WATSON’S BASIC ANALOGUE OF RAMANUJAN’S ENTRY 40
AND ITS GENERALIZATION

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Abstract. We generalize Watson’s $q$-analogue of Ramanujan’s Entry 40 continued fraction by deriving solutions to a $10\phi_9$ series contiguous relation and applying Pincherle’s theorem. Watson’s result is recovered as a special terminating case, while a limit case yields a new continued fraction associated with an $8\phi_7$ series contiguous relation.

Key words. contiguous relations, continued fractions, Pincherle’s theorem, basic hypergeometric series.

AMS subject classifications. 33D15, 39A10, 40A15

In honour of Dick Askey on the occasion of his 60th birthday.

1. Introduction. Contiguous relations for hypergeometric functions are an important source for obtaining explicit results for difference equations, continued fractions, Jacobi matrices and their corresponding orthogonal polynomials. At the top of the Askey-Wilson chart of classical orthogonal polynomials [1] one has the $4F_3$ Wilson polynomials. However the $4F_3$ label is misleading, since the properties of these polynomials and their associated case are revealed by two contiguous relations for very well poised $7F_6$ series [8], [14]. These in turn can be derived as limits of a contiguous relation for a terminating, very well poised, two balanced $9F_8$ series [18], [8]. This $9F_8$ contiguous relation is thus fundamental for the classical hypergeometric polynomials. In a previous publication [15] it was shown how this $9F_8$ contiguous relation was also related to Ramanujan’s famous Entry 40 continued fraction [16], [3].

All of the above are $q \to 1$ limits of basic hypergeometric analogues. Thus the $4\phi_3$ Askey-Wilson polynomials should be viewed in the light of very well poised $8\phi_7$ series [9] which are limits of terminating, very well poised, balanced $10\phi_9$’s. The analogous contiguous relation for $10\phi_9$’s is thus fundamental to the whole scheme of classical and basic hypergeometric orthogonal polynomials. In this paper we derive this important contiguous relation and a corresponding continued fraction. A special terminating version of this continued fraction yields the following result of Watson [17] which is the $q$-analogue of Ramanujan’s Entry 40 continued fraction [16], [3].

Theorem A (Watson [17]). Denoting the base by $q^2$ (instead of more usual $q$), let

\[
\frac{1}{G(x)} = \prod_{m=0}^{\infty} \left( 1 - q^{2m+1} \right), \quad |q| < 1,
\]

\[
P = G(\alpha\beta\gamma\delta)G(\frac{\alpha\beta}{\gamma\delta})G(\frac{\alpha\gamma}{\beta\delta})G(\frac{\alpha\delta}{\beta\gamma})G(\frac{\alpha}{\beta})G(\frac{\alpha}{\gamma})G(\frac{\alpha}{\delta})G(\frac{\alpha}{\beta\gamma\delta})
\]

\[
Q = G(\frac{\alpha\beta\gamma}{\delta})G(\frac{\alpha\beta\gamma}{\delta})G(\frac{\alpha\beta\delta}{\gamma})G(\frac{\alpha\gamma}{\beta\delta})G(\frac{\alpha\gamma}{\beta\delta})G(\frac{\alpha\gamma}{\beta\delta})G(\frac{\alpha\gamma}{\beta\delta})G(\frac{\alpha\gamma}{\beta\delta}).
\]

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Then, provided that one of the numbers $\beta, \gamma, \delta, \epsilon$ is of the form $q^{\pm n}(n = 1, 2, \ldots)$,

\[
\frac{P - Q}{P + Q} = \frac{A_0}{\beta_0 + \alpha_1 + \alpha_2 + \cdots},
\]

where

\[
A_0 = (q + q^{-1})\Pi(\alpha - \alpha^{-1})
\]

\[
\alpha_m = (q^{m+1} + q^{-m-1})(q^{m-1} + q^{-1-m})\Pi(\alpha^2 + \alpha^{-2} - q^{2m} - q^{-2m}),
\]

\[
\beta_m = (q^{2m+1} + q^{-2m-1})\left\{ (q^m + q^{-m})(q^{m+1} + q^{-m-1})(\Sigma\alpha^2 + \Sigma\alpha^{-2} + 2)
\right.
\]

\[
- \Pi(\alpha + \alpha^{-1}) - (q + q^{-1})(q^m + q^{-m})(q^{m+1} + q^{-m-1})(q^{2m+1} + q^{-2m-1}) \right\}
\]

the products and sums ranging over the numbers $\alpha, \beta, \gamma, \delta, \epsilon$.

A second special terminating version of the continued fraction obtained here will give the basic analogue of Masson’s Proposition 1 in [15], which is described as a ‘missing companion’ of Ramanujan’s Entry 40. For the sake of completeness we state Masson’s result:

**Theorem B.** (Masson [15]). Let $P' = \Pi(3 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/4$ (0, 2 or 4 minus signs) and $Q' = \Pi(1 + \alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon)/4$ (1 or 3 minus signs). Then if one of the parameters $\beta, \gamma, \delta, \epsilon$ is an odd integer,

\[
\frac{Q'}{P'} = -1 - \frac{2b_1}{a_0} - \frac{b_2}{a_2} - \frac{b_3}{a_3} - \cdots
\]

where

\[
b_n = \left( \Pi((2n - 1)^2 - \alpha^2) \right)/(16)^3(2n - 1)^2,
\]

\[
a_n = \left\{ \frac{2n^6 + n^4(5 - \Sigma\alpha^2)}{4} + n^2(-26 + (1 + \Sigma\alpha^2)^2 - 2\Sigma\alpha^4)/64 - a_0 \right\}/(4n^2 - 1),
\]

\[
a_0 = \left\{ 2(1 - \Sigma\alpha^4) + (1 - \Sigma\alpha^2)^2 - 8\Pi\alpha \right\}/(16)^2,
\]

with these products and sums ranging over the parameters $\alpha, \beta, \gamma, \delta, \epsilon$.

Masson [15] also gave the non-terminating versions of Ramanujan’s Entry 40 and Theorem B.

The object of the present study is to obtain the non-terminating versions of Watson’s theorem and the $q$-analogue of Masson’s theorem given above. They are given in Section 4 by Corollaries 7 and 8 respectively. Our approach is similar to that in several recent papers [5], [6], [12], [13] on the subject where Pincherle’s theorem [11] has been used to bring out the connection between several of Ramanujan’s Chapter 12 entries and the general theory of hypergeometric orthogonal functions (Askey and Wilson [1], Wilson [18]). For other approaches to explaining some of Ramanujan’s continued fraction entries see [3], [10], [19].

**2. Contiguous relation.** We consider a terminating very well poised balanced $10\phi_9$ basic hypergeometric function

\[
\phi = \phi(a; b, c, d, e, f, g, h)
\]

\[
 := 10\phi_9 \left( \begin{array}{c} \alpha, \sqrt{a}, -\sqrt{a}; \alpha, b, c, d, e, f, g, h; q, q \\ \sqrt{a}, -\sqrt{a}; \alpha, aq, aq, aq, aq, aq, aq, aq, aq, aq, aq \\ a^3q^2 = bcd ef gh \end{array} \right), \quad |q| < 1
\]
with say $h = q^{-n}$, $n = 0, 1, \ldots, g = sq^{n-1}$, $s := a_{bd\text{e}f}$. We follow the usual notation for variations of $\phi$ with respect to the parameters. For example $\phi(b+,c-)$ represents the $\phi$ with $b$ and $c$ replaced by $bq$ and $\frac{c}{a}$ respectively. $\phi_+$ denotes the $_{10}\phi_9$ got by replacing $a$ by $aq$ and $b, c, d, e, f, g, h$ by $bq, cq, dq, eq, fq, gq, hq$ respectively.

We need a contiguous relation basic analogue to the contiguous relation derived by Wilson [18] for the $\phi_9$ hypergeometric function. In order to work out this contiguous relation, we shall use Wilson’s method [18] using the basic hypergeometric analogues of the relevant formulas.

**Lemma 1.** Let $\phi$ be given by (2.1) (not necessarily terminating). Then

$$\phi(b-,c+)-\phi = \frac{aq^2(1 - \frac{a}{bh}q)(1 - \frac{a}{c}q)(1 - aq)(1 - aq^2)(1 - d)(1 - e)(1 - f)(1 - g)(1 - h)}{(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})(1 - \frac{aq}{h})} \times \phi_+(b-) .$$

**Proof.** A straightforward term by term subtraction on the left side of (2.2) leads to the result. 

**Lemma 2.** If $\phi$ (given by (2.1)) is terminating, then

$$\frac{b^2(1 - b)(1 - \frac{a}{b})(1 - \frac{a}{c})(1 - \frac{a}{d})(1 - \frac{a}{e})(1 - \frac{a}{f})(1 - \frac{a}{g})}{(1 - \frac{aq}{a})(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})} \phi_+(b-)$$

$$= \frac{a^2(1 - \frac{a}{h})(1 - \frac{a}{k})(1 - \frac{aq}{a})(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})}{(1 - \frac{aq}{a})(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})} \phi = 0 .$$

**Proof.** By eliminating $\phi_+(b-)$ from (2.2) and another similar relation written for $\phi(b-,d+)-\phi$ we obtain

$$c(1 - c)(1 - \frac{a}{c})(1 - \frac{dq}{b})(1 - \frac{bd}{aq})\phi(b-,c+)$$

$$- d(1 - d)(1 - \frac{a}{d})(1 - \frac{cq}{b})(1 - \frac{bc}{aq})\phi(b-,d+)$$

$$+ d(1 - \frac{b}{q})(1 - \frac{c}{d})(1 - \frac{aq}{b})(1 - \frac{cd}{a})\phi = 0 .$$

With say, $h = q^{-n}$, we can apply an iterate of Bailey’s transformation 8.5(1) [2, p. 68] to $\phi$, $\phi_+(b-)$ and $\phi_+(h-)$ (the transformation [4, exercise 2.19, p. 53] with $b, c, g$ replaced by $g, b, e$ respectively). The three transformed series are related via (2.4). Reversing the transformations in this relation we arrive at (2.3). 

**Theorem 3.** If $\phi$ (given by (2.1)) is terminating, then

$$g(1 - h)(1 - \frac{a}{h})(1 - \frac{aq}{a})(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})$$

$$\times [\phi(g-,h+)/\phi]$$

$$- h(1 - g)(1 - \frac{a}{g})(1 - \frac{aq}{a})(1 - \frac{aq}{b})(1 - \frac{aq}{c})(1 - \frac{aq}{d})(1 - \frac{aq}{e})(1 - \frac{aq}{f})(1 - \frac{aq}{g})$$

$$\times [\phi(h-,g+)/\phi]$$

$$= \frac{aq^2}{h}(1 - \frac{h}{g})(1 - \frac{gh}{aq})(1 - b)(1 - c)(1 - d)(1 - e)(1 - f) \phi = 0 .$$
Proof. We eliminate $\phi_+(b-)$ and $\phi_+(c-)$ from (2.2), (2.2) with $b \leftrightarrow c$ and (2.3) with $c \leftrightarrow h$. A final interchange of parameters $b \leftrightarrow g$, $c \leftrightarrow h$ yields the desired result.

Substituting $h = q^{-n}$, $g = sq^{n-1}$, and renormalizing, the contiguous relation (2.5) becomes the linear second order difference equation

$$X_{n+1} - a_nX_n + b_nX_{n-1} = 0, \quad n \geq 0$$

with the solution

$$X_n^{(1)} = \frac{q^{-\frac{2n}{\varphi}}}{s} \frac{(sq^{2n-1})_\infty (aq^{n+1})_\infty}{(s q^{n-1})_\infty (a q^{n+1})_\infty} \frac{(aq^{n+1})_\infty}{(aq^{n+1})_\infty} \frac{(aq^{n+1})_\infty}{(aq^{n+1})_\infty} \times \phi(a; b, c, d, e, f, sq^{n-1}, q^{-n}) .$$

Here the infinite product $(a)_\infty$ means

$$(a)_\infty = (a; q)_\infty = (1-a)(1-aq)(1-aq^2) \ldots$$

and

$$(a, b, \ldots, k)_\infty = (a)_\infty (b)_\infty \ldots (k)_\infty .$$

For the exceptional values $s = q, q^2$, the $a_n, b_n$ and $b_{n+1}$ in (2.7), (2.8) and $X_{n}^{(1)}, X_{n-1}^{(1)}$ in (2.9) are indeterminate at $n = 0$. We resolve this indeterminancy by taking limits as $n \to 0$.

Next, we proceed to find a second linearly independent solution to the second order difference equation (2.6). This can be obtained by using a $q$-analogue of [15]. Thus
from (2.6), (2.9) and a symmetry relation ((2.11), below) we are able to obtain a second terminating \(10\phi_9\) solution for the special values \(s = q, q^2, \ldots\). For general values of \(s\) the second solution will be an appropriate combination of two non-terminating \(10\phi_9\)'s which satisfy a four-term transformation (Gasper and Rahman [4], formula III.39, p. 247). We will consider the case of general \(s\) in future work.

Observe that with the replacement

\[
(a, b, c, d, e, f, sq^{n-1}, q^{-n}) \rightarrow \left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{q^{-n+2}}{s}, q^{n+1}\right)
\]

we have

\[
(a_n, b_n) \rightarrow (a_n, b_{n+1}).
\]

It is easy to check that \(b_n \rightarrow b_{n+1}\). To check \(a_n \rightarrow a_n\) we used the 'Maple' software on the computer. This meant verifying a polynomial identity in \(x = q^{-n}\) of degree fourteen.

Applying the transformation (2.10) to (2.6) and (2.9) and renormalizing, we obtain the second solution

\[
X^{(2)}_n = \frac{q^{-\frac{s^2}{16}+n}}{s^\frac{1}{2}} \frac{(s^2 q^n)_{\infty}}{(q^{n+1})_{\infty} (aq^n)_{\infty} (\frac{q^2}{a} q^{-n+1})_{\infty} (\frac{q}{b} q^{-n+1})_{\infty} (\frac{q}{c} q^{-n+1})_{\infty} (\frac{q}{d} q^{-n+1})_{\infty} (\frac{q}{e} q^{-n+1})_{\infty} (\frac{q}{f} q^{-n+1})_{\infty}}
\times \phi\left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{q^{-n+2}}{s}, q^{n+1}\right),
\]

\[s = q, q^2, \ldots.\]

Note that \(\phi\) is terminating in (2.12) because of the parameter \(q^{-n+2}/s\).

3. Asymptotics and Pincherle’s theorem. In order to obtain a minimal (subdominant) solution for (2.6) we need the large \(n\) asymptotics of (2.9) and (2.12). Applying Tannery’s theorem to the \(10\phi_9\)'s on the right side of (2.9) and (2.12) we have, as \(n \rightarrow \infty\),

\[
X^{(1)}_n \sim \frac{q^{-\frac{s^2}{16}+n}}{s^\frac{1}{2}} W(a; b, c, d, e, f), \quad \left|\frac{s}{aq}\right| < 1,
\]

and

\[
X^{(2)}_n \sim \frac{q^{-\frac{s^2}{16}+n}}{s^\frac{1}{2}} W\left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}\right), \quad \left|\frac{aq^2}{s}\right| < 1,
\]

where

\[
W(a; b, c, d, e, f) := s\phi_7 \left(\frac{a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f}{\sqrt{a}, -\sqrt{a}, \frac{qa}{b}, \frac{qa}{c}, \frac{qa}{d}, \frac{qa}{e}, \frac{qa}{f}}; q, a^2 q^2, \frac{aq^2}{bcdef}\right).
\]

We write

\[
W_1 := W(a; b, c, d, e, f), \quad |s| < |qa|
\]

and its analytic continuation otherwise; and

\[
W_2 := W\left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}\right), \quad |s| > |aq^2|
\]
and its analytic continuation otherwise.

Taking now

\[(3.3)\quad X_n^{(3)} := W_2X_n^{(1)} - W_1X_n^{(2)}\]

it follows from (3.1) and (3.2) that

\[(3.4)\quad \lim_{n \to \infty} \frac{X_n^{(3)}}{X_n^{(3)}} = 0, \quad |s| < |a| < \left|\frac{s}{q^2}\right|.

This establishes that \(X_n^{(3)}\) is a minimal solution of (2.6). An application of Pincherle’s theorem [11] then leads to the following result:

**Theorem 4.** Let \(s = q, q^2, \ldots\). Then

\[(3.5)\quad \lim_{n \to 0} \frac{W_2X_n^{(1)} - W_1X_n^{(2)}}{b_n(W_2X_n^{(1)} - W_1X_n^{(2)})}.

**Proof.** From Pincherle’s theorem \((3.5)\) is true for \(|s| < |a| < \left|\frac{s}{q^2}\right|\). For other values of \(a\) the result follows by analytic continuation. To the left side of (3.5) we can apply the ‘parabola theorem’ (see Jones and Thron [11, p. 99] and Jacobsen [10]), since from (2.6),

\[\frac{b_n}{a_n a_{n-1}} = \frac{q^2}{(1+q^2)} \left(1 + O(q^n)\right).\]

Hence the left side of (3.5) is a meromorphic function of \(a\). The right side of (3.5) involves convergent infinite products and \(s\phi_7\)’s which are each expressible in terms of convergent infinite products and convergent \(4\phi_3\)’s (Gasper and Rahman [4, (2.10.10), page 43]). Consequently the right side of (3.5) is also a meromorphic function of \(a\) and (3.5) follows by analytic continuation to all values of \(a\). Note that the exceptional cases \(s = q, q^2\) which cause indeterminacy are taken care of by the limit \(n \to 0\) on the right side of (3.5). \(\square\)

For the exceptional values \(s = q^2, q\), the above theorem gives respectively the non-terminating versions of Theorem A (Watson[17]) and the basic analogue of Theorem B (Masson [15]). We now demonstrate how to derive the terminating versions of Theorem 4. We shall need to express the ratio \(W_1/W_2\) in terms of infinite products when \(\frac{b}{a} = q^N\), \(N\) being an integer. We write

\[\hat{W}(a; b, c, d, e, f) := \left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}\right)_\infty W(a; b, c, d, e, f)\]

and

\[U(a; b, c, d, e, f) := \frac{\hat{W}(a; b, c, d, e, f)}{(aq, b, c, d, e, f)_\infty}.

**Lemma 5.** If \(\frac{b}{a} = q^N\), where \(N\) is an integer, then

\[(3.6)\quad U(a; b, c, d, e, f) = \left(\frac{s}{aq}\right)^N U\left(\frac{b^2}{a}; b, c, d, e, f\right).

**Proof.** Refer to Bailey’s three-term \(s\phi_7\) transformation (formula III.37, p. 246 of Gasper & Rahman [4]). If we apply the condition \(\frac{b}{a} = q^N\), \(N = 0, \pm 1, \pm 2, \ldots\) we obtain the desired result. \(\square\)
Lemma 6. If \( s = \frac{a^s b^s}{c^s d^s e^s f^s} = q^M \) and \( \frac{b}{a} = q^N \), \( M \) and \( N \) being integers, then

\[
W(a; b, c, d, e, f) \quad \frac{W(a; b, c, d, e, f)}{W(a; b, c, d, e, f)} = \lambda \frac{(aq, c, d, e, f, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f})_{\infty}}{(bq, \frac{bq}{c}, \frac{bq}{d}, \frac{bq}{e}, \frac{bq}{f})_{\infty}},
\]

where

\[
\lambda = (-1)^{n+1} \frac{(N + 1)q^n}{aq} \frac{N q^{(n+1)/2}}{aq} \quad \text{for} \quad \frac{aq^3}{bs} = q^{-n}, \quad n = 0, 1, 2, \ldots,
\]

and

\[
\lambda = (-1)^{n+1} \frac{(N + 1)q^n}{aq} \frac{N q^{(n+1)(n+2)/2}}{aq} \quad \text{for} \quad \frac{bs}{aq} = q^{-n}, \quad n = -1, 0, 1, 2, \ldots.
\]

Proof. We express the left side of (3.7) in terms of appropriate \( 4 \phi_3 \)'s. To the numerator \( W \) in (3.7) we first apply the identity (3.6) and then the three-term transformation formula III.36 [4, p. 246]. To the denominator \( W \) we first apply the \( s \phi_7 \) transformation formula III.24 [4, p. 243] and then the formula III.36. We also make use of the relation

\[
\lim_{s \to -a} \varphi_3(a, b, c; d, e, f; g, q) = \frac{(a, b, c, f, g, q^n, g_{q^n})}{(aq^n, bq^n, c, dq^n, f, g, q)}
\]

All this enables us to recognize and cancel a common linear combination of \( 4 \phi_3 \)'s from the numerator and the denominator yielding the desired result. We note that in the case \( \frac{bs}{aq} = q, s = q \), the limit (3.10) is not required and there is an exact cancellation.

4. Exceptional values \( s = q, q^2 \). We now restate Theorem 4 for the exceptional values \( s = q, q^2 \) and the form they take when the continued fraction terminates:

Corollary 7. If \( s = q^2 \), then (3.5) can be rewritten as

\[
\frac{1}{a_0 - a_1 - a_2 - \cdots} = \frac{2n(1 - q)}{aq^{3/2}(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)(1 - \epsilon)} \left( 1 - V \right),
\]

where

\[
a_n = \left[ q^{n+1} \Pi(\alpha^{n+1} + \alpha^{-1} - q^n) + q^{n+2} \left( q^{n+1} + q^{-n} \right) + q^{n+1} \left( q^{n+1} + q^{-n} \right) \right]
\]

\[
\times \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right)
\]

\[
- q^{n+1} \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right)
\]

\[
/ \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right),
\]

\[
b_n = \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right) \left( q^{n+1} + q^{-n} \right),
\]

\[
V = \left( \frac{\tilde{W}_1}{\tilde{W}_2} \right)_{\infty} \left( \frac{\tilde{W}_1}{\tilde{W}_2} \right)_{\infty},
\]

\[
a = (q\alpha^3\beta\delta^3 \epsilon)^{1/2},
\]
and product $\Pi$ and summation $\Sigma$ are taken over the parameters $\alpha, \beta, \gamma, \delta, \epsilon$. If one of the parameters $\beta, \gamma, \delta, \epsilon = q^N$, $N$ integer, $(\alpha, \beta, \gamma, \delta, \epsilon) \rightarrow (\alpha^2, \beta^2, \gamma^2, \delta^2, \epsilon^2)$ and the base $q$ is changed to $q^2$, then the right side of (4.1) becomes

\[
\frac{2(q^{-1} - q)}{q\Pi(\alpha - \alpha^{-1})} \frac{P - Q}{P + Q} \tag{4.4}
\]

with

\[
\frac{1}{P} = \left( \frac{\alpha \beta \gamma \delta}{\beta \gamma \delta \epsilon}, \frac{\alpha \beta \gamma}{\beta \gamma \delta \epsilon}, \frac{\alpha \beta \delta}{\beta \gamma \delta \epsilon}, \frac{\alpha \gamma \delta}{\beta \gamma \delta \epsilon}, \frac{\alpha \delta \epsilon}{\beta \gamma \delta \epsilon}, \frac{q^2}{\beta \gamma \delta \epsilon} \right) \mathcal{W} \tag{4.5}
\]

\[
\frac{1}{Q} = \left( \frac{\alpha \beta \gamma \delta}{\epsilon \delta}, \frac{\alpha \beta \gamma \delta}{\epsilon \gamma}, \frac{\alpha \beta \gamma \delta}{\epsilon \beta}, \frac{\alpha \gamma \delta \epsilon}{\beta \gamma \delta \epsilon}, \frac{\alpha \delta \epsilon \gamma}{\beta \gamma \delta \epsilon}, \frac{q^2}{\beta \gamma \delta \epsilon} \right) \mathcal{W} \tag{4.5}
\]

and $a_n, b_n$ modified accordingly. This reproduces Theorem A.

Proof. We write $\alpha = \frac{a}{q}$, $\beta = \frac{c}{q}$, $\gamma = \frac{b}{q}$, $\delta = \frac{e}{q}$, $\epsilon = \frac{f}{q}$. After that the proof is straightforward on substituting the values of $X_0^{(1)}$, $X_0^{(2)}$, $b_0X_0^{(1)}$, $b_0X_0^{(2)}$ from (2.9), (2.12) and (2.8) into Theorem 4. A lot of algebra is involved in the simplification. Also, appropriate limits are to be taken whenever indeterminates occur.

In order to obtain the terminating form (4.4) we need to use Lemma 6, after interchanging say $b$ and $f$. In both cases viz.,

\[
\frac{aq^3}{fs} = q^{-n}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad \frac{fs}{aq} = q^{-n}, \quad n = -1, 0, 1, 2, \ldots
\]

whether the termination is due to one or the other, the result works out to be the same. The above result (4.4) yields Watson’s result [17] i.e. Theorem A in Section 1.

\[\square\]

Corollary 8. If $s = q$, then (3.5) can be rewritten as

\[
\frac{1}{a_0} - \frac{b_1}{a_1} - \frac{b_2}{a_2} - \ldots = \left( \frac{a_0}{a} + \frac{a^2}{q} \frac{(\frac{1}{q})_\infty}{(\frac{a}{q})_\infty (\frac{b}{a})_\infty} \frac{W_1}{W_2} \right)^{-1} \tag{4.5}
\]

\[
a_n = \frac{q^\frac{n}{2}}{(q^n - q^{-\frac{n}{2}})(q^{n+\frac{1}{2}} - q^{-\frac{n}{2}})} \left\{ (q^{3n} + q^{3n})(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \right. \\
- (q^{2n} + q^{-2n}) \left[ q^{\frac{1}{2}} + q^{-\frac{1}{2}} + \Sigma (\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}) \right] \\
+ (q^n + q^{-n}) \left[ -q^{\frac{3}{2}} - q^{-\frac{3}{2}} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \Pi (\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}) \right] \\
+ (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \Pi (\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}) \right\} \\
+ (q^{-1} + q)(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + (q^{-1} + q) \Sigma (\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}) \\
- (q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \Pi (\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}) \\
+ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \Pi (\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}) \right\}, \tag{4.6}
\]

\[
b_n = q^{\frac{n}{2}} \frac{\Pi (q^{n} - q^{\frac{n}{2}} - \alpha \frac{q^{n}}{2} - \alpha^{-\frac{n}{2}} q^{\frac{n}{2}})}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q^{\frac{3}{2}} - q^{-\frac{3}{2}}) (q^2 - q^{-2})},
\]

\[\square\]
\[(4.7)\]
\[
\bar{W}_1 = \bar{W} \left( a; a, \sqrt{\frac{q}{q}}, a, \sqrt{\frac{q}{\alpha}}, a, \sqrt{\frac{q}{\beta}}, a, \sqrt{\frac{q}{\gamma}}, a, \sqrt{\frac{q}{\delta}}, a, \sqrt{\frac{q}{\epsilon}} \right),
\]
\[
\bar{W}_2 = \bar{W} \left( \frac{q}{a}, \frac{\sqrt{\alpha q}}{a}, \frac{\sqrt{\beta q}}{a}, \frac{\sqrt{\gamma q}}{a}, \frac{\sqrt{\delta q}}{a}, \frac{\sqrt{\epsilon q}}{a} \right),
\]
\[
a = \left( \frac{\alpha \beta \gamma \delta \epsilon}{q} \right)^\frac{1}{2},
\]

and product \( \Pi \) and summation \( \Sigma \) are taken over parameters \( \alpha, \beta, \gamma, \delta, \epsilon \). If one of the parameters \( \beta, \gamma, \delta, \epsilon \) is \( q^N \), \( N \) an odd integer, \( (\alpha, \beta, \gamma, \delta, \epsilon) \to (\alpha^4, \beta^4, \gamma^4, \delta^4, \epsilon^4) \) and the base \( q \) is changed to \( q^4 \), then (4.5) becomes

\[(4.8)\]
\[
\frac{1}{a_0 - a_1 - a_2 - \cdots} = 2 \left( a_0 - \frac{q^2}{\alpha^2} \frac{\varphi'}{\varphi} \right)^{-1},
\]

where

\[(4.9)\]
\[
\frac{1}{\varphi'} = \left( \frac{q \alpha \beta \gamma \delta, q \alpha \gamma \delta \epsilon, q \alpha \beta \epsilon \delta, q \alpha \epsilon \beta \gamma, q \alpha \epsilon \beta \delta, q \alpha \epsilon \delta \beta, q \epsilon \beta \gamma \delta, q \epsilon \beta \gamma \delta \epsilon, q \epsilon \beta \gamma \delta \epsilon \right)_\infty,
\]
\[
\frac{1}{\varphi'} = \left( q^4 \alpha \beta \gamma \delta, q^4 \alpha \gamma \delta \epsilon, q^4 \alpha \beta \epsilon \delta, q^4 \alpha \epsilon \beta \gamma, q^4 \alpha \epsilon \beta \delta, q^4 \alpha \epsilon \delta \beta, q^4 \epsilon \beta \gamma \delta, q^4 \epsilon \beta \gamma \delta \epsilon, q^4 \epsilon \beta \gamma \delta \epsilon \right)_\infty,
\]

with \( a_n \) and \( b_n \) modified accordingly.

Proof. We write \( \alpha = \frac{a^2 q}{\beta}, \beta = \frac{a^2 q}{\gamma}, \gamma = \frac{a^2 q}{\delta}, \delta = \frac{a^2 q}{\epsilon} \) and make the appropriate substitutions from (2.9), (2.12) and (2.8) into Theorem 4. A considerable amount of algebra is required to reexpress (2.7) as (4.6) for which we used 'Maple' software on the computer. We use Lemma 6 after interchanging say \( b \) and \( f \) to arrive at (4.8). The result is the same for either type of termination (3.8) or (3.9). Note that (4.8) can be reexpressed in the form

\[
- \frac{q^2}{\alpha^2} \frac{\varphi'}{\varphi} = \frac{1}{a_0 - a_1 - a_2 - \cdots}
\]

which is a \( q \)-analogue of Theorem B in Section 1. \( \square \)

5. Ordinary cases \( s = q^n, q^4, \ldots. \) By substituting \( s = q^m, m \) integer greater than 2, into (3.5) of Theorem 4 we obtain

**Corollary 9.** For \( s = q^m, m = 3, 4, \ldots \)

\[(5.1)\]
\[
\frac{1}{a_0 - a_1 - a_2 - \cdots} = \frac{q^{(m-3)/2} (1 - q^{m-1}) (1 - \frac{q}{\varphi})}{(1 - q^{m-1}) (1 - \frac{q}{\varphi}) (1 - \frac{q}{\dot{\varphi}}) (1 - \frac{q}{\ddot{\varphi}}) (1 - \frac{q}{\varphi'})}
\]
\[
\times \left[ \phi \left( \frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, q^{-m+2}, q \right) \frac{\bar{W}_2}{\bar{W}_1} \frac{(a)_\infty (aq)_\infty (q)_\infty}{(a^4)_\infty (aq^4)_\infty (q^4)_\infty} \frac{1}{(1 - q^{m-1})}
\]
\[
\times \left( \frac{b}{a} q^{m-1}, \frac{c}{a} q^{m-1}, \frac{d}{a} q^{m-1}, \frac{e}{a} q^{m-1}, \frac{f}{a} q^{m-1} \right) \right]
\]

where \( a_n, b_n \) are given by (2.7) and (2.8) for \( s = q^m, m \geq 3 \).

Note that if we substitute \( s = q^{m+1} \) into (5.1) the right side reduces to

\[(5.2)\]
\[
\frac{q}{a^2} \frac{\bar{W}_2}{\bar{W}_1} \frac{(a^2)_\infty (aq)_\infty}{(a^4)_\infty (aq^4)_\infty}
\]
and (5.1) agrees with (4.5) when the indeterminacy in $a_0$ and $b_1$ is taken into account, that is

$$
\lim_{s \to q} a_0(s) = \lim_{n \to 0} a_n(s = q)
$$

(5.3)

$$
\lim_{s \to q} b_1(s) = 2 \lim_{n \to 1} b_n(s = q).
$$

It is the $a_0$ and $b_1$ on the left of (5.3) that should occur in (5.1) with $s = q$ while it is $\lim_{n \to 0} a_n(s = q)$ and $\lim_{n \to 1} b_n(s = q)$ which are the $a_0$ and $b_1$ that occur in (4.5).

Similarly for $s = q^2$, the right side of (5.1) becomes

$$
-\frac{a}{q^2} \frac{(1 - q)}{(1 - \frac{a}{q})(1 - \frac{2}{q})(1 - \frac{4}{q})(1 - \frac{5}{q})} \left[ 1 - \frac{(aq)_\infty(\frac{q}{a})_\infty}{(\frac{q^2}{a})_\infty(\frac{2}{a})_\infty} \right].
$$

This agrees with (4.1) since

$$
\lim_{s \to q^2} a_0(s) = \lim_{n \to 0} a_n(s = q^2) + \frac{aq^2(1 - \frac{b}{a})(1 - \frac{c}{a})(1 - \frac{d}{a})(1 - \frac{e}{a})(1 - \frac{f}{a})}{2(1 - q)}
$$

$$
\lim_{s \to q^2} b_1(s) = \lim_{n \to 1} b_n(s = q^2).
$$

One can also consider the terminating case of (5.1) by taking into account Lemma 6.

Remarks

1) If in Section 2 we make the replacements $a \to \lambda a$, $b \to \lambda q$, $c \to \lambda q$, $d \to \lambda q$, $e \to ae^{i\theta}$, $f \to ae^{-i\theta}$ and let $\lambda \to \infty$, then we obtain solutions to the recurrence for Askey-Wilson polynomials [1] with $s = abcd = q^m$, $m = 1, 2, \ldots$ [7]. By applying the above limit to Corollary 7,8, and 9 we recover equations (22), (23) and (24) respectively of Gupta and Masson [7]. Note that [7, (22), (23)] give the $q$-analogue of Ramanujan’s Entries 35 and 39 [3], [13], while Corollaries 7 and 8 are the $q$-analogues of Corollaries 6 and 7 of [15].

2) The Corollary 8 case $s = q$ is particularly interesting since the approximants of the continued fraction

$$
\frac{1}{a_0} - \frac{2b_1}{a_1} - \frac{b_2}{a_2} - \cdots = \frac{q(aq)_\infty(\frac{q}{a})_\infty}{a^2(\frac{q}{a})_\infty(\frac{2}{a})_\infty} \frac{W_2}{W_1}
$$

are then given explicitly in terms of $X_n^{(1)}$ and $X_n^{(2)}$. To see this we note that the initial conditions

$$
2X_1^{(1)} - a_0X_0^{(1)} = 0
$$

$$
X_2^{(2)} - a_1X_1^{(2)} = 0
$$

which follow from (2.7), (2.8), (2.9) and (2.12) when $s = q$, together with the recurrence (2.6), imply that for $s = q$

$$
\frac{1}{a_0} - \frac{2b_1}{a_1} - \frac{b_2}{a_2} - \cdots - \frac{b_n}{a_n} = \frac{X_{n+1}^{(2)}X_0^{(1)}}{2X_{n+1}^{(1)}X_1^{(2)}}, \quad n \geq 0.
$$
3) In the limit as \( m \to \infty \) \((s = q^m \to 0)\) Corollary 9 yields a new continued fraction result given by

\[
\frac{1}{c_0} - \frac{d_1}{c_1} - \frac{d_2}{c_2} - \cdots = \frac{(1 - \frac{a}{q})}{q(1 - \frac{a}{q})(1 - \frac{a}{q})(1 - \frac{a}{q})} \\
\times \left[ W\left( \frac{a}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f} \right) - R \right],
\]

\[
R = \frac{\left( \frac{a}{q}, \frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f} \right)_\infty}{\left( \frac{a}{q}, \frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f} \right)_\infty} \times \left\{ 3\phi_2\left( \frac{q}{a}, \frac{q}{a}; q, b \right) \right\}
\]

where

\[
(5.6) \quad c_n = \left[ -\left( 1 - \frac{a}{b}q^{n+1} \right) \left( 1 - \frac{a}{c}q^{n+1} \right) \left( 1 - \frac{a}{d}q^{n+1} \right) \left( 1 - \frac{a}{e}q^{n+1} \right) - q(1 - q^n)(1 - q^n)(1 - q^n)(1 - \frac{a^2q^{n+1}}{bcde}) \right] / (1 - aq^{n+1}),
\]

\[
d_n = q(1 - q^n)(1 - \frac{a}{b}q^n)(1 - \frac{a}{c}q^n)(1 - \frac{a}{d}q^n)(1 - \frac{a}{e}q^n)(1 - \frac{a^2q^{n+1}}{bcde}).
\]

A direct proof is obtained by applying our methods to the contiguous relation

\[
q(1 - \frac{1}{f}) \left( 1 - \frac{a^2q}{bcdef} \right) (1 - \frac{a}{f}) (1 - \frac{aq}{f}) W(f+) - W
\]

\[
+ \left( \frac{a}{fB} \right) (1 - \frac{aq}{fC}) (1 - \frac{aq}{fd}) (1 - \frac{aq}{fe}) W(f-) - W
\]

\[
+ \frac{a^2q^2}{bcdef} (1 - b) (1 - c) (1 - d) (1 - e) W = 0
\]

where

\[
W = W(a; b, c, d, e, f)
\]

\[
= s\phi_7\left( a, \sqrt{a}, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \right). \frac{a^2q^2}{bcdef}.
\]

The contiguous relation (5.7) is obtained from (2.5) by taking the limit \( g \to 0 \) with \( fg = \frac{a^3q^2}{bcdef} \) and then replacing \( h = q^{-n} \) by \( f \). However the termination of the above \( s\phi_7 \) is not necessary.

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