ON SHARP THIRD HANKEL DETERMINANT FOR CERTAIN STARLIKE FUNCTIONS

NEHA VERMA AND S. SIVAPRASAD KUMAR

Abstract. In this paper, we provide an estimation for the sharp bound of the third Hankel determinant of starlike functions of order $\alpha$, where $\alpha$ ranges in the interval $[0, 1/6] \cup \{1/2\}$ and thereby extending the result of Rath et al. (Complex Anal Oper Theory: No. 65, 16(5), 8 pp 2022).

1. Introduction

Consider the set $A$, which comprises normalized analytic functions defined on the open unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$. These functions are expressed in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let $S \subset A$, where $S$ represents the class of univalent functions, and $P$ denotes the collection of analytic functions defined on $D$ with a positive real part, expressed as $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Let $h$ and $g$ are two analytic functions, then we say $h$ is subordinated to $g$, denoted as $h \prec g$, provided there exist a Schwarz function $w$, adhering to two crucial conditions: $w(0) = 0$ and $|w(z)| \leq |z|$, such that $h(z) = g(w(z))$.

In the year 1936, Robertson [15] introduced the class of starlike functions of order $\alpha$, characterized as follows:

Definition 1.1. [15] For $0 \leq \alpha < 1$, we say that a function $f \in A$ is starlike of order $\alpha$ if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in D.$$ 

The class of all such functions is represented by $S^*(\alpha)$.

In 1992, Ma and Minda [13] introduced a more general class of starlike functions through subordination, defined as follows:

$$S^*(\varphi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

where $\varphi$ is an analytic univalent function such that $\text{Re} \varphi(z) > 0$, $\varphi(D)$ is symmetric about the real axis and starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$. Through this concept, we can re-define the class $S^*(\alpha)$ as:

$$S^*(\alpha) = \left\{ f \in A : \frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad \alpha \in [0, 1) \right\}.$$

Note that $S^*(0) = S^*$ and $S^*(\varphi) \subset S^*(\alpha)$ for some $\alpha$ depending upon the choice of $\varphi$.

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The Bieberbach conjecture, as documented on [4, Page no. 17], has been a significant source of inspiration in the development of univalent function theory and in the formulation of coefficient problems. Building on this foundation, in 1966, Pommerenke [14] introduced the concept of \( q \)th Hankel determinants, denoted as \( H_q(n) \), where \( n \) and \( q \) are both natural numbers, associated with analytic functions as in (1.1), defined as follows:

\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\] (1.2)

By choosing specific values for both \( n \) and \( q \), we can examine particular cases of this concept. For instance, when we set \( q = 2 \), we obtain the expression for the second-order Hankel determinant. Numerous studies have investigated and established sharp bounds for second-order Hankel determinants and other determinants within various subclasses of \( S \), see [5,7,8] for more details. Now, if we choose \( q = 3 \) and \( n = 1 \) in (1.2), assuming \( a_1 := 1 \), we arrive at the expression for the Hankel determinant of order three, given by

\[
H_3(1) := \begin{vmatrix}
  1 & a_2 & a_3 \\
  a_2 & a_3 & a_4 \\
  a_3 & a_4 & a_5
\end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).
\] (1.3)

Determining the third-order Hankel determinant poses a greater challenge compared to the second-order, as evidenced in [9,22]. We also list some of the sharp estimates for the third-order Hankel determinant concerning functions within the class \( S^*(\phi) \), considering various selections of \( \phi(z) \) in Table 1. However, the sharp estimate of \( H_3(1) \) for \( S^*_{Ne} \) is yet to be estimated.

For the class \( S^*(\alpha) \), Krishna and Ramreddy [7] computed the bound of the second order Hankel determinant, \( |a_2a_4 - a_3^2| \leq (1 - \alpha)^2 \), \( \alpha \in [0,1/2] \) while Xu and Fang [21] calculated the sharp bounds of the Fekete and Szegö functional \( |a_3 - \lambda a_2^2| \leq (1 - \alpha) \max \{1, |3 - 2\alpha - 4\lambda(1 - \alpha)| \} \), \( \lambda \in \mathbb{C} \) and \( \alpha \in [0,1) \). We refer [3] for further information on Hankel determinants associated with the class \( S^*(\alpha) \).

The purpose of this study is to establish the sharp bound of third order Hankel determinant for functions belonging to the class, \( S^*(\alpha) \). At the end of this paper, we demonstrate the validation of our main result by considering the class \( S^*(\alpha) \) specifically for the case when \( \alpha = 0 \), and we also present some relevant applications.

### Table 1. List of sharp third order Hankel determinants

| Class | Sharp bound | Reference |
|-------|-------------|-----------|
| \( S^* := S^*(0) \) | 4/9 | [2,6] |
| \( S^*(1/2) \) | 1/9 | [11,18] |
| \( S^*_\varphi := S^*(1 + z\varphi) \) | 1/9 | [19] |
| \( SL^* := S^*(\sqrt{1 + z}) \) | 1/36 | [1] |
| \( S^*_\varepsilon := S^*(\varepsilon^z) \) | 1/9 | [16] |
| \( S^*_\rho := S^*(1 + \sinh^{-1}(z)) \) | 1/9 | [17] |
| \( S^*_{Ne} := S^*(1 + z - z^3/3) \) | — | — |
1. Preliminary. In this part of the section, we mention the initial coefficient bounds $a_i$ ($i = 2, 3, 4, 5$) in terms of the Carathéodory coefficients and a lemma which will be used in our forthcoming results. Let $f \in S^*(\alpha)$, then a Schwarz function $w(z)$ exists such that

$$\frac{zf'(z)}{f(z)} = 1 + (1 - 2\alpha)w(z) \quad (1.4)$$

Let $p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n \in \mathcal{P}$ and $w(z) = (p(z) - 1)/(p(z) + 1)$. The expressions of $a_i (i = 2, 3, 4, 5)$ are obtained in terms of $p_j (j = 1, 2, 3, 4)$ by substituting $w(z)$, $p(z)$, and $f(z)$ in equation (1.4) with suitable comparison of coefficients so that

$$a_2 = p_1(1 - \alpha),$$

$$a_3 = \frac{(1 - \alpha)}{2} \left( p_2 + p_1^2(1 - \alpha) \right),$$

$$a_4 = \frac{(1 - \alpha)}{6} \left( 2p_3 + 3p_1p_2(1 - \alpha) + p_1^3(1 - \alpha)^2 \right)$$

and

$$a_5 = \frac{(1 - \alpha)}{24} \left( 6p_4 + (1 - \alpha) \left( 3p_2^2 + 8p_1p_3 \right) + (1 - \alpha)^2 \left( 6p_1^2p_2 + p_1^4(1 - \alpha) \right) \right).$$

The formula for $p_j (j = 2, 3, 4)$, which plays a significant role in finding the sharp bound of the Hankel determinant and has been prominently exploited in the main theorem, is contained in the Lemma 1.2 below.

**Lemma 1.2.** [10, 12] Let $p \in \mathcal{P}$ has the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$2p_2 = p_1^2 + \gamma(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta,$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \gamma\eta^2 - (1 - |\eta|^2)p),$$

for some $\gamma$, $\eta$ and $p$ such that $|\gamma| \leq 1$, $|\eta| \leq 1$ and $|p| \leq 1$.

2. Sharp $H_3(1)$ for $S^*(\alpha)$

Recently, Kowalczyk et al. [6] and Banga and Kumar [2] obtained the sharp bound of the third-order Hankel determinant for functions in the class $S^* := S^*(0)$, independently whereas Rath et al. [18] determined the sharp bound of $H_3(1)$ for functions in the class $S^*(1/2)$ and corrected the proof provided in [11]. In this section, we extend our analysis to calculate the sharp bound of $H_3(1)$ for functions in the class $S^*(\alpha)$ for some additional range of $\alpha$. Below, is our main result.

**Theorem 2.1.** Let $f \in S^*(\alpha)$. Then

$$|H_3(1)| \leq \frac{4(1 - \alpha)^2}{9}, \quad \alpha \in [0, 1/6] \cup \{1/2\}.$$  

(2.1)

This result is sharp.
Proof. Since, the class $\mathcal{P}$ is invariant under rotation, we have $p_1 \in [0,2]$ and assume $p_1 = p$. The expressions of $a_i$ $(i = 2, 3, 4, 5)$ from equations (1.5)-(1.8) are substituted in equation (1.3). We get

$$H_3(1) = \frac{(1 - \alpha)^2}{144} \left( - (1 - \alpha)^4 p^6 + 3(1 - \alpha)^3 p^4 p_2 + 8(1 - \alpha)^2 p^2 p_3 + 24(1 - \alpha) p p_2 p_3 
- 18(1 - \alpha) p^2 p_4 - 9(1 - \alpha) p_3^2 - 9(1 - \alpha)^2 p^2 p_2^2 - 16 p_3^2 + 18 p_2 p_4 \right).$$

After simplifying the calculations through Lemma 1.2 we obtain

$$H_3(1) = \frac{1}{1152} \left( \Delta_1(p, \gamma) + \Delta_2(p, \gamma) \eta + \Delta_3(p, \gamma) \eta^2 + \phi(p, \gamma, \eta) \rho \right), \quad \text{for} \quad \gamma, \eta, \rho \in \mathbb{D}.$$

Here

$$\Delta_1(p, \gamma) : = (1 - \alpha)^2 \left( \alpha(1 - 2\alpha)(3 - 2\alpha)p^6 - (2 - 15\alpha + 18\alpha^2)p^2 \gamma^2(4 - p^2)^2 
+ p^2 \gamma^4(4 - p^2)^2 - (10 - 15\alpha)p^2 \gamma^2(4 - p^2)^2 + 36\alpha \gamma^3(4 - p^2)^2 
+ (3 - 12\alpha^3 + 32\alpha^2 - 19\alpha)p^4 \gamma(4 - p^2) - 9(1 - 2\alpha)p^4 \gamma^3(4 - p^2) 
+ (3 - 16\alpha^2 + 2\alpha)p^4 \gamma^2(4 - p^2) - 36(1 - 2\alpha)p^2 \gamma^2(4 - p^2) \right),$$

$$\Delta_2(p, \gamma) : = 4(1 - |\gamma|^2)(4 - p^2)(1 - \alpha)^2 \left( 8\alpha^2 - 10\alpha + 3 \right) p^3 \gamma 
+ (5 - 12\alpha)p \gamma(4 - p^2) - p \gamma^2(4 - p^2),$$

$$\Delta_3(p, \gamma) : = 4(1 - |\gamma|^2)(4 - p^2)(1 - \alpha)^2 \left( - 8(4 - p^2) - |\gamma|^2(4 - p^2) + 9(1 - 2\alpha)p^2 \gamma \right),$$

$$\phi(p, \gamma, \eta) : = 36(1 - |\gamma|^2)(4 - p^2)(1 - |\eta|^2)(1 - \alpha)^2 \left( (4 - p^2) \gamma - (1 - 2\alpha)p^2 \right).$$

Assume $x := |\gamma|$, $y := |\eta|$ and since $|\rho| \leq 1$, the above expression reduces to

$$|H_3(1)| \leq \frac{1}{1152} \left( |\Delta_1(p, \gamma)| + |\Delta_2(p, \gamma)|y + |\Delta_3(p, \gamma)|y^2 + |\phi(p, \gamma, \eta)| \right) \leq Z(p, x, y),$$

where

$$Z(p, x, y) = \frac{1}{1152} \left( z_1(p, x) + z_2(p, x)y + z_3(p, x)y^2 + z_4(p, x)(1 - y^2) \right) \quad (2.2)$$
with
\[z_1(p, x) = (1 - \alpha)^2 \left( \alpha(1 - 2\alpha)^2(3 - 2\alpha)p^6 + (2 - 15\alpha + 18\alpha^2)p^2 x^2(4 - p^2)^2 + p^2 x^4(4 - p^2)^2 + (10 - 15\alpha)p^2 x^3(4 - p^2)^2 + 36\alpha x^3(4 - p^2)^2 + (3 - 12\alpha^2 + 32\alpha^2 - 19\alpha)p^4 x(4 - p^2) + 9(1 - 2\alpha)p^4 x^3(4 - p^2) + (3 - 16\alpha^2 + 2\alpha)p^4 x^2(4 - p^2) + 36(1 - 2\alpha)p^2 x^2(4 - p^2) \right), \]
\[z_2(p, x) = 4(1 - x^2)(4 - p^2)(1 - \alpha)^2 \left( (8\alpha^2 - 10\alpha + 3)p^3 + 9(1 - 2\alpha)p^3 x + (5 - 12\alpha)p^2 x + px(4 - p^2) \right), \]
\[z_3(p, x) = 4(1 - x^2)(4 - p^2)(1 - \alpha)^2 \left( 8(4 - p^2) + x^2(4 - p^2) + 9(1 - 2\alpha)p^2 x \right), \]
\[z_4(p, x) = 36(1 - x^2)(4 - p^2)(1 - \alpha)^2 \left( (4 - p^2)x + (1 - 2\alpha)p^2 \right). \]

Note that for \(\alpha \in [0, 1/6]\), all the factors involving \(\alpha\) in \(|\Delta_1(p, \gamma)|, |\Delta_2(p, \gamma)|, |\Delta_3(p, \gamma)|\) and \(|\phi(p, \gamma, \eta)|\) are positive as 1/6 is the smallest positive root of the equation \(2 - 15\alpha + 18\alpha^2 = 0\). We maximise \(Z(p, x, y)\) within the closed cuboid \(Y : [0, 2] \times [0, 1] \times [0, 1]\), by finding the maximum values in the interior of \(Y\), in the interior of the six faces and on the twelve edges.

**Case I:**
We begin with every interior point of \(Y\) assuming \((p, x, y) \in (0, 2) \times (0, 1) \times (0, 1)\). We determine \(\partial Z/\partial y\) to examine the points of maxima in the interior of \(Y\). Thus
\[
\frac{\partial Z}{\partial y} = \frac{(4 - p^2)(1 - x^2)(1 - \alpha^2)}{288} \left( 8(8 - 9x + x^2)y - 2p^2(1 - x)y(17 - x - 18\alpha) + p^3(3 - x^2 - 10\alpha + 8\alpha^2 + x(4 - 6\alpha)) + 4xp(5 + x - 12\alpha) \right).
\]
Now, \(\partial Z/\partial y = 0\) gives
\[
y = y_0 := \frac{4xp(5 + x - 12\alpha) + p^3(3 - x^2 - 10\alpha + 8\alpha^2 + x(4 - 6\alpha))}{2(1 - x)(-4(8 - x) + p^2(17 - x - 18\alpha))}.
\]
The existence of critical points require that \(y_0 \in (0, 1)\) and can only exist when
\[
2p^2(1 - x)(17 - x - 18\alpha) > -p^3(-3 + x^2 + 10\alpha - 8\alpha^2 - x(4 - 6\alpha)) + 4px(5 + x - 12\alpha) + 8(1 - x)(8 - x). \tag{2.3}
\]
We try finding the solution satisfying the inequality (2.3) for the critical points. The possible range for which \(y_0 \in (0, 1)\), is \((0, (3 + 2\alpha)/9) \times (\hat{p}, 2)\). Here
\[
\hat{p} := \frac{2(17 - 18\alpha)}{\hat{p}_1} - \frac{2^{4/3}(1 - i\sqrt{3})(17 - 18\alpha)^2}{\hat{p}_1(\hat{p}_2 + \sqrt{-256(17 - 18\alpha)^6 + (\hat{p}_2)^2})^{1/3}} - \frac{(1 + i\sqrt{3})(\hat{p}_2 + \sqrt{-256(17 - 18\alpha)^6 + (\hat{p}_2)^2})^{1/3}}{2^{4/3}p_1}.
\]
where
\[
\begin{align*}
\tilde{p}_1 & := 3(3 - 10\alpha + 8\alpha^2); \\
\tilde{p}_2 & := 63056 - 146016\alpha + 8640\alpha^2 + 183168\alpha^3 - 110592\alpha^4.
\end{align*}
\]

Therefore, a calculation reveals that the maxima attained in the interior of \(Y\) at each such \(y_0 \in (0, 1)\) is always less than \(4(1 - \alpha)^2/9\) for \(\alpha \in [0, 1/6]\).

**Case II:**

The interior of six faces of the cuboid \(Y\), is now under consideration, for the further calculations. 

**On \(p = 0\),** \(Z(p, x, y)\) turns into
\[
s_1(x, y) := \frac{(1 - \alpha)^2((1 - x^2)((8 + x^2)y^2 + 9x(1 - y^2))) + 9x^3\alpha}{18}, \quad x, y \in (0, 1).
\]

Since
\[
\frac{\partial s_1}{\partial y} = \frac{(1 - x^2)(x + 1)(8 - x)(1 - \alpha)^2y}{9} \neq 0, \quad x, y \in (0, 1).
\]

Thus, \(s_1\) has no critical point in \((0, 1) \times (0, 1)\).

**On \(p = 2\),** \(Z(p, x, y)\) reduces to
\[
Z(2, x, y) := \frac{\alpha(1 - \alpha)^2(1 - 2\alpha)^2(3 - 2\alpha)}{18}, \quad x, y \in (0, 1).
\]

**On \(x = 0\),** \(Z(p, x, y)\) becomes
\[
s_2(p, y) := \frac{(1 - \alpha)^2}{1152} \left( (\alpha(3 - 2\alpha)(1 - 2\alpha)^2p^6 + 36(1 - 2\alpha)p^2(4 - p^2)(1 - y^2) \\
+ 32(4 - p^2)^2y^2 + 4(3 - 10\alpha + 8\alpha^2)p^3y(4 - p^2) \right)
\] (2.6)

with \(p \in (0, 2)\) and \(y \in (0, 1)\). On solving \(\partial s_2/\partial p\) and \(\partial s_2/\partial y\), to find the points of maxima. After resolving \(\partial s_2/\partial y = 0\), we get
\[
y = \frac{p^3(3 - 10\alpha + 8\alpha^2)}{2(17p^2 - 32 - 18p^2\alpha)} (=: y_0).
\] (2.7)

Upon calculations, we observe that to have \(y_0 \in (0, 1)\) for the given range of \(y, p =: p_0 \approx A(\alpha)\) (see Fig. 1) is needed with \(\alpha \in [0, \beta_0]\). This \(\beta_0 \in [0, 1)\) is the smallest positive root of \(-3 + 10\alpha - 8\alpha^2 = 0\) and no such \(p \in (0, 2)\) exists when \(\alpha \in (\beta_0, 1)\). It is to be noted that the expression of \(A(\alpha)\) is complex but the coefficient of its imaginary part for \(\alpha \in [0, 1/6]\) is highly negative (of order \(10^{-15}\)), which can be neglected and \(A(\alpha)\) can be treated as a real number. Here,
\[
A(\alpha) := \frac{1}{3(-3 + 10\alpha - 8\alpha^2)} \left( 2(18\alpha - 17) - \frac{24^{1/3}(1 - i\sqrt{3})(18\alpha - 17)^2}{B} - \frac{(1 + i\sqrt{3})B}{24^{1/3}} \right),
\]
with
\[
B := \left( C + \sqrt{-256(18\alpha - 17)^6 + C^2} \right)^{1/3}
\]
and
\[
C := -63056 + 146016\alpha - 8640\alpha^2 - 183168\alpha^3 + 110592\alpha^4.
\]

Based on computations, \(\partial s_2/\partial p = 0\) gives
\[
0 = 16p(9 - 18\alpha - y^2(25 - 18\alpha)) - 2p^2y(3 - 10\alpha + 8\alpha^2)(5p^2 - 12) \\
+ 3\alpha(1 - 2\alpha)^2(3 - 2\alpha)p^5 - 8p^3(9 - 18\alpha - y^2(17 - 18\alpha)).
\] (2.8)
After substituting equation (2.7) into equation (2.8), we have
\[
0 = p \left( 49152(1 - 2\alpha) - 3072p^2(25 - 68\alpha + 36\alpha^2) - p^8(1 - 2\alpha)^2(153 - 1437\alpha \\
+ 3118\alpha^2 - 2484\alpha^3 + 648\alpha^4) + 16p^4(2427 - 7890\alpha + 6020\alpha^2 + 616\alpha^3 - 1024\alpha^4) \\
- 128p^6(48 - 153\alpha + 20\alpha^2 + 340\alpha^3 - 352\alpha^4 + 96\alpha^5) \right), 
\]
(2.9)
A numerical calculation suggests that the solution of (2.9) in the interval (0, 2) is \(p \approx B(\alpha)\) whenever \(\alpha \in [0, \alpha_2]\), where \(\alpha_2 \in (0, 1)\) is the smallest positive root of \(153 - 1437\alpha + 3118\alpha^2 - 2484\alpha^3 + 648\alpha^4 = 0\), otherwise no such \(p \in (0, 2)\) exists, see Fig. [1]. Thus, \(s_2\) does not have any critical point in \((0, 2) \times (0, 1)\).

Here
\[
B(\alpha) := \frac{1}{\sqrt{2}} \left[ \left\{ \frac{1}{F} \left( -3072 + 3648\alpha + 6016\alpha^2 - 9728\alpha^3 + 3072\alpha^4 \\
- 4F \left\{ \frac{1}{E^2} \left( \frac{768J^2}{(1 - 2\alpha)^2} + \frac{2GE}{(1 - 2\alpha)} + \frac{HE}{I} + \frac{IE}{(1 - 2\alpha)^2} \right) \right\}^{1/2} \\
+ \frac{4F}{\sqrt{3}} \left\{ -1 \left( \frac{-1536J^2E}{(1 - 2\alpha)^2} - \frac{4GE^2}{(1 - 2\alpha)} + \frac{HE^2}{I} + \frac{IE^2}{(1 - 2\alpha)^2} \right) \right\}^{1/2} \right\}^{1/2} \right],
\]
with
\[
E := 153 - 1437\alpha + 3118\alpha^2 - 2484\alpha^3 + 648\alpha^4; \\
F := (1 - 2\alpha)E; \\
G := 2427 - 3036\alpha - 52\alpha^2 + 512\alpha^3; \\
H := 8217 - 173160\alpha + 1260312\alpha^2 - 2415264\alpha^3 + 2091664\alpha^4 - 1048576\alpha^5 + 262144\alpha^6; \\
I := \left( 127065213 - 1886889978\alpha + 12579196752\alpha^2 - 4987149952\alpha^3 + 132494582880\alpha^4 \\
- 253944918720\alpha^5 + 368411062528\alpha^6 - 410152327680\alpha^7 + 340236674304\alpha^8 \\
- 19675891584\alpha^9 + 71861010432\alpha^{10} - 14168358912\alpha^{11} + 1073741824\alpha^{12} \\
+ 288\sqrt{2} \left( (1 - 2\alpha)^8(3 - 4\alpha)^2(3604621581 - 39763739565\alpha + 202125486510\alpha^2 \\
- 633657349224\alpha^3 + 1436021769744\alpha^4 - 2516421142080\alpha^5 + 34829931648\alpha^6 \\
- 3872882513280\alpha^7 + 3466438619648\alpha^8 - 2402201403136\alpha^9 + 1198713174528\alpha^{10} \\
- 394099818496\alpha^{11} + 75581358080\alpha^{12} - 6442450944\alpha^{13} \right) \right)^{1/2} \right)^{1/3}; \\
J := -48 + 57\alpha + 94\alpha^2 - 152\alpha^3 + 48\alpha^4;
\]
Figure 1. Graphical representation of $p$ versus $\alpha$. Here, $B(\alpha)$ (Red) and $A(\alpha)$ (blue) do not intersect for any choice of $\alpha$. Dashed black line represents $p = 2$.

and

$$K := 288\sqrt{3} \left( 15963705 - 129546873\alpha + 510658314\alpha^2 - 1308834456\alpha^3 + 2415583204\alpha^4 - 3321041560\alpha^5 + 3420107120\alpha^6 - 2619528892\alpha^7 + 1464766656\alpha^8 - 575732096\alpha^9 + 147709696\alpha^{10} - 21284352\alpha^{11} + 1179648\alpha^{12} \right).$$

On $x = 1$, $Z(p, x, y)$ reduces into

$$s_3(p, y) := \frac{(1 - \alpha)^2}{576} \left( 288\alpha + 16p^2(11 - 33\alpha + 9\alpha^2) - 8p^4(5 - 13\alpha + 5\alpha^2 + 3\alpha^3) - p^6(1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4) \right), \quad p \in (0, 2). \quad (2.10)$$

While computing $\partial s_3/\partial p = 0$, $p =: p_0 \approx 2L(\alpha)$ for $\alpha \in [0, \alpha_0) \cup (\alpha_0, \alpha_1)$, comes out to be the critical point, where $\alpha_0 \in [0, 1)$ is the smallest positive root of $1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4 = 0$ and $\alpha_1 (\approx 0.370803927) \in (0, 1)$ (see Fig. 2) is the largest value so that $p \in (0, 2)$ otherwise no such real $p \in (0, 2)$ exists beyond this $\alpha_1$. Here

$$L(\alpha) := \sqrt{\frac{-10 + 26\alpha - 10\alpha^2 - 6\alpha^3 + M}{3N}};$$

$$M := \sqrt{133 - 751\alpha + 1497\alpha^2 - 1630\alpha^3 + 1666\alpha^4 - 708\alpha^5 + 144\alpha^6};$$

$$N := 1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4.$$ 

$$\begin{bmatrix}
L(\alpha) := \sqrt{\frac{-10 + 26\alpha - 10\alpha^2 - 6\alpha^3 + M}{3N}}; \\
M := \sqrt{133 - 751\alpha + 1497\alpha^2 - 1630\alpha^3 + 1666\alpha^4 - 708\alpha^5 + 144\alpha^6}; \\
N := 1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4.
\end{bmatrix} \quad (2.11)$$
Figure 2. Graphical representation of $p_0$ versus $\alpha$. Here, $\text{Re}(p_0)$ (green) and $\text{Im}(p_0)$ (red) represent the value of $p_0$ at different $\alpha$, where $\alpha_1$ (blue circle) is the point at which $p_0$ transforms from completely real to imaginary. Dashed black line represents $p_0 = 2$.

Undergoing simple calculations, $s_3$ achieves its maximum value, approximately equals $P(\alpha)$, $[0, \alpha_0) \cup (\alpha_0, \alpha_1)$ at $p_0$. Here

$$P(\alpha) := \left( \frac{1 - \alpha}{486} \right)^2 \left( \begin{array}{ccc}
243\alpha - \frac{18(11-33\alpha+9\alpha^2)(10-26\alpha+10\alpha^2+6\alpha^3-M)}{N} \\
- \frac{12(5-13\alpha+5\alpha^2+3\alpha^3)(-10+26\alpha-10\alpha^2-6\alpha^3+M)^2}{N^2}
\end{array} \right) \quad (2.12)
$$

On $y = 0$, $Z(p, x, y)$ can be seen as

$$s_4(p, x) = \left( \frac{1 - \alpha}{1152} \right)^2 \left( 576 \left( x - x^3(1 - \alpha) \right) + 16p^2 \left( 9 - 18x + x^4 + x^3(28 - 33\alpha) 
- 18\alpha + x^2(2 - 15\alpha + 18\alpha^2) \right) 
- 4p^4 \left( 9 + 2x^4 + x^3(20 - 21\alpha) 
- 18\alpha + x^2(1 - 32\alpha + 52\alpha^2) + x(-12 + 19\alpha - 32\alpha^2 + 12\alpha^3) \right) 
+ p^6 \left( x^4 + \alpha(1 - 2\alpha)^2(3 - 2\alpha) + x^3(1 + 3\alpha) 
- x^2(1 + 17\alpha - 34\alpha^2) + x(-3 + 19\alpha - 32\alpha^2 + 12\alpha^3) \right) \right).$$
Furthermore, through some calculations, such as

\[
\frac{\partial s_4}{\partial x} = \frac{(1 - \alpha)^2}{1152} \left( 576 \left( 1 - 3x^2(1 - \alpha) \right) - 16p^2 \left( 18 - 4x^3 - 3x^2(28 - 33\alpha) \right) \\
- 2x(2 - 15\alpha + 18\alpha^2) \right) + p^6 \left( 4x^3 + 3x^2(1 + 3\alpha) - 2x(1 + 17\alpha \\
- 34\alpha^2) - 3 + 19\alpha - 32\alpha^2 + 12\alpha^3 \right) - 4p^4 \left( 8x^3 + 3x^2(20 - 21\alpha) \\
+ 2x(1 - 32\alpha + 52\alpha^2) - 12 + 19\alpha - 32\alpha^2 + 12\alpha^3 \right) \right)
\]

and

\[
\frac{\partial s_4}{\partial p} = \frac{(1 - \alpha)^2}{1152} \left( 32p \left( 9 - 18x + x^4 + x^3(28 - 33\alpha) - 18\alpha + x^2(2 - 15\alpha + 18\alpha^2) \right) \\
- 16p^3 \left( 9 + 2x^4 + x^3(20 - 21\alpha) - 18\alpha + x^2(1 - 32\alpha + 52\alpha^2) \right) \\
+ x(-12 + 19\alpha - 32\alpha^2 + 12\alpha^3) + 6p^5 \left( x^4 + \alpha(1 - 2\alpha)^2(3 - 2\alpha) \\
+ x^3(1 + 3\alpha) - x^2(1 + 17\alpha - 34\alpha^2) + x(-3 + 19\alpha - 32\alpha^2 + 12\alpha^3) \right) \right),
\]

indicates that there does not exist any common solution for the system of equations \( \partial s_4/\partial x = 0 \) and \( \partial s_4/\partial p = 0 \), thus, \( s_4 \) has no critical points in \((0, 2) \times (0, 1)\).

On \( y = 1 \), \( Z(p, x, y) \) reduces to

\[
s_5(p, x) : = \frac{(1 - \alpha)^2}{1152} \left( 64px(1 - x^2)(5 + x - 12\alpha) + 64(8 - 7x^2 - x^4 + 9\alpha) \\
+ 16p^3(1 - x^2)(3 - x - 2x^2 - 10\alpha + 6x\alpha + 8\alpha^2) - 2p^6(1 - 4\alpha \\
- 16\alpha^3 + 4\alpha^4) + 16p^2 \left( 6 + 14x^2 + 2x^4 + x(9 - 18\alpha) - 66\alpha \\
+ 18\alpha^2 - 9x^3(1 - 2\alpha) \right) + 4p^5(1 - x^2) \left( x^2 - 3 + 10\alpha - 8\alpha^2 \\
- x(4 - 6\alpha) \right) - 4p^4 \left( 7x^2 + x^4 + x(9 - 18\alpha) - 9x^3(1 - 2\alpha) \\
+ 4(3 - 13\alpha + 5\alpha^2 + 3\alpha^3) \right) \right).
\]

We note that the equations \( \partial s_5/\partial x = 0 \) and \( \partial s_5/\partial p = 0 \) possess no common solution in \((0, 2) \times (0, 1)\).

**Case III:** Now, we determine the maximum values that \( Z(p, x, y) \) may obtain on the edges of the cuboid \( Y \).

From equation (2.6), we have

\[
Z(p, 0, 0) = r_1(p) := \frac{p^2(1 - \alpha)^2(1 - 2\alpha)(144 - 36p^2 + p^4\alpha(3 - 8\alpha + 4\alpha^2))}{1152}.
\]

Here, we consider the following three subcases for different choices of \( \alpha \).
(1) For \( \alpha = 0 \), \( r_1(p) \) reduces to \( p^2(4 - p^2)/32 \) and \( r_1'(p) = 0 \) for \( p = 0 \), the point of minima and \( p = \sqrt{2} \), the point of maxima. Therefore
\[
Z(p, 0, 0) \leq \frac{1}{8}, \quad p \in [0, 2].
\]

(2) For \( \alpha = 1/2 \), \( r_1(p) = 0 \).

(3) For \( \alpha = (0, 1/2) \cup (1/2, 1) \), \( r_1'(p) = p(1 - \alpha)^2(1 - 2\alpha)(48 - 24p^2 + p^4\alpha(3 - 8\alpha + 4\alpha^2)) = 0 \) for \( p = 0 \) and \( p = 2 \left( \frac{(3 - R(\alpha))/3\alpha - 8\alpha^2 + 4\alpha^3}{1/2} \right) \) as the points of minima and maxima respectively. So,
\[
Z(p, 0, 0) = R_0(\alpha) := \frac{(1 - \alpha)^2(3 - R(\alpha))(-3 + 6\alpha - 16\alpha^2 + 8\alpha^3 + R(\alpha))}{6(3 - 2\alpha)^2\alpha^2(1 - 2\alpha)},
\]
with \( R(\alpha) := \sqrt{3(3 - 3\alpha + 8\alpha^2 - 4\alpha^3)} \).

Now, equation (2.6) at \( y = 1 \), implies that \( Z(p, 0, 1) = r_2(p) := (1 - \alpha)^2(32(4 - p^2)^2 + \alpha(1 - 2\alpha)^2(3 - 2\alpha)p^6 + 4p^3(4 - p^2)(3 - 10\alpha + 8\alpha^2))/1152 \). Note that \( r_2'(p) \) is a decreasing function in \([0, 2]\) and hence \( p = 0 \) becomes the point of maxima. Thus
\[
Z(p, 0, 1) \leq \frac{4(1 - \alpha)^2}{9}, \quad p \in [0, 2].
\]

Through calculations, equation (2.6) shows that \( Z(0, 0, y) \) attains its maximum value at \( y = 1 \), which implies that
\[
Z(0, 0, y) \leq \frac{4(1 - \alpha)^2}{9}, \quad y \in [0, 1].
\]

Since, the equation (2.10) is free from \( y \), we have
\[
Z(p, 1, 1) = Z(p, 1, 0) = r_3(p) := \frac{(1 - \alpha)^2}{576} \left( 288\alpha + 16p^2(11 - 33\alpha + 9\alpha^2) \right.
\]
\[
- 8p^4(5 - 13\alpha + 5\alpha^2 + 3\alpha^3)
\]
\[
- p^6(1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4) \right).
\]

Now, \( r_3'(p) = 32p(11 - 33\alpha + 9\alpha^2) - 32p^3(5 - 13\alpha + 5\alpha^2 + 3\alpha^3) - 6p^5(1 - 4\alpha + 6\alpha^2 - 16\alpha^3 + 4\alpha^4) = 0 \)
when \( p = \delta_1 := 0 \) and \( p = \delta_2 := 2L(\alpha) \) for \( \alpha \in [0, \alpha_0) \cup (\alpha_0, \alpha_1) \), as the points of minima and maxima respectively, in the interval \([0, 2]\). The justification of \( P(\alpha) \), \( \alpha_0 \) and \( \alpha_1 \) are provided above through equation (2.11) and (2.12). Thus, from equation (2.10),
\[
Z(p, 1, 1) = Z(p, 1, 0) \leq P(\alpha), \quad p \in [0, 2] \quad \text{and} \quad \alpha \in [0, \alpha_0) \cup (\alpha_0, \alpha_1).
\]

Consider equation (2.10) at \( p = 0 \), we get
\[
Z(0, 1, y) = \frac{\alpha(1 - \alpha)^2}{2}.
\]

Equation (2.5) indicates that
\[
Z(2, 1, y) = Z(2, 0, y) = Z(2, x, 0) = Z(2, x, 1) = \frac{\alpha(1 - 2\alpha)^2(1 - \alpha)^2(3 - 2\alpha)}{18}.
\]

Using equation (2.4), \( Z(0, x, 1) = r_4(x) := (1 - \alpha)^2(8 - 7x^2 - x^4 + 9x^3\alpha)/18 \). Upon calculations, we see that \( r_4 \) is a decreasing function of \( x \) in \([0, 1]\) and therefore \( x = 0 \) is the point of maxima. Hence
\[
Z(0, x, 1) \leq \frac{4(1 - \alpha)^2}{9}, \quad x \in [0, 1].
\]
On again using equation (2.4), $Z(0, x, 0) = r_5(x) := x(1 - (1 - \alpha)x^2)(1 - \alpha)^2/2$. Moreover, $r'_5(x) = 0$ when $x = \delta_3 := 1/\sqrt{3(1 - \alpha)}$. Observe that $r_5(x)$ increases in $[0, \delta_3)$ and decreases in $(\delta_3, 1]$. Hence, $Z(0, x, 0) \leq (1 - \alpha)^2 / (3\sqrt{3(1 - \alpha)})^2$, $x \in [0, 1]$. 

We also provide a graphical representation of six upper-bounds (u.b) of $H_3(1)$ in Fig. 3. Given all the cases, the sharp inequality $|H_3(1)| \leq 4(1 - \alpha)^2 / 9$, holds for every $\alpha \in [0, 1/6] \cup \{1/2\}$.

Now, we provide remarks which incorporate the bound of $|H_3(1)|$ for the class $S^*$ and $S^*(1/2)$, which are subclasses of $S^*(\alpha)$, given as follows:

**Remark 2.2.** On substituting $\alpha = 0$ in Theorem 2.1, $S^*(0) = S^*$ and from equation (2.1), we get $|H_3(1)| \leq 4/9$. This bound is sharp and coincides with that of Kowalczyk et al. [6] and Banga and Kumar [2].

**Remark 2.3.** On substituting $\alpha = 1/2$ in Theorem 2.1, $S^*(1/2)$ and from equation (2.1), we get $|H_3(1)| \leq 1/9$. This bound is sharp and coincides with that of Rath et al. [18].

For some already known sharp bounds of $H_3(1)$, regarding various choices of $\varphi(z)$, See Table 1. We note that the same bound is not available for $\varphi(z) := 1 + z - z^3/3$. Hence, as an application of Theorem 2.1, we provide a better bound of $|H_3(1)|$ for functions belonging to the class, $S_{Ne}^* := S^*(1 + z - z^3/3)$. 

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**TABLE 1.** Sharp bounds of $H_3(1)$ for various choices of $\varphi(z)$.
Corollary 2.4. If $f \in S_{Ne}^*$. Then $|H_3(1)| \leq 32/81 \approx 0.395062$.

Proof. From [20], we have

$$\min_{|z|=r} \text{Re}(\varphi(z)) = \begin{cases} 1 - r + \frac{1}{2}r^3, & r \leq 1/\sqrt{3} \\ 1 - \frac{1}{3}(1 + r^2)^{3/2}, & r \geq 1/\sqrt{3}. \end{cases}$$

We note that $\alpha = \min_{|z|=r} \text{Re}(\varphi(z)) = 1 - 2\sqrt{2}/3$ as $r$ tends to 1. Now, substitution of $\alpha = 1 - 2\sqrt{2}/3 \approx 0.057191 \in [0, 1/6]$ in equation (2.1) implies that $|H_3(1)| \leq 32/81 \approx 0.395062$. $\blacksquare$

Open Problem:
We have attempted to provide the sharp bound of $H_3(1)$ for functions, $f \in S^*(\alpha)$ for $\alpha \in [0, 1/6] \cup \{1/2\}$ in Theorem 2.1. Further, this result is still open for the remaining range of $\alpha$ in $[0, 1)$.

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Department of Applied Mathematics, Delhi Technological University, Delhi–110042, India
Email address: nehaverma1480@gmail.com

Department of Applied Mathematics, Delhi Technological University, Delhi–110042, India
Email address: spkumar@dce.ac.in