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CORRIGENDUM

AN INVERSE PROBLEM IN CORROSION DETECTION:
STABILITY ESTIMATES, J. INV. ILL-POSED PROBLEMS
12 (4) (2004), 349-367.

MOURAD CHOULLI

Unless otherwise stated, $\Omega$ is a $C^\infty$ bounded domain of $\mathbb{R}^2$ so that its boundary $\Gamma$ is the union of two disjoint closed subsets with nonempty interior, $\Gamma = \Gamma_1 \cup \Gamma_2$.

We considered in [2] the stability issue for the problem of determining the boundary coefficient $q$, appearing in the BVP

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\partial_\nu u + qu = 0 & \text{on } \Gamma_1, \\
\partial_\nu u = f & \text{on } \Gamma_2,
\end{cases}$$

from the boundary measurement $u|_{\gamma_2}$, where $\gamma_2$ is an open subset of $\Gamma_2$.

Our proof of [2, Theorem 2.1] is partially incorrect. We rectify here this proof. We precisely establish a stability estimate of logarithmic type for the inverse problem described above. Contrary to the result announced in [2, Theorem 2.1], we do not know whether Lipschitz stability, even around a particular unknown coefficient, is true. Note that Lipschitz stability around an arbitrary unknown boundary coefficient is false in general as shows the following counter example in which $\Omega = \{1/2 < |x| < 1\}$, $\Gamma_1 = \{|x| = 1/2\}$ and $\Gamma_2 = \{|x| = 1\}$. Let, in polar coordinates system $(r, \theta)$,

$$u = 1 + \ln r, \quad u_k = u + 2^{-k}k^{-2}(r^2 + r^{-k})\cos(k\theta), \quad k \geq 1.$$

By straightforward computations we check that $u$ and $u_k$ are the solutions of the BVP (1) respectively when

$$q = \frac{2}{1 - \ln 2}, \quad q_k = \frac{2 + k^{-1}(2^{-2k+1} - 2)\sin(k\theta)}{1 - \ln 2 + k^{-2}(2^{-2n} + 1)\sin(k\theta)}, \quad k \geq 1,$$

and $f = 1$.

By simple calculations, we get $\|u - u_k\|_{L^2(\Gamma_2)} = O\left(2^{-k}k^{-2}\right)$, while $\|q - q_k\|_{L^2(\Gamma_1)} = O(k^{-1})$.

To our knowledge, the only case where Lipschitz stability holds is when $q$ is assumed to be a priori piecewise constant. We refer to [6] for more details.

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Throughout, the unit ball of a Banach space $X$ is denoted by $B_X$ and

$$L_p^e(D) = \{ h \in L^p(D); \text{supp}(h) \subset K \}, \quad 1 \leq p \leq \infty.$$ 

For sake of clarity, we start our analysis with stability around a particular boundary coefficient. To this end, fix $0 < \alpha < 1$ and, for $0 \leq f \in C^{1,\alpha}(\Gamma_2)$, denote by $w(f) \in C^{2,\alpha}(\Omega)$ the solution of the BVP

$$\begin{cases}
\Delta w = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \Gamma_1, \\
\partial_n w = f & \text{on } \Gamma_2.
\end{cases}$$

According to the strong maximum principle and Hopf’s lemma (see for instance [4]), $\partial_n w < 0$ on $\Gamma_1$.

Let $q_0 = -\partial_n w(f)_{|\Gamma_1} (> 0)$ and set $u_0 = 1 + w$. Then it is straightforward to check that $u_0$ is the unique solution of the BVP

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\partial_n u + q_0 u = 0 & \text{on } \Gamma_1, \\
\partial_n u = f & \text{on } \Gamma_2.
\end{cases}$$

For $(\varphi_1, \varphi_2) \in L^2(\Gamma_1) \oplus L^2(\Gamma_2)$, define $L(\varphi_1, \varphi_2) := y$, where $y \in H^{1/2}(\Omega)$ is the unique weak solution of the BVP

$$\begin{cases}
\Delta y = 0 & \text{in } \Omega, \\
\partial_n y + q_0 y = \varphi_1 & \text{on } \Gamma_1, \\
\partial_n y = \varphi_2 & \text{on } \Gamma_2.
\end{cases}$$

An application of Green’s formula leads

\begin{align}
\int_\Omega |\nabla y|^2 \, dx + \int_{\Gamma_1} q_0 y^2 \, d\sigma &= \int_{\Gamma_1} \varphi_1 y d\sigma + \int_{\Gamma_2} \varphi_2 y d\sigma \\
&\leq \| (\varphi_1, \varphi_2) \|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)} \| y \|_{H^1(\Omega)}. 
\end{align}

Using that $h \to \left( \int_\Omega |\nabla h|^2 \, dx + \int_{\Gamma_1} q_0 h^2 \, d\sigma \right)^{1/2}$ defines an equivalent norm on $H^1(\Omega)$, we derive from (2)

$$\| y \|_{H^1(\Omega)} \leq \kappa_0 \| (\varphi_1, \varphi_2) \|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)},$$

for some constant $\kappa_0$ depending only on $\Omega$ and $f$.

As $y$ is also the solution of the BVP

$$\begin{cases}
\Delta y = 0 & \text{in } \Omega, \\
\partial_n y + y + (1 - q_0) y + \varphi_1 & \text{on } \Gamma_1, \\
\partial_n y = \varphi_2 & \text{on } \Gamma_2,
\end{cases}$$

we get from the usual a priori estimates for non homogenous BVP’s (see [5]) that there exits a constant $\kappa_1$, depending only on $\Omega$ and $f$, so that

$$\| y \|_{H^{3/2}(\Omega)} \leq \kappa_1 \| (\varphi_1, \varphi_2) \|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)}.$$ 

In other words, we proved that $L \in \mathcal{B}(L^2(\Gamma_2), H^{1/2}(\Omega))$ and

$$\| L \| := \| L \|_{\mathcal{B}(L^2(\Gamma_1) \oplus L^2(\Gamma_2), H^{1/2}(\Omega))} \leq \kappa_1.$$
For \( q \in L^2(\Gamma_1) \), define the operator \( H_q \) as follows

\[
H_q : H^{3/2}(\Omega) \rightarrow H^{3/2}(\Omega) : H_q(u) = L (-qu|_{\Gamma_1}, 0).
\]

If \( \kappa \) is the norm of the trace operator

\[
h \in H^{3/2}(\Omega) \rightarrow u|_{\Gamma_1} \in C(\Gamma_1),
\]

then

\[
\|H_q\|_{H^{3/2}(\Omega)} \leq \kappa \|L\|_{L^2(\Gamma_1)}.
\]

Whence, for any \( q \in U = (2\kappa\|L\|)^{-1}B_{L^2(\Gamma_1)}, I - H_q \) is invertible and

\[
\| (I - H_q)^{-1} \|_{H^{3/2}(\Omega)} \leq 2, \quad q \in U.
\]

Define, for \( q \in U \) and \((\varphi_1, \varphi_2) \in L^2(\Gamma_1) \oplus L^2(\Gamma_2), \nabla\)

\[
u_q(\varphi_1, \varphi_2) = (I - H_q)^{-1}L(\varphi_1, \varphi_2).
\]

In light of the identity

\[
u_q(\varphi_1, \varphi_2) = L (-qu|_{\Gamma_1} + \varphi_1, \varphi_2),
\]

we derive that \( \nu_q(\varphi_1, \varphi_2) \in H^{3/2}(\Omega) \) is the solution of the BVP

\[
\begin{align*}
\Delta u &= 0 & \text{in } \Omega, \\
\partial_n u + (q_0 + q)u &= \varphi_1 & \text{on } \Gamma_1, \\
\partial_n u &= \varphi_2 & \text{on } \Gamma_2.
\end{align*}
\]

Note that according to (6)

\[
\|\nu_q(\varphi_1, \varphi_2)\|_{H^{3/2}(\Omega)} \leq 2\kappa_1\|L(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)}.
\]

Set \( \nu_q = \nu_q(0, f) \). That is \( \nu_q \) is the solution of the BVP

\[
\begin{align*}
\Delta u &= 0 & \text{in } \Omega, \\
\partial_n u + (q_0 + q)u &= 0 & \text{on } \Gamma_1, \\
\partial_n u &= f & \text{on } \Gamma_2.
\end{align*}
\]

Observe that (7) yields

\[
\|\nu_q\|_{H^{3/2}(\Omega)} \leq 2\kappa_1\|f\|_{L^2(\Gamma_2)}.
\]

Let \( \gamma_1 \) be a nonempty open subset of \( \Gamma_1 \) so that \( \Gamma_1 \setminus \overline{\gamma_1} \) is nonempty. Define \( L^2_{\gamma_1}(\Gamma_1) \) as the set of those functions \( p \in L^2(\Gamma) \) so that \( \text{supp}(p) \subset \overline{\gamma_1} \). We can mimic the proof of [2, Proposition 2.1] to show that the mapping

\[
\Phi : q \in U \cap L^2_{\gamma_1}(\Gamma_1) \rightarrow \chi_{\gamma_1} \left[ \partial_n u_q|_{\gamma_1} \right] \in L^2_{\gamma_1}(\Gamma_1)
\]

is continuously Fréchet differentiable and \( \Phi'(0) = N \). Here, for \( p \in L^2\big|_{\gamma_1}(\Gamma_1) \), \( Np = \chi_{\gamma_1} \left[ \partial_n v_{\gamma_1} \right] \), where \( v_p \) is the solution of the BVP

\[
\begin{align*}
\Delta v &= 0 & \text{in } \Omega, \\
\partial_n v + q_0 v &= -p & \text{on } \Gamma_1, \\
\partial_n v &= 0 & \text{on } \Gamma_2.
\end{align*}
\]

Similarly to the proof of [2, Lemma 2.1], we prove that \( N \) is an isomorphism. Therefore, by the implicit function theorem, there exists \( \overline{U} \subset U \) so that \( \Phi^{-1} \) is Lipschitz continuous, on \( \overline{U} = \Phi(\overline{U} \cap L^2\big|_{\gamma_1}(\Gamma_1)) \), with Lipschitz constant less or equal to \( 2\|N^{-1}\| \). That is

\[
\|q_1 - q_2\|_{L^2(\Gamma_1)} \leq 2\|N^{-1}\| \|\partial_n u_{q_1} - \partial_n u_{q_2}\|_{L^2(\gamma_1)}, \quad q_1, q_2 \in \overline{U} \cap L^2\big|_{\gamma_1}(\Gamma_1).
\]
Let $k$ be a positive integer, $s \in \mathbb{R}$, $1 \leq r \leq \infty$ and consider the vector space

$$B_{s,r}(\mathbb{R}^k) := \{ w \in \mathcal{S}'(\mathbb{R}^k) ; (1 + |\xi|^2)^{s/2} \hat{w} \in L^r(\mathbb{R}^k) \},$$

where $\mathcal{S}'(\mathbb{R}^k)$ is the space of tempered distributions on $\mathbb{R}^k$ and $\hat{w}$ is the Fourier transform of $w$. Equipped with the norm

$$\| w \|_{B_{s,r}(\mathbb{R}^k)} := \left\| (1 + |\xi|^2)^{s/2} \hat{w} \right\|_{L^r(\mathbb{R}^k)},$$

$B_{s,r}(\mathbb{R}^k)$ is a Banach space. Note that $B_{s,2}(\mathbb{R}^k)$ is merely the Sobolev space $H^s(\mathbb{R}^k)$. Using local charts and a partition of unity, we construct $B_{s,r}(\Gamma_1)$ from $B_{s,r}(\mathbb{R})$ similarly as $H^s(\Gamma_1)$ is built from $H^s(\mathbb{R})$.

Fix $m > 0$. If $f \in H^{3/2}(\Gamma_2)$ and $q \in mB_{3/2,1}(\Gamma_1)$, then by [1, Theorem 2.3], $u_q \in H^3(\Omega)$ and

$$\| u_q \|_{H^3(\Omega)} \leq C_0.$$  \hfill (10)

Here and henceforth, $C_0$ is a constant depending only on $\Omega$, $f$ and $m$.

But in dimension $H^3(\Omega)$ is continuously embedded in $C^2(\overline{\Omega})$. Whence, (10) entails

$$\| u_q \|_{C^2(\overline{\Omega})} \leq C_0.$$  \hfill (11)

Let

$$\Psi(\rho) = |\ln \rho|^{-1/2} + \rho, \quad \rho > 0,$$

extended by continuity at $0$ by setting $\Psi(0) = 0$.

Let $\gamma_2$ be a nonempty open subset of $\Gamma_2$. According to [3, Proposition 2.7], there exists a constant $C > 0$, depending only on $\Omega$, $f$, $m$ and $\gamma_2$, so that

$$\| \partial_v u_{q_1} - \partial_v u_{q_2} \|_{L^2(\gamma_1)} \leq C \Psi (\| u_{q_1} - u_{q_2} \|_{H^1(\gamma_2)}).$$  \hfill (12)

Set

$$\mathcal{Q}_m = mB_{3/2,1}(\Gamma_1) \cap \overline{U} \cap L^2(\gamma_1).$$

Note that $\mathcal{Q}_m \neq \emptyset$ if $m$ is chosen sufficiently large.

We can now combine (9) and (12) in order to obtain

$$\| q_1 - q_2 \|_{L^2(\Gamma_1)} \leq C \Psi (\| u_{q_1} - u_{q_2} \|_{H^1(\gamma_2)}), \quad q_1, q_2 \in \mathcal{Q}_m.$$  \hfill (13)

We sum up our analysis in the following theorem, where we used the fact that $H^{3/2}(\Gamma_2)$ is continuously embedded in $C^2(\Gamma_2)$.

**Theorem 1.** Let $0 \leq f \in H^{3/2}(\Gamma_2)$, $q_0 = -\partial_v w(f)|_{\Gamma_1}(> 0)$ and $\gamma_i$ be a nonempty open subset of $\Gamma_i$, $i = 1, 2$, with $\Gamma \setminus \overline{\gamma_i} \neq \emptyset$. There exists a neighborhood $\overline{U}$ of $q_0$ in $L^2(\Gamma_1)$, depending on $f$, $\Omega$ and $\gamma_1$ with the property that, if $m > 0$ is chosen in such a way that

$$\mathcal{Q}_m = mB_{3/2,1}(\Gamma_1) \cap \overline{U} \cap L^2(\gamma_1) \neq \emptyset,$$

we find a constant $C > 0$, depending on $f$, $m$, $\Omega$ and $\gamma_i$, $i = 1, 2$, so that

$$\| q_1 - q_2 \|_{L^2(\Gamma_1)} \leq C \Psi (\| u_{q_1} - u_{q_2} \|_{H^1(\gamma_2)}), \quad q_1, q_2 \in \mathcal{Q}_m.$$
We now discuss briefly the stability around an arbitrary $q_0$. Let then $q_0 \in L^\infty(\Gamma_1)$ be non negative and non identically equal to zero and let $f \in L^2(\Gamma_2)$ be non identically equal to zero. Denote by $u_0 \in H^{3/2}(\Omega)$ the solution of the BVP

$$
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
\partial_\nu u + q_0 u &= 0 \quad \text{on } \Gamma_1, \\
\partial_\nu u &= f \quad \text{on } \Gamma_2.
\end{align*}
$$

As it is observed in [2],

$$\Gamma_0 = \{ x \in \Gamma_1; u_0(x) \neq 0 \}$$

is an open dense subset of $\Gamma_1$.

Slight modifications of the preceding analysis allow us to prove the following result

**Theorem 2.** Let $f \in H^{3/2}(\Gamma_2)$, $f \neq 0$, $0 \leq q_0 \in L^\infty(\Gamma_1)$, $q_0 \neq 0$, $K$ a compact subset of $\Gamma_0$ so that $\Gamma_2 \setminus K \neq \emptyset$ and $\gamma_2$ be a nonempty open subset of $\Gamma_2$. There exists a neighborhood $\mathcal{U}$ of $q_0$ in $L^2(\Gamma_1)$, depending on $f$, $\Omega$ and $K$ with the property that, if $m > 0$ is chosen in such a way that

$$Q_m = mB_{B^{3/2}}(\Gamma_1) \cap \mathcal{U} \cap L^2_2(\Gamma_1) \neq \emptyset,$$

we find a constant $C > 0$, depending on $f$, $m$, $\Omega$, $K$ and $\gamma_2$, so that

$$\|q_1 - q_2\|_{L^2(\Gamma_1)} \leq C\Psi(\|u_{q_1} - u_{q_2}\|_{H^1(\gamma_2)}), \quad q_1, q_2 \in Q_m.$$

Observe that, as in [2], the last theorem can be extended to the case where $\partial \Gamma_1 \cap \partial \Gamma_2 \neq \emptyset$.

In the most general case, in dimensions two and three, we can prove a stability estimate of triple logarithmic type (see [3, Theorem 4.9]).

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