The Directed Probabilistic Powerspace*

Xiaolin Xie, Hui Kou*, Zhenchao Lyu

Department of Mathematics, Sichuan University, Chengdu, 610064, China

Abstract

Probabilistic powerdomain in domain theory plays an important role in modeling the semantics of nondeterministic functional programming languages with probabilistic choice. In this paper, we extend the notion of powerdomain to directed spaces, which is equivalent to the notion of the $T_0$ monotone-determined space [4]. We construct the probabilistic powerspace of the directed space, which is defined as a free directed space-cone. In addition, the relationships between our construction and classical probabilistic powerdomain are studied.

Keywords: topological cone, directed space-cone, probabilisitc powerspace of directed spaces, extended probabilistic powerdomain

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1. Introduction

The probabilistic powerdomain was first defined by Saheb-Djahromi [26] in 1980. Its purpose is to provide a mathematical model for the semantics of functional programming languages with probabilistic choice. Jones proved that if $X$ is a domain, then the probabilistic powerdomain of $X$ can be realized as the space of valuations, which is still a domain in her PhD thesis [16]. It has since then been studied extensively by a number of authors, such as, Plotkin[15], Graham[9], Heckmann [10, 11], Goubault and Jung [7], Tix and Keimel [17, 31], Lyu and Kou [20]. However, the category of domains together with Scott continuous maps is not cartesian closed. We are struggling to find a topological representation of the extended probabilistic powerdomain for an arbitrary dcpo $X$, but so far there has been no perfect result. Topological Cones are defined by Keimel [18] in 2006, and his intention is to provide domain theoretical tools to deal with situations where probabilistic features occur together with ordinary nondeterminism. In short, over a $T_0$ topological space $X$, he considers the structures that are close to vector spaces but asymmetric in the sense that elements do not have additive inverses, accordingly, scalar multiplication is restricted to nonnegative real numbers, besides, both these two operations are jointly continuous. Directed spaces are introduced by Kou [34] independently, which is equivalent to the $T_0$ monotone

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*Corresponding author

Email addresses: xxldannyboy@163.com (Xiaolin Xie), kouhui@scu.edu.cn (Hui Kou), zhenchaolyu@scu.edu.cn (Zhenchao Lyu)
determined space in [4]). A directed space is a $T_0$ space whose topology can be determined by monotone convergent nets. Especially, every dcpo with the Scott topology is a directed space, the category of directed spaces with continuous maps is a cartesian closed category [34]. Hence directed space can be regarded as an extended model of Domain Theory. A natural question arises: what is a directed space-version extended probabilistic powerspace over a directed space? To answer this question, we have to do two things. The first one is to define a directed space-version cone by using Keimel’s topological cone, just like the dcpo-cone. The second one is to construct a free object over any directed space. In this paper we give a concrete construction of extended probabilistic powerspace over any directed space. The paper is organized as follows: First to define the notion of directed space-cone and then give its concrete construct in section 3. In section 4, we give some relations between the extended p owerdomain over a dcpo and the extended powerspace over $X$.

2. Preliminaries

Now, we introduce the concepts needed in this article. On domain theory, topology, and category theory, see [1, 6, 21]. Let $P$ be a nonempty set. A relation $\leq$ on $P$ is called a partial order, if $\leq$ satisfies reflexivity ($x \leq x$), transitivity ($x \leq y$ & $y \leq z \Rightarrow x \leq z$) and antisymmetry ($x \leq y$ & $y \leq x \Rightarrow x = y$). $P$ is called a partial ordered set(poset) if $P$ is endowed with some partial order $\leq$. Given $A \subseteq P$, denote $\downarrow A = \{ x \in P : \exists a \in A, x \leq a \}$, $\uparrow A = \{ x \in P : \exists a \in A, a \leq x \}$. We say $A$ is a lower set (upper set) if $A = \downarrow a (A = \uparrow A)$. A nonempty set $D \subseteq P$ is called a directed set if each finite nonempty subset of $D$ has upper bound in $D$. Particularly, a poset is called a directed complete poset if each directed subset has a supremum(denoted by $\sqrt{D}$), abbreviated as dcpo. The subset $U$ of poset $P$ is called a Scott open set if $U$ is an upper set and for each directed set $D \subseteq P$, which $\sqrt{D}$ exists and belongs to $U$, then $U \cap D \neq \emptyset$. The set of all Scott open sets of poset $P$ is a topology on $P$, which is called the Scott topology and denoted by $\sigma(P)$. Suppose $P$, $E$ are two posets, a fuction $f : P \rightarrow E$ is called Scott continuous if it is continuous respect to Scott topology $\sigma(P)$ and $\sigma(E)$.

All topological spaces in this paper are required $T_0$ separation. A net of a topological space $X$ is a map $\xi : J \rightarrow X$, here $J$ is a directed set. Thus, each directed subset of a poset can be regarded as a net, and its index set is itself. Usually, we denote a net by $(x_j)_{j \in J}$ or $(x_j)$. Let $x \in X$, saying $(x_j)$ converges to $x$, denote by $(x_j) \rightarrow x$ or $x \equiv \lim x_j$, if $(x_j)$ is eventually in every open neighborhood of $x$, that is, for each given open neighborhood $U$ of $x$, there exists $j_0 \in J$ such that for every $j \in J, j \geq j_0 \Rightarrow x_j \in U$.

Let $X$ be a $T_0$ topological space, its topology is denoted by $O(X)$, the specialization order on $X$ is defined as follows:

$$\forall x, \ y \in X, \ x \subseteq y \Leftrightarrow x \in \overline{y}$$

here, $\overline{y}$ means the closure of $\{y\}$. From now on, the order of a $T_0$ topological space always indicates the specialization order "$\subseteq". Here are some basic properties of specialization order.

Proposition 2.1. [1, 6] For a $T_0$ topological space $X$, the following are always true:

1. For each open set $U \subseteq X$, $U = \uparrow U$;
2. For each closed set $A \subseteq X$, $A = \downarrow A$;
3. Suppose $Y$ is another $T_0$ topological space, and $f : X \rightarrow Y$ is a continuous function from $X$ to $Y$. Then for each $x, y \in X, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, that is every continuous function is monotone.

Suppose $X$ is a $T_0$ space, then every directed set $D \subseteq X$ can be regarded as a net of $X$, we use $D \rightarrow x$ or $x \equiv \lim D$ to represent $D$ converges to $x$. Define notation

$$D(X) = \{(D, x) : x \in X, \text{ D is a directed subset of } X \text{ and } D \rightarrow x\}.$$ 

It is easy to verify that, for each $x, y \in X$, $x \sqsubseteq y \Leftrightarrow \{y\} \rightarrow x$. Therefore, if $x \sqsubseteq y$ then $(\{y\}, x) \in D(X)$. Next, we give the concept of directed space.

**Definition 2.2.** Let $X$ be a $T_0$ space.

1. A subset $U$ of $X$ is called a directed open set if $\forall (D, x) \in D(X)$, $x \in U \Rightarrow D \cap U \neq \emptyset$.

Denote all directed open sets of $X$ by $d(X)$.

2. $X$ is called a directed space if each directed open set of $X$ is an open set, that is, $d(X) = O(X)$.

**Remark 2.3.**

1. Each open set of a $T_0$ space is directed open, but the contrary is not necessarily true. For example, suppose $Y$ is a non-discrete $T_1$ topological space, its specialization order is diagonal, that is, $\forall x, y \in Y, x \sqsubseteq y \Leftrightarrow x = y$. Thus, all subsets of $Y$ are directed open. We notice that $Y$ is non-discrete, at least one directed open set is not an open set.

2. The definition of directed space here is equivalent to the $T_0$ monotone determined space defined in [4].

3. Every poset endowed with the Scott topology is a directed space [19, 35], besides, each Alexandroff space is a directed space. Thus, the directed space extends the concept of the Scott topology.

Next, we introduce the directed continuous function.

**Definition 2.4.** Suppose $X$, $Y$ are two $T_0$ spaces. A function $f : X \rightarrow Y$ is called directed continuous if it is monotone and preserves all limits of directed set of $X$; that is, $(D, x) \in D(X) \Rightarrow (f(D), f(x)) \in D(Y)$.

Here are some characterizations of the directed continuous functions.

**Proposition 2.5.** Suppose $X$, $Y$ are two $T_0$ spaces. $f : X \rightarrow Y$ is a function between $X$ and $Y$.

1. $f$ is directed continuous if and only if $\forall U \in d(Y), f^{-1}(U) \in d(X)$.

2. If $X$, $Y$ are directed spaces, then $f$ is continuous if and only if it is directed continuous.

Now we introduce the product of directed spaces.

Suppose $X$, $Y$ are two directed spaces. Let $X \times Y$ represents the cartesian product of $X$ and $Y$, then we have a natural partial order on it: $\forall (x_1, y_1), (x_2, y_2) \in X \times Y$, 

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 \sqsubseteq x_2, y_1 \sqsubseteq y_2.$$ 

which is called the pointwise order on $X \times Y$. Now, we define a topological space $X \otimes Y$ as follows:
1. The underlying set of $X \otimes Y$ is $X \times Y$;
2. The topology on $X \times Y$ is generated as follows: for each given $\leq$ directed set $D \subseteq X \times Y$ and $(x, y) \in X \times Y$,
   $$D \to (x, y) \in X \otimes Y \iff \pi_1D \to x \in X, \pi_2D \to y \in Y,$$
   That is, a subset $U \subseteq X \times Y$ is open if and only if for every directed limit defined as above $D \to (x, y), (x, y) \in U \Rightarrow U \cap D \neq \emptyset$.

**Theorem 2.6.** \([35]\) Suppose $X$ and $Y$ are two directed spaces.
1. The topological space $X \otimes Y$ defined as above is a directed space and satisfies the following properties: the specialization order on $X \otimes Y$ equals to the pointwise order on $X \times Y$, that is $\sqsubseteq = \leq$.
2. Suppose $Z$ is another directed space, then $f : X \otimes Y \to Z$ is continuous if and only if it is continuous in each variable separately.

Denote the category of all directed spaces with continuous functions by $\mathbf{Dtop}$. It is proved in \([34, 35]\) that, $\mathbf{Dtop}$ contains all posets endowed with the Scott topology and $\mathbf{Dtop}$ is a cartesian closed category; specifically, the categorical products of two directed spaces $X$ and $Y$ are isomorphic to $X \otimes Y$. So, the directed space is an extended framework of Domain Theory.

Let $P$ be a dcpo, and $x, y \in P$. We say $x$ way below $y$, if for each given directed set $D \subseteq P$, $y \leq \bigvee D$ implies there exists some $d \in D$ such that $x \leq d$. We write $\downarrow x = \{a \in P : a \ll x\}$, $\uparrow x = \{a \in P : x \ll a\}$.

**Definition 2.7.** A dcpo $P$ is called a continuous domain if for each $x \in P$, $\downarrow x$ is directed and $x = \bigvee \downarrow x$.

**Theorem 2.8.** \([6]\) Suppose $P$ is a continuous domain. The followings hold:
1. $\forall x, y \in P, x \ll y \Rightarrow \exists z \in P, x \ll z \ll y$.
2. $\forall x \in P, \uparrow x$ is a Scott open set. Particularly, $\{\uparrow x : x \in P\}$ is a base of $(P, \sigma(P))$.

C-space was defined by Erné in \([3]\) in 1991, and it is not hard to verify that each c-space is a directed space.

**Definition 2.9.** \([3]\] A $T_0$ topological space $X$ is a c-space if each $x \in X$ and each open neighborhood $U$ of $x$, there exists some $y \in U$ such that $x \in \text{int}(\uparrow y) \subseteq U$. 

### 3. The directed probabilistic powerspaces

As mentioned above, directed space can be regarded as an extended model of Domain theory, just like the work done in article \([2]\), extending the powerdomain in the category of directed space is very meaningful. In this section, we will construct the directed probabilistic powerspace of the directed space, which is a free algebra generated by the addition and scalar multiplication of the directed space.

**Definition 3.1.** \([6]\] Let $X$ be a topological space, a map $\mu : \mathcal{O}(X) \to [0, +\infty]$ is called a continuous valuation if the followings hold:
(1) strictness: \( \mu(\emptyset) = 0 \);
(2) monotonicity: \( V \subseteq U \) implies \( \mu(V) \leq \mu(U) \);
(3) modular law: \( \mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V) \);
(4) continuity: for each directed family \( \mathcal{D} \subseteq \mathcal{O}(X) \), \( \mu(\sup \mathcal{D}) = \sup \{ \mu(U) : U \in \mathcal{D} \} \).

**Definition 3.2.** Let \( X \) be a topological space, for each \( x \in X \), we define the point valuation \( \eta_x : \mathcal{O}(X) \to [0, +\infty] \) from the lattice of open sets of \( X \), \( \eta_x(U) = 1 \) if \( x \in U \), and \( \eta_x(U) = 0 \) if \( x \notin U \). A finite linear sum \( \xi = \sum_{b \in B} r_b \eta_b \) with \( 0 < r_b < +\infty \), and defined by \( \xi(U) = \sum_{b \in U} r_b \) is called a simple valuation, and the set \( B \) is called its support.

Topological cones was defined by Keimel in [18] in 2006, here we replace the \( T_0 \) space by directed space to get the definition of directed space-cone without changing any other conditions.

**Definition 3.3.** Let \( X \) be a directed space. A directed space-cone is a directed space \( X \) equipped with a distinguished element \( 0 \in X \), an addition \( + : X \otimes X \to X \), and a scalar multiplication \( \cdot : \mathbb{R}^+ \otimes X \to X(\mathbb{R}^+ \text{ endowed with the Scott topology in its usual partial order}) \) such that both operations are jointly continuous and the followings are satisfied.

1. \( x + y = y + x, \forall x, y \in X \),
2. \( (x + y) + z = x + (y + z), \forall x, y, z \in X \),
3. \( 0 + x = x, \forall x \in X \),
4. \( (k \cdot l) \cdot x = k \cdot (l \cdot x), \forall k, l \in \mathbb{R}^+, \forall x \in X \),
5. \( (k + l) \cdot x = (k \cdot x) + (l \cdot x), \forall k, l \in \mathbb{R}^+, \forall x \in X \),
6. \( k \cdot (x + y) = (k \cdot x) + (k \cdot y), \forall k \in \mathbb{R}^+, \forall x, y \in X \),
7. \( 1 \cdot x = x, \forall x \in X \),
8. \( k \cdot 0 = 0, \forall k \in \mathbb{R}^+ \).

**Definition 3.4.** Suppose \((X, +, \cdot), (Y, \sqcup, *)\) are two directed space-cones, \( f : (X, +, \cdot) \to (Y, \sqcup, *) \) is called a directed space-cone homomorphism between \( X \) and \( Y \), if \( f \) is continuous and \( f(x + y) = f(x) \sqcup f(y), f(a \cdot x) = a * f(x) \) holds for \( \forall x, y \in X, a \in \mathbb{R}^+ \).

Denote the category of all directed space-cone and directed space-cone homomorphisms by \( \text{Dcone} \). Then \( \text{Dcone} \) is a subcategory of \( \text{Dtop} \).

Next, we give the definition of directed probabilistic powerspace.

**Definition 3.5.** Suppose \( X \) is a directed space. A directed space \( Z \) is called the directed probabilistic powerspace of \( X \) if the following two conditions are satisfied:

1. \( Z \) is a directed space-cone, that is the addition \( + \) and the scalar multiplication \( \cdot \) exist and they are continuous,
2. There is a continuous function \( i : X \to Z \) satisfies: for arbitrary directed space-cone \((Y, \sqcup, *)\) and continuous function \( f : X \to Y \), there exists a unique directed space-cone homomorphism \( f : (Z, +, \cdot) \to (Y, \sqcup, *) \) such that \( f = f \circ i \).

By the definition above, if directed space-cone \((Z_1, +, \cdot)\) and \((Z_2, \sqcup, *)\) are both the directed probabilistic powerspaces of \( X \), then there exists a topological homeomorphism which is also a directed space-cone homomorphism \( g : Z_1 \to Z_2 \). Therefore, by the means of algebraic isomorphism and topological homeomorphism, the directed probabilistic powerspaces of a directed space
is unique. Particularly, we denote the directed probabilistic powerspace of each directed space $X$ by $P_{\mathcal{P}}(X)$.

Now, we will prove the existence of the directed probabilistic powerspace of each directed space $X$ by way of concrete construction.

Let $X$ be a directed space. Let $\mathcal{S}(X)$ denote the set of all simple valuations in $X$, we have the pointwise order $\leq$ on $\mathcal{S}(X)$:

$$\xi \leq \eta \iff \xi(U) \leq \eta(U), \forall U \in \mathcal{O}(X).$$

Let $\mathcal{D} = \{\xi_i\}_{i \in I} \subseteq \mathcal{S}(X)$ be a directed set, $\xi = \sum_{i=1}^{n} r_i \eta_i \in \mathcal{S}(X)$ with support $B$. We say $\mathcal{D} \Rightarrow \rho \xi$ if the following two conditions hold:

1. For each $b_i \in B$, there exists a directed set $D_i \subseteq X$ such that in $X$ with $D_i \rightarrow b_i$, $i = 1, \ldots, n$;
2. $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i$, $\forall r_{b_i} < r_{b_i}$, $i = 1, \ldots, n$, there exists some $\xi' \in \mathcal{D}$, such that $\sum_{i=1}^{n} r_{b_i} \eta_i \leq \xi'$.

A subset $\mathcal{U} \subseteq \mathcal{S}(X)$ is called a $\Rightarrow \rho$ convergence open set of $\mathcal{S}(X)$ if and only if for each directed subset $\mathcal{D}$ of $\mathcal{S}(X)$ and $\xi \in \mathcal{S}(X)$, $\mathcal{D} \Rightarrow \rho \xi \in \mathcal{U}$ implies $\mathcal{D} \cap \mathcal{U} \neq \emptyset$. Denote all $\Rightarrow \rho$ convergence open sets of $\mathcal{S}(X)$ by $O_{\Rightarrow \rho}(\mathcal{S}(X))$.

**Proposition 3.6.** Suppose $X$ is a directed space, the following are true:

1. $(\mathcal{S}(X), O_{\Rightarrow \rho}(\mathcal{S}(X)))$ is a topological space, abbreviated as $CX$.
2. The specialization order $\preceq$ on $CX$ equals to $\leq$.
3. $C(X)$ is a directed space, that is $O_{\Rightarrow \rho}(\mathcal{S}(X)) = d(CX)$.

**Proof**

1. Obviously we have $\emptyset$, $CX \in O_{\Rightarrow \rho}(\mathcal{S}(X))$. If $U \in O_{\Rightarrow \rho}(U \cap X)$, and $\xi \leq \eta, \eta \in U$, $\xi = \sum_{i=1}^{n} r_{b_i} \eta_i$. Then it is evident that $\{\eta\} \Rightarrow \rho \xi$, since we only need to take $D_i = \{b_i\}$, $i = 1, \ldots, n$. Then, $\{\eta\} \cap U \neq \emptyset$, this means $\eta \in U$, and $U$ is an upper set with respect to order $\leq, \rho$, $U = \uparrow_{\leq, \rho} U$.

Let $U_1, U_2 \subseteq O_{\Rightarrow \rho}(\mathcal{S}(X))$, and a directed set $\mathcal{D} \subseteq CX$ with $\mathcal{D} \Rightarrow \rho \xi \in U_1 \cap U_2$, then, there exists $\xi_1 \in \mathcal{D} \cap U_1$ and $\xi_2 \in \mathcal{D} \cap U_2$, but $\mathcal{D}$ is directed, we have $\xi_3 \in \mathcal{D}, \xi_3 \geq \xi_1, \xi_2$. Then, $\xi_3 \in \mathcal{D} \cap U_1 \cap U_2$. By the same way, we can prove that $O_{\Rightarrow \rho}(\mathcal{S}(X))$ is closed under arbitrary union. It follows that $O_{\Rightarrow \rho}(\mathcal{S}(X))$ is a topology.

2. By the proof of (1), each $\Rightarrow \rho$ convergence open set is an upper set with respect to $\leq$, then $\downarrow \leq, \eta \subseteq \downarrow \leq, \eta, \forall \eta \in CX$. We only need to prove that $\downarrow \leq, \eta$ is a closed set in $CX$, since $\downarrow \leq, \eta$ is the minimal closed set containing $\eta$. Equivalently, $CX \downarrow \leq, \eta$ is a $\Rightarrow \rho$ convergence open set.

Let $\mathcal{U} = CX \downarrow \leq, \eta$. Suppose we have $\mathcal{D} \Rightarrow \rho \xi = \sum_{i=1}^{n} r_{b_i} \eta_i \in U$. By contradiction, suppose $\mathcal{U} \cap \mathcal{D} = \emptyset$, that is $\forall \xi' \in \mathcal{D}, \xi' \leq \eta$. However, by the definition of $\mathcal{D} \Rightarrow \rho \xi$, we have directed sets $D_1, \ldots, D_n \subseteq X$ with $D_i \rightarrow b_i$, $i = 1, \ldots, n$, and $\forall (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i, \forall r_{b_i} < r_{b_i}$, there exists some $\xi' \in \mathcal{D}$ such that $\sum_{i=1}^{n} r_{b_i} \eta_i \leq \xi'$. Now, we claim that $\xi \leq \eta$, which contradicts with $\xi \in U$. By the definition of pointwise order $\leq, \forall U \in \mathcal{O}(X)$, we may assume that $b_1, \ldots, b_k \in U, 0 < k \leq n$. Since $D_i \rightarrow b_i$, $i = 1, \ldots, k$, there exists $(d_1, \ldots, d_k) \in \prod_{i=1}^{k} D_i$ such that $d_i \in U$. For each $r_{b_i} < r_{b_i}, i = 1, \ldots, k$, there exists $\xi' \in \mathcal{D}$ such that $\sum_{i=1}^{k} r_{b_i} \eta_i \leq \xi'$. Note that $k \leq n$, then,

$$\sum_{i=1}^{k} r_{b_i} \eta_i (U) = \sum_{i=1}^{k} r_{b_i} \leq \xi'(U) \leq \eta(U).$$
But the supremum of the left hand side in the upper inequality is $\sum_{i=1}^{k} r_{bi} = (\sum_{i=1}^{k} r_{bi} \eta_{bi})(U) = \xi(U)$.

(3) For an arbitrary topological space $X$, $O(X) \subseteq d(X)$ holds, then $O \Rightarrow p (SV(X)) \subseteq d(CX)$. On the other hand, according to the definition of $\Rightarrow p$ convergence topology, if directed set $D \subseteq CX$ with $D \Rightarrow p \xi$, then $D$ convergents to $\xi$ respect to $O \Rightarrow p (SV(X))$. Thus, by the definition of directed open set, $D \Rightarrow p \xi \in U \in d(CX)$ will imply $U \cap D \neq \emptyset$. Then, $U \in O \Rightarrow p (SV(X))$, it follows that $O \Rightarrow p (SV(X)) = d(CX)$, that is, $CX$ is a directed space. □

**Proposition 3.7.** Suppose $X, Y$ are two directed spaces. Then function $f : CX \to Y$ is continuous if and only if for each directed set $D \subseteq CX \land \xi \in CX$, $D \Rightarrow p \xi$ implies $f(D) \to f(\xi)$.

**Proof** Since $\Rightarrow p$ convergence will lead to $O \Rightarrow p (SV(X))$ topological convergence, the necessity is obvious. We are going to prove the sufficiency. Firstly, we check that $f$ is monotone. If $\xi, \eta \in CX$ and $\xi \leq \eta$, then $\{\eta\} \Rightarrow p \xi$, by the hypothesis, $\{f(\eta)\} \to f(\xi)$, thus $f(\xi) \subseteq f(\eta)$. Suppose $U$ is an open set of $Y$ and the directed set $D \Rightarrow p \xi \in f^{-1}(U)$, then $f(D)$ is a directed set of $Y$ and $f(D) \to f(\xi) \in U$, thus there exists some $\xi' \in D$ such that $f(\xi') \subseteq U$. That is, $\xi' \in D \cap f^{-1}(U)$. According to the definition of $\Rightarrow p$ convergence open set, $f^{-1}(U) \subseteq O \Rightarrow p (SV(X))$, that means $f$ is continuous. □

Define an addition $+$ on $CX : \forall \xi, \eta \in CX, (\xi + \eta)(U) = \xi(U) + \eta(U), \forall U \in O(X)$. Define a scalar multiplication · on $CX : \forall a \in \mathbb{R}^+, \forall \xi \in CX, (a \cdot \xi)(U) = a(\xi(U))$. Next, we shall check that these two operations are both continuous. Thus, $(CX, +, \cdot)$ is a directed space-cone.

**Theorem 3.8.** Let $X$ be a directed space. Then $(CX, +, \cdot)$ is a directed space-cone.

**Proof** By Proposition 3.6, $CX$ is a directed space, and by the definition of $+$ and $\cdot$, these two operations are both monotone. According to Theorem 2.6 and Proposition 3.7 to prove the continuity of $+$, we only need to check that for arbitrary fixed $\eta = \sum_{j=1}^{m} s_{cj} \eta_{cj} \in CX$, and a directed set $D = \{\xi_{i}\}_{i \in I} \subseteq CX$ with $D \Rightarrow p \xi = \sum_{i=1}^{n} r_{bi} \eta_{bi}$, then $D + \eta = \{\xi_{i} + \eta\}_{i \in I} \Rightarrow p \xi + \eta$. By hypothesis, $D \Rightarrow p \xi$, there exist $D_{1}, \ldots, D_{m}$, with $D_{i} \to b_{i}$. Taking finite single point sets $\{c_{j}\}, j = 1, \ldots, m$. For arbitrary $(d_{1}, \ldots, d_{ni}, c_{1}, \ldots, c_{m}) \in \prod_{i=1}^{n} D_{i} \times \prod_{j=1}^{m} \{c_{j}\}$, and $\forall r_{bi} < r_{bi}', s_{cj} < s_{cj}', i = 1, \ldots, n, j = 1, \ldots, m$. Again, by the definition of $D \Rightarrow p \xi$, there exists some $\xi' \in D$ such that $\sum_{i=1}^{n} r_{bi}' \eta_{bi} \leq \xi'$, for $+$ is monotone, $\sum_{i=1}^{n} r_{bi}' \eta_{bi} + \sum_{j=1}^{m} s_{cj}' \eta_{cj} \leq \xi' + \sum_{j=1}^{m} s_{cj}' \eta_{cj}$, thus $\sum_{i=1}^{n} r_{bi}' \eta_{bi} + \sum_{j=1}^{m} s_{cj}' \eta_{cj} \leq \xi' + \eta$. This is exactly the second condition of $D + \eta \Rightarrow p \xi + \eta$.

To prove the continuity of $\cdot$, firstly, for arbitrary fixed $a \in \mathbb{R}^+$ and a directed set $D = \{\xi_{i}\}_{i \in I} \subseteq CX$ with $D \Rightarrow p \xi = \sum_{i=1}^{n} r_{bi} \eta_{bi}$, we claim that $a \cdot D = \{a \cdot \xi_{i}\}_{i \in I} \Rightarrow p \{a \cdot \xi\}$, since we have $D_{i} \to b_{i}, i = 1, \ldots, n$, and $\forall (d_{1}, \ldots, d_{ni}) \in \prod_{i=1}^{n} D_{i}, \forall r_{bi} < r_{bi}', i = 1, \ldots, n$, there exists $\xi' \in D$, such that $\sum_{i=1}^{n} r_{bi}' \eta_{bi} \leq \xi'$ by the hypothesis of $D \Rightarrow p \xi$. By the monotonicity of $\cdot$, we have $\sum_{i=1}^{n} a \cdot r_{bi}' \eta_{bi} \leq a \cdot \xi'$. Secondly, for arbitrary $\xi = \sum_{i=1}^{n} r_{bi} \eta_{bi} \in CX$ and a directed set $D \subseteq \mathbb{R}^+$ with $D \to a$. We claim that $D \cdot \xi = \{d \cdot \xi\}_{d \in D} \Rightarrow p \{a \cdot \xi\}$. Let $D_{i} = \{b_{i}\}, i = 1, \ldots, n, \forall (b_{1}, \ldots, b_{n}) \in \prod_{i=1}^{n} D_{i}$ and $\forall a r_{bi}' < ar_{bi}, i = 1, \ldots, n$, we only need to find $d \in D$ with $ar_{bi}' < dr_{bi}, i = 1, \ldots, n$. This is accessible, because for each $i = 1, \ldots, n$, $ar_{bi}/r_{bi} < a$ and $\sup D = a$.

In conclusion, $(CX, +, \cdot)$ is a directed space-cone. □

We now characterize the valuation cone as the free directed space-cone over $X$. We begin with two useful general lemmas.
Proposition 3.9. \cite{6} \cite{10} (Splitting Lemma) For two simple valuation in $\mathcal{SV}(X)$, $X$ is a $T_0$ space, we have $\zeta = \sum_{b \in B} r_b \eta_b \leq \sum_{c \in C} s_c \eta_c = \xi$ if and only if there exist $\{t_{b,c} \in [0, +\infty) : b \in B, c \in C\}$ such that for each $b \in B$, $c \in C$, 
\[ \sum_{c \in C} t_{b,c} = r_b, \quad \sum_{b \in B} t_{b,c} \leq s_c \]
and $t_{b,c} \neq 0$ implies $b \leq c$.

Lemma 3.10. \cite{6} \cite{10} Let $\zeta = \sum_{b \in B} r_b \eta_b$ and $\sum_{c \in C} s_c \eta_c = \xi$ be two simple valuations on $\mathcal{O}(X)$, where $X$ is a $T_0$ space. If $\xi$ and $\zeta$ are two distinct as linear combinations, then they are distinct as valuations.

The following theorem is the main result of this paper.

Theorem 3.11. Suppose $X$ is a directed space, then $(CX, +, \cdot)$ is the directed probabilistic powerspace over $X$, that is, endowed with topology $O_{\rightarrow p}(\mathcal{SV}(X))$, $(CX, +, \cdot) \cong P_P(X)$.

Proof Define function $i : X \to CX$ as follows: $\forall x \in X$, $i(x) = \eta_x$. By \ref{3.10} $i$ is injective. We prove the continuity of $i$. It is evident that if $x \leq y \in X$ then $\forall U \in \mathcal{O}(X)$, $x \in U$ implies $y \in U$, that is $\eta_x \leq \eta_y$. Suppose we have directed set $D \subseteq X$ and $x \in X$ with $D \to x$. Let $D = \{\eta_d : d \in D\}$, then $i(D) = D$ is a directed set in $CX$ and $D \Rightarrow_P \eta_x$. Since by hypothesis, $D \to x$, then $\forall d \in D$, $\forall k < 1$, $k \eta_d \leq \eta_d$.

Let $(Y, \triangleright, \ast)$ be a directed space-cone, $f : X \to Y$ is a continuous function. Define $\bar{f} : CX \to Y$ as follows: $\forall \xi = \sum_{i=1}^n r_i \eta_i \in CX$, 
\[ \bar{f}(\xi) = \bigcup_{i=1}^n r_i \ast f(b_i). \]

By Lemma \ref{3.10} $\bar{f}$ is well-defined,

(1) $\bar{f} = f \circ i$.

For arbitrary $x \in X$, $(f \circ i)(x) = f(i(x)) = f(\eta_x) = f(x)$.

(2) $\bar{f}$ is a directed space-cone homomorphism, that is, $\bar{f}$ is continuous and for arbitrary $\sum_{i=1}^n r_i \eta_i$, $\bar{f}(\sum_{i=1}^n r_i \eta_i) = \bigcup_{i=1}^n r_i \ast f(\eta_i)$. But this equation is evident since $\bar{f}(\eta_i) = f(b_i)$, $i = 1, \ldots, n$.

For continuity, first, $\bar{f}$ is monotone. Let $\zeta = \sum_{b \in B} r_b \eta_b \leq \sum_{c \in C} s_c \eta_c = \xi$, by Proposition \ref{3.9} there exist $\{t_{b,c} \in [0, +\infty) : b \in B, c \in C\}$ such that for each $b \in B$, $c \in C$, 
\[ \sum_{c \in C} t_{b,c} = r_b, \quad \sum_{b \in B} t_{b,c} \leq s_c \]
and $t_{b,c} \neq 0$ implies $b \leq c$. From the definition of $\bar{f}$ we have
\[ \bar{f}(\zeta) = \sum_{b \in B} r_b f(b) = \sum_{b \in B} \sum_{c \in C} t_{b,c} f(b) \leq \sum_{c \in C} \sum_{b \in B} t_{b,c} f(c) \leq \sum_{c \in C} s_c f(c) = \bar{f}(\xi). \]

Second, by Proposition \ref{3.7} we shall prove that $\bar{f}$ preserves $\Rightarrow_P$ convergence class, that is suppose we have a directed set $\mathcal{D} = \{\xi_i\}_{i \in I} \subseteq CX$ and $\xi = \sum_{i=1}^n r_i \eta_i \in CX$ with $\mathcal{D} \Rightarrow_P \xi$, then
\[ \tilde{f} \rightarrow f(\xi) \text{ in } Y. \ D \Rightarrow_{P} \xi \text{ admits } D_i \subseteq X \text{ with } D_i \rightarrow b_i, i = 1, \ldots, n. \text{ For the continuity of } f, \uplus \text{ and } *, \text{ we have} \]

\[ (r_{b_1} \uplus f(D_1)) \uplus \cdots \uplus (r_{b_n} \uplus f(D_n)) \rightarrow \bigcup_{i=1}^{n} r_{b_i} \uplus f(b_i) = \tilde{f}(\xi). \]

Here, \( (r_{b_1} \uplus f(D_1)) \uplus \cdots \uplus (r_{b_n} \uplus f(D_n)) = \{ (r_{b_1} \uplus f(d_1)) \uplus \cdots \uplus (r_{b_n} \uplus f(d_n)) : (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i \}. \)

For an arbitrary open neighborhood \( U \) of \( f(\xi) \), there exists some \( (d_1, \ldots, d_n) \in \prod_{i=1}^{n} D_i \) such that \( \bigcup_{i=1}^{n} r_{b_i} \uplus f(d_i) \in U \). Fix \( d_1, \ldots, d_n \), and for arbitrary \( r'_{b_i} < r_{b_i} \), \( i = 1, \ldots, n \), again by the definition of \( D \Rightarrow_{P} \xi \), there exists some \( \xi' \in D \) such that \( \sum_{i=1}^{n} r'_{b_i} \eta_{d_i} \leq \xi' \). Then \( \tilde{f}(\sum_{i=1}^{n} r'_{b_i} \eta_{d_i}) = \bigcup_{i=1}^{n} r'_{b_i} f(d_i) \leq \tilde{f}(\xi') \). By the continuity of \( f, \uplus \text{ and } * \), we have \( \bigcup_{i=1}^{n} r_{b_i} f(d_i) \leq \tilde{f}(\xi) \). But \( U \) is an upper set and \( \bigcup_{i=1}^{n} r_{b_i} \uplus f(d_i) \in U \), it follows that \( \tilde{f}(\xi') \in U \). \( \tilde{f} \) is continuous.

\( \text{3) Homomorphism } \tilde{f} \text{ is unique.} \)

Suppose we have a directed space-cone homomorphism \( g : (CX, +, \cdot) \rightarrow (Y, \uplus, *) \) such that \( f = g \circ i \), then \( g(\eta_x) = f(x) = \tilde{f}(\eta_x) \). For each \( \xi = \sum_{i=1}^{n} r_{b_i} \eta_{b_i} \in CX \),

\[ g(\xi) = g(r_{b_1} \eta_{b_1} + r_{b_2} \eta_{b_2} + \cdots + r_{b_n} \eta_{b_n}) = g(r_{b_1} \eta_{b_1}) \uplus g(r_{b_2} \eta_{b_2}) \uplus \cdots \uplus g(r_{b_n} \eta_{b_n}) = r_{b_1} \uplus g(\eta_{b_1}) \uplus r_{b_2} \uplus g(\eta_{b_2}) \uplus \cdots \uplus r_{b_n} \uplus g(\eta_{b_n}) = r_{b_1} \uplus f(b_1) \uplus r_{b_2} \uplus f(b_2) \uplus \cdots \uplus r_{b_n} \uplus f(b_n) = \tilde{f}(\xi). \]

Thus \( \tilde{f} \) is unique.

In conclusion, according to definition 3.5 endowed with topology \( O_{\Rightarrow_{P}}(SV(X)) \), the directed space-cone \( (CX, +, \cdot) \) is the directed probabilistic powerspace of \( X \), that is, \( P_{P}(X) \cong (CX, +, \cdot) \).

The directed probabilistic powerspace is unique in the sense of algebraic isomorphism and topological homeomorphism, so we can directly denote the directed probabilistic powerspace by \( P_{P}(X) = (CX, +, \cdot) \) of each directed space \( X \).

Suppose \( X, Y \) are two directed spaces, \( f : X \rightarrow Y \) is a continuous function. Define function \( P_{P}(f) : P_{P}(X) \rightarrow P_{P}(Y) \) as follows: \( \forall \xi = \sum_{i=1}^{n} r_{b_i} \eta_{b_i} \in CX \),

\[ P_{P}(f)(\xi) = \sum_{i=1}^{n} r_{b_i} \eta_{f(b_i)}. \]

We can check that, \( P_{P}(f) \) is well-defined and order preserving. It is easy to check that, \( P_{P}(f) \) is a directed space-cone homomorphism between these two extended probabilistic powerspaces. If \( id_X \) is the identity function and \( g : Y \rightarrow Z \) is an arbitrary continuous function from \( Y \) to a directed space \( Z \), then, \( P_{P}(id_X) = id_{P_{P}(X)}, P_{P}(g \circ f) = P_{P}(g) \circ P_{P}(f) \). Thus, \( P_{P} : Dtop \rightarrow Dcone \) is a functor from \( Dtop \) to \( Dcone \). Let \( U : Dcone \rightarrow Dtop \) be the forgetful functor. By theorem 3.11 we have the following result.

**Corollary 3.12.** \( P_{P} \) is a left adjoint of the forgetful functor \( U \), that is, \( Dcone \) is a reflective subcategory of \( Dtop \).
4. Relations Between Directed Probabilistic Powerspace And Probabilistic Powerdomain

In this section, we will discuss the relations between the extended probabilistic powerdomain of dcpo and the directed probabilistic powerspace.

According to the results in the last section, for an arbitrary directed space $X$, the directed probabilistic powerspace is the set of $CX$ endowed with the $\Rightarrow_P$ convergence topology. In general, for an arbitrary topological space and arbitrary dcpo, although the extended probabilistic powerdomain exists, their concrete structure cannot be expressed explicitly (see article [2, 12, 13]).

Definition 4.1. For a topological space $X$, the valuation powerdomain (also called the extended probabilistic powerdomain) $\mathcal{V}(X)$ of $X$ is the set of all continuous valuations on $\mathcal{O}(X)$ with the pointwise order.

Definition 4.2. For a simple valuation $\xi = \sum_{b \in B} r_b \eta_b$ and a continuous valuation $\mu$ on $X$, a domain equipped with the Scott topology, we set $\xi \prec \mu$ if for all nonempty $K \subseteq B$, we have $\sum_{b \in K} r_b < \mu(\bigcup_{b \in K} \uparrow b)$.

Theorem 4.3. For a domain $X$, the valuation powerdomain $\mathcal{V}(X)$ is a domain. Each continuous valuation $\mu$ is the directed supremum of the simple valuations way below it, and for simple valuation $\xi$, one has $\xi \ll \mu$ if and only if $\xi \prec \mu$.

Theorem 4.4. Given any dcpo-cone $C$ and a continuous function $f : X \rightarrow C$, where $X$ is a domain equipped with the Scott topology, there exists a unique continuous linear map $f^* : \mathcal{V}(X) \rightarrow C$ such that $f^* \eta_X = f$, here $\eta_X(x) = \eta_x$, $\forall x \in X$.

Definition 4.5. Suppose $X$ is a topological space, $\mu, \nu \in \mathcal{S}(X)$, and $\mu = \sum_{b \in B} r_b \eta_b$, $\nu = \sum_{c \in C} s_c \eta_c$, we say $\mu \ll \nu$ if there exist $\{t_{b,c} \in [0, +\infty) : b \in B, c \in C\}$ such that for each $b \in B, c \in C$,

$$\sum_{c \in C} t_{b,c} = r_b, \quad \sum_{b \in B} t_{b,c} < s_c$$

and $t_{b,c} \neq 0$ implies $c \in \text{int}(\uparrow b)$.

Since the soberification of each c-space is a continuous domain, by Theorem IV-9.16, Proposition IV-9.18, and IV-9.19 in [6], we have the following propositions.

Proposition 4.6. Suppose $X$ is a c-space, $\mu, \nu, \xi \in \mathcal{S}(X)$, then the followings hold:

1. $\mu \ll \nu \Rightarrow \mu \leq \nu$.
2. $\xi \ll \mu \leq \nu \Rightarrow \xi \ll \nu$.
3. $\mu, \nu \ll \xi \Rightarrow \exists \xi' \in \mathcal{S}(X), \mu, \nu \ll \xi' \ll \xi$.
4. $\mu \not\ll \nu \Rightarrow \exists \xi' \in \mathcal{S}(X), \xi' \ll \mu \& \xi' \not\ll \nu$.

According to the definition [4, 5] if we have $\mu \ll \nu$, then each $b \in B$, there exists some $c \in C$ such that $c \in \text{int}(\uparrow b)$. Let $\uparrow \mu = \{\nu \in \mathcal{S}(X) : \mu \ll \nu\}$. We claim that $\uparrow \mu$ is an open set in $CX$, and $CX$ is a c-space.

Lemma 4.7. Let $X$ be a c-space, then for each $\mu \in \mathcal{S}(X)$, $\uparrow \mu$ is open in $CX$. 

Proof We only need to check that $\uparrow \mu$ is a $\Rightarrow_p$ convergence open set. Suppose we have a directed set $\mathcal{D} \subseteq SV(X)$ and $\xi \in SV(X)$ with $\mathcal{D} \Rightarrow_p \xi \in \uparrow \mu$. Let $\mu = \sum_{i=1}^{k} r_{b_i} \eta_{b_i}$, with support $B$, $\xi = \sum_{i=1}^{n} s_{c_i} \eta_{c_i}$, with support $C$. By the definition of $\Rightarrow_p$ convergence, there exist directed sets $D_i \subseteq X$ such that $D_i \to c$, $c \in C$. Suppose we have $C \cap (\bigcup_{b \in B} \text{int}(\uparrow b)) = \{c_1, \ldots, c_n\} = K$, $1 \leq k \leq n$. For each $c \in K$ we have a finite set $B_c \subseteq B$ such that $\forall b \in B_c$, $c \in \text{int}(\uparrow b)$. Since $D_c \to c$, and eventually in $\text{int}(\uparrow b)$, we have finitely $d_{cb} \in \text{int}(\uparrow b)$, $b \in B_c$. Then we may pick the largest one $d_c \in \bigcap_{b \in B_c} \text{int}(\uparrow b)$. By hypothesis, there exist $\{t_{b,c} \in [0, +\infty) : b \in B, c \in C\}$ satisfy the definition of $\mu \ll \xi$. Define a new valuation $\nu = \sum_{c \in C} s_{c} \eta_{d_c}$. It is directly to check that $\mu \ll \nu$ with $\{t_{b,c} \in [0, +\infty) : b \in B, c \in C\}$. Again by the definition of $\mathcal{D} \Rightarrow_p \xi$, there exists some $\xi' \in \mathcal{D}$ such that $\nu \ll \xi'$. Now we have $\mu \ll \nu \ll \xi'$, according to 2 in Proposition 4.8, $\mu \ll \xi'$. It follows that $\xi' \in \mathcal{D} \cap \uparrow \mu$, $\uparrow \mu$ is open. □

**Theorem 4.8.** Let $X$ be a c-space, then $CX$ is a c-space.

Proof Suppose we have $\xi = \sum_{i=1}^{n} s_{c_i} \eta_{c_i} \in U$, where $U$ is a $\Rightarrow_p$ convergence open set. Since $X$ is a c-space, then $\mathcal{D} = \{\sum_{i=1}^{n} r_{b_i} \eta_{b_i} : r_i < s_{c_i}, c_i \in \text{int}(\uparrow b_i)\}$ is directed and $\mathcal{D} \Rightarrow_p \xi$. Thus we have some $\xi' \in \mathcal{D} \cap U$, and it is directed to check each $\mu \in \mathcal{D}$, $\mu \ll \xi$. Since $\{b_{i,c} : t_{b,c} = 0, i \neq j; t_{b,c} = 1, i = j\}$ satisfy the definition of $\mu \ll \xi$. By lemma 4.7, $\uparrow \xi'$ is open and $\xi \in \uparrow \xi' \subseteq U$, $CX$ is a c-space. □

Limited to the space, we may not generalize all the auxiliary relations from domains to c-spaces. Thus, we have the following proposition limited to domains. Let $A$ be a continuous domain. Then $(X, \sigma(X))$ is a directed space, and $CX \subseteq V(X)$. By $\sigma(V(X))|_{SV(X)}$ denote the relative topology from the Scott topology on $SV(X)$.

**Proposition 4.9.** Suppose $X$ is a domain, then $O_{\Rightarrow_p}(SV(X)) = \sigma(V(X))|_{SV(X)}$, that is $CX$ is the subspace of $(V(X), \sigma(V(X)))$.

Proof Suppose $U \in O_{\Rightarrow_p}(SV(X))$. Let $U_X = \{\mu \in V(X) : \exists \xi \in U, \xi \leq \mu\}$. Obviously, $U = U_X \cap CX$. We claim that $U_X$ is Scott open. Suppose we have a directed set of continuous valuations $\mathcal{F} = \{\mu_i \in I \subseteq V(X) \text{ with sup}_{i \in I} \mu_i = \mu \in U_X\}$. By the definition of $U_X$, there exists some $\xi = \sum_{i=1}^{n} r_{b_i} \eta_{b_i} \in U$ such that $\xi \leq \mu$. Since $X$ is a continuous domain, we have finitely directed sets $D_i = \downarrow b_i$ with $D_i \to b_i, i = 1, \ldots, n$. It is straightforward to verify that $\mathcal{D} = \{\sum_{i=1}^{n} r_{b_i} \eta_{b_i} : r_i < r_{b_i}, a_i \ll b_i, i = 1, \ldots, n\}$ is directed and $\mathcal{D} \Rightarrow_p \xi$. By the hypothesis of $U_X$, there exists some $\xi' \in \mathcal{D} \cap U$. According to the definition of $\mathcal{D}$ and definition 4.2, we have $\xi' \ll \xi$, then, by 4.3, $\xi' \ll \xi$. $V(X)$ is a continuous domain implies $\uparrow \xi'$ is a Scott open set. Since $\mathcal{F} = \{\mu_i \in I\}$ is a directed set with supremum $\mu \in \uparrow \xi'$, then we have some $\mu' \in \mathcal{F}$ with $\xi' \ll \mu'$, thus $\xi' \leq \mu$ and $\mu' \in U_X$, it follows that $U_X$ is Scott open.

On the other hand, let $V \in \sigma(V(X))$, and $\mathcal{D} \subseteq CX$ is a directed set with $\mathcal{D} \Rightarrow_p \xi = \sum_{i=1}^{n} r_{b_i} \eta_{b_i} \in V\cap CX$. We claim that $\mathcal{D} \Rightarrow_p \xi$ implies $\xi \leq \sup \mathcal{D}$. By the definition, we have finitely directed sets $D_i \subseteq X$ with $D_i \to b_i, i = 1, \ldots, n$. For arbitrary open set $U \in O(X)$ and suppose $b_1, \ldots, b_k \in U, 0 \leq k \leq n$. Thus we have $(d_1, \ldots, d_n) \in \prod_{i=1}^{k} D_i$ such that $d_i \in U, i = 1, \ldots, k$. Again by the definition of $\mathcal{D} \Rightarrow_p \xi$, $\forall r_{b_i} < r_{b_i}, i = 1, \ldots, k$, there exists some $\xi' \in \mathcal{D}$ with
\[ \sum_{i=1}^{k} r'_b \eta_{d_i} \leq \xi', \text{ then} \]
\[ \xi(U) = (\sum_{i=1}^{k} r'_b \eta_{d_i})(U) \leq \xi'(U) \leq (\sup D)(U). \]

It follows that \( \sup D \in \mathcal{V} \), thus there exists some \( \xi'' \in \mathcal{D} \cap \mathcal{V} \), then \( \xi'' \in \mathcal{D} \cap \mathcal{V} \cap \mathcal{CX} \), \( \mathcal{V} \cap \mathcal{CX} \) is open in \( \mathcal{CX} \). □

Finally, we ending this paper with an example.

**Example 4.10.** Let \( X = [0, 1] \) endowed with the Scott topology, \( \forall a \in [0, 1], (a, 1) \in \mathcal{O}(X) \), let \( \mu : \mathcal{O}(X) \to \mathbb{R}^+, \mu((a, 1]) = 1 - a \). Then \( \mu \) is a Scott continuous valuation but \( \mu \notin \mathcal{SV}(X) \), since each \( \xi = \sum_{i=1}^{n} r_b \eta_b \in \mathcal{SV}(X) \), the range of \( \eta_b, i = 1, \ldots, n, \) is 0 or 1, it follows that the range of \( \xi \) is finite. But the range of \( \mu \) is \( [0, 1] \). Thus the extended probabilistic powerspace of \( X \) is not equal to its extended probabilistic powerdomain.

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