Singular Perturbation Approximations for General Linear Quantum Systems

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Abstract—This paper considers the use of singular perturbation approximations for general linear quantum systems where the system dynamics are described in terms of both annihilation and creation operators. Results that are related to the physical realizability property of the approximate system are presented.

I. INTRODUCTION

Quantum feedback control is an active research area addressing the need to take into account quantum effects in systems that are inherently quantum in nature or when levels of accuracy approach the quantum noise limit. Examples where these effects need to be considered occur in quantum optics, quantum communications, quantum computing and precision measurement, for example gravity wave detection [1]–[19].

Recent papers (for example [1]–[3], [13]) use a consistent approach to describe quantum systems, (both plant and controller) as a combination of quantum harmonic oscillators coupled to quantum fields and described by quantum stochastic differential equations. The concept of physical realizability has been well defined (see [1]) and relates to whether a given synthesized system could be implemented as such a combination of quantum harmonic oscillators. Equivalent necessary and sufficient conditions for physical realizability are given in [1] and [13].

Physical realizability is particularly relevant in both modeling and control applications. In the case of modeling a plant (for example as part of a controller design process), it is often necessary to have a plant model that is physically realizable as this can have implications for controller design. Similarly, in the case of coherent quantum feedback control where the controller is implemented as a quantum system, it is necessary to establish whether it is in fact possible to implement a synthesized controller as a quantum system.

Examples of components occurring in physical quantum systems that are adequately described by this framework include beam splitters, phase-shift modulators and optical cavities. In [14], the physical realizability of singular perturbation approximations for a class of linear quantum systems is considered. Singular perturbation approximations are closely related to adiabatic elimination, a commonly used technique used in modeling quantum systems within the physics literature. Related results in a more general setting can be found in [16]–[19]. In [14], the class of quantum linear systems that can be describe solely in terms of annihilation operators is considered. This class corresponds to passive systems, for example optical systems containing only passive components such as optical cavities, beam-splitters and phase shifters.

In light of the results obtained in [14], this paper considers the more general class of quantum linear systems described by both annihilation and creation operators. Two results are obtained; one for a general singular perturbation, and one for a special case in which the Hamiltonian and coupling operators are singularly perturbed. In the general case a result (relevant to physical realizability) relating to the J-J unitary property of the transfer function of the approximate system is obtained. In the special case, it is shown that while in general, the system obtained from the singular perturbation approximation is not necessarily physically realizable, it is equivalent to a physically realizable system in series with a static Bogoliubov component (generalized static squeezer).

Components implementing static linear transformations, called Bogoliubov transformations (see [5]), such as a static squeezer, as encountered in quantum optics, do not belong to the class of physically realizable quantum systems, although they can be approximated by systems that are. In ([5]) a framework is developed to merge these to classes: (i) dynamical components, with linear evolution of physical variables, and (ii) static components characterized by Bogoliubov transformations. In particular, general methods for cascade, series, and feedback connections are provided, input-output maps and transfer functions for representing components are defined and the issue of convergence is addressed.

The remainder of the paper proceeds as follows. In Section II we describe the quantum system model used throughout this paper. In Section III we present our two main results related to the physical realizability property of approximate systems obtained through the use of singular perturbation approximations. The first result is for a system obtained via a general singular perturbation, whereas the second result is for a special case in which the Hamiltonian and coupling operators are singularly perturbed. An illustrative example follows in Section IV and our conclusion is given in Section V.

II. QUANTUM SYSTEM MODEL

To aid the reader, the nomenclature and variables used throughout this paper are consistent with their usage in [14].
applied to an operator, is its operator adjoint, and applied to a matrix is its complex conjugate, \((\cdot)^\dagger = (\cdot)^H\).

As in [13], we consider the class of linear quantum systems models representing \(n\) quantum harmonic oscillators coupled to \(m\) external independent quantum fields. This class of linear quantum systems is described by quantum stochastic differential equations (QSDEs) of the form

\[
\begin{align*}
\frac{da(t)}{dt} &= F a(t)^\dagger \mathrm{d}t + G \frac{du(t)}{dt}; \\
\frac{dy(t)}{dt} &= H a(t)^\dagger \mathrm{d}t + K \frac{du(t)}{dt}.
\end{align*}
\]

where \(a(t) = [a_1(t) \ldots a_n(t)]^T\) is a column vector of linear combinations of annihilation and creation operators corresponding to the harmonic oscillators. The vector \(u(t)\) represents the input to the system. It is assumed to admit the decomposition \(du(t) = \beta_a(t) \mathrm{d}t + \tilde{u}(t)\) where \(\tilde{u}(t)\) is the noise part of \(u(t)\) (with Ito products \(\tilde{u}(t) \tilde{u}(s) \mathrm{d}t = F_\varepsilon \mathrm{d}s\) where \(F_\varepsilon\) is non-negative Hermitian) and \(\beta_a(t)\) is the adapted, self adjoint part of \(u(t)\). \(F, G, H, K\) are of the form:

\[
\begin{align*}
F &= [F_1 F_2] \in \mathbb{C}^{2n \times 2n}; \\
G &= [G_1 G_2] \in \mathbb{C}^{2n \times 2m}; \\
H &= [H_1 H_2] \in \mathbb{C}^{2m \times 2n}; \\
K &= [K_1 K_2] \in \mathbb{C}^{2m \times 2m}.
\end{align*}
\]

**Definition 1:** (see [13]) A linear quantum system of the form (1) is physically realizable if there exists a commutation matrix \(\Theta = \Theta^\dagger = TT^\dagger \geq 0\), a Hamiltonian matrix \(M\), coupling matrix \(N\) and scattering matrix \(S\), with

\[
J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}; \\
T = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}; \quad T \text{ non-singular}; \\
M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1^\dagger \end{bmatrix}; \quad M^\dagger = M; \\
N = \begin{bmatrix} N_1 & N_2 \\ N_2 & N_1^\dagger \end{bmatrix}; \quad S^{-1} = S^\dagger,
\]

such that,

\[
\begin{align*}
F &= -i\Theta M - \frac{1}{2} \Theta N^\dagger JN; \\
G &= \Theta N^\dagger JK; \\
H &= N; \\
K &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}.
\end{align*}
\]

**Theorem 1:** (see [13]) Suppose the linear quantum system (1) is minimal, and that \(\lambda_i(F) + \lambda_j(F) \neq 0\) for all eigenvalues \(\lambda_i(F), \lambda_j(F)\) of \(F\). Then this linear quantum system is physically realizable if and only if the following conditions hold:

1) The system transfer function matrix \(\Gamma(s)\) is \((JJ)\)-unitary. That is, \(\Gamma^{-\dagger}(s)\Gamma(s) = J\) for all \(s \in \mathbb{C}\); where \(\Gamma^{-\dagger}(s) = \Gamma^\dagger(-s^*)\) and \(s^*\) is the complex conjugate of \(s\).

2) \(K\) is of the form \(K = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}\), where \(S^\dagger S = SS^\dagger = I\).

**Definition 2:** (see [5]) A static Bogoliubov component is a component that implements the Bogoliubov transformation:

\[
\begin{align*}
\frac{dy(t)}{dt} &= B \frac{du(t)}{dt}; \\
\frac{dy(t)}{dt} &= B \frac{du(t)}{dt}.
\end{align*}
\]

where

\[
B = \begin{bmatrix} B_1 & B_2 \\ B_2^\# & B_1^\# \end{bmatrix}; \quad JB^\dagger JB = BJB^\dagger J = I.
\]

**Remark 1:** A static Bogoliubov component is in general not physically realizable in the above sense. However, it is a useful idealization for certain devices used in quantum optics, (e.g. a static squeezer) and is correctly interpreted as a limiting situation [5].

**III. MAIN RESULT**

**A. General Singular Perturbations**

Consider the class of quantum systems of the form (1) that are dependent on a parameter \(\varepsilon \geq 0\) as follows:

\[
\begin{align*}
\frac{da_1(t)}{dt} &= \frac{1}{2} F_{1a} \frac{1}{2} F_{1b} \frac{1}{2} F_{2a} \frac{1}{2} F_{2b} \left[ a_1(t) a_2(t) \right] dt \\
\frac{da_1(t)}{dt} &= \frac{1}{2} F_{1a} \frac{1}{2} F_{1b} \frac{1}{2} F_{2a} \frac{1}{2} F_{2b} \left[ a_1(t) a_2(t) \right] dt
\end{align*}
\]

Equivalently, re-ordering partitions for convenience with new matrices as labeled, and re-writing in the more standard singularly perturbed form we obtain:

\[
\begin{align*}
\frac{da_1(t)}{dt} &= \frac{1}{2} F_{1a} \frac{1}{2} F_{1b} \frac{1}{2} F_{2a} \frac{1}{2} F_{2b} \left[ a_1(t) a_2(t) \right] dt \\
\frac{da_1(t)}{dt} &= \frac{1}{2} F_{1a} \frac{1}{2} F_{1b} \frac{1}{2} F_{2a} \frac{1}{2} F_{2b} \left[ a_1(t) a_2(t) \right] dt
\end{align*}
\]
perturbed system \(3\). From [14]:

\[
\Phi_c(s) = H(sI - F_c)^{-1}G_c + K
\]

\[
\Phi_0(s) = -\varepsilon s \left( H_0(sI - F_0)^{-1}F_b + H_bF_d^{-1} \right) F_d^{-1}
\]

\[
\times \left( F_c(sI - F_0)^{-1}G_0 + G_b \right) + O(\varepsilon^2) \tag{7}
\]

with variables defined as previously.

If the singularly perturbed system \(3\) is physically realizable for all \(\varepsilon > 0\), it follows from [13] that

\[
\Phi_c(s)^\dagger J \Phi_c(s) = J; \quad \forall s \in \mathbb{C},
\]

for all \(\varepsilon > 0\). Hence, from (7) if follows that

\[
\Phi_0(s)^\dagger J \Phi_0(s) = J
\]

for all \(s \in \mathbb{C}\). That is, the approximate system is \((J,J)\)-unitary.

\[\square\]

B. A Special Class of Singular Perturbations

We now turn our attention to a special class of singularly perturbed physically realizable quantum systems of the form \(3\) defined (as per Definition 1) in terms of \(M, N, S\) and canonical \(\Theta\) as follows:

\[
\Theta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix};
\]

\[
M = \begin{bmatrix} M_{1a} & M_{1b} & M_{2a} & M_{2b} \\ M_{1a}^\dagger & \frac{1}{\sqrt{\varepsilon}} M_{1b} & \frac{1}{\sqrt{\varepsilon}} M_{2a} & \frac{1}{\sqrt{\varepsilon}} M_{2b} \\ M_{2a}^\dagger & M_{2b} & M_{1a}^\dagger & M_{1b}^\dagger \\ M_{2a} & \frac{1}{\sqrt{\varepsilon}} M_{2b} & \frac{1}{\sqrt{\varepsilon}} M_{1a} & \frac{1}{\sqrt{\varepsilon}} M_{1b} \end{bmatrix};
\]

\[
N = \begin{bmatrix} N_{1a} & N_{1b} & N_{2a} & N_{2b} \\ N_{1a}^\dagger & \frac{1}{\sqrt{\varepsilon}} N_{1b} & \frac{1}{\sqrt{\varepsilon}} N_{2a} & \frac{1}{\sqrt{\varepsilon}} N_{2b} \\ N_{2a}^\dagger & N_{2b} & N_{1a}^\dagger & N_{1b}^\dagger \\ N_{2a} & \frac{1}{\sqrt{\varepsilon}} N_{2b} & \frac{1}{\sqrt{\varepsilon}} N_{1a} & \frac{1}{\sqrt{\varepsilon}} N_{1b} \end{bmatrix};
\]

\[
K = \begin{bmatrix} S & 0 \\ 0 & S^\dagger \end{bmatrix}; \quad SS^\dagger = I. \tag{8}
\]

For convenience define \(M_a = \begin{bmatrix} M_{1a} & M_{2a} \\ M_{2a}^\dagger & M_{1a}^\dagger \end{bmatrix}\), and likewise for \(M_b, M_c, M_d, N_a, N_b\).

From (8), we can obtain the system in the form \(3\) and thence of the form \(4\). After the change of variables \(\tau_2(t) := \frac{1}{\sqrt{\varepsilon}} \tau_2(t)\), we obtain:

\[
\begin{bmatrix} da_1(t) \\ da_1(t)^\# \end{bmatrix} = -J(iM_a + \frac{1}{2} N_a^\dagger J N_a) \begin{bmatrix} a_1(t) \\ a_1(t)^\# \end{bmatrix} dt
\]

\[
+ J(iM_b + \frac{1}{2} N_b^\dagger J N_b) \begin{bmatrix} d\tau_2(t) \\ d\tau_2(t)^\# \end{bmatrix} + J N_a^\dagger J K \begin{bmatrix} d\tau_2(t) \\ d\tau_2(t)^\# \end{bmatrix} dt
\]

\[
\varepsilon \begin{bmatrix} da_2(t) \\ da_2(t)^\# \end{bmatrix} = \begin{bmatrix} F_{1a} & F_{2a} \\ F_{1c} & F_{2c} \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_1(t)^\# \end{bmatrix} dt + \begin{bmatrix} G_{1a} & G_{2a} \\ G_{1c} & G_{2c} \end{bmatrix} \begin{bmatrix} d\tau_2(t) \\ d\tau_2(t)^\# \end{bmatrix}
\]

where:

\[
F_0 = F_a - F_b F_d^{-1} F_c;
\]

\[
G_0 = G_a - F_b F_d^{-1} G_b;
\]

\[
H_0 = H_a - F_b F_d^{-1} F_c;
\]

\[
K_0 = K - F_b F_d^{-1} G_b.
\]

Theorem 2: If the singularly perturbed linear complex quantum system \(4\) is physically realizable for all \(\varepsilon \geq 0\), and the matrix \(F_d\) is non-singular, then the corresponding reduced dimension approximate system \(5\) has transfer function matrix \(\Phi_0(s) = H_0(sI - F_0)^{-1}G_0 + K_0\) such that

\[
\Phi_0(s)^\dagger J \Phi_0(s) = J
\]

for all \(s \in \mathbb{C}\). That is, it is \((J,J)\)-unitary.

Remark 2: This result is not sufficient to prove the physical realizability of approximate system, as Theorem 2 includes additional conditions (with respect to minimality, eigenvalues, and the \(K\) matrix) that need to be met to ensure physical realizability. However, these conditions can easily be checked for the approximate system, to verify physical realizability.

Proof: Consider the transfer function of the singularly
Let 
\[
\begin{align*}
\dot{\mu} &= \mu_a + \frac{1}{2} N_b \dot{N}_b M_c \\
\dot{\tilde{N}} &= (\tilde{N}_b \dot{N}_b + 1) \dot{N}_b M_c \\
\dot{\tilde{K}} &= \tilde{K}_b + \frac{1}{2} N_b \dot{N}_b M_c
\end{align*}
\]

such that,
\[
\begin{align*}
F_0 &= -iJ\dot{\mu} - \frac{1}{2} J\dot{N}_b \tilde{N} \\
G_0 &= -J\dot{\tilde{N}}^\dagger \tilde{N} \\
H_0 &= \tilde{N} \\
K_0 &= \tilde{K}
\end{align*}
\]

Let
\[
\begin{align*}
\ddot{y}(t) &= \frac{N_b}{H_b} a(t) \frac{d}{dt} \dot{y}(t) + \frac{N_b}{H_b} \frac{\ddot{N}_b}{d(t)} \frac{d}{dt} \\
&+ K \begin{bmatrix} \frac{d}{dt} \dot{y}(t) \\
\ddot{y}(t) \end{bmatrix}
\end{align*}
\]
\begin{align*}
&+ (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \\
&- (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J N_b \\
&\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
&= J K \dagger J K \\
&- J K \dagger J N_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \\
&\times \left( i M_d + \frac{1}{2} N_b \dagger J N_b - i M_d \\
&+ \frac{1}{2} N_b \dagger J N_b - N_b \dagger J N_b \right) \\
&\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
&= J K \dagger J K \\
&\left[ I \begin{array}{c} S \dagger \\
0 \end{array} \begin{array}{c} 0 \\
S^T \end{array} \right] \begin{array}{c} I \\
0 \end{array} \begin{array}{c} S \dagger \\
0 \end{array} S^# \\
= I.
\end{align*}

We now consider $\tilde{N} = H_0$. As with $K$ it is straightforward to show that $\Sigma N^\dagger \Sigma = N$. We wish to show that $G_0 = -J \tilde{N} \dagger J K$, i.e., that $G_0 + J \tilde{N} \dagger J K = 0$. Indeed,

\begin{align*}
G_0 + J \tilde{N} \dagger J K &= \\
- J K a \dagger J K + J (i M_b + \frac{1}{2} N_a \dagger J N_b) \\
&\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ J \left( N_a \dagger - (i M_d + \frac{1}{2} N_a \dagger J N_b) \\
\times (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger \right) \\
&\times J \left( K - N_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \right) \\
= - J N_a \dagger J K \\
+ J M_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ \frac{1}{2} N_a \dagger J N_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ J N_a \dagger J K \\
- J N_b \dagger J N_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ J M_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
- \frac{1}{2} N_a \dagger J N_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
- J M_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J N_b \\
\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ \frac{1}{2} N_a \dagger J N_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J \\
\times N_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K
\end{align*}

\begin{align*}
= + J i M_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ J i M_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
- J i M_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J N_b \\
\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
- \frac{1}{2} N_a \dagger J N_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
- J i M_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J N_b \\
\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K \\
+ \frac{1}{2} N_a \dagger J N_b (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J \\
\times N_b (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J K
\end{align*}

= + J i M_b \left( (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \\
+ (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \\
- (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} N_b \dagger J N_b \\
\times (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \right) N_b \dagger J K

- \frac{1}{2} N_a \dagger J N_b \dagger i \\
\times \left( (i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} - (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \\
+ (-i M_d + \frac{1}{2} N_b \dagger J N_b)^{-1} \right) N_b \dagger J K

= 0.

Finally, we set
\[ \tilde{M} = i J \left( F_0 + \frac{1}{2} J \tilde{N} \dagger J \tilde{N} \right). \]

After simplification along similar lines to that shown for the previous equations, this yields the expression for $\tilde{M}$ given in (11) which it is straightforward to verify is hermitian.

This completes the proof of the theorem. \[\blacksquare\]

IV. ILLUSTRATIVE EXAMPLE

The following example from quantum optics demonstrates our main result. The example is similar to that in [14]. However, here we consider a cavity coupled to a squeezer as shown in Figure 1. Unlike in [14], the evolution of this system is in terms of both annihilation and creation operators.

Here, $K_1$ and $K_2$ are the coupling parameters of the first cavity, $\gamma$ is the coupling parameter of the squeezer and $\chi$
is the squeezing parameter. The system under consideration can be described by QSDE of the form (1) with:

\[
\begin{align*}
F &= \begin{bmatrix}
-\frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 & -\sqrt{K_1} & 0 & 0 \\
-\sqrt{K_2} & 0 & 0 & -\chi \\
0 & 0 & -\sqrt{K_1} & -\frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 - \sqrt{K_1} & 0 \\
0 & \chi & -\sqrt{K_1} & \left(-\sqrt{K_1} - \sqrt{K_2}\right)^2 - \sqrt{K_1} & -\frac{1}{2}
\end{bmatrix}; \\
G &= \begin{bmatrix}
-\sqrt{K_2} & 0 & 0 \\
0 & -\sqrt{K_2} & 0 \\
0 & 0 & -\sqrt{K_2} \\
\end{bmatrix}; \\
H &= \begin{bmatrix}
\sqrt{K_1 + K_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}; \\
K &= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}. \\
\end{align*}
\]

(12)

Now suppose that the dynamics of the squeezer are at a much higher frequency than that of interest. We can apply a singular perturbation approximation by letting \( \gamma = \frac{1}{\sqrt{\gamma}} \) and \( \chi = \frac{1}{\sqrt{\chi}} \). After the change of variables \( \tilde{\gamma}_2 = \gamma \tilde{a}_2 \) we obtain a system of the form (13). In fact, this system belongs to the special class in which \( M, N, S, \) and \( \Theta \) are of the form (8) as follows:

\[
\begin{align*}
M_a &= \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix}; \\
M_b &= \begin{bmatrix} \frac{1}{2}(\sqrt{K_1} - \sqrt{K_2})^2 & 0 & 0 \end{bmatrix}; \\
M_c &= \begin{bmatrix} -\frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 & 0 & 0 \\
0 & 0 & -\sqrt{K_1} \end{bmatrix}; \\
M_d &= \begin{bmatrix} 0 & 0 & 0 \\
0 & -i\chi & 0 \\
\end{bmatrix}; \\
N_a &= \begin{bmatrix} 0 \\
0 \\
\sqrt{K_1 + K_2} \end{bmatrix}; \\
N_b &= \begin{bmatrix} 0 \\
0 \\
\sqrt{K_1 + K_2} \end{bmatrix}; \\
S &= I.
\end{align*}
\]

Applying the singular perturbation approximation, we obtain a system of the form (6), and from (11) we have:

\[
\begin{align*}
F_0 &= -\frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 I + \gamma \sqrt{K_1 K_2} \begin{bmatrix} \frac{2}{K} & -\chi \chi \end{bmatrix}^{-1} \\
G_0 &= -\frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 I + \gamma \sqrt{K_2} \begin{bmatrix} \frac{2}{K} & -\chi \chi \end{bmatrix}^{-1} \\
H_0 &= \frac{1}{2}(\sqrt{K_1} + \sqrt{K_2})^2 I - \gamma \sqrt{K_1} \begin{bmatrix} \frac{2}{K} & -\chi \chi \end{bmatrix}^{-1} \\
K_0 &= I - \gamma \begin{bmatrix} \frac{2}{K} & -\chi \chi \end{bmatrix}^{-1}.
\end{align*}
\]

(13)

Finally, since the system described by (12) satisfies the conditions for Theorem 3, the approximate system described by (13) is equivalent to a physically realizable system in series with a static Bogoliubov component (static squeezer). This can be verified by obtaining \( M, N, \) and \( K \) from (11) and confirming that the physically realizable system obtained from substituting \( M = M, N = N, \) and \( S = I \) into (6) and the static Bogoliubov component described by \( K \) combine to form (13).

V. CONCLUSION

In this paper, we have considered singular perturbation approximations for the general class of quantum linear systems described by both annihilation and creation operators. Two main results were presented. We first considered a general singular perturbation approximation and obtained a result (relevant to physical realizability) relating to the J-MM unitary property of the transfer function of the approximate system. We then considered the special case in which the Hamiltonian and coupling operators are singularly perturbed. While in general the system obtained from the singular perturbation approximation for the special case is not necessarily physically realizable, it is equivalent to a physically realizable system in series with a static Bogoliubov component (generalized static squeezer).

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