DISTRIBUTION OF COMPLEX ALGEBRAIC NUMBERS

FRIEDRICH GÖTZE, DZIANIS KALIADA, AND DMITRY ZAPOROZHETS

Abstract. We study the asymptotic distribution of algebraic numbers of fixed degree in the complex plane with heights tending to infinity.

1. Introduction

The problem of investigating the distribution of algebraic numbers has many aspects and goes back more than century of history. Let us give a brief overview of known results obtained in this area.

Investigations of algebraic numbers widely involve potential theory and probabilistic methods. Here, we can mention a result obtained by Pritsker [15], who studied Schur’s problem on traces of algebraic numbers, and the asymptotic distribution of zeros of integral polynomials with growing degrees. The paper [15] also contains a number of references on this subject. Pritsker’s results are closely related to the setting of random polynomials, where the degrees of the polynomials grow to infinity. Here, one of research aspects is to study the distribution of zeros of random polynomials on the complex plane. The landmark result by Erdős and Turán [8] states that the arguments of complex roots of random polynomials are uniformly distributed as the degree tends to infinity. For some general conditions on polynomial coefficients, these roots are clustered near the unit circle [9].

There is a number of papers, in which all algebraic numbers in certain field extensions of a given degree with bounded multiplicative height are asymptotically counted as an upper bound for its heights tends to the infinity. For example, results of such type are obtained by Masser and Vaaler in [14] and [13]. References and some results related to the topic also can be found in [12, Chapter 3, §5].

Baker and Schmidt [2] introduced the concept of a regular system and proved that the set of algebraic numbers of degree at most \( n \) forms a regular system, that is, there exists a constant \( c_n \) depending on \( n \) only such that for any interval \( I \) for all sufficiently large \( Q \in \mathbb{N} \) there exist at least

\[
c_n |I| \frac{Q^{n+1}}{Q^{3n^2}}
\]

algebraic numbers \( \alpha_1, \ldots, \alpha_k \) of degree at most \( n \) and height at most \( Q \) satisfying

\[
|\alpha_i - \alpha_j| \geq \frac{(\ln Q)^{3n^2}}{Q^{n+1}}, \quad 1 \leq i < j \leq k.
\]

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Their results about regularity of the set of real algebraic numbers were improved by Beresnevich in [3], who showed that the logarithmic factors can be omitted. In the paper [5], it was also shown that complex algebraic numbers \( \alpha \) form a regular system with a function \( N(\alpha) = H(\alpha)^{-n(n+1)/2} \), in other words, there exists a constant \( c_n \) depending only on \( n \) such that in any circle \( C \) contained in the unit circle \( C_0 \subset \mathbb{C} \) for all sufficiently large \( Q \geq Q_0(C) \) there exist at least \( c_n Q^{n+1} |C| \) algebraic numbers \( \alpha_1, \ldots, \alpha_k \) of degree at most \( n \) and height at most \( Q \) such that distances between them are at least \( Q^{-(n+1)/2} \). For a more detailed discussion of the literature we refer to the excellent survey monograph of Bugeaud [6]. In [4], one can find results concerning the distribution of distances between conjugate algebraic numbers.

Note that the number of algebraic numbers of degree at most \( n \) and height at most \( Q \) is of the order \( Q^{n+1} \) as \( Q \to \infty \). Therefore these results show that for any fixed \( n \) the algebraic numbers are distributed quite regularly for sufficiently large height. However, results of such type describe the behaviour of a small part of algebraic numbers only.

An important question in this respect had been asked by K. Mahler in his letter to V. G. Sprindžuk in 1985: what is the distribution of algebraic numbers for a fixed degree \( n \geq 2 \)?

A possible answer to this question was suggested in [11] (see also [10] for the case \( n = 2 \)). Namely, fix \( n \geq 2 \) and denote by \( \Phi_Q(I) \) the number of algebraic numbers in the interval \( I \) of degree at most \( n \) and height at most \( Q \). Then

\[
\Phi_Q(I) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_I \varphi_n(x) \, dx + r_Q,
\]

where \( \zeta(x) \) denotes the value of the Riemann zeta function at \( x \), the remainder term \( r_Q \) satisfies

\[
(1) \quad r_Q = \begin{cases} 
O(Q^2 \ln Q), & n = 2, \\
O(Q^n), & n \geq 3,
\end{cases}
\]

as \( Q \to \infty \), and the limit density \( \varphi_n \) is given by the formula

\[
\varphi_n(x) = \int_{B_n(x)} \left| \sum_{k=1}^{n} k t_k x^{k-1} \right| dt_1 \ldots dt_n.
\]

The integration is performed over the region

\[
B_n(x) = \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n : \max_{1 \leq k \leq n} |t_k| \leq 1, \ |t_n x^n + \cdots + tx| \leq 1 \right\}.
\]

In particular, in some neighborhood of the origin (containing \([-1+1/\sqrt{2}, 1-1/\sqrt{2}])\) it holds

\[
\varphi_n(x) = \frac{2^{n-2}}{3} \left( 3 + \sum_{k=1}^{n-1} (k+1)^2 x^{2k} \right).
\]

Note in passing that \( \varphi_n \) coincides with the density of the real roots of a random polynomial with independent coefficients uniformly distributed in \([-1,1]\) (see, e.g., [18]).

The aim of this note is to extend this result to complex algebraic numbers.
1.1. **Notations.** Here we define all the notations which we will use in this paper.

We always assume that the degree \( n \) is some arbitrary but fixed integer number not less than 2 and the upper bound \( Q \) of the height goes to infinity. Hence the constants in different asymptotic relations (as \( Q \to \infty \)) in this paper might depend on \( n \).

As in [1], it will be typical that the case \( n = 2 \) has the extra-factor \( \log Q \). Therefore, for the sake of conciseness we put by definition

\[
l(n) = \begin{cases} 
1, & n = 2, \\
0, & n \geq 3.
\end{cases}
\]

For a complex domain \( \Omega \subset \mathbb{C} \) denote by \( \Psi_Q(\Omega) \) the number of algebraic numbers in \( \Omega \) of degree at most \( n \) and height at most \( Q \). We always assume that \( \Omega \) does not intersect real axis and that its boundary consists of a finite number of algebraic curves.

For any Borel set \( A \subset \mathbb{R}^m \) denote by \( \text{Vol}(A) \) the Lebesgue measure of \( A \), denote by \( \lambda(A) \) the number of points in \( A \) with integer coordinates, and denote by \( \lambda^*(A) \) the number of points in \( A \) with coprime integer coordinates.

The Riemann zeta function is denoted by \( \zeta(\cdot) \) and the Möbius function is denoted by \( \mu(\cdot) \).

### 2. Main result

**Theorem 2.1.** The following asymptotic approximation holds

\[
\Psi_Q(\Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_{\Omega} \psi_n(z) \nu(dz) + O \left( Q^n \log^n(l(n)) \right), \quad Q \to \infty,
\]

where \( \nu \) is the Lebesgue measure on the complex plane. The limit density \( \psi_n \) is given by the formula

\[
(3) \quad \psi_n(z) = \frac{1}{|\text{Im} z|} \int_{D_n(z)} \left| \sum_{k=1}^{n-1} t_k \left( (k+1)z^k - \frac{\text{Im} z^{k+1}}{\text{Im} z} \right) \right|^2 dt_1 \ldots dt_{n-1}.
\]

The integration is performed over the region

\[
D_n(z) = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \leq k \leq n-1} |t_k| \leq 1, \right. \\
\left. \left| z \sum_{k=1}^{n-1} t_k \left( z^k - \frac{\text{Im} z^{k+1}}{\text{Im} z} \right) \right| \leq 1, \right. \\
\left. \left| \frac{1}{\text{Im} z} \sum_{k=1}^{n-1} |t_k \text{Im} z^{k+1}| \right| \leq 1 \right\}.
\]

**Remark.** The implicit constant in the big-O-notation in (2) depends only on the degree \( n \), and on the maximal degree and the number of algebraic curves that form the boundary \( \partial \Omega \).

The proof of Theorem 2.1 is given in Section 3. Now let us derive several properties of the limit density \( \psi_n \).
Proposition 2.2. The function $\psi_n$ is positive on $\mathbb{C}$ and satisfies the following functional equations:

$$
\psi_n(-z) = \psi_n(\bar{z}) = \psi_n(z),
$$

(4)

$$
\psi_n\left(\frac{1}{z}\right) = |z|^4 \psi_n(z).
$$

Proof. The positiveness as well as the first relation are trivial. To prove (4) note that for any integral irreducible polynomial $g(z)$ of degree $n$, the polynomial $z^n g(z^{-1})$ is also irreducible and has the same degree and the same height. Hence for any domain $\Omega \subseteq \mathbb{C}$ it holds

$$
\Psi_Q(\Omega) = \Psi_Q(\Omega^{-1}),
$$

where $\Omega^{-1}$ is defined as $\Omega^{-1} = \{ z^{-1} \in \mathbb{C} : z \in \Omega \}$. When $Q$ tends to infinity, we get by applying Theorem 2.1

$$
\int_{\Omega} \psi_n(z) \nu(dz) = \int_{\Omega^{-1}} \psi_n(z) \nu(dz).
$$

On the other hand, after the substitution $z \to 1/z$, we obtain

$$
\int_{\Omega} \psi_n(z) \nu(dz) = \int_{\Omega^{-1}} \psi_n(z^{-1}) |z|^{-4} \nu(dz).
$$

Since the class of domains $\Omega$ is sufficiently large, (4) follows. □

The next statement shows that, in some sense, there is "repulsion" from the real axis for non-real algebraic numbers which increases inversely with the size of the imaginary part.

Proposition 2.3. It holds

$$
\psi_n(x_0 + iy) = A|y| \cdot (1 + o(1)), \quad y \to 0,
$$

where the constant $A$ does not depend on $y$ and can be written explicitly as follows:

$$
A = \int_{\bar{D}_n(x_0)} \left| \sum_{k=1}^{n-1} k(k+1)t_kx_0^{k-1} \right|^2 dt_1 \ldots dt_{n-1}.
$$

Here, the integration is performed over the region

$$
\bar{D}_n(x_0) = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \leq k \leq n-1} |t_k| \leq 1,
\sum_{k=1}^{n-1} k t_k x_0^{k+1} \leq 1, \quad \sum_{k=1}^{n-1} (k+1)t_k x_0^k \leq 1 \right\}.
$$

Proof. Since

$$
\frac{\text{Im } z^{k+1}}{\text{Im } z} = \frac{z^{k+1} - \bar{z}^{k+1}}{z - \bar{z}} = \sum_{j=0}^{k} z^{k-j} \bar{z}^j,
$$
it follows that
\[(k + 1)z^k - \frac{\text{Im} z^{k+1}}{\text{Im} z} = \sum_{j=0}^{k} z^{k-j} (z^j - \bar{z}^j) = (z - \bar{z}) \sum_{j=1}^{k} z^{k-j} \sum_{m=0}^{j-1} z^{j-1-m} \bar{z}^m = (z - \bar{z}) \sum_{s=1}^{k} sz^{s-1} \bar{z}^{k-s}.\]

Hence \(\psi_n(z)\) and \(D_n(z)\) can be rewritten as follows:
\[
\psi_n(z) = 4 |\text{Im} z| \int_{D_n(z)} \left| \sum_{k=1}^{n-1} t_k \sum_{s=1}^{k} sz^{s-1} \bar{z}^{k-s} \right|^2 dt_1 \ldots dt_{n-1},
\]
and
\[
D_n(z) = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \leq k \leq n-1} |t_k| \leq 1, \left| \sum_{k=1}^{n-1} t_k \sum_{j=1}^{k} z^{k-j+1} \bar{z}^j \right| \leq 1, \left| \sum_{k=1}^{n-1} t_k \sum_{j=0}^{k} z^{k-j} \bar{z}^j \right| \leq 1 \right\}.
\]

Note that \(\tilde{D}_n(x_0) = D_n(x_0 + 0 \cdot i)\). Letting \(\text{Im} z \to 0\) concludes the proof. □

When \(|z|\) is relatively small or relatively large, it is possible to write the limit density in a simpler form.

**Proposition 2.4.** If \(|z| \leq 1 - \frac{1}{\sqrt{2}}\), then
\[
\psi_n(z) = \frac{2^{n-1}}{3 |\text{Im} z|} \sum_{k=1}^{n-1} (k + 1)z^k - \frac{\text{Im} z^{k+1}}{\text{Im} z} \right|^2.
\]

If \(|z| \geq 2 + \sqrt{2}\), then
\[
\psi_n(z) = \frac{2^{n-1}}{3 |\text{Im} z|} \sum_{k=1}^{n-1} \frac{1}{|z|^{4k+4}} \left| (k + 1)z^k - \frac{\text{Im} z^{k+1}}{\text{Im} z} \right|^2.
\]

**Proof.** For \(|z| \leq 1 - \frac{1}{\sqrt{2}}\) it holds
\[
\sum_{k=2}^{n} (k-1)|z|^k \leq 1, \quad \text{and} \quad \sum_{k=2}^{n} k|z|^{k-1} \leq 1,
\]
which leads to
\[
D_n(z) = [-1, 1]^{n-1},
\]
and a straightforward integration yields the first relation. The second statement follows from the first one and □

Let us conclude the section by considering the case \(n = 2\).

**Example.** In the case of quadratic algebraic numbers, the density function takes the form
\[
\psi_2(z) = \frac{2}{|\text{Im} z|} \int_{D_2(z)} |t \text{Im} z|^2 dt,
\]
where
\[ D_2(z) = \left\{ t \in \mathbb{R} : |t| \leq \min \left(1, \frac{1}{|z|^2}, \frac{1}{2|\Re z|}\right) \right\}. \]

After a simple calculation we obtain
\[ \psi_2(x + iy) = \begin{cases} 8y, & \text{if } x^2 + y^2 \leq 1, \text{ and } |x| \leq \frac{1}{2}, \\ \frac{y}{3x^2}, & \text{if } (|x| - 1)^2 + y^2 \leq 1, \text{ and } |x| > \frac{1}{2}, \\ \frac{8y}{3(x^2 + y^2)}, & \text{if } (|x| - 1)^2 + y^2 > 1, \text{ and } x^2 + y^2 > 1. \end{cases} \]

3. Proof of Theorem 2.1

Denote by \( P_Q \) a class of all integral polynomials of degree at most \( n \) and height at most \( Q \). The cardinality of the class is \((2^n + 1)^n\). Recall that an integral polynomial is called prime, if it is irreducible over \( \mathbb{Q} \), primitive (the greatest common divisor of its coefficients equals 1), and its leading coefficient is positive.

For \( k \in \{0, 1, \ldots, n\} \) denote by \( \gamma_k \) the number of prime polynomials from \( P_Q \) that have exactly \( k \) roots lying in \( \Omega \). For any algebraic number its minimal polynomial is prime, and any prime polynomial is a minimal polynomial for some algebraic number. Therefore,
\[ \Psi_Q(\Omega) = \sum_{k=1}^{n} k \gamma_k. \]

Consider a subset \( A_k \subset [-1,1]^{n+1} \) consisting of all points \((t_0, \ldots, t_n) \in [-1,1]^{n+1}\) such that the polynomial \( t_n x^n + \cdots + t_1 x + t_0 \) has exactly \( k \) roots lying in \( \Omega \). Then the number of primitive polynomials from \( P_Q \) which have exactly \( k \) roots in \( \Omega \) is equal to \( \lambda^*(Q A_k) \). By the definition of a prime polynomial, we have
\[ \left| \gamma_k - \frac{1}{2} \lambda^*(Q A_k) \right| \leq R_Q, \]
where \( R_Q \) denotes a number of reducible polynomials (over \( \mathbb{Q} \)) from \( P_Q \). Note that the factor \( \frac{1}{2} \) arises in above inequality because prime polynomials have positive leading coefficient. It is known (see [17]) that
\[ R_Q = O \left( Q^n \log^{l(n)} Q \right), \quad Q \to \infty. \]

Hence it follows that
\[ (5) \quad \Psi_Q(\Omega) = \frac{1}{2} \sum_{k=1}^{n} k \lambda^*(Q A_k) + O \left( Q^n \log^{l(n)} Q \right), \quad Q \to \infty. \]

To estimate \( \lambda^*(Q A_k) \) we need the following lemma.

**Lemma 3.1.** Consider a domain \( A \subset \mathbb{R}^m, m \geq 2 \), with boundary consisting of a finite number of algebraic surfaces only. Then
\[ (6) \quad \lambda^*(t A) = \frac{\Vol(A)}{\zeta(m)} t^m + O \left( t^{m-1} \log^{l(m)} t \right), \quad t \to \infty. \]

Here, the implicit constant in the big-O-notation depends only on the maximal degree and the number of bounding surfaces.

**Remark.** The result of the lemma is well-known. One can find a result of this type e.g. in classical monograph by Bachmann [1], pp. 436–444] (see especially formulas (83a) and (83b) on pages 441–442). For the readers convenience we include a short proof here.
Proof. Note that
\[ \lambda(tA) = \sum_{d=1}^{[Nt]+1} \lambda^*(\frac{t}{d}A), \]
where \( N \) is chosen large enough such that \( A \subset [-N,N]^m \). Applying the classical Möbius inversion formula (see, e.g., [16]) yields
\[ (7) \lambda^*(tA) = \sum_{d=1}^{[Nt]+1} \mu(d) \lambda\left(\frac{t}{d}A\right). \]

By the Lipschitz principle (see [7]) it follows that
\[ \left|\lambda\left(\frac{t}{d}A\right) - \left(\frac{t}{d}\right)^m \text{Vol}(A)\right| \leq c \cdot \left(\frac{t}{d}\right)^{m-1} \]
for some constant \( c \) depending only on the maximal degree and the number of algebraic surfaces that compose the boundary \( \partial A \). Applying this to (7), we get
\[ (8) \left|\lambda^*(tA) - \text{Vol}(A)\right| \leq c t^{m-1} \sum_{d=1}^{[Nt]+1} \frac{1}{d^{m-1}}. \]

It is well known (see, e.g., [16]) that
\[ \sum_{d=1}^{\infty} \frac{\mu(d)}{d^m} = \frac{1}{\zeta(m)}. \]

Therefore,
\[ (9) \left|\sum_{d=1}^{[Nt]+1} \frac{\mu(d)}{d^m} - \frac{1}{\zeta(m)}\right| \leq \sum_{d=[Nt]+2}^{\infty} \frac{1}{d^m} \leq \frac{1}{(m-1)(Nt)^{m-1}}. \]

Furthermore, it holds that
\[ (10) \sum_{d=1}^{[Nt]+1} \frac{1}{d^{m-1}} \leq \begin{cases} \zeta(m-1), & m \geq 3, \\ \log([Nt]+1) + 1, & m = 2. \end{cases} \]

Combining (8), (9), and (10) completes the proof.

Since \( \Omega \) is bounded by a finite number of algebraic curves, the boundary of \( A_k \) is formed by a finite number of algebraic surfaces. It follows from Lemma (3.1) that
\[ \lambda^*(Q A_k) = \frac{\text{Vol}(A_k)}{\zeta(n+1)} Q^{n+1} + O(Q^n), \quad t \to \infty, \]
which together with (5) implies
\[ (11) \Psi_Q(\Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \sum_{k=1}^{n} k \text{Vol}(A_k) + O\left(Q^n \log^l(n) Q\right), \quad Q \to \infty. \]

To calculate \( \sum_{k=1}^{n} k \text{Vol}(A_k) \) we need the following result from the theory of random polynomials. Let \( \xi_0, \xi_1, \ldots, \xi_n \) be independent random variables uniformly distributed on \([-1,1]\). Consider the random polynomial
\[ G(x) = \xi_n x^n + \xi_{n-1} x^{n-1} + \cdots + \xi_1 x + \xi_0. \]
Denote by $N(\Omega)$ the number of the roots of $G(z)$ lying in $\Omega$. By definition,

$$\text{Vol}(A_k) = 2^{n+1} \mathbb{P}(N(\Omega) = k),$$

which implies

$$\sum_{k=1}^{n} k \text{Vol}(A_k) = 2^{n+1} \mathbb{E}N(\Omega). \quad (12)$$

The right-hand side of the latter relation was calculated in [19] in more general setup: it was shown that if the coefficients $\xi_0, \xi_1, \ldots, \xi_n$ have a joint density of distribution $p(x_0, x_1, \ldots, x_n)$, then $\mathbb{E}N(\Omega)$ is given by the formula

$$\mathbb{E}N(\Omega) = \int_{\Omega} dr \, d\alpha \int_{\mathbb{R}^{n-1}} dt_1 \ldots dt_{n-1} \frac{r^2}{\sin \alpha} \times$$

$$\left( \left[ \sum_{k=1}^{n-1} t_k r^{k-1} \left( (k+1) \cos(k+1)\alpha - \cos \alpha \frac{\sin(k+1)\alpha}{\sin \alpha} \right) \right]^2 + \left[ \sum_{k=1}^{n-1} kt_k r^{k-1} \sin(k+1)\alpha - \sin \alpha \sum_{k=1}^{n-1} t_k r^k \sin (k+1)\alpha, \ t_1, \ldots, t_{n-1} \right] \right),$$

where $r = |z|$ and $\alpha = \arg z$ are polar coordinates in the complex plane. The corresponding formula in [19] contains a typo. Here we use the correct version.

In the case when the coefficients are independent and uniformly distributed on $[-1, 1]$, their joint distribution density function is given by the normalized indicator function of the cube $[-1, 1]^{n+1}$, and after some transformation (13) takes the form

$$\mathbb{E}N(\Omega) = \frac{1}{2^{n+1}} \int_{\Omega} \psi_n(z) \nu(dz),$$

where $\psi_n$ is defined in (3). Combining this with (12) and (11) gives (2).

**Remark.** The equivalence of (3) and (13) can be shown by following computation. For the integrand in (3), we have

$$\left| \sum_{k=1}^{n-1} t_k \left( (k+1)z^k - \frac{\text{Im } z^{k+1}}{\text{Im } z} \right) \right|^2 = \frac{1}{r^2} \left| \sum_{k=1}^{n-1} t_k \left( (k+1)z^{k+1} - z - \frac{\text{Im } z^{k+1}}{\text{Im } z} \right) \right|^2 =$$

$$= \sum_{k=1}^{n-1} t_k r^k \left[ (k+1) \cos(k+1)\alpha - \cos \alpha \frac{\sin(k+1)\alpha}{\sin \alpha} \right]^2 + i \left[ k \sin(k+1)\alpha \right]^2.$$
In (3), the domain of integration $D_n(z)$ is a subset of $[-1, 1]^{n-1}$ defined by two “big” inequalities. The r.h.s. part of the first one can be transformed as follows:

$$
\left| \sum_{k=1}^{n-1} t_k \left( z^{k+1} - z \frac{\text{Im} z^{k+1}}{\text{Im} z} \right) \right| = \left| \sum_{k=1}^{n-1} t_k \left( \text{Re} z^{k+1} - \text{Re} z \frac{\text{Im} z^{k+1}}{\text{Im} z} \right) \right| = \left| \frac{1}{\sin \alpha} \sum_{k=1}^{n-1} t_k r^{k+1} \sin (k+1) \alpha \right|.
$$

For the r.h.s. part of the second “big” inequality, we immediately have

$$
\left| \frac{1}{\text{Im} z} \sum_{k=1}^{n-1} t_k \text{Im} z^{k+1} \right| = \left| \frac{1}{\sin \alpha} \sum_{k=1}^{n-1} t_k r^{k+1} \sin (k+1) \alpha \right|.
$$

For the coefficients independently and uniformly distributed on $[-1, 1]$, we have

$$
p(x_0, x_1, \ldots, x_n) = \begin{cases} 2^{-(n+1)}, & \max_{0 \leq k \leq n} |x_k| \leq 1, \\ 0, & \text{otherwise.} \end{cases}
$$

From the above calculations, it is seen that the function

$$
p \left( \frac{1}{\sin \alpha} \sum_{k=1}^{n-1} t_k r^{k+1} \sin k \alpha, -\frac{1}{\sin \alpha} \sum_{k=1}^{n-1} t_k r^{k} \sin (k + 1) \alpha, t_1, \ldots, t_{n-1} \right)
$$

is not equal to zero if and only if $(t_1, \ldots, t_n) \in D_n(z)$, where $z = r(\cos \alpha + i \sin \alpha)$.

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FRIEDRICH GÖTZE, FACULTY OF MATHEMATICS, BIELEFELD UNIVERSITY, P. O. BOX 10 01 31, 33501 BIELEFELD, GERMANY

E-mail address: goetze@math.uni-bielefeld.de

DZIANIS KALIADA, INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF BELARUS, 220072 MINSK, BELARUS

E-mail address: koledad@rambler.ru

DMITRY ZAPOROZHETS, ST. PETERSBURG DEPARTMENT OF STEKLOV INSTITUTE OF MATHEMATICS, FONTANKA 27, 191011 ST. PETERSBURG, RUSSIA

E-mail address: zap1979@gmail.com