Dissecting the qutrit

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Received 3 September 2012, in final form 21 November 2012
Published 21 December 2012
Online at stacks.iop.org/JPhysA/46/035306

Abstract
To visualize a higher dimensional object it is convenient to consider its two-dimensional cross-sections. The set of quantum states for a three level system has eight dimensions. We supplement a recent paper by Goyal et al (arXiv: 1111.4427) by considering the set of all possible two-dimensional cross-sections of the qutrit. Each such cross-section is bounded by a plane cubic curve.

PACS numbers: 03.65.-w, 03.65.Fd
(Some figures may appear in colour only in the online journal)

1. Introduction
The set of all quantum states, pure and mixed, is a very interesting object. We obviously want to know what it looks like. For a qubit the set of all quantum states is a ball, but the case of a qutrit is already complicated [1]. It is an eight-dimensional convex body. A standard way to visualize higher dimensional sets is to consider two-dimensional cross-sections, and preferably an intelligently organized Grand Tour of such 2-sections—meaning that one moves in the set of 2-sections along some curve that eventually comes close to any given 2-section [2]. Of course the idea to look at 2-sections has been pursued for the qutrit, typically by considering the finite set of cross-sections generated by the Gell-Mann matrices [3–8]. An elegant and comprehensive treatment is due to Goyal et al [9]. However, by concentrating on cross-sections generated by the Gell-Mann matrices one obtains only 28 2-sections altogether, or 56 three-dimensional 3-sections. This is quite far from a Grand Tour. On the other hand we are looking at a fairly symmetric eight-dimensional body, left invariant by an SU(3) subgroup of the rotation group SO(8), and this cuts down the size of the problem. Our purpose here is to supplement the work of Goyal et al by making this idea precise. In section 4 this will enable us to see how their selection samples the full set of 2-sections, because we will be able to visualize the set of all unitarily inequivalent two-dimensional cross-sections of the state space of the qutrit. In section 5 we will consider the shape of a general 2-section and the plane cubic curves that bound them.

Because of the duality between sections and projections of a self-dual set, a description of two-dimensional cross-sections can be translated to results on the shapes of the numerical
ranges of complex matrices \[10\]. The numerical range is an interesting tool which has found some applications in quantum information theory \[11\]. This gives some special interest to the study of 2-sections, as opposed to 3-sections. We discuss this topic in section 6.

2. Preliminaries

A 2-section of an object in an eight-dimensional vector space is its intersection with a 2-plane through the origin—which will be placed at the maximally mixed qutrit state in our case. The modeling linear subspace for such a plane is a two-dimensional subspace \(V\) of the real vector space of traceless \(3 \times 3\) Hermitian matrices equipped with the Hilbert–Schmidt scalar product

\[
M_1 \cdot M_2 = \frac{1}{2} \text{Tr} M_1 M_2 .
\]

The factor of one half is a useful convention ensuring that the distance between two orthogonal pure states equals one. The set of quantum states is the convex set of traceless Hermitian matrices \(\rho\) such that the density matrix \(\rho = \frac{1}{3} I_3 + M\) is positive. Its boundary consists of matrices with at least one zero eigenvalue. It is inscribed in a minimal sphere of radius \(R_{\text{out}}\) and contains a sphere of maximal radius \(R_{\text{in}}\), where

\[
R_{\text{out}} = \frac{1}{\sqrt{3}}, \quad R_{\text{in}} = \frac{1}{\sqrt{2}} R_{\text{out}} .
\]

\(R_{\text{out}}\) is the distance between the maximally mixed state \(I_3/3\) and any pure state, while \(R_{\text{in}}\) is the distance between \(I_3/3\) and any density matrix with two eigenvalues equal to 1/2.

We need to know the dimension of the set of all 2-sections, or equivalently of the set of all two-dimensional subspaces of a real eight-dimensional space. This set is also known as the Grassmannian \(\text{Gr}(2, 6)\). Its dimension is easily found.

**Fact 2.1.** The set of orthonormal bases in \(\mathbb{R}^n\) is isomorphic to the orthogonal group \(O(n)\).

**Fact 2.2.** The dimension of the group \(O(n)\) is \(\frac{n(n-1)}{2}\).

**Fact 2.3.** The set of two-dimensional subspaces of an eight-dimensional real vector space has 12 dimensions.

**Proof.** Any two-dimensional subspace can be obtained by choosing an orthonormal basis such that its first two vectors span the subspace, and the remaining six its orthogonal complement. But we obtain the same subspace if we rotate the bases within the subspace and within its complement. Hence the dimension of the set of 2-planes equals the dimension of \(O(8)\) minus the dimension of the subgroup \(O(2) \times O(6)\). So the answer is \(8 \cdot 7/2 - 2 \cdot 1/2 - 6 \cdot 5/2 = 12\).

A general density matrix is given by

\[
\rho = \frac{1}{3} I_3 + M ,
\]

where \(I_3/3\) is the maximally mixed state and \(M\) is a traceless Hermitian matrix chosen such that all eigenvalues of \(\rho\) are non-negative. The group \(SU(3)\) acts on our vector space through \(M \rightarrow U M U^\dagger\). This is an eight-dimensional subgroup of \(SO(8)\), the group of all rotations there. Since the action of \(SU(3)\) leaves the set of density matrices invariant, all sections with subspaces related by this action will be regarded as equivalent. Using the fact that the action of \(SU(3)\) on a generic 2-section has at most a 3-element discrete stabilizer\(^3\), we have arrived at

**Fact 2.4.** The dimension of the set of inequivalent 2-sections of the set of states of the qutrit has only \(12 - 8 = 4\) dimensions.

Equivalent 2-sections have the same shape, but the converse does not hold \[9\].

\(^3\) The generic two-dimensional subspace of traceless matrices contains at least one element of full rank; hence, it contains at least one and at most three elements of rank two. The action \(M \rightarrow U M U^\dagger\) does not change the rank, so if it leaves the subspace invariant, the operation \(U\) has order at most 3. See the next section for details.
3. Representatives of 2-sections

Let $V$ be a two-dimensional subspace of the real vector space of traceless Hermitian matrices. We begin with a simple observation.

**Lemma 1.** The subspace $V$ contains no elements of rank 1 and at least one element of rank 2.

**Proof.** Assume that the subspace $V$ is spanned by the elements $A$ and $B$. Because it is traceless a non-zero element of $V$ is either of rank 2 or of rank 3. Then assume that $A$ is of rank 3 (otherwise the proof is complete). Consider the determinant of a linear combination:

$$\det(\lambda A + B) = 0.$$ 

Because $A$ and $B$ are Hermitian, this is a real polynomial of order 3. It has at least one real root. If $\lambda$ is this root then the combination is of rank two. □

Since we are interested in picturing the set of 2-sections up to unitary equivalence, we use lemma 1 to introduce a standard form for a basis in the 2-plane. We choose the first basis vector $A$ to have rank 2 and then we choose the basis of $\mathbb{C}^3$ so that $A$ is diagonal. This fixes the basis in $\mathbb{C}^3$ up to diagonal unitary transformations, and we use this freedom to standardize the form of the orthogonal basis vector $B$. In this way we arrive at the following basis for the 2-plane:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} k & ae^{i\phi} & be^{i\phi} \\ ae^{-i\phi} & k & ce^{i\phi} \\ be^{-i\phi} & ce^{-i\phi} & -2k \end{bmatrix} \quad \text{if } abc \neq 0 \quad (4)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} k & a & b \\ a & k & c \\ b & c & -2k \end{bmatrix} \quad \text{if } abc = 0, \quad (5)$$

where we may assume that $a, b, c, k \geq 0$. We also define $d = \sqrt{3k}$. The matrices $A$ and $B$ form an orthonormal basis if and only if

$$3k^2 + a^2 + b^2 + c^2 = a^2 + b^2 + c^2 + d^2 = 1. \quad (6)$$

The set of possible tuples $(a, b, c, d)$ form 1/16 of a three-dimensional sphere, and so it is homeomorphic to a three-dimensional simplex. The set of matrices $B$ contains one representative of each unitary equivalence class of 2-sections of the set of qutrit states. We can think of it as a Cartesian product of a simplex and a circle associated with the angle $\phi$, except that the circles shrink to points at three of the four faces of the simplex.

Unfortunately, some ambiguities remain because there may be more than one matrix of rank 2 in the 2-plane, and one can then perform a unitary transformation so that another matrix takes our standard form $A$. In particular, given a matrix of rank 2 its negative also has rank 2. If we leave the origin in the direction of a rank 2 matrix, then we will hit the boundary of the set of states at a density matrix of spectrum $(2/3, 1/3, 0)$. Such a density matrix lies on a sphere of radius $R_2$, equal to its distance from the maximally mixed state. Since we also know the radii of the insphere and the outsphere (from equation (2)) we obtain

$$R_{in} < R_2 = \frac{1}{\sqrt{6}} < R_{out}. \quad (7)$$

Goyal et al [9] refer to this as the self-dual sphere, because a 2-section containing a point on this sphere will also contain its antipodal point. But this contradicts another and more common use of the word self-dual, to be discussed in section 6. Anyway the point is that the boundary of the set of quantum states intersects the sphere of radius $R_2$ in antipodal points.
Now a permutation of the first and the second vector of the basis of the Hilbert space will change the sign of $A$ and exchange the parameters $b, c$ in $B$. Thus, we are led to identify pairs of points related to each other by reflection with respect to the surface $b = c$. We reduce then the set of representatives to the halfsimplex HS of matrices $B$ where $b \geq c$ (times the circle, which shrinks to a point at two of the faces of the halfsimplex).

To see if there are further ambiguities we solve the equation $\det(\lambda A + B) = 0$. By definition of $A$ one solution is for $\lambda = \infty$. The other solutions are the roots of

$$2k \cdot \lambda^2 + \lambda (b^2 - c^2) + 2abc \cos \phi - 2k^3 + k(2a^2 - b^2 - c^2) \, .$$

(8)

First consider the situation when all elements of the subspace have rank 2. It happens if and only if $k = 0$, $b = c$ and $abc \cos \phi = 0$. The resulting 2-section is always a circular disk of radius $R_2$. As noted by Goyal et al (in a special case) these 2-sections are not unitarily equivalent even though they have the same shape. To see this, observe that

$$\frac{1}{2} \text{Tr}(AB - BA)^2 = 4a^2 + b^2 + c^2 = 3(a^2 - k^2) + 1 \, .$$

(9)

Once we have restricted ourselves to $k = 0$ it follows that the value of $a$ cannot be changed by changing the basis in the 2-plane $V$, so no ambiguities arise in this case.

If the discriminant of (8) is greater than or equal to zero, then we have a discrete ambiguity in the choice of our matrix $A$, and this does give rise to discrete ambiguities in our parametrization. In particular one can check that

$$(k, a, b, c) = \left(0, 0, \cos \frac{\theta}{2}, \sin \frac{\theta}{2}\right) \quad \text{and} \quad (k, a, b, c, \phi) = \left(\cos \frac{\theta}{2}, \cos \frac{\theta}{2}, \frac{\sin \theta}{\sqrt{2}}, \frac{\sin \theta}{\sqrt{2}}, 0\right)$$

(10)

correspond to equivalent 2-sections. This is so because the unitary transformation that transforms the representative $B$ of the first 2-section to the standard form $A$ also transforms $A$ to the representative $B$ of the second 2-section. In general the discriminant of equation (8) vanishes if either $k = 0$ or a complicated $\phi$-dependent condition holds. We decided to ignore this difficulty.

We get the following.

**Theorem 1.** Any two-dimensional section of the space of traceless Hermitian matrices in $\mathbb{C}^3$ can be represented after appropriate change of basis of $\mathbb{C}^3$ as a subspace spanned by matrices of the form given in equations (4–5), where $a^2 + b^2 + c^2 + 3k^2 = 1$, $b \geq c$ and $\phi$ is arbitrary. Topologically this set forms a simplex times a circle, with the circles shrinking to points at two of the faces of the simplex (when $abc = 0$). The parametrization determines the set of unitarily equivalent 2-sections uniquely except for discrete ambiguities that occur if the 2-section contains exactly four or exactly six traceless matrices of rank 2.

Because we impose the condition $b \geq c$ we refer to this as a halfsimplex (although it is a simplex in itself). It looks as in figure 1. All points not contained in the dashed area (with its boundary) have an additional degree of freedom—the angle $\phi$. To visualize this four-dimensional set, first deform the half-simplex homeomorphically (not diffeomorphically) such that all dashed area lie on one two-dimensional surface, and the surface $b = c$ is orthogonal to it. Now reduce the dimension of such a deformed simplex as in figure 2. Next we rotate in four dimensions using rotations that leave the surface $a = 0 \lor c = 0$ invariant. In the reduced picture this is related to rotation around the axis $y$. After rotation one gets figure 3.
Figure 1. Halfsimplex. All points not contained in the dashed area (with its boundary) has an additional degree of freedom $\phi$.

Figure 2. Halfsimplex deformed and represented in two dimensions. All points from the dashed area are marked as red (the vertical segment).

Figure 3. Deformed halfsimplex after rotation.
4. Sections spanned by Gell-Mann matrices

Goyal et al [9] consider 2-sections spanned by pairs of the eight Gell-Mann matrices, and illustrate them beautifully. The Gell-Mann matrices form an orthonormal basis of the vector space of traceless Hermitian matrices, and are defined by

$$\sum_{i=1}^{8} x_i \lambda_i = \begin{bmatrix} x_8/\sqrt{3} + x_3 & x_1 - ix_2 & x_4 - ix_5 \\ x_1 + ix_2 & x_8/\sqrt{3} - x_3 & x_6 - ix_7 \\ x_4 + ix_5 & x_6 + ix_7 & -2x_8/\sqrt{3} \end{bmatrix}. \quad (11)$$

Matrices $\lambda_1, \ldots, \lambda_7$ are of rank 2 and are unitarily equivalent to our matrix $A = \lambda_3$. This basis is also a basis for the Lie algebra of $SU(3)$, which is helpful for the parametrization of density matrices [12].

To check how a section spanned by a pair of Gell-Mann matrices is represented we bring one of them to the standard form $\lambda_3$ and call it $A$. The same operation brings then the other matrix to the form $B$, for some choice of the parameters that define $B$. This matrix is the representative of the 2-section, and to see the shape of its boundary we just calculate the determinant of $I_3/3 + xA + yB$ and set it to zero. We refer to Goyal et al [9] for illustrations of the various cases. Calculating the representative $B$ in this way one obtains the following.

- For pairs 12, 13, 23, 45, 67, the representative $B$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

The determinant of $I_3/3 + xA + yB$ is equal to $\frac{1}{3}(\frac{1}{3} - x^2 - y^2)$ and the shape of intersection is a circular disk. As a side remark we observe that the 4-section given by the quartet 1245 (say) is a round ball.

- For pairs 14, 15, 16, 17, 24, 25, 26, 27, 46, 47, 56, 57, the representative $B$ is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$  

The determinant of $I_3/3 + xA + yB$ is equal to $\frac{1}{3} - \frac{1}{6}y^2 - \frac{1}{2}x^2$ and the shape of the intersection is a circular disk, unitarily inequivalent to the above.

- For pairs 34, 35, 36, 37, the representative $B$ is

$$\frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

The determinant of $I_3/3 + xA + yB$ is equal to, respectively, $\left(\frac{1}{3} + \frac{1}{3}y - x^2\right)\left(\frac{1}{3} - y\right)$ and $\left(\frac{1}{3} - x\right)\left(\frac{1}{3} + \frac{1}{3}x - y^2\right)$ and the shape of the intersection is a parabola closed with a line segment. Note that the discrete ambiguity in our parametrization turns up here; see equation (10).

- For pairs 18, 28, 38, the representative $B$ is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$  

The determinant of $I_3/3 + xA + yB$ is equal to $\left(\frac{1}{3} - x + \frac{1}{3}y\right)\left(\frac{1}{3} + x + \frac{1}{3}y\right)\left(\frac{1}{3} - \frac{y}{\sqrt{3}}\right)$ and the shape of the intersection is an equilateral triangle.
Figure 4. The shapes of special sections and their positions in the halfsimplex $\varphi = 0$.

- For pairs $48, 58, 68, 78$, the representative $B$ is
  \[
  \frac{1}{2\sqrt{3}} \begin{bmatrix}
  1 & 3 & 0 \\
  3 & 1 & 0 \\
  0 & 0 & -2
  \end{bmatrix}.
  \]

  The determinant of $I_3/3 + xA + yB$ is equal to \((\frac{1}{3} + \frac{y}{2\sqrt{3}})^2 - x^2 - \frac{3}{2}y^2)(\frac{1}{3} - \frac{y}{\sqrt{3}})\) and the shape of the intersection is an ellipse (the line segment is tangent to it).

All these cases have $\cos \varphi = 0$ and sit at the edges of the halfsimplex, as shown (together with their shapes) in figure 4.

5. The shape of a general 2-section

So far we have sampled only some very special points in the set of all 2-sections. As we have seen, up to unitary transformations it is enough to look at density matrices of the form $\rho = I_3/3 + xA + yB$. The boundary of a 2-section is described by
\[
3 \det \rho = \frac{1}{3} - x^2 - y^2 + 3(\frac{2abc \cos \varphi - k(1 - k^2 - 3a^2)}{2}) \cdot y^3 + 6k \cdot x^2 y + 3(b^2 - c^2) \cdot x y^2 = 0.
\]

(12)

The problem of classifying all 2-sections is thereby reduced to the problem of classifying plane cubic curves of a somewhat special form. Here we will be concerned with two questions: when does the cubic curve factorize into three linear factors or into one linear and one quadratic factor? If it does not factorize, when is the boundary of the 2-section not smooth?

The cubic factorizes in the following cases.

(1) $b = c$ ∨ $(b = 0 \lor a \cos \varphi = 3k)$. There are two families of sections. One lies on the boundary $b = c = 0$ of the simplex and the second forms for any $\phi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, a line starting from the point $b = c = 1/\sqrt{2}$ and passing the surface $b = c$. For $\phi \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, the second family is only one point $b = c = 1/\sqrt{2}$. The 2-section is described by
  \[
  \left(\frac{1}{3} - 2ky\right) \left(\frac{1}{3} + 2ky - 3x^2 + 3(4k^2 - 1)y^2\right) \geq 0.
  \]

(13)
If $k = \frac{1}{2}$, then the boundary is a parabola intersecting a line on the outsphere. Otherwise the condition describes a cut hyperbola or an (possibly cut) ellipse, depending on the sign of $4k^2 - 1$:

$$\left(\frac{1}{3} - 2ky\right)\left(3\left(4k^2 - 1\right)\left(y + \frac{k}{3\left(4k^2 - 1\right)}\right)^2 - \frac{3k^2 - 1}{3\left(4k^2 - 1\right)} - 3x^2\right) \geq 0. \quad (14)$$

If $k = 0$, then this is a circle and if $k \to \frac{1}{\sqrt{3}}$, the cut hyperbola goes into a triangle. A section from the second family is always related to an uncut ellipse.

Fact 5.1. Assume that the boundary of the section contains a line segment. Then the rest of boundary is a conic which connects to the segment in two exposed points.

We now turn to the second question: if the boundary does not factorize, can it fail to be smooth? We will see that this happens if and only if the section contains a pure state. The cubic curve (12) will fail to be smooth if and only if

$$\det \rho = \partial_x \det \rho = \partial_y \det \rho = 0. \quad (18)$$

A straightforward calculation verifies that this happens if and only if

$$x^2 + y^2 = \frac{1}{3}. \quad (19)$$

This is precisely where the boundary of the 2-section touches the outsphere, of radius $\frac{1}{\sqrt{3}}$, which means that it happens if and only if the 2-section passes through a pure state. So we want to find all 2-sections containing a pure state.

Assume, that the matrix

$$\rho = I_3/3 + xA + yB = \begin{bmatrix} \frac{1}{3} + x + ky & yae^{i\phi} & ybe^{i\phi} \\ yae^{-i\phi} & \frac{1}{3} - x + ky & yce^{i\phi} \\ ybe^{-i\phi} & yce^{-i\phi} & \frac{1}{3} - 2ky \end{bmatrix} \quad (20)$$

is of rank one—it implies that all minors of size 2 are equal to zero. We will use $X, Y, Z$ to denote the diagonal entries. First observe that $y \neq 0$. Now consider the minor formed by the first and the second column and the first and the third row. We have that $X y e^{-i\phi} = aby^2$. If $\phi \notin \{0, \pi\}$, then one has $a = 0 \lor b = 0$, but in this case one can remove the phase. It is enough to consider $e^{i\phi} = \pm 1$.

We have three equations arising from non-main minors;

$$Xc = \pm aby \quad Yb = \pm cay \quad Za = \pm bcy.$$
Figure 5. Points in the simplex $\varphi = 0$ containing a pure state.

If one of the numbers $a, b, c$ is zero, then at least one other has to be zero. In this case the matrix (20) has a block structure and the determinant factorizes. The pure state lies in the points where both factors simultaneously vanish.

Observe that in the first family of sections with factorized boundary both factors can vanish simultaneously only if $k \leq \frac{1}{\sqrt{12}}$. Otherwise the boundary is an ellipse without a pure state. In the limit case $k = \frac{1}{\sqrt{12}}$, one has an ellipse with one pure state on it.

Consider now the case when $abc \neq 0$. Consider the equations for the main minors;

\[
XY = a^2y^2 \quad YZ = c^2y^2 \quad ZX = b^2y^2.
\]

One can easily calculate the following:

\[
X = \pm \frac{bc}{a} y \quad Y = \pm \frac{ab}{c} y \quad Z = \pm \frac{ca}{b} y.
\]

Using the normalization of the trace one has $y = \frac{abc}{(ab)^2 + (bc)^2 + (ca)^2}$. Applying it to the above equations one obtains: $6abc = (ab)^2 + (ac)^2 - 2(bc)^2$. Points satisfying this equation lie on the surface presented in figure 5.

An example of such a section is $k = 0, a = b = c = \frac{1}{\sqrt{3}}$. The point representing this section lies in the middle of the upper wall of the simplex. In the generic case, not falling into the previously considered special cases, the boundary is smooth, closed curve of order 3.

6. Sections, projections and numerical ranges

An important property of the set of quantum states is its self-duality. (It gives quantum logic its special flavor [13–15].) In the vector space of traceless Hermitian $n \times n$ matrices the dual $X^*$ of a set $X$ is defined by

\[
X^* = \{M : 1/n + \text{Tr}(MM') \geq 0 \} \in X \}.
\]  

(21)

If we recall that any density matrix can be written as $\rho = I_3/n + M$, where $M$ is chosen so that $\rho$ is positive, it is easily seen that the self-duality of the set of density matrices follows from the fact that $\text{Tr}\rho \rho' \geq 0$ for any pair of positive matrices $\rho$ and $\rho'$. Subsets of the set of density matrices are not self-dual. If we take for example the set of separable states, then the dual set will be a set of normalized entanglement witnesses. The properties of the duality operation are neatly summarized by

\[
X^{**} = X
\]

(22)
\[ X \subset Y \iff X^* \supset Y^* \]  
\[ \emptyset^* = \mathbb{R}^{n^2 - 1} \]  
\[ (X \cup Y)^* = X^* \cap Y^* \]  
\[ (X \cap Y)^* = \text{conv}(X^* \cup Y^*) \],

where the convex hull appears in the last line. For a nonsingular linear transformation \( A \) one has

\[ (AX)^* = (A^{-1})^T X^*. \]

In particular, if \( A \) is orthogonal, then the set and its dual transform in the same way.

**Duality of section and projection**

The reason why we bring this up here is that a cross-section of a self-dual body is dual to the orthogonal projection onto the linear subspace defining the cross-section [16].

Let \( V \) be a \( k \)-dimensional subspace of the space of traceless matrices of size \( n \). Now we want to find \((X \cap V)^*\).

Having in mind the fact (27), one can change the basis using an orthogonal operation to let \( V \) be spanned by the first \( k \) vectors of the new basis. Let \( \Pi_V \) be the projection onto the subspace \( V \). Now using the condition (21) one has

\[ (X \cap V)^* = \left\{ \eta \in \mathbb{R}^n : \forall \rho \in X \cap V \langle \rho | \eta \rangle = \sum_{i=1}^{n^2-1} \rho_i \eta_i \geq -\frac{1}{n} \right\} \]

\[ = \left\{ \eta \in \mathbb{R}^n : \forall \rho \in X \sum_{i=1}^k \rho_i \eta_i \geq -\frac{1}{n} \right\} \]

\[ = \left\{ \eta \in V : \forall \rho \in X \sum_{i=1}^{n^2-1} \rho_i \eta_i \geq -\frac{1}{n} \right\} \times V^\perp \]

\[ = \{ \eta \in V : \eta \in X^* \} \times V^\perp = \Pi_V X^* \times V^\perp. \]

The section of \( X \) with a subspace \( V \) is dual (after restriction to \( V \)) to the projection of \( X^* \) onto \( V \).

Although in this paper we deal only with projections passing the maximally mixed state a more general construction is worth mentioning.

Let now \( A \) be an affine subspace obtained by acting with a linear subspace \( V \) on vector \( v \in V^\perp \). Again, by an orthogonal operation we change the basis to let \( V \) be spanned by the first \( k \) vectors of the new basis and let \( v \) be parallel to its \( k \) + 1st vector. Now the condition (21) gives

\[ (X \cap A)^* = \left\{ \eta \in \mathbb{R}^n : \forall \rho \in X \cap V \langle \rho | \eta \rangle = \sum_{i=1}^{n^2-1} \rho_i \eta_i \geq -\frac{1}{n} \right\} \]

\[ = \left\{ \eta \in \mathbb{R}^n : \forall \rho \in X \sum_{i=1}^k \rho_i \eta_i \geq -\frac{1}{n} (1 + n|v|\eta_{k+1}) \right\} \]

\[ = \left\{ \eta \in V : \forall \rho \in X \sum_{i=1}^{n^2-1} \rho_i \eta_i \geq -\frac{1}{n} (1 + n|v|\eta_{k+1}) \right\} \times V^\perp \]
\[
\bigcup_{\eta_{k+1} \in \mathbb{R}} \{ \eta \in V : \eta \in (1 + n\eta_{k+1})X^* \} \times \{ \eta_{k+1} \} \times (V \times v)^\perp,
\]

where \( \Pi_V \left( \frac{v}{\|v\|} \right) \) denotes a perspective projection from point \(-\frac{v}{\|v\|}\) onto the subspace \( V \). When \( |v| \to 0 \), that is, when we intersect the set \( X \) with an affine subspace passing through 0, then the point of projection goes to infinity and one gets an orthographic projection.

In mathematics a cross-section of a cone of semi-positive definite matrices is called a spectrahedron, and the question of what kind of convex bodies can be obtained as projections of spectrahedra arises naturally. Once we understand the set of 2-sections of the qutrit we can answer this question in our special case [8].

There is a further connection to the notion of numerical range of a matrix [17, 11]. For a quadratic complex matrix \( A \) we define a subset of the complex plane called the numerical range of \( A \) by

\[
\mathcal{W}(A) = \{ \text{Tr}(A\rho) : \rho \in \Omega \},
\]

(28)

where \( \Omega \) denotes the set of density matrices. Let \( A = aI + B + iC \), where \( B, C \) are Hermitian and traceless. Assume first that \( B, C \) are orthogonal and of unit norm. Then one has

\[
\mathcal{W}(A) = \{ \rho_B + i\rho_C : \rho \in \Omega \} + a,
\]

where \( \rho_B \) and \( \rho_C \) denote the components of \( \rho \), respectively, in the direction of \( B \) and of \( C \). We get a translated projection of \( \Omega \) onto an affine subspace passing through the maximally mixed state. If we abandon the assumption that \( B, C \) are normalized and orthogonal, then our set will be a linearly deformed and translated projection of \( \Omega \). The numerical range of \( 3 \times 3 \) matrices is well understood [18], and the idea was used recently to explore the shadows cast on 2-planes by the set of quantum states [10].

A generic section is a third-order shape, so the generic numerical range is dual to a third-order shape (up to an affine transformation). A section whose boundary fails to be smooth at some point is related to a numerical range having an interval included in its boundary. A section whose boundary is the union of a line and a conic is dual to the convex hull of a point and a conic (the dual to a conic is a conic). This gives a connection between our characterization of possible types of shapes of two-dimensional sections and possible types of shapes of numerical ranges.

7. Conclusions

Our intention with this work was to give a parametrization of the set of all 2-sections of the qutrit. Theorem 1 is a quite satisfactory answer to this problem. In section 4 this enabled us to see at a glance how earlier works—in particular the interesting work by Goyal et al [9]—sample this set. In section 5 we explored how the shape of the 2-section changes as we move through the set of all 2-sections and in section 6 we gave a glimpse of a more general context to which the study of cross-sections of sets of positive matrices belongs.

Acknowledgments

This work was partially supported by the polish National Science Centre project DEC-2011/03/B/ST2/00136. IB was supported by the Swedish Research Council under
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