On the Possibility to Explain “The Pioneer Anomaly” within the Framework of Conformal Geometrodynamics

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Abstract

Einstein-Infeld-Hoffmann method is used to solve the problem of motion of two bodies when the equations of general relativity are of the generalized form: they have been reduced to a form invariant under conformal transformations. It is proved that not only metric degrees of freedom, but also derivatives of vector $A_\alpha$ appearing in the generalized equations can exert influence on the motion of bodies in a certain space-time domain. This influence can account for the recently observed anomalous acceleration of spacecrafts Pioneer 10, Pioneer 11. The impact of vector $A_\alpha$ on the motion of bodies is interpreted as a consequence of viscosity in geometrodynamic continuum.

Key words: EIH method; Pioneer 10, 11.

1. Introduction

The analysis of the paths of spacecrafts Pioneer 10, Pioneer 11 in refs. [1], [2] suggests that in range

$$R = (30 \div 100) \, AU = (0.45 \div 1.5) \cdot 10^{15} \, cm \quad (1)$$
the spacecrafts are subject to an anomalous acceleration component, which has come to be denoted by $a_P$ in the literature. In magnitude the acceleration $a_P$ is

$$a_P = - (8.74 \pm 1.33) \cdot 10^{-8} \text{ cm/s}^2.$$  \hspace{1cm} (2)

The minus sign indicates that the direction of the acceleration $a_P$ is close to the direction towards the Sun.

From the viewpoint of the following discussion of the issue of the acceleration $a_P$ it seems reasonable to make two estimations. First, we should compare (2) to acceleration $a_N$ gained by spacecrafts from the Sun in the Newtonian approximation. Having taken the mass of the Sun equal to $M = 2 \cdot 10^{33}$ g, gravitational constant $G = 6.67 \cdot 10^{-8}$ cm$^3$/g $\cdot$ s$^2$, we obtain for distance $R = 50$ AU $= 7.5 \cdot 10^{14}$ cm

$$a_N = \frac{GM}{R^2} = \frac{(6.67 \cdot 10^{-8} \text{ cm}^3/\text{g} \cdot \text{s}^2) (2 \cdot 10^{33} \text{ g})}{5.625 \cdot 10^{29} \text{ cm}^2} = 2.37 \cdot 10^{-4} \text{ cm/s}^2.$$  \hspace{1cm} (3)

Second, for the system (the Sun + spacecraft) we should determine smallness parameter $\lambda$, which is on the order of magnitude of the ratio between the characteristic velocity of the craft relative to the Sun and light velocity. For the above value of $R$:

$$\lambda \sim \sqrt{\frac{GM}{c^2 R}} = \sqrt{\frac{(6.67 \cdot 10^{-8} \text{ cm}^3/\text{g} \cdot \text{s}^2) (2 \cdot 10^{33} \text{ g})}{(3 \cdot 10^{10} \text{ cm/s})^2 \cdot 1.5 \cdot 10^{15} \text{ cm}}} \approx 10^{-5}. \hspace{1cm} (4)$$

The above estimations show that the $a_P$ is not predictable by the general relativity either in the Newtonian or post-Newtonian (PN) approximation. If fact, the Einstein-Infeld-Hoffmann (EIH) corrections to acceleration differ from accelerations $a_N$ by a value of the order of $\sim \lambda^2$, while relation $|a_P/a_N|$ is close to $\lambda^1$. The corrections to the acceleration in the PN approximation coincide with the distance and the $a_P$ is independent of distance.

It is our view that from the standpoint of explanation of the nature of $a_P$ the fact is essential that product $Hc$, where $H$ is Hubble constant, is equal to

$$Hc = 6.9 \cdot 10^{-8} \text{ cm/s}^2.$$  \hspace{1cm} (5)

which on the order of magnitude is close to $a_P$ (it is accepted that $H = 2.3 \cdot 10^{-18}$ 1/s), that is

$$a_P \approx -Hc.$$  \hspace{1cm} (6)
The closeness between $a_P$ and $Hc$ is noted both in ref. [1] and in many other papers. In this connection many people put forward the hypothesis that the additional acceleration $a_P$ owes its origin to the cosmological expansion of the Universe. However, no acceptable theoretical construction implementing this hypothesis has been proposed.

This paper makes an attempt to consider the motion of bodies in systems like (the Sun + spacecraft) within the framework of a so-called conformal geometrodynamics (CGD). CGD is a theoretical scheme based on equations that are a minimum conformally invariant generalization of Einstein equations for empty space. The CGD equations are derived in [3] (without $\lambda$ term) and in [4] (with $\lambda$ term). They are:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -2A_\alpha A_\beta - g_{\alpha\beta}A^2 - 2g_{\alpha\beta}A'_\mu + A_{\alpha;\beta} + A_{\beta;\alpha} + \lambda \cdot g_{\alpha\beta}.$$  \hfill (7)

Here $A_\alpha, \lambda$ are the vector and scalar fields that are attributes of the Riemannian variety dynamics. The form of equations (7) is preserved in transformations

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta} \cdot e^{2\sigma}, \quad A_\alpha \rightarrow A_\alpha - \sigma;\alpha, \quad \lambda \rightarrow \lambda \cdot e^{-2\sigma},$$  \hfill (8)

where $\sigma$ is an arbitrary scalar function of coordinates. These transformations pertain to the category of those, which effect the conformal mapping of Riemannian spaces and are termed conformal transformations (see, e.g., [5]).

To find solutions to equations (7), coordinate conditions and gauge must be specified. For the coordinate conditions the well-known de Donder conditions will be used generalized so that they acquire the form invariant under transformations (8). The conformally invariant form of the de Donder conditions is:

$$\frac{g_{\alpha\lambda}}{\sqrt{-g}} \left( \sqrt{-g} g^{\lambda\sigma} \right)_{,\sigma} + 2A_\sigma \frac{g_{\alpha\lambda}}{\sqrt{-g}} \left( \sqrt{-g} g^{\lambda\sigma} \right) = 0.$$  \hfill (9)

For the gauge condition this paper uses condition

$$\lambda = \lambda_0 = Const.$$  \hfill (10)

When (10) is valid, vector $A_\alpha$ automatically satisfies Lorentz condition:

$$A^\sigma_{,\sigma} = 0.$$  \hfill (11)

As will be demonstrated in this paper, the consideration of the body motion dynamics in a system like (the Sun + spacecraft) allows realization in a seminal manner of the hypothesis that the additional acceleration $a_P$ follows naturally from the CGD equations.
2. Equations and conditions in the EIH scheme used

We will solve the problem of finding the equations of translational motion of two point bodies in the lower orders of approximation in parameter $\lambda v/c$, where $v$ is the characteristic velocity of relative motion of the bodies. We will be solving with the Einstein-Infeld-Hoffmann (EIH) method in the form presented in refs. [8], [9]. When comparing specific expressions of this paper with the relevant expressions in refs. [8], [9] it should be kept in mind that this paper uses signature $(-+++)$ and for the coordinate condition the de Donder condition in form (9); the other notations are the same as those used in refs. [8], [9].

Assume that in our approximations $\lambda$ exerts no influence at all on the body motion and can be excluded from what follows\(^{1}\).

In the EIH method, the arrangement of orders of smallness of $\gamma_{\alpha\beta}$, $A_\alpha$, where

$$
\gamma_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} (\eta^{\mu\nu} h_{\mu\nu}), \quad h_{\alpha\beta} \equiv g_{\alpha\beta} - \eta_{\alpha\beta},
$$

$\eta_{\alpha\beta} = \text{diag}[-1,1,1,1]$ is the metric tensor of plane background space, depends on parameters of the bodies (masses, velocities) and distances between them. As applied to the system (the Sun + spacecraft) at distances $s$, this consideration adopts the following order of smallness arrangement:

$$
\begin{align*}
\gamma_{00} &= \gamma_{00}^2 + \gamma_{00}^4 + \gamma_{00}^5 + \ldots, \\
\gamma_{0k} &= \gamma_{0k}^3 + \gamma_{0k}^4 + \gamma_{0k}^5 + \ldots, \\
\gamma_{mn} &= \gamma_{mn}^4 + \gamma_{mn}^5 + \gamma_{mn}^6 + \ldots, \\
A_0 &= A_0^3 + A_0^4 + \ldots, \\
A_k &= A_k^4 + A_k^5 + \ldots.
\end{align*}
$$

As it will follow from the following, the above order of smallness arrangement rules out the influence of field $A_\alpha$ on spacecraft motion in the Newtonian approximation.

Equations (7) lead to the following equations for $\gamma_{\alpha\beta}$ (only those equations are written out which are used in what follows):

$$
\left[00; \lambda^2 \right] \Rightarrow -\frac{1}{2} \left( \Delta \gamma_{00}^2 \right) = 0. \quad (13)
$$

\(^{1}\)For this reason hereafter there will be no danger to confuse function $\lambda(x)$ with smallness parameter $\lambda$. 

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$$\begin{align*}
[0k; \lambda^3] \Rightarrow & \quad -\frac{1}{2} \left( \Delta \gamma_{0k} \right) + \frac{1}{2} \left( -\gamma_{00,0} + \gamma_{0s,s} \right) = 0. \quad (14) \\
[\lambda^4; mn] \Rightarrow & \quad \frac{1}{2} \left\{ \left( -\gamma_{0m,0n} + \gamma_{ms,sn} \right) + \left( -\gamma_{0m,0n} + \gamma_{ns,sm} \right) - \Delta \gamma_{mn} - \\
& - \delta_{mn} \left( \gamma_{00,0n} - 2 \gamma_{0s,0s} + \gamma_{pq,pq} \right) \right\} + \frac{1}{4} \gamma_{00} \gamma_{00,0n} + \\
& + \frac{1}{8} \gamma_{00,0n} - \delta_{mn} \left( \Delta \gamma_{00} \right) - \frac{3}{16} \delta_{mn} \gamma_{00,0n} = 0. \quad (15) \\
[0k; \lambda^4] \Rightarrow & \quad -\frac{1}{2} \left( \Delta \gamma_{00} \right) + \frac{1}{2} \left( -\gamma_{00,0} + \gamma_{0s,s} \right) = A_{4,0,k}. \quad (16) \\
[00; \lambda^4] \Rightarrow & \quad -\frac{1}{2} \Delta \gamma_{00} + \frac{1}{2} \gamma_{pq,pq} + \frac{3}{16} \gamma_{00,s} \gamma_{00,s} + \frac{3}{8} \gamma_{00} \left( \Delta \gamma_{2,00} \right) = 0. \quad (17) \\
[\lambda^5; mn] \Rightarrow & \quad \frac{1}{2} \left\{ \left( -\gamma_{0m,0n} + \gamma_{ms,sn} \right) + \left( -\gamma_{0m,0n} + \gamma_{ns,sm} \right) - \Delta \gamma_{mn} - \\
& - \delta_{mn} \left( -2 \gamma_{0s,0s} + \gamma_{pq,pq} \right) \right\} \\
& = A_{5,m,n} + A_{5,n,m} - 2 \delta_{mn} \left[ -A_{4,0,0} + A_{4,1,1} \right]. \quad (18)
\end{align*}$$

Coordinate conditions (13) are of form (in square brackets: C.C. = Coordinate Condition):

$$\begin{align*}
[C.C.; 0; \lambda^3] \Rightarrow & \quad \gamma_{0l,l} = \gamma_{00,0} + 2 A_{4,0}. \quad (19) \\
[C.C.; k; \lambda^4] \Rightarrow & \quad \gamma_{kl,l} = \gamma_{0k,0} - \frac{1}{4} \gamma_{00} \gamma_{00,k}. \quad (20) \\
[C.C.; 0; \lambda^4] \Rightarrow & \quad \gamma_{0l,l} = 2 A_{4,0}. \quad (21) \\
[C.C.; 0; \lambda^5] \Rightarrow & \quad \gamma_{0l,l} = \gamma_{00,0} - \frac{1}{2} \gamma_{0p} \gamma_{00,p} - \frac{1}{4} \gamma_{00} \gamma_{00,0} + \frac{1}{2} \gamma_{00} \gamma_{0s,s} + 2 A_{5,0}. \quad (22) \\
[C.C.; k; \lambda^5] \Rightarrow & \quad \gamma_{0k,0} - \frac{1}{4} \gamma_{kl,l} + 2 A_{4,k} = 0. \quad (23)
\end{align*}$$

Among all conditions (14), only the gauge conditions of the lower approximations are sufficient (in square brackets: G.C. = Gauge Condition):

$$\begin{align*}
[G.C.; \lambda^4] \Rightarrow & \quad A_{4,l,l} = 0. \quad (24)
\end{align*}$$
From equations (7) it follows that functions $A_{0,0}, A_{4,0}, A_{4,k}, A_{5,k}$ should satisfy the following equations:

\[
\begin{align*}
[0; \lambda^3] & \Rightarrow -\Delta A_3 = 0. \\
[0; \lambda^4] & \Rightarrow -\Delta A_4 = 0. \\
[k; \lambda^4] & \Rightarrow -\Delta A_4 = 0. \\
[k; \lambda^5] & \Rightarrow -\Delta A_5 = 0.
\end{align*}
\]

Thus, functions $A_{0,0}, A_{4,0}, A_{4,k}, A_{5,k}$ should be harmonic functions.

3. Centrally symmetric static solution to CGD equations

The centrally symmetric static (CSS) solution to the CGD equations is presented in different forms in several papers (see [6], [7]). Here the CSS solution will be presented in a form convenient for comparison to the Schwarzschild solution.

In the CSS problem the squared interval is

\[
ds^2 = -\exp(\gamma) \cdot dt^2 + \exp(\alpha) \cdot dz^2 + \exp(\beta) \cdot (d\theta^2 + \sin^2 \theta \cdot d\varphi^2).\]

In this case the gauge vector can have as few as two nonzero, radial coordinate dependent components:

\[
A_\alpha = (\phi, f, 0, 0).
\]

Having made use of the gauge transformation with radial variable dependent function $\sigma = \sigma(z)$, set $A_1 = f$ equal to zero. Then replace the radial coordinate so that condition $g_{00}g_{11} = -1$ be met. This transformation does not result in appearance of $A_1$-component of field $A_\alpha$. Upon the above two transformations, without loss of generality, the metric can be written in the form

\[
ds^2 = -\exp(\gamma) \cdot dt^2 + \exp(-\gamma) \cdot dz^2 + \exp(\beta) \cdot (d\theta^2 + \sin^2 \theta \cdot d\varphi^2)\]
and vector $A_\alpha$ in the form

$$A_\alpha = (\phi, 0, 0, 0).$$

Four equations are obtained for four functions $\gamma$, $\beta$, $\phi$, $\lambda$.

Equation $G^0_0 = T^0_0$ \Rightarrow

$$\exp (\gamma) \cdot \left[ \frac{3}{4} (\beta')^2 + \frac{1}{2} \gamma' \beta' \right] - \exp (-\beta) = 3 \exp (-\gamma) \cdot \phi^2 + \lambda.$$

Equation $G^1_1 = T^1_1$ \Rightarrow

$$\exp (\gamma) \cdot \left[ \frac{1}{4} (\beta')^2 + \frac{1}{2} \gamma' \beta' \right] - \exp (-\beta) = \exp (-\gamma) \cdot \phi^2 + \lambda.$$

Equation $G^2_2 = T^2_2$ \Rightarrow

$$\exp (\gamma) \cdot \left[ \frac{1}{2} \beta'' + \frac{1}{4} (\beta')^2 + \frac{1}{2} \gamma'' + \frac{1}{2} (\gamma')^2 + \frac{1}{2} \beta' \gamma' \right] = \exp (-\gamma) \cdot \phi^2 + \lambda.$$

Equation $G^1_0 = T^1_0$ \Rightarrow

$$0 = \phi' - \gamma' \phi.$$

It turns out that there are three solution types. We are concerned about that type, which contains the Schwarzschild solution as a special case. Here is the solution type with the procedure itself of its finding being omitted.

$$\begin{align*}
\phi &= p_0 \cdot \exp (\gamma) \\
\exp (\beta) &= A_0 \cdot \sh^2 (p_0 z + a_0) \\
\exp (\gamma) &= \frac{1}{p_0^2 A_0} + B_0 \cdot [p_0 z \cdot \cht (p_0 z + a_0) - 1] + b_0 p_0 \cdot \cht (p_0 z + a_0) \\
\lambda (z) &= B_0 p_0^2 
\end{align*}$$

(30)

The zero-subscripted quantities are integration constants.

With appropriately chosen constants a solution of type (30) can approximate the Schwarzschild solution as closely as is wished in a certain range of the radial variable. To obtain the approximation, set the following in (30):

$$B_0 = 0, \quad a_0 = 0, \quad A_0 p_0^2 = 1.$$

(31)

Then for $e^\gamma$:

$$e^\gamma = 1 + b_0 p_0 \cdot \cht (p_0 z).$$

(32)

Having introduced notation $b_0 \equiv -r_0$, $p_0 \equiv 1/L$, write the CSS solution.

$$e^\gamma = 1 - \frac{r_0}{L} \cdot \cht \left( \frac{z}{L} \right), \quad e^\beta = L^2 \cdot \sh^2 \left( \frac{z}{L} \right),$$

(33)
\[ \phi = \frac{1}{L} \left[ 1 - \frac{r_0}{L} \cdot \text{cth} \left( \frac{z}{L} \right) \right]. \]  

(34)

In the range of the radial variable
\[ \frac{(r_0/L)}{L} < \frac{(z/L)}{\ll 1} \]  

(35)

expressions (33), (34) assume the following form:
\[ e^\gamma = 1 - \frac{r_0}{z}, \quad e^\beta = z^2, \]  

(36)

\[ \phi = \frac{1}{L} \left[ 1 - \frac{r_0}{z} \right]. \]  

(37)

Expressions (36) are the same as the associated expressions in the Schwarzschild solution. In so doing \( r_0 \) has the meaning of gravitational radius. As for (37), there is no analog of this expression in the Schwarzschild solution. In expressions (36), (37), two facts seem essential. First, in the range of radial variable (35), for its description the CGD equation solution requires not only dimensional constant \( r_0 \), but also one more dimensional constant \( L \). Second, in the above-mentioned range of radial variable the principal term in the expansion of function \( \phi \) is constant \( 1/L \).

4. Finding functions \( \gamma_{\alpha\beta}, A_\alpha \) with the EIH method

For us it is convenient first to find the equations of translational motion of two particles under the assumption that the particle masses are commensurable in magnitude, that is, to obtain results similar to those obtained in refs. [8], [9], however not for the equations of general relativity, but for equations (7). Then from the found equations of motion the equations of motion can be obtained straightforwardly for the system, in which one of the particles is a trial particle.

The general approach to the solution of our problem is that components \( \gamma_{\alpha\beta} \) will be represented in form \( \gamma_{\alpha\beta} = \bar{\gamma}_{\alpha\beta} + \delta\gamma_{\alpha\beta} \). Here \( \bar{\gamma}_{\alpha\beta} \) are expressions that are obtained in the framework of general relativity for two point particles of masses \( M, m \) without inclusion of additional terms of \( A_\alpha \). On the other hand, \( \delta\gamma_{\alpha\beta} \) is an addition determined entirely by the \( A_\alpha \).

The structure of all quantities \( \gamma_{\alpha\beta}, A_\alpha \) is essentially predetermined in the scheme under discussion by expressions for \( \gamma_{00} \) and \( A_0 \) in the lower orders
of smallness. In our case the expressions for $\gamma_2^{00}, \gamma_3^{00}$ should coincide with expressions

$$\gamma_2^{00} = 4 \frac{M}{r_1} + 4 \frac{m}{r_2}, \quad \gamma_3^{00} = 0,$$  \hspace{1cm} (38)

that is with those expressions, which in the case of general relativity lead to the well-known equations of translational motion in the PN approximation.

As for $A_0$, certain considerations regarding its choice follow from approximate expressions (36), (37) for the exact CSS solution to the CGD equations. Clear that the expansion of $A_0$ should begin with the constant $1/L$, where $L$ is the dimensional parameter used to determine asymptotic $A_0$ at long distances. When considering multi-particle problems, the expansion of the exact one-particle solution does not allow detection of the presence of particle velocity containing terms. Appearance of this type of terms cannot also be excluded in the expression for $A_0$, when a system of at least two particles is considered. Note that a similar situation arises, for example, when one attempts to find the Kerr solution with the EIH method [9] or tries to determine radiation friction force in the Dirac-Lorentz equation with this method [10].

Reasoning from the above considerations, write the expression for $A_0$, which could be composed of the quantities present in the problem. The desired expression:

- should begin with constant $1/L$,
- can contain particle velocities $\dot{\xi}_k, \dot{\eta}_k$ in the next order of smallness,
- should be, according to (26), (27), a harmonic function,
- if it contains a part depending on coordinates, then the part should be decreasing with distance from the system,
- should be symmetric about replacement of parameters of one particle by those of the other.

It turns out that all of these requirements are satisfied, if $A_0$ is represented as

$$A_0 = A_0 + A_0 = \frac{1}{L} + \kappa \cdot \frac{\mu R^2}{(m + M) L} \left\{ \frac{(X_i \dot{\xi}_i)}{r_1^3} + \frac{(x_i - \eta_l) \dot{\eta}_l}{r_2^3} \right\}$$  \hspace{1cm} (39)

and $R/L$ is assumed to be of the third order of smallness. Here $\kappa$ is a constant numerical coefficient, $\mu = mM / (m + M)$ is scaled mass of a system of two particles. In this consideration multiplier $\mu / (m + M)$ is assumed to be of the order of unity. The need for the introduction of $\kappa$ to the $A_0$
construction is due to the lack of uniqueness that is inherent in the EIH method in construction of approximations to the exact solution with using it (for more details, see [9], [10]). The considerations on selection of the value of the constant $\kappa$ will be presented later.

$A_k$ is found from gauge conditions (24), (25). From equation (28) and condition (24) it follows that $A_k = 0$. Write the expression for $A_k$ in the vicinity of the first particle alone; in so doing make use of the equation of motion of particle in the Newtonian approximation.

$$\begin{align*}
A_k &= \frac{A_k}{4} + \frac{A_k}{5} = \\
&= \kappa \cdot \frac{\mu}{(m+M)L} \left\{ \frac{mR_k}{R_1} - \frac{r_1 (R_k \dot{r}_k)}{r_1^2} - \frac{R^2 (X_k \dot{X}_k)}{r_1^2} \right\} + (1 \Leftrightarrow 2) \tag{40}
\end{align*}$$

From here on the term with two arrows $(1 \Leftrightarrow 2)$ denotes the expression derived from the written one through replacement of parameters of one particle by those of the other.

The substitution of expression (39) for $A_0$ into equations (14), (16) with taking into account (19), (21) yields the following form of the expression for $\delta \gamma_{0k}$:

$$\begin{align*}
\delta \gamma_{0k} &= \delta \gamma_{30k} + \delta \gamma_{40k} = \\
&= \frac{2X_k}{3L} - \kappa \frac{\mu R^2}{(m+M)L} \cdot \frac{\dot{X}_k}{r_1} + (1 \Leftrightarrow 2). \tag{41}
\end{align*}$$

Found expressions (39), (40), (41) are sufficient to determine the form of the correction to the Newtonian equation of particle motion. These corrections are obtained by integration of equation (18) over infinitesimal-radius spheres surrounding the point particles. Thus we obtain the correction to the equations of motion for the first particle.

Write equation (18) with having rearranged all the terms to the left side.

$$\begin{align*}
\frac{1}{2} \left\{ \left( - \gamma_{0m,0n} + \gamma_{0n,0m} + \gamma_{ms,sm} \right) + \left( - \gamma_{40m,0n} + \gamma_{40n,0m} + \gamma_{ns,sn} \right) - \\
-\Delta \gamma_{mn} - \delta_{mn} \left( -2 \gamma_{40s,0s} + \gamma_{5pq,pq} \right) \right\} \\
- A_{4,m,n} - A_{5,n,m} + 2\delta_{mn} \left[ -A_{4,0,0} + A_{5,l,l} \right] = 0. \tag{42}
\end{align*}$$

In the integration of this equation the total contribution from the terms containing second derivatives of components $\delta \gamma_{mn}$ is zero, as they form curl.
combinations. The contribution from the following combination is zero for the same reason:

\[ \frac{1}{2} \left\{ \left( \delta_{mn} \gamma_{0s,0} \frac{1}{s} - \gamma_{0n,0} \frac{1}{m} \right) \right\}. \]

The contribution from

\[ +2\delta_{mn} \left( -A_{0,0} + A_{l,l} \right) \]

is zero by virtue of the gauge condition. As a result, to find the correction to the equation of motion, the contribution from the following terms must be calculated:

\[-\frac{1}{2} \gamma_{0m,0} + \frac{1}{2} \delta_{mn} \gamma_{0s,0} s - A_{m,n} - A_{n,m}. \]

This correction will therewith enter into the equation of motion

\[ 2M \dddot{\xi}_k = \frac{mM}{R^3} R_k + \Im_k \]  

as addition \( \Im_k \),

\[ \Im_k = \frac{1}{4\pi} \oint \left\{ -\frac{1}{2} \gamma_{0k,0} + \frac{1}{2} \delta_{kn} \gamma_{0s,0} s - A_{k,n} - A_{n,k} \right\} ds_n. \]  

The contributions of separate terms to surface integral (44) are:

\[ \frac{1}{4\pi} \oint \left\{ -\frac{1}{2} \gamma_{0k,0} \frac{1}{1} \right\} ds_n = -\kappa \cdot \frac{\mu}{2(m+M)L} \cdot \left( \frac{mR_k}{R} + 2 \left( R_l \dot{R}_l \right) \dot{\xi}_k \right). \]

\[ \frac{1}{4\pi} \oint \left\{ \frac{1}{2} \delta_{kn} \gamma_{0s,0} s \right\} ds_n = \frac{1}{4\pi} \oint \left\{ \delta_{kn} A_{0,0} \right\} ds_n = \kappa \cdot \frac{\mu}{2(m+M)L} \cdot \left( \frac{mR_k}{R} + 2 \left( R_l \dot{R}_l \right) \dot{\xi}_k \right). \]

\[ \frac{1}{4\pi} \oint \left\{ -A_{k,n} - A_{n,k} \right\} ds_n = \kappa \cdot \left( -\frac{4}{3} \cdot \frac{M}{L} \cdot \frac{R_k}{R} - \frac{8}{3} \cdot \frac{M}{mL} \cdot \left( R_l \dot{R}_l \right) \dot{\xi}_k \right). \]

Substitution of (45)-(47) into (44) gives:

\[ \Im_k = \kappa \cdot \frac{\mu}{(m+M)L} \cdot \frac{mR_k}{R} + 2\kappa \cdot \frac{\mu}{(m+M)L} \cdot \left( R_l \dot{R}_l \right) \dot{\xi}_k. \]
As a result, from (43), (44), (48) we obtain the following correction to acceleration $\delta \ddot{\xi}_k$:

$$
\delta \ddot{\xi}_k = \kappa \cdot \frac{\mu m}{2(m + M)M} \cdot \frac{R_k}{LR} + \kappa \cdot \frac{\mu}{(m + M)ML} \cdot \left( R_l \dot{R}_l \right) \dot{\xi}_k.
$$

(49)

In terms of derivatives with respect to regular time and masses in regular units of measurement equation (49) is written as:

$$
\frac{d^2 \delta \xi_k}{dt^2} = \kappa \cdot \frac{\mu m}{2(m + M)M} \cdot \frac{c^2 R_k}{LR} + \kappa \cdot \frac{\mu c^2}{(m + M)MLG} \cdot \left( R_l \frac{dR_l}{dt} \right) \frac{d\xi_k}{dt}.
$$

(50)

Assume that constant $L$ is of cosmological origin and is related with Hubble constant $H$ as

$$
L = c/H.
$$

(51)

Having substituted (51) into (50), we obtain:

$$
\frac{d^2 \delta \xi_k}{dt^2} = \frac{\kappa \mu m}{2(m + M)M} \cdot cH\frac{R_k}{R} + \frac{\kappa \mu}{(m + M)MG} \cdot cH \cdot \left( R_l \frac{dR_l}{dt} \right) \frac{d\xi_k}{dt}.
$$

(52)

Expression (52) is just the correction to the Newtonian expression for acceleration which follows from the CGD equations under the assumptions specified at the beginning of this section.

5. Equations of motion of spacecrafts Pioneer 10, 11

Write equations (52) in the special case, where mass of the first particle is much less than that of the second particle, that is where the first particle can be considered as a trial particle. In this case

$$
\frac{d^2 \delta \xi_k}{dt^2} = \frac{\kappa}{2} \cdot cH \frac{R_k}{R} - \frac{\kappa}{mG} \cdot cH \cdot \left( R_l \frac{dR_l}{dt} \right) \frac{d\xi_k}{dt}.
$$

(53)

It is interesting to note that the first term in the right-hand side of (53) depends neither on masses of the bodies nor on the distance between them whatsoever.

From (53) it follows that in the trial particle motion along circumference the correction from the second term in the right-hand side of (53) is zero, the
correction to the Newtonian expression for the acceleration is determined by
the first term.

\[ \frac{d^2 \delta x_k}{dt^2} = \frac{\kappa}{2} \cdot cH \frac{R_k}{R}. \] (54)

Consider motion of the trial particle along radius. We will omit subscript
“1” in writing the radial vector values. In this case we obtain for the radial
component of acceleration from (53):

\[ \frac{d^2 \delta \xi}{dt^2} = \frac{\kappa}{2} \cdot cH \cdot \frac{\kappa}{mG} \cdot cH \cdot R \left( \frac{d\xi}{dt} \right) \cdot \left( \frac{d\xi}{dt} \right). \] (55)

Write the squared radial velocity with making use of the law of conserva-
tion of energy, that is as

\[ \left( \frac{d\xi}{dt} \right) = \frac{2E_0}{M} + \frac{2Gm}{R}. \]

We arrive at:

\[ \frac{d^2 \delta \xi}{dt^2} = \frac{\kappa}{2} \cdot cH - \frac{\kappa}{mG} \cdot cH \cdot R \left( \frac{2E_0}{M} + \frac{2Gm}{R} \right) = -\frac{3\kappa}{2} \cdot cH - \frac{2\kappa RE_0}{mMG} \cdot cH. \] (56)

Consider the spacecrafts as trial particles. Total energy \( E_0 \) of the space-
craft leaving the system with minimum kinetic energy is close to zero. Ass-
suming \( E_0 = 0 \) in (57), we obtain

\[ \frac{d^2 \delta \xi}{dt^2} = -\kappa \cdot \frac{3}{2} \cdot cH. \] (57)

To make formula (57) consistent with formula (6), it should be assumed
that the coefficient \( \kappa \) is close to unity. More precisely,

\[ \begin{align*}
if \quad \kappa = \frac{2}{3}, \quad & \text{then } \frac{d^2 \delta \xi}{dt^2} = -cH \\
if \quad \kappa = 1, \quad & \text{then } \frac{d^2 \delta \xi}{dt^2} = -\frac{3}{2} \cdot cH.
\end{align*} \] (58)

We arrive at the conclusion that by fitting one constant multiplier CGD
allows the observed anomalous component of the acceleration of spaceships
Pioneer 10, 11 to be described. The available experimental data on the
spacecrafts Pioneer 10, 11 can be described by formula (57) with \( \kappa \) ranging
from \( \kappa = 2/3 \) to \( \kappa = 1 \).
6. Conclusion

From formulas (57), (58) it follows that as the spacecraft moves away from the Sun along radius:

1. The additional acceleration is directed oppositely to the spacecraft direction and is constant in magnitude.
2. As long as the spacecraft can be viewed as a trial body, the acceleration is independent of the spacecraft characteristics and is of universal nature.

It should be emphasized that all the consideration is valid only when the initially made assumptions are fulfilled. Below they are mentioned in the explicit form.

First, the space-time dynamics is described by the generalized equations of general relativity - conformal geometrodynamics equations (7).

Second, the principal term in the expansion of $A_0$ is a constant in the space-time domain under consideration and the next term is a quantity of the fourth order of smallness. These assumptions are nontrivial, as in this manner cosmological-origin quantity $L = c/H$ is introduced to the construction of the $A_0$ to determine asymptotic expressions for the components of vector $A_\alpha$ that satisfy equations (7). In the general relativity there is no vector $A_\alpha$, so $L$ cannot be introduced naturally in it.

In the space-time domains, where the above two assumptions are not fulfilled, first, expression (39) for $A_0$ can have a higher order of smallness, second, terms of another type (for example, pole terms) can become principal in the construction of $A_0$.

Expression (53) for the additional acceleration of the trial body admits a direct experimental verification. Thus, it predicts:

- universal nature of the additional acceleration for all bodies moving along circular orbit [only the first term in the right-hand side of (53) remains],
- possibility to change the acceleration direction in the motion along radius (in moving away - towards the Sun, in approaching - away from the Sun).

In conclusion note that in equations (52), (53) the additional acceleration owes its appearance either to the terms including derivatives of $A_\alpha$ vector components or to terms that vanish in disappearance of this derivative type. The thermodynamic analysis performed, in particular, in ref. [7] suggests that these terms determine the geometrodynamic continuum viscous stress tensor. Therefore we conclude that the additional acceleration owes its origin
to the geometrodynamic continuum viscosity.

The appearance of the anomalous component (provided that it is not explained by non-gravitational effects) raises a number of basic issues. For example, as applied to the general relativity, this is the issue of energy conservation law feasibility degree and cosmological expansion effect on the experiments within the Solar system.

Spacecraft experiments can be undertaken in the future in order to arrive at answers to fundamental questions of the space-time theory. It is not improbable that the results of this paper will prove useful in development of programs of investigations in these experiments.

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