samples. Applying Lemma 2.5 with \( \alpha = 5 \), we have the \( O(n^2 \sqrt{T} \log T) \) regret bound. Combining two parts finishes the proof.

## 4 Pandora’s Box for General \( n \)

In the Bandit Pandora’s Box problem, there are \( n \) unknown independent distributions \( D_1, \ldots, D_n \) representing the values of the \( n \) boxes. The distributions have cdfs \( F_1, \ldots, F_n \) and densities \( f_1, \ldots, f_n \). Moreover, each box/distribution \( D_i \) has a known inspection cost \( c_i \). Although in the original problem in introduction we assumed that the values and costs have support \([0, 1]\), in this
section we will scale down the costs and values by a factor of $2n$, so that they have support $[0, \frac{1}{n}]$. This scaling helps to bound the utility in each round between $[-0.5, 0.5]$. To obtain bounds for the original unscaled problem, we will multiply our bounds with this factor $2n$ in the final analysis.

Consider a $T$ rounds game where in each round we play some permutation $\pi$ representing the order of inspection and $n$ thresholds $(\tau_{\pi(1)}, \ldots, \tau_{\pi(n)})$. Our algorithm receives the following utility as feedback:

- Starting with $i = 1$, if $\max\{X_0 = 0, X_{\pi}(1), \ldots, X_{\pi}(i-1)\} \geq \tau_{\pi(i)}$ then stop and return the utility as $\max\{X_0 = 0, X_{\pi}(1), \ldots, X_{\pi}(i-1)\} - \sum_{j<i} c_{\pi(j)}$.
- Otherwise, independently draw $X_{\pi}(i) \sim D_{\pi(i)}$. Update $i \leftarrow i + 1$ and go back to the previous step.

Note that the only feedback is the utility, and we do not see any value or even the index $i$ where we stop.

In the case of known distributions, the optimal one-round policy for this problem was designed by Weitzman [Wei79]: For every distribution $D_i$, solve the equation $\mathbb{E}\left[\max\{X_i - \sigma_i, 0\}\right] = c_i$; now play permutation $\pi$ by sorting in decreasing order of $\sigma_i$ and set threshold $\tau^*_i = \sigma_i$. Let $\text{OPT}$ be the optimal expected reward according to this optimal policy. Let $\text{ALG}_i$ be the expected reward of our policy in the $t$-th round. Then, we want to design an algorithm with total regret $T \cdot \text{OPT} - \sum_{t \in T} \text{ALG}_t$ at most $\tilde{O}(\text{poly}(n)\sqrt{T})$.

Before introducing the algorithm, we define the gain function for this general case:

$$g_i(v) := -c_i + \int_v^1 (x - v) f_i(x) dx = -c_i + (1 - v) - \int_v^1 F_i(x) dx.$$  \hfill (11)

Similar to the $n = 2$ case, this gain function is the expected additional utility we get on opening $X_i$ when we already have value $v$ in hand. Note that the optimal threshold $\tau^*_i$ satisfies $g_i(\tau^*_i) = 0$.

### 4.1 High-Level Approach via Valid Policies.

We first briefly introduce the initialization algorithm. The following lemma shows what we achieve in the initialization (proved in Appendix C.1).

**Lemma 4.1.** The initialization algorithm runs $\Theta(n^2 \sqrt{T} \log T)$ samples for each distribution to output interval $[\ell_i, u_i]$, such that w.h.p. the following hold simultaneously for all $i \in [n]$:

- $\ell_i \leq \sigma_i \leq u_i$.
- $|g_i(x)| \leq T^{-\frac{1}{4}}$ simultaneously for all $x \in [\ell_i, u_i]$.

After initialization, the main part is the action set-updating algorithm. Similar to the algorithm for $n = 2$, we hope to use estimates $\tilde{F}_i(x)$ to gradually shrink the intervals $[\ell_i, u_i]$. However, one major challenge is that we don’t have a fixed order. If $n$ is a constant, we can just simply try all possible permutations and use a multi-armed bandit style algorithm to find the optimal permutation. But the number of permutations is exponential in $n$, so this approach is impossible when $n$ is a general parameter. To get a polynomial regret algorithm, we can only test poly($n$) number of different orders.

Another challenge is that the idea for $n = 2$ can bound the regret when we are playing a sub-optimal threshold, but it tells nothing about playing a sub-optimal order. We don’t have a direct way to bound the regret when playing an incorrect order.

Both difficulties imply that only keeping the confidence intervals as the constraint for the actions
is not enough. Therefore, we also introduce a set of order constraints:

**Definition 4.2 (Valid Constraint Group).** Given a set of confidence intervals \( I = \{[\ell_1, u_1], [\ell_2, u_2], \ldots, [\ell_n, u_n]\} \) and a set \( S \) of order constraints, satisfying:

- \( u_i - \ell_i \leq T^{-\frac{1}{4}} \).
- \( \sigma_i \in [\ell_i, u_i] \).
- Every constraint in \( S \) can be defined as \((i, j)\) that means \( \sigma_i > \sigma_j \).
- The constraints in \( S \) are closed, i.e., if \((i, j), (j, k) \in S\), there must be \((i, k) \in S\).
- If \((i, j) \in S\), we must have \( u_i \geq u_j \) and \( \ell_i \geq \ell_j \).

For \((I, S)\) satisfying the conditions above, we call it a valid constraint group.

The intuition of the extra order constraints is: When we are shrinking the intervals, if it is evident that \( \sigma_i > \sigma_j \), we will require \( D_i \) to be in front of \( D_j \) in the following rounds. Correspondingly, we give the following definition for a “valid” policy. During the algorithm, we will only run valid policies, according to the current constraint group we have.

**Definition 4.3 (Valid Policy).** Let \((\tau_{\pi(1)}, \tau_{\pi(2)}, \ldots, \tau_{\pi(n)})\) be a policy to play in one round, where \( \pi \) is the distribution permutation for this policy, and the threshold in front of \( \pi(i) \) is \( \tau_{\pi(i)} \). For simplicity, we use \( \pi \) to represent a policy.

For a policy \( \pi \), we say it is valid for a constraint group \((I, S)\) if the following conditions hold:

- For \( i \in [n] \), \( \tau_{\pi(i)} \in [\ell_{\pi(i)}, u_{\pi(i)}] \).
- If \((i, j) \in S\), then \( D_i \) must be in front of \( D_j \), i.e., \( \pi^{-1}(i) < \pi^{-1}(j) \).
- For \( i < j \), \( \tau_{\pi(i)} \geq \tau_{\pi(j)} \).

Notice that for a valid constraint group, we have \( \sigma_i \in [\ell_i, u_i] \) for all \( i \in [n] \), and \( \sigma_i > \sigma_j \) for all \((i, j) \in S\). Then, the optimal policy is valid. Therefore, we can always find a valid policy from the constraint group.

Now, we are ready to give the main idea of the constraint-updating algorithm. In each phase, we first update the confidence intervals and then update the order constraints as follows:

- **Step 1:** For each \( i \in [n] \), we run \( \tilde{O}(\text{poly}(n)) \) samples to update the confidence interval to \([\ell_i', u_i']\), such that for every threshold \( \tau_i, \tau_i' \in [\ell_i', u_i'] \), the moving difference is small, i.e., if we move \( \tau_i \) to \( \tau_i' \) and keep the validity, the difference of the expected reward is bounded by \( O(\text{poly}(n) \cdot \epsilon) \).

- **Step 2:** For each distribution pair \((i, j)\) without a constraint, we run \( \tilde{O}(\text{poly}(n)) \) samples to test the order between them, such that we can either clarify which one is bigger between \( \sigma_i \) and \( \sigma_j \), or we can claim that the swapping difference (the difference before and after swapping \( D_i \) and \( D_j \)) is bounded by \( O(\text{poly}(n) \cdot \epsilon) \).

Finally, we argue that for every valid policy, we can convert it into the optimal policy by using \( \text{poly}(n) \) number of moves and swaps. This is sufficient for us to give the \( O(\text{poly}(n) \cdot \epsilon) \) regret bound.

In the following analysis, we use separate sub-sections to introduce each part. Section 4.2 provides the Interval-Shrinking algorithm to bound the moving difference. Section 4.3 introduces the way to add a new order constraint to bound the swapping difference. Section 4.4 shows how to convert a valid policy to the optimal policy using a \( \text{poly}(n) \) number of moves and swaps. Finally, Section 4.5 combines the results of three sub-sections to complete the analysis.

### 4.2 Step 1: Interval-Shrinking to Bound Moving Difference

The goal of this sub-section is: Given \( i \in [n] \) and an original constraint group \((I, S)\), we want to update the confidence interval \([\ell_i, u_i]\), to make sure that moving \( \tau_i \) inside the new confidence
interval incurs a small difference. The key idea of the Interval-Shrinking algorithm is similar to the case when \( n = 2 \): For each \( i \in [n] \), we want to play two different values for \( \tau_i \), and see the difference of the expected reward. However, playing \( \tau_i = \ell_i \) and \( \tau_i = u_i \) might be impossible. The reason is: We hope to keep a decreasing threshold setting. There may not be a policy that allow \( \tau_i \) to be set to \( u_i \) and \( \ell_i \) without changing other thresholds. If we need different permutations to test \( \tau_i = u_i \) and \( \tau_i = \ell_i \), this makes the analysis involved. Therefore, we should find a policy that fixes the order and other thresholds, then test \( \tau_i \) under this fixed policy while keeping a decreasing thresholds.

When we set \( \tau_i \) to be different values, the two policies will be different only when the maximum reward before \( \tau_i \) falls between the two thresholds. Therefore, to see the largest difference, we hope the probability of this event is maximized. This intuition allows us to give the following definition:

**Definition 4.4 (MoveBound Policy).** Given \((I, S)\) and \(i \in [n]\), a MoveBound policy is a valid partial policy \( \pi^8 \), such that \( F_{\pi,i}(u) - F_{\pi,i}(\ell) \) is maximized.

In the definition, \( F_{\pi,i}(x) \) is the probability that the algorithm reaches distribution \( X_i \) with maximum value \( v < x \) in hand, i.e.,

\[
F_{\pi,i}(x) := \prod_{j < \pi^{-1}(i)} F_{\pi(j)}(x).
\]

Furthermore, \( u \) and \( \ell \) represents two possible value of \( \tau_i \) to keep a valid \( \pi \), i.e., \( \pi \) is valid when both \( \tau_i = u \) and \( \tau_i = \ell \).

A key fact of MoveBound policy is that for every different distribution, we might find a different MoveBound policy. This is different from the Prophet Inequality problem: In the Pandora’s Problem, we don’t keep a fixed order. Every order that satisfies the constraints \((I, S)\) is possible to be tested.

Now, the key idea of the Interval-Shrinking algorithm is clear: For each \( i \), find the clever threshold and run samples with \( \tau_i = u \) and \( \tau_i = \ell \). Then, use a method similar to Algorithm 3 to calculate the new interval. The following algorithm describes the details of this idea:

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**Algorithm 6: Interval-Shrinking Algorithm**

**Input:** \((I, S), \epsilon, i, \hat{F}_1(x), ... , \hat{F}_n(x)\)

1. Get an approximately clever policy \( \pi \) and \( \ell, u \) using Lemma 4.8.
2. Calculates \( \hat{F}_{\pi,i}(x) \).
3. For \( \tau \in [\ell_i, u_i] \), let
   \[
   \hat{\Delta}_i(\tau) := \hat{F}_{\pi,i}(u) \int_\tau^u (\hat{F}_i(x) - 1)dx + \hat{F}_{\pi,i}(\ell) \int_\ell^\tau (\hat{F}_i(x) - 1)dx - \int_\ell^u \hat{F}_{\pi,i}(x)(\hat{F}_i(x) - 1)dx.
   \]
4. Run \( \Theta(m \log T) \) samples with \( \tau_i = u \). Let the average reward be \( \hat{R}_u \).
5. Run \( \Theta(m \log T) \) samples with \( \tau_i = \ell \). Let the average reward be \( \hat{R}_\ell \).
6. Define \( \hat{\delta}_i(\tau) := \hat{\Delta}_i(\tau) - (\hat{R}_u - \hat{R}_\ell) \).
7. Let \( u'_i = \max_{\tau \in [\ell_i, u_i]} |\hat{\delta}_i(\tau)| < \epsilon \) and let \( \ell'_i = \min_{\tau \in [\ell_i, u_i]} |\hat{\delta}_i(\tau)| < \epsilon \).

**Output:** \( [\ell'_i, u'_i] \)

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Then, the following lemma shows the bound when modifying a threshold:

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8Here, we say \( \pi \) is a partial policy because it’s not completely fixed. We fix the permutation of the distributions and the value of all other thresholds, but the value of \( \tau_i \) is flexible.
Lemma 4.5 (Moving Difference Bound). Suppose we are given \((I, S)\), \(\epsilon > 16T^{-\frac{1}{2}}\), CDF estimates \(\hat{F}_1(x), \ldots, \hat{F}_n(x)\), and \(i \in [n]\), satisfying the following conditions for all \(j \in [n]\):

- \(|g_j(\tau)| \leq T^{-\frac{1}{4}}\) for all \(\tau \in [\ell_j, u_j]\).
- \(\tau_j^+ = \sigma_j \in [\ell_j, u_j]\).
- For any valid partial policy \(\pi'\) of \((I, S)\), we fix the order and the other thresholds except \(\tau_j\). Assume \(\pi'\) is valid when both \(\tau_j = \ell'\) and \(\tau_j = \ell\). Define \(\delta_{\pi', \pi, \ell', \ell}(\tau) = (F_{\pi', \ell'}(u') - F_{\pi', \ell'}(u'))g_i(\tau)\). Then \(|\delta_{\pi', \pi, \ell', \ell}(\tau)| \leq 3\epsilon\).
- CDF estimate \(\hat{F}_j(x)\) is constructed via \(\Theta\left(\frac{n^2 \log T}{\epsilon^2}\right)\) new i.i.d. samples of \(X_j\).

Then, Algorithm 6 runs \(\Theta\left(\frac{\log T}{\epsilon^2}\right)\) samples and calculates a new interval \([\ell'_i, u'_i]\), such that the following properties hold with high probability:

\begin{enumerate}
  \item \(\sigma_i \in [\ell'_i, u'_i]\)
  \item Let \(I'_i = (I \setminus \{[\ell_i, u_i]\}) \cup \{[\ell'_i, u'_i]\}\). For any valid partial policy \(\pi'\) of \((I'_i, S)\), we fix the order and the other thresholds. Assume \(\pi'\) is valid when both \(\tau_i = u'\) and \(\tau_i = \ell'\). Define \(\delta_{\pi', \pi, u', \ell}(\tau) = (F_{\pi', \ell'}(u') - F_{\pi', \ell'}(u'))g_i(\tau)\). Then \(|\delta_{\pi', \pi, u', \ell}(\tau)| \leq 3\epsilon\).
  \item For any valid policy of \((I'_i, S)\), if we fix the order and the other thresholds, but modify \(\tau_i\) to \(\tau_i'\), satisfying that the new policy is still valid, the difference of the expected reward between these two policies is less than \(3\epsilon\).
\end{enumerate}

Before starting the proof, we first give an accuracy bound of the distribution estimates, which is proved in Appendix C.2.

Claim 4.6. \(|\prod_{i \in S} \hat{F}_i(x) - \prod_{i \in S} F_i(x)| \leq \sqrt{\epsilon}\) simultaneously for all \(x \in [0, 1]\) and \(S \subseteq [n]\).

Proof of Lemma 4.5. We first introduce some notation. Fix the MoveBound \(\pi\). Assume we want to move \(\tau_i\) from \(\tau_i = u\) to \(\tau_i = \ell\). Since we only care about the absolute value of the difference between two expected rewards, we may assume \(u > \ell\).

If moving \(\tau_i\) from \(u\) to \(\ell\), the performance of the two policies will only be different if the previous maximum reward falls between \(\ell\) and \(u\). It will reject the previous maximum if \(\tau_i = u\), but accept it when \(\tau_i = \ell\). Besides, since \(\ell\) is greater than the next threshold in \(\pi\), when the previous maximum is inside \([\ell, u]\), the algorithm must stop before the next threshold, which means the difference only comes from \(\tau_i\) and \(X_i\).

Recall that \(F_{\pi, i}(x) = \prod_{j < \pi^{-1}(i)} F_j(x)\), i.e., \(F_{\pi(1)}(x)\) is the probability that the algorithm reaches \(\tau_i\) with \(v \leq x\) in hand. Let \(f_{\pi, i}(x) = F'_{\pi, i}(x)\). Then, the difference of the expected reward is \(\int_\ell^u f_{\pi, i}(x)g_i(x)dx = F_{\pi, i}(u)g_i(u) - F_{\pi, i}(\ell)g_i(\ell) - \int_\ell^u F_{\pi, i}(x)g'_i(x)dx\). To upper-bound this difference, define generalized bounding function

\[
\delta_i(\tau) := -(F_{\pi, i}(u) - F_{\pi, i}(\ell)) \cdot g_i(\tau).
\]

Then, to learn \(\delta_i(\tau)\), we define

\[
\Delta_i(\tau) := F_{\pi, i}(u)(g_i(u) - g_i(\tau)) - F_{\pi, i}(\ell)(g_i(\ell) - g_i(\tau)) - \int_\ell^u F_{\pi, i}(x)g'_i(x)dx
\]

\[
= F_{\pi, i}(u) \int_\tau^u (F_i(x) - 1)dx + F_{\pi, i}(\ell) \int_\ell^\tau (F_i(x) - 1)dx - \int_\ell^u F_{\pi, i}(x)(F_i(x) - 1)dx.
\]

Observe that \(\delta_i(\tau) = \Delta_i(\tau) - (R_u - R_\ell)\), where \(R_u\) and \(R_\ell\) corresponds to the expected reward in the MoveBound policy with \(\tau_i = u\) and \(\tau_i = \ell\). Then, by replacing \(F_i(x)\) with \(\hat{F}_i(x)\), we can get

\footnote{We need to guarantee that the policies are both valid when \(\tau_i = \ell\) and \(\tau_i = u\).}
\(\hat{\Delta}_i(\tau)\), which is an estimate of \(\Delta_i(\tau)\). For \(R_u\) and \(R_o\), we can learn the estimates \(\hat{R}_u\) and \(\hat{R}_o\) via running samples. Combining these estimates results in \(\hat{\delta}_i(\tau)\). Then, the following lemma shows that \(\hat{\delta}_i(\tau)\) estimates \(\delta_i(\tau)\) accurately (proved in Appendix C.3).

**Claim 4.7.** In Algorithm 6, if the conditions in Lemma 4.5 holds, then w.h.p. \(|\hat{\delta}_i(\tau) - \delta_i(\tau)| \leq \epsilon\) simultaneously for all \(\tau \in [\ell_i, u_i]\).

Now we prove the statements in Lemma 4.5. In the following proofs, we assume \(|\delta_i(\tau) - \hat{\delta}_i(\tau)| \leq \epsilon\) holds simultaneously for all \(\tau \in [\ell_i, u_i]\).

**Statement (i).** Look at Algorithm 6: It finds \(\pi, \ell, u\), gets \(\hat{\delta}_i(\tau)\), then calculates \([\ell'_i, u'_i] = \{\tau \in [\ell_i, u_i]: |\hat{\delta}_i(\tau)| < \epsilon\}\). Since \(\hat{\delta}_i(\tau^*) = 0\), there must be \(|\hat{\delta}_i(\tau^*)| \leq \epsilon\). Therefore, \(\tau^* \in [\ell'_i, u'_i]\).

**Statement (ii).** Notice that \([\ell'_i, u'_i] = \{\tau \in [\ell_i, u_i]: |\hat{\delta}_i(\tau)| \leq \epsilon\}\). Therefore, for all \(\tau \in [\ell'_i, u'_i]\), \(|\delta_i(\tau)| \leq |\hat{\delta}_i(\tau)| + |\hat{\delta}_i(\tau) - \hat{\delta}_i(\tau)| \leq 2\epsilon\). We first assume that Algorithm 6 receives a MoveBound policy. Then, when \(\pi\) is a MoveBound policy, from the definition, we have \(F_{\pi,i}(u) - F_{\pi,i}(\ell) \geq F_{\pi',i}(u') - F_{\pi',i}(\ell')\) for all valid \(\pi', u', \ell'\). Therefore, \(|\delta_i(\tau)| \leq |\hat{\delta}_i(\tau)| \leq 2\epsilon\).

**Statement (iii).** We again assume that Algorithm 6 receives a MoveBound policy. Recall that we just proved \(|\delta_i(\tau)| \leq 2\epsilon\). Combining this with (12), we have \(|g_i(\tau)| \leq \frac{2\epsilon}{(F_{\pi,i}(u) - F_{\pi,i}(\ell))}\).

Now, consider the policy \(\pi\). Assume we first have \(\tau_i = u\) and we want to move it to \(\tau_i = \ell\), satisfying \(\ell', u' \in [\ell'_i, u'_i]\) and \(\pi'\) is valid when \(\tau_i = \ell'\) and \(\tau_i = u'\). Then, the difference of the expected reward is \(\int_{\ell'}^{u'} f_{\pi',i}(x) g_i(x) dx\), and we have the following bound:

\[
\left|\int_{\ell'}^{u'} f_{\pi',i}(x) g_i(x) dx\right| \leq |F_{\pi',i}(u') - F_{\pi',i}(\ell')| \max_{x \in [\ell', u']} |g_i(x)| \leq 2\epsilon, \tag{13}
\]

where in the last inequality we use the fact that \(F_{\pi,i}(u) - F_{\pi,i}(\ell) \geq |F_{\pi',i}(u') - F_{\pi',i}(\ell')|\) when \(\pi\) is a MoveBound policy, and \(|g_i(v)| \leq \frac{2\epsilon}{(F_{\pi,i}(u) - F_{\pi,i}(\ell))}\) for all \(v \in [\ell'_i, u'_i]\). This gives an upper bound on the difference of the expected reward when we want to move \(\tau_i\).

The remaining part is to show that Algorithm 6 gets a MoveBound policy. The following Lemma shows that we can calculate an approximately MoveBound policy:

**Lemma 4.8.** There exists an algorithm with time complexity \(O(n \cdot 2^n)\) that calculates a MoveBound policy with an extra \(4\sqrt{\epsilon}\) additive error.

We leave the details of the algorithm and the proof to Appendix C.4.

Finally, we show that this \(4\sqrt{\epsilon}\) error doesn’t hurt too much for both Statement (ii) and (iii).

Define

\[
q_i := \max_{\pi} F_{\pi,i}(u) - F_{\pi,i}(\ell), \quad \text{and} \quad \hat{q}_i := F_{\pi',i}(u) - F_{\pi',i}(\ell) |\pi' = \arg\max_{\pi} \hat{F}_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell).
\]

Then \(q_i \leq \hat{q}_i + 4\sqrt{\epsilon}\). For Statement (ii), we have

\[
|\delta_{\pi',u',t',i}(\tau)| \leq q_i \max_{v \in [\ell'_i, u'_i]} |g_i(v)| \leq (\hat{q}_i + 4\sqrt{\epsilon}) \cdot \frac{\max_{v \in [\ell'_i, u'_i]} |\hat{\delta}_i(v)|}{\hat{q}_i} \leq \frac{\hat{q}_i \cdot 2\epsilon}{\hat{q}_i} + 4\sqrt{\epsilon} \cdot T^{-\frac{1}{4}} < 3\epsilon,
\]

where the last inequality holds when \(\epsilon > 16T^{-\frac{1}{4}}\). For Statement (iii), following (13), we can bound
the moving difference to
\[
\left| \int_{\tau}^{u'} f_{x',j}(x)g_i(x) \right| \leq q_i \max_{v \in [\ell'_i,u'_i]} |g_i(v)|.
\]

Therefore, the same $3\epsilon$ bound also holds.

**4.3 Step 2: Updating Order Constraints to Bound Swapping Difference**

In this section, our goal is to verify $\sigma_i$ and $\sigma_j$ which one is larger, or claiming that reversing the order of $X_i$ and $X_j$ doesn’t hurt too much. We first provide the following lemma, which shows the difference of the expected reward when we swap two distributions with a same threshold:

**Lemma 4.9.** For a policy $\pi$, such that $X_i$ and $X_j$ are consecutive with $\tau_i = \tau_j = \tau$, let $\Delta_{\pi,i,j}(\tau)$ be the change of the expected reward after swapping $X_i$ and $X_j$, then

\[
\Delta_{\pi,i,j}(\tau) = F_{\pi,i}(\tau) (g_i(\tau)(1 - F_j(\tau)) - g_j(\tau)(1 - F_i(\tau))).
\]

**Proof.** Assume we have value $v$ in hand before arriving $X_i$ and $X_j$. To pass the threshold, there must be $v \leq \tau$. If $X_i$ is in the front, the expected gain of playing $X_i$ is $g_i(v)$. After that, if $X_i < \tau$, we can play $X_j$ as well. The expected gain is $F_i(v)g_i(v) + \int_{\tau}^{u} f_i(x)g_j(x)dx = F_i(\tau)g_j(\tau) - \int_{\tau}^{u} F_i(x)g_j(x)dx$. Therefore, the total expected gain from $X_j$ and $X_i$ is $g_i(v) + F_i(\tau)g_j(\tau) - \int_{\tau}^{u} F_i(x)g_j(x)dx$. Similarly, if $X_j$ is in the front, the total expected gain from $X_i$ and $X_j$ is $g_j(v) + F_j(\tau)g_i(\tau) - \int_{\tau}^{u} F_j(x)g_i(x)dx$.

Notice that the order of $X_i$ and $X_j$ doesn’t influence the expected gain from the distributions behind $X_i$ and $X_j$. Therefore, the difference of the gain from $X_i$ and $X_j$ is exactly the difference of the expected reward:

\[
\left( g_i(v) + F_i(\tau)g_j(\tau) - \int_{\tau}^{u} F_i(x)g_j(x)dx \right) - \left( g_j(v) + F_j(\tau)g_i(\tau) - \int_{\tau}^{u} F_j(x)g_i(x)dx \right)
\]

\[
= g_i(v) + F_i(\tau)g_j(\tau) - g_j(v) - F_j(\tau)g_i(\tau) + \int_{\tau}^{u} (F_j(x)(F_i(x) - 1) - F_i(x)(F_j(x) - 1))dx
\]

\[
= \left( g_i(v) + u - \tau + \int_{\tau}^{u} F_i(x)dx \right) - \left( g_j(v) + u - \tau + \int_{\tau}^{u} F_j(x)dx \right) + F_i(\tau)g_j(\tau) - F_j(\tau)g_i(\tau)
\]

\[
= g_i(\tau)(1 - F_j(\tau)) - g_j(\tau)(1 - F_i(\tau)).
\]

Since the probability that $v$ arrives with $v < \tau$ is exactly $F_{\pi,i}(\tau)$, the expected difference is $\Delta_{\pi,i,j}(\tau) = F_{\pi,i}(\tau)(g_i(\tau)(1 - F_j(\tau)) - g_j(\tau)(1 - F_i(\tau)))$.

Lemma 4.9 shows the following properties:

1. Assume $\sigma_i > \sigma_j$. When $\tau \in [\sigma_j, \sigma_i]$, $\Delta_{\pi,i,j}(\tau) < 0$, i.e., letting $X_i$ be in the front is better. This implies: If we know the sign of $\Delta_{\pi,i,j}(\tau)$, and we are sure that $\tau$ is between $\sigma_i$ and $\sigma_j$, then we can determine that $\sigma_i$ and $\sigma_j$ which one is greater.

2. Fix $i, j, \tau$. $|\Delta_{\pi,i,j}(\tau)|$ is maximized when $F_{\pi,i}(\tau)$ is maximized.

According to Property 2, we hope to test $X_i$ and $X_j$ with a policy $\pi$ that maximizes $F_{\pi,i}(\tau)$. If the difference is bounded when $F_{\pi,i}(\tau)$ is maximized, the swapping difference is bounded in all policies. Inspired by this, we give the definition of the SwapTest policy:

**Definition 4.10 (SwapTest Policy).** Given $(I, S)$ and $i, j \in [n]$ with $i \neq j$ and $(i, j), (j, i) \notin S$. Assume we have $[\ell'_i, u'_i], [\ell'_j, u'_j] \in I$. A SwapTest policy is a pair of valid policies $(\pi, \pi')$, such that
• $\tau_i = \tau_j = \max\{\ell'_i, \ell'_j\}$.
• $X_i$ and $X_j$ are adjacent in both $\pi$ and $\pi'$, but under different orders, and this is the only difference between $\pi$ and $\pi'$. W.l.o.g. assume $X_i$ is in the front in $\pi$, while $X_j$ is in the front in $\pi'$, i.e., $\pi^{-1}(i) = \pi^{-1}(j) - 1$, and $\pi'^{-1}(j) = \pi'^{-1}(i) - 1$.
• The SwapTest policy maximizes $F_{\pi,i}(\tau)$ when the first two conditions are satisfied.

Then, the algorithm for testing $X_i$ and $X_j$ is clear: We find the SwapTest policy for $X_i$ and $X_j$, run some samples for two policies and see the difference. If the difference is too large, we can verify $\sigma_i$ and $\sigma_j$ which one is larger. Otherwise, we can bound the swapping difference. The following algorithm gives the details:

**Algorithm 7: SwapTest Algorithm**

**Input:** Distribution indices $i$ and $j$
1 Run Algorithm 8 to get SwapTest policy $(\pi, \pi')$
2 Run $\Theta\left(\frac{\log T}{m^2} \right)$ samples with policy $\pi$. Let $\hat{R}_{i,j}$ be the average reward.
3 Run $\Theta\left(\frac{\log T}{m^2} \right)$ samples with policy $\pi'$. Let $\hat{R}_{j,i}$ be the average reward.
4 if $|\hat{R}_{i,j} - \hat{R}_{j,i}| > 40n$ then
5 Add constraint $(i, j)$ into $S'$ if $\hat{R}_{i,j} > \hat{R}_{j,i}$, otherwise add constraint $(j, i)$ into $S'$.
6 Update $S'$ according to the transitivity. Update $I'$ according to the new order constraints, i.e., when adding a constraint $(a, b)$, let $u'_b \leftarrow \min\{u'_a, u'_b\}$ and $\ell'_a \leftarrow \max\{\ell'_a, \ell'_b\}$.

**Output:** Updated constraint group $(I', S')$

**Algorithm 8: Finding SwapTest Policy**

**Input:** $(I', S')$, $m$, $i, j$
1 Let $\tau = \tau_i = \tau_j = \max\{\ell'_i, \ell'_j\}$
2 Let $T = \{k|(k, i) \in S' \lor (k, j) \in S' \lor \ell'_k > \tau\}$.
3 For $k \in T$, let $\tau_k = u'_k$
4 For $k \in [n] \setminus ((i, j) \cup T)$, let $\tau_k = \ell'_k$
5 Let $\pi$ and $\pi'$ be two policies that sort the distributions in a decreasing threshold order, and break ties according to $S'$. The only difference is: $X_i$ is in front of $X_j$ in $\pi$, but $X_j$ is in front of $X_i$ in $\pi'$.

**Output:** $\pi$ and $\pi'$

Before analysing the algorithm, we point out two facts of Algorithm 7:
• Algorithm 7 relies on Algorithm 6, i.e., we need to first run Algorithm 6 to get $n$ new confidence intervals, then run Algorithm 7 to update order constraints. This is critical to the regret analysis.
• In the SwapTest algorithm, we only test the swapping difference with $\tau_i = \tau_j = \max\{\ell'_i, \ell'_j\}$, and give the difference bound only with this threshold. This is sufficient for our regret analysis.

**Lemma 4.11** (Swapping Difference Bound). Given $(I', S')$, $m$, and $i, j \in [n]$ with $i \neq j$ and $(i, j), (j, i) \notin S$, where $I'$ is generated by Algorithm 6. Assume the properties in Lemma 4.1 hold. Algorithm 7 runs $\Theta\left(\frac{\log T}{m^2} \right)$ samples and achieves one of the following:
• Clarify $\sigma_i$ and $\sigma_j$ which one is bigger w.h.p., and give a new constraint $(i, j)$ or $(j, i)$.
• Make the following claim w.h.p.: For every two valid policies of $(I', S)$, satisfying:
\[ \tau_i = \tau_j = \max \{ \ell_i, \ell_j \}, \]
\[ X_i \text{ and } X_j \text{ are consecutive in both policies but in a different order. This is the only difference between two policies.} \]

The difference of the expected reward between these two policies is no more than 60\( n\epsilon \).

**Proof.** We first prove the theorem assuming Algorithm 8 returns a SwapTest policy \((\pi, \pi')\). According to the definition of SwapTest policy, \(\pi\) and \(\pi'\) maximizes the probability of reaching \(X_i\) and \(X_j\) when \(\tau_i = \tau_j = \max \{ \ell_i', \ell_j' \}\). According to Property 2, for any valid policy, such that \(X_i\) and \(X_j\) are consecutive with \(\tau_i = \tau_j = \max \{ \ell_i', \ell_j' \}\), the swapping difference is no more than the difference between \(\pi\) and \(\pi'\). Therefore, if we are evident that the difference between \(\pi\) and \(\pi'\) is no more than 60\( n\epsilon\), we can claim that this upper bounds the swapping difference between \(X_i\) and \(X_j\) for any other policy. The proof idea is the following: We run multiple samples to estimate \(R_{i,j}\) and \(\hat{R}_{i,j}\), where \(\hat{R}_{i,j}\) is the expected reward of \(\pi\) and \(R_{i,j}\) is the expected reward of \(\pi'\). Next, we show that \(|R_{i,j} - \hat{R}_{i,j}| \leq 10n\epsilon\) and \(|R_{j,i} - \hat{R}_{j,i}| \leq 10n\epsilon\) with high probability. Then, \(|R_{i,j} - R_{j,i}| \leq 60n\epsilon\) when \(|\hat{R}_{i,j} - \hat{R}_{j,i}| \leq 40n\epsilon\).

Now, we bound \(|R_{i,j} - \hat{R}_{i,j}|\) with Hoeffding’s Inequality (Theorem A.1); bounding \(|R_{j,i} - \hat{R}_{j,i}|\) is identical. \(\hat{R}_{i,j}\) is an estimate of \(R_{i,j}\) by running \(N = C \cdot \log T / n^2\epsilon^2\) samples, and the per-round reward is bounded by \([-0.5, 0.5]\). Then, \(\Pr \left[ |R_{i,j} - \hat{R}_{i,j}| > 10n\epsilon \right] < 2 \exp(-2N \cdot 100n^2\epsilon^2 / 4) = 2T^{-50C}\).

Hence, \(|R_{i,j} - R_{j,i}| \leq 10n\epsilon\) w.h.p. when \(C\) is sufficiently large.

The concentration proof above also shows that when \(|\hat{R}_{i,j} - \hat{R}_{j,i}| > 40n\epsilon\), we can claim that w.h.p. \(|R_{i,j} - R_{j,i}| > 20n\epsilon\). Next, we show that this is evident to clarify which of \(\sigma_i\) and \(\sigma_j\) is greater. We first introduce a special case to give the intuition: Consider the case that all other confidence intervals are disjoint with \([\ell_i', u_i']\) or \([\ell_j', u_j']\). W.l.o.g., assume \(\pi\) (\(X_i\) in the front) is better than \(\pi'\) (\(X_j\) in the front). If \(\tau = \max \{ \ell_i', \ell_j' \}\) is between \(\sigma_i\) and \(\sigma_j\), we can immediately claim that \(\sigma_i > \sigma_j\) according to Property 1. If \(\tau\) doesn’t fall between \(\sigma_i\) and \(\sigma_j\), there must be \(\tau < \min \{ \sigma_i, \sigma_j \}\). Then, we adjust \(\pi\) and \(\pi'\) by increasing \(\tau_i\) and \(\tau_j\) to \(\min \{ \sigma_i, \sigma_j \}\). According to Lemma 4.5, these operations do not change the expected reward too much: Since we move two thresholds in each policy, the expected reward of \(\pi\) can decrease by at most 6\( \epsilon\), and the expected reward of \(\pi'\) can increase by at most 6\( \epsilon\). Therefore, if the original \(\pi\) is at least 20\( \epsilon\) better than \(\pi'\), we can still claim that \(\sigma_i > \sigma_j\).

However, this moving process can be invalid in the general case: \(\min \{ \sigma_i, \sigma_j \}\) might be greater than some thresholds in front of \(X_i\) and \(X_j\). To fix this issue, consider the following process:

- **Step 1:** Increase \(\tau_i\) and \(\tau_j\) until reaching \(\tau_k\), where \(X_k\) is the distribution just in front of \(X_i\) and \(X_j\).
- **Step 2:** Swap \(X_i\) and \(X_j\) with \(X_k\).
- **Repeat Step 1 and 2** until \(\tau_i = \tau_j = \min \{ \sigma_i, \sigma_j \}\).

Let \(\Delta_{\pi, \pi'}\) be the difference between expected values of \(\pi\) and \(\pi'\). We monitor the change of \(\Delta_{\pi, \pi'}\) during these operations. Step 1 can decrease \(\Delta_{\pi, \pi'}\) by at most 12\( \epsilon < 20\epsilon\). Step 2 can increase the absolute value of \(\Delta_{\pi, \pi'}\). Since there can be at most \(n\) Step 1 and 2, if initially \(\Delta_{\pi, \pi'} > 20n\epsilon\), this is sufficient to guarantee that \(\Delta_{\pi, \pi'} > 0\) at the end of the process. Then, we are evident to claim \(\sigma_i > \sigma_j\).

It remains to show that Algorithm 8 returns a SwapTest policy. Besides, this policy should also guarantee that when we are swapping \(X_i\) and \(X_j\) with \(X_k\), the policy after doing a swap is still valid. Therefore, we introduce the following lemma:

**Lemma 4.12.** Algorithm 8 calculates a SwapTest policy. Besides, it has the following property:
Let \( \tau = \max\{\ell'_i, \ell'_j\} \) and \( \tau' = \min\{\sigma_i, \sigma_j\} \). If \( \tau' > \tau \), then for all \( k \in [n] \setminus \{i, j\} \), if \( \tau_k \in [\tau, \tau'] \), there must be \((k, i) \notin S\) and \((k, j) \notin S\).

**Proof.** The first two conditions in Definition 4.10 directly follows Algorithm 8. For the objective condition, observe that no distribution in the set \( T \) can be moved behind \( X_i \) and \( X_j \). Therefore, the policy calculated by Algorithm 8 minimizes \( F_{\pi, i}(\tau) \), which means the third condition holds.

For the additional property, assume there exists \( k \) satisfying \( \tau_k = u'_k \), \( \tau_k < \min\{\sigma_i, \sigma_j\} \). Notice that if \((k, i) \in S'\), there must be \( u'_k \geq u'_i \geq \min\{\sigma_i, \sigma_j\} \), which is in contrast to the condition \( \tau_k = u'_k < \min\{\sigma_i, \sigma_j\} \). Therefore, \((k, i) \notin S'\). Similarly, \((k, j) \notin S'\). Therefore, the additional property in Lemma 4.12 holds.

Finally, applying Lemma 4.12 immediately proves Lemma 4.11.

### 4.4 Converting our Policy to the Optimal Policy in Polynomial Steps

In this section, we show that using \( \text{poly}(n) \) number of moves and swaps can convert any valid policy into the optimal policy. Since Lemma 4.5 and Lemma 4.11 already show that the difference of each move and swap is bounded by \( O(\text{poly}(n)\epsilon) \), combining these results, we can argue that the per-round loss of a valid policy is bounded by \( O(\text{poly}(n)\epsilon) \). Formally, we give the following lemma:

**Lemma 4.13.** Given a valid constraint group \((I, S)\). For a valid policy of \((I, S)\), we use a “move” to represent the action that modifies a single threshold, and guarantees that the policy after modifying the threshold is still valid. Besides, we use a “swap” to represent the action that swaps two consecutive distributions with the same threshold. This threshold should be equal to the maximum of the two lower confidence bounds, and the policy after swapping the distributions should still be valid.

For any valid policy of \((I, S)\), it can be converted into the optimal policy using \( 2n^2 \) moves and \( 2n^2 \) swaps.

**Proof.** Let \( \pi \) be the policy that \( \tau_i = \ell_i \) for all \( i \in [n] \), and sort the distributions in a decreasing order. Since for every constraint \((i, j) \in S\), we have \( \ell_i \geq \ell_j \), \( \pi \) must be a valid policy.

We can prove Lemma 4.13 by showing the following statement: Starting from the policy \( \pi \), we can move it to any valid policy \( \pi' \) using \( n^2 \) moves and \( n^2 \) swaps:

- **Step 1:** Let \( i = \arg \max_i \tau'_i \), where \( \tau'_i \) is the threshold of \( X_i \) in policy \( \pi' \).

- **Step 2:** If \( X_i \) is not the first distribution in \( \pi \), move \( \tau_i \) to \( \tau_{\pi_{n-1}(i)-1} \), then swap \( X_i \) and \( X_{\pi_{n-1}(i)-1} \).

- **Step 3:** Do Step 2 until \( X_i \) is moved to the first place. Then move \( \tau_i \) to \( \tau'_i \).

- **Step 4:** Ignore \( X_i \) in both \( \pi \) and \( \pi' \), repeat Step 1, 2 and 3 until every distribution is settled.

Each distribution only involves in \( n \) swaps and \( n \) moves, so the total number of moves and swaps are both bounded by \( n^2 \). Then, we need to show the validity of every operation. For each move, we increase \( \tau_i \) to let it be closer to \( \tau'_i \). Since \( \tau'_i \in [\ell_i, u_i] \), every move is valid. For each swap, the threshold in the front must reach its lower confidence bound. Besides, every swap happens only when there is no constraint between two distributions, so every swap is valid.

Finally, notice that every operation is bidirected. It means that starting from any valid policy \( \pi' \), we can convert it to the policy \( \pi \), and then convert it to the optimal policy using \( 2n^2 \) moves and swaps, which finishes the proof.
4.5 Putting Everything Together

In this section, we show how to combine Algorithm 6 and Algorithm 7 to generate a new valid constraint group \((I', S')\), then proves that this leads to an \(O(\text{poly}(n)\sqrt{T})\) regret algorithm. We first give the one-phase algorithm:

**Algorithm 9: Constraint Updating Algorithm for Pandora’s Box**

**Input:** \(I = \{[\ell_1, u_1], \ldots, [\ell_n, u_n]\}, S = \{(i, j)\}, \hat{F}_1(x), \ldots, \hat{F}_n(x), m\)

1. //STEP 1: Calculate new confidence interval for each distribution
   
   For \(i \in [n]\) do
   
   3. For \(j \in [n]\), construct \(\hat{F}_j(x)\) using \(\Theta(\frac{n^2 \log T}{\epsilon})\) new i.i.d. samples of \(X_j\)
   
   Run Algorithm 6 with new CDF estimates to get \(\ell'_i\) and \(u'_i\).

5. //Adjust the confidence intervals to meet constraints in \(S\).

6. for \((i, j) \in S\) do

7. Let \(\ell'_i = \max\{\ell'_i, \ell'_j\}\) and \(u'_j = \min\{u'_j, u'_i\}\).

8. Let \(I' = \{[\ell'_i, u'_i]\}\) and \(S' = S\)

9. //Add new constraints for disjoint confidence intervals

10. for \((i, j) \notin S'\) do

11. if \(\ell'_i > u'_j\) then Add \((i, j)\) into \(S'\);

12.  

13. //STEP 2: Calculate new constraints for each distribution pair

14. Let \(Q = \{(i, j) | (i, j) \notin S' \land (j, i) \notin S'\}\)

15. while \(Q \neq \emptyset\) do

16. Choose \((i, j) \in Q\) and remove \((i, j)\) from \(Q\)

17. Run Algorithm 7 with input \((i, j)\) and update \(I'\) and \(S'\)

18. //New constraints may fail some previous tests. Should add them back

19. For every \(k\) such that \(\ell'_k\) changes in Algorithm 7, if \(\exists k'\) such that \((k, k'), (k', k) \notin S'\), add \((k, k')\) into \(Q\).

**Output:** \((I', S')\)

We can directly give the following lemma according to the three lemmas above:

**Lemma 4.14** (Main Lemma). Given \((I, S)\) and \(\epsilon > 16T^{-\frac{1}{2}}\). Assume the following properties hold:

- \((I, S)\) is valid.
- For all \(j \in [n]\), for any valid partial policy \(\pi'\) of \((I, S)\), we fix the order and the other thresholds except \(\tau_j\). Assume \(\pi'\) is valid when both \(\tau_j = \ell'\) and \(\tau_j = \ell\). Define \(\delta_{\pi', \ell', \ell, j}(\tau) = (F_{\pi', j}(\ell') - F_{\pi', j}(u')) \delta(\tau)\). Then \(|\delta_{\pi', \ell', \ell, j}(\tau)| \leq 6\epsilon\).

Then Algorithm 9 runs \(O(\frac{n \log T}{\epsilon^2})\) rounds, such that the policy in each round is valid for \((I, S)\) (except Line 3), and output a new constraint group \((I', S')\), satisfying the following statements with high probability:

- \((I', S')\) is valid.
- For all \(j \in [n]\), for any valid partial policy \(\pi'\) of \((I', S')\), we fix the order and the other thresholds except \(\tau_j\). Assume \(\pi'\) is valid when both \(\tau_j = \ell'\) and \(\tau_j = \ell\). Define \(\delta_{\pi', \ell', \ell, j}(\tau) = (F_{\pi', j}(\ell') - F_{\pi', j}(u')) \delta(\tau)\). Then \(|\delta_{\pi', \ell', \ell, j}(\tau)| \leq 3\epsilon\).
- For a valid policy of \((I', S')\), the per-round regret is no more than \(126n^3\epsilon\). 

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Proof. In this proof, we assume Lemma 4.5 and Lemma 4.11 holds. Since we only use these two lemmas \( \text{poly}(n) \) times, by the union bound, our proof can success with high probability.

For the validity of \((I', S')\), the statement \( \sigma_i \in [\ell'_i, u'_i] \) follows Lemma 4.5, and the statement \( \sigma_i > \sigma_j \) for all \((i, j) \in S'\) follows Lemma 4.11. All other statements hold by definition. Therefore, \((I', S')\) is valid.

For the bound of \(|\delta_{\pi', \ell', \ell', i, j}(\tau)|\), it’s guaranteed directly by Lemma 4.5. Notice that Lemma 4.5 even provides a stronger bound for the constraint group \((I'_i, S)\). Since all possible choices of \(\pi', \ell', u'\) must be valid for \((I'_i, S)\) when it’s valid for \((I', S')\), this doesn’t hurt the statement.

For the per-round regret bound, Lemma 4.5 says that the difference of a move is bounded by \(3\epsilon\). Lemma 4.11 says that the difference of a swap is bounded by \(60n\epsilon\). Then, according to Lemma 4.13, we can convert any valid policy to the optimal policy using \(2n^2\) moves and swaps. Therefore, the per-round regret is bounded by \(126n^3\epsilon\).

Next, we argue that Algorithm 9 runs no more than \(O\left(\frac{n \log T}{\epsilon^2}\right)\) rounds. Note that Algorithm 6 is called \(n\) times, and Algorithm 6 uses \(O\left(\frac{\log T}{\epsilon^2}\right)\) rounds in one call. So the number of rounds is \(O\left(\frac{n \log T}{\epsilon^2}\right)\). For Algorithm 7, we might test a distribution pair \((X_i, X_j)\) for multiple times. The reason is the following: When using Lemma 4.13, we need to make sure that the value of the final \(\max\{\ell'_i, \ell'_j\} \) is the one that we test. Therefore, if the value of \(\ell'_i\) changes, we need to re-test some distribution pairs \((i, j)\). We can argue that the total number of tests is bounded: When doing an extra test for \((i, j)\), at least one of \(\ell'_i\) or \(\ell'_j\) must change. This can happen only when a new constraint related to \(i\) or \(j\) is added into \(S'\). There are only \(2n\) constraints related to \(i\) and \(j\), so we can test \((i, j)\) at most \(4n^3\) times. Therefore, the total number of calls of Algorithm 7 is no more than \(4n^3\), and Algorithm 7 uses \(O\left(\frac{\log T}{\epsilon^2}\right)\) samples in one call, so the number of samples is bounded by \(O\left(\frac{n \log T}{\epsilon^2}\right)\). Combining the two results finishes the proof.

Now, we are ready to show the total regret bound.

Theorem 4.15. There exists an \(O(n^{4.5} \sqrt{T \log T})\) regret algorithm for Pandora’s Box problem.

Proof. We run Algorithm 2 and then use Lemma 2.5 to bound the main part of the total regret. To run Algorithm 2, we require the conditions listed in Lemma 4.14 hold (with high probability). We discuss them separately:

- \((I, S)\) is valid: For the first phase, the condition \(\tau_i \in [\ell_i, u_i]\) is guaranteed by Lemma 4.1, and we don’t have any initial order constraints between distributions (except those distributions with disjoint confidence intervals). Therefore, \((I, S)\) is valid for the first phase. Starting from the second phase, this is guaranteed by Lemma 4.14.

- \(|\delta_{\pi', \ell', \ell', i, j}(\tau)| \leq 6\epsilon\): For the first phase, this is true because \(|\delta_{\pi', \ell', \ell', i, j}(\tau)| \leq |g(\tau)| \leq T^{-\frac{1}{4}}\), and initially we have \(\epsilon = O(1)\). Starting from the second phase, this is from Lemma 4.14 regarding the previous phase. Notice that parameter \(\epsilon\) in the new phase is exactly \(\frac{1}{2}\) in the previous phase. Therefore, there is an extra 2 factor in the condition.

Lemma 4.14 implies that after \(O\left(\frac{n^{7/2} \log T}{\epsilon^2}\right)\) rounds, the one-round regret in the new constraint group is bounded by \(\epsilon\). Applying Lemma 2.5 with \(\alpha = 7\), we have the \(O(n^{3.5} \sqrt{T \log T})\) regret bound.

Besides, there are some extra rounds not covered by Lemma 2.5, including the initialization and the CDF estimates construction (Line 3 in Algorithm 9). For the initialization, Lemma 4.1 runs \(\Theta(n^3 \sqrt{T \log T})\) samples, so the regret is \(O(n^3 \sqrt{T \log T})\). For the CDF estimates construction, let
Let $k$ be the number of phases in the doubling algorithm. Then, the total number of samples is

$$
\sum_{i=1}^{\infty} n \cdot \Theta(\frac{n^2 \log T}{\epsilon_i}) = O(\sqrt{T})
$$

Combining three parts of regret, the total regret is $O(n^{3.5}\sqrt{T}\log T)$.

Finally, recall that until now we are working on a scaled Pandora’s Box problem: We scale down the values and the costs by a factor of $2n$. Therefore, for the original problem, the final regret bound is $O(n^{4.5}\sqrt{T}\log T)$.

**4.6 Making the Algorithm Efficient**

Currently, the running time of the whole algorithm is exponential in $n$ as just Lemma 4.8 introduces an algorithm with $O(n^{2n})$ running time. If we want a polynomial time algorithm, we may need an approximation. The following lemma shows a new regret bound with approximation:

**Lemma 4.16.** Assume for every $i$, we can $\gamma$-approximate $\max_{\pi,u,\ell} F_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell)$, then there exists an $O(\max\{\gamma n^{4.5}, \gamma^2 n\} \sqrt{T}\log T)$ regret algorithm.

**Proof.** In this proof, we first discuss the problem for the scaled Pandora’s Box problem, and add the scaled $2n$ factor back at last.

We first see how the $\gamma$ approximation changes Lemma 4.5. Recall that $q_i = \max_{\pi} F_{\pi,i}(u) - F_{\pi,i}(\ell)$. We further define $\tilde{q}_i = \max_{\pi} \hat{F}_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell)$ and $\hat{q}_i = F_{\pi',i}(u) - F_{\pi',i}(\ell)$, where $\pi'$ is the chosen policy that $\gamma$-approximates $\max_{\pi,u,\ell} F_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell)$. According to Claim 4.6, we have $\hat{q}_i \leq q_i - 2\sqrt{\epsilon}$ and $\tilde{q}_i \geq \frac{\hat{q}_i}{2} - 2\sqrt{\epsilon}$. So $q_i \leq \gamma \hat{q}_i + (2\gamma + 2)\sqrt{\epsilon}$. According to (13), Statement (ii) and (iii) are both bounded by

$$
q_i \max_{v \in [\ell_i', u_i']} |g_i(v)| \leq (\gamma \hat{q}_i + (2\gamma + 2)\sqrt{\epsilon}) \max_{v \in [\ell_i', u_i']} |g_i(v)| \\
\leq \gamma \hat{q}_i \cdot \frac{2\epsilon}{\hat{q}_i} + (2\gamma + 2)\sqrt{\epsilon} \cdot T^{-\frac{1}{4}} \leq 3\gamma \epsilon.
$$

For Statement (ii), this changes the bound of $|\delta_{\pi',u',\ell',i}(\tau)|$ to $O(\gamma \epsilon)$. In our proof, we use this bound when proving Claim 4.7: The bound of $|\delta_{\pi',u',\ell',i}(\tau)|$ provides a bound for the variance of the $\Delta_i(\tau)$ function, and then we use Bernstein Inequality to show $|\Delta_i(\tau) - \hat{\Delta}_i(\tau)| \leq O(\epsilon)$. When the bound changes to $O(\gamma \epsilon)$, to get an $O(\epsilon)$ approximation of $\Delta_i(\tau)$, the number of samples for constructing CDF estimates should be multiplied by $\gamma^2$, leading to an $O(\gamma^2 \sqrt{T}\log T)$ regret bound.

For Statement (iii), notice that we need to use this moving difference to bound the swapping difference. The main idea of the original proof is: Assume we want to test $X_i$ and $X_j$. After $O(n)$ moves, we can adjust $\tau_i$ and $\tau_j$ to $\min\{\sigma_i, \sigma_j\}$, then bound the swapping difference by $O(n) \cdot O(\epsilon)$. Since there is an extra $\gamma$ factor in the new moving difference bound, the new swapping difference should be $O(\gamma ne)$.

Next, Lemma 4.13 shows that we need $2n^2$ move operations and swap operations to convert a policy to the optimal one, so the new regret bound after $O(\frac{n^2 T}{\epsilon})$ samples is $O(\gamma n^3 \epsilon)$. Then, the parameter $\alpha$ in Lemma 2.5 changes to $\gamma^2 n^7$, so the total regret from the doubling algorithm is $O(\gamma n^{3.5}\sqrt{T}\log T)$. 

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Finally, after combining these two new regret bounds and adding the scaled 2\(n\) factor back to the regret bound, we get the \(O(\max\{\gamma n^{4.5}, \gamma^2 n^2\}\sqrt{T}\log T)\) final regret bound.

Lemma 4.16 shows that: If we can get a \(\text{poly}(n)\) approximation for the \text{MoveBound} policy in polynomial time, we can still get an \(O(\text{poly}(n)\sqrt{T})\) regret algorithm. To achieve this goal, we introduce the following sub-routine:

**Definition 4.17** (sub-routine). Let Problem A be the following: Given \(n\) and real numbers \(a_1, ..., a_n, b_1, ..., b_n\), satisfying \(0 \leq a_i \leq b_i \leq 1\) for all \(i \in [n]\). The objective of Problem A is to calculate

\[
\max_{B \subseteq [n]} \prod_{i \in B} b_i - \prod_{i \in B} a_i.
\]

under a set of constraints \(\{(i, j)\}\), where a constraint \((i, j)\) means that if we have \(i \in B\), there must be \(j \in B\).

**Lemma 4.18.** If there exists an algorithm that calculates an \(\gamma\)-approximation for Problem A, then there exists an algorithm that \(\gamma\)-approximates \(\max_\pi \hat{F}_{\pi, i}(u) - \tilde{F}_{\pi, i}(\ell)\). If the running time of the algorithm for approximating Problem A is polynomial, then the algorithm for approximating \(\hat{F}_{\pi, i}(u) - \tilde{F}_{\pi, i}(\ell)\) is also polynomial.

**Proof.** Consider calculating a \text{MoveBound} policy for \(X_i\). Assume that we know the value of \(\ell\) and \(u\). Then, we only need to pick a subset \(B \subseteq [n] \setminus \{i\}\) to maximize \(\prod_{j \in B} b_j - \prod_{j \in N} a_j\), where \(b_j = \hat{F}_j(u)\), and \(a_j = \tilde{F}_j(\ell)\).

However, not all subsets \(B\) are valid. Firstly, for \(j \in B\), there must be \(\tau_j \geq u\), which means \(u_j \geq u\) is required. Similarly, we should also guarantee that \(\ell_j \leq \ell\) for all \(j \in [n] \setminus B\). Besides, if there is an order constraint \((j, k)\), then \(k \in B\) implies \(j \in B\), which can be represented as a constraint in Problem A. If all constraints are satisfied, policy \(\tau_j = u_j\) for \(j \in B\) and \(\tau_j = \ell_j\) for \(j \notin B \cup \{i\}\) is a feasible policy. Therefore, finding the optimal policy with fixed \(\ell\) and \(u\) is captured by Problem A. So, an \(\gamma\)-approximation algorithm for Problem A also \(\gamma\)-approximates \(\hat{F}_{\pi, i}(u) - \tilde{F}_{\pi, i}(\ell)\).

Notice that when maximizing \(\hat{F}_{\pi, i}(u) - \tilde{F}_{\pi, i}(\ell)\), we want to push the thresholds to the boundaries to give \(\tau_i\) enough space. Therefore, the value of \(\ell\) and \(u\) must be equal to some \(\ell_j\) or \(u_j\), which means that there are only \(O(n^2)\) candidates. Therefore, if the algorithm that \(\gamma\)-approximates Problem A runs in polynomial time, the running time of the algorithm for approximating \(\hat{F}_{\pi, i}(u) - \tilde{F}_{\pi, i}(\ell)\) is also polynomial.

It remains to give a \(\text{poly}(n)\)-approximation algorithm for Problem A, with \(O(\text{poly}(n))\) running time. The following theorem shows that this is possible:

**Lemma 4.19.** Given an instance of Problem A. Let \(B_j\) be the subset with the smallest size that contains \(j\). Let \(q_j = \prod_{i \in B_j} b_i - \prod_{i \in B_j} a_i\). Then, \(\max_j q_j\) is an \(n\)-approximation of problem A.

**Proof.** Construct a graph \(G = (V, E)\), such that \(V = [n]\), and \(E\) is the set of all constraints, i.e., a constraint \((u, v)\) is represented as a directed edge \((u, v) \in E\). Then, \(B_j\) is the set of all vertices which is reachable from \(j\).

Notice that when \(G\) contains a connected component, we can shrink the component into one single vertex, because picking any single vertex in the connected component means picking the
whole component. Therefore, we only need to prove the theorem when \( G \) is a directed acyclic graph (DAG).

Re-index the vertices in \( G \), to make sure that for every edge \((u, v) \in E\), there must be \( u > v \). Besides, make sure that \( B^* = \{1, ..., k\} \) is exactly the optimal set of Problem A. Then,

\[
\prod_{i \in [k]} b_i - \prod_{i \in [k]} a_i = \sum_{j \in [k]} \left( \prod_{i=1}^{j-1} b_i \cdot (b_j - a_j) \cdot \prod_{i=j+1}^{k} a_i \right)
\leq \sum_{j \in [k]} \left( \prod_{i \in B_j \setminus \{j\}} b_i \cdot (b_j - a_j) \cdot 1 \right)
\leq \sum_{j \in [k]} \left( \prod_{i \in B_j} b_i - \prod_{i \in T_j} a_i \right) \leq \sum_{j \in [n]} q_j,
\]

where the second-last inequality uses \( a_i \leq b_i \). Therefore, \( \max_j q_j \) is an \( n \)-approximation of \( \prod_{i \in B^*} b_i - \prod_{i \in B^*} a_i \). \( \square \)

Finally, combining Lemma 4.16, Lemma 4.18, and Lemma 4.19 gives the following main theorem:

**Theorem 1.3.** There is a polytime algorithm with \( O(n^{5.5} \sqrt{T} \log T) \) regret for the Bandit Pandora’s Box problem where we only receive utility (selected value minus total cost) as the feedback.

## 5 Lower Bounds

In this section we prove lower bounds for Online Learning Prophet Inequality and Online Learning Pandora’s Box. Our lower bounds will hold even against full-feedback.

### 5.1 \( \Omega(\sqrt{T}) \) Lower Bound for Stochastic Input

We show an \( \Omega(\sqrt{T}) \) regret lower bound for Bandit Prophet Inequality and an \( \Omega(\sqrt{nT}) \) lower bound for Pandora’s Box problem, which implies that the \( \sqrt{T} \) factor in our regret bounds is tight. We first give the lower bound for Prophet Inequality.

**Theorem 5.1.** For Bandit Prophet Inequality there exists an instance with \( n = 2 \) such that all online algorithms incur \( \Omega(\sqrt{T}) \) regret.

**Proof.** Let \( \mathcal{D}_1 \) be a distribution that always gives \( \frac{1}{2} \). Let \( \mathcal{D}_2 \) be a Bernoulli distribution. The probability of \( X_2 = 1 \) might be \( \frac{1}{2} + \frac{1}{\sqrt{T}} \) or \( \frac{1}{2} - \frac{1}{\sqrt{T}} \). Both settings appear w.p. \( \frac{1}{2} \). The online algorithm doesn’t know which is the real setting. If it chooses not to open \( X_2 \), it will lose \( \sqrt{T} \) w.p. \( \frac{1}{2} \). Otherwise, because of the variance, the algorithm needs \( \Omega(T) \) samples from \( X_2 \) to learn the real setting, and loses \( \frac{1}{2} \cdot \sqrt{T} \) for each round it runs. In both cases, the online algorithm should lose \( \Omega(\sqrt{T}) \), which finishes the proof. \( \square \)

For the Pandora’s Box problem, [GHTZ21] already shows a lower bound for the sample complexity of Pandora’s Box problem, which directly implies a lower bound for the online learning setting.
Theorem 5.2 ([GHTZ21]). For any instance of Pandora’s problem in which the rewards are bounded in $[0,1]$, running $\Omega(\frac{n}{\epsilon^2})$ samples is necessary to get an $\epsilon$-additive algorithm.

Corollary 5.3. For Pandora’s Box problem, no online algorithm can achieve $o(\sqrt{nT})$ regret.

Proof. Assume there exists an online algorithm that achieves $o(\sqrt{nT})$ regret. This implies that after $T$ rounds, we can achieve $o(\frac{1}{\sqrt{T}})$ per-round regret, which is in contradiction with Theorem 5.2. □

We remark that [GHTZ21] claims that $\Omega(\frac{n}{\epsilon^2})$ samples are necessary to get an $\epsilon$-additive algorithm for Prophet Inequality but without giving a proof. However, this claim seems incorrect since in an ongoing work we show an $\tilde{O}(\sqrt{T})$ regret algorithm for Prophet Inequality with full-feedback.

5.2 $\Omega(T)$ Lower Bound for Adversarial Input

In this paper, we study Bandit Prophet Inequality and Bandit Pandora’s Box problems under the stochastic assumption that input is drawn from unknown-but-fixed distributions. A natural extension would be: can we obtain $o(T)$ regret for adversarial inputs where the input distribution may change in each time step? The following theorems shows that sub-linear regret is impossible even for oblivious adversarial inputs with $n=2$ under full-feedback.

Theorem 5.4. For Bandit Prophet Inequality with oblivious adversarial inputs, there exists an instance with $n=2$ such that the optimal fixed-threshold strategy has total value $\frac{3}{4}T$ but no online algorithm (even under full-feedback) can obtain total value more than $\frac{1}{2}T$.

Proof. We first introduce a notation used in this proof. Let $s$ be a 01-string. Define $Bin(s)$ to be the binary decimal corresponding to $s$. For example, $Bin(1) = (0.1)_2 = \frac{1}{2}$, $Bin(0011) = (0.0011)_2 = \frac{3}{10}$.

Now, we introduce the main idea of the counter example: At the beginning, the adversary will choose a $T$-bits code $s = s_1s_2...s_T$ uniformly at random (i.e., $s_i$ is set to be 0 or 1 w.p. $\frac{1}{2}$ independently). The value of $X_1$ is $\frac{1}{2}$ plus a small bias that contains the information of the code. The value of $X_2$ is either 1 or 0, which is decided by the code. Formally, in the $i$-th round:

- $X_1 = \frac{1}{2} + \epsilon \cdot v_i$, where $\epsilon$ is an arbitrarily small constant that doesn’t affect the reward, and $v_i$ is a value between $Bin(s_1s_2...s_{i-1} + 0 + 1^{T-i})$ and $Bin(s_1s_2...s_{i-1} + 1 + 0^{T-i})$. The notation $0^k$ represents a length-$k$ string with all 0s, and $1^k$ represents a length-$k$ string with all 1s.

- $X_2 = 1$ if $s_i = 0$, otherwise $X_2 = 1$.

For an online algorithm, it only knows that the next $s_i$ can be 0 or 1 w.p. $\frac{1}{2}$. Therefore, no matter it switches to the next box or not, it can only get $\frac{1}{2}$ in expectation. So the maximum total reward it can achieve is $\frac{1}{2}T$.

However, if we know the code, playing $\tau = \frac{1}{2} + \epsilon \cdot Bin(s)$ gets $\frac{3}{4}T$: $X_2 = 1$ when $X_1 < \tau$, while $X_2 = 0$ when $X_1 \geq \tau$. Therefore, playing $\tau$ allows us to pick every 1, but stays in $X_1 = \frac{1}{2}$ when $X_2 = 0$. Since we generate the code uniformly at random, $X_2$ is 1 w.p. $\frac{1}{2}$. Therefore, the expected reward is $T \cdot (\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}) = \frac{3}{4}T$. □

Next, we use a similar proof idea to prove lower bound for Pandora’s Box. This resolves an open question of [Ger22, GT22] on whether sublinear regrets are possible for Online Learning of Pandora’s Box with adversarial inputs.

Theorem 5.5. For Bandit Pandora’s Box with oblivious adversarial inputs, there exists an instance with $n=2$ such that the optimal fixed-threshold strategy has total utility $\frac{1}{2}T$ but no online algorithm (even under full-feedback) can obtain total utility more than 0.

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Proof. At the beginning, the adversary will choose a \(T\)-bits code \(s = s_1s_2...s_T\) uniformly at random (\(s_i\) is set to be 0 or 1 w.p. \(\frac{1}{2}\) independently). The cost \(c_1\) is 0, and the value of \(X_1\) is 0 plus a small bias that contains the information of the code. The cost \(c_2\) is \(\frac{1}{2}\), and the value of \(X_2\) is either 1 or 0, which is decided by the code. Formally, in the \(i\)-th round:

- \(X_1 = 0 + \epsilon \cdot v_i\), where \(\epsilon\) is an arbitrarily small constant that doesn’t effect the reward, and \(v_i\) is a value between \(Bin(s_1s_2...s_{i-1} + 0 + 1^{T-i})\) and \(Bin(s_1s_2...s_{i-1} + 1 + 0^{T-i})\). The notation \(0^k\) represents a length-\(k\) string with all 0s, and \(1^k\) represents a length-\(k\) string with all 1s.
- \(X_2 = 1\) if \(s_i = 0\), otherwise \(X_2 = 1\).

The cost of \(X_1\) is 0, so we can always first open \(X_1\). Then, for an online algorithm, it doesn’t know whether \(X_2\) is 1 or 0. No matter it opens \(X_2\) or not, the expected reward will only be 0.

However, when we know the code, playing \(\tau = \epsilon \cdot Bin(s)\) gets \(\frac{1}{2}T\), because it will open \(X_2\) whenever \(X_2 = 1\), and skip it when \(X_2 = 0\). Since we generate the code uniformly at random, \(X_2\) is 1 w.p. \(\frac{1}{2}\). Therefore, the expected reward is \(T \cdot (\frac{1}{2} \cdot (1 - \frac{1}{2})) = \frac{1}{2}T\).

\[\square\]

A Basic Probabilistic Inequalities

**Theorem A.1** (Hoeffding’s Inequality). Let \(X_1, \ldots, X_N\) be independent random variables such that \(a_i \leq X_i \leq b_i\). Let \(S_N = \sum_{i \in [N]} X_i\). Then for all \(t > 0\), we have \(\Pr[|S_N - E[S_N]| \geq t] \leq 2 \exp(-\frac{2t^2}{\sum_{i \in [N]} (b_i - a_i)^2})\). This implies, that if \(X_i\) are i.i.d. samples of random variable \(X\), and \(a = a_i, b = b_i\) for all \(i \in [N]\), let \(\hat{X} := \frac{1}{N} \sum_{i \in [N]} X_i\), then for every \(\epsilon > 0\),

\[
\Pr \left[ |\hat{X} - E[X]| \geq \epsilon \right] \leq 2 \exp \left( -\frac{2N\epsilon^2}{(b - a)^2} \right).
\]

**Theorem A.2** (Bernstein Inequality). Given mean zero random variables \(\{X_i\}_{i=1}^N\) with \(\Pr(|X_i| \leq c) = 1\) and \(\text{Var}X_i \leq \sigma_i^2\). If \(\bar{X}_N\) denotes their average and \(\sigma^2 = \frac{1}{N} \sum_{i=1}^n \sigma_i^2\), then

\[
\Pr(|\bar{X}_N| \geq \epsilon) \leq 2 \exp \left( -\frac{N\epsilon^2}{2\sigma^2 + 2c\epsilon/3} \right).
\]

**Theorem A.3** (DKW Inequality). Given a natural number \(N\), let \(X_1, \ldots, X_n\) be i.i.d. samples with cumulative distribution function \(F(\cdot)\). Let \(\hat{F}(\cdot)\) be the associated empirical distribution function \(\hat{F}(x) := \frac{1}{N} \sum_{i \in [N]} 1_{X_i \leq x}\). Then, for every \(\epsilon > 0\), we have

\[
\Pr \left[ \sup_x |\hat{F}(x) - F(x)| > \epsilon \right] \leq 2 \exp(-2N\epsilon^2).
\]

B Missing Proofs from Section 2

B.1 Proof of Lemma 2.5

**Proof.** There are two different sources of regret. We bound them separately.

*Loss 1 from the while loop:* The main idea of the proof is to use the regret bound from the previous phase to bound the total regret in the next phase. Specifically, assume \(c_0 = O(1)\) be the maximum possible one-round regret, and assume there are \(k\) phases in the while loop. Then the total regret
can be bounded by
\[
\sum_{i=1}^{k} O\left(\frac{n^\alpha \log T}{\epsilon_i^2}\right) \cdot \epsilon_{i-1} = \sum_{i=1}^{k} O\left(\frac{n^\alpha \log T}{\epsilon_i}\right) = O(n^{\alpha/2} \sqrt{T}). \tag{14}
\]

Therefore, the total regret from the while loop is bounded by \(O(n^{\alpha/2} \sqrt{T})\).

*Loss 2 after the while loop:* After the while loop, the one-round regret is bounded by \(\epsilon_k = \Theta\left(\frac{n^\alpha \log T}{\sqrt{\tau}}\right)\), so the total regret can be bounded by \(O\left(\frac{n^\alpha \log T}{\sqrt{T}}\right) \cdot T = O(n^{\alpha/2} \sqrt{T} \log T)\).

Finally, combining the two sources of regret proves the theorem.

Besides, we should also verify that Algorithm 2 runs no more than \(O(T)\) rounds. In the while loop, the total number of rounds is
\[
\sum_{i=1}^{k} \frac{n^\alpha \log T}{\epsilon_i^2} \leq \frac{4n^\alpha \log T}{\epsilon_k^2} = O(T).
\]
Therefore, Algorithm 2 runs no more than \(O(T)\) rounds, so it’s a valid algorithm with respect to time horizon \(T\). \qed

### B.2 Missing Details of Pandora’s Box Algorithm for \(n = 2\)

#### B.2.1 Proof of Lemma 2.7

To prove Lemma 2.7, we need two claims. The first claim says that when we have a good guess \(\tau\) with a small \(\delta(\tau)\), the loss of playing \(\tau\) is bounded:

**Claim 2.8.** If \(\tau, \tau^* \in [\ell, u]\) then \(R(\tau^*) - R(\tau) \leq |\delta(\tau)|\).

**Proof.** We first upper-bound \(R(\tau^*) - R(\tau)\). The two settings are different only when \(X_1\) is between \(\tau\) and \(\tau^*\): Playing \(\tau\) will lose an extra \(|g(X_1)|\). Since \(g(x)\) is monotone, we can bound \(|g(X_1)|\) by \(|g(\tau)|\). Therefore, the extra loss of playing \(\tau\) is no more than \(|F_1(\tau^*) - F_1(\tau)|g(\tau)|\).

On the other hand,
\[
|\delta(\tau)| = (F_1(u) - F_1(\ell)) \left| \int_{\tau}^{\tau^*} g'(x)dx \right| = (F_1(u) - F_1(\ell)) \cdot |g(\tau)|.
\]
When \(\tau, \tau^* \in [\ell, u]\), \(F_1(u) - F_1(\ell) \geq |F_1(\tau^*) - F_1(\tau)|\). Therefore, \(|\delta(\tau)| \geq R(\tau^*) - R(\tau)\). \qed

The second claim shows that we can get a good estimate for function \(\delta(\tau)\):

**Claim 2.9.** In Algorithm 3, if the conditions in Lemma 2.7 hold, then w.h.p. \(|\hat{\delta}(\tau) - \delta(\tau)| \leq 4\epsilon\) simultaneously for all \(\tau \in [\ell, u]\).

**Proof.** Recall that \(\delta(\tau) = \Delta(\tau) - (R(u) - R(\ell))\). We will give the bound for \(|\Delta(\tau^*) - \Delta(\tau)|\), \(|R(u) - \hat{R}(u)|\) and \(|R(\ell) - \hat{R}(\ell)|\) separately.

For \(|\Delta(\tau^*) - \Delta(\tau)|\), we first bound the magnitude of \(\Delta(\tau):\)
\[
\Delta(\tau) = \int_{\tau}^{u} (F_1(u) - F_1(x))(F_2(x) - 1)dx - \int_{\ell}^{\tau} (F_1(x) - F_1(\ell))(F_2(x) - 1)dx,
\]
which implies

\[
|\Delta(\tau)| \leq (F_1(u) - F_1(\tau)) \int_\tau^u (1 - F_2(x))dx + (F_1(\tau) - F_1(\ell)) \int_\ell^\tau (1 - F_2(x))dx
\]

(15)

\[
= (F_1(u) - F_1(\tau))(g(\tau) - g(u)) + (F_1(\tau) - F_1(\ell))(g(\ell) - g(\tau))
\]

(16)

\[
\leq (F_1(u) - F_1(\ell))(g(\ell) - g(u))
\]

(17)

\[
\leq |\delta(u)| + |\delta(\ell)| \leq 32\epsilon,
\]

(18)

where the last equality follows from the bound $|\delta(\tau)| \leq 16\epsilon$ for all $\tau \in [\ell, u]$ in Lemma 2.7.

Now notice that the estimate $\hat{\Delta}(\tau)$ we have based on our initial estimates $\hat{F}_1$ and $\hat{F}_2$ is unbi-
ased i.e. $E[\hat{\Delta}(\tau)] = \Delta(\tau) \leq 32\epsilon$. This simply follows from exchanging interval integration and
expectation combined with the independence of $X_1$ and $X_2$:

\[
E\left[\int_\tau^u (\hat{F}_1(u) - \hat{F}_1(x))(\hat{F}_2(x) - 1)dx\right] = \int_\tau^u (E[\hat{F}_1(u)] - E[\hat{F}_1(x)])(E[\hat{F}_2(x)] - 1)dx
\]

(19)

\[
= \int_\tau^u (F_1(u) - F_1(x))(F_2(x) - 1)dx.
\]

Now let us define $\hat{\Delta}(\tau)$ per sample $i$ for each initial sample. We run $N = C \cdot \frac{\log T}{\epsilon}$ samples. Then
for $i \in [N]$, we define

\[
\hat{\Delta}^{(k)}(\tau) = \int_\tau^u (\hat{F}_1^{(k)}(u) - \hat{F}_1^{(k)}(x))(\hat{F}_2^{(k)}(x) - 1)dx - \int_\ell^\tau (\hat{F}_1^{(k)}(x) - \hat{F}_1^{(k)}(\ell))(\hat{F}_2^{(k)}(x) - 1)dx,
\]

where $\hat{F}_1^{(k)}(.)$ and $\hat{F}_2^{(k)}(.)$ are simple threshold functions at the $i$th initial sample, which are estimates
for the densities $F_1$ and $F_2$ respectively. Note that

\[
\hat{\Delta}(\tau) = \frac{1}{N} \sum_{k \in [N]} \hat{\Delta}^{(k)}(\tau).
\]

Now again similar to (19) we have

\[
E[\Delta^{(k)}(\tau)] = E[\Delta(\tau)] \leq 32\epsilon.
\]

(20)

Moreover, note that the random variable $\hat{\Delta}^{(i)}(\tau)$ is bounded by one since

\[
|\hat{\Delta}^{(k)}(\tau)| \leq \int_\tau^u |(\hat{F}_1^{(k)}(u) - \hat{F}_1^{(k)}(x))(\hat{F}_2^{(k)}(x) - 1)|dx - \int_\ell^\tau |(\hat{F}_1^{(k)}(x) + \hat{F}_1^{(k)}(\ell))(\hat{F}_2^{(k)}(x) - 1)|dx
\]

\[
\leq \int_\tau^u 1dx + \int_\ell^\tau 1dx = u - \ell \leq 1.
\]

(21)

Combining Equations (20) and (21), we have the variance bound:

\[
\text{Var}[\hat{\Delta}(\tau)] \leq E[\hat{\Delta}(\tau)^2] \leq E[\hat{\Delta}(\tau)] \leq 32\epsilon.
\]

(22)

Now, combining (21) and (22), we can apply Bernstein inequality for the random variables
\( \hat{\Delta}^{(i)}(\tau) \). We have:

\[
\Pr \left[ |\Delta(\tau) - \Delta(\tau')| \geq \epsilon \right] \leq 2 \exp\left( -\frac{N\epsilon^2}{2 \Var[\Delta(\tau)] + \frac{2}{3} \epsilon} \right) = 2T^{-\frac{3\epsilon}{2.6}}.
\]  

(23)

Therefore, \(|\Delta(\tau) - \Delta(\tau)| < \epsilon\) holds w.h.p. when \(C\) is sufficiently large.

Notice that we only prove the bound for a single \(\tau\). To strengthen this concentration bound to hold simultaneously for all \(\tau\) and \([\ell, u]\), we take a union over appropriate cover sets. In particular, consider \(C\) as a discretization of the interval \([\ell, u]\) with accuracy \(1/T\). To be able to exploit the high probability argument for the elements inside the cover for the ones outside, we need to show that \(\Delta\) is Lipschitz with respect to \(\tau, u\) and \(\ell\).

For \(\Delta\) function, we have \(|\Delta(\tau) - \Delta(\tau')| \leq 2|\tau - \tau'|\) since

\[
|\Delta(\tau) - \Delta(\tau')| = \left| \int_\tau^{\tau'} (F_1(u) - F_1(x))(F_2(x) - 1)dx \right| + \left| \int_\tau^{\tau'} (F_1(u) - F_1(x))(F_2(x) - 1)dx \right|
\]

\[
\leq 2|\tau - \tau'|.
\]

It is easy to see that the same Lipschitz bound also holds for \(\hat{\Delta}\).

Now for an arbitrary \(\tau' \in [\ell, u]\), if we consider the closest \(\tau\) to it in \(C\), we have \(|\tau' - \tau| \leq \frac{1}{T}\). Then, using the Lipschitz constant of \(\Delta\) and \(\hat{\Delta}\):

\[
|\Delta(\tau) - \Delta(\tau')| \leq \frac{2}{T} \quad \text{and} \quad |\Delta(\tau) - \hat{\Delta}(\tau')| \leq \frac{2}{T}.
\]

(24)

Now we apply a union bound over the events \(|\Delta(\tau) - \Delta(\tau)| < \epsilon\) for all \(\tau \in C\). Since running over all possibilities of \(\tau \in C\) is at most \(O(T)\) and \(|\Delta(\tau) - \Delta(\tau)| < \epsilon\) holds w.h.p. with sufficiently large \(C\), after taking a union bound we still know that all of these events happen simultaneously with high probability. We then have for \(\tau'\) and its closest element \(\tau\) in \(C\):

\[
|\Delta(\tau) - \Delta(\tau')| + |\hat{\Delta}(\tau) - \hat{\Delta}(\tau')| \leq 4|\tau - \tau'| \leq \frac{4}{T}.
\]

(25)

We simply upper-bound \(\frac{1}{T}\) by \(\epsilon\). This must be true because \(\epsilon \geq T^{-\frac{3}{2}} = \omega(\frac{1}{T})\). Then, combining the bound in (25) with (23) implies \(|\hat{\Delta}(\tau') - \hat{\Delta}(\tau')| \leq 2\epsilon\) holds w.h.p. for all \(\tau \in [\ell, u]\).

Next, we bound \(|\hat{R}_\ell - R(\ell)|\) and \(|\hat{R}_u - R(u)|\). For \(\hat{R}_\ell - R(\ell)|\), Notice that \(\hat{R}_\ell\) is an estimate of \(R(\ell)\) with \(N = C \cdot \log T\) samples, and the reward of each sample falls in \([-1, 1]\). By Hoeffding’s Inequality (Theorem A.1), the probability that \(|\hat{R}_\ell - R(\ell)| > \epsilon\) is bounded by \(2 \exp(-2N\epsilon^2/4) = 2T^{-C/2}\). So, w.h.p. \(|\hat{R}_\ell - R(\ell)| \leq \epsilon\) when \(C\) is sufficiently large. The bound for \(|\hat{R}_u - R(u)|\) is identical. Finally, combining three parts with union bound finishes the proof.

Finally, we have the tools to prove Lemma 2.7:

**Proof of Lemma 2.7.** We will assume that \(|\delta(\tau) - \delta(\tau)| \leq 4\epsilon\), which is true w.h.p. by Claim 2.9.

Observe that \(\delta(\tau)\) is a monotone increasing function, because \(\delta'(\tau) = \hat{\Delta}'(\tau) = (\hat{F}_1(u) - \hat{F}_1(\ell))(1 - \hat{F}_2(\tau)) \geq 0\). Therefore, according to the definition of \(\ell'\) and \(u'\), we have \([\ell', u'] = \{\tau \in [\ell, u] : |\delta(\tau)| \leq 4\epsilon\}\). Now, we can use this property to prove two statements separately:

For the statement that \(\tau^* \in [\ell', u']\), notice that \(\delta(\tau^*) = 0\). According to Claim 2.9, \(|\delta(\tau^*)| \leq 4\epsilon\).
Then, since $\tau^* \in [\ell, u]$ and $|\hat{\delta}(\tau^*)| \leq 4\epsilon$, there must be $\tau^* \in [\ell', u']$, because $[\ell', u'] = \{ \tau \in [\ell, u] : |\hat{\delta}(\tau)| \leq 4\epsilon \}$.

Next, we prove that $|\delta(\tau)| \leq 8\epsilon$ for all $\tau \in [\ell, u]$. This is true because $[\ell', u'] = \{ \tau \in [\ell, u] : |\hat{\delta}(\tau)| \leq 4\epsilon \}$, and we have $|\hat{\delta}(\tau) - \delta(\tau)| \leq 4\epsilon$ from Claim 2.9. Therefore, $|\delta(\tau)| \leq |\hat{\delta}(\tau)| + |\hat{\delta}(\tau) - \delta(\tau)| \leq 8\epsilon$ for all $\tau \in [\ell', u']$.

Finally, the bound $R(\tau^*) - R(\tau) \leq 8\epsilon$ directly follows Claim 2.8 and that $|\delta(\tau)| \leq 8\epsilon$. \hfill \qed

**B.2.2 Proof of Theorem 2.10**

To prove Theorem 2.10, we need to first give an initialization algorithm such that its output should satisfy the conditions listed in Lemma 2.7. Formally, we have the following lemma:

**Lemma B.1.** After running $\Theta(\sqrt{T} \log T)$ samples from $D_2$ and $D_2$, w.h.p. we can output an initial interval $[\ell, u]$ of length $u - \ell \leq T^{-\frac{1}{4}}$ and satisfying $\tau^* \in [\ell, u]$.

**Proof.** We first run $\Theta(\sqrt{T} \log T)$ extra samples for $X_2$ and calculate an estimate $\hat{F}_2(x)$. We can show that $|\hat{F}_2(x) - F_2(x)| \leq \frac{1}{4}T^{-\frac{1}{4}}$ with high probability: After running $N = C \cdot \sqrt{T} \log T$ samples, the DKW inequality (Theorem A.3) shows that $\text{Pr} \left[ |\hat{F}_2(x) - F_2(x)| > \epsilon = \frac{1}{4}T^{-\frac{1}{4}} \right] \leq 2 \exp(-2N\epsilon^2) = 2T^{-C/2}$. Then, w.h.p. $|\hat{F}_2(x) - F_2(x)| \leq \frac{1}{4}T^{-\frac{1}{4}}$ simultaneously holds for all $x \in [0, 1]$ when $C$ is sufficiently large. In the following proof, we assume this accuracy bound always holds. Then the proof succeeds with high probability.

Next, we calculate $\hat{g}(\tau)$ by replacing $F_2(x)$ with $\hat{F}_2(x)$ in (3). When $|\hat{F}_2(x) - F_2(x)| \leq T^{-\frac{1}{4}}$ holds simultaneously for all $x \in [0, 1]$, we have $|\hat{g}(\tau) - g(\tau)| \leq \int_{\tau}^{1} |\hat{F}_2(x) - F_2(x)| \leq \frac{1}{4}T^{-\frac{1}{4}}$. Then, we let $[\ell, u] := \{ \tau : |\hat{g}(\tau)| \leq \frac{1}{2}T^{-\frac{1}{4}} \}$. Since $\hat{g}(\tau) = \hat{F}_2(\tau) - 1 \leq 0$, function $\hat{g}(\tau)$ is a non-increasing. So, the set $\{ \tau : |\hat{g}(\tau)| \leq \frac{1}{2}T^{-\frac{1}{4}} \}$ must form an interval. Besides, notice that $g(\tau^*) = 0$, which means $|\hat{g}(\tau^*)| \leq \frac{1}{2}T^{-\frac{1}{4}}$, so we must have $\tau^* \in [\ell, u]$. Furthermore, for every $\tau \in [\ell, u]$, $|g(\tau)| \leq |\hat{g}(\tau)| + |\hat{g}(\tau) - g(\tau)| \leq T^{-\frac{1}{4}}$, which finishes the proof. \hfill \qed

Now, we are ready to prove Theorem 2.10:

**Proof of Theorem 2.10.** For the core part of the algorithm, we run Algorithm 2 and then use Lemma 2.5 to bound the regret. To run Algorithm 2, we let the constraints mean that the threshold played in each round is inside the interval $[\ell, u]$ given by Algorithm 3. Besides, we require the conditions listed in Lemma 2.7 hold (with high probability). We discuss them separately:

- $|g(\tau)| \leq T^{-\frac{1}{4}}$ for all $\tau \in [\ell, u]$: This is guaranteed by Lemma B.1.
- $\tau^* \in [\ell, u]$: For the first phase, this is guaranteed by Lemma B.1. Starting from the second phase, this is from Lemma 2.7 of the previous phase.
- $|\delta(\tau)| \leq 16\epsilon$: For the first phase, this is true because $\epsilon_1 = 1$. Starting from the second phase, this is from Lemma 2.7 of the previous phase. Notice that the statement in Lemma 2.7 is a little bit different: It guarantees that $|\delta(\tau)| \leq 8\epsilon$ with respect to the $[\ell, u]$ and $\epsilon$ from the previous phase. When switching to the new phase, notice that $F_2(u') - F_2(\ell') \leq F_2(u) - F_2(\ell)$, which means $|\delta(\tau)|$ drops when switching to the new phases. Besides, the parameter $\epsilon_{\text{new}}$ in the new phase is exactly $\frac{1}{2}\epsilon_{\text{old}}$. Combining these two differences shows that $|\delta(\tau)| \leq 16\epsilon$ holds in the new phase.
Therefore, Algorithm 3 satisfies algorithm Alg in Lemma 2.5. Applying Lemma 2.5 gives the $O(\sqrt{\tau} \log \tau)$ regret bound.

Besides, we also run samples for initialization and constructing CDF estimates for Algorithm 3. These are not covered by Lemma 2.5. For the initialization, Lemma B.1 states that $\Theta(\sqrt{\tau} \log \tau)$ rounds are sufficient. So the regret from the initialization is $O(\sqrt{\tau} \log \tau)$. For constructing $\hat{F}_1(x)$ and $\hat{F}_2(x)$, assume we run $k$ phases, then the total number of samples is

$$\sum_{i=1}^{k} \Theta\left(\frac{\log \tau}{\epsilon_i}\right) = O(\sqrt{\tau} \log \tau).$$

Combining three parts finishes the proof.

C Missing Proofs from Section 4

C.1 Proof of Lemma 4.1

Proof. We first prove the lemma for a single $i$. For $[\ell_i, u_i]$, we run $O(\sqrt{\tau} \log \tau)$ extra samples for $X_i$, and calculate an estimate $\hat{F}_i(x)$. We can show that $|\hat{F}_i(x) - F_i(x)| \leq \frac{1}{2} T^{-\frac{1}{2}}$ with high probability: After running $N = C \cdot \sqrt{\tau} \log \tau$ samples, the DKW inequality (Theorem A.3) shows that

$$\Pr \left[ |\hat{F}_i(x) - F_i(x)| > \varepsilon = \frac{1}{2} T^{-\frac{1}{2}} \right] \leq 2 \exp\left(-2N\varepsilon^2\right) = 2T^{-C/2}. \quad \text{Then w.h.p.} \quad |\hat{F}_i(x) - F_i(x)| \leq \frac{1}{2} T^{-\frac{1}{2}} \quad \text{for every } x \in [0, 1] \text{ when } C \text{ is sufficiently large.}$$

In the following proof, we assume this accuracy bound always holds. Then the proof succeeds with high probability.

Next, we calculate $\hat{g}_i(\tau)$ by replacing $F_i(x)$ with $\hat{F}_i(x)$ in (11). When $|\hat{F}_i(x) - F_i(x)| \leq T^{-\frac{1}{4}}$ holds for all $x \in [0, 1]$, we have $|\hat{g}_i(\tau) - g_i(\tau)| \leq \int_{\tau}^{1} |\hat{F}_i(x) - F_i(x)| \leq \frac{1}{2} T^{-\frac{1}{4}}$. Then, we let $[\ell_i, u_i] := \{\tau : |\hat{g}_i(\tau)| \leq \frac{1}{2} T^{-\frac{1}{4}}\}$. Since $\hat{g}'(\tau) = \hat{F}_i(\tau) - 1 \leq 0$, which means $\hat{g}_i(\tau)$ is a decreasing function, then the set $\{\tau : |\hat{g}_i(\tau)| \leq \frac{1}{2} T^{-\frac{1}{4}}\}$ must form an interval. Besides, notice that $g_i(\tau^*) = 0$, which means $|\hat{g}_i(\tau^*)| \leq \frac{1}{2} T^{-\frac{1}{4}}$, so there must be $\tau_i^* = \tau_i \in [\ell_i, u_i]$. Furthermore, for every $\tau \in [\ell_i, u_i], |g_i(\tau)| \leq |\hat{g}_i(\tau)| + |\hat{g}_i(\tau) - g_i(\tau)| \leq T^{-\frac{1}{4}}$.

Finally, combining the statements for all $n$ intervals finishes the proof.

C.2 Proof of Claim 4.6

Proof. We first show that $|\hat{F}_i(x) - F_i(x)| \leq \frac{\sqrt{\tau}}{2n}$ w.h.p. with $N = C \cdot \frac{2^2 \log \tau}{\epsilon_i}$ samples. Using DKW inequality (Theorem A.3), we have

$$\Pr \left[ |\hat{F}_i(x) - F_i(x)| > \frac{\sqrt{\tau}}{2n} \right] \leq 2 \exp\left(-2N\frac{\varepsilon_i^2}{\epsilon_i^2}\right) = 2T^{-C/4}. \quad \text{So the bound holds with high probability.}$$

By the union bound, w.h.p. $|\hat{F}_i(x) - F_i(x)| \leq \frac{\sqrt{\tau}}{2n}$ holds for every $i \in [n]$. Then, for the accuracy of $\prod_{i \in S} F_i(x)$, we have

$$(1 - \frac{\sqrt{\tau}}{2n})^n - 1 \leq \prod_{i \in S} \hat{F}_i(x) - \prod_{i \in S} F_i(x) \leq (1 + \frac{\sqrt{\tau}}{2n})^n - 1.$$ 

For the lower bound, we have $1 - \frac{\sqrt{\tau}}{2n} - 1 \geq 1 - \frac{\sqrt{\tau}}{2} - 1 - \sqrt{\tau}$. For the upper bound, we have $(1 + \frac{\sqrt{\tau}}{2n})^n - 1 \leq \exp(\frac{\sqrt{\tau}}{2n} \cdot n) - 1 \leq 1 + 2 \cdot \frac{\sqrt{\tau}}{2} - 1 = \sqrt{\tau}$.

Combining two bounds finishes the proof.
C.3 Proof of Claim 4.7

Proof. Since \( \delta_i(\tau) = \Delta_i(\tau) - (R_u - R_\ell) \), there are three parts in \( \delta_i(\tau) \). We show that the accuracy of each part is bounded by \( \frac{\epsilon}{2} \) with high probability.

First, similar to the derivation in Equation (18) we bound the magnitude of the \( \Delta_i \) function:

\[
|\Delta_i(\tau)| \leq (F_{\pi,i}(u) - F_{\pi,i}(\tau)) \int_\tau^u (1 - F_i(x)) \, dx + (F_{\pi,i}(\tau) - F_{\pi,i}(\ell)) \int_\ell^\tau (1 - F_i(x)) \, dx
\]

\[
= (F_{\pi,i}(u) - F_{\pi,i}(\tau))(g_i(\tau) - g_i(u)) + (F_{\pi,i}(\tau) - F_{\pi,i}(\ell))(g_i(\ell) - g_i(\tau))
\]

\[
\leq (F_{\pi,i}(u) - F_{\pi,i}(\ell))(g_i(\ell) - g_i(u))
\]

\[
\leq |\delta_{\pi,u,i}(\ell)| + |\delta_{\pi,u,\ell,i}(u)| \leq 12\epsilon,
\]

where we use the bound \( |\delta_{\pi',u',\ell',i}(\tau)| \leq 6\epsilon \) in Lemma 4.5.

Next, we hope to propose an estimator \( \hat{\Delta}_i^{(k)}(\tau) \) for the \( \Delta_i \) function which uses \( N = \frac{C \cdot \log T}{\epsilon} \) samples. For \( k \in [N] \), define

\[
\hat{\Delta}_i^{(k)}(\tau) = \int_\tau^u (\hat{F}_{\pi,i}^{(k)}(u) - \hat{F}_{\pi,i}^{(k)}(x))(\hat{F}_i^{(k)}(x) - 1) \, dx - \int_\ell^\tau (\hat{F}_{\pi,i}^{(k)}(x) - \hat{F}_{\pi,i}^{(k)}(\ell))(\hat{F}_i^{(k)}(x) - 1) \, dx,
\]

where \( \hat{F}_{\pi,i}^{(k)}(\cdot) \) and \( \hat{F}_i^{(k)}(\cdot) \) are simple threshold functions at the \( i \)th initial sample, which are estimates for the densities \( F_{\pi,i} \) and \( F_i \), respectively. This definition implies \( \hat{\Delta}_i(\tau) = \frac{\Delta_i(\tau)}{N} \sum_{k \in [N]} \hat{\Delta}_i^{(k)}(\tau) \), and Equation (26) implies

\[
\mathbb{E} \left[ \hat{\Delta}_i^{(k)}(\tau) \right] = \Delta_i(\tau) \leq 12\epsilon.
\]

Now it is easy to see that \( \hat{F}_{\pi,i}^{(k)}(x) - \hat{F}_{\pi,i}^{(k)}(y) \) is a Bernoulli random variable which are one if and only if the maximum value obtained from \( X_{\pi(1)}, \ldots, X_{\pi(n)} \) is in \([\ell, u]\). In particular, this implies that \( \hat{\Delta}_i^{(k)}(\tau) \) is bounded by 1 since

\[
|\hat{\Delta}_i^{(k)}(\tau)| \leq \int_\tau^u \left| (\hat{F}_{\pi,i}^{(k)}(u) - \hat{F}_{\pi,i}^{(k)}(x))(\hat{F}_i^{(k)}(x) - 1) \right| \, dx - \int_\ell^\tau \left| (\hat{F}_{\pi,i}^{(k)}(x) - \hat{F}_{\pi,i}^{(k)}(\ell))(\hat{F}_i^{(k)}(x) - 1) \right| \, dx
\]

\[
\leq \int_\tau^u 1 \, dx + \int_\ell^\tau 1 \, dx = u - \ell \leq 1.
\]

Combining Equations (27) and (28), we have the variance bound:

\[
\text{Var}[\hat{\Delta}_i(\tau)] \leq \mathbb{E} \left[ \hat{\Delta}_i(\tau)^2 \right] \leq \mathbb{E} \left[ \hat{\Delta}_i(\tau) \right] \leq 12\epsilon.
\]

Hence, using Bernstein inequality, we have

\[
\Pr \left[ |\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \geq \frac{\epsilon}{12} \right] \leq 2 \exp \left( -\frac{N\epsilon^2/144}{2\text{Var}[\Delta_i(\tau)] + \frac{\epsilon}{3\cdot 12}} \right) = 2T^{-\frac{C}{\epsilon^2}}.
\]

Therefore, \( |\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \leq \frac{\epsilon}{12} \) holds with high probability.

Finally, we need to take a union bound also over different such choices for \( \tau \) coming from the

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10 Although we have \( \Theta(n^2 \log T) \) samples in Algorithm 6, using \( \Theta(\log T) \) samples is sufficient.
confidence interval \([\ell_i, u_i]\). Note that since we are using fresh samples for estimating \(\hat{\Delta}_i\), we do not need to take a union bound over the choice of random variables that come before \(X_i\), and also the confidence interval \([\ell_i, u_i]\).

We can bound the Lipschitz constant of \(\Delta_i\) (and similarly \(\hat{\Delta}_i\)):

\[
|\Delta_i(\tau) - \Delta_i(\tau')| = \left| \int_\tau^{\tau'} (F_{\pi,i}(u) - F_{\pi,i}(x))(F_i(x) - 1)dx \right| + \left| \int_\tau^{\tau'} (F_{\pi,i}(u) - F_{\pi,i}(x))(F_i(x) - 1)dx \right| \\
\leq 2|\tau - \tau'|,
\]

Finally, for every \(\tau \in [\ell_i, u_i]\), let \(\tau'\) be the closest value in the discretization set. Then, we have:

\[
|\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \leq |\Delta_i(\tau) - \Delta_i(\tau')| + |\Delta_i(\tau') - \Delta_i(\tau')| + |\hat{\Delta}_i(\tau') - \hat{\Delta}_i(\tau)| \leq \frac{\epsilon}{12}\delta + \frac{\epsilon}{8} \leq \frac{\epsilon}{6},
\]

where the last inequality is true because \(\epsilon > T^{-\frac{1}{2}} = \omega(\frac{1}{T})\). Therefore, w.h.p. \(|\hat{\Delta}_i(\tau) - \Delta_i(\tau)| \leq \frac{\epsilon}{6}\) for all \(\tau \in [\ell_i, u_i]\) simultaneously.

Next, we use Hoeffding’s Inequality (Theorem A.1) to bound the accuracy of \(|R_\ell - \hat{R}_\ell|\). In each round, the reward falls in \([-0.5,0.5]\). Then, after running \(N = C \cdot m \log T\) samples, we have

\[
\Pr \left[ |R_\ell - \hat{R}_\ell| > \frac{\epsilon}{6} \right] \leq 2\exp(-2N\epsilon^2/36) = 2T^{-C/18}. \text{ Therefore, } |R_\ell - \hat{R}_\ell| \leq \frac{\epsilon}{6} \text{ w.h.p. when } C \text{ is sufficiently large. Besides, the proof for } |R_u - \hat{R}_u| \text{ is identical. Combining three parts with union bound finishes the proof.}
\]

### C.4 Proof of Lemma 4.8

In this section, we show that Algorithm 10 finds an approximately clever threshold setting. We first introduce the following lemma:

**Algorithm 10: Finding Approximately Clever Threshold**

**Input:** \((I, S), m, i, \tilde{F}_1(x), \ldots, \tilde{F}_n(x)\)

1. **for** \(P \subseteq [n] \) **do**
2.   - if \(\exists k : (k, i) \in S \land k \notin P\) or \(\exists k : (i, k) \in S \land k \in P\) or \(\exists k, j : (k, j) \in S \land k \notin P \land j \in P\)
   - then Skip this \(P\);
3.   - For \(k \in P\), let \(\tau_k = u_k\)
4.   - For \(k \in [n] \setminus (T \cup \{i\})\), let \(\tau_k = \ell_k\)
5.   - Let \(u_T = \min\{u_i, \tau_{k:k\in T}\}\), \(\ell_T = \max\{\ell_i, \tau_{k:k\notin (T \cup \{i\})}\}\)
6.   - Set partial setting \(\pi_T\) be: \(\tau_i \in [\ell_T, u_T]\). \(\pi_T\) sorts the thresholds in a decreasing order. Break the ties according to the constraints in \(S\).
7.   - Let \(\hat{F}_{\pi_T,i}(x) = \prod_{k \in T} \hat{F}_k(x)\).
8.   - Calculate \(q_T := \hat{F}_{\pi_T,i}(u_T) - \hat{F}_{\pi_T,i}(\ell_T)\)
9.   - Let \(T^* = \arg\max q_T\). **Output:** \(\pi_T^*, \ell_{T^*}, u_{T^*}, \hat{F}_{\pi_{T^*},i}\).

**Lemma C.1.** Algorithm 10 calculates a clever threshold setting, up to an \(4\sqrt{\epsilon}\) additive error. The running time of Algorithm 10 is \(O(n \cdot 2^n)\).

**Proof.** The goal of a clever threshold setting is to maximize \(F_{\pi,i}(u) - F_{\pi,i}(\ell)\). Fix \(i\). When the set \(P\), which represents the distributions in front of \(X_i\) is determined, the function \(F_{\pi,i}(x)\) is fixed.
Therefore, to maximize $F_{\pi,i}(u) - F_{\pi,i}(\ell)$, we should maximize $u$ and minimize $\ell$. This can be achieved by maximizing the thresholds in $P$ and minimizing the thresholds in $[n] \setminus (P \cup \{i\})$, which is exactly lines 8 and 9 in Algorithm 6. Then, after enumerating all valid subsets $P$, we can find a setting that maximizes $F_{\pi,i}(u) - F_{\pi,i}(\ell)$.

There is one missing detail: we only know the value of $\hat{F}_i(x)$. From Claim 4.6, we know $\hat{F}_i(x)$ is an estimate of $F_i(x)$ with accuracy $\sqrt{\epsilon}$. Therefore, $\max_\pi \hat{F}_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell)$ is at most $2\sqrt{\epsilon}$ different from $\max_\pi F_{\pi,i}(u) - F_{\pi,i}(\ell)$. After getting $\pi' = \arg \max_\pi \hat{F}_{\pi,i}(u) - \hat{F}_{\pi,i}(\ell)$, the real value of $F_{\pi',i}(u) - F_{\pi',i}(\ell)$ is at most $2\sqrt{\epsilon}$ different from $\hat{F}_{\pi',i}(u) - \hat{F}_{\pi',i}(\ell)$. Combining two errors proves the $4\sqrt{\epsilon}$ error bound.

For the running time of Algorithm 10, we need to enumerate a subset $S$, then calculate the corresponding $F_{\pi,i}(x)$ function. So the running time is $O(n \cdot 2^n)$.

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