Quasilinear integrodifferential Bernoulli-type equations

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Abstract. The equations considered in this article have the form in which the time derivative of the unknown function is expressed as a double integral over the space variables of a weighted quadratic expression of the sought function. The domain of integration is unbounded and does not depend on time but depends on the space variable. We study the Cauchy problem in the function classes accompanying the equation with initial data on the positive half-line. In application to this problem, the convergence of the successive approximation method is justified. An estimate is given of the quality of the approximation depending on the number of the iterated solution. It is proved that, on some finite time interval, the posed Cauchy problem has at most one solution in the accompanying function class. An existence theorem is proved in the same class.

1. Introduction

In the present article, we consider the following ordinary integro-differential equation with quadratic nonlinearity on the right-hand side:

$$\frac{du}{dt}(t, k) + a(t)u(t, k) = \iint_{P(k)} W(k, k_1, k_2)u(t, k_1)u(t, k_2)dk_1dk_2.$$ 

The independent variables in this equation and also the integration variables on the right-hand side are positive: \(t > 0, k > 0, k_1 > 0, k_2 > 0\). Here the function \(a(t)\) is continuous on the positive half-axis and the domain of integration \(P(k)\) is a semibounded strip in the first quadrant of the plane \((k_1, k_2)\) defined as follows

$$P(k) = \{(k_1, k_2) \mid k_2 \geq k_1 - k, \quad k_2 \leq k_1 + k, \quad k_1 + k_2 \geq k\}.$$ 

The properties of the set of solutions are largely determined by the kernel \(W(k, k_1, k_2)\) of its integral operator and also by conditions on the behavior of the solution \(u(t, k)\) as \(k \to +0\) and \(k \to +\infty\).

Results obtained under different types of numerical modeling for quadratically nonlinear integro-differential equations are presented and illustrated graphically in [1–3].
2. Statement of the problem and main results
Suppose that the kernel \( W(k, k_1, k_2) \) of the integral operator is a function continuous in the first octant for which
\[
\sup_{k \geq 0} \int\int_{P(k)} |W(k, k_1, k_2)| \, dk_1 \, dk_2 \leq M < +\infty.
\]
Here \( M \) is a given nonnegative constant. The set of all possible functions \( W(k, k_1, k_2) \) continuous in the first octant and satisfying assigned conditions will be denoted by \( \mathcal{K}(M) \). The identically zero function always belongs to \( \mathcal{K}(M) \), i.e., this class is nonempty.

For \( M > 0 \), the accompanying class \( \mathcal{K}(M) \) contains sufficiently many nontrivial elements. Suppose, for example, that \( W_1(k, k_1, k_2) \) is integrable in the first quadrant for any fixed nonnegative \( k \) and, moreover, there exists a finite constant \( M_1 \) such that the following estimates hold uniformly over \( k > 0 \):
\[
\int_0^\infty \int_0^\infty |W_1(k, k_1, k_2)| \, dk_1 \, dk_2 \leq M_1,
\]
Then we can take the following quantity as the constant \( M \):
\[
M = \sup_{k \geq 0} \int_0^\infty \int_0^\infty |W_1(k, k_1, k_2)| \, dk_1 \, dk_2 < +\infty.
\]
Pose the Cauchy problem with data on the positive half-axis for the initial equation:
\[
\begin{cases}
\frac{du}{dt}(t, k) + a(t)u(t, k) = \int\int_{P(k)} W(k, k_1, k_2)u(t, k_1)u(t, k_2) \, dk_1 \, dk_2, \\
u(t, k)|_{t=0} = \varphi(k), \quad k \geq 0.
\end{cases}
\]
Here the function \( \varphi(k) \) is continuous on the positive half-axis and satisfies the following estimate:
\[
|\varphi(k)| \leq Ce^{-\gamma k} \quad \forall k \geq 0.
\]
The exponent \( \gamma \) and the constant \( C \) on the right-hand side of this inequality are nonnegative. In particular, as initial data in the Cauchy problem, we can take any function with compact support on the positive half-axis.

Define the action of a quadratic nonlinear integral operator at a function \( U(t, k) \) continuous in the first quadrant of the plane \((t, k)\) by the relation
\[
W[U] = W[U](t, k) = \int\int_{P(k)} W(k, k_1, k_2)U(t, k_1)U(t, k_2) \, dk_1 \, dk_2.
\]
Assume that \( \mu(t) = \exp\left\{ \int_0^t a(\tau) \, d\tau \right\} \). The function \( \mu(t) \) is continuously differentiable and \( \mu(0) = 1 \). Multiplying both parts of the initial equation by a function \( \mu(t) \) write the result in the following form
\[
\frac{d}{dt} [\mu(t)u(t, k)] = \mu(t)W[u](t, k).
\]
The class $\mathbb{D} = \mathbb{D}(H, B, \gamma, \varphi)$ consists of functions $U(t, k)$ continuous in the strip $0 \leq t \leq H, k \geq 0$ and such that the product $\mu(t)U(t, k)$ differing in the strip from the initial function $\varphi(k)$ only by a quantity whose modulus does not exceed the product $Be^{-\gamma k}$:

$$\mathbb{D}(H, B, \gamma, \varphi) = \{U(t, k) : 0 \leq t \leq H, k > 0 \Rightarrow |\mu(t)U(t, k) - \varphi(k)| \leq Be^{-\gamma k}\}.$$  

Note that the function class $\mathbb{D}$ is not empty: for example, it contains the function $\varphi(k)/\mu(k)$. In case $\mu(t) = 1$ the function class $\mathbb{D}$ was introduced in [1].

Let a kernel $W(k, k_1, k_2)$ be of class $\mathbb{K}(M)$. Then for any pair of functions $U(t, k), V(t, k)$ in $\mathbb{D}(H, b, \gamma, \varphi)$ we have the following estimate:

$$\sup_{k \geq 0} e^{\gamma k}|W[U](t, k) - W[V](t, k)| \leq \frac{L}{\mu(t)} \sup_{t \geq 0} e^{\gamma \xi}|U(t, \xi) - V(t, \xi)|, \quad 0 \leq t \leq H.$$  

Here the constant $L$ is defined by the equality $L = 4BM$. The operator $W[\cdot]$ is finite at any function $U(t, k)$ in $\mathbb{D}(H, B, \gamma, \varphi)$ and

$$|W[U](t, k)| \leq Me^{-\gamma k}\left(\sup_{t \geq 0} e^{\gamma \xi}|U(t, \xi)|\right)^2 \leq \frac{4MB^2}{\mu^2(t)} e^{-\gamma k}, \quad 0 \leq t \leq H.$$

**Theorem 1.** Any two solutions $U(t, k)$ and $V(t, k)$ to the equation of class $\mathbb{D}(H, B, \gamma, \varphi)$ satisfy the a priori estimate

$$\int_0^t \sup_{k \geq 0} e^{\gamma k}|U(\tau, k) - V(\tau, k)| d\tau \leq \frac{1}{L} \int_0^t \frac{1}{e^{\gamma k}} d\tau - \sup_{k \geq 0} e^{\gamma k}|U(0, k) - V(0, k)|,$$

with $0 \leq t \leq H$.

**Corollary** (uniqueness theorem). Let $U(t, k)$ and $V(t, k)$ be two solutions to the same Cauchy problem belonging to the class $\mathbb{D}(H, B, \gamma, \varphi)$. Then $U(t, k)$ and $V(t, k)$ coincide everywhere in the domain $0 < t < H, k > 0$.

Define successive approximations to the solution to the Cauchy problem by the recurrent relations

$$u[0](t, k) = \frac{1}{\mu(t)} \varphi(k), \quad t \geq 0, \quad k \geq 0,$$

and then successively for $j = 1, 2, \ldots$:

$$\mu(t)u[j](t, k) = \varphi(k) + \int_0^t \mu(\tau) \int_0^{\tau} W(k, k_1, k_2)u[j-1](\tau, k_1)u[j-1](\tau, k_2) dk_1 dk_2 d\tau.$$  

These recurrent relations correctly define all the functions $u[j](t, k), j = 1, 2, \ldots$, on some finite interval $0 \leq t \leq T_0$.

**Theorem 2.** Let the kernel $W(k, k_1, k_2)$ of the equation be continuous in the first octant and let $W(k, k_1, k_2)$ be a member of the class $\mathbb{K}(M)$. If the estimate

$$|\mu(t)w[j-1](t, k) - \varphi(k)| \leq Be^{-\gamma k}$$

is valid for some positive number $B > 0$ for all $t$ with $0 \leq t \leq H$ then the inequality

$$|\mu(t)u[j](t, k) - \varphi(k)| \leq R\left(\int_0^t \mu(\tau) d\tau\right)e^{-\gamma k}$$

also holds for all $t$, $0 \leq t \leq H$. Here the constant $R$ is finite and defined by the relation $R = M(B + C)^2$. 


Let $T_0$ be a positive number such that $R \int_0^{T_0} \mu(\tau) d\tau \leq B$. Under the conditions of Theorem 2, take $H = T_0$. Then, for all $t$ with $0 \leq t \leq T_0$ and any number $j = 1, 2, \ldots$ we have the inequality

$$|\mu(t) u^{[j]}(t, k) - \varphi(k)| \leq R \int_0^{T_0} \mu(\tau) d\tau e^{-\gamma k} \leq Be^{-\gamma k}.$$ 

Since for $j = 0$ this estimate is fulfilled, we conclude by induction that all the successive approximations $u^{[j]}(t, k)$ are certainly defined for all $t$, $0 \leq t \leq T_0$. Moreover, each of the functions $u^{[j]}(t, k)$ belongs to the introduced class $\mathcal{D}(T_0, B, \gamma, \varphi)$.

**Theorem 3.** Suppose that the kernel $W(k, k_1, k_2)$ of the initial equation is continuous in the first octant and belongs to the class $\mathcal{K}(M)$, and $\mu(t) \geq \mu(0) = 1$. Then the successive approximations $u^{[N]}(t, k)$ converge uniformly in the half-strip $0 \leq t \leq T_0$, $0 \leq k < +\infty$ to a continuous function $u(t, k)$. The limit function $u(t, k)$ belongs to the class $\mathcal{D}(T_0, B, \gamma, \varphi)$ and satisfies the estimates

$$\mu(t) \sup_{k \geq 0} e^{\gamma k} |u(t, k) - u^{[N]}(t, k)| \leq B \sum_{j=1}^{+\infty} \frac{(Lt)^j}{j!}, \quad N = 0, 1, 2, \ldots.$$ 

Thus, the quantity $T_0$ is the length of the time interval on which the solution $u(t, k)$ to the Cauchy problem certainly exists.

**Theorem 4** (existence theorem). The limit $u(t, k)$ of the successive approximations is the function continuous for $0 \leq t \leq T_0$, $k \geq 0$ and $u(t, k)$ has continuous first derivative with respect to $t$ on its domain of definition. This function $u(t, k)$ defines the solution to the Cauchy problem on this set, which satisfies the estimates

$$\sup_{0 \leq t \leq T_0, k \geq 0} e^{\gamma k} |u(t, k)| \leq \sup_{k \geq 0} e^{\gamma k} |\varphi(k)| + B,$$

$$\sup_{0 \leq t \leq T_0} |u(t, k)| \leq (B + C)e^{-\gamma k}.$$ 

Note that the numerical parameter $B$ is strictly positive and is arbitrary in all other respects. Passage to the limit as $B \to 0$ in these estimates is useless: the length $T_0$ defining the interval of existence in time of the solution to the problem also tends to zero as $B \to 0$.

**Acknowledgments**

The authors were partially supported by the Russian Foundation for Basic Research (project no. 19-01-00422).

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