Mean-field theory for a spin-glass model of neural networks: TAP free energy and paramagnetic to spin-glass transition

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Abstract

An approach is proposed to the Hopfield model where the mean-field treatment is made for a given set of stored patterns (sample) and then the statistical average over samples is taken. This corresponds to the approach made by Thouless, Anderson and Palmer (TAP) to the infinite-range model of spin glasses. Taking into account the fact that in the Hopfield model there exist correlations between different elements of the interaction matrix, we obtain its TAP free energy explicitly, which consists of a series of terms exhibiting the cluster effect. Nature of the spin-glass transition in the model is also examined and compared with those given by the replica method as well as the cavity method.
1 Introduction

Neural networks are systems in which a great number of neurons are connected with each other by synapses. The neurons basically take the two states, i.e., the firing and non-firing states. A neuron is firing if stimuli coming from (thousands of) neighboring neurons exceed a threshold. The neuron thus firing in turn affects neighboring neurons. These features are reminiscent of an Ising spin system with long ranged interactions.

Hopfield [1] pointed out that the neural networks can be described by a mathematically equivalent model to that of spin glasses if couplings through synapses are symmetric and random. This suggests that the various methods developed for spin glasses are applicable to the neural networks. Indeed, numbers of studies have been made on this model since then [2-5]. Among others, the work made by Amit, Gutfreund and Sompolinsky (AGS) [5] is worth noticing. Applying the replica method, which is a mathematical trick to calculate the free energy, they investigated the Hopfield model to find that it exhibits a feature of the associative memory in a certain region in the $T - \alpha$ plane, where $T$ is temperature and $\alpha = p/N$ is the ratio of the number of stored patterns $p$ to that of neurons $N$. The region is called the retrieval ferromagnetic (FM) phase. It was also shown by AGS that the model has another ordered phase, called the spin-glass (SG) phase, besides a disordered paramagnetic (PM) phase at highest temperatures.

Sherrington and Kirkpatrick [6] proposed a model for spin glasses (the SK model) in which all Ising spins are coupled with each other through interactions which are given by independent Gaussian random numbers. The model was introduced to construct the mean-field theory of spin glasses. Making use of the replica method, they obtained various properties of spin glasses. Although the original SK solution involves a difficulty to yield negative entropy at low temperatures, the model is now resolved by the replica-symmetry-breaking solution due to Parisi [7].

The replica method is successful, but it is rather abstract since, by this method, the average over samples is carried out before examining thermodynamic properties of an individual sample. In order to get more direct physical insights of the SK model, Thouless, Anderson and Palmer (TAP) [8] developed the mean-field theory in the phase space, by which one first treats an individual sample and then takes the average over samples. They proposed the free-energy form which contains the effect of the 2-spin cluster besides the terms given by the conventional mean-field theory. The TAP free energy, properly derived afterwards [9,10], well works to further clarify various features of spin glasses such as the marginal stability of the SG phase [11], the many-valley structure in the free-energy landscape [12], the number of local free-energy minima [13] and so on. It is now known that the TAP free-energy approach and the replica method are consistent with each other and provide complementary understandings of spin glasses [14,15].

The present work is motivated to develop such a TAP-like approach to the Hopfield model which is expected to play roles complementary to the AGS replica theory. Such an approach has been already described by Mézard, Parisi and Virasoro (MPV) in their textbook [14]. Based on the cavity method, which they have successfully developed to derive the
TAP equations of states for the SK model, they have proposed the corresponding equations of states for the Hopfield model. We consider, however, that a part of their derivation has remained to be justified.

The main purpose of the present paper is to derive the TAP free-energy expression for the Hopfield model directly by following the method due to Plefka [10], who derived the TAP free energy of the SK model. A crucial difference between the two models is that there exist correlations between different elements of the interaction matrix in the Hopfield model [14,16], while they are not found in the SK model. Consequently the TAP free energy of the former consists of an infinite series of terms exhibiting such correlation (cluster) effects. Based on the TAP free energy derived, we analyze mostly nature of the SG phase of the model and compare the results with those obtained by the replica method as well as by the cavity method. The derived TAP free energy is valid also in the retrieval FM phase, but the solution in this phase is left for a future study.

In the next section we present the derivation of the TAP free energy of the Hopfield model. The PM-SG transition temperature $T_{SG}$ is calculated in section 3. Section 4 is devoted to some related discussions including comparisons of the present results with those obtained by AGS and MPV.

## 2 Derivation of the TAP Free Energy

Our starting Hamiltonian is

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j,$$

(1)

where $i(=1,2,\cdots,N)$ denote spin (neuron) sites, and $S_i$ stand for spins (neurons) and take the values $\pm 1$; the value +1 and −1 correspond to the neuron which is firing and is not firing, respectively. The summation is taken over all spin (neuron) pairs.

The interaction (synaptic efficiencies) $J_{ij}$ are given by

$$J_{ij} = \begin{cases} \frac{1}{N} \sum_{\mu=1}^{p} \xi_{i}^{\mu} \xi_{j}^{\mu} & \text{for } i \neq j \\ 0 & \text{for } i = j, \end{cases}$$

(2)

where $\xi_{i}^{\mu}$ take $\pm 1$ and $\{\xi_{i}^{\mu}\}$ represent the $\mu$-th stored pattern. Here we consider that $\xi_{i}^{\mu}$ are quenched, independent and random variables. This means that $J_{ij}$ are also random variables. One sees that $J_{ij}$ obey the Gaussian distribution with $\overline{J_{ij}} = 0$ and $\overline{J_{ij}^2} = p/N^2$, where the overline indicates the average over samples (different realizations of $\{J_{ij}\}$, or $\{\xi_{i}^{\mu}\}$’s). It should be noticed here that $\{J_{ij}\}$ are not independent with each other, but have correlations between different $J_{ij}$’s [14,16]; for example, we see

$$\overline{J_{ij} J_{jk} J_{ki}} = \frac{p}{N^3} = \frac{\alpha}{N^2}.$$

(3)

These non-zero correlations bring about new terms in the free energy (see below).
In order to obtain the free energy, we follow Plefka [10]. Introducing external fields \( h_{i}^{\text{ex}} \), we consider
\[
\tilde{H} = aH - \sum_{i} h_{i}^{\text{ex}} S_{i}. \tag{4}
\]
Then, we make the Legendre transformation to get the free energy as a function of \( m_{i} \),
\[
F = -T \ln \text{Tr} \, e^{-\beta \tilde{H}} + \sum_{i} h_{i}^{\text{ex}} m_{i}. \tag{5}
\]
Here \( T \) is the temperature (\( \beta = 1/T, \text{ with } k_{B} = 1 \)) and \( m_{i} = \langle S_{i} \rangle_{a} \), where \( \langle \cdots \rangle_{a} \) denotes the expectation value with respect to \( \tilde{H} \). We expand (5) with respect to \( a \), i.e.,
\[
F(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^{n} F}{\partial a^{n}} \bigg|_{a=0} a^{n}, \tag{6}
\]
and then we put \( a = 1 \). Plefka showed
\[
\frac{\partial F}{\partial a} = \langle H \rangle_{a}, \tag{7}
\]
\[
\frac{\partial^{2} F}{\partial a^{2}} = -\beta \langle H(H - \langle H \rangle_{a} - \Lambda_{1}) \rangle_{a}, \tag{8}
\]
and obtained
\[
\frac{\partial F}{\partial a} \bigg|_{a=0} = -\sum_{i,j} J_{ij} m_{i} m_{j}, \tag{9}
\]
\[
\frac{\partial^{2} F}{\partial a^{2}} \bigg|_{a=0} = -\beta \sum_{i,j} J_{ij}^{2} (1 - m_{i}^{2})(1 - m_{j}^{2}), \tag{10}
\]
where we have introduced
\[
\Lambda_{n} = \sum_{i} \frac{\partial^{n} h_{i}^{\text{ex}}}{\partial a^{n}} (S_{i} - m_{i}) \]
\[
= \sum_{i} \frac{\partial}{\partial m_{i}} \left( \frac{\partial^{n} F}{\partial a^{n}} \right) (S_{i} - m_{i}). \tag{11}
\]

Now we extend the calculation up to the 4-th order. This calculation is fairly lengthy; we have made use of the algebraic programming system REDUCE-2. The results thus obtained are as follows:
\[
\frac{\partial^{3} F}{\partial a^{3}} = \beta \langle H \rangle_{a} \frac{\partial \langle H \rangle_{a}}{\partial a} + \beta \langle H \Lambda_{2} \rangle_{a} + \beta^{2} \langle H(H - \langle H \rangle_{a} - \Lambda_{1})^{2} \rangle_{a}, \tag{12}
\]
\[
\frac{\partial^{4} F}{\partial a^{4}} = 3\beta \left( \frac{\partial \langle H \rangle_{a}}{\partial a} \right)^{2} + \beta \langle H \rangle_{a} \frac{\partial^{2} \langle H \rangle_{a}}{\partial a^{2}} + \beta \langle H \Lambda_{3} \rangle_{a} - 3\beta^{2} \langle H \Lambda_{2}(H - \langle H \rangle_{a} - \Lambda_{1}) \rangle_{a} - \beta^{3} \langle H(H - \langle H \rangle_{a} - \Lambda_{1})^{3} \rangle_{a}, \tag{13}
\]
and

\[
\frac{\partial^3 F}{\partial a^3}\bigg|_{a=0} = -4\beta^2 \sum_{(i,j)} J_{ij}^2 m_i m_j (1 - m_i^2)(1 - m_j^2) - 6\beta^2 \sum_{(i,j,k)} J_{ij} J_{jk} J_{ki} (1 - m_i^2)(1 - m_j^2)(1 - m_k^2),
\]

\[
\frac{\partial^4 F}{\partial a^4}\bigg|_{a=0} = -2\beta^3 \sum_{(i,j)} J_{ij}^4 (15m_i^2 m_j^2 - 3m_i^2 - 3m_j^2 - 1)
- 48\beta^3 \sum_{(i,j,k)} J_{ij} J_{jk} J_{ki} (1 - m_i^2)(1 - m_j^2)(1 - m_k^2)
- 24\beta^3 \sum_{(i,j,k,\ell)} J_{ij} J_{jk} J_{k\ell} J_{\ell i} (1 - m_i^2)(1 - m_j^2)(1 - m_k^2)(1 - m_\ell^2).
\]

(14)

In the above, \(\langle i, j, k \rangle\) and \(\langle i, j, k, \ell \rangle\) denote that the summation should be taken over inequivalent 3-spin clusters and 4-spin clusters, respectively.

The free energy should be of the order of \(N\), and therefore we have only to pick up terms proportional to \(N\) in (9), (10), (14) and (15). For the ferromagnetic Weiss model with \(J_{ij} = 1/N\), we can see that only (9) gives the contribution proportional to \(N\), as it should. In the SK model, the interactions \(\{J_{ij}\}\) obey the simple Gaussian distribution with \(\overline{J_{ij}} = 0\) and \(\overline{J_{ij}^2} = O(1/N)\), and there is no correlation between different \(J_{ij}\)'s. Therefore, as TAP pointed out, equations (9) and (10) give the contribution of the order of \(N\). In the Hopfield model of present interest, equations (9) and (10) are of the order of \(N\) as in the SK model. As mentioned in section 1, however, there exist correlations between different \(J_{ij}\)'s. This provides new terms to the free energy. To show this, we take the last term of (14), as an example. Its order of magnitude is estimated as

\[
\sum_{(i,j,k)} J_{ij} J_{jk} J_{ki} (1 - m_i^2)(1 - m_j^2)(1 - m_k^2)
\sim \frac{N(N - 1)(N - 2)}{6} \cdot \overline{J_{ij} J_{jk} J_{ki}}
\sim \alpha N.
\]

(16)

Similarly one can see that the last term of (15) yields the contribution of \(O(N)\). As for the other terms in (14) and (15), one can see that they can be neglected in the limit \(N \to \infty\). These analyses imply that \(\partial^n F/\partial a^n\big|_{a=0}\) for \(n \geq 5\) also provide the terms of \(O(N)\), which are written in the form,

\[
- n!\beta^{n-1} \sum_{(i_1, i_2, \ldots, i_n)} J_{i_1 j_{i_1}} J_{i_2 j_{i_2}} \cdots J_{i_n j_{i_n}} (1 - m_{i_1}^2)(1 - m_{i_2}^2) \cdots (1 - m_{i_n}^2).
\]

(17)

Their explicit derivation is given in Appendix A.

As a result, we have the following free energy,

\[
F = F_0 + F_{\text{cluster}}.
\]

(18)
with

\[ F_0 = - \sum_{(i,j)} J_{ij} m_i m_j + T \sum_i \left( \frac{1 + m_i}{2} \ln \frac{1 + m_i}{2} + \frac{1 - m_i}{2} \ln \frac{1 - m_i}{2} \right), \]  

\( \text{with} \)

\[ F_{\text{cluster}} = \frac{1}{2} \beta \sum_{(i,j)} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) \]

\[- \sum_{n=3}^{\infty} \beta^{n-1} \sum_{\{i_1,i_2,\cdots,i_n\}} J_{i_1 i_2} J_{i_2 i_3} \cdots J_{i_n i_1} \times (1 - m_{i_1}^2)(1 - m_{i_2}^2) \cdots (1 - m_{i_n}^2), \]

where the second term in (19) is the entropy, which comes from \( F(0) \) in (6). The TAP equations of states described in terms of \( \{m_i\} \) are determined by \( \partial F/\partial m_i = 0 \), i.e.,

\[ T \tanh^{-1} m_i = \sum_j J_{ij} m_j - \beta \sum_j J_{ij}^2 (1 - m_j^2) m_i \]

\[-2 \sum_{n=3}^{\infty} \beta^{n-1} \sum_{\{i_1,i_2,\cdots,i_{n-1}\}} J_{i_1 i_2} J_{j_1 j_2} \cdots J_{i_{n-1} i} \times (1 - m_{i_1}^2)(1 - m_{i_2}^2) \cdots (1 - m_{i_{n-1}}^2) m_i, \]

for \( i = 1, 2, \cdots, N \), where \( \langle i|j_1,j_2,\cdots,j_{n-1}\rangle \) means that the summation should be taken over inequivalent \( n \)-spin clusters with fixed \( i \). With the substitution of \( m_i^2 \) appearing explicitly in (21) by the spin-glass order parameter \( q = N^{-1} \sum_i m_i^2 \), equation (21) is rewritten as

\[ T \tanh^{-1} m_i = \sum_j J_{ij} m_j - \frac{\alpha \beta (1 - q)}{1 - \beta (1 - q)} m_i. \]

In deriving (22) we have used

\[ \sum_{\{i_1,i_2,\cdots,i_{n-1}\}} J_{i_1 i_2} J_{j_1 j_2} \cdots J_{i_{n-1} i} = \frac{(N - 1)(N - 2) \cdots (N - n + 1)}{2} \cdot \frac{\alpha}{N^{n-1}} \approx \frac{1}{2} \alpha, \]

in which the factor 2 has been introduced, because we have

\[ J_{i_1 i_2} J_{j_1 j_2} \cdots J_{i_{n-1} i} = J_{i_{n-1} j_{n-1}} \cdots J_{j_2 j_1} J_{i_1 j_1}. \]

### 3 Spin-Glass Transition Temperature

Let us calculate the transition temperature, \( T_{\text{SG}} \), which separates the normal (disordered) and spin-glass phases. To do so, we expand (22) up to the first order of \( m_i \) and obtain

\[ T m_i = \sum_j J_{ij} m_j - \frac{\alpha}{T - 1} m_i. \]
This implies that $T_{SG}$ is given by the equation,

$$T_{SG} + \frac{\alpha}{T_{SG} - 1} - J_{\text{max}} = 0, \quad (26)$$

where $J_{\text{max}}$ is the maximum eigenvalue of the interaction matrix $\hat{J}$. It should be mentioned here that the condition

$$J_{\text{max}} \geq 1 + 2\sqrt{\alpha} \quad (27)$$

should be satisfied to have real $T_{SG}$.

Our task is then to calculate $J_{\text{max}}$. In Appendix B, it is shown that the distribution function of eigenvalues of $\hat{J}$ is given as follows:

$$\rho(\lambda) = \begin{cases} 
\rho_0(\lambda) + (1 - \alpha)\delta(\lambda + \alpha) & \text{for } \alpha \leq 1 \\
\rho_0(\lambda) & \text{for } \alpha > 1, 
\end{cases} \quad (28)$$

with

$$\rho_0(\lambda) = \frac{1}{2\pi} \cdot \frac{\sqrt{(\lambda - 1 + 2\sqrt{\alpha})(1 + 2\sqrt{\alpha} - \lambda)}}{\lambda + \alpha}, \quad (29)$$

where $\lambda$ stands for eigenvalues of $\hat{J}$. In Fig. 1 the behavior of $\rho_0(\lambda)$, a continuous part of $\rho(\lambda)$, is shown for some $\alpha$. One notices at once that $\rho(\lambda)$ exhibits a quite different behavior from that of the independent Gaussian random matrix, for which it obeys the semi-circular law [11]. This is again the consequence of the non-zero correlations between the different matrix elements. For $\alpha < 1$ $\rho(\lambda)$ consists of a delta peak at $\lambda = -\alpha$ (whose amplitude is $1 - \alpha$) and the continuous distribution $\rho_0(\lambda)$ around $\lambda = 1$, i.e., $1 - 2\sqrt{\alpha} \leq \lambda \leq 1 + 2\sqrt{\alpha}$ (whose integrated amplitude is $\alpha$). Note that $\rho(\lambda)$ is normalized as $\int \rho(\lambda)d\lambda = 1$. At $\alpha = 1$ the delta peak merges to $\rho_0(\lambda)$, and for $\alpha > 1$ $\rho(\lambda)$ exhibits a single and broad peak. As for the shape of $\rho_0(\lambda)$, we see from (29) that it becomes semi-circular and semi-elliptic for small and large $\alpha$, respectively.

In any $\alpha$ the largest eigenvalue is given by the upper edge of $\rho_0(\lambda)$; $J_{\text{max}} = 1 + 2\sqrt{\alpha}$. Then we rewrite (26) to have

$$\frac{(T_{SG} - 1 - \sqrt{\alpha})^2}{T_{SG} - 1} = 0. \quad (30)$$

This leads to $T_{SG} = 1 + \sqrt{\alpha}$. It is noted that $J_{\text{max}}$ thus obtained is just on the boundary of the condition (27), or $T_{SG}$ is given as a double root of (26). These circumstances are the same as those of the spin-glass transition temperature extracted by the TAP equation in the SK model.

### 4 Discussion

The spin-glass transition temperature $T_{SG}$ obtained by (30) coincides with the AGS result derived by the replica method. A further interesting comparison with the AGS result is on
the expression of the entropy. To show this, let us rewrite $F_{\text{cluster}}$ of (20) in terms of the spin-glass order parameter $q$ as we have done to derive (22). We obtain

$$F_{\text{cluster}} = \frac{1}{2} \alpha N \{ 1 - q + T \ln[1 - \beta(1 - q)] \}. \quad (31)$$

The entropy coming from $F_{\text{cluster}}$ is then given by

$$S_{\text{cluster}} = -\frac{\partial F_{\text{cluster}}}{\partial T} = -\frac{1}{2} \alpha N \left\{ \ln[1 - \beta(1 - q)] + \frac{\beta(1 - q)}{1 - \beta(1 - q)} \right\}. \quad (32)$$

This is exactly the same expression as that of the entropy in the limit $T \to 0$ calculated by AGS ($S_0 = -(\partial F_0/\partial T) = 0$ in this limit). It becomes negative when it is evaluated in terms of the replica-symmetric solutions [5]. An expected proper solution is, as TAP argues for the SK model [8], that $(1 - q)$ should vanish faster than $T$ as $T \to 0$ because we should have $S_{\text{cluster}} = 0$ at $T = 0$.

In relation with the present result that $T_{SG}$ is determined as a double root of (26), let us consider the susceptibility matrix $\chi_{ij} = \partial m_i / \partial h_j$. It is known that $\hat{\chi}$ is given by $\hat{\chi} = \beta \hat{A}^{-1}$, where $\hat{A}$ is the Hessian matrix defined by $A_{ij} = \partial^2 (\beta F)/\partial m_i \partial m_j$. Then we see that $\chi_{\text{max}}$ diverges at $T_{SG}$ as $\chi_{\text{max}} \simeq (T - T_{SG})^{-2}$, where $\chi_{\text{max}}$ is the susceptibility of the eigenmode with the largest eigenvalue $J_{\text{max}}$. The spin-glass susceptibility defined by $\chi_{SG} = (1/N) \text{Tr} \hat{\chi}^2$ is calculated as

$$\chi_{SG} = \int d\lambda \frac{\rho(\lambda)}{(T + \alpha/(T - 1) - \lambda)^2} \quad (33)$$

in the PM phase. Since $\rho(\lambda) \sim (1 + 2\sqrt{\alpha} - \lambda)^{1/2}$ near its upper edge, we obtain $\chi_{SG} \sim (T - T_{SG})^{-1}$. The replica method can provide the same result. These results described here indicate that nature of the PM-SG transition in the Hopfield model, including that the replica-symmetry-breaking takes place in the SG phase [5], is almost identical to that in the SK model.

The TAP equations of state for the Hopfield model was already discussed by MPV [14]. They made use of the cavity method twice. In the first step, one spin is added to the $N$-spin system, and the relations between quantities such as the free energy and the density of states of the $N$- and $(N + 1)$-spin systems are examined to determine the distribution of field to the added spin. Then the following TAP equations are derived

$$m_i = \tanh \beta \left[ \sum_j J_{ij} m_j - \beta (r_2 - r_1) m_i \right], \quad (34)$$

where $r_2 - r_1 = N^{-1} \sum_{\mu=1}^N (\langle \eta_{\mu}^2 \rangle - \langle \eta_{\mu} \rangle^2)$ with $\eta_\mu = N^{-1/2} \sum_{i=1}^N \xi_\mu^i s_i$. For the SK model this step alone gives rise to the TAP equations of interest [14]. For the Hopfield model, on the other hand, MPV introduced another ‘cavity method’, in which the relevant relations are
those of quantities in the systems where $p$ and $(p + 1)$ patterns are stored. This yields, for the replica-symmetric solution,

$$ r_2 - r_1 = \frac{\alpha}{\beta[1 - \beta(1 - q)]}. \quad (35) $$

However equation (34) with (35) substituted does not coincide with our result, equation (22). Since the factor $\beta(1 - q)$ in the numerator of the second term of (22) is missing, the MPV equations do not reproduce the proper $T_{SG}$. We suppose that the origin of the discrepancy would lie in the second step of the cavity method in the MPV argument.

Finally we make a comment on the work by Geszti [2]. Starting from the equations $m_i = \tanh(\beta \sum J_{ij} m_j)$, he derived a set of the self-consistent equations for the retrieval FM order parameter $m$, the random overlap parameter $r$, and $q$, which coincides with those due to AGS derived by the replica theory. In his heuristic argument, however, the terms in (21) coming from $F_{\text{cluster}}$ of (20) are ignored. His argument is similar to the one by which the self-consistent equation for $q$ of the SK model is derived, and which is criticized in [15]. A proper solution of (21) in the retrieval FM phase is our next concern.

To conclude we have developed a TAP-like mean-field theory on the Hopfield model, by which we first analyze thermodynamics of individual sample with fixed $\{J_{ij}\}$, or $\{\xi^\mu_i\}$’s and then take the average over samples. In contrast to the SK model for spin glasses where only the 2-spin cluster effect is vital, it has been shown that a series of clusters, composing a large number of spins, play an important role in the Hopfield model. This gives rise to the TAP free energy which contains an infinite number of terms. Based on it we have investigated the PM-SG transition in the Hopfield model to find that its nature is almost identical to that in the SK model. We consider that the present TAP free-energy approach is useful in studying neural networks of a mean-field type since it will provide us complementary information to the replica method.
Appendix A. Derivation of equation (17)

Here we discuss $\partial^n F/\partial a^n$, from which we have terms of the order of $N$. We notice that the last terms of (14) and (15) come from the last terms of (12) and (13), respectively. Therefore we concentrate, in $\partial^n F/\partial a^n$, on the term

$$(-\beta)^{n-1} \langle H(H - \langle H \rangle_a - \Lambda_1)^{n-1} \rangle_a.$$  

(A1)

It is easily seen that $\partial^n F/\partial a^n$ contains the above term if one notices

$$\frac{\partial \langle R \rangle_a}{\partial a} = \left\langle \frac{\partial R}{\partial a} \right\rangle_a - \beta \langle R(H - \langle H \rangle_a - \Lambda_1) \rangle_a.$$  

(A2)

On the other hand, we have

$$H - \langle H \rangle_a - \Lambda_1 = -\sum_{\langle i,j \rangle} J_{ij}(S_i - m_i)(S_j - m_j),$$  

(A3)

and therefore (A1) can be written by

$$-\beta^{n-1} \left\langle \sum_{\langle i,j \rangle} J_{ij} S_i S_j \left[ \sum_{\langle i,j \rangle} J_{ij}(S_i - m_i)(S_j - m_j) \right]^{n-1} \right\rangle_a.$$  

(A4)

This provides equation (17) in the text, together with other irrelevant terms.

Appendix B. Eigenvalue Distribution of $\hat{J}$

Following Bray and Moore [11], we write down the distribution function of eigenvalues of $\hat{J}$ as

$$\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) = \frac{1}{\pi} \text{Im} \left[ \frac{1}{N} \sum_i G_{ii}(\lambda - i\epsilon) \right],$$  

(B1)

where $\epsilon$ is a positive infinitesimal and $G_{ii}$ are the diagonal elements of the matrix Green function

$$\hat{G}(\lambda) = (\lambda \cdot \hat{1} - \hat{J})^{-1},$$  

(B2)

with $\hat{1}$ being the unit matrix. Then we make use of the so-called locator expansion to have

$$G_{ii} = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_j \left( J_{ij} \frac{1}{\lambda} J_{ij} \right) \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{j,k} \left( J_{ij} \frac{1}{\lambda} J_{jk} \frac{1}{\lambda} J_{ki} \right) \frac{1}{\lambda} + \cdots$$

$$= \frac{1}{\lambda} + \Delta + \lambda \Delta^2 + \lambda^2 \Delta^3 + \cdots$$

$$= \frac{1}{\lambda(1 - \lambda \Delta)}.$$  

(B3)
Here $\Delta$ consists of an infinite series of terms due to the existence of correlations between different matrix elements of $\hat{J}$ (see section 2). Indeed, it is given by

$$\Delta = \frac{1}{\lambda} \sum_j (J_{ij} \mathcal{G} J_{ji}) \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{(i|j,k)} (J_{ij} \mathcal{G} J_{jk} \mathcal{G} J_{ki}) \frac{1}{\lambda} + \cdots$$

$$= \frac{1}{\lambda^2} \sum_{n=2}^{\infty} \mathcal{G}^{n-1} \sum_{(i_1 | i_2, i_3, \ldots, i_n)} J_{i_1 i_2} J_{i_2 i_3} \cdots J_{i_n i_1}, \quad (B4)$$

where we have introduced $\mathcal{G}$ by

$$\mathcal{G} = \frac{1}{N} \sum_i G_{ii} \quad (B5)$$

to take into account the renormalization. In the above, $(i_1 | i_2, i_3, \ldots, i_n)$ means that the summation should be taken over $n$-body cluster for fixed $i_1$; we see

$$\sum_{(i_1 | i_2, i_3, \ldots, i_n)} J_{i_1 i_2} J_{i_2 i_3} \cdots J_{i_n i_1} \approx (N - 1)(N - 2) \cdots (N - n + 1) \frac{J_{i_1 i_2} J_{i_2 i_3} \cdots J_{i_n i_1}}{J_{i_1 i_2} J_{i_2 i_3} \cdots J_{i_n i_1}} \approx \alpha. \quad (B6)$$

Then we have

$$\Delta = \frac{\alpha}{\lambda^2} \cdot \frac{\mathcal{G}}{1 - \mathcal{G}}. \quad (B7)$$

From (B3), (B5) and (B7) we obtain

$$\mathcal{G} = \frac{1}{\lambda[1 - \alpha \mathcal{G}/\lambda(1 - \mathcal{G})]}, \quad (B8)$$

which is solved as

$$\mathcal{G} = \frac{1}{2(\lambda + \alpha)} \left[ \lambda + 1 \pm \sqrt{(\lambda + 1)^2 - 4(\lambda + \alpha)} \right]. \quad (B9)$$

The above solution yields the imaginary part of $\mathcal{G}$ as follows:

$$\text{Im}\mathcal{G} = \frac{\sqrt{4(\lambda + \alpha) - (\lambda + 1)^2}}{2(\lambda + \alpha)} + \pi C \delta(\lambda + \alpha), \quad (B10)$$

where

$$C = \begin{cases} 1 - \alpha & \text{for } \alpha \leq 1 \\ 0 & \text{for } \alpha > 1. \end{cases} \quad (B11)$$

This result together with (B1) and (B5) gives us equation (28) in the text.
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Figure Captions

Fig. 1: The distribution function \(\rho_0(\lambda)\) for \(\alpha = 0.1, 0.5, 1, 1.5\) and 2.
