Noncommutative Reissner-Nordstrøm Black hole

| Journal:            | Canadian Journal of Physics |
|---------------------|-----------------------------|
| Manuscript ID       | cjp-2017-0599.R1            |
| Manuscript Type:    | Article                     |
| Date Submitted by the Author: | 21-Nov-2017               |
| Complete List of Authors: | Soto Campos, Carlos Arturo; Universidad Autonoma del Estado de Hidalgo, Physics; Valdez Alvarado, Susana; Universidad Autónoma del Estado de México., Physics |
| Keyword:            | Black Hole, Noncommutativity, Moyal bracket, Deformation, Quantization |
| Is the invited manuscript for consideration in a Special Issue? | N/A |

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Noncommutative Reissner-Nordstrøm Black hole.

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Abstract

In this work we construct a deformed embedding of the Reissner-Nordstrøm spacetime within the framework of a noncommutative Riemannian geometry. We provide noncommutative corrections to the usual Riemannian expressions for the metric and curvature tensors. For the case of the metric tensor, the expression obtained possesses terms which are valid to all orders in the deformation parameter. Then we calculate the correction to the area of the event horizon of the corresponding noncommutative R-N black-hole, obtaining an expression for the area of the black-hole which is correct up to fourth order terms in the deformation parameter. Finally we include some comments on the noncommutative version on one of the second order scalar invariants of the Riemann tensor, the so-called Kretschmann invariant, a quantity which is regularly used in order to extend gravity to quantum level.

1 Introduction

One of the most important presumptions in General Relativity is that the space, time and gravity can be modeled as a sole entity called spacetime. General Relativity analyzes spacetime as the background in which, electromagnetism, matter and their mutual influences interact with each other. That approach has been mainly used in the study of large scale phenomena.

However, it is a general belief that the picture of this spacetime as a pseudo Riemannian manifold $M$, locally modeled as the flat Minkowski spacetime, should break down at scales of the order of Planck length, $\lambda_p = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33}$ cm. A quantum theory of fields, attempting to incorporate gravitation, must then consider the limitations on the possible accuracy of the localization of events in spacetime. A lot of work has been done on the possible mechanisms which could lead to such limitations. The noncommutative geometry has become an option for the description of a quantized spacetime. The study of noncommutative geometry acquires relevancy in

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the research of the quantum nature of spacetime at high energy scales. The idea of noncommutative spacetime coordinates is an old one and it has been present in the literature from a long time ago [1]. The theory of A. Connes is a particularly successful approach to this topic, Ref. [2]. This is a theory formulated within the framework of $C^*$ algebras.

An important advance in the mathematical framework of noncommutative geometry was introduced by the deformation quantization of Poisson manifolds by M. Kontsevich Ref. [3]. His work has led to the study of new applications of noncommutative geometries on quantum theory. Besides, Seiberg and Witten Ref. [4] showed that the anti-symmetric tensor field arising from massless states of strings can be described by the noncommutativity of the coordinates of spacetime

$$[x^\mu, x^\nu] = i \Theta^{\mu\nu}$$

(1)

where $\Theta^{\mu\nu}$ is a constant antisymmetric tensor. Now the multiplication of the algebra of functions is given by the Moyal product

$$(f \star g) = f(x) \exp\left(\frac{i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu\right)g(x).$$

(2)

A lot of research had been oriented towards a formulation of General Relativity on noncommutative spacetimes. See for instance Refs. [34, 35, 36, 37, 38].

The noncommutative geometrical approach to gravity could give us some insight into a theory of gravity compatible with quantum mechanics. Much work has been done in that direction, see for instance Refs. [5, 6, 7, 23]. The noncommutative spacetime with the commutation relation given by Eq. (1) violates Lorentz symmetry but it was shown to have quantum symmetry under the twisted Poincaré algebra, see Ref. [6]. The abelian twist element

$$\mathcal{F} = \exp\left(\frac{-i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu\right)$$

(3)

was used in Refs.[6, 8] to twist the universal enveloping algebra of the Poincaré algebra providing a noncommutative multiplication for the algebra of functions which is related to the Moyal product. Then it seems natural to extend this procedure to other symmetries of noncommutative field theory.

Related to the noncommutative formulation of General Relativity, Nicolini et al. Ref. [20] found a new solution of the coupled Einstein-Maxwell field equations inspired in the noncommutative geometry. The metric they have found interpolates smoothly between a de Sitter geometry at short distances, and a Reissner-Nordstrøm (R-N) geometry far away from the origin. Contrary to the ordinary R-N spacetime, in this particular metric there is no curvature singularity at the origin, neither naked nor shielded by horizons, which seems very intriguing.

In a second paper, see Ref. [21], the same authors solved Einstein-Maxwell equations in the presence of a static spherically symmetric Gaussian distribution of mass and charge having a minimal width. They show that the coordinate fluctuations can be described, within the coherent states approach, as a smearing effect.
Basically there are two different approaches to noncommutative geometrical theories of gravity. On the one hand, we have works which are based in an intuitive approach and emphasizing the physical meaning of the deformed quantities. On the other hand, there are more formal approaches developing noncommutative counterparts of Riemannian geometric structures. For a complete series of references see for instance Refs. [32, 38, 37, 36, 34, 33]

In this paper we follow the theory of noncommutative Riemannian geometry developed by Chaichian et al. in Ref. [11] to investigate quantum aspects of gravity from a mathematical point of view. In Ref. [11] a noncommutative Riemannian geometry is constructed by developing the geometry of noncommutative n-dimensional surfaces. The notions of metric and connections are introduced on such noncommutative surfaces, giving rise to the corresponding Riemann curvature. Chaichian et al. makes use of Nash’s theorem of isometric embeddings —Refs. [24, 25]— in order to obtain the deformed versions of classical spacetimes. In the same framework of noncommutative geometry, Wang et al. [12] have constructed a quantum Schwarzschild spacetime and a quantum Schwarzschild-de Sitter spacetime with cosmological constant. They computed the metrics and curvatures and finally showed that, up to second order in the deformation parameter, the quantum spacetimes are solutions of the so called noncommutative Einstein equations.

In the present work we will construct a noncommutative deformation (which can be interpreted as quantum corrections) of the Reissner-Nordstrøm spacetime and we will look for properties of such noncommutative spacetime. The key results are the equations of the metric and the components of the connections. A preliminary version of this work (exploring a different approach) was presented in the IX Workshop of the Gravitation and Mathematical Physics Division of the Mexican Physical Society, [28]. However, in that previous work, we were trying to find noncommutative corrections to Einstein field equations that could be expressed as analytic functions of some perturbative parameter $\hbar$. In this paper we find noncommutative corrections to some classical expressions in terms of a perturbative series which will be valid at different orders of magnitude. From the point of view of Physics, this last procedure seems more appropriate.

The paper is organized as follows. In Section 2 we provide a quick review of the structure of the Moyal bracket and of the deformation of the algebra of functions on the domain of Euclidean space. In section 3 we explain how we “deform” the Riemannian geometry by calculating the relevant quantities in General Relativity. This particular section is based upon Chaichian et al and Wang et al [11, 12]. In Section 4 the Reissner-Nordstrøm spacetime is introduced. In this section we find the components of the noncommutative versions of the metric and the connections. In subsection 4.2 we compute the area of the event horizon and find its noncommutative corrections. Besides, we search for the generalization of the Kretschmann invariant (which provides us information about the embedding) and we find its noncommutative expression. Finally in Section 5 we give our conclusions and perspectives.
2 Deforming the Algebra of Functions

In traditional quantum mechanics we have the following commutation relations between the position and momentum operators

\[ [\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij} \quad \text{and} \quad [\hat{X}_i, \hat{X}_j] = [\hat{P}_i, \hat{P}_j] = 0 \] (4)

where indices \( i, j, k = 1, 2, 3 \); run over spatial coordinates. However, there is no evidence that those commutation relations remain valid at very short distances or to very high energies. A generalization that we could find “natural” — see for example Ref. [33] — in the aforementioned commutation relations is as follows

\[ [\hat{X}_i, \hat{X}_j] = i\hat{\Theta}_{ij} \] (5)

where \( \hat{\Theta}_{ij} \) is a constant antisymmetric operator with dimensions of length squared \( [L]^2 \). Obviously, when we introduce that condition in coordinates, we ruin Lorentz invariance. However, we have assumed that the commutation relations mentioned before appear at very small distances \( L \). This implies that for \( \frac{L}{\sqrt{\theta}} >> 1 \) we should recover Lorentz symmetry. In general, for a noncommutative field theory, at the classical level, in the limit \( \frac{L}{\sqrt{\theta}} >> 1 \) we must recover a previously known, commutative field theory.

Now, the above equation can be extended to the space time coordinates

\[ [x_\mu, x_\nu] = i\Theta_{\mu\nu} \] (6)

Which constitutes a definition of noncommutative space, i.e. any where the coordinates satisfy the previous commutation relations. To build the corresponding perturbative field theory, it is more convenient to use fields that are functions instead of operators. To move to these fields maintaining the validity of the commutation relations, we will define the product in the space of functionals. This new product is introduced through the —so-called— correspondence of Weyl-Wigner-Moyal:

\[ \hat{\Phi}(\hat{X}) \leftrightarrow \Phi(x) \] (7)

and as a consequence

\[ \hat{\Phi}(\hat{X}) = \int_\alpha e^{i\alpha \hat{X}} \phi(\alpha) d\alpha \] (8)

\[ \phi(\alpha) = \int e^{-i\alpha x} \phi(x) dx \] (9)

where \( \alpha \) and \( x \) are real variables. Thus

\[ (\phi_1 \star \phi_2)(x) \equiv \left[ e^{\frac{i}{\hbar} \theta_{\mu\nu} k_\mu \eta_\nu \phi_1(x + \xi) \phi_2(x + \eta)} \right]_{\xi=\eta=0} \] (10)

The latter equation suggests that we can work in a usual commutative space for which the multiplication is modified to what is known as a star product. It is easy to verify that Moyal’s bracket (the commutator in which the usual product is modified with a star product) between two coordinates, satisfies the desired commutation relation.
In the present and in the next one sections, we will introduce the deformation of the geometry. This will be done through the introduction of a noncommutative algebra over a ring. This algebra will be denoted by \( \mathcal{A}_h \) (called \textit{deformed algebra}) satisfies a \textit{correspondence principle} which establishes that we recover the commutative algebra for \( \lim_{h \to 0} \mathcal{A}_h = \mathcal{A} \).

There are different ways to deform a theory. Here we follow the approach given by Wang et al. in Ref. [12] and we use the Riemannian structures previously studied in Ref. [11], using the Moyal product of functions. The next subsection is based on the pioneering work of Gerstenhaber, Ref. [22], who studied the deformation of algebras.

### 2.1 Deformation of the Algebra

Let \( \mathcal{A} \) be a commutative ring with the unit. The Ring of formal power series \( R = \mathcal{A}[[X]] \) is the set of all the sequences \( (a_0, a_1, a_2, \ldots) \) with \( a_i \in \mathcal{A} \) for all \( i \in \mathbb{N} \) where the operations of sum and product are defined as:

1. \( (a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots) \).
2. \( (c_0, c_1, c_2, \ldots) = (a_0, a_1, a_2, \ldots) \times (b_0, b_1, b_2, \ldots) \), where \( c_k = \sum_{i+j=k} a_i b_j \) for all \( k \in \mathbb{N} \).

It is verified immediately that \( R \) is a commutative ring with a unit \( 1 = (1, 0, 0, \ldots) \). We write \( X = (0, 1, 0, 0, \ldots) \), such that \( X^2 = (0, 0, 1, 0, \ldots) \), etcetera.

We’ll say that \( \alpha = (a_0, a_1, a_2, \ldots) \) is of order \( i \) when \( a_0 = a_1 = \ldots = a_{i-1} = 0 \) and \( a_i \neq 0 \). We write \( \mathcal{O}(\alpha) = i \).

We have given a precise definition of a formal power series. From now on we are going to use the notation \( \sum_{i \geq 0} a_i X^i \) instead of \( (a_0, a_1, a_2, \ldots) \). It is clear that this is not a sum. Besides, \( \mathcal{A}[[X]] \) is a subring of \( R \).

Let us denote by \( U \) a certain domain in \( \mathbb{R}^n \) and let \( \mathcal{A} \) be the set of all the formal power series in \( h \) with coefficients on the real functions \( C^\infty \) on \( U \). Then \( \mathcal{A} \) is a \( \mathbb{R}[[h]] \)–module and its elements are formed as \( \sum_{i \geq 0} f_i h^i \), where the explicit product of two elements from the \( \mathbb{R}[[h]] \)–module is:

\[
\left( \sum_{i \geq 0} f_i h^i \right) \left( \sum_{i \geq 0} g_i h^i \right) = \sum_{n \in \mathbb{N}} \left( \sum_{k \geq 0} \sum_{i+j=k} f_k g_{n-k} h^n \right)
\]

Up to this point we have built the algebra that we are going to use in order to quantize the spacetime. To achieve this, we will proceed to deform the product of functions, substituting the usual product by a noncommutative one, the so called \textit{Moyal product}. Let \( f, g : U \to \mathbb{R} \) be two functions \( C^\infty \), we denote by \( fg \) the usual (i.e. commutative) product of functions. Then we define the star product (or Moyal product) as the operation \( \star : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) such that:

\[
(f \star g) = f(x) \exp \left( \hbar \sum_{i,j} \Theta_{ij} \frac{\partial_i}{\partial_j} \right) g(x)
\]

where the exponential function \( \exp(\hbar \sum_{i,j} \Theta_{ij} \frac{\partial_i}{\partial_j}) \) must be understand as a power series in the differential operator and \( \Theta_{ij} \) is a constant antisymmetric tensor represented
by a matrix of \((n \times n)\), where \(n\) is the dimension of \(U\). It is easy to see that the star product is associative and satisfies distributivity too. We denote by \((\mathcal{A}, \star)\)—or simply \(\mathcal{A}_h\)—the so-called deformed algebra.

From the previous discussion it is obvious that we can construct more complex structures with the deformed algebra: for example, let us denote by \(\mathcal{A}^m = \mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}, (m \text{ times})\), with elements \((X_1, X_2, \ldots, X_m)\), where \(X_i \in \mathcal{A}\), with the index \(i\) running from 1 to \(m\). In the next section we will define a deformed inner product as a map \(\bullet : \mathcal{A}^m \otimes \mathcal{A}^m \to \mathcal{A}^m\).

### 3 Deformation of the Geometry

Now we will construct an embedding \(X\) of a pseudo Riemannian manifold of dimension \(n\) to a pseudo Euclidean space of dimension \(m\) and with a metric tensor

\[
\eta_{\alpha\beta} = \text{diag}(1,1,\ldots,1,-1,-1,\ldots,-1),
\]

with \(p + q = m\), such that \(X = (X^1, X^2, \ldots, X^m) \in \mathcal{A}^m\). This section is based entirely in Ref.[12].

We can construct tangent vectors denoted by \(E_i\)

\[
E_i \equiv \partial_i X, (i = 1, 2, \ldots, n)
\]

at each point of the surface given by the parametrization \(X\). Now we will define the metric tensor as an \((n \times n)\) matrix:

\[
g_{ij} = E_i \bullet E_j,
\]

where the fat dot \(\bullet\) denotes inner product between elements of the algebra

\[
\bullet : \mathcal{A}^m \otimes \mathcal{A}^m \to \mathcal{A}^m
\]

on the open region \(U\), and is explicitly given by

\[
A \bullet B = \sum_{i=1}^{p} a_i \star b_i - \sum_{j=p+1}^{p+q} a_j \star b_j,
\]

for every \(A = (a_1, \ldots, a_m)\) and \(B = (b_1, \ldots, b_m)\) in \(\mathcal{A}^m\).

Obviously \(g_{ij}\) is invertible in \(U\) if and only if \(g_{ij}|_{\mathcal{A}} = 0\) is invertible, and we denote the inverse matrix \(g^{ij}\), as the matrix which satisfies

\[
g_{ij} \star g^{jk} = g^{kj} \star g_{ji} = \delta^k_i,
\]

with \(\delta^k_i\) the identity \(n \times n\) matrix.

In this noncommutative spacetime we can define two different connections given by

\[
\nabla_i E_j = \Gamma^k_{ij} \star E_k, \quad \text{and} \quad \tilde{\nabla}_i E_j = E_k \star \tilde{\Gamma}^k_{ij}
\]

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\]
related to the fact that there are two tangent bundles, the left and the right tangent bundle respectively. The corresponding expressions for $\Gamma^k_{ij}$ and $\tilde{\Gamma}^k_{ij}$ are

$$\Gamma^k_{ij} = \partial_i E_j \cdot \tilde{E}^k$$

and

$$\tilde{\Gamma}^k_{ij} = \tilde{E}^k \cdot \partial_i E_j$$

(17)

where $\tilde{E}^k = E_i \star g^{ik}$. The left and right connections do not coincide in general. This represents a big difference with the usual, commutative case. The noncommutative Riemann tensor associated to the left connection $\Gamma^k_{ij}$ is given by:

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk} \star \Gamma^l_{ip} - \Gamma^p_{ik} \star \Gamma^l_{jp},$$

(18)

In terms of the deformed product, two different contractions of the Riemann tensor can be defined: the Ricci curvature tensor $R^i_j$ and the $\Theta^i_j$ curvature tensor:

$$R^i_j = g^{ik} \star R^p_{kpj},$$

(19)

$$\Theta^i_p = g^{ik} \star R^i_{kp},$$

(20)

In general $R^i_j$ and $\Theta^i_j$ do not coincide, in contrast to the commutative case.

Now the noncommutative curvature scalar of the surface $X$ is:

$$R = R^i_i$$

(21)

On the other hand the noncommutative Einstein equations with cosmological constant on $U$ are:

$$R^i_j + \Theta^i_j - \delta^i_j \Lambda = 2T^i_j,$$

(22)

where $T^i_j$ is some generalized energy-momentum tensor, $\Lambda$ is the cosmological constant.

This equation reduces to the Einstein’s equation in vacuum when $T^i_j = \Lambda = 0$.

4 The Reissner-Nordstrøm Solution

4.1 Noncommutative Reissner-Nordstrøm Spacetime

The Reissner-Nordstrøm solution represents the spacetime outside a static, spherically symmetric charged body carrying an electric charge. It is the unique spherically symmetric asymptotically flat solution of the Einstein-Maxwell equations.

The metric of the R-N spacetime can be written as:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2,$$

(23)

where $m = \frac{2GM}{c^2}$ represents the gravitational mass, with $G$ the Newton’s constant and $c$ the speed of light. Here $e$ stands for the electric charge.

If $e^2 > m^2$ the metric is non-singular everywhere except for the irremovable singularity at $r = 0$. If $e^2 \leq m^2$ the metric has singularities at $r_+$ and $r_-$ where $r_\pm = m \pm (m^2 - e^2)^{1/2}$. For an especially succinct review see Ref. [14]. In this letter we
are interested only in the region outside \( r_+ \). Here, we will denote the coordinates by: \( x_1 = r, \ x_2 = \theta, \ x_3 = \varphi, \ x_4 = t \), (notice the particular choice for the coordinate \( t \)).

We are going to compute the deformed Riemannian structures introduced in section 3 for the R-N spacetime. By following the procedure depicted in the preceding section, we propose that the Reissner-Nordstrøm spacetime can be embedded in a six dimensional pseudo Euclidean space with metric \( \eta_{\mu\nu} = \text{diag}(1,1,1,1,-1,-1) \), through the map \( X = (X_1, X_2, X_3, X_4, X_5, X_6) \) given by

\[
\begin{align*}
X_1 &= g(r) \\
X_2 &= r \sin \theta \cos \varphi \\
X_3 &= r \sin \theta \sin \varphi \\
X_4 &= r \cos \theta \\
X_5 &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \sin t \\
X_6 &= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} \cos t
\end{align*}
\]

with the convention of coordinates given in the previous subsection. Here we have used the Kasner embedding already used before by Wang and Zhang in Ref.[12] for the case of a noncommutative embedding of a Schwarzschild and a Schwarzschild-de Sitter spacetime in a 6 dimensional pseudo Euclidean manifold. Using a similar procedure than that implemented by Wang and Zhang, we define the function \( g(r) \) in the first of the Eqs. (24) as a smooth function such that

\[
g'^2 + 1 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(1 + \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right)^2\right)
\]

where \( g' \) denotes the derivative of \( g \) with respect to \( r \). This requirement for \( g(r) \) simplifies the computations of the components of the noncommutative metric as will be clear very soon. From the Eqs. (23) and (24) it is easy to verify the isometry of the embedding:

\[
ds^2 = (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2 - (dX_5)^2 - (dX_6)^2
\]

We now proceed to deform the algebra of functions with the procedure mentioned in section 3 and imposing a Moyal product of functions with the matrix representation of the antisymmetric tensor \( \Theta_{\mu\nu} \)

\[
\Theta_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The tangent vectors \( E_i = \partial_i X = \frac{\partial}{\partial x_i} X \) are given by:
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\[ E_1 = \left[ g'(r), \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta, \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{-\frac{1}{2}} \right. \]
\[ \times \left( \frac{m}{r^2} - \frac{e^2}{r^3} \right) \sin t, \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{-\frac{1}{2}} \left( \frac{m}{r^2} - \frac{e^2}{r^3} \right) \cos t \right] \]

\[ E_2 = [0, r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, -r \sin \theta, 0, 0] \]

\[ E_3 = [0, -r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, 0, 0, 0] \]

\[ E_4 = [0, 0, 0, 0, \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{\frac{1}{2}} \cos t, -\left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{\frac{1}{2}} \sin t] \]

(28)

And from Eq. (14) we find Ref.[28] the nonzero components of the metric tensor:

\[ g_{11} = \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{-1} - \cos 2\theta \sinh^2 \hbar \]
\[ g_{12} = g_{21} = r \sin 2\theta \sinh^2 \hbar \]
\[ g_{13} = -g_{31} = -r \sin 2\theta \sinh \hbar \cosh \hbar \]
\[ g_{22} = r^2 + r^2 \cos 2\theta \sinh^2 \hbar \]
\[ g_{23} = -g_{32} = -r^2 \cos 2\theta \sinh \hbar \cosh \hbar \]
\[ g_{33} = r^2 \sin^2 \theta - r^2 \cos 2\theta \sinh^2 \hbar \]
\[ g_{44} = -\left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right) \]

(29)

In agreement with the observation of Chaichian et al. Ref.[9], in the sense that, for an arbitrary \( \Theta_{\mu \nu} \) given by Eq.(27), the deformed metric \( g_{\mu \nu} \) is not diagonal. In the components of \( g_{\mu \nu} \) we notice that terms containing the noncommutative parameter appear. Actually the hyperbolic functions —depending on the noncommutative parameter \( \hbar \) — arise when we compute the deformed inner product, Eq. (14), in the form of series expansions. It is a straightforward calculation to verify that the components of the noncommutative metric tensor satisfy the correspondence principle. It is important to remark that the components \( g_{12}, \ldots, g_{33} \) coincide with those computed by Wang et al [10]. That coincidence in some of the components of the metric tensor is due to the map defined by Eq. (24).

In order to make these calculations more clear, we now proceed to compute in detail one of the components of the metric, for example \( g_{11} \). From Eq.(14) we know that

\[ g_{11} = E_1 \cdot E_1, \]

and using the expressions for the generators obtained before, then after some simplifications of terms, we get
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\[ g_{11} = g^2(r) + \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \frac{\hbar^2}{2!} \left[ (\sin^2 \theta \cos^2 \varphi - \cos^2 \theta \sin^2 \varphi) + (\sin^2 \theta \sin^2 \varphi - \cos^2 \theta \cos^2 \varphi) \right] + \frac{\hbar^4}{4!} \left[ (\sin^2 \theta \cos^2 \varphi - \cos^2 \theta \sin^2 \varphi) + \sin^2 \theta \sin^2 \varphi - \cos^2 \theta \cos^2 \varphi \right] + \ldots + \cos^2 \theta \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{-1} \left( \frac{m}{r^2} - \frac{e^2}{r^3} \right)^2 \]

(30)

Arriving to this point, it is useful to introduce the function \( g(r) \) defined before in Eq. (25), such that

\[ g^2 + 1 = \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right)^{-1} \left( 1 + \left( \frac{m}{r^2} - \frac{e^2}{r^3} \right)^2 \right) \]

and identifying the series \( \sinh^2 \hbar = 2 \frac{\hbar^2}{2!} + 8 \frac{\hbar^4}{4!} + 32 \frac{\hbar^6}{6!} + \ldots \); we finally obtain the component \( g_{11} = (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1} - \cos 2\theta \sinh^2 \hbar \). Now from Eq. (14) and using the result \( g^{kj} \star g_{ji} = \delta^k_j \), we can calculate the contravariant components of the noncommutative metric tensor now. Those components become a little bit more complicated than those in Eq. (29), and that is precisely because their expressions involve the determinant of the noncommutative metric tensor

\[ \det g_{\mu\nu} = -r^4 \sin^2 \theta \left[ 1 - \sin^2 \hbar \cos 2\theta \right] + r^4 P(r) \sin^2 \hbar f(h, \theta) \]

Where we have introduced the function \( P(r) \) which is the metric coefficient of the timelike coordinate in the line element Eq. (23) defined by \( P(r) \equiv (1 - \frac{2m}{r} + \frac{e^2}{r^2}) \) and \( f(h, \theta) \) is a function involving an easy but long expression depending just on the parameter \( h \) and the coordinate \( \theta \). The last expression for \( \det g_{\mu\nu} \) differs considerably from that reported previously in Ref. [28] and in a draft version of this current paper.

It is quite easy to check that \( \det g_{\mu\nu} \) reduces to the standard commutative expression in the limit when the parameter \( h \) tends to zero. Fortunately the components of the covariant metric tensor \( g_{\mu\nu} \) just involve functions of the coordinates \( r \) and \( \theta \) and then, the Moyal products between them turn into the usual commutative products.

Now we can see that all the components of the noncommutative contravariant metric tensor will be affected by the presence of the charge in the R-N black hole, because of its dependence on the determinant of the noncommutative metric and on the function \( P(r) \) defined before.

It is worth mentioning that the present noncommutative embedding that we are using, is very similar to the approach used by C. Fronsdal in 1959, [19]. Fronsdal constructed an analytic manifold with the important distinction that it was complete. As a matter of fact, he was trying to extend the Schwarzschild line element. The manifold was represented by a Riemannian surface embedded in a six dimensional pseudo Euclidean space. The line element that he chose was given by \( ds^2 = dZ_1^2 - dZ_2^2 - dZ_3^2 - dZ_4^2 \), in a quite similar way of that used by us in subsection 4.1.

At the end of the paper mentioned before [19], Fronsdal added a note mentioning that Professor M. Kruskal had recently obtained a different solution (unpublished

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by that time). That was, of course, the now, well known Kruskal extension of the Schwarzschild solution to Einstein field equations.

Now, from Eq. (17) we can calculate the respective components of the noncommutative left connection \( \Gamma^k_{ij} = \partial_i E_j \cdot \tilde{E}^k \). This constitutes a different approach from that used in the previous paper [28]. Actually there are two different connections in this theory as we have already mention before, —see for instance Eq. (17)—. That is because in this approach, we have left and right tangent bundles to the noncommutative manifold, so, there are left and right connections respectively.

For the purposes of the present work it is enough to use the left connection defined previously. As a matter of fact, it can be proven that, both connections given in Eq. (17) can be used to calculate the same Riemann tensor (see for example Ref. [12]). Taking into account the next definitions:

\[
P' = \frac{1}{r} = \frac{2m}{r^2} - \frac{2\epsilon^2}{r^2},
\]

we will rewrite some of the expressions given before. For example, we can rewrite Eq. (25) as

\[
g'^2 + 1 = P^{-1}(1 + (P'/2)^2)
\]

Now, the modified connections will be written in the form of a perturbative series in terms of \( \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \hbar f_1(x^\mu) + \hbar^2 f_2(x^\mu) + \ldots \), up to second order in the deformation parameter by:

\[
\begin{align*}
\Gamma^1_{11} &= \Gamma^1_{11} + \frac{P'}{2} (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \hbar^2, \\
\Gamma^1_{22} &= \Gamma^1_{22} + \left[ rP(P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \\
&\quad + 2 + 4 \cos 2\theta \right] \hbar^2, \\
\Gamma^1_{33} &= \Gamma^1_{33} + rP \left[ P \sin^2 \theta (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \\
&\quad + 2(\cos^4 \theta + 1) + 4 \cos 2\theta \right] \hbar^2, \\
\Gamma^1_{44} &= \Gamma^1_{44} + \frac{PP'}{2} (P^{-1} \cos 2\theta + 1 + 2 \cos^2 \theta) \hbar + O(\hbar^2), \\
\Gamma^2_{12} &= \Gamma^2_{12} + \frac{1}{r} \left( \csc^2 \theta \cos 2\theta + 2 \cot \theta \cos 2\theta \right) \hbar + O(\hbar^2), \\
\Gamma^2_{33} &= \Gamma^2_{33} + \left[ 2P \cos^3 \theta \sin \theta - P \cos \theta ( - P^{-1} \cos 2\theta \cos^2 \theta \\
&\quad + 2 \sin^2 \theta) + 4 \cot \theta \cos 2\theta + 2 \sin 2\theta \right] \hbar^2, \\
\Gamma^3_{13} &= \Gamma^3_{13} - \frac{P}{r} (\cos 2\theta + 1) \hbar + \left[ - 3P(1 + \cos^2 \theta) + \cos 2\theta \\
&\quad \times \left( 3 - \cot^2 \theta + 2 \cot \theta \right) - 2 \cot \theta (\sin 2\theta + 2 \cos \theta) \right] \hbar^2, \\
\Gamma^3_{23} &= \Gamma^3_{23} + P \cot \theta (\cos 2\theta + 1) \hbar + \left[ \cot \theta \left( - 3P \\
&\quad - 2 \cot \theta \sin 2\theta + 3 \cos 2\theta - 2P \cos^2 \theta - 7 \cot^2 \theta \right. \\
&\quad + 2 \frac{\cos(2\theta)}{\sin \theta} + \frac{2}{\sin^2 \theta} - 2 \right] + 4P \cos \theta \sin \theta \right] \hbar^2
\end{align*}
\]

and finally,

\[
\Gamma^4_{14} = \Gamma^4_{14}.
\]
where we have used the fact that, by definition, $\Gamma^k_{ij}$ is symmetric in the covariant indices. This last statement is clear from Eq. (17). It is interesting to compare the expressions that we have just obtained in Eq. (31) with the respective commutative expressions. We can easily verify that the correspondence principle is satisfied. With the last expressions given in Eq. (31) we calculate the components of the noncommutative Riemann tensor. It is necessary to point out that in a previous publication [28] the authors used a different approach and the expressions thus obtained do not correspond to the expansions of the analytic corrections used before. This is mainly because in the previous paper the interest was to solve the noncommutative Einstein equations, which involve to provide an appropriate noncommutative energy-momentum tensor. However, that is not the objective of this paper.

In order to compare the respective commutative expressions, we focus only on some components. However there are many components of the Riemann tensor which are purely noncommutative and do not have a commutative counterpart. For the sake of clarity in the expressions we will write one of the components of noncommutative Riemann tensor in the form $R^\alpha_{\beta\mu\nu} = R^\alpha_{\beta\mu\nu} + \hbar f_1(x^\mu) + \hbar^2 f_2(x^\mu) + \ldots$, for example $R^4_{141}$. We obtain

$$R^4_{141} = R^t_{ttr} = \partial_t \Gamma^t_{rt} + \Gamma^r_{rr} \star \Gamma^t_{rt} - \Gamma^t_{rt} \star \Gamma^t_{rt} = R^t_{ttr} + \frac{P^{-1}P^2}{4} (1 + 2 \cos^2 \theta) \hbar^2$$

(32)

As in the case of the noncommutative connections, we can verify that the correspondence principle is satisfied for the limit $\hbar \to 0$. For the rest of the noncommutative Riemann tensors, the calculations are straightforward but the expressions rapidly become cumbersome in the majority of the cases. For many of the noncommutative Riemann tensors, the first corrections appear at second order in $\hbar$. However, some of the purely noncommutative Riemann tensors have expressions which depend on first order terms of $\hbar$.

### 4.2 Results

At this point we can obtain various expressions for different physical quantities in terms of the deformation parameter.

One useful quantity that we can calculate is the Kretschmann scalar $K = R^\alpha_{\beta\mu\nu} \star R_{\alpha\beta\mu\nu}$. This invariant quantity has been used in the pursuit to extend gravity to the quantum level.

In this approach to a deformed version of the R-N solution, some of the noncommutative components of the totally covariant Riemann tensor, contribute to first order in the perturbative parameter $\hbar$. For example, the components of the Riemann tensor which depend on the connections $\Gamma^1_{44}$, $\Gamma^2_{12}$, $\Gamma^3_{13}$ and $\Gamma^3_{23}$ written in Eq.(31). All of those connections exhibit noncommutative corrections to first order in $\hbar$. Thus we expect corrections of order $\hbar$ in $R^t_{1\beta\mu\nu}$, $R^t_{2\beta\mu\nu}$ and $R^t_{3\beta\mu\nu}$. So first order corrections to the Kretschmann scalar come from cross products of the usual contravariant Riemann components with their respective covariant first order corrections and vice versa. To
make this point clear, we proceed to show one of the components of \( R_{\beta \mu \nu} = g_{\beta \mu} * R^\rho_{\beta \mu \nu} \). Let us take for instance \( g_{13} * R^3_{\beta \mu \nu} \). From Eq.(29) we can see that \( g_{13} \) possess first order corrections in \( \hbar \). So when we perform the Moyal product, we will find terms that contain \( g_{13} \) multiplied by the regular commutative components \( R^3_{131}, R^3_{232}, R^3_{343} \).

After an straightforward but quite long calculation we found that

\[
K_{NC} = K + F(r, \theta, \phi) \hbar + O(\hbar^2)
\]

where \( K \) is the usual commutative expression for the Kretschmann scalar reported in Refs. [26, 27] and \( F(r, \theta, \phi) \) is a function that contains the first order corrections arising from the product of all the noncommutative Riemann tensors.

On the other hand, in the present approach we have not found corrections to the surface gravity of the noncommutative RN black hole, so \( \kappa = \frac{1}{2 \partial_1 g_{44}} \big|_{r=r_+} \) remains equal than the commutative case. Thus, the Hawking temperature

\[
T_H = \frac{\sqrt{m^2 - e^2}}{2\pi (m + \sqrt{m^2 - e^2})^2}
\]

is not modified by the embedding. This can be contrasted to Ref. [21] where a correction to \( T_H \) is given in terms of the noncommutative parameter for a variety of charged objects. However, that is a quite different approach.

The question of the stability of a R-N black hole has been studied by several authors. Since the pioneering work of Moncrief [39] to the paper of Anderson et al [40] who found that 'the stress energy tensor for a scalar field diverged strongly if the temperature was not zero'.

On the other hand, Robinson and Wilczek [41] studied the relation between the Hawking radiation of a black hole and the cancellation of anomalies of the energy momentum tensor on the event horizon on a 2-dimensional Schwarzschild background metric.

In this paper we do not find corrections to the temperature of the R-N black hole. However, if we try to find a suitable noncommutative energy momentum tensor in order to solve Einstein’s equations, we would expect to obtain nontrivial corrections to its components.

Now we proceed to calculate the area of the event horizon in the deformed R-N Black Hole. This is given by the integral

\[
A = \int_{r=r_+} \sqrt{\det g_{ab}} \, d\theta d\phi
\]

with \( a, b = 2, 3, \) and \( r_+ \) stands for the exterior radius of the event horizon \( r_+ = m + \sqrt{m^2 - e^2} \). From Eq.(29), we write

\[
g_{ab} = \begin{pmatrix}
g_{22} & g_{23} \\
g_{32} & g_{33}
\end{pmatrix}
\]

(33)

Which leads to the following result

\[
A = \int_{r=r_+} r^2 \sin \theta \sqrt{1 - \cos 2\theta \sinh^2 \hbar} \, d\theta d\phi,
\]

(34)
Performing the integral in $\theta$ we obtain the following expression in terms of the parameter $\hbar$

\[
\sqrt{(1 - \sinh^2 \hbar)} + \frac{\sqrt{2}}{2 \sinh \hbar} \arctan \left( \frac{\sqrt{2} \sinh \hbar}{\sqrt{1 - \sinh^2 \hbar}} \right) \\
+ \frac{\sqrt{2}}{2} \sinh \hbar \arctan \left( \frac{\sqrt{2} \sinh \hbar}{\sqrt{1 - \sinh^2 \hbar}} \right)
\]

Which is a smooth function of $\hbar$. In order to compare with the usual commutative result, this can be expanded in powers of the deformation parameter using the series expansion of $\sinh^2 \hbar$ given before and the corresponding expression for the function $\arctan(x)$. Then, for the area of the event horizon of the deformed R-N black hole we obtain

\[
A = 4\pi r^2 \left( 1 + \frac{h^2}{6} - \frac{h^4}{360} + \mathcal{O}(h^6) \right)
\]  

(35)

This result is similar to that obtained by Wang and Zhang in Ref.[12]. This is not surprising after all, because the determinant of the metric tensor over the sphere, $g_{ab}$ has the same expression as that reported in [12]. That is due to the particular embedding that we have used which inherits the symmetry of Schwarzschild solution over the angular coordinates. It is easy to verify that we obtain the regular — commutative — expression for the case in which $\hbar \to 0$. However, in the present case, the radius of the event horizon is obviously modified by the charge of the black hole. Then, for the expressions of the components of the Riemann tensor — see for example Eq.(32)— we have nontrivial modifications because of the presence of the functions $P(r)$ and its derivatives, which contain explicitly the charge. In the same way we would expect to obtain modifications for the deformed Einstein field equations Eq.(22).

## 5 Conclusions and perspectives

There are different ways to construct a noncommutative geometric approach to gravity, as it was pointed out previously. At the present time, there is no such a quantum theory of gravity. Despite the enormous amount of approaches to deform the geometry —some of them mentioned in the introduction of this paper— the work of Seiberg and Witten Ref. [4] remains as a very original approach. The beauty of it resides on its simplicity. The present work is in accordance with that prescription because we explicitly use the Moyal bracket in order to deform the product of functions having a dependence on coordinates. The expressions so obtained were written in almost all the cases as perturbative series depending on a perturbative parameter. In all those cases we checked that in the limit when that perturbative parameter tended to zero, we recover the usual commutative expressions.

In the present approach of a deformed R-N black hole, the Hawking temperature corresponding to this configuration is not modified by the embedding. It is not clear if the present approach modifies the stability conditions.
On the other hand, we have found an analytic expression for the noncommutative corrections to the event horizon area, a quantity that is related to the entropy of the R-N black hole.

We developed the calculations of the noncommutative Ricci tensors, but the mathematical expressions that we obtained become very heavy quite soon.

We found a first order correction to the Kretschmann scalar. This was an interesting calculation, because there appears crossed products between standard and noncommutative components of the Riemann tensor. Nevertheless, at the end the expression we got exhibits a correction to the first order in the perturbative parameter. This implies that the effects of quantization arise to the first order in our present approach.

For the case of the deformed versions of Einstein field equations, Eq.(22), it is not clear how to construct a suitable noncommutative stress tensor $T^i_j$. A possible solution to this problem would be to proceed like in Ref. [12], working with approximate expressions —given in terms of powers of the noncommutative parameter, in a similar way that we have proceeded in this paper— for the components of the Ricci tensor.

6 Acknowledgements

We would like to thank to Hugo G. Compeán for very helpful suggestions and for pointing us to some of the references of this paper. We also thank to Luis Lopez for helpful discussions.

The work of C. S. was partially supported by Prodep. S. V. was supported by a Posdoctoral Grant of Prodep, MEXICO.

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