GEODESIC INTERPOLATION ON SIERPINSKI GASKETS

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Abstract. We study the analogue of a convex interpolant of two sets on Sierpinski gaskets and an associated notion of measure transport. The structure of a natural family of interpolating measures is described and an interpolation inequality is established. A key tool is a good description of geodesics on these gaskets, some results on which have previously appeared in the literature [11,16,17,19].

The notion of a convex interpolant \( (1 - t)A + tB = \{(1 - t)a + tb : a \in A, b \in B\} \) for sets \( A, B \subset \mathbb{R}^n \) and \( t \in [0,1] \), and the Brunn-Minkowski inequality \(|(1 - t)A + tB|^{1/n} \geq (1 - t)|A|^{1/n} + t|B|^{1/n}\) for the \( n \)-volume \(|\cdot|\), have long had a central role in convex geometry. More recently, the class of functional inequalities that includes the Brunn-Minkowski inequality has been used to dramatically extend notions of curvature to more general settings, and a rich theory has developed around these advances [3,10].

The study of functional inequalities in the setting of fractal metric-measure spaces is considerably less developed. One area in which there has been a great deal of work is in relating the variation with \( \epsilon > 0 \) of the volume of an \( \epsilon \)-neighborhood of a set to the analytic and geometric properties of the set. For Euclidean sets with sufficiently smooth boundaries, such results can be obtained using the Steiner formula and inequalities of Brunn-Minkowski type. In the case of certain fractal sets and sets with fractal boundary in Euclidean space, one achievement of the theory developed by Lapidus and collaborators is a characterization of the volume of \( \epsilon \)-neighborhoods using complex dimensions, which in turn are connected to analytic structure on the set through the zeta function of its Laplacian [14]. Functional inequalities classically associated with curvature are also beginning to be considered in fractal analytic settings [1,4].

A feature of the preceding work is that it does not generally use convex interpolation. Indeed, we are not aware of previous work involving convex interpolation on fractal sets. The purpose of this paper is to consider the elementary notion of convex interpolant in the setting of one well-studied class of fractals, the Sierpinski gaskets \( S_n \) defined on regular \( n \)-simplices in \( \mathbb{R}^n \). When endowed with the Euclidean metric restricted to the set, these examples are geodesic spaces. Following [6] we can therefore define a convex interpolant \( \tilde{Z}_t(A, B) \) which generalizes the Euclidean notion of \((1 - t)A + tB\) by setting

\[
\tilde{Z}_t(a, b) = \{x : d(a,x) = td(a,b) \text{ and } d(x,b) = (1-t)d(a,b)\},
\]

(0.1)

\( \tilde{Z}_t(A, B) = \{Z_t(a,b) : a \in A, b \in B\} \).

Our goal is to study some basic properties of this interpolating set and the naturally related notion of an interpolating measure on the sets \( S_n \).

The study of \( \tilde{Z}_t(A, B) \) requires that we have a good understanding of geodesics in the Sierpinski gasket \( S_n \). These have been studied, for example in [19] and more recently [11].

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but we make some explicit constructions and reprove some fundamental theorems by methods connected to the barycentric projection in [19] because they are essential in our treatment of $Z_t$. The proofs are also, in our view, simpler than some of those in [11, 16, 17]. These results, including some which are new, are in Section 2. Our study of interpolation occupies Sections 3–5; we first replace $\tilde{Z}_t(A,B)$ with a slightly simpler but essentially equivalent set $Z_t(A,B)$, then we deal with this set in the case where $A$ is a cell and $B$ a point and when $A$ and $B$ are disjoint cells. It is easy to determine that no direct analogue of the Brunn-Minkowski inequality can hold, but we conclude with one possible interpolation inequality in Section 6.

We emphasize to the reader that this study is intentionally limited in scope. There are so many inequalities and applications of inequalities in this area that it is not practical to attempt an exhaustive treatment, even when we limit ourselves to such a simple class of examples. Moreover, the naturality of convex interpolation in our setting is not discussed, and in particular we do not consider whether the interpolation of measures is an optimizer of a transport problem. No doubt each reader will notice problems they think should perhaps have been considered, and we hope they will be inspired to do so themselves.

1. Preliminaries

The Sierpinski $n$-gasket $S_n \subseteq \mathbb{R}^n$ is the unique nonempty compact attractor of the iterated function system (IFS)

$$F_i : \mathbb{R}^n \to \mathbb{R}^n, \quad F_i(x) = \frac{1}{2}(x + q_i),$$

where $i \in \{0, 1, \ldots, n\}$, and where $\{q_0, q_1, \ldots, q_n\}$ are the vertices of an $n$-simplex with sides of unit length. We begin with some essential definitions and properties of $S_n$, mostly following the conventions of Strichartz and coauthors in [2, 20].

An $m$-level cell of the Sierpinski $n$-gasket is a set of the form $F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}(S_n)$. We call the sequence of letters $w = w_1 w_2 \cdots w_m$ from the alphabet $\{0, 1, \ldots, n\}$ a finite address of length $|w| = m$. We identify finite addresses with cells and use the notation $\langle w \rangle : = F_w(S_n)$. An $(m + k)$-level cell contained in a given $m$-level cell is called a level $k$ subcell of the $m$-level cell. A given $m$-level cell $\langle w \rangle$ has $(n + 1)$ 1-level subcells, or maximal subcells: $\langle w0 \rangle, \langle w1 \rangle, \ldots, \langle wn \rangle$. Since $S_n$ is defined on an $n$-simplex with unit length sides, the side length of an $m$-level cell is $(\frac{1}{2})^m$.

A strictly descending chain of cells $S_n \supseteq \langle w_1 \rangle \supseteq \langle w_1 w_2 \rangle \supseteq \cdots$ intersects to a point with address $w_1 w_2 \cdots$. As with cells, we identify infinite addresses with points by writing $\langle w_1 w_2 \cdots \rangle$. We use an overline to denote repeating characters in an address, so $20111 \cdots = 201 \overline{1}$. In this notation, we have $q_j = \overline{j}$ for $j = 0, \ldots, n$.

1.1. Vertices, address equivalence, and barycentric coordinates. The boundary points of a cell $\langle w \rangle$ are the vertices $\langle w0 \rangle, \langle w1 \rangle, \ldots, \langle wn \rangle$. The $m$-level vertex set of the $n$-simplex, denoted $V_n^m$, is defined recursively by

$$V_n^0 = \{q_0, q_1, \ldots, q_n\} \quad \text{and} \quad V_n^m = \bigcup_{i=0}^n F_i(V_n^{m-1}).$$

The set of all vertices of $S_n$, denoted $V_n$, is defined as $\bigcup_{m=0}^{\infty} V_n^m$; this set is dense in $S_n$.

Every vertex sits at the intersection of two neighboring cells, and consequently can be described by exactly two addresses, which are readily seen to have the form $wji$ and $wij$, where...
where $\langle wi \rangle$ and $\langle wj \rangle$ are the intersecting cells. Figure 1 demonstrates this property in $S_2$ and illustrates the addressing scheme. Each point in $S_n \setminus V^*_n$ has a unique address.

In addition to point addresses, we make considerable use of the barycentric coordinate system on $S_n$. Recall that the convex hull of $\{q_0, q_1, \ldots, q_n\}$, which contains $S_n$, consists of

\begin{equation}
 x = c_0 q_0 + c_1 q_1 + \cdots + c_n q_n,
\end{equation}

in which each $c_j \geq 0$ and $c_0 + c_1 + \cdots + c_n = 1$. The $c_j$ are called the barycentric coordinates of $x$, and we denote the $i$th barycentric coordinate $c_i$ of $x$ by $[x]_i$.

It is useful to consider the dyadic expansions $[x]_i = \sum_{j=1}^{\infty} c_j^i 2^{-j}$ of the barycentric coordinates, because of the following easy result that is well known \cite{8} page 10] and will be used frequently throughout the present work.

**Lemma 1.1.** A point $x$ is in $S_n$ if and only if there is a dyadic expansion of its barycentric coordinates with the property that for each $j$ there is a unique $i \in \{0, \ldots, n\}$ so that $c_j^i = 1$.

In fact, $x = \langle w_1 w_2 \cdots \rangle$ if and only if $c_j^i = 1$ precisely when $w_j = i$.

**Proof.** Observe that points in the 1-cell $\langle i \rangle$ have $c_1^i = 1$ and all other $c_k^j = 0$. The result then follows by self-similarity and induction.

**Remark.** Points in $V^*_n$ are those for which each $c_i$ is a dyadic rational, and the two addresses for a vertex correspond to the two (nonterminating) binary representations of the vertex. For example, the vertex $\langle 10 \rangle = \langle 01 \rangle = (\frac{1}{4}, \frac{1}{4}, 0)$ in $S_2$ can be expressed in binary as either $(0.1, 0.01, 0)$ or $(0.01, 0.1, 0)$; it may also be represented as $(0.1, 0.1, 0)$, but this latter fails to satisfy the condition that exactly one of the $c_j^k = 1$ for a given value of $k$.

**1.2. Self-similar measure.** A natural class of measures on $S_n$ are the self-similar measures. A measure $\mu_n$ of this type is a probability measure determined uniquely from a set of weights $\{\mu_n^i\}_{i=0}^{n}$, where each $\mu_n^i > 0$ and $\sum \mu_n^i = 1$, by the requirement that for any measurable $X \subseteq S_n$ one has the self-similar identity

$$
\mu_n(X) = \sum_i \mu_n^i \mu_n(F^{-1}_i(X)).
$$

The existence and uniqueness of such measures is due to Hutchinson \cite{13}. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$S_2$, with several points labelled.}
\end{figure}
The standard measure on $S_n$ is self-similar with weights $\mu'_n = \frac{1}{n+1}$ for all $i \in \{0, 1, \ldots, n\}$. The standard measure of any $m$-level cell is $(\frac{1}{m+1})^m$.

When we study interpolation in $S_n$, the interpolant sets will be determined by projections of the original sets. The measure on interpolant sets will be a projection of the original self-similar measure, and will therefore be self-similar itself. This is a consequence of the following results, which are well-known when each $F_i$ is a homothety, as it is here.

**Lemma 1.2.** Let $q_0, q_1, \ldots, q_n \in \mathbb{R}^m$, and consider the IFS $\{F_i : \mathbb{R}^m \to \mathbb{R}^m\}$ with $F_i(x) = \frac{1}{2}(x + q_i)$. Fix $\vec{v}$ a unit vector and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by $x \mapsto \langle x, v \rangle$. Then, defining $\tilde{F}_i := \phi \circ F_i \circ \phi^{-1}$, we have $\tilde{F}_i(x) = \frac{1}{2}(x + \phi(q_i))$ for $x \in \mathbb{R}$.

**Proof.** The map $\phi$ is linear, so if $y$ satisfies $\phi(y) = x$, then
\[
\phi \circ F_i(y) = \phi\left(\frac{1}{2}(y + q_i)\right) = \frac{1}{2}(\phi(y) + \phi(q_i)) = \frac{1}{2}(x\vec{v} + \phi(q_i)).
\]

**Proposition 1.3.** In the setting of Lemma 1.2, let $K$ denote the attractor of the IFS and $\mu$ be the self-similar measure on $K$ with weights $\mu'$. Then the pushforward measure $\phi_*\mu$ satisfies the self-similar identity
\[
\phi_*\mu(X) = \sum_i \mu'_i \phi_*\mu(\tilde{F}_i^{-1}(X))
\]
for measurable $X \subseteq \phi(K)$.

**Proof.** By the definition of the pushforward measure and the self-similarity of $\mu$,
\[
\phi_*\mu(X) = \mu(\phi^{-1}(X)) = \sum_i \mu'_i \mu(F_i^{-1}(\phi^{-1}(X))).
\]
However $F_i^{-1} \circ \phi^{-1}(X) = \phi^{-1} \circ \tilde{F}_i^{-1}(X)$, because, using Lemma 1.2,
\[
\phi \circ F_i(y) = \phi\left(\frac{1}{2}(y + q_i)\right) = \frac{1}{2}(\phi(y) + \phi(q_i)) = \tilde{F}_i \circ \phi(y).
\]
Thus
\[
\phi_*\mu(X) = \sum_i \mu'_i \mu(\phi^{-1} \circ \tilde{F}_i^{-1}(X)) = \sum_i \mu'_i \phi_*\mu(\tilde{F}_i^{-1}(X)).
\]

2. Geodesics

Our goal in this section is to relate barycentric coordinates to distance, and to characterize nonuniqueness of geodesics in $S_n$. We begin by considering geodesics from a point to a boundary point, and then generalize to arbitrary points in $S_n$. We prove that there exist at most five distinct geodesics between any two points in $S_2$, and at most eight geodesics between any two points in $S_n, n \geq 3$.

Let $x, y \in S_n$. A path from $x$ to $y$ in $S_n$ is a continuous function $\gamma : [0, 1] \to S_n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We say that $\gamma$ passes through a point $z$ if for some $t \in (0, 1)$ we have $\gamma(t) = z$. The length of a path $\gamma$, given by $H^1(\gamma([0, 1]))$, is denoted $|\gamma|$; a priori it may be infinite, but we will only be interested in finite paths. To avoid the usual problem of distinguishing $\gamma$ from its image we will always assume $\gamma$ is parametrized at constant speed, so $\frac{\partial H^1(\gamma([0,t]))}{\partial t} = |\gamma|$ for a.e.-t., unless some other parametrization is specified.

It is easy to see that there is always a finite path between any $x$ and $y$. We then define the intrinsic metric, $d : S_n \times S_n \to \mathbb{R}$ by
\[
d(x, y) = \inf\{|\gamma| : \gamma \text{ a path from } x \text{ to } y\}.
\]
This was previously investigated for the case of $S_2$ in [10, Section 8], and later in [7, 17]. In particular, the question of the existence of minimizing paths, or geodesics has been considered in this case. Strichartz [19] used barycentric coordinates to give a simple construction of geodesics; we follow his method, but correct his statement that the maximum number of geodesics between an arbitrary pair of points is four rather than five. Saltan et al. [17] present a formula for $d$ in terms of address representations of $x$ and $y$, and obtain the correct value for the maximum number of geodesics between $x$ and $y$, but their approach is somewhat complicated and is only done for $n = 2$. This result was generalized in [11] to the case $n \geq 3$. The rest of this section presents an alternative approach to these results for $S_n$, $n \geq 3$ and fixes notation that will be needed in our study of geodesic interpolation.

We begin by studying the problem of connecting a boundary point $x$ of a cell to a point $y$ in that cell by a geodesic. The following lemma proves that if such a geodesic exists it must lie in the cell.

**Lemma 2.1.** Two boundary points of a cell $\langle w \rangle$ are joined by a unique geodesic, namely the line segment between them. If $x$ is a boundary point of a cell $\langle w \rangle$ and $y \in \langle w \rangle$ then for any path between $x$ and $y$ that is not contained in $\langle w \rangle$ there is a strictly shorter path which is contained in $\langle w \rangle$.

**Proof.** The first statement is an obvious consequence of the fact that the line segment between two boundary points of $\langle w \rangle$ is contained in $\langle w \rangle$ and is a Euclidean geodesic. The second uses the following observation: if $\gamma$ is a path from $x$ to $y$ that exits $\langle w \rangle$ at a boundary point $z$ then either $\gamma$ re-enters at $z$ in which case it can be shortened by removing the intervening component or it re-enters at another boundary point $\tilde{z}$, in which case it can be shortened by replacing the intervening component by the line segment from $z$ to $\tilde{z}$, as the latter is geodesic. \(\square\)

This lemma suggests a substantial reduction of the problem. We fix some notation.

**Definition 2.2.** For distinct points $x$ and $y$, the unique smallest cell that contains both is called the common cell of $x$ and $y$. If $\langle w \rangle$ is a cell, the intersection points of its $n + 1$ maximal subcells, which have addresses $\langle wij \rangle$ for all pairs $i,j$, are called the bridge points of $\langle w \rangle$.

It is apparent that if $x$ is the boundary point $\langle w\tilde{i} \rangle$ of the common cell $\langle w \rangle$ of $x$ and $y$ then $y$ is in a different maximal subcell $\langle wj \rangle$ than $x$, so $j \neq i$. Any path from $x$ to $y$ must pass through a boundary point of $\langle wj \rangle$, and such have the form $\langle wjk \rangle$. The next lemma shows that we may assume $k = i$.

**Lemma 2.3.** Let $x = \langle w\tilde{i} \rangle$ and $y \in \langle wj \rangle$ with $j \neq i$. For any path $\gamma$ from $x$ to $y$ there is a path of shorter or equal length that enters $\langle wj \rangle$ through the bridge point $\langle wji \rangle$.

**Proof.** We know $\gamma$ enters $\langle wj \rangle$ at a point $\langle wjk \rangle$. Moreover, by modifying $\gamma$ as in Lemma 2.1 to remove all excursions outside $\langle wj \rangle$, we obtain a (possibly shorter) path with the property that the only portion of the path that is outside $\langle wj \rangle$ is the initial segment from $\langle wi \rangle$ to $\langle wjk \rangle$. Let $L$ denote the length of a side of a maximal subcell. If $k = i$ we are done, but if not then the point $\langle wjk \rangle$ is not in $\langle wi \rangle$, so is distance at least $L$ from $\langle wi \rangle$ and thus $2L$ from $\langle w\tilde{i} \rangle$. However the line segments from $\langle wjk \rangle$ to $\langle wji \rangle = \langle w\tilde{i} \rangle$ to $\langle w\tilde{i} \rangle$ have length exactly $2L$, so using this as the initial segment of $\gamma$ makes the path no longer and ensures it enters $\langle wj \rangle$ at the specified point. \(\square\)
The preceding lemma tells us how to construct a geodesic from a boundary point of a cell to any point inside the cell. Moreover, it allows us to write the length of this geodesic in terms of the barycentric coordinates. The latter was previously noted in [5][19].

**Proposition 2.4.** Let \( x = \langle w \rangle \) and \( y \in \langle w \rangle \). Then \( d(x, y) = [x]_i - [y]_i \) and there is a geodesic from \( x \) to \( y \).

**Proof.** It is sufficient to work on \( S_n \), because \( x, y \in \langle w \rangle \) implies the first \( |w| \) binary terms of \([x]_i\) and \([y]_i\) are equal. Let \( w_m \) be the level \( m \) truncation of an (infinite) address for \( y \) and \( x_m = \langle w_m \rangle \), so \( x_0 = x \). Define a path \( \gamma \) on \([0, 1)\) by mapping \([1 - 2^{-k}, 1 - 2^{-(k+1)}]\) to the segment from \( x_k \) to \( x_{k+1} \). Since \( \{w_m\} \) intersects to \( y \) we have \( x_m \rightarrow y \) and thus may extend \( \gamma \) continuously to \([0, 1]\) by setting \( \gamma(1) = y \). Lemma 2.3 ensures any path from \( x \) to \( y \) is at least as long as \( \gamma \), so \( \gamma \) is a geodesic.

It remains to compute the length of \( \gamma \). Observe that \( \gamma \) is constant on any segment where \( x_k = x_{k+1} \), and otherwise has length \( 2^{-(k+1)} \). Moreover, \( x_k = x_{k+1} \) if and only if the \((k+1)\)th letter in the address of \( y \) is \( i \). Now Lemma 1.1 says this occurs if and only if the \((k+1)\)th term in the binary expansion of \([y]_i\) is 1, and that \( x = \langle \tilde{i} \rangle \) implies every term in the binary expansion of \([x]_i\) is 1. Thus the binary expansion of \([x]_i - [y]_i\) has coefficient 1 multiplying \( 2^{-(k+1)} \) precisely when \( x_k \neq x_{k+1} \), proving the formula for \( d(x, y) \).

**Remark.** Note that the construction of \( \gamma \) terminates (i.e. \( x_k = y \) for all sufficiently large \( k \)) if \( y \) is a vertex which has address \( \langle \bar{w'}\bar{i} \rangle \) for some word \( w' \).

**Corollary 2.5.** For any distinct pair of points \( x, y \) there is a geodesic from \( x \) to \( y \). All geodesics are contained in the common cell of \( x \) and \( y \).

**Proof.** Take cells \( \langle w_x \rangle \) containing \( x \) and \( \langle w_y \rangle \) containing \( y \) with \( |w_x| = |w_y| = m \) large enough that the cells do not intersect. There is a geodesic from each boundary point of \( \langle w_x \rangle \) to \( x \) and from each boundary point of \( \langle w_y \rangle \) to \( y \). Moreover, between a boundary point of \( \langle w_x \rangle \) and a boundary point of \( \langle w_y \rangle \) any geodesic is composed of a finite union of edges in the graph at level \( m \) by Lemma 2.3. This reduces the problem of finding a geodesic to identifying the shortest curve in a finite collection, which may always be solved.

The fact that geodesics stay in the common cell is a consequence of Lemma 2.1, because a path that exits this cell may be written as the concatenation of a path from \( x \) to the cell boundary, a path between cell boundary points, and a path from the cell boundary to \( y \), each of which can be strictly shortened if it exits the cell.

**Corollary 2.6.** Let \( x = \langle \bar{i} \rangle \), and define a hyperplane \( H = \{(y_0, y_1, \ldots, y_n) \in S_n : y_i = a \} \) for some fixed \( a \). Then \( d(x, y) \) is constant for all \( y \in H \).

### 2.1. Uniqueness in \( S_2 \)

We now turn to the question of geodesic uniqueness in the Sierpinski \( n \)-gasket. As shown in [16], there are at most five geodesics between two points in \( S_2 \). We provide an alternate, more geometrical proof, beginning with the case of geodesics between a boundary point of a cell and a point contained in the cell.

**Proposition 2.7.** Let \( x = \langle \bar{i} \rangle \) be a boundary point of \( S_n \), and \( y \neq x \). Then there is a unique geodesic between \( x \) and \( y \) unless \( y = \langle w^j \rangle = \langle w^j_k \rangle \) for some finite word \( w \) and some \( j, k \) such that \( i, j \) and \( k \) are distinct.

**Proof.** Let \( \gamma_1 \) be a geodesic constructed as in Proposition 2.4 from a sequence of cells \( \langle \{w_m\} \rangle \) that intersect to \( y \) and \( \gamma_2 \) be any other geodesic from \( x \) to \( y \). Take the largest \( m \) such that \( \gamma_2 \) enters \( \langle w_m \rangle \) through a vertex \( z' = \langle w_m \rangle \) with \( j \neq i \), and write \( z = \langle w_m \rangle \). Using
Lemma 2.3 we can modify $\gamma_2$ to form $\hat{\gamma}_2$ which passes through $z$ and then $z'$ and has the same length as $\gamma_2$. Then

$$d(x,z) + d(z,y) = |\gamma_1| = |\hat{\gamma}_2| = d(x,z) + d(z,z') + d(z',y) = d(x,z) + 2^{-m} + d(z',y).$$

However Proposition 2.4 ensures $d(z,y) \leq 2^{-m} = d(z,z')$, so $d(z',y) = 0$ and $y = z'$ is a vertex. Moreover $\hat{\gamma}_2 = \gamma_1$.

Now by our choice of $m$ we know $\gamma_1$ and $\gamma_2$ coincide between $x$ and $(w_{m-1}\bar{i})$ because they are built from the same bridge points. Thus the only difference between these geodesics occurs on $(w_{m-1})$, and they begin at $(w_{m-1}\bar{i})$ and end at $y = (w_m j)$, where $w_m = w_{m-1}k$ for some $k$. Determining the possible paths in $(w_{m-1})$ is therefore the same as determining the geodesics in $S_n$ from $(i\bar{i})$ to $(k\bar{j}) = (j\bar{k})$ for some $k$ and some $j \neq i$. This is an easy finite computation: there is a unique such geodesic if $k = i$ or $k = j$ and exactly two geodesics if $k \neq i, j$, one through $(i\bar{k}) = (k\bar{i})$ and one through $(i\bar{j}) = (j\bar{i})$.

Remark. If $y$ is a vertex of the form identified in the proposition then we may use either of its two addresses to construct a geodesic from $x$ to $y$ by the method of Proposition 2.4. The two addresses lead to the two distinct geodesics identified in Proposition 2.7.

Corollary 2.8. Let $y \in \langle w \rangle \subseteq S_2$. If there are two distinct geodesics from $y$ to a boundary point of $\langle w \rangle$ then there is only one geodesic from $y$ to each of the other two boundary points of $\langle w \rangle$.

Proof. There are two distinct geodesics from $y$ to $\langle w\bar{i} \rangle$, so by Proposition 2.7 we have $y = (wjk)$ where $j$ and $k$ are distinct from each other and from $i$. The proposition also tells us that such a $y$ has unique geodesics to the boundary points $\langle w\bar{j} \rangle$ and $\langle w\bar{k} \rangle$, and since we are in $S_2$ with $i, j, k$ distinct, this covers all boundary points.

Our results on geodesics between a point in a cell and a boundary point of that cell have implications for geodesics between arbitrary points. Recall that the bridge points of $S_n$ are the intersection points of maximal subcells. The following lemma gives a useful classification of geodesics between points in distinct maximal cells according to the number of bridge points they contain.

Lemma 2.9. Any geodesic between two points in $S_n$ passes through at most two bridge points.

Proof. Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$. We may construct a path between them by concatenating a geodesic from $x$ to $\langle ij \rangle$ and a geodesic from $\langle ij \rangle$ to $y$. By Proposition 2.4 each such geodesic has length at most $\frac{1}{2}$, so $d(x,y) \leq 1$. However, bridge points are separated by distance $\frac{1}{2}$, so there can be at most two on a geodesic.

We note that if the geodesic from $x$ to $y$ passes through only one bridge point it must be the intersection point of the maximal cells containing them.

Definition 2.10. Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$, $i \neq j$. The geodesic $\gamma$ from $x$ to $y$ is a $P_1$ geodesic if it passes through the bridge point $\langle ij \rangle = \langle i \rangle \cap \langle j \rangle$ and a $P_2$ geodesic if it passes through two bridge points, $(i\bar{k})$ and $(j\bar{k})$, where $k \neq i, j$.

Cristea and Steinsky provide geometric criteria for $S_2$ and $S_3$ that allow one to determine whether one or both types of geodesics exist between some pair of points. The proof of the following theorem, which gives a sharp bound on the number of geodesics between $x$ and $y$ in $S_2$, recovers their results for $S_2$. The sharp bound was previously proved in [16] by a different method.
Theorem 2.11. There are at most five distinct geodesics between any two points in $S_2$, and this bound is sharp.

Proof. Fix $x$ and $y$. Corollary 2.5 tells us that all geodesics between $x$ and $y$ lie in their common cell, so we may assume the common cell is $S_2$. Thus $x \in \langle i \rangle$ and $y \in \langle j \rangle$ with $i \neq j$, and the geodesics between them are either $P_1$ geodesics through $\langle ij \rangle$ or $P_2$ geodesics through $\langle ij \rangle$ and $\langle ji \rangle$ where $k \neq i, j$.

If $\gamma$ is a geodesic from $x$ to $y$ then its restriction to $\langle i \rangle$ is a geodesic from $x$ to either $\langle ij \rangle$ or $\langle ji \rangle$. Proposition 2.7 says there are at most two geodesics to either of these points and Corollary 2.8 says that if there are two to one such point then there is only one to the other, so there are at most three options for the restriction of $\gamma$ to $\langle i \rangle$. Similarly there are at most three options for the restriction of $\gamma$ to $\langle j \rangle$.

We now consider how the pieces of geodesic previously described may be combined.

Case 1 There are two distinct geodesics between $x$ and $\langle ij \rangle$ and two between $y$ and $\langle ij \rangle$, providing four $P_1$ paths. In this case the geodesics from $x$ to $\langle ij \rangle$ and $y$ to $\langle ji \rangle$ are unique, as is that between these bridge points, so there is one $P_2$ path. If the $P_1$ and $P_2$ geodesics are the same length then there are five geodesics in total, otherwise there are four or one. Figure 2 shows that five may be achieved.

Case 2 There are two distinct geodesics between $x$ and $\langle ij \rangle$ and two between $y$ and $\langle ji \rangle$. Then there is one from $y$ to $\langle ij \rangle$ so there are two $P_1$ paths, and there is one from $x$ to $\langle ji \rangle$, so there are two $P_2$ paths. If all of these have the same length there are four geodesics, otherwise there are two.

Case 3 There is only one geodesic from $x$ to each of $\langle ij \rangle$ and $\langle ji \rangle$ and only one from $y$ to each of $\langle ji \rangle$ and $\langle ji \rangle$. Then there is one $P_1$ and one $P_2$ path; if they are the same length there are two geodesics, and otherwise there is one.

\[\begin{array}{ccc}
\text{Figure 2. Points } x \text{ and } y \text{ that are connected by five distinct geodesics:} \\
\text{four } P_1 \text{ geodesics (left) and one } P_2 \text{ geodesic (right).}
\end{array}\]

2.2. Uniqueness in $S_n$. The proof of Theorem 2.11 relies on two facts about $S_2$: there are only three bridge points, so there is at most one pair of bridge points through which a $P_2$
geodesic can pass, and nonuniqueness of a geodesic to one bridge point implies uniqueness to the other two bridge points (Corollary 2.8). Neither of these arguments is directly applicable to $S_n$, $n \geq 3$. However we can obtain a sharp bound in this more general setting by making a more detailed analysis of $P_2$ geodesics.

**Lemma 2.12.** Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$, where $i \neq j$. Then there exist $P_2$ geodesics between $x$ and $y$ passing through at most two distinct pairs of bridge points.

**Proof.** Suppose there is a $P_2$ geodesic $\gamma$ from $x$ to $y$ through $\langle ik \rangle$ and $\langle jk \rangle$. Applying Proposition 2.4 we have $d(x, \langle ik \rangle) = [(k^{-1})k - x]_k = \frac{1}{2} - [x]_k$ and similarly for $d(y, \langle jk \rangle)$, so

$$d(x, y) = |\gamma| = \frac{1}{2} + d(x, \langle ik \rangle) + d(y, \langle jk \rangle) = \frac{3}{2} - [x]_k - [y]_k.$$  

However in the proof of Lemma 2.9 we saw that $d(x, y) \leq 1$, so $[x]_k + [y]_k \geq \frac{1}{2}$.

Now $\sum_{k=0}^{n} [x]_k + [y]_k = 2$ from the definition of the barycentric coordinates, and $x \in \langle i \rangle$, $y \in \langle j \rangle$ implies $[x]_i \geq \frac{1}{2}$ and $[y]_j \geq \frac{1}{2}$, so the number of $k$ for which $[x]_k + [y]_k \geq \frac{1}{2}$ is at most 2, which implies there are at most two values of $k$ for which there is a $P_2$ geodesic through the bridge points $\langle ik \rangle$ and $\langle jk \rangle$. \hfill $\square$

It is apparent in the preceding proof that the locations of $x$ and $y$ are tightly constrained when they admit geodesics through two distinct pairs of bridge points. The following result makes this precise.

**Lemma 2.13.** Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$ with $i \neq j$. If there are $P_2$ geodesics from $x$ to $y$ through two distinct pairs of bridge points, then there are exactly two $P_2$ geodesics from $x$ to $y$.

**Proof.** Let $\gamma_1$ be a $P_2$ geodesic passing through $\langle ik \rangle$ and $\langle jk \rangle$, and let $\gamma_2$ be a $P_2$ geodesic passing through $\langle il \rangle$ and $\langle jl \rangle$. It follows from the proof of Lemma 2.9 that $|\gamma_1| = |\gamma_2| \leq 1$, so $|\gamma_1| + |\gamma_2| \leq 2$. Now by the triangle inequality

$$2 = 1 + d(\langle ik \rangle, \langle il \rangle) + d(\langle jk \rangle, \langle jl \rangle)$$

$$\leq \frac{1}{2} + d(\langle ik \rangle, x) + d(x, \langle il \rangle) + \frac{1}{2} + d(\langle jk \rangle, y) + d(y, \langle jl \rangle)$$

$$= |\gamma_1| + |\gamma_2| \leq 2.$$  

Thus equality holds in the triangle inequality and $x$ lies on the geodesic connecting $\langle il \rangle$ to $\langle ik \rangle$, and $y$ lies on the geodesic connecting $\langle jk \rangle$ to $\langle il \rangle$. This shows the geodesics from $x$ to $\langle ik \rangle$ and to $\langle il \rangle$ are unique and similarly for $y$; the result then follows from Lemma 2.12. \hfill $\square$

Lemmas 2.12 and 2.13 provide us with sufficient restrictions on the number of geodesics to prove our main result on geodesics in $S_n$, $n \geq 3$.

**Theorem 2.14.** There exist at most eight distinct geodesics between any two points in $S_n$, $n \geq 3$ and this bound is sharp.

**Proof.** Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$. As in the proof of Theorem 2.11 we may assume $i \neq j$. By Proposition 2.7 there exist at most two geodesics from $x$ to $\langle ij \rangle$ and two geodesics from $\langle ij \rangle$ to $y$. Concatenations of these pairs of geodesics yield a maximum of four $P_1$ geodesics.

By Lemma 2.12 there exist $P_2$ geodesics through at most two distinct pairs of bridge points. If there are $P_2$ geodesics through two distinct pairs of bridge points, then by Lemma 2.13 there are exactly two $P_2$ geodesics between $x$ and $y$. If, in addition, there exist $P_1$ geodesics, then there are at most six total geodesics.
If, instead, there are $P_2$ geodesics only through a single pair of bridge points $\langle i\tilde{k}, j\tilde{k} \rangle$, then they are obtained by concatenating geodesics from $x$ to $i\tilde{k}$ (of which there are at most two by Proposition 2.7) and $y$ to $j\tilde{k}$ (of which there are again at most two) with the interval from $i\tilde{k}$ to $j\tilde{k}$. This yields at most four $P_2$ geodesics between $x$ and $y$. If, in addition, there exist $P_1$ geodesics, then there are at most eight geodesics in total. Sharpness is demonstrated by Example 2.15, which is written in $S_3$ and embeds in $S_n$ for all $n \geq 3$. □

**Example 2.15.** Let $x = \langle 203\bar{1} \rangle = \langle 201\bar{2} \rangle$ and $y = \langle 303\bar{1} \rangle = \langle 301\bar{3} \rangle$. Then the following are the geodesics from $x$ to $y$:

\[
\begin{align*}
\gamma_1 &: x \rightarrow \langle 202\bar{3} \rangle \rightarrow \langle 203 \rangle \rightarrow \langle 23 \rangle = \langle 32 \rangle \rightarrow \langle 302 \rangle \rightarrow \langle 3032 \rangle \rightarrow y, \\
\gamma_2 &: x \rightarrow \langle 202\bar{3} \rangle \rightarrow \langle 203 \rangle \rightarrow \langle 23 \rangle = \langle 32 \rangle \rightarrow \langle 302 \rangle \rightarrow \langle 3012 \rangle \rightarrow y, \\
\gamma_3 &: x \rightarrow \langle 201\bar{3} \rangle \rightarrow \langle 203 \rangle \rightarrow \langle 23 \rangle = \langle 32 \rangle \rightarrow \langle 302 \rangle \rightarrow \langle 3032 \rangle \rightarrow y, \\
\gamma_4 &: x \rightarrow \langle 201\bar{3} \rangle \rightarrow \langle 203 \rangle \rightarrow \langle 23 \rangle = \langle 32 \rangle \rightarrow \langle 302 \rangle \rightarrow \langle 3012 \rangle \rightarrow y, \\
\gamma_5 &: x \rightarrow \langle 2020 \rangle \rightarrow \langle 200 \rangle = \langle 20 \rangle \rightarrow \langle 30 \rangle = \langle 300 \rangle \rightarrow \langle 3030 \rangle \rightarrow y, \\
\gamma_6 &: x \rightarrow \langle 2020 \rangle \rightarrow \langle 200 \rangle = \langle 20 \rangle \rightarrow \langle 30 \rangle = \langle 300 \rangle \rightarrow \langle 3010 \rangle \rightarrow y, \\
\gamma_7 &: x \rightarrow \langle 2010 \rangle \rightarrow \langle 200 \rangle = \langle 20 \rangle \rightarrow \langle 30 \rangle = \langle 300 \rangle \rightarrow \langle 3030 \rangle \rightarrow y, \\
\gamma_8 &: x \rightarrow \langle 2010 \rangle \rightarrow \langle 200 \rangle = \langle 20 \rangle \rightarrow \langle 30 \rangle = \langle 300 \rangle \rightarrow \langle 3010 \rangle \rightarrow y,
\end{align*}
\]

where each geodesic is composed of the edges in $S_n$ joining each pair of consecutive points.

Note that each portion of these connecting $x$ or $y$ to a bridge point is geodesic because it is constructed by the algorithm in Proposition 2.4. The first four are $P_1$ paths, with length $2\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{2}\right) = 1$. The second four are $P_2$ paths, with length $2\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{2}\right) = 1$. We have constructed all the candidates to be geodesic from $x$ to $y$ (as described in Corollary 2.5), so the fact that they are all the same length ensures all are geodesics.

For use in later sections of this paper, it is convenient to note the following consequence of our analysis of the structure of geodesics.

**Theorem 2.16.** The set of pairs of points $(x, y) \in S_n \times S_n$ such that there is more than one geodesic from $x$ to $y$ has zero $\mu_n \times \mu_n$-measure.

**Proof.** First observe that sets of the form $\{a\} \times S_n$ or $S_n \times \{b\}$ are null for $\mu_n \times \mu_n$. Taking the countable union over $a \in V_n^*$ and over $b \in V_n^*$ gives a null set. Now observe from Proposition 2.7 that $x$ is connected to any boundary of a cell containing $x$ by more than one geodesic then $(x, y)$ is in one of these null sets and similarly for $y$. Accordingly, we can assume that there is a unique geodesic from $x$ to any boundary point of a cell containing $x$, and similarly for $y$.

In this circumstance the only way $x$ and $y$ can be joined by more than one geodesic involves at least one $P_2$ geodesic. Precisely, there is a cell $\langle w_x \rangle$ containing $x$ and a cell $\langle w_y \rangle$ containing $y$; these cells are joined by distinct geodesics $\gamma$ and $\gamma'$ such that $\gamma$ enters $\langle w_x \rangle$ at $a_x$ and $\langle w_y \rangle$ at $a_y$, and $\gamma'$ enters $\langle w_x \rangle$ at $a'_x$ and $\langle w_y \rangle$ at $a'_y$. Moreover the fact that these have equal length may be written as $d(x, a_x) + d(y, a_y) + |\gamma| = d(x, a'_x) + d(y, a'_y) + |\gamma'|$. There are countably many choices of pairs $\langle w_x \rangle$, $\langle w_y \rangle$ and, for each pair, finitely many possibilities for $\gamma$ and $\gamma'$, so to prove the set of pairs $(x, y)$ joined by non-unique geodesics of this type is null we need only prove that for such a pair of cells, geodesics and boundary points, one has

\[
(\mu_n \times \mu_n)(\{(x, y) : d(x, a_x) - d(x, a'_x) + d(y, a_y) - d(y, a'_y) = |\gamma'| - |\gamma|\}) = 0.
\]
Moreover, since $\mu_n \times \mu_n$ is a product measure, by Fubini’s theorem it is sufficient that this set has zero $\mu_n$ measure for each fixed $y$. More precisely, since fixing $y$ fixes the value $d(y, a_y) - d(y, a'_y)$, it is enough that for any $s$, $\mu_n\{\{x : d(x, a_x) - d(x, a'_x) = s\}\} = 0$. Clearly the question of whether this set is null is invariant under rescaling the cell $(\bar{w}_x)$ to be $S_n$, and by symmetry we may assume $a_x = q_0$, $a'_x = q_1$; we assume this is the case.

Now we use Proposition 2.4 to write $d(x, q_0)$ as the projection on the barycentric coordinate corresponding to $q_0$, and writing $d(x, q_1)$ in the same manner we find that $d(x, q_0) - d(x, q_1)$ is the projection of $x - q_0$ on the unit vector $q_1 - q_0$ which is parallel to an edge of the simplex. Parametrizing the position along the line from $q_0$ to $q_1$ by $[0,1]$ and writing $\pi : S_n \to [0,1]$ for the projection on $q_1 - q_0$ in this parametrization, we see that the measure $\mu_n\{\{x : d(x, q_0) - d(x, q_1) \in E\}\} = \mu_n \circ \pi^{-1}(E)$ is the pushforward measure $\pi_*\mu_n$ of $\mu_n$ under $\pi$. However this is a self-similar measure on $[0,1]$ by Proposition 1.3, with the self-similarity relation

$$\pi_*\mu_n(s) = \frac{1}{n+1}(\pi_*\mu_n(2s) + \pi_*\mu_n(2s-1) + (n-1)\pi_*\mu_n(2s - \frac{1}{2})).$$

This measure is non-atomic. To see this, suppose the contrary. It is a probability measure, so there is an atom which attains the maximal mass among atoms; we let $s_0$ be the location of such an atom. Then the self-similarity relation says

$$(n + 1)\pi_*\mu_n(\{s_0\}) = \pi_*\mu_n(\{2s_0\}) + \pi_*\mu_n(\{2s_0 - 1\}) + (n-1)\pi_*\mu_n(\{2s_0 - \frac{1}{2}\})$$

but at most two of the points $2s_0$, $(2s_0 - 1)$ and $(2s_0 - \frac{1}{2})$ are in $[0,1]$, and the atoms at these points have mass not exceeding $\mu_n(\{s_0\})$, so that

$$(n + 1)\pi_*\mu_n(\{s_0\}) \leq n\pi_*\mu_n(\{s_0\})$$

and thus $\pi_*\mu_n(\{s_0\}) = 0$.

The fact that $\pi_*\mu_n$ is non-atomic says precisely that $\mu_n\{\{x : d(x, q_0) - d(x, q_1) = s\}\} = 0$ for every choice of $s$. As previously noted, this ensures the measure of the set in (2.1) is zero, and by taking the union over the countably many possible cells and geodesics connecting their boundary points we complete the proof. □

**Remark.** The paper [11] states Theorem 2.16 as their Theorem 1.3, but it appears to us that something is missing in the proof. Specifically, the authors reduce to the situation where, in our notation, $x \in (0), y \in (1)$ and there are a $P_1$ and a $P_2$ geodesic between these points (see the reasoning following their Lemma 4.3, where they say that there are vertices from our $V_1$ which they call $b_1, b_1'$, such that $d(x, b_1) + d(b_1, y) = d(x, b_1') + d(b_1', y)$ with $d(b_1, b_1') = \frac{1}{2}$). Using their Lemmas 4.4–4.7 they appear to be saying, in the proof of Proposition 4.8, that then $x$ and $y$ are points of $V_s$. (They write this as $x = \sigma^1, y = \sigma^0\sigma^\infty$.) Yet we can give an example of points $x$ and $y$ in $S_2$ that are as described above but are not from $V_s$, as follows. Consider the three line segments forming a triangle around the central hole of the gasket $S_2$. These have vertices with addresses $\langle 0\bar{1} \rangle, \langle 0\bar{2} \rangle$ and $\langle 1\bar{2} \rangle$. Take a point $x$ in $\langle 0 \rangle$ at distance $s$ from $\langle 0\bar{2} \rangle$ where $s$ is not a dyadic rational (so $x \notin V_s$) and $s < \frac{1}{4}$. Take $y$ in $\langle 1 \rangle$ at distance $\frac{1}{4} - s$ from $\langle 1\bar{2} \rangle$. Evidently, any geodesic between these points lies on the three line segments. Now the distance from $x$ to $y$ through the points $\langle 0\bar{2} \rangle = \langle 20 \rangle$ and $\langle 1\bar{1} \rangle = \langle 1\bar{2} \rangle$ is $s + \frac{1}{2} + (\frac{1}{2} - s) = \frac{3}{4}$, because it includes the edge through the cell $2$. It is equally apparent that the distance via $\langle 0\bar{1} \rangle = \langle 1\bar{0} \rangle$ is $\frac{1}{2} - s + \frac{1}{2} - (\frac{1}{4} - s) = \frac{5}{4}$. So there are two geodesics joining these points but neither point is in $V_s$. 


3. Interpolation

In Euclidean space the barycenter of sets $A$ and $B$ is $(1-t)A + tB = \{(1-t)a + tb : a \in A, b \in B\}$. In a geodesic space the natural analogue, introduced in [6], is the set defined by

$$\tilde{Z}_t(a, b) = \{x : d(a, x) = td(a, b) \text{ and } d(x, b) = (1-t)d(a, b)\},$$

$$\tilde{Z}_t(A, B) = \{\tilde{Z}_t(a, b) : a \in A, b \in B\}.$$  

(3.1)

From our results on geodesics we know that $\tilde{Z}_t(a, b)$ is a single point for almost all $a$ and $b$, but in any case contains at most 8 points.

The classical Brunn-Minkowski inequality in $\mathbb{R}^n$ says $\operatorname{vol}(\tilde{Z}_t(A, B))^{1/n} \geq (1-t)\operatorname{vol}(A)^{1/n} + t\operatorname{vol}(B)^{1/n}$. This convexity result has many applications, for which we refer to the survey [10]. Our first result makes it clear that no such result can be true for the self-similar sets $\tilde{Z}_t(A, B)$.

**Proposition 3.1.** For $A, B \subset S_n$ the set $\bigcup_{t \in (0, 1)} \tilde{Z}_t(A, B)$ has Hausdorff dimension at most 1 and hence $\mu_n$-measure zero.

**Proof.** We have shown that all geodesics are constructed as in Corollary 2.5 using the method of Proposition 2.4. In that argument, the set $\bigcup_{t \in (0, 1)} \tilde{Z}_t(a, b)$ lies entirely on the countable collection of Euclidean line segments joining vertices from $V_n^*$, so it is one-dimensional and $\mu$-measure zero.

We note in passing an amusing consequence of the preceding proof which emphasizes the difference with the Euclidean case. We call a set $A$ convex when $a, b \in A$ implies $\tilde{Z}_t(a, b) \subset A$ for all $t$, and observe that the intersection of convex sets is convex, so $A$ has a smallest closed convex superset, called its convex hull.

**Corollary 3.2.** The convex hull of a closed set $A$ has the same $\mu_n$ measure as $A$.

**Proof.** The essential idea of the proof is to take a closed convex set by adjoining to $A$ a portion of the union of line segments in Proposition 3.1 which accumulates only at $A$.

Given $A$, let $V = V_n^* \cap (\bigcup_{t \in (0, 1)} \tilde{Z}_t(A, A))$ be the set of vertices on geodesics between points of $A$. Observe that if $x \in V$ and $d(x, A) \geq \delta > 0$ then from the construction of all geodesics in Corollary 2.5 and Proposition 2.4 it must be that $x$ lies on the edge of a cell which intersects $A$ and has size at least $\delta$. Let $B$ consist of all geodesics between all pairs of points in $V$. Note that if $x \in B$ and $d(x, A) \geq \delta$ it lies on one of the finitely many edges of cells of size at least $\delta$, and thus $\{b \in B : d(b, A) \geq \delta\}$ is closed. It follows that any accumulation point of $B$ that is at a positive distance from $A$ is in $B$, and thus that $A \cup B$ is closed.

Let us consider the geodesics between points of $A \cup B$. If $a, b \in B$ then the geodesic from $a$ to $b$ is in $B$ because it is a subset of a geodesic between points of $V$. If $a, b \in A$ then the geodesic between them is the increasing union of geodesics between points of $V$ with ends that accumulate to $a$ and $b$, so is also in $A \cup B$. Similarly, if $a \in A$ and $b \in B$ the geodesic between them is a union of this type except that one part of the geodesic between points of $V$ is terminated at $b$. So $A \cup B$ is convex.

Since $A \cup B$ is closed and convex it must contain the convex hull of $A$. However, $B$ is a countable union of line segments, so it is one-dimensional and $\mu_n(A \cup B) = \mu_n(A)$.

**Proposition 3.1** tells us that there is no hope that a power of $\mu_n(\tilde{Z}_t(A, B))$ is convex in $t$, but it remains possible that the measure-theoretic properties of $\tilde{Z}_t(A, B)$ reflect some...
aspects of the geometric structure of the Sierpinski gasket $S_n$. We record some definitions and basic notions that are useful in investigating this question.

**Definition 3.3.** Let $A, B \subseteq S_n$. A common path $\gamma : [0, 1] \rightarrow S_n$ from $A$ to $B$ is a finite length path such that for each $a \in A$ and $b \in B$ there is a geodesic $\gamma_{a \rightarrow b}$ and $t_1, t_2 \in [0, 1]$, called the entry and exit times, with $\gamma = \gamma_{a \rightarrow b}(t_1, t_2)$. The initial and final entry times of $\gamma$ are, respectively, the infimum $t^I_1$ and supremum $t^F_1$ of the set of entry times over $a \in A$ and $b \in B$. The initial and final exit times $t^I_2$ and $t^F_2$ are similarly defined from the set of exit times. If $\gamma$ is a maximal common path under inclusion and $t^I_1 < t^I_2$, we call $\gamma$ a regular common path.

For arbitrary $A$ and $B$ there need not be a regular common path, but in many simple cases either there is such a path or there is a natural way to decompose $A$ and $B$ so as to obtain such paths between components. Indeed, it is easy to see that there is a regular common path between disjoint cells that are sufficiently small compared to their separation. Using this, if $A$ and $B$ are disjoint one may take a union of cells covering $A$ and another union of cells covering $B$ so that any pair of cells, one from the first union and the other from the second, admits a regular common path. Of equal importance is the fact that understanding regular common paths is sufficient for studying some aspects of the transport of measure via the set $\hat{Z}_t(A, B)$, at least for fairly simple choices of $A$ and $B$. One way to see this is as follows. Begin by deleting a nullset of $X \times X$ from Theorem 2.10 so that geodesics are unique, then observe that if $A$ and $B$ are separated by a distance $3\epsilon > 0$ then these geodesics must each contain one of finitely many edges of size bounded below by $\epsilon$. It follows that $A \times B$ can be decomposed into finitely many sets $A_j \times B_j$ so that geodesics from $A_j$ to $B_j$ have some piece of common path. See also the remark following Definition 4.1.

When there is a common path it is natural to consider only that part of $\hat{Z}_t$ that lies on the common path.

**Definition 3.4.** For $A$ and $B$ that admit a common path and $a \in A$, $b \in B$, define a modified interpolant $\hat{Z}_t(a, b) = \gamma_{a \rightarrow b}(t)$, i.e., the point in $\hat{Z}_t$ which lies on the geodesic included in the common path.

This modified interpolant is in fact a function, and for all $t \in [t^I_1, t^I_2]$ we have $\hat{Z}_t(A, B) \subseteq \gamma$, as described in the following result.

**Proposition 3.5.** Let $A, B$ be connected subsets of $S_n$ for which $\gamma$ is a regular common path. For each $t \in [t^I_1, t^I_2]$ there exists an interval $I_t \subseteq [0, 1]$ such that $\hat{Z}_t(A, B) = \gamma(I_t)$.

**Proof.** Fix $t \in [t^I_1, t^I_2]$. Continuity of $d(x, y)$ implies $\hat{Z}_t(a, b)$ is continuous on the connected set $A \times B$ and thus $\hat{Z}_t(A, B)$ is a connected subset of $\gamma$. Such subsets have the stated form. \qed

**Definition 3.6.** In the circumstances of Proposition 3.5 let $H_t : (0, 1) \rightarrow \hat{Z}_t(A, B) = \gamma(I_t)$ be the parametrization obtained from the increasing linear surjection $(0, 1) \rightarrow I_t$, which may also be defined at 0 or 1, followed by $\gamma$.

Motivated by the Brunn-Minkowski inequality, our basic object of study will be the measure on $\gamma$ that is induced by the natural measures on $A$ and $B$ via the interpolant $\hat{Z}_t$. In the next two sections we consider two basic cases: when $A$ is a cell with measure $\mu_n$, and $B = \{b\}$ is a point with Dirac mass, and when $A$ and $B$ are both cells with measure $\mu_n$. 
Lemma 4.2. as a countable (and locally finite) sum of measures of the type \( \eta \)
left with finitely many open subsets of \( b \)
deleted without affecting \( \eta \)
subset of \( \geodesic \) from ˙a to points \( \eta \)
in studying the simpler quantity For \( Z \) definition. Note, too, that one might think this could be avoided by instead studying something like \( \eta \)
Remark. The definition of \( Z \) know from Proposition 3.5 that
geodesic ˆa from points \( \eta \.)
Proof. Recall that ˆa has constant speed parametrization, so \( s \) has

\[
Z_{t,b}^{-1}(\hat{\gamma}(s)) = \left\{ a \in A : \frac{d(a, \hat{a})}{d(\hat{a}, b)} = \frac{1 - s}{1 - t} - 1 \right\},
\]
which is non-empty when \( 1 - \frac{\text{diam}(\overline{A})}{d(\hat{a}, b)} \leq \frac{1 - s}{1 - t} \leq 1 \). This is illustrated in Figure 3.

Proof. Recall that \( \hat{\gamma} \) has constant speed parametrization, so \( x = \hat{\gamma}(s) \) implies \( d(\hat{a}, x) = sd(\hat{a}, b) \) and \( d(x, b) = (1 - s)d(\hat{a}, b) \). From [5.1] the set \( Z_{t,b}(x) \) consists of those \( a \in A \) so \( x \in Z_a(a, b) \), which means \( d(a, x) = td(a, b) \) and \( d(x, b) = (1 - t)d(a, b) \). However, \( \hat{a} \) is on the geodesic from \( a \) to \( b \) and the geodesic from \( a \) to \( x \), so we have both \( d(a, b) = d(a, \hat{a}) + d(\hat{a}, b) \) and \( d(a, x) = d(a, \hat{a}) + d(\hat{a}, x) \). From this

\[
d(a, \hat{a}) + sd(\hat{a}, b) = d(a, \hat{a}) + d(\hat{a}, x) = d(a, x) = td(a, b) = td(a, \hat{a}) + td(\hat{a}, b)
\]
which may be rearranged to obtain

\[
d(a, \hat{a}) = \left( \frac{1 - s}{1 - t} - 1 \right) d(\hat{a}, b)
\]
and therefore the desired expression for \( Z_{t,b}^{-1}(\hat{\gamma}(s)) \). The condition for the set to be non-empty is a consequence of there being points \( a \in A \) with \( 0 \leq d(a, \hat{a}) \leq \text{diam}(\overline{A}) \).

From Proposition 2.4 the set of points in \( A \) at a prescribed distance from \( \hat{a} \) is a level set of the barycentric coordinate corresponding to \( \hat{a} \), see Figure 3. We define an associated projection.
**Figure 3.** The preimage of $x$ under $Z_{t,b}$ consists of points equidistant from $\hat{a}$.

**Definition 4.3.** If $A = \langle w \rangle$ and $\hat{a} = \langle w\rangle$, let $\varphi\hat{a}(y) = [F^{-1}_w(y)]$, so $\varphi\hat{a} : \langle w \rangle \to [0,1]$ is the projection of $A$ on the scaled barycentric coordinate with $\varphi\hat{a}(\hat{a}) = 1$ and $\varphi\hat{a} = 0$ at the other boundary points of $A$. Note that $d(a, \hat{a}) = 2^{-|w|}(1 - \varphi\hat{a}(a))$ for $a \in A$.

In particular, the projection allows us to use the parametrization $H_t$ from Definition 4.6 to give a more convenient version of Lemma 4.2 when $t \in [0,1]$.

**Lemma 4.4.** For $t \in [t, 1]$, we have $Z_{t,b}^{-1}(s) = \varphi_{\hat{a}}^{-1} \circ H_t^{-1}$.

**Proof.** Since both $Z_{t,b}$ and $H_t \circ \varphi\hat{a}$ map $A \to Z_t(A, b)$, are constant on level sets of $\varphi\hat{a}$ and linear with respect to distance, they are equal.

These considerations further suggest we consider a pushforward measure under the scaled barycentric projection.

**Definition 4.5.** Let $\nu_n$ be the pushforward measure $\nu_n(X) = (\varphi_{\hat{a}})_* \mu_n(X) = \mu_n \circ \varphi_{\hat{a}}^{-1}(X)$ on Borel subsets of $[0,1]$.

As $S_n$ is rotationally symmetric, we could have defined $\nu_n$ using any boundary point map $\varphi(i)$, and obtained the same measure. Moreover, the fact that $\varphi_{\hat{a}}^{-1} = F_w \circ \varphi^{-1}$ implies that $\mu_n \circ \varphi_{\hat{a}}^{-1} = (n + 1)^{-|w|} \nu_n$. It is equally important that $\nu_n$ satisfies a simple self-similarity condition.

**Lemma 4.6.** If $\tilde{F}_i = \varphi \circ F_i \circ \varphi^{-1}$ then $\nu_n = \frac{1}{n+1} \nu_n \circ \tilde{F}_0^{-1} + \frac{n}{n+1} \nu_n \circ \tilde{F}_1^{-1}$.

**Proof.** Recall that $\varphi(q_0) = 1$ and $\varphi(q_j) = 0$, for $j \neq 0$, while from Lemma 1.2 we have $\tilde{F}_j(x) = \frac{1}{2}(x + \varphi(q_j))$. Thus $\tilde{F}_0(x) = \frac{1}{2}(x + 1)$ and $\tilde{F}_j(x) = \frac{1}{2}x$ if $j \neq 0$. Proposition 1.3 says that $\nu_n$ is self-similar under the IFS $\{\tilde{F}_i\}$ with equal weights, and the result follows from the fact that $n$ of these maps are the same.

See Figure 4 for the approximate density of $\nu_2$, where the weights are $\frac{1}{3}$ and $\frac{2}{3}$.

Since the IFS $\{\tilde{F}_0, \tilde{F}_1\}$ satisfies the open set condition it is fairly elementary to compute the Hausdorff dimension of $\nu_n$, for example using the approach in Chapter 5.2 of [9]. One expression for this dimension is $\inf\{\dim_{\text{Haus}}(E) : \nu_n(E) > 0\}$.

**Proposition 4.7.** The Hausdorff dimension of $\nu_n$ is $\frac{(n+1)\log(n+1) - n \log n}{(n+1)\log 2}$. In particular it is singular with respect to Lebesgue measure on $[0,1]$.

With the pushforward measure $\nu_n$ in hand, we can give an elementary and concise description of the common path measure $\eta_t$ using Lemma 4.4; it is the main result of this section.
Figure 4. Approximate density of the self-similar measure $\nu_2$.

**Theorem 4.8.** Let $A = \langle w \rangle$ be a cell and $B = \{ b \}$ with $b \notin A$. If $t \in [t_f^1, 1]$ then $\eta_t = (n + 1)^{-|w|} \nu_n \circ H_t^{-1}$, so is singular with respect to arc length and has dimension as in Proposition 4.4.

*Proof.* We have $\eta_t = \mu_n \circ Z^1_{t,b} = \mu_n \circ \varphi_t^{-1} \circ H_t^{-1} = (n + 1)^{-|w|} \nu_n \circ H_t^{-1}$. \hfill \qedsymbol

It should be remarked that we could have described $\eta_t$ for $t \in [0, 1]$ rather than only $t \in [t_f^1, 1]$ by using Lemma 4.2 instead of Lemma 4.4, but the notation is considerably less elementary and the gain is minimal because in this case one can instead compute $\eta_t$ for the largest subcell $A' \subset A$ such that $t > t_f^1$ for $A'$.

5. **Interpolation of measures**

The general interpolation problem involves understanding $\{(a, b) : Z_t(a, b) = x\} \subset A \times B$ and its product measure. We slightly abuse notation by calling this measure $\eta_t$, as we did in the case $B = \{ b \}$.

**Definition 5.1.** Let $A$ and $B$ be sets of nonzero $\mu_n$-measure, and suppose there is a regular common path $\tilde{\gamma}$ between them. Define a measure $\eta_t$ on $\tilde{\gamma}$ to be the pushforward of $\mu_n \times \mu_n$ on $A \times B$, so that for each $t \in [0, 1]$ and Borel set $X$,

$$\eta_t(X) = (\mu_n \times \mu_n) \circ Z_t^{-1}(X).$$

As we did in the case of interpolation between a cell and a point, we take the viewpoint that interpolation between sets $A$ and $B$ should be understood as a superposition of interpolation between pairs of cells. This is by no means always possible, but it is possible for a large class of sets; for example, it is true when $A$ and $B$ are both open. Using the same considerations made when discussing point to set interpolation, we further note that from the proof of Theorem 2.16 the product $A \times B$ may be decomposed into a $\mu_n \times \mu_n$-nullset, which is obtained as a finite union of sets of the type in (2.1), and a countable union $A_j \times B_j$. 

in which $A_j$ and $B_j$ are disjoint cells joined by a unique common path. Accordingly, we focus our investigation on $\eta_i$ when $A$ and $B$ are as in Definition 5.1.

We conclude with a discussion of interpolation when $\mu_n$ is replaced with an unequally distributed self-similar measure.

### 5.1. Cell-to-cell interpolation

Let $A$ be a $k$-level cell and $B$ an $m$-level cell for which there is a common path $\gamma$ which is the unique geodesic joining the boundary points $a \in A$ and $b \in B$. From Lemma 4.2, we know that $Z_t(a,b) = Z_t(a',b)$ if $\varphi_a(a) = \varphi_a(a')$ and similarly for the second coordinate using $\varphi_b$, so it is natural to write $Z_t^{-1}(x)$ using these barycentric coordinates. Note that they are scaled differently on $A$ and $B$, as in the following definition.

**Definition 5.2.** Define $\psi_1 : [0,1] \times [0,1] \to [0,1]$ by

$$\psi_t(s,r) = \frac{2^{-k}(1-t)s + 2^{-m}t(1-r)}{2^{-k}(1-t) + 2^{-m}t}.$$ 

The following result is similar to Lemma 4.4 and is illustrated in Figure 5. Recall from Definition 4.3 that $H_t$ parametrizes $Z_t(A,B)$ when the latter is contained in $\gamma$.

**Lemma 5.3.** For all $t \in [t_1', t_2']$ we have $Z_t(a,b) = H_t \circ \psi_t(\varphi_a(a), \varphi_b(b))$.

**Proof.** Recall that $Z_t(a,b) = x$ means $d(a,x) = td(a,b)$. Suppose now that $x = H_t \circ \psi_t \circ (\varphi_a(a), \varphi_b(b))$. We establish several points that together show $d(a,x) = td(a,b)$, proving the result.

Recall from Definition 4.3 that $d(a,\hat{a}) = 2^{-k}(1 - \varphi_a(a))$ and $d(b,\hat{b}) = 2^{-m}(1 - \varphi_b(b))$. Substituting into $\psi_t$ gives

$$\psi_t(\varphi_a(a), \varphi_b(b)) = \frac{(1-t)(2^{-k} - d(a,\hat{a})) + td(\hat{b},b)}{2^{-k}(1-t) + 2^{-m}t}.$$ 

To proceed we need more information about $H_t$, the parametrization of $Z_t(A,B)$. Using Lemma 4.2, we find that the extreme points of $Z_t(A,B)$ are $x_1$ and $x_2$ satisfying $d(\tilde{a},x_1) = td(\tilde{a},\hat{a})$ and $d(x_2,\tilde{b}) = (1-t)d(\hat{a},\tilde{b})$, where $\tilde{a} \neq \hat{a}$ is a boundary point of $A$ and $\tilde{b} \neq \hat{b}$ is a boundary point of $B$. Since $d(\tilde{a},\hat{a}) = 2^{-k}$ and $d(\hat{b},\tilde{b}) = 2^{-m}$ this yields $d(\tilde{a},x_1) = t2^{-k} + td(\hat{a},\tilde{b})$ and $d(x_2,\tilde{b}) = (1-t)2^{-m} + (1-t)d(\hat{a},\tilde{b})$. Moreover $t \in [t_1', t_2']$ implies that for any $a \in A$ and $b \in B$ the geodesic from $a$ to $b$ contains the following points in order: $a,\hat{a},x_1,x_2,\hat{b},b$. We use this and the side lengths of the cells $A$ and $B$ to determine that

$$d(x_1,x_2) = d(\tilde{a},\tilde{b}) - d(\tilde{a},\hat{a}) - d(x_2,\tilde{b})$$

$$= 2^{-k} + 2^{-m} + d(\hat{a},\tilde{b}) - (1-t)d(\hat{a},\tilde{b})$$

$$= 2^{-k} + 2^{-m} + d(\hat{a},\tilde{b}) - t2^{-k} - td(\hat{a},\tilde{b}) - (1-t)2^{-m} - (1-t)d(\hat{a},\tilde{b})$$

$$= 2^{-k}(1-t) + 2^{-m}t$$

which is the denominator in $\psi_t$.

Now $H_t$ is the linear parametrization of the path from $x_1$ to $x_2$, so $x = H_t(q)$ means $d(x_1,x) = qd(x_1,x_2)$. Substituting $q = \psi_t(\varphi_a(a), \varphi_b(b))$ from (5.1) we have

$$d(x_1,x) = d(x_1,x_2)\psi_t(\varphi_a(a), \varphi_b(b)) = (1-t)(2^{-k} - d(a,\hat{a})) + td(\hat{b},b).$$
We can then compute \( d(a, x) \) as follows, using \( 2^{-k} + d(\hat{a}, x_1) = d(\hat{a}, x_1) = t2^{-k} + td(a, \hat{b}). \)

\[
d(a, x) = d(a, \hat{a}) + d(\hat{a}, x_1) + d(x_1, x)
\]

\[
= d(a, \hat{a}) + d(\hat{a}, x_1) + (1 - t)(2^{-k} - d(\hat{a}, \hat{b})) + td(\hat{b}, b)
\]

from which \( x = Z_t(a, b) \) as required. \( \Box \)

**Figure 5.** A schematic of cell-to-cell interpolation on a common path \( \hat{\gamma} \).

The function \( \psi_{t,r} \) (Definition 5.2) describes at a given \( t \) where in the interval \( Z_t(A, B) \) a point lying on the line \( \varphi_{\hat{a}}^{-1}(s) \) is as it is interpolated to a point lying on the line \( \varphi_{\hat{b}}^{-1}(r) \).

We can now prove an analogue of Theorem 4.8 for cell-to-cell interpolation.

**Theorem 5.4.** If \( A \) is a \( k \)-level cell and \( B \) an \( m \)-level cell that are joined by a regular common path \( \hat{\gamma} \) that is the unique geodesic between boundary points \( \hat{a} \in A \) and \( \hat{b} \in B \), then for all \( t \in [t_1, t_2] \)

\[
\eta_t = (n + 1)^{-k-m}(\nu_n \times \nu_n) \circ \psi_t^{-1} \circ H_t^{-1}.
\]

**Proof.** This is an immediate consequence of Lemma 5.3 applied to the definition of \( \eta_t \), because the functions \( \varphi_{\hat{a}}^{-1} \) and \( \varphi_{\hat{b}}^{-1} \) may be pulled into the product measure as follows:

\[
\eta_t = (\mu_n \times \mu_n) \circ Z_t^{-1} = (\mu_n \times \mu_n) \circ (\varphi_{\hat{a}} \times \varphi_{\hat{b}})^{-1} \circ \psi_t^{-1} \circ H_t^{-1} = (\mu_n \circ \varphi_{\hat{a}}^{-1} \times \mu_n \circ \varphi_{\hat{b}}^{-1}) \circ \psi_t^{-1} \circ H_t^{-1}
\]

so we can use \( \mu_n \circ \varphi_{\hat{a}}^{-1} = (n + 1)^{-k} \nu_n \) and similarly \( \mu_n \circ \varphi_{\hat{b}}^{-1} = (n + 1)^{-m} \nu_n \). \( \Box \)

Since it is a product of self-similar measures, the measure \( \nu_n \times \nu_n \) is self-similar. This is recorded in Proposition 5.6 after defining notation for the two-dimensional IFS. It is illustrated in Figure 6.

**Definition 5.5.** Let \( q_{00} = (0, 0), q_{01} = (0, 1), q_{10} = (1, 0), \) and \( q_{11} = (1, 1) \). For \( i, j \in \{0, 1\} \), define \( G_{ij} : [0, 1]^2 \to [0, 1]^2 \) by

\[
G_{ij}(x) = \frac{1}{2}(x + q_{ij}),
\]

and fix weights \( w_{ij} = w_i w_j \), where \( w_0 = \frac{n}{n+1} \) and \( w_1 = \frac{1}{n+1} \).

The functions \( G_{ij} \) are an IFS generating the unit square and are related to the functions \( \hat{F}_i \) by

\[
G_{ij}(x) = \left( \frac{\hat{F}_{i-1}(x)}{\hat{F}_{i-1}(y)} \right).
\]
Proposition 5.6. The measure $\nu_n \times \nu_n$ satisfies the self-similar relation

$$\nu_n \times \nu_n = \sum_{i,j} w_{ij} (\nu_n \times \nu_n) \circ G_{ij}^{-1}.$$  

Proof. We compute from the self-similarity of $\nu_n$ that

$$\nu_n \times \nu_n = \left( \sum_i w_i \nu_n \circ \tilde{F}_{1-i}^{-1} \right) \left( \sum_j w_j \nu_n \circ \tilde{F}_{1-j}^{-1} \right)$$

$$= \sum_{i,j} w_i w_j (\nu_n \times \nu_n) \circ (\tilde{F}_{1-i}^{-1} \times \tilde{F}_{1-j}^{-1})$$

$$= \sum_{i,j} w_{ij} (\nu_n \times \nu_n) \circ G_{ij}^{-1}. \quad \square$$

Theorem 5.4 establishes that $\eta_t$ depends only on the linear parametrization $H_t$ of $Z_t(A,B)$ and the pushforward measure

$$\tilde{\nu}_t^n = \nu_n \times \nu_n \circ \psi_t^{-1}. \quad (5.2)$$

This measure has a simple geometric meaning. Observe that $\psi_t$ is a scaled projection from the unit square to the unit interval along lines of slope $2^{k-m}(\frac{t}{t+1})$. The corresponding pushforward is then a generalization of a convolution; the usual convolution $\nu_n * \nu_n$ occurs when the lines have slope $-1$. From Proposition 1.3 we also find that $\tilde{\nu}_t^n$ is self-similar.

Theorem 5.7. For $t \in [0,1]$ let $\tilde{G}_{ij} = \psi_t \circ G_{ij} \circ \psi_t^{-1} : [0,1] \to [0,1]$. Then

$$\tilde{\nu}_t^n = \sum_{i,j} w_{ij} \tilde{\nu}_t^n \circ \tilde{G}_{ij}^{-1}.$$  

The maps in the IFS $\{ \tilde{G}_{ij} \}$ take $[0,1]$ to overlapping segments in $[0,1]$, with overlaps that depend on $t$. Figure 7 shows these overlapping segments, along with their corresponding weights, for one choice of $t$, and Figure 8 shows the approximate densities of $\tilde{\nu}_t^n$ for several $t$ values.

It is generally difficult to compute the dimensions of measures from overlapping IFS, but we may deduce some results from the Marstrand projection theorem [15]. First note that
The dimension of $\nu_n \times \nu_n$ is twice that of $\nu_n$, and is given by the formula

$$2 \frac{(n + 1) \log(n + 1) - n \log n}{(n + 1) \log 2}.$$  

This expression is decreasing with limit zero as $n$ increases. In particular, it is less than 1 for $n \geq 9$ and greater than 1 for $2 \leq n \leq 8$, from which we deduce the following using Theorems 6.1 and 6.3 in [12].

**Theorem 5.8.** If $2 \leq n \leq 8$ then for almost all $t \in [0,1]$ the measure $\tilde{\nu}_n^t$ is absolutely continuous with respect to Lebesgue measure on $[0,1]$. For $n \geq 9$, it is singular with respect to Lebesgue measure, and in fact has lower Hausdorff dimension given by (5.3).

We note that recent results of Shmerkin and Solomyak [18] show the set of exceptional $t$ in this theorem is not just zero measure but zero Hausdorff dimension.
5.2. Alternate weightings of self-similar measures on the gasket. We can generalize our previous results to self-similar measures other than the standard measure on $S_n$.

Consider a self-similar measure $\mu'_n$ on $S_n$ given by weights $\{\mu'_n\}_{i=1}^n$. In this case, the push-forward measure $\nu'_n = \varphi_*\mu'_n$ is self-similar, but has weights dependent on the reference point of the projection $\varphi$. In particular, if we consider the projection with respect to a vertex $\langle w_i \rangle$, the self-similarity relation is given by:

$$\nu'_n(X) = \mu'_n \nu' \circ \bar{F}_1^{-1}(X) + \left(\sum_{j \neq i} \mu'_n\right) \nu' \circ \bar{F}_0^{-1}(X).$$

This follows from Lemma 1.2 and Proposition 1.3. Self-similarity also carries over to $\nu'_n$, but the self-similarity weights depend on orientations of both the starting and ending cells with respect to the common path; if $\langle w_i \rangle$ is the entry point and $\langle w_j \rangle$ the exit point of a common path, then $\nu'_n$ has self-similarity relations as in Theorem 5.7 but with $w_{01} = \mu'_n \mu'_n$, $w_{01} = \mu'_n (\sum_{k \neq j} \mu'_n)$, $w_{10} = (\sum_{k \neq i} \mu'_n) \mu'_n$, and $w_{11} = (\sum_{k \neq j} \mu'_n) (\sum_{k \neq j} \mu'_n)$.

6. An Interpolation Inequality

The model for an inequality involving the interpolant set $Z_t$ is the classical Brunn–Minkowski inequality, which says that for sets in $\mathbb{R}^n$ the Euclidean volume $| \cdot |$ satisfies $|Z_t(A, B)|^{1/n} \geq (1-t)|A|^{1/n} + t|B|^{1/n}$. We have already noted that this inequality cannot be valid for $\mu_n$ because $Z_t(A, B)$ for $t \in (0, 1)$ has Hausdorff dimension at most 1, and thus $\mu_n$-measure zero. When seeking alternative inequalities it is not entirely clear which measures to use: $\mu_n$ is natural for $A$ and $B$, and is equivalent to $\nu_n$ for the barycentric projection of these sets, at least under the conditions considered in the previous sections, but $\nu_n$ is not natural for $Z_t(A, B)$ because it is defined on $[0, 1]$, not on the common path. Since $\cup_{t \in (0, 1)} Z_t$ is at most one-dimensional (from Proposition 3.1), the geometrically defined natural measures to consider would seem to be Hausdorff measures of dimension at most one.

In light of the work done in the previous sections, natural choices of dimension are that of $\nu_n$, that of $\nu_n \times \nu_n$, and 1. The first is likely to be uninteresting, because if $\nu_n(A)\nu_n(B) > 0$ then $Z_t(A, B)$ has dimension at least $\min\{1, 2 \dim(\nu_n)\}$ as seen in Theorem 5.8. But the others also present some issues with optimality, validity or both: for example, if $A$ and $B$ are connected then $Z_t$ contains an interval and therefore has infinite measure in dimensions less than one, so any inequality for dimension less than one will be trivially true on connected sets, while Theorem 5.8 ensures that for $n \geq 9$ one could have $\nu_n(A) = \nu_n(B) = 1$ and yet $Z_t(A, B)$ has dimension less than one, so no inequality for Hausdorff 1-measure can be true for general $A$ and $B$ in this case.

In this section we derive an inequality in the case where $A$ and $B$ are connected sets, so $Z_t(A, B)$ contains an interval and therefore the correct measure to use for $Z_t$ is the one-dimensional Hausdorff measure $\mathcal{H}^1$. There is an easy bound if $A$ and $B$ are cells.

Proposition 6.1. Suppose $A = \langle v \rangle$ and $B = \langle w \rangle$ are disjoint cells. Then we have the sharp inequality

$$\mathcal{H}^1(Z_t(A, B)) \geq (1-t)\mu_n(A)^{\log 2/\log(n+1)} + t\mu_n(B)^{\log 2/\log(n+1)}.$$  

Proof. Take $a \in A$ and $b \in B$ so that $d(a, b)$ is maximal. The geodesic from $a$ to $b$ passes through boundary points $\hat{a} \in A$ and $\hat{b} \in B$, with $d(a, \hat{a}) = 2^{-|v|}$ and $d(b, \hat{b}) = 2^{-|w|}$. Then $Z_t(A, B)$ contains an interval along this geodesic, and it is easy to compute a lower bound
for its length, which gives
\[ \mathcal{H}_1(Z_t(A, B)) \geq (1 - t)2^{-|v|} + t2^{-|w|}. \]

However \( \mu_n(A) = (n+1)^{-|v|} \) and \( \mu_n(B) = (n+1)^{-|w|} \), from which the assertion is immediate. Sharpness occurs when \( Z_t \) is equal to this interval, which is true provided \( t_1 < t < t_2 \); this can be arranged by suitably choosing \( A \) and \( B \).

If \( A \) and \( B \) are connected but are not cells then we can take minimal cells \( \langle v \rangle \supset A \) and \( \langle w \rangle \supset B \). Provided \( \langle v \rangle \) and \( \langle w \rangle \) are disjoint and joined by a common path from \( a \) to \( b \) our reasoning from the the proof of Proposition 6.1 is still useful, but the lower bound for the \( \mathcal{H}_1 \) measure must now also involve the sizes of the intervals \( \varphi_a(A) \) and \( \varphi_b(B) \) obtained by barycentric projection. (Note that these are intervals because \( A \) and \( B \) are connected.) Indeed, the geodesic between the maximally separated points \( a \in A \) and \( b \in B \) begins with a path of length at least \( 2^{-|v|} \mathcal{H}_1(\varphi_a(A)) \) in \( A \) and ends with one of length at least \( 2^{-|w|} \mathcal{H}_1(\varphi_b(B)) \) in \( B \), so that

\[ (6.1) \quad \mathcal{H}_1(Z_t(A, B)) \geq (1 - t)2^{-|v|} \mathcal{H}_1(\varphi_a(A)) + t2^{-|w|} \mathcal{H}_1(\varphi_b(B)) \]

and in order to proceed we must bound \( \mathcal{H}_1(\varphi_b(A)) \) from below using \( \mu_n(F_v^{-1}A) \). It is obvious that \( \mu_n(F_v^{-1}A) \leq \nu_n(\varphi_a(A)) \). To compare \( \mathcal{H}_1(\varphi_a(A)) \) and \( \nu_n(\varphi_a(A)) \) we establish some lemmas; the conclusion of our reasoning regarding a lower bound for \( \mathcal{H}_1(Z_t(A, B)) \) when \( A \) and \( B \) are connected is in Theorem 6.6

**Lemma 6.2.** If \([a, a + x] \subseteq [0, 1]\) then \( \nu_n([0, x]) \geq \nu_n([a, a + x]) \geq \nu_n([1 - x, 1]) \).

**Proof.** As \( \nu_n \) is non-atomic, \( \nu_n([a, a + x]) \) is continuous in \( a \) and \( x \), and it suffices to consider dyadic rationals of arbitrary scale \( m \), so \( a = \sum_{i=1}^m a_i2^{-i} \) and \( x = \sum_{i=1}^m x_i2^{-i} \).

Observe that we can assume \( x \leq a \) and \( a + x \leq 1 - x \), because if the intervals intersect then it suffices to prove the inequality for the complement of the intersection (for example, if \( x > a \) the first inequality may be proved by showing \( \nu_n([0, a]) \geq \nu_n([a, a + x]) \) because then \( \nu_n([0, x]) = \nu_n([0, a]) + \nu_n([a, x]) \geq \nu_n([a, x]) + \nu_n([x, a + x]) = \nu_n([a, a + x]) \)). Note in particular that this assumption provides \( x \leq \frac{1}{2} \).

We induct on \( m \), with the easily verifiable base case \( m = 1 \). Supposing it is true to scale \( m - 1 \), take \( a \) and \( x \) at dyadic scale \( m \) and use the self-similarity of \( \nu_n \) from Lemma 6.1. If both \( a \) and \( a + x \) are in \([0, \frac{1}{2}]\) then the scaling map is \( F_0^{-1}(y) = 2y \) and thus \( \nu_n([a, a + x]) = \frac{2}{n+1}\nu_n([2a, 2a + 2x]) \). Both \( 2a \) and \( 2a + 2x \) are dyadic of scale \( m - 1 \), so that \( \nu_n([0, 2x]) \geq \nu_n([2a, 2a + 2x]) \geq \nu_n([1 - 2x, 1]) \) from the inductive assumption. We can then use the self-similarity a second time, in the reverse direction, to obtain the desired inequality. (This latter uses \( x \leq \frac{1}{2} \).) The proof if both \( a \) and \( a + x \) are in \([\frac{1}{2}, 1]\) follows the same reasoning but uses \( F_1^{-1}(y) = 2y - 1 \) on \( y \in [\frac{1}{2}, 1] \).

For the remaining case we have \( x \leq a < \frac{1}{2} < a + x \leq 1 - x \), so we separate at \( \frac{1}{2} \) and use the self-similarity to write

\[ \nu_n([a, a + x]) = \nu_n([a, \frac{1}{2}]) + \nu_n([\frac{1}{2}, a + x]) = \frac{n}{n+1}\nu_n([2a, 1]) + \frac{1}{n+1}\nu_n([0, 2a + 2x - 1]). \]
Since $2a$ and $2a + 2x - 1$ are dyadic of scale $n - 1$ we can apply the inductive assumption to obtain $\nu_n([2a, 1]) \leq \nu_n([2a + 2x - 1, 2x])$ and $\nu_n([0, 2a + 2x - 1]) \geq \nu_n([1 - 2x, 2a])$. Thus

$$\nu_n([0, x]) = \frac{n}{n + 1} \nu_n([0, 2x]) = \frac{n}{n + 1} \left( \nu_n([2a + 2x - 1, 2x]) + \nu_n([0, 2a + 2x - 1]) \right)$$

$$\geq \frac{n}{n + 1} \nu_n([2a, 1]) + \frac{1}{n + 1} \nu_n([0, 2a + 2x - 1])$$

$$\geq \frac{1}{n + 1} \left( \nu_n([2a, 1]) + \nu_n([1 - 2x, 2a]) \right)$$

$$= \frac{1}{n + 1} \nu_n([1 - 2x, 1]) = \nu_n([1 - x, 1]),$$

where the beginning and end inequalities again use the self-similarity in reverse, which uses the earlier established fact that $x \leq \frac{1}{2}$. Comparing the middle term to (6.2) establishes the desired inequality. □

Having determined that $\nu_n([a, a + x]) \leq \nu_n([0, x])$, we next look for a minimal concave bounding function having the form found in the classical Brunn-Minkowski inequality. The fact that the following function bounds $\nu_n([0, x])$ is proved in Corollary 6.5 and illustrated in Figure 9. Note that Figure 9 makes it clear this is not the minimal concave bounding function, but only the minimal one having the classical Brunn-Minkowski form.

**Definition 6.3.** Let $d_n = \frac{\log 2}{\log \frac{a}{a - 1}}$, and $\Phi_n(x) = \left( 1 - (1 - x)^{d_n} \right)^{1/d_n}$.

**Lemma 6.4.**

\begin{align*}
(6.3) & \quad n \Phi_n(2x) \leq (n + 1) \Phi_n(x) \text{ if } x \in [0, 1/2], \\
(6.4) & \quad \Phi_n(2x - 1) \leq (n + 1) \Phi_n(x) - n \text{ if } x \in [1/2, 1].
\end{align*}

**Proof.** Dividing both sides of (6.3) and taking the $d_n$ power we find it is equivalent to

$$A_1(x) = 1 - (1 - 2x)^{d_n} \leq 2 \left( 1 - (1 - x)^{d_n} \right) = A_2(x).$$

We have $A_1(0) = 0 = A_2(0)$. Moreover $A_1'(x) = 2d_n(1 - 2x)^{d_n - 1} \leq 2d_n(1 - x)^{d_n - 1} = A_2'(x)$ because $0 \leq 1 - 2x \leq 1 - x$ and $d_n - 1 \geq 0$. The inequality (6.3) follows.

The inequality (6.4) is equivalent to

$$A_3(1 - x) = \left( 1 - (2(1 - x))^{d_n} \right)^{1/d_n} \leq (n + 1)(1 - (1 - x)^{d_n})^{1/d_n} - n = A_4(1 - x)$$

for $y = 1 - x \in [0, 1/2]$. We have $A_3(0) = 1 = A_4(0)$ and compare derivatives as follows:

$$A_3'(y) = \left( 1 - (2y)^{d_n} \right) \frac{1}{\pi_n - 1} 2^{d_n - 1} y^{d_n - 1} \leq (n + 1)(1 - y^{d_n}) \frac{1}{\pi_n - 1} y^{d_n - 1} = A_4'(y)$$

because $2^{d_n} \leq (n + 1)$ and $d_n \geq 1$ gives both $0 \leq 1 - y^{d_n} \leq 1 - (2y)^{d_n}$ on $[0, 1/2]$ and $\frac{1}{\pi_n - 1} \leq 1$. The former is easily checked using the fact that $2^{-d_n} (n + 1)$ is decreasing in $n$ and equal to 1 when $n = 1$. Thus $A_3(y) \leq A_4(y)$ on $[0, 1/2]$ and this establishes (6.4). □

**Corollary 6.5.** $\nu_n([0, x]) \leq \Phi_n(x) = \left( 1 - (1 - x)^{d_n} \right)^{1/d_n}$ on $[0, 1]$.

**Proof.** As in the proof of Lemma 6.2 it is sufficient (by continuity of the functions) to prove this for dyadic rational $x$, which we do by induction on the degree $m$ of the dyadic rational. The base case is $m = 0$ where the equalities $\nu_n(\{0\}) = 0 = \Phi_n(0)$ and $\nu_n(\{1\}) = 1 = \Phi_n(1)$ are immediate. If $x = k2^{-m}$ is a dyadic rational we use the self-similarity of $\nu_n$ from
Lemma \ref{lem:connectedness} then the fact that \(2x\) and \(2x - 1\) are dyadic rationals of lower degree so satisfy the inequality by induction, and finally Lemma \ref{lem:connectedness} to compute
\[
\nu_n([0, x]) = \begin{cases} 
\frac{n}{n+1} \nu_n([0, 2x]) \leq \Phi_n(2x) \leq \Phi_n(x) & \text{if } x \leq \frac{1}{2} , \\
\frac{n}{n+1} + \frac{1}{n+1} \nu_n([0, 2x - 1]) \leq \frac{n}{n+1} + \frac{1}{n+1} \Phi_n(2x - 1) \leq \Phi_n(x) & \text{if } x > \frac{1}{2} .
\end{cases}
\]

The function \(\Phi_n\) is concave, so applying it to both sides of (6.1) gives
\[
1 - (1 - \mathcal{H}_1(Z_t(A, B))) \geq (1 - t)\mu_n(A)\log(2/\log(n+1)) + t\mu_n(B)\log(2/\log(n+1)).
\]

**Proof.** The function \(\Phi_n^{d_n}\) is concave, so applying it to both sides of \ref{eq:concave} gives
\[
\Phi_n^{d_n}(\mathcal{H}_1(Z_t(A, B))) \geq (1 - t)\Phi_n^{d_n}(2^{-|v|}\mathcal{H}_1(\varphi_\alpha(A))) + t\Phi_n^{d_n}(2^{-|v|}\mathcal{H}_1(\varphi_\beta(B))).
\]

However we saw in Corollary \ref{cor:connectedness} that \(\Phi_n(x)\) bounds \(\nu_n([0, x])\) and in Lemma \ref{lem:connectedness} that \(\nu_n([0, x])\) bounds \(\nu_n\) of any interval of this length. The latter bound applies to \(\varphi_\alpha(A)\), which is an interval by the connectedness of \(A\). Also using that \(\nu_n([0, 2^{-m}x]) = \left(\frac{n}{n+1}\right)^m \nu_n([0, x])\) from the self-similarity of \(\nu_n\), we have
\[
\Phi_n(2^{-|v|}\mathcal{H}_1(\varphi_\alpha(A))) \geq \nu_n([0, 2^{-|v|}\mathcal{H}_1(\varphi_\alpha(A))]) \geq \left(\frac{n}{n+1}\right)^{|v|} \nu_n(\varphi_\alpha(A)).
\]

But we also know that \(\nu_n(\varphi_\alpha(A)) \geq (n + 1)^{|v|}\mu_n(A) = \mu_n(A)/\mu_n(\langle v \rangle)\) because the discussion following Definition \ref{def:connectedness} showed \(\nu_n = (n + 1)^{|v|}\mu_n \circ \varphi_\alpha^{-1}\). Using this and the definition of \(d_n\) we obtain
\[
\Phi_n^{d_n}(2^{-|v|}\mathcal{H}_1(\varphi_\alpha(A))) \geq \left(\frac{n}{n+1}\right)^{|v|d_n} \left(\frac{\mu_n(A)}{\mu_n(\langle v \rangle)}\right)^{d_n} = 2^{-|v|} \left(\frac{\mu_n(A)}{\mu_n(\langle v \rangle)}\right)^{d_n}.
\]
A similar bound applies for $B$, so we have
\[
\Phi_n^d(n \mathcal{H}_t(Z_t(A, B))) \geq (1 - t)2^{-|v|}(\frac{\mu_n(A)}{\mu_n(\langle v \rangle)})^{d_n} + t2^{-|w|}(\frac{\mu_n(B)}{\mu_n(\langle w \rangle)})^{d_n} = (1 - t)\mu_n(\langle v \rangle)^{d_n - d_m} \mu_n(A)^{d_n} + t\mu_n(\langle w \rangle)^{d_n - d_m} \mu_n(B)^{d_n}
\]
with $d_n' = \frac{\log 2}{\log (n + 1)}$. The fact that $d_n' - d_m = \frac{\log 2 \log n}{\log(\frac{n+2}{n+1}) \log(n+1)} > 0$ and both $\mu_n(\langle v \rangle) \geq \mu_n(A)$ and $\mu_n(\langle w \rangle) \geq \mu_n(B)$ then gives the result. 

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\begin{thebibliography}{9}

\bibitem{1} Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke G. Rogers, Nageswari Shanmugalingam, and Alexander Teplyaev, \textit{Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates}, arXiv e-prints (2019Mar), arXiv:1903.10078.

\bibitem{2} Jonas Azzam, Michael A Hall, and Robert S Strichartz, \textit{Conformal energy, conformal Laplacian, and energy measures on the Sierpinski gasket}, Transactions of the American Mathematical Society (2008), 2089–2130.

\bibitem{3} Franck Barthe, \textit{The Brunn-Minkowski theorem and related geometric and functional inequalities}, International Congress of Mathematicians. Vol. II, 2006, pp. 1529–1546. MR2275657

\bibitem{4} Fabrice Baudoin and Daniel J. Kelleher, \textit{Differential one-forms on Dirichlet spaces and Bakry-Émery estimates on metric graphs}, Trans. Amer. Math. Soc. \textbf{371} (2019), no. 5, 3145–3178. MR3896108

\bibitem{5} Erik Christensen, Cristina Ivan, and Michel L Lapidus, \textit{Dirac operators and spectral triples for some fractal sets built on curves}, Advances in Mathematics \textbf{217} (2008), no. 1, 42–78.

\bibitem{6} Dario Cordero-Erausquin, Robert J McCann, and Michael Schmuckenschläger, \textit{A Riemannian interpolation inequality à la Borell, Brascamp and Lieb}, Inventiones mathematicae \textbf{146} (2001), no. 2, 219–257.

\bibitem{7} Ligia L Cristea and Bertran Steinsky, \textit{Distances in Sierpiński graphs and on the Sierpinski gasket}, Aequationes mathematicae \textbf{85} (2013), no. 3, 201–219.

\bibitem{8} Gerald Edgar, \textit{Measure, topology, and fractal geometry}, Second Edition, Undergraduate Texts in Mathematics, Springer, New York, 2008. MR2356043

\bibitem{9} Gerald A. Edgar, \textit{Integral, probability, and fractal measures}, Springer-Verlag, New York, 1998. MR1484412

\bibitem{10} R. J. Gardner, \textit{The Brunn-Minkowski inequality}, Bull. Amer. Math. Soc. (N.S.) \textbf{39} (2002), no. 3, 355–405.

\bibitem{11} Jiangwen Gu, Qianqian Ye, and Lifeng Xi, \textit{Geodesics of higher-dimensional Sierpinski gasket}, Fractals \textbf{27} (2019), no. 4, 1950049, 10. MR3980704

\bibitem{12} Xiaoyu Hu and James S Taylor, \textit{Fractal properties of products and projections of measures in $\mathbb{R}^d$}, Mathematical proceedings of the Cambridge Philosophical Society, 1994, pp. 527–544.

\bibitem{13} John E Hutchinson, \textit{Fractals and self similarity}, Indiana University Mathematics Journal \textbf{30} (1981), no. 5, 713–747.

\bibitem{14} Michel L. Lapidus and Machiel van Frankenhuijsen, \textit{Fractal geometry, complex dimensions and zeta functions}, Second Edition, Springer Monographs in Mathematics, Springer, New York, 2013. MR2977849

\bibitem{15} John M Marstrand, \textit{Some fundamental geometric properties of plane sets and fractional dimensions}, Proceedings of the London Mathematical Society \textbf{3} (1954), no. 4.

\bibitem{16} Mustafa Saltan, Yunus Özdemir, and Bünymaim Demir, \textit{Geodesics of the Sierpinski gasket}, Fractals (2016), 1850024.

\bibitem{17} \textit{An explicit formula of the intrinsic metric on the Sierpinski gasket via code representation}, Turkish Journal of Mathematics \textbf{42} (2018), no. 2, 716–725.
\end{thebibliography}
[18] Pablo Shmerkin and Boris Solomyak, *Absolute continuity of self-similar measures, their projections and convolutions*, Transactions of the American Mathematical Society 368 (2016), no. 7, 5125–5151.

[19] Robert S Strichartz, *Isoperimetric estimates on Sierpinski gasket type fractals*, Transactions of the American Mathematical Society 351 (1999), no. 5, 1705–1752.

[20] , *Differential equations on fractals: a tutorial*, Princeton University Press, 2006.

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