Nonlinear chiral transport from holography

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(Dated: July 24, 2018)

Abstract

Nonlinear transport phenomena induced by the chiral anomaly are explored within a 4D field theory defined holographically as $U(1)_V \times U(1)_A$ Maxwell-Chern-Simons theory in Schwarzschild-$AdS_5$. First, in presence of external electromagnetic fields, a general form of vector and axial currents is derived. Then, within the gradient expansion up to third order, we analytically compute all (over 50) transport coefficients. A wealth of new phenomena beyond the Chiral Magnetic and Chiral Separation Effects are discovered. Particularly, the charge diffusion constant and dispersion relation of Chiral Magnetic Wave are found to receive anomaly-induced non-linear corrections due to $e/m$ background fields. Furthermore, there emerges a new gapless mode, which we refer to as Chiral Hall Density Wave, propagating along the background Poynting vector.
I. INTRODUCTION

Hydrodynamics is an effective low energy description of many interacting QFTs near thermal equilibrium. Historically, hydrodynamics has been always associated with a long wavelength limit of the underlying microscopic theory, while over the last decade or so there is an increased number of works addressing “hydronization” relaxing the long wavelength approximation. Rather, hydrodynamics is defined as an effective theory of conserved currents, such as stress tensor and/or charge currents, assuming their algebra is closed on a relevant set of near-equilibrium states.

Dynamics of the theory is governed by conservation equations (continuity equations) of the currents. The simplest example is $\partial_t \rho = -\vec{\nabla} \cdot \vec{J}$, which is a time evolution equation for the charge density $\rho$ sourced by three-current $\vec{J}$. However, this equation cannot be solved as an initial value problem without additional input, the current $\vec{J}$. In hydrodynamics, $\vec{J}$ has to be expressed in terms of thermodynamical variables, such as $\rho$ itself, temperature, and possibly external fields if present. This is known as constitutive relation. Traditionally,
in the long wavelength limit, constitutive relations are presented as a (truncated) gradient expansion. At any given order, this expansion is fixed by thermodynamic considerations and symmetries, up to a finite number of transport coefficients (TCs). The latter should be either computed from underlying microscopic theory or deduced experimentally. Diffusion constant, DC conductivity or shear viscosity are examples of the lowest order TCs.

It is well known, however, that in relativistic theory truncation of the gradient expansion at any fixed order leads to serious conceptual problems such as violation of causality. Beyond conceptual issues, causality violation results in numerical instabilities rendering the entire framework unreliable. Causality is restored when all order gradient terms are included, in a way providing a UV completion to the “old” hydrodynamic effective theory. Below we will refer to such case as all order resummed hydrodynamics \[3–8\]. The first completion of the type was originally proposed by Müller, Israel, and Stewart \[9–12\] who introduced retardation effects in the constitutive relations for the currents. Formulation of \[9–12\] is the most popular scheme employed in practical simulations. Recent ideas on the nature of the hydrodynamic expansion, gradient resummation and attractor behavior, etc. could be found in \[4, 13–21\].

In this paper we continue exploring hydrodynamic regime of relativistic plasma with chiral asymmetries. We closely follow previous works \[22, 23\] focusing on massless fermion plasma with two Maxwell gauge fields, \(U(1)_V \times U(1)_A\). As a result of chiral anomaly, which appears in relativistic QFTs with massless fermions, global \(U(1)_A\) current coupled to external electromagnetic fields is no longer conserved. The continuity equations turn into

\[
\partial_\mu J^\mu = 0, \quad \partial_\mu j_5^\mu = 12\kappa \vec{E} \cdot \vec{B},
\]

where \(J^\mu / j_5^\mu\) are vector/axial currents and \(\kappa\) is an anomaly coefficient \((\kappa = eN_c/(24\pi^2)\) for \(SU(N_c)\) gauge theory with a massless Dirac fermion in fundamental representation and \(e\) is electric charge, which will be set to unit from now on). \(\vec{E}\) and \(\vec{B}\) are vector electromagnetic fields. Non-conservation of the axial current in (1) receives extra contribution if external axial electromagnetic fields are turned on. Throughout this work, however, we will not consider external axial fields (they were considered in Ref \[22\]). Chiral plasma play a major role in a number of fundamental research areas, historically starting from primordial plasma in the early universe \[24–28\]. During the last decade, macroscopic effects induced by the chiral anomaly were found to be of relevance in relativistic heavy ion collisions \[29, 31\],
and have been searched intensively at LHC \cite{32-36}. Finally, (pseudo-)relativistic systems in condensed matter physics, such as Dirac and Weyl semimetals, display anomaly-induced phenomena, which were recently observed experimentally \cite{37-43} and can be studied via similar theoretical methods \cite{44-47}.

A hydrodynamic description of (chiral) plasma amounts to solving a set of coupled equations. As have been mentioned earlier, the continuity equations \eqref{eq:1} have to be supplemented by constitutive relations describing plasma medium effects. Generically, these are of the type

\[
\tilde{J} = \tilde{J}[\rho, \rho_5, T, \tilde{E}, \tilde{B}], \quad \tilde{J}_5 = \tilde{J}_5[\rho, \rho_5, T, \tilde{E}, \tilde{B}],
\]

where \( \rho_5 \) is the axial charge density and \( T \) stands for the temperature. In a sense, the constitutive relations are “off-shell” relations, because they treat the charge density \( \rho \) (\( \rho_5 \)) as independent of \( \tilde{J} \) (\( \tilde{J}_5 \)). Employing \eqref{eq:1}, the currents \eqref{eq:2} are put into “on-shell”. In \eqref{eq:2}, the fields \( \tilde{E}, \tilde{B} \) are assumed to be external. However, the charges and currents induce e/m fields of their own. Thus the external electromagnetic fields \( \tilde{E}, \tilde{B} \) have to be promoted into dynamical ones, satisfying Maxwell equations (in Gaussian units)

\[
\nabla \cdot \tilde{E} = 4\pi \rho_{\text{tot}}, \quad \nabla \times \tilde{B} = \frac{1}{c} \left( 4\pi \tilde{J}_{\text{tot}} + \partial_t \tilde{E} \right),
\]

\[
\nabla \cdot \tilde{B} = 0, \quad \nabla \times \tilde{E} = -\partial_t \tilde{B},
\]

where \( \rho_{\text{tot}} \) and \( \tilde{J}_{\text{tot}} \) are the total charge density and total current, a sum of external sources \( (\rho_{\text{ext}}, \tilde{J}_{\text{ext}}) \) and induced part \( (\rho, \tilde{J}) \), which is the one that enters the constitutive relations \eqref{eq:2}. The external sources could be absent when a fully isolated system is considered. A typical example would be primordial plasma in the early Universe frequently studied using magneto-hydrodynamics (MHD). MHD, along with many other effective theories of the type, also involves neutral flow dynamics. That is, in addition to the charge current sector discussed above, one simultaneously considers energy-momentum conservation. Generically, the two dynamical sectors are coupled. However, in the discussion below, we will consider isothermal systems only, ignoring back-reaction of the currents’ sector on the energy-momentum conservation. This will be referred as a probe limit.

A self-consistent evolution of the system is determined by solving together \eqref{eq:1,2,3} given some initial conditions. While the equations \eqref{eq:1,3} are exact, the constitutive relations \eqref{eq:2} are

\footnote{In principle, the axial sources \( (\rho_5, \tilde{J}_5) \), through another set of chiral anomaly-modified Maxwell’s equations, would also generate classical axial e/m fields. In their turn, the axial e/m fields would enter and modify the constitutive relations \eqref{eq:2}, see e.g. \cite{22}.}
the ones where various hydrodynamic approximations are applied. A great deal of modelling
normally enters (2), such as truncated gradient expansion, weak field approximation, etc. As
a result of a full simulation, one sometimes finds instabilities leading to exponential growths
of some quantities, such as of dynamical magnetic fields. It thus becomes mandatory to
check if the original approximations made for the constitutive relations are consistent with
the solutions found. If not, the hydrodynamical model has to be revised.

We just outlined a general setup for a hydrodynamical problem, but it is not our goal
here to carry it over for any realistic system. Instead, motivated by the discussion above we
would like to focus on the nature of the constitutive relations (2). Our objective is to ex-
perience their possible structure under various approximations, primarily zooming on transport
effects induced by the chiral anomaly, which are known to lead to a wealth of interesting
phenomena. Particularly, MHD can be affected strongly by anomalous transports [48–52],
which necessitates development of a fully self-consistent chiral MHD.

Just like in Refs. [22, 23], our playground will be a holographic model, that is $U(1)_V \times$
$U(1)_A$ Maxwell-Chern-Simons theory in Schwarzschild-$AdS_5$ [53, 54] to be introduced in
detail in Section III for which we know to compute a zoo of transport coefficients exactly.
Hoping for some sort of universality, we could learn from this model both about general
structures and relative strengths of the effects.

The constitutive relations (2) are well known to receive contributions induced by the
chiral anomaly. The most familiar example is the \textit{chiral magnetic effect} (CME) [55, 56]: a
vector current is generated along an external magnetic field when a chiral imbalance between
left- and right-handed fermions is present ($\vec{J} \sim \rho_5 \vec{B}$). There is a vast literature on CME,
which we cannot review here in full. The chiral magnetic conductivity was computed in
perturbative QCD in [57–62]. In [53, 54, 63–77] it was evaluated for the strong coupling
regime using AdS/CFT correspondence [78–80]. CME emerged via arguments based on
the second law of thermodynamics, that is positivity of entropy production [81, 82], and
also within the chiral kinetic theory (CKT) [83–87]. Finally, numerical evidence based on
lattice gauge theory for CME can be found in [88–93]. We would like to comment in passing
that CME is believed to be a strict non-equilibrium phenomenon. In other words, different
arguments indicate that CME must vanish in equilibrium \footnote{We thank Mikhail Zubkov for bringing this issue to our attention. We also thank Dmitri Kharzeev, Shu Lin, Andrey Sadofyev, and Ho-Ung Yee for stimulating discussions about this point.}
Another important transport phenomenon induced by the chiral anomaly is the *chiral separation effect* (CSE) [96, 97]: left and right charges get separated along applied external magnetic field ($\vec{J}_5 \sim \vec{B}$). Combined, CME and CSE lead to a new gapless excitation called *chiral magnetic wave* (CMW) [98]. This is a propagating wave along the magnetic field. While signature of CME/CSE has not yet been confirmed in heavy ion collision experiments [32–35], a large negative longitudinal magneto-resistance observed in Dirac/Weyl semimetals can be attributed to CME [41–43].

Beyond naive CME/CSE, there are (infinitely) many additional chiral effects induced or affected by the anomaly. Particularly, transport phenomena *nonlinear* in external fields were realised recently [99] to be of critical importance in having self-consistent evolution of chiral plasma. Combined with the causality arguments mentioned earlier, the conclusion is that the constitutive relations [12] should contain a large number of “nonlinear” transport coefficients so to guarantee applicability of the constitutive relations in a broad regime. This triggered strong interest in nonlinear chiral transport phenomena within CKT [100–103], to which we will compare some of our findings below. Previous works on the subject include [104] based on the entropy current approach and [105] based on the fluid-gravity correspondence.

In the next Section, we will review our results including connections to the previous works [22, 23] and the forthcoming publication [106]. The remaining Sections present details of the calculations.

II. SUMMARY OF THE RESULTS

A. Generalities

Following [22, 23], the charge densities and external fields are split into constant backgrounds and space-time dependent fluctuations

$$\rho(x_\alpha) = \bar{\rho} + \epsilon \delta \rho(x_\alpha), \quad \rho_5(x_\alpha) = \bar{\rho}_5 + \epsilon \delta \rho_5(x_\alpha),$$
$$\vec{E}(x_\alpha) = \vec{E} + \epsilon \delta \vec{E}(x_\alpha), \quad \vec{B}(x_\alpha) = \vec{B} + \epsilon \delta \vec{B}(x_\alpha),$$

where $\bar{\rho}$, $\bar{\rho}_5$, $\vec{E}$ and $\vec{B}$ are the backgrounds, while $\delta \rho$, $\delta \rho_5$, $\delta \vec{E}$ and $\delta \vec{B}$ stand for the fluctuations. Here $\epsilon$ is a formal expansion parameter to be used below. Furthermore, being most of the time unable to perform calculations for arbitrary background fields, we introduce an
expansion in the field strengths
\[ \vec{E} \rightarrow \alpha \vec{E}, \quad \vec{B} \rightarrow \alpha \vec{B}, \] (6)
where \( \alpha \) is a corresponding expansion parameter. Below we will introduce yet another expansion parameter \( \lambda \), which will correspond to a gradient expansion. For the purpose of gradient counting, e/m fields will be frequently considered as \( \mathcal{O}(\lambda^1) \).

The constitutive relations (2) can be formally Taylor expanded in all its arguments. This includes both the gradient (\( \lambda \)), \( \epsilon \), and \( \alpha \) expansions. Parametrically, a generic term entering (2) looks like
\[ \bar{\rho}^k \bar{\rho}^{k_5} \bar{\vec{E}}^{n_E} \bar{\vec{B}}^{n_B} \partial^{m_t} \vec{\nabla}^{m_x} \left( \delta \rho^l \delta \rho^{l_5} \delta \bar{\vec{E}}^{l_E} \delta \bar{\vec{B}}^{l_B} \right), \] (7)
multiplied by a transport coefficient \( \bar{\rho} \), where \( k, k_5, n_E, n_B, m_t, m_x, l, l_5, l_E, l_B \) are integers. The most general constitutive relations correspond to a sum of all possible terms like (7).

Obviously, we do not intend to consider all possible terms in (7). Instead, most of the results obtained in the present and early works [22, 23] can be combined in a compact constitutive relation (focusing on the vector current \( \vec{J} \)),
\[ \vec{J} = \gamma_1 \vec{\nabla} \rho + \gamma_2 \vec{\nabla} \rho_5 + \gamma_3 \bar{\vec{E}} + \gamma_4 (\rho \bar{\vec{B}}) + \gamma_5 \bar{\vec{E}} \times \bar{\vec{B}} + \gamma_6 (\bar{\vec{E}} \times \vec{\nabla} \rho) + \gamma_7 \bar{\vec{B}} \times (\rho \vec{\nabla} \rho + \rho_5 \vec{\nabla} \rho_5) + \gamma_8 (\bar{\vec{E}} \times \vec{\nabla} \rho_5) + \gamma_9 (\rho \bar{\vec{E}} \times \bar{\vec{B}}) + \gamma_{10} \vec{\nabla} \left( \bar{\vec{B}} \cdot \vec{\nabla} \rho_5 \right) + \gamma_{11} \vec{\nabla} \left( \bar{\vec{B}} \cdot \vec{\nabla} \rho \right) + \gamma_{12} (\rho \vec{\nabla} B^2) + \gamma_{13} (\rho \bar{\vec{B}}) + \gamma_{14} \vec{\nabla} (\rho \bar{\vec{E}}) + \gamma_{15} (\rho \bar{\vec{E}}) + \gamma_{16} \vec{\nabla} (\bar{\vec{E}} \cdot \vec{\nabla} \rho_5) + \gamma_{17} (\rho \vec{\nabla} E), \] (8)
where the coefficients \( \gamma_i \) are most general \( \mathcal{O}(3) \) scalars which could be constructed from three vectors \( \vec{\nabla}, \bar{\vec{E}}, \) and \( \bar{\vec{B}} \). That is, \( \gamma_i \) are scalar functions of \( E^2 \) and \( B^2 \), and pseudo-scalar functions of \( \bar{\vec{E}} \cdot \bar{\vec{B}} \). Furthermore, \( \gamma_i \) are scalar functionals of derivative operators \( \partial_t, \vec{\nabla}^2, \bar{\vec{E}} \cdot \vec{\nabla}, \) and pseudo-scalar functionals of \( \bar{\vec{B}} \cdot \vec{\nabla} \)
\[ \gamma_i = \gamma_i \left( \partial_t, \vec{\nabla}^2, \bar{\vec{E}} \cdot \vec{\nabla}, \bar{\vec{B}} \cdot \vec{\nabla}; E^2, B^2, \bar{\vec{E}} \cdot \bar{\vec{B}} \right). \] (9)

Taylor expanding \( \gamma_i \) in all their arguments (all the derivatives are assumed to act on the right of \( \gamma_i \)) corresponds to an infinite number of transport coefficients. Eq. (8) does not contain all the possible terms like in (7). Particularly, while the constitutive relation (8)

3 In fact, each term in (7) corresponds to a large number of terms obtained by different actions of the derivatives and index contractions.

4 The asymptotic nature of the gradient expansion and problems related to resummation of the series have been a hot topic over the last few years, see recent works [13, 14, 107]. In our approach, however, we never attempt to actually sum the series and thus these discussions are of no relevance to our formalism.
does contain some nonlinear in $\rho, \rho_5$ terms, it excludes most of the nonlinear terms of the third order, which appear in (74, 75).

Some of the terms in (8) are well recognisable, such as diffusion ($\gamma_1$), electrical conductivity ($\gamma_3$), or CME ($\gamma_4$). Some other terms might be less familiar and we will discuss them below in detail. $\gamma_i$ themselves are rich in structure and contain information about nonlinear corrections in the fields. Furthermore, $\gamma_i$ have infinitely many space-time derivatives, corresponding to all order gradient resummation, as mentioned in Introduction.

The objective of [22, 23] and of the present and forthcoming [106] works is to systematically explore (7) under different approximations. To help the reader to navigate between various studies and results reported in [22, 23] and here, we first briefly summarise all using concurrently notations of (7) and (8), and then deepen our presentation of the current study.

- Ref [22], study 1. No background fields, $\vec{E} = \vec{B} = 0$; all order gradient terms that are linear in the inhomogeneous fluctuations $\delta \rho, \delta \rho_5, \delta \vec{E}, \delta \vec{B}$ are resummed. This corresponds to calculating currents up to $O(\epsilon^1 \alpha^0)$. Using the notations (7) and (8) this study corresponds to

$$\gamma_{i} = \gamma_{i} (\partial_t, \vec{\nabla}, 0, 0; 0, 0, 0), \quad i = 1, 3, 4, 5. \quad (10)$$

The remaining $\gamma_i$ have not been probed in the study. $\gamma_i (\partial_t, \vec{\nabla})$ correspond to the gradient resummation. Thanks to the linearisation, the constitutive relations could be conveniently expressed in Fourier space. Then the functionals of the derivatives are turned into functions of frequency and space momenta, $\vec{\nabla} \rightarrow (-i\omega, i\vec{q})$. We refer to $\gamma_{i} (-i\omega, q^2)$ as transport coefficients functions (TCFs) [6]. TCFs contain information about infinitely many derivatives and associated transport coefficients. In practice, they are not computed as a series resummation of order-by-order hydrodynamic expansion, and are in fact exact to all orders. TCFs go beyond the hydrodynamic low frequency/momentum limit and they contain collective effects of non-hydrodynamic modes. Fourier transformed back into real space, TCFs correspond to memory functions. Diffusion and shear viscosity memory functions were previously computed in [7, 108]. Below we set $\pi T = 1$ for convenience. The dimensionful frequency and momentum should be $\pi T \omega$ and $\pi T q$.

5 In [22] we also considered transports related to axial external electromagnetic fields.
• Ref. [22], study 2. Nonlinear in $\bar{E}$ and $\bar{B}$ corrections to the vector/axial currents. The currents are derived up to $O(\epsilon^0 \alpha^3)$.

\[ n_E + n_B \leq 3, \ l = l_5 = l_E = l_B = m_t = m_x = 0, \ \forall(k, k_5) \Rightarrow \text{(analytic)} \] (11)

\[ \gamma_i = \gamma_i(0, 0, 0; E^2, B^2, \vec{E} \cdot \vec{B}), \quad i = 3, 4, 9. \]

• Ref. [23], study 1. Nonlinear corrections to vector/axial currents due to static but spatially-inhomogeneous magnetic field.

\[ n_E = l_E = 0, \ \forall(l, l_5, k_5), \ n_B + m_t + m_x + l_B \leq 2 \Rightarrow \text{(analytic)} \]

\[ n_E = l_E = 0, \ \forall(l, l_5, k_5), \ n_B + m_t + m_x + l_B = 3, \ l + l_5 + k + k_5 = 1 \Rightarrow \text{(analytic)} \]

\[ \gamma_i = \gamma_i(\partial_t, \vec{\nabla}, 0, \vec{B} \cdot \vec{\nabla}; 0, B^2, 0), \quad i = 1, 4, 5, 7, 10, 11, 12. \] (12)

• Ref. [23], study 2. Dependence of longitudinal electric conductivity on arbitrary strong constant magnetic field. A time-varying electric field is assumed to be weak.

\[ n_E = l_B = l = k = k_5 = m_x = 0, \ \forall(n_B, l_5, l_E), \ m_t + n_B + l_E \leq 3, \Rightarrow \text{(analytic)} \]

\[ n_E = l_B = l = k = k_5 = m_x = 0, \ l_E + l_5 = 1, \ \forall n_B, \ \forall m_t \Rightarrow \text{(numeric)} \]

\[ \gamma_3 = \gamma_3(\partial_t, \vec{\nabla}, 0, 0; 0, 0, \vec{E} \cdot \vec{B}), \quad \gamma_4 = \gamma_4(\partial_t, \vec{\nabla}, 0, 0; E^2, 0, 0) \] (13)

• In the present work, we relax some of the approximations made in [22, 23] and derive constitutive relations for the currents, up to third order in the gradient expansion.

\[ n_E + n_B + l_E + l_B + m_t + m_x \leq 3, \ \forall(l, l_5, k_5) \Rightarrow \text{(analytic)} \]

\[ \gamma_{1,4} = \gamma_{1,4}(\partial_t, 0, 0, 0; E^2, 0, 0), \quad \gamma_2 = \gamma_2(0, 0, 0; \vec{B} \cdot \vec{\nabla}; 0, 0, 0), \]

\[ \gamma_3 = \gamma_3(\partial_t, \vec{\nabla}, 0, 0; 0, 0, \vec{E} \cdot \vec{B}), \quad \gamma_{5,8,9} = \gamma_{5,8,9}(\partial_t, 0, 0, 0; 0, 0, 0), \]

\[ \gamma_7 = \gamma_7(0, 0, 0; 0, 0, 0). \] (14)

• The forthcoming paper [106] primarily focuses on TCFs in nonlinear terms at $O(\epsilon^1 \alpha^1)$.

\[ l_E = l_B = 0, \ n_E + n_B = 1, \ l + l_5 = 1, \ \forall(k, k_5), \ m_t + m_x \leq 3 \Rightarrow \text{(analytic)} \]

\[ l_E = l_B = 0, \ n_E + n_B = 1, \ l + l_5 = 1, \ \forall(k, k_5, m_t, m_x) \Rightarrow \text{(numeric)} \] (15)

\[ \gamma_{1,2} = \gamma_{1,2}(\partial_t, \vec{\nabla}, \vec{E} \cdot \vec{\nabla}, \vec{B} \cdot \vec{\nabla}; 0, 0, 0), \quad \gamma_i = \gamma_i(\partial_t, \vec{\nabla}, 0, 0; 0, 0, 0), \quad i = 4, 6 - 8, 15, 17. \]

Next, we expand on the main results of the present work, while all the technical details appear in the main text and Appendix.
B. Main results

A formal expression for the constitutive relations for the vector/axial currents is derived in Section IV to have the following form

\[ J^t = \rho, \quad J^5 = -D_0 \nabla \rho + \sigma^0_e \vec{E} + \sigma^0_\chi \mu_5 \vec{B} + \delta \vec{J}, \]

\[ J^t_5 = \rho_5, \quad J^5_5 = -D_0 \nabla \rho_5 + \sigma^0_\chi \mu \vec{B} + \delta \vec{J}_5, \]

where \( \rho, \rho_5 \) are generic vector/axial charge densities and \( \mu, \mu_5 \) are corresponding chemical potentials. The external e/m fields are also generic without any approximations assumed. The lowest order TCs—charge diffusion constant \( D_0 \), DC electrical conductivity \( \sigma^0_e \) and DC CME/CSE conductivity \( \sigma^0_\chi \) are

\[ D_0 = \frac{1}{2}, \quad \sigma^0_e = 1, \quad \sigma^0_\chi = 12\kappa. \]  

The corrections \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) are formally defined in (A7), which consist only of higher derivative terms starting from second order. These terms are built from powers and derivatives of \( \vec{E}, \vec{B}, \rho \) and \( \rho_5 \). Generally, \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) are not known analytically. The \( \sigma^0_\chi \)-terms in \( \vec{J} \) and \( \vec{J}_5 \) are standard CME and CSE, respectively, in agreement with “non-renormalisability” of CME [109–111]. It is important to emphasise that the structures (16,17) are exact. Non-linearity of CME/CSE in external fields \( \vec{E} \) and \( \vec{B} \) is completely absorbed into the chemical potentials \( \mu, \mu_5 \). Nevertheless, \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) introduce new effects, particularly additional contributions to the currents along the direction of \( \vec{B} \), which could be thought of as modifications to the original CME/CSE. As will be clear later, external e/m fields make corrections to \( D_0 \) and \( \sigma^0_e \), and even generalise them into tensor-type TCs. While in principle an axial analogue of \( \sigma^0_e \) (i.e., a term proportional to \( \vec{E} \)-term) in \( \vec{J}_5 \) is also possible, it does not appear in our calculations due to the probe limit [112].

We have mentioned earlier a discussion about vanishing equilibrium CME, which might appear in tension with [106]. In principle, since \( U(1)_A \) is not a symmetry, axial gauge potential \( A_\mu \) should be regarded as another external field. Our calculations, however, are performed assuming vanishing \( A_\mu \). Had we introduced a non-vanishing constant background for the time component, \( A_t \neq 0 \), CME conductivity \( \sigma^0_\chi \) would be shifted

\[ \vec{J}_{CME} \propto \kappa (\bar{\mu}_5 - A_t) \vec{B}, \]  

\[ 10 \]
due to a Chern-Simons contribution \[22, 54\]. In order to have CME vanish, it suffices to impose \( A_t = \bar{\mu}_5 \) \[44, 54\]. In \[44\] it was indeed argued that the equality \( A_t = \bar{\mu}_5 \) must be satisfied in equilibrium. While we do not have much to add to this discussion, we notice that a constant \( A_t \) does not lead to any new effect beyond the shift \[19\] in CME. This is because the bulk dynamics underlying our model is expressed entirely in terms of the vector and axial field strengths.

Within the hydrodynamic approximation of slowly varying \( \rho, \rho_5, \vec{E}, \vec{B} \), we computed \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) analytically, up to third order in the gradient expansion. At second order, the results read

\[
\begin{align*}
\delta \vec{J} &= \frac{1}{4} \sigma_m^0 (\rho^2 + \rho_5^2) \vec{\nabla} \times \vec{B} - \frac{1}{4} \mathcal{D}_H^0 \vec{B} \times \left( \rho \vec{\nabla} \rho + \rho_5 \vec{\nabla} \rho_5 \right) - \frac{1}{2} \sigma_{\alpha H}^0 \vec{E} \times \vec{\nabla} \rho_5 - \tau_e \sigma_e^0 \partial_t \vec{E} \\
&- \frac{1}{2} \tau_\chi \rho \mathcal{D} \vec{B} \times \vec{E} - \frac{1}{2} \tau_\chi \rho_5 \partial_t \vec{B} + \tau_D \partial_t \vec{B} + \tau_\chi (\partial_t \rho_5) \vec{B} + \mathcal{O}(\partial^3),
\end{align*}
\]

(20)

\[
\delta \vec{J}_5 = \frac{1}{2} \sigma_m^0 \rho \rho_5 \vec{\nabla} \times \vec{B} - \frac{1}{4} \mathcal{D}_H^0 \vec{B} \times \left( \rho \vec{\nabla} \rho_5 + \rho_5 \vec{\nabla} \rho \right) - \frac{1}{2} \sigma_{\alpha H}^0 \vec{E} \times \vec{\nabla} \rho - \frac{1}{2} \sigma_{\chi}^0 \rho_5 \vec{B} \times \vec{E} \\
- \frac{1}{2} \tau_\chi \rho \partial_t \vec{B} + \tau_D \partial_t \vec{B} + \sigma_\chi^0 (\partial_t \rho) \vec{B} + \mathcal{O}(\partial^3),
\]

(21)

where the transport coefficients take the following values

\[
\begin{align*}
\sigma_m^0 &= 72(2 \log 2 - 1) \kappa^2, & \mathcal{D}_H^0 &= 72(3 \log 2 - 2) \kappa^2, & \sigma_{\alpha H}^0 &= 6 \log 2 \kappa, \\
\sigma_{\chi}^0 &= 72 \log 2 \kappa^2, & \tau_e &= \frac{\log 2}{2}, & \tau_\chi &= 12 \log 2 \kappa, \\
\tau_D &= \frac{\pi}{8}, & \tau_\chi &= -\left(\frac{3}{2} \pi + 3 \log 2\right) \kappa.
\end{align*}
\]

(22)

The transport coefficients in (20,21) could be related to Taylor expansion of \( \gamma_i \)'s in [8], schematically indicated as follows

\[
\begin{align*}
\sigma_m^0 &\in \gamma_5, & \mathcal{D}_H^0 &\in \gamma_8, & \sigma_{\alpha H}^0 &\in \gamma_9, & \sigma_{\chi}^0 &\in \gamma_{11}, & \tau_e &\in \gamma_3, \\
\tau_\chi, \tau_\chi &\in \gamma_4, & \tau_D &\in \gamma_1.
\end{align*}
\]

(25)

The constitutive relations (20,21) are an extension of the ones obtained in \[22, 23\]. While most of the terms and respective coefficients are not new (except for \( \sigma_{\alpha H}^0 \), which is indeed new), the results are shown to have a broader range of applicability as, in contrast to \[22, 23\], they do not rely on any assumptions of \( \rho, \rho_5, \vec{E}, \vec{B} \) being static or homogeneous.

Obviously, it is necessary to give physical interpretation to every term in (20,21). Furthermore, our second order results could be compared with similar results obtained in CKT.
To this goal, we first put the currents on-shell eliminating $\partial_t \rho$ and $\partial_t \rho_5$ using the continuity equations. Second, we replace the densities by the corresponding chemical potentials. In the holographic model, the chemical potentials $\mu, \mu_5$ are computed analytically in the hydrodynamic limit. At second order in the gradient expansion,

$$\mu = \frac{1}{2} \rho + \frac{1}{16} (\pi - 2 \log 2) \nabla^2 \rho - \frac{3}{4} (\pi - 2 \log 2) \kappa (\vec{B} \cdot \nabla \rho_5) + 18 (1 - 2 \log 2) \kappa^2 \rho B^2$$

$$- \frac{1}{8} (\pi + 2 \log 2) (\nabla \cdot \vec{E}) + \mathcal{O}(\partial^3),$$

(26)

$$\mu_5 = \frac{1}{2} \rho_5 + \frac{1}{16} (\pi - 2 \log 2) \nabla^2 \rho_5 - \frac{3}{4} (\pi - 2 \log 2) \kappa (\vec{B} \cdot \nabla \rho) + 18 (1 - 2 \log 2) \kappa^2 \rho_5 B^2$$

$$+ \frac{3}{2} (\pi - 2 \log 2) \kappa (\vec{E} \cdot \vec{B}) + \mathcal{O}(\partial^3).$$

(27)

Eventually, the on-shell currents are

$$\vec{J}_{\text{on-shell}} = \sigma_0^0 \mu_5 \vec{B} - \tau_\chi \mu_5 \partial_t \vec{B} + \sigma_0^0 (\vec{E} - \nabla \mu) - \tau_e \sigma_0^0 \partial_t \vec{E} + \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho)$$

$$\vec{E} - \nabla \mu + \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho) - \mathcal{O}(\partial^3)$$

$$- D_H^0 \vec{B} \times (\mu \nabla \mu + \mu_5 \nabla \mu_5) - \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho) - \mathcal{O}(\partial^3),$$

(28)

$$\vec{J}_{\text{on-shell}}^5 = \sigma_0^0 \mu_5 \vec{B} - \tau_\chi \mu_5 \partial_t \vec{B} - \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho) - \mathcal{O}(\partial^3)$$

$$\vec{E} - \nabla \mu + \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho) - \mathcal{O}(\partial^3)$$

$$- \sigma_0^0 \kappa (\vec{B} \cdot \nabla \rho) - \mathcal{O}(\partial^3).$$

(29)

Now let’s discuss the physics of each term in (28, 29), primarily focusing on $\vec{J}_{\text{on-shell}}$. The first term in (28) is CME. The next one introduces relaxation into CME induced by time variation of the magnetic field, with $\tau_\chi$ being a relaxation time originally computed in [22]. $\tau_\chi$ was recently re-examined numerically in [113] within a quite similar holographic model but beyond probe limit. The third and fourth terms are just the classic Ohm’s and diffusion currents accompanied by another relaxation effect associated with time varying electric field. The corresponding relaxation time $\tau_e$ was originally computed in [114]. Note that in (20, 21) there are two additional relaxation time terms. The first one with $\tau_D$ enters the diffusion current [108]. Finally, $\tau_\chi$ is yet another relaxation time associated with generalised CME. Note the difference between $\tau_\chi$ and $\tau_\chi$: while the former is a TC responding to time varying external magnetic field, the latter is related to relaxation of the axial charge density. In (28, 29) both terms appear as $\mathcal{O}(\partial^3)$.

The $\vec{E} \times \vec{B}$-term in (28, 29) looks very similar to the usual Hall effect, which is, however, absent in our holographic model because of the probe limit. The term that we do find is
induced by the chiral anomaly. To distinguish it from the normal Hall effect, it is referred to as *chiral Hall effect* \[115\] with $\sigma^0_{\chi H}$ being its TC. Notice that $\sigma^0_{\chi H} \propto \kappa^2$. Contrary to purely anomaly-induced effects, which are normally odd in $\kappa$, the terms even in $\kappa$ appear as anomaly-induced corrections to normal transports \[104\]. The $D^0_H$-term generates current perpendicular to both the magnetic field and gradients of chemical potentials. In \[101\] this effect was called *Hall diffusion*. This term can be regarded as an example (we will expand on this below) in which the diffusion constant is turned into a non-trivial diffusion tensor depending on the magnetic field.

To our knowledge, in a holographic model, $\sigma^0_{a\chi H}$ is computed here for the first time. The corresponding term in \[28\] induces flow perpendicular to both the electric field and gradient of the axial chemical potential. It was referred to as *anomalous chiral Hall effect* in \[101\].

Finally, the last term in \[28\] corresponds to another anomaly-induced correction to a normal current. Normal transport due to rotor of magnetic field was first analysed in \[108\]. At second order in the gradient expansion under discussion now, the “normal” transport coefficient was found to be identically zero. Thus, the entire effect at this order arises from the anomaly alone.

Dissipative nature of each term entering \[28, 29\] is of interest. While we are not to dwell in this question here, we note in passing that the TCs $\sigma^0_e$, $\tau_e$, $\sigma^0_m$, $\sigma^0_{\chi H}$, and $\tau_D$ are all time reversal $\mathcal{T}$-even and thus non-dissipative. The remaining terms in \[28, 29\] are dissipative.

Starting from CSE, the various terms in $\vec{J}_0^{\text{on-shell}}$ could be simply understood as axial analogues of those in $\vec{J}_0^{\text{on-shell}}$.

For the sake of a more detailed comparison of our results with parallel ones in CKT, we quote here the expression for the vector current as appears in \[101\]

\[
\vec{J}_{\text{CKT}} = \frac{1}{2\pi^2\mu_5}\vec{B} - \frac{\tau\mu_5}{6\pi^2}\partial_t\vec{B} + \sigma^e_{\text{CKT}}(\vec{E} - \vec{\nabla}\mu) - \tau\sigma^e_{\text{CKT}}\partial_t\vec{E} + \sigma^H_{\text{CKT}}\mu\vec{E} \times \vec{B} \\
- D^H_{\chi} (\mu\vec{\nabla}\mu + \mu_5\vec{\nabla}\mu_5) \times \vec{B} - \sigma^a_{\chi H}\vec{E} \times \vec{\nabla}\mu_5 - D^\chi_{\chi} \mu_5\vec{\nabla}\mu_5, \tag{30}
\]

where

\[
\sigma^e_{\text{CKT}} = \frac{\tau}{9\pi^2} \left[ 1 + 3(\mu^2 + \mu^2_5) \right], \quad \sigma^H_{\text{CKT}} = \frac{\tau^2}{3\pi^2}, \quad D^H_{\chi} = \frac{\tau^2}{3\pi^2},
\]

\[
\sigma^a_{\chi H} = \frac{\tau}{6\pi^2}, \quad D^\chi_{\chi} = \frac{2\tau}{3\pi^2}. \tag{31}
\]

\[6\] Indeed, the $\tau_{\chi}, \sigma^0_{\chi H}$-terms in \[28\] could be reorganised as $-\sigma^0_{a\chi H}(\vec{E} \times \vec{\nabla}\mu_5 + \mu_5\partial_t\vec{B}) - \sigma^0_{\chi H}\mu_5\partial_t\vec{B}$. More precisely it is $(\vec{E} \times \vec{\nabla}\mu_5 + \partial_t\vec{B})$-term that was called anomalous chiral Hall effect in \[101\].
Here $\tau$ is a parameter of dimension of time introduced in relaxation time approximation (RTA) of CKT. Confronting with (28) we notice absence of $\vec{\nabla}\mu_5$ term in $\vec{J}^{\text{on-shell}}$. Similarly, there are no terms proportional to $\vec{\nabla}\mu$, $\vec{E}$, $\partial_t\vec{E}$ in $\vec{J}_5^{\text{on-shell}}$. All these terms are expected to arise beyond the probe limit. On the other hand, the magnetic conductivity term $\vec{\nabla} \times \vec{B}$ is missing in (30). All the remaining terms appear in perfect agreement, at least as far as general structures are concerned.

Because in principle the two models describe two different regimes (strong vs weak coupling), the transport coefficients are not expected to agree. It is nevertheless instructive to pursue such a comparison. For this goal, we need to fix the parameter $\tau$ of the CKT. Obviously, there is no unique way to fix $\tau$. We chose to set CME as a benchmark. That is, we equate the CME conductivities and the associated relaxation times in two models. This results in

$$\kappa = \frac{1}{24\pi^2}, \quad \tau = 3 \log 2.$$  \hspace{1cm} (32)

Then, the transport coefficients in (28) are

$$\sigma^0_e = 1, \quad \tau_e \sim \mathcal{O}(10^{-1}), \quad \sigma^0_{\chi} \sim \mathcal{O}(10^{-4}), \quad \mathcal{D}^0_H \sim \mathcal{O}(10^{-4})$$  \hspace{1cm} (33)

compared to those in (30)

$$\sigma^\text{CKT}_e \sim \mathcal{O}(10^{-2}), \quad \tau \sim \mathcal{O}(1), \quad \sigma^\text{CKT}_H = \mathcal{D}^\text{CKT}_H \sim \mathcal{O}(10^{-1}),$$
$$\sigma^\text{CKT}_{\alpha \chi} \sim \mathcal{O}(10^{-2}), \quad \mathcal{D}^\text{CKT}_\chi \sim \mathcal{O}(10^{-1}).$$  \hspace{1cm} (34)

While some of the coefficients came out to be of the same order, the electrical conductivity $\sigma^\text{CKT}_e$ in CKT is strongly suppressed (by order $10^{-2}$) compared to the holographic model. On the other hand, the anomaly-induced coefficients $\sigma^0_{\chi}$ and $\mathcal{D}^0_H$ are highly suppressed (by order $10^{-3}$) in holography.

A complimentary way of looking at (16,17,20,21) is by separately collecting terms proportional to $\vec{\nabla}\rho$ and $\vec{\nabla}\rho_5$. All these terms constitute a diffusive current, which to the lowest order in the gradients is

$$J^i_\text{diff} = -\mathcal{D}^0_{ij} \nabla_j \rho - (\mathcal{D}^0_{\chi})_{ij} \nabla_j \rho_5,$$  \hspace{1cm} (35)

where

$$\mathcal{D}^0_{ij} = \frac{1}{4}(4\mathcal{D}_0 \delta_{ij} + \mathcal{D}^0_H \epsilon_{ikj} B_k \rho), \quad (\mathcal{D}^0_{\chi})_{ij} = \frac{1}{4}(2\sigma^0_{\alpha \chi} \epsilon_{ikj} E_k + \mathcal{D}^0_H \epsilon_{ikj} B_k \rho_5).$$  \hspace{1cm} (36)
Much like in MHD, the diffusion constants are turned into tensors, which in fact depend non-linearly on the external e/m fields $E$ and $B$. Furthermore, when higher order gradients are resummed, these diffusion tensors become momenta dependent tensor functions.

Third order corrections in $\delta \vec{J}$ and $\delta \vec{J}_5$ contain a few dozens of new terms with corresponding TCs, all computed analytically; the complete listing appears in Section IV. For this summary, we focus on the gradient terms that are linear in the vector/axial charge densities only. Interestingly, we observed chiral anomaly-induced negative corrections to the charge diffusion constant $D_0$:

$$D_0 = \frac{1}{2} - 18(2 \log 2 - 1)\kappa^2 B^2 \mp \frac{3}{4} \pi^2 \kappa^2 E^2 + \cdots ,$$  \hspace{1cm} (37)

where $\cdots$ denote higher order in $E^2$, $B^2$ corrections. To the best of our knowledge, while the $\kappa^2 B^2$-correction was first calculated in [23], the $\kappa^2 E^2$ term is a new result.

The third order gradient corrections to $\vec{J}/\vec{J}_5$ modify the dispersion of CMW. For constant e/m background fields,

$$\omega = \pm \left[ 1 - 36(2 \log 2 - 1)\kappa^2 B^2 - \frac{3\pi^2}{2} \kappa^2 E^2 \right] 6\kappa (\vec{q} \cdot \vec{B}) \pm \frac{9\pi^2}{2} (\vec{E} \cdot \vec{B}) \kappa^3 (\vec{q} \cdot \vec{E})$$

$$+ (36 \log 2)\kappa^2 (\vec{q} \cdot \vec{S}) - \frac{1}{2} + 18(1 - 2 \log 2)\kappa^2 B^2 - \frac{3\pi^2}{4} \kappa^2 E^2 \right] i q^2$$

$$\pm \frac{9}{2} \log 2 \kappa (\vec{q} \cdot \vec{B}) q^2 - \frac{i}{8} q^4 \log 2 - \frac{3}{8} \pi^2 \kappa^2 (\vec{q} \cdot \vec{E})^2 + i (36 \log 2)\kappa^2 (\vec{q} \cdot \vec{B})^2 + \cdots .$$ \hspace{1cm} (38)

When $\vec{E} = 0$, the dispersion relation (38) reduces to the one obtained in [23]. The first term ($\sim \vec{q} \cdot \vec{B}$) in (38) represents nonlinear corrections to the speed of CMW, which are negative making the wave to propagate slower. The second term ($\sim \vec{q} \cdot \vec{E}$) in (38) corresponds to a wave mode propagating along the electric field. It is called density wave [31] or chiral electric wave [115]. Since the chiral electric separation effect vanishes in the probe limit, here this effect is mimicked by the second term in (38) which is induced by the chiral anomaly as a nonlinear correction. Its presence is conditional to $\vec{E}$ not being orthogonal to $\vec{B}$. We find the third term ($\sim \vec{q} \cdot \vec{S}$) of special interest because it corresponds to a new phenomenon. It corresponds to a wave propagating along the direction of the energy flux $\vec{S} = \vec{E} \times \vec{B}$, which can be referred to as chiral Hall density wave (CHDW). The remaining terms in (38) are decay rates of various wave modes.

The rest of this paper is structured as follows. In section III we present the holographic model. Section IV is devoted to the main part of our study supplemented by Appendix. Section V contains some closing remarks.
III. HOLOGRAPHIC SETUP: $U(1)_V \times U(1)_A$

The holographic model is Maxwell-Chern-Simons theory in the Schwarzschild-AdS$_5$. The bulk action is

$$S = \int d^5x\sqrt{-g}\mathcal{L} + S_{\text{c.t.}},$$

where

$$\mathcal{L} = -\frac{1}{4}(F^V)^{MN}(F^V)^{MN} - \frac{1}{4}(F^a)^{MN}(F^a)^{MN} + \frac{\kappa \epsilon^{MNPQR}}{2\sqrt{-g}}$$

$$\times [3A_M(F^V)^{NP}(F^V)^{QR} + A_M(F^a)^{NP}(F^a)^{QR}],$$

and the counter-term action $S_{\text{c.t.}}$ is

$$S_{\text{c.t.}} = \frac{1}{4}\log r \int d^4x\sqrt{-\gamma} \left[ (F^V)_{\mu\nu}(F^V)^{\mu\nu} + (F^a)_{\mu\nu}(F^a)^{\mu\nu} \right].$$

The gauge Chern-Simons terms ($\sim \kappa$) in the bulk action mimic the chiral anomaly of the boundary field theory. Note $\epsilon^{MNPQR}$ is the Levi-Civita symbol with the convention $\epsilon^{rtxyz} = +1$, while the Levi-Civita tensor is $\epsilon^{MNPQR}/\sqrt{-g}$. The counter-term action (41) is specified based on minimal subtraction, which excludes finite contribution to the boundary currents from the counter-term.

In the ingoing Eddington-Finkelstein coordinate, the Schwarzschild-AdS$_5$ is

$$ds^2 = g_{MN}dx^Mdx^N = 2dtdr - r^2 f(r)dt^2 + r^2 \delta_{ij}dx^idx^j,$$

where $f(r) = 1 - 1/r^4$. Thus, the Hawking temperature (identified as temperature of the boundary theory) is normalised to $\pi T = 1$. On the hypersurface $\Sigma$ of constant $r$, the induced metric $\gamma_{\mu\nu}$ is

$$ds^2|_{\Sigma} = \gamma_{\mu\nu}dx^\mu dx^\nu = -r^2 f(r)dt^2 + r^2 \delta_{ij}dx^idx^j.$$

It is convenient to split the bulk equations into dynamical and constraint components,

\begin{align*}
\text{dynamical equations :} & \quad EV^\mu = EA^\mu = 0, \\
\text{constraint equations :} & \quad EV^r = EA^r = 0,
\end{align*}

where

$$EV^M \equiv \nabla_N(F^V)^{NM} + \frac{3\kappa \epsilon^{MNPQR}}{\sqrt{-g}}(F^a)^{NP}(F^V)^{QR}. $$
\[ EA^M \equiv \nabla_N (F^a)^{NM} + \frac{3K\epsilon^{MNPQR}}{2\sqrt{-g}} \left[ (F^V)_{NP}(F^V)_{QR} + (F^a)_{NP}(F^a)_{QR} \right]. \] 

(47)

The boundary currents are defined as

\[ J^\mu \equiv \lim_{r \to \infty} \frac{\delta S}{\delta V_\mu}, \quad J_5^\mu \equiv \lim_{r \to \infty} \frac{\delta S}{\delta A_\mu}, \] 

(48)

which, in terms of the bulk fields, are

\[ J^\mu = \lim_{r \to \infty} \sqrt{-\gamma} \left\{ (F^V)^\mu_M n_M + \frac{6K\epsilon^{M\mu NQR}}{\sqrt{-g}} n_M A_{N}(F^V)_{QR} - \tilde{\nabla}_\nu (F^V)^{\nu\mu} \log r \right\}, \]

\[ J_5^\mu = \lim_{r \to \infty} \sqrt{-\gamma} \left\{ (F^a)^\mu_M n_M + \frac{2K\epsilon^{M\mu NQR}}{\sqrt{-g}} n_M A_{N}(F^a)_{QR} - \tilde{\nabla}_\nu (F^a)^{\nu\mu} \log r \right\}, \] 

(49)

where \( n_M \) is the outpointing unit normal vector with respect to the slice \( \Sigma \), and \( \tilde{\nabla} \) is compatible with the induced metric \( \gamma_{\mu\nu} \).

The radial gauge \( V_r = A_r = 0 \) will be assumed throughout this work. As a result, in order to determine the boundary currents (49) it is sufficient to solve dynamical equations (44) only, leaving the constraints aside. Indeed, the constraint equations (45) give rise to continuity equations (41)

\[ \partial_\mu J^\mu = 0, \quad \partial_\mu J_5^\mu = 12K E \cdot B. \] 

(50)

In this way, the currents’ constitutive relations to be derived below are off-shell.

Practically, it is more instructive to relate the currents (49) to the coefficients of near boundary asymptotic expansion of the bulk gauge fields. Near \( r = \infty \),

\[ V_\mu = V^{(1)}_\mu + \frac{V^{(2)}_\mu}{r} + \frac{2V^L_\mu}{r^2} \log r + O \left( \frac{\log r}{r^3} \right), \quad A_\mu = \frac{A^{(2)}_\mu}{r^2} + O \left( \frac{\log r}{r^3} \right), \] 

(51)

where

\[ V^{(1)}_\mu = F^V_{\mu t}, \quad 4V^L_\mu = \partial^\nu F^V_{\mu \nu}. \] 

(52)

A possible constant term for \( A_\mu \) in (51) has been set to zero, in accordance with the fact that no axial external fields is assumed to be present in the current study. \( V_\mu \) is the gauge potential of external electromagnetic fields \( \vec{E} \) and \( \vec{B} \),

\[ E_i = F^V_{it} = \partial_i V_t - \partial_t V_i, \quad B_i = \frac{1}{2} \epsilon_{ijk} F^V_{jk} = \epsilon_{ijk} \partial_j V_k. \] 

(53)

Dynamical equations (44) are sufficient to derive (51,52), where the near-boundary data \( V^{(2)}_\mu \) and \( A^{(2)}_\mu \) have to be determined by completely solving (44) from the horizon to the
As the remainder of this section, we outline the strategy for deriving the constitutive relations for $J^\mu$ and $J_5^\mu$. To this end, we turn on finite vector/axial charge densities for the dual field theory, which are also exposed to external electromagnetic fields. Holographically, the charge densities and external fields are encoded in asymptotic behaviors of the bulk gauge fields. In the bulk, we will solve the dynamical equations (44) assuming the charge densities and external fields as given, but without specifying them explicitly.

Following [108] we start with the most general static and homogeneous profiles for the bulk gauge fields satisfying the dynamical equations (44),

$$V_\mu = V_\mu - \frac{\rho}{2r^2} \delta_{\mu t}, \quad A_\mu = -\frac{\rho_5}{2r^2} \delta_{\mu t},$$

(55)

where $V_\mu, \rho, \rho_5$ are all constants for the moment. Regularity at $r = 1$ has been used to fix one integration constant for each $V_i$ and $A_i$. As explained below (52), the constant term in $A_\mu$ is set to zero. Through (54), the boundary currents are

$$J^t = \rho, \quad J^i = 0; \quad J^t_5 = \rho_5, \quad J^i_5 = 0.$$  

(56)

Hence, $\rho$ and $\rho_5$ are identified as the vector/axial charge densities.

Next, following the idea of fluid/gravity correspondence [116], we promote $V_\mu, \rho, \rho_5$ into arbitrary functions of the boundary coordinates

$$V_\mu \to V_\mu(x_\alpha), \quad \rho \to \rho(x_\alpha), \quad \rho_5 \to \rho_5(x_\alpha).$$  

(57)

As a result, (55) ceases to solve the dynamical equations (44). To have them satisfied, suitable corrections in $V_\mu$ and $A_\mu$ have to be introduced:

$$V_\mu(r, x_\alpha) = V_\mu(x_\alpha) - \frac{\rho(x_\alpha)}{2r^2} \delta_{\mu t} + \mathcal{V}_\mu(r, x_\alpha), \quad A_\mu(r, x_\alpha) = -\frac{\rho_5(x_\alpha)}{2r^2} \delta_{\mu t} + \mathcal{A}_\mu(r, x_\alpha),$$  

(58)

where $\mathcal{V}_\mu, \mathcal{A}_\mu$ will be determined by solving (44). Appropriate boundary conditions are classified into three types. First, $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ are regular over the domain $r \in [1, \infty)$. Second, at the conformal boundary $r = \infty$, we require

$$\mathcal{V}_\mu \to 0, \quad \mathcal{A}_\mu \to 0 \quad \text{as} \quad r \to \infty,$$  

(59)
which amounts to fixing external gauge potentials to be $\mathcal{V}_\mu$ and zero (for the axial fields). Additional integration constants will be fixed by the Landau frame convention for the currents,

$$J^t = \rho(x_\alpha), \quad J^5 = \rho_5(x_\alpha).$$  \hfill (60)

The Landau frame convention corresponds to a residual gauge fixing for the bulk fields.

The vector/axial chemical potentials are defined as

$$\mu = V_t(r = \infty) - V_t(r = 1) = \frac{1}{2} \rho - \mathcal{V}_t(r = 1),$$

$$\mu_5 = A_t(r = \infty) - A_t(r = 1) = \frac{1}{2} \rho_5 - A_t(r = 1).$$  \hfill (61)

Generically, $\mu, \mu_5$ are nonlinear functionals of densities and external fields.

In terms of $\mathcal{V}_\mu$ and $A_\mu$, the dynamical equations (44) are

$$0 = r^3 \partial^2_t \mathcal{V}_t + 3r^2 \partial_r \mathcal{V}_t + r \partial_r \partial_k \mathcal{V}_k + 12 \kappa \epsilon^{ijk} [\partial_r A_i (\partial_j \mathcal{V}_k + \partial_j \mathcal{V}_k) + \partial_r \mathcal{V}_t \partial_j A_k],$$

$$0 = (r^5 - r) \partial^2_r \mathcal{V}_t + (3r^4 + 1) \partial_r \mathcal{V}_t + 2r^3 \partial_r \partial_i \mathcal{V}_i - r^3 \partial_r \partial_i \mathcal{V}_t + r^2 (\partial_r \mathcal{V}_i - \partial_i \mathcal{V}_t) + r (\partial^2 \mathcal{V}_i - \partial_i \partial_k \mathcal{V}_k) + 12 \kappa r^2 \epsilon^{ijk} \left( \frac{1}{r^3 \rho} \partial_j \mathcal{V}_k + \frac{1}{r^3 \rho} \partial_k \mathcal{V}_j + \partial_r A_i \partial_j \mathcal{V}_k + \partial_r A_i \partial_k \mathcal{V}_j \right) - 12 \kappa r^2 \epsilon^{ijk} \partial_r A_j \left[ (\partial_i \mathcal{V}_k - \partial_k \mathcal{V}_i) + (\partial_i \mathcal{V}_k - \partial_k \mathcal{V}_t) + \frac{1}{2r^2} \partial_k \rho \right]$$

$$- 12 \kappa r^2 \epsilon^{ijk} \left\{ \partial_r \mathcal{V}_j \left[ (\partial_i A_k - \partial_k A_i) + \frac{1}{2r^2} \partial_k \rho \right] - \partial_j A_k \left( \partial_r \mathcal{V}_t + \frac{1}{r^3 \rho} \right) \right\},$$

$$0 = r^3 \partial^2_t A_t + 3r^2 \partial_r A_t + r \partial_r \partial_k A_k + 12 \kappa \epsilon^{ijk} [\partial_r \mathcal{V}_i (\partial_j \mathcal{V}_k + \partial_j \mathcal{V}_k) + \partial_r A_i \partial_j A_k],$$

$$0 = (r^5 - r) \partial^2_r A_i + (3r^4 + 1) \partial_r A_i + 2r^3 \partial_r \partial_i A_i - r^3 \partial_r \partial_i A_t + r^2 (\partial_r A_i - \partial_i A_t) + r (\partial^2 A_i - \partial_i \partial_k A_k) - \frac{1}{2} \partial_k \rho + 12 \kappa r^2 \epsilon^{ijk} (\partial_j \mathcal{V}_k + \partial_j \mathcal{V}_k) \left( \partial_r \mathcal{V}_t + \frac{1}{r^3 \rho} \right)$$

$$- 12 \kappa r^2 \epsilon^{ijk} \partial_r A_j \left[ (\partial_i \mathcal{V}_k - \partial_k \mathcal{V}_i) + (\partial_i \mathcal{V}_k - \partial_k \mathcal{V}_t) + \frac{1}{2r^2} \partial_k \rho \right]$$

$$- 12 \kappa r^2 \epsilon^{ijk} \left\{ \partial_r A_j \left[ (\partial_i A_k - \partial_k A_i) + \frac{1}{2r^2} \partial_k \rho \right] - \partial_j A_k \left( \partial_r A_t + \frac{1}{r^3 \rho} \right) \right\}.$$  \hfill (63)

In the next section we will present solutions to (62-65) under approximation discussed in the Introduction.
IV. NONLINEAR CHIRAL TRANSPORT

In this section, we initially explore generic structure of the vector and axial currents as emerges within the holographic model of Section III. No assumptions will be made regarding the charge densities \( \rho, \rho_5 \) and external fields \( \vec{E}, \vec{B} \). While we are not able to solve the dynamical equations (62-65) analytically, we can advance by rewriting them in integral forms and extract near-boundary asymptotic expansion for the corrections \( \mathbb{V}_\mu \) and \( A_\mu \). The procedure is rather tedious. Hence all the details are moved to Appendix A. Via (54), the near-boundary asymptotic behaviors (A1-A4) yield the results (16,17) with \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) formally given by (A7). As clear from (A5,A6), \( \delta \vec{J} \) and \( \delta \vec{J}_5 \) are composed of higher derivative terms involving \( \vec{E}, \vec{B} \) and \( \rho, \rho_5 \).

Now we continue with the gradient expansion of \( \delta \vec{J} \) and \( \delta \vec{J}_5 \). Within the hydrodynamic limit, the dynamical equations (62-65) are solved perturbatively. Let us introduce a formal expansion parameter \( \lambda \) by \( \partial_\mu \rightarrow \lambda \partial_\mu \), which counts order of the gradient expansion. Then, \( \mathbb{V}_\mu \) and \( A_\mu \) could be expanded in powers of \( \lambda \),

\[
\mathbb{V}_\mu = \sum_{n=1}^{\infty} \lambda^n \mathbb{V}_\mu^{[n]}, \quad A_\mu = \sum_{n=1}^{\infty} \lambda^n A_\mu^{[n]}.
\]

(66)

We remind the reader that for this study, \( \vec{E} \) and \( \vec{B} \) are considered to be of \( \mathcal{O}(\lambda^1) \). At each order in \( \lambda \), \( \mathbb{V}_\mu^{[n]} \) and \( A_\mu^{[n]} \) obey a system of ODEs, which could be analytically solved via direct integration over \( r \). We list the results for \( \mathbb{V}_\mu^{[n]} \) and \( A_\mu^{[n]} \) up to \( n = 2 \) in (A8-A16).

Inserting the first order results (A8-A10) into (A5,A7) produces the second order results for \( \delta \vec{J} \) and \( \delta \vec{J}_5 \), as summarised in (20,21). The results (A8,A13,A14) also lead to the expressions for the chemical potentials, as summarised in (26,27).

With the second order corrections \( \mathbb{V}_\mu^{[2]} \) and \( A_\mu^{[2]} \) (A13,A16), we obtain the third order results \( \delta \vec{J}^{[3]} \) and \( \delta \vec{J}_5^{[3]} \). However, nonlinearity makes such calculations rather involved and the number of various terms is very large. For the sake of presentation of the third order results, they are split into terms that are linear in either \( \rho \) and \( \rho_5 \) and the rest. The terms that are linear in the densities are also the ones that contribute to the gapless waves propagating in the chiral medium, such as CMW. The dispersion relation will be derived below. Eventually, the linear in the charge densities parts of \( \delta \vec{J}^{[3]} \) and \( \delta \vec{J}_5^{[3]} \), denoted as \( \delta \vec{J}^{[3]}l \) and \( \delta \vec{J}_5^{[3]}l \), are

\[
\delta \vec{J}^{[3]}l = \tau_1 \partial_t^2 (\vec{\nabla} \rho) + \tau_2 \nabla^2 (\vec{\nabla} \rho) + \tau_3 \partial_t^2 \vec{E} + \tau_4 \nabla^2 \vec{E} + \tau_5 \partial_t (\vec{\nabla} \times \vec{B}) + \tau_6 (\partial_t^2 \rho_5) \vec{B}
\]
\[ + \tau_7 \partial_\rho \rho_5 \partial_t \vec{B} + \tau_8 \rho_5 \partial_t^2 \vec{B} + \tau_9 (\vec{\nabla}^2 \rho_5) \vec{B} + \tau_{10} \rho_5 \vec{\nabla}^2 \vec{B} + \tau_{11} \left( \vec{\nabla} \rho_5 \cdot \vec{B} \right) \frac{\vec{\nabla} \rho_5}{\rho_5} \]
\[ + \tau_{12} \left( \vec{B} \cdot \vec{\nabla} \right) \vec{\nabla} \rho_5 + \tau_{12} \left( \vec{B} \cdot \vec{\nabla} \right) \rho_5 + \tau_{13} \vec{\nabla} (\rho B^2) + \tau_{14} \vec{E} \times (\vec{E} \times \vec{\nabla} \rho) \]
\[ + \tau_{15} \partial_t (\vec{E} \times \vec{\nabla} \rho_5) + \tau_{16} \partial_t (\vec{E} \times \vec{\nabla} \rho) + \tau_{17} \rho \vec{B} \times \vec{E} + \tau_{18} \partial_t (\rho \vec{B} \times \vec{E}) \]
\[ + \tau_{19} \rho_5 \vec{E} \times (\vec{E} \times \vec{B}) + \tau_{20} \vec{E} \times \partial_t (\rho \vec{B}), \]  \hspace{1cm} (67)

\[ \delta J_5^{[3]} = \tau_1 \partial_t^2 (\vec{\nabla} \rho_5) + \tau_2 \vec{\nabla}^2 (\vec{\nabla} \rho_5) + \tau_6 (\vec{\nabla}^2 \rho_5) \vec{B} + \tau_7 \partial_\rho \rho_5 \partial_t \vec{B} + \tau_{8} \rho_5 \partial_t^2 \vec{B} + \tau_{9} (\vec{\nabla}^2 \rho) \vec{B} \]
\[ + \tau_{10} \rho \vec{\nabla}^2 \vec{B} + \tau_{11} \left( \vec{\nabla} \rho \cdot \vec{B} \right) \vec{B} + \tau_{12} \left( \vec{B} \cdot \vec{\nabla} \right) \vec{\nabla} \rho + \tau_{12} \vec{\nabla} \left( \vec{B} \cdot \vec{\nabla} \right) \rho \]
\[ + \tau_{13} \vec{\nabla} (\rho_5 B^2) + \tau_{14} \vec{E} \times (\vec{E} \times \vec{\nabla} \rho_5) + \tau_{15} \partial_t (\vec{E} \times \vec{\nabla} \rho) + \tau_{16} \partial_t (\vec{E} \times \vec{\nabla} \rho) \]
\[ + \tau_{17} \rho_5 \vec{E} \times \partial_t \vec{B} + \tau_{18} \partial_t (\rho_5 \vec{B} \times \vec{E}) + \tau_{19} \rho \vec{E} \times (\vec{E} \times \vec{B}) + \tau_{20} \vec{E} \times \partial_t (\rho_5 \vec{B}) \]
\[ + \tau_{21} \vec{E} \times (\vec{\nabla} \times \vec{B}) + \tau_{22} \vec{\nabla} (\vec{B} \cdot \vec{E}), \]  \hspace{1cm} (68)

where
\[ \tilde{\tau}_{12} = \tau_{12} + \tau_{10} - \tau_9. \]  \hspace{1cm} (69)

In \cite{67,68} we have made use of the Bianchi identity \cite{11} and eliminated \( \vec{\nabla} \times \vec{E} \). The values of TCs \( \tau_{1-22} \) are collected in Appendix \ref{8} see \cite{28,50}. Apart from the \( \tau_3, \tau_4, \tau_5, \tau_{21}, \tau_{22} \) terms, one can obtain \( \delta J_5^{[3]} \) from \( \delta J_5^{[3]} \) via exchange of \( \rho \) and \( \rho_5 \). It is important to give physical interpretation for \( \tau_i \) in \cite{67,68}.

The TCs \( \tau_{1-5} \) represent the second order gradient expansion of the charge diffusion function \( \mathcal{D} \), electric and magnetic conductivity functions \( \sigma_e, \sigma_m \), and were first computed in \cite{108}. The \( \tau_8, \tau_{10} \)-terms are second order gradient expansion of CME conductivity \( \sigma_\chi \) \cite{22}. The \( \tau_{19} \)-term was first obtained in \cite{22} for constant electromagnetic fields, which, once expanded, contains nonlinear corrections to the original CME/CSE.

The underlined terms \( \tau_{13}, \tau_{14} \) include anomaly-induced \( B^2 \), \( E^2 \)-corrections to the charge diffusion constant \( \mathcal{D}_0 \). We note that both corrections are negative, see \cite{37}. \( E^2 \)-correction is new whereas \( B^2 \)-correction was first calculated in \cite{23}. Obviously, there will be higher powers in \( E^2, B^2 \) corrections to \( \mathcal{D}_0 \). In the forthcoming publication \cite{106}, we will compute the charge diffusion constant, as a function of constant e/m fields relaxing the weak field approximation.

The transport coefficients \( \tau_6, \tau_7, \tau_9 \) are due to spacetime inhomogeneity of \( \rho, \rho_5 \). \( \tau_6, \tau_9 \) correspond to second order expansion of the generalised CME/CSE conductivity function \( \sigma_\chi \) to be computed in the forthcoming paper \cite{106}.
The terms $\tau_{11}, \tau_{12}, \tilde{\tau}_{12}$ represent mixing effect between magnetic field and spatial gradients of $\rho, \rho_5$. They were first considered in [23]. The TCs $\tau_{15-18}, \tau_{20}$ have similar structure as the Hall diffusion and Hall effect, but the former are induced by time-varying densities and electromagnetic fields. The $\tau_{21}, \tau_{22}$-terms are due to spatial inhomogeneity of electromagnetic fields. Vector analogs of $\tau_{21}, \tau_{22}$ will emerge as nonlinear in $\rho, \rho_5$ terms, see $\tau_{30}, \tau_{34}$-terms in [74].

Via the criterion for dissipative/non-dissipative transports based on $T$-symmetry arguments, the TCs $\tau_{6-12}, \tilde{\tau}_{12}, \tau_{15}, \tau_{16}, \tau_{19}, \tau_{21}$ and $\tau_{22}$ are $T$-even and thus correspond to non-dissipative TCs, while the rest of terms are dissipative.

The results for $J^\mu$ and $\bar{J}_5^\mu$, up to the third order, can be used to compute dispersion relation for a collective mode propagating in the chiral plasma. We focus on the case with constant external fields only. Consider a plane wave ansatz for the charge densities

$$\delta \rho = e^{-i(\omega t - \vec{q} \cdot \vec{x})} \delta \tilde{\rho}, \quad \delta \rho_5 = e^{-i(\omega t - \vec{q} \cdot \vec{x})} \delta \tilde{\rho}_5.$$ (70)

Then the continuity equations (1) with the constitutive relations (16, 17, 20, 21, 67, 68) turn into

$$a \delta \tilde{\rho} + b \delta \tilde{\rho}_5 = 0, \quad b \delta \tilde{\rho} + a \delta \tilde{\rho}_5 = 12\kappa (\vec{E} \cdot \vec{B}).$$ (71)

The explicit expressions for $a$ and $b$ are

$$a = -i \omega + \frac{1}{2} q^2 + 18(1 - 2 \log 2) \kappa^2 q^2 B^2 - \frac{3 \pi^2}{4} \kappa^2 q^2 E^2 + 9(\pi - 2 \log 2) \kappa^2 (\vec{q} \cdot \vec{B})^2$$
$$+ \frac{3 \pi^2}{4} \kappa^2 (\vec{q} \cdot \vec{E})^2 + i \frac{\pi}{8} \omega q^2 - \frac{3 \pi^2}{4} \omega q^2 - \frac{1}{16} (\pi - 2 \log 2) q^4 + i 36 \log 2 \kappa^2 (\vec{q} \cdot \vec{S})$$
$$- (18 \kappa + \frac{21 \pi^2}{8}) \kappa^2 \omega (\vec{q} \cdot \vec{S}),$$

$$b = i 6 \kappa (\vec{q} \cdot \vec{B}) - i \frac{3}{4} (\pi - 2 \log 2) \kappa q^2 (\vec{q} \cdot \vec{B}) + i 216 (1 - 2 \log 2) \kappa^3 B^2 (\vec{q} \cdot \vec{B})$$
$$- \frac{3}{2} (\pi + 2 \log 2) \kappa \omega (\vec{q} \cdot \vec{B}) - i \frac{1}{8} (24 \kappa + 2 \log 2 \kappa^2 \omega^2 (\vec{q} \cdot \vec{B})$$
$$- i \frac{1}{2} (\pi - 2 \log 2) \kappa q^2 (\vec{q} \cdot \vec{B}) + i 19 \pi^2 \kappa^3 [(\vec{B} \cdot \vec{E})(\vec{q} \cdot \vec{E}) - E^2 (\vec{q} \cdot \vec{B})],$$ (73)

where the Poynting vector $\vec{S} = \vec{E} \times \vec{B}$. For $\omega, q \ll 1$, the dispersion equation (71) can be solved perturbatively, leading to the $\vec{B}/E$-corrected dispersion relation [38].

Finally, we turn to terms that are nonlinear in the charge densities in the third order results $\delta \tilde{J}^{[3]}$ and $\delta \tilde{J}_5^{[3]}$. We denote them as $\delta \tilde{J}^{[3]}_{nl}$ and $\delta \tilde{J}_5^{[3]}_{nl}$:

$$\delta \tilde{J}^{[3]}_{nl} = \tau_{25} (\rho^2 + \rho_5^2) \partial_t \vec{\nabla} \times \vec{B} + \tau_{24} \rho_5 (\rho_5^2 + 3 \rho^2) \vec{\nabla}^2 \vec{B} + \tau_{25} \partial_t \vec{H} + \tau_{26} (\partial_t \rho_5 \vec{\nabla} \rho_5 + \partial_t \rho \vec{\nabla} \rho) \times \vec{B}$$

22
\begin{align*}
&\tau_{25}(\rho_5 \partial_t \vec{\nabla} \rho_5 + \rho \partial_t \vec{\nabla} \rho) \cdot \vec{B} + \tau_{28}(\rho_5 \partial_t \rho_5 + \rho \partial_t \rho) \vec{\nabla} \times \vec{B} + \tau_{29}(\rho_5 \vec{\nabla} \rho_5 + \rho \vec{\nabla} \rho) \times \partial_t \vec{B} \\
&+ \tau_{30} 24 \kappa \rho \rho_5 \vec{\nabla} \times \vec{S} + \tau_{31}(\rho_5 \vec{\nabla} \rho + \rho \vec{\nabla} \rho_5) \times \vec{S} + \tau_{32}(\vec{\nabla} \rho_5 \times \partial_t \vec{\nabla} \rho + \vec{\nabla} \rho \times \partial_t \vec{\nabla} \rho_5) \\
&+ \tau_{33} 2 \rho \rho_5 \vec{B} \times \partial_t \vec{B} + \tau_{34} 2 \rho \rho_5 \vec{E} \times (\vec{\nabla} \times \vec{B}) + \tau_{35}[\rho_5 \vec{\nabla} \times (\vec{E} \times \vec{\nabla} \rho_5) + (\rho \rightarrow \rho_5)] \\
&+ \tau_{36}[\vec{\nabla} \rho \times (\vec{E} \times \vec{\nabla} \rho) + (\rho \rightarrow \rho_5)] + \tau_{37}[2 \rho \rho_5 \vec{\nabla} \rho + (\rho^2 + \rho_5^2) \vec{\nabla} \rho_5] \times (\vec{\nabla} \times \vec{B}) \\
&+ \tau_{38}(\rho_5 \vec{\nabla} \times \vec{H} + \rho \vec{\nabla} \times \vec{H}_a) + \tau_{39}(\vec{\nabla} \rho_5 \times \vec{H} + \vec{\nabla} \rho \times \vec{H}_a) + \tau_{40} \vec{E} \times \vec{H},
\end{align*}

where

\begin{equation}
\vec{S} = \vec{E} \times \vec{B}, \quad \vec{H} = \vec{B} \times (\rho_5 \vec{\nabla} \rho_5 + \rho \vec{\nabla} \rho), \quad \vec{H}_a = \vec{B} \times (\rho \vec{\nabla} \rho_5 + \rho_5 \vec{\nabla} \rho).
\end{equation}

All \( \tau_i \)’s in (74, 75) are computed analytically and the results are deposited in Appendix A, see (A51, A68). Below we give simple explanation for each term in (74, 75).

The TC \( \tau_{23} \) corresponds to anomalous corrections to the relaxation term in the magnetic conductivity \( \sigma_m \) of (22, 108). The analytical result for \( \tau_{23} \) was unknown in (22). The \( \tau_{28} \)-term is a nonlinear correction to the magnetic current (\( \sigma_m \)-term of (108)), and relies on time-varying densities. \( \tau_{37} \) corresponds to a mixing effect between the charge diffusion and magnetic current. The TC \( \tau_{30} \) is due to spatial inhomogeneity of e/m energy flux and is an analog of the magnetic conductivity.

The \( \tau_{24} \)-term stands for second order expansion of the CME conductivity \( \sigma_\chi \) of (22) and was first computed there. \( \tau_{25} \) is the relaxation term for the second order Hall diffusion current (see the \( \mathcal{D}_H^0 \)-term in (28, 29)). The \( \tau_{26}, \tau_{27}, \tau_{29} \)-terms rely on the time inhomogeneity of charge densities or magnetic field and could be thought of as extension of the Hall diffusion current. The TC \( \tau_{31} \) is related to the e/m energy flux and also generalises the Hall diffusion current.

\( \tau_{32} \) is composed of spatial gradient of charge densities and corresponds to nonlinear charge diffusion process. The \( \tau_{36}, \tau_{39} \)-terms are e/m field corrections to the nonlinear charge diffu-
The last TC $\tau_{40}$ is a nonlinear in $E, B$ correction to the charge diffusions. The terms $\tau_{33}, \tau_{34}$ are nonlinear in densities corrections to $\tau_{21}, \tau_{22}$.

$\tau_{35}$ is the third order extension of the anomalous chiral Hall effect, i.e., $\sigma_{\alpha H}^0$-term in \(28, 29\). In \(106\), we will perform a systematic resummation for certain transports and the term $\tau_{35}$ will be generalised into a TCF. Similarly, $\tau_{38}$ can be simply taken as the magnetic analogue of $\tau_{35}$ and will be extended to a TCF in \(106\).

Finally, let us mention the dissipative nature for each term in the third order results \(74, 75\). Via the criterion of $\mathcal{T}$-symmetry, the TCs $\tau_{24}, \tau_{30-34}, \tau_{37-40}$ are $\mathcal{T}$-even and are thus non-dissipative. The remaining terms are all dissipative.

V. CONCLUSION

In this work, we have continued exploration of nonlinear chiral anomaly-induced transport phenomena based on a holographic model with two $U(1)$ fields interacting via a Chern-Simons term. For a finite temperature system, we constructed off-shell constitutive relations for the vector/axial currents. We demonstrated that both CME and CSE get corrected by higher derivative terms, see \(16, 17\). In the hydrodynamic limit, we analytically calculated those gradient corrections up to third order. Most of the results and comparison with CKT are presented in the Summary section. New third order results, particularly \(67, 68\), extend those that were initially considered in \(22, 23\) and reveal new effects associated with time-dependence or inhomogeneities of the charge densities and external fields.

Among new results worth highlighting, in weak field approximation the charge diffusion constant $D_0$ was found to receive negative anomaly-induced $E^2$- and $B^2$-corrections \(37\). It is very interesting to explore the dependence of $D_0$ on the e/m fields beyond the weak field approximation, that is non-perturbatively. Of particular interest would be a strong field limit. We are pursuing this line of study in the forthcoming paper \(106\) (see also \(109\) for similar study but in a different holographic model\(^7\)).

Another result we found to be of interest is that the chiral medium is shown to support three types of collective modes: CMW (propagating along $\vec{B}$), CEW propagating along $\vec{E}$, and a new one, chiral Hall density wave, propagating orthogonal to the other two, that is

\(^7\) In \(109\) the effect of non-perturbative magnetic field on the speed of CMW and diffusion constant was induced by nonlinear DBI action, which is quite different from our model in \(106\).
along the energy flux $\mathbf{E} \times \mathbf{B}$.

We have also computed analytically several different relaxation times associated with the first and second order transport phenomena.

The forthcoming paper [106] focuses on another set of approximations. Instead of considering a fixed order gradient expansion adopted here, we compute some TCFs in nonlinear chiral transport phenomena. More specifically, the external electromagnetic fields are assumed to be constant and weak, while the charge densities are split into constant backgrounds and small inhomogeneous fluctuations. The setup is similar to that of [22], but in [106] as opposed to [22], gradient resummation is performed for terms that are linear both in the charge density fluctuations and external fields.

We have found a wealth of non-linear phenomena all induced entirely by the chiral anomaly. An important next step in deriving a full chiral MHD would be to abandon the probe limit adopted in this paper and include the dynamics of a neutral flow as well. This will bring into the picture additional effects such as thermoelectric conductivities, normal Hall current, the chiral vortical effect [117, 118], and some nonlinear effects discussed in [100]. We plan to address these in the future.

Appendix A: Supplement for Section IV

In this Appendix, we collect all calculational details omitted in section IV. Regarding the general structure of the constitutive relations of the vector/axial currents, we present the integral versions of the bulk dynamical equations and explore near boundary asymptotics. We further derive the gradient expansion, and compute analytically all TCs, up to third order.

The dynamical equations (62-65) can be directly integrated over $r$, resulting in the following integral forms

$$
\mathbb{V}_t = - \int_r^\infty \frac{dx}{x^3} \int_y^\infty dy \left\{ y \partial_y \partial_k \left( \mathbb{V}_k + \frac{E_k}{y} \right) + 12 \kappa \epsilon^{ijk} \partial_y A_i (\partial_j \mathbb{V}_k + \partial_j \mathbb{V}_k) \right. \\
\left. + 12 \kappa \epsilon^{ijk} \partial_y \mathbb{V}_i \partial_j A_k \right\} + \partial_k E_k \left( \log \frac{r}{2r^2} + \frac{1}{4r^2} \right) \\
\xrightarrow{r \to \infty} \partial_k E_k \left( \log \frac{r}{2r^2} + \frac{1}{4r^2} \right) + \mathcal{O} \left( \frac{r}{r^3} \right),
$$

(A1)
\[
\mathcal{V}_i = -\int_r^\infty \frac{xdx}{x^4-1} \left\{ -\partial_i E_i \log x + \frac{x-1}{2x} \partial_i \rho + (x-1) E_i + \epsilon^{ijk} \partial_j B_k \log x - 12\kappa B_i \right. \\
\left. \times \left[ \mu_5 + \mathcal{A}_i(x) - \frac{\rho_5}{2x^2} \right] + G_i(x) \right\} \\
\xrightarrow{r \to \infty} \left( \frac{\log r}{2r^2} + \frac{1}{4r^2} \right) \left( \partial_i E_i - \partial_i \mathcal{F}_{ik} \right) - \frac{1}{4r^2} \partial_i \rho + \left( -\frac{1}{r} + \frac{1}{2r^2} \right) E_i + \frac{6\kappa \mu_5 B_i}{r^2} \\
- \frac{G_i(x = \infty)}{2r^2} + \mathcal{O} \left( \frac{\log r}{r^3} \right),
\]

\[
\mathcal{A}_t = -\int_r^\infty \frac{dx}{x^3} \int_r^\infty \left\{ y \partial_y \partial_k \mathcal{A}_k(y) + 12\kappa \epsilon^{ijk} \partial_y \mathcal{V}_i \left( \partial_j \mathcal{V}_k + \partial_j \mathcal{V}_k \right) + 12\kappa \epsilon^{ijk} \partial_y \mathcal{A}_i \partial_j \mathcal{A}_k \right\} \\
\xrightarrow{r \to \infty} \mathcal{O} \left( \frac{1}{r^3} \right),
\]

\[
\mathcal{A}_i = -\int_r^\infty \frac{dx}{x^4-1} \left\{ \frac{x-1}{2x} \partial_i \rho_5 - 12\kappa B_i \left[ \mu + \mathcal{V}_t(x) - \frac{\rho}{2x^2} \right] + H_i(x) \right\} \\
\xrightarrow{r \to \infty} -\frac{1}{4r^2} \partial_i \rho_5 + \frac{6\kappa \mu_5 B_i}{r^2} - \frac{1}{2r^2} H_i(x = \infty) + \mathcal{O} \left( \frac{1}{r^3} \right),
\]

where \( \mu \) and \( \mu_5 \) are the chemical potentials defined in [61]. We have also provided asymptotic expansions near the boundary \( r = \infty \). The functions \( G_i(x) \) and \( H_i(x) \) are

\[
G_i(x) = -\int_1^x dy \left\{ 2y \partial_y \partial_t \left[ \mathcal{V}_i(y) + \frac{E_i}{y} \right] + \partial_t \left( \mathcal{V}_i(y) + \frac{E_i}{y} \right) - y \partial_y \partial_t \mathcal{V}_t - \partial_t \mathcal{V}_t \\
+ \frac{1}{y} (\partial^2 \mathcal{V}_i - \partial_i \partial_t \mathcal{V}_k) + 12\kappa \epsilon^{ijk} \left[ \frac{1}{y^3} \rho_5 \partial_j \mathcal{V}_k + \partial_y \mathcal{A}_t \partial_j \mathcal{V}_k \right] \\
- 12\kappa \epsilon^{ijk} \partial_y \mathcal{A}_j \left[ (\partial_t \mathcal{V}_k - \partial_k \mathcal{V}_t) + \frac{1}{2y^2} \partial_k \rho - E_k \right] \\
- 12\kappa \epsilon^{ijk} \partial_y \mathcal{V}_j \left[ (\partial_t \mathcal{A}_k - \partial_k \mathcal{A}_t) + \frac{1}{2y^2} \partial_k \rho_5 \right] \\
+ 12\kappa \epsilon^{ijk} \left( \frac{1}{y^3} \rho \partial_j \mathcal{A}_k + \partial_y \mathcal{V}_t \partial_j \mathcal{A}_k \right) \right\},
\]

\[
H_i(x) = -\int_1^x dy \left\{ 2y \partial_y \partial_t \mathcal{A}_i - y \partial_y \partial_t \mathcal{A}_t + (\partial_t \mathcal{A}_i - \partial_i \mathcal{A}_t) + \frac{1}{y} (\partial^2 \mathcal{A}_i - \partial_i \partial_t \mathcal{A}_k) + 12\kappa \epsilon^{ijk} \left[ \frac{\rho}{y^3} \partial_j \mathcal{V}_k + \partial_y \mathcal{V}_t \partial_j \mathcal{V}_k \right] \\
- 12\kappa \epsilon^{ijk} \partial_y \mathcal{A}_j \left[ (\partial_t \mathcal{A}_k - \partial_k \mathcal{A}_t) + \frac{1}{2y^2} \partial_k \rho - E_k \right] \\
- 12\kappa \epsilon^{ijk} \partial_y \mathcal{V}_j \left[ (\partial_t \mathcal{A}_k - \partial_k \mathcal{A}_t) + \frac{1}{2y^2} \partial_k \rho_5 \right] + 12\kappa \epsilon^{ijk} \left( \frac{1}{y^3} \rho_5 \partial_j \mathcal{A}_k \right) \\
+ \partial_y \mathcal{A}_t \partial_j \mathcal{A}_k \right\}.
\]

In deriving (A1-A4), all three types of the boundary conditions, as summarized in section III were used to fix the integration constants. The formal solutions (A1-A4) give rise to the
general results \cite{16,17} with $\delta J^i$ and $\delta J^i_5$ given as

$$
\delta J^i = \partial_t E_i - G_i(x = \infty), \quad \delta J^i_5 = -H_i(x = \infty). \quad (A7)
$$

For generic profiles of $\rho, \rho_5, \vec{E}, \vec{B}$, we are not able to compute $G_i(x = \infty)$ and $H_i(x = \infty)$ analytically. So, we employ the standard hydrodynamic limit and evaluate them up to third order in the gradient expansion \cite{16}. Perturbative solutions for $\nabla_\mu$ and $A_\mu$ are collected below. At first order, $n = 1$,

$$
\nabla_i^{[1]} = A_i^{[1]} = 0, \quad (A8)
$$

$$
\nabla_i^{[1]} = f_1(r) \partial_i \rho + f_3(r) E_i + f_2(r) \rho_5 B_i, \quad (A9)
$$

$$
A_i^{[1]} = f_1(r) \partial_i \rho_5 + f_2(r) \rho B_i, \quad (A10)
$$

where

$$
f_1(r) = \frac{1}{8} \left[ \log \frac{(1 + r)^2}{1 + r^2} + 2 \arctan(r) - \pi \right], \quad f_2(r) = 3 \kappa \log \frac{1 + r^2}{r^2}, \quad (A11)
$$

$$
f_3(r) = \frac{1}{4} \left[ \log \frac{1 + r^2}{(1 + r)^2} + 2 \arctan(r) - \pi \right]. \quad (A12)
$$

At second order, $n = 2$,

$$
\nabla_i^{[2]} = a_0 \partial_k E_k + a_1 \left( -\frac{1}{2} \partial^2 \rho + 6 \kappa B_k \partial_k \rho_5 \right) + a_2 72 \kappa^2 B^2 \rho, \quad (A13)
$$

$$
A_i^{[2]} = a_1 \left( -\frac{1}{2} \partial^2 \rho_5 + 6 \kappa B_k \partial_k \rho - 12 \kappa \vec{E} \cdot \vec{B} \right) + a_2 72 \kappa^2 B^2 \rho_5, \quad (A14)
$$

$$
\nabla_i^{[2]} = b_0 \epsilon^{ijk} \partial_j B_k + b_1 \partial_i \partial_i \rho + b_2 \partial_i E_i + b_3 6 \kappa \partial_i (\rho_5 B_i) + b_4 3 \kappa \rho_5 \epsilon^{ijk} \partial_j E_k + b_5 36 \kappa^2 \epsilon^{ijk} \times \left[ (\rho^2 + \rho_5^2) \partial_j B_k + \rho_5 B_j \partial_k \rho_5 + \rho B_j \partial_k \rho \right] + b_6 6 \kappa \epsilon^{ijk} \left[ E_j \partial_k \rho_5 + 12 \kappa \rho B_j E_k \right] \quad (A15)
$$

$$
\times - b_7 36 \kappa^2 \epsilon^{ijk} (\rho B_j \partial_k \rho + \rho_5 B_j \partial_k \rho_5),
$$

$$
A_i^{[2]} = b_1 \partial_i \partial_i \rho_5 + b_3 6 \kappa \partial_i (\rho B_i) + b_4 3 \kappa \rho \epsilon^{ijk} \partial_j E_k + b_5 36 \kappa^2 \epsilon^{ijk} (-2 \rho \rho_5 \partial_j B_k + \rho B_j \partial_k \rho_5 + \rho_5 B_j \partial_k \rho) + b_6 6 \kappa \epsilon^{ijk} \left[ \rho_5 B_j \partial_k \rho + \rho B_j \partial_k \rho_5 \right] \quad (A16)
$$

where

$$
a_0 = \frac{1 + 2 \log r}{4 r^2} + \int_r^\infty \frac{dx}{x^3} \int_x^\infty \frac{dy}{y(\sqrt{y^2 + 1})(y + 1)}, \quad (A17)
$$

$$
a_1 = \int_r^\infty \frac{dx}{x^3} \int_x^\infty \frac{y dy}{(y^2 + 1)(y + 1)}. \quad (A18)
$$
\[ a_2 = \int_{r}^{\infty} \frac{dx}{x^3} \int_{x}^{\infty} \frac{dy}{y(y^2 + 1)}, \quad (A19) \]

\[ b_0 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} \frac{dy}{y}, \quad (A20) \]

\[ b_1 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \left\{ -\frac{y}{(y^2 + 1)(y + 1)} - \frac{1}{8} \left[ \log \left( \frac{1+y}{1+y^2} + 2 \arctan(y) - \pi \right) \right] \right\}, \quad (A21) \]

\[ b_2 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \left\{ -\frac{2y^2}{(y^2 + 1)(y + 1)} - \frac{1}{4} \left[ \log \left( \frac{1+y^2}{(1+y^2)^2} + 2 \arctan(y) - \pi \right) \right] \right\}, \quad (A22) \]

\[ b_3 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \left\{ \frac{2}{y^2 + 1} - \frac{1}{2} \log \frac{1+y^2}{y^2} \right\}, \quad (A23) \]

\[ b_4 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \left\{ -\frac{1}{y^3} \left[ \log \frac{1+y^2}{(1+y)^2} + 2 \arctan(y) - \pi \right] \right\}, \quad (A24) \]

\[ b_5 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \frac{1+y^2}{y^3 \log \frac{1+y^2}{y^2}}, \quad (A25) \]

\[ b_6 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \frac{y}{y(y^2 + 1)}, \quad (A26) \]

\[ b_7 = -\int_{r}^{\infty} \frac{xdx}{x^4 - 1} \int_{1}^{x} dy \frac{y}{y^3(y^2 + 1)}, \quad (A27) \]

Substituting the first order solutions \( A8, A9, A10 \) into \( A5, A6 \) generates the second order results \( 20, 21 \). The chemical potentials \( 26, 27 \) are obtained similarly by substituting the results \( A8, A13, A14 \) into \( 61 \). Finally, the solutions \( A8, A16 \) give rise to the third order corrections \( 67, 68 \) with the transport coefficients \( \tau_{1-40} \) as

\[ \tau_1 = \int_{1}^{\infty} \frac{dy}{y^2} \left[ 2y \partial_y a_1(y) + a_1(y) \right] = -\frac{\pi^2}{48}, \quad (A28) \]

\[ \tau_2 = \int_{1}^{\infty} \frac{dy}{2} \left[ y \partial_y a_1(y) + a_1(y) \right] = -\frac{1}{16} \left( \pi - 2 \log 2 \right), \quad (A29) \]

\[ \tau_3 = \int_{1}^{\infty} dy \left[ 2y \partial_y b_2(y) + b_2(y) \right] = -\frac{\pi^2}{24}, \quad (A30) \]

\[ \tau_4 = -\int_{1}^{\infty} dy \left[ y \partial_y a_0(y) + a_0(y) \right] = \frac{1}{8} \left( \pi + 2 \log 2 \right), \quad (A31) \]

\[ \tau_5 = \int_{1}^{\infty} dy \left[ \frac{f_3(y)}{y} + (y \partial_y + 1) a_0(y) + (2y \partial_y + 1) b_0(y) \right] = -\frac{1}{8} \left( \pi - \frac{\pi^2}{2} + 2 \log 2 \right), \quad (A32) \]
\[
\tau_6 = \int_1^\infty dy 6 \kappa \left[ 2y \partial_y b_3(y) + b_3(y) \right] = \frac{1}{8} \kappa \left( 24C + \pi^2 + 6 \log^2 2 \right), \quad (A33)
\]

\[
\tau_7 = \int_1^\infty dy 3 \kappa (2y \partial_y + 1)[4b_3(y) + b_4(y)] = 9 \kappa C + \frac{5}{16} \kappa \pi^2 + \frac{3}{2} \kappa \log^2 2, \quad (A34)
\]

\[
\tau_8 = \int_1^\infty dy \left\{ 3 \kappa (2y \partial_y + 1)[2b_3(y) + b_4(y)] + 12 \kappa \frac{b_2(y)}{y^3} \right\} = \kappa \left( 6C + \frac{1}{4} \pi^2 \right), \quad (A35)
\]

\[
\tau_9 = \int_1^\infty dy \frac{f_2(y)}{y} = \frac{1}{8} \kappa \pi^2, \quad (A36)
\]

\[
\tau_{10} = \int_1^\infty \left[ \frac{f_2(y)}{y} - 12 \kappa \frac{b_0(y)}{y^3} \right] = \frac{1}{4} \kappa \pi^2, \quad (A37)
\]

\[
\tau_{11} = \int_1^\infty \left[ \frac{2f_2(y)}{y} + 12 \kappa \frac{\partial_y b_0(y)}{2y^2} \right] = \frac{5}{16} \kappa \pi^2, \quad (A38)
\]

\[
\tau_{12} = -\int_1^\infty \left[ 6 \kappa \left[ y \partial_y a_1(y) + a_1(y) \right] + \frac{f_2(y)}{y} \right] = \frac{1}{8} \kappa \left( 6\pi - 12 \log 2 - \pi^2 \right), \quad (A39)
\]

\[
\tau_{13} = -\int_1^\infty 72 \kappa^2 \left[ y \partial_y a_2(y) + a_2(y) \right] = 18 \kappa^2 \left( 2 \log 2 - 1 \right), \quad (A40)
\]

\[
\tau_{14} = -\int_1^\infty dy 72 \kappa^2 \partial_y b_6(y) = -\frac{3}{4} \kappa^2 \pi^2, \quad (A41)
\]

\[
\tau_{15} = \int_1^\infty \left\{ 6 \kappa \left[ 2y \partial_y b_6(y) + b_6(y) \right] - 12 \kappa \left[ f_1(y) \partial_y f_3(y) + \partial_y b_1(y) \right] \right\}
= \frac{3}{2} \kappa C + \frac{5}{32} \kappa \pi^2, \quad (A42)
\]

\[
\tau_{16} = \int_1^\infty dy 12 \kappa \left[ \partial_y b_1(y) + \partial_y (f_1(y) f_3(y)) - \frac{\partial_y b_2(y)}{2y^2} \right]
= -\frac{1}{8} \kappa \left[ 12C + \frac{\pi^2}{4} - 6 \log^2 2 \right], \quad (A44)
\]

\[
\tau_{17} = -\int_1^\infty dy 36 \kappa^2 \partial_y b_4(y) = -\frac{3}{8} \kappa^2 \left( 48C - \pi^2 \right), \quad (A45)
\]
\[ \tau_{18} = \int_{1}^{\infty} dy \left[ 72 \kappa^2 (2y \partial_y b_0(y) + b_0(y)) - 12 \kappa f_3(y) \partial_y f_2(y) \right] = \frac{3\pi^2}{2} \kappa^2, \quad (A46) \]

\[ \tau_{19} = \int_{1}^{\infty} dy 864 \kappa^3 \partial_y b_6(y) = 9 \kappa^3 \pi^2, \quad (A47) \]

\[ \tau_{20} = - \int_{1}^{\infty} dy 12 \kappa [6 \kappa \partial_y b_3(y) + \partial_y (f_2(y) f_3(y))] = -\frac{3}{8} \kappa^2 (48C + 3\pi^2), \quad (A48) \]

\[ \tau_{21} = \int_{1}^{\infty} dy 12 \kappa \partial_y b_0(y) = \frac{3}{8} \kappa \pi^2, \quad (A49) \]

\[ \tau_{22} = -2 \tau_{14} = -\frac{3}{2} \kappa (\pi - 2 \log 2), \quad (A50) \]

\[ \tau_{23} = \int_{1}^{\infty} dy \left\{ \frac{36 \kappa^2}{y^3} [2b_5(y) + b_4(y)] - 36 \kappa^2 [2 \partial_y b_5(y) + b_5(y)] \right\} = \frac{3}{2} \kappa^2 (\pi^2 - 12 \log 2), \quad (A51) \]

\[ \tau_{24} = \int_{1}^{\infty} dy 432 \kappa^3 \left[ \frac{1}{y^3} b_5(y) \right] = -108 \kappa^3 (\log 2 - 1)^2, \quad (A52) \]

\[ \tau_{25} = \int_{1}^{\infty} dy \left\{ 36 \kappa^2 (2 \partial_y + 1) [b_5(y) - b_7(y)] + 12 \kappa [f_2(y) \partial_y f_1(y) - f_1(y) \partial_y f_2(y)] \right. \]
\[ \left. - \frac{18 \kappa^2}{y^3} [y \partial_y b_4(y) + 2y \partial_y b_3(y) + 4b_3(y) + 2b_4(y)] \right\} \]
\[ = \frac{3}{16} \kappa^2 \left[ -144C + 13\pi^2 + 72 \log^2 2 + 12 \pi (9 \log 2 - 4) \right], \quad (A53) \]

\[ \tau_{26} = - \int_{1}^{\infty} dy \left\{ \frac{18 \kappa^2}{y^3} [4b_3(y) + 2b_4(y) + y \partial_y b_4(y)] + 12 \kappa f_1(y) \partial_y f_2(y) \right\} \]
\[ = \frac{9}{4} \kappa^2 [-8C + (8 + 5 \pi) \log 2], \quad (A54) \]

\[ \tau_{27} = - \int_{1}^{\infty} dy \left\{ \frac{18 \kappa^2}{y^3} [2b_4(y) + 2y \partial_y b_3(y) + y \partial_y b_4(y)] - 12 \kappa f_2(y) \partial_y f_1(y) \right\} \]
\[ = \frac{9}{16} \kappa^2 \left[ 48C + \pi^2 - 4 (8 + 7 \pi) \log 2 \right], \quad (A55) \]

\[ \tau_{28} = \int_{1}^{\infty} dy 72 \kappa^2 \left[ \frac{b_3(y)}{y^3} - [2 \partial_y + 1] b_5(y) \right] = \frac{3}{4} \kappa^2 [\pi (5\pi - 12) + 12 (\pi - 2) \log 2], \quad (A56) \]

30
\[\tau_{29} = -\int_{1}^{\infty} dy 12\kappa f_1(y) \partial_y f_2(y) = -\frac{3}{16}\kappa^2 \left[48\mathcal{C} - \pi(\pi + 12 \log 2)\right],\]  
(A57)

\[\tau_{30} = -\int_{1}^{\infty} dy 72\kappa^2 \left[\frac{1}{y^3} b_6(y)\right] = \frac{9}{2}\kappa^2 (\log 2)^2,\]  
(A58)

\[\tau_{31} = -\int_{1}^{\infty} dy \frac{432\kappa^3}{y^3} \left[y \partial_y b_6(y) + 2b_6(y)\right] = \frac{9}{2}\kappa^3 \left(\pi^2 - 24 \log^2 2\right),\]  
(A59)

\[\tau_{32} = \int_{1}^{\infty} dy 12\kappa \left[\frac{1}{2y^2} \partial_y b_1(y) - f_1(y) \partial_y f_1(y)\right] = -\frac{1}{64}\kappa \left[48\mathcal{C} + \pi(\pi - 24 \log 2)\right],\]  
(A60)

\[\tau_{33} = -\int_{1}^{\infty} dy 12\kappa f_2(y) \partial_y f_2(y) = 54\kappa^3 (\log 2)^2,\]  
(A61)

\[\tau_{34} = \int_{1}^{\infty} dy 432\kappa^3 \partial_y b_5(y) = 9\kappa^3 \left[\pi^2 - 6 \log 2(\log 2 - 2)\right],\]  
(A62)

\[\tau_{35} = \int_{1}^{\infty} dy 72\kappa^2 \left[\frac{1}{y^3} b_6(y)\right] = \frac{9}{2}\kappa^2 (\log 2)^2,\]  
(A63)

\[\tau_{36} = \int_{1}^{\infty} dy 72\kappa^2 \left[\frac{1}{2y^2} \partial_y b_6(y)\right] = \frac{3}{8}\kappa^2 \left[\pi^2 - 12 (\log 2)^2\right],\]  
(A64)

\[\tau_{37} = -\int_{1}^{\infty} dy \frac{216\kappa^3}{y^3} \left[y \partial_y b_5(y) + 4b_5(y)\right] = \frac{9}{2}\kappa^3 \left[72 - \pi^2 + 30(\log 2 - 2) \log 2\right],\]  
(A65)

\[\tau_{38} = -\int_{1}^{\infty} dy 432\kappa^3 \left[\frac{1}{y^3} b_7(y) - \frac{1}{y^3} b_5(y)\right] = \kappa^3 \left(324 - 135 \log^2 2 + 162\right),\]  
(A66)

\[\tau_{39} = \int_{1}^{\infty} dy \frac{216\kappa^3}{y^2} \left[\partial_y b_5(y) - \partial_y b_7(y)\right] = -\frac{27}{4}\kappa^3 \left[24 - \pi^2 - 16 \log 2(\log 2 - 2)\right],\]  
(A67)

\[\tau_{40} = -\int_{1}^{\infty} dy 432\kappa^3 \left[\partial_y b_5(y) - \partial_y b_7(y)\right] = -\frac{27}{2}\kappa^3 \left[\pi^2 - 4 \log 2(\log 2 - 4)\right],\]  
(A68)

where the Catalan’s constant \(\mathcal{C} \approx 0.915966\).
ACKNOWLEDGEMENTS

We would like to thank Umut Gürsoy, Dmitri Kharzeev, Nathan Kleedorin, Shu Lin, Igor Rogachevskii, Andrey Sadofyev, Ho-Ung Yee for useful discussions. YB would like to thank the hospitality of the Department of Physics at Ben-Gurion University of the Negev where this work was initialised and finalised. YB was supported by the Fundamental Research Funds for the Central Universities under grant No.122050205032 and the Natural Science Foundation of China (NSFC) under the grant No.11705037. TD and ML were supported by the Israeli Science Foundation (ISF) grant #1635/16 and the BSF grants #2012124 and #2014707.

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