APPROXIMATE FACTORIZATION AND CONCENTRATION
FOR CHARACTERS OF SYMMETRIC GROUPS

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Abstract. We prove a factorization-concentration result for characters of symmetric groups. This is then applied to the asymptotic behaviour of the decomposition of the tensor representations. There are connections with the Pastur-Marcenko distribution of random matrix theory, and freely infinitely divisible distributions.

Introduction

In [B] we determined the asymptotic behaviour of normalized irreducible characters of the symmetric groups $S_q$, as $q \to \infty$, under the hypothesis that the corresponding Young diagrams, rescaled by a factor $q^{-1/2}$, have a limit shape. It turns out that for such characters $\chi$, the order of magnitude of $\chi(\rho)$ is $q^{-|\rho|/2}$ where the length $|\rho|$ of a permutation $\rho$ is the minimal number $n$ required to write $\rho$ as a product of $n$ transpositions. Alternatively, $|\rho| = q - c(\rho)$ where $c(\rho)$ is the number of cycles of $\rho$. Furthermore these characters satisfy an approximate factorization property

\begin{equation}
\chi(\rho \sigma) = \chi(\rho) \chi(\sigma) + o(q^{-(|\rho|+|\sigma|)/2})
\end{equation}

for permutations $\rho, \sigma$ with disjoint supports. In this paper we consider normalized positive definite functions on symmetric groups. Such a function defines a probability measure on the set of Young diagrams. We shall consider functions $\psi$ satisfying bounds

\begin{equation}
|\psi(\rho)| \leq cnq^{-n/2}
\end{equation}

for $|\rho| = n$. We prove that for large $q$, the probability measure associated with a $\psi$ satisfying the approximate factorization (0.1) is concentrated on the set of Young diagrams whose normalized character takes values close to that of $\psi$. As an application of this result we shall investigate the asymptotic behaviour of the decomposition of the tensor representation $(\mathbb{C}^N)^\otimes q$ of the symmetric group $S_q$. We shall prove that in the asymptotic regime $q \to \infty$, $\frac{\sqrt{q}}{N} \to c \in [0, +\infty[$ the typical Young diagram occuring in the decomposition of this representation has a certain limit shape depending on $c$. For $c = 0$ we recover the limit shape occurring in the asymptotics of the Plancherel measure on $S_q$, as in the results of Kerov-Vershik [KV] and Logan-Shepp [LS]. For $c > 0$ this limit shape is strongly related to the so-called Pastur-Marcenko distribution occuring in the theory of random matrices, and

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in the theory of free probability, where it appears under the name of “free Poisson distribution”. We shall also consider more general measures on Young diagrams coming from tensor states on $B(H)^{\otimes q}$. By tuning the state with $q$ we will get limit diagrams which are related to freely infinitely divisible measures.

This paper is organized as follows. In Section 1 we settle notations, recall results from [B], and state the main result of the paper. In Section 2 we prove the result, and in Section 3 we discuss the decomposition of the tensor representation.

1. Notations and statement of the main result

We recall notations from [B], to which we refer for more details.

1.1 Symmetric groups and Young diagrams. We denote by $S_q$ the symmetric group on $\{1, \ldots, q\}$, by $(ij)$ the transposition exchanging $i$ and $j$, and more generally by $(i_1 i_2 \ldots i_n)$ cyclic permutations of order $n$. For $\sigma \in S_q$, let $c(\sigma)$ be its number of cycles, and $s(\sigma)$ be the number of elements not fixed by $\sigma$. Then $|\sigma| := q - c(\sigma)$ is the smallest number $n$ such that $\sigma$ can be written as a product of $n$ transpositions. We denote by $\mathcal{Y}_q$ the set of Young diagrams with $q$ boxes, and $\mathcal{Y} = \cup_{q=1}^{\infty} \mathcal{Y}_q$. If $\lambda \in \mathcal{Y}_q$ let $[\lambda]$ be the associated irreducible representation of $S_q$, and $\chi_\lambda$ be its normalized character, i.e. $\chi_\lambda(e) = 1$. Recall from [B], [K] that a Young diagram can be identified with a piecewise linear function with slopes $\pm 1$, and local minima and maxima occurring at two interlacing sequences of integer points

$$x_1 < y_1 < x_2 < \ldots < y_{n-1} < x_n$$

as in the following example

and that we can embed the set of Young diagrams in the space $CY$ of continuous Young diagrams, i.e. functions $\omega : \mathbb{R} \to \mathbb{R}$ satisfying

1. $|\omega(u_1) - \omega(u_2)| \leq |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$
2. $\omega(u) = |u|$ for sufficiently large $|u|$.

To each continuous diagram $\omega \in CY$ one can associate a probability measure $m_\omega$ with compact support on $\mathbb{R}$, determined by the equation

$$\frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{x-z} \sigma'(x) \, dx = \int_{\mathbb{R}} \frac{1}{z-x} m_\omega(dx) \quad z \in \mathbb{C} \setminus \mathbb{R}$$

where $\sigma(u) = (\omega(u) - |u|)/2; u \in \mathbb{R}$. This measure is called the transition measure of the diagram and its moments are called the moments of the corresponding
continuous diagram. We shall denote them by

\[ m_n(\omega) = \int_{\mathbb{R}} x^n \, m_\omega(dx) \]

The measure corresponding to a diagram in \( CY \) has bounded support, it is centered (i.e. \( m_1 = 0 \)), and its second moment is equal to half the area of the set

\[ \{(u,v) \in \mathbb{R}^2 | u \leq v \leq \omega(u)\} \]

Besides moments of measures we shall also consider their free cumulants. Recall ([B], [VDN]), that the free cumulants are defined as the coefficients \( R_n \) in the expansion

\[ K(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n z^{n-1} \]

where \( K(z) \) is the functional inverse of the Cauchy transform

\[ G(z) = \int_{\mathbb{R}} \frac{1}{z - x} \, m_\omega(dx) \]

1.2 Positive definite functions. Let \( \psi \) be a normalized (i.e. \( \psi(e) = 1 \)), central positive definite function on \( S_q \). By Fourier analysis on \( S_q \), one can expand \( \psi \) as a convex linear combination of normalized characters

\[ \psi = \sum_{\lambda \in \mathcal{Y}_q} p_\lambda \chi_\lambda \]

The weights \( p_\lambda \) are non negative and sum to one, therefore they define a probability measure \( \Pi_\psi \) on \( \mathcal{Y}_q \), which puts a mass \( p_\lambda \) on \( \lambda \). Let \( \mathcal{V} \subset \mathcal{Y}_q \) and \( \gamma > 0 \), we say that \( \psi \) is \( \gamma \)-supported on \( \mathcal{V} \) if \( \Pi_\psi(\mathcal{V}) > 1 - \gamma \).

Let us define a probability measure on \( \mathbb{R} \) associated with \( \psi \) by

\[ m_\psi = \sum_{\lambda \in \mathcal{Y}_q} p_\lambda m_\lambda \]

and denote its moments by

\[ m_n(\psi) = \sum_{\lambda \in \mathcal{Y}_q} p_\lambda m_n(\lambda) \]

To this probability measure we can associate a diagram, by the correspondence (1.1), which we call \( \omega_\psi \). By the GNS construction, there exists a finite dimensional complex Hilbert space \( V \), a unitary representation \( r_\psi : S_q \to \mathcal{B}(V) \) (where \( \mathcal{B}(V) \) is the space of bounded operators on \( V \)), and a state \( \tau_\psi \) on \( \mathcal{B}(V) \), tracial on \( r_\psi(S_q)^\prime\prime \) such that

\[ \psi(\rho) = \tau_\psi(r_\psi(\rho)) \]

for all \( \rho \in S_q \). As in [B] for a unitary representation \( r \) of \( S_q \) in some \( \mathcal{B}(H) \), we define the selfadjoint element

\[
\Gamma(r) = \begin{pmatrix}
0 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & r(12) & r(13) & \ldots & r(1q-1) & r(1q) \\
1 & r(12) & 0 & r(23) & \ldots & r(2q-1) & r(2q) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & r(1q-2) & r(1q-3) & r(1q-4) & \ldots & 0 & r(1q-1)
\end{pmatrix}
\]
of $B(H) \otimes M_{q+1}(\mathbb{C})$. By [B], Proposition 3.3, the measure $m_\psi$ is the distribution of the self-adjoint element $\Gamma(\psi) := \Gamma(r_\psi)$ in the non-commutative probability space
\[(B(V_\psi) \otimes M_{q+1}(\mathbb{C}), \tau_\psi \otimes |\cdot|)\]
where $\langle \cdot \rangle$ denotes the normalized trace on $M_{q+1}(\mathbb{C})$. In particular, one has
\[m_n(\psi) = \tau_\psi((\Gamma(\psi)^n)) \quad \text{for all } n \geq 1\]

1.3 Approximate factorization. We now give a precise definition of the approximate factorization property (0.1).

**Definition 1.3.** Let $n$ be a positive integer, let $c = (c_l)_{l \geq 1}$ be a sequence of positive real numbers, and let $\delta > 0$, define $F_{c,\delta}^n$ as the set of all positive definite normalized functions $\psi : S_q \to \mathbb{C}$, for some positive integer $q$, such that
- (1) $|\psi(\rho)| \leq c_l q^{-l/2}$ for all $l \leq n$, and $\rho \in S_q$ with $|\rho| = l$.
- (2) $|\psi(\rho \sigma) - \psi(\rho)\psi(\sigma)| \leq \delta q^{-l/2}$ for all $l \leq n$ and all $\rho, \sigma \in S_q$, with disjoint supports, such that $|\rho \sigma| = l$.

Note that $|\rho \sigma| = |\rho| + |\sigma|$ if $\rho$ and $\sigma$ have disjoint supports, so that (2) means that, according to (1), $\psi(\rho \sigma)$ has the same order of magnitude as $\psi(\rho)\psi(\sigma)$. The main result of this paper is the following

**Theorem 1.** Let the sequence $(c_l)_{l \geq 1}$ and the integer $n$ be given, then there exists a constant $K$, such that for all $\delta \geq 0$, every $\psi \in F_{c,\delta}^n$ is $(\delta + q^{-1})^{1/2}$-supported by the set of Young diagrams $\lambda \in \mathcal{Y}_q$ satisfying
\[|m_l(\lambda) - m_l(\psi)| \leq K(\delta + q^{-1})^{1/4} q^{l/2}\]
for all $l \leq n/2$.

Let us denote by $R_l(\psi)$ the free cumulants of the measure $m_\psi$, then using the moment-cumulant formula of [S] (see also Section 2 of [B]), we get the following corollary of the previous result

**Corollary 1.** Let the sequence $(c_l)_{l \geq 1}$ and the integer $n$ be given, then there exists a constant $K$, such that for all $\delta \geq 0$, every $\psi \in F_{c,\delta}^n$ is $(\delta + q^{-1})^{1/2}$-supported by the set of Young diagrams $\lambda \in \mathcal{Y}_q$ satisfying
\[|R_l(\lambda) - R_l(\psi)| \leq K(\delta + q^{-1})^{1/4} q^{l/2}\]
for all $l \leq n/2$.

Using the results of [B] and the above Corollary, a reformulation of Theorem 1 can be given which involves character values instead of moments.

**Theorem 2.** Let the sequence $(c_l)_{l \geq 1}$ and the integer $n$ be given, then there exists a constant $K$, such that for all $\delta \geq 0$, every $\psi \in F_{c,\delta}^n$ is $(\delta + q^{-1})^{1/2}$-supported by the set of Young diagrams $\lambda \in \mathcal{Y}_q$ satisfying
\[|\chi(\lambda)(\rho) - \psi(\rho)| \leq K(\delta + q^{-1})^{1/4} q^{-l/2}\]
for all $\rho \in S_q$, with $|\rho| = l \leq n/2$. 
We shall rely heavily on results of [B], especially Section 4. Let $\psi$ be a normalized positive definite function, then $m_n(\psi) = \sum_{\lambda \in \mathcal{Y}_q} p_\lambda m_n(\lambda)$ is the mean value for $\Pi_\psi$ of the random variable $\lambda \mapsto m_n(\lambda)$ on $\mathcal{Y}_q$. The proof of Theorem 1 consists in giving an estimate for the variance of this random variable and using Markov’s inequality. The key result is the following whose proof is analogous to the proof of Lemma 5.1.1 or 6.1 in [B].

**Lemma 2.1.** Let $n_1, n_2, \ldots n_k$ be positive integers, then for every sequence $(c_l)_{l \geq 1}$ and $\delta > 0$ there exists a constant $K$ such that for all $\psi \in \mathcal{F}_{c,\delta}$ with $n = n_1 + \ldots + n_k$, one has

$$|\tau_\psi((\Gamma(\psi)^{n_1}) (\Gamma(\psi)^{n_2}) \ldots (\Gamma(\psi)^{n_k})) - \tau_\psi((\Gamma(\psi)^{n_1}) \tau_\psi((\Gamma(\psi)^{n_2})) \ldots \tau_\psi((\Gamma(\psi)^{n_k}))|$$

$$\leq K(\delta + q^{-1})q^{n/2}$$

**Proof.** We shall only need the case $k = 2$ of this Lemma which is what we prove here. The argument can be easily extended to yield the general case, which can be used to give estimates on higher moments of $\langle \Gamma(\psi)^n \rangle - \tau_\psi(\langle \Gamma(\psi)^n \rangle)$.

One has

$$\langle \Gamma(\psi)^n \rangle = \frac{1}{q+1} \sum_{0 \leq i_1, i_2, \ldots, i_n \leq q} r_\psi((i_1 i_2) \ldots (i_{n-1} i_n))(i_n i_1)$$

and analogously

$$\langle \Gamma(\psi)^{n_1} \rangle \langle \Gamma(\psi)^{n_2} \rangle = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} r_\psi((i_1 i_2) \ldots (i_{n_1} i_{n_1}))(j_1 j_2) \ldots (j_{n_2} j_1)$$

where by convention $(ii)$ is zero and $(ij) = 1$ if either $i$ or $j$ (but not both) is 0. Applying the state $\tau_\psi$ one gets

$$\tau_\psi((\Gamma(\psi)^{n_1}) \langle \Gamma(\psi)^{n_2} \rangle) = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} \psi((i_1 i_2) \ldots (i_{n_1} i_{n_1}))(j_1 j_2) \ldots (j_{n_2} j_1)$$

(2.1.2)

We shall compare (2.1.2) to

$$\tau_\psi((\Gamma(\psi)^{n_1}) \tau_\psi((\Gamma(\psi)^{n_2})) = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} \psi((i_1 i_2) \ldots (i_{n_1} i_{n_1})) \psi((j_1 j_2) \ldots (j_{n_2} j_{n_2}))$$

(2.1.3)

In (2.1.2), since $\psi$ is a central function, the value of

$$\psi((i_1 i_2) \ldots (i_{n_1} i_{n_1}))$$

is a constant $\delta > 0$. Therefore

$$\tau_\psi((\Gamma(\psi)^{n_1}) \langle \Gamma(\psi)^{n_2} \rangle) = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} \delta \psi((j_1 j_2) \ldots (j_{n_2} j_{n_2}))$$

(2.1.4)

where

$$\tau_\psi((\Gamma(\psi)^{n_1}) \tau_\psi((\Gamma(\psi)^{n_2})) = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} \psi((i_1 i_2) \ldots (i_{n_1} i_{n_1})) \psi((j_1 j_2) \ldots (j_{n_2} j_{n_2}))$$

(2.1.5)

The difference between (2.1.4) and (2.1.5) is

$$\tau_\psi((\Gamma(\psi)^{n_1}) \langle \Gamma(\psi)^{n_2} \rangle - \tau_\psi((\Gamma(\psi)^{n_1}) \tau_\psi((\Gamma(\psi)^{n_2})) = \frac{1}{(q+1)^2} \sum_{0 \leq i_1, i_2, \ldots, i_{n_1} \leq q, 0 \leq j_1, j_2, \ldots, j_{n_2} \leq q} \delta \psi((j_1 j_2) \ldots (j_{n_2} j_{n_2}))$$

(2.1.6)
Lemma 4.3.2 of \[B\] one can see that if \( J \) is the product of two permutations with disjoint supports (2.1.7) by replacing (2.1.4) by a sum depends only on the set \( J \) of places where \( i_k \) or \( j_k \) is 0, and the partition of the set \( \{1, 2, \ldots, n_1\} \cup \{1', 2', \ldots, n_2'\} \) \( J \) given by the equivalence relation \( u \sim v \) if \( i_u = i_v \), \( u \sim v' \) if \( i_u = j_v' \), or \( u' \sim v' \) if \( j_u' = j_v' \). Given such a set \( J \subset \{1, 2, \ldots, n_1\} \cup \{1', 2', \ldots, n_2'\} \) and the partition \( \pi \), the number of corresponding terms in the sum is equal to \((q)_{\text{comp}(\pi)}\) where \((q)_r = q(q - 1)\ldots(q - r + 1)\) and \(\text{comp}(\pi)\) is the number of components of \(\pi\). Denoting by \(\psi(\pi, J)\) the common value of \(\psi\) on sequences corresponding to the set \(J\) and the partition \(\pi\), the right hand side of (2.1.2) can be rewritten as a sum of the form

\[
(2.1.4) \quad \sum_{\pi, J} \frac{1}{(q + 1)^2} (q)_{\text{comp}(\pi)} \psi(\pi, J)
\]

Analogously we can write the right hand side of (2.1.3) as

\[
(2.1.5) \quad \sum_{\pi_1: J_1 \subset \pi_2: J_2} \frac{1}{(q + 1)^2} (q)_{\text{comp}(\pi_1)} \psi(\pi_1, J_1)(q)_{\text{comp}(\pi_2)} \psi(\pi_2, J_2)
\]

Let \(h(\pi)\) denote the conjugacy class of the permutation corresponding to \(\pi\), and let \(|h(\pi)|\) denote the length of any of its elements. Hypothesis (1) of Definition 1.3 implies a bound \(|\psi(\pi, J)| \leq Cq^{-|h(\pi)|/2}\). Furthermore, by an argument similar to Lemma 4.3.2 of \[B\] one can see that if \(J \neq \emptyset\), then

\[n + |h(\pi)| \geq 2 \text{comp}(\pi) - 1\]

therefore the total contribution of terms such that \(J \neq \emptyset\) can be bounded by \(Cq^{n/2-1}\) for some constant \(C\) depending only on \(n\) and on the sequence \((c_l)_{1 \leq l \leq n}\). A similar argument using Lemma 4.3.2 of \[B\] would apply to (2.1.3), we shall therefore restrict the sums in (2.1.2) and (2.1.3) to \(i'\)'s and \(j'\)'s in the range 1, \ldots, \(q\), and replace (2.1.4) by a sum

\[
(2.1.6) \quad \sum_{\pi} \frac{1}{(q + 1)^2} (q)_{\text{comp}(\pi)} \psi(\pi)
\]

over partitions \(\pi\) of \(\{1, 2, \ldots, n_1\} \cup \{1', 2', \ldots, n_2'\}\). Consider now the contribution to (2.1.6) of partitions \(\pi\) such that some \(i \in \{1, 2, \ldots, n_1\}\) and some \(j' \in \{1', 2', \ldots, n_2'\}\) are in the same component, then using Lemma 5.1.2 from \[B\] and the estimate (1) of Definition 1.3 we can again bound this contribution by \(Cq^{n/2-1}\). It remains to consider the contribution of partitions \(\pi\) which split as the union of a partition \(\pi_1\) of \(\{1, \ldots, n_1\}\) and a partition \(\pi_2\) of \(\{1', 2', \ldots, n_2'\}\). For such a partition, any associated permutation \((i_1i_2)(i_3i_4)(j_1j_2)(j_3j_4)\) is the product of two permutations with disjoint supports \((i_1i_2)(i_3i_4)\) and \((j_1j_2)(j_3j_4)\), so that by the asymptotic factorization property, one has

\[
|\psi((i_1i_2)(i_3i_4)(j_1j_2)(j_3j_4)) - \psi((i_1i_2)(i_3i_4))\psi((j_1j_2)(j_3j_4))| \leq \delta q^{-|h(\pi)|/2}
\]

Since by \[B\], Section 4.3, one has \(n + |h(\pi)| \geq 2 \text{comp}(\pi) - 2\) we can replace (2.1.6) by

\[
(2.1.7) \quad \sum_{\pi} \frac{1}{(q + 1)^2} (q)_{\text{comp}(\pi)} \psi(\pi_1)\psi(\pi_2)
\]
making an error bounded by $C\delta q^{l/2}$. On the other hand the right hand side of (2.1.5) can be replaced by

$$
\sum_{(\pi_1, \pi_2)} \frac{1}{(q+1)^2} (q)_{\text{comp}(\pi_1)} (q)_{\text{comp}(\pi_2)} \psi(\pi_1) \psi(\pi_2)
$$

The difference between (2.1.7) and (2.1.8) is

$$
\sum_{(\pi_1, \pi_2)} \frac{1}{(q+1)^2} ((q)_{\text{comp}(\pi)} - (q)_{\text{comp}(\pi_1)} (q)_{\text{comp}(\pi_2)}) \psi(\pi_1) \psi(\pi_2)
$$

One has

$$
\text{comp}(\pi_1 \cup \pi_2) = \text{comp}(\pi_1) + \text{comp}(\pi_2)
$$

thus

$$
|(q)_{\text{comp}(\pi_1 \cup \pi_2)} - (q)_{\text{comp}(\pi_1)} (q)_{\text{comp}(\pi_2)}| \leq Cq^{\text{comp}(\pi_1) + \text{comp}(\pi_2) - 1}
$$

By Lemma 4.3.2 and 4.3.3 of [B], one has

$$
n_1 + n_2 + |h(\pi_1)| + |h(\pi_2)| \geq 2(\text{comp}(\pi_1) + \text{comp}(\pi_2)) - 4
$$

therefore each term in this difference is bounded by $Cq^{l/2 - 1}$ and one gets the stated result. □

Using the Lemma, we can now estimate the variance of $m_n(\lambda)$ under the measure $\Pi_\psi$, indeed it follows from Section 3.3 in [B] that one has

$$
\Pi_\psi((m_l(\lambda) - m_l(\psi))^2) = \tau_\psi((\Gamma(\psi)^l)^2) - \tau_\psi((\Gamma(\psi)^l)^2)
$$

therefore, using Lemma 2.1, for $\psi \in \mathcal{F}_{c,\delta}^n$ one has

$$
\Pi_\psi((m_l(\lambda) - m_l(\psi))^2) \leq K(\delta + q^{-1})q^l \quad \text{by Lemma 2.1}
$$

Theorem 1 now follows by an application of Markov’s inequality. □

Corollary 1 is a simple consequence of Theorem 1 and of the moment-cumulant formula.

Finally for the proof of Theorem 2, we can use the same arguments as in Section 4 of [B], and the hypotheses (1) and (2) of Definition 3.1. This shows that there exists a constant $K$ such that, for all $\psi \in \mathcal{F}_{c,\delta}^n$, all $l \leq n$ and all cycle $\rho$ on $l-1$ elements, one has

$$
|c_l(\psi) q^{1-l} - \psi(\rho)| \leq K(\delta + q^{-1})q^{-l/2} \quad \text{for all } l \leq n
$$

This, combined with Corollary 1 gives Theorem 2. □
3. Asymptotics of the tensor representation

3.1 The case of the canonical trace. In this section we consider the action of $S_q$ on $(\mathbb{C}^N)^{\otimes q}$ by permutation of the factors in the tensor product. This representation plays a key role in the treatment by Schur and Weyl of the representation theory of both symmetric and general linear groups [W]. We consider the decomposition of this representation into isotypic components

$$(\mathbb{C}^N)^{\otimes q} = \bigoplus_{\lambda \in \mathcal{Y}_q} E_{\lambda}$$

then the relative dimensions $\frac{\dim E_{\lambda}}{N_q}$ define a probability measure on $\mathcal{Y}_q$. Let $\Pi_q$ be the image of this probability measure on $\mathcal{Y}_q \subset \mathcal{CY}$ by the scaling map $\omega \mapsto q^{-1/2} \omega(q^{1/2})$. Thus $\Pi_q$ is a probability measure on the set $\mathcal{CY}$ of continuous Young diagrams.

Let $\arcsin$ take values in $[-\pi/2, +\pi/2]$ and $\arccos$ in $[0, \pi]$, and define

$$h(c, u) = \frac{2}{\pi} \left( u \arcsin \left( \frac{u + c}{2\sqrt{1 + uc}} \right) + \frac{1}{c} \arccos \left( \frac{2 + uc - c^2}{2\sqrt{1 + uc}} \right) + \frac{1}{2} \sqrt{4 - (u - c)^2} \right)$$

for $0 < c < \infty$, and $u \in [c - 2, c + 2]$. Let us denote by $P_c$, the continuous diagram given by the formula

$$(3.1.1)$$

$$P_0(u) = \begin{cases} \frac{2}{\pi} (u \arcsin(\frac{u}{2}) + \sqrt{4 - u^2}) & \text{if } u \in [-2, +2] \\ \frac{1}{u} & \text{if } u \notin [-2, +2] \end{cases}$$

$$P_c(u) = \begin{cases} h(c, u) & \text{if } u \in [c - 2, c + 2] \\ \frac{1}{u} & \text{if } u \notin [c - 2, c + 2] \end{cases}$$

for $0 < c < 1$

$$P_1(u) = \begin{cases} \frac{u+1}{2} + \frac{1}{\pi} \left( (u - 1) \arcsin(\frac{u-1}{2}) + \sqrt{4 - (u - 1)^2} \right) & \text{if } u \in [-1, 3] \\ \frac{1}{u} & \text{if } u \notin [-1, 3] \end{cases}$$

$$P_c(u) = \begin{cases} u + \frac{2}{c} & \text{if } u \in [-\frac{1}{c}, c - 2] \\ h(c, u) & \text{if } u \in [c - 2, c + 2] \\ \frac{1}{u} & \text{if } u \notin [-\frac{1}{c}, c + 2] \end{cases}$$

for $c > 1$.
The limit result for the $\Pi_q$ is the following.

**Theorem 3.** As $q \to \infty$ and $\sqrt{q}/N \to c \in [0, +\infty]$, the measure $\Pi_q$ converges uniformly, in probability, towards the Dirac measure at the continuous diagram $P_c$.

**Proof.** The character of the tensor representation can be computed in the basis $e_{i_1} \otimes \ldots \otimes e_{i_q}$, where $e_i$ is a basis of $\mathbb{C}^N$, one gets, for $\rho \in S_q$,

$$tr(\rho) = N^{c(\rho) - q} = N^{-|\rho|}$$

This is a positive definite function on $S_q$. Taking $N = N(q)$ such that $\sqrt{q}/N \to c \in [0, +\infty]$, we see that the corresponding functions belong to $\mathcal{F}_{c_0}^n$ for all $n$, where the sequence $(c_l = C^l)_{l \geq 1}$, with $C = \sup_q \{\sqrt{q}/N(q)\}$, therefore we can apply Theorem 1 and 2, and Corollary 1 to see that under $\Pi_q$, the sequence of cumulants of a random diagram converges in probability towards the sequence $(0, 1, c, c^2, \ldots, c^n, \ldots)$, and the sequence of moments converges towards the corresponding moments. The correspondence between measures and diagrams given by (1.1) has the following continuity property (see [K]):

if $\omega \in \mathcal{CY}$, then for all $\varepsilon > 0$ there exists $n$ and $\delta$ such that, if $\omega' \in \mathcal{CY}$ and $|m_l(\omega) - m_l(\omega')| < \delta$ for all $l \leq n$, then $|\omega(u) - \omega'(u)| < \varepsilon$ for all $u \in \mathbb{R}$.

Therefore it is enough to check that the sequence $(0, 1, c, c^2, \ldots, c^n, \ldots)$ is the sequence of cumulants of the diagram $P_c$. For $c = 0$, $P_c$ is the diagram corresponding to the semi-circle distribution (see [K]), while for $c > 0$ the constant sequence $(c^{-2}, \ldots, c^{-2}, \ldots)$ is the sequence of cumulants of the so-called Pastur-Marcenko or free Poisson distribution (see [PM], or [VDN]), with parameter $c^{-2}$. Taking the image of this measure by the mapping $x \mapsto cx - \frac{1}{c}$ yields the measure with free cumulant sequence $(0, 1, c, c^2, c^3, \ldots)$. We shall now compute the corresponding diagram, according to [K]. The $P$ transform with cumulant sequence $(0, 1, c, c^2, c^3, \ldots)$.

The diagrams are depicted below for various values of $c$. 
is
\[ R(z) = \sum_{n=1}^{\infty} z^n c^{n-1} = \frac{z}{1 - cz} \]

thus
\[ K(z) = \frac{1}{z} + R(z) = \frac{1}{z} + \frac{z}{1 - cz} \]

Inverting the function \( K \) gives the Cauchy transform
\[ G(z) = \frac{z + c - \sqrt{(z - c)^2 - 4}}{2(1 + cz)} = \frac{2}{z + c + \sqrt{(z - c)^2 - 4}} \]

where the branch of the square root on \( \mathbb{C} \setminus [0, +\infty[ \) is chosen to have positive imaginary part. We now compute the Rayleigh measure \( \tau \) according to the formula
\[ \frac{\partial}{\partial z} \log G(z) = - \int_{\mathbb{R}} \frac{\tau(du)}{z - u} \]

One has
\[ -\frac{\partial}{\partial z} \log G(z) = \frac{1 + \frac{z-c}{\sqrt{(z-c)^2-4}}}{z + c + \sqrt{(z-c)^2-4}} = \frac{1}{2\sqrt{(z-c)^2-4}} \left( \frac{2 + cz - c^2 + c\sqrt{(z-c)^2-4}}{1 + cz} \right) \]

Let
\[ k(c, u) = \frac{2 + cu - c^2}{2\pi(1 + cu)\sqrt{4 - (u - c)^2}} \quad \text{for } -2 + c < u < 2 + c. \]

Using Stieltjes inversion formula we get
\[ \tau_c(du) = \begin{cases} \frac{1}{2} \delta_{-1}(du) + \frac{du}{2\pi\sqrt{(x+1)(3-x)}} [1-1,3](u) & \text{if } 0 \leq c < 1 \\ \delta_{-\frac{1}{c}}(du) + k(c, u)1_{[-c,-c+2]}(u) du & \text{if } c = 1 \\ \delta_{-\frac{1}{c}}(du) + k(c, u)1_{[-c,-c+2]}(u) du & \text{if } c > 1 \end{cases} \]

From the Rayleigh measure \( \tau_c \) we can recover the diagram by the formula
\[ P_c(u) = \int_{-\infty}^{+\infty} |u - x| \tau_c(dx) \]

A lengthy but straightforward computation using (3.1.2) gives formula (3.1.1).

\[ \square \]

3.2 Generalization to other tensor states. Let \( H = H_0 \oplus H_1 \), be a \( \mathbb{Z}/2 \)-graded Hilbert space with grading operator \( \Delta \). Endow \( \mathcal{B}(H^\otimes q) = \mathcal{B}(H)^\otimes q \) with a tensor state \( \tau^\otimes q \) where \( \tau \) is a state on \( \mathcal{B}(H) \), of the form \( \tau(X) = Tr(TX) \), with \( T \) a positive trace class operator, with trace 1, commuting with the grading. Denote by \( t_{0,0} \geq t_{0,1} \geq \cdots \geq t_{k,0} \geq \cdots \geq 0 \) and \( t_{1,1} \geq t_{2,1} \geq \cdots \geq t_{k,1} \geq \cdots \geq 0 \) the eigenvalues of \( T \) on \( H_0 \) and \( H_1 \) respectively, and let \( p_n = Tr(\Delta(\Delta T)^n) = \sum_{j=1}^{\infty} t_{j,0}^n - (-t_{j,1})^n \) for \( n \geq 1 \).
In this last section we shall consider the representation of $S_q$ on $H^q$, given by
\[(ii + 1)(v_1 \otimes \ldots v_i \otimes v_{i+1} \ldots \otimes v_q) = (-1)^{\varepsilon(v_i)\varepsilon(v_{i+1})}v_1 \otimes \ldots v_{i+1} \otimes v_i \ldots \otimes v_q\]
where $v_k$ are graded vectors, with graduation $\varepsilon(v_k) \in \{0, 1\}$. The positive definite function on $S_q$ determined by this representation and the state $\tau^q$ is given by
\[(3.2.1) \quad \tau^q(\rho) = \prod_{c|\rho} p_{|c|+1}\]
where the product is over the non trivial cycles $c$ of $\rho$. Recall that for a cycle $c$ one has $|c| + 1 = s(c)$. This is of course a direct generalization of Section 3.1 where we had $t_{1,0} = t_{2,0} = \ldots = t_{N,0} = \frac{1}{N}$, the other eigenvalues being 0. We shall now investigate the asymptotic decomposition of the associated probability measure on Young diagrams, under the hypothesis that $\tau := \tau_q$ depends on $q$ as $q \to \infty$, and that the moments $Tr(T_q^n)$ satisfy
\[q^{(n-1)/2}Tr(T_q^n) \to_{q \to \infty} w_{n+1} \quad n \geq 1\]
for some sequence of real numbers $w_n$, with $|w_n| \leq C^n$ for some constant $C > 0$. This last condition insures that the positive definite functions (3.2.1) belong to the sets $F_{c,0}^n$ as in Section 3.1. Using the same arguments as in the preceding section, we see that the measure on rescaled diagrams converges towards the Dirac mass at the diagram with cumulants $(w_n)_{n \geq 1}$, with $w_1 = 0$, $w_2 = 1$. These cumulants have the form $w_n = \int_\mathbb{R} |t|t^{n-2}\mu(dt); \ n \geq 2$ for some positive measure $\mu$ on $[-C,C]$, satisfying $\int_\mathbb{R} |t|\mu(dt) = 1$. The $R$-transform of the measure with cumulants $w_n$ is thus
\[R(z) = \sum_{n=1}^{\infty} w_n z^{n-1} = \int_\mathbb{R} \frac{z}{1 - zt}|t|\mu(dt)\]
Comparing with the free Lévy-Khintchine formula of [BV], we see that the measure corresponding to the limit diagram is freely indefinitely divisible, with corresponding Lévy measure $\nu(dt) = \frac{1}{|t|}\mu(dt)$

**Remark.** One can give an explicit expression for the weight of a given diagram $\lambda \in \mathcal{Y}_q$, under the measure associated with the function (3.2.1), in terms of the Schur functions (see also [M] formula (7.8)). It would be interesting to rederive the results of Sections 3.1 and 3.2 directly from this explicit expression.

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