THE AREA METHOD AND APPLICATIONS

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Abstract. In this paper we develop a general method for estimating correlations of the forms

$$\sum_{n \leq x} G(n)G(x - n),$$

and

$$\sum_{n \leq x} G(n)G(n + l)$$

for a fixed $1 \leq l \leq x$ and where $G : \mathbb{N} \rightarrow \mathbb{R}^+$. To distinguish between the two types of correlations, we call the first type 2 correlation and the second type 1 correlation. As an application we estimate the lower bound for the type 2 correlation of the master function given by

$$\sum_{n \leq x} \Upsilon(n)\Upsilon(n + l_0) \geq (1 + o(1)) \frac{x}{2\mathcal{C}(l_0)} \log \log^2 x,$$

provided $\Upsilon(n)\Upsilon(n + l_0) > 0$. We also use this method to provide a first proof of the twin prime conjecture by showing that

$$\sum_{n \leq x} \Lambda(n)\Lambda(n + 2) \geq (1 + o(1)) \frac{x}{2\mathcal{C}(2)}$$

for some $\mathcal{C} := \mathcal{C}(2) > 0$.

1. Introduction and statement

Consider the sum

$$\sum_{n \leq x} G(n)G(x - n)$$

and

$$\sum_{n \leq x} G(n)G(n + l)$$

where $1 \leq l \leq x$. It is generally not easy to control sums of these forms, and unfortunately many of the open problems in number theory can be phrased in this manner. What is often required is an estimate for these sums. There are a good number of techniques in the literature for studying such sums, like the circle method of Hardy and Littlewood, the sieve method and many others.

In this paper, we introduce the area method. This method can also be used to control correlated sums of the form above. The novelty of this method is that it

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allows us to write any of these correlated sums as a double sum, which is much easier to estimate using existing tools such as the summation by part formula.

2. The area method

In this section we introduce and develop a fundamental method for solving problems related to correlations of arithmetic functions. This method is fundamental in the sense that it uses the properties of four main geometric shapes, namely the triangle, the trapezium, the rectangle and the square. The basic identity we will derive is an outgrowth of exploiting the areas of these shapes and putting them together in a unified manner.

Theorem 2.1. Let \( \{r_j\}_{j=1}^n \) and \( \{h_j\}_{j=1}^n \) be any sequence of real numbers, and let \( r \) and \( h \) be any real numbers satisfying \( \sum_{j=1}^n r_j = r \) and \( \sum_{j=1}^n h_j = h \), and

\[
(r^2 + h^2)^{1/2} = \sum_{j=1}^n (r_j^2 + h_j^2)^{1/2},
\]

then

\[
\sum_{j=2}^n r_j h_j = \sum_{j=2}^n h_j \left( \sum_{i=1}^j r_i + \sum_{i=1}^{j-1} r_i \right) - 2 \sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}.
\]

Proof. Consider a right angled triangle, say \( \triangle ABC \) in a plane, with height \( h \) and base \( r \). Next, let us partition the height of the triangle into \( n \) parts, not necessarily equal. Now, we link those partitions along the height to the hypothenus, with the aid of a parallel line. At the point of contact of each line to the hypothenus, we drop down a vertical line to the next line connecting the last point of the previous partition, thereby forming another right-angled triangle, say \( \triangle A_1B_1C_1 \) with base and height \( r_1 \) and \( h_1 \) respectively. We remark that this triangle is covered by the triangle \( \triangle ABC \), with hypothenus constituting a proportion of the hypothenus of \( \triangle ABC \). We continue this process until we obtain \( n \) right-angled triangles \( \triangle A_jB_jC_j \), each with base and height \( r_j \) and \( h_j \) for \( j = 1, 2, \ldots, n \). This construction satisfies

\[
h = \sum_{j=1}^n h_j \quad \text{and} \quad r = \sum_{j=1}^n r_j
\]

and

\[
(r^2 + h^2)^{1/2} = \sum_{j=1}^n (r_j^2 + h_j^2)^{1/2}.
\]

Now, let us deform the original triangle \( \triangle ABC \) by removing the smaller triangles \( \triangle A_jB_jC_j \) for \( j = 1, 2, \ldots, n \). Essentially we are left with rectangles and squares piled on each other with each end poking out a bit further than the one just above,
and we observe that the total area of this portrait is given by the relation

\[ A_1 = r_1 h_2 + (r_1 + r_2)h_3 + \cdots (r_1 + r_2 + \cdots + r_{n-2})h_{n-1} + (r_1 + r_2 + \cdots + r_{n-1})h_n \]

\[ = r_1 (h_2 + h_3 + \cdots + h_n) + r_2 (h_3 + h_4 + \cdots + h_n) + \cdots + r_{n-2} (h_{n-1} + h_n) + r_{n-1} h_n \]

\[ = \sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}. \]

On the other hand, we observe that the area of this portrait is the same as the difference of the area of triangle \(< ABC\) and the sum of the areas of triangles \(< A_jB_jC_j\) for \(j = 1, 2, \ldots, n\). That is

\[ A_1 = \frac{1}{2} rh - \frac{1}{2} \sum_{j=1}^{n} r_j h_j. \]

This completes the first part of the argument. For the second part, along the hypotenuse, let us construct small pieces of triangle, each of base and height \((r_i, h_i)\) \((i = 1, 2, \ldots, n)\) so that the trapezoid and the one triangle formed by partitioning becomes rectangles and squares. We observe also that this construction satisfies the relation

\[ (r^2 + h^2)^{1/2} = \sum_{i=1}^{n} (r_i^2 + h_i^2)^{1/2}, \]

Now, we compute the area of the triangle in two different ways. By direct strategy, we have that the area of the triangle, denoted \(A\), is given by

\[ A = 1/2 \left( \sum_{i=1}^{n} r_i \right) \left( \sum_{i=1}^{n} h_i \right). \]

On the other hand, we compute the area of the triangle by computing the area of each trapezium and the one remaining triangle and sum them together. That is,

\[ A = h_n/2 \left( \sum_{i=1}^{n} r_i + \sum_{i=1}^{n-1} r_i \right) + h_{n-1}/2 \left( \sum_{i=1}^{n-1} r_i + \sum_{i=1}^{n-2} r_i \right) + \cdots + 1/2 r_1 h_1. \]

By comparing the area of the second argument, and linking this to the first argument, the result follows immediately. \(\square\)

**Corollary 1.** Let \(f : \mathbb{N} \rightarrow \mathbb{C}\), then we have the decomposition

\[ \sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n+j) = \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m). \]

**Proof.** Let us take \(f(j) = r_j = h_j\) in Theorem 2.1, then we denote by \(G\) the partial sums

\[ G = \sum_{j=1}^{n} f(j) \]
and we notice that

\[
\sum_{j=1}^{n} \sqrt{h_j^2 + r_j^2} = \sum_{j=1}^{n} \sqrt{(f(j))^2 + f(j)^2} = \sum_{j=1}^{n} (f(j))^2
\]

Since \( \sqrt{(G^2 + G^2)} = G\sqrt{2} = \sqrt{2}\sum_{j=1}^{n} f(j) \) our choice of sequence is valid and, therefore the decomposition is valid for any arithmetic function. \( \square \)

**Remark 2.2.** Next we state a result for a general lower bound for any two-point correlation that captures all real arithmetic function.

**Theorem 2.3.** Let \( f : \mathbb{N} \to \mathbb{R}^+ \), a real-valued function. If

\[
\sum_{n \leq x} f(n)f(n + l_0) > 0
\]

then there exist some constant \( C := C(l_0) > 0 \) such that

\[
\sum_{n \leq x} f(n)f(n + l_0) \geq \frac{1}{C(l_0)x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).
\]

**Proof.** By Theorem 2.1, we obtain the identity by taking \( f(j) = r_j = h_j \)

\[
\sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n + j) = \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).
\]
It follows that
\[
\sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n+j) \leq \sum_{n \leq x-1} \sum_{j \leq x} f(n)f(n+j) \\
= \sum_{n \leq x} f(n)f(n+1) + \sum_{n \leq x} f(n)f(n+2) \\
+ \cdots + \sum_{n \leq x} f(n)f(n+l_0) + \cdots + \sum_{n \leq x} f(n)f(n+x) \\
\leq |M(l_0)| \sum_{n \leq x} f(n)f(n+l_0) \\
+ |N(l_0)| \sum_{n \leq x} f(n)f(n+l_0) \\
+ \cdots + |R(l_0)| \sum_{n \leq x} f(n)f(n+l_0) \\
= \left( |M(l_0)| + |N(l_0)| + \cdots + 1 \right) \sum_{n \leq x} f(n)f(n+l_0) \\
\leq C(l_0)x \sum_{n \leq x} f(n)f(n+l_0).
\]

where \( \max\{ |M(l_0)|, |N(l_0)|, \ldots, |R(l_0)|\} = C(l_0) \). By inverting this inequality, the result follows immediately. \( \square \)

The nature of the implicit constant \( C(l_0) \) could also depend on the structure of the function we are being given. The von Mangoldt function, contrary to many class of arithmetic functions, has a relatively small such constant. This behaviour stems from the fact that the Von-mangoldt function is defined on the prime powers. Thus one would expect most terms of sums of the form
\[
\sum_{n \leq x-1} \sum_{j \leq x-n} \Lambda(n)\Lambda(n+j)
\]
to fall off when \( j \) is odd for any prime power \( n = p^k \) such that \( j + p^k \neq 2^s \).

**Theorem 2.4.** Let \( f : \mathbb{N} \to \mathbb{R}^+ \). Suppose there exist some constant \( x > C(x) > 0 \) such that
\[
\sum_{n \leq x} \sum_{j \leq x-n \atop j \neq x-2n} f(n)f(n+j) = \frac{C(x)}{x} \sum_{n \leq x} \sum_{j \leq x-n} f(n)f(n+j).
\]

Then for any \( x \geq 2 \)
\[
\sum_{n \leq x-2} f(n)f(x-n) = \frac{D(x)}{x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m),
\]
where \( x - D(x) = C(x) \).
Proof. By Theorem 2.1, we obtain
\[
\sum_{n \leq x} f^2(n) = f^2(1) + \sum_{2 \leq n \leq x} f(n) \left( \sum_{m \leq n-1} f(m) + \sum_{m \leq n} f(m) \right) - 2 \sum_{n \leq x-1} f(n) \sum_{s \leq x-n} f(n + s)
\]
for \( f : \mathbb{N} \rightarrow \mathbb{R}^+ \) by taking \( r_j = h_j = f(j) \). By rearranging this identity, we obtain the identity
\[
\sum_{n \leq x-1} \sum_{j \leq x-n} f(n) f(n + j) = \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).
\]
Let \( n + j = x - n \), then it follows that \( x - 2n = j \). It follows that \( j \leq x - 2 \) if and only if \( 1 \leq n < \frac{x}{2} \). Then we can rewrite the sum on the left-hand side as
\[
\sum_{n \leq x-1} \sum_{j \leq x-n} f(n) f(n + j) = \sum_{n \leq x-1} f(n) f(x - n) + \frac{C(x)}{x} \sum_{n \leq x-1} \sum_{j \leq x-n} f(n) f(n + j)
\]
where \( 0 < \frac{C(x)}{x} < 1 \). It follows from this relation
\[
\frac{D(x)}{x} \sum_{n \leq x-1} \sum_{j \leq x-n} f(n) f(n + j) = \sum_{n < \frac{x}{2}} f(n) f(x - n)
\]
where \( 0 < \frac{D(x)}{x} = 1 - \frac{C(x)}{x} < 1 \). Using Theorem 2.1 we can write
\[
\sum_{n < \frac{x}{2}} f(n) f(x - n) = \frac{D(x)}{x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m)
\]
and the result follows immediately. \( \square \)

3. Application to the twin prime conjecture

Theorem 3.1. There exist some constant \( C := C(2) > 0 \), such that
\[
\sum_{n \leq x} \Lambda(n) \Lambda(n + 2) \geq (1 + o(1)) \frac{x}{2C(2)}.
\]

Proof. By invoking Theorem 2.3 we can write
\[
\sum_{n \leq x} \Lambda(n) \Lambda(n + 2) \geq \frac{1}{C(2)x} \sum_{2 \leq n \leq x} \Lambda(n) \sum_{m \leq n-1} \Lambda(m).
\]
Using the prime number theorem \( \# \) of the form
\[
\sum_{n \leq x} \Lambda(n) = (1 + o(1))x,
\]
the result follows immediately by using partial summation. \( \square \)

Remark 3.2. It is important to remark that with the lower bound in Theorem 3.1 we have solved the twin prime conjecture. This method not only does it solve the
twin prime conjecture, but is good in terms of its generality, for it can be used to obtain lower bounds for a general class of correlated sums of the form
\[ \sum_{n \leq x} f(n)f(n + k) \]
for a uniform \( 1 \leq k \leq x \).

4. Application to the minimum overlap problem

The minimum overlap problem was first posed by the then Hungarian mathematician Paul Erdős. The problem has the following well-known formulation:

Let \( A = \{a_i\} \) and \( B = \{b_j\} \) be any two complementary subsets, a splitting of the set \( \{1, 2, \ldots, n\} \) such that \( |A| = |B| = \frac{n}{2} \). Let \( M_k \) denotes the number of solutions to the equation \( a_i - b_j = k \), where \(-n \leq k \leq n\). Let us denote by \( M(n) := \min_{A,B} \max_k M_k \). Then the problem asks for an estimate for \( M(n) \) for sufficiently large values of \( n \). There has been significant progress in estimating from below and above the quantity \( M(n) \). Erdős [5] managed to obtain the following upper and lower bounds
\[
M(n) < (1 + o(1)) \frac{n}{2} \text{ and } M(n) > \frac{n}{4}.
\]
The lower bound was improved to (see [6])
\[
M(n) > (1 - 2^{-\frac{3}{2}})n
\]
and latter to (see [8])
\[
M(n) > \sqrt{(4 - \sqrt{15})(n - 1)}
\]
the most recent of which is [6]
\[
M(n) > \sqrt{(4 - \sqrt{15})n}.
\]
The upper bound, to the contrary, developed quite steadily overtime in the aftermath of Erdős’ result (see [5])
\[
M(n) < (1 + o(1)) \frac{2n}{5}.
\]
For the best known upper bound concerning this problem, see [7], given by
\[
M(n) < (1 + o(1)) 0.38093n.
\]
In the following sequel we obtain the following crude upper bound to the problem

**Theorem 4.1.** Let \( A = \{a_i\} \) and \( B = \{b_j\} \) be any two complementary subsets, a splitting of the set \( \{1, 2, \ldots, n\} \) such that \( |A| = |B| = \frac{n}{2} \). Let \( M_k \) denotes the number of solutions to the equation \( a_i - b_j = k \), where \(-n \leq k \leq n\). Let us denote by \( M(n) := \min_{A,B} \max_k M_k \), then for a fixed \( k \) we have the inequality
\[
M(n) < D(k) (1 - o(1)) \frac{n}{4}
\]
where \( D(k) > 1 \).
Lemma 4.2. (Area method) Let $f: \mathbb{N} \rightarrow \mathbb{C}$. If
\[ \sum_{n \leq x} f(n)f(n + l_0) > 0 \]
then there exist some constant $1 > C(l_0) > 0$ such that
\[ \sum_{n \leq x} f(n)f(n + l_0) < \frac{1}{C(l_0)x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n - 1} f(m). \]

Proof. By Corollary 1 we obtain the identity by taking $f(j) = r_j = h_j$
\[ \sum_{n \leq x - 1} \sum_{j \leq x - n} f(n)f(n + j) = \sum_{n \leq x} f(n) \sum_{m \leq n - 1} f(m). \]
Next we observe that
\[ \sum_{n \leq x - 1} \sum_{j \leq x - n} f(n)f(n + j) \gg \sum_{n \leq x} \sum_{j \leq x} f(n)f(n + j) \]
\[ = \sum_{n \leq x} f(n)f(n + 1) + \sum_{n \leq x} f(n)f(n + 2) \]
\[ + \cdots \sum_{n \leq x} f(n)f(n + l_0) + \cdots \sum_{n \leq x} f(n)f(n + x) \]
\[ \geq |M(l_0)| \sum_{n \leq x} f(n)f(n + l_0) \]
\[ + |N(l_0)| \sum_{n \leq x} f(n)f(n + l_0) \]
\[ + \cdots + \sum_{n \leq x} f(n)f(n + l_0) + \cdots + |R(l_0)| \sum_{n \leq x} f(n)f(n + l_0) \]
\[ = \left( |M(l_0)| + |N(l_0)| + \cdots + 1 \right) \sum_{n \leq x} f(n)f(n + l_0) \]
\[ + \cdots + |R(l_0)| \right) \sum_{n \leq x} f(n)f(n + l_0) \]
\[ \geq C(l_0)x \sum_{n \leq x} f(n)f(n + l_0). \]
where $\min\{|M(l_0)|, |N(l_0)|, \ldots, |R(l_0)|\} = C(l_0)$. By inverting this inequality, the result follows immediately. \qed

Definition 4.3. Let $A = \{a_i\}$ and $B = \{b_j\}$ be any two complementary subsets, a splitting of the set $\{1, 2, \ldots, n\}$ such that $|A| = |B| = \frac{n}{2}$. Then we consider the following arithmetic function
\[ \vee(c_i) = \begin{cases} 1 & \text{if } c_i \in A \cup B \\ 0 & \text{otherwise.} \end{cases} \]
Lemma 4.4. Let \( A = \{a_i\} \) and \( B = \{b_j\} \) be any two complementary subsets, a splitting of the set \( \{1, 2, \ldots, n\} \) such that \( |A| = |B| = \frac{n}{2} \), then we have
\[
\sum_{1 \leq i \leq n} \vee(a_i) = \frac{n}{2}
\]
and
\[
\sum_{1 \leq j \leq n} \vee(b_j) = \frac{n}{2}.
\]

Proof. This is an easy consequence of the size of \( |A \cup B| = n \) and the size of each complementary subset. \(\square\)

Theorem 4.5. Let \( A = \{a_i\} \) and \( B = \{b_j\} \) be any two complementary subsets, a splitting of the set \( \{1, 2, \ldots, n\} \) such that \( |A| = |B| = \frac{n}{2} \). Let \( M_k \) denotes the number of solutions to the equation \( a_i - b_j = k \), where \( -n \leq k \leq n \). Let us denote by \( M(n) := \min_{A,B} \max_k M_k \), then for a fixed \( k \) we have the inequality
\[
M(n) < D(k)(1 - o(1)) \frac{n}{4}
\]
where \( D(k) > 1 \).

Proof. Let \( k \) be fixed with \( -n \leq k \leq n \), then the underlying problem is to estimate the correlation
\[
\sum_{1 \leq i \leq n} \vee(a_i) \vee (a_i + k).
\]
Applying the area method, there exist some constant \( 0 < \mathcal{R}(k) < 1 \) such that
\[
\sum_{1 \leq i \leq n} \vee(a_i) \vee (a_i + k) < \frac{1}{\mathcal{R}(k)2n} \sum_{2 \leq i \leq n} \vee(a_i) \sum_{s \leq i-1} \vee(a_s).
\]
Applying partial summations on the right-hand side of the inequality, we have the following
\[
\sum_{2 \leq i \leq n} \vee(a_i) \sum_{s \leq i-1} \vee(a_s) \leq \sum_{1 \leq i \leq n} (i - 1) \vee (a_i)
= \sum_{1 \leq i \leq n} i \vee (a_i) - \sum_{1 \leq i \leq n} \vee(a_i)
= n \sum_{1 \leq i \leq n} \vee(a_i) - \int_{i=1}^{n} \sum_{1 \leq i \leq k} \vee(a_i)dk - \frac{n}{2}
\leq \frac{n^2}{2} - \frac{n}{2}.
\]
It follows that
\[
\sum_{1 \leq i \leq n} \vee(a_i) \vee (a_i + k) < \frac{1}{\mathcal{R}(k)2n} \left( \frac{n^2}{2} - \frac{n}{2} \right)
\]
and the claim upper bound follows, where \( 0 < \mathcal{R}(k) < 1 \). \(\square\)
5. Application to an estimate for the two-point correlation of the Liouville function

Let \( \lambda : \mathbb{N} \rightarrow \mathbb{C} \) be the Liouville function, defined by \( \lambda(n) = (-1)^{\Omega(n)} \), where

\[
\Omega(n) := \sum_{p|n} \alpha
\]

is the function counting the number of prime factors of \( n \in \mathbb{N} \) with multiplicity.

Obtaining good estimates for the correlations of the Liouville function is of much interest and has spawn extensive research in the field. It is a well-known conjecture of Chowla that the correlation of the Liouville function exhibits some cancellation. In particular that

\[
\sum_{n \leq x} \lambda(n)\lambda(n+1) = o(x)
\]

as \( x \rightarrow \infty \). A harmonic version of the conjecture (see [4]) was recently established as

\[
\sum_{n \leq x} \frac{\lambda(n)\lambda(n+1)}{n} = o(\log x).
\]

Using a different method we establish the upper bound

\[
\sum_{n \leq x} \lambda(n)\lambda(n+1) \ll_x xe^{-2c(\log x)^{\frac{4}{7}}} (\log \log x)^{-\frac{1}{5}}
\]

for \( \epsilon \in (0, 1) \). An appeal to the Riemann hypothesis improves on the claimed upper bound. This establishes the conjecture that the Liouville function exhibits some cancellation on very large intervals.

**Theorem 5.1.** Let \( f : \mathbb{N} \rightarrow \mathbb{C} \). Suppose there exists some constant \( 1 \leq \mathcal{N} := \mathcal{N}(l_o) < x \) such that

\[
\sum_{n \leq x} f(n)f(n + l_o) = \frac{\mathcal{N}(l_o)}{x} \sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n+j)
\]

where \( 1 \leq l_o < x \) then

\[
\sum_{n \leq x} f(n)f(n + l_o) = \frac{\mathcal{N}(l_o)}{x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).
\]

**Proof.** This is an easy consequence of Corollary [1].

The function \( \frac{\mathcal{N}}{x} \) in the statement of Theorem 5.1 can more be thought of as the local density function of the correlation

\[
\sum_{n \leq x} f(n)f(n + l_0)
\]

in the interval \([1, x]\). Indeed this function will always exist for any arithmetic function so long as it depends on the size of the shift \( l_0 \in \mathbb{N} \) and the range of summation \([1, x]\). We are now ready to establish the upper bound stated at the outset of the paper. We apply Theorem 5.1 tied with partial summation.
Theorem 5.2. There estimate holds
\[ \sum_{n \leq x} \lambda(n) \lambda(n + 1) \ll z x e^{-2c \log x} \left( \log \log x \right)^{\gamma - \frac{1}{2}} \]
for some constant \( z \geq 1 \).

Proof. By virtue of Theorem 5.1 there exist some \( N \geq 1 \) with \( N < x \) such that we can write
\[ \sum_{n \leq x} \lambda(n) \lambda(n + 1) = \frac{N}{x} \sum_{2 \leq n \leq x} \lambda(n) \sum_{m \leq n-1} \lambda(m). \]
Now using the estimate found in the literature
\[ \sum_{n \leq x} \lambda(n) \ll x e^{-c \log x} \left( \log \log x \right)^{\frac{1}{2}}, \]
we obtain from partial summation an upper bound for the sum
\[ \sum_{2 \leq n \leq x} \lambda(n) \sum_{m \leq n-1} \lambda(m) \ll x^2 e^{-2c \log x} \left( \log \log x \right)^{\gamma - \frac{1}{2}}. \]
By virtue of the restriction of the implicit constant \( N < x \), it follows that we can write
\[ \sum_{n \leq x} \lambda(n) \lambda(n + 1) \ll \frac{N}{x} x e^{-2c \log x} \left( \log \log x \right)^{\gamma - \frac{1}{2}}, \]
thereby establishing the upper bound. \( \square \)

5.1. Further remarks and improvements. The current paper claims an upper bound for the correlation of the Liouville function \( \lambda(n) \) of the form
\[ \sum_{n \leq x} \lambda(n) \lambda(n + 1) \ll z x e^{-2c \log x} \left( \log \log x \right)^{\gamma - \frac{1}{2}} \]
for some constant \( z \geq 1 \). This shows that the Liouville function does exhibit some cancellation on sufficiently large intervals. The estimate can be improved assuming the truth of the Riemann hypothesis by leveraging the estimate (see [1])
\[ \sum_{n \leq x} \lambda(n) = O \left( \sqrt{x} e^{C \log x} \left( \log \log x \right)^{\frac{1}{2} + \delta} \right) \]
for \( C > 0 \) and any \( \delta > 0 \) and appealing to the same approach.

6. Application to the infinitude of Sophie Germain primes
Let \( \mathbb{P} \) denotes the set of all prime numbers, then we say a prime \( p \) is a Sophie Germain prime - named after the French mathematician Sophie Germain who encountered it in her investigations of Fermat’s Last Theorem - if \( 2p + 1 \) is also a prime number. The motivation for the study of Sophie Germain primes is quite clear from a practical point of view (see [9]), as it owes its application to cryptography and primality testing [8]. There has also been a lot of computational work in verifying pushing the barrier of the largest known Sophie Germain prime, a worthwhile endeavor since the infinitude of such primes has been conjectured to hold. In the current paper we obtain a lower bound for the number of such primes less than
a given threshold, thereby confirming the infinitude of such primes. Let us denote \( \vartheta : \mathbb{N} \to \mathbb{C} \) to be function defined by
\[
\vartheta(n) := \begin{cases} 
\log p & \text{if } n = p \in \mathbb{P} \\
0 & \text{otherwise}
\end{cases}
\]
then an natural step to take to obtain an estimate for the number of such primes is to obtain an estimate for the correlation
\[
\sum_{n \leq x} \vartheta(n) \vartheta(2n + 1)
\]
or at the very least a non-trivial lower bound followed by a consequent appeal to partial summation to remove the weight \( \vartheta \). Analyzing such correlations is by no means an easy tussle but an appeal to the area method provides with at least a non-trivial lower bound.

**Theorem 6.1.** Let \( f : \mathbb{N} \to \mathbb{C} \). Suppose there exists some constant \( 1 \leq N := N(l_o) < x \) such that
\[
\sum_{n \leq x} f(n) f(n + l_o) = \frac{N(l_o)}{x} \sum_{n \leq x} \sum_{j \leq x - n} f(n) f(n + j)
\]
for arbitrary \( l_o \) with \( 1 \leq l_o < x \) then
\[
\sum_{n \leq x} f(n) f(n + l_o) = \frac{N(l_o)}{x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n - 1} f(m).
\]

**Proof.** This is an easy consequence of Corollary 1. \( \square \)

**Remark 6.2.** The function \( \frac{N(l_o)}{x} \) in the statement of Theorem 6.1 can more be thought of as the local density function of the correlation
\[
\sum_{n \leq x} f(n) f(n + l_o)
\]
for arbitrary \( l_o \) in the interval \([1, x]\). Indeed this function will always exists for any arithmetic function so long as it depends on the size of the arbitrary shift \( l_o \in \mathbb{N} \) and consequently on the range of summation \([1, x]\).

**Theorem 6.3.** Let \( \mathbb{P} \) denotes the set of all prime numbers, then we have the estimate
\[
\# \{ p \leq x \mid 2p + 1, p \in \mathbb{P} \} \geq (1 + o(1)) \frac{\mathcal{D}}{(2 + 2 \log 2) \log^2 x}
\]
where \( \mathcal{D} \geq 1 \).

**Proof.** Let us consider the function \( \vartheta : \mathbb{N} \to \mathbb{C} \) defined as
\[
\vartheta(n) := \begin{cases} 
\log p & \text{if } n = p \in \mathbb{P} \\
0 & \text{otherwise}
\end{cases}
\]
so that by virtue of Corollary 1 we obtain the decomposition

$$\sum_{n \leq x} \vartheta(n)\vartheta(n + (n + 1)) = \frac{D}{x} \sum_{2 \leq n \leq x} \vartheta(n) \sum_{m \leq n-1} \vartheta(m)$$

for $D \geq 1$. Now using the weaker estimate found in the literature

$$\sum_{n \leq x} \vartheta(n) = (1 + o(1))x$$

we obtain the following estimates by an appeal to summation by parts

$$\sum_{2 \leq n \leq x} \vartheta(n) \sum_{m \leq n-1} \vartheta(m) = (1 + o(1)) \int_{2}^{x} \left( \sum_{2 \leq n \leq t} \vartheta(n) \right) dt$$

$$= (1 + o(1))x^2 - (1 + o(1)) \int_{2}^{x} (1 + o(1)) t dt$$

$$= (1 + o(1))x^2 - (1 + o(1)) \frac{x^2}{2} + O(1)$$

(6.2)

By plugging (6.2) into (6.1) we obtain the estimate

$$\sum_{n \leq x} \vartheta(n)\vartheta(n + (n + 1)) = \frac{D}{x} (1 + o(1)) \frac{x^2}{2}$$

$$= (1 + o(1)) \frac{D}{2} x.$$ 

On the other hand, we can write

$$\sum_{n \leq x} \vartheta(n)\vartheta(n + (n + 1)) = \sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} \log p \log(2p + 1)$$

$$\approx \sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} \log^2 p + (\log 2) \sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} \log p$$

(6.3)

$$\leq (1 + \log 2) \sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} \log^2 p$$

so that by an application of partial summation we have

$$\sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} \log^2 p \leq \log^2 x \sum_{\substack{p \leq x \\ 2p+1 \in \mathbb{P}}} 1.$$  

(6.4)

By combining (6.2), (6.1) and (6.4) the lower bound follows as a consequence. □

**Corollary 2.** There are infinitely many primes $p \in \mathbb{P}$ such that $2p + 1 \in \mathbb{P}$.

**Proof.** This is a consequence of Theorem 6.3 □
7. Application to other correlated sums of type 1

In this section we apply Theorem 2.3 to provide lower estimates of other correlated sums, but with the price of an implicit constant depending on the range of shift.

**Corollary 3.** For a fixed \( l_0 > 0 \), there exist some constant \( C := C(l_0) > 0 \) such that

\[
\sum_{n \leq x} d(n)d(n + l_0) \geq (1 + o(1)) \frac{x \log^2 x}{2C(l_0)}.
\]

*Proof.* The result follows by using Theorem 2.3, using the crude estimate [10]

\[
\sum_{n \leq x} d(n) = (1 + o(1))x \log x
\]

together with partial summation. \( \square \)

**Corollary 4.** For a fixed \( k_0 > 0 \) and for \( l \geq 2 \), there exist some constant \( C := C(k) > 0 \) such that

\[
\sum_{n \leq x} d_l(n)d_l(n + k) \geq (1 + o(1)) \left( \frac{1}{(l-1)!} \right) \left( 1 - \frac{1}{2(l-1)!} \right) \frac{x \log^{2(l-1)} x}{C(k)}.
\]

*Proof.* We recall the weaker estimate for the \( l \) th divisor function [10]

\[
\sum_{n \leq x} d_l(n) = (1 + o(1)) \frac{1}{(l-1)!} x \log^{l-1} x,
\]

where

\[
d_l(n) = \sum_{n_1, n_2, \ldots, n_l = n} 1.
\]

By leveraging Theorem 2.3 and using partial summation, the lower bound follows naturally. \( \square \)

**Corollary 5.** For a fixed \( l_0 > 0 \), there exist some constant \( C := C(l_0) > 0 \) such that

\[
\sum_{n \leq x} \phi(n)\phi(n + l_0) \geq (1 + o(1)) \frac{9}{2\pi^4} \frac{x^3}{C(l_0)}.
\]

*Proof.* The result follows by applying Theorem 2.3 using the estimate [11]

\[
\sum_{n \leq x} \phi(n) = (1 + o(1)) \frac{3}{\pi^2} x^2
\]

together with partial summation. \( \square \)

**Corollary 6.** For a fixed \( l_0 > 0 \), there exist some constant \( C := C(l_0) > 0 \) such that

\[
\sum_{n \leq x} \mu^2(n)\mu^2(n + l_0) \geq (1 + o(1)) \frac{18}{\pi^4} \frac{x}{C(l_0)}.
\]
Proof. The result follows by applying Theorem 2.3 using the estimate \[ \sum_{n \leq x} \mu^2(n) = (1 + o(1)) \frac{6}{\pi^2} x \]
together with the use of partial summation. \qed

8. Application to lower bound for two-point correlation of the master function of type 1 and type 2

In this section we apply the area method developed to establish a lower bound for the two-point type 1 correlation and an estimate for the type 2 correlation of the master function. We begin with the following result:

Lemma 8.1. Let \( \Upsilon \) denote the master function, then

\[ \sum_{n \leq x} \Upsilon(n) = x \log \log x + O(x). \]

Proof. For a proof, See [2]. \qed

Theorem 8.2. The estimate is valid

\[ \sum_{n \leq x} \Upsilon(n) \Upsilon(n + l_0) \geq (1 + o(1)) \frac{x}{2C(l_0)} \log \log^2 x, \]

provided \( \Upsilon(n) \Upsilon(n + l_0) > 0 \).

Proof. Applying Theorem 2.3 and Lemma 8.1 we can write

\[ \sum_{n \leq x} \Upsilon(n) \Upsilon(n + l_0) \geq \frac{1}{xC(l_0)} \sum_{2 \leq n \leq x} \Upsilon(n) \sum_{m \leq n - 1} \Upsilon(m) = \frac{1}{xC(l_0)} (1 + o(1)) \sum_{2 \leq n \leq x} \Upsilon(n) n \log n. \]

By partial summation, we can write

\[ \sum_{2 \leq n \leq x} \Upsilon(n) n \log n = x \log \log x \sum_{n \leq x} \Upsilon(n) - \int_{2}^{x} (1 + o(1)) t (\log \log t) \left( \log \log t + \frac{1}{\log t} \right) dt \]

\[ = (1 + o(1)) x^2 \log \log^2 x - (1 + o(1)) \int_{2}^{x} t (\log \log t) \left( \log \log t + \frac{1}{\log t} \right) dt \]

\[ = (1 + o(1)) \frac{x^2}{2} \log \log^2 x. \]

The lower bound follows immediately from this estimate. \qed
Theorem 8.3. Under the assumption
\[
\sum_{n \leq x} \sum_{j \leq x-n} \Upsilon(n) \Upsilon(n + j) / \sum_{n \leq x} \sum_{j \leq x-n} \Upsilon(n) \Upsilon(n + j) < 1,
\]
then
\[
\sum_{n \leq \frac{x}{2}} \Upsilon(n) \Upsilon(x - n) = (1 + o(1)) \frac{x}{2} \mathcal{D}(x) \log \log^2 x
\]
where \( \mathcal{D} := \mathcal{D}(x) > 0 \).

Proof. The result follows by applying the area method. \( \square \)

9. Application to estimates of the number of representations of an even number as a sum of two primes

In this section we apply the area method developed in \( \text{[2.4]} \) to obtain a weaker estimate for the number of representations of an even number as a sum of two primes, under the assumption that the Goldbach conjecture is true.

Theorem 9.1. Assuming the Goldbach conjecture is true, then for any even \( x \geq 6 \)
\[
\sum_{n \leq \frac{x}{2}} \Lambda(n) \Lambda(x - n) = (1 + o(1)) \frac{x}{2} \mathcal{D}(x)
\]
where \( \mathcal{D} := \mathcal{D}(x) > 0 \).

Proof. Under the assumption that the Goldbach conjecture is true, it follows that
\[
\sum_{n \leq x} \sum_{j \leq x-n} \Lambda(n) \Lambda(n + j) / \sum_{n \leq x} \sum_{j \leq x-n} \Lambda(n) \Lambda(n + j) < 1.
\]
Applying the area method, there exist some constant \( \mathcal{D}(x) > 0 \) with \( \mathcal{D}(x) < x \), such that
\[
\sum_{n \leq \frac{x}{2}} \Lambda(n) \Lambda(x - n) = \frac{\mathcal{D}(x)}{x} \sum_{2 \leq n \leq x} \Lambda(n) \sum_{m \leq n-1} \Lambda(m).
\]
Using the prime number theorem \( \text{[3]} \) in the form
\[
\sum_{n \leq x} \Lambda(n) = (1 + o(1)) x
\]
and partial summation, we obtain
\[
\sum_{2 \leq n \leq x} \Lambda(n) \sum_{m \leq n-1} \Lambda(m) = (1 + o(1)) \frac{x^2}{2}.
\]
The result follows immediately from this rudimentary estimates. \( \square \)
10. Application to other correlated sums of type 2

In this section, we apply the area method 2.4 to obtain estimates for various correlated sums, in the following sequel. The area method is perfectly suited for functions of these forms, since they are non-vanishing on the integers.

**Theorem 10.1.** The estimate holds

\[
\sum_{n \leq \frac{x}{2}} d(n)d(x-n) = \mathcal{D}(1+o(1))\frac{x\log^2 x}{2}
\]

where \( \mathcal{D} := \mathcal{D}(x) > 0 \).

**Proof.** Since the divisor function is non-vanishing on the integers, we observe that

\[
\sum_{n \leq x} \sum_{\frac{x-n}{2}\nmid j} d(n)d(n+j) < 1.
\]

Thus by the area method 2.4 there exist some constant \( 0 < \mathcal{D}(x) < x \) such that

\[
\sum_{n \leq \frac{x}{2}} d(n)d(x-n) = \frac{\mathcal{D}(x)}{x} \sum_{2 \leq n \leq x} d(n) \sum_{m \leq n-1} d(m).
\]

Using the weaker estimate \( \|10\| \)

\[
\sum_{n \leq x} d(n) = (1+o(1))x \log x
\]

we obtain by partial summation

\[
\sum_{2 \leq n \leq x} d(n) \sum_{m \leq n-1} d(m) = (1+o(1))\frac{x^2 \log^2 x}{2}
\]

and the result follows immediately. \(\square\)

**Theorem 10.2.** The estimate holds

\[
\sum_{n \leq \frac{x}{2}} \phi(n)\phi(x-n) = (1+o(1))\mathcal{D} \frac{9}{2\pi^4} x^3
\]

where \( \mathcal{D} := \mathcal{D}(x) > 0 \).

**Proof.** Since \( \phi(n) \) is non-vanishing on the integers, we observe that

\[
\sum_{n \leq x} \sum_{\frac{x-n}{2}\nmid j} \phi(n)\phi(n+j) < 1.
\]

Then by the area method 2.4 there exist some constant \( \mathcal{D}(x) > 0 \) with \( \mathcal{D}(x) < x \) such that

\[
\sum_{n \leq \frac{x}{2}} \phi(n)\phi(x-n) = \frac{\mathcal{D}(x)}{x} \sum_{2 \leq n \leq x} \phi(n) \sum_{m \leq n-1} \phi(m).
\]
Now using the estimate $[10]$ 
\[ \sum_{n \leq x} \phi(n) = (1 + o(1)) \frac{3}{\pi^2} x^2 \]
we obtain by partial summation 
\[ \sum_{2 \leq n \leq x} \phi(n) \sum_{m \leq n-1} \phi(m) = (1 + o(1)) \frac{9}{2\pi^4} x^4, \]
and the result follows immediately. \(\square\)

**Theorem 10.3.** The estimate holds 
\[ \sum_{n \leq \frac{x}{2}} d_l(n) d_l(x - n) = (1 + o(1)) D \left( \frac{1}{(l - 1)!} \right) \left( 1 - \frac{1}{2(l - 1)!} \right) x \log^{2(l-1)} x \]
where $D := D(x) > 0$ and where 
\[ d_l(n) := \sum_{n_1 n_2 \cdots n_l = n} 1. \]

**Proof.** We observe that 
\[ \sum_{n \leq \frac{x}{2}} \sum_{j \leq x-n} d_l(n) d_l(n + j) \]
\[ \sum_{n \leq x} \sum_{j \leq x-n} d_l(n) d_l(n + j) < 1. \]
It follows from the area method $[2.4]$ there exists some constant $D(x) > 0$ with $D(x) < x$ such that 
\[ \sum_{n \leq \frac{x}{2}} d_l(n) d_l(x - n) = \frac{D(x)}{x} \sum_{2 \leq n \leq x} d_l(n) \sum_{m \leq n-1} d_l(m). \]
Using the estimate $[10]$ 
\[ \sum_{n \leq x} d_l(n) = (1 + o(1)) \frac{1}{(l - 1)!} x \log^{l-1} x, \]
It follows by partial summation 
\[ \sum_{2 \leq n \leq x} d_l(n) \sum_{m \leq n-1} d_l(m) = (1 + o(1)) \left( \frac{1}{(l - 1)!} \right) \left( 1 - \frac{1}{2(l - 1)!} \right) x^2 \log^{2(l-1)} x. \]
The claimed estimate follows immediately. \(\square\)

11. **Application to the global distribution of integers with $\Omega(n) = 2$**

The lower bounds of correlations of arithmetic functions tells us a lot about their local distributions as well as their global distribution. Theorem $8.2$ gives 
\[ \sum_{n \leq x} \Upsilon(n) \Upsilon(n + l_0) \geq (1 + o(1)) \frac{x}{2C(l_0)} \log \log^2 x, \]
provided $\sum_{n \leq x} \Upsilon(n) \Upsilon(n + l_0) > 0$. Thus for some shift in the range $[1, x]$ the correlation can be made arbitrarily large by taking the right hand side arbitrarily large.
This follows that there are infinitely many pairs of the form $(n, n + l_0)$ such that $n$ and $n + l_0$ each has exactly two prime factors.

12. Final remarks

The area method seems not particularly suited for arithmetic functions defined on a certain subsequence of the integers. As such its current form cannot be applied directly to important open problems like the Goldbach conjecture, since the implicit constant in Theorem 9.1 relies on the condition

\[
\frac{\sum_{n \leq x} \sum_{j \leq x - n, j \neq x - 2n} \Lambda(n)\Lambda(n + j)}{\sum_{n \leq x} \sum_{j \leq x - n} \Lambda(n)\Lambda(n + j)} < 1.
\]

Add to this, even if this condition were to be satisfied, we would certainly not have much information about the constant, although at the barest minimum $0 < D(x) < x$. However, we believe this method can be refined to the form applicable to functions defined on a subsequence of the integers like the primes.

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