A CENTRAL LIMIT THEOREM AND HIGHER ORDER RESULTS FOR THE ANGULAR BISPECTRUM∗

Domenico Marinucci
Department of Mathematics, University of Rome “Tor Vergata”
marinucc@mat.uniroma2.it

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Abstract

The angular bispectrum of spherical random fields has recently gained an enormous importance, especially in connection with statistical inference on cosmological data. In this paper, we provide expressions for its moments of arbitrary order and we use these results to establish a multivariate central limit theorem and higher order approximations. The results rely upon combinatorial methods from graph theory and a detailed investigation for the asymptotic behaviour of Clebsch-Gordan coefficients; the latter are widely used in representation theory and quantum theory of angular momentum.

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1. INTRODUCTION

Let $T(\theta, \varphi)$ be a random field indexed by the unit sphere $S^2$, i.e. $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. We assume that $T(\theta, \varphi)$ has zero mean, finite variance and it is mean square continuous and isotropic, i.e. its covariance is invariant with respect to the group of rotations. For isotropic fields, the following spectral representation holds in mean square sense (Yaglom (1986), Leonenko (1999)):

$$T(\theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \varphi). \quad (1)$$

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Here, we have introduced the spherical harmonics (see Varshalovich, Moskalev and Khersonskii (hereafter VMK) (1988), chapter 5), defined by

\[ Y_{lm}(\theta, \varphi) := \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P_{lm}(\cos \theta) \exp(i m \varphi) , \text{ for } m \geq 0 , \]

\[ Y_{lm}(\theta, \varphi) := (-1)^m Y_{l,-m}^*(\theta, \varphi) , \text{ for } m < 0 , \]

where the asterisk denotes complex conjugation and \( P_{lm}(\cos \theta) \) denotes the associated Legendre polynomial of degree \( l, m \), i.e.

\[ P_{lm}(x) := (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) , \quad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l , \]

\[ m = 0, 1, 2, ..., l , \quad l = 1, 2, 3, .... \]

The triangular array \( \{a_{lm}\} \) represents a set of random coefficients, which can be obtained from \( T(\theta, \varphi) \) through the inversion formula

\[ a_{lm} = \int_{-\pi}^{\pi} \int_{0}^{\pi} T(\theta, \varphi) Y_{lm}^*(\theta, \varphi) \sin \theta d\theta d\varphi , \quad m = 0, \pm 1, ..., \pm l , \quad l = 1, 2, ... ; \quad (2) \]

see for instance Kim and Koo (2002), Kim, Koo and Park (2004) for a review of Fourier analysis on \( S^2 \). The coefficients \( a_{lm} \) are complex-valued, zero-mean and uncorrelated; hence, if \( T(\theta, \varphi) \) is Gaussian, they have a complex Gaussian distribution, and they are independent over \( l \) and \( m \geq 0 \) (although \( a_{l,-m} = (-1)^m a_{lm}^* \)), with variance \( E|a_{lm}|^2 = C_l \), \( m = 0, \pm 1, ..., \pm l \). The index \( l \) is usually labeled a multipole; approximately, each multipole corresponds to an angular resolution given by \( \pi/l \).

The sequence \( \{C_l\} \) denotes the angular power spectrum: we shall always assume that \( C_l \) is strictly positive, for all values of \( l \). As well-known, if the field is Gaussian the angular power spectrum completely identifies its dependence structure. For non-Gaussian fields, the dependence structure becomes much richer, and higher order moments of the \( a_{lm} \)'s are of interest; this leads to the analysis of so-called higher order angular power spectra.

The analysis of spherical random fields has recently gained momentum, due to strong empirical motivations arising especially (but not exclusively) from cosmology and astrophysics. In particular, an enormous attention has been drawn by issues connected with the statistical analysis of Cosmic Microwave Background radiation (CMB). CMB can be viewed as a snapshot of the Universe approximately \( 3 \times 10^5 \) years after the Big Bang (Peebles (1993), Peacock (1999)). A number of experiments are aimed at measuring this radiation: we mention in particular two satellite missions, namely WMAP by NASA, which released the first full-sky of CMB fluctuations in February 2003, with much more detailed data to come in the years to come, and Planck by ESA, which is due to be launched in Spring 2007 and expected to provide maps with much greater resolution. Over the next ten years, an immense amount of cosmological information is expected from these huge data sets; at the same time, the
analysis of such data sets posits a remarkable challenge to statistical methodology. In particular, several papers have focused on testing for non-Gaussianity by a variety of nonparametric methods (to mention a few, Doré, Colombi and Bouchet (2003), Hansen, Marinucci and Vittorio (2003), Park (2004), Marinucci and Piccioni (2004), Jin et al. (2005)). The majority of efforts has focused on the angular bispectrum, which is considered an optimal statistic to verify the accuracy of the so-called inflationary scenario, the leading paradigm for the dynamics of the Big Bang. See for instance Phillips and Kogut (2000), Komatsu and Spergel (2001), Bartolo, Matarrese and Riotto (2002), Komatsu et al. (2002, 2003), Babich (2005) and several others; a review is in Marinucci (2004).

The angular bispectrum can be viewed as the harmonic transform of the three-point angular correlation function, whereas the angular power spectrum is the Legendre transform of the (two-point) angular correlation function. Write $\Omega_i = (\theta_i, \varphi_i)$, for $i = 1, 2, 3$; we have

$$ET(\Omega_1)T(\Omega_2)T(\Omega_3) = \sum_{l_1, l_2, l_3 = 1}^{\infty} \sum_{m_1, m_2, m_3} B_{l_1l_2l_3}^{m_1m_2m_3} Y_{l_1m_1}(\Omega_1) Y_{l_2m_2}(\Omega_2) Y_{l_3m_3}(\Omega_3),$$

where the bispectrum $B_{l_1l_2l_3}^{m_1m_2m_3}$ is given by

$$B_{l_1l_2l_3}^{m_1m_2m_3} = E(a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}).$$

Here, and in the sequel, the sums over $m_i$ run from $-l_i$ to $l_i$, unless otherwise indicated. Both (3) and (4) are clearly equal to zero for zero-mean Gaussian fields. Moreover, the assumption that the CMB random field is statistically isotropic entails that the right- and left-hand sides of (3) should be left unaltered by a rotation of the coordinate system. Therefore $B_{l_1l_2l_3}^{m_1m_2m_3}$ must take values ensuring that the three-point correlation function on the left-hand side of (3) remains unchanged if the three directions $\Omega_1, \Omega_2$ and $\Omega_3$ are rotated by the same angle. Careful choices of the orientations entail that the angular bispectrum of an isotropic field can be non-zero only if $l_i \leq l_j + l_k$ for all choices of $i, j, k = 1, 2, 3$; $l_1 + l_2 + l_3$ is even; and $m_1 + m_2 + m_3 = 0$. More generally, Hu (2001) shows that a necessary and sufficient condition for $B_{l_1l_2l_3}^{m_1m_2m_3}$ to represent the angular bispectrum of an isotropic random field is that there exist a real symmetric function of $l_1, l_2, l_3$, which we denote $b_{l_1l_2l_3}$, such that we have the identity

$$B_{l_1l_2l_3}^{m_1m_2m_3} = G_{l_1l_2l_3}^{m_1m_2m_3} b_{l_1l_2l_3};$$

$b_{l_1l_2l_3}$ is labeled the reduced bispectrum. In (5) we are using the Gaunt integral $G_{l_1l_2l_3}^{m_1m_2m_3}$, defined by

$$G_{l_1l_2l_3}^{m_1m_2m_3} := \int_0^{\pi} \int_0^{2\pi} Y_{l_1m_1}(\theta, \varphi) Y_{l_2m_2}(\theta, \varphi) Y_{l_3m_3}(\theta, \varphi) \sin \theta \, d\varphi \, d\theta = \left( \frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \right)^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$
where the so-called “Wigner’s 3j symbols” appearing on the second line are defined by (VMK, expression 8.2.1.5)

\[
\binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3} := (-1)^{l_3+m_3+l_2+m_2} \left( \frac{(l_1 + l_2 - l_3)!(l_1 - l_2 + l_3)!(l_1 - l_2 + l_3)!}{(l_1 + l_2 + l_3 + 1)!} \right)^{1/2} \times \frac{(l_3 + m_3)!}{(l_1 + m_1)!(l_1 - m_1)!(l_2 + m_2)!(l_2 - m_2)!} \times \frac{(-1)^z (l_2 + l_3 + m_1 - z)!(l_1 - m_1 + z)!}{z!(l_2 + l_3 - l_1 - z)!(l_3 + m_3 - z)!(l_1 - l_2 - m_3 + z)!},
\]

where the summation runs over all \(z\)'s such that the factorials are non-negative.

Note that the Wigner’s 3j are invariant with respect to permutations of the pairs \((l_i, m_i)\).

In view of (5), the dependence of the bispectrum ordinates on \(m_1, m_2, m_3\) does not carry any physical information if the field is isotropic; hence it can be eliminated by focussing on the angular averaged bispectrum, defined by

\[
B_{l_1 l_2 l_3} := \sum_{m_1 = -l_1}^{l_1} \sum_{m_2 = -l_2}^{l_2} \sum_{m_3 = -l_3}^{l_3} \binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3} B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \left( \frac{4\pi}{2l_1 + 1)(2l_2 + 1)(2l_3 + 1)} \right)^{1/2} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \cdot \left( \begin{array}{ccc} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \end{array} \right),
\]

(6)

where we have used the orthogonality condition

\[
\sum_{m_1 = -l_1}^{l_1} \sum_{m_2 = -l_2}^{l_2} \sum_{m_3 = -l_3}^{l_3} \binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3}^2 = 1.
\]

The minimum mean square error estimator of the bispectrum is provided by (Hu 2001)

\[
\hat{B}_{l_1 l_2 l_3} := \sum_{m_1 = -l_1}^{l_1} \sum_{m_2 = -l_2}^{l_2} \sum_{m_3 = -l_3}^{l_3} \binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3} (a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}).
\]

The statistic \(\hat{B}_{l_1 l_2 l_3}\) is called the (sample) angle averaged bispectrum; for any realization of the random field \(T\), it is a real-valued scalar, which does not depend on the choice of the coordinate axes and it is invariant with respect to permutation of its arguments \(l_1, l_2, l_3\).

Under Gaussianity, the bispectrum can be easily made model-independent, namely we can focus on the normalized bispectrum, which we define by

\[
I_{l_1 l_2 l_3} := (-1)^{(l_1 + l_2 + l_3)/2} \frac{\hat{B}_{l_1 l_2 l_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3}}},
\]

(7)

The factor \((-1)^{(l_1 + l_2 + l_3)/2}\) is usually not included in the definition of the normalized bispectrum; it corresponds, however, to the sign of the Wigner’s coefficients.
for $m_1 = m_2 = m_3 = 0$, and thus it seems natural to include it to ensure that $I_{l_1l_2l_3}$ and $b_{l_1l_2l_3}$ share the same parity (see (6)).

In practice $I_{l_1l_2l_3}$ is unfeasible because $C_l$ is unknown. A natural estimator for $C_l$ is

$$\hat{C}_l := \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2, \ l = 1, 2, \ldots,$$

which is clearly unbiased (see also Arjunwadkar et al. (2004)). Thus $I_{l_1l_2l_3}$ can be replaced by the feasible statistic

$$\hat{I}_{l_1l_2l_3} := (-1)^{(l_1+l_2+l_3)/2} \frac{\hat{B}_{l_1l_2l_3}}{\sqrt{\hat{C}_{l_1}\hat{C}_{l_2}\hat{C}_{l_3}}}.$$

Although the bispectrum has been the object of an enormous attention in the cosmological literature, very few analytic results are so far available on its probabilistic properties. In a previous paper (Marinucci (2005)), we established some bounds on the behaviour of its first eight moments, and we used these results to establish the asymptotic behaviour of some functionals of the bispectrum array; such functionals were proposed to build nonparametric tests for non-Gaussianity. The results on these moments where established by means of a direct analysis of cross-products of Wigner’s $3j$ coefficients; this analysis was performed by means of explicit summation formulae originated from the quantum angular momentum literature (see VMK for a very detailed collection of results). In the present paper, these results are made sharper and extended to moments of arbitrary orders by means of a more general argument. More precisely, we show how it is possible to associate to higher moments the coefficients of unitary matrices transforming alternative bases of tensor product spaces generated by the spherical harmonics. We are then able to obtain combinatorial expressions for (cross-)moments of arbitrary orders. These results are then exploited to obtain multivariate central limit theorems and higher order approximations. It should be noted that the asymptotic theory presented in this work is of a fixed-domain type, a framework which has become increasingly popular in recent years, see for instance Stein (1999) or Loh (2005).

The structure of the paper is as follows: in Section 2 we review some basic combinatorial material on diagrams and graphs; in Section 3 we present our general results on moments and we exploit it to obtain multivariate central limit theorems for the angular bispectrum with known or unknown $C_l$; in Section 4, we discuss higher-order approximations.

2. DIAGRAMS AND GRAPHS

We shall review here some elementary notions from graph theory, which is widely used in physics when handling Wigner’s $3j$ coefficients (see VMK,
Take $i = 1, \ldots, p$ and $j = 1, \ldots, q_i$, and consider the set of indexes:

$$T = \left\{ (1, 1) \quad \ldots \quad (1, q_1) \\
\quad \ldots \quad \ldots \quad \ldots \\
\quad (p, 1) \quad \ldots \quad (p, q_p) \right\};$$

we stress that the number of columns $q_i$ need not be the same for each row $i$. A diagram $\gamma$ is any partition of the elements of $T$ into pairs like $\{(i_1, j_1), (i_2, j_2)\}$; these pairs are called the edges of the diagram. We label $\Gamma(T)$ the family of these diagrams. We also note that if we identify each row $i_k$ with a vertex (or node), and view these vertexes as linked together by the edges $\{(i_k, j_k), (i_{k'}, j_{k'})\} = i_k i_{k'}$, then it is possible to associate to each diagram a graph. As it is well known, a graph is an ordered pair $(I, E)$ where $I$ is non-empty (in our case the set of the rows of the diagram), and $E$ is a set of unordered pairs of vertexes (in our cases, the pair of rows that are linked in a diagram). We consider only graphs which are not directed, that is, $(i_1 i_2)$ and $(i_2 i_1)$ identify the same edge; however, we do allow for repetitions of edges (two rows may be linked twice), in which case the term multigraph is more appropriate. A graph carries less information than a diagram (the information on the “columns”, i.e. the second element $j_k$, is neglected) but it is much easier to represent pictorially. We shall use some result on graphs below; with a slight abuse of notation, we denote the graph $\gamma$ with the same letter as the corresponding diagram.

We say that

a) A diagram has a flat edge if there is at least a pair $\{(i_1, j_1), (i_2, j_2)\}$ such that $i_1 = i_2$; we write $\gamma \in \Gamma_F(T)$ for a diagram with at least a flat edge, and $\gamma \in \Gamma_F(T)$ otherwise. A graph corresponding to a diagram with a flat edge includes an edge $i_k i_{k'}$ which arrives in the same vertex where it started; for these circumstances the term pseudograph is preferred by some authors (e.g. Foulds (1992)).

b) A diagram $\gamma \in \Gamma_F(T)$ is connected if it is not possible to partition the $i$'s into two sets $A, B$ such that there are no edges with $i_1 \in A$ and $i_2 \in B$. We write $\gamma \in \Gamma_C(T)$ for connected diagrams, $\gamma \in \Gamma_F(T)$ otherwise. Obviously a diagram is connected if and only if the corresponding graph is connected, in the standard sense.

c) A diagram $\gamma \in \Gamma_F(T)$ is paired if, considering any two set of edges $\{(i_1, j_1), (i_2, j_2)\}$ and $\{(i_3, j_3), (i_4, j_4)\}$, then $i_1 = i_3$ implies $i_2 = i_4$; in words, the rows are completely coupled two by two. We write $\gamma \in \Gamma_P(T)$ for paired diagrams.
d) We shall say a diagram has a \( k \)-loop if there exist a sequence of \( k \) edges
\[
\{(i_1, j_1), (i_2, j_2)\}, \ldots, \{(i_k, j_k), (i_{k+1}, j_{k+1})\} = (i_1 i_2, \ldots, i_k i_{k+1})
\]
such that \( i_1 = i_{k+1} \); we write \( \gamma \in \Gamma_{L(k)}(T) \) for diagrams with a \( k \)-loop and no loop of order smaller than \( k \).

Note that \( \Gamma_P(T) = \Gamma_{L(1)}(T) \) (a flat edge is a 1-loop); also, we write
\[
\Gamma_{CL(k)}(T) = \Gamma_C(T) \cap \Gamma_{L(k)}(T)
\]
for connected diagrams with \( k \)-loops, and \( \Gamma_{CL(k)}(T) \) for connected diagrams with no loops of order \( k \) or smaller. For instance, a connected diagram belongs to \( \Gamma_{CL(k)}(T) \) if there are neither flat edges nor two edges \( \{(i_1, j_1), (i_2, j_2)\} \) and \( \{(i_3, j_3), (i_4, j_4)\} \) such that \( i_1 = i_3 \) and \( i_2 = i_4 \); in words, there are no pairs of rows which are connected twice.

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e) A tree is a graph with no loops (written \( \gamma \in \Gamma_T(T) \)).

Graphs and diagrams play a key role to evaluate the behaviour of moments of the bispectrum; to this issue we devote the next section.

3. A CENTRAL LIMIT THEOREM FOR THE BISPECTRUM

In this section, we shall investigate the behaviour of the higher order moments for the normalized bispectrum \( \hat{\mathcal{B}}_n(T) \), under the assumption of Gaussianity; to this aim, we define
\[
\Delta_{l_1 l_2 l_3} := 1 + \delta_{l_1}^{l_2} + \delta_{l_2}^{l_3} + 3\delta_{l_1}^{l_3} = \begin{cases} 1 & \text{for } l_1 < l_2 < l_3 \\ 2 & \text{for } l_1 = l_2 < l_3 \text{ or } l_1 < l_2 = l_3 \\ 6 & \text{for } l_1 = l_2 = l_3 \end{cases}
\]
here and in the sequel, \( \delta_{a}^{b} \) denotes Kronecker’s delta, that is \( \delta_{a}^{b} = 1 \) for \( a = b \), zero otherwise.
Under Gaussianity, it is obvious that the expectation of all odd powers of $I_{l_1l_2l_3}$ is zero. To analyze the behaviour of even powers, we first recall that, for a multivariate Gaussian vector $(x_1, ..., x_{2k})$, we have the following diagram formula

$$
E(x_1 \times x_2 \times ... \times x_{2k}) = \sum (E x_{i_1} x_{i_2}) \times ... \times (E x_{i_{2k-1}} x_{i_{2k}}) , \tag{9}
$$

where the sum is over all the $(2k)!/(k!2^k)$ different ways of grouping $(x_1, ..., x_{2k})$ into pairs (see for instance Giraitis and Surgailis (1987)). Even powers of $I_{l_1l_2l_3}$ yield even powers of the $a_{lm}$’s, which have a complex Gaussian distribution, weighted by Wigner’s $3j$ coefficients.

In the sequel, unless otherwise specified, we rearrange terms so that $l_{i_1} \leq l_{i_2} \leq l_{i_3}$ for all $i$.

**Theorem 3.1** Assume that $(l_{i_1}, l_{i_2}, l_{i_3}) \neq (l_{i_1'}, l_{i_2'}, l_{i_3'})$ whenever $i \neq i'$. There exist an absolute constant $K_{p_1...p_I}$ such that, for $p_i \geq 1$, $i = 1, ..., I$

$$
\left| E \left\{ \prod_{i=1}^{I} t_{l_{i_1}l_{i_2}l_{i_3}}^{2p_i} \right\} - \prod_{i=1}^{I} \left( (2p_i - 1)!! \Delta_{l_{i_1}l_{i_2}l_{i_3}}^{p_i} \right) \right| \leq \frac{K_{p_1...p_I}}{2l_{i_1} + 1} \quad \tag{10}
$$

always, where $(2p - 1)!! = (2p - 1) \times (2p - 3) \times ... \times 1$.

**Remark 3.1** The condition that $(l_{i_1}, l_{i_2}, l_{i_3}) \neq (l_{i_1'}, l_{i_2'}, l_{i_3'})$ whenever $i \neq i'$ is merely notational and entails no loss of generality; indeed, whenever $(l_{i_1}, l_{i_2}, l_{i_3}) = (l_{i_1'}, l_{i_2'}, l_{i_3'})$ it suffices to identify the two indexes and change the values of $p_i$ accordingly.

**Proof** The proof is lengthy and computationally burdensome, so before we proceed we find it useful to sketch heuristically its main features. The first step is to notice that, in view of (9), higher order moments can be associated with sums over all possible graphs configurations of cross-products of Wigner’s $3j$ coefficients. Our aim below will be to show that the contribution of each of these components is determined by its degree of connectivity. Indeed, the leading term will be provided by paired graphs $\gamma \in \Gamma_p(T)$, where the nodes are partitioned into disjoint pairs. Next to that, we shall show that the components where at least $p$ nodes are connected are bounded by $O(l_{i_1}^{-\tilde{p}})$ (a bound that can be improved for some values of $p$). This bound can be obtained by partitioning these connected graphs into trees, that is subgraphs with no loops, and then associating these trees to the coefficients of some unitary matrices arising in tensor spaces generated by spherical harmonics. The proof of (10) can then be simply concluded by a direct graph-counting argument. Let us now make this argument rigorous.

We start by introducing some notation, which is to some extent the same as in Marinucci (2005). We need first to introduce a new set of triples $L = \{ \ldots \}$. Now, let $\mathcal{L} = \{ \ldots \}$.
for brevity’s sake, we write \( \{ l_{11}, l_{12}, l_{13} \} \), defined by

\[
\ell_{r1}, \ell_{r2}, \ell_{r3} := (l_{11}, l_{12}, l_{13}) \text{ for } r = 1, \ldots, 2p_1 \\
\ell_{r1}, \ell_{r2}, \ell_{r3} := (l_{21}, l_{22}, l_{23}) \text{ for } r = p_1 + 1, \ldots, 2p_2 \\
\ldots
\]

more explicitly, the set \( \mathcal{L} \) is obtained by replicating \( 2p_i \) times each of the \((l_{11}, l_{12}, l_{13})\) triples. Let \( T \) be a set of indexes \( \{(r, k)\} \), where \( k = 1, 2, 3 \) and \( r = 1, 2, \ldots, \sum_{i=1}^l p_i \); for any \( \gamma \in \Gamma(T) \), we can define

\[
\delta(\gamma; \mathcal{L}) := \prod_{\{(r, k), (r', k')\} \in \gamma} (-1)^{m_{r'k'a} - m_{r'k''a}} \delta_{\hat{\ell}_{r'k'a}} \delta_{\hat{\ell}_{r'k''a}} ;
\]

for brevity’s sake, we write \( \delta(\gamma) \) rather than \( \delta(\gamma; \mathcal{L}) \) whenever this causes no ambiguity. Recall that

\[
\text{E}
\left.
\prod_{(r,k) \in T} \frac{\alpha_{\ell_{r1}m_{r1}}}{\sqrt{C_{\ell_r}}} \right|_{\ell_{r1} = \ell_{r2} = \ell_{r3} = 1} = \sum_{\gamma \in \Gamma(T)} \delta(\gamma) .
\]

In view of (1), and because the spherical harmonic coefficients are (complex) Gaussian distributed, the following formula holds:

\[
E \left. \prod_{(r,k) \in T} \frac{\alpha_{\ell_{r1}m_{r1}}}{\sqrt{C_{\ell_r}}} \right|_{\ell_{r1} = \ell_{r2} = \ell_{r3}} = \sum_{\gamma \in \Gamma(T)} \delta(\gamma) .
\]

Write \( \{(r, k), \} \in \gamma \) to signify that the pair \( \{(r, k), (r', k')\} \) belongs to \( \gamma \), for some \( (r', k') \); for any diagram \( \gamma \), we can hence define

\[
D(\gamma) := \sum_{m_{rk} = -\ell_{rk}}^{\ell_{rk}} \prod_{r:(r,k) \in T} \left( \begin{array}{ccc} \ell_{r1} & \ell_{r2} & \ell_{r3} \\ m_{r1} & m_{r2} & m_{r3} \end{array} \right) \delta(\gamma) .
\]

We define also

\[
D(A) := \sum_{\gamma \in A} D(\gamma) = \sum_{\gamma \in A} \sum_{m_{rk} = -\ell_{rk}}^{\ell_{rk}} \prod_{r:(r,k) \in T} \left( \begin{array}{ccc} \ell_{r1} & \ell_{r2} & \ell_{r3} \\ m_{r1} & m_{r2} & m_{r3} \end{array} \right) \delta(\gamma) ;
\]

in words, \( D(.) \) represents the component of the expected value that corresponds to a particular set of diagrams. Notice that

\[
E \left. \prod_{i=1}^l T_{l_{11}l_{12}l_{13}}^{2p_i} \right|_{l_{11}l_{12}l_{13}} = \sum_{m_{rk} = -\ell_{rk}}^{\ell_{rk}} \prod_{r:(r,k) \in T} \left( \begin{array}{ccc} \ell_{r1} & \ell_{r2} & \ell_{r3} \\ m_{r1} & m_{r2} & m_{r3} \end{array} \right) \sum_{k=1}^3 \frac{\alpha_{\ell_{rk}m_{rk}}}{\sqrt{C_{\ell_k}}} \right|_{\ell_{r1} = \ell_{r2} = \ell_{r3} = 1} = \sum_{\gamma \in \Gamma(T)} \prod_{\{(r, k), (r', k')\} \in \gamma} \delta(\gamma; \mathcal{L}) = D[\Gamma(T); \mathcal{L}] .
\]
Now
\[ D[\Gamma(T); L] = D[\Gamma_p(T); L] + D[\Gamma(T) \setminus \Gamma_p(T); L] \]
by an identical combinatorial argument as in Marinucci (2005), it is simple to show that
\[ D[\Gamma_p(T); L] = \prod_{i=1}^{l} \{(2p_i - 1)!\Delta_{\ell_{i1}\ell_{i3}}^p\} . \]

To complete the proof, it is then sufficient to establish that
\[ D[\Gamma(T) \setminus \Gamma_p(T); L] = O \left((2l_{11} + 1)^{-1}\right). \]

It is shown in Lemmas 3.1-3.3 in Marinucci (2005) that diagrams with a 1-loop (\(\gamma \in \Gamma_{L(1)}(T)\)) correspond to summands identically equal to zero, whereas diagrams with \(p\) nodes and loops of orders 2 or 3 can be reduced to terms corresponding to diagrams with \(p - 2\) nodes times a factor \(O((2l_{11} + 1)^{-1})\). In the sequel, it is hence sufficient to focus only on graphs which have no loops of orders 1, 2 or 3.

Now call \(R\) the set of nodes of the graphs, and partition it into subsets such that
\[ R = R_1 \cup R_2 \cup \ldots \cup R_g. \]

Then it will also possible to partition \(\gamma\) into subdiagrams \(\gamma_1, \gamma_2, \ldots, \gamma_g, \gamma_{12}, \ldots, \gamma_{g-1,g}\) such that \(\gamma_1\) includes the pairs with both row indexes in \(R_1\), \(\gamma_2\) includes the pairs with both rows in \(R_2\), \(\gamma_{12}\) includes the pairs with one row in \(R_1\) and the other in \(R_2\), and so on; we assume all internal subdiagrams \(\gamma_i\) to be non-empty, whereas this need not be the case for \(\gamma_{ij}\). In terms of edges, \(\gamma_1\) includes the edges that are internal to \(R_1\), \(\gamma_2\) includes the edges that are internal to \(R_2\), \(\gamma_{12}\) includes the edges that connect \(R_1\) to \(R_2\), and so forth. Note that,
\[ \gamma = (\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_g \cup \gamma_{12} \cup \ldots \cup \gamma_{g-1,g}) \]
and we can write
\[
D(\gamma) = \sum_{i=1}^{g} \sum_{j=i+1}^{g} \sum_{m_{r,k}=-\ell_{r,k}}^{\ell_{r,k}} \left\{ \prod_{i=1}^{l} \left( \frac{\ell_{r,k}}{m_{r,1}} \frac{\ell_{r,2}}{m_{r,2}} \frac{\ell_{r,3}}{m_{r,3}} \right) \delta(\gamma_i) \right\} \delta(\gamma_{ij})
\]
\[
= \sum_{i=1}^{g} \sum_{j=i+1}^{g} \sum_{m_{r,k}=-\ell_{r,k}}^{\ell_{r,k}} \left\{ \prod_{i=1}^{g} X_{R_i; \gamma_i} \right\} \delta(\gamma_{ij}),
\]
where
\[
X_{R_i; \gamma_i} := \prod_{m_{r,k}=-\ell_{r,k}}^{\ell_{r,k}} \left( \frac{\ell_{r,1}}{m_{r,1}} \frac{\ell_{r,2}}{m_{r,2}} \frac{\ell_{r,3}}{m_{r,3}} \right) \delta(\gamma_i).
\]
$X_{R_i;\gamma_i}$ can be viewed as a vector whose elements are indexed by $m_{r_i,k_i}$, where $r_i \in R_i$ and $\{(r_i, k_i)\} \notin \gamma_i$ (indeed those indexes $m_{r_i,k_i}$ such that $\{(r_i, k_i)\} \in \gamma_i$ have been summed up internally). For instance, for $g = 2$ we have

$$D(\gamma) = \sum_{\ell_{rk} = -\ell_{rk}} \sum_{\ell_{rk} = -\ell_{rk}} \prod_{r \in R_1} \left( \frac{\ell_{r_1}}{m_{r_1}} \frac{\ell_{r_2}}{m_{r_2}} \frac{\ell_{r_3}}{m_{r_3}} \right) \delta(\gamma_1)$$

$$\times \left\{ \sum_{\ell_{rk} = -\ell_{rk}} \prod_{r \in R_2} \left( \frac{\ell_{r_1}}{m_{r_1}} \frac{\ell_{r_2}}{m_{r_2}} \frac{\ell_{r_3}}{m_{r_3}} \right) \delta(\gamma_2) \right\} \delta(\gamma_{12}).$$

In Figure II, we provide a graph with eight nodes $(R) = 8$ (right), and then (left) we partition it with $g = 2$, $(R_1) = (R_2) = 4$; the nodes in $R_1$ are labelled with a circle, the nodes in $R_2$ are labelled with a cross, the edges in $\gamma_1$ and $\gamma_2$ have a solid line while those in $\gamma_{12}$ are dashed. Here we have $3 + 3 = 6$ internal sums and six external ones.

Assume now that $\gamma_i$ does not include any loop, for $i = 1, ..., g$. Our point will be to show that

$$|D(\gamma)| \leq \prod_{i=1}^{g} \|X_{R_i;\gamma_i}\|$$

where $\|\cdot\|$ denotes Euclidean norm, and

$$\|X_{R_i;\gamma_i}\| \leq \prod_{\{(r,k)\} \in \gamma_i} (2\ell_{rk} + 1)^{-1/2} \leq (2 \min_{\{(r,k)\} \in \gamma_i} \ell_{rk} + 1)(\#(R_i) - 1)/2;$$

(note that if $\gamma_i$ does not include any loop the number of edges it contains must be identically equal to $\#(R_i) - 1$, where $\#(.)$ denotes the cardinality of a set.)
Let us consider (15) first. It is clear that we can choose new indexes such that
\[ X_{R_i;\gamma_i} =: X^{(i)} \], where \( X^{(i)} \) is a vector with elements
\[ X^{(i)} = \left\{ X^{(i)}_{m_{i1}, \ldots, m_{iv_i}} \mid -\ell_{ij} \leq m_{ij} \leq \ell_{ij}, j = 1, \ldots, v_i \right\}, \quad i = 1, \ldots, g; \]
we write \( \tilde{T} \) for this new set of indexes, namely
\[
\tilde{T} = \begin{cases}
(1, 1) & \ldots & (1, v_1) \\
\vdots & \ddots & \vdots \\
g & \ldots & (g, v_g)
\end{cases};
\]
here \( g \) can be viewed as the number of trees and \( v_i \) as the number of vertexes which are in a given tree \( i \). Clearly the vectors \( X^{(1)}, \ldots, X^{(g)} \) have dimensions 
\[ \#(X^{(i)}) = \prod_{j=1}^{v_i} (2\ell_{ij} + 1). \]

The following Lemma can be viewed as an extension of the Cauchy-Schwartz inequality.

**Lemma 3.1 (Generalized Cauchy-Schwartz inequality)** Let \( \tilde{\gamma} \) be a partition of \( \tilde{T} \) with no flat edges.

\[
\left\{ g \prod_{i=1}^{v_i} \sum_{j=1}^{\ell_{ij}} \right\} \left\{ \prod_{i=1}^{g} \left| X^{(i)}_{m_{i1}, \ldots, m_{iv_i}} \right| \right\} |\delta(\tilde{\gamma})| \leq \prod_{i=1}^{g} \left\| X^{(i)} \right\|
\]
and
\[
\left\{ g \prod_{i=1}^{v_i} \sum_{j=1}^{\ell_{ij}} \right\} = \sum_{m_{i1}=-\ell_{i1}}^{\ell_{i1}} \ldots \sum_{m_{v_{vg}}=-\ell_{vg}}^{\ell_{vg}}.
\]

**Proof** The result follows from the iterated application of the Cauchy-Schwartz inequality; we shall argue by induction. Without loss of generality, we can assume that the diagram \( \gamma \) is connected (if it is not, argue separately for the connected components). It is trivial to show that the result holds for \( g = 2 \), indeed in that case it just the standard Cauchy-Schwartz result. Let us now show that if the result holds for the product of \( g - 1 \geq 2 \) components, it must hold for \( g \) components as well. Recall we consider diagrams with no flat edges, so the indexes cannot match on the same vector \( X^{(i)} \). Relabel terms so that there exist (at least) a link (that is, a common index \( m \)) between the first two vectors \( X^{(1)}, X^{(2)} \). Without loss of generality we can order terms in such a way that the matching is internal for the first \( v^* \) indexes and external (that is, with the remaining nodes (3, 4, ..., \( g \))) for \( m_{1j} : j = v^* + 1, \ldots, v_1 \) and \( m_{2j} : j = v^* + 1, \ldots, v_2 \). We write also \( \tilde{\gamma} = \tilde{\gamma}_{12} \cup \tilde{\gamma}_{T_{12}} \), where \( \tilde{\gamma}_{12} \) is the set of edges.
linking node 1 to node 2 and $\tilde{\gamma}_{\Pi_2} = \tilde{\gamma}_{12}$. We have

$$
\left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma})| 
\leq \left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\times \left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\frac{1}{2}.
$$

The last step follows again by standard Cauchy–Schwarz inequality. Now define

$$
X_{m_{1,v^*_i+1} \ldots m_{2v_2}}^{(1;2)} : = \left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\times \left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\frac{1}{2}.
$$

so that $\mathbb{W}$ becomes

$$
\sum_{m_{1,v^*_i+1}=-\ell_{v^*_i}}^{\ell_{v^*_i}} \sum_{m_{2v_2}=-\ell_{2v_2}}^{\ell_{2v_2}} \sum_{m_{31}=-\ell_{31}}^{\ell_{31}} \sum_{m_{vq}=-\ell_{vq}}^{\ell_{vq}} \left\{ X_{m_{1,v^*_i+1} \ldots m_{2v_2}}^{(1;2)} \prod_{i=3}^g X_{m_{i1} \ldots m_{iv_i}}^{(i)} \prod_{i=3}^g \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\leq \left\| X_{m_{1,v^*_i+1} \ldots m_{2v_2}}^{(1;2)} \right\| \left\| \prod_{i=3}^g X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\| 
$$

by the inductive step (indeed within the curly brackets we have the product of $g - 2 + 1 = g - 1$ components). Now notice that

$$
\left\| X_{m_{1,v^*_i+1} \ldots m_{2v_2}}^{(1;2)} \right\| = \left\{ \prod_{i=1} g v_i \sum_{j=1}^{\ell_{ij}} \prod_{m_{ij} = -\ell_{ij}} g \right\} \left\{ \prod_{i=1} X_{m_{i1} \ldots m_{iv_i}}^{(i)} \right\} |\delta(\tilde{\gamma}_{\Pi_2})| 
\frac{1}{2}.
$$
Remark 3.2 We provide an example to make the statement of Lemma 3.1 more transparent. Take
\[ X^{(1)}_{m_{11}m_{12}}, X^{(2)}_{m_{21}m_{22}m_{23}}, X^{(3)}_{m_{31}m_{32}m_{33}}, - \ell_i \leq m_{ij} \leq \ell_i, \ i, j = 1, 2, 3 ; \]
here
\[ \tilde{T} = \left\{ (1, 1) \ (1, 2) \ (2, 1) \ (2, 2) \ (2, 3) \ (3, 1) \ (3, 2) \ (3, 3) \right\} . \]
Now take for instance
\[ \bar{\gamma} = \left[ \{(1, 1), (2, 1)\}, \{(1, 2), (3, 1)\}, \{(2, 2), (3, 2)\}, \{(2, 3), (3, 3)\} \right] , \]
and consider the sum
\[
\left\{ \prod_{i=1}^{3} \prod_{j=1}^{m_{ij}} \sum_{l_{ij}=-\ell_{ij}}^{\ell_{ij}} \right\} |X^{(1)}_{m_{11}m_{12}}X^{(2)}_{m_{21}m_{22}m_{23}}X^{(3)}_{m_{31}m_{32}m_{33}}||\delta(\bar{\gamma})||
\]
\[
= \sum_{m_{11}} \sum_{m_{12}} \sum_{m_{22}} \sum_{m_{23}} |X^{(1)}_{m_{11}m_{12}}X^{(2)}_{m_{21}m_{22}m_{23}}X^{(3)}_{m_{31}m_{32}m_{33}}|
\]
\[
\leq \left\| X^{(1)} \right\| \left\| X^{(2)} \right\| \left\| X^{(3)} \right\| ,
\]
by Lemma 3.1, where
\[
\left\| x^{(1)} \right\| = \sqrt{\sum_{m_{11}m_{12}} (x^{(1)}_{m_{11}m_{12}})^2}, \quad \left\| x^{(2)} \right\| = \sqrt{\sum_{m_{21}m_{22}m_{23}} (x^{(2)}_{m_{21}m_{22}m_{23}})^2},
\]
\[
\left\| x^{(3)} \right\| = \sqrt{\sum_{m_{31}m_{32}m_{33}} (x^{(3)}_{m_{31}m_{32}m_{33}})^2}.
\]
It is not difficult to see that (15) is an immediate consequence of Lemma 3.1; in particular, note that \( \overline{\gamma} \) can be viewed as the diagram which is obtained from \( \gamma \) by identifying all nodes that belong to the same set \( R_i \), for \( i = 1, \ldots, g \). Now let us consider (16). The following result uses the previous inequality to bound the components of a connected graph. Without loss of generality, we re-order terms so that for all \( r \), we have \( \ell_{r1} \leq \ell_{r2} \leq \ell_{r3} \) and

\[
\ell_{11} \leq \ell_{21} \leq \ldots \leq \ell_{R_1}.
\]

Note that the same inequality need not be satisfied for the sequences \( \{\ell_{rk}\}_{r=1,\ldots,R; k=2,3} \).

**Lemma 3.2**

a) Every connected graph \( \gamma \in \Gamma \) with no loops of orders 1,2 or 3 can be partitioned as

\[
\gamma = (\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_g \cup \gamma_{12} \cup \ldots \cup \gamma_{g-1,g})
\]

where \( \gamma_i \) has no loops of any order and is such that \( \#(R_i) \geq 2 \) (in other words, \( \gamma \) can be broken into into binary trees with at least two nodes)

b) For all \( \gamma_i, i = 1, \ldots, g \) we have

\[
\|X_{R_i;\gamma_i}\| \leq \prod_{\{(r,k)\} \in \gamma_i} (2\ell_{rk} + 1)^{-1/2} \leq (2 \min_{\{(r,k)\} \in \gamma_i} \ell_{rk} + 1)^{\#(R_i)-1/2}
\]

that is, every tree with \( p \) nodes corresponds to summands which are \( O((2\ell_{11} + 1)^{-(p-1)/2}) \), where \( p \) is the number of nodes in the trees.

c) For all connected graphs \( \gamma \) with \( p \) nodes, we have

\[
|D[\gamma; L]| \leq \prod_{r=1}^{p/4} (2\ell_{r1} + 1)^{-1}. \tag{19}
\]

**Proof**

a) We drop edges till we reach the point where there are only binary trees or isolated points. Each of these isolated points can be connected to either another isolated point, in which case we simply have a tree with two nodes, or to a binary tree. The graph which is obtained by linking this point to the tree is itself a tree if its paths have at most two edges: recall there are no loops of order 2 or 3. On the other hand, if the graph has a path which covers four nodes, then we delete one edge and obtain two trees with two nodes. These procedures can be iterated until no isolated point remains.

b) We recall the identities (see VMK, chapter 8)

\[
\left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{array} \right) = (-1)^{\ell_3+m_3} \frac{1}{\sqrt{2\ell_3 + 1}} C_{\ell_1,-m_1,\ell_2,-m_2}^{\ell_3,m_3}
\]

\[
C_{\ell_1,m_1,\ell_2,m_2}^{\ell_3,m_3} = (-1)^{\ell_1-\ell_2+m_3} \sqrt{2\ell_3 + 1} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{array} \right),
\]

15
where the coefficients \( C_{\ell_1,m_1,\ell_2,m_2}^{\ell_3,m_3} \) (Clebsch-Gordan coefficients) are the elements of a unitary matrix which implements the change of basis from a tensor product to a direct sum representation for a space spanned by spherical harmonics. More precisely, denote by \( \{Y_{\ell_1} \otimes Y_{\ell_2}\}_{m_1,m_2} \) the elements of a basis for the tensor product \( Y_{\ell_1} \otimes Y_{\ell_2} \); here, \( Y_{\ell} \) is the vector space generated by \( Y_{\ell m} \), \( m = -\ell, \ldots, \ell \): see for instance Vilenkin and Klimyk (1991), Chapter 8 for a full discussion of tensor products and their properties. For our purposes, it suffices to recall the identity

\[
\{Y_{\ell_1} \otimes Y_{\ell_2}\}_{m_1,m_2} = \sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} \sum_{m=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} Y_{\ell m}.
\]

More compactly, we might refer to the \((2\ell_1 + 1)(2\ell_2 + 1) \times (2\ell_1 + 1)(2\ell_2 + 1)\) matrix \( \mathcal{C} \), whose elements \( \{C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m}\} \) are indexed by \( m_1, m_2 \) over the rows and \( \ell, m \) over the columns. The matrix \( \mathcal{C} \) is unitary: it transforms a basis the orthonormality relationships read

\[
\sum_{m_1,m_2} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m'} = \delta_{\ell}^{\ell'} \delta_{m}^{m'},
\]

\[
\sum_{\ell,m} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m'} = \delta_{m_1}^{m_1'} \delta_{m_2}^{m_2'}.
\]

Now the argument can be iterated to higher-order tensor products, to obtain

\[
\{Y_{\ell_1} \otimes Y_{\ell_2} \otimes Y_{\ell_3}\}_{m_1,m_2,m_3} = \sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} \sum_{m=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} \{Y_{\ell} \otimes Y_{\ell_3}\}_{m,m_3}
\]

\[
= \sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} \sum_{\ell_3=|\ell_3-\ell|}^{\ell_3+\ell} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m'} Y_{\ell m} Y_{\ell_3 m'}.
\]

The orthonormality conditions now read

\[
\sum_{m_1,m_2,m_3} \left\{ \sum_{m=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell,m_3,m_3}^{\ell,m'} \right\}^2 = \sum_{\ell\ell'} \left\{ \sum_{m=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell,m_3,m_3}^{\ell,m'} \right\}^2 = 1.
\]

More precisely, the coefficients

\[
C_{\ell_1,m_1,\ell_2,m_2,\ell_3,m_3}^{\ell,m} := \sum_{m=-\ell}^{\ell} C_{\ell_1,m_1,\ell_2,m_2}^{\ell,m} C_{\ell,m_3,m_3}^{\ell,m'}
\]

are the elements of a unitary matrix with \((2\ell_1 + 1)(2\ell_2 + 1)\) rows and columns; the columns are indexed by \( \ell, \ell', m \) and they are

\[
\sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} \sum_{\ell'=|\ell_3-\ell|}^{\ell_3+\ell} (2\ell'+1) = (2\ell_3+1) \sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} (2\ell+1) = (2\ell+1)(2\ell_2+1)(2\ell_3+1).
\]
In general, we have the coefficients

\[
C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} := \sum_{\mu_1 = -\lambda_1}^{\lambda_1} \cdots \sum_{\mu_{p-2} = -\lambda_{p-2}}^{\lambda_{p-2}} C_{\ell_1, m_1; \ell_2, m_2}^{\lambda_1, \mu_1} \times C_{\lambda_1, \mu_1; \ell_3, m_3}^{\lambda_2, \mu_2} \times \cdots \times C_{\lambda_1, \mu_1; \ell_p, m_p}^{\lambda_{p-1}, \mu_{p-1}}
\]

such that

\[
\{Y_{\ell_1} \otimes \cdots \otimes Y_{\ell_p}\}_{m_1, \ldots; m_p} = \sum_{\lambda_1} \cdots \sum_{\lambda_{p-2}} \sum_{\lambda_{p-1} \mu = -\lambda_{p-1}}^{\lambda_{p-1}} C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} Y_{\lambda_{p-1}, \mu}.
\]

whence the orthonormality conditions yield

\[
\sum_{m_1, \ldots, m_p} \left\{ C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} \right\}^2 = \sum_{\lambda_1} \cdots \sum_{\lambda_{p-2}} \sum_{\lambda_{p-1} \mu = -\lambda_{p-1}}^{\lambda_{p-1}} \left\{ C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} \right\}^2 = 1.
\]

It follows that any cross-product of \( p \) Wigner’s \( 3j \) coefficients corresponding to a binary tree can be bounded by a term like

\[
\sum_{m_1, \ldots, m_p, m} \left| \sum_{\lambda_1} \cdots \sum_{\mu_{p-1} = -\lambda_{p-1}}^{\lambda_{p-1}} C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} \right|^2 \leq (2\ell + 1) \prod_{j=1}^{p-1} (2\lambda_j + 1)^{-1}
\]

\[
\times \sum_{m_1, \ldots, m_p, m} \left| \sum_{\lambda_1} \cdots \sum_{\mu_{p-1} = -\lambda_{p-1}}^{\lambda_{p-1}} C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \mu} \right|^2 \leq (2\ell + 1) \prod_{j=1}^{p-1} (2\lambda_j + 1)^{-1} \sum_{m_1, \ldots, m_p, m} \left\{ C_{\ell_1, m_1; \ldots; \ell_p, m_p}^{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \ell, m} \right\}^2 \leq \prod_{j=1}^{p-1} (2\lambda_j + 1)^{-1}.
\]

\( c \) Connected components with loops of order 1, 2 or 3 can be reduced as shown in Marinucci (2005). From part \( b \), we have

\[
|D[\gamma; L]| \leq \prod_{r=1}^{(p-g)/2} (2\ell_r + 1)^{-1}, \tag{20}
\]

whereas from \( a \) we learn that it is always possible to choose a partition such that \( g \leq p/2 \); the result follows immediately.

\( \square \)

**Remark 3.3** An inspection of our argument (especially in \( c \)) reveals that the bound \( (10) \) can be improved in those cases where it is possible to partition the
graph $\gamma$ into a minimal number of trees. For instance, it is known (see for instance Biederharn and Louck (1981)) that all connected graphs with up to 8 nodes can be partitioned into two binary trees; for such cases, we hence obtain the bound

$$|D[\gamma; \mathcal{L}]| \leq \prod_{r=1}^{p/2-1} (2\ell_{r+1} + 1)^{-1}, \ p \leq 8.$$  \hspace{1cm} (21)$$

For $p \geq 10$ it is no longer true that such a partition necessarily exists; we leave for future research, however, to ascertain whether (20) can be improved by a more efficient use of graphical arguments.

We consider now the case where the angular power spectrum is unknown and estimated from the data. Define

$$u_{lm} := \frac{a_{lm}}{\sqrt{C_l}} , \hat{u}_{lm} := \frac{a_{lm}}{\hat{C}_l} , \ m = 0, 1, ..., l ;$$  \hspace{1cm} (22)

from Marinucci (2005) we recall the following simple result.

**Lemma 3.3** Let $l$ and $p$ be positive integers, and define

$$g(l; p) := \prod_{k=1}^{p} \left\{ \frac{2l + 1}{2l + 2k - 1} \right\} .$$  \hspace{1cm} (23)

For $u$ and $\hat{u}$ defined by (22), we have

$$E \left\{ \frac{u_{00}\hat{u}_{00}u_{11}\hat{u}_{11}...u_{l1}\hat{u}_{l1}...u_{lk}\hat{u}_{lk}...u_{lk}\hat{u}_{lk}}{q_0 \text{ times } q_1 \text{ times } q_1 \text{ times } q_2 \text{ times } q_\ell \text{ times } q_\ell \text{ times}} \right\}$$

$$= E \left\{ \frac{u_{00}u_{01}...u_{l1}u_{l1}...u_{lk}u_{lk}...u_{lk}u_{lk}}{q_0 \text{ times } q_1 \text{ times } q_1 \text{ times } q_2 \text{ times } q_\ell \text{ times } q_\ell \text{ times}} \right\} \times g(l; q_0 + q_1 + ... + q_k) .$$

Now partition the indexes $\{\ell_{11}, \ell_{12}, \ell_{13}, ..., \ell_{R1}, \ell_{R2}, \ell_{R3}\}$ into equivalence classes where $\ell_{rk}$ takes the same value; we can then relabel them as $\ell_u, u = 1, ..., c$ where $c$ is the number of such classes, i.e. the number of different values of $\ell$; clearly $c \leq 3R$. For convenience, we write

$$[\{(l_{11}, l_{12}, l_{13}), ... (l_{11}, l_{12}, l_{13}): p_1, ..., p_1\}] := \{\ell_{11}, \ell_{12}, \ell_{13}, ..., \ell_{R1}, \ell_{R2}, \ell_{R3}\} ,$$

and define

$$G[\{(l_{11}, l_{12}, l_{13}), ... (l_{11}, l_{12}, l_{13}): p_1, ..., p_1\}] := \prod_{u=1}^{c} g(\ell_u; \#(u)) / 2$$

$$= \prod_{u=1}^{c} \prod_{j=1}^{(\#(u))/2} \left\{ \frac{2\ell_u + 1}{2\ell_u + 2j - 1} \right\} ,$$
where \( \#(u) \) denotes the cardinality of the class where \( \overline{\ell}_u \) belongs, or in other words the number of times that each particular value is repeated in \( \{ \ell_{11}, \ell_{12}, \ell_{13}, \ldots, \ell_{R1}, \ell_{R2}, \ell_{R3} \} \).

We give some example to make the previous definition more transparent. Consider for notational simplicity the univariate case \( I = 1 \). For \( I_3 < I_2 < I_1 \) we have \( c = 3 \) and

\[
G \left[ \{ (l_{11}, l_{12}, l_{13}), \ldots (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right] = \frac{3^R \prod_{u=1}^{\bar{R}} \left\{ \frac{2l_{u} + 1}{2l_{u} + 2j - 1} \right\}}{j=1}
\]

for \( I_3 < I_2 < I_1 \) we have \( c = 2 \) and

\[
G \left[ \{ (l_{11}, l_{12}, l_{13}), \ldots (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right] = \frac{2^R \prod_{j=1}^{l_{I_1, I_2, I_3}} \left\{ \frac{2l_{I_1, I_2, I_3} + 1}{2l_{I_1, I_2, I_3} + 2j - 1} \right\}}{j=1}
\]

for \( I_3 = I_2 = I_1 \) we have \( c = 1 \) and

\[
G \left[ \{ (l_{11}, l_{12}, l_{13}), \ldots (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right] = 3^R \prod_{j=1}^{\frac{2l_{I_1, I_2, I_3} + 1}{2l_{I_1, I_2, I_3} + 2j - 1}}
\]

It is shown in Marinucci (2005) that (see (23))

\[
E \left[ T_{l_1, l_2, l_3} \right] = E \left[ T_{l_1, l_2, l_3}^p \right] \cdot g(l_1, l_2, l_3; p)
\]

for any \( p \in \mathbb{N} \); likewise, take \( p_1, \ldots, p_I \in \mathbb{N} \). For exactly the same argument it is immediate to obtain

\[
E \left[ \prod_{i=1}^{l_{I_1, I_2, I_3}} \left[ \prod_{i=1}^{l_{I_1, I_2, I_3}} \right] G \left[ \{ (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right]
\]

Notice that

\[
\lim_{l_{I_1} \to \infty} G \left[ \{ (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right] = 1.
\]

The following result extends Theorem 3.1 to the case where the angular power spectrum is estimated from the data.

**Theorem 3.2** Assume that \( (l_{i_1}, l_{i_2}, l_{i_3}) \neq (l_{i'1}, l_{i'2}, l_{i'3}) \) whenever \( i \neq i' \). There exist an absolute constant \( K_{p_1, \ldots, p_I} \) such that, for \( p_i \geq 1 \), \( i = 1, \ldots, I \)

\[
\left| E \left[ \prod_{i=1}^{l_{I_1, I_2, I_3}} \left( \prod_{i=1}^{l_{I_1, I_2, I_3}} (2p_i - 1)!! \Delta_{l_{I_1, I_2, I_3}} \right) G \left[ \{ (l_{I_1, I_2, I_3}); p_1, \ldots, p_I \} \right] \right| \leq \frac{K_{p_1, \ldots, p_I}}{2l_{I_1} + 1}.
\]

**Proof** It is sufficient to combine Theorem 3.1 and (23). \( \square \)
An immediate consequence of the Theorems 3.1 and 3.2 is the following

**Theorem 3.3 (Multivariate Central Limit Theorem)** For any $k \in \mathbb{N}$, as $l_{11} \to \infty$,

$$
\left( \frac{I_{l_{11}l_{12}l_{13}}}{\sqrt{\Delta_{l_{11}l_{12}l_{13}}}}, \ldots, \frac{I_{l_{k1}l_{k2}l_{k3}}}{\sqrt{\Delta_{l_{k1}l_{k2}l_{k3}}}} \right), \left( \frac{\hat{I}_{l_{11}l_{12}l_{13}}}{\sqrt{\Delta_{l_{11}l_{12}l_{13}}}}, \ldots, \frac{\hat{I}_{l_{k1}l_{k2}l_{k3}}}{\sqrt{\Delta_{l_{k1}l_{k2}l_{k3}}}} \right) \to_d N(0, I_k),
$$

where $I_k$ denotes the $(k \times 1)$ identity matrix.

**Proof** The results follow immediately from Theorems 3.1, 3.2 and the method of moments. \hfill \Box

**Remark 3.4** It is immediate to see that, for some $K'_{p_1 \ldots p_I}$, $K''_{p_1 \ldots p_I} > 0$

$$
|G \{ (l_{11}, l_{12}, l_{13}), \ldots, (l_{I1}, l_{I2}, l_{I3}); p_1, \ldots, p_I \} | - 1 | \leq \frac{K'_{p_1 \ldots p_I}}{2l_{11} + 1}
$$

whence Theorem 3.1 can be also formulated as

$$
\left| E \left\{ \prod_{i=1}^I \hat{I}_{l_{i1}l_{i2}l_{i3}}^{2p_i} \right\} - \prod_{i=1}^I \left\{ (2p_i - 1)! \Delta_{l_{i1}l_{i2}l_{i3}}^{p_i} \right\} \right| \leq \frac{K''_{p_1 \ldots p_I}}{2l_{11} + 1}.
$$

A careful inspection of the proofs reveals that the rates of the bounds are exact (we shall come back to this point in the next Section); in other words, the bispectrum converges to a Gaussian distribution with the same rate when either the angular power spectrum is known or unknown (the bounding constants differ, however). This result settles some questions raised in Komatsu et al. (2002), where the distributions of $I_{l_{i1}l_{i2}l_{i3}}$ and $\hat{I}_{l_{i1}l_{i2}l_{i3}}$, where compared by means of Monte Carlo simulations.

**Remark 3.5** It is interesting to note that the angular bispectrum ordinates at different multipoles are asymptotically independent, for any triples $(l_{11}, l_{12}, l_{13}) \neq (l_{21}, l_{22}, l_{23})$.

### 4. HIGHER ORDER RESULTS

The results of the previous sections can be further developed to provide some higher order approximation on the moments of the angular bispectrum. For notational simplicity, in this Section we shall focus on the univariate case, that is, we do not consider cross-moments; the multivariate generalization is straightforward, however, and no new ideas are required. In the statement of
Theorem 4.1
There exist an absolute constant $K_p$ such that

$$\left| \frac{E^p_{i_1 i_2 i_3}}{\Delta^p_{i_1 i_2 i_3}} - (2p - 1)!! - \left\{ \frac{p(p-1)}{6}\kappa_4(l_1, l_2, l_3) \right\} (2p - 1)!! \right| \leq \frac{K_p}{(2l_1 + 1)^2},$$

(27)

where

$$\kappa_4(l_1, l_2, l_3) := \frac{6}{2l_1 + 1} + \frac{6}{2l_2 + 1} + \frac{6}{2l_3 + 1} + 6 \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1 & l_2 & l_3 \end{array} \right\}, \text{ for } l_1 < l_2 < l_3,$$

$$\kappa_4(l_1, l_2, l_3) := \frac{96}{2l_2 + 1} + \frac{24}{2l_3 + 1} + 48 \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1 & l_2 & l_3 \end{array} \right\}, \text{ for } l_1 = l_2 < l_3,$$

$$\kappa_4(l_1, l_2, l_3) := \frac{96}{2l_1 + 1} + \frac{24}{2l_3 + 1} + 48 \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1 & l_2 & l_3 \end{array} \right\}, \text{ for } l_1 < l_2 = l_3,$$

$$\kappa_4(l, l, l) := \frac{6 \times 18^2}{2l + 1} + 6^4 \left\{ \begin{array}{ccc} l & l & l \\ l & l & l \end{array} \right\}, \text{ for } l_1 = l_2 = l_3 = l.$$

It holds that (see (26))

$$|\kappa_4(l_1, l_2, l_3)| \leq \frac{C}{2l_1 + 1}, \text{ some } C > 0.$$

Proof For all $p \in \mathbb{N}$, a direct combinatorial argument yields the following

$$E^p_{i_1 i_2 i_3} = (2p - 1)!!\Delta^p_{i_1 i_2 i_3} (=: D_1(l_1, l_2, l_3))$$

$$+ \left( \begin{array}{c} \frac{2p}{4} \end{array} \right) \kappa_4(l_1, l_2, l_3)(2p - 5)!!\Delta^p_{i_1 i_2 i_3} (=: D_2(l_1, l_2, l_3))$$

$$+ \frac{1}{2} \left( \frac{2p}{4} \right) \left( \frac{2p - 4}{4} \right) \kappa_4(l_1, l_2, l_3)(2p - 9)!!\Delta^p_{i_1 i_2 i_3} (=: D_3(l_1, l_2, l_3))$$

$$+ \left( \begin{array}{c} \frac{2p}{6} \end{array} \right) \kappa_6(l_1, l_2, l_3)(2p - 7)!!\Delta^p_{i_1 i_2 i_3} (=: D_3(l_1, l_2, l_3))$$
where the sum runs over all elements such that the factorials are nonnegative; also, we take $(2p-k)!! = 1$ when $k > 2p$ and

$$\kappa_u(l_1, l_2, l_3) := \sum_{\gamma} D[\gamma; l_1, l_2, l_3] ;$$

where the sum runs over all possible connected graphs with $u$ nodes, that is $\kappa_u(l_1, l_2, l_3)$ represents the expected value corresponding to connected components with $u$ nodes. The term $D_1(l_1, l_2, l_3)$ corresponds to the sums over all graphs with exactly $p$ connected components, each of them with exactly two nodes. The term $D_2(l_1, l_2, l_3)$ correspond to the sums over all graphs with exactly $p-1$ connected components: the number of such graphs corresponds to the possible ways to select 4 nodes out of $p$ and then partition the remaining $2p-4$ into pairs, that is.

$$\left( \frac{2p}{4} \right) \times (2p-5)!! \Delta_{l_1l_2l_3}^{p-2} .$$

The argument for the remaining terms is entirely analogous. Now from the previous section we know that $\kappa_u(l_1, l_2, l_3) = O(l_1^{-\min(2, \frac{3}{2})})$ for $u \geq 6$, whence the result will follow from an explicit evaluation of $\kappa_4(l_1, l_2, l_3)$. For the latter, note that connected graphs with four nodes must include a loop of order 1, 2 or 3. Terms with a 1-loop are identically zero (Marinucci (2005), Lemma 3.1). Graphs associated to terms with 2- or 3-loops ($\gamma \in \Gamma_{L(2)}$ or $\gamma \in \Gamma_{L(3)}$) are represented in Figure III.
We consider the three cases separately.

a) For $l_1 < l_2 < l_3$, we consider first the graphs $\gamma \in \Gamma_{L(2)}$ (on the left-hand side of the figure). By means of Lemma 3.2 in Marinucci (2005), it is simple to ascertain that each corresponding term produces a factor $(2l_i + 1)$, where the $l_i$ correspond to the index which is not in the two-loop (the dashed line). Direct counting of permutations shows that there are 6 such graphs for each fixed $l_i$. Likewise, the Wigner’s $6j$ coefficients arise in connection with the graphs $\gamma \in \Gamma_{L(3)}$ where each node is linked to all three others, compare (25) and Lemma 3.3 in Marinucci (2005); direct counting of possible permutations shows that there can be six distinct combinations of this form (fix for instance node 1, which by assumption is linked to all the other three nodes: there are three degrees of freedom to choose the connection with node 2, then two left for node 3, for a total of six, as claimed).

b) For $l_1 = l_2 \neq l_3$, there are two possible types of graphs $\gamma \in \Gamma_{L(2)}$, that is, those where both 2-loops involve $l_2$ and those where one 2-loop involves $l_2$ and the other $l_3$. In the former case, there are three ways to choose the pairs, two ways in each pair to choose the links, and two ways to choose the way to match the edges corresponding to $l_3$: the total is 24. In the latter case, it can be checked that there are 6 possible choices of pairs and 4 possible matchings for the 2-loop which involves $l_2$ and $l_3$. The remaining term is similar.

c) For $l_1 = l_2 = l_3$, again we choose in three possible ways the pairs, plus the single link in two possible ways; within each pair, we can choose $3 \times 3$ couples and two possible ways to link them. The remaining term is similar.

\[\Box\]

**Remark 4.1** Theorem 4.1 can be used to establish (in a merely formal sense) Edgeworth or Cornish-Fisher type approximations for the asymptotic behaviour of the angular bispectrum (see Hall (1991)). We provide here only some heuristic discussion, and leave for future research the possibility to establish rigorously a valid Edgeworth expansion. Let

\[\mu_{123}(2p) := E \left( \frac{I^{2p}_{l_1,l_2,l_3}}{\Delta^2_{l_1,l_2,l_3}} \right),\]

whence

\[
\sum_{p=1}^{\infty} \mu_{123}^{(2p)} \frac{(it)^{2p}}{2p!} \approx \sum_{p=0}^{\infty} \frac{(it)^{2p}}{2p!} (2p-1)!! + \frac{\kappa_4(l_1,l_2,l_3)}{24} \sum_{p=2}^{\infty} \frac{(it)^{2p}}{2p!} 2p(2p-2)(2p-1)!! + O(l_1^{-2})
\]

\[
= \sum_{p=0}^{\infty} \frac{(it)^{2p}}{2p!} (2p-1)!! + \frac{\kappa_4(l_1,l_2,l_3)}{24} \sum_{p=2}^{\infty} \frac{(it)^{2p}}{(2p-4)!} (2p-5)!! + O(l_1^{-2})
\]

\[
= \sum_{p=0}^{\infty} \frac{(it)^{2p}}{2p!} (2p-1)!! + \frac{\kappa_4(l_1,l_2,l_3)}{24} (it)^4 \sum_{p=0}^{\infty} \frac{(it)^{2p}}{2p!} (2p-1)!! + O(l_1^{-2})
\]

\[
= \exp \left( -\frac{l^2}{2} \right) \left\{ 1 + \frac{\kappa_4(l_1,l_2,l_3)}{24} l^4 \right\} + O(l_1^{-2}).
\]
Again in a formal sense, we can take Fourier transforms on both sides, leading to the conjecture that

$$\Pr \left\{ \frac{l_{123}}{\sqrt{\Delta_{123}}} \leq x \right\} = \int_{-\infty}^{x} \phi(z)dz + \frac{\kappa_4(l_1,l_2,l_3)}{24} \int_{-\infty}^{x} (z^4-6z+3)\phi(z)dz + O(l_1^{-2})$$

where $\phi(z)$ denotes the density function of a standard Gaussian variable. It is remarkable that the first term in this expansion involves the fourth Hermite polynomial $H_4(z) = (z^4 - 6z + 3)$ rather than the second $H_2(z) = (z^2 - 1)$ as it is more commonly the case.

**Remark 4.2** By means of (24), it is immediate to extend Theorem 4.1 to cover the case where the angular power spectrum is estimated from the data; we provide below a detailed calculation for $2p = 4$. Also, higher order moments can also be evaluated iteratively, according to the following expression which holds for $p \geq 3$

$$EI_{123}^{2p} = (2p-1)\Delta_{123}^p + \left( \frac{2p}{4} \right) (2p-5)\Delta_{123}^{p-2} \times [EI_{123}^4 - 3\Delta_{123}^2]$$

$$\quad + \left( \frac{2p}{6} \right) (2p-7)\Delta_{123}^{p-3} \times [EI_{123}^6 - 15\Delta_{123}^3] + \ldots$$

$$\quad + \left( \frac{2p}{2k} \right) (2p-2k-1)\Delta_{123}^{p-k} \times [EI_{123}^{2k} - (2k-1)\Delta_{123}^k] + \ldots$$

$$\quad + \left( \frac{2p}{2p-2} \right) \Delta_{123}^2 \left[ EI_{123}^{2p-2} - (2p-3)\Delta_{123}^{p-1} \right] + \kappa_{2p}(l_1,l_2,l_3)$$

$$= (2p-1)\Delta_{123}^p + \sum_{k=2}^{p-1} \left( \frac{2p}{2k} \right) (2p-2k-1)\Delta_{123}^{p-k} \times [EI_{123}^{2k} - (2k-1)\Delta_{123}^k] + \kappa_{2p}(l_1,l_2,l_3).$$

This expression can also be exploited to derive approximations of moments, where the term $\kappa_{2p}(l_1,l_2,l_3)$ is simply neglected.

**Remark 4.3** It is interesting to note that for $p = 2$ (24) holds with $K_p \equiv 0$. We provide hence an explicit evaluation of these moments. First recall that (VMK, Chapter 9)

$$\left\{ \begin{array}{l} l_1 \ l_2 \ l_3 \\ l_1 \ l_2 \ l_3 \end{array} \right\} = \left[ \frac{(l_1 + l_2 - l_3)!}{(l_1 + l_2 + l_3 + 1)!} \right]^{2} \times$$

$$\sum_{n=l_1+l_2+l_3}^{2l_2+2l_3} \frac{(-1)^n(n+1)!}{((n-l_1-l_2-l_3)!)((2l_1+2l_2-n)!(2l_1+2l_3-n)!)}$$
This expression can be simplified for some values of the triple \((l_1, l_2, l_3)\). More precisely, we have
\[
\left\{ \begin{array}{lll}
1 & l & l \\
1 & l & l
\end{array} \right\} = \sum_{n=3l}^{4l} \frac{(-1)^n(n+1)!}{((n-3l)!)^3((4l-n)!)^3}
\]
and (VMK, eq. 9.5.2.10)
\[
\left\{ \begin{array}{llll}
l_1 & l_2 & l_1+l_2 \\
l_1 & l_2 & l_1+l_2
\end{array} \right\} = \frac{(2l_1)! (2l_2)!}{(2l_1+2l_2+1)!}.
\]
Thus we have
\[
E I^4_{l_1,l_2,l_1+l_2} = 3 + \frac{6}{2l_1+1} + \frac{6}{2l_2+1} + \frac{6}{2l_1+2l_2+1} + \frac{6}{2l_1+2l_2+1} \frac{(2l_1)! (2l_2)!}{(2l_1+2l_2+1)!}, \quad l_1 \neq l_2,
\]
\[
E I^4_{l_1,l_2} = 3 \times 2^2 + \frac{96}{2l+1} + \frac{24}{4l+1} + \frac{48 ((2l)!)^2}{(4l+1)!},
\]
and
\[
E I^4_{l_1} = 3 \times 2^2 + \frac{6 \times 18^2}{2l+1} + 6^4 \frac{(ll)^6}{((3l+1)!)^2} \sum_{n=3l}^{4l} \frac{(-1)^n(n+1)!}{((n-3l)!)^3((4l-n)!)^3}.
\]
The validity of formulas (28)-(30) has been confirmed with a remarkable accuracy by a Monte Carlo experiment (not reported here), where 200 replications of the sample bispectrum at various multipoles \((l_1, l_2, l_3)\) were generated for Gaussian fields; their sample moments were then evaluated and found to be in excellent agreement with our theoretical results.

By means of (24) these results can be immediately extended to the case where the normalization is random. More precisely, we obtain for \(l_1 \neq l_2\)
\[
\hat{E} I^4_{l_1,l_2,l_1+l_2} = \left\{ \begin{array}{lll}
3 & 6 & 6 \\
2l_1+1 & 2l_2+1 & 2l_1+2l_2+1
\end{array} \right\} \frac{(2l_1)! (2l_2)!}{(2l_1+2l_2+1)!},
\]
\[
\hat{E} I^4_{l_1,l_2} = \left\{ \begin{array}{llll}
12 & 96 & 24 \\
2l+1 & 4l+1 & (4l+1)!
\end{array} \right\} \frac{(2l!)^2}{(4l+1)!},
\]
and
\[
\hat{E} I^4_{l_1} = \left[ 3 \times 36 + \frac{6 \times 18^2}{2l+1} + 6^4 \frac{(ll)^6}{((3l+1)!)^2} \sum_{n=3l}^{4l} \frac{(-1)^n(n+1)!}{((n-3l)!)^3((4l-n)!)^3} \right] \times \prod_{k=1}^{6} \left\{ \begin{array}{l}
2l+1 \\
2l+2k-1
\end{array} \right\}.
\]
These expressions can be of practical interest for statistical inference on cosmological data. For instance, the square bispectrum is often used in goodness-of-fit statistics to test the validity of the Gaussian assumption. The previous results yield immediately its exact variance, which so far has been typically evaluated by Monte Carlo simulations (see for instance Komatsu et al. (2002)).

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Address for correspondence:
Dipartimento di Matematica, Universita' di Roma Tor Vergata
via della Ricerca Scientifica 1, Roma, Italy. Postal Code: 00133