Computable Operations on Compact Subsets of Metric Spaces with Applications to Fréchet Distance and Shape Optimization

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Abstract—We extend the Theory of Computation on real numbers, continuous real functions, and bounded closed Euclidean subsets, to compact metric spaces \((X,d)\); thereby generically including computational and optimization problems over higher types, such as the compact ‘hyper’ spaces of \((i)\) nonempty closed subsets of \(X\) w.r.t. Hausdorff metric, and of \((ii)\) equicontinuous functions on \(X\). The thus obtained Cartesian closure is shown to exhibit the same structural properties as in the Euclidean case, particularly regarding function pre/image. This allows us to assert the computability of \((iii)\) Fréchet Distances between curves and between loops, as well as of \((iv)\) constrained/Shape Optimization.

I. INTRODUCTION, MOTIVATION, BACKGROUND

Identifying (and justifying) the ‘right’ concepts and notions is crucial for the foundation of a theory. The classical Theory of Computing is based on Turing machines with data encoded in binary and runtime taken in the worst-case over all inputs of length \(n \to \infty\) asymptotically — for discrete data. The Theory of Computing with continuous data also dates back to Turing (1937) for single reals and to Grzegorczyk (1957) for real functions; yet the quest for the right notions over higher types is still in progress [Roeg97, Schr09, KaCo10, LoNo15] since an input here contains infinite information and cannot even be read in full before having to start producing output. The present work continues the pursuit [KSZ16] for a uniform treatment of computability and complexity on general compact metric spaces \((X,d)\); thus generically including operations on the ‘higher type’ spaces of \((i)\) nonempty closed subsets of \(X\) w.r.t. the Hausdorff distance, and of \((ii)\) equicontinuous functions from \(X\) to another compact metric space \(Y\) w.r.t. the supremum norm. We are guided by the structural properties exhibited in Computational Logic [Esca13] and by the well-established Euclidean case [Ko91, BrWe99, Weih00].

A. Computing Real Numbers, Functions, Closed Subsets

Computing a real number \(r\) means to produce an integer sequence \(a_m\) of numerators of dyadic rationals \(a_m/2^m\) approximating \(r\) up to absolute error \(\leq 2^{-m}\). And computing a (possibly partial) function \(f : \mathbb{R}^d \to \mathbb{R}\) means:

Convert any sequence \((\bar{a}_m) \subseteq \mathbb{Z}^d\) satisfying
\[
|\bar{x} - \bar{a}_m/2^m| \leq 2^{-m}, \quad \bar{x} := \lim_m \bar{a}_m/2^m \in \text{dom}(f),
\]

to some \((b_n) \subseteq \mathbb{Z}\) s.t. \(|y - b_n/2^n| \leq 2^{-n}\) for \(y = f(\bar{x})\) while the behaviour on other sequences \((\bar{a}_m)\) is arbitrary [Weih00, §4.3]. For example addition \(\mathbb{R}^2 \to \mathbb{R}\) is computed by converting \((\bar{a}_m) \subseteq \mathbb{Z}^2\) to \(b_n := [a_{n+2,x}/4 + a_{n+2,y}/4]\). A computation according to Equation (1) runs in time \(T(n)\) if \(b_n\) appears after at most \(T(n)\) steps, regardless of \(\bar{x} \in \text{dom}(f)\) or \((\bar{a}_m)\). Following [BrWe99, Definition 4.8] and [Weih00 Exercise 5.2.1], call a non-empty compact \(W \subseteq \mathbb{R}^d\) computable iff there exists a family \(A_n \subseteq \mathbb{Z}^d\) such that \(W\) has Hausdorff distance \(\leq 2^{-m}\) to \(\{a_m/2^m : a_m \in A_m\}\), where \((A_m)\) is required to be uniformly recursive in the sense that \(\prod_{\mathbb{N}} \{\{n\} \times A_n \subseteq \mathbb{N} \times \mathbb{Z}^d\}^d\) is decidable. \(W\) is co-computable if there exists a uniformly co-r.e. such family. We report:

Fact 1: Fix non-empty compact \(W \subseteq [0;1]^d\) and computable total \(\Lambda : W \to \mathbb{R}^e\).

a) \([0;1]^d\) itself is computable. The union of two co-l/ computable subsets is again co-/computable, and the intersection of two co-computable sets is co-computable; cmp. [Weih00 Theorem 5.1.13].

b) A point \(\bar{x} \in [0;1]^d\) is computable iff the compact singleton \(\{\bar{x}\} \subseteq [0;1]^d\) is co-computable iff \(\{\bar{x}\}\) is computable; cmp. [Weih00 Example 5.1.12.1].

c) If \(W\) is computable, then it contains some computable point; cmp. [Weih00 Exercise 5.1.13].

d) A admits a runtime bound \(T = T(n)\), i.e., depending only on the output precision \(n\); and for any such bound \(T, n \to T(n+1) + 1\) is a binary modulus of continuity in that \(|\bar{x} - \bar{x}'| \leq 2^{-T(n+1)}\) implies \(|\Lambda(\bar{x}) - \Lambda(\bar{x}')| \leq 2^{-n+1}\); cmp. [Ko91, Theorem 2.19] and [Weih00 Theorem 7.2.7].

e) If \(W\) is co-computable, then the set \(\{(\bar{a}_0, \ldots, \bar{a}_n) : n \in \mathbb{N}, \exists \bar{w} \in W \forall j \leq n : \bar{a}_j \in \mathbb{Z}^d \land |\bar{a}_j/2^j - \bar{w}| \leq 2^{-j}\}\) (of finite initial sequences of dyadic sequences converging to some \(\bar{w} \in W\)) is co-r.e. and \(\Lambda\) has a recursive runtime bound; cmp. [Weih00 Theorems 2.4.7+7.2.5+7.2.7].

f) If \(W\) and non-empty compact \(V \subseteq [0;1]^e\) are co-computable, then so is \(\Lambda^{-1}[V] \subseteq W\); cmp. [Weih00 Example 5.1.19.2].

g) If \(W\) is computable, then the image \(\Lambda[W] \subseteq [0;1]^e\) is again co-computable [Weih00 Example 5.2.11].

h) If compact \(W \subseteq [0;1]^d\) coincides with the closure of its interior, \(W^\circ\), and both \(W\) and \([0;1]^d\) are co-computable, then they are computable [Zieg02, Theorem 3.1].

i) If non-empty compact \(W \subseteq [0;1]^d\) is computable, then so are max \(W \in [0;1]\) and min \(W \in [0;1]\); cmp. [Weih00 Lemma 5.2.6].
Item d) follows from careful continuity and compactness considerations. It corresponds to, and generalizes the (trivial) observation in discrete complexity theory that any total computation on \( \{0,1\}^n \) admits a worst-case runtime bound depending only on the length \( n \) of, but not on the input \( \vec{x} \in \{0,1\}^n \); itself. This complexity-theoretic property in turn is the key to prove Items e) to h) although the latter are only concerned with computability.

Item a) is optimal in that there exist computable compact \( V,W \subseteq [0;1] \) such that \( V \cap W \) is not computable [Wei00 EXERCISE 5.2.11]; and, regarding Item f), there exists a computable \( \Lambda : [0;1] \rightarrow [0;1] \) such that \( \Lambda^{-1}[0] \neq \emptyset \) contains no computable point [Weih00 EXERCISE 6.3.12]. Also Ernst Specker constructed a recursive and increasing sequence of integer fractions whose supremum is not computable [Wei00 EXAMPLE 1.3.2].

The former condition is known as lower semi-computability [AlBu10 DEFINITION 2] or left-computability; in fact a real number \( x \) is computable iff it is both left and right computable [Wei00 LEMMA 4.2.5]. We also record that, with the notation from Equation (1), strict inequality \( \"f(x) > 0\" \) is equivalent to \( \exists n : b_n > 1\" \) and thus r.e. (recursively enumerable, aka semi-decidable), but in general undecidable [Wei00 EXERCISE 4.2.9]. We use \( \\text{computable} \) for the continuous realm, \( \\text{decidable/co-/recursive/enumerable} \) for the discrete one.

### B. Overview, Previous and Related Work

The present work generalizes the Theory of Computation from Euclidean unit cubes to compact metric spaces \( (X,d) \). Being separable, computing here naturally means approximation up to error \( 2^{-n} \) by a sequence (of indices w.r.t. a fixed partial enumeration \( \xi : \mathbb{N} \rightarrow X \)) of some countable dense subset, thus generalizing the dyadic rationals \( \mathbb{D} = \{a/2^n : a, n \in \mathbb{Z}\} \) canonically employed the real case; cmp. [PeRi89 §2] or [Wei00 DEFINITION 8.1.2]. Of course the particular choice of said enumeration \( \xi \) heavily affects the computational properties it induces [BrPr03, Shr04].

We propose in Definition 4.2.3 weak conditions on \( \xi \) that assert the entire Fact [1] to carry over; see Theorem 7. They permit the categorical construction of dense enumerations, again satisfying said conditions, for (i) the compact space of non-empty closed subsets of \( X \) equipped with the Hausdorff distance, and for (ii) the compact space of equicontinuous functions from \( X \) to another compact metric space, identified with their graph: Theorem 10. We demonstrate the relevance and applicability of these conditions by asserting (Theorem 14) the computability of the Fréchet Distance between curves and between loops; and by asserting computability of the generic nonlinear optimization problem \( \max \{\Lambda(x) : \Phi(x) \leq 0\} \) for every computable cost function \( \Lambda : X \rightarrow \mathbb{R} \) and every computable, feasible, and open constraint \( \Phi : X \rightarrow \mathbb{R} \) including the contemporary case of Shape Optimization: Theorem 17.

The so-called Type-2 Theory of Effectivity considers computability on second countable topological T₀ spaces [Wei00 LEMMA 3.2.6] by means of partial encodings as infinite binary sequences, that is, over Cantor space. It establishes Cartesian closure by constructing generic encodings of countable products [Wei00 DEFINITION 3.3.3] and of the space of (relatively) continuous functions [Wei00 DEFINITION 3.3.13] as well as, for the case of a complete metric space, of its induced Hausdorff hyperspace [Wei00 EXERCISE 8.1.10]. However, lacking (local/sigma) compactness, properties (d)+(f)+(h) from Fact 1 do not hold in general. Indeed several notions of computability, equivalent in the Euclidean case [BrWe99], have been shown distinct for separable metric spaces [BrPr03]. In fact, compactness is well-known crucial in Pure as well as in computable analysis [Schr04, Esca13, Stei16].

### C. Recap: Continuous Functions and Compact Metric Spaces

We presume a basic comprehension of mathematical calculus and properties of compact metric spaces \( (X,d) \). Write \( B(x,r) = \{x' \in X : d(x,x') < r\} \) for the open ball in \( X \) with center \( x \in X \) and radius \( r \geq 0 \), \( \overline{B}(x,r) \) for its closure (unless \( r = 0 \)). More generally abbreviate \( B(S,r) := \bigcup_{x \in S} B(x,r) \) and similarly for \( \overline{B}(S,r) \subseteq X \). \( S^\circ \) denotes the interior (=largest open subset), \( S \) the closure (=least closed superset) of \( S \); \( \partial S := S \setminus S^\circ \) is the boundary of \( S \). Write \( \overline{B}^{d} = \{x \in \mathbb{R}^d : x_1^2 + \ldots + x_d^2 \leq 1\} \) for the closed Euclidean \( d \)-dimensional unit ball, \( \mathbb{R}^{d-1} := \overline{B}^{d-1} \) for the unit sphere. Let \( C(X,Y) \) denote the space of continuous functions \( f : X \rightarrow Y; C(X) \) in case \( Y = \mathbb{R} \).

**Definition 2**: a) A modulus of continuity of \( f : X \rightarrow Y \) is a non-decreasing right-continuous mapping \( \omega : [0;\infty) \rightarrow [0;\infty) \) such that \( \omega(t) \rightarrow 0 \) as \( t \rightarrow 0 \) and \( e(f(x),f(x')) \leq \omega(d(x,x')) \) holds for all \( x,x' \in X \).

b) A binary modulus of continuity of \( f : X \rightarrow Y \) is a non-decreasing mapping \( \mu : \mathbb{N} \rightarrow \mathbb{N} = \{0,1,\ldots\} \) such that \( e(f(x),f(x')) \leq 2^{-\mu(n)} \) whenever \( d(x,x') \leq 2^{-n(n)} \).

c) Abbreviate \( C_\mu(X,Y) = \{f : X \rightarrow Y : f \text{ has binary modulus of continuity } \mu\} \); \( \text{Lip}_1 := C_{id} \) is the space of non-expansive functions, \( \text{Lip}_2 := C_{id-n+1} \).

d) Let \( \mathbb{N}^n = \{(v_1,\ldots,v_n) : v_i \in \mathbb{N}\} \) denote Baire space, equipped with the metric \( \beta(\bar{v},\bar{w}) = 2^{-\min\{n:v_i\neq w_i\}} \).

e) For \( \emptyset \subseteq V,W \subseteq X \) consider the distance function \( d_V : X \ni x \mapsto \inf\{d(x,v) : v \in V\} \geq 0 \) and Hausdorff distance \( d_H(W,V) = \max\{\sup\{d_V(w) : w \in W\},\sup\{d_V(v) : v \in V\}\} \). The sup-metric on \( C(X,Y) \) is denoted \( c_{\infty}(f,g) = \max\{e(f(x),g(x)) : x \in X\} \).

f) \( \text{Ex} : \mathbb{N} \rightarrow \mathbb{N} \) denotes Kolmogorov’s metric entropy, also known as modulus of uniform boundedness [Ko87 LEMMA 18.52]: It is defined such that \( X \) can be covered by \( 2^{\text{Ex}(n)} \) open balls of radius \( 2^{-n} \), but not by \( 2^{\text{Ex}(n)-1} \).

If the entropy grows linearly, its asymptotic slope coincides with the Minkowski-Bouligand or box-counting dimension of \( X \); otherwise the latter is infinite. Compare also [KSZ16, Mayo16]. The central mathematical tool of the present work is compactness, so let us recall some aspects of this concept relevant in the sequel:

**Fact 3**: Suppose \( (X,d) \) is a compact metric space; i.e., every sequence in \( X \) admits a convergent subsequence; equiv-
alently: every cover \( \bigcup_{n} B(x_{n}, r_{n}) \supseteq X \) by open balls contains a finite subcover \( \bigcup_{n \leq N} B(x_{n}, r_{n}) \supseteq X \). Fix another compact metric space \( (Y, \varepsilon) \).

a) \( X \) is complete, that is, every Cauchy sequence converges. A non-empty subset of \( X \) is closed iff the restriction of \( d \) turns it into a compact space. Every continuous \( f: X \to \mathbb{R} \) attains its infimum and supremum.

b) Every \( f \in C(X,Y) \) has compact image \( f[X] \subseteq Y \), is uniformly continuous and thus admits a (binary) modulus of continuity.

c) By the Arzela Ascoli Theorem, a set \( F \subseteq C(X,Y) \) has compact closure iff it is a subset of \( C_{\mu}(X,Y) \) for some binary modulus of continuity \( \mu \).

d) By König’s Lemma, a non-empty subset \( C \subseteq \mathbb{N}^{n} \) of Baire space has compact closure if each node has finite degree in the tree of finite initial segments

\[ C^{*} := \{ \overrightarrow{u} = (u_{0}, \ldots, u_{n}) \mid n \in \mathbb{N}, \exists \overrightarrow{v}: \overrightarrow{u} \circ \overrightarrow{v} \in C \} \subseteq \mathbb{N}^{*} \]

e) The set \( K(X) \) of non-empty closed subsets of \( X \) equipped with the Hausdorff distance \( d_{H} \) (from Definition 1) constitutes again a compact Hausdorff hyper-space over \( X \).

f) Fix \( W \in K(X), D: \mathbb{N} \to \mathbb{N} \) a sequence, and \( \xi: \mathbb{N} \to X \) some (possibly partial) enumeration. Then the subsets \( \bar{x}_{\xi,D} = \bigcap_{m} x_{\xi,D,m} \) and \( W_{\xi,D} := \bigcup_{x \in W} x_{\xi,D} \) of Baire space \( \mathbb{N}^{n} \) are compact, where we use the abbreviations \( x_{\xi,D,m} := \{ u \in [2^{D(m)}] \cap \text{dom}(\xi) : d(x, \bar{x}(u)) \leq 2^{-m} \} \) and \( [M] := \{ 0, 1, \ldots, M - 1 \} \).

Consider the compact metric space \( (X \times Y, d \times e) \), where \( (d \times e)(x,y), (x',y') \) := \( \max\{ d(x,x'), e(y,y') \} \). A total function \( f: X \to Y \) is continuous iff \( graph(f) := \{ (x,f(x)) : x \in X \} \) is compact, i.e., an element of \( K(X \times Y) \).

Moreover it holds \( (d \times e)_{H}(graph(f), graph(g)) \leq \omega_{\omega_{0}}(f,g) \leq (\omega+id)(d \times e)_{H}(graph(f), graph(g)) \)

for \( \omega \) any modulus of continuity of \( f \) or \( g \) and \( F \subseteq C(X,Y) \) is compact iff \( graph(F) := \{ graph(f) : f \in F \} \subseteq K(X \times Y) \).

Item a) justifies the minimum and maximum in Definition 2).

II. Computing on a Compact Metric Space

Here we extend Fact 1 from Euclidean \([0;1]^{d} \) to arbitrary compact metric spaces. Generalizing the real case with dyadic rationals \( \mathbb{D} \) as canonical countable dense subset, computation on a metric space is commonly defined by operating on (sequences of indices wrt.) some fixed countable partial dense enumeration; cmp. [PER89] §2 or [Weih00] DEFINITION 8.1.2). Of course, the particular choice of said enumeration heavily affects whether, and which items of, Fact 1 carry over [Schr04]. In Definition 4 below we propose conditions that formalize and generalize numerical (i.e. Euclidean) grids to (i) more general compact metric spaces by considering a relaxation of Hierarchical Space Partitioning whose (ii)

subsets are closed balls that however (iii) may overlap as long as (iv) their centers keep distance \( \eta \) from each other:

**Definition 4**: Fix a compact metric space \( (X,d) \) of diameter \( \text{diam}(X) := \max\{ d(x,x') : x,x' \in X \} \leq 1 \).

a) For \( m \in \mathbb{Z}, \) an \( m \)-covering of \( X \) is a subset \( X_{m} \subseteq X \) such that \( X \supseteq \bigcup_{x \in X_{m}} B(x, 2^{-m-1}) \). For \( \eta \in \mathbb{N}, X_{m} \subseteq X \) is \( \eta \)-separated if it holds \( d(x,x') \geq 2^{-\eta} \) for all distinct \( x,x' \in X_{m} \) with \( \eta \)-rectangular if \( \forall x,x' \in X_{m} : 2^{n}d(x,x') \in \mathbb{N} \).

b) \( (X,d,\xi,D) \) is a presented (compact) metric space if \( \xi: \mathbb{N} \to X \) is a partial dense enumeration and \( D: \mathbb{N} \to \mathbb{N} \) strictly increasing such that, for every \( m \in \mathbb{N}, \) the image \( \xi([2^{D(m)}]) \cap \text{dom}(\xi) \) constitutes an \( m \)-covering of \( X \). For strictly increasing \( \eta: \mathbb{N} \to \mathbb{N} \) and injective \( \xi: (X,d,\xi,D) \) is \( \eta \)-rectangular/or \( \eta \)-separated if, for every \( m \in \mathbb{N}, \xi([2^{D(m)}]) \) constitutes an \( \eta(m) \)-rectangular/or \( \eta(m) \)-separated \( m \)-covering of \( X \), respectively.

c) Presented metric space \( (X,d,\xi,D) \) is computably compact if \( \text{dom}(\xi) \) and \( D: \mathbb{N} \to \mathbb{N} \) areursive and the following is semi-decidable: \( \{ (a,b,n,u,v) \mid u,v \in \text{dom}(\xi), a/2^{n} < d(\xi(u),\xi(v)) < b/2^{n} \} \subseteq \mathbb{N}^{5} \)

d) For presented \( (X,d,\xi,D), \) a name of a point \( x \in X \) is a sequence \( \bar{u} = (u_{a}), u_{m} \in \text{dom}(\xi) \cap [2^{D(m)}] \), with \( d(\xi(u_{m}),x) \leq 2^{-m} \). The point \( x \) is computable if it admits a recursive name. It is polynomial-time computable if there exists a Turing machine which prints a name \( \bar{u} \) such that \( u_{m} \) appears after a number of steps bounded by some polynomial in \( m \).

e) Fix presented \( (X,d,\xi,D) \) and \( (Y,e,v,E) \) and recall that \( W_{\xi,D} \subseteq \text{dom}(\xi)^{\ast} \) denotes the set of finite initial sequences \( \bar{u} \in W_{\xi,D} \). A name of a partial mapping \( \Lambda: \subseteq X \to Y \) is a (w.r.t. initial substrings) monotonic, total mapping \( \Lambda^{\ast}: X_{x,D}^{\ast} \to Y_{x,E}^{\ast} \) such that, for every \( \bar{u} \in x_{\xi,D} \), \( x \in \text{dom}(\Lambda) \), the (w.r.t. initial substrings) non-decreasing sequence \( (\Lambda^{\ast}(u_{0},\ldots,u_{m}))_{m} \) has bounded length and supremum \( \bar{u} \in \Lambda(x,v,E) \). We write \( \Lambda_{n}(\bar{u}) \) for the \( n \)-th element \( v_{n} \) of said supremum. Computing \( \Lambda \) means (for a Turing machine with write-only right-moving tape) to compute some name \( \Lambda^{\ast} \). Such a computation runs in time \( t(n) \) if it takes at most that many steps to output \( \Lambda_{n}(\bar{u}) \), independently of \( \bar{u} \in \text{dom}(\Lambda)(x) \).

f) For presented \( (X,d,\xi,D), \) a name of compact non-empty \( W \subseteq X \) is a sequence \( \bar{A} = (A_{m}) \) of finite sets \( A_{m} \subseteq [2^{D(m)}] \cap \text{dom}(\xi) \) such that, for every \( m \in \mathbb{N}, \) the set \( \xi[A_{m}] \subseteq X \) has Hausdorff distance (Definition 1) at most \( 2^{-m} \) to \( W \). The empty sequence is a name of \( 0 \). \( W \) is computable if it has a name \( \bar{A} = (A_{m}) \) which is uniformly recursive in the sense that the set \( \bigcap_{m} [m] \times A_{m} \subseteq \mathbb{N} \times \mathbb{N} \) is decidable. If said set is co-re, \( W \) is called co-computable. A standard name \( \bar{A} \) of \( W \) satisfies \( d_{W}(\xi(a)) < 2^{-m} \) for every \( a \in A_{m} \subseteq [2^{D(m)}] \cap \text{dom}(\xi) \), and \( d_{W}(\xi(a)) > 2^{-m-1} \) for every \( a \in [2^{D(m)}] \cap \text{dom}(\xi) \setminus A_{m} \).

g) A rounding function for presented \( (X,d,\xi,D) \) is a mapping \( R: \text{dom}(\xi) \times \mathbb{N} \to \text{dom}(\xi) \) such that it holds \( R(u,m) \in [2^{D(m)}] \) and \( d(\xi(R(u,m))),\xi(u)) \leq 2^{-m-1} \).
[Weih03] Definition 6.2.2] calls an \( m \)-covering \((m + 1)\)-spanning. Various common notions of computability for closed subsets, equivalent over \(\mathbb{R}^d\) [BrPr03, Theorem 3.6], are known to become distinct over more general spaces [BrPr03, Theorems 3.9(3)+3.11(4)+3.15(2)]. Definition \(4\) thus has been crafted with great care. For instance, although computations according to e) operate on approximations up to absolute error \(\leq 2^{-m}\), the requirement in a) of an \( m \)-covering to provide strictly better approximations is crucial. Item c) strengthens [Weih03, Definition 8.1.2.3] in requiring \(\text{dom}(\xi)\) to be recursive, thus asserting a computably compact space \(X\) to be a computable subset of itself in the sense of Item f); see Theorem \(7\) below. It also guarantees the set \(X_{\xi,D} \subseteq \mathbb{N}^*\) to be co-r.e.; see Theorem \(2\). The rest of this subsection will provide further justification by comparison to the Euclidean Fact \(1\). Indeed, Definition \(4\) generalizes the real case:

**Example 5:** a) Let \(\mathbb{D}_m := \{a/2^m : a \in \mathbb{N}, 0 \leq a < 2^m\}\) and \(\mathbb{D} := \bigcup_m \mathbb{D}_m\) denote the set of dyadic rationals in \((0, 1)\). Define \(D(m) := m + 1\) as well as \(\varrho(0): = 0\) and inductively \(\varrho : [2^{m+1} \setminus 2^m] \to a + 2^m + 1 \mapsto (2a+1)/2^m \in \mathbb{D}_{m+1} \setminus \mathbb{D}_m\). Then \((0, 1], \cdot, \varrho, \text{id}+1\) constitutes a computably \(m\)-rectangular (formally: \(id\)-rectangular) compact space. It admits a computable rounding function, namely

\[
R(\cdot, m) : [2^{m+n+1}] \setminus [2^m] \ni a + 2^m \mapsto 2^m + \left(\frac{2a+1}{2^m} - \frac{1}{2}\right) \in [2^{m+1}] .
\]

b) The ‘circle’ \((0, 1] /\mathbb{N}\) with the metric \(d(x, y) = \min\{x - y, y - x\}\), equipped with the enumeration \(\varrho\) from a) but now taking \(D(m) = m\), is also computably \(id\)-rectangular compact.

c) Consider Cantor space \(\{0, 1\}^\mathbb{N}\) equipped with the metric \(\beta\) inherited from Baire space; recall Definition \(4\)). We turn this into a computably \(m\)-rectangular compact space \(\{0, 1\}^\mathbb{N} \times \{\beta, \gamma, \text{id}+1\}\) as follows: Let \(\tilde{\gamma} : \mathbb{N} \to \{0, 1\}^*\) enumerate all finite binary strings in order of length, define \(\gamma(0) := 0^\omega\), \(\gamma(u+1) := \tilde{\gamma}(u) \circ 10 0^\omega\); and truncation constitutes a computable rounding function.

d) Suppose \(\psi : X \to Y\) is a homeomorphism between computably compact \((X, d, \xi, D)\) and topological space \(Y\). Then \((Y, d \circ \psi^{-1}, \psi \circ \xi, D)\) constitutes a computable compact metric space (thus justifying the notation in b). If the former is \(\eta\)-separated/rectangular/has rounding function \(R\), then so does the latter.

e) For computably compact metric spaces \((X, d, \xi, D)\) and \((Y, e, v, E)\), their Cartesian product \((X \times Y, d \times e, \xi \times v, D \times E)\) becomes again a computably compact metric space by defining \(\xi \times v : \mathbb{N} \to X \times Y\) inductively on \([D(m) \cdot E(m); D(m+1) \cdot E(m+1) - 1] \cap \mathbb{N}\):

\[
D(m) \cdot E(m) + (D(m+1) - D(m)) \cdot v + u \mapsto \left(\xi(D(m) + u), v(E(m) + v)\right),
\]

\[\mathbb{N} \ni u < D(m+1) - D(m), \quad \mathbb{N} \ni v < E(m)\]

\[
D(m+1) \cdot E(m) + D(m+1) \cdot v + u \mapsto \left(\xi(u), v(E(m) + v)\right),
\]

\[\mathbb{N} \ni u < D(m+1) - D(m), \quad \mathbb{N} \ni v < E(m)\]

If \(X\) and \(Y\) are \(\eta(m)\)-separated/rectangular, then so is \(X \times Y\). Recursive rounding functions for \(X\) and \(Y\) give rise to one for \(X \times Y\).

Items a+e) generalize the case of real vectors. A computable compact subset need not in turn constitute a computably compact metric space: consider for example \(\{1/\pi\} \subseteq [0; 1]\).

**Remark 6:** a) To every compact \((X, d)\) with partial dense enumeration \(\xi : \subseteq \mathbb{N} \to X\), there exists some \(D : \mathbb{N} \to \mathbb{N}\) rendering \((X, d, \xi, D)\) a presented metric space. If the restricted distance

\[
\mathbb{N} \times \mathbb{N} \ni (u, v) \mapsto d(\xi(u), \xi(v)) \in \mathbb{R}
\]

is computable and the natively \(\Pi_2\) set

\[
\{ (u, v, v_1, \ldots, v_j) \mid j \in \mathbb{N}, u, v_1, \ldots, v_j \in \text{dom}(\xi), \xi(u) = B(\xi(v_1), 2^{-n_1}) \cup \cdots \cup B(\xi(v_j), 2^{-n_j}) \}
\]

\[
\text{with } \forall i \leq j : d(\xi(w), \xi(v_i)) < 2^{-n_i}
\]

is actually semi-decidable, then there exists a recursive such \(D\). [BrPr03, Definition 2.6] calls Equation \(2\), with dyadic radii generalized to arbitrary rationals, the effective covering property.

d) Every compact metric space \((X, d)\) admits a partial dense enumeration \(\xi\) such that \(D(m) := \varepsilon_X(m+1)\) turns \((X, d, \xi, D)\) into an \((m+1)\)-separated space: Let \(\xi : [2^{D(m)}] \to [2^{D(m)}] \cap \text{centers of open balls of radius } 2^{-m-1}\)

Conversely whenever \((X, d, \xi, D)\) is \(\eta\)-separated, it holds \(D(m) \leq \varepsilon_X(\eta(m) + 1)\): Consider some choice of \(2^{\varepsilon_X(\eta(m)+1)}\) centers of open balls of radius \(2^{-\eta(m)-1}\)

\(\mathbb{N}\); since the latter have pairwise distance \(\geq 2^{-\eta(m)}\); cmp. [KSTZ16, §3].

c) Not every convex compact metric space admits a rectangular enumeration, though: Consider the geodesic distance on a circle of irrational circumference. Even in the Euclidean case, it has been conjectured since Erdős and Ulam (1946) that no open subset of \(\mathbb{R}^2\) admits a dense sequence of pairwise rational distances; cmp. [http://terrytao.wordpress.com/2014/12/20](http://terrytao.wordpress.com/2014/12/20) [d) Suppose \((X, d, \xi, D)\) is computably compact. Then computably compact \((X, d, \xi, m \mapsto D(m+1))\) admits a recursive rounding function: Given \(u \in \text{dom}(\xi)\) and \(m \in \mathbb{N}\) there exists, and Definition \(5\)\) asserts enumeration of \([2^{D(m+1)}] \cap \text{dom}(\xi)\) to find, some \(v := R(u, m)\) with \(d(\xi(v), \xi(u)) < 2^{-m-1}\). (In \([2^{D(m)}] \cap \text{dom}(\xi)\), distance \(2^{-m-1}\) is feasible, but not necessarily computably so...)}
e) \((A_m)\) is a name of non-empty compact \(W \subseteq X\) iff (i) every \(\bar{a} = (a_m)\) with \(a_m \in A_m\) satisfying
\[
\forall a, m : d(\xi(a_m), \xi(a_n)) \leq -2^{-m} + 2^{-n} \tag{3}
\]
constitutes a name of some \(x \in W\) and if (ii) conversely to every \(x \in W\) there exists a name \(\bar{a}\) such that Equation (3) holds: If \((A_m)\) constitutes a name of \(W\), then (i) every \(\bar{a}\) satisfying Equation (3) gives rise to \(x \in \min_m \xi(a_m) \in X\) by completeness, and \(d_W(\xi(a_m)) \leq 2^{-m}\) shows \(x \in W\); and (ii) to \(x \in W\) and every \(m \in \mathbb{N}\) there exists some \(a_m \in A_m\) with \(d(\xi(a_m), x) \leq 2^{-m}\), hence satisfying Equation (3).

f) Every standard name of some non-empty compact \(W \subseteq X\) is also a name of \(W\): By Definition (4) there exists, to every \(x \in W\), some \(a \in \text{dom}(\xi) \cap \{2^{D(m)}\}\) with \(d(\xi(a), x) \leq 2^{-m-1}\); and \((A_m)\) being a standard name of \(W\) requires \(a \in A_m\) whenever \(d_W(\xi(a)) \leq 2^{-m-1}\): thus \((A_m)\) is a name of \(W\).

Conversely, to every co-r.e./recursive name \((A_m)\) of non-empty compact \(W \subseteq X\), there exists a co-r.e./recursive standard name of \(W\): Indeed every \(a_m\) with
\[
a \in A_m + 3 \land d(\xi(a), \xi(a')) \leq 5 \cdot 2^{-m-3}
\]
constitutes such a standard name: To \(a \in A_m + 3\) there exists some \(x \in W\) with \(d(\xi(a), x) \leq 2^{-m-3}\), hence \(a' \in A_m'\) implies \(d(\xi(a), \xi(a')) < 7 \cdot 2^{-m-3}\) and in turn \(d_W(\xi(a')) < (7 + 1) \cdot 2^{-m-3} = 2^{-m}\); while \(a' \in \text{dom}(\xi(a)) \cap \{2^{D(m)}\}\) \(A_m'\) implies \(d(\xi(a), \xi(a')) \leq 5 \cdot 2^{-m-3}\) for every \(a \in A_m + 3\), and in turn \(d_W(\xi(a')) > (5-1) \cdot 2^{-m-3} = 2^{-m}\). Now recall that strict inequality of distances is r.e., and non-strict is co-r.e. Hence, if \((A_m)\) is uniformly co-r.e., then so is the left-hand side of Equation (4); and if \((A_m)\) is even uniformly recursive, then the right-hand side is uniformly r.e.: now apply the next item to its complement.

g) Fix pairwise disjoint families \(X_m\) and \(Z_m\) of uniformly co-r.e. subsets of integers. Then there exists a uniformly recursive family \(Y_m\) disjoint from \(Z_m\) with \(X_m \subseteq Y_m\): For given \(x, m\) search in parallel for a witness that \(x \notin X_m\) and for one that \(x \notin Z_m\) and report the (negation of the) first to succeed.

h) Suppose \((X, d, \xi, D)\) and \((Y, e, \upsilon, E)\) are computably compact with rounding function \(R : \text{dom}(\xi) \times \mathbb{N} \to \text{dom}(\xi)\). Let \((A_m)\) denote a name of non-empty compact \(W \subseteq X\), and \(\Lambda^*\) a name of \(\Lambda : W \subseteq X \to Y\), computable in time \(t(n)\). Recall that \(W_{\xi, D}^* \subseteq \mathbb{N}^*\) denotes the set of all finite initial segments of sequences in \(W_{\xi, D}\) and observe that, for every \(n \in \mathbb{N}\) and \(a \in A_{\xi, D}(n)\), even though \(\xi(a)\) itself might not even lie in \(W\), the closed ball \(\overline{B}(\xi(a), 2^{-t(n)})\) does intersect \(W\). Moreover, for every \(x \in W \cap \overline{B}(\xi(a), 2^{-t(n)})\), the finite sequence \(\bar{a} := (R(a, 0), R(a, 1), \ldots, R(a, t(n) - 1), a)\) extends to some name \(\bar{a}\) of \(x\), i.e., belongs to \(W_{\xi, D}^*\). Abbreviating \(y := \Lambda(x)\), the machine computing \(\Lambda^*\) will on input \(\bar{a}\) produce \(\Lambda_n(\bar{a}) =: b_n \in [\text{dom}(e) \cap \{2^{E(n)}\}]\) with \(e(v(b_n), y) \leq 2^{-n}\) for \(y := \Lambda(x)\). However, in time bound \(t(n)\) it cannot even read past \(\bar{a}\); hence \(\Lambda_n(\bar{a})\) depends only on \(a\): \(\Lambda_n(\bar{a}) = \Lambda_n(\bar{a})\) holds whenever \(\bar{a} \subseteq \Lambda_n(\bar{a})\) is computable. Hence, if \(\Lambda_n(\bar{a})\) is computable, then the image \(\Lambda_n(\bar{a})\) is co-r.e.; and \(\Lambda\) admits a computable time bound \(T = T(n)\) depending only on the output precision \(n\): for any such bound \(T\), \(n \to T(n+1) + 1\) is a binary modulus of continuity of \(\Lambda\).

A. Computable Operations on a Compact Metric Space

For presented metric spaces, the composition of two computable functions is again computable; computable functions map computable points to computable points; a constant function is computable iff its value is computable. Moreover, similarly to Fact (1) we have:

Theorem 7: Let \((X, d, \xi, D)\) and \((Y, e, \upsilon, E)\) be computably compact spaces with recursive rounding functions \(R\) and \(S\) according to Definition (2). Suppose compact non-empty \(W \subseteq X\) and total \(\Lambda : W \to Y\) are computable.

a) \(X\) is a computable subset of itself. The union of two co-recursive sets is again co-recursive; the intersection of two co-recursive sets is co-recursive.

b) A point \(x \in X\) is computable iff the compact singleton \(\{x\} \subseteq X\) is computable.

c) If \(W\) is computable, it contains some computable point.

d) \(\Lambda\) admits a computable time bound \(T = T(n)\) depending only on the output precision \(n\): for any such bound \(T\), \(n \to T(n+1) + 1\) is a binary modulus of continuity of \(\Lambda\).

e) If \(W\) is co-computable, then \(W_{\xi, D}^*\) is co-recursive; and \(\Lambda\) has a recursive binary modulus of continuity and runtime bound.

f) If \(W\) and non-empty compact \(W \subseteq Y\) are co-computable, then so is \(\Lambda^{-1}[V] \subseteq W\).

g) If non-empty compact \(W \subseteq Y\) is computable, then the image \(\Lambda[W] \subseteq Y\) is again (compact and) computable.

h) If non-empty compact \(W \subseteq X\) coincides with \(\overline{\mathbb{F}}\) and both \(W\) and \(X \setminus \mathbb{F}\) are co-computable, then they are computable.

j) A total \(\Lambda : W \to Y\) is computable (in the sense of Def (2)) iff the compact set \(\text{graph}(\Lambda) \subseteq X \times Y\) is computable.

Item g) can be regarded as a discrete counterpart to Fact (1). Item h) is based on the continuity/adversary argument underlying the sometimes so-called Main Theorem [Weih00] §2.2, Fact (1)+e), and Theorem (2)+e) below.
graph as a ‘static’ object, and justifies our encoding of compact function spaces in Subsection II-C below.

B. Proof of Theorem 7

a) By Definition 4), \(A_m := \text{dom}(\xi) \cap [2^{D(m)}] \) is a name of \(X\); and uniformly recursive according to Definition 4). For \(A_m, B_m \subseteq [2^{D(m)}] \cap \text{dom}(\xi)\) with 
\[
d_{H}(\xi(A_m), W), d_{H}(\xi(B_m), V) \leq 2^{-m}, A_m \cup B_m \subseteq [2^{D(m)}] \cap \text{dom}(\xi)\] 
and is uniformly recursive/co-r.e. whenever both \(A_m\) and \(B_m\) are. The cases \(W = \emptyset\) or \(V = \emptyset\) are easily treated separately. Finally, for uniformly co-r.e. \(A_m, B_m\) subsets of recursive \(\text{dom}(\xi) \cap [2^{D(m)}]\),

\[
C_m := \{ c \in A_m \mid \forall n', n' \in N \exists a \in A_n \exists b \in B_{n'} : d(\xi(a), b) \leq 2^{-n} + 2^{-n'} \}
\]
is co-r.e. (since \(\text{dom}(\xi)\) is) and a name of \(V \cap W\): To every \(x \in V \cap W\) there exist \(a_n\) and \(b_n\) in \(B_{n'}\) with 
\[
d(\xi(a_n), b_n), \xi(x) \leq 2^{-n} \text{ and } d(\xi(b_n), \xi(x)) \leq 2^{-n'}, \text{ so } d(\xi(c), \xi(x)) \leq 2^{-n} + 2^{-n'}; \text{ and, since } x \in V, \text{ there exists some } c \in A_m \text{ with } (c, x) \subseteq \text{dom}(\xi), x \subseteq 2^{-m}; \text{ so } d(\xi(c), \xi(x)) \leq 2^{-n} + 2^{-m}; \text{ resulting in } c \in C_m.\]

b) For computable \(x\) with recursive name \(\bar{u} = (u_m)_{m \in N\} \in x_{\xi,D}\), the uniformly recursive sequence of singletons \(A_m := \{ u_m \}\) constitutes a name of \(x\). Conversely suppose \((A_m)_{m \in N}\) is a co-r.e. name of \(x\). Then the sets

\[
A'_m := \{ R(a', m) \mid a' \in \text{dom}(\xi) \cap [2^{D(m+3)}] \text{, } \forall a \in [2^{D(m+3)}] : a \not\in A_{m+3} \text{ or } d(\xi(a), \xi(a')) < 2^{-m-2} \}
\]
are (i) uniformly semi-decidable, (ii) non-empty, and (iii) any sequence \(a'_m \in A'_m\) constitutes a name of \(x\). Indeed (ii) to \(x \in X\) there exists \(a \in \text{dom}(\xi) \cap [2^{D(m+3)}]\) with 
\[
d(\xi(a'), x) \leq 2^{-m-4}, \text{ while every } a \in A_{m+3} \text{satisfies } d(\xi(a), x) \leq 2^{-m-2}. \text{ Conversely (iii) every } a \in A_{m+3} \text{satisfies } d(\xi(a), x) \leq 2^{-m-3}; \text{ hence any } R(a', m) \in A'_m \text{ has } d(\xi(a'), x) < 3 \cdot 2^{-m-3} \text{ and } d(\xi(R(a'), m), x) < 2^{-m-3}.\]

c) Let \(A_m \subseteq [2^{D(m)}] \cap \text{dom}(\xi)\) be a uniformly recursive name of compact non-empty \(W \subseteq X\). Then, starting with any \(a_0 \in A_2,\) one can iteratively computationally search for, and according to Remark 5), is guaranteed to find, some \(a_m \in A_{m+2}\) with 
\[
d(\xi(a_m), \xi(a)) < 2^{-m-1} + 2^{-n} \text{ for all } n < m; \text{ recall that strict inequality of distances is semi-decidable according to Definition 4). Then } u_m := R(a_m, m) \in \text{dom}(\xi) \cap [2^{D(m)}]  \text{ satisfies } d(\xi(u_m), \xi(u_n)) < 2^{-m} + 2^{-n} \text{ and } d_W(\xi(u_m)) \leq 3 \cdot 2^{-m-2} < 2^{-m} \text{ by triangle inequality; hence } x := \lim_m \xi(u_m) \in W.\]

d) Fix \(n \in N\) and recall from Remark 5) that a machine \(\mathcal{A}\) computing a name \(\Lambda^*\) of \(\Lambda\), when presented with any sequence \(u = (u_m)_{m \in N} \in W_{\xi,\xi}\) for some \(w \in W\), produces according to Definition 4) some \(v_m = \Lambda_u \in \text{dom}(\nu)\) with \(e(\Lambda(x), v(\nu_m)) \leq 2^{-n}\); after a finite number \(T = T_{\lambda}(n, \bar{u})\) of steps and in particular ‘knowing’ no more than the first entries \(T\) of \(u\). The thus defined function \((T(n, \cdot)) : W_{\xi,\xi} \to N\) (implicitly depending on \(\mathcal{A}\)) is therefore locally constant, that is, continuous: \(T(n, \bar{u}) = T(n, \bar{u}')\) whenever \(\beta(\bar{u}, \bar{u}') \leq 2^{-T(n, \bar{u})} \).

Now Fact 5) asserts \(W_{\xi,\xi} \subseteq N^N\) to be compact; hence, by Fact 3), \((T, n, \cdot)\) is bounded by some integer \(T(n)\) (again depending also on \(\mathcal{A}\)). We show that \(n \to T(n + 1) + 1\) constitutes a binary modulus of continuity of \(\Lambda|_{W} : \text{Fix } x \in W \text{ and consider for each } m \in N \text{ some } u_m \in [2^{E(m)}] \text{ with } d(x, \xi(u_m)) \leq 2^{-m-1}\) according to Definition 4). For this particular \(\bar{u} \in x_{\xi,\xi}\), every \(x' \in \mathcal{B}(x, 2^{-m-1})\) has \(d(x', \xi(u_m)) \leq 2^{-m}\) for all \(m' \leq m\); indeed \(x'\) admits a sequence \(\bar{u}' \in x'_{\xi,\xi}\), which coincides with \(\bar{u}\) on the first \(m\) entries. And for \(m \geq T(n) \geq T(n, \bar{u})\), by definition, \(A\)'s output \(\bar{v}\) on input \(\bar{u}\) coincides up to position \(n\) with its output \(\bar{v}'\) on input \(\bar{u}'\). Triangle inequality thus yields 
\[
d(x, x') \leq 2^{-T(n - 1)} \Rightarrow e(\Lambda(x), \Lambda(x')) \leq 2^{-n - 1}.
\]
e) Let \((A_m)_{m \in N}\) denote a co-r.e. name for \(W\). According to Remark 5), \(W_{\xi,\xi}\) coincides with the set

\[
\{ \bar{u} = (u_0, \ldots, u_n) \mid n \in N, \forall j \leq n: u_j \in \text{dom}(\xi) \cap [2^{E(j)}] \wedge d(\xi(u_i), \xi(u_j)) \leq 2^{-i} + 2^{-j} \wedge \forall m \exists a_m \in A_m : d(\xi(a_m), \xi(u_j)) \leq 2^{-m} + 2^{-j} \}
\]
which is clearly co-r.e. with “\(\forall m\)” as only unbounded quantifier and co-r.e. inequality “\(d(\xi(u_i), \xi(u_j)) \leq 2^{-i} + 2^{-j}\)”. Indeed, for every \(\bar{u} = (u_0, \ldots, u_n)\) according to Equation 5) and fixed \(j \leq n\), there exists a sequence \(a_m \in A_m\) such that, one the one hand, \(d_W(\xi(u_m)) \leq 2^{-m} \to 0\) hence \(\xi(a_m) =: x \in W\) for some subsequence; while on the other hand \(2^{-j} \to 2^{-m} + 2^{-j} \geq d(\xi(a_m), \xi(u_j)) \to d(x, \xi(u_j))\). This asserts \(\bar{u}\) to extend to some \(\bar{u} \in x_{\xi,\xi}\).

To find a recursive time bound \(T' \geq T\) to \(d\), extend the partial algorithm \(\mathcal{A}\) computing \(\Lambda^*\) from compact \(W_{\xi,\xi}\) to \(A'_m\), accepting inputs \(\bar{u}\) from the entire set \(\prod_{m \geq 0}[2^{E(m)}]\) by simultaneously to executing \(\mathcal{A}(\bar{u})\) until it prints the \(n\)-th output symbol, trying to refute \(\bar{u} \in W_{\xi,\xi}\) just established as co-r.e. Noting that \(A'_m\) indeed terminates on all possible inputs from compact \(\prod_{m \geq 0}[2^{E(m)}] \subseteq N^N\) and therefore \(d\) in some time bound \(T'(n) \geq T(n)\) depending only on \(n\), the following algorithm computes such \(T'(n)\):

Initialize \(T' := 1\). Simulate \(A'_m\) on each \(\bar{u} \in \prod_{m = 1}^{n-1}[2^{E(m)}] \subseteq N^T\) until either (i) it terminates or
(ii) reads past the finite input. In case (ii), increase $T'$ and restart; else output $T'$ and terminate.

g) The image $\Lambda[W] \subseteq Y$ is compact by Fact [3b]. To see its computability, fix a recursive name $(A_m)_m$ of $W$, a recursive binary modulus of continuity $\mu : \mathbb{N} \to \mathbb{N}$ according to (e), and uniformly recursive $\Lambda^{\mu}_\mu : A_{\mu(m)} \to \text{dom}(v) \cap [2^E[n]]$ according to Remark [8]. Now consider the set

$$B_m := \{ \Lambda^{\mu}_\mu(a) \mid a \in A_{\mu(m)} \}$$

which is clearly uniformly decidable since $\mu$ and $A_{\mu(m)}$ is. We show that $(B_m)$ constitutes a name of $\Lambda[W]$. To every $y = \Lambda(x)$ with $x \in W$ there exists by hypothesis some $a \in A_{\mu(m)}$ with $d(\Lambda(x), a) \leq 2^{-m(m)}$, hence $e\left(v(\Lambda^{\mu}_\mu(a)), y\right) \leq 2^{-m}$ by choice of $\mu$ and $\Lambda^{\mu}_\mu$. Conversely, every $\Lambda^{\mu}_\mu(a) \in B_m$ arises from some $a \in A_{\mu(m)}$ and in turn some $x \in W$ with $d(\xi(a), x) \leq 2^{-m(m)}$, hence $e\left(v(\Lambda^{\mu}_\mu(a)), y\right) \leq 2^{-m}$ for $y := \Lambda(x) \in \Lambda[W]$.

h) By Remark [8], suppose w.l.o.g. that $(A_m)$ is a co-r.e. standard name of $W$ and $(B_n)$ one of $X \setminus R^c$. We show that every family $(C_m)$ with

$$A_m \subseteq C_m \subseteq \{ u \in \text{dom}(\xi) \cap [2^{D(m)}] \mid \exists n > m \exists v \in \text{dom}(\xi) \cap [2^{D(n)}] \mid B_n \cdot d(\xi(u), \xi(v)) < 2^{-m} \}$$

constitutes a name of $W$: Since the right-hand side of Equation (6) is r.e., the claim then follows with Remark [9]. Indeed, every element $u$ of the right-hand side arises from some $v \notin B_n$ with $d(\xi(u), x) > 2^{-m}$ for $x := \xi(v)$. $(B_n)$ being a standard name of $X\setminus X^0$ implies $d_{W^X}(x) > 2^{-m}$ and in particular $x \in W^\circ \subseteq W$.

On the other hand for every $u \in A_m$, being a standard name implies $d(\xi(u), x) < 2^{-m}$ for some $x \in W = W^\circ$. Hence there exists some $y \in W^\circ$ with $d(x, y) < \varepsilon := 2^{-m} - d(x, \xi(u))$; and in turn some integer $n > m$ such that $2^{-n-1} < \varepsilon - d(x, y)$ and $B(y, 3 \cdot 2^{-n-1}) \subseteq W^\circ$ holds; and in turn some $v \in \text{dom}(\xi) \cap [2^{D(n)}]$ with $d(y, \xi(v)) < 2^{-n-1}$. Then $d(\xi(v), \xi(u)) < 2^{-n-1}$ and $B(\xi(v), 2 \cdot 2^{-n-1}) \subseteq B(y, 3 \cdot 2^{-n-1}) \subseteq W^\circ$ implies $d_{W^X}(\xi(v)) \leq 2 \cdot 2^{-n-1} = 2^{-n}$ hence $v \notin B_n$, as the latter is a standard name of $X \setminus W^\circ$. This demonstrates that every $u \in A_m$ is an element of the right-hand side of Equation (6).

j) Fix a recursive name $(A_{\mu(m)})_m$ of $W$, a joint recursive time bound and binary modulus of continuity $\mu : \mathbb{N} \to \mathbb{N}$ of $\Lambda$ according to Theorem [4], and uniformly recursive $\Lambda^{\mu}_\mu : A_{\mu(m)} \to \text{dom}(v) \cap [2^E[n]]$ according to Remark [9]. Then the sets $$\text{graph}(\Lambda)_m := \left\{ \left( \xi(R(u, m)), v\left(S\left(\Lambda^{\mu}_\mu(u), m\right)\right) \right) \mid u \in A_{\mu(m + 2)} \right\} \subseteq X \times Y$$

are of the form $(\xi \times v)[C_m]$ for uniformly recursive $C_m \subseteq [D(m) \cdot E(m)]$ and satisfy $(d \times e)_{H}\left(\text{graph}(\Lambda), \text{graph}(\Lambda)_m\right) \leq 2^{-m}$. The converse claim follows from Theorem [10].

C. Exponential Objects and Higher-Type Computation

This subsection generalizes Theorem [7] uniformly, that is, with $(W, \Lambda)$ not fixed but given as input: taken from the Cartesian product (Example [3b]) of the Hausdorff hyper-space $K(X)$ over $X$ for $W$, and for $\Lambda : X \to Y$ from some closed hyper-space of equicontinuous functions to another compact metric space $Y$: such as to render this new input space in turn compact (Fact [3]). The buzzword ‘hyper’ here stresses our climbing up the continuous type hierarchy:

Remark 8: For $(X, d)$ a compact metric space, and borrowing notation to hint at the dual of a topological linear space, write $(X', d_\omega)$ for the compact hyper-space $X' := \text{Lip}_1(X, [0; 1])$ of non-expansive real functions.

a) If $\text{diam}(X) = 1$, then $X$ embeds isometrically into $X'$ via $i : x \mapsto d(x, \cdot)$. 

b) If $X'$ maps onto $X$ as $f : X' \to X$ with $d_\omega(f(x, y), f(x', y')) < d_\omega(f(x, y), f(x', y'))$, then $f$ is an isometry.
b) In this sense, \( X \) is a proper subset of \( X' \) since there exists no isometry from \( X' \) to \( X \) for reasons of entropy:

Consider \( Z \subseteq X \) (non-empty and finite but) of maximum cardinality such that it holds \( \forall z', z \in Z : z = z' \lor d(z, z') \geq 1 \). Then every \( F : Z \rightarrow \{0, 1\} \) is 1-Lipschitz and extends to some \( \tilde{F} \in X' \). [Juut02], thus having mutual supremum distance \( \geq 1 \). This gives rise to \( \text{Card}(Z) \geq \text{Card}(Z) \) distinct such \( \tilde{F} \); Mapping them isometrically to \( X \) would violate maximality of \( Z \subseteq X \).

c) On the other hand every compact space, and in particular \( X' \), is well-homeomorphic to some compact subset of the Hilbert Cube \( \prod_{j \in \mathbb{N}} [0; 2^{-j}] =: X \). So in this topological (rather than metric) sense \( X' \) may actually admit an embedding into \( X \).

d) For \( X = [0; 1]^d \), however, \( X' \) is not homeomorphic to (a subset of) \( X \): Fix \( k \in \mathbb{N} \) and for \((y_1, \ldots, y_k) \in [0; 1]^k \) let \( f_g : [0; 1]^d \rightarrow [0; 1] \) denote the piecewise linear function with \( f_g(x, y) \equiv y_j/k \). Then \( \Psi_k : [0; 1]^k \rightarrow [0; 1]^d \) is well-defined, injective, and continuous: an embedding \( \Phi : X' \rightarrow X \) would thus yield a continuous injective \( \Phi \circ \Psi_k : [0; 1]^k \rightarrow [0; 1]^d \); contradicting Invariance of Domain for \( k > d \).

We now turn compact hyper-space \( \mathcal{K}(X) \) into a computably compact metric space, such that any name of \( W \in \mathcal{K}(X) \) in the sense of Definition 3) is the binary encoding of a name of \( W \subseteq X \) in the sense of Definition 2), and vice versa:

Definition 9: a) For computably compact metric space \((X, d, \xi, D)\), consider \((\mathcal{K}(X), d_H, \xi_H, D^2)\) with

\[
\xi_H : \subseteq \mathbb{N} \ni \sum_{j \geq 0} b_j \cdot 2^j \mapsto \{\xi(j) : b_j = 1\} \in \mathcal{K}(X)
\]

for \( b_j \in \{0, 1\} \) in case \( \emptyset \neq \{j : b_j = 1\} \subseteq \text{dom}(\xi), \sum_{j \geq 0} b_j \cdot 2^j \not\in \text{dom}(\xi_H) \) otherwise.

b) Let \( \mathcal{C} \subseteq X \times Y, \mathcal{C}(W, Y) \) denote the set of partial functions \( \lambda : X \rightarrow Y \) with compact domain; similarly for \( \mathcal{C}_\mu \). c) Consider the continuous embedding

\[
\mathcal{C}(\subseteq X, Y) \ni \lambda \mapsto \text{graph}(\lambda) \in \mathcal{K}(X \times Y),
\]

justified by Fact 2(h), by Theorem 7), and particularly by Item e) of the following uniform result:

Theorem 10: Let \((X, d, \xi, D), (Y, e, v, E)\) be computably compact metric spaces with recursive rounding functions.

a) The union mapping \( \mathcal{K}(X) \times \mathcal{K}(X) \ni (V, W) \mapsto V \cup W \in \mathcal{K}(X) \) is computable.

b) The mappings \( X \ni x \mapsto \{x\} \in \mathcal{K}(X) \) and \( \mathcal{K}(X) \ni \{\{x\} : x \in X\} \ni \{x\} \mapsto \{x\} \in \mathcal{K}(X) \) are computable.

c) There is a computable mapping converting any given name of some \( W \in \mathcal{K}(X) \) into a standard name of the same \( W \).

d) For computable \( W \in \mathcal{K}(X) \) and recursive strictly increasing \( \mu : \mathbb{N} \rightarrow \mathbb{N} \), graph \((\mathcal{C}_\mu(W, Y))\) is a computable compact subset of \( \mathcal{K}(X \times Y) \), i.e. a computable point in \( \mathcal{K}(\mathcal{K}(X \times Y)) \).

e) Partial function evaluation is computable, that is, the mapping

\[
\mathcal{K}(X \times Y) \times X \ni \{\text{graph}(\lambda), x\} \mapsto \mathcal{K}(W, Y) \ni \lambda \in \mathcal{C}(W, Y), W \in \mathcal{K}(X), x \in W \}
\]

f) The evaluation algorithm from (e) admits a uniformly computable multivalued runtime bound \( T(\lambda, n) \), i.e., depending only on \( \lambda \) and the output precision \( n \), that is simultaneously a binary modulus of continuity:

\[
T : \mathcal{K}(X \times Y) \times \mathbb{N} \ni \{\text{graph}(\lambda), x\} \mapsto \mathcal{K}(W, Y) \ni \lambda \in \mathcal{C}(W, Y), x \in W \}
\]

\[
\forall x, x' \in \text{dom}(\lambda) : d(x, x') \leq 2^{-m} \lor e(\lambda(x), \lambda(x')) \leq 2^{-n}
\]

g) Function restriction is computable, i.e. the mapping

\[
\mathcal{K}(X \times Y) \times \mathcal{K}(X) \ni \{\text{graph}(\lambda), W\} \mapsto \mathcal{K}(W, Y) \ni \lambda \in \mathcal{C}(W, Y), W \in \mathcal{K}(X), V \subseteq W \}
\]

\[
\forall x \in \text{dom}(\lambda) : d(x, \lambda(x)) \leq 2^{-m} \lor e(\lambda(x) \lambda(x')), \lambda(x')) \leq 2^{-n}
\]

h) Type conversion is also computable: partial evaluation

\[
\mathcal{C}(\subseteq X \times Y, Z) \times X \ni (\Lambda, x) \mapsto \lambda(x, \cdot) \in \mathcal{K}(\subseteq Y, Z)
\]
as well as the converse, \( \text{un-‘Schönfinkeling’} \) [Stra00, p.21].

j) And so is function image

\[
\mathcal{C}(\subseteq X \times Y, X) \ni (\Lambda, W) \mapsto \lambda(W) \in \mathcal{K}(Y)
\]

k) Suppose \( \Phi : X \rightarrow Y \) is computable and open in that images \( \Phi(U) \subseteq Y \) of open \( U \subseteq X \) are open again. Then the restricted pre-image mapping

\[
\mathcal{K}(X \times Y) \ni V \mapsto \Phi^{-1}(V) \in \mathcal{K}(X)
\]
is well-defined and computable.

Here we denote by \( \mathcal{K}(X) = \{W \subseteq X : W = W^\circ\} \subseteq \mathcal{K}(X) \) the family of so-called regular subsets of \( X \); recall Fact 1(h) and Theorem 7).

Proof of Theorem 12): Preimage of a continuous open mapping commutes with topological closure and interior: \( \Phi^{-1}[S^\circ] = (\Phi^{-1}[S])^\circ \) and \( \Phi^{-1}[S] = \Phi^{-1}[\overline{S}] \); cmp. [Zieg02, Lemma 4.4ab], \( \Phi^{-1}[V] \) is thus regular. Moreover both \( W := \Phi^{-1}[V] \) and \( \Phi^{-1}[Y \setminus V^\circ] = Y \setminus W^\circ \) are co-computable according to Theorem 7; hence \( W \) is computable by virtue of Theorem 7(h). This argument is non-uniform, but closer inspection shows it to hold uniformly.

III. APPLICATIONS

We apply the above considerations to two computational problems over compact metric spaces beyond the classical Euclidean case: a space of homeomorphisms (Subsection III-A), and the space of compact subsets (Subsection III-B).
A. Fréchet Distance

In 1906 Maurice Fréchet introduced a pseudo-metric for parameterized continuous curves and, in 1924, for parameterized surfaces that in various ways improves over both supremum and Hausdorff Norm:

**Definition 11**: Let \((X, d), (Y, e)\) be compact metric spaces.

a) The Fréchet Distance of two continuous mappings \(A, B : X \rightarrow Y\) is given by \(F(A, B) = \inf_{\varphi} F_{\text{id}, \varphi}(A, B)\), where

\[
F_{\alpha, \beta}(A, B) := \sup_{x \in X} e \left( A(\alpha(x)), B(\beta(x)) \right)
\]

with infimum ranging over the set \(\text{Aut}(X)\) of all homeomorphisms (i.e., continuous bijections) \(\varphi : X \rightarrow X\).

b) For \(X = [0; 1]\), the oriented Fréchet Distance \(F'(A, B)\) of continuous (not necessarily simple) curves \(A, B : [0; 1] \rightarrow Y\) is defined similarly with the infimum ranging over \(\text{Aut}'([0; 1])\): the set of all strictly increasing continuous \(\varphi : [0; 1] \rightarrow [0; 1]\) with \(\varphi(0) = 0\) and \(\varphi(1) = 1\).

c) For \(X = S^1\) the unit circle, the oriented Fréchet Distance \(F'(A, B)\) of continuous loops \(A, B : S^1 \rightarrow Y\) is defined similarly with infimum ranging over \(\text{Aut}'(S^1)\): the set of all clockwise continuous bijections \(\varphi : S^1 \rightarrow S^1\).

d) For \(X = \mathbb{B}^2\) the Euclidean unit disc, the oriented Fréchet Distance \(F'(A, B)\) of continuous 2D surfaces \(A, B : \mathbb{S}^1 \rightarrow Y\) is defined similarly with infimum ranging over \(\text{Aut}'(\mathbb{B}^2)\): the set of all continuous bijections \(\varphi : \mathbb{T}^2 \rightarrow \mathbb{T}^2\) mapping some/all clockwise simple curves in \(\mathbb{B}^2\) to clockwise image(s) [Alt09, Definition 2].

e) More generally fix a \(d\)-dimensional orientable compact manifold \(X\), i.e., with \(d\)-th homology group \(H_d(X, \mathbb{Z}) \cong \mathbb{Z}\) [Munk84, Corollary 65.4]. For any homeomorphism \(\varphi : X \rightarrow X\), the action of composition with \(\varphi\) induces an isomorphism of the \(k\)-th homology group; which for \(k = d\) can only be multiplication either by \(-1\) or by \(+1\); and the latter \(\varphi\) by definition comprise \(\text{Aut}'(X)\).

The above notions have recently received much attention — in Computational Geometry, that is, for polygonal curves and triangulated surfaces; cf. for instance [AAB*16, AHK*13, BDS14, Alt09, Goda91] and both the references and motivating examples therein — as well as for the important

**Question 12**: Without restricting to piecewise/combinatorial inputs, can the Fréchet Distance(s) be computed in the sense of Recursive Analysis, that is, by approximation up to guaranteed absolute error \(2^{-n}\) for every given \(n \in \mathbb{N}\) and every given/fixd pair of continuous/computable functions \(A, B\)?

Theorem [14] below gives a positive answer for curves \((X = [0; 1])\) and loops \((X = S^1)\) but also shows that an optimal reparameterization \(\varphi\) cannot in general be computable.

Recall (Fact[3]) that compactness and continuity guarantee infimum (e.g. in Definition [2]) to exist, be attained, and computable according to Fact [1] and Theorem [2]. Our goal is to argue similarly in Equation [7], only that the ground space here consists of functions \(\varphi\). A first naive attempt fails since \(\text{Aut}(X) \subseteq C(X, X)\) is not compact and the infimum thus not necessarily attained:

**Remark 13**: a) The pseudo-metric in Equation [7] is symmetric: \(F_{\alpha, \varphi}(A, B) = F_{\text{id}, \alpha \circ \varphi \circ \beta}(A, B)\) holds for all bijections \(\alpha, \beta : X \rightarrow X\), since the set \(\{(x, \varphi(x)) : x \in X\}\) agrees with \(\{(\alpha \circ \beta(y), \alpha \circ \varphi \circ \beta(y)) : y \in X\}\). However the inf over \(\varphi \in \text{Aut}(X)\) in Definition [11] is in general ‘attained’ only by non-injective reparameterizations (Figure [1]).

b) On the other hand, the mapping \((\alpha, \varphi) \mapsto F_{\alpha, \varphi}(A, B)\) is uniformly continuous; namely has modulus of continuity the sum of those of \(A\) and \(B\). The sets \(\text{Aut}(X)\) and \(\text{Aut}'(X)\) may thus be replaced by their topological closures, \(\text{Aut}(X)\) and \(\text{Aut}'(X)\) in \(C(X, X)\) as proper supersets, without affecting the value of \(F\) and \(F'\), respectively. However those closures still lack equicontinuity.

c) For the smooth simple curves \(A, B : [0; 1] \rightarrow \mathbb{R}^2\) depicted in Figure [1b], their (non-oriented) Fréchet Distance is attained by a continuum of homeomorphisms \(\varphi : [0; 1] \rightarrow [0; 1]\), i.e., non-uniquely.

d) \(\text{Aut}'([0; 1])\) is the disjoint union of the path-connected subspace of increasing homeomorphisms, i.e. those in \(\text{Aut}'([0; 1])\), and the decreasing ones; similarly for \(\text{Aut}(S^1)\). More generally, for any \(d\)-dimensional orientable compact manifold \(X\), \(\text{Aut}(X)\) decomposes into the locally arc-connected subspace \(\text{Aut}'(X)\) of orientation-preserving homeomorphisms and that of orientation-reversing ones [Sand60].

e) To every \(\varphi \in \text{Aut}'([0; 1])\) there exist 2-Lipschitz \(\psi, \chi \in \text{Aut}'([0; 1])\) such that \(\varphi = \psi \circ \chi^{-1}\); similarly for the non-oriented case.

f) There exists a constant \(K \geq 2\) such that every \(\varphi \in \text{Aut}'(S^1)\) admits a decomposition \(\varphi = \psi \circ \chi^{-1}\) with \(K\)-Lipschitz bijections \(\psi, \chi \in \text{Aut}'(S^1)\); again, similarly for the non-oriented case.

g) There exists a constant \(K \geq 2\) such that every Lipschitz-continuous \(\varphi \in \text{Aut}'(\mathbb{B}^2)\) admits a decomposition \(\varphi = \chi^{-1} \circ \beta \circ \gamma^{-1}\) with \(K\)-Lipschitz \(\alpha, \beta, \gamma \in \text{Aut}'(2\mathbb{B}^2)\); similarly for the non-oriented case. Here, using Minkowski operations, \(\mathbb{B}^2 + \mathbb{B}^2 = 2\mathbb{B}^2 = \mathbb{B}(0, 2) \subseteq \mathbb{R}^2\) denotes the closed Euclidean ball around center 0 with radius 2.

h) Picking up on b), extend the definition of \(F_{\alpha, \beta}(A, B)\) according to Equation [7] from continuous functions \(\alpha, \beta : X \rightarrow X\) to compact relations \(\alpha, \beta \subseteq X \times X\) as

\[
\sup \left\{ e(A(a), B(b)) \mid \exists x \in X : (x, a) \in \alpha, (x, b) \in \beta \right\}.
\]

Then \((\alpha, \varphi) \mapsto F_{\alpha, \varphi}(A, B)\) still remains continuous w.r.t.
the Hausdorff metric on \( \mathcal{K}(X \times X) \times \mathcal{K}(X \times X) \), and it holds
\[ F(A, B) = \inf_{\varphi} F_{\varphi}(A, B, \alpha, \beta) = \inf_{\alpha, \beta} F_{\alpha, \beta}(A, B) \]
with infimum taken over graph(\( \text{Aut}(X) \)) \( \subseteq \mathcal{K}(X \times X) \); similarly for \( F' \) and \( \text{Aut}'(X) \); see Figure 2.

For motivation, consider a bounded uniformly continuous functional \( \Phi : \mathbb{P} \to \mathbb{R} \) on non-compact \( \mathbb{P} = \{ (\alpha, \varphi) : \alpha, \varphi > 0 \} \) but satisfying ‘scaling invariance’ \( \Phi(\alpha, \varphi) = \Phi(1, \varphi/\alpha) \).

Then it obviously suffices to consider \( (\alpha, \varphi) \in [0, 1]^2 \); a compact space. Items e+f are similar to a similar property for \( (\alpha, \varphi) \to e_{\alpha, \varphi}(A \circ \alpha, \beta \circ \varphi) \), but without computability.

**Theorem 14:** Let \( (Y, e, v, E) \) denote a computably compact space of \( \text{diam}(Y) \leq 1 \).

a) The compact set graph \( \text{graph}(\text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1])) \subseteq \mathcal{K}([0, 1]^2) \) of graphs of non-decreasing 2-Lipschitz \( \varphi : [0, 1] \to [0, 1] \) with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) is computable.

b) Non/oriented Fréchet Distances between continuous paths \( F, F' : C([0, 1], Y)^2 \to [0, 1] \) are computable.

c) The same holds for non/oriented Fréchet Distances between continuous loops \( F, F' : C(S^1, Y)^2 \to [0, 1] \) are computable.

d) There exist computable smooth \( A, B : [0, 1] \to [0, 1] \) and strictly increasing homeomorphism \( \varphi : [0, 1] \to [0, 1] \) such that \( A \circ \varphi \) holds but no computable non-decreasing surjection \( \varphi \) satisfies \( A = B \circ \varphi \); nor does any computable non-increasing surjection \( \varphi \).

e) There exist computable smooth \( \text{simple (}=\text{injective}) \tilde{A}, \tilde{B} : [0, 1] \to [0, 1]^2 \), codomain equipped with the 2D maximum norm, such that \( F'(\tilde{A}, \tilde{B}) = 1 = F(\tilde{A}, \tilde{B}) \) is attained by some strictly increasing homeomorphism \( \varphi : [0, 1] \to [0, 1] \) but by no computable non-decreasing/non-increasing surjection \( \varphi \).

Regarding higher dimensions, [AlBu10] **Theorem 1** has asserted at least left/upper semi-computability; recall the paragraph following Fact 1. Computationally enumerating a sequence \( \varphi_n \) dense in separable \( \text{Aut}(X) \subseteq C(X, X) \), together with computability and continuity of \( A, B, \epsilon_{\alpha, \varphi} \) yields a computable sequence \( F_{\alpha, \varphi}(A, B) \) whose infimum coincides with \( F(A, B) \); and for \( \varphi_n \) ranging over a compact space, its covering property asserts that finitely many (balls centered around) them suffice to approximate \( F(A, B) \) also from below.

**Proof of Theorem 7.**

a) Recall that a name of continuous \( \varphi : [0, 1] \to [0, 1] \) is a family of finite sets \( C_m \subseteq \mathbb{D}_m \times \mathbb{D}_m \) approximating graph(\( \varphi \)) in Hausdorff metric. Now it is easy to enumerate, uniformly in \( m \in \mathbb{N} \), all those \( C_m \) satisfying the following condition: \( C_m \) is a ‘chain’ of points in the sense of Go (aka [Mb]), starting at the lower left corner and proceeding to the upper right such that at least every second step ‘up’ is followed by one ‘right’; see Figure 2, illustrating the idea (that we deliberately refrain from formalizing further). Then the graph of every \( \varphi \in \text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1]) \) has Hausdorff distance at most \( 2^{-m} \) to some such \( C_m \); and conversely every such \( C_m \) has distance at most \( 2^{-m} \) to the graph of some \( \varphi \in \text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1]) \). The collection \( C_m \) of all those \( C_m \subseteq [0, \infty] \times [0, \infty] \) with \( m \in \mathbb{N} \) thus constitutes a name of graph \( \text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1]) \) \( \subseteq \mathcal{K}([0, 1]^2) \).

b) By Remark 13b+e, \( F'(A, B) \) coincides with \( \inf_{\chi, \psi} F_{\chi, \psi}(A, B) \), where the infimum ranges over the closet subset \( \text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1]) \times \text{Lip}_2([0, 1]; [0, 1]) \cap \text{Aut}'([0, 1]) \) of computably compact \( \text{Lip}_2([0, 1]; [0, 1]) \times \text{Lip}_2([0, 1]; [0, 1]) \). Moreover said subset is computable by a); and so is the mapping \( (\chi, \psi) \to F_{\chi, \psi}(A, B) \) on it. Hence Theorem 7 asserts its image to be a computable subset of \( [0, 1] \), whose minimum is computable according to Fact 1. The non-oriented case proceeds similarly according to Remark 13i).

c) By Remark 13b+t and regarding that \( \text{Lip}_K(S^1, S^3) \times \text{Lip}_K(S^3, S^1) \times \text{Lip}_K(S^1, S^3) \cap \text{Aut}'(S^1) \) is a computable subset of compactly \( \text{Lip}_K(S^1, S^3) \times \text{Lip}_K(S^3, S^1) \) as domain of computable mapping \( \{ \chi, \psi \to F_{\chi, \psi}(A, B) \} \). The non-oriented case proceeds similarly.

**Proof of Remark 13.**

e) Let \( \varphi \) be non-decreasing. Then the continuous and surjective mapping \( \tilde{\varphi} : [0, 1] \ni t \mapsto (t + \varphi(t))/2 \in [0, 1] \) satisfies, for \( t \geq t' \),
\[
\tilde{\varphi}(t) - \tilde{\varphi}(t') = (t - t')/2 + (\varphi(t) - \varphi(t'))/2 \geq (t - t')/2
\]
and hence is strictly increasing with 2-Lipschitz inverse \( \chi \in \text{Aut}'([0, 1]) \). It remains to observe that \( \psi := \varphi \circ \chi \) is 2-Lipschitz since, again for \( t \geq t' \),
\[
\psi(t) - \varphi(t') \leq \varphi(t) - \varphi(t') \leq \varphi(t) - \varphi(t') + t - t' = 2 \cdot (\varphi(t) - \varphi(t')/2).
\]
f) Appyling an isometric rotation we may w.l.o.g. suppose \( \varphi(1) = 1 \). Since \( S^1 \) is homeomorphic to \( [0, 1] \mod 1 \), this reduces to e).

g) According to f), the restriction of \( \varphi \) to the ball’s boundary admits a decomposition \( \varphi|_{S^1} = \psi \circ \chi^{-1} \) with \( K \)-Lipschitz homeomorphisms \( \psi, \chi : S^1 \to S^1 \). Note that \( \bar{x} \mapsto \bar{x}/|\bar{x}|_2 \) is \( L \)-Lipschitz outside the Euclidean disk \( B^d(0, 1/L) \) of radius \( 1/L \). Thus, applying Alexander’s Trick, \( \chi \) extends radially to a \( 2K \)-Lipschitz homeomorphism of entire \( B^d \).
and of $2\overline{B}$, via $\chi(\overline{x}) := \chi(\overline{x}/|\overline{x}|_2)\cdot |\overline{x}|$. So $\varphi \circ \chi|_{\overline{B}} \in Aut(\overline{B})$ coincides on $S^1$ with $\psi$ and in particular is $2K$-Lipschitz there. Now abbreviate $L := 2K$ and define

$$
\beta : \overline{x} \mapsto \left\{ \begin{array}{ll}
\varphi \circ \chi(\overline{x})/L : |\overline{x}|_2 \leq 1 \\
bl(|\overline{x}|_2) \cdot \varphi \circ \chi(\overline{x}/|\overline{x}|_2) : |\overline{x}|_2 \geq 1
\end{array} \right.
$$

for affine $bl(r) := (2 - 1/L)\cdot r + 2/L$ and $aL(s) := L\cdot s + 2L - 2$; cmp. Figure 5 Then, $bl_L : [1; 2] \to [\frac{1}{L}; 2]$ constituting an increasing bijection, implies that $\beta : 2\overline{B} \to 2\overline{B}$ is well-defined, continuous, injective, surjective, and $2L$-Lipschitz. Similarly, $aL : [\frac{1}{L}; 2] \to [1; 2]$ constituting an increasing bijection, implies that $\tilde{\alpha} : 2\overline{B} \to 2\overline{B}$ is well-defined, continuous, injective, and surjective with $2L$-Lipschitz inverse $\alpha := \tilde{\alpha}^{-1}$ since $aL \circ bl_L = id_{[1;2]}$. Finally, $\varphi \circ \chi = \tilde{\alpha} \circ \beta|_{\overline{B}}$ is easily verified.

B. Shape Optimization

This subsection exhibits weak conditions that assert computability of the following generic optimization problem:

**Definition 15**: For fixed $X$ and given $\Lambda, \Phi : X \to \mathbb{R}$, determine the real number $r := \max \{x \in \overline{\Phi}^{-1}([-\infty; 0])\}$, provided it exists. $\Lambda$ is the objective function, $r$ its optimum w.r.t. constraint $\Phi \leq 0$, the latter called feasible if $0 \in \Phi[X]$.

To guarantee said existence, it suffices that (i) $X$ be compact, (ii) $\Lambda$ be continuous, and (iii) $\Phi$ be lower semi-continuous: Condition (iii) asserts that the non-empty domain $\Phi^{-1}([-\infty; 0]) \subseteq X$ is closed and thus compact by (i), hence (ii) $\Lambda$ attains its maximum on it. Conversely simple counterexamples show that none of the three conditions can hence (ii) be omitted.

**Example 16**: a) Linear Optimization refers, up to scaling, to the case $X = [0; 1]^d$ with $\Lambda : X \to \mathbb{R}$ linear together with a finite conjugation of (w.l.o.g. non-constant) linear constraints $\Phi_j : \overline{x} \mapsto b_j + \overline{x} \cdot \overline{a}_j$, $\overline{a}_j \neq 0$, collected into the single $\Phi := \max_j \Phi_j$. Note that $\Lambda$ and $\Phi$ are continuous, and $\Phi$ is furthermore open.

b) Convex Optimization refers, again up to scaling, to the case $X = [0; 1]^d$ but now permits the generalization to convex negated objective functions $\Lambda : X \to \mathbb{R}$ and non-constant constraints $\Phi_j$, again subsumed in the (still convex) single $\max_j \Phi_j$.

c) In Discrete Optimization, $X$ is a finite but ‘large’ set. Equipped with the discrete topology, it renders every $\Lambda, \Phi : X \to \mathbb{R}$ continuous.

d) *Shape Optimization* refers, again up to scaling, to the case $X = K([0; 1]^d)$ with $\Lambda, \Phi \in C(X)$; cmp. [HepPos].

e) The (Lebesgues) measure, considered as mapping $\Vol_d : K\mathcal{R}([0; 1]) \to [0; 1]$, is upper semi-continuous but not continuous nor computable.

Recall $K\mathcal{R}(X) = \{W \subseteq X : W = \overline{W}\}$. For a fixed convex compact subset $X$ of a Fréchet (i.e. complete translation invariant metric) space, write $K\mathcal{C}(X) \subseteq K(X)$ for the family of its convex compact non-empty subsets.

Shape Optimization has recently grown a hot topic in Numerical Engineering [SoZo99] but generally lacks mathematical specification and rigorous algorithmic analysis. We establish that this, as well as the generic optimization problem from Definition 15 is computable for (i’) computably compact metric space $X$, (ii*) computable objective function and (iii*) open computable constraint:

**Theorem 17**: Let $(X, \delta, \xi, D)$ and $(Y, v, E)$ denote computably compact metric spaces.

a) For $X$ any convex compactly computable subset of Euclidean space with non-empty interior, $K\mathcal{C}(X) \subseteq K(X)$ is in turn computably compact.

b) The generic optimization problem

$$
\{X, [0; 1]\} \times (C(X, [0; 1]) \cap O_0(X)) \ni (\Lambda, \Phi) \mapsto \max \Lambda \left[\Phi^{-1}([-1; 0])\right] \in [0; 1]
$$

is computable, where $O_0(X)$ denotes the family of open $\Phi : X \to \mathbb{R}$ with $0 \in \Phi[X]$.

c) As opposed to Example 16b), the mapping $\Vol_d : K\mathcal{C}([0; 1]^d) \to [0; 1]$ is (continuous and) computable.

d) The mapping $\Vol_d \left[\{0; 1]^d\right] \ni W \mapsto A_{d-1}(\partial W) \in [0; 2d]$ is well-defined, open, (continuous and) computable, where $A_{d-1}(\partial W)$ denotes the area measure of $W$’s boundary.

Claim a) is a minor strengthening of Blaschke’s Selection Theorem; b) follows by combining Theorem 10b+k) with Fact 1) since $[-1; 0]$ is regular.

As an example ‘application’, consider the classical Isoperimetric Problem asking to maximize $\Vol_d(W)$ subject to the constraint $A_d(\partial W) - 1 \leq 0$, say. Combining the items of Theorem 17 we conclude that the solution is computable!

On second thought this comes at no surprise, though: Knowing that the optimal shape is a Euclidean ball, the solution is easily calculated explicitly as $1/(4\pi)$ in dimension 2, $1/(6\sqrt{\pi})$ in dimension 3, and similar expressions can be derived in any dimension $d$ involving the gamma function at half-integral arguments.

IV. CONCLUSION AND PERSPECTIVES

For computably compact metric spaces $(X, \delta, \xi, D)$ and $(Y, v, E)$ in the sense of Definition 4 we have (i) turned the hyper-space $K(X)$ of non-empty compact subsets of $X$ into computably compact metric space, again; and (ii) similarly for the space $C_0(W, Y)$ of partial equicontinuous functions $\Lambda : W \to Y$ having non-empty compact domain $W \subseteq X$. The latter proceeds by identifying such $\Lambda$ with $\text{graph}(\Lambda) \in K(X \times Y)$; and was shown to render evaluation uniformly computable. This realizes well-known results for the Euclidean, to arbitrary compact metric, spaces – including a hierarchy of higher types.
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Proof of Fact [3]:

a) See [Rudi76 THEOREMS 3.11+2.35+4.16].

b) See [Rudi76 THEOREMS 4.14+4.19].

c) See [Rudi76 THEOREM 7.25], cmp. also [KSZ16 LEMMA 13d].

e) See [Prie40].

f) Observe that Theorem 7d+e) relativizes: For any oracle A,B of W and (B_m) one of graph(\Lambda) \subseteq X \times Y, where A : W \to Y has binary modulus of continuity \mu according to f), Abbreviat h_n := \mu(m+2) + 1 \geq m + 2 and verify

C_m := \{ (R(u,m), S(v,m)) | (u,v) \in B_n, u \in A_n \}

Proof of Remark [7b]:

\| e_\infty(A \circ \alpha, B \circ \varphi) - e_\infty(A \circ \check{\alpha}, B \circ \check{\varphi}) \| \leq \| e_\infty(A \circ \check{\alpha}, B \circ \check{\varphi}) - e_\infty(A \circ \check{\alpha}, B \circ \varphi) \| + \| e_\infty(A \circ \alpha, B \circ \varphi) - e_\infty(A \circ \alpha, B \circ \check{\varphi}) \| \leq e_\infty(B \circ \varphi, B \circ \check{\varphi}) + e_\infty(A \circ \alpha, A \circ \check{\alpha}) \leq \omega_B(e_\infty(\varphi, \check{\varphi}) + \omega_A(e_\infty(\alpha, \check{\alpha}))

by reverse triangle inequality for moduli of continuity \omega_A, \omega_B of A, B respectively. ■

Proof of Theorem [7d]:

a) similarly to the proof of Theorem [7b].

b) similarly to the proof of Theorem [7f+c].

c) similarly to the proof of Remark [7f].

d) similarly to the proof of Theorem [7f].

e) Let (u_m) with u_m \in dom(\xi) \cap [2^{D(m)}] be a given name of x \in W and (C_m) with C_m \subseteq (dom(\xi) \cap [2^{D(m)}] \times (dom(v) \cap [2^{E(m)}])) one of graph(\Lambda) \subseteq X \times Y for some continuous \Lambda : W \to Y. For m \in \mathbb{N}, seach for, and output, some v_n \in dom(v) \cap [2^{E(n)}] such that there exists an m \in \mathbb{N} satisfying the following:

\forall (u', v') \in C_m : \quad d(\xi(u_m), \xi(u')) > 2^{-m+1} \lor e(v(v_m), v(v')) < 2^{-n-2m-m} \quad (8)

First, Property (8) is obviously r.e.. Second, any v_n found satisfies e(v(v_n), y) < 2^{-n} for y := \Lambda(x) with x \in W s.t. d(\xi(u_m), x) \leq 2^{-m}; (d \times e)H(\Lambda(\alpha,C_m)) \leq 2^{-m} with (x,y) \in \Lambda(\alpha) yields some (u', v') \in C_m with d(\xi(u'), x), e(v'(v'), y) \leq 2^{-m}; hence d(\xi(u_m), \xi(u')) \leq 2^{-m+1} by triangle inequality, so the second part of Property (8) applies. Third, there exists some v_m \in dom(v) \cap [2^{E(n)}] with e(v(v_m), y) \leq 2^{-n-1}; and, as \Lambda is defined and continuous at x, there is some m \geq n + 3 such that every x' \in W with d(x,x') \leq 2^{-m+2} satisfies e(g, \Lambda(x')) \leq 2^{-n-2}. In particular every (u', v') \in C_m with d(\xi(u_m), \xi(u')) \leq 2^{-m-1}, having d(\xi(u'), x'), e(v'(v'), \Lambda(x')) \leq 2^{-m} for some x' \in W implies e(v(v_m), v(v')) \leq e(v(v_m), y) + e(g, \Lambda(x')) + e(A(x'), v') \leq 2^{-n-1} + 2^{-n-2} + 2^{-m} - 2^{-2m-2}. ■

f) Observe that Theorem [7d+e] relativizes: For any oracle \Omega, if W \in \mathcal{K}(X) is (co-)computable with \Omega and \Lambda : W \to Y is computable with \Omega, then \Lambda has a runtime bound/binary modulus of continuity \tau(\Lambda,) : \mathbb{N} \to \mathbb{N} computable with \Omega. Now for \Omega encoding names of W and \Lambda, any query made by the oracle machine computing said \tau(\Lambda,) can be answered by performing a look-up on the input graph(\Lambda). For instance the projection (of a name of) graph(\Lambda) \subseteq X \times Y onto X yields (a name of) dom(\Lambda) = W.

g) By c) w.l.o.g. let (A_m) denote a given standard name of V \subseteq W and (B_m) one of graph(\Lambda) \subseteq X \times Y, where A : W \to Y has binary modulus of continuity \mu according to f). Abbreviat e n := \mu(m+2)+1 \geq m+2 and verify

C_m := \{ (R(u,m), S(v,m)) | (u,v) \in B_n, u \in A_n \}

In particular, uniform convergence \mathcal{F} \ni f_n \to f implies graph(f_n) \to graph(f) in Hausdorff distance; and graph(f_n) \to graph(f) with f \in \mathcal{C}(X,Y) implies f_n \to f in uniform norm according to b).■
constituting a name of graph(Λ|v); To every \((x, y) \in \text{graph}(Λ|v)\) there exist \(u \in \text{dom}(ξ) \cap \{2^{D(n)}\}\) and \(v \in \text{dom}(v) \cap \{2^{E(n)}\}\) with \(d(x, ξ(u)), e(y, v(v)) \leq 2^{-n-1} \leq 2^{-m-1}\); hence \(u \in A_n\) and \((u, v) \in B_n\): since these are standard names. Then \(u' := R(u, m)\) and \(v' := S(v, m)\) with \((u', v') \in C_m\) have \(d(ξ(u'), ξ(u)), e(v(v'), v(v)) \leq 2^{-m-1}\), so \((d \times e)((u', v'(v')), (x, y)) \leq 2^{-m}\). Conversely, to every \((u', v') \in C_m\) there exist by definition \((u, v) \in B_n\) with \(u \in A_n\) such that \(d(ξ(u'), ξ(u)), e(v'(v'), v(v)) \leq 2^{-m-1}\); and \((x, y) \in \text{graph}(A)\) as well as \(x' \in V\) with \(d(x', ξ(u')), d(x, ξ(u)), e(y, v(v)) \leq 2^{-m}\). Hence \(d(x, x') \leq 2^{-n+1} = 2^{-d(m)}\) implies \(e(y', y) \leq 2^{-m-2}\) for \(y' := Λ(x')\); and therefore it holds \(d(x', ξ(u')) \leq 2^{-n-2}\) for \(x' \in X\), as well as \(e(y', v(v')) \leq 2^{-m-2} + 2^{-n-2} \leq 2^{-m}\). By Theorem 44.

j) Similar to g) and Theorem 44., respectively.

Proof of Theorem 44.

d) Let \(α, β : \mathbb{N} \to \mathbb{N}\) denote total recursive injective enumerations of recursively inseparable sets \(α[N], β[N] \subseteq \mathbb{N}\); cmp. Soar[6] EXERCISE 1.6.26]. Intuitively, although inputs \(m \notin α[N] \cup β[N]\) may be accepted or rejected arbitrarily, provably no total algorithm can make such a decision.

Now for each \(m \in \mathbb{N}\) abbreviate \(z(m) := 0\) for \(m \notin α[N] \cup β[N]\) and \(z(m) := 2^{-m} - \min(k:α(k)=m\land β(k)=m)\) otherwise: a computable sequence of real numbers; and so is \(x(m)\), defined as \(z(m)\) if \(m \in α[N]\) and \(x(m) := 0\) otherwise; and \(y(m) := z(m)\) if \(m \in β[N]\) and \(y(m) := 0\) otherwise. Next consider the computable 1-Lipschitz ‘hat’ function \(ψ(x) = \max\{0, 1 - |x|\}\) or the computable smooth ‘pulse’ function \(ψ(x) = \exp\left(\frac{1}{x^2}\right)\) for \(|x| \leq 1\), \(ψ(x) := 0\) for \(|x| \geq 1\): both have support \([-1, 1]\), see Figure 3.

Finally define \(A(x)\) and \(B(x)\), respectively, as

\[
\begin{align*}
\sum_m ψ(2^{m+3} \cdot x - 7)/2^{m+3} + ψ(2^{m+4} \cdot x - 10) \cdot z(m), \\
\sum_m ψ(2^{m+3} \cdot x - 7)/2^{m+3} + ψ(2^{m+5} \cdot x - 19) \cdot x(m) \\
+ ψ(2^{m+5} \cdot x - 21) \cdot y(m)
\end{align*}
\]

and note that all terms in each sum have disjoint supports. Moreover, the thus well-defined \(A, B : [0; 1] \to [0; 1]\) are continuous (particularly at 0) and even smooth, since the \(d\)-th derivative of the \(m\)-th term is bounded by \(z_m \cdot (2^{m+4})^d \leq 2^{-m^2 + m + 4d} \to 0\) as \(m \to \infty\). Next, it holds \(A = B \circ φ\) for the increasing continuous surjection \(φ : [0; 1] \to [0; 1]\) mapping the closed interval \([6 \cdot 2^{-m-3}; 11 \cdot 8 \cdot 2^{-m-3}]\) isometrically to \([6 \cdot 2^{-m-3}; 11 \cdot 8 \cdot 2^{-m-3}]\) to match the ‘large’ pulses as well as the ‘small’ ones by mapping \([17 \cdot 2^{-m-5}; 21 \cdot 2^{-m-5}]\) in case \(m \in α[N]\) and to \([19 \cdot 2^{-m-5}; 23 \cdot 2^{-m-5}]\) in case \(m \in β[N]\); arbitrarily in case \(m \notin α[N] \cup β[N]\). Conversely, no non-increasing surjective \(φ\) can make the large pulses match nor satisfy \(B = A \circ φ\); and any non-decreasing \(φ\) that does, must necessarily map \(10 \cdot 2^{-m-4}\) to \(19 \cdot 2^{-m-5}\) in case \(m \in α[N]\), to \(21 \cdot 2^{-m-5}\) in case \(m \in β[N]\), and anywhere in \([8 \cdot 2^{-m-4}; 12 \cdot 2^{-m-4}]\) in case \(m \notin α[N] \cup β[N]\). A computable such (total) \(φ\) would allow for a recursive separation of \(α[N]\) from \(β[N]\): Given \(m \in \mathbb{N}\), accept if the \(2^{-m-6}\)-approximation to \(φ(10 \cdot 2^{-m-4})\) is less than \(20 \cdot 2^{-m-5}\) and reject otherwise: Contradiction.

e) Continuing d), consider computable injective \(A : [0; 1] \ni x \mapsto (x, A(x)) \in [0; 1]^2\) and \(B(x) := (x, B(x) + 1) \in [0; 1]^2\): The offset in \(y\)-coordinate dominates their pointwise distance in the 2D maximum norm over local reparametrisation in \(x\)-coordinate.

Proof of Theorem 44.

c) \(d\)-dimensional volume of convex subsets of \([0; 1]^d\) is \(2^d\)-Lipschitz continuous, worst-case depicted in Figure 4. The unit hypercube has volume 1, shrinking it by \(ε := 2^{-m}\) on each side yields volume \((1 - 2ε)^d \approx 1 - 2dε\). Thus, for \(C_m \subseteq \mathbb{D}_m^d\) approximating \(W \in \text{KC}([0; 1]^d)\) up to error \(2^{-m}\) in Hausdorff distance, the volume of the convex hull of \(C_m\) approximates the volume of \(W\) up to error \(2d \cdot 2^{-m}\).

d) Slightly ‘pulling’ any extreme point of convex compact \(W \subseteq [0; 1]^d\) in/outwards, yields a proper sub/superset \(W'\) which is again convex compact but with area of the boundary strictly larger/smaller than that of \(W\), cmp. math.stackexchange.com/questions/262568: This shows that surface area measure is open a mapping; cmp. Figure 4. The latter also depicts a worst-case to
4d(d − 1)-Lipschitz continuity of $K\mathcal{C}([0; 1]^d) \ni W \mapsto$

$$\Rightarrow \text{Area}(\partial W) = \liminf_{\delta \to 0} \frac{\text{Vol}_d(W + \delta \cdot \mathcal{B}^d) - \text{Vol}_d(W)}{\delta}$$

according to Minkowski-Steiner: The unit hypercube has surface area $2d$, shrinking it by $\varepsilon := 2^{-m}$ yields surface area $2d \cdot (1 - 2\varepsilon)^{d-1} \approx 2d - 4d \cdot (d - 1)$. Thus, for $C_m \subseteq D_d$, approximating $W \in K\mathcal{C}([0; 1]^d)$ up to error $2^{-m}$ in Hausdorff distance, the surface area of the convex hull of $C_m$ approximates that of $W$ up to $4d(d - 1) \cdot 2^{-m}$.

Fig. 5. Illustrating the proof of Remark [13k]