The $D_s^+ \to \pi^+ K_s^0 K_s^0$ reaction

and the $I = 1$ partner of the $f_0(1710)$ state

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Abstract

We have identified the decay modes of the $D_s^+ \to \pi^+ K^+ K^-, \pi^+ K_s^0 K_s^0$ reactions producing two vector mesons and one pseudoscalar. The posterior vector-vector interaction generates two resonances that we associate to the $f_0(1710)$ and the $a_0(1710)$ recently claimed. We find two acceptable scenarios that give results for the ratio of the branching ratios of these two reactions in agreement with experiment. With these two scenarios we make predictions for the branching ratios of the $D_s^+ \to \pi^0 K_s^0$ reaction, finding values within the range of $(2.0 \pm 0.7) \times 10^{-3}$. Comparison of these predictions with coming experimental results on that latter reaction will be very most useful to deepen our understanding on the nature of these two resonances.

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I. INTRODUCTION

The $f_0(1710)$ is a well established meson in the PDG [1]. In the relativized quark model of Godfrey and Isgur [2] it appears as an $I = 0, J^{PC} = 0^{++}$ state at 1780 MeV with the $2\,^3\!P_1$ configuration. In the same work a state with the same mass and configuration appears for $I = 1$. Similar results are also reported in [3]. The $f_0(1710)$ is also obtained in [4] with the same configuration, but no mention is made of the possible $I = 1$ partner. These models consider the excitation of $u, d$ quarks. However, the fact that the $f_0(1710)$ decays mostly in $K\bar{K}, \eta\eta$ with only about 4% branching ratio to $\pi\pi$ decay [1] indicates that this state should have large components of $s\bar{s}$ quarks.

A different picture for the $f_0(1710)$ comes from the work of [5], where the interaction of vector mesons studied in [6] for the $\rho\rho$ case is extended to the SU(3) space. The $f_0(1710)$ is found around 1726 MeV and couples mostly to $K^*\bar{K}^*$, but also to $\omega\phi, \phi\phi, \omega\omega, \rho\rho$ in that order. The vector-vector interaction is taken from the local hidden gauge approach [7–10], which stems from a contact term plus vector exchange. Considering box diagrams with intermediate two pseudoscalar mesons, decay rates to $K\bar{K}, \eta\eta, \pi\pi$ were evaluated in [5] and found consistent with experimental data. Interestingly, in [5] a partner state of the $f_0(1710)$ with $I = 1, J^{PC} = 0^{++}$ a$_0$ state is also found at 1780 MeV with $\Gamma \sim 130$ MeV. This state couples mostly to $K^*\bar{K}^*$ but also to $\rho\omega$ and $\rho\phi$. The picture of [5] for the $f_0(1710)$ has been tested in different processes. In [11] the $\gamma\gamma$ decay rate is evaluated and found consistent with the PDG information [1]. In [12] it is also suggested that the peak observed at the $\phi\omega$ threshold in the $\phi\omega$ mass distribution of the $J/\psi \rightarrow \gamma\phi\omega$ decay [13] is due to the $f_0(1710)$ resonance. Predictions for other decay modes, and rates for $f_0(1710)$ production decays of other particles are done in [14–19].

The success of the predictions for other vector-vector molecules obtained in [5] discussed in the former references gives us confidence in that model and concretely about the existence of the $I = 1$ partner of the $f_0(1710)$ state, which we will call the $a_0(1710)$ by analogy to the $f_0(1710)$. Yet, this state is not reported in the PDG [1]. The situation has changed suddenly with the appearance of two works showing evidence for this state. One of these works is the clear observation of a peak around 1710 MeV in the $\pi^+\eta$ mass distribution in the $\eta_c \rightarrow \eta\pi^+\pi^-$ decay [20]. The other work is the study of the $D_s^+ \rightarrow \pi^+ K^0_s K^0_s$ decay [21] showing a peak around 1710 MeV in the $K^0_s K^0_s$ mass distribution with an abnormally large
strength compared to the one of a similar peak seen in the $D^+_s \to \pi^+K^+K^-$ decay [22]. This cannot be explained from a $KK$ state in $I = 0$, implying that there must also be an $I = 1$ state with a similar mass. The $I = 0$ and $I = 1$ states have relative opposite sign in the $K^+K^-$ or $K^0\bar{K}^0$ components and the contribution of the two states around 1710 MeV will give differences in the $K^+K^-$ or $K^0\bar{K}^0$ production rates.

Our aim in the present work is to show that from the perspective of Ref. [5] for the $f_0(1710)$ and $a_0(1710)$ it is natural to reproduce the experimental data on $KK$ production in $D^+_s \to \pi^+K\bar{K}$ decay. At the same time we can make predictions for the rate of the $I = 1 a_0(1710)$ production in the $K^+K^0_s$ invariant mass distribution of the $D^+_s \to \pi^0K^+K^0_s$ reaction, which, as mentioned in [21] is in the process of being analyzed at BESIII.

II. FORMALISM

We look at the mechanism for $D^+_s \to \pi^+KK$ production at the quark level starting from the dominant external emission process and then considering the internal emission, both in the Cabibbo-favored mode [23]. The external emission with $\pi^+$ production is shown in Fig. 1.

Since we wish to have three mesons in the final state we must hadronize a pair of quarks introducing an extra $\bar{q}q$ with the vacuum quantum numbers ($\bar{q}q = \bar{u}u + \bar{d}d + \bar{s}s$). Also, the hadronization must produce a pair of vector mesons, such that their interaction can produce
the $f_0(1710)$ and $a_0(1710)$ resonances. Then, hadronizing $s\bar{s}$ we will have

$$s\bar{s} \to \sum_i s \bar{q}_i q_i \bar{s} = V_{3i} V_{i3} = (V^2)_{33}, \quad (1)$$

where $V$ is the $q_i\bar{q}_j$ matrix written in terms of the vector meson

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^0 & \phi \end{pmatrix} \quad (2)$$

Then we have the hadronic state

$$H_1 = (V^2)_{33} \pi^+ = (K^{*-} K^{*+} + \bar{K}^0 K^{*0} + \phi\phi) \pi^+. \quad (3)$$

With the implicit isospin phase convention of this matrix, with the $(K^{*+}, K^{*0})$, $(\bar{K}^{*0}, -K^{*-})$ isospin doublets, the former combination represents $K^*\bar{K}^*$ in isospin $I = 0$, as it should be, plus $\phi\phi$, also $I = 0$, since originally we had the $I = 0$ $s\bar{s}$ state.

The two isospin states of $K^*\bar{K}^*$ are given by

$$|K^*\bar{K}^*, I = 0\rangle = -\frac{1}{\sqrt{2}} (K^{*+} K^{*-} + K^{*0} \bar{K}^0)$$

$$|K^*\bar{K}^*, I = 1, I_3 = 0\rangle = -\frac{1}{\sqrt{2}} (K^{*+} K^{*-} - K^{*0} \bar{K}^0) \quad (4)$$

We could also think of hadronizing the $u\bar{d}$ pair with $VV$ but then $s\bar{s}$ should be a pseudoscalar, in this case a combination of $\eta$ and $\eta'$, and we do not get the $\pi^+K\bar{K}$ mode. Note that if we wish to have $\pi^+K\bar{K}$, we could also have the $s\bar{s}$ as the $\phi$ meson and then $\phi \to K\bar{K}$, but the invariant mass of $K\bar{K}$ will peak at the $\phi$ mass and we are only concerned about the vicinity of 1710 MeV, where the $f_0(1710)$ and $a_0(1710)$ resonances appear. This decay mode and the $K^+K^-\pi^+$ spectrum at low $K^+K^-$ invariant mass has been studied in detail in [24]. We shall concentrate here only in the region of $f_0(1710)$ and $a_0(1710)$ production.

We have then another possibility which is to hadronize the $u\bar{d}$ component with $VP$ or $PV$ ($P$ for pseudoscalar meson). Similarly to Eq. (2) we have the $q_i\bar{q}_j$ matrix for pseudoscalars

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} \end{pmatrix} \quad (5)$$
where we have used the $\eta$ and $\eta'$ mixing of Ref. [25] and neglected the $\eta'$ which does not play any role here.

In this case we obtain the contribution

$$ud \to \sum_i u \bar{q}_i q_i d = M_{i1} M'_{i2} = (MM')_{12}$$ (6)

but now $M, M'$ can be vector or pseudoscalar. Hence, we obtain the combinations

$$(VP)_{12} = \left( \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) \pi^+ + \rho^+ \left( -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) + K^{*+} K^0$$ (7)

and the $s\bar{s}$ pair will provide the $\phi$ meson.

We aim at getting $\pi^0 f_0(1710)$ and $\pi^0 a_0(1710)$ which have $G$-parity negative and positive respectively. Neither $VP$ or $PV$ of Eqs. (7), (8) have good $G$-parity but the combinations $VP \pm PV$ have. Thus, we construct

$$H_2 = \phi [(VP)_{12} + (PV)_{12}] = \left[ 2 \frac{\omega}{\sqrt{2}} \pi^+ + \frac{2}{\sqrt{3}} \rho^+ \eta + K^{*+} K^0 + K^+ \bar{K}^0 \right] \phi$$ (9)

$$H_3 = \phi [(VP)_{12} - (PV)_{12}] = \left[ 2 \frac{\rho^0}{\sqrt{2}} \pi^+ - \frac{2}{\sqrt{3}} \rho^+ \pi^0 + K^{*+} \bar{K}^0 - K^+ K^0 \right] \phi$$ (10)

The combination of Eq. (9) has $G$-parity positive while the case of Eq. (10) has $G$-parity negative\(^1\). With the latter combination we are able to reach the $\pi^0 f_0(1710)$ state, while from Eq. (9) we can produce $\pi^0 a_0(1710)$.

So far we have relied upon external emission. Suppressed by a color factor $\frac{1}{N_c}$ we have internal emission, which is depicted in Fig. 2. We could hadronize the $s \bar{d}$ or $u \bar{s}$ components with $VV$ but then we neither get a pion nor the $VV$ combination to give the nonstrange $f_0(1710)$ or $a_0(1710)$. We must hadronize with $VP$ and $PV$ combinations and we get

$$(VP)_{32} = K^{*-} \pi^+ + \bar{K}^{*0} \left( \frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) + \phi \bar{K}^0$$

$$(PV)_{32} = K^- \rho^+ + \bar{K}^0 \left( \frac{-\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) - \frac{\eta}{\sqrt{3}} \bar{K}^{*0}$$

$$(VP)_{13} = \left( \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} \right) K^+ + \rho^0 K^0 - K^{*+} \frac{\eta}{\sqrt{3}}$$

$$(PV)_{13} = \left( \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) K^{*+} + \pi^0 K^{*0} + K^+ \phi$$ (11)

\(^1\) To facilitate testing the $G$-parity of the $K, K^*$ states we note that we have $G K^+ = \bar{K}^0$, $G K^0 = -K^-$, $G \bar{K}^0 = -\bar{K}^+$, $G K^- = K^0$, and the same with a global minus sign for $K^*$, since we have the convention $C \rho^0 = -\rho^0$, for the charge conjugation of vector mesons.
FIG. 2. Internal emission. (a) with hadronization of the $s\bar{d}$ pair; (b) with hadronization of the $u\bar{s}$ pair.

We must form the good $G$-parity combination from the former terms and we find

$$H_4 = K^{*+}(VP)_{32} + \bar{K}^{*0}(PV)_{13}$$
$$= \pi^+(K^{*+}K^{*-} + K^{*0}K^{*0}) + \frac{2}{\sqrt{3}}\eta K^{*+}\bar{K}^{*0} + \phi(K^{*+}\bar{K}^{*0} + \bar{K}^{*0}K^{*+})$$ (12)

$$H_5 = K^{*+}(VP)_{32} - \bar{K}^{*0}(PV)_{13}$$
$$= \pi^+(K^{*+}K^{*-} - \bar{K}^{*0}K^{*0}) - \sqrt{2}\pi^0 K^{*+}\bar{K}^{*0} + \phi(K^{*+}\bar{K}^{*0} - \bar{K}^{*0}K^{*+})$$ (13)

Note that $(PV)_{32}$ and $(VP)_{31}$ combinations have no pions and do not lead to our desired final state. We see again that $H_4$ has $G$-parity negative and can lead to $\pi^+ f_0(1710)$, while $H_5$ has $G$-parity positive and can lead to $\pi^+ a_0(1710)$. The different mechanisms have different weights and we shall give weights:

$$H_1 : A \quad H_2 : A\alpha \quad H_3 : A\beta \quad H_4 : A\gamma \quad H_5 : A\delta .$$

We will evaluate ratios of $\pi^+ K^0_s K^0_s$ and $\pi^+ K^+ K^-$ production and the global factor $A$ disappears. Then we have 4 parameters to adjust an experimental ratio, which seems an excessive freedom, but we are very limited since $|\alpha| \sim 1, |\beta| \sim 1, |\gamma| \sim \frac{1}{3}, |\delta| \sim \frac{1}{3}$ and then the freedom is drastically reduced.

The hadronic states $H_i$ ($i = 1, 2, 3, 4, 5$) do not have $K\bar{K}$ in the final state. We must produce the $f_0(1710)$ and $a_0(1710)$ and then let them decay into $K\bar{K}$ . The mechanisms for $K\bar{K}$ decay are explained in [5], but since we only care about ratios, all that is needed are the Clebsch-Gordan coefficients, and considering the wave functions $|K\bar{K}, I = 0\rangle = -\frac{1}{\sqrt{2}}(K^+ K^- + K^0 K^0)$, $|K\bar{K}, I = 1, I_3 = 0\rangle = -\frac{1}{\sqrt{2}}(K^+ K^- - K^0 K^0)$, $|K\bar{K}, I = 1, I_3 = 1\rangle = \cdots$
we will have the weights

\[
f_0(1710) \rightarrow \begin{cases} 
K^+ K^- = -\frac{1}{\sqrt{2}}g_{K\bar{K}} \\
K^0 \bar{K}^0 = -\frac{1}{\sqrt{2}}g_{K\bar{K}} 
\end{cases}
\]

\[
a_0(1710) \rightarrow \begin{cases} 
K^+ K^- = -\frac{1}{\sqrt{2}}g_{K\bar{K}}, \ I_3 = 0 \\
K^0 \bar{K}^0 = \frac{1}{\sqrt{2}}g_{K\bar{K}}, \ I_3 = 0 
\end{cases}
\]

\[
a_0(1710) \rightarrow K^+ \bar{K}^0 = g_{K\bar{K}}, \ I_3 = 1 
\]

and we also need the couplings of the resonances to the different vector-vector channels, which we take from [5] and show in Table I.

**TABLE I. Couplings of $f_0(1710)$ and $a_0(1710)$ to $VV$ channels. All quantities are in units of MeV.**

| $g[f_0(1710)]$ | $K^*\bar{K}^*$ | $\rho\rho$ | $\omega\omega$ | $\omega\phi$ | $\phi\phi$ |
|----------------|----------------|-----------|----------------|--------------|------------|
| (7124, $i\cdot96$) | $(-1030, i\cdot1086)$ | $(-1763, i\cdot108)$ | $(3010, -i\cdot210)$ | $(-2493, -i\cdot204)$ |

| $g[a_0(1710)]$ | $K^*\bar{K}^*$ | $\rho\rho$ | $\rho\omega$ | $\rho\phi$ |
|----------------|----------------|-----------|--------------|------------|
| (7525, $-i\cdot1529$) | 0 | $(-4042, i\cdot1391)$ | $(4998, -i\cdot1872)$ |

We can see how the different $H_i$ terms contribute to $\pi^+ f_0(1710), \pi^+ a_0(1710)(I_3 = 0)$ and $\pi^+ a_0(1710)(I_3 = 1)$.

\[
H_1 : \pi^+ f_0(1710) \text{ with } \pi^+ K^*\bar{K}^* \text{ and } \phi\phi \text{ terms.}
\]

\[
H_2 : \pi^+ f_0(1710) \text{ with } \omega\phi\pi^+ \text{ term.}
\]

\[
H_3 : \pi^+ a_0(1710) \ (I_3 = 0) \text{ with } \pi^+ \rho^0\phi \text{ term;}
\]

\[
\pi^+ a_0(1710) \ (I_3 = 1) \text{ with } \pi^0\rho^+\phi \text{ term.} \quad (15)
\]

\[
H_4 : \pi^+ f_0(1710) \text{ with } \pi^+ K^*\bar{K}^* \text{ term.}
\]

\[
H_5 : \pi^+ a_0(1710) \ (I_3 = 0) \text{ with } \pi^+ K^*\bar{K}^* \text{ term;}
\]

\[
\pi^+ a_0(1710) \ (I_3 = 1) \text{ with } \pi^0 K^*\bar{K}^* \text{ term.}
\]

The mechanism for $f_0(1710)$ and $a_0(1710)$ production and $K\bar{K}$ final state are depicted in Fig. 3.
FIG. 3. Mechanisms for $D_s^+ \rightarrow \pi^+ K^+ K^-(K^0 \bar{K}^0)$ and $D_s^+ \rightarrow \pi^0 K^+ K^0$. For $\pi^+ f_0(1710)$ production $V_i V'_i \equiv K^* \bar{K}^*, \omega \phi, \phi \phi$; for $\pi^+ a_0(1710) (I_3 = 0)$ production $V_i V'_i \equiv K^* \bar{K}^*, \rho^0 \phi$; for $\pi^+ a_0(1710) (I_3 = 1)$ production $V_i V'_i \equiv K^* \bar{K}^*, \rho^+ \phi$.

All this said, and with the weights of the different mechanisms we can write

$$\tilde{t}_f = A\left\{ -\sqrt{2} (1 + \gamma) G_{K^+ K^-} (M_{\text{inv}}) g_{f_0, K^+ \bar{K}^+} + 2 \times \frac{1}{2} G_{\phi \phi} (M_{\text{inv}}) \sqrt{2} g_{f_0, \phi \phi} \right\}$$

$$\tilde{t}_{a_0} (I_3 = 0) = A\left\{ \sqrt{2} \beta G_{\rho \phi} (M_{\text{inv}}) g_{a_0, \rho \phi} - \sqrt{2} \delta G_{K^* \bar{K}^*} (M_{\text{inv}}) g_{a_0, K^* \bar{K}^*} \right\}$$

$$\tilde{t}_{a_0} (I_3 = 1) = A\left\{ \sqrt{2} \beta G_{\rho \phi} (M_{\text{inv}}) g_{a_0, \rho \phi} - \sqrt{2} \delta G_{K^* \bar{K}^*} (M_{\text{inv}}) g_{a_0, K^* \bar{K}^*} \right\}$$

where we have taken into account the $K^* \bar{K}^*$ wave functions of Eqs. (4), and, considering the isospin multiplet $(-\rho^+, \rho^0, \rho^-)$, the $\rho \phi$ wave functions

$$|\rho \phi; I = 1, I_3 = 0\rangle = \rho^0 \phi$$

$$|\rho \phi; I = 1, I_3 = 1\rangle = -\rho^+ \phi .$$

The $G$ functions in Eqs. (16), (17), (18) are the loop functions for pairs of vector mesons, which are calculated using a cutoff method with $q_{\text{max}} = 960$ MeV, giving similar results as those found in [5], where dimensional regularization was used.

Since in all mechanisms we started from $s \bar{d}$, $u \bar{s}$, this state is $I = 1, I_3 = 1$. Considering the phase of $\pi^+$, the components $\pi^+ a_0$ ($I_3 = 0$) and $\pi^0 a_0$ ($I_3 = 1$) have the same weights in $I = 1, I_3 = 1$, which means that Eqs. (17), (18) should be the same, which is indeed the case.

Considering the weights of Eqs. (14) of the resonances to $K \bar{K}$, we can then write:
\[
\begin{align*}
 t_{K^+K^-} &= -\tilde{t}_{f_0} \frac{1}{M_{\text{inv}}^2 - M_{f_0}^2 + iM_{f_0}\Gamma_{f_0}} \frac{1}{\sqrt{2}} g_{K\bar{K}} - \tilde{t}_{a_0} \frac{1}{M_{\text{inv}}^2 - M_{a_0}^2 + iM_{a_0}\Gamma_{a_0}} \frac{1}{\sqrt{2}} g_{K\bar{K}} \\
 t_{K^0\bar{K}^0} &= -\tilde{t}_{f_0} \frac{1}{M_{\text{inv}}^2 - M_{f_0}^2 + iM_{f_0}\Gamma_{f_0}} \frac{1}{\sqrt{2}} g_{K\bar{K}} + \tilde{t}_{a_0} \frac{1}{M_{\text{inv}}^2 - M_{a_0}^2 + iM_{a_0}\Gamma_{a_0}} \frac{1}{\sqrt{2}} g_{K\bar{K}} \\
 t_{K^+\bar{K}^0} &= \tilde{t}_{a_0} \frac{1}{M_{\text{inv}}^2 - M_{a_0}^2 + iM_{a_0}\Gamma_{a_0}} g_{K\bar{K}} \\
 t_{K^+K^0} &= -\frac{1}{\sqrt{2}} t_{K^+\bar{K}^0}
\end{align*}
\]  

(20)

where in the last equation we have taken into account that \( K^0_s = \frac{1}{\sqrt{2}} (K^0 - \bar{K}^0) \).

We can see how \( \tilde{t}_{f_0}, \tilde{t}_{a_0} \) appear with opposite relative signs in \( K^+K^- \) or \( K^0\bar{K}^0 \) production, which explains why there can be differences in these production rates.

Finally we have to calculate the differential decay width given by

\[
\frac{d\Gamma}{dM_{\text{inv}}(KK)} = \frac{1}{(2\pi)^3} \frac{1}{4M_D^2} p_\pi \bar{p}_k |\tilde{t}|^2 \tag{21}
\]

with

\[
p_\pi = \frac{\lambda^{1/2}(M_D^2, m_\pi^2, M_{KK}^2)}{2M_D}, \quad \bar{p}_k = \frac{\lambda^{1/2}(M_{KK}^2, m_K^2, m_K^2)}{2M_{KK}} \tag{22}
\]

We integrate from \( M_{\text{inv}} = 1600 \text{ MeV} \) to \( 1870 \text{ MeV} \) to obtain the integrated width into \( \pi f_0, \pi a_0 \) and use the data of the PDG for the \( f_0(1710) \) and from [5] for the \( a_0 \),

\[
M_{f_0} = 1732 \text{ MeV}; \quad \Gamma_{f_0} = 147 \text{ MeV} \\
M_{a_0} = 1777 \text{ MeV}; \quad \Gamma_{a_0} = 148 \text{ MeV} \tag{23}
\]

### III. RESULTS

As we have pointed out, the \( A, A\alpha, A\beta \) parameters are associated to external emission and hence should all have a similar strength. Since \( A \) disappear in ratios, we shall take \( A = 1 \) and then \( |\alpha| \approx |\beta| \approx 1 \). On the other hand \( \gamma \) and \( \delta \) come from internal emission and should have a strength of around \( \frac{1}{3} \), hence \( |\gamma| \approx |\delta| \approx \frac{1}{3} \). But we do not know the signs. Hence we make a table of the results that we obtained using all possible signs. This makes 16 combinations. We define the ratio

\[
R_1 = \frac{\Gamma(D_s^+ \to \pi^+ K^0 \bar{K}^0)}{\Gamma(D_s^+ \to \pi^+ K^+ K^-)} \tag{24}
\]
using the results of Eqs. (20) and (21). Similarly we also define

\[ R_2 = \frac{\Gamma(D_s^+ \to \pi^0 K^+ K^0)}{\Gamma(D_s^+ \to \pi^+ K^0 K^-)} \]  

(25)

and the results of \( R_1, R_2 \) are given in Table II for the 16 different combinations of the parameters (we integrate \( d\Gamma/dM_{\text{inv}} \) over the range of \( M_{\text{inv}} \in [1600 - 1870] \) MeV).

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( R_1 \) | \( R_2 \) | \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( R_1 \) | \( R_2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 1.85 | 0.15 | -1 | 1 | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 1.53 | 0.05 |
| 1 | 1 | \( \frac{1}{3} \) | -\( \frac{1}{3} \) | 4.94 | 1.44 | -1 | 1 | \( \frac{1}{3} \) | -\( \frac{1}{3} \) | 4.11 | 0.62 |
| 1 | 1 | -\( \frac{1}{3} \) | \( \frac{1}{3} \) | 2.49 | 0.73 | -1 | 1 | -\( \frac{1}{3} \) | \( \frac{1}{3} \) | 1.90 | 0.12 |
| 1 | 1 | -\( \frac{1}{3} \) | -\( \frac{1}{3} \) | 2.53 | 1.51 | -1 | 1 | -\( \frac{1}{3} \) | -\( \frac{1}{3} \) | 5.92 | 1.49 |
| 1 | -1 | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 0.20 | 0.29 | -1 | -1 | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 0.24 | 0.15 |
| 1 | -1 | \( \frac{1}{3} \) | -\( \frac{1}{3} \) | 0.54 | 0.08 | -1 | -1 | \( \frac{1}{3} \) | -\( \frac{1}{3} \) | 0.66 | 0.03 |
| 1 | -1 | -\( \frac{1}{3} \) | \( \frac{1}{3} \) | 0.40 | 0.60 | -1 | -1 | -\( \frac{1}{3} \) | \( \frac{1}{3} \) | 0.17 | 0.25 |
| 1 | -1 | -\( \frac{1}{3} \) | -\( \frac{1}{3} \) | 0.40 | 0.29 | -1 | -1 | -\( \frac{1}{3} \) | -\( \frac{1}{3} \) | 0.53 | 0.06 |

We should compare these results with experiment. Recalling the discussion in Ref. [21], we see that \( \text{Br}[D_s^+ \to \pi^+ f_0(1710)] \) from [22] is \(^2\)

\[
\text{Br}[D_s^+ \to \pi^+ "f_0(1710)"; "f_0(1710)" \to K^+ K^-] = (1.0 \pm 0.2 \pm 0.3) \times 10^{-3} \]  

(26)

and from [21]

\[
\text{Br}[D_s^+ \to \pi^+ "f_0(1710)"; "f_0(1710)" \to K^0_s K_s^0] = (3.1 \pm 0.3 \pm 0.1) \times 10^{-3} \]  

(27)

This gives

\[
R_1 = 2 \times \frac{\Gamma(D_s^+ \to \pi^+ K^0_s K_s^0)}{\Gamma(D_s^+ \to \pi^+ K^+ K^-)} = 6.20 \pm 0.67 \]  

(28)

\(^2\) Recall that the message in Ref. [21] is that both in [21] and [22] what is assumed \( f_0(1710) \) is actually a combination of \( f_0(1710) \) and \( a_0(1710) \). This is what mean by ".".
where we have added experimental errors in quadrature. We have a bracket of experimental values of $R_1 \in [5.57 - 6.83]$.

From Table II we observe that among all the 16 possible combinations of parameters only two give results for $R_1$ comparable to those of Eq. (28). These are

$$\begin{cases} 
\alpha = 1, \beta = 1, \gamma = \frac{1}{3}, \delta = -\frac{1}{3} & \text{(set I)} \\
\alpha = -1, \beta = 1, \gamma = -\frac{1}{3}, \delta = -\frac{1}{3} & \text{(set II)} 
\end{cases}$$

(29)

with some preference for the second set which gives $R_1$ compatible with experiment. In both cases we see that the ratio of $R_1$ is of the order of $1.4$, indicating a branching ratio for

$$\text{Br}[D_s^+ \to \pi^0 a_0(1710); a_0(1710) \to K^+ K_s^0] \simeq 1.4 \times 10^{-3}$$

(30)

There are yet no data for this decay which is in the process of analysis, as indicated in [21]. This number should be considered a neat prediction of our approach, together with the fact that the large ratio of $R_1$ observed in [21] finds a natural explanation within our theoretical framework.

So far we have taken sharp values for $\alpha, \beta, \gamma, \delta$ and the results obtained are reasonably close to experimental values. We can make small changes in the parameters to reach values closer to the experiment. For this we take the solutions of set I and set II and make small increments of the parameters in the direction of the gradient of $R_1$, $\vec{\nabla} R_1 \equiv \left( \frac{\partial R_1}{\partial \alpha_i}; \alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3 = \gamma, \alpha_4 = \delta \right)$ till we find a value in the range of the experimental value of Eq. (28). With these parameters we evaluate $R_2$. We find the following results

$$\begin{cases} 
R_1 = 5.95, \quad R_2 = 1.88 & \text{set I: } \alpha = 0.739, \beta = 0.764, \gamma = 0.483, \delta = -0.783 \\
R_1 = 6.23, \quad R_2 = 2.02 & \text{set II: } \alpha = -0.996, \beta = 1.011, \gamma = -0.361, \delta = -0.382 
\end{cases}$$

(31)

The values of the parameters have changed little and then we still can conclude that with natural values of the parameters for the external and internal emission we obtain results for the $R_1$ ratio compatible with the experiment of [21]. We can not decide on which of the two solutions is the correct one, although the proximity of the parameters of set II to the starting ones of Eq. (29) reinforces our preference for this set. Interestingly, the predictions for the $R_2$ ratio are very similar with two sets. We can safely say that our approach predicts a ratio

$$R_2 \in [1.8 - 2.0]$$

(32)
or what is the same, using the experimental data of Eq. (26), and summing errors in quadrature,

\[ \text{Br}[D_s^+ \to \pi^0 a_0(1710); a_0(1710) \to K^+ K_0^0] = (2.0 \pm 0.7) \times 10^{-3} \]  

(33)

This is a neat prediction of our approach and it will be most interesting to see what the coming experiment mentioned in [21] reports.

To finalize our study we plot in Fig. 4 the results of \( d\Gamma/dM_{\text{inv}}(K\bar{K}) \) of Eq. (21) with the different amplitudes \( t_i \) of Eq. (20), with

\[
\begin{align*}
1) & \quad t_i = \tilde{t}_{f_0} \text{ part of } t_{K^+K^-} \\
2) & \quad t_i = \tilde{t}_{a_0} \text{ part of } t_{K^0\bar{K}^0} \\
3) & \quad t_i = t_{K^+K^-} \\
4) & \quad t_i = t_{K^0\bar{K}^0}
\end{align*}
\]  

(34)

As we can see, the differential mass distribution for the case of only the \( a_0(1710) \) contribution has a strength a bit bigger than the one with only the \( f_0(1710) \) contribution. The peak of the \( a_0 \) contribution is displaced to the right relative to the \( f_0 \) contribution because we use the theoretical mass of [5] [see Eq. (23)]. This distribution should be the same seen in the \( D_s^+ \to \pi^0 K^+ K_0^0 \) reaction. It will be most interesting to see the peak position in the coming experiment of the \( D_s^+ \to \pi^0 K^+ K_0^0 \) decay. Note that uncertainties of 30 – 40 MeV in the predicted position of the resonance are normal in the approach of [5].

What is clear from the figure is that in the \( K^0\bar{K}^0 \) mass distribution there has been a constructive interference of the \( f_0 \) and \( a_0 \) resonances, while in the \( K^+K^- \) mass distribution the interference has been destructive. This is exactly the reason given in the analysis of the \( D_s^+ \to \pi^+ K^+ K^- \) and \( D_s^+ \to \pi^+ K_0^0 K_0^0 \) reactions [21] to justify the existence of the \( a_0(1710) \) resonance, which should give the same \( K^+K^- \) or \( K^0\bar{K}^0 \) mass distributions should there be only the \( f_0(1710) \) state.

**IV. CONCLUSIONS**

The appearance of the \( D_s^+ \to \pi^+ K_0^0 K_0^0 \) experiment [21] contrasting the results with those observed in the \( D_s^+ \to \pi^+ K^+ K^- \) reaction in [22] and claiming the existence of an \( a_0 \)
FIG. 4. Mass distributions $d\Gamma/dM_{\text{inv}}$ for the cases of Eq. (34). The result for $K^+K^0_s$ is the same as for the case of $t_{a_0}$ in the figure. The results correspond to the value of the parameters of set II of Eq. (31).

resonance around 1710 MeV, the isospin partner of the $f_0(1710)$ (with mass 1735 MeV in the PDG), motivated us to perform this work, since indeed such a resonance had been predicted in Ref. [5] as a molecular state of $K^*\bar{K}^*$ and other vector-vector coupled channels.

We looked into the possible ways that the $f_0(1710)$ and $a_0(1710)$ could be produced in the $D_s^+ \to \pi^+K^+K^-, \pi^+K^0\bar{K}^0, \pi^0K^+\bar{K}^0$ decays and we identified five different modes in which they could be produced. Three of them associated with external emission, to which we gave weights $A, A\alpha, A\beta$ and two modes associated with internal emission with weights $A\gamma, A\delta$. The value of $A$ is irrelevant since it is related to the global strength and disappears when we perform ratios of rates. While it might look like we have four parameters free, this is not the case, since taking $A = 1, \alpha$ and $\beta$, corresponding to external emission like the case $A$, will also have weight around 1, but with unknown sign. Similarly, the $\gamma$ and $\delta$ parameters corresponding to decay modes of internal emission, which are suppressed by a color factor $N_c$, will have a weight around $\frac{1}{3}$, again with unknown signs. We calculated the ratio of the $D_{s_1}^+ \to \pi^+K^0\bar{K}^0$ and $D_{s_1}^+ \to \pi^+K^+K^-$ decay widths with the 16 possible sign combinations with the strength of the parameters discussed above, and found that only two
of them gave an acceptable value of the ratios compared with experiment. A fine tuning of
the parameters with small deviations from the standard values gave us values of the ratio in
agreement with the range of the experimental value found in [21]. In this way we have shown
that the picture of [5] for the $a_0(1710)$ state provides results for this ratio in agreement with
the findings of the experiment.

Another interesting result of our study is that we make predictions for the branching
ratio of the, yet, unknown results for the $D_s^+ \rightarrow \pi^0 K^+ K_s^0$ reaction. For either of the two
sets of parameters that we found acceptable we found branching ratios for this reaction in
the range of $(2.0 \pm 0.7) \times 10^{-3}$. This is a neat prediction of our theoretical approach which
is only tied to the theoretical couplings of the $f_0(1710)$ and $a_0(1710)$ resonances found in
[5] to the different coupled channels that build up the resonance, and to the experimental
value of the ratio of branching ratios of $D_s^+ \rightarrow \pi^+ K_s^0 K_s^0$ and $D_s^+ \rightarrow \pi^0 K^+ K^-$ found in [21].
An agreement of the coming results of the $D_s^+ \rightarrow \pi^0 K^+ K_s^0$ reaction with the predictions
made here would give a boost to the molecular interpretation on the nature of these two
resonances, while a deviation by one order of magnitude would certainly pose serious trouble
to that picture. The present work makes then very valuable the expected results for the
$D_s^+ \rightarrow \pi^0 K^+ K_s^0$ reaction.

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