PSEUDORANDOM VECTORS GENERATION USING ELLIPTIC CURVES OVER FINITE FIELDS AND APPLICATIONS

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Abstract. In this paper we present, using the arithmetic of elliptic curves over finite fields, an algorithm for the efficient generation of sequence of uniform pseudorandom vectors in high dimension with long period, that simulates sample sequence of a sequence of independent identically distributed random variables, with values in the hypercube $[0, 1]^d$ with uniform distribution. As an application, we obtain, in the discrete time simulation, an efficient algorithm to simulate, uniformly distributed sample path sequence of a sequence of independent standard Wiener processes. This could be employed for use, in the full history recursive multi-level Picard approximation method, for numerically solving the class of semilinear parabolic partial differential equations of the Kolmogorov type.

1. Introduction

In numerical integration via the Monte Carlo method, and in the simulation of stochastic processes, an important role is played by the generation of pseudorandom numbers, and other more general pseudorandom variates. Arguably the most fundamental one is that of sequence of uniform pseudorandom numbers in the unit interval $[0, 1]$, that simulates a sample sequence of a sequence of independent identically distributed random variables, with values in $[0, 1]$ with uniform distribution (recall that by the Strong Law of Large Numbers and the Weyl Criterion for uniform distribution, for a sequence $\{X_n(\cdot)\}_{n \geq 0}$ of independent identically distributed random variables on a probability space $\Omega$, with values in $[0, 1]$ with uniform distribution, a sample sequence $\{X_n(\omega)\}_{n \geq 0}$, for $\omega \in \Omega$, is almost surely a uniformly distributed sequence in $[0, 1]$). The linear congruential generator is an efficient algorithm to generate such a sequence of uniform pseudorandom numbers.

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In section 2 of the paper, we are concerned with the generation of sequences of uniform pseudorandom vectors in high dimension, that simulate samples of a sequence of independent identically distributed random variables with values in the hypercube $[0, 1]^d$ with uniform distribution. These can be generated by using the matrix version of the linear congruential generator; nevertheless, it is well known that, in the higher dimensional case, the sequence of pseudorandom vectors produced by using the linear congruential generator (or its matrix version thereof) could exhibit lattice structures, which sometimes make them not suitable for use in Monte Carlo simulations.

Nonlinear versions of congruential generators could be constructed using the arithmetic of finite fields, for instance the inversive congruential generator (see for example [Ni]); the sequence of pseudorandom vectors thus constructed is observed to be free of lattice structures in general. In this paper we present an algorithm, which relies on the arithmetic of elliptic curves over finite fields, to construct sequence of uniform pseudorandom vectors; the algorithm is a variation of that of Hallgren ([Ha], [GBS]), having origin in elliptic curve cryptography [Ka]. Whereas a finite field is uniquely determined up to isomorphism by its cardinality, one has an ample supply of isomorphism classes of elliptic curves over a given (large) finite field to work with, making it all the more appealing from the perspective of Monte Carlo methods. In applications it is important to obtain sequence of pseudorandom vectors with long period. We give the analogue of the Hull-Dobell Theorem for our algorithm to yield the maximum period.

Either the inverse transform method or the Box-Muller method transforms a sequence of independent identically distributed random vectors in $[0, 1]^d$ with uniform distribution, to a sequence of independent identically distributed Gaussian random vectors in $\mathbb{R}^d$ with standard normal distribution; in addition, by normalizing these Gaussian random vectors so that they lie on $S^{d-1}$, the $d-1$ dimensional sphere of radius one, what one obtains then is a sequence of independent identically distributed random vectors in $S^{d-1}$ with uniform distribution with respect to $S^{d-1}$ (the uniform measure of $S^{d-1}$ is being normalized so that the measure of $S^{d-1}$ is equal to one, i.e. a probability measure). Now we use the result of Cutland-Ng [CN], which says that the sequence of uniform probability measures on $S^{d-1}$ converges weakly to the Wiener measure as $d \to \infty$; more precisely they gave a nonstandard analysis interpretation of the “Wiener sphere”: essentially, by taking $d$ to be a nonstandard infinite integer $d_{ns}$, the uniform probability measure on
$S^{d_{ns}-1}$ corresponds to the Wiener measure, and a random variable taking values in $S^{d_{ns}-1}$ with uniform distribution with respect to $S^{d_{ns}-1}$, corresponds to a standard Wiener process. Thus a sequence of independent identically distributed random vectors in $S^{d_{ns}-1}$ with uniform distribution with respect to $S^{d_{ns}-1}$, corresponds to a sequence of independent standard Wiener processes (the distribution law of standard Wiener process being the Wiener measure). The precise statements will be given in section 3.

Consequently, when being applied to the sequence of uniform pseudorandom vectors in $[0, 1]^d$ as constructed in section 2 of the paper, with $d$ being a large integer, this construction gives us a discrete time simulation of, uniformly distributed sample path sequence of a sequence of independent standard Wiener processes (uniform distribution with respect to discrete time simulation of the Wiener measure). These will be discussed in section 3 of the paper.

For the class of semilinear second order parabolic partial differential equations of the Kolmogorov type, one has stochastic representation of the viscosity solutions given by Feynman-Kac type formulas [BHJ], namely as a suitable expectation value against the Wiener measure; in a Monte Carlo style, using the full history recursive multi-level Picard approximation method (see for example, [EHJK1], [EHJK2], [HJvW], [HK], [HJKNvW]), these expectation values could be evaluated numerically by employing, the discrete time simulation of uniformly distributed sample path sequence of sequence of independent Wiener processes, as given in this paper. Explicit numerical studies will be the subject of a future investigation.

2. Construction of sequence of uniform pseudorandom vectors in $[0, 1]^d$

2.1. Résumé on elliptic curves over finite fields. In this subsection we recall some facts concerning elliptic curves over finite fields. For details we refer to chapter V of Silverman’s book [Si].

Notations: let $p$ be a prime, and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with $p$ elements. For convenience fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$; for $n \in \mathbb{Z}_{\geq 1}$, denote by $\mathbb{F}_{p^n}$ the unique subfield of $\overline{\mathbb{F}}_p$ consisting of $p^n$ elements ($\mathbb{F}_p$ is the prime subfield of $\mathbb{F}_{p^n}$ with $n = [\mathbb{F}_{p^n} : \mathbb{F}_p]$). Recall that all finite fields of the same cardinality are isomorphic.
Consider a finite field $F$ of characteristic $p$, with $q = p^m$ being the cardinality of $F$ (thus $F \cong \mathbb{F}_q$). An elliptic curve over $F$ could be specified by an affine Weierstrass equation:

$$(2.1) \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

with $a_1, a_2, a_3, a_4, a_6 \in F$, whose discriminant $\Delta \in F$ is nonzero (see section III.1 of [Si] for the explicit formulas for the discriminant and also the $j$-invariant associated to an affine Weierstrass equation). The elliptic curve $E$ over $F$ associated to (2.1) is the non-singular projective algebraic curve over $F$ of genus one, defined as the Zariski closure of (2.1) in the projective plane $\mathbb{P}^2$ over $F$. The affine part of $E$ is as given by (2.1), while there is a distinguished point of $E$, the unique point at infinity $O$ of $E$, that does not belong to the affine part (thus strictly speaking, the elliptic curve is the pair $(E, O)$, but we often refer to it just as $E$ for simplicity). For any field extension $L$ of $F$, we denote by $E(L)$ the set of points of $E$ whose coordinates belong to $L$. We have in particular that $O \in E(F)$.

**Remark 2.1.**

When $p \geq 5$, any elliptic curve over $F$ is isomorphic over $F$ to one whose affine Weierstrass equation is of the form:

$$y^2 = x^3 + Ax + B$$

with $A, B \in F$, such that the discriminant $\Delta = -16 \cdot (4A^3 + 27B^2) \in F$ is nonzero.

The elliptic curve $E$ is a commutative group variety over $F$ with identity element $O$. In particular, for any field extension $L$ of $F$, the set $E(L)$ is naturally an abelian group with identity element $O$. The abelian group addition law on $E$ is given by the chord-tangent law, and the formulas for the addition law on $E$ is given by rational functions of the affine coordinates $x, y$ with coefficients in $F$ (for the explicit formulas see section III.2 of [Si]).

For any field extension $L$ of $F$ and $P, Q \in E(L)$, we denote by $P + Q \in E(L)$ the sum of $P$ and $Q$ with respect to the addition law on $E$, and similarly denote by $-P \in E(L)$ the additive inverse of $P$ with respect to the addition law on $E$. For $n \in \mathbb{Z}_{\geq 1}$ and $P \in E(L)$, we define $[n](P) \in E(L)$ to be the point given by adding $P$ to itself $n$ times (with respect to the addition law on $E$), and we define $[0](P) := O$, and if $n \in \mathbb{Z}_{<0}$, then $[n](P) := [[n]](-P)$. 
Since $F$ is a finite field, one has that $E(F)$ is a finite abelian group. By the Hasse bound (c.f. Theorem 2.3.1 of Chapter V of [Si]), one has:

$$|\#E(F) - (q + 1)| \leq 2q^{1/2}$$

while in terms of group structure, one has:

$$E(F) \cong \mathbb{Z}/M_1\mathbb{Z} \times \mathbb{Z}/M_2\mathbb{Z}$$

with $M_1, M_2 \in \mathbb{Z}_{\geq 1}$, $M_1 | M_2$. Thus $E(F)$ is cyclic if and only if $M_1 = 1$, in which case we say that $E$ is cyclic over $F$.

As we will see in the next subsection, elliptic curves that are cyclic over $F$ allow us to construct sequences of uniform pseudorandom vectors with maximum period. We recall some of the results of Vladut [Vl].

Firstly recall that the elliptic curve $E$ over $F$ is supersingular (s.s.) if:

$$\#E(F) \equiv q + 1 \equiv 1 \mod p.$$  

If $E$ is supersingular with $j \in F$ being its $j$-invariant, then one has $[\mathbb{F}_p(j) : \mathbb{F}_p] = 1$ or 2. See section V.3 of [Si] for other equivalent definitions of supersingularity (in particular, the supersingular property only depends on $E$ over the algebraic closure of $F$); see also section V.4 of loc. cit. for examples of supersingular elliptic curves.

Supersingular elliptic curves exist over any finite field (c.f. [Br]), and the number of $F$-isomorphism classes of supersingular curves over $F$ is of the order $O(q^{1/2})$, in comparison with the total number of $F$-isomorphism classes of elliptic curves over $F$ which is close to $2q$ (c.f. [Vl], p. 19). In addition, define:

$$c(F)_{ss} = \frac{\text{# of F-isom. classes of s.s. elliptic curves over F cyclic over F}}{\text{# of F-isom. classes of s.s. elliptic curves over F}}$$

We have the following result of Vladut [Vl]. Recall that the cardinality of $F$ is $q = p^m$; the result depends on whether $m$ is even (i.e. on whether $\mathbb{F}_{p^2}$ could be embedded into $F$):

**Theorem 2.2.** (Proposition 3.1 of [Vl])

(i) If $\#F = q = p^m$ is not a square, then

- $c(F)_{ss} = 1$ for $p = 2$ or $p \equiv 1 \mod 4$.
- $c(F)_{ss} = 1/2$ for $p \equiv 3 \mod 4$. 

(ii) If \( \#F = q = p^m \) is a square, then

- \( \text{c}(F)_{ss} = 0 \) for \( p \equiv 1 \) mod 12.
- \( \text{c}(F)_{ss} = 24/(p + 31) \) for \( p \equiv 5 \) mod 12.
- \( \text{c}(F)_{ss} = 24/(p + 29) \) for \( p \equiv 7 \) mod 12.
- \( \text{c}(F)_{ss} = 36/(p + 49) \) for \( p \equiv 11 \) mod 12.
- \( \text{c}(F)_{ss} = 5/7 \) for \( p = 2 \).
- \( \text{c}(F)_{ss} = 2/3 \) for \( p = 3 \).

In particular, unless we both have \( \#F \) being a square and \( p \equiv 1 \) mod 12, there always exists supersingular elliptic curve over \( F \) that is cyclic over \( F \); in the case where \( \#F \) is not a square, we have that all supersingular elliptic curves over \( F \) are cyclic over \( F \), unless \( p \equiv 3 \) mod 4.

We also refer to [Vl] for discussions of the general case where one considers elliptic curves that are not necessarily supersingular.

2.2. The algorithm, part I. We follow the formalism of L’Ecuyer [LE]. Given the elliptic curve \( E \) over the finite field \( F \), the set of states is taken to be \( E_p \). Fix: a nonzero integer \( k \), and \( Q \in E(F) \). Define the transition function:

\[
T = T_{k,Q} : E(F) \to E(F)
\]

to be the following affine transformation on \( E(F) \):

\[
T(P) = [k](P) + Q, \text{ for } P \in E(F).
\]

Given an initial state \( P_0 \in E(F) \), define the sequence \( \{P_n\}_{n \geq 0} \) of points in \( E(F) \) recursively by the rule: \( P_{n+1} = T(P_n) = [k](P_n) + Q \). This is the elliptic curve version of the linear congruential generator [Ha], [GBS]. The general formula for \( P_n \) for \( n \geq 1 \) is as follows: firstly if \( k = 1 \), then one has:

\[
P_n = [n](Q) + P_0;
\]

on the other hand, if \( k \neq 1 \), then one has:

\[
P_n = [(k^n - 1)/(k - 1)](Q) + [k^n](P_0).
\]

Put \( N := \#E(F) \). Recall that by the Hasse bound, one has \( N = q + O(q^{1/2}) \). The maximum period for the sequence \( \{P_n\}_{n \geq 0} \) is \( N \) (in the language of dynamical systems, the maximum period condition amounts to saying that, the dynamical system on the finite state space \( E(F) \) defined by the transition function \( T_{k,Q} \) is ergodic). We have the following:
Theorem 2.3. The period for the sequence \( \{P_n\}_{n \geq 0} \) attains the maximum value \( N = \#E(F) \), if and only if the following holds:

1. \( E \) is cyclic over \( F \).
2. The point \( Q \) has order \( N \).
3. For each prime factor \( \ell \) of \( N \), we have \( k \equiv 1 \mod \ell \).
4. If \( 4 | N \) then \( k \equiv 1 \mod 4 \).

Proof. For the “if” part of the proof, assume conditions (1) - (4) hold. Since \( E \) is cyclic over \( F \), we can fix an isomorphism:

\[
E(F) \cong \mathbb{Z}/N\mathbb{Z}
\]

and let \( \gamma \mod N \) corresponds to the point \( Q \in E(F) \) under the isomorphism (2.3). Condition (2) is then equivalent to \( \gamma \) being relatively prime to \( N \). In addition, under the isomorphism (2.3), the iteration in \( E(F) \):

\[
P_{n+1} = [k](P_n) + Q
\]

is isomorphic to the following iteration in \( \mathbb{Z}/N\mathbb{Z} \):

\[
\alpha_{n+1} = k\alpha_n + \gamma \mod N
\]

which is the iteration appearing in the usual linear congruential generator. By the Hull-Dobell Theorem ([HD] Theorem 1, [Ku] Chapter 3, Theorem A), the conditions (3), (4) on \( k \), together with the condition that \( \gamma \) is relatively prime to \( N \), is equivalent to the condition that, the sequence \( \{\alpha_n \mod N\}_{n \geq 0} \) generated by the iteration (2.5) has the maximum period \( N \). This finishes the “if” part of the proof.

For the converse, we first show that \( E \) must be cyclic over \( F \). Fix an isomorphism:

\[
E(F) \cong \mathbb{Z}/M_1\mathbb{Z} \times \mathbb{Z}/M_2\mathbb{Z}
\]

with \( M_1 | M_2 \) and \( N = M_1 M_2 \). Let \( (\gamma \mod M_1, \delta \mod M_2) \) corresponds to the point \( Q \in E(F) \) under the isomorphism (2.6). Then the iteration (2.4) in \( E(F) \) corresponds, under the isomorphism (2.6), to the following iteration in \( \mathbb{Z}/M_1\mathbb{Z} \times \mathbb{Z}/M_2\mathbb{Z} \):

\[
(\alpha_{n+1} \mod M_1, \beta_{n+1} \mod M_2) = (k\alpha_n + \gamma \mod M_1, k\beta_n + \delta \mod M_2).
\]

The period of the sequence:

\( \{(\alpha_n \mod M_1, \beta_n \mod M_2)\}_{n \geq 0} \)

generated by the iteration (2.7), is at most:

\[
\max_{1 \leq a \leq M_1, 1 \leq b \leq M_2} \text{LCM}(a, b)
\]
and this is strictly less than \( N \), unless we have \( M_1 = 1, M_2 = N \). It follows that \( E(F) \) must be cyclic.

Now with \( E(F) \) being cyclic, we argue as in the “if” part: by using the Hull-Dobell Theorem again, the condition that the period for the sequence \( \{P_n\}_{n \geq 0} \) is \( N \), implies that conditions (2), (3), (4) must be satisfied. This finishes the “only if” part of the proof.

\[\square\]

**Remark 2.4.**

Efficient algorithms to compute \( N = \#E(F) \) were given by Schoof [Sc] and Elkies [El].

**Remark 2.5.**

One can consider more general higher order linear recursive sequence of points on elliptic curves over finite fields, as in [GL].

**2.3. The algorithm, part II.** As before \( F \) is a finite field with cardinality \( q = p^m \). Let \( m = a \cdot r \) be a factorization of \( m \), with \( a, r \in \mathbb{Z}_{\geq 1} \). Again we follow the formalism of [LE]. The set of outputs is going to be a subset of \( [0, 1]^{2r} \), and we define the output function \( G : E(F) \to [0, 1]^{2r} \) in this subsection.

Since \( a|m \), there is a unique finite subfield \( K \) of \( F \) with cardinality \( p^a \); thus \( [K : F_p] = a \) and \( [F : K] = r \). Fix a basis \( \alpha = \{\kappa_1, \cdots, \kappa_a\} \) of \( K \) over \( F_p \). For each element \( \zeta \in K \) and \( i = 1, \cdots, a \), define \( \phi_i(\zeta) \in \{0, 1, \cdots, p-1\} \subset \mathbb{Z}_{\geq 0} \), to be the coordinates of \( \zeta \) with respect to the basis \( \alpha \) (thus \( \zeta = \phi_1(\zeta) \cdot \kappa_1 + \cdots + \phi_a(\zeta) \cdot \kappa_a \)); here we fix \( \{0, 1, \cdots, p-1\} \) to be the set of representatives of elements of \( F_p = \mathbb{Z}/p\mathbb{Z} \).

Define (the “digit method”):

\[
\Phi : K \to [0, 1) \subset [0, 1] \\
\Phi(\zeta) = \sum_{i=1}^{a} \frac{\phi_i(\zeta)}{p^i}, \; \zeta \in K.
\]

Note that \( \Phi \) is injective.

Fix also a basis \( \beta = \{\lambda_1, \cdots, \lambda_r\} \) of \( F \) over \( K \). For \( \eta \in F \) and \( j = 1, \cdots, r \), define \( \langle \eta \rangle_j \in K \) to be the coordinates of \( \eta \) with respect to the basis \( \beta \) (thus we have \( \eta = \langle \eta \rangle_1 \cdot \lambda_1 + \cdots + \langle \eta \rangle_r \cdot \lambda_r \)).
We now define the output function $G$ (which depends on the choices of the bases $\alpha$ and $\beta$): for $P \in E(F)$ with $P \neq O$, let $x(P), y(P) \in F$ be the affine Weierstrass coordinates of the point $P$; define $G(P) \in [0, 1]^{2r}$ to be the vector:

$$G(P) = \left( \Phi(\langle x(P) \rangle_1), \cdots, \Phi(\langle x(P) \rangle_r), \Phi(\langle y(P) \rangle_1), \cdots, \Phi(\langle y(P) \rangle_r) \right)$$

(which actually lies in $[0, 1)^{2r}$). Finally we define $G(O) \in [0, 1]^{2r}$ to be the vector where all the coordinates are 1. It is clear that the output function $G : E(F) \to [0, 1]^{2r}$ thus defined is injective.

Given a value of initial state $P_0 \in E(F)$, we then compute the sequence of vectors $G(P_n) \in [0, 1]^{2r}$ for $n \geq 0$ (with the sequence of points $P_n \in E(F)$ for $n \geq 0$, being defined as in the previous subsection). The sequence $\{G(P_n)\}_{n \geq 0}$ is our sequence of pseudorandom vectors in $[0, 1]^{2r}$.

Remark 2.6.

In computations, it is useful to note that the quantities appearing in the definition of the vector $G(P)$, could be rewritten as follows. With notations as above, denote by $\beta' = \{\lambda'_1, \cdots, \lambda'_r\}$ the basis of $F$ over $K$ that is dual to $\beta = \{\lambda_1, \cdots, \lambda_r\}$ with respect to $Tr_{F/K}$, the trace from $F$ to $K$; thus for $1 \leq j, j' \leq r$, one has $Tr_{F/K}(\lambda_j \cdot \lambda'_{j'}) = 1$ (as elements of $K$) if $j = j'$, and is equal to 0 if $j \neq j'$. Then for $\eta \in F$ and $j = 1, \cdots, r$, one has

$$\langle \eta \rangle_j = Tr_{F/K}(\eta \cdot \lambda'_j).$$

Similarly denote by $\alpha' = \{\kappa'_1, \cdots, \kappa'_a\}$ the basis of $K$ over $F_p$ that is dual to $\alpha = \{\kappa_1, \cdots, \kappa_a\}$ with respect to $Tr_{K/F_p}$, the trace from $K$ to $F_p$. Then for $\zeta \in K$ and $i = 1, \cdots, a$, one has

$$\phi_i(\zeta) = Tr_{K/F_p}(\zeta \cdot \kappa'_i)$$

where as before we take $\{0, 1, \cdots, p-1\}$ as representatives for elements of $F_p$. 
Now for any \( \eta \in F \), and \( 1 \leq j \leq r \), one has:
\[
\Phi(\langle \eta \rangle_j) = \sum_{i=1}^{a} \frac{Tr_{K/F_p}(Tr_{F/K}(\eta \cdot \lambda_j^i \cdot \kappa_i^j))}{p^i} = \sum_{i=1}^{a} \frac{Tr_{F_p/F}(\eta \cdot \lambda_j^i \cdot \kappa_i^j)}{p^i}.
\]

And so the quantities appearing in the definition of the vector \( G(P) \), could be computed by using trace from \( F \) to \( F_p \).

In applications it is important to obtain sequence of pseudorandom vectors with long period. For instance, under the conditions of Theorem 2.3, we have, for any value of initial state \( P_0 \in E(F) \), that the sequence \( \{G(P_n)\}_{n \geq 0} \) has maximum period (equal to \( N = \#E(F) \)).

Finally, one way to justify the claim that the sequence \( \{G(P_n)\}_{n \geq 0} \) simulates a sample sequence, of a sequence of random variables with values in \([0, 1]^{2^r}\) with uniform distribution, is to estimate the discrepancy of the sequence \( \{G(P_n)\}_{n \geq 0} \).

In general the discrepancy of a finite set \( \mathcal{M} \subset [0, 1]^h \) (for \( h \in \mathbb{Z}_{\geq 1} \)) is defined as:
\[
\mathcal{D} = \mathcal{D}(\mathcal{M}) = \sup_{\mathcal{B} \subset [0, 1]^h} \left| \frac{\#(\mathcal{B} \cap \mathcal{M})}{\#\mathcal{M}} - \text{volume}(\mathcal{B}) \right|
\]
where the sup is taken over all the boxes \( \mathcal{B} \subset [0, 1]^h \).

Now for simplicity we consider the case where the integer \( k \) in the definition of the transition function \( T = T_{k,Q} \) is equal to 1. Let \( t \) be the period of the sequence \( \{P_n\}_{n \geq 0} \) (hence also that of \( \{G(P_n)\}_{n \geq 0} \)). Using the Erdös-Turán-Koksma inequality for Walsh function system, due to Hellekalek (Theorem 1 of [He]), the same argument as in the proof of Theorem 1 of El Mahassni-Shparlinski [ES] gives the bound for the discrepancy \( \mathcal{D} \) of the sequence \( \{G(P_n)\}_{n \geq 0} \):
\[
(2.8) \quad \mathcal{D} = O(t^{-1}q^{1/2}(\ln(p^a))^{2r}).
\]

More generally, for \( s \geq 2 \), the \( s \)-discrepancy \( \mathcal{D}_s \) of the sequence \( \{G(P_n)\}_{n \geq 0} \), which is a measure of its statistical independence of \( s \)
successive terms in \( \{G(P_n)\}_{n \geq 0} \), is defined to be the discrepancy of the following set of points:

\[
(G(P_n), \cdots, G(P_{n+s-1})), \quad n \geq 0
\]

regarded as vectors in \([0, 1]^{2rs}\). Then similarly as in Theorem 3 of Hess-Shparlinski [HS], we have the bound:

\[
D_s = O(t^{-1}q^{1/2}(\ln(p^s))^{2rs}).
\]

Thus for \( t \geq q^{1/2+\epsilon} \) (with \( \epsilon > 0 \)), the sequence \( \{G(P_n)\}_{n \geq 0} \) can be regarded as a good quality sequence of uniform pseudorandom vectors in \([0, 1]^{2r}\); in view of the Law of the Iterated Logarithm (c.f. Chapter 7 of [Ni]), the (pseudo-)randomness of the sequence \( \{G(P_n)\}_{n \geq 0} \) is pronounced, if one has \( t = q^{1+o(1)} \) (which certainly holds when \( t = N \), by the Hasse inequality).

For other values of \( k \), if the period is equal to \( N \), then the above discrepancy estimates (2.8) and (2.9) again hold with \( t = N \). An interesting problem is to obtain discrepancy estimates for the sequence \( \{G(P_n)\}_{n \geq 0} \) in the situation when \( k \) is not necessarily equal to 1 and when the period is not necessarily equal to \( N \); in the particular case where \( Q = \mathbf{O} \) (also known as the elliptic curve version of the power generator), we refer to the papers [LS, AS].

Summarizing, our algorithm for the construction of sequence of uniform pseudorandom vectors in \([0, 1]^d\) for \( d \in \mathbb{Z}_{\geq 1} \) is as follows:

- Fix a prime \( p \).
- Fix \( a, r \in \mathbb{Z}_{\geq 1} \) with \( d \leq 2r \). Put \( m = a \cdot r \).
- Fix finite field \( F \) with \( \#F = q = p^m \), and let \( K \) be the unique subfield \( F \) with \( \#K = p^r \). Fix a basis \( \alpha \) of \( K \) over \( \mathbb{F}_p \), and a basis \( \beta \) of \( F \) over \( K \).
- Fix an elliptic curve \( E \) over \( F \).
- Fix nonzero integer \( k \) and \( Q \in E(F) \) to define the transition function \( T = T_{k, Q} : E(F) \to E(F) \). The output function \( G : E(F) \to [0, 1]^{2r} \) is defined as above using the bases \( \alpha \) and \( \beta \).
- Given an initial state \( P_0 \in E(F) \), compute, for \( n \geq 0 \), the sequence of points \( P_n \in E(F) \) by iterating the transition function \( T \).
- For \( n \geq 0 \), compute the vectors \( G(P_n) \in [0, 1]^{2r} \).
- For \( n \geq 0 \), project the vectors \( G(P_n) \) to \([0, 1]^d\), by taking the first \( d \) coordinates of \( G(P_n) \).
3. Discrete time simulation of uniformly distributed sample path sequence of a sequence of independent Wiener processes

3.1. The algorithm, part I. To construct discrete time simulation of uniformly distributed sample path sequence of a sequence of independent standard Wiener processes, we first transform a sequence \( \{u_n\}_{n \geq 0} \) of uniform pseudorandom vectors in the unit hypercube \([0, 1]^d\), as constructed by the algorithm of the previous section, to a sequence of Gaussian pseudorandom vectors in \( \mathbb{R}^d \) with standard normal distribution (i.e. mean vector = the zero vector in \( \mathbb{R}^d \), variance matrix = the \( d \times d \) identity matrix). Firstly, we delete any vectors from the sequence \( \{u_n\}_{n \geq 0} \) whose any coordinate is either 0 or 1. When this is done, we may then assume without loss of generality that \( \{u_n\}_{n \geq 0} \) is a sequence of uniform pseudorandom vectors in \( \mathbb{R}^d \) with values in \( [p, q]^d \) for \( i = 1, \cdots, d \).

1. The Inverse Transform Method: let

\[
\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt
\]

be the (normalized) error function. Put for \( n \geq 0 \) and \( i = 1, \cdots, d \):

\[
v_n^{(i)} = \text{erf}^{-1}(u_n^{(i)}).
\]

The sequence \( \{v_n\}_{n \geq 0} \), with \( v_n = (v_n^{(1)}, \cdots, v_n^{(d)}) \), is then a sequence of Gaussian pseudorandom vectors in \( \mathbb{R}^d \), with standard normal distribution on \( \mathbb{R}^d \).

Justification: Let \( \{U_n(\cdot)\}_{n \geq 0} \) be a sequence of independent identically distributed random variables with values in \((0,1)^d\) with uniform distribution, with \( U_n(\cdot) = (U_n^{(1)}(\cdot), \cdots, U_n^{(d)}(\cdot)) \). Put for \( n \geq 0 \) and \( i = 1, \cdots, d \):

\[
V_n^{(i)}(\cdot) = \text{erf}^{-1}(U_n^{(i)}(\cdot)).
\]

Then the sequence \( \{V_n(\cdot)\}_{n \geq 0} \), with \( V_n(\cdot) = (V_n^{(1)}(\cdot), \cdots, V_n^{(d)}(\cdot)) \), is a sequence of independent identically distributed random variables with values in \( \mathbb{R}^d \), with standard normal distribution on \( \mathbb{R}^d \).
2. The Box-Muller Method: (without loss of generality) assume that $d$ is even: $d = 2g$. Put for $n \geq 0$ and $j = 1, \cdots, g$:

$$v_{n}^{(2j-1)} = \sqrt{-2 \ln(u_{n}^{(2j-1)})} \cos(2\pi v_{n}^{(2j)})$$

$$v_{n}^{(2j)} = \sqrt{-2 \ln(u_{n}^{(2j-1)})} \sin(2\pi v_{n}^{(2j)}).$$

The sequence $\{v_{n}\}_{n \geq 0}$, with $v_{n} = (v_{n}^{(1)}, \cdots, v_{n}^{(d)})$, is then a sequence of Gaussian pseudorandom vectors in $\mathbb{R}^d$, with standard normal distribution on $\mathbb{R}^d$.

Justification: Let $\{U_{n}(\cdot)\}_{n \geq 0}$ be a sequence of independent identically distributed random variables with values in $(0,1)^d$ with uniform distribution, with $U_{n}(\cdot) = (U_{n}^{(1)}(\cdot), \cdots, U_{n}^{(d)}(\cdot))$. Put for $n \geq 0$ and $j = 1, \cdots, g$:

$$V_{n}^{(2j)}(\cdot) = \sqrt{-2 \ln(U_{n}^{(2j)}(\cdot))} \cos(2\pi U_{n}^{(2j)}(\cdot))$$

$$V_{n}^{(2j)}(\cdot) = \sqrt{-2 \ln(U_{n}^{(2j)}(\cdot))} \sin(2\pi U_{n}^{(2j)}(\cdot))$$

then the sequence $\{V_{n}(\cdot)\}_{n \geq 0}$, with $V_{n}(\cdot) = (V_{n}^{(1)}(\cdot), \cdots, V_{n}^{(d)}(\cdot))$, is a sequence of independent identically distributed random variables with values in $\mathbb{R}^d$, with standard normal distribution on $\mathbb{R}^d$ ([BM], [OG]).

Now let $S^{d-1} \subset \mathbb{R}^d$ be the $d - 1$ dimensional unit sphere consisting of elements whose norm is equal to one. The uniform measure on $S^{d-1}$ (i.e. rotationally invariant measure) is normalized to be equal to one, i.e. a probability measure.

Then given a sequence of Gaussian pseudorandom vectors in $\mathbb{R}^d$ with standard normal distribution, we delete any vector from the sequence that is equal to the zero vector. When this is done, we may then assume that none of the $v_{n}$ is equal to the zero vector. Put for $n \geq 0$:

$$w_{n} = v_{n}/||v_{n}|| \in S^{d-1}$$

(assuming that $V_{n}(\cdot)$ does not take the zero vector as value for any $n \geq 0$), is a sequence of independent identically distributed random variables with values in $S^{d-1}$, with uniform distribution with respect to $S^{d-1}$.

Justification: Let $\{V_{n}(\cdot)\}_{n \geq 0}$ be a sequence of independent identically distributed random variables with values in $\mathbb{R}^d$ with standard normal distribution. Then the sequence $\{W_{n}(\cdot)\}_{n \geq 0}$, with $W_{n}(\cdot) := V_{n}(\cdot)/||V_{n}(\cdot)||$ (assuming that $V_{n}(\cdot)$ does not take the zero vector as value for any $n \geq 0$), is a sequence of independent identically distributed random variables with values in $S^{d-1}$, with uniform distribution respect to $S^{d-1}$.
(this follows from the fact that the standard normal distribution on $\mathbb{R}^d$

is invariant with respect to rotation about the origin).

3.2. The algorithm, part II. We can now construct the discrete
time simulation of uniformly distributed sample path sequence, of a
sequence of independent standard Wiener processes (to be precise, uni-
form distribution with respect to discrete time simulation of the Wiener
measure). Our construction relies on work of Cutland-Ng [CN] (which
is based in turn on [Cu]). We consider (one dimensional) Wiener pro-
cesses for the time interval $[0, T]$ with $T \in \mathbb{R}_{\geq 0}$. For simplicity we take
$T = 1$. Denote by $C_0([0, 1])$ the set of $\mathbb{R}$-valued continuous functions $c(t)$ on $[0, 1]$ such that $c(0) = 0$. The set $C_0([0, 1])$ is equipped with
the Wiener measure (and hence is a probability space).

With $d \geq 1$ as before, put $t_i = i/d$ for $i = 0, 1, \ldots, d$ (discretization
of the time interval $[0, 1]$). Define the map:

$$
\Sigma_d : S^{d-1} \rightarrow C_0([0, 1])
$$

as follows: for $w = (w^{(1)}, \ldots, w^{(d)}) \in S^{d-1}$, define $\Sigma_d(w) : [0, 1] \rightarrow \mathbb{R}$
to be the polygonal path in $C_0([0, 1])$, such that:

$$
\begin{align*}
(\Sigma_d(w))(0) &= 0 \\
(\Sigma_d(w))(t_i) &= \sum_{k=1}^{i} w^{(k)} \text{ for } i = 1, \ldots, d
\end{align*}
$$

and $(\Sigma_d(w))(t)$ is linearly interpolated between $(i-1)/d \leq t \leq i/d$ for
$i = 1, \ldots, d$. The map $\Sigma_d$ is clearly injective and measurable.

Firstly, we have:

**Theorem 3.1.** (Theorem 2.4 of [CN]) For $d \geq 1$ let $m_d$ be the uni-
form probability measure on $S^{d-1}$. Then the sequence of measures on
$C_0([0, 1])$ given by the push-forward of $m_d$ to $C_0([0, 1])$:

$$
m_d \circ \Sigma_d^{-1}
$$

converges weakly to the Wiener measure on $C_0([0, 1])$, as $d \rightarrow \infty$.

The measure $m_d \circ \Sigma_d^{-1}$ could thus be regarded as a discrete time
simulation of the Wiener measure on $C_0([0, 1])$.

Now if $\{W_n(\cdot)\}_{n \geq 0}$ is a sequence of independent identically distributed
random variables with values in $S^{d-1}$, with uniform distribution with
respect to $S^{d-1}$, then by the Strong Law of Large Numbers and the
Weyl Criterion for uniform distribution, a sample sequence $\{W_n(\omega)\}_{n \geq 0}$
of \( \{W_n(\cdot)\}_{n \geq 0} \) is almost surely a sequence in \( S^{d-1} \) with uniform distribution with respect to \( S^{d-1} \); as in the previous subsection, this could be simulated by a sequence \( \{w_n\}_{n \geq 0} \) of pseudorandom vectors in \( S^{d-1} \) with uniform distribution with respect to \( S^{d-1} \). By taking \( d \) to be a large integer, the sequence \( \{\Sigma_d(w_n)\}_{n \geq 0} \) could be considered as discrete time simulation of uniformly distributed sample path sequence, of a sequence of independent standard Wiener processes. Here since we are dealing with discrete time simulation, uniform distribution here is in fact meant to be with respect to discrete time simulation of the Wiener measure, namely \( m_d \circ \Sigma_d^{-1} \).

To justify this we use the more precise form of the result of Cutland-Ng [CN], which used the tools of nonstandard analysis. We refer to [Go] for the background on nonstandard analysis.

Let \( ^*\mathbb{R} \) be nonstandard extension of \( \mathbb{R} \), and for \( x \in ^*\mathbb{R} \) define \(^x \) to be the standard part of \( x \). This means that, if \( x \) is finite, then \(^x \) is the unique element in \( \mathbb{R} \) that is infinitesimally close to \( x \); otherwise if \( x \) is not finite, then we simply define \(^x \) to be \( \pm \infty \). Fix a nonstandard infinite integer \( d_{ns} \). Inside the \(^*\)Euclidean space \(^*\mathbb{R}^{d_{ns}} \), denote by \( \mathcal{S} \subset ^*\mathbb{R}^{d_{ns}} \) the internal subset of \(^*\mathbb{R}^{d_{ns}} \) consisting of elements whose norm is equal to one. In [CN] the internal set \( \mathcal{S} \) is referred to as the Wiener sphere.

For \( w \in \mathcal{S} \), define the internal polygonal path \( \Sigma_{d_{ns}}(w) : [0, 1] \rightarrow ^*\mathbb{R} \) in a similar way as before: put \( \tau_i = i/d_{ns} \) for \( i = 0, 1, \ldots, d_{ns} \), then:

\[
(\Sigma_{d_{ns}}(w))(0) = 0
\]
\[
(\Sigma_{d_{ns}}(w))(\tau_i) = \sum_{k=1}^i w^{(k)} \text{ for } i = 1, \ldots, d_{ns}
\]

and in general for \( \tau \in [0, 1] \), the value \( (\Sigma_{d_{ns}}(w))(\tau) \) is linearly interpolated between \((i - 1)/d_{ns} \leq \tau \leq i/d_{ns} \) for \( i = 1, \ldots, d_{ns} \).

Define the internal map:

\[
B : \mathcal{S} \times [0, 1] \rightarrow ^*\mathbb{R}
\]

by the rule: for \( w \in \mathcal{S} \) and \( \tau \in [0, 1] \):

\[
B(w, \tau) = (\Sigma_{d_{ns}}(w))(\tau).
\]

Put \( b(w, t) := \circ B(w, t) \) for \( w \in \mathcal{S} \) and \( t \in [0, 1] \).

Now denote by \( \mathfrak{m} \) the uniform (i.e. rotationally invariant) internal probability measure on \( \mathcal{S} \), and by \( \mathfrak{m}_L \) its Loeb extension; thus \( \mathcal{S} \)
equipped with the Loeb measure \( m_L \) is a probability space in the usual sense, and \( b(\cdot, \cdot) \) is a stochastic process.

We then have:

**Theorem 3.2.** (Theorem 2.1 and Corollary 2.2 of [CN]) We have, for almost all \( w \in S \) (with respect to the Loeb measure \( \mu_L \)), that the sample path \( b(w, \cdot) \) defines an element in \( C_0([0, 1]) \). Thus by restricting to a subset \( S' \subset S \) with \( S \setminus S' \) being of measure zero (with respect to \( m_L \)), we have a map:

\[
\Sigma_{\infty} : S' \rightarrow C_0([0, 1])
\]

\[
(\Sigma_{\infty}(w))(t) := b(w, t)
\]

\[
= \circ ((\Sigma_{\delta_n}(w))(t)), \ w \in S', \ t \in [0, 1].
\]

The map \( \Sigma_{\infty} \) is measurable. Furthermore, the measure on \( C_0([0, 1]) \) given by the push-forward of \( m_L \) by \( \Sigma_{\infty} \):

\[
m_L \circ (\Sigma_{\infty})^{-1}
\]

is the Wiener measure on \( C_0([0, 1]) \). In addition \( b(\cdot, \cdot) \) is a standard Wiener process (whose distribution law is the Wiener measure).

Note that a sample path \( b(w, \cdot) \) for \( w \in S' \), of the standard Wiener process \( b(\cdot, \cdot) \), is \( \Sigma_{\infty}(w) \). In addition, it also follows from Theorem 3.2 that, if \( \{W_n(\cdot)\}_{n \geq 0} \) is a sequence of independent identically distributed random variables on a probability space \( \Omega \), with values in \( S' \) with uniform distribution with respect to the Loeb measure \( m_L \) of \( S \), then \( \{b(W_n(\cdot), \cdot)\}_{n \geq 0} \) is a sequence of independent standard Wiener processes, whose sample path sequence is \( \{\Sigma_{\infty}(W_n(\omega))\}_{n \geq 0} \), for \( \omega \in \Omega \).

Now having recalled the results of [CN], we finally let \( d \) be a large integer (simulation of a nonstandard infinite integer), and as in the end of the previous subsection, let \( \{w_n\} \) be a sequence of pseudorandom vectors in \( S^{d-1} \) with uniform distribution with respect to \( S^{d-1} \) (simulation of a sample sequence of a sequence of independent identically distributed random variables, with values in \( S^{d-1} \) with uniform distribution). The previous discussion thus justifies the procedure of taking the sequence \( \{\Sigma_d(w_n)\}_{n \geq 0} \) as, discrete time simulation of uniformly distributed sample path sequence, of a sequence of independent standard Wiener processes (uniform distribution with respect to discrete time simulation of the Wiener measure).

**Remark 3.3.**
The theorem of Cutland-Ng [CN] can be extended directly to the case of Wiener processes in $\mathbb{R}^D$ (for $D \in \mathbb{Z}_{\geq 1}$), by working with the Cartesian product of $D$ copies of the Wiener sphere $S$. Our algorithm can thus be extended to this setting as well, to construct discrete time simulation of uniformly distributed sample path sequence, of a sequence of independent standard Wiener processes in $\mathbb{R}^D$.

4. Conclusion

In this paper we present, using the arithmetic of elliptic curves over finite fields, an efficient algorithm for the generation of sequence uniform pseudorandom vectors in the unit hypercube. Criterion for the algorithm to generate sequences with maximum period is also given.

We have shown how these could be transformed to construct, discrete time simulation of uniformly distributed sample path sequence, of a sequence of independent standard Wiener processes. In a Monte Carlo style, these could be used for the numerical evaluation of expectation values against the Wiener measure, for example those occurring in Feynman-Kac type formulas.

For the class of semilinear parabolic partial differential equations of the Kolmogorov type, their viscosity solutions have stochastic representation given by non-linear Feynman-Kac formulas ([BHJ]). In the full history recursive multi-level Picard approximation (MLP) method, these non-linear Feynman-Kac formulas could be evaluated numerically, using as input a denumerable set of independent standard Wiener processes (see for example [EHJK1], [EHJK2], [HJvW], [HK], [HJKNvW]).

Our construction of discrete time simulation of uniformly distributed sample path sequence of a sequence of independent Wiener processes, using the algorithm as given in sections 2 and 3 of this paper, could thus be employed as inputs for the MLP method in the numerical approximation of solutions to these class of equations. Explicit numerical studies will be carried out in a future investigation.

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