Combinatorial Parameters on Matchings in Complete and Complete Bipartite Graphs

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Abstract

Take $2m$ labeled points on a circle and join them in disjoint pairs by $m$ chords. The resulting diagram is called a chord diagram.

In our research we study the graph of chord diagrams $\mathcal{C}_m$, a graph that has, as vertices, all chord diagrams, and two diagrams are connected by an edge in the graph if their symmetric difference is a cycle of length four.

We show that the diameter as well as the eccentricity of every vertex in $\mathcal{C}_m$ are $m - 1$. We prove the following surprising result: The number of shortest paths between every two antipodes is exactly $m^{m-2}$. Note, that the expression $m^{m-2}$ appears in the well known Cayley Formula for the number of labeled trees on $m$ points. We then give an explicit formula for the number of shortest paths between any two diagrams in $\mathcal{C}_m$.

We conclude by showing there is exactly one pair of antipodes in the graph of non-crossing diagrams $\mathcal{M}_m$ with $m^{m-2}$ shortest paths between them. All other pairs have a smaller number of shortest paths.

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1 Introduction

1.1 Background

Take $2m$ labeled points on a circle and join them in disjoint pairs by $m$ chords. The resulting diagram is called a chord diagram (see Fig. 1). A classical result is that the number of diagrams, where the chords do not cross, is a Catalan number $\frac{1}{m+1}(\frac{2m}{m})$.

The enumeration of chord diagrams is relevant to other fields, like the combinatorial approach to the Ising model initiated by M. Kac and J. C. Ward [7], the analysis of data structures in computer science [4] [5], or the study of invariants in knot theory [13].

In 1975, J. Riordan [11] gave a remarkable exact formula for the natural extension of this problem to the enumeration of all possible pairings of $2m$ points on a circle by number of crossings of the chords. This result is based on Touchard’s work from 1950-1952 [14] [15] [16]. He stated (without proof) formulas for the mean and variance of this parameter. In 1979 R. C. Read [10] translated this problem into one of counting trees, proving an alternate recursive formula to the one of Riordan, and then used continued fractions to derive Riordan’s formula.

In 2000, Noy and Flajolet [3] proved the formulas suggested for the mean and variance by Riordan and analyzed statistical properties of other combinatorial parameters over chord diagrams, in particular, the number of connected components in the diagram and the size of the largest connected component. A. Khruzin [8] enumerated the number of orbits of chord diagrams under the
natural action of $D_{2m}$ and $C_{2m}$. In 2006, Chen, Deng, Du, Stanley and Yan \[2\] showed that \textit{cross} and \textit{nesting} are equidistributed, over chord diagrams.

Hernando, Hurtado and Noy defined in their article \[6\] the graph of non-crossing perfect matchings. This is the graph with non-crossing chord diagrams as vertices, where two diagrams are connected by an edge in the graph if their symmetric difference is a cycle of length four. They showed that the diameter and the eccentricity of every vertex in this graph are $m - 1$, where $m$ is the number of chords in the diagrams.

In our research we study the graph of chord diagrams, a graph that has, as vertices, all chord diagrams, and two diagrams are connected by an edge in the graph if their symmetric difference is a cycle of length four. Although we do not need the geometric representation of a chord diagram in this work, and could define a diagram as a perfect matching of the complete graph, we keep the terminology consistent with the geometric representation of chord diagrams.

1.2 Main Results

In our research we study the graph of chord diagrams $C_m$. We show that the diameter as well as the eccentricity of every vertex in $C_m$ remain $m - 1$ although it is larger and denser than the graph of of non-crossing chord diagrams $M_m$, which is an induced subgraph.

We then prove a surprising result (see Corollary \[3.22\]):

\textbf{Theorem 1.1} The number of shortest paths between every two antipodes is exactly $m^{m-2}$.

This result does not hold for $M_m$, although antipodes in $M_m$ with exactly $m^{m-2}$ shortest paths between them exist. No analog of this result is known for closely related graphs of matchings.

We then give an explicit formula for the number of shortest paths between any two diagrams in $C_m$. The expression $m^{m-2}$ appears in the well known Cayley Formula for the number of labeled trees on $m$ points. A combinatorial explanation for this equality is desired - for example, an explicit bijection between the set of shortest paths and the set of labeled trees.
2 Preliminaries

A simple graph $G$ is an ordered pair $(V, E)$ comprising a set $V$ of vertices or nodes together with a set $E \subseteq V \times V$ of edges with no loops: \( \forall v \in V ((v, v) \notin E) \) and \( \forall v, u \in V ((v, u) \in E \iff (u, v) \in E) \). If $(v, u) \in E$ then $u$ and $v$ are called adjacent vertices, denoted $v \sim u$.

A subgraph $H$ of a simple graph $G$, is a pair of subsets $V(H) \subseteq V(G)$ (of vertices) and $E(H) \subseteq E(G)$ (of edges) so that $H = (V(H), E(H))$ is a simple graph on its own. $H$ being a subgraph of $G$ is denoted by $H \leq G$.

A subgraph $H$ of a graph $G$ is said to be an induced subgraph if it has all the edges that appear in $G$ over the same vertex set.

A bipartition of a graph $G$ is an ordered pair $(U, V)$ of complementary subsets of $V(G)$ such that each edge of $G$ has one end in $U$ and one end in $V$. A bipartite graph is one that has a bipartition. A complete bipartite graph $K_{m,n}$ is one with bipartition $(U, V)$ so that $|U| = m, |V| = n$ and $E$ consists of all the edges between vertices of $U$ and vertices of $V$.

A matching $M$ in a graph $G$ is a set of vertex-disjoint edges. A maximal matching of a graph is one that no edges can be added to it. A perfect matching is one that covers all vertices of $G$.

The complete graph of order $n$, denoted by $K_n$, is the graph of order $n$ in which all vertices are adjacent.

Notice that two maximal matchings of a graph are not necessarily of the same size, however, all maximal matchings of complete graphs of even order and complete bipartite graphs, $K_{n,n}$, are perfect and all maximal matchings of complete bipartite graphs $K_{m,n}$, $m \leq n$, contain $m$ edges.

The distance between two vertices in a graph $d(u, v)$ is the number of edges in a shortest path between them.

The eccentricity of a vertex is the maximal distance of the vertex from any other vertex in the graph.
The diameter of a graph is the maximal distance between any two vertices in the graph.

A combinatorial parameter (or statistic) on a set $S$ is a function $f : S \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers.

The distribution of a combinatorial parameter $f$ on $S$ is captured by its generating function, the formal sum (polynomial in $q$)

$$\sum_{x \in S} q^{f(x)}.$$

Two combinatorial parameters are equidistributed if they have identical generating functions. The explicit calculation of these generating functions is of interest on its own.

3 The Graph of Chord Diagrams

3.1 Basic Definitions

Definition 3.1 Let $n = 2m$ be an even positive integer. A chord diagram is a perfect matching in the complete graph $K_n$. We call the edges of $K_n$ chords. Denote the set of all chord diagrams with $m$ chords $C_m$.

Observation 3.2 The number of chord diagrams with $m$ chords ($n = 2m$) is the double factorial:

$$|C_m| = (n - 1)!! = (n - 1) \cdot (n - 3) \cdot \ldots \cdot 1$$

Proof The number of ways to choose a match for an arbitrary first vertex is $n - 1$, then a match for another arbitrary vertex is $n - 3$ etc.

Definition 3.3 For any chord $e$ in the complete graph $K_{2m}$ and a chord diagram $M$ in $C_m$ we define the insertion of the chord $e$ into the chord diagram $M$, denoted by: $M * e$, as follows:

If $e$ is already in $M$ then $M * e = M$; otherwise, if $a$ and $c$ are the vertices of $e$ and $(a, b), (c, d)$ are the chords in $M$ incident with $a$ and $c$,

$$M * (a, c) = M - (a, b) - (c, d) + (a, c) + (b, d)$$
And recursively:

\[ M * (e_1, \ldots, e_n) = (M * (e_1, \ldots, e_{n-1})) * e_n \]

![Figure 2: Insertion of a chord](image)

**Observation 3.4** Notice that if the chords \((a, b)\) and \((c, d)\) are in \(M_1\) and \(M_1 * (a, c) = M_2\) then:

- The symmetric difference of \(M_1\) and \(M_2\) is the alternating cycle of length four containing the chords \(\{(a, b), (b, c), (c, d), (d, a)\}\).

- \(M_1 * (a, c) = M_1 * (b, d) = M_2\)
- \(M_2 * (a, b) = M_2 * (c, d) = M_1\) and
- \(M_1 * (a, d) = M_1 * (b, c) = M_2 * (a, d) = M_2 * (b, c)\).

**Definition 3.5** We define the graph of chord diagrams \(C_m\) to be the graph having the set of all chord diagrams \(C_m\) as vertices; two distinct chord diagrams \(M_1\) and \(M_2\) are connected by an edge if \(M_1 * e = M_2\) for some chord \(e\) in \(M_2\).

**Observation 3.6** Given a pair of distinct chords \((a, b), (c, d)\) in a chord diagram \(M\), they determine two chord diagrams: \(M_1\), which is obtained by insertion of \((a, c)\) or \((b, d)\), and \(M_2\), which is obtained by insertion of \((a, d)\) or \((b, c)\). Then

- Any two of \(M, M_1, M_2\) are neighbors in \(C_m\).

- The only neighbors of \(M\) which contain the chords \(M \setminus \{(a, b), (c, d)\}\) are \(M_1\) and \(M_2\).
By Observation 3.6

**Observation 3.7** \( C_m \) is regular, and every chord diagram in \( C_m \) has exactly \( 2\binom{m}{2} = m(m-1) \) neighbors.

**Observation 3.8** Given two chord diagrams \( M_1 \) and \( M_2 \) in \( C_m \) and assuming that \( (e_1, \ldots, e_m) \) are the chords of \( M_2 \) arranged in any order, \( M_2 = M_1 \ast (e_1, \ldots, e_m) \).

**Proof** The chords inserted to \( M_1 \) are vertex disjoint, and therefore they all belong to the final chord diagram \( M_1 \ast (e_1, \ldots, e_m) \).

**Corollary 3.9** The graph of chord diagrams \( C_m \) is connected.

### 3.2 Shortest Path and Diameter

Hernando, Hurtado and Noy [6] proved that the distance between two diagrams in the graph of non-crossing chord diagrams with \( m \) chords is determined by the number of components in the union of the two graphs with the result that the diameter and the eccentricity of every vertex in the graph is \( m - 1 \). In this sub-section we show that, surprisingly, this result extends to the graph of all chord diagrams. In particular, we show that although the general graph of chord diagrams with \( m \) chords is larger and denser than the graph of non-crossing chord diagrams, where the latter is an induced subgraph, the diameter and the eccentricity of every vertex remain \( m - 1 \) (see Corollary 3.16) and the distance between any two vertices of the graph of chord diagrams is determined by the number of components in the union of the two diagrams.

**Fact 3.10** The union of two chord diagrams is a vertex-disjoint union of alternating cycles of even length (a common chord is considered as a cycle of length two):

\[
M_1 \cup M_2 = C_1 \oplus C_2 \oplus \cdots \oplus C_l .
\]

The main result in this sub-section is

**Theorem 3.11** Given \( M_1 \cup M_2 = C_1 \oplus C_2 \oplus \cdots \oplus C_l \) where \( M_1, M_2 \in C_m \),

\[
d(M_1, M_2) = m - l
\]
The proof of this theorem will be given below.

It should be noted that an analog of Theorem 3.11 for the graph of non-crossing chord diagrams appears in Hernando, Hurtado and Noy’s article [6] in the equivalent form: \( d(M_1, M_2) = \frac{1}{2} \sum_{i=1}^{d} (\text{length}(C_i) - 2) \). Our proof simplifies ideas which appear in [6].

**Lemma 3.12** For two adjacent chord diagrams \( M_1 \sim M_2 \) in \( \mathcal{C}_m \) and any chord \( e \), either \( M_1 * e = M_2 * e \) or \( M_1 * e \sim M_2 * e \).

**Proof** If \( e \) is a common chord of \( M_1 \) and \( M_2 \) then insertion of \( e \) into the diagrams, by definition, does not change them and they remain adjacent. Otherwise:

1. Notice that, by assumption, the symmetric difference of \( M_1 \) and \( M_2 \) is a cycle of length four. If both vertices of \( e \) are incident with this cycle then, by Observation 3.4, \( M_1 * e = M_2 * e \).

2. If \( e \) is vertex-disjoint from the symmetric difference of the diagrams then, denoting by \( S_1 \) the two chords of \( M_1 \) in the symmetric difference, \( e \) is incident only with vertices from \( M_1 \setminus S_1 = M_2 \setminus S_2 \), so \( (M_1 \setminus S_1) * e = (M_2 \setminus S_2) * e \) and adding back both \( S_1 \) to \( M_i * e \) will make them adjacent again:
   \[
   M_1 * e = (M_1 \setminus S_1) * e \cup S_1 \sim (M_2 \setminus S_2) * e \cup S_2 = M_2 * e.
   \]

3. If exactly one of the vertices of \( e \) is incident with a chord from the symmetric difference of the diagrams then the other vertex must be incident with a common chord. Now, we can denote the chords in the symmetric difference by \( (a, b), (c, d) \in M_1 \) and \( (a, c), (b, d) \in M_2 \), the inserted chord \( e = (a, v) \) and the common chord containing \( v \) by \( (v, w) \). The symmetric difference of \( M_1 * (a, v) \) and \( M_2 * (a, v) \) is now \( \{(b, w), (w, c), (c, d), (b, d)\} \) and thus \( M_1 * e \sim M_2 * e \).

\[ \blacksquare \]

**Lemma 3.13** For two chord diagrams \( M', M'' \) in \( \mathcal{C}_m \) and a common chord \( e \) that belongs to both, every chord diagram, in every shortest path between the two chord diagrams \( M', M'' \) in \( \mathcal{C}_m \), contains \( e \).
Proof Let $d(M', M'') = d$ and let

$$M' = M_0 \sim M_1 \sim \cdots \sim M_d = M''$$

be a shortest path between $M'$ and $M''$. Suppose that $e$ does not belong to every chord diagram in the path and let $M_i$ be the first diagram in the path that $e$ does not belong to.

Consider now the path

$$M' = M_0 * e \sim M_1 * e \sim \cdots \sim M_d * e = M''$$

This is still a path between $M'$ and $M''$ because, by Theorem 3.12, two adjacent diagrams after insertion of $e$ are either still adjacent or equal. Because $M_{i-1} * e = M_i * e$ it is a shorter path than $d$ in contradiction to the assumption that $d(M', M'') = d$.

Lemma 3.14 Let $(e_1, \ldots, e_m)$ be the chords of $M \in \mathcal{C}_m$ given in any order, then for any $M' \in \mathcal{C}_m$

$$M' * e_1 \simeq M' * (e_1, e_2) \simeq M' * (e_1, e_2, e_3) \simeq \cdots \simeq M' * (e_1, \ldots, e_m) = M$$

is a shortest path between them (where some of the neighborhood symbols can be equalities).

Proof Let $(e_1, \ldots, e_m)$ be the chords of $M \in \mathcal{C}_m$ given in any order, and

$$M' = M_0 \sim M_1 \sim \cdots \sim M_d = M$$

be a shortest path between $M'$ and $M$. We shall find a path of the same length starting with the insertion of $e_1$.

If $e_1$ belongs to $M'$ then $M' = M' * e_1$ and we have the same path

$$M' = M' * e_1 = M_0 \sim M_1 \sim \cdots \sim M_d = M$$

If $e_1$ does not belong to $M'$ then let $M_i$ be the first in the path that contains $e_1$. Notice that because both $M_i$ and $M'$ contain $e_1$ then, by Theorem 3.13 for any $i \leq j \leq d$ the chord diagram $M_j$ contains $e_1$.

Now consider the following path between $M'$ and $M$:

$$M' \sim M_0 * e_1 \sim \cdots \sim M_{i-1} * e_1 = M_i \sim \cdots \sim M_d = M$$
The first equality in the path has turned into an adjacency $M' \sim M_0 \ast e_1$ and, by Theorem 3.12, the adjacency $M_{i-1} \sim M_i$ turned into an equality $M_{i-1} \ast e_1 = M_i \ast e_1$. Thus, the length of the new path, starting with the insertion of $e_1$ is still $d$.

The same argument shows that there is a path of length $d - 1$ from $M' \ast e_1$ to $M$ starting with the insertion of $e_2$. Now, By Observation 3.8, $d < m$ and the repetition of the process proves the claim. 

**Fact 3.15** This is a linear time algorithm (in $m$) to find a shortest path between two diagrams in $C_m$.

Following is the proof of Theorem 3.11: Given $M_1 \cup M_2 = C_1 \oplus C_2 \oplus \cdots \oplus C_l$ where $M_1, M_2 \in C_m$

\[ d(M_1, M_2) = m - l \]

**Proof** We will use induction on the number of chords $m$:

For $m = 2$ it is clear.

Assume that the statement is true for all $t < m$. If $M_1, M_2$ have a common chord $e$, then it is a component, say $C_i$, in $M_1 \cup M_2$. Now $M_1 \setminus e \cup M_2 \setminus e = C_1 \oplus C_2 \oplus \cdots \oplus C_{i-1}$ and, by induction, $d(M_1 \setminus e, M_2 \setminus e) = (m - 1) - (l - 1) = m - l$.

And by Theorem 3.13, $d(M_1 \setminus e, M_2 \setminus e) = d(M_1, M_2)$.

Otherwise, assume that $e$ is a chord in $M_2$ but not in $M_1$. By the proof of Theorem 3.14, there is a shortest path from $M_1$ to $M_2$ starting with the insertion of $e$. The alternating cycle $C_j$ in $M_1 \cup M_2$ that contains $e$ turns in $(M_1 \ast e) \cup M_2$ into two components, the component of the common chord $e$ and another cycle shorter than $C_j$ by two, that is, insertion of $e$ increased the number of components by one, and by induction, $d((M_1 \ast e) \setminus e, M_2 \setminus e) = (m - 1) - l$.

**Corollary 3.16** The eccentricity of every diagram in $C_m$ and the diameter of $C_m$ are $m - 1$ and the number of antipodes of every diagram is $(2m - 2)! = 2^{m-1}(m-1)!$ (the number of ways to choose a diagram which has a symmetric difference of one big cycle).

### 3.3 Number of Shortest Paths

In this sub-section we show that the number of shortest paths between two diagrams with a symmetric difference of one cycle of length $2k$ is $k^{k-2}$. According to Theorem 3.11 the distance between two diagrams depends only
on the number of components in their union. This leads to the following observation:

**Observation 3.17** For $M_1, M_2 \in \mathcal{C}_m$ and $e \notin M_1$, let $M' = M_1 \ast e$. Then exactly one of the following cases holds:

1. $d(M', M_2) = d(M_1, M_2) + 1$. This happens iff the vertices of $e$ belong to two different cycles in the union $M_1 \cup M_2$.

2. $d(M', M_2) = d(M_1, M_2) - 1$. This happens iff the vertices of $e$ belong to the same cycle in the union $M_1 \cup M_2$ and the insertion of $e$ into $M_1$ splits the cycle into two cycles.

3. $d(M', M_2) = d(M_1, M_2)$. This happens iff the vertices of $e$ belong to the same cycle in the union $M_1 \cup M_2$ and the insertion of $e$ into $M_1$ does not split the cycle.

**Definition 3.18** Let $P_{2k}$ $(k \geq 2)$ be the number of shortest paths between two diagrams whose symmetric difference is one cycle of length $2k$. Define $P_2 = 1$.

**Observation 3.19** It follows from Observation 3.17 that $P_{2k}$ depends only on $k$, and not on the total number $m$ of chords in the diagram.

**Proof** Let $M_1, M_2 \in \mathcal{C}_m$ be two diagrams with a symmetric difference of one cycle of length $2k$. By Observation 3.17, an insertion of a chord will decrease the distance between the two diagrams only if the vertices of the chord that is inserted belong to the same cycle in the symmetric difference of $M_1$ and $M_2$. Thus, the number of shortest paths between $M_1$ and $M_2$ does not depend on chords not within the cycle of the symmetric difference.

**Theorem 3.20**

$$P_{2k} = \frac{k}{2} \sum_{i=1}^{k-1} \binom{k-2}{i-1} P_{2i} P_{2k-2i} \ (k \geq 2)$$

and $P_2 = 1$. 
Proof Let $M_1, M_2 \in \mathcal{C}_m$ be two diagrams with a symmetric difference of one cycle of length $2k$, denoted hereafter $C$. Therefore, $M_1 \cup M_2$ has $m - k + 1$ components: $m - k$ of size 2 and one of size $2k$. By Theorem 3.11, $d(M_1, M_2) = k - 1$. By Observation 3.17, in any shortest path from $M_1$ to $M_2$, all inserted chords will be incident with vertices of $C$, and every insertion of a chord must split $C$.

Let $e_1$ and $e_2$ be a pair of distinct chords in $C$ that belong to $M_1$. $M_1$ has two unique neighbors in $\mathcal{C}_m$, denoted $M'$ and $M''$, resulting from insertions of chords incident with vertices of $e_1$ and $e_2$ (see Observation 3.6). The union of $M_2$ with one of the two neighbors, $M_2 \cup M'$, has one more component than $M_1 \cup M_2$ ($C$ is split) and the union with the other, $M_2 \cup M''$, has the same number of components. Therefore, every pair of distinct chords in $C$, that belongs to $M_1$, defines exactly one neighbor $M'$ of $M_1$ in a shortest path to $M_2$: $d(M_1, M') = 1$, $d(M', M_2) = k - 2$. Conversely, each neighbor $M'$ with this property is obtained from a unique choice of $e_1, e_2$.

Let $0 < i < \frac{k}{2}$ and let $e_1, e_2$ be a pair of chords in $C$ that belong to $M_1$ with exactly $i$ chords from $M_2$ between them. Note that there are exactly $k - i$ chords from $M_2$ between $e_1$ and $e_2$ on the other side of $C$ (we shall deal with the case $i = \frac{k}{2}$ later). Denote by $M'$ the neighbor that $e_1$ and $e_2$ define, as above. The symmetric difference of $M'$ and $M_2$ has two cycles, of lengths $2i$ and $2k - 2i$. Therefore, the number of shortest paths from $M_1$ to $M_2$, beginning with $M'$, is \((k-i-1)\binom{k-2}{i} P_{2i} P_{2k-2i}\). The binomial coefficient \((k-i-1)\binom{k-2}{i}\) counts the ways to interlace insertions between the two cycles, the other factor is the number of paths that each such order of insertions defines.

The number of possible pairs of chords $e_1, e_2$ with a given $0 < i < \frac{k}{2}$ is $k$; and they also correspond to $\frac{k}{2} < k - i < k$. For $i = \frac{k}{2}$, there are $\frac{k}{2}$ such pairs. All in all, one gets:

$$P_{2k} = \frac{k}{2} \sum_{i=1}^{k-1} \left( \binom{k-2}{i-1} P_{2i} P_{2k-2i} \right) (k \geq 2).$$
Lemma 3.21 \( P_{2k} \) above also satisfies the following recursion:

\[
P_{2k} = \sum_{i=1}^{k-1} i \binom{k-2}{i-1} P_{2i} P_{2k-2i}
\]

with \( P_2 = 1 \).

Proof Noting the symmetry of the binomial coefficients \( \binom{k-2}{i-1} = \binom{k-2}{k-i-1} \) and denoting \( a_i = \binom{k-2}{i-1} P_{2i} P_{2k-2i} \):

\[
2 \sum_{i=1}^{k-1} a_i = \sum_{i=1}^{k-1} a_i + \sum_{i=1}^{k-1} a_{k-i} = \sum_{i=1}^{k-1} \left( i \binom{k-2}{i-1} P_{2i} P_{2k-2i} + (k - i) \binom{k-2}{k-i-1} P_{2(k-i)} P_{2k-2(k-i)} \right)
\]

\[
= \sum_{i=1}^{k-1} \left( i \binom{k-2}{i-1} P_{2i} P_{2k-2i} + (k - i) \binom{k-2}{i-1} P_{2k-2i} P_{2i} \right)
\]

\[
= k \sum_{i=1}^{k-1} \binom{k-2}{i-1} P_{2i} P_{2k-2i} = 2 P_{2k}
\]

The last equation is essentially Theorem 3.20.

Corollary 3.22 For \( k \geq 1 \),

\[
P_{2k} = k^{k-2}
\]

Proof By the well known Cayley Formula \( \prod \), the number of labeled trees on \( k \) vertices is \( k^{k-2} \). By [9] ex. 6, p. 34 and pp. 249-250] the number \( T_k \) of labeled trees on \( k \) points satisfies \( T_k = \sum_{i=1}^{k-1} i \binom{k-2}{i-1} T_i T_{k-i} \) with \( T_1 = 1 \). Comparison with Lemma 3.21 gives the desired result.

More generally, the number of paths for any two diagrams is as follows:

Corollary 3.23 Let \( M_1, M_2 \in C_m \). By Fact 3.10, \( M_1 \cup M_2 = C_1 \oplus C_2 \oplus \cdots \oplus C_l \) where \( C_i \) are disjoint alternating cycles of even length (a common chord is considered as a cycle of length two).

The number of shortest paths between \( M_1 \) and \( M_2 \) is:

\[
\left( \binom{m-l}{n(C_1), \ldots, n(C_l)} \right) \prod_{i=1}^{l} (n(C_i) + 1)^{n(C_i)-1}
\]

Where \( n(C_i) = \frac{\text{length}(C_i)}{2} - 1 \) is the number of insertions within the cycle \( C_i \).

Proof The binomial coefficient \( \binom{m-l}{n(C_1), \ldots, n(C_l)} \) counts the ways to interlace insertions between the cycles. The other factor is the number of paths each such order of insertions defines.
Corollary 3.24 There is exactly one pair of antipodes in the graph of non-crossing diagrams $\mathcal{M}_m$ with $m^{m-2}$ shortest paths between them. All other pairs have a smaller number of shortest paths.

Proof Let $M_1$ and $M_2$ be two antipodes in $\mathcal{M}_m$. They are also antipodes (at distance $m - 1$) in $\mathcal{C}_m$. Thus, by Theorem 3.11, they have a symmetric difference of one cycle of length $2m$. Denote the convex hull of the $2m$ points of the diagrams in $\mathcal{M}_m$ by $H$.

If $M_1$ has a chord, denoted by $e$, which is contained in the interior of $H$ (except for its two end points), then: The set $H \setminus e$ has two connected components $H_1$ and $H_2$. $M_1$ must have at least one chord $e_1$ contained in $H_1$ and another, $e_2$, contained in $H_2$. By the proof of Theorem 3.20 every pair of chords from $M_1$ defines a distinct neighbor of $M_1$ in a shortest path in $\mathcal{C}_m$ to $M_2$. Let $M' \in \mathcal{C}_m$ be the neighbor of $M_1$ defined by $e_1$ and $e_2$ in a shortest path in $\mathcal{C}_m$ to $M_2$. $M'$ must have two chords intersecting $e$. Therefore, $M' \notin \mathcal{M}_m$ and the number of paths between a diagram in $\mathcal{M}_m$ with a chord contained in the interior of $H$ and any other diagram in $\mathcal{M}_m$, is smaller than $m^{m-2}$.

There are only two diagrams $M_1, M_2$ in $\mathcal{M}_m$ having all of their chords on the boundary of $H$. These two diagrams are antipodes in $\mathcal{M}_m$. Let $M' \in \mathcal{C}_m$ be a neighbor of $M_1$ in a shortest path to $M_2$. By the proof of Theorem 3.20 $M' \cup M_2$ is a union of two vertex-disjoint alternating cycles, in this case, two disjoint convex polytopes (or a convex polytope and a shared edge), therefore, $M'$ is also in $\mathcal{M}_m$. Now, by the proof of Theorem 3.20 similarly to the first insertion, every insertion on a shortest path to $M_2$ will split some convex polytope into two disjoint convex polytopes (or shared edges). Thus, every chord diagram in a shortest path to $M_2$ in $\mathcal{C}_m$ is also in $\mathcal{M}_m$ and the number of shortest paths between $M_1$ and $M_2$ in $\mathcal{M}_m$ is $m^{m-2}$.
Figure 3: Examples of antipodes in $\mathcal{M}_m$

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