A symmetry of silting quivers

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Received: 2 December 2022; Revised: 22 June 2023; Accepted: 23 June 2023; First published online: 26 July 2023

Keywords: silting object; silting mutation; silting quiver; support τ-tilting module; support τ-tilting quiver; anti-automorphism

2020 Mathematics Subject Classification: Primary - 16G20; Secondary - 16G60

Abstract

We investigate symmetry of the silting quiver of a given algebra which is induced by an anti-automorphism of the algebra. In particular, one shows that if there is a primitive idempotent fixed by the anti-automorphism, then the 2-silting quiver (= the support τ-tilting quiver) has a bisection. Consequently, in that case, we obtain that the cardinality of the 2-silting quiver is an even number (if it is finite).

1. Introduction

In this paper, we study symmetry of the silting quiver of a finite dimensional algebra Λ over an algebraically closed field; the silting quiver is a quiver whose vertices are (basic) silting objects and arrows \( T \rightarrow U \) are drawn whenever \( U \) is an irreducible left mutation of \( T \), and it coincides with the Hasse quiver of the poset \( \text{silt} \Lambda \) of silting objects [5].

The main theorem (Theorem 1.2) of this paper shows that an anti-automorphism of \( \Lambda \) (i.e., an algebra isomorphism \( \Lambda^{\text{op}} \cong \Lambda \)) induces a symmetry of \( \text{silt} \Lambda \). Here, \( \Lambda^{\text{op}} \) stands for the opposite algebra of \( \Lambda \). Focusing on 2-term silting objects, which bijectively correspond to support \( \tau \)-tilting modules [3], we obtain a bisection of the poset \( 2\text{silt} \Lambda \) of 2-term silting objects if there is a fixed primitive idempotent by the anti-automorphism (Theorem 1.4). Thus, in that case, it turns out that the cardinality of \( 2\text{silt} \Lambda \) is even (if it is finite).

When \( \Lambda \) is 2-silting finite (= \( \tau \)-tilting finite); i.e., \( |2\text{silt} \Lambda| < \infty \), counting the number of elements in \( 2\text{silt} \Lambda \) is one of the important problems in this area; see [1, 2, 4, 9, 15]. In this context, Theorem 1.4 gives a very useful method to reduce the whole pattern to half of \( 2\text{silt} \Lambda \). Indeed, this may be applied to such works on Hecke algebras and Schur algebras, see [8], [17], etc.

For example, the following admit anti-automorphisms fixing a primitive idempotent:

- enveloping algebras (Theorem 2.1);
- preprojective algebras of Dynkin type (Theorem 2.5);
- cellular algebras (Theorem 2.6);
- symmetric algebras with radical cube zero, which contain multiplicity-free Brauer line/cycle algebras (Theorem 2.7);
- selfinjective Nakayama algebras, which contain Brauer star algebras with an exceptional vertex in the center (Theorem 2.8);
- group algebras (Theorem 2.10);
- the trivial extensions of algebras with an anti-automorphism fixing a primitive idempotent (Theorem 2.12).

TA was partly supported by JSPS Grant-in-Aid for Young Scientists 19K14497. QW was partly supported by JSPS Grant-in-Aid for Young Scientists 20J10492.

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Here is an illustration of the symmetry of $2\text{silt} \Lambda$ for the preprojective algebra $\Lambda$ of Dynkin type $A_3$, in which $\square$ and $\blacklozenge$ correspond:

![Diagram of symmetry]

**Notation.** Throughout this paper, let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $K$, and $\mathcal{K}_\Lambda := K^b(\text{proj} \Lambda)$ denote the perfect derived category of $\Lambda$. The $\Lambda$-dual is denoted by $(\_)^* := \text{Hom}(\_ \Lambda)$ for $\_ = \mathcal{K}_\Lambda$ or $\mathcal{K}_\Lambda^{\text{op}}$.

### 2. Results

We say that an object $T$ of $K/\Lambda$ is silting (tilting) if it satisfies $\text{Hom}_{K/\Lambda}(T, T[i]) = 0$ for every integer $i > 0$ ($i \neq 0$) and $\mathcal{K}_\Lambda = \text{thick} T$. Here, $\text{thick} T$ stands for the smallest thick subcategory of $\mathcal{K}_\Lambda$ containing $T$. We denote by $\text{silt} \Lambda$ (tilt $\Lambda$) the set of isomorphism classes of basic silting (tilting) objects in $\mathcal{K}_\Lambda$.

Let us recall silting mutation and a partial order on $\text{silt} \Lambda$.

**Definition-Theorem 1.1.** [5, Theorem 2.11, 2.31, 2.35 and Definition 2.41]

1. Let $T$ be a silting object of $\mathcal{K}_\Lambda$ with decomposition $T = X \oplus M$. Taking a minimal left $\text{add} M$-approximation $f : X \rightarrow M'$ of $X$, we construct a new object $\mu_X^-(T) := Y \oplus M$, where $Y$ is the mapping cone of $f$. Then $\mu_X^-(T)$ is also silting, and we call it the left mutation of $T$ with respect to $X$. Dually, we define the right mutation $\mu_X^+(T)$ of $T$ with respect to $X$.
2. For objects $T$ and $U$ of $\mathcal{K}_\Lambda$, we write $T \geq U$ if $\text{Hom}_{\mathcal{K}_\Lambda}(T, U[i]) = 0$ for $i > 0$. Then $\geq$ gives a partial order on $\text{silt} \Lambda$.
3. We construct the silting quiver $\mathcal{H}$ of $\mathcal{K}_\Lambda$ as follows.
   - The vertices of $\mathcal{H}$ are basic silting objects of $\mathcal{K}_\Lambda$;
   - We draw an arrow $T \rightarrow U$ if $U$ is a left mutation of $T$ with respect to an indecomposable direct summand.

Then $\mathcal{H}$ coincides with the Hasse quiver of the partially ordered set $\text{silt} \Lambda$.

We define a subset of $\text{silt} \Lambda$ by

$$2\text{silt} \Lambda := \{ T \in \text{silt} \Lambda \mid \Lambda \geq T \geq \Lambda[1] \}.$$

This bijectively corresponds to the poset of support $\tau$-tilting modules [3, Theorem 3.2].
We say that $\Lambda$ admits an anti-automorphism if there is a $K$-linear automorphism $\zeta : \Lambda \to \Lambda$ satisfying $\zeta(xy) = \zeta(y)\zeta(x)$, or equivalently if an algebra isomorphism $\sigma : \Lambda^{op} \to \Lambda$ exists. Here, $\Lambda^{op}$ stands for the opposite algebra of $\Lambda$. In this case, we obtain an equivalence $\mathcal{K}_{\Lambda^{op}} \to \mathcal{K}_{\Lambda}$, also denoted by $\sigma$.

We now investigate that an anti-automorphism of $\Lambda$ induces a symmetry of $\text{silt}\, \Lambda/\text{2silt}\, \Lambda$.

**Theorem 1.2.** Assume $\Lambda$ admits an anti-automorphism $\sigma$. Then we have the following.

1. The functor $S_\sigma := \sigma \circ (-)^*$ induces an anti-automorphism of the poset $\text{silt}\, \Lambda$.
2. Let $T$ be a silting object. Then there is an algebra isomorphism $\text{End}_{\mathcal{K}_{\Lambda}}(T)^{op} \simeq \text{End}_{\mathcal{K}_{\Lambda}}(S_\sigma(T))$.
   Moreover, if $\Gamma$ is derived equivalent to $\Lambda$, then so is $\Gamma^{op}$; hence, $\Gamma$ and $\Gamma^{op}$ are also derived equivalent.
3. The functor $S_\sigma := [1] \circ S_\sigma$ induces an anti-automorphism of the poset $2\text{silt}\, \Lambda$.

**Proof.** (1)(3) It is evident that $(-)^*$ and $\sigma$ yield an anti-isomorphism $\text{silt}\, \Lambda \to \text{silt}\, \Lambda^{op}$ and an isomorphism $\text{silt}\, \Lambda^{op} \to \text{silt}\, \Lambda$, respectively. Composing them makes an anti-automorphism of $\text{silt}\, \Lambda$. This immediately implies that $S_\sigma := [1] \circ S_\sigma$ is also an anti-automorphism of $2\text{silt}\, \Lambda$.

(2) Clearly, $\text{End}_{\mathcal{K}_{\Lambda}}(S_\sigma(T)) \simeq \text{End}_{\mathcal{K}_{\Lambda}}(T)^{op} \simeq \text{End}_{\mathcal{K}_{\Lambda}}(T)^{op}$. If $\Gamma$ is derived equivalent to $\Lambda$, then there is a tilting object $T$ of $\mathcal{K}_{\Lambda}$ with $\Gamma \simeq \text{End}_{\mathcal{K}_{\Lambda}}(T)$. Since $S_\sigma(T)$ is also tilting, it is seen by (1) that $\Gamma^{op} \simeq \text{End}_{\mathcal{K}_{\Lambda}}(S_\sigma(T))$ is derived equivalent to $\Lambda$. □

We discuss a benefit derived from the symmetry $S_\sigma$ of $2\text{silt}\, \Lambda$.

Let $P$ be an indecomposable projective $\Lambda$-module. We define subsets of $2\text{silt}\, \Lambda$ by

$$\mathcal{T}_P^- := \{ T \in 2\text{silt}\, \Lambda \mid \mu_T^-(\Lambda) \geq T \geq \Lambda[1] \} \quad \text{and} \quad \mathcal{T}_P^+ := \{ T \in 2\text{silt}\, \Lambda \mid \Lambda \geq T \geq \mu_T^+(\Lambda[1]) \}.$$

Denote by $X^i$ the $i$th term of a complex $X$. We make the following observation.

**Lemma 1.3.** We have $\mathcal{T}_P^- = \{ T \in 2\text{silt}\, \Lambda \mid P \in \text{add}\, T^{-1} \}$ and $\mathcal{T}_P^+ = \{ T \in 2\text{silt}\, \Lambda \mid P \in \text{add}\, T^0 \}$. In particular, $\mathcal{T}_P^- \cup \mathcal{T}_P^+ = 2\text{silt}\, \Lambda$.

**Proof.** Let $T \in 2\text{silt}\, \Lambda$. We know that $T$ is of the form $[T^{-1} \to T^0]$ with $T^{-1}, T^0 \in \text{add}\, \Lambda$. By [5, Lemma 2.25], we have $\text{add}\, T^{-1} \cap \text{add}\, T^0 = 0$. It is easily seen that $\text{add}\, (T^{-1} \oplus T^0) = \text{add}\, \Lambda$. Now, we obtain from [5, Theorem 2.35] (and its dual) that:

(i) $P \in \text{add}\, T^{-1} \iff \mu_T^-(\Lambda) \geq T$;

(ii) $P \in \text{add}\, T^0 \iff T \geq \mu_T^+(\Lambda[1])$.

This completes the proof. □

The symmetry $S_\sigma$ is useful to analyze the cardinality of $2\text{silt}\, \Lambda$ as follows.

**Theorem 1.4.** Let $e$ be a primitive idempotent of $\Lambda$ and put $P := e\Lambda$. Assume that $\Lambda$ admits an anti-automorphism $\sigma$. If $\sigma(e) = e$, then we have a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$, i.e., $\mathcal{T}_P^- \simeq \mathcal{T}_P^+$. In particular, $|2\text{silt}\, \Lambda| = 2 \cdot |\mathcal{T}_P^-| = 2 \cdot |\mathcal{T}_P^+|$.

**Proof.** We see that $S_\sigma$ gives a one-to-one correspondence between $\mathcal{T}_P^-$ and $\mathcal{T}_{S_\sigma(P)}$. As $S_\sigma(P) \simeq P$ by assumption, the assertion follows from Lemma 1.3. □
Let \( T := [T_1 \to T_2] \) be a 2-term silting object of \( \mathcal{K}_\Lambda \); i.e., \( T \in \text{2silt}_\Lambda \), and \( E \) denote a complete list of pairwise orthogonal primitive idempotents of \( \Lambda \). Recall that the \( g \)-vector \( g_T \) of \( T \) is the vector \( (g_e)_{e \in E} \) which is given by \( g_e := c_e - c_e' \). Here, \( c_e' \) stands for the multiplicity of \( e \Lambda \) in \( T_i \).

We immediately obtain the following corollary.

**Corollary 1.5.** Suppose that \( \Lambda \) admits an anti-automorphism \( \sigma \) satisfying \( \sigma(e) = e \) for every primitive idempotent \( e \) of \( \Lambda \). Then \( S_\sigma \) reverses the directions of the \( g \)-vectors of all 2-term silting objects in \( \mathcal{K}_\Lambda \).

**Proof.** Let \( T := [e_1 \Lambda \to e_0 \Lambda] \) be a 2-term silting object of \( \mathcal{K}_\Lambda \), where \( e_0 \) and \( e_1 \) are idempotents of \( \Lambda \). Since any idempotent is fixed by \( \sigma \), we observe that \( S_\sigma \) sends \( T \) to the 2-term silting object \([e_0 \Lambda \to e_1 \Lambda]\), which immediately tells us the fact that \( g_{S_\sigma(T)} = -g_T \). \( \square \)

### 3. Applications and examples

We explore when \( \Lambda \) admits an anti-automorphism \( \sigma \) with \( \sigma(e) = e \) for some primitive idempotent \( e \) of \( \Lambda \), and give applications and examples of Theorem 1.4.

Let us start with enveloping algebras.

**Theorem 2.1.** The enveloping algebra \( \Lambda^{\text{op}} \otimes_K \Lambda \) has an anti-automorphism \((a \otimes b \mapsto b \otimes a)\) fixing the primitive idempotent \( e \otimes e \) for a primitive idempotent \( e \) of \( \Lambda \). In particular, there is a bijection between \( \mathcal{T}_P^- \) and \( \mathcal{T}_P^+ \), where \( P := (e \otimes e)\Lambda^{\text{op}} \otimes_K \Lambda \).

Let \( Q := (Q_0, Q_1) \) be a (finite) quiver, where \( Q_0 \) and \( Q_1 \) are the sets of vertices and arrows, respectively. For a vertex \( v \) of \( Q \), we denote by \( e_v \) the primitive idempotent of \( KQ \) corresponding to \( v \). The opposite quiver of \( Q \) is denoted by \( Q^{\text{op}} \); that is, it consists of the same vertices as \( Q \) and reversed arrows \( a^* \) for arrows \( a \) of \( Q \), i.e., \( a^* \) is obtained by swapping the source and target of \( a \). For an admissible ideal \( I \) of \( KQ \), reversing arrows makes the admissible ideal \( I^{\text{op}} \) of \( KQ^{\text{op}} \); for example, \( ab \in I \) implies \( b^*a^* \in I^{\text{op}} \).

We consider the case that an isomorphism \( \iota : Q^{\text{op}} \to Q \) of quivers exists; \( \iota \) gives rise to an algebra isomorphism \( KQ^{\text{op}} \to KQ \), which will be also written by \( \iota \).

**Proposition 2.2.** Let \( \Lambda \) be an algebra presented by a quiver \( Q \) and an admissible ideal \( I \) of \( KQ \). Suppose that there is an isomorphism \( \iota : Q^{\text{op}} \to Q \) of quivers satisfying \( I^{\text{op}} = \iota^{-1}(I) \) and fixing a vertex \( v \); put \( P := e_v \Lambda \). Then we have a bijection between \( \mathcal{T}_P^- \) and \( \mathcal{T}_P^+ \). In particular, \(|\text{2silt}_\Lambda| = 2 \cdot |\mathcal{T}_P^-|\).

**Proof.** As \( I^{\text{op}} = \iota^{-1}(I) \), we get isomorphisms

\[
\Lambda^{\text{op}} = (KQ/I)^{\text{op}} \cong KQ^{\text{op}}/I^{\text{op}} \cong KQ/I = \Lambda;
\]

write the composition by \( \sigma : \Lambda^{\text{op}} \to \Lambda \). Since \( \iota(v) = v \) by assumption, we have \( \sigma(e_v) = e_v \). Thus, the assertion follows from Theorem 1.4. \( \square \)

**Example 2.3.** Let \( \Lambda \) be the algebra given by the \( A_n \)-quiver \( Q : 1 \xrightarrow{x} 2 \xrightarrow{x} \cdots \xrightarrow{x} n \) and the admissible ideal \( I = 0 \) or \( I := \langle x^r \rangle \) for some \( r > 0 \). We have an isomorphism \( Q^{\text{op}} \to Q \) of quivers which assigns \( i \mapsto n - i + 1 \) (i.e., \( x^r \mapsto x \) (i.e., \( x \in Q_i \)). The equalities \( I^{\text{op}} = \langle (x^r)^r \rangle = \iota^{-1}(I) \) imply that \( \Lambda \) admits an anti-automorphism \( \sigma \). If \( n \) is even, then we apply Theorem 1.2. If \( n \) is odd, then the vertex \( v := \frac{n+1}{2} \) is fixed by \( \sigma \), whence we can apply Proposition 2.2: we get \( \mathcal{T}_{\iota, \Lambda} \cong \mathcal{T}_{\iota', \Lambda} \). The following are the Hasse quivers of \( \text{2silt}_\Lambda \) for \( n = 2 \) and \( n = 3 \), in which \( \square \) and \( \bigcirc \) correspond and \( \bullet \) is stable by \( S_{\sigma} \).
Here, in the RHS, $\square$ and $\bigcirc$ are the members of $\mathcal{T}_{P^+}$ and $\mathcal{T}_{P^-}$, respectively.

### 3.1. Algebras presented by double quivers

Recall that the double quiver $\overline{Q}$ of $Q$ is the quiver constructed by $\overline{Q}_0 := Q_0$ and $\overline{Q}_1 := Q_1 \sqcup \{a^* \mid a \in Q_1\}$, where $a^*$ is obtained by swapping the source and target of $a$. Clearly, the assignments $v \mapsto v$ ($v \in Q_0$), $a^* \mapsto a^*$ and $(a^*)^* \mapsto a$ ($a \in Q_1$) make an isomorphism $\iota: \overline{Q}^{op} \rightarrow \overline{Q}$ of quivers; note that $\iota$ fixes all vertices.

Let us give examples of algebras presented by a double quiver.

**Example 2.4.**

1. [11] The preprojective algebra $\Pi_1$ of a Dynkin quiver $Q$ is defined as the quotient $K\overline{Q}/I$ of $K\overline{Q}$ by $I := \langle aa^* - a^*a \mid a \in Q_1 \rangle$. Then, it is finite dimensional and selfinjective.

2. [18, Example 1.6] Let $Q$ be a quiver and $I$ an admissible ideal of $KQ$. For a path $p = a_1a_2 \cdots a_\ell$ in $Q$, write $p^* := a_\ell^* \cdots a_2^*a_1^*$; extending it linearly, we also use the terminology $p^*$ for a linear combination $p$ in $KQ$. We define an ideal $\overline{I}$ of $K\overline{Q}$ which is generated by $p, p^*$ ($p \in I$) and $ab^*$ ($a, b \in Q_1$). Then the algebra $\Lambda(Q, I) := K\overline{Q}/\overline{I}$ is finite dimensional. If $Q$ contains no oriented cycle, then $\Lambda(Q, I)$ is a quasi-hereditary algebra with a duality.

Now, an application of Proposition 2.2 is obtained.

**Theorem 2.5.** Let $\Lambda = \Pi_1$ for a Dynkin quiver $Q$ or $\Lambda(Q, I)$ for a quiver $Q$ and an admissible ideal $I$ of $KQ$. Then we have a bijection between $\mathcal{T}_{P^-}$ and $\mathcal{T}_{P^+}$ for any indecomposable projective module $P$ of $\Lambda$. In particular, $|2\text{silt } \Lambda| = 2 \cdot |\mathcal{T}_{P^-}|$.

**Proof.** We can easily check the equality $\overline{I}^{op} = \iota^{-1}(\overline{I})$ holds, and apply Proposition 2.2. \qed

### 3.2. Cellular algebras

Cellular algebras were introduced by Graham and Lehrer [10]. An algebra $\Lambda$ is called cellular if it admits a cellular basis; that is, a basis with certain nice multiplicative properties. We refer to
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[14] for more details. By the definition, each cellular basis of \( \Lambda \) admits an involution \( \sigma \); i.e., an anti-automorphism \( \sigma \) of \( \Lambda \) with \( \sigma^2 = 1 \). It is shown in [14, Proposition 5.1] that the involution \( \sigma \) fixes all simples of a cellular algebra. Hence, we have the following result.

**Theorem 2.6.** Let \( \Lambda \) be a cellular algebra. Then there exists a bijection between \( \mathcal{T}_P^- \) and \( \mathcal{T}_P^+ \) for any indecomposable projective module \( P \) of \( \Lambda \). In particular, \( |\text{2silt } \Lambda| = 2 \cdot |\mathcal{T}_P^-| \).

Nowadays, a lot of interesting algebras have been found to be cellular, for example, Ariki–Koike algebras, \((q)\)-Schur algebras as well as various generalizations, block algebras of category \( O \), and various diagram algebras. We hope that Theorem 2.6 will be useful to verify the finiteness of \(|\text{2silt } \Lambda|\) for the aforementioned algebras, especially for Hecke algebras [18], Schur algebras [17], etc.

3.3. Symmetric algebras with radical cube zero

We get the following result.

**Theorem 2.7.** Let \( \Lambda \) be a symmetric algebra with radical cube zero. Then there exists a bijection between \( \mathcal{T}_P^- \) and \( \mathcal{T}_P^+ \) for any indecomposable projective module \( P \) of \( \Lambda \). In particular, \( |\text{2silt } \Lambda| = 2 \cdot |\mathcal{T}_P^-| \).

**Proof.** By [2, Proposition 3.3], it turns out that the Gabriel quiver of \( \Lambda \) is given by adding loops to the double quiver of a quiver \( Q \); denote by \( \hat{Q} \) the quiver of \( \Lambda \). We also observe that \( aa^* \neq 0 \neq a^*a \) for any arrow \( a \) of \( \hat{Q} \) and \( ab = 0 \) unless \( b = a^* \) and \( a = b^* \); if \( a \) is an added loop, write \( a^* = a \). Thus, we get an isomorphism \( \iota : \hat{Q}^\text{op} \rightarrow \hat{Q} \) of quivers which fixes all vertices.

Let \( i \) be a vertex of \( \hat{Q} \) and \( a \) an arrow starting at \( i \). Since \( \Lambda \) is symmetric, it is seen that \( aa^* \) spans the socle of \( P_i := e_i\Lambda \) as a vector space. Applying changes of basis, we have \( aa^* = bb^* \) for every arrow \( b \) of \( \hat{Q} \) starting from \( i \). Let \( I \) denote the ideal of \( K\hat{Q} \) consisting of such relations; so \( \Lambda \simeq K\hat{Q}/I \). Then, we obtain the equality \( I^\text{op} = \iota^{-1}(I) \), whence the assertion follows from Proposition 2.2.

### 3.4. Selfinjective Nakayama algebras

It is well known that a self-injective Nakayama algebra is presented by a cycle quiver

\[
\bullet \xrightarrow{x} \bullet \xrightarrow{x} \ldots \xrightarrow{x} \bullet
\]

with relations \( x^r = 0 \) for some \( r > 0 \). Here is an easy application of Proposition 2.2.

**Theorem 2.8.** Let \( \Lambda \) be a self-injective Nakayama algebra and \( P \) an indecomposable projective module of \( \Lambda \). Then we have a bijection between \( \mathcal{T}_P^- \) and \( \mathcal{T}_P^+ \). In particular, \( |\text{2silt } \Lambda| = 2 \cdot |\mathcal{T}_P^-| \).

**Remark 2.9.** Let \( \Lambda \) be a self-injective Nakayama algebra given by a cycle quiver \( Q \). Whenever we choose a vertex \( i \) of \( Q \), one gets an isomorphism \( Q^\text{op} \rightarrow Q \) of quivers fixing \( i \). So, a bijection between \( \mathcal{T}_{e_i\Lambda}^- \) and \( \mathcal{T}_{e_i\Lambda}^+ \) depends on the choice of vertices.

3.5. Group algebras

Let \( G \) be a finite group and \( p \) the characteristic of \( K \). While the group algebra \( KG \) is, in general, neither basic nor ring-indecomposable,\(^1\) it admits an anti-automorphism by \( g \mapsto g^{-1} \); we can then apply Theorem 1.2 to \( KG \).

The following situation enables us to apply Theorem 1.4.

\(^1\)It is well known that if there is a normal \( p \)-subgroup of \( G \) containing its centralizer, then \( KG \) is ring-indecomposable; see [16, Exercise V. 2. 10] for example.
Let $G$ be a semidirect product $E \rtimes D$ of a $p'$-group $E$ (i.e., $p \nmid |E|$) on a $p$-group $D$. Then there exists a primitive idempotent $e$ of $\Lambda := KG$ such that $T_{e\Lambda}^{-}$ bijectively corresponds to $T_{e\Lambda}^{+}$. In particular, $|2\text{silt}\Lambda|$ is double $|T_{e\Lambda}^{-}|$.

**Proof.** As the argument above, we know that $\Lambda$ admits an anti-automorphism $\sigma$ ($g \mapsto g^{-1}$). Since $|E|$ is invertible in $K$, we put $e := \frac{1}{|E|} \sum_{g \in E} g$; clearly, it is an idempotent fixed by $\sigma$. It is seen that $e\Lambda = eKG = eKD \simeq KD$ (as $KD$-modules), which implies that $e$ is primitive. Thus, we deduce the assertion from Theorem 1.4. □

We obtain an interesting observation.

**Corollary 2.11.** Let $\Lambda$ be a $p$-block of $KG$ with a normal defect group $D$ and $E$ its inertial quotient. If $E$ has trivial Schur multiplier (i.e., $H^{2}(E, K^{*}) = 1$), then the number of 2-term silting objects is even if it is finite.

**Proof.** Thanks to Külshammer’s theorem [13, Theorem A], we see that $\Lambda$ is Morita equivalent to the twisted group algebra $K^{\alpha}[E \rtimes D]$ for some 2-cocycle $\alpha$, which is just $K[E \rtimes D]$ by assumption. Thus, we find out that $|2\text{silt}\Lambda|$ is even by Theorem 2.10. □

It is known that groups of deficiency zero have the trivial Schur multiplier; see [12]. Here, the deficiency of a group $G$ is defined to be the maximum of the integers $|X| - |R|$ for all presentations $G = \langle X \mid R \rangle$ of $G$, which is nonpositive if $G$ is a finite group. Typical examples of deficiency-zero finite groups are cyclic groups $\langle g \mid g^{n} = 1 \rangle$ and quaternion groups $\langle a, b \mid a^{2n} = 1, a^{n} = b^{2}, ba = a^{-1}b \rangle = \langle a, b \mid bab = a^{-1}, aba = b \rangle$. Thus, the first example of Corollary 2.11 should be the case that $D$ is cyclic; then, $E$ is automatically cyclic, $\Lambda$ is a symmetric Nakayama algebra [6, Theorem 17.2], and so $|2\text{silt}\Lambda| = \binom{2n}{n}$ (even), where $n := |E|$ [1, Corollary 2.29]. Moreover, the equality $|2\text{silt}\Lambda| = \binom{2n}{n}$ holds even if we drop the assumption of $D$ being normal in $G$; then, $\Lambda$ is still a Brauer tree algebra [6, Theorem 17.1], whence the equality is obtained from [7, Theorem 5.1].

### 3.6. Trivial extension algebras

The trivial extension $T(\Lambda)$ of an algebra $\Lambda$ (by its minimal cogenerator $D\Lambda$) is defined to be $\Lambda \oplus D\Lambda$ as a $K$-vector space with multiplication given by $(a, f) \cdot (b, g) := (ab, ag + fb)$. Here, $D$ denotes the $K$-dual. We can easily verify that there is a one-to-one correspondence between simple modules of $\Lambda$ and $T(\Lambda)$; so we use the same symbol $e$ as a primitive idempotent of $\Lambda$ and $T(\Lambda)$ (via the correspondence).

We state that a bisection of $2\text{silt}\Lambda$ can be extended to that of $2\text{silt}T(\Lambda)$.

**Theorem 2.12.** An anti-automorphism $\sigma$ of $\Lambda$ induces one on $T(\Lambda)$, say $\bar{\sigma}$. If $\sigma$ fixes a primitive idempotent $e$ of $\Lambda$, then the corresponding idempotent $e\bar{\sigma}$ of $T(\Lambda)$ is stable by $\bar{\sigma}$. In the case, we have a bisection of $2\text{silt}T(\Lambda)$ with respect to $P := eT(\Lambda)$.

**Proof.** Note that $T(\Lambda)^{\op} = T(\Lambda^{\op})$. Since $\sigma^{-1} : \Lambda \to \Lambda^{\op}$ is an algebra isomorphism, we have a $K$-linear automorphism $t_{\sigma} := \text{Hom}_{K}(\sigma^{-1}, K) : D(\Lambda^{\op}) \to D\Lambda$ of $D\Lambda$. For any $a, b \in \Lambda^{\op}$ and $f \in D(\Lambda^{\op})$, we get equalities

$$
t_{\sigma}(a \bullet f \bullet b)(x) = (a \bullet f \bullet b)(\sigma^{-1}(x)) = f(b \bullet \sigma^{-1}(x) \bullet a) = f(\sigma^{-1}(\sigma(b)\sigma(a)))
$$

$$
= t_{\sigma}(f)(\sigma(b)\sigma(a)) = (\sigma(a)t_{\sigma}(f)(\sigma(b))(x).
$$

Here, $\bullet$ stands for the multiplication or the action of $\Lambda^{\op}$. It turns out that

$$
t_{\sigma}(a \bullet f \bullet b) = \sigma(a)t_{\sigma}(f)(\sigma(b).
$$
Now, we define a $K$-linear automorphism $\bar{\sigma}: T(\Lambda^{\text{op}}) \rightarrow T(\Lambda)$ by $(a,f) \mapsto (\sigma(a), t_\sigma(f))$. Let us check that $\bar{\sigma}$ is an anti-automorphism of $T(\Lambda)$; for any $a, b \in \Lambda^{\text{op}}$ and $f, g \in D(\Lambda^{\text{op}})$,

$$\bar{\sigma}((a,f) \bullet (b,g)) = \bar{\sigma}(a \bullet b, a \bullet g + f \bullet b) = (\sigma(a \bullet b), t_\sigma(a \bullet g + f \bullet b))$$

$$= (\sigma(a) \sigma(b), \sigma(a)t_\sigma(g) + t_\sigma(f) \sigma(b))$$

$$= (\sigma(a), t_\sigma(f)) \cdot (\sigma(b), t_\sigma(g))$$

$$= \bar{\sigma}(a,f) \cdot \bar{\sigma}(b,g).$$

Thus, the first assertion holds. As the second assertion is clear, the last one immediately follows from Theorem 1.4.

\[\square\]

**Remark 2.13.** Theorem 2.12 does not imply that taking trivial extensions transmits the $\tau$-tilting finiteness. In fact, the radical-square-zero self-injective Nakayama algebra with 2 simple modules is $\tau$-tilting finite, but its trivial extension is not so.

### 3.7. Applying the main theorem twice

In this subsection, we try applying Theorem 1.4 twice in a row. Let us show the following.

**Theorem 2.14.** Assume that $\Lambda$ is basic and admits an anti-automorphism $\sigma$ fixing a primitive idempotent $e$ of $\Lambda$; write $P := e\Lambda$. Let $P'$ be the mapping cone of a minimal left $\text{add}(\Lambda/P)$-approximation of $P$; that is, $\mu_P^-(\Lambda) = P' \oplus \Lambda/P$. Putting $\Gamma := \text{End}_{\Lambda/\Lambda} (\mu_P^-(\Lambda))$, $e'$ denotes the idempotent of $\Gamma$ corresponding to $P'$. Assume that the following hold:

1. $\mu_P^-(\Lambda)$ is tilting;
2. There is an anti-automorphism $\sigma'$ of $\Gamma$ satisfying $\sigma'(e') = e'$.

Then, we have a poset isomorphism $T_P^{-} \simeq T_{\Gamma'}^{+}$ and $|2\text{silt}\, \Lambda| = |2\text{silt}\, \Gamma|$.

**Proof.** As $\mu_P^-(\Lambda)$ is tilting, we identify $2\text{silt}\, \Gamma$ with $\{T \in \text{silt}\, \Lambda \mid \mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\}$. By Lemma 1.3, we have an equality:

$$\{T \in \text{silt}\, \Lambda \mid \mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\} = \{T \in \text{silt}\, \Lambda \mid \mu_P^+\mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\} \cup \{T \in \text{silt}\, \Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\},$$

in which the components of RHS have the same cardinality by Theorem 1.4. Thus, the cardinality of LHS in the equality is the double of that of $T_P^-$, which is equal to the cardinality of $2\text{silt}\, \Lambda$.

We give two examples; one illustrates Theorem 2.14, and the other explains that a derived equivalence does not necessarily preserve the cardinality of the poset $2\text{silt}\, (-)$ even if a given algebra is a symmetric algebra which admits an anti-automorphism fixing a primitive idempotent.

**Example 2.15.** Let $\Lambda$ be the algebra presented by the quiver with relations as follows:

```
  1
 / \   \beta
/ \   /  \\
\gamma^* \gamma \ 3
  \gamma
```

\[
\begin{align*}
\beta \gamma \alpha &= 0 = \gamma (\gamma^* \gamma)^3 \\
\alpha \beta &= \gamma^* \gamma \gamma^*
\end{align*}
\]
Note that $\Lambda$ is symmetric and admits an anti-automorphism which fixes the vertex 1 and switches the vertices 2 and 3. Set $P_1 := e_1\Lambda$.

1. Let $T_1$ be the left mutation of $\Lambda$ with respect to $P_1$. By hand, we can check that the endomorphism algebra $\Gamma_1$ of $T_1$ is given by the quiver with relations:

$$
\begin{array}{c}
2 \xrightarrow{\alpha} 1 \xrightarrow{\beta} 3 \\
\end{array}
$$

$$
\begin{cases}
\alpha\beta^*\beta = \beta^*\beta\alpha^* = \alpha = 0 \\
\alpha^*\alpha = (\beta\beta^*)^2 \\
\end{cases}
$$

It is obtained that $\Gamma_1$ has an anti-automorphism fixing the vertex 1. Thus, we derive from Theorem 2.14 that $\text{2silt} \, \Lambda$ and $\text{2silt} \, \Gamma_1$ has the same cardinality; it is illustrated by (anti-) isomorphisms $\mathcal{T}_{(P_1)_{1,2}} \cong \mathcal{T}_{(P_1)_{1,2}} \cong \mathcal{T}_{(P_1)_{1,2}} \cong \mathcal{T}_{(P_1)_{1,2}}$. Actually, $\text{2silt} \, \Lambda$ and $\text{2silt} \, \Gamma_1$ are finite sets and the numbers are 32 [4, Theorem 2].

2. Let $T_2$ be the left mutation of $\Lambda$ with respect to $P_2$. We have the endomorphism algebra $\Gamma_2$ presented by the quiver with relations:

$$
\begin{array}{c}
1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \\
\end{array}
$$

$$
\begin{cases}
\beta\gamma = \beta^* = 0 = \gamma^*\beta^* = \gamma^* \alpha \\
\alpha\gamma = 0 \\
\alpha^2 = \beta^* \beta \\
\alpha^3 = \gamma\gamma^* \\
\end{cases}
$$

Unfortunately, the cardinality of $\text{2silt} \, \Gamma_2$ is 28 by [4, Theorem 2]. Since $\Gamma_2$ admits an anti-automorphism fixing the vertex 2, a similar argument as the proof of Theorem 2.14 explains that $\mathcal{T}_{(P_2)_{1,2}} \cong \mathcal{T}_{(P_2)_{1,2}} \cong \mathcal{T}_{(P_2)_{1,2}} \cong \mathcal{T}_{(P_2)_{1,2}}$, and so we obtain $|\mathcal{T}_{(P_2)_{1,2}}| = 14$ and $|\mathcal{T}_{(P_2)_{1,2}}| = 18$. (Note that $\mathcal{T}_{(P_2)_{1,2}} \cong \mathcal{T}_{(P_1)_{1,2}}$, so, $|\mathcal{T}_{(P_1)_{1,2}}| = 18$ and $|\mathcal{T}_{(P_1)_{1,2}}| = 14$.) When $U_3$ is the left mutation of $\Gamma_2$ with respect to $P_3$, the endomorphism algebra of $U_3$ is isomorphic to $\Lambda$. This says that a derived equivalence does not necessarily preserve the number of $\text{2silt} \, (\Lambda)$, although $\Gamma_2$ is symmetric and admits an anti-automorphism fixing all vertices.

There are some special derived equivalence classes of algebras for which the cardinalities of $\text{2silt} \, (\Lambda)$ are constant, but the proofs are case by case for each algebra. Using Theorem 2.14, we may give an explicit example of such classes.

**Example 2.16.** Let $\Lambda$ be the multiplicity-free Brauer triangle algebra; that is, it is given by the quiver with relations as follows.

$$
\begin{array}{c}
1 \xrightarrow{a} 2 \xrightarrow{a} 3 \\
\end{array}
$$

$$
\begin{cases}
aa^* = a^*a \\
a^2 = 0 = (a^*)^2 \\
\end{cases}
$$

We see that $\Lambda$ admits an anti-automorphism fixing every vertex; cf. Theorem 2.7.

Let $P := e_1\Lambda$ and $\Gamma$ denote the endomorphism algebra of the left mutation $\mu_P(\Lambda)$; note that $\mu_P(\Lambda)$ is a tilting object in $\mathcal{K}_\Lambda$, and so $\Lambda$ and $\Gamma$ are derived equivalent. By hand, we obtain that $\Gamma$ is presented by the quiver with relations:

$$
\begin{array}{c}
2 \xrightarrow{a} 1 \xrightarrow{b^*} 3 \\
\end{array}
$$

$$
\begin{cases}
aa^* = 0 = bb^* \\
a^*ab^*b = b^*ba^*a \\
\end{cases}
$$

Observe that $\Gamma$ admits an anti-automorphism fixing all vertices.
Thus, it turns out by Theorem 2.14 that $2\text{silt }\Lambda$ and $2\text{silt }\Gamma$ have the same cardinality; actually, they are finite sets and the numbers are 32. See $D(3K)$ and $D(3A)$, in Table 1 of [9]. Moreover, the class $\{\Lambda, \Gamma\}$ forms a derived equivalence class.

Acknowledgments. The second author is grateful to Susumu Ariki for various conversations and lectures. He also thanks Sota Asai and Kengo Miyamoto for useful discussions.

Conflict of interest. Not applicable.

References

[1] T. Adachi, The classification of $\tau$-tilting modules over Nakayama algebras, *J. Algebra* **452** (2016), 227–262.
[2] T. Adachi and T. Aoki, The number of two-term tilting complexes over symmetric algebras with radical cube zero, *Ann. Comb.* **27**(1) (2023), 149–167.
[3] T. Adachi, O. Iyama and I. Reiten, $\tau$-tilting theory, *Compos. Math.* **150**(3) (2014), 415–452.
[4] T. Aihara, T. Honma, K. Miyamoto and Q. Wang, Report on the finiteness of silting objects, *Proc. Edinb. Math. Soc.* **2**(2) (2021), 64–233.
[5] T. Aihara and O. Iyama, Silting mutation in triangulated categories, *J. Lond. Math. Soc.* **2**(3) (2012), 85–668.
[6] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11 (Cambridge University Press, Cambridge, 1986).
[7] H. Asashiba, Y. Mizuno and K. Nakashima, Simplicial complexes and tilting theory for Brauer tree algebras, *J. Algebra* **551** (2020), 119–153.
[8] S. Ariki and L. Speyer, Schurian-finiteness of blocks of type $A$ Hecke algebras, Preprint (2022), arXiv: 2112.11148.
[9] F. Eisele, G. Janssens and T. Raedschelders, A reduction theorem for $\tau$-rigid modules, *Math. Z.* **290**(3-4) (2018), 1377–1413.
[10] J. J. Graham and G. I. Lehrer, Cellular algebras, *Invent. Math.* **123**(1) (1996), 1–34.
[11] I. M. Gelfand and V. A. Ponomarev, Model algebras and representations of graphs, *Funktsional. Anal. i Prilozhen.* **13**(3) (1979), 1–12.
[12] D. L. Johnson, *Presentations of groups*, London Mathematical Society Student Texts, vol. 15 (Cambridge University Press, Cambridge, 1976).
[13] B. Kulshammer, Crossed products and blocks with normal defect groups, *Commun. Algebra* **13**(1) (1985), 147–168.
[14] S. Konig and C. Xi, On the structure of cellular algebras, in *Algebra and modules, II*, CMS Conf. Proc., vol. 24 (Amer. Math. Soc., Providence, RI, 1998), 365–386.
[15] Y. Mizuno, Classifying $\tau$-tilting modules over preprojective algebras of Dynkin type, *Math. Z.* **277**(3-4) (2014), 665–690.
[16] H. Nagao and Y. Tsushima, *Representations of finite groups* (Academic Press, Inc., Boston, MA, 1989).
[17] Q. Wang, On $\tau$-tilting finiteness of the Schur algebra, *J. Pure Appl. Algebra* **226**(1) (2022), 106818.
[18] C. Xi, Quasi-hereditary algebras with a duality, *J. Reine Angew. Math.* **449** (1994), 201–215.