Quantum speed limits for adiabatic evolution, Loschmidt echo and beyond

N. Il’in
Skolkovo Institute of Science and Technology,
Skolkovo Innovation Center 3, Moscow 143026, Russia

O. Lychkovskiy\textsuperscript{1,2}
\textsuperscript{1}Skolkovo Institute of Science and Technology,
Skolkovo Innovation Center 3, Moscow 143026, Russia
\textsuperscript{2}Steklov Mathematical Institute of Russian Academy of Sciences,
Gubkina str., 8, Moscow 119991, Russia

Abstract. One often needs to estimate how fast an evolving state of a quantum system can depart from some target state or target subspace of a Hilbert space. Such estimates are known as quantum speed limits. We derive a quantum speed limit for a general time-dependent target subspace. When the target subspace is an instantaneous invariant subspace of a time-dependent Hamiltonian, the obtained quantum speed limit bounds the adiabatic fidelity, which is a figure of merit of quantum adiabaticity. We also compare two states evolving under two different Hamiltonians and derive a bound on the Loschmidt echo.

1. Introduction

Consider a quantum system with a time–dependent Hamiltonian $H_t$ and a density matrix $\rho_t$ evolving under the Schrödinger equation

$$i\dot{\rho}_t = [H_t, \rho_t]$$

(here and in what follows dot stands for the time derivative and $\hbar = 1$). It is often of interest to assess the probability to find the state of the system in some “target” subspace $\mathcal{L}_t$ of the Hilbert space. The subspace $\mathcal{L}_t$ can be described by a projector

$$\Pi_t = \Pi_t^2.$$  \hspace{1cm} (2)

The above-mentioned probability is given by

$$F_t \equiv \text{tr}\ \rho_t \Pi_t.$$ \hspace{1cm} (3)

For a pure state, $\rho_t = |\psi_t\rangle\langle\psi_t|$, and a one-dimensional target subspace with $\Pi_t = |\phi_t\rangle\langle\phi_t|$ the above probability fits a standard definition of quantum fidelity of two states, $F_t = |\langle\phi_t|\psi_t\rangle|^2$. We will refer to $F_t$ as fidelity also in a general case.
Often $F_t$ is estimated for some particular $\Pi_t = \Pi$ which actually does not depend on time. In particular, a popular object of study is $\Pi = |\psi_0\rangle\langle\psi_0|$, in which case $F_t$ quantifies how far the dynamical state $\psi_t$ departs from the initial state $\psi_0$ in time $t$. A bunch of inequalities bounding $F_t$ from below (and, sometimes, also from above) are known as quantum speed limits (see Refs. [1, 2] for reviews). A rigorous formulation of the time-energy uncertainty relation by Mandelstam and Tamm [3] can be viewed as the earliest quantum speed limit.

A very general quantum speed limit valid for an arbitrary, possibly time-dependent target space has been derived by Pfeifer and Fröhlich [1]. Here we also focus on time-dependent target spaces but use a different method. The paper is organized as follows. In the next Section we derive a general bound for an arbitrary dependence of $\mathcal{L}_t$ on time which differs from the bound of Ref. [1]. In particular, we derive a quantum speed limit for the case when $\mathcal{L}_t$ is an instantaneous invariant subspace of the Hamiltonian or its eigenspace, than $F_t$ quantifies to what extent the evolution is adiabatic, and the derived bound on $F_t$ constitutes an adiabatic condition. In Section 3 we compare our method and result with those of Ref. [1] and detail the differences. In Section 4 we consider states evolving under two different Hamiltonians and prove a bound on the corresponding Loschmidt echo.

Throughout the paper we employ the following conventions. We consider finite-dimensional Hilbert spaces. We assume that all time-dependent operators and vectors in the Hilbert space are continuous and differentiable. An operator norm of an operator $A$ is denoted as $||A||$. The Hamiltonian $H_t$ is a self-adjoint operator smoothly depending on the parameter $t$. Occasionally we use bra-ket notations for vectors and projectors. We use symbol $\Pi$ for a general projection operator and $P_\psi \equiv |\psi\rangle\langle\psi|$ for a one-dimensional projector on a pure state $\psi$. The norm of a vector $\psi$ is denoted as $||\psi|| \equiv \langle\psi|\psi\rangle^{1/2}$.

### 2. Generalized quantum speed limit

We assume that $\mathcal{L}_t$ is an arbitrary subspace of the Hilbert space which smoothly varies with time. The respective projector $\Pi_t$ is also arbitrary (in particular, it need not commute with the Hamiltonian). The initial state of the system, $\rho_0$, is arbitrary as well (in particular, it need not belong to $\mathcal{L}_0$). Under these conditions the following lower bounds on the fidelity $F_t$ can be proven:

**Theorem 1.**

$$F_t \geq \cos^2 \left\{ g_0 + \int_0^t \|i[H_\tau, \Pi_\tau] + \dot{\Pi}_\tau\| d\tau \right\} \quad \text{for } t \in [0, t^+],$$

$$F_t \leq \cos^2 \left\{ g_0 - \int_0^t \|i[H_\tau, \Pi_\tau] + \dot{\Pi}_\tau\| d\tau \right\} \quad \text{for } t \in [0, t^-],$$

where

$$g_0 = \arccos \left\{ \sqrt{\text{tr} \rho_0 \Pi_0} \right\},$$

(4)

(5)

(6)
t^+ is the single root of the equation \( g_0 + \int_0^{t^+} \|i[H_\tau, \Pi_\tau]| + \dot{\Pi}_\tau \|d\tau = \pi/2 \),
t^− is the single root of the equation \( g_0 - \int_0^{t^-} \|i[H_\tau, \Pi_\tau]| + \dot{\Pi}_\tau \|d\tau = 0 \).

In a particular case of a one-dimensional projector \( \Pi_t = |\phi_t\rangle\langle\phi_t| \), where \( \phi_t \) is an arbitrary vector smoothly dependent on time, one immediately obtains a quantum speed limit first derived in ref. [1]:

**Corollary.**

\[
F_t \geq \cos^2 \left\{ g_0 + \int_0^t \sqrt{\|iH_\tau\phi_\tau + \dot{\phi}_\tau\|^2 - |\langle iH_\tau\phi_\tau + \dot{\phi}_\tau|\phi_\tau\rangle|^2} d\tau \right\} \quad \text{for } t \in [0, t^+],
\]

\[
F_t \leq \cos^2 \left\{ g_0 - \int_0^t \sqrt{\|iH_\tau\phi_\tau + \dot{\phi}_\tau\|^2 - |\langle iH_\tau\phi_\tau + \dot{\phi}_\tau|\phi_\tau\rangle|^2} d\tau \right\} \quad \text{for } t \in [0, t^-],
\]

where

\[
g_0 = \arccos \left\{ \sqrt{\langle \phi_0|\rho_0|\phi_0\rangle} \right\},
\]

t^+ is the single root of \( g_0 + \int_0^{t^+} \sqrt{\|iH_\tau\phi_\tau + \dot{\phi}_\tau\|^2 - |\langle iH_\tau\phi_\tau + \dot{\phi}_\tau|\phi_\tau\rangle|^2} d\tau = \pi/2 \),
t^− is the single root of \( g_0 - \int_0^{t^-} \sqrt{\|iH_\tau\phi_\tau + \dot{\phi}_\tau\|^2 - |\langle iH_\tau\phi_\tau + \dot{\phi}_\tau|\phi_\tau\rangle|^2} d\tau = 0 \).

**Proof of Theorem 1. Special case of** \([H_t, \Pi_t] = 0 \). For an arbitrary \( \Pi_t \) one obtains

\[
\dot{\Pi}_t = \text{tr } \rho_t \dot{\Pi}_t + \text{tr } \rho_t \dot{\Pi}_t = i \text{tr } (\rho_t [H_t, \Pi_t]) + \text{tr } \rho_t \dot{\Pi}_t.
\]

We first assume that \([H_t, \Pi_t] = 0 \). This is the case, in particular, when \( \Pi_t \) projects on an instantaneous invariant subspace of the Hamiltonian. Under this assumption the first term in the right hand side (r.h.s.) of the above equation is zero, and we are left with

\[
\dot{\Pi}_t = \text{tr } \rho_t \dot{\Pi}_t.
\]

Recall some well-known properties of \( \Pi_t \) [4]. Differentiating the equality \( \Pi_t = \Pi_t^2 \) one obtains \( \dot{\Pi}_t = \dot{\Pi}_t \Pi_t + \Pi_t \dot{\Pi}_t \), which implies \( \Pi_t \dot{\Pi}_t \Pi_t = 0 \) and

\[
\dot{\Pi}_t = (\mathbb{I} - \Pi_t) \dot{\Pi}_t + \Pi_t \dot{\Pi}_t (\mathbb{I} - \Pi_t) = \left[ \dot{\Pi}_t, \Pi_t, \Pi_t \right].
\]

This equation along with eq. (10) leads to

\[
\dot{\Pi}_t = 2 \text{Re } \text{tr } (\rho \Pi_t \dot{\Pi}_t (\mathbb{I} - \Pi_t)).
\]

This can be bounded from above as

\[
|\dot{F}_t| \leq 2 |\text{tr } ((\Pi_t \sqrt{\rho})^\dagger \dot{\Pi}_t (\mathbb{I} - \Pi_t) \sqrt{\rho})| \\
\leq 2 \sqrt{\text{tr } (\rho \Pi_t) \text{tr } (\Pi_t^2 (\mathbb{I} - \Pi_t) \rho (\mathbb{I} - \Pi_t))} \\
\leq 2 \|\Pi_t\| \sqrt{\text{tr } \rho \Pi_t \sqrt{\text{tr } \rho (\mathbb{I} - \Pi_t)},
\]
where we use the Schwartz inequality $|\text{tr}(X^\dagger Y)| \leq \sqrt{\text{tr}(X^\dagger X)\text{tr}(Y^\dagger Y)}$ in the second line, the inequality $|\text{tr}(AB)| \leq \|A\|\text{tr} B$ and the equality $\|A^2\| = \|A\|^2$ for any $A = A^\dagger, B \geq 0$ in the third line. Noting that $\sqrt{\text{tr} \rho_t \Pi_t} = \sqrt{F_t}$ and $\sqrt{\text{tr} \rho_t (I - \Pi_t)} = \sqrt{1 - F_t}$, we obtain

$$|\dot{F}_t| \leq 2\|\dot{\Pi}_t\|\sqrt{F_t(1 - F_t)}. \quad (15)$$

Next we employ a substitution

$$F_t = \cos^2 g_t \quad (16)$$

with $g_t \in [0, \frac{\pi}{2}]$. This way we obtain

$$|\dot{g}_t| \leq \|\dot{\Pi}_t\|. \quad (17)$$

Integrating $\dot{g}_t$ one obtains

$$|g_t - g_0| = \left| \int_0^t \dot{g}_\tau d\tau \right| \leq \int_0^t \left| \dot{g}_\tau \right| d\tau \leq \int_0^t \|\dot{\Pi}_\tau\| d\tau. \quad (18)$$

In view of eq. (16), this leads to the bounds

$$F_t \geq \cos^2 \left\{ g_0 + \int_0^t \|\dot{\Pi}_\tau\| d\tau \right\} \quad \text{for} \quad t \in [0, t^+], \quad (19)$$

$$F_t \leq \cos^2 \left\{ g_0 - \int_0^t \|\dot{\Pi}_\tau\| d\tau \right\} \quad \text{for} \quad t \in [0, t^-], \quad (20)$$

where

$$g_0 = \arccos \left\{ \sqrt{\text{tr} \rho_0 \Pi_0} \right\}, \quad (21)$$

t$^+$ is the single root of the equation $g_0 + \int_0^{t^+} \|\dot{\Pi}_\tau\| d\tau = \pi/2$,

t$^-$ is the single root of the equation $g_0 - \int_0^{t^-} \|\dot{\Pi}_\tau\| d\tau = 0$.

**Proof of Theorem 1. General case.**

To proceed with a case of a general $\Pi_t$ we define a new projector $\Pi_t^U \equiv U_t^\dagger \Pi_t U_t$, where the unitary operator $U_t$ satisfies the Schrödinger equation

$$i\dot{U}_t = H_t U_t. \quad (22)$$

Since $F_t \equiv \text{tr}(\rho_t \Pi_t) = \text{tr}(\rho_0 \Pi_t^U)$, one obtains

$$\dot{F}_t = \text{tr}(\rho_0 \dot{\Pi}_t^U). \quad (23)$$

This equation is analogous to eq. (11). Thus one can simply substitute $\dot{\Pi}_t$ in eqs. (19) and (20) by

$$\dot{\Pi}_t^U = U_t^\dagger (i[\Pi_t, H_t] + \dot{\Pi}_t) U_t. \quad (24)$$

This way one obtains estimates (4) and (5).
Appendix A

Proof of the Corollary.
Consider a one-dimensional \( L_t \) with \( \Pi_t = P_{\phi_t} \equiv |\phi_t\rangle\langle\phi_t| \). The key idea is to use the auxiliary projector \( \Pi_t^U \) introduced above. In the case under consideration it reads

\[
\Pi_t^U = U_t^† P_{\psi_t} U_t = P_{\psi_t},
\]

where the unitary operator \( U_t \) satisfies the Schrödinger equation (22), \( \psi_t = U_t^† \phi_t \) and \( \dot{\psi}_t = U_t^† (i H_t \phi_t + \dot{\phi}_t) \). The Corollary follows from eq. (24) and the equality

\[
\| \hat{P}_{\phi_t} \| = \sqrt{\langle \dot{\psi}_t | \dot{\psi}_t \rangle - |\langle \dot{\psi}_t | \psi_t \rangle|^2}.
\] (25)

The latter inequality is valid for any normalized vector \( \psi_t \) smoothly dependent on time, as is shown in Appendix A.

Several remarks are in order.

Remark 1. Observe that the bounds (7, 8) are invariant under the transformations \( \phi_t \rightarrow e^{i \theta_t} \phi_t \), where \( \theta_t \) is an arbitrary smooth real function of time.

Remark 2. When \( \phi_t \) is an instantaneous eigenvector of the Hamiltonian \( H_t \), \( H_t \phi_t = E_t \phi_t \), the integrand in (7, 8) does not contain \( E_t \) explicitly, since

\[
\| i E_t \phi_t + \dot{\phi}_t \|^2 - |\langle i E_t \phi_t + \dot{\phi}_t | \phi_t \rangle|^2 = \| \dot{\phi}_t \|^2 - |\langle \dot{\phi}_t | \phi_t \rangle|^2.
\] (26)

Remark 3. Inequality (19) with \( g_0 = 0 \) represents a sufficient adiabatic condition. Indeed, assume that \( \Pi_t \) projects on an instantaneous invariant subspace of the Hamiltonian, and the support of \( \rho_0 \) belongs to this subspace (the latter implies \( F_0 = 1 \) and \( g_0 = 0 \)). Quantum evolution is said to be adiabatic with a precision \( \varepsilon \) as long as \( 1 - F_t < \varepsilon \), and \( F_t \) is referred to as adiabatic fidelity in this context. Inequality (19) guarantees this for times smaller than the smallest positive root \( t_0 \) of the equation

\[
\varepsilon = \cos^2 \left( \int_0^{t_0} \| \Pi_t \|^2 d\tau \right).
\]

In the case of a one-dimensional projector eqs. (19) and (25) lead to the following bound on the adiabatic fidelity:

\[
|\langle \phi_t | \psi_t \rangle|^2 \geq \cos^2 \left\{ \int_0^t \sqrt{\| \dot{\phi}_t \|^2 - |\langle \dot{\phi}_t | \phi_t \rangle|^2} d\tau \right\} \text{ for } t \in [0, t^+],
\] (27)

where \( t^+ \) is the single root of the equation

\[
\int_0^{t^+} \sqrt{\| \dot{\phi}_t \|^2 - |\langle \dot{\phi}_t | \phi_t \rangle|^2} d\tau = \pi/2.
\]

Here \( \phi_t \) is an instantaneous eigenvector smoothly varying with time, \( H_t \phi_t = E_t \phi_t \), and \( \psi_t \) is a solution of the Schrödinger equation \( i \dot{\psi}_t = H_t \psi_t \) with the initial condition \( \psi_0 = \phi_0 \).

It should be stressed that the adiabatic conditions (19) and (27) do not allow one to diminish the adiabatic error arbitrarily by rescaling the time (such rescaling corresponds to evolving along the same path in the parameter space with a different pace). Thus they are very different from the sufficient adiabatic conditions which are used to prove the adiabatic theorem [4, 5]. In fact, bounds (19) and (27) work best at small times. In particular, they capture the quadratic scaling of \( 1 - F_t \) with time, which is characteristic for the initial stage of evolution starting form an instantaneous eigenstate.
Remark 4. For an instantaneous eigenstate $\phi_t$ of the Hamiltonian $H_t$ one can prove [6] that

$$\| \dot{P}_{\phi_t} \| = \sqrt{\langle \dot{\phi}_t | \dot{\phi}_t \rangle - |\langle \dot{\phi}_t | \phi_t \rangle|^2} \leq \| \dot{H}_t \| / \Delta_t,$$  
(28)

where $\Delta_t$ is the energy gap between $\phi_t$ and the closest other eigenstate of $H_t$. A similar but more tight bound can be obtained under additional assumptions [7]. Eq. (27) can be supplemented by these bounds in cases when the direct calculation of $\| \dot{P}_{\phi_t} \|$ is not possible.

Remark 5. One can always find a (nonunique) unitary operator $W_t$ which generates the subspace $L_t$ from $L_0$, i.e. $\Pi_t = W_t \Pi_0 W_t^\dagger$. If calculating $\| \dot{\Pi}_t \|$ is for some reason easier than $\| \dot{\Pi}_t \|$, one can proceed as follows. First, note that

$$\| \dot{\Pi}_t \| \leq \| \dot{W}_t \|. \quad (29)$$

We prove this bound and elaborate upon it in Appendix B. For $\Pi_t^U = U_t^\dagger \Pi_0 U_t$, $Y_t = U_t^\dagger W_t$ we have $\dot{Y}_t = U_t^\dagger (iH_t + \dot{W}_t W_t^\dagger) W_t$. So we can plug the bound

$$\| i[H_t, \Pi_t] + \dot{\Pi}_t \| = \| \dot{\Pi}_t^U \| \leq \| H_t - iW_t W_t^\dagger \|$$  
(30)

to eqs. (4) and (5).

Remark 6. While we have considered finite-dimensional Hilbert spaces, a generalisation of our results to the infinite-dimensional case is well conceivable. However, in the latter case one should take care of the fact that some relevant operators may become unbounded. In particular, it can happen that $\| \dot{P}_{\phi_t} \|$ is finite but $\| \dot{H}_t \| = \infty$ (such situation has been encountered in recent studies of a driven system consisting of a one-dimensional quantum fluid with an impurity particle immersed in it [8, 9, 10]). In this case one can not use eq. (28) with the bound of the type (27). This issue calls for tighter estimates of $\| \dot{P}_{\phi_t} \|$.

3. Comparison to the approach by Pfeifer and Fröhlich

A different approach to obtaining quantum speed limits for time-dependent target subspaces was elaborated by Pfeifer and Fröhlich [11, 1]. Here we review their approach and show that our method provides tighter bounds when the dimension of the target subspace is large.

Following Ref. [1], we define a function $f(R, A)$ of a self-adjoint operator $A = A^\dagger$ and a self-adjoint positive operator $R = R^\dagger \geq 0$ which generalises the notion of quantum uncertainty. Let $R = \sum_n \lambda_n \Pi_n$ be the spectral decomposition of $R$, where $\lambda_n$ are distinct eigenvalues and $\Pi_n$ are corresponding eigenprojectors. Then

$$f(R, A) \equiv \sqrt{\sum_n \lambda_n \text{tr} (\Pi_n A^2 - (\Pi_n A)^2)}.$$  
(31)

Note that the rank of $\Pi_n$ is equal to the degeneracy of the corresponding eigenvalue. As a consequence, $f(R, A)$ is not continuous with respect to $R$. 

Consider the case of $R = \Pi$, where $\Pi$ is a projector. Then
\[
f^2(\Pi, A) = -\frac{1}{2} \text{tr}[\Pi, A]^2,
\]
where $A$ is an arbitrary self-adjoint operator, and
\[
f^2(\Pi, \rho) \leq \sqrt{\text{tr} \rho \Pi (1 - \text{tr} \rho \Pi)},
\]
where $\rho$ is an arbitrary density matrix [1].

Importantly, for any self-adjoint positive $R$ and any two self-adjoint operators $A$ and $B$ a generalized uncertainty relation holds [1]:
\[
|\text{tr} R[A, B]| \leq 2 f(R, A) f(R, B).
\]

Now we are prepared to review the approach of ref. [1] and compare it to ours. For simplicity, we consider the case $[H_t, \Pi] = 0$ and $\Pi_0 \rho_0 = \rho_0$. Due to eq. (12) $\Pi_t$ is a solution of the Schrödinger-like equation $i\dot{\Pi}_t = \mathcal{H}_t \Pi_t$ with a fictitious Hamiltonian $\mathcal{H}_t = i[\Pi_t, \Pi_t]$. Using the inequality (34), we obtain
\[
|\dot{F}_t| = |\text{tr} \rho_t \dot{\Pi}_t| = |\text{tr} \rho_t [\mathcal{H}_t, \Pi_t]| = |\text{tr} \Pi_t [\mathcal{H}_t, \rho_t]| \leq 2 f(\Pi_t, \mathcal{H}_t) f(\Pi_t, \rho_t).
\]
Following (32,33) we get $f(\Pi_t, \rho_t) \leq \sqrt{\text{tr} \rho_t \Pi_t (1 - \text{tr} \rho_t \Pi_t)} = \sqrt{\text{tr} \Pi_t^2} / 2$. The inequality (35) then reduces to
\[
|\dot{F}_t| \leq 2 \sqrt{\frac{\text{tr} \Pi_t^2}{2}} \sqrt{F_t(1 - F_t)}.
\]

Analogously to the proof of Theorem 1, this leads to the following inequality for $F_t$:
\[
F_t \geq \cos^2 \left\{ \frac{1}{\sqrt{2}} \int_0^t \sqrt{\text{tr} \dot{\Pi}_t^2} d\tau \right\} \quad \text{for} \quad t \in [0, t^*],
\]
where $t^*$ is the single root of the equation $\int_0^{t^*} \sqrt{\text{tr} \dot{\Pi}_t^2} d\tau / \sqrt{2} = \pi / 2$.

Our aim is to compare the bound (37) obtained along the lines of ref. [1] to our bound (19) (with $g_0 = 0$). First we note that for a one-dimensional projector $P_{\phi_t} = |\phi_t\rangle \langle \phi_t|$ these bounds coincide, since $\sqrt{\text{tr} \dot{P}_{\phi_t}^2} = \sqrt{2} \| \dot{P}_{\phi_t} \|$. For higher-dimensional projectors our bound (19) tends to be tighter than the bound (37). Below we construct an example which makes this apparent.

Consider $\Pi_t = \sum_n P_{n,t}$, where $N$ orthogonal one-dimensional projectors $P_{n,t}$ satisfy $\dot{P}_{n,t} \dot{P}_{m,t} = 0$ for $n \neq m$ and $\| P_{n,t} \| = \| \dot{P}_{n,t} \|$, $\text{tr} \dot{P}_{n,t}^2 = \text{tr} \dot{P}_{m,t}^2$ for any $n$ and $m$. This can be the case e.g. when the corresponding vectors evolve each in its own subspace orthogonal to all other subspaces, the evolution of all vectors being identical otherwise. It is easy to verify that the bound (37) reduces to
\[
F_t \geq \cos^2 \left\{ \sqrt{N} \int_0^t \| \dot{P}_{1,t} \| d\tau \right\},
\]
while our bound (19) reads

\[ F_t \geq \cos^2 \left\{ \int_0^t \left\| \dot{P}_{1,\ell} \right\| \, d\tau \right\}. \tag{39} \]

The latter inequality is obviously tighter than the former, the difference becoming dramatic for large \( N \). We believe that this simple example captures the general tendency for high-dimensional target subspaces. We expect that the improvement provided by our result over the prior work \([1,11]\) can prove particularly important for studies of adiabaticity in many-body systems \([12,13,14,15]\).

4. Evolution under two different Hamiltonians

Here we consider a problem of comparing states of two quantum systems evolving under two different Hamiltonians. We are interested in pure states \( \psi^{(1)}_t \) and \( \psi^{(2)}_t \) evolving under Hamiltonians \( H^{(1)}_t \) and \( H^{(2)}_t \), respectively. We assume that initially the states coincide, \( \psi^{(1)}_0 = \psi^{(2)}_0 \). In this context \( F_t = |\langle \psi^{(1)}_t | \psi^{(2)}_t \rangle|^2 \) can be interpreted as the Loschmidt echo which plays an important role in quantum chaos \([16]\) and elsewhere \([17]\). We assume that \( \psi^{(2)}_t \) is known (e.g. due to the integrability of \( H^{(2)}_t \)) but \( \psi^{(1)}_t \) is not, so the direct evaluation of the Loschmidt echo is not possible. It can be estimated, however, due to following theorem.

**Theorem 2.**

\[ F_t \geq \cos^2 \left\{ \int_0^t \sqrt{\left\| (H^{(1)}_\tau - H^{(2)}_\tau) \psi^{(2)}_\tau \right\|^2 - \langle \psi^{(2)}_\tau | H^{(1)}_\tau - H^{(2)}_\tau | \psi^{(2)}_\tau \rangle^2} \, d\tau \right\} \text{ for } t \in [0,t^*], \tag{40} \]

where 

\[ t^* \text{ is the single root of the equation} \]

\[ \int_0^{t^*} \sqrt{\left\| (H^{(1)}_\tau - H^{(2)}_\tau) \psi^{(2)}_\tau \right\|^2 - \langle \psi^{(2)}_\tau | H^{(1)}_\tau - H^{(2)}_\tau | \psi^{(2)}_\tau \rangle^2} \, d\tau = \pi/2. \]

**Proof of Theorem 2.**

The inequality (40) follows from the bound (7) and the Schrödinger equation \( \dot{\psi}_t^{(2)} = -i H^{(2)}_t \psi_t^{(2)} \). \( \square \)

5. Summary

We have proven a quantum speed limit, eqs. (4) and (5), valid for an arbitrary time-dependent target subspace. While for one-dimensional target subspaces this quantum speed limit reduces to eqs. (7), (8) which had been obtained in ref. [1], for multidimensional target subspaces it is tighter than the results of ref. [1]. We have used the obtained quantum speed limit to derive a sufficient adiabatic condition, eqs. (19) and (27), as well as a bound (40) on the Loschmidt echo.
From (A.1) it follows that
\[ \|\hat{P}_{\phi t}\| = \sqrt{\langle \hat{\phi}_t | \hat{\phi}_t \rangle - |\langle \hat{\phi}_t | \phi_t \rangle|^2}. \]
valid for any normalised vector \( \phi_t \). To this end we introduce a normalised vector \( \phi_t^\perp = (1 - P_{\phi}) \hat{\phi}_t / (1 - P_{\phi}) \| \hat{\phi}_t \| \) which is orthogonal to \( \phi_t \), and expand \( \hat{\phi}_t \):
\[ \hat{\phi}_t = P_{\phi} \hat{\phi}_t + (1 - P_{\phi}) \phi_t = \langle \phi_t | \phi_t \rangle \phi_t + \| (1 - P_{\phi}) \| \| \hat{\phi}_t \| \phi_t^\perp. \] (A.2)
This implies
\[ \hat{P}_{\phi} = \langle \hat{\phi}_t | \phi_t \rangle P_{\phi} + |\phi_t^\perp|^2 \langle \phi_t | (1 - P_{\phi}) \hat{\phi}_t \rangle + h.c.. \] (A.3)
Due to the normalization condition \( \langle \phi_t | \phi_t \rangle = 1 \) the first term and its complex conjugate cancel: \( \langle \phi_t | \phi_t \rangle P_{\phi} + \langle \phi_t | \hat{\phi}_t \rangle P_{\phi} = (\frac{1}{2} \langle \phi_t | \phi_t \rangle) P_{\phi \phi} = 0 \). Thus
\[ \|\hat{P}_{\phi t}\| = \| (1 - P_{\phi}) \hat{\phi}_t \| = \sqrt{\langle \hat{\phi}_t | \hat{\phi}_t \rangle - |\langle \hat{\phi}_t \phi_t \rangle|^2}. \] (A.4)

**Appendix B. Proof of the bound (29)**

Consider \( \Pi_t \) generated by some unitary \( W_t \), \( \Pi_t = W_t \Pi_0 W_t^\dagger \). This evolution can be described by a Schrödinger equation with a fictitious Hamiltonian \( \mathcal{H}_t = i \dot{W}_t W_t^\dagger \),
\[ i \dot{\Pi}_t = [\mathcal{H}_t, \Pi_t]. \] (B.1)
To derive this equation one should use the fact that \( W_t W_t^\dagger = W_t^\dagger W_t = 1 \) and, hence, \( \dot{W}_t W_t^\dagger + W_t \dot{W}_t^\dagger = W_t^\dagger W_t + W_t^\dagger W_t = 0 \). Observe that
\[ \|\mathcal{H}_t\| = \|\dot{W}_t\|. \] (B.2)
From (12) and (B.1) we obtain
\[ \dot{\Pi}_t = i \Pi_t \mathcal{H}_t (1 - \Pi_t) - i (1 - \Pi_t) \mathcal{H}_t \Pi_t. \] (B.3)
Since \( \dot{\Pi}_t \) is self-adjoint operator we can estimate its norm as
\[ \|\dot{\Pi}_t\| = \sup_{\|\varphi\| = 1} \langle \varphi | \dot{\Pi}_t | \varphi \rangle = 2 \sup_{\|\varphi\| = 1} \text{Im} \langle \varphi | (1 - \Pi_t) \mathcal{H}_t \Pi_t | \varphi \rangle \leq 2 \sup_{\|\varphi\| = 1} \| (1 - \Pi_t) | \mathcal{H}_t \| \| \Pi_t | \varphi \rangle \|. \] (B.4)
Further, since \( \Pi_t \) is a projector and \( \|\varphi\| = 1 \), we have \( \| (1 - \Pi_t) | \varphi \rangle \|. \) Therefore
\[ \|\dot{\Pi}_t\| \leq 2 \|\mathcal{H}_t\| \sup_{\|\varphi\| = 1} \sqrt{\|\Pi_t | \varphi \rangle \|^2 (1 - \|\Pi_t | \varphi \rangle \|^2)}. \] (B.5)
As \( \sup_{x \in [0,1]} \sqrt{x(1-x)} = 1/2 \) then
\[
\|\dot{\Pi}_t\| \leq \|\mathcal{H}_t\|. \tag{B.6}
\]
In view of eq. (B.2) this proves the bound (29).

We note that the equality in (B.6) can be reached for \( \mathcal{H}_t = \mathcal{H}_t \equiv i[\dot{\Pi}_t, \Pi_t] \) introduced in Section 3. Let us prove this fact. First, one verifies that \( \mathcal{H}_t \) indeed generates \( \Pi_t \) via eq. (B.1), see eq. (12), and hence
\[
\|\dot{\Pi}_t\| \leq \|\mathcal{H}_t\|. \tag{B.7}
\]
On the other hand
\[
\mathcal{H}_t = i(\mathbb{I} - \Pi_t)\dot{\Pi}_t \Pi_t - i\Pi_t \dot{\Pi}_t (\mathbb{I} - \Pi_t) \tag{B.8}
\]
and we get
\[
\|\mathcal{H}_t\| \leq \|\dot{\Pi}_t\| \tag{B.9}
\]
analogously to eq. (B.6). Inequalities (B.7) and (B.9) imply
\[
\|\dot{\Pi}_t\| = \|\mathcal{H}_t\|. \tag{B.10}
\]

References

[1] Peter Pfeifer and Jürg Fröhlich. Generalized time-energy uncertainty relations and bounds on lifetimes of resonances. *Reviews of Modern Physics*, 67(4):759, 1995.
[2] Sebastian Deffner and Steve Campbell. Quantum speed limits: from heisenberg’s uncertainty principle to optimal quantum control. *Journal of Physics A: Mathematical and Theoretical*, 50(45):453001, 2017.
[3] L Mandelstam and IG Tamm. The uncertainty relation between energy and time in non-relativistic quantum mechanics. In *Selected Papers*, pages 115–123. Springer, 1991.
[4] Tosio Kato. On the adiabatic theorem of quantum mechanics. *Journal of the Physical Society of Japan*, 5(6):435–439, 1950.
[5] Tameem Albash and Daniel A. Lidar. Adiabatic quantum computation. *Rev. Mod. Phys.*, 90:015002, Jan 2018.
[6] Sergio Boixo, Emanuel Knill, and Rolando D Somma. Eigenpath traversal by phase randomization. *Quantum Information & Computation*, 9(9):833–855, 2009.
[7] Hao-Tien Chiang, Guanglei Xu, and Rolando D Somma. Improved bounds for eigenpath traversal. *Physical Review A*, 89(1):012314, 2014.
[8] Oleg Lychkovskiy, Oleksandr Gamayun, and Vadim Cheianov. Quantum many-body adiabaticity, topological thoulless pump and driven impurity in a one-dimensional quantum fluid. *AIP Conf. Proc.*, 1936(1):020024, 2018.
[9] Oleksandr Gamayun, Oleg Lychkovskiy, Evgeni Burovski, Matthew Malcomson, Vadim V. Cheianov, and Mikhail B. Zvonarev. Impact of the injection protocol on an impurity’s stationary state. *Phys. Rev. Lett.*, 120:220605, Jun 2018.
[10] Oleg Lychkovskiy, Oleksandr Gamayun, and Vadim Cheianov. Necessary and sufficient condition for quantum adiabaticity in a driven one-dimensional impurity-fluid system. *Phys. Rev. B*, 98:024307, Jul 2018.
[11] Peter Pfeifer. How fast can a quantum state change with time? *Physical review letters*, 70(22):3365, 1993.
[12] A. Polkovnikov and V. Gritsev. Breakdown of the adiabatic limit in low-dimensional gapless systems. *Nature Phys.*, 4(6):477–481, 2008.
[13] A. Altland and V. Gurarie. Many body generalization of the Landau-Zener problem. *Phys. Rev. Lett.*, 100(6):063602, 2008.

[14] Sven Bachmann, Wojciech De Roeck, and Martin Fraas. The adiabatic theorem for many-body quantum systems. *Phys. Rev. Lett.*, 119:060201, 2017.

[15] Oleg Lychkovskiy, Oleksandr Gamayun, and Vadim Cheianov. Time scale for adiabaticity breakdown in driven many-body systems and orthogonality catastrophe. *Phys. Rev. Lett.*, 119(20):200401, 2017.

[16] Asher Peres. Stability of quantum motion in chaotic and regular systems. *Physical Review A*, 30(4):1610, 1984.

[17] Thomas Gorin, Tomáš Prosen, Thomas H Seligman, and Marko Žnidarič. Dynamics of loschmidt echoes and fidelity decay. *Physics Reports*, 435(2-5):33–156, 2006.