Integrable semi-discretization of a multi-component short pulse equation

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In the present paper, we mainly study the integrable semi-discretization of a multi-component short pulse equation. Firstly, we briefly review the bilinear equations for a multi-component short pulse equation proposed by Matsuno (J. Math. Phys. 52 123705) and reaffirm its $N$-soliton solution in terms of pfaffians. Then by using a Bäcklund transformation of the bilinear equations and defining a discrete hodograph (reciprocal) transformation, an integrable semi-discrete multi-component short pulse equation is constructed. Meanwhile, its $N$-soliton solution in terms of pfaffians is also proved.

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation, as one of the universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear dispersive media, has been very successful in many applications such as nonlinear optics and water waves\textsuperscript{1–4}. The NLS equation is integrable and can be solved by the inverse scattering transform\textsuperscript{5}. However, in the regime of ultra-short pulses where the width of optical pulse is in the order of femtosecond ($10^{-15}$ s), the NLS equation becomes less accurate\textsuperscript{6}. Description of ultra-short processes requires a modification of going beyond the standard slow varying envelope approximation (SVEA). Recently, Schäfer and Wayne derived a short pulse (SP) equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

(1)

in attempting to describe the propagation of ultra-short optical pulses in nonlinear media\textsuperscript{2}. Here, \(u = u(x, t)\) is a real-valued function, representing the magnitude of the electric field, the subscripts \(t\) and \(x\) denote partial differentiation. It has been shown that the SP equation performs better than NLS under this case\textsuperscript{8}.

Apart from the context of nonlinear optics, the SP equation has also been derived as an integrable differential equation associated with pseudospherical surfaces\textsuperscript{9}. The SP equation has been shown to be completely integrable\textsuperscript{9–13}. The loop soliton solutions as well as smooth soliton solutions of the SP equation were found in\textsuperscript{14,15}. The connection between the SP equation and the sine-Gordon equation through the hodograph transformation was clarified, and then the \(N\)-soliton solutions including multi-loop and multi-breather ones were given in\textsuperscript{16,17} by using Hirota’s bilinear method\textsuperscript{18}. An integrable discretization of the SP equation was constructed by the authors in\textsuperscript{19}, and its geometric interpretation was given in\textsuperscript{20}.

A major simplification made in the derivation of the short pulse equation is to assume that the polarization is preserved during its propagating inside an optical fiber. However, this is not always the case in practice. For example, we have to take into account the effects of polarization or anisotropy in birefringent fibers\textsuperscript{21}. Therefore, an extension to a two-component version of the short pulse equation is needed in order to describe the propagation of ultra-short pulse in birefringent fibers. In fact, several integrable coupled short pulse have been proposed in the literature\textsuperscript{22–27}. The bi-Hamiltonian structures for several coupled short pulse equations were obtained in\textsuperscript{28}.

In the present paper, we are concerned with the integrable semi-discretization of a multi-
component short pulse (MCSP) equation

\[ u_{i,xt} = u_i + \frac{1}{2} \left( \sum_{1 \leq j < k \leq n} c_{jk} u_j u_k \right) u_{i,x}, \quad i = 1, 2, \ldots, n, \tag{2} \]

where the coefficients \( c_{jk} \) are arbitrary constants with symmetry \( c_{jk} = c_{kj} \). Eq. (2) was proposed by Matsuno through Hirota’s bilinear method, meanwhile, multi-soliton solution was given as well.

The remainder of the present paper is organized as follows. In Section 2, the MCSP equation is briefly reviewed. We provide its \( N \)-soliton solution in an alternative pfaffian form and prove it by the pfaffian technique. In Section 3, by using a Bäcklund transformation of the bilinear equations and defining a discrete hodograph transformation, we construct a semi-discrete analogue of the MCSP equation. Meanwhile, \( N \)-soliton solution in terms of pfaffian is provided and proved. In Section 4, we investigate in detail the one- and two-soliton solutions to the semi-discrete complex short pulse equation, which can be reduced from the MCSP equation. The paper is concluded by several remarks in Section 5.

II. REVIEW OF THE MULTI-COMPONENT SHORT PULSE EQUATION AND ITS MULTI-SOLITON SOLUTION

It was shown by Matsuno that the SP equation (1) is derived from bilinear equations

\[
\begin{align*}
D_s D_y \tilde{f} \cdot \tilde{f} &= \frac{1}{2} \left( \tilde{f}^2 - \tilde{f}'^2 \right), \\
D_s D_y \tilde{f}' \cdot \tilde{f}' &= \frac{1}{2} \left( \tilde{f}'^2 - \tilde{f}^2 \right),
\end{align*}
\tag{3}
\]

through transformations

\[ u = 2i \left( \ln \frac{\tilde{f}'}{\tilde{f}} \right)_s, \quad x = y - 2(\ln \tilde{f} \tilde{f}')_s, \quad t = s. \tag{4} \]

Here \( D \) is called Hirota \( D \)-operator defined by

\[
D^n_s D^m_y f \cdot g = \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial s'} \right)^n \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m f(y, s) g(y', s')|_{y=y', s=s'}. \]

Recently, in view of the fact that the SP equation (1) can also be derived from another set of bilinear equations

\[
\begin{align*}
D_s D_y f \cdot g &= f g, \\
D^2_s f \cdot f &= \frac{1}{2} g^2,
\end{align*}
\tag{5}\]
through transformations
\[ u = \frac{g}{f}, \quad x = y - 2(\ln f)_s, \quad t = s, \] (6)

Matsuno\textsuperscript{24} constructed a multi-component generalization of the short pulse equation (1) based on a multi-component generalization of bilinear equations (5), which reads
\[
\begin{aligned}
D_s D_y f \cdot g_i &= f g_i, \quad i = 1, 2, \cdots, n, \\
D_s^2 f \cdot f &= \frac{1}{2} \sum_{1 \leq j < k \leq n} c_{jk} g_j g_k.
\end{aligned}
\] (7)

**Remark 1.** The set of bilinear equations (5) is actually obtained from a 2-reduction of the KP-Toda hierarchy, which basically delivers only two tau-functions out of a sequence of the tau-functions. Furthermore, when these two tau-functions are made complex conjugate to each other, the bilinear equations (5) is converted into the sine-Gordon equation \( \phi_{ys} = \sin \phi \) via a transformation \( \phi = 2i \ln(\tilde{f}'/\tilde{f}) \), which is further converted into the SP equation (1) by a hodograph transformation.

**Remark 2.** In\textsuperscript{29}, Hirota and one of the authors have shown that both the bilinear equations (3) and (5) derive the sine-Gordon equation. Furthermore, the relations between tau-functions, which read
\[
f = \tilde{f}' \tilde{f}, \quad g = 2i D_s \tilde{f}' \cdot \tilde{f},
\] (8)
were also presented.

**Remark 3.** As mentioned previously, Eqs. (3) originate from 2-reduction of single component KP-Toda hierarchy, whereas, Eqs. (5) come from \((1 + 1)\)-reduction of two-component KP-Toda hierarchy. Since they both belong to \(A_1^{(1)}\) of the Euclidean Lie algebra\textsuperscript{30}, it is natural that they both derive the SP equation. However, the latter can be easily extended to \((1 + \cdots + 1)\)-reduction of multi-component KP-Toda hierarchy, which gives rise to the multi-component generalization of the short pulse equation.

In what follows, we will briefly review how the bilinear equations (7) determines a multi-component generalization of the SP equation (1). Dividing both sides by \(f^2\), the bilinear equations (7) can be cast into
\[
\begin{aligned}
\left( \frac{g_i}{f} \right)_{sy} + 2 \frac{g_i}{f} (\ln f)_s &= \frac{g_i}{f}, \quad i = 1, 2, \cdots, n, \\
(\ln f)_{ss} &= \frac{1}{4} \sum_{1 \leq j < k \leq n} c_{jk} \frac{g_j g_k}{f^2}.
\end{aligned}
\] (9)
Introducing a hodograph transformation

\[ x = y - 2(\ln f)_s, \quad t = s, \quad (10) \]

and a dependent variable transformation

\[ u_i = \frac{g_i}{f}, \quad (i = 1, 2, \cdots, n), \quad (11) \]

we then have

\[ \frac{\partial x}{\partial s} = -2(\ln f)_{ss} = -\frac{1}{2} \sum_{1 \leq j < k \leq n} c_{jk} u_j u_k, \quad \frac{\partial x}{\partial y} = 1 - 2(\ln f)_{sy}, \]

which implies

\[ \partial_y = \rho^{-1} \partial_x, \quad \partial_x = \partial_t - \frac{1}{2} \left( \sum_{1 \leq j < k \leq n} c_{jk} u_j u_k \right) \partial_x \quad (12) \]

by letting \( 1 - 2(\ln f)_{sy} = \rho^{-1} \).

Notice that the first equation in (9) can be rewritten as

\[ \left( \frac{g_i}{f} \right)_{sy} = (1 - 2(\ln f)_{sy}) \frac{g_i}{f}, \]

or

\[ \rho \left( \frac{g_i}{f} \right)_{sy} = \frac{g_i}{f}, \]

which is converted into

\[ \partial_x \left( \partial_t - \frac{1}{2} \left( \sum_{1 \leq j < k \leq n} c_{jk} u_j u_k \right) \partial_x \right) u_i = u_i, \quad (13) \]

by the conversion relation (12). Obviously, Eq. (13) is nothing but the MCSP (2).

Next, we give an alternative representation of the \( N \)-solution to the MCSP equation (2) in the form of pfaffians. To this end, let us define a class of set \( B_\mu, \mu = 1, 2, \cdots, n \), which satisfies the following condition,

\[ B_\mu \cap B_\nu = \emptyset, \quad \text{if} \quad \mu \neq \nu, \quad \bigcup_{\mu=1}^n B_\mu = \{ b_1, b_2, \cdots, b_{2N} \}. \]

Then we define the elements of the pfaffians (others not mentioned below are zeros)

\[ \text{Pf}(a_j, a_k) = \frac{p_j - p_k}{p_j + p_k} e^{\xi_j + \xi_k}, \quad \text{Pf}(a_j, b_k) = \delta_{j,k}, \quad (14) \]

\[ \text{Pf}(b_j, b_k) = \frac{1}{4} \frac{c_{\mu\nu}}{p_j^2 - p_k^2}, \quad (b_j \in B_\mu, b_k \in B_\nu), \quad (15) \]
\[ \text{Pf}(d_l, a_k) = p_k^l e^{\xi_k}, \quad (16) \]

\[ \text{Pf}(b_j, \beta_\mu) = \begin{cases} 
1 & b_j \in B_\mu \\
0 & b_j \notin B_\mu
\end{cases}. \quad (17) \]

Here \( j, k = 1, 2, \ldots, 2N, \mu, \nu = 1, 2, \ldots, n, \xi_j = p_j y + p_j^{-1} s + \xi_{i0} \) and \( l \) is an integer.

By defining the elements of the pfaffians, we can give the pfaffian solutions satisfying bilinear equations (7).

**Theorem 1.** The bilinear equations (7) have the following pfaffian solution

\[ f = \text{Pf}(a_1, \ldots, a_{2N}, b_1, \ldots, b_{2N}), \quad (18) \]

\[ g_i = \text{Pf}(d_0, \beta_i, a_1, \ldots, a_{2N}, b_1, \ldots, b_{2N}), \quad (19) \]

where \( i = 1, 2, \ldots, n \) and the elements of the pfaffians are given in Eqs. (14)–(17).

**Proof.** Since

\[ \frac{\partial}{\partial y} \text{Pf}(a_j, a_k) = (p_j - p_k) e^{\xi_j + \xi_k} = \text{Pf}(d_0, d_1, a_j, a_k), \]

\[ \frac{\partial}{\partial s} \text{Pf}(a_j, a_k) = (p_j^{-1} - p_k^{-1}) e^{\xi_j + \xi_k} = \text{Pf}(d_{-1}, d_0, a_j, a_k), \]

\[ \frac{\partial^2}{\partial s^2} \text{Pf}(a_j, a_k) = (p_j^{-2} - p_k^{-2}) e^{\xi_j + \xi_k} = \text{Pf}(d_{-2}, d_0, a_j, a_k), \]

\[ \frac{\partial^2}{\partial y \partial s} \text{Pf}(a_j, a_k) = (p_j p_k^{-1} - p_k p_j^{-1}) e^{\xi_j + \xi_k} = \text{Pf}(d_{-1}, d_1, a_i, a_j), \]

where \( \text{Pf}(d_l, d_m) = 0 \) for integers \( l \) and \( m \), we then have

\[ \frac{\partial f}{\partial y} = \text{Pf}(d_0, d_1, \ldots), \]

\[ \frac{\partial f}{\partial s} = \text{Pf}(d_{-1}, d_0, \ldots), \]

\[ \frac{\partial^2 f}{\partial s^2} = \text{Pf}(d_{-2}, d_0, \ldots), \]

\[ \frac{\partial^2 f}{\partial y \partial s} = \text{Pf}(d_{-1}, d_1, \ldots). \]
Here \( \text{Pf}(d_0, d_1, a_1, \ldots, a_{2N}, b_1, \ldots, b_{2N}) \) is abbreviated by \( \text{Pf}(d_0, d_1, \ldots) \), so as other similar pfaffians.

Furthermore, it can be shown

\[
\frac{\partial g_i}{\partial y} = \frac{\partial}{\partial y} \left[ \sum_{j=1}^{2N} (-1)^j \text{Pf}(d_0, a_j) \text{Pf}(\beta_i, \ldots, \hat{a}_j, \ldots) \right]
\]

\[
= \sum_{j=1}^{2N} (-1)^j \left[ (\partial_y \text{Pf}(d_0, a_j)) \text{Pf}(\beta_i, \ldots, \hat{a}_j, \ldots) + \text{Pf}(d_0, a_j) \partial_y \text{Pf}(\beta_i, \ldots, \hat{a}_j, \ldots) \right]
\]

\[
= \sum_{j=1}^{2N} (-1)^j \left[ \text{Pf}(d_1, a_j) \text{Pf}(\beta_i, \ldots, \hat{a}_j, \ldots) + \text{Pf}(d_0, a_j) \text{Pf}(\beta_i, d_0, d_1, \ldots, \hat{a}_j, \ldots) \right]
\]

\[
= \text{Pf}(d_1, \beta_i, \ldots) + \text{Pf}(d_0, \beta_i, d_0, d_1, \ldots)
\]

\[
= \text{Pf}(d_1, \beta_i, \ldots).
\]

Here \( \hat{a}_j \) means that the index \( j \) is omitted. Similarly, we can show

\[
\frac{\partial g_i}{\partial s} = \text{Pf}(d_{-1}, \beta_i, \ldots)
\]

An algebraic identity of pfaffian\(^{18}\)

\[
\text{Pf}(d_{-1}, \beta_i, d_0, d_1, \ldots) \text{Pf}(\ldots) = \text{Pf}(d_{-1}, d_0, \ldots) \text{Pf}(d_1, \beta_i, \ldots)
\]

\[
-\text{Pf}(d_{-1}, d_1, \ldots) \text{Pf}(d_0, \beta_i, \ldots) + \text{Pf}(d_{-1}, \beta_i, \ldots) \text{Pf}(d_0, d_1, \ldots),
\]

implies

\[
(\partial_s \partial_y g_i - g_i) \times f = \partial_s f \times \partial_y g_i - \partial_s \partial_y f \times g_i + \partial_s g_i \times \partial_y f,
\]

which is actually the first bilinear equation.
The second bilinear equation is proved in a similar way as the one used by Iwao and Hirota\cite{31} in connection with a different system. We start from the r.h.s of the bilinear equation.

\[
\frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} g_\mu g_\nu = \frac{1}{4} \sum_{1 \leq \mu, \nu \leq n} c_{\mu \nu} Pf(d_0, \beta_\mu, \cdots) Pf(d_0, \beta_\nu, \cdots)
\]

\[
= \frac{1}{4} \sum_{1 \leq \mu, \nu \leq n} c_{\mu \nu} \sum_{i,j} (-1)^{i+j} Pf(\beta_\mu, b_i) Pf(d_0, \cdots, \hat{b}_i, \cdots) Pf(\beta_\nu, b_j) Pf(d_0, \cdots, \hat{b}_j, \cdots)
\]

\[
= \sum_{i,j} (-1)^{i+j} \sum_{1 \leq \mu, \nu \leq n} \frac{1}{4} c_{\mu \nu} Pf(\beta_\mu, b_i) Pf(\beta_\nu, b_j) Pf(d_0, \cdots, \hat{b}_i, \cdots) Pf(d_0, \cdots, \hat{b}_j, \cdots)
\]

\[
= \sum_{i,j} (-1)^{i+j} (p_i^{-2} - p_j^{-2}) Pf(b_i, b_j) Pf(d_0, \cdots, \hat{b}_i, \cdots) Pf(d_0, \cdots, \hat{b}_j, \cdots)
\]  

(20)

Next, the expansion of the vanishing pfaffian \( Pf(b_i, d_0, \cdots) \) on \( b_i \) yields

\[
\sum_{j=1}^{2N} (-1)^{i+j} Pf(b_i, b_j) Pf(d_0, \cdots, \hat{b}_j, \cdots) = Pf(d_0, \cdots, \hat{a}_i, \cdots),
\]

which subsequently leads to

\[
\sum_{i,j} (-1)^{i+j} p_i^{-2} Pf(b_i, b_j) Pf(d_0, \cdots, \hat{b}_i, \cdots) Pf(d_0, \cdots, \hat{b}_j, \cdots)
\]

\[
= \sum_{i} p_i^{-2} Pf(d_0, \cdots, \hat{a}_i, \cdots) Pf(d_0, \cdots, \hat{b}_i, \cdots). \tag{21}
\]

Similarly, we can show

\[
- \sum_{i,j} (-1)^{i+j} p_j^{-2} Pf(b_i, b_j) Pf(d_0, \cdots, \hat{b}_i, \cdots) Pf(d_0, \cdots, \hat{b}_j, \cdots)
\]

\[
= \sum_{j} p_j^{-2} Pf(d_0, \cdots, \hat{a}_j, \cdots) Pf(d_0, \cdots, \hat{b}_j, \cdots). \tag{22}
\]

Substituting Eqs. (21)–(22) into Eq. (20), we arrive at

\[
\frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} g_\mu g_\nu = 2 \sum_{i} p_i^{-2} Pf(d_0, \cdots, \hat{a}_i, \cdots) Pf(d_0, \cdots, \hat{b}_i, \cdots). \tag{23}
\]
Now we work on the l.h.s. of the second bilinear equation

\[
\frac{\partial^2 f}{\partial s^2} \times 0 - \frac{\partial f}{\partial s} \frac{\partial f}{\partial s} = \text{Pf}(d_{-2}, d_0, \ldots) \text{Pf}(d_0, d_0, \ldots) - \text{Pf}(d_{-1}, d_0, \ldots) \text{Pf}(d_{-1}, d_0, \ldots)
\]

\[
= \sum_{i=1}^{2N} (-1)^i \text{Pf}(d_{-2}, a_i) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \sum_{j=1}^{2N} (-1)^j \text{Pf}(d_0, a_j) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
- \sum_{i=1}^{2N} (-1)^i \text{Pf}(d_{-1}, a_i) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \sum_{j=1}^{2N} (-1)^j \text{Pf}(d_{-1}, a_j) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
= \sum_{i,j=1}^{2N} (-1)^{i+j} \left[ \text{Pf}(d_{-2}, a_i) \text{Pf}(d_0, a_j) - \text{Pf}(d_{-1}, a_i) \text{Pf}(d_{-1}, a_j) \right]
\]

\times \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
= \sum_{i,j=1}^{2N} (-1)^{i+j+1} \left[ p_i^{-2} + p_i^{-1} p_j^{-1} \right] \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

The summation over the second term within the bracket vanishes due to the fact that

\[
\sum_{i,j=1}^{2N} (-1)^{i+j+1} p_i^{-1} p_j^{-1} \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
= \sum_{j=1}^{2N} (-1)^{j+i+1} p_i^{-1} p_j^{-1} \text{Pf}(a_j, a_i) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots)
\]

\[
= - \sum_{i,j=1}^{2N} (-1)^{i+j+1} p_i^{-1} p_j^{-1} \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

Therefore,

\[
\frac{\partial f}{\partial s} \frac{\partial f}{\partial s} = \sum_{i,j=1}^{2N} (-1)^{i+j+1} p_i^{-2} \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
= \sum_{i=1}^{2N} (-1)^{i+1} p_i^{-2} \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \left[ \sum_{j=1}^{2N} (-1)^j \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots) \right]
\]

Further, we note that the following identity can be substituted into the term within bracket

\[
\sum_{j=1}^{2N} (-1)^j \text{Pf}(a_i, a_j) \text{Pf}(d_0, \ldots, \hat{a}_j, \ldots)
\]

\[
= \text{Pf}(d_0, a_i) \text{Pf}(\ldots) + (-1)^{i+1} \text{Pf}(d_0, \ldots, \hat{b}_i, \ldots)
\]

which is obtained from the expansion of the following vanishing pfaffian \( \text{Pf}(a_i, d_0, \ldots) \) on \( a_i \).
Consequently, we have

\[-\frac{\partial f}{\partial s} \frac{\partial f}{\partial s} = \sum_{i=1}^{2N} (-1)^{i+1} p_i^{-2} \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \left[ \text{Pf}(d_0, a_i) \text{Pf}(\cdot \ldots) + (-1)^{i+1} \text{Pf}(d_0, \ldots, \hat{b}_i, \ldots) \right],\]

\[= -\text{Pf}(\cdot \ldots) \text{Pf}(d_{-2}, d_0, \ldots) + \sum_{i=1}^{2N} p_i^{-2} \text{Pf}(d_0, \ldots, \hat{a}_i, \ldots) \text{Pf}(d_0, \ldots, \hat{b}_i, \ldots),\]

\[= -\frac{\partial^2 f}{\partial s^2} f + \frac{1}{4} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} g_\mu g_\nu,\]

which can be rewritten as

\[2 \frac{\partial^2 f}{\partial s^2} f - 2 \frac{\partial f}{\partial s} \frac{\partial f}{\partial s} = \frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} g_\mu g_\nu.\]

The above equation is nothing but the second bilinear equation. Thus, the proof is complete. \(\square\)

III. SEMI-DISCRETE ANALOGUE OF THE MULTI-COMPONENT SHORT PULSE EQUATION

In this section, we attempt to construct an integrable semi-discretization of the MCSP equation (2). Firstly, we propose a semi-discrete analogue of bilinear equations (7)

\[\begin{cases}
\frac{1}{a} D_s (g_{k+1}^{(j)} \cdot f_k - g_k^{(j)} \cdot f_{k+1}) = g_{k+1}^{(j)} f_k + g_k^{(j)} f_{k+1}, & j = 1, 2, \ldots, n, \\
D_s^2 f_k \cdot f_k = \frac{1}{2} \sum_{1 \leq i < j \leq n} c_{ij} g_k^{(i)} g_k^{(j)}. & 
\end{cases}\]

By introducing a dependent variable transformation

\[u_k^{(i)} = \frac{g_k^{(i)}}{f_k}, \quad i = 1, 2, \ldots, n,\]

and a discrete version of the hodograph transformation

\[x_k = 2ka - 2(\ln f_k)_s, \quad t = s,\]

the second bilinear equation in (26) is rewritten as

\[(\ln f_k)_{ss} = \frac{1}{4} \sum_{1 \leq i < j \leq n} c_{ij} \frac{g_k^{(i)} g_k^{(j)}}{f_k^2} = \frac{1}{4} \sum_{1 \leq i < j \leq n} c_{ij} u_k^{(i)} u_k^{(j)}.\]

\[\text{(29)}\]
From the discrete hodograph transformation, we can define an nonuniform mesh

\[ \delta_k = x_{k+1} - x_k = 2a - 2 \left( \ln \frac{f_{k+1}}{f_k} \right) s, \quad (30) \]

it then immediately follows

\[ \frac{d\delta_k}{ds} = -\frac{1}{2} \sum_{1 \leq i < j \leq n} c_{ij} \left( u^{(i)}_{k+1}u^{(j)}_{k+1} - u^{(i)}_k u^{(j)}_k \right) \]

from Eq. (29). Next, dividing both sides by \( f_{k+1}f_k \), the first bilinear equation in (26) can be calculated out by

\[ \left( \frac{g^{(i)}_{k+1,s} - g^{(i)}_{k,s}}{f_{k+1}/f_k} \right) - \frac{g^{(i)}_{k+1}f_{k,s} - g^{(i)}_k f_{k+1,s}}{f_{k+1}f_k} = a \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} + \frac{g^{(i)}_k}{f_k} \right) \]

or

\[ \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} - \frac{g^{(i)}_k}{f_k} \right) \frac{1}{s} + \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} + \frac{g^{(i)}_k}{f_k} \right) \left( \frac{f_{k+1,s}}{f_{k+1}} - \frac{f_{k,s}}{f_k} \right) = a \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} + \frac{g^{(i)}_k}{f_k} \right) \]

which is recast into

\[ \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} - \frac{g^{(i)}_k}{f_k} \right) = \left( a - \left( \ln \frac{f_{k+1}}{f_k} \right) \frac{1}{s} \right) \left( \frac{g^{(i)}_{k+1}}{f_{k+1}} + \frac{g^{(i)}_k}{f_k} \right). \quad (32) \]

With the use of Eqs. (27) and (30), we finally arrive at

\[ \frac{d(u^{(i)}_{k+1} - u^{(i)}_k)}{ds} = \frac{1}{2} (x_{k+1} - x_k) (u^{(i)}_{k+1} + u^{(i)}_k). \quad (33) \]

Eqs. (31), (33) constitute the semi-discrete analogue of the MCSP equation. We summarize the results by the following Theorem.

**Theorem 2.** The bilinear equations (26) yield a semi-discrete multi-component short pulse equation

\[
\begin{cases}
\frac{d(u^{(j)}_{k+1} - u^{(j)}_k)}{ds} = \frac{1}{2} \delta_k (u^{(j)}_{k+1} + u^{(j)}_k), \\
\frac{d\delta_k}{ds} = -\frac{1}{2} \sum_{1 \leq i < j \leq n} c_{ij} \left( u^{(i)}_{k+1}u^{(j)}_{k+1} - u^{(i)}_k u^{(j)}_k \right) 
\end{cases} \quad (34)
\]

through dependent variable transformation

\[ u^{(i)}_k = \frac{g^{(i)}_k}{f_k}, \quad i = 1, 2, \cdots, n, \]

and discrete hodograph transformation

\[ x_k = 2ka - 2(ln f_k)_s, \quad t = s, \]

where \( \delta_k = x_{k+1} - x_k. \)
To assure its integrability, we provide its multi-soliton solution in terms of pfaffians by the following theorem. The elements of the pfaffians are defined as follows:

\[
Pf(a_i, a_j)_k = \frac{p_i - p_j}{p_i + p_j} \varphi^{(0)}_i(k) \varphi^{(0)}_j(k), \quad Pf(a_i, b_j)_k = \delta_{i,j},
\]

\[
Pf(b_i, b_j)_k = \frac{1}{4p_i^2 - p_j^2} (b_i \in B_\mu, b_j \in B_\nu),
\]

\[
Pf(d_i, a_i)_k = \varphi_i^{(0)}(k), \quad Pf(a_i, d^k)_k = \varphi_i^{(0)}(k + 1),
\]

\[
Pf(b_j, \beta_\mu)_k = \begin{cases} 
1 & b_j \in B_\mu \\
0 & b_j \notin B_\mu
\end{cases},
\]

\[
Pf(d_0, d^k)_k = 1, \quad Pf(d_{-1}, d^k)_k = -a,
\]

where

\[
\varphi_i^{(n)}(k) = p_i^n \left( \frac{1 + ap_i}{1 - ap_i} \right)^k e^{\xi_i}, \quad \xi_i = p_i^{-1} s + \xi_{i0}.
\]

Here \(i, j = 1, 2, \ldots, 2N, \mu, \nu = 1, 2, \ldots, n\) and \(k, l\) are arbitrary integers. Other pfaffian elements not mentioned above are all zeros. Note that \(\varphi_i^{(n)}(k)\) has the following property

\[
\frac{\varphi_i^{(n)}(k + 1) - \varphi_i^{(n)}(k)}{a} = \varphi_i^{(n+1)}(k + 1) + \varphi_i^{(n+1)}(k),
\]

which is used in the proof of the theorem.

**Theorem 3.** The bilinear equations (26) have the following pfaffian solution

\[
f_k = Pf(a_1, \ldots, a_{2N}, b_1, \ldots, b_{2N})_k,
\]

\[
g_k^{(j)} = Pf(d_0, \beta_j, a_1, \ldots, a_{2N}, b_1, \ldots, b_{2N})_k,
\]

where \(j = 1, 2, \ldots, n\), the elements of pfaffians are defined in eqs. (35)-(39).

**Proof.** Since

\[
\frac{\partial}{\partial s} Pf(a_i, a_j)_k = \varphi_i^{(0)}(k) \varphi_j^{(-1)}(k) - \varphi_i^{(-1)}(k) \varphi_j^{(0)}(k) = Pf(d_{-1}, d_0, a_i, a_j)_k,
\]

\[
Pf(a_i, a_j)_{k+1} = Pf(a_i, a_j)_k + \varphi_i^{(0)}(k + 1) \varphi_j^{(0)}(k) - \varphi_i^{(0)}(k) \varphi_j^{(0)}(k + 1)
\]

\[= Pf(d_0, d^k, a_i, a_j)_k,
\]
\[(\partial_s - a) \text{Pf}(a_i, a_j)_{k+1} \]
\[= \varphi_i^{(0)}(k + 1) \varphi_j^{(-1)}(k + 1) - \varphi_i^{(-1)}(k + 1) \varphi_j^{(0)}(k + 1)\]
\[-a \left( \text{Pf}(a_i, a_j)_k + \varphi_i^{(0)}(k + 1) \varphi_j^{(0)}(k) - \varphi_i^{(0)}(k) \varphi_j^{(0)}(k + 1) \right)\]
\[= -a \text{Pf}(a_i, a_j)_k + \varphi_i^{(0)}(k + 1) \left( \varphi_j^{(-1)}(k + 1) - a \varphi_j^{(0)}(k) \right)\]
\[= -a \text{Pf}(a_i, a_j)_k + \varphi_i^{(0)}(k + 1) \left( a \varphi_j^{(0)}(k + 1) + \varphi_j^{(-1)}(k) \right)\]
\[-\varphi_j^{(0)}(k + 1) \left( \varphi_i^{(-1)}(k + 1) - a \varphi_i^{(0)}(k) \right)\]
\[= -a \text{Pf}(a_i, a_j)_k + \varphi_i^{(0)}(k + 1) \varphi_j^{(-1)}(k) - \varphi_i^{(-1)}(k) \varphi_j^{(0)}(k + 1)\]
\[= \text{Pf}(d_{-1}, d^k, a_i, a_j)_k,\]

we have
\[\partial_s f_k = \text{Pf}(d_{-1}, d_0, \ldots)_k,\]
\[f_{k+1} = \text{Pf}(d_0, d^k \ldots)_k,\]
\[\partial_s - a) f_{k+1} = \text{Pf}(d_{-1}, d^k \ldots)_k.\]

Furthermore, we can verify
\[\partial_s g_k^{(\mu)} = \partial_s \left( \sum_{i=1}^{2N} (-1)^i \text{Pf}(d_0, a_i)_k \text{Pf}(\beta_\mu, \cdots, \hat{a}_i, \cdots)_k \right)\]
\[= \sum_{i=1}^{2N} (-1)^i \left( \partial_s \text{Pf}(d_0, a_i)_k \text{Pf}(\beta_\mu, \cdots, \hat{a}_i, \cdots)_k + \text{Pf}(d_0, a_i)_k \partial_s \text{Pf}(\beta_\mu, \cdots, \hat{a}_i, \cdots)_k \right)\]
\[= \sum_{i=1}^{2N} (-1)^i (\text{Pf}(d_{-1}, a_i)_k \text{Pf}(\beta_\mu, \cdots, \hat{a}_i, \cdots)_k + \text{Pf}(d_0, a_i)_k \text{Pf}(\beta_\mu, d_{-1}, d_0, \cdots, \hat{a}_i, \cdots)_k)\]
\[= \text{Pf}(d_{-1}, \beta_\mu, \cdots)_k + \text{Pf}(d_0, \beta_\mu, d_{-1}, d_0, \cdots)_k\]
\[= \text{Pf}(d_1, \beta_\mu, \cdots)_k,\]


\[ g_{k+1}^{(\mu)} = \sum_{i=1}^{2N} (-1)^i \text{Pf} (d_0, a_i)_{k+1} \text{Pf} (\beta\mu, \cdots, \hat{a}_i, \cdots)_{k+1} \]

\[ = \sum_{i=1}^{2N} (-1)^{i-1} \text{Pf} (d^k, a_i)_k \text{Pf} (\beta\mu, d_0, d^k, \cdots, \hat{a}_i, \cdots)_k \]

\[ = \sum_{i=1}^{2N} (-1)^{i-1} \text{Pf} (d^k, a_i)_k \left( \text{Pf} (\beta\mu, \cdots, \hat{a}_i, \cdots)_k + \sum_{j=1}^{i-1} (-1)^j \text{Pf} (\beta\mu, d^k, \cdots, \hat{a}_j, \cdots, \hat{a}_i, \cdots)_k \right) \]

\[ + \sum_{j=i+1}^{2N} (-1)^{j-1} \text{Pf} (\beta\mu, d^k, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots)_k \]

\[ = \text{Pf}(d^k, \beta\mu, \cdots)_k, \]

\[ (\partial_s - a) g_{k+1}^{(\mu)} = \sum_{i=1}^{2N} (-1)^{i-1} \left( (\partial_s - a) \text{Pf} (d^k, a_i)_k \right) \text{Pf} (\beta\mu, \cdots, \hat{a}_i, \cdots)_k \]

\[ + \sum_{1 \leq i < j \leq 2N} (-1)^{i+j-1} \left( \partial_s \text{Pf}(a_i, a_j)_k \right) \text{Pf} (\beta\mu, d^k, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots)_k \]

\[ = \sum_{i=1}^{2N} (-1)^{i-1} \left( \text{Pf}(d_{-1}, a_i)_k + a \text{Pf}(d_0, a_i)_k \right) \text{Pf} (\beta\mu, \cdots, \hat{a}_i, \cdots)_k \]

\[ + \sum_{1 \leq i < j \leq 2N} (-1)^{i+j-1} \text{Pf}(d_{-1}, d_0, a_i, a_j)_k \text{Pf} (\beta\mu, d^k, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots)_k \]

\[ = \text{Pf}(d_{-1}, \beta\mu, d_0, d^k, \cdots)_k. \]

Therefore, an algebraic identity of pfaffian

\[ \text{Pf}(d_{-1}, \beta\mu, d_0, d^k, \cdots)_k \text{Pf}(\cdots)_k = \text{Pf}(d_{-1}, d_0, \cdots)_k \text{Pf}(d^k, \beta\mu, \cdots)_k \]

\[ - \text{Pf}(d_{-1}, d^k, \cdots)_k \text{Pf}(d_0, \beta\mu, \cdots)_k + \text{Pf}(d_{-1}, \beta\mu, \cdots)_k \text{Pf}(d_0, d^k, \cdots)_k, \]

together with above pfaffian relations gives

\[ (\partial_s - a) g_{k+1}^{(\mu)} \times f_k = g_{k+1}^{(\mu)} \times \partial_s f_k - (\partial_s - a) f_{k+1} \times g_{k+1}^{(\mu)} + \partial_s g_{k+1}^{(\mu)} \times f_{k+1}, \]

which is nothing but the first bilinear equation. The second bilinear equation can be proved in a similar way as in the continuous case.

\[ \Box \]

**Remark 4. Bilinear equations (26) can be viewed as a Bäcklund transformation of bilinear equations (7), which yield the MCSP equation. In other words, if \( f_k \) and \( g_k^{(j)} \) satisfy (7), so do \( f_{k+1} \) and \( g_{k+1}^{(j)} \). Based on (26), we propose a semi-discrete analogue of the MCSP equation. The integrability of the semi-discrete MCSP equation is guaranteed by the existence of \( N \)-soliton solution.**
Finally, let us show that in the continuous limit, \( a \to 0 \) \( (\delta_k \to 0) \), the proposed semi-discrete multi-component short pulse equation recovers the continuous one (13). The dependent variable \( u \) is regarded as a function of \( x \) and \( t \), where \( x \) is the space coordinate of the \( k \)-th lattice point and \( t \) is the time, defined by

\[
x = x_0 + \sum_{j=0}^{k-1} \delta_j, \quad t = s,
\]

where \( \delta_j = x_{j+1} - x_j \). In the continuous limit, \( a \to 0 \) \( (\delta_k \to 0) \), we have

\[
\frac{1}{2} (u_{k+1}^{(i)} + u_k^{(i)}) \to u_i, \quad \frac{\partial_s (u_{k+1}^{(i)} - u_k^{(i)})}{\delta_k} \to u_{i,xx},
\]

\[
\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{k-1} \frac{\partial \delta_j}{\partial s}
\]

\[
= \frac{\partial x_0}{\partial s} - \frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} \sum_{j=0}^{k-1} \left( u_{j+1}^{(\mu)} u_{j+1}^{(\nu)} - u_j^{(\mu)} u_j^{(\nu)} \right)
\]

\[
\to -\frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} u_{\mu} u_{\nu},
\]

hence

\[
\partial_s = \partial_t + \frac{\partial x}{\partial s} \partial_x \to \partial_t - \frac{1}{2} \left( \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} u_{\mu} u_{\nu} \right) \partial_x,
\]

where the origin of space coordinate \( x_0 \) is taken so that \( \frac{\partial x_0}{\partial s} \) cancels \( \frac{1}{2} \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} u_{0}^{(\mu)} u_{0}^{(\nu)} \). Thus the first semi-discrete multi-component SP equation converges to

\[
\partial_x \left( \partial_t - \frac{1}{2} \left( \sum_{1 \leq \mu < \nu \leq n} c_{\mu \nu} u_{\mu} u_{\nu} \right) \partial_x \right) u_i = u_i,
\]

which is exactly the MCSP equation (2).

IV. TWO-COMPONENT SHORT PULSE EQUATION

Since the two-component short pulse equation is of particular importance for applications in nonlinear optics, we provide a detailed study for this two-component system, together with its semi-discrete analogue in this section. For the continuous case of \( n = 2 \), we can take \( c_{12} = 1 \) without loss of generality and arrive at the following two-component system\(^{24}\)

\[
u_{xt} = v + \frac{1}{2} (uvw)_x, \quad \text{(43)}
\]
where \( u = u_1, v = u_2 \). Furthermore, if we assume \( u \) is a complex-values function and impose a complex conjugate condition \( v = \bar{u} \), where \( \bar{u} \) means the complex conjugate of \( u \). Eq. \( (42) \) leads to a \textbf{complex short pulse equation} studied in\textsuperscript{27,33}

\[
  u_{xt} = u + \frac{1}{2} \left( |u|^2 u_x \right)_x . 
\]  

(44)

Since the complex short pulse equation \( (44) \) is a special case of two-component system \( (42)-(43) \), its \( N \)-soliton solution can be obtained from the \( N \)-soliton solution of system \( (42)-(43) \) by requiring \( f = \bar{f}^{(1)} = \bar{f}^{(2)} \). These requirements can be achieved by putting \( p_k = \bar{p}_j \) and \( \xi_k = \bar{\xi}_j \), where \( k = j + N, j = 1, 2, \ldots, N \).

In particular, the tau-functions for one-soliton solution \( (N = 1) \) are found to be

\[
  f = -1 - \frac{1}{4} \frac{(p_1 \bar{p}_1)^2}{(p_1 + \bar{p}_1)^2} e^{m + \bar{m}}, 
\]  

(45)

\[
  g = -e^{m}. 
\]  

(46)

Let \( p_1 = p_{1R} + ip_{1I} \), and we assume \( p_{1R} > 0 \) without loss of generality, then the one-soliton solution can be expressed in the following parametric form

\[
  u = \frac{2p_{1R}}{|p_1|^2} e^{i\eta_1} \sech \left( \eta_{1R} + \eta_{10} \right), 
\]  

(47)

\[
  x = y - \frac{2p_{1R}}{|p_1|^2} \left( \tanh \left( \eta_{1R} + \eta_{10} \right) + 1 \right), \quad t = s, 
\]  

(48)

where

\[
  \eta_{1R} = p_{1R} y + \frac{p_{1R}}{|p_1|^2} s, \quad \eta_{1I} = p_{1I} y - \frac{p_{1I}}{|p_1|^2} s, \quad \eta_{10} = \ln \left( \frac{|p_1|^2}{4p_{1R}} \right). 
\]  

(49)

Eq. \( (47) \) represents an envelope soliton of amplitude \( 2p_{1R}/|p_1|^2 \) and phase \( \eta_{1I} \). The details analysis concerning its property was carried out in\textsuperscript{27}. In summary three types can be classified.

- when \( |p_{1R}| < |p_{1I}| \), it is a \textbf{smooth soliton} solution, which is similar to the envelope soliton solution for the nonlinear Schrödinger equation.

- when \( |p_{1R}| > |p_{1I}| \), it is a \textbf{loop soliton} solution, which admits multi-valued property.

- when \( |p_{1R}| = |p_{1I}| \), it is a \textbf{cuspon soliton}, this is a case which divides the single-valued and multi-valued solution.
A. Semi-discrete two-component system

Based on the results in previous section, we have an integrable semi-discrete analogue of two-component system (42)–(43)

\[
\begin{cases}
\frac{d(u_{k+1} - u_k)}{dt} = \frac{1}{2} \delta_k (u_{k+1} + u_k), \\
\frac{d(v_{k+1} - v_k)}{dt} = \frac{1}{2} \delta_k (v_{k+1} + v_k), \\
\frac{d\delta_k}{dt} = -\frac{1}{2} (u_{k+1} v_{k+1} - u_k v_k),
\end{cases}
\]  

(50)

which admit the following \(N\)-soliton solution in parametric form

\[
\begin{align*}
    u_k &= \frac{g^{(1)}_k}{f_k}, \\
    v_k &= \frac{g^{(2)}_k}{f_k}
\end{align*}
\]

(51)

and hodograph transformation

\[
x_k = 2ka - 2(\ln f_k)s, \quad t = s,
\]

(52)

with

\[
f_k = \text{Pf}(a_1, \cdots, a_{2N}, b_1, \cdots, b_N, c_1, \cdots, c_N)_k,
\]

(53)

\[
g^{(i)}_k = \text{Pf}(\beta_i, d_0, a_1, \cdots, a_{2N}, b_1, \cdots, b_N, c_1, \cdots, c_N)_k, \quad i = 1, 2
\]

(54)

where the elements of pfaffians are (others not mentioned are all zeros)

\[
\text{Pf}(a_i, a_j)_k = \frac{p_i - p_j}{p_i + p_j} \varphi_i^{(0)}(k)\varphi_j^{(0)}(k), \quad \text{Pf}(a_i, b_j)_k = \delta_{i,j},
\]

\[
\text{Pf}(b_i, c_j) = -\frac{1}{4} \frac{(p_ip_{N+j})^2}{p_i^2 - p_{N+j}^2}, \quad \text{Pf}(a_i, c_j)_k = \delta_{i,j+N},
\]

\[
\text{Pf}(b_i, \beta_j) = \text{Pf}(c_i, \beta_2) = 1, \quad \text{Pf}(d_0, a_i)_k = \varphi_i^{(0)}(k).
\]

By imposing complex conjugate conditions \(p_k = \bar{p}_j, \xi_{k0} = \bar{\xi}_{j0} (k = j + N, j = 1, 2, \cdots, N)\), it then follows \(f_k = \bar{f}_k, g^{(1)}_k = \bar{g}^{(2)}_k\), thus \(u_k = \bar{v}_k\), which leads to a semi-discrete analogue of the complex short equation (44)

\[
\begin{cases}
\frac{d(u_{k+1} - u_k)}{dt} = \frac{1}{2} \delta_k (u_{k+1} + u_k), \\
\frac{d\delta_k}{dt} = -\frac{1}{2} (|u_{k+1}|^2 - |u_k|^2).
\end{cases}
\]

(55)
Its $N$-soliton solution immediately follows from the $N$-soliton solution of Eqs. (53)–(54) under complex conjugate conditions mentioned above. In what follows, we list the one- and two-soliton solutions.

**One-soliton solution:** The tau-functions for one-soliton solution to Eq. (55) are

$$f_k = -1 - \frac{1}{4} \left( \frac{p_1 p_2}{p_1 + p_2} \right)^2 \varphi^{(0)}_{12} (k), \quad g_k^{(1)} = -\varphi_1^{(0)} (k),$$

with

$$\varphi^{(0)}_i (k) = \left( 1 + ap_i \right)^k e^{-i \xi a}, \quad \varphi^{(0)}_{ij} (k) = \varphi^{(0)}_i (k) \varphi^{(0)}_j (k), \quad i, j = 1, 2,$$

where $p_1 = \bar{p}_2$, $\xi_{10} = \bar{\xi}_{20}$. Similar to the continuous case, if $p_1 = p_{1R} + ip_{1I}$, we then arrive at the one-soliton solution of semi-discrete complex short pulse equation (55)

$$u_k = \frac{2p_{1R}}{|p_1|^2} e^{i \chi_k} \text{sech}(\theta_k + \theta_0),$$

$$x_k = 2ka - \frac{2p_{1R}}{|p_1|^2} (\tanh(\theta_k + \theta_0) + 1),$$

where

$$\theta_k = kd_1 + \frac{p_{1R}}{|p_1|^2} s, \quad \chi_k = kd_2 - \frac{p_{1R}}{|p_1|^2} s, \quad 1 + \frac{ap_1}{1 - ap_1} = e^{d_1 + id_2}.$$ 

In Fig. 1 (a)–(c), we illustrate the envelope soliton for $p_1 = 1 + 1.5i$, $1 + i$, $1 + 0.5i$, which correspond to the smooth, cuspon and loop soliton, respectively in the continuous case. **Two-soliton:** The tau-functions for two-soliton solutions to semi-discrete complex short pulse equation (55) can be obtained by

$$f_k = 1 + a_{13} \varphi^{(0)}_{13} (k) + a_{14} \varphi^{(0)}_{14} (k) + a_{23} \varphi^{(0)}_{23} (k) + a_{24} \varphi^{(0)}_{24} (k)$$

$$+ 16a_{12}a_{34}a_{14}a_{23} (p_1^{-1} - p_2^{-1})^2 (p_3^{-1} - p_4^{-1})^2 \varphi^{(0)}_{12} (k) \varphi^{(0)}_{34} (k),$$

$$g_k^{(1)} = \varphi_1^{(0)} (k) + \varphi_2^{(0)} (k) + 4 \left( a_{13} a_{23} \varphi^{(0)}_{3} (k) + a_{14} a_{24} \varphi^{(0)}_{4} (k) \right) \left( p_1^{-1} - p_2^{-1} \right)^2 \varphi^{(0)}_{12} (k).$$

where

$$a_{ij} = \frac{p_i - p_j}{p_i + p_j}, \quad p_1 = \bar{p}_3, \quad p_2 = \bar{p}_4$$

and $\xi_{10} = \bar{\xi}_{30}$ and $\xi_{20} = \bar{\xi}_{40}$. In Fig. 2 (a)–(d), we show the process of interaction between a smooth envelop soliton and a cuspon envelop soliton with $p_1 = 1 + 1.5i$, $p_2 = 1 + i$, $\xi_{10} = -15$, $\xi_{20} = -25$.
FIG. 1. Envelope soliton for the semi-discrete complex short pulse equation (a) smooth soliton with $p_1 = 1 + 1.5i$, (b) cuspon soliton with $p_1 = 1 + i$, (c) loop solitn with $p_1 = 1 + 0.5i$.

V. CONCLUDING REMARKS

We have derived an integrable semi-discrete analogue of the multi-component short pulse equation proposed by Matsuno\textsuperscript{24} based on a Bäcklund transform and Hirota’s bilinear method. We find its $N$-soliton solution in terms of pfaffians and prove it. Moreover, a complex short pulse equation, which possess smooth, cuspon or loop type envelop soliton, is proposed and its semi-discrete analogue is constructed as well. We conclude the present paper by the following remarks.

- The $N$-solution for multi-component short pulse equation given in the present paper agrees with the one given by Matsuno in\textsuperscript{24}. This solution is a benchmark for the study of soliton interactions.

- Similar to our previous results\textsuperscript{19,33,34}, the semi-discrete multi-component short pulse equa-
The integrable fully discretization of the multi-component short pulse equation is a further topic deserve to study.

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