Bayesian Estimation of Gumbel Type-II Distribution under Type-II Censoring with Medical Applications

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1. Introduction

In medical research, data supporting the time until the occurrence of a particular event, such as the death of a patient, are frequently encountered. Such data are referred to as survival time data which has generally right-skewed distribution, and Gumbel type-II distribution can be used for this purpose. It was introduced by the German mathematician Gumbel in [1] and is useful to model "extreme values" such as floods, earthquakes, and natural disasters and also used in life expectancy tables, hydrology, and rainfall. The probability density function (PDF) of Gumbel type-II distribution is

$$f(x|\alpha, \beta) = \alpha \beta x^{-(\alpha + 1)} \exp(-\beta x^{-\alpha}), \quad x > 0, \alpha, \beta > 0, \quad (1)$$

where "\alpha" is the shape and "\beta" is the scale parameter of the distribution. The corresponding cumulative distribution function (CDF) is

$$F(x|\alpha, \beta) = \exp(-\beta x^{-\alpha}). \quad (2)$$

A common feature of lifetime data is that the data points are possibly censored. In manifold reliability and life-testing,
studies, experiments are generally windup before failure times of all items are observed. Therefore, adequate information and results on failure times of all objects cannot be obtained. During experimentation, these situations occur due to loss or removal of objects before they fail. Therefore, generally, such experiments are preplanned and purposeful to save time and cost of these testing. Data obtained from such experiments are called censored. The type-I and type-II censoring are two well-known censoring schemes. In type-II censoring scheme, the number of failure units are fixed in advanced. For example, the investigator may decide to terminate the study after four of the six rats have developed tumors. There is an enormous literature accessible on estimation of parameters of distributions using type-II censoring, for example, Abbas and Tang [2] considered ML and least square estimators of Frechet distribution using type-II censored samples. Okasha [3] estimated the unknown parameters, reliability, and hazard functions of Lomax distribution under type-II censoring using Bayesian and E-Bayesian estimation. Abu-Zinadah [4] studied on exponentiated Gompertz distribution based on type-II and complete censored data. El-Sagheer [5] studied the generalized pareto distribution under the different censoring schemes.

Recently, many authors have worked on Gumbel type-II distribution and Bayesian estimation using different loss functions. Abbas et al. [6] worked on Gumbel type-II distribution and obtained the Bayes estimators under different loss functions. Feroze and Aslam [7] obtained Bayes estimators of two components of Gumbel type-II distribution. Malinowska and Szynal [8] also derived Bayes estimators for Gumbel type-II distribution on kth lower record values. Sultan et al. [9] worked on a three-component mixture of Gumbel type-II distribution using Bayesian estimation under different priors such as informative and noninformative. Moreover, Metiri et al. [10] worked on the properties of the Lindley distribution. The Bayes estimates were derived under LINEX loss function using informative and noninformative priors (Reyad and Ahmed [11]). Preda et al. [12] developed Bayes estimators of modified Weibull distribution under squared error loss function (SELF) and LINEX loss function.

However, Bayesian estimation of Gumbel type-II distribution based on type-II censoring is not frequently discussed; therefore, we are interested in estimating the unknown parameters of Gumbel type-II distribution under type-II censored data. Including this introduction section, the rest of the paper is arranged as follows: in Section 2, maximum likelihood estimators (MLEs) for the parameters are obtained. In Section 3, Bayesian estimators based on different loss functions by taking noninformative and gamma priors are derived. The proposed estimators are compared in terms of their mean squared error (MSE) in Section 4. Section 5 illustrates the applications of proposed estimators with two examples, namely, data set of remission times for bladder cancer and survival times of inoperable adenocarcinoma of the lung. Finally, conclusions and recommendations are presented in Section 6.

2. Maximum Likelihood Estimation

Suppose that \( X_1 < X_2 < \ldots < X_n \) is a type-II censored sample of size \( r \) obtained from a life test on \( n \) items whose life times have the Gumbel type-II distribution with parameters \( \alpha \) and \( \beta \). The likelihood function of \( r \) failures and \((n - r)\) censored values may be written as

\[
L = \left( \prod_{i=1}^{r} f(x_i, \theta) \right) (1 - F(x_r, \theta))^{n-r},
\]

\[
L = \left( \prod_{i=1}^{r} a\beta x_i^{-(\alpha + 1)} \exp(-\beta x_i^{-\alpha}) \right) (1 - \exp(-\beta x_r^{-\alpha}))^{n-r}.
\]

It is more convenient to work with log-likelihood. The log-likelihood function is

\[
\ln L = r \ln \alpha + r \ln \beta - (\alpha + 1) \sum_{i=1}^{r} \ln x_i - \beta \sum_{i=1}^{r} x_i^{-\alpha} + (n - r) \ln(1 - \exp(-\beta x_r^{-\alpha})).
\]

To get the ML estimator of \( \alpha \) and \( \beta \), differentiate equation (5) with respect to \( \alpha \) and \( \beta \) and the resulting equations are

\[
\frac{\partial \ln L}{\partial \alpha} = r - \sum_{i=1}^{r} \ln x_i + \beta \sum_{i=1}^{r} x_i^{-\alpha} \ln x_i + (n - r)
\]

\[
\frac{\partial \ln L}{\partial \beta} = r \sum_{i=1}^{r} x_i^{-\alpha} + (n - r)
\]

Equations (6) and (7) cannot be written in closed form. Therefore, here, we use the Laplace approximation to get the point estimates of \( \alpha \) and \( \beta \).

3. Bayesian Estimation

In Bayesian estimation, we consider different loss functions such as squared error loss function (SFL) proposed by Legendre [13] and Gauss [14], LINEX (Varian [15]), and general entropy loss function (GELF) introduced by Calabria and Pulcini [16]. As both parameters are unknown, independent noninformative form of priors can be used. Supposed that \( \alpha \) and \( \beta \) have independent Gamma \((a, b)\) and Gamma \((c, d)\) priors, respectively, for \( a, b, c, d > 0 \), i.e.,

\[
\pi_1(\alpha) \propto \alpha^{a-1} \exp(-ba),
\]

\[
\pi_2(\beta) \propto \beta^{d-1} \exp(-d\beta).
\]

The joint prior distribution of parameters is
\[ \Phi'(\alpha, \beta | x) = K(\alpha^{(\alpha+r-1)})(\beta^{(\beta+r-1)}) \]
\[ \cdot \left( \exp \left( -\alpha - d\beta - \beta \sum_{i=1}^{r} x_i^{-\alpha} \right) \right) \left( \prod_{i=1}^{r} x_i^{-\alpha-1} \right) \]
\[ \cdot \left( 1 - \exp \left( \sum_{i=1}^{r} x_i^{-\alpha} \right) \right), \]
\tag{10} \]

where \( K \) is the normalizing constant that makes \( \Phi'(\alpha, \beta | x) \) a proper PDF. Thus,
\[ K^{-1} = \int_{\alpha, \beta} \left( \alpha^{(\alpha+r-1)})(\beta^{(\beta+r-1)}) \right) \left( \exp \left( -\alpha - d\beta - \beta \sum_{i=1}^{r} x_i^{-\alpha} \right) \right) \left( \prod_{i=1}^{r} x_i^{-\alpha-1} \right) \left( 1 - \exp \left( \sum_{i=1}^{r} x_i^{-\alpha} \right) \right) d\alpha d\beta. \]
\tag{11} \]

Therefore, the joint posterior density under any loss function is
\[ \bar{\alpha}_{\text{SELF}} = \tilde{a} + \frac{1}{2} \left\{ \frac{2r}{\tilde{a}^3} + \tilde{\beta} \sum_{i=1}^{r} x_i^{-\alpha} (\ln x_i)^3 - \left( \frac{(n-r)(1-A)\tilde{\beta} (\ln x_i)^3 x_i^{-\alpha}}{A^3} \right) \left( \tilde{\beta} x_i^{-\alpha} - A \right) \left( 2 - A - A(1 + A) \right) + A^2 \right\} \]
\[ S_{11}^{S2} + \left\{ \frac{2r}{\tilde{\beta}} - \frac{(n-r)x_i^{-\alpha} (1-A)A(2-A)}{A^3} \right\} S_{21} S_{22} \]
\[ + 3 \left\{ -\sum_{i=1}^{r} x_i^{-\alpha} (\ln x_i)^2 - \left( \frac{(1-A)(n-r)x_i^{-\alpha} (\ln x_i)^2}{A^3} \right) \left( \tilde{\beta} x_i^{-\alpha} - A \right) \left( A - 2 + A(A + 1) - A^3 \right) \right\} \]
\[ S_{11} S_{12} + \left\{ \frac{(n-r)(1-A)x_i^{-\alpha} \ln x_i}{A^3} \left( \tilde{\beta} x_i^{-\alpha} (2 - A) - 2A \right) \right\} \left( S_{22} S_{21} + 2 S_{21}^2 \right) \]
\[ + \left( \frac{a - 1}{\tilde{a}} - b \right) S_{11} + \left( \frac{c - 1}{\tilde{\beta}} - d \right) S_{21}, \]
\tag{14} \]

Posterior distribution (12) takes a ratio form that cannot be reduced to a closed form. Therefore, we use Lindley approximation [17] to get the Bayesian estimates, which can be written as
\[ \hat{\theta} = g(\tilde{a}, \tilde{\beta}) + \frac{1}{2} \left[ \sum_{i=1}^{r} l_i S_i + l_0 A_{12} + l_0 A_{21} + l_2 B_{13} + l_2 B_{21} \right] \]
\[ + q_1 C_{12} + q_2 C_{21}, \]
\tag{13} \]

where \( q_1 = (\partial \ln \pi(\alpha, \beta)/\partial a); \ q_2 = (\partial \ln \pi(\alpha, \beta)/\partial \beta); \ l_{11} = (\partial^2 g(\alpha, \beta)/\partial a^2); \ l_{12} = (\partial^2 g(\alpha, \beta)/\partial a \partial \beta); \ l_2 = (\partial^2 g(\alpha, \beta)/\partial \beta^2); \ l_i = (\partial g(\alpha, \beta)/\partial a); \ l_j = (\partial g(\alpha, \beta)/\partial \beta); \ \ A_{ij} = (l_i S_i + l_j S_j) S_{ij}; \ B_{ij} = 3l_i S_i S_j + l_i (S_i S_j + 2S_{ij}^2); \ C_{ij} = l_i S_i + l_j S_j; \ j = 1, 2. \ The \ detail \ of \ equation \ (13) \ is \ provided \ in \ Appendix. \ The \ approximate \ Bayesian \ estimators \ of \ “a” and “b” based on SELF are
\[ \tilde{\beta}_{BSELF} = \tilde{\beta} + \frac{1}{2} \left\{ \frac{2r}{\tilde{\alpha}} + \tilde{\beta} \sum_{i=1}^{r} x_i^{-\tilde{\alpha}} (\ln x_i)^3 - \frac{(n-r)(1-A)\tilde{\beta}(\ln x_i)^3 x_i^{-\tilde{\alpha}}}{A^3} \right\} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - A \right) \left( 2 - A - A (1 + A) \right) + A^2 } \]

\[ S_{12} S_{11} + \frac{2r}{\tilde{\beta}} \frac{(n-r)x_i^{-3\tilde{\alpha}} (1-A)(A-2)}{A^3} \left\{ S_{22} + \frac{1}{2} \sum_{i=1}^{r} x_i^{-\tilde{\alpha}} (\ln x_i)^3 - \frac{(1-A)(n-r)x_i^{-\tilde{\alpha}} (\ln x_i)^2}{A^3} \right\} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - A \right) \left( 2 - A - A (1 + A) \right) - A^2 } \]

\[ \left( S_{11} S_{22} + 2S_{12}^2 \right) + 3 \left\{ \frac{(n-r)(1-A)x_i^{-\tilde{\alpha}} \ln x_i}{A^3} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - 2A \right) \right\} S_{22} S_{21} \]

\[ + \left( \frac{a-1}{\tilde{\alpha}} - b \right) S_{21} + \left( \frac{c-1}{\tilde{\beta}} - d \right) S_{22}. \]

(15)

Similarly, the Bayesian estimators of "\( \alpha \)" and "\( \tilde{\beta} \)" under LINEX loss function are

\[ \tilde{\alpha}_{BLINEX} = \frac{1}{k} \ln \left( e^{-\kappa_\alpha} + \frac{1}{2} \left\{ k e^{-\kappa_\alpha} S_{11} - ke^{-\kappa_\alpha} \left( \frac{2r}{\tilde{\alpha}} + \tilde{\beta} \sum_{i=1}^{r} x_i^{-\tilde{\alpha}} (\ln x_i)^3 - \frac{(n-r)(1-A)\tilde{\beta}(\ln x_i)^3 x_i^{-\tilde{\alpha}}}{A^3} \right) \right\} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - A \right) \left( 2 - A - A (1 + A) \right) + A^2 } \]

\[ \left( \frac{2r}{\tilde{\beta}} \frac{(n-r)x_i^{-3\tilde{\alpha}} (1-A)(A-2)}{A^3} \right) S_{11} S_{22} \]

\[ - ke^{-\kappa_\alpha} \left( \frac{2r}{\tilde{\beta}} \frac{(n-r)x_i^{-3\tilde{\alpha}} (1-A)(A-2)}{A^3} \right) S_{11} S_{22} \]

\[ - 3ke^{-\kappa_\alpha} \left( \frac{1}{2} \sum_{i=1}^{r} x_i^{-\tilde{\alpha}} (\ln x_i)^3 - \frac{(1-A)(n-r)x_i^{-\tilde{\alpha}} (\ln x_i)^2}{A^3} \right) \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - A \right) \left( 2 - A - A (1 + A) \right) \left( 2 - A - A (1 + A) \right) - A^2 } \]

\[ \left( S_{11} S_{22} + 2S_{12}^2 \right) + 3 \left\{ \frac{(n-r)(1-A)x_i^{-\tilde{\alpha}} \ln x_i}{A^3} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} - 2A \right) \right\} S_{22} S_{21} \]

\[ -ke^{-\kappa_\alpha} \left( \frac{(n-r)(1-A)x_i^{-\tilde{\alpha}} \ln x_i}{A^3} \left( \tilde{\beta} x_i^{-\tilde{\alpha}} (2 - A) \right) \right) \left( S_{22} S_{21} + 2S_{21}^2 \right) \]

\[ -ke^{-\kappa_\alpha} \left( \frac{a-1}{\tilde{\alpha}} - b \right) S_{21} - ke^{-\kappa_\alpha} \left( \frac{c-1}{\tilde{\beta}} - d \right) S_{22} \]

(16)
\[ \hat{\beta}_{BLINEX} = -\frac{1}{k} \ln \left[ e^{-\hat{\beta}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\beta}^0} S_{22} - k e^{-\hat{\beta}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\beta} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 - \frac{(n-r)(1-A)\tilde{\beta}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \left( \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right) \right] \]

\[ = -\frac{k \alpha^{-1/k}}{k} + \left( \frac{2r}{\beta^3} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 - \frac{(n-r)(1-A)\tilde{\beta}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \left( \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right) S_{12} S_{11} \]

\[ = -\frac{k \alpha^{-1/k}}{k} + \frac{2r}{\beta^3} \frac{(n-r)x_r^{\tilde{\alpha}} (\ln x_r)^3}{A^3} x_r^{\tilde{\alpha}} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) + 3 \frac{k \alpha^{-1/k}}{A^3} \left( \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 - \frac{(n-r)(1-A)\tilde{\beta}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \left( \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right) S_{12} S_{11} \]

The Bayesian estimators of "α" and "β" under GELF are

\[ \hat{\alpha}_{GELF} = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]

\[ = \frac{1}{k} \ln \left[ e^{-\hat{\alpha}^0} + \frac{1}{2} \left( k^2 e^{-\hat{\alpha}^0} S_{22} - k e^{-\hat{\alpha}^0} \cdot \left( \frac{2r}{\alpha^2} + \tilde{\alpha} \sum_{i=1}^{r} x_i^{\tilde{\alpha}} (\ln x_i)^3 \right) - \frac{(n-r)(1-A)\tilde{\alpha}(\ln x_r)^3 x_r^{\tilde{\alpha}}}{A^3} \beta x_r^{\tilde{\alpha}} \left( 2A - (A + 1) \right) + A^2 \right) \right] \]
\( \hat{\beta}_{BGELF} = \left[ \frac{1}{2} k (k + 1) \bar{\beta} \right]_{22}^{-k} \left\{ k \left( \frac{1}{2} k \right) \bar{\beta} \right\}_{22}^{-(k+2)} S_{22} \)

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{\bar{\alpha}} + \frac{\beta}{\bar{\alpha}} \sum_{i=1}^{r} x_i^\bar{\alpha} (\ln x_i)^3 \right) \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{\bar{\alpha}} (n - r)(1 - A) \bar{\beta} (\ln x_r)^3 x_r^\bar{\alpha} \left( \bar{\beta} x_r^\bar{\alpha} (2 - A) - A (A + 1) + A^2 \right) \right) S_{13} S_{11} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

\[- k \bar{\beta}^{-(k+1)} \left( \frac{2r}{3} \left( n - r \right) x_r^\bar{\alpha} (1 - A)(A - 2) \right) S_{22}^{22} - k \bar{\beta}^{-(k+1)} \]

where \( \alpha \) and \( \beta \) are the ML estimators of \( \alpha \) and \( \beta \) which can be obtained from equations (6) and (7), respectively.

### 4. Simulation Study

The performance of the proposed Bayesian estimators with their ML counterpart in terms of MSE, different sample sizes, and different values of parameters are considered using Monte Carlo simulation based on prespecified different percentages of failures, i.e., 40\%, 60\%, and 80\%. Monte Carlo simulation is conducted as follows:

(i) Take the initial values of \( \alpha \) and \( \beta \), respectively, and the samples are generated from the Gumbel type-II distribution using inverse transformation technique, i.e., \( X(F) = (-\log U/\beta)^{-(1/\alpha)} \), where \( U \sim \text{uniform} \( (0, 1) \).

(ii) First, calculate the MLE using Laplace approximation, and then Bayesian estimates under non-informative priors are obtained via Lindley approximation.

(iii) The process is replicated 5000 times for each sample size and averages of these estimates and the corresponding MSEs (within parenthesis) were calculated for each method using the R software version (i386 3.6.1), which approximately takes around half an hour.

The results are reported in Tables 1–4 for comparison purposes. Tables 1 and 2 contain simulation results for the case where loss function parameter \( k = 1 \) and values of hyperparameters are considered as \( a = b = c = d = 2 \), whereas Tables 3 and 4 comprise the results for the case where \( k = 1.5 \) and values of hyperparameters are \( a = 1, b = 2, c = 2.25 \), and \( d = 1.5 \) for the simulation study. From the results of the simulation study, conclusions are drawn regarding the behavior of the estimators, which are summarized as follows:

(i) In terms of MSEs, the ML and Bayesian estimators become closer by increasing the sample sizes.

(ii) For fixed percentage of failures, as sample size increases, it is observed that the MSEs of all the estimators decrease because as for large sample sizes, prior has minimal effect on the posterior.

(iii) For fixed values of \( \alpha \) and \( \beta \), the MSEs of ML and Bayesian estimators decrease when both increase the sample size and percentage of failures.

(iv) When \( k = 1 \) and \( a = b = c = d = 2 \), the Bayesian estimators based on GELF and LINEX loss function are smaller as compared to ML estimators in terms of MSEs. Therefore, Bayes estimators are much stable than ML estimators.

(v) Generally, the ML and Bayesian estimators are closed for the large sample in terms of MSE.

### 5. Data Analysis

In this section, we consider two examples for illustration purposes.

#### 5.1. Example 1

The real data about remission times (in months) of a random sample of 128 bladder cancer patients presented in Table 5 were reported by Lee and Wang [20]. A total of 128 patients with different prespecified percentages of events, i.e., 40\%, 50\%, 60\%, and 80\%, represented patients whose treatment was terminated and rest of the percentages are censored. Clearly, Figure 1 confirms that the histogram is slightly skewed to the right and is leptokurtic. Moreover, ML and Bayesian estimates can also be envisioned in Figure 1, in which the \( x \)-axis represents the remission times (in months) of bladder cancer patients, while the Gumbel type-II density function is taken on the \( y \)-axis. Therefore, it would be appropriate to select positively skewed distributions for describing the behavior of remission times of bladder cancer patients. Amongst the skewed distributions, Gumbel type-II distribution is fitted and the parameter estimates using ML and Bayesian methods are presented in Table 6 for comparison purposes.
Table 1: Average ML and Bayesian estimates with corresponding MSEs (within parenthesis) using different percentages of failures for $\alpha = 1.5$ (when $a = b = c = d = 2$, $k = 1$).

| $n$ | Percent | ML       | BSELF     | BGELF     | BLINEX    |
|-----|---------|----------|-----------|-----------|-----------|
|     |         | (0.2808) | (0.3524)  | (0.9989)  | (0.9811)  |
| 25  | 40      | 1.7162   | 1.7948    | 1.5450    | 1.6466    |
|     | 60      | 1.6408   | 1.6893    | 1.5633    | 1.5478    |
|     | 80      | 1.6160   | 1.7527    | 1.5883    | 1.5647    |
| 50  | 40      | 1.6263   | 1.5858    | 1.5064    | 1.5179    |
|     | 60      | 1.5610   | 1.5856    | 1.4836    | 1.4891    |
|     | 80      | 1.5667   | 1.5840    | 1.5084    | 1.5126    |
| 80  | 40      | 1.5708   | 1.5889    | 1.5042    | 1.5095    |
|     | 60      | 1.5516   | 1.5648    | 1.5041    | 1.5080    |
|     | 80      | 1.5269   | 1.5373    | 1.4926    | 1.4956    |
| 100 | 40      | 1.5540   | 1.5681    | 1.5023    | 1.5069    |
|     | 60      | 1.5345   | 1.5448    | 1.4975    | 1.5008    |
|     | 80      | 1.5292   | 1.5374    | 1.5020    | 1.5045    |
| 150 | 40      | 1.5327   | 1.5448    | 1.4992    | 1.5026    |
|     | 60      | 1.5234   | 1.5381    | 1.4991    | 1.5016    |
|     | 80      | 1.5143   | 1.5208    | 1.4977    | 1.4995    |
| 200 | 40      | 1.5317   | 1.5384    | 1.5067    | 1.5092    |
|     | 60      | 1.5220   | 1.5271    | 1.5040    | 1.5058    |
|     | 80      | 1.5119   | 1.5160    | 1.4987    | 1.5001    |

ML: maximum likelihood estimators, BSELF: Bayesian estimators under squared error loss function, BGELF: Bayesian estimators under general entropy loss function, and BLINEX: Bayesian estimators under linear exponential loss function.

Table 2: Average ML and Bayesian estimates with corresponding MSEs (within parenthesis) using different percentages of failures for $\beta = 1.5$ (when $a = b = c = d = 2$, $k = 1$).

| $n$ | Percent | ML       | BSELF     | BGELF     | BLINEX    |
|-----|---------|----------|-----------|-----------|-----------|
|     |         | (0.1538) | (0.1852)  | (0.1131)  | (0.1042)  |
| 25  | 40      | 1.5418   | 1.6030    | 1.3879    | 1.3988    |
|     | 60      | 1.5907   | 1.6648    | 1.4357    | 1.4448    |
|     | 80      | 1.5725   | 1.6280    | 1.4255    | 1.4346    |
| 50  | 40      | 1.5211   | 1.5473    | 1.4455    | 1.4521    |
|     | 60      | 1.5396   | 1.5644    | 1.4677    | 1.4735    |
|     | 80      | 1.5358   | 1.5609    | 1.4658    | 1.4715    |
| 80  | 40      | 1.5147   | 1.5303    | 1.4676    | 1.4721    |
|     | 60      | 1.5215   | 1.5364    | 1.4774    | 1.4814    |
|     | 80      | 1.5262   | 1.5415    | 1.4830    | 1.4869    |
| 100 | 40      | 1.5026   | 1.5149    | 1.4654    | 1.4691    |
|     | 60      | 1.5220   | 1.5338    | 1.4868    | 1.4901    |
|     | 80      | 1.5158   | 1.5278    | 1.4817    | 1.4850    |
| 150 | 40      | 1.5033   | 1.5114    | 1.4784    | 1.4809    |
|     | 60      | 1.5118   | 1.5194    | 1.4886    | 1.4909    |
|     | 80      | 1.5067   | 1.5145    | 1.4842    | 1.4865    |
| 200 | 40      | 1.5060   | 1.5121    | 1.4873    | 1.4892    |
|     | 60      | 1.5105   | 1.5162    | 1.4931    | 1.4949    |
|     | 80      | 1.5110   | 1.5169    | 1.4940    | 1.4958    |

Table 3: Average ML and Bayesian estimates with corresponding MSEs (within parenthesis) using different percentages of failures for $\alpha = 1.5$ (when $a = 1$, $b = 2$, $c = 2.25$, $d = 1.5$, and $k = 1$).

| $n$ | Percent | ML       | BSELF     | BGELF     | BLINEX    |
|-----|---------|----------|-----------|-----------|-----------|
|     |         | (0.9023) | (0.1025)  | (0.0629)  | (0.0605)  |
| 50  | 40      | 1.5879   | 1.6185    | 1.4239    | 1.4251    |
|     | 60      | 1.5775   | 1.5975    | 1.4604    | 1.4615    |
|     | 80      | 1.5404   | 1.5575    | 1.4576    | 1.4590    |
| 80  | 40      | 1.5434   | 1.5607    | 1.4439    | 1.4454    |
|     | 60      | 1.5704   | 1.5838    | 1.4973    | 1.4983    |
|     | 80      | 1.5493   | 1.5597    | 1.4479    | 1.4982    |
| 100 | 40      | 1.5280   | 1.5419    | 1.4599    | 1.4495    |
|     | 60      | 1.5145   | 1.5247    | 1.4973    | 1.4612    |
|     | 80      | 1.5389   | 1.5472    | 1.4971    | 1.4982    |
| 200 | 40      | 1.5370   | 1.5439    | 1.4959    | 1.4968    |
|     | 60      | 1.5184   | 1.5248    | 1.4903    | 1.4910    |
|     | 80      | 1.5216   | 1.5257    | 1.5007    | 1.5013    |
It is concluded that the proposed estimators of Gumbel type-II distribution fit the data well. Therefore, it is recommended that the Bayesian estimators can be more beneficial to address the uncertainty in medical-related censored data.

5.2. Example 2. The survival times, in weeks, of 61 patients with unoperable lung cancer treated with cyclophosphamide considered in Lagakos and Williams ([18]) and in Lee and Wolfe ([19]) are presented in Table 7. Here are 33 uncensored observations and 28 censored observations, representing the patients whose treatment was terminated because of a devolving condition. The point estimates of $\alpha$ and $\beta$ obtained by all the methods are summarized in Table 8. Figure 2 shows the results of different estimation methods and depicts that Gumbel type-II distribution fits the data better, in which $x$-axis comprises the survival times in weeks of 61 patients with inoperable adenocarcinoma of the lung.

Table 4: Average ML and Bayesian estimates with corresponding MSEs (within parenthesis) using different percentages of failures for $\beta = 1.5$ (when $a = 1, b = 2, c = 2.25, d = 1.5,$ and $k = 1.5$).

| Percent | ML         | BSELF       | BGELF       | BLINEX      |
|---------|------------|-------------|-------------|-------------|
| 40      | 1.5341 (0.0661) | 1.5818 (0.0734) | 1.5067 (0.0546) | 1.5085 (0.0521) |
| 60      | 1.5564 (0.0627) | 1.5719 (0.0693) | 1.4987 (0.0524) | 1.5006 (0.0501) |
| 80      | 1.5464 (0.0494) | 1.5599 (0.0536) | 1.4829 (0.0430) | 1.4856 (0.0408) |
| 40      | 1.5052 (0.0382) | 1.5459 (0.0407) | 1.5008 (0.0342) | 1.5024 (0.0332) |
| 60      | 1.5308 (0.0319) | 1.5208 (0.0330) | 1.4741 (0.0302) | 1.4763 (0.0292) |
| 80      | 1.5187 (0.0285) | 1.5337 (0.0301) | 1.4901 (0.0260) | 1.4919 (0.0254) |

Table 5: Remission times (in months) of a random sample of 128 bladder cancer patients.

| Percent | ML         | BSELF       | BGELF       | BLINEX      |
|---------|------------|-------------|-------------|-------------|
| 40      | 0.5433     | 0.5452      | 0.5287      | 0.5329      |
| 50      | 0.5909     | 0.5921      | 0.5779      | 0.5812      |
| 60      | 0.6339     | 0.6348      | 0.6225      | 0.6252      |
| 80      | 0.6988     | 0.6994      | 0.6898      | 0.6917      |

Table 6: Point estimates of $\alpha$ and $\beta$ using different percentages of failures when $a = 1, b = 2, c = 2.25, d = 1.5,$ and $k = 1.5$.

| Percent | ML         | BSELF       | BGELF       | BLINEX      |
|---------|------------|-------------|-------------|-------------|
| 40      | 0.5433     | 0.5452      | 0.5287      | 0.5329      |
| 50      | 0.5909     | 0.5921      | 0.5779      | 0.5812      |
| 60      | 0.6339     | 0.6348      | 0.6225      | 0.6252      |
| 80      | 0.6988     | 0.6994      | 0.6898      | 0.6917      |

Table 7: Survival times in weeks of 61 patients with inoperable adenocarcinoma of the lung.

| Percent | ML         | BSELF       | BGELF       | BLINEX      |
|---------|------------|-------------|-------------|-------------|
| 40      | 2.4491     | 2.4702      | 2.3960      | 2.3915      |
| 50      | 2.4252     | 2.4440      | 2.3729      | 2.3689      |
| 60      | 2.4139     | 2.4312      | 2.3623      | 2.3594      |
| 80      | 2.4095     | 2.4253      | 2.3588      | 2.3551      |

28 Censored observations
0.14, 0.14, 0.29, 0.43, 0.57, 0.57, 0.86, 1.00, 1.00, 1.00, 1.29, 1.60, 2.00, 2.76, 3.00, 3.00, 3.29, 3.29, 3.60, 6.00, 6.00, 6.14, 8.71, 10.57, 11.86, 15.77, 15.77, 17.29, 18.71, 21.29, 23.86, 26.00, 27.57, 32.14, 33.14, 47.29

33 Uncensored observations
0.43, 2.86, 3.14, 3.14, 3.43, 3.43, 3.71, 3.86, 6.14, 6.86, 9.00, 9.43, 10.71, 10.86, 11.14, 13.00, 14.43, 15.71, 18.43, 18.57, 20.71, 29.14, 29.71, 30.47, 48.57, 49.43, 53.86, 61.86, 66.57, 68.71, 68.96, 72.86, 72.86
the lung as the Gumbel type-II density function is taken on the $y$-axis.

6. Conclusion and Recommendations

In medical decision-making, Bayesian tools incorporate the state of uncertainty and provide a rational framework for studying such problems. Usually, medical data are generally skewed to the right, and positively skewed distributions can be most suitable for describing unimodal medical data. In this study, an attempt has been made to develop the Bayesian estimators for Gumbel type-II distribution based on type-II censored data using squared error loss, GELF, and LINEX loss functions via Lindley’s approximation. It is concluded that ML and Bayesian estimators become closer by increasing the sample sizes and prespecified percentages of failures. Based on the outcomes of this research study, we may suggest that this study can be further extended by using other skewed distributions considering the Bayesian framework with other loss functions using medical data.

Appendix

Observed Fisher Information Matrix

The observed Fisher information matrix (FIM) is computed by taking the $2^{nd}$ partial derivatives with respect to “$\alpha$” and “$\beta$,” respectively. Therefore, the matrix may be defined as

$$
I_{(\alpha, \beta)} = \begin{bmatrix}
\frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\
\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2}
\end{bmatrix}.
$$

(A.1)

The components of observed FIM are

$$
\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{r}{\alpha^2} - \beta \sum_{i=1}^{r} x_i^{-\alpha} \left( \ln x_i \right)^3 - (n-r)\beta \ln x_i,
$$

(A.2)

$$
\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = -(n-r)x_i^{-\alpha} \ln x_i \left[ \frac{x_i^{-\alpha} \beta - A}{A^2} \right] + r \alpha^{-2} + \beta \sum_{i=1}^{r} x_i^{-\alpha} \left( \ln x_i \right)^2 + (n-r)\beta \ln x_i,
$$

$$
\frac{\partial^2 \ln L}{\partial \beta^2} = \sum_{i=1}^{r} x_i^{-\alpha} \ln x_i + (n-r) \left[ (1-A)x_i^{-\alpha} \ln x_i \left( \beta x_i^{-\alpha} - A \right) \right] - \sum_{i=1}^{r} x_i^{-\alpha} \ln x_i - (n-r) \left[ \frac{(1-A)x_i^{-\alpha} \ln x_i \left( \beta x_i^{-\alpha} - A \right)}{A^2} \right] + W.
$$

(A.3)

The observed FIM matrix is rewritten as

$$
I_{(\alpha, \beta)} = \begin{bmatrix}
U & W \\
W & V
\end{bmatrix}.
$$

(A.5)

The inverse of $I_{(\alpha, \beta)}$ is

$$
I^{-1}_{(\alpha, \beta)} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix},
$$

(A.6)

$$
S_{11} = \frac{V}{UV - W^2},
$$

$$
S_{12} = \frac{W}{UV - W^2},
$$

$$
S_{21} = \frac{W}{UV - W^2},
$$

$$
S_{22} = \frac{U}{UV - W^2}.
$$
Data Availability

This work is mainly a methodological development and has been applied on secondary data, but if required, data will be provided.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.
[4] H. H. Abu-Zinadah, “Bayesian estimation on the exponentiated Gompertz distribution under type II censoring,” *International Journal of Contemporary Mathematical Sciences*, vol. 9, no. 11, pp. 497–505, 2014.

[5] R. M. El-Sagheer, “Bayesian prediction based on general progressive censored data from generalized Pareto distribution,” *Journal of Statistics Applications & Probability*, vol. 5, no. 1, pp. 43–51, 2016.

[6] K. Abbas, J. Fu, and Y. Tang, “Bayesian estimation of Gumbel type-II distribution,” *Data Science Journal*, vol. 12, pp. 33–46, 2013.

[7] N. Feroze and M. Aslam, “Bayesian estimation of two-component mixture of gumbel type II distribution under informative priors,” *International Journal of Advanced Science and Technology*, vol. 53, pp. 11–30, 2013.

[8] I. Malinowska and D. Szyndra, “On characterization of certain distributions of kth lower (upper) record values,” *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 338–347, 2008.

[9] T. Sultana, M. Aslam, and M. Raftab, “Bayesian estimation of 3-component mixture of Gumbel type-II distributions under non-informative and informative priors,” *Journal of the National Science Foundation of Sri Lanka*, vol. 45, no. 3, pp. 287–306, 2017.

[10] F. Metiri, H. Zeghdoudi, and M. R. Remita, “On Bayes estimates of Lindley distribution under Linux loss function: informative and non informative priors,” *Global Journal of Pure and Applied Mathematics*, vol. 12, no. 1, pp. 391–400, 2016.

[11] H. Reyad and S. O. Ahmed, “E-Bayesian analysis of the Gumbel type-II distribution under type-II censored scheme,” *International Journal of Advanced Mathematical Sciences*, vol. 3, no. 2, pp. 108–120, 2015.

[12] V. Preda, E. Panaitescu, and A. Constantinescu, “Bayes estimators of modified- weibull distribution parameters using Lindleys approximation,” *WSEAS Transactions on Mathematics*, vol. 9, no. 7, pp. 539–549, 2010.

[13] A. Legendre, *New Method for the Dermination of Orbits of Comets*, Courcier, Paris, France, 1805.

[14] C. F. Gauss, *Least Squares Method for the Combinations of Observation*, (Translated by J. Bertrand 1955), Mallet-Bachelier, Paris, France, 1810.

[15] H. R. Varian, *A Bayesian Approach to Real Estate Assessment*, pp. 195–208, North Holland, Amsterdam, Netherlands, 1975.

[16] R. Calabria and G. Pulcini, “Point estimation under asymmetric loss functions for left-truncated exponential samples,” *Communications in Statistics—Theory and Methods*, vol. 25, no. 3, pp. 585–600, 1996.

[17] D. V. Lindley, “Approximate bayesian methods,” *Trabajos de Estadistica Y de Investigacion Operativa*, vol. 31, no. 1, pp. 223–245, 1980.

[18] E. T. Lee and J. W. Wang, Eds., *Statistical Methods for Survival Data Analysis*, John Wiley and Sons, New York, NY, USA, 3rd edition, 2003.

[19] S. W. Lagakos and J. S. Williams, “Models for censored survival analysis: a cone class of variable-sum models,” *Biometrika*, vol. 65, pp. 181–189, 1978.

[20] S. Lee and R. A. Wolfe, “A simple test for independent censoring under the proportional hazards model,” *Biometrics*, vol. 54, pp. 1176–1182, 1998.