TRANSLATION INVARIANT PURE STATE ON $\otimes_{\mathbb{Z}^d} \mathbb{M}_d(\mathbb{C})$ AND HAAG DUALITY

ANILESH MOHARI

Abstract

We prove Haag duality property of any translation invariant pure state on $B = \otimes_{\mathbb{Z}^d} \mathbb{M}_d(\mathbb{C})$, $d \geq 2$, where $\mathbb{M}_d(\mathbb{C})$ is the set of $d \times d$ dimensional matrices over the field of complex numbers. We also prove a necessary and sufficient condition for a translation invariant factor state to be pure on $B$.

1. Introduction

A state $\omega$ on a $C^*$-algebra $B$ is called a factor if the center of the von-Neumann algebra $\pi_\omega(B)''$ is trivial, where $(H_\omega, \pi_\omega, \Omega)$ is the GNS space associated with $\omega$ on $B$ [BR vol-I]. A state $\omega$ on $B$ is called pure if $\pi_\omega(B)'' = B(H_\omega)$, the algebra of all bounded operators on $H_\omega$. Here we fix our convention that Hilbert spaces that are considered here equipped always with inner product $<\cdot,\cdot>$ which is linear in the second variable and conjugate linear in the first variable. In this paper our primary objective is to study states on $C^*$-algebra that naturally arise in quantum spin chain models on a lattice.

Let $B = \otimes_{\mathbb{Z}_k} \mathbb{M}_d(\mathbb{C})$ be the uniformly hyper-finite $C^*$-algebra over the lattice $\mathbb{Z}_k$ of dimension $k \geq 1$, where $\mathbb{M}_d(\mathbb{C})$ denote the algebra of $d \times d$-matrices over the field of complex number $\mathbb{C}$. A state $\omega$ on $B$ is called translation invariant if $\omega(x) = \omega(\theta_m(x))$ where $m = (m_1, m_2, \ldots, m_k)$ and $\theta_m$ is the translation induced by $\bar{Z} \to \bar{Z} + \bar{m}$ for all $\bar{z} \in \mathbb{Z}^k$. It is well known since late 60's [Pow] that a translation invariant state $\omega$ on $B$ is a factor state if and only if

$$\sup_{x \in B, \|x\| \leq 1} |\omega(xy) - \omega(x)\omega(y)| \to 0$$

for all $y \in B$ as $n \to \infty$, where $\Lambda_n$ is the local algebra with support in the finite set $\{\bar{m} : -n \leq m_k \leq n\}$.

Such a criterion is used extensively to show that KMS states of a translation invariant Hamiltonian on the lattice form a simplex and its extreme points are translation invariant factor states. For more details and an account until 1980 we refer to [BR vol-II] and also [Sim vol-I] for a later edition. Such an elegant asymptotic criterion is missing for a translation invariant pure state. Here one of our objectives is to do so.

1991 Mathematics Subject Classification. 46L.

Key words and phrases. Uniformly hyperfinite factors. Cuntz algebra, Popescu dilation, Kolmogorov’s property, Arveson’s spectrum, Haag duality.

... This paper has grown over the years starting with initial work in the middle of 2005. The author gratefully acknowledge discussion with Ola Bratteli and Palle E. T. Jorgensen for inspiring participation in sharing the intricacy of the present problem. Finally the author is indebted to Taku Matsui for valuable comments on an earlier draft of the present problem where the author made an attempt to prove Haag duality.
For the sake of simplicity, we will consider the simplest situation namely one lattice dimensional quantum mechanical spin systems. Now onwards we consider the lattice to be one dimensional. We briefly set the standard notations and known relations in the following text. The quantum spin chain that we consider here is described by a UHF $C^*$-algebra denoted by $B = \oplus_{\mathbb{Z}} M_d(\mathbb{C})$. Here $B$ is the $C^*$-completion of the infinite tensor product of the algebra $M_d(\mathbb{C})$ of $d \times d$ complex matrices $[Sa]$, each component of the tensor product element is indexed by an integer $j$. Let $Q$ be a matrix in $M_d(\mathbb{C})$. We denote the element $Q^{(j)} = \cdots \otimes 1 \otimes 1 \otimes Q \otimes 1 \otimes \cdots$ where $Q$ appears in the $j$-th component. Given a subset $\Lambda$ of $\mathbb{Z}$, $B_\Lambda$ is defined as the $C^*$-sub-algebra of $B$ generated by all $Q^{(j)}$ with $Q \in M_d(\mathbb{C}), j \in \Lambda$. We also set

$$B_{loc} = \bigcup_{\Lambda: |\Lambda| < \infty} B_\Lambda$$

where $|\Lambda|$ is the cardinality of $\Lambda$. Let $\omega$ be a state on $B$. The restriction of $\omega$ to $B_\Lambda$ is denoted by $\omega_\Lambda$. We also set $\omega_R = \omega_{[1,\infty)}$ and $\omega_L = \omega_{(-\infty,0]}$. The translation $\theta_k$ is an automorphism of $B$ defined by $\theta_k(Q^{(j)}) = Q^{(j+k)}$. Thus $\theta_1$ and $\theta_{-1}$ are unital $\ast$-endomorphisms on $B_R$ and $B_L$ respectively. We say $\omega$ is translation invariant if $\omega \circ \theta_k = \omega$ on $B$ ($\omega \circ \theta_1 = \omega$ on $B$). In such a case $(B_R, \theta_1, \omega_R)$ and $(B_L, \theta_{-1}, \omega_L)$ are two unital $\ast$-endomorphisms with invariant states.

We will consider a Hamiltonian in one dimensional lattice of the following form

$$H = \sum_{k \in \mathbb{Z}} \theta^k(h_0)$$

for $h_0 = h_0 \in B_{loc}$ where the formal sum gives an auto-morphism $\alpha = (\alpha_t: t \in \mathbb{R})$ via the thermodynamic limit of $\alpha^L_t(x) = e^{itH} x e^{-itH}$ for a net of finite subsets of the lattice $\Lambda \uparrow \mathbb{Z}$ whose surface energies are uniformly bounded, where $H_\Lambda = \sum_{k \in \Lambda} \theta^k(h_0)$ $[Ru,BR2]$. Such a thermodynamic limit automorphism $\alpha$ is uniquely determined by $H$. In such a case, i.e. translation invariant Hamiltonian $H$ having finite range interaction, KMS state at a given inverse temperature exists and is unique $[Ara1],[Ara2],[Ki]$ and inherits translation and other symmetry of the Hamiltonian. Thus low temperature limit points of unique KMS states give ground states for the Hamiltonian $H$ inheriting translation and other symmetry of Hamiltonian. It is a well known fact that ground states of a translation invariant Hamiltonian form a face in the convex set of states on $B$ and its extreme points are pure. In general ground states need to be unique and there are other non translation invariant ground states for a translation invariant Hamiltonian $[Ma4]$. Ising model admits non translation invariant ground states known as Néel state $[BR2]$. However, ground states that appear as low temperature limit of KMS states of a translation invariant Hamiltonian inherit translation and other symmetry (that we would consider in a follow up paper in more details) of the Hamiltonian. In particular if ground state for a translation invariant Hamiltonian model of type (2) is unique, then the ground state is a translation invariant pure state.

Unlike classical spin chain problem, any translation invariant state $\omega$ on $B$ gives rise to a quantum Markov state in the sense of Luigi Accardi $[Ac]$ and more specifically finite or infinitely correlated translation invariant state $([FNW1]$, see also $[BJ],[BJKW],[Mo3]$ for their natural generalization $)$. Here we briefly recall now explaining these two related concepts and explain the basic setup of the present problem and some difficulties that crop up. A detailed account is given in section 2 and then section 3 holds key results.
First we recall that the Cuntz algebra $O_d (d \in \{2, 3, \ldots\})$ [Cun] is the universal unital $C^*$-algebra generated by the elements \( \{s_1, s_2, \ldots, s_d\} \) subject to the following Cuntz relations:

\[
  s_i^* s_j = \delta_{ij} 1, \quad \sum_{1 \leq i \leq d} s_i s_i^* = 1
\]

There is a canonical action of the group $U(d)$ of unitary $d \times d$ matrices on $O_d$ given by

\[
  \beta_g(s_i) = \sum_{1 \leq j \leq d} g_{ij}^* s_j
\]

for $g = (g_{ij}) \in U(d)$. In particular the gauge action is defined by

\[
  \beta_z(s_i) = z s_i, \quad z \in \mathbb{I} = S^1 = \{z \in \mathbb{C} : |z| = 1\}.
\]

If $\text{UHF}_d$ is the fixed point sub-algebra under the gauge action, then $\text{UHF}_d$ is the closure of the linear span of all Wick ordered monomials of the form

\[
  s_{i_1} \ldots s_{i_k} s_{j_1}^* \ldots s_{j_m}^*
\]

which is also isomorphic to the $\text{UHF}_d$ algebra

\[
  B_R = \mathcal{M}_d(\mathbb{C})
\]

so that the isomorphism carries the Wick ordered monomial above into the matrix element

\[
  e^{i \theta_{i_1}(1)} \otimes e^{i \theta_{i_2}(2)} \otimes \ldots \otimes e^{i \theta_{j_m}(k)} \otimes 1 \otimes 1
\]

and the restriction of $\beta_g$ to $UHF_d$ is then carried to action

\[
  \text{Ad}(g) \otimes \text{Ad}(g) \otimes \text{Ad}(g) \otimes \ldots
\]

We also define the canonical endomorphism $\lambda$ on $O_d$ by

\[
  \lambda(x) = \sum_{1 \leq i \leq d} s_i x s_i^*
\]

The isomorphism carries $\lambda$ restricted to $\text{UHF}_d$ into the one-sided shift

\[
  y_1 \otimes y_2 \otimes \ldots \rightarrow 1 \otimes y_1 \otimes y_2 \ldots
\]

on $\mathcal{M}_d(\mathbb{C})$. Note that $\lambda \beta_g = \beta_g \lambda$ on $\text{UHF}_d$.

Let $d \in \{2, 3, \ldots\}$ and $\mathbb{Z}_d$ be a set of $d$ elements. $I$ be the set of finite sequences $I = (i_1, i_2, \ldots, i_m)$ where $i_k \in \mathbb{Z}_d$ and $m \geq 1$. We also include empty set $\emptyset \in I$ and set $s_{\emptyset} = 1 = s_{\emptyset}^*$, $s_I = s_{i_1} \ldots s_{i_m} \in O_d$ and $s_I^* = s_{i_m}^* \ldots s_{i_1}^* \in O_d$.

We fix a translation invariant state $\omega$ on $\mathcal{B}$ and denote by $\omega_R$ the restriction of $\omega$ to $B_R$. Using weak* compactness of the convex set of states on a $C^*$-algebra, a standard averaging method ensures that the set

\[
  K_\omega = \{ \psi \in \mathcal{S}(O_d) : \psi \lambda = \psi, \psi \text{ UHF}_d = \omega_R \}
\]

is a non-empty compact subset of $\mathcal{S}(O_d)$, where $\mathcal{S}(O_d)$ is the weak* compact convex set of states on $O_d$. Further extremal elements in $K_\omega$ is a factor state if and only if $\omega_R$ is a factor state and any two such extremal elements $\psi, \psi'$ are related by $\psi' = \psi \beta_z$ for some $z \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ by Lemma 7.4. in [BJKW] where $\beta_z(s_i) = z s_i$ is the automorphism on $O_d$ determined uniquely by universal property of Cuntz algebra.

Irrespective of the factor property of $\omega$, we may choose an element $\psi$ of $K_\omega$ and consider the GNS space $(\mathcal{H}, \pi, \Omega)$ associated with state $\psi$ on $O_d$. We set $P \in \pi(O_d)^\prime$
to be the support projection of $\psi$ i.e. $P = [\pi(O_d)^\dagger \Omega]$. Invariance property of the state $\psi = \psi_\Lambda$ will ensure that $PA(I - P)P = 0$ where

$$\Lambda(X) = \sum_i S_i X S_i^*$$

is the canonical endomorphism on $\pi_\psi(O_d)^\dagger$ with $S_i = \pi_\psi(s_i)$. This verifies that

$$S_i^* P = P S_i^* P, 1 \leq i \leq d$$

We define a family of contractions $\{v_i : 1 \leq i \leq d\}$ in $\mathcal{M}$ by $v_i = P S_i^* P$, $1 \leq i \leq d$ where we set von-Neumann algebra $\mathcal{M} = P \pi_\psi(O_d)^\dagger P$ acting on Hilbert subspace $\mathcal{K}$ where $\mathcal{K}$ is range of $P$. Thus we get $\mathcal{M} = \{v_i : 1 \leq i \leq d\}$ and a unital completely positive map $\tau(x) = PA(Px)P = \sum_i v_i x v_i^*$ for all $x \in \mathcal{M}$. Furthermore a crucial point to be noted that the support projection of $\psi$ in $\pi(O_d)^\dagger$ being equal to $P$, by our construction we have

$$\{x \in B(\mathcal{K}) : \sum_i v_i x v_i^* = x\} = \mathcal{M}$$

Conversely let $\mathcal{M}$ be a von-Neumann algebra acting on a Hilbert space $\mathcal{K}$. A family of contractions $\{v_i : 1 \leq i \leq d\}$ in $\mathcal{M}$ is called Popescu’s elements if $\sum_i v_i v_i^* = 1$. Given a Popescu’s elements $\mathcal{P} = \{K, M, v_i, 1 \leq i \leq d, \sum_i v_i v_i^* = 1\}$, the map

$$s_i s_j^* \rightarrow v_i v_j^*$$

is unital completely positive from $O_d$ to $\mathcal{M}$ and thus Stinespring minimal dilation gives a representation $\pi : O_d \rightarrow B(\mathcal{H})$, a Hilbert space $\mathcal{H}$ with a projection $P$ with range equal to $\mathcal{K}$ such that

$$P \pi(s_i)^* P = \pi(s_i)^* P = v_i^*$$

and $\{\pi(s_i) \mathcal{K} : |i| < \infty\}$ is total in $\mathcal{H}$. For a faithful normal state $\phi$ on $\mathcal{M}$ we define state $\psi$ on $O_d$ by

$$\psi(s_i s_j^*) = \phi(v_i v_j^*)$$

The crucial point that we arrive at Proposition 2.4 that $P$ is the support projection for $\pi(O_d)^\dagger$ if and only if (6) holds.

We verify also with $v_i^* = P S_i^* P$ that

$$\omega_R([e_{i_1} \ldots e_{i_n}] \otimes [e_{j_1} \ldots e_{j_n}]) = \phi(v_i v_j^*)$$

where $v_i = v_{i_1} \ldots v_{i_n}$ and $v_j^* = v_{j_1}^* \ldots v_{j_n}^*$. The relation (7) can now be recast as a quantum Markov state as follows: Let $\mathcal{K}$ be the Hilbert subspace $P$ of $\mathcal{H}$ and $\mathcal{M}$ be a von-Neumann sub-algebra of $B(\mathcal{K})$. Let $V^* : \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{C}^d$ be an isometry and in an orthonormal basis $(e_i)$ for $\mathcal{C}^d$, we have $V^* = (v_1^*, v_2^*, \ldots, v_d^*)$ with $v_i \in \mathcal{M}$. We define

$$E : M_d(\mathcal{M}) \rightarrow \mathcal{M}$$

by

$$E(X) = V X V^* = \sum_{1 \leq i, j \leq d} v_i x_j^* y_j^*$$

where $X = ((x_{i j}))$. Let $\phi$ be a state on $\mathcal{M}$ such that

$$\phi(E(B \otimes I_d)) = \phi(B), \forall B \in \mathcal{M}$$

For each $A \in M_d(\mathcal{C})$, define $E_A : \mathcal{M} \rightarrow \mathcal{M}$ by

$$B \rightarrow E(B \otimes A)$$

Then

$$\omega(A_1 \otimes A_2 \otimes \ldots \otimes A_m) = \phi(E_{A_1} \circ E_{A_2} \circ \ldots \circ E_{A_m}(I_{\mathcal{K}}))$$
defines a $\lambda$-invariant state on $B_R$ and the inductive limit state of $B_R \to ^\lambda B_R$ [Sa] gives a translation invariant state on $B$. $\mathcal{E}$ naturally gives a Markov map i.e. a unital completely positive map on $\mathcal{M}$ defined by $\tau(x) = \mathcal{E}(x \otimes I_d) = \sum_i v_i x v_i^*$. We have $\phi \sigma = \phi$ on $\mathcal{M}$.

The state $\phi(x) = \langle \Omega, x\Omega \rangle$ on $\mathcal{M}$ being faithful and invariant of $\tau : \mathcal{M} \to \mathcal{M}$ we find a unique unital completely positive map $\tilde{\tau} : \mathcal{M} \to \mathcal{M}'$ satisfying the duality relation

$$< y\Omega, \tau(x)\Omega > = \langle \tilde{\tau}(y)\Omega, x\Omega \rangle$$

for all $x \in \mathcal{M}$ and $y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$. For a proof we refer to section 8 in the monograph [OP] or [Mo1].

$\phi$ being also a faithful state, $\Omega \in \mathcal{K}$ is a cyclic and separating vector for $\mathcal{M}$ and the closure of the close-able operator $S_0 : x\Omega \to x^*\Omega$, $S$ possesses a polar decomposition $S = J_0 \Delta^{1/2}$, where $J$ is an anti-unitary and $\Delta$ is a non-negative self-adjoint operator on $\mathcal{K}$. Tomita's [BR] theorem says that $\Delta'' \mathcal{M} \Delta''^{-it} = \mathcal{M}$, $t \in \mathbb{R}$ and $J_0 \mathcal{M} J_0 = \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. We define the modular automorphism group $\sigma = (\sigma_t, t \in \mathbb{R})$ on $\mathcal{M}$ by

$$\sigma_t(x) = \Delta'' x \Delta''^{-it}$$

which satisfies the modular relation

$$\phi(x \sigma_{-\frac{1}{2}}(y)) = \phi(\sigma_{-\frac{1}{2}}(y)x)$$

for any two analytic elements $x, y$ for the automorphism. A more useful form for modular relation here

$$\phi(\sigma_{-\frac{1}{2}}(x^*) \sigma_{-\frac{1}{2}}(y^*)) = \phi(y^* x)$$

which shows that $J_0 x\Omega = \sigma_{-\frac{1}{2}}(x^*)\Omega$. $J$ and $\sigma = (\sigma_t, t \in \mathbb{R})$ are called Tomita’s conjugation operator and modular automorphisms associated with $\phi$. Since $\tau(x) = v_k x v_k^*$ is an inner map i.e. each $v_k \in \mathcal{M}$, we have an explicit formula for $\tilde{\tau}$ as follows.

We set $\tilde{v}_k = J_0 \sigma_{\frac{1}{2}}(v_k^*) J$ $\in \mathcal{M}'$. That $\tilde{v}_k$ is indeed well defined as an element in $\mathcal{M}'$ given in section 8 in [BJKW]. By KMS or modular relation [BR vol-I] we verify that

$$\sum_k \tilde{v}_k \tilde{v}_k^* = 1$$

and

$$\tilde{\tau}(y) = \sum_k \tilde{v}_k y \tilde{v}_k^*$$

and

$$\phi(v_I v_I^*) = \phi(\tilde{v}_I \tilde{v}_I^*)$$

where $I = (i_n, \ldots, i_2, i_1)$ if $I = (i_1, i_2, \ldots, i_n)$. Moreover $\tilde{v}_I^* \Omega = J_0 \sigma_{\frac{1}{2}}(v_I^*) J_0 \Omega = J_0 \Delta^\frac{1}{2} v_I \Omega = v_I^* \Omega$. We also set $\mathcal{M}$ to be the von-Neumann algebra generated by $\{ \tilde{v}_k : 1 \leq k \leq d\}$. Thus $\mathcal{M} \subseteq \mathcal{M}'$. The major problem that we will address in the text when do we have the following equality:

$$\{ x \in \mathcal{B}(\mathcal{K}) : \sum_k \tilde{v}_k x \tilde{v}_k^* = x \} = \mathcal{M}$$

Equality in (13) will ensure that $P : \tilde{\mathcal{H}} \to \mathcal{K}$ is also the support projection of $\tilde{\pi}(\mathcal{O}_d)''$ where $\tilde{\pi}$ is the Popescu’s prescription of Stinespring representation $\tilde{\pi} : \mathcal{O}_d \to \mathcal{B}(\mathcal{H})$. 

\[T_{\text{R \& H dual}}\]
associated with the completely positive map \( s_\ell s_\ell^* \rightarrow \tilde{v}_\ell \tilde{v}_\ell^* \), \(|J|, |\bar{J}| < \infty\) so that \( P \tilde{\pi}(s_\ell^*) P = \tilde{\pi}(s_\ell^*) P = s_\ell^* \). Details has been worked out in Proposition 2.4.

Thus so far we have taken an arbitrary element \( \psi \in K_0 \) and worked with its support projection to arrive at a representation of \( \omega \) given in (7) or (8) by Popescu’s elements \( \mathcal{P} = \{(K, v_i \in \mathcal{M}, 1 \leq i \leq d, \Omega) : \sum_k v_i v_i^* = I\} \). However by Lemma 7.4 in [BJKW] for a factor state \( \omega \), if we choose an extreme point \( \psi \in K_0 \), two such extreme points \( \psi \) and \( \psi' \) in \( K_0 \) are related by \( \psi' = \phi \beta_z \) for some \( z \in S^1 \), \( \mathcal{P} \) is uniquely determined modulo a unitary conjugation. In other words we find a one-one correspondence between

\[
\omega \Leftrightarrow \omega_R \Leftrightarrow K_0^c \Leftrightarrow \mathcal{P}_e
\]

modulo unitary conjugation where \( K_0^c \) denotes the set of extreme points in \( K_0 \) and \( \mathcal{P}_e \) the set of Popescu’s elements associated with extreme points of \( K_0 \) on support projection of the state as described above. Further in such a case \( \mathcal{M} = \{v_k : 1 \leq k \leq d\} \) is a factor and \( (M, \tau, \phi) \) is an ergodic quantum dynamical system [La,Ev]. A unital completely positive map \( \tau \) on a von-Neumann algebra \( \mathcal{M} \) with an invariant normal state \( \phi \) [BJKW,Mo1] is called ergodic if

\[
\frac{1}{N} \sum_{0 \leq k \leq N-1} \tau^k(x) \rightarrow \phi(x)I \quad \text{as } N \rightarrow \infty \text{ in weak* topology for all } x \in \mathcal{M}.
\]

Thus any symmetry of \( \omega \) will act on Popescu elements \( \mathcal{P}_e \) via this correspondence. It would be worthwhile to have a result generalizing this correspondence in a more general situation that Ruy Exel developed [Ex].

For a translation invariant factor state \( \omega \) on \( B \), we say it admits Haag duality property if

\[
\pi_{\omega}(B_R)' = \pi_{\omega}(B_L)''
\]

It is clear that such a factor state is pure. A pure mathematical question that arises here whether converse is true? i.e. Do we always have Haag duality property for a translation invariant pure state of \( B \)?

In case \( \pi_{\omega}(B_R)'' \) is a type-I factor state, Haag duality property follows easily. In fact we can find Hilbert spaces \( \mathcal{H}_{-}^{\omega}, \mathcal{H}_{+}^{\omega} \) such that \( \mathcal{H}_{\omega} \) is unitary equivalent to \( \mathcal{H}_{-}^{\omega} \otimes I_{\mathcal{H}_{+}} \) so that \( \pi_{\omega}(B_R)'' = B(\mathcal{H}_{-}^{\omega}) \otimes I_{\mathcal{H}_{+}} \) and \( \pi_{\omega}(B_L)'' = I_{\mathcal{H}_{-}} \otimes B(\mathcal{H}_{+}^{\omega}) \). A simple proof goes as follows: \( \pi_{\omega}(B_R)'' \) being a type-I factor, its commutant is also a type-I factor. Thus we have an inclusion of type-I sub-factor \( \pi_{\omega}(B_L)'' \leq \pi_{\omega}(B_R)' \). Without loss of generality we assume now that \( \pi_{\omega}(B_R)' = B(\mathcal{H}_{-}^{\omega}) \) and \( \pi_{\omega}(B_L)'' \leq B(\mathcal{H}_{+}^{\omega}) \). Now we again use type-I factor property of \( \pi_{\omega}(B_L)'' \) to write \( \mathcal{H}_{-}^{\omega} \otimes \mathcal{H}_{-}^{\omega} (1) \otimes \mathcal{H}_{-}^{\omega} (2) \) for two Hilbert spaces \( \mathcal{H}_{-}^{\omega} (k), k = 1, 2 \) so that \( \pi_{\omega}(B_L)'' \leq B(\mathcal{H}_{-}^{\omega} (1)) \otimes I_{\mathcal{H}_{-}^{\omega} (2)} \). Since \( \omega \) is pure we have \( \pi_{\omega}(B_R)' \cap \pi_{\omega}(B_L)' = \mathbb{C} \) and so \( \mathcal{H}_{-}^{\omega} (2) = \mathbb{C} \). Thus we arrive at our conclusion. The real trouble lies in the fact that the factors \( \pi_{\omega}(B_R)'' \) and \( \pi_{\omega}(B_L)'' \) could be of type-III and for a type-III factor we may have non-trivial inclusion with trivial relative commutant. Of course, such a splitting relation is not true since tensor product of two type-III factors will give a type-III factor. In [Mo5] we will explain in detail how Haag duality property finds profound importance in studying reflection symmetry of a pure translation invariant state and its split property.

A notion of duality appeared first in the framework of local field theory in Minkowski’s space-time formulated by Rudolf Haag [Hag]. We also refer [DHR] for a detailed historical account and its subsequent adaptation in conformal field
Before we go further into the results proven in this paper, besides Haag duality property (16), we give a brief history of the present topic and related results. Functional relation (9) is called quantum Markov state by Luigi Accardi [Ac] as a generalization of classical Markov state or more generally of a classical Gibbs state. He has shown that this functional property holds good for unique KMS state for a class of Hamiltonian in a one-dimensional infinite quantum spin lattice with a finite range interaction studied previously by [Ara1]. In [FNW1] M. Fannes, Bruno Nachtergaele and R.F. Werner investigated mathematical structures of valence bound states introduced in [AKLT] and found its close relation with quantum Markov states which they have unified in a framework that we have discussed above where $\mathcal{M}$ is a matrix algebra. When $\mathcal{M}$ is a matrix algebra in relations (9), it is called finitely correlated state and the general mathematical structures of such a translation invariant states on $\mathcal{B}$ were further investigated in details in [FNW2] [FNW3], [Ma1],[Ma2],[Ma3] and there afterward in [BJ] and [BJKW] for more general translation invariant states on $\mathcal{B}$. For a brief account on the historical notes about its relevance to more deeper problems in statistical mechanics, we refer interested readers to Bruno Nachtergaele’s expository paper [Br]. If $K = \mathbb{C}$, then Popescu elements are just some complex numbers i.e. $\nu_k = \lambda_k$, then $\omega$ is a pure tensor product state. We call such a state a Bernoulli state. Thus Bernoulli state once restricted to the the diagonal algebra $\{S_I S_I^* : |I| < \infty\}$ will give a classical Bernoulli state. On the other hand Gelu Popescu develops a dilation theory, analogous to that of B. Sz.-Nagy and C. Foia for a single contraction, for an infinite sequence $v_i$ of non-commuting operators satisfying the condition $\sum v_i v_i^* \leq I$. Here we closely follow the presentation of Popescu’s dilation as given in [BJKW] and Theorem 2.1 in section 2 is a finer version of Popescu’s theorem [Po] and commutant lifting theorem in the present form is a new feature that we explore to an extent in this text, particularly while giving proof of Theorem 3.6.

A finitely correlated pure state $\omega$ gives a type-I factor state once restricted to $\mathcal{B}_R$. On the other hand Araki and Matsui [AMa] found that the unique ground state of XY model is not finitely correlated and in fact once restricted to $\mathcal{B}_R$ gives a type-III$_1$ factor state. In a recent paper [Mo3] it had been shown that for any translation invariant pure state $\omega$, $\omega_R$ is either a type-I or a type-III factor state. This feature makes classification of translation dynamics an interesting problem which we now describe briefly. Given two translation invariant states $\omega_1$ and $\omega_2$ on $\mathcal{B}$, when can we say their translation dynamics $(\mathcal{B}, \theta, \omega_1)$ and $(\mathcal{B}, \theta, \omega_2)$ are isomorphic? i.e. When can we say that there exists an automorphism $\alpha$ on $\mathcal{B}$ such that $\omega_2 \alpha = \omega_1$ and $\theta \alpha = \alpha \theta$?

Let $\Theta_k : \pi_k(\mathcal{B})'' \to \pi_k(\mathcal{B})''$ be the associated automorphisms where $(\mathcal{H}, \pi_k, \Omega_k)$ are GNS spaces for $(\mathcal{B}, \omega_k)$, $k = 1, 2$. $(\mathcal{B}, \theta, \omega_1)$ and $(\mathcal{B}, \theta, \omega_2)$ are said to be weakly isomorphic if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ so that $U \Omega_1 = \Omega_2$ and $U \Theta_1(X) U^* = \Theta_2(U X U^*)$ for all $X \in \pi_1(\mathcal{B})''$. Kolmogorov-Sinai dynamical entropy [Pa] is an invariance for translation dynamics on classical spin chain on lattice $\mathbb{Z}$ and a celebrated result due to D. Ornstein [Or1] also says that Kolmogorov-Sinai dynamical entropy is a complete invariance for classical Markov states. However it is not a complete invariance [Or2] for translation invariants states on classical spin chain on $\mathbb{Z}$. Thus the main obstruction comes from the fact that a classical
translation invariant state need not be a classical stationary Markov state. However such an obstruction is absent in the full quantum situation as we have shown above that any translation invariant state is a stationary quantum Markov state. In [Mo6] we have achieved a partial success in proving that two such states gives weakly isomorphic dynamics if both satisfy Kolmogorov’s property [AM,Mo2] described below. In that proof the simplicity property of $C^*$-algebra $B_R$ played a vital role. This result gives a rare hope for a complete classification for translation dynamics hopefully with some new symmetries and spacial correlation properties of the states under consideration.

We now briefly recall the Kolmogorov property [Mo2]. Given a translation invariant state $\omega$ on $B$, we set increasing sequence of projections $e_n = |\pi(\theta^n(B_R))\rangle\langle\Omega|$, $n \in \mathbb{Z}$ in the GNS space $(H, \pi, \Omega)$ associated with state $\omega$ on $B$. It is simple to check that

\[
S_m e_n S_m^* = e_{n+m}
\]

where $S_m$ is the unitary operator implementing $\theta^m$. The family of operators $(S_n, e_n - |\Omega><\Omega|)$ gives rise to a system of imprimitivity if and only if $e_n \downarrow |\Omega><\Omega|$ as $n \downarrow -\infty$ [Mac]. We say $\omega$ admits Kolmogorov property if $e_n \downarrow |\Omega><\Omega|$ as $n \downarrow -\infty$. In particular Kolmogorov property implies purity of $\omega$. But the converse statement is not true in general [Mo6,Appendix]. This makes classification of translation dynamics an interesting mathematical problem. As a next step of our goal, we would be aiming to classify translation dynamics with pure states. Such a problem demands a comprehensive understanding about translation invariant pure states on $B$. In [Mo4] we have shown that a translation invariant pure state $\omega$ on $B$ can give only type-I or type-III factor states once we restrict to $B_R$ (Theorem 3.4 in [Mo4]). One natural question that arises now for two such translation dynamics with pure states. How does restrictions of those states to $B_R$ determine whether their dynamics are isomorphic or weakly isomorphic? If both give type-I factor states, answer is affirmative for weak isomorphism as type-I property gives Kolmogorov property (Theorem 3.4 in [Mo4]). One related important question that also arises here how Kolmogorov property which is little stronger then purity can ensure existence of free energy density for a translation invariant state? For the definition of free energy state and its existence for finitely correlated state, we refer to [HMOP]. We will not address this classification problem here by studying known invariance. Rather we will confine our interest to investigate translation invariant pure states with additional symmetry by studying associated quantum Markov state $\psi$ and Markov map $(M, \tau, \phi)$.

Now we explain the basic ingredients in the proof of Haag duality property (16). To that end for the time being we fix a translation invariant factor state $\omega$ on $B$ and an extreme point $\psi \in K_\omega$. We consider the GNS space $(H, \pi, \Omega)$ associated with the state $\psi$ on $O_d$ and associated Popescu’s elements $P = (K, M, v_k, 1 \leq k \leq d, \Omega)$ arises on support projection $P = |\pi(O_d)\rangle\langle\Omega|$. Now consider the dual Popescu’s elements $\bar{P} = (K, \bar{M}, \bar{v}_k; 1 \leq k \leq d, \Omega)$ and the completely positive map from $O_d$ to $B(K)$ defined by

\[
\bar{s}_I \bar{s}_J^* \to \bar{v}_I \bar{v}_J^*, |I|, |J| < \infty
\]

Let $\pi : \hat{O}_d \to B(\hat{H})$ be the minimal Stinespring representation so that $P\pi(\bar{s}_I \bar{s}_J^*)P = \bar{v}_I \bar{v}_J^*$ for all $|I|, |J| < \infty$. In particular we have

\[
P\pi(\bar{s}_I^*)P = \pi(\bar{s}_I^*)P = \bar{v}_I^*
\]

We also consider the state $\tilde{\psi}$ on $\hat{O}_d$ given by

\[
\tilde{\psi}(\bar{s}_I \bar{s}_J^*) = \phi(\bar{v}_I \bar{v}_J^*)
\]

where $\phi$ played a vital role.
By relation (12) we check that \( \tilde{\psi} |\text{UHF}_d = \omega | B_L \) (see section 3). Such relations are perfectly symmetric while moving from \( \psi \) to \( \tilde{\psi} \) except the fact that though \( P \) is the support projection of \( \psi \) in \( \pi (\mathcal{O}_d)^n \), it is not guaranteed that \( P \) equals to \( [\pi (\mathcal{O}_d)^n] \). We give an explicit example to support this claim in the note that follows the proof of Proposition 3.1.

We consider the GNS space \((H_0, \pi, \Omega)\) associated with \( \omega \) on \( B \). Let \( e_0 = [\pi_\omega (B_R)^n] \) and \( \tilde{e}_0 = [\pi_\omega (B_L)^n] \) be the support projections of \( \omega \) in \( (B_R)^n \) and \( (B_L)^n \) respectively. We set projection \( q_0 = e_0 \tilde{e}_0 \) and take \( K_0 \) to be the subspace of \( H \) determined by the projection \( q_0 \). Also set von-Neumann algebras \( M_0^1 = q_0 \pi_\omega (B_R)^n q_0 \) and \( M_0^2 = q_0 \pi_\omega (B_L)^n q_0 \). So by our construction we have \( M_0^1 \subseteq (M_0^2)^\prime \). Further the vector state \( \phi(x) = \langle \Omega, x \Omega \rangle \) on \( B(K_0) \) is faithful and normal on \( M_0^1 \) and \( M_0^2 \). Further \( \omega \) being a factor state, we will also have factor property of \( M_0^1 \) and \( M_0^2 \) (Theorem 2.4). What is less obvious is when we can expect cyclic property i.e. \([M_0^1] = [M_0^2] = I_{K_0}\), identity of \( K_0 \). The following theorem answers all non trivial questions that we have raised so far.

**Theorem 1.1.** Let \( \omega \) be a translation invariant factor state on \( B \) and \( \psi \) be an extremal element in \( K_\omega \). Then the following statements are equivalent:

(a) \([M_0^1] = I_{K_0}, [M_0^2] = I_{K_0}\);
(b) \((M_0^1)' = M_0^2\);
(c) \(\pi_\omega (B_R)' = \pi_\omega (B_L)^n\);
(d) \([\pi_\omega (B_R)^n] = [\pi_\omega (B_L)^n] ;\)
(e) \(\{ x \in B(K) : \sum_k \beta_k x \beta_k^* = x \} = M \);
(f) \( \omega \) is pure.

In such a case \( M_0^1 = \{ x \in \tilde{M} : \beta_z (x) = x : z \in H \} \) where \( H = \{ z \in S^1 : \psi = \psi \beta_z \} \).

Before we elaborate further on equivalence of above statements we briefly recall results on translation invariant pure state on \( B = \otimes \mathbb{Z} M_d (\mathbb{C}) \) that finds its relevance while proving Haag duality property. There is a one to one affine map between translation invariant states on \( B \) and translation invariant states on \( B_R = \otimes \mathbb{Z}_+ M_d (\mathbb{C}) \) by \( \omega \rightarrow \omega_R = \omega | B_R \). The inverse map is the inductive limit state of \( (B_R, \psi_R) \rightarrow \lambda_n (B_R, \psi_R) \) where \( (\lambda_n : n \geq 0) \) is the canonical semi-group of right shifts on \( B_R \). Pure states on a UHF algebra are studied in the general framework of [Pow]. Such a situation has been investigated also in detail at various degrees of generality in [BJP] and [BJKW] motivated by the development of C*-algebraic method in the study of iterative function systems and its associated wavelet theory. One interesting result in [BJP] says that any translation invariant pure state on \( B_R \) is also a product state and the canonical endomorphism associated with two such states are unitary equivalent. However such a statement is not true for two translation invariant pure states on \( B \) as their restriction to \( B_R \) need not be isomorphic. Theorem 3.4 in [Mo4] says that \( \omega_R \) is either a type-I or a type-III factor state on \( B_R \). Both type of factors are known to exist in the literature of quantum statistical mechanics [BR vol-II, Si, Ma1]. Thus the classification problem of translation dynamics on \( B \) with invariant pure states on \( B \) up to unitary isomorphism is
a delicate one. In this context one interesting problem that remain open is whether mean entropy $[Ru]$ is an invariance for translation dynamics.

Since a $\theta$ invariant state $\omega$ on $B$ is completely determined by its restriction $\omega_R$ to $B_R$, in principle it is possible to describe various properties of $\omega$ including purity by studying their restriction $\omega_R$. Theorem 3.2 in [Mo3] gives a precise answer: $\omega$ is pure if and only if there exists a sequence of positive contractive elements $x_n \in M$ such that

$$x_n \to I, \quad x_{m+n}\tau_n(x) \to \phi(x)I$$

in strong operator topology for all $x \in M$ and $m \geq 1$. As an application of this result, we prove that (e) implies (f). This statement can be taken as the correct version of Theorem 7.1 of [BJKW]. Theorem 7.1 in [BJKW] has aimed towards a sufficient condition on Popescu elements $\mathcal{P}$ for purity of the translation invariant state. However the statement and its proof are faulty as certain argument used in the proof is not time reversal symmetric and a factor state with Popescu elements on support projection satisfies the conditions of the statement of Theorem 7.1. One natural remedy to add additional hypothesis that (e) holds. In particular Lemma 7.6 in [BJKW] needs that additional assumption related to the support projection of the dual state $\tilde{\psi} \in K_\omega$. Besides this additional structure proof of Lemma 7.8 in [BJKW] is also not complete unless we find a proof for $M = M'$ (we retained same notations here in the text) for such a factor state $\omega$. Such a problem could have been solved if there were any method which shows directly that Takesaki’s conditional $[Ta]$ expectation exists from $M'$ onto $M$. For von-Neumann algebras $\mathcal{N} \subseteq M$, a normal unital completely positive map $E_c : M \to \mathcal{N}$ is called normal conditional expectation if

$$(20) \quad E_c(yxz) = yE_c(x)z$$

for all $y, z \in \mathcal{N}$ and $x \in M$. A theorem of Takesaki’s $[Ta$, also see AcC] says that conditional expectation $E_c$ preserving a faithful normal state $\phi$ on $M$ exists if and only if the modular group $\sigma = (\sigma_t : t \in \mathbb{R})$ associated with $\phi$, which preserves $\mathcal{M}$, also preserves $\mathcal{N}$, i.e. $\sigma_t(x) \in \mathcal{N}$ for all $t \in \mathbb{R}$ and $x \in \mathcal{N}$. Thus main body of the proof for Theorem 7.1 even with the additional natural hypothesis $p_0 = q_0$, in [BJKW] is incomplete.

Besides the proof of (e) implies (f), other implication follows along with while we will prove the hardest part of this theorem namely (f) implies (e) i.e. purity implies Haag duality property (16). For the proof we have explored the set of representation of $B$ quasi-equivalent to $\pi$ and equip it with a strict partial ordering depending on our situation to prove Haag duality. Mackey’s system of imprimitivity $[Mac]$ plays a crucial role even though a pure state not necessarily give rise to a Mackey’s system of imprimitivity generated by the support projection $e_0$ with respect to shift. Though we have worked here with amalgamated representation of $\mathcal{O}_d \otimes \mathcal{O}_d$ in [BJKW], it seems that just for Haag duality one can avoid doing so. It seems that the underlining group $\mathbb{Z}$ can easily be replaced by $\mathbb{Z}^k$ for some $k \geq 2$ and wedge duality for a pointed cone can be proved by following the same ideas. We defer this line of analysis leaving it for work as its relation with problems in quantum spin chain in higher dimensional lattice needs some additional structure. We also defer application of Haag duality property in studying symmetry and correlation of a translation invariant pure state to another paper [Mo5].

The paper is organized as follows. In section 2 we study Popescu’s dilation associated with a translation invariant state on Cuntz algebra $\mathcal{O}_d$ and review ‘commutant lifting theorem’ investigated in [BJKW]. The proof presented here remove
the murky part of the proof of Theorem 5.1 in [BJKW]. In section 3 we explore both the notion of Kolmogorov’s shift and its intimate relation with Mackey’s imprimitivity system to explore a duality argument introduced in [BJKW]. We find a useful necessary and sufficient condition (Theorem 1.1 (a) ) in terms of support projection of Cuntz’s state for a translation invariant factor state \( \omega \) on \( B \) to be pure. The criterion on support projection is crucial for our main mathematical result Theorem 3.6.

**Remark 1.2.** The paper “On Haag Duality for Pure States of Quantum Spin Chain” by authors: M. Keyl, Taku Matsui, D. Schlingemann, R. F. Werner, Rev. Math. Phys. 20:707-724,2008 has an incomplete proof for Haag duality property as Lemma 4.3 in that paper has a faulty argument.

### 2. States on \( O_d \) and the commutant lifting theorem

In this section we essentially recall results from [BJKW] and organize it with additional remarks and arguments as it needed to understand the present problem investigated in section 3. In the following we recall a commutant lifting theorem (Theorem 5.1 in [BJKW] ), crucial for our purpose.

**Theorem 2.1.** Let \( v_1, v_2, \ldots, v_d \) be a family of bounded operators on a Hilbert space \( K \) so that \( \sum_{1 \leq k \leq d} v_k v_k^* = I \). Then there exists a unique up to isomorphism Hilbert space \( H \), a projection \( P \) on \( K \) and a family of isometries \( \{ S_k : 1 \leq k \leq d \} \) satisfying Cuntz’s relation so that

\[
PS_k^*P = S_k^*P = v_k^*
\]

for all \( 1 \leq k \leq d \) and \( K \) is cyclic for the representation i.e. the vectors \( \{ S_i K : |I| < \infty \} \) are total in \( H \).

Moreover the following holds:

(a) \( \Lambda^n(P) \uparrow I \) as \( n \uparrow \infty \) where

\[
\Lambda(X) = \sum_k S_k X S_k^* \leq \Lambda^n(D) \to \chi' \text{ weakly as } n \to \infty \text{ for some } \chi' \text{ in the commutant } \{ S_k, S_k^* : 1 \leq k \leq d \}' \text{ so that } PX'P = D. \text{ Moreover the self adjoint elements in the commutant } \{ S_k, S_k^* : 1 \leq k \leq d \}' \text{ is isometrically order isomorphic with the self adjoint elements in } \mathcal{B}_c(K) \text{ via the surjective map } X' \to PX'P, \text{ where } \mathcal{B}_c(K) = \{ x \in \mathcal{B}(K) : \sum_{1 \leq k \leq d} v_k x v_k^* = x \}.
\]

(c) \( \{ v_k, v_k^*, 1 \leq k \leq d \}' \subseteq \mathcal{B}_c(K) \) and equality holds if and only if \( P \in \{ S_k, S_k^* \} \).

If \( (w_i) \) be another such an Popescu elements on a Hilbert space \( K \) such that there exists an operator \( u : K \to K' \) so that \( \sum_k w_k v_k^* = u \) then there exists an operator \( U : \mathcal{H}_u \to \mathcal{H}_w \) so that \( \pi'(x)U = U \pi(x) \) where \( (\mathcal{H}_u, \pi', S_1) \) are Popescu dilation of \( (w_i) \) and \( \pi' \) is the associated minimal representation of \( O_d \). In particular \( U \) is isometry, unitary if \( u \) is so respectively. If \( u \) is unitary and \( \mathcal{K} = \mathcal{K}' \) then we can as well take \( \mathcal{H}_u = \mathcal{H}_w \).

**Proof.** Following Popescu [Po] we define a completely positive map \( R : O_d \to \mathcal{B}(K) \) by

\[
R(s_i s_j^*) = v_i v_j^*
\]

for all \( |I|, |J| < \infty \). The representation \( S_1, \ldots, S_d \) of \( O_d \) on \( \mathcal{H} \) thus may be taken to be the Stinespring dilation of \( R \) [BR, vol-2] and uniqueness up to unitary equivalence follows from uniqueness of the Stinespring representation. That \( \mathcal{K} \) is cyclic for the representation follows from the minimality property of the Stinespring dilation.
For (a) let $Q$ be the limiting projection of $\Lambda^n(P)$ as $n \uparrow \infty$. Then we have $\Lambda(Q) = Q$ i.e. $QA(I - Q)Q = 0$ and so $(I - Q)S^n_QQ = 0$. Interchanging the role of $Q$ with $I - Q$, we get $QS^n_Q(I - Q) = 0$. This shows $QS^n_Q = S^n_QQ$ for all $1 \leq k \leq d$. Since $Q$ is a projection, taking adjoint in the relation, we get $Q \in \{S_k, S^n_k\}$. That $Q \geq P$ is obvious since $\Lambda^n(P) \geq P$ for all $n \geq 1$. In particular $QSf = f$ for all $f \in K$ and $|f| = 1$. Hence $Q = I$ by the cyclicity of $K$.

For (b) essentially we defer from the argument used in Theorem 5.1 in [BJKW]. We fix any $D \in B_r(K)$ and note that $P\Lambda_n(D) = \tau_n(D) = D$ for any $k \geq 1$. Thus for any integers $n > m$ we have

$$\Lambda_m(P)\Lambda_n(D)\Lambda_m(P) = \Lambda_m(\Lambda_n(D)m(P)P) = \Lambda_m(D)$$

Hence for any fixed $m \geq 1$ limit $< f, \Lambda_m(D)g >$ exists for all $f, g \in \Lambda_m(D)$. Since the family of operators $\Lambda_n(D)$ is uniformly bounded and $\Lambda_m(D) \uparrow I$ as $m \to \infty$, a standard density argument guarantees that the weak operator limit of $\Lambda_n(D)$ exists as $n \to \infty$. Let $X'$ be the limit. So $\Lambda(X') = X'$, by Cuntz's relation, $X' \in \{S_k, S^n_k : 1 \leq k \leq k\}$. Since $P\Lambda_n(D) = D$ for all $n \geq 1$, we also conclude that $PX'P = D$ by taking limit $n \to \infty$. Conversely, it is obvious that $P\{S_k, S^n_k : k \geq 1\}P \subseteq B_r(K)$. Hence we can identify $P\{S_k, S^n_k : k \geq 1\}P$ with $B_r(K)$.

Further it is obvious that $X'$ is self-adjoint if and only if $D = PX'P$ is self-adjoint (since $\Lambda^n(P) \uparrow I$ as $n \uparrow \infty$, by taking limit $\Lambda^n(PX) = \Lambda^n(PX^*P)$ shows that $X^* = X$ if $D^* = D$). Now fix any self-adjoint element $D \in B_r(K)$. Since the identity operator on $K$ is an element in $B_r(K)$ for any $\alpha \geq 0$ for which $-\alpha P \leq D \leq \alpha P$, we have $\alpha \Lambda_n(D) \leq \Lambda_n(D) \leq \alpha \Lambda_n(D)$ for all $n \geq 1$. By taking limit $n \to \infty$ we conclude that $-\alpha I \leq X' \leq \alpha I$, where $PX'P = D$. Since operator norm of a self-adjoint element $A$ in a Hilbert space is given by

$$||A|| = \inf_{\alpha \geq 0}\{\alpha : -\alpha I \leq A \leq \alpha I\}$$

we conclude that $||X'|| \leq ||D||$. That $||D|| = ||PX'P|| \leq ||X'||$ is obvious, $P$ being a projection. Thus the map is isometrically order isomorphic taking self-adjoint elements of the commutant to self-adjoint elements of $B_r(K)$.

We are left to prove (c). Inclusion is trivial. For the last part, we assume first that $P \in \pi(O_d)''$. For any element $D$ in $B_r(K)$ there exists an element $X'$ in $\{S_k, S^n_k : 1 \leq k \leq d\}$ so that $PX'P = D$. Since $P$ commutes with $X'$, we verify that

$$Dv_k = PX'P = PX'S^n_kP = PS^n_kX'P = PS^n_kPX'P = v_kD$$

We also have $D^* \in B_r(K)$ and thus $D^*v_k = v_kD^*$. Hence $D \in \{v_k, v_k^* : 1 \leq k \leq d\}''$. Since $P\pi(O_d)''P = B(K)_r$, we conclude that $B(K)_r \subseteq \mathcal{M}'$. Thus equality holds whenever $P \in \{S_k, S^n_k, 1 \leq k \leq d\}''$.

For the converse note that by commutant lifting property self-adjoint elements of the commutant $\{S_k, S^n_k, 1 \leq k \leq d\}$ is order isometric with the algebra $\mathcal{M}'$ via the map $X' \to PX'P$. Hence $P \in \{S_k, S^n_k, 1 \leq k \leq d\}''$ by Proposition 4.2 in [BJKW].

For the proof of intertwining relation and their property we refer to main body of the proof of Theorem 5.1 in [BJKW]. ■
A family \((v_k, 1 \leq k \leq d)\) of contractive operators on a Hilbert space \(K\) is called Popescu’s elements if \(\sum_k v_k v_k^* = I\) and the dilation \((\mathcal{H}, P, K, S_1, 1 \leq k \leq d)\) in Theorem 2.1 is called Popescu’s dilation to Cuntz elements. In the following proposition we deal with a family of minimal Popescu elements for a state on \(\mathcal{O}_d\).

**Proposition 2.2.** There exists a canonical one-one correspondence between the following objects:

(a) States \(\psi\) on \(\mathcal{O}_d\)

(b) Function \(C : \mathcal{I} \times \mathcal{I} \to \mathbb{C}\) with the following properties:
   (i) \(C(\emptyset, \emptyset) = 1\);
   (ii) for any function \(\lambda : \mathcal{I} \to \mathbb{C}\) with finite support we have
   \[
   \sum_{I, J \in \mathcal{I}} \lambda(I) C(I, J) \lambda(J) \geq 0
   \]
   (iii) \(\sum_{i \in \mathcal{I}_d} C(I_i, J_i) = C(I, J)\) for all \(I, J \in \mathcal{I}\).

(c) Unitary equivalence class of objects \((K, \Omega, v_1, \ldots, v_d)\) where
   (i) \(K\) is a Hilbert space and \(\Omega\) is a unit vector in \(K\);
   (ii) \(v_1, \ldots, v_d \in \mathcal{B}(K)\) so that \(\sum_{i \in \mathcal{I}_d} v_i v_i^* = 1\);
   (iii) the linear span of the vectors of the form \(v_i^* \Omega\), where \(I \in \mathcal{I}\), is dense in \(K\).

The correspondence is given by a unique completely positive map \(R : \mathcal{O}_d \to \mathcal{B}(K)\), where

(i) \(R(s_I s_I^*) = v_I v_I^*\);

(ii) \(\psi(x) = \langle \Omega, R(x) \Omega \rangle\);

(iii) \(\psi(s_I s_I^*) = C(I, J) = \langle v_I^* \Omega, v_J^* \Omega \rangle\);

(iv) For any fixed \(g \in U_d\), the completely positive map \(R_g : \mathcal{O}_d \to \mathcal{B}(K)\) defined by
   \(R_g = R \circ \beta_g\) give rises to a Popescu system given by \((K, \Omega, \beta_g(v_1), \ldots, \beta_g(v_d))\) where
   \[
   \beta_g(v_i) = \sum_{1 \leq j \leq d} g_{ij} v_j.
   \]

**Proof.** For a proof we simply refer to Proposition 2.1 in [BJKW].

The following is a simple consequence of Theorem 2.1 valid for a \(\lambda\)-invariant state \(\psi\) on \(\mathcal{O}_d\). This proposition will have very little application in the main body of this paper but this gives a clear picture explaining the delicacy of the present problems.

**Proposition 2.3.** Let \(\psi\) be a state on \(\mathcal{O}_d\) and \((\mathcal{H}, \pi, \Omega)\) be the GNS space associated with \((\mathcal{O}_d, \psi)\). We set \(S_i = \pi(s_i)\) and normal state \(\psi_\Omega\) on \(\pi(\mathcal{O}_d)''\) defined by

\[
\psi_\Omega(X) = \langle \Omega, X \Omega \rangle
\]

Let \(P\) be the projection on the closed subspace \(K\) generated by the vectors \(\{S_i^* \Omega : |I| < \infty\}\) and

\[
v_k = P S_k P
\]

for \(1 \leq k \leq d\). Then following holds:

(a) \(\{v_i^* \Omega : |I| < \infty\}\) is total in \(K\).

(b) \(\sum_{1 \leq k \leq d} v_k v_k^* = 1\).

(c) \(S_k^* P = P S_k^* P\) for all \(1 \leq k \leq d\).
Thus \( \psi(s_j s_i^*) = \langle \Omega, v_i v_j^* \Omega \rangle \)
and the vectors \( \{S_f f : f \in K, |I| < \infty \} \) are total in the GNS Hilbert space associated with \((\mathcal{O}_d, \psi)\). Further such a family \((K, \psi_k, 1 \leq k \leq d, \omega)\) satisfying (a) to (d) is determined uniquely up to isomorphism.

Conversely given a Popescu system \((K, \psi_k, 1 \leq k \leq d, \Omega)\) satisfying (a) and (b) there exists a unique state \(\psi\) on \(\mathcal{O}_d\) so that (c) and (d) are satisfied.

Furthermore the following statements are valid:

(e) If the normal state \(\phi(x) = \langle \Omega, x \Omega \rangle\) on the von-Neumann algebra \(\mathcal{M} = \{v_i, v_i^*\}''\) is invariant for the Markov map \(\tau(x) = \sum_{1 \leq k \leq d} v_i x v_i^*\), \(x \in \mathcal{M}\) then \(\psi\) is \(\lambda\) invariant and \(\phi\) is faithful on \(\mathcal{M}\).

(f) If \(P \in \pi(\mathcal{O}_d)''\) then following are equivalent:
(i) \(\psi\) is an ergodic state for \((\mathcal{O}_d, \lambda)\);
(ii) \((\mathcal{M}, \tau, \phi)\) is ergodic;
(iii) \(\mathcal{M}\) is a factor.

Proof. We fix a state \(\psi\) and consider the GNS space \((\mathcal{H}, \pi, \Omega)\) associated with \((\mathcal{O}_d, \psi)\) and set \(S_i = \pi(s_i)\). It is obvious that \(S_i^* P \subseteq P\) for all \(1 \leq k \leq d\), thus \(P\) is the minimal subspace containing \(\Omega\) and invariant by all \(\{S_k^*; 1 \leq k \leq d\}\) i.e.

\[
PS_i^* P = S_i^* P
\]

Thus \(v_i^* = P S_i^* P = S_i^* P\) and so \(\sum_k v_k v_k^* = \sum_k P S_k^* S_k^* P = P\) which is identity operator in \(\mathcal{K}\). This completes the proof of (a) (b) and (c).

For (d) we note that

\[
\psi(s_j s_i^*) = \langle \Omega, S_j S_i^* \Omega \rangle = \langle \Omega, P S_i S_j^* P \Omega \rangle = \langle \Omega, v_i v_j^* \Omega \rangle.
\]

Since \(\mathcal{H}\) is spanned by the vectors \(\{S_j S_i^* \Omega : |I|, |J| < \infty\}\) and \(\mathcal{K}\) is spanned by the vectors \(\{S_i^* \Omega = v_i^* \Omega : |I| < \infty\}\), \(\mathcal{K}\) is cyclic for \(S_j\) i.e. the vectors \(\{S_j K : |I| < \infty\}\) spans \(\mathcal{H}\). Uniqueness up to isomorphism follows as usual by total property of vectors \(v_j^* \Omega\) in \(\mathcal{K}\).

Conversely for a Popescu’s elements \((K, \psi, \Omega)\) satisfying (a) and (b), we consider the family \((\mathcal{H}, S_k, 1 \leq k \leq d, P)\) of Cuntz’s elements defined as in Theorem 2.1. We claim that \(\Omega\) is a cyclic vector for the representation \(\pi(s_i) \rightarrow S_i\). Note that by our construction vectors \(\{S_f f : f \in K : |I| < \infty\}\) are total in \(\mathcal{H}\) and \(v_j^* \Omega = S_j^* \Omega\) for all \(|J| < \infty\). Thus by our hypothesis that vectors \(\{v_j^* \Omega : |I| < \infty\}\) are total in \(\mathcal{K}\), we verify that vectors \(\{S_j S_i^* \Omega : |I|, |J| < \infty\}\) are total in \(\mathcal{H}\). Hence \(\mathcal{O}\) is cyclic for the representation \(s_i \rightarrow S_i\) of \(\mathcal{O}_d\).

We are left to prove (e) and (f). It simple to note by (d) that \(\psi \lambda = \psi\) i.e.

\[
\sum_i \langle \Omega, S_i S_j S_i^* S_i^* \Omega \rangle = \sum_i \langle \Omega, v_i v_j v_i^* v_i^* \Omega \rangle = \langle \Omega, v_i v_j v_i^* \Omega \rangle = \langle \Omega, S_i S_j \Omega \rangle
\]

for all \(|I|, |J| < \infty\) where in the second equality we have used our hypothesis that the vector state \(\phi\) on \(\mathcal{M}\) is \(\tau\)-invariant. In such case we aim now to show that
Thus by a standard theorem \[La,Ev,Fr,BJKW\] ergodic property is equivalent to 

\[p' \tau (1 - p') = 0\] 

Since \(p' \tau (1 - p') \geq 0\) and an element in \(M\), by minimality of support projection, we conclude that \(p' \tau (1 - p') = 0\). Hence \(p' \tau = \Omega\) and \(p' v_i^* v_i = e_i^* e_i\) for all \(1 \leq k \leq d\). Thus \(p' v_i^* \tau = v_i^* \Omega\) for all \(|I| < \infty\). As \(K\) is the closed linear span of the vectors \(\{v_i^* \Omega : |I| < \infty\}\), we conclude that \(p' = p\). In other words \(\phi\) is faithful on \(M\). This completes the proof for (e).

We are left to show (f) where we assume that \(P \in \pi(O_d)^\prime\prime\). \(\Omega\) being a cyclic vector for \(\pi(O_d)^\prime\prime\), the weak* limit of the increasing projection \(\Lambda^k(P)\) is \(I\). Thus by Theorem 3.6 in [Mo1] we have \((\pi(O_d)^\prime\prime, \Lambda, \psi_\Omega)\) is ergodic if and only if the reduced dynamics \((M, \tau, \phi)\) is ergodic. For the last part of the statement, we need to show for a projection \(e \in M\), \(\tau(e) = e\) if and only if \(e \in M \cap M'\). It is an easy consequence since \(\tau(e) = e\) says that \(\tau(I - e) = 0\) and so \((1 - e)v_i^* e = 0\). Changing the role of \(e\) by \(I - e\), we also get \(ev_i^* (I - e) = 0\) for all \(1 \leq k \leq d\). Thus we get that \(e\) commutes with each \(v_k\). \(P\) being in \(\pi(O_d)^\prime\prime\), \(v_k \in M\). So \(e \in M'\). Thus by a standard theorem [La,Ev,Fr,BJKW] ergodic property is equivalent to factor property of \(M\).

Before we move to our next result, we comment here that in general for a \(\lambda\) invariant state on \(O_d\), the normal state \(\phi\) on \(M = \{v_k, v_k^* : 1 \leq k \leq d\}\) need not be invariant for \(\tau\). To that end we consider ([BR] vol-II page 110) the unique KMS state \(\psi = \psi_\beta\) for the automorphism \(\alpha_t(s_i) = e^{it}s_i\) on \(O_d\). \(\psi\) is \(\lambda\) invariant and \(\psi\mid_{UHF}\) is the unique faithful trace. \(\psi\) being a KMS state for an automorphism, the normal state induced by the cyclic vector on \(\pi(O_d)^\prime\prime\) is also separating for \(\pi(O_d)^\prime\prime\). As \(\psi_\beta = \psi\) for all \(z \in S^1\) we have \(<\Omega, \pi(s_i)\Omega > = z|I| > <\Omega, \psi_\beta(s_i)\Omega > = z|I| > \psi_\beta(s_i)\Omega >\) for all \(z \in S^1\) and so \(<\Omega, \pi(s_i)\Omega > = 0\) for all \(|I| \geq 1\). In particular \(<\Omega, v_i^* \Omega > = 0\) where \((v_i)\) are defined as in Proposition 2.3 and thus \(<v_i \Omega, v_i^* \Omega > = <\Omega, v_i^* v_i \Omega > = 0\) for all \(1 \leq i \leq d\). Hence \(v_i \Omega = 0\). By Proposition 2.3 (f), \(\Omega\) is separating for \(M\) and so we get \(v_i = 0\) for all \(1 \leq i \leq d\) and this contradicts that \(\sum_i v_i v_i^* = 1\). Thus we conclude by Proposition 2.3 (f) that \(\phi\) is not \(\tau\) invariant on \(M\). This example also indicates that the support projection of a \(\lambda\) invariant state \(\psi\) in \(\pi(O_d)^\prime\prime\) need not be equal to the minimal sub-harmonic projection \(P\) i.e. the closed span of vectors \(\{S_I^* \Omega : |I| < \infty\}\) containing \(\Omega\) and \(\{v_I v_I^* : |I|, |J| < \infty\}\) need not be even an algebra.

Now we aim to deal with another class of Popenescu elements associated with an \(\lambda\)-invariant state on \(O_d\). In fact this class of Popenescu elements will play a significant role for the rest of the text and we will repeatedly use this proposition in section 3.

**PROPOSITION 2.4.** Let \((\mathcal{H}, \pi, \Omega)\) be the GNS representation of a \(\lambda\) invariant state \(\psi\) on \(O_d\) and \(P\) be the support projection of the normal state \(\psi_\Omega(X) = <\Omega, XX\Omega >\) in the von-Neumann algebra \(\pi(O_d)^\prime\prime\). Then the following holds:

(a) \(P\) is a sub-harmonic projection for the endomorphism \(\Lambda(X) = \sum_k S_k X S_k^*\) on \(\pi(O_d)^\prime\prime\) i.e. \(\Lambda(P) \geq P\) satisfying the following:

(i) \(\Lambda_n(P) \uparrow I\) as \(n \uparrow \infty\);

(ii) \(PS_k^* P = S_k^* P\), \(1 \leq k \leq d\);

(iii) \(\sum_{1 \leq k \leq d} v_k v_k^* = I\) where \(S_k = \pi(s_k)\) and \(v_k = PS_k P\) for \(1 \leq k \leq d\);
(b) For any \( I = (i_1, i_2, \ldots, i_k), J = (j_1, j_2, \ldots, j_l) \) with \( |I|, |J| < \infty \) we have \( \psi(s_is_j^*) = < \Omega, \psi(v_i^*v_j^*) \Omega > \) and the vectors \( \{ s_if : f \in \mathcal{K}, |I| < \infty \} \) are total in \( \mathcal{H} \).

(c) The von-Neumann algebra \( \mathcal{M} = P\pi(\mathcal{O}_d)''P \), acting on the Hilbert space \( \mathcal{K} \), i.e. the range of \( P \), is generated by \( \{ v_k, v_k^* : 1 \leq k \leq d \}'' \) and the normal state \( \phi(x) = < \Omega, x\Omega > \) is faithful on the von-Neumann algebra \( \mathcal{M} \).

(d) The self-adjoint part of the commutant of \( \pi(\mathcal{O}_d)' \) is norm and order isomorphic to the space of self-adjoint fixed points of the completely positive map \( M \).

Thus we have \( \Lambda(\tau, \phi) \) satisfying (a)-(d).

Conversely let \( \mathcal{M} \) be a von-Neumann algebra generated by a family \( \{ v_k : 1 \leq k \leq d \} \) of bounded operators on a Hilbert space \( \mathcal{K} \) so that \( \sum_k v_kv_k^* = 1 \) and the commutant

\[
\mathcal{M}' = \{ x \in \mathcal{B}(\mathcal{K}) : \sum_k v_kxv_k^* = x \}\n
Then the Popescu dilation \( (\mathcal{H}, P, S_k, 1 \leq k \leq d) \) described in Theorem 2.1 satisfies the following:

(i) \( P \in \{ S_k, S_k^*, 1 \leq k \leq d \}'' \);

(ii) For any faithful normal \( \tau \)-invariant state \( \phi \) on \( \mathcal{M} \) there exists a state \( \psi \) on \( \mathcal{O}_d \) defined by

\[
\psi(s_is_j^*) = \phi(v_i^*v_j^*), |I|, |J| < \infty \]

such that the GNS space associated with \( (\mathcal{M}, \phi) \) is the support projection for \( \psi \) in \( \pi(\mathcal{O}_d)'' \) satisfying (a)-(d).

Further for a given \( \lambda \)-invariant state \( \psi \), the family \( (\mathcal{K}, \mathcal{M}, v_k : 1 \leq k \leq d, \phi) \) satisfying (a)-(d) is determined uniquely up to unitary conjugation.

(e) \( \phi \) is a faithful normal \( \tau \)-invariant state on \( \mathcal{M} \). Furthermore the following statements are equivalent:

(i) \( \mathcal{O}_d, \lambda, \psi \) is ergodic;

(ii) \( (\mathcal{M}, \tau, \phi) \) is ergodic;

(iii) \( \mathcal{M} \) is a factor.

**Proof.** \( \Lambda(P) \) is also a projection in \( \pi(\mathcal{O}_d)'' \) so that \( \psi_{\Omega}(\Lambda(P)) = 1 \) by invariance property. Thus we have \( \Lambda(P) \geq P \) i.e. \( PA(I - P)P = 0 \). Hence we have

\[
PS_k^*P = S_k^*P \]

Moreover by \( \lambda \) invariance property we also note that the faithful normal state \( \phi(x) = < \Omega, x\Omega > \) on the von-Neumann algebra \( \mathcal{M} = P\pi(\mathcal{O}_d)''P \) is invariant for the reduced Markov map \( [M_0] \) on \( \mathcal{M} \) given by

\[
\tau(x) = PA(PxP)P \]

We claim that \( \lim_{n \to \infty} \Lambda^n(P) = I \). That \( \{ \Lambda^n(P) : n \geq 1 \} \) is a sequence of increasing projections follows from sub-harmonic property of \( P \) and endomorphism property of \( \Lambda \). Let the limiting projection be \( Y \). Then \( \Lambda(Y) = Y \) and so \( Y \in \{ S_k, S_k^* \} \). Since by our construction, the GNS Hilbert space \( \mathcal{H}_{\pi_{\phi}} \) is generated by \( S_kS_k^*\Omega \), \( Y \) is a scalar. \( Y \) being a non-zero projection, it is the identity operator in \( \mathcal{H}_{\pi_{\phi}} \).
Now it is routine to verify (a) (b) and (c). For the first part of (d) we appeal to Theorem 2.1. For the last part note that for any invariant element \( D \) in \( B(\mathcal{K}) \) there exists an element \( X' \) in \( \pi(O_d)'' \) so that \( PXX'P = D \). Since \( P \in \pi(O_d)'' \) we note that \((1 - P)X'P = 0\). Now since \( X' \in \{ S_k, S_k' \} \), we verify that \( \pi(Dv_k) = PXS_k'P = PS_kX'P = PS_kXP = v_k^*D \). Since \( D^* \) is order isometric with the algebra \( M' \) via the map \( X' \mapsto PXX'P, P \in \{ S_k, S_k', 1 \leq k \leq d \}'' \) by Proposition 4.2 in [BJKW]. For (ii) without loss of generality, we assume that \( \phi(x) =< \Omega, x\Omega > \) for all \( x \in M \) and \( \Omega \) is a cyclic and separating vector for \( M \).

To that end let \( Y \in \pi(O_d)'' \) be the projection on the subspace generated by the vectors \( \{ S_k S_k' \Omega : |k|, |J| < \infty \} \). Note that, \( P \) being an element in \( \pi(O_d)'' \), \( Y \) commutes with all the elements of \( \pi(O_d)'' \). Hence \( Yx\Omega = x\Omega \) for all \( x \in M \). Thus \( Y \geq P \). Since \( \Lambda_n(P) \uparrow I \) as \( n \uparrow \infty \) by our construction, we conclude that \( Y = \Lambda_n(Y) \geq \Lambda_n(P) \uparrow I \) as \( n \uparrow \infty \). Hence \( Y = I \). In other words \( \Omega \) is cyclic for the representation \( \Lambda_1 \rightarrow S_1 \). This completes the proof for (ii).

Uniqueness up to unitary isomorphism follows as GNS representation is determined uniquely up to unitary conjugation and so its support projection.

The first part of (e) we note that \( PS_k S_k' = v_k v_k^* \) for all \( I, |J| < \infty \) and thus \( M = P\pi(O_d)'' \) is the von-Neumann algebra generated by \( \{ v_k, v_k^* : 1 \leq k \leq d \} \) and thus \( \tau(x) = P\lambda(P\pi(O_d)''P) \) for all \( x \in M \). That \( \phi \) is \( \tau(x) = \sum v_k x v_k^* \) invariant follows as \( \psi \) is \( \lambda \)-invariant. We are left to prove equivalence of statements (i)-(iii).

By Theorem 3.6 in [Mo1] the Markov semi-group \( (M, \tau, \phi) \) is ergodic if and only if \( (\pi(O_d)'' \Lambda, \psi_{\Omega}) \) is ergodic (here we need to recall by (a) that \( \Lambda_n(P) \uparrow I \) as \( n \uparrow \infty \)). By a standard result [Ev, also BJKW] \( (M, \tau, \phi) \) is ergodic if and only if there is no non trivial projection \( e \) invariant for \( \tau \) i.e. \( T^c = \{ e \in M : e^* = e, e^2 = e, \tau(e) = e \} = \{ 0, 1 \} \). If \( \tau(e) = e \) for some projection \( e \in M \) then \( (1 - e)\tau(e)(1 - e) = 0 \) and so \( e v_k^*(1 - e) = 0 \). Same is true if we replace \( e \) by \( 1 - e \) as \( \tau(1 - e) = \tau(1) - \tau(e) = 1 - e \) and so \( (1 - e) v_k e = 0 \). Thus \( e \) commutes with \( v_k, v_k^* \) for all \( 1 \leq k \leq d \). Hence \( T^c \subseteq M \). Inequality in the reverse direction is trivial and thus \( T^c \) is trivial if and only if \( M \) is a factor. This shows equivalence of (ii) and (iii) follows by a standard result [La,Fr] in non-commutative ergodic theory. This completes the proof.

The following two propositions are essentially easy adaptations of results proved in [BJKW, Section 6 and Section 7]. These results are crucial in our present framework.

**Proposition 2.5.** Let \( \psi \) be a \( \lambda \) invariant factor state on \( O_d \) and \( (H, \pi, \Omega) \) be its GNS representation. Then the following holds:

(a) The closed subgroup \( H = \{ z \in S^1 : \psi z = \psi \} \) is equal to

\[ \{ z \in S^1 : \beta_z \text{ extends to an automorphism of } \pi(O_d)'' \} \]
(b) Let $O^H_d$ be the fixed point sub-algebra in $O_d$ under the action of gauge group $\{\beta_z : z \in H\}$. Then $\pi(O^H_d)'' = \pi(UHF_d)''$.

(c) If $H$ is a finite cyclic group of $k$ many elements and $\pi(UHF_d)''$ is a factor, then $\pi(O_d)'' \cap \pi(UHF_d)'' \equiv \mathbb{C}^m$ where $1 \leq m \leq k$.

**Proof.** It is simple that $H$ is a closed subgroup. For any fix $z \in H$ we define unitary operator $U_z$ extending the map $\pi(x)\Omega \rightarrow \pi(\beta_z(x))\Omega$ and check that the map $X \rightarrow U_zUX_z^* \beta_z$ to an automorphism of $\pi(O_d)''$. For the converse we will use the hypothesis that $\psi$ is a $\lambda$-invariant factor state and $\beta\psi\lambda = \lambda\beta\psi$ to guarantee that $\psi\beta_z(X) = \frac{1}{m} \sum_{1 \leq k \leq n} \psi\lambda^k\beta_z(X) = \frac{1}{m} \sum_{1 \leq k \leq n} \psi\beta\lambda^k(X) \rightarrow \psi(X)$ as $n \rightarrow \infty$ for any $X \in \pi(O_d)''$, where we have used the same symbol $\beta_z$ for the extension. Hence $z \in H$.

In the following instead of working with $O_d$ we should be working with the inductive limit $C^*$ algebra and their inductive limit states. For simplicity of notation we still use $UHF_d, O_d$ for its inductive limit of $O_d \rightarrow^\lambda O_d$ and $UHF_d \rightarrow^\lambda UHF_d$ respectively and so for its inductive limit states.

Now we aim to prove (b). $H$ being a closed subgroup of $S^1$, it is either entire $S^1$ or a finite subgroup $\{\exp(2\pi ik) | k = 0, 1, ..., k-1\}$ where the integer $k \geq 1$. If $H = S^1$ we have nothing to prove for (b). When $H$ is a finite closed subgroup, we identify $[0, 1)$ with $S^1$ by the usual map and note that if $\beta$ is restricted to $t \in [0, \frac{1}{k})$, then by scaling we check that $\beta$ defines a representation of $S^1$ in automorphisms of $O^H_d$. Now we consider the direct integral representation $\pi'$ defined by

$$\pi' = \int_{[0, 1)} dt \pi_{\omega_t^H} \beta_t$$

of $O^H_d$ on $\mathcal{H}_{\omega_t^H} \otimes L^2([0, \frac{1}{k})$, where $\mathcal{H}_{\omega_t^H}$ is the cyclic space of $\pi(O^H_d)$ generated by $\Omega$. That it is indeed direct integral follows as states $\psi\beta_t$, and $\psi\beta_t$ are either same or orthogonal for a factor state $\psi$ (see the proof of Lemma 7.4 in [BJKW] for further details). Interesting point here to note that the new representation $\pi'$ is $(\beta_t)$ co-variant i.e. $\pi'\beta_t = \beta_t\pi'$, hence by simplicity of the $C^*$ algebra $O_d$ we conclude that $\pi'(UHF_d)'' = \pi'(O_d)''\beta_t$.

By exploring the hypothesis that $\psi$ is a factor state, we also have as in Lemma 6.11 in [BJKW] $I \otimes L^\infty([0, \frac{1}{k}) \subset \pi'(O_d)''$. Hence we also have

$$\pi'(O^H_d)'' = \pi(O^H_d)'' \otimes L^\infty([0, \frac{1}{k}))$$

Since $\beta_t$ is acting as translation on $I \otimes L^\infty([0, \frac{1}{k})$ which being an ergodic action, we have

$$\pi'(UHF_d)'' = \pi(UHF_d)'' \otimes 1$$

Since $\pi'(UHF_d)'' = \pi(UHF_d)'' \otimes 1$, we conclude that $\pi(UHF_d)'' = \pi(O^H_d)''$.

A proof for the statement (c) follows from Lemma 7.12 in [BJKW]. The original idea of the proof can be traced back to Arveson’s work on spectrum of an automorphism of a commutative compact group [Ar1].

Let $\omega'$ be an $\lambda$-invariant state on the UHF$_d$ sub-algebra of $O_d$. Following [BJKW, section 7], we consider the set

$$K_{\omega'} = \{\psi : \psi \text{ is a state on } O_d \text{ such that } \psi\lambda = \psi \text{ and } \psi_{UHF_d} = \omega'\}$$
By taking invariant mean on an extension of $\omega'$ to $\mathcal{O}_d$, we verify that $K_{\omega'}$ is nonempty and $K_{\omega'}$ is clearly convex and compact in the weak topology. In case $\omega'$ is an ergodic state (extremal state) $K_{\omega'}$ is a face in the $\Lambda$ invariant states. Before we proceed to the next section here we recall Lemma 7.4 of [BJKW] in the following proposition.

**Proposition 2.6.** Let $\omega'$ be ergodic. Then $\psi \in K_{\omega'}$ is an extremal point in $K_{\omega'}$ if and only if $\psi$ is a factor state and moreover any other extremal point in $K_{\omega'}$ has the form $\psi_{\beta_z}$ for some $z \in S^1$.

**Proof.** Though Proposition 7.4 in [BJKW] appeared in a different set up, the same proof goes through for the present case. We omit the details and refer to the original work for a proof.

3. **Dual Popescu system and pure translation invariant states:**

In this section we review the amalgamated Hilbert space developed in [BJKW] and prove a powerful criterion for a translation invariant factor state to be pure. Finally we will give proof of Theorem 1.1.

To that end let $M$ be a von-Neumann algebra acting on a Hilbert space $K$ and \( \{v_k, 1 \leq k \leq d\} \) be a family of bounded operators on $K$ so that $M = \{v_k, v_k^*; 1 \leq k \leq d\}$ and $\sum_k v_k v_k^* = 1$. Furthermore let $\Omega$ be a cyclic and separating vector for $M$ so that the normal state $\phi(x) = \langle \Omega, x\Omega \rangle > 0$ on $M$ is invariant for the Markov map $\tau$ on $M$ defined by $\tau(x) = \sum_k v_k x v_k^*$ for $x \in M$. Let $\omega$ be the translation invariant state on $UHF_d = \otimes_Z M_d$ defined by

$$\omega(e^{i \theta}_j(l) \otimes e^{i \theta}_j(l + 1) \otimes \ldots \otimes e^{i \theta}_j(l + n - 1)) = \phi(v_j^* v_j^*)$$

where $e_j^*(l)$ is the elementary matrix at lattice site $l \in \mathbb{Z}$.

We set $\tilde{v}_k = J\sigma^j_2(v_k^*)J \in M'$ (see [BJKW] for details) where $J$ and $\sigma = (\sigma_t, t \in \mathbb{R})$ are Tomita’s conjugation operator and modular automorphisms associated with $\phi$.

By KMS or modular relation [BR vol-1] we verify that

$$\sum_k \tilde{v}_k \tilde{v}_k^* = 1$$

and

$$\phi(v_j^* v_j^*) = \phi(\tilde{v}_j^* \tilde{v}_j^*)$$

where $\tilde{I} = (i_n, ..., i_2, i_1)$ if $I = (i_1, i_2, ..., i_n)$. Moreover $\tilde{v}_I^* \Omega = J\sigma^j_2(v_I)^*J\Omega = J\Delta^j_2 v_I \Omega = v_I^* \Omega$. We also set $M$ to be the von-Neumann algebra generated by $\{\tilde{v}_k: 1 \leq k \leq d\}$. Thus $M \subseteq M'$. A major problem that we will have to address is: when equality holds and its relation to Haag duality property (16).

Let $(H, P, S_k, 1 \leq k \leq d)$ and $(\tilde{H}, P, \tilde{S}_k, 1 \leq k \leq d)$ be the Popescu dilation described as in Theorem 2.1 associated with $(K, v_k, 1 \leq k \leq d)$ and $K, \tilde{v}_k, 1 \leq k \leq d$ respectively. Following [BJKW] we consider the amalgamated tensor product $H \otimes_K H$ of $H$ with $\tilde{H}$ over the joint subspace $K$. It is the completion of the quotient of the set

$$\mathcal{O} \otimes \mathcal{O} \subseteq \mathcal{O} \otimes \mathcal{O}.$$
where $\hat{I}, I$ both consist of all finite sequences with elements in $\{1, 2, \ldots, d\}$, by the equivalence relation defined by a semi-inner product defined on the set by requiring
\[
< I \otimes I \otimes f, \hat{I} \otimes I \otimes g > = < f, v_I v_J g >,
\]
\[
< I \otimes I \otimes f, \hat{I} \otimes I \otimes g > = < \hat{v}_I f, v_J g >
\]
and all inner product that are not of these form are zero. We also define two commuting representations $(\hat{S}_i)$ and $(\tilde{S}_i)$ of $O_d$ on $\mathcal{H} \otimes K \hat{\mathcal{H}}$ by the following prescription:
\[
\tilde{S}_i \lambda (J \otimes J \otimes f) = \lambda (J \otimes J \otimes f),
\]
\[
\tilde{S}_i \lambda (J \otimes J \otimes f) = \lambda (J \otimes J \otimes f),
\]
where $\lambda$ is the quotient map from the index set to the Hilbert space. Note that the commuting representations $S_i$ and $\tilde{S}_i$ are identified with the restriction of $S_i$ and $\tilde{S}_i$ defined here. Same is valid for $\tilde{S}_f$. The subspace $K$ is identified here with $\lambda (0 \otimes 0 \otimes K)$. Thus $K$ is a cyclic subspace for the representation
\[
\tilde{s}_j \otimes s_i \rightarrow \tilde{S}_j S_i
\]
of $\hat{O}_d \otimes O_d$ in the amalgamated Hilbert space. Let $P$ be the projection onto $K$. Then we have
\[
S_i^* P = P S_i^* P = v_i^*
\]
\[
\tilde{S}_i^* P = P \tilde{S}_i^* P = \tilde{v}_i^*
\]
for all $1 \leq i \leq d$.

We start with a simple proposition.

**Proposition 3.1.** The following holds:
(a) For any $1 \leq i, j \leq d$ and $|I|, |J| < \infty$ and $|\hat{I}|, |\hat{J}| < \infty$
\[
< \Omega, \tilde{S}_i \tilde{S}_j \hat{S}_i S_i S_j S_i^* S_j^* \Omega > = < \Omega, \tilde{S}_i \tilde{S}_j \hat{S}_i S_i S_j S_i^* S_j^* \Omega >;
\]
(b) The vector state $\psi_\Omega$ on
\[
UHF_d \otimes UHF_d = \mathcal{B}(\mathcal{H} \otimes K \mathcal{H})
\]
is equal to $\omega$;
(c) $\pi(O_d \otimes O_d)^n = \mathcal{B}(\mathcal{H} \otimes K \mathcal{H})$ if and only if $\{x \in \mathcal{B}(K) : \tau(x) = x, \tilde{\tau}(x) = x\} = \{zI : z \in \mathcal{C}\}$.

**Proof.** By our construction $\tilde{S}_i^* \Omega = \tilde{v}_i^* \Omega = v_i^* \Omega = S_i^* \Omega$. Now (a) and (b) follow by repeated application of $\tilde{S}_i^* \Omega = S_i^* \Omega$ and the commuting property of the two representation $\pi(O_d \otimes I)$ and $\pi(I \otimes O_d)$. The last statement (c) follows from a more general fact proved below that the commutant of $\pi(O_d \otimes O_d)^n$ is order isomorphic with the set $\{x \in \mathcal{B}(K) : \tau(x) = x, \tilde{\tau}(x) = x\} = \{zI : z \in \mathcal{C}\}$ via the map $X \rightarrow PXP$ where $X$ is the weak* limit of $\Lambda_m \Lambda_n(x)$ as $(m, n) \rightarrow (\infty, \infty)$. For details let $Y$ be the strong limit of increasing sequence of projections $(\Lambda \Lambda)^n (P)$ as $n \rightarrow \infty$. Then $Y$ commutes with $\tilde{S}_i \tilde{S}_j, S_i S_j$ for all $1 \leq i, j \leq d$. As $\Lambda(P) \supseteq P$, we also have $\Lambda(Y) \supseteq Y$. Hence $(1 - Y)S_i Y = 0$. As $Y$ commutes with $S_i \tilde{S}_j$ we get $(1 - Y) S_i \tilde{S}_j Y = 0$ i.e. $(1 - Y) \tilde{S}_j Y = 0$ for all $1 \leq j \leq d$. By symmetry of the argument we also get $(1 - Y) S_i Y = 0$ for all $1 \leq i \leq d$. Hence $Y$ commutes with $\pi(O_d)^n$ and by symmetry of the argument $Y$ commutes as well with $\pi(O_d)^n$. As $Yf = f$ for all $f \in K$ and $K$ is cyclic for the representation $\pi(O_d \otimes O_d)$ we conclude that $Y = I$ on $\mathcal{H} \otimes K \mathcal{H}$.

Let $x \in \mathcal{B}(K)$ so that $\tau(x) = x$ and $\tilde{\tau}(x) = x$ then as in the proof of Theorem 2.1 we also check that $(\Lambda \Lambda)^k (P) \Lambda_m \Lambda_n(x) (\Lambda \Lambda)^k (P)$ is independent of $m, n$ as long as
\(m, n \geq k\). Hence the weak* limit \(\Lambda^n \Lambda^m(x) \to X\) exists as \(m, n \to \infty\). Furthermore the limiting element \(X \in \pi(O_d \otimes \tilde{O}_d)^\prime\) and \(PXP = x\). That the map \(X \to PXP\) is an order-isomorphic on the set of selfadjoint elements follows as in Theorem 2.1. This completes the proof.

In short Proposition 3.1 also says that \((\tilde{H} \otimes_K \mathcal{H}, S_i S_j, 1 \leq i, j \leq d, P)\) is the Popescu dilation associated with Popescu elements \((K, v_i \tilde{v}_j, 1 \leq i, j \leq d)\). Now we will be more specific in our starting Popescu’s elements in order to explore the representation \(\pi\) of \(\tilde{O}_d \otimes O_d\) in the amalgamated Hilbert space \(\tilde{H} \otimes_K \mathcal{H}\).

Let \(\omega\) be a translation invariant factor state on \(\mathcal{B}\) and \(\omega'\) be its restriction to \(\mathcal{B}_R\) which we identified with \(\text{UHF}_d\) with respect to an orthonormal basis \((e_i)_j\) of \(\mathbb{C}^d\) (see statement before equation (4)). Let \(\psi\) be an extremal point in \(K_{\omega'}\). We consider the Popescu’s elements \((K, M, v_k, 1 \leq k \leq d, \Omega)\) described as in Proposition 2.4 associated with support projection of the state \(\psi\) in \(\pi_\psi(O_d)^\prime\) and also consider associated dual Popescu’s elements \((K, M, \tilde{v}_k, 1 \leq k \leq d)\) where \(M\) is the von-Neumann algebra generated by \(\{\tilde{v}_k : 1 \leq k \leq d\}\). Thus in general \(M \subseteq M'\) and an interesting question: when do we have \(M' = M\)? Going back to our starting example of unique KMS state for the automorphisms \(\beta_t(s_i) = t s_i, t \in S^1\), we check that \(v_t^* = S_k^\perp, J v_t^* J = \frac{1}{2} S_k\) and thus equality holds i.e. \(M = M'\). But the corner vector space \(M_c = P \pi(\tilde{O}_d)^\prime P\) generated by the elements \(\{\tilde{v}_t \tilde{v}_t^* : |I|, |J| < \infty\}\) fails to be an algebra. Thus two questions sounds reasonable here.

(a) Does the equality \(M' = M\) hold in general for an extremal element \(\psi \in K_{\omega'}\) and a factor state \(\omega'\)?

(b) When can we expect \(M_c\) to be a \(*\)-algebra and so equal to \(M\)?

The dual condition on support projection and equality \(M = M'\) are rather deep and will lead us to a far reaching consequence on the state \(\omega\). In the paper [BJKW] these two conditions are implicitly assumed to give a criterion for a translation invariant factor state to be pure. Apart from this refined interest, we will address the converse problem that turns out to be crucial for our main results. In the following we prove a crucial step towards that goal fixing the basic structure which will be repeatedly used in the computation using Cuntz relations.

**Proposition 3.2.** Let \(\omega\) be a translation invariant factor state on \(\mathcal{B}\) and \(\psi\) be an extremal point in \(K_{\omega'}\). We consider the amalgamated representation \(\pi\) of \(\tilde{O}_d \otimes O_d\) in \(\tilde{H} \otimes_K \mathcal{H}\) where the Popescu’s elements \((K, M, v_k, 1 \leq k \leq d)\) are taken as in Proposition 2.4. Then the following statements hold:

(a) \(\pi(\tilde{O}_d \otimes O_d)^\prime = \mathcal{B}(\tilde{H} \otimes_K \mathcal{H})\). Furthermore \(\pi(O_d)^\prime\) and \(\pi(\tilde{O}_d)^\prime\) are factors and the following sets are equal:

(i) \(H = \{z \in S^1 : \psi \beta_z = \psi\}\);

(ii) \(H_\pi = \{z : \beta_z\) extends to an automorphisms of \(\pi(O_d)^\prime\}\);

(iii) \(H_\pi = \{z : \beta_z\) extends to an automorphisms of \(\pi(\tilde{O}_d)^\prime\}\). Moreover \(\pi(\text{UHF}_d \otimes I)^\prime\) and \(\pi(I \otimes \text{UHF}_d)^\prime\) are factors.

(b) \(z \to U_z\) is the unitary representation of \(H\) in the Hilbert space \(\tilde{H} \otimes_K \mathcal{H}\) defined by \(U_z(\pi(s_j \otimes s_i)\Omega = \pi(z s_j \otimes s_i)\Omega\).
(c) The commutant of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ is invariant by the canonical endomorphisms $\Lambda(X) = \sum_i S_i X S_i^*$ and $\tilde{\Lambda}(X) = \sum_i \tilde{S}_i X \tilde{S}_i^*$. Same is true for each $i$ that the surjective map $X \to S_i^* X S_i$ keeps the commutant of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ invariant. Same holds for the map $X \to \tilde{S}_i^* X \tilde{S}_i$.

(d) The centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ is invariant by the canonical endomorphisms $\Lambda(X) = \sum_i S_i X S_i^*$ and $\tilde{\Lambda}(X) = \sum_i \tilde{S}_i X \tilde{S}_i^*$. Moreover for each $i$, the surjective map $X \to S_i^* X S_i$ keeps the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ invariant. Same holds for the map $X \to S_i^* X S_i$.

**Proof.** $P$ being the support projection by Proposition 2.4 we have \( \{ x \in \mathcal{B}(\mathcal{K}) : \sum_k v_k x v_k^* = x \} = \mathcal{M}' \). That \((\mathcal{M}', \tilde{\tau}, \tilde{\phi})\) is ergodic follows from a general result [Mo1] (see also [BJKW] for a different proof) as \((\mathcal{M}, \tau, \phi)\) is ergodic for a factor state $\psi$ being an extremal element in $\mathcal{K}_{\tilde{\tau}}$ (Proposition 2.6). Hence \( \{ x \in \mathcal{B}(\mathcal{K}) : \tilde{\tau}(x) = \tilde{\tau}(x) = x \} = \mathbb{C} \). Hence by Proposition 3.1, we conclude that $\pi(\mathcal{O}_d \otimes \mathcal{O}_d)' = \mathcal{B}(\mathcal{H} \otimes \mathcal{K} \mathcal{H})$. That both $\pi(\mathcal{O}_d)''$ and $\pi(\tilde{\mathcal{O}}_d)''$ are factors follows trivially as $\pi(\tilde{\mathcal{O}}_d \otimes \tilde{\mathcal{O}}_d)'' = \mathcal{B}(\tilde{\mathcal{H}} \otimes \mathcal{K} \mathcal{H})$ and $\pi(\mathcal{O}_d)'' \subseteq \pi(\tilde{\mathcal{O}}_d)'$.

By the discussion above we first recall that $\Omega$ is a cyclic vector for the representation of $\pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)$. Let $G = \{ z = (z_1, z_2) \in S^1 \times S^1 : \beta_z \text{ extends to an automorphism on } \pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)'' \}$ be the closed subgroup where

\[ \beta_{(z_1, z_2)}(\tilde{s}_j \otimes s_i) = z_1 \tilde{s}_j \otimes z_2 s_i. \]

By repeated application of the fact that $\pi(O_d)''$ commutes with $\pi(\tilde{O}_d)''$ and $\tilde{S}_i^* \Omega = \tilde{S}_i^* \Omega$ as in Proposition 3.1 (a) we verify that $\psi_\beta(z, z) = \psi$ on $\mathcal{O}_d \otimes \mathcal{O}_d$ if $z \in H$. For $z \in H$ we set unitary operator $U_z \pi(x \otimes y) \Omega = \pi(\beta_z(x) \otimes \beta_z(y)) \Omega$ for all $x \in \mathcal{O}_d$ and $y \in \mathcal{O}_d$. Thus we have $U_z \pi(s_i) U_z^* = z \pi(s_i)$ and also $U_z \pi(\tilde{s}_i) U_z^* = z \tilde{s}_i$. By taking its restriction to $\pi(\mathcal{O}_d)''$ and $\pi(\tilde{\mathcal{O}}_d)''$ respectively we check that $H \subseteq \tilde{H}_\pi$ and $H \subseteq H_\pi$.

For the converse let $z \in H_\pi$ and we use the same symbol $\beta_z$ for the extension to an automorphism of $\pi(O_d)''$. By taking the inverse map we check easily that $\tilde{z} \in H_\pi$ and in fact $H_\pi$ is a subgroup of $S^1$. Since $\lambda$ commutes with $\beta_z$ on $\mathcal{O}_d$, the canonical endomorphism $\Lambda$ defined by $\Lambda(X) = \sum_k \tilde{S}_k X \tilde{S}_k^*$ also commutes with the extension of $\beta_z$ on $\pi(O_d)''$. Note that the map $\pi(x)|_{\mathcal{H}} \rightarrow \beta_z(x)|_{\mathcal{H}}$ for $x \in \mathcal{O}_d$ is a well defined $*$-homomorphism. Since same is true for $\tilde{z}$ and $\beta_{\tilde{z}} \beta_z = I$, the map is an isomorphism. Hence $\beta_z$ extends uniquely to an automorphism of $\pi(O_d)|_{\mathcal{H}}''$ commuting with the restriction of the canonical endomorphism on $\pi(O_d)|_{\mathcal{H}}''$. Since $\pi(O_d)|_{\mathcal{H}}''$ is a factor, we conclude as in Proposition 2.5 (a) that $z \in H$. Thus $H_\pi \subseteq H$. As $\pi(\tilde{\mathcal{O}}_d)''$ is also a factor, we also have $\tilde{H}_\pi \subseteq H$. Hence we have $H = H_\pi = \tilde{H}_\pi$ and $\{(z, z) : z \in H\} \subseteq G \subseteq H \times H$.

For the second part of (a) we will adopt the argument used for Proposition 2.5. To that end we first note that $\Omega$ being a cyclic vector for the representation $\mathcal{O}_d \otimes \mathcal{O}_d$ in the Hilbert space $\tilde{\mathcal{H}} \otimes \mathcal{K} \mathcal{H}$, by Lemma 7.11 in [BJKW] (note that the proof only needs the cyclic property) the representation of UHF$_d$ on $\tilde{\mathcal{H}} \otimes \mathcal{K} \mathcal{H}$ is quasi-equivalent to its sub-representation on the cyclic space generated by $\Omega$. On the other hand, by our hypothesis that $\omega$ is a factor state, Power’s theorem [Pow] ensures that the state $\omega'$ (i.e. the restriction of $\omega$ to $\mathcal{B}_R$ which is identified here with UHF$_d$) is also a factor state on UHF$_d$. Hence quasi-equivalence ensures that $\pi(I \otimes \text{UHF}_d)''$ is a factor. We also note that the argument used in Lemma 7.11 in [BJKW] is symmetric i.e. same argument is also valid for UHF$_d$. Thus $\pi(\text{UHF}_d \otimes I)''$ is also
a factor. This completes the proof of (a). We have proved (b) while giving proof of (a).

For $X \in \mathcal{B}(\tilde{\mathcal{H}} \otimes \mathcal{K})$, as $\Lambda(X)$ commutes with $\pi(\Lambda(\text{UHF}_d \otimes \text{UHF}_d))''$ and $\{S_iS_i^* : 1 \leq i, j \leq d\}$ we verify by Cuntz’s relation that $\Lambda(X)$ is also an element in the commutant of $\pi(\Lambda(\text{UHF}_d \otimes \text{UHF}_d))''$ once $X$ is so. It is also obvious that $\Lambda(X)$ is an element in $\pi(\Lambda(\text{UHF}_d \otimes \text{UHF}_d))''$ if $X$ is so. Thus $\Lambda(X)$ is an element in the commutant/centre of $\pi(\Lambda(\text{UHF}_d \otimes \text{UHF}_d))''$ once $X$ is so. For the last statement consider the map $X \to S_i^*XS_i$ on $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ which is clearly onto by Cuntz relation (3). Hence we need to show that $S_i^*XS_i$ is an element in the commutant whenever $X$ is so. To that end note that $S_i^*XS_iS_i^*YS_i = S_i^*XS_iXYS_i = S_i^*YS_iS_i^*XS_i$ since $X$ commutes with $S_iS_i^*$. Thus onto property of the map ensures that $S_i^*XS_i$ is an element in the commutant/centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ once $X$ is so. This completes the proof of (c) and (d). 

One interesting problem here how to describe the von-Neumann algebra $\mathcal{I}$ which consists of invariant elements of the gauge action $\{\beta_z : z \in H\}$ in $\mathcal{B}(\tilde{\mathcal{H}} \otimes \mathcal{K})$. A general result due to E. Stormer [So] says that the algebra of invariant elements is a von-Neumann algebra of type-I with centre completely atomic. Here the situation is much simple because we know explicitly that $\mathcal{I} = \{U_z : z \in H\}'$ and we write spectral decomposition as

$$U_z = \sum_{k \in \hat{H}} z^k F_k$$

for $z \in H$, $\hat{H}$ is the dual group of $H$, either $\hat{H} = \{z : z^n = 1\}$ or $\mathbb{Z}$. Thus the centre of $\mathcal{I}$ is equal to $\{F_k : k \in \hat{H}\}$.

As a first step we describe the center $Z$ of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ by exploring Cuntz relation, that it is also non-atomic even for a factor state $\omega$. In fact we will show that the centre $Z$ is a sub-algebra of the centre of $\mathcal{I}$. In the following proposition we give an explicit description.

**Proposition 3.3.** Let $\omega, \psi$ be as in Proposition 3.2 with Popescu system $(\mathcal{K}, \mathcal{M}, v_0, \Omega)$ be taken as in Proposition 2.4 i.e. on support projection. Then the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ is completely atomic and the element $E_0 = [\pi(\text{UHF}_d \otimes \text{UHF}_d)'' \vee \pi(\text{UHF}_d \otimes \text{UHF}_d)'' \Omega]$ is a minimal projection in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ and the centre is invariant for both $\Lambda$ and $\tilde{\Lambda}$. Furthermore the following holds:

(a) The centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ has the following two disjoint possibilities:

(i) There exists a positive integer $m \geq 1$ such that the centre is generated by the family of minimal orthogonal projections $\{\Lambda_k(E_0) : 0 \leq k \leq m - 1\}$ where $m \geq 1$ is the least positive integer so that $\Lambda^m(E_0) = E_0$. In such a case $\{z : z^m = 1\} \subseteq H$;

(ii) The family of minimal nonzero orthogonal projections $\{E_k : k \in \mathbb{Z}\}$ where $E_k = \Lambda^k(E_0)$ for $k \geq 0$ and $E_k = S_i^*E_0S_i$ for $k < 0$ where $|I| = -k$ and independent of multi-index $I$ generates the centre and $H = S^I$;

(b) $\Lambda(E) = \tilde{\Lambda}(E)$ for any $E$ in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$

(c) If $\Lambda(E_0) = E_0$ then $E_0 = 1$. 

PROOF. Let $E' \in \pi(\text{UHF}_d \otimes \text{UHF}_d)'$ be the projection onto the subspace generated by the vectors $\{S_1S_2^*, \ldots, S_{d^n}, \Omega, \ |I| = |J|, |J'| = |J'| < \infty \}$ and $\pi_\Omega$ be the restriction of the representation $\pi$ of $\text{UHF}_d \otimes \text{UHF}_d$ to the cyclic subspace $H_\Omega$ generated by $\Omega$. Identifying $B$ with $\text{UHF}_d \otimes \text{UHF}_d$ we check that $\pi_\omega$ is unitary equivalent to $\pi_\Omega$. Thus $\pi_\Omega$ is a factor representation.

For any projection $E$ in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$, via the unitary equivalence, we note that $EE' = E'E$ is an element in the centre of $\pi_\Omega(\text{UHF}_d \otimes \text{UHF}_d)''$. $\omega$ being a factor state we conclude that $EE'$ is a scalar multiple of $E'$ and so we have

$$EE' = \omega(E)E'$$

Thus we also have $YE'E = \omega(E)YE'$ for all $Y \in \pi(\text{UHF}_d \otimes \text{UHF}_d)'$ and so

$$EE_0 = \omega(E)E_0$$

Since $EE'$ is a projection and $E' \neq 0$, we have $\omega(E) = \omega(E)^2$. Thus $\omega(E) = 1$ or $0$. So, for such an element $E$, the following is true:

(i) If $E \leq E_0$ then either $E = 0$ or $E = E_0$ i.e. $E_0$ is a minimal projection in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$.
(ii) $\omega(E) = 1$ if and only if $E \geq E_0$
(iii) $\omega(E) = 0$ if and only if $EE_0 = 0$.

As $\Lambda(E_0)$ is a projection in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ by our last proposition i.e. Proposition 3.2 (c), we have either $\omega(\Lambda(E_0)) = 1$ or $0$. Since $\Lambda(E_0) \neq 0$ by the injective property of the endomorphism, we have either $\Lambda(E_0) \geq E_0$ or $\Lambda(E_0)E_0 = 0$. In case $\Lambda(E_0) \geq E_0$ we have $S_i^*E_0S_i \leq S_i^*\Lambda(E_0)S_i = E_0$ for all $1 \leq i \leq d$. So, if $S_i^*E_0S_i$ is a non-zero projection in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ (Proposition 3.2 (c) ), by (i) we have $E_0 = \Lambda(E_0)$. Thus we have either $\Lambda(E_0) = E_0$ or $\Lambda(E_0)E_0 = 0$.

If $\Lambda(E_0)E_0 = 0$, we have $\Lambda(E_0) \leq I - E_0$ and by Cuntz’s relation we check that $E_0 \leq I - S_i^*E_0S_i$ and $S_i^*E_0S_i \leq I - S_i^*E_0S_i$ for all $1 \leq i,j \leq d$. So we also have $E_0S_i^*S_j^*E_0S_jS_iE_0 \leq E_0 - E_0S_i^*E_0S_jE_0 = E_0$. Thus we have either $E_0S_i^*S_j^*E_0S_jS_iE_0 = 0$ or $E_0S_i^*S_j^*E_0S_jS_iE_0 = E_0$ as $S_i^*S_j^*E_0S_jS_i$ is an element in the centre by Proposition 3.2 (c). So either we have $\Lambda^2(E_0)E_0 = 0$ or $\Lambda^2(E_0) \leq E_0$. $\Lambda$ being an injective map we either have $\Lambda^2(E_0)E_0 = 0$ or $\Lambda^2(E_0) = E_0$.

More generally we check that if $\Lambda(E_0)E_0 = 0$, $\Lambda^2(E_0)E_0 = 0, \ldots, \Lambda^{k+1}(E_0)E_0 = 0$ for some $k \geq 1$ then either $\Lambda^{k+1}(E_0)E_0 = 0$ or $\Lambda^{k+1}(E_0) = E_0$. To verify that first we check that in such a case $E_0 \leq I - S_i^*E_0S_i$ for all $|I| = n$ and then following the same steps as before to check that $S_i^*S_j^*E_0S_jS_i \leq I - S_i^*E_0S_i$ for all $i$. Thus we have $E_0S_i^*S_j^*E_0S_jS_iE_0 \leq E_0$ and arguing as before we complete the proof of the claim that either $\Lambda^{k+1}(E_0)E_0 = 0$ or $\Lambda^{k+1}(E_0) = E_0$.

We summarize now by saying that $E_0, \Lambda(E_0), \ldots, \Lambda^{m-1}(E_0)$ are mutually orthogonal projections with $m \geq 1$ possibly be infinite, if not then $\Lambda^m(E_0) = E_0$.

Let $\pi_k, k \geq 0$ be the representation $\pi$ of $\text{UHF}_d \otimes \text{UHF}_d$ restricted to the subspace $\Lambda^k(E_0)$. The representation $\pi_0$ of $\text{UHF}_d \otimes \text{UHF}_d$ is isomorphic to the representation $\pi$ of $\text{UHF}_d \otimes \text{UHF}_d$ restricted to $E'$ and thus quasi-equivalent. For a general discussion on quasi-equivalence we refer to section 2.4.4 in [BR vol-1]. Since $\omega$ is a factor state, $\pi_0$ is a factor representation. We claim now that each $\pi_k$ is a
factor representation. We fix any $k \geq 1$ and let $X$ be an element in the centre of $\pi_k(\text{UHF}_d \otimes \text{UHF}_d)$. Then for any $|I| = k$, $S_I^* E_k S_I = E_0$ and so $S_I^* X S_I$ is an element in the centre of $\pi_0(\text{UHF}_d \otimes \text{UHF}_d)$ by Proposition 3.2 (d). Further $S_I^* X S_I = S_J^* X S_I S_J^* S_I = S_J^* X S_J$ for all $|J| = |I| = k$. $\pi_0$ being a factor representation, we have $S_I^* X S_I = cE_0$ for some scalar $c$ independent of the multi-index we choose $|I| = k$. Hence $c\Lambda_k(E_0) = \sum_{|J|=k} S_J^* X S_J = \sum_{|J|=k} S_J S_J^* X = X$ as $X$ is an element in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)$. Thus for each $k \geq 1$, $\pi_k$ is a factor representation as $\pi_0$ is so.

We also note that $\Lambda(E_0)\Lambda(E_0) \neq 0$. Otherwise we have $<S\Omega, \tilde{S}\Omega> = 0$ for all $i, j$ and so $<\Omega, \tilde{S}_i^* \tilde{S}_j> = 0$ for all $i, j$ as $\pi(\tilde{O}_d)^\prime$ commutes with $\pi(\tilde{O}_d)^\prime$. However $\tilde{S}_i^* \tilde{S}_j^* = \tilde{S}_i^* \tilde{S}_j$ and $\sum_i \tilde{S}_i \tilde{S}_i^* = 1$ which leads a contradiction. Hence $\Lambda(E_0)\Lambda(E_0) \neq 0$.

As $\pi$ restricted to $\Lambda(E_0)$ is a factor state and both $\Lambda(E_0)$ and $\tilde{\Lambda}(E_0)$ are elements in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime$, by Proposition 3.2 (d), we conclude that $\Lambda(E_0) = \tilde{\Lambda}(E_0)$. Using the commuting property of the endomorphisms $\Lambda$ and $\tilde{\Lambda}$, we verify by a simple induction method that $\Lambda^k(E_0) = \tilde{\Lambda}^k(E_0)$ for all $k \geq 1$. Thus the sequence of orthogonal projections $E_0, \tilde{\Lambda}(E_0), ...$, are also periodic with same period or aperiodic according as the sequence of orthogonal projections $E_0, \Lambda(E_0), ...$ is.

If $\Lambda^m(E_0) = E_0$ for some $m \geq 1$ then we check that $\sum_{0 \leq k \leq m-1} \Lambda^k(E_0)$ is a $\Lambda$ and as well $\tilde{\Lambda}$-invariant projection and thus equal to 1 by the cyclic property of $\Omega$ for $\pi(\tilde{O}_d \otimes \tilde{O}_d)^\prime$. In such a case we set $V_z = \sum_{0 \leq k \leq m-1} z^k E_k$ for $z \in S^1$ for which $z^m = 1$ and check that $\Lambda(V_z) = \sum_{0 \leq k \leq m-1} z^k \Lambda(E_k) = \sum_{0 \leq k \leq m-1} z^k E_{k+1} = z V_z$ where $E_m = E_0$ and so by the Cuntz relations we have $V_z^* S_i V_z = z S_i$ for all $1 \leq i \leq d$. Following the same steps we also have $\tilde{\Lambda}(V_z) = z V_z$ and so $V_z^* \tilde{S}_i V_z = z \tilde{S}_i$ for $1 \leq i \leq d$.

Now we consider the case where $E_0, \Lambda(E_0), ... \Lambda^k(E_0), ...$ is a sequence of aperiodic orthogonal projections. We extend the family of projections $\{E_k : k \in \mathbb{Z}\}$ to all integers by

$E_k = \Lambda^k(E_0)$ for all $k \geq 1$

and

$E_k = S_I^* E_0 S_I$ for all $k \leq 1$, where $|I| = -k$

We claim that the definition of $\{E_k : k \leq -1\}$ does depends only on length of the multi-index $I$ that we choose. We may choose any other $J$ so that $|J| = |I|$ and check the following identity:

$S_I^* E_0 S_J = S_I^* E_0 S_I S_J^* S_J = S_I^* S_I S_J^* E_0 S_J = S_I^* E_0 S_J$

where $E_0$, being an element in the centre of $\pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime$, commutes with $S_I S_J^*$ as $|I| = |J|$. Further $\Lambda^k(E_0) = \tilde{\Lambda}^k(E_0)$ ensures that $S_I S_J^*$ commutes with $E_0$ for all $|I| = |J| = k$ and $k \geq 1$. Hence we also have

$E_{-k} = S_I^* E_0 S_I S_J^* S_J = S_I^* E_0 S_J$

for all $|J| = |I| = k$ and $k \geq 1$. Now we claim that

$\Lambda(E_k) = \tilde{\Lambda}(E_k) = E_{k+1}$

for all $k \in \mathbb{Z}$. For $k \geq 0$ we have nothing to prove. For $k \leq -1$ we check that the following steps

$\Lambda(S_I^* E_0 S_J) = \sum_j S_j S_I^* S_J^* E_0 S_I S_J S_j^*$
Thus by Proposition 3.2 (c) we get that \( \tilde{\Lambda}(E_k) = E_{k+1} \) may follow the same steps as \( E_k = S_j^* X S_j \) for \( |I| = -k \) and \( k \leq -1 \).

We also claim that \( \{ E_k : k \in \mathbb{Z} \} \) is an orthogonal family of non-zero projections. To that end we choose any two elements say \( E_k, E_m \), \( k \neq m \) and use endomorphism \( \Lambda \) for \( n \) large enough so that both \( n + k \geq 0, n + m \geq 0 \) to conclude that \( \Lambda^n(E_k E_m) = E_{k+n} E_{k+m} = 0 \) as \( k + n \neq k + m \). \( \Lambda \) being an injective map we get the required orthogonal property. Thus \( \sum_{k \in \mathbb{Z}} E_k \) being an invariant projection for both \( \Lambda \) and \( \tilde{\Lambda} \) we get by cyclic property of \( \Omega \) that \( \sum_{k \in \mathbb{Z}} E_k = I \). Let \( \pi_k \), \( k \leq -1 \) be the representation \( \pi \) of \( UHF_d \otimes UHF_d \) restricted to the subspace \( E_k \). Going along the same line as above, we verify that for each \( k \leq -1 \), \( \pi_k \) is a factor representation of \( UHF_d \otimes UHF_d \). We also set \( V_z = \sum_{-\infty < k < \infty} \tilde{z}^k E_k \) for all \( z \in S^1 \) and check that \( \Lambda(V_z) = \lambda V_z \) and also \( \tilde{\Lambda}(V_z) = \tilde{z} V_z \). Hence \( S^1 = H \) as \( H \) is a closed subset of \( S^1 \). This completes the proof of (a). Proof of (b) and (c) are now simple consequence of the proof of (a).

It is clear that \( I \) contains \( I_0 := \text{def} \pi(UHF_d \otimes UHF_d)'' \), \( \{ U_z : z \in H \}'' \). By the last proposition the centre of \( I \), which is equal to \( \{ U_z : z \in H \}'' \), contains the centre of \( \pi(UHF_d \otimes UHF_d)'' \) and thus by taking the commutant we also have \( I \subseteq \pi(UHF_d \otimes UHF_d)'' \otimes \pi(UHF_d \otimes UHF_d)'' \). In the last proposition we have described explicitly the factor decomposition of the representation \( \pi(UHF_d \otimes UHF_d)'' \). One central issue is when such an factor decomposition is also an extremal decomposition. A clear answer at this stage seems to be somewhat hard. However the following proposition makes an attempt for our purpose. To that end we set few more notations and elementary properties.

For each \( k \) \in \( H \), let \( \pi_k' \) be the representation \( \pi \) of \( UHF_d \otimes UHF_d \) restricted to \( F_k \). We claim that each \( \pi_k' \) is pure if \( \pi_0' \) is pure. Fix any \( k \) \in \( H \) and let \( X \) be an element in the commutant of \( \pi_k'(UHF_d \otimes UHF_d)'' \), then \( S_j^* S_k S_l = S_j^* F_k S_l = F_k \) as \( S_j^* S_k \) commutes with \( F_0 \) for \( |I| = |J| \) and further for any \( |I| = |J| \), \( S_j^* X S_l S_j S_j^* X S_j S_l S_j X S_l = X S_j S_j^* X S_j S_l = X S_j S_j^* X S_l \) as \( X \) commutes with \( S_j S_j^* \) with \( |I| = |J| \). Thus by Proposition 3.2 (c) \( S_j^* X S_l \) is an element in commutant of \( \pi_0(UHF_d \otimes UHF_d)'' \) for any \( |I| = k \) and thus \( S_j^* X S_l = c F_0 \) for some scalar \( c \) independent of \( |I| = k \) as \( \pi_k' \) is pure. We use the commuting property of \( X \) with \( \pi(UHF_d \otimes UHF_d)'' \) to conclude that \( X = c \Lambda^k(E_0) \) for some scalar \( c \). If \( k \leq -1 \) we employ the same method but with the endomorphism \( \Lambda^{-k} \) so that \( \Lambda^{-k}(X) \) is an element in the commutant of \( \pi_0(UHF_d \otimes UHF_d)'' \). Thus \( \sum_{|I| = -k} S_j^* X S_l = c F_0 \) and by the injective property of the endomorphism we get \( X \) is a scalar. Thus we conclude that each \( \pi_k' \) is pure if \( \pi_0' \) is pure.

Next we claim that for each fixed \( k \) \in \( H_0 \), the representation \( \pi_k \) of \( UHF_d \otimes UHF_d \) defined in the proof of Proposition 3.3 is quasi-equivalent to representation \( \pi_k' \) ( here we recall \( H_0 \subseteq H \) as \( H_0 \subseteq H \) ). That \( \pi_k' \) is quasi-equivalent to \( \pi_0' \) follows as they are isomorphic by construction. In the proof of Proposition 3.3 we defined representation \( \pi_k \) of \( UHF_d \otimes UHF_d)'' \) for all \( k \) \in \( H_0 \) associated with minimal projections \( \{ E_k : k \in H_0 \} \). More generally for any \( k \) \in \( H \), we denote \( \pi_k \) for the restriction of \( \pi \) to the minimal central projections \( E_k \) on the subspace span
by \( \{\pi(\tilde{UHF}_d \otimes UHF_d) f : f \in \mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}}, \ F_k f = f \} \). So each \( E_k \) is a minimal central element containing \( F_k \). However two such elements i.e. \( E_k \) and \( E_j \) are either equal or mutually orthogonal being minimal. Thus \( \{ E_k : k \in H \} = \{ E_k : k \in \tilde{H}_0 \} \) and quasi-equivalence follows as \( \pi_k \) is isomorphic with \( \pi_k^e \) for all \( k \in \tilde{H}_0 \).

We now set
\[
F'_0 = \{ \pi(\tilde{UHF}_d \otimes UHF_d)'' \Omega \}
\]

For a vector \( f \) if \( F'_0 f = f \) then \( U_z f = f \) for all \( z \in H \) and thus \( F'_0 \leq F_0 \). We prove in following text that equality holds if \( \omega \) is pure.

First we consider the case when \( H = \{ z : z^n = 1 \} \). Projections \( \Lambda(F'_0) \) and \( \tilde{\Lambda}(F'_0) \) are elements in \( \pi(\tilde{UHF}_d \otimes UHF_d)'' \) by Proposition 3.3. The representation \( \pi(\tilde{UHF}_d \otimes UHF_d)'' \) restricted to both the projections \( \Lambda(F'_0), \tilde{\Lambda}(F'_0) \) are pure as well.

A pull back by the map \( X \rightarrow S^*_k X S_i \) with any \( 1 \leq i \leq d \) will do the job for the projection \( \Lambda(F'_0) \). Thus \( \Lambda(F'_0) \Lambda(F'_0) \Lambda(F'_0) = c \Lambda(F'_0) \) for some scalar. By pulling back with the action \( X \rightarrow S^*_k X S_i \) we get \( F'_0 | S^*_k F'_0 S_i F'_0 = c F'_0 \) and so
\[
c = \sum_k < \Omega, S^*_k F'_0 S_i S^*_k \Omega >
\]
as \( S^*_k \Omega = S^*_k \Omega \) and further \( F'_0 \) commutes with \( \pi(\tilde{UHF}_d) \) and thus
\[
c = \sum_k < \Omega, S^*_k S^*_k \Omega > = 1
\]

This shows that \( \tilde{\Lambda}(F'_0) \geq \Lambda(F'_0) \). Interchanging the role of \( \Lambda \) and \( \tilde{\Lambda} \) we conclude that \( \Lambda(F'_0) = \tilde{\Lambda}(F'_0) \). Now it essentially follows along the same line \( \Lambda^k(F'_0) = \tilde{\Lambda}^k(F'_0) \) for all \( k \geq 1 \). By Proposition 2.5 we also note that
\[
\Lambda^n(F'_0) = F'_0 = \tilde{\Lambda}^n(F_0)
\]
as \( H = \{ z : z^n = 1 \} \). Thus \( F' = \sum_{0 \leq k \leq n-1} \Lambda(F'_0) \) is a \( \Lambda \) and as well \( \tilde{\Lambda} \) invariant projection. Since \( F' \Omega = \Omega \) we conclude by the cyclic property of \( \Omega \) for \( \pi(\tilde{O}_d \otimes \tilde{O}_d)'' \) that \( F' = 1 \). Since \( \Lambda^k(F'_0) \leq F_k \) and \( \sum_k F_k = 1 \), we conclude that \( \Lambda^k(F'_0) = F_k \).

In such a case we may check that
\[
F_k = \{ \pi(\tilde{UHF}_d \otimes UHF_d)'' S^*_i \Omega : |I| = n - k \}
\]
for \( 1 \leq k \leq n - 1 \).

Similarly in case \( H = S^1 \) and \( \omega \) is pure we also have \( F_0 = F'_0 \) and for \( k \geq 1 \)
\[
F_k = \{ \pi(\tilde{UHF}_d \otimes UHF_d)'' S^*_i \Omega : |I| = k \}
\]
\[
F_{-k} = \{ \pi(\tilde{UHF}_d \otimes UHF_d)'' S^*_i \Omega : |I| = k \}
\]
Thus we have got an explicit description of the complete atomic centre of \( \mathcal{I} \) when \( \omega \) is a pure state.

**Proposition 3.4.** Let \( \omega, \psi \) and Popescu system \((\mathcal{K}, \mathcal{M}, v_\psi, \Omega)\) be as in Proposition 3.3. Then
(a) \( \{ \beta_{z_k} : z \in H \} \) invariant elements in \( \pi(\tilde{UHF}_d \otimes \mathcal{O}_d)'' \) (as well as in \( \pi(\tilde{O}_d \otimes \tilde{UHF}_d)'' \) are equal to \( \pi(\tilde{UHF}_d \otimes \tilde{UHF}_d)'' \).
(b) \( \mathcal{I} = \mathcal{I}_0 \) if and only if \( \omega \) is pure.

Further the following statements are equivalent:
(c) \( \mathcal{I} = \pi(\tilde{UHF}_d \otimes UHF_d)'' ; \)
(d) \( \pi(\tilde{UHF}_d \otimes \mathcal{O}_d)'' = \mathcal{B}(\tilde{\mathcal{H}} \otimes_{\mathcal{K}} \mathcal{H}) ; \)
(e) \( \pi(\tilde{O}_d \otimes UHF_d)'' = \mathcal{B}(\tilde{\mathcal{H}} \otimes_{\mathcal{K}} \mathcal{H}) ; \)
In such a case (i.e. if any of (c),(d) and (e) is true) the following statements are also true:

(f) \( \pi(\tilde{UHF}_d \otimes UHF_d)'' \) is a type-I von-Neumann algebra with centre equal to \( \{U_z : z \in H\}'' \) where \( U_z \) is defined in Proposition 3.2.

(g) \( \omega \) is a pure state on \( B \).

Conversely if \( \omega \) is a pure state then \( \pi(\tilde{UHF}_d \otimes UHF_d)'' \) is a type-I von-Neumann algebra with centre equal to \( \{U_z : z \in H_0\}'' \) where \( H_0 \) is a subgroup of \( H \).

Proof. Along the same line of the proof of Proposition 2.5 (b) we get (\( \beta_z : z \in H \}) \) invariant elements in \( \pi(O_d \otimes UHF_d)'' \) which are also true:

(z) \( \{U_z : z \in H\} \) is a commuting family of unitaries such that \( \beta_z(X) = U_z X U_z^* \) and thus by (z) \( \{U_z : z \in H\}'' \subseteq \pi(UHF_d \otimes UHF_d)'' \). Let \( X \) be an element in the commutant of \( \pi(UHF_d \otimes UHF_d)'' \) \( \pi(O_d \otimes UHF_d)' \). Then \( X \) commutes also with \( \{U_z : z \in H\}'' \) and thus \( X \in \pi(UHF_d \otimes UHF_d)'' \) by (z). Hence \( X \) is an element in the centre of \( \pi(UHF_d \otimes UHF_d)'' \) and so \( X = \sum_{k} c_k E_k \) where \( E_k \) are the minimal projections in the centre of \( \pi(UHF_d \otimes UHF_d)'' \) given in Proposition 3.3. However \( X \) also commutes with \( \pi(O_d)'' \) by our assumption (c) and \( \Lambda(E_k) = E_{k+1} \) for \( k \in H \). So \( c_k = c_{k+1} \) and \( X \) is a scalar multiple of unit operator. Hence (d) follows from (c). Along the same line we prove (c) implies (e). For a proof for (d) implies (c) and (e) implies (c), we simply apply (a).

Now we will prove (f) and (g). That \( \pi(\tilde{UHF}_d \otimes UHF_d)'' \) is a type-I von-Neumann algebra (with completely atomic centre) follows by a theorem of [So] once we use (c). In the proof of Proposition 3.3 we have proved that the centre of \( \pi(UHF_d \otimes UHF_d)'' \) is \( \{U_z : z \in H_0\}'' \) where \( H_0 \subseteq H \). For equality in the present situation we simply use (c), as \( \beta_w(U_z) = U_z \) for all \( w, z \in H \), to conclude that \( U_z \) is in the centre of \( \pi(UHF_d \otimes UHF_d)'' \).

If (c) holds then \( \mathcal{I}_0 = \mathcal{I} \) and thus (g) follows by (b). Here we will give another proof using the same idea to prove (f). Let \( X \) be an element in the commutant of
π₀(UHFₜ ⊗ UHFₜ)''', where π₀ is the factor representation on the minimal central projection E₀ defined in Proposition 3.3. Then X commutes with \{U_z : z ∈ ℋ\}'' and so by (c) X is an element in π(UHFₜ ⊗ UHFₜ)'''. So X is in the centre of \π₀(UHFₜ ⊗ UHFₜ)''', π₀ being a factor representation X is a scalar multiple of E₀. Thus π₀ is an irreducible representation and so ω is pure.

By Proposition 3.1 we recall that π₀ is unitary equivalent to the GNS representation of \( (\mathcal{B}, \omega) \). Thus π₀ is irreducible if and only if ω is pure. So for a pure state ω, for each k ∈ ℋ₀, πₖ being quasi-equivalent to π₀, πₖ is a type-I factor representation of π(UHFₜ ⊗ UHFₜ)'''. This completes the proof.

The following theorem is the central step that will be used repeatedly.

**Proposition 3.5.** Let ω be an extremal translation invariant state on \( \mathcal{B} \) and ψ be an extremal element \( \psi \) in \( K_\omega \). We consider the Popescu elements \( (\mathcal{K}, v_k : 1 ≤ k ≤ d, M, \Omega) \) as in Proposition 2.4 for the dual Popescu elements and associated amalgamated representation \( \pi \) of \( O_d ⊗ \mathcal{O}_d \) as described in Proposition 3.1. Let \( E \) and \( \tilde{E} \) be the support projections of the state \( \psi \) in \( \pi(O_d)^'' \) and \( \pi(\tilde{O}_d)^'' \) respectively. Let \( F \) and \( \tilde{F} \) be the projections \( [\pi(O_d)^'' \Omega] \) and \( [\pi(\tilde{O}_d)^'' \Omega] \) respectively. Then the following holds:

(a) \( \pi(\tilde{O}_d ⊗ O_d)^'' = \mathcal{B}(\tilde{H} ⊗ K \mathcal{H}) \);

(b) \( \pi(\tilde{O}_d)^'' = \pi(O_d)^'' \) if and only if \( \pi(\tilde{O}_d)^'' E = \pi(O_d)^'' E \);

(c) \( Q = E\tilde{E} \) is the support projection of the state \( \psi \) in \( \pi(O_d)^'' \tilde{E} \) and also in \( \pi(\tilde{O}_d)^'' E \). Further \( P = E\tilde{F} \leq Q \);

(d) If \( E\tilde{F} \) then \( \tilde{E} = \tilde{F} \), \( \tilde{F} = P \), \( P = Q \);

(e) If \( P = Q \) then the following statements are true:

(i) \( M' = \tilde{M} \) where \( \tilde{M} = \{P S_i P : 1 ≤ i ≤ d\}'' \);

(ii) \( \pi(O_d)^' = \pi(\tilde{O}_d)^'' \);

(f) If \( P = \{\tilde{M} \Omega]\) then \( M' = \tilde{M} \);

(g) \( \omega \) is pure on \( \mathcal{B} \) if and only if there exists a sequence of elements \( x_n ∈ M \) such that for each \( m ≥ 0 \) \( x_{n+m}τ_n(x(x) → φ(x)1 \) as \( n → ∞ \) in strong operator topology, equivalently \( \phi(τ_n(x)x_{n+m}τ_n(y)) → φ(x)φ(y) \) as \( n → ∞ \) for all \( x, y ∈ M \) where \( M = \{v_i = PS_i P : 1 ≤ i ≤ d\}'' \) and \( τ(x) = \sum_{1 ≤ k ≤ d} \beta_k(x) = x ∈ M; \) Same holds true if we replace \( M_0 \) for \( M \) where \( M_0 = \{x ∈ M : β(x) = x ∈ H\} \).

**Proof.** (a) is a restatement of Proposition 3.2 (a), \( E \) ( \( \tilde{E} \) ) being the support projection of the state \( ψ \) in \( π(O_d)^'' \) ( \( π(\tilde{O}_d)^'' \) ) and ψ = ψΛ we have \( Λ(E) ≥ E \) and further we have \( E = [π(O_d)^'' Ω] ≥ [π(\tilde{O}_d)^'' Ω] \) and hence increasing projections in \( π(O_d)^'' \), \( Λ''(E) \uparrow I \) as \( n → ∞ \) because \( Ω \) is cyclic for \( π(O_d ⊗ \tilde{O}_d)^'' \) in \( \mathcal{H} ⊗ K \tilde{H} \) and limiting projection being in the commutant of \( π(O_d)^'' \) as well i.e. since \( Λ(Y) = Y \) implies \( Y ∈ π(O_d)^'' \) by Cuntz relations.

We set von-Neumann algebras \( N_1 = π(O_d)^' E \) and \( N_2 = π(\tilde{O}_d)^'' E \). By our construction we have \( π(\tilde{O}_d)^'' ⊆ π(O_d)^' \) and so \( N_2 ⊆ N_1 \). Since \( Λ''(E) \uparrow I \) as
For (c) we note that \( Q = \varepsilon \tilde{E} \in \mathcal{N}_2 \subseteq \mathcal{N}_1 \) and claim that \( Q \) is the support projection of the state \( \psi \in \mathcal{N}_2 \). To that end let \( x\varepsilon \geq 0 \) for some \( x \in \pi(\mathcal{O}_d)'' \) so that \( \psi(QxQ) = 0 \). As \( \Lambda^k(\varepsilon) \geq 0 \) for all \( k \geq 1 \) and \( \Lambda^k(\varepsilon) \to I \) we conclude that \( x \geq 0 \). As \( \varepsilon \Omega = \Omega \) and thus \( \psi(\varepsilon x\varepsilon) = \psi(QxQ) = 0 \), we conclude \( \varepsilon x\varepsilon = 0 \), \( \varepsilon \) being the support projection for \( \pi(\mathcal{O}_d)'' \). Hence \( QxQ = 0 \). As \( \psi(Q) = 1 \), we complete the proof of the claim that \( Q \) is the support of \( \psi \) in \( \mathcal{N}_2 \). Similarly \( Q \) is also the support projection of the state \( \psi \in \pi(\mathcal{O}_d)''\tilde{E} \). This completes the proof of (c).

Thus if \( \varepsilon F = \tilde{E} \tilde{F} \), we get \( \Lambda^k(\varepsilon)^F = \tilde{E} \Lambda^k(\tilde{F}) \) and \( \varepsilon \Lambda(\tilde{F}) = \Lambda(\tilde{E})\tilde{F} \) and thus taking limit we get \( F = \tilde{E} \) and \( \varepsilon = \tilde{F} \). This completes the proof of (d).

As \( \varepsilon \in \pi(\mathcal{O}_d)'' \) and \( \tilde{E} \in \pi(\mathcal{O}_d)'' \) we check that von-Neumann algebras \( \mathcal{M}^1 = Q\pi(\mathcal{O}_d)''Q \) and \( \tilde{M}^1 = Q\pi(\mathcal{O}_d)''Q \) acting on \( Q \) satisfy \( \mathcal{M}^1 \subseteq \tilde{M}^1 \). Now we explore that \( \pi(\mathcal{O}_d)''Q = B(\mathcal{H} \otimes \mathcal{K}) \) and note that in such a case \( Q\pi(\mathcal{O}_d)''Q \) is the set of all bounded operators on the Hilbert subspace \( Q \). As \( \varepsilon \in \pi(\mathcal{O}_d)'' \) and \( \tilde{E} \in \pi(\mathcal{O}_d)'' \) we check that together \( \mathcal{M}^1 = Q\pi(\mathcal{O}_d)''Q \) and \( \tilde{M}^1 = Q\pi(\mathcal{O}_d)''Q \) generate all bounded operators on \( Q \). Thus both \( \mathcal{M}^1 \) and \( \tilde{M}^1 \) are factors. The canonical states \( \psi \) on \( \mathcal{M}^1 \) and \( \tilde{M}^1 \) are faithful and normal. We set \( \tilde{l}_k = Q\tilde{S}_kQ, 1 \leq k \leq d \) and recall that \( v_k = PS_kP \) and \( \tilde{v}_k = \tilde{P}\tilde{S}_k\tilde{P}, 1 \leq k \leq d \). We note that \( P\tilde{l}_kP = v_k \) and \( P\tilde{l}_k\tilde{l}_kP = \tilde{v}_k \) where we recall that, by our construction, \( P \) is the support projection of the state \( \psi \) in \( \pi(\mathcal{O}_d)''[\pi(\mathcal{O}_d)\Omega] \). \( Q \) being the support projection of \( \pi(\mathcal{O}_d)\tilde{E} \), by Theorem 2.4 applied to Cuntz elements \( \{S_i\tilde{E} : 1 \leq i \leq d\} \), \( \varepsilon \pi(\mathcal{O}_d)\tilde{E} \) is order isomorphic to \( \mathcal{M}^1 \) via the map \( X \to QXQ \). As the projection \( F = [\pi(\mathcal{O}_d)''\Omega] \in \pi(\mathcal{O}_d)' \), we check that the element \( QF\tilde{E}Q \in \mathcal{M}^1 \). However \( QF\tilde{E}Q = \varepsilon F\tilde{E}E = QPQ = P \) and thus \( P \in \mathcal{M}^1 \). We also check that \( \mathcal{M}^1 = \mathcal{M}^1 P\Omega = P\mathcal{M}^1 \Omega = \mathcal{M} \Omega \) and thus \( P = [\mathcal{M}^1 \Omega] \). We set \( \tilde{M} \) for the von-Neumann algebra generated by \( \{\tilde{v}_k : 1 \leq k \leq d\} \).

So far our analysis did not use the hypothesis on statement (e) i.e. \( P = Q \). For \( P = Q \), we have \( \mathcal{M}^1 = \mathcal{M} \) and \( \tilde{M}^1 = \tilde{M} \). By order isomorphic property we get (i) is equivalent to \( \varepsilon \pi(\mathcal{O}_d)\tilde{E} = \varepsilon \pi(\mathcal{O}_d)''\tilde{E} \) and taking commutant again we get \( \pi(\mathcal{O}_d)''\tilde{E} = \pi(\mathcal{O}_d)''\tilde{E} \). Now we invoke the first part of the argument changing the role or using the endomorphism \( \Lambda \) we conclude that \( \pi(\mathcal{O}_d)'' = \pi(\mathcal{O}_d)' \). This completes the proof of (e) provided we find a independent proof for (i) which is not so evident and this crucial point was not noticed in the proof given for Lemma 7.8 in [BJKW].

Now we will analyze the representation \( \pi \) of \( \mathcal{O}_d \otimes \mathcal{O}_d \) which is pure to prove (i). To that end we note since \( P = Q \) by our assumption, \( \Omega \) is a common cyclic and separating vector for \( \mathcal{M} \) and \( \mathcal{M}' \). Thus we can get an endomorphism \( \alpha : \mathcal{M}' \to \tilde{M} \) defined by

\[
\alpha(y) = \tilde{J} y \tilde{J} \tilde{J} \tilde{J}
\]

where \( \tilde{J} \) is the Tomita’s conjugate operator associated with the cyclic and separating vector \( \Omega \) for \( \mathcal{M} \) (i.e. for \( y \in \mathcal{M}' \), we have \( JyJ \in \mathcal{M} \subseteq \mathcal{M}' \) since \( \mathcal{M} \subseteq \mathcal{M}' \)). Since \( \tilde{J} \mathcal{M}' \tilde{J} = \tilde{M} \), we have \( \alpha(y) \in \tilde{M} \). We note that the general theory does not guarantee [AcC] that the endomorphism be Takesaki’s canonical conditional expectation associated with \( \phi \). If so then the modular automorphism group \( (\sigma_t) \) of
\( \mathcal{M}' \) also preserves \( \hat{\mathcal{M}} \). Thus \( \sigma_x(x) \in \hat{\mathcal{M}} \) for \(-1 \leq \text{Im}(z) \leq 1 \) if \( x \) is an analytic element in \( \hat{\mathcal{M}} \). Thus we would have got \( J v_k(\delta) J = \sigma_{\delta}(\hat{v}_k(\delta)) \in \hat{\mathcal{M}} \) where \( \hat{v}_k(\delta) \) is an analytic element with respect to Gaussian measure with variance \( \delta > 0 \). That \( \hat{v}_k(\delta) \) is analytic element follows from the general Tomita-Takesaki theory [BR1]. Since \( v_k(\delta) \to v_k \) in strong operator topology as \( \delta \to 0 \) and \( \mathcal{M} = \{ v_k : 1 \leq k \leq d \}'' \) together with \( J \mathcal{M} J = \mathcal{M}' \) we arrived at \( \hat{\mathcal{M}} = \mathcal{M}' \). In the following we avoid this tempting route and aim to explore the general representation theory of \( C^* \)-algebras [BR1, chapter 2].

We claim that \( \mathcal{M}' = \hat{\mathcal{M}} \). Suppose not i.e. \( \hat{\mathcal{M}} \subseteq \mathcal{M}' \). Then \( \alpha(\hat{\mathcal{M}}) \) is a proper von-Neumann subalgebra of \( \alpha(\mathcal{M}') \subseteq \mathcal{M} \) being an into map and hence \( \alpha(\mathcal{M}) \) is a proper von-Neumann sub-algebra of \( \hat{\mathcal{M}} \). Now consider the Popescu elements \( (K, \alpha(\hat{v}_i), \Omega) \) and its dilation as in Theorem 2.1. Then by the commutant lifting theorem applied to pairs \( (\hat{v}_i), \alpha(\hat{v}_i) \) we find a unitary operator \( U \) on \( \hat{\mathcal{H}} \) so that \( U \pi(\hat{\mathcal{O}}_d)^{\prime\prime} U^* \) is strictly contained in \( \pi(\hat{\mathcal{O}}_d)^{\prime\prime} \) (Without loss of generality we can take the dilated Hilbert space for \( (K, \alpha(\hat{v}_i), \Omega) \) to be same as \( \mathcal{H} \) as there exists an isomorphism preserving \( K \), see the remark that follows after Theorem 2.1 ). We extend \( U \) to a unitary operator on \( \mathcal{H} \otimes K \mathcal{H} \) and denote \( \pi_u(x) = U \pi(x) U^* \) for \( x \in \mathcal{O}_d \otimes \mathcal{O}_d \) which is unitary equivalent to the pure representation \( \pi \) and \( \pi_u(\mathcal{O}_d)^{\prime\prime} \) is strictly contained in \( \pi(\mathcal{O}_d)^{\prime\prime} \). Now \( \pi_u \) is also an amalgamated representation over the subspace \( K \) with \( P_u = Q_u \). Thus we can repeat now same procedure with \( \pi_u \) and so on. Note that the process won’t terminate in finite time. Our aim is to find a contradiction from this using formal set theory.

To that end we use temporary notation \( \pi_0 \) for \( \pi \) defined in last paragraph and \( \pi \) will be used for a generic representation. Let \( \mathcal{P} \) be the collection of representation \( (\pi, H_\pi, \Omega) \) quasi-equivalent to \( \pi_0 : \mathcal{O}_d \otimes \mathcal{O}_d \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K} \mathcal{H}) \) with a shift invariant vector state \( \omega(x) = < \Omega, \pi(x) \Omega > \) i.e. \( \omega(\pi(\theta(x))) = \omega(\pi(x)) \). So there exists cardinal numbers \( n_{\pi}, n_0(\pi) \) so that \( n_{\pi} H_\pi \) is unitary equivalent to \( n_0(\pi) \pi_0 \). Thus given an element \( (\pi, H_\pi, \Omega) \) we can associate two cardinal numbers \( n_{\pi} \) and \( n_0(\pi) \) and without loss of generality we assume that \( H_\pi \subseteq n_0(\pi) H_0 \) and \( n_{\pi} H_\pi = n_0(\pi) H_0 \). \( \pi_0 \) being a pure representation, any element \( \pi \in \mathcal{P} \) is a type-I factor representation of \( \mathcal{O}_d \otimes \mathcal{O}_d \). The interesting point here that \( \pi_{\pi \in \mathcal{P}} \) is also an element in \( \mathcal{P} \) with associated cardinal numbers \( \sum_\pi n_{\pi} \) and \( \sum_\pi n_0(\pi) \). We say \( (\pi_1, H_{\pi_1}, \Omega_1) \not< (\pi_2, H_{\pi_2}, \Omega_2) \) if there exists an isometry \( U : n_{\pi_1} H_{\pi_1} \to n_{\pi_2} H_{\pi_2} \) so that

(C1) For each \( 1 \leq \alpha \leq n_{\pi_1} \) we have \( U \Omega_1^\alpha = \Omega_2^\alpha \), for some \( 1 \leq \alpha' \leq n_{\pi_2} \);
(C2) \( n_{\pi_2} \pi_2(x) E_2' = U n_{\pi_1} \pi_1(x) U^* \) where \( E_2' \in n_{\pi_2} \pi_2(\mathcal{O}_d)^' \);
(C3) \( U \oplus_1 \leq \alpha \leq n_{\pi_1} \sum_\pi_1^{\pi_1}(\mathcal{O}_d)^{\prime\prime} U^* \subseteq \oplus_1 \leq \alpha \leq n_{\pi_2} \sum_\pi_2^{\pi_2}(\mathcal{O}_d)^{\prime\prime} E_2'. \)

That the partial order is non-reflexive follows as \( (\pi, H_\pi, \Omega) \not< (\pi, H_\pi, \Omega) \) contradicts (C3) as \( I = E_2' \). By our starting assumption that \( \mathcal{M}' \neq \hat{\mathcal{M}} \) we check that \( \pi_0 \not< \pi_u \). Thus going via the isomorphism we also check that for a given element \( \pi \in \mathcal{P} \) there exists an element \( \pi' \in \mathcal{P} \) so that \( \pi \not< \pi' \). Thus \( \mathcal{P}_0 \) is a non empty set and has at least one infinite chain. Partial order property follows easily. If \( \pi_1 \not< \pi_2 \) and \( \pi_2 \not< \pi_3 \) then \( \pi_1 \not< \pi_3 \). If \( U_{12} \) and \( U_{23} \) are isometric operators that satisfies (C1)-(C3) respectively, then \( U_{13} = U_{23} U_{12} \) will do the job for \( \pi_1 \) and \( \pi_3 \).

However by Hausdorff maximality theorem there exists a non-empty maximal totally ordered subset \( \mathcal{P}_0 \) of \( \mathcal{P} \). We claim that \( \pi_{\max} = \oplus_{\pi \in \mathcal{P}_0} \pi \) on \( H_{\pi_{\max}} = \oplus_{\pi \in \mathcal{P}_0} H_{\pi} \) is an upper bound in \( \mathcal{P}_0 \). That \( \pi_{\max} \in \mathcal{P} \) is obvious. Further given an element \( (H_1, \pi_1, \Omega_1) \in \mathcal{P}_0 \) there exists an element \( (H_2, \pi_2, \Omega_2) \in \mathcal{P}_0 \) so that \( \pi_1 \not< \pi_2 \) by
our starting remark as \( \pi_0 < \pi_u \). By extending isometry \( U_{12} \) to an isometry from \( H_1 \to n_{\pi_{\text{max}}} H_{\pi_{\text{max}}} \) trivially we get the required isometry that satisfies (C1),(C2) and (C3) where cardinal numbers \( n_{\pi_{\text{max}}} = \sum_{\pi \in \mathcal{P}_0} n_{\pi} \in N_0 \). Thus by maximal property of \( \mathcal{P}_0 \) we have \( \pi_{\text{max}} \in \mathcal{P}_0 \). This brings a contradiction as by our construction \((\pi_{\text{max}}, H_{\pi_{\text{max}}}, \Omega) < (\pi_{\text{max}}, H_{\pi_{\text{max}}}, \Omega) \) as \( \pi_{\text{max}} \in \mathcal{P}_0 \) but partial order is strict. This contradicts our starting hypothesis that \( \mathcal{M} \) is a proper subset of \( \mathcal{M}' \). This completes the proof for (i) of (e) \( \mathcal{M}' = \mathcal{M} \) when \( P = Q \).

In the proof of \( \mathcal{M}' = \tilde{\mathcal{M}} \) in (e), we have used equality \( P = Q \) just to ensure that \( \Omega \) is also a cyclic for \( \tilde{\mathcal{M}} \) and \( P = Q \) is used to prove \( \pi(\mathcal{O}_d)^\prime = \pi(\mathcal{O}_d)^\prime \prime \). So (f) follows by the proof of (e).

A proof for (g) is given in [Mo3] with \( \mathcal{M}_0 \). Here we will also give an alternative proof relating the criteria obtained in Proposition 3.4. To that end we claim that

\[
\bigcap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)^\prime) = \pi(UHF_d)^\prime \bigcap \pi(UHF_d)^\prime.
\]

That \( \tilde{\Lambda}^n(\pi(UHF_d)^\prime) \subseteq \{ \tilde{S}_1 \tilde{S}_1 : |I| = |J| < \infty \} \) follows by Cuntz relation and thus \( \bigcap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)^\prime) \subseteq \pi(UHF_d)^\prime \bigcap \pi(UHF_d)^\prime \). For the reverse inclusion let \( X \in \pi(UHF_d)^\prime \bigcap \pi(UHF_d)^\prime \). For \( n \geq 1 \), we choose \( |I| = n \) and set \( Y_n = \tilde{S}_1 \tilde{S}_I \). We check that it is independent of the index that we have chosen as \( Y_n = \tilde{S}_1 \tilde{S}_I \tilde{S}_J \tilde{S}_I = \tilde{S}_1 \tilde{S}_I \tilde{S}_J \tilde{S}_I \tilde{S}_J = \tilde{S}_1 \tilde{S}_I \tilde{S}_J \tilde{S}_I \tilde{S}_J \tilde{S}_J \) where in second equality we have used \( X \in \pi(UHF_d)^\prime \) and also \( \tilde{\Lambda}^n(Y_n) = \sum_{|I|=n} \tilde{S}_1 \tilde{S}_I \tilde{S}_J \tilde{S}_I \tilde{S}_J = X \). This proves the equality in the claim. Going along the same line we also get

\[
\bigcap_{n \geq 1} \tilde{\Lambda}^n(\pi(O_d)^\prime) = \pi(UHF_d)^\prime \bigcap \pi(O_d)^\prime = \pi(UHF_d \otimes O_d)^\prime.
\]

By Proposition 3.4 \( \omega \) is pure if and only if the set above is trivial. Thus once more by Proposition 1.1 in [Ar2] and Theorem 2.4 in [Mo2], purity is equivalent to asymptotic relation \( ||\tilde{\psi}^n - \phi|| \to 0 \) as \( n \to \infty \) for any normal state \( \psi \) on \( \mathcal{M}' \). ( Here we recall by Proposition 2.4 \( P\pi(O_d)^\prime P = \mathcal{M}' \) as \( P \) is also the support projection in \( \pi(O_d)^\prime F \) and support projection of \( \phi_\Omega \) in \( \pi(O_d)^\prime \) is \( F = [\pi(O_d)^\prime \Omega] \), where commutant is taken in \( B(K) \). ) By a duality argument Theorem 2.4 in [Mo3] we conclude that \( \omega \) is pure if and only if there exists a sequence of elements \( x_n \in \mathcal{M} \) so that for each \( m \geq 0, x_{m+n} + \tau_n(x) \to \phi(x)1 \) as \( n \to \infty \) for all \( x \in \mathcal{M} \subseteq B(K) \). This completes the proof of (d) with \( \mathcal{M} \). For the proof with \( \mathcal{M}_0 \) we need to show if part as only if part follows \( \mathcal{M}_0 \) being a subset of \( \mathcal{M} \) and \( \tau \) takes elements of \( \mathcal{M}_0 \) to itself. For if part we refer to Theorem 3.2 in [Mo3].

We set

\[
(\mathcal{M}')_0 = \{ x \in \mathcal{M}' : \beta_z(x) = x, \quad z \in H \}.
\]

Similarly we also set \( \mathcal{M}_0 \) and \( (\mathcal{M}')_0 \) as \( \{ \beta_z : z \in H \} \) invariant elements of \( \mathcal{M}_0 \) and \( (\mathcal{M}')_0 \) respectively. We note that as a set \( (\mathcal{M}_0)' \) could be different from \( (\mathcal{M}')_0 \).

We note also that \( P\mathcal{M}P' \subseteq \mathcal{M} \) and unless \( P \) is an element in \( \mathcal{M}_0 \), equality is not guaranteed for a factor state \( \omega \). The major problem is to show that \( P \) is indeed an element in \( \mathcal{M}_0 \) when \( \omega \) is a pure state.

We warn here an attentive reader that in general for a factor state \( \omega \), the set \( \mathcal{F} \eta(\mathcal{O}_d)^\prime \mathcal{F} \), which is a subset of \( \mathcal{F} \eta(\mathcal{O}_d)^\prime \mathcal{F} \), need not be an algebra. However by commutant lifting theorem applied to dilatation \( v_i \to S_i \mathcal{F}, \pi(\mathcal{O}_d)^\prime \mathcal{F} \) is order isomorphic to \( \mathcal{M}' \) as \( P = \mathcal{F} \mathcal{E} \) is the support projection. Thus the von-Neumann
sub-algebra generated by the elements \( F \) is order isomorphic to \( \tilde{M} \). However \( \mathcal{M}_0 \) may properly include \( \mathcal{M}_{00} = \{ P\pi(\mathbb{UHF}_d)P \}''' \) (as an example take \( \psi = \text{tr} \) to be the unique KMS state on \( \mathcal{O}_d \) and \( \omega \) be the unique trace on \( \mathcal{B} \) for which we get \( \mathcal{M}_{00} = \mathbb{C} \) and \( P\pi(\mathbb{UHF}_d)P \)''' is the linear span of \( \{ v_j, 1, \bar{v}_j : |j| < \infty \}. \)

Existence of a \( \phi \) preserving normal conditional expectation \( \int_{\mathcal{H}} \beta_z d\omega : \mathcal{M} \to \mathcal{M}_0 \) by Proposition 2.5 ensures that modular operator of \( \phi \) preserves \( \mathcal{M}_0 \) \cite{T} and so does on \( (\mathcal{M}')_0 \). However there is no reason to take it granted for \( \tilde{M}_0 \) to be invariant by the modular group of \((\mathcal{M}', \phi)\). By Takesaki’s theorem such a property is true if and only if there exists a \( \phi \)-invariant norm one projection from \( (\mathcal{M}')_0 \) onto \( \tilde{M}_0 \). In the following we avoid this tempted route.

At this stage it is not clear how we can ensure existence of a normal conditional expectation from \( \mathcal{M}' \) onto \( \tilde{M} \) directly and so the equality \( \mathcal{M}' = \tilde{M} \) when \( \omega \) is a pure state. Further interesting point here that the equality \( \mathcal{M} = \mathcal{M}' \) holds when \( \omega \) is the unique trace on \( \mathcal{B} \) as \( v_1 = S_{\pi} \) and \( J\bar{v}_k, J = \frac{1}{2}S_{\pi} \) for all \( 1 \leq k \leq d \) where \( P \neq Q \) and \( \pi(\mathbb{O}_d)'' \supset \pi(\tilde{O}_d)'' \). In the last proposition we have also proved if \([\mathcal{M}\Omega] = P\) then \( \mathcal{M}' = \tilde{M} \). Thus a natural question that arises here: how the equality \( P = Q \) is related to purity of \( \omega \)? We are now in a position to state the main mathematical result of this section.

**Theorem 3.6.** Let \( \omega \) be as in Theorem 3.5. Then the following holds:

\( (a) \) \( P \) is also the support projection of \( \psi \) in \( \pi(\bar{\mathcal{O}}_d)'''\tilde{\mathcal{H}} \) if and only if \( \omega \) is pure.

\( (b) \) If \( \omega \) is pure then the following holds:

\( (i) \) \( \mathcal{M}' = \tilde{M} \) where \( \tilde{M} = \{ P\bar{S}_i P : 1 \leq i \leq d \}''' \);

\( (ii) \) \( \pi(\mathbb{O}_d)' = \pi(\tilde{O}_d)''' \);

\( (iii) \) \( \pi_\omega(B_R)' = \pi_\omega(B_L)''' \);

**Proof.** First we will prove that \( \omega \) is pure if \( P \) is also the support projection of the state \( \psi \) in \( \pi(\bar{\mathcal{O}}_d)'\bar{\mathcal{F}} \), where \( \bar{\mathcal{F}} = [\pi(\bar{\mathcal{O}}_d)'\Omega] \). The support projection of \( \psi \) in \( \pi(\bar{\mathcal{O}}_d)'\bar{\mathcal{F}} \) is \( \bar{\mathcal{E}}\bar{\mathcal{F}} \) and thus we also have \( P = \bar{\mathcal{E}}\bar{\mathcal{F}} \) by our hypothesis. Since \( \Lambda''(P) = \Lambda''(\mathcal{E}\bar{\mathcal{F}}) \uparrow \mathcal{F} \) and now \( \Lambda'(P) = \mathcal{E}\Lambda''(\mathcal{E}\bar{\mathcal{F}}) \uparrow \mathcal{E} \) as \( n \uparrow \infty \), we also have \( \mathcal{F} = \mathcal{E} \). Similarly we also have for each \( n, \mathcal{E}\Lambda''(\mathcal{F}) = \Lambda''(\mathcal{E}\bar{\mathcal{F}}) \) and thus taking limit we also get \( \mathcal{E} = \mathcal{F} \).

So we have \( P = \mathcal{E}\mathcal{F} = \mathcal{E}\bar{\mathcal{E}} = Q \). \( \tilde{M} = P\pi(\bar{\mathcal{O}}_d)'''P \) is cyclic in \( K \) i.e. \( [\tilde{M}\Omega] = [P\pi(\bar{\mathcal{O}}_d)'''P\Omega] = P\bar{\mathcal{F}} = P\mathcal{E} = P \) as \( \bar{\mathcal{E}} = \mathcal{E} \).

However \( \bigcap_{n \to \infty} \Lambda''(\pi(\mathbb{UHF}_d)) = \pi(\mathbb{UHF}_d)''' \cap \pi(\mathbb{UHF}_d)' \) (for a proof which is a simple application of Cuntz relation, we refer to section 5 of \cite{Mo}). Further \( \psi \) being a factor state in \( K_\omega \), by Proposition 3.2 \( \pi(\mathbb{UHF}_d)''' \) is a factor. In particular we have \( \bigcap_{n \to \infty} \Lambda''(\pi(\mathbb{UHF}_d)) = \mathbb{C}\bar{\mathcal{F}} \). Thus by Proposition 1.1 in \cite{Ar} we conclude that \( ||\psi_\pi - \phi|| \to 0 \) as \( n \to \infty \) for all normal state \( \Psi \) on \( \tilde{M}_0 \) where \( \tilde{M}_0 = P\pi(\mathbb{UHF}_d)'''P \) as \( \bar{\mathcal{F}} = \mathcal{E} \) and support projection of \( \psi \) in \( \pi(\mathbb{UHF}_d)''' \) is \( \tilde{P} \) and \( \bar{\mathcal{E}} = \mathcal{E} \).

Note that \( \tilde{M}_0 \subseteq \tilde{M}_0' \) where \( \tilde{M}_0 = P\pi(\mathbb{UHF}_d)'''P \). Further by Proposition 2.5 \( \tilde{M}_0 = \{ x \in \mathcal{M} : \beta_x(x) = x, z \in H \} \) and \( \tilde{M}_0 = \{ x \in \mathcal{M} : \bar{\beta}_x(x) = x, z \in H \} \). Once \( \tilde{P} = [\tilde{M}_0\Omega] \) then we also have \( \tilde{P} = [\tilde{M}_0\Omega] \) as \( [\tilde{M}\Omega] = P = [\tilde{M}_0\Omega] \) by expending \( u_z = \sum_{k \in H} z^k \bar{P}_k \) where \( z \to u_z = PU_zP \) is a unitary representation of group \( H \).
For \( x \in \hat{\mathcal{M}}, y \in \hat{\mathcal{M}}' \) we have
\[
\phi(\hat{\tau}(x)y) = \sum_k <\hat{v}_k^*\Omega, x\hat{v}_k^*y\Omega> = \sum_k <v_k^*\Omega, xyv_k^*\Omega>
\]
(as \( v_k^*\Omega = \hat{v}_k^*\Omega \))
\[
= \sum_k <\Omega, xuv_kv_k^*\Omega> = \phi(x\tau(y))
\]
The dual group of \((\hat{\mathcal{M}}, \hat{\tau}, \phi)\) is given on the commutant by \((\hat{\mathcal{M}}', \tau, \phi)\) where \(\tau(x) = \sum_k u_kv_kv_k^*\) for \(x \in \hat{\mathcal{M}}'\). where commutant is taken in \(\mathcal{B}(\mathcal{K})\). Now moving to \(\{\beta_z : z \in H\}\} invariant elements in the duality relation above, we verify that adjoint Markov map of \((\hat{\mathcal{M}}_0, \hat{\tau}, \phi)\) is given by \((\hat{\mathcal{M}}_0', \tau, \phi)\) where \(\mathcal{M}_0\), the commutant of \(\mathcal{M}_0\) is taken in \(\mathcal{B}(\mathcal{K}_0)\) and \(\mathcal{K}_0\) is the Hilbert subspace \(P_0\) with \(\Omega\) as cyclic and separating vector for \(\mathcal{M}_0\) in \(\mathcal{K}_0\). Thus by Theorem 2.4 in [Mo3], there exists a sequence of elements \(y_n \in \hat{\mathcal{M}}_0'\) such that for each \(m \geq 1\), \(y_{m+n} \rightarrow \phi(y)\) as \(n \rightarrow \infty\) for all \(y \in \mathcal{M}_0\) \(\subseteq \mathcal{B}(\mathcal{K}_0)\). Thus \(\omega\) is pure by Proposition 3.5 (e) once we recall \(\mathcal{M}' = \hat{\mathcal{M}}\) as \(P = Q\) (and so \(\mathcal{M}_0 = \mathcal{M}_0\) ) by Proposition 3.5 (d). This completes the proof of purity property of \(\omega\).

In the following we now prove the hardest part of the theorem namely \(\hat{\mathcal{F}} = \mathcal{E}\) and \(\mathcal{F} = \hat{\mathcal{E}}\) if \(\omega\) is pure. Proof uses extensively the general theory of quasi-equivalent representation of a \(C^*\) algebra and we refer [BR1, Chapter 2.4.4] as a general reference.

We set unitary operator \(V = \sum_k S_k \hat{S}_k^*\). That \(V\) is a unitary operator follows by Cuntz’s relations and commuting property of \((S_i)\) and \((\hat{S}_i)\). Further a simple computation shows that \(V\tau(x)V^* = \pi(\theta(x))\) for all \(x \in \mathcal{B} = \mathcal{B}_k \otimes \mathcal{B}_R\), identified with \(\text{UHF}_d \otimes \text{UHF}_d\) and \(\theta\) is the right shift. We also have
\[
(33) \quad \theta(\mathcal{E}) = V\mathcal{E}V^* = \sum_{k,k'} S_k \hat{S}_{k'}^* \mathcal{E} S_{k'} \hat{S}_k^* = \Lambda(\mathcal{E}) \geq \mathcal{E}
\]
Similarly we also have
\[
(34) \quad \theta^{-1}(\hat{\mathcal{F}}) \geq \hat{\mathcal{F}}
\]
So in particular we have \(V(I - \mathcal{E})V^* \leq I - \mathcal{E}\), i.e. \((I - \mathcal{E})V^*\mathcal{E} = 0\). Also for any \(X \in \pi(\mathcal{O}_u)'\) we have \(V^*\hat{\mathcal{F}}X\Omega = \hat{\mathcal{F}}\sum_k \hat{S}_k X \hat{S}_k^*\Omega\) as \(\hat{S}_k^*\Omega = \hat{S}_k^*\Omega\). Thus \((I - \hat{\mathcal{F}})V^*\hat{\mathcal{F}} = 0\) i.e. \(V\hat{\mathcal{F}}V^* = \Lambda(\hat{\mathcal{F}}) \geq \hat{\mathcal{F}}\). Similarly we also have
\[
(35) \quad \theta^{-1}(\mathcal{E}) = V^*\mathcal{E}V \geq \mathcal{E} and V^*\mathcal{F}V \geq \mathcal{F}
\]
We also set two family of increasing projections for all natural numbers \(n \in \mathbb{Z}\) as follows
\[
(36) \quad \mathcal{E}_n = V^n\mathcal{E}V^n*, \quad \hat{\mathcal{F}}_n = V^n\hat{\mathcal{F}}(V^n)^*
\]
Since \(\beta_z(V) = V\) for all \(z \in H\), \(V \in \pi(\text{UHF}_d \otimes \text{UHF}_d)'\) by Proposition 3.4 as \(\omega\) is pure. \(\omega\) being also a factor state, we have \(<f, V_n g> \rightarrow <f, \Omega> <g, \Omega>\) as \(n \rightarrow \infty\) for any \(f, g \in \pi(B_{\text{loc}})\Omega\) by Power’s criteria [Pow] given in (1). Since such vectors are dense in the Hilbert space topology and the family \(\{V^n : n \geq 1\}\) is uniformly bounded, we get \(V^n \rightarrow |\Omega> <\Omega|\) in weak operator topology as \(n \rightarrow \infty\).

For the time being we assume that \(H\) is trivial. Otherwise the argument that follows here we can use for the representation \(\pi_0\) of \(\text{UHF}_d \otimes \text{UHF}_d\) i.e. \(\pi\) restricted to \(\pi(\text{UHF}_d \otimes \text{UHF}_d)\).
We have the following cases:

**Case 1.** $\mathcal{E} \neq I (\tilde{\mathcal{E}} \neq I)$. Let $\mathcal{E}_n \to \mathcal{E}_{-\infty}$ as $n \to -\infty$ and thus $\mathcal{V} \mathcal{E}_{-\infty} V^* = \mathcal{E}_{-\infty}$. We claim that either $\mathcal{E}_{-\infty} = [\Omega] > [\Omega]$ or $\mathcal{E}_{-\infty}$ is a proper infinite dimensional projection i.e. if $\mathcal{E}_{-\infty}$ is a finite projection then $\mathcal{E}_{-\infty} = [\Omega] > [\Omega]$. Suppose not then the finite subspace is shift invariant. In particular there exists a unit vector $f$ orthogonal to $\Omega$ such that $Vf = zf$ for some $z \in \mathbb{C}$ and this contradicts weak mixing property i.e. $V^n \to [\Omega] > [\Omega]$ in weak operator topology proved above as point spectrum of $V$ has only 1 with spectral multiplicity 1.

If $\mathcal{E}_{-\infty}$ is infinite dimensional we can get a unitary operator $U_0$ from $F_0 = \mathcal{H} \otimes K \mathcal{H}$ onto $\mathcal{E}_{-\infty}$ and via the unitary map we can get a sequence of increasing projections $U_0 \mathcal{E}_n U_0^*$ in $\mathcal{E}_{-\infty}$ and note that $U_0 \mathcal{E}_n U_0' = V^n U_0' \mathcal{E} V^n (V^n)^*$. Note that if $\mathcal{E}_{-\infty}$ is infinite dimension the process will not stop in finite step. Thus we have $F_0 \otimes \Omega = \oplus_{1 \leq k \leq n} F(k)$ where the index set is either singleton or infinity and each $F(k)$ will give a system of imprimitivity with respect to $\mathcal{V}$, where $F(1) = F_0 - \mathcal{E}_{-\infty}$. Further UHF$_d$ being a simple C$^*$-algebra, each such imprimitivity system is of Mackey index $N_0$ [Mo3, section 4]: We fix a nonzero $f \in \mathcal{E} - \theta^{-1}(\mathcal{E})$ 0 otherwise $\mathcal{E} = I$ as $\theta^{-1}(\mathcal{E}) \cap I$ as $n \to -\infty$. $\pi(x) : x \to \theta^{-1}(x)f$ gives a representation of UHF$_d = B_L$ and we check that $[\pi(x)B_L \mathcal{E}] \subseteq \mathcal{E} - \theta^{-1}(\mathcal{E})$ as $f \perp [\theta^{-1}(\pi(B_L))\mathcal{E}]$. Thus simplicity ensures that $\mathcal{E} - \theta^{-1}(\mathcal{E})$ is a projection of dimension $N_0$. Further $n_\mathcal{E}$ is either 1 or $N_0$ since $F_0$ is separable.

Since $\hat{\mathcal{F}}$ is also a proper projection, same argument is valid for $\hat{\mathcal{F}}$ with $\tilde{\mathcal{F}}_{-\infty} = \lim_{n \to -\infty} \theta^n(\hat{\mathcal{F}})$ i.e. we can write $F_0 \otimes \Omega = \oplus_{1 \leq k \leq n} \hat{H}(k)$, where each $\hat{H}(k)$ gives rise to a system of imprimitivity with respect to $\mathcal{V}$ where each system of imprimitivity is of Mackey index $N_0$ where $\hat{\mathcal{F}}(1) = F_0 - \tilde{\mathcal{F}}_{-\infty}$ and $n_{\hat{\mathcal{F}}}$ is either 1 or $N_0$.

In the following we use temporary notation $H$ for Hilbert subspace $F_0$. For a cardinal number $n$, we amplify a representation $\pi : B \to B(H)$ of the C$^*$ algebra $B$ to $n$ fold direct sum $n \pi(x) = \oplus_{1 \leq k \leq n} \pi_k(x)$ acting on $nH = \oplus_{1 \leq k \leq n} H_k$ defining by

$$n \pi(x)(\oplus_{i=1}^{m} \xi_i) = \oplus_{i=1}^{m} \pi_k(x) \xi_i$$

where $\pi_k = \pi$ is the representation of $B = UHF_d \otimes \mathcal{U}HF_d$ on $H_k = H$ where $H = [\pi(UHF_d \otimes \mathcal{U}HF_d)]\Omega$. We also extend $\hat{\mathcal{F}} = \oplus \hat{F}_n$, $\hat{\mathcal{E}} = \oplus \hat{E}_n$ and $\mathcal{V} = \oplus_{1 \leq k \leq n} \hat{V}_k$ respectively. We also set notation $\Omega_k = \oplus_{1 \leq k \leq n} \delta_{\Omega_k}$.  

Thus by Mackey’s theorem, there exists a cardinal number $n \in N_0$ and a unitary operator $U : nH \to nH$ so that $\mathcal{V} = \overline{U} \mathcal{V} U^*$ and $\hat{\mathcal{E}} = \overline{UF} U^*$. We set a representation $\pi^U : B \to B(nH)$ by $\pi^U(x) = Un \pi(x) U^*$ and rewrite the above identity as

$$\oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger \Omega_k] = \oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger n \Omega_k]$$

where $\pi_k^U(x) = U \pi_k(x) U^*$. Note that by our construction we can ensure $U \Omega_k = \Omega_k$ for all $1 \leq k \leq n$ as the operator intertwining between two imprimitivity systems are acting on the orthogonal subspace of the projection generated by vectors $\{\Omega_k : 1 \leq k \leq n\}$.

We claim $\mathcal{E} = \hat{\mathcal{E}}$. Suppose not i.e. $\hat{\mathcal{F}} \subset \mathcal{E}$. In such a case we have

$$\oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger \Omega_k] < \oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger n \Omega_k]$$

Alternatively

$$\oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger \Omega_k] < \oplus_{1 \leq k \leq n} [\pi_k(UHF_d)^\dagger n \Omega_k]$$
Thus in principle we can repeat our construction now with $\pi^U$ and so we get a strict partial ordered set of quasi-equivalent representation of $B$. In the following we now aim to employ formal set theory to bring a contradiction on our starting assumption that $\tilde{F} < \tilde{E}$.

To that end we need to deal with more then one representation of $B$. For the rest of the proof we reset notation $\pi_0$ for $\pi$ used for the pure representation of $B$ in $H_0 = [\pi_0(B)\Omega_0]$ where $\Omega_0$ is the cyclic vector, the reset notation for $\Omega$. Let $P$ be the collection of representation $(\pi, H_\pi, \Omega)$ quasi-equivalent to $\pi_0 : B \to B(H_0)$ with a shift invariant vector state $\omega(x) = \langle \Omega, \pi(x)\Omega \rangle$ i.e. $\omega(\pi(\theta(x))) = \omega(\pi(x))$. So there exists minimal cardinal numbers $n_\pi, n_0(\pi)$ so that $n_\pi H_\pi$ is unitary equivalent to $n_0(\pi)\pi_0$. Thus for such an element $(\pi, H_\pi, \Omega_\pi)$ we can associate two cardinal numbers $n_\pi$ and $n_0(\pi)$ and without loss of generality we assume that $H_\pi \subseteq n_0(\pi)H_0$ and $n_\pi H_\pi = n_0(\pi)H_0$. $\pi_0$ being a pure representation, any element $\pi \in P$ is a type-I factor representation of $B$. The interesting point here that $\oplus_{\pi \in P} \pi$ is also an element in $P$ with associated cardinal numbers $\sum_\pi n_\pi$ and $\sum_\pi n_0(\pi)$. We say $(\pi_1, H_{\pi_1}, \Omega_1^\pi) < (\pi_2, H_{\pi_2}, \Omega_2^\pi)$ if there exists an isometry $U : n_{\pi_1} H_{\pi_1} \to n_{\pi_2} H_{\pi_2}$ so that

(C1) For each $1 \leq \alpha \leq n_{\pi_2}$ we have $U\Omega_1^\alpha = \Omega_2^\alpha$, for some $1 \leq \alpha' \leq n_{\pi_1}$;
(C2) $n_{\pi_2}\pi_2(x)E_2' = UN\pi_1\pi_1(x)U^*$ where $E_2' \in n_{\pi_2}\pi_2(B)^*$;
(C3) $\oplus_{1 \leq \alpha \leq n_{\pi_2}} [\pi_1^U(UHF_d)^\Omega_1^\alpha] \ominus \oplus_{1 \leq \alpha \leq n_{\pi_2}} [\pi_2^U(UHF_d)^\Omega_2^\alpha] E_2'$.

In the inequality we explicitly used that both Hilbert spaces are subspaces of $nH_0$ for some possibly larger cardinal number $n$. That the partial order is non-reflexive follows as $(\pi, H, \Omega) < (\pi, H, \Omega)$ contradicts (C3) as $I = E_2'$. Partial order property follows easily. If $\pi_1 \prec \pi_2$ and $\pi_2 \prec \pi_3$ then $\pi_1 \prec \pi_3$. If $U_{12}$ and $U_{23}$ are isometric operators that satisfies (C1)-(C3) respectively, then $U_{13}U_{23}U_{12}$ will do the job for $\pi_1$ and $\pi_3$. Thus $\pi^U \in P$ and by our starting assumption that $\tilde{F} \neq \tilde{E}$ we also check that $\pi_0 \prec \pi^U$. Thus going via the isomorphism we also check that for a given element $\pi \in P$ there exists an element $\pi' \in P$ so that $\pi \prec \pi'$. Thus $P_0$ is a non empty set and has at least one infinite chain containing $\pi_0$.

However by Hausdorff maximality theorem there exists a non-empty maximal totally ordered subset $P_0$ of $P$ containing $\pi_0$. We claim that $\pi_{\max} = \oplus_{\pi \in P_0} \pi$ on $H_{\pi_{\max}} = \oplus_{\pi \in P_0} H_\pi$ is an upper bound in $P_0$. That $\pi_{\max} \in P$ is obvious. Further given an element $(H_{\pi_1}, \pi_1, \Omega_1) \in P_0$ there exists an element $(H_{\pi_2}, \pi_2, \Omega_2) \in P_0$ so that $\pi_1 \prec \pi_2$ by our starting remark as $\pi_0 \prec \pi^U$. By extending isometry $U_{12}$ to an isometry from $H_{\pi_1} \to n_{\pi_{\max}} H_{\pi_{\max}}$ trivially we get the required isometry that satisfies (C1),(C2) and (C3) where cardinal numbers $n_{\pi_{\max}} = \sum_\pi n_\pi \pi \subseteq 8$. Thus by maximal property of $P_0$ we have $\pi_{\max} \in P_0$. This brings a contradiction as by our construction $(\pi_{\max}, H_{\pi_{\max}}, \Omega) \prec (\pi_{\max}, H_{\pi_{\max}}, \Omega)$ as $\pi_{\max} \in P_0$ but partial order is strict. This contradicts our starting hypothesis that $\tilde{F} < \tilde{E}$. This completes the proof that $\tilde{F} = \tilde{E}$ when $\tilde{E} \neq I$. By symmetry of the argument we also get $\tilde{F} = \tilde{E}$ when $\tilde{E} < 1$.

Case 2: $\tilde{E} = I$ ($\tilde{E} = I$). We need to show $\tilde{F} = I$ ($\tilde{F} = I$) respectively. Suppose not and assume that both $\tilde{F}$ is a proper non-zero projection.

We set projection $G$ on the closed linear span of elements in the subspaces $[\theta^{-n}(\mathcal{F})\pi(UHF_d)^\Omega]$ for all $n \geq 0$. We recall that $\theta(X) = VXV^*$ where $V = \sum_k S_k^\pi S_k^\pi$ and $\theta^{-1}(X) = \tilde{\Lambda}(X)$ for $X \in (UHF_d)^\pi$. Thus we have $V^*\theta^{-n}(\mathcal{F})\pi(UHF_d)^\Omega$
Thus $(1 - G)V^*G = 0$ i.e. $\theta(G) \geq G$. It is also clear that $\tilde{F} \leq G$ as the defining sequence of subspaces of $G$ goes to precisely $\tilde{F}$ as $n \to \infty$ (recall that $\theta^n(F) = \hat{\Lambda}_n(F) \uparrow I$ strongly as $n \to \infty$). Once more we have $\theta^n(G) \geq \theta^n(\tilde{F}) = \Lambda^n(\tilde{F}) \uparrow I$ as $n \uparrow \infty$.

If $G$ is a proper projection we can follow the steps as in the case 1 to find a unitary operator $U : nH \to nH$ with $U\Omega_k = \Omega_k$ and $UVU^* = \tilde{V}$. We consider the subset $P_G$ of elements in $\mathcal{P}$ for which $\mathcal{E}_x = 1$ and $\{\theta^{-n}(\mathcal{F}_x) : n \geq 0\}$ commutes with $\tilde{F}_x$ and modify the strict partial ordering by modifying (C3) as

$$(\text{C3'}) \oplus_{1 \leq n \leq n_n} [\pi^0(UHF_d)^n\Omega^1_n] < \oplus_{1 \leq n \leq n_n} [\pi^2(UHF_d)^n\Omega^2_n]E'_2$$

So we also get $\pi^U \in P_G$ and $\pi_0 \prec \pi^U$ and going along the same line we conclude that $G = \tilde{F}$. Thus we conclude that $G$ is either equal to 1 or $G = \tilde{F}$.

**Sub-case 1 of case 2:** If $G = I$ then $FG = F$ and so $[\mathcal{F}_{\pi(UHF_d)}\Omega] = F$ as $\theta^{-n}(F) \geq F$. Thus $\tilde{F} \geq F$. So $\tilde{F} \geq \Lambda^n(F)$ for all $n \geq 1$ and taking limit we get $\tilde{F} \geq I$ i.e. $\tilde{F} = I$. This contradicts our starting assumption that $\tilde{F}$ is a proper projection.

**Sub-case 2 of case 2:** Now we consider the case $G = \tilde{F} < I$. In such a case we have $(1 - \tilde{F})\theta^{-n}(F)\tilde{F} = 0$ and so $\theta^{-n}(F)$ commutes with $\tilde{F}$ for all $n \geq 0$.

First we rule out the simplest possibility in the present situation for $\mathcal{F}\tilde{F} = [\Omega \prec \Omega]$. If so then $\omega$ is a Bernoulli state and a proof follows once we compute the following using the property $\Lambda(\tilde{F}) \geq \tilde{F}$ i.e. $\tilde{F}\pi(s_i)^*\tilde{F} = \pi(s_i)^*\tilde{F}$ and commuting property of $\mathcal{F}$ with $\pi(s_i)^*$ to get some scalars $\tilde{\lambda}_i$ such that

$$\tilde{\lambda}_i = \mathcal{F}\tilde{F}\pi(s_i)^*\tilde{F} = \pi(s_i)^*\tilde{F}\mathcal{F}$$

and so

$$\omega(s_is_j) = <\Omega, \pi(s_i)^*\pi(s_j)^*\Omega>$$

$$= <\Omega, \mathcal{F}\tilde{F}\pi(s_i)^*\pi(s_j)^*\tilde{F}\mathcal{F}\Omega>$$

$$= \tilde{\lambda}_i \tilde{\lambda}_j$$

The inductive limit state $\omega$ on $B$, give rises to a pure state once restricted to $B_R$ and thus we have $\mathcal{E} = \tilde{F}$ by Haag duality when $\pi(B_R)^\vee$ is a type-I factor. This contradicts our starting assumption that $\mathcal{E} = I$.

Now we set projection $\mathcal{F}'$ defined by

$$\mathcal{F}' = \mathcal{F} - \mathcal{F}\tilde{F} + [\Omega \prec \Omega]$$

and check by commuting property of $\tilde{F}$ with $\mathcal{F}$ that

$$\mathcal{F}'(\theta^{-1}(F))\mathcal{F}' = (I - \tilde{F})\mathcal{F}(I - \theta^{-1}(\tilde{F}))(I - \tilde{F})\mathcal{F} + [\Omega \prec \Omega]$$

$$= \mathcal{F}(I - \tilde{F}) + [\Omega \prec \Omega] = \mathcal{F}'$$

where we have used $\theta^{-1}(F) = \Lambda(F) \geq F$, $\theta(F) \leq F$ and $\theta(\tilde{F}) = \Lambda(\tilde{F}) \geq \tilde{F}$. Thus we get

$$\theta^{-1}(\mathcal{F}') \geq \mathcal{F}'$$
We also rule out the possibility that \( \hat{F} \leq \hat{F} \) and \( \hat{A}^n(F) \leq \hat{A}^n(\hat{F}) = \hat{F} \). Taking limit we get \( \hat{F} = I \) as \( \Lambda^n(F) \uparrow I \) as \( n \to \infty \). This brings a contradiction to our hypothesis.

So we have in particular \( F' < F \leq I \) and \( F' - |\Omega| > \Omega | \neq 0 \). Now we will rule out the possibility of \( \hat{F} \leq I \) under our hypothesis \( \hat{F} \leq E = I \). Suppose so i.e. \( F = I \), then \( \hat{E} = I \) since \( \hat{E} \geq \hat{F} \). Then \( Q = E\hat{E} = E = E\hat{F} = F \).

Thus by Proposition 3.5 (e) we get \( \pi(O_d)' = \pi(O_d)' \) and so we have in particular \( \hat{F} = [\pi(O_d)'/\Omega] = [\pi(O_d)'/\Omega] = \hat{E} \). This contradicts our starting assumption once more that \( \hat{F} < E = I \).

We also have \( \theta^{-1}(F') \geq F' \) and \( \theta^{-1}(F) \geq F \). Thus we can follow the steps of Case-1 with elements \( F', F, \theta^{-1} \) replacing the role of \( \hat{F}, E, \theta \) to get a unitary operator \( U : nH \to nH \) so that \( U\hat{V} = \hat{V}U \) and \( UF^* = F^* \) for a cardinal number \( n \).

Now we consider a further subset \( P_G' \) of \( P_G \) consist of quasi-equivalent representations \( \pi \) to \( \pi_0 \) of \( B \) where \( \pi \) admits the additional property: \( \mathcal{F}_\pi \leq I, \mathcal{E}_\pi = I \) and \( \{ \theta^{-n}(\mathcal{F}_\pi) : n \geq 0 \} \) commutes with \( \mathcal{F}_\pi \) with the strict partial ordering \( (\mathcal{H}_\pi, \pi_1, \Omega, \pi_2) \prec (\mathcal{H}_{\pi_2}, \pi_2, \Omega_{\pi_2}) \) given by modifying condition (C3') as

\[
(C3') \odot_{1 \leq a \leq n_2} \pi_2(\mathcal{H}_d)'[\Omega_1] > \odot_{1 \leq a \leq n_2} \pi_2(\mathcal{H}_d)'[\Omega_2]E' \]

Since \( \pi_U \) also satisfies the conditions that of \( \pi_0 \in P_G \) by covariance relation of \( U \) with respect to shifts once more we get \( \pi_U \in P_G' \) and \( \pi^U < \pi_U \). Thus we can repeat the process and so \( P_G' \) has at least one infinite chain of totally ordered containing \( \pi_0 \). Once more by Hausdorff maximality principle we bring a contradiction to our starting assumption that \( F' < F \). In other words this brings a contradiction to our starting hypothesis that \( \hat{F} \) is a proper projection i.e. \( \hat{F} < E = I \). Thus we arrive at \( \hat{F} = \hat{E} \) when \( E = I \).

By symmetry of argument used here it also follows that \( \hat{F} = \hat{E} \) when \( E = I \). This completes the proof of \( \hat{F} = \hat{E} \), \( \hat{F} = \hat{E} \) for the case when \( H \) is the trivial closed subgroup of \( S^1 \).

Now we will remove the assumption that \( H \) is trivial using Proposition 3.4. Let \( (H, \pi_0, \Omega) \) be the GNS space of the state \( \omega \) on \( B \). Let \( e_0 \) and \( \hat{e}_0 \) be the support projections of \( \omega \) in \( \pi_0(BR)' \) and \( \pi_0(BL)' \) respectively. Similarly we also set projections \( F_0 = [\pi_0(A_R)'/\Omega] \) and \( \hat{F}_0 = [\pi_0(A_L)'/\Omega] \). We also set projections \( q_0 = [\pi_0(UHd)'\Omega][\pi_0(UHd)'\Omega] \) and \( p_0 = [\pi_0(UHd)'\Omega][\pi_0(UHd)'\Omega] \).

\( \omega \) being pure, by Theorem 3.4 any \( \{ \beta_z : z \in H \} \) invariant element of \( \mathcal{B}(H \otimes \mathcal{K}H) \) is an element in \( \pi(UHd) \otimes UHd) \). \( \hat{\mathcal{E}}, \hat{\mathcal{F}}, \hat{\mathcal{F}} \) are \( \{ \beta_z : z \in H \} \) elements. Thus once we identify cyclic space \( [\pi_0(B)]\Omega \) with \( F_0 = [\pi(UHd) \otimes UHd)'/\Omega] \), we get obvious relations

\[
(39) \quad \mathcal{E}F_0 = e_0, \hat{\mathcal{E}}F_0 = \hat{e}_0, \mathcal{F}F_0 = f_0, \hat{\mathcal{F}}F_0 = \hat{f}_0
\]

and

\[
(40) \quad PF_0 = p_0 \quad \text{and} \quad QF_0 = q_0
\]

as \( \mathcal{E} = [\pi(UHd)'\Omega], \hat{\mathcal{E}} = [\pi(UHd)'\Omega], Q = \mathcal{E}\hat{\mathcal{E}} \) and \( P = \mathcal{E}\mathcal{F} \). Further \( V \) is also \( \{ \beta_z : z \in H \} \) invariant and \( V\pi(x)V^* = (\pi(\theta(x))) \) for all \( x \in B \) which we have identified with \( UHd \otimes UHd \).
By applying the first part of the argument with representation \( \pi_0 \), for pure \( \omega \), we have \( e_0 = f_0 \) and \( \bar{e}_0 = \bar{f}_0 \) and \( p_0 = q_0 \).

Now we write the equality \( p_0 = q_0 \) as \( E F_0 = E \hat{E} F_0 \) and apply \( \Lambda \) on both sides to conclude that \( \Lambda(E)F_1 = \Lambda(E)\hat{E} F_1 \) and multiplying by \( \mathcal{E} \) from left we get \( \mathcal{E} \Lambda(E)F_1 = \mathcal{E}\hat{E} F_1 \) as \( \Lambda(\mathcal{E})\mathcal{E} = \mathcal{E} \) and thus we get \( PF_1 = QF_1 \). By repeated application of \( \Lambda \), we get \( PF_m = QF_m \).

If \( H = \{z : z^n = 1\} \) then we get \( P = \sum_k PF_k = \sum_k QF_k = Q \). This completes the proof for \( P = Q \). Similarly \( \mathcal{F} = \sum_k \mathcal{F} F_k = \sum_k \mathcal{E} F_k = \mathcal{E} \) and also \( \mathcal{F} = \hat{\mathcal{E}} \).

If \( H = S^1 \) then \( \hat{H} = \mathbb{Z} \) and for \( m \geq 0 \), we have \( PF_m = QF_m \). For \( m < 0 \) we take \( k = -m \) and check that \( \Lambda^k(QF_m - PF_m) = \Lambda^k(Q)F_{m+k} - \Lambda^k(P)F_{m+k} = \Lambda^k(\mathcal{E})\hat{F}_0 - \hat{\Lambda}(\mathcal{E})F_0 = \hat{\Lambda}(\mathcal{E})(\bar{e}_0 - f_0) = 0 \) Since \( \Lambda \) is an injective map, we get \( QF_m = PF_m \) for all \( m < 0 \).

Now we are left to prove those three statements given in (b). \( \omega \) being pure we have \( P = Q \) and thus by Proposition 3.5 we have \( M = M' \) and \( \pi(\mathcal{O}_d)' = \pi(\mathcal{O}_d)' \). We are left to show \( \pi_\omega(B_R)' = \pi_\omega(B_L)'' \). For that we recall \( F_0 \) and check few obvious relation \( F_0\pi(\mathcal{O}_d)'F_0 = F_0\pi(\mathcal{O}_d)'F_0 \) and \( \pi_\omega(B_R)' \subseteq F_0\pi(\mathcal{O}_d)'F_0 \). Since \( F_0\pi(\mathcal{U}HF_d)''F_0 \) is equal to \( \{\beta_1 : \beta \in H\} \) invariant elements in \( F_0\pi(\mathcal{O}_d)'F_0 \) and elements in \( \pi_\omega(B_R)' \) are \( \{\beta_2 : \beta \in H\} \) invariant we conclude that \( \pi_\omega(B_R)' \subseteq \pi_\omega(B_L)'' \). Inclusion in other direction is obvious and thus Haag duality property (iii) holds.

Proof. (of Theorem 1.1:) (a) implies that \( q_0 = p_0 \). (d) also says that \( p_0 = q_0 \). Thus in either case, following last part of the proof of Theorem 3.6 we get \( P = Q \). The statement (e) also implies \( P = Q \). That shows now that (a),(d) as well as (e) implies (f) by the if part of Theorem 2.6 (a). That (b) implies (a) is trivial as \( \Omega \) is separating for \( \mathcal{M}_1 = \mathcal{M} \) by faithful property. That (c) implies (f) is trivial as \( \pi_\omega(B_R)'' \) is a factor. Thus we have showed so far any of the statement (a),(b),(c),(d),(e) implies (f). For the converse we appeal to the only if part of Theorem 3.6.

REFERENCES

- [Ac] Accardi, L.: A non-commutative Markov property, (in Russian), Functional. anal. i Prilozhen 9, 1-8 (1975).
- [AcC] Accardi, Luigi; Cecchini, Carlo: Conditional expectations in von Neumann algebras and a theorem of Takesaki. J. Funct. Anal. 45 (1982), no. 2, 245273.
- [AM] Accardi, L., Mohari, A.: Time reflected Markov processes. Infin. Dimens. Anal. Quantum Probab. Relat. Top., vol-2, no-3, 397-425 (1999).
- [AKLT] Affleck, L., Kennedy, T., Lieb, E.H., Tasaki, H.: Valence Bond States in Isotropic Quantum Antiferromagnets, Commun. Math. Phys. 115, 477-528 (1988).
- [AHP] Akihio, Nobuyuki; Hiai, Fumio; Petz, Dnes Equilibrium states and their entropy densities in gauge-invariant C-systems. Rev. Math. Phys. 17 (2005), no. 4, 365-389.
- [Ara1] Araki, H.: Gibbs states of a one dimensional quantum lattice. Comm. Math. Phys. 14 120-157 (1969).
• [Ara2] Araki, H.: On uniqueness of KMS-states of one-dimensional quantum lattice systems, Comm. Maths. Phys. 44, 1-7 (1975).
• [AMA] Araki, H., Matsui, T.: Ground states of the XY model, Commun. Math. Phys. 101, 213-245 (1985).
• [Ar1] Arveson, W.: On groups of automorphisms of operator algebras, J. Func. Anal. 15, 217-243 (1974).
• [Ar2] Arveson, W.: Pure $E_0$-semigroups and absorbing states. Comm. Math. Phys 187, no.1, 19-43, (1997)
• [BBN] Baumgartner, Bernhard; Benatti, Fabio; Narnhofer, Heide: Translation invariant states on twisted algebras on a lattice. J. Phys. A 43 (2010), no. 11, 115301, 13 pp.
• [BR] Bratteli, Ola; Robinson, D.W.: Operator algebras and quantum statistical mechanics, I,II, Springer 1981.
• [BJP] Bratteli, Ola.; Jorgensen, Palle E.T. and Price, G.L.: Endomorphism of $\mathcal{B}(\mathcal{H})$, Quantisation, nonlinear partial differential equations, Operator algebras, (Cambridge, MA, 1994), 93-138, Proc. Sympos. Pure Math 59, Amer. Math. Soc. Providence, RT 1996.
• [BJKW] Bratteli, Ola.; Jorgensen, Palle E.T., Kishimoto, Akitaka and Werner Reinhard F.: Pure states on $O_d$, J.Operator Theory 43 (2000), no-1, 97-143.
• [BJ] Bratteli, Ola: Jorgensen, Palle E.T. Endomorphism of $\mathcal{B}(\mathcal{H})$, II, Finitely correlated states on $O_N$, J. Functional Analysis 145, 323-373 (1997).
• [Cun] Cuntz, J.: Simple $C^*$-algebras generated by isometries. Comm. Math. Phys. 57, no. 2, 173–185 (1977).
• [DHR] Doplicher,S., Haag, R. and Roberts, J.: Local observables and particle statistics I, II, Comm. Math. Phys. 23 (1971) 119-230 and 35, 4985 (1974)
• [Ev] Evans, D.E.: Irreducible quantum dynamical semigroups, Commun. Math. Phys. 54, 293-297 (1977).
• [Ex] Exel, Ruy: A new look at the crossed-product of a $C^*$-algebra by an endomorphism. (English summary) Ergodic Theory Dynam. Systems 23 (2003), no. 6, 1733-1750.
• [FNW1] Fannes, M., Nachtergaele, B., Werner, R.: Finitely correlated states on quantum spin chains, Commun. Math. Phys. 144, 443-490(1992).
• [FNW2] Fannes, M., Nachtergaele, B., Werner, R.: Finitely correlated pure states, J. Funct. Anal. 120, 511- 534 (1994).
• [FNW3] Fannes, M., Nachtergaele, B. Werner, R.: Abundance of translation invariant states on quantum spin chains, Lett. Math. Phys. 25 no.3, 249-258 (1992).
• [Fr] Frigerio, A.: Stationary states of quantum dynamical semigroups, Comm. Math. Phys. 63 (1978) 269-276.
• [Hag] Haag, R.: Local quantum physics, Fields, Particles, Algebras, Springer 1992.
• [HMHP] Hiai, Fumio; Mosonyi, Miln; Ohno, Hiromichi; Petz, Dnes, Free energy density for mean field perturbation of states of a one-dimensional spin chain. Rev. Math. Phys. 20 (2008), no. 3, 335-365.
• [Ki] Kishimoto, A.: On uniqueness of KMS-states of one-dimensional quantum lattice systems, Comm. Maths. Phys. 47, 167-170 (1976).
• [La] Christopher, Lance, E.: Ergodic theorems for convex sets and operator algebras, Invent. Math. 37, no. 3, 201-214 (1976).
• [Mac] Mackey, George W.: Imprimitivity for representations of locally compact groups. I. Proc. Nat. Acad. Sci. U. S. A. 35, 537-545 (1949).
• [Ma1] Matsui, A.: Ground states of fermions on lattices, Comm. Math. Phys. 182, no.3 723-751 (1996).
• [Ma2] Matsui, T.: A characterization of pure finitely correlated states. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1, no. 4, 647–661 (1998).
• [Ma3] Matsui, T.: The split property and the symmetry breaking of the quantum spin chain, Comm. Maths. Phys vol-218, 293-416 (2001)
• [Ma4] Matsui, Taku, On the absence of non-periodic ground states for the antiferromagnetic XXZ model. Comm. Math. Phys. 253 (2005), no. 3, 585-609.
• [Na] Nachtergae, B. Quantum Spin Systems after DLS1978, “Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon’s 60th Birthday” Fritz Gesztesy et al. (Eds), Proceedings of Symposia in Pure Mathematics, Vol 76, part 1, pp 47–68, AMS, 2007.
• [Mo1] Mohari, A.: Markov shift in non-commutative probability, J. Funct. Anal. vol-199 , no-1, 190-210 (2003) Elsevier Sciences.
• [Mo2] Mohari, A.: Pure inductive limit state and Kolmogorov’s property, J. Funct. Anal. vol 253, no-2, 584-604 (2007) Elsevier Sciences.
• [Mo3] Mohari, A: Jones index of a completely positive map, Acta Applicandae Mathematicae, Vol 108, Number 3, 665-677 (2009).
• [Mo4] Mohari, A.: Pure inductive limit state and Kolmogorov’s property-II, [http://arxiv.org/abs/1101.5961] To appear Journal of Operator Theory.
• [Mo5] Mohari, A.: Translation invariant pure states in quantum spin chain and its split property, [http://arxiv.org/abs/0904.2104].
• [Mo6] Mohari, A.: A complete weak invariance for Kolmogorov states on $B = \otimes_{k \in \mathbb{Z}} M^{(k)}_d(C)$, [http://arxiv.org/abs/]
• [OP] Ohya, M., Petz, D.: Quantum entropy and its use, Text and monograph in physics, Springer 1995.
• [Or1] Ornstein, D. S.: Bernoulli shifts with the same entropy are isomorphic, Advances in Math. 4 1970 337-352 (1970).
• [Or2] Ornstein, D. S.: A K-automorphism with no square root and Pinsker’s conjecture, Advances in Math. 10, 89-102. (1973).
• [Pa] Parry, W.: Topics in Ergodic Theory, Cambridge University Press, 1981.
• [Po] Popescu, G.: Isometric dilations for infinite sequences of non-commutating operators, Trans. Amer. Math. Soc. 316 no-2, 523-536 (1989).
• [Pow1] Powers, Robert T.: Representations of uniformly hyper-finite algebras and their associated von Neumann. rings, Annals of Math. 86 (1967), 138-171.
• [Ru] Ruelle, D. : Statistical Mechanics, Benjamin, New York-Amsterdam (1969).
• [Sa] Sakai, S. : Operator algebras in dynamical systems. The theory of unbounded derivations in $C^*$-algebras. Encyclopedia of Mathematics and its Applications, 41. Cambridge University Press, Cambridge, 1991.
• [So] Stormer E.: On projection maps of von Neumann algebras, Math. Scand. 30, 46-50 (1972).
• [Ta] Takesaki, M.: Conditional Expectations in von Neumann Algebras, J. Funct. Anal., 9, pp. 306-321 (1972)