STABILITY OF $C^\infty$ CONVEX INTEGRANDS

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Abstract. In this paper, it is shown that the set consisting of stable convex integrands $S^n \to \mathbb{R}_+$ is open and dense in the set consisting of $C^\infty$ convex integrands with respect to Whitney $C^\infty$ topology. Moreover, an application of the proof of this result is also shown.

1. Introduction

In the celebrated series [14, 15, 16, 17, 18, 19], J. Mather gave a complete answer to the problem on density of proper stable mappings in a surprising form. For proper $C^\infty$ mappings of special type, it is natural to ask the similar question, namely to ask “Are generic proper mappings of special type stable?” Such investigations, for instance, can be found in [20] for generic projections of submanifolds, in [6] for generic projections of stable mappings and in [8, 9, 10, 11] for generic distance-squared mappings and their generalizations.

Motivated by these researches, in this paper, it is investigated the density problem for $C^\infty$ convex integrands. The notion of convex integrand was firstly introduced in [23], which is defined as follows. For a positive integer $n$, let $S^n$ be the unit sphere of $\mathbb{R}_n^{n+1}$. The set consisting of positive real numbers is denoted by $\mathbb{R}_+$. Then, a continuous function $\gamma: S^n \to \mathbb{R}_+$ is called a convex integrand if the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$ is exactly the same set as $\text{inv}(\text{graph}(\gamma))$, where $\text{graph}(\gamma)$ is the set $\{ (\theta, \gamma(\theta)) | \theta \in S^n \}$ with respect to the polar plot expression for $\mathbb{R}_n^{n+1} - \{0\}$ and $\text{inv}: \mathbb{R}_n^{n+1} - \{0\} \to \mathbb{R}_n^{n+1} - \{0\}$ is the inversion defined by $\text{inv}(\theta, r) = (\theta, \frac{1}{r})$. The notion of convex integrand is closely related with the notion of Wulff shape, which was firstly introduced in [24] as a geometric model of crystal at equilibrium. Integration of a convex integrand $\gamma$ over $S^n$ represents the surface energy of the Wulff shape associated with $\gamma$. Hence, $\gamma$ is called a convex integrand. For more details on convex integrands, see for instance [22, 23].

Set

$$C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) = \{ \gamma \in C^\infty(S^n, \mathbb{R}_+) | \gamma \text{ is a convex integrand} \},$$

where $C^\infty(S^n, \mathbb{R}_+)$ is the set consisting of $C^\infty$ functions $S^n \to \mathbb{R}_+$. The set $C^\infty(S^n, \mathbb{R}_+)$ is endowed with Whitney $C^\infty$ topology (for details on Whitney $C^\infty$ topology, see for instance [2, 7]); and the set $C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$ is a topological subspace of $C^\infty(S^n, \mathbb{R}_+)$. Two $C^\infty$ functions $\gamma_1, \gamma_2 : S^n \to \mathbb{R}_+$ are said to be $A$-equivalent if there exist $C^\infty$ diffeomorphisms $h : S^n \to S^n$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that the equality $\gamma_2 = H \circ \gamma_1 \circ h^{-1}$ holds. A $C^\infty$ function $\gamma \in C^\infty(S^n, \mathbb{R}_+)$ is said to be stable if the $A$-equivalence class $A(\gamma)$ is an open subset of the topological space $C^\infty(S^n, \mathbb{R}_+)$. By definition, any function $A$-equivalent to

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a stable function is stable. Set
\[ S^\infty(S^n, \mathbb{R}_+) = \{ \gamma \in C^\infty(S^n, \mathbb{R}_+) \mid \gamma \text{ is stable} \}. \]

By definition, \( S^\infty(S^n, \mathbb{R}_+) \) is open. The following proposition is one of corollaries of Mather’s series \([14, 15, 16, 17, 18, 19]\).

**Proposition 1.**

1. A \( C^\infty \) function \( \gamma \in C^\infty(S^n, \mathbb{R}_+) \) is stable if and only if all critical points of \( \gamma \) are non-degenerate and \( \gamma(\theta_1) \neq \gamma(\theta_2) \) holds for any two distinct critical points \( \theta_1, \theta_2 \in S^n \).

2. The open subset \( S^\infty(S^n, \mathbb{R}_+) \) is dense in \( C^\infty(S^n, \mathbb{R}_+) \).

The assertion (2) of Proposition 1 asserts that any \( C^\infty \) function \( \gamma : S^n \to \mathbb{R}_+ \) can be perturbed to a stable function \( \tilde{\gamma} \) by a sufficiently small perturbation; and for any sufficiently small \( \varepsilon > 0 \), any continuous mapping \( \Phi : (-\varepsilon, \varepsilon) \to C^\infty(S^n, \mathbb{R}_+) \) such that \( \Phi(0) = \tilde{\gamma} \) and any two \( t_1, t_2 \in (-\varepsilon, \varepsilon) \), there exist \( C^\infty \) diffeomorphisms \( h : S^n \to S^n \) and \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the equality \( \Phi(t_2) = H \circ \Phi(t_1) \circ h^{-1} \) holds.

The main purpose of this paper is to show the following:

**Theorem 1.** The open subset \( S^\infty(S^n, \mathbb{R}_+) \cap C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) \) is dense in \( C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) \).

Similarly as the assertion (2) of Proposition 1, Theorem 1 asserts that any \( C^\infty \) convex integrand \( \gamma : S^n \to \mathbb{R}_+ \) can be perturbed to a stable convex integrand \( \tilde{\gamma} \) by a sufficiently small perturbation; and for any sufficiently small \( \varepsilon > 0 \), any continuous mapping \( \Phi : (-\varepsilon, \varepsilon) \to C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) \) such that \( \Phi(0) = \tilde{\gamma} \) and any two \( t_1, t_2 \in (-\varepsilon, \varepsilon) \), there exist \( C^\infty \) diffeomorphisms \( h : S^n \to S^n \) and \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the equality \( \Phi(t_2) = H \circ \Phi(t_1) \circ h^{-1} \) holds.

In Section 2 preliminaries for the proof of Theorem 1 are given. Theorem 1 is proved in Section 3. In Section 4 an application of the proof of Theorem 1 is given.

2. Preliminaries

Let \( \phi : S^n \to \mathbb{R}^{n+1} \) be a \( C^\infty \) embedding. Consider the family of functions \( F : \mathbb{R}^{n+1} \times S^n \to \mathbb{R} \) defined by
\[ F(v, z) = \frac{1}{2} \| \phi(z) - v \|^2. \]

Notice that \( F \) may be regarded as a mapping from \( \mathbb{R}^{n+1} \) to \( C^\infty(S^n, \mathbb{R}) \) which maps each \( v \in \mathbb{R}^n \) to the function \( f_v(z) = F(v, z) \in C^\infty(S^n, \mathbb{R}) = \{ g : S^n \to \mathbb{R} \mid C^\infty \} \). The set of values \( v \) for which \( f_v(z) \) has a degenerate critical point, denoted by \( \text{Caust}(\phi) \), is called the Caustic of \( \phi \) (for details on caustics, for instance see \([11, 12, 13]\)). The set of values \( v \) for which \( f_v(z) \) has a multiple critical value forms the Symmetry set of \( \phi \), denoted by \( \text{Sym}(\phi) \) (for details on symmetry sets, see for instance \([3, 4, 5]\)). By the assertion (1) of Proposition 1, these two sets \( \text{Caust}(\phi) \) and \( \text{Sym}(\phi) \) constitute the set of points \( v \) for which the function \( f_v \in C^\infty(S^n, \mathbb{R}) \) is not stable.

**Proposition 2.** Let \( \phi : S^n \to \mathbb{R}^{n+1} \) be a \( C^\infty \) embedding. Then, \( \text{Caust}(\phi) \) has Lebesgue measure zero in \( \mathbb{R}^{n+1} \).

For the proof of Proposition 2 see \([21]\), §6 “Manifolds in Euclidean space”.

**Proposition 3.** Let \( \phi : S^n \to \mathbb{R}^{n+1} \) be a \( C^\infty \) embedding. Then, \( \text{Sym}(\phi) \) has Lebesgue measure zero in \( \mathbb{R}^{n+1} \).
Proof. Since $\phi$ is an embedding, the complement of $\phi(S^n)$ constitutes two connected components. Denote the bounded connected component by $V_\theta$. Let $M = \phi(S^n)$. For each $\theta \in S^n$, consider the normal vector space $N_{\phi(\theta)}(M)$ to $M$ at $\theta$. Notice that $N_{\phi(\theta)}(M)$ is a 1-dimensional vector space. Thus, we can uniquely specify the unit vector $N_{\phi(\theta)}(M)$ so that $\phi(\theta) + \varepsilon n(\theta)$ belongs to $V_{\theta}$ for any sufficiently small $\varepsilon > 0$. For any $t \in \mathbb{R}$, let $\phi_t : S^n \to \mathbb{R}^{n+1}$ be the $C^\infty$ mapping defined by $\phi_t(\theta) = \phi(\theta) + tn(\theta)$. The mapping $\phi_t$ is called a wave front of $\phi$ (for details on wave fronts, see for instance [1, 2, 12, 13]).

Notice that, since $N_{\phi(\theta)}(M)$ is of Lebesgue measure zero. Take one point $t \in \mathbb{R}$, the set $\text{Sym}(\phi)$ can be characterized as follows:

$$\text{Sym}(\phi) = \bigcup_{t \in \mathbb{R}} \{ \phi_t(\theta_1) = \phi_t(\theta_2) \mid \theta_1 \neq \theta_2 \}.$$

By Proposition 2, the intersection $\text{Sym}(\phi) \cap \text{Caust}(\phi)$ is of Lebesgue measure zero. Thus, in order to show Proposition 3, it is sufficient to show that $\text{Sym}(\phi) \cap (\mathbb{R}^{n+1} - \text{Caust}(\phi))$ is of Lebesgue measure zero. Take one point $\phi_{\theta_0}(\theta_1) = \phi_{\theta_0}(\theta_2)$ of $\text{Sym}(\phi) \cap (\mathbb{R}^{n+1} - \text{Caust}(\phi))$, where $\theta_1, \theta_2$ are two distinct point of $S^n$. Set $x_0 = \phi_{\theta_0}(\theta_1) = \phi_{\theta_0}(\theta_2)$, and let $U_0$ be a sufficiently small open neighborhood of $x_0$. Notice that, since $\text{Caust}(\phi)$ is compact, $U_0$ may be chosen so that $U_0 \cap \text{Caust}(\phi) = \emptyset$.

Let $i$ be 1 or 2. For the $i$, define the mapping $(t_i, \tilde{\theta}_i) : U_0 \to \mathbb{R} \times S^n$ as follows:

$$x = \phi_{t_i(x)}(\tilde{\theta}_i(x)), \quad \left( t_i(x_0) = t_0, \tilde{\theta}_i(x_0) = \theta_i \right).$$

Notice that, since $U_0 \cap \text{Caust}(\phi) = \emptyset$, both of the following two are well-defined $C^\infty$ diffeomorphisms.

$$\begin{align*}
\left( t_1, \tilde{\theta}_1 \right) & : U_0 \to \left( t_1, \tilde{\theta}_1 \right) (U_0), \\
\left( t_2, \tilde{\theta}_2 \right) & : U_0 \to \left( t_2, \tilde{\theta}_2 \right) (U_0).
\end{align*}$$

Set $T = t_1 - t_2$. Then, it is clear that $\text{Sym}(\phi) \cap U_0 = T^{-1}(0)$.

For any $i = 1, 2$, let $\nabla t_i(x_0)$ be the gradient vector of $t_i$ at $x_0$. Since both $t_1, t_2$ are non-singular functions, it follows that neither $\nabla t_1(x_0)$ nor $\nabla t_2(x_0)$ is the zero vector. Moreover, from the construction, it is easily seen that even when $\nabla t_1(x_0)$ and $\nabla t_2(x_0)$ are linearly dependent, $\nabla T(x_0) = \nabla t_1(x_0) - \nabla t_2(x_0)$ is a non-zero vector. Therefore, taking a smaller open neighborhood $\tilde{U}_0$ of $x_0$ if necessary, it follows that $T^{-1}(0) \cap \tilde{U}_0$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Therefore, Proposition 3 follows.



Corollary 1. Let $\phi : S^n \to \mathbb{R}^{n+1}$ be a $C^\infty$ embedding. Then, the union $\text{Caust}(\phi) \cup \text{Sym}(\phi)$ is a subset of Lebesgue measure zero in $\mathbb{R}^{n+1}$.

3. Proof of Theorem 4

Let $\gamma : S^n \to \mathbb{R}^+$ be a $C^\infty$ convex integrand, and let $V$ be a neighborhood of $\gamma$ in $C_{\text{conv}}(S^n, \mathbb{R}^+)$. It is sufficient to show that $V \cap S^\infty(S^n, \mathbb{R}^+) \neq \emptyset$. In order to construct an element of $V \cap S^\infty(S^n, \mathbb{R}^+)$, we consider the $C^\infty$ embedding $\phi : S^n \to \mathbb{R}^{n+1} - \{0\}$ defined as follows:

$$\phi(\theta) = \left( \theta, \frac{1}{\gamma(-\theta)} \right).$$
Let $W$ be the convex hull of $\phi(S^n)$. Then, since $\gamma$ is a convex integrand, it follows that
\begin{equation}
\phi(S^n) = \partial W,
\end{equation}
where $\partial W$ stands for the boundary of $W$.

Next, for any $v \in \text{int}(W)$ consider the parallel translation $T_v : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $T_v(x) = x - v$, where $\text{int}(W)$ means the set consisting of interior points of $W$. Moreover, for any $\theta \in S^n$ set $L_\theta = \{(\theta, r) \in \mathbb{R}^{n+1} - \{0\} | r \in \mathbb{R}_+\}$ and for any $v \in \text{int}(W)$ define $\gamma_v : S^n \to \mathbb{R}_+$ as follows.
\[ (\theta, \gamma_v(\theta)) = T_v(\partial W) \cap L_\theta. \]
Notice that, by $(\ast)$ and $v \in \text{int}(W)$, $\gamma_v$ is a well-defined function. Notice also that $\text{graph}(\gamma_v) = T_v(\partial W)$. By $(\ast)$ and $v \in \text{int}(W)$ again, it follows that $||\phi(\theta) - v|| > 0$ for any $\theta \in S^n$. Thus, it follows that the mapping $h_v : S^n \to S^n$ defined by
\[ h_v(\theta) = \frac{\phi(\theta) - v}{||\phi(\theta) - v||} \]
is a $C^\infty$ diffeomorphism and the following holds:
\[ (\gamma_v \circ h_v)(\theta) = ||\phi(\theta) - v||. \]
Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be the $C^\infty$ diffeomorphism defined by $H(X) = \frac{1}{2}X^2$. Then, we have the following:
\[ F(v, \theta) = \frac{1}{2}||\phi(\theta) - v||^2 = (H \circ \gamma_v \circ h_v)(\theta). \]
Hence, we have
\[ \text{Caust}(\phi) = \{ v \circ \exists \theta : \nabla(H \circ \gamma_v \circ h_v)(\theta) = 0 \text{ and } \text{det}(Hess(H \circ \gamma_v \circ h_v)(\theta)) = 0 \} = \{ v \circ \exists \theta : \nabla(\gamma_v \circ h_v)(\theta) = 0 \text{ and } \text{det}(Hess(\gamma_v \circ h_v)(\theta)) = 0 \} \]
and
\[ \text{Sym}(\phi) = \{ v \circ \exists \theta_1 \neq \theta_2 : \nabla(H \circ \gamma_v \circ h_v)(\theta_1) = \nabla(H \circ \gamma_v \circ h_v)(\theta_2) = 0 \text{ and } (H \circ \gamma_v \circ h_v)(\theta_1) = (H \circ \gamma_v \circ h_v)(\theta_2) \} = \{ v \circ \exists \theta_1 \neq \theta_2 : \nabla(\gamma_v \circ h_v)(\theta_1) = \nabla(\gamma_v \circ h_v)(\theta_2) = 0 \text{ and } (\gamma_v \circ h_v)(\theta_1) = (\gamma_v \circ h_v)(\theta_2) \}. \]
For any $r \in \mathbb{R}_+$, let $B(0, r)$ be the open disk with radius $r$ centered at 0. Then, by Corollary 1 for any sufficiently small $\varepsilon > 0$ there exists a point $v \in B(0, \varepsilon)$ such that $\gamma_v \circ h_v$ is stable. This implies that there exists a sequence $\{v_n \in \text{int}(W)\}_{n=1,2,...}$ converging to the origin such that $\gamma_{v_n}$ is stable for any $n \in \mathbb{N}$.

For any $v \in \text{int}(W)$, define the convex integrand $\gamma_v : S^n \to \mathbb{R}_+$ as follows:
\[ \gamma_v(\theta) = \frac{1}{\gamma_v(-\theta)}. \]
Since $S^n$ is compact, the mapping $\Phi : B(0, \varepsilon) \to C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$ defined by $\Phi(v) = \gamma_v$ is continuous. Since $\gamma_0 = \gamma$, it follows that if $n$ is sufficiently large, then the convex integrand $\gamma_{v_n}$ must be inside the given neighborhood $V$ of $\gamma$. \hfill \square
4. Application of the proof of Theorem 1

Theorem 1 guarantees that $C_{\text{conv}}^\infty (S^n, \mathbb{R}^+)$ has sufficiently many stable functions. As in [21], $C^\infty$ convex integrands $\gamma$ having only non-degenerate critical points can be a useful tool to investigate $\text{inv}(\text{graph}(\gamma))$. For instance, we have the following:

**Proposition 4.** Let $n$ be an integer satisfying $n \geq 2$. Let $\gamma : S^n \to \mathbb{R}^+$ be a $C^\infty$ convex integrand such that any critical point is non-degenerate and the index of $\gamma$ at any critical point is zero or $n$. Then, $\gamma$ must be stable.

Notice that Proposition 4 does not hold in the case $n = 1$.

**Proof.** The following lemma is needed.

**Lemma 4.1.** Let $n$ be an integer satisfying $n \geq 2$. Let $\gamma : S^n \to \mathbb{R}^+$ be a convex integrand having only non-degenerate critical points such that the index of $\gamma$ at any critical point is zero or $n$. Then, $\gamma$ has only two critical points, the index is zero at one point and it is $n$ at another point.

Lemma 4.1 can be proved by using the Morse inequalities as follows. For any non-negative integer $\lambda$, denote by $C_\lambda$ the number of critical points of index $\lambda$ and by $R_\lambda$ the $\lambda$-th Betti number of $S^n$. Set

$$S_\lambda = R_\lambda - R_{\lambda-1} + \cdots + R_0.$$

Then, the following inequalities, called the **Morse inequalities**, hold (21):

$$(i_\lambda) \quad S_\lambda \leq C_\lambda - C_{\lambda-1} + \cdots + C_0.$$ 

The inequality $(i_0)$ implies $1 \leq C_0$. Since $n \geq 2$, the inequality $(i_1)$ implies $0 - 1 \leq 0 - C_0$, which is equivalent to $C_0 \leq 1$. Thus, we have $C_0 = 1$. In the case $\lambda > n$, the inequalities $(i_\lambda), (i_{\lambda+1})$ imply the following:

$$\sum_{\lambda=0}^{n} (-1)^\lambda R_\lambda = \sum_{\lambda=0}^{n} (-1)^\lambda C_\lambda.$$

By using this equality, it is easily seen that $C_n = 1$; and thus the proof of Lemma 4.1 completes.

Now we prove Proposition 4. It is sufficient to show the following inclusion:

$$\text{Sym}(\phi) \cap \text{int}(W) \subset \text{Caust}(\phi) \cap \text{int}(W).$$

Here, $\phi : S^n \to \mathbb{R}^{n+1}$ is the $C^\infty$ embedding defined by $\phi(\theta) = \left( \theta, \frac{1}{\gamma(\theta)} \right)$; $\text{Sym}(\phi)$ is the symmetry set of $\phi$, $\text{Caust}(\phi)$ is the caustic of $\phi$ and $\text{int}(W)$ is the set consisting of interior points of the convex hull of $\text{inv}(\text{graph}(\gamma))$. Suppose that the following set is non-empty.

$$\text{Sym}(\phi) \cap \text{int}(W) - \text{Caust}(\phi) \cap \text{int}(W).$$

Then, take one element $v$ of this set. For the $v$, as in Section 3 we construct a $C^\infty$ convex integrand $H \circ \gamma_v \circ h_v$. Since $v$ is outside $\text{Caust}(\phi) \cap \text{int}(W)$, it follows that all critical point of $H \circ \gamma_v \circ h_v$ are non-degenerate. By Lemma 4.1 the function $H \circ \gamma_v \circ h_v$ has only two critical points, the index is zero at one point and it is $n$ at another point. Let $\theta_1$ (resp., $\theta_2$) be the critical point with index zero (resp., index $n$). Then, it follows that $H \circ \gamma_v \circ h_v(\theta_1)$ is the minimal value and $H \circ \gamma_v \circ h_v(\theta_2)$ is the maximal value. On the other hand, since $v$ is inside $\text{Sym}(\phi) \cap \text{int}(W)$, there exist two distinct critical points $\tilde{\theta}_1, \tilde{\theta}_2 \in S^n$ such that $H \circ \gamma_v \circ h_v(\tilde{\theta}_1) = H \circ \gamma_v \circ h_v(\tilde{\theta}_2)$. Since there are no critical points of $H \circ \gamma_v \circ h_v$
except for $\theta_1, \theta_2$, it follows that $\{\theta_1, \theta_2\} = \{\tilde{\theta}_1, \tilde{\theta}_2\}$. Therefore, $H \circ \gamma_v \circ h_v$ must be a constant function, and this implies that $\phi(S^n)$ is a sphere centered at $v$. Hence, it follows that $\text{Caust}(\phi) = \text{Sym}(\phi) = \{v\}$, which contradicts the assumption that $v \notin \text{Caust}(\phi)$. Therefore, the following inclusion holds:

$$\text{Sym}(\phi) \cap \text{int}(W) \subset \text{Caust}(\phi) \cap \text{int}(W).$$

□

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