Free Boson Representation of $U_q(\hat{sl}_2)$

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ABSTRACT

A representation of the quantum affine algebra $U_q(\hat{sl}_2)$ of an arbitrary level $k$ is realized in terms of three boson fields, whose $q \to 1$ limit becomes the Wakimoto representation. An analogue of the screening current is also obtained. It commutes with the action of $U_q(\hat{sl}_2)$ modulo total difference of some fields.
1. Introduction

Recently the anti-ferroelectric spin 1/2 XXZ-Hamiltonian was exactly diagonalized [1] by using the technique of $q$-vertex operators [2]. (See [3] for higher spin cases.) Further, in [4], an integral formula for correlation functions of local operators was found in the case of spin 1/2 with the help of a boson representation of $U_q(\hat{sl}_2)$ of level 1 [5].

However, for the higher spin XXZ model, it seems difficult to get such a formula because we lack the boson representation of $U_q(\hat{sl}_2)$ for higher level. The aim of this paper is to show that $U_q(\hat{sl}_2)$ is also bosonized for an arbitrary level. As in [5], we shall use an analogue of the currents defined by the Drinfeld realization [6] of $U_q(\hat{sl}_2)$ and express them by three boson fields. In the $q \to 1$ limit, this new representation tends to the Wakimoto representation with bosonized $\beta - \gamma$ system [7][8].

We obtain an analogue of the screening current in terms of boson fields. This operator has the property that it commutes with the currents modulo total differences of some operators. Hence a suitable Jackson integral of the screening current should commute exactly with the currents. A Jackson integral formula for the solution to $q$-deformed Knizhnik-Zamolodchikov equation [2] was found by Matsuo [9][10] and Reshetikhin [12]. We think that there is a deep connection between the existence of our screening current and these Jackson integral formulas.

After finishing the work we learned from Atsushi Matsuo that he got another bosonization [11]. The relation between the two bosonizations is yet to be clarified.

2. Free boson fields $a, b, c$

In this article we consider bosonization of the Drinfeld realization of $U_q(\hat{sl}_2)$. We construct Drinfeld’s generators in terms of three free boson fields. Hereafter let $q$ be a generic complex number such that $|q| < 1$. We will frequently use the following standard notation:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}},$$  \hspace{1cm} (1)

for $m \in \mathbb{Z}$.

Let $k$ be a complex number. Let $\{a_n, b_n, c_n, Q_a, Q_b, Q_c|n \in \mathbb{Z}\}$ be a set of operators satisfying the following commutation relations:

$$[a_n, a_m] = \delta_{n+m,0} \frac{[2n][2n][(k + 2)n]}{n}, \quad [\tilde{a}_0, Q_a] = 2(k + 2),$$
$$[b_n, b_m] = -\delta_{n+m,0} \frac{[2n][2n]}{n}, \quad [\tilde{b}_0, Q_b] = -4,$$
$$[c_n, c_m] = \delta_{n+m,0} \frac{[2n][2n]}{n}, \quad [\tilde{c}_0, Q_c] = 4,$$  \hspace{1cm} (2)

where

$$\tilde{a}_0 = \frac{q - q^{-1}}{2 \log q} a_0, \quad \tilde{b}_0 = \frac{q - q^{-1}}{2 \log q} b_0, \quad \tilde{c}_0 = \frac{q - q^{-1}}{2 \log q} c_0.$$  \hspace{1cm} (3)
and the other commutators vanish.

Let us introduce three free boson fields \( a, b, \) and \( c \) carrying parameters \( L, M, N \in \mathbb{Z}_{>0}, \alpha \in \mathbb{R}. \) Define \( a(L; M, N | z; \alpha) \) by

\[
a(L; M, N | z; \alpha) = - \sum_{n \neq 0} \frac{[Ln][Mn][Nn]}{n} q^{-|n\alpha|} z^{-n} [Ln] a_n [Mn] [Nn] z^{-n} q^{|n\alpha|} + L\tilde{a}_0 MN \log z + \frac{LQ_0}{MN}. \tag{4}
\]

We define \( b(L; M, N | z; \alpha) \), \( c(L; M, N | z; \alpha) \) in the same way. In the case \( L = M \) we also write

\[
a(N | z; \alpha) = a(L; L, N | z; \alpha)
\]

\[
= - \sum_{n \neq 0} \frac{a_n}{[Nn]} z^{-n} q^{|n\alpha|} + \tilde{a}_0 N \log z + \frac{Q_0}{N}, \tag{5}
\]

and likewise for \( b(N | z; \alpha) \), \( c(N | z; \alpha) \).

3. \( q \)-difference operator and the Jackson integral

We define a sort of \( q \)-difference operator with a parameter \( n \in \mathbb{Z}_{>0} \) by

\[
n\partial_z f(z) \equiv \frac{f(q^n z) - f(q^{-n} z)}{(q - q^{-1}) z}. \tag{6}
\]

Then we have the following chain rule:

\[
n\partial_z (f(z)g(z)) = (n\partial_z f(z))g(q^n z) + f(q^{-n} z)(n\partial_z g(z)) \tag{7}
\]

We get formulas for the \( q \)-difference of boson fields, for example

\[
N\partial_z a(N | z; \alpha) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} q^{|n\alpha|}. \tag{8}
\]

Note that this is independent of \( N \). The operators \( \tilde{a}_0, \tilde{b}_0, \tilde{c}_0 \) are so normalized that the formula (8) becomes simple.

Now let \( p \) be a complex number such that \(|p| < 1 \) and \( s \in \mathbb{C}^\times \). We define the Jackson integral by

\[
\int_0^{s\infty} f(t)dp t = s(1 - p) \sum_{m=-\infty}^{\infty} f(sp^m)p^m, \tag{9}
\]

whenever it is convergent \cite{9}.

If the integrand \( f(t) \) is a total difference of some function \( F(t) \):

\[
f(t) = n\partial_t F(t), \tag{10}
\]

then by taking \( p = q^{2n} \), we have

\[
\int_0^{s\infty} f(t)dp t = 0. \tag{11}
\]
4. Wick’s Theorem  Let \( \{ a_n, b_n, c_n | n \in \mathbb{Z}_{\geq 0} \} \) be annihilation operators, and \( \{ a_n, b_n, c_n, Q_a, Q_b, Q_c | n \in \mathbb{Z}_{<0} \} \) creation operators. We denote by \( \cdots : \) the corresponding normal ordering of operators. For example,

\[
: \exp \left\{ b(2|z; \alpha) \right\} : = \exp \left\{ - \sum_{n<0} \frac{b_n}{2n} z^{-n} q^{|n| \alpha} \right\} \exp \left\{ - \sum_{n>0} \frac{b_n}{2n} z^{-n} q^{|n| \alpha} \right\} e^{Q_b/2} z^{Q_b/2}.
\]

The formal power series in \( w/z \) for boson fields \( a \) reads as follows:

\[
\langle a(L; M, N|z; \alpha) \ a(L'; M', N'|w; \beta) \rangle = - \sum_{n>0} \frac{[L_n][L'_n][a_n,a_{-n}]}{[M_n][N_n][M'_n][N'_n]} \left( \frac{w}{z} \right)^n q^{(\alpha+\beta)n} + \frac{LL'[a_0, Q_a]}{MM'M'} \log z.
\]

The formal power series in \( w/z \) is convergent if \( |w/z| < 1 \). We introduce the propagators for boson fields \( b \) and \( c \) in the same manner. Occasionally one can rewrite it simply by using the logarithm. For example,

\[
\langle b(2|z; \alpha) \ b(2|w; \beta) \rangle = - \log (z - q^{\alpha+\beta} w), \quad |z| < |q^{\alpha+\beta} w|.
\]

Using these propagators, we obtain Wick’s Theorem in the following form:

**Proposition 1** (Wick’s Theorem)

1) \[
: \exp \left\{ a(L; M, N|z; \alpha) \right\} : \exp \left\{ a(L'; M', N'|w; \beta) \right\} :
\]

\[
= \exp \left\{ \langle a(L; M, N|z; \alpha) \ a(L'; M', N'|w; \beta) \rangle \right\}
\]

\[
\times : \exp \left\{ a(L; M, N|z; \alpha) + a(L'; M', N'|w; \beta) \right\} :.
\]

2) \[
: \frac{\partial}{\partial z} : \exp \left\{ a(L; M, N|z; \alpha) \right\} : \exp \left\{ a(L'; M', N'|w; \beta) \right\} :,
\]

\[
= \frac{\partial}{\partial z} \exp \left\{ \langle a(L; M, N|z; \alpha) \ a(L'; M', N'|w; \beta) \rangle \right\}
\]

\[
\times : \exp \left\{ a(L; M, N|z; \alpha) + a(L'; M', N'|w; \beta) \right\} :.
\]

3) \[
: \frac{\partial}{\partial w} : \exp \left\{ a(L; M, N|z; \alpha) \right\} : \exp \left\{ a(L'; M', N'|w; \beta) \right\} :
\]

\[
= \frac{\partial}{\partial w} \exp \left\{ \langle a(L; M, N|z; \alpha) \ a(L'; M', N'|w; \beta) \rangle \right\}
\]

\[
\times : \exp \left\{ a(L; M, N|z; \alpha) + a(L'; M', N'|w; \beta) \right\} :.
\]

There are similar formulas for \( b \) and \( c \).
5. Current algebra

Now we define the currents $J^3(z), J^\pm(z)$ as follows:

$$
J^3(z) = k_+ 2 \partial_z a \left( k + 2 \left| q^{-2} z; -1 \right| \right) + 2 \partial_z b \left( 2 \left| q^{-k-2} z; -\frac{k+2}{2} \right| \right),
$$

$$
J^+ (z) = - : \left[ 1 \partial_z \exp \left\{ - c \left( 2 \left| q^{-k-2} z; 0 \right| \right) \right\} \times \exp \left\{ - b \left( 2 \left| q^{-k-2} z; 1 \right| \right) \right\} : ;
$$

$$
J^- (z) = : \left[ k_+ 2 \partial_z \exp \left\{ a \left( k + 2 \left| q^{-2} z; -\frac{k+2}{2} \right| \right) + b \left( 2 \left| q^{-k-2} z; -1 \right| \right) + c \left( k + 1; 2, k + 2 \left| q^{-k-2} z; 0 \right| \right) \right\} \times \exp \left\{ - a \left( k + 2 \left| q^{-2} z; \frac{k+2}{2} \right| \right) + c \left( 1; 2, k + 2 \left| q^{-k-2} z; 0 \right| \right) \right\} : .
$$

Define further the auxiliary fields $\psi(z), \varphi(z)$ as

$$
\psi(z) = \exp \left\{ (q - q^{-1}) \sum_{n>0} (q^n a_n + q^{(k+2) n} b_n) z^{-n} + (\tilde{a}_0 + \tilde{b}_0) \log q \right\} :,
$$

$$
\varphi(z) = - (q - q^{-1}) \sum_{n<0} (q^n a_n + q^{3(k+2) n} b_n) z^{-n} - (\tilde{a}_0 + \tilde{b}_0) \log q \right\} :.
$$

By the definition of the boson fields $a, b, c$, and the $q$-difference operator, we can recast these fields as

$$
J^3(z) = \sum_{n \in \mathbb{Z}} \left( q^{2n-|n|} a_n + q^{(k+2) n} b_n \right) z^{-n-1},
$$

$$
J^+ (z) = \frac{-1}{(q - q^{-1}) z} \left[ : \exp \left\{ - b \left( 2 \left| q^{-k-2} z; 1 \right| \right) - c \left( 2 \left| q^{-k-1} z; 0 \right| \right) \right\} : \right.

- \exp \left\{ - b \left( 2 \left| q^{-k-2} z; 1 \right| \right) - c \left( 2 \left| q^{-k-3} z; 0 \right| \right) \right\} : \right.

$$

$$
J^- (z) = \frac{1}{(q - q^{-1}) z} \left[ : \exp \left\{ a \left( k + 2 \left| q^k z; -\frac{k+2}{2} \right| \right) - a \left( k + 2 \left| q^{-2} z; \frac{k+2}{2} \right| \right)

+ b \left( 2 \left| z; -1 \right| \right) + c \left( 2 \left| q^{-1} z; 0 \right| \right) \right\} :.

- \exp \left\{ a \left( k + 2 \left| q^{k-4} z; -\frac{k+2}{2} \right| \right) - a \left( k + 2 \left| q^{-2} z; \frac{k+2}{2} \right| \right)

+ b \left( 2 \left| q^{2k-4} z; -1 \right| \right) + c \left( 2 \left| q^{-2k-3} z; 0 \right| \right) \right\} : \right.

$$

$$
\psi(z) = \exp \left\{ a \left( k + 2 \left| q^{\frac{1}{2}} z; -\frac{k+2}{2} \right| \right) - a \left( k + 2 \left| q^{-\frac{1}{2}} z; \frac{k+2}{2} \right| \right)

+ b \left( 2 \left| q^{-\frac{1}{2}} z; -1 \right| \right) - b \left( 2 \left| q^{-\frac{1}{2}} z; 1 \right| \right) \right\} :,
$$

$$
\varphi(z) = \exp \left\{ a \left( k + 2 \left| q^{\frac{1}{2} - 4} z; -\frac{k+2}{2} \right| \right) - a \left( k + 2 \left| q^{\frac{1}{2} - 2} z; \frac{k+2}{2} \right| \right)

+ b \left( 2 \left| q^{-\frac{k+2}{2}} z; -1 \right| \right) - b \left( 2 \left| q^{-\frac{k+2}{2}} z; 1 \right| \right) \right\} :.
$$
These manageable expressions would be convenient for calculation. Using Wick’s Theorem, we get the following formulas.

**Proposition 2** The following relations hold in the sense of analytic continuation:

\[
\varphi(z)\varphi(w) = \varphi(w)\varphi(z),
\]

\[
\psi(z)\psi(w) = \psi(w)\psi(z),
\]

\[
\varphi(z)\psi(w) = \frac{(q^2 z - q^k w)(z - q^{k+2} w)}{(z - q^{k+2} w)(q^2 z - q^k w)} \psi(w)\varphi(z),
\]

\[
\psi(z)J^\pm(w) = \left(\frac{q^2 z - q^{\pm \frac{3}{2}} w}{z - q^{\pm \frac{3}{2}} w}\right)^{\pm 1} J^\pm(w)\psi(z),
\]

\[
\varphi(z)J^\pm(w) = \left(\frac{q^2 w - q^{\pm \frac{3}{2}} z}{w - q^{\pm \frac{3}{2}} z}\right)^{\mp 1} J^\pm(w)\psi(z),
\]

\[
J^\pm(z)J^\pm(w) = \frac{q^{\pm 2} z - w}{z - q^{\pm 2} w} J^\pm(w)J^\pm(z),
\]

and

\[
J^+(z)J^-(w) \sim \frac{1}{q - q^{-1}} \left( \frac{1}{(z - q^k w)w} \psi(q^k w) - \frac{1}{(z - q^{-k} w)w} \varphi(q^{-k} w) \right).
\]

The symbol \(\sim\) in the last formula means that both sides of the formula are equal modulo some fields which are regular at \(z = q^{\pm k} w\).

We consider the mode expansions of these fields as

\[
\sum_{n \in \mathbb{Z}} J^3_n z^{-n-1} = J^3(z), \quad \sum_{n \in \mathbb{Z}} J^\pm_n z^{-n-1} = J^\pm(z),
\]

\[
\sum_{n \in \mathbb{Z}} \psi_n z^{-n} = \psi(z), \quad \sum_{n \in \mathbb{Z}} \varphi_n z^{-n} = \varphi(z).
\]

Putting \(K = q^{\delta_0 + \delta_0}\) we obtain our main proposition:

**Proposition 3** The operators \(\{J^3_n|n \in \mathbb{Z}_{\neq 0}\}, \{J^\pm_n|n \in \mathbb{Z}\}\) and \(K\) satisfy the following relations.

\[
[J^3_n, J^3_m] = \delta_{n+m,0} \frac{1}{n} [2n] [kn], \quad n \neq 0,
\]

\[
[J^3_n, K] = 0,
\]

\[
K J^\pm_n K^{-1} = q^{\pm 2} J^\pm_n,
\]

\[
[J^3_n, J^\pm_m] = \pm \frac{1}{n} [2n] q^{\pm \frac{k(n)}{2}} J^\pm_{n+m},
\]

\[
J^\pm_{n+1} J^\pm_m - q^{\pm 2} J^\pm_m J^\pm_{n+1} = q^{\pm 2} J^\pm_m J^\pm_{n+1} - J^\pm_{m+1} J^\pm_n,
\]

\[
[J^\pm_n, J^\pm_m] = \frac{1}{q - q^{-1}} (q^{\frac{k(n+m)}{2}} \psi_{n+m} - q^{\frac{k(m-n)}{2}} \varphi_{n+m}).
\]
These are exactly the relations of the Drinfeld realization of $U_q(\hat{sl}_2)$ for level $k$ \cite{[1]}. Thus (18) yields the required bosonization. One can immediately find that this representation goes to the Wakimoto representation in the $q \to 1$ limit.

6. screening current

Let us define the screening current $J^S(z)$ as follows:

\[
J^S(z) = - : \left[ 1 \partial_z \exp \left\{ - c \left( \frac{2}{q^{k-2}z}; 0 \right) \right\} \right] 
\times \exp \left\{ - b \left( \frac{2}{q^{k-2}z}; -1 \right) - a \left( k + 2 \left| \frac{q^{-2}z; -k+\frac{2}{2}}{2} \right| \right) \right\} :.
\]

(23)

Then we get the following proposition.

**Proposition 4** The commutation relations

\[
\begin{align*}
\left[ J^3_n, J^S(z) \right] &= 0, \\
\left[ J^+_n, J^S(z) \right] &= 0, \\
\left[ J^-_n, J^S(z) \right] &= k + 2 \partial_z \left[ z^n : \exp \left\{ - a \left( k + 2 \left| \frac{q^{-2}z; -k+\frac{2}{2}}{2} \right| \right) \right\} : \right].
\end{align*}
\]

(24)

hold for any $n \in \mathbb{Z}$.

Therefore, if the Jackson integral of the screening current

\[
\int_0^{\infty} J^S(t)dt, \quad p = q^{2(k+2)}
\]

(25)

is convergent, then it commutes with $U_q(\hat{sl}_2)$ exactly.

We note that the operator $\tilde{b}_0 + \tilde{c}_0$ has the following interesting property:

\[
\begin{align*}
\left[ J^3(z), \tilde{b}_0 + \tilde{c}_0 \right] &= 0, \\
\left[ J^\pm(z), \tilde{b}_0 + \tilde{c}_0 \right] &= 0, \\
\left[ J^S(z), \tilde{b}_0 + \tilde{c}_0 \right] &= 0,
\end{align*}
\]

(26)

This operator is useful when we construct the $U_q(\hat{sl}_2)$ modules in the Fock module of bosons \cite{[13]}.

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