The Second Variation of the Ricci Expander Entropy

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1 Introduction

In [10], Perelman discovered two important functionals, the $F$-functional and the $W$ functional. The corresponding entropy functionals $\lambda$ and $\nu$ are monotone along the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ and constant precisely on steady and shrinking solitons. In [2], H.-D. Cao, R. Hamilton and T. Ilmanen presented the second variations of both entropy functionals and studied the linear stabilities of certain closed Einstein manifolds of nonnegative scalar curvature.

To find the corresponding variational structure for the expanding case, M. Feldman, T. Ilmanen and L. Ni [9] introduced the $W^+$ functional. Let $(M^n, g)$ be a compact Riemannian manifold, $f$ a smooth function on $M$, and $\sigma > 0$. Define

$$W^+(g, f, \sigma) = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} [\sigma (|\nabla f|^2 + R) - f + n]dV,$$

$$\mu^+(g, \sigma) = \inf \{W^+(g, f, \sigma) | f \in C^\infty(M), \text{ and } (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}dV = 1\},$$

and

$$\nu^+(g) = \sup_{\sigma > 0} \mu^+(g, \sigma).$$

Then $\nu^+$ is nondecreasing along the Ricci flow and constant precisely on expanding solitons.

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In this note, analogous to [2], we present the first and second variations of the entropy $\nu_+$. By computing the first variation of $\nu_+$, one can see that the critical points are expanding solitons, which are actually negative Einstein manifolds (see e.g. [3]). Our main result is the following

**Theorem 1.** Let $(M^n, g)$ be a compact negative Einstein manifold. Let $h$ be a symmetric 2-tensor. Consider the variation of metric $g(s) = g + sh$. Then the second variation of $\nu_+$ is

$$
\frac{d^2\nu_+(g(s))}{ds^2}|_{s=0} = \frac{\sigma}{Vol(g)} \int_M <N_+ h, h>,
$$

where

$$
N_+ h := \frac{1}{2} \Delta h + \text{div}^* \text{div} h + \frac{1}{2} \nabla^2 v_h + Rm(h, \cdot) + \frac{g}{2n\sigma\text{vol}(g)} \int_M \text{tr} h,
$$

and $v_h$ is the unique solution of

$$
\Delta v_h - \frac{v_h}{2\sigma} = \text{div} (\text{div} h), \quad \int_M v_h = 0.
$$

In this case, we may still define the concept of linear stability. We say that an expanding soliton is **linearly stable** if $N_+ \leq 0$, otherwise it is **linearly unstable**. Similar to [2], the $N_+$ operator is nonpositive definite if and only if the maximal eigenvalue of the Lichnerowicz Laplacian acting on the space of transverse traceless 2-tensors has certain upper bound. Then using the results in [4] and [5], one can see that compact hyperbolic spaces are linearly stable. But unlike the positive Einstein case, it seems hard to find other examples of negative Einstein manifolds which are either linear stable or linear unstable.

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## 2 The First Variation of the Expander Entropy

Recall that in [10], the $\mathcal{F}$ functional is defined by

$$
\mathcal{F}(f, g) = \int_M (|\nabla f|^2 + R)e^{-f}dV,
$$

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and its entropy $\lambda(g)$ is

$$\lambda(g) = \inf \{ \mathcal{F}(f, g) : f \in C^\infty(M), \int_M e^{-f} = 1 \},$$

where $R$ is the scalar curvature. By Theorem 1.7 in [9], we know that $\mu_+(g, \sigma)$ is attained by some function $f$. Moreover, if $\lambda(g) < 0$, then $\nu_+(g)$ can be attained by some positive number $\sigma$.

**Lemma 1.** Assume that $\nu_+(g)$ is realized by some $f$ and $\sigma$, then it is necessary that the pair $(f, \sigma)$ solves the following equations,

$$\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n + \nu_+ = 0, \quad (1)$$

and

$$(4\pi\sigma)^{-\frac{n}{2}} \int_M f e^{-f} dV = \frac{n}{2} - \nu_+. \quad (2)$$

**Proof:** For fixed $\sigma > 0$, suppose that $\mu_+(g, \sigma)$ is attained by some function $f$. Using Lagrange multiplier method, consider the following functional

$$L(g, f, \sigma, \lambda) = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}[\sigma(|\nabla f|^2 + R) - f + n]dV - \lambda((4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}dV - 1).$$

Denote by $\delta f$ the variation of $f$. Then the variation of $L$ is

$$0 = \delta L = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \{(-\delta f)[\sigma(|\nabla f|^2 + R) - f + n] + [2\nabla f \nabla (\delta f) - \delta f]\}dV$$

$$- (4\pi\sigma)^{-\frac{n}{2}} \int_M \lambda(\delta f)e^{-f}dV$$

$$= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}(\delta f)[\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n - 1 - \lambda]dV$$

Therefore, we have

$$\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n - 1 - \lambda = 0.$$ 

Integrating both sides with respect to the measure $(4\pi\sigma)^{-\frac{n}{2}} e^{-f}dV$, we get

$$-\lambda - 1 = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}[\sigma(|\nabla f|^2 + R) - f + n]dV = \mu_+(g, \sigma).$$
When $\sigma$ and $f$ realize $\nu_+(g)$, the above formula is just equation (1).

Now we consider the variations $\delta\sigma$ and $\delta f$ of both $\sigma$ and $f$. We have

$$0 = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}(-\frac{n}{2\sigma} \delta\sigma - \delta f)[\sigma(|\nabla f|^2 + R) - f + n]dV$$

$$+ (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}[\delta\sigma(|\nabla f|^2 + R) + 2\nabla f \nabla (\delta f) - \delta f]dV$$

and

$$(4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}(-\frac{n}{2\sigma} \delta\sigma - \delta f)dV = 0. \quad (4)$$

Using (1) and (4), we can write (3) as

$$0 = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}[\delta\sigma(|\nabla f|^2 + R) - \delta f]dV$$

$$= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}[\frac{1}{\sigma} \delta\sigma(\nu_+ + f - n) + \frac{n}{2\sigma} \delta\sigma]dV$$

$$= (\delta\sigma) \frac{1}{\sigma} (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f}(\nu_+ + f - \frac{n}{2})dV$$

Hence, we get equation (2). Q.E.D.

Before computing the variations of $\nu_+$ functional, let's recall some variation formulas of curvatures. By direct computation, we can get the following lemma

**Lemma 2.** Suppose that $h$ is a symmetric 2-tensor, and $g(s) = g + sh$ is a variation of $g$. Then

$$\frac{\partial R}{\partial s}|_{s=0} = -h_{kl}R_{kl} + \nabla_p \nabla_k h_{pk} - \Delta tr h,$$  \hspace{1cm} (5)

and
Here, $\nabla$ is the Levi-Civita connection of $g$, and $\text{tr} h$ is the trace of $h$ taken with respect to $g$.

Now we are ready to compute the first variation of $\nu_+(g)$.

**Proposition 1.** Let $(M^n, g)$ be a compact Riemannian manifold with $\lambda(g) < 0$. Let $h$ be any symmetric covariant 2-tensor on $M$, and consider the variation $g(s) = g + sh$. Then the first variation of $\nu_+(g(s))$ is

$$\left. \frac{d\nu_+(g(s))}{ds} \right|_{s=0} = (4\pi \sigma)^{-\frac{n}{2}} \int_M \sigma e^{-f} \left( -R_{ij} - \nabla_i \nabla_j f - \frac{1}{2\sigma} g_{ij} h_{ij} \right) dV,$$

where the smooth function $f$ and $\sigma > 0$ realize $\nu_+(g)$.

**Proof:**

$$\frac{\partial \nu_+(g(s))}{\partial s} = (4\pi \sigma)^{-\frac{n}{2}} \int_M e^{-f} \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} g^{ij} h_{ij} \right) \left[ \sigma (|\nabla f|^2 + R) - f + n \right] dV$$

$$+ (4\pi \sigma)^{-\frac{n}{2}} \int_M e^{-f} \frac{\partial \sigma}{\partial s} \left( |\nabla f|^2 + R \right) dV$$

$$+ (4\pi \sigma)^{-\frac{n}{2}} \int_M e^{-f} \left[ \sigma (-g^{ip} g^{jq} h_{pq} \nabla_i f \nabla_j f + 2 g^{ij} \nabla_i f \nabla_j f \frac{\partial f}{\partial s} + \frac{\partial R}{\partial s}) - \frac{\partial f}{\partial s} \right] dV. \quad (7)$$

From

$$(4\pi \sigma)^{-\frac{n}{2}} \int_M e^{-f} dV = 1,$$

we have

$$(4\pi \sigma)^{-\frac{n}{2}} \int_M \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} g^{ij} h_{ij} \right) e^{-f} dV = 0. \quad (8)$$
Substituting (1), (2) and (8) in (7), we obtain

$$\frac{\partial \nu_+(s)}{\partial s} \bigg|_{s=0} = (4\pi \sigma)^{-\frac{n}{2}} \int_M \left[ 2\sigma(|\nabla f|^2 - \Delta f) + \nu_+(0)\right] \left(-\frac{n}{2} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2}g^{ij}h_{ij}\right) e^{-f} dV$$

$$+ (4\pi \sigma)^{-\frac{n}{2}} \int_M \left[ \frac{\partial \sigma}{\partial s} (|\nabla f|^2 + R) - \frac{\partial f}{\partial s} + \sigma(-h_{ij} \nabla_i f \nabla_j f + 2 \frac{\partial f}{\partial s} (|\nabla f|^2 - \Delta f)$$

$$+ \nabla_i \nabla_j h_{ij} - \Delta \text{tr} h - h_{ij} R_{ij})\right] e^{-f} dV$$

$$= (4\pi \sigma)^{-\frac{n}{2}} \int_M \left[ \frac{\partial \sigma}{\partial s} (|\nabla f|^2 + R) - \frac{\partial f}{\partial s} - \sigma(h_{ij} \nabla_i \nabla_j f + h_{ij} R_{ij})\right] e^{-f} dV$$

$$= (4\pi \sigma)^{-\frac{n}{2}} \int_M \left[ \frac{1}{\sigma} \frac{\partial \sigma}{\partial s} [f(0) - \frac{n}{2} + \nu_+(0) - 2\sigma(|\nabla f|^2 - \Delta f)] e^{-f}\right.$$

$$\left.- \sigma h_{ij} (R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\sigma} g_{ij}) e^{-f} dV\right]$$

$$= (4\pi \sigma)^{-\frac{n}{2}} \int_M -\sigma h_{ij} (R_{ij} + \nabla_i \nabla_j f + \frac{1}{2\sigma} g_{ij}) e^{-f} dV.$$

Hence, the first variation of $\nu_+$ is

$$\frac{d\nu_+(g(s))}{ds} \bigg|_{s=0} = (4\pi \sigma)^{-\frac{n}{2}} \int_M \sigma e^{-f} (-R_{ij} - \nabla_i \nabla_j f - \frac{1}{2\sigma} g_{ij}) h_{ij} dV.$$ Q.E.D.

From the above proposition, we can see that a critical point of $\nu_+(g)$ satisfies

$$Rc + \nabla^2 f + \frac{1}{2\sigma} g = 0,$$

which means that $(M, g)$ is a gradient expanding soliton.

3 The Second Variation

Now we compute the second variation of $\nu_+$. Since any compact expanding soliton is Einstein (e.g. see [3]), it implies that $f$ is a constant. After adding $f$ by a
constant we may assume that \( f = \frac{n}{2} \).

In the following, as in [24], we denote \( Rm(h,h) = R_{ijkl}h_{ik}h_{jl} \), \( \text{div}\omega = \nabla_i\omega_i \), \( (\text{div}h)_i = \nabla_jh_{ji} \), \( (\text{div}^*\omega)_{ij} = - (\nabla_i\omega_j + \nabla_j\omega_i) = -\frac{1}{2}L_{\omega^#}g_{ij} \), where \( h \) is a symmetric 2-tensor, \( \omega \) is a 1-tensor, \( \omega^# \) is the dual vector field of \( \omega \), and \( L_{\omega^#} \) is the Lie derivative.

**Proof of Theorem 1**: Let \((M,g)\) be a compact negative Einstein manifold with \( f = \frac{n}{2} \) and \( R_{ij} = -\frac{1}{2g}g_{ij} \). For any symmetric 2-tensor \( h \), consider the variation \( g(s) = g + sh \). Then by proposition 1, we know that \( \frac{d\nu}{ds}|_{s=0} = 0 \).

From (1) and (2), we can get

\[
\frac{n}{2}\sigma \frac{\partial f}{\partial s}(0) - 2\sigma \Delta f(0) - \sigma \frac{\partial R}{\partial s}(0) + \frac{\partial f}{\partial s}(0) = 0,
\]

and

\[
(4\pi\sigma)^{-\frac{n}{2}} \int_M \frac{n}{2} e^{-\frac{n}{2}} (-\frac{n}{2}\sigma \frac{\partial f}{\partial s}(0) - \frac{\partial f}{\partial s}(0) + \frac{1}{2}trgh)dV + (4\pi\sigma)^{-\frac{n}{2}} \int_M \frac{\partial f}{\partial s}(0)e^{-\frac{n}{2}}dV = 0.
\]

It follows by (8) that

\[
(4\pi\sigma)^{-\frac{n}{2}} \int_M \frac{\partial f}{\partial s}(0)e^{-\frac{n}{2}}dV = 0,
\]

and

\[
\frac{n}{2}\sigma \frac{\partial f}{\partial s}(0) = \frac{1}{Vol(g)} \int_M \frac{1}{2}trhdV,
\]

where \( (4\pi\sigma)^{-\frac{n}{2}}e^{-\frac{n}{2}} = \frac{1}{Vol(g)} \). Thus
\[
\frac{d\nu}{ds} = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \left(-n \frac{\partial \sigma}{2\sigma \partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} g^{ij} h_{ij} \right) \sigma(\nabla^2 f + R) - f + n) dV
\]

\[
+ (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \left[ \frac{\partial f}{\partial s} \right] (|\nabla f|^2 + R) + \sigma(-g^{ij} g^{pq} \nabla_i f \nabla_j f + 2g^{ij} \nabla_i f \nabla_j \frac{\partial f}{\partial s} + \frac{\partial R}{\partial s})
\]

\[
\frac{\partial f}{\partial s} dV
\]

\[
= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \left(-n \frac{\partial \sigma}{2\sigma \partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} g^{ij} h_{ij} \right) [2\sigma(\nabla^2 f - \Delta f) + \nu_+] dV
\]

\[
+ (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \left[ \frac{\partial f}{\partial s} \right] (|\nabla f|^2 + R) + \sigma(-g^{ij} g^{pq} \nabla_i f \nabla_j f + 2g^{ij} \nabla_i f \nabla_j \frac{\partial f}{\partial s} + \frac{\partial R}{\partial s})
\]

\[
- \frac{\partial f}{\partial s} dV
\]

\[
= (4\pi\sigma)^{-\frac{n}{2}} \int_M \sigma e^{-f} g^{ij} h_{ij} (|\nabla f|^2 - \Delta f) dV
\]

\[
+ (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} [\sigma(-g^{ij} g^{pq} \nabla_i f \nabla_j f + \frac{\partial R}{\partial s}) - \frac{1}{2} g^{ij} h_{ij} ] dV,
\]

where we note that

\[
(4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \cdot 2\sigma g^{ij} \nabla_i f \nabla_j \frac{\partial f}{\partial s} dV = (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \cdot 2\sigma \frac{\partial f}{\partial s} (|\nabla f|^2 - \Delta f) dV,
\]

and

\[
(4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \frac{\partial \sigma}{\partial s} (|\nabla f|^2 + R) - \frac{\partial f}{\partial s} dV
\]

\[
= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \frac{\partial \sigma}{\partial s} (|\nabla f|^2 + R) + \frac{n}{2} \frac{\partial \sigma}{\partial s} - \frac{1}{2} g^{ij} h_{ij} dV
\]

\[
= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \frac{1}{\sigma} \frac{\partial \sigma}{\partial s} [\sigma(\nabla^2 f + R) + \frac{n}{2}] dV - (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \cdot \frac{1}{2} g^{ij} h_{ij} dV
\]

\[
= (4\pi\sigma)^{-\frac{n}{2}} \int_M e^{-f} \cdot \frac{1}{2} g^{ij} h_{ij} dV.
\]

Since \( f(0) = \frac{n}{2} \), we have
\[
\frac{d^2 \nu_+}{ds^2} \big|_{s=0} = -\frac{1}{\text{Vol}(g)} \int_M \sigma trh \Delta \frac{\partial f}{\partial s} dV \\
+ \frac{1}{\text{Vol}(g)} \int_M \left( -\frac{n}{2} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} trh \left( \sigma \frac{\partial R}{\partial s} - \frac{1}{2} trh \right) \right) dV \\
+ \frac{1}{\text{Vol}(g)} \int_M \left( \frac{\partial \sigma}{\partial s} + \sigma \frac{\partial^2 R}{\partial s^2} + \frac{1}{2} |h_{ij}|^2 \right) dV.
\]

(12)

In the following, all quantities are evaluated at \( s = 0 \).

Firstly, we have

\[
\frac{1}{\text{Vol}(g)} \int_M \sigma \frac{\partial^2 R}{\partial s^2} dV = \frac{\sigma}{\text{Vol}(g)} \int_M \left[ -\frac{1}{\sigma} |h_{ij}|^2 - h_{kl} \left( 2 \nabla_p \nabla_k h_{pl} - \Delta h_{kl} - \nabla_k \nabla_l trh \right) \right. \\
- \nabla_p \left[ h_{pq} \left( 2 \nabla_k h_{kq} - \nabla_q trh \right) \right] + \nabla_k \left( h_{pq} \nabla_k h_{pq} \right) \\
\left. + \frac{1}{2} \nabla_p trh \left( 2 \nabla_k h_{kp} - \nabla_k trh \right) \right. \\
\left. + \frac{1}{2} \left( \nabla_k h_{pq} \nabla_k h_{pq} - 2 \nabla_p h_{kp} \nabla_q h_{kp} \right) \right] dV \\
= \frac{\sigma}{\text{Vol}(g)} \int_M \left[ -\frac{1}{\sigma} |h_{ij}|^2 - h_{kl} \nabla_p \nabla_k h_{pl} - \frac{1}{2} |\nabla h|^2 - \frac{1}{2} |\nabla trh|^2 \right] dV \\
= \frac{\sigma}{\text{Vol}(g)} \int_M \left[ -\frac{1}{\sigma} |h_{ij}|^2 - h_{kl} \nabla_k \nabla_l h_{kl} + R_{kq} h_{ql} + R_{pkql} h_{pq} \right. \\
\left. - \frac{1}{2} |\nabla h|^2 - \frac{1}{2} |\nabla trh|^2 \right] dV \\
= -\frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} |h_{ij}|^2 dV \\
+ \frac{\sigma}{\text{Vol}(g)} \int_M |\text{div} h|^2 + Rm(h, h) - \frac{1}{2} |\nabla h|^2 - \frac{1}{2} |\nabla trh|^2 dV.
\]

(13)

Moreover,

\[
\frac{1}{\text{Vol}(g)} \int_M \frac{\partial \sigma}{\partial s} \frac{\partial R}{\partial s} dV = \frac{\sigma}{n \text{Vol}(g)} \int_M trh dV \frac{1}{\text{Vol}(g)} \int_M \frac{\partial R}{\partial s} dV \\
= \frac{1}{2n} \left( \frac{1}{\text{Vol}(g)} \int_M trh dV \right)^2.
\]

(14)
Let $v_h$ be the solution to the following equation,
\[
\Delta v_h - \frac{v_h}{2\sigma} = \text{div}\text{div}h = \nabla_p \nabla_q h_{pq}, \quad \int_M v_h = 0.
\]

Then
\[
\frac{1}{\text{Vol}(g)} \int_M \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} \text{tr} h \right) \frac{\partial R}{\partial s} dV

= \frac{\sigma}{\text{Vol}(g)} \int_M \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} \text{tr} h \right) \Delta v_h - \frac{v_h}{2\sigma} + \frac{1}{2\sigma} \text{tr} h - \Delta \text{tr} h dV

= - \left( \frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h dV \right)^2 + \frac{\sigma}{\text{Vol}(g)} \int_M v_h \left( -\Delta \frac{\partial f}{\partial s} + \frac{1}{2\sigma} \frac{\partial f}{\partial s} \right) dV

+ \frac{\sigma}{\text{Vol}(g)} \int_M \text{tr} h \left( \frac{\partial f}{\partial s} - \frac{1}{2\sigma} \frac{\partial f}{\partial s} \right) dV + \frac{\sigma}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h \left( \Delta v_h - \frac{v_h}{2\sigma} + \frac{1}{2\sigma} \text{tr} h - \Delta \text{tr} h \right) dV,
\]
where we have used (11) to derive the first term in the last equality.

Meanwhile,
\[
- \frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} \text{tr} h \right) = - \frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h \left( -2\sigma \frac{\partial f}{\partial s} - \frac{\partial R}{\partial s} + \frac{1}{2} \text{tr} h \right).
\]

It follows that
\[
\frac{1}{\text{Vol}(g)} \int_M \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} \text{tr} h \right) \frac{\partial R}{\partial s} dV

= \frac{1}{\text{Vol}(g)} \int_M \sigma \text{tr} h \frac{\partial f}{\partial s} dV - \frac{1}{\text{Vol}(g)} \int_M \frac{1}{4} (\text{tr} h)^2 dV - \left( \frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h dV \right)^2

+ \frac{\sigma}{\text{Vol}(g)} \int_M v_h \left( -\Delta \frac{\partial f}{\partial s} + \frac{1}{2\sigma} \frac{\partial f}{\partial s} \right) dV + \frac{\sigma}{\text{Vol}(g)} \int_M \text{tr} h \frac{\partial f}{\partial s} - \frac{1}{2\sigma} \frac{\partial f}{\partial s} dV

+ \frac{\sigma}{\text{Vol}(g)} \int_M \text{tr} h \left( \Delta v_h - \frac{v_h}{2\sigma} + \frac{1}{2\sigma} \text{tr} h - \Delta \text{tr} h \right) dV.
\]
Now since
\[
\frac{\sigma}{\text{Vol}(g)} \int_M v_h \left( -\Delta \frac{\partial f}{\partial s} + \frac{1}{2} \frac{\partial f}{\partial s} \right) dV = \frac{\sigma}{\text{Vol}(g)} \int_M v_h \left( -\frac{n}{4\sigma^2} \frac{\partial \sigma}{\partial s} + \frac{1}{2} \frac{\partial R}{\partial s} \right) dV
\]
\[
= \frac{\sigma}{\text{Vol}(g)} \int_M \frac{1}{2} v_h (\Delta v_h - \frac{v_h}{2\sigma} + \frac{1}{2} \text{tr} h - \Delta \text{tr} h) dV
\]
\[
= \frac{\sigma}{\text{Vol}(g)} \int_M -\frac{1}{2} |\nabla v_h|^2 - \frac{n^2}{4\sigma} + \frac{v_h}{4\sigma} \text{tr} h - \frac{1}{2} v_h \Delta \text{tr} h dV,
\]
and
\[
\frac{\sigma}{\text{Vol}(g)} \int_M \text{tr} h \left( \frac{\partial f}{\partial s} - \frac{1}{2} \frac{\partial f}{\partial s} \right) dV = \frac{\sigma}{\text{Vol}(g)} \int_M \text{tr} h (\frac{n}{4\sigma^2} \frac{\partial \sigma}{\partial s} - \frac{1}{2} \frac{\partial R}{\partial s}) dV
\]
\[
= \left( \frac{1}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h \ dV \right)^2
\]
\[
- \frac{\sigma}{\text{Vol}(g)} \int_M \frac{1}{2} \text{tr} h (\Delta v_h - \frac{v_h}{2\sigma} + \frac{1}{2} \text{tr} h - \Delta \text{tr} h) dV,
\]
we have
\[
\frac{1}{\text{Vol}(g)} \int_M \left( -\frac{n}{2\sigma} \frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2} \text{tr} h \right) (\frac{\sigma}{\partial s} - \frac{1}{2} \text{tr} h) dV
\]
\[
= \frac{1}{\text{Vol}(g)} \int_M \sigma \text{tr} h \Delta \frac{\partial f}{\partial s} dV + \frac{\sigma}{\text{Vol}(g)} \int_M -\frac{1}{2} |\nabla v_h|^2 - \frac{n^2}{4\sigma} + \frac{1}{2} |\nabla \text{tr} h|^2 dV. \tag{15}
\]
Substituting (13), (14) and (15) in (12), we get
\[
\frac{d^2 \nu_+}{ds^2} \big|_{s = 0} = \frac{\sigma}{\text{Vol}(g)} \left( \int_M |\text{div} h|^2 + Rm(h, h) - \frac{1}{2} |\nabla h|^2 - \frac{1}{2} |\nabla v_h|^2 - \frac{n^2}{4\sigma} dV \right)
\]
\[
+ \frac{1}{2n} \left( \frac{1}{\text{Vol}(g)} \int_M \text{tr} h dV \right)^2
\]
\[
= \frac{\sigma}{\text{Vol}(g)} \int_M < N_+ h, h > . \quad \text{Q.E.D.}
\]

As a simple application, we may briefly discuss the linear stability of negative Einstein manifolds. Analogue to [2], we say that a negative Einstein manifold is
linearly stable if \( N_+ \leq 0 \), otherwise it is linearly unstable. As in [2], decompose the space of symmetric 2-tensors as
\[
\ker \text{div} \oplus \text{im} \text{div}^*,
\]
and further decompose \( \ker \text{div} \) as
\[
(\ker \text{div})_0 \oplus \mathbb{R}g,
\]
where \( (\ker \text{div})_0 \) is the space of divergence free 2-tensors \( h \) with \( \int_M trh = 0 \). It is easy to see that \( N_+ \) vanishes on \( \text{im} \text{div}^* \), and on \( (\ker \text{div})_0 \)
\[
N_+ = \frac{1}{2}(\Delta_L - \frac{1}{\sigma}),
\]
where \( \Delta_L = \Delta + 2\text{Rm}(\cdot, \cdot) - 2\text{Rc} \) is the Lichnerowicz Laplacian on symmetric 2-tensors.

Moreover, we may write \( (\ker \text{div})_0 \) as
\[
(\ker \text{div})_0 = S_0 \oplus S_1,
\]
where \( S_0 \) is the subspace of trace free 2-tensors, and \( S_1 = \{h \in (\ker \text{div})_0 : h_{ij} = (-\frac{1}{2\sigma}u + \Delta u)g_{ij} - \nabla_i \nabla_j u, u \in C^\infty(M) \text{ and } \int_M u = 0 \} \) (see e.g. [1]).

Define
\[
Tu := (-\frac{1}{2\sigma}u + \Delta u)g_{ij} - \nabla_i \nabla_j u.
\]
Since \( \Delta_L(Tu) = T(\Delta u) \) for all smooth functions \( u \) and \( \ker T = \{0\} \), we can see that the Lichnerowicz Laplacian and the Laplacian on function space have the same eigenvalues. Thus \( N_+ \) is always negative on \( S_1 \). Therefore, to study the linear stability of negative Einstein manifolds, it remains to look at the behavior of \( \Delta_L \) acting on \( S_0 \) which is the space of transverse traceless 2-tensors.

**Example** Suppose that \( M \) is an \( n \) dimensional compact real hyperbolic space with \( n \geq 3 \). By [4] or [8], the biggest eigenvalue of \( \Delta_L \) on trace free symmetric 2-tensors on real hyperbolic space is \( \frac{-(n-1)(n-9)}{4} \). Since on \( M \) we have \( \text{Rc} = -(n-1)g \), \( \frac{1}{\sigma} = 2(n - 1) \). Thus the biggest eigenvalue of \( N_+ \) on \( S_0 \) is not greater than \( \frac{-(n-1)^2}{8} \). It implies that \( M \) is linearly stable for \( n \geq 3 \).
Remark When $n = 3$, D. Knopf and A. Young ([7]) proved that closed 3-folds with constant negative curvature are geometrically stable under certain normalized Ricci flow. R. Ye obtained a more powerful stability result earlier in [12].

Remark For $n=2$, R. Hamilton([6]) proved that when the average scalar curvature is negative, the solution of the normalized Ricci flow with any initial metric converges to a metric with constant negative curvature. In particular, they are linearly stable. On the other hand, in [5] we see that the biggest eigenvalue of the Lichnerowicz Laplacian on trace free symmetric 2-tensors is 2. Thus $N_+$ is nonpositive definite on $(\ker \text{div})_0$, which also implies the linear stability.

Remark For noncompact case, in [11], V. Suneeta proved certain geometric stability of $\mathbb{H}^n$ using different methods.

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