DISK COMPLEXES AND GENUS TWO HEEGAARD SPLITTINGS FOR NON-PRIME 3-MANIFOLDS

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Abstract. Given a genus two Heegaard splitting for a non-prime 3-manifold, we define a special subcomplex of the disk complex for one of the handlebodies of the splitting, and then show that it is contractible. As applications, first we show that the complex of Haken spheres for the splitting is contractible, which refines the results of Lei and Lei-Zhang. Secondly, we classify all the genus two Heegaard splittings for non-prime 3-manifolds, which is a generalization of the result of Montesinos-Safont. Finally, we show that the mapping class group of the splitting, called the Goeritz group, is finitely presented by giving its explicit presentation.

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Introduction

Every closed orientable 3-manifold $M$ can be decomposed into two handlebodies $V$ and $W$ by cutting $M$ along a closed orientable surface $\Sigma$ embedded in it. This is called a Heegaard splitting for the manifold $M$, and denoted by the triple $(V, W; \Sigma)$. The surface $\Sigma$ is called a Heegaard surface and its genus is called the genus of the splitting. A separating 2-sphere $P$ in $M$ is called a Haken sphere for the splitting $(V, W; \Sigma)$ if $P$ intersects the Heegaard surface $\Sigma$ in a single essential circle. If $(V, W; \Sigma)$ is a genus two Heegaard splitting for $M$ that admits a Haken sphere, then $M$ is one of the 3-sphere, $S^2 \times S^1$, lens spaces or their connected sums. In particular, if the manifold $M$ is non-prime, then $M$ is a connected sum whose summands are lens spaces or $S^2 \times S^1$.

In this paper, we study the genus two Heegaard splittings for non-prime 3-manifolds. Given a genus two Heegaard splitting $(V, W; \Sigma)$ for a closed orientable non-prime 3-manifold $M$, we define a special subcomplex of the disk complex for each of the handlebodies $V$ and $W$, which we will call the semi-primitive disk complex, and then show that it is contractible. The semi-primitive disk complex is an analogue of the primitive disk complexes studied in the authors’ previous works [4, 5, 6, 7, 8, 19] to find presentations of certain kinds of mapping class groups, including some Goeritz groups.

Understanding the structure of the semi-primitive disk complexes with their properties, we produce several applications. First, we prove that the complex of Haken spheres

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is contractible for the genus two Heegaard splitting for any non-prime 3-manifold. The complex of Haken spheres is the simplicial complex whose vertices are isotopy classes of Haken spheres, and it has been an interesting problem to understand the structure of it since Scharlemann [26] showed that the complex for the genus two Heegaard splitting for the 3-sphere is connected. In Lei [20] and Lei-Zhang [21], it was shown that the complexes of Haken spheres are connected for genus two Heegaard splittings for non-prime 3-manifolds. In Theorem 3.1 in this work, we refine their results in an alternative way, showing that those complexes are actually contractible.

Secondly, we classify all the genus two Heegaard splittings for non-prime 3-manifolds. Indeed, any non-prime 3-manifold $M$ admits at most two different genus two Heegaard splittings, and it is known from Montesinos-Safont [23] that, if $M$ is the connected sum of two lens spaces $L(p, q_1)$ and $L(p, q_2)$, then there exists a unique genus two Heegaard surface for $M$ up to homeomorphism if and only if $q_1^2 \equiv 1$ or $q_2^2 \equiv 1 \pmod{p}$. Including this result, we determine all the non-prime 3-manifolds that admit unique Heegaard surfaces up to homeomorphism, which is stated in Theorem 4.2.

The final application is to obtain a presentation of the mapping class group of a genus two Heegaard splitting for a non-Haken 3-manifold, using the semi-primitive disk complex. Such a group is called a (genus two) Goeritz group. Precisely, the Goeritz group of a Heegaard splitting $(V, W; \Sigma)$ for a manifold $M$ is the group of isotopy classes of orientation-preserving homeomorphisms of $M$ that preserve $V$ and $W$ setwise. In Theorem 5.1 in this work, we show that the genus two Goeritz groups for any non-prime 3-manifolds are all finitely presented by giving their explicit presentations.

The Goeritz groups have been interesting objects in the study of Heegaard splittings. For example, some interesting questions on Goeritz groups were proposed by Minsky in [11]. A Goeritz group will be “small” when the gluing map of the two handlebodies that defines the Heegaard splitting is sufficiently complicated. Indeed, Namazi [24] showed that the Goeritz group is actually a finite group when the Heegaard splitting has “high” Hempel distance. Here, we just simply mention that the Hempel distance is a measure of complexity of the gluing map that defines the splitting. We refer to [14] for its precise definition. Namazi’s result is improved by Johnson in [16] showing that the Goeritz group is finite if the Hempel distance of the splitting is at least four. We refer the reader to [17, 18] for related topics. The Goeritz groups of Heegaard splittings of low Hempel distance are not as “small” as in the case of the high Hempel distance.

For example, it is easy to see that the Goeritz group of the genus $g$ Heegaard splitting for $\#_g(S^2 \times S^1)$, which is the double of the genus $g$ handlebody $V$, is isomorphic to the mapping class group of $V$. We note that the Hempel distance of this splitting is zero. The mapping class group of a handlebody of genus at least two is, of course, not finite. A finite generating set of this group is obtained by Suzuki [29], and its finite presentation is obtained by Grasse [12] and Wajnryb [30] independently. See also [22, 15].

It is natural to ask if a given Goeritz group is finitely generated or presented, and so finding a generating set or a presentation of it has been an important problem. But beyond the case of $\#_g(S^2 \times S^1)$, the generating sets or the presentations of the groups have been obtained only for few manifolds with their splittings of small genus. In the case of the 3-sphere, it is known that the Goeritz group for the genus two splitting is finitely presented from the works [10, 26, 1, 4]. Further, a finite presentation of the
Goeritz group of the genus two Heegaard splitting is obtained for each of the lens spaces $L(p, 1)$ in [5] and $S^2 \times S^1$ in [7]. In addition, finite presentations of the genus two Goeritz groups of some other lens spaces are given in [8]. For the higher genus Goeritz groups of the 3-sphere and lens spaces, it is conjectured that they are all finitely presented but it is still known to be an open problem.

This paper is organized as follows. In Sections 1 and 2 we introduce semi-primitive disks with their various properties, and then show that the semi-primitive disk complexes are contractible, by giving explicit description of them. In Section 3 the complex of Haken spheres are shown to be contractible (Theorem 3.1), and in Section 4 we give a classification of the genus two Heegaard splittings for non-prime 3-manifolds (Theorem 4.2). In the final section, a finite presentation is given for the Goeritz group of each non-prime 3-manifold with its genus two Heegaard splitting (Theorem 5.1).

By disks, pairs of disks, triples of disks properly embedded in a handlebody, we often mean their isotopy classes throughout the paper. Also, we often speak of Haken spheres of a Heegaard splitting to mean their isotopy classes preserving the Heegaard splitting. When we choose representatives of their isotopy classes, we assume implicitly that they intersect each other minimally and transversely. Moreover, by homeomorphisms we often mean their isotopy classes when it is obvious from context.

We use the standard notation of lens spaces as follows. Let $V$ and $W$ be oriented solid tori. Let $(m, l)$ be the pair of a meridian and a longitude of $V$. We orient $m$ and $l$ in such a way that the pair $(m, l)$ yields the orientation of $\partial V$ induced by that of $V$. The homology classes $[m]$ and $[l]$ of $m$ and $l$ induce a basis of $H_1(\partial V)$. In the same manner, we have the pair $(m', l')$ of a meridian and a longitude of $W$. The lens space $L(p, q)$ is a 3-manifold obtained by identifying the boundaries of $V$ and $W$ using an orientation-reversing homeomorphism $\varphi: \partial V \to \partial W$ that induces an isomorphism $\varphi_*: H_1(\partial V) \to H_1(\partial W)$ represented by $\begin{pmatrix} q & p \\ s & -r \end{pmatrix}$, where $qr + ps = 1$. In particular, $\varphi$ maps $m'$ to a $(p, q)$-curve with respect to $(m, l)$ on $\partial V$, that is, $\varphi_*[m'] = p[l] + q[m]$ in $H_1(\partial V)$. We note that the image of $m$ by $\varphi^{-1}$ is a $(p, r)$-curve with respect to $(m', l')$ on $\partial W$. By definition, a lens space is equipped with a canonical orientation induced from those of $V$ and $W$. This orientation induces a canonical orientation of the connected sum of two lens spaces. Throughout the paper, we will not regard $S^3 = L(1, 0)$ nor $S^2 \times S^1 = L(0, 1)$ as lens spaces.

1. Semi-primitive disks

An element of a free group $\mathbb{Z} * \mathbb{Z}$ of rank 2 is said to be primitive if it is a member of a generating pair of the group. Primitive elements of $\mathbb{Z} * \mathbb{Z}$ have been well-understood. For example, we refer [25] to the reader. A key property of the primitive elements is that, fixing a generating pair $\{x, y\}$ of $\mathbb{Z} * \mathbb{Z}$, any primitive element has a cyclically reduced form which is a product of terms each of the form $x^\epsilon y^n$ and $x^\epsilon y^{n+1}$, or else a product of terms each of the form $y^\epsilon x^n$ and $y^\epsilon x^{n+1}$, for some $\epsilon \in \{1, -1\}$ and some $n \in \mathbb{Z}$. The following is a direct consequence of this property.
Lemma 1.1. Fix a generating pair \( \{x, y\} \) of \( \mathbb{Z} \times \mathbb{Z} \). Let \( w \) be a cyclically reduced word on \( \{x, y\} \). If \( w \) contains both \( x \) and \( x^{-1} \), both \( y \) and \( y^{-1} \) or both \( x^{\pm 2} \) and \( y^{\pm 2} \) simultaneously, then the element represented by \( w \) is neither trivial nor a power of a primitive element.

Let \( V \) be a genus two handlebody, and let \( D \) and \( E \) be disjoint disks in \( V \) such that \( D \cup E \) cuts \( V \) into a 3-ball. We fix an orientation on each of \( \partial D \) and \( \partial E \), and then assign letters \( x \) and \( y \) to \( \partial D \) and \( \partial E \) respectively. Let \( l \) be an oriented simple closed curve on \( \partial V \) which intersects \( \partial D \cup \partial E \) minimally and transversely. Then \( l \) determines a word on \( \{x, y\} \) that can be read off by the intersections of \( l \) with \( \partial D \) and \( \partial E \). We note that this word is well-defined up to cyclic conjugation. The following is a simple criterion for triviality and primitiveness of the elements represented by \( l \), which can be considered as a simpler version of Lemma 2.3 in [6].

Lemma 1.2. In the above setting, if a word \( w \) determined by the simple closed curve \( l \) contains a subword of the form \( xy^p x^{-1} \) for some \( p \in \mathbb{N} \), or \( x^2 y^2 \), then any word determined by \( l \) is cyclically reduced. Moreover, the element represented by \( w \) is neither trivial nor a power of a primitive element.

The idea of the proof is that, if \( w \) contains one of those subwords, then any word determined by \( l \) cannot contain \( x^{\pm 1} x^{\mp 1} \) and \( y^{\pm 1} y^{\mp 1} \), and any cyclically reduced word containing both \( x \) and \( x^{-1} \) or both \( x^2 \) and \( y^2 \) cannot represent a power of a primitive element by Lemma 1.1.

Let \( (V, W; \Sigma) \) be a genus two Heegaard splitting for a non-prime 3-manifold. Recall that, by [13], the splitting \( (V, W; \Sigma) \) admits a Haken sphere. A non-separating disk \( D \) in \( V \) is said to be semi-primitive if there exists a Haken sphere \( P \) of \( (V, W; \Sigma) \) disjoint from \( D \). The next lemma follows from the definition.

Lemma 1.3. Let \( (V, W; \Sigma) \) be a genus two Heegaard splitting for a non-prime 3-manifold. Let \( D \) be a semi-primitive disk in \( V \). Then an element of \( \pi_1(W) \) determined by \( \partial D \) is either trivial or a power of a primitive element.

We remark that there is a semi-primitive disk \( D \) in \( V \) such that \( \partial D \) represents the trivial element of \( \pi_1(W) \) if and only if the manifold has a \( S^2 \times S^1 \) summand. In this case, \( \partial D \) also bounds a disk in \( W \).

Lemma 1.4. Let \( (V, W; \Sigma) \) be a genus two Heegaard splitting for a non-prime 3-manifold. Let \( D \) be a non-separating disk in \( V \). Then \( D \) is semi-primitive if and only if there exists a non-separating disk \( E \) in \( W \) disjoint from \( D \).

Proof. The necessity is trivial. For the sufficiency, let \( \Sigma' \) be the 4-holed sphere obtained by cutting \( \Sigma \) along \( \partial D \cup \partial E' \). Let \( d^+ \) and \( d^- (e^+ \) and \( e^- \), respectively) be the two boundary circles of \( \Sigma' \) coming from \( \partial D \cup \partial E' \), respectively. Let \( \alpha \) be an arbitrary simple arc in \( \Sigma' \) connecting \( d^+ \) and \( d^- \). Then, up to isotopy, there exists a unique simple arc \( \alpha' \) in \( \Sigma' \) connecting \( e^+ \) and \( e^- \) such that \( \alpha \cap \alpha' = \emptyset \). We note that the frontier \( \gamma \) of a regular neighborhood of \( d^+ \cup \alpha \cup \partial D \cup d^- \) coincides with the frontier of a regular neighborhood of \( e^+ \cup e^- \) in \( \Sigma' \). It follows that \( \gamma \) bounds a disk in each of \( V \) and \( W \). This implies that there exists a Haken sphere \( P \) of \( (V, W; \Sigma) \) such that \( P \cap \Sigma = \gamma \). □

In the proof above, every simple closed curve \( \gamma \) in \( \Sigma' \) that separates \( d^+ \cup d^- \) and \( e^+ \cup e^- \) is the frontier of a regular neighborhood of the union of \( d^+ \cup d^- (e^+ \cup e^- \).
respectively) and a simple arc \( \alpha_Q (\alpha_Q', \text{respectively}) \) in \( \Sigma' \) connecting \( d^+ \) and \( d^- (e^+, \text{respectively}) \). Thus every essential, separating, simple closed curve in \( \Sigma \) disjoint from \( \partial D \cup \partial E' \) bounds separating disks in both \( V \) and \( W \).

1.1. Connected sum of two lens spaces. Throughout this subsection, we always assume that \((V, W; \Sigma)\) is a genus two Heegaard splitting for the connected sum of two lens spaces.

**Lemma 1.5.** Let \( D \) be a semi-primitive disk in \( V \). Then there is a unique non-separating disk \( E' \) in \( W \) disjoint from \( D \).

**Proof.** By Lemma 1.4, such a disk \( E' \) exists. To see the uniqueness, assume that there exist non-isotopic, non-separating disks \( E'_1 \) and \( E'_2 \) in \( W \) disjoint from \( D \). We assume that \( E'_1 \) and \( E'_2 \) intersect each other transversely and minimally. If they have non-empty intersection, a disk obtained from \( E'_1 \) by a surgery along an outermost subdisk of \( E'_2 \) cut off by \( E'_1 \cap E'_2 \) is also a non-separating disks in \( W \) disjoint from \( D \). This disk has fewer intersection with \( E'_1 \) than \( E'_2 \) had, and so by repeating surgeries if they still have intersection, we obtain a non-separating disk \( E' \) in \( W \) disjoint from \( E'_1 \) and from \( D \). Since \( \partial D \) does not intersects \( E'_1 \cup E'_2 \), the circle \( \partial D \) bounds a disk \( D' \) in \( W \). This implies that \( D \cup D' \) is a non-separating sphere in the connected sum of two lens spaces, whence a contradiction. \( \square \)

The next theorem will play an important role in Section 2.

**Theorem 1.6.** Let \( D \) and \( E \) be semi-primitive disks in \( V \) that intersect each other transversely and minimally. Then at least one of the two disks obtained from \( E \) by a surgery along an outermost subdisk of \( D \) cut off by \( D \cap E \) is semi-primitive.

**Proof.** Let \( C \) be an outermost subdisk of \( D \) cut off by \( D \cap E \). Each Haken sphere \( P \) of \((V, W; \Sigma)\) disjoint from \( E \) cuts the handlebody \( V \) into two solid tori \( V_1 \) and \( V_2 \), and \( W \) into \( W_1 \) and \( W_2 \). We assume that \( E \) is the meridian disk of \( V_1 \), and that \( V_1 \cup W_1 \) and \( V_2 \cup W_2 \) are punctured lens spaces. Let \( E_0 \), \( E' \) and \( E'_0 \) be the meridian disks of solid tori \( V_2 \), \( W_1 \) and \( W_2 \), respectively, which are disjoint from \( P \). We choose a Haken sphere \( P \) among all Haken spheres disjoint from \( E \) so that \( |C \cap E_0| \) is minimal. Assume that \( \partial E' \) (\( \partial E'_0 \), respectively) is a \((p_2, q_2)\)-curve \((p_1, q_1)\)-curve, respectively) with respect to the meridian \( \partial E \) (\( \partial E_0 \), respectively) and a fixed longitude on \( \partial V_1 \) (\( \partial V_2 \), respectively). Each element of \( \pi_1(W) \) can be represented by a word on \( \{x, y\} \), where \( x \) and \( y \) are determined (up to sign) by the meridian disks \( E' \) and \( E'_0 \) respectively. If \( E_0 \) is disjoint from \( C \), then \( E_0 \) is one of the disks obtained from \( E \) by a surgery along \( C \), which is semi-primitive, and so we are done.

Assume that \( C \cap E_0 \neq \emptyset \). Let \( C_0 \) be an outermost subdisk of \( C \) cut off by \( C \cap E_0 \) such that \( C_0 \cap E = \emptyset \). Let \( \Sigma_0 \) be the 4-holed sphere obtained by cutting \( \Sigma \) along \( \partial E \cup \partial E_0 \). Let \( e^+ \) and \( e^- (e_0^+ \text{ and } e_0^- \), respectively) be the boundary circles of \( \Sigma_0 \) coming from \( \partial E \) (\( \partial E_0 \), respectively). Then \( C_0 \cap \Sigma_0 \) is the frontier of a regular neighborhood of the union of one of \( e^+ \) and \( e^- \), say \( e^+ \), and and a simple arc \( \alpha_0 \) connecting \( e^+ \) and one of \( e_0^+ \) and \( e_0^- \), say \( e_0^+ \). Up to isotopy, the arc \( \alpha_0 \) does not intersect \( \partial E_0 \), otherwise a word of \( \partial D \) would contain the subword \( yxy^{-1} \) (after changing the orientations if necessary), which contradicts Lemmas 1.2 and 1.3. We denote by \( E_1 \) the disk obtained from \( E_0 \) by
a surgery along $C_0$ that is not $E$. We remark that $|C \cap E_1| < |C \cap E_0|$ and that $\partial E_1$ determines a word of the form $x^{p_2}y^{p_1}$. See Figure 1.

![Figure 1](image_url)

**Figure 1.** The case where $(p_1, q_1) = (3, 1)$ and $(p_2, q_2) = (2, 1).

We define inductively a sequence of disks $E_2, E_3, \ldots, E_{p_1}$ in $V$ as follows. For $i \in \{1, 2, \ldots, p_1 - 1\}$ let $\Sigma_i$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial E \cup \partial E_i$. Let $e^+$ and $e^-$ ($e_i^+$ and $e_i^-$, respectively) be the boundary circles of $\Sigma_i$ coming from $\partial E$ ($\partial E_i$, respectively) by abuse of notation. Let $\alpha_i$ be a simple arc connecting $e^+$ and $e_i^+$ that does not intersect $\partial E_{0}$. Let $E_{i+1}$ be the disk obtained by the band sum of $E$ and $E_i$ along $\alpha_i$. See Figure 1.

The circle $\partial E_{p_1}$ determines a word of the form $(x^{p_2}y^{p_1})$, which is apparently a power of a primitive element of $\pi_1(W)$. We show that $E_{p_1}$ is in fact a semi-primitive disk. Let $\Sigma_{p_1}$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial E \cup \partial E_{p_1}$. By the construction, the two boundary circles $e^+$ and $e^-$ of $\Sigma_{p_1}$ coming from $\partial E$ are contained in the same component of $\Sigma_{p_1}$ cut off by $\Sigma_1 \cap \partial E_0$. Hence there exists an arc $\alpha_Q$ in $\Sigma_{p_1}$ connecting $e^+$ and $e^-$ such that $\alpha_Q \cap \partial E_0 = \emptyset$. We denote by $\gamma_Q$ the frontier of a regular neighborhood of $e^+ \cup \alpha_Q \cup e^-$. Apparently, $\gamma_Q$ is disjoint from $E \cup E_0$. Thus it follows from the remark right after Lemma 1.4 that there exists a Haken sphere $Q$ in $(V, W, \Sigma)$ such that $Q \cap \Sigma = \gamma_Q$. In particular, $Q$ is disjoint from $E_{p_1}$, and hence $E_{p_1}$ is semi-primitive.

Now, we claim that, for $i \in \{1, 2, \ldots, p_1 - 1\}$, $C \cap E_i \neq \emptyset$, and $E_{i+1}$ is obtained by surgery along an outermost subdisk $C_i$ of $C$ cut off by $C \cap E_i$ such that $C_i \cap E = \emptyset$. The latter claim follows immediately from the former one, since, if $C$ intersects $E_i$, then $C \cap \Sigma_i$ is the frontier of a regular neighborhood of $e^+ \cup \alpha_i$ in $\Sigma_i$, and so the same reason to the case of $\alpha_0$ implies the latter claim. Suppose that $E_i$ is the first disk disjoint from $C$ for contradiction. First, assume that $i \in \{1, 2, \ldots, p - 2\}$. Since $E_{i-1}$ intersects $C$ and $i < p_1 - 1$, $C \cap \Sigma_i$ gives a word containing $x^2y^2$. This implies that $D$ is not semi-primitive by Lemmas 1.2 and 1.3 whence a contradiction. Next, assume that $i = p - 1$. In this case, $C$ is disjoint from $E_{p-1}$ and intersects $E_{p-2}$. Then it is easy to see that one of the resulting disks obtained by surgery on $E$ along $C$ is $E_{p-1}$, and the other one is $E_p$. In particular, $C$ is disjoint from $E_p$. This contradicts the minimality of $|C \cap E_0|$ since we are assuming that $C \cap E_0 \neq \emptyset$. Hence we get the claim. However, this is impossible since now we have the inequalities $|C \cap E_{p_1}| < |C \cap E_{p_1-1}| < \cdots < |C \cap E_0|$ and this contradicts, again, the minimality of $|C \cap E_0|$.

□
Lemma 1.7. Let $D$ and $E$ be disjoint, non-isotopic semi-primitive disks in $V$. Then there exists a unique Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$.

Proof. The uniqueness follows immediately from Lemma 1.5. To show the existence of a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$, we choose a Haken sphere $P$ among all Haken spheres disjoint from $E$ so that $|D \cap E_0|$ is minimal as in the proof of Theorem 1.6. Also, we take the disks $E'$ and $E'_0$ in $W$ as in the proof of Theorem 1.7. Each element of $\pi_1(W)$ is represented by a word on $\{x, y\}$, where $x$ and $y$ are determined (up to sign) by the meridian disks $E'$ and $E'_0$. If $D = E_0$, we are done. Assume that $D \neq E_0$ and $D \cap E_0 = \emptyset$. Then a word determined by $\partial D$ contains a subword of the form $x^2y^2$, (after changing the orientations if necessary). This contradicts Lemmas 1.2 and 1.3. Assume that $D \cap E_0 \neq \emptyset$. Then by the same argument as the proof of Theorem 1.6 for the disk $D$ instead of an outermost subdisk $C$, we can deduce a contradiction. □

Lemma 1.8. Let $D$, $E$ and $F$ be pairwise disjoint, pairwise non-isotopic essential disks in $V$. If $D$ and $E$ are semi-primitive, then $F$ is not semi-primitive.

Proof. Let $P$ be a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$. Let $D'$ and $E'$ be the meridian disks of the solid tori $W$ cut off by $P \cap W$ that are disjoint from $P$. Each element of $\pi_1(W)$ can be determined by a word on $\{x, y\}$, where $x$ and $y$ are determined (up to sign) by the meridian disks $D'$ and $E'$. Then, by a similar argument in the proof of Lemma 1.7, a word determined by $\partial F$ would contain a subword of the form $x^2y^2$ (after changing the orientations if necessary), which means $F$ is not semi-primitive. □

1.2. Connected sum of $S^2 \times S^1$ and a lens space. Throughout this subsection, we always assume that $(V, W; \Sigma)$ is a genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. A non-separating disk $D$ in $V$ is called a reducing disk if $\partial D$ bounds a disk in $W$. We remark that a reducing disk is also semi-primitive and the boundary circle of a reducing disk represents the trivial element of $\pi_1(W)$.

Lemma 1.9. Let $D$ be a reducing disk in $V$. Let $E$ be a non-separating disk in $V$ that is not isotopic to $D$.

(1) If $E$ is disjoint from $D$, then there exists a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$. In particular, $E$ is a semi-primitive disk but is not a reducing disk.

(2) If $E$ intersects $D$, then $E$ is not a semi-primitive disk.

Proof. (1) Let $\Sigma'$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial D \cup \partial E$. Let $d^+$ and $d^-$ be the two boundary circles of $\Sigma'$ coming from $\partial D$. Let $\alpha_P$ be an arbitrary simple arc in $\Sigma'$ connecting $d^+$ and $d^-$. Since $D$ is a reducing disk, the frontier $\gamma_P$ of a regular neighborhood of $d^+ \cup \alpha_P \cup d^-$ bounds a disk in each of $V$ and $W$. This implies that there exists a Haken sphere $P$ of $(V, W; \Sigma)$ such that $P \cap \Sigma = \gamma_P$, which is disjoint from $D \cup E$.

(2) Let $D'$ be a disk in $W$ bounded by $\partial D$. Let $C$ be an outermost subdisk of $E$ cut off by $D \cap E$. Then a standard cut-and-paste argument allows us to have a non-separating disk $E_1$ in $V$ that is not isotopic to $D$ and disjoint from $C \cup D$. By (1), $E_1$ is a semi-primitive disk. Let $P$ be a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E_1$. Let $E'_1$ be the semi-primitive disk in $W$ disjoint from $P$ that is not isotopic to $D'$. Let $\Sigma'$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial D \cup \partial E_1$. Let $d^+$ and $d^-$ ($e^+_1$ and $e^-_1$,
respectively) be the two boundary circles of $\Sigma'$ coming from $\partial D$ ($\partial E_1$, respectively). We note that $\partial E_1' \cap \Sigma'$ cuts $\Sigma'$ into a finite number of rectangles and a single rectangle with two holes $d^+$ and $d^-$. See Figure 2 (i). Then $C \cap \Sigma'$ is the frontier of a regular neighborhood of the union of an arc $\alpha_0$ in $\Sigma'$ connecting one of $d^+$ and $d^-$, say $d^+$ and one of $e_1^+$ and $e_1^-$, say $e_1^+$, and the boundary circle $e_1^+$. Assume that $\alpha_0$ meets $\partial E_1'$. Let $\alpha_1$ be a subarc of $\alpha_0$ connecting $d^+$ and $\partial E_1'$ such that the interior of $\alpha_1$ is disjoint from $\partial E_1'$. Let $E''_1 \subset W$ be the band sum of $E'_1$ and $D'$ along $\alpha_1$. $E''_1$ is a semi-primitive disk and we have $|\alpha_0 \cap \partial E''_1| < |\alpha_0 \cap \partial E'_1|$. See Figure 2 (ii). Repeating this process finitely many times, we obtain a semi-primitive disk $\hat{E}'_1$ in $W$ disjoint from both $D$ and $\alpha_0$.

We give letters $x$ and $y$ to the circles $\partial D'$ and $\partial \hat{E}'_1$, respectively, after fixing an orientation of each of them. Then a word on $\{x, y\}$ determined by $\partial E$ contains a subword of the form $xy^px^{-1}$, which is determined by the subarc $C \cap \Sigma'$ after changing the orientations if necessary. See Figure 2 (iii). by Lemma 1.2, $E$ is neither a reducing disk nor a semi-primitive disk. □

By Lemma 1.9 $(V, W; \Sigma)$ admits a unique reducing disk. The next lemma follows immediately from the definition of a reducing disk and the proof of Lemma 1.4.

**Lemma 1.10.** Let $D$ be the reducing disk in $V$. Then any non-reducing, semi-primitive disk in $V$ is disjoint from $D$ up to isotopy.

2. The complex of semi-primitive disks

Let $V$ be a handlebody. The disk complex $K(V)$ of $V$ is the simplicial complex whose vertices are the isotopy classes of essential disks in $V$ such that the collection of distinct $k + 1$ vertices spans a $k$-simplex if they admit a set of pairwise disjoint representatives. The full-subcomplex $D(V)$ of $K(D)$ spanned by the vertices corresponding to non-separating disks is called the non-separating disk complex of $V$. In [22], it is shown that both $K(V)$ and $D(V)$ are contractible. Moreover, we have the following theorem.
Theorem 2.1 (22 [4]). A full subcomplex $L$ of the disk complex $K(V)$ is contractible if, given any two representative disks $E$ and $D$ of vertices of $L$ intersecting each other transversely and minimally, at least one of the disks from surgery on $E$ along an outermost subdisk of $D$ cut off by $D \cap E$ represents a vertex of $L$.

Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. The semi-primitive disk complex $SP(V)$ of $V$ is the full subcomplex of $D(V)$ spanned by the vertices corresponding to semi-primitive disks of $V$. We remark that the Goeritz group $G$ of $(V,W;\Sigma)$ acts on $SP(V)$ simplicially.

Theorem 2.2. Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

1. If $M_1$ is a lens space, then $SP(V)$ is a tree.
2. If $M_1 = S^2 \times S^1$, then $SP(V)$ is the cone of a tree.

Proof. (1) is a straightforward consequence of Theorems 1.6 and 2.1 and Lemma 1.8.

(2) Let $D$ be the unique reducing disk in $V$. Let $SP_D(V)$ denote the full subcomplex of $D(V)$ spanned by the vertices corresponding to non-reducing semi-primitive disks. By Lemmas 1.9 and 1.10, the complex $SP_D(V)$ is the link of the vertex corresponding to $D$ in $D(V)$. It is shown in [4, 22] that the link of any vertex of $D(V)$ is a tree, and hence $SP_D(V)$ is a tree.

3. The complex of Haken spheres

Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for a closed orientable 3-manifold $M$. The complex $H$ of Haken spheres of the splitting $(V,W;\Sigma)$ is defined to be the simplicial complex whose vertices consists of the isotopy classes of Haken spheres such that the collection $P_0, P_1, \ldots, P_k$ of distinct $k + 1$ vertices spans a $k$-simplex if $|P_i \cap \Sigma \cap P_j| = 4$ for all $0 \leq i < j \leq k$. It is shown that the complex of Haken spheres of the genus two splitting for $S^3$ is connected by Scharlemann [26], and it turns out that the complex actually deformation retracts to a tree from the works [1] and [4]. Lei [20] and Lei-Zhang [21] showed that the complex of Haken spheres of the genus two splitting for a non-prime 3-manifold is connected. In this section, we refine the results of Lei and Lei-Zhang. That is, we show that the complexes of Haken spheres for non-prime 3-manifolds are connected in a new way, and further show that they are actually contractible. We use the results on the semi-primitive disk complexes developed in the previous section.

Theorem 3.1. Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Then the complex $H$ of Haken spheres of the splitting $(V,W;\Sigma)$ is contractible. The dimension of $H$ is 1, that is, $H$ is a tree, if $M_1$ is a lens space, and is 3 if $M_1$ is $S^2 \times S^1$.

Proof. Let us assume first that $M_1$ is a lens space. In Theorem 2.2 we have seen that the semi-primitive disk $SP(V)$ is a tree. Let $SP'(V)$ be the first barycentric subdivision of the tree $SP(V)$. The tree $SP'(V)$ is bipartite, of which we call the vertices of countably infinite valence (the vertices of the original $SP(V)$) the black vertices, and the vertices of valence 2 the white ones. By Lemma 1.7 the set of the white vertices one-to-one corresponds to the set of Haken spheres.
Let $D$ be a semi-primitive disk in $V$. We note that $D$ represents a black vertex of the tree $\mathcal{SP}'(V)$. By Lemma 1.5, there exists the unique semi-primitive disk $E'$ in $W$ disjoint from $D$. The set of white vertices in the link of $D$ in $\mathcal{SP}'(V)$ one-to-one correspond of the set of the Haken spheres disjoint from $D \cup E'$. Let $\Sigma'$ be the 4-holed sphere obtained by cutting $\Sigma$ off along $\partial D \cup \partial E'$. Let $d^+$ and $d^-$ ($e'^+\text{ and } e'^-$, respectively) be the two boundary circles of $\Sigma'$ coming from $\partial D$ ($\partial E'$, respectively). Let $\mathcal{H}_D$ be the full subcomplex of the complex $\mathcal{H}$ spanned by the vertices corresponding to Haken spheres disjoint from $D$. We assign each vertex of $\mathcal{H}_D$ an element of $\mathbb{Q} \cup \{\infty\}$ as in the following way. Fix a Haken sphere $P$ of $(V, W; \Sigma)$ disjoint from $D \cup E'$. Set $\mu = P \cap \Sigma'$ and fix a separating simple closed curve $\nu$ in $\Sigma'$ such that $\nu$ separates $d^+ \cup e'^+$ and $d^- \cup e'^-$, and that $|\mu \cap \nu| = 2$ after minimizing the intersection. Let $\tilde{\Sigma}'$ be the covering space of $\Sigma'$ such that

- the components of the preimage of $\mu$ (or $\nu$, respectively) are the vertical (horizontal, respectively) lines in the Euclidean plane;
- the set of components of the preimage of $\partial D$ correspond to the set of points whose coordinates consist of integers.

See Figure 3. For each arc connecting $d^+$ and $d^-$, we assign the slope $s/t \in \mathbb{Q} \cup \{\infty\}$ of its preimage with respect to the above covering map. Since the set of Haken spheres disjoint from $D \cup E'$ one-to-one corresponds to the set of simple arcs in $\Sigma'$ connecting $d^+$ and $d^-$ as in the proof of Lemma 1.4, the above assignment provides an assignment of each vertex of $\mathcal{H}_D$ to an element of $\mathbb{Q} \cup \{1/0\}$.

We now briefly review some well-known facts on the Farey complex. The Farey complex $\mathcal{F}$ is the flag complex whose vertices is $\mathbb{Q} \cup \{1/0\}$. Two vertices $s_1/t_1$ and $s_2/t_2$ are connected by an edge if and only if $s_1t_2 - s_2t_1 = \pm 1$. See the left-hand side of Figure 4. The assignment of each vertex of $\mathcal{H}_D(V)$ with an element of $\mathbb{Q} \cup \{1/0\}$ described above allows us to get an embedding of $\mathcal{H}_D(V)$ into $\mathcal{F}$. The image of $\mathcal{H}_D(V)$ is the full subcomplex $\mathcal{F}_{\text{odd}}$ of $\mathcal{F}$ spanned by $1/0$ and irreducible rational numbers having odd numerators. See the right-hand side of Figure 4. It is easy to check that $\mathcal{F}_{\text{odd}}$ is a tree.
It follows that there exists a natural simplicial isomorphism from $H$ to the simplicial complex obtained from $SP'(V)$ by replacing the star of each black vertex with the tree simplicially isomorphic to $F_{\text{odd}}$. See Figure 5. Consequently, $H$ is a tree.

Next, assume that $M_1 = S^2 \times S^1$. Recall that, by Lemma 1.9, there exists the unique reducing disk $D$ in $V$. Let $\Sigma_D$ be the 2-holed torus obtained by cutting $\Sigma$ along $\partial D$. Let $d^+$ and $d^-$ be the two components of $\partial \Sigma_D$. Let $A(\Sigma_D)$ be the simplicial complex whose vertices are isotopy classes of simple arcs in $\Sigma_D$ connecting $d^+$ and $d^-$ such that the collection of distinct $k+1$ vertices spans a $k$-simplex if they admits a set of pairwise disjoint representatives. Each simple arc $\alpha_P$ in $\Sigma_D$ connecting $d^+$ and $d^-$ determine a unique Haken sphere $P$ of $(V,W; \Sigma)$. By the uniqueness of $D$, this correspondence gives a simplicial isomorphism $A(\Sigma_D) \to H$. It is shown that $A(\Sigma_D)$ is a contractible 3-dimensional simplicial complex in [9, 27], and so is $H$.

We remark that the argument developed in [22] allows us to show easily that $H$ is also a tree for the genus two Heegaard splitting $(V,W; \Sigma)$ for $(S^2 \times S^1)\#(S^2 \times S^1)$.

In the remaining of this section, we analyze the action of the Goeritz group on the set of Haken spheres of genus two Heegaard splittings for later works.

Lemma 3.2. Let $(V,W; \Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces $L(p_1,q_1)$ and $L(p_2,q_2)$. For any two Haken spheres $P$ and $Q$ of $(V,W; \Sigma)$
with \(|P \cap \Sigma \cap Q| = 4\), there exists an element of the Goeritz group of \((V, W; \Sigma)\) that maps \(P\) to \(Q\).

**Proof.** The Haken sphere \(P\) cuts \(V\) into two solid tori \(V_1\) and \(V_2\), and \(W\) into \(W_1\) and \(W_2\). We may assume that \(V_1 \cup W_1\) and \(V_2 \cup W_2\) are punctured lens spaces. Let \(D\) and \(E\) be the meridian disks of \(V_1\) and \(V_2\), respectively, disjoint from \(P\). Similarly, let \(D'\) and \(E'\) be the meridian disks of \(W_1\) and \(W_2\), respectively, disjoint from \(P\).

**Claim.** Up to isotopy, \(Q\) is disjoint from \(D \cup E'\) or \(E \cup D'\).

**Proof of Claim.** Let \(C_0\) be an outermost sub-disk of the disk \(Q \cap V\) cut off by \(P \cap Q \cap V\), which is contained in either \(V_1\) or \(V_2\). Assume first that \(C_0\) is contained in \(V_1\). Then there exists exactly one more such a sub-disk \(C_1\) of \(Q \cap V\), and it is also contained in \(V_1\). Since \(|P \cap \Sigma \cap Q| = 4\), we have \(V_1 \cap Q = C_1 \cup C_2\), and hence \(Q\) is disjoint from \(D\). Further, if \(D_0\) is an outermost sub-disk of the disk \(Q \cap W\) cut off by \(P \cap Q \cap W\), then \(D_0\) must be contained in \(W_2\), otherwise \(\partial D\) would bound a meridian disk in \(W_1\), which forms a non-separating sphere with the disk \(D\) in the punctured lens space \(V_1 \cup W_1\), a contradiction. Further, by the same reason to the case of \(C_0\) and \(C_1\), there exists exactly one more sub-disk \(D_1\) of \(Q \cap W\), and it is also contained in \(W_2\). Thus \(Q\) is also disjoint from \(E'\). If \(C_0\) is contained in \(V_2\), then, by the same argument, \(Q\) is disjoint from \(E \cup D'\).

By the claim, we assume that \(Q\) is disjoint from \(D \cup E'\) without loss of generality. Let \(\Sigma'\) be the 4-holed sphere obtained by cutting \(\Sigma\) along \(\partial D \cup \partial E'\). Let \(d^+\) and \(d^-\) (\(e'^+\) and \(e'^-\), respectively) be the two boundary circles of \(\Sigma'\) coming from \(\partial D\) (\(\partial E'\), respectively). Then \(P \cap \Sigma'\) \((Q \cap \Sigma'\), respectively\) is the frontier of a regular neighborhood of the union of \(d^+\) and \(d^-\) and a simple arc \(\alpha_P\) (\(\alpha_Q\), respectively) in \(\Sigma'\) connecting \(d^+\) and \(d^-\). Since \(|P \cap \Sigma \cap Q| = 4\), we may assume that \(\alpha_P \cap \alpha_Q = \emptyset\). Set \(\mu = P \cap \Sigma'\) and let \(\nu\) be a simple closed curve in \(\Sigma\) such that \(\nu\) separates \(d^+ \cup e'^+\) and \(d^- \cup e'^-\). See Figure 6(i). We note that a half-Dehn twist about \(\mu\) extends to an orientation-preserving homeomorphism of \(L(p_1, q_1) \# L(p_2, q_2)\) that preserves \(V\). Up to a finite number of half-Dehn twists about \(\mu\) and isotopy, a single Dehn twist \(\tau_\nu\) about \(\nu\) maps \(\alpha_P\) to \(\alpha_Q\). See also Lemma 5.2. However, \(\tau_\nu\) extends to a homeomorphism of neither of \(V\) nor \(W\), since the algebraic intersection number of \(\partial E\) and \(\tau_\nu(\partial E)\) (\(\partial D'\) and \(\tau_\nu(\partial D')\), respectively) is \(\pm p_2^2\) (\(\pm p_1^2\), respectively), and so \(\tau_\nu(\partial E)\) (\(\tau_\nu(\partial D')\), respectively) no longer bounds a disk in \(V\) (\(W\), respectively). See Figure 6(ii). We compose \(\tau_\nu\) with a Dehn twist \(\tau_\partial D\) about \(\partial D\) and a Dehn twist \(\tau_{\partial E'}\) about \(\partial E'\) in such a way that the geometric intersection number of

![Figure 6](image-url)
\[\partial E \text{ and } \tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial E) (\partial D') \text{ and } \tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial D'), \text{ respectively} \text{ is zero. This implies that } \tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial E) \text{ is null-homologous in } V. \text{ We note that the Dehn twists } \\
\tau_{\partial D} \text{ and } \tau_{\partial E'} \text{ does not change the isotopy class of } Q \cap \Sigma. \text{ Since a handlebody contains no incompressible surface with a connected boundary except a disk, } \\
\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial E) \text{ bounds a disk in } V. \text{ Apparently, } \tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial D) \text{ and } \\
\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial E) \text{ bound disks in } V. \text{ Therefore by Alexander’s trick, this composition extends to a homeomorphism of } V. \text{ Similarly, } \tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu}(\partial D') \text{ bounds a disk in } W \text{ and hence this composition extends to a homeomorphism of } W. \text{ As a consequence, the map } \\
\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_{\nu} \text{ extends to an orientation-preserving homeomorphism of } L(p_1, q_1) \# L(p_2, q_2) \text{ that preserves } V. \]

Lemma 3.3. Let \(M_1\) be a lens space or \(S^2 \times S^1\), and let \(M_2\) be a lens space. Let \((V, W; \Sigma)\) be a genus two Heegaard splitting for \(M_1 \# M_2\). Then the Goeritz group of \((V, W; \Sigma)\) acts transitively on the set of Haken spheres of \((V, W; \Sigma)\).

Proof. The case where \(M_1\) is a lens space follows from Theorem 3.1 and Lemma 3.2. Assume \(M_1 = S^2 \times S^1\). Let \(P\) be a Haken sphere of \((V, W; \Sigma)\). Then \(P\) cuts \(V\) into two solid tori \(V_1\) and \(V_2\), and \(W\) into \(W_1\) and \(W_2\). We may assume that \(V_1 \cup W_1\) is a punctured \(S^2 \times S^1\). Let \(D\) and \(E\) be the meridian disks of \(V_1\) and \(V_2\), respectively, disjoint from \(P\). Similarly, let \(D'\) and \(E'\) be the meridian disks of \(W_1\) and \(W_2\), respectively, disjoint from \(P\). In this case, we may assume that \(\partial D = \partial D'\). As we have seen in Subsection 1.2, \(D\) is the unique reducing disk in \(V\). Let \(\Sigma_D\) be a 2-holed torus obtained by cutting \(\Sigma\) along \(\partial D\). We denote the boundary circles of \(\Sigma_D\) by \(d^+\) and \(d^-\). Then there exists a simple arc \(\alpha_P\) in \(\Sigma_D\) connecting \(d^+\) and \(d^-\) such that \(P \cap \Sigma_D\) is the frontier of a regular neighborhood of \(d^+ \cup \alpha_P \cup d^-\). Let \(Q\) be another Haken sphere of \((V, W; \Sigma)\).

By the same argument as above, there exists a simple arc \(\alpha_Q\) in \(\Sigma_D\) connecting \(d^+\) and \(d^-\) such that \(Q \cap \Sigma_D\) is the frontier of a regular neighborhood of \(d^+ \cup \alpha_Q \cup d^-\). Then there exists a homeomorphism \(\varphi\) of \(\Sigma_D\) defined by pushing \(d^+\) in such a way that \(\varphi\) maps \(\alpha_P\) to \(\alpha_Q\). See Figure 7. The homeomorphism \(\varphi\) extends to a slide of a foot of a handle of each of

\[
\begin{align*}
&d^+ \\
&d^- \\
&\Sigma_D
\end{align*}
\]

Figure 7. Pushing \(d^+\) along the guide arc maps \(\alpha_P\) to \(\alpha_Q\).

\[V \text{ and } W, \text{ and so } \varphi \text{ extends to a homeomorphism of } M_1 \# M_2 \text{ that preserves } V. \]

4. Classification of genus two Heegaard splittings

For \(i \in \{1, 2\}\), let \(M_i\) be a lens space \(L(p_i, q_i)\) or \(S^2 \times S^1\), and let \((V_i, W_i; \Sigma_i)\) be a genus one Heegaard splitting for \(M_i\). By 3, \(\Sigma_i\) is the unique Heegaard surface for \(M_i\) up to isotopy, and there exists an orientation-preserving homeomorphism of \(M_i\) that
reversible Haken sphere. By Lemma 3.3, if the splitting $(V, W; \Sigma)$ for $M_1 \# M_2$ is created by gluing $V_1$ and $V_2$ to obtain $V$, and $W_1$ and $W_2$ to obtain $W$, by an appropriate orientation-reversing map $\partial B_1 \to \partial B_2$ after removing the interiors of $B_1$ and $B_2$ from $M_1$ and $M_2$, respectively. Also, another genus two Heegaard splitting $(V', W'; \Sigma')$ for $M_1 \# M_2$ is created by gluing $V_1$ and $W_2$ to obtain $V'$, and $W_1$ and $V_2$ to obtain $W'$ in the same way. From [13], it is known that each genus two Heegaard surface for $M_1 \# M_2$ is one of the above two Heegaard surfaces $\Sigma$ and $\Sigma'$ modulo the homeomorphisms of $M_1 \# M_2$. However, it is shown in [2] that $\Sigma$ and $\Sigma'$ do not always coincide modulo homeomorphisms of $M$. In [23], genus two Heegaard surfaces for $L(p_1, q_1) \# L(p_2, q_2)$ modulo the homeomorphisms of $L(p_1, q_1) \# L(p_2, q_2)$ are classified when $p_1 = p_2$ as follows.

**Theorem 4.1** ([23]). Let $M$ be the connected sum of two lens spaces $L(p, q_1)$ and $L(p, q_2)$. Then there exists a unique genus two Heegaard surface for $M$ modulo homeomorphisms of $M$ if and only if $q_1^2 \equiv 1 \pmod{p}$, and two Heegaard surfaces otherwise.

The following is a generalization of Theorem 4.1 to the case of all non-prime 3-manifolds which admit genus two Heegaard splittings.

**Theorem 4.2.** Let $M$ be a connected sum of $M_1$ and $M_2$, where $M_i$ is a lens space $L(p_i, q_i)$ or $S^2 \times S^1$ for $i \in \{1, 2\}$. Then there exists a unique genus two Heegaard surface for $M$ modulo homeomorphisms of $M$ if and only if one of $M_i$ is $S^2 \times S^1$ or a lens spaces $L(p_i, q_i)$ with $q_i^2 \equiv 1 \pmod{p_i}$, and two Heegaard surfaces otherwise.

**Proof.** The sufficiency is trivial. Suppose that both of $M_i$ are lens spaces $L(p_i, q_i)$ with $q_i^2 \not\equiv 1 \pmod{p_i}$. Let $(V, W; \Sigma)$ and $(V', W'; \Sigma')$ be the two Heegaard splittings of $M = L(p_1, q_1) \# L(p_2, q_2)$ obtained from the genus one Heegaard splittings $(V_i, W_i; \Sigma_i)$ of $L(p_i, q_i)$, $i = 1, 2$, as described in the beginning of this section. The construction provides the Haken spheres $P$ and $P'$ for the splittings $(V, W; \Sigma)$ and $(V', W'; \Sigma')$, respectively. We give an orientation of $P$ and $P'$ so that the $L(p_1, q_1)$-summand lies in the negative side. Suppose that there exists a homeomorphism $f$ of $M$ that maps $\Sigma'$ to $\Sigma$. Then, by Lemma 3.3, there exists a homeomorphism $g$ of $M$ that preserves $\Sigma$ and that maps $f(P')$ to $P$. We may assume that $P$ and $g \circ f(P')$ have the same orientation. Moreover, we may assume that $g \circ f$ induces a homeomorphism of $L(p_1, q_1)$ that preserves $V_1$ and $W_1$. Since $q_1^2 \not\equiv 1 \pmod{p_1}$, $g \circ f$ is orientation-preserving. Now, $g \circ f$ induces an orientation-preserving homeomorphism of $L(p_2, q_2)$ that interchanges $V_2$ and $W_2$, contradicting $q_2^2 \equiv 1 \pmod{p_2}$. 

5. Genus two Goeritz groups

Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces. A Haken sphere $P$ of $(V, W; \Sigma)$ is said to be reversible if there exists an element $g$ of $\mathcal{G}$ fixing $P$ setwise such that $g$ restricted to $P$ is an orientation-reversing homeomorphism on $P$. We say that the splitting $(V, W; \Sigma)$ is symmetric if it admits a reversible Haken sphere. By Lemma 3.3 if the splitting $(V, W; \Sigma)$ admits a reversible Haken sphere, then every Haken sphere of $(V, W; \Sigma)$ is reversible.
For a genus two Heegaard splitting \((V, W; \Sigma)\) for the connected sum of two lens spaces, we fix the following notations throughout the section.

- Disjoint, non-parallel semi-primitive disks \(D\) and \(E\) in \(V\),
- the disjoint semi-primitive disks \(D'\) and \(E'\) in \(W\) such that \(D \cap E' = E \cap D' = \emptyset\) (such \(D'\) and \(E'\) are determined uniquely by Lemma 1.5),
- the Haken sphere \(P\) of \((V, W; \Sigma)\) disjoint from \(D \cup E\) (the existence and uniqueness of \(P\) follows from Lemma 1.7), and
- a Haken sphere \(Q_1\) \((Q_2, \text{ respectively})\) of \((V, W; \Sigma)\) disjoint from \(D \cup E'\) \((E \cup D', \text{ respectively})\) such that \(|P \cap \Sigma \cap Q_1| = 4\) \(|P \cap \Sigma \cap Q_2| = 4\), \(\text{ respectively}\) (the existence of \(Q_1\) and \(Q_2\) follows from the proof of Lemma 1.4).

See Figure 8. In the figure, the 4-holed sphere \(\Sigma'\) is obtained by cutting \(\Sigma\) along \(\partial D \cup \partial E'\) and the boundary circles \(d^+\) and \(d^-\) \((e^+\) and \(e^-, \text{ respectively})\) come from \(\partial D\) \((\partial E', \text{ respectively})\). By \(\alpha \in \mathcal{G}\), we denote the hyperelliptic involution of both \(V\) and \(W\). By \(\beta \in \mathcal{G}\) \((\tau \in \mathcal{G}, \text{ respectively})\), we denote the extension of a half-Dehn twist \((\text{Dehn twist}, \text{ respectively})\) about the disk \(P \cap V\) \((D, \text{ respectively})\). By \(\gamma \in \mathcal{G}\), we denote an element of order 2 that interchanges \(P\) and \(Q\) (the existence of this element is proved in Lemma 5.4). When \(P\) is reversible, we denote by \(\delta \in \mathcal{G}\) an element of order 2 that reverses \(P\).

Also, for a genus two Heegaard splitting \((V, W; \Sigma)\) for the connected sum of \(S^2 \times S^1\) and a lens space, we fix the following notations throughout the section.

- The reducing disk \(D\) in \(V\) and the disk \(D'\) in \(W\) bounded by \(\partial D\) \((D\) is unique by Lemma 1.9),
- disjoint, non-isotopic, semi-primitive disks \(E_1\) and \(E_2\) in \(V\),
- a semi-primitive disk \(E'\) in \(W\) such that \(\partial E'\) has the same type with respect to \(E_1\) and \(E_2\) \((\text{the existence of } E'\text{ follows from the proof of Lemma 5.6}), and
- Haken spheres \(P\) and \(Q\) of \((V, W; \Sigma)\) disjoint from \(D \cup E_1\) such that \(|P \cap \Sigma \cap Q| = 4\) \((\text{the existence of } P\text{ and } Q\text{ follows from the proof of Lemma 1.4}(1))\).

See Figure 9. In the figure, the 4-holed sphere \(\Sigma'\) is obtained by cutting \(\Sigma\) along \(\partial D \cup \partial E_1\) and the boundary circles \(d^+\) and \(d^-\) \((e_1^+\) and \(e_1^-\), \(\text{ respectively})\) come from \(\partial D\) \((\partial E_1, \text{ respectively})\). By \(\alpha \in \mathcal{G}\), we denote the hyperelliptic involution of both \(V\) and \(W\). By \(\beta \in \mathcal{G}\ \((\tau \in \mathcal{G}, \text{ respectively})\), we denote the extension of a half-Dehn twist \((\text{Dehn twist, respectively})\) about the disk \(P \cap V\) \((D, \text{ respectively})\). By \(\gamma \in \mathcal{G}\), we denote an element of order 2 that interchanges \(P\) and \(Q\) \((\text{the existence of this element is proved in Lemma 5.4})\).
By \(\sigma \in G\), we denote an element of order 2 that preserves \(E'\) and that interchanges \(E_1\) and \(E_2\) (the existence of this element will be proved in Lemma 5.6).

Now we are ready to state the main theorem, which provides presentations of genus two Goeritz groups of all non-prime 3-manifolds. (Recall that the genus two Goeritz group of \((S^2 \times S^1) \# (S^2 \times S^1)\) is the mapping class group of the genus two handlebody and its presentation is already known.)

**Theorem 5.1.** Let \(M_1\) be a lens space or \(S^2 \times S^1\), and let \(M_2\) be a lens space. Let \((V, W; \Sigma)\) be a genus two Heegaard splitting for \(M_1 \# M_2\). Then the Goeritz group \(G\) of \((V, W; \Sigma)\) has the following presentation:

1. If \(M_1\) is a lens space,
   - (a) \(\langle \alpha \mid \alpha \rangle \oplus \langle \beta, \gamma_1, \gamma_2 \mid \gamma_1^2, \gamma_2^2 \rangle\) if \((V, W; \Sigma)\) is not symmetric;
   - (b) \(\langle \alpha \mid \alpha \rangle \oplus \langle \beta, \gamma_1, \delta \mid \gamma_1^2, \delta^2, \delta \beta \delta = \alpha \beta \rangle\) if \((V, W; \Sigma)\) is symmetric;

2. If \(M_1 = S^2 \times S^1\), \(\langle \alpha \mid \alpha \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma_2^2, \sigma^2 \rangle \oplus \langle \tau \rangle\).

We remark that, from Section 4, once a genus two Heegaard splitting for \(M = L(p_1, q_1) \# L(p_2, q_2)\) is given, we may easily determine whether the splitting is symmetric or not. If \(L(p_1, q_1) \not\cong L(p_2, q_2)\) (as oriented manifolds), no genus two Heegaard splitting of \(M\) is symmetric. If \(L(p_1, q_1) \cong L(p_2, q_2)\), exactly one genus two Heegaard splitting of \(M\) is symmetric and the other, if any, is not.

Throughout the section, for suitable subsets \(A_1, A_2, \ldots, A_k\) of \(M_1 \# M_2\), we denote by \(G_{\{A_1, A_2, \ldots, A_k\}}\) the subgroup of its Goeritz group \(G\) consisting of elements that preserve each of \(A_1, A_2, \ldots, A_k\) setwise.

**Lemma 5.2.** Let \(M_1\) be a lens space or \(S^2 \times S^1\), and let \(M_2\) be a lens space. Let \((V, W; \Sigma)\) be a genus two Heegaard splitting for \(M_1 \# M_2\).

1. If \(M_1\) is a lens space, then \(G_{\{D, P\}} = \langle \alpha \mid \alpha \rangle \oplus \langle \beta \rangle\).
2. If \(M_1 = S^2 \times S^1\), then \(G_{\{D, P\}} = \langle \alpha \mid \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \tau \rangle\).

**Proof.** Let \(g\) be an element of \(G_{\{D, P\}}\).

1. Since \(g\) preserves \(D\), \(g\) is orientation-preserving on \(P\). We may assume that \(g\) maps each of the disks \(D, D', E\) and \(E'\) to itself. Moreover if \(g\) is orientation-preserving on \(D\) (\(E\), respectively), then so is on \(D'\) (\(E'\), respectively). Hence by taking a composition with \(\alpha\) and \(\beta\), if necessary, we may assume that \(g\) fixes \(D \cup D' \cup E \cup E'\). Now, \(\Sigma\) cut off by \(D \cup D' \cup E \cup E'\) consists of several disks and a single annulus. By Alexander’s trick,
boundary-preserving homeomorphisms on a disk is unique up to isotopy. Also, boundary-preserving homeomorphisms on an annulus are determined by Dehn twist about its core circle up to isotopy. This implies that $g$ is a power of $\beta$.

(2) Let $l$ be a simple closed curve in $\Sigma$ disjoint from $P$ that intersects $\partial D$ in a single point. Let $g$ be an element of $\mathcal{G}_{(D,P)}$. Since $g$ preserves $D$, $g$ is orientation-preserving on $P$. We may assume that $g$ maps each of the disks $D$, $D'$, $E$ and $E'$ to itself. Moreover if $g$ is orientation-preserving on $D$ ($E$, respectively), then so is on $D'$ and $l$ ($E'$ and $l$, respectively). Hence modulo the action of $\alpha$ and $\tau$, $g$ fixes $D \cup D' \cup l \cup E \cup E'$. The remaining argument is exactly the same as (1).

**Lemma 5.3.** Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

1. Suppose that $M_1$ is a lens space. Let $Q'_1$ be a Haken sphere of $(V,W;\Sigma)$ disjoint from $D \cup E'$ such that $|P \cap \Sigma \cap Q'_1| = 4$. Then a power of $\beta$ maps $Q_1$ to $Q'_1$.

2. Suppose that $M_1$ is $S^2 \times S^1$. Let $Q'$ be a Haken sphere of $(V,W;\Sigma)$ disjoint from $D \cup E$ such that $|P \cap \Sigma \cap Q'| = 4$. Then a power of $\beta$ maps $Q$ to $Q'$.

**Proof.** (1) Let $\Sigma'$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial D \cup \partial E'$. Let $d^+$ and $d^-$ ($e^+$ and $e^-$, respectively) be the two boundary circles of $\Sigma'$ coming from $\partial D$ ($\partial E'$, respectively). Let $\alpha_P$, $\alpha_Q$, and $\alpha_Q'$ be simple arcs in $\Sigma'$ connecting $d^+$ and $d^-$ such that the frontiers of regular neighborhoods of $d^+ \cup \alpha_P \cup d^-$, $d^+ \cup \alpha_Q \cup d^-$, and $d^+ \cup \alpha_Q' \cup d^-$ are $P \cap \Sigma$, $Q \cap \Sigma$, and $Q' \cap \Sigma$, respectively. We may assume that $\alpha_P \cap \alpha_Q = \alpha_P \cap \alpha_Q' = \emptyset$ since $|P \cap \Sigma \cap Q| = |P \cap \Sigma \cap Q'| = 4$. Since $\alpha_P$ cuts $\Sigma'$ into a pair of pants, a certain power of $\beta$ carries $\alpha_Q$ to $\alpha_Q'$. The proof of (2) is exactly the same as (1).

**Lemma 5.4.** Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

1. If $M_1$ is a lens space, then $\mathcal{G}_{(D,P,Q)} = \langle \alpha | \alpha^2, \gamma \rangle$, and $\mathcal{G}_{(D,P,Q)} = \langle \alpha | \alpha^2 \rangle \oplus \langle \gamma | \gamma^2 \rangle$.

2. If $M_1$ is $S^2 \times S^1$, then $\mathcal{G}_{(D,P,Q)} = \langle \alpha | \alpha^2 \rangle \oplus \langle \tau \rangle$, and $\mathcal{G}_{(D,P,Q)} = \langle \alpha | \alpha^2 \rangle \oplus \langle \gamma | \gamma^2 \rangle \oplus \langle \tau \rangle$.

**Proof.** (1) We first show the existence of the element $\gamma_1 \in \mathcal{G}$. Let $\beta'_1$ denote a half-Dehn twist about the sphere $Q_1$. By Lemma 5.3 there exists an element $g \in \mathcal{G}$ that carries $P$ to $Q_1$. We may assume without loss of generality that $g$ maps $D$ to $D$ and $E'$ to $E'$. By Lemma 5.3 a certain power $\beta''_1$ of $\beta'_1$ carries $g(Q_1)$ to $P$. We remark that $\beta''_1 \circ g$ interchanges $P$ and $Q_1$ and this map carries $D$ to $D$ and $E'$ to $E'$. Up to isotopy, we may assume that $(\beta''_1 \circ g)^2$ fixes $D \cup E' \cup P \cup Q_1$. Then by cutting $\Sigma$ along $\partial D \cup \partial E'$ and considering simple arcs connecting the two holes coming from $\partial D$ as in the proof of Lemma 5.3 we can easily check that $(\beta''_1 \circ g)^2$ restricted to $\Sigma$ is a power of Dehn twist along $\partial E'$. Hence $(\beta''_1 \circ g)^2$ is isotopic to the identity. This implies that $\beta''_1 \circ g$ is the required element $\gamma_1$. Since $\tau$ is commutative with any element of $\mathcal{G}$ that preserves $D$, (2) follows from the same argument as (1).

**Lemma 5.5.** Let $M_1$ be a lens space or $S^2 \times S^1$, and let $M_2$ be a lens space. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Let $D$ be a semi-primitive disk in $V$. 
(1) If $M_1$ is a lens space, then $G_{(D)} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 \rangle$.

(2) If $M_1$ is $S^2 \times S^1$, then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 \rangle \oplus \langle \tau \rangle$.

Proof. (1) By Lemma 1.5, $E'$ is the unique non-separating disk in $W$ disjoint from $D$. This implies that each element of $G_D$ preserves $E'$. Let $\Sigma'$ be the 4-holed sphere obtained by cutting $\Sigma$ along $\partial D \cup \partial E'$. Let $d^+$ and $d^-$ ($e^+$ and $e^-$, respectively) be the two boundary circles of $\Sigma'$ coming from $\partial D$ ($\partial E'$, respectively). As in the proof of Theorem 5.1, let $H_D$ be the full subcomplex of the complex $H$ of Haken spheres of $(V,W;\Sigma)$ spanned by the vertices corresponding to Haken spheres disjoint from $D$. Then $H_D$ is a tree as we have seen in Lemma 3.1. Let $D, E$ be disjoint, non-isotopic, semi-primitive and non-reducing disks in $V$. Then there exists an element of the Gorenst group $g$ of the Heegaard splitting $(V,W;\Sigma)$ that preserves $E_1$ and $E_2$.

Lemma 5.6. Let $(V,W;\Sigma)$ be the genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. Let $E_1$ and $E_2$ be disjoint, non-isotopic, semi-primitive and non-reducing disks in $V$. Then there exists an element of the Goeritz group $G$ of the Heegaard splitting $(V,W;\Sigma)$ that interchanges $E_1$ and $E_2$.

Proof. It is easy to see that there exists a non-reducing semi-primitive disk $\hat{E}_2$ in $V$ such that $E_1$ and $\hat{E}_2$ can be interchanged by an element of $G$. Thus it suffices to show that there exists an element of $G$ that preserves $E_1$ and that maps $E_2$ to $\hat{E}_2$. Let $\Sigma_D$ be a 2-holed torus obtained by cutting $\Sigma$ along $\partial D$. We denote the boundary circles of $\Sigma_D$ by $d^+$ and $d^-$. Since both $E_2$ and $\hat{E}_2$ are meridian disks of the solid torus obtained by cutting $V$ along $D$, there exists a pushing of $d^+$ in $\Sigma_D$ that preserve $\partial E_1$, and that maps $\partial E_2$ to $\partial \hat{E}_2$. As we have seen in Lemma 3.3, every pushing map of $d^+$ extends to a slide of a foot of a handle of each of $V$ and $W$, thus it extends to a homeomorphism of $(S^2 \times S^1)\# L(p,q)$ that preserves $V$. 

Finally, the following two lemmas follow from Lemmas 1.7 and 5.1.

Lemma 5.7. Let $(V,W;\Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces.

(1) If $(V,W;\Sigma)$ is not symmetric, then $G_{(D,E)} = G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$.

(2) If $(V,W;\Sigma)$ is symmetric, then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$. Let $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$.

Lemma 5.8. Let $(V,W;\Sigma)$ be the genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. Then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$ and $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$.

Proof of Theorem 5.1. By Theorem 2.2, $SP(V)$ is a tree. By Lemmas 3.3, the vertices modulo the action of $G$ consists of two classes, one contains $D$ and the other contains $E$. Also, any edge of $SP(V)$ is equal to the edge $\{D, E\}$ modulo the action of $G$. Theorem 5.1 is proved.

Proof of Theorem 5.2. If $(V,W;\Sigma)$ is symmetric, then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$ and $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$. Let $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$.

Proof of Theorem 5.3. If $(V,W;\Sigma)$ is not symmetric, then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle$. Let $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle$.

Proof of Theorem 5.4. If $(V,W;\Sigma)$ is symmetric, then $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$ and $G_{(D,E)} = \langle \alpha \mid \alpha^2 \rangle$.
Therefore the quotient of $\mathcal{SP}(V)$ by the action of $\mathcal{G}$ is an edge. Now by the Bass-Serre theory and Lemmas 5.5 and 5.7, we have

$$
\mathcal{G} = \mathcal{G}_{\{D\}} * \mathcal{G}_{\{D,E\}} \mathcal{G}_{\{E\}}
$$

(by Theorem 2.2), $\mathcal{G}_{\{D\}}$ is a tree. Let $\mathcal{SP}'(V)$ be the first barycentric subdivision of $\mathcal{SP}(V)$. By Lemma 5.6 the vertices of $\mathcal{SP}'(V)$ consists of the vertices of $\mathcal{SP}(V)$ and the barycenters of the edges of $\mathcal{SP}(V)$, each of which corresponds to an unordered pair of vertices. By Lemmas 5.6 and 5.7, every vertex of $\mathcal{SP}'(V)$ is equal to the vertex $D$ or the barycenter $\{D, E\}$, and any edge of $\mathcal{SP}'(V)$ is equal to the edge $\{D, \{D, E\}\}$ modulo the action of $\mathcal{G}$. Therefore the quotient of $\mathcal{SP}'(V)$ by the action of $\mathcal{G}$ is an edge.

By the Bass-Serre theory and Lemmas 5.5 and 5.7, we have

$$
\mathcal{G} = \mathcal{G}_{\{D\}} * \mathcal{G}_{\{D,E\}} \mathcal{G}_{\{D,E\}}
$$

(by Theorem 2.2), $\mathcal{G}_{\{D\}}$ is a tree. Let $\mathcal{SP}'(V)$ be the first barycentric subdivision of $\mathcal{SP}(V)$. By Lemma 5.6 the quotient of $\mathcal{SP}'(V)$ by the action of $\mathcal{G}$ consists of an edge. By the Bass-Serre theory and Lemmas 5.5 and 5.7, we have

$$
\mathcal{G} = \mathcal{G}_{\{D\}} * \mathcal{G}_{\{D,E\}} \mathcal{G}_{\{D,E\}}
$$

This completes the proof.

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References

[1] Akbas, E., A presentation for the automorphisms of the 3-sphere that preserve a genus two Heegaard splitting, Pacific J. Math. 236 (2008), 201–222.

[2] Birman, J., On the equivalence of Heegaard splittings of closed, orientable 3-manifolds, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), pp. 137–164, Ann. of Math. Studies 84, Princeton Univ. Press, Princeton, N.J., 1975.

[3] Bonahon, F., Otal, J.-P., Scindements de Heegaard des espaces lenticulaires, Ann. Sci. Éc. Norm. Sup. (4) 16 (1983), 451–466.

[4] Cho, S., Homeomorphisms of the 3-sphere that preserve a Heegaard splitting of genus two, Proc. Amer. Math. Soc. 136 (2008), 1113–1123.

[5] Cho, S., Genus two Goeritz groups of lens spaces, to appear in Pacific J. Math.

[6] Cho, S., Koda, Y., Primitive disk complexes for lens spaces, arXiv:1206.6243
[7] Cho, S., Koda, Y., The genus two Goeritz group of $S^2 \times S^1$. [arXiv:1303.7145]
[8] Cho, S., Koda, Y., Connected primitive disk complexes and Goeritz groups of lens spaces, in preparation.
[9] Cho, S., McCullough, D., Seo, A., Arc distance equals level number, Proc. Amer. Math. Soc. 137 (2009), 2801–2807.
[10] Goeritz, L., Die Abbildungen der Brezeläbre und der Vollbrezel vom Geschlecht 2. Abh. Math. Semin. Hamb. Univ. 9 (1933), 244–259.
[11] Gordon, C., Problems. Workshop on Heegaard Splittings, pp. 401–411, Geom. Topol. Monogr. 12, Geom. Topol. Publ., Coventry, 2007.
[12] Grasse, P., Finite presentation of mapping class groups of certain three-manifolds, Topology Appl. 32 (1989) 295–305.
[13] Haken, W., Some results on surfaces in 3-manifolds. Studies in Modern Topology, pp. 39–98, Math. Assoc. Amer., distributed by Prentice-Hall, Englewood Cliffs, N.J., 1968.
[14] Hempel, J., 3-manifolds as viewed from the curve complex, Topology 40 (2001), 631–657.
[15] Johannson, K., Topology and combinatorics of 3-manifolds, Lecture Notes in Mathematics 1599, Springer-Verlag, Berlin, 1995.
[16] Johnson, J., Mapping class groups of medium distance Heegaard splittings, Proc. Amer. Math. Soc. 138 (2010), 4529–4535.
[17] Johnson, J., Heegaard splittings and open books. [arXiv:1108.6302]
[18] Johnson, J., Mapping class groups of Heegaard splittings of surface bundles. [arXiv:1201.2628]
[19] Koda, Y., Automorphisms of the 3-sphere that preserve spatial graphs and handlebody-knots, [arXiv:1106.4777]
[20] Lei, F., Haken spheres in the connected sum of two lens spaces, Math. Proc. Camb. Philos. Soc. 138 (2005), 97–105.
[21] Lei, F., Zhang, Y., Haken spheres in genus 2 Heegaard splittings of nonprime 3-manifolds, Topology Appl. 142 (2004) 101–111.
[22] McCullough, D., Virtually geometrically finite mapping class groups of 3-manifolds, J. Diff. Geom. 33 (1991), 1–65.
[23] Montesinos, J., Safont, C., On the Birman invariants of Heegaard splittings, Pacific J. Math. 132 (1988), 113–142.
[24] Namazi, H., Big Heegaard distance implies finite mapping class group, Topology Appl. 154 (2007), 2939–2949.
[25] Osborne, R. P., Zieschang, H., Primitives in the free group on two generators, Invent. Math. 63 (1981), no. 1, 17–24.
[26] Scharetgmann, M., Automorphisms of the 3-sphere that preserve a genus two Heegaard splitting, Bol. Soc. Mat. Mexicana 10 (2004), 503–514.
[27] Seo, A., Torus leveling of (1,1)-knots, dissertation at the University of Oklahoma (2008).
[28] Seo, J.-P., Arbres, amalgames, SL_2, Astérisque 46, Société Mathématique de France, Paris, 1977.
[29] Suzuki, S., On homeomorphisms of a 3-dimensional handlebody, Can. J. Math. 29 (1977), 111–124.
[30] Wajnryb, B., Mapping class group of a handlebody, Fund. Math. 158 (1998), 195–228.

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