Precise adiabatic transport and geometry of quantum Hall states

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We argue that in addition to the Hall conductance and the non-dissipative component of the viscous tensor there exists a third independent kinetic coefficient which is precise on the quantum Hall plateaus. We show that the new coefficient is the Chern number over moduli space of surfaces of genus two or higher and therefore is precise. As such it does not transpire on a sphere or a torus. In the linear response theory this coefficient determines intensive forces exerted on electronic fluid by adiabatic deformations of geometry and represents the effect of the gravitational anomaly. We also present the method of computing the transport coefficients for quantum Hall states.

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1. Introduction Quantum Hall states are distinguished by a precise quantization of the Hall conductance in materials with imprecise characteristics. A natural question is whether the Hall conductance is a unique precise characteristic. Are there any other independent transport coefficients which are also precise on the QH-plateau?

Precise quantization in materials occurs when the transport is a conservative adiabatic process. Quantum Hall states is a system where adiabatic conditions are in place. There the ground state is separated by a spectral gap (in the fractional case) or by a cyclotron energy (in the integer case) from the rest of the bulk spectrum so that an adiabatic change of parameters instigates a flux without igniting bulk excited states.

From this standpoint the quest for the quantized transport is a search for independent adiabatic parameters. However, not all adiabatic processes yield the precise transport, but only those with non-trivial first Chern class. We argue that apart from the Hall conductance, QH states are characterized by two more coefficients possessing the same degree of universality, albeit up to now only the former is experimentally accessible. One such coefficient, known for quite some time, is the non-dissipative component of viscous tensor introduced in \cite{1,2}. Indications at the existence of another precise coefficient appeared recently in connection with the gravitational anomaly \cite{3–11}.

In this paper we discuss the precise transport in QH-states on compact surfaces. Here we present a general method to compute all three transport coefficients at once, with the emphasis on the third coefficient, which is most subtle. We explain which topological invariants are responsible for their preciseness. Our method sheds light on the relations between the adiabatic transport in QHE and its counterparts in conformal field theory. Our analysis is based on the fundamental principle behind the precise adiabatic transport. That is the holomorphic nature of QH-states.

Non-dissipative transport coefficients could be understood from two complementary points of view. First one is through their relation to topological invariants, such as Chern numbers of a vector bundle over the parameter space \cite{12,13}. This relation prompts the precise quantization on QH plateaus and suggests a stability to a model wave function chosen for calculations.

The parameter space for the Hall conductance is known to be the space of Aharonov-Bohm fluxes piercing through the handles of the surface. For the non-dissipative viscosity, as well as for the third transport coefficient, the relevant parameters are complex moduli. However, there is an important distinction – while the viscosity coefficient can be expressed as a rational first Chern number already on the moduli space of the torus \cite{1,2}, the preciseness of the third coefficient transpires on surfaces of genus 2 and higher \cite{14}. We show this in the paper.

The second view is via the linear response theory. Although the linear response does not establish quantization \cite{12,13}, it often provides a clearer physical interpretation. As such, the third coefficient describes a part of non-dissipative viscosity, which does not depend on the fluid density – an analog of Casimir forces.

2. Electromotive adiabatic transport An example illustrating the quantization of a non-dissipative adiabatic transport and its relation to the linear response theory goes back to papers \cite{12,13,13}. The QHE can be abstracted as a charge transport on a torus with Aharonov-Bohm (AB) fluxes $\varphi_a$ and $\varphi_b$ along its cycles. In the absence of a dissipative diagonal components of the conductance matrix, the electromotive force (emf) $\varphi_b$ drives the current along the other cycle $I_a = \frac{1}{2\pi} \sigma \varphi_b$. An adiabatic increase of the AB-flux by the flux unit $h/e$ transports the charge $Q_a = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \sigma d\varphi_b$. The transported charge defines the adiabatic transport coefficient $\sigma_H = Q_a$ as an average of the Hall conductance over the flux period $\sigma_H = \frac{1}{2\pi} \int_0^{2\pi} \sigma d\varphi_b$. More generally the adiabatic transport is defined as an average of the non-dissipative conductance 2-form \cite{13}

$$ \Omega = \frac{1}{2\pi} \sigma \delta \varphi_b \wedge \delta \varphi_a, $$

where $\sigma = \sigma(\varphi)$, over the entire manifold of parameters.
In this case, a torus $T_\phi : 0 \leq \phi_a, \phi_b < 2\pi$

$$\sigma_H = \frac{1}{2\pi} \int_{T_\phi} \Omega$$  \hspace{1cm} (2)

Following the well known arguments of \cite{1, 2, 13, 14}, the conductance 2-form (1) is proportional to the adiabatic curvature

$$\hbar \Omega = -i\nu \text{tr}(\delta \psi | \delta \psi),$$  \hspace{1cm} (3)

where we trace over all normalized degenerate states. On the torus, this number is the inverse filling fraction $1/\nu$. Then the Hall conductance $\sigma_H$ equals $\nu$ times the first Chern number of the vector bundle over the parameter space. Since the Chern number is integer it follows that the conductance is quantized in units of the filling fraction $\nu$. Throughout the paper we adopt the units in which adiabatic parameters, transport coefficients and adiabatic curvature are dimensionless.

A subtle difference between adiabatic transport and the conductance matrix has been emphasized in \cite{12, 17}: while the conductance 2-form (1) may fluctuate in mesoscopic systems, the adiabatic transport does not. The conductance 2-form (1) consists of a universal part that saturates the adiabatic transport \cite{2}, and a non-universal exact form which does not affect it. We emphasize the difference by labeling the precise adiabatic transport by a subscript $H$, like $\sigma_H$ in (2) to distinguish it from, a non-precise linear response coefficient $\sigma$ in (1).

This point reflects the difference of recently developed approaches of ”effective action” \cite{4, 8, 11} and the generating functional \cite{5, 7, 9} with the adiabatic transport \cite{2}. The effective action, or adiabatic phase, is given by the integral of the adiabatic curvature \cite{4} over surface in parameter space enclosed by the adiabatic process, so that the entire conductance form (1) is relevant. Conversely, the adiabatic transport is given by the integration of the adiabatic curvature over a closed 2-cycle as in (2), and therefore is determined by the universal part of the conductance form $\frac{1}{\pi} \sigma_H \delta \phi_b \wedge \delta \phi_a$.

3. Geometric adiabatic transport Apart from the charge transport driven by emf, another set of adiabatic parameters comes from geometry. In seminal papers Avron, Seiler and Zograf \cite{1} and Lévy \cite{2} computed the adiabatic transport caused by a deformation of the complex modulus $\tau = \tau_1 + i\tau_2$ of the torus. The latter defines the complex structure through the conformal coordinate $z = x + iy$. In this coordinate system the metric has the form $ds^2 = g_{zz}|dz|^2$, where $g_{zz} = \frac{V}{\tau_2}$ and $V$ is the area of the surface.

An infinitesimal change of the modulus $\tau \to \tau + \delta \tau$ preserves the area. Then the metric transforms as

$$\delta (ds^2) = \delta g_{zz}|dz|^2 + \delta g_{zz}(dz)^2 + \delta g_{z\bar{z}}(d\bar{z})^2,$$

where

$$g_{zz}^{-1} \delta g_{zz} = 2|\delta \mu|^2, \quad g_{zz}^{-1} \delta g_{z\bar{z}} = \delta \mu, \quad g_{zz}^{-1} \delta g_{\bar{z}z} = \delta \mu, \quad (4)$$

where $\delta \mu$ is called Beltrami differential. In the case of the torus $\delta \mu = \frac{i\delta \tau}{2\tau_2}$ is uniform.

According to \cite{1, 2} for the torus the adiabatic curvature is proportional to invariant area form on the moduli space

$$\Omega = -2i\eta_H (\delta \mu \wedge \delta \mu), \quad \delta \mu = \frac{i\delta \tau}{2\tau_2}, \quad (5)$$

where $\eta_H$ is, yet another, universal transport coefficient.

The authors of \cite{1} interpreted the transport coefficient $(\hbar/V)\eta_H$ as a non-dissipative viscosity.

The computations of \cite{1, 2} have been carried out for the integer QHE and on the torus. They have been extended in \cite{15, 16} to fractional QH-states states, also on the torus. It was shown that on the torus the transport coefficient $\eta_H$ is extensive, i.e., proportional to the number of flux quanta $N_F = \frac{1}{2\pi} \int BdV$

$$\text{torus: } \eta_H = \varsigma_H N_F, \quad (6)$$

For the Laughlin states the coefficient was found to $\varsigma_H = 1/4$ independent from the fraction.

The proper domain $\mathcal{M}$ of the parameter space is determined by the modular transformation properties of the quantum state. For the Laughlin states the relevant transformation is the congruence subgroup (usually called $\Gamma(2)$), generated by the transformations $\tau \to \tau + 2$ and $\tau \to -\frac{1}{\tau}$. This group acts in a unitary fashion, and the parameter space is the fundamental domain of this subgroup. This space, an orbifold, is 3 times larger than fundamental domains of $SL(2, \mathbb{Z})$. The integral of the adiabatic curvature \cite{5} over this space is the Chern number

$$\frac{1}{2\pi} \int_{\mathcal{M}} \Omega = -\frac{\eta_H}{\pi} \text{vol} \mathcal{M} \quad (7)$$

and the volume of $\mathcal{M}$ is $\text{vol} \mathcal{M} = i \int d\mu \wedge \delta \bar{\mu} = \pi$. Albeit non-integer it is a topological invariant which warrants the preciseness of $\eta_H$.

Now we turn to another universal coefficient, also related to deformations of geometry. We will show that the relation (7) acquires an intensive quantum correction \cite{9}, which cannot be read off from the torus. It becomes visible only on surfaces with genus two and higher. Relation (7) establishes its preciseness. The integer QHE on compact surfaces with a constant negative curvature was first studied in the important paper of Lévy \cite{21}. We first state the main result and then sketch its derivation.

4. Geometric adiabatic transport - the main result We recall the basic notions of the moduli space of complex structures \cite{21}. We consider deformations of the metric \cite{4} which exclude diffeomorphisms. Such deformations are given by holomorphic differentials in the sense

$$\partial_z (g_{zz} \delta \bar{\mu}) = 0.$$  \hspace{1cm} (8)

For surfaces of a genus $g \geq 2$ there are $3g - 3$ independent holomorphic differentials $\eta_i$. Accordingly, the Beltrami differential $\delta \mu = \frac{1}{2\tau_2} \sum_{i=1}^{3g-3} \eta_i \delta y_i$ is characterized by $3g - 3$ complex coordinates $\delta y_1, \ldots, \delta y_{3g-3}$ on the
tangent space to the moduli space. On the torus there is only one complex modulus. There is none on the sphere.

We recall the notion of the Weil-Petersson form on the moduli space. It is the form invariant with respect to a coordinate choice of the moduli space

\[ \Omega_{WP} = i \int_{\Sigma} (\delta \mu \wedge \delta \bar{\mu}) dV. \]

Here \( dV = g_{zz} dz d\bar{z} \) is the volume element of the surface \( \Sigma \). We will show that the universal part of the adiabatic curvature of QH-states on the moduli space is

\[ \Omega = -2 \eta_H \Omega_{WP}, \quad \eta_H = \zeta_H N \Phi - \frac{c_H}{12} \chi(\Sigma), \quad (9) \]

where \( c_H \) is a new precise transport coefficient, and \( \chi(\Sigma) = 2 - 2g \) is the Euler characteristic of the surface.

We list the value of all three precise coefficients for the spin-\( j \) Laughlin states defined in \([7, 9]\)

\[ \sigma_H = \nu, \quad \zeta_H = \frac{1}{4}(1 - 2j\nu), \quad c_H = 1 - \frac{3}{\nu}(1 - 2j\nu)^2 \quad (10) \]

and compute them below at once. We comment that methods of \([5, 7, 9]\) allow to compute these coefficients for other FQH states. Later in the paper we identify the coefficient \( \zeta_H \) and \( c_H \) with the background change and the central charge of the relevant conformal field theory. Notice that the value of \( c_H \) for \( \nu = 1/3 \) Laughlin state is \( c_H = -8 \), and that \( \zeta_H \) may have any sign and even vanish for spin-\( j \) states \([22]\).

The formula \((10)\) generalizes the result of \([1, 2, 18–20]\) to fractional QH states on an arbitrary surface. Note that as the adiabatic transport, the coefficient \( c_H \) cannot be seen on the torus, since Eq. \((9)\) then reduces to \((6)\), \((23)\).

In \([19]\) it was argued that the extensive part \( \zeta_H N \Phi \) of \( \eta_H \) in \((1)\) is linked to the difference between the admissible number of electrons and the magnetic flux \([21]\),

\[ N = \sigma_H N \Phi + 2 \zeta_H \chi(\Sigma). \quad (11) \]

With the help of \((11)\) we can also write the non-dissipative viscosity coefficient in \((9)\) as

\[ \eta_H = \frac{1}{4\nu}(1 - 2j\nu)N - \frac{\chi(\Sigma)}{12}. \quad (12) \]

We observe that a kinematic viscosity \( h \eta_H / N \) receives a universal finite size correction \(-\chi(\Sigma)/12\). This is analogous to the Casimir effect, where forces posses a contribution not proportional to volume. The origin of this correction is the gravitational anomaly. We show how to obtain it in the end of the paper.

The same arguments as in Sec.3 establish the preciseness of the coefficient \( c_H \). Since the integral of the left hand side of Eq. \((9)\) over any closed 2-cycle in the moduli space is a topological invariant and the volume of these cycles in Weil-Petersson metric is in general, a rational number \([22]\), the coefficients \( \zeta_H \) and \( c_H \) are precise \([20]\).

We emphasize that deformations of the metric which do not change the moduli, such as a deformation of the conformal factor \( g_{zz} \) or diffeomorphisms do not incur any precise adiabatic transport. Same holds for the variations of gauge potential, which leave the total flux and AB-fluxes intact. Unlike the geometric transport, the emf transport on higher genus surfaces is a direct generalization of \((11)\), therefore it does not yield new precise coefficients. The two transports decouple \([27]\).

5. Defining relation for holomorphic states

The foundational principle behind the precise adiabatic transport is the holomorphic properties of states on the lowest Landau level.

The space of these states written in complex coordinates (where the metric is \( ds^2 = g_{zz} |dz|^2 \)) is spanned by one-particle states annihilated by the operator

\[ D^\dagger = g_{zz}^{-1/2} (-i\partial_{\bar{z}} - A_\bar{z} + j \omega_\bar{z}). \]

Here \( A_\bar{z} \) and \( \omega_\bar{z} \) are complex components of the gauge field and the spin connection, and \( j \) is the spin of the state. The states are holomorphic functions of particle coordinates if the gauge field and the spin connection are treated as adiabatic parameters. But there is more to it. Unnormalized states are holomorphic functions of properly chosen adiabatic parameters, in our case they are the moduli.

Under a deformation of metric \((1)\) the operator \( D^\dagger \) deforms holomorphically with \( \mu \) as \( \delta D^\dagger = \delta \mu D^\dagger \) and do so unnormalized wave-functions. The unnormalized states depend only on \( \mu \) not on \( \bar{\mu} \)

\[ \psi(z_1, \ldots, z_N | \mu) = (Z[\mu, \bar{\mu}])^{-\frac{1}{2}} F(z_1, \ldots, z_N | \mu). \quad (13) \]

For genus \( g \geq 1 \) fractional QH-states are degenerate. On the torus the degenerate Laughlin states form a unitary representation of the modular group, which may not be the case for more structured states. Therefore in the Laughlin case the modular invariant normalization factor is the same for each state. We assume this property to hold also for Laughlin states on higher genus surfaces.

Under this assumption the common normalization factor determines the adiabatic curvature

\[ \Omega = \int_{\Sigma} (d\bar{d} \log Z) dV, \quad (14) \]

where \( d = \delta \mu \frac{\partial}{\partial \mu} \) and \( \bar{d} = \delta \bar{\mu} \frac{\partial}{\partial \bar{\mu}} \) and similar for AB fluxes. The formula \((14)\) follows directly from the definition \((9)\) and the property \((13)\).

The defining relation \((14)\) is valid for any states with the holomorphic dependence on complex parameters. Such states occur in a broad scope of physical systems, notably in conformal field theory (e.g., \([28, 29]\)).

7. Generating functional

The generating functional for the Laughlin states has been obtained in Ref. \([3]\) (cf. \([5, 7]\)). It consists of two parts

\[ \log Z = \log Z_H + \mathcal{F}[B, R]. \quad (15) \]

The first term is the bilinear form of the gauge and spin connections. The second term is a local functional of the magnetic field, scalar curvature and their derivatives.
We recall that the spin connection is defined through the relation \( de = \omega \wedge e \), where \( e \) the zweibein. In complex coordinates \( e = \sqrt{g_{zz}} (dz + \mu dz) \). The exterior derivatives of spin connection is the (scalar) curvature \( d \omega = \frac{1}{2} RdV \), similarly \( dA = BdV \).

We write the precise part arranging \( z \)-components of connections to a vector and row-column and assume the transversal gauge \( \partial_z A_z = -\partial_z A_z, \quad \partial_z \omega_z = -\partial_z \omega_z \).

\[
\log Z_H = \frac{2}{\pi} \int (A_z \omega_z) \left( \sigma \frac{2 \kappa_H}{2 \kappa_H - \kappa_H} \right) dz d\bar{z}.
\]

(16)

Now we posses all necessary data to compute the adiabatic transport. The variation over AB-fluxes varies the functional (15) encodes the linear response. In order to obtain it we choose \( A_z \) and \( \omega_z \) components of connections to be holomorphic adiabatic parameters and compute the conductance matrix. The differential in (14) becomes

\[
d = \omega_{z} \frac{\delta}{\delta \omega_{z}} + \delta A_z \frac{\delta}{\delta A_z} \text{ and the conductance 2-form reads}
\]

\[
\Omega = \frac{2}{\pi} \int \left( \delta A_z \delta \omega_z \right) \left( \sigma \frac{2 \kappa}{\kappa - \frac{12}{2}} \right) dz d\bar{z}.
\]

(19)

The conductance matrix in (19) is the Hessian

\[
\sigma = \frac{\pi}{2} \delta^2 \log Z = \frac{2 \pi}{2} \log Z, \quad 2 \kappa = \frac{\pi}{2} \delta^2 \log Z - \frac{c}{12} = \frac{\pi}{2} \delta^2 \log Z.
\]

We used a short cut notation for a non-uniform conductances e.g., \( \sigma(\xi, \xi') = \frac{\delta}{\delta \xi} \delta \log Z \).

With the help of this formulas we can find the current and the stress incurred by the external forces. In complex coordinates where the stress is \( \sigma_{ij} dx^i dx^j = \pi_{zz}(dz)^2 + \pi_{\bar{z} \bar{z}}(d\bar{z})^2 \), the response is

\[
I_z = \frac{\pi}{2} \frac{d}{dt} \frac{\delta \log Z}{\delta A_z}, \quad \pi_{zz} = \hbar \frac{\pi}{2} \frac{d}{dt} \frac{\delta \log Z}{\delta \omega_z}.
\]

(20)

The term \( \log Z_H \) in (15) determines the precise part of the transport. The non-uniform, non-universal part comes from \( F[B, R] \). It can be explicitly computed from \( F[B, R] \) but vanishes under space-averaging. We observe that averaging over the space of parameters (2) or space averaging yield the same results \( \sigma_H = V^{-2} \int \sigma(\xi, \xi') dV \).

The universal parts of currents and stress caused by the electric and gravitational fields for the Laughlin state follow from (20) and (19)

\[
I_z = i(\nu E_z + \xi_z), \quad \pi_{zz} = -i \frac{\hbar}{4 \nu} \partial_z I_z - \frac{\hbar}{12} \partial_z \xi_z.
\]

(21)

These formulas extend the notion of the Hall conductance: the e.m. current is a sum of Lorentz forces caused by the electric and the gravitational fields. The last term in (21) appears because the conductance matrix is non-degenerate. It has the same origin as the finite size correction in \( \nu \) and is, yet another manifestation of the gravitational anomaly.

Using the continuity equation \( \dot{\rho} + \nabla \cdot I = 0 \), we obtain the extension of the Str"edel formula \( \rho = \frac{1}{2 \pi} \left[ \sigma_H B + \xi_H R \right] \) connecting electronic density to the magnetic field and the curvature. Integrating the density with the help of the Gauss-Bonnet formula \( \int_{\Sigma} R = 4 \pi \chi(\Sigma) \) we obtain the relation (11) connecting \( \chi_{\Sigma} \) to the number of particles and the number of fluxes.

7. Linear response The Hessian of \( \log Z \) with respect to adiabatic parameters is the generalized conductance matrix describing a linear response of QH-states to emf and geometric forces. Consider a general process where connections evolve along a (non-closed) path \( A(t), \omega(t) \) but keep the complex structure unchanged. Such process produces an electric field \( E_z = \dot{A}_z \) and its gravitational counterpart \( \xi_z = \frac{\dot{\omega}_z}{2} \), which in their turn cause an electric current \( I_z \) and a stress \( \pi_{zz} \). The generating functional (13) encodes the linear response. In order to obtain it we choose \( A_z \) and \( \omega_z \) components of connections
We seek a conformal field theory which represents the unnormalized part of the Laughlin wave function. Since this state consists of one type of particles it is described by one Gaussian field \( \Phi \) coupled to magnetic field and curvature

\[
S[\Phi] = \frac{\sigma_H}{4\pi} \int (\nabla \Phi)^2 dV + \frac{i}{2\pi} \int (\sigma_H B + \zeta_H R) \Phi dV. \tag{22}
\]

The couplings \( \sigma_H, \zeta_H \) are fixed by the requirements:

(i) An electron in the Laughlin state to be represented by a holomorphic primary operator \( V(z) \) with the electric charge 1. This fixes the vertex operator \( e^{i\Phi(z, \bar{z})} = V(z)V(\bar{z}) \) and also fixes the coupling to the gauge field.

(ii) The OPE of two electronic operators requires to be \( V(z_1)V(z_2) \sim (z_1 - z_2)^{1/\nu} \). This condition determines \( \sigma_H = \nu \) in (22).

(iii) A particle in the spin-\( j \) Laughlin state has conformal spin \( j \). This is a generalization of the traditional \( j = 0 \) Laughlin state \([7, 9]\). Since the state is holomorphic, its conformal dimension also equals to \( j \). We recall that the conformal dimension of the vertex operator \( e^{ia\Phi} \) with respect to the action (22) is

\[
h_a = \frac{a}{2\sigma_H}(a - 2\sqrt{\chi_H}).
\]

Choosing \( a = 1 \) and \( h_1 = j \) we obtain \( \chi_H = \frac{j}{4}(1 - 2j\nu) \) as in (10) These conditions fix the parameters of the spin-\( j \) Laughlin state. The central charge of such theory is given by (10), (33)

\[
d_H = 1 - 48\frac{\Delta^2}{\sigma_H}.
\]

Now let us compute the unnormalized correlation function of a string of vertex operators \( e^{i\Phi} \), following the method of [7]. The result of the calculations is

\[
\int \left[ \prod_{i=1}^{N} e^{iD(z_i, \bar{z}_i)} \right] e^{-S[\Phi]} D\Phi = Z_G|F(z_1, \ldots, z_N)|^2, \tag{23}
\]

where \( F \) is the unnormalized Laughlin wave-function. For example, on the sphere it reads

\[
F(z_1, \ldots, z_N) = \prod_{i < j}^{N} (z_i - z_j)^{\frac{1}{2}j^2} e^{\sum_{i=1}^{N} Q(z_i, \bar{z}_i)}, \tag{24}
\]

where the potential \( Q \) is defined as \( \partial_i Q = 2i(A_z - j\omega_z) \).

The factor \( Z_G \) in (23) is given by

\[
Z_G = \left| \frac{\text{Det}(-\Delta)}{\pi} \right|^{-\frac{1}{2}} e^{-\pi^2 H_1(\nu, 2j\nu)} |F(z_1, \ldots, z_N)|^2 dz d\bar{z},
\]

where \( \text{Det}(-\Delta) \) is the spectral determinant of the Laplace-Beltrami operator.

The next step is to integrate over positions of particles and use the relation \( \int |F|^2 dV_N = \sum \int_{N} |F|^2 dV = Z \), where \( Z \) is the normalization factor in (13). The result of the calculations is

\[
\int \left[ \prod_{i=1}^{N} e^{iD(z_i, \bar{z}_i)} \right] e^{-S[\Phi]} D\Phi = Z_G = Z, \tag{25}
\]

where \( Z \) is the spectral determinant of the Laughlin state. The central charge of such theory is given by (10), (33)

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h_a = \frac{a}{2\sigma_H}(a - 2\sqrt{\chi_H}).
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[27] The complex structure in AB-fluxes parameter space is \( \varphi_i = \tau_{ij}\varphi_{0j} - \varphi_{ij} \), where \( \tau_{ij} \) is the period matrix of the Riemann surface. The adiabatic curvature is \( \Omega = \frac{1}{4\pi}\sigma_H \sum_{i,j=1}^8 (\text{Im}\tau_{ij})^{-1}\delta\varphi_i \wedge \delta\varphi_j \) [13].

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[31] The leading \( 1/N \) order of the functional \( F \) computed in [3] reads \( F = \frac{1}{4\pi} \int (\frac{1}{2}(1 - 2\nu^2)(\frac{1}{2}\Delta \log B - R) + 1 - 2\nu) b \log b \, dv \), where we denote \( B = B + \frac{1}{2}(1 - j) R \).

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