General heavenly equation governs anti-self-dual gravity

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Abstract
We show that the general heavenly equation, suggested recently by Doubrov and Ferapontov (2010 arXiv:0910.3407v2 [math.DG]), governs anti-self-dual (ASD) gravity. We derive ASD Ricci-flat vacuum metric governed by the general heavenly equation, null tetrad and basis of 1-forms for this metric. We present algebraic exact solutions of the general heavenly equation as a set of zeros of homogeneous polynomials in independent and dependent variables. A real solution is obtained for the case of a neutral signature.

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1. Introduction
There are several four-dimensional scalar equations of Monge–Ampère type that determine potentials of anti-self-dual (ASD) Ricci-flat metrics: first and second heavenly equations of Plebański \cite{2}, complex Monge–Ampère equation (CMA) as a real version of the first heavenly equation and the Husain equation \cite{3} together with the closely related mixed heavenly equation \cite{4}. All these equations appear as canonical forms of the general second-order four-dimensional equation which admits partner symmetries and turns out to be of the Monge–Ampère type with the additional constraints that it has a two-dimensional divergence form (instead of four-dimensional divergence form which it will have in general) and contains only second partial derivatives of the unknown \cite{4}. In a recent paper \cite{1}, Doubrov and Ferapontov classified all integrable four-dimensional Monge–Ampère equations with no restriction of admitting a two-dimensional divergence form. Among resulting normal forms of these equations they presented
only one equation that is not of a two-dimensional divergence form (it has a three-dimensional
divergence form), which they called the general heavenly equation
\[
\alpha u_{12} u_{34} + \beta u_{13} u_{24} + \gamma u_{14} u_{23} = 0, \quad \alpha + \beta + \gamma = 0, \tag{1.1}
\]
where \(\alpha\), \(\beta\) and \(\gamma\) are the arbitrary constants with one linear dependence between them. Here
and further on, subscripts mean partial derivatives with respect to variables \(z^1, z^2, z^3, z^4\).
All other normal forms are either well-known equations that govern self-dual gravity or
the ‘modified heavenly equation’ which is a particular case of our ‘asymmetric heavenly
equation’ [4]. Equation (1.1) also appeared in our recent paper [5], where it was shown that by
using partner symmetries one can obtain its particular solutions, which also satisfy the CMA
equation.

Our original motivation for this study of the general heavenly equation was just to make
sure that all four-dimensional equations of the Monge–Ampère type can be used to describe
ASD gravity. Then we realized an importance of the characteristic feature of the general
heavenly equation that being homogeneous, in contrast to other heavenly equations mentioned
above, it enables us to construct easily algebraic solutions, sets of zeros of polynomials in
dependent and independent variables. By ‘homogeneous’, we mean that the equation admits
independent scaling transformations of each independent (and also dependent) variable. Any
such algebraic solution, being a manifold of zeros of a homogeneous polynomial in a complex
space, determines a compact complex smooth algebraic manifold. It is known for a long time
that the CMA equation has a solution which determines a compact \(K3\) surface, which is
called \(K3\) instanton by physicists [6]. However, it is extraordinarily difficult to find algebraic
solutions, and \(K3\) in particular, by directly solving CMA because of its inhomogeneity. Here
we construct a first example of algebraic solutions that determine Ricci-flat ASD metrics by
solving the general heavenly equation.

In section 2, we modify the Lax pair of operators from [1] so that these operators commute
on the solutions of (1.1).

In section 3, we construct a null tetrad for ASD Ricci-flat metric governed by the general
heavenly equation.

In section 4, we construct basis one-forms and ASD metric determined by solutions of
the general heavenly equation.

Though the results of sections 3 and 4 are straightforward consequences of the general
Ashtekar–Jacobson–Smolin and Mason–Newman theorems [7, 8], we need to specify these
results for the general heavenly equation and thus obtain an explicit description of ASD gravity
in terms of solutions of this equation.

In section 5, following the idea from Maciej Dunajski’s book [9], we show how a Lax pair
of commuting operators for the general heavenly equation could be derived in a straightforward
way.

In section 6, we perform a complete study and obtain explicitly one class of algebraic
solutions of this equation and also show more general solutions.

In section 7, we consider such a real cross-section of the general heavenly equation,
which specifies the signature of the corresponding real metric to be either Euclidean or neutral
(ultrahyperbolic). By solving reality conditions imposed on the complex solutions obtained in
section 6, we obtain real solutions as sets of zeros of homogeneous polynomials in dependent
and independent variables. These solutions determine a real ASD Ricci-flat metric with a
neutral signature. More generally, we also have non-polynomial and functionally invariant
solutions with the same property.

In section 8, we present all point symmetries of a real version of the general heavenly
equation and find that our solutions, for any values of the parameters that they depend on, are
invariant solutions with respect to a certain pair of symmetries of the equation. Hence, they determine ASD vacuum metrics which have two Killing vectors.

2. Lax pair

We start from the Lax pair for equation (1.1) presented in [1]

\[ X_1 = u_{34} \partial_1 - u_{13} \partial_4 + \gamma \lambda (u_{34} \partial_1 - u_{14} \partial_3), \]
\[ X_2 = u_{23} \partial_4 - u_{34} \partial_2 + \beta \lambda (u_{34} \partial_2 - u_{24} \partial_3), \]

(2.1)

where \( \partial_1 \) means \( \partial/\partial z \) and so on. The commutator of these operators does not vanish on solutions of equation (1.1):

\[ u_{34} [X_1, X_2] = \{ u_{34} u_{234} - u_{23} u_{344} \} X_1 + \{ u_{34} u_{134} - u_{13} u_{344} \} X_2. \]

(2.2)

For our purposes, we need a Lax pair that commutes on solutions. It has the form

\[ L_0 = \frac{1}{u_{34}} X_1, \quad M_0 = \frac{1}{u_{34}} X_2, \]

(2.3)

so that

\[ [L_0, M_0] = \lambda \frac{1}{u_{34}^2} \left\{ \left( \Phi_4 - \frac{u_{344}}{u_{34}} \Phi \right) \partial_1 - \left( \Phi_3 - \frac{u_{343}}{u_{34}} \Phi \right) \partial_4 \right\}, \]

(2.4)

where \( \Phi = \alpha u_{12} u_{34} + \beta u_{13} u_{24} + \gamma u_{14} u_{23} \) is the left-hand side of the general heavenly equation, \( \Phi_3, \Phi_4 \) are partial derivatives of \( \Phi \) with respect to \( z^3, z^4 \) and \( [L_0, M_0] = 0 \) on solutions.

3. Null tetrad for ASD vacuum metric

This section is based on the Ashtekar–Jacobson–Smolin theorem [7] (see also Mason and Newman [8]). We will use here the notation and formulation of results from the book of Mason and Woodhouse [10]. Let \( \Omega \) be a holomorphic function of \( z^1, z^2, z^3, z^4 \). We denote a null tetrad for the general heavenly equation by \( W, Z, \bar{W}, \bar{Z} \) and set

\[ L = W - \lambda \bar{Z}, \quad M = Z - \lambda \bar{W}. \]

(3.1)

Define \( \Omega \) by the relations \( L_0 = \Omega L \) and \( M_0 = \Omega M \) with \( \Omega \) yet unknown. Then

\[ [L_0, M_0] = [\Omega L, \Omega M] = 0. \]

(3.2)

Let \( \nu \) be a holomorphic 4-form on a four-dimensional complex manifold with the coordinates \( \{ z^i \} \), which should satisfy the conditions

\[ \mathcal{L}_L (\Omega^{-1} \nu) = \mathcal{L}_M (\Omega^{-1} \nu) = 0, \]

(3.3)

where \( \mathcal{L} \) denotes Lie derivative.

We note that

\[ \mathcal{L}_L (u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = \mathcal{L}_M (u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = 0, \]

equivalent to

\[ \mathcal{L}_L (u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = \mathcal{L}_M (u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = 0. \]

(3.4)

Comparing (3.3) and (3.4) we deduce that

\[ \nu = \Omega^2 u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \]

(3.5)
satisfies condition (3.3). According to proposition 13.4.8 in [10], if \( L \) and \( M \) satisfy conditions (3.2) and (3.3) and the normalization condition
\[
2\nu(W, Z, \bar{W}, \bar{Z}) = 1, \tag{3.6}
\]
then \( W, Z, \bar{W}, \bar{Z}, \) defined in (3.1), is a null tetrad for an ASD vacuum metric. Substituting \( \nu \) from (3.5) into (3.6), we obtain
\[
\Omega^2 = \frac{\beta \gamma \Delta}{u_{34}}, \tag{3.7}
\]
The 4-form \( \nu \) becomes
\[
\nu = \frac{\beta \gamma \Delta}{u_{34}} \, dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4. \tag{3.8}
\]
Since \( L = \Omega^{-1} L_0, \, M = \Omega^{-1} M_0 \) with \( L_0 \) and \( M_0 \) defined by (2.3), from the definition of \( W, Z, \bar{W}, \bar{Z} \) in (3.1) we obtain the explicit form of an ASD tetrad frame
\[
W = \frac{u_{34} \partial_1 - u_{13} \partial_4}{\sqrt{\beta u_{34} \Delta}}, \quad Z = \frac{u_{23} \partial_4 - u_{34} \partial_2}{\sqrt{\beta u_{34} \Delta}}, \quad \bar{W} = \frac{\sqrt{\beta} (u_{23} \partial_3 - u_{34} \partial_2)}{\sqrt{\gamma u_{34} \Delta}}, \quad \bar{Z} = \frac{-\sqrt{\beta} (u_{14} \partial_3 - u_{34} \partial_1)}{\sqrt{\gamma u_{34} \Delta}}, \tag{3.9}
\]
which is governed by solutions of the general heavenly equation (1.1) expressed solely in terms of the independent parameters \( \beta \) and \( \gamma \)
\[
(\beta + \gamma) u_{12} u_{34} = \beta u_{13} u_{24} + \gamma u_{14} u_{23}. \tag{3.10}
\]

4. Basis one-forms and ASD metric governed by the general heavenly equation

The corresponding coframe consists of four 1-forms \( \omega^i = \omega_j^i \, dz^j \) which satisfy the following normalization conditions:
\[
\omega^1(W) = \omega^3(Z) = \omega^3(\bar{W}) = \omega^3(\bar{Z}) = 1 \tag{4.1}
\]
with all other \( \omega^1(W), \omega^3(Z), \omega^3(\bar{W}), \omega^3(\bar{Z}) \) vanishing. By solving these bi-orthogonality relations, we obtain the following coframe 1-forms:
\[
\omega^1 = \sqrt{\frac{\beta \gamma}{u_{34} \Delta}} \{u_{23}(u_{14} \, dz^1 + u_{24} \, dz^2) + u_{34}(u_{23} \, dz^3 + u_{24} \, dz^4)\}
\]
\[
\omega^2 = \sqrt{\frac{\beta \gamma}{u_{34} \Delta}} \{u_{13}(u_{14} \, dz^1 + u_{24} \, dz^2) + u_{34}(u_{13} \, dz^3 + u_{14} \, dz^4)\}\tag{4.2}
\]
\[
\omega^3 = -\sqrt{\frac{\gamma}{u_{34} \Delta}} \{u_{14}(u_{13} \, dz^1 + u_{23} \, dz^2) + u_{34}(u_{13} \, dz^3 + u_{14} \, dz^4)\}
\]
\[
\omega^4 = \sqrt{\frac{\beta}{u_{34} \Delta}} \{u_{24}(u_{13} \, dz^1 + u_{23} \, dz^2) + u_{34}(u_{23} \, dz^3 + u_{24} \, dz^4)\}.
\]

On solutions of (3.10) the corresponding ASD vacuum metric reads
\[
ds^2 = 2(\omega^2 \omega^4 - \omega^1 \omega^3) = \frac{2(\beta + \gamma)}{\Delta}
\]
\[
\times \{u_{12} u_{14} u_{13} (dz^1)^2 + u_{23} u_{24} (dz^2)^2 + u_{34} [u_{13} u_{23} (dz^3)^2 + u_{42} u_{24} (dz^4)^2] \}
\]
\[
+ (u_{13} u_{24} + u_{14} u_{23})(u_{12} dz^1 dz^2 + u_{34} (dz^3)^2 + u_{42} u_{24} (dz^4)^2)
\]
\[
+ (u_{12} u_{34} + u_{14} u_{32})(u_{13} dz^1 dz^3 + u_{24} dz^2 dz^4)
\]
\[
+ (u_{12} u_{34} + u_{14} u_{32})(u_{23} dz^2 dz^3 + u_{14} dz^3 dz^4). \tag{4.3}
\]
Using the program EXCALC run by REDUCE, we have computed the Riemann curvature 2-forms and checked vanishing of the Ricci tensor on solutions of the general heavenly equation in the form (3.10), so our metric is indeed Ricci-flat. The expressions for the Riemann curvature 2-forms are too lengthy to be presented here.

5. Derivation of the Lax pair

Here we will show that the Lax pair (defined by (2.1) and (2.3)) of commuting operators \( L_0 \) and \( M_0 \) together with the equation generated by it could be derived in a straightforward way. In doing this, we will follow the idea and notation from the book of Dunajski [9].

We search for Lax operators linear in the spectral parameter \( \lambda \): \[
L_0 = e_{00} - \lambda e_{10} \quad \text{and} \quad M_0 \equiv L_1 = e_{01} - \lambda e_{11},
\]
with the spinor indices. We require \( [L_0, M_0] = 0 \) for operators (5.1) which by splitting in \( \lambda \) yields three equations
\[
[e_{00}, e_{10}] = 0, \quad [e_{01}, e_{11}] = 0, \quad [e_{01}, e_{10}] = 0.
\]

We look for \( e_{A{A}'} \) as linear combinations of partial derivatives operators \( \partial_i \) with variable coefficients:
\[
e_{00} = a\partial_1 + b\partial_2 + c\partial_3 + d\partial_4, \quad e_{10} = e\partial_1 + f\partial_2 + g\partial_3 + h\partial_4, \quad e_{01} = e'\partial_1 + f'\partial_2 + g'\partial_3 + h'\partial_4.
\]
Substituting expressions (5.3) into equations (5.2), computing the commutators and separately equating to zero coefficients of the operators \( \partial_i \), we obtain 3 groups, each of 4 equations, for the coefficients in (5.3). The first group of equations reads
\[
AE_1 + BE_2 + CE_3 + DE_4 - EA_1 - FA_2 + GA_3 - HA_4 = 0,
AF_1 + BF_2 + CF_3 + DF_4 - EB_1 - FB_2 + GB_3 - HB_4 = 0,
AG_1 + BG_2 + CG_3 + DG_4 - EC_1 - FC_2 + GC_3 - HC_4 = 0,
AH_1 + BH_2 + CH_3 + DH_4 - ED_1 - FD_2 + GD_3 - HD_4 = 0.
\]

We can always normalize \( e_{00} \) by choosing \( A = 1 \). To make the first three equations in this group to be identically satisfied, we can choose \( B = C = E = G = 0 \) and \( F = \text{const} \). Later we note that a convenient choice is \( F = -1 \). The last equation in (5.4) becomes
\[
H_1 + D_2 + DH_4 - HD_4 = 0
\]
whereas the first two vector fields \( e_{A{A}'} \) reduce to
\[
e_{00} = \partial_1 + D\partial_4, \quad e_{10} = -\partial_2 + H\partial_4.
\]
With our choice of coefficients \( A, B, C, E, F, G \) the second group of four equations in (5.2) becomes
\[
e_1 + a_2 + De_4 - Ha_4 = 0, \quad f_1 + b_2 + Df_4 - Hb_4 = 0, \quad (5.7)
g_1 + c_2 + Dg_4 - Hc_4 = 0, \quad (5.8)
h_1 + d_2 + Dh_4 - hD_4 - hD_4 + aH_1 + bH_2 + cH_3 - eD_1 - fD_2 - gD_3 = 0. \quad (5.9)
\]
To satisfy equations (5.7) identically, we set constant values to \( a = -\gamma \) and \( f = -\beta \) and we choose \( b = d = e = h = 0 \). Equation (5.9) becomes
\[
-\gamma H_1 + \beta D_2 + cH_3 - gD_3 = 0.
\]

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The third commutator equation in (5.2) for the chosen values of coefficients has only one identically nonvanishing component: the term with $\partial_3$. Equating it to zero, we obtain

$$-\gamma g_1 + \beta c_2 + cg_3 - gc_3 = 0.$$  

(5.11)

The last two vector fields $e_{AA'}$ in (5.3) become

$$e_{01}' = -\gamma \partial_1 + c \partial_3, \quad e_{11}' = -\beta \partial_2 + g \partial_3.$$  

(5.12)

Consider now a holomorphic 4-form on a four-dimensional complex manifold with the coordinates $\{z_i\}$

$$\rho = \Lambda(z) \frac{dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4}{\Lambda_1}$$  

(5.13)

with $\Lambda(z)$ such that $\rho$ should satisfy the conditions

$$L_{L_0}(\rho) = L_{M_0}(\rho) = 0,$$  

(5.14)

where $L$ denotes the Lie derivative and $L_0, M_0$ have the form (5.1). With our previous choices of the coefficients in $e_{AA'}$ equations (5.14), being split in $\lambda_1, \ldots, \lambda_4$, produce the four equations

$$\Lambda_1 + (D\Lambda)_4 = 0, \quad -\Lambda_2 + (H\Lambda)_4 = 0, \quad \gamma \Lambda_1 - (c\Lambda)_3 = 0, \quad \beta \Lambda_2 - (g\Lambda)_3 = 0.$$  

(5.15)

Equations (5.15) are satisfied by introducing the potentials $V$ and $W$

$$\Lambda = V_4, \quad D\Lambda = -V_1, \quad H\Lambda = V_2, \quad \Lambda = W_3, \quad c\Lambda = \gamma W_1, \quad g\Lambda = \beta W_2,$$  

(5.16)

which implies $V_4 = W_3$. Hence, there exists a potential $u$ such that $V = u_3, W = u_4$, so that $\Lambda = u_34$. Eliminating $\Lambda$ in (5.16), we obtain

$$H = \frac{V_2}{V_4} = \frac{u_{23}}{u_{34}}, \quad D = -\frac{V_1}{V_4} = -\frac{u_{13}}{u_{34}},$$  

(5.17)

$$c = \gamma \frac{W_1}{W_3} = \gamma \frac{u_{14}}{u_{34}}, \quad g = \beta \frac{W_2}{W_3} = \beta \frac{u_{24}}{u_{34}},$$  

(5.18)

which solve equations (5.5) and (5.11). The 4-form (5.13) becomes

$$\rho = u_{34} \frac{dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4}{\Omega_2}$$  

with the 4-form $\nu$ defined in (3.5).

The two remaining equations (5.8) and (5.9) expressed in terms of the potential $u$ can be put in the form $\partial_i (\Phi/u_{34}) = 0$ and $\partial_i (\Phi/u_{34}) = 0$, respectively, so that $\Phi = u_{34}$. Here, same as at the end of section 2, $\Phi$ denotes the left-hand side of the general heavenly equation (1.1). By a suitable redefinition of $u$ this becomes $\Phi = 0$, which is exactly the general heavenly equation (1.1). Using formulae (5.17) and (5.18), we obtain the vector fields $e_{AA'}$ (5.6) and (5.12) in the final form

$$e_{00}' = \partial_1 + \frac{u_{13}}{u_{34}} \partial_4, \quad e_{10}' = -\partial_2 + \frac{u_{23}}{u_{34}} \partial_4,$$  

$$e_{01}' = -\gamma \left( \partial_1 - \frac{u_{14}}{u_{34}} \partial_3 \right), \quad e_{11}' = -\beta \left( \partial_2 - \frac{u_{24}}{u_{34}} \partial_3 \right)$$  

(5.19)

and the operators $L_0, M_0$ in (5.1) coincide with the Lax pair (2.3) for the general heavenly equation (1.1).
6. Algebraic solutions of the general heavenly equation

We look for solutions of the general heavenly equation (1.1), which are algebraic surfaces of even order \(2m, m = 1, 2, 3, \ldots\) of the special form

\[
u^{2m} - [a_1(z^1)^{2m} + a_2(z^2)^{2m} + a_3(z^3)^{2m} + a_4(z^4)^{2m} + a_5(z^1)^m(z^2)^m
+ a_6(z^3)^m + a_7(z^2)^m(z^3)^m + a_8(z^3)^m(z^4)^m
+ a_9(z^1)^m(z^3)^m + a_{10}(z^5)^m(z^4)^m + C] = 0, \tag{6.1}
\]

where \(C\) is an arbitrary constant and the constant coefficients \(a_i\) are determined by equation (1.1). A complete description of this class of solutions is given in the following statement.

**Proposition 1.** Solution of the form (6.1) with \(m \neq 1\) or \(2\) to equation (1.1), which satisfies the condition: \(\Delta = u_{13}u_{24} - u_{14}u_{23} \neq 0\) in the denominators of coframe 1-forms (4.2) and metric (4.3), exists if its coefficients are determined by one of the two sets of relations

\[
\begin{align*}
a_1 &= \frac{a_6(a_6a_10 + \beta a_5a_0) + 2\gamma a_4a_9a_9}{2\gamma(2a_4a_7 - a_8a_{10})}, \quad a_7 = -\frac{a_4a_6 + \beta a_5a_{10}}{\gamma a_6}, \\
a_2 &= \frac{a_4a_{10}a_9 - a_9a_{10} + \gamma a_5(2a_4a_7 - a_8a_{10})}{2\gamma(2a_4a_9 - a_8a_{10})}, \\
a_3 &= \frac{\beta a_8(a_9 - a_8a_{10}) - a_9(2a_4a_7 - a_8a_{10})}{2\gamma(a_6a_{10} - 2a_4a_5)},
\end{align*}
\tag{6.2}
\]

where seven coefficients \(a_4, a_5, a_6, a_8, a_9, a_{10}, C\) are free parameters, or

\[
\begin{align*}
a_1 &= \frac{\beta a_4a_9a_5^2}{a_5^2a_6 + 2\beta a_4a_9a_5}, \quad a_2 = \frac{a_5(2\beta^2a_4a_9a_5 - \alpha a_5a_6^2)}{4\beta^2a_5a_5^2}, \\
a_3 &= \frac{2\beta a_4a_9 - \gamma a_5a_6^2}{4\beta a_5a_5}, \quad a_6 = 0, \quad a_{10} = -\frac{a_5a_8}{\beta a_9},
\end{align*}
\tag{6.3}
\]

with six free parameters \(a_4, a_5, a_7, a_8, a_9, C\).

The proof of this statement has been obtained as an output of an appropriate REDUCE program where solutions with \(\Delta = 0\) have been eliminated. It is remarkable that these coefficients do not depend on the choice of power \(m\). In the special cases \(m = 1, m = 2\) solutions have more general form which will be studied elsewhere.

Let \(z = \{z^1, z^2, z^3, z^4\}\). The proof of the following statement is straightforward.

**Proposition 2.** If \(P(z)\) satisfies equation (1.1) and the differential constraint

\[
\alpha(P_1P_2P_3 + P_3P_4P_2) + \beta(P_1P_3P_4 + P_2P_4P_3) + \gamma(P_1P_4P_3 + P_2P_3P_4) = 0, \tag{6.4}
\]

then the function \(u(z)\), implicitly determined by the equation \(G(u, \nu) = 0\) where \(G\) is an arbitrary smooth function, is also a solution of (1.1).

Thus, we have obtained a functionally invariant solution. An example of \(P(z)\), that satisfies both conditions of this theorem, is given in curly braces in (6.1) with coefficients determined by (6.2). We also note that solutions of the form (6.1) are valid more generally for an arbitrary real or complex parameter \(m\) but then they are not algebraic manifolds.

**Remark 1.** The left-hand side of differential constraint (6.4) with \(P\) replaced by \(u\) is the Lagrangian density for the general heavenly equation (1.1):

\[
L = \alpha(u_{14}u_{23} + u_{13}u_{24}) + \beta(u_{14}u_{23} + u_{24}u_{13}) + \gamma(u_{14}u_{23} + u_{24}u_{13}), \tag{6.5}
\]
i.e. equation (1.1) is the Euler–Lagrange equation for Lagrangian (6.5). Thus, the differential constraint (6.4) selects those solutions of (1.1) for which \( L = 0 \).

The following remark is relevant to a characterization of the joint solution space of the two equations involved in proposition 2. To be specific, we will use equations (1.1) and (6.4) algebraically solved with respect to \( u_{12} \) and \( u_{13} \).

**Remark 2.** The system of two equations (1.1) and (6.4) (with \( P \) replaced by \( u \)) has two third-order integrability conditions

\[
A(u_2u_{34} - u_3u_{24}) + B(u_4u_{23} - u_3u_{24}) = 0, \\
D(u_2u_{34} - u_3u_{24}) + E(u_4u_{23} - u_3u_{24}) = 0,
\]

where

\[
A = u_4(u_2u_{34} - u_3u_{24}) - u_3(u_4u_{23} - u_3u_{24}) + u_2(u_2u_{34} - u_3u_{23}), \\
B = u_3(u_2u_{34} - u_3u_{24}) - u_2(u_4u_{23} - u_3u_{24}) + u_4(u_2u_{34} - u_3u_{24}), \\
D = 2u_3u_4u_{24} - u_3u_2u_{34} + u_2u_4u_{34} - u_2u_3u_{24} - u_4u_3u_{24}, \\
E = u_3(u_3u_{24} - u_4u_{34}) - u_2(u_3u_{34} - u_4u_{34}) + u_4(u_2u_{34} - u_3u_{24}).
\]

Applying the algorithms of [11, 12], one can check that the system of four equations (1.1), (6.4) and (6.6) is involutive (generates no further integrability conditions) and that the main part of arbitrariness of its solution manifold is determined by three arbitrary functions of two variables, whereas the general solution manifold of equation (1.1) without differential constraints depends on two arbitrary functions of three variables.

In the generic case, when \( (u_2u_{34} - u_3u_{24}, u_4u_{23} - u_3u_{24}) \neq (0, 0) \), we obtain the constraint

\[
\det \begin{bmatrix} A & B \\ D & E \end{bmatrix} = 0.
\]

Two equations (6.6) become linearly dependent, so we still have two third-order integrability conditions: one equation in (6.6) and (6.7). No further integrability conditions are generated for the system (1.1), (6.4), (6.6) and (6.7).

**7. Real cross-section of the general heavenly equation and its real algebraic solutions**

For applications to self-dual gravity we need real cross-sections of the general heavenly equation and its solutions. We specify the real cross-section by the requirement that the corresponding real metric should have a certain signature. Then we have to make the following identifications: \( z^2 = \bar{z}^1 \) and \( z^4 = \bar{z}^3 \) (\( z^2 = -\bar{z}^1 \) would also do) and then replace everywhere index 3 by 2. Here the bar means complex conjugation.

The real general heavenly equation takes the form

\[
a\omega_{11} + \beta\omega_{12} + \gamma\omega_{12} = 0,
\]

while \( \Delta = u_{12}u_{12} - u_{13}u_{13} \). In the following we assume that \( \gamma \neq 0 \).

The signature of the metric depends on the sign of \( \beta/\gamma \). If \( \beta/\gamma > 0 \), we set \( \beta = \gamma \delta^2 \) with \( \delta > 0 \). Then basis 1-forms (4.2) become

\[
\omega^1 = \frac{|\gamma/\delta|}{\sqrt{u_{22}\Delta}} \tilde{l}_1, \quad \omega^2 = \frac{|\gamma/\delta|}{\sqrt{u_{22}\Delta}} \tilde{l}_2, \\
\omega^3 = -\frac{1}{\delta\sqrt{u_{22}\Delta}} \tilde{l}_1, \quad \omega^4 = \frac{\delta}{\sqrt{u_{22}\Delta}} \tilde{l}_2.
\]
where 1-forms \( l_1 \) and \( l_2 \) are defined as
\[
\begin{align*}
l_1 &= u_{12}(u_{12} \, dz^1 + u_{12} \, dz^1) + u_{23}(u_{12} \, dz^2 + u_{12} \, dz^2), \\
l_2 &= u_{12}(u_{12} \, dz^1 + u_{12} \, dz^1) + u_{23}(u_{12} \, dz^2 + u_{12} \, dz^2).
\end{align*}
\] (7.3)

The metric becomes
\[
d s^2 = 2(\omega^2 - \omega^1) = \frac{2|\Gamma|}{|u_{22}\Delta|} (|l_1|^2 + \delta^2 |l_2|^2),
\] (7.5)

which has obviously a Euclidean signature. It is determined by solutions of the real version (7.1) of the general heavenly equation in the form
\[
(\delta^2 + 1)u_{11}u_{22} = \delta^2 u_{12}u_{12} + u_{12}u_{21}.
\] (7.6)

If \( \beta/\gamma < 0 \), we set \( \beta = -\gamma \delta^2 \) and then the real cross-section of the general heavenly equation (7.1) becomes
\[
(\delta^2 - 1)u_{11}u_{22} = \delta^2 u_{12}u_{12} - u_{12}u_{21}.
\] (7.7)

Basis 1-forms (4.2) become
\[
\begin{align*}
\omega^1 &= \frac{i|\gamma|}{\sqrt{u_{22}\Delta}} \tilde{l}_1, \\
\omega^2 &= \frac{i|\gamma|}{\sqrt{u_{22}\Delta}} \tilde{l}_2, \\
\omega^3 &= \frac{1}{i\delta \sqrt{u_{22}\Delta}} \tilde{l}_1, \\
\omega^4 &= \frac{i\delta}{\sqrt{u_{22}\Delta}} \tilde{l}_2.
\end{align*}
\] (7.8)

Metric (4.3) takes the form
\[
d s^2 = 2(\omega^2 - \omega^1) = \frac{2|\Gamma|}{|u_{22}\Delta|} (|l_1|^2 - \delta^2 |l_2|^2),
\] (7.9)

which obviously has a neutral signature. This metric is determined by solutions of equation (7.7).

So far, we were able to obtain a real solution only in the case of a neutral signature. This solution of equation (7.7) is obtained by solving reality conditions imposed on the complex algebraic solution (6.1) with coefficients (6.2). A relatively simple particular solution has the following form:
\[
\begin{align*}
F &= \frac{A}{2} \left( 1 + \frac{\delta(AB + F^2) - FR}{\delta F - R} \right) [\exp(2i\theta)(\bar{z}^1)^m + \exp(-2i\theta)(\bar{z}^1)^m] \\
&\quad + \frac{1}{2A} \left[ \delta(AB + F^2) - FR \right] [(\bar{z}^1)^m + (\bar{z}^1)^m] - A(z^1)^m (\bar{z}^1)^m \\
&\quad + R[\exp(i\theta)(z^1)^m (z^1)^m + \exp(-i\theta)(z^1)^m (z^1)^m] + B(z^2)^m (\bar{z}^2)^m \\
&\quad + F[\exp(i\theta)(z^1)^m (z^1)^m + \exp(-i\theta)(z^1)^m (z^1)^m] + C = 0,
\end{align*}
\] (7.10)

where \( R = \sqrt{(\delta^2 - 1)AB + \delta^2 F^2} \) and the constants \( m, A, B, F, \theta, C \) are free real parameters (obviously, from now on we use \( A, B, C, E, F \) in a different sense than in the preceding sections). Parameter \( \theta \) is inessential, since it can be scaled out by using scaling symmetries of the general heavenly equation (7.7), so that we can set \( \theta = 0 \) in (7.10) and our solution depends only on five essential parameters. For \( m = 1, 2, 3, \ldots \) this solution determines an algebraic hypersurface in the real five-dimensional space.

The real cross-section of the second complex solutions (6.1), (6.3) can be transformed to a particular case of solution (7.10) by using scaling symmetries of equation (7.7) and therefore does not yield an essentially new real solution.
A general solution of reality conditions for (6.1) with coefficients (6.2) reads

\begin{equation}
\begin{aligned}
u^{2m} - \{a_1(z^1)^{2m} + \tilde{a}(z^1)^{2m} + \bar{E}[\tilde{z}^{2m} + (\tilde{z}^{2m})] - A(z^1)^{m}(\tilde{z}^{1}) - R[(\tilde{z}^{1})^{m}(\tilde{z}^{2})^{m} + (\tilde{z}^{2}m)(\tilde{z}^{1})^m) + B(\tilde{z}^{2})^m(\tilde{z}^{2})^m] + F[\cos 2(\tilde{z}^{1})m(\tilde{z}^{2})^m + e^{-i\phi}m(\tilde{z}^{2})^m] + C\} = 0, \\
\end{aligned}
\end{equation}

(7.11)

where all inessential parameters are scaled out by scaling symmetries of the general heavenly equation (7.7), the coefficients satisfy the relations

\begin{equation}
a_1 = \frac{2AEF\cos\phi}{2(BFe^{-i\phi} - 2ER)},
\end{equation}

(7.12)

\begin{equation}
\cos\phi = \frac{B(R^2 + AB + F^2) - 4AE^2}{4EFR},
\end{equation}

(7.13)

\(R\) is defined above and \(\tilde{a}_1\) is complex conjugate to (7.12). This solution at \(m = 1, 2, 3, \ldots\) and fixed \(\delta\) depends on five real parameters, four of which, \(A, B, E, F\), should satisfy the relation

\begin{equation}
|B(R^2 + F^2 + AB) - 4AE^2| \leq 4|EF|R
\end{equation}

(7.14)

to ensure \(|\cos\phi| < 1\). It is not difficult to check that inequality (7.14) can indeed be satisfied for the considered case of neutral signature, when \(\beta/\gamma < -\delta^2 < 0\), while for \(\beta/\gamma = \delta^2 > 0\) in the case of the Euclidean signature it cannot be satisfied for real solutions.

The particular solution (7.10) with \(\theta = 0\) is obtained from (7.11) at \(\phi = 0\), which implies the relation \(E = \frac{4A\bar{A} + BF^2 - FR}{2A}\).

Using our solutions (7.10) and (7.11) in metric (7.9) together with definitions (7.4) of 1-forms \(l_1, l_2\), we obtain real ASD Ricci-flat metric with the neutral signature, which depends on five real parameters. We have checked that \(\Delta \neq 0\) as far as \(A \cdot (AB + F^2) \cdot (\delta^2 - 1) \neq 0\) (for \(m = 1\) also \(C \neq 0\) while \(w_{22} \neq 0\) obviously implies \(B \neq 0\). Under these conditions, metric (7.9) has no identically vanishing denominators.

### 8. Symmetries of general heavenly equation

We have determined all point symmetries of the general heavenly equation. For its real version (7.7), which governs the ASD metrics with the neutral signature, symmetry generators have the form

\begin{equation}
\begin{aligned}
X_1 &= a(z^1)\partial_1, \quad \tilde{X}_1 = \tilde{a}((z^1))\partial_1, \\
X_2 &= b(z^2)\partial_2, \quad \tilde{X}_2 = \tilde{b}(\tilde{z}^2)\partial_2, \quad X_3 = u\partial_u, \\
X_4 &= c(z^1)\partial_3, \quad \tilde{X}_4 = \tilde{c}(\tilde{z}^2)\partial_3, \\
X_5 &= d(z^2)\partial_3, \quad \tilde{X}_5 = \tilde{d}(\tilde{z}^2)\partial_3,
\end{aligned}
\end{equation}

(8.1)

where \(a, b, c, d\) and their complex conjugates are the arbitrary functions of one variable. Finite symmetry transformations generated by \(X_3, X_4, X_5\) and \(\tilde{X}_4, \tilde{X}_5\), given in (8.1), incorporate scaling \(\tilde{u} = \lambda u\) and translations \(\tilde{u} = u + \epsilon c(z^i)\) for \(i = 1, 2\), together with their complex conjugates. Symmetry transformations generated by \(X_1, X_2\) have the form \(\tilde{a}(\tilde{z}^i) = \tilde{a}(z^i) + \epsilon\), where we have introduced the notation \(\tilde{a}(z) = \int \frac{A(z)}{B(z)}\), plus complex conjugate equations. In particular, if either \(a(z) = 1\) or \(\tilde{a}(z) = z\) and similarly for their complex conjugates, we obtain translations and scaling in each variable \(z^1, \tilde{z}^2\), respectively. We note that our solutions (7.10) and (7.11) with \(C \neq 0\) are noninvariant under these particular cases of symmetry transformations. However, consider the invariance condition of our solution (7.11) under the symmetry generator \(X = X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2\) in (8.1), with the choice
\[ a(z^1) = c_1/(z^1)^m, b(z^2) = c_2/(z^2)^m \] and their complex conjugates:

\[
\begin{align*}
2\alpha_1 \tilde{c}_1 &- A\tilde{c}_1 + Fe^{i\phi}c_2 + R\tilde{c}_2 = 0, \\
-Ac_1 + 2\alpha_1\tilde{c}_1 + Rc_2 + Fe^{-i\phi}\tilde{c}_2 &= 0, \\
Fe^{i\phi}c_1 + Rc_1 + 2Ec_2 + B\tilde{c}_2 &= 0, \\
Rc_1 + Fe^{-i\phi}\tilde{c}_1 + Bc_2 + 2E\tilde{c}_2 &= 0.
\end{align*}
\] (8.2)

This system of linear algebraic equations admits nonzero solution for the coefficients \( c_i, \tilde{c}_i \) since the determinant of this system turns out to be 0. Moreover, the rank of this system equals 2. This proves the existence of two independent symmetries of our equation, such that our real solution of the general form is invariant with respect to these symmetries for any choice of parameters in the solution and hence it is an invariant solution. Therefore, the corresponding metric (7.9) will have two Killing vectors.

9. Conclusion

Applying the general theorem of Ashtekar–Jacobson–Smolin–Mason–Newman, we have obtained an explicit description of anti-self-dual (ASD) gravity in terms of solutions to the general heavenly equation, introduced by Doubrov and Ferapontov [1]. We have derived the ASD Ricci-flat vacuum metric, determined by solutions of the general heavenly equation, together with the corresponding null tetrad and basis 1-forms. Following the ideas of Dunajski’s book [9], we have also been able to derive straightforwardly a Lax pair of commuting operators. Unlike other heavenly equations that describe ASD gravity, the general heavenly equation is homogeneous, i.e. admit independent scaling transformations of each independent (and also dependent) variable. This property allows us to obtain algebraic solutions to this equation in the form of homogeneous polynomials in independent and dependent variables, which can be modified by adding an arbitrary constant \( C \). In this respect, the homogeneity seems to be the most important property of the considered equation, so that we would suggest to call it the homogeneous heavenly equation. Although for \( C \neq 0 \) these solutions do not admit scaling or any other obvious symmetries of the equation, we have proved that they are invariant solutions with respect to certain two symmetries of this equation, so that the corresponding metric will have two Killing vectors. The work on noninvariant algebraic solutions to a heavenly equation, such that their real form will determine metrics with the Euclidean signature, is currently in progress. Such a solution may be relevant to the search of the famous \( K^3 \) instanton [6].

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