DOES MOMENTUM HELP? A SAMPLE COMPLEXITY ANALYSIS.

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ABSTRACT

Heavy ball momentum is a popular acceleration idea in stochastic optimization. There have been several attempts to understand its perceived benefits, but the complete picture is still unclear. Specifically, the error expression in the presence of noise has two separate terms: the bias and the variance, but most existing works only focus on bias and show that momentum accelerates its decay. Such analyses overlook the interplay between bias and variance and, therefore, miss important implications. In this work, we analyze a sample complexity bound of stochastic approximation algorithms with heavy-ball momentum that accounts for both bias and variance. We find that for the same step size, which is small enough, the iterates with momentum have improved sample complexity compared to the ones without. However, by using a different step-size sequence, the non-momentum version can nullify this benefit. Subsequently, we show that our sample complexity bounds are indeed tight for a small enough neighborhood around the solution and large enough noise variance. Our analysis also sheds some light on the finite-time behavior of these algorithms. This explains the perceived benefit in the initial phase of momentum-based schemes.

1 Introduction

Stochastic Approximation (SA) algorithms is a class of recursive algorithms that are used to find zeros of a function using noisy observations. Here, the function need not necessarily be a gradient and thus SA algorithms are applicable beyond optimisation problems. There is a huge literature that deals with the asymptotic convergence of SA algorithms under both martingale difference noise and Markov noise (see Borkar [2008], Benveniste et al. [1990], Kushner and Yin [2003]). In addition, there has been a surge of interest in finite time analysis of these algorithms (see Srikant and Ying [2019], Dalal et al. [2020], Dalal et al. [2018], Kaledin et al. [2019]). Here, we consider linear SA of the form

\[ \theta_{n+1} = \theta_n + \alpha_n (b - A \theta_n + M_{n+1}), \]

where, \( b \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d} \) and \( M_{n+1} \in \mathbb{R}^d \) is a martingale difference noise. It has been shown that with appropriate choice of the step-size sequence \( \{\alpha_n\} \), the iterate given in (1) converges to \( \theta^* = A^{-1}b \) (see Borkar [2008]). In particular, stochastic gradient descent (SGD) with quadratic objective function can be cast as a linear SA given by (1). There are several methods used to accelerate the iterates of stochastic gradient descent. One of the most widely used methods is Polyak’s heavy ball momentum (Polyak [1964]) and takes the following form:

\[ \theta_{n+1} = \theta_n + \alpha_n (b - A \theta_n + M_{n+1}) + \eta_n (\theta_n - \theta_{n-1}). \]

In the stochastic setting (i.e., in the presence of noise), the iterates can be decomposed into two terms, a bias term (formed by setting the noise to 0) and a variance term (formed by starting the iterate at the solution). Can et al. [2019]

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When the step-size and momentum parameter are constant, the iterates do not converge almost surely to the solution. However, all these results show an improvement corresponding to the bias term but study the variance term separately. To understand the behaviour of these iterates with momentum in the presence of noise, one must study both terms and their effect on each other. Two recent works undertake such a study. In Yuan et al. [2016], the authors establish that both heavy ball and Nesterov’s momentum methods are equivalent to standard stochastic gradient descent method with a re-scaled step size value. In Kidambi et al. [2018], the authors show that for all choice of the parameters, the performance of heavy ball momentum does not improve over SGD by more than a constant.

When the step-size and momentum parameter are constant, the iterates do not converge almost surely to the solution. Instead, the iterates converge in a mean squared sense to a neighbourhood around the solution. Under such a setting, this work intends to shed more light on the finite time behaviour of the heavy ball momentum scheme by computing the number of iterations required to hit an $\epsilon > 0$ boundary of the solution (typically called the sample complexity of the algorithm). We emphasize that such an analysis takes into account both the bias and the variance term, because to reach a small enough neighbourhood around the solution, both the terms need to be small enough. We first compute an upper-bound on the error term for both the momentum and vanilla schemes. We observe that the step-size for the momentum scheme is required to be smaller than that in the vanilla scheme. This results in a sample complexity bound of the same order when the best allowed step-size is chosen in both the cases. To understand if these results are tight, we undertake an analysis of the lower bound on the error. In other words, we quantify the number of iterations until which the error is greater than $\epsilon$, i.e., the iterates remain outside the $\epsilon$ neighbourhood of the solution. We observe that for small enough $\epsilon$ and large enough noise variance, the sample complexity bound on the momentum scheme is indeed tight.

Our observations agree with the available results in the literature. Indeed in our analysis also, the bias term decays faster in the momentum scheme than in the vanilla scheme. When the variance of the noise is not very high or the neighbourhood around the solution considered is large, the sample complexity bound achieved is also better. This explains the perceived improvement in performance of the momentum schemes in the initial part of the algorithm. However, in the presence of persistent noise with large enough variance, and when the neighbourhood considered is small enough, the improvement is lost. The presence of noise forces the step-size in case of momentum scheme to be small. However, the vanilla scheme has the liberty to choose a larger step-size and reach the $\epsilon$ boundary with same number of samples.

Recently, Avrachenkov et al. [2020] studied the asymptotic behaviour of linear SA with momentum in the uni-variate setting and established a.s. convergence when $\alpha_n \to 0$ and $\eta_n \to 1$. Under such a setting, we show that the optimal convergence rate of the momentum scheme matches that of the vanilla scheme asymptotically.

### 1.1 Key Contributions

- **Upper Bound for MSE:** For a constant step size $\alpha$ below a certain threshold and $\epsilon > 0$, we show that SA without momentum reaches an $\epsilon$ boundary of the optimum value in $\tilde{O} \left( \frac{1}{\alpha \lambda_{\text{min}}(A)} \right)$ iterations. Next, for SA with momentum we choose the momentum parameter $\eta = \left(1 - \sqrt{\frac{\alpha \lambda_{\text{min}}(A)}{2}}\right)^2$ to show that the sample complexity then is only $\tilde{O} \left( \frac{1}{\sqrt{\alpha \lambda_{\text{min}}(A)}} \right)$. Here, $\tilde{O}$ hides the logarithmic terms and the associated constants, and $\lambda_{\text{min}}(A)$ is the smallest eigenvalue of the driving matrix $A$. For some choice of $\alpha$ (small enough), the sample complexity bound for SA with momentum is better. However, if we choose the optimal $\alpha$ in both the cases separately, then the sample complexity bound is $\tilde{O} \left( \frac{K}{\lambda_{\text{min}}(A)^2} \right)$ for both schemes (See Assumption 2 for definition of $K$ and Remark 2 for details).

- **Lower Bound for MSE:** Next, to show that the above bounds are indeed tight, we lower bound the MSE when there is persistent noise and show that for $n = \Theta \left( \frac{1}{\lambda_{\text{min}}(A)^2} \right)$, $\exists \epsilon > 0$ such that $\mathbb{E}[\|\theta_n - \theta^*\|^2] \geq \epsilon$ (See Remark 5).

- **Convergence rate:** Here, we consider (i) and (2) with step size $\alpha_n \to 0$ and momentum sequence $\eta_n \to 1$. The convergence of iterates with this choice of $\alpha_n$ and $\eta_n$ has been studied in (Gitman et al. [2019], Avrachenkov et al. [2020], Gadat et al. [2016]). It is also known that the best possible convergence rate of (1) is $O(1/\sqrt{n})$ with step size choice of $\alpha_n = O(1/n)$ (see Chung [1954], Derman [1956], Fabian [1968]). Here, we show that, under some standard assumptions, the optimal convergence rate of the iterate in (2) is $\Omega(1/\sqrt{n})$. Thus the optimal convergence rate of SA with momentum is not better than the one without (See Remark 6).
2 Related Work

Momentum methods such as Polyak’s heavy ball [Polyak 1964] and Nesterov’s accelerated gradient [Nesterov 1983] were proposed to accelerate the convergence of gradient descent. In the deterministic case (when exact gradient is available) these methods indeed accelerate the convergence of the iterates $\theta_t$ towards the solution $\theta^*$ at a faster rate:

$$\|\theta_t - \theta^*\|^2 \leq \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{max}} + \lambda_{\text{min}}} \right) \|\theta_{t-1} - \theta^*\|^2,$$

in comparison to gradient descent whose convergence rate is given by:

$$\|\theta_t - \theta^*\|^2 \leq \left( \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{max}} + \lambda_{\text{min}}} \right)^2 \|\theta_{t-1} - \theta^*\|^2.$$

However, in the stochastic case (when noisy estimates of the gradient is available), there has been some dispute whether momentum is indeed helpful. Mou et al. [2020] shows an improvement in the mixing time of the iterates with a heavy ball momentum term in the context of linear regression with an additional Polyak Ruppert averaging [Polyak 1990], Ruppert 1988]. Alternatively, Can et al. [2019] shows an accelerated linear convergence in Wasserstein distance. The mixing time and the Wasserstein distance indicate how fast the distributions of the iterates converge to its stationary distribution. The claimed acceleration concerns only the bias term in the error estimate which corresponds to setting the noise to zero in (2). The final conclusion on the improvement is therefore similar to the one in the deterministic setting, and does not analyze the variance simultaneously. Furthermore, there is a non-trivial gap in the arguments leading to our bound. A consequence of this is that their analysis does not provide a fair means for comparing the momentum case with the one without (See Appendix A.1 for details). Similarly, in Can et al. [2019] we show that although there is an improvement in the convergence rate, it comes at the cost of making the term corresponding to the variance larger than SGD (See Appendix A.2 for details). In Gitman et al. [2019], it was shown that the stationary distribution of normalised SHB improves by increasing the momentum parameter. However, when translated to our setting, it appears that the step-size decreases as the momentum parameter increases in normalised SHB. This results in a worse convergence rate, which offsets the improvement in the stationary distribution (See Appendix A.3 for details).

In stochastic approximation setting Devraj et al. [2019] introduce matrix momentum and study their asymptotic variance. However, the resulting algorithms are not equivalent to heavy ball momentum and the authors do not study the finite time behaviour. Deb and Bhatnagar [2021] studied heavy ball momentum in the context of gradient TD algorithms but again analyzes only the asymptotic convergence. Separately, in settings where the noise in the iterates diminish to zero asymptotically, it has been show that momentum methods indeed improve the convergence rate Loizou and Richtárik 2020], Allen-Zhu [2018], Nitanda 2014]. In this work, we show that this improvement is not achieved when the noise is persistent i.e., the variance in the noise is lower bounded by a constant.

Some early works such as Roy and Shynk [1990], Sharma et al. [1998] analyzed momentum schemes in Least Mean Squares setting and showed a worse steady state behaviour than SGD in the presence of noise. Two recent works have also cast some doubt on whether heavy ball momentum helps in the stochastic setting and our work comes closest to the ideas explored in these. Kidambi et al. [2018] shows that heavy ball does not provide an improvement in the convergence rate over SGD by more than a constant. However, the result is true for large enough iterations and does not explain the algorithms behaviour in the initial iterations. Additionally, Yuan et al. [2016] show that for a small enough step size for heavy ball algorithm, a re-scaled (larger) step-size for SGD makes the two iterates move close to each other. The idea of using a larger step-size for SGD to obtain similar performance is something that we also observe in our analysis. Although the result holds for all time steps, nevertheless the result does not explain why heavy ball schemes seem to perform empirically better in the initial iterations and in the presence of small amount of noise. Our analysis does explain these observations.

3 Main Results

In Section 3.1 and 3.2 we analyze the sample complexity of heavy ball momentum algorithm and compare with the one without. Specifically, in Section 3.1 we compute the number of samples required that ensures that the mean squared error $\mathbb{E}[\|\theta_t - \theta^*\|^2]$ is bounded above by some $\epsilon > 0$. Here, $\epsilon$ is the radius of the ball around the solution that determines how close we want our iterates to be to the solution. Subsequently, in Section 3.2 we show that the bound on the sample complexity obtained in Section 3.1 is tight for small neighbourhoods. More specifically, we show that there exists some $\epsilon > 0$ such that the sample complexity of the momentum iterates are indeed tight. In section 3.3 we analyze the convergence rate of the linear SA with momentum when $\alpha_n \to 0$ and $\eta_n \to 1$ and show that asymptotically the rate is equal to the iterates without momentum.
3.1 Upper bound on Mean Squared Error

Unless specified, \( \| \cdot \| \) denotes the \( L_2 \) norm. We start by stating some assumptions on the driving matrix \( A \) and the noise sequence \( M_{n+1} \).

**Assumption 1.** The eigenvalues of \( A \), \( \{\lambda_i(A)\}_{i=1}^d \), are distinct, real and positive.

**Assumption 2.** The noise sequence \( M_{n+1} \) is a martingale difference sequence w.r.t the filtration \( \{F_n\} \), where \( F_n = \sigma(\theta_m, M_m; m \leq n) \). Furthermore, \( \mathbb{E}[\|M_{n+1}\|^2 | F_n] \leq K(1 + \|\theta_n - \theta^*\|^2) \) a.s.

**Remark 1.** (Discussion on Assumption 1): The distinct eigenvalues assumption can be justified from the fact that matrices belonging to \( \mathbb{R}^{d \times d} \) that have repeated eigenvalues form a set of zero Lebesgue measure in \( \mathbb{R}^{d \times d} \). We also point out that, assuming eigenvalues of \( A \) to be real does limit the applicability of SA algorithms. Specifically, it excludes its application to the widely used setting of Temporal Difference (TD) in Reinforcement Learning. The assumption is made to ease out the proof of the theorems and presently it is not clear how to choose the momentum parameter \( \eta \) when the eigenvalues of \( A \) are complex. However, we do not assume that the matrix \( A \) is symmetric. Thus, the results in this paper are applicable beyond gradient descent problems.

**Remark 2.** (Discussion on Noise): Several results in stochastic gradient literature assume that the noise sequence is i.i.d and/or has bounded variance. Here, we only assume that the noise is a martingale difference sequence and that \( \mathbb{E}[\|M_{n+1}\|^2 | F_n] \leq K(1 + \|\theta_n\|^2) \) a.s. for some \( K \in \mathbb{R} \). The bound on the noise is quite standard in the stochastic approximation literature (see Borkar [2008]; Chapter 2).

We now present our first result that characterizes a bound on the sample complexity of the standard SA iterate given by (1) to reach an \( \epsilon \) neighbourhood of the solution in expectation.

**Proposition 3.1.** Consider the SA without momentum iterate given in (1) and suppose Assumption 1 and 2 hold. Let \( \Lambda = \|\theta_0 - \theta^*\|^2 \) and \( \epsilon \) be the given degree of accuracy. Choose \( \alpha \) such that

\[
\alpha \leq \min \left( \frac{2}{\lambda_{\min}(A)}, \frac{2}{\epsilon \lambda_{\min}(A) + \lambda_{\max}(A)} \right),
\]

where \( \lambda_{\min}(A) = \min_i \lambda_i(A) \) and \( C \) is defined in (11). Then, the expected error \( \mathbb{E}[\|\theta_n - \theta^*\|^2] \leq \epsilon \) for some \( n \in \mathcal{O}\left(\frac{1}{\alpha \lambda_{\min}(A)}\right) \).

**Proof.** See Appendix B.1

The next theorem characterizes the sample complexity of SA with momentum.

**Theorem 3.2.** Consider the SA with momentum iterate in (2) and suppose Assumption 1 and 2 hold. Let \( \Lambda = \|\theta_0 - \theta^*\|^2 \) and \( \epsilon \) be the given degree of accuracy. Choose \( \alpha \) and \( \eta \) as follows:

\[
\alpha \leq \min \left( \left\lceil \frac{\epsilon \lambda_{\min}(A) + \lambda_{\max}(A)}{2} \right\rceil, \left( \frac{2}{\epsilon \lambda_{\min}(A) + \lambda_{\max}(A)} \right)^{3/2}, \frac{2}{\sqrt{\lambda_{\min}(A) + \lambda_{\max}(A)}} \right),
\]

where \( \lambda_{\min}(A) = \min_i \lambda_i(A) \) and \( C \) is defined in (11). Then, the expected error \( \mathbb{E}[\|\theta_n - \theta^*\|^2] \leq \epsilon \) for some \( n \in \mathcal{O}\left(\frac{1}{\alpha \lambda_{\min}(A)}\right) \).

**Proof.** See Appendix B.2

**Remark 3.** The first bound on the step-size \( \alpha \) in (3) and (4) is because of the fact that the variance in the noise is not bounded by a constant but could also depend on the iterates itself. The second bound is to ensure that the noise term in the decomposition of the error is within an \( \epsilon \) boundary of the solution. The third condition is the best choice of step-size in the deterministic case.

The sample complexity bound achieved by Theorem 3.2 is better than that achieved by Proposition 3.1 when either of the following is satisfied:
The error $\|\theta_n - \theta^*\|^2$ averaged over 1000 runs is plotted against time-steps. For a given step-size $\alpha$, the best momentum parameter is chosen (See Figure 2). For the same step-size, the momentum algorithm outperforms the standard algorithm. However, the same improvement can be achieved by choosing a larger step-size in the standard algorithm.

1. When the same value of $\alpha$ is chosen for both the algorithms, that satisfies both (3) and (4) (See Remark 4).
2. When the variance in the noise is small enough making the first two terms in (3) and (4) greater than the third term (See Remark 5).
3. When the $\epsilon$ boundary under consideration is large enough and the noise variance is uniformly bounded (See Remark 6).

**Remark 4.** Observe that for the same choice of $\alpha$ in both the cases, there is an improvement in the sample complexity bound of the algorithm by a factor of $\tilde{O}(\frac{1}{\sqrt{\lambda_{\min}(A)}})$. However, for SA without momentum when $K$ is large enough, the optimal choice of step-size $\alpha \in \Theta(\lambda_{\min}(A))$, which produces a sample complexity bound of $\tilde{O}(\frac{1}{\lambda_{\min}(A)^2})$. For SA with momentum the optimal choice of $\alpha \in \Theta(\lambda_{\min}(A)^3)$ the sample complexity bound is again $\tilde{O}(\frac{1}{\lambda_{\min}(A)^2})$. The dependence on $\epsilon$ and $K$ also remains the same. This shows that for the standard linear SA iterates we can choose a higher value of $\alpha$ to obtain the same sample complexity bound.

**Remark 5.** When the noise is small enough, the third condition on $\alpha$ is the minimum and again we observe an improvement in the sample complexity bound of $\tilde{O}(\sqrt{\lambda_{\max}(A)+\lambda_{\min}(A)})^2$ as in the deterministic case. This explains the observed improvement in performance when the noise variance is small.

**Remark 6.** When the variance in the noise is uniformly bounded and does not depend on the iterates, then again the minimum on the step-size vanishes. Under such a setting, the neighbourhood $\epsilon$ considered is large enough, then again the minimum on the step-size is achieved by the third condition in (3) and (4), and we observe an improvement in sample complexity as in Remark 5. This explains the perceived improvement in performance in the initial phase as the neighbourhood under consideration is large. However, when the variance in the noise depends on the iterates itself, then it is far more difficult for the momentum algorithm to outperform the standard algorithm. The error $\|\theta_n - \theta^*\|^2$ in Assumption 2 during the initial phase could potentially lead to large variance in the noise. Then the first bound in (3) and (4) still forces the momentum algorithm to pick a smaller step-size and the sample complexity bounds are same as in Remark 5. However, when $K = \mathcal{O}(\lambda_{\min}(A)^2)$, then again the minimum is achieved by the third term and the improvement resurfaces.

**Remark 7.** To emphasize on the idea that the benefits obtained by using a heavy ball momentum in Linear SA can be nullified by using a larger step-step in the standard algorithm, we run both the algorithms on a simple linear SA problem. We consider the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$ in (5) and an additive Gaussian noise with zero mean and bounded variance. The results are plotted in Figure 1.

### 3.2 Lower bound on Mean Squared Error

For ease of exposition, consider the univariate iterates for standard linear SA in (5) and its momentum variant in (6):

$$\theta_{n+1} = \theta_n + \alpha(b - \lambda \theta_n + M_{n+1}), \quad \theta_{n+1} = \theta_n + \alpha(b - \lambda \theta_n + M_{n+1})$$

$$\theta_{n+1} = \theta_n + \alpha(b - \lambda \theta_n + M_{n+1}),$$

(5)
Theorem 3.2 is tight up to constants to ensure that
\[ \theta_{n+1} = \theta_n + \alpha(b - \lambda \theta_n + M_{n+1}) + \eta(\theta_n - \theta_{n-1}), \]
with constant step-size \( \alpha \) and momentum parameter \( \eta \). This can be extended to the multivariate case by analyzing each block separately (See Appendix B.3.1 for details). Here we additionally make the assumption of persistent noise, i.e., we assume that the variance in the noise is lower bounded by a constant.

**Assumption 3.** The noise sequence \( M_{n+1} \in \mathbb{R} \) is a martingale difference sequence w.r.t. the filtration \( \{F_n\} \), where \( F_n = \sigma(\theta_m, M_m; m \leq n) \). Furthermore, \( \mathbb{E}[||M_{n+1}||^2 | F_n] \geq K \) a.s.

**Theorem 3.3.** Consider the SA with momentum iterate in (6) and suppose Assumption 1 and 3 hold. Then \( \exists \epsilon > 0 \) such that \( \forall \alpha > 0, \forall \eta \in [0, 1], \mathbb{E}[\|\theta_n - \theta_*\|^2] \geq \epsilon \) for some \( n \in \Theta \left( \frac{1}{\epsilon^2} \right) \).

**Proof.** See Appendix B.3. \( \square \)

**Remark 8.** The above theorem shows that for \( \alpha = \Theta(c^2 \lambda^3) \), \( \mathbb{E}[\|\theta_n - \theta_*\|^2] \geq \epsilon \). Therefore the condition on \( \alpha \) in Theorem 3.2 is tight up to constants to ensure that \( \mathbb{E}[\|\theta_n - \theta_*\|^2] \leq \epsilon \).

### 3.3 Convergence Rate

It can be shown that for a step-size \( \alpha_n = 1/(n+1)^{\alpha}, \alpha \in (0, 1) \), the convergence rate \( \mathbb{E}[\|\theta_n - \theta_*\|^2] = \Theta(n^{-\alpha/2}) \) (See Avrachenkov et al. (2020)). The optimal convergence rate is therefore given by \( \Theta(1/\sqrt{n}) \). Now consider (2) and choose momentum parameter as \( \eta_n = \frac{\beta_n-wa_n}{b_n} \) and \( \gamma_n = \frac{a_n}{b_n} \), where \( \beta_n \) is a positive sequence. Then, (2) can be re-written as (See Appendix B.4):

\[ \theta_{n+1} = \theta_n + \beta_n(u_n + \xi_n) \]
\[ u_{n+1} = u_n + \gamma_n(b - wu_n - A\theta_n + M_{n+1}), \]

where, \( \xi_n = u_{n+1} - u_n \) is a perturbation term and \( w > 0 \) is a constant. We now show that under some assumptions the convergence rate of the iterate with momentum is \( \Omega(n^{-\beta/2}) \).

**Assumption 4.** There exists a positive definite matrix \( \Gamma_{22} \) such that
\[ \lim_{n \to \infty} \mathbb{E}[M_{n+1}M_n^T | F_n] = \Gamma_{22} \]

**Assumption 5.** The step-size sequences satisfy \( \beta_n = \frac{1}{(n+1)^{\gamma}}, \gamma_n = \frac{1}{(n+1)^{\gamma}}, \) where \( \frac{1}{2} < \gamma < \beta \leq 1 \).

**Theorem 3.4.** Under Assumptions 1, 2, 4 and 5 it follows that, \( \mathbb{E}[\|\theta_n - \theta_*\|^2] = \Omega(n^{-\beta/2}) \). Here, the relation \( X_n = \Omega(n^p) \) means \( \exists n_0 \in \mathbb{N} \) and a constant \( c > 0 \) such that \( X_n > cn^p, \forall n \geq n_0 \).

**Proof.** See Appendix B.4. \( \square \)

**Remark 9.** **(Discussion on convergence rate):** The lower bound in Theorem 3.4 is minimised for \( \beta = 1 \), whence we have \( \mathbb{E}[\|\theta_n - \theta_*\|^2] = \Omega(n^{-1/2}) \). Now since the optimal convergence rate for the algorithm given in (4) is \( \Omega(n^{-1/2}) \), it can be said that addition of momentum doesn’t help improve the convergence rate, for large values of \( n \).

### 4 Proof Outline

We briefly outline the proof of Theorem 3.2 in Section 4.1 and Theorem 3.3 in Section 4.2.

#### 4.1 Proof Outline of Theorem 3.2

We start by defining the error term at time-step \( n \) as \( \hat{\theta}_n = \theta_n - \theta^* \) and rewriting equation (10):

\[ \hat{\theta}_{n+1} = (I - \alpha A)\hat{\theta}_n + \alpha(M_{n+1}) + \eta(\hat{\theta}_n - \hat{\theta}_{n-1}) \]

We convert the above iterate into an iterate of double dimension as follows:

\[ \bar{X}_n = P\bar{X}_{n-1} + \alpha W_n \]

where, \( \bar{X}_n \triangleq \begin{bmatrix} \hat{\theta}_n \\ \hat{\theta}_{n-1} \end{bmatrix}, W_n \triangleq \begin{bmatrix} M_n \\ 0 \end{bmatrix} \), and

\[ P \triangleq \begin{bmatrix} I - \alpha A + \eta I & -\eta I \\ \eta I & 0 \end{bmatrix} \]

(7)
Using Assumption 2, we can obtain the following bound on the mean squared error:

\[ \mathbb{E}[\|\hat{X}_n\|^2] \leq \|P^n\|^2 \|\hat{X}_0\|^2 + \alpha^2 K \sum_{i=0}^{n-1} \|P^{n-i}\|^2 (1 + \mathbb{E}[\|\hat{X}_i\|^2]) \]

Here, the first term corresponds to the bias and the second term correspond to the variance. A spectral analysis of \( P \) tells us that the best choice of \( \eta \) is given by \( \eta = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2}\right)^2 \) (Figure 2 shows how \( p(P) \) varies with \( \eta \) in the univariate case \( A \equiv \lambda \)) with \( \alpha \leq \left(\frac{1}{\sqrt{\lambda_{\min}(A) + \sqrt{\lambda_{\max}(A)}}}\right)^2 \). However, to simplify the analysis we choose the parameter \( \eta = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2}\right)^2 \), which maintains the same rate of decay of the bias term. Using Assumption 1 and Lemma B.1, we define a sequence \( \{V_n\} \),

\[ V_n = \hat{C}^2 \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2}\right)^{2n} \Lambda + \alpha^2 \hat{C}^2 K \sum_{i=0}^{n-1} \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2}\right)^{2(n-i)} (1 + V_i) \]

and choose \( \alpha \leq \left(\frac{e(\lambda_{\min}(A))^{3/2}}{200C^2K}\right)^2 \) to ensure

\[ \mathbb{E}[\|\hat{X}_n\|^2] \leq V_n \leq e^{-n\sqrt{\lambda_{\min}(A)\alpha}} \hat{C}^2 \Lambda + \frac{4\alpha^2 \hat{C}^2 K}{\sqrt{\alpha \lambda_{\min}(A)}}. \]

Here we have used the fact that \( \alpha \leq \frac{1}{\lambda_{\min}(A)} \) and therefore,

\[ \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{4}\right)^2 \leq e^{-\sqrt{\lambda_{\min}(A)\alpha}}. \]

Using Lemma B.2 we bound \( \hat{C} \) in terms of \( C \) used in Proposition 3.1 and obtain

\[ V_n \leq e^{-n\sqrt{\lambda_{\min}(A)\alpha}} \frac{25C^2}{\lambda_{\min}(A)\alpha} \Lambda + \alpha^2 \frac{100C^2 K}{(\lambda_{\min}(A)\alpha)^{3/2}} \]

for \( n \geq \frac{4}{\sqrt{\alpha \lambda_{\min}(A)}} \log \left(\frac{1}{\lambda_{\min}(A)\alpha}\right) \),

\[ V_n \leq 25C^2 \Lambda e^{-\frac{1}{2} \sqrt{\lambda_{\min}(A)\alpha}} + \sqrt{\alpha} \frac{100C^2 K}{(\lambda_{\min}(A)\alpha)^{3/2}} \]

Choose \( \alpha \) small enough as \( \alpha \leq \left(\frac{e(\lambda_{\min}(A))^{3/2}}{200C^2K}\right)^2 \) to ensure that the second term in the above inequality is within an \( \epsilon \) boundary and \( n \) large enough:

\[ n = \frac{4}{\sqrt{\alpha \lambda_{\min}(A)}} \log \left(\frac{50C^2 \Lambda}{\epsilon}\right), \]
to ensure that the first term is within an \( \epsilon \) boundary. Now,

\[
    n = \max \left( \frac{4}{\sqrt{\alpha \lambda_{\min}(A)}} \log \left( \frac{50C^2 \Lambda}{\epsilon} \right), \frac{4}{\sqrt{\alpha \lambda_{\min}(A)}} \log \left( \frac{1}{\lambda_{\min}(A) \alpha} \right) \right)
\]

gives the desired bound on the sample complexity.

![Figure 2: Convergence rate \( \rho(P) \) as a function of the momentum parameter \( \eta \). Here, each curve is plot using fixed \( \alpha \) and \( \lambda \). The minimum for each curve is attained at \( \eta = (1 - \sqrt{\alpha \lambda})^2 \).](image)

### 4.2 Proof Outline of Theorem 3.3

In order to show that the sample complexity bounds are tight, we lower bound the MSE. We outline the proof for the univariate case, \( A \equiv \lambda \) below. The proof is split into three parts, where the first part deals with \( \eta \in \left[0, \left(1 - \sqrt{\alpha \lambda}\right)^2 \right) \), the second with \( \eta = \left(1 - \sqrt{\alpha \lambda}\right)^2 \) and finally \( \eta \in \left( \left(1 - \sqrt{\alpha \lambda}\right)^2, 1 \right] \). For all three cases, we have from the proof of Theorem 3.2 that:

\[
    \hat{X}_n = P\hat{X}_{n-1} + \alpha W_n = P^n \hat{X}_0 + \alpha \sum_{j=0}^{n-1} P^n - 1 - j W_{j+1},
\]

where,

\[
    \hat{X}_n \triangleq \begin{bmatrix} \hat{\theta}_n \\ \hat{\theta}_{n-1} \end{bmatrix}, P \triangleq \begin{bmatrix} 1 - \alpha \lambda + \eta & -\eta \\ \alpha & 0 \end{bmatrix}
\]

and \( W_n \triangleq \begin{bmatrix} M_n \\ 0 \end{bmatrix} \).

Thus,

\[
    \mathbb{E}[\|\hat{X}_n\|^2] = \mathbb{E}[\|P^n \hat{X}_0 + \alpha \sum_{j=0}^{n-1} P^n - 1 - j W_{j+1}\|^2]
\]

\[
    = \|P^n \hat{X}_0\|^2 + \mathbb{E}[\alpha^2 \sum_{j=0}^{n-1} \|P^n - 1 - j W_{j+1}\|^2]
\]

\[
    \geq \|P^n \hat{X}_0\|^2 + \alpha^2 K \sum_{j=0}^{n-1} \|P^n - 1 - j \epsilon_1\|^2,
\]
where the second equality follows since each $W_{j+1}$ is a martingale difference sequence and the inequality follows from Assumption 3. We can choose $\tilde{X}_0$ suitably such that $\|P^n\tilde{X}_0\|^2 \geq \rho(P)^{2n}\|\tilde{X}_0\|^2$.

Now, we require lower bounds on $\|P^je_1\|^2$ in order to find lower bounds on the MSE. From this point on, the analyses for the three cases take different turns due to differences in the nature of matrix $P$. In the first case, $P$ is diagonalisable with complex eigenvalues, in the second case $P$ is not diagonalisable with repeated real eigenvalues and in the third case, $P$ is diagonalisable with real eigenvalues. For the first and third cases, we find that

$$P^j = \frac{1}{\mu_+ - \mu_-} \begin{pmatrix} \mu_+^{j+1} - \mu_-^{j+1} & -\mu_+^{j+1} \mu_- + \mu_-^{j+1} \mu_+ \\ \mu_+^j - \mu_-^j & -\mu_+^j \mu_- + \mu_-^j \mu_+ \end{pmatrix}$$

and

$$P^j e_1 = \frac{1}{\mu_+ - \mu_-} \begin{pmatrix} \mu_+^{j+1} - \mu_-^{j+1} \\ \mu_+^j - \mu_-^j \end{pmatrix},$$

where

$$\mu_+ = -\frac{(\lambda \alpha - 1 - \eta) + \sqrt{(\lambda \alpha - 1 - \eta)^2 - 4\eta}}{2}$$

and

$$\mu_- = -\frac{(\lambda \alpha - 1 - \eta) - \sqrt{(\lambda \alpha - 1 - \eta)^2 - 4\eta}}{2}.$$ 

Thus,

$$\|P^je_1\|^2 = \left(\frac{\mu_+^{j+1} - \mu_-^{j+1}}{\mu_+ - \mu_-}\right)^2 + \left(\frac{\mu_+^j - \mu_-^j}{\mu_+ - \mu_-}\right)^2 \geq \left(\frac{\mu_+^{j+1} - \mu_-^{j+1}}{\mu_+ - \mu_-}\right)^2 = \left(\frac{\sin(j+1)\omega}{\sin(\omega)}\right)^2.$$ 

In the first case, since $\mu_1$ and $\mu_2$ are complex, we let $\mu_+ = r(\cos \omega + i \sin \omega)$ and $\mu_- = r(\cos \omega - i \sin \omega)$, where $r = \rho(P)$ and $\omega \in [0, \pi/2]$. Using this, we get,

$$\|P^je_1\|^2 \geq \left(\frac{\mu_+^{j+1} - \mu_-^{j+1}}{\mu_+ - \mu_-}\right)^2 = \left(\frac{\sin(j+1)\omega}{\sin(\omega)}\right)^2.$$ 

In the third case, we simply use that

$$\|P^je_1\|^2 \geq \left(\frac{\mu_+^{j+1} - \mu_-^{j+1}}{\mu_+ - \mu_-}\right)^2 = \frac{1}{(\mu_+ - \mu_-)^2} \left(\mu_+^{2(j+1)} + \mu_-^{2(j+1)} - 2(\mu_+ \mu_-)^j\right).$$

For the second case, since $P$ is not diagonalisable, we can only obtain its Jordan decomposition. From this, we get:

$$P^j = \begin{pmatrix} (j + 1)\mu^j - j\mu^{j+1} \\ j\mu^{j-1} \end{pmatrix},$$

and

$$P^j e_1 = \begin{pmatrix} (j + 1)\mu^j \\ j\mu^{j-1} \end{pmatrix},$$

where $\mu = \mu_+ = \mu_-$. It follows that,

$$\|P^je_1\|^2 \geq j^2 \mu^{2j}.$$

Using these bounds on $\|P^je_1\|^2$, we can lower bound the MSE for $\forall \eta \in [0, 1]$. Thus, we can show that $\exists \epsilon > 0$ and $n = \tilde{O} \left(\frac{1}{\epsilon^2}\right)$ such that the variance term $\geq \Theta(\epsilon)$. 


5 Concluding Remarks

In this work we analyzed the sample complexity of heavy ball momentum algorithm in the context of linear SA and compared it with the the standard linear SA algorithm. The sample complexity analysis takes into account both the bias and variance terms in the error decomposition and puts into perspective some recent works that claim an improvement by using momentum. Specifically, we show that although the bias term could decrease faster when the same step-size is chosen, however the improvement gets lost when the variance terms are forced to be of the same order in both the algorithms. Our analysis reasons why momentum algorithms appear to perform better when the variance in the noise is small or when the neighbourhood around the solution considered is large. Additionally, we show that when considering a small neighbourhood, the momentum algorithm is forced to choose a smaller step-size which offsets the improvement in the bias decay. We also provide an asymptotic convergence rate for the momentum algorithm when the step-size $\alpha_n \to 0$ and $\eta_n \to 1$.

Although we did not analyze the Nesterov’s accelerated method in this work, we suspect that a similar behaviour would also be observed in such a setting. Assran and Rabbat [2020] showed an improved convergence rate for Nesterov’s accelerated gradient on quadratic objectives. However a careful look into their results shows an additional $C_\epsilon$ term multiplied with the variance term. Appendix B in Assran and Rabbat [2020] suggest that $C_\epsilon$ might grow linearly with the number of iterations and therefore the variance term also grows linearly with the number of iterations. This might lead to the nullification of the accelerated rate if we enforce that the variance terms are of the same order in both the algorithms. Can et al. [2019] also show an improved convergence rate with Nesterov’s accelerated gradient. However as we showed in the heavy ball case (See Appendix A.2 of the current work), it can be shown that the term corresponding to the variance is worse in this case also. Our speculations are further supported by the result in Yuan et al. [2016] which shows an equivalence of the two schemes by re-scaling the step-size. A rigorous analysis of the Nesterov’s method in the spirit of this work is therefore an important future direction to explore.

6 Future Work

To the best of our understanding, it is necessary to kill the noise asymptotically to make the best use of momentum schemes. This idea is supported by the improvement results in Loizou and Richtárik [2020], Allen-Zhu [2018], Nitanda [2014] where the noise in the iterates diminish to zero asymptotically. In the presence of persistent noise, a natural way to kill it asymptotically is to use decreasing step-size sequence. Although, under such a setting, our work shows that the optimal convergence rate does not improve asymptotically by using momentum, but it does not explain its finite time behaviour. The driving matrix $P$ defined in (7) would then depend on the time-step. Analyzing the behaviour of the algorithm under such a setting becomes quite challenging as it gives rise to a linear time invariant system. Nevertheless, such an analysis might lead to an improved finite time behaviour of the existing momentum algorithms.

Finally, another possible way to kill the noise asymptotically is to use iterate averaging of the type suggested in Polyak [1990], Ruppert [1988]. However, some of our preliminary work suggests that the fast decay of the bias term might get lost if iterate averaging is performed from the start of the algorithm. Jain et al. [2018] use the idea of tail averaging to reduce the variance and Jain et al. [2018b] introduce a new accelerated stochastic gradient descent algorithm with iterate averaging that achieves the minimax optimal statistical risk in linear regression problems. The latter work does a more fine grained analysis and also takes into account properties of the input distribution. Extending these ideas to the stochastic approximation setting where the driving matrix need not be symmetric would have far reaching consequences in reinforcement learning. Subsequently, analyzing such extensions to multiple timescale algorithms as in Sutton et al. [2009b,a], Borkar [2005], Bhatnagar [2010], Bhatnagar and Panigrahi [2006], Deb and Bhatnagar [2022] would also be worth exploring.

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Appendix

A Comparison with recent works

A.1 Comparison with Mou et al. (2020)

Claim 1 in (p. 20, Mou et al. [2020]) analyzes the asymptotic covariance of the heavy ball momentum algorithm (with Polyak averaging) and claims a correction term that satisfies:

\[ \text{Tr}(L_\eta) \lesssim O(\eta \frac{\kappa^2(U)}{\lambda_{\min}(A)^{3/2}}) \]

where \( \tilde{A} = \left( \begin{array}{cc} 0 & I_d \\ -\bar{A} & \alpha I_d + \eta \bar{A} \end{array} \right) = U D U^{-1} \) as in the decomposition of \( \tilde{A} \) in Lemma 1 of Mou et al. [2020] and \( \kappa(U) = \|U\|_{op}\|U^{-1}\|_{op} \).

However in the proof of claim 1, we are not sure if the following bound holds, since the matrix \( \tilde{A} \) is not symmetric:

\[ \text{Tr}(\tilde{A}^{-1}E(\tilde{\Xi}_A \Lambda_\eta^* (\tilde{\Xi}_A)^T)(\tilde{A}^{-1})^T) \leq (\min_i |\lambda_i(\tilde{A}))|^{-2}(1 + \eta^2) \nu_A^2 \mathbb{E}_{\pi_\eta}\|\theta_t - \theta^*\|^2 \]

Our calculation points towards the following bound:

\[ \text{Tr}(L_\eta) \lesssim O(\eta \frac{\kappa^2(U)}{\lambda_{\min}(A)^{3/2}}) \]

We outline the proof for the uni-variate case, when \( \tilde{A} = \lambda \) for some \( \lambda > 0 \). Then,

\[ \tilde{A} = \left( \begin{array}{cc} 0 & 1 \\ -\lambda & \alpha + \eta \lambda \end{array} \right) \text{, and } \tilde{A}^{-1} = \frac{1}{\lambda} \left( \begin{array}{cc} \alpha + \eta \lambda & -1 \\ \lambda & 0 \end{array} \right). \]

Observe that \( \tilde{A}^{-1}(\tilde{A}^{-1})^T = \frac{1}{\lambda^2} \left[ \begin{array}{cc} 1 + (\alpha + \eta \lambda)^2 & \lambda(\alpha + \eta \lambda) \\ \lambda(\alpha + \eta \lambda) & \lambda^2 \end{array} \right] \) and therefore \( \text{Tr}(\tilde{A}^{-1}(\tilde{A}^{-1})^T) = O\left(\frac{1}{\lambda^2}\right) \). Using this we have,

\[ \text{Tr}(\tilde{A}^{-1}E(\tilde{\Xi}_A \Lambda_\eta^* (\tilde{\Xi}_A)^T)(\tilde{A}^{-1})^T) \leq O\left(\frac{1}{\lambda^2}\right)\text{Tr}(E(\tilde{\Xi}_A \Lambda_\eta^* (\tilde{\Xi}_A)^T)) \]

\[ \lesssim O(\eta \frac{\kappa^2(U)}{\lambda_{\min}(A)^{3/2}}) \]  

(8)

The second inequality follows as in Mou et al. [2020]. Next we analyze the dependence of \( \kappa^2(U) \) on \( \lambda \). Again for simplicity we consider the uni-variate case where \( \tilde{A} = \lambda \). Let \( \tilde{A} = \left( \begin{array}{cc} 0 & 1 \\ -\lambda & \alpha + \eta \lambda \end{array} \right) \) be diagonalizable. Therefore,

\[ \tilde{A} = U \left[ \begin{array}{cc} \mu_+ & 0 \\ 0 & \mu_- \end{array} \right] U^{-1}, \]

where \( \mu_+ \) and \( \mu_- \) are the eigenvalues of \( \tilde{A} \). Let \( U = \left[ \begin{array}{cc} u_1 & u_2 \\ u_3 & u_4 \end{array} \right] \). We therefore have,

\[ \left[ \begin{array}{cc} 0 & 1 \\ -\lambda & \alpha + \eta \lambda \end{array} \right] \left[ \begin{array}{cc} u_1 & u_2 \\ u_3 & u_4 \end{array} \right] = \left[ \begin{array}{cc} u_1 & u_2 \\ u_3 & u_4 \end{array} \right] \left[ \begin{array}{cc} \mu_+ & 0 \\ 0 & \mu_- \end{array} \right]. \]

Solving the system of equations, we get:

\[ U = \left[ \begin{array}{cc} 1 & 1 \\ \mu_+ & \mu_- \end{array} \right] \text{ and } U^{-1} = \frac{1}{\mu_+ - \mu_-} \left[ \begin{array}{cc} 1 & 1 \\ \mu_+ & \mu_- \end{array} \right]. \]

Now, \( \mu_+ - \mu_- = \sqrt{(\alpha + \eta \lambda) - 4\lambda} \). Using the choice of \( \alpha = \sqrt{\lambda} \) as in Mou et al. [2020], we have:

\[ \mu_+ - \mu_- = \sqrt{\lambda + \eta^2 \lambda^2 + 2\alpha \eta \lambda - 4\lambda} \]
\[ \sqrt{\lambda} \sqrt{\eta^2 \lambda + 2\eta \sqrt{\lambda} - 3} \]

For \( \lambda \ll 1 \) (which is the case where the momentum algorithm is claimed to improve the mixing time in [Mou et al. (2020)], \( \mu_+ - \mu_- \geq O(\sqrt{\lambda}) \). As in proof of Lemma B.2 in Appendix C.2 and using the fact that \( \|U\|_{op}\|U^{-1}\|_{op} = \sigma_{max}(U)\sigma_{max}(U^{-1}) \), we can show that \( \kappa^2(U) \leq O(\frac{1}{\lambda}) \). Combining with [1], we have:

\[ \text{Tr}(L_\eta) \lesssim O(\eta \lambda^{-7/2}) . \]

A similar analysis can be carried for the multivariate case to show that

\[ \text{Tr}(L_\eta) \lesssim O(\eta \lambda_{\text{min}}(\bar{A})^{-7/2}) . \]

The correction term in SGD is \( O(\eta \lambda_{\text{min}}(\bar{A})^{-3}) \) (See Mou et al. [2020], Claim 1). The stationary distribution for the momentum algorithm is larger than that of SGD when \( \lambda_{\text{min}}(A) \ll 1 \). Indeed if we enforce that the two asymptotic covariances are of the same size by choosing the step-size for momentum iterate \( \eta^m \) in terms of the step-size of SGD, i.e.,

\[ O\left(\frac{1}{\lambda_{\text{min}}(A)^3}\right) = O\left(\frac{1}{\lambda_{\text{min}}(A)^{7/2}}\right), \]

then we must choose \( \eta^m = O(\sqrt{\lambda_{\text{min}}(A)} \eta) \). In Appendix C.1. of [Mou et al. 2020], the mixing time of momentum iterate is shown to be \( O\left(\frac{1}{\eta \lambda_{\text{min}}(A)}\right) \), while the mixing time of SGD is \( O\left(\frac{1}{\eta \lambda_{\text{min}}(A)}\right) \). When we choose \( \eta^m = O(\sqrt{\lambda_{\text{min}}(A)} \eta) \), then the mixing time of momentum algorithm turns out to be the same as SGD. This behaviour is identical to what we observe in Theorem 3.1 and Theorem 3.2 where if we choose the same step-size then there is improvement by a square root factor.

### A.2 Comparison with Can et al. (2019)

For strongly convex quadratic functions of the form:

\[ f(x) = \frac{1}{2} x^T Q x + a^T x + b, \]

where \( x \in \mathbb{R}^d \), \( Q \in \mathbb{R}^{d \times d} \) is p.s.d, \( a \in \mathbb{R}^d \), \( b \in R \) and \( \mu I_d \preceq Q \preceq L I_d \), Can et al. (2019) shows acceleration in Wasserstein distance by a factor of \( \sqrt{\kappa} = \sqrt{\frac{\mu}{L}} \). The trace of the asymptotic covariance matrix \( X_{HB} \) is given by (See Appendix C.2 of Can et al. 2019):

\[ \text{Tr}(X_{HB}) = \sum_{i=1}^{d} \frac{2\alpha(1 + \beta)}{(1 - \beta)\lambda_i(2 + 2\beta - \alpha\lambda_i)}, \]

where, \( \alpha \) is the step size, \( \beta \) is the momentum parameter and \( \lambda_i \) is the \( i^{th} \) eigen-value of \( Q \). To compare the size of the stationary distribution with the iterates of SGD we set \( \beta = 0 \) and \( \alpha = \frac{2}{\mu + L} \) and get:

\[ \text{Tr}(X_{SGD}) = \sum_{i=1}^{d} \frac{2\lambda_i^2}{\lambda_i(2 - \frac{2}{\mu + L}\lambda_i)} \]
\[ = \sum_{i=1}^{d} \frac{2\lambda_i^2}{\mu + L(\mu + L - \lambda_i)} \]
\[ = \sum_{i=1}^{d} \lambda_i \frac{2}{\mu + L - \lambda_i} \]
To compute the size of the stationary distribution with the iterates of heavy ball we set the step size \( \alpha = \frac{4}{(\sqrt{\mu} + \sqrt{L})^2} \) and momentum parameter \( \beta = \left( \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2 \) and get:

\[
\text{Tr}(X_{SGD}) = \sum_{i=1}^{d} \frac{2}{(1 - \beta)\lambda_i} \left( 2 + 2 \left( \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2 - \frac{4}{(\sqrt{\mu} + \sqrt{L})^2} \lambda_i \right) (1 + \beta) \]

\[
= \sum_{i=1}^{d} \frac{2}{(1 - \beta)\lambda_i} \left( \frac{2(\sqrt{L} + \sqrt{\mu})^2 + 2(\sqrt{L} - \sqrt{\mu})^2 - 4\lambda_i}{(\sqrt{L} + \sqrt{\mu})^2} \right) (1 + \beta) \]

\[
= \sum_{i=1}^{d} \frac{2}{(1 - \beta)\lambda_i} \left( \frac{4}{(\sqrt{\mu} + \sqrt{L})^2} \right) \left( \mu + L - \lambda_i \right) (1 + \beta) \]

\[
= \sum_{i=1}^{d} \frac{2(1 + \beta)}{(1 - \beta)\lambda_i(\mu + L - \lambda_i)} \]

\[
= \sum_{i=1}^{d} \frac{2}{\lambda_i(\mu + L - \lambda_i)} \left( \frac{1 + \left( \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2}{1 - \left( \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2} \right) \]

\[
= \sum_{i=1}^{d} \frac{2}{\lambda_i(\mu + L - \lambda_i)} \left( \frac{L + \mu}{4\sqrt{\mu L}} \right) \]

\[
= \sum_{i=1}^{d} \frac{2}{\lambda_i(\mu + L - \lambda_i)} \left( \sqrt{\kappa} + \frac{1}{\sqrt{\kappa}} \right) \]

\[
= \sum_{i=1}^{d} \frac{2}{\lambda_i(\mu + L - \lambda_i)} \mathcal{O}(\sqrt{\kappa}) \]

\[
= \text{Tr}(X_{SGD})\mathcal{O}(\sqrt{\kappa}) \]

The above calculation shows that the size of the asymptotic covariance for the momentum iterates is \( \mathcal{O}(\sqrt{\kappa}) \) larger than that of SGD iterates.

### A.3 Comparison with Gitman et al. (2019)

The iterates for QHM are as follows:

\[
d_k = (1 - \beta_k)g_k + \beta_k d_{k-1} \]

\[
x_{k+1} = x_k - \alpha_k (1 - \nu_k)g_k + \nu_k d_k. \]

When \( \nu_k \equiv 1 \), QHM recovers normalised SHB. Letting \( \alpha_k \equiv \alpha \) and \( \beta_k \equiv \beta \), the iterates for normalised SHB can be expressed as follows:

\[
x_{k+1} = x_k - \alpha' g_k + \beta(x_k - x_{k-1}), \]

where \( \alpha' = \alpha(1 - \beta) \). As \( \beta \to 1 \), the step-size of the SHB iterates, \( \alpha' \), goes to 0. This explains the improvement obtained by Gitman et al. (2019) in the stationary distribution of normalised SHB over SGD. Using Theorem 3.3, we find that the sample complexity of normalised SHB iterates turns out to be \( \tilde{\Theta}(\frac{K}{\lambda_2^2}) \). Thus, the improvement in the stationary distribution in normalised SHB is neutralised by the worsening of the bias term which leads to the same sample complexity as in SGD.
B  Proof of Main Results

B.1 Proof of Theorem 3.1

We start by transforming the iterates by defining $\tilde{\theta}_n = \theta_n - \theta^*$ where $\theta^* = A^{-1}b$, setting the step-size $\alpha_n$ to a constant $\alpha$ and momentum parameter $\eta_n$ to $\eta$. The iterates (1) and (2) can be written as:

\[
\begin{align*}
\theta_{n+1} - \theta^* &= \theta_n - \theta^* + \alpha (A\theta^* - A\theta_n + M_{n+1}) \\
\theta_{n+1} - \theta^* &= \theta_n - \theta^* + \alpha (A\theta^* - A\theta_n + M_{n+1}) + \eta((\theta_n - \theta^*) - (\theta_{n-1} - \theta^*))
\end{align*}
\]

(9) (10)

Let $\tilde{\theta}_n = \theta_n - \theta^*$. Then, equation (9) can be rewritten as:

\[
\tilde{\theta}_n = \tilde{\theta}_{n-1} + \alpha (-A\tilde{\theta}_{n-1} + M_n) = (I - \alpha A)\tilde{\theta}_{n-1} + \alpha M_n
\]

Let

\[
\tilde{\eta}_n = \theta_n - \theta^* = \max \{ I, M_{i+1} \} = \max \{ I, M_{i+1} \} = \max \{ I, M_{i+1} \} = \max (I, A) \theta_0 + \alpha \sum_{i=0}^{n-1} [(I - \alpha A)^{n-1-i} M_{i+1}]
\]

Taking the square of the norm on both sides of the above equation, we obtain

\[
\| \tilde{\theta}_n \|^2 = \| (I - \alpha A)^{\ast} \tilde{\theta}_0 \|^2 + 2\alpha \left( (I - \alpha A)^{\ast} \tilde{\theta}_0 \right)^T \left( \sum_{i=0}^{n-1} (I - \alpha A)^{n-1-i} M_{i+1} \right)
\]

\[
+ \alpha^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((I - \alpha A)^{n-1-i} M_{i+1})^T ((I - \alpha A)^{(n-1-j)} M_{j+1})
\]

Now we take expectation on both sides to obtain

\[
E [\| \tilde{\theta}_n \|^2] \leq \| (I - \alpha A)^{\ast} \tilde{\theta}_0 \|^2 + 2\alpha \left( (I - \alpha A)^{\ast} \tilde{\theta}_0 \right)^T \left( \sum_{i=0}^{n} (I - \alpha A)^{(n-1-i)} E [M_{i+1}] \right)
\]

\[
+ \alpha^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E [(I - \alpha A)^{(n-1-i)} M_{i+1})^T ((I - \alpha A)^{(n-1-j)} M_{j+1})]
\]

Now, from Assumption 2 $E [M_{i+1}] = E [E [M_{i+1} | F_i]] = 0$. Therefore the second term becomes 0. Next consider the term inside the double summation. First consider the case $i \neq j$. Without loss of generality, suppose $i < j$.

\[
E \left[ M_{i+1}^T (I - \alpha A)^{(n-1-i)} (I - \alpha A)^{(n-1-j)} M_{j+1} \right]
\]

\[
= E \left[ M_{i+1}^T (I - \alpha A)^{(n-1-i)} (I - \alpha A)^{(n-1-j)} M_{j+1} E [F_j] \right]
\]

\[
= E \left[ M_{i+1}^T (I - \alpha A)^{(n-1-i)} (I - \alpha A)^{(n-1-j)} E [M_{j+1} | F_j] \right] = 0
\]

The last equality follows from Assumption 2. When $i = j$,

\[
E \left[ M_{i+1}^T (I - \alpha A)^{(n-1-i)} (I - \alpha A)^{(n-1-i)} M_{i+1} \right]
\]

\[
\leq E \left[ \| (I - \alpha A)^{(n-1-i)} \|^2 E [\| M_{i+1} \|^2 | F_i] \right] \leq \| (I - \alpha A)^{(n-1-i)} \|^2 K \left( 1 + E [\| \tilde{\theta}_i \|^2] \right)
\]

Substituting the above values and using $\Lambda = \| \tilde{\theta}_0 - \theta^* \|^2$ we get

\[
E [\| \tilde{\theta}_n \|^2] \leq \| (I - \alpha A)^n \|^2 \Lambda + \alpha^2 K \sum_{i=0}^{n-1} \| (I - \alpha A)^{(n-1-i)} \|^2 (1 + E [\| \tilde{\theta}_i \|^2])
\]

We next use the following lemma to bound $\| (I - \alpha A)^i \|$.

**Lemma B.1.** Let, $M \in \mathbb{R}^{d \times d}$ be a matrix and $\lambda_i(M)$ denote the $i^{th}$ eigen-value of $M$. Then, $\forall \delta > 0$

\[
\| M^n \| \leq C_\delta (\rho(M) + \delta)^n
\]

where $\rho(M) = \max_i | \lambda_i(M) |$ is the spectral radius of $M$ and $C_\delta$ is a constant that depends on $\delta$. Furthermore, if the eigen-values of $M$ are distinct, then

\[
\| M^n \| \leq C (\rho(M))^n
\]

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Proof. See Appendix C.1

We let $\lambda_{\min}(A) = \min_i \lambda_i(A)$. Using Assumption 1 and Lemma B.1 we have

$$E[||\hat{\theta}_n||^2] \leq C^2(1 - \alpha \lambda_{\min}(A))^2nK + C^2\alpha^2K \sum_{i=0}^{n-1} (1 - \alpha \lambda_{\min}(A))^{2(n-1-i)}(1 + E[||\hat{\theta}_i||^2])$$

where, $C = \frac{\sqrt{d}}{\sigma_{\min}(S)\sigma_{\min}(S^{-1})}$.

$S$ is the matrix in Jordan decomposition of $A$ and $\sigma_{\min}(S)$ is the smallest singular value of $S$. We define the sequence $\{U_k\}$ as below:

$$U_k = C^2(1 - \alpha \lambda_{\min}(A))^2kK + C^2\alpha^2K \sum_{i=0}^{k-1} (1 - \alpha \lambda_{\min}(A))^{2(k-1-i)}(1 + U_i)$$

Observe that $E[||\hat{\theta}_n||^2] \leq U_n$ and that the sequence $\{U_k\}$ satisfies

$$U_{k+1} = (1 - \alpha \lambda_{\min}(A))^2U_k + C^2K\alpha^2(1 + U_k); \quad U_0 = C^2\Lambda$$

Therefore, we have

$$U_{k+1} = ((1 - \alpha \lambda_{\min}(A))^2 + C^2K\alpha^2)U_k + \alpha^2C^2K$$

To ensure that $(1 - \alpha \lambda_{\min}(A))^2 + C^2K\alpha^2 \leq (1 - \alpha \lambda_{\min}(A)/2)^2$, choose $\alpha$ as follows:

$$\alpha^2\lambda_{\min}(A)^2 - 2\alpha \lambda_{\min}(A) + C^2K\alpha^2 \leq \frac{\alpha^2\lambda_{\min}(A)^2}{4} - \alpha \lambda_{\min}(A)$$

or $\alpha \leq \frac{\lambda_{\min}(A)}{\frac{3}{4}\lambda_{\min}(A)^2 + C^2K}$

$$U_n \leq \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^2 U_{n-1} + \alpha^2C^2K$$

$$\leq \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^{2n} U_0 + \alpha^2C^2K \sum_{i=0}^{n-1} \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^{2i}$$

$$\leq \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^{2n} U_0 + \alpha^2C^2K \frac{1}{1 - \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^2}$$

$$\leq \left(1 - \frac{\alpha \lambda_{\min}(A)}{2}\right)^{2n} U_0 + \alpha^2C^2K \frac{2}{\alpha \lambda_{\min}(A)}$$

We assume that $\alpha \leq \frac{1}{\lambda_{\min}(A)}$ and therefore $(1 - \frac{\alpha \lambda_{\min}(A)}{2})^2 \leq e^{-\alpha \lambda_{\min}(A)}$.

$$U_n \leq e^{-n\alpha \lambda_{\min}(A)}C^2\Lambda + \frac{2\alpha^2C^2K}{\alpha \lambda_{\min}(A)}$$

Choose $\alpha$ as below:

$$\alpha \leq \frac{\epsilon \lambda_{\min}(A)}{4C^2K}$$

Then,

$$\frac{2\alpha^2C^2K}{\alpha \lambda_{\min}(A)} \leq \frac{\epsilon}{2} \Rightarrow E[||\hat{\theta}_n||^2] \leq U_n \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

when the sample complexity is:

$$n = \frac{1}{\alpha \lambda_{\min}(A)} \log \left(\frac{2C^2\Lambda}{\epsilon}\right)$$
B.2 Proof of Theorem 3.2

Equation (10) can be re-written as
\[ \tilde{\theta}_{n+1} = (I - \alpha A)\tilde{\theta}_n + \alpha (M_{n+1}) + \eta (\tilde{\theta}_n - \tilde{\theta}_{n-1}) \]

This can be re-written as:
\[ \begin{bmatrix} \tilde{\theta}_{n+1} \\ \tilde{\theta}_n \end{bmatrix} = \begin{bmatrix} I - \alpha A + \eta I & -\eta I \\ I & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta}_n \\ \tilde{\theta}_{n-1} \end{bmatrix} + \alpha \begin{bmatrix} M_{n+1} \\ 0 \end{bmatrix} \]

Let us define
\[ \tilde{X}_n \triangleq \begin{bmatrix} \tilde{\theta}_n \\ \tilde{\theta}_{n-1} \end{bmatrix}, P \triangleq \begin{bmatrix} I - \alpha A + \eta I & -\eta I \\ I & 0 \end{bmatrix} \] and \( W_n \triangleq \begin{bmatrix} M_{n+1} \\ 0 \end{bmatrix} \).

Note that \( E[W_{n+1}|\mathcal{F}_n] = 0 \) and \( E[\|W_{n+1}\|^2|\mathcal{F}_n] = E[\|M_{n+1}\|^2|\mathcal{F}_n] \leq K(1 + E[\|\tilde{\theta}_n\|^2]) \leq K(1 + E[\|\tilde{X}_n\|^2]) \). It follows that,
\[ \tilde{X}_n = P \tilde{X}_{n-1} + \alpha W_n = P^n \tilde{X}_0 + \alpha \sum_{i=0}^{n-1} P^{n-1-i} W_{i+1} \]

The norm square of the above equation gives:
\[ \|\tilde{X}_n\|^2 = \|P^n \tilde{X}_0\|^2 + \alpha (P^n \tilde{X}_0)^T \sum_{i=0}^{n-1} P^{n-1-i} W_{i+1} \]
\[ + \alpha^2 (\sum_{i=0}^{n-1} P^{n-1-i} W_{i+1})^T \sum_{i=0}^{n-1} P^{n-1-i} W_{i+1} \]

Taking expectation on both sides as well as using the fact that \( E[W_{k+1}|\mathcal{F}_k] = 0 \) and \( E[\|W_{k+1}\|^2|\mathcal{F}_k] \leq K(1 + E[\|\tilde{X}_k\|^2]) \), we have
\[ E[\|\tilde{X}_n\|^2] \leq \|P^n\|^2 \|\tilde{X}_0\|^2 + \alpha^2 K \sum_{i=0}^{n-1} \|P^{n-1-i}\|^2 (1 + E[\|\tilde{X}_i\|^2]) \]

Without loss of generality assume \( \tilde{\theta}_{-1} = 0 \). Therefore, \( \|\tilde{X}_0\|^2 = \|\tilde{\theta}_0\|^2 = \Lambda \). As before, for a matrix \( M \), let \( \rho(M) = \max_i |\lambda_i(M)| \) denote the spectral radius of \( M \). Next, we compute \( \rho(P) \). Consider the characteristic equation of \( P \):
\[ \det \left( I - \alpha A + \eta I - \mu I - \eta I \right) = 0 \]

When \( A_{21} \) and \( A_{22} \) commute, we have the following formula for determinant of a block matrix (Horn and Johnson (1990)):
\[ \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det (A_{11} A_{22} - A_{12} A_{21}) \]

Using this, the characteristic equation of \( P \) simplifies to:
\[ \det (-\mu I + \alpha \mu A - \eta I + \mu^2 I + \eta I) = 0 \]

We note that when \( \mu = 0 \), the LHS of the above equation becomes \( \det(\eta I) \). Thus, \( \mu = 0 \) can never be a solution of the characteristic equation of \( P \) whenever \( \eta \neq 0 \). We now further simplify the characteristic equation of \( P \) to a more convenient form:
\[ \det \left( A - I \left( \frac{\mu + \eta \mu - \mu^2 - \eta}{\alpha \mu} \right) \right) = 0 \]

The only zeros of the characteristic equation of a matrix are its eigenvalues. Let \( \lambda_i(A) \) be the eigen-value of \( A \) with \( \lambda_i(A) = \frac{\mu + \eta \mu - \mu^2 - \eta}{\alpha \mu} \) so that
\[ \mu^2 + \mu(\lambda_i(A) - 1 - \eta) + \eta = 0 \]

The above is a quadratic equation in \( \mu \) and the solution is given by
\[ \mu = \frac{-(\lambda_i(A)\alpha - 1 - \eta) \pm \sqrt{(\lambda_i(A)\alpha - 1 - \eta)^2 - 4\eta}}{2} \]
When \((\lambda_i(A)\alpha - 1 - \eta)^2 - 4\eta \leq 0\), the absolute value of eigenvalues of \(P\) are independent of \(\alpha\) and

\[
|\mu_j(P)| = \frac{1}{2} \left( \sqrt{(\lambda_i(A)\alpha - 1 - \eta)^2 + (\lambda_i(A)\alpha - 1 - \eta)^2 - 4\eta} \right) = \sqrt{\eta}
\]

To ensure that \((\lambda_i(A)\alpha - 1 - \eta)^2 - 4\eta \leq 0\), we must have

\[
(\alpha\lambda_i(A) + 1) - 2\sqrt{\lambda_i(A)\alpha \leq \eta \leq (\alpha\lambda_i(A) + 1) + 2\sqrt{\lambda_i(A)\alpha}
\]

\[
\left(1 - \sqrt{\lambda_i(A)\alpha} \right)^2 \leq \eta \leq \left(1 + \sqrt{\lambda_i(A)\alpha} \right)^2
\]

For the spectral radius of \(P\) to be \(\sqrt{\eta}\), the above must hold for all \(i\). We choose \(\alpha\) as:

\[
\alpha \leq \left( \frac{2}{\sqrt{\lambda_{\min}(A)} + \sqrt{\lambda_{\max}(A)}} \right)^2
\]

and \(\eta\) as:

\[
(1 - \sqrt{\lambda_{\min}(A)\alpha} \leq \eta \leq (1 + \sqrt{\lambda_{\min}(A)\alpha})^2
\]

Observe that if we choose the momentum parameter \(\eta = \left(1 - \sqrt{\lambda_{\min}(A)\alpha} \right)^2\), then \(P\) has two repeated roots since

\[
\sqrt{\lambda_i(A)\alpha - 1 - \eta}^2 - 4\eta = 0.
\]

To ensure that \(P\) does not have any repeated root we choose the momentum parameter \(\eta = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^2\). Therefore, \(\rho(P) = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^2\). Using Assumption 1 and Lemma B.1 we get

\[
\mathbb{E}[\|\tilde{X}_n\|^2] \leq \hat{C}^2 \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^{2n} \Lambda + \alpha^2 \hat{C}^2 K \sum_{i=0}^{n-1} \left(1 - \frac{\lambda_{\min}(A)\alpha}{2} \right)^{2(n-1-i)} (1 + \mathbb{E}[\|\tilde{X}_i\|^2])
\]

However, unlike in Theorem 3.1, here the constant \(\hat{C}\) is not independent of \(\alpha\). The following lemma specifies an upper bound on \(\hat{C}\).

**Lemma B.2.** \(\hat{C} \leq C \frac{5}{\sqrt{\alpha \lambda_{\min}(A)}}\), where \(C\) is as defined in [11].

**Proof.** See Appendix C.2.

We define the sequence \(\{V_n\}\) as follows

\[
V_n = \hat{C}^2 \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^{2n} \Lambda + \alpha^2 \hat{C}^2 K \sum_{i=0}^{n-1} \left(1 - \frac{\lambda_{\min}(A)\alpha}{2} \right)^{2(n-1-i)} (1 + V_i)
\]

Observe that \(\mathbb{E}[\|\tilde{X}_n\|^2] \leq V_n\), and that \(\{V_k\}\) satisfies

\[
V_{k+1} = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^2 V_k + \hat{C}^2 K\alpha^2 (1 + V_k); \quad V_0 = \hat{C}^2 \Lambda
\]

Therefore, we have

\[
V_{k+1} = \left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^2 V_k + \hat{C}^2 K\alpha^2
\]

To ensure that \(\left(1 - \frac{\sqrt{\lambda_{\min}(A)\alpha}}{2} \right)^2 + \hat{C}^2 K\alpha^2 \leq \left(1 - \frac{\lambda_{\min}(A)\alpha}{4} \right)^2\) we need to choose \(\alpha\) such that

\[
1 + \frac{\lambda_{\min}(A)\alpha}{4} - \sqrt{\lambda_{\min}(A)\alpha} + \hat{C}^2 K\alpha^2 \leq 1 + \frac{\lambda_{\min}(A)\alpha}{16} - \frac{\lambda_{\min}(A)\alpha}{2}
\]

or

\[
\frac{3\alpha \lambda_{\min}(A)}{16} + \hat{C}^2 K\alpha^2 \leq \frac{\lambda_{\min}(A)}{2}
\]
Next, using Lemma B.2, the above can be ensured by choosing $\alpha$ such that
\[
\frac{3\sqrt{\alpha} \lambda_{\text{min}}(A)}{16} + C^2 \frac{25}{\alpha \lambda_{\text{min}}} K \alpha^2 \leq \frac{\sqrt{\lambda_{\text{min}}(A)}}{2}
\]

or
\[
\alpha \leq \left( \frac{\sqrt{\lambda_{\text{min}}(A)}}{3 \lambda_{\text{min}}(A) + \frac{25CK}{\lambda_{\text{min}}(A)}} \right)^2.
\]

The recursion for the sequence $\{V_{k+1}\}$ then follows
\[
V_{k+1} \leq \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^2 V_k + \hat{C}^2 K \alpha^2
\]

Unrolling the recursion, we get
\[
V_n \leq \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^{2n} V_0 + \hat{C}^2 K \alpha^2 \sum_{i=0}^{n-1} \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^{2i}
\]
\[
\leq \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^{2n} V_0 + \hat{C}^2 K \alpha^2 \frac{1}{1 - \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^2}
\]
\[
\leq \left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^{2n} V_0 + \hat{C}^2 K \alpha^2 \frac{4}{\sqrt{\alpha \lambda_{\text{min}}(A)}}
\]

Further it follows from $\alpha \leq \left( \frac{2}{\sqrt{\lambda_{\text{min}}(A) + \sqrt{\lambda_{\text{max}}(A)}}} \right)^2$ that $\alpha \leq \frac{1}{\lambda_{\text{min}}(A)}$ and $\left( 1 - \frac{\sqrt{\lambda_{\text{min}}(A)} \alpha}{4} \right)^2 \leq e^{-\frac{\lambda_{\text{min}}(A)n}{2}}$.

\[
V_n \leq e^{-\frac{\sqrt{\lambda_{\text{min}}(A)n}}{2}} \hat{C}^2 K + \frac{4 \alpha^2 \hat{C}^2 K}{\sqrt{\alpha \lambda_{\text{min}}(A)}}
\]

Again using Lemma B.2
\[
V_n \leq e^{-\frac{\sqrt{\lambda_{\text{min}}(A)n}}{2}} \frac{25C^2}{\lambda_{\text{min}}(A)} \alpha^2 \Lambda + \alpha^2 \frac{100C^2 K}{(\lambda_{\text{min}}(A))^{3/2}}
\]

Observe that
\[
\frac{e^{-\frac{\sqrt{\lambda_{\text{min}}(A)n}}{2}} \lambda_{\text{min}}(A) \alpha}{\lambda_{\text{min}}(A)} \leq e^{-\frac{\sqrt{\lambda_{\text{min}}(A)n}}{2}} \leq \frac{4}{\sqrt{\alpha \lambda_{\text{min}}(A)}} \log \left( \frac{1}{\lambda_{\text{min}}(A) \alpha} \right).
\]

Let $n$ be as above. Then,
\[
V_n \leq 25C^2 \Lambda e^{-\frac{\sqrt{\lambda_{\text{min}}(A)n}}{2}} + \sqrt{\alpha} \frac{100C^2 K}{(\lambda_{\text{min}}(A))^{3/2}}
\]

Choose $\alpha$ as below:
\[
\alpha \leq \left( \frac{\epsilon (\lambda_{\text{min}}(A))^{3/2}}{200C^2 K} \right)^2
\]

Then,
\[
\sqrt{\alpha} \frac{100C^2 K}{(\lambda_{\text{min}}(A))^{3/2}} \leq \frac{\epsilon}{2} \Rightarrow E[\|\hat{\theta}_n\|^2] \leq E[\|\hat{X}_n\|^2] \leq V_n \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

when $n$ is as follows:
\[
n = \frac{4}{\sqrt{\alpha \lambda_{\text{min}}(A)}} \log \left( \frac{50C^2 \Lambda}{\epsilon} \right), \quad \frac{4}{\sqrt{\alpha \lambda_{\text{min}}(A)}} \log \left( \frac{1}{\lambda_{\text{min}}(A) \alpha} \right)
\]

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B.3 Proof of Theorem 3.3

As in proof of Theorem 3.2 we have:

\[ \hat{X}_n = P\tilde{X}_{n-1} + \alpha W_n = P^n\tilde{X}_0 + \alpha \sum_{j=0}^{n-1} P^{n-1-j} W_{j+1}, \]

where,

\[ \tilde{X}_n \triangleq \begin{bmatrix} \hat{\theta}_n \\ \theta_{n-1} \end{bmatrix}, \quad P \triangleq \begin{bmatrix} 1 - \alpha \lambda + \eta & -\eta \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad W_n \triangleq M_n \begin{bmatrix} \eta \\ 0 \end{bmatrix}. \]

We prove the claim by showing that the expected error \( \mathbb{E}[\|\tilde{X}_n\|^2] \neq \epsilon \) for some \( \alpha \in \tilde{O}(\frac{K}{\epsilon^2 \lambda^2}). \) We show that this holds for small enough \( \epsilon \) and all choices of \( \alpha > 0 \) and \( \eta \in [0, 1]. \) We separately handle the case when \( \eta \in \left[ 0, \left(1 - \sqrt{\alpha \lambda}\right)^2 \right) \), \( \eta = \left(1 - \sqrt{\alpha \lambda}\right)^2 \) and \( \eta \in \left( (1 - \sqrt{\alpha \lambda})^2, 1 \right]. \)

First consider \( \eta \in \left( (1 - \sqrt{\alpha \lambda})^2, 1 \right]. \) We have that,

\[
\mathbb{E}[\|\tilde{X}_n\|^2] = \mathbb{E}[\|P^n\tilde{X}_0 + \alpha \sum_{j=0}^{n-1} P^{n-1-j} W_{j+1}\|^2]
\[
= \|P^n\tilde{X}_0\|^2 + \mathbb{E}[\alpha^2 \sum_{j=0}^{n-1} \|P^{n-1-j} W_{j+1}\|^2]
\[
\geq \|P^n\tilde{X}_0\|^2 + \alpha^2 K \sum_{j=0}^{n-1} \|P^{n-1-j} e_1\|^2,
\]

where \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), the second equality follows since each \( W_{j+1} \) is a martingale difference sequence and the inequality follows from Assumption 3. The bias term:

\[
\|P^n\tilde{X}_0\|^2 = \|P^n\|^2 \|\tilde{X}_0\|^2 \quad \text{for some } \tilde{X}_0
\[
\geq \rho(P)^{2n} \|\tilde{X}_0\|^2
\]

for \( \eta = \left(1 - \sqrt{\alpha \lambda} + \delta\right)^2 \), \( 0 < \delta \leq \sqrt{\alpha \lambda}, \)

\[
\|P^n\tilde{X}_0\| \geq (1 - \sqrt{\alpha \lambda} + \delta)^2 \|\tilde{X}_0\|^2
\[
\geq (1 - \sqrt{\alpha \lambda})^2 \|\tilde{X}_0\|^2
\[
\geq e^{-\frac{\delta}{\sqrt{\alpha \lambda}}} \cdot 2^n \|\tilde{X}_0\|^2
\]

Next we consider the variance term. Let \( P = \hat{S}D\hat{S}^{-1}, \) where \( D = \begin{pmatrix} \mu_- & 0 \\ 0 & \mu_+ \end{pmatrix} \) and \( \mu_- \) and \( \mu_+ \) are the eigenvalues of \( P. \) One choice of \( \hat{S} \) and \( \hat{S}^{-1} \) are (See proof of Lemma B.2):

\[
\hat{S}^{-1} = \frac{1}{\mu_+ - \mu_-} \begin{bmatrix} 1 & -\mu_- \\ -1 & \mu_+ \end{bmatrix}, \quad \text{and} \quad \hat{S} = \begin{bmatrix} \mu_+ & \mu_- \\ 1 & -1 \end{bmatrix}
\]

Since \( P^j = \hat{S}D^j\hat{S}^{-1}, \) therefore

\[
P^j = \frac{1}{\mu_+ - \mu_-} \begin{pmatrix} \mu_+ & \mu_- \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_+ & \mu_- \\ 0 & \mu_- \end{pmatrix} \begin{pmatrix} 1 & -\mu_- \\ -1 & \mu_+ \end{pmatrix}
\]

\[
21
\]
Therefore,

\[
\sin \omega \leq (6 + 5) \frac{\eta}{\rho} = 2 \rho \sin \omega \leq 2 \sin(j + 1) \omega.
\]

Using these we get:

\[
P^i e_1 = \frac{r^j}{\sin \omega} \left( \frac{\sin(j + 1) \omega}{\sin j \omega} \right).
\]

Using these results and the fact that \( r = |\mu_\omega| \leq 1 \) for the choice of \( \eta \), the variance term can be lower bounded as follows:

\[
\alpha^2 K \sum_{j=0}^{n-1} \|P^j e_1\|^2 \geq \alpha^2 K \frac{\rho(P)^{2n}}{\sin^2 \omega} \sum_{j=0}^{n-1} \sin^2(j + 1) \omega.
\]

Let \( k = j + 1 \), and define the set \( I = \{ k \in \{1, 2, \ldots, n\} : \sin^2 k \omega \geq \frac{1}{4\sin^2 \omega} \} \). We now show that \( |I| = \Theta(n) \).

\[
\sin^2 k \omega \geq \frac{1}{4\sin^2 \omega} \Rightarrow \sin^2 k \omega \geq \frac{1}{4}
\]

\[
\Rightarrow \sin k \omega \geq \frac{1}{2} \text{ or } \sin k \omega \leq -\frac{1}{2}
\]

Therefore, \( I = \{ k \in \mathbb{N} \cap I_m : I_m = \left[ \frac{(6m+1)\pi}{6\omega}, \frac{(6m+5)\pi}{6\omega} \right], m \in \mathbb{N}, \frac{(6m+1)\pi}{6\omega} \leq n \leq \frac{(6m+5)\pi}{6\omega} \} \). Now, \( \frac{(6m+1)\pi}{6\omega} \leq n \leq \frac{(6m+5)\pi}{6\omega} \)

implies \( \frac{n\omega}{\pi} - \frac{5}{6} \leq m \leq \frac{n\omega}{\pi} - \frac{1}{6} \). Also note that \( |I_m| = \frac{2\pi}{\pi} \) and therefore \( |I| = m |I_m| = \Theta(n) \). With this we have:

\[
\alpha^2 K \sum_{j=0}^{n-1} \left\| P^j e_1 \right\|^2 \geq \alpha^2 K \frac{\rho(P)^{2n}}{4\sin^2 \omega} \Theta(n).
\]

Finally we lower bound \( \frac{1}{\sin^2 \omega} \). Recall that \( \mu_+ = r(\cos \omega + i \sin \omega) \) and \( \mu_- = r(\cos \omega - i \sin \omega) \). Therefore,

\[
\sin \omega = \frac{\sqrt{|\Delta|}}{2\rho(P)}, \text{ where } |\Delta| = 4\eta - (1 - \alpha \lambda + \eta)^2.
\]

For \( \eta = (1 - \sqrt{\alpha \lambda + \delta})^2 \),

\[
1 - \alpha \lambda + \eta = 1 - \alpha \lambda + (1 - \sqrt{\alpha \lambda + \delta})^2
\]

\[
= 1 - \alpha \lambda + (1 + \alpha \lambda + \delta^2 - 2\sqrt{\alpha \lambda + 2\delta} - 2\sqrt{\alpha \lambda \delta})
\]

\[
= (2 - 2\sqrt{\alpha \lambda + 2\delta} + (\delta^2 - 2\sqrt{\alpha \lambda \delta})
\]

\[
= 2\sqrt{\eta} + (\delta^2 - 2\sqrt{\alpha \lambda \delta})
\]

This implies that,

\[
|\Delta| = (2\sqrt{\eta})^2 - (2\sqrt{\eta} + (\delta^2 - 2\sqrt{\alpha \lambda \delta})^2
\]

\[
= (2\sqrt{\alpha \lambda \delta} - \delta^2 - \delta^2 - 2\sqrt{\alpha \lambda \delta})
\]

This gives the desired lower bound.
We next consider the case when $\eta$. We already have shown that to ensure that the bias term is the 
Any value of $\eta$ larger than this would imply the error term to be an order larger than $\epsilon$. Any value smaller than this makes the sample complexity $n$ an order larger than $\tilde{\Theta}(\frac{K}{\sqrt{\alpha \lambda}})$, immediately proving the Theorem. This shows that $\exists \epsilon > 0$ such that for some $n \in \tilde{\Theta}(\frac{K}{\sqrt{\alpha \lambda}})$, $\mathbb{E}[\|\theta_t - \theta^*\|] \geq \Theta(\epsilon)$.

We next consider the case when $\eta = (1 - \sqrt{\alpha \lambda})^2$. As in the previous case, to reach the $\epsilon$ boundary around the solution, it is necessary that $n = \tilde{\Theta}(\frac{K^3}{\epsilon^2\lambda^2})$ and we next seek to find a lower bound on $\|P^ne_1\|^2$. In this case, since the matrix $P$ has repeated eigen values, it is not diagonalizable and therefore we consider the Jordan decomposition as follows:

$$P = \tilde{S} \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \tilde{S}^{-1},$$

where $\mu$ is the eigen value of $P$ with multiplicity 2. Let $\tilde{S} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$ and therefore,

$$\begin{bmatrix} 1 - \alpha \lambda + \eta & -\eta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}$$
Solving the above system of equations, we get:

\[
\hat{S} = \left( \begin{array}{cc} \mu & 1 + \mu \\ 1 & 1 \end{array} \right), \hat{S}^{-1} = \left( \begin{array}{cc} -1 & 1 + \mu \\ 1 & -\mu \end{array} \right) .
\]

Therefore,

\[
P^n = \left( \begin{array}{cc} \mu & 1 + \mu \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} \mu^n & n\mu^{n-1} \\ 0 & \mu^n \end{array} \right) \left( \begin{array}{cc} -1 & 1 + \mu \\ 1 & -\mu \end{array} \right)
\]

\[
= \left( \begin{array}{cc} \mu & 1 + \mu \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} -\mu^n + n\mu^{n-1} & \mu^n(1 + \mu) - n\mu^n \\ -\mu^n + n\mu^{n-1} + \mu^n & \mu^2(1 + \mu) - n\mu^{n+1} \end{array} \right)
\]

\[
= \left( \begin{array}{cc} (n+1)\mu^n - n\mu^{n-1} & \frac{\mu^n}{(1-n)} \end{array} \right)
\]

From the above result we have \( P^n e_1 = \left( \begin{array}{c} (n+1)\mu^n \\ n\mu^{n-1} \end{array} \right) \) and therefore,

\[
\| P^n e_1 \|^2 \geq n^2 \mu^{2n}.
\]

Using the fact that \( \sum_{j=0}^{n-1} j^2 = \Theta(n^3) \) and \( n = \tilde{\Theta} \left( \frac{1}{\sqrt{\alpha}} \right) \), we get the following bound on the variance term:

\[
\alpha^2 K \sum_{j=0}^{n-1} \| P^j e_1 \|^2 \geq K (1 - \sqrt{\alpha})^{2n} \tilde{\Theta} \left( \frac{\sqrt{\alpha}}{\lambda^{3/2}} \right)
\]

Using similar arguments as in the previous case, we can show that for some \( n \in \tilde{\Theta}(\frac{K}{\sqrt{\lambda}^2}) \), \( \mathbb{E}[\| \theta - \theta^* \|] \geq \Theta(\epsilon) \).

Finally, we consider the case where \( \eta \in [0, (1 - \sqrt{\alpha})^2) \). Let \( n = \frac{K}{64\lambda^2} \log(\frac{1}{\epsilon}) \) and fix \( \eta \in [0, (1 - \sqrt{\alpha})^2) \). From the previous calculation for \( \eta \in ((1 - \sqrt{\alpha})^2, 1] \)

\[
\mathbb{E}[\| \hat{X}_n \|^2] \geq \| P^n e_1 \|^2 + \alpha^2 K \sum_{j=0}^{n-1} \| P^j e_1 \|^2 .
\]

If \( \alpha \) is such that \( \| P^n e_1 \|^2 \geq \epsilon \), then the claim follows for this choice of \( \alpha \) and \( \eta \). If \( \alpha \) is such that \( \| P^n e_1 \|^2 < \epsilon \), then we have \( \rho(P)^{2n} < \epsilon \). This is because

\[
\| P^n e_1 \|^2 = \left( \frac{\mu_+^{n+1} - \mu_-^{n+1}}{\mu_+ - \mu_-} \right)^2 + \left( \frac{\mu_+^n - \mu_-^n}{\mu_+ - \mu_-} \right)^2
\]

\[
> \left( \frac{\mu_+^{n+1} - \mu_-^{n+1}}{\mu_+ - \mu_-} \right)^2
\]

\[
= \mu_+^{2n} \left( \frac{1 - (\mu_-/\mu_+)^{n+1}}{1 - (\mu_-/\mu_+)} \right)^2
\]

\[
> \mu_+^{2n}
\]

and \( \mu_+ = \rho(P) \), since now the eigenvalues of \( P \) are all real and positive. Then,

\[
\| P^n e_1 \|^2 \geq \rho(P)^{2n} \geq e^{-2n \frac{1 - \rho(P)}{\rho(P)}}
\]

Thus, when \( \| P^n e_1 \| < \epsilon \), we have \( e^{-2n \frac{1 - \rho(P)}{\rho(P)}} < \epsilon \) and \( n = \frac{K}{64\lambda^2} \log(\frac{1}{\epsilon}) \geq \frac{\rho(P)}{\pi(1 - \rho(P))} \log(\frac{1}{\epsilon}) \). Thus, by re-arranging, and choosing \( \epsilon \) such that \( 1 \leq \frac{K}{32\lambda^2} \), we have

\[
\frac{1}{1 - \rho(P)} \leq \frac{K}{32\lambda^2} + 1 \leq \frac{K}{16\epsilon^2} .
\] (17)
We shall now show that the second term in the lower bound on the expected error, $\alpha^2 K \sum_{j=0}^{n-1} \|P^j e_1\|^2 > \epsilon$. For this purpose, define $h(\eta, \alpha \lambda) := \frac{(1-\mu^2_1)(1-\mu^2_\alpha)(1-\mu_+\mu_-)}{(\alpha \lambda)^2}$ ($\mu_+$ and $\mu_-$ are functions of only $\eta$ and $\alpha \lambda$). We have the following lemma:

**Lemma B.3.** If $\|P^n e_1\|^2 < \epsilon$, for a sufficiently small $\epsilon$, then

$$\alpha^2 K \sum_{j=0}^{n-1} \|P^j e_1\|^2 \geq \frac{(K/\lambda^2)}{2h(\eta, \alpha \lambda)}$$

for all $\alpha, \lambda > 0$ and $\eta \in [0, (1 - \sqrt{\alpha \lambda})/2]$.

*Proof.* See Appendix C.3  

Additionally, we also have the following:

**Lemma B.4.** For all $\eta \in [0, (1 - \sqrt{\alpha \lambda})/2)$, $h(\eta, \alpha \lambda) \leq \frac{8}{1 - \rho(P)}$.

*Proof.* See Appendix C.4  

From Lemma B.4 and (17) it follows that $h(\eta, \alpha \lambda) \leq \frac{K}{2n^2}$. Combining this bound and Lemma B.3

$$\alpha^2 K \sum_{j=0}^{n-1} \|P^j e_1\|^2 \geq \frac{(K/\lambda^2)}{2h(\eta, \alpha \lambda)} \geq \epsilon.$$  

This completes the proof for the univariate case.

**B.3.1 Multivariate Case**

Finally, for the multivariate case, we study the transformed iterates $\tilde{Y}_n = Z \tilde{X}_n$, where $Z := E_{d \times d} \left( \begin{array}{cc} S & 0 \\ 0 & S \end{array} \right)$ (see Appendix C.2). Since

$$\tilde{X}_n = P \tilde{X}_{n-1} + \alpha W_n$$

we get,

$$\tilde{Y}_n = Z \tilde{X}_n$$

$$= ZP \tilde{X}_{n-1} + \alpha ZW_n$$

$$= ZP \tilde{X}_{n-1} + \alpha ZW_n$$

$$= ZP \tilde{X}_{n-1} + \alpha ZW_n$$

$$= BP_{n-1} + \alpha ZW_n,$$  

where

$$B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_k \end{pmatrix},$$

and

$$B_i = \begin{pmatrix} 1 + \eta - \alpha \lambda_i & -\eta \\ 1 & 0 \end{pmatrix}.$$  

We also point out that $ZW_n = E_{d \times d} \left( \begin{array}{cc} S & 0 \\ 0 & S \end{array} \right) \left( \begin{array}{c} M_n \\ 0 \end{array} \right)$ is a vector with 0’s in all its even co-ordinates. As a result, we notice that $\tilde{Y}_n$ can be split into $d$ iterates, each corresponding to the univariate case. Thus, results in the univariate case can be extended to that of the multivariate case.
We use Theorem 1 from Mokkadem and Pelletier [2006] and Proposition 4 from Dalal et al. [2020] to prove our result.

\[ \xi \]

\[ \text{Let } u_k = \frac{\beta_k - w\alpha_k}{\beta_{k-1}}, \text{ where } w > 0 \text{ is a constant and } \beta_k \text{ is a positive sequence. Then,} \]

\[ \begin{align*}
\theta_{k+1} &= \theta_k + \alpha_k (b - A\theta_k + M_{k+1}) + \frac{\beta_k - w\alpha_k}{\beta_{k-1}} (\theta_k - \theta_{k-1}) \\
\theta_{k+1} - \theta_k &= \frac{\beta_k}{\beta_{k-1}} (\theta_k - \theta_{k-1}) + \alpha_k \left( b - A\theta_k + M_{k+1} - w \left( \frac{\theta_k - \theta_{k-1}}{\beta_{k-1}} \right) \right) \\
\theta_{k+1} - \theta_k &= \frac{\theta_k - \theta_{k-1}}{\beta_{k-1}} + \alpha_k \left( b - A\theta_k + M_{k+1} - w \left( \frac{\theta_k - \theta_{k-1}}{\beta_{k-1}} \right) \right)
\end{align*} \]

Let \( u_{k+1} = \frac{\theta_{k+1} - \theta_k}{\beta_k} \) and \( \gamma_k = \frac{\alpha_k}{\beta_k} \). Then \( u_k \) can be re-written with the following two equations:

\[ \begin{align}
\theta_{k+1} &= \theta_k + \beta_k (u_k + \xi_k), \\
u_{k+1} &= u_k + \gamma_k (b - A\theta_k - wu_k + M_{k+1}).
\end{align} \]

where \( \xi_k = u_{k+1} - u_k \).

### B.4.2 Proof of Theorem 4

We use Theorem 1 from Mokkadem and Pelletier [2006] and Proposition 4 from Dalal et al. [2020] to prove our result. We first show that the assumptions in Mokkadem and Pelletier [2006] hold for the iterates given by \( (19) \) and \( (18) \).

1. To show that A1 holds, we use Lakshminarayanan and Bhatnagar [2017] to ensure that the iterates are stable and Theorem 6.2 from Borkar [2008] to ensure convergence to \( \theta^* = A^{-1}b \) and \( u^* = \frac{b - A\theta^*}{w} = 0 \).

2. A2 (i) holds with

\[ \begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix} = \begin{pmatrix}
0 & I \\
-A & -wI
\end{pmatrix} \]

and A2 (ii) holds due to Assumption 2.

3. A3 holds due to Assumption 1.

4. For A4(i) compare \( (18) \) and \( (19) \) with Mokkadem and Pelletier [2006] to see that \( V_{n+1} = 0 \) and \( W_{n+1} = M_{n+1} \). A4 (i) holds due to Assumption 2. Next, observe that A4 (ii) in Mokkadem and Pelletier [2006] can be relaxed to

\[ \lim_{n \to \infty} E \left[ \left( \frac{V_{n+1}}{W_{n+1}} \right) (V_{n+1}^T W_{n+1}^T) | \mathcal{F}_n \right] = \Gamma = \begin{pmatrix}
\Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & \Gamma_{22}\end{pmatrix} \]

where, \( \Gamma_{22} \) is a positive definite matrix. A4 (ii) in Mokkadem and Pelletier [2006] is required to ensure that the matrices \( \Gamma_{\theta}, \Sigma_{\theta} \) and \( \Sigma_{\mu} \) are well defined. Specifically, \( \Sigma_{\theta} \) and \( \Sigma_{\mu} \) are solutions of two lyapunov equations and require \( \Gamma_{\theta} \) and \( \Gamma_{22} \) to be positive definite. This follows from the fact that \( \Gamma_{22} \) is positive definite, irrespective of \( \Gamma_{11}, \Gamma_{12} \) and \( \Gamma_{21} \). Hence, the relaxed version of A4 (ii) holds due to Assumption 3. A4 (iii), holds due to Assumption 2. Finally, for A4 (iii), observe that the perturbation term in our case \( \xi_k = \gamma_k (b - A\theta_k - wu_k + M_{k+1}) = o(\sqrt{\beta_k}) \). This follows from the fact that \( \gamma > \frac{1}{2} \geq \frac{\beta}{2} \) and therefore A4 (iv) holds.

From Theorem 1 of Mokkadem and Pelletier [2006] we have

\[ \begin{align}
n^\beta/2 (\theta_n - \theta^*) & \Rightarrow \mathcal{N}(0, \Sigma_{\theta})
\end{align} \]

for some co-variance matrix \( \Sigma_{\theta} \). It follows that, for every \( \epsilon \), there are constants \( c' \) and \( n_0 \) such that \( P(\|\theta_n - \theta^*\| < c'n^{-\beta/2}) \leq \epsilon \forall n \geq n_0 \). (see Theorem 13 in Dalal et al. [2020]). Therefore we have, \( P(\|\theta_n - \theta^*\| \geq c'n^{-\beta/2}) \geq 1 - \epsilon \forall n \geq n_0 \). Observe that

\[ \|\theta_n - \theta^*\| = \|\theta_n - \theta^*\| I_{\{\|\theta_n - \theta^*\| \leq c'n^{-\beta/2}\}} + \|\theta_n - \theta^*\| I_{\{\|\theta_n - \theta^*\| > c'n^{-\beta/2}\}} \]

and let \( c = c'(1 - \epsilon) \). It then follows that, \( E[\|\theta_n - \theta^*\|] \geq cn^{-\beta/2}(1 - \epsilon) \forall n \geq n_0 \).
C Proof of Auxiliary Lemmas

C.1 Proof of Lemma B.1

As in Foucart [2012], we first construct a matrix norm \( \| \cdot \| \) such that \( \| M \| = \rho(M) + \delta \). Consider the Jordan canonical form of \( M \)

\[
M = S \begin{bmatrix}
J_{n_1}(\lambda_1(M)) & 0 & \cdots & 0 \\
0 & J_{n_2}(\lambda_2(M)) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & J_{n_k}(\lambda_k(M))
\end{bmatrix} S^{-1}
\]

We define

\[
D(\delta) = \begin{bmatrix}
D_{n_1}(\delta) & 0 & \cdots & 0 \\
0 & D_{n_2}(\delta) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D_{n_k}(\delta)
\end{bmatrix},
\]

where

\[
D_j(\delta) = \begin{bmatrix}
\delta & 0 & \cdots & 0 \\
0 & \delta^2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \delta^j
\end{bmatrix}
\]

Therefore,

\[
D\left(\frac{1}{\delta}\right)S^{-1}MSD(\delta) = \begin{bmatrix}
B_{n_1}(\lambda_1(M), \delta) & 0 & \cdots & 0 \\
0 & B_{n_2}(\lambda_2(M), \delta) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & B_{n_k}(\lambda_k(M), \delta)
\end{bmatrix}
\]

where

\[
B_i(\lambda, \delta) = D_i\left(\frac{1}{\delta}\right)J_i(\lambda)D_i(\delta) = \begin{bmatrix}
\lambda & \delta & 0 & \cdots & 0 \\
0 & \lambda & \delta & \ddots & \vdots \\
0 & \cdots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \lambda \\
0 & \cdots & 0 & 0 & \lambda
\end{bmatrix}
\]

We define the matrix norm \( \| \cdot \| \) as

\[
\| M \| \triangleq \left\| D\left(\frac{1}{\delta}\right)S^{-1}MSD(\delta) \right\|_1
\]

where \( \| \cdot \|_1 \) is the matrix norm induced by the vector \( L_1 \)-norm. Using the fact that \( \| M \|_1 = \max_{j \in [1:d]} \sum_{i=1}^d |m_{i,j}| \), where \( m_{i,j} \) is the \( i,j \)-th entry of \( M \), we have

\[
\| M \| = \max_{j \in [1:d]} (|\lambda_j| + \delta) = \rho(M) + \delta.
\]
We conclude the first half of the lemma by defining $C$ where $D$ is a diagonal matrix with eigenvalues of $A$. For $\delta < 1$, and $r$ defined as the size of largest Jordan block of $M$

$$
\sigma_{\min}\left(D\left(\frac{1}{\delta}\right)\right) \sigma_{\min}(D(\delta)) = \delta^{-\frac{1}{2}} = \delta^{-1}.
$$

We conclude the first half of the lemma by defining $C_\delta$ as $\frac{\sqrt{2}}{\sigma_{\min}(S)\sigma_{\min}(S^{-1})}$.

In the case that the eigenvalues are distinct, $r = 1$ and $C_\delta$ defined above becomes independent of $\delta$. Moreover, when all eigen-values are distinct, each Jordan block is $J_{n_i}(\lambda_i(M)) = [\lambda_i(M)]$ and the second half follows.

### C.2 Proof of Lemma [B.2]

Since the eigenvalues of $P$ are distinct, it follows that $\hat{C} = \frac{\sqrt{2}}{\sigma_{\min}(S^{-1})\sigma_{\min}(S)}$, where $S$ is the matrix in Jordan decomposition of $P$ (see proof of Lemma [B.1]). Let $S$ be the matrix in Jordan decomposition of $A$, i.e., $SAS^{-1} = D$, where $D$ is a diagonal matrix with eigenvalues of $A$ as its diagonal elements. Then,

$$
\begin{pmatrix}
S & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
S^{-1} & 0 \\
0 & S^{-1}
\end{pmatrix}
= \begin{pmatrix}
I - \alpha SAS^{-1} + \eta I & -\eta SS^{-1} \\
-\eta SS^{-1} & I
\end{pmatrix}
= \begin{pmatrix}
I - \alpha D + \eta I & -\eta I \\
-\eta I & I
\end{pmatrix},
$$

where $0_d$ is the zero matrix of dimension $d \times d$. For ease of exposition, suppose $A$ is a $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$. Then

$$
\begin{pmatrix}
S & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
S^{-1} & 0 \\
0 & S^{-1}
\end{pmatrix}
\begin{pmatrix}
1 + \eta & 0 & -\eta & 0 \\
0 & 1 + \eta & -\eta & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
1 + \eta & 0 & -\eta & 0 \\
0 & 1 + \eta & -\eta & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
$$

Suppose $E$ is the elementary matrix associated with the exchange of row-2 and row-3. It is easy to see that $E = E^T = E^{-1}$ and that,

$$
E \begin{pmatrix}
S & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
S^{-1} & 0 \\
0 & S^{-1}
\end{pmatrix} E^{-1}
= \begin{pmatrix}
1 + \eta & 0 & -\eta & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 + \eta & -\eta \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
$$

where,

$$
B_i = \begin{pmatrix}
1 + \eta & -\eta \\
1 & 0
\end{pmatrix}
$$

Suppose, $X_i = \begin{pmatrix}x_{i,1} & x_{i,2} \\ x_{i,3} & x_{i,4}\end{pmatrix}$ and,

$$
X_i^{-1}B_i X_i = \begin{pmatrix}\mu_{i,+} & 0 \\ 0 & \mu_{i,-}\end{pmatrix}.
$$

Here $\mu_{i,+} = \frac{(1-\alpha\lambda_i+\eta) + \sqrt{\Delta_i}}{2}$ and $\mu_{i,-} = \frac{(1-\alpha\lambda_i+\eta) - \sqrt{\Delta_i}}{2}$, where $\Delta_i = (1 + \eta - \alpha\lambda_i)^2 - 4\eta$. Solving the above equation we get,

$$
X_i = \begin{pmatrix}x_{i,3}\mu_{i,+} & x_{i,4}\mu_{i,+} \\ x_{i,3} & x_{i,4}\end{pmatrix}
$$
Setting \( x_{i,3} = x_{i,4} = 1 \),
\[
X_i = \begin{pmatrix} \mu_{i,+} & \mu_{i,-} \\ 1 & 1 \end{pmatrix}
\text{ and } X_i^{-1} = \frac{1}{\mu_{i,+} - \mu_{i,-}} \begin{pmatrix} 1 & -1 \\ -\mu_{i,-} & \mu_{i,+} \end{pmatrix}
\]

For a general \( d \times d \) matrix \( A \), using a similar procedure, it can be shown that
\[
\begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & X_d \end{pmatrix} E_{d \times d} \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \end{pmatrix} P \begin{pmatrix} S^{-1} & 0 & 0 \\ 0 & S^{-1} & 0 \end{pmatrix} E_{d \times d}^{-1} \begin{pmatrix} X_1^{-1} & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & X_d^{-1} \end{pmatrix}
\]

= \[
\begin{pmatrix} \mu_{1,+} & 0 & 0 \cdots & 0 \\ 0 & \mu_{1,-} & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{d,+} \end{pmatrix}
\]

where \( E_{d \times d} \) and \( E_{d \times d}^{-1} \) are permutation matrices that transform the matrix between them to a block diagonal matrix.

Let \( \hat{S} = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & X_d \end{pmatrix} E_{d \times d} \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \). Therefore,
\[
\hat{C} = \frac{\sqrt{d}}{\sigma_{\min}(\hat{S})\sigma_{\min}(\hat{S}^{-1})}
\]

In order to simplify the expression for \( \hat{C} \), we require the following lemma:

Lemma C.1. For all invertible matrices \( M \) of order \( d \times d \), the following identity holds:
\[
\frac{1}{\sigma_d(M)\sigma_d(M^{-1})} = \sigma_1(M)\sigma_1(M^{-1}),
\]
where \( \sigma_1(X) \geq \ldots \geq \sigma_d(X) \) denote the singular values of \( X \).

Proof. By definition, \( \sigma_d^2(M) \geq \ldots \geq \sigma_1^2(M) \) are the eigenvalues of \( M^TM \). Then, the eigenvalues of \( (M^TM)^{-1} = M^{-1}(M^{-1})^T \) are \( \frac{1}{\sigma_d(M)^2} \geq \ldots \geq \frac{1}{\sigma_1(M)^2} \). Note that \( M^{-1}(M^{-1})^T \) and \( (M^{-1})^TM^{-1} \) are similar since \( (M^{-1})^TM^{-1} = M(M^{-1}M^{-1})M^{-1} \). Consequently, \( M^{-1}(M^{-1})^T \) and \( (M^{-1})^TM^{-1} \) have the same set of eigenvalues and we find that the singular values of \( M^{-1} \) are \( \frac{1}{\sigma_d(M)} \geq \ldots \geq \frac{1}{\sigma_1(M)} \).

Thus,
\[
\sigma_1(M^{-1}) = \frac{1}{\sigma_d(M)},
\]
\[
\sigma_d(M^{-1}) = \frac{1}{\sigma_1(M)}
\]

and the result follows. \( \square \)

Using Lemma C.1, we have
\[
\hat{C} = \sqrt{d}\sigma_{\max}(\hat{S})\sigma_{\max}(\hat{S}^{-1})
\leq \sqrt{d} \max_i \{ \sigma_{\max}(X_i) \} \sigma_{\max}(S)\sigma_{\max}(S^{-1}) \max_i \{ \sigma_{\max}(X_i^{-1}) \}
= C \max_i \{ \sigma_{\max}(X_i) \} \max_i \{ \sigma_{\max}(X_i^{-1}) \},
\]

where \( C \) is as defined in (11). Now, for any matrix \( X \) of order \( d \times d \),
\[
\sigma_{\max}(X) = \|X\|_2 \leq \|X\|_F = (\sum_{i,j} |x_{ij}|^2)^{1/2} \leq d \max_{i,j} |x_{ij}|,
\]

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where \( \| \cdot \|_F \) denotes the Frobenius norm. Using the above inequality, \( \sigma_{\max}(X_i) \leq 2 \) and \( \sigma_{\max}(X_i^{-1}) \leq \frac{2}{|\mu_{i,+}-\mu_{i,-}|} = \frac{2}{|\sqrt{\Delta_i}|} \). Next we lower bound \( |\sqrt{\Delta_i}| \).

For a complex number \( z \), observe that \( |\sqrt{z}| = \sqrt{|z|} \). Now,
\[
|\Delta_i| = 4\eta - (1 + \eta - \alpha \lambda_{\min})^2
\]
Using \( \eta = \left(1 - \frac{\sqrt{\alpha \lambda_{\min}}}{2}\right)^2 \),
\[
|\Delta_i| = 4\eta - \left(1 + 1 + \frac{\alpha \lambda_{\min}}{4} - \sqrt{\alpha \lambda_{\min}} - \alpha \lambda_{\min}\right)^2
= 4\eta - \left(2\sqrt{\eta} - \frac{3\alpha}{4} \lambda_{\min}\right)^2
= 4\left[(\sqrt{\eta})^2 - \left(\sqrt{\eta} - \frac{3\alpha \lambda_{\min}}{8}\right)^2\right]
= \frac{3\alpha \lambda_{\min}}{2} \left(2\sqrt{\eta} - \frac{3\alpha \lambda_{\min}}{8}\right)
= \frac{3\alpha \lambda_{\min}}{2} \left(2 - \sqrt{\frac{\alpha \lambda_{\min}}{3}} - \frac{3\alpha \lambda_{\min}}{8}\right)
\geq \frac{3\alpha \lambda_{\min}}{2} \left(2 - 1 - \frac{3}{8}\right)
= \frac{15}{16} \alpha \lambda_{\min}
\]

Using this, we get \( \max_i \{\sigma_{\max}(X_i^{-1})\} \leq 2 \sqrt{\frac{15}{16} \frac{1}{\sqrt{\alpha \lambda_{\min}}}} \), and therefore
\[
\hat{C} \leq \frac{5C}{\sqrt{\alpha \lambda_{\min}}}
\]

C.3 Proof of Lemma B.3

We see that
\[
\sum_{j=0}^{n-1} \left\| P^j e_1 \right\|^2 \geq \sum_{j=0}^{n-1} \left(\frac{\mu_{+}^j - \mu_{-}^j}{\mu_{+} - \mu_{-}}\right)^2
\]
\[
= \frac{1}{(\mu_{+} - \mu_{-})^2} \sum_{j=0}^{n-1} (\mu_{+}^j + \mu_{-}^j - 2(\mu_{+} \mu_{-})^j)
\]
\[
= \frac{1}{(\mu_{+} - \mu_{-})^2} \left(\sum_{j=0}^{n-1} \mu_{+}^j + \sum_{j=0}^{n-1} \mu_{-}^j - 2 \sum_{j=0}^{n-1} (\mu_{+} \mu_{-})^j\right)
\]
\[
= \frac{1}{(\mu_{+} - \mu_{-})^2} \left(\frac{1 - \mu_{+}^n}{1 - \mu_{+}} + \frac{1 - \mu_{-}^n}{1 - \mu_{-}} - 2 \frac{1 - (\mu_{+} \mu_{-})^n}{1 - \mu_{+} \mu_{-}}\right)
\]
The sum of fractions in the bracket can be expressed as a single fraction with denominator \((1 - \mu_{+}^n)(1 - \mu_{-}^n)(1 - \mu_{+} \mu_{-})\).
The numerator turns out to be:
\[
num = (\mu_{+} - \mu_{-})^2 + \mu_{+} \mu_{-}(\mu_{+} - \mu_{-})^2 + \mu_{+} \mu_{-}(\mu_{+}^n - \mu_{-}^n)^2 + (\mu_{+} \mu_{-})^2 (\mu_{+}^n - \mu_{-}^n)^2 -
(\mu_{+} \mu_{-})^3 (\mu_{+}^{n-1} - \mu_{-}^{n-1})^2 - 2(\mu_{+} \mu_{-})^n (\mu_{+} - \mu_{-})^2 - (\mu_{+} - \mu_{-})^2.
\]
Observe that
\[
\mu_{+} \mu_{-}(\mu_{+} - \mu_{-})^2 \geq 0,
\mu_{+} \mu_{-}(\mu_{+}^n - \mu_{-}^n)^2 \geq 0,
(\mu_{+} \mu_{-})^2 (\mu_{+}^{n-1} - \mu_{-}^{n-1})^2 - (\mu_{+} \mu_{-})^n (\mu_{+} - \mu_{-})^2 - (\mu_{+} - \mu_{-})^2 \geq 0.
\]

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The third inequality follows because \( \mu_+ \mu_- = \frac{(1-\alpha \lambda + \eta)^2 - (1-\alpha \lambda + \eta -4\eta)}{4} = \eta \) and therefore \((\mu_+ \mu_-)^2 \geq (\mu_+ \mu_-)^3\).

Next,
\[
2(\mu_+ \mu_-)^n (\mu_+ - \mu_-)^2 \leq 2(\mu_+)^2(\mu_+ - \mu_-)^2
= 2(\rho(P))^{2n}(\mu_+ - \mu_-)^2
\leq 2 \epsilon(\mu_+ - \mu_-)^2
\]
The last inequality follows because \( \rho(P)^{2n} \leq \|P^n e_1\|^2 < \epsilon \). Also, note that \((\frac{\mu_+ - \mu_-}{n_+ - n_-})^2 \leq \|P^n e_1\|^2 < \epsilon \) and therefore \((\mu_+^n - \mu_-^n)^2 \leq \epsilon(\mu_+ - \mu_-)^2 \). It follows that \( \text{num} \geq (1-3\epsilon)(\mu_+ - \mu_-)^2 \). Using these bounds, taking \( \epsilon \leq \frac{1}{6} \) and noting that \( h(\eta, \alpha \lambda) := \frac{(1-\mu_+^2)(1-\mu_-^2)}{(\alpha \lambda)^2} \), we get
\[
\alpha^2 K \sum_{j=0}^{n-1} \left\| P^j e_1 \right\|^2 \geq \frac{\alpha^2 K}{(\mu_+ - \mu_-)^{2}} \left( \frac{1 - \mu_+^2}{1 - \mu_+^2} + \frac{1 - \mu_-^2}{1 - \mu_-^2} - 2 \frac{1 - (\mu_+ \mu_-)^2}{1 - \mu_+ \mu_-} \right)
\geq \frac{\alpha^2 K}{2(1 - \mu_+^2)(1 - \mu_-^2)(1 - \mu_+ \mu_-)}
\geq \frac{(K/\lambda^2)}{2h(\eta, \alpha \lambda)}
\]

C.4 Proof of Lemma B.4

We prove this by showing \( h(\eta, \alpha \lambda)(1-\rho(P)) \leq 8 \). Note that \( 1 - \mu_+^2 = (1 - \mu_+)(1 + \mu_+) \leq 2(1 - \mu_+) \) since \( \mu_+ \leq 1 \).
Similarly, \( 1 - \mu_-^2 \leq 2(1 - \mu_-) \). Thus, \( h(\eta, \alpha \lambda)(1-\mu_+) \leq \frac{4(1-\mu_+^2)(1-\mu_-)(1-\mu_+ \mu_-)}{(\alpha \lambda)^2} \). Since \( \mu_+ + \mu_- = 1 - \alpha \lambda + \eta \), it follows that \( (1 - \mu_+)(1 - \mu_-) = 1 + \mu_+ \mu_- - \mu_+ - \mu_- = 1 - (1 - \alpha \lambda + \eta) \) and \( \eta = \alpha \lambda \). Thus,
\[
h(\eta, \alpha \lambda)(1-\mu_+) \leq \frac{4(1-\mu_+^2)(1-\eta)}{\alpha \lambda}
= \frac{4(1-\eta)}{\alpha \lambda} \left( 1 - (1 + \eta - \lambda \alpha) + \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} \right) \triangleq g(\eta, \alpha \lambda).
\]

We see that
\[
\frac{\partial g(\eta, \alpha \lambda)}{\partial \eta} = \frac{2 \left( \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} + \eta - 1 \right) \left( \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} + \eta - \alpha \lambda - 1 \right)}{\alpha \lambda \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta}}
\]
Observe that the denominator in the above expression is positive. We consider the following two cases: (i) \( \alpha \lambda \geq 1 \) and (ii) \( \alpha \lambda < 1 \). When \( \alpha \lambda \geq 1 \), we can directly bound \( g(\eta, \alpha \lambda) \leq \frac{4(1-\mu_+^2)(1-\eta)}{\alpha \lambda} \). Since \( \mu_+ \geq -1 \) and \( \eta \geq 0 \), we get \( g(\eta, \alpha \lambda) \leq 8 \).

Now consider the case \( \alpha \lambda < 1 \). We have that,
\[
\sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} + \eta - 1 = (1 - \eta)^2 + (\alpha \lambda)^2 - 2\alpha \lambda (1 + \eta) - (1 - \eta)
\]
When \( \alpha \lambda < 1 \), \( (\alpha \lambda)^2 - 2\alpha \lambda (1 + \eta) < 0 \) for all \( \eta \geq 0 \), thus \( \left( \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} + \eta - \alpha \lambda - 1 \right) \leq \left( \sqrt{(1 - \alpha \lambda + \eta)^2 - 4\eta} + \eta - 1 \right) < 0 \). This implies that the numerator of the partial derivative is also positive and we have that \( \frac{\partial g(\eta, \alpha \lambda)}{\partial \eta} > 0 \). Since the partial derivative is positive \( g(\eta, \alpha \lambda) \) is an increasing function of \( \eta \) and thus the maximum is achieved at \( \eta = (1 - \sqrt{\alpha \lambda})^2 \) and is given by:
\[
g((1 - \sqrt{\alpha \lambda})^2, \alpha \lambda) = \frac{4}{\alpha \lambda} \left( 1 - (1 - \sqrt{\alpha \lambda}) \right) \left( 1 - (1 - \sqrt{\alpha \lambda})^2 \right)
= \frac{4\alpha \lambda(2 - \sqrt{\alpha \lambda})}{\alpha \lambda}
\leq 8