New fractional integral unifying six existing fractional integrals

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Abstract

In this paper we introduce a new fractional integral that generalizes six existing fractional integrals, namely, Riemann-Liouville, Hadamard, Erdélyi-Kober, Katugampola, Weyl and Liouville fractional integrals in to one form. Such a generalization takes the form

\[
\left( \rho I_{a+;\eta,\kappa}^{\alpha,\beta} f \right) (x) = \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^\rho (\eta+1) - 1}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty.
\]

A similar generalization is not possible with the Erdélyi-Kober operator though there is a close resemblance with the operator in question. We also give semigroup, boundedness, shift and integration-by-parts formulas for completeness.

Keywords: Riemann-Liouville integral, Hadamard integral, Erdélyi-Kober integral, Katugampola integral

Until very recently, the fractional calculus had been a purely mathematical subject without apparent applications. Nowadays, it plays a major role in modeling anomalous behavior and memory effects and appears naturally in modeling long-term behaviors, especially in the areas of viscoelastic materials and viscous fluid dynamics [25, 26]. Fractional integrals alone, without its counterpart, naturally appear in certain modeling and theoretical problems, for example, probability theory [1], surface-volume reaction problems [11], anomalous diffusion [28], porous medium equations [29, 30], and numerical analysis [2], among other applications. Now, consider the following generalized integral.

Definition 1.1. Let \( f \in X^\rho_c(a,b) \) \[10\], \( \alpha > 0, \rho, \eta, \kappa \in \mathbb{R} \). The left-sided generalized fractional integral is defined by,

\[
\left( \rho I_{a+;\eta,\kappa}^{\alpha,\beta} f \right) (x) = \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^\rho (\eta+1) - 1}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau, \quad (0 \leq a < x < b \leq \infty),
\]

if the integral exists.

It can be seen that this integral generalizes four existing integrals. For \( \eta = 0, \kappa = 0 \), the Riemann-Liouville fractional integral is obtained when \( \rho = 1 \), while the Katugampola integral is obtained if \( \beta = \alpha \), further, in this case, when \( \rho \to 0^+ \), the integral coincides with Hadamard integral, which can be easily verified using L’Hospital rule. Now, for \( \beta = 0, \kappa = -\rho(\alpha + \eta) \), and any \( \eta \), it gives the Erdélyi-Kober(type) operator. It should be remarked that \( \rho^{1-\beta} \) is complex when \( \beta \notin \mathbb{Z} \) and \( \rho < 0 \) and can be treated using theory of complex analysis considering appropriate branches.

Fractional integrals sometimes work in pairs, specially in variational calculus [2]. The corresponding right-sided fractional integral can be defined as,

\[
\left( \rho I_{b-;\eta,\kappa}^{\alpha,\beta} f \right) (x) = \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_x^b \frac{\tau^\rho (\eta+1) - 1}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau, \quad (0 \leq a < x < b \leq \infty),
\]

if the integral exists.

In the Definitions [1] and [2], we can also consider the cases \( a = -\infty \) and \( b = \infty \), respectively, and are known in the literature as Weyl and Liouville type integrals, respectively. Such integrals are corresponding to infinite memory effects and have applications in financial mathematics and diffusion models [5, 33].

Let us note that, using the change of variable \( u = (\tau/x)^\rho \), the integral [1] can be rewritten in the Riemann-Liouville form as

\[
\left( \rho I_{a+;\eta,\kappa}^{\alpha,\beta} f \right) (x) = \frac{x^{\kappa+\rho(\alpha+\eta)}}{\rho^\beta \Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^\rho f(xu^{1/\rho}) du, \quad \rho \neq 0.
\]

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Preprint submitted for publication December 28, 2016
Further the Riemann-Liouville fractional integral is used to define both the Riemann-Liouville and the Caputo fractional derivatives [23, 32]. To justify the claim about the Hadamard integral, when $\kappa = 0$ and $\beta = \alpha > 0$, using L’Hospital rule and taking $\rho \to 0^+$, we have

$$\lim_{\rho \to 0^+} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau = \frac{1}{\Gamma(\alpha)} \int_a^x \lim_{\rho \to 0^+} \left( \frac{x^\rho - \tau^\rho}{\rho} \right)^{\alpha-1} \tau^{\rho-1} f(\tau) d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x^{\rho}}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau,$$

which is the Hadamard fractional integral [23, p.110]. It should be pointed out that a similar result is not possible with the Erdélyi-Kober operator though there is a close resemblance with the operator in [1]. Recent results about the Hadamard and Hadamard-type integrals such as Hadamard-type fractional calculus [21], composition and semigroup properties [7], Mellin transforms [6], integration by parts formulae [8], G-transform representations [22], impulsive differential equations with Hadamard fractional derivative [36, 37], and Hadamard type fractional differential systems [13] can be found in the literature, among others.

The Katugampola fractional integral was first introduced in [12, 15] as a generalization of $n$–fold integral, and then a simpler version was discussed in [16] along with the corresponding fractional derivatives. The Mellin transforms of it were given in [17]. The same reference also discusses a new class of generalized Stirling numbers of 2nd kind and a recurrence formula for such sequences. Further applications of Katugampola fractional integrals or derivatives are in probability theory [1], variational calculus [2], inequalities [9], Langevin equations [31], Fourier and Laplace transforms [20] fractional differential equations [4] and Numerical analysis [3], among others.

2. Main Results

For simplicity, we give the following results without proofs. The proofs of the similar results for the Erdélyi-Kober type operators can be found in the classical books by Kiryakova [24], Yakubovich and Luchko [35, p.54], McBride [27, p.123], Kilbas et al. [23, Section 2.6] and Samko et al. [32, Section 18.1], and can be generalized to the present case.

For “sufficiently good” functions $f, g$ we have the following results.
a) Shift formulae

\[ p I_{a+\eta,\kappa}^{\alpha,\beta} x^\gamma f(x) = \left(p I_{a+\eta,\kappa+\gamma}^{\alpha,\beta} f \right)(x), \]
\[ p I_{b-\eta,\kappa}^{\alpha,\beta} x^\gamma f(x) = \left(p I_{b-\eta,\kappa+\gamma}^{\alpha,\beta} f \right)(x). \]  

(4)

b) Composition (index) formulae

\[ p I_{a+\eta,\kappa}^{\alpha_1,\beta_1} p I_{a+\eta,\kappa}^{\alpha_2,\beta_2} f = p I_{a+\eta,\kappa}^{\alpha_1+\alpha_2,\beta_1+\beta_2} f, \]
\[ p I_{b-\eta,\kappa}^{\alpha_1,\beta_1} p I_{b-\eta,\kappa}^{\alpha_2,\beta_2} f = p I_{b-\eta,\kappa}^{\alpha_1+\alpha_2,\beta_1+\beta_2} f. \]  

(5)

which hold in the corresponding spaces of the functions \( f \) if \( \alpha_2 > 0, \alpha_1 + \alpha_2 \geq 0 \) or \( \alpha_2 < 0, \alpha_1 + \alpha_2 > 0 \) or \( \alpha_1 < 0, \alpha_1 + \alpha_2 \leq 0 \). (See Theorem 2.5 of [32]).

c) Fractional product-integration formulae

\[ \int_a^b x^{p-1} f(x) \left(p I_{a+\eta,\kappa}^{\alpha,\beta} g \right)(x) dx = \int_a^b x^{p-1} g(x) \left(p I_{b-\eta,\kappa}^{\alpha,\beta} f \right)(x) dx \]  

(6)

Similar results are also valid when \( a = 0 \) and \( b = \infty \) and in particular for \( p = 2 \) and \( p = 1 \).

The proof of part (a) is straightforward and for completeness we shall prove parts (b) and (c) later in the paper. Here we mean by a “sufficiently good” is that The proof is similar to the case of Katugampola integral [15, Theorem 3.1]. First consider the case \( f \in X_c^p(a, b) \) [16], that is \( t^{\alpha-1/p} f(t) \in L_p(a, b) \). We need such additional conditions to guarantee convergence. Further results on such conditions can be found, for example, in [24] and [32]. In such a space we have the following boundedness result.

**Theorem 2.1.** Let \( \alpha > 0, 1 \leq p \leq \infty, 0 < a < b < \infty \) and let \( \rho \in \mathbb{R} \) and \( c \in \mathbb{R} \) be such that \( \rho \geq c \). Then the operator \( p I_{a+\eta,\kappa}^{\alpha,\beta} \) is bounded in \( X_c^p(a, b) \) and

\[ \left\| \left(p I_{a+\eta,\kappa}^{\alpha,\beta} f \right) \right\|_{X_c^p} \leq K \left\| f \right\|_{X_c^p} \]  

(7)

where

\[ K = \rho^{\alpha-\beta} \rho^{(\alpha+\eta)+\kappa} \Gamma(\alpha) \int_1^\infty u^{\rho-\rho(\alpha+\eta)-1} \frac{du}{(u^\rho - 1)^{1-\alpha}}, \quad \rho \neq 0, \kappa \in \mathbb{R}, \kappa \geq 0. \]  

(8)

**Proof.** The proof is similar to the case of Katugampola integral [15, Theorem 3.1]. First consider the case \( 1 \leq p \leq \infty \). Since \( f \in X_c^p(a, b) \), then \( t^{\alpha-1/p} f(t) \in L_p(a, b) \) and we can apply the generalized Minkowsky inequality. We thus have

\[ \left\| p I_{a+\eta,\kappa}^{\alpha,\beta} f \right\|_{X_c^p} \leq \left( \int_a^b x^{p-1} t^{\alpha-1} t^{\rho(\alpha+\eta)+\kappa} \left| f(t) \right| dt \right)^{\frac{1}{p}} \]  

\[ \leq \rho^{\alpha-\beta} \rho^{(\alpha+\eta)+\kappa} \Gamma(\alpha) \left( \int_a^b x^{p-1} \left( x^{\rho-1} + x^{\rho(\alpha+\eta)+\kappa} \right) \right)^{\frac{1}{p}} \]  

\[ = \rho^{\alpha-\beta} \rho^{(\alpha+\eta)+\kappa} \Gamma(\alpha) \left( \int_a^b x^{p-1} \left( x^{\rho-1} + x^{\rho(\alpha+\eta)+\kappa} \right) \right)^{\frac{1}{p}} \]  

and hence

\[ \left\| p I_{a+\eta,\kappa}^{\alpha,\beta} f \right\|_{X_c^p} \leq K \left\| f \right\|_{X_c^p}. \]
where

\[
K = \rho^{1-\beta}\frac{b^{\rho(\alpha + \eta) + \kappa}}{\Gamma(\alpha)} \int_1^b u^c (u^\rho - 1)^{\alpha - 1}\,du, \quad 1 \leq p < \infty
\]  

(9)

thus, Theorem 2.1 is proved for \(1 \leq p < \infty\). For \(p = \infty\), by taking into account the essential supremum \[5\] Eq. (3.2)], we have

\[
|xe^{\rho I_{a+\eta}^{\alpha+\beta} x^\kappa} f(x)| \leq \rho^{1-\beta}\frac{b^{\rho(\alpha + \eta) + \kappa}}{\Gamma(\alpha)} \int_1^b u^c (u^\rho - 1)^{\alpha - 1}\,du \cdot \|f\|_{X^\kappa},
\]

(10)

after the substitution \(u = x/\tau\). This agrees with [3]. above. This completes the proof of the theorem. \(\square\)

For simplicity, we have only considered the cases of \(0 \leq a \leq b < \infty\). For \(a = -\infty\) and \(b = \infty\), we obtain a different \(K\) in [6].

Next we give the semigroup properties of the integral operator.

**Theorem 2.2.** Let \(a_1, a_2 > 0\), \(\beta_1, \beta_2 \in \mathbb{R}\), \(1 \leq p \leq \infty\), \(0 < a < b < \infty\), and \(c \in \mathbb{R}\) be such that \(\rho \geq c\). Then for \(f \in X^\kappa_c(a,b)\) the semigroup properties hold. That is,

\[
\rho I_{a+\eta}^{\alpha+\beta} \rho I_{a+\eta}^{\alpha+\beta} f = \rho I_{a+\eta}^{\alpha+\beta} f \quad \text{and} \quad \rho I_{b-\eta}^{\alpha+\beta} \rho I_{b-\eta}^{\alpha+\beta} f = \rho I_{b-\eta}^{\alpha+\beta} f.
\]

(11)

In particular, we have

\[
\rho I_{a+\eta}^{\alpha+\beta} \rho I_{a+\eta}^{\alpha+\beta} f = \rho I_{a+\eta}^{\alpha+\beta} f \quad \text{and} \quad \rho I_{b-\eta}^{\alpha+\beta} \rho I_{b-\eta}^{\alpha+\beta} f = \rho I_{b-\eta}^{\alpha+\beta} f.
\]

(12)

**Proof.** For brevity we only prove the first result. The proof of the other identity is similar. Using Fubini’s theorem, for “sufficiently good” function \(f\), and Dirichlet technique [51, p.64], we have

\[
\rho I_{a+\eta}^{\alpha+\beta} \rho I_{a+\eta}^{\alpha+\beta} f(x) = \frac{\rho^{2-\beta_1-\beta_2} x^{\kappa_1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \frac{x^\rho \tau^{\nu(\eta_1+\kappa_2+1)} - 1}{(x^\rho - \tau^\rho)^{1-\alpha_1}} \int_a^\tau \frac{\tau^\rho \tau^{\nu(\eta_2+1)-1}}{(\tau^\rho - t^\rho)^{1-\alpha_2}} f(t)\,dt\,d\tau
\]

(13)

The inner integral is evaluated by the change of variable \(u = (\tau^\rho - t^\rho)/(x^\rho - t^\rho)\) and taking \(\kappa_2 = -\rho\eta_1\) into account,

\[
\int_0^\tau \frac{\tau^\rho \tau^{\nu(\eta_1+\kappa_2+1)} - 1}{(\tau^\rho - t^\rho)^{1-\alpha_1}(\tau^\rho - u^\rho)^{1-\alpha_2}} \,dt = \frac{(x^\rho - t^\rho)^{\alpha_1+\alpha_2-1}}{\rho \Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{1-u^\rho} (1-u)^{\alpha_1-1} u^{\alpha_2-1} \,du
\]

(14)

according to the known formulae for the beta function [23]. Substituting (14) into (13) we obtain

\[
\rho I_{a+\eta}^{\alpha+\beta} \rho I_{a+\eta}^{\alpha+\beta} f(x) = \frac{\rho^{1-(\beta_1+\beta_2)x^{\kappa_1}}}{\Gamma(\alpha_1+\alpha_2)} \int_a^x (x^\rho - t^\rho)^{\alpha_1+\alpha_2-1} t^{\rho(\eta_2+1)-1} f(t)\,dt,
\]

(15)

and thus, (11) is proved for “sufficiently good” functions \(f\). If \(\rho \geq c\) then by Theorem 2.1 the operators \(\rho I_{a+\eta}^{\alpha+\beta} \rho I_{a+\eta}^{\alpha+\beta}\) and \(\rho I_{a+\eta}^{\alpha+\beta}\) are bounded in \(X^\kappa_c(a,b)\), hence the relation (11) is true for \(f \in X^\kappa_c(a,b)\). This completes the proof of the theorem 2.2. \(\square\)

We have the following corollary.

**Corollary 2.3.** Let \(a > 0\), \(\beta > 0\), \(1 \leq p \leq \infty\), \(0 < a < b < \infty\) and let \(\rho \in \mathbb{R}\) be such that \(\rho \geq 1/p\). Then for \(f \in L^p(a,b)\) the semigroup property (17) holds.

Now we shall prove the fractional product-integration formulae [6] for the generalized integral. A similar result is referred by some authors as fractional integration by parts formula, but in our opinion this is not similar to the integration by parts formula due to the absence of a derivative term and we shall use the former to identify it.
**Theorem 2.4.** Let $\alpha > 0$, $\beta \in \mathbb{R}$, $1 \leq \rho \leq \infty$, $0 \leq a < b \leq \infty$ and let $\rho \in \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\rho \geq c$. Then for $f, g \in \mathcal{X}^\alpha_\rho(a, b)$ the fractional product-integration formula hold. That is,

$$
\int_a^b x^{\rho-1} f(x) \left( \rho I_{a+;\eta,\kappa,\omega}^{\alpha, \beta} g \right)(x) \, dx = \int_a^b x^{\rho-1} g(x) \left( \rho I_{a-;\eta,\kappa,\omega}^{\alpha, \beta} f \right)(x) \, dx 
$$

(16)

Similar results are also valid when $a = 0$ and $b = \infty$ and in particular for $\rho = 2$ and $\rho = 1$. 

**Proof.** The proof is straightforward. Using Dirichlet technique, we have

$$
\int_a^b x^{\rho-1} f(x) \left( \rho I_{a+;\eta,\kappa,\omega}^{\alpha, \beta} g \right)(x) \, dx = \frac{\rho^{1-\beta}}{\Gamma(\alpha)} \int_a^b x^{\alpha+\rho-1} f(x) \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^- - \tau^{\rho})^{1-\eta}} g(\tau) \, d\tau \, dx 
$$

$$
= \frac{\rho^{1-\beta}}{\Gamma(\alpha)} \int_a^b x^{\alpha+\rho-1} g(x) \int_x^b \frac{x^{\kappa+\rho-1}}{(x^+ - \tau^{\rho})^{1-\kappa}} f(\tau) \, d\tau \, dx 
$$

$$
= \int_a^b x^{\rho-1} g(x) \left( \rho I_{a-;\eta,\kappa,\omega}^{\alpha, \beta} f \right)(x) \, dx,
$$

which completes the proof. \(\square\)

**Remark 2.5.** Instead of Eq. (2), we can also consider a more general right- fractional integral given by

$$
\left( \rho I_{a-;\eta,\kappa,\omega}^{\alpha, \beta} f \right)(x) = \frac{\rho^{1-\beta} x^\omega}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(x^+ - \tau^{\rho})^{1-\kappa}} f(\tau) \, d\tau, \quad (0 \leq a < x < b \leq \infty).
$$

(17)

The drawback is that the results in Theorem 2.4 would take a more complicated form. This explains the rationale behind the choice of the parameter(s) $\rho$ of Eq. (2).

**Remark 2.6.** The generalized fractional integral introduced in this paper has a corresponding generalized fractional derivative which unifies the six fractional derivatives, namely, the Riemann-Liouville, Hadamard, Erdélyi-Kober, Katugampola, Weyl and Liouville fractional derivatives in to one form and will be discussed in another article.

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