Magnetic properties of doped Heisenberg chains

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The magnetic susceptibility of systems from a class of integrable models for doped spin-$S$ Heisenberg chains is calculated in the limit of vanishing magnetic field. For small concentrations $x_h$ of the mobile spin-$(S - 1/2)$ charge carriers we find an explicit expression for the contribution of the gapless mode associated to the magnetic degrees of freedom of these holes to the susceptibility which exhibits a singularity for $x_h \to 0$ for sufficiently large $S$. We prove a sum rule for the contributions of the two gapless magnetic modes in the system to the susceptibility which holds for arbitrary hole concentration. This sum rule complements the one for the low temperature specific heat which has been obtained previously.

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1 Introduction

Essential insights into the properties of low-dimensional correlated electron systems have been gained based on studies of integrable models. Exact results on the low energy spectrum of the one-dimensional Hubbard and $t$–$J$ models have made possible their identification as microscopic realizations of so-called Tomonaga-Luttinger liquids allowing a complete classification of their critical exponents [1–3]. Recently, the $t$–$J$ model – introduced to describe the effect of hole doping on an $S = 1/2$ Heisenberg antiferromagnet – has been generalized to larger values of $S$ [4, 5] giving the framework for detailed studies of quantum effects in the double exchange model. Fine tuning of the coupling constants in these models describing spin-$S$ chains doped with mobile spin-$S' = (S - 1/2)$ carriers (called ‘holes’ below) an exactly solvable model has been constructed [6, 7] which interpolates smoothly between the spin-$S$ (spin-$S'$) Takhtajan-Babujian chain [8–10].

An important open question is the proper identification of effective field theories describing the low energy/low temperature behaviour of these systems [3, 6, 11, 12]. In the limit of low hole concentration the low temperature specific heat indicates that the two gapless magnetic modes found for vanishing magnetic field can be described in terms of a level-2$S$ $SU(2)$ Wess-Zumino-Witten (WZW) model (which is the effective field theory of the undoped Takhtajan-Babujian spin-$S$ chain [8, 9, 10]) and the minimal model $M_{2S+1}$, respectively [6, 7]. Upon variation of the doping a continuous transition to a product $SU(2)_{2S-1} \otimes SU(2)_1$ satisfying a sum rule in the coefficients of the low temperature specific heat is observed [7]. For $S = 1$ a field theoretical description of these degrees of freedom has been proposed based on four Majorana fermions [3] (see also [11, 12]). However, to determine the coupling constants present in this continuum theory additional insight into the physical properties of the massless excitations, in particular their response to an external magnetic field, is necessary.

In this paper we analyze the Bethe Ansatz equations to compute the zero temperature magnetic susceptibility of the integrable doped Heisenberg chains in the limit of vanishing magnetic field $H \to 0$. Below we briefly review the thermodynamic Bethe Ansatz for the model to provide the foundation for the subsequent analysis. In Sect. 3 we concentrate on the limit of small magnetic fields and derive a particular sum rule for the contributions $\chi_1$ and $\chi_{2S}$ of the two gapless magnetic excitations to the susceptibility, similar to the one for the specific heat mentioned above. In Sect. 4 we formulate a matrix Riemann-Hilbert problem (RHP) whose solution determines $\chi_1$ and $\chi_{2S}$ individually. It is shown that the symmetries of this RHP immediately imply the sum rule mentioned above. In Sect. 5 we present the main result of this
paper which is the explicit calculation of $\chi_1$ and $\chi_{2S}$ in the limit of small hole concentration.

The Hamiltonians considered in this paper are of the form

$$\mathcal{H}^{(S)} = \sum_{n=1}^{L} \left\{ T_{n,n+1}^{(S)} + X_{n,n+1}^{(S)} \right\}.$$  \hspace{1cm} (1.1)

Here $T_{ij}^{(S)}$ and $X_{ij}^{(S)}$ describe the hopping of the holes and the (antiferromagnetic) exchange between sites $i$ and $j$ of the lattice, respectively. Similar Hamiltonians arise as effective spin models obtained from the double exchange (DE) model \[13,14\] in the limit of strong ferromagnetic Hund’s rule coupling between the spins of the itinerant electrons and the local moments with spin-$S'$ \[4,5\]. In the basis of the relevant spin multiplets the kinetic part of the Hamiltonian reads

$$T_{ij}^{(S)} = -P_{ij} Q_{S}(S_i \cdot S_j)$$ \hspace{1cm} (1.2)

where $S_i$ is a spin-$S$ ($S'$) operator describing the particle (hole) on site $i$. The operator $P_{ij}$ permutes the states on sites $i$ and $j$ thereby allowing the holes to move and $Q_{S}(x)$ is a polynomial of degree $2S - 1$ leading to a hopping amplitude depending on the total spin $S_T = \frac{1}{2}, \frac{3}{2}, \ldots 2S - \frac{1}{2}$ on sites $i$ and $j$. In the DE model these hopping amplitudes favour the formation of a ferromagnetically ordered state, this feature is not shared by the integrable chains \[8,9\]. In terms of a microscopic model the electronic hopping has to be dressed by bilinears in the localized magnetic moments, similar to bond-charge type interactions considered in generalizations of the Hubbard model.

As a consequence of $SU(2)$ invariance the exchange operators $X_{ij}^{(S)}$ can also be written as a polynomial in $S_i \cdot S_j$, the precise form of these polynomials depends on the local configuration, i.e. particle-particle, hole-hole and particle-hole exchange are different. As mentioned above, the pure limits are the integrable spin-$S$ and $S'$ Takhtajan-Babujian chains determining the two former processes.

### 2 Bethe Ansatz for the doped spin-$S$ chain

Starting from the ferromagnetically polarized eigenstate of the undoped chain we consider excitations obtained by adding $N_h$ holes and – in addition – lowering $N_\downarrow$ spins. The spectrum of these states can be studied using the algebraic Bethe Ansatz. They are parameterized in terms of $N_h$ real numbers $\nu_\alpha$ and $N_h + N_\downarrow$ complex numbers $\lambda_j$ solving the Bethe Ansatz.
equations (BAE) \[3, 4\]

\[
\left( \frac{\lambda_j + iS}{\lambda_j - iS} \right)^L = \prod_{k \neq j}^{N_h + N_i} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \prod_{\alpha=1}^{N_h} \frac{\lambda_j - \nu_\alpha - i \frac{1}{2}}{\lambda_j - \nu_\alpha + i \frac{1}{2}}, \quad j = 1, \ldots, N_h + N_i
\]

\[
1 = \prod_{\alpha=1}^{N_h + N_i} \frac{\nu_\alpha - \lambda_k + i \frac{1}{2}}{\nu_\alpha - \lambda_k - i \frac{1}{2}}, \quad \alpha = 1, \ldots, N_h.
\]

In the thermodynamic limit $L \to \infty$ general solutions of these equations are known to consist of real hole rapidities $\nu_\alpha$ and complex $n$-strings of spin-rapidities $\lambda_j^{n,k} = \lambda_j^{(n)} + i \frac{1}{2} (n + 1 - 2k)$, $k = 1, \ldots, n$ with real centers $\lambda_j^{(n)}$. Considering solutions of (2.1) built from $N_h$ hole rapidities and $M_n \lambda$-strings of length $n$ we can rewrite the BAE in terms of the real variables $\nu_\alpha$ and $\lambda_j^{(n)}$.

Taking the logarithm of (2.1) we obtain

\[
L \theta_{n,2S} \left( \lambda_j^{(n)} \right) = 2\pi J_j^{(n)} + \sum_{m=1}^{M_n} \sum_{k=1}^{\infty} \Xi_{nm} \left( \lambda_j^{(n)} - \lambda_k^{(m)} \right) - \sum_{\alpha=1}^{N_h} \theta_n \left( \lambda_j^{(n)} - \nu_\alpha \right)
\]

\[
0 = 2\pi I_\alpha - \sum_{n=1}^{\infty} \sum_{j=1}^{M_n} \theta_n \left( \nu_\alpha - \lambda_j^{(n)} \right)
\]

(2.2)

where $\theta_n(x) = 2 \arctan(2x/n)$ and

\[
\theta_{nm}(x) = \theta_{m+n-1}(x) + \theta_{m+n-3}(x) + \ldots + \theta_{|m-n|+1}(x),
\]

\[
\Xi_{nm}(x) = \theta_{n+m}(x) + 2\theta_{n+m-2}(x) + \ldots + 2\theta_{|m-n|+2}(x) + (1 - \delta_{nm}) \theta_{|n-m|}(x).
\]

(2.3)

From Eqs. (2.2) the allowed values of the quantum numbers $I_\alpha$ and $J_j^{(n)}$ are found to be ($t_{nm} = \min(n, m) - \delta_{nm}/2$)

\[
|I_\alpha| \leq \frac{1}{2} \sum_{m=1}^{\infty} M_n, \quad |J_j^{(n)}| \leq \frac{1}{2} L \min(n, 2S) - \sum_{m=1}^{\infty} t_{nm} M_m + \frac{1}{2} (N_h - 1).
\]

(2.4)

A given choice of these numbers the solution of (2.2) uniquely determines a particular eigenstate of the system with energy

\[
E = \sum_{n=1}^{\infty} \sum_{j=1}^{M_n} \left\{ \epsilon_n^{(0)} \left( \lambda_j^{(n)} \right) + nH \right\} - \left( \mu + \frac{1}{2} H \right) N_h - LSH
\]

(2.5)

and magnetization

\[
S_{\text{tot}}^z = LS - \sum_{n=1}^{\infty} n M_n + \frac{1}{2} N_h.
\]

(2.6)
In Eq. (2.5) $H$ is the external magnetic field, $\mu$ is the chemical potential for the holes and the “bare energies” of the $\lambda$-strings are $\epsilon_n^{(0)}(x) = -2\pi (A_{n,2S} \ast s)(x)$ with $s(x) = 1/(2 \cosh \pi x)$ and

$$A_{nm}(x) = \frac{1}{2\pi} \Xi'_{nm}(x) + \delta_{nm} \delta(x)$$

(2.7)

($(f \ast g)(x)$ denotes a convolution).

In the thermodynamic limit $L \to \infty$ with $N_h/L$ and $M_n/L$ held fixed one can introduce densities $\rho(x)$ for the hole rapidities and $\sigma_n(x)$ for the $\lambda$-strings of length $n$. The BAE (2.2) turn into linear integral equations for these functions

$$\tilde{\sigma}_n(x) = (A_{n,2S} \ast s)(x) - \sum_m (A_{nm} \ast \sigma_m)(x) + (a_n \ast \rho)(x) ,$$

$$\rho(x) + \tilde{\rho}(x) = \sum_n (a_n \ast \sigma_n)(x) .$$

(2.8)

Here, $2\pi a_n(x) = \theta_n'(x) = 4n/(4x^2 + n^2)$, and $\tilde{\rho}(x), \tilde{\sigma}_n(x)$ are the densities associated with the distribution of vacancies of quantum numbers $I_\alpha$ and $J_j^{(\alpha)}$ in the intervals (2.4), respectively. Similarly, the energy (2.5) and magnetization (2.6) expressed in terms of these densities read

$$\frac{E}{L} = \sum_{n=1}^{\infty} \int dx \epsilon_n^{(0)}(x) \sigma_n(x) - \mu \int dx \rho(x) - H S^z_{\text{tot}} ,$$

(2.9)

and

$$\frac{1}{L} S^z_{\text{tot}} = S - \sum_{n=1}^{\infty} n \int dx \sigma_n(x) + \frac{1}{2} \int dx \rho(x) = \frac{1}{2} \lim_{n \to \infty} \int dx \tilde{\sigma}_n(x) .$$

(2.10)

Analysis of these equations shows that the ground state configuration of the system at temperature $T = 0$ consists of up to three condensates formed by the hole rapidities and $\lambda$-strings of length 1 and 2S only. These “Fermi seas” are characterized by their end points $Q$ for $\rho(x)$ and $\Lambda_j$ for $\sigma_j(x)$, $j = 1, 2S$. In the following we drop the distinction of particle and vacancy densities by identifying $\rho(x)$ with $\rho(x) + \tilde{\rho}(x)$ and similar for the densities $\sigma_n(x)$ of the $\lambda$-strings. This is possible since

$$\rho(x) + \tilde{\rho}(x) \equiv \begin{cases} \rho(x) & \text{for } |x| < Q \\ \tilde{\rho}(x) & \text{for } |x| > Q \end{cases} .$$

(2.11)

Using $M_j = 0$ for $j \neq 1, 2S$ in Eqs. (2.4) and (2.6) one can express the magnetization in terms of $\sigma_{2S}$ only

$$\frac{1}{L} S^z_{\text{tot}} = \int_{\Lambda_{2S}}^{\infty} dx \sigma_{2S}(x) .$$

(2.12)

1The second expression is obtained by integrating Eq. (2.8) over all $x$. 

4
3 Thermodynamic properties in a small magnetic field

For vanishing magnetic field one can show that $\Lambda_j \to \infty$. Hence, in the limit of a small magnetic field $H$ which we are interested in below one has $\Lambda_j \gg Q$ allowing to eliminate the $\sigma_j$ from the integral equation for the density $\rho$ of hole rapidities $[7]$. This results in

$$\rho(x) - \int_{-Q}^{Q} dy R(x-y)\rho(y) = (a_{2S} * s)(x).$$

(3.1)

The kernel of this integral equation is $R(x) = a_2 * (1 + a_2)^{-1}$ and the concentration of the holes is $x_h = N_h/L = \int_{-Q}^{Q} dx \rho(x)$. In this regime the boundaries of integration or “Fermi points” $\pm Q$ are a function of the hole chemical potential $\mu$ alone: they are determined through the condition $\kappa_0(\pm Q) = 0$ for the dressed energy $\kappa(x)$ of the excitations associated with the hole rapidities which in turn is given as solution of a linear integral equation similar to (3.1)

$$\kappa_0(x) - \int_{-Q}^{Q} dy R(x-y)\kappa_0(y) = -\{2\pi a_{2S} * s(x) + \mu\}.$$

(3.2)

Excitations with charge rapidities near $\pm Q$ are massless. The effective low energy theory for the charge mode has been identified as that of a free boson. The velocity of the charge mode can be obtained from the dispersion (3.2)

$$v = \frac{1}{2\pi \rho(Q)} \frac{\partial \kappa_0}{\partial x}\bigg|_{x=Q}.$$

(3.3)

The integral equations for the relevant densities of $\lambda$-strings $\sigma_1$ and $\sigma_{2S}$ can be rewritten as

$$\sigma_1(x) = \int_{-Q}^{Q} dy s(x-y)\rho(y) + \int_{|y|<\Lambda_1} dy K_{11}(x-y)\sigma_1(y) + \int_{|y|>\Lambda_{2S}} dy K_{12}(x-y)\sigma_{2S}(y),$$

$$\sigma_{2S}(x) = s(x) + \int_{|y|>\Lambda_1} dy K_{21}(x-y)\sigma_1(y) + \int_{|y|>\Lambda_{2S}} dy K_{22}(x-y)\sigma_{2S}(y).$$

(3.4)

The kernels of the integral operators are easiest given in terms of their Fourier transforms

$$\hat{K}^{(S)}(\omega) = \frac{1}{2 \cosh \frac{\omega}{2} \sinh(S - \frac{1}{2})\omega} \begin{pmatrix} \sinh(S - 1)\omega & \sinh\frac{1}{2}\omega \\ \sinh\frac{1}{2}\omega & \sinh(S - 1)\omega + e^{-|y|/2} \sinh(S - \frac{1}{2})\omega \end{pmatrix}.$$

(3.5)

Note that for any given positive $x$ the contributions from the intervals $y < -\Lambda_j$ in Eqs. (3.4) can be neglected in the limit $H \to 0$. After replacing $\int_{|y|>\Lambda_j}$ by $\int_{\Lambda_j}^{\infty}$ these equations are a system of Wiener-Hopf (WH) integral equations. Furthermore, the replacement of the driving terms by the large-$x$ asymptotics will not change the behaviour of the solutions of these equations near
the Fermi points \( x \approx \Lambda_j \gg 1 \) which contains all the relevant information for the low energy properties of the system. Hence we arrive at the following set of equations for the densities of the \( \lambda \)-strings valid for \( x \gg 1 \)

\[
\sigma_1(x) = B e^{-\pi|x|} + \int_{\Lambda_1}^{\infty} dy \, K_{11}(x-y)\sigma_1(y) + \int_{\Lambda_2}^{\infty} dy \, K_{12}(x-y)\sigma_2S(y),
\]

\[
\sigma_{2S}(x) = e^{-\pi|x|} + \int_{\Lambda_1}^{\infty} dy \, K_{21}(x-y)\sigma_1(y) + \int_{\Lambda_2}^{\infty} dy \, K_{22}(x-y)\sigma_2S(y),
\]

(3.6)

with \( B = \int_{-Q}^{Q} dx \, \exp(\pi x)\rho(x) \). In the same spirit we obtain a system of WH equations for the dressed energies of the magnetic excitations

\[
\epsilon_1(x) = -2\pi A e^{-\pi|x|} + \int_{\Lambda_1}^{\infty} dy \, K_{11}(x-y)\epsilon_1(y) + \int_{\Lambda_2}^{\infty} dy \, K_{12}(x-y)\epsilon_2S(y),
\]

\[
\epsilon_{2S}(x) = \frac{1}{2} H - 2\pi e^{-\pi|x|} + \int_{\Lambda_1}^{\infty} dy \, K_{21}(x-y)\epsilon_1(y) + \int_{\Lambda_2}^{\infty} dy \, K_{22}(x-y)\epsilon_2S(y),
\]

(3.7)

with \( 2\pi A = -\int_{-Q}^{Q} dy \, \exp(\pi x)\kappa_0(y) \). The solution of (3.7) determines the values of \( \Lambda_j \) as a function of the external magnetic field \( H \) by the condition \( \epsilon_j(\Lambda_j) = 0 \).

Both magnetic modes \( \epsilon_1(x) \) and \( \epsilon_{2S}(x) \) allow for massless excitations near these Fermi points, in the limit \( H \to 0 \) their velocities are

\[
\nu_{2S} = \lim_{x \to \infty} \frac{\epsilon_{2S}(x)}{2\pi \sigma_{2S}(x)} \equiv \pi, \quad \nu_1 = \lim_{x \to \infty} \frac{\epsilon_1(x)}{2\pi \sigma_1(x)} = \pi \frac{A}{B}.
\]

(3.8)

Their critical properties have been studied using the thermodynamic Bethe Ansatz. In the limiting cases of the undoped (completely doped) model where the model reduces to the integrable spin-\( S \) (spin-(\( S - 1/2 \)) Takhtajan-Babujian spin chains, respectively, \( \epsilon_{2S}(x) \) reduces to the massless spinon mode of these models which can be described by a \( SU(2) \) level-\( 2S \) (level-(\( 2S - 1 \)) WZW model. For arbitrary doping the contribution of the magnetic modes to the low temperature specific heat has been found to be

\[
C \simeq C_{2S} + C_1 \equiv \frac{\pi T}{3} \left( \frac{c_{2S}}{\nu_{2S}} + \frac{c_1}{\nu_1} \right)
\]

(3.9)

where the parameters \( c_j \) satisfy the “sum rule”

\[
c_{2S} + c_1 \equiv 2 \frac{4S - 1}{2S + 1}
\]

(3.10)

independent of the doping, i.e. the values of the parameters \( A, B \) in Eqs. (3.6) and (3.7). In the limiting cases mentioned above \( c_{2S} \) becomes the conformal central charge of the \( SU(2)_{2S} \)
and the $SU(2)_{2S-1}$ WZW models, respectively. From the other parameter, $c_1$, the low energy theory for $\epsilon_1$ is identified as the minimal unitary model $M_{2S+1}$ in the limit of vanishing hole concentration ($x_h \to 0$) and a free boson with $c_1 = 1$ and compactification radius $R = 1/\sqrt{2}$ as $x_h \to 1$\footnote{Note that these arguments are easily extended to the case of non-zero temperatures.}. The latter is equivalent to the level-1 $SU(2)$ WZW model.

The magnetic field dependent part of ground state energy of the system for small $H$ expressed in terms of the solutions of (3.7) is

$$e_s \equiv \frac{1}{L} E = -2\pi \left( \frac{1}{v_{2S}} \int_{\Lambda_{2S}}^{\infty} \! dx \ e^{-\pi x} \epsilon_{2S}(x) + \frac{1}{v_1} \int_{\Lambda_1}^{\infty} \! dx \ A e^{-\pi x} \epsilon_1(x) \right) = -\frac{1}{2} \chi H^2. \quad (3.11)$$

This implies, that the zero field magnetic susceptibility $\chi$ will have contributions from both massless magnetic modes – very similar to the feature found for the specific heat (3.9)

$$\chi = \chi_{2S} + \chi_1 \quad (3.12)$$

A sum rule for the contributions $\chi_j$ can be obtained by considering formally the case $A = B$ which implies $v_1 = v_{2S} = \pi$: then the solutions of the integral equations (3.6) and (3.7) are related through

$$\sigma_j(x) = \frac{1}{2\pi^2} \frac{\partial \epsilon_j(x)}{\partial x}. \quad (3.13)$$

Using this relation with the asymptotic behaviour $\lim_{x \to \infty} \epsilon_{2S}(x) = 2S H$ in (2.12) one immediately finds\footnote{Note that these arguments are easily extended to the case of non-zero temperatures.}

$$v_{2S} \chi_{2S} + v_1 \chi_1 = \frac{S}{\pi}. \quad (3.14)$$

From the known susceptibilities for the limiting cases of the spin-$S$ and spin-$(S-1/2)$ Takhtajan-Babujian spin \cite{10} chains we expect that $(v_{2S} \chi_{2S})$ decreases continuously from $S/\pi$ for vanishing hole concentration $x_h$ to $(S - 1/2)/\pi$ for $x_h = 1$. This implies $v_1 \chi_1 = 0$ for $x_h = 0$. However, since $v_1$ vanishes in this limit as well, these simple considerations do not rule out a finite or even singular contribution $\chi_1$ to the susceptibility (3.12).

## 4 Calculation of the susceptibility

To calculate $\chi_j$ we have to analyze the Wiener-Hopf equations (3.7) for the dressed energies in the limit $H \to 0$. In terms of the functions $f_j(x) = \epsilon_j(\Lambda_j + x)$ the WH equations can be
rewritten as \( t \equiv \Lambda_2 S - \Lambda_1 \)

\[
f_1(x) = -2\pi A e^{-x|\Lambda_1 + x|} + \int_0^\infty dy K_{11}(x-y)f_1(y) + \int_0^\infty dy K_{12}(x-y-t)f_{2S}(y),
\]
\[
f_{2S}(x) = \frac{1}{2} H - 2\pi e^{-\pi|\Lambda_2 S + x|} + \int_0^\infty dy K_{21}(x-y+t)f_1(y) + \int_0^\infty dy K_{22}(x-y)f_{2S}(y).
\]

Again, the \( \Lambda_j \) have to be determined such that \( f_j(0) = 0 \). By Fourier transformation of (4.1) we obtain

\[
G^T(\omega)\hat{F}_+(\omega) - \hat{F}_-(\omega) = T(\omega).
\]

Here, \( \hat{F}_\pm \) are the two-component vectors

\[
\hat{F}_\pm(\omega) = \pm \begin{pmatrix}
\int_0^\infty dx e^{\pm i\omega x} f_1(x) \\
\int_0^\infty dx e^{\pm i\omega x} f_2(x)
\end{pmatrix}
\]

of analytical functions of \( \omega \) in the upper (lower) half-planes, respectively. \( G(\omega) \) is a matrix given by

\[
G(\omega) = U^{-1}(\omega) \left( I - \hat{K}^{(S)}(\omega) \right) U(\omega), \quad \det G(\omega) = (1 + e^{-|\omega|})^{-2} \frac{(1 - e^{-|\omega|})}{(1 - e^{-(2S-1)|\omega|})}
\]

(here \( U(\omega) = \text{diag}(\exp(-i\omega \Lambda_1), \exp(-i\omega \Lambda_2 S)) \) and \( I \) is the \( 2 \times 2 \) unit matrix) and \( T(\omega) \) is the vector

\[
T(\omega) = \pi H \delta(\omega) \begin{pmatrix}
0 \\
1
\end{pmatrix} - \frac{4\pi^2}{\omega^2 + \pi^2} \Omega(\omega), \quad \Omega(\omega) = U(\omega) \begin{pmatrix}
A \\
1
\end{pmatrix}.
\]

Now the solution of the equation (4.2) can be given in terms of the one of the regular matrix Riemann–Hilbert problem (RHP):

\[
Z(\omega) \to I, \quad \omega \to \infty,
\]
\[
Z(\omega) \quad \text{is analytical for} \quad \omega \notin \mathbb{R},
\]
\[
Z_- (\omega) = Z_+(\omega) G(\omega), \quad \omega \in \mathbb{R}.
\]

Here and below the subscripts \( \pm \) denote the limit values of functions from the left (right) of the contour.

If the solution of the RHP (4.6) is found, then Eq. (4.2) can be written in the form

\[
\left[ Z_+^T(\omega) \right]^{-1} \hat{F}_+(\omega) - \left[ Z_-^T(\omega) \right]^{-1} \hat{F}_-(\omega) = \left[ Z_-(\omega) \right]^{-1} T(\omega),
\]
which is evidently solved by
\[ \hat{F}_+ (\omega) = \frac{1}{2\pi i} Z_+^T (\omega) \int_{-\infty}^{\infty} \frac{Z_+^T (u)}{u - \omega_+} \, du. \] (4.8)

Now we can substitute here the explicit form (4.5) of \( T (\omega) \). The integration of \( \delta \)-function is trivial. For the second term one can shift the integration contour in (4.8) to the lower half-plane. Then the only singularity of the integrand is simple pole at the point \( u = -i\pi \) and we obtain
\[ \hat{F}_+ (\omega) = \frac{iH}{2\omega_+} Z_+^T (\omega) \left[ Z_+^T (0) \right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{2\pi i}{\omega + i\pi} Z_+^T (\omega) \left[ Z_+^T (-i\pi) \right]^{-1} \Omega (-i\pi). \] (4.9)

The condition \( f_j (x = 0) = 0 \) implies \(-i \lim_{\omega \to \infty} \omega \hat{F}_+ (\omega) = 0\). Hence the boundaries of integration \( \Lambda_j \) are given as a function of the magnetic field \( H \) and the parameter \( A \) by
\[ \Omega (-i\pi) = \frac{H}{4\pi} \left[ Z_+^T (-i\pi) \right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (4.10)

Similarly, the ground state energy (3.11) is now
\[ e_s = -2\pi \left( \frac{e^{-\pi A_2 S}}{v_2 S} \int_0^\infty e^{-\pi x} f_2 S (x) + \frac{A e^{-\pi A_1}}{v_1} \int_0^\infty e^{-\pi x} f_1 (x) \right) = -2\pi \Omega (-i\pi)^T V^{-1} \hat{F}_+ (i\pi), \] (4.11)

\((V = \text{diag}(v_1, v_2 S))\) is a diagonal matrix containing the Fermi velocities (3.8) of the magnetic modes. Using (4.10) in (4.9) we can express \( F_+ (i\pi) \) through the solution of the RHP
\[ \hat{F}_+ (i\pi) = \frac{H}{4\pi} Z_+^T (i\pi) \left[ Z_+^T (0) \right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (4.12)

and finally have the magnetic susceptibility in terms of \( Z \) as
\[ \chi = \frac{1}{4\pi} \left\{ \left[ Z_- (0) \right]^{-1} Z_- (-i\pi) V^{-1} Z_+^T (i\pi) \left[ Z_+^T (0) \right]^{-1} \right\}_{22}. \] (4.13)

The symmetries the matrix \( G (\omega) \) allow to deduce some general properties of the solution \( Z \) of the matrix Riemann–Hilbert problem (4.3). As a consequence of the identity \( G^T (-\omega) = G (\omega) \) one has \( Z_+ (\omega) Z_+^T (-\omega) = I \). This allows to rewrite the expression (4.13) in terms of \( Z_+ (0) \) and \( Z_+ (i\pi) \) alone
\[ \chi = \frac{1}{4\pi} \left\{ G^{-1} (0) \left[ Z_+ (0) \right]^{-1} Z_+^T (i\pi) V^{-1} \left[ Z_+ (0) \right]^{-1} Z_+^T (i\pi) \right\}_{22}. \] (4.14)
Eqs. (4.4) and (3.3) yield
\[ G(0) = \frac{1}{2(2s-1)} \begin{pmatrix} 2s & -1 \\ -1 & 1 \end{pmatrix}, \quad G^{-1}(0) = 2 \begin{pmatrix} 1 & 1 \\ 1 & 2s \end{pmatrix}. \] (4.15)

Replacing \( V \) by a unit matrix this immediately reproduces the susceptibility sum rule (3.14). In Figure 1 we present numerical results from the integration of Eqs. (4.1) for \( S = 1 \) and \( S = 3 \), respectively. In the limiting cases \( x_h \to 0 \) and \( x_h \to 1 \) corresponding to undoped spin-\( S \) and -(\( S - 1/2 \)) chains, the velocity \( v_1 \) vanishes. The singular behaviour of the susceptibility near \( x_h = 1 \) is a consequence of the contribution \( \chi_1 \sim 1/(2\pi v_1) \) to the susceptibility.

5 Susceptibility at low doping

Interestingly, however, the dependence of the susceptibility on the hole concentration is found to depend strongly on the value of \( S \) for \( x_h \to 0 \): for all values of \( S \) the velocity \( v_1 \) vanishes with \( x_h \). In fact, since \( x_h \ll 1 \) corresponds to small \( Q \) in the integral equations (3.1) and (3.2) for the densities and dressed energies of the charge excitations, the latter can be solved by iteration giving \( (\psi(x) \text{ and } \psi^{(2)}(x) \text{ are the digamma and polygamma functions, respectively}) \)
\[ A = \frac{\pi^2}{12} \frac{\psi^{(2)} \left( \frac{2S+3}{4} \right) - \psi^{(2)} \left( \frac{2S+1}{4} \right)}{\psi \left( \frac{2S+3}{4} \right) - \psi \left( \frac{2S+1}{4} \right)} x_h^3 \] (5.1)

and \( v_1 = \pi A/x_h \propto x_h^2 \). As discussed above, we expect \( v_1 \chi_1 \to 0 \) in the limit \( x_h \to 0 \) which agrees with numerical findings. The contribution \( \chi_1 \) of the ‘minimal’ mode \( \epsilon_1(x) \) alone to the susceptibility, however, is singular for sufficiently large value of \( S \).

In general, however, we have \( v_1 \neq v_{2S} \) and to find the susceptibility we need to solve matrix RHP (4.6). As in the studies of the low temperature thermodynamics [3, 4] we make use of the fact that we have \( A \ll 1 \) for small hole concentration \( x_h \): for small \( H \) one finds one has \( 0 \ll \Lambda_1 \ll \Lambda_{2S} \) in this regime. Thus, the jump matrix \( G(\omega) \) (4.4) contains the large parameter \( t \to +\infty \). This allows to find the solution of the RHP (4.6) asymptotically and thus to determine the susceptibility as a function of \( A \). Since the treatment for \( S = 1 \) and \( S > 1 \) differs in we present the main results of this asymptotic analysis separately below. Further technical details are presented in the Appendix.
5.1 $S = 1$

To solve the RHP (4.6) we factorize its solution into a product of two matrices

$$Z(\omega) = \Phi(\omega) \begin{pmatrix} 1 & 0 \\ 0 & \alpha(\omega) \end{pmatrix}. \quad (5.2)$$

Here the function $\alpha(\omega)$ is analytical for $\omega \not\in \mathbb{R}$, $\alpha(\omega) \to 1$ as $\omega \to \infty$, and solves the regular scalar RHP on the real axis:

$$\alpha_-(\omega) = \alpha_+(\omega) \det G(\omega) = \alpha_+(\omega) \left(1 + e^{-|\omega|}\right)^{-2} \text{ for } \omega \in \mathbb{R}, \quad (5.3)$$

It is easy to see that

$$\alpha_+(\omega) = \frac{2\pi}{\Gamma^2 \left(\frac{1}{2} - \frac{\omega}{2\pi}\right)} \left(-\frac{i\omega}{2\pi}\right)^{-\frac{\omega}{\pi}}, \quad (5.4)$$

and $\alpha_-(\omega) = \alpha_+^{-1}(-\omega)$. In particular, we have $\alpha_+(0) = 2$ and $\alpha_+(i\pi) = \pi/e$.

From (4.6) we find that the matrix $\Phi(\omega)$ in (5.2) is analytical for $\omega \not\in \mathbb{R}$, it approaches identity at $\omega \to \infty$ and its boundary values on the real axis are related by

$$\Phi_-(\omega) = \Phi_+(\omega)G_\Phi(\omega), \quad \omega \in \mathbb{R}. \quad (5.5)$$

The matrix

$$G_\Phi(\omega) = \begin{pmatrix} 1 & -\frac{\alpha_+(\omega)}{2 \cosh \frac{\omega}{2}} e^{i\omega t} \\ -\frac{\alpha_+(\omega)}{2 \cosh \frac{\omega}{2}} e^{i\omega t} & 1 + e^{-|\omega|} \end{pmatrix} \quad (5.6)$$

can be factorized into the product $G_\Phi(\omega) = M_+(\omega)M_-(\omega)$ with ($\sigma_\pm$ are Pauli matrices)

$$M_+(\omega) = I - \frac{\alpha_+(\omega)}{2 \cosh \frac{\omega}{2}} e^{i\omega t} \sigma_-, \quad M_-(\omega) = I - \frac{\alpha_+(\omega)}{2 \cosh \frac{\omega}{2}} e^{-i\omega t} \sigma_+. \quad (5.7)$$

Obviously $M_\pm(\pm \omega)$ are analytical in the strip $0 < \Im \omega < \pi$. In particular $M_\pm(\omega)$ can be analytically continued into the strip $0 < \pm \Im \omega < a$ for some $a \in (0, \pi)$ where the off-diagonal entries of $M_\pm$ vanish in the limit $t \to \infty$. Following the asymptotic analysis of a similar problem in Ref. [15] we introduce a deformation of the RHP (5.3) which can be solved in terms of an absolutely convergent series in $\exp(-\pi t)$. In Appendix A.1 the asymptotic behaviour of the matrix $\Phi(\omega)$ in the points $\omega = i\pi$, 0 is computed from this deformed RHP ($p$ and $q$ are functions of $t$ defined in Eq. (A.17))

$$\Phi_+(i\pi) = \begin{pmatrix} 1 + p^2(q + 1) & p \\ qp & 1 - p^2 \end{pmatrix} + o(p^2), \quad (5.8)$$
\[ \Phi_+(0) = \begin{pmatrix} 1 + 2p - 2p^2 & 2p \\ 1 - 2p - 2p^2 & 1 - 2p^2 \end{pmatrix} + o(p^2). \]  

Using Eqs. (4.10), (4.14) we obtain \( A = 2\pi p^2 + o(p^2) \) which finally gives the zero field limit of the magnetic susceptibility for small doping (or, equivalently, \( A \ll 1 \))

\[ \chi = \frac{1}{\pi} \left( \frac{1 - \xi}{v_2} + \frac{\xi}{v_1} \right), \]

where

\[ \xi = -\frac{A}{2\pi} \left( \log \frac{A}{2\pi} + 2C \right) + o(A). \]  

This result refines the estimate given in Ref. [6]. In terms of the hole concentration the contribution \( \chi_1 \) to the magnetic susceptibility is

\[ \chi_1 = \frac{3x_h}{2\pi^3} \left( \log x_h + 0.217927 \ldots \right) \]  

which is in excellent agreement with the numerical data in Figure 1(a).

### 5.2 \( S > 1 \)

Similar to the case \( S = 1 \) the solution of the corresponding RHP is given by the product of two matrices

\[ Z(\omega) = \Phi(\omega) \begin{pmatrix} \beta_1(\omega) & 0 \\ 0 & \beta_2(\omega) \end{pmatrix}, \]

where \( \beta_j(\omega) \) solve scalar regular RHPs with canonical normalization condition and

\[ \beta_{1-}(\omega) = \beta_{1+}(\omega) G_{11}(\omega), \quad \omega \in \mathbb{R}, \]

\[ \beta_{2-}(\omega) = \beta_{2+}(\omega) \frac{\det G(\omega)}{G_{11}(\omega)}, \quad \omega \in \mathbb{R}. \]  

Below, we will need only the ratio \( \beta_2(\omega)/\beta_1(\omega) \equiv \alpha^{(S)}(\omega) \), which is equal to

\[ \alpha^{(S)}_+(\omega) = \frac{(2S - 1)^{i\pi}(2S - 1)^{i\pi}}{(2S)^{i\pi}} \frac{\Gamma(-\frac{i\omega}{2\pi}) \Gamma(-\frac{i\omega}{2\pi}(2S - 1))}{\Gamma^2(-\frac{i\omega}{2\pi})} \left( -\frac{i\omega}{2\pi e} \right)^{-\frac{i\omega S}{\pi}}. \]  

As before \( \alpha^{(S)}_+(\omega)\alpha^{(S)}_-(\omega) = 1 \) and in particular

\[ \alpha^{(S)}_+(0) = 2S\eta, \quad \alpha^{(S)}_+\left(\frac{i\pi}{S}\right) = \frac{\pi}{e \sin \frac{\pi}{2S}} \eta^{\frac{S-1}{S}}, \quad \eta = (2S - 1)^{-1/2}. \]
The asymptotic analysis of the RHP for $\Phi(\omega)$ is again based on the factorization of the corresponding jump-matrix $G_{\Phi}(\omega)$ and subsequent deformation of the original jump-contour (see Appendix A.2 for details). The matrices $\Phi_+(0)$ and $\Phi_+(i\pi)$ needed for the calculation of the susceptibility are

$$\Phi_+(0) = \begin{pmatrix} 1 + u_1p\eta - u_1^2p^2/2 & u_1p \\ (1 - u_1^2p^2/2)\eta - u_1p & 1 - u_1^2p^2/2 \end{pmatrix} + o(p^2),$$

and

$$\Phi_+(i\pi) = \begin{pmatrix} 1 + \frac{u_1^2p^2}{2(S-1)} & \frac{u_1p}{S+1} - \frac{u_2p^2}{S+2} \\ \frac{u_1p}{S-1} - \frac{u_2p^2}{S-2} & 1 - \frac{u_1^2p^2}{2(S+1)} \end{pmatrix} + o(p^2).$$

Here we have used the following notations

$$u_k = \frac{1}{\pi} \alpha_+^{(S)}(i\pi k) \frac{\sin(\pi k)}{S}, \quad p = e^{-\frac{\pi^2}{S^2}}.$$  

In the case $S = 2$ the entry $\Phi_{21+}(i\pi)$ should be replaced by

$$\Phi_{21+}(i\pi) = u_1p - \pi u_2p^2 \left( t - i \frac{\partial}{\partial z} \log \alpha_+^{(S)}(z) \bigg|_{z=\pi} \right) + o(p^2), \quad S = 2.$$  

This difference, however, does not affect the final results for $A$ which is

$$A = \frac{S\alpha_+^{(S)}(i\pi)}{(S+1)e} \left( \frac{\xi}{v_2} \right)^{S+1} p^{S+1} + o(p^{S+1}).$$

For susceptibility of the spin-$S$ model at small hole concentration we obtain

$$\chi = \frac{S}{\pi} \left( \frac{1-\xi}{v_2} + \frac{\xi}{v_1} \right),$$

with

$$\xi = \frac{S^2}{S^2 - 1} (u_1p)^2 + o(p^2).$$

Substituting here $u_1$ from (5.19) and $p$ from (5.21) we finally obtain

$$\xi = \frac{1}{S-1} \left( \frac{\Gamma^2(S)}{\Gamma(1/2)\Gamma(S-1/2)} \right)^{\frac{2}{S+1}} \left( \frac{2S-1}{4S^2(S+1)} \right)^{\frac{2}{S+1}} A^{\frac{2}{S+1}} + o(A^{\frac{2}{S+1}}).$$

As a function of the hole concentration the contribution of the minimal mode $\epsilon_1$ to the susceptibility is $\chi_1 \propto x_h^{\frac{2S+1}{S+1}}$, which will give a singularity for $S > 2$. For $S = 3$, the case depicted in Fig. 1(b), $\chi_1$ diverges as $1/\sqrt{x_h}$ for small $x_h$. 

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6 Summary and Conclusions

In this paper we have studied the magnetic properties of a class of integrable models for doped spin-$S$ Heisenberg chains in the limit of very small magnetic field. Of particular interest was the singular behaviour of the susceptibility as a function of the hole concentration for small $x_h$. The response of the two magnetic modes present in these systems to an external field is essential for the proper identification of an effective field theoretical description of the system which in turn will allow for analytical investigations of the properties of the metallic phases of quasi-one dimensional transition metal oxides. The susceptibility due to the ‘background mode’ $\epsilon_{2S}(x)$, which interpolates smoothly between the spinons of the spin-$S$ and $-(S-1/2)$ Takhtajan-Babujian models when the hole concentration is varied between $x_h = 0$ and $x_h = 1$, decreases monotonically from $S/\pi$ to $(S-1/2)/\pi$ as a function of $x_h$. The contribution $\chi_1$ of the second magnetic mode $\epsilon_1(x)$ can be singular near $x_h = 0, 1$ as a consequence of the vanishing of the corresponding Fermi velocity. For $x_h \to 1$, where few spin-$S$ particles propagate in a background of the spin-$(S-1/2)$ holes this singularity may be understood in a similar way as for an underscreened Kondo impurity (although the net impurity moment vanishes in the ground state). For small doping the effect of the doping on the susceptibility is more subtle. Our analysis shows that $\lim_{x_h \to 0} \chi_1$ is non-zero only for $S \geq 2$. The effect of the (non-universal) velocities on the singular behaviour of the susceptibility (3.12) and the specific heat (3.9) can be eliminated by introducing a “Wilson Ratio” for the relative contributions of the two magnetic modes

$$R_W = \frac{\chi_1/\chi_{2S}}{C_1/C_{2S}}. \quad (6.1)$$

With this definition the contribution from $\epsilon_{2S}$ is interpreted as that of the host while associating the mode $\epsilon_1$ with the impurities which is certainly justified in the limits $x_h \to 0, 1$. In these limits we find

$$R_W \approx \begin{cases} \frac{3S(2S+1)}{(S+2)(2S-1)} \xi & x_h \to 0, \\ \frac{3}{2S+1} & x_h \to 1 \end{cases}, \quad (6.2)$$

where according to (5.11) and (5.24) $\xi$ vanishes as $x_h^3 \log x_h$ for $S = 1$ and as $x_h^{S+1}$ for $S > 1$. Extending the interpretation of the two magnetic modes to arbitrary hole concentration one finds a continuous change of $R_W$ with $x_h$ (see Figure 3): as was to be expected from the low temperature specific heat (3.9) the ratio defined in (6.1) is not universal.

Note that our results are the leading contributions to the susceptibility for small magnetic field. Due to approximations such as that of Eqs. (3.4) by (3.6) additional terms $\propto 1/\log H$
to the susceptibility have been neglected. For the undoped systems the precise form of these contributions (which should coincide with the ones present in \(\lim_{x_2 \to 0} \chi_{2S} \)) is given in Ref. [10].

Finally, we mention that the solution of the matrix Riemann-Hilbert problem (4.3) presented in this paper opens the possibility to study the critical exponents in the asymptotics of correlation functions in the \(H = 0\) phase of the doped Heisenberg chains. In the case of a simple Tomonaga-Luttinger liquid the spin part of correlation functions is completely determined by the \(SU(2)\) symmetry and only the compactification radius of the boson related to the charge mode gives rise to anomalous exponents [1]. In the doped spin-\(S\) chains considered here one should expect an additional doping dependence of the critical exponents due to the presence of two magnetic modes: in spite of the \(SU(2)\) invariance of the system a subtle balance between these modes is preserved which is manifest in the sum rules for the low temperature specific heat (3.10) and the zero field magnetic susceptibility (3.14). Further investigations are also necessary to elucidate the relation of these quantities in the finite field phase and the one with vanishing magnetic field.

**Acknowledgements**

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A. Asymptotic analysis of the RHP for large $t$

A.1 $S = 1$

Using the factorization of the jump matrix $G_\Phi(\omega)$ in Eq. (5.5) we follow Ref. [13] by defining a new matrix $U(\omega)$ as

$$U(\omega) = \begin{cases} 
\Phi(\omega), & |\Im \omega| > a, \\
\Phi(\omega)M_+(\omega), & 0 < \Im \omega < a, \\
\Phi(\omega)M_-^1(\omega), & -a < \Im \omega < 0,
\end{cases}$$

(A.1)

where $a \in (0, \pi)$ (see Fig. 2). Then $U(\omega)$ solves the following regular RHP:

$$U(\omega) \to I, \quad \omega \to \infty,$$

$$U(\omega) \text{ is analytical for } \omega \notin \Gamma,$$

$$U_-(\omega) = U_+(\omega)G_U(\omega), \quad \omega \in \Gamma,$$

where we have introduced

$$G_U(\omega) = \begin{cases} 
M_+(\omega), & \omega \in \Gamma_+ \\
M_-^{-1}(\omega), & \omega \in \Gamma_-
\end{cases}.$$ (A.3)

The contour $\Gamma = \Gamma_+ \cup \Gamma_-$ is shown in Fig. 2 (the arrows show positive direction). It is easy to see that $U(\omega)$ has no cut on the real axis. Thus $U_-(\omega)$ can be analytically continued inside the strip $|\Im \omega| < a$. Similarly, $U_+(\omega)$ is analytical in the remaining domain $|\Im \omega| > a$, where it coincides with the matrix $\Phi(\omega)$ (more precisely $U_+(\omega) = \Phi_+(\omega)$ for $\Im \omega > a$, but $U_+(\omega) = \Phi_-(\omega)$ for $\Im \omega < -a$).

From the explicit expression (5.7) we have $M_\pm - I \sim e^{-at}$ on the contour $\Gamma_\pm$. This implies

$$U(\omega) = I + o(e^{-at}), \quad t \to \infty,$$

(A.4)

for the solution of (A.2) where $a$ is arbitrary from the interval $(0, \pi)$. Thus, for $t \to \infty$ the asymptotic behaviour of the solution to (5.5) is

$$\Phi(\omega) = I + O(e^{-at}), \quad |\Im \omega| > \pi,$$

$$\Phi(\omega) = I + O(e^{-\omega_0 t}), \quad 0 < |\Im \omega| = \omega_0 \leq \pi.$$ (A.5)

This accuracy, however, is not sufficient to calculate the leading doping dependence of the susceptibility. In particular this estimate can not be used to determine $\Phi(0)$ entering Eqs. (4.10) and (4.14). To obtain the subleading contributions we can use singular integral equations.
equivalent to the RHP \( A.2 \). Note that only \( \Phi_+ (i\pi) = U_+(i\pi) \) and \( \Phi_+ (0) \) are needed for the susceptibility.

From (5.6) we find \( \det G_\Phi (\omega) = 1 \) and \( G_\Phi (\omega) = G_\Phi^T (-\omega) \), hence we conclude that

\[
\det \Phi (\omega) = 1, \quad \Phi_+ (\omega) \Phi_+^T (-\omega) = I.
\] (A.6)

Hence we have

\[
\begin{align*}
\Phi_{11+} (i\pi) &= U_{11+} (i\pi) = U_{22+} (-i\pi), \\
\Phi_{12+} (i\pi) &= U_{12+} (i\pi) = -U_{21+} (-i\pi), \\
\Phi_{21+} (i\pi) &= U_{21+} (i\pi) = -U_{22+} (-i\pi), \\
\Phi_{22+} (i\pi) &= U_{22+} (i\pi) = U_{11+} (-i\pi)
\end{align*}
\] (A.7)

at \( \omega = i\pi \) while \( \Phi (\omega = 0) \) is parameterized by a single parameter \( \varphi \)

\[
\Phi_+ (0) = \begin{pmatrix} \cos \varphi + \sin \varphi & \sin \varphi \\ \cos \varphi - \sin \varphi & \cos \varphi \end{pmatrix}
\] (A.8)
as a consequence of Eq. (A.6) together with jump condition (5.5).

Consider now the singular integral equation for \( U_+(\omega) \):

\[
U_+ (\omega) = I - \frac{1}{2\pi i} \int_\Gamma \frac{U_+(z) (G_U (z) - I)}{z - \omega_+} dz,
\] (A.9)

where \( \omega_+ \) means that \( \omega \) is shifted from the integration contour to the left. In components, the equations for the entries \( U_{21+} (\omega) \) and \( U_{22+} (\omega) \) read:

\[
\begin{align*}
U_{21+} (\omega) &= \frac{1}{2\pi i} \int_{\Gamma_+} \frac{U_{22+} (z) \alpha_+ (z) e^{izt}}{2 \cosh \frac{z}{2}} \frac{dz}{z - \omega_+}, \\
U_{22+} (\omega) &= 1 - \frac{1}{2\pi i} \int_{\Gamma_-} \frac{U_{21+} (z) e^{-izt}}{2 \alpha_- (z) \cosh \frac{z}{2}} \frac{dz}{z - \omega_+}.
\end{align*}
\] (A.10)

Let \( \omega \in \Gamma_- \) in the first of (A.10). Then shifting the integration contour into the upper half-plane, we obtain

\[
U_{21+} (\omega) = i \sum_{k=0}^{\infty} (-1)^k \frac{U_{22+} (i\pi (2k+1)) \alpha_+ (i\pi (2k+1)) e^{-t\pi (2k+1)}}{\omega - i\pi (2k+1)}.
\] (A.11)

Since \( U_{22+} (\omega) \to 1 \) and \( \alpha_+ (\omega) \to 1 \) at \( \omega \to \infty \), the series (A.11) is absolutely convergent in the domain \( \Im \omega < \pi \), and we arrive at

\[
U_{21+} (\omega) = i \frac{U_{22+} (i\pi) \alpha_+ (i\pi) e^{-t\pi}}{\omega - i\pi} + O \left( e^{-3\pi t} \right), \quad \Im \omega < \pi.
\] (A.12)
Substituting (A.12) into the equation for $U_{22+}$, putting $\omega = -i\pi$ and shifting the integration contour to the lower half-plane, we obtain

$$U_{22+}(-i\pi) = U_{11+}(i\pi) = 1 + \frac{\pi}{2} U_{22+}(i\pi) e^{-2\pi t - 2} \left( t + \gamma + \frac{1}{2\pi} \right) + o\left(e^{-2\pi t}\right),$$

(A.13)

where ($C \approx 0.57721 \ldots$ is Euler’s constant)

$$\gamma = -i \partial_z \log \alpha_+ (z) \big|_{z=i\pi} = \frac{1}{\pi} \left( \log 2 - C \right).$$

(A.14)

Similarly, we analyse the equations for $U_{11+}$ and $U_{12+}$ and obtain

$$U_{11+}(-i\pi) = U_{22+}(i\pi) = 1 - \frac{1}{2} U_{12+}(i\pi) e^{-\pi t - 1} + o\left(e^{-2\pi t}\right),$$

$$U_{12+}(-i\pi) = -U_{21+}(i\pi) = -\pi e^{-\pi t - 1}(t + \gamma) + o\left(e^{-2\pi t}\right).$$

(A.15)

Finally, using relations (A.7) with Eqs. (A.12), (A.13) and (A.13) we find

$$U_+(i\pi) = \Phi_+(i\pi) = \left( 1 + p^2(q + 1) \begin{array}{c} 1 \end{array} \begin{array}{c} p \\
pq \\
1 - p^2 \end{array} \right) + o\left(p^2\right),$$

(A.16)

where $p$ and $q$ are defined as

$$p = \frac{1}{2} e^{-\pi t - 1}, \quad q = 2\pi t + 2(\log 2 - C).$$

(A.17)

Since the estimate (A.12) is uniform in the domain $\Im \omega < \pi$, we have with (A.1)

$$\Phi_{21-}(0) = U_{21+}(0) = -U_{22+}(i\pi) e^{-t\pi - 1} + O\left(e^{-3\pi t}\right).$$

(A.18)

Comparing with (A.8) and using (A.14) we find $\sin \varphi = 2p$, and thus we arrive at (5.9).

### A.2 $S > 1$

The asymptotic analysis of the RHP (4.6) for $S > 1$ is quite similar to the case $S = 1$. Now the boundary values of the matrix $\Phi(\omega)$ introduced in (5.13) on the real axis satisfy

$$\Phi_- (\omega) = \Phi_+ (\omega) G_\Phi (\omega), \quad \omega \in \mathbb{R}$$

(A.19)

with

$$G_\Phi (\omega) = \left( \begin{array}{cc} 1 & -\frac{1}{\alpha_+^{(S)}(\omega)} \frac{\sinh \frac{\omega}{2}}{\sinh \omega S} e^{-i\omega t} \\
-\alpha_+^{(S)}(\omega) \frac{\sinh \frac{\omega}{2}}{\sinh \omega S} e^{i\omega t} & e^{-\frac{|\omega|}{2} \frac{\sinh \omega S}{\sinh (S-1/2)}} \end{array} \right).$$

(A.20)
As for $S = 1$ this matrix can be factorized into the product of matrices $M_+$ and $M_-$, which now are

$$M_+(\omega) = I - \alpha_+^{(S)}(\omega) \frac{\sinh \frac{\omega}{2} \sinh \omega S}{\sinh \omega S} e^{i\omega t} \sigma_-, \quad M_-(\omega) = I - \frac{1}{\alpha_-^{(S)}(\omega)} \frac{\sinh \frac{\omega}{2} \sinh \omega S}{\sinh \omega S} e^{-i\omega t} \sigma_+ \quad (A.21)$$

The following considerations almost completely repeat the corresponding part of the subsection $S = 1$. First, we again come to the new RHP on the contour $\Gamma$. The only difference is that now $\alpha \in (0, \frac{\pi}{S})$. Thus we obtain that $\Phi(\omega) \approx I$ up to exponentially small corrections everywhere except $\omega \in \mathbb{R}$. In order to improve this estimate we use the corresponding singular integral equations. Now the asymptotic behaviour of $\Phi(\omega)$ is defined by residues of the integrands in the points $\omega = i\pi S k$. Respectively the solution of the RHP is given by the asymptotic series of $\exp\{-\frac{\pi}{S} k\}$. Note also that just like in the case $S = 1$ the matrix $\Phi(0)$ can be parametrized by a single parameter. The explicit expression now has the form

$$\Phi_+(0) = \begin{pmatrix}
\cos \varphi + \eta \sin \varphi & \sin \varphi \\
\eta \cos \varphi - \sin \varphi & \cos \varphi
\end{pmatrix}. \quad (A.22)$$

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Figure 1: Magnetic susceptibility of the doped $S = 1$ (a) and $S = 3$ (b) system as a function of the concentration $x_h$ of holes (full line). The broken line is the contribution of the ‘background’ mode $\epsilon_{2S}$ to $\chi$. 
Figure 2: The jump-contour for the new RHP (A.2).

Figure 3: Wilson ratio (6.1) as a function of the hole concentration $x_h$ for the $S = 1$ and $S = 3$ system.