Analytical and Numerical Modeling of Tsunami Wave Propagation for double layer state in Bore

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Abstract. Tsunami wave enters into the river bore in the landslide. Tsunami wave propagation are described in two-layer states. The velocity and amplitude of the tsunami wave propagation are calculated using the double layer. The numerical and analytical solutions are given for the nonlinear equation of motion of the wave propagation in a bore.

1. Introduction
Since the 2004 Indian Ocean tsunami disasters, tsunami related research gain momentum among the countries bordering the Indian Ocean such as India, Sri Lanka and Indonesia. The disasters was one of the deadliest natural disasters in human history, more than 2, 30,000 people were killed or missing nearly in 14 countries if Indian Ocean border and it was most deadliest earth quake tsunami was ever recorded in past 25decades or 250 years [Figure 1].

Tsunami ( Japanese word in the meaning of harbor waves) waves or some called as a seismic sea waves, is a series of waves generated by the displacement of the large amount of water, due natural disasters such as Earthquakes, volcanic eruptions and land sliding. Underwater nuclear explosions is one of artificial reason behind the generation of tsunami waves. Sea shores are less than 25 feet within the 1-2 km are at greater risk even before the warning is issued. Tsunami waves and the receding water are more dangerous to the building in the run-up zone. After tsunami, the drinking water was completely contaminated.
Tsunami waves are often referred to as tidal waves (sea or ocean waves), but the major differences between this two are tsunami waves are Earthquakes, volcanic eruptions and land sliding, whereas, tidal waves are generated by wind and gravitation force of moon. Another important different is the size of tsunami waves incomparably bigger than the tidal waves. Out in the depths of the ocean, the high of tsunami are not increasing drastically. When the travels to land, the size rapidly increase to very high value. The speed and height of tsunami waves inversely proportional to the depth of the ocean and it is independent to the distance from the source of the wave. The velocity of Tsunami waves is very high, it may travel as fast as a jet plane (1000 km /hrs) over deep waters, only slowing down when reaching shallow waters.

2. The motion of the wave fluid as a layer

From the sea floor, Tsunami waves are originated. The arbitrary change of free wave surface and the depth of the ocean are considered as a main variable for equation of motion and the continuity of wave.

The bottom landslide generates water motion from the bottom the ocean. The wave motion is assembled using the variables space, time, density and the layer of the pressure in the defined region. The General equation of wave motion and their continuity are expressed of a two-layer water surface. The single layer water surface flow also established with same parameters.

The amplitude of wave surface for their different states are $\gamma_1$ and $\gamma_2$, the continuity of wave surface break between the different states. Generated wave density of lower state and upper state $\tau_1$ and $\tau_2$. The motion of the water is the vertical of the surface of the ocean and also the surface of the ocean is considered as homogeneous.

The equation of the wave motion in the upper direction as

$$\frac{\partial P_s}{\partial K_z} = -f\lambda$$

(2.1)

The sum of the depth of the defined region in the two states with the amplitude $\gamma_1$ and $\gamma_2$ for the water surface $K_z = \gamma_1(K_x, K_y, t)$

$$-\int_{K_z}^{\gamma_1} dP_s = \int_{K_z}^{\gamma_1} f\lambda dK_z$$

(2.2)

The wave surface pressure $P_s(\gamma_1)$ is same as the pressure in the outer surface of ocean $P_0$ and pressure on the surface of water by wave motion in any depth $P(K_z)$ is considered as $P_{K_z}$.

$$P_{K_z} = P_0 + \lambda f(\gamma_1 - K_z)$$

(2.3)

Figure 1. Six of the deadliest Earthquake-tsunami

Figure 2. Waves in two states of fluid motion
For the upper surface state
\[ P_{S1} = \lambda_1 f(\gamma_1 - K_x) + P_0 \] (2.4)

For the lower surface state
\[ P_{S1} = \lambda_1 f(\gamma_1 + M_1 - \gamma_2) + \lambda_2 f(\gamma_2 - K_x) + P_0 \] (2.5)

The combined upper surface equation of continuity wave is
\[ \frac{\partial (\gamma_1 - \gamma_2)}{\partial t} = -\frac{\partial (N_1 x_1)}{\partial K_x} - \frac{\partial (N_1 y_1)}{\partial K_y} \] (2.6)

The overall depth of the upper state is
\[ N_1 = M_1 + \gamma_1 - \gamma_2 \] (2.7)

And \[ N_2 = M_2 + \gamma_2 \]

The water wave motion for the upper state in \( K_x \) axis direction can be expressed as
\[ -f \frac{\partial \gamma_1}{\partial K_x} - \frac{1}{\lambda_1} \frac{\partial P_0}{\partial K_x} + D_h \left( \frac{\partial^2 x_1}{\partial K_x^2} + \frac{\partial^2 y_1}{\partial K_y^2} \right) + \frac{1}{\lambda_1 M_1} [\zeta_{S,K_x} - r_1 x \sqrt{x^2 + y^2}] \] (2.8)

the upper state in \( K_y \) axis direction is
\[ -f \frac{\partial \gamma_2}{\partial K_y} - \frac{1}{\lambda_1} \frac{\partial P_0}{\partial K_y} + D_h \left( \frac{\partial^2 x_1}{\partial K_x^2} + \frac{\partial^2 y_1}{\partial K_y^2} \right) + \frac{1}{\lambda_1 M_1} [\zeta_{S,K_y} - r_1 y \sqrt{x^2 + y^2}] \] (2.9)

The water wave motion for the upper state in \( K_y \) axis direction can be expressed as
\[ \frac{\partial x_2}{\partial t} + x_1 \frac{\partial x_2}{\partial K_x} + y_1 \frac{\partial x_2}{\partial K_y} - g y_2 = \frac{\lambda_1}{\lambda_2} f \frac{\partial \gamma_1}{\partial K_x} - \nabla \lambda \frac{\partial \gamma_2}{\partial K_y} \\
- \frac{1}{\lambda_2} \frac{\partial P_0}{\partial K_x} + D_h \left( \frac{\partial^2 x_2}{\partial K_x^2} + \frac{\partial^2 y_2}{\partial K_y^2} \right) + \frac{1}{\lambda_2 M_2} [r_1 x_2 \sqrt{x^2 + y^2} - r_1 x \sqrt{x^2 + y^2}] \] (2.10)

the lower state in \( K_y \) axis direction is
\[ \frac{\partial y_2}{\partial t} + x_2 \frac{\partial y_2}{\partial K_x} + y_2 \frac{\partial y_2}{\partial K_y} + g y_2 = \frac{\lambda_1}{\lambda_2} f \frac{\partial \gamma_1}{\partial K_x} - \nabla \lambda \frac{\partial \gamma_2}{\partial K_y} \\
- \frac{1}{\lambda_2} \frac{\partial P_0}{\partial K_y} + D_h \left( \frac{\partial^2 y_2}{\partial K_x^2} + \frac{\partial^2 y_2}{\partial K_y^2} \right) + \frac{1}{\lambda_2 M_2} [r_1 y_2 \sqrt{x^2 + y^2} - r_1 y \sqrt{x^2 + y^2}] \] (2.11)

Velocity of the upper state and lower state are denoted as \( x_1 \) and \( x_2 \)

The circumference of the upper and lower state is \( M_1 \) and \( M_2 \)

The stress of the water surface along the \( x \) direction and \( y \) direction as \( \zeta_{S,K_x} \) and \( \zeta_{S,K_y} \)

\[ r_1 x \sqrt{x^2 + y^2} \] and \[ r_1 y \sqrt{x^2 + y^2} \] are the elements of the upper state water surface pressure.

\[ r_1 x_2 \sqrt{x_2^2 + y_2^2} \] and \[ r_1 y_2 \sqrt{x_2^2 + y_2^2} \] are the elements of the upper state water surface pressure.

\( \nabla \lambda = \lambda_2 - \lambda_1 \)

\( r_1 \) and \( r \) are the coefficient of friction force.

f- gravitational friction.
3. Tsunami wave generation for single layer state

For the single state tsunami wave generation by seafloor displacement after the first state of water wave, the convening equation are expressed as

$$\frac{\partial (y_1 - y_2)}{\partial t} = \frac{\partial (N_1 x_1)}{\partial K_x} - \frac{\partial (N_1 y_1)}{\partial K_y} = - \frac{\partial (N_2 x_2)}{\partial K_x} - \frac{\partial (N_2 y_2)}{\partial K_y}$$  \hspace{2cm} (3.1)

Where $N_1 = M + \gamma_1 - \gamma_2 = N = M + \gamma - \mu$ is the overall depth of the water surface to the seafloor movement.

The wave motion equation for the upper state of the water surface in the $K_x$ axis direction is defined as

$$\frac{\partial x}{\partial t} + x_1 \frac{\partial x}{\partial K_x} + y_1 \frac{\partial x}{\partial K_y} - gy = -f \frac{\partial y}{\partial K_x} - \frac{1}{\lambda N} r x \sqrt{x^2 + y^2}$$ \hspace{2cm} (3.2)

Lower state of the water surface in the $K_y$ axis direction is defined as

$$\frac{\partial y}{\partial t} + x_1 \frac{\partial y}{\partial K_x} + y_1 \frac{\partial y}{\partial K_y} + gx = -f \frac{\partial y}{\partial K_y} - \frac{1}{\lambda N} r y \sqrt{x^2 + y^2}$$ \hspace{2cm} (3.3)

The surface of water is considered in the wind free surface and the friction is omitted in horizontal direction. The tsunami wave is considered along the $K_x$ surface plane direction friction and the small waves occurred in the ocean surface without surface plane direction friction.

3.1 Tsunami wave generation for double layer state

The two-state water movement in the wave propagation after the seafloor movement. The equation of the double layer state in the upper and lower state is expressed as

$$\frac{\partial \gamma_1}{\partial t} - \frac{\partial \mu}{\partial t} = \frac{\partial (N_2 x_2)}{\partial K_x} - \frac{\partial (N_2 y_2)}{\partial K_y}$$ \hspace{2cm} (3.4)

3.2 Non-linear shallow water equation approximation

The water wave equation for shallow water surface of the ocean is explained with the motion in the wave channel

$$\frac{\partial x}{\partial t} + x_1 \frac{\partial x}{\partial K_x} + f \frac{\partial y}{\partial K_x} = \frac{\zeta_{u,K_x} \zeta_{u,K_x}}{N \lambda}$$ \hspace{2cm} (3.5)

$$\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} = 0$$ \hspace{2cm} (3.6)

The resistance of the motion of wave in the sea floor gives the velocity as nonlinear functions

$$\zeta_{u,K_x} = r x |x|$$ \hspace{2cm} (3.7)

The value of 'r' taken in the range of (1/100) \*4 based on the coefficient of the resistance of the wave motion

The numerical approximation w.r.t to time is given as

$$\frac{x_i^{a+1} - x_i^a}{R} + \frac{x_i^a - x_{i-1}^a}{w} xp + \frac{x_i^a - x_{i-1}^a}{w} xq = 2r \frac{x_i^a |x_i^a|}{(N_i^a + N_{i-1}^a)} - f \frac{(y_i^a - y_{i-1}^a)}{w}$$ \hspace{2cm} (3.8)

Consider $xp = \frac{1}{2} (x_i^a + |x_i^a|)$ and $xq = \frac{1}{2} (x_i^a - |x_i^a|)$

$$\frac{x_i^{a+1} - x_i^a}{R} = -x_i^a \frac{1}{2} \frac{(x_{i-1}^a + x_{i+1}^a)}{w} - x_i^a \frac{1}{2} \frac{(x_{i-1}^a + x_{i+1}^a)}{w}$$ \hspace{2cm} (3.9)

The above equation gives clear construction of the wave front from the source wave. The numerical approximation is constructed with stable parameters and higher order nonlinear gives
\[
\frac{x_i^a - x_{i-1}^a}{w} x_p + \frac{x_{i+1}^a - x_i^a}{w} x_q = \frac{1}{2}(x_i^a + |x_i^a|) \frac{x_i^a - x_{i-1}^a}{w} + \frac{1}{2}(x_i^a - |x_i^a|) \frac{x_{i+1}^a - x_i^a}{w}
\]

This is named as the equation of the vertical wave motion derivative.

The wave propagation direction characterised as \(u - xt\) if the direction is considered in positive direction. The wave propagation direction characterised as \(u + xt\) if the direction is considered in negative direction.

The adjoining equation for the geometric solution given as \(x = g(u - xt)\). The rate of movement is calculated by \(x_p\) and \(x_q\) assigned for cells in each state of mesh point \(i\).

The distance \(d_h\) assigned for equidistance of the cells for each layer state.

![Wave state](image)

**Figure 3.** The wave movement direction for the velocity at the \(i\) point and the \(a+1\) time step

The rate of movement of the wave motion along the defined directions at the time step non-linear estimation for the points between \(i\) and \(i+1\)

\[
x^+_h = \frac{d_h^+ x_{i-1}^a + (w - d_h^+) x_i^a}{w} = x_i^a - \frac{d_h^+}{w} (x_i^a - x_{i-1}^a)
\]

\(d_h^+\) is the length for each state in positive way

The Length is approximated using the formula \(d_h^+ = \frac{1}{2}(x_i^a + |x_i^a|)R\) and the \(x_h^+\) is used for the \(xp\) equation for the next level of equations.

For the negative direction of the wave motion for each cell state \(i\) also constructed as

\[
x^-_h = \frac{d_h^- x_{i+1}^a + (w - d_h^-) x_i^a}{w} = x_i^a - \frac{d_h^-}{w} (x_{i+1}^a - x_i^a)
\]

The vertical wave motion applied in the first order equations. This method gives the stable solution for the second and higher order with the initial derivative. The initial derivative is constructed using the three -point cell and the four-point cell approach.

\[
x \frac{\partial x}{\partial u} = x_p \left(\frac{x_{i-2}^a + 3x_i^a - 4x_{i-1}^a}{2w}\right) + x_q \left(\frac{-x_{i+2}^a + 4x_{i+1}^a - 3x_i^a}{2w}\right) + \Delta(w^3)
\]

(3.12)
The depth of the ocean changes due to their rate of water displacement from the seafloor and the density of the water. The sea level N is kept unchanged w.r.to time. The level depends on the variable assigned to each cell of states using the points between $i$ and $i + 1$.

Figure 4. Sea level oscillation by upward motion variable assigned to each cell of states using the points between $i$ and $i+1$

Consider the modulation in the left side of the region defined in state. The modulation taken in positive motion of displacement $x_p = \frac{1}{2}(x_i^{a+1} + |x_i^{a+1}|)$. When the modulation considered in the negative motion of displacement $x_q = \frac{1}{2}(x_i^{a+1} - |x_i^{a+1}|)$.

The whole modulation of the wave motion in left side of the region is

$$MODL_i = (x_p^{a+1} \gamma_{i-1}^a + x_q^{a+1} \gamma_i^a + \gamma_{i,2}^{a+1} (M_i, M_{i-1}, j))$$

The finite difference of the wave motion equation with their continuous progress of wave propagation in the region, the state defined in the right side of cell is

$$\gamma_i^{a+1} = \gamma_i^a - \frac{R}{w} (MODL_{i+1} - MODL_i)$$

The equations are improvised by the parameters associated with the continuous motion of the wave equation. The positive and negative rate of wave motion is at left side of the state to their region. The mean of the depth of the ocean from the length $d_R^+ = xR$. The positive modulation of the wave from the seafloor and mean of their cell points $i - 1$ and $i$ is

$$\gamma_p^a = \gamma_{i-1}^a + \frac{1}{2}(1 - 2x_p^{a+1} \frac{R}{w}) (\gamma_i^a - \gamma_{i-1}^a)$$

Negative modulation of the depth of the ocean gives

$$\gamma_q^a = \gamma_{i}^a - \frac{1}{2}(1 + 2x_p^{a+1} \frac{R}{w}) (\gamma_i^a - \gamma_{i-1}^a)$$

It can be written as
4. NUMERICAL AND ANALYTICAL COMPARISON OF WAVE PROPAGATION IN BORE

The numerical and analytical solutions of the wave propagation along the bore are experimented. In this the linear and nonlinear equation parameters are used to calculate the depth, amplitude and velocity of the wave propagation in bore. From the results of the amplitude and velocity the distance of the wave generation and wavelength are estimated. Considered the sea level and the traveling of wave are constant for the analytical approach, $M = 300 + 4.5 \times 10^{-4} u$, $M = s + tu$. The co ordinates applied in this methodology are considered as rectangular form with major axis along the river bore channel. The Equations of wave motion in the bore are considered as continuity for the resistance of the wave movement is given as

$$\frac{\partial x}{\partial t} = -f \frac{\partial y}{\partial u}$$
$$\frac{\partial y}{\partial t} + f \frac{\partial x M}{\partial u} = 0$$

The Velocity of the bore is not considered for the equation of the ocean depth

$$\frac{\partial^2 y}{\partial t^2} - f \frac{\partial}{\partial u} M \frac{\partial y}{\partial u} = 0$$

After the simplification and substitution

$$\phi^2 y - f \frac{\partial}{\partial u} M \frac{\partial y}{\partial u} = 0$$

The equation rewritten with the variable $u_1$ where $u_1 = s + tu$

$$\phi^2 y - f t^2 \frac{\partial}{\partial u_1} u_1 \frac{\partial y}{\partial u_1} = 0$$

The solution of the above equation is given as

$$y = H B_0 (2 \sqrt{\phi} \sqrt{u_1})$$

The amplitude of the wave for the defined state is H and $\chi = \phi^2 / f t^2$.

The wave propagation for the time period 10 sec and amplitude for 150 meters the tsunami wave in to the bore at the 100-meter depth. The solution for both numerical and analytical for the bore in the lower state is given below.

The approximate numerical solution calculated for the non-linear equation for every 0.2 seconds and upper state is at the distance of 40 meters.
Figure 5. Comparison of solutions of Numerical and Analytical method

The relation between the wavelength and the amplitude is explored in the above figure in a certain the distance of wave motion. The solution of the equation for the wave motion by numerical and analytical method shown between the 100 to 200 kilometers based on their amplitude and the wave motion period. The solution by the numerical method shown as the sine curve for the wave front generations.

Figure 6. Solution by Numerical and analytical for the length between 50 to 200 kilometers

4.1 The non-linear parameters for the wave propagation in a river bore approximations

The outcome of the wave motion equation with the linear parameter in the above figure illustrated. The non-linear parameter is included for the extension of the above result. The resistance of the wave motion in sea floor in considered as the nonlinear parameter for the same amplitude and the length of the wave. In the lower state of the wave motion the resistance is very high and considered as important nonlinear parameter.
Conclusion

The floor movement under the sea originate the Tsunami wave propagation. The sea surface movement is considered in two-layer states. The upper and lower layer state are defined with nonlinear equation after the movement in seafloor. The upward motion of the wave propagation in identified and given the wave equation as first order equation. The results of the non-linear equations are given in terms of the extended parameters. The wave length and amplitude are estimated with nonlinear extended parameters. Numerical solution and analytical solution of wave propagation analyzed for the various parameters depth of the ocean, amplitude of the wave and speed of the wave propagation in the bore. The comparison of wave generation and wave length are discussed in numerical and analytical manner. The floor resistance during the Tsunami wave propagation is compared by numerical solution.

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