A Generalization of the Convex Kakeya Problem

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Abstract Given a set of line segments in the plane, not necessarily finite, what is a convex region of smallest area that contains a translate of each input segment? This question can be seen as a generalization of Kakeya's problem of finding a convex region of smallest area such that a needle can be rotated through 360 degrees within...
this region. We show that there is always an optimal region that is a triangle, and we give an optimal $\Theta(n \log n)$-time algorithm to compute such a triangle for a given set of $n$ segments. We also show that, if the goal is to minimize the perimeter of the region instead of its area, then placing the segments with their midpoint at the origin and taking their convex hull results in an optimal solution. Finally, we show that for any compact convex figure $G$, the smallest enclosing disk of $G$ is a smallest-perimeter region containing a translate of every rotated copy of $G$.

**Keywords** Computational geometry · Discrete geometry · Algorithms · Kakeya

1 Introduction

Let $\mathfrak{F}$ be a family of objects in the plane. A translation cover for $\mathfrak{F}$ is a set $K$ such that any object in $\mathfrak{F}$ is contained in a translate of $K$ [28]. We are interested in determining a convex translation cover for $\mathfrak{F}$ of smallest possible area or perimeter.

Since the convex hull of a set of objects is the smallest convex figure that contains them, this problem can be reformulated as translating the objects in $\mathfrak{F}$ such that the perimeter or the area of their convex hull is minimized. When $\mathfrak{F}$ consists of $n$ objects, we can fix one object and translate the remaining $n - 1$ objects. Therefore we can use a vector in $\mathbb{R}^{2(n-1)}$ to represent the translations of $n - 1$ objects. Consider the functions $\mathbb{R}^{2(n-1)} \to \mathbb{R}$ that take a vector in $\mathbb{R}^{2(n-1)}$ and return the perimeter and the area of the convex hull of the fixed object and the translated copies of the $n - 1$ other objects. Ahn and Cheong [1] showed that for the perimeter case, this function is convex. They also showed that for the area case, the function is convex if $n = 2$. However, this is no longer true when $n > 2$, as the following example shows. Let $s_1$ be a vertical segment of length one, and let $s_2$ and $s_3$ be copies of $s_1$ rotated by $60^\circ$ and $120^\circ$. Then the area of their convex hull is minimized when they form an equilateral triangle, so there are two isolated local minima, as shown in Fig. 1. This explains why minimizing the perimeter appears to be a much easier problem than minimizing the area of a translation cover.

As a special case of translation covers, we can consider the situation where the family $\mathfrak{F}$ consists of copies of a given compact convex figure $G$, rotated by all angles in $[0, 2\pi)$. In other words, we are asking for a smallest possible convex set $K$ such that $G$ can be placed in $K$ in every possible orientation. We will call such a translation cover a keyhole for $G$ (since a key can be turned fully in a keyhole, it can certainly be placed in every possible orientation).

A classical keyhole or translation cover problem is the Kakeya needle problem. It asks for a minimum area region in the plane, a so-called Kakeya set, in which a needle of length 1 can be rotated through $360^\circ$ continuously, and return to its initial position. (See Fig. 2.) This question was first posed, for convex regions, by Soichi Kakeya in 1917 [15]. Pál [18] showed that the solution of Kakeya’s problem for convex sets is the equilateral triangle of height one, having area $1/\sqrt{3}$. With our terminology, he characterized the smallest-area keyhole for a line segment.

For the general case, when the Kakeya set is not necessarily convex or even simply connected, the answer was thought to be a deltoid with area $\pi/8$. However, Besicovitch gave the surprising answer that one could rotate a needle using an arbitrary small area [3, 4].
Fig. 1 The area function \( \omega : \mathbb{R}^{2(n-1)} \rightarrow \mathbb{R} \) of the convex hull of \( n \geq 3 \) segments is not necessarily convex.

Fig. 2 Within a Kakeya set (shaded), a needle can be rotated through 360°.

Fig. 3 (a) A needle can be translated to any location using arbitrarily small area. (b) There is an open subset of the plane of arbitrary small area which contains a unit line segment in every direction.

Besicovitch’s solution builds upon two basic observations [25]. The first observation is that one can translate any needle to any location using arbitrarily small area. The idea is to slide the needle, rotate it, slide it back and then rotate it back, as illustrated in Fig. 3(a). The area can be made arbitrarily small by sliding the needle over a large distance. The second observation is that one can construct an open subset of the plane of arbitrary small area, which contains a unit line segment in every direction, as illustrated in Fig. 3(b). The original construction by Besicovitch [3, 4] has been simplified by Perron [19], Rademacher [20], Schoenberg [22, 23], Besicovitch [5, 6] and Fisher [13].

Bezdek and Connelly [7] surveyed results on minimum-perimeter and minimum-area translation covers. For the family of closed curves of length at most one, they proved that smallest-perimeter translation covers are exactly the convex sets of constant width 1/2. The corresponding problem for minimizing the area, known as Wetzel’s problem, is still open [7, 28]. For the family of sets of diameter at most one, Bezdek and Connelly [8] proved that the unique minimum-perimeter translation cover is the circle of radius 1/\( \sqrt{3} \). More precisely, they proved that this circle is the unique smallest-perimeter keyhole for the equilateral triangle of side length one. By
Jung’s theorem [14], this circle contains any set of diameter one, and so the translation cover result follows.

Recently, Kakeya-type problems have received considerable attention due to their many applications. There are strong connections between Kakeya-type problems and problems in number theory [9], geometric combinatorics [29], arithmetic combinatorics [16], oscillatory integrals, and the analysis of dispersive and wave equations [25].

In this paper, we first generalize Pál’s result [18] in the following way: For any family $\mathcal{F}$ of line segments in the plane, there is a triangle that is a minimum-area translation cover for $\mathcal{F}$.

**Theorem 1** Let $\mathcal{F}$ be a set of line segments in the plane, and let $P$ be a convex translation cover for $\mathcal{F}$. Then there is a translation cover $T$ for $\mathcal{F}$ which is a triangle, and such that the area of $T$ is less than or equal to the area of $P$.

With this characterization in hand, we can efficiently compute a smallest area translation cover for a given family of $n$ line segments. Our algorithm runs in time $O(n \log n)$, which we prove to be optimal in the algebraic computation tree model. It is based on the problem of finding a smallest-area affine-regular hexagon containing a given centrally symmetric polygon, a problem that is interesting in its own right. As far as we know, except for some trivial cases, previously known algorithms for finding smallest-area translation covers have a running time exponential in $n$, the number of input objects [1, 27].

As observed above, minimizing the perimeter of a translation cover is much easier. Let $\mathcal{F}$ be a family of centrally symmetric convex figures. We prove that if we translate each figure such that its center of symmetry is the origin, then the convex hull of their union is a smallest-perimeter translation cover for $\mathcal{F}$.

This immediately implies that a circle with diameter 1 is a smallest-perimeter keyhole for the unit-length segment. For figures $G$ that are not centrally symmetric, this argument no longer works. We generalize the result by Bezdek and Connelly [8] mentioned above and prove the following theorem (Bezdek and Connelly’s result is the special case where $G$ is an equilateral triangle):

**Theorem 2** Let $G$ be a compact convex set in the plane, and let $\mathcal{G}$ be the family of all the rotated copies of $G$ by angles in $[0, 2\pi)$. Then the smallest enclosing disk of $G$ is a smallest-perimeter translation cover for $\mathcal{G}$.

## 2 Preliminaries

An oval is a compact convex figure in the plane. For an oval $P$, let $w_P : [0, \pi] \to \mathbb{R}$ denote the width function of $P$. The value $w_P(\theta)$ is the length of the projection of $P$ on a line with slope $\theta$ (that is, a line that makes angle $\theta$ with the $x$-axis). Let $|P|$ denote the area of $P$.

For two ovals $P$ and $Q$, we write $w_P \geq w_Q$ or $w_Q \leq w_P$ to mean pointwise domination, that is for every $\theta \in [0, \pi)$ we have $w_P(\theta) \geq w_Q(\theta)$. We also write $w_P = w_Q$ if and only if both $w_P \leq w_Q$ and $w_Q \leq w_P$ hold.
The Minkowski symmetrization of an oval $P$ is the oval $\bar{P} = \frac{1}{2}(P - P) = \{\frac{1}{2}(x - y) \mid x, y \in P\}$. It is well known and easy to show that $\bar{P}$ is centrally symmetric around the origin, and that $w(\bar{P}) = w_P$.

An oval $D$ is a trigonal disk if there is a centrally symmetric hexagon $AUBVCW$ such that $D$ contains the triangle $ABC$ and is contained in the hexagon $AUBVCW$, as illustrated in Fig. 4(a). Trigonal disks were called “relative Reuleaux triangles” by Ohmann [17] and Chakerian [10], the term “trigonal disk” being due to Fejes Tóth [12] who used it in the context of packings by convex disks. A trigonal disk has three “main” vertices and three arcs connecting these main vertices. For example, the trigonal disk $D$ in Fig. 4(a) consists of three vertices $A$, $B$, and $C$, and three arcs connecting them.

Ohmann [17] and Chakerian [10] studied sets with a given fixed width function, and obtained the following result (see for instance Theorem 3’ in [10] for a proof):

**Fact 1** Given an oval $P$, there is a trigonal disk $D$ with $|D| \leq |P|$ such that $w_D = w_P$.

### 3 Minimum Area for a Family of Segments

In this section we will prove Theorem 1. The proof contains two parts. First we prove that for every oval $P$ there exists a triangle $T$ with $|T| \leq |P|$ and $w_T \geq w_P$ (Theorem 3). The second part is to prove that for an oval $P$ and a closed segment $s$, if $w_s \leq w_P$ then $P$ contains a translated copy of $s$ (Lemma 1).

**Theorem 3** Given an oval $P$, there exists a triangle $T$ with $|T| \leq |P|$ and $w_T \geq w_P$. 

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Proof Let $\mathcal{D}$ be the set of trigonal disks $D$ such that we have $|D| \leq |P|$ and $w_D \geq w_P$. The set $\mathcal{D}$ is nonempty by Fact 1. Consider the three arcs connecting the main vertices of a trigonal disk in $\mathcal{D}$. Each arc can be straight, or not. We choose a trigonal disk $D \in \mathcal{D}$ with a maximum number of straight arcs. We show that $D$ is a triangle.

Let $AUBVCW$ be the hexagon from the definition of the trigonal disk $D$, and assume for a contradiction that $D$ is not a triangle, that is, there is at least one non-straight arc among the three arcs connecting $A$, $B$, and $C$. See Fig. 4(a). Without loss of generality, we assume that the arc connecting $A$ and $B$ is not straight.

Let the sides $AW$ and $BV$ be vertical, with $C$ above the line $AB$. Let $X$ be the point of $D$ below $AB$ with the largest vertical distance $d$ from the line $AB$. Let $C'$ be the point vertically above $C$ at distance $d$ from $C$. Let $D'$ be the convex hull of the part of $D$ above the line $AB$ and the point $C'$. It is not difficult to see that $D'$ is also a trigonal disk: Let $U'$ be the point vertically below $U$ at distance $d$ from $U$. Then the hexagon $AU'BVCW$ is centrally symmetric and contains $D'$. Clearly $D'$ contains the triangle $ABC'$. See Fig. 4(b). We observe that if $AC$ is a straight arc in $D$, then $A' = A$ and $A'C'$ is a straight arc in $D'$. Similarly, if $BC$ is a straight arc in $D$, then $B'C'$ is a straight arc in $D'$. Since $AB$ is a straight arc in $D'$, the trigonal disk $D'$ has at least one straight arc more than $D$.

We show next that $|D'| \leq |D|$. The area of $D' \setminus D$ is bounded by the area of the two triangles $A'C'C$ and $B'C'C$, where $A'$ and $B'$ are points on $D$ such that $A'C'$ and $B'C'$ are tangent to $D$. This area is equal to $d/2$ times the horizontal distance between $A'$ and $B'$. But the horizontal distance between $A'$ and $B'$ is at most the horizontal distance between $A$ and $B$, so the area of $D' \setminus D$ is bounded by the area of the triangle $AXB$, and we have $|D'| \leq |D|$. We also need to argue that $w_{D'} \geq w_D$. Consider a minimal strip $\mathcal{S}$ containing $D$, and a minimal strip $\mathcal{S}'$ with the same orientation for $D'$. If $\mathcal{S}$ does not touch $D$ from below between $A$ and $B$, then $\mathcal{S}'$ is at least as wide as $\mathcal{S}$. Otherwise, $\mathcal{S}$ touches $D$ from below at a point $Y$ between $A$ and $B$, and touches from above at $C$, as $C$ is the only antipodal point of $D$ for $Y$. The strip $\mathcal{S}'$ is determined either by $A$ and $C'$, or by $B$ and $C'$. Since the top side of $\mathcal{S}'$ meets $C'$, it lies above the top side of $\mathcal{S}$ at distance $d$. The vertical distance between the bottom sides of $\mathcal{S}$ and $\mathcal{S}'$ is at most $d$. Hence, the width of $\mathcal{S}'$ is not less than the width $\mathcal{S}$.

Since $w_{D'} \geq w_D \geq w_P$ and $|D'| \leq |D| \leq |P|$ the trigonal disk $D'$ is a member of $\mathcal{D}$, but has more straight arcs than $D$. This contradicts our choice of $D$, and so our assumption that $D$ is not a triangle must be false.

This finishes the first part. We need the following lemma, which shows that whether or not an oval $P$ contains a translated copy of a given segment $s$ can be determined by looking at the width functions of $P$ and $s$ alone:

**Lemma 1** Let $s$ be a segment in the plane, and let $P$ be an oval. $P$ contains a translated copy of $s$ if and only if $w_s \leq w_P$.

*Proof* Without loss of generality, let $s$ be a horizontal segment. The “only if” part follows immediately from the definition of the width function of an oval.
It remains to prove the “if” direction. Let $pq$ be a horizontal segment of maximal length contained in $P$. Then $P$ has a pair of parallel tangents $\ell_1$ and $\ell_2$ through $p$ and $q$. By the assumption $w_s \leq w_P$, the distance between $\ell_1$ and $\ell_2$ must be large enough to place $s$ in between the two lines. But this implies that the segment $pq$ is at least as long as $s$, and $s$ can be placed on the segment $pq$ in $P$. □

To prove Theorem 1, let $P$ be an oval that contains a translated copy of every $s \in \mathcal{F}$. By Theorem 3 there is a triangle $T$ such that $|T| \leq |P|$ and $w_T \geq w_P$. Let $s \in \mathcal{F}$. Using the result in Lemma 1 we can argue as follows. Since there is a translated copy of $s$ contained in $P$, we must have $w_s \leq w_P \leq w_T$. Thus, there is then a translated copy of $s$ contained in $T$.

4 From Triangles to Hexagons

We now turn to the computational problem: Given a family $\mathcal{F}$ of line segments, find a smallest-area convex set that contains a translated copy of every $s \in \mathcal{F}$.

By Theorem 1 we can choose the answer to be a triangle. In this section we show that this problem is equivalent to finding a smallest-area affine-regular hexagon enclosing some centrally symmetric convex figure. An affine-regular hexagon is the image of a regular hexagon under a non-singular affine transformation. An affine-regular hexagon is centrally symmetric, but not every centrally symmetric hexagon is affine-regular: For instance, in an affine-regular hexagon, the diagonals are parallel to a hexagon side, and have twice the length of that side. In this paper, we only consider affine-regular hexagons that are centrally symmetric about the origin, so by abuse of terminology, we will write affine-regular hexagon for an affine-regular hexagon that is centrally symmetric about the origin.

The basic insight is that for centrally symmetric figures, comparing width-functions is equivalent to inclusion:

Lemma 2 Let $P$ and $Q$ be ovals centrally symmetric about the origin. Then $w_P \leq w_Q$ if and only if $P \subset Q$.

Proof One direction is trivial, so consider for a contradiction the case where $w_P \leq w_Q$ and $P \not\subset Q$. Then there is a point $p \in P \setminus Q$. Since $Q$ is convex, there is a line $\ell$ that separates $p$ from $Q$. Since $P$ and $Q$ are centrally symmetric, this means that $Q$ is contained in the strip bounded by the lines $\ell$ and $-\ell$, while $P$ contains the points $p$ and $-p$ lying outside this strip. This implies that for the orientation $\theta$ orthogonal to $\ell$ we have $w_P(\theta) > w_Q(\theta)$, a contradiction. □

Recall that $\bar{P}$ denotes the Minkowski symmetrization of an oval $P$.

Lemma 3 Let $T$ be a non-degenerate triangle. Then $\bar{T}$ is an affine-regular hexagon, and $|\bar{T}| = \frac{3}{2} |T|$. Every affine-regular hexagon $H$ can be expressed in this form.
Proof Since every non-degenerate triangle is the affine image of an equilateral triangle, it suffices to observe this relationship for the equilateral triangle and the regular hexagon. □

Since \( w_P = w_{\bar{P}} \), \( w_T = w_{\bar{T}} \), and by Lemmas 2 and 3, we immediately have

**Lemma 4** Given an oval \( P \), a triangle \( T \) is a smallest-area triangle with \( w_T \geq w_P \) if and only if \( \bar{T} \) is a smallest-area affine regular hexagon with \( \bar{P} \subset \bar{T} \).

This leads us to the following algorithm. In Sect. 6, we will show that the time bound is tight.

**Theorem 4** Let \( \mathcal{F} \) be a set of \( n \) line segments in the plane. Then we can find in \( O(n \log n) \) time a triangle \( T \) that is a minimum-area convex translation cover for \( \mathcal{F} \).

Proof Given a family \( \mathcal{F} \) of \( n \) line segments, place every \( s \in \mathcal{F} \) with its center at the origin. Let \( P \) be the convex hull of these translated copies. \( P \) can be computed in \( O(n \log n) \) time, and is a centrally symmetric convex polygon with at most \( 2n \) vertices. We then compute a smallest area affine-regular hexagon \( H \) containing \( P \). In the next section we will show that this can be done in time \( O(n) \). Finally, we return a triangle \( T \) with \( \bar{T} = H \). The correctness of the algorithm follows from \( w_P(\theta) = \max_{s \in \mathcal{F}} w_s(\theta) \) and Lemma 4. □

### 5 Algorithm for Computing the Smallest Enclosing Affine-Regular Hexagon

In this section we discuss the following problem: Given a convex polygon \( P \), centrally symmetric about the origin, find a smallest-area affine-regular hexagon \( H \) such that \( P \subset H \).

Let us first sketch a simple quadratic-time algorithm: The affine-regular hexagons centered at the origin are exactly the images of a regular hexagon centered at the origin under a non-singular linear transformation. Instead of minimizing the hexagon, we can fix a regular hexagon \( H \) with center at the origin, and find a linear transformation \( \sigma \) such that \( \sigma P \subset H \) and such that the determinant of \( \sigma \) is maximized. The transformation \( \sigma \) can be expressed as a \( 2 \times 2 \) matrix with coefficients \( a, b, c, d \). The condition \( \sigma P \subset H \) can then be written as a set of \( 6n \) linear inequalities in the four unknowns \( a, b, c, d \). We want to find a feasible solution that maximizes the determinant \( ad - bc \), a quadratic, non-concave expression. This can be done by computing the 4-dimensional polytope of feasible solutions, and considering every facet of this polytope in turn. We triangulate each facet, and solve the maximization problem on each simplex of the triangulation.

In the following, we show that the problem can in fact be solved in linear time.

For a set \( S \subset \mathbb{R}^2 \), let \( S^\circ = -S \) denote the mirror image with respect to the origin. A **strip** is the area bounded by a line \( \ell \) and its mirror image \( \ell^\circ \).

For the below discussion we fix \( P \). An affine-regular hexagon \( H \) is the intersection of three strips \( \mathcal{G}_1, \mathcal{G}_2, \) and \( \mathcal{G}_3 \), as in Fig. 5, where the sides of \( H \) are supported by \( \mathcal{G}_1, \mathcal{G}_2, \) and \( \mathcal{G}_3 \).
The hexagon $H$ is defined by three strips $S_1$, $S_2$, and $S_3$ in counter-clockwise order. The intersection $S_1 \cap S_2$ is a parallelogram $Q = ABA^\circ B^\circ$. Since $H$ is affine-regular, the sides supported by $S_3$ must be parallel to and half the length of $BB^\circ$, and so $S_3$ is uniquely defined by $S_1$ and $S_2$: It supports the sides $UV$ and $U^\circ V^\circ$ of $H$, where $U$ is the midpoint of $BA^\circ$ and $V$ is the midpoint of $A^\circ B^\circ$. Note that $|H| = 3|Q|/4$.

It is easy to see that if $H$ is a minimum-area affine-regular hexagon containing $P$, then two of the three strips must be touching $P$. Without loss of generality, we can assume these to be strips $S_1$ and $S_2$. Let us now observe what happens when we keep $S_1$ fixed and let the orientation of $S_2$ rotate through its possible range.

For convenience of presentation, let us choose a coordinate system where $S_1$ is horizontal. Since $S_1$ touches $P$, there is a vertex $p$ of $P$ on the side $V^\circ B$. For $S_2$ to touch $P$, there must be a vertex $q \in P$ on the side $BU$, and so $S_2$’s orientation can range from horizontal to the orientation of the edge of $P$ that intersects the $x$-axis.

For each orientation, we choose $S_2$ to be the smallest enclosing strip of this orientation. This implies that when we rotate the orientation counter-clockwise, then one side of $S_2$ rotates about a vertex $q$ of $P$, while the opposite side rotates about $q^\circ$, see Fig. 6. (When the strip side becomes aligned with an edge of $P$, then rotation continues around the next vertex of $P$.) Consider now the effect of a small rotation about a vertex $q$: The triangles $qBB'$ and $qA^\circ A'^\circ$ are similar, and since $q$ lies above or on the $x$-axis, we have $|qA^\circ A'^\circ| \geq |qBB'|$. This implies that the area of $Q$ is non-increasing during this rotation (it remains constant exactly if $q = U$). Since $|H| = 3|Q|/4$, the area of $H$ either decreases or remains constant as well.
Furthermore, the point $U$ moves horizontally along the $x$-axis to the right (or not at all, when $q = U$). The point $A^\circ$ moves horizontally to the right with at least twice the speed of point $U$. As $V$ is the midpoint of $A^\circ$ and $B^\circ$, this implies that $V$ moves horizontally to the right with at least the speed of $U$, and so the line $UV$ is rotating counter-clockwise. It follows that while strip $S_2$ rotates counter-clockwise, the part of $H$ lying below the $x$-axis and to the left of the line $pp^\circ$ is strictly shrinking.

We have shown:

**Lemma 5** Keeping $S_1$ horizontal and in contact with a pair of vertices $p, p^\circ$ of $P$ and rotating $S_2$ counter-clockwise through its possible range of orientations while in contact with $P$, the area $|H|$ is non-increasing. The quadrilateral $oUVp^\circ$ is strictly shrinking by inclusion.

By construction, the parallelogram $Q$ contains $P$ at any stage of this rotation. However, since $oUVp^\circ$ is strictly shrinking, there is a unique maximal orientation where $P \subset H$. At this orientation, each side of $S_3$ contains a vertex of $P$. We denote these uniquely defined strips as $S_2(S_1)$ and $S_3(S_1)$, see Fig. 7. By Lemma 5, the affine-regular hexagon defined by $S_1, S_2(S_1), S_3(S_1)$ has minimum area among all the affine-regular hexagons supported by $S_1$ and containing $P$.

This implies immediately that there exists a minimum-area affine-regular hexagon $H$ such that every side of $H$ contains a vertex of $P$. We can now show that we can in fact choose $H$ such that one of its sides contains an edge of $P$:

**Lemma 6** There exists a minimum-area affine-regular hexagon $H$ containing $P$ such that a side of $H$ contains an edge of $P$. In addition, if no minimum-area affine-regular hexagon containing $P$ shares a vertex with $P$, then every minimum-area affine-regular hexagon has a side containing an edge $P$.

**Proof** By the discussion above, we can assume that there is a minimum-area affine-regular hexagon $H$ where each side of $H$ touches $P$. If a side of $H$ contains an edge of $P$, then we are done. In the following, we thus assume that every side of $H$ intersects $P$ in a single point. Also, we assume the vertices of $H$ are $(1, 0), (1, 1), (0, 1)$ and their antipodal points $(-1, 0), (-1, -1), (0, -1)$. This can be done by applying a non-singular linear transformation, see Fig. 8.
First, we consider the case where no vertex of $H$ coincides with a vertex of $P$. We claim that in this case, there exists a non-singular linear transformation $\sigma$ such that $\sigma P \subset H$ and $|\sigma P| > |P|$ hold, implying that the inverse image $\sigma^{-1}H$ of $H$ also contains $P$ and its area $|\sigma^{-1}H|$ is strictly smaller than $|H|$, a contradiction. We denote by $a, b, c$ the three contact points as in Fig. 8. Considering $a, b, c$ as vectors in $\mathbb{R}^2$, we can write $c$ as a linear combination $c = \alpha a + \beta b$ of $a$ and $b$. We have $a = (1, s), b = (t, 1), and c = (\alpha + \beta t, \alpha s + \beta)$. Since $c$ lies in the quadrant $xy < 0$, we have $\alpha \beta \neq 0$ and $\alpha / \beta < 0$. As the point $c$ lies on the line with equation $y = x + 1$, we have $t = \frac{\alpha}{\beta} s + \frac{\beta - \alpha - 1}{\beta}$. Then the area of the triangle $oab$ is given by

$$2|oab| = 1 - \frac{\alpha}{\beta} s^2 - \frac{\beta - \alpha - 1}{\beta} s. \quad (1)$$

Assume we apply a linear transformation $\sigma$ to $P$ such that each point in $a, b, c$ moves along the side of $H$ that currently contains it. Thus, $s$ changes, and $t$ changes in such a way that $c$ remains on the same side of $H$. Then the area of $P$ is proportional to the area $|oab|$. As we observed that $\alpha / \beta < 0$, the coefficient of $s^2$ in Eq. (1) is positive, and so the area $|oab|$ cannot be at a local maximum, a contradiction.

Consider now the case where at least one of the contact points $a, b, c$ lies at a vertex of $H$. Since each side of $H$ has a single-point intersection with $P$, two of $a, b, c$ are identical. Using a suitable linear transformation, we can assume $b = c$ in Fig. 8. We now use the shear transformation $(x, y) \mapsto (x, y + \rho x)$, where $\rho > 0$ is chosen such that the edge of $P$ incident to $b$ on the right becomes horizontal. Then $\sigma P \subset H$, $|\sigma P| = |P|$, and the horizontal sides of $H$ contain an edge of $P$. \hfill $\square$

We have now reduced the search for a minimum-area affine-regular hexagon to a small set of candidates:

**Lemma 7** There exists a minimum-area affine-regular hexagon $H$ defined by three strips $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, where $\mathcal{S}_1$ supports an edge of $P$, and $\mathcal{S}_2 = \mathcal{S}_2(\mathcal{S}_1)$ and $\mathcal{S}_3 = \mathcal{S}_3(\mathcal{S}_1)$. 

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It remains to find an efficient way to compute $S_2(S_1)$ for the $n$ candidate strips $S_1$. We will give a linear-time rotating calipers [26] type algorithm. Our algorithm enumerates the possible strips $S_1$ by enumerating the edges of $P$ in counter-clockwise order, and maintains the strip $S_2(S_1)$ during the process.

The correctness of the algorithm hinges on the following description of the changes in the orientation of $S_2$ and $S_3$ as $S_1$ rotates around $P$.

**Lemma 8** As $S_1$ rotates counter-clockwise, the corresponding strips $S_2(S_1)$ and $S_3(S_1)$ rotate (non-strictly) counter-clockwise.

**Proof** We start with a configuration where $S_2 = S_2(S_1)$ (and therefore $S_3 = S_3(S_1)$). We rotate $S_1$ slightly, while keeping $S_2$ fixed. The top side of $S_1$ rotates around a vertex $p \in P$. The points $B$ and $A^\circ$ move downwards along the line $BA^\circ$, see Fig. 9. The point $V$ moves downwards along the line $oV$, parallel to $BA^\circ$ (or doesn’t move at all if $p^\circ = V$). It follows that the new side $U'V'$ lies outside the old hexagon $H$ and touches it only in $V'$, and so $U'V'$ cannot possibly intersect the interior of $P$. By Lemma 5, this implies that strip $S_2$ now needs to rotate counter-clockwise or not at all to obtain $S_2(S_1)$.

Furthermore, similar to the arguments above, we observe that $A^\circ$ moves with speed at least twice the speed of $V$. Since $U$ is the midpoint of $BA^\circ$, it moves with at least half the speed of $A^\circ$, so $U$ moves with speed at least equal to the speed of $V$. Since $U$ and $V$ move on parallel lines, it follows that the line $UV$ is rotating counter-clockwise, and so $S_3$ rotates counter-clockwise during the rotation of $S_1$.

We are now ready to describe our algorithm.

**Theorem 5** Given a centrally-symmetric convex 2n-gon $P$, a smallest-area affine-regular hexagon enclosing $P$ can be found in time $O(n)$.

**Proof** As mentioned above, we use a rotating calipers [26] type algorithm. It maintains an edge $e$ of $P$ defining $S_1$, a second strip $S_2$ and the vertex $q$ of $P$ where $S_2$ touches $P$, and a vertex $r$ of $P$. Let $H = BUVB^\circ U^\circ V^\circ$ be the hexagon defined by $S_1$ and $S_2$, as in Fig. 5. The algorithm maintains the invariant that $P$ has a supporting line in $r$ that is parallel to $UV$. The algorithm proceeds by rotating $S_2$ counter-clockwise (as in Fig. 6) until $S_2 = S_2(S_1)$, and then proceeds to the next edge defining the next $S_1$.
We initialize \( e \) to an arbitrary edge of \( P \). Let \( e \) be horizontal for ease of presentation, with \( P \) below \( e \). Let \( q \) be the left endpoint of \( e \), and let \( r \) be the leftmost vertex of \( P \) (the lower one, if \( P \) has a vertical edge).

In the initial configuration, \( \mathcal{S}_2 \) is obtained from \( \mathcal{S}_1 \) by a counter-clockwise rotation around \( q \) by an infinitely small amount. This implies that \( B = q \), and that the side \( UV \) is nearly vertical, so the invariant for \( r \) holds.

We now rotate \( \mathcal{S}_2 \) counter-clockwise around the vertex \( q \), until one of the following events occurs:

- If \( r \) no longer supports a tangent to \( P \) parallel to \( UV \), replace \( r \) by the counter-clockwise next vertex of \( P \), and continue rotating \( \mathcal{S}_2 \).
- If \( \mathcal{S}_2 \) supports an edge of \( P \), then replace \( q \) by the counter-clockwise next vertex of \( P \), and continue rotating \( \mathcal{S}_2 \).
- If \( UV \) touches \( r \), or if \( q = U \) and \( UV \) supports the edge of \( P \) that starts counter-clockwise from \( q \), then we have found the strip \( \mathcal{S}_2(\mathcal{S}_1) \). We compute its area and update a running minimum. Then we replace \( e \) with the counter-clockwise next edge of \( P \). As long as \( r \) does not support a tangent to \( P \) parallel to \( UV \), we replace \( r \) by the counter-clockwise next vertex of \( P \). Then continue rotating \( \mathcal{S}_2 \).

The algorithm ends when \( n \) edges have been considered. Its running time is clearly linear. We still need to prove correctness. By Lemma 7, it suffices to enumerate all the strips \( \mathcal{S}_1 \) that support an edge of \( P \), and the corresponding \( \mathcal{S}_2(\mathcal{S}_1) \). Our algorithm considers all such strips \( \mathcal{S}_1 \) in counterclockwise order, and Lemma 8 shows that it suffices to consider the strips \( \mathcal{S}_2 \) in counter-clockwise order, as our algorithm does. □

## 6 Lower Bound for Computing a Translation Cover

In this section, we prove an \( \Omega(n \log n) \) lower bound for the problem of computing a minimum-area translation cover for a set of \( n \) line segments. We first need the following result on regular 6n-gons (see Fig. 10(a)).

![Fig. 10](image_url) Proof of Lemma 9. (a) An optimal enclosing hexagon \( H^* \) and the regular 18-gon \( R \). (b) When \( H \) and \( R \) share two vertices, the area of \( H \) is larger than the area of \( H^* \). (c) An affine-regular enclosing hexagon \( H \). (d) The hexagon \( H' \)

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Lemma 9 Let \( R \) denote a regular \( 6n \)-gon centered at the origin, for some integer \( n \geq 1 \). Then any minimum-area affine-regular hexagon enclosing \( R \) is a regular hexagon such that every side of this hexagon contains an edge of \( R \).

Proof The statement is trivial for \( n = 1 \), so assume \( n \geq 2 \). Let \( H^* \) denote a regular hexagon enclosing \( R \), and such that each side of \( H^* \) contains an edge of \( R \). Let \( H \) denote another smallest affine-regular hexagon enclosing \( R \). We will argue that \( H \) is also a regular hexagon whose sides contain edges of \( R \).

We first rule out the case where \( H \) shares a vertex with \( R \). For sake of contradiction, assume that \( H \) shares two opposite vertices \( V \) and \( V' \) of \( R \). Without loss of generality, we assume that \( V, V' \) are on the \( x \)-axis. The sides \( e, e' \) of \( H \) that are not adjacent to \( V, V' \) are parallel to \( VV' \) and have half the length of \( VV' \). In addition, the sides of \( H \) that are adjacent to \( V \) and \( V' \) make an angle at most \( \pi/12 \) with the \( y \)-axis. (See Fig. 10(b).) Then a direct calculation shows that \(|H| > |H^*|\), a contradiction.

Thus, by Lemma 6, we know that a side \( e \) of \( H \) contains an edge of \( R \). Without loss of generality we assume that this side is parallel to the \( x \)-axis. (See Fig. 10(c).) For sake of contradiction, assume that \( e \) is not symmetric with respect to the \( y \)-axis. Consider the hexagon \( H' \) that is obtained from \( H \) by a horizontal shear transformation that moves \( e \) and the opposite side parallel to the \( x \)-axis, until they are centered at the \( y \)-axis. Then \( H' \) (see Fig. 10(d)) is an affine-regular hexagon containing \( R \) that is symmetric with respect to the \( y \)-axis and that only touches \( R \) along its top and bottom edges. This implies that \( H' \) strictly contains a regular hexagon \( H^* \) enclosing \( R \), and hence \(|H| = |H'| > |H^*|\), a contradiction.

Therefore, \( e \) is symmetric with respect to the \( y \)-axis, and thus \( H \) is symmetric with respect to the \( y \)-axis. Only one such affine-regular hexagon is circumscribed to \( R \), so \( H = H^* \). □

We are now able to prove our lower bound.

Theorem 6 In the algebraic computation tree model, and in the worst case, it takes \( \Omega(n \log n) \) time to compute a minimum-area translation cover for a family \( F \) of \( n \) line segments in the plane.

Proof For an interval \( I \subset \mathbb{R} \), we denote by \( C_I \) the arc of the unit circle corresponding with polar angles in the interval \( I \), that is \( C_I = \{(\cos \theta, \sin \theta) \mid \theta \in I \} \). As \( C_I \) is the intersection of a circle and a cone, a node of an algebraic computation tree can decide whether a point lies in \( C_I \).

We use a reduction from the following problem. The input is a set of points \( p_1, \ldots, p_n \in C_{[0, \pi/3]} \). The goal is to decide whether there exists an integer \( 0 \leq k < n \) such that \( C_{(k\pi/3n, (k+1)\pi/3n)} \) is empty, that is, this arc does not contain any point \( p_i \). It follows from Ben-Or’s bound [2] that any algebraic computation tree that decides this problem has depth \( \Omega(n \log n) \). (The set of negative instances has at least \( n! \) connected components: To each permutation \( \sigma \) of \( 1, \ldots, n \), we associate a negative instance where each \( p_i \) lies in the \( \sigma_i \)'s arc. In order to move continuously from one of these configuration to another, we must have a crossing \( p_i = p_j \), which implies that one interval is empty by the pigeonhole principle, and thus the instance is positive.)
Our construction is as follows. Consider the (fixed) regular $6n$-gon $R$, whose vertices are $r_k = (\cos(k\pi/3n), \sin(k\pi/3n))$ for $k = 1, \ldots, 6n$. Let $P$ denote the convex $12n$-gon whose vertices are the vertices of $R$ and all the rotated copies of the points $p_1, \ldots, p_n$ by angles $0, \pi/3, \ldots, 5\pi/3$ around the origin.

If there is an integer $k = 0, \ldots, n - 1$ such that $C(k\pi/3n, (k+1)\pi/3n)$ is empty, then by Lemma 6, the regular hexagon containing $R$ whose sides contain the edge $r_k r_{k+1}$ and its rotated copies by angles $0, \pi/3, \ldots, 5\pi/3$ is a minimum area affine-regular hexagon containing $P$.

If on the other hand, for every integer $k \in \{0, \ldots, n - 1\}$ the arc $C(k\pi/3n, (k+1)\pi/3n)$ is nonempty, then by Lemma 6, any minimum-area affine hexagon containing $R$ is a regular hexagon whose sides contain edges of $R$, and thus it cannot contain $P$.

So we have proved that, when some arc $C(k\pi/3n, (k+1)\pi/3n)$ is empty, then a minimum-area hexagon containing $P$ has area $|H^*|$, where $H^*$ is a minimum-area hexagon containing $R$. Otherwise, if all these arcs are non-empty, then the minimum area is larger than $|H^*|$.

Thus, if we could compute in $o(n \log n)$ time a minimum-area convex translation cover for the diagonals of $P$, then by Lemma 4 we would also get in $o(n \log n)$ time the area of a smallest enclosing affine-regular hexagon containing $P$, and then we would be able to decide in $o(n \log n)$ time whether there exists an empty arc $C(k\pi/3n, (k+1)\pi/3n)$, a contradiction.

\section{Minimizing the Perimeter}

If we wish to minimize the perimeter instead of the area, the problem becomes much easier: it suffices to translate all segments so that their midpoints are at the origin, and take the convex hull of the translated segments. This follows from the following more general result.

\textbf{Theorem 7} Let $C$ be a family of centrally symmetric convex figures. Under translations, the perimeter of the convex hull of their union is minimized when the centers coincide.

\textbf{Proof} By the Cauchy-Crofton formula [11], the perimeter is the integral of the width of the projection over all directions. We argue that the width is minimized when the centers coincide, for all directions simultaneously, implying the claim.

When the figures are not symmetric, our proof of Theorem 7 breaks down. However, we are able to solve the problem for a family consisting of all the rotated copies of a given oval. (Remember that an oval is a compact convex set.) The following theorem was already stated in the introduction.

\textbf{Theorem 2} Let $G$ be an oval, and let $\mathcal{G}$ be the family of all the rotated copies of $G$ by angles in $[0, 2\pi)$. Then the smallest enclosing disk of $G$ is a smallest-perimeter translation cover for $\mathcal{G}$.
Proof We observe first that, if $G$ is a segment, then by Theorem 7, the smallest enclosing disk of $G$ is a smallest-perimeter translation cover for $G$.

Consider next the case where $G$ is an acute triangle. Choose a coordinate system with origin at the center of the circumcircle of $G$, and such that the circumcircle has radius one. We wish to prove that any translation cover for $G$ must have perimeter at least $2\pi$, implying that the circumcircle is optimal.

We borrow an idea of Bezdek and Connelly [8]. Let $v_1, v_2, v_3$ be the three vertices of $G$. By our assumptions, the origin lies in the interior of their convex hull, and the three vectors have length one. The origin can be expressed as a convex combination $0 = \sum_{i=1}^{3} \alpha_i v_i$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{3} \alpha_i = 1$. Let $\delta_i$, for $i = 1, 2, 3$, be the angle formed by $ovi$ and the positive $x$-axis.

Let $K$ be a translation cover for $G$ and let $h$ be the support function [21] of $K$. That is, $h(u) = \sup \{ \langle x, u \rangle \mid x \in K \}$ for any unit vector $u$. We denote by $u_\theta = (\cos \theta, \sin \theta)$ the unit vector making angle $\theta$ with the positive $x$-axis, so that $v_i = u_\delta_i$.

The length $\lambda$ of the perimeter of $K$ is equal to the integral over the support function [24]

$$\lambda = \int_{0}^{2\pi} h(u_\theta)d\theta.$$ 

Since $\theta \mapsto h(u_\theta)$ is a periodic function with period $2\pi$, we have

$$\lambda = \int_{0}^{2\pi} h(u_\theta)d\theta = \int_{0}^{2\pi+\delta_i} h(u_\theta)d\theta = \int_{0}^{2\pi} h(u_\theta+\delta_i)d\theta.$$

It follows that

$$\lambda = \sum_{i=1}^{3} \alpha_i \lambda = \sum_{i=1}^{3} \alpha_i \int_{0}^{2\pi} h(u_\theta+\delta_i)d\theta = \int_{0}^{2\pi} \left( \sum_{i=1}^{3} \alpha_i h(u_\theta+\delta_i) \right)d\theta. \quad (2)$$

Consider now a fixed orientation $\theta$. The translation cover $K$ must contain a rotated copy $G(\theta)$ of $G$ such that, for some translation vector $c(\theta)$, the vertices of $G(\theta)$ are the points $v_i(\theta) = c(\theta) + u_\theta+\delta_i$ for $i = 1, 2, 3$.

Since $v_i(\theta)$ lies in $K$, the value of the support function $h(u_\theta+\delta_i)$ is lower bounded by

$$h(u_\theta+\delta_i) \geq \langle v_i(\theta), u_\theta+\delta_i \rangle = \langle c(\theta) + u_\theta+\delta_i, u_\theta+\delta_i \rangle = \langle c(\theta), u_\theta+\delta_i \rangle + 1 \quad (3)$$

and thus

$$\sum_{i=1}^{3} \alpha_i h(u_\theta+\delta_i) \geq 1 + \left( c(\theta), \sum_{i=1}^{3} \alpha_i u_\theta+\delta_i \right) = 1 + \langle c(\theta), 0 \rangle = 1.$$ 

Plugging this into Eq. (2) gives $\lambda \geq 2\pi$.

Consider finally the general case where $G$ is an arbitrary compact convex figure, and let $D$ be the smallest enclosing disk of $G$. Either $D$ touches $G$ in two points that form a diameter of $D$, or $D$ touches $G$ in three points that form an acute triangle. In both cases, our previous results imply that $D$ is a smallest-perimeter translation cover for either the segment or the triangle, and therefore for $G$. \qed
The minimum enclosing circle is not always the unique minimum-perimeter keyhole: For instance, when $G$ is a unit line segment, then any set of constant width is a solution. In the theorem below, we show that when $G$ is an acute triangle, then its circumcircle is the unique solution. This generalizes directly to any figure $G$ that touches its circumcircle at 3 points.

**Theorem 8** If $G$ is an acute triangle, then its smallest enclosing disk is the unique smallest-perimeter translation cover for the family of all rotated copies of $G$.

**Proof** We use the same notations as in the proof of Theorem 2: $K$ is a smallest-perimeter translation cover for $G$. For any $\theta$, it contains at least one copy of $G$ rotated by angle $\theta$. We call $G(\theta)$ one such rotated copy. The vertices of $G(\theta)$ are the points $v_i(\theta) = c(\theta) + u_{\theta+i\delta}$, for $i = 1, 2, 3$.

We will prove that all the triangles $G(\theta)$ have the same circumcircle. Our strategy is to show that the function $\theta \mapsto c(\theta)$ is differentiable and its derivative is 0. Without loss of generality, we only prove that $c'(0) = 0$, and we assume that $c(0) = 0$.

For sake of contradiction, assume that $c$ is not differentiable at 0, or it is differentiable at 0 and its derivative is nonzero. This means that we do not have $\lim_{\theta \to 0} c(\theta)/\theta = 0$. Hence, there exists an $\varepsilon > 0$ such that for any integer $n$, there exists $\theta_n \in (-1/n, 0) \cup (0, 1/n)$ with $\|c(\theta_n)/\theta_n\| > \varepsilon$. This implies $c(\theta_n) \neq 0$, and so $c(\theta_n)/\|c(\theta_n)\|$ is a sequence of unit vectors. Since the set of unit vectors is compact, there is a subsequence $(\theta_{n_k})$ such that $c(\theta_{n_k})/\|c(\theta_{n_k})\|$ converges to a unit vector $c_0$.

We denote this subsequence again as $(\theta_{n_k})$. Since $u_{\delta_1}, u_{\delta_2}, u_{\delta_3}$ span $\mathbb{R}^2$, there exists $i \in \{1, 2, 3\}$ such that $\langle c_0, u_{\delta_i} \rangle > 0$. So

$$\lim_{n \to \infty} \frac{1}{\|c(\theta_n)\|} \langle c(\theta_n), u_{\delta_i} \rangle = \langle c_0, u_{\delta_i} \rangle > 0.$$  

As $\|c(\theta_n)\| > \varepsilon \|\theta_n\|$ for all $n$, this implies that for $n$ large enough,

$$\langle c(\theta_n), u_{\delta_i} \rangle > \frac{\varepsilon \|\theta_n\|}{2} \langle c_0, u_{\delta_i} \rangle,$$

hence

$$\langle v_i(\theta_n), u_{\delta_i} \rangle = \langle c(\theta_n) + u_{\theta_n+i\delta}, u_{\delta_i} \rangle$$

$$\geq \frac{\varepsilon \|\theta_n\|}{2} \langle c_0, u_{\delta_i} \rangle + \cos(\theta_n)$$

$$= 1 + \frac{\varepsilon \|\theta_n\|}{2} \langle c_0, u_{\delta_i} \rangle - \frac{\theta_n^2}{2} + o(\theta_n^3).$$

Thus, for large enough $n$, we have $\langle v_i(\theta_n), u_{\delta_i} \rangle > 1$. Since $v_i(\theta_n) \in K$ for all $n$, this implies $h(u_{\delta_i}) > 1$. But since $c(0) = 0$, this means $h(u_{\delta_i}) > 1 + \langle c(0), u_{\delta_i} \rangle$, and so Inequality (3) in the proof of Theorem 2 is not tight. Since the support function $h$ is continuous [21], this implies that $\lambda > 2\pi$, a contradiction. \qed
8 Conclusions

In practice, it is an important question to find the smallest convex container into which a family of ovals can be translated. For the perimeter, this is answered by Theorem 7 for centrally symmetric ovals. For general ovals, it is still not difficult, as the perimeter of the convex hull is a convex function under translations [1]. This means that the problem can be solved in practice by numerical methods.

For minimizing the area, the problem appears much harder, as there can be multiple local minima. The following lemma solves a very special case.

Lemma 10 Let \( R \) be a family of axis-parallel rectangles. The area of their convex hull is minimized if their bottom left corners coincide (or equivalently if their centers coincide).

Proof Let \( C \) be the convex hull of some placement of the rectangles. For any \( x \), let \( \ell(x) \) be the length of the intersection of the vertical line at coordinate \( x \) with \( C \). The function \( x \mapsto \ell(x) \) is concave (by the Brunn-Minkowski theorem in two dimensions). For any \( z \geq 0 \), we define \( w(z) \) to be the length of the interval of all \( x \) where \( \ell(x) \geq z \).

We observe that the area of \( C \) is equal to \( \int \ell(x)dx \), which is again equal to \( \int_0^\infty w(z)dz \). We will now argue that \( w(z) \) is minimized for every \( z \) when the bottom left corners of the rectangles coincide, implying the claim.

To see this, consider the placement with coinciding bottom left corners at the origin, and the line \( y = z \). It intersects the convex hull at \( x = 0 \) and at some convex hull edge defined by two rectangles \( R_1 \) and \( R_2 \). \( w(z) \) is equal to the length of this intersection. It remains to observe that for any placement of \( R_1 \) and \( R_2 \), the convex hull of these two rectangle already enforces this value of \( w(z) \). \( \square \)

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