Non-Euclidean Monotone Operator Theory
with Applications to Recurrent Neural Networks

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Abstract—We provide a novel transcription of monotone operator theory to the non-Euclidean finite-dimensional spaces $\ell_1$ and $\ell_{\infty}$. We first establish properties of mappings which are monotone with respect to the non-Euclidean norms $\ell_1$ or $\ell_{\infty}$. In analogy with their Euclidean counterparts, mappings which are monotone with respect to a non-Euclidean norm are amenable to numerous algorithms for computing their zeros. We demonstrate that several classic iterative methods for computing zeros of monotone operators are directly applicable in the non-Euclidean framework. We present a case-study in the equilibrium computation of recurrent neural networks and demonstrate that casting the computation as a suitable operator splitting problem improves convergence rates.

I. INTRODUCTION

In the last few years, monotone operator methods have become prevalent to solve problems in optimization and control [4], [26], game theory [23], systems analysis [5], and to better understand machine learning models [9], [28]. However, monotone operator techniques are primarily based on the theory of Hilbert and Euclidean spaces, while many problems are well-posed or better-suited for analysis in a Banach space or finite-dimensional non-Euclidean space. For example, in machine learning, it is known that robustness analysis of artificial neural networks is naturally performed via the $\ell_{\infty}$ norm and that such a norm is most appropriate for high-dimensional input data such as images. Additionally, in the field of robust control, $H_{\infty}$ techniques are naturally stated over an infinite-dimensional Banach space, so monotone operator techniques do not apply.

Problem description and motivation: In this paper, we aim to provide a natural transcription of many monotone operator techniques for computing zeros of monotone operators for operators which are naturally “monotone” with respect to an $\ell_1$ or $\ell_{\infty}$ norm in a finite-dimensional space.

Monotone operator theory is a fertile field of nonlinear functional analysis that generalizes the notion of monotone functions on $\mathbb{R}$ to mappings on arbitrary Hilbert spaces and examines the properties of such maps. In particular, an integral component of monotone operator theory is the design of algorithms to compute zeros of monotone operators. This aspect makes monotone operator theory compatible with convex optimization since the subdifferential of any convex function is monotone and minimizing a convex function is synonymous with finding a zero of its subdifferential. To this end, there has been an extensive amount of work in the last decade in applying monotone operator theory to convex optimization; e.g., see [7], [24], [25].

Through the lens of duality theory, the theory of dissipative and accretive operators on Banach spaces mirrors monotone operators on Hilbert spaces to a degree [15]. Despite these parallels, the theory of dissipative and accretive operators has largely focused on iteratively computing solutions of integral equations and PDEs in $L_p$ spaces for $p \neq 2$; see [6] for a relevant textbook. Moreover, many works in this direction focus on Banach spaces that additionally have a uniformly smooth or uniformly convex structure; this structure is not possessed by the finite-dimensional $\ell_1$ and $\ell_{\infty}$ spaces. Ultimately, in contrast to monotone operator theory over Hilbert spaces, the theory of dissipative and accretive operators has found far fewer direct applications to systems, control, and machine learning.

A notion similar to a monotone operator in a Hilbert space is that of a contracting vector field [21]. In fact, a vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is contracting with respect to an $\ell_2$ norm if and only if the negative vector field $-F$ is monotone when thought of as an operator. However, vector fields are not restricted to being contracting with respect to a Euclidean norm. In general, a vector field may be contracting with respect to a non-Euclidean norm but not a Euclidean one [1]. Recently, there has been an increased interest in studying vector fields that are contracting with respect to the non-Euclidean norms $\ell_1$ and $\ell_{\infty}$ [2], [11], [13]. Due to the connection between monotone operators and contracting vector fields, it is of interest to explore the properties of operators that may be thought of as monotone with respect to an $\ell_1$ or $\ell_{\infty}$ norm.

Contributions: To facilitate the application of monotone operator theory techniques to problems naturally arising in finite-dimensional non-Euclidean spaces, we propose a novel non-Euclidean monotone operator framework based on the theory of logarithmic norms [27] (also known as matrix measures). We use the logarithmic norm as a direct substitute for inner-products in Hilbert spaces and we demonstrate that many classic results from monotone operator theory directly carry over to their non-Euclidean counterparts. Specifically, we show that the resolvent and reflected resolvent operators of a non-Euclidean monotone operator have properties anal-
ogous to those arising in Euclidean spaces.

Second, building upon the non-Euclidean monotone operator framework, we demonstrate that classical iterative algorithms such as the forward step method and proximal point method allow us to compute zeros of non-Euclidean monotone operators in a manner identical to the procedure for traditional monotone operators. These results build upon both classical and modern works on iterative methods for computing fixed points of non-expansive maps in Banach spaces [10], [17]. We present estimates for Lipschitz constants of these iterative methods and demonstrate that, for diagonally weighted \( \ell_1 \) and \( \ell_\infty \) norms, these algorithms achieve improved rates of convergence compared to their Euclidean counterparts. As a clear distinction from the classical theory, we prove that the forward step method is convergent for an operator which is (weakly) monotone with respect to an \( \ell_1 \) or \( \ell_\infty \) norm, but that the method need not converge if the operator is monotone with respect to a Euclidean norm. This result is analogous to the result on weakly-contracting ODEs as in [18, Theorem 21].

Third, we study operator splitting methods. We prove that the standard forward-backward, Peaceman-Rachford, and Douglas-Rachford splitting algorithms all apply in our framework and that improved convergence may be achieved for these non-Euclidean norms compared to their Euclidean counterparts.

Fourth, as an application, we present methods to compute equilibria for recurrent neural networks. We extend the recent work of [14], [19] to demonstrate that our non-Euclidean monotone operator theory is readily applicable and can provide accelerated convergence of iterations when viewing the problem of computing an equilibrium as an appropriate operator splitting problem. We highlight several iterations for the computation of the equilibrium and discuss the trade-off between computation, allowable range of stepsizes, and rate of convergence between the iterations. Finally, we present numerical simulations presenting rates of convergence of the different iterations when applied to this problem.

Since this document is an arXiv technical report, it contains proofs of additional technical lemmas that are not presented in the conference version.

II. PRELIMINARIES

A. Notations

For differentiable \( F : \mathbb{R}^n \to \mathbb{R}^n \), we let \( DF(x) := \frac{\partial F(x)}{\partial x} \in \mathbb{R}^{n \times n} \) denote its Jacobian evaluated at \( x \). For an arbitrary mapping \( F \), we let \( \text{Dom}(F) \) be its domain. For \( F : \mathbb{R}^n \to \mathbb{R}^n \), we let \( \text{Zero}(F) := \{ x \in \mathbb{R}^n \mid F(x) = 0 \} \) and \( \text{Fix}(F) = \{ x \in \mathbb{R}^n \mid F(x) = x \} \) be the sets of zeros of \( F \) and fixed points of \( F \), respectively. We let \( I_n : \mathbb{R}^n \to \mathbb{R}^n \) be the identity map and \( I_n \in \mathbb{R}^{n \times n} \) be the \( n \times n \) identity matrix.

B. Norms and Logarithmic Norms

Instrumental to the theory of non-Euclidean monotone operator theory are logarithmic norms (also referred to as matrix measures), henceforth called log norms, independently discovered by Dahlquist and Lozinskii in 1958 [12], [22].

**Definition 1** (Logarithmic norm). Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^n \) and its corresponding induced norm on \( \mathbb{R}^{n \times n} \). The logarithmic norm of a matrix \( A \in \mathbb{R}^{n \times n} \) is

\[
\mu(A) := \lim_{h \to 0^+} \frac{\| I_n + hA \| - 1}{h}. \tag{1}
\]

It is well known that this limit is well posed because the right-hand side of (1) is non-increasing in \( h \), due to the convexity of the norm. We refer to [16] for properties enjoyed by log norms, which include subadditivity, positive homogeneity, convexity, and \( \alpha(A) \leq \mu(A) \leq \| A \| \).

We will be specifically interested in diagonally weighted \( \ell_1 \) and \( \ell_\infty \) norms defined by

\[
\| x \|_{1,[\eta]} = \sum_i \eta_i |x_i| \quad \text{and} \quad \| x \|_{\infty,[\eta]} = \max_i \frac{1}{\eta_i} |x_i|,
\]

where, given a positive vector \( \eta \in \mathbb{R}^n_+ \), we use \( [\eta] \) to denote the diagonal matrix with diagonal entries \( \eta \). For \( A \in \mathbb{R}^{n \times n} \), the corresponding induced and log norms are

\[
\| A \|_{\infty,[\eta]} := \max_{i \in \{1, \ldots, n\}} \sum_{j=1}^n \eta_{ij} |a_{ij}|,
\]

\[
\mu_{\infty,[\eta]}(A) : = \max_{i \in \{1, \ldots, n\}} \left( \eta_i + \sum_{j=1,j\neq i}^n |a_{ij}| \frac{\eta_j}{\eta_i} \right),
\]

\[
\| A \|_{1,[\eta]} = \| A^\top \|_{\infty,[\eta]}^{-1}, \quad \mu_{1,[\eta]}(A) = \mu_{\infty,[\eta]}^{-1}(A^\top).
\]

We note also that for the Euclidean norm \( \| \cdot \|_2 \), the corresponding log norm is \( \mu_2(A) = \frac{1}{2} \lambda_{\max}(A + A^\top) \).

C. Contractions, nonexpansive maps, Banach-Picard and Krasnosel’skii–Mann iterations

For the remainder of the paper, we assume all mappings are continuously differentiable unless otherwise stated.

**Definition 2** (Lipschitz continuity). Let \( \| \cdot \| \) be a norm and \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a map. \( F \) is Lipschitz continuous with constant \( \text{Lip}(F) \in \mathbb{R}_{\geq 0} \) if for all \( x_1, x_2 \in \mathbb{R}^n \)

\[
\| F(x_1) - F(x_2) \| \leq \text{Lip}(F) \| x_1 - x_2 \|. \tag{2}
\]

Equivalently, \( F \) is Lipschitz continuous with constant \( \text{Lip}(F) \) if and only if

\[
\| DF(x) \| \leq \text{Lip}(F) \quad \text{for all } x \in \mathbb{R}^n. \tag{3}
\]

**Definition 3** (One-sided Lipschitz functions [13, Definition 26]). Given a norm \( \| \cdot \| \) with corresponding log norm \( \mu(\cdot) \), a map \( F : \mathbb{R}^n \to \mathbb{R}^n \) is one-sided Lipschitz with constant \( \text{osL}(F) \in \mathbb{R} \) if

\[
\mu(DF(x)) \leq \text{osL}(F) \quad \text{for all } x \in \mathbb{R}^n. \tag{4}
\]

Note that (i) the one-sided Lipschitz constant is upper bounded by the Lipschitz constant, (ii) a Lipschitz map is always one-sided Lipschitz, and (iii) the one-sided Lipschitz constant may be negative.

**Definition 4** (Contractions and nonexpansive maps). Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be Lipschitz with respect to a norm \( \| \cdot \| \). We say

(i) \( T \) is a contraction if \( \text{Lip}(T) \in [0,1] \),

(ii) \( T \) is nonexpansive if \( \text{Lip}(T) = 1 \).
**Definition 5** (Averaged maps). We say a nonexpansive map \( T : \mathbb{R}^n \to \mathbb{R}^n \) is averaged provided that there exists a nonexpansive map \( N : \mathbb{R}^n \to \mathbb{R}^n \) such that for some \( \theta \in [0, 1] \),
\[
T = (1 - \theta)I + \theta N.
\] (5)

**Remark 6.** When the norm is induced by an inner product, the composition of two averaged mappings yields another averaged mapping; see [3, Proposition 4.44]. This, however, need not hold for non-Euclidean spaces.

**Definition 7** (Krasnosel’ski–Mann iterations [3, Section 5.2]). Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be nonexpansive with respect to a norm \( \| \cdot \| \). The Krasnosel’ski–Mann iterations applied to \( T \) defines the sequence \( \{x_k\}_{k=0}^\infty \) by
\[
x_{k+1} = (1 - \theta)x_k + \theta T(x_k),
\] (6)
where \( \theta \in [0, 1] \).

**Lemma 8** (Convergence and asymptotic regularity of Krasnosel’ski–Mann iterations [10]). Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be nonexpansive with respect to a norm \( \| \cdot \| \) and consider the Krasnosel’ski–Mann iterations as in (6). Suppose \( \operatorname{Fix}(T) \neq \emptyset \) and let \( x^* \in \operatorname{Fix}(T) \). Then
\[
\|x_k - T(x_k)\| \leq \frac{2\|x_0 - x^*\|}{\sqrt{k\pi}(1 - \theta)}.
\] (7)
In particular, \( \|x_k - T(x_k)\| \to 0 \) as \( k \to \infty \) with \( \|x_k - T(x_k)\| \sim \mathcal{O}(1/\sqrt{k}) \). Moreover, the convergence rate is optimized with \( \theta = 1/2 \).

III. NON-EUCLIDEAN MONOTONE OPERATORS

A. Definitions and Properties

**Definition 9** (Non-Euclidean monotone operator). A continuously differentiable operator \( F : \mathbb{R}^n \to \mathbb{R}^n \) is strongly monotone with monotonicity parameter \( c > 0 \) with respect to a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) provided for all \( x, y \in \mathbb{R}^n \),
\[
-\mu(-DF(x)) \geq c.
\] (8)
If the inequality holds with \( c = 0 \), we say \( F \) is monotone (or weakly monotone) with respect to \( \| \cdot \| \).

Note that this condition is equivalent to \( -\cos L(-F) \geq c \). Moreover, if \( F \) is only locally Lipschitz, we ask that (8) holds almost everywhere.

**Remark 10** (Comparison to the Euclidean case). For an operator \( F : \mathbb{R}^n \to \mathbb{R}^n \), let \( \| \cdot \|_2 \) be the Euclidean norm with corresponding inner product \( \langle \cdot, \cdot \rangle \). Then following [3, Definition 20.1], \( F \) is monotone with respect to \( \| \cdot \|_2 \) if
\[
\langle F(x) - F(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in \mathbb{R}^n.
\]
If \( F \) is continuously differentiable, this condition is known to be equivalent to (e.g., [24]) \( DF(x) + DF(x)^\top \preceq 0 \), or equivalently \( -\mu_2(-DF(x)) \geq 0 \) or \( \frac{1}{2} \lambda_{\min}(DF(x) + DF(x)^\top) \geq 0 \), which coincides with Definition 9.

By subadditivity of \( \mu \), a sum of operators which are monotone with respect to the same norm is also monotone.

Additionally, if \( F \) is (strongly) monotone with monotonicity parameter \( c \geq 0 \), then for any \( \alpha \geq 0 \), \( \Id + \alpha F \) is strongly monotone with monotonicity parameter \( 1 + \alpha c \).

**Remark 11** (Connection with contracting vector fields [21]). A continuously differentiable mapping \( F : \mathbb{R}^n \to \mathbb{R}^n \) is strongly contracting with rate \( c > 0 \) with respect to a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) provided for all \( x, y \in \mathbb{R}^n \),
\[
\mu(DF(x)) \leq -c.
\] (9)
If this inequality holds with \( c = 0 \), we say \( F \) is weakly contracting with respect to \( \| \cdot \| \). Clearly, \( F \) is (strongly) monotone if and only if \( -F \) is (strongly) contracting.

**Example 12.** An affine mapping \( F(x) = Ax + b \) is monotone if and only if \( -\mu(-A) \geq 0 \) and strongly monotone with parameter \( c \) if and only if \( -\mu(-A) \geq c \). This implies that the spectrum of \( A \) lies in the portion of the complex plane given by \( \{ z \in \mathbb{C} \mid |\Re(z)| \geq c \} \).

**Lemma 13** (Lipschitz constants of inverses of strongly monotone operators). Suppose \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a strongly monotone operator with parameter \( c > 0 \). Then \( F^{-1} \) is Lipschitz with constant \( \ell = 1/c \).

To prove Lemma 13, we leverage the following useful property of log norms.

**Proposition 14** (Product property of log norms [16]). Let \( A \in \mathbb{R}^{n \times n} \) and \( \| \cdot \| \) be a norm on \( \mathbb{R}^n \) with corresponding log norm \( \mu(\cdot) \). Then for all \( x \in \mathbb{R}^n \),
\[
\|Ax\| \geq \max\{-\mu(A), -\mu(-A)\}\|x\|.
\] (10)

**Proof of Lemma 13.** Note that by the mean-value theorem and Proposition 14,
\[
\|F(x) - F(y)\| \geq \left| \int_0^1 DF(y + \tau(x - y))d\tau \right| (x - y) \geq \int_0^1 \mu(-DF(y + \tau(x - y)))d\tau \|x - y\| \geq \int_0^1 -\mu(-DF(y + \tau(x - y)))d\tau \|x - y\| \geq c\|x - y\|.
\]
where second inequality is by subadditivity and continuity of \( \mu \) and the final inequality is by the assumption of strong monotonicity. We can then immediately see that if \( F(x) = F(y) \), then necessarily \( x = y \), which implies that \( F^{-1} \) is a mapping. Then write \( x = F^{-1}(u), y = F^{-1}(v) \) (and therefore \( F(x) = u, F(y) = v \)). Then
\[
\|u - v\| \geq c\|x - y\| = c\|F^{-1}(u) - F^{-1}(v)\|,
\] (11)
which shows that \( F^{-1} \) has Lipschitz constant \( 1/c \).

**Lemma 15.** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be globally Lipschitz with respect to a diagonally-weighted \( \ell_1 \) or \( \ell_\infty \) norm \( \| \cdot \| \) with constant \( \text{Lip}(F) = \ell \). If \( F \) is (possibly strongly) monotone...
with respect to $\| \cdot \|$ with monotonicity parameter $c \geq 0$, then
\[
\text{Lip}(\text{Id} - \alpha F) = 1 - \alpha c, \quad \text{for all } \alpha \in \left[0, \frac{1}{\text{diagL}(F)}\right],
\]
where $\text{diagL}(F) := \sup_{x \in \mathbb{R}^n} \max_{i \in \{1, \ldots, n\}} (DF(x))_{ii} \leq \ell$.
\[(12)\]

**Proof.** The result follows from [19, Theorem 2].

Note that for Euclidean norms, if $F$ is monotone, but not strongly monotone, then $(\text{Id} - \alpha F)$ need not be expansive for any $\alpha > 0$. Indeed, consider $F(x) = \left( \frac{1}{2}, 0 \right) x$, which is monotone with respect to the $\ell_2$ norm, but $(\text{Id} - \alpha F)$ is expansive for every $\alpha > 0$.

**B. Resolvent and reflected resolvent operators**

**Definition 16** (Resolvent and reflected resolvent). Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a mapping and $\alpha > 0$. The resolvent of $\alpha F$ is defined as
\[
J_{\alpha F} = (\text{Id} + \alpha F)^{-1}.
\]
(13)
The reflected resolvent, also called the Cayley operator of $\alpha F$ is
\[
R_{\alpha F} = 2J_{\alpha F} - \text{Id}.
\]
(14)

**Theorem 17** (A non-Euclidean Minty-Browder theorem). Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is monotone. Then for every $\alpha > 0$, $\text{Dom}(J_{\alpha F}) = \text{Dom}(R_{\alpha F}) = \mathbb{R}^n$.

**Proof.** Note that
\[
(\text{Id} + \alpha F)(x) = 0 \iff -\alpha F(x) = x.
\]
However, since $F$ is continuously differentiable and $F$ is monotone,
\[
-\mu(-DF(x)) \geq 0, \quad \text{for all } x \iff \mu(-\alpha DF(x)) < 1,
\]
for all $x \in \mathbb{R}^n$. Then by [19, Theorem 1], $-\alpha F(x) = x$ has a unique solution, so $(\text{Id} + \alpha F)(x) = 0$ has a unique solution. Moreover, for every $u \in \mathbb{R}^n$, the mapping $x \mapsto \alpha F(x) - u$ is continuously differentiable and monotone and thus has a unique fixed point, implying for every $u \in \mathbb{R}^n$, there exists an $x \in \mathbb{R}^n$ such that $(\text{Id} + \alpha F)(x) = u$. This proves that $\text{Dom}(J_{\alpha F}) = \mathbb{R}^n$. The proof for the reflected resolvent is a straightforward consequence of $\text{Dom}(J_{\alpha F}) = \mathbb{R}^n$.

**Lemma 18** (Lipschitz constant of the resolvent operator). Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is (strongly) monotone with parameter $c \geq 0$. Then for every $\alpha > 0$, $\text{Lip}(J_{\alpha F}) = \frac{1}{1 + \alpha c}$.

**Proof.** We observe that $\text{Id} + \alpha F$ is strongly monotone with parameter $1 + \alpha c$. Then by Lemma 13, the result holds.

**Lemma 19** (Reflected resolvent characterization [24]). Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is monotone and $\alpha \geq 0$. Then
\[
R_{\alpha F} = (\text{Id} - \alpha F)(\text{Id} + \alpha F)^{-1}.
\]

**Theorem 20** (Lipschitz constant of the Cayley operator). Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is globally Lipschitz with constant $\ell$ with respect to a diagonally weighted $\ell_1$ or $\ell_\infty$ norm. Moreover, suppose $F$ is (strongly) monotone with respect to $\| \cdot \|$ with monotonicity parameter $c \geq 0$. Then for $\alpha \in \left[0, \frac{1}{\text{diagL}(F)}\right]$,
\[
\text{Lip}(R_{\alpha F}) = \frac{1 - \alpha c}{1 + \alpha c} \leq 1.
\]
(16)

**Proof.** By Lemma 15, $\text{Lip}(\text{Id} - \alpha F) = 1 - \alpha c$ for $\alpha \in \left[0, \frac{1}{\text{diagL}(F)}\right]$. Therefore, the result follows from Lemmas 18 and 19 since the Lipschitz constant of a composition of Lipschitz maps is the product of the Lipschitz constants.

**Lemma 21** (Averagedness of resolvent). Suppose $F$ is Lipschitz and monotone with respect to a diagonally weighted $\ell_1$ or $\ell_\infty$ norm. Then for every $\alpha > 0$, $J_{\alpha F}$ is averaged.

**Proof.** Consider the auxiliary operator
\[
C_{\alpha}^\theta := \frac{\text{Id} - \frac{1 - \theta}{\theta} F}{\theta} - \frac{1}{\theta} \frac{\text{Id} + \alpha F}{(\text{Id} + \alpha F)^{-1}}
\]
for $\theta \in [0, 1]$. Note that the reflected resolvent corresponds to $\theta = \frac{1}{2}$. Then it is straightforward to compute
\[
C_{\alpha}^\theta = \frac{\text{Id} + \alpha F}{\theta} - \frac{1 - \theta}{\theta} \frac{\text{Id} + \alpha F}{(\text{Id} + \alpha F)^{-1}} = \frac{\text{Id} - \frac{1 - \theta}{\theta} F}{\theta} J_{\alpha F}.
\]
Since $F$ is monotone, $J_{\alpha F}$ is nonexpansive, and by Lemma 15, $\text{Lip}\left(\frac{\text{Id} - (1 - \theta) F}{\theta}\right) = 1$, for all $\alpha \in \left[0, \frac{1}{\theta \text{diagL}(F)}\right]$, which implies that $C_{\alpha}^\theta$ is nonexpansive for all $\alpha$ in this range.

Let $\alpha > 0$ be arbitrary. Then for any $\theta \leq \frac{1}{1 + \alpha \text{diagL}(F)} \in [0, 1[$, we have that
\[
J_{\alpha F} = (1 - \theta) \text{Id} + \theta C_{\alpha}^\theta
\]
and $C_{\alpha}^\theta$ is nonexpansive. This proves that $J_{\alpha F}$ is averaged.

**Example 22.** Consider the linear operator
\[
F(x) = Ax = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} x.
\]
Then clearly $F$ is monotone with respect to the $\ell_\infty$ norm since $-\mu(\text{diag}(-A)) = -\mu(\frac{2}{1}) = 0$. Then for $\alpha = 1$, we compute
\[
J_{\alpha F}(x) = \begin{pmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \end{pmatrix} x, \quad R_{\alpha F}(x) = \begin{pmatrix} -1/2 & 1/2 \\ -1/4 & -1/4 \end{pmatrix} x.
\]
Thus, $\text{Lip}(J_{\alpha F}) = 1/2$ and $\text{Lip}(R_{\alpha F}) = 1$. In other words, for $\alpha = 1$, $J_{\alpha F}$ is a contraction and $R_{\alpha F}$ is nonexpansive. For $\alpha = 2$, we compute
\[
J_{\alpha F}(x) = \begin{pmatrix} 3/2 & 4/3 \\ -2/3 & -8/3 \end{pmatrix} x, \quad R_{\alpha F}(x) = \begin{pmatrix} -17/23 & 8/23 \\ -2/23 & -12/23 \end{pmatrix} x.
\]
Thus, $\text{Lip}(J_{\alpha F}) = 7/23$ and $\text{Lip}(R_{\alpha F}) = 25/23$. In other words, for $\alpha = 2$, $J_{\alpha F}$ is a contraction and $R_{\alpha F}$ is expansive.
IV. Finding Zeros of Non-Euclidean Monotone Operators

Consider the problem of finding an \( x \in \mathbb{R}^n \) that satisfies
\[
F(x) = 0, \tag{17}
\]
where \( F \) is monotone. This problem shows up in the computation of equilibrium points of contracting vector fields as noted in Remark 11. We present several well-known algorithms for finding zeros of monotone operators (see, e.g., [24]) and show how the non-Euclidean monotone operator framework allows the same algorithms to compute zeros of non-Euclidean monotone operators.

Algorithm 23 (Forward step method). The forward step method corresponds to the fixed point iteration
\[
x_{k+1} = (\Id - \alpha F)(x_k). \tag{18}
\]

Theorem 24 (Convergence guarantees for the forward step method). Suppose \( F : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz with constant \( \ell \) with respect to a diagonally-weighted \( \ell_1 \) or \( \ell_\infty \) norm \( \| \cdot \| \) and

(i) \( F \) is strongly monotone with respect to \( \| \cdot \| \) with monotonicity parameter \( c > 0 \). Then the iteration (18) converges to the unique zero, \( x^* \), of \( F \) for every \( \alpha \in ]0, \frac{1}{\text{diagL}(F)}[ \). Moreover, for every \( k \in \mathbb{Z}_{\geq 0} \), the iteration satisfies
\[
\| x_{k+1} - x^* \| \leq (1 - \alpha c) \| x_k - x^* \|,
\]
with the convergence rate being optimized at \( \alpha = 1/\text{diagL}(F) \).

(ii) \( F \) is monotone with respect to \( \| \cdot \| \). Then if Zero(\( F \)) \( \neq \emptyset \), the iteration (18) converges to an element of Zero(\( F \)) for every \( \alpha \in ]0, \frac{1}{\text{diagL}(F)}[ \).

Proof. Statement (i) follows from Lemma 15. Regarding Statement (ii), since \( F \) is monotone with respect to a diagonally weighted \( \ell_1 \) or \( \ell_\infty \) norm, \( (\Id - \alpha F) \) is nonexpansive for \( \alpha \in ]0, \frac{1}{\text{diagL}(F)}[ \) by Lemma 15. Moreover, for every \( \alpha \in ]0, \frac{1}{\text{diagL}(F)}[ \), there exists \( \theta \in ]0, 1[ \) such that
\[
\Id - \alpha F = (1 - \theta)\Id + \theta(\Id - \tilde{\alpha} F),
\]
for some \( \tilde{\alpha} \in ]0, \frac{1}{\text{diagL}(F)}[ \). Therefore \( \Id - \alpha F \) is averaged and by Lemma 8, if Zero(\( F \)) \( \neq \emptyset \), the forward step method converges to an element of Zero(\( F \)).

Algorithm 25 (Proximal point method). The proximal point method corresponds to the fixed point iteration
\[
x_{k+1} = J_\alpha F(x_k) = (\Id + \alpha F)^{-1}(x_k). \tag{19}
\]

Theorem 26 (Convergence guarantees for the proximal point method). Suppose \( F : \mathbb{R}^n \to \mathbb{R}^n \) is

(i) strongly monotone with respect to a norm \( \| \cdot \| \) with monotonicity parameter \( c > 0 \). Then the iteration (19) converges to the unique zero, \( x^* \), of \( F \) for every \( \alpha \in ]0, \infty[ \). Moreover, for every \( k \in \mathbb{Z}_{\geq 0} \), the iteration satisfies
\[
\| x_{k+1} - x^* \| \leq \frac{1}{1 + \alpha c} \| x_k - x^* \|,
\]
and

(ii) monotone and globally Lipschitz with respect to a diagonally weighted \( \ell_1 \) or \( \ell_\infty \) norm. Then if Zero(\( F \)) \( \neq \emptyset \), the iteration (19) converges to an element of Zero(\( F \)) for every \( \alpha \in ]0, \infty[ \).

Proof. Statement (i) holds due to Lemma 18. Statement (ii) holds since Lipschitzness of \( F \) implies that \( J_\alpha F \) is averaged together with Lemma 8.

Algorithm 27. The Cayley method corresponds to the fixed point iteration
\[
x_{k+1} = R_\alpha F(x_k) = 2(\Id + \alpha F)^{-1}(x_k) - x_k. \tag{20}
\]

Theorem 28 (Convergence guarantees for the Cayley method). Suppose \( F : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz with constant \( \ell \) with respect to a diagonally-weighted \( \ell_1 \) or \( \ell_\infty \) norm \( \| \cdot \| \) and

(i) \( F \) is strongly monotone with respect to \( \| \cdot \| \) with monotonicity parameter \( c > 0 \). Then the iteration (20) converges to the unique zero, \( x^* \), of \( F \) for every \( \alpha \in ]0, \frac{1}{\text{diagL}(F)}[ \). Moreover, for every \( k \in \mathbb{Z}_{\geq 0} \), the iteration satisfies
\[
\| x_{k+1} - x^* \| \leq \frac{1 - \alpha c}{1 + \alpha c} \| x_k - x^* \|,
\]
with the convergence rate being optimized at \( \alpha = 1/\text{diagL}(F) \).

(ii) \( F \) is monotone with respect to \( \| \cdot \| \). Then if Zero(\( F \)) \( \neq \emptyset \), the averaged iterations
\[
x_{k+1} = \frac{1}{2} x_k + \frac{1}{2} R_\alpha F(x_k)
\]
correspond to the proximal point iterations (19), which are guaranteed to converge to an element of Zero(\( F \)) for every \( \alpha \in ]0, \infty[ \).

Proof. Statement (i) follows from Theorem 20. Statement (ii) holds since \( \frac{1}{2}\Id + \frac{1}{2}(J_\alpha F - \Id) = J_{\frac{1}{2}\alpha} F \), and convergence follows by Theorem 26(ii) since Zero(\( F \)) \( \neq \emptyset \).

We provide a comparison of the range of step sizes and Lipschitz constants as provided by the classical monotone operator theory [24] and Theorems 24, 26, and 28 in Table I. Note that in Table I we do not assume that the strongly monotone \( F \) is the gradient of a strongly convex function.

V. Finding Zeros of a Sum of Non-Euclidean Monotone Operators

In many instances, one may wish to execute the proximal point method, Algorithm 25, to compute a zero of a monotone operator \( N : \mathbb{R}^n \to \mathbb{R}^n \). However, in general, the implementation of the iteration (19) may be hindered by the difficulty in evaluating \( J_\alpha N \). To remedy this issue, it is often assumed that \( N \) can be expressed as the sum of two monotone operators \( F \) and \( G \) where the resolvent \( J_\alpha G \) may be easy to compute and \( F \) satisfies some regularity condition. Alternatively, in some situations, decomposing \( N = F + G \) and finding \( x \in \mathbb{R}^n \) such that \( (F + G)(x) = 0 \) provides additional flexibility in choice of algorithm and may improve convergence rates.
Motivated by the above, we consider the problem of finding an \( x \in \mathbb{R}^n \) such that
\[
(F + G)(x) = 0,
\]
where \( F, G : \mathbb{R}^n \to \mathbb{R}^n \) are monotone with respect to a diagonally weighted \( \ell_1 \) or \( \ell_\infty \) norm.

**Algorithm 29** (Forward-backward splitting). Assume \( \alpha > 0 \). Then by [24, Section 7.1]
\[
(F + G)(x) = 0 \iff x = J_\alpha G(Id - \alpha F)(x).
\]
The forward-backward splitting method corresponds to the fixed point iteration
\[
x_{k+1} = J\alpha G(Id - \alpha F)(x_k). \tag{22}
\]
Additionally, if both \( F \) and \( G \) are monotone, define the averaged forward-backward splitting iterations
\[
x_{k+1} = \frac{1}{2} x_k + \frac{1}{2} J\alpha G(Id - \alpha F)(x_k). \tag{23}
\]

**Theorem 30** (Convergence guarantees for forward-backward splitting method). Suppose \( F : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz with respect to a diagonally weighted \( \ell_1 \) or \( \ell_\infty \) norm \( \| \cdot \| \) and \( G : \mathbb{R}^n \to \mathbb{R}^n \) is monotone with respect to the same norm.

(i) If \( F \) is strongly monotone with respect to \( \| \cdot \| \) with monotonicity parameter \( c \) > 0, then the iteration (22) converges to the unique zero, \( x^* \), of \( F + G \) for every \( \alpha \in [0, \frac{1}{\text{DiagL}(F)}] \). Moreover, for every \( k \in \mathbb{Z}_{\geq 0} \), the iteration satisfies
\[
\|x_{k+1} - x^*\| \leq (1 - \alpha c)\|x_k - x^*\|,
\]
with the convergence rate being optimized at \( \alpha = 1/\text{DiagL}(F) \).

(ii) If \( F \) is monotone with respect to \( \| \cdot \| \) and \( \text{Zero}(F + G) \neq \emptyset \), then the iteration (23) converges to an element of \( \text{Zero}(F + G) \) for every \( \alpha \in [0, \frac{1}{\text{DiagL}(F)}] \).

**Proof.** Statement (i) follows from the fact that the Lipschitz constant of a composition of maps is the product of the Lipschitz constants together with Lemma 15. Statement (ii) follows from Lemma 8.

Compared to the Euclidean case, if both \( F \) and \( G \) are monotone, then the averaged iterations (23) must be applied to compute a zero of \( F + G \). In the Euclidean case, both \( J_\alpha G \) and \( (Id - \alpha F) \) are averaged and therefore the composition is also averaged.

**Algorithm 31** (Peaceman-Rachford and Douglas-Rachford splitting). Let \( \alpha > 0 \). Then by [24, Section 7.3],
\[
(F + G)(x) = 0 \iff R_\alpha F R_\alpha G z = z \text{ and } x = J_\alpha G z. \tag{24}
\]
The Peaceman-Rachford splitting method corresponds to the fixed point iteration
\[
x_{k+1} = J\alpha G(z_k),
\]
\[
z_{k+1} = 2x_{k+1/2} - z_k,
\]
\[
x_{k+1} = J_\alpha F(z_{k+1/2}),
\]
\[
z_{k+1} = 2x_{k+1} - z_{k+1/2}. \tag{25}
\]
If both \( F \) and \( G \) are monotone, the term \( R_\alpha F R_\alpha G \) in (24) is averaged to yield the fixed point equation
\[
(F + G)(x) = 0 \iff \frac{1}{2}(Id + R_\alpha F R_\alpha G) z = z \text{ and } x = J_\alpha G z. \tag{26}
\]

The fixed point iteration corresponding to (26) is called the Douglas-Rachford splitting method and is given by
\[
x_{k+1/2} = J_\alpha G(z_k),
\]
\[
z_{k+1/2} = 2x_{k+1/2} - z_k,
\]
\[
x_{k+1} = J_\alpha F(z_{k+1/2}),
\]
\[
z_{k+1} = z^k + x_{k+1} - x_{k+1/2}. \tag{27}
\]

**Theorem 32** (Convergence guarantees for Peaceman-Rachford and Douglas-Rachford splitting methods). Suppose both \( F : \mathbb{R}^n \to \mathbb{R}^n \) and \( G : \mathbb{R}^n \to \mathbb{R}^n \) are globally Lipschitz with respect to a diagonally weighted \( \ell_1 \) or \( \ell_\infty \) norm \( \| \cdot \| \) and (without loss of generality) \( G \) is monotone with respect to the same norm.
(i) If $F$ is strongly monotone with respect to $\| \cdot \|$ with monotonicity parameter $c > 0$, then the sequence of $\{x_k\}_{k=0}^\infty$ generated by the iteration (25) converges to the unique zero, $x^*$, of $F + G$ for every $\alpha \in \left[0, \min \left\{ \frac{1}{\text{diag}(L)}, \frac{1}{\text{diag}(G)} \right\} \right]$. Moreover, for every $k \in \mathbb{Z}_{\geq 0}$, the iteration satisfies
\[
\|x_{k+1} - x^*\| \leq \frac{1 - \alpha c}{1 + \alpha c} \|x_k - x^*\|,
\]
with the convergence rate being optimized at $\alpha = \min \left\{ \frac{1}{\text{diag}(L)}, \frac{1}{\text{diag}(G)} \right\}$.

(ii) If $F$ is monotone with respect to $\| \cdot \|$ and $\text{Zero}(F + G) \neq \emptyset$, then the sequence of $\{x_k\}_{k=0}^\infty$ generated by the iteration (27) converges to an element of $\text{Zero}(F + G)$ for every $\alpha \in \left[0, \min \left\{ \frac{1}{\text{diag}(L)}, \frac{1}{\text{diag}(G)} \right\} \right]$, with rate $\gamma = \frac{\alpha}{\text{diag}(G)}$.

Proof. Statement (i) holds by Theorem 20, while statement (ii) holds by Theorem 20 and Lemma 8.

VI. APPLICATION TO RECURRENT NEURAL NETWORKS

A. Analysis and various iterations

Consider the continuous-time recurrent neural network
\[
\dot{x} = -x + \Phi(Ax + Bu + b) = F(x, u),
\]
where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n$, and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an activation function applied entrywise, i.e., $\Phi(x) = (\phi(x_1), \ldots, \phi(x_n))^T$. In this example, we consider the case that $\Phi$ is a LeakyReLU activation function, i.e., $\phi(x) = \max(x, ax \eta)$ for some $a \in [0, 1]$. In [14], it was shown that a sufficient condition for the contractivity of this neural network is the existence of weights $\eta \in \mathbb{R}_{>0}^n$ such that $\mu_{\infty,|\eta|^{-1}}(A) < 1$. If this condition holds, then the recurrent neural network (28) is contracting with respect to $\| \cdot \|_{\infty,|\eta|^{-1}}$ with rate $1 - \phi(\gamma)$. In what follows, we define $\gamma := \mu_{\infty,|\eta|^{-1}}(A) < 1$.

Suppose that, for fixed $u$, we are interested in efficiently computing the unique equilibrium point $x^*(u)$ of $F(x, u)$. Since $F(x, u)$ is contracting with respect to $\| \cdot \|_{\infty,|\eta|^{-1}}$, $F(x, u)$ is strongly monotone with monotonicity parameter $1 - \phi(\gamma)$. As a consequence, applying the forward step method, Algorithm 18 to compute $x^*(u)$ yields the iteration
\[
x_{k+1} = (1 - \alpha)x_k + \alpha \phi(Ax_k + Bu + b),
\]
which is the iteration proposed in [19]. This iteration is guaranteed to converge for every $\alpha \in \left[0, \frac{1}{1 - \min_{i \in \{1, \ldots, n\}} (a \cdot (A)_{ii}, (A)_{ii})} \right]$ with contraction factor $1 - \alpha(1 - \phi(\gamma))$.

However, rather than viewing finding an equilibrium of (28) as finding a zero of a non-Euclidean monotone operator, it is also possible to view it as an operator splitting problem. In particular, in the spirit of [28, Theorem 1], we prove that finding a fixed point of $\Phi(Ax + Bu + b)$ corresponds to an appropriate operator splitting problem under suitable assumptions on $\Phi$. However, first we must define the proximal operator.

**Definition 33** (Proximal operator [3, Definition 12.23]). Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a proper lower semicontinuous convex function. Then the proximal operator of $f$ evaluated at $x \in \mathbb{R}^n$ is
\[
\text{prox}_f(x) = \arg \min_{z \in \mathbb{R}^n} \frac{1}{2} \|x - z\|^2 + f(z).
\]

**Proposition 34.** Suppose $\phi$ is the proximal operator of a continuously differentiable convex function $f$. Then finding an equilibrium point $x^*(u)$ of (28) is equivalent to the operator splitting problem $(F + G)(x^*(u)) = 0$, where $F(z) = (I_n - A)(z) - (Bu + b)$, $G(z) = df(z)$, (31) where we denote $df(z) = (f'(z_1), \ldots, f'(z_n))^T$.

Proof. First, we note that computing an equilibrium point of (28) is equivalent to computing a fixed-point $x = \Phi(Ax + Bu + b)$. Since $\phi(x_i) = \text{prox}_{f_i}(x_i)$, by [3, Proposition 16.44] we have that $\Phi(x) = J_{df}(x)$. Thus, the fixed-point problem is equivalent to
\[
x = J_{df}(Ax + Bu + b),
\]
which implies the result since this corresponds to Algorithm 29 with $\alpha = 1$. Therefore, any equilibrium point of (28) is a zero of the splitting problem $(F + G)(x) = 0$ where $F$ and $G$ are defined as in (31).

We note that although this assumption on $\phi$ appears restrictive, if the assumption of smoothness is relaxed, many common activation functions satisfy the assumption, as is noted in the following little-known proposition.

**Proposition 35** ([8, Proposition 2.4]). Let $\phi : \mathbb{R} \to \mathbb{R}$. Then $\phi$ is the proximal operator of a proper lower semicontinuous convex function $f : \mathbb{R} \to \mathbb{R}$ if and only if $\phi$ satisfies
\[
0 \leq \frac{\phi(x) - \phi(y)}{x - y} \leq 1, \text{ for all } x, y \in \mathbb{R}, x \neq y.
\]

For the LeakyReLU activation function, it is known that the $f$ corresponding to $\phi$ is given by $f(z_i) = \frac{1 - a}{2a} \min\{z_i, 0\}^2$, [20, Table 1] which is continuously differentiable and which can be written in vector form $df(z) = \frac{1 - a}{2a} \min\{z, 0\}$. Moreover, $df$ is Lipschitz with constant $(1 - a)/a$. Now we will show that under the sufficient condition $\gamma < 1$, $F$ is strongly monotone with respect to the norm $\| \cdot \|_{\infty,|\eta|^{-1}}$ and $G$ is monotone with respect to the same norm.

Since $\gamma < 1$,
\[
-\mu_{\infty,|\eta|^{-1}}(-(I_n - A)) = 1 - \mu_{\infty,|\eta|^{-1}}(A) = 1 - \gamma > 0,
\]
which implies $F$ is strongly monotone with monotonicity parameter $1 - \gamma$. Moreover, checking that $G$ is monotone is straightforward since $df$ is Lipschitz and $Df(z)$ is diagonal for every $z \in \mathbb{R}^n$ for which it exists and has diagonal entries in $[0, (1 - a)/a]$. As a consequence, for almost every $z \in \mathbb{R}^n$, $\mu_{\infty,|\eta|^{-1}}(-Df(z)) \leq 0$, which implies monotonicity of $G$ with respect to $\| \cdot \|_{\infty,|\eta|^{-1}}$.

Therefore, we can consider different operator splitting algorithms to compute the equilibrium of (28). First, the
forward-backward splitting method as applied to this problem is

\begin{equation}
    x_{k+1} = \text{prox}_{\alpha f}((1-\alpha)x_k + \alpha(Ax_k + Bu + b)).
\end{equation}

Since F is Lipschitz, this iteration is guaranteed to converge to the unique fixed point of (28). Moreover, the contraction factor for this iteration is \(1 - \alpha(1 - \gamma)\) for \(\alpha \in [0, \frac{1}{\min_i(A_{ii})}]\), with contraction factor being maximized at \(\alpha^* = \frac{1}{\min_i(A_{ii})}\). Note that compared to the iteration (29), the forward-backward iteration has a larger allowable range of step sizes and improved contraction factor at the expense of computing a proximal operator at each iteration.

Alternatively, the fixed point may be computed by means of the Peaceman-Rachford splitting algorithm, which can be written

\begin{equation}
    x_{k+1/2} = \frac{I_n + \alpha(I_n - A)}{1 + \alpha(1-\gamma)}(z_k + \alpha(Bu + b)),
    z_{k+1/2} = 2x_{k+1/2} - z_k,
    x_{k+1} = \text{prox}_{\alpha f}(z_{k+1/2}),
    z_{k+1} = 2x_{k+1} - z_{k+1/2}.
\end{equation}

Since both F and G are Lipschitz, this iteration converges to the unique fixed point of (28). Moreover, the contraction factor is \(1 - \alpha(1 - \gamma)\) for \(\alpha \in \left[0, \min \left\{ \frac{1}{1 - \min_i(A_{ii})}, \frac{a}{1 - a} \right\} \right]\), which comes from the Lipschitz constants of F and G. In other words, the contraction factor is improved for Peaceman-Rachford compared to the forward-backward splitting, but the stepsize is additionally limited by the Lipschitz constant of df. For recurrent neural networks where A has large negative diagonal entries and \((I_n + \alpha(I_n - A))\) may be easily inverted, this splitting method may be preferred.

B. Numerical implementations

To assess the efficacy of the iterations in (29), (33), and (34), we generated \(A, B, b, u\) in (28) and applied the iterations to compute the equilibrium. We generate \(A \in \mathbb{R}^{200 \times 200}, B \in \mathbb{R}^{200 \times 50}, u \in \mathbb{R}^{50}, b \in \mathbb{R}^{200}\) with entries normally distributed as \(A_{ij}, B_{ij}, b_i \sim \mathcal{N}(0, 1/\sqrt{200})\) and \(u_i \sim \mathcal{N}(0, 1/\sqrt{50})\). To ensure that \(A \in \mathbb{R}^{200 \times 200}\) satisfies the constraint \(\mu_{\infty, \eta}^{-1}(A) < 1\) for some \(\eta \in \mathbb{R}^n_{\geq 0}\), we pick \([\eta] = I_n\) and project A onto the convex polytope \(\{A \in \mathbb{R}^{n \times n} | \mu_{\infty}(A) \leq 0.99\}\). We additionally computed \(\mu_2(A) \approx 1.0034\), so F is not strongly monotone with respect to \(|| \cdot ||_2\).

For all iterations, we initialize \(x_0\) at the origin and for the Peaceman-Rachford iteration, we additionally initialize \(z_0\) at the origin. We set \(a = 0.1\) in LeakyReLU and for each iteration pick the largest theoretically allowable stepsize, which for the forward-step method and forward-backward splitting methods for computing the equilibrium of (28) converge at the same rate. This result agrees with the theory since \(\gamma = 0.99 > 0\), so that \(\phi(\gamma) = \gamma\) and the estimated contraction factor for both the forward step method and forward-backward splitting is \(1 - \alpha(1 - \gamma) \approx 0.9910\). For the Peaceman-Rachford splitting method, for the theoretically largest allowable \(\alpha = 1/9\), the estimated contraction factor is \(1 - \alpha(1 - \gamma) \approx 0.9978\), which is very close to 1 and thus justifies the slow rate of convergence for the iterations in this case. However, if we let \(\alpha = 0.9015\) as in the other methods, we observe a significant acceleration in the convergence of these iterations. Increasing the range of allowable stepsizes and the tightness of the Lipschitz constants to be more consistent with the empirical results remains an interesting topic of future research.

VII. Conclusion

We develop a non-Euclidean monotone operator framework with an emphasis on operators which are monotone with respect to finite-dimensional \(\ell_1\) and \(\ell_\infty\) norms. Many classical algorithms for computing zeros of monotone operators including the forward step method, proximal point method, and splitting methods such as forward-backward splitting and Peaceman-Rachford splitting are directly applicable in our framework and can exhibit improved convergence rates compared to their corresponding algorithms in Euclidean spaces. We apply our results to recurrent neural network equilibrium computation and empirically demonstrate that applying splitting methods yields improved rates of convergence to the equilibria as compared to other methods.

Topics of future research include (i) tightening the Lipschitz estimates of the operator splitting techniques, (ii) extending the results to include infinite-dimensional Banach spaces and set-valued operators F, and (iii) applying this...
framework for robustness analysis of control systems and machine learning models.

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