Effective action and linear response of compact objects in Newtonian gravity

Sayan Chakrabarti,1,2 Térence Delsate,1,3 and Jan Steinhoff1,4,*

1Centro Multidisciplinar de Astrofísica — CENTRA, Departamento de Física, Instituto Superior Técnico — IST, Universidade de Lisboa — ULisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal, EU
2Department of Physics, Indian Institute of Technology Guwahati, Assam 781039, India
3UMons, Université de Mons, Place du Parc 20, 7000 Mons, Belgium, EU
4ZARM, University of Bremen, Am Fallturm, 28359 Bremen, Germany, EU

(Dated: May 7, 2014)

We apply an effective field theory method for the gravitational interaction of compact stars, developed within the context of general relativity, to Newtonian gravity. In this effective theory a compact object is represented by a point particle possessing generic gravitational multipole moments. The time evolution of the multipoles depends on excitations due to external fields. This can formally be described by a response function of the multipoles to applied fields. The poles of this response correspond to the normal oscillation modes of the star. This gives rise to resonances between modes and tidal forces in binary systems. The connection to the standard formalism for tidal interactions and resonances in Newtonian gravity is worked out. Our approach can be applied to more complicated situations. In particular, a generalization to general relativity is possible.

PACS numbers: 04.25.-g,04.40.Dg, 97.60.Jd, 11.10.Gh

I. INTRODUCTION

The Newtonian description of gravitating systems was formulated in the 17th century and is nowadays part of most classical mechanics textbooks. Remarkably enough, it is still possible to understand new facets of this model. At the same time, Newton’s formulation of gravity is mathematically simple and allows straightforward analytic treatment. In particular, the Poisson equation admits a well understood Green function solution. This is why the Newtonian theory is also a good starting point to investigate some fundamental aspects of general relativity.

One of these aspects, among many others, is tidal interaction in a binary system. In the beginning of the last century, A. E. H. Love introduced two numbers named after him in order to characterize the shape change of the Earth due to an external tidal potential [1]. A third Love number was introduced later by Shida [2]. These numbers essentially encode the mass redistribution of a planet due to tidal forces, including those generated by the tidal bulges themselves. A tide-generating potential arises, for example, in binary systems, such as binary stars or a planet and its satellite, e.g., the Earth and the moon. The latter example comprises ocean tides, which motivated the first studies of the subject and its naming. Tidal forces also play a crucial role for the concept of local inertial frames in general relativity. Despite this close interrelation between tidal interaction and general relativity, a relativistic definition of Love numbers was worked out only a century after its Newtonian counterpart [3–5]; see also Refs. [6, 7]. It should be emphasized that modeling tidal interactions through constant Love numbers assumes a slowly (adiabatically) varying tidal field.

On another hand, binary systems possess a typical frequency due to the revolution of the bodies around each other. In some situations, particularly for binary stars, the orbital frequency can become comparable to the normal oscillation frequencies of one of the binary’s components. Therefore, a resonance can take place, possibly leading to large energy and angular momentum transfers. The analytic framework to treat dynamical (time-dependent) tidal interactions in Newtonian gravity goes back to Press and Teukolsky [8]. Though they focus on tidal capture as an application, resonances in bound binaries can be treated as well. Tidal heating, tidal disruption, and tidal locking are other astrophysically important effects belonging into this domain, see, e.g., Refs. [9–12]. A more complete list of references for generic binaries is given in, e.g., Refs. [13–14]. Also, resonances in neutron star binaries have been considered using these methods [10, 14–20], though one should be careful to draw final conclusions from such investigations, as general relativistic corrections can be large in this case.

Our motivation is indeed to improve the situation for neutron star binaries by formulating a general model for time-dependent tides in general relativity. But the problem is highly nontrivial due to the nonlinear nature of general relativity. Here we focus on the Newtonian formulation of the problem instead, because it is described in terms of simple functions, and most of the technical difficulties of the relativistic case are avoided. Though excellent formalisms for dynamical tidal interactions already exist [8, 13, 21], we devise yet another formalism based on quantities that allow a general relativistic generalization. Our reformulation is thus also interesting in its own right. For instance, it can turn out to be advanta-
geous for more complicated situations in the Newtonian regime, too, e.g., when nonlinear tidal perturbations and mode coupling are considered (see Refs. [22, 23] for recent investigations on neutron stars). The relativistic case is presented in another publication [24].

Our method relies on an effective field theory approach to gravitational interaction of compact objects in classical general relativity [25], which is cut down to the Newtonian case for the present work. The principle is to effectively represent a compact source, with all its potentially complicated internal dynamics, by a point-particle source decorated by multipolar degrees of freedom [25–27]. The dynamics of these multipoles can be encoded through a response function to external gravitational fields [26]. The game is then to match the gravitational field of the effective source to the gravitational field of the actual compact object, which fixes the response. The response function, which is in fact the retarded propagator of the multipolar degrees of freedom, then encodes all the (macroscopically relevant) time-dependent internal dynamics. The advantage is that it is sufficient, at least in linear perturbation theory, to know the gravitational field of a single perturbed compact object in order to make predictions about a gravitationally interacting many-body system. These predictions can be derived from the effective theory, e.g., with the help of diagrammatic techniques developed for quantum field theory (Feynman diagrams), see Refs. [25, 28] and references therein for the case of relativistic gravitational interaction.

The normal oscillation modes of a compact object play a crucial role in our approach. A compact configuration can possess a series of normal modes, and each mode can be excited by an external field of the appropriate frequency. Indeed, the mode frequencies maximize the response to external perturbations, similar to a forced oscillator. In fact, the multipolar degrees of freedom turn out to be sums of these forced oscillators. From this point of view, multipoles are composite, and the fundamental composing quantities are the deformation amplitude modes (this will be detailed later). All this treatment can be carried out explicitly; i.e., an analytic expression for the response function can be given in terms of oscillation mode frequencies and overlap integrals, which are defined in the standard approach [8, 13, 21]. But the concepts of effective action, response functions/propagators, matching, and perturbations of single compact objects can be generalized to the general relativistic case [24], which is less obvious for some elements of the standard formalism. As we work with an effective action here, it is most natural to make connections to the standard approach through the variational principle in Ref. [21], which was developed to describe mode resonances in binary white dwarfs systems [29, 30].

Resonances between oscillation modes and orbital motion can also be interesting for future gravitational wave astronomy, in particular for binaries comprising neutron stars. Past investigations suggest that these resonances are likely not detectable through gravitational waves using observatories currently under construction [16, 31]. However, it should be stressed again that such an analysis did not model the internal dynamics entirely within general relativity. In fact, numerical relativity simulations suggest that resonances can even be driven into the nonlinear regime in certain cases (eccentric orbits) and thus contribute substantially to gravitational waves [32]. Of course, tidal interactions are important during the inspiral also away from a resonance, and they can encode important information on the nuclear equation of state in the gravitational wave signal, which is supported by both analytic models [33–35] and numerical relativity [36–39]. Also, the merger of binary neutron stars [40] and the details of the tidal disruption in mixed (neutron star – black hole) binaries [41–46] allows crucial statements on the equation of state. Furthermore, resonances can have other observable effects in neutron star binaries besides gravitational waves. It was suggested in Ref. [47] that oscillations powered by a resonance can be strong enough to shatter the neutron star crust. This could explain precursor flares in short gamma ray bursts, with the main burst produced by the merger of the binary. Similar violent processes may be observable in the electromagnetic spectrum if instabilities develop. Indeed, some oscillation modes—including the f-mode—of rotating neutron stars can become unstable [48].

In spite of these interesting prospects, an entirely satisfactory analytical tidal interaction model incorporating a general relativistic internal dynamics is still missing. So far resonances could only be accounted for by investigating a point mass orbiting the star; see Refs. [49–52] and also Ref. [53] for the black hole case. In between a fully relativistic and a Newtonian treatment is a post-Newtonian approximation of the internal dynamics. A corresponding formalism was developed in Refs. [54–57] and further elaborated in Refs. [19, 58]. The first post-Newtonian approximation was applied to binary neutron stars in Refs. [59–61]. Another good dynamical model was developed around the Newtonian limit in Refs. [62, 63]. Still, it would be optimal to find the description of tidal interaction based on quantities derived from perturbation theory around nonlinear background solutions. So far this succeeded only for the definition of relativistic Love numbers [3–5, 64], which represent the adiabatic limit of the tidal response. (Tidal coefficients beyond the adiabatic case were formally introduced in Refs. [65, 66].) Further, relativistic perturbation theory was used to study absorptive effects of black holes in tidal fields; see Refs. [66–71] and references therein.

In this paper, we start in Sec. II by reviewing and motivating the effective field theory approach in general relativity and argue that the same formalism applies in the Newtonian theory. We explain in detail how the tidal interaction can be encoded using time-dependent multipolar degrees of freedom and work out the connection to tidal coefficients and absorption coefficients. Section III reviews a variational principle for perturbations of a gravitating system in the Newtonian theory. We improve on
previous presentations by performing all transformations at the level of the action. Section IV discusses some useful tools, such as the symmetric trace-free (STF) tensor formalism, which is useful for the present work. Next, we show how to cast stellar perturbations in an effective field theory formulation in Sec. V and show how to read off the overlap integrals from the external field of a perturbed Newtonian star (which generalizes to the relativistic case and thus defines relativistic overlap “integrals” [24]). Subsequently, we extract the full time-dependent dynamics of the quadrupolar degrees of freedom in Sec. VI by analyzing the perturbation field of a single compact object. We adapt the matching to a numerical setup in Sec. VII and illustrate our results with a simple (analytic) background solution corresponding to a particular polytropic star. Section VIII discusses the conclusions and outlook. Finally, the Appendix provides more details on some mathematical aspects of our results.

Our conventions are the following. Greek indices refer to 4-dimensional spacetime, while lowercase Latin indices denote spatial components. (But sometimes $l$ and $m$ are used for angular momentum quantum numbers, which should be obvious from the context.) Uppercase Latin indices are multi-indices of spatial components. Einstein’s summation convention is applied to these types of indices. Our sign convention for the metric $g_{\mu\nu}$ is taken to be $(-, +, +, +)$ and the Riemann tensor is

$$R^\mu_{\nu\lambda\sigma} = \Gamma^\mu_{\nu,\lambda\sigma} - \Gamma^\mu_{\nu\lambda,\sigma} + \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\delta} - \Gamma^\mu_{\nu\delta} \Gamma^\lambda_{\lambda\sigma}, \quad (1.1)$$

where $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol. We use units such that the speed of light is $c = 1$. The Newton constant is denoted $G$.

II. EFFECTIVE ACTION

The goal of this paper is to apply the ideas of effective field theory developed in general relativity to the Newtonian theory of gravity. In fact, we find later on that the nonrelativistic limit of the effective action and an action derived for perturbations in pure Newtonian gravity are essentially equivalent. This should of course be the case if the effective action is constructed from the correct symmetries. We first discuss various forms of effective actions for tidal interactions (and their relations) in the relativistic case, since the discussion is essentially the same in the nonrelativistic case. Afterward, we determine the Newtonian limit and construct interaction potentials.

A. Effective field theory in gravity

Some of the most important sources of gravitational waves are the inspiral, merger, and ringdown phases of compact objects in NS-NS, NS-BH, or BH-BH binaries, where NS stands for neutron star and BH for black hole. These waves will contain important information about the compact object. Such binary systems exhibit a number of different length scales, for example, the size of the compact objects, the orbital radius, and the wavelength of the emitted radiation. On general grounds, attacking a two-body problem in general relativity needs numerical methods. However, there exist certain regimes with a clear separation of length scales, and one can track the problem analytically using approximate methods. For instance, the post-Newtonian expansion or the extreme mass ratio inspiral limits are two such cases. These two methods are efficient in a certain regime but break down above some limits.

There exists a systematic way to account for effects that arise at different length scales, which is the effective field theory approach; see, e.g., Refs. [72–74] for reviews. These methods are often associated with quantum theory, and in fact the first application to the nonrelativistic limit of gravity [75] was focused on quantum corrections. Nevertheless, the methods are very useful for astrophysical binaries and their gravitational waves in classical gravity, too, which was put forward in Refs. [25, 76, 77]. In this situation, the compact object is described by a worldline action which includes all the possible terms consistent with the diffeomorphism invariance of general relativity [and eventually symmetries inherited from the single unperturbed object, like SO(3) rotational invariance]. The method is particularly helpful to systematically account for tidal and other finite size effects, which become important in some regime, e.g., the late inspiral stage of a compact binary system. The effective action in Refs. [25, 27] can be used to find the point particle description of nondissipative finite size effects and can be extended in order to include dissipation [26]. In the latter case, the gravitational multipoles enter the action as worldline degrees of freedom, for which the dynamics is encoded in a propagator or response function. This manifestly covariant extension of the point-mass action is adopted as a model for extended objects here.

Two approaches to effective theories can be contrasted [72]: A Wilsonian approach of integrating out short-scale physics from a full theory and a continuum effective field theory approach, where the form of the effective action is first constructed by hand (e.g., using symmetries) and afterward specialized by a matching to the full theory. The latter method is usually much simpler. However, the setting of the present work is simple enough to follow an approach along the lines of Wilson’s ideas, too. This is elaborated in Sec. V. An analytic matching to the full theory (variational fluid dynamics) in the spirit of continuum effective theories is discussed in Sec. VI. Next, we discuss in Sec. VII a matching procedure to solutions of the full theory obtained numerically. This in an important shift in paradigm, as many complicated systems cannot be treated analytically, but numeric simulations are usually possible. This includes stars with more complicated (realistic) internal structure and nonlinear perturbations. Not surprisingly, the numeric matching is the
approach followed for neutron stars in the general relativistic generalization of our work \cite{24}, while the black hole case still allows analytic treatment.

The more standard method to handle different scales in classical general relativity is the matched asymptotic expansion; see Refs. \cite{78, 79} and references therein. For a comparison of matched asymptotic expansion and matching in effective field theory, see Ref. \cite{80}. Another aspect of the effective field theory approach in \cite{25} is the consequent formulation of perturbative calculations through Feynman diagrams. However, diagrammatic approaches were used in classical gravity before, see \cite{28} and references therein. Effective worldline actions as a model for finite size effects were first discussed in the context of scalar-tensor theories of gravity \cite{81}, since they are potentially more relevant in this class of alternative theories compared to the general relativistic case.

\section{Action for dynamical multipoles}

In the following, we intend to illustrate the usefulness of the effective action in Ref. \cite{26} in the context of tidal interactions. We simply start with the action constructed in Ref. \cite{26}.

Introducing a dynamical quadrupole degree of freedom in the form of a STF tensor $Q^{ab}$, the point-particle (PP) worldline action proposed in Ref. \cite{26} reads

\[ S_{\text{PP}} = \int d\tau \left[ -m - \frac{1}{2} E^{ab} Q_{ab} + \ldots \right], \quad (2.1) \]

where $m$ is a constant mass parameter, $E_{\mu\nu}$ is the electric part of the Weyl tensor $C_{\mu\nu\rho\sigma}$, $E_{\mu\nu} = C_{\mu\nu\rho\sigma} u^{\rho} u^{\sigma}$, and $u^{\mu}$ is the 4-velocity with respect to the proper time $\tau$. (In vacuum, $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$ holds.) $E_{\mu\nu}$ is evaluated at the position of the object. In this section, Latin indices denote spatial components of the comoving frame in a local Cartesian basis. The dots denote higher multipole corrections as well as magnetic-type multipoles. For the sake of simplicity, we restrict to interactions of the mass quadrupole $Q^{ab}$ for now, which can be classified as electric type.

Within the approach proposed in Ref. \cite{26}, the dynamics of $Q^{ab}$ is introduced via its two-point function, or propagator, which is obtained from a matching procedure in the frequency regime of interest. This provides a model for the quadrupole, which is required in order to complete the description of the system. As the outcome of the matching is yet unknown, the complete effective action including the quadrupole dynamics cannot be given explicitly. However, if we restrict to linear tidal effects, the pure quadrupole part of the action can always be written in terms of an invertible Hermitian linear operator $\mathcal{O}^{ab}_{\text{cd}}$ in the form

\[ S_Q = -\frac{1}{2} \int d\tau Q_{ab} \mathcal{O}^{ab}_{\text{cd}} Q^{cd}. \quad (2.2) \]

The complete effective action for the compact object including the mass quadrupole finally reads

\[ S_{\text{eff}} = S_Q + S_{\text{PP}}. \quad (2.3) \]

\section{Quadrupole response}

As $Q^{ab}$ is assumed to be a dynamical variable here, its equation of motion follows from $S_{\text{eff}}$ as

\[ \mathcal{O}^{ab}_{\text{cd}} Q^{cd} = -\frac{1}{2} \mathcal{E}^{ab}_{\text{cd}}(\tau), \quad (2.4) \]

where we used that $\mathcal{O}^{ab}_{\text{cd}}$ should be Hermitian. If this is not the case, then one must complete the quadrupole model by giving its equation of motion (2.4) instead of an action $S_Q$. The formal solution to Eq. (2.4) is given by

\[ Q^{ab}(\tau) = -\frac{1}{2} \int d\tau' \mathcal{F}^{ab}_{\text{cd}}(\tau, \tau') \mathcal{E}^{cd}(\tau'), \quad (2.5) \]

where $F$ is a Green function or propagator of the operator $\mathcal{O}^{ab}_{\text{cd}}$. One can make the ansatz

\[ \mathcal{F}^{ab}_{\text{cd}}(\tau, \tau') = F(\tau - \tau') \mathcal{\hat{O}}^{ab}_{\text{cd}}, \quad (2.6) \]

where $\mathcal{\hat{O}}^{ab}_{\text{cd}}$ is the STF projector of the form

\[ \mathcal{\hat{O}}^{ab}_{\text{cd}} = \frac{1}{2} (\delta^{ab} \delta_{cd} + \delta^{ac} \delta_{bd}) - \frac{1}{3} \delta^{ab} \delta_{cd}. \quad (2.7) \]

We require that $\mathcal{F}^{ab}_{\text{cd}}(\tau, \tau') = 0$ if $\tau' > \tau$, so we consider the retarded Green function. Then, $\mathcal{F}^{ab}_{\text{cd}}(\tau, \tau')$ provides the response function of the quadrupole under external gravitational forces. This solution implements the boundary condition that the quadrupole vanishes in the absence of external forces and for $\tau \to -\infty$.

It is often useful to consider the response functions in the frequency domain. In particular, one can make further transformations of the effective action by expanding around small frequencies $\omega$. Our convention for the Fourier transform from time to frequency domain is

\[ \tilde{F}(\omega) = \mathcal{F}(F) = \int d\tau F(\tau) e^{-i\omega \tau}, \quad (2.8) \]

\[ F(\tau) = \mathcal{F}^{-1}(\tilde{F}) = \frac{1}{2\pi} \int d\omega \tilde{F}(\omega) e^{i\omega \tau}, \quad (2.9) \]

where $\mathcal{F}$ denotes the Fourier transformation operator and $\mathcal{F}^{-1}$ its inverse. Note that we use a tilde to denote Fourier transformed quantities. In the frequency domain, the formal solution for the quadrupole (2.5) reads

\[ Q^{ab} = -\frac{1}{2} \tilde{F}(\omega) \tilde{E}^{ab}. \quad (2.10) \]
D. Relativistic Love numbers

In this section, we make contact between certain Love numbers and the small frequency regime of the response function. In fact, we will eliminate (integrate out) the quadrupole degrees of freedom from the action. Inserting the formal solution (2.10) into Eqs. (2.1) and (2.2), we find that the action turns into

\[ S_{\text{eff}} = \int d\tau \left[ -m + \frac{1}{8} F^{-1}(\tilde{F} E_{ab}) E^{ab} + \ldots \right]. \]  

(2.11)

The connection to the relativistic Love numbers and other constants defined in previous literature [3–5, 65] becomes apparent if we Taylor-expand the response

\[ \tilde{F}(\omega) = 2\mu_2 + i\lambda \omega + 2\mu_2'\omega^2 + O(\omega^3), \]  

(2.12)

and explicitly perform the inverse Fourier transform in Eq. (2.11). The \( \lambda \)-term is related to absorption [26]. In this case, the action (2.11) reduces to

\[ S_{\text{eff}} = \int d\tau \left[ -m + \frac{\mu_2}{4} E_{ab} E^{ab} + \frac{\mu_2'}{4} \dot{E}_{ab} E^{ab} + \ldots \right]. \]  

(2.13)

The contribution from \( \lambda \) formally turns into a total derivative and drops out. Indeed, insertions of solutions of equations of motion into action principles project onto the time-symmetric part of the dynamics [82], and therefore one must handle dissipative effects more carefully. (In Ref. [82], a classical variant of the closed time path formalism of quantum field theory was developed for this purpose.) Here, \( \mu_2 \) is the relativistic tidal Love number [3–5], and \( \mu_2' \) represents the tidal response of the neutron star beyond the adiabatic approximation [65]. It should be noted that \( \mu_2 \) and \( \mu_2' \) are in fact defined by an effective action of the form (2.13) and should be determined numerically through a matching procedure, just like \( \tilde{F} \) (this has not yet been undertaken for \( \mu_2' \)).

From a historical perspective, the Love numbers of the first and second kind were introduced in Ref. [1] in the context of tides on Earth. These are dimensionless numbers describing the response of the tidal deformation (first Love number or shape Love number) and tidal potential (second Love number) of an elastic body. In general relativity, there exist electric-type \( (k_i) \), magnetic-type \( (j_i) \), and shape \( (h_i) \) Love numbers. The shape and electric-type Love numbers are directly analogous to the first and second Newtonian Love numbers. The relation to \( \mu_2 \) is given by \( k_2 = 3\mu_2 G/2R^5 \), where \( R \) is the radius of the star. (The subscript 2 in \( \mu_2 \) denotes \( l = 2 \) multipole contribution and not the second Love number.) The magnetic-type Love numbers in general relativity were first introduced in Ref. [55]. The relativistic shape Love numbers were considered in Ref. [4] for neutron stars and in Refs. [83, 84] for black holes.

E. Newtonian limit

Finally, for our purpose, we give here the nonrelativistic limit of the effective action (2.1). First, we note that

\[ E_{ab} \approx \partial_a \partial_b \Phi, \]  

(2.14)

in the Newtonian limit, where \( \Phi \) is the Newtonian gravitational potential. This is obtained from evaluating the electric part of the Weyl tensor to linear order in the potential with the metric

\[ ds^2 = \left( 1 + \frac{2\Phi}{c^2} \right) c^2 dt^2 + \left( 1 + \frac{2\Phi}{c^2} \right)^{-1} dr^2 + r^2 d\Omega^2 + O \left( \frac{1}{c} \right)^3, \]  

(2.15)

where \( x^a = (r, \theta, \varphi) \) and \( \Phi \) is the Newtonian potential. The speed of light \( c \) is introduced as a bookkeeping parameter here and can be removed at the end of the calculation. The Newtonian limit of \( E_{ab} \) is then obtained straightforwardly by expanding around large \( c \).

Next, the proper time is related to the Newtonian absolute time by

\[ d\tau = dt \sqrt{-g_{\mu\nu} u^\mu u^\nu} \approx dt \sqrt{1 + 2\Phi - z^2}, \]  

(2.16)

where \( t \) is the absolute Newtonian time and \( z \) is the location of the star. Finally, the Newtonian limit of Eq. (2.1) leads to the effective Newtonian action

\[ S_{\text{PP}} \approx \int dt \left[ -m + \frac{1}{2} m z^2 - m \Phi - \frac{1}{2} Q^{ab} \partial_a \partial_b \Phi + \ldots \right]. \]  

(2.17)

F. Potentials

We are going to illustrate how the effective action can be applied to the binary problem and explain the advantages. Effects due to the internal structure just enter via the quadrupole here. For simplicity, we assume that the particle described by Eq. (2.17) moves in a fixed gravitational field of a (heavy) point mass \( m_{\text{heavy}} \) located at the coordinate origin,

\[ \Phi = -\frac{G m_{\text{heavy}}}{r}, \]  

(2.18)

where \( r = |x| \). Insertion into the quadrupole interaction in Eq. (2.17) leads to the monopole-quadrupole interaction potential

\[ V_{\text{mono-quad}} = -\frac{3G m_{\text{heavy}}}{2|z|^3} Q^{ab} n^a n^b, \]  

(2.19)

where \( n^a \) is the unit vector pointing toward the quadrupole particle. Notice that the derivation is rather simple. Further, the gravitational interaction and the internal dynamics are logically separated. This can be very
advantageous for model building, particularly in complicated situations.

The derivation of Eq. (2.19) is formally valid for large $m_{\text{heavy}}$ only since we omit backreaction. However, the result is actually correct for generic binaries. In general, each body is represented by a copy of Eq. (2.17),

$$S_{\text{binary}} = S_{\Phi}^{\Phi 1} + S_{\Phi}^{\Phi 2} + S_{\Phi}, \quad (2.20)$$

where the gravitational action is given by

$$S_{\Phi} = \frac{1}{8\pi G} \int dt \, d^3x \, \Phi \Delta \Phi. \quad (2.21)$$

Insertion of the solution for $\Phi$ in fact leads to the same result (2.19) for the monopole-quadrupole potential. This requires us to drop singular self-interactions, as $\Phi$ is singular at the position of each particle now. An extension of the potential to the first post-Newtonian approximation to general relativity can be found in Ref. [85].

In effective field theory parlance, we have in fact just “integrated out” the gravitational field $\Phi$, and we will occasionally make use of this phrase. Indeed, on a classical level, integrating out a field translates to obtaining its field equations from the action, solving them, and inserting the solution back into the action (also into the pure-field part $S_{\Phi}$). The combinatorial aspect suggests to elegantly organize the computation in a diagrammatic manner à la Feynman, in particular in a perturbative context.

III. ACTION FOR STELLAR PERTURBATIONS

In this section, we briefly review the variational treatment of linear perturbations in Newtonian gravity and cast it into the form needed for our investigation. This will be the starting point for the derivation of the effective action later on. Variational principles for nonradial stellar oscillations in the Newtonian context were discussed in previous literature [86, 87] and subsequently extended to tidal excitations in binary systems [21].

In the following, we refer to linear stellar perturbations as the “full theory.” This is inaccurate in at least two aspects. First, linear perturbations are just an approximation to generic perturbations, which can usually only be tackled by 3-dimensional numerical simulations. Second, on a more fundamental level, a fluid description for stars is just an approximation, too. Fluids can be thought of as an effective theory for atomic or subatomic particles, which themselves may be described by an effective theory of an even more fundamental theory, and so forth and so on. This is the effective field theory viewpoint of physical model building, namely through a tower of effective theories.

A. Variational principle for perturbations

We consider a gravitating ideal fluid configuration described by the following quantities: the velocity field $u$ of the fluid elements, usual thermodynamic variables, like mass density $\rho$, pressure $P$, and equations of state relating them. But we disregard temperature here, which is a good approximation for neutron stars. We analyze the system in a perturbative setting, starting from a background configuration denoted by an index 0, a first perturbation denoted by an index 1, and so forth and so on.

For the background, we take a static nonrotating (spherically symmetric) star in equilibrium, and we restrict to first order perturbations here. Then, the unperturbed fluid velocity vanishes, $u_0 = 0$. The other background variables are functions of the radial coordinate $r$ only and determined by

$$P_{\rho 0}' = -\rho_0 \Phi_0', \quad (3.1)$$

$$\Delta \Phi_0 = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho_0 \right) = 4\pi G \rho_0, \quad (3.2)$$

where $' = d/dr$. Notice that these are just the equation of hydrostatic equilibrium and the Newtonian field equation for spherically symmetric configurations. Together with a barotropic equation of state $P = P(\rho)$, the system of equations is closed. In the following, the background variables are not varied in the action principle but are considered as functions of $r$ given by the solution to this system of equations. Except in simple cases, the solution is obtained from numeric integration.

The fundamental variable of stellar perturbations is the displacement vector field $\xi$ of the fluid elements. If $x'$ denotes the position of a fluid element in the perturbed star and $x$ denotes the same in the unperturbed background, then the physical displacement is $\xi = x' - x$. The perturbation variables can be expressed in terms of the displacement as

$$u_i = \xi_i, \quad (3.3)$$

$$\rho_1 = -\nabla \cdot (\rho_0 \xi), \quad (3.4)$$

$$P_1 = c_s^2 \rho_1, \quad (3.5)$$

where $c_s$ is the (unperturbed) adiabatic speed of sound given by $c_s^2 = dP_0/d\rho_0$. Notice that Eq. (3.4) follows from an integration of the continuity equation $\dot{\rho} + \nabla \cdot (\rho u) = 0$ (conservation of mass) together with $\rho_0 = 0 = u_0$. We further assume that the perturbation is caused by some external mass density $\rho^{\text{ext}}$ that equals $\rho_1^{\text{ext}}$, which can represent another close-by star.

The Lagrangian of the “full theory” (stellar perturbations) can now be written as a sum of gravitational, kinetic, star’s interior, and coupling parts,

$$L_{\text{full}} = L_{\Phi 1} + L_{\text{kin}} + L_{\text{star}} + L_{\text{coup}}, \quad (3.6)$$

where

$$L_{\Phi 1} = \frac{1}{8\pi G} \int d^3x \, \Phi_1 \Delta \Phi_1, \quad L_{\text{kin}} = \frac{1}{2} m \xi^2, \quad (3.7)$$
\[
L_{\text{star}} = \int d^3x \left[ \frac{1}{2} \rho_0 \dot{\xi}^2_{\text{COM}} - (\rho E)_2 - \rho_0 \xi \cdot (\nabla \Phi_1 + \ddot{z}) \right],
\]
and \( E \) is the specific internal energy. The second-order perturbation of the internal energy \( \rho E(\rho) \) is given by
\[
(\rho E)_2 \approx \rho_1 \frac{dE_0}{d\rho_0} \rho_1 + \frac{1}{2} \rho_0 \frac{d^2E_0}{d\rho_0^2} \rho_1^2 = \frac{\dot{\xi}^2}{2 \rho_0} \rho_1^2. \tag{3.10}
\]
Remember that the first law of thermodynamics reads \( dE = -P d\rho^{-1} \) in our case. Also notice that in order to obtain the first-order perturbation equations, some second-order contributions must be included in the Lagrangian (but \( \rho_2 \) may be omitted). At this point, the dynamical variables are \( \dot{z}, \xi, \) and \( \Phi_1 \). Here, \( z \) is the center of mass (COM) of the background solution. The subscript on \( \xi_{\text{COM}} \) indicates that the time derivative is taken in center-of-mass frame. As no further time derivatives of fields are present, we assume that all volume integrals are performed in this frame from now on. Then, the background fields are spherically symmetric around the origin of integration \( x = 0 \).

A derivation of \( L_{\text{star}} \) starting from variational fluid dynamics can be obtained by essentially following Ref. [21] and is therefore not repeated here. This derivation assumes a potential for the fluid velocity and a homentropic equation of state \( \rho = \rho \rho_0 \). As no further time derivatives of fields are present, we assume that all volume integrals are performed in this frame from now on. Then, the background fields are spherically symmetric around the origin of integration \( x = 0 \).

In the following, we are actually not going to substitute \( \rho_{\text{ext}} \) by another compact object. Instead, we view it as an external “current,” similar as for generating functionals in quantum field theory. The important observation is that
\[
\Phi = \Phi_0 + \Phi_1 = \frac{\delta L_{\text{full}}}{\delta \rho_{\text{ext}}}. \tag{3.11}
\]
This enables us to recover the gravitational field from the Lagrangian even after it was integrated out.

### B. Normal modes

To define the normal oscillation modes of the star, we need to integrate out the gravitational field perturbation. The formal solution to its field equation reads
\[
\Phi_1 = 4 \pi G \Delta^{-1} [\rho_1 + \rho_{\text{ext}}], \tag{3.12}
\]
where the inverse Laplacian \( \Delta^{-1} \) is defined for usual boundary conditions at spatial infinity. After insertion into the Lagrangian, we reorder the Lagrangian into normal mode (NM), interaction, and pure external parts (keeping only the kinetic part untouched),
\[
L_{\text{full}} = L_{\text{kin}} + L_{\text{NM}} + L_{\text{int}} + L_{\text{ext}}, \tag{3.13}
\]
where
\[
L_{\text{NM}} = \int d^3x \left[ \frac{\dot{\xi}^2}{2 \rho_0} \xi \cdot \nabla \xi \right], \tag{3.14}
\]
\[
L_{\text{int}} = -\int d^3x [\rho_0 \Phi_{\text{ext}} + \rho_1 (\Phi_{\text{ext}} + \mathbf{x} \cdot \ddot{z})], \tag{3.15}
\]
\[
L_{\text{ext}} = -\frac{1}{2} \int d^3x \rho_{\text{ext}} \Phi_{\text{ext}}. \tag{3.16}
\]
with the abbreviation \( \Phi_{\text{ext}} = 4 \pi G \Delta^{-1} \rho_{\text{ext}} \) and the linear (nonlocal, integro-differential) operator \( D \) defined by
\[
D \xi = -\nabla \left\{ \left[ \frac{\dot{\xi}^2}{\rho_0} + 4 \pi G \Delta^{-1} \right] \nabla \cdot (\rho_0 \xi) \right\}. \tag{3.17}
\]
It is easy to see that \( D \) is Hermitian with respect to the compact integration measure \( dm_0 = \rho_0 d^3x \), which was first recognized by Chandrasekhar [86]. This fact guarantees that it possesses a complete set of eigenfunctions \( \xi_{\text{nlm}}^{\text{NM}} \) with real eigenvalues \( \omega_{nl}^2 \),
\[
D \xi_{\text{nlm}}^{\text{NM}} = \omega_{nl}^2 \xi_{\text{nlm}}^{\text{NM}}, \tag{3.18}
\]
which are orthogonal and can be normalized,
\[
\int d^3x \rho_0 \xi_{\text{nlm}}^{\text{NM}} \xi_{l'm'}^{\text{NM}} = \delta_{nl} \delta_{l'm'}. \tag{3.19}
\]
This normalization is convenient for calculations but implies unusual units for \( \xi_{\text{nlm}}^{\text{NM}} \).

This defines the normal modes of the star. The modes are discrete due to the compact integration measure. As \( D \) respects rotational symmetry (in the center-of-mass system), one can label the normal modes by the usual angular momentum “quantum” numbers \( l \) and \( m \). (When \( n \) is not used as an index, it always refers to the mass.) In particular, the perturbation quantities can be expanded over the relevant (scalar, vector,...) spherical harmonics and inherit the associated angular symmetries. Notice that \( \omega_{nl} \) is independent of \( m \). The label \( n \) denotes the radial structure of the modes. Usually these are given as a letter \( (f, p, g, \text{ etc.}) \), labeling a general class to which the modes belong by considering the primary restoring mechanism (corresponding to fundamental, pressure, gravity modes), together with a number counting its radial nodes (analogous to the overtone number for a vibrating string).

### C. Amplitude formulation

A general physical displacement \( \xi \) can now be decomposed into a sum of discrete normal modes \( \xi_{\text{nlm}}^{\text{NM}} \) with
corresponding time-dependent amplitudes $A_{nlm}$,
\begin{equation}
\xi = \sum_{nlm} A_{nlm}(t) \xi_{nlm}^\text{NM}(x),
\end{equation}
thanks to the completeness of the $\xi_{nlm}^\text{NM}$. The reality condition $\xi^* = \xi$ leads to
\begin{equation}
A_{nlm}^* = (-1)^m A_{nl-m},
\end{equation}
which follows from the property $Y^{lm*} = (-1)^m Y^{l-m}$ of the spherical harmonics. Additionally, according to Eq. (3.4), it holds
\begin{equation}
\rho_1 = \sum_{nlm} A_{nlm} \rho_{nlm}^\text{NM}, \quad \rho_{nlm}^\text{NM} := -\nabla \cdot (\rho_0 \xi_{nlm}^\text{NM}).
\end{equation}
Notice that we can separate the angular dependence as
\begin{equation}
\rho_{nlm}^\text{NM} = \rho_{nlm}^\text{NM}(r) Y^{lm}(\Omega),
\end{equation}
and that the radial part $\rho_{nlm}^\text{NM}$ must be a real function. This property allows the use of the orthogonality properties of spherical harmonics under angular integration $d\Omega$, which will become important below.

The normal mode and interaction Lagrangians turn into
\begin{align}
L_{\text{NM}} &= \sum_{nlm} \frac{1}{2} \left[ |\ddot{A}_{nlm}|^2 - \omega_{nl}^2 |A_{nlm}|^2 \right], \\
L_{\text{int}} &= \sum_{nlm} A_{nlm} f_{nlm}^* - \int d^3x \rho_0 \Phi_{\text{ext}},
\end{align}
where the force term is given by
\begin{equation}
f_{nlm} = -\int d^3x \rho_{nlm}^\text{NM*} (\Phi_{\text{ext}} + x \cdot \ddot{z}).
\end{equation}
The discrete amplitudes $A_{nlm}$ now take on the role of dynamic variables in place of $\xi$. Their equations of motion have the form of a forced harmonic oscillator,
\begin{equation}
\ddot{A}_{nlm} + \omega_{nl}^2 A_{nlm} = f_{nlm},
\end{equation}
where we used $f_{nlm}^* = (-1)^m f_{nl-m}$ after variation of the action. This amplitude formulation is the starting point for investigations in Refs. [11, 13, 14, 29]; see also Refs. [8, 90, 86, 87].

IV. FURTHER DEVELOPMENTS

So far we have used the angular momentum numbers $l$ and $m$ to characterize the perturbation. Alternatively, one can transform the spherical harmonics to Cartesian tensors, specifically STF tensors. This is useful because the multipoles appearing in the effective action are STF tensors. We are also going to relate the multipoles and the amplitudes in this section. A brief account on the STF formalism is given in Appendix A.

A. STF basis

When written in terms of STF tensors, the expressions above are essentially the same, but the index $m$ is replaced by STF tensor indices of rank $l$. This is essentially just a basis change of a vector space: The index $m$ can take on values between $-l$ and $l$ and thus labels components of a $2l + 1$-dimensional vector, while the independent components of a STF rank-$l$ tensor exactly matches $2l + 1$. This is not a coincidence but deeply rooted in properties of rotation group representations: STF tensors transform irreducibly under rotations, just like spherical harmonics.

We denote the basis transformation matrix by $\mathcal{Y}$, such that it holds
\begin{align}
\mathcal{Y}^{00} &= \mathcal{Y}^{00}, \\
\mathcal{Y}^{1m} &= \mathcal{Y}^{k_1m} n^{k_11}, \\
\mathcal{Y}^{2m} &= \mathcal{Y}^{k_2m} n^{k_11} n^{k_22}, \\
&\vdots \\
\mathcal{Y}^{lm} &= \mathcal{Y}^{k_lm} \hat{n}^{K_l},
\end{align}
where $n^k = n^k(\Omega)$ is a unit vector. In the last line, we adopted a multi-index $K_i = \{k_1, k_2, \ldots, k_l\}$ and the abbreviation
\begin{equation}
\hat{n}^{K_i} = [n^{k_1} n^{k_2} \ldots n^{k_l}]^\text{STF}.
\end{equation}
For instance, $\hat{n}^{ij} = n^i n^j - \delta^{ij}/3$. Because of the use of Cartesian multipoles in the effective action, it is best to transform the oscillator amplitudes and force integrals to the STF basis from now on, which is denoted by a hat,
\begin{align}
\hat{A}_{nK_i} &= \sum_{m} A_{nlm} N_i \mathcal{Y}_{nlm}^{1m}, \\
\hat{f}_{nK_i} &= \sum_{m} f_{nlm} N_i \mathcal{Y}_{nlm}^{1m},
\end{align}
where the normalization factor $N_i$ is defined and explained in Appendix A.

An immediate consequence is that the acceleration term in Eq. (3.26) contributes only for $l = 1$, which is due to $x = r n \sim Y^{1m}$ and the orthogonality of the spherical harmonics. We are going to analyze the contributions of $\Phi_{\text{ext}}$ in a similar manner now.

B. Multipoles and overlap integrals

The Cartesian mass multipole moments of the perturbation are defined by
\begin{equation}
Q^{K_i} = \int d^3x \rho_1 r^l \hat{n}^{K_i}.
\end{equation}
It should be emphasized that the multipoles $Q^{K_i}$ are not the fundamental dynamical variables of the theory, but
are composed of the mode amplitudes. This becomes explicit by making use of Eqs. (3.22) and (3.23) in Eq. (4.8),

$$Q^K_l = \sum_{n l m} A_{nlm} \int dr \, r^{l+2} \rho_{nl}^{NM} (r) \int d\Omega \, Y_{l m} \hat{\eta}^{K_l},$$

$$= \sum_n I_{nl} \hat{A}_{nl}^{K_l},$$

(4.9)

where

$$I_{nl} = N_l \int dr \, r^{l+2} \rho_{nl}^{NM} (r).$$

(4.10)

The angular integration is given by Eq. (A17). However, the result is very plausible. Only modes with angular momentum number \( l \) can contribute to the 2l-pole \( Q^K_l \).

The quantity \( I_{nl} \) is better known as the overlap integral. (We included the factor \( N_l \) in contrast to other publications, see Appendix B.) It describes to which extend an external field excites the mode. To understand this, we need to analyze the force terms \( f_{nlK_l} \) defined by Eq. (4.7). Here, the external field enters through Eq. (3.26). As the source of \( \Phi^{ext} \) is located outside of the star, we can expand it by a Taylor series around the center \( z \),

$$\Phi^{ext} = \Phi^{ext}(z) + x^i (\partial_i \Phi^{ext})(z)$$

$$+ \frac{1}{2} x^i x^j (\partial_i \partial_j \Phi^{ext})(z) + \ldots.$$  

(4.11)

Because of the Laplace equation \( \Delta \Phi^{ext} = 0 \) valid inside the star, the traces of the multiple partial derivatives can be removed. This results in

$$\Phi^{ext} = \sum_l \frac{1}{l!} \int dr \, r^{l+2} \hat{\eta}^{K_l} (\partial_K \Phi^{ext})(z).$$

(4.12)

It is important that \( \hat{\eta}^{K_l} \) is orthogonal to all spherical harmonics with angular momentum number different from \( l \).

We insert Eq. (4.12) into the integral in Eq. (3.26) to arrive at

$$f_{nlK_l} = \frac{I_{nl}}{l!} (\partial_K \Phi^{ext})(z),$$

(4.13)

for \( l \neq 1 \), where we made heavy use of the formulas in Appendix A. The hat over \( \partial_K \) again denotes STF projection. For \( l = 1 \) it holds

$$f_{n1k} = -I_{n1} (\partial_k \Phi^{ext})(z) + \hat{\pi} \approx 0,$$

(4.14)

where we used the leading-order equation of motion for \( z \). This means that dipole oscillations are not excited in binary systems (by linear perturbations).

It should be noted that the theory is invariant under \( A_{nlm} \rightarrow -A_{nlm} \) and \( \epsilon^{NM} \rightarrow -\epsilon^{NM} \). This also implies that \( \rho_{nl}^{NM} \rightarrow -\rho_{nl}^{NM} \) and consequently \( I_{nl} \rightarrow -I_{nl} \). Therefore, we can always make \( I_{nl} \) positive, \( I_{nl} \geq 0 \). We assume this to be the case.

V. DERIVATION OF THE EFFECTIVE ACTION

The standard formalism for tidal interactions was cast into an action approach in the last section. Based on this result, the effective action can be derived in an elegant manner. This action will indeed be of the form (2.17), which was obtained from an effective action in general relativity by taking the nonrelativistic limit.

A. Multipole expansion

We are going to show that the interaction Lagrangian can be written in the same form as the effective action (2.17). For this purpose, we insert Eq. (4.12) into the interaction Lagrangian (3.15). Because of the spherical symmetry of \( \rho_0 \sim Y_0^0 \), it is easy to see that

$$- \int d^3 x \, \rho_0 \Phi^{ext} = -m \Phi^{ext}(z),$$

(5.1)

where

$$m = \int d^3 x \, \rho_0 = \text{const.}$$

(5.2)

Furthermore, it holds

$$- \int d^3 x \, \rho_1 \Phi^{ext} = - \sum_{l} \frac{1}{l!} Q^K_l (\partial_K \Phi^{ext})(z),$$

(5.3)

where the \( Q^K_l \) are just the usual mass multipoles of the perturbation (4.8). We assume that

$$Q = \int d^3 x \, \rho_1 = 0,$$

(5.4)

which implies that the mass of the compact object is not modified by the perturbation. Further, the terms from Eq. (3.15) involving the mass dipole \( Q^1 \) combine to \( -[(\partial_0 \Phi^{ext})(z) + \hat{\pi}]Q^1 \), which vanishes by virtue of the leading-order equation of motion for \( z \) and is of higher order. That is, this term can be removed at the Lagrangian level by a redefinition of \( z \), see Ref. [88], which can produce further terms only at quadratic perturbation order.

The result for the interaction Lagrangian (3.15) is

$$L_{int} = -m \Phi^{ext}(z) - \sum_{l \geq 2} \frac{1}{l!} Q^K_l (\partial_K \Phi^{ext})(z).$$

(5.5)

The applied method of deriving the multipole expansion was suggested and developed further in Ref. [89] in the relativistic case. Notice that we have localized the Lagrangian on the center-of-mass position \( z \), which means that we represent the extended object by a point particle comprising various multipole moments \( Q^K_l \). By cutting off the multipole summation at some value of \( l \), we can neglect effects of small-scale structure in a controlled manner. That is, the summation in Eq. (5.5) is a sum
over interaction energies and one only needs to include the terms relevant for the desired accuracy. Here a clear separation of scales is crucial.

Remember that the multipoles \(Q^K\) are not the fundamental dynamical variables of the theory but are composed of the mode amplitudes; see Eq. (4.9). The most explicit form of the interaction Lagrangian thus reads

\[
L_{\text{int}} = -m\rho_{\text{ext}}(\mathbf{z}) - \sum_{n,l \geq 2} \frac{I_{nl}}{l!} (\partial^l K_n \rho_{\text{ext}})(\mathbf{z}) \dot{A}_{nK_l},
\]

(5.6)

From this result, we can also easily obtain \(f_{nlK_l}\) as the coefficient of \(\dot{A}_{nK_l}\), cf. Eq. (4.13). The kinematic terms for the STF amplitudes \(\dot{A}_{nK_l}\) simply read

\[
L_{NM} = \sum_{n,l} \frac{1}{2} \left[ (\dot{A}_{nK_l})^2 - \omega_{nl}^2 (\dot{A}_{nK_l})^2 \right],
\]

(5.7)

which is clear from the unitarity of the transformation to the STF basis.

**B. Effective action**

The final step is to construct the effective action according to its definition in quantum field theory. That is, we reintroduce the gravitational field as a dynamical variable through a Legendre transformation using Eq. (5.8). The effective Lagrangian is the Legendre transform

\[
L_{\text{eff}} = \int d^3 x \rho_{\text{ext}} \Phi + L_{\text{fall}},
\]

(5.8)

where the solution for \(\rho_{\text{ext}}\) from Eq. (3.11) must be inserted. [The usual sign in Eq. (5.8) is consistent with Eq. (3.11).] Recalling Eqs. (3.13), (3.16), and (5.5) and the abbreviation \(\Phi_{\text{ext}} = 4\pi G \Delta^{-1} \rho_{\text{ext}}\), we can evaluate Eq. (3.11) and solve it for \(\rho_{\text{ext}}\),

\[
\rho_{\text{ext}} = \frac{1}{4\pi G} \Delta \Phi - m\delta - \sum_{l \geq 2} \frac{(-1)^l}{l!} Q^K \partial^l K_l \delta,
\]

(5.9)

where \(\delta = \delta(\mathbf{x} - \mathbf{z})\) and \(\mathbf{x}\) is the field coordinate in an inertial frame now. Also notice that the Legendre transformation (5.8) involves the full field \(\Phi\), while Eq. (3.16) comprises the external part only. As an intermediate step, we notice that

\[
L_{\text{eff}} = L_{\text{kin}} + L_{\text{NM}} + \frac{1}{2} \int d^3 x \rho_{\text{ext}} \Phi + L_{\text{fall}},
\]

(5.10)

where \(L_{\text{NM}}\) is still given by Eq. (3.24) or (5.7). Here, we still need to insert Eq. (5.9), which produces singular self-interactions, like \(\delta \Delta^{-1} \delta\). These are simply dropped here. The physical origin of these singularities is the inability of the multipole expansion to reproduce the gravitational field inside the body. The situation is completely analogous to the electrostatic energy of a charge distribution in the point-charge (monopole) limit and discussed in many textbooks. Ignoring these self-interactions, the result of the Legendre transformation reads

\[
L_{\text{eff}} = L_{\Phi} + L_{\text{NM}} + L_{\text{PP}},
\]

(5.11)

where

\[
L_{\text{PP}} = \frac{1}{2} m\dot{\mathbf{z}}^2 - m\Phi - \sum_{l \geq 2} \frac{1}{l!} Q^K \partial^l K_l \Phi,
\]

(5.12)

\[
L_{\Phi} = \frac{1}{8\pi G} \int d^3 x \Phi \Delta \Phi.
\]

(5.13)

The argument \(\mathbf{z}\) of the fields was omitted in \(L_{\text{PP}}\) for simplicity. The general relativistic (covariant) generalization of \(L_{\text{PP}}\) is given by Ref. [26, Eq. (20)] to quadrupole order \(l = 2\), see also Sec. II here, and our generic result (5.12) can be compared immediately to Ref. [27, Eq. (1)].

It is interesting to interpret the derivation given in the present section in the context of Wilson’s effective action. In the standard construction, the field \(\Phi\) is split into short-scale (ultraviolet) and long-scale (infrared) parts. This is best done in spatial Fourier domain \(\mathbf{k}\), where the operator \(\Delta^{-1}\) is a local one. Then, the procedure is as follows:

1. The first step is to integrate out only the ultraviolet part of \(\Phi\).

2. Next, \(\mathbf{k}\) is rescaled such that the now vacant ultraviolet regime is repopulated by the infrared contributions. At this point, one has basically zoomed out and views the system at a larger scale.

3. In a final step, the dynamical variables are renormalized in order to recover the original normalization of the kinematic terms in the action.

Here, the procedure is different but essentially analogous:

1. The field is first integrated out entirely (not just the ultraviolet part). This is beneficial for defining the operator \(\mathcal{D}\) and subsequent definition of the oscillation modes.

2. The subsequent Taylor or multipole expansion shrinks the object to a point, which projects onto the infrared scales larger than the size of the object (remember that the multipole approximation breaks down in the interior). This basically zooms out, but without the need for an explicit rescaling.

3. Finally, in the course of Legendre transformation to the effective action, divergent terms were dropped. This corresponds to an implicit renormalization of the dynamical variables.

The similarities to the standard construction are obvious. There exists a shortcut to Eq. (5.12), which is characterized as the continuum effective field theory in Ref.
It is an intuitive assumption that \( L_{PP} \) should be local. Further, invariance under rotations is even strictly required. It is also possible to absorb certain terms by variable redefinitions. Such considerations allow one to restrict the most generic possible form of \( L_{PP} \) considerably. This is how the general relativistic version of Eq. (5.12) was constructed in Ref. [27]; see also Refs. [25, 26]. The idea is then to fix the remaining arbitrariness of the Lagrangian (here given by the coefficients \( Q^{K_l} \)) through a matching against the full theory. This approach will be followed in the next section. It is usually simpler than an explicit derivation of the effective action, especially for nonlinear theories like general relativity. In the present Newtonian case, the advantage is not incredible but still illustrative.

VI. ANALYTIC MATCHING

In this section, we assume that the generic form of the effective action was constructed, e.g., from symmetry arguments. In our case, this leads to Eq. (2.17) at quadrupole order or Eq. (5.12) to all multipole orders. The precise arguments are given in Refs. [25–27], even for the relativistic case, and are not repeated here. However, in this section, we pretend that we are completely uninformed about the fact that the multipoles are composed of mode amplitudes (4.9) and about the Lagrangian \( L_{NM} \) for them. Instead, we establish the connection between the constructed effective action and the full theory from Sec. III through a matching procedure. The result will be the response function introduced in Sec. II C.

A. Matching condition

We are going to fit together the gravitational field \( \Phi \) predicted by the effective theory (5.11) to the desired one of the full theory (3.13). Without loss of generality, we assume \( \mathbf{z} = 0 \) from now on. The matching condition can be formulated as

\[
\frac{\delta L_{\text{eff}}}{\delta \rho_{\text{ext}}} = -\Phi = \frac{\delta L_{\text{full}}}{\delta \rho_{\text{ext}}},
\]

which should hold at large scales, i.e., for \( r \gg R \), where \( R \) is the radius of the compact object. We first evaluate the left-hand side,

\[
\Phi_{\text{eff}} = 4\pi G \Delta^{-1} \left[ \rho_{\text{ext}} + m \delta + \sum_{l \geq 2} \frac{(-1)^l}{l!} Q^{K_l} \partial_{K_l} \delta \right],
\]

which is just the inverse of Eq. (5.9). Using \( 4\pi \Delta^{-1} \delta = -r^{-1} \) and Eq. (A18), we arrive at to the potential generated by the multipoles,

\[
\Phi_{\text{eff}} = -\frac{Gm}{r} - \sum_{l \geq 2} \frac{G(2l-1)!!}{l!} Q^{K_l} \frac{\tilde{n}^{K_l}}{r^{l+1}} + \Phi_{\text{ext}}.
\]

Remember that we act as if the composition of the \( Q^{K_i} \), Eq. (4.9), and the form of \( L_{NM} \) in the effective theory are unknown to us.

On the other hand, in the full theory, it holds

\[
\Phi_{\text{full}} = -\frac{\delta L_{\text{full}}}{\delta \rho_{\text{ext}}} = 4\pi G \Delta^{-1} [\rho_0 + \rho_1] + \Phi_{\text{ext}}.
\]

After writing \( \Delta^{-1} \) as an integral operator involving the usual gravitational Green function, the contributions can be analyzed for \( r \gg R \geq r' \) through a standard multipole expansion. The matching to Eq. (6.3) then implies the identifications

\[
m = \int d^3x \rho_0, \quad Q^{K_i} = \int d^3x \rho_1 r^l \tilde{n}^{K_i},
\]

as before. We assume that the dipole vanishes, as it cannot be excited in binary systems; see Eq. (4.14). In any case, one can otherwise redefine the center in the full theory such that the dipole vanishes exactly. Notice that here the multipoles were defined as formal coefficients in the effective action, and only the matching related them to integrals over the mass density of the source.

The conceptual problem here is that the dynamics of the multipoles is still unknown. Just the coupling between the multipoles and the gravitational field is fixed for now. Therefore, we come back to the idea from Sec. II C to describe the dynamical reaction of the multipoles to gravitational interaction by a response function; see Eq. (2.10) for the quadrupole. This is almost trivial in the present case, but in general relativity the definition of source multipoles is fully clarified for test bodies only. The matching in the effective field theory can potentially generalize this to the self-gravitating case.

B. Response function

The generalization of Eqs. (2.10) and (2.14) to arbitrary multipole order reads

\[
\tilde{Q}^{K_i}(\omega) = \frac{1}{i}\tilde{F}(\omega) \mathcal{F}(\partial_{K_i} \Phi)(\mathbf{z}, \omega),
\]

which is suggested by the coupling of the multipoles in the action (5.12). Remember that \( \mathcal{F} \) denotes the Fourier transform. The response function \( \tilde{F}(\omega) \) is the important ingredient of the effective theory that we need to obtain from the matching. Notice that a response function offers a very generic way to encode the dynamics, which extends far beyond the specific “full” theory considered here.

On the other hand, we have seen that the multipoles in our full theory are composed as Eq. (4.9),

\[
\tilde{Q}^{K_i}(\omega) = \sum_n I_{nt} \mathcal{F} \tilde{A}_n^{K_i}(\omega).
\]

Now, the amplitudes \( \tilde{A}_n^{K_i} \) satisfy a forced harmonic oscillator equation (3.27), in the STF basis and Fourier
domain given by
\[ (-\omega^2 + \omega_n^2) F A_n K_l = F \tilde{F}_n K_l = -\frac{I_n}{l!} F (\delta K_l \Phi_{\text{ext}}), \] (6.8)
where we used Eq. (4.13). Combining both equations, we find the solution for the quadrupole in the frequency domain,
\[ \tilde{Q} K_l = -\frac{1}{l!} \left[ \sum_n \frac{I_n^2}{\omega_n^2 - \omega^2} \right] F (\delta K_l \Phi_{\text{ext}})(z, \omega). \] (6.9)

Next, we match Eqs. (6.6) and (6.9). This can be done by noting that \( \Phi(z) = \Phi_{\text{ext}}(z) \) if the singular self-field is dropped (and similarly for partial derivatives). The analytic result for the response function finally reads
\[ \tilde{F}_l = \sum_n \frac{I_n^2}{\omega_n^2 - \omega^2}. \] (6.10)

Because of the normalization in Eq. (3.19), one must be careful when analyzing units. One can check that
\[ [I_n] = \sqrt{ML^{l-1}}, \quad [\tilde{F}_l] = ML^{2l}, \] (6.11)
where \( M \) is units of mass and \( L \) of length. Remember that we have \( c = 1 \). We will show that this response function can be directly obtained from numerical solutions for the exterior gravitational field. Interestingly, this gives an alternative to determine the overlap integrals \( I_{nl} \), Eq. (4.10), which appear here as the coefficients of the poles of the response function.

**VII. NUMERIC MATCHING**

Effective field theories not only allow a matching to a known full theory but also a matching to experimental data. In the present, section we explore a matching to numerical simulations, which may be regarded as numerical experiments. We argue that the method immediately applies to more complicated scenarios.

**A. Numerical setup**

In this section, we compute the response of a compact configuration to time-dependent external excitation numerically. In the Newtonian case, the response should be “almost trivial,” in the sense that the equation for the Newtonian potential is linear and not time dependent. In contrast to this, the concept of a “time-dependent” relativistic response function is much more involved.

We first discuss the solution of the background equations (3.1) and (3.2). For the sake of simplicity, we focus on the particular polytrope
\[ P = k \rho^2, \] (7.1)
since in this case, an exact solution is available. The density profile is given by
\[ \rho_0(r) = \frac{\rho_c}{K r} \sin Kr, \quad K = \left( \frac{2\pi G}{k} \right)^{\frac{1}{2}}, \] (7.2)
where \( \rho_c \) is the central density.

The two perturbation equations encoded by the Lagrangian from Sec. III A read
\[ \rho_0 \ddot{\xi} = -\rho_0 \nabla \cdot (\rho_0 \xi) + \Phi_1 + \mathbf{x} \cdot \ddot{\mathbf{z}}, \] (7.3)
\[ \Delta \Phi_1 = -4\pi G \nabla \cdot (\rho_0 \xi). \] (7.4)

As we consider a single compact object fixed at the coordinate origin, it holds \( z = 0 \), and we dropped the subscript COM. Remember that the fluid velocity perturbation can be expressed in terms of a potential,
\[ \dot{\xi} = \mathbf{u}_1 = \nabla \phi_1. \] (7.5)

Integrating Eq. (7.3) along an arbitrary line, we can conclude that
\[ \dot{\phi}_1 - \frac{c^2}{\rho_0} \nabla \cdot (\rho_0 \xi) + \Phi_1 = \text{const.} \] (7.6)

We can further set the integration constant to zero without loss of generality, since it can be absorbed into \( \dot{\phi}_1 \).

The next step is to separate time, radial, and angular dependence. This is achieved using Fourier modes and spherical harmonics \( Y^{lm} \),
\[ \Phi_1 = \frac{1}{2\pi} \int d\omega \sum_{lm} e^{i\omega t} \frac{1}{2} h^{(m)}_l (r, \omega) Y^{lm}, \] (7.7)
\[ \phi_1 = \frac{1}{2\pi} \int d\omega \sum_{lm} e^{i\omega t} \frac{1}{\omega} U^{(m)}_l (r, \omega) Y^{lm}. \] (7.8)

The prefactors are chosen to allow an easy comparison to the relativistic generalization [24]. The subscript on \( h_0 \) does not denote the perturbation order but the component of the metric. We drop the indices \( l, m \) and the arguments of \( h_0 \) and \( U \) from now on. The perturbation equations are then
\[ h''_0 + 2h'_0 + h_0 \left[ \frac{4\pi G \rho_0}{c_s^2} - \frac{(l+1)}{r^2} \right] = \frac{8\pi G \rho_0}{c_s^2} U, \] (7.9)
\[ U'' + U' \left[ \frac{2}{r} + \frac{\rho'_0}{\rho_0} \right] + U \left[ \frac{\omega^2}{c_s^2} - \frac{(l+1)}{r^2} \right] = \frac{\omega^2}{2c_s^2} h_0. \] (7.10)

Here, Eq. (7.9) arises from Eq. (7.4) with the right-hand side replaced using Eq. (7.6), while Eq. (7.10) directly derives from Eq. (7.6). After choosing boundary conditions at \( r = 0, \) \( r = \bar{R} \), and \( r = \infty \), these equations can be readily integrated numerically.

Regularity at the origin imposes the boundary conditions
\[ h_0 \propto r^l + O(r^{l+1}), \quad U \propto r^l + O(r^{l+1}), \] (7.11)
for the perturbation fields. Regularity at the surface of the star imposes
\[ 2U(R) - h_0(R) + \frac{2Gm}{R^2\omega^2}U'(R) = 0. \] (7.12)
We have given three boundary conditions. The remaining arbitrariness of the solution is just its overall normalization, which has no physical significance here (due to linearity of the perturbation equations). It follows that we have enough boundary conditions to uniquely solve the problem numerically. No further conditions need to be imposed at \( r = \infty \).

B. Matching the exterior field

We have derived in Eq. (6.3) the generic gravitational field predicted by the effective theory. This field should of course match the numerically obtained exterior field of the neutron star. For simplicity, we consider specific values of \( l \geq 2 \) and \( m \), as the numerical integration decomposes into such sectors. Inserting Eq. (4.12) into Eq. (6.3), we then obtain for the field perturbation
\[ \Phi_1^{\text{eff}} = \frac{\partial K_l}{\partial r} \left[ -r^{-l-1}GQK_l(2l-1)!! + r^l(\partial \Phi_{\text{ext}})(z) \right]. \] (7.13)
Translated to the function \( h_0 \) by Eq. (7.7) and inserting the definition of the response (6.6), this reads
\[ h_0^{\text{eff}} = C_{lm} \left[ r^{-l-1}G(2l-1)!! + r^l(\partial \Phi_{\text{ext}})(z) \right], \] (7.14)
where \( C_{lm} \) is an overall normalization factor that can be related to the magnitude of the external field. The explicit expression reads \( C_{lm} = \frac{2|\mathbf{\Omega}|^2}{l!}Y_{lm}^{*}F(\partial \Phi_{\text{ext}})(z, \omega) \), where Eq. (A7) was used. We dropped the summation over \( m \), as we focus on a specific value. Remember that only the external field part contributes to Eq. (6.6), as the other part leads to singularities, which are dropped.

As the analytic result (7.14) represents a generic vacuum solution, it is clear that the exterior part of the numeric solution can be written as
\[ h_0^{\text{numeric}} = C \left[ a_l \left( \frac{Gm}{r} \right)^{l+1} + \left( \frac{r}{Gm} \right)^l \right] \] (7.15)
for \( r > R \).

In other words, \( a_l \) is proportional to the ratio of the regular and irregular parts of the potential as \( r \to \infty \). The part of \( h_0 \) diverging for large \( r \) is coming from the external gravitational field, whereas the part approaching zero is due to the multipole of the object. Notice that \( a_l \) is dimensionless, and its definition corresponds to the one in Ref. [4] in the relativistic case, except that here it is a function of \( \omega \). The numeric construction from the last section leads to a unique numeric value for \( a_l \).

Matching the effective potential (7.14) to the exterior numerics (7.15) results in
\[ \tilde{F}_l(\omega) = \frac{(Gm)^{2l+1}!!}{G(2l-1)!!}a_l(\omega). \] (7.16)

FIG. 1. Quadrupolar \( l = 2 \) response function for a star with \( R = 8.89 \text{ km}, m = 1.2M_\odot \), and polytropic index 1 obtained numerically. The dots are just some selected data points. More many were used for the fit, with higher density around the poles.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Mode & \( f \) & \( p_1 \) & \( p_2 \) & \( q_{nl} \) \\
\hline
\( \nu_{nl} \) [kHz] & 2.95 & 8.3 & 13.0 & 0.32 \\
\hline
\( \omega_{nl}R/2\pi \) & 0.0873 & 0.25 & 0.385 & 0.016 & 0.002 \\
\hline
\end{tabular}
\caption{Frequencies \( \nu_{nl} = \omega_{nl}/2\pi \) and overlap integrals for a star with \( R = 8.89 \text{ km}, m = 1.2M_\odot \), and polytropic index 1 obtained from fitting the response function of the quadrupole, \( l = 2 \).}
\end{table}

Comparing with Ref. [4, Eq. (48)] we see that \( \tilde{F}_l = l\mu_1 + O(\omega) \), where \( \mu_1 \) is related to the dimensionless Love number of the second kind \( k_l \) by Ref. [4, Eq. (47)].

C. Numerical results

We integrated the system of equations (7.9) and (7.10) with suitable boundary conditions for a typical mass and radius of a neutron star. More specifically, we chose parameters such that the radius is \( R \approx 8.89 \text{ km} \) and the mass is \( m \approx 1.2M_\odot \). We considered the quadrupolar case \( l = 2 \) here.

The result is summarized in Fig. 1, where three poles can distinctly be seen. Notice that we formed dimensionless quantities using the size of the object, which is most natural for effective field theories due to scaling arguments [77]. Figure 1 further suggests that the response function is given by the superposition of response functions for harmonic oscillators,
\[ \frac{G\tilde{F}_l}{R^{3/2}} = \sum_n \frac{\omega_{nl}^2}{R^2(\omega_{nl}^2 - \omega^2)}, \] (7.17)
which we know must hold exactly from our analytic result (6.10). The dimensionless overlap integrals \( q_{nl} \) are
related to the $I_{nl}$ through
\[ q_{nl}^2 = \frac{G}{R^3} I_{nl}^2. \] (7.18)

In contrast to this definition, other publications often define dimensionless overlap integrals based on the central density of the star, while our convention is adapted to the Love number $k_l$. We fitted our numerical results using the first three terms of Eq. (7.17) and found the first frequencies of the normal modes ($f_k$, $p_1$, and $p_2$-modes) and the associated overlap integrals; see Table I. Remember that $I_{nl} \geq 0$, and we also assume $q_{nl} \geq 0$.

The numerical matching described here is not only a useful alternative to obtain mode frequencies and overlap integrals. The method is applicable in more complicated situations, too, even for full-fledged 3-dimensional simulations. This is possible as no presuppositions on the response function are made. Instead, it always comes out as a numeric function, whether one can find a good and interpretable fit or not. The exterior potential is always the linear combination (7.15) (though sectors with different $l$ do not decouple in general).

VIII. CONCLUSIONS AND OUTLOOK

In this paper, we considered astrophysically relevant perturbations of compact objects in the Newtonian framework. We showed that the effective field theoretical approach available in the relativistic case applies naturally in the nonrelativistic case, too. This was expected, since Newtonian gravitation comes out of general relativity in the appropriate limit. However, we see here explicitly the connection between an effective description in one case and an exact rewriting at the level of the action in the other case.

Following the effective field theory approach, we showed how to describe a generic compact body deformed in a time-dependent way by a point particle with multipolar degrees of freedom. We argued that the numeric matching can in principle be applied to arbitrary complicated structured objects. The method allows a systematic way to understand the impact of the internal structure on the dynamics of a binary system. This is due to the fact that the effective theory is matched to a single object first, which allows one to model a single object (e.g., by mechanical models like oscillators) as a building block before proceeding to the binary case. The potentials of a binary system can then be characterized as monopole-monopole, monopole-quadrupole (2.19), quadrupole-quadrupole interactions, and so forth and so on. This method has interesting analogies to thermodynamics, where systems are characterized on a macroscopic level by state functions. Indeed, multipoles encode the macroscopic gravitational interaction of compact objects. Predictions require the knowledge of correlation functions between state variables, which is analogous to the response functions here.

The effective description is explicit once we compare the Newton potential of the actual compact object with the potential of a multipole alone. Actually, this matching procedure further provides a prescription to compute the response of a compact source to an external perturbation. The perturbation comes as a regular part in the Newtonian potential, while the backreaction of the central object is irregular at the origin of the object. From a mathematical perspective, the irregular solution to the Poisson equation is sourced by a delta distribution located at the center. This is precisely interpreted as being the potential generated by the source multipole we are considering. The (properly normalized) ratio of the regular and irregular contributions is then understood as the response of the object to the external perturbation. The response function encodes the tidal coefficients of the central object, which come out as coefficients of a Taylor series in frequency.

Furthermore, our formalism gives a straightforward way to compute the normal modes of compact objects in Newtonian gravity. Indeed, only regularity conditions at the origin and at the surface of the star are required. The generic solution should then be continuously connected to the regular and irregular solution of the source-free Poisson equation (Laplace equation) in order to extract the response function. The poles of the response function are then precisely the normal modes of the compact object, while the overlap integrals are related to the width of the poles and can be obtained from a simple fit.

One obvious extension of the present work is to generalize the problem to the relativistic case. This will be presented in another publication [24] and is based on the numerical matching method. The results are essentially the same, in the sense that in the end, the response function turns out to be related to the ratio of external-field and quadrupole-reaction solutions to the source-free perturbation equation. However, the problem is much more difficult to attack since in this case, the equation itself admits singular points, the solution is expressed as a series of special functions, and the singular effective source has to be regularized with a suitable renormalization scheme. The Newtonian case is then very enlightening since it is much easier and not plagued by the same amount of technical difficulties. For instance, the Hermitian operator $\mathcal{D}$ gives rise to a complete system of modes. No general relativistic analog is known to us.

Finally, it should be noted that also in the relativistic case, the fit for the response is to a good approximation a sum of harmonic oscillators (7.17), at least for the simple neutron star model considered in Ref. [24]. This implies that the quadrupole can be written as a sum of oscillator amplitudes (4.9) in the relativistic case, with the internal dynamics given by Eq. (3.24).
ACKNOWLEDGMENTS

We gratefully acknowledge fruitful discussions with P. Pani and V. Cardoso. We also acknowledge V. Vitagliano for useful expressions. This work was supported by DFG (Germany) through Projects No. STE 2017/1-1 and No. STE 2017/2-1, FCT (Portugal) through Projects No. PTDC/C/CTEAST/098034/2008, No. PTDC/FIS/098032/2008, No. SFRH/BI/52132/2013, and No. PCOFUND-GA-2009-246542 (cofunded by Marie Curie Actions), and CERN through Project No. CERN/FP/123593/2011.

Appendix A: STF formalism

In this appendix, we summarize some relations for STF tensors. Reviews of the STF-tensor formalism and the relation to spherical harmonics can be found in Sec. II of Ref. [90], Appendix A of Re. [91], and Sec. II of Ref. [92].

1. Basic relations

With the help of a multi-index $K_l = \{k_1, k_2, \ldots, k_l\}$, we introduce the notation

$$n^{K_l} = n^{k_1} n^{k_2} \ldots n^{k_l},$$  
(A1)

$$\hat{n}^{K_l} = [n^{k_1} n^{k_2} \ldots n^{k_l}]^\text{STF}.$$  
(A2)

The orthogonality property [90, Eq. (2.5)] can be written in the convenient form

$$\int d\Omega \, \hat{n}^{K_{l'}} \hat{n}^{K_l} = N_l^2 \delta_{l l'} \delta^{K_l}_{K_{l'}}, \quad N_l^2 = \frac{4 \pi l!}{(2l + 1)!!},$$  
(A3)

where $\delta^{K_l}_{K_{l'}}$ is the STF projector, i.e.,

$$[B^{K_l}]^\text{STF} = \delta^{K_l}_{K_{l'}} B^{K_{l'}},$$  
(A4)

for an arbitrary $B^{K_l}$. Using the normalization

$$\int d\Omega Y_{lm} Y_{l'm'}^* = \delta_{mm'} \delta_{ll'},$$  
(A5)

the scalar spherical harmonics can be expressed as

$$Y_{lm} = Y_{lm}^{\hat{n}^{K_l}},'$$  
(A6)

where $Y_{lm}$ is given by [90, Eq. (2.12)]. A bijection between the STF-$l$ tensor basis and $m$-basis is provided through [90, Eq. (2.13)] by virtue of $Y_{lm}^{\hat{n}^{K_l}}$, e.g., we can invert Eq. (A6) as

$$\hat{n}^{K_l} = \sum_m N_l^2 Y_{lm}^{\hat{n}^{K_l}} Y_{lm}.'$$  
(A7)

From Eq. (A3), it follows that

$$N_l^2 Y_{lm}^{\hat{n}^{K_l}} Y_{lm}.' = \delta_{mm'},$$  
(A8)

$$N_l^2 \sum_m Y_{lm}^{\hat{n}^{K_l}} Y_{lm}' = \delta^{K_l}_{K_{l'}}.$$  
(A9)

We then define the transformation between STF components and $lm$ components by

$$\hat{B}^{K_l} = \sum_m N_l \gamma^{\hat{n}^{K_l}}_{m,l} B^{lm}, \quad B^{lm} = N_l \gamma^{\hat{n}^{K_l}}_{m,l} \hat{B}^{K_l}.$$  
(A10)

Components in the STF basis are often denoted by a hat. Sometimes this notation is also used for STF projection, but this is always clear from the context.

2. Normal modes in STF basis

We now transform $\xi_{nlm}$ and $A_{nlm}$, reading

$$\hat{\xi}_{nlk_l}(x) = \sum_m N_l \gamma^{\hat{n}^{K_l}}_{m,l} \xi_{nlm}(x),$$  
(A11)

$$\hat{A}_{nlk_l}(t) = \sum_m N_l \gamma^{\hat{n}^{K_l}}_{m,l} A_{nlm}(t).$$  
(A12)

Notice that the complex conjugation in the first equation is due to the fact that $\xi_{nlm}$ gives a basis, while $A_{nlm}$ are components. This allows us to write

$$\xi = \sum_{nlm} A_{nlm}(t) \xi_{nlm}(x),$$  
(A13)

$$= \sum_{nl} \hat{A}_{nlk_l}(t) \hat{\xi}_{nlk_l}(x),$$  
(A14)

Notice that $\hat{\xi}_{nlk_l}$ is real, while $\xi_{nlm}$ and $Y_{lm}$ are complex. It holds

$$\mathcal{D} \hat{\xi}_{nlk_l} = \omega^2 \xi_{nlk_l},$$  
(A15)

where we used that $Y_{lm}$ provides a bijection between the STF-$l$ tensors basis and $m$-basis. The $\xi_{nlm}$ are orthonormal,

$$\int d^3 x \bar{\xi}_{nm}^{\text{NM}} \xi_{nlm} = \delta_{nm} \delta_{ll'} \delta_{ll'},$$  
(A16)

This follows from Eq. (3.19) and the properties of $Y_{lm}$ listed above.

3. Useful formulas

Using above formulas, the angular integration in Eq. (4.9) immediately follows as

$$\int d\Omega Y_{lm} Y_{l'm'} \hat{n}^{K_l} = \delta_{l'l} \int d\Omega \, \hat{n}^{K_l} = \delta_{l'l} N_l^2 Y_{lm}.$$  
(A17)

Another useful relation is [91, Eq. (A 34)],

$$\partial_{K_l} \frac{1}{r} = (-1)^l \frac{(2l + 1)!!}{r^{l+1}} \hat{n}^{K_l}.$$  
(A18)
Appendix B: Overlap in terms of displacement

The angular dependence of the displacement vector can be separated as

\[ \xi_{nlm}^{NM} = \xi_{nl}^{R}(r)Y_{lm}^{R}(\Omega) + \xi_{nl}^{E}(r)Y_{lm}^{E}(\Omega) + \xi_{nl}^{B}(r)Y_{lm}^{B}(\Omega). \]  

(B1)

The three parts correspond to radial (R), electric-type (E), and magnetic-type (B) modes with corresponding orthogonal (but un-normalized) vector spherical harmonics,

\[ Y_{lm}^{E}(\Omega) = r \nabla Y_{lm}^{l}(\Omega), \]  

(B2)

\[ Y_{lm}^{B}(\Omega) = n \times Y_{lm}^{E}(\Omega), \]  

(B3)

Notice that the radial functions are independent of \( m \).

As \( \xi \) can be related to a scalar potential (7.5), we must have \( \xi_{nl}^{B} = 0 \). From Eqs. (3.22) and (3.23), we then obtain

\[ \rho_{nl}^{NM} = - \frac{1}{r^2} \frac{d}{dr} \left[ \frac{r^2 \rho_{nl}^{R}}{\Omega} \right] + \frac{1}{r} \rho_{nl}(l + 1) \xi_{nl}^{E}. \]  

(B5)

Our convention for the overlap integrals (4.10) now read

\[ I_{nl} = N_{nl} \int dr r^{l+1} \rho_{nl} \left[ \xi_{nl}^{R} + (l + 1)\xi_{nl}^{E} \right]. \]  

(B6)

[1] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, Cambridge, England, 1911.

[2] T. Shida, “On the elasticity of the Earth and the Earth’s crust,” in *Memoirs of the College of Science and Engineering*, vol. 4, pp. 1–296. Kyoto Imperial University, Kyoto, 1912.

[3] T. Hinderer, “Tidal Love numbers of neutron stars,” *Astrophys. J.* 677 (2008) 1216–1220, arXiv:0711.2420 [astro-ph].

[4] T. Damour and A. Nagar, “Relativistic tidal properties of neutron stars,” *Phys. Rev. D* 80 (2009) 084035, arXiv:0906.0096 [gr-qc].

[5] T. Binnington and E. Poisson, “Relativistic theory of tidal Love numbers,” *Phys. Rev. D* 80 (2009) 084018, arXiv:0906.1366 [gr-qc].

[6] É. É. Flanagan and T. Hinderer, “Constraining neutron star tidal Love numbers with gravitational wave detectors,” *Phys. Rev. D* 77 (2008) 021502, arXiv:0709.1915 [astro-ph].

[7] E. Berti, S. Iyer, and C. M. Will, “A post-Newtonian diagnosis of quasi-equilibrium configurations of neutron star-neutron star and neutron star-black hole binaries,” *Phys. Rev. D* 77 (2008) 024019, arXiv:0709.2589 [gr-qc].

[8] W. H. Press and S. A. Teukolsky, “On formation of close binaries by two-body tidal capture,” *Astrophys. J.* 213 (1977) 183–192.

[9] P. Mészáros and M. J. Rees, “Tidal heating and mass loss in neutron star binaries: Implications for gamma-ray burst models,” *Phys. Rev. D* 58 (1998) 024019, arXiv:astro-ph/9704163.

[10] D. Lai, “Resonant oscillations and tidal heating in coalescing binary neutron stars,” *Mon. Not. R. Astron. Soc.* 270 (1994) 611, arXiv:astro-ph/9404062.

[11] R. A. Gingold and J. J. Monaghan, “The Roche problem for polytropes in central orbits,” *Mon. Not. Roy. Astron. Soc.* 191 (1980) 897–924.

[12] L. Bildsten and C. Cutler, “Tidal interactions of inspiraling compact binaries,” *Astrophys. J.* 400 (1992) 175–180.

[13] M. E. Alexander, “Tidal resonances in binary star systems,” *Mon. Not. Roy. Astron. Soc.* 227 (1987) 843–861. http://adsabs.harvard.edu/abs/1987MNRAS.227..843A.

[14] É. É. Flanagan and É. Racine, “Gravitomagnetic resonant excitation of Rossby modes in coalescing neutron star binaries,” *Phys. Rev. D* 75 (2007) 044001, arXiv:gr-qc/0610029 [gr-qc].

[15] M. Shibata, “Effects of tidal resonances in coalescing compact binary systems,” *Prog. Theor. Phys.* 91 (1994) 871–884.

[16] A. Reisenegger and P. Goldreich, “Excitation of neutron star normal modes during binary inspiral,” *Astrophys. J.* 426 (1994) 688–691.

[17] K. D. Kokkotas and G. Schäfer, “Tidal and resonant effects in coalescing binaries,” *Mon. Not. R. Astron. Soc.* 275 (1995) 301–308, arXiv:gr-qc/9502034.

[18] W. C. Ho and D. Lai, “Resonant tidal excitations of rotating neutron stars in coalescing binaries,” *Mon. Not. Roy. Astron. Soc.* 308 (1999) 153, arXiv:astro-ph/9812116 [astro-ph].

[19] É. É. Flanagan, “General-relativistic coupling between orbital motion and internal degrees of freedom for inspiraling binary neutron stars,” *Phys. Rev. D* 58 (1998) 124030, arXiv:gr-qc/9706045.

[20] D. Lai and Y. Wu, “Resonant tidal excitations of inertial modes in coalescing neutron star binaries,” *Phys. Rev. D* 74 (2006) 024007, arXiv:astro-ph/0604163.

[21] Y. Rathore, A. E. Broderick, and R. Blandford, “A variational formalism for tidal excitation: Non-rotating, homentropic stars,” *Mon. Not. Roy. Astron. Soc.* 339 (2003) 25–32, arXiv:astro-ph/0209003 [astro-ph].

[22] N. N. Weinberg, P. Arras, and J. Burkart, “An instability due to the nonlinear coupling of p-modes to g-modes: Implications for coalescing neutron star binaries,” *Astrophys. J.* 760 (2013) 121, arXiv:1302.2292 [astro-ph.SR].

[23] N. N. Weinberg, P. Arras, E. Quataert, and J. Burkart, “Nonlinear tides in close binary systems,” *Astrophys. J.* 751 (2012) 136, arXiv:1107.0946 [astro-ph.SR].

[24] S. Chakrabarti, T. Delsate, and J. Steinhoff, “New
perspectives on neutron star and black hole spectroscopy and dynamic tides,”

[25] W. D. Goldberger and I. Z. Rothstein, “An effective field theory of gravity for extended objects,” Phys. Rev. D 73 (2006) 104029,

[26] W. D. Goldberger and I. Z. Rothstein, “Dissipative effects in the worldline approach to black hole dynamics,” Phys. Rev. D 73 (2006) 104030,

[27] W. D. Goldberger and A. Ross, “Gravitational radiative corrections from effective field theory,” Phys. Rev. D 81 (2010) 124015,

[28] T. Damour and G. Esposito-Farese, “Testing gravity to second post-Newtonian order: A field theory approach,” Phys. Rev. D 53 (1996) 5541–5578,

[29] Y. Rathore, R. D. Blandford, and A. E. Broderick, “Resonant excitation of white dwarf oscillations in compact object binaries — I. The no back reaction approximation,” Mon. Not. Roy. Astron. Soc. 357 (2005) 834,

[30] Y. Rathore, Resonant excitation of white dwarf oscillations in compact object binaries. PhD thesis, Caltech, 2005.

http://resolver.caltech.edu/CaltechETD:etd-05272005-17

[31] P. Balachandran and É. É. Flanagan, “Detectability of mode resonances in coalescing neutron star binaries,”

[32] R. Gold, S. Bernuzzi, M. Thierfelder, B. Brügmann, and F. Pretorius, “Eccentric binary neutron star mergers,” Phys. Rev. D 86 (2012) 121501,

[33] L. Baiotti, T. Damour, B. Giacomazzo, A. Nagar, and L. Rezzolla, “Analytic modeling of tidal effects in the relativistic inspiral of binary neutron stars,” Phys. Rev. Lett. 105 (2010) 261101,

[34] T. Damour, A. Nagar, and L. Villain, “Measurability of the tidal polarizability of neutron stars in late-inspiral gravitational-wave signals,” Phys. Rev. D 85 (2012) 123007,

[35] T. Hinderer, B. D. Lackey, R. N. Lang, and J. S. Read, “Tidal deformability of neutron stars with realistic equations of state and their gravitational wave signatures in binary inspiral,” Phys. Rev. D 81 (2010) 123016,

[36] M. Bejger, D. Gondek-Rosinska, E. Gourgoulhon, P. Haensel, K. Taniguchi, et al., “Impact of the nuclear equation of state on the last orbits of binary neutron stars,” Astron. Astrophys. 431 (2005) 297,

[37] M. Shibata and K. Kyutoku, “Constraining nuclear-matter equations of state by gravitational waves from black hole-neutron star binaries,” Prog. Theor. Phys. Suppl. 186 (2010) 17–25.

[38] F. Pannarale, L. Rezzolla, F. Ohme, and J. S. Read, “Will black hole-neutron star binary inspirals tell us about the neutron star equation of state?,” Phys. Rev. D 84 (2011) 104017,

[39] S. Bernuzzi, A. Nagar, M. Thierfelder, and B. Brügmann, “Tidal effects in binary neutron star coalescence,” Phys. Rev. D 86 (2012) 044030,

[40] A. Bauswein and H.-T. Janka, “Measuring neutron-star properties via gravitational waves from binary mergers,” Phys. Rev. Lett. 108 (2012) 011101,

[41] P. Wiggins and D. Lai, “Tidal interaction between a fluid star and a Kerr black hole: Relativistic Roche-Riemann model,” Astrophys. J. 532 (2000) 530.

[42] M. Ishii, M. Shibata, and Y. Mino, “Black hole tidal problem in the Fermi normal coordinates,” Phys. Rev. D 71 (2005) 044017,

[43] V. Ferrari, L. Gualtieri, and F. Pannarale, “Neutron star tidal disruption in mixed binaries: The imprint of the equation of state,” Phys. Rev. D 81 (2010) 064026,

[44] V. Ferrari, L. Gualtieri, and F. Pannarale, “A semi-relativistic model for tidal interactions in BH-NS coalescing binaries,” Class. Quant. Grav. 26 (2009) 125004.

[45] M. Kyutoku, M. Shibata, and K. Taniguchi, “Gravitational waves from nonspinning black hole-neutron star binaries: Dependence on equations of state,” Phys. Rev. D 82 (2010) 044049,

[46] K. Kyutoku, H. Okawa, M. Shibata, and K. Taniguchi, “Gravitational waves from spinning black hole-neutron star binaries: Dependence on black hole spins and on neutron star equations of state,” Phys. Rev. D 84 (2011) 064018,

[47] D. Tsang, J. S. Read, T. Hinderer, A. L. Piro, and R. Bondarescu, “Resonant shattering of neutron star crusts,” Phys. Rev. Lett. 108 (2012) 011102,

[48] E. Gaertig, K. Glampedakis, K. D. Kokkotas, and B. Zink, “The f-mode instability in relativistic neutron stars,” Phys. Rev. Lett. 107 (2011) 101102,

[49] Y. Kojima, “Stellar resonant oscillations coupled to gravitational waves,” Prog. Theor. Phys. 77 (1987) 297–309.

[50] J. Ruoff, F. Laguna, and J. Pullin, “Excitation of neutron star oscillations by an orbiting particle,” Phys. Rev. D 63 (2001) 064019,

[51] L. Gualtieri, E. Berti, J. A. Pons, G. Miniutti, and V. Ferrari, “Gravitational signals emitted by a point mass orbiting a neutron star: A perturbative approach,” Phys. Rev. D 64 (2001) 104007,

[52] J. A. Pons, E. Berti, L. Gualtieri, G. Miniutti, and V. Ferrari, “Gravitational signals emitted by a point mass orbiting a neutron star: Effects of stellar structure,” Phys. Rev. D 65 (2002) 104021.
[53] H. Fang and G. Lovelace, “Tidal coupling of a Schwarzschild black hole and circularly orbiting moon,” Phys. Rev. D72 (2005) 124016, arXiv:gr-qc/0505156 [gr-qc].

[54] T. Damour, M. Soffel, and C. Xu, “General relativistic celestial mechanics. I. Method and definition of reference systems,” Phys. Rev. D 43 (1991) 3273–3307.

[55] T. Damour, M. Soffel, and C. Xu, “General relativistic celestial mechanics. II. Translational equations of motion,” Phys. Rev. D 45 (1992) 1017–1044.

[56] T. Damour, M. Soffel, and C. Xu, “General relativistic celestial mechanics. III. Rotational equations of motion,” Phys. Rev. D 47 (1993) 3124–3135.

[57] T. Damour, M. Soffel, and C. Xu, “General relativistic celestial mechanics. IV. Theory of satellite motion,” Phys. Rev. D 49 (1994) 618–635.

[58] E. Racine and E. É. Flanagan, “Post-1-Newtonian equations of motion for systems of arbitrarily structured bodies,” Phys. Rev. D 71 (2005) 044010, arXiv:gr-qc/0404101.

[59] M. Shibata, K. Taniguchi, and T. Nakamura, “Location of the innermost stable circular orbit of binary neutron stars in the post-Newtonian approximations of general relativity,” Prog. Theor. Phys. Suppl. 128 (1997) 295–333, arXiv:gr-qc/9801004 [gr-qc].

[60] K. Taniguchi, H. Asada, and M. Shibata, “Irrotational and incompressible ellipsoids in the first post-Newtonian approximation of general relativity,” Prog. Theor. Phys. 100 (1998) 703–735, arXiv:gr-qc/9806039 [gr-qc].

[61] K. Taniguchi and M. Shibata, “Gravitational radiation from corotating binary neutron stars of incompressible fluid in the first post-Newtonian approximation of general relativity,” Phys. Rev. D 58 (1998) 084012, arXiv:gr-qc/9807005 [gr-qc].

[62] A. Maselli, L. Gualtieri, F. Pannarale, and V. Ferrari, “On the validity of the adiabatic approximation in compact binary inspirals,” Phys. Rev. D 86 (2012) 044032, arXiv:1205.7006 [gr-qc].

[63] V. Ferrari, L. Gualtieri, and A. Maselli, “Tidal interaction in compact binaries: A post-Newtonian affine framework,” Phys. Rev. D 85 (2012) 044045, arXiv:1111.6607 [gr-qc].

[64] T. Damour and A. Nagar, “Effective one body description of tidal effects in inspiraling compact binaries,” Phys. Rev. D 81 (2010) 084016, arXiv:0911.5041 [gr-qc].

[65] D. Bini, T. Damour, and G. Faye, “Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description,” Phys. Rev. D 85 (2012) 124034, arXiv:1202.3565 [gr-qc].

[66] E. Poisson, “Absorption of mass and angular momentum by a black hole: Time-domain formalisms for gravitational perturbations, and the small-hole / slow-motion approximation,” Phys. Rev. D 70 (2004) 084044, arXiv:gr-qc/0407050 [gr-qc].

[67] E. Poisson, “Metric of a tidally distorted, nonrotating black hole,” Phys. Rev. Lett. 94 (2005) 161103, arXiv:gr-qc/0501032 [gr-qc].

[68] S. Taylor and E. Poisson, “Nonrotating black hole in a post-Newtonian tidal environment,” Phys. Rev. D 78 (2008) 084016, arXiv:0806.3052 [gr-qc].

[69] E. Poisson, “Tidal interaction of black holes and Newtonian viscous bodies,” Phys. Rev. D 80 (2009) 064029, arXiv:0907.0874 [gr-qc].

[70] S. Comeau and E. Poisson, “Tidal interaction of a small black hole in the field of a large Kerr black hole,” Phys. Rev. D 80 (2009) 087501, arXiv:0908.4518 [gr-qc].

[71] K. Chatzioannou, E. Poisson, and N. Yunes, “Tidal heating and torque of a Kerr black hole to next-to-leading order in the tidal coupling,” Phys. Rev. D 87 (2013) 044022, arXiv:1211.1686 [gr-qc].

[72] H. Georgi, “Effective field theory,” Ann. Rev. Nucl. Part. Sci. 43 (1993) 209–252.

[73] C. P. Burgess, “Introduction to effective field theory,” Ann. Rev. Nucl. Part. Sci. 57 (2007) 329–362, arXiv:hep-th/0701053 [hep-th].

[74] I. Z. Rothstein, “TASI lectures on effective field theories,” arXiv:hep-ph/0308266 [hep-ph].

[75] J. F. Donoghue, “Introduction to the effective field theory description of gravity,” arXiv:gr-qc/9512024 [gr-qc].

[76] W. D. Goldberger and I. Z. Rothstein, “Towers of gravitational theories,” Gen. Rel. Grav. 38 (2006) 1537–1546, arXiv:hep-th/0605238 [hep-th].

[77] W. D. Goldberger, “Les Houches lectures on effective field theories and gravitational radiation,” arXiv:hep-ph/0701129.

[78] L. Blanchet, “Gravitational radiation from post-Newtonian sources and inspiralling compact binaries,” Living Rev. Relativity 9 (2006) 4. http://www.livingreviews.org/lrr-2006-4.

[79] L. Blanchet, “Post-Newtonian theory and the two-body problem,” Fundam. Theor. Phys. 162 (2011) 125–166, arXiv:0907.3598 [gr-qc].

[80] B. Kol and M. Smolkin, “Classical effective field theory and caged black holes,” Phys. Rev. D 77 (2008) 044033, arXiv:0712.2822 [hep-th].

[81] T. Damour and G. Esposito-Farese, “Gravitational-wave versus binary-pulsar tests of strong-field gravity,” Phys. Rev. D 58 (1998) 042001, arXiv:gr-qc/9803031.

[82] C. R. Galley, “The classical mechanics of non-conservative systems,” Phys. Rev. Lett. 110 (2013) 174301, arXiv:1210.2745 [gr-qc].

[83] T. Damour and O. M. Lecian, “On the gravitational polarizability of black holes,” Phys. Rev. D 80 (2009) 044017, arXiv:0906.3003 [gr-qc].

[84] I. Vega, E. Poisson, and R. Massey, “Intrinsic and extrinsic geometries of a tidally deformed black hole,” Class. Quant. Grav. 28 (2011) 175006, arXiv:1106.0510 [gr-qc].

[85] J. E. Vines and É. É. Flanagan, “Post-1-Newtonian quadrupole tidal interactions in binary systems,” Phys. Rev. D 88 (2013) 024046, arXiv:1009.4919 [gr-qc].
[86] S. Chandrasekhar, “A general variational principle governing the radial and the non-radial oscillations of gaseous masses,” Astrophys. J. 139 (1964) 664–674.

[87] D. Lynden-Bell and J. P. Ostriker, “On the stability of differentially rotating bodies,” Mon. Not. Roy. Astron. Soc. 136 (1967) 293–310. http://adsabs.harvard.edu/abs/1967MNRAS.136..293L.

[88] T. Damour and G. Schäfer, “Redefinition of position variables and the reduction of higher order Lagrangians,” J. Math. Phys. 32 (1991) 127–134.

[89] A. Ross, “Multipole expansion at the level of the action,” Phys. Rev. D 85 (2012) 125033.

[90] K. S. Thorne, “Multipole expansions of gravitational radiation,” Rev. Mod. Phys. 52 (1980) 299–339.

[91] L. Blanchet and T. Damour, “Radiative gravitational fields in general relativity I. General structure of the field outside the source,” Phil. Trans. R. Soc. A 320 (1986) 379–430. http://www.jstor.org/stable/37878.

[92] T. Damour and B. R. Iyer, “Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors,” Phys. Rev. D 43 (1991) 3259–3272.