THERMODYNAMIC PROPERTIES OF THE
PIECEWISE UNIFORM STRING

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Abstract

The thermodynamic free energy $F$ is calculated for a gas whose particles are the quantum excitations of a piecewise uniform bosonic string. The string consists of two parts of length $L_I$ and $L_{II}$, endowed with different tensions and mass densities, adjusted in such a way that the velocity of sound always equals the velocity of light. The explicit calculation is done under the restrictive condition that the tension ratio $x = T_I/T_{II}$ approaches zero. Also, the length ratio $s = L_{II}/L_I$ is assumed to be an integer. The expression for $F$ is given on an integral form, in which $s$ is present as a parameter. For large values of $s$, the Hagedorn temperature becomes proportional to the square root of $s$. 
1 Introduction

In conventional theories of the bosonic string in D-dimensional spacetime the string is taken to be uniform, throughout the whole of its length $L$. The composite string model, in which the (relativistic) string is assumed to consist of two or more separately uniform pieces, is a variant of the conventional theory. The composite model was introduced by the present authors in 1990 [1]; the string was there assumed to consist of two pieces $L_I$ and $L_{II}$. The dispersion equation was derived, and the Casimir energy calculated for various integer values of the length ratio $s = L_{II}/L_I$. Later on, the composite string model has been generalized and further studied from various points of view [2] - [8].

It may be useful to summarize some reasons why the composite string model turns out to be an attractive model. First, if one does Casimir energy calculations one finds that the system is remarkable easy to regularize: one has access to the cutoff method [1], the complex contour integration method [3] - [5], or the Hurwitz $\zeta$-function method [2], [4], [5], [7]. (Ref. [8] contains a review of the various regularization methods.) As a physical result of the Casimir energy calculations it is also worth noticing that the energy is in general nonpositive, and is more negative the larger is the number of
uniform pieces in the string. Second, the composite string model may serve as
a useful two-dimensional field theoretical model of the early universe. These
aspects have recently been discussed more closely in [6]. Finally, as a possible
practical application of the composite string model, we mention the recent
attempt [9] that has been done to analyse the influence from the Casimir
effect for uniform strings on the swimming of micro-organisms. Probably
would here a composite string model be more adequate than the very simple
uniform string model.

The purpose of the present paper is to calculate thermodynamic quantities
- the free energy (the one-loop partition function) for the composite two-piece
string. It ought to be emphasized that the present string model is relativistic,
in the sense that the velocity of sound is always taken to be equal to c. As far
as we are aware, this task has not been undertaken before. When we come to
concrete calculations, we shall put $D = 26$. Moreover we shall limit ourselves
to the limiting case in which the tension ratio, defined as $x = T_I / T_{II}$, goes
to zero. As shown earlier [1], this case leads to significant simplifications in
the formalism, but is yet nontrivial enough to show the essential physical
behaviour of the system. Another simplification is that we shall assume $s$ to
be an integer.
In the next section we discuss briefly the essentials of the theory of the classical planar string: the general dispersion equation, and the junction conditions as well as the eigenvalue spectrum in the case of $x \to 0$. In Sec. \ref{sec:2} we develop the classical theory of the composite string in flat $D$-dimensional spacetime. The string coordinates $X^\mu$ are expanded into oscillator coordinates, and formulas are derived for the Hamiltonian and the string mass. In Sec. \ref{sec:3} we derive the quantum theory, assuming $D = 26$, focusing attention on the free energy $F$ and the critical Hagedorn temperature. Finally we end with some conclusions in Sec. 5.

2 Planar Oscillations of the Classical String in the Minkowski Space

We begin by considering the classical theory of the oscillating two-piece string in the Minkowski space. The total length of the string is $L$. For later purpose we shall set $L = \pi$. With $L_I$, $L_{II}$ denoting the length of the two pieces, we thus have $L_I + L_{II} = \pi$. As mentioned the string is relativistic, in the sense that the velocity $v_s$ of transverse sound is everywhere required to be equal
to the velocity of light ($\hbar = c = 1$):

$$v_s = (T_I/\rho_I)^{1/2} = (T_{II}/\rho_{II})^{1/2} = 1. \quad (1)$$

Here $T_I, T_{II}$ are the tensions and $\rho_I, \rho_{II}$ are the mass densities of the two pieces. We let $s$ denote the length ratio and $x$ the tension ratio:

$$s = L_{II}/L_I, \quad x = T_I/T_{II}. \quad (2)$$

Assume now that the transverse oscillations of the string, called $\psi(\sigma, \tau)$, are linear, and take place in the plane of the string. (We employ usual notation, so that $\sigma$ is the position coordinate and $\tau$ the time coordinate of the string.)

We can thus write in the two regions

$$\psi_I = \xi_I e^{i\omega(\sigma-\tau)} + \eta_I e^{-i\omega(\sigma+\tau)},$$

$$\psi_{II} = \xi_{II} e^{i\omega(\sigma-\tau)} + \eta_{II} e^{-i\omega(\sigma+\tau)}, \quad (3)$$

with the $\xi$ and $\eta$ being constants. Taking into account the junction conditions at $\sigma = 0$ and $\sigma = L_I$, meaning that $\psi$ itself as well as the transverse force $T \partial \psi/\partial \sigma$ be continuous, we obtain the dispersion equation

$$\frac{4x}{(1-x)^2} \sin^2 \frac{\omega \pi}{2} + \sin \left( \frac{\omega \pi}{1+s} \right) \sin \left( \frac{\omega s \pi}{1+s} \right) = 0 \quad (4)$$

(more details are given in [1]). From this equation the eigenvalue spectrum can be calculated, for arbitrary values of $x$ and $s$. Because of the invariance
under the substitution $x \rightarrow 1/x$, one can restrict the ratio $x$ to lie in the interval $0 < x \leq 1$. Similarly, because of the invariance under the interchange $L_I \leftrightarrow L_{II}$ one can take $L_{II}$ to be the larger of the two pieces, so that $s \geq 1$.

In the following we shall impose two simplifying conditions: (i) We take the tension ratio limit to approach zero,

$$x \rightarrow 0.$$  \hspace{1cm} (5)

Assuming $T_{II}$ to be a finite quantity, this limit implies that $T_I \rightarrow 0$. From the junction conditions given in [1] we obtain in this limit the equations

$$\xi_I + \eta_I = \xi_{II}e^{i\pi \omega} + \eta_{II}e^{-i\pi \omega},$$  \hspace{1cm} (6)

$$\xi_I e^{2\pi i \omega/(1+s)} + \eta_I = \xi_{II} e^{2\pi i \omega/(1+s)} + \eta_{II},$$  \hspace{1cm} (7)

$$\xi_{II} e^{2\pi i \omega} = \eta_{II},$$  \hspace{1cm} (8)

$$\xi_{II} e^{2\pi i \omega/(1+s)} = \eta_{II}.$$  \hspace{1cm} (9)

According to the dispersion equation (4) we obtain now two sequences of modes. The eigenfrequencies are seen to be proportional to integers $n$, and will for clarity be distinguished by separate symbols $\omega_n(s)$ and $\omega_n(s^{-1})$:

$$\omega_n(s) = (1 + s)n,$$  \hspace{1cm} (10)
\( \omega_n(s^{-1}) = (1 + s^{-1})n, \)  \( (11) \)

with \( n = \pm 1, \pm 2, \pm 3, \ldots \), corresponding to the first and the second branch.

(ii) Our second condition is that the length ratio \( s \) is an integer, \( s = 1, 2, 3, \ldots \).

3 Classical String in Flat \( D \)-Dimensional Space-time

3.1 Oscillator Coordinates. The Hamiltonian

We are now able to generalize the theory. We consider henceforth the motion of a two-piece classical string in flat \( D \)-dimensional space-time. Following the notation in [10] we let \( X^\mu(\sigma, \tau) \) (\( \mu = 0, 1, 2, \cdots (D - 1) \)) specify the coordinates on the world sheet. For each of the two branches - corresponding to Eqs. [10] and [11] respectively - we can write the general expression for \( X^\mu \) in the form

\[
X^\mu = x^\mu + \frac{p^\mu}{\pi T(s)} + \theta(L_I - \sigma)X^\mu_I + \theta(\sigma - L_I)X^\mu_{II}, \quad (12)
\]

\[ \]
where $x^\mu$ is the center of mass position and $p^\mu$ is the total momentum of the string. Besides $\bar{T}(s)$ denotes the mean tension,

$$\bar{T}(s) = \frac{1}{\pi} (L_I T_I + L_{II} T_{II}) \rightarrow \frac{s}{1 + s} T_{II}. \quad (13)$$

The second term in (12) implies that the string’s translational energy $p^0$ is set equal to $\pi \bar{T}(s)$. This generalizes the relation $p^0 = \pi T$ that is known to hold for a uniform string [10]. The two last terms in (12) contain the step function, $\theta(x > 0) = 1$, $\theta(x < 0) = 0$. To show the structure of the decomposition of $X^\mu$ into fundamental model we give here the expressions for $X_I^\mu$ for each of the two branches: for the first branch

$$X_I^\mu = \frac{i}{2} l(s) \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^\mu(s) e^{i(1+s)n(\sigma - \tau)} + \tilde{\alpha}_n^\mu(s) e^{-i(1+s)n(\sigma + \tau)} \right], \quad (14)$$

where the $\alpha_n, \tilde{\alpha}_n$ are oscillator coordinates of the right- and left-moving waves respectively. The sum over $n$ goes over all positive and negative integers except from zero. The factor $l(s)$ is a constant. For the second branch in region I, analogously

$$X_I^\mu = \frac{i}{2} l(s^{-1}) \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n^\mu(s^{-1}) e^{i(1+s^{-1})n(\sigma - \tau)} + \tilde{\alpha}_n^\mu(s^{-1}) e^{-i(1+s^{-1})n(\sigma + \tau)} \right], \quad (15)$$
where \( l(s^{-1}) \) is another constant, which in principle can be different from \( l(s) \). Since \( X^\mu \) is real, we must have

\[
\alpha^\mu_{-n} = (\alpha^\mu_n)^*, \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*.
\]

(16)

When writing expressions (14) and (15), we made use of Eqs. (10) and (11) for the eigenfrequencies. The condition \( x \to 0 \) was thus used. The condition that \( s \) be an integer has however not so far been used. This condition will be of importance when we construct the expression for \( X^\mu_{II} \). Before doing this, let us however consider the constraint equation for the composite string. Conventionally, when the string is uniform the two-dimensional energy-momentum tensor \( T_{\alpha\beta} (\alpha, \beta = 0, 1) \), obtainable as the variational derivative of the action \( S \) with respect to the two-dimensional metric, is equal to zero. In particular, the energy density component is then \( T_{00} = 0 \) locally. In the present case the situation is more complicated, due to the fact that the presence of the junctions restricts the freedom of the variations \( \delta X^\mu \). We cannot put \( T_{\alpha\beta} = 0 \) locally anymore. What we have at our disposal, is the expression for the action

\[
S = -\frac{1}{2} \int d\tau d\sigma T(\sigma) \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu,
\]

(17)
where $T(\sigma)$ is the position-dependent tension

$$T(\sigma) = T_I + (T_{II} - T_I)\theta(\sigma - L_I). \quad (18)$$

The momentum conjugate to $X^\mu$ is $P^\mu(\sigma) = T(\sigma)\dot{X}^\mu$. The Hamiltonian of the two-dimensional sheet becomes accordingly (here $L$ is the Lagrangian)

$$H = \int_0^\pi \left[ P_\mu(\sigma)\dot{X}^\mu - L \right] d\sigma = \frac{1}{2} \int_0^\pi T(\sigma)(\dot{X}^2 + X'^2) d\sigma. \quad (19)$$

The basic condition that we shall impose, is that $H = 0$ when applied to the physical states. This is a more weak condition than the strong condition $T_{\alpha\beta} = 0$ applicable for a uniform string.

### 3.2 Classical Mass Formula. The First Branch

Assume that $s$ is an arbitrary integer, $s = 1, 2, 3, \ldots$. When $s$ is different from 1, we have to distinguish between the eigenfrequencies $\omega_n(s)$ and $\omega_n(s^{-1})$ for the first and the second branch. Let us consider the first branch. In region I, the representation for the right- and left- moving modes was given above, in Eq. (14). For reasons that will become clear from the quantum mechanical discussion later, we will choose $l(s)$ equal to

$$l(s) = \frac{1}{\sqrt{\pi T_I}}. \quad (20)$$
Since we have assumed $T_I$ to be small, the expression (20) will tend to infinity. This is actually a delicate point: as we will see later, in Eq. (31), the expression for the Hamiltonian $H_I$ in region I derived with the use of Eq. (20) is independent of the tension. Thus $H_I$ behaves in a ‘normal’ way. The coordinates $X'_I$ in region I themselves diverge, but after multiplication with the very small string tension $T_I$ they become suppressed and lead to a finite expression for $H_I$. Accordingly, if one quantizes the system starting from the canonical commutation relations in region I, one arrives at a standard quantum mechanical picture in this region. (We will return to this point in Sec. 4.)

When writing the analogous mode expansion in region II, we have to observe the junction conditions (6) - (9), which hold for all $s$. For the first branch $\omega_n(s)$, and for odd values of $s$, it is seen that the junction conditions impose no restriction on the values of $n$. All frequencies, corresponding to $n = \pm 1, \pm 2, \pm 3, \cdots$, permit the waves to propagate from region I to region II. Equations (6) - (9) reduce in this case to the equations

$$\xi_I + \eta_I = 2\xi_{II} = 2\eta_{II}, \quad (21)$$

which show that the right- and left-moving amplitudes $\xi_I$ and $\eta_I$ in region I
can be chosen freely and that the amplitudes $\xi_{II}, \eta_{II}$ in region II are thereafter fixed. Transformed into oscillator coordinate language, this means that $\alpha^\mu_n$ and $\tilde{\alpha}^\mu_n$ can be chosen freely.

If $s$ is an even integer, then the validity of Eqs.(21) requires $n$ in Eq.(10) to be even. If $n$ is odd, the junction conditions reduce instead to

$$\xi_I + \eta_I = 0, \quad \xi_{II} = \eta_{II} = 0,$$

which show that the waves are now unable to penetrate into region II. The oscillations in region I are in this case standing waves.

The expansion for the first branch in region II can in view of (21) be written

$$X^{\mu}_{II} = \frac{i}{2\sqrt{\pi} T_I} \sum_{n \neq 0} \frac{1}{n} \gamma^\mu_n(s) e^{-i(1+s)n\tau} \cos[(1 + s)n\sigma],$$

where we have defined $\gamma_n(s)$ as

$$\gamma^\mu_n(s) = \alpha^\mu_n(s) + \tilde{\alpha}^\mu_n(s), \quad n \neq 0.$$  

The oscillations in region II are thus standing waves; this being a direct consequence of the condition $x \to 0$.

It is useful to introduce light-cone coordinates, $\sigma^- = \tau - \sigma$ and $\sigma^+ = \tau + \sigma$.  

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The derivatives conjugate to \( \sigma^\mp \) are \( \partial_\mp = \frac{1}{2}(\partial_\tau \mp \partial_\sigma) \). In region I we find

\[
\partial_- X^\mu = \frac{1 + s}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \alpha_\mu^\nu(n) e^{i(1+s)n(\sigma-\tau)},
\]

\[
\partial_+ X^\mu = \frac{1 + s}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \tilde{\alpha}_\mu^\nu(n) e^{-i(1+s)n(\sigma+\tau)},
\]

where we have defined

\[
\alpha_\mu^\nu_0(s) = \tilde{\alpha}_\mu^\nu_0(s) = \frac{p^\mu}{T_{II}s} \sqrt{\frac{T_I}{\pi}}.
\]

Further, in region II we find

\[
\partial_\mp X^\mu = \frac{1 + s}{4\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \gamma_\mu^\nu(s) e^{\pm i(1+s)n(\sigma^\mp)},
\]

with

\[
\gamma_\mu^\nu_0(s) = \frac{2p^\mu}{T_{II}s} \sqrt{\frac{T_I}{\pi}} = 2\alpha_\mu^\nu_0(s).
\]

Inserting Eqs. (24) and (27) into the Hamiltonian

\[
H = \int_0^\pi T(\sigma)(\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X) d\sigma
\]

we get, for the full first branch

\[
H = H_I + H_{II},
\]

where

\[
H_I = T_I \int_I (\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X) d\sigma
\]
\[ H_{II} = \frac{s(1+s)}{8x} \sum_{-\infty}^{\infty} \gamma_n(s) \cdot \gamma_n(s), \quad (32) \]

with \( x = T_I/T_{II} \) as before.

The case \( s = 1 \) is of particular interest. The string is then divided into two pieces of equal length. We have then

\[ H_I(s = 1) = \frac{1}{2} \sum_{-\infty}^{\infty} (\alpha_n \cdot \alpha_n + \tilde{\alpha}_n \cdot \tilde{\alpha}_n), \quad (33) \]

\[ H_{II}(s = 1) = \frac{1}{4x} \sum_{-\infty}^{\infty} \gamma_n \cdot \gamma_n. \quad (34) \]

It is notable that Eq. (33) is formally the same as the standard expression for a closed uniform string, of length \( \pi \). See, for instance, Eq. (2.1.76) in Ref. [10]. The fact that we recover the characteristics of a closed string in region I is understandable, since this part of our composite string permits both right-moving and left-moving waves. The absence of any tension-dependent factor in front of the expression (33) is related to our choice (20) for the length \( l(s) \). Moreover, Eq. (34) is essentially the standard expression for an open uniform
string, corresponding to standing waves. The presence of the inverse tension ratio $1/x$ in front of the expression is caused by the junction conditions, Eqs. (21).

The condition $H = 0$ enables us to calculate the mass $M$ of the string. It must be given by $M^2 = -p^\mu p_\mu$, similarly as in the uniform string case [10]. From Eqs. (31) and (32) we obtain, taking into account that $x << 1$ and that $\alpha_0(s) \cdot \alpha_0(s) = -M^2 x/(\pi T_{II} s^2)$,

$$M^2 = \pi T_{II} s \sum_{n=1}^{\infty} \left[ \alpha_{-n}(s) \cdot \alpha_n(s) + \tilde{\alpha}_{-n}(s) \cdot \tilde{\alpha}_n(s) + \frac{s}{2x} \gamma_{-n}(s) \cdot \gamma_n(s) \right].$$  \hspace{1cm} (35)

This holds for the first branch, for odd/even values of $s$.

### 3.3 The Second Branch

For the second branch whose eigenfrequencies are $\omega(s^{-1})$ the mode expansion in region I becomes

$$X_I^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n(s^{-1}) e^{i(1+s^{-1})n(\sigma-\tau)} + \tilde{\alpha}_n(s^{-1}) e^{-i(1+s^{-1})n(\sigma+\tau)} \right].$$  \hspace{1cm} (36)

Analogously in region II
where

\[ \gamma^\mu_n(s^{-1}) = \alpha^\mu_n(s^{-1}) + \tilde{\alpha}^\mu_n(s^{-1}), \quad n \neq 0. \]  

(38)

The expansions (36) and (37) hold for all integers \( s \). This is so because the basic expressions (10) and (11) for the eigenfrequencies hold for all values of \( s \). However it may be noted that if the junction conditions are required to imply nonvanishing oscillations in region II, corresponding to nonvanishing right hand sides in Eq.(21), then further restrictions come into play. Namely, if \( s \) is odd, the index \( n \) in Eqs.(36) and (37) has to be a multiple of \( s \). If \( s \) is even, then \( n \) has to be an even integer times \( s \). We recall that analogous considerations were made in the case of the first branch. When we later shall consider the quantum mechanical free energy, it becomes necessary to include all modes, including those that lead to zero oscillations in region II according to the classical theory.

Let us calculate the light-cone derivatives: in region I they are

\[
\partial_\perp X^\mu = \frac{1 + s^{-1}}{2\sqrt{\pi I_I}} \sum_{n=1}^{\infty} \alpha^\mu_n(s^{-1}) e^{i(1+s^{-1})n\sigma}.
\]
\[ \partial_\pm X^\mu = \frac{1 + s^{-1}}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \tilde{\alpha}_n^\mu (s^{-1}) e^{-i(1+s^{-1})n(\sigma+\tau)}, \tag{39} \]

and in region II

\[ \partial_\pm X^\mu = \frac{1 + s^{-1}}{4\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \gamma_n^\mu (s^{-1}) e^{\pm i(1+s^{-1})n(\sigma+\tau)}, \tag{40} \]

where

\[ \alpha_0^\mu (s^{-1}) = \tilde{\alpha}_0^\mu (s^{-1}) = \frac{1}{2} \gamma_0^\mu (s^{-1}) = \frac{p^\mu}{T_{II}} \sqrt{\frac{T_I}{T_{II}}}. \tag{41} \]

Thus \( \alpha_0 (s^{-1}) \) differs from \( \alpha_0 (s) \), Eq. (26). Again writing the Hamiltonian as

\[ H = H_I + H_{II}, \]

we now find

\[ H_I = \frac{1 + s^{-1}}{4s} \sum_{-\infty}^{\infty} \left[ \alpha_{-n} (s^{-1}) \cdot \alpha_n (s^{-1}) + \tilde{\alpha}_{-n} (s^{-1}) \cdot \tilde{\alpha}_n (s^{-1}) \right], \tag{42} \]

\[ H_{II} = \frac{1 + s^{-1}}{8x} \sum_{-\infty}^{\infty} \gamma_{-n} (s^{-1}) \cdot \gamma_n (s^{-1}). \tag{43} \]

If \( s = 1 \), we recover the same expressions for \( H_I \) and \( H_{II} \), Eqs. (33) and (34), as for the first branch.

Thus from the condition \( H = 0 \) we calculate the mass, observing that \( \alpha_0 (s^{-1}) \cdot \alpha_0 (s^{-1}) = -M^2 x/(\pi T_{II}) \):

\[ M^2 = \frac{\pi T_I s}{s} \sum_{n=1}^{\infty} \left[ \alpha_{-n} (s^{-1}) \cdot \alpha_n (s^{-1}) + \tilde{\alpha}_{-n} (s^{-1}) \cdot \tilde{\alpha}_n (s^{-1}) \right] \]

\[ + \frac{\pi T_{II}}{2x} \sum_{n=1}^{\infty} \gamma_{-n} (s^{-1}) \cdot \gamma_n (s^{-1}). \tag{44} \]
4 Quantum Theory. The Free Energy of the String

4.1 Quantization

We shall consider the free energy of the quantum fields with masses given by the mass formula corresponding to the piecewise bosonic string. Our starting point is the following expression for the free energy $F$, at finite temperature $T$, of free fields of mass $M$ in $D$-dimensions:

$$\beta F = -\ln Z = \frac{1}{2} \beta \sum_{n=-\infty}^{\infty} \omega_n - \beta \sum_{m=1}^{\infty} \int_0^{\infty} \frac{du}{u} (2\pi u)^{-D/2} \exp \left( -\frac{M^2 u}{2} - \frac{m^2 \beta^2}{2u} \right).$$

(45)

Here $\beta = 1/k_B T$, and $Z$ is the partition function. We follow the formalism of Ref. [11]; some other related references are [12] - [18]. The constituent “fields” of the quantum gas are the excitations associated with the modes of a single string.

In Eq.(45) we thus need to know the expression for $M^2$ in the quantum theory. We quantize the system according to conventional methods as found, for instance, in Ref. [10], Chapter 2.2. In accordance with the canonical prescription in region I the equal-time commutation rules are required to be
\[ T_I[\dot{X}^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i\delta(\sigma - \sigma')\eta^{\mu\nu}, \] (46)

and in region II

\[ T_{II}[\dot{X}^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i\delta(\sigma - \sigma')\eta^{\mu\nu}, \] (47)

where \( \eta^{\mu\nu} \) is the \( D \)-dimensional metric. These relations are in conformity with the fact that the momentum conjugate to \( X^\mu \) is in either region equal to \( T(\sigma)\dot{X}^\mu \). The remaining commutation relations vanish:

\[ [X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = [\dot{X}^\mu(\sigma, \tau), \dot{X}^\nu(\sigma', \tau)] = 0. \] (48)

The quantities to be promoted to Fock state operators are \( \alpha_{\mp n}(s) \) and \( \tilde{\alpha}_{\mp n}(s) \) (first branch, region I), \( \gamma_{\mp n}(s) \) (first branch, region II), \( \alpha_{\mp n}(s^{-1}) \) and \( \tilde{\alpha}_{\mp n}(s^{-1}) \) (second branch, region I), and \( \gamma_{\mp n}(s^{-1}) \) (second branch, region II). These operators satisfy

\[
\begin{align*}
\alpha_{-n}(s) &= \alpha_n^{\dagger}(s), \quad \gamma_{-n}(s) = \gamma_n^{\dagger}(s), \\
\alpha_{-n}(s^{-1}) &= \alpha_n^{\dagger}(s^{-1}), \quad \gamma_{-n}(s^{-1}) = \gamma_n^{\dagger}(s^{-1})
\end{align*}
\] (49)

for all \( n \). We insert our previous expansions for \( X^\mu \) and \( \dot{X}^\mu \) in the commutation relations in regions I and II for the two branches, and make use of the
effective relationship

\[
\sum_{-\infty}^{\infty} e^{i(1+s)n(\sigma - \sigma')} = 2 \sum_{-\infty}^{\infty} \cos(1 + s)n\sigma \cos(1 + s)n\sigma' \to \frac{2\pi}{1+s}\delta(\sigma - \sigma'). \quad (50)
\]

For the first branch we then get in region I

\[
[\alpha_n^\mu(s), \alpha_m^\nu(s)] = n\delta_{n+m,0}\eta^{\mu\nu}, \quad (51)
\]

with a similar relation for the \(\tilde{\alpha}_n\). In region II

\[
[\gamma_n^\mu(s), \gamma_m^\nu(s)] = 4nx\delta_{n+m,0}\eta^{\mu\nu}. \quad (52)
\]

For the second branch we get analogously

\[
[\alpha_n^{\mu(-1)}(s), \alpha_m^{\nu(-1)}(s)] = n\delta_{n+m,0}\eta^{\mu\nu}, \quad [\gamma_n^{\mu(-1)}(s), \gamma_m^{\nu(-1)}(s)] = 4nx\delta_{n+m,0}\eta^{\mu\nu}. \quad (53)
\]

By introducing annihilation and creation operators for the first branch in the following way:

\[
\alpha_n^\mu(s) = \sqrt{n}a_n^\mu(s), \quad \alpha_n^{\mu(-1)}(s) = \sqrt{n}a_n^\mu(s), \\
\gamma_n^\mu(s) = \sqrt{4nx}c_n^{\mu\nu}(s), \quad \gamma_n^{\mu(-1)}(s) = \sqrt{4nx}c_n^{\mu\nu}(s), \quad (54)
\]

we find for \(n \geq 1\) the standard form

\[
[a_n^\mu(s), a_m^{\nu\dagger}(s)] = \delta_{nm}\eta^{\mu\nu}, \quad (55)
\]
\[ [c^\mu_n(s), c^\nu_m(s)] = \delta_{nm} \eta^{\mu\nu}. \] (56)

The commutation relations for the second branch are analogous, only with the replacement \( s \rightarrow s^{-1} \).

4.2 The Free Energy and the Hagedorn Temperature

In the following we shall limit ourselves to the first branch only. Using Eqs. (54) in Eqs.(31) and (32) we may write the two parts of the Hamiltonian as

\[ H_I = -\frac{M^2 x}{2st(s)} + \frac{1}{2} \sum_{n=1}^{\infty} \omega_n(s) [a^\dagger_n(s) \cdot a_n(s) + \bar{a}^\dagger_n(s) \cdot \bar{a}_n(s)], \] (57)

\[ H_{II} = -\frac{M^2}{2t(s)} + s \sum_{n=1}^{\infty} \omega_n(s) c^\dagger_n(s) \cdot c_n(s), \] (58)

where we for convenience have introduced the symbol \( t(s) \) defined by

\[ t(s) = \pi \bar{T}(s). \] (59)

(Observe the notation \( c^\dagger_n \cdot c_n \equiv c^\mu_n c_{n\mu} \)). The extra factor \( s \) in Eq. (58) is related to the fact that the relative length of region II is equal to \( s \). From the condition \( H = H_I + H_{II} = 0 \) in the limit \( x \rightarrow 0 \) we obtain, either from
Eqs. (57) and (58) or directly from Eq. (35),

\[
M^2 = t(s) \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n(s) [a_{ni}^\dagger(s)a_{ni}(s) + \tilde{a}_{ni}^\dagger \tilde{a}_{ni}(s) - A_1] \\
+ 2st(s) \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n(s) [c_{ni}^\dagger(s)c_{ni}(s) - A_2].
\] (60)

We have here put \( D = 26 \), the commonly accepted space-time dimension for the bosonic string. As usual, the \( c_{ni} \) denote the transverse oscillator operators (here for the first branch). Further, we have introduced in Eq.(60) two constants \( A_1 \) and \( A_2 \), in order to take care of ordering ambiguities.

In Eq. (45), there occurs a zero-point energy \( \frac{1}{2} \sum \omega_n \), summed over all eigenfrequencies. Apart from an infinite constant of no physical significance, this is actually the Casimir energy, which was calculated in [1]. When \( x \to 0 \) we have, for arbitrary \( s \), when the string length equals \( \pi \),

\[
\frac{1}{2} \sum_{-\infty}^{\infty} \omega_n \to -\frac{1}{24} (s + \frac{1}{s} - 2).
\] (61)

The constraint for the closed string (expressing the invariance of the theory in the region I under shifts of the origin of the co-ordinate) has the form

\[
\sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n(s) \left[ a_{ni}^\dagger(s)a_{ni}(s) - \tilde{a}_{ni}^\dagger \tilde{a}_{ni}(s) \right] = 0.
\] (62)

The commutation relations for above operators are given by Eqs.(55) and (56). The mass of state (obtained by acting on the Fock vacuum \(|0>\) with
creation operators) can be written as follows (mass)\(^2 \sim a_{n_1} \ldots a_{n_i} c_{n_1} \ldots c_{n_i}|0 \rangle.

As usual the physical Hilbert space consists of all Fock space states obeying the condition (62), which can be implemented by means of the integral representation for Kronecker deltas. Thus the free energy of the field content in the "proper time" representation becomes

\[
F = -\frac{1}{24} (s + \frac{1}{s} - 2) - 2^{-14} \pi^{-13} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \left[ \theta_3 \left( 0 \middle| \frac{i \beta^2}{2 \pi \tau_2} \right) - 1 \right] \text{Tr} \exp \left\{ -\frac{\tau_2 M^2}{2} \right\} \times \int_{-\pi}^{\pi} \frac{d\tau_1}{2\pi} \text{Tr} \exp \left\{ i \tau_1 \sum_{i=1}^{24} \sum_{n=1}^\infty \omega_n(s) [a_{n_i}^\dagger(s)a_{n_i}(s) - \tilde{a}_{n_i}^\dagger(s)\tilde{a}_{n_i}(s)] \right\}. \tag{63}
\]

Performing the trace over the entire Fock space (note that \([H_I, H_{II}] = 0\) and \(\text{Tr} y a_i^\dagger a_i = (1 - y)^{-1}\) we have

\[
\text{Tr} \exp \left\{ \sum_{i=1}^{24} \sum_{n=1}^\infty \omega_n(s)a_{n_i}^\dagger(s)a_{n_i}(s) \left( -\frac{1}{2} t(s) \tau_2 \pm i \tau_1 \right) \right\} = \prod_{n=1}^{\infty} \left[ 1 - e^{\omega_n(s)(-\frac{1}{2} t(s)\tau_2 \pm i \tau_1)} \right]^{-24}, \tag{64}
\]

\[
\text{Tr} \exp \left\{ -st(s)\tau_2 \sum_{i=1}^{24} \sum_{n=1}^\infty \omega_n(s)c_{n_i}^\dagger(s)c_{n_i}(s) \right\} = \prod_{n=1}^{\infty} \left[ 1 - e^{-st(s)\tau_2 \omega_n(s)} \right]^{-24}. \tag{65}
\]

Working out the sums in Eq. (63) for \(A_1 = 2, A_2 = 1\), and changing variables
to \( \tau_1 \to \tau_1 2\pi, \tau_2 \to \tau_2 4\pi/t(s) \) one can finally get

\[
F = -\frac{1}{24}(s + \frac{1}{s} - 2) - 2^{-40}\pi^{-26}t(s)^{-13} \int_0^{\infty} \frac{d\tau_2}{\tau_2^{1/2}} \int_{-1/2}^{1/2} d\tau_1 \left[ \theta_3 \left( 0 \mid \frac{i\beta^2 t(s)}{8\pi^2 \tau_2} \right) - 1 \right] \\
\times \left| \eta[(1 + s)\tau] \right|^{-48} \left| s(1 + s)(\tau - \overline{\tau}) \right|^{-24},
\]

where we integrate over all possible non-diffeomorphic toruses which are characterized by a single Teichmüller parameter \( \tau = \tau_1 + i\tau_2 \). In Eq. (66) the Dedekind \( \eta \)-function and the Jacobi \( \theta_3 \)-function

\[
\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi i n\tau} \right),
\]

\[
\theta_3(v|x) = \sum_{n=-\infty}^{\infty} e^{ixn^2 + 2\pi im},
\]

and the condition \( \eta(-\overline{\tau}) = \overline{\eta(\tau)} \) has been used.

Once the free energy has been found, the other thermodynamic quantities can readily be calculated. For instance, the energy \( U \) and the entropy \( S \) of the system are

\[
U = \frac{\partial(\beta F)}{\partial \beta}, \quad S = k_B \beta^2 \frac{\partial F}{\partial \beta}.
\]

What is the Hagedorn temperature, \( T_c = 1/(k_B \beta_c) \), of the composite string? This critical temperature, introduced by Hagedorn in the context of
strong interactions a long time ago \[19\], is the temperature above which the
free energy is ultraviolet divergent. In the ultraviolet limit \((\tau_2 \to 0)\),
\[
\eta^{-24}(i\tau) = \tau^{12} e^{2\pi/\tau} \left[ 1 + O\left(e^{-2\pi/\tau}\right) \right],
\]
\[
\theta_3\left(0, \frac{i\beta^2 t(s)}{8\pi^2 \tau_2}\right) - 1 = 2 \exp\left(-\frac{\beta^2 t(s)}{8\pi^2 \tau_2}\right) + O\left(\exp\left(-\frac{\beta^2 t(s)}{2\pi^2 \tau_2}\right)\right),
\]
which upon insertion into Eq. (66) shows that the integrand is ultraviolet
finite if
\[
\beta > \beta_c = \frac{4}{s} \sqrt{\frac{\pi(1+s)}{T_{II}}}. \tag{71}
\]

For a fixed value of \(T_{II}\) the Hagedorn temperature is thus seen to depend
on \(s\). We may mention here that the physical meaning of the Hagedorn
temperature is still not clear. There are different interpretations possible:
(i) one may argue that \(T_c\) is the maximum obtainable temperature in string
systems, this meaning, when applied to cosmology, that there is a maximum
temperature in the early Universe. Or, (ii) one may take \(T_c\) to indicate some
sort of phase transition to a new stringy phase. Some further discussion on
these matters is given, for instance, in Refs. \[11, 16, 17, 18\].

Finally, let us consider the limiting case in which one of the pieces of
the string is much shorter than the other. Physically this case is of interest,
since it corresponds to a point mass sitting on a string (note, however, that
the "mass" concept must be taken in a peculiar sense, since the relativistic condition of Eq. (1) is satisfied always). Since we have assumed that \( s \geq 1 \), this case corresponds to \( s \to \infty \). We let the tension \( T_{II} \) be fixed, though arbitrary. It is seen, first of all, that the Hagedorn temperature (71) goes to infinity so that \( F \) is always ultraviolet finite,

\[
\beta_c \to 0, \quad T_c \to \infty. \tag{72}
\]

Next, since \( \exp\left(-\beta^2 t(s)/8\pi^2 \tau_2\right) \) can be taken to be small we obtain, when using again the expansion (70) for \( \theta_3(0|\beta^2 t(s)/8\pi^2 \tau_2) \),

\[
F(\beta\to 0) = -\frac{s}{24} - (8\pi^3 T_{II})^{-13} \int_{-1/2}^{1/2} d\tau_1 \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^{1/4}} \times \exp\left(-\frac{\beta^2 T_{II}}{8\pi \tau_2}\right) |\eta[(1 + s)\tau]|^{-48} |\eta[s(1 + s)(\tau - \tau)]|^{-24}. \tag{73}
\]

Physically speaking, the linear dependence of the first term in (73) reflects that the Casimir energy of a little piece of string embedded in an essentially infinite string has for dimensional reasons to be inversely proportional to the length \( L_I = \pi/(1 + s) \simeq \pi/s \) of the little string. The first term in (73) is seen to outweigh the second, integral term, which goes to zero when \( s \to \infty \).
5 Summary

For the two-piece relativistic string, the starting point in classical theory is the dispersion equation (4), valid for arbitrary tension ratios $x = T_1/T_{II}$ and length ratios $s = L_{II}/L_I$. In our calculations we have made two simplifying assumptions: first, we have considered only the limit $x \to 0$. Taking $T_{II}$ to be finite, this means that $T \to 0$. Second, we have assumed $s$ to be an integer.

The string’s eigenvalue spectrum is given by Eqs. (10) and (11), meaning that there are in general (for $s \neq 1$) two different branches. The boundary conditions at the two junctions are given by Eqs. (6) - (9). As for the first branch, Eq. (10), there is for odd $s$ no restriction on $n$. All frequencies are permitted to propagate from region I to region II. The junction conditions reduce to one single equation, Eq. (21). If $s$ is even, the waves are unable to propagate into region II. The oscillations in region I are then standing waves.

As for the second branch, Eq.(11), and for odd values of $s$, the integer $n$ in Eqs.(36) and (37) has to be a multiple of $s$ in order to permit nonvanishing oscillations in region II. If $s$ is even, then $n$ has correspondingly to be an even integer times $s$. 

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In some sense the behaviour of the string in region II is similar to that of a conventional open uniform string of length $L_{II}$. The physical conditions in the two cases are however not the same, since Eqs. (3) - (6) are different from the boundary conditions $X'_{\mu} = 0$ at the ends of an open string [10].

Our string model is actually some kind of a hybrid model. In the quantum mechanical formulation the starting point for the free energy $F$ is Eq.(45), into which we have to insert the expression (62) for the quadratic mass. The final result for $F$ is given by Eq.(66). Once $F$ is known, other thermodynamic quantities can easily be calculated. The inverse Hagedorn temperature is given by Eq.(71), and is dependent on $s$. If $s \to \infty$, corresponding to a point "mass" sitting on the string, the Hagedorn temperature goes to infinity.

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