The differential transformation method and Miller’s recurrence

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Abstract
The differential transformation method (DTM) enables the easy construction of a power-series solution to a nonlinear differential equation. The exponentiation operation has not been specifically addressed in the DTM literature, and constructing it iteratively is suboptimal. The recurrence for exponentiating a power series by J.C.P. Miller provides a concise implementation of exponentiation by a positive integer for DTM. An equally-concise implementation of the exponential function is also provided.

Keywords: differential transformation method, DTM, power series, exponentiation, exponential function, differential equations

1. Introduction

Constructing power-series solutions to differential equations, especially those which do not admit a closed-form solution, has long been an important, and widely-used, solution technique. Traditionally, computing power-series solutions required a fair amount of “boiler-plate” symbolic manipulation, especially in the setup of the power-matching phase. The differential transformation method (DTM) enables the easy construction of a power-series solution by specifying a conversion between the differential equation and a recurrence relation for the power-series coefficients [1].

The table in the current literature which specifies the translation between the terms of the differential equation and the recurrence relation has a striking omission: it contains no exponentiation operation. Exponentiation by a positive integer can be constructed iteratively using the table entry for multiplication (i.e. multiplying the function with itself \(n\) times), but such a construction is suboptimal because it leads to \(n-1\) nested sums. Using a recurrence for exponentiating a power series by J.C.P. Miller, a table entry for positive-integer exponentiation can be provided which introduces only a single sum. A single-sum recurrence for the exponential function of a power series can be similarly constructed.
2. Preliminaries

2.1. Differential transformation

The differential transform of the function \( w(x) \), called \( W(k) \), is defined as [1]

\[
W(k) = \frac{1}{k!} \left[ \frac{\partial^k}{\partial x^k} w(x) \right]_{x=0}
\]  
(1)

where \( \frac{\partial^k}{\partial x^k} \) is the \( k \)th derivative with respect to \( x \). The inverse transformation is

\[
w(x) = \sum_{k=0}^{\infty} W(k) x^k.
\]  
(2)

Combining Equations 1 and 2 it is clear that the differential transform is derived from the Taylor-series representation of \( w(x) \) about \( x = 0 \)

\[
w(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^k}{\partial x^k} w(x) \right]_{x=0} x^k.
\]  
(3)

Table 1 shows the traditional list of operations and their differential transforms.

| Original Function | Transformed Function |
|-------------------|----------------------|
| \( w(x) = y(x) \pm z(x) \) | \( W(k) = Y(k) \pm Z(k) \) |
| \( w(x) = \lambda y(x) \), \( \lambda \) a constant | \( W(k) = \lambda Y(k) \) |
| \( \frac{\partial^k}{\partial x^k} y(x) \) | \( W(k) = \frac{(k+m)!}{k!} Y(k + m) \) |
| \( w(x) = y(x) z(x) \) | \( W(k) = \sum_{l=0}^{k} Y(l) Z(k - l) \) |
| \( w(x) = x^m \) | \( W(k) = \delta_{k,m} \), \( \delta \) is the Kronecker delta |

Table 1: Traditional operations under DTM

2.2. J.C.P. Miller’s recurrence

In Knuth’s classic book on seminumerical algorithms [4], he provides a recurrence relation attributed to J.C.P. Miller for the coefficients of a power series raised to some integer power. Unfortunately, the recurrence is not well known [6]. Given some formal power series

\[
w(x) = \sum_{k=0}^{N} a_k x^k
\]  
(4)

where \( N \) could be \( \infty \) and \( a_0 \neq 0 \), then

\[
w(x)^m = \left( \sum_{k=0}^{N} a_k x^k \right)^m = \sum_{k=0}^{Nm} c_k x^k
\]  
(5)

\[1\] Recent literature has also included tables with entries for additional elementary functions (e.g. \( \sin(x) \), \( \cos(x) \), \( e^x \)) and integral relations [3]
where the \( \{c_k\} \) are given by the recurrence relation \[6\]

\[
\begin{align*}
c_0 &= a_0^n \\
c_k &= \frac{1}{N_k} \sum_{j=1}^{N_k} [(m+1)j - k] a_j c_{k-j}.
\end{align*}
\]

A simple proof by Zeilberger is as follows \[6\]: \( c_k \) is the coefficient of \( x^0 \) in the Laurent series expansion of \( w(x)^m/x^k \). For any Laurent series, \( f(x) \), the coefficient of \( x^0 \) in \( x^m f(x) \) is zero. So:

\[
\begin{align*}
0 &= [x^0] \frac{d}{dx} \frac{w(x)^m}{x^k} \\
&= [x^0] \left(-k [a_0 + a_1 x + \ldots + a_N x^N] \frac{w(x)^m}{x^k}\right) \\
&= [x^0] \left(-k \left[ a_0 \frac{w(x)^m}{x^m} + a_1 \frac{w(x)^m}{x^{m+1}} + \ldots + a_N \frac{w(x)^m}{x^{m+N}} \right]\right) \\
&= -k [a_0 c_{k-1} + a_1 c_{k-2} + \ldots + a_N c_{k-N} + (m+1)[a_0 c_{k-1} + a_1 c_{k-2} + \ldots + a_N c_{k-N}]].
\end{align*}
\]

From which the recurrence follows. The standard notation \[5\] that \( [x^0] f(x) \) is the coefficient of \( x^0 \) in \( f(x) \) has been used.

### 3. Exponentiation in the DTM

Combining the recurrence relation with the DTM formalism is straightforward. Note that \( W(k) \) is the the \( k^{th} \) Taylor-series coefficient of \( w(x) \), and this yields the table entry in Table 2. Care must be taken, however, to insure that if \( Y(0) \) is zero then the replacement \( \bar{y}(x) = \frac{w(x)}{x^k} \) is made.

| Original Function | Transformed Function |
|-------------------|----------------------|
| \( w(x) = y(x)^m, Y(0) \neq 0 \) | \( W(k) = \frac{W(0)}{Y(0)^m} \sum_{j=1}^{k} [(m+1)j-k] Y(j) W(k-j) \) |
| \( w(x) = e^{y(x)} \) | \( W(0) = Y(0)^m \) |

\[
W(k) = \sum_{k=0}^{\infty} \sum_{k_{m-1}=0}^{k_{m-2}=0} \cdots \sum_{k_2=0}^{k_1=0} Y(k_1) Y(k_2 - k_1) \cdots Y(k_m - k_{m-1}) Y(k - k_{m-1}).
\]

From which the recurrence follows. The standard notation \[5\] that \( [x^0] f(x) \) is the coefficient of \( x^0 \) in \( f(x) \) has been used.

### Table 2: Exponentiation operations under DTM

\( n \) is a positive integer.

Using this formulation greatly simplifies the expressions resulting from exponentiating a function compared to the previously-available method \[2\]. Specifically, for \( w(x) = y(x)^m \) the representation constructed iteratively from the multiplication rule was

\[
W(k) = \sum_{k=0}^{\infty} \sum_{k_{m-1}=0}^{k_{m-2}=0} \cdots \sum_{k_2=0}^{k_1=0} Y(k_1) Y(k_2 - k_1) \cdots Y(k_m - k_{m-1}) Y(k - k_{m-1}).
\]

Numerically, there is a significant advantage to using the recurrence in Table 2 which requires \( O(N) \) computations, compared to the iterative method (Equation 8), which requires \( O(mN \log N) \) computations.
4. The exponential function in DTM

Using Zeilberger’s method, it is possible to derive a recurrence for $e^{ω(x)}$:

$$0 = [x^0] \frac{e^{ω(x)}}{ω(x)}$$
$$= [x^0] \left( -k \frac{e^{ω(x)}}{ω(x)} + (a_1 + 2a_2 x + \ldots + Na_N x^{N-1} \frac{e^{ω(x)}}{ω(x)} \right)$$
$$= [x^0] \left( -k \frac{e^{ω(x)}}{ω(x)} + [a_1 e^{ω(x)} + 2a_1 e^{ω(x)} + \ldots + Na_N e^{ω(x)}] \right)$$

$$= -kc_k + [a_1 c_{k-1} + 2a_2 c_{k-2} + \ldots + Na_N c_{k-N}]$$

The resulting recurrence is also listed in Table[2]. Again, it is much simpler than the iterative exponentiation construction in combination with the Taylor-series expansion of the exponential function[3].

$$W(k) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_0}^{\infty} \sum_{k_{m-1}=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} Y(k_0)Y(k_2-k_1)\cdots Y(k_{m-1}-k_{m-2})Y(k-k_{m-1})$$

5. An example: The One-Dimensional Planar Bratu Problem

To exemplify the formulation presented here, the simplified recurrence will be applied to the one-dimensional planar Bratu problem which had previously been studied using DTM by Hassan and Ertürk[2]. The differential equation is

$$u'' + λu = 0 \quad 0 ≤ x ≤ 1$$
$$u(0) = 0, u(1) = 0$$

and the previously-derived DTM solution is

$$U(k + 2) = -\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_0}^{\infty} \sum_{k_{m-1}=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} U(k_0)U(k_2-k_1)\cdots U(k_{m-1}-k_{m-2})U(k-k_{m-1})$$

for which the coefficients $[U(k)]$ are computed starting from

$$U(0) = 0$$
$$U(1) = \gamma$$

where $γ$ is a constant fixed by the $x = 1$ boundary condition

$$\sum_{k=0}^{\infty} U(k) = 0.$$\n
This can be compared to the analytic solution[2]

$$u(x) = -2 \ln \left[ \frac{\cosh \left( \frac{x - \frac{1}{2}}{\theta} \right)}{\cosh \left( \frac{\theta}{4} \right)} \right]$$

where $θ$ solves

$$θ = \sqrt{\frac{1}{4} \cosh \left( \frac{θ}{4} \right)}.$$
Depending on the value of $\lambda$, Equation 16 has zero, one or two solution(s). See the above-referenced paper by Hassan and Ertürk, and references cited therein, for further details.

Applying the recurrence for $e^{\theta(x)}$ provided here yields the following DTM solution

$$U(k + 2) = -\frac{1}{(k + 1)(k + 2)}W(k)$$

$$W(k) = \frac{1}{k} \sum_{j=1}^{k} jU(j)W(k - j)$$

$$W(0) = e^{\theta(0)}$$

which can be simplified by writing $W(k) = -\frac{(k + 1)(k + 2)}{\lambda}U(k + 2)$ to

$$U(k + 2) = -\frac{1}{2\lambda} \sum_{j=1}^{k} j(k - j)(k - j + 1)U(k - j + 2) \quad k \geq 1$$

The first few values of the recurrence are

$$U(0) = 0$$
$$U(1) = \gamma$$
$$U(2) = -\frac{1}{\gamma}$$
$$U(3) = -\frac{1}{\gamma^2}$$

and these are the same regardless of whether Equation 12 or Equation 18 is used.

6. Conclusion

The rule derived from J.C.P. Miller’s recurrence, and a similar recurrence derived for the exponential function, given in Table 2, are far simpler than the representations previously used in the literature (Equations 8 and 10). It should prove easier to apply DTM, and any power-series-solution technique, to nonlinear differential equations using the recurrence relations presented here.

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