Portfolio analysis with mean-CVaR and mean-CVaR-skewness criteria based on mean–variance mixture models

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Abstract

The paper [Zhao et al. (2015)] shows that mean-CVaR-skewness portfolio optimization problems based on asymmetric Laplace (AL) distributions can be transformed into quadratic optimization problems for which closed form solutions can be found. In this note, we show that such a result also holds for mean-risk-skewness portfolio optimization problems when the underlying distribution belongs to a larger class of normal mean–variance mixture (NMVM) models than the class of AL distributions. We then study the value at risk (VaR) and conditional value at risk (CVaR) risk measures of portfolios of returns with NMVM distributions. They have closed form expressions for portfolios of normal and more generally elliptically distributed returns, as discussed in [Rockafellar & Uryasev (2000) and in Landsman & Valdez (2003)]. When the returns have general NMVM distributions, these risk measures do not give closed form expressions. In this note, we give approximate closed form expressions for the VaR and CVaR of portfolios of returns with NMVM distributions. Numerical tests show that our closed form formulas give accurate values for VaR and CVaR and shorten the computational time for portfolio optimization problems associated with VaR and CVaR considerably.

Keywords: Portfolio selection · Normal mean–variance mixtures · Risk measure · Mean-risk-skewness · EM algorithm
1 Introduction

Numerous empirical studies of asset returns suggest that their distributions deviate from the normal distribution, see Cont & Tankov (2004) and Schoutens (2003). In fact, it has been demonstrated in many papers that asset returns have fat tails and skewness, see Rachev et al. (2005) and Campbell, Lo & MacKinlay (1997) for example. The leptokurtic features of empirical asset return data suggest that there are more realistic models than the normal distribution. It has been empirically demonstrated in numerous papers that the multivariate Generalized Hyperbolic (mGH) distributions and their sub-classes can describe multivariate financial data returns very well , see McNeil et al. (2015) and the references therein. The class of mGH distributions is a subclass of general normal mean–variance mixture (NMVM) models.

A return vector $X = (X_1, X_2, \cdots, X_n)^T$ (here and from now on $T$ denotes transpose) of $n$ assets follows an NMVM distribution if

$$X \overset{d}{=} \mu + \gamma Z + \sqrt{Z} AN_n,$$  

(1)

where $N_n$ is an $n-$dimensional standard normal random variable $N(0, I)$ (here $I$ is the $n$-dimensional identity matrix), $Z \geq 0$ is a non-negative scalar-valued random variable which is independent of $N_n$, $A \in \mathbb{R}^{n \times n}$ is a matrix, $\mu \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^n$ are constant vectors which describe the location and skewness of $X$ respectively.

In general, $Z$ in (1) can be any non-negative valued random variable. But if $Z$ follows a Generalized Inverse Gaussian (GIG) distribution, the distribution of $X$ is called an mGH distribution, see McNeil et al. (2015) and Prause et al. (1999) for the details of the GH distributions. When $Z$ is an exponential random variable with parameter $\lambda = 1$, $X$ follows an asymmetric Laplace (AL) distribution, see Mittnik & Rachev (1993), Kozubowski & Rachev (1994), and Kozubowski & Podgórski (2001) for the definition and financial applications of AL distributions.

A GIG distribution $W$ has three parameters $\lambda, \chi, \psi$ and its density is given by

$$f_{GIG}(w; \lambda, \chi, \psi) = \left\{ \begin{array}{ll} \frac{\chi^{\lambda} (\chi \psi)^{\frac{\lambda}{2}}}{2 K_{\lambda}(\sqrt{\chi \psi})} w^{\lambda-1} e^{-\frac{w}{2} + \frac{\psi w}{\chi} + \frac{\chi w}{2}}, & w > 0, \\
0, & w \leq 0, \end{array} \right. $$  

(2)

where $K_{\lambda}(x) = \frac{1}{\pi} \int_0^\infty y^{\lambda-1} e^{-x y} dy$ is the modified Bessel function of the third kind with index $\lambda$ for $x > 0$. The parameters in (2) satisfy $\chi > 0$ and $\psi \geq 0$ if $\lambda < 0$; $\chi > 0$ and $\psi > 0$ if $\lambda = 0$; and $\chi \geq 0$ and $\psi > 0$ if $\lambda > 0$. The expected value of $W$ is given by

$$E(W) = \frac{\sqrt{\chi/\psi} K_{\lambda+1}(\chi \psi)}{K_{\lambda}(\sqrt{\chi \psi})}. $$  

(3)

With $Z \sim GIG$ in (1), the density function of $X$ has the following form

$$f_X(x) = \frac{(\sqrt{\psi/\chi})^{\lambda}(\psi + \gamma^T \Sigma^{-1} \gamma)^{\frac{\lambda}{2} - \lambda}}{(2\pi)^{\frac{3}{2}} |\Sigma|^{\frac{3}{2}} K_{\lambda}(\sqrt{\chi \psi})} \times \frac{K_{\lambda-\frac{3}{2}}(\sqrt{(\psi + Q(x))(\psi + \gamma^T \Sigma^{-1} \gamma)e^{(x-\mu)^T \Sigma^{-1} \gamma}})}{\left(\sqrt{(\chi + Q(x))(\psi + \gamma^T \Sigma^{-1} \gamma)}\right)^{\frac{1}{2} - \lambda}}, $$  

(4)

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where \( Q(x) = (x - \mu)^T \Sigma^{-1}(x - \mu) \) denotes the Mahalanobis distance.

The GH distribution contains several special cases that are used frequently in financial modelling. For example, the case \( \lambda = \frac{n+1}{2} \) corresponds to a multivariate hyperbolic distribution, see Eberlein & Keller (1995) and Bingham & Kiesel (2001) for applications of this case in financial modelling. When \( \lambda = -\frac{1}{2} \), the distribution of \( X \) is called a Normal Inverse Gaussian (NIG) distribution and Barndorff-Nielsen (1997) proposes NIG as a good model for finance. If \( \chi = 0 \) and \( \lambda > 0 \), the distribution of \( X \) is a Variance Gamma (VG) distribution, see Madan & Seneta (1990) for the details of this case. If \( \nu = 0 \) and \( \lambda < 0 \), the distribution of \( X \) is called the generalized hyperbolic Student \( t \) distribution and Aas & Haff (2006) shows that this distribution matches the empirical data very well.

The mean-risk portfolio optimization problems based on NMVM models were discussed in the recent paper Shi & Kim (2021). They provided a method that transforms a high dimensional problem into a two-dimensional problem in their proposition 3.1. This enabled them to solve portfolio optimization problems associated with NMVM distributed returns more efficiently. When the returns have Gaussian distributions or more generally elliptical distributions, the VaR and CVaR risk measures have closed form expressions for portfolios of such returns, as shown in Owen & Rabinovitch (1983) and Landsman & Valdez (2003). For returns with more general NMVM distributions, these risk measures do not give closed form formulas. One of the goals of this paper is to provide approximate closed form formulas for risk measures for portfolios of NMVM distributed returns.

**Remark 1.1.** We remark here that the NMVM models, which are more general than the mGH models, allow for models obtained by the multiple subordination techniques applied in Shirvani et al. (2021a) and Shirvani et al. (2021a) for example. The distributional class of mGHs has a specific structure of the subordinators and thus fixes the shape of the distributional tail behavior. NMVM includes the class of all distributions representing the unit increment of multivariate subordinated Lévy processes, see Shirvani et al. (2021b) for example. The number of subordinations changes the behavior of the tails of the distribution and can be used as a model parameter. In particular, the class of NMVM models includes multiple subordinated mGH. For example, if the first subordination determines the heavy tailedness and the skewness of the stock returns, the second subordination can be used to track transaction volume, and the third the ESG score of the company issuing the stock. As shown in Shirvani et al. (2021a) and Shirvani et al. (2021b), one subordination is not necessarily sufficient in financial modeling. These facts necessitate the need for the discussion of the general class of NMVM models.

Measuring the risk of a financial position is a complex process. It is standard to assess the riskiness of a financial position by means of convex risk measures, see Frittelli & Gianin (2002), Föllmer & Schied (2002), Artzner et al. (1999). A convex risk measure is a convex function \( \rho : L^p \to (-\infty, +\infty] \) which is

(a) Monotone: \( \rho(X) \leq \rho(Y) \) if \( X \geq Y \) almost surely.

(b) Cash-invariant: \( \rho(X + m) = \rho(X) - m \) for all \( m \in \mathbb{R} \).
Here $L^p$ denotes the class of random variables with finite $p$ moments (here, $p \in [1, +\infty)$). A convex risk measure $\rho$ satisfies the convexity property $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$, $\lambda \in [0, 1]$. A convex risk measure $\rho$ is called coherent if it satisfies the positive homogeneity property $\rho(\lambda X) = \lambda \rho(X)$ for any $\lambda > 0$. The quantity $\rho(X)$ can be viewed as the minimal capital that has to be added to the financial position $X$ to make it acceptable. A risk measure $\rho$ with the above two properties (a) and (b) is called a monetary risk measure, see Föllmer & Schied (2002). Monetary risk measures are also defined through an associated acceptable set. A subset $A$ of $X$ is called acceptable if $X \in A$ and $Y \geq X$ implies $Y \in A$. A risk measure $\rho(X)$ associated with an acceptable set $A$ is defined to be the minimum capital that has to be added to $X$ to make it acceptable, i.e.,

$$\rho(X) := \inf \{ a \in \mathbb{R} | X + a \in A \}. \tag{5}$$

A risk measure can also be defined by (5) for a given acceptable set $A$. For any risk measure $\rho$, its associated acceptance set is given by $A_{\rho} = \{ X \in \mathcal{X} | \rho(X) \leq 0 \}$. The risk measure $\rho$ can be recovered from $A_{\rho}$ as in (5). The concept of coherent risk measures was first introduced in the seminal paper Artzner et al. (1999), also see Malevergne & Sornette (2006). Convex risk measures were introduced and studied in Frittelli & Gianin (2002), Heath (2000), and Föllmer & Schied (2002).

A monetary risk measure $\rho$ is law invariant if $\rho(X) = \rho(Y)$ whenever $X$ and $Y$ are equal in distribution. The value at risk (VaR) is a law invariant monetary risk measure. For any $\alpha \in (0, 1)$, the value at risk at significance level $\alpha$ is denoted by $\text{VaR}_\alpha$ and its acceptance set is given by

$$A_{\text{var}_\alpha} = \{ X \in \mathcal{X} | P(X < 0) \leq \alpha \}.$$

The VaR is given by

$$\text{VaR}_\alpha(X) = \inf \{ m \in \mathbb{R} | X + m \in A_{\text{var}_\alpha} \}
= -\inf \{ c \in \mathbb{R} | P(X \leq c) > \alpha \} = -q^+_X(\alpha), \tag{6}$$

where $q^+_X(\alpha)$ is the upper $\alpha$-quantile of $X$. As pointed out in [Artzner et al.] (1999), the VaR lacks the subadditivity property. In particular, VaR is a non-convex function and portfolio optimization problems with it lead to multiple local extremes. Therefore portfolio optimization with VaR is computationally expensive.

Recently, the risk measure CVaR has become popular both in academia and in finance. The CVaR is a coherent risk measure (see, e.g., Acerbi & Tasche (2002)) and therefore has some favorable properties that VaR lacks. The conditional value at risk (average value at risk, tail value at risk, or expected shortfall) is defined for any $\alpha \in (0, 1)$ as follows

$$\text{CVaR}_\alpha(X) =: -\frac{1}{\alpha} \int_0^\alpha \text{Var}_\beta(X) d\beta = -\frac{1}{\alpha} E[1_{\{X \leq -\text{VaR}_\alpha(X)\}}].$$

Since $\text{VaR}_\alpha$ is decreasing in $\alpha$, we clearly have $\text{CVaR}_\alpha(X) \geq \text{VaR}_\alpha(X)$. In fact, CVaR is the smallest coherent risk measure which is law invariant and dominates the VaR.
It is well known that a convex measure \( \rho \) is law invariant if and only if
\[
\rho(X) = \sup_{Q \in \mathcal{M}_1((0,1])} \left[ \int_{[0,1]} \text{CVaR}_\alpha(X)Q(d\alpha) - \beta_{\min}(Q) \right],
\]
where
\[
\beta_{\min}(Q) = \sup_{X \in \mathcal{A}_p} \int_{[0,1]} \text{CVaR}_\alpha(X)Q(d\alpha),
\]
and \( \mathcal{M}_1((0,1]) \) is the space of probability measures on \((0,1]\). A coherent risk measure \( \rho \) is law invariant if and only if
\[
\rho(X) = \sup_{Q \in \mathcal{M}} \int_{[0,1]} \text{CVaR}_\alpha(X)Q(d\alpha)
\]
for a subset \( \mathcal{M} \) of \( \mathcal{M}_1((0,1]) \). For the details of these results see [Kusuoka (2001)], [Dana (2005)], and [Frittelli & Gianin (2005)].

Remark 1.2. We remark here that the relations (7), (8), and (9) above show that any law invariant convex risk measure can be expressed in terms of CVaR. In Section 3 of this paper, we will discuss approximate closed form formulas for CVaR for portfolios of NMVM distributed returns. One can then use these formulas in place of CVaR in (7), (8), and (9) above and construct simpler expressions for any law invariant convex risk measure.

This paper is organized as follows. In Section 2, we study a mean-risk-skewness multi-objective optimization problem and provide closed form solutions for optimal portfolios when the return vectors follow a certain class of NMVM models. In Section 3, we give approximate closed form expressions for the risks of the portfolios of NMVM returns when the risk measures are coherent ones. In Section 4, we present the results of numerical tests of our results.

2 Closed form solutions for mean-risk-skewness optimal portfolios

As mentioned in the Abstract, [Zhao et al. (2015)] studies mean-CVaR-skewness optimization problems for AL distributions. The class of AL distributions considered in their paper have the mean–variance mixture form
\[
X = \gamma Z + \tau \sqrt{Z} N
\]
for two real numbers \( \gamma \in \mathbb{R} \) and \( \tau > 0 \), where \( Z \sim \text{Exp}(1) \), and it is independent of the standard normal random variable \( N \). The characteristic function of \( X \) is given by the formula (1) in their paper with \( \mu \) replaced by \( \gamma \). In this section and for the rest of the paper, we use “\( \sim \)” to denote
the equivalence in distribution of two random variables. "Skew" denotes the skewness of a random variable and "Kurt" denotes the kurtoses of a random variable.

In the multi-dimensional case, the AL distributions they have considered have the form $X = \gamma Z + \sqrt{Z} AN_n$, where now $\gamma \in \mathbb{R}^n$ and $A$ is an $n \times n$ matrix and $N_n$ is a $n-$dimensional standard normal random variable. For any portfolio $\omega = (\omega_1, \cdots, \omega_n)^T$, they calculated $CVaR(\omega^T X)$ and $SKew(\omega^T X)$ as

$$CVaR(\omega^T X) = (1 - \ln \alpha)g(\gamma_p, \tau_p) - g(\gamma_p, \tau_p) \ln(2 + \frac{\gamma_p}{g(\gamma_p, \tau_p)}),$$

$$SKew(\omega^T X) = \frac{2\gamma_p^3 + 3\gamma_p \tau_p^2}{(\gamma_p^2 + \tau_p^2)^{3/2}},$$

where $\gamma_p, \tau_p,$ and $g(\mu_p, \tau_p)$ are given as in (9), (11), and in (16) in their paper respectively. Note that here $\gamma_p$ represents $\mu_p$ in their paper.

In their paper, they considered the mean-CVaR-skewness multi-objective problem

$$\begin{cases}
\min_{\omega \in \mathbb{R}^n} CVaR_\beta(\omega^T X), \\
\max_{\omega \in \mathbb{R}^n} SKew(\omega^T X), \\
s.t. \ E(\omega^T X) = r, \ \omega^T e = 1.
\end{cases}$$

(12)

for any return $r > 0$. This problem can be written under general law invariant risk measure $\rho$ as follows:

$$\begin{cases}
\min_{\omega \in \mathbb{R}^n} \rho(\omega^T X), \\
\max_{\omega \in \mathbb{R}^n} SKew(\omega^T X), \\
s.t. \ E(\omega^T X) = r, \ \omega^T e = 1.
\end{cases}$$

(13)

Remark 2.1. We remark here that the problem (13) is a multi-objective problem, i.e., optimize both $\rho(\omega^T X)$ and $SKew(\omega^T X)$ simultaneously under the constraints $E(\omega^T X) = r, \ \omega^T e = 1$. In comparison, the relevant paper Akturk & Ararat (2020) discusses a static portfolio optimization problem also and their optimization problem is a single-objective problem.

Remark 2.2. Such mean-risk-skewness portfolio optimization problem as in (13) have been studied in many papers in the past, see Konno & Suzuki (1995), Zhao et al. (2015), and the references therein, for example. This is because, as stated in the introduction of Konno & Suzuki (1995), many investors prefer a positively skewed distributions to a negative one and also larger skewed distributions compared to smaller skewed distributions if the mean and the risk are the same.

In their paper, they observe that CVaR, as was shown in the proof of their Theorem 2, is an increasing function of $\tau_p$ and Skew is a decreasing function of $\tau_p$. They concluded, therefore, that
minimizing $\tau_p$ gives the same solution as (12) above and also it translates into the quadratic optimization problem (17) in their paper.

In this paper, we take a different approach than Zhao et al. (2015) for the solution of the problem (12) above and obtain similar results for $X = \gamma Z + \sqrt{Z}AN_n$ when $Z$ is more general than $Exp(1)$ and also when CVaR is replaced by a general law-invariant risk measure as in (13).

We first fix some notation. We denote by $A_1^c, A_2^c, \ldots, A_n^c$ the column vectors of $A$. We assume that $A$ is an invertable matrix. We write $\gamma$ as linear combinations of $A_1^c, A_2^c, \ldots, A_n^c$, i.e., $\gamma = \sum_{i=1}^n \gamma_i A_i^c$. We denote by $\gamma_0 = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ the vector of the corresponding coefficients of such linear combination. As usual, we denote by $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ the Euclidean norm of the vector $x$. Also, for each NMVM distribution $X$ as above, we define

$$Y = \gamma_0 Z + \sqrt{Z}N_n,$$

and we call $Y$ the NMVM distribution associated with $X$.

For any portfolio $\omega$ we define a transformation $T : R^n \to R^n$ by $T \omega = x$, where $x = (x_1, x_2, \ldots, x_n)^T$ is given by

$$x_1 = \omega^T A_1^c, \quad x_2 = \omega^T A_2^c, \ldots, \quad x_n = \omega^T A_n^c. \quad (15)$$

In matrix form, this is written as $x^T = \omega^T A$. From now on, for any domain $D$ of portfolios $\omega$, we denote by $R_D$ the image of $D$ under $T$.

We start our discussions with the following simple lemma. In the following lemma, $N$ denotes a standard normal random variable that independent of the mixing distribution $Z$.

**Lemma 2.3.** We have $\omega^T X = x^T Y$ and $x^T Y \overset{d}{=} x^T \gamma_0 Z + \sqrt{x^T x} \sqrt{Z} N$. Therefore we have $\rho(\omega^T X) = \rho(x^T Y) = \sqrt{x^T x} \rho(\cos(\theta(\gamma_0, x))Z + \sqrt{Z}N)$ for any law invariant coherent risk measure $\rho$ whenever $\omega \in R^n$ and $x \in R^n$ are related by (15). Here $\cos(\theta(\gamma_0, x))$ is the cosine of the angle between $\gamma_0$ and $x$.

**Proof.** Note that $\omega^T X = \omega^T \gamma Z + \sqrt{Z}(\omega^T A)N_n$. We have $\omega^T \gamma = x^T \gamma_0$ and $x = \omega^T A$. Therefore $\omega^T X = x^T Y$. Since $x^T N_n \overset{d}{=} \sqrt{x^T x} N$ and $\rho$ is a law invariant risk measure, we have $\rho(\omega^T X) = \rho(x^T Y) = \rho(x^T \gamma_0 Z + \sqrt{x^T x} \sqrt{Z} N)$. Since $x^T \gamma_0 = \sqrt{x^T x} ||\gamma_0|| \cos(\theta(\gamma_0, x))$ and $\rho$ is positive homogeneous, we have $\rho(x^T Y) = \sqrt{x^T x} \rho(\cos(\theta(\gamma_0, x))Z + \sqrt{Z}N).$ \hfill $\Box$

The above lemma shows that optimization problems under mean-risk-skewness criteria can be studied in the $x$-coordinate system rather than the original $\omega$-coordinate system. This has some advantage as will be seen shortly.

Below, we write the problem (13) in the $x$-coordinate system as follows

$$\begin{align*}
\min_{x \in R^n} & \rho(x^T Y), \\
\max_{x \in R^n} & Skew(x^T Y), \\
s.t. & x^T m = r, \ x^T e_A = 1,
\end{align*} \quad (16)$$

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where \( e_A = A^{-1} e \) and \( m = \gamma_0 E Z \).

Next, we will show that the problem (16) can be reduced to a quadratic optimization problem when \( X \) satisfies certain conditions which are satisfied by AL distributions. We first need to calculate \( \text{SKew}(x^T Y) \).

**Proposition 2.4.** For any \( x = \omega^T A \), we have

\[
\begin{align*}
\text{StD}(x^T Y) &= ||x|| \sqrt{||\gamma_0||^2 \phi^2 \text{Var}(Z) + EZ}, \\
\text{SKew}(x^T Y) &= \frac{||\gamma_0||^3 \phi^3 m_3(Z) + 3||\gamma_0|| \phi \text{Var}(Z)}{(||\gamma_0||^2 \phi^2 \text{Var}(Z) + EZ)^\frac{3}{2}}, \\
\text{Kurt}(x^T Y) &= \frac{||\gamma_0||^4 \phi^4 m_4(Z) + 6||\gamma_0||^2 \phi^2 [E Z^3 - 2E Z^2 E Z + (E Z)^3] + 3E Z^2}{(||\gamma_0||^2 \phi^2 \text{Var}(Z) + EZ)^2},
\end{align*}
\]

where \( m_3(Z) = E(E - EZ)^3 \), \( m_4(Z) = E(Z - EZ)^4 \), and \( \phi = \cos(x, \gamma_0) \).

**Proof.** Note that \( x^T Y \overset{d}{=} x^T Y_0 + ||x|| \sqrt{Z N} = ||x|| (||\gamma_0|| \phi Z + \sqrt{Z N}) \). Therefore, on putting \( Y_0 = ||\gamma_0|| \phi Z + \sqrt{Z N}, \) we have \( \text{StD}(x^T Y) = ||x|| \text{StD}(Y_0), \text{SKew}(x^T Y) = \text{SKew}(Y_0), \) and \( \text{Kurt}(x^T Y) = \text{Kurt}(Y_0). \) Since \( Z \) is independent of \( N \), a straightforward calculation gives

\[
\text{StD}(Y_0) = \sqrt{||\gamma_0||^2 \phi^2 \text{Var}(Z) + EZ}.
\]

We have \( \text{SKew}(Y_0) = E(Y_0 - EY_0)^3/[\text{StD}(Y_0)]^3 \) and again a straightforward calculation shows that \( E(Y_0 - EY_0)^3 \) equals the numerator of \( \text{SKew}(x^T Y) \) in (17). The kurtosis equals \( \text{Kurt}(Y_0) = E(Y_0 - EY_0)^4/[\text{StD}(Y_0)]^4 \). It can be easily checked that \( E(Y_0 - EY_0)^4 \) equals the numerator of \( \text{Kurt}(x^T Y) \) in (17).

**Remark 2.5.** From Proposition 2.4, we observe that \( \text{StD} \) depends on \( x \) through both \( \cos(x, \gamma_0) \) and \( ||x|| \). However, both \( \text{SKew} \) and \( \text{Kurt} \) depend on \( x \) only through the angle \( \cos(x, \gamma_0) \).

We need to calculate the derivative of \( \text{SKew}(x^T Y) \) with respect to \( \phi \). A straightforward calculation gives us

\[
\frac{\partial \text{SKew}(x^T Y)}{\partial \phi} = \frac{3||\gamma_0||^3 (m_3(Z) EZ - 2 \text{Var}^2(Z)) \phi^2 + 3||\gamma_0|| \text{Var}(Z) EZ}{(||\gamma_0||^2 \phi^2 \text{Var}(Z) + EZ)^\frac{3}{2}}.
\]

From (18) we see that if the mixing distribution \( Z \) satisfies

\[
m_3(Z) EZ \geq 2 \text{Var}^2(Z),
\]

then \( \frac{\partial}{\partial \phi} \text{SKew}(x^T Y) \geq 0. \)
Remark 2.6. Clearly, if the mixing distribution $Z$ is such that $\frac{\partial}{\partial \phi} S\text{Kew}(x^T Y) \geq 0$ for all $\phi \in [-1, 1]$, a condition which is weaker than (18), then Skew is an increasing function of $\phi$. Here we singled out the condition (19) for its simplicity as a sufficient condition for $\frac{\partial}{\partial \phi} S\text{Kew}(x^T Y) \geq 0$.

Now we state the main result of this paper.

Theorem 2.7. Let $\rho$ be any law invariant coherent risk measure. Assume $Z$ in (10) is such that $\frac{\partial}{\partial \phi} S\text{Kew}(x^T Y) \geq 0$ for all $\phi \in [-1, 1]$. Then the solution of the problem (16) is equal to the solution of the following quadratic optimization problem:

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} & \quad x^T x, \\
\text{s.t.} & \quad x^T m = r, \quad x^T e_A = 1,
\end{aligned} \tag{20}$$

and is given explicitly by $x = sm + te_A$, where

$$
\begin{aligned}
s &= 2 \left| \begin{array}{c}
r & e_{A}^{T} m \\
1 & e_{A}^{T} e_A
\end{array} \right| / \left| \begin{array}{cc}
m_{A} & e_{A}^{T} m \\
m_{A} & e_{A}^{T} e_A
\end{array} \right| \quad \text{and} \quad t = 2 \left| \begin{array}{c}
m_{A} & r \\
1 & e_{A}^{T} e_A
\end{array} \right| / \left| \begin{array}{cc}
m_{A} & m_{A} \\
m_{A} & e_{A}^{T} e_A
\end{array} \right|.
\end{aligned} \tag{21}
$$

Proof. From Lemma (2.3) above we have $\rho(x^T Y) = \rho(\frac{x^T m}{||x||}Z + ||x||\sqrt{Z}N)$. Since $m = \gamma_0 EZ$ from the constraint (20) we have $x^T \gamma_0 = r/EZ$. We conclude that $\rho(x^T Y) = \rho(\frac{r}{||x||}EZ + ||x||\sqrt{Z}N)$ is a function of $||x||$ under the constraint (20). From part (3) of Theorem 3.1 of Shi & Kim (2021) we conclude that $\rho$ is a non-decreasing function of $||x||$. On the other hand, the condition on Skew ensures that Skew is an increasing function of $\phi = \frac{x^T \gamma_0}{||x|| \gamma_0}$. With the constraint $x^T \gamma_0 = r/EZ$, $\phi$ becomes a decreasing function of $||x||$. Therefore decreasing $||x||$ minimizes $\rho$ and maximizes Skew simultaneously. To show (59), we form the Lagrangian $L = x^T x + s(r - x^T m) + t(1 - x^T e_A)$. The first order condition gives

$$\frac{dL}{dx} = 2x - sm - te_A = 0. \tag{22}$$

From this we get $x^T = \frac{1}{2} sm^T + \frac{1}{2} te_A^T$ and since $x^T m = r$ and $x^T e_A = 1$, we obtain two equations $sm^T m + te_A^T m = 2r$ and $sm^T e_A + te_A^T e_A = 2$. The solution of these two equations give (21). \hfill \Box

For the remainder of this section, we give examples of the mixing distributions $Z$ that satisfy condition (19).

Example 2.8. We consider the case of gamma distributions $G(\lambda, \frac{\gamma^2}{2})$ with the density function

$$f(x) = \left(\frac{\gamma^2}{2}\right)^{\lambda-1} x^{\lambda-1} e^{-\frac{\gamma^2 x^2}{2}}1_{\{x \geq 0\}}.$$

In this case we have $EZ = \lambda^{2/\gamma^2}, Var(Z) = \lambda (\gamma^2)^2$, and $m_3(Z) = 2\lambda (\gamma^2)^3$. We obtain

$$-2 Var^2(Z) + m_3(Z)EZ = -2\lambda^2 \left(\frac{1}{\gamma^2}\right)^4 + 2\lambda \left(\frac{1}{\gamma^2}\right)^3 \lambda \left(\frac{1}{\gamma^2}\right) = 0.$$
Thus, in this case $Z$ satisfies the condition (19). Note that $G(\lambda, 1)$ is an $\text{Exp}(1)$ random variable and therefore $\text{Exp}(1)$ also satisfies the condition (19). Thus for asymmetric Laplace distributions the problem (16) can be reduced to a quadratic optimization problem (20). This also shows that our Theorem 2.7 above extends Theorem 2 in Zhao et al. (2015) to the case of more general mean–variance mixture models.

Example 2.9. Now consider inverse Gaussian random variables $Z \sim IG(\delta, \gamma)$ with probability density function

$$f_{IG}(z) = \frac{\delta}{\sqrt{2\pi}}e^{\delta\gamma z^{-3/2}e^{-\frac{1}{2}(\delta^2 z^{-1}+\gamma^2 z)}}1_{\{z>0\}}.$$ 

The moments of these random variables are calculated in Hammerstein (2010):

$$E[Z] = K_{-\frac{1}{2}}(\delta \gamma) \left( \frac{\delta}{\gamma} \right), \quad \text{for } \delta, \gamma > 0,$$

where

$$K_{-\frac{1}{2}}(x) = K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}, \quad K_{\lambda+1}(x) = \frac{2\lambda}{x}K_{\lambda}(x) + K_{\lambda-1}(x).$$

From these, we have

$$E[Z] = \frac{K_{\frac{1}{2}}(\delta \gamma)}{K_{-\frac{1}{2}}(\delta \gamma)} \left( \frac{\delta}{\gamma} \right) = \frac{\delta}{\gamma},$$

$$E[Z^2] = \frac{K_{\frac{3}{2}}(\delta \gamma)}{K_{-\frac{1}{2}}(\delta \gamma)} \left( \frac{\delta}{\gamma} \right)^2 = \frac{1}{\delta \gamma}K_{\frac{1}{2}}(\delta \gamma) + \frac{1}{\gamma}K_{-\frac{1}{2}}(\delta \gamma) \left( \frac{\delta}{\gamma} \right)^2 = \frac{\delta}{\gamma^3} + \left( \frac{\delta}{\gamma} \right)^2,$$

$$E[Z^3] = \frac{K_{\frac{5}{2}}(\delta \gamma)}{K_{-\frac{1}{2}}(\delta \gamma)} \left( \frac{\delta}{\gamma} \right)^3 = \frac{3}{\delta \gamma}K_{\frac{1}{2}}(\delta \gamma) + \frac{3}{\gamma}K_{-\frac{1}{2}}(\delta \gamma) \left( \frac{\delta}{\gamma} \right)^3 = \left( \frac{3}{\delta \gamma} \left( \frac{1}{\delta \gamma} + 1 \right) + 1 \right) \left( \frac{\delta}{\gamma} \right)^3 = \frac{3\delta}{\gamma^5} + 3\delta^2 \gamma^4 + \left( \frac{\delta}{\gamma} \right)^3,$$

and

$$\text{Var}(Z) = E[Z^2] - (E[Z])^2 = \frac{\delta}{\gamma^3},$$

$$m_3(Z) = E[Z^3] - 3E[Z^2]E[Z] + 2(E[Z])^3 = \frac{3\delta}{\gamma^5}.$$ 

Therefore

$$-2\text{Var}^2(Z) + m_3(Z)EZ = -2\frac{\delta^2}{\gamma^6} + 3\frac{\delta^2}{\gamma^6} = \frac{\delta^2}{\gamma^6} > 0.$$

This shows that inverse Gamma random variables also satisfy (19).
Remark 2.10. We should mention that in the above examples we only gave two classes of random variables that satisfy (13). However, the class of random variables that satisfy (13) is not restricted to these two types of random variables. One can check that the class of generalized inverse Gaussian random variables GIG also satisfies (13) for certain parameter values.

3 Portfolio optimization under mean-risk criteria

3.1 Recent results

The mean–variance portfolio theory was first introduced by Markowitz (1959) and since then it has been very popular for practitioners and scholars. However, the Markowitz portfolio theory neglects downside risk. It has been shown in many papers that downside risk can affect returns significantly, see Bollerslev & Todorov (2011) and Bali & Cakici (2004) for example. Because of this, many alternative risk measures have been the focus of academic research, see Chekhlov et al. (2005), Konno et al. (1993). Among downside risk measures, the VaR and CVaR have been extensively studied, see Rockafellar & Uryasev (2000) and Kolm et al. (2014).

The risk measure CVaR was first studied in the context of portfolio optimization problems by Rockafellar & Uryasev (2000, 2002). They showed that a mean-CVaR optimization problem can be transformed into a linear programming problem that improves the efficiency of solving portfolio optimization problems associated with CVaR significantly. However, closed form approximate formulas are still more convenient and efficient and therefore we will discuss such formulas for them below.

For elliptical distributions, the VaR and CVaR have closed form expressions, as studied in Landsman & Valdez (2003). For the general class of NMVM models, these risk measures do not give closed form expressions. In this section, we will present approximate closed form formulas for them. Below, we start by discussing the properties of portfolios of NMVM distributed returns.

Let $X$ denote the return of $n$ assets and assume that $X$ has the NMVM distribution

$$X = \mu + \gamma Z + \sqrt{Z} \Lambda N_n,$$

where $\mu \in \mathbb{R}^n$ is a constant vector and the other symbols are as in the multi-dimensional case in (10) above. For any portfolio $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$, we have

$$\omega^T X \overset{d}{=} \omega^T \mu + \omega^T \gamma Z + \sqrt{Z} \sqrt{\omega^T \Sigma \omega} N,$$

where $N$ is a standard normal random variable and $\Sigma = AA^T$. We will denote the density and distribution functions of the scalar valued random variables $\omega^T X$ by $f_\omega$ and $F_\omega$ respectively for each portfolio $\omega$. Then, according to formula (7) of Hellmich & Kassberger (2011), the conditional value at risk of $-\omega^T X$ is given by

$$CVaR_\beta(-\omega^T X) = -\frac{1}{1 - \beta} \int_{-\infty}^{F^{-1}(1-\beta)} y f_\omega(y) dy,$$
where the quantile-function \( F^{-1}(\cdot) \) is calculated by a root-finding method and the integral in (25) is calculated by numerical integration. Therefore, optimization problems like
\[
\min_{\omega \in D} CVaR(-\omega^T X)
\]
(26)
in some domain \( D \) of the set of portfolios are time consuming and computationally heavy. Another approach to solving the problem (26) was proposed in Rockafellar & Uryasev (2000, 2002). They introduced an auxiliary function
\[
F_{\beta}(\omega, \alpha) = \alpha + \frac{1}{1 - \beta} \int_{y \in \mathbb{R}^m} [-\omega^T y - \alpha]_+ p(y) dy,
\]
(27)
where \( \alpha \) is a real number and \( p(y) \) is the \( d \)-dimensional probability density function of asset returns. They showed that
\[
CVaR_{\beta}(-\omega^T X) = \min_{\alpha \in \mathbb{R}} F_{\beta}(\omega, \alpha).
\]
(28)
Since \( \omega^T X \) follows \( f_\omega \), the above \( F_{\beta}(\omega, \alpha) \) can also be written as
\[
F_{\beta}(\omega, \alpha) = \alpha + \frac{1}{1 - \beta} \int_{x \in \mathbb{R}^n} [-x - \alpha]_+ f_\omega(x) dx.
\]
(29)
With these, the optimization problem (26) becomes
\[
\min_{\omega \in D} CVaR(-\omega^T X) = \min_{\omega \in D} \min_{\alpha \in \mathbb{R}} F_{\beta}(\omega, \alpha).
\]
(30)
Solving the problem (30) is also time consuming as one needs to calculate (29) numerically for each \( \omega \in D \), and then for each \( \omega \in D \) the right-hand side of (30) needs to be minimized for \( \alpha \).

The recent paper by Shi & Kim (2021) studies portfolio optimization problems with NMVM distributed returns. More specifically, they studied the optimization problem
\[
\begin{align*}
\min_{\omega \in D} \rho(\omega^T X) \\
s.t. \quad \omega^T e = 1 \\
E[\omega^T X] \geq k
\end{align*}
\]
(31)
for any coherent risk measure \( \rho \), where \( D \) is a subset of the feasible portfolio set, \( k \in \mathbb{R} \), and \( e = (1, \cdots, 1)^T \) is an \( n \)-dimensional column vector in which each component equals one. They showed that the optimal solution \( \omega^* \) to (31) can be expressed as
\[
\omega^* = \Sigma^{-1}(\mu, \gamma, e)D^{-1}(\tilde{\mu}^*, \tilde{\gamma}^*, 1)^T,
\]
(32)
where

\[
(\tilde{\mu}^{*}, \tilde{\gamma}^{*}) = \arg\min_{\tilde{\mu}, \tilde{\gamma}} \rho(\tilde{\mu} + \tilde{\gamma}Z + \sqrt{g(\tilde{\mu}, \tilde{\gamma})ZN}) 
\]  
\text{s.t. } \tilde{\mu} + \tilde{\gamma}EZ \geq k  
\tag{33} \]

with

\[
g(\tilde{\mu}, \tilde{\gamma}) = (\tilde{\mu}, \tilde{\gamma}, 1)D^{-1}(\tilde{\mu}, \tilde{\gamma}, 1)^T,  
\tag{34} \]

and

\[
D = \begin{pmatrix}
\mu^T \Sigma^{-1} \mu & \gamma^T \Sigma^{-1} \mu & e^T \Sigma^{-1} \mu \\
\mu^T \Sigma^{-1} \gamma & \gamma^T \Sigma^{-1} \gamma & e^T \Sigma^{-1} \gamma \\
\mu^T e & \gamma^T e & e^T e
\end{pmatrix}.  
\tag{35} \]

While this approach gives an expression for the optimal portfolio as in (32), one still needs to solve another optimization problem, namely (33).

In the following, we will show that (32) and (33) can be simplified further. We first fix some notation. Similar to the case of \(\gamma\) in Section 2, we write \(\mu\) as a linear combination of \(A_1, A_2, \ldots, A_n\) as \(\mu = \sum_{i=1}^{n} \mu_i A_i\). We denote by \(\mu_0 = (\mu_1, \mu_2, \ldots, \mu_n)^T\) the column vector formed by the corresponding coefficients of this linear combination. Also, for each NMVM distribution \(X\) as in (23), we define \(Y = \mu_0 + \gamma_0 Z + \sqrt{Z}N\).

and we call \(Y\) the NMVM distribution associated with \(X\). In the \(x\)-coordinate system, problem (31) can be written as

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \rho(x^T Y) \\
\text{s.t.} & \quad x^T e_A = 1 \\
& \quad x^T m \geq k
\end{align*}  
\tag{37} \]

where \(e_A = A^{-1}e\) and now \(m = \mu_0 + \gamma_0 EZ\).

The solution of (37) can be characterized by using the same idea as in Proposition 3.1 of Shi & Kim (2021). Below, we write down the optimal solution of (37) as a corollary of their result.

**Corollary 3.1.** For any law invariant coherent risk measure \(\rho\), the solution of (37) is given by

\[
x^* = (\mu_0, \gamma_0, e_A)G^{-1}(\tilde{\mu}_0^{*}, \tilde{\gamma}_0^{*}, 1)^T,  
\tag{38} \]

where

\[
(\tilde{\mu}_0^{*}, \tilde{\gamma}_0^{*}) = \arg\min_{\tilde{\mu}_0, \tilde{\gamma}_0} \rho(\tilde{\mu}_0 + \tilde{\gamma}_0 Z + \sqrt{g(\tilde{\mu}_0, \tilde{\gamma}_0)ZN}),  
\tag{39} \]

\text{s.t. } \tilde{\mu}_0 + \tilde{\gamma}_0 EZ \geq k,  
\]
with
\[ g(\tilde{\mu}_0, \tilde{\gamma}_0) = (\tilde{\mu}_0, \tilde{\gamma}_0, 1)G^{-1}(\tilde{\mu}_0, \tilde{\gamma}_0, 1)^T, \] (40)
and
\[ G = \begin{pmatrix} \mu_0^T & \gamma_0^T \mu_0 & e_A^T \mu_0 \\ \mu_0^T \gamma_0 & \gamma_0^T \gamma_0 & e_A^T \gamma_0 \\ \mu_0^T e_A & \gamma_0^T e_A & e_A^T e_A \end{pmatrix}. \] (41)

Proof. The proof follows from the same argument as in the proof of Proposition 3.1 of Shi & Kim (2021). In our case here, we have \( \Sigma = I \), and with this, their formula reduces to the solution in the corollary.

We remark that the optimal solutions \( \omega^* \) in (32) and \( x^* \) in (38) are related by \( x^* = (\omega^*)^T A \).

3.2 Closed form approximations for risks

The result (32) reduces the high dimensional nature of the portfolio optimization problem (31) to two dimensions, as claimed in Shi & Kim (2021). In particular, this approach reduces the computational time of determining the optimal portfolio significantly, as claimed in their paper. In this section, we take an alternative approach and attempt to reduce the computational time of such optimization problems also. For this, we linearly transform the portfolio space into a different coordinate system.

The following simple lemma shows that to determine the optimal portfolio, one only needs to solve a low-dimensional optimization problem in, as explained in Remark 3.3 below.

Lemma 3.2. For any coherent risk measure \( \rho \), we have
\[ \rho(x^T Y) = -x^T \mu_0 + \rho(aZ + \sqrt{ZN}) \sqrt{x^T x}, \] (42)
where
\[ a = ||\gamma_0|| \cos[\theta(\gamma_0, x)], \]
and \( \theta(\gamma_0, x) \) is the angle between \( \gamma_0 \) and \( x \).

Proof. We have \( x^T Y = x^T \mu_0 + x^T \gamma_0 Z + ||x|| \sqrt{ZN} \). Since \( \rho \) is law invariant and cash invariant, we have \( \rho(x^T Y) = -x^T \mu_0 + \rho(x^T \gamma_0 Z + ||x|| \sqrt{ZN}) \). Now, since \( x^T \gamma_0 = ||x|| ||\gamma_0|| \cos[\theta(x, \gamma_0)] \) and \( \rho \) is positive homogeneous, (42) follows.

Remark 3.3. We can also write \( x^T \mu_0 = ||x|| ||\mu_0|| \cos[\theta(\mu_0, x)] \) in (42). This shows that optimization problems like \( \min_{R_D} \rho(x^T Y) \) on the image \( R_D \) of some portfolio set \( D \) under the transformation defined in (15) depend on the norm \( ||x|| \) of \( x \), the angle between \( x \) and \( \mu_0 \), and the angle between \( x \) and \( \gamma_0 \).
Lemma (42) shows that the risk $\rho(x^T Y)$ is dependent on $\rho(aZ + \sqrt{Z}N)$. From now on we write
\[ h(a) =: \rho(aZ + \sqrt{Z}N), \quad \text{for} \quad a = ||\gamma_0||\cos[\theta(x, \gamma_0)]. \tag{43} \]
In the following lemma, we state some properties of $h(a)$.

**Lemma 3.4.** For any coherent risk measure $\rho$, the function $h(a) = \rho(aZ + \sqrt{Z}N)$ is a decreasing, convex, and continuous function of $a$ in $(-||\gamma_0||, ||\gamma_0||)$.

**Proof.** The decreasing property of $h(a)$ follows from Theorem 3.1 of Shi & Kim (2021). Here, for the sake of being self-contained, we present its proof. Take two $a_2 \geq a_1$. Then the subadditivity and monotonicity of $\rho$ implies
\[
\begin{align*}
    h(a_2) &= \rho(a_2Z + \sqrt{Z}N) = \rho(a_1Z + (a_2 - a_1)Z + \sqrt{Z}N) \\
    &\leq \rho(a_1Z + \sqrt{Z}N) + \rho((a_2 - a_1)Z) \\
    &\leq \rho(a_1Z + \sqrt{Z}N) + \rho(0) \\
    &= \rho(a_1Z + \sqrt{Z}N) = h(a_1). \tag{44}
\end{align*}
\]

The convexity of $h(a)$ follows from
\[
\begin{align*}
    h(\lambda a + (1 - \lambda)b) &= \rho(\lambda[aZ + \sqrt{Z}N] + (1 - \lambda)[bZ + \sqrt{Z}N]) \\
    &\leq \rho(\lambda[aZ + \sqrt{Z}N]) + \rho((1 - \lambda)[bZ + \sqrt{Z}N]) \\
    &= \lambda h(a) + (1 - \lambda)h(b), \tag{45}
\end{align*}
\]
for any $1 \geq \lambda \geq 0$. Since $h(a)$ is a convex function, it is continuous in the interior of its domain, which is $(-||\gamma_0||, ||\gamma_0||)$. \qed

**Remark 3.5.** Note that from Lemma 3.2, we have $\rho(x^T Y) = -x^T \mu_0 + \sqrt{x^T x}h(a)$. Since $h(a)$ is a continuous function in $(-||\gamma_0||, ||\gamma_0||)$, as was shown in Lemma 3.4, we have
\[
    h(a) \approx \sum_{i=1}^{n-1} 1_{[a_i, a_{i+1})}(a)\rho(a_iZ + \sqrt{Z}N)
\]
for an appropriately chosen partition $a_1 \leq a_2 \leq \cdots \leq a_n$ of the interval $(-||\gamma_0||, ||\gamma_0||)$. Therefore we have the approximation
\[
    \rho(x^T Y) \approx -x^T \mu_0 + \sqrt{x^T x} \sum_{i=1}^{n-1} 1_{[a_i, a_{i+1})}(a)\rho(a_iZ + \sqrt{Z}N) \tag{46}
\]
with $a = -||\gamma_0||\cos[\theta(\gamma_0, x)]$ when the mesh $\max_{1 \leq i \leq n-1}(a_{i+1} - a_i)$ of the partition $-||\gamma_0|| = a_1 \leq a_2 \leq \cdots \leq a_n = ||\gamma_0||$ is sufficiently small.
Remark 3.6. We remark that in order to obtain high precision, the points $a_1, a_2, \ldots, a_n$ need to be chosen to make $\max_i (a_i - a_{i-1})$ as small as possible. However, this comes with a cost, as we need to calculate $\rho(a_i Z + \sqrt{ZN})$ for each $a_i$ to obtain an approximate value of $\rho(x^T Y)$.

This proposition gives an accurate approximation for the value of the risk measure when the mesh of the corresponding partition is very small. However, as stated earlier, we need to calculate $h(a_i) = \rho(a_i Z + \sqrt{ZN})$ for many $i$.

When the value of $||\gamma_0||$ is relatively small, we can get an even simpler approximation, as stated in the following proposition. The next proposition gives an approximation for the risk measure in which one needs to calculate $h(a_i)$ for only two values of $a$ to obtain a good approximation. In the following, we use the following notation $g = \sqrt{\sum_{i=1}^{d} \gamma_i^2}$.

Proposition 3.7. Any law invariant coherent risk measure $\rho$ can be approximated by

$$\bar{\rho}(\omega^T X) = -x^T \mu_0 + \sqrt{x^T x} \left[ \frac{1 - \cos[\theta(x, \gamma_0)]}{2} h(-g) + \frac{1 + \cos[\theta(x, \gamma_0)]}{2} h(g) \right],$$

(47)

where $\theta[x, \gamma_0]$ is the angle between $x^T = \omega^T A$ and $\gamma_0$.

Proof. We have $\rho(x^T Y) = -x^T \mu_0 + \sqrt{x^T x} h(a)$ from Lemma 3.2. We approximate $h(a)$ by the line that passes through $(-g, h(-g))$ and $(g, h(g))$. The equation of this line is

$$y(a) - h(-g) = \frac{h(-g) - h(g)}{2g}(a + g) = \frac{h(-g) + h(g)}{2} - \frac{h(-g) + h(g)}{2} \cos[\theta(x, \gamma_0)].$$

Replacing $h(a)$ by $y(a)$ in the expression for $\rho(x^T Y)$ above and combining the terms $h(-g)$ and $h(g)$ gives (47).

Remark 3.8. The precision of the approximation (47) clearly depends on the properties of the risk measure $\rho$. More specifically, the properties of $h(a)$ in (43) determine the degree of accuracy of our approximation. The only information on $h(a)$ that we know is its continuity, convexity, and the decreasing property as stated in Lemma 3.4. These are not sufficient to study the accuracy of the approximation for this proposition. However, our extensive numerical tests show that when $g$ is relatively small, which is usually the case for EM estimates of the NMVM models from empirical data, these approximations work pretty well.

The above proposition simplifies the computations of optimal portfolios considerably, as one only needs to evaluate $h(a)$ at the two points $-||\gamma_0||$ and $-||\gamma_0||$.

Remark 3.9. If $\gamma = 0$, then $\gamma_0 = 0$ and the right-hand side of (47) reduces to

$$-x^T \mu_0 + \sqrt{x^T x} \rho(\sqrt{ZN}).$$

(48)

The expression (48) is exactly the formula for risk measures for elliptical distributions, as discussed in [Landsman & Valdez] (2003).
Remark 3.10. We remark that in optimization problems like\[ \min_{\omega} \rho(\omega^T X), \]
we can optimize \( \bar{\rho}(\omega^T X) \) instead and obtain an approximately optimal portfolio. Observe that \( \bar{\rho}(x^T Y) \) can also be written as
\[
\bar{\rho}(x^T Y) = -x^T \mu_0 - \frac{h(-g) - h(g)}{2||\gamma_0||} x^T \gamma_0 + \frac{h(g) + h(-g)}{2} \sqrt{x^T x}. \tag{49}
\]
Here, \( \frac{h(-g) - h(g)}{2||\gamma_0||} \) is a positive number, as \( h(a) \) is a decreasing function, as stated in Lemma 2.8. The minimization and maximization of functions of the form (49) were discussed in Landsman (2008) in detail, see also Owadally (2011) and Landsman & Makov (2016).

Remark 3.11. We should mention that (47) was obtained under the assumption that the portfolio space is the whole of \( \mathbb{R}^n \). For problems associated with some small portfolio space, the value of \( ||\gamma_0|| \cos[\theta(x, \gamma_0)] \) does not cover the whole interval \( [||\gamma_0||, -||\gamma_0||] \). Therefore, in such cases, (47) needs to be adjusted appropriately.

As mentioned earlier, calculating the VaR and CVaR needs numerical procedures or Monte Carlo approaches for most models of asset returns. In the past, calculations of the VaR relied on linear approximations of the portfolio risks, see, e.g., Duffie & Pan (1997) and Jorion (1996), or Monte Carlo simulation-based tools, see, e.g., Uryasev (2000), Bucay & Rosen (1999), and Pritsker (1997). Next, we present closed form approximations for them, as a corollary to Proposition 3.7.

Acerbi and Tasche (2010) defines
\[ \text{VaR}_\alpha(X) = -q^\alpha(X) \]
of \( X \), i.e., \( \text{VaR}_\alpha(X) = -q^\alpha(X) \). When the random variable \( X \) has a positive probability density function, we have \( q^\alpha(X) = F_X^{-1}(\alpha) \) and so \( \text{VaR}_\alpha(X) = -F_X^{-1}(\alpha) \). With this definition, VaR is a positive homogeneous monetary risk measure. The conditional value at risk is defined by \( \text{CVaR}_\alpha(X) = -E[X|X \leq -\text{VaR}_\alpha] \). Then, for any portfolio \( \omega \), the CVaR of the loss \(-\omega^T X\) and return \( \omega^T X \) are given by
\[
\text{CVaR}_\beta(-\omega^T X) = \frac{1}{\beta} \int_{-\infty}^{+\infty} y f_{\omega^T}(y) dy, \quad \text{CVaR}_\beta(\omega^T X) = -\frac{1}{\beta} \int_{F_{\omega^T}^{-1}(\beta)}^{+\infty} y f_{\omega^T}(y) dy \tag{50}
\]
respectively. When \( X \sim N(\mu, \sigma^2) \), a straightforward calculation shows that these risk measures have the closed form expressions
\[
\text{VaR}_\alpha(X) = -\mu - \sigma \Phi^{-1}(\alpha), \quad \text{CVaR}_\alpha(X) = -\mu + \sigma \frac{e^{-[\Phi^{-1}(\alpha)]^2/2}}{\alpha \sqrt{2\pi}}, \tag{51}
\]
where \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal random variable. When \( X \) is elliptically distributed, one can also express these risk measures in closed form, as in equation (2) of Landsman and Valdez (2000).
Below, we discuss this risk measure for the class $Y_a = aZ + \sqrt{Z}N$ of random variables as this is sufficient to obtain expressions for these risk measures for general NMVM models, due to (42).

Denoting the probability density function of $Z$ by $g(s)$, the probability density function of $Y_a$ is given by

$$g_a(y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} g(s) \frac{1}{\sqrt{s}} e^{-\frac{(y-as)^2}{2s}} ds. \quad (52)$$

If $Z \sim GIG(\lambda, \chi, \psi)$, then $Y_a$ has the following density function

$$f_a(y) = \frac{(\sqrt{\chi/\lambda})^\lambda (\psi + a^2)^{\frac{\lambda}{2} - \lambda}}{2\pi K_\lambda(\sqrt{\chi/\lambda})} \times \frac{K_{\lambda-\frac{1}{2}}(\sqrt{(\chi + y^2)(\psi + a^2)}) e^{\psi y}}{(\sqrt{(\chi + y^2)(\psi + a^2)})^{\frac{\lambda}{2} - \lambda}}, \quad (53)$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind. Write $y_\beta(a) =: VaR_\beta(Y_a)$ and note that $-y_\beta(a)$ is the $\beta$ quantile of $Y_a$. By using the definitions of VaR and CVaR, we obtain the following relations:

**Lemma 3.12.** We have

$$\beta = \int_{0}^{+\infty} g(s) \Phi\frac{-y_\beta(a) - as}{\sqrt{s}} ds, \quad (54)$$

and

$$CVaR_\beta(Y_a) = -\frac{1}{\beta} \int_{0}^{+\infty} \left[ as \Phi\frac{-y_\beta(a) - as}{\sqrt{s}} - \sqrt{s} \frac{1}{2\pi} e^{-\frac{(y_\beta(a) + as)^2}{2s}} \right] g(s) ds, \quad (55)$$

where $\Phi$ is the cumulative distribution function of the standard normal random variable.

**Proof.** By the definition of $y_\beta(a)$, we have $P(Y_a \leq -y_\beta(a)) = \beta$. We have $P(Y_a \leq -y_\beta(a)) = \int_{0}^{+\infty} P(N \leq \frac{-y_\beta(a) - as}{\sqrt{s}}) g(s) ds = \int_{0}^{+\infty} \Phi\left(\frac{-y_\beta(a) - as}{\sqrt{s}}\right) g(s) ds$. This completes the proof of (54). To obtain (55), note that $CVaR_\beta(Y_a) = -E[Y_a / Y_a \leq -y_\beta(a)] = -\frac{E[Y_a 1_{Y_a \leq -y_\beta(a)}]}{P(Y_a \leq -y_\beta(a))}$. We have $P(Y_a \leq -y_\beta(a)) = \beta$ by the definition of $y_\beta(a)$. The numerator equals

$$E[Y_a 1_{Y_a \leq -y_\beta(a)}] = \int_{0}^{+\infty} E[N(\mu, s) 1_{N(\mu, s) \leq -y_\beta(a)}] g(s) ds.$$

We can easily calculate $E[N(\mu, s) 1_{N(\mu, s) \leq -y_\beta(a)}] = as \Phi\frac{-y_\beta(a) - as}{\sqrt{s}} - \sqrt{s} \frac{1}{2\pi} e^{-\frac{(y_\beta(a) + as)^2}{2s}}$ and this completes the proof.

Now, if we apply (42) to the risk measures VaR and CVaR, we obtain

$$VaR_\beta(\omega^T X) = Var_\beta(x^T Y) = -x^T \mu_0 + VaR_\beta(Y_a) \sqrt{x^T x}$$

$$= -x^T \mu_0 + y_\beta(a) \sqrt{x^T x}, \quad (56)$$

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and
\[
CVaR_\beta(\omega^T X) = CVaR_\beta(x^T Y) = -x^T \mu_0 + CVaR_\beta(Y_0) \sqrt{x^T x},
\] (57)

where \( x^T = \omega^T A \), \( a = ||\gamma_0|| \cos(x, \gamma_0) \), and \( y_0 \) satisfies (54). Therefore, optimization problems like \( \min_D CVaR_\beta(\omega^T X) \) or \( \min_D VaR_\beta(\omega^T X) \) for some domain \( D \) of portfolios involve computing (54) or (55) for each \( x^T = \omega^T A \).

Below, we apply Proposition 2.8 and obtain simpler expressions for the VaR and CVaR.

**Theorem 3.13.** The VaR_\beta(\cdot) and CVaR_\beta(\cdot) can be approximated by the following V_\beta(\cdot) and CV_\beta(\cdot)

\[
V_\beta(\omega^T X) =: -x^T \mu_0 + \sqrt{x^T x} \left[w_+ + w_- \cos[\theta(x, \gamma_0)]\right],
\] (58)

and
\[
CV_\beta(\omega^T X) =: -x^T \mu_0 + \sqrt{x^T x} \left[v_+ + v_- \cos[\theta(x, \gamma_0)]\right],
\] (59)

where \( x = \omega^T A \) and the constants \( w_+, w_-, v_+, v_- \), are given by
\[
w_+ = \frac{VaR_\beta(Y_b) + VaR_\beta(Y_{-b})}{2}, \quad v_+ = \frac{CVaR_\beta(Y_b) + CVaR_\beta(Y_{-b})}{2},
\]
\[
w_- = \frac{VaR_\beta(Y_b) - VaR_\beta(Y_{-b})}{2}, \quad v_- = \frac{CVaR_\beta(Y_b) - CVaR_\beta(Y_{-b})}{2},
\] (60)

where \( Y_b = bZ + \sqrt{Z}N \) and \( Y_{-b} = -bZ + \sqrt{Z}N \) with \( b = ||\gamma_0|| \).

**Proof.** The proof follows from Proposition 47. \( \square \)

We remark that optimization problems like \( \min_R CVaR_\beta(\omega^T X) \) or \( \min_R VaR_\beta(\omega^T X) \) can be replaced by \( \min_R V_\beta(\omega^T X) \) and \( \min_R V_\beta(\omega^T X) \) respectively by using the expressions in (58) and (59). The latter ones are computationally much efficient as they involve the values of the VaR and CVaR of \( Y_b = bZ + \sqrt{Z}N \) and \( Y_{-b} = -bZ + \sqrt{Z}N \) only for \( b = ||\gamma_0|| \).

### 4 Numerical results

In this section, we numerically check the performance of our results. First, we fit the GH distribution to empirical data of five stocks by using the EM algorithm. For this, we use five years of price history of the stocks AMD, CZR, ENPH, NVDA, and TSLA from the 2nd of January 2015 to the 30th of December 2020. The following table gives a summary of this data.

We apply the modified EM scheme to fit the daily log-returns of these stocks to 5-dimensional GH distributions. The algorithm is called multi-cycle, expectation, conditional estimation (MCECM)
Table 1: Data Describe

| Name | AMD    | CZR    | ENPH   | NVDAC  | TSLA   |
|------|--------|--------|--------|--------|--------|
| mean | 0.003121 | 0.002738 | 0.003218 | 0.002568 | 0.002432 |
| std  | 0.040087 | 0.04008 | 0.056228 | 0.028707 | 0.034737 |
| min  | -0.242291 | -0.37505 | -0.373656 | -0.187559 | -0.210628 |
| max  | 0.522901 | 0.441571 | 0.424446 | 0.298067 | 0.198949 |

procedure, see McNeil et al. (2015) Meng & Rubin (1993) for the details of this algorithm. First, we fit the return data to the model $\gamma Z + \sqrt{Z \Lambda Z_n}$. In this case, the estimated parameters for $Z$ are

$$\lambda = -\frac{1}{2},$$

$$\chi = 0.87953198,$$

$$\psi = 0.645169932,$$

and

$$\gamma = \begin{bmatrix} 0.00268318, 0.00147543, 0.00273905, 0.00145453, 0.00180711 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix}
0.001341 & 0.000253 & 0.000398 & 0.000529 & 0.000333 \\
0.000253 & 0.001034 & 0.0003 & 0.00025 & 0.000269 \\
0.000398 & 0.0003 & 0.00285 & 0.000274 & 0.000321 \\
0.000529 & 0.00025 & 0.000274 & 0.000675 & 0.000311 \\
0.000333 & 0.000269 & 0.000321 & 0.000311 & 0.00109
\end{bmatrix}.$$

With these parameters, we have $EZ = \frac{\chi}{\psi}, \text{Var}(Z) = \frac{\chi^2}{\psi^2}, m_3(Z) = \frac{3\chi}{\psi^3}$. We also have $m = \gamma_0 EZ = \frac{\chi}{\psi} \gamma_0, e_A = \Sigma^{\frac{1}{2}}e$. As in Zhao et al. (2015), we consider returns $r \in [0, 0.2]$. We take $r_i = \frac{0.2}{100}i, i = 0, 1, 2, \ldots, 100$. For each $r_i$, we calculate $x^i, i = 1, 2, \ldots, 20$, from $x^i = s_im + t_ie_A$, where $s_i$ and $t_i$ are given by (59). Then for each $i$, we calculate $\text{SKew}((x^i)^TY)$ from (17), $\text{CVaR}_\beta((x^i)^TY)$ from (50) and we plot $(\text{SKew}((x^i)^TY), r_i, \text{CVaR}_\beta((x^i)^TY))$ in three-dimensional space. Table 2 gives a summary of the optimal portfolios and the corresponding skewness. Figure 1 is the mean-CVaR-skewness efficient frontier.
In the second case, we fit the return data of the above listed five stocks to the model

\[ X = \mu + \gamma Z + \sqrt{Z} N_n. \]

The following table lists the numerical values for the parameters of our fit.

| Expect Return | 0.0020 | 0.0022 | 0.0024 | 0.0027 | 0.0029 |
|---------------|--------|--------|--------|--------|--------|
| \( \omega_1 \) | 0.077077 | 0.194069 | 0.31106 | 0.428051 | 0.545042 |
| \( \omega_2 \) | 0.252863 | 0.22433 | 0.195798 | 0.167265 | 0.138732 |
| \( \omega_3 \) | 0.067729 | 0.101723 | 0.135716 | 0.169709 | 0.203703 |
| \( \omega_4 \) | 0.399764 | 0.26734 | 0.134915 | 0.00249 | -0.12994 |
| \( \omega_5 \) | 0.202566 | 0.212539 | 0.222512 | 0.232485 | 0.242458 |
| Skewness       | 0.34231 | 0.370487 | 0.383957 | 0.385706 | 0.380047 |
\[ \lambda = -0.378655004, \]
\[ \chi = 0.379275063, \]
\[ \psi = 0.371543387, \]

The values of \( \gamma \) and \( \Sigma \) are as follows.

\[ \mu = [0.00041332, 0.00152207, 0.00058012, 0.00156685, 0.0006603], \]
\[ \gamma = [0.00163631, 0.00073499, 0.00159418, 0.000605, 0.00107086], \]
\[ \Sigma = \begin{bmatrix}
0.001341 & 0.000253 & 0.000398 & 0.000529 & 0.000333 \\
0.000253 & 0.001034 & 0.0003 & 0.000259 & 0.000321 \\
0.000398 & 0.0003 & 0.00285 & 0.000274 & 0.000311 \\
0.000529 & 0.000259 & 0.0003 & 0.000269 & 0.000333 \\
0.000333 & 0.000269 & 0.0003 & 0.000311 & 0.00109
\end{bmatrix}. \]

By using the definitions of \( \mu_0 \) and \( \gamma_0 \), we calculate them as follows.

\[ \mu_0 = [-0.0064934, 0.0357838, 0.0046229, 0.0524977, 0.0067647], \]
\[ \gamma_0 = [0.0339663, 0.0125813, 0.0211417, 0.0018441, 0.0207596]. \]

Table 3 tests the performance of our Theorem 3.12. The value of the VaR is calculated by a numerical root finding method and also by using Theorem 3.12. It can be seen that the results match very well.

| Portfolio Weights | \( \beta = 0.1 \) | \( \beta = 0.05 \) | \( \beta = 0.01 \) | \( \beta = 0.1 \) | \( \beta = 0.05 \) | \( \beta = 0.01 \) |
|-------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| [0.1, 0.4, 0.2, 0.1, 0.2] | 0.027729 | 0.042165 | 0.082055 | 0.029828 | 0.044277 | 0.084206 |
| [0.2, 0.1, 0.5, 0.1, 0.1] | 0.038508 | 0.058196 | 0.112613 | 0.040006 | 0.059714 | 0.114193 |
| [0.1, 0.4, 0.1, 0.3, 0.1] | 0.02568 | 0.039183 | 0.076513 | 0.02816 | 0.041678 | 0.079054 |
| [0.3, 0.1, 0.3, 0.1, 0.2] | 0.03131 | 0.047384 | 0.091771 | 0.032775 | 0.048857 | 0.093264 |
| [0.1, 0.3, 0.1, 0.3, 0.2] | 0.025091 | 0.038256 | 0.074639 | 0.027397 | 0.040574 | 0.076994 |
In Table 4 we compare the numerical calculation of the CVaR with the performance of the approximation of the CVaR in Theorem 3.12. Again, it can be seen that the results match very well.

| Portfolio Weights | CVaR $\beta = 0.1$ | CVaR $\beta = 0.05$ | CVaR $\beta = 0.01$ | CV $\beta = 0.1$ | CV $\beta = 0.05$ | CV $\beta = 0.01$ |
|-------------------|---------------------|---------------------|---------------------|------------------|------------------|------------------|
| [0.1, 0.4, 0.2, 0.1, 0.2] | 0.050643 | 0.067315 | 0.111296 | 0.050654 | 0.067341 | 0.111366 |
| [0.2, 0.1, 0.5, 0.1, 0.1] | 0.069764 | 0.092505 | 0.152508 | 0.0698 | 0.092567 | 0.15264 |
| [0.1, 0.4, 0.1, 0.3, 0.1] | 0.04712 | 0.062719 | 0.103883 | 0.047136 | 0.062755 | 0.103972 |
| [0.3, 0.1, 0.3, 0.1, 0.2] | 0.056816 | 0.075369 | 0.124296 | 0.056815 | 0.075376 | 0.124327 |
| [0.1, 0.3, 0.1, 0.3, 0.2] | 0.04599 | 0.061195 | 0.10131 | 0.046 | 0.06122 | 0.101378 |

5 Conclusion

Zhao et al. (2015) showed that mean-CVaR-skewness portfolio optimization problems based on asymmetric Laplace distributions can be transformed into quadratic optimization problems. In this note, we extended their result and showed that mean-risk-skewness portfolio optimization problems based on a larger class of NMVM models can also be transformed into quadratic optimization problems under any law invariant risk measure. The critical step to achieve this was to transform the original portfolio space into another space by an appropriate linear transformation, a step which enabled us to express both the risk and skewness as functions of a single variable. By showing that any law invariant coherent risk measure is an increasing function and skewness is a decreasing function of this variable, we were able to transform the original optimization problem into a quadratic optimization problem as in Zhao et al. (2015). In the rest of this paper, we made use of this transformation to come up with approximate closed form formulas for law invariant risk measures and hence also for the VaR and CVaR. Our numerical tests show that such closed form approximations are accurate.

References

Aas, K. & Haff, I. H. (2006). The generalized hyperbolic skew Student’s t-distribution. Journal of Financial Econometrics, 4, 275–309

Acerbi, C. & Tasche, D. (2002). On the coherence of expected shortfall. Journal of Banking & Finance, 26, 1487–1503

Akturk, T. D. & Ararat, C. (2020). Portfolio optimization with two coherent risk measures. Journal of Global Optimization, 597–626
Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance, 9*, 203–228

Bali, T. G. & Cakici, N. (2004). Value at risk and expected stock returns. *Financial Analysts Journal, 60*, 57–73

Barndorff-Nielsen, O. E. (1997). Processes of normal inverse Gaussian type. *Finance and Stochastics, 2*, 41–68

Bingham, N. H. & Kiesel, R. (2001). Modelling asset returns with hyperbolic distributions. Return Distributions in Finance, 1–20. Elsevier

Bollerslev, T. & Todorov, V. (2011). Tails, fears, and risk premia. *The Journal of Finance, 66*, 2165–2211

Bucay, N. & Rosen, D. (1999). Credit risk of an international bond portfolio: A case study. *ALGO Research Quarterly, 2*, 9–29

Chekhlov, A., Uryasev, S., & Zabarankin, M. (2005). Drawdown measure in portfolio optimization. *International Journal of Theoretical and Applied Finance, 8*, 13–58

Cont, R. & Tankov, P. (2004). Nonparametric calibration of jump-diffusion option pricing models. *The Journal of Computational Finance, 7*, 1–49

Dana, R. A. (2005). A representation result for concave Schur concave functions. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 15*, 613–634

Duffie, D. & Pan, J. (1997). An overview of value at risk. *Journal of Derivatives, 4*, 7–49

Eberlein, E. & Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli, 281–299

Föllmer, H. & Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics, 6*, 429–447

Frittelli, M. & Gianin, E. R. (2002). Putting order in risk measures. *Journal of Banking & Finance, 26*, 1473–1486

Frittelli, M. & Gianin, E. R. (2005). Law invariant convex risk measures. Advances in Mathematical Economics, 33–46. Springer

Hammerstein, E. (2010). Generalized hyperbolic distributions: Theory and applications to CDO pricing. Ph.D. thesis
Heath, D. (2000). Back to the future. Plenary lecture. First World Congress of the Bachelier Finance Society, Paris

Hellmich, M. & Kassberger, S. (2011). Efficient and robust portfolio optimization in the multivariate generalized hyperbolic framework. *Quantitative Finance*, 11, 1503–1516

Jorion, P. (1996). Risk2: Measuring the risk in value at risk. *Financial Analysts Journal*, 52, 47–56

Kolm, P. N., Tütüncü, R., & Fabozzi, F. J. (2014). 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research*, 234, 356–371

Konno, H., Shirakawa, H., & Yamazaki, H. (1993). A mean–absolute deviation–skewness portfolio optimization model. *Annals of Operations Research*, 45, 205–220

Konno, H. & Suzuki, K. (1995). A mean–variance–skewness portfolio optimization model. *Journal of the Operation Research Society of Japan*, 38

Kozubowski, T. J. & Podgorcki, K. (2001). Asymmetric Laplace laws and modeling financial data. *Mathematical and Computer Modelling*, 34, 1003–1021

Kozubowski, T. J. & Rachev, S. T. (1994). The theory of geometric stable distributions and its use in modeling financial data. *European Journal of Operational Research*, 74, 310–324

Kusuoka, S. (2001). On law invariant coherent risk measures. Advances in Mathematical Economics, 83–95. Springer

Landsman, Z. (2008). Minimization of the root of a quadratic functional under a system of affine equality constraints with application to portfolio management. *Journal of Computational and Applied Mathematics*, 220, 739–748

Landsman, Z. & Makov, U. (2016). Minimization of a function of a quadratic functional with application to optimal portfolio selection. *Journal of Optimization Theory and Applications*, 308–322

Landsman, Z. M. & Valdez, E. A. (2003). Tail conditional expectations for elliptical distributions. *North American Actuarial Journal*, 7, 55–71

Lo, A. W. & MacKinlay, A. C. (1997). Maximizing predictability in the stock and bond markets. *Macroeconomic Dynamics*, 1, 102–134

Madan, D. B. & Seneta, E. (1990). The variance gamma (VG) model for share market returns. *Journal of Business*, 511–524
Malevergne, Y. & Sornette, D. (2006). Extreme financial risks: From dependence to risk management. Springer-Verlag

Markowitz, H. M. (1959). Portfolio Selection: Efficient Diversification of Investments, volume 16

McNeil, A. J., Frey, R., & Embrechts, P. (2015). Quantitative Risk Management: Concepts, Techniques and Tools–Revised Edition. Princeton University Press

Meng, X. L. & Rubin, D. B. (1993). Maximum likelihood estimation via the ECM algorithm: A general framework. *Biometrika*, 80, 267–278

Mittnik, S. & Rachev, S. T. (1993). Modeling asset returns with alternative stable distributions. *Econometric Reviews*, 12, 261–330

Owadally, I. (2011). An improved closed-form solution for the constrained minimization of the root of a quadratic functional. *Journal of Computational and Applied Mathematics*, 4428–4435

Owen, J. & Rabinovitch, R. (1983). On the class of elliptical distributions and their applications to the theory of portfolio choice. *The Journal of Finance*, 38, 745–752

Prause, K. *et al.* (1999). The generalized hyperbolic model: Estimation, financial derivatives, and risk measures. Ph.D. thesis

Pritsker, M. (1997). Evaluating value at risk methodologies: Accuracy versus computational time. *Journal of Financial Services Research*, 12, 201–242

Rachev, S. T., Stoyanov, S. V., Biglova, A., & Fabozzi, F. J. (2005). An empirical examination of daily stock return distributions for US stocks. Data Analysis and Decision Support, 269–281. Springer

Rockafellar, R. T. & Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of risk*, 2, 21–42

Rockafellar, R. T. & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26, 1443–1471

Schoutens, W. (2003). Lévy processes in finance: Pricing financial derivatives. Wiley Online Library

Shi, X. & Kim, Y. S. (2021). Coherent risk measures and normal mixture distributions with applications in portfolio optimization. *International Journal of Theoretical and Applied Finance (IJTAF)*, 24, 1–18

Shirvani, A., Rachev, T. S., & Fabozzi, F. J. (2021a). Multiple subordinated modeling of asset returns: Implications for option pricing. *Rev Quant Finan Acc*, 156, 1329–1342
Shirvani, A., Stoyanov, S., & Fabozzi, F. e. a. (2021b). Equity premium puzzle or faulty economic modelling? *Rev Quant Finan Acc*, 156, 1329–1342

Uryasev, S. (2000). Conditional value-at-risk: Optimization algorithms and applications. Proceedings of the IEEE/IAFE/INFORMS 2000 Conference on Computational Intelligence for Financial Engineering (CIFEr) (Cat. No. 00TH8520), 49–57. IEEE

Zhao, S., Lu, Q., Han, L., Liu, Y., & Hu, F. (2015). A mean-CVaR-skewness portfolio optimization model based on asymmetric Laplace distribution. *Annals of Operations Research*, 226, 727–739