Density of disc algebra functions in de Branges-Rovnyak spaces

Alexandru Aleman and Bartosz Malman

Abstract

We prove that functions continuous up to the boundary are dense in de Branges-Rovnyak spaces induced by extreme points the unit ball of $H^\infty$.

1 Introduction

Let $H^\infty$ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$ in the complex plane, and denote by $\mathcal{A}$ the disc algebra, i.e. the subalgebra of $H^\infty$ consisting of functions which extend continuously to the closed disk. The Hardy space $H^2$ consists of power series in $\mathbb{D}$ with square-summable coefficients. If $T$ denotes the unit circle, we identify as usual $H^2$ with the closed subspace of $L^2(T)$ consisting of functions whose negative Fourier coefficients vanish. The orthogonal projection from $L^2(T)$ onto $H^2$ is denoted by $P_+$. For $\phi \in L^\infty(T)$ let $T_\phi$ denote the Toeplitz operator on $H^2$ defined by $T_\phi f = P_+ \phi f$. Given $b \in H^\infty$ with $\|b\|_\infty \leq 1$ we define the corresponding de Branges-Rovnyak space $\mathcal{H}(b)$ as

$$\mathcal{H}(b) = (1 - T_b T_b^*)^{1/2} H^2.$$

$\mathcal{H}(b)$ is endowed with the unique norm which makes the operator $(1 - T_b T_b^*)^{1/2}$ a partial isometry from $H^2$ onto $\mathcal{H}(b)$. Alternatively, $\mathcal{H}(b)$ is defined as the reproducing kernel Hilbert space with kernel

$$k_b(z, \lambda) = \frac{1 - b(\lambda) \overline{b(z)}}{1 - \overline{\lambda} z}.$$

$\mathcal{H}(b)$-spaces are naturally split into two classes with fairly different structures according to whether the quantity $\int_T \log(1 - |b|) \, dm$ is finite or not. Here $m$ denotes the normalized arc-length measure on $T$. The present note concerns the approximation of $\mathcal{H}(b)$-functions by functions in $\mathcal{A} \cap \mathcal{H}(b)$ and from the technical point of view there is a major difference between the two classes, which we shall briefly explain.

If $\int_T \log(1 - |b|) \, dm < \infty$, or equivalently, if $b$ is a non-extreme point of the unit ball of $H^\infty$ (see [8]), then $\mathcal{H}(b)$ contains all functions analytic in a neighborhood of the closed unit disk. By a theorem of Sarason, the polynomials form a norm-dense subset of the space $\mathcal{H}(b)$. An interesting feature of the proofs of density of polynomials in an $\mathcal{H}(b)$-space is that the usual approach of approximating a function $f$ first by its dilations $f_r(z) = f(rz)$, and then by their
truncated Taylor series, or by their Cesàro means, does not work. Sarason’s
initial proof of density of polynomials is based on a duality argument. In recent
years a more involved constructive polynomial approximation scheme has been
obtained in [6].

The picture changes dramatically in the case when
\[ \int_T \log(1 - |b|) \, dm = \infty, \]
or equivalently when \( b \) is an extreme point of the unit ball of \( H^\infty \). Then it
is in general a difficult task to identify any functions in the space other than
the reproducing kernels, and it might happen that \( \mathcal{H}(b) \) contains no non-zero
function analytic in a neighborhood of the closed disk. A special class of extreme
points are the inner functions. If \( b \) is inner then \( \mathcal{H}(b) = H^2 \ominus bH^2 \) with equality
of norms, and it is a consequence of a celebrated theorem of Aleksandrov [1]
that in this case the intersection \( \mathcal{A} \cap \mathcal{H}(b) \) is dense in the space. The result is
surprising since, as pointed out above, in most cases it is not obvious at all that
\( \mathcal{H}(b) \) contains any non-zero function in the disk algebra \( \mathcal{A} \).

Motivated by the situation described here, E. Fricain [4], raised the natural
question whether Aleksandrov’s result extends to all other \( \mathcal{H}(b) \)-spaces induced
by extreme points \( b \) of the unit ball of \( H^\infty \). It is the purpose of this note to
provide an affirmative answer to this question, contained in the main result
below.

**Theorem 1.** If \( b \) is an extreme point of the unit ball of \( H^\infty \) then \( \mathcal{A} \cap \mathcal{H}(b) \) is
a dense subset of \( H^b \).

Together with Sarason’s result [8] on the density of polynomials in the non-
extreme case, it follows that the intersection \( \mathcal{A} \cap \mathcal{H}(b) \) is dense in the space
\( \mathcal{H}(b) \) for any \( b \) in the unit ball of \( H^\infty \). Our proof Theorem 1 is deferred to
Section 3 and relies on a duality argument. Therefore, just as the earlier proofs
of Sarason and Aleksandrov, our approach is non-constructive. Section 2 serves
to a preliminary purpose.

## 2 Preliminaries

### 2.1 The norm on \( \mathcal{H}(b) \)

An essential step is the following useful representation of the norm in \( \mathcal{H}(b) \). The authors have originally deduced the result using
the techniques in [3] (see also [2, Chapter 3]), but once the goal is identified,
several available techniques provide simpler proofs. For example, the proposition
below can be deduced from results in [8]. For the sake completeness, we
include a new shorter proof.

**Proposition 2.** Let \( b \) be an extreme point of the unit ball of \( H^\infty \) and let
\[ E = \{ \zeta \in T : |b(\zeta)| < 1 \}. \]

Then for \( f \in \mathcal{H}(b) \) the equation
\[ P_p \overline{bf} = -P_+ \sqrt{1 - |b|^2} g. \]
has a unique solution \( g \in L^2(E) \), and the map \( J : \mathcal{H}(b) \to H^2 \oplus L^2(E) \) defined
by
\[ Jf = (f, g), \]
is an isometry. Moreover,
\[ J(\mathcal{H}(b))^\perp = \left\{ (bh, \sqrt{1 - |b|^2} h) : h \in H^2 \right\}. \]
Proof. Let
\[ K = \left\{ (bh, \sqrt{1 - |b|^2} h) : h \in H^2 \right\}, \]
and let \( P_1 \) be the projection from \( H^2 \oplus L^2(E) \) onto the first coordinate \( H^2 \), i.e., \( P_1(f, g) = f \). We observe first that \( P_1|K^\perp \) is injective. Indeed, if \( K^\perp \) contains a tuple of the form \((0, g) \in H^2 \oplus L^2(E)\), it follows that
\[
\int_T \zeta^n g(\zeta) \sqrt{1 - |b(\zeta)|^2} \, dm(\zeta) = 0, \quad n \geq 0,
\]
and consequently the function \( g \sqrt{1 - |b|^2} \) coincides a.e. with the boundary values of the complex conjugate of a function \( f \in H^2 \). But the assumption that \( b \) is an extreme point then implies that
\[
\int_T \log |f| \, dm = -\infty,
\]
and since \( f \in H^2 \), we conclude that \( f = 0 \), i.e., \( g = 0 \). Thus, the space \( \mathcal{H} = P_1 K^\perp \) with the norm \( \|f\|_{\mathcal{H}} = \|P_1^{-1} f\|_{H^2 \oplus L^2(E)} \) is a Hilbert space of analytic functions on \( \mathbb{D} \), contractively contained in \( H^2 \), in particular, it is a reproducing kernel Hilbert space. We now show that \( \mathcal{H} \) equals \( \mathcal{H}(b) \) by verifying that the reproducing kernels of the two spaces coincide. This follows from a simple computation. For \( \lambda \in \mathbb{D} \), the tuple
\[
(f_\lambda, g_\lambda) = \left( \frac{1 - \overline{b}(\lambda) b(z)}{1 - \overline{\lambda} z}, \frac{-\overline{b}(\lambda) \sqrt{1 - |b(z)|^2}}{1 - \overline{\lambda} z} \right)
\]
is obviously orthogonal to \( K \), while the last tuple on the right hand side is in \( \mathcal{H} \), so that \( f_\lambda \) is the reproducing kernel in \( \mathcal{H} \), which obviously equals the reproducing kernel in \( \mathcal{H}(b) \). The first assertion in the statement is now self-explanatory. \( \square \)

2.2 The Khintchin-Ostrowski theorem. Recall that analytic functions \( f \) in \( \mathbb{D} \) satisfy \( \sup_{0 < r < 1} \int_T \log^+ |f_r| \, dm < \infty \) if and only if they are quotients of \( H^\infty \)-functions, in particular they have finite nontangential limits a.e. on \( \mathbb{T} \) which define a boundary function denoted also by \( f \). The class \( N^+(\mathbb{D}) \) consists of quotients of \( H^\infty \)-functions such that the denominator can be chosen to be outer, it contains all Hardy spaces \( H^p \), \( p > 0 \). The Khintchin-Ostrowski theorem reads as follows. A proof can be found in [7].

Theorem 3. Let \( \{f_n\} \) be a sequence of functions analytic in the unit disk satisfying the following conditions:

(i) There exists a constant \( C > 0 \) such that
\[
\int_T \log^+ (|f_n(\rho e^{it})|) \, dt \leq C.
\]

(ii) On some set \( E \) of positive measure, the sequence \( f_n \) converges in measure to a function \( \phi \).

Then the sequence \( f_n \) converges uniformly on compact subsets of the unit disk to a function \( f \) which satisfies \( f = \phi \) a.e. on \( E \).
3 Proof of the main result

Due to Proposition 2 we can now implement Aleksandrov's strategy from [1] which will then be combined with the Khintchin-Ostrowski theorem. Recall that the dual $A'$ of the disk algebra $A$ can be identified with the space $C$ of Cauchy transforms of finite measures on $\mathbb{T}$ ([5]) via the pairing

$$\langle f, C\mu \rangle = \lim_{r \to 1^-} \int_{\mathbb{T}} f(\zeta) \overline{C\mu(r\zeta)} \, dm(\zeta) = \int_{\mathbb{T}} f \, d\mu,$$

where

$$C\mu(z) = \int_{\mathbb{T}} \frac{1}{1 - z\zeta} \, d\mu(\zeta)$$

is the Cauchy transform of $\mu$. The space $C$ is endowed with the obvious quotient norm and is continuously contained in all $H^p$ spaces for $0 < p < 1$. The following result extends Alexandrov’s approach to the context of $H(b)$-spaces, when $b$ is extremal in the unit ball of $H^\infty$.

Lemma 4. Let $E = \{ \zeta \in \mathbb{T} : |b(\zeta)| < 1 \}$, $B = A \oplus L^2(E)$ and $B' = C \oplus L^2(E)$. Then the set

$$S = \{(C\mu, h) : C\mu/b \in N^+(D), C\mu/b = h/\sqrt{1 - |b|^2} \text{ a.e. on } E\}$$

is weak-* closed in $B'$.

Proof. Since $A \oplus L^2(E)$ is separable, it will be sufficient to show that $S$ is weak-* sequentially closed. Let $(C\mu_n, h_n)$ converge weak-* to $(C\mu, h)$, where $(C\mu_n, h_n) \in S$ for $n \geq 1$. Equivalently, $h_n \to h$ weakly in $L^2(E)$, and

$$\sup_n \|C\mu_n\| < \infty, \quad \lim_{n \to \infty} C\mu_n(z) = C\mu(z), \quad z \in D.$$

Now by passing to a subsequence and the Cesàro means of that subsequence we can assume that $h_n \to h$ in the $L^2$-norm. Finally using another subsequence we may also assume that $h_n \to h$ pointwise a.e. on $E$. Let $I_b$ be the inner factor of $b$. Since $C\mu_n/I_b \in N^+(D)$, it follows by Vinogradov’s theorem ([5, Theorem 6.5.1]) that $(C\mu_n/I_b)_n$ is a bounded sequence in $C$ converging pointwise on $D$ to $C\mu/I_b$. This implies weak-* convergence in $C$, in particular, $C\mu/I_b \in C \subset N^+(D)$, and consequently, $C\mu/b \in N^+(D)$. Moreover, we have a.e. on $E$ that $C\mu_n/b = h_n/\sqrt{1 - |b|^2}$ which converges pointwise to $h/\sqrt{1 - |b|^2}$, hence we conclude that the sequence $C\mu_n$ converges in measure to some function $\phi$ on $E$. Finally, if $p \in (0, 1)$ and $\| \cdot \|_p$ denotes the $H^p$-norm, then

$$\int_{\mathbb{T}} \log^+(|C\mu_n(re^{it})|) \, dt \lesssim \int_{\mathbb{T}} \|C\mu_n(re^{it})\|^p \, dt \lesssim \|C\mu_n\|^p_p \lesssim \sup_n \|C\mu_n\| < \infty.$$

Thus the assumptions of Theorem 3 are satisfied, and so (a subsequence of) $C\mu_n$ converges a.e. on $E$ to $C\mu$. This clearly implies $C\mu/b = h/\sqrt{1 - |b|^2}$ a.e. on $E$, i.e. $(C\mu, h) \in S$.

We are now ready to complete the proof of the main theorem.
Proof of Theorem 1. Let $J$ denote the embedding in Proposition 2. Based on the pairing described at the beginning of this section, a direct application of Proposition 2 gives

$$J(\mathcal{A} \cap \mathcal{H}(b)) = \cap_{h \in H^2} \ker l_h,$$

where the functionals $l_h$ are identified with elements of $C \oplus L^2(E)$ as

$$l_h = \left( hh, h\sqrt{1-|h|^2} \right).$$

It is a consequence of the Hahn-Banach theorem that the annihilator $J(\mathcal{A} \cap \mathcal{H}(b))^\perp$ is the weak-* closure of the set of the functionals $l_h$. Since for all $h \in H^2$ we have $l_h \in S$, the set considered in Lemma 4, by the lemma we conclude that $J(\mathcal{A} \cap \mathcal{H}(b))^\perp \subset S$. Thus if $f \in \mathcal{H}(b)$ is orthogonal to $\mathcal{A} \cap \mathcal{H}(b)$, we must have $Jf \in S$, that is

$$Jf = \langle hh, h\sqrt{1-|h|^2} \rangle$$

for some $h \in H^2$. But then by Proposition 2, $Jf \in J(\mathcal{H}(b))^\perp$, which gives $Jf = 0$ and the proof is complete.

References

[1] A. B. Aleksandrov, Invariant subspaces of shift operators. An axiomatic approach, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 113 (1981), pp. 7–26, 264.

[2] A. Aleman, N. S. Feldman, and W. T. Ross, The Hardy space of a slit domain, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2009.

[3] A. Aleman, S. Richter, and C. Sundberg, Beurling’s theorem for the Bergman space, Acta Math., 177 (1996), pp. 275–310.

[4] C. Bénétanet, A. Condori, C. Liaw, W. T. Ross, and A. Sola, Some open problems in complex and harmonic analysis: report on problem session held during the conference Completeness problems, Carleson measures, and spaces of analytic functions, in Recent progress on operator theory and approximation in spaces of analytic functions, vol. 679 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2016, pp. 207–217.

[5] J. Cima, A. Matheson, and W. T. Ross, The Cauchy transform, vol. 125 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2006.

[6] O. El-Fallah, E. Fricain, K. Kellay, J. Mashreghi, and T. Ransford, Constructive approximation in de Branges-Rovnyak spaces, Constr. Approx., 44 (2016), pp. 269–281.

[7] V. P. Havin and B. Jörıccke, The uncertainty principle in harmonic analysis, vol. 72 of Encyclopaedia Math. Sci., Springer, Berlin, 1995.

[8] D. Sarason, Sub-Hardy Hilbert spaces in the unit disk, vol. 10 of University of Arkansas Lecture Notes in the Mathematical Sciences, John Wiley & Sons, Inc., New York, 1994.
