Quantum Error Correcting Subsystem Codes From Two Classical Linear Codes

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Abstract—The essential insight of quantum error correction was that quantum information can be protected by suitably encoding this quantum information across multiple independently erred quantum systems. Recently it was realized that, since the most general method for encoding quantum information is to encode it into a subsystem, there exists a novel form of quantum error correction beyond the traditional quantum error correcting subspace codes. These new quantum error correcting subsystem codes differ from subspace codes in that their quantum correcting routines can be considerably simpler than related subspace codes. Here we present a class of quantum error correcting subsystem codes constructed from two classical linear codes. These codes are the subsystem versions of the quantum error correcting subspace codes which are generalizations of Shor’s original quantum error correcting subspace codes. For every Shor-type code, the codes we present give a considerable savings in the number of stabilizer measurements needed in their error recovery routines.

Real quantum systems are open quantum systems which can couple in an unwanted manner to an environment or control system and lose their intrinsic quantum nature through the processes of decoherence, quantum noise, and imprecise measurement, preparation, and control. These problems serve as an obstacle towards the eventual construction of a robust large scale quantum computer[1], [2], [3], [4]. If left unchecked, these problems turn a quantum computer into a classical information processing device, or even worse, into a machine which can enact no computation at all. Fortunately, however, soon after the discovery that quantum computers were more efficient at solving certain problems than classical computers[5], [6], it was discovered that, under specific reasonable physical assumptions, a fault-tolerant quantum computer could be built. In particular, a set of threshold theorems for fault-tolerant quantum computation were established[7], [8], [9], [10], [11], [12]. These theorems proved (or gave a heuristic proof) that if decoherence, quantum noise, and lack of control were all small enough in comparison to the ability to control the quantum system (below some threshold or thresholds), then these noisy imprecise bare devices could be efficiently put together in a fashion which decreased the failure probability of a quantum computer to any desired level.

A central insight used in the theory of fault-tolerant quantum computation is that quantum information can be encoded into what is known as a quantum error correcting code[13], [14]. Such codes spread quantum information across many physical quantum bits (qubits) of a system and protect the quantum information so encoded from the undesired effects which cause a loss of quantum coherence. In the original theory of quantum error correction, quantum information was encoded into a subspace of the Hilbert space of many quantum systems. Such subspace codings, however, are not the most general way to encode quantum information into a quantum system. The most general method to encode quantum information is to encode it into a subsystem[15]. This realization has recently led to the discovery of a new method for performing quantum error correction using quantum error correcting subsystems[16], [17], [18], [19]. While every quantum error correcting subsystem can be turned into a subspace code, subsystems codes differ significantly in how quantum error correction is performed on the encoded quantum information. Of particular significance is that quantum error correcting subsystems can significantly reduce the number of stabilizer measurements needed during their quantum error recovery routines[20], [21], [22]. This in turn can lead to significantly improved thresholds for fault-tolerant quantum computation[23], [24].

In this paper we construct a class of new quantum error correcting subsystem codes. These codes are a generalization of a code presented by one of us in [21] where they were put to use in an attempt to construct self-correcting quantum memories. The codes we describe are in the class of stabilizer subsystem codes described by Poulin[20]. In particular they can be described as the subsystem version of the class of codes arising from generalizing Shor’s original quantum error correcting subspace codes[13]. The subsystems codes we describe are constructed by taking two classical linear codes and, instead of concatenating them in the manner of Shor, constructing a new quantum code which is a subsystem quantum error correcting code directly from these classical linear codes. In Shor’s original construction a bit flip error correcting code is concatenated with a phase flip error correcting code (or vice versa.) The order of these concatenations presents an asymmetry in the recovery routine for these codes. If, for example a bit flip code is used on the lowest level, then the recovery procedure for the bit flip code must be enacted for every lowest level code, whereas the phase flip code recovery routine need only be enacted.
on the next level of the code. In the codes we construct, this asymmetry between bit flips and phase flips in the recovery routine is removed. This leads to codes which have substantially simpler error recovery routines, but provide the same amount of protection as the generalized Shor-type codes. Since the threshold for fault-tolerant quantum computation depends significantly on the complexity of the error recovery routine, these codes offer the hope of a substantial improvement in the threshold for fault-tolerant quantum computation[23], [24].

I. CLASSICAL AND QUANTUM LINEAR CODES

We begin by briefly reviewing classical and quantum linear codes in order to establish conventions for our future constructions.

We will work with classical codes over the finite field \( \mathbb{F}_2 \). \( \mathbb{F}_2 \) is the field with two elements 0 and 1 with addition defined as \( 0 + 0 = 1 + 1 = 0 \) and \( 0 + 1 = 1 + 0 = 1 \) and multiplication \( 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \) and \( 1 \cdot 1 = 1 \). Ordered \( n \)-tuples of \( \mathbb{F}_2 \) form a linear space, \( (\mathbb{F}_2)^n \). A binary linear code on \( n \) bits is a subspace \( C \) of \( (\mathbb{F}_2)^n \). A subspace can be described by a set of \( k \) basis vectors. We can list such a set of \( k \) basis vectors in a \( k \times n \) matrix, \( G \), with elements from \( \mathbb{F}_2 \). This matrix is called the generator matrix for the code \( C \). The generator matrix defines the encoding procedure for the code. In particular if we left multiply the \( k \times n \) matrix \( G \) by a \( k \)-tuple row vector of bits to be encoded, then we obtain the encoded length \( n \) string for these bits. For a code \( C \), the parity check matrix, \( P \) is an \( (n - k) \times n \) matrix with elements from \( \mathbb{F}_2 \) such that \( Pv = 0 \) for all \( v \in C \). Note that we use a slightly non-conventional row and column ordering for the parity and generator matrices. The (Hamming) distance between two elements of \( \mathbb{F}_2 \), \( v \) and \( w \), is given by \( H(v,w) = \sum_{i=1}^{n} \delta_{v_i,w_i} \), i.e. the number of places where the two \( n \) bit strings disagree. We say that a classical linear code is a \([n,k,d]\) code if it encodes \( k \) bits into length \( n \) bits and the minimum distance between any two elements of the code space is \( d \).

We now discuss quantum linear codes which are also known as stabilizer codes. We will assume that the reader is familiar with stabilizer codes at the level of [25] and provide the following review to set our definitions.

Suppose we have \( n \) qubits. Then \( \mathcal{P} \) is the group, commonly known as the Pauli group, whose elements are made up of tensor products of single qubit Pauli operators \( (I,X,Y,Z) \) along with a global phase of \( i^s \), for \( s \in \{0,1,2,3\} \). Every Pauli operator can be written in the form \( i^s Z^{a_1} X^{b_1} \otimes Z^{a_2} X^{b_2} \otimes \cdots \otimes Z^{a_n} X^{b_n} \) where \( s \in \{0,1,2,3\} \) and \( a_i, b_i \in \{0,1\} \). The weight of a Pauli group element is the number of non-identity single qubit operators in the above expansion. All Pauli group elements either commute with each other \( [P,Q] = PQ - QP = 0 \) or anticommute with each other \( [P,Q] = PQ + QP = 0 \).

A stabilizer group is an abelian subgroup of the Pauli group which does not contain the Pauli group element \(-I^\otimes n\). All of the elements of a stabilizer square to \( I^\otimes n \) and hence have eigenvalues \( \pm 1 \). Given a stabilizer group \( S \), we can define a stabilizer code as the subspace of the Hilbert space on \( n \) qubits which is stabilized by all elements in \( S \), i.e. the subspace \( \mathcal{H}_S \) defined as all states \( |\psi\rangle \) such that \( S |\psi\rangle = |\psi\rangle \) for all \( S \in S \). If the stabilizer group has a minimal set of generators \( S_1, \ldots, S_{n-k} \), then the dimension of the stabilized subspace (the code subspace) is \( 2^k \), i.e. it encodes \( k \) qubits of quantum information. The normalizer \( \mathcal{N} \) of the stabilizer \( S \) in \( \mathcal{P} \) is the set of elements of \( P \in \mathcal{P} \) such that \( P S P^\dagger \in \mathcal{S} \) for all \( S \in \mathcal{S} \). Note that \( \mathcal{S} \subseteq \mathcal{N} \). Elements of the normalizer preserve the stabilizer subspace, \( S_i (|\psi\rangle) = N S_j |\psi\rangle = |\psi\rangle \) for \( \in \mathcal{N} \). The group \( \mathcal{N}/\mathcal{S} \) is a Pauli group on \( k \) qubits, i.e. \( \mathcal{N}/\mathcal{S} \) acts as encoded operators on the stabilizer code.

If an element \( E \) of the Pauli group anticommutes with at least one stabilizer element \( S_k \) \( (\{E,S_k\} = 0) \) then this element takes a state encoded into the stabilizer subspace into one which is outside of this subspace. This can be verified by noting that for \( |\psi\rangle \in \mathcal{H}_S \), \( E |\psi\rangle = ES_k |\psi\rangle = -S_k E |\psi\rangle \) which tells us that \( E \) flips the sign of the eigenvalue of \( S_k \). This implies that if we encode quantum information into our stabilizer subspace and then a Pauli operator \( E \) which anticommutes with at least one \( S_k \) acts on the system, then we can detect this event by measuring the eigenvalue of \( S_k \). Elements of the Pauli group which anticommute with at least a single \( S_k \) are elements of the Pauli group which are not in \( \mathcal{N} \).

This leads to the following characterization of the capabilities of a stabilizer code to correct errors. Let \( \{E_i\} \) be a set of Pauli elements such that \( E_i \neq E_j \in \mathcal{N} - \mathcal{S} \) for all \( i,j \). Then \( \{E_i\} \) is a set of correctable errors for this stabilizer code. The weight of the smallest \( E_i \) which is in \( \mathcal{N} - \mathcal{S} \) is the distance \( d \) of the stabilizer code. A quantum code which can correct up to \( t \) errors must have a distance at least \( 2t + 1 \). A quantum error correcting code which uses \( n \) qubits, encodes \( k \) qubits, and has a distance \( d \) is called a \([n,k,d]_q\) quantum code.

II. GENERALIZED SHOR CODES

Here we review codes which are simple generalizations of Shor’s original \([9,1,3]_q\) quantum error correcting code. The codes we construct later in this paper will be subsystem versions of these codes, achieving the same error correcting properties as a generalized Shor code, but with significantly reduction in the number of stabilizers which need to be measured in order to perform quantum error correction.

To construct generalized Shor codes we first consider what a classical linear code looks like in the stabilizer formalism. Let \( \mathcal{C} \) be a classical linear code with generator matrix \( G \) and parity check matrix \( P \). Suppose that we wish to construct a quantum error correcting code which corrects bit flips (X errors) just as this classical code corrects bit flips. The stabilizer for this code is then obtained simply from the rows of the parity check matrix. In particular we can construct the stabilizer generators

\[
S_i = \bigotimes_{j=1}^{n} \mathcal{Z}_{P_{i,j}}.
\]
Clearly these stabilizer elements all commute with each other because they are all made up of either $I$ or $Z$ Pauli operators, and they are independent of each other since the rows of $P_{i,j}$ are linearly independent. It is clear that measuring these stabilizer generators is equivalent to using the parity check matrix to calculate the syndrome for this stabilizer code.

What are the logical operators for this code? The logical $X$ operators for the $k$ encoded qubits can be obtained directly from the generator matrix, $G$. In particular we can choose the $i$th logical $X$ operator to be

$$\bar{X}_i = \bigotimes_{j=1}^n X^{G_{i,j}}. \quad (2)$$

These operators are certainly not in the stabilizer since they are not made up entirely of $X$ operators and they commute with all of the elements of the stabilizer because

$$\bar{X}_i S_j = \bigotimes_{k=1}^n X^{G_{i,k}} \bigotimes_{l=1}^n Z^{P_{j,l}} \quad \Rightarrow \quad (-1)^{\sum_{k=1}^n G_{i,k} P_{j,k}} \bar{X}_i S_j = S_j \bar{X}_i \quad (3)$$

where we have used the fact that $\sum_{k=1}^n G_{i,k} P_{j,k} = 0$. Further we see, as expected, that there are $k$ of these encoded $\bar{X}$ operators. What about the encoded $\bar{Z}$ operators? Well it is always possible to construct these operators from tensor products of $Z$ and $I$ operators. In particular we let the $i$th logical $Z$ operator be

$$\bar{Z}_i = \bigotimes_{j=1}^n Z^{P^c_{i,j}}. \quad (4)$$

What is $P^c_{i,j}$? Let $S$ be the subspace of $(\mathbb{F}_2)^n$ spanned by the rows of the parity check matrix $P$. Then $P^c$ is a $k$ by $n$ matrix whose row vectors are linearly independent from the rows of $P$, are themselves linearly independent, and which together with the rows of $P$ span the entire space $(\mathbb{F}_2)^n$. In other words $P^c$ is made up of rows which together with the rows of $P$ form a basis for the full space $(\mathbb{F}_2)^n$. The choice of a generator matrix $G$ forces a particular choice (up to row multiplications) for $P^c$ such that it is a logical $Z$ operator. Encoded $Y$ operators can be obtained directly from $\bar{Y}_i = i\bar{X}_i \bar{Z}_i$.

In a similar vein to the construction of $P^c$, we can define a $n-k$ by $n$ matrix $G^c$ such that the rows of this matrix are linearly independent of the other rows of $G$, linearly independent of each other, and together with the rows of $G$ form a basis for $(\mathbb{F}_2)^n$. We can then define the operators

$$E_i = \bigotimes_{j=1}^n X^{G^c_{i,j}}. \quad (5)$$

Now the set of detectable bit flip errors is easily defined. It is any error which can be expressed as a product of at least one $E_i$ and any number (including zero) of $\bar{X}_i$ operators. The reason these are detectable is that products of $E_i$ operators are guaranteed to anticommute with at least one $S_j$.

In this table, each box shows the matrices used to construct a group for the linear classical code and the number of independent generators of the group is listed at the top of each column. In addition to the stabilizer and encoded $X$ and $Z$ operators, another group is formed from $G^c$. We have labelled this latter group the group of “pure errors.” These are errors which do not affect the encoded information and are detectable errors.

| TABLE I                                                                 |
|-------------------------------------------------------------|
| **CLASSICAL LINEAR CODES IN THE STABILIZER FORMALISM**       |
| **Tensor product of Z and I operators**                     |
| $P$ (Stabilizer)                                            |
| $P^c$ (Encoded Z)                                           |
| **Tensor product of X and I operators**                     |
| $G^c$ (Pure Errors)                                         |
| $G$ (Encoded X)                                             |

Finally let us note that the above construction can be used to correct phase flip errors instead of bit flip errors. This can be done by interchanging the $Z$ Pauli operators in the stabilizer with the $X$ Pauli operators. Replacing $Z$ Pauli operators in the logical $Z$ operator with the $X$ Pauli operators will produce a new encoded $Z$ operator for this code. Replacing $X$ Pauli operators in the logical $X$ operator with the $Z$ Pauli operators will produce a new encoded $X$ operator for this code.

We are now ready to describe generalized Shor codes. Shor’s basic idea was that one could construct a quantum error correcting code for errors on qubits by concatenating a code designed to deal with bit flips ($X$ errors) with one which is designed to deal with phase flips ($Z$ errors). Notice that there is an asymmetry in this construction: one should decide which of these errors to be dealt with on the lowest level of the concatenation and which should be dealt with at the second level of the concatenation.

Let $C_1$ and $C_2$ be two $[n_1, k_1, d_1]$ and $[n_2, k_2, d_2]$ linear codes. Let $C_1$ and $C_2$ have generator matrices, $G_1$ and $G_2$ respectively, and have parity check matrices $P_1$ and $P_2$ respectively. In generalized Shor codes we use these two codes to construct a $[n_1 n_2, k_1 k_2, \min(d_1, d_2)]$ quantum error correcting code. To do this we proceed as follows. Take $n_1 n_2$ qubits and partition them into $n_2$ blocks of size $n_1$. For each of these blocks of $n_1$ qubits we can define a quantum error correcting code from $C_1$ which is designed to correct bit flip errors as above. Each of the $n_1$ blocks encodes $k_1$ qubits. We can now use these encoded qubits to protect against phase flip errors. The first observation we need is that any tensor product of $Z$ and $I$ acting as an error on one of our blocks will act as a phase error on one or more of the encoded qubits times an element of the stabilizer. The reason for this is that every tensor product of $Z$ and $I$ is either in the stabilizer of quantum error correcting or is a product of a stabilizer operator and encoded $Z$ operators. This allows us to consider phase errors on blocks as phase errors on the encoded qubit. To protect against phase flip errors we begin by picking the $i$th encoded qubit from every block. We can then use each of these from all blocks to construct a quantum error correcting
code for phase flips using the $C_2$ code. Since for each choice of $i$ we obtain a code with $k_2$ encoded qubits, this code can be used to store $k_1k_2$ qubits. Further, the code will be of distance $d_1$ for bit flip errors and $d_2$ for phase flip errors. For $Y$ errors the code will be of distance $\min(d_1,d_2)$. Thus the distance of the code will be $\min(d_1,d_2)$.

Generalized Shor codes are easy to construct, but do not have nice asymptotic error correcting properties. However, they are conceptually extremely easy to understand and their error recovery routines are easily obtained from their constituent classical linear code recovery routines. One important point about these codes, however, is that they have an asymmetry in their recovery routines due to the choice of whether to concatenate bit flip codes with phase flip codes or vice versa. In particular for each the lowest level of concatenation, error correction should be performed for every single block. Indeed it is easy to see that the stabilizer for the generalized Shor code constructed above is generated by a set of $(n_1-k_1)n_2+(n_2-k_2)$ independent operators. We will show below that using the notion of a subsystem, this can be reduced, for every generalized Shor code, to $(n_1-k_1)k_2+(n_2-k_2)k_1$.

III. QUANTUM ERROR CORRECTING SUBSYSTEM CODES

In stabilizer codes one encodes information into a subspace of a quantum system. However the most general way to encode information is not to encode it into a subspace, but instead to encode it into a subsystem[15]. Let us briefly review this concept and describe the notion of quantum error correcting subsystem codes.

Suppose we have a Hilbert space $H$. Then one method for encoding quantum information is to encode this information into a subspace of $H$. In particular if $H$ is the direct sum of two subspaces, $H_C$ and $H_D$, $H = H_C \oplus H_D$, then we can encode quantum information into one of the subspaces, $H_C$. In addition to the notion of a direct sum, $\oplus$, of two Hilbert spaces, another notion for combining two Hilbert spaces is to construct the tensor product of these two Hilbert spaces. Thus, for example, we can combine two Hilbert space $H_C$ and $H_D$ as $H = H_C \otimes H_D$. Then we can encode quantum information into the subsystem $H_C$. Notice that such an encoding, for a fixed encoding into $H_D$ is a subspace encoding, but, without such a specification, the encoding is not a subspace encoding.

By repeatedly constructing subsystems and subspaces on Hilbert spaces, we can, most generally decompose a Hilbert space into a multiple direct sum of multiple tensor products of Hilbert spaces (since the process of direct sum and tensor product obey a distributive law.) Further if we single out a single one of these Hilbert spaces, call it $H_C$, then we may collect the other terms in such a decomposition so that the Hilbert space decomposes as

$$H = (H_C \otimes H_D) \oplus H_E.$$  

(6)

The decomposition described above, which is the most general for encoding a single Hilbert space, can be described as taking a Hilbert space $H$ and decomposed it into a subspace $H_E$ and a perpendicular subspace, $H_E^\perp$. On this perpendicular subspace we have further decomposed this into a tensor product of two subsystem Hilbert spaces, $H_E^\perp = H_C \otimes H_D$. Thus if we are going to encode quantum information to the subsystem $H_C$ we can do this by preparing the quantum state

$$\rho = (\rho_C \otimes \rho_D) \oplus 0_E$$  

(7)

where $\rho_C$ is the density matrix of the encoded quantum information, $\rho_D$ is information encoded into the subsystem $D$ (which can be arbitrary) and $0_E$ is the all zero matrix on the subspace $H_E$. At this point we can see one of the particular features of subsystem encodings: if we act nontrivially on the subsystem $D$ then the quantum information encoded into the subsystem $C$ is not affected. In other words, information encoded into a subsystem is not affected by information encoded into different subsystems. Encoding into subsystems has been used most notably in noiseless subsystems[15], [26], [27] and communicating without a shared reference frame[28], as well as being essential to an important transform in quantum information theory, the quantum Schur transform[29], [30].

What does this mean for the theory of quantum error correction? Suppose that we encode quantum information into a subsystem $H_C$ of a Hilbert space which has been decomposed as $H = (H_C \otimes H_D) \oplus H_E$. Next suppose that a quantum operation, corresponding to some quantum error, occurs on our system. Then the goal of quantum error correction is to restore the information encoded into the subsystem $H_C$. In the case where we have a subspace code, i.e. when $H_D = 0$, then we must apply an operation which correctly restores the quantum information encoded into the subspace $H_C$. If on the other hand we have a subsystem code, $H_D \neq 0$, then we must apply an operation which correctly restores the information encoded into the subsystem $H_C$, but we do not care what happens in this procedure to the information encoded into $H_D$. In other words, if we are to perform quantum error correction on a subsystem code, then the error recovery routine need only correct the error modulo the subsystem structure. We need not be worried if information encoded into $H_D$ is destroyed by the entire error/recovery routine, as long as the information in $H_C$ is correctly restored.

Suppose that we wish to protect our quantum information from a set of errors $E_a$ after we have encoded the quantum information into a subsystem as described above. Suppose that $|i\rangle \otimes |j\rangle$ is a basis for the subsystem $H_C \otimes H_D$, then a necessary and sufficient condition[16], [19] for the set of errors $E_a$ to be correctible is that

$$\langle i\rangle \otimes \langle k\rangle|E_a^\dagger E_b|\langle j\rangle \otimes \langle l\rangle = \delta_{i,j}c_{a,b}.$$  

(8)

Notice that this condition does not depend on the $|k\rangle$ and $|l\rangle$.

For a more complete description of quantum error correcting subsystems we refer the reader to [16], [17] which details not just the notion of a quantum error correcting subsystem, but also the notion of a operator quantum error correction,
which is a complete method for dealing with quantum error correcting subsystems.

IV. SUBSYSTEM CODES FROM TWO LINEAR CODES

We now turn to the construction of a new class of quantum error correcting subsystems. We will begin by detailing the construction and then proving that our construction has the error correcting properties which we claim. Our construction follows, in rough outline, that presented in [21].

A. Subsystem Code Construction

Suppose we are given two classical linear codes, \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), which are a \([n_1, k_1, d_1]\) code and a \([n_2, k_2, d_2]\) code, respectively. We will now show how to use these codes to construct a quantum error correcting subsystem code which is a \([n_1 n_2, k_1 k_2, \min(d_1, d_2)]\) code. Let \( P_1 \) and \( P_2 \) denote the parity check matrices and \( G_1 \) and \( G_2 \) denote the generator matrices for the two codes \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) respectively. From the rows of \( P_1 \) we can construct a stabilizer code on \( n_1 \) qubits. In particular we can define the \( n_1 - k_1 \) stabilizer operators \( S_1 = \otimes_{i=1}^{n_1} Z^{(P_1)_{ij}} \). Call the stabilizer group generated by this set of operators \( S_1 = \langle S_1, \ldots, S_{n_1-k_1} \rangle \).

In a similar manner we can use \( P_2 \) to construct a stabilizer code. In particular define the \( n_2 - k_2 \) stabilizer operators \( T_j = \otimes_{j=1}^{n_2} X^{(P_2)_{ij}} \). This forms a stabilizer group \( S_2 = \langle T_1, \ldots, T_{n_2-k_2} \rangle \).

These codes are classical codes except that the second code is not in the computational basis \([0], [1]\), but is instead in the dual \([+], [-]\) basis. The first code is designed to correct \( \left[ \frac{d_1 - 1}{2} \right] \) bit flip errors (Pauli \( X \) errors) while the second code is designed to correct \( \left[ \frac{d_2 - 1}{2} \right] \) phase flip errors (Pauli \( Z \) errors.) For both codes, we can follow our construction of classical codes and construct encoded operators. For the first code call the encoded \( X \) and \( Z \) operators \( (\bar{X}_1)_j \) and \( (\bar{Z}_1)_j \) and for the second code call the encoded \( X \) and \( Z \) operators \( (\bar{X}_2)_j \) and \( (\bar{Z}_2)_j \).

Now we will show how to produce a subsystem code using these codes. Put \( n = n_1 n_2 \) qubits on a rectangular \( n_1 \times n_2 \) lattice (this lattice is for illustrative purposes only, and is not a necessary part of the code.) We will now use the stabilizer codes operators \( S_1 \) in the columns and the stabilizer code operators \( S_2 \) in the rows to construct a nonabelian group \( \mathcal{T} \) on these \( n^2 \) qubits. More specifically, let \( T_1 \) be the stabilizer group made up of letting \( S_1 \) operators acting on all of the columns of the lattice and let \( T_2 \) be the stabilizer group made up of letting \( S_2 \) operators acting on all of the rows of the lattice. Then the group we are considering, \( \mathcal{T} \), is the group generated by the elements of \( T_1 \) and \( T_2 \).

The group \( \mathcal{T} \) is clearly nonabelian. From \( \mathcal{T} \) we can further construct an abelian invariant subgroup (invariant meaning that all elements of \( \mathcal{T} \) commute with the elements of the subgroup.) To do this, we do the following. Take one of the stabilizer operators from \( S_1 \). Take one of the codewords from \( C_2 \), call it \( w \). Now construct an operator on our \( n \) qubits which has \( S_{1j}^w \) acting on each column \( j \) (where \( S_{1j}^w = I \).) Clearly these elements are in \( \mathcal{T} \). Further they commute with all of the elements of \( \mathcal{T} \) since in a particular row they are made up of \( Z \) operators which have the form \( \otimes_{j=1}^{n_2} Z^{v_j} \) and in a particular column they are made up of elements of \( S_1 \). We can similarly take stabilizer operators form \( S_2 \) and codewords \( w \) from \( C_1 \) and construct operators which act like \( S_2^{v_j} \) acting on each row \( j \). These operators will also commute with all of the elements of \( \mathcal{T} \). So now we can form our abelian invariant subgroup \( \mathcal{S} \) as the group generated by these stabilizer operators for all possible stabilizer codeword combinations in both rows and columns.

So now we have a structure set up where we have a non-abelian group \( \mathcal{T} \) and an abelian invariant subgroup of this group, \( \mathcal{S} \). There is another set of operators which are important, which will correspond to the logical operators on the code \( \mathcal{L} \). Suppose we take an encoded \( X \) operator for the stabilizer code \( S_1 \), call it \( (\bar{X}_1)_i \), and take an encoded operator for the stabilizer \( \mathcal{S}_2 \), call it \( (\bar{X}_2)_j \). Then we can form an operator \( \bar{X}_{i,j} \) acting on our \( n^2 \) qubits by putting \( (\bar{X}_1)_i \) in each column \( j \) where \( (\bar{X}_2)_j \) acts non-trivially as \( X \). Similarly from \( (\bar{Z}_1)_i \) and \( (\bar{Z}_2)_j \) we can construct an operator \( (\bar{Z}_{i,j}) \) which is \( (\bar{Z}_1)_i \) in each column \( j \) where \( (\bar{Z}_2)_j \) acts non-trivially as \( Z \). It is easy to see that \( \bar{X}_{i,j} \) and \( \bar{Z}_{i,j} \) commute with the group \( \mathcal{T} \) since in each row or column they look like encoded operators. Further these operators also anticommute with each other if and only if their indices match, \( \{\bar{X}_{i,j}, \bar{Z}_{k,l}\} = \delta_{i,k}\delta_{j,l} \), and hence commute with each other otherwise. These operators further form a group which is isomorphic to a Pauli group on \( k_1 k_2 \) qubits.

Next we need to discuss how to put \( \mathcal{T}, \mathcal{S} \) and \( \mathcal{L} \) together to form a subsystem code. This is nearly identical to the procedure described in [21].

To do this, it is convenient to adopt explicit forms for the operators we have described above in a simple notation. Let \( M \) be a \( n \times n \) matrix with entries either 0 or 1. Then we define \( P^M \) as the operator on our \( n \) by \( n \) qubits which is a tensor product of \( P \) and \( I \) operators which acts on the qubit at the \( i \)th row and \( j \)th column as \( P^M_{i,j} \). In this notion it is easy to see that every element of the Pauli group on our \( n^2 \) qubits can be expressed as \( i^k Z^l X^B \) where \( k \in \{0,1,2,3\} \) and \( A \) and \( B \) are \( n \times n \) matrices. We will now proceed to use this notation to express the operators in \( \mathcal{T}, \mathcal{S} \) and \( \mathcal{L} \).

First consider \( \mathcal{T} \). Consider operators in \( \mathcal{T} \) which have elements of \( S_1 \) which lie in a column and are products of \( Z \) operators. In our new notation, these operators can be written as \( Z^A \) where \( A_{i,j} = (p_1^T P_1)_{ij} \delta_{j,0} \) where \( p_1^T \) is a \( n_1 - k_1 \) binary row vector and \( j_0 \) is the column where this operator acts. Similarly, we can construct operators from \( S_2 \) lie in a row and are products of \( X \) operators. They can be expressed as \( X^B \) where \( B_{i,j} = \delta_{i,0} (p_2^T P_2)_{ij} \) where \( i_0 \) is the row where this operator acts and \( p_2^T \) is a \( n_2 - k_2 \) binary row vector. In order to make a distinction which will be useful later, it is useful to express the delta functions in the above expression as follows. Since the rows of \( G_1 \) and \( G_2 \) form a basis for the entire space \( (F^2)^n \) we can express \( \delta_{j,0} \) as \( (g_j^T G_1 + (g_j^T)^T G_1^T) \) for some choice of length \( k \) binary row vector \( g_j^T \) and length \( n - k \) binary row vector \( (g_j^T)^T \). We can perform a similar decomposition for the \( X \) operators. Since multiplication now corresponds, up to a phase factor,
to addition, it is then easy to see that every element of the group $T$ can be expressed as $t(Q, Q^c, R, R^c, p) = i^p Z^A X^B$ with

$$A = P_1^T(Q) G_2 + P_1^T(Q^c) G_2^c$$

and

$$B = G_1^T(R) P_2 + (G_1^c)^T(R^c) P_2$$

where $Q$ is a $n_1 - k_1$ by $k_2$ 0/1 matrix, $Q^c$ is a $n_1 - k_1$ by $n_2 - k_2$ 0/1 matrix, $R$ is a $k_1$ by $n_2 - k_2$ 0/1 matrix, $R^c$ is a $n_1 - k_1$ by $n_2 - k_2$ 0/1 matrix, and $p \in \{0, 1, 2, 3\}$. Note that we have extended the group slightly by adding in a phase factor of $i$. This is for convenience sake when describing certain Pauli subgroups of $T$.

Having described $T$ in our notation, we now turn to $S$. $S$ is a subgroup of $T$ and thus we can express it again as a $t(Q, Q^c, R, R^c, p)$. In particular from the definition of the $S$ we see that its elements are of the form $s(Q, R) = t(Q, 0, R, 0, 0)$, i.e. they are elements of $T$ with $Q^c = 0$, $R^c = 0$, and $p = 0$.

Next we express elements of $L$ in this notation. Elements of $L$ are formed from encoded operations from the two codes. In particular it is easy to see that they can be expressed as $l(U, V, p) = i^p Z^A X^B$ with

$$A = (P_1^c)^T(U) G_2$$

and

$$B = G_1^T(V) P_2$$

where $U$ and $V$ are $k_1$ by $k_2$ 0/1 matrices and $p \in \{0, 1, 2, 3\}$. Again we have added an extra phase factor of $i$.

Finally let us notice that every element of the Pauli group on our $n^2$ qubits can be expressed as $o(Q, Q^c, R, R^c, U, U^c, V, V^c, p) = i^p Z^A X^B$ with

$$A = P_1^T(Q) G_2 + P_1^T(Q^c) G_2^c + (P_1^c)^T(U) G_2 + (P_1^c)^T(U^c) G_2^c$$

and

$$B = G_1^T(R) P_2 + (G_1^c)^T(R^c) P_2 + G_1^T(V) P_2^c + (G_1^c)^T(V^c) P_2^c$$

where $Q$, $Q^c$, $R$, $R^c$, $U$, and $V$ have the dimensions listed above, $U^c$ is a $k_1$ by $n_2 - k_2$ 0/1 matrix, $V^c$ is a $n_1 - k_1$ by $k_2$ 0/1 matrix, and $p \in \{0, 1, 2, 3\}$.

We can now define our subsystem code. We begin with $S$. $S$ is an abelian subgroup of the Pauli group which does not contain $-I^\otimes n^2$. Thus it is a stabilizer group. From this stabilizer group we can from a stabilizer code $S$. Operators which are in $T$ and $L$ are then in the normalizer of this stabilizer code, since all of the elements of $T$ and $L$ commute with all of the elements of $S$. Further we can think of the operators from $L$ and $T$ acting on different logical qubits for the stabilizer code, since all of the elements of $L$ commute with all of the elements of $T$.

The subsystem code can now be defined. Since $S$ is a stabilizer code, we can label subspaces of the $n^2$ qubits Hilbert space by the $\pm 1$ eigenvalues of the stabilizer generators. There are $(n_1 - k_1)k_2$ such stabilizer generators made up of tensor products of $I$ and $Z$ operators and $k_1(n_2 - k_2)$ stabilizer generators made up of tensor products of $I$ and $X$ operators. Thus the number of generators for this stabilizer group is $(n_1 - k_1)k_2 + k_1(n_2 - k_2)$. Next notice that $L$ forms an encoded Pauli group acting on $k_1k_2$ qubits. To see this notice that $l(U, v, p)$ follows the multiplication rules of a Pauli group as if for $k_1 k_2$ independent qubits. Now consider $T/S$, the group $T$ after we divide out the stabilizer group. This group is made up of operators $t(0, Q^c, 0, R^c, p)$. This group is an encoded Pauli group on $(n_1 - k_1)(n_2 - k_2)$ qubits. Putting this together we have a stabilizer with $(n_1 - k_1)k_2 + k_1(n_2 - k_2)$ generators, encoded $T/S$ operators which act as a Pauli group on $(n_1 - k_1)(n_2 - k_2)$ qubits, and encoded $L$ operators which act as a Pauli group on $k_1 k_2$ qubits. The total of these generators and the number of encoded qubits is $(n_1 - k_1)(n_2 - k_2) + (n_1 - k_1)k_2 + k_1(n_2 - k_2) + k_1k_2 = n_1 n_2$. This implies that the $T/S$ encoded operators and the $L$ encoded operators form an exhaustive list of encoded operators for the stabilizer code $S$.

So how do we define our subsystem code? Since there are $(n_1 - k_1)k_2 + k_1(n_2 - k_2)$ stabilizer generators, we can label subspaces of dimension $2^{n_1 n_2} - (n_1 - k_1)k_2 + k_1(n_2 - k_2)$ by the $\pm 1$ eigenvalues, call them $s_i$, of these stabilizer generators. Call $s$ the $(n_1 - k_1)k_2 + k_1(n_2 - k_2)$ tuple of these values. Further for each such subspace there is now a tensor product between encoded logical operators from $L$ and those from $T/S$. Thus we can find a basis for our Hilbert space on $n_1 n_2$ qubits such that it decomposes as

$$H = \bigoplus_{s \in \{0, 1\}^{(n_1 - k_1)k_2 + k_1(n_2 - k_2)}} H_{sT/S} \otimes H_s^L$$

where $\dim H_s^L = 2^{k_1k_2}$ and $\dim H_s^{T/S} = 2^{(n_1 - k_1)(n_2 - k_2)}$. It is now useful to describe how elements of $S$, $T/S$ and $L$ operate on this decomposition. Elements of $S$ act as either $\pm I$ on each subspace. In particular they act as

$$\bigoplus_{s \in \{0, 1\}^{(n_1 - k_1)k_2 + k_1(n_2 - k_2)}} (-1)^{\sum_{i=1}^{(n_1 - k_1)k_2 + k_1(n_2 - k_2)} s_i} I \otimes I$$

Elements of $L$ act as encoded qubits on the $H_s^L$ subspaces

$$\bigoplus_{s \in \{0, 1\}^{(n_1 - k_1)k_2 + k_1(n_2 - k_2)}} I \otimes L(s)$$

while elements of $T$ act as encoded qubits on the $H_s^{T/S}$ subspaces

$$\bigoplus_{s \in \{0, 1\}^{(n_1 - k_1)k_2 + k_1(n_2 - k_2)}} T(s) \otimes I$$

Our subsystem code can now be defined. We will encode our quantum information into the all $s_i = 1$ subspace and the corresponding $H_s^L$ subsystem. This encoding will encoded $k_1k_2$ qubits and the logical Pauli operators on this code come from $L$. Note that elements of $T$ can always be expressed as operators which do not act on this subsystem. This is a subsystem degree of freedom which makes our code a quantum error correcting subsystem code.
B. Subsystem Error Correcting Routine

Having identified the subsystem we are encoding into and the representation theoretic structure of operators in $S$, $T$ and $L$ we can now turn to the error correcting procedure for this code. We will show that this code can detect single qubit errors and describe the error recovery routine for this code. This error recovery routine corrects the information in our subsystem but may act nontrivially on the subsystem $H^T$.

Suppose that $C_1$ and $C_2$ can correct the sets of errors $E_1$ and $E_2$ respectively. We will now show how this allows us to correct bit flip and phase flip errors modulo the subsystem structure.

First consider an $X$ error on our code. Suppose that $e_1$ is a correctable error for code $C_1$. Let $E_1$ be a $n_1$ by $n_2$ matrix. We will show that every error $X^B$ with $B_{i,j} = (e_1)_i (E_1)_{i,j}$ is a correctable error for our subsystem code. To see this we first note that if we express this error as in Eq. (14)

$$B = G_1^T (R) P_2 + (G_1^T)^T (R^c) P_2 + G_1^T (V) P_2^c + (G_1^T)^T (V^c) P_2^c$$

then we can turn this into a product of an element of $T$ and one which is not in $T$, $X^{B_1} X^{B_2}$ where

$$B_1 = G_1^T (V) P_2^c + (G_1^T)^T (V^c) P_2^c$$

and

$$B_2 = G_1^T (R) P_2 + (G_1^T)^T (R^c) P_2.$$  

Since $X^{B_2}$ is an element of $T$ it cannot act as an error on the information encoded into our subsystem. Therefore we can consider the error to be purely of the form $X^{B_1}$. Now since the errors we are considering have $B$ matrices of the form $X^B$ with $B_{i,j} = (e_1)_i (E_1)_{i,j}$ this means that we can restrict the rows of $E_1$ to be from the subspace spanned by $P_2^c$.

Now suppose that we measure the elements of $S$ made up of tensor product of $I$ and $Z$ operators. These operators are of the form $Z^A$ where

$$A = (P_1)^T (U) G_2$$

and $G_2$ act as encoded $X$ and $Z$ operators, respectively, for the quantum version of the code $C_2$. Thus by measuring the stabilizer generators which are tensor products of $I$ and $Z$ operators, we can, for each encoded qubit in the second code, make a measurement of the $P_1$ for these encoded qubits. If the error $e_1$ is a correctable error for $C_1$, then we can apply the appropriate $e_1 p_1^T$ operators where $p_i$ are the appropriate row vectors from $P_2$. The effect of this correction procedure will be to restore the information encoded into the subsystem up to the operator $X^{B_2}$ which is an element of $T$. Thus we see that, as claimed, that we can correct errors which are of the form $X^B$ with $B_{i,j} = (e_1)_i (E_1)_{i,j}$.

A similar prescription applies for $Z$ errors by measuring the stabilizer generators which are made up of tensor products of $I$ and $X$ operators. If $e_2$ is a correctable error for the code $C_2$, then this will correct errors of the form $Z^A$ with $A_{i,j} = (e_2)_j (E_2)_{i,j}$. If these two procedures are carried out one after another they will also correct errors which $Z^A X^B$ with $A$ and $B$ correctable as above.

Thus we see that by measuring the generators of $S$ we can correct errors related to the original codes $C_1$ and $C_2$. What is the distance of this code? For bit flip errors the distance will be $d_1$ since we the smallest error of the form $X^B$ with $B_{i,j} = (e_1)_i (E_1)_{i,j}$ has only single $X$ errors in a row. Similarly the distance for phase flip errors will be $d_2$. The full distance must include $Z^A X^B$ errors and thus the distance is $\min(d_1, d_2)$.

C. Savings Over Generalized Shor Codes

In a generalized $[[n_1 n_2, k_1 k_2, \min(d_1, d_2)]]$ Shor code we have seen that error correction is achieved by measuring $(n_1 - k_1) n_2 + (n_2 - k_2)$ stabilizer generators. In our construction of a $[[n_1 n_2, k_1 k_2, \min(d_1, d_2)]]$ subsystem code above we have achieved the same parameters for the code but now using only $(n_1 - k_1) k_2 + k_1 (n_2 - k_2)$ stabilizer measurements. Both methods use two classical linear error correcting codes to construct a new quantum error correcting code. Indeed, the subsystem codes we present are nothing more than generalized Shor codes with certain stabilizers which do not add to the error correcting distance removed\[20\]. The subsystem code versions of generalized Shor codes have considerable advantages over the subspace code when it comes to the complexity of the error recovery routine, providing a quadratic savings in the number of stabilizers which need to be measured.

V. EXAMPLES

In this section we present a few examples of our code construction.

A. Redundancy Code

This is the construction presented in \[21\]. Let $C_1$ and $C_2$ both be a simple $n$ qubit redundancy code with generator matrix $G = (1,1, \ldots, 1)$ and parity check matrix

$$P = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 1
\end{bmatrix}$$

This redundancy code is a $[[n, 1, n]]$ code. The resulting subsystem code is a $[[n^2, 1, n]]$ code. Properties of this code are described further in \[21\]. Notice that a generalized Shor code constructed from this redundancy code requires the measurement of $n^2 - 1$ stabilizer operators. Our subsystem code achieves the same parameters for the code but using measurements of only $2(n - 1)$ stabilizer operators. This increase in efficiency combined with other nice properties of this code has recently been shown to improve the threshold for fault-tolerant quantum computation\[24\].
B. A [[49,1,5]] Code Which Outperforms a Concatenated Steane Code

Consider using the Hamming [7,4,3] code for the codes $C_1$ and $C_2$. Our construction will yield a quantum error correcting subsystem code with parameters $[[16,1,4]]$. Now consider using those 16 encoded qubits in a redundancy $[[16,1,4]]$ code as described in the last subsection. If we use the error recovery for our $[[16,1,4]]$ code followed by error correction for the $[[16,1,4]]$ code this will produce a code which can correct arbitrary 2 qubit errors and is thus effectively a $[[49,1,5]]$ code. The number of stabilizers which need to be measured for this use of these codes is $24 + 6 = 30$. Another option is to use on 9 of the 16 encoded operators in the redundancy subsystem code described above (allowing any error to occur on the other subsystem.) In this case one achieves a subsystem code with parameters $[[49,1,5]]$ but with a number of stabilizers given by $24 + 4 = 28$.

This should be compared with the normal (not optimal) use of concatenating the Steane code $[[7,1,3]]$ code with itself. There one uses error correction on the Steane code $[[7,1,3]]$ for multiple levels of the concatenation. This results in a code which is effectively a $[[49,1,5]]$ code. This concatenation scheme will require the measurement of $42 + 6 = 48$ stabilizer operators. Thus we see that a considerable savings of 18 (or 20) less stabilizer measurements.

VI. CONCLUSION

We have shown that for every generalized Shor code there is a subsystem code with the same parameters but which requires significantly fewer stabilizer measurements in order to perform quantum error correction. These codes are generalization of the codes presented in [21] and are in the class of stabilizer subsystem codes described in [20]. Recently Aliferis and Cross [24] have used the subsystem codes described in [21] to significantly improve the provable threshold for fault-tolerant quantum computation. A major open question is whether the subsystem codes described in the work described here can lead to similar and perhaps greater increases in the threshold for fault-tolerant quantum computation.

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