Left-Right Symmetric Models in Noncommutative Geometries?

Florian Girelli
Centre de Physique Théorique, CNRS - Luminy, Case 907, 13288 Marseille Cedex, France
girelli@cpt.univ-mrs.fr
January 4, 2019

Abstract

Left-right symmetric models are analyzed in the context of noncommutative geometry where we show that spontaneous parity violation is ruled out...

1 Introduction

In the Standard Model of electro-weak and strong forces, parity is broken explicitly by the choice of inequivalent representations for left- and right-handed fermions. Within the framework of Yang-Mills-Higgs models this is certainly an aesthetic drawback that physicists have tried to correct by the introduction of left-right symmetric models. These are Yang-Mills-Higgs models where parity is broken spontaneously together with the gauge symmetry. However the price to pay for this aesthetic surgery is high, especially on an aesthetic scale: the simplest left-right symmetric model for electro-weak forces has a $SU(2)_L \times SU(2)_R \times U(1)$ group and four scalar representations transforming like two $2_L \otimes 2_R$ representations, a $3_L \otimes 1_R$ and a $1_L \otimes 3_R$. The Higgs potential contains some twenty coupling constants [11]. For a tiny class of Yang-Mills-Higgs models, noncommutative geometry derives the Higgs mechanism, i.e. the scalar representation and symmetry breaking Higgs potential, from first principles. Parity violation is crucial here in the sense that vector-like models are not in this tiny class, the Standard Model, however, with its explicit parity violation qualifies for noncommutative geometry. Left-right symmetric models are halfway in between vector-like models and models with explicit parity breaking and its is natural to ask whether they do qualify for noncommutative geometry. In the Connes-Lott models [4], a first try was made but unfortunately did not qualify [3]. This was, however, before the setting of a precise notion of noncommutative geometry (Connes’ Axioms [1]) and the introduction of real structure which gave a very rigid structure to construction of finite spectral triples. A complete classification of these latter was made in [8], enabling to construct some very general finite spectral triples in the framework of these axioms.

Here we shall use this general classification in order to study the most general LRS models, with the aim to check if they can be expected to be physical.

We shall first define the LRS models in the context of almost commutative geometries, by just specializing the usual LRS models to this approach. However, Connes’Axioms imply a number of conditions that the model has to fulfill in order to be well-defined. In particular, we will see that having 2 groups acting respectively
on the left and the right fermions will not be enough and we will need a third. It will appear also that the Poincaré duality cannot be satisfied.

Secondly, we shall study some examples of LRS models as the updated (Spectral action and respecting the Axioms) chiral electromagnetism. These examples will show that no parity breaking occurs and that, furthermore, the physical gauge bosons are either axial or vectorial. Finally, in the last part, we shall give a general proof of this result.

## 2 Left Right Symmetric Models

The LRS models are a particular type of YMH models. We recall that a LRS model is based on a product of 3 Lie groups, $G_L$, $G_R$ and $G_V$. $G_L$ and $G_R$ acting respectively on $\psi_L, \psi_R$ and $G_V$ acts vectorially. LR symmetry means that $G_L$ and $G_R$ are isomorphic, $G_L \simeq G_R$ and that the representations on the left-handed fermions $\rho_L$ and on right-handed fermions $\rho_R$ are identical, $\rho_L(G_L, G_V) = \rho_R(G_R, G_V) := \rho(G_L, G_R, G_V)$.

Another condition is needed. The Higgs vacuum $V_{\text{vac}}$ generates the fermionic masses as well the gauge bosons masses.

$$<\psi, V_{\text{vac}} \psi> = <\psi_R, M \psi_L> + <\psi_L, M^* \psi_R>$$  \hspace{1cm} (1)

The matrix $M$ is in general not hermitian, and we get the masses by a biunitary transformation $(U_L, U_R)$ acting such:

$$U_R^{-1} MU_L = \begin{pmatrix} m_1 & \cdots \\ \vdots & \ddots \end{pmatrix}$$  \hspace{1cm} (2)

However, in order to keep the symmetry between the Left and the Right fermions, we must have only one unitary transformation: $U_R = U_L := U$. By definition, a LRS model has a Higgs vacuum (i.e. the fermionic mass matrix) generated by a hermitian matrix $M$.

The specialization to almost commutative models is then straightforward (see also a slightly different approach in [11]). The main object in almost commutative geometries is the associative algebra $A$ (say for example a real one) equipped with a representation, noted $\pi$ (i.e. the fundamental one). The gauge bosons are the connections: $\Omega^1(M, \text{Lie}U(A))$ where $M$ is the space time manifold, and $\text{Lie}U(A)$ the Lie algebra of the unitaries of $A$. The real structure $J$ generates a bimodule structure on $\mathcal{H}$, the Hilbert space of fermions. The group action on $\mathcal{H}$ is then $\rho(U)\psi = \pi(U)J\pi(U)J^{-1}\psi$ for $U \in U(A)$. To get the interactions of the gauge bosons with fermions, we have to go to the Lie algebra. Taking an infinitesimal element of $U(A)$, that is $u \in \text{Lie}U(A)$, the action of $\text{Lie}U(A)$ on $\mathcal{H}$ (noted $\pi^s$ for symmetrized representation) is then $\pi^s(u)\psi = (\pi(u) + J\pi(u)J^{-1})\psi$.

Applying this to LRS model, we see that the finite algebra must be decomposed in $A_L, A_R$ and $A_V$ acting respectively on $\psi_L, \psi_R$ and vectorially, such that $\pi^s(A_L, A_V) = \pi^s(A_R, A_V)$ and $A_L \simeq A_R$.

The other condition about the Higgs vacuum (or the fermionic mass matrix) is not changed and can be taken as it is.

This being set, let us check how Connes’ axioms constrain our model. We shall deal with $A_L$ and $A_R$ being simple algebras as these are the simplest to deal with and that the case of semi simple algebras can be easily recovered in the same way.
The fact that $\pi^*_L = \pi^*_R$ sets some very strong conditions on the matrix of multiplicities $[3]$. First, let us consider the case $A = A_L \oplus A_R$, that is with no $A_V$. $A_L, A_R$ acting respectively on the left and right-handed fermions, are respectively affected by a sign $\ominus$ and $\oplus$. The matrix of multiplicities is therefore:

\[
\begin{pmatrix}
A_L & A_R \\
A_L & \ominus & \ominus, \oplus \text{ are resp. for } \pi^*(A_L), \pi^*(A_R) \\
A_R & & \\
A_V & \ominus_3 & \oplus_4 \\
\ominus_2 & \oplus_4 & ?
\end{pmatrix}
\]

In $[3]$, it was shown that to construct the most general Dirac operator, we have to draw vertical or horizontal lines between different signs, and that each line was representing a non-zero element in the Dirac operator. It is clear here that we can not draw any, therefore the Dirac operator is zero. This is not interesting us as we want to have nonzero fermionic masses. Therefore we see that we need a third algebra $A_V$ which would act on both left and right fermions. This is the same reason as in the Standard Model were we need the $SU(3)$ color implemented to have a well-defined model.

Bearing this in mind, we can easily establish the most general matrix of multiplicities:

\[
\begin{pmatrix}
A_L & A_R & A_V \\
A_L & \ominus_1 & \ominus_2 \\
A_R & \ominus_3 & \oplus_4 \\
A_V & \ominus_2 & \oplus_4 & ?
\end{pmatrix}
\]

$\ominus_1$ stands for $\pi^*(A_L)$, $\ominus_3$ for $\pi^*(A_R)$, $\ominus_2$ for $\pi^*(A_L, A_V)$, $\ominus_4$ for $\pi^*(A_R, A_V)$, and $?$ for $\pi^*(A_V)$.

$?$ can be $\oplus$ or $\ominus$ or void (here $A_V$ is supposed simple but nothing forbids us to take it semi-simple).

However if the action $A_V$ was affected by a sign, it would mean that this algebra act only on a particular fermion chirality. This would break the LR symmetry so the ? has to be void, that is the algebra $A_V$ acts on fermions of both chiralities, in the same way i.e. vectorially.

The signs in the matrix of multiplicities can be affected by relative numbers. Indeed, the reality condition implies that $\pi$ and $J\pi J^{-1}$ commute. Therefore, they can be decomposed into irreducible representations: $\pi(A) = \oplus_i M_{n_i} \otimes I_{m_i}$. Let us take $x_i \in M_{n_i}(\mathbb{C})$ and

\[
\begin{align*}
\pi(A) & \ni \oplus_{ij} x_i \otimes I_{m_{ij}} \otimes I_{n_j} \\
J\pi(A)J^{-1} & \ni \oplus_{ij} I_{n_i} \otimes I_{m_{ij}} \otimes \overline{x_j}
\end{align*}
\]

The (symmetric) matrix of natural numbers $(m_{ij})$ with each $m_{ij}$ affected by our previous sign is by definition the matrix of multiplicities $[3]$ $\mu_{ij} = \text{sign}_{ij} m_{ij}$. So now the most general matrix of multiplicities is:

\[
\mu = \begin{pmatrix}
\mu_{11} & \mu_{13} \\
\mu_{22} & \mu_{23} \\
\mu_{31} & \mu_{32}
\end{pmatrix} = \begin{pmatrix}
\ominus & \ominus \\
\ominus & \oplus \\
\ominus & \oplus
\end{pmatrix}
\]

Obviously, we need $\mu_{11} = \mu_{22}$ and $\mu_{13} = \mu_{23}$ in order to have $\pi^*_L = \pi^*_R$. But this is a problem as we can check that the matrix then has a zero determinat! This means that the Poincaré duality cannot be satisfied in this model $[3]$. However we shall not dismiss the LRS models for this “mathematical” reason, indeed we are going to see that also for a physical reason the model is not viable: parity remains unbroken.
The most general Dirac operator can be constructed and from it the Higgs fields. The computation of the Higgs potential then depends on which scheme we are considering. However, our main interest concerns the Higgs vacuum which will give both fermionic and gauge bosons masses. It is calculated by minimizing the Higgs potential. In the case of Connes-Lott models, we will get as minimum the Dirac, but in the case of the Spectral Action, we do not know what we will get as the potential is a tricky polynomial to minimize.

However, in both cases the Higgs vacuum will be given in terms of the fermionic mass matrix $M$ and once again, we must have $M$ hermitian, in order to have the LR symmetry preserved.

3 Examples

3.1 Chiral electromagnetism

Let us study a simple example to see what is going on. The ancestor of the almost commutative Standard Model is the two sheeted model: $C^\infty(M) \otimes (C_L \oplus C_R)$ \cite{1}. This first model showed that on the discrete part some spontaneous symmetry breaking occurred, and therefore at the end, we get one massive and one massless gauge bosons. The massive acting axially and the latter vectorially, this model was called chiral electromagnetism. However vectorial or axial interactions do not break parity. Indeed, in the mass Lagrangian $L_m = m_a^2 W_a^2 + m_v^2 \gamma_v$, to exchange the algebras $C_L$ and $C_R$ would change nothing as $W_a \to -W_a$ and $\gamma_v \to \gamma_v$.

Chiral electromagnetism was introduced in the Connes-Lott scheme, and before the setting of the Axioms and the Spectral action \cite{8}\cite{4}. Here we are going to calculate the updated model (that is taking account of Connes’ Axioms and in the Spectral action scheme) and check explicitly that parity remains unbroken.

As we saw in the last section, we need to add another algebra to $C_L$ and $C_R$ in order to have the first order condition verified.

We choose to take the simplest one: $C$. So our algebra is now $C_L \oplus C_R \oplus C_V$ where $C_V$ acts vectorially. Taking $(a, b, c) \in C_L \oplus C_R \oplus C_V$, the chosen representation is (one family):

$$
\begin{pmatrix}
  a \\
  b \\
  c
\end{pmatrix}
\begin{pmatrix}
  e_L \\
  e_R
\end{pmatrix}
\in \mathcal{H} \sim \mathbb{C}^4.
$$

Chirality and charge conjugaison are respectively given by:

$$
\Gamma = \begin{pmatrix}
  -1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\quad J = \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0
\end{pmatrix}
\circ \text{complex conjugation}
$$

Then the most general Dirac operator is

$$
\mathcal{D} = \begin{pmatrix}
  0 & m & 0 & 0 \\
  m^* & 0 & 0 & 0 \\
  0 & 0 & 0 & m^* \\
  0 & 0 & m & 0
\end{pmatrix}
\quad \text{with } m \in \mathbb{C}.
$$

The Higgs field is parametrized by a complex field $\phi \in \mathbb{C}$:
Using the Spectral action, we get the Higgs potential:

\[ V(\phi) = -\text{tr}\Phi^2 + \frac{1}{2}\text{tr}\Phi^4 = -2|m|^2|\phi|^2 + |m|^4|\phi|^4. \]  

(4)

So, we get the Higgs vacuum given by:

\[ \Phi_{\text{vac}} = \begin{pmatrix} 0 & V_{\text{vac}} & 0 & 0 \\ V_{\text{vac}}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

\( V_{\text{vac}} \) is chosen real. Next we want to compute the mass matrix of the gauge bosons which is given by:

\[ \mathcal{M} = \text{tr}(\pi(A), \Phi_{\text{vac}} + \mathcal{J}\Phi_{\text{vac}}\mathcal{J}^{-1})[\pi(A), \Phi_{\text{vac}} + \mathcal{J}\Phi_{\text{vac}}\mathcal{J}^{-1}]^* \]

\[ = 4\text{tr}(\pi(A), \Phi_{\text{vac}})[\pi(A), \Phi_{\text{vac}}]^* \]

Expanding in our case, we get:

\[ \mathcal{M} = (b^2 + a^2)V_{\text{vac}}^2 - 2ab V_{\text{vac}}^2 \]  

(5)

and this can be written as

\[ (a, b) \begin{pmatrix} V_{\text{vac}}^2 & -V_{\text{vac}}^2 \\ -V_{\text{vac}}^2 & V_{\text{vac}}^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \]

To know the mass of the physical bosons, we need to diagonalize this mass matrix. The rotation needed is \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) and we get the mass eigenvalues \( 2V_{\text{vac}}, 0 \) associated respectively with the eigenvectors \( W_A = a - b \) and \( \gamma_V = a + b \). We get the announced result, that is an axial massive boson and a vectorial massless boson, the photon. This means of course that parity is unbroken. However this property is not linked to the fact that we took commutative algebras. Indeed other models have been studied: \( \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C} \) \(^3\) and \( \mathbb{H} \oplus \mathbb{H} \) \(^2\). The latter was studied before the Connes’Axioms setting. \( \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C} \) is the updated version of \( \mathbb{H} \oplus \mathbb{H} \): the \( \mathbb{C} \) is there to make the model compatible with the Axioms. Both models were done in the Connes-Lott scheme, and both gave that no parity breaking occurred and that we still had that the eigenvectors of the gauge bosons mass matrix were either vectorial or axial. Just as a last example, we can have a look of what goes on in the Spectral Action scheme with the same algebra, i.e. \( \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C} \). (We recall that \( \mathbb{H} \) is considered as a real algebra.)

### 3.2 Case of \( \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C} \)

We take the same representation as in \(^3\): \( (a, b, c) \in \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C} \) is represented as:

\[
\begin{pmatrix}
a \otimes 1_2 \\
b \otimes 1_2 \\
\bar{c}1_2 \otimes 1_2 \\
\bar{c}1_2 \otimes 1_2
\end{pmatrix}
\]
acting on
\[
\begin{pmatrix}
\lambda_L \\
\nu_{\lambda L} \\
\lambda_R \\
\nu_{\lambda R} \\
\lambda_L \\
\nu_{\lambda L} \\
\lambda_R \\
\nu_{\lambda R}
\end{pmatrix}
\otimes \begin{pmatrix}
\lambda = \text{electron} \\
\lambda = \text{muon}
\end{pmatrix} \in \mathcal{H} \sim \mathbb{C}^{16}
\]

Chirality and charge conjugaison are respectively given by:
\[
\Gamma = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Then the Dirac operator is taken to be [6]:
\[
\mathcal{D} = \begin{pmatrix}
0 & M & 0 & 0 \\
M^* & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{M} \\
0 & 0 & \overline{M^*} & 0
\end{pmatrix}
\]

with \( M = p_1 \otimes M_e + p_2 \otimes M_\nu \in M_4(\mathbb{C}) \)

\( p_i \) are given by \( (p_i)_{kl} = \delta_{ki} \delta_{il} \) and \( M_e, M_\nu \in M_2(\mathbb{C}) \). The Higgs is then constructed in the usual way.

\[
\Phi = \begin{pmatrix}
0 & \phi_1 \otimes M_e + \phi_2 \otimes M_\nu & 0 & 0 \\
\phi_1^* \otimes M_e^* + \phi_2^* \otimes M_\nu^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( \phi_i \) is defined by \( \phi_i = h_i - p_i \) with \( h_i \in M_2(\mathbb{C}) \). One can show that the \( \phi_i \)'s are not independent [6]: \( \phi_2 = P_0 \overline{\phi_1} P_0^{-1} \) with \( P_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The Higgs potential constructed from the Spectral action gives:
\[
V(\Phi) = -\mu^2 tr|\Phi|^2 + \lambda tr|\Phi|^4 \quad (6)
\]

In order to simplify the calculations we take \( M_\nu = 0 \), and then we want to minimize this potential. If we take \( \Phi \Phi^* \) as the variable then this minimum is obtained when \( \phi \) is unitary, i.e. \( \phi \in U(2) \) (modulo a renormalization factor \( \kappa = \frac{\mu^2 tr|M_e|^2}{2 tr|M_e|^4} \))

\[
V_{\text{vac}} = \kappa u \otimes M_e \quad u \in U(2) \quad (7)
\]

\[
\Phi_{\text{vac}} = \kappa \begin{pmatrix}
0 & u \otimes M_e & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

In order to keep the LR symmetry, we have to choose \( V_{\text{vac}} \) to be hermitian, and fermionic masses are obtained by diagonalizing \( V_{\text{vac}} \).

This is a striking result, indeed the Dirac operator is not a minimum of the Higgs potential, as one could expect. To the best of my knowledge, this is the first example of this type.

We consider now the mass matrix for the gauge bosons:
\[
\mathcal{M}_b = tr([\pi(\mathcal{A}), \Phi_{\text{vac}}] [\pi(\mathcal{A}), \Phi_{\text{vac}}]^*) \quad (8)
\]
The gauge bosons (elements of the complexification of the Lie algebra) are respectively for $A_L$ and $A_R$ $a_\mu = \left( \begin{array}{cc} A^3_\mu & A^+_\mu \\ A^-_\mu & -A^3_\mu \end{array} \right)$ and $b_\mu = \left( \begin{array}{cc} B^3_\mu & B^+_\mu \\ B^-_\mu & -B^3_\mu \end{array} \right)$ and they can be taken hermitian. So we get for the mass matrix:

$$M_b = \text{tr}(V_{vac}V_{vac}^*(a_\mu^2 \otimes 1_2) + V_{vac}^*V_{vac}(b_\mu^2 \otimes 1_2)) - 2((a_\mu \otimes 1_2)V_{vac}(b_\mu \otimes 1_2)V_{vac}^*)$$

First we can take the simple value for $V_{vac}, u = 1_2$. Straightforward calculations (diagonalization of $M_b$) show that there are 3 massless bosons, all of them vectorial, one being neutral, the two others charged. All massive bosons are axial and have the same mass: $m_Z = m_{W^\pm}$. Another way of counting massless bosons, is to calculate the little group of the potential: It is given by the $u_L, u_R \in SU(2)$ such that $u_L u R^{-1} = u$. For the simple vacuum expectation value $u = 1_2$, the little group is $SU(2)$ with $u_L = u_R$ and there are 3 massless bosons.

If now we take $u$ to be a general element of $U(2)$, the mass matrix is more complicated to calculate and we compute then the little group. Indeed a general $u \in U(2)$ is diagonalizable by a biunitary transformation $P_L, P_R \in U(2)$ such that $P_L u P_R^{-1} = 1_2$. Then the equation $u_L u R^{-1} = u$ becomes

$$u_L P_L^{-1} P_R u_R^{-1} = P_L^{-1} P_R$$

Therefore $u_R = (P_L^{-1} P_L) u_L (P_R^{-1} P_L)^{-1}$ and if $u_L$ is of determinant one, so is $u_R$. The little group is $SU(2)$ again. One checks also that the massless bosons are still vectorial and the massive ones, all of the same mass, are axial.

Parity is still not broken.

Let us compare our $\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C}$ example in the Spectral action scheme to the $\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{C}$ example in the Connes-Lott scheme [3]:

- Firstly we saw that starting with a Dirac operator with rank one in the isospin sector, we ended up with a fermionic mass matrix of rank 2 (maximal rank) in the isospin sector, whereas in the Connes-Lott scheme, the initial Dirac operator used to construct 1-forms is also the minimum of the Higgs potential.

- Secondly, this special feature implies a difference between gauge bosons masses; instead of the unique massless neutral boson in the Connes-Lott scheme [3], we get 2 more charged massless bosons. In terms of little groups, Connes-Lott has $U(1)$ as little group, whereas the Spectral action has $SU(2)$.

Let us note that the Standard Model does not exhibit these differences. Indeed, here the Dirac operator also minimizes the Higgs potential derived from the Spectral action, and the little groups coincide in both schemes. This is another remarkable feature of the Standard Model in noncommutative geometry.

Instead of calculating all masses by hand, there is a quicker way to show that parity is not broken. Indeed, we do not even need to calculate the minimum explicitly: let us take it as

$$\Phi_{vac} = \begin{pmatrix}
0 & V_{vac} & 0 & 0 \\
V_{vac}^* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

We consider now the mass matrix for the gauge bosons: $M = \text{tr}(V_{vac}V_{vac}^*(a_\mu^2 \otimes 1_2) + V_{vac}^*V_{vac}(b_\mu^2 \otimes 1_2)) - 2((a_\mu \otimes 1_2)V_{vac}(b_\mu \otimes 1_2)V_{vac}^*)$. Here, we can notice that by
interchanging \( a_\mu \) and \( b_\mu \) the matrix \( \mathcal{M} \) is not modified. This switching is just the action of the matrix \( \Sigma \) which can be written in the chosen basis as \( \Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes^3 \).

We have of course that \( \Sigma = \Sigma^{-1} = \Sigma^t \). Therefore

\[
\mathcal{M} = \Sigma \mathcal{M} \Sigma. \tag{10}
\]

We can see then that \( \Sigma \) and \( \mathcal{M} \) are commuting and so diagonalized in the same eigenbasis. By construction, \( \Sigma \) has eigenvalues \(-1, 1\). Let us take \( W_\mu \) an eigenvector of \( \mathcal{M} \) with \( m \) as eigenvalue:

\[
\mathcal{M} W_\mu = m W_\mu. \tag{11}
\]

This last equation is just asserting that \( W_\mu \) is either axial or vectorial and parity remains unbroken!

These results suggest us the following theorem.

4 Theorem

In almost commutative YMH models, parity cannot be spontaneously broken, moreover the physical gauge bosons, mass eigenstates, are always vectorial or axial.

Let us prove this.

Proof

Let us take the general case of \( \mathcal{A} = \mathcal{A}_L \oplus \mathcal{A}_R \oplus \mathcal{A}_V \). We always have the possibility to reorder the Hilbert basis such that \( \pi_t(\mathcal{A}_{L,R}, \mathcal{A}_V) = \pi(\mathcal{A}_{L,R}) \oplus \pi - V(\mathcal{A}_V) \). The mass matrix of the gauge bosons is then given:

\[
\mathcal{M} = \text{tr}
\left((\pi_t(\mathcal{A}), V_{vac})[\pi_t(\mathcal{A}), V_{vac}]^*\right) = \text{tr}
\left((\pi(\mathcal{A}_L) M - M \pi(\mathcal{A}_R))(\pi(\mathcal{A}_L) M - M \pi(\mathcal{A}_R))^*\right)
\]

This formula holds in any scheme, Connes-Lott or Spectral action, this is why our proof is independent of the used scheme. We note also that \( \mathcal{A}_V \) does not occur in our calculations: vector-like groups are never broken in almost commutative geometries. The matrix \( \mathcal{M} \) is by construction hermitian and has positive (or zero) eigenvalues, indeed \( \mathcal{M} \) represents a positive bilinear form.

In order to check if parity breaking can occur, we have to see what happens upon exchange of \( \mathcal{A}_L \) and \( \mathcal{A}_R \), that is under the permutation of left and right gauge bosons. The permutation, noted \( \Sigma \), verifies obviously:

\[
\Sigma = \Sigma^t = \Sigma^{-1} \tag{12}
\]

and its eigenvalues are 1 and \(-1\).

The exchange of \( \mathcal{A}_L \) and \( \mathcal{A}_R \) is thus given by:

\[
\Sigma \mathcal{M} \Sigma = \text{tr}
\left((\pi(\mathcal{A}_R) M - M \pi(\mathcal{A}_L))(\pi(\mathcal{A}_R) M - M \pi(\mathcal{A}_L))^*\right) \tag{13}
\]

Working on it:

\[
\mathcal{M} = \text{tr}
\left((M^* \pi^*(\mathcal{A}_R) - \pi^*(\mathcal{A}_L) M)^*(M^* (\pi^*(\mathcal{A}_R) - \pi^*(\mathcal{A}_L) M)^*) \right) \tag{14}
\]

Recall now that \( M \) is hermitian, and that \( \pi(\mathcal{A}_{L,R}) \) is anti hermitian since the gauge bosons belong to the Lie algebra \( \text{Lie} U(\mathcal{A}) \), so we get:

\[
\Sigma \mathcal{M} \Sigma = \text{tr}
\left((\pi(\mathcal{A}_R) M - M \pi(\mathcal{A}_L))(\pi(\mathcal{A}_R) M - M \pi(\mathcal{A}_L))^*\right) = \mathcal{M} \tag{15}
\]
Since $\Sigma = \Sigma^{-1}$, we see that $\Sigma$ and $\mathcal{M}$ commute and can be diagonalized in the same eigenbasis. Let $W$ be such a simultaneous eigenvector of $\mathcal{M}$:

$$\Sigma W = \pm W.$$  \hspace{1cm} (16)

This last expression just expresses that $W$ is either vectorial or axial! Of course, by construction, we have as many vectorial bosons as axial ones. It is obvious now that in any case there can be no parity breaking!

\section{Conclusion}

In conclusion, explicit parity violation is a must in noncommutative geometry. On the mathematical side, this follows from Poincaré duality. On the more physical side, it follows from the above theorem which rules out spontaneous parity breaking. There is an impressing list of intricate features of the Standard Model, that remain completely ad-hoc in the context of Yang-Mills-Higgs models and that are unavoidable in noncommutative geometry:

- the gauge group is non-simple,
- fermions transform according to fundamental or trivial representations under isospin and color,
- strong forces couple vectorially,
- color is unbroken,
- isospin is broken spontaneously by one doublet of scalars,
- the gauge group is reduced by $\mathbb{Z}_2$.

We may now add to this list:

- Parity violation is explicit.

\textbf{Acknowledgements}

T. Schücker is warmly thanked for suggesting the subject of this article as well as for his many advices. B. Iochum and T. Krajewski are also very much thanked for all their comments and advices.

\textbf{References}

[1] A. Connes, J. Lott The metric aspect of Non commutative geometry, Proceedings of the 1991 Cargèse summer conference, eds J. Fröhlich et al, Plenum Press (1992)

[2] B. Iochum, T. Schücker, A Left-Right Symmetric model a la Connes-Lott \texttt{hep-th/9401048} Lett. Math. Phys. 32 (1994) 153

[3] T. Krajewski, Classification of finite Spectral Triples \texttt{hep-th/9701081} J. Geom. Phys. 28 1998 1

[4] T. Schücker, Geometries and forces \texttt{hep-th/9712093}

[5] T. Krajewski, Constraints on scalar potential from Spectral Action principle \texttt{hep-th/9803199}

[6] R Wiechmann, Diplomarbeit 1999 Universität Dortmund (unpublished) Almost commutative geometries and a Left-Right Symmetric model?
[7] A. Connes, J. Lott Nucl Phys. Supp B18 (1990) 295

[8] A. Chamseddine, A. Connes, The spectral action principle, Com. Math. Phys. 186 (1997) 731; Universal formula for non commutative geometry actions: Unification of gravity and the Standard Model Phys. Rev. Lett. 24 (1996) 4868

[9] A. Connes, Gravity coupled with matter and the foundation of non Commutative Geometry, hep-th/9603053 Com. Math Phys. 182 (1996) 155

[10] G. Konisi, T. Saito, Z. Maki, M. Nakahara, Left-Right Symmetric Model from Geometric Formulation of Gauge Theory in $M_4 \times Z_2 \times Z_2$, hep-th/9812065, Prog. Theor. Phys. 101, 5(1999) 1105

[11] P. Binetruy, M. Gaillard, Z. Kunszt, The price of natural parity conservation in the neutral current sector, Nucl. Phys. B 144 (1978) 141