A CLASSICAL AND A RELATIVISTIC LAW OF MOTION FOR SPHERICAL SUPERNOVAE

LORENZO ZANINETTI

Dipartimento di Fisica, Via Pietro Giuria 1, I-10125 Torino, Italy; zaninetti@ph.unito.it

Received 2014 March 22; accepted 2014 September 13; published 2014 October 15

ABSTRACT

In this paper we derive some first order differential equations which model the classical and the relativistic thin layer approximations. The circumstellar medium is assumed to follow a density profile of the Plummer type, the Lane–Emden (n = 5) type, or a power law. The first order differential equations are solved analytically, numerically, by a series expansion, or by recursion. The initial conditions are chosen in order to model the temporal evolution of SN 1993J over 10 yr and a smaller chi-squared is obtained for the Plummer case with n = 6. The stellar mass ejected by the SN progenitor prior to the explosion, expressed in solar mass, is identified with the total mass associated with the selected density profile and varies from 0.217 to 0.402 when the central number density is 10^7 particles per cubic centimeter. The FWHM of the three density profiles, which can be identified with the size of the pre-SN 1993J envelope, varies from 0.0071 pc to 0.0092 pc.

Key words: ISM: supernova remnants – supernovae: general – supernovae: individual (SN 1993J)

Online-only material: color figures

1. INTRODUCTION

The absorption features of supernovae (SN) allow the determination of their expansion velocity, v. We select, among others, some results. The spectropolarimetry (CA II IR triplet) of SN 2001el gives a maximum velocity of \( \approx 26,000 \text{ km s}^{-1} \); see Wang et al. (2003). The same triplet, when searched in seven SNe of type Ia, gives \( 10,400 \text{ km s}^{-1} < v < 17,700 \text{ km s}^{-1} \); see Table 1 in Mazzali et al. (2005). A time series of eight spectra in SN 2009ig allows us to assert that the velocity at the CA II line, for example, decreases in 12 days from 32,000 \text{ km s}^{-1} to 21,500 \text{ km s}^{-1}; see Figure 9 in Marion et al. (2013). A recent analysis of 58 type Ia SNe gives \( 9660 \text{ km s}^{-1} < v < 14,820 \text{ km s}^{-1} \) in Si II; see Table 2 in Childress et al. (2014). The previous analysis allows us to say that the maximum velocity observed so far for SNe is \( (v / c) \approx 0.1 \), where \( c \) is the speed of light; this observational fact points to a relativistic equation of motion.

We now briefly review the shocks and the Kompaneyets approximation in special relativity (SR). A similar solution for strong relativistic shocks in a circumstellar medium (CSM) which varies with the radius was found by Blandford & McKee (1976). Relativistic shocks are commonly used for gamma-ray bursts (GRB) in order to explain the production of non-thermal electrons; see Baring (2011). The interactions between the shock and the ambient density fluctuations can produce turbulence with a significant component of magnetic energy; see Inoue et al. (2011). Relativistic radiation-mediated shocks can produce GRBs with typical parameters similar to those observed; see Nakar & Sari (2012). Trans-relativistic shocks have been used to produce high-energy neutrinos and gamma-rays in SNe; see Kashiyama et al. (2013). The Kompaneyets approximation is usually developed in a Newtonian framework (see Kompaneyets 1960; Olano 2009) and has been derived in SR by Shapiro 1979; Lyutikov 2012. The temporal observations of SNe such as SN 1993J establish a clear relation between the instantaneous radius of expansion \( r \) and the time \( t: r \propto t^{0.82} \) (see Marcaide et al. 2009) and therefore allows the variants of the thin layer approximation to be explored. The previous observational facts excludes SN propagation in a CSM with constant density: two solutions of this type are the Sedov solution, which scales as \( r \propto t^{0.4} \) (see Sedov 1959; McCray 1987), and the momentum conservation in a thin layer approximation, which scales as \( r \propto t^{0.25} \) (see Dyson & Williams 1997; Padmanabhan 2001).

Previous efforts to model these observations in the framework of the thin layer approximation in a CSM governed by a power law (see Zaninetti 2011) or in the framework in which the CSM has a constant density but swept mass regulated by a parameter called porosity (see Zaninetti 2012) have been successfully explored. An important feature of the various models is based on the type of CSM that surrounds the expansion. As an example in the framework of classical shocks, Chevalier (1982a 1982b) analyzed self-similar solutions with a CSM of type \( r^{-\gamma} \), which means an inverse power-law dependence. In the framework of Kompaneyets’ equation (see Kompaneyets 1960), for the motion of a shock wave in different plane-parallel stratified media, Olano (2009) considered four types of CSM. It is therefore interesting to take into account a self-gravitating CSM, which gives a physical basis to the considered model. The relativistic treatment has concentrated on the determination of the Lorentz factor, \( \gamma \), for the ejecta in GRB; we report some research in this regard: Granot & Kumar (2006) found \( 30 < \gamma < 50 \) for a significant number of GRBs; Pe’er et al. (2007) found \( \gamma = 305 \) for GRB 970828 and \( \gamma = 384 \) for GRB 990510; Zou & Piran (2010) found high values for the sample of the GRBs considered 30.5 < \( \gamma < 900 \); Aoi et al. (2010), in the framework of a high-energy spectral cutoff originating from the creation of electron–positron pairs, found \( \gamma \approx 600 \) for GRB 080916C; and Muccino et al. (2013) found \( \gamma \approx 6.7 \times 10^2 \) for GRB 090510. The last phase of the stellar evolution predicts the production of \( ^{56}\text{Ni} \) (see Truran et al. 1967; Bodansky et al. 1968; Matz & Share 1990; Truran et al. 2012) and therefore this type of decay has been used to model the light curve of SNe (see among others Mazzali et al. 1997; Elmhamdi et al. 2003; Stritzinger et al. 2006; Magkotsios et al. 2010; Krisciunas et al. 2011; Okita et al. 2012; Chen et al. 2013) as well the reddening measurements of the supernova remnant (SNR) Cassiopeia A (see Eriksen et al. 2009). These theoretical and observational efforts are interested in exploring the modification of \( ^{56}\text{Ni} \) decay due to time dilation.

In this paper, we review the standard two-phase model for the expansion of an SN (Section 2), and three density profiles

1 http://www.ph.unito.it/~zaninetti
Figure 1. Theoretical radius as given by the two-phase solution (full line) and astronomical data of SN 1993J with vertical error bars.

(Section 3). In Section 4, we derive the differential equations that model the thin layer approximation for an SN in the presence of three types of media. Section 4 also contains a model in which the center of the explosion does not coincide with the center of the polytrope. A relativistic treatment is carried out in Section 5. The application of the developed theory to SN 1993J is split into the classical case (Section 6), and the relativistic case (Section 7).

2. THE STANDARD MODEL

An SN expands at constant velocity until the surrounding mass is of the order of the solar mass. This time, \( t_M \), is

\[
 t_M = 186.45 \frac{\sqrt{M}}{\sqrt{n_0 v_{10000}}} \text{ yr},
\]

where \( M \) is the amount of matter in solar masses in the volume occupied by the SN, \( n_0 \) is the number density expressed in particles cm\(^{-3}\), and \( v_{10000} \) is the initial velocity expressed in units of 10,000 km s\(^{-1}\); see McCray (1987). A first law of motion for the SN is the Sedov solution

\[
 R(t) = \left( \frac{25}{4} \frac{E t^2}{\pi \rho} \right)^{1/5},
\]

where \( E \) is the energy injected into the process and \( t \) is the time; see Sedov (1959) and McCray (1987). Our astrophysical units are: time, \( (t) \), which is expressed in years; \( E_{51} \), the energy in 10\(^{51}\) erg; \( n_0 \), the number density expressed in particles cm\(^{-3}\) (density \( \rho = n_0 m \), where \( m = 1.4 m_{1H} \)). In these units, Equation (2) becomes

\[
 R(t) \approx 0.313 \sqrt[5]{\frac{E_{51} t^2}{n_0}} \text{ pc}.
\]

The Sedov solution scales as \( r^{0.4} \). We are now ready to couple the Sedov phase with the free expansion phase

\[
 R(t) = \begin{cases} 
 0.0157 t \text{ pc} & \text{if } t \leq 2.5 \text{ yr} \\
 0.0273 \sqrt{t} \text{ pc} & \text{if } t > 2.5 \text{ yr}.
\end{cases}
\]

This two-phase solution is obtained with the following parameters \( M_0 = 1, n_0 = 1.127 \times 10^5, E_{51} = 0.567 \) and Figure 1 presents its temporal behavior as well as the data. A similar model is reported in Spitzer (1978) with the difference that the first phase ends at \( t = 60 \text{ yr} \) against our \( t = 2.5 \text{ yr} \). A careful analysis of Figure 1 reveals that the standard two-phase model does not fit the observed radius–time relation for SN 1993J.

3. DENSITY PROFILES FOR THE CSM

This section introduces three density profiles for the CSM: a Plummer-like profile, a self-gravitating profile of Lane–Emden type, and a power-law profile.

3.1. The Plummer Profile

The Plummer-like density profile, after Plummer (1911), is

\[
 \rho(r; R_{flat}) = \rho_c \left( \frac{R_{flat}}{R_{flat} + r^2} \right)^n,
\]

where \( r \) is the distance from the center, \( \rho \) is the density, \( \rho_c \) is the density at the center, \( R_{flat} \) is the distance before which the density is nearly constant, and \( n \) is the power-law exponent at large values of \( r \); see Whitworth & Ward-Thompson (2001) for more details. The following transformation, \( R_{flat} = \sqrt{3} b \), gives a Plummer-like profile, which can be compared with the Lane–Emden profile

\[
 \rho(r; b) = \rho_c \left( \frac{1}{1 + \frac{r}{b}} \right)^n.
\]

At low values of \( r \), the Taylor expansion of the Plummer-like profile can be performed:

\[
 \rho(r; b) \approx \rho_c \left( 1 - 1/6 \frac{n r^2}{b^2} \right),
\]

and at high values of \( r \), the behavior of the Plummer-like profile is

\[
 \rho(r; b) \sim \rho_c \left( \frac{\sqrt{3} b^n}{r} \right)^n.
\]

The total mass \( M(r; b) \) between 0 and \( r \) is

\[
 M(r; b) = \int_0^r 4\pi r^2 \rho(r; b) \, dr = \frac{\text{PN}}{\text{PD}},
\]

where

\[
 \text{PN} = -3 b \rho \pi \left( \frac{4 E_{51}}{\eta} \right)^{\frac{1}{2}} \left( \frac{3^{1/2} + 5/2}{\eta} \right) \Gamma(-\eta/2 + 5/2) \Gamma(\eta/2) \cos(\eta \pi b^3 r^{-\eta} 3^{1/2} + \eta b^{\eta-1}) - 9 \pi^{3/2} b^2 \eta + 27 \pi^{3/2} b^2
\]

and

\[
 \text{PD} = 3 \left( 1/2 \pi \eta \right) \Gamma(-\eta/2 + 5/2)(\eta - 3) \Gamma(\eta/2),
\]

where \( \frac{1}{2} F_1(a; b; c; z) \) is the regularized hypergeometric function (Abramowitz & Stegun 1965; Olver et al. 2010). When \( \eta = 6, M(r; b)_6 \), the above expression simplifies to

\[
 M(r; b)_6 = \frac{27 \rho_c \pi b^3 \sqrt{3}}{2 (3 b^2 + r^2)^2} \arctan \left( \frac{1}{3} \frac{r \sqrt{3}}{b} \right) + 9 \rho_c \pi b^3 \sqrt{3} \frac{2}{(3 b^2 + r^2)^2} \arctan \left( \frac{1}{3} \frac{r \sqrt{3}}{b} \right)
\]

\[
 + \frac{3}{2} \frac{\rho_c \pi b^3 \sqrt{3} \eta}{(3 b^2 + r^2)^2} \arctan \left( \frac{1}{3} \frac{r \sqrt{3}}{b} \right)
\]

\[
 - \frac{27 \rho_c \pi b^3 r^3}{2 (3 b^2 + r^2)^2} + 9/2 \frac{\rho_c \pi b^3 r^3}{(3 b^2 + r^2)^2}.
\]
The astrophysical version of the total mass is

\[ M(r_{\text{pc}}; b_{\text{pc}}) = \frac{\text{PNA}}{\text{PDA}} M_\odot, \]

with

\[
\begin{align*}
\text{PNA} &= 2.47 \times 10^{-10} b_{\text{pc}}^3 n_0 \left[ 1.02 \times 10^{10} \arctan \left( 1.73 \frac{b_{\text{pc}}}{r_{\text{pc}}} \right) b_{\text{pc}}^4 \right. \\
&\quad + 6.8 \times 10^9 \arctan \left( 1.73 \frac{b_{\text{pc}}}{r_{\text{pc}}} \right) b_{\text{pc}}^2 r_{\text{pc}}^2 \\
&\quad + 1.13 \times 10^9 \arctan \left( 1.73 \frac{b_{\text{pc}}}{r_{\text{pc}}} \right) r_{\text{pc}}^4 - 1.6 \times 10^9 b_{\text{pc}}^4 \\
&\quad + 5.89 \times 10^9 b_{\text{pc}}^3 r_{\text{pc}} - 1.06 \times 10^9 b_{\text{pc}}^2 r_{\text{pc}}^2 \\
&\quad - 1.96 \times 10^9 b_{\text{pc}} r_{\text{pc}}^3 - 1.78 \times 10^9 r_{\text{pc}}^4 \\n\left. \right] \\
\text{PDA} &= (3.0 b_{\text{pc}}^2 + r_{\text{pc}}^2)^2,
\end{align*}
\]

where \( b_{\text{pc}} \) is \( b \) expressed in pc, \( r_{\text{pc}} \) is \( r \) expressed in pc, and \( n_0 \) is the same as in Equation (3). The relationship between the FWHM and \( b_{\text{pc}} \) is

\[ \text{FWHM} = 1.766 b_{\text{pc}}. \]

### 3.2. The Lane–Emden Profile

A self-gravitating sphere of polytropic gas is governed by the Lane–Emden differential equation of the second order

\[ \frac{d^2 Y(x)}{dx^2} + \frac{2}{x} \frac{dY(x)}{dx} + (Y(x))^n = 0, \]

where \( n \) is an integer; see Lane (1870), Emden (1907), Chandrasekhar (1967), Binney & Tremaine (2011), and Zwillinger (1989).

The solution \( Y(x)_n \) has the density profile

\[ \rho = \rho_c Y(x)_n^n, \]

where \( \rho_c \) is the density at \( x = 0 \). The pressure \( P \) and temperature \( T \) scale as

\[ P = K \rho^{1+\frac{n}{2}}, \]

\[ T = K' Y(x), \]

where \( K \) and \( K' \) are two constants; for more details, see Hansen & Kawaler (1994).

Analytical solutions exist for \( n = 0, 1, \) and \( 5; \) that for \( n = 0 \) is

\[ Y(x) = \frac{\sin(x)}{x}, \]

and therefore has an oscillatory behavior. The analytical solution for \( n = 5 \) is

\[ Y(x) = \frac{1}{\left(1 + \frac{x^2}{3}\right)^{1/2}}, \]

and the density for \( n = 5 \) is

\[ \rho(x) = \rho_c \frac{1}{\left(1 + x^2\right)^{5/2}}. \]

The variable \( x \) is non-dimensional and we now introduce the new variable \( x = r/b \)

\[ \rho(r; b) = \rho_c \frac{1}{\left(1 + \frac{r^2}{3b^2}\right)^{5/2}}. \]

This profile is a particular case, \( n = 5 \), of the Plummer-like profile as given by Equation (4). At low values of \( r \), the Taylor expansion of this profile is

\[ \rho(r; b) \approx \rho_c \left(1 - 5/6 \frac{r^2}{b^2}\right), \]

and at high values of \( r \), its behavior is

\[ \rho(r; b) \sim 9 \rho_c \frac{\sqrt{3} b^5}{r^9}. \]

The FWHM is

\[ \text{FWHM} = 1.95 b_{\text{pc}}. \]

The gradient here is assumed to be local: it covers lengths smaller than 1 pc, and is not connected with the gradient which regulates the equilibrium of a galaxy that extends over a region of a couple of kpc. The interaction of the progenitor star with the CSM, through stellar winds, creates CSM. The dynamics of this medium can be far from equilibrium. The main astrophysical assumption adopted here is that the density of the CSM decreases smoothly at low values of distance, as \( \rho \sim A - Br^2 \), due to previous stellar winds, and \( \rho \sim C r^{-3} \) in the far regions is not contaminated by previous activity, with \( A, B, \) and \( C \) being constants. In view of the behavior of this self-gravitating profile at high \( r \), in Section 3.3 we will analyze a power-law dependence for the CSM.

The total mass \( M(r; b) \) between 0 and \( r \) is

\[ M(r; b) = \int_0^r 4\pi r^2 \rho(r; b) dr = \frac{4 b^3 r^3 \rho_c \pi^{3/2}}{(3 b^2 + r^2)^{3/2}}, \]

or in solar units

\[ M(r_{\text{pc}}; b_{\text{pc}}) = \frac{2.2 \times 10^{55} b_{\text{pc}}^3 r_{\text{pc}}^3 n_0}{(2.85 \times 10^{37} b_{\text{pc}}^2 + 9.52 \times 10^{36} r_{\text{pc}}^2)^{3/2}} M_\odot. \]

The total mass of the profile can be found by calculating the limit \( r \to \infty \) in Equation (10)

\[ M(\infty; b) = \lim_{r \to \infty} M(r; b) = 4 b^3 \rho_c \pi \sqrt{3}. \]

Another interesting physical quantity deduced in the framework of the virial theorem is the mean square speed of the system, which according to formula (4.249a) in Binney & Tremaine (2011) is

\[ \langle v^2 \rangle = \frac{GM}{r_g}, \]

where \( M \) is the total mass, \( r_g \) is the gravitational radius as defined in Equation (2.42) in Binney & Tremaine (2011), and \( G \) is the Newtonian gravitational constant. In the case of a Lane–Emden profile as given by Equation (9), the gravitational radius is

\[ r_g = \frac{32 b \sqrt{3}}{3 \pi}. \]
The mean square speed of the system according to formulae (13) and (11) is
\[
\langle v^2 \rangle = \frac{3 GB^2 \rho_c \pi^2}{8}.
\] (14)

The relationship between gravitational radius, \( r_g \), and half-mass radius, \( r_h \), is
\[
\frac{r_h}{r_g} = 0.16647,
\] (15)
and this allows a more simple definition for the mean square speed
\[
\langle v^2 \rangle = 0.16647 \frac{GM}{r_h}.
\] (16)

The astrophysical version of the square root of the mean square speed as given by formula (14) is
\[
\sqrt{\langle v^2 \rangle} = 233.81 \sqrt{b_{pc}^2 n_7} \text{ km s}^{-1},
\] (17)
where \( b_{pc} \) is the scale parameter expressed in pc, \( n_7 \) represents the number density expressed in \( 10^7 \text{ cm}^{-3} \) units, and \( G = 6.67384 \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \); see Mohr et al. (2012).

3.3. A Power Law for the CSM

We now assume that the CSM around the SN scales with the SN with the following piecewise dependence (which avoids a pole at \( r = 0 \))
\[
\rho(r; r_0, d) = \begin{cases} \rho_c & \text{if } r \leq r_0 \\ \frac{\rho_c (r_0/r)^d}{\frac{d}{3} - 1} & \text{if } r > r_0. \end{cases}
\] (18)

The mass swept, \( M_0 \), in the interval \([0, r_0]\) is
\[
M_0 = \frac{4}{3} \rho_0 \pi r_0^3.
\]
The total mass swept, \( M(r; r_0, d) \), in the interval \([0, r]\) is
\[
M(r; r_0, d) = -4 r^3 \rho_c \pi \left( \frac{r_0}{r} \right)^d (d - 3)^{-1}
\]
\[
+ \frac{4 \rho_c \pi r_0^3}{d - 3} + \frac{4}{3} \rho_c \pi r_0^3,
\]
or in solar units
\[
M(r_{pc} r_0_{pc}, d) = 3.14 n_0 \left( 0.137 r_{pc}^3 \left( \frac{0.0459 r_0_{pc}^3}{d} \right) - 0.0459 r_0_{pc}^3 \right) / (d - 3) M_\odot,
\]
where \( r_{0,pc} \) is \( r_0 \) expressed in pc. The FWHM is
\[
\text{FWHM} = \frac{2 r_0_{pc}}{e^{-\frac{1}{2}\pi d}}.
\]

4. CLASSICAL CONSERVATION OF MOMENTUM

This section reviews the standard equation of motion in the case of the thin layer approximation in the presence of a CSM with constant density and derives the equation of motion under the conditions of each of the three density profiles for the density of the CSM. A simple asymmetrical model is introduced.

4.1. Motion with Constant Density

In the case of a constant density, \( \rho_c \), CSM, the differential equation which models momentum conservation is
\[
\frac{4}{3} \pi (r(t))^3 \rho_c \frac{dv}{dt} (r(t)) - \frac{4}{3} \pi r_0^3 \rho_c v_0 = 0,
\]
where \( v \) is the initial conditions at \( r = r_0 \) and \( v = v_0 \) when \( t = 0 \). The variables can be separated, and the radius as a function of the time is
\[
r(t) = \frac{2}{3} r_0^3 v_0 (t - t_0) + r_0^4,
\]
and its behavior as \( t \to \infty \) is
\[
r(t) = \frac{2}{3} r_0^3 v_0 (t - t_0) + r_0^4
\]
\[
= \frac{16}{16647} \frac{\sqrt{v_0}}{r_0^3} \sqrt{t - t_0} + 1.
\]
The velocity as a function of time is
\[
v(t) = \frac{r_0^3 v_0}{4 (r_0^3 v_0 (t - t_0) + r_0^4)^{3/4}}.
\]

4.2. Motion with Plummer Profile

The case of CSM with a Plummer-like profile as given by (4) when \( \eta = 6 \) produces the differential equation
\[
\frac{d}{dt} r(t) = \frac{\text{NDEP}}{\text{DNDEP}},
\] (19)
where
\[
\text{NDEP} = \left( 9 \sqrt{3} \arctan \left( \frac{1}{3} \frac{r_0 \sqrt{3}}{b} \right) \right) b^4
\]
\[
+ 6 \sqrt{3} \arctan \left( \frac{1}{3} \frac{r_0 \sqrt{3}}{b} \right) b^2 \rho_0^3
\]
\[
+ \sqrt{3} \arctan \left( \frac{1}{3} \frac{r_0 \sqrt{3}}{b} \right) r_0^4 - 9 b^3 r_0 + 3 b r_0^3
\]
\[
\times v(t) (3 b^2 + (r(t))^2)^2,
\]
and
\[
\text{DNDEP} = (3 b^2 + r_0^2) \left( 9 \sqrt{3} \arctan \left( \frac{1}{3} \frac{r(t) \sqrt{3}}{b} \right) \right) b^4
\]
\[
+ 6 \sqrt{3} \arctan \left( \frac{1}{3} \frac{r(t) \sqrt{3}}{b} \right) b^2 (r(t))^2
\]
\[
+ \sqrt{3} \arctan \left( \frac{1}{3} \frac{r(t) \sqrt{3}}{b} \right) (r(t))^4
\]
\[
- 9 b^3 r(t) + 3 b r(t)^3.
\]

There is no analytical solution to this differential equation, but the solution can be found numerically.

4.3. Motion with the Lane–Emden Profile

In the case of variable density for the CSM as given by the profile (8), the differential equation which models momentum conservation is
\[
\frac{4 b^3 (r(t))^3 \rho_c \pi \sqrt{3} \frac{d}{dt} r(t)}{(3 b^2 + (r(t))^2)^{3/2}} - \frac{4 b^3 r_0^3 \rho_c \pi \sqrt{3} v_0}{(3 b^2 + r_0^2)^{3/2}} = 0.
\] (20)
The variables can be separated and the solution is
\[ r(t; r_0, v_0, t_0, b) = \frac{N}{D}, \] (21)
where
\[ N = \sqrt{2} r_0^{3/4}(t_0^{13/2} + 2 r_0^{11/2}(t - t_0)v_0
+ r_0^{9/2}(t - t_0)^2 v_0^2 + 6 b^2 r_0^{9/2}
+ 18 b^2 r_0^7(t - t_0)v_0 + \sqrt{A} r_0^4 + \sqrt{A} r_0^3(t - t_0)v_0
+ 9 b_4 r_0^{5/2} + 36 b^3 r_0^{3/2}(t - t_0)
+ 9 \sqrt{A} b^2 r_0^2 + 18 \sqrt{A} b^4 \frac{1}{2}), \]
and
\[ D = 2(3 b^2 + r_0^2)^{3/2}, \]
with
\[ A(t - t_0) = r_0^3(t - t_0)^2 v_0^2 + 36 b^4(t - t_0)v_0
+ 18 b^2 r_0^7(t - t_0)v_0
+ 2 r_0^6(t - t_0)v_0 + 9 b^4 r_0 + 6 b^2 r_0^3 + r_0^5. \]

This is the first solution and has an analytical form. The analytical solution for the velocity can be found from the first derivative of the analytical solution as represented by Equation (21),
\[ v(t; r_0, v_0, t_0, b) = \frac{d}{dt} r(t; r_0, v_0, t_0, b), \] (22)
The previous differential Equation (20) can be organized as
\[ \frac{d}{dt} r(t) = f(r; r_0, v_0, t_0, b), \] (23)
and we seek a power series solution of the form
\[ r(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3 + \cdots; \] (24)
see Tenenbaum & Pollard (1963) and Ince (2012). The Taylor expansion of Equation (23) gives
\[ f(r; r_0, v_0, t_0, b) = b_0 + b_1(t - t_0) + b_2(t - t_0)^2 + b_3(t - t_0)^3 + \cdots, \]
where the values of \( b_n \) are
\[ b_0 = f(r_0; r_0, v_0, t_0, b) \]
\[ b_1 = \frac{\partial}{\partial t} f(r_0; r_0, v_0, t_0, b) \]
\[ b_2 = \frac{1}{2!} \frac{\partial^2}{\partial t^2} f(r_0; r_0, v_0, t_0, b) \] (25)
\[ b_3 = \frac{1}{3!} \frac{\partial^3}{\partial t^3} f(r_0; r_0, v_0, t_0, b) \]
\[ \cdots \]
The relation between the coefficients \( a_n \) and \( b_n \) is
\[ a_1 = b_0 \]
\[ a_2 = \frac{b_1}{2} \]
\[ a_3 = \frac{b_2}{3} \]
\[ \cdots \]
The higher-order derivatives plus the initial conditions give
\[ a_0 = r_0 \]
\[ a_1 = v_0 \]
\[ a_2 = -\frac{9 v_0^2 b^2}{2(3 b^2 + r_0^2)^2} \]
\[ a_3 = \frac{9 v_0^3 b^2 (7 b^2 + r_0^2)}{2 r_0^2 (3 b^2 + r_0^2)^3} \] (26)
These are the coefficients of the second solution, which is a power series.

A third solution can be represented by a difference equation which has the following type of recurrence relation
\[ r_{n+1} = r_n + v_n \Delta t \]
\[ v_{n+1} = r_n^3 v_0 \left( \frac{3 b^2 + r_n^2}{b^3 + r_n^2} \right)^{3/2} \] (27)
where \( r_n, v_n, \) and \( \Delta t \) are the temporary radius, the velocity, and the interval of time.

The physical units have not yet been specified: pc for length and yr for time are the units most commonly used by astronomers. With these units, the initial velocity \( v_0 \) is expressed in pc yr\(^{-1}\), 1 yr = 365.25 days, and should be converted into \( \text{km s}^{-1} \); this means that \( v_0 = 1.02 \times 10^{10} v_1 \) where \( v_1 \) is the initial velocity expressed in \( \text{km s}^{-1} \). In these units, the speed of light is \( c = 0.306 \) pc yr\(^{-1}\). The previous analysis covers the case of symmetrical expansion. In the framework of the spherical coordinates \((r, \theta, \varphi)\) where \( r \) is the distance \((r = 0 \text{ refers to the center of the expansion})\), \( \theta \) is the polar angle with range \([0 – \pi]\), and \( \varphi \) is the azimuthal angle with range \([0 – 2\pi]\). The symmetrical expansion is characterized by the independence of the advancing radius from \( \theta \) and \( \varphi \). We now cover the case of asymmetrical expansion in which the center of the expansion is at \( r = 0 \) in spherical coordinates but does not coincide with the center of the polytrope, which is at the distance \( z_{\text{asym}} \). The density profile of the polytrope, which now has a shifted origin, now depends on the direction of the radius vector. So the supernova’s shell would evolve in non-spherical forms. We align the polar axis with the line \( 0 - z_{\text{asym}} \). The symmetry is now around the \( z \)-axis and the expansion will be independent of the azimuthal angle \( \varphi \). The advancing radius will conversely depend on the polar angle \( \theta \). The case of an expansion that starts from a given distance, \( z_{\text{asym}} \), from the center of the polytrope cannot be modeled by the differential Equation (20), which is derived for a symmetrical expansion. It is not possible to find \( R \) analytically and a numerical method should be implemented. The advancing expansion is computed in a 3D Cartesian coordinate system \((x, y, z)\) with the center of the explosion at \((0, 0, 0)\). The degree of asymmetry is evaluated by introducing \( R_{\text{up}}, R_{\text{up}} \), and \( R_{\text{down}} \) which are the momentary radii in the equatorial plane, polar direction up, and polar direction down. The percentage of asymmetry is defined by the two ratios
\[ a_{\text{up}} = \left| \frac{R_{\text{up}} - R_{\text{eq}}}{R_{\text{eq}}} \right| \times 100, \] (28)
and
\[ a_{\text{down}} = \left| \frac{R_{\text{down}} - R_{\text{eq}}}{R_{\text{eq}}} \right| \times 100. \] (29)
As a reference the measured asymmetry of SN 1993J is under 2%; see Marcaide et al. (2009).
4.4. Motion with a Power-law Profile

The differential equation that models momentum conservation when the density has a power-law behavior, as given by (18), is

\[
\left(-4 \frac{r'(t)^3 \rho_0 \pi}{d - 3} \left(\frac{r_0}{r(t)}\right)^d + \frac{4 \rho_0 \pi r_0^3}{d - 3} + \frac{4/3 \rho_0 \pi r_0^3}{d - 3} \right) \frac{dr}{dt} = \frac{4}{3 \rho_0 \pi r_0^3 v_0}. 
\]  

(30)

A first solution can be found numerically; see Zaninetti (2011) for more details. A second solution is a truncated series about the ordinary point \( t = t_0 \), which to fourth order has coefficients

\[
a_0 = r_0, \quad a_1 = v_0, \quad a_2 = \frac{-3 \sqrt{t_0^3}}{2r_0}, \quad a_3 = \frac{(d + 7)v_0^3}{2r_0^2}. \quad (31)
\]

A third approximate solution can be found assuming that

\[
3 \sqrt{r_0^d - \left(4r_0^d - r_0^3 d^2 \right)r_0^3} \rightarrow 0.
\]

\[
r(t) = \left(r_0^{4-d} - \frac{1}{3} dr_0^{4-d} (4-d) + \frac{1}{3} (4-d) v_0 r_0^{3-d} (3-d) (t-t_0)\right)^{\frac{1}{d-3}}.
\]

This is an important approximate result because, given the astronomical relation \( r(t) \propto t^\eta \), we have \( d = 4 - (1/\alpha) \).

5. CONSERVATION OF THE RELATIVISTIC MOMENTUM

The thin layer approximation assumes that all mass swept during the travel from the initial time, \( t_0 \), to the time \( t \) resides in a thin shell of radius \( r(t) \) with velocity \( v(t) \). Assuming a Lane–Emden dependence \( (n = 5) \), the total mass \( M(r; b) \) between 0 and \( r \) is given by Equation (10). The relativistic conservation of momentum (see French 1968; Zhang 1997; Guéry-Odelin & Lahaye 2010) is formulated as

\[
M(r_0; b)\gamma_0 \beta_0 = M(r; b)\gamma \beta,
\]

where

\[
\gamma_0 = \frac{1}{\sqrt{1 - \beta_0^2}}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}},
\]

and

\[
\beta_0 = \frac{v_0}{c}; \quad \beta = \frac{v}{c}.
\]

The relativistic conservation of momentum is easily solved for \( \beta \) as a function of the radius:

\[
\beta = \frac{\gamma}{\gamma_0} \beta_0 = \sqrt{\frac{A(3b^2 + r^2)(3b^2 + r^2)r_0^3 \beta_0}{A}},
\]

with

\[
A = -27b^6 r^6 \beta_0^2 - 27b^6 r^2 \beta_0^2 r_0^6 + 27b^4 r^6 \beta_0^2 r_0^2
+ 27 b^4 r^6 \beta_0^2 r_0^2 - 9 b^6 r^4 \beta_0^2 r_0^4 + 9 b^2 r^4 \beta_0^2 r_0^4 + 27 b^2 r^6
+ 27 b^4 r^6 r_0^2 + 9 b^2 r^4 r_0^4 + r_0^6 
\]

Inserting

\[
\beta = \frac{1}{c} \frac{d}{dt} r(t),
\]

the relativistic conservation of momentum can be written as the differential equation

\[
\frac{4b^3 (r(t))^3 \rho_0 \pi \sqrt{\frac{5}{r(t)}} r(t)}{(3b^2 + (r(t))^2)^{3/2}c^2 + \frac{(r(t))^3}{c^2} + 1} = \frac{4b^3 r_0^3 \rho_0 \pi \sqrt{3} \beta_0}{(3b^2 + r_0^2)^{3/2} - \beta_0^2 + 1}.
\]

(32)

This first order differential equation can be solved by separating the variables:

\[
\int_{r_0}^{r} \frac{A}{\sqrt{A(3b^2 + r^2)(3b^2 + r^2)r_0^3 \beta_0}} \frac{dr}{r} = c(t - t_0).
\]

(33)

The previous integral does not have an analytical solution and we treat the previous result as a non-linear equation to be solved numerically. The differential equation has a truncated series solution about the ordinary point \( t = t_0 \) which to fifth order is

\[
r_s(t) = \sum_{n=0}^{4} a_n(t - t_0)^n.
\]

(34)

The coefficients are

\[
a_0 = r_0, \quad a_1 = c\beta_0, \quad a_2 = \frac{9b^2 \beta_0^2 c^2 (\beta_0^2 - 1)}{2r_0 (3b^2 + r_0^2)}, \quad a_3 = \frac{9c^3 (\beta_0^2 - 1)(\beta_0^2 + 1)(12b^2 \beta_0^2 - 7b^2 - r_0^2) \beta_0^3 b^2}{2(3b^2 + r_0^2)^3 r_0^3},
\]

\[
a_4 = \frac{9b^2 (\beta_0^2 - 1)(\beta_0^2 + 1)B \beta_0^4 c^4}{8r_0^3 (3b^2 + r_0^2)^5},
\]

where \( B = 756b^4 \beta_0^4 - 927b^4 \beta_0^2 - 117b^2 \beta_0^2 r_0^2 + 231b^4 + 69b^2 r_0^2 + 4r_0^4 \).

The velocity approximated to the fifth order is

\[
v_s(t) = \sum_{n=1}^{4} a_n(t - t_0)^n \frac{n}{(t - t_0)}. \quad (36)
\]

The presence of an analytical expression for \( \beta \) as given by Equation (32) allows the recursive solution

\[
r_{n+1} = r_n + c\beta_n \Delta t
\]

\[
\beta_{n+1} = \sqrt{A(3b^2 + r_{n+1}^2)(3b^2 + r_{n+1}^2)r_0^3 \beta_n} \quad (37)
\]

with \( A = 27b^6 \beta_0^2 r_0^6 - 27b^6 \beta_0^2 r_{n+1}^6 + 27b^4 \beta_0^2 r_{n+1}^2 r_0^2 
- 27 b^6 \beta_0^2 r_{n+1}^2 r_0^6 + 9 b^2 \beta_0^2 r_{n+1}^4 r_0^4
- 9 b^2 \beta_0^2 r_{n+1}^4 r_0^4 + 27 b^4 r_{n+1}^6 + 27 b^2 r_{n+1}^6 r_0^2 + 9 b^2 r_{n+1}^6 r_0^4 + r_{n+1}^6 r_0^6 \)

where \( r_n, \beta_n \), and \( \Delta t \) are the temporary radius, the relativistic \( \beta \) factor, and the interval of time, respectively. Up to now we have taken the time interval \( t - t_0 \) to be that seen by an observer
on Earth. For an observer moving on the expanding shell, the proper time \( \tau^* \) is

\[
\tau^* = \int_{t_0}^t dt = \int_{t_0}^t \sqrt{1 - \beta^2} dt;
\]

see Larmor (1897), Lorentz (1904), Einstein (1905), and Macrossan (1986). In the series solution framework, \( \beta = (v_i/c) \), where \( v_i \) is given by Equation (36). A measure of the time dilation is given by

\[
D = \frac{\tau^*}{t - t_0},
\]

with \( 0 < D < 1 \). It is interesting to point out that the time dilation analyzed here is not connected with the “cosmological time dilation in GRB” which, conversely, is related to the cosmological redshift; see Kocevski & Petrosian (2013) and Zhang et al. (2013) for two diametrically opposed views. An application of the time dilation is in modeling the decay of a radioactive isotope using the following law for remnant particles in the laboratory framework

\[
N(t) = N_0 e^{-t/\tau^*},
\]

where \( \tau \) is the proper lifetime, \( N_0 \) is the number of nuclei at \( t = t_0 \), and the half life is \( T_{1/2} = \ln(2) \tau \). In a frame that is moving with the shell, the decay law is

\[
N(t) = N_0 e^{-t/\tau}.
\]

This theory is used to explain the lifetime of the muons in cosmic rays: in that case, \( \gamma \approx 8.4 \) and \( \tau = 2.196 \times 10^{-6} \) s; see Rossi & Hall (1941); Frisch & Smith (1963); Beringer et al. (2012). In SR, the total energy of a particle is

\[
E = mc^2 = m_0 \gamma c^2,
\]

where \( m_0 \) is the rest mass. The relativistic kinetic energy is

\[
KE = m_0 c^2 (\gamma - 1),
\]

where the rest energy has been subtracted from the total energy. In order to have a simple expression for the velocity as a function of time, we deduce a series expansion, limited to the third order, for the radius as a function of time. The relativistic kinetic energy is therefore

\[
KE = m_0 c^2 \left( \frac{1}{\sqrt{1 - (c\beta_0 + 9 \frac{b\beta_0}{N_0} c_{\text{th}}^2 \gamma^2 (1 - \beta_0^2)^{1/2} c^{-2} - 1)} - 1} \right).
\]

6. CLASSICAL ASTROPHYSICAL APPLICATIONS

This section introduces: the SN chosen for testing purposes, the astrophysical environment connected with the selected SN, two types of fit commonly used to model the radius–time relation in SNe, the application of the results obtained for the Lane–Emden density profile to the selected SN, the asymmetric explosion, and the case of CSM characterized by a power law.

6.1. The Data

The data of SN 1993J, radius in parsecs and elapsed time in years, can be found in Table 1 of Marcaide et al. (2009). The instantaneous velocity of expansion can be deduced from the formula

\[
v_i = \frac{r_{i+1} - r_i}{t_{i+1} - t_i},
\]

where \( r_i \) is the radius and \( t_i \) is the time at the position \( i \). The uncertainty in the instantaneous velocity is found by implementing the error propagation equation; see Bevington & Robinson (2003). A discussion of the thickness of the radio shell in SN 1993J in the framework of a reverse shock Chevalier (1982a, 1982b) can be found in Bartel et al. (2007). The thickness of the radio shell can also be explained in the framework of image theory; see Section 6.3 in Zaninetti (2011).

6.2. Astrophysical Scenario

The progenitor of SN 1993J was a K-supergiant star (see Aldering et al. 1994) and probably formed a binary system with a B-supergiant companion star; see Maund et al. (2004). These massive stars have strong stellar winds and blow huge bubbles (of \( \approx 20 \) to \( 40 \) pc in size) in their lifetimes. From an analytic approximation Weaver et al. (1977) obtained a formula for the radius of the bubble, their Equation (21); see too their Figure 3. The inside of the bubble has very low density, and the border of the bubble is the wall of a relatively dense shell which is in contact with the ISM. The circumstellar envelope of pre-SN 1993J with which the SN shock front is interacting is a small structure within the big bubble created by the strong stellar winds of the SN progenitor (and probably of its binary companion) during its lifetime. Therefore this envelope of pre-SN 1993J would be the product of a recent stellar mass ejection suffered pre-SN 1993J. That is to say, the SN shock wave interacts with a CSM created by pre-supernova mass loss. In this respect, Schmidt et al. (1993) gave evidences that significant mass loss had taken place before the explosion; see also Smith (2008). In the scenario where the pre-SN 1993J formed an interacting binary system, this can be interpreted in terms of mass transfer. It is possible that this type of supernova originates in interacting binary systems.

6.3. Two Types of Fit

The quality of the fits is measured by the merit function \( \chi^2 \)

\[
\chi^2 = \sum_j \frac{(r_{\text{th},j} - r_{\text{obs},j})^2}{\sigma_{\text{obs},j}^2},
\]

where \( r_{\text{th},j} \), \( r_{\text{obs},j} \), and \( \sigma_{\text{obs},j} \) are the theoretical radius, the observed radius, and the observed uncertainty, respectively. A first fit can be done by assuming a power-law dependence of the type

\[
r(t) = r_p t^{\alpha_p},
\]

where the two parameters \( r_p \) and \( \alpha_p \) as well their uncertainties can be found using the recipes suggested in Zaninetti (2011). A second fit can be done by assuming a piecewise function as in Figure 4 of Marcaide et al. (2009)

\[
r(t) = \begin{cases} 
  r_{\text{br}} \left( \frac{t}{t_{\text{br}}} \right)^{\alpha_1} & \text{if } t \leq t_{\text{br}}, \\
  r_{\text{br}} \left( \frac{t}{t_{\text{br}}} \right)^{\alpha_2} & \text{if } t > t_{\text{br}}.
\end{cases}
\]

This type of fit requires determining four parameters: \( t_{\text{br}} \) the break time, \( r_{\text{br}} \) the radius of expansion at \( t = t_{\text{br}} \), and the exponents \( \alpha_1 \) and \( \alpha_2 \) of the two phases. The parameters of these two fits as well the \( \chi^2 \) can be found in Table 1.
The astronomical data of SN 1993J are represented with vertical error bars.

Figure 3. Theoretical radius as given by Equation (21) (full line) and series solution as given by Equation (24) (dashed line). Data as in Table 1.

Table 1

| N  | Values                                                                 | \( \chi^2 \) |
|----|------------------------------------------------------------------------|-------------|
| 2  | Power law as a fit
  \( \alpha_p = 0.82 \pm 0.0048 \)
  \( r_p = (0.015 \pm 0.00011) \) pc | 6364        |
| 4  | Piecewise fit
  \( \alpha_1 = 0.83 \pm 0.01 \)
  \( \alpha_2 = 0.78 \pm 0.0077 \)
  \( r_{01} = 0.05 \) pc; \( r_{02} = 4.10 \) yr | 32          |
| 2  | Plummer profile, \( \eta = 6 \)
  \( b = 0.0045 \) pc; \( r_0 = 0.008 \) pc; \( v_0 = 19,500 \) \( \) km s\(^{-1} \) | 265         |
| 2  | Lane–Emden profile
  \( b = 0.00367 \) pc; \( r_0 = 0.008 \) pc; \( v_0 = 19,500 \) \( \) km s\(^{-1} \) | 471         |
| 2  | Power-law profile
  \( d = 2.93 \)
  \( r_0 = 0.0022 \) pc;
  \( t_0 = 0.249 \) yr; \( v_0 = 100,000 \) \( \) km s\(^{-1} \) | 276         |

6.4. The Lane–Emden Case

The numerical solution of the differential equation connected with the Plummer-like profile, \( \eta = 6 \), is reported in Figure 8 when the data of Table 1 are adopted.

A comparison with the power-law behavior for the CSM is reported in Figure 9 which is built from the data in Table 1. The series solution for the power-law dependence of the CSM with coefficients as given by Equation (31) is not reported because the range in time of reliability is limited to \( t - t_0 \approx 0.0003 \) yr.
7. RELATIVISTIC ASTROPHYSICAL APPLICATIONS

We now apply the relativistic solutions derived so far to SN 1993J. The initial observed velocity, $v_0$, as deduced from radio observations (see Marcaide et al. 2009), is $v_0 \approx 20,000 \text{ km s}^{-1}$ at $t_0 \approx 0.5 \text{ yr}$. We now reduce the initial time $t_0$ and we increase the velocity up to the relativistic regime, $t_0 = 10^{-4} \text{ yr}$ and $v_0 = 100,000 \text{ km s}^{-1}$. This choice of parameters allows the observed radius–time relation that should be reproduced to be fitted. The data used in the simulation are shown in Table 2. The relativistic numerical solution of Equation (33) is reported in Figure 10, the relativistic series solution as given by Equation (34) is reported in Figure 11, and the

| Table 2 |
|---|
| Numerical Values of the Parameters Used in Three Relativistic Solutions |
| Parameters |
| $n_0 = 10^{-4} \text{ yr}$; $r_0 = 0.0033 \text{ pc}$; $\beta_0 = 0.3333$; $b = 0.004 \text{ pc}$ |
Figure 11. Theoretical relativistic radius as a solution of the nonlinear Equation (33) (full line), and series solution as given by Equation (34) (dashed line). Data as in Table 2. The time is expressed in days.

Figure 12. Theoretical relativistic radius as solution of the nonlinear Equation (33) (full line), and recursive solution as given by Equation (38) when $\Delta t = 0.053 \text{ yr}$ (dashed line). Data as in Table 2.

Figure 13. Map of time dilation as represented by $D$ as a function of time (in days) and $\beta_0$. Data as in Table 2.

(A color version of this figure is available in the online journal.)

recursive solution as given by Equation (38) is contained in Figure 12.

Figure 14 reports a 2D map of the parameter $D$ which parameterizes the time dilation.

Figure 14. Number of nuclei of $^{56}\text{Ni}$ in the inertial frame of the laboratory (full line) and in the frame that is moving with the SN (dashed line). Data as in Table 2.

Figure 15. Relativistic kinetic energy of a proton in Mev as a function of time (in days). Data as in Table 2.

8. CONCLUSIONS

Classic case. The thin layer approximation, which models the expansion in a self-gravitating medium of the Lane–Emden type ($n = 5$), can be modeled by a differential equation of the first order for the radius as a function of time. This differential equation has an analytical solution represented by Equation (21). A power-law series (see Equation (24)) can model the solution of Lane–Emden type for a limited range of time; see Figure 3. Conversely, a recursive solution for the first order differential equation, as represented by Equation (27), approximates quite well the analytical solution of the Lane–Emden type and at the time $t = 10 \text{ yr}$ a precision of four digits is reached when $\Delta t = 10^{-3} \text{ yr}$. The goodness of the results as given by the solutions of the three differential equations is evaluated in Table 1. The smallest $\chi^2$ is obtained by the Plummer-like ($\eta = 6$) profile, followed by the power-law profile and the Lane–Emden type ($n = 5$) profile. The two-piece fit has the smallest $\chi^2$ but requires four parameters and does not have a physical basis. The disadvantages of the power-law dependence in the CSM are the following. (1) There is a two-piece dependence at $r = r_0$ which was introduced in order to avoid a pole, (2) there is no analytical solution, and (3) the series solution has a narrow range of reliability.

Self-gravitating medium. The previous analysis raises a question: is the CSM around an SN really self-gravitating? In order to answer this question, the CSM should be carefully analyzed by astronomers in order to detect the presence
of gradients in density or pressure or temperature. In the case of a Lane–Emden \((n = 5)\) CSM, the density of the medium around SN 1993J, see Equation (8), decreases by a factor \(\approx 21\) going from \(r_0 = 0.008\) pc to \(r = 0.1\) pc. Over the same distance, the pressure (see Equation (6)) decreases by a factor \(\approx 40\) and the temperature (see Equation (7)) by a factor \(\approx 1.8\). The presence or absence of a magnetic field should be also confirmed.

**Nature of the CSM.** The total mass obtained in the three models can be interpreted as the stellar mass ejected by the pre-SN right before the explosion or the stellar mass involved in the interaction of the binaries; see Table 3. The size of the pre-SN 1993J envelope, i.e., the FWHM, is an important parameter. Note. The previous table allows a fast evaluation of the CSM, the density of the medium around SN 1993J, see Equation (8), decreases by a factor \(\approx 21\) going from \(r_0 = 0.008\) pc to \(r = 0.1\) pc. Over the same distance, the pressure (see Equation (6)) decreases by a factor \(\approx 40\) and the temperature (see Equation (7)) by a factor \(\approx 1.8\). The presence or absence of a magnetic field should be also confirmed.

**REFERENCES**

Abramowitz, M., & Stegun, I. A. 1965, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (New York: Dover)

Aldering, G., Humphreys, R. M., & Richmond, M. 1994, AJ, 107, 662

Aoi, J., Murase, K., Takahashi, K., Ioka, K., & Nagataki, S. 2010, ApJ, 722, 440

Baring, M. G. 2011, AdSpR, 47, 1427

Bartel, N., Bietenholz, M. F., Rufen, M. P., & Dwarkadas, V. V. 2007, ApJ, 686, 924

Beringer, J., Arquín, J.-F., Barnett, R. M., & Copic, K. 2012, PHVd, 86, 010001

Bevington, P. R., & Robinson, D. K. 2003, Data Reduction and Error Analysis for the Physical Sciences (New York: McGraw-Hill)

Binney, J., & Tremaine, S. 2011, Galactica Dynamics, Second Edition (Princeton, NJ: Princeton University Press)

Blandford, R. D., & McKee, C. F. 1976, PhFl, 19, 1130

Bodansky, D., Clayton, D. D., & Fowler, W. A. 1968, ApJS, 16, 299

Chandrasekhar, S. 1967, An Introduction to the Study of Stellar Structure (New York: Dover Publications)

Chevalier, R. A. 1982a, ApJ, 258, 790

Chevalier, R. A. 1982b, ApJ, 259, 302

Childress, M. J., Filippenko, A. V., Ganeshalingam, M., & Schmidt, B. P. 2011, MNRAS, 437, 338

Dyson, J. E., & Williams, D. A. 1997, The Physics of the Interstellar Medium (Bristol: Institute of Physics Publishing)

Einstein, A. 1905, AnP, 322, 891

Emden, R. 1907, Gaskugeln: anwendungen der mechanischen warmetheorie (Braunschweig: Vieweg)

Fisk, R. H., & Smith, J. H. 1963, AmPPh, 31, 342

Granot, J., & Kumar, P. 2005, MNRAS, 366, L13

Guéry-Odelin, D., & Lahaye, T. 2010, Classical Mechanics Illustrated by Modern Physics: 42 Problems with Solutions (London: Imperial College Press)

Hansen, C. J., & Kawaler, S. D. 1994, Stellar Interiors. Physical Principles, Structure, and Evolution (Berlin: Springer)

Ince, E. L. 2012, Ordinary Differential Equations (New York: Dover)

Inoue, T., Asano, K., & Ioka, K. 2011, ApJ, 734, 77

Kashiwaya, K., Murase, K., Horiiuchi, S., Gao, S., & Mészaros, P. 2013, ApJL, 769, L6

Kocevski, D., & Petrosian, V. 2013, ApJ, 765, 116

Kompayne, J. A. 1960, SPDS, 5, 46

Krisciunas, K., Li, W., Matheson, T., et al. 2011, AJ, 142, 74

Lane, H. J. 1870, AmJSS, 148, 57

Larmor, J. 1897, Philos. Trans. R. Soc. Lond., Ser. A, Contain, Pap. Math. Phys. Character, 190, 205

Lorentz, H. A. 1904, Proc. Section of Sciences, Vol. 6 (Amsterdam: Academy of the Sciences of Amsterdam), 809

Lyu, K. M. 2012, MNRAS, 421, 522

Macrossan, M. N. 1986, Br. J. Phil. Sci., 37, 232

Magni, G., Timmes, F. X., Hungerford, A. L., et al. 2010, ApJS, 191, 66

Marcade, J. M., Martí-Vidal, I., Alberdi, A., & Pérez-Torres, M. A. 2009, A&A, 505, 927

Marion, G. H., Vinko, J., & Wheeler, J. C. 2013, ApJ, 777, 40

Matz, S. M., & Share, G. H. 1990, ApJ, 362, 235

Maund, J. R., Smartt, S. J., Kudritzki, R. P., Podsadiowski, P., & Gilmore, G. F. 2004, Natur, 427, 129

Mazzali, P. A., Benetti, S., Altavilla, G., Blanc, G., & Cappellaro, E. 2005, ApJL, 623, L37

Mazzali, P. A., Chugai, N., Turatto, M., et al. 1997, MNRAS, 284, 151

McCray, R. A. 1987, in Spectroscopy of Astrophysical Plasmas, ed. A. Dalgarno & D. Layzer (Cambridge: Cambridge Univ. Press), 255

**Table 3**

| Model                | Total Mass | FWHM   |
|----------------------|------------|--------|
| Plummer profile, \(\eta = 6\) | \(M(0.1; 0.0045) = 0.402 n_7 M_{\odot}\) | 0.0079 pc |
| Emden profile        | \(M(0.008; 0.00367) = 0.178 n_7 M_{\odot}\) | 0.0071 pc |
| Emden profile        | \(M(0.1; 0.00367) = 0.368 n_7 M_{\odot}\) | 0.0071 pc |
| Power-law profile    | \(M(0.1; 0.0022, 2.93) = 0.217 n_7 M_{\odot}\) | 0.0092 pc |

Note. The parameter \(n_7\) represents the number density expressed in \(10^7\) cm\(^{-3}\) units.

(see Equation (38)) in which the desired accuracy is reached by decreasing the time step \(\Delta t\). The relativistic results here presented model SN 1993J and are obtained with an initial velocity of \(v_0 = 100,000\) km s\(^{-1}\), \(\beta_0 = 0.333\), or \(\gamma = 1.06\). The time dilation is evaluated and then applied to the decay of \(^{56}\)Ni; see Figure 13. The relativistic kinetic energy of a proton is computed and the temporal evolution in MeV outlined; see Figure 15.
Mohr, P. J., Taylor, B. N., & Newell, D. B. 2012, Rev. Mod. Phys., 84, 1527
Muccino, M., Ruffini, R., Bianco, C. L., & Izzo, L. 2013, ApJ, 772, 62
Nakar, E., & Sari, R. 2012, ApJ, 747, 88
Okita, S., Umeda, H., & Yoshida, T. 2012, in AIP Conf. Proc. 1484, Mass and Spatial Distribution of 56Ni Induced by Aspherical Explosion of Massive CO Star, ed. S. Kubono, T. Hayakawa, T. Kajino, H. Miyatake, T. Motobayashi, & K. Nomoto (Melville, NY: AIP), 418
Olano, C. A. 2009, A&A, 506, 1215
Olver, F. W. J., Lozier, D. W., Boisvert, R. F., & Clark, C. W. 2010, NIST Handbook of Mathematical Functions (Cambridge: Cambridge Univ. Press)
Padmanabhan, P. 2001, Theoretical Astrophysics, Vol. II: Stars and Stellar Systems (Cambridge: Cambridge University Press)
Pe’er, A., Ryde, F., Wijers, R. A. M. J., & Mészáros, P. 2007, ApJL, 664, L1
Plummer, H. C. 1911, MNRAS, 71, 460
Rossi, B., & Hall, D. B. 1941, PhyR, 59, 223
Schmidt, B. P., Kirshner, R. P., Eastman, R. G., et al. 1993, Natur, 364, 600
Sedov, L. I. 1959, Similarity and Dimensional Methods in Mechanics (New York: Academic Press)
Shapiro, P. R. 1979, ApJ, 233, 831
Smith, N. 2008, Rev. Mex. Astron. Astrofís. Ser. Conf., 33, 154
Spitzer, L. 1978, Physical Processes in the Interstellar Medium (New York: Wiley)
Stritzinger, M., Leibundgut, B., Walch, S., & Contardo, G. 2006, A&A, 450, 241
Suzuki, T., & Nomoto, K. 1995, ApJ, 455, 658
Tenenbaum, M., & Pollard, H. 1963, Ordinary Differential Equations: An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences (New York: Dover)
Truran, J. W., Arnett, W. D., & Cameron, A. G. W. 1967, CaJPh, 45, 2315
Truran, J. W., Glasner, A. S., & Kim, Y. 2012, JPhCS, 337, 012040
Wang, L., Baade, D., Hoflich, P., Khokhlov, A., & Wheeler, J. C. 2003, ApJ, 591, 1110
Weaver, R., McCray, R., Castor, J., Shapiro, P., & Moore, R. 1977, ApJ, 218, 377
Whitworth, A. P., & Ward-Thompson, D. 2001, ApJ, 547, 317
Woosley, S. E., Eastman, R. G., Weaver, T. A., & Pinto, P. A. 1994, ApJ, 429, 300
Zaninetti, L. 2011, Ap&SS, 333, 99
Zaninetti, L. 2012, CEJPh, 10, 32
Zhang, F.-W., Fan, Y.-Z., Shao, L., & Wei, D.-M. 2013, ApJL, 778, L11
Zhang, Y. 1997, Special Relativity and Its Experimental Foundations (Singapore: World Scientific)
Zou, Y.-C., & Piran, T. 2010, MNRAS, 402, 1854
Zwillinger, D. 1989, Handbook of Differential Equations (New York: Academic Press)