Reconstructing braneworld inflation

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Abstract

The reconstruction of a braneworld inflationary universe considering the parametrization (or attractor) of the scalar spectral index $n_s(N)$ in terms of the number of $e$-folding $N$ is developed. We also study the possibility that the reconstruction for the scenario of braneworld inflation, can be realized in terms of the tensor to scalar ratio $r(N)$. For both reconstruction methodologies, we consider a general formalism in order to obtain the effective potential as function of the cosmological parameters $n_s(N)$ or $r(N)$. For both reconstruction methods, we consider the specific examples for large $N$ in the framework of the slow roll approximation as; the attractor $n_s - 1 \propto N^{-1}$ for the scalar spectral index and the attractor $r \propto N^{-2}$ for the tensor to scalar ratio. In this context and depending on the attractors used, we find different expressions for the effective potential $V(\phi)$, as also the constraints on the parameters present in the reconstruction.

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I. INTRODUCTION

It is well known that during the early universe, the introduction of the inflationary stage or inflation, is to date a possible solution to many long-standing problems of the hot big bang model (horizon, flatness, monopoles, etc.)\cite{1,3}. However, the most significant characteristic of the inflationary model is that inflation gives account of a causal interpretation of the origin of the observed anisotropy of the cosmic microwave background radiation (CMB), as also the distribution of large scale structure observed today\cite{4,6}.

In order to describe the inflationary epoch for the early universe, different inflationary models have been proposed in the framework of General Relativity (GR) as in modified gravity or alternative to Einstein’s General Relativity. In this context, implications of string/M-theory to Friedmann-Robertson-Walker (FRW) cosmological models have attracted a great deal of attention in the last years and in particular some models with brane-antibranes configurations like so space-like branes, together with their applications to the inflationary cosmology\cite{7}. In this framework, the introduction of extra dimensions generates extra terms in the Friedmann equation product of the dimensional reduction (embed) to four-dimensions\cite{8,10} and the standard model of particles is confined to the brane, whilst the gravitation propagates into the bulk space-time\cite{9}. In this respect, the inflationary model of Randall-Sundrum (RS) type II scenario has taken great attentiveness in the last years\cite{11} and this modification to GR for the cosmological models has been widely studied. In particular the chaotic model on the brane in the framework of slow roll was analyzed in Ref.\cite{12}. In Ref.\cite{13} an inverse power law potential was studied, where a single scalar field can act as an inflaton field and quintessence for an appropriate value of the brane tension. In the case of a tachyonic potential considering the power law inflation in the frame of brane-world cosmology was developed in Ref.\cite{14}. For a comprehensible review of brane-cosmology, see e.g. Refs.\cite{15,17} and recent articles, see the list in\cite{18}.

On the other hand, the reconstruction of the background and in particular the effective potential associated to scalar field in the context of inflation from observational data such as the scalar spectrum, scalar spectral index $n_s$ and the tensor to scalar ratio $r$, have been analyzed by several authors\cite{19,25}. Originally, considering a single scalar the reconstruction of inflationary potentials from the primordial scalar spectrum was proposed in Ref.\cite{19}.

An attractive mechanism to reconstruct the effective potential of the scalar field assuming
the slow roll approximation is through the parametrization in terms of the number of e-folds $N$. In this respect, by considering the scalar spectral index $n_s(N)$ and the tensor to scalar ratio $r(N)$ (commonly called attractors) it is possible to reconstruct the background during the inflationary epoch. From the observational point of view, the attractors given by $n_s - 1 \propto N^{-1}$ and $r \propto N^{-2}$, by considering the number of e-folding $N \simeq 50 - 70$ at the end of the inflationary epoch are agreed with the Planck results [6]. In particular and considering the framework of GR the scalar spectral index given by $n_s(N) - 1 \propto N^{-1}$, it is possible to build different effective potentials such as; the T-model [26], E-model [27], Staronbisky $R^2$-model [1], the chaotic model [28] and the model of Higgs inflation with non minimal coupling [29, 30]. In the framework of warm inflation unlike cold inflation, it was necessary to consider jointly the attractors $n_s(N)$ and $r(N)$, in order to reconstruct the effective potential and the dissipation coefficient [31]. Analogously, the reconstruction of an inflationary model in the context of the Galileon model or G-model, considering as attractors the scalar spectral index and the tensor to scalar ratio as a function of the number of e-folding jointly was studied in Ref. [32].

We also mentioned that another way to reconstruct the background is related with the slow-roll parameter $\epsilon$ and its parametrization in terms of $N$. In this sense, considering $\epsilon(N)$ is possible to find the scalar spectral index and the tensor to scalar ratio for inflationary models in GR, see [33, 34]. In particular assuming different slow-roll parameter $\epsilon(N)$, it was possible the reconstruction of the effective potential associated to scalar field and the observational parameters [17]. Following this idea in Refs. [35–37] were found the effective potential and the consistency relation $r = r(n_s)$, but considering the two slow roll parameters $\epsilon(N)$ and $\eta(N)$.

The goal of this study is to reconstruct the braneworld inflation, through the parametrization of the scalar spectral index or the tensor to scalar ratio, as function of the number of e-folding. In fact, we analyze how the brane model changes the reconstructions of the scalar potential, considering as attractors the spectral index $n_s(N)$ or the tensor to scalar ratio $r(N)$. In order to choose the observable $n_s$ or the tensor to scalar ratio $r$ in terms of the number of e-folds for large $N$, we will show the possibility to reconstruct the effective potential $V(\phi)$, in the frame of braneworld inflation. In this respect, we will consider the domination of the brane effect, in order to obtain analytical solutions in the reconstruction of the background. We will also formulate a general formalism to find the effective potential, by
assuming the parametrization $n_s(N)$ or $r(N)$, in the context of the slow roll approximation.

As an application of the formulated formalism, we will analyze two different reconstructions. Following the standard reconstruction of the background from $n_s(N)$, we shall consider the specific case in which the scalar spectral index is given by $n_s = 1 - 2/N$. As a second reconstruction, we shall regard the reconstruction from the point of view of the tensor to scalar ratio $r(N)$ and as it modifies the reconstruction of the effective potential. In these reconstructions, we will derive different constraints on the parameters present in the models.

The outline of the paper is as follows. The next section presents a brief review of the background and the cosmological perturbations on brane world. In Section II, we develop the reconstruction in our model. In Section III, we consider the high energy limit and the reconstruction, considering as attractor the scalar spectral index $n_s(N)$. Here, we formulate a general formalism to find the effective potential and in subsection IV A we also apply our results to a specific example for the spectral index $n_s(N)$. In Section IV, we formulate the reconstruction from the tensor to scalar ratio $r(N)$ under a general formalism and in subsection V A we consider as example the attractor $r(N) \propto N^{-2}$. Finally, in Section VI we summarize our finding. We chose units so that $c = \hbar = 1$.

II. BRANEWORLD INFLATION: BASIC EQUATIONS

In this section we give a brief review of the background equations and cosmological perturbations on the brane. In this respect, we consider the five-dimensional brane scenario, in which the flat Friedmann equation is modified from its common form results [9, 38]

$$3H^2 = \kappa \rho \left[1 + \frac{\rho}{2\tau}\right] + \Lambda_4 + \frac{\xi}{a^4},$$

(1)

where the quantity $H = \dot{a}/a$ denotes the Hubble rate, $a$ corresponds to the scale factor and $\rho$ denotes the matter field confined to the brane. Here, $\Lambda_4$ corresponds to the four dimensional cosmological constant and the constant $\kappa = 8\pi/m_p^2$, where $m_p$ is the four-dimensional Planck mass. The quantity $\xi/a^4$ has a form of dark radiation and it indicates the influence of the bulk gravitons on the brane, in which $\xi$ corresponds to an integration constant. Also, the brane tension $\tau$ is relates with the four and five dimensional Planck masses by the relation $m_p^2 = 3M_5^6 / (4\pi \tau)$, see Ref.[38]. An constraint on the value of the brane tension is found from nucleosynthesis given by $\tau > (1\text{MeV})^4$ [39]. However, a different constraint for the brane
tension from current tests for deviation from Newton’s law was obtained in Refs. [40, 41] in which it restricts to \( \tau \geq (10 \text{ TeV})^4 \).

In the following, we will consider that the constant \( \Lambda_4 = 0 \), and once that the inflation epoch initiates, the quantity \( \xi/a^4 \) will rapidly become unimportant, with which the modified Friedmann Eq. (1) becomes:

\[
3H^2 = \kappa \rho \left[ 1 + \frac{\rho}{2 \tau} \right].
\]

In order to describe the matter, we consider that the energy density \( \rho \) corresponds to a standard scalar field \( \phi \), where the energy density \( \rho(\phi) \) and the pressure \( P(\phi) \) are defined as:

\[
\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \quad P = \frac{\dot{\phi}^2}{2} - V(\phi),
\]

Here, the quantity \( V(\phi) = V \) denotes the scalar potential. We also consider that the scalar field \( \phi \) is a homogeneous scalar field i.e., \( \dot{\phi} = \dot{\phi}(t) \) and also this field is confined to the brane [9, 38]. In this context, the dynamics of the scalar field can be written as:

\[
\dot{\rho} + 3H(\rho + P) = 0,
\]

or equivalently

\[
\ddot{\phi} + 3H\dot{\phi} + V' = 0,
\]

where \( V' = \partial V(\phi)/\partial \phi \). Here the dots mean derivatives with respect to the cosmological time.

By assuming the slow roll approximation in which the energy density \( \rho \sim V(\phi) \), then the Eq. (2) reduces to [9, 38]

\[
3H^2 \approx \kappa V \left[ 1 + \frac{V}{2 \tau} \right],
\]

and Eq. (4) can be written as

\[
3H\dot{\phi} \approx -V'.
\]

Following Ref. [38], we can introduce the slow roll parameters \( \epsilon \) and \( \eta \) defined as:

\[
\epsilon = \frac{1}{2\kappa} \left( \frac{V'}{V} \right)^2 \left( 1 + \frac{V}{\tau} \right)^2, \quad \text{and} \quad \eta = \frac{1}{\kappa V(1 + V/2\tau)}.
\]

On the other hand, introducing the number of e-folding \( N \) between two different values of the time \( t \) and \( t_e \) as:

\[
N = \int_t^{t_e} H \, dt \simeq \kappa \int_{\phi_e}^{\phi} \frac{V}{V'} \left( 1 + \frac{V}{2\tau} \right) \, d\phi,
\]
where $t_e$ corresponds to the end of inflationary stage and here we have considered the slow roll approximation.

In the context of the brane world the power spectrum $\mathcal{P}_R$ of the curvature perturbations assuming the slow-roll approximation is given by \cite{15}.

$$\mathcal{P}_R = \left( \frac{H^2}{\dot{\phi}^2} \right) \left( \frac{H}{2\pi} \right)^2 \simeq \frac{\kappa^2}{12\pi^2} \frac{V^3}{V'2} \left( 1 + \frac{V}{2\tau} \right)^3. \quad (9)$$

The scalar spectral index $n_s$ is defined as $n_s - 1 = \frac{d\ln \mathcal{P}_R}{d\ln k}$ and in terms of the slow roll parameters $\epsilon$ and $\eta$ can be written as \cite{15}

$$n_s - 1 = -6\epsilon + 2\eta. \quad (10)$$

Here we have used Eqs.(7) and (9), respectively.

It is well known that the tensor-perturbation during inflation would produce gravitational waves. In the brane-world the tensor perturbation is more complicated than the standard expression obtained in GR, where the amplitude of the tensor perturbations $\mathcal{P}_g \propto H^2$. Because the brane-world gravitons propagate in the bulk, the amplitude of the tensor perturbation suffers a modification \cite{42}, where

$$\mathcal{P}_g = 8\kappa \left( \frac{H}{2\pi} \right)^2 F^2(x), \quad (11)$$

where the quantity $x = Hm_p \sqrt{3/(4\pi\tau)}$ and the function $F(x)$ is defined as

$$F(x) = \left[ \sqrt{1 + x^2} - x^2 \sinh^{-1}(1/x) \right]^{-1/2}, \quad (12)$$

in which the correction given by the function $F(x)$, appeared from the normalization of a zero-mode \cite{42}. In particular in the limit in which the tension $\tau \gg V$, the function $F(x) \to 1$ and then $\mathcal{P}_g \propto H^2$.

An important observational quantity is the tensor to scalar ratio $r$, defined as $r = \frac{\mathcal{P}_g}{\mathcal{P}_R}$. Thus, combining Eqs.(9) and (11), the tensor-scalar ratio, $r$, is given by

$$r = \left( \frac{\mathcal{P}_g}{\mathcal{P}_R} \right) \simeq \frac{8}{\kappa} \left( \frac{V'}{V} \right)^2 \left( 1 + \frac{V}{2\tau} \right)^{-3} F^2(V). \quad (13)$$

Here, we have considered that the quantity $x$ can be rewritten in terms of the effective potential from Eq.(5).
In this section we consider the methodology in order to reconstruct the background variables, considering the scalar spectral index in terms of the number of e-folds in the framework of brane-world. As a first part, we rewrite the scalar spectral index given by Eq. (10), as function of the number of e-folds \( N \) and its derivatives. In this form, obtaining the index \( n_s = n_s(N) \), we should find the potential \( V = V(N) \) in terms of the number of e-folding \( N \). Subsequently, utilizing the relation given by Eq. (8), we should obtain the e-folds \( N \) as function of the scalar field \( \phi \) i.e., \( N = N(\phi) \). Finally, considering these relations, we can reconstruct the effective potential \( V(\phi) \) in order to satisfy a specific attractor \( n_s(N) \).

In this way, we start by rewriting the standard slow roll parameters \( \epsilon \) and \( \eta \) in terms of the number of e-folds \( N \). Thus, the derivative of the scalar potential \( V' \) from Eq. (8) can be rewritten as

\[
V' = \frac{dV}{d\phi} = V'_{,N} \frac{dN}{d\phi}, \quad \text{in which} \quad V'^2 = \kappa V \left(1 + \frac{V}{2\tau}\right) V_{,N},
\]

and this suggests that \( V_{,N} \) is a positive quantity. In the following, we will consider that the notation \( V_{,N} \) corresponds to \( dV/dN \), \( V_{,NN} \) denotes \( d^2V/dN^2 \), etc.

Analogously, we can rewrite \( V'' \) as

\[
V'' = \frac{\kappa}{2V_{,N}} \left[V_{,N}^2 \left(1 + \frac{V}{\tau} \right) + V \left(1 + \frac{V}{2\tau}\right) V_{,NN}\right].
\]

In this form, the slow roll parameter \( \epsilon \) can be rewritten as

\[
\epsilon = \frac{1}{2} \frac{(1 + \frac{V}{\tau})}{V \left(1 + \frac{V}{2\tau}\right)} V_{,N},
\]

and the parameter \( \eta \) as

\[
\eta = \frac{1}{2V} \frac{(1 + \frac{V}{\tau})}{\left(1 + \frac{V}{2\tau}\right)} V_{,N} + \frac{V_{,NN}}{2V_{,N}},
\]

respectively. Here, we have considered that \( V' > 0 \).

Also, from Eqs. (8) and (14) we can rewritten \( dN/d\phi \) as

\[
\frac{dN}{d\phi} = \sqrt{\frac{\kappa V}{V_{,N}}} \sqrt{\left(1 + \frac{V}{2\tau}\right)}.
\]

In this way, by using Eq. (10) we find that the scalar spectral index can be rewritten as

\[
n_s - 1 = -2 \frac{(1 + \frac{V}{\tau})}{V \left(1 + \frac{V}{2\tau}\right)} V_{,N} + \frac{V_{,NN}}{V_{,N}},
\]
or equivalently
\[ n_s - 1 = -2 \left[ \ln \left( V \left[ 1 + \frac{V}{2\tau} \right] \right) \right]_{\mathcal{N}} + \ln V,_{\mathcal{N}} = \left[ \ln \left( \frac{V,_{\mathcal{N}}}{V^2 \left( 1 + \frac{V}{2\tau} \right)^2} \right) \right]_{\mathcal{N}}. \] \tag{19}

We also note that in the limit in which \( \tau \to \infty \), Eq.~(19) reduces to GR, in which \( n_s - 1 = \left( \frac{\ln V,_{\mathcal{N}}}{V^2} \right)_{\mathcal{N}} \), see Ref.~[24].

From Eq.~(19) we have
\[ \frac{V,_{\mathcal{N}}}{V^2 (1 + V/2\tau)^2} = e^{\int (n_s - 1) d\mathcal{N}}. \] \tag{20}

This equation gives us the effective potential \( V(N) \) for a specific attractor \( n_s(N) \). Thus, integrating we have
\[ \frac{1}{\tau} \ln \left( \frac{1 + V/2\tau}{V/2\tau} \right) - \frac{(1 + V/\tau)}{V(1 + V/2\tau)} = \int \left[ e^{\int (n_s - 1) d\mathcal{N}} \right] d\mathcal{N}. \] \tag{21}

However, this equation results in a transcendental equation for the scalar potential \( V \) and this result does not permit to obtain the relation \( V = V(N) \).

We also note that by combining Eqs.~(17) and (20), we obtain that the relation between the number of \( e \)-folds \( N \) and the scalar field \( \phi \) can be written as
\[ \left[ \sqrt{V \left( 1 + \frac{V}{2\tau} \right)} e^{\int \frac{n_s - 1}{2} d\mathcal{N}} \right] d\mathcal{N} = d\phi. \] \tag{22}

On the other hand, from Eq.~(13) the tensor-scalar ratio, \( r \) can be rewritten as
\[ r(N) \simeq \left( \frac{4}{7} \right) V,_{\mathcal{N}} \left( 1 + \frac{V}{2\tau} \right)^{-3} F^2(V). \] \tag{23}

In the following we will consider the high energy limit in which \( \rho \simeq V \gg \tau \), in order to obtain analytical solution in the reconstruction of the scalar potential in terms of the scalar field \( V(\phi) \).

**IV. HIGH ENERGY: RECONSTRUCTION FROM THE ATTRACTOR \( n_s(N) \)**

In this section we consider the high energy limit \( (V \gg \tau) \) in order to reconstruct the scalar potential, considering as attractor the scalar spectral index in terms of the number of \( e \)-folds i.e., \( n_s = n_s(N) \). In this limit, the derivatives \( V' \) and \( V'' \) can be rewritten as
\[ V'^2 = \frac{\kappa}{2\tau} V^2 V,_{\mathcal{N}}, \quad \text{and} \quad V'' = \frac{\kappa}{2\tau} V \left[ V,_{\mathcal{N}} + \frac{V V,_{\mathcal{N}}}{2V,_{\mathcal{N}}} \right]. \] \tag{24}
In this way, the relation between the number $N$ and the scalar field $\phi$ in this limit becomes
\[ \frac{dN}{d\phi} = \frac{\kappa}{2\tau} \left( \frac{V^2}{V'} \right) = \left( \frac{\kappa}{2\tau} \right)^{1/2} \frac{V}{\sqrt{V',N}}. \] (25)

From Eq. (10) we find that the scalar spectral index $n_s$ results
\[ n_s - 1 = \frac{4\tau}{\kappa} \left[ V'' - 3V'^2 V^{-1} \right] \frac{1}{V^2} = -4 \frac{V_N V}{V} + \frac{V_{NN}}{V',N}, \] (26)

or equivalently
\[ n_s - 1 = -4 \left[ \ln V \right]_{,N} + \left[ \ln V_{,N} \right]_{,N} = \left[ \ln \left( \frac{V_{,N}}{V^4} \right) \right]_{,N}. \] (27)

We note that the relation between the scalar potential and the scalar spectral index given by Eq. (27) becomes independent of the brane tension $\tau$ in the high energy limit.

From Eq. (27), the scalar potential in terms of the number of $e$-folding can be written as
\[ V = V(N) = \left[ -3 \int \left( e^{\int (n_s-1)dN} \right) dN \right]^{-1/3}. \] (28)

Now, by combining Eqs. (25) and (27), we find that the relation between $N$ and $\phi$ is given by general expression
\[ \left[ Ve^{\int \frac{(n_s-1)}{2}dN} \right] dN = \left( \frac{\kappa}{2\tau} \right)^{1/2} d\phi, \] (29)

where $V$ is given by Eq. (28).

In this form, Eqs. (28) and (29) are the fundamental relations in order to build the scalar potential $V(\phi)$ for an attractor point $n_s(N)$, in the framework of the high energy limit in brane world inflation.

On the other hand, in the high energy limit in which $V \gg \tau$, the function $F^2(x)$ given by Eq. (13) becomes $F^2(x) \approx \frac{3}{2} x = \frac{3}{2} \frac{V_{,N}}{V}$. In this form, in the high energy limit the tensor to scalar ratio $r$ becomes
\[ r \approx 48\tau \left( \frac{V_{,N}}{V^2} \right). \] (30)

Here we have considered Eq. (23).

A. An example of $n_s = n_s(N)$.

In order to develop the reconstruction of the scalar potential $V(\phi)$ in the brane world inflation, we consider the famous attractor $n_s(N)$ given by
\[ n_s(N) = n_s = 1 - \frac{2}{N}, \] (31)
as example.

From the attractor \((31)\), we find that considering Eq.(27) we have \(V_N = \alpha/N^2\), in which \(\alpha\) corresponds to a constant of integration (with units of \(m_p^{-12}\)) and since \(V_N > 0\), then the constant of integration \(\alpha > 0\). In this form, the effective potential as function of the number of e-folding \(N\) from Eq.(28) becomes

\[
V(N) = 3^{-1/3} \left[ \frac{\alpha}{N} + \beta \right]^{-1/3},
\]

where \(\beta\) denotes a new constant of integration. Here, the new constant of integration \(\beta\) with units of \(m_p^{-12}\), can be considered \(\beta = 0\) or \(\beta \neq 0\).

In the high energy limit, we find that the power spectrum \(\mathcal{P}_R\) given by Eq.(9) can be rewritten as

\[
\mathcal{P}_R \simeq \frac{1}{12\pi^2} \left( \frac{\kappa V^2}{2\tau} \right)^3 \frac{1}{V'^2} = \frac{\kappa^2}{48\pi^2\tau^2} \left( \frac{V^4}{V_N} \right) = \frac{\kappa^2}{48\pi^2\tau^2} \left( \frac{N^2}{\alpha} \right).
\]

Note that this result does not depend of the constant of integration \(\beta\). From Eq.(33), it is possible to write the constant of integration \(\alpha\) in terms of the number \(N\), \(\mathcal{P}_R\) and the tension \(\tau\) as

\[
\alpha = \frac{\kappa^2}{48\pi^2\tau^2} \left( \frac{N^2}{\mathcal{P}_R} \right).
\]

In particular by considering \(N = 60\) and \(\mathcal{P}_R = 2.2 \times 10^{-9}\), we obtain that the constant of integration \(\alpha \simeq 3 \times 10^9(\kappa/\tau)^2\).

On the other hand, from Eq.(33) the tensor to scalar ratio can be rewritten as

\[
r \simeq 48\tau \left( \frac{V_N}{V^2} \right) = 48\tau \alpha \left( \frac{V^2}{N^2} \right) = 48\tau \alpha \left( \frac{3^{-2/3}}{N^2(\alpha/N + \beta)^{2/3}} \right).
\]

Note that considering the attractor \(n_s\) given by Eq.(31), we can find a relation between the tensor to scalar ratio \(r\) with the scalar spectral index or the consistency relation given by

\[
r(n_s) \simeq \left( \frac{12\alpha\tau}{3^{2/3}} \right) \left[ \frac{\alpha(1 - n_s)}{2} + \beta \right]^{-2/3} (1 - n_s)^2.
\]

In the following, we will analyze the cases separately in which the constant of integration \(\beta\) takes the values \(\beta = 0\) and \(\beta \neq 0\), in order to reconstruct the effective potential \(V(\phi)\).

For the case \(\beta = 0\), we obtain that the relation between the number of e-folding \(N\) and scalar field \(\phi\) considering Eqs.(29), (31) and (32) becomes

\[
N(\phi) = N = \frac{1}{3^2\alpha^{1/2}} \left( \frac{\kappa}{2\tau} \right)^{3/2} (\phi - \phi_0)^3,
\]
where $\phi_0$ corresponds to a constant of integration. In this way, in the high energy limit we find that the reconstruction of the effective potential as function of the scalar field for the case $\beta = 0$ and assuming the attractor $n_s - 1 = -2/N$ is given by

$$V(\phi) = V_0 (\phi - \phi_0), \text{ where } V_0 = \left(\frac{\kappa}{18\tau\alpha}\right)^{1/2}. \quad (38)$$

Also, we note that for the case $\beta = 0$, the consistency relation $r = r(n_s)$ has a dependence $r(n_s) \propto (1 - n_s)^{4/3}$. In particular, by considering $n_s = 0.964$, $N = 56$ and $P_R = 2.2 \times 10^{-9}$, we find an upper bound for the brane tension given by $\tau < 10^{-13}m_p^4$, from the condition $r < 0.07$. For this bound on $\tau$, we have used Eq.(36). Now, from Eq.(34) and considering $N = 60$ and $P_R = 2.2 \times 10^{-9}$, together with the upper limit on $\tau$, we obtain a lower limit for the constant $\alpha$ given by $\alpha > 1.9 \times 10^{38}m_p^{-12}$.

On the other hand, in the reconstruction for the situation in which the constant of integration $\beta \neq 0$, we find that considering Eq.(29) the relation between $dN$ and $d\phi$ can be written as

$$\frac{dN}{[\alpha N^2 + \beta N^3]^{1/3}} = \frac{dN}{[\beta^{1/3}[\alpha_0 N^2 + N^3]^{1/3}} = C_1 d\phi, \text{ where } C_1 = 3^{1/3} \left(\frac{\kappa}{2\alpha\tau}\right)^{1/2}, \quad (39)$$

and the quantity $\alpha_0 = \alpha/\beta$. In the following, we will consider for simplicity the case in which the constant of integration $\beta > 0$ i.e, $\alpha_0 > 0$. We also note that the integration of Eq.(39) does not permit to obtain an analytical solution for the number of $e$-folds as function of the scalar field i.e., $N = N(\phi)$. In this sense, the solution of Eq.(39) can be written as

$$\sqrt{3} \arctan \left[ \left(1 + 2 \left[1 + \frac{\alpha_0}{N}\right]^{-1/3}\right)/\sqrt{3} \right] + \frac{1}{2} \ln \left[1 + \left(1 + \frac{\alpha_0}{N}\right)^{-2/3} + \left(1 + \frac{\alpha_0}{N}\right)^{-1/3}\right]$$

$$- \ln \left[1 - \left(1 + \frac{\alpha_0}{N}\right)^{-1/3}\right] = \beta^{1/3} C_1 (\phi - \phi_0), \quad (40)$$

where $\phi_0$ denotes a constant of integration.

Numerically, we note that in the limit in which $\alpha/\beta = \alpha_0 < N$, the first two terms of Eq.(40) are approximately constants and the dominant term corresponds to (see Fig.2)

$$- \ln \left[1 - \left(1 + \frac{\alpha_0}{N}\right)^{-1/3}\right] \approx \beta^{1/3} C_1 (\phi - \phi_0). \quad (41)$$

Thus, we find that the reconstruction of the effective potential $V(\phi)$ considering the specific case in which $\alpha_0 < N$ is given by

$$V(\phi) \approx \frac{1}{(3 \beta)^{1/3}} \left[1 - \exp(-\beta^{1/3} C_1 [\phi - \phi_0])\right]. \quad (42)$$
FIG. 1: The upper and lower panels show the tensor-to-scalar ratio $r$ as a function of the scalar spectral index $n_s$, for three different values of the brane tension $\tau$. In both panels we have considered the two-marginalized constraints joint 68% and 95% CL at $k = 0.002 \text{ Mpc}^{-1}$ from the Planck 2018 results \cite{Planck2018}. Also, in both panels the solid, dashed and dotted lines correspond to the values of brane tension $\tau/m_p^4 = 10^{-12}, 10^{-13}$ and $10^{-14}$, respectively. In the upper panel we show the consistency relation for the specific case in which the constant $\beta = 0$ and we have used $\alpha = 10^{38} m_p^{-12}$. In the lower panel we show the case in which $\beta \neq 0$ and we have considered $\beta = \alpha/60$ and $\alpha = 3 \times 10^9 (\kappa/\tau)^2$, respectively.

Here, we have combined Eqs.\((32)\) and \((41)\). Curiously, we observe that this effective potential is similar to the obtained in the Starobinsky model \cite{Starobinsky1980} in which $\beta^{1/3} C_1 = \sqrt{2/3} m_p^{-1}$ i.e., $\beta^{1/3} C_1 m_p \approx O(1)$. Also, in the limit $\beta^{1/3} C_1 |\phi - \phi_0| \gg 1$, the effective potential corresponds
to a constant potential i.e., a solution of de Sitter.

For the inverse case in which $\alpha_0 > N$, we note that the first term of Eq.(40) dominates with which (see Fig.2)

$$\sqrt{3} \arctan \left[ \left( 1 + 2 \left[ 1 + \frac{\alpha_0}{N} \right]^{-1/3} \right) / \sqrt{3} \right] \approx \beta^{1/3} C_1 (\phi - \phi_0).$$

(43)

In this form, we obtain that the reconstruction in the limit in which $\alpha_0 > N$ becomes

$$V(\phi) \approx \frac{1}{2(3\beta)^{1/3}} \left[ \sqrt{3} \tan(\beta^{1/3} C_1 (\phi - \phi_0) / \sqrt{3}) - 1 \right].$$

(44)

Here the range for the scalar field is given by $\sqrt{3} \pi/6^{1/4} C_1 + \phi_0 \lesssim \phi \lesssim \sqrt{3} \pi/2^{1/4} C_1 + \phi_0$.

![FIG. 2: The evolution of the three terms on the right given by Eq.(40) versus the dimensionless quantity $\frac{\alpha_0}{N} = \frac{\alpha}{N}$. Here, the dotted, dashed, and solid lines denote the first, second and third terms of Eq.(40), respectively.](image)

In Fig.1 we show the ratio $r$ versus the spectral index $n_s$, for three different values of the brane tension $\tau$. In both panels we consider the two-marginalized constraints for the consistency relation $r = r(n_s)$ (at 68% and 95% CL at $k = 0.002$ Mpc$^{-1}$) from the new Planck data [6]. In the upper panel we consider the special case in which the constant of integration $\beta = 0$, where the consistency relation is given by Eq.(36). Here, we take the value $\alpha = 10^{38} m_p^{-12}$. In the lower panel we take into account the case in which $\beta \neq 0$ and for the relation $r = r(n_s)$ we have used Eq.(36). In this case we have considered the specific value of $\beta$ at $N = 60$ (point limit $\alpha_0/N = 1$ or $\beta = \alpha/N$) wherewith $\beta = \alpha/60$ and $\alpha = 3 \times 10^9 (\kappa/\tau)^2$, respectively. Also, in both panels the solid, dashed and dotted lines correspond to the values of brane tension $\tau/m_p^4 = 10^{-12}, 10^{-13}$ and $10^{-14}$, respectively. In
particular for the case $\beta = 0$ we find that the brane tension has an upper limit given by $\tau < 10^{-13}m_p$, as can be seen on the upper panel of Fig.1. For the case in which $\beta \neq 0$ we find that in the particular case in which $\beta = \alpha/60$, the value of the brane tension $\tau < 10^{-12}m_p^4$ is well corroborated by Planck 2018 results, see lower panel of Fig.1. This suggests that the value of the constant of integration $\beta$ modifies the the upper bound on the brane tension. We note that in the case in which the constant $\beta > \alpha/60$ the upper limit on the brane tension increases and in the opposite case ($\beta < \alpha/60$) the upper limit on $\tau$ decreases.

In Fig.2 we show the behavior of the three terms on the right of Eq.(40) versus the dimensionless quantity $\frac{\alpha}{\beta N} = \frac{\alpha_0}{N}$. We note that for the limit in which $\alpha_0 < N$ dominates the third term of Eq.(40), see solid line of Fig.2. However, for the case in which $\alpha_0 > N$ the dominant term corresponds to the first expression of Eq.(40) given by dotted line on Fig.2.

In order to clarify our above results, we can study some specific limits for the ratio $\alpha/(\beta N) = \alpha_0/N$ in which $\alpha_0/N \ll 1$ and $\alpha_0/N \gg 1$. As first approximation we consider the case in which $\alpha_0/N \ll 1$ or $\alpha_0 \ll N$. For this limit we find from Eq.(39) that the relation $N = N(\phi)$ is given by

$$N(\phi) = \exp[\beta^{1/3}C_1(\phi - \phi_0)],$$

(45)

where $\phi_0$ denotes a constant of integration. Thus, considering the limit $\alpha/(\beta) = \alpha_0 \ll N$ we obtain that the effective potential $V(\phi)$ given by Eq.(32) becomes a constant and equal to $V(\phi) = (3\beta)^{-1/3}$. In fact, this result indicates an accelerated expansion de Sitter or de Sitter inflation, since in the high energy limit and considering the slow-roll approximation, we have $H \propto V =\text{constant.}$ Note that this constant potential coincides with the potential given by Eq.(44) when $\beta^{1/3}C_1 [\phi - \phi_0] \gg 1$. We also observe that for the consistency relation $r = r(n_s)$, we get $r = 48\tau\alpha/[(3\beta)^{2/3}N^2] \propto (1 - n_s)^2$ (see Eq.(36)).

For the case in which $\alpha/(\beta N) \gg 1$ or $\alpha_0 \gg N$, we find from Eq.(39) that the relation $N = N(\phi)$ coincides with the case $\beta = 0$ i.e., Eq.(37) and then the effective potential $V(\phi)$ changes linearly with the scalar field according to Eq.(38) in which $V(\phi) \propto \phi$. This effective potential agrees with the potential given by Eq.(44) assuming that the argument $\beta^{1/3}C_1(\phi - \phi_0)/\sqrt{3} < 1$. 

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V. HIGH ENERGY: RECONSTRUCTION FROM THE ATTRACTOR $r(N)$

In this section we consider the high energy limit, in order to reconstruct the effective potential $V(\phi)$, but from a different point of view. In order to reconstruct the scalar potential, we consider as attractor the tensor to scalar ratio in terms of the number of e folding $N$ i.e., $r = r(N)$. In this sense, considering Eq.(30) we obtain the potential effective $V(N)$ can be written as

$$V = V(N) = -48\tau \left[ \int r \, dN \right]^{-1}. \quad (46)$$

Now from Eq.(25) we find that the relation between the number $N$ and the scalar field $\phi$ is given by

$$r^{1/2} \frac{dN}{d\phi} = (24\kappa)^{1/2}. \quad (47)$$

Here, we have considered Eq.(30).

In this context, we can obtain the scalar spectral index $n_s$ as function of the number of e-folds $N$, combining the expressions given by Eqs.(27) and (46) for a specific attractor $r = r(N)$. Thus, the scalar spectral index can be rewritten as

$$n_s - 1 = \left[ \ln \left( \frac{r}{48\tau V^2} \right) \right]_{N}. \quad (48)$$

Here, the potential $V$ is given by Eq.(46).

A. An example of $r = r(N)$.

In order to develop the reconstruction of the scalar potential $V(\phi)$ in the brane world inflation, we consider that the attractor for the tensor to scalar ratio as function of the number of e- folds $r(N)$ is given by

$$r(N) = \frac{\alpha_1}{N^2}. \quad (49)$$

where $\alpha_1 > 0$ corresponds to a constant (dimensionless). For this attractor the cases in which $\alpha_1 = 12$, was analyzed in Ref.[26] and the specific value $\alpha_1 = 8$, was obtained in Ref.[24].

In particular considering $N = 60$ and $r < 0.07$, we find that the value of the constant $\alpha_1 < 252$. 
By combining Eqs. (46) and (49) we obtain that the scalar potential in terms of the number of e-folding becomes

\[ V(N) = \frac{1}{\alpha_2 + \beta_1}, \quad \text{where} \quad \alpha_2 = \frac{\alpha_1}{48\pi}. \]  

(50)

Here the quantity \( \beta_1 \) corresponds to a constant of integration with units of \( m_p^{-4} \).

In order to obtain the relation between the number \( N \) and the scalar field \( \phi \), we consider Eq. (47) together with the attractor given by Eq. (49) obtaining

\[ N = \exp \left[ \sqrt{\frac{24\kappa}{\alpha_1}} (\phi - \phi_0) \right], \]  

(51)

where \( \phi_0 \) denotes a new constant of integration. Thus, the reconstruction of the scalar potential in terms of the scalar field can be written as

\[ V(\phi) = \left( \alpha_2 \exp \left[ -\sqrt{\frac{24\kappa}{\alpha_1}} (\phi - \phi_0) \right] + \beta_1 \right)^{-1}. \]  

(52)

In particular assuming that \( \beta_1 > 0 \) and \( \alpha_2/\beta_1 \gg N \), the effective potential has the behavior of an exponential potential i.e., \( V(\phi) \propto e^{(24\kappa/\alpha_1) \phi} \) (recalled the we have considered that \( V' > 0 \)). In the inverse case in which \( N \gg \alpha_2/\beta_1 \), the scalar potential corresponds to a constant potential \( V(\phi) = \text{constant} \).

In the context of the cosmological perturbations, we have in the high energy limit the power spectrum becomes

\[ P_R \simeq \frac{1}{12\pi^2} \left( \frac{\kappa V^2}{2\tau} \right)^3 \frac{1}{V' \tau^2} = \frac{\kappa^2}{48\pi^2 \tau^2} \left( \frac{V^4}{V_N} \right) = \frac{\kappa^2}{48\pi^2 \tau^2} \frac{N^2}{\alpha_2} \left( \frac{\alpha_2}{N} + \beta_1 \right)^{-2}. \]  

(53)

Here we have used Eqs. (9) and (50), respectively. Thus, we can write the constant \( \beta_1 \) in terms of the scalar spectrum \( P_R \), the number of e-folds \( N \) and the constant \( \alpha_2 \) as

\[ \beta_1 = \sqrt{\frac{1}{3 \alpha_2 P_R} \left( \frac{\kappa N}{4\pi \tau} \right)} - \frac{\alpha_2}{N} = \sqrt{\frac{\alpha_2}{3 P_R} \left( \frac{12\kappa N}{\pi \alpha_1} \right)} - \frac{\alpha_2}{N}. \]  

(54)

On the other hand, from Eq. (48) we find that the relation between the scalar index \( n_s \) and the number of e-folding is given by

\[ n_s - 1 = \frac{2}{N} \left[ \frac{\beta_1}{\alpha_2/N + \beta_1} - 2 \right]. \]  

(55)

Note that in the specific case in which \( N \gg \alpha_2/\beta_1 \), the scalar spectral index \( n_s \) gives the famous attractor \( n_s - 1 = -2/N \).
Now, from Eq. (55) we can find the constant $\beta_1$ in terms of the $n_s$, $N$ and $\alpha_2$ as

$$\beta_1 = \frac{[N(n_s - 1) + 4]}{N[(1 - n_s)N - 2]} \alpha_2. \tag{56}$$

In this form, combining Eqs. (54) and (56) we find that the tension $\tau$ as function of the observables $n_s$ and $\mathcal{P}_R$ together with the number of e-folding $N$ and $\alpha_1$ becomes

$$\tau = \left( \frac{\mathcal{P}_R \pi^2 \alpha_1^3}{4 \times 12^3 \kappa^2 N^4} \right) \left[ \frac{[N(n_s - 1) + 4]}{[(1 - n_s)N - 2]} + 1 \right]^2. \tag{57}$$

Here we have used that $\alpha_2 = \alpha_1 / (48 \tau)$.

In particular assuming that the spectral index $n_s = 0.964$, the spectrum $\mathcal{P}_R \simeq 2.2 \times 10^{-9}$ and $N = 60$, we obtain that the constraint on the brane tension $\tau$ is given by

$$\tau \simeq 6 \times 10^{-20} \alpha_1^3 m_p^4. \tag{58}$$

Note that Eq. (58) gives a relation between the brane tension and the parameter $\alpha_1$. Now, by assuming that $\alpha_1 < 252$ in order to obtain $r < 0.07$ at $N = 60$, we find that the upper bound for the brane tension becomes

$$\tau < 9.6 \times 10^{-13} m_p^4 \simeq 10^{-12} m_p^4.$$

On the other hand, from Eq. (55) we find that the relation between the scalar index and the tensor to scalar ratio, can be written as

$$n_s - 1 = -\frac{2 r^{1/2}}{\alpha_1^{1/2}} \left[ \frac{2 \alpha_2 + \beta_1 \sqrt{\alpha_1/r}}{\alpha_2 + \beta_1 \sqrt{\alpha_1/r}} \right]. \tag{59}$$

Here we have used the attractor given by Eq. (49).

In Fig. 3 we show the tensor to scalar ratio versus the scalar spectral index for three different values of the brane tension considering the attractor $r(N) = \alpha_1 N^{-2}$. Here we have used Eq. (59) and the solid, dotted and dashed lines correspond to the values of brane tension $\tau / m_p^4 = 10^{-11}, 10^{-12}$ and $10^{-13}$, respectively. From this plot we check that the upper limit for the brane tension given by $\tau < 10^{-12} m_p^4$ is well corroborated from Planck data.

VI. CONCLUSIONS

In this article we have analyzed the reconstruction of the background in the context of braneworld inflation. Considering a general formalism of reconstruction, we have obtained
FIG. 3: As before, we show the tensor-to-scalar ratio $r$ as a function of the scalar spectral index $n_s$ from Planck 2018 results\textsuperscript{[6]} for three different values of the brane tension $\tau$ but assuming the attractor $r(N) \propto N^{-2}$ as starting point. Here the solid, dotted and dashed lines correspond to the values of brane tension $\tau/m_p^4 = 10^{-11}, 10^{-12}$ and $10^{-13}$, respectively.

an expression for the effective potential under the slow roll approximation. In order to obtain analytical solutions in the reconstruction on the brane, we have considered the high energy limit in which the energy density $\rho \simeq V \gg \tau$. In this analysis for the reconstruction of the background, we have considered the parametrization of the scalar spectral index or the tensor to scalar ratio as function of the number of e-foldings $N$. In this general description we have found from the cosmological parameter $n_s(N)$ or the parameter $r(N)$, integrable solutions for the effective potential depending on the cosmological attractor $n_s(N)$ or $r(N)$.

For the reconstruction from the attractor associated to scalar spectral index $n_s(N)$, we have assumed the famous attractor $n_s = 1 - 2/N$ as example. From this attractor, we have obtained that the consistency relation $r = r(n_s)$ is given by Eq.(36) and from the power spectrum we have found that the integration constant $\alpha$ depends on the brane tensor, see Eq.(34). On the other hand, depending of value of the second constant of integration $\beta$, we have found different results for the reconstruction of the effective potential $V(\phi)$. In particular for the specific case in which the constant $\beta = 0$, we have obtained that the reconstruction of the effective potential corresponds to a potential $V(\phi) \propto \phi$. Also, assuming that the observational constraint on the tensor to scalar ratio $r < 0.07$, we have found an
upper limit for the brane tension given by $\tau < 10^{-13}m_p^4$, wherewith the brane model is well supported by the Planck data, see upper panel of Fig.1. In this same context, for the case in which the constant of integration $\beta \neq 0$ we have found an transcendental equation for the number of e-folds as function of the scalar field $N = N(\phi)$ and the reconstruction does not work. However, as a first approximation we have analyzed the dominant terms of the transcendental equation in order to give an approach to the reconstruction of the effective potential, see Fig.2. Also, we have considered the extreme limits $\alpha_0/N \ll 1$ and $\alpha_0/N \gg 1$ in order to find analytical expressions for the potential $V(\phi)$. In this approach we have obtained that in the limit in which $\alpha_0/N \gg 1$ the effective potential coincides with the case in which the constant of integration $\beta = 0$, where the effective potential changes linearly with the scalar field.

On the other hand, we have explored the possibility of the reconstruction in the framework of braneworld inflation, considering as attractor the tensor to scalar ratio in terms of the number of e-folding i.e., $r = r(N)$. Here we have found general relation in order to build the effective potential. As a specific example we have considered the attractor $r(N) \propto N^{-2}$. Here, we have obtained that the reconstruction of the effective potential is given by Eq. (52). In particular considering the limit in which $\alpha_2/\beta_1 \gg N$, we have obtained that the effective potential corresponds to an exponential potential i.e., $V(\phi) \propto e^\phi$. In the inverse limit we have found that the effective potential $V(\phi) = \text{constant}$. Also, utilizing the observables as the scalar spectral index and the power spectrum together with the number of e-folds we have found a relation between the brane tension and the associated parameter $\alpha_1$ to the attractor $r(N)$. Thus, by considering that $\alpha_1 < 252$ in order to obtain $r < 0.07$ at $N = 60$ we have found an upper bound on the brane tension given by $\tau < 10^{-12}m_p^4$ and this constraint is well corroborated with Planck data, see Fig.1V.

We have also found that in the framework of braneworld inflation, the incorporation of the additional term in Friedmann’s equation affects substantially the reconstruction of the effective potential $V(\phi)$, considering the simplest attractors, such as $n_s(N) - 1 \propto N^{-1}$ or $r(N) \propto N^{-2}$. In this respect, we have shown that in order to obtain analytical solutions for the reconstruction of $V(\phi)$, the attractor $r(N)$ is an adequate methodology to be considered.
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