Invariant measures under random integral mappings and marginal distributions of fractional Lévy processes.*

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Abstract. It is shown that some convolution semigroups of infinitely divisible measures are invariant under the random integral mappings $I_{a,b}^{h,r}$ defined in (⋆) below. The converse implication is specified for the semigroups of generalized s-selfdecomposable and selfdecomposable distributions. Some application are given to the moving average fractional Lévy process (MAFLP).

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Abbreviated title: Invariant measures under random integral mappings

Let us recall that the moving average fractional Lévy process (in short, MAFLP) $(Z(t), t \in \mathbb{R})$ is given as follows

$$Z(t) := \int_{\mathbb{R}} \left( (t - s)_{+}^{\alpha} - (-s)_{+}^{\alpha} \right) dY_{\nu}(s), \quad t \geq 0,$$

where $(Y_{\nu}(t), t \in \mathbb{R})$ is a Lévy process in $\mathbb{R}^{d}$, $\nu$ is the probability distribution of the process at time $t = 1$, the parameter $\alpha$ is from the interval $(0, 1/2)$ and

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\( a_+ := \max(0, a) \) is the positive part of \( a \). We study here how the laws of \( Z(t) \), in (1), is related to the law of \( Y_\nu(1) \). Our approach to that questions is based on the so-called random integral representation (or random integral mapping). This is a technique that represents an infinitely divisible distribution, say \( \rho \), as a law of a random integral of the following form:

\[
\rho = I_{(a,b)}^{h,r}(\nu) := \mathcal{L}\left( \int_{[a,b]} h(t)dY_\nu(r(t)) \right), \quad (\star)
\]

where \((a,b] \subset \mathbb{R}^+, h: \mathbb{R}^+ \to \mathbb{R} \), \( Y_\nu(\cdot) \) is a Lévy process such that \( \mathcal{L}(Y_\nu(1)) = \nu \) and \( r: \mathbb{R}^+ \to \mathbb{R}^+ \) is a piecewise monotone time change, \((2)\)

and its limit as \( b \to \infty \); cf. Jurek (2011) for a review of the random integral mapping method and its application to characterizations of classes of infinitely divisible laws. In this context one might look at the conjectured "meta-theorem" in The Conjecture on www.math.uni.wroc.pl/~zjjurek or in Jurek (1985), p. 607 and Jurek (1988), p. 474.

We investigate classes of probability measures that are invariant under random integral mappings \( I_{(a,b)}^{h,r} \) (Proposition 1), then we characterize those generalized \( s \)-selfdecomposable measures that are, indeed, selfdecomposable ones (Proposition 2), and finally we specify our results to the moving average fractional Lévy processes (MAFLP). [ Note the remark at the end of this paper.]

1. Notations and the results. Let \( ID \) and \( ID_{\log} \) denote the class of all infinitely divisible probability measures on \( \mathbb{R}^d \) and those that integrate the logarithmic function \( \log(1 + ||x||) \), respectively. Further, let \(*\) and \( \Rightarrow \) stand for the convolution and the weak convergence of measures, respectively. Thus \((ID, *, \Rightarrow)\) becomes closed convolution subsemigroup of the semigroup of all probability measures \( \mathcal{P} \) (on \( \mathbb{R}^d \)).

Let \((Y_\nu(t), t \geq 0)\) denotes a Lévy process, i.e., a stochastic process with stationary independent increments, starting from zero, and with paths that continuous from the right and with finite left limits (in short: cadlag), such that \( \nu \) is its probability distribution at time 1: \( \mathcal{L}(Y_\nu(1)) = \nu \), where \( \nu \) can be any \( ID \) probability measure. Throughout the paper \( \mathcal{L}(X) \) will denote the probability distribution of an \( \mathbb{R}^d \)-valued random vector \( X \). Furthermore, for a probability Borel measures \( \mu \) its characteristic function \( \hat{\mu} \) is defined as

\[
\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{iy<x,x>} \mu(dx), \quad y \in \mathbb{R}^d,
\]

where \( <\cdot, \cdot> \) denotes the scalar product (or a bilinear form in case of Banach space; cf. the concluding remark).
In Section 3, formula (16), the Lévy-Khintchine representation for \( \hat{\mu} \) of \( \mu \in ID \) is recalled.

For the three parameters in (2) (i.e., the functions \( h, r \) and the interval \( (a, b] \)), let \( D_{(a,b]}^{h,r} \) denotes the domain of definition of the mapping \( I_{(a,b]}^{h,r} \). That is, the set of all infinitely divisible measures \( \nu \) (Lévy processes \( (Y_\nu(t), t \geq 0) \)) such that the integral (2) is well defined. Then the random integral mapping

\[
I_{(a,b]}^{h,r} : D_{(a,b]}^{h,r} \longrightarrow ID,
\]

is a homomorphism between the corresponding convolution semigroups because approximating \((\ast)\) in (2) by the Riemann-Stieltjes sums we get

\[
\log(I_{(a,b]}^{h,r}(\nu))(y) = \int_{(a,b]} \log \hat{\nu}(h(t)y)dr(t), \ y \in \mathbb{R}^d;
\]

cf. for more details Jurek-Vervaat (1983), Lemma 1.1 or Jurek and Mason (1993), Chapter 3.

Remark 1. Cohen and Maejima (2011) defined the integral (1) in the same way as it was in Marquardt (2006); see also the reference therein. In particular they worked in the framework of Lévy processes with finite variance, square integrable functions and Euclidean spaces. However, using the formal integration by parts we are able to define random integrals for larger class of integrands \( h \) and Lévy processes \( Y \). Moreover, still having the crucial equality (4).

From our definition of random integrals, in particular from (4), we infer the following properties:

\[
I_{(a,b]}^{h,r}(\nu_1) \ast I_{(a,b]}^{h,r}(\nu_2) = I_{(a,b]}^{h,r}(\nu_1 \ast \nu_2), \quad I_{(a,b]}^{h,r}(T_u \nu) = I_{(a,b]}^{h,r}(\nu) = T_u(I_{(a,b]}^{h,r}(\nu)) \quad (5)
\]

\[
I_{(a,b]}^{h,r}((\nu)^s) = (I_{(a,b]}^{h,r}(\nu))^s = (I_{(a,b]}^{h,s}(\nu)), \quad I_{(a,b]}^{h,r}((\nu) \lambda (b,c)) = I_{(a,b]}^{h,r}(\nu) \ast I_{(b,c]}^{h,r}(\nu) \quad (6)
\]

if \( \nu_n \Rightarrow \nu \) then \( I_{(a,b]}^{h,r}(\nu_n) \Rightarrow I_{(a,b]}^{h,r}(\nu) \), \quad (7)

where \( T_u \) is the dilation, i.e., \( T_u(x) := ux, \ u \in \mathbb{R}, \ x \in \mathbb{R}^d \) and \( s > 0 \). [Replacing the dilation \( T_u \) by a matrix (or a bounded linear operator) \( A \) in (5) we get \( A(I_{(a,b]}^{h,r}(\nu)) = I_{(a,b]}^{h,r}(A\nu) \).]

Random integrals over half-lines are defined as limits almost surely (or in distribution or in probability) over finite intervals \( (a,b] \) as \( b \to \infty \); cf. Jurek and Vervaat (1983).
Proposition 1. Let $\mathcal{K}$ be a closed convolution subsemigroup of the semigroup $ID$ (of all infinitely divisible measures) that is also closed under dilations and convolution powers (i.e., if $a \in \mathbb{R}$ and $\nu \in \mathcal{K}$ then $T_a \nu \in \mathcal{K}$ and for $c > 0$ also $\nu^{*c} \in \mathcal{K}$). Then if $\nu \in \mathcal{K} \cap D_{h,r}^{h,r}$ then $I_{(a,b)}^{h,r} (\nu) \in \mathcal{K}$.

The same holds also for improper random integrals (over half-lines or lines) provided they are well-defined.

Using the properties of (5)-(7) we get

Corollary 1. Domains of definition $D_{h,r}^{h,r}$ of random integrals $I_{(a,b)}^{h,r}$ are examples of semigroups $\mathcal{K}$ from Proposition 1.

Other, more explicite, examples of classes $\mathcal{K}$ are given in Example 1 below, after introducing some auxiliary notions and notations.

For the purpose of this note we will consider two specific random integral mappings and their corresponding semigroups.

Firstly, for $\beta > 0$ and $\nu \in ID$, let us define

$$I_{(0,1)}^{\nu,\beta} (\nu) \equiv J_{\nu}^{\beta} (\nu) := \mathcal{L} \left( \int_{(0,1]} t \, dY_{\nu}(t^\beta) \right), \quad \text{and} \quad \mathcal{U}_\beta := J_{\nu}^{\beta} (ID). \quad (8)$$

To the distributions from the semigroups $\mathcal{U}_\beta$ we refer to as the generalized $s$-selfdecomposable distributions.

Remark 2. The classes $\mathcal{U}_\beta$ were already introduced in Jurek (1988) as the limiting distributions in some schemes of summing of independent variables. The terminology has its origin in the fact that distributions from the class $\mathcal{U}_1 \equiv \mathcal{U}$ were called $s$-selfdecomposable distribution (the "s-", stands here for the shrinking operations that were used originally in the definition of $\mathcal{U}$); cf. Jurek (1981), (1985), (1988) and references therein.

Secondly, for $\nu \in ID_{\log}$ let us put

$$I_{(0,\infty)}^{0,\nu} (\nu) \equiv I(\nu) := \mathcal{L} \left( \int_{(0,\infty]} e^{-s} \, dY_{\nu}(s) \right) \quad \text{and} \quad L := \mathcal{L} (ID_{\log}). \quad (9)$$

The distributions from the semigroup $L$ are called selfdecomposable ones or Lévy class $L$ distributions. Let us stress here that the logarithmic moment guarantees the existence of the improper random integral (7); cf. Jurek-Vervaat (1983), Theorem 2.3 or Jurek-Mason (1993, Chapter III).

Remark 3. In classical probability theory the selfdecomposability is usually defined via some decomposability property or by scheme of limiting distributions. However, since Jurek-Vervaat (1983) we know that the class $L$ coincides with the class of distributions of random integrals given in (9). Hence it is used here as its definition.
Between the classes \( L, U_\beta(\beta > 0) \), the class \( G \) of all Gaussian measures and the class \( S \) of all stable probability measures we have the following proper inclusions:

\[
G \subset S \subset L \subset U_\beta \subset ID, \quad \text{i.e.,} \quad \mathcal{I}(ID_{\log}) \subset \mathcal{J}^\beta(ID).
\] (10)

Remark 4. It might be of an interest to recall here that many classical distributions in mathematical statistics such as gamma, t-Student, Fisher F etc. are in the class \( L \) but, of course, they are not stable; cf. the survey article Jurek (1997) or Jurek-Yor (2004) or the book by Bondesson (1992).

Example 1. The classes \( L \) (of the selfdecomposable distributions), \( U_\beta \) (of the generalized s-selfdecomposable distributions) and \( G \) (of the Gaussian measures) are examples of the above class \( K \). Also the Urbanik class \( L_\infty \), that coincides with the smallest closed convolution semigroup generated by all stable distributions, is an example of the class \( K \); cf. Urbanik (1973), or Jurek (2004).

From the inclusions in (10) we get that all selfdecomposable measures are generalized s-selfdecomposable ones whenever \( \beta > 0 \). With the notations described below the formula (16), we give conditions for the converse claim.

Proposition 2. Let \( \nu = [b, S, N] \in ID \) and \( \rho = [a, R, M] \in ID_{\log} \). Then the following conditions are equivalent:

(i) \( \mathcal{J}^\beta(\nu) = \mathcal{I}(\rho) \), i.e., a generalized s-selfdecomposable measure is in fact a selfdecomposable one;

(ii) \( \nu = \rho^{1/\beta} \ast \mathcal{I}(\rho) \) and, consequently, \( \nu \in ID_{\log} \);

(iii) \( \mathbb{R}^d \ni y \rightarrow \exp \beta \left[ \log \hat{\nu}(y) - \beta \int_0^1 \log \hat{\nu}(ty)t^{\beta-1}dt \right] \) is a Fourier transform of an \( ID_{\log} \) measure;

(iv) \( \int_0^1 \left( N(A) - N(s^{-1}A) \right)s^{\beta-1}ds \geq 0 \) for all Borel sets \( A \) such that \( 0 \notin A \) and \( \int_{\{||x||>1\}} \log ||x||N(dx) < \infty \).

Here we have the above condition (i) in terms of the triples from Lévy-Khintchine representation:

Corollary 2. In order that \( \mathcal{J}^\beta([b, S, N]) = \mathcal{I}([a, R, M]) \) it is necessary and sufficient that

\[
b = (\beta + 1)\beta^{-1}a + \int_{\{||x||>1\}} x ||x||^{-1}M(dx) \quad \text{and} \quad S = (\beta + 2)(2\beta)^{-1}R
\]

and \( N(A) = \beta^{-1}M(A) + \int_0^1 M(t^{-1}A)t^{-1}dt \) for all \( A \in \mathcal{B}_0 \).
2. The case of MAFLP. Now we will specify our considerations to the case of MAFLP $Z(t)$ given in (1). First of all, note that similarly as in (1), for a Lévy process $(Y_\nu(t), t \geq 0)$, putting

$$U^{(\nu)}(t) := \int_{-\infty}^{0} ((t-s)\alpha - (-s)\alpha)dY_\nu(s) \quad \text{and} \quad V^{(\nu)}(t) := \int_{0}^{t} (t-s)\alpha dY_\nu(s) \quad (11)$$

we get that

$$Z(t) = U^{(\nu)}(t) + V^{(\nu)}(t) \quad \text{and the summands are independent.} \quad (12)$$

This is so because two-sided Lévy process (i.e., with time index in $\mathbb{R}$) is defined by taking independent copies of Lévy processes on both half-lines; cf. Marquardt (2006), p. 1102.

Furthermore, using the invariance principle for Lévy processes, that is, the property that for each fixed positive $t$ we have

$$( - Y_\nu(-s), 0 \leq s \leq t ) \overset{d}{=} (Y_\nu(s), 0 \leq s \leq t) \overset{d}{=} (Y_\nu(t) - Y_\nu(t-s) - , 0 \leq s \leq t)$$

(the equality in distribution of three Lévy processes) we infer that that

$$U^{(\nu)}(t) \overset{d}{=} \int_{0}^{t} ((t+s)\alpha - s\alpha)dY_\nu(s), \quad V^{(\nu)}(t) \overset{d}{=} \int_{0}^{t} s\alpha dY_\nu(s). \quad (13)$$

Of course, from (13) and (8) we have that

$$1^{s,t/\alpha}_{(0,1]}(\nu) = I_{(0,1]}^{s,t/\alpha}(\nu) = \mathcal{L}(V^{(\nu)}(1)) \in \mathcal{U}_{1/\alpha} \quad \text{and} \quad 2 < 1/\alpha.$$  

Then for $t > 0$, the above with (5), (6) and Example 1 (for the class $\mathcal{U}_{3}$) give

$$T_{\nu}^{t}[I_{(0,1]}^{s,t/\alpha}(\nu)] = T_{\nu}^{t}[I_{(0,1]}^{s,t}(\nu)] = I_{(0,1]}^{(t,s)^{\alpha},(t,s)^{\alpha}}(\nu) = I_{(0,1]}^{(t,s)^{\alpha},(t,s)^{\alpha}}(\nu) = \mathcal{L}(V^{(\nu)}(t)) \in \mathcal{U}_{1/\alpha}, \quad (14)$$

and consequently we get

**Corollary 3.** For all infinitely divisible measures $\nu$, probability distributions of $V^{(\nu)}(t)$ are in the class $\mathcal{U}_{1/\alpha}$ of generalized $s$-selfdecomposable probability measures with $1/\alpha > 2$.

In (13) integrals $U^{(\nu)}(t)$ over half-line are defined as limits, i.e.,

$$U^{(\nu)}(t) = \lim_{b \to \infty} U^{(\nu),b}(t) := \lim_{b \to \infty} \int_{(0,b]} ((t+s)\alpha - s\alpha)dY_\nu(s) \quad (15)$$
a.s. or in distribution. Because of (11) and (13), $U^{(\nu),b}(t)$ and $V^{(\nu)}(t)$ are stochastically independent and \( \lim_{b \to \infty} [U^{(\nu),b}(t) + V^{(\nu)}(t)] = Z(t) \).

Since an integral $U^{(\nu),b}(t)$ is of the form $I_{(a,b]}^{k,r}$ we may apply Proposition 1 and get properties of marginal distributions of MAFLP summarized as follows:

**Corollary 4.** Let $\mathcal{K}$ be a closed convolution semigroup of infinitely divisible measures that is also closed under dilations and convolution powers (i.e., if $c > 0$ and $\nu \in \mathcal{K}$ then $T_c \nu \in \mathcal{K}$ and $\nu^{*c} \in \mathcal{K}$). Then

(a) if $\nu \in \mathcal{K}$ then $L[U^{(\nu),b}(t) + V^{(\nu)}(t)] \in \mathcal{K}$ for all $t > 0$;

(b) if $\nu \in \mathcal{K}$ and MAFLP $Z(.)$ is well defined then its marginal distributions $L(Z(t)) \in \mathcal{K}$ for all $t > 0$.

**Remark 5.** The above corollary (part (b)) for the case of selfdecomposable measures was also noted in Cohen and Maejima (2011).

### 3. Proofs

Recall that for infinitely divisible measures $\mu$ their characteristic functions admit the following Lévy-Khintchine representation:

$$\hat{\mu}(y) = e^{i \Phi(y)}\text{, } y \in \mathbb{R}^d,$$

and the Lévy exponents $\Phi$ are of the form

$$\Phi(y) = i < y, a > - \frac{1}{2} < y, Ry > +$$

$$\int_{\mathbb{R}^d \setminus \{0\}} [e^{i < y, x >} - 1 - i < y, x > 1_B(x)] M(dx),$$

(16)

where $a$ is a shift vector, $R$ is a covariance operator corresponding to the Gaussian part of $\mu$, $1_B$ is the indicator function of the unit ball $B$ and $M$ is a Lévy spectral measure. Since there is a one-to-one correspondence between a measure $\mu \in ID$ and the triple $a, R$ and $M$ in its Lévy-Khintchine formula (10) we will write $\mu = [a, R, M]$; also cf. the remark at the end of this paper.

From (13) and an appropriate version of (4) we get the following

**Lemma 1.** (i) If $\nu = [b, S, N]$ and $J^\beta(\nu) = [b^{(\beta)}, S^{(\beta)}, N^{(\beta)}]$ then

$$b^{(\beta)} := \frac{\beta}{\beta + 1} (b + \int_{\{||x|| > 1\}} x ||x||^{-1-\beta} N(dx)); \quad S^{(\beta)} := \frac{\beta}{2\beta+1} S;$$

$$N^{(\beta)}(A) := \int_0^1 N(t^{-1/\beta} A) \, dt = \beta \int_0^1 N(s^{-1} A) s^{\beta-1} \, ds \text{ for each } A \in \mathcal{B}_0.$$
(ii) If \( \mu = [a, R, M] \) and \( \mathcal{I}(\nu) = [a^\sim, R^\sim, M^\sim] \) then

\[
a^\sim := a + \int_{||x|| > 1} x ||x||^{-1} M(dx); \quad R^\sim := \frac{1}{2} R
\]

\[
M^\sim(A) := \int_0^\infty M(e^t A)dt = \int_0^1 M(t^{-1} A) t^{-1} dt \quad \text{for each } A \in \mathcal{B}_0;
\]

cf. Czyżewska-Jankowska and Jurek (2011), Lemma 2, and Jurek and Vervat (1983) on p. 250 for more details.

**Proof of Proposition 1.** For \( h \) of bounded variation, cadlag Lévy process \( Y \) and montone \( r \) we define here the random integral \( \ast \) as follows:

\[
\int_{(a,b]} h(t)dY(r(t)) := h(b)Y(r(b)) - h(a)Y(r(a)) - \int_{(a,b]} Y(r(t) -)dh(t)
\]

\[
= h(b)(Y(b) - Y(a)) - \int_{[a,b]} (Y(r(t) -) - Y(r(a)))dh(t), \quad (17)
\]

where \( Y(r(t) -) \) denotes the left-hand limit. Consequently, for the partition \( a = t_0 < t_1 < t_2 < \ldots < t_n = b \), the random integral (17) can be approximated by the Riemann-Stieltjes sums

\[
h(b)[Y(r(b)) - h(a)Y(r(a))] - \sum_{j=1}^n Y(r(t_j))(h(t_j) - h(t_{j-1}))
\]

\[
\quad = \sum_{j=1}^n h(t_j)(Y(r(t_j) - Y(r(t_{j-1}))). \quad (18)
\]

The summands in (18) are independent and since \( \nu = \mathcal{L}(Y(1)) \in \mathcal{K} \) we get that

\[
\mathcal{L}[h(t_j)(Y(r(t_j) - Y(r(t_{j-1})))] = T_{h(t_j)}(\mathcal{L}(Y(1)))^{s(r(t_j) - r(t_{j-1}))} \in \mathcal{K},
\]

if \( r(t_j) - r(t_{j-1}) \geq 0 \). Similarly, \( \mathcal{L}[-h(t_j)(Y(r(t_{j-1}) - Y(r(t_j)))] \in \mathcal{K} \) when \( r(t_j) - r(t_{j-1}) \leq 0 \). Closeness and semigroup property of \( \mathcal{K} \) guarantees that \( I_{[a,b]}^{h,r}(\nu) \in \mathcal{K} \), which gives the proof of Proposition 1 for finite intervals \( (a, b] \).

Since integrals on half-lines are given as weak limits of those over \( (a, b) \) as \( b \to \infty \) and \( \mathcal{K} \) is closed in weak topology we get Proposition 1 for half-lines, which completes a proof.

**Proof of Proposition 2.** (i) \( \equiv \) (ii). Let us put \( \Phi(y) := \hat{\nu}(y) \) and \( \Psi(y) := \hat{\rho}(y) \), i.e., they are the corresponding Lévy exponents. Then using (4), (8) and (9) we infer that (i) is equivalent the following identity

\[
\beta \int_0^1 \Phi(ty)t^{\beta - 1}dt = \int_0^\infty \Psi(e^{-s}y)ds = \int_0^1 \Psi(ty) \frac{dt}{t}, \quad \text{for all } y \in \mathbb{R}^d. \quad (19)
\]
Putting into above $sy$ for $y$, where $s \in \mathbb{R}$ varies and $y$ is fixed, and then substituting $w := st$ we get
\[
\int_0^s \Phi(wy)w^{\beta - 1}dw = \beta^{-1}s^\beta \int_0^s \Psi(wy)\frac{dw}{w}.
\]
Differentiating with respect to $s$ and then putting $s = 1$ we arrive at
\[
\Phi(y) = \int_0^1 \Psi(wy)\frac{dw}{w} + \beta^{-1}\Psi(y), \quad \text{for all } y,
\]
and after exponentiating both sides we get the equality (ii) in terms of Fourier transforms.

Conversely, starting with (20) and substituting $ty$ for $y$ and then integrating both sides over the unit interval with respect $dt^\beta$ we arrive at
\[
\beta \int_0^1 \Phi(ty)t^{\beta - 1}dt = \int_0^1 \Psi(ty)\frac{dt}{t},
\]
which means that $\mathcal{F}^\beta(\nu) = \mathcal{I}(\rho)$.

$(i) \equiv (ii) \Rightarrow (iii)$. Substituting $\mathcal{F}^\beta(\nu)$ for $\mathcal{I}(\rho)$ in (ii) and then taking Fourier transforms both sides we get that $\hat{\rho}$ has the form as in (iii).

$(iii) \Rightarrow (iv)$. Let $\rho \in ID_{\log}$ has Fourier transform given by (iii). Then $\rho^{\ast \beta / \beta} \in ID_{\log}$ and its Lévy spectral measure is of the form
\[
N(A) - N^{(\beta)}(A) = \beta \int_0^1 (N(A) - N(t^{-1}A))t^{\beta - 1}dt \geq 0 \quad \text{for all } A \in \mathcal{B}_0
\]
which is the claim (iv).

$(iv) \Rightarrow (i)$. Multiplying (iv) by $\beta$, and using the notation from Lemma 1 (i), we have that $0 \leq N^{(\beta)} \leq N$. Consequently, $N - N^{(\beta)}$ is a Lévy spectral measure with finite log-moment; cf. Czyżewska-Jankowska and Jurek (2011), Lemma 2 (ii). Furthermore, from Lemma 1(ii),
\[
(\beta(N - N^{(\beta)}))(A) = \beta(\int_0^1 N(t^{-1}A)t^{-1}dt - \int_0^1 \int_0^1 N(t^{-1/\beta}s^{-1}A)dt \, s^{-1}ds)
\]
\[
= \beta(\int_0^1 N(t^{-1}A)t^{-1}dt - \int_0^1 (\int_0^s N(w^{-1}A)\beta w^{\beta - 1}dw)s^{-(\beta + 1)}ds)
\]
\[
= \beta(\int_0^1 N(t^{-1}A)t^{-1}dt - \int_0^1 N(w^{-1}A)w^{\beta - 1}(w^{-\beta} - 1)dw)
\]
\[
= \beta \int_0^1 N(w^{-1}A)w^{\beta - 1}dw = N^{(\beta)}(A).
\]
Similarly, using Lemma 1 (ii), for Gaussian covariance operator we have

\[(\beta(S - S(\beta)))^\sim = (2\beta(\beta + 2)^{-1}S)^\sim = \beta(\beta + 2)^{-1}S = S(\beta).\]

Finally, applying (ii) in Lemma 1 for the shift vectors we get

\[(\beta(b - b(\beta)))^\sim = \beta(b - b(\beta)) + \beta \int_{\|x\| > 1} \frac{x}{\|x\|} N(dx) - \beta \int_{\|x\| > 1} \frac{x}{\|x\|} N(\beta)(dx)\]

\[= \beta(b - b(\beta)) + \beta \int_{\|x\| > 1} \frac{x}{\|x\|} N(dx) - \beta^2 \int_0^1 \int_{\|x\| > 1} \frac{x}{\|x\|} N(s^{-1}dx)s^{\beta-1}ds\]

\[= \beta(b - b(\beta)) + \beta \int_{\|x\| > 1} \frac{x}{\|x\|} N(dx) - \beta \int_{\|w\| > 1} \frac{w}{\|w\|} \int_0^1 \beta s^{\beta-1}N(dw)\]

\[= \beta(b - b(\beta)) + \beta \int_{\|x\| > 1} \frac{x}{\|x\|} N(dx) - \beta \int_{\|w\| > 1} \frac{w}{\|w\|} N(dw) - \beta \int_{\|w\| > 1} \frac{w}{\|w\|^{1+\beta}} N(dw)\]

\[= \beta(b + \int_{\|w\| > 1} \frac{w}{\|w\|^{1+\beta}} N(dw) - b(\beta)) = \beta((\beta + 1)\beta^{-1}b(\beta) - b(\beta)) = b(\beta).\]

All in all we have that \(\rho = [\beta(b - b(\beta)), \beta(S - S(\beta)), \beta(N - N(\beta))] \in ID_{\log}\) and \(I(\rho) = J(\nu), which completes the proof of (iv) \(\Rightarrow (i)\) and thus the proof of Proposition 2.

**Concluding remark.** Last but not least, although we presented our results for the Euclidean space \(\mathbb{R}^d\), our methods are applicable for processes and random variables with values in any real separable Banach space. For an exposition of probability on Banach spaces see Araujo-Giné (1980) or Ledoux-Talagrand (1991) and for a case of Hilbert spaces we recommend Parthasarathy (1967), Chapter VI. In particular, the crucial Lévy-Khintchine representation (16) holds true in the generality of separable infinite dimensional Banach spaces. But there is no integrability criterion for Lévy (spectral) measures \(M\) analogous that we have on Euclidean and Hilbert spaces.

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