Crystalline aspects of geography of low dimensional varieties I: numerology

Kirti Joshi

In memoriam Torsten Ekedahl and Michel Raynaud

Abstract
This is a modest attempt to study, in a systematic manner, the structure of low dimensional varieties in positive characteristics using \( p \)-adic invariants. The main objects of interest in this paper are surfaces and threefolds. There are many results we prove in this paper and not all can be listed in this abstract. Here are some of the results. We prove inequalities related to the Bogomolov–Miyaoka–Yau inequality: in Corollary 4.7 that \( c_1^2 \leq \max(5c_2 + 6b_1, 6c_2) \) holds for a large class of surfaces of general type. In Theorem 4.17 we prove that for a smooth, projective, Hodge–Witt, minimal surface of general type (with additional assumptions such as slopes of Frobenius on \( H^{2 \text{ cris}}(X) \)) are \( \geq 1/2 \) that \( c_1^2 \leq 5c_2 \). We do not assume any lifting, and novelty of our method lies in our use of slopes of Frobenius and the slope spectral sequence. We also construct new birational invariants of surfaces. Applying our methods to threefolds, we characterize Calabi–Yau threefolds with \( b_3 = 0 \). We show that for any Calabi–Yau threefold \( b_2 \geq c_3/2 - 1 \) and that threefolds which lie on the line \( b_2 = c_3/2 - 1 \) are precisely those with \( b_3 = 0 \) and threefolds with \( b_2 = c_3/2 \) are characterized as Hodge–Witt rigid (included are rigid Calabi–Yau threefolds which have torsion-free crystalline cohomology and whose Hodge–de Rham spectral sequence degenerates).

Keywords Crystalline cohomology · de Rham–Witt complex · Domino numbers · Hodge–Witt numbers · Chern number inequalities · Bogomolov–Miyaoka–Yau inequality · Calabi–Yau varieties · Quintic threefolds · Hypersurfaces · Frobenius split varieties · Algebraic surfaces · Projective surfaces

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Kirti Joshi
kirti@math.arizona.edu

1 Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA
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K. Joshi
1 Introduction

This is a modest attempt to study, in a systematic manner, the structure of low dimensional varieties using $p$-adic invariants. The main objects of interest in this paper are surfaces and threefolds. It is known that there are many examples of unexpected behavior, including the failure of the famous Bogomolov–Miyaoka–Yau inequality, in surfaces and even exotic behavior in threefolds. Our focus here is on obtaining general positive results.
We will take as thoroughly understood, the theory of algebraic surfaces over complex numbers. In Sect. 2 we recall results in crystalline cohomology, and the theory of de Rham–Witt complex which we use in this paper. The reader familiar with [10–12,17,20] is strongly advised to skip this section. Here are some questions which this paper attempts to address.

1.1 The problem of Chern number inequalities for surfaces

In Sect. 3 we begin with one of the main themes of the paper. Let \( X/k \) be a smooth, projective surface over an algebraically closed field \( k \), of characteristic \( p > 0 \). There has been considerable work, of great depth and beauty, on the Bogomolov–Miyaoka–Yau inequality in positive characteristic (as well as in characteristic zero). Despite this, our understanding of the issue remains, at the best, rather primitive. It has been known for some time that the famous inequality of Bogomolov–Miyaoka–Yau (see [2,32,45]):

\[
c^2_1 \leq 3c_2
\]

for Chern numbers of \( X \) fails in general. In fact it is possible to give examples of surfaces of general type in characteristic \( p \geq 5 \) such that

\[
c^2_1 > pc_2,
\]

and even surfaces for which

\[
c^2_1 > p^nc_2,
\]

for suitable \( n \geq 1 \).

The following questions arise at this point:

- What is the weakest inequality for Chern numbers which holds for a large class of surfaces?
- Is there a class of surfaces for which one can prove some inequality of the form \( c^2_1 \leq Ac_2 \) (with \( A > 0 \))?
- Where does the obstruction to Chern class inequality for \( X \) originate?

The questions are certainly quite vague, but also quite difficult: as the examples of surfaces of general type with \( c^2_1 > pc_2 \) illustrate, and are not new (for example the first was proposed in [40]). In this paper we provide some answers to all of these questions. Our answers are not satisfactory, at least to us, but should serve as starting point for future investigations and to pose more precise questions.

Much of the existent work on this subject has been carried out from the geometric point view. However as we point out in this paper, the problem is not only purely of geometric origin, but rather of arithmetic origin. More precisely, infinite \( p \)-torsion of the slope spectral sequence intervenes in a crucial way, already, in any attempt to
prove the weaker inequality first considered in [44]:

\[ c_1^2 \leq 5c_2. \]

Before proceeding further, let us dispel the notion that infinite torsion in the slope spectral sequence is in any way “pathological.” Indeed any supersingular K3 surface, any abelian variety of dimension \( n \) and of \( p \)-rank at most \( n - 2 \), and any product of smooth, projective curves with supersingular Jacobians, and even Fermat varieties of large degree (for \( p \) satisfying a suitable congruence modulo the degree), all have infinite torsion in the slope spectral sequence (this list is by no means exhaustive or complete) and as is well-known, this class of varieties is quite reasonable from every other geometric and cohomological point of view. So we must view the presence of infinite torsion as the entry of the subtler arithmetic of the slope spectral sequence of the variety into the question of its geometry rather than as a manifestation of any pathological behavior.

Consider the first question. In Sect. 4 we take up the topic of Chern class inequalities of the type \( c_1^2 \leq 5c_2 \) and \( c_1^2 \leq 5c_2 + 6b_1 \) (such inequalities were first considered in [44]). We show in Theorem 4.6 that

\[ c_1^2 \leq 5c_2 + 6b_1 \]

for a large class of surfaces. In particular one has for such surfaces (Corollary 4.7):

\[ c_1^2 \leq \max(5c_2 + 6b_1, 6c_2), \]

and if \( b_1 = 0 \) then \( c_1^2 \leq 5c_2 \). Surfaces covered in Theorem 4.6 include surfaces whose Hodge–de Rham spectral sequence degenerates at \( E_1 \) or which lift to \( W_2 \) and have torsion-free crystalline cohomology (i.e., Mazur–Ogus surfaces), or surfaces which are ordinary or more generally Hodge–Witt. Moreover there also exist surfaces which do not satisfy these inequalities.

The reader familiar with geography of surfaces over \( \mathbb{C} \) will recall that geography of surfaces (see [36]) over \( \mathbb{C} \) is planar—with the numbers \( c_1^2, c_2 \) serving as variables in a plane and the Bogomolov–Miyaoka–Yau line \( c_1^2 = 3c_2 \) representing the absolute boundary beyond which no surfaces of general type can live. On the other hand, we show that the geography of surfaces in positive characteristic is non-planar and three-dimensional, involving variables \( c_1^2, c_2, b_1 \) with the plane \( c_1^2 = 5c_2 + 6b_1 \) serving as a natural boundary and unlike the classical case of surfaces over \( \mathbb{C} \), the region \( c_1^2 > 5c_2 + 6b_1 \) is also populated. In our view surfaces which live in the region \( c_1^2 \leq 5c_2 + 6b_1 \) are the ones which can hope to be understood.

Proposition 4.1 provides the following important criterion:

\[ c_1^2 \leq 5c_2 \iff m^{1,1} - 2T^{0,2} \geq b_1, \]

here the term \( T^{0,2} \) is of de Rham–Witt origin and is a measure of infinite torsion in the slope spectral sequence, and the term \( m^{1,1} \) is of crystalline origin involving slopes of Frobenius in the second crystalline cohomology of \( X \). This criterion translates the
problem of Chern number inequality to an inequality involving slopes of Frobenius, de Rham–Witt contributions, and is the reason for our contention that the problem of the Bogomolov–Miyaoka–Yau inequality in positive characteristic is intimately related to understanding the influence of the infinite torsion in the slope spectral sequence.

But the inequality $m^{1,1} - 2T^{0,2} \geq b_1$ is still difficult to deal with in practice: the $m^{1,1}$ term and the $b_1$ terms live in two different cohomologies ($H^1_{\text{cris}}$ and $H^2_{\text{cris}}$ resp.) and these two cohomologies are not as correlated as they are in characteristic zero (the proof of [44] can be viewed as establishing a correlation between $H^1_{\text{dR}}$ and $H^2_{\text{dR}}$ in characteristic zero).

The next step is to resolve this difficulty. This is carried out in Proposition 4.12 (resp. Proposition 4.14) for Hodge–Witt surfaces (resp. for Mazur–Ogus surfaces), which show that

$$m^{1,1} \geq 2p_g \implies c_1^2 \leq 5c_2 \quad \text{(resp. } m^{1,1} \geq 4p_g \implies c_1^2 \leq 5c_2).$$

(Here $m^{1,1}$ is the slope number (see Definition 2.22) and $p_g$ is the geometric genus.) The criterion of Proposition 4.12 is used in the proof of one of the main results of this paper: Theorem 4.17 which we discuss next.

**Theorem** If $X/k$ is a smooth, projective, minimal surface of general type over a perfect field of characteristic $p > 0$ which satisfies the following conditions:

(i) $c_2 > 0$,
(ii) $p_g > 0$,
(iii) $X$ is Hodge–Witt,
(iv) $\text{Pic}(X)$ is reduced or $H^2_{\text{cris}}(X/W)$ is torsion-free, and
(v) $H^2_{\text{cris}}(X/W)$ has no slope $< 1/2$.

Then

$$c_1^2 \leq 5c_2.$$

The novelty of our method lies in our use of slopes of Frobenius to prove such an inequality (when de Rham–Witt torsion is controlled—by the Hodge–Witt hypothesis). It seems to us that this is certainly not the most optimal result which can be obtained by our methods, but should serve as a starting point for understanding the Bogomolov–Miyaoka–Yau type inequality for surfaces using slopes of Frobenius. For a detailed discussion of our hypothesis and simple examples of surfaces of general type which satisfy all of the above hypotheses the reader is referred to the remarks preceding and following the theorem, but here we point out that in Theorem 4.23 we show that surfaces which have no slope zero part in $H^2_{\text{cris}}$ have slopes bounded from below by $1/(p_g + 1)$. At any rate we demonstrate that the class of surfaces which satisfy assumptions (i)–(v) is a non-empty (in general) and form a locally closed subset in moduli if we assume fixed Chern classes.

Let us also point out that after posting the 2012 version of this paper on the preprint arxiv, I was informed by Adrian Langer that he obtained a proof of the Bogomolov–Miyaoka–Yau inequality for surfaces lifting to $W_2$ (see [27]). This is certainly an
important development in the subject. However note that in Theorem 4.17 we do not assume any lifting hypothesis and our methods have no overlap with those of [27].

Let us remark that for ordinary surfaces $m^{1,1} - 2 T^{0,2} \geq b_1$, as $T^{0,2} = 0$ by a theorem of [20], one reduces to the inequality

$$m^{1,1} \geq b_1$$

and this is still non-trivial to prove (and we do not know how to prove it). If we assume in addition to ordinarity that $X$ has torsion-free crystalline cohomology, then $m^{1,1} = h^{1,1}$ and hence the inequality is purely classical—involving Hodge and Betti numbers. But the proof of the inequality in this case would, nevertheless be non-classical. In this sense the ordinary case is closest to the classical case, but it appears to us that this case nevertheless lies beyond classical geometric methods. In Sect. 4.10 we describe a recurring fantasy to prove

$$c_1^2 \leq 5 c_2 + 6$$

for ordinary surfaces (this was not included in any version prior to the 2012 version of this paper) and also

$$c_1^2 \leq 6 c_2$$

for all ordinary surfaces except for those with $0 \leq c_1^2 < 36$, $0 \leq c_2 < 6$ (which form a bounded family at any rate). This recurring fantasy should be considered the de Rham–Witt avatar of van de Ven’s Theorem [44].

Surfaces which do not satisfy $c_1^2 \leq 5 c_2 + 6 b_1$ are particularly extreme cases of failure of the Bogomolov–Miyaoka–Yau inequality and their properties are studied in Theorem 3.1 and the remarks following it. They all exhibit the following properties: are not Hodge–Witt, exhibit non-degeneration of Hodge de Rham or presence of crystalline torsion, and if $c_2 > 0$ then $\Omega^1_X$ is unstable.

The key tool in these and other results proved in this paper are certain invariants of non-classical nature, called Hodge–Witt numbers, which were introduced by T. Ekedahl (see [12]) and use a remarkable formula of R. Crew (see loc. cit.), and in particular for surfaces the key tool is the Hodge–Witt number $h^{1,1}_W$ of surfaces. This integer can be negative (and its negativity signals failure of Chern class inequalities) and in fact it can be arbitrarily negative. In Proposition 3.4 we note that for a smooth, projective surface $h^{1,1}_W$ is the only one which can be negative, the rest of $h^{i,j}$ are non-negative. In Sect. 3.3 we use the Enriques classification of surfaces to prove that if $h^{1,1}_W$ is negative then $X$ is of general type or quasi-elliptic (these occur only for $p = 2, 3$). In Sect. 4.6 we investigate lower bounds on $h^{1,1}_W$. For instance we note that if $X$ is of general type then $-c_1^2 \leq h^{1,1}_W \leq h^{1,1}$, except possibly for $p \leq 7$ and $X$ is fibered over a curve of genus at least two and the generic fiber is a singular rational curve of arithmetic genus at most four. It seems rather optimistic to conjecture that if $b_1 \neq 0$ and $h^{1,1}_W < 0$ then $X \to \text{Alb}(X)$ has one dimensional image (and so such surfaces admit a fibration with an irrational base).
1.2 A Chern inequality for Calabi–Yau threefolds

In Sect. 7 we take up the study of Hodge–Witt numbers of smooth projective threefolds. Applying our methods to Calabi–Yau threefolds, we show that up to symmetry, the only possibly negative Hodge–Witt number is \( h_{1,2}^W \) and in Theorem 7.4 we obtain a complete characterization (in any positive characteristic) of Calabi–Yau threefolds with \( h_{1,2}^W < 0 \). These are precisely the threefolds for which the Betti number \( b_3 = 0 \). As a corollary we deduce (in Corollary 7.5) that if \( X \) is any smooth, projective Calabi–Yau threefold then \( h_{1,2}^W \geq -1 \), which is equivalent to the inequality (valid for all projective Calabi–Yau threefolds in positive characteristic):

\[
b_2 \geq \frac{1}{2} c_3 - 1.
\]

On the other hand, the condition \( h_{1,2}^W \geq 0 \) (for Calabi–Yau threefolds) is equivalent to the geometric inequality (always valid in characteristic zero):

\[
b_2 \geq \frac{1}{2} c_3.
\]

In particular such threefolds which do not satisfy this inequality cannot lift to characteristic zero. Most known examples of non-liftable Calabi–Yau threefolds have this property \( (b_3 = 0) \) and it is quite likely, at least if \( H_{\text{cris}}^2(X/W) \) is torsion-free, that all non-liftable Calabi–Yau threefolds are of this sort. In Sect. 7.6 we establish some results in geography of Calabi–Yau threefolds. In particular we note that unlike characteristic zero, in characteristic \( p > 0 \) there is a new region in the geography of such threefolds which is unconnected from the realm of classical Calabi–Yau threefolds and many non-liftable Calabi–Yau threefolds live on this island (see the section for the definition).

We expect that if a Calabi–Yau threefold has non-negative Hodge–Witt numbers and torsion-free crystalline cohomology then it lifts to \( W_2 \) (for \( p \geq 5 \)). Our conjecture is that if \( H_{\text{cris}}^2(X/W) \) is torsion-free, equivalently if \( H^0(X, \Omega^1_X) = 0 \), for a Calabi–Yau threefold, and if this inequality holds then \( X \) lifts to \( W_2 \) (for \( p \geq 5 \)). In Sect. 7.8 we describe our formulation of a remarkable recent result of Yobuko (see [46]) which provides some evidence towards our conjecture. In Proposition 7.11 we show that if \( X \) is Mazur–Ogus then

\[
c_3 = 2b_2 \iff h_{1,2}^W = 0 \iff X \text{ is a rigid Calabi–Yau threefold.}
\]

1.3 Other results proved in the paper

Here are some of the other results we prove in this paper.
1.3.1 Computing domino numbers of Mazur–Ogus varieties

We prove (see Theorem 2.5) that the domino numbers of a smooth, projective variety whose Hodge–de Rham spectral sequence degenerates and whose crystalline cohomology is torsion-free are completely determined by the Hodge numbers and the slope numbers (in other words they are completely determined by the Hodge numbers and the slopes of Frobenius). This had been previously proved by Ekedahl [19] for abelian varieties. In Sect. 2.38 we compute domino numbers of smooth hypersurfaces in $\mathbb{P}^n$ (for $n \leq 4$).

1.3.2 Birational invariance of domino numbers of smooth, projective surfaces

The other question of interest, especially for surfaces is: how do the various $p$-adic invariants reflect in the Enriques–Kodaira classification? Of course, the behavior of cohomological invariants is quite well-understood. But in positive characteristic there are other invariants which are of a non-classical nature. These new invariants of surfaces are at moment defined only for smooth, projective surfaces. But they are of birational nature in the sense that: two smooth, projective surfaces which are birational surfaces have the same invariants. These are: the $V$-torsion, the Néron–Severi torsion, the exotic torsion and in Proposition 5.2 we prove that if $X'$ and $X$ are two smooth surfaces which are birational then the dominos associated to the corresponding differentials

$$H^2(X, W(0_X)) \rightarrow H^2(X, W\Omega^1_X)$$

and

$$H^2(X', W(0_{X'})) \rightarrow H^2(X', W\Omega^1_{X'})$$

are naturally isomorphic. In particular the dimension of the domino, denoted by $T^{0.2}(X)$, is a new birational numerical invariant of smooth, projective surfaces which lives only in positive characteristic. We also study torsion in $H^2_{\text{cris}}(X/W)$ in terms of the Enriques classification, making precise several results found in the existing literature.

1.3.3 Crystalline torsion and a question of V.B. Mehta

In Sect. 5 we digress a little from our main themes. This section may be well-known to the experts. We consider torsion in the second crystalline cohomology of $X$. It is well-known that torsion in the second cohomology of a smooth projective surface is a birational invariant, and in fact the following variant of this is true in positive characteristic (see Proposition 5.3): torsion of every species (i.e., Néron–Severi, the $V$-torsion, and the exotic torsion) is a birational invariant. We also note that surfaces of Kodaira dimension at most zero do not have exotic torsion. The section ends with a criterion for absence of exotic torsion which is often useful in practice.
In Theorem 5.11 we show that if $X$ is a smooth, projective surface of general type with exotic torsion then $X$ has Kodaira dimension $\kappa(X) \geq 1$. A surface with $V$-torsion must either have $\kappa(X) \geq 1$ or $\kappa(X) = 0$, $b_2 = 2$ and $p_g = 1$ or $p = 2$, $b_2 = 10$ and $p_g = 1$. This theorem is proved via Proposition 5.9 where we describe torsion in $H^2_{\text{cris}}(X/W)$ for surfaces with Kodaira dimension zero.

In Sect. 6 we answer a question of Mehta (for surfaces) about crystalline torsion. We show that any smooth, projective surface $X$ of Kodaira dimension at most zero has a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X'/W)$ is torsion-free. Thus crystalline torsion in these situations, can in some sense, be uniformized, or controlled. We do not know if this result should be true without the assumption on Kodaira dimension.

In [21] which is a thematic sequel (if we get around to completing it) to this paper we will study the properties of a refined Artin invariant of families of Mazur–Ogus surfaces and related stratifications. While bulk of this paper was written more than a decade ago, and after a few rejections from journals where I thought (rather naively) that the paper could (or perhaps should) appear, I lost interest in its publication and the paper has gestated at least since 2007. Over time there have been many additions which I have made to this paper (and I have rewritten the introduction to reflect my current thinking on this matter), but theme of the paper remains unchanged. Notable additions are: Theorem 4.17 and Propositions 4.12 and 4.14 which are of later vintage (being proved around 2012–2013); and Sect. 4.26 has matured for many years now, but has been added more recently; I also added a computation of domino numbers of hypersurfaces in Sect. 2.38 around 2009. In 2017 during the course of revisions suggested by the referee I also proved and added Theorem 4.23 (which sheds some light on the hypothesis of Theorems 4.17, 2.7). Theorem 7.14 was added upon reading Yobuko’s preprint which also appeared in 2017.

1.4 Acknowledgements

Untimely death of Torsten Ekedahl in 2011 reminded me of this manuscript again and I decided to revive it from its slumber (though it takes a while to wake up after such a long slumber). Unfortunately, Michel Raynaud also passed away while this paper was being revised for publication.

*I dedicate this paper to the memory of Torsten Ekedahl and Michel Raynaud. I never had the opportunity to meet Ekedahl, but his work has been a source of inspiration for a long time. The few meetings I had with Raynaud during my visits to Orsay, I recall with great pleasure.*

This paper clearly owes its existence to the work of Richard Crew, Torsten Ekedahl, Luc Illusie, and Michel Raynaud. I take this opportunity to thank Luc Illusie and Michel Raynaud for encouragement. Thanks are also due to Minhyong Kim for constant encouragement while early versions of this paper were being written (he was at Arizona at the time the paper was written). I would like to thank the Korea Institute of Advanced Study and especially thank the organizers of the International Workshop on Arithmetic Geometry in the fall of 2001, where some of the early results were announced, for support.
I am also deeply indebted to the referee for a number of suggestions and corrections which have vastly improved this manuscript and also to the Editors of this Journal, especially Fedor Bogomolov, for their patience in face of the long delay in revising this manuscript.

I have tried to keep this paper as self-contained as possible. This may leave the casual reader the feeling that these results are elementary, and to a certain extent they are; but we caution the reader that this feeling is ultimately illusory as we use deep work of Crew, Ekedahl, Illusie, and Raynaud on the slope spectral sequence which runs into more than four hundred pages of rather profound and beautiful mathematics.

2 Notations and preliminaries

2.1 Witt vectors

Let $p$ be a prime number and let $k$ be a perfect field of characteristic $p$. Let $\bar{k}$ be an algebraic closure of $k$. Let $W = W(k)$ be the ring of Witt vectors of $k$ and let $W_n = W/p^n$ be the ring of Witt vectors of $k$ of length $n \geq 1$. Let $K$ be the quotient field of $W$. Let $\sigma$ be the Frobenius morphism $x \mapsto x^p$ of $k$ and let $\sigma : W \rightarrow W$ be its canonical lift to $W$. We will also write $\sigma : K \rightarrow K$ for the extension of $\sigma : W \rightarrow W$ to $K$.

2.2 The Cartier–Dieudonné–Raynaud algebra

Following [20, p. 90] we write $R$ for the Cartier–Dieudonné–Raynaud algebra or more simply the Raynaud algebra of $k$. Recall that $R$ is a $W$-algebra generated by symbols $F, V, d$ with the following relations:

\[
FV = p, \\
VF = p, \\
\text{and for all } a \in W, \quad Fa = \sigma(a)F, \\
aV = V\sigma^{-1}(a), \\
d^2 = 0, \\
FdV = d, \\
da = ad.
\]

The Raynaud algebra is graded $R = R^0 \oplus R^1$ where $R^0$ is the $W$-subalgebra generated by symbols $F, V$ with relations above and $R^1$ is generated as an $R^0$ bi-module by $d$ (see [20, p. 90]).

2.3 Explicit description of $R$

Every element of $R$ can be written uniquely as a sum
\[
\sum_{n > 0} a_{-n} V^n + \sum_{n \geq 0} a_n F^n + \sum_{n > 0} b_{-n} dV^n + \sum_{n \geq 0} b_n F^n d
\]

where \(a_n, b_n \in W\) for all \(n \in \mathbb{Z}\) (see [20, p. 90]).

2.4 Graded modules over \(R\)

Any graded \(R\)-module \(M\) can be thought of as a complex \(M = M^*\) where \(M^i\) for \(i \in \mathbb{Z}\) are \(R^0\) modules and the differential \(M^i \rightarrow M^{i+1}\) is given by \(d\) with \(FdV = d\) (see [20, p. 90]). From now on we will assume that all \(R\)-modules are graded.

2.5 Canonical filtration

On any \(R\)-module \(M\) we define a filtration (see [20, p. 92]) by

\[
\text{Fil}^n M = V^n M + dV^n M, \\
\text{gr}^n M = \text{Fil}^n M / \text{Fil}^{n+1} M.
\]

In particular we set \(R_n = R / \text{Fil}^n R\).

2.6 Topology on \(R\)

We topologize an \(R\)-module \(M\) by the linear topology given by \(\text{Fil}^n M\) (see [20, p. 92]).

2.7 Complete modules

We write \(\hat{M} = \text{proj lim}_n M / \text{Fil}^n M\) and call \(\hat{M}\) the completion of \(M\). We say \(M\) is complete if \(\hat{M} = M\). Note that \(\hat{M}\) is complete and one has \((\hat{M})^i = \text{proj lim}_n M^i / \text{Fil}^n M^i\) (see [20, Section 1.3, p. 90]).

2.8 Differential

Let \(M\) be an \(R\)-module, for all \(i \in \mathbb{Z}\) we let

\[
Z^i M = \ker (d : M^i \rightarrow M^{i+1}), \\
B^i M = d(M^{i-1}).
\]

The \(W\)-module \(Z^i M\) is stable by \(F\) but not by \(V\) in general and we let

\[
V^{-\infty} Z^i = \bigcap_{r \geq 0} V^{-r} Z^i
\]
where \( V^{-r} Z^i = \{ x \in M^i \mid V^r(x) \in Z^i \} \). Then \( V^{-\infty} Z^i \) is the largest \( R^0 \)-submodule of \( Z^i \) and \( M^i/V^{-\infty} Z^i \) has no \( V \)-torsion (see [20, p. 93]).

The \( W \)-module \( B^i \) is stable by \( V \) but not in general by \( F \). We let

\[
F^\infty B^i = \bigcup_{s \geq 0} F^s B^i.
\]

Then \( F^\infty B^i \) is the smallest \( R^0 \)-submodule of \( M^i \) which contains \( B^i \) (see [20, p. 93]).

### 2.9 Canonical factorization

The differential \( M^i-1 \to M^i \) factors canonically as

\[
M^i-1 \to M^i-1/V^{-\infty} Z^i-1 \to F^\infty B^i \to M^i
\]

(see [20, p. 93, 1.4.5]). We write \( \tilde{H}^i(M) = V^{-\infty} Z^i/F^\infty B^i \), it is called the heart of the differential \( d : M^i \to M^{i+1} \) and we will say that the differential is heartless if its heart is zero.

### 2.10 Profinite modules

A (graded) \( R \)-module \( M \) is profinite if \( M \) is complete and for all \( n, i \) the \( W \)-module \( M^i/Fil^n M^i \) is of finite length (see [20, Definition 2.1, p. 97]).

### 2.11 Coherence

A (graded) \( R \)-module \( M \) is coherent if it is of bounded degree, profinite and all the hearts \( \tilde{H}^i(M) \) are of finite type over \( W \) (see [20, Theorem 3.8, 3.9, p. 118]).

### 2.12 Dominos

An \( R \)-module \( M \) is a domino if \( M \) is graded, profinite, concentrated in two degrees (say) 0, 1 and \( V^{-\infty} Z^0 = 0 \) and \( F^\infty B^1 = M^1 \).

If \( M \) is any graded, profinite \( R \)-module then the canonical factorization of \( d : M^i \to M^{i+1} \), given in (1), gives a domino: \( M^i/V^{-\infty} Z^i \to F^\infty B^{i+1} \) (see [20, 2.16, p. 110]), which we call the domino associated to the differential \( d : M^i \to M^{i+1} \).

### 2.13 Dimension of a domino

Let \( M \) be a domino, then we define \( T(M) = \dim_k M^0/V M^0 \) and call it the dimension of the domino. If \( M \) is any \( R \)-module we write \( T^i(M) \) for the dimension of the domino associated to the differential \( M^i \to M^{i+1} \). It is standard that \( T^i(M) \) is finite (see [20, Proposition 2.18, p. 110]).
2.14 Filtration on dominos

Any domino $M$ comes equipped with a finite decreasing filtration by $R$-submodules such that the graded pieces are certain standard one-dimensional dominos $U_j$ (see [20, Proposition 2.18, p. 110]). Note that the filtration may not be unique but the number of factors is the same for all such filtrations.

2.15 One-dimensional dominos

The $U_j$, one for each $j \in \mathbb{Z}$, provide a complete list of all the one-dimensional dominos (see [20, Proposition 2.19, p. 111]).

2.16 Domino devissage lemmas

Further by [12, Lemma 4.2, p. 12] one has $\text{Hom}_R(U_i, U_j) = 0$ if $i > j$, and $\text{Hom}_R(U_i, U_i) = k$.

2.17 Geometrically connectedness assumption

To avoid tedious repetition we will assume throughout this paper that all schemes which appear in this paper are geometrically conneced. We caution the reader that this assumption may not always appear in the statements of the theorems but is tacitly used in many of the proofs. (We thank the referee for reminding us of this.)

2.18 Slope spectral sequence

Let $X$ be a smooth, projective scheme over $k$, we will write $H^*_\text{cris}(X/W)$ for the crystalline cohomology of $X$. If $X$ is smooth and proper, in [17], one finds the construction of the de Rham–Witt complex $W\Omega^*_X$. The construction of this complex is functorial in $X$. This complex computes $H^*_\text{cris}(X/W)$ and one has a spectral sequence (the slope spectral sequence of $X$)

$$E_1^{i,j} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}_{\text{cris}}(X/W).$$

The construction slope spectral sequence is also functorial in $X$ and for each $j \geq 0$, $H^j(X, W\Omega^*_X)$ is a graded, profinite and coherent $R$-module. Modulo torsion, the slope spectral sequence always degenerates at $E_1$.

2.19 Classical invariants of surfaces

Let $X/k$ be a smooth projective surface. Then recall the following standard notation for numerical invariants of $X$. We will write $b_1 = \dim_{\mathbb{Q}_l} H^1_{\text{et}}(X, \mathbb{Q}_l), q = \dim \text{Alb}(X) = \dim \text{Pic}^0(X)_{\text{red}}; 2q = b_1, h^{i,j} = \dim_k H^j(X, \Omega^i_X)$; and $p_g(X) = h^{0,2} = h^{2,0}$. Then
one has the following form of Noether’s formula:

\[ 10 + 12p_g = c_1^2 + b_2 + 8q + 2(h^{0,1} - q), \]

(2)

one has by definition \( c_1^2 = K_X^2 \), the self-intersection of the canonical bundle of \( X \). The point is that all the terms are non-negative except possibly \( c_1^2 \). Further by [3, p. 25] we have

\[ 0 \leq h^{0,1} - q \leq p_g. \]

Formula (2) is easily seen to be equivalent to the usual form of Noether’s formula

\[ 12\chi(\mathcal{O}_X) = c_1^2 + c_2 = c_1^2 + \chi_\text{\'et}(X). \]

(3)

2.20 Hodge–Noether formula

In addition, when the ground field \( k = \mathbb{C} \), we get yet another form of Noether’s formula which is a consequence of (3) and the Hodge decomposition:

\[
\begin{align*}
  h^{1,1} &= 10\chi(\mathcal{O}_X) - c_1^2 + b_1, \quad \text{or equivalently,} \\
  h^{1,1} &= \frac{5c_2 - c_1^2}{6} + b_1.
\end{align*}
\]

(4)

We will call this the Hodge–Noether formula.

2.21 Hodge–Witt invariants and other invariants

In the next few subsections we recall results on Hodge–Witt numbers [12, p. 85] of surfaces and threefolds. We recall the definition of Hodge–Witt numbers and their basic properties.

2.22 Slope numbers

Let \( X \) be a smooth projective variety over a perfect field \( k \). The slope numbers of \( X \) are defined by (see [12, p. 85]):

\[
m^{i,j} = \sum_{\lambda \in [i-1,i)} (\lambda - i + 1) \dim K H^{i+j}_{\text{cris}}(X/W)_{[\lambda]} + \sum_{\lambda \in [i,i+1)} (i + 1 - \lambda) \dim K H^{i+j}_{\text{cris}}(X/W)_{[\lambda]},
\]

where the summation is over all the slopes of Frobenius \( \lambda \) in the indicated intervals and \( H^{i+j}_{\text{cris}}(X/W)_{[\lambda]} \) denotes the slope \( \lambda \) part of \( H^{i+j}_{\text{cris}}(X/W) \otimes_W K \). Let us note that slope numbers of a smooth, projective variety are non-negative integers (see [6]).
2.23 Dominos in the slope spectral sequence

Let $X$ be a smooth projective variety. For all $i, j \geq 0$ the differential

$$d : H^j(X, W\Omega^i_X) \rightarrow H^j(X, W\Omega^{i+1}_X)$$

admits a canonical factorization

$$H^j(X, W\Omega^i_X) \xrightarrow{d} H^j(X, W\Omega^{i+1}_X)$$

where the lower arrow is a domino, called the *domino associated with the differential*

$$d : H^j(X, W\Omega^i_X) \rightarrow H^j(X, W\Omega^{i+1}_X)$$

and is denoted by $\text{Dom}^{i,j}(H^\bullet(X, W\Omega^\bullet_X))$.

The *domino numbers* $T^{i,j}$ of $X$ are defined by (see [12, p. 85]):

$$T^{i,j} = \dim_k \text{Dom}^{i,j}(H^\bullet(X, W\Omega^\bullet_X))$$

in other words, $T^{i,j}$ is the dimension of the domino associated to the differential

$$d : H^j(X, W\Omega^i_X) \rightarrow H^j(X, W\Omega^{i+1}_X).$$

Note that these are non-negative integers.

2.24 Hodge–Witt numbers

Let $X$ be a smooth projective variety over a perfect field $k$. The *Hodge–Witt numbers* of $X$ are defined by the formula (see [12, p. 85]):

$$h^{i,j}_W = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}.$$  

In particular one sees from the definition that Hodge–Witt numbers are also integers.

2.25 Formulaire

The Hodge–Witt numbers, domino numbers and the slope numbers satisfy the following properties which we will now list. See [12] for details.
2.25.1 Slope number symmetry

For all $i, j$ one has the symmetries (see [12, Lemma 3.1, p. 112]):

$$m^{i,j} = m^{j,i} = m^{n-i,n-j},$$

here the first is a consequence of the Hard Lefschetz Theorem and the second is a consequence of Poincaré duality. Further these numbers are obviously non-negative:

$$m^{i,j} \geq 0.$$

2.25.2 Theorems of Ekedahl

The following formulae give relations to Betti numbers (see [12, Theorem 3.2, p. 85]):

$$\sum_{i+j=n} m^{i,j} = \sum_{i+j=n} h^{i,j}_W = b_n$$

and one has Ekedahl’s upper bound (see [12, Theorem 3.2, p. 86]):

$$h^{i,j}_W \leq h^{i,j},$$

and if $X$ is Mazur–Ogus (see Sect. 2.30 for the definition of a Mazur–Ogus variety) then we have

$$h^{i,j}_W = h^{i,j}.$$

2.25.3 Ekedahl’s duality

One has the following fundamental duality relation for dominos due to Ekedahl (see [11, Corollary 3.5.1, p. 226]): for all $i, j$, the domino $\text{Dom}^{i,j}$ is canonically dual to $\text{Dom}^{n-i-2,n-j+2}$ and in particular one has the equality of the domino numbers

$$T^{i,j} = T^{n-i-2,n-j+2}. \quad (5)$$

2.26 Crew’s formula

The Hodge and Hodge–Witt numbers of $X$ satisfy a relation known as Crew’s formula which we will use often in this paper. The formula is the following:

$$\sum_j (-1)^j h^{i,j}_W = \chi(\Omega^i_X) = \sum_j (-1)^j h^{i,j}. \quad (6)$$
2.26.1 Hodge–Witt symmetry

For any smooth proper variety of dimension at most three we have for all \( i, j \) (see [12, Corollary 3.3 (iii), p. 113]):

\[
h^i,j_W = h^{j,i}_W.
\]

Note this is proved in loc. cit. first for smooth projective varieties of dimension at most three (projectivity is essential part as one makes use of Deligne’s Hard Lefschetz Theorem [8, Théorème 4.1.1]) and then Ekedahl appeals to resolution of singularities (known for \( p \geq 5 \) for threefolds) to deduce the proper case. The argument of loc. cit. also works if we replace the use of resolution of singularities by de Jong’s theorem on alterations.

2.26.2 Hodge–Witt duality

For any smooth projective variety of dimension \( n \) we have for all \( i, j \) (see [12, Corollary 3.2 (i), p. 113]):

\[
h^i,j_W = h^{n-i,n-j}_W.
\]

2.27 Crew’s formula for surfaces

For surfaces, the formulas in (6) take more explicit forms (see [12, p. 85] and [18, p. 64]). We recall them now as they will play a central role in our investigations. Let \( X/k \) be a smooth projective surface, \( K_X \) be its canonical divisor, \( T_X \) its tangent bundle, and let \( c_1^2, c_2 \) be the usual Chern invariants of \( X \) (so \( c_i = c_i(T_X) = (-1)^i c_i(\Omega^1_X) \)). Then the Hodge–Witt numbers of \( X \) are related to the other numerical invariants of \( X \) by means of the following formulae [12, p. 114]:

\[
\begin{align*}
    h^{0,1}_W &= h^{1,0}_W, \\
    h^{0,1}_W &= b_1/2, \\
    h^{0,2}_W &= h^{2,0}_W, \\
    h^{0,2}_W &= \chi(\mathcal{O}_X) - 1 + \frac{b_1}{2}, \\
    h^{1,1}_W &= b_1 + \frac{5}{6} c_2 - \frac{1}{6} c_1^2.
\end{align*}
\]

2.28 Hodge–Witt–Noether formula

The formula for \( h^{1,1}_W \) above and Noether’s formula give the following variant of the Hodge–Noether formula of (4). We will call this variant the Hodge–Witt–Noether formula.
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formula:
\[ h^{1,1}_W = 10\chi(O_X) - c_1^2 + b_1, \]
or equivalently,
\[ h^{1,1}_W = \frac{5c_2 - c_1^2}{6} + b_1. \] (8)

This paper began with the realization that the above formula is the de Rham–Witt analogue of the Hodge–Noether formula (4) and will be central to our study of surfaces in this paper.

2.29 Hodge–Witt and ordinary varieties

Let \( X/k \) be a smooth, projective variety over a perfect field \( k \) of characteristic \( p > 0 \). Then

- We say that \( X \) is Hodge–Witt if for each \( i, j \geq 0 \), \( H^j(X, W\Omega^i_X) \) is a finite type module over \( W \).
- We say that \( X \) is ordinary if \( H^j(X, BW\Omega^i_X) = 0 \) for all \( i, j \geq 0 \).

Here are some examples (all of these assertions require proofs which are far from elementary). Any smooth, projective curve is Hodge–Witt. An abelian variety is ordinary if and only if it has \( p \)-rank equal to its dimension. An abelian variety is Hodge–Witt if and only if its \( p \)-rank is at least as big as its dimension minus one. Any ordinary variety is a Hodge–Witt variety. A K3 surface is Hodge–Witt if and only if it is of finite height and a K3 surface is ordinary if and only if its height is one. In particular one sees that in all dimensions there exist Hodge–Witt varieties which are not ordinary.

2.30 Mazur–Ogus and Deligne–Illusie varieties

In the next few subsections we enumerate the properties of a class of varieties known as Mazur–Ogus varieties. We will use this class of varieties at several different points in this paper as well as its thematic sequels so we elaborate some of the properties of this class of varieties here.

2.31 Mazur–Ogus varieties

A smooth, projective variety over a perfect field \( k \) is said to be a Mazur–Ogus variety if it satisfies the following conditions:

- the Hodge–de Rham spectral sequence of \( X \) degenerates at \( E_1 \), and
- crystalline cohomology of \( X \) is torsion-free.

2.32 Deligne–Illusie varieties

A smooth, projective variety over a perfect field \( k \) is said to be a Deligne–Illusie variety if it satisfies the following conditions:

- \( X \) admits a flat lifting to \( W_2(k) \), and
Differentiate with respect to \( x \).
X. Using iterated Cartier operators (or their inverses), we get (see [17, Chapter 0, 2.2, p.519]) a sequence of sheaves \( B_n \Omega^1_X \subset Z_n \Omega^1_X \) and the exact sequence

\[
0 \to B_n \Omega^1_X \to Z_n \Omega^1_X \to \Omega^1_X \to 0,
\]

and sequence of sheaves \( Z_{n+1} \Omega^1_X \subset Z_n \Omega^1_X \). We will write

\[
Z^\infty \Omega^1_X = \bigcap_{n=0}^{\infty} Z_n \Omega^1_X.
\]

This is the sheaf of indefinitely closed one-forms. We will say that a global one-form is indefinitely closed if it lives in \( H^0(X, Z^\infty \Omega^1_X) \subset H^0(X, \Omega^1_X) \). In general the inclusions \( H^0(Z^\infty \Omega^1_X) \subset H^0(Z_1 \Omega^1_X) \subset H^0(\Omega^1_X) \) may all be strict.

### 2.35 Mazur–Ogus explicated for surfaces

For a smooth, projective surface, the condition that \( X \) is Mazur–Ogus takes more tangible geometric forms which are often easier to check in practice. Part of our next result is implicit in [17]. We will use the class of Mazur–Ogus surfaces in extensively in this paper as well as its sequel and in particular the following result will be frequently used.

**Theorem 2.4** Let \( X \) be a smooth, projective surface over a perfect field \( k \) of characteristic \( p > 0 \). Consider the following assertions:

(i) \( X \) is Mazur–Ogus.
(ii) \( H^2_{\text{cris}}(X/W) \) is torsion-free and \( h^{1,1}_W = h^{1,1}_{\text{dR}} \).
(iii) \( H^2_{\text{cris}}(X/W) \) is torsion-free and every global one-form on \( X \) is closed.
(iv) \( \text{Pic}(X) \) is reduced and every global one-form on \( X \) is indefinitely closed.
(v) The differentials \( H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X) \) and \( H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^2_X) \) are zero.
(vi) The Hodge–de Rham spectral sequence

\[
E_1^{p,q} = H^q(X, \Omega^p_X/k) \Rightarrow H^{p+q}_{\text{dR}}(X/k)
\]

degenerates at \( E_1 \).

Then (i) \( \iff \) (ii) \( \iff \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) \( \iff \) (vi).

**Proof** It is clear from Ekedahl’s Theorems (see Sect. 2.25.2) that (i) \( \Rightarrow \) (ii). Consider the assertion (ii) \( \Rightarrow \) (iii). By [17, Proposition 5.16, p.632] the assumption of (ii) that \( H^2_{\text{cris}}(X/W) \) is torsion-free implies that \( \text{Pic}(X) \) is reduced, the equality

\[
\dim H^0(Z_1 \Omega^1_X) = \dim H^0(Z^\infty \Omega^1_X),
\]

and also the equality

\[
b_1 = h^{1,1}_{\text{dR}} = h^{0,1} + \dim H^0(Z_1 \Omega^1_X).
\]
Now our assertion will be proved using Crew’s formula (6). We claim that the following equalities hold:

\[
\begin{align*}
 h^{0,0}_W &= h^{0,0}, \\
 h^{0,1}_W &= h^{0,1}, \\
 h^{0,2}_W &= h^{0,2}, \\
 h^{2,0}_W &= h^{2,0}, \\
 h^{1,1}_W - h^{1,1}_W &= 2(h^{1,0} - h^{0,1}).
\end{align*}
\]

The first of these is trivial as both the sides are equal to one for trivial reasons. The second follows from the explicit form of Crew’s formula for surfaces (and the fact that \(\text{Pic}(X)\) is reduced). The third formula follows from Crew’s formula and the first two computations as follows. Crew’s formula (6) says that

\[
 h^{0,0}_W - h^{0,1}_W + h^{0,2}_W = h^{0,0} - h^{0,1} + h^{0,2}.
\]

By the first two equalities we deduce that \(h^{0,2}_W = h^{0,2}\). By Hodge–Witt symmetry and Serre duality we deduce that

\[
 h^{0,2}_W = h^{2,0}_W = h^{0,2} = h^{2,0}.
\]

Again Crew’s formula also gives

\[
 h^{1,0}_W - h^{1,1}_W + h^{1,2}_W = h^{1,0} - h^{1,1} + h^{1,2}.
\]

So we get on rearranging that

\[
 h^{1,1}_W - h^{1,1}_W = h^{1,0} - h^{1,0} + h^{1,2} - h^{1,2}.
\]

By Serre duality \(h^{1,2} = h^{1,0}\) and on the other hand

\[
 h^{1,2}_W = h^{1,0}_W = h^{0,1}_W = h^{0,1}
\]

by Hodge–Witt symmetry (see Sects. 2.26.1, 2.26.2) and the last equality holds as \(\text{Pic}(X)\) is reduced. Thus we see that

\[
 h^{1,1}_W - h^{1,1}_W = 2(h^{1,0} - h^{0,1}).
\]

Thus the hypothesis of (ii) implies that

\[
 h^{1,0} = h^{0,1},
\]

and

\[
 h^{1,0} = \dim H^0(Z_1 \Omega^1_X) = \dim H^0(Z_{\infty} \Omega^1_X).
\]
So we have deduced that every global one-form on $X$ is closed and hence $(ii) \Rightarrow (iii)$ is proved.

Now $(iii) \Rightarrow (i)$ is proved as follows. The only condition we need to check is that the hypotheses of $(iii)$ imply that Hodge–de Rham spectral sequence degenerates. The only non-trivial part of this assertion is that $H^1(\Omega^1_X) \to H^1(X, \Omega^1_X)$ is zero. By [17, Proposition 5.16, p. 632] we know that if $H^2_{\text{cris}}(X/W)$ is torsion-free then the differential $H^1(\Omega^1_X) \to H^1(X, \Omega^1_X)$ is zero. Further by hypotheses of $(iii)$ we see that $H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$ also is zero. The other differentials in the Hodge–de Rham spectral sequence are either zero for trivial reasons or are dual to one of the above two differentials and hence Hodge–de Rham degenerates. So $(iii) \Rightarrow (i)$.

Now let us prove that $(iii) \Rightarrow (iv)$. The first assertion is trivial after [17, Proposition 5.16, p. 632]. Indeed the fact that $H^2_{\text{cris}}(X/W)$ is torsion-free implies that Pic$(X)$ is reduced and $H^0(X, Z_\infty \Omega^1_X) = H^0(X, Z_1 \Omega^1_X)$ and by the hypotheses of $(iii)$ we have further that $H^0(X, Z_1 \Omega^1_X) = H^0(X, \Omega^1_X)$. Thus we have deduced $(iii) \Rightarrow (iv)$.

The remaining assertions are well-known and are implicit in [17, Proposition 5.16, p. 632] but we give a proof for completeness. Now assume $(iv)$ we want to prove $(v)$. By the hypothesis of $(iv)$ and [17, Proposition 5.16, p. 632] we see that the differential $H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$ is zero. So we have to prove that the differential $H^1(\Omega^1_X) \to H^1(\Omega^1_X)$ is zero. We use the method of proof of [17, Proposition 5.16, p. 632] to do this. Let $f : X \to \text{Alb}(X)$ be the Albanese morphism of $X$. Then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & H^0(A, \Omega^1_A) & \to & H^1_{\text{dR}}(A/k) & \to & H^1(\Omega^1_A) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(Z_1 \Omega^1_X) & \to & H^1_{\text{dR}}(X/k) & \to & H^1(\Omega^1_X) & & 
\end{array}
$$

with exact rows and the vertical arrows are injective and cokernel of the middle arrow is $\rho H^2_{\text{cris}}(X/W)_{\text{tor}}$ (the $p$-torsion of the torsion of $H^2_{\text{cris}}(X/W)$). Moreover the image of $H^1_{\text{dR}}(X/k) \to H^1(\Omega^1_X)$ is the $E^{0,1}_{\infty}$ term in the Hodge–de Rham spectral sequence. Thus the hypothesis of $(iv)$ that Pic$(X)$ is reduced implies that $H^1(\Omega^1_A) = H^1(\Omega^1_X)$, so we have $H^1(\Omega^1_A) = H^1(\Omega^1_X) \subset E^{0,1}_{\infty} = H^1(\Omega^1_X)$. Hence $E^{0,1}_{\infty} = H^1(X, \Omega^1_X)$, so that the differential $H^1(\Omega^1_X) \to H^1(\Omega^1_X)$ is zero. This proves $(iv)$ implies $(v)$.

Now let us prove $(v) \Leftrightarrow (vi)$. It is trivial that $(vi)$ implies $(v)$. So we only have to prove $(v) \Rightarrow (vi)$. This is elementary, but we give a proof. The differential $H^0(\Omega^1_X) \to H^0(\Omega^1_X)$ is trivially zero, so by duality $H^2(\Omega^1_X) \to H^2(\Omega^2_X)$ is zero. The differential $H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ is zero by hypothesis of $(v)$. This is dual to $H^2(\Omega^1_X) \to H^2(\Omega^2_X)$ hence which is also zero. The differential $H^1(\Omega^1_X) \to H^1(\Omega^1_X)$ is dual to the differential $H^1(\Omega^1_X) \to H^1(\Omega^1_X)$ which is zero by hypothesis of $(v)$. Hence we have proved $(v) \Leftrightarrow (vi)$. 

\( \square \)
2.36 Domino numbers of Mazur–Ogus varieties

Let $X$ be a smooth projective variety over a perfect field. The purpose here is to prove the following. This was proved for abelian varieties in [12].

**Theorem 2.5** Let $X$ be a smooth projective Mazur–Ogus variety over a perfect field. Then for all $i, j \geq 0$ the domino numbers $T^{i,j}$ are completely determined by the Hodge numbers of $X$ and the slope numbers of $X$.

**Proof** This is proved by an inductive argument. The first step is to note that by the hypothesis and [12] one has

$$h^{i,j} = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}$$

and so we get for all $j \geq 0$

$$T^{0,j} = h^{0,j} - m^{0,j}$$

so the assertion is true for $T^{0,j}$ for all $j \geq 0$. Next we prove the assertion for $T^{i,n}$ for all $i$. From the above equation we see that $T^{i,n} = h^{i,n} - m^{i,n} + 2T^{i-1,n+1} - T^{i-2,n+2}$ and the terms involving $n + 1, n + 2$ are zero. Now do a downward induction on $j$ to prove the result for $T^{i,j}$: for each fixed $j$, the formula for $T^{i,j}$ involves $T^{i-1,j+1}, T^{i-2,j+2}$ and by induction hypothesis on $j$ (for each $i$) these two domino numbers are completely determined by the Hodge and slope numbers. Thus the result follows.

In particular as complete intersections in projective space are Mazur–Ogus, we have the following.

**Corollary 2.6** Let $X$ be a smooth projective complete intersection in projective space. Then $T^{i,j}$ are completely determined by the Hodge numbers of $X$ and the slope numbers of $X$.

2.37 Hodge and Newton polygons of Hodge–Witt surfaces

Let $X/k$ be a smooth, projective surface over a perfect field of characteristic $p > 0$ with $H^*_{\text{cris}}(X/W)$ torsion-free. Let $\beta_2 = \dim H^2_{\text{dR}}(X/k) = \dim_K H^2_{\text{cris}}(X/W) \otimes W K$ be the second Betti number of $X$. Recall from [25] that the Hodge polygon of $H^2_{\text{dR}}(X/k)$ (resp. the Newton polygon of $H^2_{\text{cris}}(X/W)$) is the graph of a continuous, piecewise linear, $\mathbb{R}$-valued function denoted here by $\text{Hod}(X) : [0, \beta_2] \to \mathbb{R}$ (resp. $\text{New}(X) : [0, \beta_2] \to \mathbb{R}$). We refer the reader to [1,25] for the definition of these polygons and functions.

Readers may find the following result which is, as far as we are aware, the most explicit result of its kind, useful in understanding the influence of Hodge–Witt condition in the more familiar realm of Hodge and Newton polygons.

**Theorem 2.7** Let $X/k$ be a smooth, projective surface over a perfect field $k$ of characteristic $p > 0$. Suppose $X$ is Hodge–Witt and $H^*_{\text{cris}}(X/W)$ is torsion-free. Then

$$\text{Hod}(x) = \text{New}(x) \text{ on the interval } \sum_{\lambda < 1} m_{\lambda} \leq x \leq \sum_{\lambda < 1} m_{\lambda} + m_1 = \sum_{\lambda \leq 1} m_{\lambda}.$$
In other words the Hodge and Newton polygons of \( X \) touch over the closed interval

\[
\sum_{\lambda < 1} m_\lambda \leq x \leq \sum_{\lambda < 1} m_\lambda + m_1 = \beta_2 - \sum_{\lambda < 1} m_\lambda.
\]

**Remark 2.8** Here is an explicit example which illustrates this theorem. Let \( X \) be a Hodge–Witt K3 surface (equivalently a K3 surface of finite height). Then \( H^2_{\text{ cris}} \) has only one slope \( \lambda = 1 - 1/n < 1 \) of multiplicity \( m_\lambda = n \) where \( 1 \leq n \leq 10 \), and one sees easily that \( m_1 = 22 - 2n \). Hence the theorem asserts that Hodge and Newton functions of \( X \) agree on the interval \([n, n + 22 - 2n] = [n, 22 - n] \), an interval of length at least two as \( n \geq 1 \).

**Proof** Note that projectivity hypothesis implies \( m_1 \geq 1 \) and hence the interval in the assertion has length at least one. By the definition of the Hodge-function one has \( \text{Hod}(x) = 0 \) on \([0, h^{0,2}] \) and \( \text{Hod}(x) \) is linear of slope one on \([h^{0,2}, h^{0,2} + h^{1,1}] \) and of slope two on the interval \([h^{0,2} + h^{1,1}, h^{0,2} + h^{1,1} + h^{2,0}] = [h^{0,2} + h^{1,1}, \beta_2] \). From this one has

\[
\text{Hod}(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq h^{0,2}, \\
n_1 \times \lambda_1, \ldots, n_2 \times \lambda_2, \ldots, n_r \times \lambda_r & \text{if } h^{0,2} \leq x \leq h^{0,2} + h^{1,1}, \\
2x - \beta_2 & \text{if } h^{0,2} + h^{1,1} \leq x \leq h^{0,2} + h^{1,1} + h^{2,0} = \beta_2.
\end{cases}
\]

Now let us determine the Newton function of \( H^2_{\text{ cris}}(X/W) \). Let \((\lambda_1, n_1), \ldots, (\lambda_r, n_r)\) be all the distinct slopes of \((H^2_{\text{ cris}}(X/W), F)\). So the slope sequence looks like this:

\[
\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_r, \ldots, \lambda_r,
\]

and one of these slopes is equal to 1 and its multiplicity we will denote by \( m_1 \). Now by definition of the Newton polygon (or the Hodge-function) one has

\[
\text{New}(x) = \begin{cases} 
\lambda_1 x & \text{for } 0 \leq x \leq n_1, \\
\lambda_2(x - n_1) + \lambda_1 n_1 & \text{for } n_1 \leq x \leq n_1 + n_2, \\
\vdots & \text{for } \sum_{j=1}^{i} n_j \leq x \leq \sum_{j=1}^{i+1} n_j, \\
\lambda_{i+1}(x - \sum_{j=1}^{i} n_j) + \sum_{j=1}^{i} \lambda_j n_j & \text{for } \sum_{j=1}^{i} n_j \leq x \leq \sum_{j=1}^{i+1} n_j, \\
\vdots & \text{for } \sum_{j=1}^{i} n_j \leq x \leq \sum_{j=1}^{i+1} n_j,
\end{cases}
\]

Now suppose \( \lambda_1 < \lambda_2 < \cdots < \lambda_{i_0} < \lambda_{i_0+1} = 1 < \lambda_{i_0+2} < \cdots < \lambda_r \). Then on the closed interval

\[
\sum_{j \leq i_0} n_j \leq x \leq \sum_{j \leq i_0} n_j + n_{i_0+1} = \sum_{j \leq i_0} n_j + m_1
\]
one has

$$\text{New}(x) = 1 \cdot \left( x - \sum_{j=1}^{i_0} n_j \right) + \sum_{j=1}^{i_0} \lambda_j n_j$$

$$= x - \sum_{j=1}^{i_0} (1 - \lambda_j) n_j = x - m^{0.2}. $$

Now $X$ is Hodge–Witt, one has $T^{0.2} = 0$, and as $H^2_{\text{cris}}(X/W)$ is torsion-free one has from [6, Corollary 5] that

$$m^{0.2} = h^{0.2},$$

and hence

$$\text{New}(x) = x - m^{0.2} = x - h^{0.2} = \text{Hod}(x)$$

for $\sum_{\lambda < 1} m_{\lambda} \leq x \leq \sum_{\lambda < 1} m_{\lambda} + m_1$. Further by Poincaré duality and our hypothesis, one has $\beta_2 = 2 \sum_{\lambda < 1} m_{\lambda} + m_1$. Hence

$$\beta_2 - \sum_{\lambda < 1} m_{\lambda} = \sum_{\lambda < 1} m_{\lambda} + m_1.$$ 

This proves the assertion. $\square$

### 2.38 Explicit formulae for domino numbers of hypersurfaces

For $X \subset \mathbb{P}^{n+1}$ a smooth hypersurface we can make the formulas quite explicit using [28,37,42]. We do this here for the reader’s convenience. We begin with the remark that we have the following explicit bound for $T^{0,j}$:

$$T^{0,j} \leq h^{0,j}, \quad (9)$$

and if $\dim(X) = n = 2$ then we have $m^{0.2} = 0$ if and only if $H^2_{\text{cris}}(X/W) \otimes \mathbb{K}$ is pure slope one. In particular, if this is the case, we have $T^{0.2} = h^{0.2}$ and so we have $T^{0.2} = p_g(X)$. If $\deg(X) = d$ then we have from [28] that $h^{0.2} = \dim H^0(X, \Omega^2_X) = H^0(X, \mathcal{O}_X(d - 4))$ because $\Omega^2_X = \mathcal{O}_X(d - 4)$ by the adjunction formula.

If $\dim(X) = 3$ one sees from [42] that there are at most two, possibly nonzero domino numbers, namely, $T^{0.2}$ and $T^{0.3}$; the remaining domino numbers are either zero or equal to these two by Ekedahl’s duality. This is seen as follows: as $X$ is a hypersurface $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$ and hence $T^{0,i} = 0$ for all $i$ except possibly $i = 3$ and $T^{1.2} = T^{0.3}$ by Ekedahl’s duality. So we have $T^{0.3} = h^{0.3} - m^{0.3}$. If $H^3_{\text{cris}}(X/W)$ has no slopes in $[0, 1)$ then we see that $m^{0.3} = 0$ and so again $T^{0.3} = h^{0.3} = \dim H^3(X, \mathcal{O}_X)$.
If \( \dim(X) = 4 \) then from [42] one sees that there are two possibly non-trivial domino numbers to determine \( T^{0,4} \) and \( T^{1,3} \) and we may determine \( T^{0,4} \) easily as above. The formula for \( h_{W}^{1,3} = m^{1,3} + T^{1,3} - 2T^{0,4} \) gives \( T^{1,3} = h^{1,3} - m^{1,3} + 2T^{0,4} \). If \( H_{\text{cris}}^{4}(X/W) \) is pure slope two, then \( m^{1,3} = 0 \) and so we have \( T^{1,3} = h^{1,3} + 2T^{0,4} \).

\[ \dim(X) = 2: \text{ We have } T^{0,2} \leq h^{0,2} \text{ and equality holds if and only if } H_{\text{cris}}^{2}(X/W) \text{ is pure slope one.} \]

\[ \dim(X) = 3: \text{ We have } T^{0,3} \leq h^{0,3} \text{ with equality if and only if } H_{\text{cris}}^{3}(X/W) \text{ has no slopes in } [0, 1) \text{ (equivalently by Poincaré duality } H_{\text{cris}}^{3}(X/W) \text{ has no slopes in } (2, 3)). \]

\[ \dim(X) = 4: \text{ We have } T^{0,4} \leq h^{0,4} \text{ and } T^{1,3} \leq h^{1,3} + 2T^{0,4} \text{ and both are equal if and only if } H_{\text{cris}}^{4}(X/W) \text{ is of pure slope two.} \]

Thus we have proved the following:

**Corollary 2.9** Let \( X \subset \mathbb{P}^{n+1} \) be a smooth, projective hypersurface over a perfect field of characteristic \( p > 0 \) with \( 2 \leq \dim(X) = n \leq 4 \). Then we have \( T^{0,i} = 0 \) unless \( i = n \) and in that case

\[ T^{0,n} \leq h^{0,n}. \quad (10) \]

Further \( T^{1,i} = 0 \) unless \( i = 4 \) and if \( i = 4 \) then

\[ T^{1,4} \leq h^{1,3} + 2h^{0,4}. \]

Further equality holds if \( H_{\text{cris}}^{i}(X/W) \) satisfies certain slope conditions which are summarized in Table 1. The table records the only, possibly non-trivial, domino numbers and the crystalline condition which is necessary and sufficient for these domino numbers to achieve their maximal value.

The following table records the standard formulae (see [7,28,37]) for the non-trivial numerical invariants of a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \) with \( 2 \leq n \leq 4 \) and of degree \( d \). These can be extracted as the coefficients of the power series expansion of the function

\[ H_{d}(y, z) = \frac{(1 + z)^{d-1} - (1 + y)^{d-1}}{z(1 + y)^{d} - y(1 + z)^{d}} = \sum_{p,q \geq 0} h_{0}^{p,q} y^{p} z^{q}, \]

**Table 1** Slope condition(s) for maximal domino numbers

| \( \dim(X) \) | 2                  | 3                  | 4                  |
|----------------|--------------------|--------------------|--------------------|
| \( T^{0,2} \)  | \( H_{\text{cris}}^{2} = H_{\text{cris}}^{2,[1]} \) | 0                  | 0                  |
| \( T^{0,3} \)  | 0                  | \( H_{\text{cris}}^{3} = H_{\text{cris}}^{3,[1,2]} \) | 0                  |
| \( T^{0,4} \)  | 0                  | 0                  | \( H_{\text{cris}}^{4} = H_{\text{cris}}^{4,[1,3]} \) |
| \( T^{1,3} \)  | 0                  | 0                  | \( H_{\text{cris}}^{4} = H_{\text{cris}}^{4,[2]} \) |
where \( h_{0}^{p,q} = h_{p,q} - \delta_{p,q} \) (where \( \delta_{p,q} \) is the Kronecker delta symbol); we may also calculate the Betti number of \( X \) using the Hodge numbers of \( X \) as Hodge–de Rham spectral sequence degenerates for a hypersurface in \( \mathbb{P}^{n+1} \).

For computational purposes, we can write the above generating function, following [7], as

\[
H_d(y, z) = \sum_{i, j \geq 0} \frac{(d-1)}{1 - \sum_{i, j \geq 1} \binom{d}{i+j} y^i z^j}.
\]

Now it is possible, after a bit of work (which we suppress here) to arrive at the formulae for Hodge numbers for low dimensions (such as the ones we need). Our results are summarized in the following table.

Assuming that the slope conditions for maximal domino numbers hold (see Table 1) we can use the Hodge number calculation to calculate domino numbers. Table 2 gives formulae for Hodge numbers \( h_{i,j} \).

We record for future use:

\[
h_{1,1} - 2p_g = \frac{d^3 - 4d + 6}{3} \quad (11)
\]

and hence

\[
b_2 - 4p_g = \frac{d^3 - 4d + 6}{3} \quad (12)
\]

We summarize formulae for maximal values of \( T^{0,i} \) which we can obtain using this method in the following.

| \( \text{dim}(X) \) | 2          | 3          | 4          |
|-------------------|------------|------------|------------|
| \( h_{0,2} \)     | \( \frac{(d-1)(d-2)(d-3)}{3} \) | \( \frac{(d-1)(d-2)(11d^3-51d^2+56d-30)}{5!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{1,1} \)     | \( \frac{d(2d^2-6d+7)}{3} \) | \( \frac{(d-1)(d-2)(d-3)(d-4)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{0,3} \)     | \( \frac{(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(11d^2-17d+12)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{1,2} \)     | \( \frac{(d-1)(d-2)(13d^3-51d^2+56d-30)}{5!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{0,4} \)     | \( \frac{(d-1)(d-2)(d-3)(d-4)(d-5)}{5!} \) | \( \frac{(d-1)(d-2)(13d^3-51d^2+56d-30)}{5!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{1,3} \)     | \( \frac{2(d-1)(d-2)(13d^3-51d^2+56d-30)}{5!} \) | \( \frac{(d-1)(d-2)(13d^3-51d^2+56d-30)}{5!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( h_{2,2} \)     | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
| \( b_2 \)         | \( d^3 - 4d^2 + 6d - 2 \) | \( d^3 - 4d^2 + 6d - 2 \) | \( d^3 - 4d^2 + 6d - 2 \) |
| \( b_3 \)         | \( (d-1)(d-2)(d^2-2d+2) \) | \( (d-1)(d-2)(d^2-2d+2) \) | \( (d-1)(d-2)(d^2-2d+2) \) |
| \( b_4 \)         | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) | \( \frac{(d-1)(d-2)(3d^3-12d^2+15d-10)}{4!} \) |
Proposition 2.10 Assume that $X \subseteq \mathbb{P}^n$ is a smooth, projective hypersurface of degree $d$ and $\dim(X) \leq 4$. Suppose that the crystalline cohomology of $X$ satisfies the slope condition of for maximal domino numbers given in (Table 1). Then the domino numbers $T^{0,i}$ for $i = 2, 3, 4$ (resp. $\dim(X) = 2, 3, 4$) are given by Table 3.

### 3 Enriques classification and negativity of $h_{W}^{1,1}$

#### 3.1 Main result of this section

The main theorem we want to prove is Theorem 3.1. The proof of Theorem 3.1 is divided into several parts and it uses the Enriques classification of surfaces. We do not know how to prove the assertion without using Enriques classification [3,4].

**Theorem 3.1** Let $X/k$ be a smooth, projective surface over a perfect field $k$ of characteristic $p > 0$. Suppose $h_{W}^{1,1} < 0$ then the following dichotomy holds:

(i) $X$ is quasielliptic, or

(ii) $X$ is of general type and further

(a) either $c_2 \leq 0$, or

(b) $\Omega_X^1$ is Bogomolov unstable.

**Remark 3.2** Recall that the quasi-elliptic surfaces exist if and only if $p = 2$ or $p = 3$ and such surfaces are of Kodaira dimension at most one. In particular if $p \geq 5$ and $h_{W}^{1,1} < 0$ then $X$ is of general type. For $p = 2, 3$ using [26, Corollary, p. 480] and the formula for $h_{W}^{1,1}$ it is possible to write down examples of quasi-elliptic surfaces of Kodaira dimension one where this invariant is negative.

#### 3.2 Reduction to minimal model

Before proceeding further we record the following elementary lemma which allows us to reduce the question of $h_{W}^{1,1} < 0$ to minimal surfaces (when such a model exists).

**Lemma 3.3** If $X$ is a smooth, projective surface over a perfect field with $h_{W}^{1,1}(X) < 0$, and if $X \to X'$ is a proper birational morphism with $X'$ minimal, then $h_{W}^{1,1}(X') < 0$. 

---

Table 3 Maximal domino numbers

| $i$ | $T^{0,i}$ |
|-----|-----------|
| 2   | $(d-1)(d-2)(d-3)$ |
| 3   | $(d-1)(d-2)(d-3)(d-4)$ |
| 4   | $(d-1)(d-2)(d-3)(d-4)(d-5)$ |
| $T^{1,3}$ | $(d-1)(d-2)(14d^3 - 63d^2 + 103d - 90)$ |
**Proof** Since every proper birational morphism $X \to X'$ is a composition of a finite number of blowups at closed points, it is enough to prove the assertion for a blowup at a closed point. Now the lemma follows from the easily established fact that under blowup at a closed point, $h^{1,1}_W$ increases by the degree of the point (this fact can be easily established by using standard properties of Chern classes and Betti numbers under blowups as applied to (7)), so passing from $X$ to $X'$ involves a decrease in $h^{1,1}_W$ by this degree. Thus we see that $h^{1,1}_W(X') < h^{1,1}_W(X) < 0$. □

### 3.3 Enriques classification

We briefly recall Enriques classification of surfaces (see [3,4,33]). Let $X/k$ be a smooth, projective, minimal surface over $k$. Then Enriques classification is carried out by means of the Kodaira dimension $\kappa(X)$. All surfaces with $\kappa(X) = -\infty$ are ruled surfaces; the surfaces with $\kappa(X) = 0$ comprise of K3 surfaces, abelian surfaces, Enriques surfaces, non-classical Enriques surfaces (in characteristic two), bielliptic surfaces and non-classical hyperelliptic surfaces (in characteristic two and three). The surfaces with $\kappa(X) = 1$ are (properly) elliptic surfaces and finally the surfaces with $\kappa(X) = 2$ are surfaces of general type.

### 3.4 Two proofs of non-negativity

In this subsection we give two proofs of the following:

**Proposition 3.4** Let $X/k$ be a geometrically connected, smooth projective surface. Then for $(i, j) \neq (1, 1)$ we have $h^{i,j}_W \geq 0$.

**Proof** (First proof) The assertion is trivial for $h^{0,1}_W = h^{1,0}_W = b_1/2$. So we have to check it for $h^{2,0}_W = h^{0,2}_W = \chi(O_X) - 1 + b_1/2$. Writing out this explicitly we have

$$h^{0,2}_W = h^{0,0} - h^{0,1} + \frac{b_1}{2}$$

or as $h^{0,0} = 1$ (as $X$ is geom. connected) we get

$$h^{0,2}_W = h^{0,2} - (h^{0,1} - q)$$

where $q = b_1/2 = \dim_k \text{Alb}(X)$ is the dimension of the Albanese variety of $X$. By [3, p. 25] we know that $h^{0,1} - q \leq p_g = h^{0,2}$ and so the non-negativity assertion follows. □

**Proof** (Second Proof) We use the definition of

$$h^{i,j}_W = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}.$$ 

To prove the result it suffices to show that $T^{i-1,j+1}$ is zero for all $(i, j) \neq (1, 1)$. This follows from the fact that in the slope spectral sequence of a smooth projective
surface, there is at most one non-trivial differential (see [34], [17, Corollary 3.14, p. 619]) and this gives vanishing of the domino numbers except possibly $T^{0,2}$, and if $(i - 1, j + 1) = (0, 2)$ then $(i, j) = (1, 1)$. □

3.5 The case $\kappa(X) = -\infty$

We begin by stepping through the Enriques classification (see Sect. 3.3) and verifying the non-negativity of the Hodge–Witt number in all the cases.

**Proposition 3.5** Let $X/k$ be a smooth, projective surface. If $\kappa(X) = -\infty$ then $h_{W}^{1,1} \geq 0$.

**Proof** Since every smooth, projective surface admits a birational morphism to a smooth, minimal surface, and every such morphism is composed of a finite number of blowups at closed points (which increase $h_{W}^{1,1}$), to prove non-negativity of $h_{W}^{1,1}$ we may assume that $X$ is minimal with $\kappa(X) = -\infty$. As $\kappa(X) = -\infty$, we know from [4] that either $X$ is rational or it is ruled (irrational ruled). Assume $X$ is irrational ruled. Then one has $c_{1}^{2} = 8 - 8q$ and $\chi(O_{X}) = 1 - q$. By Noether’s formula $\chi(O_{X}) = (c_{1}^{2} + c_{2})/12$ we get

$$h_{W}^{1,1} = b_{1} + \frac{5}{6} 4(1 - q) - \frac{1}{6} 8(1 - q) = b_{1} + 2(1 - q) = 2 \geq 0$$

where we have used the fact that for a ruled surface $\text{Pic}(X)$ is reduced (which follows from $p_{g} = 0$ for a ruled surface so $H_{2}(X, W(O_{X})) = 0$), and the fact that $b_{1} = 2q$.

If $X$ is rational, then either $X = \mathbb{P}^{2}$ or $X$ is ruled, rational. In the first case $c_{1}^{2} = 9$ and in the second case $c_{1}^{2} = 8$. In both the cases $\chi(O_{X}) = 1$ and we are done by an explicit calculation. □

3.6 The case $\kappa(X) = 0$

**Proposition 3.6** Let $X/k$ be a smooth projective, minimal surface with $\kappa(X) = 0$. Then $h_{W}^{1,1} \geq 0$.

**Proof** This is easy: when $\kappa(X) = 0$, we know that $c_{1}^{2} = 0$ and so it suffices to show that $\chi(O_{X}) \geq 0$. This follows from the table in [3, p. 25]. □

3.7 The case $\kappa(X) = 1$

The following proposition shows that $h_{W}^{1,1} \geq 0$ holds for surfaces of $\kappa(X) = 1$ unless the surface is quasi-elliptic. There are examples of quasi-elliptic surfaces for which the result fails.

**Proposition 3.7** Assume $X$ is a smooth projective, minimal surface over a perfect field of characteristic $p > 0$ with $\kappa(X) = 1$. If $p = 2, 3$, assume that $X$ is not quasi-elliptic. Then $h_{W}^{1,1} \geq 0$.
Proof Under our hypothesis, $X$ is a properly elliptic surface (i.e., the generic fibre is smooth curve of genus 1 and $c_1^2 = 0$). Hence it suffices to verify that $c_2 \geq 0$. As $c_2 = \chi_{\text{et}}(X)$, the required inequality is equivalent to proving $\chi_{\text{et}}(X) \geq 0$. This inequality is implicit in [3]; it can also be proved directly using the Euler characteristic formula (see [5, p. 290, Proposition 5.1.6] and the paragraph preceding it).

3.8 Proof of Theorem 3.1

Now we can assemble various components of the proof. Assume that $X$ has $h^{1,1}_W < 0$. By Proposition 3.5 we have $h^{1,1}_W \geq 0$ for $\kappa(X) = -\infty$, so we may assume that $\kappa(X) \geq 0$. Then by Lemma 3.3 we may assume that $X$ is already minimal. Now by Propositions 3.6 and 3.7 we have $h^{1,1}_W \geq 0$ if $0 \leq \kappa(X) \leq 1$ and $X$ is not quasi-elliptic. So if $h^{1,1}_W < 0$ and $X$ is not quasi-elliptic then one has $\kappa(X) > 1$ and so $X$ is of general type.

So suppose $X$ is of general type with $h^{1,1}_W < 0$. Observe that one has the tautology $c_2 \leq 0$ or $c_2 > 0$. If the first of these holds then there is nothing to prove. So suppose that $X$ is of general type with $h^{1,1}_W < 0$ and $c_2 > 0$ then we claim that $\Omega^1_X$ is Bogomolov unstable (see [40] for the definition of Bogomolov stability). This follows from [40, Corollary 15]. To see that the conditions of that corollary are valid it suffices to verify that $c_1^2 > \frac{16p^2}{(4p^2-1)}c_2$. By (7), one has $h^{1,1}_W < 0 \Rightarrow c_1^2 > 5c_2 + 6b_1 > 5c_2$. On the other hand by [40, Corollary 15] if $c_2 > 0$ and $c_1^2 > \frac{16p^2}{4p^2-1}c_2$, then $\Omega^1_X$ is Bogomolov unstable. As

$$4 < \frac{16x^2}{4x^2 - 1} \leq 4.26 < 5 \cdots ,$$

for $x \geq 2$ our claim of Bogomolov unstability of $\Omega^1_X$ follows from Shepherd–Barron’s result.

Remark 3.8 Surfaces of general type with $c_2 \leq 0$ are studied in [40] where it is shown, for instance that if $c_2 < 0$ then $X$ is also uniruled (a result conjectured by Raynaud).

4 Chern class inequalities

4.1 Elementary observations

In this section we study the Chern class inequality $c_1^2 \leq 5c_2$ and a weaker variant $c_1^2 \leq 5c_2 + 6b_1$. These were studied in characteristic zero in [44]. It is, of course, well-known that $c_1^2 \leq 5c_2$ fails for some surfaces in positive characteristic. The first observation we have, albeit an elementary one, is that the obstructions to proving $c_1^2 \leq 5c_2$ are of de Rham–Witt (i.e., involving torsion in the slope spectral sequence) and crystalline (i.e., involving slopes of Frobenius on $H_{\text{cris}}^2(X/W)$) in nature. This has not been noticed before.

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Let us begin by recording some trivial but important consequences of the remarkable formula for $h_{W}^{1,1}$ (see (7)). The main reason for writing them out explicitly is to illustrate the fact that obstructions to Chern class inequalities for surfaces are of crystalline (involving slope of Frobenius) and de Rham–Witt (involving the domino number $T^{0,2}$).

In what follows we will write

$$
c_i = c_i(T_X),
$$

$$
b_1 = \dim_K H^1_{\text{cris}}(X/W) \otimes K,
$$

$$
T^{0,2} = \dim_k \text{Dom}^{0,2}(H^2(X, W(0)_X)) \to H^2(X, W\Omega^1_X)).
$$

**Proposition 4.1** Let $X$ be a smooth, projective surface over a perfect field of characteristic $p > 0$.

(i) Then the following conditions are equivalent:

(a) the inequality $c^2_1 \leq 5c_2$ holds,

(b) the inequality $h_{W}^{1,1} \geq b_1$ holds (if $X$ is Mazur–Ogus this is equivalent to $h_{W}^{1,1} \geq b_1$),

(c) the inequality $2T^{0,2} + b_1 \leq m^{1,1}$ holds.

(ii) If $X$ is a Hodge–Witt surface. Then the following assertions are equivalent:

(a) the inequality $c^2_1 \leq 5c_2$ holds,

(b) the inequality $m^{1,1} \geq 2m^{0,1}$ holds.

**Proof** All the assertions are trivial consequences of the following formulae, and the fact that if $X$ is Hodge–Witt then $T^{0,2} = 0$:

$$
h_{W}^{1,1} = m^{1,1} - 2T^{0,2},
$$

$$
h_{W}^{1,1} = \frac{5c_2 - c^2_1}{6} + b_1,
$$

$$
b_1 = m^{0,1} + m^{1,0},
$$

$$
m^{0,1} = m^{1,0},
$$

and are left to the reader. \(\square\)

**4.2 Consequences of $h_{W}^{1,1} \geq 0$**

We also record the main reason for our interest in $h_{W}^{1,1} \geq 0$.

**Proposition 4.2** Let $X/k$ be a smooth projective surface over a perfect field $k$. Then $h_{W}^{1,1} \geq 0$ holds if and only if the inequality

$$
c^2_1 \leq 5c_2 + 6b_1
$$

(13)
holds. On the other hand if $h_{W}^{1,1} < 0$, then

$$c_1^2 \geq 5c_2.$$  \hfill (14)

**Proof** The asserted inequalities follow easily from Ekedahl’s formula (7) for $h_{W}^{1,1}$:

$$h_{W}^{1,1} = b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2.$$  

Hence we see that $h_{W}^{1,1} \geq 0$ gives $5c_2 - c_1^2 \geq -6b_1$ or $c_1^2 \leq 5c_2 + 6b_1$. This proves the first part of the assertion.

If $h_{W}^{1,1} < 0$ then we see that

$$b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2 < 0.$$  

As $b_1 \geq 0$ the term on the left is not less than $\frac{5}{6}c_2 - \frac{1}{6}c_1^2$ and so

$$\frac{5}{6}c_2 - \frac{1}{6}c_1^2 \leq h_{W}^{1,1} < 0,$$  \hfill (15)

and the result follows. \hfill \Box

**Remark 4.3** Let $X$ be a smooth projective surface of general type. Clearly when $h_{W}^{1,1} < 0$ the Bogomolov–Miyaoka–Yau inequality also fails. On the other hand if $X$ satisfies $c_1^2 \leq 3c_2$ then $h_{W}^{1,1} \geq 0$. Thus the point of view which seems to emerge from the results of this section is that surfaces with $h_{W}^{1,1} < 0$ are somewhat more exotic than the ones for which $h_{W}^{1,1} \geq 0$. Indeed as was pointed out in [12], $h_{W}^{1,1}$ is a deformation invariant so surfaces with $h_{W}^{1,1} < 0$ do not even admit deformations which lift to characteristic zero.

**Corollary 4.4** If $X$ is a smooth projective surface for which (13) fails to hold, then the slope spectral sequence of $X$ has infinite torsion and does not degenerate at $E_1$.

**Proof** Indeed, this follows from the formula

$$h_{W}^{1,1} = m^{1,1} - 2T^{0,2},$$  \hfill (16)

which is just the definition of $h_{W}^{1,1}$. The claim now follows as $m^{1,1} \geq 0$ and hence $h_{W}^{1,1} < 0$ implies that $T^{0,2} \geq 1$. \hfill \Box

**Remark 4.5** Thus we see that the counter examples to the Bogomolov–Miyaoka–Yau inequality given in [43] are not Hodge–Witt.
4.3 The surfaces for which \( c_1^2 \leq 5c_2 + 6b_1 \) holds or equivalently \( h_{W}^{1,1} \geq 0 \) holds

Our next result provides a large class of surfaces for which \( h_{W}^{1,1} \geq 0 \) does hold.

**Theorem 4.6** Let \( X/k \) be a smooth, projective surface over a perfect field of characteristic \( p > 0 \). Assume \( X \) satisfies any one of the following hypothesis:

(i) the surface \( X \) is Hodge–Witt (in the sense of Sect. 2.29),
(ii) or \( X \) is ordinary (in the sense of Sect. 2.29),
(iii) or \( X \) is a Mazur–Ogus surface,
(iv) or assume \( p \geq 3 \) and \( X \) is a Deligne–Illusie surface,
(v) or assume \( p = 2 \), and \( X \) lifts to \( W_2 \).

Then \( X \) satisfies (13):

\[
c_1^2 \leq 5c_2 + 6b_1.
\]

**Proof** The assertion that (13) holds is equivalent to \( h_{W}^{1,1} \geq 0 \) where:

\[
h_{W}^{1,1} = m^{1,1} - 2T^{0,2} = \frac{5c_2 - c_1^2}{6} + b_1.
\]

Thus it suffices to prove that \( h_{W}^{1,1} \geq 0 \) under any of the assumptions (i)–(v). If (i) holds then the asserted inequality follows from the fact that \( T^{0,2} = 0 \) as \( X \) is Hodge–Witt and \( m^{1,1} \geq 0 \) by definition. If (ii) holds then \( X \) is ordinary and in particular \( X \) is Hodge–Witt and so the result follows from the implication (i) \( \Rightarrow \) (13). If (iii) holds we can simply invoke [12, Corollary 3.3.1, p. 86] which gives us \( h_{W}^{1,1} = h^{1,1} \). However we give an elementary proof in the spirit of this paper. We will use the formulas \( \chi(\mathcal{O}_X) = 1 - h^{0,1} + h^{0,2} \) and \( c_2 = \chi_{et}(X) = 1 - b_1 + b_2 - b_3 + b_4 = 2 - 2b_1 + b_2 \).

By (2) we have

\[
12\chi(\mathcal{O}_X) = c_1^2 + c_2,
\]

or equivalently \( c_1^2 = 12\chi(\mathcal{O}_X) - c_2 \). Now the assertion will follow if we prove that \( c_1^2 \leq 5c_2 + 6b_1 \). But

\[
5c_2 - c_1^2 + 6b_1 = 5c_2 - (12\chi(\mathcal{O}_X) - c_2) + b_1
= 6c_2 - 12\chi(\mathcal{O}_X) + 6b_1
= 6(2 - 2b_1 + b_2) - 12\chi(\mathcal{O}_X) + 6b_1
= 12 - 12b_1 + 6b_2 - 12(1 - h^{0,1} + h^{0,2}) + 6b_1
= 6b_1 - 12h^{0,1} + 6b_2 - 12h^{0,2}.
\]

Thus we see that \( 5c_2 - c_1^2 + 6b_1 = 6(b_1 - 2h^{0,1}) + 6(b_2 - 2h^{0,2}) \). By [9] and the hypothesis that the crystalline cohomology of \( X \) is torsion-free we have \( b_2 = h^{0,2} + h^{1,1} + h^{2,0} \). Or equivalently by Serre duality we get \( b_2 = 2h^{0,2} + h^{1,1} \) and
again by the hypothesis that the crystalline cohomology of \( X \) is torsion-free we see that \( \text{Pic}(X) \) is reduced and so \( b_1 = 2h^{0,1} \). Thus \( 5c_2 - c_1^2 + 6b_1 = 6h^{1,1} \) and so is non-negative and in particular we have deduced that \( h^{1,1}_W = h^{1,1} \). Now the implication (iv) \( \Rightarrow \) (13) follows from the implication (iii) \( \Rightarrow \) (13) (via [9])—see Theorem 2.2. The fifth assertion (v) \( \Rightarrow \) (13) falls into two cases: assume \( X \) is not ruled, then this follows from [40] as the hypothesis (v) implies that \( c_1^2 \leq 3c_2 \) by [40]. If \( X \) is ruled one deduces this from our earlier result on surfaces with Kodaira dimension \(-\infty\). \( \square \)

The following corollary is immediate:

**Corollary 4.7** Under the hypothesis of Theorem 4.6 one has

\[
c_1^2 \leq \max (5c_2 + 6b_1, 6c_2).
\]

**Remark 4.8** For this remark assume that the characteristic \( p \geq 3 \). In the absence of crystalline torsion, \( h^{1,1}_W \) detects obstruction to lifting to \( W_2 \). More precisely, if \( X \) has torsion-free \( H^2_{\text{cris}}(X/W) \), and \( h^{1,1}_W < 0 \), then \( X \) does not lift to \( W_2 \).

### 4.4 Examples of Szpiro and Ekedahl

As was pointed out in [12] the counter examples constructed by Szpiro in [43] also provide examples of surfaces which are beyond the (13) faultline. We briefly recall these examples. In [43] Szpiro constructed examples of smooth projective surfaces \( S \) together with a smooth, projective and non-isotrivial fibration \( f : S \to C \) where the fibres have genus \( g \geq 2 \) and \( C \) has genus \( q \geq 2 \). Let \( f_n : S_n \to C \) be the fibre product of \( f \) with the \( n \)th-iterate of Frobenius \( F_{C/k} : C \to C \). Then

\[
\begin{align*}
c_2(S_n) &= 4(g - 1)(q - 1), \\
c_1^2(S_n) &= p^n d + 8(g - 1)(q - 1),
\end{align*}
\]

where \( d = \deg (f_\ast (\Omega^1_{X/C})) \) is a positive integer. Thus in this case, as was pointed in [12], \( h^{1,1}_W \to -\infty \) as \( n \to \infty \). Further observe, as \( d \geq 1 \), that

\[
c_1^2 > pc_2
\]

for \( n \) large; and also that for any given integer \( m \geq 1 \), there exists a smooth, projective, minimal surface of general type such that \( c_1^2 > p^mc_2 \).

### 4.5 Weak Bogomolov–Miyaoka–Yau inequality holds in characteristic zero

Assume for this remark that \( k = \mathbb{C} \), and that \( X \) is a smooth, projective surface. Then using the Hodge decomposition for \( X \), Noether’s formula can be written as

\[
h^{1,1} = 10\chi(O_X) - c_1^2 + b_1,
\]
and as the left-hand side of this formula is always non-negative we deduce that

\[ c_1^2 \leq 10 \chi(\mathcal{O}_X) + b_1. \]

This is easily seen to be equivalent to

\[ c_1^2 \leq 5c_2 + 6b_1. \]

### 4.6 Lower bounds on \( h^{1,1}_W \)

In this subsection we are interested in lower bounds for \( h^{1,1}_W \). It turns out that unless we are in characteristic \( p \leq 7 \), the situation is not too bad thanks to a conjecture of Raynaud (which is a theorem of Shepherd–Barron).

**Proposition 4.9** Let \( X \) be a smooth projective surface of general type. Then

(i) except when \( p \leq 7 \) and \( X \) is fibred over a curve of genus at least two and the generic fibre is a singular rational curve of arithmetic genus at most four we have

\[ -c_1^2 \leq h^{1,1}_W \leq h^{1,1}. \]

(ii) If \( c_2 > 0 \) then \( h^{1,1}_W > -\frac{1}{6} c_1^2 \).

(iii) If \( X \) is not uniruled then

\[ -\frac{1}{6} c_1^2 \leq h^{1,1}_W \leq h^{1,1}. \]

(iv) If \( h^{1,1}_W < -\frac{1}{6} c_1^2 \) then there exists a morphism \( X \to C \) with connected fibres and \( C \) has genus at least one.

**Proof** We prove (i). Assume if possible that \( h^{1,1}_W < -c_1^2 \). Then by using the formula \( h^{1,1}_W = b_1 + 10 \chi(\mathcal{O}_X) - c_1^2 \) we get \( b_1 + 10 \chi(\mathcal{O}_X) < 0 \). As \( b_1 \geq 0 \) this implies that \( \chi(\mathcal{O}_X) < 0 \). By [40, Theorem 8] we know that for any surface of general type with negative \( \chi(\mathcal{O}_X) \) we have \( p \leq 7 \); and whenever \( \chi(\mathcal{O}_X) < 0 \) the surface \( X \) is fibred over a curve of genus at least two and the generic fibre is singular rational curve of genus at most four. Next we prove (ii) and (iii) which are really a consequence of Raynaud’s conjecture which was proved in [40], using the formula for \( h^{1,1}_W \) in terms of \( c_1^2, c_2, b_1 \). So suppose that \( X \) is not uniruled and assume, if possible, that

\[ h^{1,1}_W < -\frac{1}{6} c_1^2. \]

Then writing out Ekedahl’s formula for \( h^{1,1}_W \) we get

\[ h^{1,1}_W = b_1 + \frac{5}{6} c_2 - \frac{1}{6} c_1^2 < -\frac{1}{6} c_1^2, \]
and so this forces:

\[ b_1 + \frac{5}{6} c_2 < 0 \]

and as \( b_1 \geq 0 \) we see that \( c_2 \) is negative. Now by [40, Theorem 7, p. 263] we see that \( X \) is uniruled which contradicts our hypothesis. Now we prove (iv). This is a part of the proof of Raynaud’s conjecture in [40]. It is clear that the hypothesis implies that \( c_2 < 0 \). So by loc. cit. we know that the map \( X \to \text{Alb}(X) \) has one-dimensional image, and this finishes the proof.

\[ \square \]

**Remark 4.10** 1. By a result of [26], exceptions in Theorem 4.9 (i) do occur.

2. Thus the examples of surfaces given in Sect. 4.4 satisfy the inequality in Proposition 4.9.

The following is rather optimistic expectation (because of the paucity of examples) and it would not surprise us if it turns out to be false.

**Conjecture 4.11** If \( X \) is a smooth projective surface with \(-\frac{1}{6} c_1^2 \leq h^{1,1}_W < 0\) and \( b_1 \neq 0 \) then the image of the Albanese map \( X \to \text{Alb}(X) \) is one-dimensional.

### 4.7 A class of surfaces general type surfaces for which \( c_1^2 \leq 5c_2 \)

Let us begin with the following proposition.

**Proposition 4.12** Let \( X \) be a smooth, projective, minimal surface of general type such that

(i) \( X \) is Hodge–Witt,

(ii) \( c_2 > 0 \),

(iii) \( m^{1,1} \geq 2p_g \).

Then \( c_1^2 \leq 5c_2 \) holds for \( X \).

**Proof** Since \( X \) is minimal of general type so \( c_1^2 > 0 \). By the formula for \( h^{1,1}_W \) we have

\[ 6h^{1,1}_W = 6(m^{1,1} - 2T^{0,2}) = 5c_2 - c_1^2 + 6b_1. \]

As \( X \) is Hodge–Witt we see that \( T^{0,2} = 0 \) and so

\[ 6(m^{1,1} - b_1) = 5c_2 - c_1^2. \]

Hence the asserted inequality holds if \( m^{1,1} - b_1 \geq 0 \). Writing

\[ m^{1,1} - b_1 = (m^{1,1} - 2p_g) + (2p_g - 2q), \]

where we have used \( b_1 = 2q \). Thus to prove the proposition it will suffice to prove that each of the two terms in the parenthesis are non-negative. The first holds by the
hypothesis of the proposition and for the second, we see, as \(2q \leq 2h^{0.1}\), that

\[
2p_g - 2q \geq 2p_g - 2h^{0.1} = 2(\chi(\mathcal{O}_X) - 1).
\]

Thus to prove the proposition, it suffices to show that we have \(\chi(\mathcal{O}_X) \geq 1\). This is immediate, from Noether’s formula (3) and our hypothesis that \(c_2 > 0\).

**Remark 4.13** Let us remark that in characteristic \(p > 0\), \(\chi(\mathcal{O}_X)\) may be non-positive and likewise \(c_2\) can be non-positive. However it has been shown in [39] that if \(p \geq 7\), then \(\chi(\mathcal{O}_X) > 0\) for any smooth, projective minimal surface of general type. Moreover if \(\chi(\mathcal{O}_X) = 0\), then \(c_2 < 0\) by Noether’s formula (2). It was shown in [40], if \(c_2 < 0\) then \(X\) is uniruled (and in any case if \(c_2 < 0\), then the inequality \(c_2^2 \leq 5c_2\) is false). In contrast if \(k = \mathbb{C}\), a well-known result of Castelnuovo says \(c_2 \geq 0\) and \(\chi(\mathcal{O}_X) > 0\) (\(X\) minimal of general type).

The following proposition is a variant of Proposition 4.12 and is valid for the larger class of Mazur–Ogus surfaces. This proposition gives a sufficient condition (in terms of slopes of Frobenius and the geometric genus of the surface) for \(c_1^2 \leq 5c_2\) to hold.

**Proposition 4.14** Let \(X\) be a smooth, projective, minimal surface of general type such that

(i) \(X\) is Mazur–Ogus,
(ii) \(c_2 > 0\),
(iii) \(m^{1,1} \geq 4p_g\).

Then \(c_1^2 \leq 5c_2\) holds for \(X\).

**Proof** We argue as in the proof of Proposition 4.12. Since \(X\) is minimal of general type so \(c_1^2 > 0\). By the formula for \(h^{1,1}_W\) we have

\[
6h^{1,1}_W = 6(m^{1,1} - 2T^{0.2}) = 5c_2 - c_1^2 + 6b_1.
\]

As \(X\) is Mazur–Ogus we see that \(h^{1,1}_W = h^{1,1}\) and so we have

\[
6(m^{1,1} - 2T^{0.2} - b_1) = 5c_2 - c_1^2.
\]

Hence the asserted inequality holds if \(m^{1,1} - 2T^{0.2} - b_1 \geq 0\). Writing

\[
m^{1,1} - 2T^{0.2} - b_1 = (m^{1,1} - 4p_g) + (2p_g - 2T^{0.2}) + (2p_g - 2q),
\]

where we have used \(b_1 = 2q\). Thus to prove the proposition it will suffice to prove that each of the three terms in the parentheses are non-negative. The first holds by the hypothesis of the proposition and for the second we argue as follows: as \(X\) is Mazur–Ogus, so

\[
b_2 = h^{0.2} + h^{1.1} + h^{2,0} = h^{1,1} + 2p_g
\]
by degeneration of Hodge–de Rham at $E_1$. Further

$$b_2 - 2g = h^{1,1} = h^{1,1}_W = m^{1,1} - 2T^{0,2}.$$ 

Hence $b_2 - 2g = m^{1,1} - 2T^{0,2}$. So we get $b_2 - m^{1,1} = 2(p_g - T^{0,2})$. Now

$$b_2 = m^{0,2} + m^{1,1} + m^{2,0},$$

which shows that $b_2 - m^{1,1} \geq 0$ and so $p_g - T^{0,2} \geq 0$. Hence this term is non-negative. For the third term we see, as $2q \leq 2h^{0,1}$, that

$$2p_g - 2q \geq 2p_g - 2h^{0,1} = 2(\chi(\mathcal{O}_X) - 1).$$

Thus the proposition follows as we have $\chi(\mathcal{O}_X) \geq 1$ from our hypothesis that $c_2 > 0$ and Noether’s formula (3).

We do not know how often the inequality $m^{1,1} \geq 4p_g$ holds. But the following proposition shows that for surfaces of large degree in $\mathbb{P}^3$ the inequality $m^{1,1} \geq 4p_g$ holds.

**Proposition 4.15** Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree $d$. Then if $d \geq 5$, $X$ satisfies all the hypotheses of Proposition 4.14. Hence the class of surfaces to which Proposition 4.14 applies is non-empty.

**Proof** Let us assume $X$ is a smooth, projective surface of degree $d$ in $\mathbb{P}^3$. From the formulae for Hodge numbers in [28,37] it is clear for smooth, projective surfaces in $\mathbb{P}^3$, the numbers $b_2, h^{1,1}, h^{0,2} = h^{2,0} = p_g$ depend only on $d$ and are constant in the family of smooth surfaces. Further Hodge–de Rham spectral sequence for $X$ degenerates at $E_1$ and crystalline cohomology of $X$ is torsion-free. Since $b_1 = 0$ we see that $c_2 > 0$. Thus all the hypotheses of Proposition 4.14 are satisfied except possibly the hypothesis that $m^{1,1} \geq 4p_g$.

From the formula for Hodge–Witt numbers we have

$$h^{1,1} = h^{1,1}_W = m^{1,1} - 2T^{0,2},$$

and as $h^{1,1}$ is constant in this family (as it depends only on the degree), while $T^{0,2}$ can only increase under specialization (this is a result of Crew [6]), so we see that $m^{1,1}$ must also increase under specialization. At any rate we have $h^{1,1} \leq m^{1,1}$. So it suffices to prove that $h^{1,1} \geq 4p_g$. Now by Table 2, we see that $h^{1,1} = b_2 - 2p_g$ and simple calculation shows

$$h^{1,1} - 4p_g = 2d^2 - 5d + 4$$

which is positive for $d \geq 5$. Hence for $d \geq 5$, $m^{1,1} \geq h^{1,1} > 4p_g > 2p_g$. ☐

**Remark 4.16** It seems reasonable to expect that for large $c_1^2, c_2$, in the moduli of smooth, projective surfaces of general type, there is a Zariski open set of an irreducible component(s) consisting of Mazur–Ogus surfaces where $m^{1,1} \geq 4p_g$ holds.

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The following result, while not best possible, shows how we can use preceding ideas to obtain a Chern class inequality under reasonable geometric hypothesis. We do not know a result of comparable strength which can be proved by purely geometric means.

**Theorem 4.17** Let $X$ be a smooth, projective, minimal surface of general type. Assume

(i) $c_2 > 0$,
(ii) $p_g > 0$,
(iii) $X$ is Hodge–Witt,
(iv) $\text{Pic}(X)$ is reduced or $H^2_{\text{cris}}(X/W)$ is torsion-free,
(v) $H^2_{\text{cris}}(X/W)$ has no slope $< 1/2$.

Then $c_1^2 \leq 5c_2$.

**Remark 4.18** Before proving the theorem let us remark that in a smooth family of surfaces the Hodge–Witt locus is open (this is a result of Crew, see [6]); the slope condition in our hypothesis is a closed condition in Newton strata (Newton polygon of $X$ lies on or above a finite list of polygons). So these two conditions provide a locally closed subset of base space. If we consider moduli of surfaces with fixed $c_1^2, c_2$, then by [29] if $p$ is larger than a constant depending only on $c_2$, then $\text{Pic}(X)$ is reduced. So our hypothesis while not generic in the moduli are relatively harmless.

**Proof of Theorem 4.17** Before proceeding let us make two remarks. Firstly the assumption that the slopes of Frobenius on $H^2_{\text{cris}}(X/W)$ are $\geq 1/2$, together with the assumption that $p_g > 0$ says that $X$ is Hodge–Witt but not ordinary (by Mazur’s proof of Katz’s conjecture); secondly we see that $\chi(\mathcal{O}_X) \geq 1$. This follows from Noether’s formula (2), and the fact that for $X$ minimal of general type, $c_1^2 \geq 1$. Observe that $c_2 > 0$ is necessary for $c_1^2 \leq 5c_2$ to hold (as $c_2^2 \geq 1$). Thus this hypothesis (that $c_2 > 0$) is at any rate required if we wish to consider Chern class inequality of Bogomolov–Miyaoka type to hold. Moreover our hypothesis that $H^2_{\text{cris}}(X/W)$ is torsion-free implies $\text{Pic}(X)$ is reduced (see [17, Remark 6.4, p. 641]).

Proposition 4.12 shows that we have to prove that $m_{1,1} \geq 2p_g$ (under our hypothesis). We now prove this inequality under our hypothesis $\text{Pic}(X)$ is reduced and $X$ is Hodge–Witt and $H^2_{\text{cris}}(X/W)$ satisfies the slope condition stated in hypothesis (v).

We begin by recalling the formula for $m_{1,1}$ (see Sect. 2.22).

$$m_{1,1} = \sum_{\lambda \in [0,1]} \lambda m_\lambda + \sum_{\lambda \in [1,2]} (2 - \lambda) m_\lambda,$$

which, on writing $m_0 = \dim H^2_{\text{cris}}(X/W)_{[0]}$, $m_1 = \dim H^2_{\text{cris}}(X/W)_{[1]}$, and noting that $m_0$ does not contribute to $m_{1,1}$, can be written as

$$m_{1,1} = \sum_{\lambda \in [0,1]} \lambda m_\lambda + m_1 + \sum_{\lambda \in [1,2]} (2 - \lambda) m_\lambda.$$

Poincaré duality says that if $\lambda$ is a slope of $H^2_{\text{cris}}(X/W)$ then $\lambda' = 2 - \lambda$ is also a slope of $H^2_{\text{cris}}(X/W)$. Further under Poincaré duality we get $m_\lambda = m_{\lambda'}$. Hence for any
\( \nu \in ]1, 2[ \) we have
\[
(2 - \nu)m_\nu = (2 - \nu)m_\nu' = \nu'm_\nu',
\]
and for any \( \nu \in ]1, 2[ , \nu' \in ]0, 1[ . \) Thus we see that
\[
\sum_{\lambda \in ]0, 1[} \lambda m_\lambda + \sum_{\lambda \in ]1, 2[} (2 - \lambda)m_\lambda = 2 \sum_{\lambda} \lambda m_\lambda .
\]
Thus we get
\[
m^{1,1} = m_1 + 2 \sum_{\lambda \in ]0, 1[} \lambda m_\lambda .
\]
Under our hypothesis we claim that the sum on the right in the above equation is \( \geq 2p_g \). This will involve the assumption that \( \text{Pic}(X) \) is reduced and the assumption that \( X \) is Hodge–Witt. The assumption that \( X \) is Hodge–Witt says that \( H^2(W(\mathcal{O}_X)) \) is a finite type \( W \)-module (see [17]). Our hypothesis that \( \text{Pic}(X) \) is reduced means \( V \) is injective on \( H^2(W(\mathcal{O}_X)) \). We claim that \( H^2(W(\mathcal{O}_X)) \) is a free, finite type \( W \)-module. If \( H^2_{\text{cris}}(X/W) \) is torsion-free, then this is an immediate consequence of the existence of the Hodge–Witt decomposition of \( H^2_{\text{cris}}(X/W) \) (see [20, Theorem 4.5, p. 202]):
\[
H^2_{\text{cris}}(X/W) = H^2(X, W(\mathcal{O}_X)) \oplus H^1(X, W\Omega^1_X) \oplus H^0(X, W\Omega^2_X),
\]
which implies that \( H^2(W(\mathcal{O}_X)) \) is torsion-free.

Now suppose instead that \( \text{Pic}(X) \) is reduced. We want to prove that \( H^2(X, W(\mathcal{O}_X)) \) is torsion-free. We note that \( X \) is Hodge–Witt so we see that \( H^2(X, W(\mathcal{O}_X)) \) is finite type as a \( W \)-module and profinite (as a \( W[F, V] \)-module). Now \( \text{Pic}(X) \) is reduced, by [17, Remark 6.4, p.641], gives us the injectivity of \( V \) on \( H^2(X, W(\mathcal{O}_X)) \). Hence \( H^2(X, W(\mathcal{O}_X)) \) is a Cartier module (see [20, Definition 2.4]). So by [20, Proposition 2.5(d), p.99], \( H^2(X, W(\mathcal{O}_X)) \) is free of finite type over \( W \). In particular we have an exact sequence of
\[
0 \longrightarrow H^2(W(\mathcal{O}_X)) \xrightarrow{V} H^2(W(\mathcal{O}_X)) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow 0
\]
which shows that the \( W \)-rank of the former is at least \( p_g > 0 \).

Let us remind the reader that the degeneration of the slope spectral sequence modulo torsion (see [17]) or the existence of the Hodge–Witt decomposition of \( H^2_{\text{cris}}(X/W) \) (as above) shows that the slopes \( 0 \leq \lambda < 1 \) of \( H^2_{\text{cris}}(X/W) \) live in \( H^2(W(\mathcal{O}_X)) \).

Next note that for \( \lambda \) satisfying \( 1/2 \leq \lambda < 1 \) we have
\[
\lambda \geq 1 - \lambda > 0.
\]
Thus we have
\[ \sum_{\lambda \in [0, 1]} \lambda m_\lambda \geq \sum_{\lambda \in [0, 1]} (1 - \lambda) m_\lambda. \]
We claim now that
\[ \sum_{\lambda \in [0, 1]} (1 - \lambda) m_\lambda = \dim H^2(W(\mathcal{O}_X))/V H^2(W(\mathcal{O}_X)). \]
This is standard and is a consequence of the proof of [6, Lemma 3]—we state Crew’s result explicitly as Lemma 4.19 (below) for convenient reference. Finally the above exact sequence shows that
\[ H^2(W(\mathcal{O}_X))/V H^2(W(\mathcal{O}_X)) \cong H^2(\mathcal{O}_X), \]
and hence that the sum is \( \geq p_g > 0 \), and therefore
\[ m^{1,1} \geq m_1 + 2p_g > 2p_g, \]
as \( m_1 \geq 1 \) (by projectivity of \( X \) so we are done by Proposition 4.12.

**Lemma 4.19** ([6, Lemma 3]) Let \( M \) be an \( R_0 = W[F, V] \)-module (with \( F V = p \)) and such that \( M \) is finitely generated and free as a \( W \)-module. Assume that the slopes of Frobenius on \( M \) satisfy \( 0 \leq \lambda < 1 \). Then
\[ \text{length}(M/VM) = \sum_{\lambda} (1 - \lambda) m_\lambda. \]

**Remark 4.20** Let us give examples of surfaces of general type which satisfy the hypotheses of our theorem. In general construction of Hodge–Witt but non-ordinary surfaces is difficult. Suppose \( C, C' \) are smooth, proper curves over \( k \) (perfect) with genus \( \geq 2 \), and that \( C \) is ordinary and \( C' \) has slopes of Frobenius \( \geq 1/2 \) in \( H^1_{\text{cris}}(C'/W) \). Such curves exist—for example there exist curves of genus \( \geq 2 \) whose Jacobian is a supersingular abelian variety. Now let \( X = C \times_k C' \). Then by a well-known theorem of Katz and Ekedahl [11, Proposition 2.1 (iii)], \( X \) is Hodge–Witt and by Kunneth’s formula, the slopes of Frobenius on \( H^2_{\text{cris}}(X/W) \) are \( \geq 1/2 \). The other hypotheses of Theorem 4.17 are clearly satisfied.

### 4.8 On \( c_1^2 \leq 5c_2 \) for supersingular surfaces

In this subsection, we will say that \( X \) is a supersingular surface if \( H^2(X, W(\mathcal{O}_X)) \otimes_W K = 0 \) (here \( K \) is the quotient field of \( W \)). This means that \( H^2_{\text{cris}}(X/W) \) has no slopes in \([0, 1)\) and hence by Poincaré duality, it has no slopes in \((1, 2]\). Thus \( H^2_{\text{cris}}(X/W) \) is pure slope one. Under reasonable assumptions the following dichotomy holds:
Proposition 4.21 Let $X$ be a smooth, projective surface over a perfect field of characteristic $p > 0$. Assume

(i) $X$ is a minimal surface of general type,
(ii) $p_g > 0$,
(iii) $c_2 > 0$,
(iv) $X$ is Mazur–Ogus,
(v) $X$ is supersingular.

Then either $c_1^2 \leq 5c_2$, or $c_2 < 2\chi(\mathcal{O}_X)$, and in the second case no smooth deformation of $X$ admits any flat lifting to characteristic zero.

Proof Since $X$ is Mazur–Ogus (i.e., Hodge–de Rham spectral sequence of $X$ degenerates at $E_1$, and crystalline cohomology of $X$ is torsion-free), and supersingular, we see that

$$b_2 = h^{2,0} + h^{1,1} + h^{0,2}, \quad m^{1,1} = b_2.$$

Thus (6) gives

$$h^{1,1}_W = h^{1,1} = b_2 - 2p_g = m^{1,1} - 2T^{0,2} = b_2 - 2T^{0,2},$$

which gives $T^{0,2} = p_g > 0$, so $X$ is not Hodge–Witt. Further we have

$$h^{1,1}_W = h^{1,1} = b_2 - 2p_g = \frac{5c_2 - c_1^2}{6} + b_1.$$

This gives

$$6(b_2 - b_1 - 2p_g) = 5c_2 - c_1^2.$$

Thus $c_1^2 \leq 5c_2$ if and only if

$$b_2 - b_1 - 2p_g \geq 0.$$

If $b_2 - b_1 - 2p_g \geq 0$ then $c_1^2 \leq 5c_2$ and the first assertion holds and we are done. Now suppose $b_2 - b_1 - 2p_g < 0$. So we get

$$b_2 < b_1 + 2p_g.$$

Now we get from the fact that $c_2 = b_2 - 2b_1 + 2$,

$$c_2 = b_2 - 2b_1 + 2 < b_1 + 2p_g - 2b_1 + 2 = 2p_g - b_1 + 2 = 2\chi(\mathcal{O}_X).$$

Hence

$$c_2 < 2\chi(\mathcal{O}_X). \quad (17)$$
On the other hand note that if $k = \mathbb{C}$ and $X/\mathbb{C}$ is smooth, minimal of general type then $c_2 \geq 3\chi(\mathcal{O}_X)$ (by the Bogomolov–Miyaoka–Yau inequality and Noether’s formula) so $c_2 < 2\chi(\mathcal{O}_X)$ never happens over $\mathbb{C}$. Hence no smooth deformation of a surface with $c_2 < 2\chi(\mathcal{O}_X)$ can be liftable to characteristic zero (as $c_2, \chi(\mathcal{O}_X)$ are deformation invariants).

**Remark 4.22** We do not know if the condition (on minimal surfaces of general type) that $c_2 < 2\chi(\mathcal{O}_X)$ is relatively rare or even bounded.

### 4.9 A lower bound on slopes of Frobenius

In this section we prove a lower bound on slopes Frobenius on $H^2_{\text{cris}}(X/W)$ for smooth, projective surfaces. This theorem shows that the assumption on the smallest slope of Frobenius in Theorem 4.17 is perhaps not too unreasonable. The theorem is the following:

**Theorem 4.23** Let $X/k$ be a smooth, projective surface over a perfect field $k$ of characteristic $p > 0$. Let

$$\lambda^2_{\text{min}} = \min \{ \lambda : \lambda \text{ is a slope of Frobenius on } H^2_{\text{cris}}(X/W) \}.$$

Assume that $p_g \neq 0$. Then exactly one of the following holds:

(i) either $\lambda^2_{\text{min}} = 0$, or

(ii) $\lambda^2_{\text{min}} \geq 1/(p_g + 1)$.

**Remark 4.24** In [30] it was shown that if $X$ is a smooth, projective surface with $p_g = 1$ and $X$ has torsion-free crystalline cohomology then

$$\lambda^2_{\text{min}} = \begin{cases} 0 & \text{or,} \\ 1 - 1/n & \text{with } n \geq 2, \end{cases}$$

and so if $\lambda^2_{\text{min}} \neq 0$ one has $\lambda^2_{\text{min}} \geq 1/2$.

**Remark 4.25** While Theorem 4.23 asserts that $\lambda_{\text{min}} \geq 1/(p_g + 1)$, in Theorem 4.17 we had assumed that $\lambda_{\text{min}} \geq 1/2$. It is tempting to hope that perhaps surfaces of general type with $1/(p_g + 1) \leq \lambda_{\text{min}} < 1/2$ do not occur or occur in bounded families (for each fixed characteristic). But again we have no evidence for this if $p_g > 1$.

**Proof** If $\lambda^2_{\text{min}} = 0$ then there is nothing to prove. So assume $\lambda^2_{\text{min}} > 0$.

Recall that $m^{0,2} = \sum_{\lambda \in [0,1]} (1 - \lambda)m_\lambda$, and

$$0 \leq m^{0,2} \leq m^{0,2} + T^{0,2} = h^{0,2}_W \leq h^{0,2} = p_g \neq 0. \quad (18)$$

For notational convenience let $\nu = \lambda^2_{\text{min}}$ and let $N = p_g + 1$. Suppose if possible that $0 < \nu < 1/N$. 

\[ \text{Springer} \]
Then \( 1/v > N \) and \(-v > -1/N\) so \( 1 - v > 1 - 1/N \). Thus one has

\[
m_v(1 - v) > m_v\left(1 - \frac{1}{N}\right).
\]

Now \( vm_v \geq 1 \) (as \( 0 < vm_v \in \mathbb{Z} \) by Dieudonné Theory) so \( m_v \geq 1/v > N \). Hence one sees that

\[
m_v(1 - v) > m_v\left(1 - \frac{1}{N}\right) > \left(1 - \frac{1}{N}\right)N = N - 1 = p_g,
\]

in particular

\[
m_v(1 - v) > p_g.
\]

Thus one has combining (18) and (19) that

\[
p_g < (1 - v)m_v \leq m^{0.2} \leq p_g
\]

which is a contradiction. Thus \( v \geq 1/N = 1/(p_g + 1) \). \( \square \)

4.10 Recurring fantasy for ordinary surfaces

In this subsection we sketch a very optimistic conjectural program (in fact we are still somewhat reluctant to call it a conjecture—perhaps, following Spencer Bloch, it would be better to call it a recurring fantasy) to prove the analog of van de Ven’s inequality [44] for ordinary surfaces of general type and which satisfy Assumptions 4.10.1. Unfortunately we do not know how to prove our conjecture (see Conjecture 4.26 below). We will make the following assumptions on \( X \) smooth, projective over an algebraically closed field \( k \) of characteristic \( p > 0 \).

4.10.1 Assumptions

For the entire Sect. 4.10 we make the following assumptions on a smooth, projective surface \( X \):

1. \( X \) is minimal of general type,
2. \( X \) is not fibred over a smooth, projective curve of genus \( g > 1 \),
3. \( X \) is an ordinary surface, and
4. \( X \) has torsion-free crystalline cohomology.

The last two assumptions imply (see [20]) that

1. Hodge and Newton polygons of \( X \) coincide, and
2. we have a Newton–Hodge decomposition:

\[
H^{n}_{\text{cris}}(X/W) = \bigoplus_{i+j=n} H^i(X, W\Omega^j_X),
\]
3. the Hodge–de Rham spectral sequence of \( X \) degenerates at \( E_1 \).

In particular we have

\[
H^1_{\text{cris}}(X/W) = H^0(X, W\Omega^1_X) \oplus H^1(X, W\Omega_X),
\]

and

\[
\text{rk}_W H^0(X, W\Omega^1_X) = \text{rk}_W H^1(X, W\Omega_X).
\]

By the usual generalities (see [17]) we have a cup product pairing of \( W[F, V] \)-modules (here \( FV = p \)):

\[
\langle \cdot, \cdot \rangle : H^1(X, W\Omega_X) \otimes H^0(X, W\Omega^1_X) \to H^1(X, W\Omega^1_X). \tag{20}
\]

**Conjecture 4.26** For any surface \( X \) satisfying assumptions of 1.–4. of Sect. 4.10.1, the cup product paring of (20) satisfies the following properties:

(i) for each fixed \( 0 \neq v \in H^1(X, W\Omega_X) \), the map \( \langle v, - \rangle \) is injective;

(ii) for each fixed \( 0 \neq v' \in H^0(X, W\Omega^1_X) \), the mapping \( \langle -, v' \rangle \) is injective.

**Corollary 4.27** Assume Conjecture 4.26 and let

\[
h^{0,1} = \text{rk}_W H^1(X, W\Omega_X),
\]

\[
h^{1,0} = \text{rk}_W H^0(X, W\Omega^1_X),
\]

\[
h^{1,1} = \text{rk}_W H^1(X, W\Omega^1_X).
\]

Then \( h^{1,1} \geq 2h^{1,0} - 1 = b_1 - 1 \).

It is clear that, assuming Conjecture 4.26, this can be proved in a manner similar to van de Ven’s proof of the above inequality (see [44]). The conjecture and the above inequality have the following immediate consequence.

**Theorem 4.28** Under the assumptions of Sect. 4.10.1 and Conjecture 4.26 on \( X \), the Chern classes of \( X \) satisfy \( c_1^2 \leq 5c_2 + 6 \).

**Proof** Recall the formula of Crew and Ekedahl (6)

\[
6h_W^{1,1} = 6(m^{1,1} - T^0) = 5c_2 - c_1^2 + 6b_1,
\]

where \( h_W^{1,1} \) is the Hodge–Witt number of \( X \) and \( m^{1,1} \) is the slope number of \( X \). If \( X \) is ordinary, we see that \( T^0 = 0 \) and so \( m^{1,1} = h^{1,1} \) is the dimension of the slope one part of \( H^2_{\text{cris}}(X/W) \).

Hence

\[
5c_2 - c_1^2 + 6b_1 = 6m^{1,1} = 6h^{1,1},
\]
so that
\[ 5c_2 - c_1^2 = 6m^{1,1} - 6b_1 = 6h^{1,1} - 6b_1. \]

Now by the corollary we have \( h^{1,1} \geq b_1 - 1 \) so that \( h^{1,1} - b_1 \geq -1 \) and so
\[ 5c_2 - c_1^2 \geq -6, \]
or equivalently,
\[ c_1^2 \leq 5c_2 + 6. \]

This completes the proof. \( \square \)

**Theorem 4.29** Assume Conjecture 4.26. Then except for a bounded family of surfaces satisfying assumptions from Sect. 4.10.1, we have \( c_1^2 \leq 6c_2 \).

**Proof** By Theorem 4.28, \( c_1^2 \leq 5c_2 + 6 \) holds for all surfaces satisfying assumptions from Sect. 4.10.1. Now consider surfaces for which 4.10.1 hold and \( c_2 < 6 \). Then, for these surfaces \( c_1^2 \leq 5c_2 + 6 < 36 \). Thus surfaces which satisfy \( c_2 < 6 \) also satisfy \( c_1^2 < 36 \) (under 4.10.1). Now surfaces of general type which satisfy \( c_1^2 < 36 \) and \( c_2 < 6 \) form a bounded family. For surfaces which do not belong to this family \( c_2 \geq 6 \). Hence \( c_1^2 \leq 5c_2 + 6 < 5c_2 + c_2 = 6c_2 \). \( \square \)

### 5 Enriques classification and torsion in crystalline cohomology

The main aim of this section is to explore geographical aspects of torsion in crystalline cohomology. It is well-known that if \( X/\mathbb{C} \) is a smooth, projective surface then the torsion in \( H^2(X, \mathbb{Z}) \) is invariant under blowups. We will see a refined version of this result holds in positive characteristic (see Theorem 5.3). One of the main results of this section provides a new birational invariant of smooth surfaces in characteristic \( p > 0 \).

#### 5.1 Gros’ blowup formula

In the next few subsections we will use the formulas which describe the behavior of cohomology of the de Rham–Witt complex under blowups. We recall these from [14]. Let \( X \) be a smooth projective variety and let \( Y \subset X \) be a closed, smooth subscheme, pure of codimension \( d \). Let \( X' \) denote the blowup of \( X \) along \( Y \), and let \( f : X' \to X \) be the blowing up morphism. Then one has an isomorphism:

\[
H^j(X, W\Omega^i_X) \oplus \left( \bigoplus_{0<n<d} H^{j-n}(Y, W\Omega^{i-n}_Y) \right) \cong H^j(X', W\Omega^i_{X'}). \quad (21)
\]
5.2 Birational invariance of the domino of a surface

The de Rham–Witt cohomology of a smooth, projective surface has only one, possibly non-trivial, domino. This is the domino associated to the differential $H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega^1_X)$. In this section we prove the following.

**Theorem 5.1** Let $X, X'$ be two smooth, projective surfaces over an algebraically closed field $k$ of characteristic $p > 0$ and let $X' \to X$ be a birational morphism. Then the dominos $\text{Dom}^{0,2}(X)$ (resp. $\text{Dom}^{0,2}(X')$) associated to the differentials $H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega^1_X)$ (resp. $H^2(X, W(\mathcal{O}_X)) \to H^2(X', W\Omega^1_{X'})$) are naturally isomorphic.

5.3 Proof of Theorem 5.1

As any birational morphism $X' \to X$ as above factors as a finite sequence of blowups at closed points, we may assume that $X' \to X$ is the blowup of $X$ at a single point. In what follows, to simplify notation, we will denote objects on $X'$ by simply writing them as primed quantities and the unprimed quantities will denote objects on $X$. We will use the notation of Sect. 2.8.

The construction of the de Rham–Witt complex $W\Omega^*_X$ is functorial in $X$. The properties of the de Rham–Witt complex (in the derived category of complexes of sheaves of modules over the Cartier–Dieudonné–Raynaud algebra) under blowing up have been studied extensively in [14], and using [14, Chapter 7, Theorem 1.1.9], and the usual formalism of de Rham–Witt cohomology, we also have a morphism of slope spectral sequences. The blowup isomorphisms described in the blowup formula fit into the following diagram:

$$
\begin{array}{cccc}
H^2(W(\mathcal{O}_{X'})) & \xrightarrow{d'} & H^2(W\Omega^1_{X'}) & \xrightarrow{d'} & H^2(W\Omega^2_{X'}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(W(\mathcal{O}_X)) & \xrightarrow{d} & H^2(W\Omega^1_X) & \xrightarrow{d} & H^2(W\Omega^2_X).
\end{array}
$$

By Gros’ blowup formula (21) all the vertical arrows are isomorphisms. This induces an isomorphism $Z' = \ker(d') \to \ker(d) = Z$.

Now the formula for blowup for cohomology of the de Rham–Witt complex also shows that we have isomorphisms for $i = 1, 2$,

$$
H^i(X', W(\mathcal{O}_{X'})) \xrightarrow{\sim} H^i(X', W(\mathcal{O}_X))
$$

and these fit into the following commutative diagram:

$$
\begin{array}{cccc}
H^1(W(\mathcal{O}_{X'})) & \to & H^1(X', \mathcal{O}_{X'}) & \to & H^2(W(\mathcal{O}_{X'})) & \to & H^2(W(\mathcal{O}_{X'})) & \to & H^2(\mathcal{O}_{X'}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(W(\mathcal{O}_X)) & \to & H^1(X, \mathcal{O}_X) & \to & H^2(W(\mathcal{O}_X)) & \to & H^2(W(\mathcal{O}_X)) & \to & H^2(\mathcal{O}_X).
\end{array}
$$
In the above commutative diagram we claim that all the vertical arrows are isomorphisms. Indeed [15, Proposition 3.4, V.5] shows that \( H^i(X', \mathcal{O}_{X'}) \to H^i(X, \mathcal{O}_X) \) are isomorphisms for \( i \geq 0 \). The other vertical arrows are isomorphisms by [14]. Thus from the diagram we deduce an induced isomorphism
\[
\ker(H^2(W(\mathcal{O}_{X'}))) \xrightarrow{V} H^2(W(\mathcal{O}_{X'})) \xrightarrow{\sim} \ker(H^2(W(\mathcal{O}_X))) \xrightarrow{V} H^2(W(\mathcal{O}_X))).
\]
Thus the \( V \)-torsion in \( H^2(W(\mathcal{O}_X)) \) of \( X \) and \( X' \) in \( H^2(W(\mathcal{O}_{X'})) \) are isomorphic (we will use this in the proof of our next theorem as well).

Now these two arguments combined also give the corresponding assertions for the composite maps \( dV^n \) (resp. \( d'V''^n \)). Thus we also have from a similar commutative diagram (with \( dV^n \) etc.) from which we deduce that we have isomorphisms \( \ker(d'V''^n) = V'^{-n}Z' \to V^{-n}Z = \ker(dV^n) \). Thus we have an isomorphism of the intersection of
\[
V'^{-\infty}Z' = \bigcap_n V'^{-n}Z' \to V'^{-\infty}Z' = \bigcap_n V^{-n}Z.
\]
Thus we have in particular, isomorphisms
\[
\frac{H^2(X', W(\mathcal{O}_{X'}))}{V'^{-\infty}Z'} \xrightarrow{\sim} \frac{H^2(X', W(\mathcal{O}_X))}{V^{-\infty}Z}.
\]
Now in the canonical factorization of \( d \) (resp. \( d' \)) in terms of their dominos we have a commutative diagram
\[
\begin{array}{cccccc}
H^2(W(\mathcal{O}_{X'})) & \xrightarrow{H^2(W(\mathcal{O}_{X'}))} & H^2(W(\mathcal{O}_{X'})) & \xrightarrow{F'^{-\infty}B'} & H^2(W\Omega^1_{X'}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(W(\mathcal{O}_X)) & \xrightarrow{H^2(W(\mathcal{O}_X))} & H^2(W\Omega^1_{X}) & \xrightarrow{F^{-\infty}B} & H^2(W\Omega^1_X).
\end{array}
\]
The first two vertical arrows and the last are isomorphisms. Hence so is the remaining arrow. This completes the proof of the theorem.

### 5.4 \( T_{0,2} \) is a birational invariant

The following corollary is immediate from the above theorem, but we also provide a simple and direct proof of this fact using properties of numerical invariants.

**Corollary 5.2** Let \( X, X' \) be smooth, projective surfaces over a perfect field and suppose that \( X' \to X \) is a birational morphism. Then \( T_{0,2}(X) = T_{0,2}(X') \).

**Proof** (Another proof) It is enough to assume that the ground field is algebraically closed. Using the fact that any birational morphism \( X' \to X \) of surfaces factors as
finite sequence of blowups at closed points, we reduce to proving this assertion for the case when \( \nu' \) is the blowup at one closed point.

As \( c_2 \) increases by 1 and \( c_1^2 \) decreases by 1 under blowups, the formula for \( h_{W}^{1,1} \) shows that \( h_{W}^{1,1}(X') = h_{W}^{1,1}(X) + 1 \) while using the formula for blowups for crystalline cohomology and a slope computation shows that the slope numbers of \( X' \) and \( X \) satisfy

\[
m^{1,1}(X') = m^{1,1}(X) + 1,
\]

here the “1” is the contribution coming from the cohomology in degree two of the exceptional divisor which is one-dimensional, so the result follows as \( h_{W}^{1,1} = m^{1,1} - 2T^{0,2} \).

\[ \square \]

5.5 Crystalline torsion

We begin by quickly recalling Illusie’s results about crystalline torsion. By crystalline torsion we will mean torsion in the \( W \)-module \( H_{\text{cris}}^2(X/W) \), which we will denote by \( H_{\text{cris}}^2(X/W)_{\text{Tor}} \). Let \( X/k \) be a smooth projective variety. According to [17], torsion in \( H_{\text{cris}}^1(X/W) \) arises from several different sources (see [17, Section 6]). Torsion in the Néron–Severi group of \( X \), denoted \( \text{NS}(X/k)_{\text{Tor}} \) in this paper, injects into \( H_{\text{cris}}^2(X/W) \) via the crystalline cycle class map (see [17, Proposition 6.8, p. 643]). The next species of torsion one finds in the crystalline cohomology of a surface is the \( V \)-torsion, denoted by \( H_{\text{cris}}^2(X/W)_{V} \). It is the inverse image of \( V \)-torsion in \( H^2(X, W(\mathcal{O}_X)) \), denoted here by \( H^2(X, W(\mathcal{O}_X))_{V\text{-tors}} \), under the map \( H^2(X/W) \to H^2(X, W(\mathcal{O}_X)) \). It is disjoint from the Néron–Severi torsion (see [17, Proposition 6.6, p. 642]). Torsion of these two species is collectively called the divisorial torsion in [17, p. 643] and denoted by \( H_{\text{cris}}^2(X/W)_{d} \). The quotient

\[
H_{\text{cris}}^2(X/W)_e = H_{\text{cris}}^2(X/W)_{\text{Tor}}/H_{\text{cris}}^2(X/W)_{d}
\]

is called the exotic torsion of \( H_{\text{cris}}^2(X/W) \), or if \( X \) is a surface then simply by the exotic torsion of \( X \).

5.6 Torsion of all types is invariant under blowups

Our next result concerns the torsion in the second crystalline cohomology of a surface.

**Theorem 5.3** Let \( X' \to X \) be a birational morphism of smooth projective surfaces. Then

(i) we have an isomorphism

\[
H_{\text{cris}}^2(X/W)_{\text{Tor}} \to H_{\text{cris}}^2(X'/W)_{\text{Tor}}.
\]

(ii) and this isomorphism induces an isomorphism on the Néron–Severi, the \( V \)-torsion, and the exotic torsion.
Proof As every $X' \to X$ as in the hypothesis factors as a finite sequence of blowups at closed points, it suffices to prove the assertion for the blowup at one closed point. So let $X' \to X$ be the blowup of $X$ at one closed point $x \in X$. The formula for blowup for crystalline cohomology induces an isomorphism

$$H^2_{\text{cris}}(X/W)_{\text{Tor}} \cong H^2_{\text{cris}}(X'/W)_{\text{Tor}}.$$ 

This proves assertion (i). As remarked earlier, the proof of Theorem 5.1, also shows that the $V$-torsion of $H^2(W(\mathcal{O}_X))$ and $H^2(W(\mathcal{O}_{X'}))$ are isomorphic. Then by [17, Proposition 6.6, p. 642] we see that the $V$-torsion of $X$ and $X'$ are isomorphic. Thus we have an isomorphism on the $V$-torsion $H^2_{\text{cris}}(X/W)_v \cong H^2_{\text{cris}}(X'/W)_v$. Further it is standard that the Néron–Severi group of $X$ does not acquire any torsion under blowup $X' \to X$. So we have an isomorphism

$$H^2_{\text{cris}}(X/W)_d \to H^2_{\text{cris}}(X'/W)_d,$$

of the divisorial torsion of $X$ and $X'$. Therefore in the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^2_{\text{cris}}(X/W)_d & \longrightarrow & H^2_{\text{cris}}(X/W)_{\text{Tor}} & \longrightarrow & H^2_{\text{cris}}(X/W)_e & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2_{\text{cris}}(X'/W)_d & \longrightarrow & H^2_{\text{cris}}(X'/W)_{\text{Tor}} & \longrightarrow & H^2_{\text{cris}}(X'/W)_e & \longrightarrow & 0
\end{array}
$$

the first two columns are isomorphisms and the rows are exact so that the last arrow is an isomorphism. \qed

Remark 5.4 It is clear from Proposition 5.3 that while studying torsion in the crystalline cohomology of a surface that we can replace $X$ by its smooth minimal model (when it exists).

5.7 $\kappa \leqslant 0$ means no exotic torsion

The next result we want to prove is probably well-known to the experts. But we will prove a more precise form of this result in Theorem 5.11 and Proposition 5.9. We begin by stating the result in its coarsest form.

Theorem 5.5 Let $X/k$ be a smooth projective surface over a perfect field. If $\kappa(X) \leqslant 0$ then $H^2_{\text{cris}}(X/W)$ does not have exotic torsion.

5.8 The case $\kappa(X) = -\infty$

The case $\kappa(X) = -\infty$ is the easiest of all. If $\kappa(X) = -\infty$, then $X$ is rational or ruled. If $X$ is rational, by the birational invariance of torsion we reduce to the case $X = \mathbb{P}^2$ or $X$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and in this case one deduces the result following result by inspection. Thus one has to deal with the case that $X$ is ruled.
Proposition 5.6 Let $X$ be a smooth ruled surface over $k$. Then $H^2_{\text{cris}}(X/W)$ is torsion-free and $X$ is Hodge–Witt.

Proof The first assertion follows from the formula for crystalline cohomology of a projective bundle over a smooth projective scheme. The second assertion follows from the following lemma which is of independent interest and will be of frequent use to us.

Lemma 5.7 Let $X$ be a smooth, projective variety over a perfect field $k$.

(i) If $H^i(X, \mathcal{O}_X) = 0$ then $H^i(X, W(\mathcal{O}_X)) = 0$.

(ii) If $X/k$ is a surface with $p_g(X) = 0$ then $X$ is Hodge–Witt.

Proof This is well-known and was also noted in [23]. We include it here for completeness. Clearly, it is sufficient to prove the first assertion. We have the exact sequence

$$0 \rightarrow W_{n-1}(\mathcal{O}_X) \rightarrow W_n(\mathcal{O}_X) \rightarrow \mathcal{O}_X \rightarrow 0.$$ 

The result now follows by induction on $n$ and the fact that $H^i(X, \mathcal{O}_X) = 0$.

Lemma 5.8 Let $X$ be a smooth projective variety over a perfect field. If $H^2(X, \mathcal{O}_X) = 0$ then there is no exotic or $V$-torsion in $H^2_{\text{cris}}(X/W)$.

Proof By Illusie’s description of exotic torsion (see [17]) one knows that it is the quotient of a part of $p$-torsion in $H^2(X, W(\mathcal{O}_X))$, but this group is zero by Lemma 5.7, so its quotient by the $V$-torsion is zero as well.

5.9 Surfaces with $\kappa(X) = 0$

Let $X$ be a smooth projective surface with $\kappa(X) = 0$. We can describe the crystalline torsion of such surfaces completely. The description of surfaces with $\kappa(X) = 0$ breaks down into the following cases based on the value of $b_2$ of the surface $X$ (see [4]).

Proposition 5.9 Let $X/k$ be a smooth projective surface of Kodaira dimension zero. Then one has the following:

(i) if $b_1(X) = 4$, then $X$ is an abelian surface and $H^2_{\text{cris}}(X/W)$ is torsion-free so all species of torsion are zero; moreover $X$ is Hodge–Witt if and only if $X$ has $p$-rank one.

(ii) If $b_2(X) = 22$, then $X$ is a $K3$-surface and $H^2_{\text{cris}}(X/W)$ is torsion-free and $X$ is Hodge–Witt if and only if the formal Brauer group of $X$ is of finite height.

(iii) Assume $b_2 = 2$. Then $b_1 = 2$ and there are two subcases given by the value of $p_g$:

(a) if $p_g = 0$, then $H^2_{\text{cris}}(X/W)$ has no torsion and $X$ is Hodge–Witt;

(b) if $p_g = 1$ then $H^2_{\text{cris}}(X/W)$ has $V$-torsion and $\text{Pic}(X)$ is not reduced.

(iv) if $b_2 = 10$, then $p_g = 0$ and unless $\text{char}(k) = 2$ and in the latter case $p_g = 1$; in the former case $X$ is Hodge–Witt and $X$ has no $V$-torsion; if $p_g = 1$ then $H^2_{\text{cris}}(X/W)$ has $V$-torsion.
Proof The assertion (i) is well-known. The assertion (ii) is due to [34]. The cases when $X$ has $p_g = 0$ can be easily dealt with by using Lemmas 5.7 and 5.8.

Corollary 5.10 Let $X$ be a smooth projective surface over a perfect field. Assume $\kappa(X) = 0$ then $H^2_{\text{cris}}(X/W)$ has no exotic torsion.

Proof The cases when $p_g = 0$ are treated by means of Lemma 5.8. The remaining cases follow from Suwa’s criterion (see [41]) as in all these cases one has by [4] that $q = -p_a$ so Suwa’s criterion applies and in this situation $H^2(X, W(\mathcal{O}_X))$ is $V$-torsion, and therefore there is no exotic torsion.

Corollary 5.11 Let $X$ be a smooth, projective surface over an algebraically closed field $k$ of characteristic $p > 0$.

(i) If $X$ has exotic torsion the $\kappa(X) \geq 1$.

(ii) If $X$ has $V$-torsion then

(a) $\kappa(X) \geq 1$ or,

(b) $\kappa(X) = 0$ and $X$ has $b_2 = 2$, $p_g = 1$ or $p = 2$, $b_2 = 10$, $p_g = 1$.

5.10 A criterion for non-existence of exotic torsion

Apart from [22,41] we do not know any useful general criteria for ruling out existence of exotic torsion. The following trivial result is often useful in dealing with exotic torsion in surfaces of general type.

Proposition 5.12 Let $X/k$ be a smooth, projective surface over a perfect field. Assume Pic$(X)$ is reduced and $H^2(X, W(\mathcal{O}_X))$ is of finite type. Then $H^2_{\text{cris}}(X/W(k))$ does not contain exotic torsion.

Proof Recall from [34] that a smooth projective surface is Hodge–Witt if and only if $H^2(X, W(\mathcal{O}_X))$ is of finite type. Then as Pic$(X)$ is reduced, we see that $V$ is injective on $H^2(X, W(\mathcal{O}_X))$. Thus $H^2(X, W(\mathcal{O}_X))$ is a Cartier module of finite type. By [20, Proposition 2.5, p. 99] we know that any $R^0$-module which is a finite type $W(k)$-module is a Cartier module if and only if it is a free $W(k)$-module. Thus $H^2(X, W(\mathcal{O}_X))$ is a free $W(k)$-module of finite type. By [17, Section 6.7, p. 643] we see that the exotic torsion of $H^2_{\text{cris}}(X/W(k))$ is zero as it is a quotient of the image of torsion in $H^2_{\text{cris}}(X/W(k))$ (under the canonical projection $H^2_{\text{cris}}(X/W(k)) \to H^2(X, W(\mathcal{O}_X)))$ by the $V$-torsion of $H^2(X, W(\mathcal{O}_X))$. But as $H^2(X, W(\mathcal{O}_X))$ is torsion-free, we see that the exotic torsion is zero.

6 Mehta’s question for surfaces

6.1 Is torsion uniformizable?

In this section we answer the following question of Mehta (see [22]):

Questions 6.1 Let $X/k$ be a smooth, projective, Frobenius split variety over a perfect field $k$. Then does there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X/W)$ is torsion-free.

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6.2 Absence of exotic torsion

In [22] it was shown that the second crystalline cohomology of smooth, projective, Frobenius split surface does not have exotic torsion in the second crystalline cohomology. In [23] it was shown that any smooth, projective Frobenius split surface is ordinary.

6.3 The case $\kappa(X) = 0$

We will prove now that the answer to the above question is affirmative and in fact the assertion is true more generally for $X$ with $\kappa(X) \leq 0$. The main theorems of this section are

**Theorem 6.2** Let $k$ be perfect of characteristic $p \geq 5$. Let $X/k$ be a smooth, projective surface of Kodaira dimension at most zero, then there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X/W)$ is torsion-free.

**Theorem 6.3** Let $k$ be perfect of characteristic $p \geq 5$. Let $X/k$ be a smooth, projective surface. Assume $X$ is Frobenius split. Then there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X'/W)$ is torsion-free.

**Proof of Theorem 6.2** We now note that Mehta’s question is trivially true for ruled surfaces as these have torsion-free crystalline cohomology. So we may assume that $\kappa(X) = 0$. In this case we have a finite number of classes of surfaces for which the assertion has to be verified. These classes are classified by $b_2$. When $X$ is a $K3$ or an Enriques surface or an abelian surface then we can take $X' = X$ as such surfaces have torsion-free crystalline cohomology. When $b_2 = 2$ the surface is bielliptic and by explicit classification of these we know that we may take the Galois cover to be the product of elliptic curves and so we are done in these cases as well. $\blacksquare$

**Proof of Theorem 6.3** After Theorem 6.2 it suffices to prove that the Kodaira dimension of a Frobenius split surface is at most zero. This follows from Proposition 6.4 below (and is, in any case, well known to experts). $\blacksquare$

**Proposition 6.4** Let $k$ be perfect of characteristic $p \geq 5$. Let $X/k$ be a smooth projective surface. If $X$ is Frobenius split, then $X$ has Kodaira dimension at most zero and is in the following list:

1. $X$ is either rational or ruled over an ordinary curve,
2. $X$ is either an ordinary $K3$, or an ordinary abelian surface or $X$ is bielliptic with an ordinary elliptic curve as its Albanese variety, or $X$ is an ordinary Enriques surface.

**Proof** We first control the Kodaira dimension of a Frobenius split surface. By [31] we know that if $X$ is Frobenius split, then $H^2(X, K_X) \to H^2(X, K^n_X)$ is injective, or by duality, $H^0(X, K^{1-p}_X)$ has a non-zero section and hence in particular, $H^0(X, K^{n}_X)$ has sections for large $n$. Hence, if $\kappa(X) \geq 1$, then as the pluricanonical system $P_n$ is also non-zero for large $n$, so we can choose an $n$ large enough such that both $K^n_X$ and...
$K_X^{-n}$ have sections and so $K_X^n = \emptyset$ for some integer $n$. But this contradicts the fact that $\kappa(X) = 1$, for in that case $K_X$ is non-torsion, so we deduce that $X$ has $\kappa(X) \leq 0$. Now the result follows from the classification of surfaces with $\kappa(x) \leq 0$.

\section*{7 Hodge--Witt numbers of threefolds}

In this section we compute Hodge--Witt numbers of smooth proper threefolds. In Theorem 7.4 we characterize Calabi--Yau threefolds with negative Hodge--Witt numbers and in Proposition 7.6 Calabi--Yau threefolds constructed by [16,38] appear as examples of Calabi--Yau threefolds with negative Hodge--Witt numbers.

\subsection*{7.1 Non-negative Hodge--Witt numbers of threefolds}

We begin by listing all the Hodge--Witt numbers of a smooth, proper threefolds which are always non-negative.

\begin{proposition}
Let $X/k$ be a smooth, proper threefold over a perfect field of characteristic $p > 0$.

(i) Then $h_i^{i,j}_W \geq 0$ except possibly when $(i,j) \in \{(1,1), (2,1), (1,2), (2,2)\}$.

(ii) All the Hodge--Witt numbers except $h_1^{1,1} = h_2^{2,2}, h_1^{1,2} = h_2^{2,1}$ coincide with the corresponding slope numbers.

(iii) For the exceptional cases we have the following formulas:

\[ h_1^{1,2} = m_1^{1,2} - T_0^{0,3}, \]
\[ h_1^{1,1} = m_1^{1,1} - 2T_0^{0,2}. \]

\end{proposition}

\begin{proof}
Let us prove (i). This uses the criterion for degeneration of the slope spectral sequence given in [22]. The criterion shows that $T_{i,j} = 0$ unless $(i,j) \in \{(0,3), (0,2), (1,2), (3,1)\}$. By Definition 2.24 of $h_i^{i,j}_W$ it suffices to verify that $T_{i-1,j+1} = 0$ except possibly in the four cases listed in the proposition. This completes the proof of (i). To prove (ii), we begin by observing that Hodge--Witt symmetry 2.26.1 gives $h_2^{2,1} = h_1^{1,2}$ and we also have $h_1^{1,1} = h_3^{3,3} = h_2^{2,2}$. So this proves the first part of (ii). Next the criterion for degeneration of the slope spectral sequence shows that in all the cases except the listed ones, the domino numbers which appear in the definition of $h_i^{i,j}_W$ are zero. This proves (ii). The second formula of (iii) now follows again from the definition of $h_i^{i,j}_W$ (see 2.24 and the criterion for the degeneration of the slope spectral sequence). The first formula of (iii) follows from the definition of $h_1^{1,2} = m_1^{1,2} + T_1^{1,2} - 2T_0^{0,3}$, and by duality for domino numbers, (5), we have $T_1^{1,2} = T_0^{0,3}$.

\hfill \Box

\end{proof}
7.2 Hodge–Witt formulaire for Calabi–Yau threefolds

The formulas for Hodge–Witt numbers can be made even more explicit in the case of Calabi–Yau varieties.

Proposition 7.2 Let $X$ be a smooth, proper Calabi–Yau threefold. Then the Hodge–Witt numbers of $X$ are given by:

$$
\begin{align*}
    h_{W}^{0,0} &= 1, \quad h_{W}^{0,1} = 0, \quad h_{W}^{0,2} = 0, \quad h_{W}^{0,3} = 1, \\
    h_{W}^{1,1} &= b_2, \quad h_{W}^{1,2} = b_2 - \frac{c_3(X)}{2}, \quad h_{W}^{1,3} = 0.
\end{align*}
$$

The remaining numbers are computed from these by using Hodge–Witt symmetry and the symmetry $h_{W}^{i,j} = h_{W}^{3-i,3-j}$.

Proof We first note that $h_{W}^{0,0} = h_{W}^{0,0} = 1$ is trivial. The Hodge–Witt numbers in the first four equations are also non-negative by the previous proposition as $T_{0,2}^{0,2} = 0$. Moreover, by [12] it suffices to note that $h_{W}^{i,j} \leq h_{W}^{i,i}$ and in the second and the third formulas we have by non-negativity of $h_{W}^{i,j}$ that $0 \leq h_{W}^{0,1} \leq h_{W}^{0,1} = 0$ (by the definition of Calabi–Yau threefolds) and similarly for the third formula. The fourth formula is a consequence of Crew’s formula and first three equations:

$$
0 = \chi(O_X) = h_{W}^{0,0} - h_{W}^{0,1} + h_{W}^{0,2} - h_{W}^{0,3}. \tag{22}
$$

In particular we deduce from the fourth formula and

$$
0 \leq h_{W}^{0,3} = m_{0}^{0,3} + T_{0,3}^{0,3} \leq 1
$$

that if $T_{0,3}^{0,3} = 0$, so that $X$ is Hodge–Witt, then the definition of $m_{0}^{0,3}$ shows that

$$
m_{0}^{0,3} = \sum_{\lambda} (1 - \lambda) \dim H_{3,\text{cris}}^{3}(X/W)_{[\lambda]}.
$$

So the inequality shows that $H_{3,\text{cris}}^{3}(X/W)$ contains at most one slope $0 \leq \lambda < 1$ with $\lambda = (h - 1)/h$ (with $h$ allowed to be 1, to include $\lambda = 0$), and so if $T_{0,3}^{0,3} = 0$ then $m_{0}^{0,3} = 1$. Thus it remains to prove the formulas for $h_{W}^{1,1}$ and $h_{W}^{2,1}$. We first note that by definition:

$$
h_{W}^{1,1} = m_{0}^{1,1} + T_{1,1}^{1,1} - 2T_{0,2}^{0,2}. \tag{23}
$$

Now as $h_{W}^{0,2} = 0$ we get $T_{0,2}^{0,2} = 0$, and $T_{1,1}^{1,1} = 0$ by [20, Corollaire 3.11, p. 136]. Thus we get $h_{W}^{1,1} = m_{0}^{1,1}$. Next

$$
m_{0}^{0,2} + m_{1}^{1,1} + m_{2}^{2,0} = b_2
$$

and as $m_{0}^{0,2} = 0 = m_{2}^{2,0}$ we see that $h_{W}^{1,1} = m_{1}^{1,1} = b_2$. The remaining formula is also a straightforward application of Crew’s formula

$$
\chi(O_X^{1}) = h_{W}^{1,0} - h_{W}^{1,1} + h_{W}^{1,2} - h_{W}^{1,3}. \tag{24}
$$
and the Grothendieck–Hirzebruch–Riemann–Roch for $\Omega^1_X$, which we recall in the following lemma.

**Lemma 7.3** Let $X$ be a smooth proper threefold over a perfect field. Then

$$\chi(\Omega^1_X) = -\frac{23}{24} c_1 \cdot c_2 - \frac{c_3}{2}. \quad (25)$$

**Proof** This is trivial from the Grothendieck–Hirzebruch–Riemann–Roch theorem. We give a proof here for completeness. We have

$$\chi(\Omega^1_X) = \left[ 3 - c_1 + \frac{c_1^2 - 2c_2}{2} + \frac{-c_3 - 3c_1 \cdot c_2 - 3c_3}{6} \right] \times \left[ 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 \cdot c_2}{24} \right].$$

This simplifies to the claimed equation. $\square$

### 7.3 Calabi–Yau threefolds with negative $h^{1,2}_W$

In this section we investigate Calabi–Yau threefolds with negative Hodge–Witt numbers. From the formulas (7.2) it is clear that the only possible Hodge–Witt number which might be negative is $h^{1,2}_W$. We begin by characterizing such surfaces (see Theorem 7.4 below). Then we verify (in Proposition 7.6) that in the characteristic $p = 2, 3$, there do exist Calabi–Yau threefolds with negative Hodge–Witt numbers. These are the Hirokado and Schröer–Calabi–Yau threefolds (which do not lift to characteristic zero).

**Theorem 7.4** Let $X$ be a smooth, proper Calabi–Yau threefold over a perfect field of characteristic $p > 0$. Then the following conditions are equivalent:

(i) the Hodge–Witt number $h^{1,2}_W = -1$,

(ii) the Hodge–Witt number $h^{1,2}_W < 0$,

(iii) the $W$-module $H^3_{\text{cris}}(X/W)$ is torsion,

(iv) the Betti number $b_3 = 0$,

(v) the threefold $X$ is not Hodge–Witt and the slope number $m^{1,2}_1 = 0$.

**Proof** It is clear that (i) implies (ii), and similarly it is clear that (iii) $\Leftrightarrow$ (iv). So the only assertions which need to be proven are the assertions (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), and the assertion (iv) $\Rightarrow$ (i). So let us prove that (ii) implies (iii). By the proof of Proposition 7.2 we see that $h^{1,2}_W = m^{1,2} - T^{0,3}$ and as $T^{0,3} \leq 1$, we see that if $h^{1,2}_W < 0$ then we must have $h^{1,2}_W = -1$, $T^{0,2} = 1$, $m^{1,2} = 0$ (the first of these equalities of course shows that (ii) $\Rightarrow$ (i)). So the hypothesis of (ii) implies in particular that $T^{0,3} = 1$, in other words, $X$ is non-Hodge–Witt, and so $H^3(X, W(\mathcal{O}_X))$ is $p$-torsion. Hence the number $m^{0,3} = 0$. Now by the symmetry (2.25.1) we see that $m^{0,3} = m^{3,0} = m^{1,2} = m^{1,2} = 0$. From this and the formula (2.25.2) we see that

$$b_3 = m^{0,3} + m^{1,2} + m^{2,1} + m^{3,0} = 0.$$
This completes the proof of (ii) ⇒ (iii). Let us prove that (iv) implies (v). The hypothesis of (iv) and preceding equation shows that $m^{0,3} = m^{1,2} = 0$. So we have to verify that $X$ is not Hodge–Witt. Assume that this is not the case. The vanishing $m^{0,3} = 0$ says that $H^3(X, W(\mathcal{O}_X)) \otimes W K = 0$ and as $H^2(X, \mathcal{O}_X) = 0$ we see that $V$ is injective on $H^3(X, W(\mathcal{O}_X))$. If $X$ is Hodge–Witt, then so this $W$-module is a finite type $W$-module with $V$ being injective on it. Therefore it is a Cartier module of finite type. By [20] such a Cartier module is a free $W$-module. Hence $H^3(X, W(\mathcal{O}_X))$ is free and torsion so we deduce that $H^3(X, W(\mathcal{O}_X))$ is zero. This is a contradiction. So we see that (iv) implies (v). So now let us prove that (v) implies (i). The first hypothesis of (v) implies that $X$ is a non-Hodge–Witt Calabi–Yau threefold so that $T^{0,3} = 1$ and hence we see that $h^{1,2}_W = m^{1,2} - T^{0,3} = -1 < 0$. This completes the proof of the theorem. \[ \Box \]

Let us record the following trivial but important corollary.

**Corollary 7.5** Let $X$ be any smooth, proper, Calabi–Yau threefold. Then one has

$$h^{1,2}_W \geq -1 \quad \text{or, equivalently,} \quad b_2 \geq \frac{c_3}{2} - 1.$$ 

The next assertion shows that there do exist Calabi–Yau threefolds with $h^{1,2}_W < 0$:

**Proposition 7.6** Let $k$ be an algebraically closed field of characteristic $p = 2, 3$. Then there exists smooth, proper Calabi–Yau threefold $X$ such that $h^{1,2}_W < 0$.

**Proof** Hirokado [16] and Schröer [38] have constructed examples of Calabi–Yau threefolds (and in fact, families of such threefolds in the latter case) in characteristic $p = 2, 3$ which are not liftable to characteristic zero. We claim that these Calabi–Yau threefolds are the examples we seek. It was verified in loc. cit. that these threefolds have $b_3 = 0$. So we are done by Theorem 7.4. \[ \Box \]

**Corollary 7.7** The Hirokado and Schröer threefolds are not Hodge–Witt.

**Remark 7.8** The Hirokado and Schröer threefolds have been investigated in detail by Ekedahl (see [13]) who has proven their arithmetical rigidity. One should note that the right hand side of (7.2) is non-negative if $X$ lifts to characteristic zero without any additional assumptions on torsion of $H^*_{\text{cris}}(X/W)$ as the following proposition shows.

**Proposition 7.9** Let $X$ be a smooth, proper Calabi–Yau threefold. If $X$ lifts to characteristic zero then $c_3 \leq 2b_2$.

**Proof** Under the hypothesis, we know that $b_1 = b_3 = 0$ so that $c_3 = 2 + 2b_2 - b_3$ and in turn $b_2 - c_3/2 = b_3/2 - 1$, and by the Hodge decomposition, $b_3 \geq 2$ is even and so the assertion holds. \[ \Box \]

**Remark 7.10** The examples (for $p = 2, 3$) constructed in [38] have $b_2 = 23, c_3 = 48$ so $c_3 > 2b_2$. For the example of [16] we have also have $c_3 > 2b_2$.  

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7.4 Classical and non-classical Calabi–Yau threefolds

The above results motivate the following definition. Let \( X \) be a Calabi–Yau threefold over a perfect field of characteristic \( p > 0 \). Then we say that \( X \) is a classical Calabi–Yau threefold if \( h^{1,2}_W \geq 0 \), otherwise we say that \( X \) is non-classical Calabi–Yau threefold. Note that by (7.2) any non-classical Calabi–Yau threefold is not Hodge–Witt, i.e., \( T^{0,3} \neq 0 \). It is clear from the definition and the above results that the examples of [16,38] are non-classical Calabi–Yau threefolds. In particular non-classical Calabi–Yau threefolds exist.

7.5 Hodge–Witt rigidity

For Calabi–Yau threefolds recall that, by Serre duality, and using the perfect pairing \( \Omega^1 \otimes \Omega^3_X \to \Omega^3_X = 0 \), one has

\[
h^{1,2} = 0 \iff \dim H^1(X, T_X) = 0.
\]

If \( X \) satisfies \( h^{1,2} = 0 \) (equivalently \( H^1(X, T_X) = 0 \)) then one says that \( X \) is a rigid Calabi–Yau threefold. Using the Hodge decomposition theorem, it is easy to verify that rigidity of a Calabi–Yau threefold over complex numbers is equivalent to the condition \( b_2 = c_3/2 \). Motivated by the above definition of rigidity we introduce a weaker notion which is valid over fields of characteristic \( p > 0 \). Let \( X/k \) be a smooth, Calabi–Yau threefold. We say that \( X \) is Hodge–Witt rigid if \( h^{1,2}_W = 0 \). From Proposition 7.2 we see that

\[
X \text{ is Hodge–Witt rigid} \iff b_2 = \frac{c_3}{2}.
\]

If \( X \) is a classical Calabi–Yau threefold that is rigid then \( 0 = h^{1,2} \geq h^{1,2}_W \geq 0 \) shows \( h^{1,2}_W = 0 \). In other words,

\[
X \text{ is classical and rigid} \implies X \text{ is Hodge–Witt rigid}.
\]

But in general the two notions are different: for example the Hirokado threefold is rigid but not Hodge–Witt rigid. However, the following result shows that if \( X \) is a Mazur–Ogus threefold then Hodge–Witt rigidity and rigidity are equivalent.

**Proposition 7.11**  Let \( X \) be a Mazur–Ogus, Calabi–Yau threefold over a perfect field of characteristic \( p > 0 \). Then the following are equivalent:

(i) \( X \) is Hodge–Witt rigid, i.e., \( h^{1,2}_W = 0 \).
(ii) \( X \) is rigid, i.e., \( h^{1,2} = 0 \).
(iii) \( b_3 = 2 \).
(iv) \( c_3 = 2b_2 \).

**Proof**  The assertion is clear from Proposition 7.2 and the fact that if \( X \) is Mazur–Ogus then \( h^{1,2} = h^{1,2}_W \). \( \square \)
Remark 7.12 Let us note that there exist Calabi–Yau threefolds that are Hodge–Witt rigid. Indeed, by the above proposition it is enough to find Calabi–Yau threefolds that are Mazur–Ogus and rigid. This is not too difficult as the reduction modulo any sufficiently large, unramified prime of good reduction of any rigid Calabi–Yau threefold is both rigid and Mazur–Ogus (as it arises from characteristic zero).

### 7.6 Geography of Calabi–Yau threefolds

Note that for any Calabi–Yau threefold, $b_2$ and $c_3$ determine $b_3$. Thus one should view $b_2, c_3$ as variables for studying the geography of Calabi–Yau threefolds. Preceding results establish some results in the subject of geography of Calabi–Yau threefolds. These results are best summarized in the two maps shown in Fig. 1 (for $k = \mathbb{C}$) and in Fig. 2 (for $k$ perfect of characteristic $p > 0$). In either of these maps a quintic Calabi–Yau threefold in $\mathbb{P}^4$ such as $x_0^5 + \cdots + x_4^5 = 0$, which is a classical, Mazur–Ogus Calabi–Yau threefold, corresponds to the point $(-200, 1)$ on the line $b_2 = \max(c_3/2c_3, 1)$. Rigid Calabi–Yau threefolds live on the line $b_2 = c_3/2$. Over complex numbers the geography of Calabi–Yau threefolds does not have a non-classical component as $h^{1,2} \geq 0$.

Now suppose that we are in positive characteristic. The geography now looks different from that over $k = \mathbb{C}$. One can have non-classical Calabi–Yau threefolds and if these exist then such threefolds live on the line $b_2 = c_3/2 - 1$. In particular the Hirokado and Schröer threefolds are on this line. The region $c_3/2 - 1 < b_2 < c_3/2$ is unpopulated as in this region one would have $-1 < h^{1,2}_W = b_2 - c_3/2 < 0$ which is impossible as $h^{1,2}_W$ is an integer. Note that if $X$ is a non-classical Calabi–Yau threefold then $h^{1,2}_W = -1 = b_2 - c_3/2$ from which we see that $c_3 > 0$ and as $b_2 \geq 1$ by projectivity, so one has $c_3 \geq 4$. In particular one deduces that the half line $b_2 = c_3/2 - 1, c_3 \geq 4$, which corresponds to non-classical Calabi–Yau threefolds, is an island in the geography of Calabi–Yau threefolds as it is not connected to the classical region $b_2 \geq \frac{c_3}{2}$.
continent $b_2 \geq c_3/2$ which is the realm of classical Calabi–Yau threefolds. The line $b_2 = c_3/2$ corresponds to Hodge–Witt rigidity $h_{W}^{1,1} = 0$ and is populated, for instance, by Mazur–Ogus rigid Calabi–Yau threefolds.

### 7.7 A conjecture about Calabi–Yau threefolds

Let $X$ be a smooth proper Calabi–Yau threefold over a perfect field of characteristic $p > 0$. The following conjecture provides a necessary and sufficient condition for $X$ to admit a lifting to characteristic zero. Note that in characteristic zero any Calabi–Yau threefold does not have non-vanishing global one-forms (this is a consequence of Hodge symmetry); on the other hand in positive characteristic we do not know if this vanishing assertion always holds. On the other hand, if $H^{*}_{cris}(X/W)$ is torsion-free then certainly $H^{0}(X, \Omega_{X}^{1}) = 0$, while if $b_3 = 0$ then $H^{3}_{cris}(X/W)$ is torsion (possibly zero). As had been pointed out in the preceding discussion, any Calabi–Yau threefold which lifts to characteristic zero is classical. So the following is a very optimistic conjecture:

**Conjecture 7.13** Let $X$ be a smooth, proper Calabi–Yau threefold. Then $X$ lifts to characteristic zero if and only if

(i) $H^{0}(X, \Omega_{X}^{1}) = 0$, and

(ii) $h_{W}^{1,2} \geq 0$, equivalently:

(iii) $c_3 \leq 2b_2$, equivalently:

(iv) $X$ is classical.

Note that equivalence of (ii) and (iii), i.e., $h_{W}^{1,2} \geq 0 \Leftrightarrow c_3 \leq 2b_2$ is clear from Proposition 7.2. Also note that if $X$ is Hodge–Witt then $h_{W}^{1,2} \geq 0$. If $X$ is Hodge–Witt and also has torsion-free crystalline cohomology then $X$ is Mazur–Ogus by [20, Theorem 4.7, p. 204]. Thus Conjecture 7.13 predicts that any Hodge–Witt Calabi–Yau threefold with torsion-free $H^{*}_{cris}(X/W)$ lifts to characteristic zero.

![Fig. 2](image-url)  
**Fig. 2** Geography of Calabi–Yau threefolds
7.8 A remarkable theorem of Yobuko

At the time we made this conjecture there was not much evidence for it. But recently Yobuko [46] has proven the following remarkable theorem (the formulation provided below is our reformulation of [46] for Calabi–Yau threefolds) which provides some evidence to Conjecture 7.13.

**Theorem 7.14** Suppose $X$ is a smooth, proper Hodge–Witt Calabi–Yau threefold. Consider the following assertions:
(a) $X$ is quasi-Frobenius split.
(b) $X$ has finite height.
(c) $X$ is Hodge–Witt (hence classical).
(d) $X$ lifts to $W_2$.

Then one has (a) $\iff$ (b) $\iff$ (c) $\Rightarrow$ (d). In particular, any Hodge–Witt Calabi–Yau threefold lifts to $W_2$.

**Proof** The implication (a) $\iff$ (b) is due to [46] and the implication (a) $\Rightarrow$ (d) is the main theorem of [46]. The assertion (c) $\Rightarrow$ (b) is standard: Hodge–Witt hypothesis implies $H^3(W(O_X))$ is free of finite type (as $H^2(X, O_X) = 0$) so $X$ is of finite height. Now (b) $\Rightarrow$ (c) follows from [12,22] as there is only one possibly non-trivial domino number $T_0$. If $X$ has finite height then $H^3(W(O_X))$ is free of finite type and hence $T_0 = 0$. So $X$ is Hodge–Witt.

**Remark 7.15** Let us remark that for Calabi–Yau variety $X$ of dimension $\leq 3$, $X$ is Hodge–Witt if and only if $X$ is of finite height. If $\dim(X) > 3$ and $X$ is Hodge–Witt then $X$ is of finite height but the converse may not hold. The main theorem of [46] proves more generally that if $X$ is of finite height then $X$ lifts to $W_2$. In particular it follows that any Hodge–Witt Calabi–Yau variety (of any dimension) lifts to $W_2$.

**Remark 7.16** Let us remark that if $k = \mathbb{C}$, then $c_3 = 2b_2$ holds if and only if $X$ is a rigid Calabi–Yau threefold. Indeed if $k = \mathbb{C}$, $h^{1,2} = b_2 - c_3/2 = 0$ if and only if $H^2(X, \Omega^2_X) = 0$. By Serre duality and the fact that $X$ is a Calabi–Yau threefold we get $H^2(X, \Omega^1_X) = H^1(X, T_X) = 0$.

So $X$ is rigid. Converse is clear as these steps are reversible.

**Remark 7.17** Let us point out that the hypothesis $h^{1,2}_W \geq 0$ implies $b_3 \neq 0$. To see this we use our formula

$$ h^{1,2}_W = m^{1,2} - T^{0,3}. $$

We see that the assertion is immediate if $m^{1,2} \geq 1$, as $b_3 = m^{0,3} + m^{1,2} + m^{2,1} + m^{3,0}$. So assume $m^{1,2} = 0$. As $h^{1,2}_W \geq 0$, we see that $m^{1,2} = 0$ gives $T^{0,3} = 0$. Thus $X$ is a Hodge–Witt Calabi–Yau threefold. Thus $H^3(W(O_X))$ is of finite type and $V$ is injective on it and hence $H^3(W(O_X))$ is torsion-free and its $W$-rank is at least the length of $H^3(W(O_X)) / \text{Tor} H^3(W(O_X)) = H^3(O_X) \neq 0$, so $H^3(W(O_X)) \otimes W K \neq 0$, so $H^{3,\text{cris}}(X/W) \otimes K \neq 0$ which gives $b_3 \neq 0$.  

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