Geometry of Maurer-Cartan Elements on Complex Manifolds

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Abstract: The semi-classical data attached to stacks of algebroids in the sense of Kashiwara and Kontsevich are Maurer-Cartan elements on complex manifolds, which we call extended Poisson structures as they generalize holomorphic Poisson structures. A canonical Lie algebroid is associated to each Maurer-Cartan element. We study the geometry underlying these Maurer-Cartan elements in the light of Lie algebroid theory. In particular, we extend Lichnerowicz-Poisson cohomology and Koszul-Brylinski homology to the realm of extended Poisson manifolds; we establish a sufficient criterion for these to be finite dimensional; we describe how homology and cohomology are related through the Evens-Lu-Weinstein duality module; and we describe a duality on Koszul-Brylinski homology, which generalizes the Serre duality of Dolbeault cohomology.

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1. Introduction

Due to their close connection to mirror symmetry, noncommutative deformations of complex manifolds have recently generated increasing interest [5,20]. The Kashiwara-Kontsevich’s stacks of algebroids are one way of substantiating the abstract concept of quantum complex manifolds (or noncommutative deformations of complex manifolds) [6,7,16–18,20,35]. The quantization of the sheaf of holomorphic functions $\mathcal{O}_X$ of a complex manifold $X$ may no longer produce a sheaf of algebras but, instead, lead to a nonabelian gerbe over the complex manifold $X$ [6,34] or, in Kontsevich’s terminology, a stack of algebroids. Roughly speaking, an algebroid à la Kontsevich consists of an open cover $\{U_i\}_{i \in I}$ of the complex manifold $X$, a sheaf of associative unital algebras $\mathcal{A}_i$ on each $U_i$, an isomorphism of algebras $g_{ij} : \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ for each nonempty intersection $U_{ij}$, and an invertible element $a_{ijk} \in \Gamma(U_{ijk}, \mathcal{A}_i \wedge \mathcal{A}_j \wedge \mathcal{A}_k)$ for each triple intersection $U_{ijk}$. The isomorphisms $g_{ij}$ do not satisfy the usual cocycle condition. Instead, the equations $g_{ij} \circ g_{jk} \circ g_{ki} = \text{Ad}_{a_{ijk}}^{-1}$ are satisfied as well as other compatibility conditions (among which a “tetrahedron equation”). In the terminology of [25], an algebroid à la Kontsevich would be described as an extension of a Čech groupoid by algebras. A stack of algebroids can be thought of as a Morita equivalence class (see [25]) of algebroids. A canonical abelian category of coherent sheaves can be defined on a quantum complex manifold using its stack of algebroids description [16–18,20].

It is well known that the semi-classical data attached to quantum real manifolds (i.e. star-algebras) are Poisson structures [1,2]. The cotangent bundle of a real Poisson manifold $(M, \pi)$ is endowed with a canonical Lie algebroid structure denoted by $(\mathcal{T}^*M)_\pi$. This Lie algebroid structure plays a central role in Poisson geometry. For instance, the Lichnerowicz-Poisson cohomology is simply the Lie algebroid cohomology of $(\mathcal{T}^*M)_\pi$ with trivial coefficients. Evens-Lu-Weinstein discovered a procedure for constructing a canonical module over a given Lie algebroid. With the canonical module of $(\mathcal{T}^*M)_\pi$ at hand, they interpreted Koszul-Brylinski homology as a Lie algebroid cohomology. According to Kontsevich’s formality theorem and Tsygan’s chain formality theorem, the Hochschild cohomology and Hochschild homology of a star algebra are isomorphic to the Lichnerowicz-Poisson cohomology and Koszul-Brylinski homology of the underlying Poisson manifold.

In the context of complex geometry, the semiclassical data associated to quantum complex manifolds are solutions of the Maurer-Cartan equation in the derived global sections $R\Gamma(X, \wedge^\bullet \mathcal{T}X[1])$ of the sheaf of graded Lie algebras $\wedge^\bullet \mathcal{T}X[1]$ of polyvector fields on $X$, which, according to Kontsevich’s formality theorem, classify the deformations of stacks of algebroids up to gauge transformations [6,20,35]. More precisely, a Maurer-Cartan element is an

$$H = \pi + \theta + \omega \in \Omega^{0,0}(\wedge^2 \mathcal{T}^{1,0}X) \oplus \Omega^{0,1}(\wedge^1 \mathcal{T}^{1,0}X) \oplus \Omega^{0,2}(\wedge^0 \mathcal{T}^{1,0}X)$$
(where $\Omega^0, p(\wedge^q T^{1,0} X)$ denotes the space of $\wedge^q T^{1,0} X$-valued $(0, p)$-forms on $X$) satisfying the following equations:

$$\bar{\partial}\omega + [\omega, \theta] = 0, \quad \bar{\partial}\pi + [\theta, \pi] = 0,$$

$$\bar{\partial}\theta + [\omega, \pi] + \frac{1}{2}[\theta, \theta] = 0, \quad [\pi, \pi] = 0.$$

Holomorphic Poisson bivector fields are special cases of such Maurer-Cartan elements, as are holomorphic $(0, 2)$-forms. For this reason, complex manifolds endowed with such a Maurer-Cartan element $H$ will be called extended Poisson manifolds. In a recent paper [30], one of the authors studied the Koszul-Brylinski homology of holomorphic Poisson manifolds, and established a duality on it using the general theory developed by Evens-Lu-Weinstein [12].

In this paper, in order to study the geometry of extended Poisson manifolds, we apply the Evens-Lu-Weinstein theory to complex Lie algebroids. Indeed, considering Maurer-Cartan elements as Hamiltonian operators (in the sense of [26]) deforming a Lie bialgebroid [27], we define a complex Lie algebroid, which mimics the role played by the cotangent Lie algebroid in real Poisson geometry. It is not surprising that, for a holomorphic Poisson structure, this complex Lie algebroid is the derived Lie algebroid of the holomorphic cotangent Lie algebroid $(T^* X)_\pi$, i.e. the matched pair $T^{0,1} X \bowtie (T^* X)^{(1,0)}_{\pi}$ studied in [24,30]. Using this complex Lie algebroid, we introduce a Lichnerowicz-Poisson cohomology and a Koszul-Brylinski homology for extended Poisson manifolds, and study the relation between them. We extend the notion of coisotropic submanifolds of holomorphic Poisson manifolds to the “extended” setting. We give a criterion on the ellipticity of the complex Lie algebroid (in the sense of Block [4]) induced by a Maurer-Cartan element. And in the elliptic case, we obtain a duality, which we call Evans-Lu-Weinstein duality, on the Koszul-Brylinski homology groups. As was pointed out in [30] for the holomorphic Poisson case, this duality generalizes the Serre duality on Dolbeault cohomology.

Note that, modulo gauge equivalences, our extended Poisson structures and Yekutieli’s Poisson deformations (see [35]) are equivalent. It would be interesting to explore the connection between our results on Poisson homology and Berest-Etingof-Ginzburg’s [3]. It would also be interesting to investigate if one can extend the method in this paper to study the Bruhat-Poisson structures of Evens-Lu on flag varieties [11] and the toric Poisson structures of Caine [8].

2. Preliminaries

2.1. Lie bialgebroids. A complex Lie algebroid [32] consists of a complex vector bundle $A \to M$, a bundle map $a : A \to T^*_C M$ called anchor, and a Lie algebra bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$ such that $a$ induces a Lie algebra homomorphism from $\Gamma(A)$ to $\mathfrak{X}_C(M)$ and the Leibniz rule

$$[u, f v] = (a(u) f) v + f[u, v]$$

is satisfied for all $f \in C^\infty(M, \mathbb{C})$ and $u, v \in \Gamma(A)$.

It is well-known that a Lie algebroid $(A, [\cdot, \cdot], a)$ is equivalent to a Gerstenhaber algebra $(\Gamma(\wedge^* A), \wedge, [\cdot, \cdot])$ [33]. On the other hand, for a Lie algebroid structure on a
vector bundle $A$, there is also a degree 1 derivation $d$ of the graded commutative algebra $(\wedge^* A^*, \wedge)$ such that $d^2 = 0$. The differential $d$ is given by

$$(d\alpha)(u_0, u_1, \ldots, u_n) = \sum_{i=0}^{n} (-1)^i a(u_i)\alpha(u_0, \ldots, \widehat{u_i}, \ldots, u_n)$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([u_i, u_j], u_0, \ldots, \widehat{u_i}, \ldots, \widehat{u_j}, \ldots, u_n).$$

Indeed, a Lie algebroid structure on $A$ is also equivalent to a differential graded algebra $(\wedge^* A^*, \wedge, d)$.

Let $A \to M$ be a complex vector bundle. Assume that $A$ and its dual $A^*$ both carry Lie algebroid structures with anchor maps $a : A \to T_CM$ and $a_* : A^* \to T_CM$, brackets on sections $\Gamma(A) \otimes_C \Gamma(A) \to \Gamma(A) : u \otimes v \mapsto [u, v]$ and $\Gamma(A^*) \otimes_C \Gamma(A^*) \to \Gamma(A^*) : \alpha \otimes \beta \mapsto [\alpha, \beta]_*$, and differentials $d : \wedge^* A^* \to \wedge^* A^*$ and $d_* : \Gamma(\wedge^* A) \to \Gamma(\wedge^* A^*)$.

This pair of Lie algebroids $(A, A^*)$ is a Lie bialgebroid $[22, 28, 27]$ if $d_*$ is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^* A), \wedge, [\cdot, \cdot])$ or, equivalently, if $d$ is a derivation of the Gerstenhaber algebra $(\Gamma(\wedge^* A^*), \wedge, [\cdot, \cdot]_*)$. Since the bracket $[\cdot, \cdot]_*$ (resp. $[\cdot, \cdot]$) can be recovered from the derivation $d_*$ (resp. $d$), one is led to the following alternative definition.

**Proposition 2.1** ([33]). A Lie bialgebroid $(A, A^*)$ is equivalent to a differential Gerstenhaber algebra structure on $(\Gamma(\wedge^* A), \wedge, [\cdot, \cdot], d_*)$ (or, equivalently, on $(\Gamma(\wedge^* A^*), \wedge, [\cdot, \cdot]_*, d)$).

### 2.2. Hamiltonian operators

Let $(A, A^*)$ be a complex Lie bialgebroid, and $H \in \Gamma(\wedge^2 A)$. We now replace the differential $d_* : \Gamma(\wedge^* A) \to \Gamma(\wedge^* A^*)$ by a twist by $H$:

$$d_*^H : \Gamma(\wedge^* A) \to \Gamma(\wedge^* A^*)$$

$$d_*^H u = d_* u + [H, u].$$

It follows from a simple verification that if $H$ satisfies the Maurer-Cartan equation:

$$d_*^H H + \frac{1}{2} [H, H] = 0,$$

then $(d_*^H)^2 = 0$ and $(\Gamma(\wedge^* A), \wedge, [\cdot, \cdot], d_*^H)$ is again a differential Gerstenhaber algebra. Thus one obtains a Lie bialgebroid $(A, A_*^H)$. A solution $H \in \Gamma(\wedge^2 A)$ to Eq. (2) is called a **Hamiltonian operator** [26]. The Lie algebroid structure on $A_*^H$ can be described explicitly: the anchor and the Lie bracket are given, respectively, by

$$a_*^H = a_* + a \circ H^2$$

and

$$[\alpha, \beta]_*^H = [\alpha, \beta]_* + [\alpha, \beta]_H.$$

Here

$$[\alpha, \beta]_H = L_{H^2(\alpha)} \beta - L_{H^2(\beta)} \alpha - d_*([H^2(\alpha)] \beta),$$

for all $\alpha, \beta \in \Gamma(A^*)$. We shall use $A_*^H$ to denote such a Lie algebroid and call it the $H$-twisted Lie algebroid of $A^*$. Thus we obtain the following theorem, which was first proved in [26] by a different method.
Theorem 2.2. If \((A, A^*)\) constitutes a Lie bialgebroid, and \(H \in \Gamma(\wedge^2 A)\) is a Hamiltonian operator, then \((A, A^*_H)\) is a Lie bialgebroid.

3. Maurer-Cartan Elements

3.1. The Lie bialgebroid stemming from a complex manifold. We fix a complex manifold \(X\) of complex dimension \(n\) with almost complex structure \(J\). We regard the tangent bundle \(TX\) as a real vector bundle over \(X\). The complexification of \(TX\) is denoted \(T_CX\), namely: \(T_CX = TX \otimes \mathbb{C}\). Similarly, \(T_C^*X = T^*X \otimes \mathbb{C}\). Let \(\mathbb{J}: T_CX \to T_CX\) be the \(\mathbb{C}\)-linear extension of the almost complex structure \(J\), and \(T^{1,0}X\) and \(T^{0,1}X\) its \(+i\) and \(-i\) eigenbundles, respectively. We adopt the following notations:

\[
T^{p,q}X = \wedge^p T^{1,0}X \otimes \wedge^q T^{0,1}X, \\
(T^{p,q}X)^* = \wedge^p (T^{1,0}X)^* \otimes \wedge^q (T^{0,1}X)^*.
\]

Consider the following two vector bundles which are obviously mutually dual:

\[
A = T^{1,0}X \oplus (T^{0,1}X)^*, \quad A^* = T^{0,1}X \oplus (T^{1,0}X)^*.
\]  

(3)

We can endow \(A\) with a complex Lie algebroid structure. The anchor is the projection onto the first component:

\[
a\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^i} \quad a(d\bar{z}^j) = 0.
\]

The bracket of two sections of \(T^{1,0}X\) is their bracket as vector fields; the bracket of any pair of sections of \((T^{0,1}X)^*\) is zero; and the bracket of a holomorphic vector field (i.e. a holomorphic section of the holomorphic vector bundle \(T^{1,0}X\)) and an anti-holomorphic 1-form (i.e. an anti-holomorphic section of the holomorphic vector bundle \((T^{0,1}X)^*\)) is also zero. Thus

\[
\left[\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right] = 0, \quad [d\bar{z}^i, d\bar{z}^j] = 0, \quad \text{and} \quad \left[\frac{\partial}{\partial z^i}, d\bar{z}^j\right] = 0.
\]

Together with the Leibniz rule, the above three rules completely determine the bracket of any two arbitrary sections of \(A\). Similarly, one endows \(A^*\) with a complex Lie algebroid structure as well. It is simple to see that \((A, A^*)\) constitutes a Lie bialgebroid. Indeed \(A\) and \(A^*\) are transversal Dirac structures of the Courant algebroid \(T_CX \oplus T_C^*X\), for they are the eigenbundles of the generalized complex structure on \(X\) induced by its complex manifold structure \([15, 13]\). In the sequel we will use the symbols

\[
T^{1,0}X \rightsquigarrow (T^{0,1}X)^* \quad \text{and} \quad T^{0,1}X \rightsquigarrow (T^{1,0}X)^*
\]  

(4)

to refer to \(A\) and \(A^*\) when seen as Lie algebroids \([24]\).

Moreover, one has

\[
\wedge^k A \cong \bigoplus_{i+j=k} T^{i,0}X \otimes (T^{0,j}X)^*, \quad \wedge^k A^* \cong \bigoplus_{i+j=k} T^{0,i}X \otimes (T^{j,0}X)^*.
\]

The Lie algebroid differentials associated to the Lie algebroid structures on \(A^*\) and \(A\) are the usual \(\tilde{\partial}\)- and \(\partial\)-operators, respectively:
\[ d_* = \tilde{\partial} : \Omega^{0,i}(T^i,0) \to \Omega^{0,i+1}(T^i,0), \]
\[ d = \partial : \Omega^{i,0}(T^0,i) \to \Omega^{i+1,0}(T^0,i). \]

### 3.2. Extended Poisson structures.

**Definition 3.1.** An extended Poisson manifold \((X, H)\) is a complex manifold \(X\) equipped with an \(H \in \Gamma(\wedge^2 A)\) which is an Hamiltonian operator with respect to \((A, A^*)\), i.e.
\[
\tilde{\partial} H + \frac{1}{2}[H, H] = 0. \tag{5}
\]

In this case, \(H\) is called an extended Poisson structure.

Any \(H \in \Gamma(\wedge^2 A)\) decomposes as
\[
H = \pi + \theta + \omega,
\]
where \(\pi \in \Gamma(T^{2,0} X), \theta \in \Gamma(T^{1,0} X \otimes (T^{0,1} X)^*)\) and \(\omega \in \Gamma((T^{0,2} X)^*)\). We will use the following notations to denote the bundle maps induced by natural contraction:
\[
\theta^b : T^{0,1} X \to T^{1,0} X,
\]
\[
\theta^z : (T^{1,0} X)^* \to (T^{0,1} X)^*,
\]
\[
\pi^z : (T^{1,0} X)^* \to T^{1,0} X,
\]
\[
\omega^b : T^{0,1} X \to (T^{0,1} X)^*.
\]

Note that \(\theta^z = -(\theta^b)^*\).

The following lemma is immediate.

**Lemma 3.2.** An element \(H = \pi + \theta + \omega\) is an extended Poisson structure if and only if the following equations are satisfied:
\[
\tilde{\partial} \omega + [\omega, \theta] = 0, \tag{6}
\]
\[
\tilde{\partial} \theta + [\omega, \pi] + \frac{1}{2}[\theta, \theta] = 0, \tag{7}
\]
\[
\tilde{\partial} \pi + [\theta, \pi] = 0, \tag{8}
\]
\[
[\pi, \pi] = 0. \tag{9}
\]

**Remark 3.3.** When only one of the three terms of \(H\) is not zero, we are left with one of the following three special cases:

(a) \(H = \pi\) is an extended Poisson if and only if \(\pi\) is a holomorphic Poisson bivector field.

(b) \(H = \theta\) is an extended Poisson if and only if \(\tilde{\partial} \theta + \frac{1}{2}[\theta, \theta] = 0\). Moreover, if \(\theta^b \circ \theta^b - \text{id}\) is invertible, \(\theta\) is equivalent to a deformed complex structure [19].

(c) \(H = \omega\) is an extended Poisson if and only if \(\tilde{\partial} \omega = 0\).

In fact, if \([\omega, \pi] = 0\), Eq. (7) implies that \(\theta\) defines a deformed complex structure (under the assumption that \(\theta^b \circ \theta^b - \text{id}\) is invertible). Then, according to Lemma 3.15 below, Eq. (6) is equivalent to \(\tilde{\partial} \theta \omega = 0\), where \(\tilde{\partial} \theta = \tilde{\partial} + [\theta, \cdot]\), and Eqs. (8)–(9) mean that \(\pi\) is a holomorphic Poisson tensor with respect to the deformed complex structure.

**Corollary 3.4.** If \(H = \pi + \theta + \omega\) is an extended Poisson structure, then so is
\[
\lambda \pi + \theta + \lambda^{-1} \omega,
\]
for any $\lambda \in \mathbb{C}^\times$. In particular,

$$H^\vee = -\pi + \theta - \omega$$

is an extended Poisson structure.

Note that Maurer-Cartan elements as deformations of Lie bialgebroids or differential Gerstenhaber algebras were already considered by Cleyton-Poon [10] in their study of nilpotent complex structures on real six-dimensional nilpotent algebras.

A natural question is: when will $(A, A_H^*)$ arise from a generalized complex structure in the sense of Hitchin [15,13]? Let us recall the following:

**Lemma 3.5.** (Lemma 6.1 in [29]). The graph $\{H^\sharp \xi + \xi \in A \oplus A^*\}$ of $H$, which is clearly isomorphic to $A_H^*$ as a vector bundle, is the $+i$- (or $-i$-) eigenbundle of a generalized complex structure on $X$ if and only if $\overline{H^\sharp} \circ H^\sharp - \text{id}_{A^*}$ is invertible. Here the map $\overline{H^\sharp} : A \to A^*$ is defined by $\overline{H^\sharp}(u) = \overline{H^\sharp(\overline{u})}$, $\forall u \in A$.

Again we let $H = \pi + \theta + \omega$ be an extended Poisson structure on $X$. Relative to the direct sum decompositions of $A$ and $A^*$, the endomorphisms $H^\sharp$ and $\overline{H^\sharp}$ are represented by the block matrices

$$H^\sharp = \begin{pmatrix} \theta^\flat & \pi^\sharp \\ \omega^\flat & \theta^\sharp \end{pmatrix} \quad \text{and} \quad \overline{H^\sharp} = \begin{pmatrix} \bar{\theta}^\flat & \bar{\pi}^\sharp \\ \bar{\omega}^\flat & \bar{\theta}^\sharp \end{pmatrix}.$$  

In turn, we have

$$\overline{H^\sharp} H^\sharp = \begin{pmatrix} \bar{\theta}^\flat \circ \theta^\flat + \bar{\pi}^\sharp \circ \omega^\flat & \bar{\theta}^\flat \circ \pi^\sharp + \bar{\pi}^\sharp \circ \theta^\sharp \\ \bar{\omega}^\flat \circ \theta^\flat + \bar{\omega}^\sharp \circ \omega^\flat & \bar{\omega}^\flat \circ \pi^\sharp + \bar{\omega}^\sharp \circ \theta^\sharp \end{pmatrix}. \quad (10)$$

**Proposition 3.6.** Given an extended Poisson manifold $(X, H)$, let $A = T^{1,0}X \bowtie (T^{0,1}X)^*$. Then $A_H^*$ is the $(\pm i)$-eigenbundle of a generalized complex structure if and only if $\overline{H^\sharp} H^\sharp - \text{id}_{A^*}$ is invertible.

**Example 3.7.** If $H = \pi$ (i.e. $H$ is a holomorphic Poisson bivector field) or $H = \omega$, it is clear that $\overline{H^\sharp} H^\sharp$ is zero. Hence, in these two situations, the extended Poisson structure on $X$ is actually a generalized complex structure.

Here is a simple example of extended Poisson structure, which does not arise from a generalized complex structure.

**Example 3.8.** Consider the torus $T = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with its standard complex structure. Let $z$ be the standard coordinate on $T$. Obviously, any

$$\theta = f(z, \bar{z}) \frac{d}{dz} \wedge d\bar{z}, \quad (11)$$

where $f$ is a smooth $\mathbb{C}$-valued function, is an extended Poisson structure. In this case, $\overline{H^\sharp} H^\sharp = |f|^2 \text{id}$. Hence $A^*_\theta$ does not stem from a generalized complex structure provided that $|f| = 1$. 

3.3. Elliptic Lie algebroids. As in [4], we say that a complex Lie algebroid $B$ is elliptic if $\text{Re} \circ a_B : B \to TX$ is surjective. Here $a_B : B \to T_CX$ is the anchor map of $B$ and $\text{Re} : T_CX \to TX$ is the projection onto the real part.

**Theorem 3.9 ([4]).** If $B$ is an elliptic Lie algebroid over a compact complex manifold $X$, and $E$ a finite rank complex vector bundle with a $B$-action as in [12], then all cohomology groups $H^\bullet(B, E)$ are finite dimensional.

It is therefore natural to ask when $A^*_H$ is elliptic. An easy calculation shows the following:

**Proposition 3.10.** Let $a^*_H$ denote the anchor of $A^*_H$ and $C : T^{0,1}X \to T^{1,0}X$ the complex conjugation. The bundle maps $\text{Re} \circ a^*_H$ and $F = (C + \theta^\flat) \oplus \pi^\flat : T^{0,1}X \oplus (T^{1,0}X)^* \to T^{1,0}X$,

![Diagram](13)

and the isomorphism of real vector bundles $\text{Re} : T^{1,0}X \to TX$ fit into the commutative diagram

As a consequence, $A^*_H$ is an elliptic Lie algebroid if and only if $F$ is surjective.

**Example 3.11.** When $H = \pi$, or $\omega$, it is clear that $A^*_H$ is elliptic. On the other hand, if we consider the torus $T$ endowed with the bivector field $\theta$ of Example 3.8, the Lie algebroid $A^*_H$ is elliptic if and only if $f$ is not identically 1.

3.4. Poisson cohomology.

**Definition 3.12.** Given an extended Poisson manifold $(X, H)$, the cohomology of the Lie algebroid $A^*_H$ is called the Poisson cohomology of the extended Poisson structure, and denoted $H^\bullet(X, H)$. In other words, it is the cohomology of the cochain complex:

$$
\cdots \to \Gamma(\wedge^k A) \xrightarrow{\delta_H} \Gamma(\wedge^{k+1} A) \xrightarrow{\delta_H} \cdots,
$$

where $\Gamma(\wedge^k A) = \oplus_{i+j=k} \Omega^{0,i}(T^{1,0}X)$ and $\delta_H = \bar{\delta} + [H, \cdot]$.

Poisson cohomology is also called tangent cohomology by Kontsevich [21].

As an immediate consequence of Theorem 3.9 and Proposition 3.10, we have

**Corollary 3.13.** If $H$ is an extended Poisson structure on a compact complex manifold $X$ and the map $F$ (given by Eq. (12)) is surjective, then all Poisson cohomology groups are finite dimensional.

**Remark 3.14.** When $H$ is a holomorphic Poisson bivector field $\pi$, the cochain complex (14) is the total complex of the double complex as discussed in Corollary 4.26 in [24].

On the other hand, if $H = \theta \in \Omega^{0,1}(T^{1,0}X)$ is a Maurer-Cartan element such that $\bar{\theta}^\flat \circ \theta^\flat - \text{id}$ is invertible, then $\theta$ defines a new complex structure on $X$ according to Kodaira [19].
The following lemma can be verified directly.

**Lemma 3.15.** Let $H = \theta \in \Omega^{0,1}(T^{1,0}X)$ be a Maurer-Cartan element such that $\theta^b \circ \theta^b - \text{id}$ is invertible. Then the Lie algebroid $A^*_H$ is isomorphic to $T^{1,0}_\theta X \bowtie (T^{0,1}_\theta X)^*$, where $T^{1,0}_\theta X$ and $T^{0,1}_\theta X$ are, respectively, the $+i$ and $-i$ eigenbundles of the deformed almost complex structure $J_\theta : TX \to TX$. As a consequence, the differential operator $d_H^*$ in Eq. (1) is equal to $\overline{\partial}_\theta$, the new $\overline{\partial}$-operator of the deformed complex structure.

Thus we have

**Proposition 3.16.** If $H = \theta \in \Omega^{0,1}(T^{1,0}X)$ is a Maurer-Cartan element such that $\theta^b \circ \theta^b - \text{id}$ is invertible, then

$$H^k(X, H) \cong \bigoplus_{i+j=k} H^i(X, \wedge^j T_\theta X),$$

where $T_\theta X$ denotes the holomorphic tangent bundle of the deformed complex manifold $X$.

### 3.5. Coisotropic submanifolds

Suppose that $Y \subseteq X$ is a complex submanifold [19]. Set

$$N^{1,0}Y = \left\{ \xi \in (T^{1,0}X|_Y)^* \text{ s.t. } \langle \xi|Y \rangle = 0, \forall Y \in T^{1,0}Y \right\},$$

and consider the subbundle $K = T^{0,1}Y \oplus N^{1,0}Y$ of $A^*$.

**Definition 3.17.** A complex submanifold $Y$ of $X$ is called coisotropic if $H(u, v) = 0$, for all $u, v \in K$.

**Example 3.18.** If $H = \pi$ is a holomorphic Poisson bivector field, then $Y$ is coisotropic if and only if it is coisotropic in the usual sense, i.e. $\pi(\xi_1, \xi_2) = 0$, $\forall \xi_1, \xi_2 \in N^{1,0}Y$, or $\pi^c(N^{1,0}Y) \subseteq T^{1,0}Y$.

**Example 3.19.** If $H = \omega$, then $Y$ is coisotropic if and only if $\iota^*\omega = 0$, where $\iota : Y \to X$ is the embedding map.

**Example 3.20.** If $H = \theta$, then $Y$ is coisotropic if and only if $\theta^b(T^{0,1}Y) \subseteq T^{1,0}Y$.

It is well known that given a coisotropic submanifold $C$ of a real Poisson manifold $(P, \pi)$, the conormal bundle $NC = \left\{ \xi \in T^*_cP \text{ s.t. } c \in C; \langle \xi|X \rangle = 0, \forall X \in T_cC \right\}$ is a Lie subalgebroid of the cotangent Lie algebroid $(T^*P)\pi$ [31]. The following proposition can be considered as an analogue of this fact in the extended Poisson setting.

**Proposition 3.21.** Let $Y$ be a coisotropic submanifold of the extended Poisson manifold $(X, H)$. Then the vector subbundle $K = T^{0,1}Y \oplus N^{1,0}Y$ is a Lie subalgebroid of $A^*_H$. That is, $a^H_\xi$ maps $K$ into $T_\xi Y$ and for any smooth extensions $\bar{u}, \bar{v} \in \Gamma(A^*_H)$ to $X$ of any two sections $u, v \in \Gamma(K)$, the restriction to $Y$ of $[\bar{u}, \bar{v}]_\xi^H$ is a section of $K$ which does not depend on the choice of extensions.
3.6. Poisson relations. Following Weinstein [31], we introduce the following

**Definition 3.22.** Let \((X_1, H_1)\) and \((X_2, H_2)\) be extended Poisson manifolds. A Poisson relation from \((X_2, H_2)\) to \((X_1, H_1)\) is a coisotropic submanifold of the product manifold \(X_1 \times X_2^\vee\) (i.e. \(X_1 \times X_2\) endowed with the extended Poisson structure \((H_1, H_2^\vee)\), see Corollary 3.4).

We call a holomorphic map \(f : X_2 \to X_1\) between extended Poisson manifolds \((X_1, H_1)\) and \((X_2, H_2)\) an **extended Poisson map** if its graph

\[
G_f = \{(f(x), x) \text{ s.t. } x \in X_2\} \subset X_1 \times X_2^\vee
\]

is a Poisson relation.

**Proposition 3.23.** Let \((X_1, H_1)\) and \((X_2, H_2)\) be extended Poisson manifolds, where the extended Poisson structures decompose as \(H_i = \pi_i + \theta_i + \omega_i\) \((i = 1, 2)\). Then a holomorphic map \(f : X_2 \to X_1\) is an extended Poisson map if and only if \(f_*\pi_2 = \pi_1\); \(f^*\omega_1 = \omega_2\); and \(f_* \circ \theta_2 = \theta_1 \circ f_*\).

The proof is a direct verification and is left to the reader. As a consequence, we have

**Corollary 3.24.** The composition of two extended Poisson maps is again an extended Poisson map.

4. Koszul-Brylinski Poisson Homology

In this section we will introduce homology groups for extended Poisson manifolds based on the Evens-Lu-Weinstein module of a Lie algebroid.

4.1. Koszul-Brylinski cochain complex. First we recall the notion of Clifford algebras and spin representation. Let \(V\) be a vector space of dimension \(n\) endowed with a non-degenerate symmetric bilinear form \((\cdot, \cdot)\). Its Clifford algebra \(\mathcal{C}(V)\) is defined as the quotient of the tensor algebra \(\bigoplus_{k=0}^n V \otimes^k V\) by the relations \(x \otimes y + y \otimes x = 2(x, y)\), with \(x, y \in V\). It is naturally an associative \(\mathbb{Z}_2\)-graded algebra. Up to isomorphisms, there exists a unique irreducible module \(S\) of \(\mathcal{C}(V)\) called spin representation [9]. The vectors of \(S\) are called spinors.

An operator \(O\) on \(S\) is called even (or of degree 0) if \(O(S^i) \subset S^i\) and odd (or of degree 1) if \(O(S^i) \subset S^{i+1}\). Here \(i \in \mathbb{Z}_2\). If \(O_1\) and \(O_2\) are operators of degree \(d_1\) and \(d_2\) respectively, then their commutator is the operator

\[
[O_1, O_2] = O_1 \circ O_2 - (-1)^{d_1d_2} O_2 \circ O_1.
\]

**Example 4.1.** Let \(W\) be a vector space of dimension \(r\). We can endow \(V = W \oplus W^*\) with the non-degenerate pairing

\[
(u_1 + \xi_1, u_2 + \xi_2) = \frac{1}{2} \left( (\xi_1(u_2) + \xi_2(u_1)) \right),
\]

where \(u_1, u_2 \in W\) and \(\xi_1, \xi_2 \in W^*\). The representation of \(\mathcal{C}(V)\) on \(S = \bigoplus_{k=0}^r \wedge^k W\) defined by \(u \cdot w = u \wedge w\) and \(\xi \cdot w = \iota_\xi w\), where \(u \in W, \xi \in W^*\) and \(w \in S\), is the spin representation. Note that \(S\) is \(\mathbb{Z}\)- and thus also \(\mathbb{Z}_2\)-graded.

Recall that \(E = T_CX \oplus T_C^*X\) admits the standard pseudo-metric

\[
(X_1 + \xi_1, X_2 + \xi_2) = \frac{1}{2} \left( \langle \xi_1 | X_2 \rangle + \langle \xi_2 | X_1 \rangle \right),
\]
where $X_i \in T_C X$ and $\xi_i \in T^*_C X$. The corresponding Clifford bundle $\mathcal{C}(E)$ can be identified with the vector bundle $(\wedge^* T_C X) \otimes (\wedge^* T^*_C X)$, under which the Clifford action of $\mathcal{C}(E)$ on the spinor bundle

$$\wedge^* T^*_C X = \bigoplus_{p,q} (T^{p,q} X)^*$$

is given by

$$(W \otimes \xi) \cdot \lambda = (-1)^{\frac{w(w-1)}{2}} \iota_W (\xi \wedge \lambda).$$

Here $W \in \wedge^w T_C X$, $\xi, \lambda \in \wedge^\bullet T^*_C X$, and the symbol $\iota_W$ denotes the standard contraction

$$\langle \iota_W \xi | X \rangle = \langle \xi | W \wedge X \rangle,$$

for $\xi \in \wedge^p T^*_C X$ and $X \in \wedge^{n-p} T_C X$ with $p \geq w$.

Let $(X, \mathcal{H})$ be an extended Poisson manifold of complex dimension $n$. Then $A^*_H$ is a Lie algebroid and the Evens-Lu-Weinstein module [12] of $A^*_H$ is the complex line bundle

$$Q_{A^*_H} = \wedge^{2n} A^*_H \otimes \wedge^{2n} T^*_C X.$$

The representation of $A^*_H$ on $Q_{A^*_H}$ is given by

$$\nabla^H_{\alpha} (\alpha_1 \wedge \cdots \wedge \alpha_{2n} \otimes \mu) = \sum_{i=1}^{2n} \left( \alpha_1 \wedge \cdots \wedge [\alpha, \alpha_i]^H \wedge \cdots \wedge \alpha_{2n} \otimes \mu \right)$$

$$+ \alpha_1 \wedge \cdots \wedge \alpha_{2n} \otimes L^{H}_{\alpha}(\alpha) \mu,$$

where $\alpha, \alpha_1, \ldots, \alpha_{2n} \in \Gamma(A^*_H)$, $\mu \in \Gamma(\wedge^{2n} T^*_C X)$.

A simple computation yields that $Q_{A^*_H} \cong \wedge^n (T^{1,0} X)^* \otimes \wedge^n (T^{0,1} X)^*$. Accordingly,

$$\mathcal{L} = Q_{A^*_H}^{\frac{1}{2}} \cong \wedge^n (T^{1,0} X)^* = (T^{n,0} X)^*$$

is also an $A^*_H$-module and we use $\nabla^H$ again to denote the representation. Equivalently, we have an operator

$$D^H : \Gamma(\mathcal{L}) \rightarrow \Gamma(A \otimes \mathcal{L}),$$

such that

$$\iota_{\alpha} D^H s = \nabla^H_{\alpha} s, \quad \forall \alpha \in \Gamma(A^*), \quad s \in \Gamma(\mathcal{L}),$$

which allows us to define a differential operator

$$\tilde{d}^H_* : \Gamma(\wedge^k A \otimes \mathcal{L}) \rightarrow \Gamma(\wedge^{k+1} A \otimes \mathcal{L})$$

by

$$\tilde{d}^H_*(u \otimes s) = (\tilde{d}^H u) \otimes s + (-1)^k u \wedge D^H s,$$

for all $u \in \Gamma(\wedge^k A)$ and $s \in \Gamma(\mathcal{L})$.

The following lemma is needed later.
Lemma 4.2. The relation
\[ \tau(X \otimes s) = X \cdot s, \]
where in the r.h.s. \( X \in \wedge^k A \) is regarded as an element of the Clifford algebra \( \mathbb{C}(E) \) and \( s \in \mathcal{L} \) is regarded as an element in \( \wedge^\bullet T^*_C X \), defines an isomorphism of vector bundles
\[ \tau : \wedge^k A \otimes \mathcal{L} \to \bigoplus_{i-j=n-k} (T^{i,j} X)^*. \]
Equivalently,
\[ \tau ((W \wedge \xi) \otimes s) = (-1)^{w(w-1)/2} \iota_W (\xi \wedge s) = (-1)^{w(w-1)+n(k-w)} (\iota_W s) \wedge \xi, \]
for \( W \in T^{w,0} X, \xi \in (T^{0,k-w} X)^* \) and \( s \in \mathcal{L} \).
We define the inner product of \( H \in \Gamma(\wedge^2 A) \) with \( \lambda \in \Gamma(\wedge^\bullet T^*_C X) \) as
\[ \iota_H \lambda = -H \cdot \lambda. \]
This coincides with the usual inner product of bivector fields with differential forms.
Introduce
\[ [\partial, \iota_H] = \partial \circ \iota_H - \iota_H \circ \partial : \Gamma(\wedge^\bullet T^*_C X) \to \Gamma(\wedge^\bullet T^*_C X). \]
Let us denote \( \Omega^{i,j}(X) = \Gamma((T^{i,j} X)^*) \). The following theorem is the main result in this section.

Theorem 4.3. The diagram
\[ \begin{CD}
\Gamma(\wedge^k A \otimes \mathcal{L}) @>{\tilde{\partial}^H}>> \Gamma(\wedge^{k+1} A \otimes \mathcal{L})
\end{CD} \]
\[ \begin{CD}
\bigoplus_{i-j=n-k} \Omega^{i,j}(X) @>{\bar{\partial} + [\partial, \iota_H]}>> \bigoplus_{i-j=n-k-1} \Omega^{i,j}(X)
\end{CD} \]
(17)
commutes.

Definition 4.4. The cohomology of the cochain complex \( \bigoplus_{i-j=n-k} \Omega^{i,j}(X) \) is called the \textbf{Koszul-Brylinski Poisson homology} of the extended Poisson manifold \((X, H)\), and denoted \( H_* (X, H) \).

Remark 4.5. (a) If \( H = \pi \) is a holomorphic Poisson bivector field, the cochain complex \( \bigoplus_{i-j=n-k} \Omega^{i,j}(X) \) is the total complex of a double complex. Its cohomology is the usual Koszul-Brylinski Poisson homology of a holomorphic Poisson manifold, as studied in detail by one of the authors [30].

(b) If \( H = \omega \in \Omega^{0,2}(X) \) with \( \bar{\partial} \omega = 0 \), the complex \( \bigoplus_{i-j=n-k} \Omega^{i,j}(X), \bar{\partial} + [\partial, \iota_H] \) becomes \( \bigoplus_{i-j=n-k} \Omega^{i,j}(X), \bar{\partial} + (\partial \omega) \wedge \). Its cohomology is the twisted Dolbeault cohomology.

(c) If \( H = \theta \in \Omega^{0,1}(T^{1,0} X) \) is a Maurer-Cartan element such that \( \bar{\partial}^b \circ \theta^b - \text{id} \) is invertible, then \( \theta \) defines a new complex structure on \( X \). According to Lemma 3.15, the cochain complex \( \bigoplus_{i-j=n-k} \Omega^{i,j}(X), \bar{\partial} + [\partial, \iota_H] \) is isomorphic to \( \bigoplus_{i-j=n-k} \Omega^{i,j}_\theta(X), \bar{\partial} \), where \( \bar{\partial} \) is the \( \bar{\partial} \)-Dolbeault operator of the deformed complex structure. As a consequence, we have \( H_k(X, \theta) \cong \bigoplus_{i-j=n-k} H^{i,j}_\theta(X) \), where \( H^{i,j}_\theta(X) \) is the Dolbeault cohomology of the deformed complex structure.
4.2. Evens-Lu-Weinstein duality. Consider a compact complex (and therefore orientable) manifold $X$ with $\dim_{\mathbb{C}} X = n$, a complex Lie algebroid $B$ over $X$ with $\text{rk}_{\mathbb{C}} B = r$. According to [12], the complex line bundle $Q_B = \wedge^{r} B \otimes \wedge^{2n} T^*_{\mathbb{C}} X$ is a module over the complex Lie algebroid $B$. If $Q^\frac{1}{2}$ exists as a complex vector bundle, $Q^\frac{1}{2}$ becomes a $B$-module as well. There is a natural map

$$\phi : \Gamma(\wedge^k B^* \otimes Q^\frac{1}{2}_B) \otimes \Gamma(\wedge^{r-k} B^* \otimes Q^\frac{1}{2}_B) \to \Gamma(\wedge^r B^* \otimes Q_B) \cong \Gamma(\wedge^{2n} T^*_{\mathbb{C}} X).$$

Integrating, we get the pairing

$$\Gamma(\wedge^k B^* \otimes Q^\frac{1}{2}_B) \otimes \Gamma(\wedge^{r-k} B^* \otimes Q^\frac{1}{2}_B) \to \mathbb{C}, \quad \xi \otimes \eta \mapsto \int_X \phi(\xi \otimes \eta).$$

(18)

The following result is essentially due to Evens-Lu-Weinstein [12] for the pairing, and to Block [4] for the non-degeneracy (see also [30]).

**Theorem 4.6.** For a complex Lie algebroid $B$, with $\text{rk}_{\mathbb{C}} B = r$, over a compact manifold $X$, the pairing (18) induces a pairing

$$H^k(B, Q^\frac{1}{2}_B) \otimes H^{r-k}(B, Q^\frac{1}{2}_B) \to \mathbb{C}.$$ 

Moreover, if $B$ is an elliptic Lie algebroid, this pairing is non-degenerate.

Let $(X, H)$ be a compact extended Poisson manifold of complex dimension $n$. Consider the Lie algebroid $B = (T^{0,1} X \rightrightarrows (T^{1,0} X)^*)_H$. Applying Theorem 4.6 and Proposition 3.10, we obtain

**Theorem 4.7.** Let $(X, H)$ be a compact extended Poisson manifold of complex dimension $n$, with $H = \pi + \theta + \omega$. Then the map

$$\Omega^{i,j}(X) \otimes \Omega^{k,l}(X) \to \mathbb{C} : \zeta \otimes \eta \mapsto \int_X (\zeta \wedge \eta)^{top}$$

induces a pairing on the Koszul-Brylinski Poisson homology:

$$H_k(X, H) \otimes H_{2n-k}(X, H) \to \mathbb{C}.\quad (19)$$

Moreover, if the bundle map $F = (\mathbb{C} + \theta^0) \oplus \pi^*$ maps $T^{0,1} X \oplus (T^{1,0} X)^*$ surjectively onto $T^{1,0} X$, then all homology groups $H_\bullet(X, H)$ are finite dimensional vector spaces and the pairing (19) is non-degenerate.

4.3. Proof of Theorem 4.3. The following lemmas are needed.

**Lemma 4.8.** For any $u \in \Gamma(\wedge^p A), \lambda \in \Omega^{\bullet, \bullet}(X)$, one has

$$\bar{\partial}(u \cdot \lambda) = (\bar{\partial} u) \cdot \lambda + (-1)^p u \cdot \bar{\partial} \lambda.$$ 

(20)

**Lemma 4.9.** For any $u \in \Gamma(\wedge^p A), v \in \Gamma(\wedge^q A)$, the Schouten bracket $[u, v]$ is determined by

$$[u, v] \cdot \lambda = (-1)^{q+1} [u, [v, \partial]] \lambda, \quad \forall \lambda \in \Omega^{\bullet, \bullet}(X).\quad (21)$$

Both lemmas can be proved by induction; this is left to the reader.
Lemma 4.10. For any \( u \in \Gamma(\wedge^i A) \) and \( \lambda \in \Omega^{\bullet,\bullet}(X) \), one has
\[
[\partial, \iota_H](u \cdot \lambda) = [H, u] \cdot \lambda + (-1)^i u \cdot ([\partial, \iota_H] \lambda).
\] (22)

In particular, for any smooth function \( f \in \mathcal{C}^\infty(X, \mathbb{C}) \), one has
\[
[\partial, \iota_H](f \lambda) = [H, f] \cdot \lambda + f[\partial, \iota_H] \lambda.
\] (23)

Proof. According to Eq. (21), we have
\[
[H, u] \cdot \lambda = (-1)^i u \cdot ([\partial, \iota_H] \lambda) = (-1)^i (u \cdot (\partial(H \cdot \lambda)) - H \cdot u\cdot (\partial \lambda)) + (H \cdot (\partial(u \cdot \lambda))) - \partial(u \cdot H \cdot \lambda))
\]
\[
= (-1)^i (u \cdot (\partial(H \cdot \lambda)) - u \cdot H \cdot (\partial \lambda)) + (H \cdot (\partial(u \cdot \lambda))) - \partial(H(u \cdot \lambda))
\]
\[
= -(-1)^i u \cdot ([\partial, \iota_H] \lambda) + [\partial, \iota_H](u \cdot \lambda).
\]
\[\square\]

A straightforward (though lengthy) computation shows the following:

Lemma 4.11. Suppose that \((z^1, \ldots, z^n)\) is a local holomorphic chart and \( H = \pi + \theta + \omega \) is given by
\[
H = \pi^{i,j} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} + \theta^p_q \frac{\partial}{\partial z^p} \wedge dz^q + \omega_{k, l} d\bar{z}^k \wedge d\bar{z}^l,
\] (24)

where \(\pi^{i,j}, \theta^p_q\), and \(\omega_{k, l}\) are complex valued smooth functions on \(X\). Then the \(H\)-twisted Lie algebroid structure on \(A_H^* \cong T^{0,1}X \oplus (T^{1,0}X)^*\) can be expressed by:
\[
a^H_a \left( \frac{\partial}{\partial z^i} \right) = \frac{\partial}{\partial z^i} - \theta^p_i \frac{\partial}{\partial z^p}, \quad a^H_{dz^j} = 2\pi^{i,q} \frac{\partial}{\partial z^q},
\] (25)
\[
\left[ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right]^H_s = 2\partial \omega_{i,j}, \quad \left[ dz^i, dz^j \right]^H_s = 2\partial \pi^{i,j}, \quad \left[ dz^i, \frac{\partial}{\partial z^j} \right]^H_s = \partial \bar{\theta}^j_i.
\] (26)

Lemma 4.12. Making the same assumptions as in Lemma 4.11, consider the local section
\[
s = dz^1 \wedge \cdots \wedge dz^n
\] (27)

of \(\mathcal{L} = Q^1_{A_H^*}\). The representation of \(A_H^*\) on \(\mathcal{L}\) is given by
\[
\nabla^H_{\frac{\partial}{\partial z^i}} s = -\frac{\partial \theta^p_i}{\partial z^p} s, \quad \nabla^H_{dz^j} s = 2\frac{\partial \pi^{i,p}}{\partial z^p} s.
\] (28)

Proof. Using Eq. (25), we compute
\[
L_{a^H_{dz^j}} dz^i = -d\theta^i_j, \quad L_{a^H_{\frac{\partial}{\partial z^i}}} dz^j = 0,
\] (29)
\[
L_{a^H_{\frac{\partial}{\partial z^i}}} dz^j = 2d\pi^{i,j}, \quad L_{a^H_{dz^j}} dz^i = 0.
\]

Write
\[
s^2 = (\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n} \wedge dz^1 \wedge \cdots \wedge dz^n) \otimes (dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n).
\]
Then, using Eqs. (26) and (29), one obtains
\[ \nabla_H \partial s^2 = -2 \frac{\partial \theta_i^p}{\partial z^p} s^2, \quad \nabla_H dz^s s^2 = 4 \frac{\partial \pi^{i,p}}{\partial z^p} s^2. \]
The conclusion thus follows immediately. □

**Corollary 4.13.** Locally, the operator \( D^H \) in Eq. (15) is given by
\[ D^H s = \left( 2 \frac{\partial \pi^{i,p}}{\partial z^p} \frac{\partial}{\partial z^i} - \frac{\partial \theta_i^p}{\partial z^p} dz^i \right) \otimes s, \] (30)
where \( s \) is defined in Eq. (27).

We are now ready to prove Theorem 4.3.

**Proof of Theorem 4.3.** We adopt an inductive approach. First we prove the commutativity of Diagram (17) for \( k = 0 \).

Note that for any \( f \in C^\infty(X, \mathbb{C}) \), \( u \in \Gamma(\land^k A) \) and \( s \in \Gamma(L) \), one has
\[ \tau \tilde{d}_s^H (fu \otimes s) = \tau (f \tilde{d}_s^H (u \otimes s) + ([\tilde{\partial}, f] + [H, f]) \wedge u \otimes s) \] by Eq. (16)
\[ = f \tau \tilde{d}_s^H (u \otimes s) + (\tilde{\partial} f + [H, f]) \cdot \tau (u \otimes s). \]

On the other hand, if we write \( \lambda = \tau (u \otimes s) \), one has
\[ (\tilde{\partial} + [\partial, \iota_H]) \tau (fu \otimes s) \]
\[ = (\tilde{\partial} + [\partial, \iota_H]) (f \lambda) \]
\[ = f (\tilde{\partial} + [\partial, \iota_H]) \tau (u \otimes s) + (\tilde{\partial} f + [H, f]) \cdot \tau (u \otimes s). \]

It thus follows that the map \( \tau \circ \tilde{d}_s^H - (\tilde{\partial} + [\partial, \iota_H]) \circ \tau \) is \( C^\infty(X) \)-linear. Take a local holomorphic chart \((z^1, \ldots, z^n)\) and write \( H \) locally as in Eq. (24) in Lemma 4.11. Again take \( s \) as in Eq. (27). For \( k = 0 \), we have \( \tilde{d}_s^H s = D^H s \), which is given locally by Eq. (30). Then, we compute
\[ \tau (\tilde{d}_s^H s) = \left( 2 \frac{\partial \pi^{i,p}}{\partial z^p} \frac{\partial}{\partial z^i} - \frac{\partial \theta_i^p}{\partial z^p} dz^i \right) \cdot (dz^1 \wedge \cdots \wedge dz^n) \]
\[ = 2 \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial \pi^{i,p}}{\partial z^p} dz^1 \wedge \cdots \wedge \hat{dz}^i \wedge \cdots \wedge dz^n - \frac{\partial \theta_i^p}{\partial z^p} d\hat{dz}^i \wedge dz^1 \wedge \cdots \wedge dz^n. \]
Thus we have
\[ (\tilde{\partial} + [\partial, \iota_H]) s = \partial \iota_H (dz^1 \wedge \cdots \wedge dz^n) \]
\[ = \partial \left( 2 \sum_{i<j} (-1)^{i+j-1} \pi^{i,j} dz^1 \wedge \cdots \wedge \hat{dz}^i \wedge \cdots \wedge \hat{dz}^j \wedge \cdots \wedge dz^n \right). \]
\[
+ \sum_{p=1}^{n} (-1)^{p+1} \theta_i^p d\bar{z}^i \wedge dz^1 \wedge \cdots \wedge dz^p \wedge \cdots \wedge dz^n \\
- \omega_{k,l} d\bar{z}^k \wedge d\bar{z}^l \wedge dz^1 \wedge \cdots \wedge dz^n \right)
\]

\[= \tau(\tilde{d}_H s).\]

It thus follows that Diagram (17) indeed commutes when \(k = 0\).

Now assume that we have proved the commutativity of Diagram (17) when \(k \leq m\) (where \(0 \leq m \leq 2n - 1\)). To prove the \(k = m+1\) case, we consider a section \((u \wedge w) \otimes s \in \Gamma(\wedge^{m+1} A \otimes \mathcal{L})\), where \(u \in \Gamma(A), w \in \Gamma(\wedge^m A)\) and \(s \in \Gamma(\mathcal{L})\). Then

\[
(\tilde{\partial} + [\partial, \iota_H])(u \wedge w) \otimes s
\]

\[= (\tilde{\partial} + [\partial, \iota_H])(u \cdot \lambda) \quad \text{where } \lambda = w \cdot s
\]

\[= \tilde{\partial} u \cdot \lambda - u \cdot \tilde{\partial} \lambda + [H, u] \cdot \lambda - u \cdot ([\partial, \iota_H] \lambda) \quad \text{by Eqs. (20) and (22)}
\]

\[= \tilde{\partial} H u \cdot \lambda - u \cdot (\tilde{\partial} + [\partial, \iota_H]) \lambda
\]

\[= \tau \left( (\tilde{\partial} H u \wedge w) \otimes s \right) - u \cdot \tau \tilde{d}_H^*(w \otimes s) \quad \text{by assumption}
\]

\[= \tau \tilde{d}_H^*((u \wedge w) \otimes s).
\]

This concludes the proof. \(\Box\)

### 4.4. Modular classes.

The modular class of a Lie algebroid was introduced by Evens-Lu-Weinstein [12]. The following version for complex Lie algebroids appeared in the preprint version of [12] but not in the published paper. It is also implied in [14]. The presentation which we give below was communicated to us by Camille Laurent-Gengoux [23].

Let \(B\) be a complex Lie algebroid over a real manifold \(M\), with \(\text{rk}_C B = r\) and \(\text{dim } M = m\). Its Evens-Lu-Weinstein module is \(Q_B = \wedge^r B \otimes \wedge^m T^*_C M\).

Consider the complex of sheaves

\[
\tilde{S}^0 \xrightarrow{\tilde{d}_B} S^1 \xrightarrow{d_B} S^2 \cdots \xrightarrow{d_B} S^r, \tag{31}
\]

where \(\tilde{S}^0\) is the sheaf of nowhere vanishing smooth complex valued functions on \(M\); \(S^\bullet\) is the sheaf of sections of \(\wedge^\bullet B^\bullet\); \(d_B\) is the usual Lie algebroid cohomology differential; and \(\tilde{d}_B f = d_B \log f = \frac{d_B f}{f}\), for all \(f \in C^\infty(U, \mathbb{C}^\times)\), where \(U\) is an arbitrary open subset of \(M\). We denote its hypercohomology by \(\tilde{H}^\bullet(B, \mathbb{C})\). Note that in Eq. (31), if we replace \(\tilde{S}^0\) by \(S^0\), the sheaf of smooth complex valued functions on \(M\), and \(\tilde{d}_B\) by the usual Lie algebroid differential \(d_B\), the hypercohomology of the resulting complex of sheaves

\[
S^0 \xrightarrow{d_B} S^1 \xrightarrow{d_B} S^2 \cdots \xrightarrow{d_B} S^r, \tag{32}
\]

is isomorphic to the usual Lie algebroid cohomology \(H^\bullet(B, \mathbb{C})\) of the complex Lie algebroid \(B\) with trivial coefficients \(\mathbb{C}\) since each \(S^\bullet\) is a soft sheaf. The exponential sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow S \rightarrow \tilde{S} \rightarrow 0,
\]
where \( \mathcal{S} \) (resp. \( \mathcal{S}^\prime \)) stands for the complex of sheaves (32) (resp. (31)) and the locally constant sheaf \( \mathbb{Z} \) is regarded as a complex of sheaves concentrated in degree 0, induces the long exact sequence

\[
\cdots \to H^i(M, \mathbb{Z}) \to H^i(B, \mathbb{C}) \to \tilde{H}^i(B, \mathbb{C}) \to H^{i+1}(M, \mathbb{Z}) \to \cdots.
\]

Note that \( \tilde{H}^*(B, \mathbb{C}) \) can be computed as the total cohomology of the Čech double complex

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\delta & \delta & \delta \\
\check{C}^2(\mathcal{U}; \mathcal{S}^0) & \check{C}^2(\mathcal{U}; \mathcal{S}^1) & \check{C}^2(\mathcal{U}; \mathcal{S}^2) \\
\delta & \delta & \delta \\
\check{C}^1(\mathcal{U}; \mathcal{S}^0) & \check{C}^1(\mathcal{U}; \mathcal{S}^1) & \check{C}^1(\mathcal{U}; \mathcal{S}^2) \\
\delta & \delta & \delta \\
\check{C}^0(\mathcal{U}; \mathcal{S}^0) & \check{C}^0(\mathcal{U}; \mathcal{S}^1) & \check{C}^0(\mathcal{U}; \mathcal{S}^2) \\
\delta & \delta & \delta \\
\cdots & \cdots & \cdots
\end{array}
\]  

(33)

where \( \mathcal{U} = \{U_i\}_{i \in I} \) is a good open cover of \( M \) and \( \delta \) is the usual Čech coboundary operator.

Let \( (U_i)_{i \in I} \) be a good open cover of \( M \), and \( \omega_i \) a nowhere vanishing section of \( Q_B \) over \( U_i \). For all \( i, j \in I \), there exists a unique nowhere vanishing function \( f_{ij} \in C^\infty(U_{ij}, \mathbb{C}^\times) \) such that \( \omega_i = f_{ij} \omega_j \). It is clear from the construction that

\[ f_{ij} f_{jk} f_{ki} = 1. \]

Let \( \xi_i \in \Gamma(B^*|U_i) \) be the modular 1-form on \( U_i \) corresponding to \( \omega_i \). That is, we have \( \nabla_X \omega_i = (\xi_i|X) \omega_i \) for all \( X \in \Gamma(B|U_i) \), where \( \nabla \) denotes the canonical representation of \( B \) on \( Q_B \) of [12]. It thus follows that

\[ \xi_i = \xi_j + \frac{d_B f_{ij}}{f_{ij}} = \xi_j + \tilde{d}_B f_{ij}. \]

As a consequence, \((\xi_i, f_{ij})\) is a 1-cocycle of the double complex (33), and therefore defines a class in \( \tilde{H}^1(B, \mathbb{C}) \).

**Definition 4.14.** The class in \( \tilde{H}^1(B, \mathbb{C}) \) defined by \((\xi_i, f_{ij})\) is called the **modular class** of the complex Lie algebroid \( B \), and denoted \( \text{mod}(B) \).

**Lemma 4.15.** Consider the long exact sequence

\[
\cdots \to H^1(B, \mathbb{C}) \to \tilde{H}^1(B, \mathbb{C}) \xrightarrow{\tau} H^2(M, \mathbb{Z}) \to \cdots.
\]

The image of the modular class \( \text{mod}(B) \) under \( \tau \) is the first Chern class \( c_1(Q_B) \) of \( Q_B \). When \( c_1(Q_B) = 0 \), the modular class \( \text{mod}(B) \) is the image of a class in \( H^1(B, \mathbb{C}) \), which is defined exactly in the same way using a global nowhere vanishing section, as the usual modular class in [12].

A complex Lie algebroid \( B \) is said to be **unimodular** if its modular class vanishes. The following result follows immediately from Lemma 4.15.
Corollary 4.16. A complex Lie algebroid $B$ is unimodular if and only if $c_1(Q_B) = 0$ and for any fixed nowhere vanishing section $\omega \in \Gamma(Q_B)$, the modular section $\xi \in \Gamma(B^*)$ defined by

$$\nabla_X \omega = \langle \xi | X \rangle \omega \quad (\forall X \in \Gamma(B))$$

is a coboundary, i.e. $\xi = d_B f$ for some $f \in C^\infty(M, \mathbb{C})$.

As a consequence, a complex Lie algebroid $B$ is unimodular if and only if $Q_B$ is isomorphic to the trivial module $\mathbb{C}$.

Proposition 4.17. When $B = T^{0,1} X \bowtie A^{1,0}$ is the derived complex Lie algebroid [24,30] of a holomorphic Lie algebroid $A$ over $X$, $B$ is a unimodular complex Lie algebroid if and only if $A$ is a unimodular holomorphic Lie algebroid, i.e. $Q_A$ is trivial as a holomorphic line bundle and there exists a holomorphic global section $\omega$ of $Q_A$ such that $\nabla_X \omega = 0$ for all $X \in A$.

Definition 4.18. An extended Poisson manifold $(X, H)$ is unimodular if its corresponding complex Lie algebroid $A^*_H$ is unimodular.

According to Theorem 4.3, we have

Proposition 4.19. An extended Poisson manifold $(X, H)$ is unimodular if and only if there exists a nowhere vanishing $(n, 0)$-form $\omega \in \Omega^{n,0}(X)$ such that

$$\bar{\partial} \omega + [\partial, \iota_H] \omega = \bar{\partial} \omega + \partial \iota_H \omega = 0.$$ 

Remark 4.20. It is clear that, when $H = 0$, $(X, H)$ is unimodular means that $X$ is Calabi-Yau. Thus one can consider a unimodular extended Poisson manifold $(X, H)$ as a generalized Calabi-Yau manifold.

As an immediate consequence of the discussion above, we have

Corollary 4.21. For any unimodular extended Poisson manifold $(X, H)$ of complex dimension $n$, we have

$$H_k(X, H) \cong H^{2n-k}(X, H).$$

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