BERNSTEIN-SZEGÖ MEASURES IN THE PLANE

JEFFREY S. GERONIMO AND PLAMEN ILIEV∗

Abstract. We define a class of Bernstein-Szegő measures on $\mathbb{R}^2$ and we establish their spectral properties, providing a natural extension of the one-dimensional theory. We also derive conditions involving finitely many moments, which are new in the two-dimensional setting, and which completely characterize these measures. An important part of the paper is devoted to a self-contained development of the Bernstein-Szegő theory for matrix-valued functionals. We also show how recent results in the Fejér-Riesz factorization problem for bivariate trigonometric polynomials can be used to construct explicit bases of the spaces associated with the Bernstein-Szegő measures on $\mathbb{R}^2$. The proofs use a mixture of techniques from real analysis, complex analysis and algebra.

Contents

1. Introduction 2
1.1. Bernstein-Szegő measures on $\mathbb{R}$ 2
1.2. Bernstein-Szegő measures on $\mathbb{R}^2$ 3
1.3. Notations 3
2. One-dimensional Bernstein-Szegő weights, Hankel matrices and reproducing kernels 4
3. Matrix orthogonal polynomials 8
4. Two variable Bernstein-Szegő measures 13
4.1. One-sided factorization: spectral properties and characterization 21
4.2. Bernstein-Szegő measures: spectral properties and characterization 28
5. Orthogonal decompositions for Bernstein-Szegő measures on the torus 33
6. Szegő map 35
6.1. Szegő map for the weight in Theorem 4.1 35
6.2. Szegő map for the Bernstein-Szegő weight in Theorem 4.11 41
7. Explicit examples 43
7.1. Example 1: Bernstein-Szegő weight 43
7.2. Example 2: one-sided factorization 48
References 53

Date: August 26, 2022.

2020 Mathematics Subject Classification. 42C05, 47A57, 30E05.

Key words and phrases. Bernstein-Szegő measures, bivariate polynomials, Fejér-Riesz factorizations, matrix polynomials, stability.

∗PI gratefully acknowledges the support of a Simons Foundation Grant #635462 and a CRM-Simons Professorship at the Centre de Recherches Mathématiques, Université de Montréal.
1. Introduction

1.1. Bernstein-Szegő measures on $\mathbb{R}$. An important class of measures on $\mathbb{R}$ introduced by Bernstein and Szegő [26] are the measures of the form

$$d\mu = \frac{2}{\pi} \sqrt{1-x^2} Q(x) \chi_{(-1,1)}(x) \, dx,$$  \hspace{1cm} (1.1)

where $Q(x)$ is a polynomial nonvanishing on $(-1,1)$, with at most simple zeros at $x = \pm 1$ and $\chi_J$ denotes the characteristic function of a set $J$. Recall that if $\{p_k(x)\}_{k=0}^{\infty}$ are orthonormal polynomials with respect to a measure $\mu$ on the real line, then the multiplication by $x$ can be represented by a three-term operator

$$a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x) = xp_k(x).$$  \hspace{1cm} (1.2)

Suppose that $Q(x)$ is a polynomial of degree at most $2n$ for some positive integer $n$, and let $q(z)$ denote the stable Fejér-Riesz factor of $Q(x)$, i.e. $q(z)$ is the unique polynomial with real coefficients with no zeros in the closed unit disk, except possibly for simple zeros at $z = \pm 1$, such that

$$Q(x) = q(z) q(1/z), \quad \text{where} \quad x = \frac{1}{2} \left( z + \frac{1}{z} \right),$$  \hspace{1cm} (1.3)

normalized so that $q(0) > 0$. Then we can define orthonormal polynomials with respect to $\mu$ in (1.1) by

$$p_k(x) = \frac{z^{k+1} q(1/z) - z^{-k-1} q(z)}{z - 1/z} \quad \text{for} \quad k \geq n.$$  \hspace{1cm} (1.4)

The last equation implies that $a_{k+1} = \frac{1}{2}$ and $b_k = 0$ for $k \geq n$. \hspace{1cm} (1.5)

Conversely, suppose now that (1.5) holds. We can restrict our attention to $p_0(x), \ldots, p_n(x)$, or more generally, we can consider a positive linear functional $\mathcal{L}$ defined on the space $\mathbb{R}_{2n}[x]$ of polynomials of degree at most $2n$, with orthonormal polynomials $p_0(x), \ldots, p_n(x)$. Since every positive linear functional $\mathcal{L}$ on $\mathbb{R}_{2n}[x]$ can be extended to a positive linear functional on the space of all polynomials using (1.5), the natural question one might ask is what conditions on $\mathcal{L}$ guarantee the existence of a polynomial $Q(x) \in \mathbb{R}_{2n}[x]$ such that

$$\mathcal{L}(f) = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) \sqrt{1-x^2}}{Q(x)} \, dx \quad \text{for all} \quad f \in \mathbb{R}_{2n}[x].$$  \hspace{1cm} (1.6)

Using (1.4) as hint, we can give a simple answer as follows: for a positive linear functional $\mathcal{L} : \mathbb{R}_{2n}[x] \to \mathbb{R}$ with orthonormal polynomials $\{p_k(x)\}_{k=0}^{n}$ equation (1.6) holds for some $Q(x) \in \mathbb{R}_{2n}[x]$ if and only if

$$q(z) = z^n (p_n(x) - 2za_n p_{n-1}(x)) \neq 0 \quad \text{for} \quad z \in (-1,1),$$  \hspace{1cm} (1.7)

see [3, 10, 13, 17] and the references therein. Moreover, in this case $Q(x)$ is unique and can be computed using the polynomial $q(z)$ in (1.7) and (1.3).
1.2. Bernstein-Szegő measures on $\mathbb{R}^2$. The goal of this article is to address the following questions:

(I) Are there bivariate extensions of the Bernstein-Szegő weights on the real line which possess spectral properties similar to (1.3)?

(II) How do we characterize the measures in (I)?

(III) Are there analogs of (1.4)?

In the remaining part of the introduction, we will go over some of the main results in the paper which answer the questions raised above. We begin by explaining how we define orthonormal polynomials and recurrence coefficients. Unlike the one-variable theory, there is no canonical way to introduce orthonormal polynomials in the plane. Starting with [18], it is customary to replace the orthonormal polynomial $p_k(x)$ in the one-dimensional setting with the space $\Pi_k[x,y]$ consisting of all polynomials of total degree $k$ which are orthogonal to all polynomials of total degree at most $k-1$. Then, by picking an orthonormal basis of each $\Pi_k$, we replace the three-term recurrence relation with two matrix relations which correspond to the multiplications by $x$ and $y$, respectively. Since $\dim(\Pi_k) = k + 1$ the coefficient $a_{k+1}$ in (1.2) is replaced by rectangular $(k+1) \times (k+2)$ matrices $A_{x,k+1}$ and $A_{y,k+1}$, see [6, 8, 25, 27] for a detailed account. For instance, for the Chebyshev measure

$$d\mu = \frac{4}{\pi^2} \sqrt{1 - x^2} \sqrt{1 - y^2} \chi((-1,1)^2)(x,y) \, dx \, dy$$

we can choose bases so that the matrices have the block structure $A_{x,k+1} = \frac{1}{2}[I_{k+1} | 0]$, $A_{y,k+1} = \frac{1}{2}[0 | I_{k+1}]$, $B_{x,k} = 0 = B_{y,k}$, where $I_{k+1}$ is the identity $(k + 1) \times (k + 1)$ matrix. However, these formulas do not hold beyond the trivial weight in (1.8) for any $k$, even if we consider simple examples like products $w_1(x)w_2(y)$ of two Bernstein-Szegő weights on $\mathbb{R}$.

A different way to analyze two-dimensional measures stems from the work of Geronimo and Woerdeman [15, 16] on the bivariate Fejér-Riesz factorization. Within the context of $\mathbb{R}^2$ this approach was developed in [4] where the spaces $\Pi_k[x,y]$ were replaced by the following spaces of orthogonal polynomials with respect to a measure, or more generally a positive linear functional $\mathcal{L}$

$$P_{k,l;\mathcal{L}}[x,y] = \mathbb{R}_{k,l}[x,y] \cap \mathbb{R}_{k-1,l}[x,y],$$

$$\bar{P}_{k,l;\mathcal{L}}[x,y] = \mathbb{R}_{k,l}[x,y] \cap \mathbb{R}_{k,l-1}[x,y].$$

In the above equations $k$ and $l$ are nonnegative integers and $\mathbb{R}_{k,l}[x,y]$ denotes the space of polynomials with real coefficients in $x$ and $y$ of degrees at most $k$ in $x$ and $l$ in $y$. For every $l$ we fix a basis

$$E_l = (\beta_0(y), \beta_1(y), \ldots, \beta_l(y))$$

of the space $\mathbb{R}_l[y]$ of polynomials of degree at most $l$ in $y$ and we define an orthonormal basis $\{p_{k,l}^j(x,y) : 0 \leq j \leq l\}$ of the spaces $P_{k,l;\mathcal{L}}[x,y]$ for all $k$ by applying the Gram-Schmidt process to the elements in the set $f_{0,l}^k, \ldots, f_{l,l}^k$ where

$$f_{k,l}^j = x^k \beta_j(y) - \text{Proj}_{\mathbb{R}_{k-1,l}[x,y]}(x^k \beta_j(y)),$$

and $\text{Proj}_V$ denotes the orthogonal projection onto the space $V$. We set

$$P_{k,l}(x,y) = [p_{k,l}^0(x,y), p_{k,l}^1(x,y), \ldots, p_{k,l}^l(x,y)]^t.$$
Similarly, we fix a basis $B_k$ of the space $\mathbb{R}_k[x]$ of polynomials of degree at most $k$ in $x$, which leads to an orthonormal basis $\{\tilde{p}_{k,l}^j(x,y) : 0 \leq j \leq k\}$ for $P_{k,l}[x,y]$, and we set

$$\tilde{P}_{k,l}(x,y) = [\tilde{p}_{k,l}^0(x,y), \tilde{p}_{k,l}^1(x,y), \ldots, \tilde{p}_{k,l}^k(x,y)]^t.$$  

With these notations, it is easy to see that the above vector polynomials satisfy the following recurrence relations

$$\begin{align*}
x P_{k,l} &= A_{k+1,l} P_{k+1,l} + B_{k,l} P_{k,l} + A_{k,l}^t P_{k-1,l}, \\
y \tilde{P}_{k,l} &= \tilde{A}_{k+1,l} \tilde{P}_{k+1,l} + \tilde{B}_{k,l} \tilde{P}_{k,l} + \tilde{A}_{k,l}^t \tilde{P}_{k-1,l},
\end{align*}$$

where $A_{k,l}$ and $B_{k,l}$ are $(l+1) \times (l+1)$ matrices while $\tilde{A}_{k,l}$ and $\tilde{B}_{k,l}$ are $(k+1) \times (k+1)$ matrices. The vector polynomials $P_{k,l}(x,y)$ can be written as

$$P_{k,l}(x,y) = P_k^l(x)[\beta_0(y), \beta_1(y), \ldots, \beta_l(y)]^t,$$

where $P_k^l(x)$ is a polynomial of degree $k$ in $x$ whose coefficients are $(l+1) \times (l+1)$ matrices, and the highest coefficient is a lower-triangular matrix with positive diagonal entries. The recurrence relation (1.11a) is equivalent to the three-term relation for the matrix orthogonal polynomials $\{P_k^l(x)\}_{k \in \mathbb{N}_0}$, and we can relate (1.11b) to the theory of matrix orthogonal polynomials in a similar manner.

The vector polynomials $P_{k,l}$ and the matrices $A_{k,l}$ and $B_{k,l}$ in (1.11a) depend on the basis $B_l$ in (1.10), but it is easy to see that the condition

$$A_{k+1,l} = \frac{1}{2} I_{l+1} \quad \text{and} \quad B_{k,l} = 0$$

is independent of $B_l$ and the same is true for the coefficients in (1.11b). Thus, we can formulate question (I) precisely by replacing (1.5) with

$$\begin{align*}
A_{k+1,l} &= \frac{1}{2} I_{l+1}, & B_{k,l} &= 0, & \text{for all } & k \geq n, \ l \geq m, \\
\tilde{A}_{k+1,l} &= \frac{1}{2} I_{k+1}, & \tilde{B}_{k,l} &= 0, & \text{for all } & k \geq n, \ l \geq m,
\end{align*}$$

for $n$ and $m$ sufficiently large. Moreover, if we consider a product measure $\mu_1(x) \times \mu_2(y)$ on $\mathbb{R}^2$, where each $\mu_j$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then one can show that (1.12) holds for $n$ and $m$ sufficiently large if and only if both $\mu_1(x)$ and $\mu_2(y)$ are Bernstein-Szegő measures of the form (1.13).

The first nontrivial example of a measure on $\mathbb{R}^2$ for which equations (1.12) hold was discovered in [11, Section 3] as a one-parameter deformation of the Chebyshev measure (1.3). It was built in a rather roundabout way from an algorithm in [4] which guaranteed that (1.12a)-(1.12b) hold for $n = m = 1$, and inverse scattering techniques were used to derive the corresponding measure which can be written as follows

$$d\mu(x,y) = \frac{4}{\pi^2} \frac{\chi_{(-1,1)^2}(x,y)}{\sqrt{1-x^2} \sqrt{1-y^2}} \omega(z,w) \omega(1/z,w) \omega(z,1/w) \omega(1/z,1/w) \, dx \, dy,$$

where $x = \frac{1}{2} (z + \frac{1}{z})$, $y = \frac{1}{2} (w + \frac{1}{w})$, 

$$\omega(z,w) = \frac{1 - czw}{\sqrt{1 - c^2}}, \quad \text{and } c \in (-1,1) \text{ is a free parameter.}$$

This was extended later in [5, Theorem 4.2] where it was shown that equations (1.12) hold for $n$ and $m$ sufficiently large if we consider measures of the form (1.13).
with
\[ \omega(z, w) = \prod_{j=1}^{N} (1 + c_j zw), \quad \text{where } c_j \in (-1, 1) \text{ are free parameters.} \]

While it seemed natural to believe that (1.12) must hold for arbitrary polynomials \( \omega(z, w) \) nonvanishing when \(|z| \leq 1\) and \(|w| \leq 1\), the proof in [5] was rather involved, using several technical intermediate steps, see Lemmas 3.7-3.8 and Theorems 3.9-3.11 in [5], and it was clear that this approach could not be easily extended.

Our first result settles this conjecture for arbitrary polynomials \( \omega(z, w) \).

**Theorem 1.1** (Bernstein-Szegő measures on \( \mathbb{R}^2 \)). Suppose that \( n_0, n_1, m_0, m_1 \) are nonnegative integers and
- \( \omega(z, w) \in \mathbb{R}_{n_0,m_0}[z, w] \) is nonzero for \(|z| < 1, |w| < 1\),
- \( q_1(x) \in \mathbb{R}_{2n_1}[x] \) is positive for \( x \in (-1, 1) \),
- \( q_2(y) \in \mathbb{R}_{2m_1}[y] \) is positive for \( y \in (-1, 1) \),

and

\[ Q(x, y) = q_1(x)q_2(y)\omega(z, w)\omega(1/z, w)\omega(z, 1/w)\omega(1/z, 1/w) \quad (1.14) \]

is such that

\[ \iint_{(-1,1)^2} \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{Q(x, y)} \, dx \, dy < \infty. \quad (1.15) \]

Then the recurrence coefficients for the measure

\[ d\mu(x, y) = \frac{4}{\pi^2} \frac{\chi(-1,1)^2(x, y)\sqrt{1-x^2}\sqrt{1-y^2}}{q_1(x)q_2(y)\omega(z, w)\omega(1/z, w)\omega(z, 1/w)\omega(1/z, 1/w)} \, dx \, dy, \quad (1.16) \]

satisfy equations (1.12) with \( n = n_0 + n_1 \) and \( m = m_0 + m_1 \).

If \( \omega(z, w) = 1 \) we obtain the trivial examples corresponding to product measures discussed earlier. Note that we only require \( \omega(z, w) \) to be nonzero inside the bi-disk \(|z| < 1, |w| < 1\) as long as the moments are finite. Thus, the theorem applies also to examples like \( \omega(z, w) = 2 + z + w \) when \( q_1(x) \) and \( q_2(y) \) are positive on \([-1, 1]\).

If \( \tilde{q}_2(w) \) denotes the stable Fejér-Riesz factors of \( q_2(y) \) and if we set \( p(x, w) = \tilde{q}_2(w)\omega(z, w)\omega(1/z, w) \), then \( Q(x, y) \) in (1.16) can be written as

\[ Q(x, y) = q_1(x)p(x, w)p(x, 1/w). \quad (1.17) \]

Our proof of Theorem 1.1 actually establishes a stronger statement, namely that the factorization in (1.17) implies (1.12a).

**Theorem 1.2.** Suppose that for some \( n_1 \leq n \), the polynomials \( q(x) \in \mathbb{R}_{2n_1}[x] \) and \( p(x, w) \in \mathbb{R}_{n-n_1,2m}[x, w] \) are such that
- \( (i) \ q(x) > 0 \) and \( p(x, w) \neq 0 \) when \( x \in (-1, 1), |w| < 1 \),
- \( (ii) \ \int_{(-1,1)^2} \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{q(x)p(x, w)p(x, 1/w)} \, dx \, dy < \infty. \)

Then the recurrence coefficients of the measure

\[ d\mu(x, y) = \frac{4}{\pi^2} \frac{\chi(-1,1)^2(x, y)\sqrt{1-x^2}\sqrt{1-y^2}}{q(x)p(x, w)p(x, 1/w)} \, dx \, dy, \quad (1.18) \]
satisfy (1.12a) and the orthonormal vector polynomials \( \{P_{k,i}(x,y)\} \) are related to \( q(x) \) and \( p(x,w) \) via
\[
\begin{align*}
q(x) & = -\frac{(ww_1)^{-m-1}p(x,w)p(x,w_1) - (ww_1)^{m+2}p(x,1/w)p(x,1/w_1)}{(w_1 - 1/w)(w_1 - 1/w_1)} \\
& + \frac{(w_1/w)^{m+1}p(x,w)p(x,1/w_1) - (w/w_1)^{m+1}p(x,1/w)p(x,1/w_1)}{w - w_1} \\
& = P_{n,m}(x,y)^tP_{n,m}(x,y) - 4xp_{n,m}(x,y)^tA_{k,m}^tP_{n-1,m}(x,y) \\
& + 4P_{n-1,m}(x,y)^tA_{k,m}A_{n,m}^tP_{n-1,m}(x,y).
\end{align*}
\] (1.19)

Clearly, Theorem 1.2 implies Theorem 1.1 since we can exchange the roles of \( x \) and \( y \), and apply Theorem 1.2 for the analogous factorization of \( (1.17) \) to get (1.12a). Another key ingredient of this theorem is (1.19) which provides a necessary condition for the opposite direction. This condition is missing in the one-dimensional case thanks to the Fejér-Riesz lemma [7, 23], but the extension of this classical result to several variables leads to a series of difficult questions, closely related to the famous 17th problem of Hilbert which lie at the interface of analysis, algebra and algebraic geometry. The identity (1.19) allows us to completely characterize the measures in (1.13) by working with a finite-dimensional space of orthogonal polynomials, and by imposing nonvanishing conditions similar to (1.7). The precise statement is as follows.

**Theorem 1.3.** Let \( \mathcal{L} : \mathbb{R}_{2n,2m}[x,y] \to \mathbb{R} \) be a positive linear functional with vector orthonormal polynomials \( \{P_{k,m}(x,y)\}_{k \leq n} \), and suppose that for some \( n_1 \leq n \) there exist polynomials \( q(x) \in \mathbb{R}_{2n}[x], p(x,w) \in \mathbb{R}_{n-n_1,2m}[x,w] \) satisfying (1.19). If
(a) \( p(x,w) \neq 0 \) for \( x \in (-1,1) \), \( w \in (-1,1) \), and
(b) \( \Psi_n(z) = z^n(P^m_n(x) - 2zA_{k,m}^tP_{n-1,m}(x)) \) is invertible for \( z \in (-1,1) \),

then \( q(x) \) and \( p(x,w) \) satisfy conditions (i)-(ii) in Theorem 1.2 and
\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{[-1,1]^2} f(x,y) \sqrt{1 - x^2} \sqrt{1 - y^2} q(x)p(x,w)p(x,1/w) \, dx \, dy, \quad \text{for all } f \in \mathbb{R}_{2n,2m}[x,y].
\]

Given a functional \( \mathcal{L} \), there is a constructive way to check whether (1.19) holds, and thus the above theorem can be used in practice to reconstruct the polynomials \( p(x,w) \) and \( q(x) \) from the moments. We illustrate this with several examples at the end of the paper. Moreover, one can extend Theorem 1.3 and characterize the Bernstein-Szegő measures (1.16); we omit the details and we refer the reader to Theorem 1.12 for the precise statement.

These theorems provide a complete answer to questions (I–III) for the two-dimensional Bernstein-Szegő measures discussed above. We now turn to the last question. Let us start with the measure in (1.15) possessing the one-sided factorization. Using the one-dimensional theory discussed briefly in Section 1.1, one can easily see that if \( \{U_j^q(x)\}_{j \in \mathbb{N}_0} \) are the orthonormal polynomials with respect to the measure \( \frac{2}{\pi q(x)} \chi_{(-1,1)}(x) \, dx \) on \( \mathbb{R} \), and if we set
\[
p_M(y;x) = \frac{w^{M+1}p(x,1/w) - w^{-M-1}p(x,w)}{w - 1/w},
\]
then the polynomials
\[
\{p_M(y;x)U_j^q(x)\}_{j=0}^{N-n_0}, \quad \text{where } n_0 = n - n_1,
\] (1.20)
are orthonormal elements in the space $\tilde{P}_{N,M}[x,y]$ when $N \geq n$, $M \geq m$. The natural question now is whether we can extend this set to a complete orthonormal basis of the space $\tilde{P}_{N,M}[x,y]$. Note that we have already used the stable Fejér-Riesz factor for the inverse of the weight, and we need $n_0$ new quantities to construct the complement. It turns out that the spaces introduced in [12] for the Bernstein-Szegő measures on the bi-circle can be used to answer this question. If $p(x, w) \neq 0$ for $x \in [-1,1]$ and $|w| \leq 1$, we can look at the spaces $\tilde{P}_{k,l;\Gamma^2}[z,w]$ of orthogonal polynomials for the measure

$$
\frac{1}{(2\pi)^2} \frac{|dz| |dw|}{p(x, w)p(x, 1/w)} \quad \text{on} \quad \Gamma^2 = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\} \quad (1.21)
$$

which are defined just like the ones for $\mathbb{R}^2$ in (1.9)). However, in order to connect the two sets of polynomials, we need to choose $k = 2n_0 - 1$ and $l = 2M + 1$ which means that on the bi-circle we get a space whose dimension is twice the dimension we need to complete the orthonormal set of real polynomials. A key fact discovered in [12] and further analyzed in [14] is that the space $\tilde{P}_{2n_0-1,2M+1,\Gamma^2}[z,w]$ has a subtle decomposition as a direct sum of two subspaces $\tilde{P}_{2n_0-1,2M+1,\Gamma^2}[z,w]$ and $\tilde{P}_{2n_0-1,2M+1;\Gamma^2}[z,w]$ which possess extra orthogonality conditions and “nice” spectral properties. Moreover, if we start with the measure (1.21), these two subspaces are simply related to each other by a reflection in $z$, and thus we can focus on one of them, say $\tilde{P}_{2n_0-1,2M+1,\Gamma^2}[z,w]$ having exactly the dimension $n_0$ we need. The extra orthogonality and spectral properties of this space allow us to construct a Szegő map $S : P_{1,2n_0-1,2M+1,\Gamma^2}[z,w] \to P_{N,M}[x,y]$ which completes the set in (1.20) to an orthonormal basis. But there is an interesting twist in this story: while $S$ is a linear isomorphism of $\tilde{P}_{2n_0-1,2M+1,\Gamma^2}[z,w]$ onto the complement of the set in (1.20), this map is not an isometry, and we need to modify the inner product on $P_{1,2n_0-1,2M+1,\Gamma^2}[z,w]$ in order to get an orthonormal basis of polynomials in the plane. For the Bernstein-Szegő measures (1.16) many simplifications take place and we can construct explicit orthonormal bases on both sides in terms of fixed orthonormal polynomials on the bi-circle contained in the finite-dimensional space $\mathbb{R}_{n_0,m_0}[z,w]$. The precise statement for the measures in (1.18) and (1.16) are given in Theorems 6.2 and 6.6, respectively.

It would be interesting to see if the absolutely continuous measures of the form given in (1.16) are completely characterized by (1.12). A more difficult question is to drop the absolute continuity condition and to describe all possible measures on $\mathbb{R}^2$ for which equations (1.12) hold. This is well-known in the one-variable case, see for instance [3,13] and the references therein. A similar spectral characterization for Bernstein-Szegő measures on the bi-circle can be found in [12]. The problem in $\mathbb{R}^2$ is significantly more difficult since the weight of the absolutely continuous part of the measure is no longer simply related to a single orthonormal polynomial. Moreover, if we drop the absolute continuity, the measures can have singular continuous and discrete components. Interesting examples which illustrate this beyond the case of product measures are presented in the last section.

The paper is organized as follows. The next section provides a short introduction to the one-dimensional Bernstein-Szegő theory for the measures in (1.1), emphasizing the interplay between orthogonal polynomials, Hankel matrices and reproducing kernels which are important here. Most of the results in this section are known, but we have included enough details to make the presentation self-contained and
to stress some subtle points which play a crucial role in the proofs. In Section 3 we develop the Bernstein-Szegő theory for matrix-valued measures. This section is of independent interest and can be used as a self-contained account for the matrix theory. In Section 4 we prove Theorems 1.1-1.3 above together with the characterization of the Bernstein-Szegő measures (1.16). In Section 5 we have collected some of the constructions and the results from [12, 14] for measures on the bi-circle together with several new facts needed for the measures in the plane. In Section 6 we define a bivariate extension of the Szegő map and we use it to construct bases of orthogonal polynomials for the spaces associated with Bernstein-Szegő measures on $\mathbb{R}^2$. Section 7 contains two explicit examples illustrating different constructions in the paper and the importance of some of the conditions imposed in the main theorems.

1.3. Notations. Throughout the paper, we use the following notations and conventions.
- $\mathbb{R}$ and $\mathbb{C}$ will denote the fields of real and complex numbers, respectively.
- $\mathbb{Z}$ and $\mathbb{N}_0$ will denote the ring of integers and the set of nonnegative integers.
- $D = \{ z \in \mathbb{C} : |z| < 1 \}$, $\overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and $T = \{ z \in \mathbb{C} : |z| = 1 \}$ will denote the open disk, the closed disk and the unit circle in $\mathbb{C}$, respectively.
- We will use $x$ and $y$ to denote real variables, which will be related to the complex variables $z$ and $w$ via the formulas
  \[ x = \frac{1}{2} \left( z + \frac{1}{z} \right) \quad \text{and} \quad y = \frac{1}{2} \left( w + \frac{1}{w} \right). \]
- For a ring $F$, we denote by $F[a_1, \ldots, a_k]$ the ring of polynomial in $a_1, \ldots, a_k$ with coefficients in $F$, and by $F_{m_1, \ldots, m_k}[a_1, \ldots, a_k]$ the space of polynomials such that $a_j$ has degree at most $m_j$ for every $j = 1, \ldots, k$.
- $\mathbb{R}^{l \times l}$ denotes the ring of all $l \times l$ matrices with real coefficients, and $\text{Sym}(\mathbb{R}^{l \times l})$ is the space of all symmetric matrices. In particular, for $N \in \mathbb{N}_0$ we denote by $\mathbb{R}^{l \times l}_N[x]$ the space of polynomials of degree at most $N$ in $x$ with coefficients in $\mathbb{R}^{l \times l}$.
- To unify the notation, we develop the matrix and the bivariate Bernstein-Szegő theory working with positive linear functionals rather than measures.

2. One-dimensional Bernstein-Szegő weights, Hankel matrices and reproducing kernels

In this section we review the connection between Bernstein-Szegő measures on $\mathbb{R}$, inverses of Hankel matrices, reproducing kernels and the Christoffel-Darboux formula.

Recall that $y = \frac{w^{1/w} + 1}{2}$ and consider measures of the form

\[ d\mu = \frac{2}{\pi} \frac{\sqrt{1 - y^2}}{|q(w)|^2} \chi_{(-1,1)}(y)dy \quad (2.1) \]

where $q(w) = q_m w^m + \cdots + q_0$ is a polynomial of degree $m$ with real coefficients nonvanishing for $w \in \overline{D}$, except for possible simple zeros at $w = \pm 1$. Set

\[ p_k(y) = \frac{w^{k+1} q(1/w) - w^{-k-1} q(w)}{w - 1/w}, \quad (2.2) \]

and note that

\[ w^{k+1} q(1/w) \in \text{span}\{w^{k+1}, \ldots, w^{k+m+1}\}. \]
This shows that if \( k \geq m - 1 \) then \( p_k(y) \) has degree \( k \), while if \( k < m - 1 \) the degree of \( p_k(y) \) is \( \leq \max(k, m - k - 2) \). In both cases, since \( q(w) \) has real coefficients, it follows that \( p_k(y) \) has real coefficients which depend linearly on the coefficients of \( q(w) \). Using the method in Szegő, see [26, Theorem 2.6, (2.6.3)] we have

**Proposition 2.1.** If \( 2k + 1 \geq m \) then \( p_k(y) \) is a polynomial of degree \( k \) orthonormal with respect to \( \mu \).

**Proof.** Set \( w = e^{i\varphi} \) and note that for every polynomial \( f(y) \) we have

\[
\int f(y) p_k(y) d\mu(y) = \int_0^{2\pi} f(y) \left( w^{k+1} q(1/w) - w^{-k-1} q(w) \right) \frac{\sin \varphi}{\pi i |q(w)|^2} d\varphi
= \frac{1}{\pi i} \int_{-\pi}^{\pi} f(y) w^{k+1} \frac{\sin \varphi}{q(w)} \sin d\varphi
= -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(y) w^k}{q(w)} \left( w - \frac{1}{w} \right) dw.
\]

Since \( q(w) \) is nonzero for \( w \in \mathbb{T} \) except possibly for simple zeros at \( w = \pm 1 \), the Cauchy-Goursat theorem shows the last integral is 0 when \( f(y) \) is a polynomial of degree less than \( k \). When \( f(y) = p_k(y) \) we find

\[
\int p_k^2(y) d\mu(y) = -\frac{1}{2\pi i} \int_{\mathbb{T}} \left( w^{2k+1} q(1/w) - w^{-k-1} q(w) \right) w^k dw
= -\frac{1}{2\pi i} \int_{\mathbb{T}} \left( \frac{w^{2k+1} q(1/w)}{q(w)} - \frac{1}{w} \right) dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dw}{w} = 1,
\]

completing the proof. \( \square \)

**Remark 2.2.** Note that if \( m = 2k + 2 \), the comments before Proposition 2.1 and the proof above show that \( p_k(y) \) defined in (2.2) is a polynomial of degree \( k \), which is orthogonal, but not orthonormal with respect to \( \mu \), since

\[
\int p_k^2(y) d\mu(y) = \frac{q_0 - q_m}{q_0}. \tag{2.3}
\]

Let \( h_j = \int y^j d\mu \) be the moments with respect to the measure in (2.1), and for \( k \in \mathbb{N}_0 \) let \( H_k \) be the Hankel matrix

\[
H_k = \begin{bmatrix}
h_0 & h_1 & \cdots & h_k \\
h_1 & h_2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
h_k & \cdots & \cdots & h_{2k}
\end{bmatrix}. \tag{2.4}
\]

Recall that if \( \{p_k^\circ(y)\}_{k=0}^{\infty} \) are orthonormal polynomials with respect to \( \mu \), then for \( k \in \mathbb{N}_0 \) the reproducing kernel \( K_k(y, y_1) \) is defined as follows

\[
K_k(y, y_1) = p_k^\circ(y) p_k^\circ(y_1) + \cdots + p_k^\circ(y) p_k^\circ(y_1).
\]

On one hand, it is easy to see that

\[
K_k(y, y_1) = [1, y, \ldots, y^k] H_k^{-1} [1, y_1, \ldots, y_1^k]^t. \tag{2.6}
\]

Indeed, the function \( K_k(y, y_1) \) defined in (2.6) satisfies

\[
\int K_k(y, y_1) y^s d\mu(y) = y_1^s. \tag{2.7}
\]
for every \( s = 0, 1, \ldots, k \), hence it must coincide with the reproducing kernel in (2.5). On the other hand, if we set
\[
p_k^o(y) = \sum_{j=0}^{k} \alpha_k y^j,
\]
then by the Christoffel-Darboux formula [26, formula (3.2.3)] we have
\[
K_k(y, y_1) = \alpha_{kk} \frac{p_{k+1}(y)p_k^o(y_1) - p_k^o(y)p_{k+1}^o(y_1)}{y - y_1}.
\] (2.8)
As an immediate corollary of Proposition 2.1 and Remark 2.2 we obtain the following explicit formula for the reproducing kernel.

**Lemma 2.3.** If \( 2k + 2 \geq m \), then the reproducing kernel \( K_k(y, y_1) \) of the measure \( \mu \) in (2.1) is given by the formula
\[
K_k(y, y_1) = \frac{p_{k+1}(y)p_k(y_1) - p_k(y)p_{k+1}(y_1)}{2(y - y_1)},
\] (2.9)
where \( p_j(y) \) are the polynomials defined in (2.2).

**Proof.** If \( 2k + 1 \geq m \), then we can take \( p_k^o(y) = p_j(y) \) for \( j = k \) and \( j = k + 1 \) by Proposition 2.1. Equation (2.8) implies that \( \alpha_{jj} = 2^{j}q_0 \), which combined with (2.8) establishes (2.9).

If \( 2k + 2 = m \), then we can take \( p_k^o(y) = \sqrt{q_0/(q_0 - q_m)}p_k(y) \) by Remark 2.2 and \( p_{k+1}^o(y) = p_{k+1}(y) \) by Theorem 2.1. This shows that \( \alpha_{kk} = 2^k \sqrt{q_0(q_0 - q_m)} \), \( \alpha_{k+1,k+1} = 2^{k+1}q_0 \) and substituting these formulas in (2.8) we obtain (2.9). \( \square \)

The next lemma tells that the entries of the matrix \( H_k^{-1} \) coincide with coefficients in the expansion of reproducing kernel \( K_k(y, y_1) \) in powers of \( y \) and \( y_1 \).

**Lemma 2.4.** The entries \( (H_k^{-1})_{j\ell} \) of the Hankel matrix (2.4) can be determined from the reproducing kernel (2.5) as follows
\[
K_k(y, y_1) = \sum_{j,\ell=0}^{k} (H_k^{-1})_{j\ell} y^j y_1^\ell.
\] (2.10)

**Proof.** If we expand \( K_k(y, y_1) \) in powers of \( y \) and \( y_1 \)
\[
K_k(y, y_1) = \sum_{j,\ell=0}^{k} \beta_{j\ell} y^j y_1^\ell,
\]
then for every \( s = 0, \ldots, k \), equation (2.10) holds and therefore
\[
\sum_{j,\ell=0}^{k} \beta_{j\ell} h_{j+s} y_1^\ell = y_1^s.
\]
Comparing the coefficients of \( y_1^\ell \) in both sides of the last formula we see that
\[
\sum_{j=0}^{k} h_{j+s} \beta_{j\ell} = \delta_{s,\ell}.
\] (2.11)
Since the ℓ-th column of $H_k^{-1}$ is the unique solution $(z_0, \ldots, z_k)^t$ of the linear system of equations
\[
\sum_{j=0}^k h_{s+j} z_j = \delta_{s,\ell}
\]
where $s = 0, 1, \ldots, k$, we conclude from (2.11) that $z_j = \beta_{j,\ell}$.

From Lemma 2.3 and Lemma 2.4 we obtain the following important corollary.

**Corollary 2.5.** If $2k + 2 \geq m$, then the entries of $H_k^{-1}$ are homogeneous quadratic polynomials of the coefficients of $q(w)$.

The next proposition provides a characterization of Hankel matrices in terms of reproducing kernels, or the stable Fejér-Riesz factors of Bernstein-Szegő weights.

**Proposition 2.6.** For a real positive definite $(k+1) \times (k+1)$ matrix $H$ the following conditions are equivalent.

(i) $H$ is a Hankel matrix.

(ii) There exist $p_k(y) \in \mathbb{R}_k[y]$ and $p_{k-1}(y) \in \mathbb{R}_{k-1}[y]$ such that
\[
\frac{p_k(y)p_{k-1}(y) - p_k(y_1)p_{k-1}(y)}{2(y - y_1)} + p_k(y)p_k(y_1) = [1, y, \ldots, y^k] H^{-1} \begin{bmatrix} 1 \\ y_1 \\ \vdots \\ y_1^k \end{bmatrix}.
\] (2.12)

(iii) There exists $q(w) \in \mathbb{R}_{2k}[w]$ such that
\[
\frac{(ww_1)^{-k-1} q(w)q(w_1) - (ww_1)^{k+2} q(1/w)q(1/w_1)}{1 - ww_1}
+ \frac{(w_1/w)^{k+1} w_1 q(w)q(1/w_1) - (w_1/w)^{k+1} wq(1/w)q(w_1)}{w - w_1}
= [1, y, \ldots, y^k] H^{-1} [1, y_1, \ldots, y_1^k]^t.
\] (2.13)

**Proof.** The equivalence (i)$\iff$(ii) follows from the work of Lander [21] on Bezoutians, but we outline a direct proof in our setting, which allows us to reconstruct the polynomials $p_k(y)$, $p_{k-1}(y)$ and $q(w)$ uniquely from $H$, up to an overall sign.

Suppose first that (i) holds, i.e. $H = H_k$ has the Hankel structure in (2.4). Then $L(y^j) = h_j$ for $j = 0, 1, \ldots, 2k$ extends to a positive linear functional $L : \mathbb{R}_{2k}[y] \to \mathbb{R}$. If we denote by \{p_{j}^k(y)\}_{0 \leq j \leq k}$ the orthonormal polynomials with respect to $L$, then using formulas (2.5), (2.6) and (2.8) we see that (ii) holds if we set $p_k(y) = p_k^k(y)$ and $p_{k-1}(y) = (2\alpha_{k-1,k-1}/\alpha_{kk}) p_{k-1}^k(y)$.

Conversely, suppose now that (ii) holds. Let $H = LL^t$ be the Cholesky factorization of $H$, and let $p_0^k(y), \ldots, p_k^k(y)$ be the components of the vector $L^{-1}[1, y, \ldots, y^k]^t$. Then (2.12) can be rewritten as
\[
\frac{p_k(y)p_{k-1}(y) - p_k(y_1)p_{k-1}(y)}{2(y - y_1)} + p_k(y)p_k(y_1) = p_0^k(y)p_0^k(y_1) + \cdots + p_k^k(y)p_k^k(y_1).
\]

One can show now by induction on $k$ that if the last equation holds, then we can define a positive linear functional $L : \mathbb{R}_{2k}[y] \to \mathbb{R}$ with orthonormal polynomials $p_0^k(y), \ldots, p_k^k(y)$. Then the right-hand side of (2.12) will be the reproducing kernel for this functional. Therefore $H$ will be a Hankel matrix of the form given in (2.4), where $h_j = L(y^j)$. This completes the proof of the equivalence (i)$\iff$(ii).
To see that (ii)⇒(iii), note that the linear mapping
\[
\mathbb{R}_k[y] \times \mathbb{R}_{k-1}[y] \to \mathbb{R}_{2k}[w]
\]
\[
(p_k(y), p_{k-1}(y)) \to q(w) = w^k(p_k(y) - wp_{k-1}(y))
\] (2.14)
is an isomorphism with inverse \( p_j(y) = \frac{w^{j+1}q(1/w) - w^{j+1}q(w)}{w^{j+1} - w^{j+1}} \), where \( j = k \) and \( j = k-1 \). It is straightforward to check that \((p_k(y), p_{k-1}(y))\) satisfy (2.12) if and only if \( q(w) \) satisfies (2.13), completing the proof. \( \square \)

**Remark 2.7.** Suppose that the equivalent conditions in Proposition 2.6 hold. If we set \( p_j(y) = \sum_{t=0}^j \beta_t y^t \) for \( j = k \) and \( j = k-1 \), and if we compare the coefficients of \( y_k \) on both sides of (2.12), it follows that
\[
\beta_{kk}p_k(y) = [1, y, \ldots, y^k] H^{-1}[0, 0, \ldots, 0, 1]^t. \quad (2.15)
\]
Comparing now the coefficients of \( y^k \) we see that \( \beta_{kk}^2 = (H^{-1})_{k+1,k+1} \). In particular, this shows that \( \beta_{kk} \neq 0 \) and is uniquely determined from the right-hand side of (2.12), up to a sign. We can use now (2.15) to compute \( p_k(y) \) and it is determined uniquely from \( H \), up to an overall sign. Without a restriction, we can normalize \( \beta_{kk} \) to be positive, and then \( p_k(y) \) will coincide with the orthonormal polynomial \( p^*_k(y) \) of degree \( k \) with respect to the positive linear functional \( L \) whose moment matrix is \( H \). Once we compute \( p_k(y) \), we can multiply both sides of (2.12) by \( 2(y-y_i) \), and comparing the coefficients of \( y^k \) we can compute \( p_{k-1}(y) \). It will be a polynomial of degree \( k-1 \) with positive highest coefficient, which up to a positive factor is equal to the orthonormal polynomial \( p^*_k(y) \) of degree \( k-1 \) with respect to \( L \). Moreover, if \( wp^*_k(y) = a_k p^*_k(y) + b_k p^*_k(y) + a_{k-1} p^*_{k-2}(y) \) is the three-term recurrence relation for the orthonormal polynomials, then
\[
p^*_k(y) - 2a_k wp^*_k(y) = p_k(y) - wp_{k-1}(y). \quad (2.16)
\]
This shows that \( q(w) = w^k(p_k(y) - wp_{k-1}(y)) \) in (2.13) is uniquely determined from (2.12) up to an overall sign and \( q(0) \neq 0 \). In particular, if we normalize the highest coefficient of \( p_k(y) \) to be positive as above, then the linear mapping (2.14) will fix the unique solution of (2.13) for which \( q(0) > 0 \).

**Remark 2.8.** One can use the computation in the proof of Proposition 2.1 to show that if \( q(w) = q_m w^m + \cdots + q_0 \) is a polynomial of degree \( m \) which is nonzero for \( w \in \mathbb{D} \), then the moments
\[
h_j(q) = \frac{2}{\pi} \int_{-1}^{1} y^j \sqrt{1 - y^2} \frac{1}{|q(w)|^2} \, dy, \quad j \in \mathbb{N}_0 \quad (2.17)
\]
for the measure in (2.1) are rational functions of the coefficients \( q_0, \ldots, q_m \). In particular, this shows that they can be computed explicitly using algebraic manipulations. Indeed, rewriting the integral above as an integral over the unit circle we see that
\[
h_j(q) = -\frac{1}{2^{j+2\pi i}} \oint_{|q(w)|=\infty} \frac{(w + \frac{1}{w})^j (w - \frac{1}{w})^2}{q(w)q(1/w)} \frac{dw}{w}, \quad j \in \mathbb{N}_0, \quad (2.18)
\]
and using Cauchy’s residue theorem we can evaluate the contour integral by computing the residues of the integrand at \( w = 0 \) and at the zeros of the polynomial \( \frac{q(w)}{q(1/w)} \) which all lie in \( \mathbb{D} \). This will give a rational symmetric function of the zeros \( \frac{q}{q} \) of \( q(w) \), which can be expressed in terms of the coefficients of \( q(w) \) by
Vieta’s formulas. The next proposition provides a more detailed information about these formulas.

We denote by $\Re(f, g)$ the resultant of polynomials $f(w)$ and $g(w)$.

**Proposition 2.9.** Suppose that $q(w) = q_m w^m + \cdots + q_0 \in \mathbb{R}[w]$ is a polynomial of degree $m$ which is nonzero for $w \in \mathbb{R}$ and let $\overline{q}(w) = w^m q(1/w)$. If $w_1, \ldots, w_m$ denote the zeros of $q(w)$, then

$$
\Re(q, \overline{q}) = (-1)^m q(1) q(-1) \Delta_m(q)^2,
$$

where

$$
\Delta_m(q) = (-q_m)^{m-1} \prod_{1 \leq k < l \leq m} (w_k w_l - 1) \in \mathbb{Z}[q_0, \ldots, q_m]
$$

is a homogeneous polynomial of degree $m - 1$ in the coefficients $q_0, \ldots, q_m$. Moreover, for the moments $h_j(q)$ in (2.17) we have

$$
2^j \Delta_m(q) h_j(q) \in \mathbb{Z}[q_0^{m+1}, q_1, q_2, \ldots, q_m].
$$

**Proof.** Since $\overline{q}(w)$ has roots $w_1^{-1}, \ldots, w_m^{-1}$ and the highest coefficient is $q_0$, the formula for the resultant yields

$$
\Re(q, \overline{q}) = q_m^m q_0^{m-1} \prod_{k,l=1}^m (w_k - 1/w_l) = q_m^m q_0^{m-1} \prod_{k,l=1}^m (w_k w_l - 1)
$$

$$
= (-1)^m \left( q_m^2 \prod_{k=1}^m (w_k^2 - 1) \right) q_m^{2m-2} \prod_{1 \leq k \neq l \leq m} (w_k w_l - 1)
$$

$$
= (-1)^m q(1) q(-1) \left( \prod_{1 \leq k < l \leq m} (w_k w_l - 1) \right)^2,
$$

establishing the factorization in (2.19). The sign in (2.20) is chosen so that $q_0^{m-1}$ has coefficient 1 in the expansion of $\Delta_m(q)$.

Let $\varphi_j(w) \in \mathbb{Z}[w]$ be such that

$$
\left( w + \frac{1}{w} \right)^j \left( w - \frac{1}{w} \right)^2 = \varphi_j(w) + \varphi_j(1/w).
$$

Clearly, $\varphi_j(w)$ has zeros at $w = \pm 1$ and from (2.18) we see that

$$
2^j h_j(q) = -\frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{\varphi_j(w)}{q(w)q(1/w)} \frac{dw}{w}.
$$

Without any restriction, we can assume that the zeros $w_1, \ldots, w_m$ of $q(w)$ are simple. Applying Cauchy’s residue theorem it follows that

$$
2^j h_j(q) = \sum_{k=1}^m \frac{\varphi_j(w_k) w_k}{q(1/w_k) q'(w_k)} - \text{Res}_{w=0} \frac{\varphi_j(w)}{w q(w) q(1/w)}.
$$

The right-hand side of the last equation is a rational symmetric function of $w_1, \ldots, w_m$. Our next goal is to see what common denominator we can choose so that the numerator becomes a Laurent polynomial from the space $\mathbb{Z}[w_1^\pm 1, \ldots, w_m^\pm 1]$. 
Note that
\[ q(1/w_k) = q_m \prod_{j=1}^{m} (1/w_k - w_j) = \frac{q_m}{w_k^m} \prod_{j \neq k} (1 - w_k w_j), \]
and the term \((1 - w_k^2)\) will cancel with a factor in \(\varphi_j(w_k)\) since \(\varphi_j(\pm 1) = 0\). The term \(\prod_{j \neq k} (1 - w_k w_j)\) will cancel if we multiply \(h_j(q)\) by \(\Delta_m(q)\). Next, note that \(q'(w_k) = q_m \prod_{j \neq k} (w_k - w_j)\), so all these terms will be canceled if we multiply \(h_j(q)\) by the Vandermonde determinant \(V(w_1, \ldots, w_m) = \prod_{1 \leq j < k \leq m} (w_k - w_j)\). Finally, note that the residue at \(w = 0\) belongs to \(\mathbb{Z}[w_{\pm 1}^1, \ldots, w_{m}^{\pm 1}]\). Thus, we conclude that
\[ 2^j h_j(q) \Delta_m(q) = \frac{F(w_1, \ldots, w_m)}{V(w_1, \ldots, w_m)} \text{ for some } F(w_1, \ldots, w_m) \in \mathbb{Z}[w_{\pm 1}^1, \ldots, w_{m}^{\pm 1}]. \]
The left-hand side \(2^j h_j(q) \Delta_m(q)\) of the last equation is a symmetric function of \(w_1, \ldots, w_m\) and therefore \(\frac{F(w_1, \ldots, w_m)}{V(w_1, \ldots, w_m)}\) must also be symmetric. Since \(V(w_1, \ldots, w_m)\) is skew-symmetric, it follows that \(F(w_1, \ldots, w_m)\) is also skew-symmetric and therefore divisible by each \((w_k - w_j)\) for \(j < k\). This shows that \(2^j h_j(q) \Delta_m(q) \in \mathbb{Z}[w_{\pm 1}^1, \ldots, w_{m}^{\pm 1}]\), and applying Vieta’s formulas we see that
\[ 2^j h_j(q) \Delta_m(q) \in \mathbb{Z}[q_0^{\pm 1}, q_1, q_2, \ldots, q_{m-1}, q_m^{\pm 1}]. \]
Finally, note that for arbitrary \(q_0, q_1, \ldots, q_{m-1}\) such that \(|q_0| > |q_1| + \cdots + |q_{m-1}|\), the function \(2^j h_j(q) \Delta_m(q)\) must be analytic in a neighborhood of \(q_m = 0\), which shows that (2.21) holds. \(\square\)

The formulas obtained in Proposition 2.9 extend to case when \(q(w)\) has simple roots at \(w = \pm 1\).

**Example 2.10.** Suppose \(q(w) = q_4 w^4 + q_3 w^3 + q_2 w^2 + q_1 w + q_0\) is a polynomial of degree 4, which is nonzero for \(w \in \mathbb{D}\). A straightforward computation shows that (2.19) and (2.20) hold with
\[ \Delta_4(q) = q_0^3 - q_0^2 q_2 + q_0 q_1 q_3 - q_0 q_2 q_3 - q_2^2 q_4 - q_1^2 q_4 + 2 q_0 q_2 q_4 + q_1 q_3 q_4 - 9 q_0^2 q_4 - q_2 q_4^2 + q_4^3, \]
and the first few moments (2.17) can be computed from the coefficients by the following formulas
\[ h_0(q) = \frac{q_0 - q_4}{\Delta_4(q)}, \]
\[ h_1(q) = \frac{q_3 - q_4}{2 \Delta_4(q)}, \]
\[ h_2(q) = \frac{q_0^2 + q_1^2 - q_0 q_2 - q_1 q_3 + q_2 q_4 - q_4^2}{4 q_0 \Delta_4(q)}. \]
From this computation, we can also obtain the first few moments for any polynomial of degree \(j \leq 4\) simply by setting \(q_{j+1} = \cdots = q_4 = 0\) above. For instance, if
\[ \hat{q}(w) = q_2 w^2 + q_1 w + q_0 \]
is a polynomial of degree 2, we set \(q_3 = q_4 = 0\) and we see that \(\Delta_4(q) = q_0^2 (q_0 - q_2) = q_0^2 \Delta_2(\hat{q})\), where \(\Delta_2(\hat{q}) = q_0 - q_2\). The first few moments for \(\hat{q}(w)\) follow easily from
the formulas above

\[ h_0(\hat{q}) = \frac{1}{q_0(q_0 - q_2)} \]
\[ h_1(\hat{q}) = -\frac{2q_0}{q_0(q_0 - q_2)} \]
\[ h_2(\hat{q}) = \frac{q_0^2 + q_1^2 - q_0q_2}{4q_0^2(q_0 - q_2)} \]

and illustrate Proposition 2.9 in the case of a quadratic polynomial.

3. Matrix orthogonal polynomials

The theory of matrix orthogonal polynomials has an extensive literature \[1\] \[2\] and the works \[9\] \[19\] will be of the most use for us. We will review these results and make more precise some of the statements found in \[19\]. The connection between the two variable problem and the theory of matrix orthogonal polynomials on the square has been developed in \[4\] and the connection with scattering theory was sketched out in \[11\]. We include most of the results in this section to make the paper self contained and for the convenience of the reader.

Recall that \( \mathbb{R}^{l \times l} \) denotes the space of all \( l \times l \) matrices with real coefficients and \( \text{Sym}(\mathbb{R}^{l \times l}) \) is the subspace of all symmetric matrices. For \( N \in \mathbb{N}_0 \) we denote by \( \mathbb{R}_N^{l \times l}[x] \) the space of polynomials of degree at most \( N \) in \( x \) with coefficients in \( \mathbb{R}^{l \times l} \).

A linear transformation \( \mathcal{L}_I : \mathbb{R}_{2N}[x] \to \text{Sym}(\mathbb{R}^{l \times l}) \) will be called a \textit{matrix-valued functional} and we define a matrix-valued inner product on \( \mathbb{R}_N^{l \times l}[x] \) as follows: if \( Q(x) = \sum_{j=0}^N Q_j x^j \) and \( R(x) = \sum_{j=0}^N R_j x^j \) are two elements in \( \mathbb{R}_N^{l \times l}[x] \), then

\[ \langle Q(x), R(x) \rangle_{\mathcal{L}_I} = \sum_{j,k=0}^N Q_j \mathcal{L}_I(x^{j+k}) R_k. \]

We say that \( \mathcal{L}_I \) is a \textit{positive matrix-valued functional} if \( \langle Q(x), Q(x) \rangle_{\mathcal{L}_I} \) is a positive definite matrix for every matrix-valued polynomial \( Q(x) = x^k + \sum_{j=0}^{k-1} Q_j x^j \) with highest coefficient \( I_k \). With this in hand, we can apply the Gram-Schmidt process to \( I_1, xI_1, \ldots, x^N I_1 \) to construct a sequence of matrix orthonormal polynomials \( \{P_n(x)\}_{n=0}^N \), such that \( P_n(x) \in \mathbb{R}_N^{l \times l}[x] \) whose coefficient of \( x^n \) is a lower triangular matrix with positive diagonal entries, and

\[ \langle P_n(x), P_m(x) \rangle_{\mathcal{L}_I} = I_l \delta_{n,m}. \]

It is not difficult to see that the above conditions uniquely define \( P_n(x) \).

Standard arguments show that these polynomials satisfy the following recurrence relation

\[ xP_n(x) = A_{n+1} P_{n+1}(x) + B_n P_n(x) + A_n P_{n-1}(x), \quad (3.1) \]

where \( A_n = \langle x P_{n-1}(x), P_n(x) \rangle_{\mathcal{L}_I} \in \mathbb{R}^{l \times l} \) is a lower triangular with positive diagonal entries, and \( B_n = \langle x P_n(x), P_n(x) \rangle_{\mathcal{L}_I} \in \text{Sym}(\mathbb{R}^{l \times l}) \). If \( \mathcal{L}_I \) is defined on \( \mathbb{R}_{2N}[x] \), then \( (3.1) \) holds for \( n = 0, 1, \ldots, N-1 \) with the convention \( P_{-1}(x) = 0 \). If \( \mathcal{L}_I \) extends to a positive matrix-valued linear functional on \( \mathbb{R}[x] \), then \( (3.1) \) holds for all \( n \in \mathbb{N}_0 \).

Our first goal is to characterize the linear functionals on \( \mathbb{R}[x] \) for which the recurrence coefficients satisfy

\[ A_{n+1} = \frac{1}{2} I_l \quad \text{and} \quad B_n = 0 \quad \text{for all} \quad n \geq N, \quad (3.2) \]
for some \( N \in \mathbb{N} \). Following [9], we introduce the matrix-valued function

\[
\Psi_n(z) = P_n(x) - 2zA_n^tP_{n-1}(x),
\]

(3.3)

where \( z = x - \sqrt{x^2 - 1} \) and the branch of the square root is chosen so that \( z \to 0 \) as \( x \to +\infty \). Using (3.1) and (3.3) one can deduce that for \( n \geq 1 \) we have

\[
\Psi_n(z) = \frac{1}{2z}A_n^{-1}\Psi_{n-1}(z) + \frac{1}{2} A_n^{-1} [(I - 4A_nA_n^t)z - 2B_n] P_{n-1}(x).
\]

(3.4)

Combing the last equation with (3.2) we see that

\[
\hat{\Psi}_n(z) = z^n\Psi_n(z) = \hat{\Psi}_N(z) \quad \text{for all } N,
\]

and then using (3.3) we find

\[
P_n(x) = \frac{z^{n+1}\hat{\Psi}_N(1/z) - z^{-n-1}\hat{\Psi}_N(z)}{z - 1/z} \quad \text{for } n \geq N.
\]

(3.5)

If \( X_n(x) \) and \( Y_n(x) \) are two solutions of equation (3.1) with boundary conditions \( X_{-1}(x) = Y_{-1}(x) = 0 \), then routine manipulations yield

\[
X_{n-1}(x)^tA_nY_n(y) - X_n(x)^tA_n^tY_{n-1}(y) = (y - \bar{x}) \sum_{j=0}^{n-1} X_j(x)^tY_j(y),
\]

(3.6)

where \( M^t = \overline{M^t} \) denotes the Hermitian conjugate of \( M \). We set \( X_n(x) = Y_n(x) = P_n(x) \) and \( x \in \mathbb{R} \) in the last equation, and by taking the limit \( y \to x \) we find

\[
P_{n-1}(x)^tA_nP_n(x) = P_n(x)^tA_n^tP_{n-1}(x) = \sum_{j=0}^{n-1} P_j(x)^tP_j(x).
\]

(3.7)

Since the right-hand side of the above equation is positive definite for all \( x_0 \in \mathbb{R} \), it follows that there is no nonzero vector \( a \in \mathbb{C}^t \) so that both \( P_{n-1}(x_0)a = 0 \) and \( P_n(x_0)a = 0 \). This implies that there is no \( x_0 \in \mathbb{R} \) for which there is a vector \( a \in \mathbb{C}^t \setminus \{0\} \) so that both \( P_n(x_0)a = 0 \) and \( \Psi_n(z_0)a = 0 \).

Taking \( X_n(x) = Y_n(x) = P_n(x) \) and \( y = x \in \mathbb{R} \) in (3.6) we see that

\[
P_{n-1}(x)^tA_nP_n(x) = P_n(x)^tA_n^tP_{n-1}(x),
\]

(3.8)

hence using (3.3) we find

\[
\Psi_n(z)^tP_n(x) - P_n(x)^t\Psi_n(z) = 0.
\]

(3.9)

Finally equations (3.3) and (3.8) yield

\[
\Psi_n(z)^t\Psi_n(1/z) = \Psi_n(1/z)^t\Psi_n(z).
\]

(3.10)

With \( X_n(x) = Y_n(x) = P_n(x) \) and \( y = x \) in equation (3.6) we find after using equation (3.6)

\[
\frac{1}{2i} \left( P_n(x)^t \frac{\psi_n(z)}{z} - \frac{\psi_n(z)^t}{z} P_n(x) \right) = \left( \text{Im}(z) + \text{Im} \left( \frac{1}{z} \right) \right) \sum_{j=0}^{n-1} P_j(x)^tP_j(x)
\]

\[
\quad + \text{Im} \left( \frac{1}{z} \right) P_n(x)^tP_n(x).
\]

(3.11)

This shows that there is no vector \( a \neq 0 \) so that \( \Psi_n(z)a = 0 \) for \( z \in \mathbb{D} \setminus [-1, 1] \).

If \( \Phi(z) \) is an \( l \times l \) matrix-valued meromorphic function defined in a neighborhood of \( z_0 \) such that \( \det \Phi(z) \) is not identically 0, then
• the order of a pole of \( \Phi \) at \( z_0 \) is defined to be the minimal \( k > 0 \) such that 
\[
\lim_{z \to z_0} (z - z_0)^k \Phi(z) \text{ is a finite nonzero matrix, and a simple pole is a pole of order 1.}
\]

• \( \Phi \) has a simple zero at \( z_0 \) if \( \Phi(z)^{-1} \) has a simple pole at \( z_0 \).

A lemma of Newton and Jost [22] that will be of use is,

**Lemma 3.1.** Let \( \Phi(z) \) be an \( l \times l \) matrix which is analytic in the open disk \( \mathbb{D} \), such that 
\[
\det \Phi(0) = 0 \text{ but } \det \Phi(z) \neq 0 \text{ for } z \in \mathbb{D} \setminus \{0\}.
\]
Then the matrix \( \Phi(z) \) has a simple zero at \( z = 0 \) if and only if the relations 
\[
\Phi(0)a = 0, \quad (\Phi(0)b + \Phi'(0))a = 0, \tag{3.12}
\]

where \( a \) and \( b \) are constant vectors imply \( a = 0 \).

Suppose that \( \Phi(z) \) has a simple zero at \( z = 0 \) and write 
\[
\Phi(z) = \Phi(0) + \Phi'(0)z + \cdots, \tag{3.13}
\]

and 
\[
\Phi(z)^{-1} = \frac{\Phi^{-1}}{z} + \Phi_0 + \Phi_1 z + \cdots. \tag{3.14}
\]

Since \( \Phi(z)\Phi(z)^{-1} = I_l = \Phi(z)^{-1}\Phi(z) \) the above equations give 
\[
\Phi(0)\Phi_1 = 0 = \Phi_1(0), \tag{3.15}
\]

and 
\[
\Phi(0)\Phi_0 + \Phi'(0)\Phi_1 = I_l = \Phi_0(0) + \Phi_1(0). \tag{3.16}
\]

If equation (3.13) is used in (3.17) we find 
\[
\begin{align*}
P_n(x)^t \left( \frac{\Psi_n(z)}{z} \right)^t - \frac{\Psi_n(z)^t dP_n(x)}{dz} &= \left( 1 - \frac{1}{z^2} \right) \sum_{j=0}^{n-1} P_j(x)^t P_j(x) \\
&\quad - \frac{1}{z^2} P_n(x)^t P_n(x). \tag{3.17}
\end{align*}
\]

Thus at a zero \( z_0 \in \overline{\mathbb{D}} \setminus \{0\} \) of \( \det \Psi_n(z) \), which must be real, there is a nonzero real vector \( a \) so that \( \Psi_n(z_0)a = 0 \). Equation (3.17) implies that 
\[
a^t P_n(x_0)^t \Psi_n'(z_0) a = \left( 1 - \frac{1}{x_0^2} \right) \sum_{j=0}^{n-1} a^t P_j(x_0)^t P_j(x_0) a - \frac{1}{x_0^2} a^t P_n(x_0)^t P_n(x_0) a < 0. \tag{3.18}
\]

Now suppose that besides \( \Psi_n(z_0)a = 0 \) there is vector \( b \) so that \( \Psi_n(z_0)b + \Psi'_n(z_0)a = 0 \). Hence, 
\[
a^t P_n(x_0)^t \Psi_n(z_0)b + a^t P_n(x_0)^t \Psi'_n(z_0) a = 0.
\]

An application of (3.9) yields that \( a^t P_n(x_0)^t \Psi'_n(z_0)a = 0 \) which contradicts equation (3.18). From Lemma 3.1 we find that \( \Psi_n(z) \) has a simple zero at \( z_0 \).

We summarize the above discussion with the following

**Lemma 3.2.** Suppose that \( L_t : \mathbb{R}[x] \to \text{Sym}(\mathbb{R}^{l 	imes l}) \) is a positive matrix-valued functional and let \( \Psi_n(z) = z^n (P_n(x) - 2z A_n^0 P_{n-1}(x)) \). Then \( \Psi_n(z) \) is nonsingular for \( z \in \overline{\mathbb{R}} \setminus \{0\} \cup [0, 1] \), and can have only simple zeros when \( z \in [-1, 0) \cup (0, 1] \).
Note that \( \hat{\Psi}_N(z) \in \mathbb{R}^{J_N}[z] \) for every \( N \in \mathbb{N}_0 \) and from (3.10) it follows that
\[
W_N(x) = \hat{\Psi}_N(1/z)^t \hat{\Psi}_N(z) \in \text{Sym}(\mathbb{R}^{J_N}[x]).
\] (3.19)
Moreover, since \( \hat{\Psi}_N(0) \) is an invertible matrix, it is easy to see that \( \hat{\Psi}_N(z) \) and \( W_N(x) \) have the same degrees. The degree of a matrix-valued polynomial \( \Phi(z) \) is the minimal \( k \) for which \( \Phi(z) \in \mathbb{R}^{J_N}[z] \) (however, the coefficient of \( z^k \) need not be an invertible matrix).

Let \( z_0 \in \mathbb{D} \setminus \{0\} \) be a zero of \( \hat{\Psi}_N(z) = z^N \hat{\Psi}_N(z) \), which must be real and simple. Let \( E_0 \) be the orthogonal projection onto \( \ker(\hat{\Psi}_N(z_0)) \), and let
\[
\hat{\Psi}_N(z)^{-1} = \frac{M_{-1}}{z - z_0} + M_0 + \cdots,
\]
be the Laurent expansion of \( \hat{\Psi}_N(z)^{-1} \) in a neighborhood of \( z_0 \). From equations (3.15) and (3.16) we have
\[
\hat{\Psi}_N(z_0)M_{-1} = 0 = M_{-1}\hat{\Psi}_N(z_0)
\] (3.20)
and
\[
\hat{\Psi}_N(z_0)M_{-1} + \hat{\Psi}_N(z_0)M_0 = I_l = M_{-1}\hat{\Psi}_N(z_0) + M_0\hat{\Psi}_N(z_0).
\] (3.21)
We now use an argument of Newton and Jost [22] to show that
\[
\ker(\hat{\Psi}_N(z_0)) = \text{range}(M_{-1}).
\] (3.22)
The inclusion range\( (M_{-1}) \subset \ker(\hat{\Psi}_N(z_0)) \) follows immediately from (3.20). Conversely, if \( a \in \ker(\hat{\Psi}_N(z_0)) \), then (3.21) shows that \( a = M_{-1}\hat{\Psi}_N(z_0)a \in \text{range}(M_{-1}) \) which gives (3.22). This implies that \( E_0M_{-1} = M_{-1} \). If (3.2) holds we see from equation (3.15) that
\[
P_n(x_0)A = \frac{z_0^{n+1}\hat{\Psi}_N(1/z_0)}{z_0 - 1/z_0} A, \quad \text{when } n \geq N,
\] (3.23)
for \( A = E_0 \) or \( A = M_{-1} \). Thus taking the limit \( n \to \infty \) in equation (3.17) and using the fact that \( E_0 \) is symmetric gives,
\[
\frac{1}{z_0 - 1/z_0} E_0 \hat{\Psi}_N(1/z_0)^t \hat{\Psi}_N'(z_0) A = (1 - 1/z_0^2) \sum_{j=0}^{\infty} E_0 P_j(x_0)^t P_j(x_0) A,
\]
since \( P_n(x_0)A \to 0 \) as \( n \to \infty \) by (3.23). If \( A = E_0 \) then the left hand side of the above equation is negative semi-definite. If \( A = M_{-1} \) then equation (3.21) shows that \( \hat{\Psi}_N(z_0)M_{-1} = I_l - \hat{\Psi}_N(z_0)M_0 \), so that
\[
\frac{1}{z_0 - 1/z_0} E_0 \hat{\Psi}_N(1/z_0)^t \hat{\Psi}_N'(z_0)M_{-1} = \frac{1}{z_0 - 1/z_0} E_0 \hat{\Psi}_N(1/z_0)^t (I_l - \hat{\Psi}_N(z_0)M_0)
\]
\[
= \frac{1}{z_0 - 1/z_0} E_0 \hat{\Psi}_N(1/z_0)^t,
\]
where equation (3.10) has been used to obtain the last equality. This implies that
\[
\frac{1}{z_0 - 1/z_0} E_0 \hat{\Psi}_N(1/z_0)^t = (1 - 1/z_0^2) \sum_{j=0}^{\infty} E_0 P_j(x_0)^t P_j(x_0) M_{-1}
\]
\[
= (1 - 1/z_0^2) \sum_{j=0}^{\infty} E_0 P_j(x_0)^t P_j(x_0) E_0 M_{-1}
\]
\[
= (1 - 1/z_0^2) L_0 M_{-1},
\] (3.24)
where

\[ L_0 = \sum_{j=0}^{\infty} E_0 P_j(x_0)^t P_j(x_0) E_0 + (I_t - E_0) \]
\[ = \frac{z_0}{(z_0 - 1/z_0)^2} E_0 \hat{\Psi}_N(1/z_0)^t \hat{\Psi}'_N(z_0) E_0 + (I_t - E_0) \]  \hspace{1cm} (3.25)

is an invertible positive definite matrix that commutes with $E_0$. Note that if we use the polynomial $W_N(x)$ in (3.19), we have

\[ E_0 \hat{\Psi}_N(1/z_0)^t \hat{\Psi}'_N(z_0) E_0 = \frac{z_0 - 1/z_0}{2z_0} E_0 W_N(x_0) E_0. \]

and therefore $L_0 = \frac{1}{z_0 - 1/z_0} E_0 W_N(x_0) E_0 + (I_t - E_0)$.  

Definition 3.3. If $z_0 \in (-1, 0) \cup (0, 1)$ is zero of $\hat{\Psi}_N(z)$ and if $E_0$ denotes the orthogonal projection onto $\ker(\hat{\Psi}_N(z_0))$, then the $l \times l$ matrix

\[ \rho_0 = E_0 \left( \frac{1}{2(z_0 - 1/z_0)} E_0 W_N(x_0) E_0 + (I_t - E_0) \right)^{-1} \]  \hspace{1cm} (3.26)

is called a canonical weight at $x_0 = \frac{1}{2}(z_0 + 1/z_0)$.

A definition for canonical weights can be found in [19] which has been modified in [20], due to the incorrect Lemma 2.19, to be similar to the one given in above. From equations (3.25) and (3.26), we see that $\rho_0 = E_0 L_0^{-1}$ which combined with (3.24) shows that

\[ \hat{\Psi}_N(1/z_0) = \frac{z_0 - 1/z_0}{z_0} M^{-1}. \]  \hspace{1cm} (3.27)

We have the following,

Theorem 3.4. Let $\mathcal{L}_l : \mathbb{R}[x] \rightarrow \text{Sym}(\mathbb{R}^{l \times l})$ be a positive matrix-valued functional with orthonormal polynomials \{\(P_n(x)\)\}_{n=0}^{\infty}, normalized so that the highest coefficients are lower triangular matrices with positive diagonal entries. Let $N \in \mathbb{N}_0$ and let $\hat{\Psi}_N(z) = z^N (P_N(x) - 2z A_N P_{N-1}(x))$. Then $\hat{\Psi}_N(z)$ is nonsingular for $z \in \mathbb{P} \setminus \{[-1, 0) \cup (0, 1]\}$, and can have only simple zeros when $z \in [-1, 1] \setminus \{0\}$. Moreover, $A_n = \frac{1}{2} I_1$ and $B_n = 0$ for all $n \geq N$ if and only if

\[ \mathcal{L}_l(f) = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1 - x^2} W_N(x)^{-1} dx + \sum_{j=1}^{k} f(x_j) \rho_j, \text{ for } f \in \mathbb{R}[x] \]  \hspace{1cm} (3.28)

where $W_N(x) = \hat{\Psi}_N(1/z)^t \hat{\Psi}_N(z)$, $\rho_j$ are the canonical weights at the zeros \{\(z_j\)\}_{j=1}^{k} of $\hat{\Psi}_N(z)$ in $(-1, 1)$, and $x_j = \frac{1}{2}(z_j + 1/z_j)$. If

- $A_N \neq \frac{1}{2} I_1$, then $\hat{\Psi}_N(z)$ is a polynomial of degree $2N$;
- $A_N = \frac{1}{2} I_1$, but $B_{N-1} \neq 0$, then $\hat{\Psi}_N(z)$ is a polynomial of degree $2N - 1$.

Proof. We have already established the properties of $\hat{\Psi}_N(z)$, and the statements about its degree follow from (3.4). In particular, these properties show that the integral in (3.28) is well defined. We show next that the assumptions on the recurrence coefficients imply that the polynomials $P_n$ are orthonormal with respect to
the functional given in (3.28). To this end, take \( s < n \) and consider
\[
\mathcal{L}_l(x^sP_n(x)) = \frac{2}{\pi} \int_{-1}^{1} x^sP_n(x)\sqrt{1-x^2}W_N(x)^{-1} \, dx + \sum_{j=1}^{k} x_j^sP_n(x_j)\rho_j. \tag{3.29}
\]
If we set \( z = e^{i\theta} \) and \( x = \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \), then for \( n \geq N \) the integral in the above expression can be rewritten using (3.5) as
\[
\frac{2}{\pi} \int_{-1}^{1} x^sP_n(x)\sqrt{1-x^2}W_N(x)^{-1} \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} z^{n+1}x^s(z)(\hat{\Psi}_N(z)^{-1})^t \, d\theta
\]
\[
= -\frac{1}{2\pi i} \oint T z^n(x-1/z)(\hat{\Psi}_N(z)^{-1})^t \, dz.
\]
We see that the integrand is analytic except at the zeros of \( \hat{\Psi}_N(z) \) so the residue theorem shows,
\[
\frac{2}{\pi} \int_{-1}^{1} x^sP_n(x)\sqrt{1-x^2}W_N(x)^{-1} \, dx = -\sum_{j=1}^{k} z_j^n x_j^s(z_j - 1/z_j)\text{res}_{z=z_j}(\hat{\Psi}_N(z)^{-1})^t
\]
\[
= -\sum_{j=1}^{k} x_j^sP_n(x_j)\rho_j,
\]
where we have used the definition of \( \rho_j \), equation (3.27), and equation (3.5). The above equation proves (3.29). The fact that \( \langle P_n(x), P_n(x) \rangle_{\mathcal{L}_l} = I_l \) follows by the above argument after using equation (3.28) to eliminate \( P_n \) and utilizing the fact that the residue at \( z = 0 \) is equal to \( I_l \). For \( n < N \) the result follows by induction with the aid of the three term recurrence formula.

To show the other direction assume that \( \mathcal{L}_l \) has the form in (3.28). Construct the polynomials \( P_n \) for \( n \geq N \) from equation (3.3). Then the argument above shows that \( P_n \) is an orthonormal polynomial with respect to \( \mathcal{L}_l \). The polynomials satisfy the recurrence formula
\[
\frac{1}{2}P_{n+2}(x) + \frac{1}{2}P_n(x) = xP_{n+1}(x)
\]
for \( n \geq N \) so that \( A_{n+1} = \frac{1}{2}I_l \) and \( B_{n+1} = 0 \) for \( n \geq N \).

We also have using (3.3)
\[
xP_N(x) - \frac{1}{2}P_{N+1}(x) = \frac{1}{2} \frac{z^N\hat{\Psi}_N(1/z) - (1/z)^N\hat{\Psi}_N(z)}{z - 1/z},
\]
which is polynomial of degree \( N - 1 \) implying that \( B_N = 0 \). This completes the proof. \( \square \)

We will also need the following theorem, see [28, Theorem 1].

**Theorem 3.5** (Matrix Fejér-Riesz factorization). Let \( W(x) \in \text{Sym}(\mathbb{R}_N^{s_{\ell}[x]}) \) be a positive definite matrix for a.e. \( x \in (-1, 1) \). Then there exists \( \Psi(z) \in \mathbb{R}_N^{s_{\ell}[z]} \) which is invertible for \( z \in \mathbb{D} \) such that \( W(x) = \Psi(1/z)^t\Psi(z) \). Moreover, \( \Psi \) can be normalized so that \( \Psi(0) \) is a lower triangular matrix with positive diagonal entries.

Note that since \( \Psi(0) \) is invertible, the matrix-valued polynomials \( W(x) \) and \( \Psi(z) \) in the Fejér-Riesz factorization theorem above will have the same degree. Combining the last theorem with Theorem 3.3 we can establish the following useful theorem.
Theorem 3.6. For a positive matrix-valued functional \( \mathcal{L}_t : \mathbb{R}_{2N}[x] \to \text{Sym}(\mathbb{R}^{l \times l}) \) the following conditions are equivalent:

(i) There exists \( W(x) \in \text{Sym}(\mathbb{R}^{l \times l}_{2N}[x]) \) which is positive definite for \( x \in (-1, 1) \) and has at most simple zeros at \( x = \pm 1 \) such that for \( f \in \mathbb{R}_{2N}[x] \) we have

\[
\mathcal{L}_t(f) = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1 - x^2} W(x)^{-1} \, dx.
\] (3.30)

(ii) \( \hat{\Psi}(z) = z^N(P_N(x) - 2zA_t^tP_{N-1}(x)) \) is invertible for \( z \in (-1, 1) \).

Moreover, if the equivalent conditions above hold, then \( W(x) \) in (i) is uniquely determined from \( \mathcal{L}_t \) by \( W(x) = \hat{\Psi}_N(1/z)^t\hat{\Psi}_N(z) \), and \( (3.30) \) defines the unique extension of \( \mathcal{L}_t \) to a positive matrix-valued functional on \( \mathbb{R}[x] \) for which \( A_{n+1} = \frac{1}{2}I_l \) and \( B_n = 0 \) for all \( n \geq N \).

Proof. Note first that if \( \mathcal{L}_t \) is a positive matrix-valued functional defined on \( \mathbb{R}_{2N}[x] \), then there exists a unique extension of \( \mathcal{L}_t \) to a positive matrix-valued functional on the space \( \mathbb{R}[x] \) of all polynomials such that (3.2) holds. Indeed, we can use (3.1) to define \( P_n(x) \) for \( n > N \), and it follows that the highest coefficient of \( P_n(x) \) will be \( 2^n - N \) times the highest coefficient of \( P_N(x) \). It is easy to see that there is a unique extension of \( \mathcal{L}_t \) so that these polynomials are orthonormal: if \( \mathcal{L}_t \) is extended to \( \mathbb{R}_{2N}[x] \), then the equation \( (xP_n(x), P_n(x))_{\mathcal{L}_t} = 0 \) defines uniquely \( \mathcal{L}(x^{2n+1}) \in \text{Sym}(\mathbb{R}^{l \times l}) \), and \( (xP_n(x), P_{n+1}(x))_{\mathcal{L}_t} = \frac{1}{2}I_l \) defines uniquely \( \mathcal{L}_t(x^{2n+2}) \), and the resulting functional is positive on \( \mathbb{R}_{2n+2}[x] \).

Suppose first that (i) holds. By the Fejér-Riesz factorization theorem, the matrix-valued polynomial \( W(x) \) can be factored as \( W(x) = \Psi(1/z)^t\Psi(z) \), where \( \Psi(z) \in \mathbb{R}^{l \times l}_{2N}[z] \) is nonzero for \( z \in \mathbb{D} \) and \( \Psi(0) \) is a lower triangular matrix with positive diagonal entries. Since \( W(x) \) is invertible for \( x \in (-1, 1) \) and has at most simple zeros at \( x = \pm 1 \), it follows that \( \Psi(z) \) is also invertible for \( z \in \mathbb{T} \setminus \{\pm 1\} \) and has at most simple zeros at \( z = \pm 1 \). We now set \( P_n(x) = \frac{z^{n+1}\Psi(1/z) - z^{-n-1}\Psi(z)}{z^{1/2}} \) for \( n \geq N \) and the contour integral given in Theorem [3.4] shows that \( P_n(x) \) is an orthonormal matrix-valued polynomial of degree \( n \) with respect to the functional \( \mathcal{L} \) in (3.30). The explicit formula for \( P_n \) also shows that \( A_{n+1} = \frac{1}{2}I_l \) and \( B_n = 0 \) for \( n \geq N \) and therefore \( \Psi(z) = \Psi_n(z) \) for \( n \geq N \). From this, we deduce (ii) and the formula for \( W(x) \). The implication (ii) \( \Rightarrow \) (i) follows immediately from Theorem [3.4].

4. Two variable Bernstein-Szegő measures

Recall that \( \mathbb{R}_{n,m}[x, y] \) denotes the space of polynomials with real coefficients in \( x \) and \( y \) of degrees at most \( n \) in \( x \) and \( m \) in \( y \). A linear functional \( \mathcal{L} : \mathbb{R}_{2n,2m}[x, y] \to \mathbb{R} \) is said to be positive if \( \mathcal{L}(p^2) > 0 \) for every nonzero polynomial \( p \in \mathbb{R}_{n,m}[x, y] \). Using \( \mathcal{L} \), we define an inner product on \( \mathbb{R}_{n,m}[x, y] \) by

\[
\langle p, q \rangle_{\mathcal{L}} = \mathcal{L}(pq), \quad p, q \in \mathbb{R}_{n,m}[x, y],
\]

and we consider the spaces of orthogonal polynomials in (1.9) with respect to this inner product. For every \( l \leq m \), we fix a basis \( \mathcal{B}_l \) of \( \mathbb{R}_l[y] \) which leads to an orthonormal basis \( \{p_{k,l}(x, y) : 0 \leq j \leq l\} \) of \( \mathbb{P}_{k,l}[x, y] \) as explained in Section 1.2 and we set

\[
P_{k,l}(x, y) = [p^0_{k,l}(x, y), p^1_{k,l}(x, y), \ldots, p^l_{k,l}(x, y)]^t.
\]
Similarly, for \( k \leq n \), we fix a basis \( \tilde{B}_k \) of the space \( \mathbb{R}_k[x] \) of polynomials of degree at most \( k \) in \( x \), which leads to an orthonormal basis \( \{ \tilde{p}_{k,l}^j(x,y) : 0 \leq j \leq k \} \) for \( \tilde{P}_{k,l}[x,y] \), and we set
\[
\tilde{P}_{k,l}(x,y) = [\tilde{p}_{k,l}^0(x,y), \tilde{p}_{k,l}^1(x,y), \ldots, \tilde{p}_{k,l}^k(x,y)]^t.
\]
With these notations, the vector polynomials satisfy the recurrence relations in (1.11), where \( A_{k,l} = \mathcal{L}(xP_{k-1,l}P_{1,l}) \) and \( B_{k,l} = \mathcal{L}(xP_{k,l}P_{1,l}) \) are \( (l+1) \times (l+1) \) matrices, and \( \tilde{A}_{k,l} \) and \( \tilde{B}_{k,l} \) are \( (k+1) \times (k+1) \) matrices defined in a similar manner. The vector polynomials \( P_{k,l}(x,y) \), \( \tilde{P}_{k,l}(x,y) \) and the recurrence relations (1.11) can be naturally related to the theory of matrix orthogonal polynomials which we explore in this section. Indeed, the vector polynomials \( P_{k,l}(x,y) \) can be represented as
\[
P_{k,l}(x,y) = P_{k,l}^t(x)[\beta_0(y), \beta_1(y), \ldots, \beta_l(y)],
\]
where \( P_{k,l}^t(x) \) is a polynomial of degree \( k \) in \( x \), whose coefficients are \( (l+1) \times (l+1) \) matrices, and the highest coefficient is a lower-triangular matrix with positive diagonal entries. Moreover, these polynomials are orthonormal with respect to the positive matrix-valued functional \( \mathcal{L}_{t+1} \) defined as follows
\[
\mathcal{L}_{t+1}(f) = \mathcal{L}(f(x)H_{l,\beta}(y)),
\]
where \( H_{l,\beta}(y) \) is an \( (l+1) \times (l+1) \) matrix with entries \( (H_{l,\beta}(y))_{i,j} = \beta_i(y)\beta_j(y) \), for \( 0 \leq i,j \leq l \). The recurrence relation (1.11) is equivalent to the three-term recurrence relation for these matrix polynomials. Equation (1.11) has a similar interpretation obtained by exchanging the roles of \( x \) and \( y \) in the above construction. In particular, if \( B_1 = (1, y, \ldots, y^l) \) and \( \tilde{B}_k = (1, x, \ldots, x^k) \) are the standard bases of \( \mathbb{R}_l[y] \) and \( \mathbb{R}_k[x] \), respectively, we obtain the vector polynomials defined and studied in [I]. Note that in this case \( H_{l,\beta}(y) \) is a Hankel matrix.

### 4.1. One-sided factorization: spectral properties and characterization.

We fix below \( n, m \in \mathbb{N}_0 \) and we can formulate the first main result of the paper as follows.

**Theorem 4.1.** (I) Suppose that for some \( n_1 \leq n \), the polynomials \( q(x) \in \mathbb{R}_{2n_1}[x] \) and \( p(x, w) \in \mathbb{R}_{n-n_1, 2m}[x, w] \) are such that
1. \( q(x) > 0 \) and \( p(x, w) \neq 0 \) when \( x \in (-1, 1) \), \( w \in \mathbb{D} \),
2. \( \int_{(-1,1)^2} \sqrt{1-x^2} \sqrt{1-y^2} \frac{q(x)p(x, w)p(x, 1/w)}{q(x)p(x, w)p(x, 1/w)} \) \( dx \) \( dy \) \( < \infty \).

Then the orthonormal vector polynomial \( \{P_{k,l}(x,y)\} \) for the linear functional
\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} f(x,y) \sqrt{1-x^2} \sqrt{1-y^2} \frac{q(x)p(x, w)p(x, 1/w)}{q(x)p(x, w)p(x, 1/w)} \) \( dx \) \( dy \),
\]
are related to \( q(x) \) and \( p(x, w) \) via
\[
\frac{q(x)}{(w-1/w)(w_1-1/w_1)} \left( (ww_1)^{m-1}p(x, w)p(x, w_1) - (ww_1)^{m+2}p(x, 1/w)p(x, 1/w_1) \right)
+ \frac{q(x)}{(w_1/w)^{m+1}w_1p(x, w)p(x, 1/w_1)} \left( (w/w_1)^{m+1}p(x, 1/w)p(x, 1/w_1) - (w/w_1)^{m+2}wp(x, 1/w)p(x, 1/w_1) \right)
= P_{n,m}(x,y)^tP_{n,m}(x,y) - 4xP_{n,m}(x,y)^tA_{n,m}^tP_{n-1,m}(x,y_1)
+ 4P_{n-1,m}(x,y)^tA_{n,m}^tP_{n-1,m}(x,y_1),
\]

\[ (4.4) \]
and the recurrence coefficients of \( \mathcal{L} \) satisfy
\[
A_{k+1,l} = \frac{1}{2} I_{l+1} \quad \text{and} \quad B_{k,l} = 0 \quad \text{for all} \quad k \geq n, \quad l \geq m. \tag{4.5}
\]

(II) Conversely, let \( \mathcal{L} : \mathbb{R}_{2n,2m}[x,y] \to \mathbb{R} \) be a positive linear functional with vector orthonormal polynomials \( \{P_{k,m}(x,y)\}_{k \leq n} \), and suppose that there exist polynomials \( q(x) \in \mathbb{R}_{2n}[x] \), \( p(x,w) \in \mathbb{R}_{n-n_1,2m}[x,w] \) for some \( n_1 \leq n \) satisfying (4.4). If \( q(x) = q_1(x)r(x) \). If we want to have uniqueness, we can add \( q_1(x) = x^n \). If we want to have uniqueness, we can add \( q_1(x) = x^n \) and \( q(x) \) and \( p(x,w) \) satisfy conditions (i)-(ii) in part (I), and (4.3) holds for all \( f \in \mathbb{R}_{2n,2m}[x,y] \). In particular, (4.3) extends \( \mathcal{L} \) to a positive linear functional on \( \mathbb{R}[x,y] \) and (4.5) holds.

**Remark 4.2.** Recall that the construction of the polynomials \( P_{k,l}(x,y) \), and the recurrence coefficients \( A_{k,l} \) depend on the basis \( \mathcal{B}_l \) of the space \( \mathbb{R}_l[y] \). However, it is easy to see that
- the right-hand side of (4.4) is independent of the basis we choose, and thus depends only on \( \mathcal{L} \);
- if (4.5) holds for one basis \( \mathcal{B}_l \) of \( \mathbb{R}_l[y] \), then (4.5) holds for any basis \( \mathcal{B}_l \) of \( \mathbb{R}_l[y] \).

Moreover, if (4.5) holds and if \( P_{k,l}(x,y) \) and \( \hat{P}_{k,l}(x,y) \) denote the vectors polynomials constructed using the bases \( \mathcal{B}_l \) and \( \mathcal{B}_l \), respectively, then there exists an \( (l+1) \times (l+1) \) orthogonal matrix \( U_l \) such that \( P_{k,l}(x,y) = U_l P_{k,l}(x,y) \) for all \( k \geq n \).

**Remark 4.3.** We can naturally extend the isomorphism (2.11) to an isomorphism
\[
\mathbb{R}_{n-n_1,m}[x,y] \times \mathbb{R}_{n-n_1,m-1}[x,y] \to \mathbb{R}_{n-n_1,2m}[x,w]
\]
with inverse given by
\[
p_j(y;x) = \frac{w^{j+1}p(x,1/w) - w^{j+1}p(x,w)}{w - 1}, \quad \text{for } j = m \text{ and } j = m - 1. \tag{4.7}
\]
Using this correspondence and the arguments in Proposition 2.1, we see that there exists \( p(x,w) \in \mathbb{R}_{n-n_1,2m}[x,w] \) satisfying (4.4) if and only if there exist \( p_m(y;x) \in \mathbb{R}_{n-n_1,m}[x,y] \) and \( p_{m-1}(y;x) \in \mathbb{R}_{n-n_1,m-1}[x,y] \), satisfying
\[
q(x) \left( \frac{p_m(y_1;x)p_{m-1}(y;y_1) - p_m(y;x)p_{m-1}(y_1;x)}{2(y_1 - y)} + p_m(y;x)p_m(y_1;x) \right) = P_{n,m}(x,y)^t A_{n,m} P_{n-1,m}(x,y) + 4P_{n,m}(x,y)^t A_{n,m} P_{n-1,m}(x,y). \tag{4.8}
\]
Moreover, if the last equation holds, then \( \sqrt{q(x)}p_m(y;x) \) and \( \sqrt{q(x)}p_{m-1}(y;x) \) are uniquely determined from the right-hand side of (4.8), up to a simultaneous sign change, and can be computed from the right-hand side following the steps in Remark 2.7. This shows that if \( q(x) \) and \( p(x,w) \) satisfy (4.4), then \( \sqrt{q(x)}p(x,w) \) is uniquely determined from the right-hand side, up to a sign. Clearly, if \( q(x) \) factors in \( \mathbb{R}[x] \) as \( q(x) = q_1(x)r(x)^2(x) \), then we can replace the pair \( (q(x),p(x,w)) \) with \( (q_1(x),r(x)p(x,w)) \), and up to such transformations \( q(x) \) and \( p(x,w) \) are uniquely determined from the right-hand side of (4.4). If we want to have uniqueness, we can normalize \( p(x,w) \) so that it does not have nonconstant factors depending only on \( x \), then to fix the constant multiples, we can add \( q(0) = 1 \), and finally, we can
fix the sign of \( p(x, w) \) by asking that \( p(0, 0) > 0 \). With this normalization, \( q(x) \) and \( p(x, w) \) will be uniquely determined from \((4.4)\).

**Proof of Theorem 4.1, Part (I).** We show first that if (i) and (ii) hold, then

\[
\int_{(-1, 1)} \frac{1 - y^2}{p(x, w)p(x, 1/w)} dy < \infty
\]

for all \( x \in (-1, 1) \). Indeed, from Tonelli’s theorem we know that \((4.9)\) holds for a.e. \( x \in (-1, 1) \). Note that if we fix such \( x \), then \( p(x, w) \neq 0 \) for \( w \in \mathbb{D} \setminus \{ \pm 1 \} \), and \( p(x, w) \) can have only simple zeros at \( w = \pm 1 \). This means that \( R(x) = \int_{(-1, 1)} \frac{1 - y^2}{p(x, w)p(x, 1/w)} dy \) is a rational function for a.e. \( x \in (-1, 1) \) for which \((4.9)\) holds, see Remark 2.8. If \( x_0 \in (-1, 1) \) is such that \( \int_{(-1, 1)} \frac{1 - y^2}{p(x_0, w)p(x_0, 1/w)} dy = \infty \), then by Fatou’s lemma we conclude that \( \lim_{x \to x_0} R(x) = \infty \). This implies that \( R(x) \) has a pole of even order at \( x_0 \), hence \( \int_{(-1, 1)} \frac{1 - y^2}{q(x)} R(x) dx = \infty \), which contradicts (ii).

We fix now \( x \in (-1, 1) \), and applying Lemma 2.3 we see that the left-hand side of \((4.8)\) is equal to the reproducing kernel \( K_m^x(y, y_1) \) for the measure \( d\mu_x(y) = \frac{2\sqrt{1 - y^2}}{\pi |q(x)p(x, w)|} (y_1 - y) dy \). We can also compute \( K_m^x(y, y_1) \) using formula \((2.6)\), where

\[
H_m(x) = \frac{2}{\pi} \int_{(-1, 1)} \begin{bmatrix} 1 & y & y^2 & \cdots & y^m \\ y & y^3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ y^m & \cdots & \cdots & y^{2m} \end{bmatrix} \frac{\sqrt{1 - y^2}}{|q(x)p(x, w)|} dy.
\]

Clearly, \( H_m(x) \) is a positive definite matrix for every \( x \in (-1, 1) \) and from Corollary 2.3 we conclude that \( H_m(x)^{-1} \) is a matrix polynomial in \( x \) of degree at most \( 2n \). If we use the standard basis \( B_m = (1, y, \ldots, y^m) \) of \( \mathbb{R}_m[y] \), then the vector polynomials \( P_{j, m}(x, y) \) can be represented as

\[
P_{j, m}(x, y) = P_j^m(x)[1, y, \ldots, y^m]^t,
\]

where \( \{P_j^m(x)\} \) are the matrix orthogonal polynomials associated with the positive matrix functional \( L_{m+1} : \mathbb{R}[x] \to \text{Sym}(\mathbb{R}^{(m+1) \times (m+1)}) \) defined as follows

\[
L_{m+1}(f) = \mathcal{L} \left( f(x) \begin{bmatrix} 1 & y & \cdots & y^m \\ y & y^2 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ y^m & \cdots & \cdots & y^{2m} \end{bmatrix} \right).
\]

This combined with \((4.3)\) and \((4.10)\) shows that

\[
L_{m+1}(f) = \frac{2}{\pi} \int_{(-1, 1)} f(x) \sqrt{1 - x^2} H_m(x) dx.
\]

From Theorem 4.6 it follows that \((4.3)\) holds and that \( H_m(x)^{-1} = \Psi_m^x(1/z)^t \Psi_m^x(z) \). The latter combined with \((4.11)\) and \((2.6)\) shows that the reproducing kernel \( K_m^x(y, y_1) \) is equal to the right-hand side of \((4.8)\), thus establishing \((4.4)\), and completing the proof of Part (I). \( \square \)
Proof of Theorem 4.1, Part (II). Starting with \( L \), we define the matrix orthogonal polynomials \( \{ P_n^m(x) \} \) associated with the positive matrix functional \( L_{m+1} : \mathbb{R}_{2n}[x] \to \text{Sym}(\mathbb{R}^{(m+1) \times (m+1)}) \) in (4.12) and related to the vector polynomials by (4.11). If we set \( H_m(x) = [\hat{\Psi}^m_n(1/z) \hat{\Psi}^m_n(z)]^{-1} \), then from condition (b) and Theorem 3.6 we see that \( H_m(x) \) is positive definite for \( x \in (-1, 1) \) and

\[
L_{m+1}(f) = \frac{2}{\pi} \int_{(-1,1)} f(x) \sqrt{1 - x^2} H_m(x) \, dx, \quad \text{for all } f \in \mathbb{R}_{2n}[x]. \tag{4.13}
\]

Moreover, the right-hand side of (4.13) can be rewritten as

\[
[1, y, \ldots, y^m] H_m(x)^{-1}[1, y_1, \ldots, y_1^m].
\]

Therefore, for fixed \( x \in (-1, 1) \), Proposition 2.6 tells us that \( H_m(x) \) is a Hankel matrix, i.e.

\[
H_m(x) = \begin{bmatrix}
h_0(x) & h_1(x) & \cdots & h_m(x) \\
h_1(x) & h_2(x) & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
h_m(x) & \cdots & \cdots & h_{2m}(x)
\end{bmatrix}.	ag{4.14}
\]

Thus, for fixed \( x \in (-1, 1) \), we can think of \( H_m(x) \) as the moment matrix of a positive linear functional \( L_x : \mathbb{R}_{2m}[y] \to \mathbb{R} \), and (4.3) is its reproducing kernel \( K^x_m(y, y) \). If \( p_m(x) \) denotes the coefficient of \( y^m \) in \( p_m(y; x) \), then comparing the coefficients of \( y^m y_1^m \) on both sides of equation (4.13) we obtain

\[
q(x)p_{m,m}^2(x) = [0, 0, \ldots, 1] H_m(x)^{-1}[0, 0, \ldots, 1]^t \quad \text{for } x \in (-1, 1),
\]

establishing the first condition in (i). Using Theorem 3.6 for the functional \( L^x \) (note that \( l = 1 \) here, and \( z \) and \( \hat{\Psi}_N(z) \) there are replaced by \( w \) and \( \sqrt{q(x)p(x, w)} \), respectively, in view of Remark 2.3 and (2.16)) and condition (a) we conclude that

\[
L^x(g) = \frac{2}{\pi} \int_{(-1,1)} \frac{\sqrt{1 - y^2}}{q(x)p(x, w)p(x, 1/w)} g(y) \, dy, \quad \text{for all } g \in \mathbb{R}_{2m}[y]. \tag{4.15}
\]

Moreover, for every fixed \( x \in (-1, 1) \), Lemma 3.2 tells us that \( p(x, w) \neq 0 \) when \( w \in \mathbb{T} \setminus \{ \pm 1 \} \), thus establishing condition (i) in Part (I). If we take \( g(y) = y^j \) in (4.15) we see that

\[
h_j(x) = L^x(y^j) = \frac{2}{\pi} \int_{(-1,1)} \frac{\sqrt{1 - y^2}}{q(x)p(x, w)p(x, 1/w)} y^j \, dy, \quad \text{for } j = 0, \ldots, 2m. \tag{4.16}
\]

Using now equations (4.12)-(4.13) with \( f(x) = x^k, k = 0, 1, \ldots, 2n \), and then (4.14) and (4.16), we see that

\[
L(x^ky^j) = \frac{2}{\pi} \int_{(-1,1)} x^k \sqrt{1 - x^2} h_j(x) \, dx
\]

\[
= \frac{4}{\pi^2} \int_{(-1,1)} \left( \int_{(-1,1)} x^ky^j \sqrt{1 - x^2} \sqrt{1 - y^2} \, dy/q(x)p(x, w)p(x, 1/w) \right) \, dx, \tag{4.17}
\]

for all \( k = 0, 1, \ldots, 2n \) and \( j = 0, 1, \ldots, 2m \). In particular, if we take \( k = j = 0 \) in the last formula and by Tonelli’s theorem we deduce that

\[
\int_{(-1,1)^2} \sqrt{1 - x^2} \sqrt{1 - y^2} \, dx \, dy = \pi^2 \mathcal{L}(1) < \infty,
\]

which yields condition (ii) in Part (I). Using now equation (4.17), Fubini’s theorem and the linearity of \( L \) we see that (4.3) holds. \( \square \)
Remark 4.4. Note that if either conditions (i)-(ii) in Part (I) hold, or conditions (a)-(b) in Part (II) hold, then the proof shows that:

- $p(x, w) \neq 0$ for $x \in (-1, 1) \times \mathbb{D} \setminus \{\pm 1\}$, and
- $\Psi_n(z)$ is invertible for all $z \in \mathbb{D} \setminus \{\pm 1\}$.

In the next lemma, we show that these conditions also imply that $p(x, w) \neq 0$ when $(x, w) \in \{\pm 1\} \times \mathbb{D}$. This means that when we look at the closure $[-1, 1] \times \mathbb{D}$, the polynomial $p(x, w)$ can only vanish on $\{\pm 1\} \times \mathbb{T}$ and $(-1, 1) \times \{\pm 1\}$. There are examples where $p(x, w)$ vanishes on these parts of the boundary and Theorem 4.1 can be applied with appropriate polynomials $q(x)$. For instance:

- $p(x, w) = 2 + x + w^2$ vanishes when $(x, w) = (-1, \pm i)$, and
  
  $\frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{p(x, w)p(x/1 \pm i)} \, dy = \frac{1}{(2 + x)(1 + x)}$,
  
  by Example 2.10, hence Theorem 4.1 can be applied if $q(x) > 0$ for $x \in [-1, 1)$, possibly having a simple zero at $x = 1$.

- $p(x, w) = (x - x_0)^2 + 1 + w$ vanishes when $(x, w) = (x_0, -1)$, and
  
  $\frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-y^2}}{p(x, w)p(x/1 \pm i)} \, dy = \frac{1}{((x - x_0)^2 + 1)^2}$,
  
  by Example 2.10, hence Theorem 4.1 can be applied if $q(x) > 0$ for $x \in (-1, 1)$, possibly having simple zeros at $x = \pm 1$.

Lemma 4.5. Suppose that $p(x, w) \in \mathbb{R}[x, w]$ is such that

1. $p(x, w) \neq 0$ when $(x, w) \in (-1, 1) \times \mathbb{D}$,
2. $\int_{(-1,1)^2} \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{p(x, w)p(x/1 \pm i)} \, dx \, dy < \infty$.

Then

$p(x, w) \neq 0$ for $(x, w) \in \left((-1, 1) \times \mathbb{D} \setminus \{\pm 1\}\right) \cup \left(\{\pm 1\} \times \mathbb{D}\right)$.

Proof. The proof of Theorem 4.1 shows that $p(x, w) \neq 0$ when $x \in (-1, 1)$ and $w \in \mathbb{D} \setminus \{\pm 1\}$, so we need to show that this is also true when $(x, w) \in \{\pm 1\} \times \mathbb{D}$. Suppose that $p(x_0, w_0) = 0$ for some $x_0 \in \{\pm 1\}$ and $w_0 \in \mathbb{D}$. For $k \in \mathbb{N}$, set $x_k = x_0 - x/0/k \in (-1, 1)$ and consider the sequence of polynomials $f_k(w) = p(x_k, w)$ which do not vanish on $\mathbb{D}$. Since $f_k(w)$ converges to $f(w) = p(x_0, w)$ uniformly on $\mathbb{D}$ and $f(w_0) = 0$, Hurwitz’s theorem tells us that $f(w) = p(x_0, w)$ must be identically equal to 0. This means that $(x - x_0)$ divides $p(x, w)$ and therefore condition (2) cannot hold.

As we explained in Remark 4.3, the polynomial $q(x)p(x, w)p(x/1 \pm i)$ in the denominator of (4.3) is uniquely determined from $L$. More generally, the lemma below shows that there exists at most one polynomial $Q(x, y) \in \mathbb{R}_{2n, 2m}[x, y]$ such that $L(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} f(x, y) \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{Q(x, y)} \, dx \, dy$.

Lemma 4.6. Let $Q_1(x, y), Q_2(x, y) \in \mathbb{R}_{2n, 2m}[x, y]$ be positive for $(x, y) \in (-1, 1)^2$ and such that $\int_{(-1,1)^2} \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{Q_1(x, y)} \, dx \, dy < \infty$ for $j = 1$ and $j = 2$. If

$$\int_{(-1,1)^2} f(x, y) \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{Q_1(x, y)} \, dx \, dy = \int_{(-1,1)^2} f(x, y) \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{Q_2(x, y)} \, dx \, dy$$

(4.18)
for all \( f(x, y) \in \mathbb{R}_{2n,2m}[x,y] \), then \( Q_1(x,y) = Q_2(x,y) \).

Proof. Set \( d\nu(x,y) = \frac{1}{\pi^2} \sqrt{1-x^2} \sqrt{1-y^2} \chi_{(-1,1)^2} \, dx \, dy \), and note that (4.13) implies that

\[
\left( \int_{(-1,1)^2} \frac{Q_2(x,y)}{Q_1(x,y)} d\nu(x,y) \right)^2 = \left( \int_{(-1,1)^2} \frac{Q_1(x,y)}{Q_2(x,y)} d\nu(x,y) \right)^2 = 1.
\]

Applying the Cauchy-Schwarz inequality in the space \( L^2(\nu) \), we see that

\[
1 = \left( \left( \int_{(-1,1)^2} \frac{Q_2(x,y)}{Q_1(x,y)} d\nu(x,y) \right)^2 \right)^{1/2} \left( \left( \int_{(-1,1)^2} \frac{Q_1(x,y)}{Q_2(x,y)} d\nu(x,y) \right)^2 \right)^{1/2} \leq \int_{(-1,1)^2} \frac{Q_2(x,y)}{Q_1(x,y)} d\nu(x,y) \int_{(-1,1)^2} \frac{Q_1(x,y)}{Q_2(x,y)} d\nu(x,y) = 1.
\]

Thus, the polynomials \( Q_1(x,y) \) and \( Q_2(x,y) \) must be equal a.e. on \((-1,1)^2\), and therefore they coincide. \( \square \)

**Remark 4.7.** Equation (4.5) in Theorem 4.1 tells us that for every fixed \( M \geq m \), we can construct orthonormal bases of the spaces \( P_{N,M} \mathcal{L}[x,y] \) for the functional \( \mathcal{L} \) in (4.3) satisfying a “Chebyshev relation” in \( N \) when \( N \geq n \). With the notations in the theorem and using the one-dimensional theory outlined in Section 2, it follows that if \( \{U^q_j(x)\}_{j \in \mathbb{N}_0} \) are the orthonormal polynomials with respect to the measure

\[
\frac{2 \sqrt{1-x^2}}{\pi q(x)} \chi_{(-1,1)}(x)dx \text{ on } \mathbb{R}
\]

and if we set

\[
p_M(y;x) = \frac{w^{M+1}p(x,1/w) - w^{-M-1}p(x,w)}{w-1/w}, \tag{4.19}
\]

then \( p_M(y;x) \in \mathbb{R}_{n-1-M}[x,y] \) and the polynomials \( \{p_M(y;x)U^q_j(x)\}_{j=0}^{N-(n-1)} \) are orthonormal elements of \( \widehat{P}_{N,M} \mathcal{L}[x,y] \) when \( N \geq n \), \( M \geq m \).

(i) We explain in Section 6 how this orthonormal set can be extended to an orthonormal basis of \( \widehat{P}_{N,M} \mathcal{L}[x,y] \) which also satisfies a Chebyshev relation in \( N \) in the case when \( p(x,w) \) does not vanish on \([-1,1] \times \mathbb{D} \) using an appropriate basis of spaces associated with Bernstein-Szegő measures on the torus \( \mathbb{T} \), see Theorem 6.2.

**Remark 6.3.** (ii) and Corollary 6.4.

(ii) Note that the coefficient of \( y^M \) in \( p_M(y;x) \) is \( 2^M p(x,0) \). This means that the basis constructed in (i) above can be characterized as follows. We start with the set \( \{p(x,0), xp(x,0), \ldots, x^{N-(n-1)}p(x,0)\} \) in \( \mathbb{R}_N[x] \) and we complete it to a basis \( \mathcal{B}_N \) of \( \mathbb{R}_N[x] \). Then the basis of \( \widehat{P}_{N,M} \mathcal{L}[x,y] \) in (i) corresponds to the orthonormal polynomials built using \( \mathcal{B}_N \) as explained in Section 1.2. In particular, if \( p(x,0) \) is a positive constant, we can take \( \mathcal{B}_N \) to be the standard basis \( (1,x,\ldots,x^N) \) of \( \mathbb{R}_N[x] \).

**Remark 4.8.** The explicit examples in Section 7 show that equation (4.4) does not imply conditions (a) and (b) in Theorem 4.1 (II).

**Remark 4.9.** Clearly, we can reverse the roles of \( x \) and \( y \) in Theorem 4.1 and consider functionals of the form

\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} f(x, y) \sqrt{1-x^2} \sqrt{1-y^2} q(y)\tilde{p}(z,y)\tilde{p}(1/z,y) \, dx \, dy, \tag{4.20}
\]

where \( q(y) \in \mathbb{R}_{2m_1}[y] \) and \( \tilde{p}(z,y) \in \mathbb{R}_{2n,m-m_1}[z,y] \) are such that

(i) \( q(y) > 0 \) and \( \tilde{p}(z,y) \neq 0 \) when \( (z,y) \in \mathbb{D} \times (-1,1) \),
Lemma, it follows that (4.9) holds for all $x = x_D$ of Lemma 4.10. Let

Using the same argument but reversing the roles of $z$ and $w$, we can restrict our attention to $T$ on just one of the variables vanishing on $T(T \times T)$ only simple zeros at $x = 0$. Thus we can restrict our attention to $T$ and need to show that $\omega(z, w)$ can vanish there only at $(\pm 1, \pm 1)$. This follows by applying arguments similar to the ones we used at the beginning of the proof of Theorem 4.1. (I).

We set

and thus the denominator in the integrand in (4.23) is equal to $p(x, w)p(x, 1/w)$. By Tonelli’s theorem, equation (4.23) holds for a.e. $x \in (-1, 1)$. Moreover, if $x = x_0$ is such that (4.9) holds then $p(x_0, w) \neq 0$ for $w \in \overline{T} \setminus \{\pm 1\}$ and $p(x_0, w)$ can have only simple zeros at $w = \pm 1$. But if $z_0 \in T$ is such that $x_0 = \frac{1}{2} \left( z_0 + \frac{1}{z_0} \right)$, then (4.25) shows that the real zeros of $p(x_0, w)$ have even multiplicities. Therefore, if $x = x_0$ is such that (4.9) holds then $p(x_0, w) \neq 0$ for $w \in \overline{T}$. Applying Fatou’s lemma, it follows that (4.9) holds for all $x \in (-1, 1)$ and therefore

Using the same argument but reversing the roles of $z$ and $w$, we see that

The proof follows from equations (4.26a) and (4.26b).
Note that \( \omega_{\varepsilon_1 \varepsilon_2}(z, w) = 2 + \varepsilon_1 z + \varepsilon_2 w \) where \( \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \) provide examples of polynomials satisfying (4.29) and vanishing at the four points \((\pm 1, \pm 1)\).

The next theorem describes the characteristic properties of the Bernstein-Szegö measures on \( \mathbb{R}^2 \) for which both Theorem 4.1 and Remark 4.9 apply.

**Theorem 4.11** (Bernstein-Szegö measures on \( \mathbb{R}^2 \)). Suppose that
- \( \omega(z, w) \in \mathbb{R}_{n_0,m_0}[z, w] \) is nonzero for \((z, w) \in \mathbb{D}^2 \),
- \( q_1(x) \in \mathbb{R}_{2n_1}[x] \) is positive for \( x \in (-1, 1) \),
- \( q_2(y) \in \mathbb{R}_{2m_1}[y] \) is positive for \( y \in (-1, 1) \),

and

\[
Q(x, y) = q_1(x)q_2(y)\omega(z, w)\omega(1/z, w)\omega(1, 1/w)\omega(1/z, 1/w)
\]

is such that

\[
\iint_{(-1,1)^2} \frac{1-x^2}{Q(x, y)} \frac{1-y^2}{Q(x, y)} \, dx \, dy < \infty.
\]

Then the recurrence coefficients of the linear functional \( \mathcal{L} : \mathbb{R}[x, y] \to \mathbb{R} \) defined by

\[
\mathcal{L}(f) = \frac{4}{\pi^2} \iint_{(-1,1)^2} f(x, y) \frac{1-x^2}{Q(x, y)} \frac{1-y^2}{Q(x, y)} \, dx \, dy,
\]

satisfy

\[
A_{k+1,l} = \frac{1}{2} I_{k+1}, \quad B_{k,l} = 0, \quad \text{for all} \quad k \geq n, \quad l \geq m,
\]

\[
\tilde{A}_{k+1,l} = \frac{1}{2} \tilde{I}_{k+1}, \quad \tilde{B}_{k,l} = 0, \quad \text{for all} \quad k \geq n, \quad l \geq m,
\]

where \( n = n_0 + n_1, \quad m = m_0 + m_1 \). Moreover, if \( \tilde{q}_1(z) \in \mathbb{R}_{2n_1}[z], \quad \tilde{q}_2(w) \in \mathbb{R}_{2m_1}[w] \) denote the stable Fejér-Riesz factors of \( q_1(x) \) and \( q_2(y) \), respectively, and if we set

\[
p(x, w) = \tilde{q}_2(w)\omega(z, w)\omega(1/z, w),
\]

\[
\tilde{p}(z, y) = \tilde{q}_1(z)\omega(z, w)\omega(1, 1/w),
\]

then equation (4.1) with \( q(x) \) replaced by \( q_1(x) \) and equation (4.21) with \( q(y) \) replaced by \( q_2(y) \) hold.

**Proof.** From Lemma 4.10 we know that \( \omega(z, w) \) is nonzero on \( \mathbb{D}^2 \), except possibly at the four points \((\pm 1, \pm 1)\). Combining this with (4.31a) we see that \( p(x, w) \neq 0 \) for \((x, w) \in (-1, 1) \times (\mathbb{D} \setminus \{-1, 1\})\). Note also that \( p(x, w) \in \mathbb{R}_{n_0,2m}[x, w] \) and the denominator on the right-hand side of (4.29) can be rewritten as

\[
q_1(x)p(x, w)p(x, 1/w)
\]

Thus (4.30a) and (4.31) with \( q(x) \) replaced by \( q_1(x) \) follow from Theorem 4.1. The proof of (4.30b) and (4.21) with \( q(y) \) replaced by \( q_2(y) \) can be obtained by applying the same arguments, exchanging the roles of \( z \) and \( w \). \( \square \)

The next theorem goes in the opposite direction and characterizes the Bernstein-Szegö measures in (4.27)-(4.29).

**Theorem 4.12.** Let \( \mathcal{L} : \mathbb{R}_{2n,2m}[x, y] \to \mathbb{R} \) be a positive linear functional with vector orthonormal polynomials \( \{P_{k,m}(x, y)\}_{k \leq n} \) and \( \{\tilde{P}_{n,j}(x, y)\}_{j \leq m} \), such that \( \tilde{\Psi}^m_n(z) = z^n(P^m_n(x) - 2z P_{n,m}^m(x)) \) is invertible for \( z \in (-1, 1) \), while \( \tilde{\Psi}^m_n(w) = w^m(P^m_n(y) - 2w \tilde{P}_{n,m}^m(y)) \) is invertible for \( w \in (-1, 1) \). Suppose that there exist polynomials...
(a) \( p(x, w) \in \mathbb{R}_{n-n_1, 2m}[x, w] \) which is nonzero for \((x, w) \in (-1, 1)^2\) and \(q_1(x) \in \mathbb{R}_{2m, 1}[x]\) for some \(n_1 \leq n\) such that eq. (4.11) with \(q(x)\) replaced by \(q_1(x)\) holds,
(b) \( p(z, y) \in \mathbb{R}_{2m, m-1}[z, y] \) which is nonzero for \((z, y) \in (-1, 1)^2\) and \(q_2(y) \in \mathbb{R}_{2m_1, 1}[y]\) for some \(m_1 \leq m\) such that eq. (4.21) with \(q(y)\) replaced by \(q_2(y)\) holds.

Then \(q_1(x) > 0\), \(q_2(y) > 0\) for \(x, y \in (-1, 1)\), and there exists \(\omega(z, w) \in \mathbb{R}_{n-n_1, m-m_1}[z, w]\) which is nonzero for \((z, w) \in \mathbb{D}^2\) and such that if we set

\[ Q(x, y) = q_1(x)q_2(y)\omega(z, w)\omega(1/z, w)\omega(1, 1/w)\omega(1/z, 1/w), \]

(4.32)

then (4.29) holds for all \(f \in \mathbb{R}_{2m, 2m}[x, y]\).

Besides Theorem 4.1(II) and Remark 4.9 for the proof of the above theorem we need a few lemmas. The first one provides a natural extension of a well-known criterion [2] Theorem 1] for stability of bivariate polynomials on \(\mathbb{D}^2\), by replacing \(\mathbb{D}^2\) with \(\mathbb{D}^2\) and by allowing vanishing at some points on the boundary of \(\mathbb{D}^2\).

**Lemma 4.13.** Let \(\omega(z, w) \in \mathbb{C}[z, w]\).

(a) If \(z_0 \in \bar{\mathbb{D}}\) and if \(\hat{T}_2\) is a dense subset of \(\mathbb{T}\) such that

\[ \omega(z_0, w) \neq 0, \quad \text{for all } w \in \mathbb{D}, \]

(4.33a)

\[ \omega(z, w) \neq 0, \quad \text{for } (z, w) \in \mathbb{D} \times \hat{T}_2, \]

(4.33b)

then

\[ \omega(z, w) \neq 0 \quad \text{for } (z, w) \in \mathbb{D}^2. \]

(4.34)

(b) In particular, if \(T_1\) and \(T_2\) are countable subsets of \(\mathbb{T}\) and

\[ \omega(z, w) \neq 0, \quad \text{for } (z, w) \in (\mathbb{T} \setminus T_1) \times (\mathbb{D} \setminus T_2), \]

(4.35a)

\[ \omega(z, w) \neq 0, \quad \text{for } (z, w) \in \mathbb{D} \times (\mathbb{T} \setminus T_2), \]

(4.35b)

then (4.34) holds.

**Proof.** The statements in both parts are trivial if \(\omega(z, w)\) depends only on one of the variables. Moreover, if \(\omega(z, w)\) factors in \(\mathbb{C}[z, w]\) then equations (4.33) in (a) or (4.35) in (b) will hold for each of the factors and (4.34) will be true for \(\omega(z, w)\) if and only if it is true for each of the factors. Thus, without any restriction, we can assume in the proof that \(\omega(z, w)\) has no factors depending only on \(z\) or \(w\).

We start by proving (a). It is easy to see that \(\omega(z, w) \neq 0\) for all \((z, w) \in \mathbb{D} \times \mathbb{T}\). By (4.33b), we just need to show that this is true if \(w \notin \hat{T}_2\). We fix \(w_0 \in \mathbb{T} \setminus \hat{T}_2\) and we consider a sequence \(\hat{w}_k\) of points in \(\hat{T}_2\) such that \(\hat{w}_k \to w_0\) as \(k \to \infty\). Since \(\omega(z, \hat{w}_k) \to \omega(z, w_0)\) uniformly on \(\mathbb{D}\), the proof follows from Hurwitz’s theorem.

Let \(\{K_n\}_{n=1}^\infty\) be a sequence of connected, compact subsets of \(\mathbb{D} \cup \{z_0\}\) such that \(z_0 \in K_n\) and

\[ \bigcup_{n=1}^\infty K_n = \mathbb{D} \cup \{z_0\}. \]

(4.36)

For instance, we can take \(K_n\) to be the convex hull of the closed disk \(\overline{B}_{n/(n+1)}\)

centered at the origin with radius \(n/(n+1)\) and \(\{z_0\}\). Then for every fixed \(z \in K_n\), \(\omega(z, w) \neq 0\) when \(w \in \mathbb{T}\) and the number of zeros of \(\omega(z, w)\) for \(w \in \mathbb{D}\) is

\[ Z(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\partial_w \omega(z, w)}{\omega(z, w)} dw. \]

By the dominated convergence theorem, \(Z : K_n \to \mathbb{N}_0\) is continuous and since \(K_n\) is connected, \(Z\) must be a constant. Since \(Z(z_0) = 0\), we see that \(Z(z) = 0\) for all
$z \in K_n$, i.e. \( \omega(z, w) \neq 0 \) when \((z, w) \in K_n \times \mathbb{D} \). Equation (4.34) now follows from (4.36), completing the proof of (a).

If we set $T_2 = T \setminus T_1$, then clearly (4.35a) implies (4.33a), hence (b) will follow from (a) if we can show that equation (4.33a) implies (4.35a). For each $w_0 \in T_2$ the polynomial $\omega(z, w_0)$ is not identically 0 and has finitely many roots in $\mathbb{C}$. Let $z_0 \in T \setminus T_1$ be such that $\omega(z_0, w_0) \neq 0$ for all $w_0 \in T_2$. Then by construction and using (4.35a) we see that (4.33a) holds.

**Lemma 4.14.** Let $\xi(z, w) \in \mathbb{R}[z, w]$ be an irreducible polynomial which depends on both $z$ and $w$, such that

$$\xi(z, w) = cz^k \xi(1/z, w), \quad (4.37)$$

for some $k \in \mathbb{N}$ and $c \in \mathbb{R}$. Then $c = 1$, $k$ is even and there exists $(z_0, w_0) \in \mathbb{D}^2$ such that $\xi(z_0, w_0) = 0$.

**Proof.** Iterating (4.37) we see that $\xi(z, w) = cz^k \xi(1/z, w) = c^2 \xi(z, w)$ which shows that $c = \pm 1$. If we assume that $c = -1$ then setting $z = 1$ in (4.37) gives $\xi(1, w) = 0$ which is impossible because this would imply that $\xi(z, w)$ is divisible by $z - 1$. Thus $c = 1$ and it is easy to see that $k$ must be even since otherwise (4.37) with $z = -1$ would imply that $\xi(z, w)$ is divisible by $z + 1$.

Equation (4.37) and the irreducibility of $\xi(z, w)$ show that $\xi(z, w)$ has the expansion

$$\xi(z, w) = \sum_{j=0}^{k} \xi_j(w)z^j,$$

where $\xi_j(w) = \xi_{k-j}(w)$ for $j = 0, \ldots, k$ and the highest coefficient $\xi_k(w)$ is not identically equal to 0. If we take any $w_1 \in \mathbb{D}$ such that $\xi_k(w_1) \neq 0$, the product of the roots of $\xi(z, w_1) = 0$ will be 1 and therefore, if none of them is in $\mathbb{D}$, they all must lie on $\mathbb{T}$. The proof will follow if we can show that this cannot be true for all such $w_1 \in \mathbb{D}$. Fix $w_1 \in \mathbb{D}$ which is neither a zero of the highest coefficient $\xi_k(w)$ nor a zero of the discriminant of $\xi(z, w)$ viewed as a polynomial of $z$. If $z_1$ is any root of the equation $\xi(z, w_1) = 0$, there exists an open disk $D_1(w_1) \subset \mathbb{D}$ containing $w_1$ and a nonconstant holomorphic function $f : D_1(w_1) \to \mathbb{C}$ such that $f(w_1) = z_1$ and $\xi(f(w), w) = 0$. Since the set $f(D_1(w_1))$ is open, it cannot be a subset of $\mathbb{T}$.

**Lemma 4.15.** Let $Q(x, y) \in \mathbb{R}[x, y]$ be a polynomial which is positive for $(x, y) \in (-1, 1)^2$. If there exist polynomials $q_1(x)$, $q_2(y)$, $p(x, w)$ and $\tilde{p}(z, y)$ with real coefficients such that

(i) $p(x, w) \neq 0$ for $(x, w) \in (-1, 1) \times (\mathbb{D} \setminus \{-1, 1\})$,

(ii) $\tilde{p}(z, y) \neq 0$ for $(z, y) \in (\mathbb{D} \setminus \{-1, 1\}) \times (-1, 1)$,

and

$$Q(x, y) = q_1(x)p(x, w)p(x, 1/w) = q_2(y)\tilde{p}(z, y)\tilde{p}(1/z, y), \quad (4.38)$$

then there exists a polynomial $\omega(z, w) \in \mathbb{R}[z, w]$ which is nonzero for $(z, w) \in \mathbb{D}^2$ and equation (4.27) holds.

**Proof.** Equation (4.38) shows that $q_1(x) > 0$ for $x \in (-1, 1)$ and $q_1(x)$ divides $Q(x, y)$. Moreover, if $\tilde{q}_1(z) \in \mathbb{R}[z]$ is the stable Fejér-Riesz factor of $q_1(x)$, then it is easy to see that $\tilde{q}_1(z)$ divides $\tilde{p}(z, y)$. Thus, we can cancel $q_1(x)$ in (4.38) and the remaining polynomials will satisfy the conditions in the lemma. This means that we can assume that $q_1(x) = 1$ and a similar argument shows that we can take
$q_2(y) = 1$. In view of this, we assume below that $q_1(x) = q_2(y) = 1$ and therefore (4.38) reduces to

$$Q(x, y) = p(x, w)p(x, 1/w) = \hat{p}(z, y)\hat{p}(1/z, y).$$

We can further assume that $p(x, w)$ and $\hat{p}(z, y)$ have no factors depending on just one of the variables. Indeed, $p(x, w)$ has a factor $q_0(x)$ if and only if $\hat{q_0(x)}$ divides $Q(x, y)$ and $\hat{p}(z, y)$ has a factor $\hat{q_0(z)}$ such that $\hat{q_0}(x) = \hat{q_0(z)}\hat{q_0}(1/z)$. This means that if we know how to construct $\omega(z, w)$ satisfying (4.32) with $Q$ replaced by $Q(x, y)/\hat{q_0}(x)$, we can multiply it by $\hat{q_0(z)}$ to obtain $\omega(z, w)$ for $Q(x, y)$. A similar argument can be applied for factors depending only on $y$ and $w$. We can also assume that $\hat{p}$ and $\hat{p}$ are normalized so that $p(0, 0) > 0$ and $\hat{p}(0, 0) > 0$.

Next, we replace $x$ by $\frac{1}{2}(z + \frac{x}{w})$ and $p(x, w)$ into a product of irreducible factors in the unique factorization domain $\mathbb{R}[z^{\pm 1}, w]$. With the normalization above, we know that in this factorization, $p(x, w)$ cannot have irreducible factors depending only on $z$ or $w$. We show next that this factorization cannot contain an irreducible factor $\xi(z, w)$ which is associated to $\xi(1/z, w)$ in $\mathbb{R}[z^{\pm 1}, w]$. Indeed, assume that $p(x, w)$ has such irreducible factor, i.e. (4.37) holds and by multiplying by a unit element of the form $a z^j$ where $a \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{Z}$ we can assume that $\xi(z, w)$ is an irreducible element in $\mathbb{R}[z, w]$. Condition (i) shows that

$$\xi(z, w) \neq 0 \quad \text{when} \quad (z, w) \in (\mathbb{T} \setminus \{−1, 1\}) \times (\mathbb{D} \setminus \{−1, 1\}).$$

We replace $y$ by $\frac{1}{2}(w + \frac{1}{y})$ and we consider (4.39) in the ring $\mathbb{R}[z^{\pm 1}, w^{\pm 1}]$ of Laurent polynomials of $z$ and $w$ which is also a unique factorization domain. This shows that $\xi(z, w)$ must also be an irreducible factor in both $\hat{p}(z, y)$ and $\hat{p}(1/z, y)$ in the ring $\mathbb{R}[z^{\pm 1}, w^{\pm 1}]$. Therefore, condition (ii) tells us that

$$\xi(z, w) \neq 0 \quad \text{when} \quad (z, w) \in (\mathbb{D} \setminus \{−1, 1\}) \times (\mathbb{T} \setminus \{−1, 1\}).$$

Using equations (4.40a)-(4.40b) and Lemma 4.13(b) with $T_1 = T_2 = \{−1, 1\}$ it follows that $\xi(z, w) \neq 0$ for $(z, w) \in \mathbb{D}^2$ which contradicts Lemma 4.13.

Since $p(x, w)$ is invariant under the involution $z \rightarrow 1/z$ and has no irreducible factor $\xi(z, w)$ which is associated to $\xi(1/z, w)$, we can factor it as follows

$$p(x, w) = \prod_{j=1}^l \omega_j(z, w)\omega_j(1/z, w), \quad \text{where} \quad \omega_j(z, w) \in \mathbb{R}[z, w] \text{ are irreducible. (4.41)}$$

From condition (i) it follows that for all $j = 1, \ldots, l$ we have

$$\omega_j(z, w) \neq 0 \quad \text{for} \quad (z, w) \in (\mathbb{T} \setminus \{−1, 1\}) \times (\mathbb{D} \setminus \{−1, 1\}).$$

Note that we still have some freedom in choosing the polynomials $\omega_j(z, w)$: if $n_j \in \mathbb{N}$ is the minimal positive integer such that $z^{n_j}\omega_j(1/z, w) \in \mathbb{R}[z, w]$ (i.e. $n_j$ is the degree of $\omega_j(z, w)$ viewed as a polynomial in $z$ for generic $w$), then we can replace $\omega_j(z, w)$ by $z^{n_j}\omega_j(1/z, w)$ and both the representation in (4.41) and the nonvanishing condition in (4.42) will still hold. For each $j$, we will pick one of these two terms below and include it in $\omega(z, w)$.

Similarly to the constructions above, we replace $y$ by $\frac{1}{2}(w + \frac{1}{y})$ and we can factor $\hat{p}(z, y)$ as follows

$$\hat{p}(z, y) = \prod_{k=1}^l \hat{\omega}_k(z, w)\hat{\omega}_k(1/w), \quad \text{where} \quad \hat{\omega}_k(z, w) \in \mathbb{R}[z, w] \text{ are irreducible. (4.43)}$$
Using condition (ii) we see that for \( k = 1, \ldots, l \) we have
\[
\hat{\omega}_k(z, w) \neq 0 \quad \text{for} \quad (z, w) \in (\overline{D} \setminus \{1\}) \times (\mathbb{T} \setminus \{-1, 1\}).
\] (4.44)

If \( \hat{m}_k \in \mathbb{N} \) is the minimal positive integer such that \( w^{\hat{m}_k} \hat{\omega}_k(z, 1/w) \in \mathbb{R}[z, w] \), then we can replace \( \hat{\omega}_k(z, w) \) by \( w^{\hat{m}_k} \hat{\omega}_k(z, 1/w) \) and both the representation in (4.43) and the nonvanishing condition in (4.44) will still hold.

We substitute (4.41) and (4.43) into (4.39) and consider this equation in the unique factorization domain \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \). It follows that for every \( j \), there exists a unique \( k \) such that exactly one of the following holds:

(a) \( \omega_j(z, w) \) is associate to \( \hat{\omega}_k(z, w) \) or \( \hat{\omega}_k(z, 1/w) \) in \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \);

(b) \( \omega_j(z, w) \) is associate to \( \hat{\omega}_k(1/z, w) \) or \( \hat{\omega}_k(1/z, 1/w) \) in \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \).

If (b) holds for some \( j \), then we will use the freedom to replace \( \omega_j(z, w) \) by \( z^{n_j} \omega_j(1/z, w) \), and thus we can assume that (a) holds for all \( j = 1, \ldots, l \). Next, if \( \omega_j(z, w) \) is associate to \( \hat{\omega}_k(z, 1/w) \) then we replace \( \hat{\omega}_k(z, w) \) by \( w^{\hat{m}_k} \hat{\omega}_k(z, 1/w) \).

With this normalization, for every \( j \in \{1, \ldots, l\} \), there exists a unique \( k \in \{1, \ldots, l\} \) such that \( \omega_j(z, w) \) is associate to \( \hat{\omega}_k(z, w) \) in \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \). Note that the units in \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \) are of the form \( cz^j w^s \), where \( c \in \mathbb{R} \setminus \{0\} \), \( j, s \in \mathbb{Z} \) and the polynomials \( \omega_j(z, w) \), \( \hat{\omega}_k(z, w) \) are not divisible by \( z \) or \( w \), so we must have
\[
\omega_j(z, w) = c_k \hat{\omega}_k(z, w), \quad \text{where} \quad c_k \text{ is a nonzero constant}.
\]

Combining the last equation with equations (4.42) and (4.44), and applying Lemma 4.13(b) with \( T_1 = T_2 = \{-1, 1\} \), we see that \( \omega_j(z, w) \neq 0 \) for \( (z, w) \in \mathbb{D}^2 \). Since this is true for every \( j \), the proof follows by setting \( \omega(z, w) = \prod_{j=1}^l \omega_j(z, w) \).

**Proof of Theorem 4.12** Theorem 4.11(H) and Remark 4.3 tell us that \( q_1(x) > 0 \) for \( x \in (-1, 1) \),
\[
p(x, w) \neq 0 \quad \text{for} \quad (x, w) \in (-1, 1) \times (\overline{D} \setminus \{-1, 1\})
\]
and if we set
\[
Q(x, y) = q_1(x)p(x, w)p(x, 1/w),
\]
then (4.29) holds for all \( f \in \mathbb{R}_{2n,2m}[x,y] \). Similarly, reversing the roles of \( x \) and \( y \) using Remark 4.9 we see that \( q_2(y) > 0 \) for \( y \in (-1, 1) \),
\[
\hat{p}(z, y) \neq 0 \quad \text{for} \quad (z, y) \in (\overline{D} \setminus \{-1, 1\}) \times (-1, 1)
\]
and if we set
\[
Q_1(x, y) = q_2(y)\hat{p}(z, y)\hat{p}(1/z, y),
\]
then (4.29) holds for all \( f \in \mathbb{R}_{2n,2m}[x,y] \) with \( Q \) replaced by \( Q_1 \). By Lemma 4.6
\[
Q(x, y) = q_1(x)p(x, w)p(x, 1/w) = q_2(y)\hat{p}(z, y)\hat{p}(1/z, y).
\]
The proof now follows from Lemma 4.15.

**5. Orthogonal decompositions for Bernstein-Szegő measures on the torus**

We denote by \( \mathbb{C}[z^{\pm 1}, w^{\pm 1}] = \mathbb{C}[z, z^{-1}, w, w^{-1}] \) the space of Laurent polynomials with complex coefficients in \( z \) and \( w \). For every polynomial \( g(z, w) \) we set
\[
\bar{g}(z, w) = \overline{g(\bar{z}, \bar{w})},
\]
Lemma 5.2. \[ V[z, w] = \{ \tilde{g}(z, w) : g(z, w) \in V[z, w] \}. \]

Recall that \( \mathbb{C}_{\tilde{n}, \tilde{m}}[z, w] \) denotes the space of polynomials with complex coefficients in \( z \) and \( w \) of degrees at most \( \tilde{n} \) in \( z \) and \( \tilde{m} \) in \( w \). Throughout this section, we fix \( p_c(z, w) \in \mathbb{C}_{\tilde{n}, \tilde{m}}[z, w] \) which is nonzero for \( (z, w) \in \mathbb{T} \times \mathbb{D} \) and we define a positive linear functional on \( \mathbb{C}[z^{\pm 1}, w^{\pm 1}] \) by

\[ L_{p_c}(z^k w^l) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}} \int_{\mathbb{D}} z^k w^l |dz| |dw| |p_c(z, w)|^2. \]

The functional \( L_{p_c} \) induces an inner product on \( \mathbb{C}[z, w] \) by

\[ \langle f, g \rangle_{L_{p_c}} = L_{p_c}(f(z, w)\tilde{g}(1/z, 1/w)), \quad \text{where } f, g \in \mathbb{C}[z, w], \]

and we consider the spaces

\[ P_{k,l} : L_{p_c}[z, w] = \mathbb{C}_{k,l}[z, w] \oplus \mathbb{C}_{k-1,l}[z, w], \quad (5.1a) \]
\[ \tilde{P}_{k,l} : L_{p_c}[z, w] = \mathbb{C}_{k,l}[z, w] \oplus \mathbb{C}_{k,l-1}[z, w], \quad (5.1b) \]

of orthogonal polynomials with respect to this inner product.

With these notations, we can summarize some of the results in [12][14] as follows.

**Theorem 5.1.** For \( M \geq \tilde{m} \), there exist unique sub-spaces \( \tilde{P}_{n-1,M;L_{p_c}}, \tilde{P}_{n-1,M;L_{p_c}} \) of \( \tilde{P}_{n-1,M;L_{p_c}}[z, w] \) such that

\[ \tilde{P}_{n-1,M;L_{p_c}}[z, w] = \tilde{P}_{n-1,M;L_{p_c}}^{1} \oplus \tilde{P}_{n-1,M;L_{p_c}}^{2}, \quad (5.2a) \]
\[ \tilde{P}_{n,M;L_{p_c}}[z, w] = \tilde{P}_{n-1,M;L_{p_c}}^{1} \oplus \mathbb{C}_{n,M}[z, w] \oplus \mathbb{C}_{n-1,M}[z, w], \quad (5.2b) \]

where \( \tilde{p}_c(z, w) = z^n w^{\tilde{m}} p_c(1/z, 1/w) \). Moreover, these spaces satisfy the following orthogonality conditions

\[ \tilde{P}_{n-1,M;L_{p_c}}^{1} \perp z^k w^l, \quad \text{for all } 0 \leq k, \quad 0 \leq l \leq M - 1, \quad (5.3a) \]
\[ \tilde{P}_{n-1,M;L_{p_c}}^{1} \perp z^k w^l \tilde{p}_c(z, w), \quad \text{for all } 0 \leq k, \quad (5.3b) \]
\[ \tilde{P}_{n-1,M;L_{p_c}}^{2} \perp z^k w^l, \quad \text{for all } k \leq n - 1, \quad 0 \leq l \leq M - 1, \quad (5.3c) \]
\[ z^k \tilde{P}_{n-1,M;L_{p_c}}^{2} \perp w^{M-n} \tilde{p}_c(z, w), \quad \text{for all } 1 \leq k, \quad (5.3d) \]
\[ \tilde{P}_{n-1,M;L_{p_c}}^{1} \perp z^k \tilde{P}_{n-1,M;L_{p_c}}^{2}, \quad \text{for all } 0 \leq k. \quad (5.3e) \]

Note that the orthogonality conditions [13][14] tell us that for all \( N \geq \tilde{n} \) and \( M \geq \tilde{m} \) we have

\[ \tilde{P}_{N,M;L_{p_c}}[z, w] = \tilde{P}_{n-1,N;L_{p_c}}^{1} \oplus \mathbb{C}_{N-n-1,M}[z, w] \oplus \bigoplus_{k=0}^{N-n} \mathbb{C}_{z^k w^{M-n} \tilde{p}_c(z, w)} \]

In the lemma below we prove another orthogonality relation which will be useful later.

**Lemma 5.2.** For \( k \geq 0 \) we have

\[ z^{k+n-1} w^{M-k} \tilde{P}_{n-1,M;L_{p_c}}^{1} [1/z, 1/w] \perp \tilde{P}_{n-1,M;L_{p_c}}^{1}[z, w]. \]
Proof. In the proof, it will be convenient to use the fact that the inner product
\( \langle \cdot, \cdot \rangle_{\mathcal{L}_p} \) extends to the Hilbert space \( L^2(T^2, \mu_c) \) where \( d\mu_c = \frac{|dz| |d\omega|}{(2\pi)^2 |p_c(z,w)|} \). Suppose that
\[
p_c(z,0) = p_s(z)p_u(z),
\]
where
- \( p_s(z) \) is nonzero for \(|z| \leq 1\);
- \( p_u(z) \) is nonzero for \(|z| \geq 1\),
and let \( \tilde{n} = \text{deg}_z p_u(z) \). If \( f \in \mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p} \), then by [14] Theorem 2.3 we have
\[
f(z,w) = h(z)z^{\tilde{n}0}\mathcal{P}_u(1/z)w^M + g(z,w),
\]
where \( h(z) \in \mathbb{C}_{\tilde{n}-\tilde{n}_0-1}[z] \) and \( g(z,w) \in \mathbb{C}_{\tilde{n}-1,M-1}[z,w] \).

Therefore,
\[
z^{k+\tilde{n}-1}w^{-1}f(1/z,1/w) = z^{k+\tilde{n}-1}\tilde{h}(1/z)p_u(z)\frac{1}{w} + z^{k+\tilde{n}-1}w^{M-1}\tilde{g}(1/z,1/w).
\]

Note that \( z^{k+\tilde{n}-1}w^{M-1}\tilde{g}(1/z,1/w) \parallel \mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p} \) by (5.3a), and therefore it is enough to show that \( w^{-1}p_u(z)\mathbb{C}[z] \perp \mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p} \). Moreover,
\[
p_u(z) = \frac{p(z,0)}{wp_s(z)} = \frac{p(z,w)}{wp_s(z)} - \frac{p(z,w) - p(z,0)}{wp_s(z)},
\]
where \( r(z,w) \) is a polynomial in \( w \) of degree at most \( \tilde{n} - 1 \), whose coefficients are holomorphic on the closed disk \(|z| \leq 1\), and therefore \( r(z,w)\mathbb{C}[z] \perp \mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p} \) by (5.3a). Finally, it is easy to prove \( (5.5) \) using the definition of the inner product and by computing the \( w \)-integral. \qed

**Proposition 5.3.** If \( \tilde{n} \) is even and
\[
z^{\tilde{n}}p_c(1/z,w) = p_c(z,w),
\]
then
\[
\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}[z,w] = z^{\tilde{n}-1}\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}[1/z,w].
\]

Proof. For \( k \in \mathbb{N}_0 \), consider the involution \( \mathcal{R}_s^k \) on \( \mathbb{C}[z^\pm 1, w^\pm 1] \) defined by
\[
\mathcal{R}_s^k(g(z,w)) = z^kg(1/z,w).
\]
Equation (5.5) shows that \( \mathcal{R}_s^k \) preserves the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{L}_p} \), and since
\[
\mathcal{R}_s^k(\mathbb{C}_{k,\ell}[z,w]) = \mathbb{C}_{k,\ell}[z,w],
\]
we see that \( \mathcal{P}_{k,\ell;\mathcal{L}_p}[z,w] \) is an invariant subspace of \( \mathcal{R}_s^k \). In particular, applying \( \mathcal{R}_s^{\tilde{n}-1} \) to (5.2a) yields
\[
\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p} = \mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}) \oplus \mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}).
\]
Equation (5.3a) implies that \( \mathcal{R}_s^{\tilde{n}}(\mathcal{P}_c(z,w)) = \mathcal{P}_c(z,w) \) and since
\[
\mathcal{R}_s^{\tilde{n}}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}) = z\mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}), \quad \mathcal{R}_s^{\tilde{n}}(z\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}) = z\mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}),
\]
applying \( \mathcal{R}_s^{\tilde{n}} \) to (5.2b) shows that
\[
\mathcal{P}_{\tilde{n},M;\mathcal{L}_p} = z\mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}) \oplus \mathcal{R}_s^{\tilde{n}-1}(\mathcal{P}_{\tilde{n}-1,M;\mathcal{L}_p}) \oplus \text{span}_\mathbb{C}\{w^M-\tilde{m}\mathcal{P}_c(z,w)\}.
\]
From equations (5.7a)-(5.7b) and the uniqueness of the subspaces \( \hat{P}_{n-1,M;C_{Pc}}^j \) satisfying (5.2a)-(5.2b) it follows that (5.6) holds.

Note that (5.5) cannot hold if \( \tilde{n} \) is odd because \( p_c(z,w) \) will vanish when \( z = -1 \).

**Remark 5.4.** An important subclass of measures on \( \mathbb{T}^2 \) associated with polynomials \( p_c(z,w) \) which are nonzero for \( (z,w) \in \mathbb{D}^2 \) were introduced and studied in \cite{15}. In this case, there are several simplifications:

\[
\hat{P}_{n-1,M;C_{Pc}}^2 = \{0\}, \quad \hat{P}_{n-1,M;C_{Pc}}^1 [z,w] = \hat{P}_{n-1,M;C_{Pc}}^1 [z,w] = w^{M-\tilde{m}} \hat{P}_{n-1,\tilde{m};C_{Pc}} [z,w]
\]

and

\[
\hat{P}_{n-1,\tilde{m};C_{Pc}} \perp z^k w^l, \quad \text{for all} \quad 0 \leq k, \quad l \leq \tilde{m} - 1. \tag{5.8}
\]

We will also use the following theorem established in \cite[Theorem 2.7]{12}.

**Theorem 5.5.** Suppose that \( p_1(z,w) \in \mathbb{C}_{\tilde{n}_1,\tilde{n}_2}[z,w], \) \( p_2(z,w) \in \mathbb{C}_{\tilde{n}_2,\tilde{n}_2}[z,w] \) are polynomials nonvanishing for \( (z,w) \in \mathbb{D}^2 \), and

\[
p_c(z,w) = p_1(z,w)\tilde{n}_2 \cdot p_2(1/z,w), \text{ where } \tilde{n} = \tilde{n}_1 + \tilde{n}_2 \text{ and } \tilde{m} = \tilde{m}_1 + \tilde{m}_2.
\]

Then we have

\[
\hat{P}_{n-1,\tilde{m};C_{Pc}}^1 [z,w] = (w^{\tilde{m}_2} p_2(z,1/w)) \hat{P}_{n-1,\tilde{m}_1;C_{Pc}} [z,w], \tag{5.9a}
\]

\[
\hat{P}_{n-1,\tilde{m};C_{Pc}}^2 [z,w] = (z^{\tilde{n}_1} w^{\tilde{m}_2} p_2(1/z,1/w)) z^{\tilde{n}_2 - 1} p_1(1/z,w), \tag{5.9b}
\]

Moreover, for \( M \geq \tilde{m} \) we have

\[
\hat{P}_{n-1,M;C_{Pc}}^1 [z,w] = w^{M-\tilde{m}} \hat{P}_{n-1,\tilde{m};C_{Pc}}^1 [z,w], \tag{5.10a}
\]

\[
\hat{P}_{n-1,M;C_{Pc}}^2 [z,w] = w^{M-\tilde{m}} \hat{P}_{n-1,\tilde{m};C_{Pc}}^2 [z,w]. \tag{5.10b}
\]

If \( p_c(z,w) \) has a factor depending only on \( z \) we have the following decomposition.

**Theorem 5.6.** Suppose that \( p_1(z,w) \in \mathbb{C}_{\tilde{n}_1,\tilde{n}_2}[z,w] \) is nonzero for \( (z,w) \in \mathbb{T} \times \mathbb{D} \), \( p_2(z) \in \mathbb{C}_{\tilde{n}_2}[z] \) is nonzero for \( z \in \mathbb{D} \), and let

\[
p_c(z,w) = p_1(z,w)p_2(z), \text{ where } \tilde{n} = \tilde{n}_1 + \tilde{n}_2.
\]

If we set \( \hat{P}_1(z,w) = z^{\tilde{n}_1} w^{\tilde{m}_2} p_2(1/z,1/w) \) and \( \hat{P}_2(z) = z^{\tilde{n}_2} p_2(1/z) \), then for \( M \geq \tilde{m} \) we have

\[
\hat{P}_{n-1,1,M;C_{Pc}} [z,w] = p_2(z) P_{n-1,1,M;C_{Pc}}^1 [z,w] + w^{M-\tilde{m}} \hat{P}_1(z,w) C_{n_2-1}[z], \tag{5.11a}
\]

\[
\hat{P}_{n-1,2,M;C_{Pc}} [z,w] = \hat{P}_2(z) P_{n-1,2,M;C_{Pc}}^2 [z,w]. \tag{5.11b}
\]
Proof. We can check that with respect to $L_{pc}$ we have:

$$
p_{2}(z)\tilde{P}_{n-1,M;L_{pc}}^{1}[z, w] \perp z^{k}w^{l}, \quad \text{for all} \quad 0 \leq k, 0 \leq l \leq M-1,
$$

$$
w^{M-n}\tilde{P}_{1}(z, w)C_{n-1}[z] \perp z^{k}w^{l}, \quad \text{for all} \quad 0 \leq k, 0 \leq l \leq M-1,
$$

$$
p_{2}(z)\tilde{P}_{2n-1,M;L_{pc}}^{1}[z, w] \perp w^{M-n}\tilde{P}_{1}(z, w)C_{n-1}[z],
$$

$$
\tilde{P}_{2}(z)\tilde{P}_{2n-1,M;L_{pc}}^{2}[z, w] \perp z^{k}w^{l}, \quad \text{for all} \quad k \leq n-1, 0 \leq l \leq M-1,
$$

$$
p_{2}(z)\tilde{P}_{1}(z, w)C_{n-1}[z] \perp w^{M-n}\tilde{P}_{1}(z, w),
$$

$$
z\tilde{P}_{2}(z)\tilde{P}_{2n-1,M;L_{pc}}^{2}[z, w] \perp w^{M-n}\tilde{P}_{1}(z, w),
$$

and apply Theorem 5.1. □

Remark 5.7. If $p_{c}(z, w)$ has real coefficients then the spaces $\tilde{P}_{n-1,M;L_{pc}}^{1}$, $\tilde{P}_{2n-1,M;L_{pc}}^{2}$ in Theorem 5.1 have bases consisting of polynomials with real coefficients, and we can replace the complex field with $\mathbb{R}$ in all statements above. More precisely, if

$$
p_{c}(z, w) \in \mathbb{R}_{n,m}[z, w]
$$

and if we set

$$
P_{k,l;L_{pc};R}[z, w] = P_{k,l;L_{pc}}[z, w] \cap \mathbb{R}[z, w], \quad \tilde{P}_{k,l;L_{pc};R}[z, w] = \tilde{P}_{k,l;L_{pc}}[z, w] \cap \mathbb{R}[z, w],
$$

then we can replace equations (5.2) with

$$
\tilde{P}_{n-1,M;L_{pc};R}^{1} = \tilde{P}_{n-1,M;L_{pc};R}^{1} \oplus \tilde{P}_{2n-1,M;L_{pc};R}^{1}, \quad \text{(5.12a)}
$$

$$
\tilde{P}_{n,M;L_{pc};R}^{1} = \tilde{P}_{n-1,M;L_{pc};R}^{1} \oplus z\tilde{P}_{2n-1,M;L_{pc};R}^{1} \oplus \text{span}_{\mathbb{R}} \{w^{M-n}\tilde{P}_{1}(z, w)\}. \quad \text{(5.12b)}
$$

6. Szegő map

In this section, we explain how the decomposition of the spaces of polynomials on $T^{2}$ in the previous section can be used to construct bases of orthonormal polynomials for the Bernstein-Szegő measures in the plane. The section is divided in two parts. In the first one, we consider in detail the functional $L$ in Theorem 4.1 and the construction of the orthonormal complement on the tilde side of the space spanned by product polynomials obtained in analogy with the one-dimensional theory, cf. Remark 4.7. The spaces associated with the Bernstein-Szegő measures in Theorem 4.1 are analyzed in the second subsection. In this case, the constructions simplify significantly and provide explicit orthonormal bases on both sides in terms of fixed orthonormal polynomials on the bi-circle contained in the finite-dimensional space $\mathbb{R}_{n_{0}, m}[z, w]$.

6.1. Szegő map for the weight in Theorem 4.1 In this subsection, we consider the functional $L$ in (4.3) where $q(x) \in \mathbb{R}_{2n_{1}}[x], \ p(x, w) \in \mathbb{R}_{n_{0}, 2m}[x, w]$ are fixed.
polynomials such that $q(x) > 0$ and $p(x, w) \neq 0$ when $(x, w) \in [-1, 1] \times \mathbb{T}$. Let $\tilde{q}(z)$ be the stable Fejér-Riesz factor of $q(x)$ and let

\[ p_c(z, w) = \tilde{q}(z)\tilde{p}(z, w) \in \mathbb{R}_{2n,2m}[z,w], \text{ where } n = n_1 + n_0 \text{ and } \tilde{p}(z, w) = z^{n_0}p(x, w). \tag{6.1} \]

With these notations, the denominator of the weight in (6.3) can be written simply as $p_c(z, w)p_c(1/z, 1/w)$. Note that $p_c(z, w)$ is nonzero for $(z, w) \in \mathbb{T} \times \mathbb{T}$, and since it has real coefficients, we can use Remark 5.7 and work with the spaces introduced in previous section over $\mathbb{R}$ where $\tilde{n} = 2n$, $\tilde{m} = 2m$, but we will omit the explicit $\mathbb{R}$ dependence in order to simplify the notation. Using Theorem 5.6 we see that for $M \geq m$ we have

\[ \tilde{p}^1_{2n-1,2M+1;\mathcal{L}_p}[z,w] = \tilde{q}(z)\tilde{p}^1_{2n_0-1,2M+1;\mathcal{L}_p}[z,w] \oplus w^{2M+1}\tilde{p}(z,1/w)\mathbb{R}_{2n_1-1}[z], \tag{6.2a} \]

\[ \tilde{p}^2_{2n-1,2M+1;\mathcal{L}_p}[z,w] = z^{2n_1}\tilde{q}(1/z)\tilde{p}^2_{2n_0-1,2M+1;\mathcal{L}_p}[z,w]. \tag{6.2b} \]

Since $z^{2n_0}\tilde{p}(1/z, w) = \tilde{p}(z, w)$, Proposition 5.3 tells us that the spaces $\tilde{p}^1_{2n_0-1,2M+1;\mathcal{L}_p}[z,w]$ and $\tilde{p}^2_{2n_0-1,2M+1;\mathcal{L}_p}[z,w]$ in (6.2a) and (6.2b), respectively, are related by an appropriate reflection in $z$. In particular, this implies that

\[ \dim(\tilde{p}^1_{2n_0-1,2M+1;\mathcal{L}_p}[z,w]) = \dim(\tilde{p}^2_{2n_0-1,2M+1;\mathcal{L}_p}[z,w]) = n_0. \]

Let $S_{z,N} : \mathbb{R}[z] \to \mathbb{R}[x]$ denote the (linear) Szegő map

\[ S_{z,N}(f(z)) = \frac{z^{N+1}f(1/z) - z^{-N-1}f(z)}{z - 1/z}. \]

It is easy to see that $S_{z,N} : \mathbb{R}_{2n-1}[z] \to \mathbb{R}_N[x]$ for all $N \geq n$. Similarly, we have the Szegő map $S_{w,M-1} : \mathbb{R}[w] \to \mathbb{R}[y]$ defined by

\[ S_{w,M-1}(g(w)) = \frac{w^{M}g(1/w) - w^{-M}g(w)}{w - 1/w}, \]

and $S_{w,M-1} : \mathbb{R}_{2M+1}[w] \to \mathbb{R}_M[y]$. Finally, we define the bivariate Szegő map

\[ \mathcal{S} = S_{N,M-1} = S_{z,N} \circ S_{w,M-1} : \mathbb{R}[z,w] \to \mathbb{R}[x,y], \]

and therefore

\[ \mathcal{S} : \mathbb{R}_{2n-1,2M+1}[z,w] \to \mathbb{R}_{N,M}[x,y]. \]

Explicitly, we have

\[ \mathcal{S}(f(z,w)) = \frac{1}{(z - 1/z)(w - 1/w)} \left[ z^{N+1}w^{M}f(1/z,1/w) - z^{-N-1}w^{M}f(z,1/w) \right. \]

\[ \left. - z^{N+1}w^{-M}f(1/z, w) + z^{-N-1}w^{-M}f(z,w) \right]. \]

To simplify the notation, we fix in this subsection $N \geq n$, $M \geq m$, and we will write $\mathcal{S}$ instead of $S_{N,M-1}$, unless the $(N,M)$ dependence is important. Repeating the computation in Proposition 2.4 for the $z$-integral and the $w$-integral it follows that for $g(x,y) \in \mathbb{R}[x,y]$ and $f(z,w) \in \mathbb{R}[z,w]$ we have

\[ \langle g(x,y), \mathcal{S}(f(z,w)) \rangle_{\mathcal{L}} = \langle z^Nw^{-M-1}(z^2 - 1)(w^2 - 1)g(x,y), f(z,w) \rangle_{\mathcal{L}_p}. \tag{6.3} \]

Using this equation, it is easy to see that

\[ \mathcal{S} : \tilde{P}^1_{2n_1-1,2M+1;\mathcal{L}_p}[z,w] \to \tilde{P}_{N,M;\mathcal{L}}[x,y]. \tag{6.4} \]
Indeed, if \( g(x, y) \in \mathbb{R}_{N,M-1}[x, y] \), then \( z^N w^{M-1} (z^2 - 1)(w^2 - 1)g(x, y) \in \mathbb{R}[z, w] \) is of degree at most \( 2M \) in \( w \), hence (5.3a) shows that \( z^N w^{M-1} (z^2 - 1)(w^2 - 1)g(x, y) \perp \tilde{P}_{2n-1,2M+1,L_p}[z, w] \) with respect to \( L_{p_r} \). Thus, \( S(\tilde{P}_{2n-1,2M+1,L_p}[z, w]) \) is orthogonal to \( \mathbb{R}_{N,M-1}[x, y] \) with respect to \( L_\cdot \), establishing (6.4).

Note that if \( f(z, w) = w^{2M+1} z^{n_0} p(x, 1/w) g(z) \in w^{2M+1} \tilde{p}(z, 1/w) \mathbb{R}_{2n_1-1}[z] \) is an element of the second space in (6.2a), then

\[
S(f(z, w)) = S_{w,M-1}(w^{2M+1} p(x, 1/w)) S_{z,N}(z^{n_0} g(z))
\]

\[
= -S_{w,M}(p(x, w)) S_{z,N-n_0}(g(z)) \in S_{w,M}(p(x, w)) \mathbb{R}_{N-n_0}[x],
\]

(6.5)

where \( S_{w,M}(p(x, w)) = p_M(y; x) \) is the polynomial in Remark 4.7. We show next that the restriction of \( S \) to the first space in (6.2a) builds the orthogonal complement of the orthonormal set \( \{p_M(y; x)U^j_N(x)\}_{j=0}^{N-n_0} \) in \( \tilde{P}_{N,M;L}[x, y] \).

First, note that if \( f(z, w) = \tilde{q}(z) h(z, w) \in \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \) and \( g(x, y) = p_M(y; x)r(x) \) in \( p_M(y; x) \mathbb{R}_{N-n_0}[x] \), then by (6.3) we have

\[
\langle g(x, y), S(f(z, w)) \rangle_L = \left\langle \frac{(z^2-1)z^{N-n_0}r(x) w^{2M+1} z^{n_0} p(x, 1/w)}{\tilde{q}(z)}, h(z, w) \right\rangle_{L_p}.
\]

(6.6a)

The right-hand side of (6.6a) is equal to 0 by (5.3b), and it is easy to see that the right-hand side of (6.6b) is equal to 0 by computing the \( w \)-integral. This shows that the left-hand side of (6.6a) is also equal to 0, and therefore

\[
S : \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \rightarrow \tilde{P}_{N,M;L}[x, y] \oplus (p_M(y; x) \mathbb{R}_{N-n_0}[x]).
\]

Note that both spaces in the last equation have the same dimension \( n_0 \), and we will prove next that \( S \) is an isomorphism. However, in general, \( S \) will not be an isometry if we use the natural inner product \( \langle \cdot, \cdot \rangle_{L_p} \) on \( \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \).

Lemma 6.1. The map \( S : \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \rightarrow \tilde{P}_{N,M;L}[x, y] \) is injective. Moreover, for \( f, g \in \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \) we have

\[
(S(f(z, w)), S(g(z, w)))_{L_p} = \langle f(z, w), g(z, w) \rangle_{L_p} - \langle w^{2M} f(z, 1/w), g(z, w) \rangle_{L_p}.
\]

(6.7)

Proof. Suppose first that \( f(z, w) \in \tilde{q}(z) \tilde{P}_{2n_0-1,2M+1;L_p}[z, w] \) is a nonzero element. We will show the coefficient of \( y^M \) in \( S(f(z, w)) \) is a nonzero polynomial, thus proving that \( S \) is injective. If \( h(z) \) is the stable Fejér–Riesz factor of \( p(x, 0) \), then by (5.3a) we have

\[
f(z, w) = \tilde{q}(z) h(z) \gamma(z) w^{2M+1} \mod \mathbb{R}_{2n-1,2M}[z, w], \quad \text{where } 0 \neq \gamma(z) \in \mathbb{R}_{n_0-1}[z].
\]

Since \( S_{w,M-1}(\mathbb{R}_{2M}[w]) \subset \mathbb{R}_{M-1}[y] \), we see that

\[
S(f(z, w)) = S_{z,N}(\tilde{q}(z) h(z) \gamma(z)) S_{w,M-1}(w^{2M+1}) \mod \mathbb{R}_{N,M-1}[x, y],
\]

i.e. up to a nonzero constant factor, the coefficient of \( y^M \) is \( S_{z,N}(\tilde{q}(z) h(z) \gamma(z)) \).

For simplicity, we set \( \alpha(z) = \tilde{q}(z) h(z) \in \mathbb{R}_{n_0+2n_1}[z] \) which has no zeros in \( \mathbb{D} \). If we assume that \( S_{z,N}(\alpha(z) \gamma(z)) = 0 \), then

\[
\alpha(z) \gamma(z) = z^{2N+2} \alpha(1/z) \gamma(1/z) = (z^{2N+3-n_0} \alpha(1/z))(z^{n_0-1} \gamma(1/z)).
\]

(6.8)
The polynomials $\alpha(z)$ and $(z^{2N+3-n_0} \alpha(1/z))$ are relatively prime in $\mathbb{R}[z]$, because the first one has no zeros in $\mathbb{D}$, while all the zeros of the second one are in $\mathbb{D}$. From (6.8) we see that $\alpha(z)$ must divide $z^{n_0-1} \gamma(1/z)$ and therefore we can rewrite this equation as

$$\gamma(z) = \left(z^{2N+3-n_0} \alpha(1/z)\right) \frac{z^{n_0-1} \gamma(1/z)}{\alpha(z)},$$

where the right-hand side is a polynomial of degree at least $2N+3-n_0 > n_0$ which is impossible. The contradiction shows that the coefficient of $y^M$ in $\mathcal{S}(f(z,w))$ is nonzero and therefore $\mathcal{S}$ is injective.

For the second part, note that if we replace $g(x,y)$ in (6.3) by $\mathcal{S}(g(z,w))$ we obtain

$$\langle \mathcal{S}(g(z,w)), \mathcal{S}(f(z,w)) \rangle_{\mathcal{E}} = \langle g(z,w), f(z,w) \rangle_{\mathcal{E}_\rho} - \langle w^{2M} g(z,1/w), f(z,w) \rangle_{\mathcal{E}_\rho}$$

(6.9a)

$$+ \langle z^{2N+2} w^{2M} g(1/z,1/w), f(z,w) \rangle_{\mathcal{E}_\rho}$$

(6.9b)

$$- \langle z^{2N+2} g(1/z,w), f(z,w) \rangle_{\mathcal{E}_\rho}. \quad (6.9c)$$

If $f,g \in \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$, then the term in (6.9b) is 0 by Lemma 5.2. If we take $f(z,w) = \tilde{g}(z)f_1(z,w)$ and $g(z,w) = \tilde{g}(z)g_1(z,w)$ to be two elements in $\tilde{g}(z)\tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$, then the term in (6.9c) can be rewritten as follows

$$\langle z^{2N+2} g(1/z,w), f(z,w) \rangle_{\mathcal{E}_\rho} = \langle z^{2N+2} g_1(1/z,w), f_1(z,w) \rangle_{\mathcal{E}_\rho}. \quad (6.10)$$

Since $z^{n_0} \tilde{p}(1/z,w) = \tilde{p}(z,w)$, Proposition 5.3 tells us that

$$z^{2n_0-1} g_1(1/z,w) \in \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w].$$

This shows that $z^{2N+2} g_1(1/z,w) \in z^{2(N-n_0)+3} \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$ which combined with (5.3c) proves that the term in (6.10) is 0. Since the terms in (6.9b), (6.9c) are zero, equation (6.9a) reduces to (6.7).

For $f(z,w) \in \mathbb{R}[z,w]$ we denote by $\mathcal{M}_{f(z,w)} : \mathbb{R}[z,w] \to \mathbb{R}[z,w]$ the multiplication by $f(z,w)$, i.e.

$$\mathcal{M}_{f(z,w)}(g(z,w)) = f(z,w)g(z,w).$$

Note that $\mathcal{M}_{\tilde{g}(z)} : \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w] \to \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$ is an isometry. Summarizing the statement so far we obtain the following theorem.

**Theorem 6.2.** Let $\mathcal{E}$ be the functional defined by (1.3) where $q(x) \in \mathbb{R}_{2n_1}[x], p(x,w) \in \mathbb{R}_{n_0,2m}[x,w]$ are such that $q(x) > 0, p(x,w) \neq 0$ when $(x,w) \in [-1,1] \times \mathbb{D}$ and let $\tilde{p}(z,w) = z^{n_0} p(x,w)$. Let $N \geq n_0 + n_1$ and $M \geq m$. For $f,g \in \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$, the bilinear form

$$\langle f,g \rangle := \langle f(z,w), g(z,w) \rangle_{\mathcal{E}_\rho} - \langle w^{2M} f(z,1/w), g(z,w) \rangle_{\mathcal{E}_\rho}, \quad (6.11)$$

defines an inner product on $\tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w]$, and with this inner product the map

$$\mathcal{S}_{N,M-1} \circ \mathcal{M}_{\tilde{g}(z)} : \tilde{\mathcal{P}}_{2n_0-1,2M+1;\mathcal{E}_\rho}[z,w] \to \tilde{\mathcal{P}}_{N,M;\mathcal{E}}[x,y],$$
is an isometry. Moreover, if \( \{U_j^q(x)\}_{j=0}^{N-n_0} \) are orthonormal polynomials with respect to the measure \( \frac{\chi(-1,1)(x)}{q(x)} \chi(-1,1) \) on \( \mathbb{R} \), and if we set
\[
p_M(y; x) = \frac{w^{M+1}p(x,1/w) - w^{-M-1}p(x,w)}{w - 1/w},
\]
then
\[
\tilde{P}_{N,M;L}[x,y] = S_{N,M-1}(\tilde{q}(z)\tilde{P}_{2n_0-1,2M+1;L}[z,w]) \bigoplus_{k=0}^{N-n_0} \text{span}_R \{p_M(y; x)U_k^q(x)\}.
\]
(6.12)

**Remark 6.3.** (i) We can now relax the conditions on \( q(x) \) and show that the decomposition (6.12) holds even if \( q(x) \) has simple zeros at \( x = \pm 1 \). Indeed, suppose that \( q(x) = (1-x)\nu_1(1+x)^{\nu_2}q_0(x) \), where \( \nu_1, \nu_2 \in \{0, 1\} \) and \( q_0(x) \) is nonzero for \( x \in [-1,1] \). Then, for \( \epsilon > 0 \), we can consider (6.12) with \( q(x) \) replaced by \( q(x; \epsilon) = (1+\epsilon-x)^{\nu_1}(1+\epsilon+x)^{\nu_2}q_0(x) \). Note that \( \epsilon \) will not appear in the inner product (6.11), and the only terms depending on \( \epsilon \) on the right-hand side of (6.12) are \( \tilde{q}(z) \) and \( U_k^q(x) \), so taking the limit \( \epsilon \to 0 \) is straightforward.

(ii) From Theorem 6.2, with \( l = 1 \) we know that the recurrence coefficients \( a_{k+1} \) and \( b_k \) of the polynomials \( U_k^q(x) \) will be equal to \( \frac{1}{2} \) and 0, respectively, for \( k \geq n_1 \). Clearly, the orthonormal elements \( p_M(y; x)U_k^q(x) \) of \( \tilde{P}_{N,M;L}[x,y] \) will satisfy the same recurrence relation. Note that the inner product in (6.11) depends only on \( M \), but not on \( N \). From (6.12) it follows that the orthonormal set \( \{p_M(y; x)U_k^q(x)\} \) can be extended to an orthonormal basis of \( \tilde{P}_{N,M;L}[x,y] \) by adding elements which also satisfy a Chebyshev relation. The precise statement is given below.

**Corollary 6.4.** With the notations in Theorem 6.2, let \( \{\tilde{q}_j^M(z,w)\}_{j=1}^{n_0} \) be an orthonormal basis of \( \tilde{P}_{2n_0-1,2M+1;L}[z,w] \) with respect to the inner product in (6.11), and let
\[
r_{j,N}^M(x,y) = S_{N,M-1}(\tilde{q}(z)\tilde{q}_j^M(z,w)).
\]
The set
\[
\{p_M(y; x)U_k^q(x) : k = 0, \ldots, N - n_0\} \cup \{r_{j,N}^M(x,y) : j = 1, \ldots, n_0\}
\]
is an orthonormal basis of \( \tilde{P}_{N,M;L}[x,y] \) and the elements \( r_{j,N}^M(x,y) \) satisfy the Chebyshev relation
\[
r_{j,N}^M(x,y) + r_{j,N+2}^M(x,y) = 2x \tau_{j,N+1}^M(x,y), \quad \text{for } j = 1, \ldots, n_0.
\]
(6.13)

**Remark 6.5.** In general, the last term in (6.11) need not be zero, and therefore, \( S \) is not an isometry, see for instance the example in Section 7.2 and the discussion in Subsection 7.2.1.

6.2. Szegő map for the Bernstein-Szegő weight in Theorem 4.11. In this subsection, we assume that

- \( \omega(z,w) = 1_{n_0,n_0}[z,w] \) is nonzero for \( (z,w) \in \mathbb{D}^2 \),
- \( q_1(x) \in \mathbb{R}_{2n_1}[x] \) is positive for \( x \in [-1,1] \),
- \( q_2(y) \in \mathbb{R}_{2m_1}[y] \) is positive for \( y \in [-1,1] \),

and we set \( n = n_0 + n_1, m = m_0 + m_1 \). The conditions on \( q_1(x) \) and \( q_2(y) \) will be relaxed later, see Remark 6.7(i). Let \( \tilde{q}_1(z) \in \mathbb{R}_{2n_1}[z] \) and \( \tilde{q}_2(w) \in \mathbb{R}_{2m_1}[w] \) be the
stable Fejér-Riesz factors of \( q_1 \) and \( q_2 \), respectively. We can apply the constructions in the previous subsection with \( g(x) \) replaced by \( q_1(x) \) and

\[
\hat{p}(z, w) = \omega(z, w) (\tilde{q}_2(w) z^{n_0} \omega(1/z, w)).
\]  

By Theorem 5.5, we have

\[
\tilde{P}_{2n_0-1,2m;L_\beta}[z, w] = w^{2m_1+m_0} \tilde{q}_2(1/w) \omega(z, 1/w) \tilde{P}_{n_0-1,m_0;L_\beta}[z, w],
\]  

and for \( M \geq m \)

\[
\tilde{P}_{2n_0-1,2M+1;L_\beta}[z, w] = w^{2M+1-m_0} \tilde{q}_2(1/w) \omega(z, 1/w) \tilde{P}_{n_0-1,m_0;L_\beta}[z, w].
\]

Using the last equation we can show that the last term in (6.11) vanishes, and therefore we do not need to modify the inner product on the space \( \tilde{P}_{2n_0-1,2M+1;L_\beta} \).

Indeed, if \( f, g \in \tilde{P}_{2n_0-1,2M+1;L_\beta} \), then from (6.11) we have

\[
f(z, w) = w^{2M+1-m_0} \tilde{q}_2(1/w) \omega(z, 1/w) f_1(z, w)
\]

\[
g(z, w) = w^{2M+1-m_0} \tilde{q}_2(1/w) \omega(z, 1/w) g_1(z, w),
\]

where \( f_1, g_1 \in \tilde{P}_{n_0-1,m_0;L_\beta}[z, w] \subset \mathbb{R}_{n_0-1,m_0}[z, w] \). Substituting these formulas in \( \langle g(z, w), w^{2M} f(z, w) \rangle_{L_\beta} \) and computing just the \( w \)-integral, up to a constant factor, we obtain

\[
f \int \frac{w^{2(M-m)+2} g_1(z, w) (w^{2m_1} \tilde{q}_2(1/w) f_1(1/z, w)) \omega(z, 1/w) \omega(z, w)}{w}.
\]

Since \( M \geq m \), the numerator is a polynomial of \( w \) while the denominator is nonzero for \( w \in \mathbb{R} \), and therefore the integral is 0 by the Cauchy-Goursat theorem.

If we use the involution \( R_{w_0}^{m_0} \) on \( \mathbb{C}[z^{\pm 1}, w^{\pm 1}] \) defined by

\[
R_{w_0}^{m_0}(g(z, w)) = w^{m_0} g(z, 1/w),
\]

then a computation similar to the one in (6.5) shows that

\[
S_{w_0,M-1} \circ M_{w,2M+1-m_0} g(z, 1/w) = -S_{w,M} \circ M_{g(z, w)} \circ R_{w_0}^{m_0}.
\]

Combining the above with formula (6.10) and Theorem 6.2 we obtain the following theorem for the Bernstein-Szegő weight defined in (6.4) and (6.24).

**Theorem 6.6.** Let \( L \) be the functional defined by (6.14) and (6.24) where

- \( \omega(z, w) \in \mathbb{R}_{n_0,m_0}[z, w] \) is nonzero for \( (z, w) \in \mathbb{D}^2 \),
- \( q_1(x) \in \mathbb{R}_{2n_1}[x] \) is positive for \( x \in [-1, 1] \),
- \( q_2(y) \in \mathbb{R}_{2m_1}[y] \) is positive for \( y \in [-1, 1] \),

and let \( \tilde{q}_1(z) \) and \( \tilde{q}_2(w) \) be the stable Fejér-Riesz factors of \( q_1(x) \) and \( q_2(y) \), respectively. For \( N \geq n_0 + n_1 \) and \( M \geq m_0 + m_1 \), define

\[
\tilde{T}_{N,M} = S_{N,M} \circ M_{\tilde{q}_1(z) \tilde{q}_2(w)}(z,w) \circ R_{w_0}^{m_0}
\]

\[
\tilde{T}_{N,M} = S_{N,M} \circ M_{\tilde{q}_1(z) \tilde{q}_2(w)}(z,w) \circ R_{w_0}^{m_0}.
\]

Then

\[
\tilde{T}_{N,M} : \tilde{P}_{n_0-1,m_0;L_\beta}[z, w] \to \tilde{P}_{N,M;L}[x, y]
\]

\[
\tilde{T}_{N,M} : \tilde{P}_{n_0,m_0-1;L_\beta}[z, w] \to \tilde{P}_{N,M;L}[x, y]
\]

are isometries. Moreover, if

- \( \{U_j^N(x)\}_{j=0}^{N-n_0} \) are orthonormal polynomials with respect to \( \frac{2 x (1-x)}{\pi q_1(x) (x^{-1} - 1)} \)
\( \{U_j^{q_2}(y)\}_{j=0}^{M-m_0} \) are orthonormal polynomials with respect to \( \frac{2\sqrt{1-y^2}}{q_2(y)} \chi(-1,1)(y)dy \), and if we set
\[
p(x, w) = \hat{q}_2(w)\omega(z, w)\omega(1/z, w), \quad p_M(y; x) = \frac{w^{M+1}p(x, 1/w) - w^{-M-1}p(x, w)}{w - 1/w},
\]
(6.19a)
\[
\hat{p}(z, y) = \hat{q}_1(z)\omega(z, w)\omega(1, w), \quad \hat{p}_N(x; y) = \frac{z^{N+1}\hat{p}(1/z, y) - z^{-N-1}\hat{p}(z, y)}{z - 1/z},
\]
(6.19b)
then
\[
\hat{p}_{N;M,L}[x, y] = \hat{T}_{N,M}(\hat{p}_{n_0-1,m_0;L_\omega}[z, w]) \bigoplus_{k=0}^{N-m_0} \text{span}_R \{ p_M(y; x)U_k^{q_1}(x) \},
\]
(6.20a)
\[
\hat{P}_{N;M,L}[x, y] = \hat{T}_{N,M}(P_{n_0,m_0-1;L_\omega}[z, w]) \bigoplus_{k=0}^{M-m_0} \text{span}_R \{ \hat{p}_N(x; y)U_k^{q_2}(y) \}.
\]
(6.20b)

**Remark 6.7.** (i) Similarly to Remark 6.3(i), we can relax the conditions on \( q_1(x) \) and \( q_2(y) \) and show that the equations (6.18)-(6.20) hold even when \( q_1(x) \) and \( q_2(y) \) have simple zeros at \( x = \pm 1 \) and \( y = \pm 1 \).

(ii) Note that the spaces \( P_{n_0-1,m_0;L_\omega}[z, w] \) and \( P_{n_0,m_0-1;L_\omega}[z, w] \) in (6.18) and (6.20) are independent now of \( N \) and \( M \). Therefore, we can use the decompositions in equations (6.20) to construct orthonormal bases of \( \hat{P}_{N;M,L}[x, y] \) and \( P_{N;M,L}[x, y] \) so that the multiplications by \( x \) and \( y \) are represented by Chebyshev relations. In particular, this provides a new proof of equations (4.30) in Theorem 4.11 in the case when \( \omega(z, w) \neq 0 \) for \((z, w) \in \mathbb{R}^2\).

### 7. Explicit Examples

In this section we include examples which illustrate different aspects of the main results. In both examples, we start with a positive functional \( \mathcal{L} : \mathbb{R}_{2n,2m} \to \mathbb{R} \) by displaying its moment matrix \( \mathcal{M} = [h_{k,l}] \) where \( h_{k,l} = \mathcal{L}(x^k y^l) \) for \( 0 \leq k \leq 2n \), \( 0 \leq l \leq 2m \).

#### 7.1. Example 1: Bernstein-Szegő weight

Let \( \mathcal{L} : \mathbb{R}_{4,2} \to \mathbb{R} \) be the functional with moment matrix
\[
\mathcal{M} = \begin{bmatrix}
1 & c_1 c_2 & c_1 c_2 \frac{c_2^2}{4} & c_1 c_2 \frac{c_2^4}{16} & c_1 c_2 \frac{c_2^6}{64}
\frac{c_2}{c_2+1} & \frac{c_2}{c_2+2} & \frac{c_2}{c_2+3} & \frac{c_2}{c_2+5} & \frac{c_2}{c_2+7}
\frac{c_2^2}{c_2+2} & \frac{c_2^2}{c_2+3} & \frac{c_2^2}{c_2+5} & \frac{c_2^2}{c_2+7} & \frac{c_2^2}{c_2+9}
\frac{c_2^3}{c_2+3} & \frac{c_2^3}{c_2+5} & \frac{c_2^3}{c_2+7} & \frac{c_2^3}{c_2+9} & \frac{c_2^3}{c_2+11}
\frac{c_2^4}{c_2+4} & \frac{c_2^4}{c_2+5} & \frac{c_2^4}{c_2+7} & \frac{c_2^4}{c_2+9} & \frac{c_2^4}{c_2+11}
\frac{c_2^5}{c_2+5} & \frac{c_2^5}{c_2+7} & \frac{c_2^5}{c_2+9} & \frac{c_2^5}{c_2+11} & \frac{c_2^5}{c_2+13}
\frac{c_2^6}{c_2+6} & \frac{c_2^6}{c_2+8} & \frac{c_2^6}{c_2+10} & \frac{c_2^6}{c_2+12} & \frac{c_2^6}{c_2+14}
\end{bmatrix},
\]
(7.1)
where \( c_1, c_2 \) are free real parameters. This example was considered in [11], Section 4, using the parameters \( s_{i,j} \) in [4] related to \( c_1 \) and \( c_2 \) above as follows: \( c_1 = s_{1,1}, c_2 = 2s_{1,0} \). In particular, the results in these papers imply that if
\[
c_1 \in (-1, 1) \quad \text{and} \quad c_2 \in \mathbb{R},
\]
then the functional \( \mathcal{L} \) extends to a positive linear functional on the space \( \mathbb{R}[x, y] \) of all polynomials and equations (4.30) hold with \( n = 2 \) and \( m = 1 \). By abuse of
Theorem 4.1(II) holds. For (b), we compute \( \hat{\Psi} \)

\[
P_{0,1}(x, y) = \begin{bmatrix} 1 \\ 2y - c_1 c_2 \end{bmatrix},
\]

(7.3a)

\[
P_{1,1}(x, y) = \begin{bmatrix} \frac{2x - 2c_1 y + c_2^2 c_1 - c_2}{\sqrt{1 - c_1^2}} \\ 4xy - 2c_2 y - c_1 \end{bmatrix},
\]

(7.3b)

\[
P_{2,1}(x, y) = \begin{bmatrix} \frac{4x^2 - 4c_1 xy - 2c_2 x + 2c_1 c_2 y + c_2 - 1}{\sqrt{1 - c_1^2}} \\ 8x^2 y - 4c_2 xy - 2c_1 x - 2y + c_1 c_2 \end{bmatrix},
\]

(7.3c)

and therefore

\[
A_{2,1} = \frac{1}{2} t_2.
\]

(7.3d)

Substituting the above formulas into the right-hand side of (1.8), we see that (1.8) holds when \( n = 2, m = 1 \) with

\[
q(x) = 1 + c_2^2 - 2c_2 x = (1 - c_2 z)(1 - c_2 / z)
\]

(7.4a)

and

\[
p_1(y; x) = \frac{2}{\sqrt{1 - c_1^2}} (y - c_1 x), \quad p_0(y; x) = \sqrt{1 - c_1^2}.
\]

(7.4b)

Equivalently, this means that (1.4) holds if we set \( p(x, w) = w (p_1(y; x) - wp_0(y; x)) \), or explicitly

\[
p(x, w) = \frac{1}{\sqrt{1 - c_1^2}} (c_2 w^2 - 2c_1 x w + 1) = \frac{1}{\sqrt{1 - c_1^2}} (c_1 w - z)(c_1 w - 1/z).
\]

(7.5)

Equation (7.2) shows that \( p(x, w) \neq 0 \) for \( (x, w) \in (-1, 1)^2 \), i.e. condition (a) in Theorem 4.1(II) holds. For (b), we compute \( \hat{\Psi} \) and find

\[
\hat{\Psi}(z) = (1 - c_2 z) \begin{bmatrix} \frac{z^2 + 1}{\sqrt{1 - c_1^2}} - \frac{2z + c_1}{\sqrt{1 - c_1^2}} \\ -c_1 z \end{bmatrix}, \text{ hence } \det(\hat{\Psi}(z)) = \frac{2(1 - c_2 z)^2}{\sqrt{1 - c_1^2}}.
\]

(7.6)

Therefore, \( \hat{\Psi}(z) \) will be invertible for \( z \in (-1, 1) \) if and only if \( c_2 \in [-1, 1] \). We continue by dividing this subsection into 2 parts. In the first part, we consider the case when \( c_2 \in [-1, 1] \) and we illustrate Theorems 4.1 and 6.6. In the second part, we describe the measure for the extension of \( \mathcal{L} \) when \( c_2 \notin [-1, 1] \), which completes, extends and corrects the results in [11] Theorem 4.1] where one range of the parameters was overlooked.

7.1.1. The case \( c_2 \in [-1, 1] \). Throughout Subsection 7.1.1 we assume that

\[
(c_1, c_2) \in (-1, 1) \times [-1, 1].
\]

Since both conditions in Theorem 4.1(II) are satisfied, it follows that (1.3) holds where \( q(x) \) and \( p(x, w) \) are given in equations (7.4a) and (7.5), respectively. Thus \( \mathcal{L} \) can be extended to a positive functional on \( \mathbb{R}[x, y] \), which we will denote also by \( \mathcal{L} \). If we set

\[
\omega(z, w) = \frac{1 - c_1 zw}{\sqrt{1 - c_1^2}},
\]

(7.7)
then we can apply Theorem 4.11 with \( q_1(x) = q(x) \) given in (7.4a) and \( q_2(y) = 1 \) to conclude that equations (4.30) hold with \( n = 2 \) and \( m = 1 \). Moreover, we can use Theorem 6.6 to construct simple explicit bases of \( \mathcal{P}_{N,M;L}[x,y] \) and \( \tilde{\mathcal{P}}_{N,M;L}[x,y] \) for all \( N \geq 2, M \geq 1 \), using \( p(x, w) \) in (7.5) and

\[
\hat{p}(z, y) = \frac{(1 - c_2z)(1 - 2c_1zy + c_1^2z^2)}{\sqrt{1 - c_1^2}}.
\] (7.8)

Since \( p(x, 0) = \frac{1}{\sqrt{1 - c_1^2}} \) and \( \hat{p}(0, y) = \frac{1}{\sqrt{1 - c_1^2}} \) are positive constants, we can obtain orthonormal bases of \( \mathcal{P}_{N,M;L}[x,y] \) and \( \tilde{\mathcal{P}}_{N,M;L}[x,y] \) corresponding to the standard bases of \( \mathbb{R}_M[y] \) and \( \mathbb{R}_N[x] \), see Remark 4.7(ii). We illustrate this for the space \( \mathcal{P}_{N,M;L}[x,y] \) which also shows how to obtain (7.3c) from Theorem 6.6. From (7.8) and (6.10b) it follows that

\[
\hat{p}_N(x; y) = \frac{1}{\sqrt{1 - c_1^2}}(U_N(x) - (c_2 + 2c_1y)U_{N-1}(x) + c_1(c_1 + 2c_2y)U_{N-2}(x) - c_1^2c_2U_{N-3}(x)),
\] (7.9)

where \( U_k(x) = \frac{\sin((k+1)x)}{\sin(x)} \) are the Chebyshev polynomials of the second kind, with the convention that \( U_{-1}(x) = 0 \) and \( U_k(x) = -U_{-k-2}(x) \) if \( k < -1 \). Note next that

\[
\mathcal{L}_\omega(z^k w^l) = \frac{1 - c_1^2}{(2\pi)^2} \iint_{T^2} z^k w^l \frac{|dz| |dw|}{(1 - c_1zw)(1 - c_1/(zw))},
\]

and the space \( \tilde{\mathcal{P}}_{1,0;L,\omega}[z,w] \) is spanned by the unit element \( \phi(z) = \sqrt{1 - c_1^2}z \). Since \( \tilde{q}(z) = 1 - c_2z \) is the stable Fejér-Riesz factor of \( q(x) \), we have

\[
\tilde{T}_{N,M}(\phi(z)) = \mathcal{S}_{N,M}((1 - c_2z)(1 - c_1zw))
\]

\[
= U_N(x)U_M(y) - c_2U_{N-1}(x)U_M(y) - c_1U_{N-1}(x)U_{M-1}(y) + c_1c_2U_{N-2}(x)U_{M-1}(y).
\]

From the above computations and equation (6.20b) it follows that for \( N \geq 2 \) and \( M \geq 1 \) the entries of the vector polynomials \( \tilde{p}_{N,M}(x, y) \) with respect to the standard basis \( \mathcal{B}_M = (1, y, \ldots, y^M) \) of \( \mathbb{R}_M[y] \) are given by

\[
p^k_{N,M}(x, y) = \frac{(U_N(x) - (c_2 + 2c_1y)U_{N-1}(x) + c_1(c_1 + 2c_2y)U_{N-2}(x) - c_1^2c_2U_{N-3}(x))U_k(y)}{\sqrt{1 - c_1^2}},
\] (7.10a)

for \( k = 0, 1, \ldots, M - 1 \), and

\[
p^M_{N,M}(x, y) = U_N(x)U_M(y) - c_2U_{N-1}(x)U_M(y) - c_1U_{N-1}(x)U_{M-1}(y) + c_1c_2U_{N-2}(x)U_{M-1}(y).
\] (7.10b)

It is easy to see that equations (7.10) with \( N = 2 \) and \( M = 1 \) lead to (7.3c). Note that the formulas in equations (7.10) work also when \( N = M = 1 \) and give (7.3d), which explains (7.3c) (cf. with the last part of Theorem 3.4). Similar formulas for the orthonormal basis \( \{\tilde{p}_{N,M}(x, y)\} \) of \( \tilde{\mathcal{P}}_{N,M;L}[x,y] \) stem from (6.20a).

7.1.2. The case \( c_2 \notin [-1,1] \). In this subsection we analyze the possible cases when \( c_1 \in (-1,1) \) while \( c_2 \notin [-1,1] \). We set

\[
x_0 = \frac{1}{2} \left( c_2 + \frac{1}{c_2} \right), \quad y_0 = \frac{1}{2} \left( c_1c_2 + \frac{1}{c_1c_2} \right), \text{ when } c_1 \neq 0,
\]

and we show that
(i) if \(|c_1c_2| \leq 1 < |c_2|\), then
\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} \frac{f(x, y) \sqrt{1 - x^2} \sqrt{1 - y^2}}{q(x)p(x, w)p(x, 1/w)} dx dy
+ \frac{2(c_2^2 - 1)}{\pi c_2^2} \int_{(-1,1)} \frac{f(x_0, y)}{p(x_0, w)p(x_0, 1/w)} \sqrt{1 - y^2} dy,
\tag{7.11}
\]
(ii) if \(|c_1| < 1 < |c_2|\), then
\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} \frac{f(x, y) \sqrt{1 - x^2} \sqrt{1 - y^2}}{q(x)p(x, w)p(x, 1/w)} dx dy
+ \frac{2(c_2^2 - 1)}{\pi c_2^2} \int_{(-1,1)} \frac{f(x_0, y)}{p(x_0, w)p(x_0, 1/w)} \sqrt{1 - y^2} dy + \frac{c_1^2 c_2^2 - 1}{c_1^2 c_2^2} f(x_0, y_0).
\tag{7.12}
\]
For \(c_1, c_2\) satisfying (7.12) we define the functional \(\mathcal{L}_0 : \mathbb{R}[x, y] \to \mathbb{R}\) by
\[
\mathcal{L}_0(f) = \frac{4}{\pi^2} \int_{(-1,1)^2} \frac{f(x, y) \sqrt{1 - x^2} \sqrt{1 - y^2}}{q(x)p(x, w)p(x, 1/w)} dx dy,
\]
which coincides with \(\mathcal{L}\) when \((c_1, c_2) \in (-1, 1) \times [-1, 1]\) by the results in Subsection 7.1.1
Write \(z = e^{i\theta}, w = e^{i\phi}\), or equivalently \(x = \cos \theta, y = \cos \phi\), and use the invariance of the integrand of \(\mathcal{L}_0\) under the transformations \(z \rightarrow 1/z\) and \(w \rightarrow 1/w\) to obtain
\[
\mathcal{L}_0(f) = \frac{1}{\pi^2} \int_{(-\pi,\pi)^2} \frac{f(\cos \theta, \cos \phi) \sin^2 \theta \sin^2 \phi}{q(x)p(x, w)p(x, 1/w)} d\theta d\phi.
\]
If \(c_1 = 0\), equation (7.11) follows easily from the one-dimensional theory, so we will assume below that \(c_1 \neq 0\). We begin with the region \(0 < |c_1| < |c_2| < 1\) and evaluate the above integral by residues. Thus we write
\[
\mathcal{L}_0(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathcal{I}(f, w) \sin^2 \phi d\phi
\tag{7.13}
\]
where
\[
\mathcal{I}(f, w) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(\cos \theta, \cos \phi) \sin^2 \theta}{q(x)p(x, w)p(x, 1/w)} d\theta = \frac{1 - c_1^2}{4\pi i} \oint_{\Gamma} \frac{f(x, y)w^2(1 - z^2)^2}{\tau(z, w)} dz,
\tag{7.14}
\]
and
\[
\tau(z, w) = (c_1zw - 1)(c_1 - w)(zw - c_1)(c_1w - z)(c_2z - 1)(z - c_2).
\]
The residues of the above contour integral are at \(z = 0, z = c_1w, z = c_1/w, \) and \(z = c_2\). Hence
\[
\mathcal{I}(f, w) = R_0(w) + R_1(w) + R_2(w) + R_3(w)
\]
where
\[
R_0(w) = -\frac{1 - c_1^2}{2} \text{res}_{z=0} \frac{f(x, y)w^2(1 - z^2)^2}{\tau(z, w)},
\tag{7.15a}
R_1(w) = -\frac{1}{2c_1} \frac{f\left(\frac{1}{2}(c_1w + c_1w), \frac{1}{c_2} (c_2z - 1)\right)}{w^2 c_1^2 - 1},
R_2(w) = R_1(1/w),
\tag{7.15b}
\]
and
and
\[ R_3(w) = -\frac{(1 - c_1^2)(1 - c_2^2)f(x_0, y)w^2}{2(c_1c_2w - 1)(w - c_1c_2)(c_2^2w - c_1)(c_1w - c_2)}. \] (7.15d)
Thus \( \mathcal{L}(f) = \mathcal{L}_0(f) \) when \(|c_1| < |c_2| < 1 \) and
\[ \mathcal{L}(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} (R_0(w) + 2R_1(w) + R_3(w)) \sin^2 \phi \, d\phi \]
\[ = -\frac{1}{4\pi i} \oint (R_0(w) + 2R_1(w) + R_3(w)) \frac{(1 - w^2)^2}{w^3} \, dw. \] (7.16)
Applying the Cauchy’s residue theorem we obtain
\[ \mathcal{L}(f) = -\frac{1}{2} \text{Res}_{w=0} \left( R_0(w) \frac{(1 - w^2)^2}{w^3} \right) - \text{Res}_{w=0} \left( R_1(w) \frac{(1 - w^2)^2}{w^3} \right) \]
\[ - \frac{1}{2} \sum_{j=0}^{1} \text{Res}_{w=w_j} \left( R_3(w) \frac{(1 - w^2)^2}{w^3} \right), \] (7.17)
where \( w_0 = c_1c_2, \ w_1 = c_1/c_2. \) Equation (7.17) provides an explicit formal extension of the functional \( \mathcal{L} \) for all \( c_1, c_2 \) satisfying (7.2). In the remaining part, we explain how this formula coincides with the integral formulas given in (7.11)-(7.12), depending on the values of the parameters.

We consider now the region \( |c_1c_2| \leq |c_2| \) in (i). In this case the residue at \( z = c_2 \) in the evaluation of (7.14) is replaced by the residue at \( z = 1/c_2 \) which gives
\[ \hat{R}_3(w) = \frac{(1 - c_1^2)(1 - c_2^2)f(x_0, y)w^2}{2(c_1c_2w - 1)(w - c_1c_2)(c_2^2w - c_1)(c_1w - c_2)} = -R_3(w). \] (7.18)

Since,
\[ -\text{Res}_{w=0} \left( R_3(w) \frac{(1 - w^2)^2}{w^3} \right) - \sum_{j=0}^{1} \text{Res}_{w=w_j} \left( R_3(w) \frac{(1 - w^2)^2}{w^3} \right) = \frac{c_2^2 - 1}{c_2^2} f(x_0, y), \]
equations (7.18), (7.14), (7.17) and (7.18) yield
\[ \mathcal{L}(f) - \mathcal{L}_0(f) = \frac{2(c_2^2 - 1)}{\pi c_2} \int_{(-1, 1)} \frac{f(x_0, y)}{p(x_0, w)p(x_0, 1/w)} \sqrt{1 - y^2} \, dy, \]
proving (7.11).

Finally, we consider the case when \( 1 < |c_1c_2| \). We denote by \( \mathcal{L}_1 \) the functional in (7.11), which coincides with \( \mathcal{L} \) and \( \mathcal{L}_1 \) when \( |c_1c_2| \leq 1 < |c_2| \). Note that if we apply the residue theorem for \( \mathcal{L}_1 \) in (7.16) when \( 1 < |c_1c_2| \), we will have to compute also the residues of \( R_1(w) \) at \( w = 1/w_0 \) and replace the residue of \( R_3(w) \) with the residue at \( w = 1/w_0 \). Since
\[ \text{Res}_{w=1/w_0} \left( R_1(w) \frac{(1 - w^2)^2}{w^3} \right) = \frac{c_1^2 - 1}{2c_1^2 c_2} f(x_0, y_0) \]
\[ \text{Res}_{w=1/w_0} \left( R_3(w) \frac{(1 - w^2)^2}{w^3} \right) = -\text{Res}_{w=1/w_0} \left( R_3(w) \frac{(1 - w^2)^2}{w^3} \right) = \frac{c_2^2 - 1}{2c_1^2 c_2} f(x_0, y_0), \]
we see that

\[
\mathcal{L}(f) - \mathcal{L}_1(f) = -\frac{1}{2} \text{Res}_{w=w_0} \left( R_3(w) \frac{(1-w^2)^2}{w^3} \right) + \frac{1}{2} \text{Res}_{w=1/w_0} \left( R_4(w) \frac{(1-w^2)^2}{w^3} \right) \\
+ \text{Res}_{w=1/w_0} \left( R_1(w) \frac{(1-w^2)^2}{w^3} \right) = \frac{c_1^2 c_2^2 - 1}{c_1 c_2} f(x_0, y_0),
\]

completing the proof of (7.12).

7.2. Example 2: one-sided factorization. Let \( \mathcal{L} : \mathbb{R}_{2,2} \to \mathbb{R} \) be the functional with moment matrix

\[
\mathfrak{M} = \begin{pmatrix}
1 & 13c_0^2 & 13c_0^2 + 731c_1c_0 & 38509c_0^2 + 3996576c_1^2 + \frac{2}{3} \\
\frac{6}{113} & \frac{173c_0}{2688} & \frac{731c_0}{48384} & \frac{13c_0^2}{3096576} \\
\frac{1}{25909} & \frac{70429c_0}{1204224} & \frac{1548288}{43713900827c_0^2} & \frac{2}{21} \\
\frac{1}{9078} & \frac{5394252}{35994252} & \frac{3996576}{4530466326528} & \frac{68}{71}
\end{pmatrix},
\]

where \( c_0, c_1 \) are real free parameters. To simplify the notation, we set

\[
d_0 = \sqrt{(28c_0 + 53c_1)^2 + 112896} \quad \text{and} \quad d_1 = \sqrt{(28c_0 + 53c_1)^2 + 1382976}.
\]

Using the standard basis \( B_1 = (1, y) \) of \( \mathbb{R}_1[y] \) we compute the vector orthonormal polynomials and we obtain

\[
P_{0,1}(x, y) = \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{8d_0}(2688y + 448c_0 + 173c_1) \end{pmatrix}, \quad (7.20a)
\]

\[
P_{1,1}(x, y) = \begin{pmatrix} 112c_0^2 x^2 + 30240(28c_0 + 53c_1)y + (28c_0 + 53c_1)(4249c_0 + 449c_1) - 3189412 \\ 896d_1^2 x^4 + 45(9408(448c_0 + 173c_1) - 5c_1d_1^2 x^2 - 32(49d_0^2 + 4d_1^2)y - 45(5c_0 d_1^2 + 2688(14c_0 - 311c_1)) \end{pmatrix}^{1/2},
\]

and therefore

\[
A_{1,1} = \begin{pmatrix} \frac{\sqrt{3}\sqrt{d_0}}{28d_0} & 0 \\ \frac{3\sqrt{3}(28c_0 + 53c_1)}{14d_1} & \frac{252\sqrt{3}}{d_1} \end{pmatrix}.
\]

In particular, these equations show that \( \mathcal{L} \) is a positive linear functional. Substituting the above formulas into the right-hand side of (4.8), we see that (4.8) holds when \( n = m = 1 \) with \( q(x) = 1 \) and

\[
p_1(y; x) = \frac{32(53 - 28x)y + 225(c_1x + c_0)}{224\sqrt{15}}, \quad p_0(y; x) = \frac{1073 - 448x}{224\sqrt{15}}.
\]

Equivalently, this means that (4.8) holds if we take \( p(x, w) = w(p_1(y; x) - wp_0(y; x)) \), or explicitly

\[
p(x, w) = \frac{16(53 - 28x) + 225(c_1x + c_0)w - 225w^2}{224\sqrt{15}}.
\]

Note that for \( x \in (-1, 1) \), the product of the roots \( w_1, w_2 \) of the equation \( p(x, w) = 0 \) is

\[
w_1 w_2 = -\frac{16(53 - 28x)}{225} < -\frac{16}{9}.
\]
This means that for \( x \in (-1, 1) \), the roots are real, have opposite signs and \( |w_1 w_2| > 1 \), i.e. at most one of them is in \( \mathbb{D} \). Since

\[
p(x, w) = -\frac{15\sqrt{15}}{224} (w - w_1)(w - w_2), \tag{7.24}
\]

it follows that condition (a) in Theorem 4.1 (II) holds if and only if \( p(x, 1) \geq 0 \) and \( p(x, -1) \geq 0 \) for \( x \in (-1, 1) \). These conditions can be rewritten as

\[
|c_0 + c_1| \leq \frac{7(89 - 64x)}{225} \quad \text{for all} \quad x \in (-1, 1).
\tag{7.25}
\]

It is easy to see that the last condition holds if and only if the inequality is true at \( x = \pm 1 \). This shows that \( p(x, w) \neq 0 \) for \( (x, w) \in (-1, 1)^2 \) if and only if the parameters \( c_0 \) and \( c_1 \) satisfy the following conditions:

\[
|c_0 + c_1| \leq \frac{7}{9} \quad \text{and} \quad |c_0 - c_1| \leq \frac{119}{25}.
\tag{7.26}
\]

For (b), we compute \( \hat{\Psi}_1^4(z) \) and we find

\[
\det(\hat{\Psi}_1^4(z)) = \frac{1}{105} \left( \frac{7}{2} - z \right)^3 \left( \frac{32}{7} - z \right).
\tag{7.27}
\]

Therefore, \( \hat{\Psi}_1^4(z) \) is invertible for all \( z \in (-1, 1) \). Summarizing, we see that the conditions in Theorem 4.1 (II) hold with \( q(x) = 1 \) and \( p(x, w) \) given in equation (7.22) if and only if the parameters \( c_0, c_1 \) satisfy (7.25). In particular, this means that if equation (7.25) holds, then \( L \) extends to a positive linear functional on \( \mathbb{R}[x, y] \) and the recurrence coefficients satisfy (3.5) with \( n = m = 1 \). Note that if we have equality in one or both inequalities in (7.25), then \( p(x, w) \) will vanish at some of the points \( (\pm 1, \pm 1) \) on the boundary of \( (-1, 1) \times \mathbb{D} \) and we obtain an example similar to the ones discussed in Remark 4.3.

If \( M \geq 1 \), we can use (4.19) to construct \( N \) orthonormal elements in \( \tilde{P}_{N,M;L}[x, y] \) for every \( N \geq 1 \). For instance, when \( M = 1 \) we can take the polynomial \( p_1(y; x) \) in (7.21) and the polynomials \( \{p_1(y; x) U_j(x)\}_{j=0}^{N-1} \) will form an orthonormal set in \( \tilde{P}_{N,1;L}[x, y] \), which can be completed to a basis of \( \tilde{P}_{N,1;L}[x, y] \) by adding just one element. If \( p(x, w) \neq 0 \) for \( (x, w) \neq [-1, 1] \times \mathbb{D} \) we can construct this element using Theorem 6.2. We illustrate this below in the case \( M = 1 \). A direct computation shows that when \( N = 1 \) the polynomial

\[
\tilde{p}_{1,1}^0(x, y) = \frac{\sqrt{7}(18816xy + 7(448c_0 + 173c_1)x - 2688y + 2(311c_1 - 14c_0))}{28d_0}
\tag{7.28}
\]

is orthogonal to \( \tilde{p}_{1,1}^0(x, y) = p_1(y; x) \) and has norm 1. Therefore, \( (\tilde{p}_{1,1}^0(x, y), \tilde{p}_{1,1}^0(x, y)) \) is an orthonormal basis of \( \tilde{P}_{1,1;L}[x, y] \) with vector polynomial

\[
\tilde{P}_{1,1}(x, y) = \begin{bmatrix} \tilde{p}_{1,1}^0(x, y) \\ \tilde{p}_{1,1}^0(x, y) \end{bmatrix}
\]

corresponding to the basis \( \tilde{R}_{1} = (53 - 28x, x) \) of \( \mathbb{R}_1[x] \) by Remark 4.7 (ii). Note that \( p(x, 0) \) is not a constant, and thus the polynomials are not associated with the standard basis \( (1, x) \) of \( \mathbb{R}_1[x] \). If we define the orthogonal matrix

\[
\tilde{R}_{1,1} = \frac{1}{d_1} \begin{bmatrix} 504\sqrt{5} & -d_0 \\ d_0 & 504\sqrt{5} \end{bmatrix},
\]
then \( \tilde{P}_1(x, y) \) yields the orthonormal basis of \( \tilde{P}_{1,1}[x, y] \) with respect to the standard basis \((1, x)\) of \( \mathbb{R}_1[x] \).

### 7.2.1. Illustration of Theorem 6.2

In this subsection we illustrate Theorem 6.2 when \( M = 1 \). For simplicity, we consider first the case

\[
c_1 = c_2 = 0
\]

which will be sufficient to explain some of the subtle points very explicitly, and in particular, the need to modify the inner product on \( \tilde{P}_{1,3;L_{p_c}}[z, w] \). Then we apply the constructions for arbitrary parameters \( c_1, c_2 \) satisfying the strict inequalities in (7.25), but we omit some of the details since the explicit formulas for the polynomials on the torus \( T^2 \) become very involved.

**Special case: \( c_1 = c_2 = 0 \).** Note that for \( c_1 = c_2 = 0 \) the polynomial \( p(x, w) \) in (7.22) reduces to

\[
p(x, w) = \frac{16(53 - 28x) - 225w^2}{224\sqrt{15}},
\]

and therefore

\[
p_c(z, w) = zp(x, w) = -\frac{16(2z - 7)(7z - 2) + 225zw^2}{224\sqrt{15}}.
\]

With the above polynomial, we define the positive linear functional

\[
\mathcal{L}_{p_c}(f) = \frac{1}{(2\pi)^2} \int_{T^2} f(z, w) |dz| |dw| \left| \frac{p_c(z, w)}{|p_c(z, w)|^2} \right|,
\]

on \( \mathbb{R}[z^{\pm 1}, w^{\pm 1}] \). Note that here \( \tilde{q}(z) = 1 \) and therefore \( \tilde{p}(z, w) = p_c(z, w) \) in the notations of Theorem 6.2. A straightforward computation shows that the space \( \tilde{P}_{1,3;L_{p_c}}[z, w] \) is spanned by the element

\[
\tilde{\phi}(z, w) = \frac{\sqrt{3}w(64w^2(2z - 7) + (7\sqrt{17} - 33)z + 32(9 - \sqrt{17}))}{896}.
\]

The polynomial \( \tilde{\phi}(z, w) \) is normalized so that

\[
\langle \tilde{\phi}(z, w), \tilde{\phi}(z, w) \rangle_{p_c} = \frac{3}{56}(23 - \sqrt{17}),
\]

\[
\langle w^2\tilde{\phi}(z, 1/w), \tilde{\phi}(z, w) \rangle_{p_c} = \frac{1}{56}(13 - 3\sqrt{17}).
\]

This means that for the inner product defined in (6.11) we have

\[
\langle \tilde{\phi}(z, w), \tilde{\phi}(z, w) \rangle = \langle \tilde{\phi}(z, w), \tilde{\phi}(z, w) \rangle_{p_c} - \langle w^2\tilde{\phi}(z, 1/w), \tilde{\phi}(z, w) \rangle_{p_c} = 1.
\]

Note, in particular that the term in (7.30b) is nonzero, and therefore this term cannot be omitted. By Theorem 6.2 for every \( N \geq 1 \), the element

\[
\tilde{p}_{N,1}(x, y) = S_{N,0}(\tilde{\phi}(z, w)) = \frac{\sqrt{3}}{2} S_{N,0} \left( w^3 \left( -1 + \frac{2}{7}z \right) \right) = \sqrt{3} \left( U_N(x) - \frac{2}{7}U_{N-1}(x) \right) y,
\]

completes the set \( \{p_1(y; x), p_1(y; x)U_1(y), \ldots, p_1(y; x)U_{N-1}(y)\} \) to an orthonormal basis of \( P_{1,1}[x, y] \). It is straightforward to check that for \( N = 1 \) the above formula coincides with \( \tilde{p}_{1,1}(x, y) \) in (7.27) when \( c_1 = c_2 = 0 \).
General case. Suppose now that $c_0$ and $c_1$ are real constants, such that

$$|c_0 + c_1| < \frac{7}{9} \quad \text{and} \quad |c_0 - c_1| < \frac{119}{25}. \quad (7.31)$$

We set $\tilde{p}(z, w) = p_c(z, w) = zp(x, w)$ where $p(x, w)$ is the polynomial in (7.22) and we consider the basis

$$\mathcal{L}_{p_c}(f) = \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} f(z, w) \frac{|dz| |dw|}{|p_c(z, w)|^2},$$
on $\mathbb{R}[z^{\pm 1}, w^{\pm 1}]$. A straightforward computation shows that the space $\tilde{P}_{1, \mathcal{L}_{p_c}}[z, w]$ endowed with the inner product in (6.11) is spanned by a unit element of the form

$$\tilde{\phi}(z, w) = (\tilde{\phi}_{0,0} + \tilde{\phi}_{1,0}z)(1 + w^2) + (\tilde{\phi}_{0,1} + \tilde{\phi}_{1,1}z)w + \tilde{\phi}_0(z, w)$$
on

where

$$\tilde{\phi}_0(z, w) = \frac{\sqrt{3}}{56d_0} (4(14c_0 - 311c_1)z - 7(448c_0 + 173c_1)) w^2 + \frac{24\sqrt{3}(2z - 7)w^3}{d_0},$$

with $\tilde{\phi}_{0,0}, \tilde{\phi}_{1,0}, \tilde{\phi}_{0,1}, \tilde{\phi}_{1,1} \in \mathbb{R}$. Since $S_{w,0}(w) = S_{w,0}(1 + w^2) = 0$, we have

$$\tilde{p}_{N,1}^N(x, y) = \mathcal{F}_{N,0}(\tilde{\phi}(z, w)) = \mathcal{F}_{N,0}(\tilde{\phi}_0(z, w))$$

$$= \frac{\sqrt{3}}{56d_0} (7(448c_0 + 173c_1)U_N(x) - 4(14c_0 - 311c_1)U_{N-1}(x) + 2688(7U_N(x) - 2U_{N-1}(x))y).$$

(7.33)

It is easy to see that when $N = 1$ the above formula agrees with (7.27). Recall that $p_1(y; x)$ is given in (7.21). With these formulas, Theorem 6.2 tells us that if $N \geq 1$ and we consider the basis

$$\tilde{B}_N = (53 - 28x, (53 - 28x)x, (53 - 28x)x^2, \ldots, (53 - 28x)x^{N-1}, x^N) \text{ of } \mathbb{R}[x],$$

then

$$\tilde{P}_{N,1}(x, y) = [p_1(y; x), p_1(y; x)U_1(x), p_1(y; x)U_2(x), \ldots, p_1(y; x)U_{N-1}(x), \tilde{p}_{N,1}^N(x, y)]^t.$$

7.2.2. Extension. As in the previous example, there are extensions to regions where $p(x, w)$ can vanish when $(x, w) \in (-1, 1)^2$. We will consider one such extension,

$$c_0 + c_1 \geq \frac{7}{9} \quad \text{and} \quad c_0 - c_1 \geq \frac{119}{25}, \quad (7.34)$$

and we will assume that at least one of these inequalities is strict, so that (7.22) does not hold. For fixed $x \in \mathbb{R}$, we denote by $w_1 = w_1(x)$ and $w_2 = w_2(x)$ the roots of $p(x, w) = 0$

$$w_1 = \frac{1}{2} \left( c_0 + c_1x - \sqrt{\frac{64}{225}(53 - 28x) + (c_0 + c_1x)^2} \right),$$

$$w_2 = \frac{1}{2} \left( c_0 + c_1x + \sqrt{\frac{64}{225}(53 - 28x) + (c_0 + c_1x)^2} \right),$$

and we set

$$y_1(x) = \frac{1}{2} \left( w_1(x) + \frac{1}{w_1(x)} \right).$$
With these notations, we show below that if \((7.34)\) holds, then

\[-1 < w_1(x) < 0 < w_2(x) \quad \text{for} \quad x \in (-1, 1),\]  

and

\[
\mathcal{L}(f) = \frac{4}{\pi^2} \int_{(-1, 1)^2} f(x, y) \sqrt{1 - x^2} \sqrt{1 - y^2} \frac{dx \, dy}{p(x, w)p(x, 1/w)} + \frac{6272}{15\pi} \int_{-1}^{1} f(x, y_1(x)) (1 - w_1(x)^2) w_2(x) \sqrt{1 - x^2} \frac{dx}{(53 - 28x)(1 - w_1(x)w_2(x))(w_2(x) - w_1(x))}
\]

extends \(\mathcal{L}\) from \(\mathbb{R}_{x,y}\) with moment matrix in \((7.19)\) to a positive linear functional on \(\mathbb{R}[x, y]\). Note that \(y_1(x)\) in the last line of the above equation depends on \(x\) unlike the formulas in the previous example.

Since \(c_0 + c_1x > 0\) for \(x \in (-1, 1)\), it follows that \(w_1 < 0 < w_2\) when \(x \in (-1, 1)\). Furthermore \(|w_1w_2| > 1\) for all real \(c_0, c_1\) when \(x \in (-1, 1)\) by \((7.23)\), so that if one root is in magnitude less than one for all \(x \in (-1, 1)\) the other must be in magnitude greater than one. We look to see if \(-1 < w_1\) for \(x \in (-1, 1)\). This leads to the inequality

\[
\frac{623 - 448x}{225} < c_0 + c_1 x.
\]

This will be true if and only if \(\frac{623 - 448x}{225} \leq c_0 + c_1 x\) for \(x = \pm 1\) and at least one of these inequalities is strict. It is straightforward to see that these two inequalities coincide with the ones in \((7.34)\), thus completing the proof of \((7.35)\).

For arbitrary \(c_0, c_1 \in \mathbb{R}\) and with \(p(x, w)\) in \((7.22)\) we define the functional \(\mathcal{L}_0 : \mathbb{R}[x, y] \to \mathbb{R}\) by

\[
\mathcal{L}_0(f) = \frac{4}{\pi^2} \int_{(-1, 1)^2} f(x, y) \sqrt{1 - x^2} \sqrt{1 - y^2} \frac{dx \, dy}{p(x, w)p(x, 1/w)} = \frac{2}{\pi} \int_{-1}^{1} \mathcal{I}(f, x) \sqrt{1 - x^2} \, dx
\]

where

\[
\mathcal{I}(f, x) = \int_{-\pi}^{\pi} f(x, \cos \phi) \sin^2 \phi \frac{dx}{p(x, w)p(x, 1/w)} d\phi
\]

\[
= \frac{784}{15(53 - 28x)\pi i} \oint f(x, \frac{1}{2}(w + \frac{1}{w}))(w - 1/w)^2w \frac{dw}{\tau(x, w)},
\]

and

\[
\tau(x, w) = (w - w_1)(w - w_2)(w - 1/w_1)(w - 1/w_2).
\]

In the region where equation \((7.23)\) holds \(\mathcal{L} = \mathcal{L}_0\) and the residues of the above contour integral are at \(w = 0, w = 1/w_1,\) and \(w = 1/w_2\). Hence

\[
\mathcal{I}(f, x) = \frac{784}{15(53 - 28x)} (R_0(x) + R_1(x) + R_2(x))
\]

where

\[
R_0(x) = 2 \text{res}_{w=0} f(x, \frac{1}{2}(w + \frac{1}{w}))(w - 1/w)^2w \frac{dw}{\tau(z, w)},
\]

\[
R_1(x) = 2 \frac{f(x, \frac{1}{2}(w_1 + \frac{1}{w_1}))(1 - w_1^2)(1 - \frac{w_1}{w_2})}{(1 - w_1 w_2)(1 - \frac{w_1}{w_2})},
\]

and

\[
R_2(x) = 2 \frac{f(x, \frac{1}{2}(w_2 + \frac{1}{w_2}))(1 - w_2^2)(1 - \frac{w_2}{w_1})}{(1 - w_1 w_2)(1 - \frac{w_2}{w_1})}.
\]
Therefore
\[
R_2(x) = 2 \frac{f(x, \frac{1}{2}(w_2 + \frac{1}{w_2}))(1 - w_2^2)}{(1 - w_1 w_2)(1 - \frac{w_1}{w_2})}.
\]

We now consider the region where (7.34) holds. In this case \(|w_1| < 1 < |w_2|\) and the residue at \(w = 1/w_1\) is replaced by the residue at \(w = w_1\) which gives
\[
\hat{R}_1(x) = -2 \frac{f(x, \frac{1}{2}(w_1 + \frac{1}{w_1}))(1 - w_1^2)}{(1 - w_1 w_2)(1 - \frac{w_1}{w_2})} = -R_1(x).
\]

Therefore
\[
\mathcal{L}(f) - \mathcal{L}_0(f) = \frac{6272}{15\pi} \int_{-1}^{1} \frac{f(x, y_1(x))(1 - w_1(x)^2)\sqrt{1 - x^2}}{(53 - 28x)(1 - w_1(x)w_2(x))(1 - \frac{w_1(x)}{w_2(x)})} dx,
\]
establishing (7.30).

References

[1] Ju. M. Berezans’kii, *Expansions in eigenfunctions of selfadjoint operators*, Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, R.I., 1968.

[2] D. Damanik, A. Pushnitski and B. Simon, *The analytic theory of matrix orthogonal polynomials*, Surv. Approx. Theory 4 (2008), 1–85.

[3] D. Damanik and B. Simon, *Jost functions and Jost solutions for Jacobi matrices. II. Decay and analyticity*, Int. Math. Res. Not. 2006, Art. ID 19396, 32 pp.

[4] A. Delgado, J. Geronimo, P. Iliev and F. Marcellán, *Two variable orthogonal polynomials and structured matrices*, SIAM J. Matr. Anal. Appl. 28 (2006), no. 1, 118–147.

[5] A. Delgado, J. Geronimo, P. Iliev and Y. Xu, *On a two variable class of Bernstein-Szegő measures*, Constr. Approx. 30 (2009), no. 1, 71–91.

[6] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, 2nd edition, Encyclopedia of Mathematics and its Applications 155, Cambridge University Press, 2014.

[7] L. Fejér, *Über trigonometrische Polynome*, J. Reine Angew. Math. 146 (1916), 53–82.

[8] M. I. Gekhtman and A. A. Kalyuzhny, *On the orthogonal polynomials in several variables*, Integral Equations Operator Theory 19 (1994), no. 4, 404–418.

[9] J. S. Geronimo, *Scattering theory and matrix orthogonal polynomials on the real line*, Circuits Systems and Signal Processing 1 (1982), 471–495.

[10] J. S. Geronimo and K. M. Case, *Scattering theory and polynomials orthogonal on the real line*, Trans. Amer. Math. Soc. 258 (1980), no. 2, 467–494.

[11] J. S. Geronimo and P. Iliev, *Two variable deformations of the Chebyshev measure*, in: “InTEGRABLE SYSTEMS AND RANDOM MATRICES: IN HONOR OF PERCY DEIFT”, pp. 197–213, Contemp. Math. 458, Amer. Math. Soc., Providence, RI, 2008.

[12] J. S. Geronimo and P. Iliev, *Fejér-Riesz factorizations and the structure of bivariate polynomials orthogonal on the bi-circle*, J. Eur. Math. Soc. (JEMS) 16 (2014), 1849–1880.

[13] J. S. Geronimo and P. Iliev, *Bernstein-Szegő measures, Banach algebras, and scattering theory*, Trans. Amer. Math. Soc. 369 (2017), no. 8, 5581–5600.

[14] J. S. Geronimo, P. Iliev and G. Knese, *Polynomials with no zeros on a face of the bidisk*, J. Funct. Anal. 270 (2016), 3505–3558.

[15] J. S. Geronimo and H. Woerdeman, *Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables*, Ann. of Math. (2) 160 (2004), 839–906.

[16] J. Geronimo and H. J. Woerdeman, *Two variable orthogonal polynomials on the bicircle and structured matrices*, SIAM J. Matrix Anal. Appl. 29 (2007), no. 3, 796–825.

[17] L. Ya. Geronimus and G. Szegő, *Two papers on special functions*, American Mathematical Society Translations, Ser. 2, Vol. 108, 1977, ii+130 pp.

[18] D. Jackson, *Formal properties of orthogonal polynomials in two variables*, Duke Math. J. 2 (1936), 423–434.

[19] R. Kozhan, *Jost asymptotics for matrix orthogonal polynomials on the real line*, Constr. Approx. 36 (2012), 267–309.

[20] R. Kozhan, *Erratum to: Jost asymptotics for matrix orthogonal polynomials on the real line*. 
[21] F. I. Lander, *The Bezoutiant and the inversion of Hankel and Toeplitz matrices*, Mat. Issled. 9 (1974), 69–87, 249–250, in Russian.
[22] R. G. Newton and R. Jost, *The construction of potentials from the S-matrix for systems of differential equations*, Nuovo Cimento (10) 1 (1955), 590–622.
[23] F. Riesz, *Über ein Problem des Herrn Carathéodory*, J. Reine Angew. Math. 146 (1916), 83–87.
[24] M. G. Strintzis, *Tests of stability of multidimensional filters*, IEEE Trans. Circuits and Systems CAS-2 (1977), no. 8, 432–437.
[25] P. K. Suetin, *Orthogonal polynomials in two variables*, translated from the 1988 Russian original by E. V. Pankratiev, Analytical Methods and Special Functions, vol. 3, Gordon and Breach Science Publishers, 1999.
[26] G. Szegő, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Coll. Publ. Vol. 23, Providence, RI, 1975.
[27] Y. Xu, Orthogonal polynomials of several variables. *Encyclopedia of special functions: the Askey-Bateman project*. Vol. 2. Multivariable special functions, 19–78, Cambridge Univ. Press, Cambridge, 2021.
[28] D. C. Youla and N. N. Kazanjian, *Bauer-type factorization of positive matrices and the theory of matrix polynomials orthogonal on the unit circle*, IEEE Trans. Circuits and Systems CAS-2 (1978), no. 2, 57–69.

JG, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332–0160, USA
Email address: geronimo@math.gatech.edu

PI, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332–0160, USA
Email address: iliev@math.gatech.edu