Abstract

It is shown that the de Rham complex of a symplectic manifold $M$ satisfying the hard Lefschetz condition is formal. Moreover, it is shown that the differential Gerstenhaber-Batalin-Vilkoviski algebra associated to such a symplectic structure gives rise, along the lines explained in the papers of Barannikov and Kontsevich [alg-geom/9710032] and Manin [math/9801006], to the structure of a Frobenius manifold on the de Rham cohomology of $M$.

§0. Introduction

It was shown in [1] (see also [4] for detailed exposition and proofs) that the formal moduli space of solutions to the Maurer-Cartan equations modulo gauge equivalence associated to a very special class of differential Gerstenhaber-Batalin-Vilkoviski (dGBV) algebras, carries a natural structure of a Frobenius manifold.

To author’s knowledge, only one example of such a special dGBV algebra was known, the one constructed out of the Dolbeault complex of an arbitrary Calabi-Yau manifold by Barannikov and Kontsevich [1].

In this note we produce another example of a special dGBV-algebra, this time the one associated with an arbitrary symplectic manifold $(M, \omega)$ satisfying the hard Lefschetz condition which says that the cup product

$$[\omega^k] : H^{m-k}(M) \to H^{m+k}(M)$$
is an isomorphism for any \( k \leq m = \frac{1}{2} \dim M \). Applying then the machinery developed in [1, 4] to the moduli space of solutions of the associated Maurer-Cartan equation we get a structure of a Frobenius manifold on the de Rham cohomology of \( M \).

§1. Formality of the de Rham complex

Let \( M \) be a \( 2m \)-dimensional manifold equipped with a symplectic 2-form \( \omega \). The associated \((2m|2m)\)-dimensional supermanifold \( \mathcal{M} = \Pi TM \), \( \Pi \) being the parity change functor and \( TM \) the tangent bundle to \( M \), comes equipped canonically with an odd vector field \( d \) and a second order even differential operator \( L^* : \mathcal{O}_M \rightarrow \mathcal{O}_M \), where \( \mathcal{O}_M \) is the (complexified) structure sheaf on \( \mathcal{M} \). They are most easily described in a local coordinate chart \((x^a, \psi^b = dx^b)\), \( a, b = 1, \ldots, 2m \), on \( \mathcal{M} \),

\[
d = \sum_{a=1}^{2m} \psi^a \frac{\partial}{\partial x^a},
\]

and

\[
L^* = \sum_{a,b=1}^{2m} \omega^{ab} \frac{\partial^2}{\partial \psi^a \partial \psi^b},
\]

where \( \omega^{ab} \) is the \( 2m \times 2m \) matrix inverse to the matrix, \( \omega_{ab} \), of coefficients of \( \omega \) in the basis \( dx^a \). Under the canonical isomorphism \( \Gamma(\mathcal{M}, \mathcal{O}_M) = \Omega^* M \) the vector field \( d \) goes into the usual de Rham differential.

1.1. Lemma The second order differential operator \( \Delta := [L^*, d] \) satisfies \( \Delta^2 = 0 \) and \([\Delta, d] = 0\).

Proof. In a local coordinate chart,

\[
\Delta = \sum_{a,b} \omega^{ab} \frac{\partial^2}{\partial \psi^a \partial x^b} = \sum_{a,b,c} \frac{\partial \omega^{ab}}{\partial x^c} \psi^c \frac{\partial^2}{\partial \psi^a \partial \psi^b}.
\]

Under the assumption (without loss of generality) that \( x^a \) are Darboux coordinates the required statements become obvious. \( \square \)

The isomorphism \( \Gamma(\mathcal{M}, \mathcal{O}_M) = \Omega^* M \) sends \( \Delta \) into a differential \( \Delta : \Omega^* M \rightarrow \Omega^* M \) of degree -1 on forms.
1.2. Remark. Clearly, for any manifold $M$ and any section $\nu \in \Gamma(M, \Lambda^2 TM)$ we can define operators $d$, $L^*$ and $\Delta = [L^*, d]$ on $\Pi TM$ as above. Koszul [3] showed that Lemma 1.1 still holds true if the pair $(M, \nu)$ is a Poisson manifold. He suggested to call the cohomology of the resulting complex $(\Omega^* M, \Delta)$ the \textit{canonical cohomology}. Brylinski [2] showed that for a symplectic manifold the canonical and de Rham cohomologies coincide. He also showed that $\Delta$, when viewed as a degree $-1$ differential on $\Omega^* M$, satisfies
\[ \Delta|_{\Omega^k M} = (-1)^{k+1} \ast d \ast, \]
where $\ast : \Omega^k M \to \Omega^{2m-k}$ is the symplectic analogue of the Hodge duality operator defined by the condition $\beta \wedge (\ast \alpha) = \langle \beta, \alpha \rangle \omega^m / m!$, with $\langle \cdot, \cdot \rangle$ being the pairing between $k$-forms induced by the symplectic form. This star operator satisfies $\ast (\ast \alpha) = \alpha$ and $\beta \wedge (\ast \alpha) = (\ast \beta) \wedge \alpha$.

1.3. Symplectic harmonic forms. A differential form $\alpha \in \Omega^* M$ is called \textit{symplectic harmonic} if it satisfies $d\alpha = \Delta \alpha = 0$. Mathieu [5] proved that the following three statements are equivalent
(i) the symplectic manifold $M$ satisfies the Hard Lefschetz condition;
(ii) the morphism of differential complexes $(\Omega^* M, \Delta) \to (\Omega^* M / d\Omega^* M, \Delta)$ induces an isomorphism in cohomology;
(iii) any class in the de Rham cohomology $H^*(M, \mathbb{C})$ contains a symplectic harmonic representative.

We use these results to prove the following

1.4. Proposition. Let $M$ be a symplectic manifold satisfying the Hard Lefschetz condition. Then the differentials $d$, $\Delta : \Omega^* M \to \Omega^* M$ satisfy
\[ \text{Im} d \Delta = \text{Im} d \cap \text{Ker} \Delta = \text{Im} \Delta \cap \text{Ker} d. \]

\textbf{Proof.} It follows immediately from 1.3(ii) that $\text{Im} d \cap \text{Im} \Delta = \text{Im} d \cap \text{Ker} \Delta = \text{Im} \Delta \cap \text{Ker} d$. Thus it remains to show that $\text{Im} d \cap \text{Im} \Delta = \text{Im} d \Delta$ which will follow from the following

\textbf{Claim.} For any $p$-form $\alpha_p$ such that $\alpha_p = d\gamma_{p-1} = \Delta \beta_{p+1}$ for some $\gamma_{p-1} \in \Omega^{p-1} M$ and $\beta_{p+1} \in \Omega^{p+1} M$ there exists a $p$-form $\tau_p$ such that $\alpha_p = d \Delta \tau_p$. 
We shall prove this Claim by induction. It is trivially true for \( p = 2m \) (and \( p = 0 \)). Let us show that it is true for \( p = 2m - 1 \). Since \( d\beta_{2m} \) is trivially 0, then, by 1.3(iii), there is a representation \( \beta_{2m} = \beta_{2m}^0 + d\tau_{2m-1} \) for some \( \tau_{2m-1} \in \Omega^{2m-1}M \) and \( \beta_{2m}^0 \in \Omega^{2m}M \) satisfying \( \Delta \beta_{2m}^0 = 0 \). Hence \( \alpha_{2m-1} = d\Delta \tau_{2m-1} \).

Assume now that the Claim is true for \( p = k + 2 \). Let us show that it is true for \( p = k \). If \( \alpha_k = d\gamma_{k-1} = \Delta \beta_{k+1} \), then, due to the fact that \( d \) and \( \Delta \) commute, \( \alpha_{k+2} := d\beta_{k+1} \in \text{Ker} \Delta \). Since \( \text{Im} d \cap \text{Ker} \Delta = \text{Im} d \cap \text{Im} \Delta \), \( \alpha_{k+2} = d\beta_{k+1} = \Delta \mu_{k+3} \) for some \( \mu_{k+3} \in \Omega^{k+3}M \) and hence, by the induction hypothesis, \( \alpha_{k+2} = d\Delta \nu_{k+2} \) for some \( \nu_{k+2} \in \Omega^{k+2}M \). Then \( d(\beta_{k+1} - \Delta \nu_{k+2}) = 0 \) and, by 1.3(iii), there is a decomposition

\[
\beta_{k+1} = \beta_{k+1}^0 + d\tau_k + \Delta \nu_{k+2}
\]

for some \( k \)-form \( \tau_k \) and \((k+1)\)-form \( \beta_{k+1}^0 \) satisfying \( d\beta_{k+1}^0 = \Delta \beta_{k+1}^0 = 0 \). Thus \( \alpha_k = \Delta \beta_{k+1} = d\Delta \tau_k \). This completes the proof of the Claim and hence of the Proposition. \( \square \)

A differential complex is called \textit{formal} if it is quasi-isomorphic to its cohomology.

1.5. \textbf{Theorem.} The de Rham complex \((\Omega^* M, d)\) on a symplectic manifold satisfying the Hard Lefschetz condition is formal.

\textit{Proof.} It follows immediately from Proposition 1.4 above and Lemma 5.4.1 in [4] that the natural inclusion

\[(\text{Ker} \Delta, d) \longrightarrow (\Omega^* M, d)\]

and the projection

\[(\text{Ker} \Delta, d) \longrightarrow (H^*(M, \mathbb{C}), 0)\]

induced from the map \( \text{Ker} \Delta \to H^*(\Omega^* M, \Delta) = H^*(M, \mathbb{C}) \), are quasi-isomorphisms. \( \square \)

\section{dGBV algebra of a symplectic manifold}

In this section we plug in the data of \( \S 1 \) into the general machinery developed in [1] (see also [4]) and produce the structure of a Frobenius manifold on the
de Rham cohomology of a symplectic manifold satisfying the Hard Lefschetz condition. We shall give only a very short outline of the construction and refer to [4] for full details.

Let \((M, \omega)\) be a symplectic manifold. For a moment we switch back to the interpretation of \(\Delta\) and \(d\) as an odd second order derivation and, respectively, an odd vector field on the supermanifold \(\mathcal{M} = \Pi TM\).

### 2.1. Odd Poisson structure on \(\mathcal{M}\).

For any \(f, g \in \mathcal{O}_\mathcal{M}\) we define the odd brackets

\[
[f \bullet g] = (-1)^{\tilde{f} + 1} \Delta(fg) - (-1)^{\tilde{f}} \Delta(f)g - a \Delta b.
\]

where \(\tilde{\cdot}\) stands for the parity of the kernel symbol. It is not hard to check that the conditions \(\tilde{\Delta} = 1\) and \(\Delta^2 = 0\) imply

a) odd anticommutativity: 
\[
[f \bullet g] = -(-1)^{(\tilde{f} + 1)(\tilde{g} + 1)}[g \bullet f];
\]

b) odd Jacobi identity:
\[
[f \bullet [g \bullet h]] = [[f \bullet g] \bullet h] + (-1)^{(\tilde{f} + 1)(\tilde{g} + 1)}[g \bullet [f \bullet h]];
\]

c) odd Poisson identity:
\[
[f \bullet gh] = [f \bullet g]h + (-1)^{\tilde{g}(\tilde{f} + 1)}g[f \bullet h];
\]

d) two odd differentials:
\[
\Delta[f \bullet g] = [\Delta f \bullet g] + (-1)^{(\tilde{f} + 1)}[f \bullet \Delta g],
\]
\[
d[f \bullet g] = [df \bullet g] + (-1)^{(\tilde{f} + 1)}[f \bullet dg].
\]

Thus \((\Gamma(\mathcal{M}, \mathcal{O}_\mathcal{M}) = \Omega^* M, \bullet, \Delta, d)\) is an odd Lie superalgebra with two commuting differentials. Note, however, that the roles of \(d\) and \(\Delta\) are not symmetric: \(d\) is a derivation of the associative multiplicative structure in \(\Omega^* M\), while \(\Delta\) is not. Such a structure is often called a differential Gerstenhaber-Batalin-Vilkoviski algebra.

### 2.2. A normalised solution to the Maurer-Cartan equation.

From now on we assume that \(M\) satisfies the Hard Lefschetz condition. Let \([c_i]\) be a basis and \(x^i\) the associated linear coordinates in \(H^*(M, \mathbb{C})\). We define \(K = \mathbb{C}[[x^i]]\) and consider the odd Lie superalgebra \((K \otimes_{\mathbb{C}} \Omega^* M, \bullet)\) equipped with
the differentials \( d_K = 1 \otimes d \) and \( \Delta_K = 1 \otimes \Delta \). It follows from Proposition 1.4 above and Proposition 6.1.1 in [4] that there exists a generic even formal solution \( \Gamma \in K \otimes \Omega^* M \) to the Maurer-Cartan equation

\[
d\Gamma + \frac{1}{2} [\Gamma \cdot \Gamma] = 0
\]

such that \( \Gamma_0 = 0, \Gamma_1 = \sum_i x^i c_i \) and \( \Gamma_n \in K \otimes \text{Im} \Delta \) for all \( n \geq 2 \), where \( c_i \) is a symplectic harmonic harmonic representative of \([c_i]\) (with \( c_0 = 1 \)), and \( \Gamma_n \) is the homogeneous component of \( \Gamma \) of degree \( n \) in \((x^i)\). Moreover, \( \Gamma \) can be chosen in such a way that all \( \Gamma_n \) for \( n \geq 2 \) do not depend on \( x^0 \).

The operator

\[
d\Gamma : \ K \otimes \Omega^* M \longrightarrow \ K \otimes \Omega^* M
\]

\[
f \quad \longrightarrow \quad d\Gamma f = d_K f + [\Gamma \cdot f]
\]

commutes with \( \Delta \) and satisfies \( d\Gamma^2 = 0 \). Actually, all the results of §1 hold true after the replacements \( \Omega^* M \to K \otimes \Omega^* M, \Delta \to \Delta_K \) and \( d \to d_K \).

2.3. Integral. Since \( M \) satisfies the Hard Lefschetz condition, \( H^{2\text{m}}(M, \mathbb{C}) = \mathbb{C}[\omega^m] \) and hence \( M \) is compact. Then the integral

\[
\int_M : \ \Omega^* M \longrightarrow \ \mathbb{C}
\]

\[
\lambda \quad \longrightarrow \quad \int_M \lambda := \lambda \cap [M]
\]

is well-defined.

2.3.1. Lemma. For any \( \alpha, \beta \in \Omega^* M \),

\[
\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta,
\]

\[
\int_M \Delta \alpha \wedge \beta = (-1)^k \int_M \alpha \wedge \Delta \beta.
\]

Proof. The first statement follows immediately from the Stokes theorem, while the second one requires a small computation (in which we assume, for definiteness, that \( \alpha \in \Omega^k M \) and hence \( \beta \in \Omega^{2m-k+1}M \):

\[
\int_M \Delta \alpha \wedge \beta = (-1)^{k+1} \int_M (* d * \alpha) \wedge \beta
\]
\[
= (-1)^{k+1} \int_M (d \ast \alpha) \wedge (\ast \beta)
= \int_M (\ast \alpha) \wedge (d \ast \beta)
= \int_M \alpha \wedge (\ast d \ast \beta)
= (-1)^k \int_M \alpha \wedge \Delta \beta. \quad \square
\]

2.4. From symplectic structures to Frobenius manifolds. Consider a map
\[
\psi : \quad H_K := K \otimes H^*(M, \mathbb{C}) \quad \longrightarrow \quad K \otimes \Omega^* M
_X \quad \longrightarrow \quad \bar{X} \Gamma
\]
which, by definition, acts on the basis vectors \([c_i]\) of \(H_K\) as follows
\[
\psi([c_i]) = \frac{\partial \Gamma}{\partial x^i}.
\]
Using the isomorphism \(\text{Ker} d\Gamma/\text{Im} d\Gamma = H_K\), one introduces a supercommutative structure into \(H_K\),
\[
\overline{X \circ Y} := \overline{X} \Gamma \cdot \overline{Y} \Gamma \mod \text{Im} d\Gamma.
\]
From Proposition 1.4 and Lemma 2.3.1 it easily follows that the data \((\Omega^* M, d, \Delta, \int_M)\) satisfies the Assumptions 1-3 of Manin in [4]. Then his Theorems 6.2.3, 6.4.1 and Proposition 6.3.1 [4] immediately imply that the above product is potential,
\[
[c_i] \circ [c_j] = \sum_{k,l} \frac{\partial^3 \Phi}{\partial x^i \partial x^j \partial x^k} g^{kl} [c_l]
\]
with
\[
\Phi = \int_M \left( \frac{1}{6} \Gamma^3 - \frac{1}{2} dB \Delta B \right),
\]
associative and admits an Euler vector field. Here \(\Gamma = \Gamma_1 + \Delta B\) with \(B_0 = B_1 = 0\), and \(g_{ij} = \int_M [c_i] \wedge [c_j]\) is the standard Poincare metric.
Thus $H^*(M, \mathbb{C})$ carries the structure of a Frobenius manifold.

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