ESTIMATES OF THE BERGMAN KERNEL ON TEICHMÜLLER SPACE

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Abstract. In this paper, a comparison between the Bergman kernel form and the pushforward measure of the Masur-Veech measure on the Teichmüller space of genus \( g \geq 2 \) is obtained.

1. Introduction

1.1. Background and Main theorem. The Teichmüller space \( T_g \) of genus \( g \geq 2 \) is realized as a bounded domain in \( \mathbb{C}^{3g-3} \) (cf. [6]). In view of complex analysis, there is an important invariant \( K_{T_g} \), the Bergman kernel form, on \( T_g \) (cf. [18] and [25]). From the transformation law, the Bergman kernel form is thought of as a non-negative \((6g-6)\)-form on \( T_g \) which is defined from the reproducing kernel of the space of \( L^2 \)-holomorphic \( N \)-forms on \( T_g \) (cf. §3.1).

The Teichmüller geodesic flow on the unit tangent bundle \( UT_g \) on \( T_g \) (in terms of the Teichmüller metric) admits a natural invariant measure, named the Masur-Veech measure (cf. [21] and [28]). By descending the Masur-Veech measure to \( T_g \) via the natural projection \( UT_g \to T_g \), we get a measure \( m_g \) on \( T_g \) which is invariant under the action of the mapping class group (cf. [3]).

In this short paper, we shall show the following global comparison.

Theorem 1.1 (Bergman kernel and Masur-Veech measure). There are two positive constants \( C_1, C_2 \) depending only on \( g \) such that

\[
C_1 \, dm_g \leq K_{T_g} \leq C_2 \, dm_g
\]

on \( T_g \).

Notice that the inequality \((1.1)\) is understood as comparisons of non-negative \((6g-6)\)-forms on \( T_g \) (cf. \([3,2]\) ). We will also prove that the Bergman kernel form on \( T_g \) is comparable with the \((6g-6)\)-dimensional Hausdorff measure with respect to the Teichmüller distance (cf. Corollary \([11]\) ). Our comparisons give global geometric informations of the Bergman kernel form in view of the Teichmüller theory.

As a prior work, B. Chen \([11]\) studied the asymptotic behavior of the Bergman kernel function at the Bers boundary. However, to the authors’ knowledge, there are less informations on the Bergman kernel form on the Teichmüller space. On
the other hand, there is an enormous amount of studies of the Bergman kernel in
the function theory of several complex variables, and many applications to several
fields in mathematics. The authors believe that the study of the Bergman kernel
on the Teichmüller space is important for developing the complex analytical aspect
in the Teichmüller theory.

1.2. Two corollaries. For \( x \in T_g \), let \( B_T(x, R) \) be the open \( R \)-ball of the center at
\( x \) with respect to the Teichmüller distance. Applying [3, Theorem 1.3], we obtain

**Corollary 1.1 (Volume estimate).** There are two positive constants \( D_1, D_2 \)
depending only on \( g \geq 2 \) such that

\[
D_1 \leq \liminf_{R \to \infty} e^{- (3g - 3)R} \int_{B_T(x, R)} K_{T_g} \leq \limsup_{R \to \infty} e^{- (3g - 3)R} \int_{B_T(x, R)} K_{T_g} \leq D_2
\]

for any \( x \in T_g \).

It is an interesting problem to determine whether the limit of \( e^{- (3g - 3)R} \int_{B_T(x, R)} K_{T_g} \)
as \( R \to \infty \) exists or not, like Athreya, Bufetov, Eskin, and Mirzakhani’s work on
\( \mathfrak{m}_g \) in [3].

Since the Bergman kernel form is a biholomorphic invariant, \( K_{T_g} \) descends to
a non-negative \((6g - 6)\)-form on the moduli space \( \mathcal{M}_g \) of genus \( g \geq 2 \). Since the
volume of \( \mathcal{M}_g \) with respect to \( \mathfrak{m}_g \) is finite (cf. [21] and [3]), we obtain

**Corollary 1.2 (Integral is finite).** For any \( g \geq 2 \),

\[
\int_{\mathcal{M}_g} K_{T_g} < \infty.
\]

1.3. About the proof of Theorem 1.1. As discussed in §4.6, Theorem 1.1 follows
from a comparison between the Bergman kernel form and the Busemann volume form as given in Theorem 4.1 and Dowdall-Duchin-Masur’s comparisons of several measures on the Teichmüller space in [13]. As remarked after the statement of
Theorem 4.1, the thought of the proof of Theorem 4.1 is given by mimicking that of
the proof by Z. Błocki [8] for convex domains.

Theorem 4.1 is stated for the Teichmüller space \( T_{g,m} \) of type \((g, m)\) with \( 2g - 2 + m > 0 \). The authors hope that Dowdall-Duchin-Masur’s comparisons of volumes also hold for all \((g, m)\) with \( 2g - 2 + m > 0 \), and it implies the estimate (1.1) for
any \((g, m)\).

### 2. Teichmüller theory

2.1. Teichmüller space. The **Teichmüller space** \( T_{g,m} \) of type \((g, m)\) is the equivalence classes of marked Riemann surfaces of type \((g, m)\). A **marked Riemann surface** \((M, f)\) of type \((g, m)\) is a pair of a Riemann surface \( M \) of analytically finite type \((g, m)\) and an orientation preserving homeomorphism \( f : \Sigma_{g,m} \to M \). Two marked Riemann surfaces \((M_1, f_1)\) and \((M_2, f_2)\) of type \((g, m)\) are (**Teichmüller**) equivalent if there is a conformal mapping \( h : M_1 \to M_2 \) such that \( h \circ f_1 \) is homotopic to \( f_2 \). For simplicity, we write \((M, f)\) the Teichmüller equivalence class of the marked Riemann surface \((M, f)\).

Let \( x = (M, f) \in T_{g,m} \). Let \( L^\infty(M) \) be the space of measurable \((-1, 1)\)-forms \( \mu = \mu(z)dz \) on \( M \) with

\[
\|\mu\|_\infty = \text{ess.sup}\{||\mu(p)|| \mid p \in M\} < \infty.
\]
Let $B_x = \{ \mu \in L^\infty(M) \mid \|\mu\|_\infty < 1 \}$. For $\mu \in B_x$, let $f_\mu = f_{\mu; x}$ be the $\mu$-quasiconformal mapping on $M$. We define the Bers projection $\Phi_x : B_x \to \mathcal{T}_{g,m}$ by $\Phi_x(\mu) = (f_\mu(M), f_\mu \circ f^{-1}) \in \mathcal{T}_{g,m}$.

For $x = (M, f) \in \mathcal{T}_{g,m}$, we denote by $Q_x$ the complex Banach space of holomorphic quadratic differentials $q = q(z)dz^2$ on $M$ with

$$
\|q\| = \int_M |q(z)|dxdy < \infty,
$$

where $z = x + iy$. The holomorphic tangent space $T_x \mathcal{T}_{g,m}$ at $x \in \mathcal{T}_{g,m}$ is identified with the quotient space of $L^\infty(M)$ by the equivalence relation, where $\mu_1$, $\mu_2 \in L^\infty(M)$ are equivalent if

$$
\int_M \mu_1 q = \int_M \mu_2 q
$$

for all $q \in Q_x$. In other words, the kernel of the differential $(\Phi_x)_*$ at $x$ consists of $\mu \in L^\infty(M)$ with $\int_M \mu q = 0$ for all $q \in Q_x$ (cf. [16 Theorem 7.6]). We denote by $[\mu] \in T_x \mathcal{T}_{g,m}$ the tangent vector at $x$ associated to $\mu \in L^\infty(M)$. By definition, $(\Phi_x)_*(\mu)$ represents the equivalence class of $\mu$. Namely,

$$
(\Phi_x)_*(\mu) = [\mu] \in T_x \mathcal{T}_{g,m}
$$

for $\mu \in L^\infty(M) = T_0 B_x$.

2.2. Teichmüller distance. The Teichmüller distance $d_T$ is a complete distance on $\mathcal{T}_{g,m}$ defined by

$$
d_T(x_1, x_2) = \frac{1}{2} \log \inf_h K(h)
$$

for $x_i = (M_i, f_i) \ (i = 1, 2)$, where the infimum runs over all quasiconformal mappings $h: M_1 \to M_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K(h)$ is the maximal dilatation of $h$.

The Teichmüller distance is known to be a Finsler distance with the Finsler metric $F_x(v) = \sup_{\|q\| = 1, q \in Q_x} \left| \int_M \mu q \right|$ for $x = (M, f) \in \mathcal{T}_{g,m}$, $v = [\mu] \in T_x \mathcal{T}_{g,m}$ and $\mu \in L^\infty(M)$ (cf. [15 §7]). We call $F_x$ the Teichmüller metric. It is known that the Teichmüller metric coincides with the Kobayashi-Royden metric (cf. [20] and [13]).

The Kobayashi-Teichmüller indicatrix $I_T(x)$ at $x \in \mathcal{T}_{g,m}$ is defined by

$$
I_T(x) = \{ v \in T_x \mathcal{T}_{g,m} \mid F_x(v) \leq 1 \}.
$$

Since the Teichmüller metric $F_x$ is a norm on $T_x \mathcal{T}_{g,m}$, the indicatrix $I_T(x)$ is a convex set in $T_x \mathcal{T}_{g,m}$.

2.3. Bers slice. Let $x_0 = (M_0, f_0) \in \mathcal{T}_{g,m}$. Let $\Gamma_0$ be the Fuchsian group of $M_0$ acting on the upper-half plane $\mathbb{H}^2$. Let $A_2 = A_2(\mathbb{H}^*, \Gamma_0)$ be the set of holomorphic functions $\varphi$ on the lower-half plane $\mathbb{H}^*$ such that $\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)$ ($\gamma \in \Gamma_0$) and

$$
\|\varphi\|_\infty = \sup_{z \in \mathbb{H}^*} 4 \text{Im}(z)^2|\varphi(z)| < \infty.
$$

Let $\varphi \in A_2$, we define a locally univalent function $W_{\varphi}$ on $\mathbb{H}^*$ such that $W_{\varphi}(z) = (z + i)^{-1} + o(1)$ as $z \to -i$ and the Schwarzian derivative of $W_{\varphi}$ coincides with $\varphi$. The Teichmüller space $\mathcal{T}_{g,m}$ is canonically identified with the Bers slice $\mathcal{T}^B_{g,m}$ which consists of $\varphi \in A_2$ such that $W_{\varphi}$ admits a quasiconformal extension on the Riemann
sphere, via Bers’ simultaneous uniformization (cf. [5] and [19]). The biholomorphic identification \( \beta_{x_0} : T_{g,m} \to T^B_{x_0} \subset A_2 \) is called the Bers embedding.

### 2.4. Ahlfors-Weill section
Let \( x_0 = (M_0, f_0) \in T_{g,m} \). We define \( H_{x_0} : A_2 \to L^\infty(M_0) \) by

\[
H_{x_0}(\psi)(z) = -2 \text{Im}(z)^2 \psi(z).
\]

Then \( \|H_{x_0}(\psi)\|_\infty = \|\psi\|_\infty/2 \).

Let \( B^\infty_{r;x_0} \) be the closed \( r \)-ball in \( A_2 \) with respect to \( \|\cdot\|_\infty \). The restriction of \( H_{x_0} \) to the ball \( \text{Int}(B^\infty_{2;x_0}) \) is called the Ahlfors-Weill section which satisfies

\[
\beta_{x_0} \circ \Phi_{x_0} \circ H_{x_0}(\varphi) = \varphi \quad (\varphi \in \text{Int}(B^\infty_{2;x_0}))
\]

\[
(\beta_{x_0})_*([H_{x_0}(\psi)]) = \psi \quad (\psi \in T_B A_2 = A_2)
\]

(cf. [1] and [16, Theorem 6.9]). Via the linear isomorphism \( H_{x_0} \), the Kobayashi-Teichmüller indicatrix is realized in \( A_2 = T_B A_2 \) as

\[
H_{x_0}^{-1}(I_T(x)) = \{ \psi \in A_2 \mid F_x([H_{x_0}(\psi)]) \leq 1 \}.
\]

We claim

**Lemma 2.1.** \( B^\infty_{2;x_0} \subset H_{x_0}^{-1}(I_T(x_0)) \subset B^\infty_{6;x_0} \).

**Proof.** Since \( \psi \in \partial H_{x_0}^{-1}(I_T(x_0)) \) satisfies \( F_{x_0}([H_{x_0}(\psi)]) = 1 \),

\[
1 = F_{x_0}([H_{x_0}(\psi)]) = \sup_{\|q\| = 1} \left| \int_{M_0} H_{x_0}(\psi) q \right| \leq \|H_{x_0}(\psi)\|_\infty = \|\psi\|_\infty/2
\]

for all \( \psi \in \partial H_{x_0}^{-1}(I_T(x_0)) \). This means that \( B^\infty_{2;x_0} \subset H_{x_0}^{-1}(I_T(x_0)) \).

Let \( \psi \in H_{x_0}^{-1}(I_T(x_0)) \subset A_2 = T_B A_2 \). By Nehari-Kraus’ theorem, the image of the Bers embedding is contained in the ball \( B^\infty_{6;x_0} \) (cf. [16]). Since the Teichmüller metric coincides with the Kobayashi metric, by the distance decreasing property and (2.5), we have

\[
\|\psi\|_\infty = \|(\beta_{x_0})_*([H_{x_0}(\psi)])\|_\infty \leq F_{x_0}([H_{x_0}(\psi)]) \leq 1,
\]

where the left-hand side of the above calculation is the Kobayashi-Finsler norm of \( \psi \in T_0 B^\infty_{6;x_0} = T_B A_2 \) on \( B^\infty_{6;x_0} \) (e.g. [12]). This means that \( \psi \in B^\infty_{6;x_0} \).

\[ \qed \]

### 2.5. Differential of maximal dilatation
For \( x, y \in T_{g,m} \), we set

\[
k_0(x, y) = \tan d_T(x, y).
\]

The following lemma immediately follows from the discussion in the proof of [15 §6.6, Theorem 7]. For the completeness, we shall give a proof.

**Lemma 2.2.** Let \( x_0 \in T_{g,m} \). For \( v = [\mu] \in T_{x_0} T_{g,m} \), let \( x_t \) be the quasiconformal deformation of \( x_0 \) associated to the Beltrami differential \( \mu \). Then,

\[
|k_0(x_0, x_t) - tF_{x_0}(v)| \leq 4t^2 \|\mu\|_\infty^2
\]

when \( t < (2\|\mu\|_\infty)^{-1} \).
Proof. We may assume that $\mu \neq 0$. By definition, $k_0(x_0, x_t) \leq t\|\mu\|_\infty$ when $t < 1/\|\mu\|_\infty$. Following [15, §6.4], we set
\[
I[\mu] = \sup_{\|q\| = 1} \left| \int_{M_0} \frac{\mu q}{1 - |\mu|^2} \right|,
\]
\[
J[\mu] = \sup_{\|q\| = 1} \int_{M_0} \frac{|\mu|^2 |q|}{1 - |\mu|^2}.
\]
By a simple calculation, we have
\[
|I[\mu] - tF_{x_0}(v)| \leq \frac{t^3\|\mu\|_\infty^3}{1 - t^3\|\mu\|_\infty^2}.
\]
Combining with [15, Theorem 4], we obtain
\[
|k_0(x_0, x_t) - tF_{x_0}(v)| \leq \left| k_0(t) - \frac{k_0(t)}{1 - k_0(t)^2} \right| + \left| \frac{k_0(t)}{1 - k_0(t)^2} - I[\mu] \right| + |I[\mu] - tF_{x_0}(v)|
\]
\[
\leq \frac{k_0(t)^3}{1 - k_0(t)^2} + \frac{k_0(t)^2}{1 - k_0(t)^2} + \frac{t^3\|\mu\|_\infty^3}{1 - t^3\|\mu\|_\infty^2}
\]
\[
= \frac{2t^2\|\mu\|_\infty^2 + 2t^3\|\mu\|_\infty^3}{1 - t^3\|\mu\|_\infty^2} \leq 4t^2\|\mu\|_\infty
\]
when $t \leq (2\|\mu\|_\infty)^{-1}$.
\[
\square
\]

3. Complex analysis

3.1. Bergman Kernel. Let $\Omega$ be an $N$-dimensional complex manifold. We denote by $\mathcal{O}L^2_{N,0}(\Omega)$ the Hilbert space of holomorphic $N$-forms $f = f(z)d\bar{z}$ ($d\bar{z} = dz_1 \wedge \cdots \wedge dz_N$) with the inner product
\[
(f_1, f_2) = \frac{i^{N^2}}{2^{N^2}} \int_{\Omega} f_1 \wedge \overline{f_2} = \int_{\Omega} f_1(z) \overline{f_2(z)} dV_E(z),
\]
where $dV_E = dz_1 dy_1 \cdots dz_N dy_N$ is the standard Euclidean measure (Lebesgue measure) on local charts $(z_1, \ldots, z_N)$ and $z_k = x_k + iy_k$ ($1 \leq k \leq N$). The reproducing kernel form on $\mathcal{O}L^2_{N,0}(\Omega)$ is a bi-form on $\Omega$ defined by
\[
K_\Omega(z, w) = \sum_{k=1}^{\infty} f_k(z) \overline{f_k(w)} dZ \otimes d\bar{W},
\]
where $d\bar{W} = d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_N$ and $\{f_k = f_k(z)dz\}_{k=1}^{\infty}$ is a complete orthonormal basis of $\mathcal{O}L^2_{N,0}(\Omega)$ (cf. [15] and [25, Chapter 4]). We call $K_\Omega = K_\Omega(z, z)$ the Bergman kernel on $\Omega$:
\[
K_\Omega = K_\Omega(z) d\bar{Z} \otimes d\bar{Z} = \sum_{k=1}^{\infty} |f_k(z)|^2 dZ \otimes d\bar{Z}
\]
on $\Omega$. The transformation law (cf. [25, (4.9)])
\[
K_{\Omega'}(F(z)) |\det F'(z)|^2 = K_\Omega(z) \quad (z \in \Omega)
\]
with a biholomorphic mapping (or a local chart) $F: \Omega \to \Omega'$ ($F'$ is the complex Jacobian of $F$) implies that
\[
K_\Omega = K_\Omega(z) dV_E
\]
is a well-defined non-negative $2N$-form on $\Omega$. It is known that

\begin{equation}
K_\Omega \leq K_{\Omega'}
\end{equation}

for any open set $\Omega' \subset \Omega$ (cf. [25, Corollary 4.1]).

When $\Omega$ is a domain in $\mathbb{C}^N$, the space $OL^2_{0,0}(\Omega)$ is isometrically identified with the space $OL^2_{0,0}(\Omega)$ of $L^2$-holomorphic functions with respect to the Lebesgue measure by

$$\begin{align*}
OL^2_{0,0}(\Omega) \ni f = f(z) \, dZ \mapsto f(\cdot) \in OL^2_{0,0}(\Omega),
\end{align*}$$

(cf. [25, Example 4.1]).

### 3.2. Pluricomplex Green function

Let $\Omega$ be a domain in $\mathbb{C}^N$. The pluricomplex Green function $g_\Omega$ with a pole at $w \in \Omega$ is defined by

$$g_\Omega(w, z) = \sup \{ u(z) \in \text{PSH}(\Omega)^- \mid \limsup_{z \to w} (u(z) - \log |z - w|) < \infty \}$$

where PSH(\Omega)^- denotes the class of negative plurisubharmonic functions on $\Omega$ (cf. [17]). In [22], it is shown that

\begin{equation}
g_{T_g,m}(x, y) = \log \tanh d_T(x, y) = \log k_0(x, y)
\end{equation}

for $x, y \in T_g,m$.

### 4. Estimates of the Bergman Kernel

#### 4.1. Busemann volume forms

We first recall the Busemann volume form on an $N$-dimensional Finsler manifold $(M, F)$ after [2] and [13, §4]. Usually, the Finsler norm is assumed to be smooth. However, we only assume here the Finsler norm to be continuous.

Let $x \in M$ and $B_x = \{ v \in T_{x}M \mid F_x(v) \leq 1 \}$ be the unit ball (the $F$-indicatrix) with respect to the Finsler norm $F$. For an identification $T_x M \cong \mathbb{R}^N$ induced by a local coordinate around $x$, we define the Busemann volume form on $M$ by

\begin{equation}
d\mu_{M;B} = \frac{\epsilon_N}{V_E(B_x)} dV_E,
\end{equation}

where $\epsilon_N$ is the volume of the unit ball in $\mathbb{R}^N$ and $V_E$ is the standard Euclidean measure (Lebesgue measure) on $\mathbb{R}^N$ as the previous section (“B” in the subscription of the notation stands for the initial letter of “Busemann”).

#### 4.2. Comparison

In this section, we show

**Theorem 4.1** (Bergman kernel and Busemann volume form).

\begin{equation}
\frac{1}{\epsilon_{6g-6+2m}} d\mu_{T_g,m;B} \leq K_{T_g,m} \leq \frac{3^{6g-6+2m}}{\epsilon_{6g-6+2m}} d\mu_{T_g,m;B}
\end{equation}

on $T_g,m$.

As discussed in (3.2), the inequality (4.2) is thought of as comparisons of non-negative $(6g - 6 + 2m)$-forms on $T_g,m$.

In the pluripotential theory, the relation (4.2) is first observed by Z. Błocki for convex domains in $\mathbb{C}^n$ (cf. [8, Theorem 2]). Hence, the inequality (4.2) is closely related to the inequality conjectured by Suita [27] (cf. [9]).

We give two proofs of the lower bound of Theorem 4.1. The first proof is based on the same line as his proof in [8], meanwhile we apply Teichmüller theory in the essential part of the proof. The second proof is given by characterizing the
Azukawa metric on the Teichmüller space and applying Blocki-Zwonek’s result in [9]. The discussion of the upper estimate is a mimic of the discussion by Blocki in [8, Theorem 5] for the convex domains (see also [24, Corollary 4]).

4.3. First proof of lower estimate. It suffices to confirm the equation 4.2 for a local chart at $x_0$.

We fix a complex linear identification $L: A_2 \cong \mathbb{C}^{3g-3+m}$. Then $L \circ \beta_{x_0}: \mathcal{T}_{g,m} \to \mathbb{C}^{3g-3+m}$ is a complex local chart at $x_0$. We denote by $V_E$ the Euclidean volume form (Lebesgue measure) on $\mathbb{C}^{3g-3+m}$ as above. From [24], the coordinate $L \circ \beta_{x_0}$ induces a complex linear isomorphism

$$L \circ ((\Phi_{x_0})_* \circ H_{x_0})^{-1}: T_{x_0} \mathcal{T}_{g,m} \to \mathbb{C}^{3g-3+m},$$

which induces the Euclidean volume form on $T_{x_0} \mathcal{T}_{g,m}$. For simplicity, we denote by $V_E$ the volume form on $T_{x_0} \mathcal{T}_{g,m}$.

As remarked in the previous section, the following lemma is first observed by Blocki [8, Proposition 3] for convex domains in the complex Euclidean space by applying Lempert’s theory [20].

Lemma 4.1 (Volume of sublevel sets of pluricomplex Green function). Under the above identifications $T_{x_0} \mathcal{T}_{g,m} \cong \mathbb{C}^{3g-3+m}$ and $\mathcal{T}_{g,m} \cong \mathcal{T}^B_{x_0} \subset A_2 \cong \mathbb{C}^{3g-3+m}$, we have

$$\lim_{a \to \infty} e^{-2(3g-3+m)a} V_E(\{y \in \mathcal{T}_{g,m} \mid g_{\mathcal{T}_{g,m}}(x_0, y) < -a\}) = V_E(\mathcal{I}_T(x_0)).$$

Proof. Let $S_{x_0} = \partial(H_{x_0}^{-1}(\mathcal{I}_T(x_0))) \subset A_2 \cong \mathbb{C}^{3g-3+m}$. Since $F_\psi([H_{x_0}(\psi)]) = 1$ for $\psi \in \partial H_{x_0}^{-1}(\mathcal{I}_T(x_0))$, from Lemmas 2.1 and 2.2

$$|k_0(x_0, \Phi_{x_0}(tH_{x_0}(\psi))) - t| \leq 144t^2,$$

when $t < 1/12$. From (3.2),

$$\{t \psi \mid 0 \leq t \leq \delta_2(a), \psi \in S_{x_0}\} \subset \beta_{x_0}(\{y \in \mathcal{T}_{g,m} \mid g_{\mathcal{T}_{g,m}}(x_0, y) < -a\}) \subset \{t \psi \mid 0 \leq t < \delta_1(a), \psi \in S_{x_0}\},$$

when $a > 0$ is sufficiently large, where $\delta_1(a) = (1 - \sqrt{1 - 96e^{-a}})/48$ and $\delta_2(a) = (\sqrt{1 + 96e^{-a}} - 1)/48$. Since $\delta_i(a) = e^{-a} + O(e^{-2a})$ as $a \to \infty$ for $i = 1, 2$, we have

$$V_E(\{g_{\mathcal{T}_{g,m}}(x_0, y) < -a\}) = e^{-2(3g-3+m)a} V_E(\mathcal{I}_T(x_0)) + o(e^{-2(3g-3+m)a})$$

as $a \to \infty$. \hfill $\square$

Let us finish proving Theorem [4.1]. Blocki [8, Theorem 1] observed that

$$K_{\Omega}(z) \geq \frac{1}{e^{2Na} V_E(\{g_0(z, \cdot) < -a\})}$$

for any pseudoconvex domain $\Omega \subset \mathbb{C}^N$ and $z \in \Omega$. Notice that Blocki [8] considered the Bergman kernel as a reproducing kernel function on the space of $L^2$-holomorphic functions with respect to the Lebesgue measure $dV_E$. Since our inner product is defined as (3.1), by (3.2), the Bergman kernel function on $T_{x_0}^B \subset \mathbb{C}^{3g-3+m}$ appears as the coefficient of the $(6g - 6 + 2m)$-form $K_{\mathcal{T}_{g,m}}$. Hence, we get the desired inequality (4.2) from Lemma 4.1 by letting $a \to \infty$. 

4.4. Second proof of lower estimate. Let $\Omega$ be an $N$-dimensional complex manifold. The Azukawa metric on $\Omega$ is defined by

$$A_\Omega(p,v) = \limsup_{t \to 0} \frac{\exp(g_\Omega(p,\varphi(t)))}{|t|}$$

for $v \in T_p\Omega$ and $\varphi: \{|t| < \epsilon\} \to \Omega$ is a holomorphic map with $\varphi(0) = p$ and $\varphi_*(\partial/\partial t)|_{t=0} = v$ (cf. [3, 52]).

Lemma 4.2. The Azukawa metric on $T_{g,m}$ coincides with the Teichmüller metric.

Proof. Let $x_0 \in T_{g,m}$ and $v \in T_{x_0}T_{g,m}$. Let $\varphi: \{|t| < \epsilon\} \to T_{g,m}$ with $\varphi(0) = p$ and $\varphi_*(\partial/\partial t)|_{t=0} = v$. By Lemma 2.2

$$\exp(g_{T_{g,m}}(x_0, \varphi(t))) = k_0(x_0, \varphi(t)) = |t|F_{x_0}(v) + o(|t|),$$

and $A_{T_{g,m}}(x_0; v) = F_{x_0}(v)$. \qed

Let us finish the second proof of Theorem 1.1. In [9], Blocki and Zwonek proved that the Bergman kernel function is at least the reciprocal of the volume of the Azukawa indicatrix for pseudoconvex domains. This implies the desired estimate.

Remark 4.1. The second proof can be applied for general situations. Indeed, the inequality (4.2) holds for pseudoconvex domains when the Busemann volume form

$$\lambda_\psi \leq \int_{B_{x_0}} (1 - |\lambda(\psi)|)^2 \mu_{x_0} \, d\mu_{x_0}(\psi).$$

4.5. An upper estimate of the Bergman kernel. We fix a local chart $L \circ \beta_{x_0}$ on $T_{g,m}$ as [7, 11]. From Lemma 2.1 and Nehari’s theorem (cf. [10]),

$$H_{x_0}^{-1}(I_T(x_0)) \subset 3T_{x_0},$$

where $rE = \{r\psi \in A_2 \mid \psi \in E\}$ for $E \subset A_2$. Notice that $H_{x_0}^{-1}(I_T(x_0))$ is balanced in the sense that $\lambda \psi \in H_{x_0}^{-1}(I_T(x_0))$ for $\psi \in H_{x_0}^{-1}(I_T(x_0))$ and $|\lambda| \leq 1$. Furthermore, $H_{x_0}^{-1}(I_T(x_0))$ is convex, and hence is pseudoconvex. Therefore

$$K(1/3)H_{x_0}^{-1}(I_T(x_0)) = \frac{1}{V_E((1/3)H_{x_0}^{-1}(I_T(x_0)))}$$

(e.g. [24]). From (3.2) and (3.3), we conclude

$$K_{T_{g,m}} \leq K(1/3)H_{x_0}^{-1}(I_T(x_0)) = \frac{\mu T_{g,m}}{\mu T_g} \frac{\mu T_g}{\mu T_{T_{g,m}}} \frac{dV_E}{V_E((1/3)H_{x_0}^{-1}(I_T(x_0)))}.$$ 

4.6. Proof of Theorem 1.1. In [13] Corollary 4.4, Dowdall, Duchin, and Masur observed that the pushforward measure $m_g$ of the Masur-Veech measure via the projection $\mathcal{U}T_g \to T_g$ is comparable with the Busemann volume form. Therefore, we obtain the estimate in Theorem 1.1.

4.7. Hausdorff measure on $T_{g,m}$. In [13], Dowdall, Duchin, and Masur also noticed that the Busemann volume form on $T_g$ associated to the Teichmüller metric coincides with the $(6g-6)$-dimensional Hausdorff measure $H_{T_g}$ associated to the Teichmüller metric. This coincidence also holds for $T_{g,m}$ since Busemann proved that the Busemann volume coincides with the top-dimensional Hausdorff measure with respect to the Finsler distance for arbitrary (continuous) Finsler manifolds (cf. [10] and [2 Theorem 3.23]).
Corollary 4.1 (Hausdorff measure).
\[
\frac{1}{\epsilon_{6g-6+2m}} d\mathcal{H}_{\mathcal{T}_{g,m}} \leq K_{\mathcal{T}_{g,m}} \leq \frac{3^{6g-6+2m}}{\epsilon_{6g-6+2m}} d\mathcal{H}_{\mathcal{T}_{g,m}}
\]
on \mathcal{T}_{g,m}.

Remark 4.2. The coincidence $\mu_{\Omega;B} = \mathcal{H}_{\Omega}$ for the Kobayashi distance is also directly observed by Bland and Graham [7, Theorem 1] for Kobayashi hyperbolic manifolds $\Omega$ with continuous infinitesimal Kobayashi-Royden metrics whose Kobayashi-indicatrices are convex. (In our case, the Teichmüller metric $F_{x_0}$ is a norm on the tangent space.)

Remark 4.3. As remarked in Remark 4.1, the inequality in Corollary 4.1 also holds for pseudoconvex domains with the continuous Azukawa metrics since the coincidence $\mu_{\Omega;B} = \mathcal{H}_{\Omega}$ holds for continuous Finsler manifolds $\Omega$.

REFERENCES
[1] L. Ahlfors and G. Weill. A uniqueness theorem for Beltrami equations. Proc. Amer. Math. Soc., 13:975–978, 1962.
[2] J. C. Álvarez Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In A sampler of Riemann-Finsler geometry, volume 50 of Math. Sci. Res. Inst. Publ., pages 1–48.
Cambridge Univ. Press, Cambridge, 2004.
[3] Jayadev Athreya, Alexander Bufetov, Alex Eskin, and Maryam Mirzakhani. Lattice point asymptotics and volume growth on Teichmüller space. Duke Math. J., 161(6):1055–1111, 2012.
[4] Kazuo Azukawa. The invariant pseudometric related to negative plurisubharmonic functions. Kodai Math. J., 10(1):83–92, 1987.
[5] Lipman Bers. Simultaneous uniformization. Bull. Amer. Math. Soc., 66:94–97, 1960.
[6] Lipman Bers. Correction to "Spaces of Riemann surfaces as bounded domains". Bull. Amer. Math. Soc., 67:465–466, 1961.
[7] J. Bland and Ian Graham. On the Hausdorff measures associated to the Carathéodory and Kobayashi metrics. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 12(4):503–514 (1986), 1985.
[8] Zbigniew Błocki. A lower bound for the Bergman kernel and the Bourgain-Milman inequality. In Geometric aspects of functional analysis, volume 2116 of Lecture Notes in Math., pages 53–63.
Springer, Cham, 2014.
[9] Zbigniew Błocki and Włodzimierz Zwonek. Estimates for the Bergman kernel and the multidimensional Suita conjecture. New York J. Math., 21:151–161, 2015.
[10] Herbert Busemann. Intrinsic area. Ann. of Math. (2), 48:234–267, 1947.
[11] Bo-Yong Chen. The Bergman metric on Teichmüller space. Internat. J. Math., 15(10):1085–1091, 2004.
[12] Seán Dineen. The Schwarz lemma. Oxford Mathematical Monographs.
The Clarendon Press, Oxford University Press, New York, 1989. Oxford Science Publications.
[13] Spencer Dowdall, Moon Duchin, and Howard Masur. Statistical hyperbolicity in Teichmüller space. Geom. Funct. Anal., 24(3):748–795, 2014.
[14] Clifford J. Earle and Irwin Kra. On holomorphic mappings between Teichmüller spaces. pages 107–124, 1974.
[15] Frederick P. Gardiner. Teichmüller theory and quadratic differentials. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1987. A Wiley-Interscience Publication.
[16] Yoichi Imayoshi and Masahiko Taniguchi. An introduction to Teichmüller spaces. Springer-Verlag, Tokyo, 1992.
[17] M. Klimek. Extremal plurisubharmonic functions and invariant pseudodistances. Bull. Soc. Math. France, 113(2):231–240, 1985.
[18] Shoshichi Kobayashi. Geometry of bounded domains. Trans. Amer. Math. Soc., 92:267–290, 1959.
[19] Irwin Kra. On Teichmüller spaces for finitely generated Fuchsian groups. *Amer. J. Math.*, 91:67–74, 1969.
[20] László Lempert. La métrique de Kobayashi et la représentation des domaines sur la boule. *Bull. Soc. Math. France*, 109(4):427–474, 1981.
[21] Howard Masur. Interval exchange transformations and measured foliations. *Ann. of Math. (2)*, 115(1):169–200, 1982.
[22] Hideki Miyachi. Pluripotential theory on Teichmüller space I: Pluricomplex Green function. *Conform. Geom. Dyn.*, 23:221–250, 2019.
[23] Subhashis Nag. The complex analytic theory of Teichmüller spaces. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1988.
[24] Nikolai Nikolov and Pascal J. Thomas. Comparison of the Bergman kernel and the Carathéodory-Eisenman volume. *Proc. Amer. Math. Soc.*, 147(11):4915–4919, 2019.
[25] Takeo Ohsawa. $L^2$ approaches in several complex variables. Springer Monographs in Mathematics. Springer, Tokyo, 2018. Towards the Oka-Cartan theory with precise bounds, Second edition of [MR3443603].
[26] Halsey L. Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces* (Proc. Conf., Stony Brook, N.Y., 1969), pages 369–383. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.
[27] Nobuyuki Suita. Capacities and kernels on Riemann surfaces. *Arch. Rational Mech. Anal.*, 46:212–217, 1972.
[28] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)*, 115(1):201–242, 1982.

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