ON THE CONSIDERATIONS ADOPTED BY BREIDZE AND TRACZYK TOWARDS THE FAITHFULNESS OF BURAU REPRESENTATION FOR $n = 4$

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Abstract. This work discusses the open problem of the faithfulness of the reduced Burau representation for $n = 4$. Birman showed that in order to prove this representation is faithful, it is sufficient to find two matrices $A$ and $B$ that generate a free group of rank 2. Breidze and Traczyk gave a simple proof to the result found by Witzel and Zaremsky, which shows that $A^3$ and $B^3$ generate the free group of rank 2. In our work, we use the same procedure of the work of Breidze and Traczyk and show that $A^2$ and $B^2$ generate the free group of rank 2.

1. Introduction

Let $B_n$ be the braid group on $n$ strands. This braid group has many linear representations. The earliest was Artin’s representation, which is an embedding $B_n \to \text{Aut}(F_n)$ [3], the automorphism group of a free group on $n$ generators $x_1, x_2, \cdots, x_n$. Other than Artin, Burau considered representations of $B_n$ of degrees $n$ and $n-1$, known as Burau and reduced Burau representation respectively [3].

Magnus and Peluso [5] showed that the Burau representation is faithful for $n \leq 3$. Long and Paton [4] showed that this representation is not faithful for $n \geq 6$. Moreover, Bigelow [2] proved that $B_n$ is not faithful for $n = 5$. Therefore, the only case that remains open is for $n = 4$.

J. Birman in [3] showed that the Burau representation $B_4$ is faithful if and only if there are two matrices $A$ and $B$ that generate a free group of rank 2. They considered the matrices $A$ and $B$ to be $A = \psi(\sigma_3^{-1}\sigma_1) \cdot \psi(\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})$ and $B = \psi(\sigma_3\sigma_1^{-1})$. More precisely, we have

$$A = \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & -t & -t^{-1} + t \\ -1 & 0 & -t^{-1} + 1 \end{pmatrix}, \quad B = \begin{pmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{pmatrix},$$

$$a = A^{-1} \begin{pmatrix} 1 - t & 0 & -1 \\ t^{-1} - t & -t^{-1} & 0 \\ -t & 0 & 0 \end{pmatrix}, \quad b = B^{-1} \begin{pmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{pmatrix}.$$
In [6], Witzel and Zaremsky found two matrices $A$ and $B$, where $A^3$ and $B^3$ generate a free group of rank 2. Later in [1], Breidze and Traczyk gave a simple proof that $A^3$ and $B^3$ generate the free group of rank 2. The authors wondered in their work whether their considerations could be refined to prove that $A^2$ and $B^2$ generate a free group of rank two. As a matter of fact, we follow the same procedure as in [1] to show that $A^2$ and $B^2$ generate the free group of rank 2. In addition to that, we show a generalization for the main result.

2. $A^2$ and $B^2$ Generate A Free Group Of Rank Two

In this section, we give a simple proof that $A^2$ and $B^2$ generate the free group of rank 2. We consider a non-reducible word $w$ with letters $A, B, a, b$, where each letter is repeated at least two times. Our aim is to show that the product of these matrices is not equal to the identity matrix. In order to make the computations easier, we will show that the first column of the product of the matrices contains terms with negative exponent, say $t^d$. In particular, the whole matrix is not equal to the identity matrix.

In order to discuss the positions of lowest terms in the first column, we use schematical information like this: $\begin{bmatrix}✓&✓&✓\end{bmatrix}$ means that the terms in the first column with the smallest degree are in positions $(1,1)$ and $(3,1)$. We will refer to these schemes as the s-pattern of the matrix. Moreover, it has the following form:

$$\begin{bmatrix}
at^d + at^{d+1} + R_1 \\
b^t + \beta t^{d+1} + R_2 \\
c^t + \gamma t^{d+1} + R_3
\end{bmatrix}, a \neq 0, c \neq 0.$$

Here $R_1, R_2, \text{ and } R_3$ are polynomials that contain terms of degree $\geq d + 2$.

**Proposition 2.1.** Let $xw$ be a non-reducible word in letters $A, B, a, b$ where each letter in $w$ appears in a sequence of length $\geq 2$, except possibly for the first letter of $w$ (from the left) which can be $x$. Then,

1. The possible s-patterns of $xw$ are:
   
   $\begin{bmatrix}✓&✓&✓\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, for $x = B$,
   
   $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, for $x = A$,
   
   $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, for $x = b$,
   
   $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, $\begin{bmatrix}✓\circ✓\circ\end{bmatrix}$, for $x = a$.

2. In passing from $w$ to $xw$, the lowest degree exponent decreases if the s-pattern of $w$ is the one given in the first column, and it remains the same if the s-pattern is the one given in the second or third column (exceptional case).
ON THE CONSIDERATIONS ADOPTED BY BREIDZE AND TRACZYK TOWARDS THE FAITHFULNESS OF BURAU REPRESENTATION FOR

(3) The exceptional cases occur for \(xw\) if the leftmost letter of \(w\) is \(y\) not \(x\), and \(xy\) is equal to \(ba\), or \(Ab\). Also exceptional cases occur if the leftmost letters of \(w\) are \(yy\) with s-pattern \(\sqrt{✓}\) and \(xy\) is equal to \(BAA\), or if the

leftmost letters of \(w\) are \(yy\) with s-pattern \(\circ \circ \) and \(xy\) is equal to \(aBB\).

Proof. (1) Let us consider the word \(Bw\). The leftmost letters of \(w\) could be \(A^2w'\) or \(a^2w'\). First, let us consider the cases of \(A^2w'\), namely \(A^2.A, A^2.b^2\), and \(A^2.B^2\).

\[
B.A^2.A = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
-t^{-2} + t^{-1} & \\
-t^{-2} + t^{-1} & t^{-2} + t^{-1} \\
1 + & \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
-t^2 + & \\
-t^2 + & t^2 + \ldots \\
1 + & 1 + \ldots
\end{bmatrix} = \begin{bmatrix}
✓ & ✓ \\
✓ & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
B.A^2.b^2 = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
t^{-3} & \\
t^{-3} - t^{-1} & t^{-3} - t^{-2} & t^{-3} - t^{-2} \\
1 + & \\
1 + & 1 + & 1 +
\end{bmatrix} = \begin{bmatrix}
-t^4 & ✓ \\
-t^4 & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
B.A^2.B^2 = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
t & \\
t - t^3 & t - t^2 & t - t^2 \\
1 + & \\
1 + & 1 + & 1 +
\end{bmatrix} = \begin{bmatrix}
-t^2 + & \\
-t^2 + & t^2 + \ldots \\
1 + & 1 +
\end{bmatrix} = \begin{bmatrix}
✓ & ✓ \\
✓ & ✓ \\
✓ & ✓
\end{bmatrix}
\]

Now, let us cover the cases of \(a^2w'\), which are \(a^2.B^2, a^2.b^2\), and \(a^2.a\).

\[
B.a^2.B^2 = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
t^2 - t^3 + t^4 & \\
-t + t + & -t^3 + t^4 \\
-t^4 + t^3 & t^{-1} + & t^{-1} +
\end{bmatrix} = \begin{bmatrix}
-t^4 + & ✓ \\
-t^4 + & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
B.a^2.b^2 = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
t^{-2} - t^{-1} & \\
-t^{-4} + t^{-3} & -t^{-1} + & -t^{-1} +
\end{bmatrix} = \begin{bmatrix}
-t^3 + & ✓ \\
-t^3 + & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
B.a^2.a = \begin{bmatrix}
-t & t & 0 \\
0 & 1 & 0 \\
t^{-1} & -t^{-1}
\end{bmatrix} \cdot \begin{bmatrix}
1 - t + t^2 & \\
-t^3 - t^2 - t^3 & t^{-4} + \ldots & t^{-4} + \ldots
\end{bmatrix} = \begin{bmatrix}
-t^3 - & ✓ \\
-t^3 - & ✓ \\
✓ & ✓
\end{bmatrix}
\]

Now, for the word \(A.w\). The leftmost letters of \(w\) could be \(B^2w'\) or \(b^2w'\). Let us consider the words \(B^2.B, B^2.a^2\), and \(B^2.A^2\).

\[
A.B^2.B = \begin{bmatrix}
0 & 0 & -t^{-1} \\
0 & -t & -t^{-1} + t \\
-1 & 0 & -t^{-1} + 1
\end{bmatrix} \cdot \begin{bmatrix}
t^{-3} & \\
0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & ✓ \\
0 & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
A.B^2.a^2 = \begin{bmatrix}
0 & 0 & -t^{-1} \\
0 & -t & -t^{-1} + t \\
-1 & 0 & -t^{-1} + 1
\end{bmatrix} \cdot \begin{bmatrix}
-t^{-1} + & \\
-t^{-2} + & -t^{-2} + \ldots \\
-t^{-4} + & -t^{-4} + \ldots
\end{bmatrix} = \begin{bmatrix}
-t^{-5} + & ✓ \\
-t^{-5} + & ✓ \\
✓ & ✓
\end{bmatrix}
\]

\[
A.B^2.A^2 = \begin{bmatrix}
0 & 0 & -t^{-1} \\
0 & -t & -t^{-1} + t \\
-1 & 0 & -t^{-1} + 1
\end{bmatrix} \cdot \begin{bmatrix}
1 - t & \\
t^{-1} - t & t^{-1} - t \\
\end{bmatrix} = \begin{bmatrix}
-t^2 + & ✓ \\
-t^2 + & ✓ \\
✓ & ✓
\end{bmatrix}
\]

Now, we cover the cases of \(b^2w'\), which are \(b^2.b, b^2.A^2\), and \(b^2.a^2\).
As above, we consider \( \underline{A.b^2} \), \( \underline{A.b^2.A^2} \), \( \underline{A.b^2.a^2} \), \( \underline{A.b^2.A} \), \( \underline{A.b^2} \), \( \underline{A.b^2.a} \). In a similar way as above, we consider \( A^2.A, A^2.b^2, \) and \( A^2.B^2 \).

The leftmost letters of \( w \) in \( bw \) could be \( A^2w' \) or \( a^2w' \). In a similar way as above, we consider \( A^2.A, A^2.b^2, \) and \( A^2.B^2 \).

Now, we cover the cases of \( a^2w' \), which are \( a^2.B^2, a^2.b^2, \) and \( a^2.a \).

The leftmost letters of \( w \) in \( aw \) could be \( B^2w' \) or \( b^2w' \). We consider \( B^2.B, B^2.a^2, \) and \( B^2.A^2 \).
Now, we cover the cases of $b^2w'$, which are $b^2b$, $b^2A^2$, and $b^2a^2$.

\[ a.b^2b = \begin{bmatrix} 1-t & 0 & -1 \\ t^{-1} - t & -t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix} \begin{bmatrix} -t^{-3} \\ 0 \\ t^{-2} \end{bmatrix} = \begin{bmatrix} -t^{-3} + t^{-2} \\ -t^{-4} + t^{-2} \\ t^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ a.b^2A^2 = \begin{bmatrix} 1-t & 0 & -1 \\ t^{-1} - t & -t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix} \begin{bmatrix} t^{-3} + \cdots \\ t^{-1} - t \\ t^{-1} - 1 \end{bmatrix} = \begin{bmatrix} t^{-3} - t^{-2} + \cdots \\ t^{-4} - t^{-3} + \cdots \\ -t^{-2} + t^{-1} + \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ a.b^2a^2 = \begin{bmatrix} 1-t & 0 & -1 \\ t^{-1} - t & -t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix} \begin{bmatrix} t^{-3} - t^{-2} + \cdots \\ -t^{-2} + t^{-1} + \cdots \\ -t^{-2} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-3} - t^{-2} \\ -t^{-4} + t^{-2} \\ -t^{-2} + \cdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

(2) The computations in (1) show that by moving from $w$ to $Bw$, the lowest degree exponent decreases when the s-pattern of $w$ is equal to $\begin{bmatrix} \circ \\ \circ \end{bmatrix}$ and it remains the same if the s-pattern of $w$ is equal to $\begin{bmatrix} \circ \\ \circ \end{bmatrix}$ or $\begin{bmatrix} \circ \\ \circ \end{bmatrix}$.

Similar calculations show that by moving from $w$ to $xw$, the lowest degree exponent decreases if the s-pattern of $w$ is the one given in the first column which is $\begin{bmatrix} \checkmark \\ \checkmark \\ \circ \end{bmatrix}$, and $\begin{bmatrix} \circ \\ \checkmark \\ \checkmark \end{bmatrix}$ for $x = A, b, a$ respectively. Otherwise, the lowest degree exponent remains the same if the s-pattern of $w$ is the one given in the second and third column.

(3) We consider a word of the form $baaw$. We assume by induction that the s-pattern of $aw$ is $\begin{bmatrix} \checkmark \\ \circ \end{bmatrix}$ or $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$. Multiplying by $a$ from the left gives in both cases s-pattern of the form $\begin{bmatrix} \checkmark \\ \circ \end{bmatrix}$. Therefore, the first column of $aaw$ is of the form:

\[ \begin{bmatrix} \alpha t^{d+1} + \cdots \\ bt^d + \beta t^{d+1} + \cdots \\ \gamma t^{d+1} + \cdots \end{bmatrix} \]

where $b \neq 0$. Now, we need to consider the s-pattern of $baaw$.

\[ baaw = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix} \begin{bmatrix} \alpha t^{d+1} + \cdots \\ bt^d + \beta t^{d+1} + \cdots \\ \gamma t^{d+1} + \cdots \end{bmatrix} = \begin{bmatrix} (b - \alpha) t^d + \cdots \\ bt^d + \beta t^{d+1} + \cdots \\ bt^d + \beta t^{d+1} + \cdots \end{bmatrix} \]

Therefore, the lowest degree exponent remains the same by passing from $aaw$ to $baaw$.

Similar calculations shows that the s-pattern of $bw$ is $\begin{bmatrix} \checkmark \\ \circ \end{bmatrix}$. Moreover, it is shown that the lowest degree exponent remains the same after multiplying $A$ with $bw$. 
Now, we consider a word of the form $aBBw$. We assume by induction that the s-pattern of $Bw$ is $\begin{bmatrix} \circ & \circ \\ \vee & \checkmark \end{bmatrix}$, or $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$. Multiplying by $B$ from the left gives $\begin{bmatrix} \circ & \circ \\ \checkmark & \checkmark \end{bmatrix}$, or $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$. Now, we need to consider the s-pattern of $aBBw$.

\[
\begin{align*}
a \begin{bmatrix} \circ \\ \checkmark \end{bmatrix} &= \begin{bmatrix} 1 - t & 0 & -1 \\ t^{-1} & t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix}, \\
a \begin{bmatrix} \circ & \circ \\ \checkmark \checkmark \end{bmatrix} &= \begin{bmatrix} \alpha t^d + \cdots \\ \beta t^d + \cdots \\ \gamma t^d + \cdots \end{bmatrix}, \\
a \begin{bmatrix} \circ & \circ \\ \checkmark & \checkmark \end{bmatrix} &= \begin{bmatrix} -ct^d + \cdots \\ (\alpha - \beta)t^d + \cdots \\ -at^d + \cdots \end{bmatrix}.
\end{align*}
\]

Therefore, the lowest degree exponent remains the same if the s-pattern of $BBw$ is $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$, and it decreases if the s-pattern of $BBw$ is $\begin{bmatrix} \circ & \circ \\ \checkmark & \checkmark \end{bmatrix}$, or $\begin{bmatrix} \circ \circ \checkmark \checkmark \end{bmatrix}$.

Same calculations show that if the s-pattern of $AAw$ is $\begin{bmatrix} \circ \circ \checkmark \checkmark \end{bmatrix}$ and $B$ is multiplied from the left, then the lowest degree remains the same. Otherwise it decreases for $BAAw$ if the s-pattern of $AAw$ is either $\begin{bmatrix} \circ \circ \checkmark \checkmark \end{bmatrix}$ or $\begin{bmatrix} \circ \circ \circ \checkmark \checkmark \end{bmatrix}$.

\section*{Theorem 2.2.} Let $w$ be a non-reducible word in letters $A, B, a, b$ such that each letter is repeated twice, once it appears. Then, the product of matrices is not equal to the identity matrix. Therefore, $A^2$ and $B^2$ generate the free group of rank 2.

\textbf{Proof.} Our aim is to prove that the product of matrices is not equal to the identity matrix. In other words, it is sufficient to show that the first column contains a negative exponent. So, we may assume that the rightmost letter of $w$ is $a$ or $b$ to get negative exponent in the first column. Moreover, Proposition [2.1] says that the lowest degree decreases except in some cases where it remains the same. So by that, the product of matrices will always contain negative exponent which leads to the proof of Theorem [2.2].

We illustrate the proof of Theorem [2.2] by the following example.

\section*{Example 2.3.} Let us consider the word $BBAAbbAABBaa$. The s-pattern of a is $\begin{bmatrix} \circ & \circ \\ \checkmark & \checkmark \end{bmatrix}$, the lowest degree decreases by 1 after multiplying it with $a$ and the s-pattern is $\begin{bmatrix} \circ \circ \checkmark \checkmark \end{bmatrix}$. More precisely,

\[
aa = \begin{bmatrix} 1 - t & 0 & -1 \\ t^{-1} & t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 - t & 0 & -1 \\ t^{-1} & t^{-1} & 0 \\ -t & 0 & 0 \end{bmatrix} = \begin{bmatrix} t^2 - t + 1 \\ t^2 - t^{-1} - t \\ t^2 - t \end{bmatrix}.
\]
Note that we took only the first column of the product of the matrices. Now, after multiplying $aa$ by $B$, the lowest degree decreases and the s-pattern is $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$.

$$Baa = \begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix} \cdot \begin{bmatrix} t^2 - t + 1 \\ t^2 - t^{-1} - t \\ t^2 - t - 1 \end{bmatrix} = \begin{bmatrix} t^{-1} - 1 + \cdots \\ t^{-2} - t^{-1} - t \\ t^{-3} - t^{-2} + \cdots \end{bmatrix}.$$  

Multiplying $B$ by $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$ gives $\begin{bmatrix} \circ \\ \checkmark \end{bmatrix}$ and the lowest degree decreases by 1,

$$BBaa = \begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix} \cdot \begin{bmatrix} t^{-1} - 1 + \cdots \\ t^{-2} - t^{-1} - t \\ t^{-3} - t^{-2} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-1} + t + \cdots \\ t^{-2} - t^{-1} - t \\ -t^{-4} + 2t^{-3} - t^{-2} + \cdots \end{bmatrix}.$$  

After multiplying $A$ by $BBaa$, the lowest degree decreases by 1 and the s-pattern is $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$,

$$ABBaa = \begin{bmatrix} 0 & 0 & -t^{-1} \\ -t & -t^{-1} + t & 0 \\ -1 & 0 & -t^{-1} + 1 \end{bmatrix} \cdot \begin{bmatrix} t^{-1} + t + \cdots \\ t^{-2} - t^{-1} - t \\ -t^{-4} + 2t^{-3} - t^{-2} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-5} - 2t^{-4} + t^{-3} + \cdots \\ t^{-5} - 2t^{-4} + 2t^{-2} + \cdots \\ t^{-5} - 3t^{-4} + 3t^{-3} + \cdots \end{bmatrix}.$$  

Multiplying $A$ by $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ gives $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ and the lowest degree decreases by 1,

$$AABBaa = \begin{bmatrix} 0 & 0 & -t^{-1} \\ -t & -t^{-1} + t & 0 \\ -1 & 0 & -t^{-1} + 1 \end{bmatrix} \cdot \begin{bmatrix} t^{-5} - 2t^{-4} + t^{-3} + \cdots \\ t^{-5} - 2t^{-4} + 2t^{-2} + \cdots \\ t^{-5} - 3t^{-4} + 3t^{-3} + \cdots \end{bmatrix} = \begin{bmatrix} -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 3t^{-5} + 4t^{-4} + \cdots \end{bmatrix}.$$  

Multiplying $b$ by $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ gives $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ and the lowest degree decreases by 1,

$$bAABBaa = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix} \cdot \begin{bmatrix} -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 3t^{-5} + 4t^{-4} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-7} - 4t^{-6} + 6t^{-5} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 4t^{-5} - 6t^{-4} + \cdots \end{bmatrix}.$$  

Now, multiplying $b$ by $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ gives $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ and the lowest degree decreases by 1,

$$bbAABBaa = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix} \cdot \begin{bmatrix} t^{-7} - 4t^{-6} + 6t^{-5} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 4t^{-5} - 6t^{-4} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-8} + 4t^{-7} - 7t^{-6} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 4t^{-5} - 7t^{-4} + \cdots \end{bmatrix}.$$  

Let $w$ be $bbAABBaa$, the leftmost letter of $w$ is $b$. After multiplying $w$ by $A$, the lowest degree remains the same which is one of the exceptional cases that are mentioned in Proposition 2.4. Now, multiplying $b$ by $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ gives $\begin{bmatrix} \checkmark \\ \checkmark \end{bmatrix}$ and the lowest degree decreases by 1,

$$AbbAABBaa = \begin{bmatrix} 0 & 0 & -t^{-1} \\ -t & -t^{-1} + t & 0 \\ -1 & 0 & -t^{-1} + 1 \end{bmatrix} \cdot \begin{bmatrix} -t^{-8} + 4t^{-7} - 7t^{-6} + \cdots \\ -t^{-6} + 3t^{-5} - 3t^{-4} + \cdots \\ -t^{-6} + 4t^{-5} - 7t^{-4} + \cdots \end{bmatrix} = \begin{bmatrix} t^{-7} - 4t^{-6} + 7t^{-5} + \cdots \\ t^{-7} - 4t^{-6} + 7t^{-5} + \cdots \\ t^{-8} - 3t^{-7} + 2t^{-6} + \cdots \end{bmatrix}.$$
the letters
that it is not necessary for
does not matter if it remains the same, the most important is not to in crease.

Therefore, we show a pattern of pluses and spaces, where the plus sign indicates that

\[ BAAbbAABBaa = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & t & 0 & 0 \\ 0 & t^{-1} & -t^{-1} & 0 \end{bmatrix} \begin{bmatrix} -t^{-9} + 3t^{-8} - 2t^{-7} + \\
\cdots \\
t^{-8} + \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\
\cdots \\
\cdots \end{bmatrix} \]

Therefore, the lowest degree decreases. Now, for the word \( BAAbbAABBaa \)

\[ + + + + + + + + \]

Therefore, the lowest degree decreases by 1 and the space means that the lowest degree remains
the same. \( BAAbbAABBaa \)

Now, we could easily obtain the following generalization following the same strat-

\[ \text{Theorem 2.4. Let } w \text{ be a non-reducible word in letters } A, B, a, b \text{ such that wherever}
\text{the letters } ba, aB, BA, \text{ or } Ab \text{ appears, then it is a part of bigger sequence of at least}
ba^2, aB^2, BA^2, \text{ or } Ab^2. \text{ In particular, the product of the considered matrices is not}
\text{the identity matrix.} \]

\[ \text{Proof. Without loss of generality, we consider the case of } aB^2. \text{ Our goal is to show}
\text{that it is not necessary for } a \text{ to be repeated twice, and at the same time the lowest}
\text{degree could decreases or remains the same. In addition to that, we check how the}
\text{lowest degree changes after multiplying } aB^2 \text{ with the other matrices, } B \text{ and } b. \text{ It}
\text{does not matter if it remains the same, the most important is not to increase.} \]

We consider the word \( aB^2w' \). The leftmost letters of \( w' \) could be \( a^2, A^2, \) or \( B. \)
We multiply the matrices \( B \) and \( b \) by \( aB^2w' \) for each \( w' \). More precisely, for the
word, \( aB^2a^2 \) we have

\[ aB^2a^2 = \begin{bmatrix} -t^{-4} + \\
t^{-3} - t^{-2} + \\
1 + t^2 + \cdots \end{bmatrix} \]

Now, we multiply it by \( B \) to get

\[ BaB^2a^2 = \begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix} \begin{bmatrix} -t^{-4} + \\
t^{-3} - t^{-2} + \\
1 + t^2 + \cdots \end{bmatrix} = \begin{bmatrix} t^{-3} + t^{-2} + \\
t^{-3} - t^{-2} + \\
t^{-3} - t^{-3} + \cdots \end{bmatrix} \]

Therefore, the lowest degree decreases. Now, multiplying \( aB^2a^2 \) by \( b \) gives

\[ baB^2a^2 = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix} \begin{bmatrix} -t^{-4} + \\
t^{-3} - t^{-2} + \\
1 + t^2 + \cdots \end{bmatrix} = \begin{bmatrix} t^{-5} + t^{-3} + \\
t^{-3} - t^{-2} + \\
t^{-3} - t^{-2} + \cdots \end{bmatrix} \]

Therefore, the lowest degree decreases. Now, for the word \( aB^2A^2 \) we have
ON THE CONSIDERATIONS ADOPTED BY BREIDZE AND TRACZYK TOWARDS THE FAITHFULNESS OF BURAU REPRESENTATION FOR

Therefore, the lowest degree remains the same. Now, for the last case \( aB^2B \) we have

\[
aB^2B = \begin{bmatrix} -t^3 + t^4 \\ -t^2 + t^4 \\ t^4 \end{bmatrix}. \]

We multiply it by the matrix \( B \) to get

\[
BaB^3 = \begin{bmatrix} -t & t & 0 \\ 0 & 1 & 0 \\ 0 & t^{-1} & -t^{-1} \end{bmatrix} \begin{bmatrix} -t^3 + t^4 \\ -t^2 + t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} -t^3 + t^4 + \cdots \\ -t^2 + t^4 \\ -t \end{bmatrix}.
\]

Therefore, the lowest degree decreases. For the case of \( baB^3 \), we have

\[
baB^3 = \begin{bmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix} \begin{bmatrix} -t^3 + t^4 \\ -t^2 + t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} -t^3 + t^4 + \cdots \\ -t^2 + t^4 \\ -t^2 + t^4 \end{bmatrix}.
\]

Therefore, the lowest degree remains the same. Moreover, all other cases, \( ba^2, BA^2 \) and \( Ab^2 \), are calculated in a similar manner which completes the proof of the theorem.

As an example, let us consider the word \( bABAbbABAABabb \). We will show a pattern of pluses and spaces, where the plus sign indicates that the lowest degree decreases by 1 and the space means that the lowest degree remains the same.

\[
+++ +++ +++ ++++++
\]

3. CONCLUSION

Birman proved that the Burau representation for \( n = 4 \) is faithful if and only if there are two matrices \( A \) and \( B \) that generate a free group of rank 2. Breidze and Traczyk used this strategy to show that \( A^3 \) and \( B^3 \) generate a free group of rank 2.

In our paper, we do a similar work to discuss the open problem of faithfulness of the reduced Burau representation for \( n = 4 \) by proving that \( A^2 \) and \( B^2 \) generate a free group of rank 2.

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