Concrete Fock representations of Mickelsson-Faddeev-like algebras

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Abstract
The Mickelsson-Faddeev (MF) algebra can naturally be embedded in a non-Lie algebra, which suggests that it has no Fock representations. The difficulties are due to the inhomogeneous term in the connection’s transformation law. Omitting this term yields a “classical MF algebra”, which has other abelian extensions that do possess Fock modules. I explicitly construct such modules and the intertwining action of the higher-dimensional Virasoro algebra.

1 Introduction
A most challenging and potentially important problem is to construct the representation theory of local Lie algebras in $N$-dimensional spacetime. The name indicates that the generators are localized in spacetime, i.e. that the structure constants are proportional to finitely many derivatives of delta functions. This class includes the current algebra $map(N, g)$, the diffeomorphism algebra $diff(N)$, as well as algebras of divergence free, Hamiltonian, or contact vector fields. When $N = 1$, the projective representations are described by affine Kac-Moody and Virasoro algebras, respectively, but much less is known in higher dimensions. The reason for this is the fundamental observation that functions and normal ordering are incompatible except in one dimension. Typically, the classical representations of local Lie algebras
are functions over spacetime, with values in some finite-dimensional vector space such as \( \mathfrak{g} \) or \( \mathfrak{gl}(N) \) modules. A naïve strategy to construct projective representations would be to start from such functions, add canonically conjugate momenta, normal order, and hope to obtain a realization on Fock space. However, this approach only works in one dimension; in higher dimensions, new infinities arise.

In view of this fundamental incompatibility between functions and normal ordering, there are two philosophically distinct strategies. One is to keep functions and do something about normal ordering; papers following this route typically contain the keywords “further regularization”. The most ambitious program in this direction has been carried out by Mickelsson and collaborators, targeting the Mickelsson-Faddeev (MF) algebra \([1, 7, 14, 15, 16, 17, 18]\). Although representations in an abstract sense have been reported, concrete representations (on a separable Hilbert space) seem to be missing \([19]\).

The logical alternative is to keep normal ordering and do something about functions; more precisely, functions can be replaced by trajectories in the space of finite jets, which can be viewed as the coefficients of truncated Taylor expansions. This route was first entered by Moody, Eswara-Rao and Yokonoma \([4, 5, 13]\), whereas the geometrical understanding was provided by myself \([9, 10]\). This approach immediately leads to concrete Fock representations of abelian extensions of current algebras. However, these cocycles are not of MF type, but rather of the higher-dimensional Kac-Moody type described by Kassel \([6]\) and rediscovered in \([8, 13]\). In particular, they involve one-chains rather than three-chains. Applied to the diffeomorphism algebra, the same method leads to the higher-dimensional Virasoro algebras of \([8, 12]\).

It is thus natural to ask if the MF algebra could also be represented using such methods. The answer appears to be negative. Cederwall et al. \([2]\) found a natural realization of a “classical MF algebra”, where the inhomogeneous term in the connection’s transformation law has been dropped. However, this term is not recovered by normal ordering; even worse, it spoils the Jacobi identities for the realization mentioned above. It is shown in the present paper that the classical MF algebra also admits other cocycles, but these are of Kac-Moody type. These new MF-like algebras possess lowest-energy representations, described in section 3. Moreover, they can be intertwined with the diffeomorphism algebra, but the extensions are then no longer central, since they do not commute with diffeomorphisms.
2 Embeddings

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra with basis $J^a$, totally skew structure constants $f^{abc}$, Killing metric $\delta^{ab}$, and brackets $[J^a, J^b] = f^{abc}J^c$. As usual, set $d^{abc} = \text{tr}J^a\{J^b, J^c\} \propto \text{tr}J^{(a}J^{b}J^{c)}$, where the trace is evaluated in some representation and paranthesized indices are symmetrized. The following identities hold:

\[ f^{aed}f^{bcd} + f^{acd}f^{bed} + f^{abd}f^{ced} = 0, \]
\[ f^{aed}d^{bcd} + f^{acd}d^{bed} + f^{abd}d^{ced} = 0, \]
\[ f^{bac} = -f^{abc}, \quad f^{bca} = f^{abc}, \quad d^{bac} = d^{abc}, \quad \text{and} \quad d^{bca} = d^{abc}. \]

Denote by $J^a(m) = \exp(\im \cdot x)J^a$ the generators of $\text{map}(N, \mathfrak{g})$, the algebra of maps from $\mathbb{R}^N$ to $\mathfrak{g}$, where $x = (x^\mu)$ and $m = (m_\mu)$. Moreover, let $A^a_\mu = A^a_\mu(x)dx^\mu$ be the connection one-form, with Fourier coefficients $A^a_\mu(m)$. The Mickelsson-Faddeev (MF) algebra \cite{1, 14} is the following Lie algebra extension of $\text{map}(3, \mathfrak{g})$:

\[ [J^a(m), J^b(n)] = f^{abc}J^c(m + n) + d^{abc}m_\mu n_\nu \epsilon^{\mu\nu\rho}A^c_\rho(m + n), \]
\[ [J^a(m), A^b_\sigma(n)] = f^{abc}A^c_\sigma(m + n) + \delta^{ab}m_\rho \delta_\sigma(m + n), \]
\[ [A^a_\mu(m), A^b_\nu(n)] = 0, \]

where

\[ \epsilon^{\mu\nu\rho} = \begin{cases} +1, & \mu\nu\rho \text{ positive permutation of 123}, \\ -1, & \mu\nu\rho \text{ negative permutation of 123}, \\ 0, & \text{otherwise}, \end{cases} \]

is the totally anti-symmetric symbol in three dimensions.

Set $H^{a\mu\nu}(m) = \epsilon^{\mu\nu\rho}A^a_\rho(m)$. Then the MF algebra takes the form

\[ [J^a(m), J^b(n)] = f^{abc}J^c(m + n) + d^{abc}m_\mu n_\nu H^{c\mu\nu}(m + n), \]
\[ [J^a(m), H^{b\mu}(n)] = f^{abc}H^{c\mu}(m + n) + \delta^{ab}m_\rho S^{\mu\nu\rho}_3(m + n), \]

and all other brackets vanish. Moreover, $S^{\mu\nu\rho}_3$ is totally antisymmetric and subject to the additional condition

\[ m_\rho S^{\mu\nu\rho}_3(m) \equiv 0, \]

which geometrically means that it is a closed three-chain. In three dimensions, the unique solution to (4) is $S^{\mu\nu\rho}_3(m) = \epsilon^{\mu\nu\rho}\delta(m)$, but the present formulation holds in any number of dimensions $\geq 3$, and in two dimensions with $S^{\mu\nu\rho}_3(m) \equiv 0$. 


Eq. (3) can be embedded in the following algebra:

\[
\begin{align*}
[J^a(m), J^b(n)] &= f^{abc} J^c(m + n), \\
[J^a(m), G^{\mu n}(n)] &= f^{abc} G^{\mu}(m + n), \\
[J^a(m), H^{b\mu\nu}(n)] &= f^{abc} H^{c\mu\nu}(m + n) + \delta^{ab} m_\rho S^\rho_3(m + n), \\
[G^\mu(m), G^{\nu n}(n)] &= d^{abc} H^{c\mu\nu}(m + n),
\end{align*}
\]

(5)

and all other brackets vanish. Explicitly, this is accomplished by means of the redefinition

\[
J^a(m) \mapsto J^a(m) + m_\mu G^{\mu}(m).
\]

(6)

In the absence of the closed three-chain $S^\mu_3(m)$, this embedding was first described by [2].

However, there is one big problem with (5): in the presence of the three-chain, it is not a Lie algebra, because the following Jacobi identity fails:

\[
[J^a(m), [G^{\mu n}(n), G^{\nu r}(n)]] + \text{cycl.} = d^{abc} m_\rho S^\rho_3(m + n + r) \neq 0.
\]

(7)

I consider this as a strong indication that the MF algebra has no Fock representations with $S^\mu_3(m)$ non-zero. At least, it can not be possible to isolate operators $G^{\mu}(m)$ as in (6), since that would violate the Jacobi identities.

If we skip the three-chain, we obtain the “classical MF algebra”, which has another abelian extension:

\[
\begin{align*}
[J^a(m), J^b(n)] &= f^{abc} J^c(m + n) - k \delta^{ab} m_\rho S^\rho_1(m + n) + d^{abc} m_\mu n_\nu H^{c\mu\nu}(m + n), \\
[J^a(m), H^{b\mu\nu}(n)] &= f^{abc} H^{c\mu\nu}(m + n),
\end{align*}
\]

(8)

where $S^\rho_1(m)$ is a closed one-form, satisfying

\[
\begin{align*}
[J^a(m), S^\rho_1(n)] &= [H^{a\mu\nu}(m), S^\rho_1(n)] = 0, \\
m_\rho S^\rho_1(m) &\equiv 0.
\end{align*}
\]

(9)

This algebra can be embedded into the following algebra by means of the same redefinition (6).

\[
\begin{align*}
[J^a(m), J^b(n)] &= f^{abc} J^c(m + n) - k \delta^{ab} m_\rho S^\rho_1(m + n), \\
[J^a(m), G^{\mu n}(n)] &= f^{abc} G^{\mu}(m + n), \\
[J^a(m), H^{b\mu\nu}(n)] &= f^{abc} H^{c\mu\nu}(m + n), \\
[G^\mu(m), G^{\nu n}(n)] &= d^{abc} H^{c\mu\nu}(m + n),
\end{align*}
\]

(10)

where $G^{\mu}(m)$ also commutes with the one-chain $S^\rho_1$. 

4
3 Fock representations

Consider the following Lie algebra:

\[
\begin{align*}
[J^a(s), J^b(t)] &= f^{abc} J^c(s) \delta(s - t) + \frac{k}{2\pi i} \delta^{ab} \delta(s - t), \\
[J^a(s), G^{ab}(t)] &= f^{abc} G^{ac}(s) \delta(s - t), \\
[G^{a\mu}(s), G^{b\mu}(t)] &= d^{abc} H^{c\mu}(s) \delta(s - t), \\
[J^a(s), H^{j\mu\nu}(t)] &= f^{abc} H^{c\mu\nu}(s) \delta(s - t), \\
[G^{a\mu}(s), H^{b\rho\nu}(t)] &= [H^{a\mu\nu}(s), H^{b\sigma\tau}(t)] = 0,
\end{align*}
\]

(11)

where \( s, t \in S^1 \). Note that the first relation is the affine Kac-Moody algebra with central charge \( k \). Moreover, introduce \( N \) bosonic oscillators \( q^\mu(t) \). Then the following expressions yield a realization of (11) and thus of the modified MF algebra (8).

\[
\begin{align*}
\mathcal{J}^a(m) &= \int dt \ e^{imq(t)} J^a(t), \\
\mathcal{G}^{a\mu}(m) &= \int dt \ e^{imq(t)} G^{a\mu}(t), \\
\mathcal{H}^{a\mu\nu}(m) &= \int dt \ e^{imq(t)} H^{a\mu\nu}(t), \\
S^a_1(m) &= \frac{1}{2\pi} \int dt \ q^\mu(t) e^{imq(t)}.
\end{align*}
\]

(12)

Moreover, the value of the central charge \( k \) is the same in both formulas.

The problem of finding Fock representations of the modified MF algebra has thus been reduced to representing (11). This may be done e.g. by introducing oscillators \( \phi^a(t), \psi^{a\mu}(t), \) and \( \zeta^{a\mu\nu}(t) \), together with their canonical conjugate momenta \( \bar{\phi}^a(t), \bar{\psi}^{a\mu}(t), \) and \( \bar{\zeta}^{a\mu\nu}(t) \). Moreover, \( \zeta^{a\mu\nu}(t) \) and \( \zeta^{a_{\mu\nu}}(t) \) are assumed to be symmetric in \( \mu, \nu \). The canonical commutation relation read

\[
\begin{align*}
[\bar{\phi}^a(s), \phi^b(t)] &= \delta^{ab} \delta(s - t), \\
[\bar{\psi}_\mu^a(s), \psi_{\mu}^b(t)] &= \delta^{ab} \delta_s^\mu \delta(s - t), \\
[\bar{\zeta}_{\mu\nu}^a(s), \zeta_{\rho\tau}^{b}(t)] &= \delta^{ab} \delta_s^{\mu\nu} \delta(\tau - \sigma) \delta(s - t),
\end{align*}
\]

(13)

and all other brackets vanish. Then the following operators

\[
\begin{align*}
J^a(t) &= f^{abc} (\phi^b(t) \bar{\phi}^a(t) : + \psi^{\mu\nu}(t) \bar{\psi}_\mu^b(t) : + \zeta^{\mu\nu}(t) \bar{\zeta}_{\mu\nu}^b(t) :),
\end{align*}
\]

5
\[ G^{a\mu}(t) = f^{abc} \phi^{c\mu}(t) \phi^{b}(t) + d^{abc} \zeta^{c\mu}(t) \bar{\phi}^{b}(t), \]
\[ H^{\alpha\nu}(t) = f^{abc} \zeta^{c\mu}(t) \bar{\phi}^{b}(t), \]

satisfy (11). The double dots in the first expression indicate standard one-dimensional normal ordering with respect to frequency.

4 Diffeomorphisms

The algebra (10), and thus also the modified MF algebra (8), admits an intertwining action of an extension of the diffeomorphism algebra \( \text{diff}(N) \). The additional brackets read

\[ [\mathcal{L}_\mu(m), \mathcal{L}_\nu(n)] = n_\mu \mathcal{L}_\nu(m + n) - m_\nu \mathcal{L}_\mu(m + n) + (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu)m_\rho S_1^{\rho}(m + n), \]
\[ [\mathcal{L}_\mu(m), \mathcal{J}^a(n)] = n_\mu \mathcal{J}^a(m + n), \]
\[ [\mathcal{L}_\mu(m), \mathcal{G}^{\alpha\nu}(n)] = n_\mu \mathcal{G}^{\alpha\nu}(m + n) + \delta^\alpha_\mu m_\rho \mathcal{G}^{\alpha\rho}(m + n), \]
\[ [\mathcal{L}_\mu(m), \mathcal{H}^{\alpha\nu}(n)] = n_\mu \mathcal{H}^{\alpha\nu}(m + n) + \delta^\alpha_\mu m_\sigma \mathcal{H}^{\alpha\sigma}(m + n), \]
\[ [\mathcal{L}_\mu(m), S_1^{\nu}(n)] = n_\mu S_1^{\nu}(m + n) + \delta^\nu_\mu m_\rho S_1^{\rho}(m + n), \]

where \( \mathcal{L}_\mu(m) \) are the \( \text{diff}(N) \) generators and the cocycles multiplied by \( c_1 \) and \( c_2 \) were first found by Eswara-Rao and Moody [5] and myself [8], respectively. For the classification of \( \text{diff}(N) \) cocycles, see [3] and [12]. To construct a representation of (15), introduce \( N \) oscillators \( p_\mu(t) \) which are the canonical momenta of \( q^\mu(t) \), i.e.

\[ [p_\mu(s), q^\nu(t)] = \delta^\nu_\mu \delta(s - t), \quad [p_\mu(s), p_\nu(t)] = [q^\mu(s), q^\nu(t)] = 0. \quad (16) \]

The \( \text{diff}(N) \) generators have the realization

\[ \mathcal{L}_\mu(m) = \int dt \left( -i :e^{imq(t)}p_\mu(t): + m_\nu e^{imq(t)}T_\nu^\mu(t) \right), \quad (17) \]

where \( T_\nu^\mu(t) \) are the generators of the Kac-Moody algebra \( \widehat{gl(N)} \). The relevant relations read

\[ [T_\nu^\mu(s), T_\rho^\sigma(t)] = (\delta_\nu^\rho T_\nu^\mu(s) - \delta_\sigma^\mu T_\sigma^\nu(s)) \delta(s - t) - \frac{1}{2\pi i} (k_1 \delta^\mu_\nu \delta^\rho_\sigma + k_2 \delta^\mu_\sigma \delta^\rho_\nu) \delta(s - t), \]
\[ [T^\mu_\nu(s), q^\rho(t)] = [T^\mu_\nu(s), p_\rho(t)] = 0, \]
\[ [T^\mu_\nu(s), J^\alpha(t)] = 0, \]
\[ [T^\mu_\nu(s), G^{\alpha\sigma}(t)] = \delta^\alpha_\sigma G^{\mu\nu}(s) \delta(s - t), \]
\[ [T^\mu_\nu(s), H^{\alpha\sigma\tau}(t)] = (\delta^\alpha_\sigma H^{\mu\nu\tau}(s) + \delta^\tau_\nu H^{\alpha\sigma\mu}(s)) \delta(s - t), \]

and the abelian charges in (15) take the values
\[ c_1 = 1 + k_1, \quad c_2 = k_2. \]

The \( \hat{gl}(N) \) acts on the same Fock space as the operators in (14), by means of the following expression:
\[ T^\mu_\nu(t) = \delta^\mu_\nu (\cdots \phi^a(t) \bar{\phi}^a(t) \cdots) + \psi^a_\sigma(t) \bar{\phi}^a(t) \cdots + \zeta^a_\sigma\tau(t) \bar{\psi}^a_{\sigma\tau}(t) \cdots + \psi^{a\mu}(t) \psi^a_\nu(t) \cdots + \zeta^{a\mu\nu}(t) \bar{\psi}^a_{\mu\nu}(t) \cdots. \]  

As was noted in [10], we can actually represent a larger algebra on the same Fock space. Namely, there is a natural action of an additional \( \text{diff}(1) \) algebra, which classically commutes with both \( \text{diff}(N) \) and the MF algebra. Geometrically, this algebra describes reparametrizations of the observer’s trajectory.

5 Conclusion

Originally, the present study had two goals: to construct projective Fock representations of the classical MF algebra, and to find representations where the cocycle had precisely the MF form (3).

The first goal was easily reached using the formalism developed in [9, 10, 11]. It is clear that much more complicated representations can be written down along the same lines. Geometrically, the oscillators can be viewed as zero-jets, i.e. the value of fields like \( \phi^a(x) \) on the observer’s trajectory \( x^\mu = q^\mu(t) \). One can generalize to \( p \)-jets, with basis \( \partial_{\nu_1} \cdots \partial_{\nu_r} \phi^a(q(t)) \) for all \( r \leq p \). This is a genuine \( N \)-dimensional object which probes not only the value of the fields along the trajectory, but also finitely many transverse derivatives. On the other hand, the presence of the MF term \( H^{\alpha\mu\nu}(m) \) is quite uninteresting, since it can be disentangled using (3). The interesting quantum (normal ordering) effect is the Kac-Moody extension for the \( J J \) bracket in (10).

However, my second goal failed. Indeed, the fact that the true MF algebra (3) can be naturally embedded into the non-Lie algebra (3) is a serious obstruction against the existence of Fock modules. I am convinced that the technique of combining normal ordering with jet space trajectories
can only produce one-chain (Kac-Moody) cocycles. This is a serious problem because this technique has so far been the only viable method to produce concrete Fock modules in more than one dimension. Two conclusions are possible: either the MF algebra lacks Fock modules altogether, or it points to a new type of representation theory which is not yet understood.

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