GENERATORS AND RELATIONS FOR SCHUR ALGEBRAS

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Abstract. We obtain a presentation of Schur algebras (and $q$-Schur algebras) by generators and relations, one which is compatible with the usual presentation of the enveloping algebra (quantized enveloping algebra) corresponding to the Lie algebra $\mathfrak{gl}_n$ of $n \times n$ matrices. We also find several new bases of Schur algebras.

1. Introduction

The classical Schur algebra $S(n, d)$ (over $\mathbb{Q}$) may be defined as the algebra $\text{End}_{\Sigma_d}(V \otimes^d)$ of linear endomorphisms on the $d$th tensor power of an $n$-dimensional $\mathbb{Q}$-vector space $V$ commuting with the action of the symmetric group $\Sigma_d$, acting by permutation of the tensor places (see [6]). Schur algebras determine the polynomial representation theory of general linear groups, and they form an important class of quasi-hereditary algebras.

We identify $V$ with $\mathbb{Q}^n$. Then $V_\mathbb{Z} = \mathbb{Z}^n$ is a lattice in $V$. One can define a $\mathbb{Z}$-order $S_\mathbb{Z}(n, d)$ (the integral Schur algebra) in $S(n, d)$ by setting $S_\mathbb{Z}(n, d) = \text{End}_{\Sigma_d}(V_\mathbb{Z} \otimes^d)$. For any field $K$, one then obtains the Schur algebra $S_K(n, d)$ over $K$ by setting $S_K(n, d) = S_\mathbb{Z}(n, d) \otimes_{\mathbb{Z}} K$. Moreover, $S_\mathbb{Q}(n, d) \cong S(n, d)$.

In the quantum case one can replace $\mathbb{Q}$ by $\mathbb{Q}(v)$ ($v$ an indeterminate), $V$ by an $n$-dimensional $\mathbb{Q}(v)$-vector space, and $\Sigma_d$ by the corresponding Hecke algebra $H(\Sigma_d)$. Then the resulting commuting algebra, $S(n, d)$, is known as the $q$-Schur algebra, or quantum Schur algebra. It appeared first in work of Dipper and James, and, independently, Jimbo. Beilinson, Lusztig, and MacPherson [1] have given a geometric construction of $S(n, d)$ in terms of orbits of flags in vector spaces. (See also [5].)

In this situation $\mathbb{Z}$ is replaced by the ring $A = \mathbb{Z}[v, v^{-1}]$, and there is a corresponding “integral” form $S_A(n, d)$ in $S(n, d)$. The above quantized objects specialize to their classical versions when $v = 1$. We have more detailed information about the Schur algebras in rank 1; see [2] and [3]. Proofs of the main results will appear in [4].

2. Serre’s Presentation of $U$

Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$ be the root system of type $A_{n-1}$, with simple roots $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ denotes the standard
basis of $\mathbb{Z}^n$. The corresponding set $\Phi^+$ of positive roots is the set $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$.

The abelian group $\mathbb{Z}^n$ has a bilinear form $(\ , \ )$ given by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ (Kronecker delta). We write $\alpha_j := \varepsilon_j - \varepsilon_{j+1}$.

The enveloping algebra $U = U(\mathfrak{gl}_n)$ is the associative algebra (with 1) on generators $e_i, f_i$ ($i = 1, \ldots, n - 1$) and $H_i$ ($i = 1, \ldots, n$) with relations

(R1) $H_i H_j = H_j H_i$

(R2) $e_i f_j - f_j e_i = \delta_{ij}(H_j - H_{j+1})$

(R3) $H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j)e_j, \quad H_i f_j - f_j H_i = - (\varepsilon_i, \alpha_j)f_j$

(R4) $e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1)$

$e_i e_j - e_j e_i = 0 \quad$ (otherwise)

(R5) $f_i^2 f_j - 2 f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1)$

$f_i f_j - f_j f_i = 0 \quad$ (otherwise).

For $\alpha \in \Phi$, let $x_{\alpha}$ denote the corresponding root vector of $\mathfrak{gl}_n$ viewed as an element of $U$. We have in particular $e_i = x_{\alpha_i}$ and $f_i = x_{-\alpha_i}$.

3. THE QUANTIZED ENVELOPING ALGEBRA

The Drinfeld-Jimbo quantized enveloping algebra $U = U_q(\mathfrak{gl}_n)$, by definition, is the $\mathbb{Q}(v)$-algebra with generators $E_i, F_i$ ($1 \leq i \leq n - 1$), $K_i^{\pm 1}$ ($1 \leq i \leq n$) and relations

(Q1) $K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$

(Q2) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_i^{-1} - K_i^{-1} K_i + 1}{v - v^{-1}}$

(Q3) $K_i E_j = v^{(\varepsilon_i, \alpha_j)} E_j K_i, \quad K_i F_j = v^{-(\varepsilon_i, \alpha_j)} F_j K_i$

(Q4) $E_i^2 E_j - (v + v^{-1})E_i E_j E_i + E_j E_i^2 = 0 \quad (|i - j| = 1)$

$E_i E_j - E_j E_i = 0 \quad$ (otherwise)

(Q5) $F_i^2 F_j - (v + v^{-1})F_i F_j F_i + F_j F_i^2 = 0 \quad (|i - j| = 1)$

$F_i F_j - F_j F_i = 0 \quad$ (otherwise).

For $\alpha \in \Phi^+$, let $E_\alpha$ and $F_\alpha$ be the positive and negative quantum root vectors of $U$ as defined by Jimbo [3]. We have in particular $E_i = E_{\alpha_i}$ and $F_i = F_{-\alpha_i}$.

4. MAIN RESULTS: CLASSICAL CASE

We now give a precise statement of our main results in the classical case. The first result describes a presentation by generators and relations of the Schur algebra over the rational field $\mathbb{Q}$. This presentation is compatible with the usual presentation (see section 3) of $U = U(\mathfrak{gl}_n)$. 
Theorem 1. Over \( \mathbb{Q} \), the Schur algebra \( S(n, d) \) is isomorphic to the associative algebra (with 1) on the same generators as for \( U \) subject to the same relations \( (R_1) - (R_5) \) as for \( U \), together with the additional relations:

\[
\begin{align*}
(R_6) & \quad H_1 + H_2 + \cdots + H_n = d \\
(R_7) & \quad H_k (H_k - 1) \cdots (H_k - d) = 0, \quad (k = 1, \ldots, n).
\end{align*}
\]

The next result gives a basis for the Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) basis of \( U \).

Theorem 2. The algebra \( S(n, d) \) has a “truncated PBW” basis (over \( \mathbb{Q} \)) which can be described as follows. Fix an integer \( k_0 \) with \( 1 \leq k_0 \leq n \) and set

\[
G = \{ x_\alpha \mid \alpha \in \Phi \} \cup \{ H_k \mid k \in \{1, \ldots, n\} - \{k_0\} \}
\]

and fix some ordering for this set. Then the set of all monomials in \( G \) (with specified order) of total degree at most \( d \) is a basis for \( S(n, d) \).

Our next result constructs the integral Schur algebra \( S_\mathbb{Z}(n, d) \) in terms of the generators given above. We need some more notation. For \( B \in \mathbb{N}^n \), we write

\[
H_B = \prod_{k=1}^n \binom{H_k}{b_k}.
\]

Let \( \Lambda(n, d) \) be the subset of \( \mathbb{N}^n \) consisting of those \( \lambda \in \mathbb{N}^n \) satisfying \( |\lambda| = d \); this is the set of \( n \)-part compositions of \( d \). Given \( \lambda \in \Lambda(n, d) \) we set \( 1_\lambda := H_\lambda \). One can show that the collection \( \{1_\lambda\} \) as \( \lambda \) varies over \( \Lambda(n, d) \) forms a set of pairwise orthogonal idempotents in \( S_\mathbb{Z}(n, d) \) which sum to the identity element.

For \( m \in \mathbb{N} \) and \( \alpha \in \Phi \), set \( x_\alpha^{(m)} := x_\alpha^m / (m!) \). Any product in \( U \) of elements of the form

\[
x_\alpha^{(m)}, \quad \binom{H_k}{m} \quad (m \in \mathbb{N}, \alpha \in \Phi, k \in \{1, \ldots, n\}),
\]

taken in any order, will be called a Kostant monomial. Note that the set of Kostant monomials is multiplicatively closed, and spans \( U_\mathbb{Z} \). We define a function \( \chi \) (content function) on Kostant monomials by setting

\[
\chi(x_\alpha^{(m)}) := m \varepsilon_{\max(i, j)}, \quad \chi\left(\binom{H_k}{m}\right) := 0
\]

where \( \alpha = \varepsilon_i - \varepsilon_j \ (i \neq j) \), and by declaring that \( \chi(X Y) = \chi(X) + \chi(Y) \) whenever \( X, Y \) are Kostant monomials.

We write any \( A \in \mathbb{N}_\Phi^+ \) in “multi-index” form \( A = (a_\alpha)_{\alpha \in \Phi}^+ \) and set \( |A| := \sum_{\alpha \in \Phi^+} a_\alpha \). For \( A, C \in \mathbb{N}_\Phi^+ \) we write

\[
e_A = \prod_{\alpha \in \Phi^+} x_\alpha^{(a_\alpha)}, \quad f_C = \prod_{\alpha \in \Phi^-} x_\alpha^{(c_\alpha)}
\]

where the products in \( e_A \) and \( f_C \) are taken relative to any fixed orders on \( \Phi^+ \) and \( \Phi^- \).

Theorem 3. The integral Schur algebra \( S_\mathbb{Z}(n, d) \) is the subring of \( S(n, d) \) generated by all divided powers \( e_j^{(m)}, f_j^{(m)} (j \in \{1, \ldots, n-1\}, m \in \mathbb{N}) \). Moreover, this algebra has a \( \mathbb{Z} \)-basis consisting of all

\[
(B_1) \quad e_A 1_\lambda f_C \quad (\chi(e_A f_C) \leq \lambda)
\]
and another such basis consisting of all
\[(B2) \quad f_A 1_\lambda e_C \quad (\chi(f_A e_C) \leq \lambda)\]
where \(A, C \in \mathbb{N}^{n^+}, \lambda \in \Lambda(n,d),\) and where \(\leq\) denotes the componentwise partial ordering on \(\mathbb{N}^n.\)

Finally, we have another presentation of the Schur algebra by generators and relations. This presentation has the advantage that it possesses a quantization of the same form, in which we can set \(v = 1\) to recover the classical version.

**Theorem 4.** For each \(i \in \{1, \ldots, n-1\}\) write \(\alpha_i := \varepsilon_i - \varepsilon_{i+1}.\) The algebra \(S(n, d)\) is the associative algebra (with 1) given by generators \(1_\lambda (\lambda \in \Lambda(n,d)), e_i, f_i (i \in \{1, \ldots, n-1\})\) subject to the relations
\[
(1) \quad 1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda \\
2 \quad e_i 1_\lambda = \begin{cases} 
1_{\lambda+\alpha_i} e_i & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\
0 & \text{otherwise}
\end{cases}
3 \quad f_i 1_\lambda = \begin{cases} 
1_{\lambda-\alpha_i} f_i & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\
0 & \text{otherwise}
\end{cases}
4 \quad 1_\lambda e_i = \begin{cases} 
e_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\
0 & \text{otherwise}
\end{cases}
5 \quad 1_\lambda f_i = \begin{cases} 
f_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\
0 & \text{otherwise}
\end{cases}
\]
along with the Serre relations \((R4), (R5),\) for \(i, j \in \{1, \ldots, n-1\}.\)

5. **Main results: quantized case**

Our main results in the quantized case are similar in form to those in the classical case. The first result describes a presentation by generators and relations of the quantized Schur algebra over the rational function field \(\mathbb{Q}(v).\) This presentation is compatible with the usual presentation (see section 3) of \(U = U_v(\mathfrak{g}_n).\)

**Theorem 1'.** Over \(\mathbb{Q}(v),\) the \(q\)-Schur algebra \(S(n, d)\) is isomorphic with the associative algebra (with 1) given by generators \(1_\lambda (\lambda \in \Lambda(n,d)), e_i, f_i (i \in \{1, \ldots, n-1\})\) subject to the same relations \((Q1)-(Q5)\) as for \(U,\) together with the additional relations:
\[
(1) \quad K_1 K_2 \cdots K_n = v^d \\
2 \quad (K_j - 1)(K_j - v)(K_j - v^2) \cdots (K_j - v^d) = 0, \quad (j = 1, \ldots, n).
\]

The next result gives a basis for the \(q\)-Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) type basis of \(U,\) given in Lusztig [4, Prop. 1.13].
Theorem 2'. The algebra $S(n,d)$ has a "truncated PBW type" basis which can be described as follows. Fix an integer $k_0$ with $1 \leq k_0 \leq n$ and set
$$G' = \{ E_\alpha, F_\alpha \mid \alpha \in \Phi^+ \} \cup \{ K_k \mid k \in \{1, \ldots, n\} - \{k_0\} \}$$
and fix some ordering for this set. Then the set of all monomials in $G'$ (with specified order) of total degree at most $d$ is a basis for $S(n,d)$.

Note that setting $v = 1$ in the basis of Theorem 2 does not yield the basis of Theorem 3 since $K_i$ acts as the identity when $v = 1$.

Our next result constructs the integral $q$-Schur algebra $S_A(n,d)$ in terms of the generators given above. For $B$ in $\mathbb{N}^n$, we write
$$K_B = \prod_{j=1}^n \left[ \frac{K_j}{b_j} \right].$$
where for indeterminates $X, X^{-1}$ satisfying $XX^{-1} = X^{-1}X = 1$ and any $t \in \mathbb{N}$ we formally set
$$\left[ X^t \right] := \prod_{s=1}^t \frac{X^{v^{-s+1}} - X^{-1}v^{-s-1}}{v^s - v^{-s}},$$
an expression that makes sense if $X$ is replaced by any invertible element of a $\mathbb{Q}(v)$-algebra.

Given $\lambda \in \Lambda(n,d)$ we set (when we are in the quantum case) $1_{\lambda} := K_\lambda$. It turns out that, just as in the classical case, the collection $\{1_{\lambda}\}$ as $\lambda$ varies over $\Lambda(n,d)$ forms a set of pairwise orthogonal idempotents in $S_A(n,d)$ which sum to the identity element.

Let $[m]$ denote the quantum integer $[m] := (v^m - v^{-m})/(v - v^{-1})$ and set $[m]! := [m][m-1]\cdots[1]$. Then the $q$-analogues of the divided powers of root vectors are defined to be $E^{(m)}_\alpha := E_\alpha/[m]!$ and $F^{(m)}_\alpha := F_\alpha/[m]!$. Any product in $U$ of elements of the form
$$E^{(m)}_\alpha, \ F^{(m)}_\alpha, \ \left[ \frac{K_j}{m} \right] \quad (m \in \mathbb{N}, \alpha \in \Phi, j \in \{1, \ldots, n\}),$$
taken in any order, will be called a Kostant monomial. As before, the set of Kostant monomials is multiplicatively closed, and spans $U_A$. The definition of content $\chi$ of a Kostant monomial is obtained similarly, by setting
$$\chi(E^{(m)}_\alpha) = \chi(F^{(m)}_\alpha) := m \varepsilon_{\max(i,j)}, \ \chi(\left[ \frac{K_j}{m} \right]) := 0$$
where $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$, and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever $X, Y$ are Kostant monomials. For $A, C \in \mathbb{N}^{\Phi^+}$ we write
$$E_A = \prod_{\alpha \in \Phi^+} E^{(a_\alpha)}_\alpha, \ F_C = \prod_{\alpha \in \Phi^+} F^{(c_\alpha)}_\alpha$$
where the products in $E_A$ and $F_C$ are taken relative to any two specified orderings on $\Phi^+$.

Theorem 3'. The integral $q$-Schur algebra $S_A(n,d)$ is the subring of $S(n,d)$ generated by all quantum divided powers $E^{(m)}_j, F^{(m)}_j$ ($j \in \{1, \ldots, n-1\}, m \in \mathbb{N}$), along
with the elements \([K_j^m] (j \in \{1, \ldots, n\}, m \in \mathbb{N})\). Moreover, this algebra has a basis over \(A\) consisting of all 
\[(B1') \quad E_{A1_\lambda}F_C \quad (\chi(e_Af_C) \preceq \lambda)\]
and another such basis consisting of all 
\[(B2') \quad F_{A1_\lambda}E_C \quad (\chi(f_Ae_C) \preceq \lambda)\]
where \(A, C \in \mathbb{N}_0^+\), \(\lambda \in \Lambda(n, d)\), and where \(\preceq\) denotes the componentwise partial ordering on \(\mathbb{N}^n\).

Unlike the truncated PBW basis, the bases of Theorem 3 do specialize when \(v = 1\) to their classical analogues given in Theorem 3.

Finally, we have another presentation of the \(q\)-Schur algebra by generators and relations. This presentation has the advantage that by setting \(v = 1\), we recover the classical version given in Theorem 4. The relations of the presentation are similar to relations that hold for Lusztig’s modified form \(\dot{U}\) of \(U\). (See \([9, \text{Chap. 23}]\).)

**Theorem 4'.** For each \(i \in \{1, \ldots, n-1\}\) write \(\alpha_i := \varepsilon_i - \varepsilon_{i+1}\). The algebra \(S(n, d)\) is the associative algebra (with 1) given by generators \(1_\lambda (\lambda \in \Lambda(n, d)), E_i, F_i\) \((i \in \{1, \ldots, n-1\})\) subject to the relations
\[
(S1') \quad 1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda \quad (\lambda, \mu \in \Lambda(n, d))
\]
\[
(S2') \quad E_i 1_\lambda = \begin{cases} 
1_{\lambda + \alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda(n, d) \\
0 & \text{otherwise}
\end{cases}
\]
\[
F_i 1_\lambda = \begin{cases} 
1_{\lambda - \alpha_i} F_i & \text{if } \lambda - \alpha_i \in \Lambda(n, d) \\
0 & \text{otherwise}
\end{cases}
\]
\[
1_\lambda E_i = \begin{cases} 
E_i 1_{\lambda - \alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n, d) \\
0 & \text{otherwise}
\end{cases}
\]
\[
1_\lambda F_i = \begin{cases} 
F_i 1_{\lambda + \alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n, d) \\
0 & \text{otherwise}
\end{cases}
\]
\[
(S3') \quad E_iF_j - F_jE_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} [\lambda_j - \lambda_{j+1}] 1_\lambda
\]
along with the \(q\)-Serre relations \([Q4], [Q5]\) for \(i, j \in \{1, \ldots, n-1\}\).

6. Other results

6.1. **Triangular decomposition.** One can define the plus, minus, and zero parts of Schur algebras in terms of the generators, as follows. (These subalgebras have been studied before.) The plus part \(S^+(n, d)\) (resp., the minus part \(S^-(n, d)\)) is the subalgebra of \(S(n, d)\) generated by all \(x_\alpha, \alpha \in \Phi^+\) (resp., \(\alpha \in \Phi^-\)). The zero part \(S^0(n, d)\) is the subalgebra generated by all \(H_j, j = 1, \ldots, n\). We also have the Borel Schur algebras \(S^{\geq0}(n, d)\) (resp., \(S^{\leq0}(n, d)\)), the subalgebra generated by \(S^+(n, d)\) (resp., \(S^-(n, d)\)) together with \(S^0(n, d)\).
Moreover, from our main results we see easily that the set of all \( e_{s} \) satisfying
\[
|s| \leq 1.
\]
for \( \alpha \in \Phi^{+} \) (resp., \( \alpha \in \Phi^{-} \)) and \( m \in \mathbb{N} \). We give in [4] a presentation of \( S^{0}(n, d) \) by generators and relations. In particular, we prove that \( H_B = 0 \) whenever \( |B| > d \) (\( B \in \mathbb{N}^{n} \)) and that the set of (pairwise orthogonal) idempotents \( 1_{\lambda} \), \( \lambda \in \Lambda(n, d) \), is a \( \mathbb{Z} \)-basis of \( S_{Z}^{0}(n, d) \).

The algebra \( S = S(n, d) \) has a triangular decomposition \( S = S^{+}S^{0}S^{-} \). We show that in this decomposition one can permute the three factors in any order. Moreover, the same result holds over \( \mathbb{Z} \).

The zero part \( S_{Z}^{0}(n, d) \) is the algebra generated by all \( \left( \frac{H_{j}}{m} \right) \) for \( j = 1, \ldots, n \) and \( m \in \mathbb{N} \). We give in [3] a presentation of \( S^{0}(n, d) \) by generators and relations. In particular, we prove that \( H_B = 0 \) whenever \( |B| > d \) (\( B \in \mathbb{N}^{n} \)) and that the set of (pairwise orthogonal) idempotents \( 1_{\lambda} \), \( \lambda \in \Lambda(n, d) \), is a \( \mathbb{Z} \)-basis of \( S_{Z}^{0}(n, d) \).

\( S_{Z}^{+}(n, d) \) (resp., \( S_{Z}^{-}(n, d) \)) is the algebra generated by all divided powers \( x_{a}^{(m)} \) for \( \alpha \in \Phi^{+} \) (resp., \( \alpha \in \Phi^{-} \)) and \( m \in \mathbb{N} \). It is an easy consequence of the commutation formulas [2] that each generator \( x_{a} \) is nilpotent of index \( d + 1 \); see [3] for details. Moreover, from our main results we see easily that the set of all \( e_{A} \) (resp., \( f_{A} \)) such that \( |A| \leq d \) is a \( \mathbb{Z} \)-basis for the algebra \( S_{Z}^{+}(n, d) \) (resp., \( S_{Z}^{-}(n, d) \)).

It also follows immediately from our results that the set of all \( e_{A}1_{\lambda} \) (resp., \( 1_{\lambda}f_{A} \)) satisfying \( \chi(e_{A}) \leq \lambda \) is an integral basis for the Borel Schur algebra \( S_{Z}^{0}(n, d) \) (resp., \( S_{Z}^{\leq 0}(n, d) \)).

Similar statements to the above hold in the quantum case. In particular, as a corollary of the commutation formulas [2], one can give a simple proof of [3], Prop. 2.3.

6.2. Explicit reduction formulas. Fix a positive root \( \alpha \) and write \( \alpha = \varepsilon_{i} - \varepsilon_{j} \) for \( i < j \).

Then from [3] we have the following reduction formulas in \( S_{Z}(n, d) \), for any \( a, b, c \in \mathbb{N} \):

\[
\begin{align*}
(1) & \quad f_{a}^{(a)} \left( \frac{H_{j}}{b} \right) e_{b}^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f_{c}^{(a-k)} \left( \frac{H_{j}}{b+k} \right) e_{c}^{(c-k)} \\
(2) & \quad e_{a}^{(a)} \left( \frac{H_{i}}{b} \right) f_{b}^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} e_{c}^{(a-k)} \left( \frac{H_{i}}{b+k} \right) f_{c}^{(c-k)}
\end{align*}
\]

where \( s = a + b + c - d \) and \( s \geq 1 \).

We do not have a \( q \)-analogue of these formulas.

The results of [3] give another type of reduction formula for \( S_{A}(n, d) \). If \( b_{1}, b_{2} \in \mathbb{N} \) satisfy \( b_{1} + b_{2} = d \), set \( \lambda := b_{1}\varepsilon_{i} + b_{2}\varepsilon_{j} \in \Lambda(n, d) \). Then for all \( s \geq 1 \) we have:

\[
\begin{align*}
(3) & \quad E_{a}^{(a)}_{\lambda} F_{a}^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{a} b_{1+k}^{(a-k)} E_{a}^{(a-k)}_{1+ka} F_{a}^{(c-k)} \\
(4) & \quad F_{a}^{(a)}_{\lambda} E_{a}^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{a} b_{2+k}^{(a-k)} F_{a}^{(a-k)}_{1-ka} E_{a}^{(c-k)}
\end{align*}
\]

where \( s = a + b_{1} + c - d \) in [3] and \( s = a + b_{2} + c - d \) in [4].

The classical analogues of formulas [3] and [4] hold in \( S(n, d) \).
6.3. **Another presentation** (for $n = 2$). We have the following result from [2], which presents $S(2, d)$ as a quotient of $U(\mathfrak{sl}_2)$.

**Theorem 5.** Over $\mathbb{Q}$, the Schur algebra $S(2, d)$ is isomorphic with the associative algebra (with 1) generated by $e$, $f$, $h$ subject to the relations:

$$he - eh = 2e; \quad ef - fe = h; \quad hf - fh = -2f$$

$$(h + d)(h + d - 2) \cdots (h - d + 2)(h - d) = 0.$$ 

Moreover, this algebra has a “truncated PBW” basis over $\mathbb{Q}$ consisting of all $f^ah^be^c$ such that $a + b + c \leq d$.

Note that if we eliminate the last relation we have the usual presentation of $U(\mathfrak{sl}_2)$ over $\mathbb{Q}$. The last relation is the minimal polynomial of $h$ in the representation on tensor space. The problem of presenting $S(n, d)$ as a quotient of $U(\mathfrak{sl}_n)$ seems to be more difficult for $n > 2$.

We also have from [3] the following $q$-version of the above, which presents the $q$-Schur algebra $S(2, d)$ as a quotient of the quantized enveloping algebra $U(\mathfrak{sl}_2)$.

**Theorem 5'.** Over $\mathbb{Q}(v)$, the quantum Schur algebra $S(2, d)$ is isomorphic to the algebra generated by $E$, $F$, $K^{\pm 1}$ subject to the relations:

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = v^2E \quad KFK^{-1} = v^{-2}F$$

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}}$$

$$(K - v^d)(K - v^{d-2}) \cdots (K - v^{-d+2})(K - v^{-d}) = 0.$$ 

6.4. **Hecke algebras.** Suppose that $n \geq d$. Let $\omega = (1^d)$. Then the subalgebra $1_\omega S(n, d)1_\omega$ is isomorphic with the Hecke algebra $H(\Sigma_d)$. The nonzero elements of the basis (B1) of the form $1_\omega E_i F_i C_\omega$ form a basis of the Hecke algebra; similarly for elements of the basis (B2) of the form $1_\omega F_i E_i C_\omega$.

Moreover, taking $d = n$, we can see that $H = H(\Sigma_n)$ is generated by the elements $1_\omega E_i F_i 1_\omega$ ($1 \leq i \leq n - 1$). Alternatively, $H$ is generated by the $1_\omega F_i E_i 1_\omega$ ($1 \leq i \leq n - 1$). One can easily describe the relations on these generators, thus obtaining a presentation of $H$ which is closely related to one in [4].

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