Disassortative mixing accelerates consensus in the naming game

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Abstract. In this paper, we study the role of degree mixing in the naming game. It is found that consensus can be accelerated on disassortative networks. We provide a qualitative explanation of this phenomenon based on cluster statistics. Compared with assortative mixing, disassortative mixing can promote the merging of different clusters, thus resulting in a shorter convergence time. Other quantities, including the evolution of the success rate, the number of total words and the number of different words, are also studied.

Keywords: network dynamics, random graphs, networks
1. Introduction

Language dynamics, as an important issue in social dynamics [1], has been extensively studied, focusing on the origins and evolution of language [2–6]. To account for the emergence of shared vocabularies or conventions in a community of interacting agents, a Naming Game (NG) model was proposed [7]. The NG has been widely applied in the study of semiotic dynamics. A typical example is the so-called Talking Heads experiment [8], in which a robot assigns names to objects observed through cameras and negotiates with other robots about these names. The NG can achieve global consensus from a multi-opinion state, which is apparently different from other opinion models, such as the majority rule model [9] and voter model [10].

Recently, a minimal version of the NG based on principles of statistical physics was proposed [11]. This model simplifies the original NG model but can reproduce the same experimental phenomena. The minimal NG model has been studied in fully connected graphs [12], regular lattices [13], complex networks (e.g. random networks, small-world networks and scale-free networks) [14–18] and dynamic networks [19]. Some modified versions of the minimal naming-game model have been proposed to better characterize the convergent behavior, such as connectivity-induced weighted words [20], finite memory [21], local broadcast [22–24], asymmetric negotiation [25], reputation [26], n object [27] and a preference for multi-word agents [28].

Many real-world networks have various degree mixing patterns [29]: A network is said to show assortative mixing if the nodes in the network that have many connections tend to be connected to other nodes with many connections. A network is said to show disassortative mixing if high-degree nodes tend to be attached to low-degree ones. A measure of degree mixing for networks is defined by the so-called assortativity coefficient [29]:

\[ r = \frac{M^{-1} \sum_i j_i k_i - [M^{-1} \sum_i \frac{1}{2}(j_i + k_i)]^2}{M^{-1} \sum_i \frac{1}{2}(j_i^2 + k_i^2) - [M^{-1} \sum_i \frac{1}{2}(j_i + k_i)]^2}, \]

\[ (1) \]
where $j_i$, $k_i$ are the degrees of the nodes at the ends of the $i$th edge, $M$ is the number of edges in the network and $i = 1 \ldots M$. It was found that social networks are often assortatively mixed ($r$ is positive), but that technological and biological networks tend to be disassortative ($r$ is negative) [29]. For some celebrated network models, such as the Erdős–Rényi random graphs and the Barabási–Albert (BA) scale-free networks [30], the assortativity coefficient $r$ tends to be zero, indicating the lack of degree correlation. It has been shown that the degree mixing plays an important role in various dynamics such as the spread of epidemics [31] and the evolution of cooperation [32]. However, the effect of the degree mixing on the NG has not yet been studied in the NG. In the following, we will show that disassortative mixing can accelerate consensus in the NG.

2. Model

First, we generate a scale-free network according to the Barabási–Albert (BA) model [30]. Then we use the algorithm proposed by Xulvi-Brunet and Sokolov (XS) to obtain networks with expected degree mixing patterns [33]. In order to get an assortative network, each step randomly chooses two different edges with four different ends and then purposefully swaps the two edges by linking the vertices with higher degrees and lower degrees, respectively. By repeating this procedure, forbidding multiple connections and disconnected components, a network will attain degree assortativity without altering the degree distribution of the original network. Through the opposite operation whereby one edge links the highest and the lowest nodes and the other edge connects the two remaining nodes, the network will attain disassortative mixing.

After a network with the expected assortativity coefficient is constructed, we play the NG. In the game, each node of a network represents an agent. $N$ agents observe a single object and try to communicate its name with the others. Each agent is endowed with an internal inventory to store an unlimited number of names. Initially, each agent has an empty memory. Then the system evolves as follows:

(i) At each time step, a speaker $i$ is chosen at random and then $i$ randomly chooses one of its neighbors $j$ as the hearer. This is referred to as the directed NG [15].

(ii) If the speaker $i$’s inventory is empty, it invents a new word and records it. Otherwise, if $i$ already knows one or more names of the object, with equal probability it randomly choose one word from its inventory. The invented or selected word is then transmitted to the hearer.

(iii) If the hearer $j$ already has this transmitted word in its inventory, negotiation is regarded as successful and both agents keep this common word and delete all other words in their inventories; otherwise, the negotiation fails and the new word is included in the memory of the hearer without any deletion, i.e. the hearer learns the new word.

(iv) The above process is repeated until consensus is reached, that is, all agents have only one word in their inventories and there are no different words in the system.

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Figure 1. The average degree of nearest neighbors $k_{\text{nn}}(k)$ as a function of degree $k$ for networks with different assortativity coefficients $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each curve is an average of 50 different realizations.

Figure 2. Evolution of the success rate $S(t)$ for different values of the assortativity coefficient $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each curve is an average of 2000 different realizations.

3. Simulation results

Figure 1 shows the average degree of nearest neighbors $k_{\text{nn}}(k)$ as a function of degree $k$ for networks with different values of the assortativity coefficients $r$. From figure 1, one can see that $k_{\text{nn}}(k)$ is almost independent of $k$ when $r = 0$. For assortative (disassortative) networks, $k_{\text{nn}}(k)$ is an increasing (decreasing) function of $k$, indicating that the hubs tend to connect with large(small)-degree nodes.

Figure 2 shows the evolution of the success rate $S(t)$ for different values of the assortativity coefficient $r$. From figure 2, one can see that at first $S(t)$ is lowest for...
Figure 3. Evolution of $N_w(t)/N$, the average number of words per agent, for different values of the assortativity coefficient $r$. The inset shows the maximum total number of words $N_w^{\text{max}}/N$ as a function of $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each curve is an average of 2000 different realizations.

Disassortative mixing ($r = -0.24$) and $S(t)$ is highest for assortative mixing ($r = 0.2$). But later on $S(t)$ increases most quickly to 1 for disassortative networks, as compared to that for assortative and uncorrelated ($r = 0$) networks.

Figure 3 shows the average number of words per agent $N_w(t)/N$ as time evolves. One can see that $N_w(t)/N$ grows until it reaches a maximum (hereafter denoted by $N_w^{\text{max}}/N$) and then it starts decreasing due to an increase in successful interactions. The inset of figure 3 displays $N_w^{\text{max}}/N$ as a function of the assortativity coefficient $r$. We find that $N_w^{\text{max}}/N$ is not a monotonic function of $r$. In fact, $N_w^{\text{max}}/N$ attains its minimum for $r \approx 0.04$. The non-monotonic relationship between the convergence time and the maximum total memory was also found in [25].

Figure 4 shows the time evolution of the number $N_d$ of different words in the system. One can see that there exists a peak of $N_d$ during the evolution for each value of assortativity coefficient $r$. The inset of figure 4 shows the maximum number of different words $N_d^{\text{max}}$ as a function of $r$. It is found that $N_d^{\text{max}}$ decreases with the increase of $r$.

Next we study the most important quantity, the convergence time $t_c$ defined as the time steps for reaching the final consensus. Figure 5 shows $t_c$ as a function of the assortativity coefficient $r$. One can see that $t_c$ increases with $r$, indicating that disassortative mixing accelerates consensus in the NG.

To understand the process of convergence to consensus, we study the evolution of clusters of words. A cluster is a connected component (subgraph) fully occupied by nodes sharing a common unique word. It has been shown that the dynamics of the NG proceeds by the formation of such clusters [13]. Figure 6 shows the number of clusters $N_{cl}$ and the normalized size of the largest cluster $s_1 = S_L/N$ as a function of the rescaled time $t/N$ for different values of $r$, where $S_L$ is the size of the largest cluster. From figure 6(a), we see that the number of clusters $N_{cl}$ reaches a plateau in the beginning, but then it rapidly falls to one. Meanwhile, the normalized size of the largest cluster $s_1$ remains very close to zero.
Figure 4. Evolution of the number of different words $N_d(t)$ for different values of the assortativity coefficient $r$. The inset shows the maximum number of different words $N_d^{\text{max}}$ as a function of $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each curve is an average of 2000 different realizations.

Figure 5. Convergence time $t_c$ as a function of the assortativity coefficient $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each data point is obtained by averaging over 2000 different realizations.

during the plateau, subsequently it increases to one with a similar sudden transition (see figure 6(b)). The sharp transitions for $N_{cl}$ and $s_1$ reflect the merging of different clusters. Previous studies have shown that this merging accelerates consensus in the NG [20, 25] and other opinion models [34, 35]. From figure 6, one can see that the merging of different clusters is earlier and quicker in the case of disassortative mixing ($r = -0.24$), compared to the cases of random mixing ($r = 0$) and assortative mixing ($r = 0.2$), thus leading to the faster convergence.

Figure 7 shows the convergence time $t_c$ as a function of the network size $N$ for different values of the assortativity coefficient $r$. One can see that $t_c$ scales as $N^\beta$. The value of exponent $\beta$ increases with $r$. For $r = -0.1, 0$ and 0.1, $\beta$ is about 1.35, 1.38 and 1.40 respectively. This result manifests that disassortative mixing still accelerates consensus for the large network size.
Figure 6. (a) The number of clusters $N_{cl}$ and (b) the normalized size of the largest cluster $s_1$ as a function of the rescaled time $t/N$ for different values of the assortativity coefficient $r$. The BA network size $N = 5000$ and the average degree $\langle k \rangle = 4$. Each curve is an average of 2000 different realizations.

The above studies are conducted on BA scale-free networks which follow a power-law degree distribution $P(k) \sim k^{-\gamma}$ with $\gamma = 3$. However, the finding that disassortative mixing accelerates consensus is not restricted to BA networks. In fact, we have observed a similar behavior for scale-free networks constructed by the configuration model (CM) [36]. Figure 8 shows the convergence time $t_c$ as a function of the assortativity coefficient $r$ for CM networks with different values of the power-law exponent $\gamma$. From figure 8, we also observe that $t_c$ increases with $r$ for each value of $\gamma$.

Apart from the directed NG, there are other updating strategies of NG, such as the reverse NG [14] and the broadcasting NG [22]. In the reverse NG, we first choose the hearer at random and then one of its neighbors as the speaker. In the broadcasting NG, a speaker transmits its word to all its neighbors at the same time, rather than to a randomly selected one. It has been found that the updating strategy greatly affects the process of convergence to consensus [15, 23]. From figure 9, we observe that the convergence time $t_c$ increases with the assortativity coefficient $r$ in the reverse NG and the broadcasting NG.

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Figure 7. Convergence time $t_c$ as a function of the BA network size $N$ for different values of the assortativity coefficient $r$. The average degree $\langle k \rangle = 4$. The slope of the fitted line is 1.35, 1.38, 1.40 for $r = -0.1, 0, 0.1$ respectively. Each data point is an average of 2000 different realizations.

Figure 8. Convergence time $t_c$ as a function of the assortativity coefficient $r$ for CM networks with different values of the power-law exponent $\gamma$. The CM network size $N = 5000$ and the minimal degree $k_{\text{min}} = 2$. Each data point is obtained by averaging over 2000 different realizations.

4. Conclusions and discussions

In conclusion, we have studied the impact of degree mixing on consensus in the naming game. We adjusted the assortativity coefficient of a network by swapping the ends of the two edges while keeping the degree of each node unchanged. We have found that the convergence time decreases with the increase of the assortativity coefficient. Compared with uncorrelated and assortative networks, disassortative networks can accelerate the convergence to global consensus. This finding is robust with respect to different types of scale-free networks including the Barabási–Albert model and the configuration model and to different kinds of updating strategies including the directed, reverse and broadcasting.
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Figure 9. Convergence time \( t_c \) as a function of the assortativity coefficient \( r \) in the reverse and broadcasting NG respectively. The BA network size \( N = 5000 \) and the average degree \( \langle k \rangle = 4 \). Each data point is obtained by averaging over 2000 different realizations.

naming games. We have explained such a phenomenon in terms of the evolution of the word clusters. The merging of different clusters and the formation of big clusters become easier in the case of disassortative mixing, compared to that in the cases of random mixing and assortative mixing. We expect our work to provide new insights into agreement dynamics on degree-correlated networks.

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References

[1] Castellano C, Fortunato S and Loreto V 2009 Rev. Mod. Phys. 81 591
[2] Nowak M and Krakauer D 1999 Proc. Natl Acad. Sci. USA 96 8028
[3] Abrams D and Strogatz S H 2003 Nature 424 900
[4] Schulze C, Stauffer D and Wichmann S 2008 Commun. Comput. Phys. 3 271
[5] Loreto V, Baronchelli A, Mukherjee A, Puglisi A and Tria F 2011 J. Stat. Mech. P04006
[6] Loreto V, Mukherjee A and Tria F 2012 Proc. Natl Acad. Sci. USA 109 6819
[7] Steels L 1995 Artif. Life 2 319
[8] Steels L 1998 Auton. Agent Multi Agent Syst. 1 169
[9] Krapivsky P L and Redner S 2003 Phys. Rev. Lett. 90 238701
[10] Sood V and Redner S 2005 Phys. Rev. Lett. 94 178701
[11] Baronchelli A, Felici M, Loreto V, Caglioti E and Steels L 2006 J. Stat. Mech. P06014
[12] Baronchelli A, Loreto V and Steels L 2008 Int. J. Mod. Phys. C 19 785
[13] Baronchelli A, Dall’Asta L, Barrat A and Loreto V 2006 Phys. Rev. E 73 015102

doi:10.1088/1742-5468/2015/01/P01009
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[14] Dall’Asta L, Baronchelli A, Barrat A and Loreto V 2006 Europhys. Lett. 73 969
[15] Dall’Asta L, Baronchelli A, Barrat A and Loreto V 2006 Phys. Rev. E 74 036105
[16] Barrat A, Baronchelli A, Dall’Asta L and Loreto V 2007 Chaos 17 026111
[17] Liu R R, Jia C X, Yang H X and Wang B H 2009 Physica A 388 3615
[18] Liu R R, Wang W X, Lai Y C, Chen G and Wang B H 2011 Phys. Lett. A 375 363
[19] Nardini C, Kozma B and Barrat A 2008 Phys. Rev. Lett. 100 158701
[20] Tang C L, Lin B Y, Wang W X, Hu M B and Wang B H 2007 Phys. Rev. E 75 027101
[21] Wang W X, Lin B Y, Tang C L and Chen G R 2007 Eur. Phys. J. B 60 529
[22] Lu Q, Korniss G and Szymanski B K 2008 Phys. Rev. E 77 016111
[23] Baronchelli A 2011 Phys. Rev. E 83 046103
[24] Baronchelli A and Díaz-Guilera A 2012 Phys. Rev. E 85 016113
[25] Yang H X, Wang W X and Wang B H 2008 Phys. Rev. E 77 027103
[26] Brigatti E 2008 Phys. Rev. E 78 046108
[27] Lipowski A and Lipowska D 2009 Phys. Rev. E 80 056107
[28] Lipowska D and Lipowski A 2014 J. Stat. Mech. P08001
[29] Newman M E J 2002 Phys. Rev. Lett. 89 208701
[30] Barabási A L and Albert R 1999 Science 286 509
[31] Boguñá M and Pastor-Satorras R 2002 Phys. Rev. E 66 047104
[32] Rong Z and Wu Z X 2009 Europhys. Lett. 87 30001
[33] Xulvi-Brunet R and Sokolov I M 2004 Phys. Rev. E 70 066102
[34] Yang H X, Wu Z X, Zhou C, Zhou T and Wang B H 2009 Phys. Rev. E 80 046108
[35] Yang H X, Wang W X, Lai Y C and Wang B H 2012 Phys. Lett. A 376 282
[36] Molloy M and Reed B 1995 Random Struct. Algorithms 6 161

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