ON THE LEAST COMMON MULTIPLE OF POLYNOMIAL SEQUENCES AT PRIME ARGUMENTS

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Abstract. Cilleruelo conjectured that if $f \in \mathbb{Z}[x]$ is an irreducible polynomial of degree $d \geq 2$ then, $\log \text{lcm}\{f(n) \mid n < x\} \sim (d-1)x \log x$. In this article, we investigate the analogue of prime arguments, namely, $\text{lcm}\{f(p) \mid p < x\}$, where $p$ denotes a prime and obtain non-trivial lower bounds on it. Further, we also show some results regarding the greatest prime divisor of $f(p)$.

1. Introduction

For a polynomial $f \in \mathbb{Z}[x]$, define $L_f(x) = \text{lcm}\{f(n) \mid n < x \text{ and } f(n) \neq 0\}$, where the lcm of an empty set is taken to be 1. The Prime Number Theorem is equivalent to $\log \text{lcm}\{1, 2, \ldots, n\} \sim n$.

Therefore, we expect similar rate of growth for the case when $f$ is a product of linear polynomials; see the article by Hong, Qian, and Tan [7] for a thorough analysis of this case. However, the growth is not the same for higher degree polynomials. Cilleruelo in [2] conjectured that $\log L_f(x) \sim (d-1)x \log x$ for irreducible polynomials $f$ of degree $d \geq 2$ and proved it for $d = 2$. For some time, $\log L_f(x) \gg x$ proven by Hong, Luo, Qian, and Wang in [6], for polynomials with non-negative integer coefficients, was the strongest bound known. Recently, the conjectured order of growth was obtained by Maynard and Rudnick in [10] and the bound was improved to $x \log x$ by Sah in [12]. For a thorough survey on the least common multiple of polynomial sequences, see [1].

In this article, we study the analogous problem at prime arguments. From the Prime Number Theorem, we know that $\log \text{lcm}\{p \mid p < x\} \sim x$.

This motivates us to consider $\text{lcm}\{f(p) \mid p < x\}$ for an arbitrary polynomial $f \in \mathbb{Z}[x]$. For simplicity, we will only consider irreducible polynomials $f$.

Theorem 1.1. Let $f \in \mathbb{Z}[x]$ be an irreducible polynomial of degree $d$. Then,$$\log \text{lcm}\{f(p) \mid p < x\} \gg x^{1-\varepsilon(d)},$$where $\varepsilon(1) = 0.3735$, $\varepsilon(2) = 0.153$ and $\varepsilon(d) = \exp\left(\frac{-d-0.9788}{2}\right)$ for $d \geq 3$.
We remark that \( \log \text{lcm}\{f(p) \mid p < x\} \leq (d + o(1))x \ll x \) follows from the Prime Number Theorem.

There is a lot of literature on the subject of largest prime divisor of \( p + a \) for some fixed integer \( a \). Goldfeld in [4] showed that there is a positive proportion of primes \( p \) such that \( p + a \) has a prime divisor greater than \( p^\delta \) for \( \delta = 0.5 \). The strongest known result in this regard is \( \delta = 0.677 \) proven by Baker and Harman in [5, Theorem 8.3], an improvement of \( \delta = 0.6687 \) obtained by Fouvry in [3]. Luca in [9] obtained lower bounds on the proportion of such primes \( p \) for \( \delta \in [\frac{1}{3}, \frac{1}{2}] \). Similar work is also done for quadratic polynomials. Wu and Xi in [14] proved that there exist infinitely many primes \( p \) such that \( p^2 + 1 \) has a prime divisor greater than \( p^{0.847} \) by virtue of the Quadratic Brun-Titchmarsh theorem (see Theorem 2.4) developed by the authors.

We obtain a result of a similar flavor for general polynomials which we state as follows.

**Theorem 1.2.** Let \( f \in \mathbb{Z}[x] \) be an irreducible polynomial of degree \( d \). Then, there is a positive proportion of primes \( p \) such that \( f(p) \) has a prime divisor greater than \( p^{1-\epsilon(d)} \), where \( \epsilon(1) = 0.3735, \epsilon(2) = 0.153 \) and \( \epsilon(d) = \exp\left(-\frac{d-0.9788}{2}\right) \) for \( d \geq 3 \).

The following table shows some values of \( 1 - \epsilon(d) \) for various \( d \).

| \( d \) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( 1 - \epsilon(d) \) | 0.6265 | 0.847 | 0.8632 | 0.9170 | 0.9496 | 0.9694 | 0.9814 | 0.9887 |

**Notations.** We employ Landau-Bachmann notations \( O \) and \( o \) as well as their associated Vinogradov notations \( \ll \) and \( \gg \). We say that \( a(x) \sim b(x) \) if

\[
\lim_{x \to \infty} \frac{a(x)}{b(x)} = 1.
\]

As usual, define \( \pi(x; m, a) \) to be the number of primes \( p < x \) such that \( p \equiv a \pmod{m} \). Throughout the article, \( p \) and \( q \) will denote primes, and we fix an irreducible polynomial \( f \in \mathbb{Z}[x] \) of degree \( d \geq 1 \). We will often suppress the dependence of constants on \( f \). At places, we may use Mertens’ first theorem without commentary.

2. Background

**Theorem 2.1** (Brun-Titchmarsh, [11]). Let \( \theta = \frac{\log m}{\log x} \), where \( \theta \in (0, 1) \). Then,

\[
\pi(x; m, a) \leq (C(\theta) + o(1)) \cdot \frac{x}{\phi(m) \log x}
\]

where

\[
C(\theta) = \frac{2}{1 - \theta}.
\]
Corollary 2.2. Let $\varepsilon > 0$ be a constant. Then,
\[ \pi(x; m, a) \ll_{\varepsilon} x \frac{x}{\phi(m) \log x} \]
for all positive integers $m < x^{1-\varepsilon}$.

Theorem 2.3 (Iwaniec, [8]). Let $\theta = \frac{\log m}{\log x}$ where $\theta \in \left[ \frac{9}{10}, \frac{2}{3} \right]$. Then,
\[ \pi(x; m, a) < (C(\theta) + o(1)) \cdot \frac{x}{\phi(m) \log x}, \]
where
\[ C(\theta) = \frac{8}{6 - 7\theta}. \]

Theorem 2.4 (Wu and Xi, [15]). Let $A > 0$ and $f(x)$ be an irreducible quadratic polynomial. Define $\varsigma(m) = \# \{ p < x \mid f(p) \equiv 0 \pmod{m} \}$ and $\rho(m)$ to be the number of solutions of the congruence $f(x) \equiv 0 \pmod{m}$. For large $L = x^\theta$ with $\theta \in \left[ \frac{3}{2}, \frac{18}{17} \right)$, we have
\[ \varsigma(m) \leq (C(\theta) + o(1))\rho(m) \cdot \frac{x}{\phi(m) \log x}, \]
for all $m \in [L, 2L]$ with at most $O_A(L/(\log L)^A)$ exceptions, where
\[ C(\theta) = \begin{cases} 
\frac{124}{91 - 89\theta}, & \text{if } \theta \in \left[ \frac{1}{2}, \frac{64}{97} \right), \\
\frac{86 - 89\theta}{19}, & \text{if } \theta \in \left[ \frac{64}{97}, \frac{32}{41} \right), \\
\frac{28}{19 - 18\theta}, & \text{if } \theta \in \left[ \frac{32}{41}, \frac{16}{17} \right). 
\end{cases} \]

Theorem 2.5 (Bombieri-Vinogradov). Let $A \geq 6$ and $Q \leq x^{\frac{4}{5}}/(\log x)^A$. Then,
\[ \sum_{q \leq Q} \max_{2 \leq y \leq x} \max_{(a, q) = 1} \left| \pi(y; q, a) - \frac{y}{\phi(q) \log y} \right| \ll_A \frac{x}{(\log x)^B}, \]
where $B = A - 5$.

Lemma 2.6. Let $f$ be an irreducible integer polynomial and $\rho(m)$ be the number of roots of the congruence $f(x) \equiv 0 \pmod{m}$. Then,
\[ \sum_{p < x} \frac{\rho(p) \log p}{p - 1} = \log x + R + o(1) \]
for some constant $R$.

Proof. By [13, 3.3.3.5], we have that
\[ \sum_{p < x} \rho(p) = \text{Li}(x) + O\left( \frac{x}{(\log x)^2} \right), \]
where $\text{Li}(x)$ is the logarithmic integral. Applying Abel summation formula,

$$\sum_{p<x} \frac{\rho(p) \log p}{p} = \frac{\log x}{x} \sum_{p<x} \rho(p) + \int_2^x \frac{\log x - 1}{x^2} \left( \sum_{p<u} \rho(p) \right) \, du + C_0$$

$$= C_0 + 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{\log u - 1}{u^2} \log \text{Li}(u) \, du + O\left(\int_2^x \frac{\log u - 1}{u \log u} \, du\right)$$

$$= \log x + C_1 + O\left(\frac{1}{\log x}\right)$$

for some constants $C_0$ and $C_1$. And the sum

$$\sum_{p<x} \frac{\rho(p) \log p}{p - 1} - \sum_{p<x} \frac{\rho(p) \log p}{p} = \sum_{p<x} \frac{\rho(p) \log p}{p(p-1)}$$

is $C_2 + o(1)$ for some constant $C_2$. Hence, our lemma is proved. \qed

3. Proof of Theorem 1.1

3.1. Setup. We study the product defined by

$$Q(x) = \prod_{q<x} |f(q)| = \prod_p p^{\alpha_p(x)}$$

and exploit the fact that the contribution of prime factors less than $x^\delta$ is negligible compared to that of prime factors greater than $x^\delta$, where $\delta$ is a parameter in $(\frac{1}{2}, 1)$ to be chosen later. For some large enough constant $B$, set $x_B = x^{1/2} (\log x)^{-B}$ for brevity.

Define $\varrho(m)$ to be the set of residues modulo $m$ which satisfy the congruence $f(x) \equiv 0 \pmod{m}$ and $\rho(m)$ to be the cardinality of $\varrho(m)$. Note that we have $\rho(m) \leq d$ by Lagrange’s theorem and that if $p \nmid \text{disc } f$ then $\rho(p) = \rho(p^n)$ for all $n \geq 2$ by Hensel’s lemma. Also define $\varsigma(m)$ to be the sum

$$\sum_{r \in \varrho(m)} \pi(x; m, r),$$

the number of elements in $\{f(p) \mid p < x\}$ divisible by $m$.

3.2. Estimating small primes. We define

$$Q_S(x) = \prod_{p<x_B} p^{\alpha_p(x)}$$

the part of $Q(x)$ consisting of small prime divisors. The main result here is the following.

**Proposition 3.1.** $\log Q_S(x) = \frac{x}{2} - \frac{B x \log \log x}{\log x} + O\left(\frac{x}{\log x}\right)$

The proof uses an estimate on $\alpha_p(x)$ making it easy to directly apply the Bombieri-Vinogradov theorem (Theorem 2.5) in the end. The following result is proved by standard analysis involving Hensel’s lemma and the Brun-Titchmarsh theorem (Corollary 2.2).
Lemma 3.2. Let $p$ be a prime. If $p \nmid \text{disc } f$, then

$$
\alpha_p(x) = \sum_{p^n < x_b} \zeta(p^n) + O \left( \frac{x}{\max\{p, x_b\} \log x} + \frac{(\log x)^2 B}{\log p} \right);
$$

else if $p \mid \text{disc } f$, we have

$$
\alpha_p(x) = \zeta(p).
$$

Proof. The case when $p \mid \text{disc } f$ is easy to solve. So, let us assume $p \nmid \text{disc } f$. Observe that

$$
\alpha_p(x) = \sum_{n=1}^{\infty} \zeta(p^n).
$$

When $p^n \geq x$, we see that $\zeta(p^n) \leq \rho(p^n) \leq d$. If $p^n$ divides $f(k)$ for some $1 \leq k \leq x$, we have $p^n \leq f(k) \leq f(x) < x^{d+1}$, which implies that $n < (d+1) \log x/\log p$. Thus,

$$
\alpha_p(x) = \sum_{n=1}^{\infty} \zeta(p^n) = \sum_{p^n < x} \zeta(p^n) + O \left( \frac{\log x}{\log p} \right).
$$

We split the summation into three intervals: $p^n \in [1, x_b] \cup (x_b, x^{0.9}] \cup (x^{0.9}, x)$. The last summation is

$$
\sum_{p^n \in (x^{0.9}, x)} \zeta(p^n) \leq \sum_{p^n \in (x^{0.9}, x) \cap \rho(p^n)} \left( \frac{x}{p^n} + 1 \right) \leq \sum_{p^n \in (x^{0.9}, x)} \rho(p^n) (x^{0.1} + 1) \ll x^{0.2}.
$$

By Corollary 2.2, the second summation is

$$
\sum_{p^n \in (x_b, x^{0.9}]} \zeta(p^n) \ll \frac{\rho(p)x}{\log x} \sum_{x_b < p^n \leq x^{0.9}} \frac{1}{\varphi(p^n)}
$$

$$
\ll \frac{x}{\max\{p, x_b\} \log x} + \frac{x}{\log x} \sum_{x_b < p^n \leq x^{0.9}} \frac{1}{p^n}
$$

$$
\ll \frac{x}{\max\{p, x_b\} \log x} + \frac{x}{\log x} \log x \frac{1}{p^2}
$$

$$
\ll \frac{x}{\max\{p, x_b\} \log x} + \frac{(\log x)^2 B}{\log p}.
$$

Thus, our lemma is proved. \qed

Proof of Proposition 3.1. Using Lemma 3.2,

$$
\log Q_S(x) = \sum_{p < x_b} \alpha_p(x) \log p
$$

$$
= \sum_{p < x_b} \left( \sum_{p^n < x_b} \zeta(p^n) + O \left( \frac{x}{x_b \log x} + \frac{(\log x)^2 B}{\log p} \right) \right) \log p
$$

$$
= \sum_{m < x_b} \zeta(m) \Lambda(m) + O \left( \frac{x}{\log x} \right).
$$
Using Theorem 2.5 and Lemma 2.6, we can estimate the above sum as
\[
\sum_{m < x^B} \zeta(m)\Lambda(m) = \frac{x}{\log x} \sum_{m < x^B} \frac{\rho(m)\Lambda(m)}{\phi(m)} + O\left(\frac{x}{(\log x)^{B-5}}\right)
\]
\[
= \frac{x}{\log x} \left(\frac{1}{2} \log x - B \log \log x\right) + O\left(\frac{x}{(\log x)^{B-5}}\right)
\]
\[
= \frac{x}{2} - \frac{Bx \log \log x}{\log x} + O\left(\frac{x}{\log x}\right),
\]
proving the result. □

3.3. Removing medium-sized primes. Define the product
\[
Q_M(x) = \prod_{x^B \leq p \leq x^{1/2}} p^{\alpha_p(x)},
\]
the part of \(Q(x)\) consisting of medium-sized primes. The main result of this section is the following.

**Proposition 3.3.** \(\log Q_M(x) \ll \frac{x \log \log x}{\log x}\).

This means we can just remove medium-sized primes from \(\log Q(x)\) and only lose a sublinear portion. The proof is a simple computation using Lemma 3.2.

**Proof of Proposition 3.3.** From Lemma 3.2, it follows that
\[
\log Q_M(x) = \sum_{x^B \leq p \leq x^{1/2}} \alpha_p(x) \log p
\ll \sum_{x^B \leq p \leq x^{1/2}} \left(\frac{x}{p \log x} + \frac{(\log x)^{2B}}{\log p}\right) \log p
\]
\[
= \frac{x}{\log x} \left(\sum_{x^B \leq p \leq x^{1/2}} \frac{\log p}{p}\right) + O\left(x^{1/2} (\log x)^{2B}\right)
\ll \frac{x \log \log x}{\log x},
\]
as desired. □

3.4. Bounding large primes. Define the product
\[
Q_L(x) = \prod_{x^{1/2} < p < x^3} p^{\alpha_p(x)},
\]
the part of \(Q(x)\) consisting of large primes. The main result of this section is the following.

**Proposition 3.4.** \(\log Q_L(x) \leq (1 + o(1)) x \int_{1/2}^{\delta} C(\theta) \, d\theta\).

The proof uses the Brun-Titchmarsh theorem (Theorem 2.1 and 2.3) and involves standard procedures to convert sums over primes to integrals.
Proof of Proposition 3.4. Let $p$ be a prime in $(x^{1/2}, x^\delta)$. Similar to the proof of Lemma 3.2, we have

$$\alpha_p(x) = \sum_{n=1}^{\infty} \varsigma(p^n) = \varsigma(p) + \mathcal{O}(\log x / \log p) = \varsigma(p) + \mathcal{O}(1)$$

as $p^2 > x$. Therefore,

$$\log Q_L(x) = \sum_{x^{1/2} < p < x^\delta} \alpha_p(x) \log p$$

$$= \sum_{x^{1/2} < p < x^\delta} \varsigma(p) \log p + \mathcal{O}(x^\delta).$$

By Theorem 2.1, 2.3 and Lemma 2.6, we have

$$\sum_{x^{1/2} < p < x^\delta} \varsigma(p) \log p \leq \sum_{x^{1/2} < p < x^\delta} \frac{(C(\theta) + o(1))x}{\phi(p) \log x} \rho(p) \log p$$

$$= \frac{x}{\log x} \sum_{x^{1/2} < p < x^\delta} \frac{C(\theta) + o(1)}{\phi(p)} \rho(p) \log p$$

$$= \frac{x}{\log x} \sum_{x^{1/2} < p < x^\delta} C(\theta) \frac{\rho(p) \log p}{p - 1} + o\left(\frac{x \log \log x}{\log x}\right).$$

It can be verified that the above inequality is true even when $f$ is an irreducible quadratic polynomial and we apply Theorem 2.4 instead of Theorem 2.3. By standard techniques to convert sums over primes into integrals, we have

$$\sum_{x^{1/2} < p < x^\delta} \varsigma(p) \log p \leq (1 + o(1))x \int_{1/2}^{\delta} C(\theta) \, d\theta,$$

proving the lemma.

3.5. The main bound. It is easy to see that

$$\log Q(x) = \sum_{p < x} (d \log p + \mathcal{O}(1)) = dx + \mathcal{O}(x / \log x).$$

Define

$$Q_{VL}(x) = \prod_{p \geq x^\delta} p^{\alpha_p(x)},$$

the part of $Q(x)$ consisting of primes at least $x^\delta$ (very large primes). Using Propositions 3.1, 3.3, and 3.4, we obtain

$$\log Q_{VL}(x) = \log \frac{Q(x)}{Q_S(x)Q_M(x)Q_L(x)} \geq \left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta) \, d\theta + o(1)\right) x.$$

Proposition 3.5. $\log Q_{VL}(x) \geq \left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta) \, d\theta + o(1)\right) x.$
3.6. **Bounding the integral.** The strategy will be to make $\delta$ as large as possible while keeping Proposition 3.5 non-trivial. Thanks to Theorem 2.1 and 2.3, we are able to bound the integral effortlessly. For $d \geq 2$,

$$
\int_{1/2}^{\delta} C(\theta) \, d\theta = \int_{1/2}^{2/3} C(\theta) \, d\theta + \int_{2/3}^{\delta} C(\theta) \, d\theta < \int_{1/2}^{2/3} \frac{8}{6 - 7\theta} \, d\theta + \int_{2/3}^{\delta} \frac{2}{1 - \theta} \, d\theta < -1.4788 - 2\log(1 - \delta).
$$

The case $d = 1$ is a little special because we cannot make $\delta$ greater than $2/3$. For $d = 1$,

$$
\int_{1/2}^{\delta} C(\theta) \, d\theta < \int_{1/2}^{\delta} \frac{8}{6 - 7\theta} \, d\theta < 1.0472 - \frac{8}{7} \log(6 - 7\delta).
$$

3.7. **Choosing $\delta$.** To preserve the linear lower bound in Proposition 3.5, we want to have

$$
d - \frac{1}{2} \geq -1.4788 - 2\log(1 - \delta)
$$

if $d \geq 2$. This reduces to $\delta \leq 1 - \exp\left(\frac{d - 0.9788}{2}\right)$. And for $d = 1$,

$$
1 - \frac{1}{2} \geq 1.0472 - \frac{8}{7} \log(6 - 7\delta) \implies \delta \leq 0.62656.
$$

However, we can do a lot better for $d = 2$, thanks to Theorem 2.4. The following numerical computation, also performed in [14], shows that

$$
\int_{1/2}^{\delta} C(\theta) \, d\theta < \int_{1/2}^{\delta} \frac{8}{6 - 7\theta} \, d\theta < 1.0472 - \frac{8}{7} \log(6 - 7\delta).
$$

with $\delta = 0.847$. Thus, we set $\delta = 1 - \varepsilon(d)$ for the rest of the argument, where $\varepsilon(1) = 0.3735$, $\varepsilon(2) = 0.153$, and $\varepsilon(d) = \exp\left(\frac{d - 0.9788}{2}\right)$ for $d \geq 3$.

3.8. **Finishing the argument.** Define $L(x) = \lcm\{f(p) \mid p < x\}$. Let $p$ be a prime such that $p \geq x^\delta$. Note that the exponent of $p$ in $Q(x)$ is $O(x^{1-\delta})$. We know that $\log Q_{VL}(x) \gg x$. Therefore,

$$
x \ll \log Q_{VL}(x) \ll x^{1-\delta} \sum_{p \geq x^\delta \atop p \mid Q(x)} \log p.
$$

Thus,

$$
\log L(x) > \sum_{p \geq x^\delta \atop p \mid Q(x)} \log p \gg x^\delta,
$$

as desired.

**Remark 3.6.** It is worth noting that the same method gives $\log \rad \lcm\{f(p) \mid p < x\} \gg x^{1-\varepsilon(d)}$, similar to that obtained by Sah in [12].
4. Digression on the Greatest Prime Divisor of \( f(p) \)

The main ingredient in proving Theorem 1.2 is Proposition 3.5, which provides us a good handle on large primes dividing \( Q(x) \).

**Proof of Theorem 1.2.** By Proposition 3.5,

\[
\log Q_{VL}(x) = \sum_{q < x} \sum_{\substack{p > \delta \cdot x^d \mid f(q) \mid}} \log p \gg x.
\]

Set \( \delta = 1 - \varepsilon(d) \). Let the number of primes \( p \) less than \( x \) such that \( f(p) \) has a prime divisor greater than \( x \) be \( N(x) \). Note that if \( p \mid Q(x) \), then \( p < x^{d+1} \) for all large \( x \). Thus,

\[
N(x) \gg \sum_{q < x} \sum_{\substack{p > \delta \cdot x^d \mid f(q) \mid}} 1 \gg \sum_{q < x} \sum_{\substack{p > \delta \cdot x^d \mid f(q) \mid}} \frac{\log p}{\log x^{d+1}} \gg \frac{1}{\log x} \sum_{q < x} \sum_{\substack{p > \delta \cdot x^d \mid f(q) \mid}} \log p \gg \frac{x}{\log x},
\]

which completes the proof. \( \square \)

**Remark 4.1.** It can be seen that the Elliott–Halberstam conjecture allows us to take \( \varepsilon(d) \) to be any positive constant. For completeness, a formulation of the Elliott-Halberstam conjecture is as follows:

**Elliott-Halberstam Conjecture.** Define the error function

\[
E(x; q) = \max_{\gcd(a, q) = 1} \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right|,
\]

where the max is taken over all \( a \) relatively prime to \( q \). For every \( \theta < 1 \) and \( A > 0 \), we have

\[
\sum_{1 \leq q \leq x^\theta} E(x; q) \ll_{\theta, A} x \frac{\log A}{\log x}.
\]

We end the article with the following question for readers.

**Question 4.2.** Let \( f \) be an irreducible integer polynomial. Is it true that \( \log \text{lcm}\{f(p) \mid p < x\} \gg x \)?

**References**

1. D. Bazzanella and C. Sanna, Least common multiple of polynomial sequences, *Rend. Semin. Mat. Univ. Politec. Torino* **78**(1) (2020) 21–25.
2. J. Cilleruelo, The least common multiple of a quadratic sequence. *Compos. Math.* **147**(4) (2011) 1129–1150.
3. E. Fouvry, Théorème de Brun-Titchmarsh; application au théorème der Fermat, *Invent. Math.* **79** (1985) 383–408.
4. M. Goldfeld, On the number of primes \( p \) for which \( p + a \) has a large prime factor, *Mathematika* **16**(1) (1969) 23–27.
5. G. Harman, *Prime-Detecting Sieves* (Princeton University Press, 2007).
6. S. Hong, Y. Luo, G. Qian and C. Wang, Uniform lower bound for the least common multiple of a polynomial sequence, *C. R. Math. Acad. Sci. Paris* **351**(21-22) (2013) 781–785.
7. S. Hong, G. Qian and Q. Tan, The least common multiple of a sequence of products of linear polynomials, *Acta Math. Hungar.* **135**(1-2) (2012) 160–167.
8. H. Iwaniec, On the Brun-Titchmarsh theorem, *J. Math. Soc. Japan* **34**(1) (1982) 95–123.
9. F. Luca, R. Menares and A. Pizarro-Madariaga, On shifted primes with large prime factors and their products, *Bull. Belg. Math. Soc. Simon Stevin* **22** (2015) 39–47.
10. J. Maynard and Z. Rudnick, A lower bound on the least common multiple of polynomial sequences, *Riv. Mat. Univ. Parma* **12**(1) (2021) 143–15.
11. H. L. Montgomery and R. C. Vaughan, The Large Sieve. *Mathematika* **20**(02) (1973) 119.
12. A. Sah, An improved bound on the least common multiple of polynomial sequences, *J. Théor. Nombres Bordeaux* **32**(3) (2020) 891–899.
13. J.-P. Serre, *Lectures on $N_X(p)$* (CRC Press Book, Research Notes in Mathematics, 2011).
14. J. Wu and P. Xi, Quadratic polynomials at prime arguments, *Math. Z.* **285** (2017) 631–646.
15. J. Wu and P. Xi, Arithmetic exponent pairs for algebraic trace functions, *to appear in Algebra Number Theory.*

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