Cosmic Structure Formation

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Abstract

This article reviews the prevailing paradigm for how galaxies and larger structures formed in the universe: gravitational instability. Basic observational facts are summarized to motivate the standard cosmological framework underlying most detailed investigations of structure formation. The observed universe approaches spatial uniformity on scales larger than about $10^{26}$ cm. On these scales gravitational dynamics is almost linear and therefore relatively easy to relate to observations of large-scale structure. On smaller scales cosmic structure is complicated not only by nonlinear gravitational clustering but also by nonlinear nongravitational gas dynamical processes. The complexity of these phenomena makes galaxy formation one of the grand challenge problems of the physical sciences. No fully satisfactory theory can presently account in detail for the observed cosmic structure. However, as this article summarizes, significant progress has been made during the last few years.

Keywords: astrophysics, cosmology, gravitation, nonlinear dynamics

1 Introduction: Basic Cosmological Facts and Principles

If the universe began in a state of near-perfect homogeneity and isotropy, then how did it become so inhomogeneous on small scales? This is the puzzle facing cosmologists sorting through the fossil relics of the early universe: the cosmic microwave background radiation, the chemical elements, the mass both visible and invisible, and the complex patterns — galaxies, clusters and superclusters of galaxies, voids, filaments, and fluctuations — that organize these ingredients. In the following we shall examine these fossils and the models that are being used to try to explain their origin.
Before getting to details let us size up the problem by considering the length scales under investigation. I shall use cgs units; those who prefer may translate to parsecs or light-years (1 pc = 3.09 \times 10^{18} \text{ cm} = 3.26 \text{ lt-yr}), SI units, or anything else. Starting with the familiar, we note that the solar system, defined by the major axis of Pluto’s orbit, is almost $10^{15} \text{ cm}$ in extent. It lies a distance of $2 \times 10^{21} \text{ cm}$ from the center of our Galaxy. Our Galaxy is part of the Local Group of galaxies; the nearest galaxy as large as our own is M31, the Andromeda nebula, at a distance of $2 \times 10^{24} \text{ cm}$. The Local Group is about $5 \times 10^{25} \text{ cm}$ from the Virgo Cluster, which lies at the center of the Local Supercluster of galaxies.

Cosmic structures are arranged, almost hierarchically — like a fractal on small scales, although not large scales — up to a size not much larger than the Local Supercluster. For comparison, the radius of the presently observable universe is about $10^{28} \text{ cm}$, i.e., $10^{10} \text{ lt-yr}$ (assuming that the age of the universe is about $10^{10} \text{ yrs}$). Astrophysical cosmologists seek to understand structure on scales from roughly $10^{22}$ to $10^{28} \text{ cm}$ [93, 87].

1.1 Five Observations about the Universe

Models of structure formation must take into account basic empirical properties of the universe averaged over large scales. Five sets of empirical facts seem especially relevant:

1. **The isotropy of distant objects.** A standard statistical measure of anisotropy (deviations from spherical symmetry around us) is the angular 2-point correlation function $w(\theta)$, giving the relative excess number of pairs of objects separated by angle $\theta$ compared with the mean number for a Poisson distribution. For $\theta$ in the range of 1 to $3^\circ$, $|w| \lesssim 10^{-3}$ for faint radio sources (primarily distant galaxies and quasars) [82]. For $\theta = 10^\circ$, $w < 10^{-4}$ for X-ray sources (fig. 10 of ref. [64]), which are also primarily distant galaxies and quasars. The cosmic microwave background radiation, after subtraction of a dipole (cosine) variation over the whole sky, shows fluctuations of rms amplitude $1.1 \times 10^{-5}$ [108] averaged over circular patches of size $10^\circ$.

2. **Hubble’s linear velocity-distance relation.** The celebrated discovery of cosmic expansion was made by Hubble in 1929 [62]. Galaxies shine by starlight (with some emission from rarefied gas) and thereby display well-known spectral lines owing to radiative transitions between quantum states of abundant chemical elements. The wavelengths of these lines are found to be shifted relative to their laboratory values, generally to larger values, in rough proportion to the distance $r$: $c\Delta \lambda/\lambda \equiv cz \approx H_0r$, where $c$ is the speed of light and $H_0$ is the Hubble constant, $H_0 = h/(10^{10} \text{ yr})$, with $h = 0.75 \pm 0.25$. The linear relation is modified for distances so large that $H_0r/c$ approaches 1. The Doppler interpretation of cosmological
Table 1: Abundances (mass fractions) of the light nuclei.

| Nuclide | Solar     | Primordial |
|---------|-----------|------------|
| $^1$H   | 0.73      | 0.76       |
| $^2$H   | $2.3 \times 10^{-4}$ | $\sim 3 \times 10^{-5}$ |
| $^3$He  | $5.0 \times 10^{-5}$ | $\sim 2 \times 10^{-5}$ |
| $^4$He  | 0.25      | 0.24       |
| $^7$Li  | $6.5 \times 10^{-9}$ | $\sim 10^{-10}$ |
| C,N,O   | 0.017     | $\ll 10^{-4}$ |

redshifts is well established: $v = cz[1 + O(z)]$ is the recession speed. Of much greater difficulty are distance measurements \[98\]; until recently, the accuracy of relative distance measurements was limited to about 20%; absolute distances are even more uncertain. (This fact explains the large error bar on the dimensionless Hubble parameter $h$.)

3. \textbf{The cosmic microwave background radiation}. In 1965, Penzias & Wilson \[96\] discovered that the sky glows brightly and uniformly in the microwave at wavelengths of about 1 cm. In the more than 25 years since then, the spectrum (wavelength-dependence) and isotropy of this radiation have been found to match a blackbody (Planck spectrum) with temperature $T_0 = 2.73$ K. The \textit{Cosmic Background Explorer} satellite (COBE) has placed limits on the deviations from the Planck spectrum of less than $3 \times 10^{-4}$ relative to the peak intensity \[83\]. COBE also was used to make the discovery of anisotropy cited above.

4. \textbf{The abundances of light nuclei}. The most abundant nuclei in the universe are $^1$H and $^4$He. Abundances of the light nuclei are shown in Table 1 using data compiled by refs. \[1\], \[15\]. All heavier elements — primarily carbon, nitrogen, and oxygen — are grouped together as “metals” by astronomers. The relative abundances of the metals — but not the light nuclei — can be explained by nuclear fusion (“nucleosynthesis” in the jargon of astrophysicists) occurring in stars and supernovae \[19\]. In older stars the mass fraction of $^4$He is less than in the sun, but in no case is convincingly below 0.22. Nucleosynthesis in massive stars produces a much larger ratio of metals to He than 1:10, and stars effectively destroy $^2$H and Li. (Deuterium is probably enhanced in the solar system by chemical fractionation, while some Li can be produced by cosmic ray spallation.) Thus, the light nuclei could not have been produced in stars. The only satisfactory explanation known was proposed by Gamow, Alpher and Herman in the late 1940s \[17\], \[2\]: the light elements were produced by nucleosynthesis at relatively low temperatures (\(\lesssim 10^9\) K) and high densities for a duration of several tens of seconds.
5. **The existence of large amounts of dark matter.** In the 1930s it was recognized by Zwicky \[123\] that galaxies in clusters move too rapidly for the clusters to remain bound, assuming that the galaxies weigh no more than the visible stars and gas they contain. In the 1980s, Rubin and others \[99\] discovered that stars and gas clouds in the outskirts of spiral galaxies also orbit too quickly around the center to be held in place by the gravity of the visible matter. The simplest explanation is that there is unseen mass present in galaxies and clusters. A wide range of evidence supports the conclusion that most (perhaps 90-99\%) of the mass in the universe is much less luminous (per unit mass) than stars. Ordinary luminous matter dominates in the central regions of galaxies, with the dark matter forming an extended “halo” around the luminous parts. Unfortunately, little else is known about the dark matter. Three outstanding questions are: (1) What is the dark matter? (2) How much is there? (3) How is it distributed through space? Speculative answers have been given to all of these questions, e.g., the dark matter is some new type of elementary particle, abundant enough to just close the universe, and it is distributed somewhat more uniformly than galaxies. However, this is speculation. The dark matter problem posed by these three questions is currently one of the most outstanding puzzles in all of science.

### 1.2 Simple Cosmological Models

The five sets of facts summarized above underly the cosmological models considered tenable by astrophysicists. In particular, the first two items, large-scale isotropy and the Hubble expansion, motivate the *Cosmological Principle* introduced by Einstein and Milne \[84\]: The universe is approximately homogeneous and isotropic on large scales with a uniformly expanding mass distribution.

Spatial homogeneity is difficult to establish because we cannot travel to a distant galaxy to see whether from there the universe looks similar to our vicinity. However, it is a natural extension of the Copernican principle, which asserts that our vantage point is not special. If the universe is isotropic (in the large) around every point, then it is necessarily homogeneous. The available data are consistent with large-scale homogeneity.

Uniform expansion means that the galaxies separate with time with all distances scaling in proportion to a universal expansion scale factor \(a(t)\) where \(t\) is the proper time measured by observers in each galaxy. The galaxies themselves do not expand, nor do any other bound systems such as galaxy clusters (or, on a much smaller scale, the solar system). Actually, there are slight departures from perfectly uniform expansion even on large scales. Figure \[\] illustrates the concept of perturbed Hubble expansion.

There are several widespread misconceptions about the Hubble expansion. The first is associated with the question, “What is the universe expanding into?” It is not expanding *into* anything. The universe is all of space, so none is left to accommodate the expansion, nor is any more space necessary. Another misconception is that Hub-
Figure 1: Perturbed Hubble expansion.
ble expansion is a purely general relativistic effect due to the stretching of space. It is more accurate to say that galaxies are moving apart because they were set in motion by some initial mechanism; it then follows automatically that the distances between them increase with time. This Newtonian interpretation is valid for \( H_0 \rho/c \ll 1 \); general relativistic models simply extend the expansion to the universe as a whole. Finally, there is misunderstanding about how the universe can have a finite volume — a possibility that cannot be excluded (but neither is favored) by observations. If the volume is finite, we expect the space to be compact (like an ordinary 2-sphere but with one more dimension) and not embedded in anything else. Practically all cosmological models — including the ones discussed in this paper — are founded on general relativity (or some modified metric theory), which gives a precise description of space, time, and cosmological expansion.

Simple models of a homogeneous and isotropic, uniformly expanding universe were introduced during the period 1917–1940 by de Sitter, Friedman, Lemaître, Robertson, Milne, and others.\(^1\) Although Einstein proposed the first cosmological model as a solution to his field equations of general relativity in 1917, he assumed, incorrectly in hindsight, that the universe was static. To force his field equations to yield a static (as opposed to expanding or contracting) solution he added an extra term called the cosmological constant.

In all of these models except Einstein’s, all separations between objects scale with time in proportion to the universal scale factor \( a(t) \). Thus, the position of each galaxy relative to some origin (in fact, any location, such as the Earth, may be taken as origin) may be written \( \vec{r} = a(t)\vec{x} \), where \( \vec{x} \) is a constant vector for that galaxy, called the comoving position. The Hubble law follows at once: \( \vec{v} = d\vec{r}/dt = H\vec{r} \) where \( H(t) = d\ln a/dt \). This result implies that \( H \) need not be independent of time; only its present value, \( H_0 \), is called the Hubble constant.

We know that \( a(t) \) is presently increasing with time and will double in about \( 10^{10} \) years. Therefore it was smaller in the past, and may have been very small (possibly even zero) at some finite proper time in the past. In fact, points (iii) and (iv) of section 1.1 support the notion that about \( 1.5 \times 10^{10} \) years ago, the expansion scale factor was very much smaller than it is now, and that the universe began expanding tremendously rapidly in an event that has come to be called the “big bang.” The reasoning is simple: the temperature of an expanding gas (such as fills the universe) decreases adiabatically if there is no heat input. The heat content of the microwave background radiation is far too large to have been produced at low energies except in highly contrived models \[94, 3\]. Therefore the mass and radiation in the universe must have been hotter and denser in the past.

When the temperature was above \( 10^{10} \) K, atomic nuclei were dissociated into protons and neutrons. The universe expanded and cooled rapidly, requiring only a few minutes to cool through the era of cosmic nucleosynthesis. About \( 3 \times 10^5 \) years later the temperature dropped below 3500 K, the temperature at which hydrogen ionizes at cosmic density. The

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\(^1\) Many of the key papers appear, in English, in ref. 5.
hot gas filling space glowed bright red at this time with a Planck spectrum because the gas and radiation were in thermal equilibrium — the radiation was scattered, absorbed, and reemitted rapidly by the plasma. After this time the protons and electrons joined to form neutral hydrogen gas, which is almost completely transparent to radiation. At the present time we look out in distance, and therefore back in time, to see the radiation left over from this “recombination” era. Because of the large distance and relativistic recession velocity, the radiation is redshifted by a factor of about 1100, so that we detect it as the microwave background radiation. The big bang theory accurately predicts both the nuclear abundances and nearly perfect Planck spectrum and isotropy of the cosmic microwave background radiation in an approximately homogeneous and isotropic expanding universe.

To my knowledge, no alternative to the big bang has been able to account for points (i)–(iv) in section 1.1. Therefore I shall assume the basic scenario outlined in the previous paragraph. This does not require me to ask what preceded or caused the big bang. Those remain metaphysical questions in the absence of any empirical data.

The basic laws of a uniformly expanding universe obeying general relativity theory were first set out by Friedman in 1922 [46]. Just as one would expect from Newtonian ideas, the cosmic expansion decelerates due to gravity. Consider a cosmological model with uniform mass density \( \bar{\rho}(t) \) decreasing with time owing to the expansion. (The equivalence of mass and energy implies that \( \bar{\rho} \) includes all forms of mass and energy. For nonrelativistic matter \( \bar{\rho} \propto a^{-3} \) from mass conservation, while for a relativistic gas \( \bar{\rho} \propto a^{-4} \) because the energy decreases due to the work done by pressure during the expansion.) The expansion rate \( H = d \ln a / dt \) obeys the Friedman equation

\[
H^2(t) = \frac{8 \pi G}{3} \frac{\rho}{a^2(t)},
\]

where \( G \) is Newton’s constant and \( K \) is the cosmic curvature constant, related to the curvature of three-dimensional hypersurfaces of constant time in spatially homogeneous and isotropic (or Robertson-Walker) models. Euclidean space has \( K = 0 \), while a closed (compact) universe with finite volume (a three-sphere) has \( K > 0 \) and an open universe (a three-hyperboloid) has \( K < 0 \).

The Friedman equation relates the geometry of space to the mean mass density. By combining \( H \) and \( G \rho \) we can define a dimensionless density parameter \( \Omega \): \[
\Omega(t) \equiv \frac{\bar{\rho}(t)}{\rho_c(t)} \equiv \frac{8 \pi G \bar{\rho}(t)}{3H^2(t)}, \quad \rho_c(t_0) = 2 \times 10^{-29} h^2 \mathrm{g} \mathrm{cm}^{-3},
\]

where \( t_0 \) refers to the present. The Friedman equation now reads \( K = (\Omega - 1)(aH)^2 \), showing that the spatial curvature depends on the mean density of matter. The mean density also determines the evolution of \( a(t) \). For a gas with pressure \( p = \alpha pc^2 \) with \( \alpha > -\frac{4}{3} \) (e.g., \( \alpha = +\frac{1}{3} \) for a relativistic gas of photons while \( \alpha \approx 0 \) for nonrelativistic matter), the Friedman equation may be integrated to show that the universe will continue
expanding forever if $K \leq 0$ while it will eventually cease expanding and will recollapse if $K > 0$. Moreover, all of these models begin with a singularity $a = 0$ at some finite time in the past. Figure 2 illustrates the solutions for a matter-dominated universe with $p \ll \rho c^2$.

In past decades, astrophysicists took the Friedman models rather literally, and cosmology was widely regarded as a search for two numbers, $H_0$ and $\Omega_0 = \Omega(t_0)$. However, that time is past. We recognize that the universe need not be homogeneous and isotropic on scales much larger than the $10^{10}$ lt-yrs we can see (in principle) today and that there is no way to know if the Friedman models are globally correct. These questions, like the one of what preceded the big bang, are metaphysical. All that we know about the global spacetime geometry is that within our observable patch the universe looks, to a good approximation, like a Friedman model with $0 < \Omega_0 \sim 2$. Some observational cosmologists may go further and narrow the range of $\Omega_0$, or advocate a nonzero cosmological constant $\Lambda$, which adds a term $\Lambda/3$ to the right-hand side of the Friedman equation. However, there is no consensus on these issues [102, 22].

Interest in the value of $\Omega_0$ remains strong for several reasons. First, during the last 15 years astrophysicists have realized increasingly that large amounts of dark matter exist in halos surrounding galaxies, increasing estimates of $\Omega_0$ beyond those from the visible matter. Second, certain theories of the early universe make predictions for the value of $\Omega_0$, in particular, the inflation theory of Guth [56]. According to this theory, which supplements the standard big bang model, at very early times the cosmic expansion was accelerated tremendously by a temporary large cosmological constant, causing $a(t)$ to increase exponentially so that the curvature term in eq. (11) became strongly suppressed relative to the other terms. As a result, spatial curvature should be negligible today so that $\Omega_0 = 1$. (Any remaining cosmological constant term can be included in $\bar{\rho}$ as a constant “vacuum” energy density.) The inflation theory is attractive because it can explain the large-scale homogeneity and isotropy of the universe, but it remains speculative.

Direct observation of stars, with a reasonable extrapolation for faint and burned out stars, yields only $\Omega_0 \lessapprox 0.01$, far less than predicted by inflation. For cosmic nucleosynthe-
sis to produce the observed abundances of the light nuclei, the abundance of “baryonic” matter (made of protons, neutrons, and electrons) is relatively tightly constrained \[ \Omega_B h^2 = 0.0125 \pm 0.0025 \]. From the gravitating mass in galaxies and clusters of galaxies one infers \( \Omega_0 \sim 0.2 \) \[92, \ 113\], implying the existence of much more dark than luminous matter. However, these are all lower limits to the global mean value of \( \Omega_0 \), because dark matter may exist between galaxies and clusters. A recent large-scale measurement of the gravity field implies \( \Omega_0 > 0.3 \) \[36\].

Dark matter is a complication of simple big bang cosmology. Its nature is important for galaxy formation but only its abundance is important for homogeneous cosmology. Many searches for dark matter are presently underway \[55\], but it is possible that most of the dark matter interacts so weakly with ordinary matter and radiation as to be undetectable aside from gravitational effects. An alternative way to investigate dark matter is to test specific theories of dark matter for their predictions for the formation of galaxies and large-scale structure, which are sensitive to the gravitational clustering of dark matter. Before doing that, we shall discuss another important and possibly related fact about the universe: It is not perfectly homogeneous and isotropic.

### 1.3 The Perturbed Universe

Like Darwin’s theory of the evolution of species and Alfred Wegener’s theory of continental drift, the big bang theory is only a starting point for more detailed models of the universe. It is a framework that requires additional ideas for a complete and consistent physical cosmological theory. Chief among the missing ideas are those relating to departures from strict homogeneity and isotropy.

The universe is *extremely* inhomogeneous: the density of this paper (or the computer screen displaying these characters) is about 30 orders of magnitude denser than intergalactic space. This fact is not necessarily incompatible with the Cosmological Principle stated in section 1.2, because the inhomogeneity is much less when measured on larger scales (see Table 2). Nevertheless, it begs the question: How did these inhomogeneities develop?

Among the four known fundamental forces, gravity would seem, *a priori*, to be the most likely agent responsible for the formation of cosmic structure. After all, we know that gravity holds together the Earth and the solar system, and that the purely attractive
and long-range nature of gravity cause it to be more important on large scales than the other forces (electromagnetic, strong, and weak nuclear forces).

To test whether gravity might create structure it is useful to consider the evolution of small-amplitude perturbations of a homogeneous medium. As in plasma physics, optics, and other disciplines, we consider the propagation of waves in a uniform background, with relative density fluctuation \( \delta \rho / \bar{\rho} \propto \exp[i(\vec{k} \cdot \vec{x} - \omega t)] \). For simplicity we shall neglect the expansion of the universe, which is a good approximation over a short period of time if \( \omega^2 \gg H^2 \). Linearizing the equations of motion for the matter (here, approximated by the perfect fluid equations), one obtains the dispersion relation

\[
\omega^2 = k^2 c_s^2 - 4\pi G \bar{\rho} = c_s^2(k^2 - k_j^2), \quad k_j \equiv \left(\frac{4\pi G \bar{\rho}}{c_s^2}\right)^{1/2}.
\]

This result is similar to the dispersion relation for high-frequency electromagnetic waves in a plasma, where the sound speed \( c_s \) is replaced by the speed of light \( c \) and the term \(-4\pi G \bar{\rho}\) is replaced by the square of the plasma frequency, \( \omega_p^2 = +4\pi \bar{n}_e e^2/m_e \) for electrons of charge \( e \), mass \( m_e \), and number density \( \bar{n}_e \). Gravity differs from electricity in two essential ways. First, the gravitational “charge-to-mass” ratio (gravitational mass divided by inertial mass) is 1 for all objects as was discovered by Galileo almost 400 years ago. Second, gravity has the opposite sign: all masses attract each other. The sign difference is crucial: it leads to gravitational instability of long-wavelength fluctuations, as was first pointed out by Jeans nearly a century ago [66]. When \( k < k_j \) (\( k_j \) is known as the Jeans wavenumber), \( \omega^2 < 0 \) so that one of the two roots of the dispersion relation corresponds to exponential growth.

In reality, the growth is exponential only for a static medium with \( H = 0 \) and \( \bar{\rho} = \) constant. In an expanding universe \( \bar{\rho} \) decreases with time and therefore so does the growth rate. In this case the linear growth of perturbations is proportional to a power of \( t \) rather than being exponential.

But how do we know this instability is physical? After all, the other imaginary root of the dispersion relation for \( k < k_j \) leads to damping; perhaps the growing solution should be discarded.

Gravitational instability occurs because because it is energetically favorable for perturbations to grow in amplitude. For example, dense regions gain negative gravitational energy by collapsing, more than compensating for the increase in positive kinetic energy. Another way to view this process is that overdense regions in an \( \Omega = 1 \) universe are like small portions of closed (\( \Omega > 1 \)) universes; therefore they expand less rapidly than their surroundings and eventually collapse (fig. 2). Conversely, underdense regions are like small portions of open (\( \Omega < 1 \)) universes which expand more rapidly. The result is that matter is transferred from underdense to overdense regions. Figure 3 illustrates this process in a small cosmological N-body simulation beginning from a slightly perturbed zero-pressure Friedman model. The evolution of one two-dimensional layer of particles

\[
\omega^2 = k^2 c_s^2 - 4\pi G \bar{\rho} = c_s^2(k^2 - k_j^2), \quad k_j \equiv \left(\frac{4\pi G \bar{\rho}}{c_s^2}\right)^{1/2}.
\]
is shown; the total simulation used $16^3$ particles and integrated Newton's laws in three dimensions with periodic boundary conditions.

Granted that gravity leads to a linear instability — but perhaps this instability saturates at moderate amplitude like so many others do in the nonlinear regime. This is not so. In fact, gravitational instability strengthens with collapse because the gravitational energy of a given mass, roughly $-GM^2/R$, diverges as $R \to 0$. The instability ceases only after a gravitationally bound object forms with enough internal kinetic energy to support itself against further gravitational collapse, as in the case of the Earth orbiting the sun or the gas in the sun itself. Bertschinger & Jain [12] have recently proven that gravitational instability in a cold, initially homogeneous medium inevitably drives mass elements with density exceeding the critical density to collapse to arbitrarily high density, until pressure, vorticity, or other non-gravitational forces inhibit further collapse.

Dissipative processes drive self-gravitating systems to still higher density. Self-gravitating systems are peculiar in that they have negative specific heat and therefore no stable thermal equilibrium state in general. This behavior follows from the classical virial theorem [22], which relates the kinetic energy $K$ and the gravitational potential energy $W$ of an equilibrium self-gravitating system: $2K + W = K + E = 0$ where $E$ is the total energy. The specific heat is thus $\partial E/\partial K = -1$. If energy is removed from the system (by atomic radiative processes, for example), $E$ decreases but $K$ increases; the system shrinks and gains twice as much gravitational binding energy as it loses to radiation, requiring the kinetic energy to increase to maintain equilibrium.

2 Large-Scale Structure: Is Gravity Responsible?

In the preceding section we have outlined the main ingredients needed to investigate the formation of cosmic structure. Now we consider the evidence for and implications of cosmic large-scale structure, defined by the distribution of galaxies and dark matter on scales larger than about $10^{26}$ cm. Averaged over these scales, the number density of galaxies (and, we believe, the net mass density) is relatively smoothly varying, with fluctuations $|\delta \rho/\bar{\rho}| \approx 1$. On these scales gravitational instability is therefore still relatively mild so that we can hope to infer the primeval fluctuations and test specific theories for their origin and evolution. The main question we address is this: Is gravity responsible for the formation of large-scale structure?

In gravitational instability models the structure of spacetime is perturbed by small-amplitude fluctuations in the gravitational potential $\phi(\vec{x}, t)$. The description of this system in general relativity is not difficult. The spacetime geometry is given by the line element of a perturbed Robertson-Walker model:

$$ds^2 = -(1 + 2\phi/c^2)c^2 dt^2 + (1 - 2\phi/c^2)a^2(t)(dx_1^2 + dx_2^2 + dx_3^2). \quad (4)$$

[We are assuming $(\phi/c^2)^2 \ll 1$, the background spatial curvature $K$ is negligible on scales of interest, and gravitational radiation and other purely relativistic effects are
Figure 3: Gravitational instability in an expanding universe.
unimportant. These are very good approximations for the problem at hand.] In general relativity, as in special relativity, time and space are not independent; invariant distances can be defined only by combining all four coordinates. However, we do not use the same coordinate system as in special relativity. Besides including the gravitational potential in the line element, we factor out the cosmic expansion scale factor $a(t)$ from our coordinates $x_i$ — they are the comoving coordinates introduced in section 1.2.

The presence of the gravitational potential $\phi$ in eq. (4) represents the variations in the spacetime geometry caused by density fluctuations. (In general relativity theory, gravitational “forces” are caused by variations in the spacetime curvature. Freely falling particles follow geodesics — the “straightest” possible curves in a curved spacetime.) The perturbed Einstein field equations imply a cosmological version of the classical Poisson equation of Newtonian gravity (assuming nonrelativistic sources and length scales small compared with the Hubble distance $c/H$):

$$a^{-2} \nabla^2 \phi = 4\pi G \bar{\rho} \left( \frac{\delta \rho}{\bar{\rho}} \right).$$

(5)

The factor $a^{-2}$ is necessary because the spatial Laplacian is with respect to the comoving coordinates. The chief difference from Newton’s version is that the source is not the total mass density $\rho$; rather it is the density fluctuation $\delta \rho = \rho - \bar{\rho}$. This difference is insignificant within the solar system but is very important in cosmology. If the source were $\rho$ then $\phi$ would diverge in an infinite universe and the gravity field would not be well-defined. Newton recognized this problem but its solution had to wait for Einstein. If the density field approaches homogeneity on large scales, $\delta \rho$ has vanishing spatial average and $\phi$ is convergent. A rigorous proof requires full general relativity theory, but the result stated here is correct for the types of perturbations of Friedman models under consideration.

Note that $\delta \rho / \bar{\rho}$ may be large yet $\phi / c^2$ small. For a mass fluctuation of proper wavelength $\lambda$, the solution of eq. (5) has characteristic amplitude

$$\phi \sim \left( \frac{\lambda}{\pi c/H_0} \right)^2 \frac{\delta \rho}{\bar{\rho}} \sim \left( \frac{\lambda}{10^{28} \text{ cm}} \right)^2 \frac{\delta \rho}{\bar{\rho}}.$$

(6)

Thus, mildly nonlinear structures of size $10^{26}$ cm do not produce large potential fluctuations. This is fortunate, because otherwise the Einstein equations would be much more complicated than eq. (5) and large-scale dynamics would be harder to relate to initial conditions.

In many models for the formation of large-scale structure, gravitational potential fluctuations were generated by some physical process occurring in the early universe. For example, in the inflation scenario, quantum fluctuations in the field driving the inflationary expansion lead to large-scale density fluctuations. Without some source of potential fluctuations it is difficult to understand how large-scale structure could have
formed. Thus, we shall examine the consequences of assuming a nonzero $\phi(\vec{x}, t_i)$ as our initial conditions for structure formation. At this stage we are not concerned with what produced these fluctuations; the first question is whether they can produce the large-scale structure.

Primeval potential fluctuations have three major observable effects: (1) They produce microwave background anisotropy; (2) They imply inhomogeneities in the distribution of mass; and (3) They induce nonzero velocities relative to the uniform Hubble flow. Observations of microwave background anisotropy, galaxy clustering, and galaxy motions therefore can be combined to test the consistency of gravitational amplification of primordial fluctuations.

2.1 Cosmic Microwave Background Anisotropy

A very important step in testing of the gravitational instability paradigm follows from the exciting discovery last year of anisotropy in the cosmic microwave background radiation by Smoot et al.\cite{108} using the COBE satellite. How is this anisotropy related to the primeval potential fluctuations?

Photons traveling to us from the recombination layer (the cosmic photosphere occurring when the temperature dropped below 3500 K) did work against gravity in climbing out of the gravitational potential minima. Consequently, the microwave background temperature should be slightly smaller in the direction of potential minima on this photosphere compared with its average value and higher toward potential maxima. The magnitude of this gravitational redshift effect in terms of the radiation brightness temperature is $\Delta T / T = \Delta \phi / c^2$. However, it is partially offset by the fact that the density is higher in the potential minima from the Poisson equation, so the temperature is also higher and recombination occurred there a little later than elsewhere. As a result these photons have suffered less cosmic Doppler shift in traveling toward us. The net anisotropy is \cite{101}

$$\frac{\Delta T}{T_0}(\vec{n}) = \frac{1}{3c^2} \Delta \phi(\vec{x}, t_r) \; ,$$

where $\vec{x}$ lies at the cosmic photosphere (nearly at the edge of the presently observable universe) in direction $\vec{n}$, and $t_r$ is the time of recombination. This simple formula is valid for isentropic (constant entropy) fluctuations on sufficiently large scales (larger than about one angular degree) so that acoustic waves in the coupled photon-baryon fluid have not modified the temperature. The simplest models of fluctuation generation predict isentropic fluctuations, with constant ratios of the fluctuations for all components — photons, baryons, neutrinos, etc. The microwave background anisotropy generally is larger if these ratios vary. Eq. (7) also assumes that the gravitational potential does not evolve in time after recombination.

Figure 4 shows the angular correlation function of the cosmic microwave background
Figure 4: The angular correlation function of cosmic microwave background temperature anisotropy. Points with error bars are taken from [119]. Dotted curves show three Monte Carlo samples for a scale-invariant primeval density fluctuation spectrum including realistic sampling and noise. The mean correlation function for this model, convolved with the COBE beam and sampled like the observations, is drawn as the smooth heavy curve.

temperature, denoted here by $C(\theta)$ instead of the $w(\theta)$ referred to earlier, defined by

$$C(\theta) \equiv \langle \Delta T(\vec{n}_1) \Delta T(\vec{n}_2) \rangle_{\vec{n}_1, \vec{n}_2 = \cos \theta},$$

where the angle brackets denote an average over all pairs of directions separated by angle $\theta$. Also shown in fig. 4 are several Monte Carlo simulations characteristic of a scale-invariant spectrum of primeval potential fluctuations [8, 105], as predicted by the simplest inflationary cosmology theories. Although the statistical fluctuations due to receiver noise as well as to finite sample size are appreciable, the detected signal is statistically highly significant and indicates the presence of very long-wavelength fluctuations in the universe.

The COBE anisotropy measurements are important for showing that large-scale potential perturbations indeed exist, as required by gravitational instability models for cosmic structure formation. However, they have an important limitation: the COBE measurements probe only very large scales. At cosmological distances, an angle of 10° (approximately the minimum scale probed by COBE) corresponds to a linear size of 1.05 $(\Omega_0 h)^{-1}$ comoving Gpc (or $4 \times 10^{27}$ cm), more than seven times larger than the biggest known galaxy superclusters (e.g., the “great wall” of galaxies, cf. ref. [50]). COBE alone cannot test structure formation theories. Other measures of structure are needed. Moreover, these measures must allow us to relate the amplitude of fluctuations on different size scales.
2.2 Gravitational Potential Fluctuations

In all models, the primeval potential field is a stochastic quantity whose statistical properties, but not its actual value, may be predicted a priori. Assuming that the potential has a well-defined mean value (which we may take to be zero without loss of generality), the most important simple statistic is its two-point correlation function. In order to express the scale-dependence of the potential it is most convenient to work in the Fourier transform space such that fluctuations are expanded in plane waves \( \exp(i\vec{k} \cdot \vec{x}) \). (If \( K \neq 0 \) the background space is non-Euclidean and plane waves must be replaced by the appropriate eigenfunctions of the spatial Laplacian.) Note that here we take \( \vec{x} \) to be the comoving position defined in section [12] therefore \( \vec{k} \) is the comoving wavevector. The two-point correlation function in Fourier space is given by the power spectral density (power spectrum) \( P_\phi(k, t) \), defined so that the variance of \( \phi \) is

\[
\langle \phi^2(t) \rangle = \int d^3k P_\phi(k, t).
\]

Here the angle brackets denote either an average over an ensemble of universes or a spatial average for a single universe; they are equivalent for ergodic processes including all widely studied cosmic fluctuation models. Note that the variance is independent of position because we assume the potential to be a homogeneous random process, and the power spectral density is independent of the direction of the wavevector because we assume the potential to be statistically isotropic. In making these assumptions we are restricting the class of models for the perturbations.

The power spectral density is a useful quantity for comparing the amplitudes of fluctuations on different size scales. It is common in cosmology to represent the fluctuations in terms of \( \delta \rho/\bar{\rho} \), but we have seen above that the gravitational potential is more natural because of its simple relation to spacetime curvature fluctuations (eq. [4]) and to cosmic microwave background fluctuations (eq. [7]). Another reason for preferring \( \phi \) is that its time derivative (at fixed comoving position) vanishes in a matter- or radiation-dominated universe for wavelengths exceeding the distance sound waves can travel in the age of the universe. (Acoustic waves produce damped oscillations of the potential.) Given the potential, it is easy to get the density fluctuation from the Poisson eq. (5).

In place of the power spectral density we introduce a quantity called the potential amplitude function [8]:

\[
A(k, t) = c^{-2}[k^3 P_\phi(k, t)]^{1/2}.
\]

This quantity measures the root-mean-square potential fluctuation (divided by \( c^2 \)) on scale \( k^{-1} \); it is dimensionless (usually we set \( c = 1 \) anyway). The potential amplitude function has two advantages over the spectral density. First, it more naturally indicates the amplitude of fluctuations as a function of scale. Second, it is constant for a scale-invariant spectrum. A scale-invariant spectrum is defined to be one such that \( A(k, t) \) is independent of \( k \) and is a good approximation to the simplest inflationary models of
the early universe, which predict only a weak (logarithmic) dependence on \( k \). Often such a spectrum, also known as the Harrison-Zel’dovich-Peebles spectrum \([60, 122, 95]\), is described by saying that the power spectral density of \( \delta \rho/\bar{\rho} \) is proportional to \( k \). Complicated arguments about the growth of perturbations are cited to explain why \( P_\delta \propto k \) is “scale-invariant.” It is much simpler to say that the potential fluctuations have the same amplitude on all scales so that \( P_\phi \propto k^{-3} \), and then to note that the Poisson equation implies \( P_\delta \propto k^4 P_\phi \propto k \).

2.3 Galaxy Redshifts and Peculiar Velocities

As noted above, the anisotropy of the cosmic microwave background radiation has been measured on scales larger than galaxy superclusters. To probe smaller scales cosmologists use the galaxies themselves as tracers of the density and velocity fields.

Under the assumption that galaxies are distributed like mass on large scales (“light traces mass”), the galaxy number density distribution \( n_g(\vec{x}) \) is related to the mass density distribution. This relationship is complicated in several ways. For example, we cannot measure accurately the three-dimensional distribution of galaxies because distances are difficult to measure and are highly uncertain. Because of the difficulty of obtaining accurate extragalactic distances, astronomers generally settle for redshifts and assume them to be related approximately to distances by the Hubble law.

Catalogs of redshifts and angular positions are called redshift surveys. With the advent about a decade ago of high quantum-efficiency detectors, galaxy redshift surveys have grown by a large factor and now encompass about \( 10^5 \) galaxies. Figure 5 shows part of the well-known second Center for Astrophysics redshift survey \([38, 50, 63]\) showing evidence for a bubble-like topology of galaxy clustering.

Large redshift surveys have been conducted and analyzed by many groups \([18, 103, 110, 113, 78, 44, 104]\); see ref. \([51]\) for a review of earlier work. Smoothed over the intergalactic spacing, the surveys provide a measure of the density fluctuation field \( \delta n_g/\bar{n}_g \) whose spectral density may be compared with theoretical models. A simple formula fitting most of the results very well was proposed by Peacock \([88]\); in terms of the potential amplitude function this formula is

\[
A(k, t_0) = \frac{A_0 \Omega_0}{[1 + (k/k_c)^\gamma]^{1/2}}.
\]

The factor \( \Omega_0 \) is included because the potential is proportional to the mean mass density. Ref. \([8]\) discusses the estimates of the parameters in eq. (11). A good fit over the range \( 0.02 \lesssim k \text{ Mpc}/h \lesssim 1.0 \) is provided by \( A_0 = 1.27 \times 10^{-5} \), \( \gamma = 2.4 \), and \( k_c = 0.024 \text{ h Mpc}^{-1} \).

One should be cautious in extrapolating the results of galaxy surveys to the mass distribution as a whole because there is no proof that dark matter is distributed like galaxies. Indeed, it is known that dark matter is less concentrated toward the centers of galaxies than luminous matter. It is also plausible that averaged over larger scales galaxy
number density fluctuations are an amplified version of the mass density fluctuations due to a process called biased galaxy formation: $\delta n_g/\bar{n}_g = b\delta \rho/\bar{\rho}$. This relation, suggested by Kaiser on the basis of the statistical properties of peaks of a gaussian random field [67], is highly schematic; because galaxies are effectively a point process while the mass density is effectively continuous, the relationship only makes sense for the smoothed fields. Possible but speculative physical mechanisms leading to this bias have been suggested [37, 17].

The “biasing factor” $b$ may also be a function of smoothing scale and position, as well as galaxy type. However, in the absence of a detailed theory of galaxy formation having real predictive power, it is reasonable to use such phenomenological models to interpret galaxy redshift surveys.

The second probe of potential fluctuations suffers little from uncertainty about how galaxies trace mass. This method is based on the so-called “peculiar” velocities of galaxies | their residuals from the Hubble flow [20]. We noted before that the proper position is $\vec{r} = a(t)\vec{x}$ and the Hubble velocity is $H\vec{r} = (da/dt)\vec{x}$. The peculiar velocity is then $\vec{v} \equiv d\vec{r}/dt - H\vec{r} = a\vec{d}\vec{x}/dt$. The key idea is that galaxies acquire their peculiar velocities as test-bodies falling in the large-scale perturbed gravitational field. If peculiar velocities initially are small, so that objects fall from rest — relative to the background cosmic expansion! — the peculiar velocities initially grow in proportion to the gravitational field: $\vec{v} \propto \vec{g} \equiv -a^{-1}\nabla \phi$. The constant of proportionality is not exactly equal to the cosmic time, because the gravity field may change with time (as well as varying in space). Assuming galaxies have not moved far (in comoving coordinates) so that a linear relation still applies, the constant of proportionality still depends on $\Omega$. Taking the divergence of
this velocity-gravity relation and using the Poisson equation, we can estimate the mass density fluctuations giving rise to the peculiar velocities [7]:

$$\frac{\delta \rho}{\bar{\rho}} \approx -\frac{\vec{\nabla} \cdot \vec{v}_{H}}{H \Omega^{0.6}}. \tag{12}$$

The factor $\Omega^{0.6}$ approximately expresses the $\Omega$-dependence of the peculiar velocities for a wide range of cosmologies [90, 76]. Eq. (12) is valid only if $|\delta \rho / \bar{\rho}| \lesssim 1$, but modifications work well in the quasi-linear regime ($|\delta \rho / \bar{\rho}| \lesssim 5$) [85, 54]. The most important points are that galaxies provide an unbiased tracer of the gravity field because all bodies fall the same way in a gravitational field (Galileo), and the inferred density fluctuation field applies to all the mass and not just the galaxies. Thus, peculiar velocities offer an excellent means for testing theories of large-scale structure.

These ideas have been used by my collaborators and I to reconstruct the large-scale mass density fluctuation field [9, 35, 10] by applying a method called POTENT to a large sample of estimated galaxy distances [79, 43]. Recently we have compared the mass density fluctuations from POTENT with the galaxy density fluctuations from a complete redshift survey of galaxies selected from the infrared survey made by the IRAS satellite [110]. Figure 6 shows the results in one plane through the local universe [36]. The results of this comparison show consistency within the measurement uncertainties, which are dominated by galaxy distance errors, provided $\Omega_0 \gtrsim 0.3$. Unless galaxies are much less clustered than the dark matter (the opposite of what is usually assumed), our results strongly exclude low-density models, even spatially flat ones in which most of the energy density is in the form of a cosmological constant. Similar conclusions were reached in an earlier comparison of the radial velocity and gravity fields [69].

The velocity-density comparison relies on galaxy distance measurements (for the peculiar velocity) that may be prone to systematic errors in addition to the large statistical uncertainties. Large-scale measurements of $\Omega_0$ also can be made using galaxy redshift surveys alone, assuming that the spatial clustering is statistically isotropic and that the observed radial distortions are due to peculiar velocities [58]. Present results from this method have large observational uncertainties [59], but the size of galaxy redshift surveys will increase ten-fold by the end of the decade, enabling a powerful comparison to be made of different large-scale dynamical measurements of $\Omega_0$.

### 2.4 Can Gravity Account for Cosmic Structure?

In the preceding sections we described three ways to estimate the gravitational potential field: cosmic microwave background fluctuations, galaxy redshift surveys, and gravity field measurements. By combining estimates of the potential amplitude function made using these different techniques we can test whether the available data are consistent with one mechanism for inducing large-scale structure.
Figure 6: Contours of smoothed density for galaxies (left) and mass (right) in the plane of the Local Supercluster of galaxies [36]. Contours are spaced by 0.1 in $\delta\rho/\bar{\rho}$, with positive values solid, negative values dashed, and the zero contour slightly thicker. Outside the heavy solid line the standard error of the mass density fluctuation exceeds 0.2, while outside the heavy dashed line the sampling of galaxies is too sparse for reliable estimation of the mass density.
Figure 7: The potential amplitude function from measurements of cosmic microwave background anisotropy (COBE band), galaxy redshift surveys (dashed-dotted curve marked IRAS, CfA galaxies; its extrapolation to longer wavelengths is indicated as a dotted line), and peculiar velocities (crosses marked POTENT) [7]. Scaling density and velocity to potential requires assuming $\Omega_0$; two different choices are shown in parts (a) and (b). Solid curves going through the COBE band is the linear theory prediction for flat cold dark matter (CDM) models normalized to COBE. The dashed curve shows the CDM models smoothed the same way as POTENT.

Figure 7 shows the present-day potential amplitude function inferred from the measurements described above [8]. The COBE points depend on $\Omega_0$ because if $\Omega_0 \neq 1$, the gravitational potential fluctuations change with time, increasing the microwave anisotropy for a given potential amplitude [53] and therefore requiring smaller-amplitude potential fluctuations today for a given measured anisotropy. The estimates based on galaxy number density and peculiar velocities also depend on $\Omega_0$. The spectrum estimates from peculiar velocities are not corrected for noise power. Seljak & Bertschinger [106] recently have carefully analyzed the peculiar velocity data and showed that they are compatible with the cold dark matter model (discussed in the next section) for $\Omega_0 = 1$ but not for $\Omega_0 = 0.2$, just as one would infer from fig. [7].

Remarkably, the various measurements of the gravitational potential amplitude are in rough agreement with each other and with the cold dark matter theory over more than three decades of spatial frequency. Given the difficulty of making these measurements, agreement to within a factor of two is tremendously reassuring. If gravity were not responsible for cosmic structure, there is no a priori reason one should expect any agreement at all.
3 Small-scale Structure: How did Galaxies Form?

Even if gravity alone is responsible for cosmic structure formation, one may fear that nonlinear evolution is so complicated that it is impossible to say anything about the initial conditions from the present-day distribution of mass. As we saw in the previous section, on large scales (i.e., when one spatially averages the mass) this assessment is too pessimistic. When smoothed sufficiently the density fluctuations have small amplitude and they evolve in a simple way. Consequently, the relation between the initial (i.e., immediately after recombination) and present density, velocity and potential fields is approximately linear on large scales. Eqs. (7) and (12) reflect this linearity.

However, on small scales cosmic gravitational dynamics is strongly nonlinear and chaotic. The Lyapunov time for individual particle trajectories is about one orbital time in the dense bound clumps that form by gravitational instability [70]. Even if we could specify the position and velocity of every particle in the universe today, the strong chaos of gravitational dynamics would make it impossible to integrate the trajectories backwards in time. Rather than go backwards, therefore, cosmologists try different theories for the initial conditions and integrate forward in time using gravitational N-body simulations [11] in which each particle represents a cloud of dark matter. More recently some workers have also begun to include gas dynamics in their simulations in order to follow the baryonic matter — a trace contaminant in many theories, but all that we can see!

Galaxy formation is very difficult to simulate for two reasons. The first one is the large required dynamic range in length and mass scales (cf. Table 2). Individual luminous galaxies are smaller than $10^{23}$ cm but they reside in structures thousands of times larger. Realistic simulations would require a dynamic range of at least $10^4$ in length and $10^9$ in mass. Simulations of this size are beyond the current state-of-the-art, although they should be possible within five years.

The second difficulty is the complexity of gas dynamical processes, which only exacerbate the dynamic range problems. Impressive calculations of cosmological gas dynamics have recently been made (refs. [120, 71, 72, 100, 26, 42], and references therein), but the greater computational cost (and more severe timestep restriction) of gas dynamics relative to gravitational simulation has severely limited the dynamic range in mass and/or length. Even when these problems are ameliorated by much more powerful computers, the complexity of radiative processes, magnetic fields, etc., will continue to challenge the computational astrophysicist. Star formation — which is known to be important in determining the appearance and evolution of galaxies — can be treated only in a phenomenological manner at best.

Is galaxy formation so complicated then as to defeat our attempts to test theories of cosmic structure formation? Probably not, for several reasons. First, because all theories of the initial conditions are stochastic, it is unnecessary for calculations to correctly reproduce every detail of the evolution beginning from a specific state. In effect, we
demand only that the coarse-grained distribution of mass, averaged over galactic or larger scales, have the correct statistical properties. Second, it is plausible that gas dynamical effects are important only within galaxies and that galaxies form at minima of the gravitational potential field of the dark matter. (Gas dynamical simulations at least should be able to test this hypothesis.) If so, then gravity is sufficient to identify the sites of galaxy formation, if not the internal properties of the galaxies themselves. Third, analytical arguments suggest and numerical simulations confirm several simple scaling relations for the evolution of self-gravitating dark matter, allowing us to extend the dynamic range of simulations in a statistical sense.

In the following sections we shall present these scaling relations, followed by a case study of a particular structure formation model, the standard cold dark matter model. As we shall see, that model appears to make predictions at variance with observations, leading cosmologists to explore alternatives.

3.1 Hierarchical Clustering

Let us begin with visual examination of the mass distributions produced by strongly nonlinear gravitational instability. Figure 8 illustrates gravitational clustering beginning from white noise perturbations of a homogeneous and uniformly expanding $\Omega = 1$ universe: $128^3$ particles were simply placed uniformly and independently at random in a cube with zero peculiar velocities at the initial time. Unlike fig. 3, comoving coordinates are used here to factor out the mean cosmic expansion. The top three and bottom left two panels show the time evolution of the entire volume while the last panel is a ten times magnification of the lower right-hand corner of the last output. The largest clump in the bottom right panel contains about 7500 particles. The spatial resolution is about $10^{-3}$ of the simulation size and the entire simulation consumed about 100 Cray Y-MP hours for 630 timesteps.

Poisson initial conditions are not believed to be realistic for our universe. For our purposes, however, this N-body simulation provides an excellent illustration of the process of hierarchical clustering. In this process, mass is gathered by gravity into dense clumps, which merge successively to form larger clumps. The universe remains homogeneous on the largest scales, but the transition length scale between homogeneity and strong clustering — called the clustering length $\ell_c(t)$ — increases with time in comoving coordinates.

It is straightforward to understand the increase in the clustering length by extrapolating linear theory with a simple model of nonlinear effects. In fig. 8 we see that the mass in an overdense region expands until the relative density contrast with its surroundings becomes of order unity. At that point the mass collapses to form a gravitationally bound system that ceases expanding with the universe. At any given time, therefore, the clustering length is roughly the length scale on which the rms density fluctuation is about unity. Since linear theory is obeyed to a reasonable approximation until the
Figure 8: A mosaic showing time evolution of the projected mass distribution in an ω = 1 universe with particles initially distributed as a Poisson process. The first five panels show the projection of all the mass (colors represent logarithm of projected mass density) at expansion factors 8, 23, 64, 125, and 250 after the start of the simulation. The bottom right panel is a ten times magnification of the lower right-hand corner of the last output.
density contrast equals unity, we may estimate the clustering length using linear theory. This is most naturally done using Fourier analysis to decompose the density field into its spatial frequency components. The rms density fluctuation on comoving scale \( \ell = k^{-1} \) is, by analogy with eq. (11), \([k^3 P_\delta(k, t)]^{1/2}\). Setting this to unity gives an implicit equation for \( \ell_c^{-1} = k_c(t) \).

In an \( \Omega = 1 \) universe, as a result of gravitational instability small-amplitude (linear) density fluctuations evolve in proportion with the expansion scale factor: \( \delta \rho/\bar{\rho} \propto a(t) \) with fixed spatial dependence in comoving coordinates [10, 7]. (If \( \Omega < 1 \) the growth is less rapid, while if \( \Omega > 1 \) it is more rapid.) This behavior is manifested by the first three panels of fig. 8, in which the contrast increases but the spatial pattern changes little on the scales resolved in this image. As a result, the power spectrum evolves as \( P_\delta(k, t) \propto [a^2(t)/a^2(t_i)] P_\delta(k, t_i) \), where \( t_i \) may be taken to be any time after recombination while the fluctuations are still evolving linearly.

If \( \Omega = 1 \) and the initial power spectrum is a power law \( P_\delta(k, t_i) \propto k^n \), then \( \ell_c \propto a^{2/(3+n)} \). If \( n > -3 \), the initial fluctuation amplitude is small on large scales and large on small scales. The resulting behavior, with the comoving clustering length growing with time, is called hierarchical clustering. The same qualitative behavior results even if the initial power spectrum is not a power law, as long as \( k^3 P_\delta(k, t) \) grows with \( k \). Such a field is not smooth — it is infinitely spiky on arbitrarily small scales. In ideal hierarchical models, the density field is nonlinear from the beginning on sufficiently small scales. The standard cold dark matter model (section 3.2) is a hierarchical model.

The N-body simulation shown in fig. 8 exhibits hierarchical clustering with initial density spectrum exponent \( n = 0 \). Our simple theory then predicts that the clustering scale should increase by factors of 2, 4, 6, and 10 for the last four outputs \( (a = 23, 64, 125, \text{ and } 250, \text{ respectively}) \) relative to the first output \( (a = 8) \). This scaling is plausible visually from the mean spacing of typical dense clumps and it is borne out by a detailed analysis of the nonlinear power spectrum.

In models where \( k^3 P_\delta(k, t_i) \) decreases as \( k \rightarrow \infty \), by contrast, small objects do not collapse first. Indeed, nothing collapses until \( k^3 P_\delta(k, t) \approx 1 \) first has a solution at \( t_c \) on scale \( \ell_c = k_c^{-1} \). The initial density field is smooth with a coherence scale \( \ell_c \). A coherence scale can be built in by physical processes that suppress linear growth on small scales, such as the collisionless (free-streaming) damping that occurs if the dark matter has large thermal velocities (e.g., light massive neutrinos [16]). In this case the initial stages are described by the quasilinear Lagrangian theory of Zel’dovich [121, 107], modified for the effects of shear and tides [12]. Historically such models were called “pancake” models after Zel’dovich’s description of the generic shapes of the first objects to collapse. In realistic models, however, the power in the density field decreases on larger scales (this is required by fig. 7), so that pancake models at late times look like hierarchical models.

Hierarchical clustering is complicated in detail because it involves the gravitational interaction of infinitely many degrees of freedom. Nevertheless, by applying simple scaling arguments similar to the one given above for the evolution of the clustering length,
cosmologists have been able to devise simple analytical theories for various properties of the nonlinear density field, such as the nonlinear power spectrum, the distribution of clumps by mass, the typical internal density profiles of clumps, the clustering properties of mass and clumps, etc. \cite{71, 72, 42, 118}.

One is tempted to identify the dense dark matter clumps formed by hierarchical clustering with the dark matter dominated halos surrounding galaxies. This idea is supported by high-resolution cosmological simulations including gas, which show that dense gas indeed collects in the dark matter clumps \cite{71, 72, 42}. However, the dark matter clumps in most $\Omega = 1$ models appear to merge excessively compared with luminous galaxies. The discrepancy may plausibly be solved by the radiation of energy by shock heated gas, allowing the gas to sink toward the centers of the dark matter potential wells \cite{118}.

3.2 The Cold Dark Matter Model

The cold dark matter (CDM) model was the most popular specific model for cosmic structure formation during the 1980s \cite{45, 31, 77, 86}. First proposed by Peebles \cite{91}, it soon replaced the pancake model with light massive neutrinos as the leading theory for structure formation \cite{4, 30}. The ingredients of the CDM model include the standard big bang theory plus nonbaryonic dark matter (denoted “X” and supposed to be some new elementary particle) with

1. $\Omega_X = 0.95$ and $\Omega_B = 0.05$ (inflation predicts $\Omega = 1$, while primordial nucleosynthesis models favor small baryonic abundance);

2. $h = 0.5$ (small Hubble constant, implying for $\Omega = 1$ a cosmic age of 13.2 billion years);

3. scale-invariant ($n = 1$) gaussian isentropic density fluctuations.

Before the measurement of the large scale microwave background anisotropy made last year, the CDM model had one free parameter, the normalization of the scale-invariant spectrum. Conventionally this was described, based on measurements from galaxy redshift surveys \cite{32}, by the rms relative fluctuation $\sigma_8$ in mass in randomly placed spheres of radius $8 h^{-1}$ Mpc. If linear theory is used to compute $\sigma_8$, it is simply related to the potential amplitude $A_0 = A(k \to 0, t_0)$ of eqs. (10) and (11) by $\sigma_8 = 1.6 \times 10^5 A_0$. The COBE normalization for the standard CDM model implies linear $\sigma_8 = 1.05 \pm 0.17$ \cite{40, 103}. Observations yield $\sigma_8 = 1.0$ for galaxies, but, as mentioned in section 2.3, it is possible that galaxies are more strongly clustered than mass, with $\sigma_8(\text{galaxies}) = b\sigma_8(\text{mass})$ with $b > 1$.

The CDM model became widely known after the first high-resolution N-body simulations of nonlinear gravitational clustering were published in 1985 by Davis \textit{et al.} \cite{30}.
These authors concluded that one cannot simultaneously fit galaxy clustering and the relative peculiar velocities of galaxies (essentially the “temperature” of the galaxy distribution) for any $\sigma_8$ if $b = 1$. The essential problem is that galaxy thermal motions (small-scale relative velocities) grow too rapidly with $\sigma_8$ so that when the clustering is sufficient, the galaxy distribution is far too hot. Davis et al. proposed a solution: set $\sigma_8 = 0.4$ (making the thermal velocities of galaxies acceptable) and require that the galaxies cluster more strongly than the mass by assuming that $b = 2.5$. Unless the anisotropy measured by COBE is mostly due to something other than the gravitational potential fluctuations of eq. (7), this model is now ruled out. However, clever theoreticians can find other ways to produce microwave anisotropy (e.g., deflection of the microwave radiation by long-wavelength gravitational radiation [73, 84, 77]), so the $\sigma_8 = 0.4$ model is worth examining further based on its predictions for galaxy clustering and velocities.

One problem with the biasing idea is that it appears ad hoc; why should galaxies cluster 2.5 times more strongly than dark matter? A higher-resolution N-body simulation in 1987 performed by White et al. [117] showed that some “bias” indeed arose naturally as a result of preferential formation of galaxies in dense regions.

During the pre-COBE period 1986–1991, many authors concluded that the $b = 2.5$ ($\sigma_8 = 0.4$) CDM model has too little large-scale power to account for large-scale clustering and motions (e.g., [13, 103]). Achieving adequate large-scale power evidently requires a larger $\sigma_8$. In retrospect this is not surprising, as fig. 6 shows that roughly consistent normalizations are implied by COBE and large-scale structure.

In 1989, Carlberg & Couchman made the important point that galaxy velocities may be “biased” (relative to the mass) as well as the galaxy number density field [21]. Their simulations — with resolution comparable to the best previous calculations [117] — showed a significant “velocity bias:” the temperature of the galaxy distribution was about four times less than that of the mass. Thus, if a high amplitude normalization is assumed ($\sigma_8 = 1.0$), the galaxy clustering may be strong enough without the small-scale peculiar velocities being excessive. However, this velocity bias does not appear to persist on the larger scales probed by POTENT discussed above in section 2.3.

More recent numerical work indicates that velocity bias is probably inadequate to reconcile the simulated velocities with observations [71, 23, 24, 49]. Most workers therefore reject the CDM model. However, given that the simulations are limited in dynamic range and the treatment of physical processes, this conclusion should be considered tentative and subject to reexamination as simulations improve.

### 3.3 Alternative Models

The search for alternative models is guided by analysis of the problems of cold dark matter combined with measurements of large-scale structure (fig. 7). The chief problem with the cold dark matter model appears to be excessive power on small scales. This is not
apparent in fig. 7, which appears to show a good match between the theoretical (solid) and measured (dashed-dotted) curves at large $k$. However, the theoretical curves are based on the linear power spectrum; N-body and analytical calculations for $\Omega = 1$ show that the nonlinear spectrum is enhanced by mode-coupling effects [53]. The excessive small-scale motions predicted by the theory imply the need to decrease the masses of galactic-scale clumps. This would also bring the theory better into agreement with observational estimates of the mass in galaxies and clusters [113].

Three ways have been suggested to decrease the masses of galaxies and clusters in the CDM model without sacrificing the relatively good agreement of the model with large-scale structure when normalized to the microwave background anisotropy [113, 10, 112]. The first is simply to decrease the mean mass density of the universe, parametrized by $\Omega$, so that there is less mass everywhere. Such models are further subdivided by whether or not they include a cosmological constant (Einstein’s “blunder” resurrected [1]) in order to make the cosmic curvature $K$ (eq. 1) vanish as predicted by inflation theory. Open universe models without a cosmological constant require fluctuations from a source other than quantum fluctuations during inflation, and such models are more complicated and uncertain than inflation [109, 28].

Simulations of cold dark matter with $\Omega_X = 0.2$ and the remainder made of a cosmological constant have been performed by several different groups [111, 111, 24]. These authors find that the model looks very promising from the viewpoint of galaxy formation and clustering. However, as fig. 7 indicates, it is in conflict with measured large-scale peculiar velocities [106]. Given the possible systematic errors of galaxy distance estimates, however, it may be prudent not to reject the theory without further examination.

Another way to reduce galaxy masses is to “tilt” the primeval spectral index away from the scale-invariant slope $n = 1$, i.e., to modify the scale-invariant form of the primeval potential amplitude function $A(k, t)$. Because the COBE measurement constrains $A$ at small $k$ (fig. 1), decreasing the small-scale power requires decreasing $n$. With less power on small scales, galaxies should be less massive. However, such a model must still be compatible with galaxy clustering and peculiar velocities. Unfortunately, even $n = 0.723$ still leads to excessive thermal motions of galaxies [18]. This problem can be ameliorated by decreasing $n$ further or reducing the normalization of potential fluctuations by making some of the cosmic microwave background anisotropy with gravitational radiation (refs. 75, 34, 77), but then the models have too little power in the wavenumber range $0.01 \lesssim k \text{ Mpc}/h \lesssim 0.1$ (fig. 1, 106). Positive tilts ($n > 1$) are a nonstarter because they produce excessive small-scale structure.

The third way potentially to repair the cold dark matter model is to replace some of the cold dark matter with a mass component that clusters less strongly. The cosmological constant model mentioned above is one extreme version, but a simpler method (from the viewpoint of fundamental physics) is to suppose that one type of neutrino has a very small mass, equivalent to a rest-mass energy of several eV (refs. 61, 114, 83, 112, 73 and references therein). Neutrinos were created in the big bang by thermal processes
but they effectively ceased interacting with the rest of the matter and radiation shortly before the era of primordial nucleosynthesis \[74\]. Since that time their comoving number density has been conserved and is comparable today with the number density of photons in the microwave background radiation. They have a mean thermal energy comparable with the mean energy of microwave background photons. In the past the thermal energy per neutrino was high; consequently, massive neutrinos are often referred to as hot dark matter. There are three types of neutrinos (electron, muon, and tau), but it is likely that at most one of them (presumably the \(\tau\)) has a cosmologically interesting mass. A neutrino of mass \(97 \, h^2 \, \text{eV}\) would close the universe by itself with \(\Omega_\nu = 1\).

The thermal velocities of neutrinos cause them to cluster less strongly than cold dark matter: gravity cannot confine hot dark matter in a shallow potential well. If there is no cold dark matter at all, then the massive neutrinos stream out of the potentials, which are thereby erased. The resulting model has too little power on small scales and, as mentioned in section 3.2, was rejected before cold dark matter became popular. If, however, the dark matter is an admixture of hot and cold dark matter, perhaps one can adjust \(\Omega_\nu\) (which is proportional to the neutrino mass) so as to suppress small-scale clustering sufficiently while retaining the successes of the COBE-normalized cold dark matter model for large-scale structure.

The mixed dark matter model with \(\Omega_\nu = 0.3\) and \(h = 0.5\) (requiring a neutrino of mass 7 eV) has been studied recently by several groups with N-body simulations \[33, 73\] and gravity plus gas dynamics \[27\]. The first two groups concluded that this model looks very promising, while the latter authors find that the model is unsatisfactory because the suppression of small-scale power is inadequate to solve the small-scale clustering problems of CDM but at the same time too much small-scale power is removed to form galaxies sufficiently early. Because all of these simulations are based on moderate-resolution grid methods that fall far short of the dynamic range desired for realistic simulations, their conclusions should be considered highly tentative.

To test models with resolution adequate to study the formation and clustering of galactic halo-sized dark matter clumps, I have performed large N-body simulations of the CDM, CDM plus cosmological constant, and mixed dark matter models. The standard CDM (\(\Omega = 1\) and \(h = 0.5\)) simulation has \(144^3\) particles and is described and analyzed in \[49\]. The simulation required 770 IBM 3090 hours to evolve to \(a = 1.0\) (the present time) with 1200 timesteps. The model with a cosmological constant has \(\Omega_0 = 0.2\) and \(h = 0.8\), with \(128^3\) particles, and required 250 Cray Y-MP hours to evolve to \(a = 1.2\) with 2061 timesteps. The mixed dark matter model has \(128^3\) cold and \(10 \times 128^3\) hot particles with initial conditions (for \(\Omega_\nu = 0.3\) and \(h = 0.5\)) generated as described in \[80\]. It required 500 Convex C-3880 hours to evolve to \(a = 0.5\) with 265 timesteps. In all cases, \(a = 1.0\) corresponds to the COBE normalization. All simulations (as well as the one shown in fig. 5) were performed using the adaptive mesh-refined particle-particle/particle-mesh algorithm \[29, 49\], had spatial resolution smaller than \(10^{-3}\) of the simulation size, and conserved energy to a fraction of a percent.
Figure 9: The projected mass density for four different models, all in a cube of comoving length $50h^{-1}$ Mpc at a redshift $z = 1$ (when the universe was about one-third its present age). The upper left panel is cold dark matter with linear $\sigma_8 = 0.5$. The upper right panel is the same model at the smaller amplitude linear $\sigma_8 = 0.2$. The lower left panel is cold dark matter with $\Omega_0 = 0.2$ plus a cosmological constant added to cancel the background spatial curvature. The lower right panel is mixed (hot plus cold) dark matter with $\Omega_\nu = 0.3$. All models except the upper right one are normalized to COBE as in fig. 7. Colors represent the logarithm of the projected mass density ranging from 1 to 20 times the cosmic mean.
Figure 9 compares the mass distributions in these models at redshift $z = 1$. One sees that both the abundance and clustering of dense dark matter clumps varies widely among the models. The clumps in the CDM model grow significantly between $\sigma_8 = 0.2$ and 0.5 (top two panels), by which time there are already too many massive clumps to represent galaxy halos. The large-scale clustering in this model is rather weak. When $\Omega_0$ is decreased to 0.2 (bottom left panel), the large-scale structure increases significantly and the numbers of dense clumps becomes more reasonable. Although the most massive clumps have larger density contrasts than in the CDM model, their masses are smaller because the model has 5 times less mass overall ($\Omega_0 = 0.2$). Finally, it is evident that the mixed dark matter model with $\Omega_0 = 0.3$ has much less nonlinear structure at $z = 1$ than the other models. The low abundance of dense dark matter clumps suggests that this model may not form enough objects to match observations of galaxies and quasars at high redshift. Quantitative analysis of these simulations will be published elsewhere.

4 Conclusions

We have divided the whole of cosmic structure formation theory into three broad areas: (1) homogeneous big bang cosmology, (2) large-scale structure, and (3) galaxy formation. This division may be regarded as a sequence of physical scale — ranging from the greatest distances that we can possibly see, to the merely astronomically large — or as one of complexity. On the largest scales (greater than $10^{27}$ cm) the universe appears remarkably uniform, yet on much smaller scales the universe is richly textured, with virtually all of the luminous matter concentrated into objects — galaxies — a million times denser than the cosmic mean. The major challenge facing cosmologists is to account for this spatial progression from order to disorder.

It is clear that cosmic structure formation is not going to be explained by a single simple physical theory in the way that elementary particle interactions are explained by the standard model of particle physics. A hierarchy of theoretical models is required, with fundamental physics — a theory of gravity and spacetime plus theories of the behavior of matter and radiation at extreme energies in the early universe — required for the universe as a whole, progressing to theories, in effect, of gravitational and hydrodynamical turbulence on galactic scales and below. However, if this subject is to be a hard science, it must have theories that make specific predictions that can be tested by data. In this article we have given an overview of these theories: general relativity, the Robertson-Walker spacetime models, gravitational instability, and nonlinear gravitational and gas dynamics.

How do these theories stack up against observations? On the largest scales, the agreement between the predictions of the big bang theory and four different types of observations is extremely good (section 2). The remarkably accurate measurements of the spectrum and isotropy of the cosmic microwave background radiation made by the COBE satellite strengthen an already solid foundation. No crises have shaken the big
bang theory lately (occasional news reports notwithstanding) and no alternative theory has appeared as a serious rival. However, the nature, amount, and distribution of dark matter remain important outstanding questions.

What about theories of large-scale structure? The prevailing paradigm is that structure evolved as a result of the gravitational instability of the homogeneous big bang model, seeded by some source of small-amplitude fluctuations of very large scale (ranging in wavelength from at least $10^{22}$ to $10^{28}$ cm). As we have seen, the gravitational instability paradigm predicts relations for three dynamical fields — the gravitational potential, the velocity, and the density — that can be tested by combining measurements of cosmic microwave background anisotropy, galaxy motions, and galaxy clustering. A major success has occurred during the last two years, as data in all these areas are being combined (fig. 7) to test our paradigm of large-scale structure. Although the measurement uncertainties are still large and difficult to quantify, the good agreement (better than a factor of two over three decades of wavelength) has given strong encouragement to cosmologists to pursue more detailed modeling. The observational situation is expected to improve further during the next few years as major projects are underway in all areas of large-scale structure.

On smaller scales, however, no theory presently stands out as a clear leader, in large part because of the uncertainty in the nature, amount, and distribution of dark matter. This is a reversal of the situation five years ago, when the cold dark matter theory was favored by many cosmologists. High resolution computer simulations combined with improved observations have raised serious problems for the cold dark matter model. In fact, the demise of this theory is a mark of progress, showing that theorists can usefully calculate the consequences following from a few precepts of early universe cosmology, with enough accuracy to be contradicted by observations. Indeed, we now have strong guidance as to how the model should be changed: the small-scale power (or the masses of galaxy halos and clusters) must be diminished while retaining or slightly boosting the large-scale structure needed for \textit{COBE}, peculiar velocities, and large-scale clustering.

It is no mark of shame that, even with this guidance, we do not yet have a standard model of galaxy formation to replace cold dark matter. Any other theory is more complicated — cold dark matter has virtually no free parameters — and the recognition of the importance of large-scale structure has boosted the dynamic range requirements to a level straining our present computational capabilities. Future tests will rely heavily on supercomputer simulations of nonlinear gravitational clustering and complex gas dynamical interactions. Cosmic structure formation is a grand challenge computational problem of the physical sciences. Given the rate of progress in this field, I am optimistic that before the end of this decade we will have a well-tested standard model of galaxy formation.

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References

[1] C.W. Allen, Astrophysical Quantities (Athlone, London, 1973).
[2] R.A. Alpher and R.C. Herman, Rev. Mod. Phys. 22 (1950) 53.
[3] H.C. Arp, G. Burbidge, F. Hoyle, J.V. Narlikar and N.C. Wickramasinghe, Nature 346 (1992) 807.
[4] R. Balian and R. Schaeffer, Astron. Astrophys. 220 (1989) 1; 226 (1989) 373.
[5] J. Bernstein and G. Feinberg, Cosmological Constants (Columbia University, New York, 1986).
[6] E. Bertschinger, Nature 348 (1990) 675.
[7] E. Bertschinger, in: New Insights into the Universe, eds. V.J. Martinez, M. Portilla and D. Sáez, (Springer, New York, 1992) 65.
[8] E. Bertschinger, Ann. N.Y. Acad. Sci. 688 (1993) 297.
[9] E. Bertschinger and A. Dekel, Astrophys. J. 336 (1989) L5.
[10] E. Bertschinger, A. Dekel, S.M. Faber, A. Dressler and D. Burstein, Astrophys. J. 364 (1990) 370.
[11] E. Bertschinger and J.M. Gelb, Comp. Phys. 5 (1991) 164.
[12] E. Bertschinger and B. Jain, preprint (1993) MIT-CSR–93-12.
[13] E. Bertschinger and R. Juszkiewicz, Astrophys. J. (Lett.) 334 (1988) L59.
[14] G.R. Blumenthal, S.M. Faber, J.R. Primack and M.J. Rees, Nature 311 (1984) 517.
[15] A.M. Boesgaard and G. Steigman, Ann. Rev. Astron. Astrophys. 23 (1985) 319.
[16] J.R. Bond and A.S. Szalay, Astrophys. J. 274 (1983) 443.
[17] R.G. Bower, P. Coles, C.S. Frenk and S.D.M. White, Astrophys. J. 405 (1993) 403.
[18] T.J. Broadhurst, R.S. Ellis, D.C. Koo and A.S. Szalay, Nature 343 (1990) 726.
[19] E.M. Burbidge, G.R. Burbidge, W.A. Fowler and F. Hoyle, Rev. Mod. Phys. 29 (1957) 547.
[20] D. Burstein, Rep. Prog. Phys. 53 (1990) 421.
[21] R.G. Carlberg and H.M.P. Couchman, Astrophys. J. 340 (1989) 47.
[22] S.M. Carroll, W.H. Press and E.L. Turner, Ann. Rev. Astron. Astrophys. 30 (1992) 499.
[23] R. Cen, N.Y. Gnedin, L. Kofman and J.P. Ostriker, Astrophys. J. (Lett.) 399 (1992) L11.
[24] R. Cen, N.Y. Gnedin and J.P. Ostriker, Astrophys. J. 417 (1993) 387.
[25] R. Cen and J.P. Ostriker, Astrophys. J. (Lett.) 399 (1992) L113.
[26] R. Cen and J.P. Ostriker, Astrophys. J. 417 (1993) 415.
[27] R. Cen and J.P. Ostriker, preprint (1993) POP-534.
[28] R. Cen, J.P. Ostriker and P.J.E. Peebles, Astrophys. J. 415 (1993) 423.
[29] H.M.P. Couchman, Astrophys. J. (Lett.) 368 (1991) L23.
[30] M. Davis, G. Efstathiou, C.S. Frenk and S.D.M. White, Astrophys. J. 292 (1985) 371.
[31] M. Davis, G. Efstathiou, C.S. Frenk and S.D.M. White, Nature 356 (1992) 489.
[32] M. Davis and P.J.E. Peebles, Astrophys. J. 267 (1983) 465.
[33] M. Davis, F.J. Summers and D. Schlegel, Nature 359 (1992) 393.
[34] R. Davis, H.M. Hodges, G.R. Smoot, P.J. Steinhardt and M.S. Turner, Phys. Rev. Lett. 69 (1992) 1856.
[35] A. Dekel, E. Bertschinger and S.M. Faber, Astrophys. J. 364 (1990) 349.
[36] A. Dekel, E. Bertschinger, A. Yahil, M.A. Strauss, M. Davis and J.P. Huchra, Astrophys. J. 412 (1993) 1.
[37] A. Dekel and M.J. Rees, Nature 326 (1987) 455.
[38] V. de Lapparent, M.J. Geller and J.P. Huchra, Astrophys. J. (Lett.) 302 (1986) L1.
[39] G. Efstathiou, in: Physics of the Early Universe, eds. J.A. Peacock, A.F. Heavens and A.T. Davies (IOP, Bristol, 1990) 361.
[40] G. Efstathiou, J.R. Bond and S.D.M. White, Mon. Not. R. Astron. Soc. 258 (1992) 1p.
[41] G. Efstathiou, W.J. Sutherland and S.J. Maddox, Nature 348 (1990) 705.
[42] A.E. Evrard, F.J. Summers and M. Davis, Astrophys. J. (1994) in press.
[43] S.M. Faber, G. Wegner, D. Burstein, R.L. Davies, A. Dressler, D. Lynden-Bell and R.J. Terlevich, Astrophys. J. Suppl. 69 (1989) 763.

34
[44] K.B. Fisher, M. Davis, M.A. Strauss, A. Yahil and J.P. Huchra, Astrophys. J. 402 (1993) 42.

[45] C.S. Frenk, Phys. Scripta T36 (1991) 70.

[46] A. Friedman, Zeits. f. Phys. 10 (1922) 377.

[47] G. Gamow, Phys. Rev. 70 (1946) 572.

[48] J.M. Gelb, B.-A. Gradwohl and J.A. Frieman, Astrophys. J. (Lett.) 403 (1993) L5.

[49] J.M. Gelb and E. Bertschinger, preprint (1992) FERMILAB Pub-92/74-A.

[50] M.J. Geller and J.P. Huchra, Science 246 (1989) 897.

[51] R. Giovanelli and M.P. Haynes, Ann. Rev. Astron. Astrophys. 29 (1991) 499.

[52] H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, 1980).

[53] K. Górski, J. Silk and N. Vittorio, Phys. Rev. Lett. 68 (1992) 733.

[54] M. Gramann, Astrophys. J. 405 (1993) L47.

[55] K. Griest, Ann. N.Y. Acad. Sci. 688 (1993) 390.

[56] A.H. Guth, Phys. Rev. D23 (1981) 347.

[57] A.J.S. Hamilton, P. Kumar, E. Lu and A. Matthews, Astrophys. J. (Lett.) 374 (1991) L1.

[58] A.J.S. Hamilton, Astrophys. J. (Lett.) 385 (1992) L5.

[59] A.J.S. Hamilton, Astrophys. J. (Lett.) 406 (1993) L47.

[60] E.R. Harrison, Phys. Rev. D1 (1970) 2726.

[61] J.A. Holtzman, Astrophys. J. Suppl. 71 (1989) 1.

[62] E. Hubble, Proc. Nat. Acad. Sci. 15 (1929) 168.

[63] J.P. Huchra, M.J. Geller, V. de Lapparent and H.C. Corwin, Astrophys. J. Suppl. 72 (1990) 433.

[64] K. Jahoda, preprint (1993) GSFC-LHEA-93-13.

[65] B. Jain and E. Bertschinger, preprint (1993).

[66] J.H. Jeans, Phil. Trans. 199A (1902) 1.

[67] N. Kaiser, Astrophys. J. (Lett.) 284 (1984) L9.

[68] N. Kaiser, Mon. Not. R. Astron. Soc. 227 (1987) 1.
[69] N. Kaiser, G. Efstathiou, R.S. Ellis, C.S. Frenk, A. Lawrence, M. Rowan-Robinson and W. Saunders, Mon. Not. R. Astron. Soc. 252 (1991) 1.

[70] H.E. Kandrup and H. Smith, Astrophys. J. 374 (1991) 255.

[71] N. Katz, L. Hernquist and D.H. Weinberg, Astrophys. J. (Lett.) 399 (1992) L109.

[72] N. Katz and S.D.M. White, Astrophys. J. 412 (1993) 455.

[73] A. Klypin, J.A. Holtzman, J.R. Primack and E. Regos, Astrophys. J. 416 (1993) 1.

[74] E.W. Kolb and M.S. Turner, The Early Universe (Addison-Wesley, Redwood City, 1990).

[75] L.M. Krauss and M. White, Phys. Rev. Lett. 69 (1992) 869.

[76] O. Lahav, P.B. Lilje, J.R. Primack and M.J. Rees, Mon. Not. R. Astron. Soc. 251 (1991) 128.

[77] A.R. Liddle and D.H. Lyth, Phys. Rep. 231 (1993) 1.

[78] J. Loveday, G. Efstathiou, B.A. Peterson and S.J. Maddox, Astrophys. J. (Lett.) 400 (1992) L43.

[79] D. Lynden-Bell, S.M. Faber, D. Burstein, R.L. Davies, A. Dressler, R.J. Terlevich and G. Wegner, Astrophys. J. 326 (1988) 19.

[80] C.-P. Ma and E. Bertschinger, preprint (1993) MIT-CSR–93-14.

[81] H. Martel, Astrophys. J. 366 (1991) 353.

[82] C.A. Masson, Mon. Not. R. Astron. Soc. 188 (1989) 261.

[83] J.C. Mather et al., preprint (1993) COBE 93-01.

[84] E.A. Milne, Relativity, Gravitation and World Structure (Clarendon Press, Oxford, 1935).

[85] A. Nusser, A. Dekel, E. Bertschinger and G.R. Blumenthal, Astrophys. J. 379 (1991) 6.

[86] J.P. Ostriker, Ann. Rev. Astron. Astrophys. 31 (1993) 689.

[87] T. Padmanabhan, Structure Formation in the Universe (Cambridge University Press, 1993).

[88] J.A. Peacock, Mon. Not. R. Astron. Soc. 253 (1991) 1p.

[89] J.A. Peacock, in: New Insights into the Universe, eds. V.J. Martinez, M. Portilla and D. Sáez, (Springer, New York, 1992) 1.
[90] P.J.E. Peebles, The Large Scale Structure of the Universe (Princeton University Press, 1980).

[91] P.J.E. Peebles, Astrophys. J. (Lett.) 263 (1982) L1.

[92] P.J.E. Peebles, Nature 321 (1986) 27.

[93] P.J.E. Peebles, Principles of Physical Cosmology (Princeton University Press, 1992).

[94] P.J.E. Peebles, D.N. Schramm, E.L. Turner and R.G. Kron, Nature 352 (1991) 769.

[95] P.J.E. Peebles and J.T. Yu, Astrophys. J. 162 (1970) 815.

[96] A.A. Penzias and R.W. Wilson, Astrophys. J. 142 (1965) 419.

[97] W.H. Press and P. Schechter, Astrophys. J. 187 (1974) 425.

[98] M. Rowan-Robinson, The Cosmological Distance Ladder (Freeman, New York, 1985).

[99] V.C. Rubin, W.K. Ford and N. Thonnard, Astrophys. J. 238 (1980) 471.

[100] D. Ryu, J.P. Ostriker, H. Kang and R. Cen, Astrophys. J. 414 (1993) 1.

[101] R.K. Sachs and A.M. Wolfe, Astrophys. J. 147 (1967) 73.

[102] A. Sandage, Ann. Rev. Astron. Astrophys. 26 (1988) 561.

[103] W. Saunders, C. Frenk, M. Rowan-Robinson, G. Efstathiou, A. Lawrence, N. Kaiser, R. Ellis, J. Crawford, X.-Y. Xia and I. Parry, Nature 349 (1991) 32.

[104] S.A. Schectman, P.L. Schechter, A.A. Oemler, D. Tucker, R.P. Kirshner and H. Lin, in: Clusters and Superclusters of Galaxies, ed. A.C. Fabian (Kluwer, Dordrecht: Kluwer).

[105] U. Seljak and E. Bertschinger, Astrophys. J. (Lett.) 417 (1993) L9.

[106] U. Seljak and E. Bertschinger, preprint (1993).

[107] S.F. Shandarin and Ya.B. Zel’dovich, Rev. Mod. Phys. 61 (1989) 185.

[108] G.F. Smoot et al., Astrophys. J. (Lett.) 396 (1992) L1.

[109] D.N. Spergel, Astrophys. J. (Lett.) 412 (1993) L5.

[110] M.A. Strauss, J.P. Huchra, M. Davis, A. Yahil, K.B. Fisher and J. Tonry, Astrophys. J. Suppl. 83 (1992) 29.

[111] T. Suginohara and Y. Suto, Astrophys. J. 396 (1992) 395.

[112] A.N. Taylor and M. Rowan-Robinson, Nature 359 (1992) 396.
[113] S. Tremaine, Phys. Today 45 (1992) 28.
[114] A. van Dalen and R.K. Schaeffer, Astrophys. J. 398 (1992) 33.
[115] M.S. Vogeley, C. Park, M.J. Geller and J.P. Huchra, Astrophys. J. (Lett.) 391 (1992) L5.
[116] T.P. Walker, G. Steigman, D.N. Schramm, K.A. Olive and H.-S. Kang, Astrophys. J. 376 (1991) 51.
[117] S.D.M. White, M. Davis, G. Efstathiou and C.S. Frenk, Nature 330 (1987) 451.
[118] S.D.M. White and M.J. Rees, Mon. Not. R. Astron. Soc. 183 (1978) 341.
[119] E.L. Wright et al., Astrophys. J. (Lett.) 396 (1992) L13.
[120] W. Yuan, J.M. Centrella and M.L. Norman, Astrophys. J. (Lett.) 376 (1991) L29.
[121] Ya.B. Zel’dovich, Astron. Astrophys. 5 (1970) 84.
[122] Ya.B. Zel’dovich, Mon. Not. R. Astron. Soc. 160 (1972) 1p.
[123] F. Zwicky, Helv. Phys. Acta 6 (1933) 110.