Stability of normal quantum-fluid mixtures

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Abstract

Mixtures of quantum fluids, that is gases or liquids, are considered with the emphasis on the conditions characterizing the stability of the mixtures. The mixtures, that can be formed by cold atoms or molecules, are assumed to be quantum requiring the description using quantum techniques, but not so cold that to exhibit superfluidity or superconductivity. Reviewing the stability conditions of such normal quantum systems is important for the comparison of these conditions with the stability conditions of, e.g., Bose-condensed mixtures. The behavior of observable quantities under the stratification of quantum mixtures is discussed.

Keywords: quantum mixtures, normal quantum systems, stability conditions, response functions, dynamic form-factors, structure factors, isothermal compressibility, sum rules, role of temperature
1 Introduction

Quantum systems composed of atoms or molecules are nowadays intensively studied due to the developed techniques of cooling and trapping [1]. The major interest has been paid to the study of Bose systems in the condensed state, which several books [2–4] and review articles [5–16] are devoted to. Of particular interest are the mixtures of quantum superfluids, Bose-Bose mixtures [17], Bose-Fermi mixtures [18], and Fermi-Fermi superfluid mixtures [19].

One of the main points in considering quantum mixtures is the problem of their stability with respect to their stratification into separate components occupying different spatial locations. This type of stability, for brevity, is called mixture stability. The mixture stability of superfluid Bose-Bose mixtures is usually analyzed in the coherent approximation [20, 21] or Bogolubov approximation [22]. The Hugenholtz-Pines relation [23] for superfluid systems can be generalized for superfluid mixtures [24, 25]. The stability of superfluid mixture in the Hartree-Fock-Bogolubov approximation has been recently analyzed [26].

The mixture stability for normal (not superfluid) quantum fluids has been considered to a much lesser extent. The aim of the present paper is to compensate this deficiency and to survey what is known on the stability conditions of normal quantum mixtures.

The principal difference in the study of mixture stability for superfluid and normal fluids is in the following. The single-particle spectrum of a superfluid, where the gauge symmetry is broken, coincides with the spectrum of collective excitations [27, 28]. Therefore, to investigate the dynamic stability of a superfluid spectrum it is sufficient to consider the poles of the single-particle Green function. Contrary to this, for a normal quantum system, where the gauge symmetry is preserved, it is necessary to study the poles of the two-particle Green function or of the response function (dynamic susceptibility). For a normal system, the single-particle Green function does not define the spectrum of collective excitations.

It is interesting and important to compare the conditions of mixture stability for superfluid and normal quantum systems. Also, it is useful to compare dynamic and thermodynamic stability [29, 30]. Dynamic stability requires that the spectrum of collective excitations be positive for finite momenta. And thermodynamic stability studies conditions where the thermodynamic potential of a mixture is minimal.

We consider the systems composed of neutral atoms or molecules, whose density fluctuations enjoy sound modes. The particles can be bosons or fermions, provided that sound modes are well defined for them. The instability of a mixture with respect to the spatial separation of the components can be caused by the interspecies interactions. The instability can arise when the repulsive interspecies interactions become sufficiently strong. The derivation of conditions characterizing the mixture stability of normal quantum-fluid mixtures is the main topic of the present survey.

Throughout the paper, the system of units is employed where the Planck and Boltzmann constants are set to one.

2 Quantum-fluid mixture

Let us start with the analysis of dynamic stability of a quantum-fluid mixture. We consider particles without internal degrees of freedom, such as spin. In the case of particles with spin, this could imply that their spins are frozen in one direction and thus are switched off the particle
dynamics.

The grand Hamiltonian of the mixture of several components, enumerated by the index \( i \), has the form

\[
H = \sum_{i} \int \psi_{i}^{\dagger}(\mathbf{r}) \left( -\frac{\nabla^2}{2m} + U_{i} - \mu_{i} \right) \psi_{i}(\mathbf{r}) \, d\mathbf{r} + \\
+ \frac{1}{2} \sum_{ij} \psi_{i}^{\dagger}(\mathbf{r}) \psi_{j}^{\dagger}(\mathbf{r}') \Phi_{ij}(\mathbf{r} - \mathbf{r}') \psi_{j}(\mathbf{r}') \psi_{i}(\mathbf{r}) \, d\mathbf{r} \, d\mathbf{r}',
\]

where \( \psi_{i}(\mathbf{r}) = \psi_{i}(\mathbf{r}, t) \) are field operators (either Bose or Fermi), \( U_{i} = U_{i}(\mathbf{r}, t) \) is an external potential, \( \mu_{i} \) are chemical potentials, and \( \Phi_{ij}(\mathbf{r}) \) is an interaction potential.

The response functions are the variational derivatives

\[
\chi_{ij}(\mathbf{r}, t, \mathbf{r}', t') = \frac{\delta \rho_{i}(\mathbf{r}, t)}{\delta U_{j}(\mathbf{r}', t')}
\]

of the density

\[
\rho_{i}(\mathbf{r}, t) = \langle \psi_{i}^{\dagger}(\mathbf{r}, t) \psi_{i}(\mathbf{r}, t) \rangle = \pm \mathcal{G}_{i}(\mathbf{r}, t, \mathbf{r}, t),
\]

where we assume the causal Green functions

\[
\mathcal{G}_{i}(\mathbf{r}, t, \mathbf{r}', t') = -i \langle \hat{T} \psi_{i}(\mathbf{r}, t) \psi_{i}^{\dagger}(\mathbf{r}', t') \rangle.
\]

The Green functions at coinciding times are defined as

\[
\mathcal{G}_{i}(\mathbf{r}, t, \mathbf{r}', t) = \lim_{t' \to t+0} \mathcal{G}_{i}(\mathbf{r}, t, \mathbf{r}', t').
\]

Some interaction potentials are known to be non-integrable and not allowing for their Fourier transformation. This obstacle can be overpassed by accepting an effective interaction potential smoothed by the pair correlation function \([31]\). It has been shown \([7,32,33]\) that, starting from such a correlated approximation, it is possible to formulate a self-consistent iterative theory of all orders. Keeping this in mind, we assume that the interaction potential is integrable and allows for the Fourier transformation

\[
\Phi_{ij}(\mathbf{r}) = \int \Phi_{ij}(k) e^{ikr} \frac{dk}{(2\pi)^3}, \quad \mathbf{\Phi}_{ij}(k) = \int \Phi_{ij}(r) e^{-ikr} \, d\mathbf{r} ,
\]

with the Fourier transform \([6]\) existing for all \( k \), including \( k = 0 \).

Considering a uniform system in volume \( V \), we have the constant densities of the components

\[
\rho_{j} \equiv \frac{N_{j}}{V} \quad \left( N = \sum_{j} N_{j} \right) ,
\]

hence the constant total density

\[
\rho \equiv \frac{N}{V} = \sum_{j} \rho_{j} .
\]

Assuming that the single-particle spectrum is real, for the Fourier transform of the Green function \([11]\) we have

\[
\mathcal{G}_{i}(k, \omega) = \frac{1 \pm n_{i}(k)}{\omega - \omega_{i}(k) + i0} + \frac{n_{i}(k)}{\omega - \omega_{i}(k) - i0} ,
\]
with the single-particle spectrum given by the equation
\[ \omega_i(k) = \frac{k^2}{2m_i} + \Sigma_i(k, \omega) - \mu_i, \quad (10) \]
where \( \Sigma_i(k, \omega) \) is a self-energy and the momentum distribution is
\[ n_i(k) = \frac{1}{\exp\{\beta \omega_i(k)\} + 1} \quad (\beta T = 1), \quad (11) \]
with \( T \) being temperature. Here the upper sign is for Bose-Einstein statistics, while the lower
sign, for Fermi-Dirac statistics. The chemical potentials are defined by the normalization to
the density
\[ \rho_i = \int n_i(k) \frac{dk}{(2\pi)^3}. \quad (12) \]

3 Random-phase approximation

The response functions are usually calculated in the random-phase approximation \[34\] with the
self-energy taken in the Hartree form
\[ \Sigma_i(k, \omega) = \sum_j \Phi_{ij} \rho_j, \quad (13) \]
where
\[ \Phi_{ij} \equiv \int \Phi_{ij}(r) \, dr = \Phi_{ij}(0). \quad (14) \]

Then the Fourier transforms of the mixture response functions satisfy the equations
\[ \chi_{ij}(k, \omega) = \Pi_{ij}(k, \omega) + \Pi_{ii}(k, \omega) \sum_n \Phi_{in}(k) \chi_{nj}(k, \omega), \quad (15) \]
with the polarization functions
\[ \Pi_{ij}(k, \omega) = \pm i \delta_{ij} \int G_i(k + k', \omega + \omega') G_i(k', \omega') \frac{dk' d\omega'}{(2\pi)^4}. \quad (16) \]

For the Green function \[9\], we get
\[ \Pi_{ii}(k, \omega) = \frac{k^2}{m_i} \int \frac{n_i(k')}{(\omega - k \cdot k'/m_i)^2 - (k^2/2m_i)^2} \frac{dk'}{(2\pi)^3}. \quad (17) \]

In the case of two components, the solutions to these equations are
\[ \chi_{11} = \frac{\Pi_{11}(1 - \Pi_{22} \Phi_{12})}{(1 - \Pi_{11} \Phi_{11})(1 - \Pi_{22} \Phi_{22}) - \Pi_{11} \Pi_{22} \Phi_{12}^2}, \quad (18) \]
\[ \chi_{12} = \frac{\Pi_{11} \Pi_{22} \Phi_{12}}{(1 - \Pi_{11} \Phi_{11})(1 - \Pi_{22} \Phi_{22}) - \Pi_{11} \Pi_{22} \Phi_{12}^2}. \]
Here, for compactness, the variables $k$ and $\omega$ are omitted. The response functions $\chi_{21}$ and $\chi_{22}$ can be obtained by exchanging the indices 1 and 2 in the above expressions.

We consider here an arbitrary potential $\Phi(\mathbf{r})$. In particular, for dilute systems of trapped atoms, one has

$$
\Phi_{ii} = 4\pi \frac{a_i}{m_i}, \quad \Phi_{ij} = 2\pi \frac{a_{ij}}{m_{ij}} \quad \left( m_{ij} \equiv \frac{m_im_j}{m_i+m_j} \right),
$$

where $a_i$ and $a_{ij}$ are scattering lengths.

4 Excitation spectrum

The spectrum of collective excitations is given by the poles of the response functions, that is by the equation

$$
\chi_{ij}^{-1}(\mathbf{k}, \omega) = 0.
$$

(19)

It is convenient to introduce the function

$$
f_i(\mathbf{k}, \omega) \equiv \frac{1}{\rho_i} \int \frac{[\omega^2 - (k^2/2m_i)^2]n_i(k')}{(\omega - k \cdot k'/m_i)^2 - (k^2/2m_i)^2} \frac{dk'}{(2\pi)^3}
$$

(20)

satisfying the limits

$$
\lim_{k \to 0} f_i(\mathbf{k}, \omega) = \lim_{k \to \infty} f_i(\mathbf{k}, \omega) = 1.
$$

(21)

For the mixture of two components, the spectrum of collective excitations can be represented as

$$
\varepsilon_{\pm}^2 = \frac{1}{2} \left[ \varepsilon_1^2 + \varepsilon_2^2 \pm \sqrt{(\varepsilon_1^2 - \varepsilon_2^2)^2 + 4\varepsilon_{12}^4} \right],
$$

(22)

where

$$
\varepsilon_i^2(\mathbf{k}) = s_i^2(\mathbf{k}) k^2 + \left( \frac{k^2}{2m_i} \right)^2,
$$

(23)

with the effective sound velocity

$$
s_i^2(\mathbf{k}) = \frac{\rho_i}{m_i} \Phi_{ii}(\mathbf{k}) f_i(\mathbf{k}, \varepsilon_{\pm}),
$$

(24)

and where

$$
\varepsilon_{12}^2(\mathbf{k}) = s_{12}^2(\mathbf{k}) k^2,
$$

(25)

with

$$
s_{12}^2(\mathbf{k}) = \sqrt{\frac{\rho_1 \rho_2}{m_1 m_2} f_1(\mathbf{k}, \varepsilon_{\pm}) f_2(\mathbf{k}, \varepsilon_{\pm}) \Phi_{12}(\mathbf{k})}.
$$

(26)

In the long-wave limit, the spectra of a binary mixture reduce to the sound-type form

$$
\varepsilon_{\pm}(\mathbf{k}) \simeq s_{\pm} k \quad (k \to 0),
$$

(27)

with the sound velocities given by the equation

$$
s_{\pm}^2 = \frac{1}{2} \left[ s_1^2 + s_2^2 \pm \sqrt{(s_1^2 - s_2^2)^2 + 4s_{12}^4} \right],
$$

(28)
where
\[ s_i^2 \equiv s_i^2(0) = \frac{\rho_i}{m_i} \Phi_{ii} , \quad s_{12}^2 \equiv s_{12}^2(0) = \sqrt{\frac{\rho_1\rho_2}{m_1m_2}} \Phi_{12} \,. \] (29)

From here, it is seen that for the separate stability of each of the components, it should be
\[ \Phi_{ii} > 0 \,. \] (30)

However, the interaction of different components, \( \Phi_{12} \) can be of both signs.

In the short-wave limit, one has
\[ \varepsilon_{\pm}(k) \simeq \frac{k^2}{2m_i} \quad (k \to \infty) \,. \] (31)

As is clear, the spectrum branch \( \varepsilon_{\pm}(k) \), describing the total density oscillations, is non-negative, and the branch \( \varepsilon_{-}(k) \), characterizing the oscillations of the components with respect to each other, is non-negative under the condition
\[ \varepsilon_{12}^2(k) < \varepsilon_1(k) \varepsilon_2(k) \,, \] (32)

which is the condition of mixture stability. In the long-wave limit, we have
\[ s_{12}^2 < s_1 s_2 \,, \] (33)

which gives the stability condition
\[ \Phi_{12}^2 < \Phi_{11} \Phi_{22} \,. \] (34)

It is interesting that this is the same condition as for a coherent, completely Bose-condensed mixture [20]. In general, depending on the interaction potentials, there can appear a finite momentum where the stability condition becomes broken, as under the roton instability in dipolar systems [36].

5 Moving fluids

The stability conditions also depend on the motion of fluids [20,30,35]. Let us consider a binary mixture, such that the components move with velocities \( v_i \). Then, acting similarly to the above analysis, we come to the equation for the spectrum of collective excitations
\[ \left[ (\omega - \alpha_1)^2 - \varepsilon_1^2 \right] \left[ (\omega - \alpha_2)^2 - \varepsilon_2^2 \right] = \varepsilon_{12}^4 \,, \] (35)

in which
\[ \alpha_i(k) \equiv v_i \cdot k = v_i k \cos(\vartheta_i) \,. \] (36)

By the Descartes polynomial theorem [37] this equation enjoys two real roots provided that
\[ \varepsilon_{12}^4(k) < \left[ \varepsilon_1^2(k) - \alpha_1^2(k) \right] \left[ \varepsilon_2^2(k) - \alpha_2^2(k) \right] \] (37)

for all angles \( \vartheta_i \). Minimizing the right-hand side of the above inequality yields
\[ \varepsilon_{12}^4(k) < \left[ \varepsilon_1^2(k) - v_1^2 k^2 \right] \left[ \varepsilon_2^2(k) - v_2^2 k^2 \right] \] (38)
for the angles $\theta_i = 0, \pi$. For the broken condition \(^{(38)}\), under the angles $\theta_1 = 0$ and $\theta_2 = \pi$ or $\theta_1 = \pi$ and $\theta_2 = 0$, this is called the counter-flow instability, and when $\theta_1 = \theta_2$, this is the instability of motion.

In the long-wave limit, inequality \(^{(38)}\) reduces to the form

$$s_{12}^4 < (s_1^2 - v_1^2)(s_2^2 - v_2^2) . \tag{39}$$

In terms of the interaction potentials, this is equivalent to the condition

$$\Phi_{12}^2 < \left( \Phi_{11} - \frac{m_1 v_1^2}{\rho_1} \right) \left( \Phi_{22} - \frac{m_2 v_2^2}{\rho_2} \right) . \tag{40}$$

### 6 Dynamic form-factor

The measured quantity under neutron or x-ray scattering is the double-differential cross section (see \([5,38]\))

$$\frac{1}{N} \frac{d^2 \sigma}{d\Omega_k d\omega} = b_{sc}^2 k_{sc} k_{in} S_{tot}(k, \omega) , \tag{41}$$

in which the transferred momentum and energy are

$$k \equiv k_{in} - k_{sc} , \quad \omega \equiv E_{in} - E_{sc} ,$$

$\Omega_k$ is a solid angle around $k$, and $b_{sc}$ is the scattering length of a neutron on an atom. The dynamic form-factor at equilibrium is the Fourier transform

$$S_{tot}(k, \omega) = \frac{1}{N} \sum_{ij} \int D_{ij}(r, t, r', 0) e^{-i k \cdot (r - r') + i \omega t} dr dr' dt \tag{42}$$

of the density-density correlation function

$$D_{ij}(r, t, r', t') \equiv \langle \hat{n}_i(r, t) \hat{n}_j(r', t') \rangle , \tag{43}$$

where the density operator is

$$\hat{n}_i(r, t) \equiv \psi_i^\dagger(r, t) \psi_i(r, t) . \tag{44}$$

The total dynamic form-factor is the sum of two terms,

$$S_{tot}(k, \omega) = S_{el}(k, \omega) + S(k, \omega) . \tag{45}$$

The first term

$$S_{el}(k, \omega) = \frac{1}{N} \sum_{ij} \int \rho_i(r, t) \rho_j(r', 0) e^{-i k \cdot (r - r') + i \omega t} dr dr' dt , \tag{46}$$

where

$$\rho_i(r, t) \equiv \langle \hat{n}_i(r, t) \rangle , \tag{47}$$

describes elastic scattering. The second term

$$S(k, \omega) = \frac{1}{N} \sum_{ij} \int [ D_{ij}(r, t, r', 0) - \rho_i(r, t) \rho_j(r', 0) ] e^{-i k \cdot (r - r') + i \omega t} dr dr' dt , \tag{48}$$

7
also called the van Hove function, describes inelastic scattering.

For a uniform system, where

\[ \rho_i(r, t) = \rho_i = \frac{N_i}{V}, \]

the elastic scattering term is

\[ S_{\text{el}}(k, \omega) = (2\pi)^4 \sum_{ij} \frac{\rho_i \rho_j}{\rho} \delta(k) \delta(\omega). \] (49)

The inelastic term takes the form

\[ S(k, \omega) = \sum_{ij} S_{ij}(k, \omega), \] (50)

where

\[ S_{ij}(k, \omega) = \frac{1}{N} \int \left[ D_{ij}(r, t, r', 0) - \rho_i(r, t) \rho_j(r', 0) \right] e^{-ik(r-r')-\omega t} \, dr \, dr' \, dt. \] (51)

Defining the Fourier transform of the density-density correlation function

\[ D_{ij}(k, \omega) = \int D_{ij}(r, t, 0, 0) e^{-ikr+i\omega t} \, dr \, dt ] \] (52)

gives for a uniform system

\[ S_{ij}(k, \omega) = \frac{1}{\rho} \left[ D_{ij}(k, \omega) - (2\pi)^4 \rho_i \rho_j \delta(k) \delta(\omega) \right]. \] (53)

The density-density correlation function \((43)\) is connected with the density-density Green function

\[ -i \langle \hat{T} \hat{n}(r, t) \hat{n}(r, t') \rangle. \]

Using the properties of this Green function \([39]\) results in the Fourier transform

\[ D_{ij}(k, \omega) = \frac{2\text{Im} \chi_{ij}(k, \omega)}{1 + e^{-\beta \omega}} + (2\pi)^4 \rho_i \rho_j \delta(k) \delta(\omega), \] (54)

where \(\chi_{ij}(k, \omega)\) is the Fourier transform of the response function \([2]\). Respectively, for the dynamic form-factor \([51]\), we get

\[ S_{ij}(k, \omega) = - \frac{2\text{Im} \chi_{ij}(k, \omega)}{\rho(1 + e^{-\beta \omega})}. \] (55)

The response function enjoys the spectral representation

\[ \chi_{ij}(k, \omega) = \int_{-\infty}^{\infty} \frac{\gamma_{ij}(k, \omega')}{\omega - \omega'} \frac{d\omega'}{2\pi}, \] (56)

with the spectral function

\[ \gamma_{ij}(k, \omega) = i \left[ \chi_{ij}(k, \omega + i0) - \chi_{ij}(k, \omega - i0) \right], \] (57)
and the dispersion relation
\[ \text{Im} \chi_{ij}(k, \omega) = -\frac{1}{2} \gamma_{ij}(k, \omega) \coth \left( \frac{\beta \omega}{2} \right). \] (58)

This gives
\[ S_{ij}(k, \omega) = \frac{\gamma_{ij}(k, \omega)}{\rho(1 - e^{-\beta \omega})}. \] (59)

The dynamic form-factor possesses the properties
\[ S_{ij}(-k, \omega) = S_{ij}(k, \omega), \quad S_{ij}(k, -\omega) = S_{ij}(k, \omega) e^{-\beta \omega}. \] (60)

It can be written as the sum
\[ S_{ij}(k, \omega) = S_{ij}^+(k, \omega) + S_{ij}^-(k, \omega) \] (61)

of the symmetric term
\[ S_{ij}^+(k, \omega) \equiv \frac{1}{2} \left[ S_{ij}(k, \omega) + S_{ij}(k, -\omega) \right] = \frac{1}{2\rho} \gamma_{ij}(k, \omega) \coth \left( \frac{\beta \omega}{2} \right) = S_{ij}^+(k, -\omega) \] (62)

and the antisymmetric term
\[ S_{ij}^-(k, \omega) \equiv \frac{1}{2} \left[ S_{ij}(k, \omega) - S_{ij}(k, -\omega) \right] = \frac{1}{2\rho} \gamma_{ij}(k, \omega) = -S_{ij}^-(k, -\omega) \] (63)

with respect to the change of the sign of \( \omega \).

### 7 Response functions

The response functions (18) can be written in the form

\[ \chi_{11}(k, \omega) = \left( \frac{\rho_1}{m_1} \right) \frac{[\omega^2 - \varepsilon_2^2(k)] k^2}{[\omega^2 - \varepsilon_1^2(k)] [\omega^2 - \varepsilon_2^2(k)]}, \]
\[ \chi_{12}(k, \omega) = \sqrt{\frac{\rho_1 \rho_2}{m_1 m_2}} \frac{\varepsilon_{12}^2(k) k^2}{[\omega^2 - \varepsilon_1^2(k)] [\omega^2 - \varepsilon_2^2(k)]}. \] (64)

Then the spectral functions (57) become

\[ \gamma_{11}(k, \omega) = \frac{\rho_1}{m_1} \left[ \omega^2 - \varepsilon_2^2(k) \right] k^2 \gamma(k, \omega), \]
\[ \gamma_{12}(k, \omega) = \sqrt{\frac{\rho_1 \rho_2}{m_1 m_2}} \varepsilon_{12}^2(k) k^2 \gamma(k, \omega), \] (65)

where

\[ \gamma(k, \omega) = 2\pi \delta \left( \left[ \omega^2 - \varepsilon_2^2(k) \right] \left[ \omega^2 - \varepsilon_2^2(k) \right] \right) = \]
\[ = \frac{\pi}{\varepsilon_+^2(k) - \varepsilon_-^2(k)} \left\{ \frac{1}{\varepsilon_+(k)} \left[ \delta(\omega - \varepsilon_+(k)) - \delta(\omega + \varepsilon_+(k)) \right] - \right\} \]
\[ - \frac{1}{\varepsilon_-^2(k)} \left[ \delta(\omega - \varepsilon_-(k)) - \delta(\omega + \varepsilon_-(k)) \right] \}. \] (66)
8 Structure factor

The system structure factor

\[ S(k) = \sum_{ij} S_{ij}(k) \]  

(67)

is the sum of the partial structure factors

\[ S_{ij}(k) = \int_{-\infty}^{\infty} S_{ij}(k, \omega) \frac{d\omega}{2\pi}. \]  

(68)

For the latter, we have

\[ S_{ij}(k) = \frac{1}{\rho} \int_{0}^{\infty} \gamma_{ij}(k, \omega) \coth\left(\frac{\beta\omega}{2}\right) d\omega. \]  

(69)

Explicitly, for a binary mixture, this reads as

\[ S_{11}(k) = \left(\frac{\rho_1}{m_1}\right) \frac{k^2}{2\rho\left[\varepsilon_+^2(k) - \varepsilon_-^2(k)\right]} \times \]

\[ \times \left[ \frac{\varepsilon_+^2(k) - \varepsilon_-^2(k)}{\varepsilon_+(k)} \coth\left(\frac{\beta\varepsilon_+(k)}{2}\right) - \frac{\varepsilon_-^2(k) - \varepsilon_+^2(k)}{\varepsilon_-(k)} \coth\left(\frac{\beta\varepsilon_-(k)}{2}\right) \right]. \]  

(70)

and

\[ S_{12}(k) = \sqrt{\frac{\rho_1\rho_2}{m_1m_2}} \frac{\varepsilon_{12}^2(k)k^2}{2\rho\left[\varepsilon_+^2(k) - \varepsilon_-^2(k)\right]} \times \]

\[ \times \left[ \frac{1}{\varepsilon_+(k)} \coth\left(\frac{\beta\varepsilon_+(k)}{2}\right) - \frac{1}{\varepsilon_-(k)} \coth\left(\frac{\beta\varepsilon_-(k)}{2}\right) \right]. \]  

(71)

It is possible to check that in the short-wave limit

\[ \lim_{k \to \infty} \sum_{ij} S_{ij}(k) = 1. \]  

(72)

The limiting behavior of the partial structure factors depends on the relation between the sound modes and temperature. If the sound-mode energy is much smaller than temperature, then the structure factors tend to the limits

\[ S_{11}(0) = \left(\frac{\rho_1}{m_1}\right) \frac{s_2^2(0)T}{\rho s_+^2(0) s_-^2(0)}; \]

\[ S_{12}(0) = \sqrt{\frac{\rho_1\rho_2}{m_1m_2}} \frac{s_{12}^2(0)T}{\rho s_+^2(0) s_-^2(0)} \left( \frac{s_+k}{T} \ll 1 \right). \]  

(73)

In the opposite limit of temperature much smaller than the sound-mode energy, one has

\[ S_{11}(k) \approx \left(\frac{\rho_1}{m_1}\right) \frac{[\varepsilon_+(k)\varepsilon_-(k) + \varepsilon_+^2(k)]k^2}{2\rho\varepsilon_+(k)\varepsilon_-(k)[\varepsilon_+(k) + \varepsilon_-(k)]}; \]

\[ S_{12}(k) \approx -\sqrt{\frac{\rho_1\rho_2}{m_1m_2}} \frac{\varepsilon_{12}^2(k)k^2}{2\rho\varepsilon_+(k)\varepsilon_-(k)[\varepsilon_+(k) - \varepsilon_-(k)]}. \]  

(74)
provided that
\[ \frac{T}{s \pm k} \ll 1. \]

If both temperature and the sound-mode energy tend to zero so that \( T/s \pm k \) tends to zero, then
\[
S_{11}(k) \simeq \left( \frac{\rho_1}{m_1} \right) \frac{[s_+(0)s_-(0) + s_2^2(0)]k}{2\rho s_+(0)s_-(0)[s_+(0) + s_-(0)]},
\]
\[
S_{12}(k) \simeq -\sqrt{\frac{\rho_1 \rho_2}{m_1 m_2}} \frac{s_{12}^2(0)k}{2\rho s_+(0)s_-(0)[s_+(0) + s_-(0)]},
\]
under the condition
\[ T \to 0, \quad k \to 0, \quad \frac{T}{s \pm k} \to 0. \]

In any case, when the mixture looses stability and starts separating, with \( s_-(0) \) tending to zero, the structure factors diverge.

9 Sum rules

One considers the integral
\[
K_{ij}(k) = \int_{-\infty}^{\infty} \omega S_{ij}(k, \omega) \frac{d\omega}{2\pi}
\]
that, using the properties of the dynamic form-factor, can be written as
\[
K_{ij}(k) = \frac{1}{\rho} \int_{0}^{\infty} \omega \gamma_{ij}(k, \omega) \frac{d\omega}{2\pi}.
\]

With the spectral functions (65), this yields
\[
K_{ij}(k) = \delta_{ij} \frac{\rho_i}{\rho} \left( \frac{k^2}{2m_i} \right).
\]

The other important integral is
\[
Q_{ij}(k) = \int_{-\infty}^{\infty} \frac{1}{\omega} S_{ij}(k, \omega) \frac{d\omega}{2\pi},
\]
which can be rewritten as
\[
Q_{ij}(k) = \frac{1}{\rho} \int_{0}^{\infty} \frac{1}{\omega} \gamma_{ij}(k, \omega) \frac{d\omega}{2\pi}.
\]

Taking into account the spectral representation (56) gives
\[
Q_{ij}(k) = -\frac{1}{\rho} \text{Re} \chi_{ij}(k, 0).
\]

Since the response function has the spectral representation
\[
\chi_{ij}(k, \omega) = \int_{-\infty}^{\infty} \gamma_{ij}(k, \omega') \left[ \frac{1 + n(\omega')}{\omega - \omega' + i0} - \frac{n(\omega')}{\omega - \omega' - i0} \right] \frac{d\omega'}{2\pi},
\]
where \( n(\omega) \equiv (e^{\beta \omega} - 1)^{-1} \), then using the property
\[
\frac{1}{\omega \pm i0} = P \frac{1}{\omega} \mp i\pi \delta(\omega),
\]
where \( P \) implies the principal value, we obtain the real part
\[
\text{Re} \chi_{ij}(k, \omega) = P \int_{-\infty}^{\infty} \frac{\gamma_{ij}(k, \omega')}{\omega - \omega'} \frac{d\omega'}{2\pi}
\]
and the imaginary part
\[
\text{Im} \chi_{ij}(k, \omega) = -\frac{1}{2} \gamma_{ij}(k, \omega) \left[ 1 + 2n(\omega) \right].
\]

Here
\[
1 + 2n(\omega) = \frac{e^{\beta \omega} + 1}{e^{\beta \omega} - 1} = \coth \left( \frac{\beta \omega}{2} \right).
\]

10 Isothermal compressibility

The response functions define the compressibility functions
\[
\kappa_i(r, t, r', t') \equiv -\frac{\chi_{ii}(r, t, r', t')}{\rho_i(r, t)\rho_i(r', t')},
\]
whose Fourier transforms are
\[
\kappa_i(k, \omega) = \int \chi_i(r, t, 0, 0) e^{-i kr + i \omega t} \, dr dt.
\]

For an equilibrium uniform system, we have
\[
\kappa_i(k, \omega) = -\frac{\chi_{ii}(k, \omega)}{\rho_i^2}.
\]

The coefficient of isothermal compressibility is defined by the expression
\[
\kappa_i \equiv \text{Re} \kappa_i(0, 0) = -\frac{\text{Re} \chi_{ii}(0, 0)}{\rho_i^2}.\]

Hence it is given by the relation
\[
\kappa_i = \frac{\rho}{\rho_i^2} Q_{ii}(0).
\]

For a binary mixture, we get
\[
\text{Re} \chi_{11}(0, 0) = -\frac{\rho_1 s_{2}^2(0)}{m_1 s_{+}^2(0)s_{-}^2(0)},
\]
\[
\text{Re} \chi_{12}(0, 0) = \sqrt{\frac{\rho_1 \rho_2}{m_1 m_2}} \frac{s_{12}^2(0)}{s_{+}^2(0)s_{-}^2(0)}.\]
Therefore
\[ \kappa_1 = \frac{s_2^2(0)}{\rho_1 m_1 s_+^2(0)s_+^2(0)} , \quad \kappa_2 = \frac{s_1^2(0)}{\rho_2 m_2 s_+^2(0)s_+^2(0)} . \] (91)

Thus we find another relation for the isothermal compressibilities
\[ \kappa_i = \frac{\rho S_{ii}(0)}{\rho_i^2 T} . \] (92)

When the mixture loses its stability and \( s_-(0) \) tends to zero, the compressibility coefficients diverge.

## 11 Thermodynamic stability

Thermodynamic stability conditions are based on the comparison of the free energies of the mixed, \( F_{\text{mix}} \), and separated, \( F_{\text{sep}} \), systems. The mixture is stable when
\[ F_{\text{mix}} < F_{\text{sep}} . \] (93)

In what follows, we consider a uniform system with the component densities \( \rho_i(\mathbf{r}) = N_i/V \) and use the Hartree approximation.

The free energy of a mixed system writes as
\[ F_{\text{mix}} = E_{\text{mix}} - TS_{\text{mix}} . \] (94)

The energy
\[ E_{\text{mix}} = \langle H \rangle + \sum_i \mu_i N_i = K_{\text{mix}} + E^{\text{ext}}_{\text{mix}} + E^{\text{int}}_{\text{mix}} \] (95)
includes the kinetic energy term \( K_{\text{mix}} \), the energy due to external potentials,
\[ E^{\text{ext}}_{\text{mix}} = \sum_i \int_V U_i(\mathbf{r}) \rho_i(\mathbf{r}) \, d\mathbf{r} = \sum_i U_i \, N_i , \] (96)
where
\[ U_i \equiv \frac{1}{V} \int_V U_i(\mathbf{r}) \, d\mathbf{r} , \]
and the interaction energy
\[ E^{\text{int}}_{\text{mix}} = \frac{1}{2} \sum_{ij} \int_V \rho_i(\mathbf{r}) \Phi_{ij}(\mathbf{r} - \mathbf{r}') \rho_j(\mathbf{r}') \, d\mathbf{r}d\mathbf{r}' . \] (97)

For a uniform system, the latter is
\[ E^{\text{int}}_{\text{mix}} = \frac{1}{2} \sum_{ij} \Phi_{ij} \frac{N_i N_j}{V} , \] (98)
where
\[ \Phi_{ij} \equiv \int_V \Phi_{ij}(\mathbf{r}) \, d\mathbf{r} . \] (99)
The Hamiltonian of a separated system is

\[ H_{\text{sep}} = \sum_i \int \psi_i^\dagger(r) \left( -\frac{\nabla^2}{2m} + U_i - \mu_i \right) \psi_i(r) \, dr + \]

\[ + \frac{1}{2} \sum_i \int \psi_i^\dagger(r) \psi_i^\dagger(r') \Phi_{ii}(r-r') \psi_i(r') \psi_i(r) \, dr \, dr'. \tag{100} \]

The free energy writes as

\[ F_{\text{sep}} = E_{\text{sep}} - T S_{\text{sep}}. \tag{101} \]

The energy

\[ E_{\text{sep}} = \langle H_{\text{sep}} \rangle + \sum_i \mu_i N_i = K_{\text{sep}} + E_{\text{sep}}^{\text{ext}} + E_{\text{sep}}^{\text{int}} \tag{102} \]

contains the kinetic energy \( K_{\text{sep}} \), the energy caused by external potentials \( E_{\text{sep}}^{\text{ext}} \), and the interaction energy \( E_{\text{sep}}^{\text{int}} \).

The assumption of the system uniformity presupposes that the system volumes are sufficiently large, so that the equalities are valid:

\[ \frac{1}{V} \int_V U_i(r) \, dr = \frac{1}{V_i} \int_{V_i} U_i(r) \, dr, \quad \int_V \Phi_{ii}(r) \, dr = \int_{V_i} \Phi_{ii}(r) \, dr. \]

Also, we assume that the difference in the kinetic energy, under the same given conditions, can be neglected as compared to the difference in the potential energy, thus setting \( K_{\text{mix}} - K_{\text{sep}} \approx 0 \).

The component densities of the uniform separated system are

\[ \rho_{\text{loc}}^i(r) = \langle \psi_i^\dagger(r) \psi_i(r) \rangle = \frac{N_i}{V_i}. \tag{103} \]

Then the energy due to external potentials is

\[ E_{\text{sep}}^{\text{ext}} = \sum_i \int_{V_i} U_i(r) \rho_{\text{loc}}^i(r) \, dr = \sum_i U_i N_i, \tag{104} \]

while the interaction energy of the separated system becomes

\[ E_{\text{sep}}^{\text{int}} = \frac{1}{2} \sum_i \int_{V_i} \rho_{\text{loc}}^i(r) \Phi_{ii}(r-r') \rho_{\text{loc}}^i(r') \, dr \, dr', \tag{105} \]

which gives

\[ E_{\text{sep}}^{\text{int}} = \frac{1}{2} \sum_i \Phi_{ii} \frac{N_i^2}{V_i}. \tag{106} \]

The condition of thermodynamic stability takes the form

\[ E_{\text{mix}}^{\text{int}} - E_{\text{sep}}^{\text{int}} < N T \Delta S_{\text{mix}}, \tag{107} \]

in which

\[ \Delta S_{\text{mix}} = \frac{1}{N} (S_{\text{mix}} - S_{\text{sep}}) \tag{108} \]
is the entropy of mixing per particle.

Mechanical equilibrium implies the equality of the pressures

\[ P_i = -\frac{\partial F_{sep}}{\partial V_i} = P_j = -\frac{\partial F_{sep}}{\partial V_j}. \]  

(109)

From the latter condition, we have

\[ \frac{V_j}{V_i} = \frac{N_j}{N_i} \sqrt{\frac{\Phi_{jj}}{\Phi_{ii}}}. \]

This allows us to rewrite the stability condition (107) as the inequality

\[ \sum_{ij} \frac{N_i N_j}{2V} \left( \Phi_{ij} - \sqrt{\Phi_{ii} \Phi_{jj}} - \frac{2T}{\rho} \Delta S_{mix} \right) < 0. \]

(110)

Assuming that this inequality is valid for any \( N_i \) and \( N_j \), we come to the stability condition

\[ \Phi_{ij} - \sqrt{\Phi_{ii} \Phi_{jj}} < \frac{2T}{\rho} \Delta S_{mix}. \]

(111)

The entropy of mixing per particle can be represented as

\[ \Delta S_{mix} = -\sum_i n_i \ln n_i \left( n_i \equiv \frac{N_i}{N} \right). \]

(112)

The mixture stability condition (111) shows that finite temperatures facilitate the component mixing. The components could be immiscible at zero temperature but become miscible at finite temperature. This is also in agreement with the miscibility criterion for Bose-condensed mixtures, taking into account strong interactions and temperature [26]. The miscibility condition of type (34) can be invalid, but the components, anyway, can be miscible at finite temperatures and interactions.

### 12 Conclusion

Mixtures of quantum fluids, liquids or gases, are considered. The studied systems are treated as normal quantum systems, that is requiring quantum techniques for their description, but exhibiting no superfluidity or superconductivity. The main point of interest in the present review is the conditions of quantum mixture stability with respect to the spatial separation into different components. The mixture stability can be studied from the point of view of dynamic or thermodynamic stability.

The study of dynamic stability is based on the investigation of spectra of collective excitations. The principal difference between Bose-condensed and normal fluids is in the following. In a Bose-condensed system, the global gauge symmetry is broken. This symmetry breaking is conveniently characterized by the Bogolubov shift [40–42]. Then the vacuum state is described by a coherent state formed by completely condensed particles, while the uncondensed particles play the role of excitations above the vacuum [43]. Because of this, the single-particle Green
functions of uncondensed particles, actually, describe excitations above the coherent-state vac-
uum. This is why to study the collective spectrum of a Bose-condensed system it is sufficient
to consider solely the single-particle Green functions of uncondensed particles, whose poles give
the required collective spectrum.

On the other hand, the vacuum for a normal quantum system is an empty state. The
poles of single-particle Green functions describe the single-particle spectra, but the spectrum
of collective excitations is characterized by the poles of two-particle Green functions or by the
poles of response functions.

For the description of response functions, one usually involves the random-phase approx-
imation. This approximation describes well the low-temperature states, but the influence of
temperature on the spectrum of collective excitations is not as important.

The role of temperature can be investigated by resorting to the study of thermodynamic
stability, when one compares the free energies of a mixed and separated systems. The found
stability criterion demonstrates that temperature facilitates the mixing of different quantum
components. At low temperature, the system could be immiscible, while becomes miscible
when rising temperature.

This effect also is in agreement with the study of stability of Bose-condensed systems with
strong interactions and finite temperature. For the correct description of such systems, it is
necessary to take into account anomalous averages and the existence of two chemical potentials
for each of the components, which is caused by the global gauge symmetry breaking [44–46].
Including into the description the corrections above the coherent approximation implies the
inclusion into consideration of quantum and thermal fluctuations. These fluctuations make
the mixing easier. The Bose-condensed mixture stability, accurately taking account of fluctua-
tions, has been recently analyzed [26], confirming that fluctuations facilitate the mixing of the
components.
References

[1] Letokhov V 2007 Laser Control of Atoms and Molecules (New York: Oxford University)

[2] Lieb E H, Seiringer R, Solovej J P and Yngvason J 2005 The Mathematics of the Bose Gas and its Condensation (Basel: Birkhauser)

[3] Pethick C J and Smith H 2008 Bose-Einstein Condensation in Dilute Gases (Cambridge: Cambridge University)

[4] Ueda M 2010 Fundamentals and New Frontiers of Bose-Einstein Condensation (Singapore: World Scientific)

[5] Courteille P W, Bagnato V S and Yukalov V I 2001 Laser Phys. 11 659

[6] Andersen J O 2004 Rev. Mod. Phys. 76 599

[7] Yukalov V I 2004 Laser Phys. Lett. 1 435

[8] Bongs K and Sengstock K 2004 Rep. Prog. Phys. 67 907

[9] Yukalov V I and Girardeau M D 2005 Laser Phys. Lett. 2 375

[10] Posazhennikova A 2006 Rev. Mod. Phys. 78 1111

[11] Morsch O and Oberthaler M 2006 Rev. Mod. Phys. 78 179

[12] Yukalov V I 2007 Laser Phys. Lett. 4 632

[13] Moseley C, Fialko O and Ziegler K 2008 Ann. Phys. (Berlin) 17 561

[14] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80 885

[15] Proukakis N P and Jackson B 2008 J. Phys. B 41 203002

[16] Yurovsky V A, Olshanii M and Weiss D S 2008 Adv. At. Mol. Opt. Phys. 55 61

[17] Castilho P C M, E Pedrozo-Penafiel E, Gutierrez E M, Mazo P L, Roati G, Farias K M and Bagnato V S 2019 Laser Phys. Lett. 16 035501

[18] Saarela M and Taipaleenmaki T 2003 Int. J. Mod. Phys. B 17 5227

[19] Adhikari S K 2007 Phys. Rev. A 76 053609

[20] Yukalov V I and Yukalova E P 2004 Laser Phys. Lett. 1 50

[21] Abad M, Recati1 A, Stringari1 S and Chevy F 2015 Eur. Phys. J. D 69 126

[22] Nepomnyashchii Y A 1974 Theor. Math. Phys. 20 399

[23] Hugenholtz N M and Pines D 1959 Phys. Rev. 116 489

[24] Nepomnyashchii Y A 1976 J. Exp. Theor. Phys. 43 559
[25] Watabe S 2021 Phys. Rev. A **103** 053307

[26] Rakhimov A, Abdurakhmonov T, Narzikulov Z and Yukalov V I 2022 Phys. Rev. A **106** 033301

[27] Gavoret T and Nozières P 1964 Ann. Phys. (N.Y.) **28** 349

[28] Watabe S 2020 New J. Phys. **22** 103010

[29] Kürten K E and Campbell C E 1982 Phys. Rev. B **26** 124

[30] Yukalov V I 2016 Laser Phys. **26** 062001

[31] Kirkwood J G 1965 *Quantum Statistics and Cooperative Phenomena* (New York: Gordon and Breach)

[32] Yukalov V I 1990 Phys. Rev. A **42** 3324

[33] Yukalov V I 2016 Phys. Rev. E **94** 102106

[34] Kadanoff L P and Baym G 1962 *Quantum Statistical Mechanics* (New York: Benjamin)

[35] Yukalov V I 1980 Acta Phys. Pol. A **57** 295

[36] Yukalov V I 2018 Laser Phys. **28** 053001

[37] Henrici P 1988 *Applied and Computational Complex Analysis* Vol. 1 (New York: Wiley)

[38] Yukalov V I 2007 J. Phys. Stud. **11** 55

[39] Yukalov V I 1998 *Statistical Green’s Functions* (Kingston: Queen’s University)

[40] Bogolubov N N 1967 *Lectures on Quantum Statistics* (New York: Gordon and Breach) Vol. 1

[41] Bogolubov N N 1970 *Lectures on Quantum Statistics* (New York: Gordon and Breach) Vol. 2

[42] Bogolubov N N 2015 *Quantum Statistical Mechanics* (Singapore: World Scientific)

[43] Yukalov V I 2006 Laser Phys. **16** 511

[44] Yukalov V I 2005 Phys. Rev. E **72** 066119

[45] Yukalov V I 2006 Phys. Lett. A **359** 712

[46] Yukalov V I 2008 Ann. Phys. (N.Y.) **323** 461