Chaotic dynamics of a classical radiant cavity

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The statistical properties of a classical electromagnetic field in interaction with matter are numerically investigated on a one–dimensional model of a radiant cavity, conservative and with finite total energy. Our results suggest a trend towards equipartition of energy, with the relaxation times of the normal modes of the cavity increasing with the mode frequency according to a law, the form of which depends on the shape of the charge distribution.

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The study of quantum deviations from classical ergodicity has occupied much of Quantum Chaology since its origins. Remarkably enough, the historical development of quantum mechanics started with the Blackbody problem, which displays a deviation as blatant as possible from classical ergodicity. When a classical radiation field interacts with matter inside an enclosure with perfectly reflecting walls, approach to statistical equilibrium – if at all possible – appears to entail unending escape of energy towards higher and higher frequencies, in sharp contrast to the Planck distribution law \[ \frac{\hbar}{k_B T} \]. Thus the problem of blackbody radiation was at once the first problem in quantum mechanics, the first problem in quantum field theory, and the first problem in quantum chaos. Considerable progress has been meanwhile attained in understanding the complex behaviour of nonlinear classical dynamical systems and its quantum counterparts, so a re-examination of the blackbody problem in the light of such developments appears necessary. One would like, first, to build a Hamiltonian model of a radiant cavity, which does indeed exhibit the sort of tendency to equipartition expected in Jeans’s time; second, to understand how does Planck’s law emerge from the quantal dynamics of that very model. Neither of these issues seems to have been satisfactorily dealt with as yet. In this Letter we accomplish the first part of the task by presenting a model whose classical dynamics leads to equipartition in the sense of Jeans.

Our model is a variant of one which was introduced years ago \[ \dagger \] to this purpose, and which was later investigated in several papers \[ \S \]. None of those investigations was able to detect a tendency towards energy equipartition among the normal modes of the cavity. Analogous results were obtained on different models \[ \S \], such as a one–dimensional linear string interacting with nonlinear oscillators etc.

The general picture which emerges from all these works, in which the Newton–Maxwell equations were numerically solved, is that there is no tendency to energy equipartition among the field normal modes. The reason lies with two fully general aspects of the classical field–matter interaction. First, the total energy is finite, whereas the number of freedoms is infinite; second, field modes can only exchange energy via interaction with finitely many mechanical freedoms. Thus mechanical nonlinearities become less and less effective as energy flows from the matter to the field: this fact prevents the appearance of an altogether chaotic dynamics, thus causing high frequency modes to be nonergodically “frozen”. One therefore needs a model, giving rise to chaotic behavior of the mechanical freedoms, no matter how small their energy is.

Though extremely simplified, our model displays this property. Let us first consider an electromagnetic field confined inbetween two parallel, perfectly reflecting plane mirrors, a distance \[ 2l \] apart. We take Cartesian coordinates \( XYZ \) with the \( X \) axis normal to the mirrors, and restrict to excitations only dependent on \( X \), thus getting a 1–dimensional radiant cavity, the normal modes of which have angular frequencies \( \omega_n = (\pi c/2l) n, n = 1, 2, \ldots \). Then we introduce a uniformly charged, infinite plate of thickness \( 2\delta \), situated midway between the mirrors and parallel to them, bound to move along the \( Z \)–direction only. We denote \( z \) its displacement in that direction, \( \sigma \) and \( m \) the charge and mass densities per unit surface of the plate, \( f(x) \) the normalized (transverse) distribution of charge in the plate. Finally, the plate is subject to a mechanical restoring force per unit surface, \( F(z) = -m \omega_n^2 z \). Using the Coulomb gauge, plus zero boundary conditions on the mirrors for the \( Z \) component of the vector potential, we obtain the following Hamiltonian for the full system plate plus field:
\[ H_0 = \frac{1}{2m} \left( p_z - 2 \left( \frac{\pi}{2} \right)^{1/2} \sigma \sum_{n=1}^{\infty} a_n q_n \right)^2 \]
\[ + \frac{1}{2} m \omega_0^2 z^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left( p_n^2 + \omega_n^2 q_n^2 \right), \]

where \((z, p_z)\) and \((q_n, p_n)\) are canonical conjugated variables for the plate and the \(n\)-th mode of the field respectively. In particular, \(q_n(t)\) is the amplitude of the \(Z\) component of the vector potential on the \(n\)-th normal mode of the free field, and the coefficients \(a_n\) are given by \(a_n = \int_{-\delta}^{\delta} dx f(x) \cos(\omega_n x/\epsilon)\). \(\sum\) means the sum over odd \(n\)'s only, because, with the chosen boundary conditions, even modes do not interact with the plate.

Finite-energy states of our hamiltonian system correspond to vectors in the Hilbert space \(\mathcal{H}_0\) of square–summable, \(\infty\)-dimensional vectors
\[ q = \left\{ m^{1/2} \omega_0 z, m^{1/2} \dot{z}, \ldots, \omega_{2n-1} q_{2n-1}, p_{2n-1}, \ldots \right\} \]
whose squared norm is just twice the energy. Since Hamilton's equations are linear, the evolution of states whose squared norm is just twice the energy. Since \(H_{new}\) is a Hilbert space summable, the evolution between collisions is again described by adding to the vector \(q\) one more component \(m^{1/2}w\), with \(w\) the velocity of the uncharged plate. The new Hilbert space \(\mathcal{H}\) of such \(q\)-vectors is the phase space of our model. The evolution between collisions is again unitary, given by \(\exp(iHt)\), with a generator \(H\), and a complete set of normal modes, which are trivially related to the above described ones. At collisions the two plates exchange their velocities, thus mixing all the amplitudes in the expansion of the state vector over normal modes.

The evolution from immediately after one collision to immediately after the next is given by a map, which, in Hilbert space notations, has the following simple form:
\[ S(q) = (Id - 2P)e^{iHt(q)}q. \]

The 1st (operator) factor describes a collision: \(Id\) is the identity operator, \(P\) is a one–dimensional projection: \(Pq = <e|q > e\), where \(e\) is the unit vector such that the scalar product \(<e|q >\) yields the relative velocity of the two plates in the state \(q\). The 2nd factor describes evolution over the free-flight time \(t(q)\), which is the smallest positive root of the equations
\[ |z(e^{iHt}q) - z(q)| - w(q)t(\lambda \rightarrow 0, 2R). \]

The map allows for efficient numerical simulation: e.g., in the case of 200 oscillators, we were able to follow a trajectory up to \(5 \times 10^6\) bounces with a relative error in energy conservation less than \(10^{-10}\).

In numerical simulations, one has of course to consider a finite number \(N\) of field oscillators. In our computations we have varied \(N\) and all other parameters, except \(l = \pi, m = 1, \epsilon = 1\); moreover, since the dynamics depends on energy only via the scaled parameter \(R/\sqrt{E}\), we have always taken \(E = 1\) and varied the “free path” \(R\) instead.

The choice of the charge density \(f(x)\) is important, because the coupling of individual modes to the charged plate is scaled by the coefficients \(a_n\) of the Fourier expansion of \(f(x)\). Choosing a singular density \(f(x)\), as in earlier studies, results, at all times, in a power law decay of the distribution of energy over the field modes, so that truncation effects are already significant at small integration times. We have therefore chosen \(f(x) = k \exp(-\delta^2/(\delta^2 - x^2))\) (the standard compactly supported \(C^\infty\) function), with the constant \(k\) fixed from normalization; this ensures a faster than algebraic, albeit nonexponential, decay of the distributions.

Even though we do not have rigorous results, the dynamics of this billiard–type model appears to be completely chaotic independently of the total energy. Moreover, the finite–dimensional reduced dynamics has positive maximal Lyapunov exponents \(\lambda, \lambda_c\) (the former being defined with respect to real time, the latter to the number of collisions). These were numerically computed by multiplying matrices obtained from linearization of the map (3) along a trajectory. The exponent \(\lambda\) decreases with the number of normal modes taken into account, because bounces become less frequent, the two plates going to rest, in time average, for \(N \rightarrow \infty\) (see below). On the contrary \(\lambda_c\) was observed to increase with \(N\); indeed,
as collisions become more distant in time, phases change more drastically in between them, and their randomization is faster. If instead \( N \) is increased keeping the energy per mode \( E/N \) fixed, both \( \lambda \) and \( \lambda_n \) appear to saturate, suggesting that Lyapunov exponents converge to a finite non-zero value in the thermodynamic limit. The maximal Lyapunov exponents remain positive on reducing \( \sigma \), with no stochasticity threshold displayed. However, the time required to reach a converged value becomes larger, because trajectories need more time to fill the phase space.

\[
T = \lim_{t \to \infty} \frac{\langle z(t) \rangle}{\langle z \rangle} \quad \text{for all normal modes,}
\]

\[
n_{\text{eff}}(t) = \exp \left\{ - \sum_{n=0}^{N+1} \frac{E_n(t)}{\bar{E}_n(t)} \ln \bar{E}_n(t) \right\},
\]

where \( \bar{E}_n(t) \) indicates the normalized time average energy (up to the time \( t \)) of the \( n \)-th normal mode (\( E_{N+1} \) refers to the energy of the neutral plate). The parameter \( n_{\text{eff}} \) is a measure of the number of modes significantly excited at time \( t \); if only a finite number of modes is considered, it also measures the degree of equipartition, because \( n_{\text{eff}} \) if only one normal mode is excited, whereas the maximal value \( n_{\text{eff}} \sim N + 3/2 \) is only attained in the presence of complete equipartition. As far as the numerical simulation is truly representative of the infinite-dimensional system, \( n_{\text{eff}} \) appears to increase with \( t \) slower than any power, but faster than logarithmically (Fig. 2).

Most of our numerical experiments were meant to understand how the energy is distributed among all the degrees of freedom (in time average). In Fig. 1 the time-average kinetic energy \( \bar{T} = \lim_{t \to \infty} T(t) \) of the charged plate is shown as a function of the number \( N \) of field modes considered. In the above definition, \( T(t) \) is the time average up to time \( t \) of \( \frac{1}{2}m\dot{z}^2(t) \), while the limit means that the motion has been followed until stabilization of the time-average. Results are in accordance with the equipartition theorem: the total energy is equally shared between the \( 2N + 3 \) relevant canonical variables. For \( N \to \infty \) we can extrapolate \( \bar{T} = 0 \), that is, the electromagnetic field acts as a friction force on the plate.

The approach to equilibrium is not uniform, because the relaxation time associated with the \( n \)-th overall normal mode increases with \( n \). To analyze this increase we have used the equipartition indicator

\[
E_k(\tau) = (1 - 4|e_k|^2)E_k(\tau - 1) + 4W(\tau - 1)|e_k|^2,
\]

where \( W(\tau) \) is the random phase approximation.
\[ e_k = \langle u_k | e \rangle; \quad W(\tau) = \sum_k |e_k|^2 E_k(\tau). \quad (9) \]

Recalling the meaning of \( e \), one easily realizes that \( W(\tau) \) is proportional to the average kinetic energy of the relative motion of the plates. Equvs. (8) can be solved numerically, to find how \( n_{eff} \) increases with the number of collisions. The result (shown by the dashed line in the insert of Fig.(2)) matches quite well with the numerical solution of the exact equations of motion, confirming the validity of the random phase approximation, hence the chaotic nature of dynamics. One can also solve (3) analytically, by implementing a continuous time approximation, plus standard Laplace transform techniques. Omitting details, one finds that the large-\( \tau \) asymptotics of the solution is determined by the large-\( k \) asymptotics of the coefficients \( a_k \). In case of algebraic decay \( a_k \sim |k|^{-\alpha} \), dispensing with prefactors which depend on \( \epsilon, l, c \) and estimating the average time delay between the \( \tau + 1 \)-th and the \( \tau \)-th collision as \( t \sim R/\sqrt{2|E_k|} \), one finds \( n_{eff}(t) \sim t^{2+\alpha} \). With the charge distribution \( f(x) \) used in our numerical simulations, we cannot give likewise explicit formulas, due to the complicated decay of coefficients \( a_k \). However, it is possible to prove that \( n_{eff} \) increases with \( t \) faster than logarithmically, but slower than any power of \( t \), as found in Fig.2. Thus the way the relaxation time of modes increases with their frequencies is determined by the choice of the charge density.

To verify that the truncated system numerically investigated here really represents (up to a certain time) the real, infinite-dimensional, model, we plotted in Fig.3 \( n_{eff} \) as a function of the number \( N \) of field oscillators taken into account. We found that, at any fixed time, \( n_{eff} \) converges on increasing \( N \), its limit value giving the number of overall normal modes significantly excited in the infinite-dimensional system. As this value increases with time, an equilibrium state is never reached.

In summary, our numerical experiments have exposed a chaotic dynamics in the unusual case of an infinite-dimensional, conservative system with a finite total energy. The dependence of time-averages on initial conditions gets lost, and, for any finite-dimensional reduction, the system reaches an equilibrium state, with equipartition of energy among the degrees of freedom of the field and of the matter.

From our results we infer that, in the real infinite-dimensional problem, there is a trend towards equipartition, the finite energy of matter being removed to higher and higher frequencies of the field. However, this process takes place at a nonuniform rate, as relaxation times of normal modes increase with their frequency. Therefore, as in an old hypothesis of Jeans [1], a real equilibrium state is never reached.

Artificial though it may appear, our model is actually the simplest, one-dim. model with charged particles undergoing elastic collisions inside a reflecting enclosure.

We believe that more realistic models displaying the same basic features will display a similar behaviour. We also submit that quantization of this, or similar, field models may open a challenging new direction in the field of Quantum Chaos, starting from a very old problem.

![Equi-partition parameter \( n_{eff} \) as a function of the number \( N \) of odd field modes considered, at \( t = 10^3 \) (triangles), \( t = 10^5 \) (squares), \( t = 10^7 \) (circles), \( t = 10^8 \) (diamonds), with initial conditions and parameters chosen as in Fig.2. The full line indicates equipartition values \( n_{eff} \sim N + 3/2 \).](image)

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