BOUND FOR THE COCHARACTERS OF THE IDENTITIES OF IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}_2(\mathbb{C})$

MÁTYÁS DOMOKOS

Dedicated to Vesselin Drensky on his 70th birthday

Abstract. For each irreducible finite dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $2 \times 2$ traceless matrices, an explicit uniform upper bound is given for the multiplicities in the cocharacter sequence of the polynomial identities satisfied by the given representation.

1. Introduction

Let $\rho : g \rightarrow \mathfrak{gl}(V)$ be a finite dimensional representation of the Lie algebra $g$ over a field $K$ of characteristic zero; that is, $\mathfrak{gl}(V) = \text{End}_K(V)$, the space of all $K$-linear transformations of the finite dimensional $K$-vector space $V$, viewed as a Lie algebra with Lie product $[A,B] := A \circ B - B \circ A$ for $A,B \in \text{End}_K(V)$, and $\rho$ is a homomorphism of Lie algebras. Denote by $F_m := K\langle x_1,\ldots,x_m \rangle$ the free associative $K$-algebra with $m$ generators. Consider $F_m$ a subalgebra of $F_{m+1}$ in the obvious way, and write $F := \bigcup_{m=1}^{\infty} F_m$ for the free associative algebra of countable rank. We say that $f = 0$ is an identity of the representation $\rho$ of $g$ (or briefly, of the pair $(g,\rho)$) for some $f \in F_m$ if for any elements $A_1,\ldots,A_m \in g$ we have the following equality in the associative $K$-algebra $\text{End}_K(V)$:

$$f(\rho(A_1),\ldots,\rho(A_m)) = 0 \in \text{End}_K(V).$$

Note that an identity of the representation $\rho$ of the Lie algebra $g$ is also called in the literature a weak polynomial identity for the pair $(\text{End}_K(V),\rho(g))$. This notion was introduced and powerfully applied first by Razmyslov [13, 14, 15, 16] (see Drensky [8] for a recent survey on weak polynomial identities). Set

$$I(g,\rho) := \{ f \in F \mid f = 0 \text{ is an identity of } (g,\rho) \}.$$

Clearly $I(g,\rho)$ is an ideal in $F$ stable with respect to all $K$-algebra endomorphisms of $F$ of the form $x_i \mapsto u_i$, where $u_i$ for $i = 1,2,\ldots$ is an element of the Lie subalgebra of $F$ generated by $x_1,x_2,\ldots$. In particular, the general linear group $\text{GL}_m(K)$ acts on $F_m$ via $K$-algebra automorphisms: for $g = (g_{ij})_{i,j=1}^{m}$ we have

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\[ g \cdot x_j = \sum_{i=1}^m g_{ij}x_i, \text{ and } I(g, \rho) \cap F_m \text{ is a } GL_m(K)\text{-invariant subspace of } F_m. \] The multilinear component of \( F_m \) is
\[ P_m := \text{Span}_K \{ x_{\pi(1)} \cdots x_{\pi(m)} \mid \pi \in S_m \}, \]
where \( S_m \) is the symmetric group of degree \( m \). It is well known that when \( \text{char}(K) = 0 \), the ideal \( I(g, \rho) \) is determined by the multilinear components \( I(g, \rho) \cap P_m, m = 1, 2, \ldots \). Identifying \( S_m \) with the subgroup of permutation matrices in \( GL_m(K) \) we get its action on \( F_m \) via \( K \)-algebra automorphisms (more explicitly, \( \pi \in S_m \) is the automorphism of \( F_m \) given by \( x_i \mapsto x_{\pi(i)} \)), and the subspaces \( P_m \) and \( I(g, \rho) \cap P_m \) are \( S_m \)-invariant. Define the \( m \text{th cocharacter of } (g, \rho) \) as
\[ \chi_m(g, \rho) := \text{the character of the } S_m\text{-module } P_m/(I(g, \rho) \cap P_m). \]
We call
\[ \chi(g, \rho) := (\chi_m(g, \rho) \mid m = 1, 2, \ldots) \]
the cocharacter sequence of \( (g, \rho) \). The irreducible \( S_m \)-modules are labeled by partitions of \( m \); let \( \chi^\lambda \) denote the character of the irreducible \( S_m \)-module associated to the partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \vdash m \). We have
\[ \chi_m(g, \rho) = \sum_{\lambda \vdash m} \text{mult}_\lambda(g, \rho)\chi^\lambda, \]
and we are interested in the multiplicities \( \text{mult}_\lambda(g, \rho) \) of the irreducible \( S_m \)-characters in the cocharacter sequence. Note that the value of \( \chi_m(g, \rho) \) on the identity element of \( S_m \) is
\[ c_m(g, \rho) := \dim_K (P_m/(I(g, \rho) \cap P_m)), \]
and
\[ (c_m(g, \rho) \mid m = 1, 2, \ldots) \]
is called the codimension sequence of \( (g, \rho) \). It was proved by Gordienko [10] that \( \lim_{m \to \infty} \sqrt[m]{c_m(g, \rho)} \) exists and is an integer. As is observed in [10, Example 3], an obvious upper bound for \( c_m(g, \rho) \) can be obtained from the fact that there is a natural \( K \)-linear embedding
\[ (1) \quad P_m/(I(g, \rho) \cap P_m) \hookrightarrow \text{Hom}_K(\rho(g)^{\otimes m}, \text{End}_K(V)). \]
Our starting observation is that the adjoint representation of \( g \) on itself induces a natural representation of \( g \) on \( \rho(g)^{\otimes m} \) (the \( m \text{th tensor power of } \rho(g) \)) and on \( \text{End}_K(V) \), such that the image of the embedding (1) is contained in the subspace of \( g \)-module homomorphisms from \( \rho(g)^{\otimes m} \) to \( \text{End}_K(V) \). So (1) can be refined as
\[ (2) \quad P_m/(I(g, \rho) \cap P_m) \hookrightarrow \text{Hom}_g(\rho(g)^{\otimes m}, \text{End}_K(V)). \]
This will be used to give an upper bound for the multiplicities in the cocharacter sequence \( \chi(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \) of the \( d \)-dimensional irreducible representation
\[ \rho^{(d)}: \mathfrak{sl}_2(\mathbb{C}) \to gl(\mathbb{C}^d) = \mathbb{C}^{d \times d} \]
of \( \mathfrak{sl}_2(\mathbb{C}) \) for \( d = 1, 2, \ldots \). Note that throughout the paper we shall identify \( gl(\mathbb{C}^d) \) with the associative algebra \( \mathbb{C}^{d \times d} \) of \( d \times d \) complex matrices, viewed as a Lie algebra with Lie bracket \([A, B] = AB - BA\).

**Theorem 1.1.** The multiplicity \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \) in \( \chi(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \) is non-zero only if \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) (i.e., \( \lambda \) has at most 3 non-zero parts), and in this case we have the inequality
\[ \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \leq 3^{d-2}. \]
Remark 1.2. (i) The exact values of \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \) are known for \( d \leq 3 \). For \( d = 1 \) all the multiplicities are obviously zero. It was proved in [12] (see also [7, Exercise 12.6.12]) that

\[
\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(2)}) = 1 \quad \text{for all } \lambda = (\lambda_1, \lambda_2, \lambda_3).
\]

The multiplicities \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(3)}) \) are computed in [3, Theorem 3.7, Proposition 3.8]. It turns out that \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(3)}) \in \{1, 2, 3\} \) for each \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \).

(ii) Theorem 1.1 shows in particular that for each dimension \( d \), there is a uniform bound (depending on \( d \) only) for the multiplicities \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \). For comparison we mention that the multiplicities in the cocharacter sequence of the ordinary polynomial identities of \( 2 \times 2 \) matrices are unbounded: see [6] and [9]. For example, for any partition \( \lambda = (\lambda_1, \lambda_2) \) with \( \lambda_2 > 0 \), the multiplicity is \( (\lambda_1 - \lambda_2 + 1)\lambda_2 \). On the other hand, the cocharacter multiplicities of any PI algebra are polynomially bounded by [2].

(iii) There is no uniform upper bound independent of \( d \) for the multiplicities \( \text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \), because by Proposition 1.1, \( \max\{\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \mid m = 1, 2, \ldots, \lambda^t \} \geq d - 1 \) for \( d \geq 2 \).

(iv) The irreducible representations of \( \mathfrak{sl}_2(\mathbb{C}) \) are defined over \( \mathbb{Q} \). For any field \( K \) of characteristic zero and any positive integer \( d \), the Lie algebra \( \mathfrak{sl}_2(K) \) has a unique (up to isomorphism) \( d \)-dimensional irreducible representation \( \rho^{(d)}_K \) over \( K \). By well-known general arguments, the multiplicities \( \text{mult}_\lambda(\mathfrak{sl}_2(K), \rho^{(d)}_K) \) do not depend on \( K \). Therefore Theorem 1.1 implies that \( \text{mult}_\lambda(\mathfrak{sl}_2(K), \rho^{(d)}_K) \leq 3^{d-2} \) for any field \( K \) of characteristic zero.

(v) A different interpretation and approach to the study of \( \text{Hom}_g(\rho(g)^n \otimes_m, \text{End}_K(V)) \) for \( g = \mathfrak{sl}_2(\mathbb{C}) \) and \( \rho = \rho^{(d)} \) is given in our parallel preprint [4], using classical invariant theory.

We close the introduction by mentioning the recent paper of da Silva Macedo and Koshlukov [3, Theorem 3.7], where the codimension growth of polynomial identities of representations of Lie algebras is studied. In particular, in [3, Theorem 3.7] the identities of representations of \( \mathfrak{sl}_2(\mathbb{C}) \) play a decisive role.

2. MATRIX COMPUTATIONS

Denote by \( \overline{\rho}^{(d)} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^{d \times d}) \) the representation given by

\[
\overline{\rho}^{(d)}(A)(L) = \rho^{(d)}(A)L - L\rho^{(d)}(A) \quad \text{for } A \in \mathfrak{sl}_2(\mathbb{C}), \ L \in \mathbb{C}^{d \times d}.
\]

We have \( \overline{\rho}^{(d)} \cong \rho^{(d)} \otimes \rho^{(d)^*} \). The representations of \( \mathfrak{sl}_2(\mathbb{C}) \) are self-dual, and so by the Clebsch-Gordan rules we have

\[
\overline{\rho}^{(d)} \cong \rho^{(d)} \otimes \rho^{(d)^*} \cong \bigoplus_{n=1}^{d} \rho^{(2n-1)}.
\]

We shall need an explicit decomposition of \( \mathbb{C}^{d \times d} \) as a direct sum of minimal \( \overline{\rho}^{(d)} \)-invariant subspaces.

Set

\[
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

so \( e, f, h \) is a \( \mathbb{C} \)-vector space basis of \( \mathfrak{sl}_2(\mathbb{C}) \), with \([h, e] = 2e, [h, f] = -2f\), and \([e, f] = h\).
Recall that given a representation $\psi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$, by a highest weight vector we mean a non-zero element $w \in V$ such that $\psi(e)(w) = 0 \in V$ and $\psi(h)(w) = nw$ for some non-negative integer $n$ (the non-negative integer $\lambda$ is called the weight of $w$); in this case $w$ generates a minimal $\mathfrak{sl}_2(\mathbb{C})$-invariant subspace in $V$, on which the representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $\rho^{(n+1)}$. Moreover, any finite dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module contains a unique (up to non-zero scalar multiples) highest weight vector.

**Lemma 2.1.** Consider the $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^{d \times d}$ via the representation $\rho^{(d)}$. To simplify notation set $\rho := \rho^{(d)}$ and $\tilde{\rho} := \tilde{\rho}^{(d)}$.

(i) $\rho(e)^n$ is a highest weight vector in $\mathbb{C}^{d \times d}$ of weight $2n$ for $n = 0, 1, \ldots, d - 1$.

(ii) $\rho(e)^{n-1}$ generates a minimal $\tilde{\rho}$-invariant subspace $V_n$ on which $\mathfrak{sl}_2(\mathbb{C})$ acts via $\rho^{(2n-1)}$ for $n = 1, \ldots, d$.

(iii) $\mathbb{C}^{d \times d} = \bigoplus_{n=1}^{d} V_n$.

(iv) For $L_1 \in V_{n_1}$ and $L_2 \in V_{n_2}$ with $1 \leq n_1 \neq n_2 \leq d$ we have $\text{Tr}(L_1L_2) = 0$.

**Proof.** (i) We have $\rho(e)(\rho(e)^n) = \rho(e)\rho(e)^n - \rho(e)^n\rho(e) = 0$ and $\rho(h)(\rho(e)^n) = \rho([h, e])\rho(e)^{n-1} + \rho(e)(\rho([h, e])\rho(e)^{n-2} + \cdots + \rho(e)^{n-1}\rho([h, e]) = 2n\rho(e)^n$.

This shows that $\rho(e)^n$ is a highest weight vector of weight $2n$ for the representation $\tilde{\rho}$.

(ii) Statement (i) implies that $\rho(e)^{n-1}$ generates an irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of $\tilde{\rho}$ isomorphic to $\rho^{(2n-1)}$ for $n = 1, \ldots, d$.

(iii) follows from (ii) and (i).

(iv) Consider the symmetric non-degenerate bilinear form

$$\beta : \mathbb{C}^{d \times d} \times \mathbb{C}^{d \times d} \to \mathbb{C}, \quad (L, M) \mapsto \text{Tr}(LM).$$

Note that $\beta$ is $\tilde{\rho}$-invariant:

$$\beta([\rho(A), L], M) + \beta(L, [\rho(A), M]) = \text{Tr}([\rho(A), L]M) + \text{Tr}(L[\rho(A), M]) = \text{Tr}([\rho(A), LM]) = 0 \quad \text{for any } A \in \mathfrak{sl}_2(\mathbb{C}).$$

The radical of the bilinear form $\beta_{V_n} : V_n \times V_n \to \mathbb{C}$ (the restriction of $\beta$ to $V_n \times V_n$) is a $\tilde{\rho}$-invariant subspace in $V_n$, so it is either $V_n$ or $\{0\}$. We claim that it is not $V_n$. Indeed, $V_n$ contains a non-zero diagonal matrix $D$ with real entries, since the zero weight subspace in $\mathbb{C}^{d \times d}$ (with respect to $\tilde{\rho}(h)$) is the subspace of diagonal matrices, and $V_n$ intersects the zero-weight space in a 1-dimensional subspace (defined over the reals). Now being a sum of squares of non-zero real numbers, $0 \neq \text{Tr} (D^2) = \beta(D, D)$. Thus $\beta_{V_n}$ is non-degenerate. The representation $\tilde{\rho}$ is multiplicity free by (i), and by (ii) and (iii), every $\tilde{\rho}$-invariant subspace is of the form $\sum_{j \in J} V_j$ for some subset $J \subseteq \{1, 2, \ldots, d\}$. As we showed above, the orthogonal complement of $V_n$ (with respect to $\beta$) is disjoint from $V_n$, so it is the sum of the other minimal invariant subspaces $V_j$, $j \in \{1, \ldots, d\} \setminus \{n\}$. $\square$

The representation $\rho^{(2)}$ is the defining representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$, and $\rho^{(d)}$ is the $(d - 1)$th symmetric tensor power of $\rho^{(2)}$. Denote by $x, y$ the standard basis vectors in $\mathbb{C}^2$, and take the basis $x^{d-1}, x^{d-1}y, \ldots, y^{d-1}$ in the $(d - 1)$th symmetric tensor power of $\mathbb{C}^2$. Then denoting by $E_{i,j}$ the matrix unit with entry 1 in the $(i, j)$ position and zeros in all other positions, the representation $\rho^{(d)}$ as a matrix
representation \( \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{C}^{d \times d} \) is given as follows:

\[
\rho^{(d)}(e) = \sum_{i=1}^{d-1} i E_{i,i+1}, \quad \rho^{(d)}(f) = \sum_{i=1}^{d-1} (d-i) E_{i+1,i}, \quad \rho^{(d)}(h) = \sum_{i=1}^{d} (d+1-2i) E_{i,i}
\]

Lemma 2.2. For \( d \geq 3 \) the \( \mathbb{C} \)-vector space \( \mathbb{C}^{d \times d} \) is spanned by

\[
\{ \rho^{(d)}(A_1) \cdots \rho^{(d)}(A_{d-1}) \mid A_1, \ldots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C}) \}.
\]

Proof. To simplify the notation write \( \rho := \rho^{(d)} \) and \( \tilde{\rho} := \tilde{\rho}^{(d)} \). Let \( \mathcal{L} \) be the subspace of \( \mathbb{C}^{d \times d} \) spanned by the products \( \rho(A_1) \cdots \rho(A_{d-1}) \), where \( A_1, \ldots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C}) \).

Clearly \( \mathcal{L} \) is a \( \tilde{\rho} \)-invariant subspace of \( \mathbb{C}^{d \times d} \). Since the representation \( \tilde{\rho} \) is multiplicity free by (ii), we have \( \mathcal{L} = \sum_{J \subseteq \{1,2,\ldots,d\}} V_J \) for some subset \( J \subseteq \{1,2,\ldots,d\} \) by Lemma 2.1 (ii) and (iii). Therefore to prove the equality \( \mathcal{L} = \mathbb{C}^{d \times d} \) it is sufficient to show that \( \mathcal{L} \cap V_n \neq \{0\} \) for each \( n = 1, \ldots, d \), or equivalently, that \( \mathcal{L} \) is not contained in \( \sum_{j \in J \setminus \{n\}} V_j \). Since \( V_d \) is generated by \( \rho(e)^{d-1} \) in \( \mathcal{L} \), we have \( V_d \subseteq \mathcal{L} \). Moreover, to prove \( \mathcal{L} \nsubseteq \sum_{j \in J \setminus \{n\}} V_j \) for \( n = 0, 1, \ldots, d-2 \), it is sufficient to present an element \( L_n \in \mathcal{L} \) with \( \text{Tr}(\rho(e)^n L_n) \neq 0 \) by Lemma 2.1 (i), (ii) and (iv). We shall give below such elements \( L_n \in \mathcal{L} \) for \( n = 0, 1, \ldots, d-2 \).

For \( n = 1, \ldots, d-1 \) we have

\[
\rho(e)^n = \sum_j j \cdot (j+1) \cdots (j+n-1) E_{j,j+n}
\]

\[
\rho(f)^n = \sum_j (d-j) \cdot (d-j-1) \cdots (d-j-n+1) E_{j,j+n}
\]

and \( \rho(e)^0 = I_d = \rho(f)^0 \), where \( I_d \) is the \( d \times d \) identity matrix. It follows that for \( n = 1, \ldots, d-1 \),

\[
\rho(e)^n \rho(f)^n = \sum_j j(j+1) \cdots (j+n-1) \cdot (d-j)(d-j-1) \cdots (d-j-n+1) E_{j,j}
\]

is a diagonal matrix with non-negative integer entries, and the \((1,1)\)-entry is positive. The same holds for \( \rho(e)^0 \rho(f)^0 = I_d \). For \( n \) with \( d-1-n \) even, \( \rho(h)^{d-1-n} \) is the square of a diagonal matrix with integer entries, and its \((1,1)\)-entry is positive. Hence \( \text{Tr}(\rho(e)^n \rho(f)^n \rho(h)^{d-1-n}) \neq 0 \), being a positive integer. So in this case we may take \( L_n := \rho(f)^n \rho(h)^{d-1-n} \). For \( n < d-2 \) with \( d-1-n \) odd, note that \( \rho(e)\rho(f) - \rho(f)\rho(e) = \rho([e,f]) = \rho(h) \), and thus

\[
\rho(f)^n \rho(h)^{d-2-n} = \rho(f)^n \rho(h)^{d-3-n}(\rho(e)\rho(f) - \rho(f)\rho(e))
\]

also belongs to \( \mathcal{L} \). Since \( \rho(h)^{d-2-n} \) is a diagonal matrix with non-negative integer entries, and with a positive \((1,1)\)-entry, we may take \( L_n := \rho(f)^n \rho(h)^{d-2-n} \) in this case. It remains to deal with the case \( n = d-2 \). Then

\[
\rho(e)^{d-2} \rho(f)^{d-2} = (d-1)((d-2)!)^2 \cdot (E_{1,1} + E_{2,2}),
\]

hence taking \( L_{d-2} := \rho(f)^{d-2} \rho(h) \) we get

\[
\text{Tr}(\rho(e)^{d-2} L_{d-2}) = \text{Tr}((d-1)((d-2)!)^2 \cdot ((d-1)E_{11} + (d-3)E_{22})
\]

\[
= (2d-4)(d-1)((d-2)!)^2,
\]

which is non-zero for \( d \geq 3 \). This finishes the proof of the equality \( \mathcal{L} = \mathbb{C}^{d \times d} \). \( \square \)
3. Adjoint invariants

Denote by $\text{ad} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$ the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ on itself, so $\text{ad}(A)(B) = [A, B]$ for $A, B \in \mathfrak{sl}_2(\mathbb{C})$. Take the $n$-fold direct sum $\text{ad}^{\oplus n} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})^{\oplus n})$ of the adjoint representation, and write $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ for the algebra of $\text{ad}^{\oplus n}$-invariant polynomial functions on $\mathfrak{sl}_2(\mathbb{C})^{\oplus n}$. There is a right action of $\text{GL}_n(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})^n$ that commutes with $\text{ad}^{\oplus n}$: for $g = (g_{ij})_{i,j=1}^n$ and $(A_1, \ldots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have

$$(A_1, \ldots, A_n) \cdot g := \left( \sum_{i=1}^n g_{i1} A_i, \ldots, \sum_{i=1}^n g_{in} A_i \right).$$

This induces a left $\text{GL}_n(\mathbb{C})$-action on the coordinate ring $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$: for $g \in \text{GL}_n(\mathbb{C})$, $f \in \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ and $(A_1, \ldots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have $(g \cdot f)(A_1, \ldots, A_n) = f((A_1, \ldots, A_n) \cdot g)$.

**Lemma 3.1.** Consider the linear map $\iota : F_m = \mathbb{C} \langle x_1, \ldots, x_m \rangle \to \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ given by

$$\iota(f)(A_1, \ldots, A_{m+d-1}) = \text{Tr}(f(\rho^{(d)}(A_1), \ldots, \rho^{(d)}(A_m)) \cdot \rho^{(d)}(A_{m+1}) \cdots \rho^{(d)}(A_{m+d-1}))$$

for $f \in F_m$ and $(A_1, \ldots, A_{m+d-1}) \in \mathfrak{sl}_2(\mathbb{C})^{m+d-1}$. It has the following properties:

(i) The image of $\iota$ is contained in the subalgebra $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]^{\mathfrak{sl}_2(\mathbb{C})}$ of $\mathfrak{sl}_2(\mathbb{C})$-invariants.

(ii) For $d \geq 3$ the kernel of $\iota$ is the ideal $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_m$.

(iii) The map $\iota$ is $\text{GL}_m(\mathbb{C})$-equivariant, where we restrict the $\text{GL}_{m+d-1}(\mathbb{C})$-action on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ to the subgroup $\text{GL}_m(\mathbb{C}) \cong \{ \begin{pmatrix} g & 0 \\ 0 & I_{d-1} \end{pmatrix} \mid g \in \text{GL}_m(\mathbb{C}) \}$ in $\text{GL}_{m+d-1}(\mathbb{C})$.

**Proof.** For notational simplicity we shall write $\rho$ instead of $\rho^{(d)}$.

(i) By linearity of $\iota$ it is sufficient to show that $\iota(x_1 \cdots x_k)$ is an $\mathfrak{sl}_2(\mathbb{C})$-invariant for any $k \leq m$. Setting $n = k + d - 1$, $B_1 = A_1$, $B_k = A_{i_k}$, $B_{k+1} = A_{m+1}$, $B_{n+2} = A_{m+d-1}$ we have

$$(5) \quad \iota(x_1 \cdots x_k)(A_1, \ldots, A_{m+d}) = \text{Tr}(\rho(B_1) \cdots \rho(B_n)).$$

For any $X \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$0 = \text{Tr}([\rho(X), \rho(B_1) \cdots \rho(B_n)])$$

$$= \text{Tr}(\sum_{j=1}^n \rho(B_1) \cdots \rho(B_{j-1})[\rho(X), \rho(B_j)] \rho(B_{j+1}) \cdots \rho(B_n))$$

$$= \text{Tr}(\sum_{j=1}^n \rho(B_1) \cdots \rho(B_{j-1})[X, B_j] \rho(B_{j+1}) \cdots \rho(B_n))$$

$$= \sum_{j=1}^n \text{Tr}(\rho(B_1) \cdots \rho(B_{j-1})[X, B_j] \rho(B_{j+1}) \cdots \rho(B_n)).$$

The equalities [5] and

$$\sum_{j=1}^n \text{Tr}(\rho(B_1) \cdots \rho(B_{j-1})[X, B_j] \rho(B_{j+1}) \cdots \rho(B_n)) = 0$$

hold.
mean that $\iota(x_i \cdots x_{i_k})$ is $\mathfrak{sl}_2(\mathbb{C})$-invariant, so (i) holds.

(ii) Suppose that $f \in \ker(\iota)$. Then $\text{Tr}(f(\rho(A_1), \ldots, \rho(A_m))B) = 0$ for all $B \in \mathbb{C}^{d \times d}$ by Lemma 2.2. By non-degeneracy of the trace we get $f(\rho(A_1), \ldots, \rho(A_m)) = 0$. That is, $f \in I(\mathfrak{sl}_2(\mathbb{C}), \rho)$. Thus $\ker(\iota) \subseteq I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m$. The reverse inclusion $I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m \subseteq \ker(\iota)$ is obvious.

(iii) Take $g = (g_{ij})_{i,j=1}^m \in \text{GL}_m(\mathbb{C})$. For $f \in F_m$ and $(A_1, \ldots, A_m) \in \mathfrak{sl}_2(\mathbb{C})^m$ we have (by linearity of $\rho$)

$$
\iota(g \cdot f)(A_1, \ldots, A_m)
= \text{Tr}(f(\sum_{i=1}^m g_{i1}\rho(A_i), \ldots, \sum_{i=1}^m g_{im}\rho(A_i))) \cdot \rho(A_{m+1}) \cdots \rho(A_{m+d})
= \text{Tr}(f(\rho(A_1), \ldots, \rho(A_m))) \cdot \rho(A_{m+1}) \cdots \rho(A_{m+d})
= (g \cdot \iota(f))(A_1, \ldots, A_m).
$$

This shows (iii).

Restricting the action of $\text{GL}_n(\mathbb{C})$ on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ to the subgroup of diagonal matrices we get an $\mathbb{N}_0^n$-grading on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$, preserved by the action of $\mathfrak{sl}_2(\mathbb{C})$. Denote by $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ the multihomogeneous component of multidegree $(1, \ldots, 1)$; this is the space of $n$-linear functions on $\mathfrak{sl}_2(\mathbb{C})$. The spaces $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ and $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$ are $S_n$-invariant (where we restrict the $\text{GL}_n(\mathbb{C})$-action to its subgroup $S_n$ of permutation matrices). Lemma 3.1 has the following immediate consequence:

**Corollary 3.2.** For $d \geq 3$ the restriction of $\iota$ to the multilinear component $P_m$ of $\mathbb{C}(x_1, \ldots, x_m)$ factors through an $S_m$-equivariant $\mathbb{C}$-linear embedding

$$
\iota : P_m/(I(\mathfrak{sl}_2(\mathbb{C})) \cap P_m) \to \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^{m+d-1})}
$$

where on the right hand side we consider the restriction of the $S_{m+d-1}$-action to its subgroup $S_m$ (the stabilizer in $S_{m+d-1}$ of the elements $m+1, m+2, \ldots, m+d-1$).

For a partition $\lambda \vdash m$ denote by $r(\lambda)$ the multiplicity of $\chi^\lambda$ in the restriction to $S_m$ of the $S_{m+d-1}$-module $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]_{(1^{m+d-1})}^{\mathfrak{sl}_2(\mathbb{C})}$. Corollary 3.2 immediately implies the following:

**Corollary 3.3.** For $d \geq 3$ and any partition $\lambda \vdash m$ we have the inequality

$$\text{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \leq r(\lambda).$$

The $S_n$-character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^n)}$ is known:

**Proposition 3.4.** For a partition $\lambda \vdash n$ denote by $\nu(\lambda)$ the multiplicity of $\chi^\lambda$ in the $S_n$-character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^n)}$. Then we have

$$
\nu(\lambda) = \begin{cases} 1 \text{ for } \lambda = (\lambda_1, \lambda_2, \lambda_3) \text{ with } \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \text{ modulo } 2 \\
0 \text{ otherwise.}
\end{cases}
$$

**Proof.** The $\text{GL}_n(\mathbb{C})$-module structure of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ is given for example in [12, Theorem 2.2]. The isomorphism types of the irreducible $\text{GL}_n(\mathbb{C})$-module
direct summands of the degree $n$ homogeneous component of $O[sl_2(\mathbb{C})^n]$ are labeled by partitions of $n$ with at most 3 non-zero parts. The multiplicity $\mu(\lambda)$ of the irreducible $GL_n(\mathbb{C})$-module $W_\lambda$ in the degree $n$ homogeneous component of $O[sl_2(\mathbb{C})^n]^{sl_2(\mathbb{C})}$ is 1 if $\lambda_1, \lambda_2, \lambda_3$ have the same parity and is zero otherwise. Note finally that the multilinear component of $W_\lambda$ is $S_n$-stable, and its $S_n$-character is $\chi^\lambda$ (see for example [1 Corollary 6.3.11]).

Following [11 Section I.1] for partitions $\lambda \vdash n$ and $\mu \vdash k$ we write $\lambda \subset \mu$ is $\lambda_i \leq \mu_i$ for all $i$. Moreover, given $\lambda \vdash m$ and $\mu \vdash m+d-1$ with $\lambda \subset \mu$, by a standard tableau of shape $\mu/\lambda$ we mean a sequence $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(d-1)}$ of partitions $\lambda^{(i)} \vdash m+i$, where $\lambda^{(0)} = \lambda, \lambda^{(d-1)} = \mu$. By the well-known branching rules for the symmetric group, for $\lambda \vdash m$ the multiplicity of $\chi^\lambda$ in the restriction to $S_m$ of the irreducible $S_{m+d-1}$-character $\chi^\mu$ equals the number of standard tableaux of shape $\mu/\lambda$ (see for example [1 Theorem 6.4.11]). Therefore Proposition 3.4 has the following consequence.

**Corollary 3.5.** We have the equality

$$r(\lambda) = |\{T \mid T \text{ is a standard skew tableau of shape } \mu/\lambda, \mu \vdash m+d-1, \mu = (\mu_1, \mu_2, \mu_3), \mu_1 \equiv \mu_2 \equiv \mu_3 \mod 2\}|.$$

**Corollary 3.6.** For $d \geq 3$ we have the inequality $r(\lambda) \leq 3^{d-2}$.

**Proof.** Associate to a standard skew tableau $T = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(d-1)}$ of shape $\mu/\lambda$, where $\mu = (\mu_1, \mu_2, \mu_3) \vdash m+d-1$ and $\mu_1 \equiv \mu_2 \equiv \mu_3 \mod 2$ the function $f_T : \{1, \ldots, d-1\} \to \{1, 2, 3\}$, which maps $j \in \{1, \ldots, d-1\}$ to the unique $i \in \{1, 2, 3\}$ such that the $i$th component of the partition $\lambda^{(j)}$ is 1 greater than the $i$th component of $\lambda^{(j-1)}$. The assignment $T \mapsto f_T$ is obviously an injective map from the set of standard skew tableaux of shape $\mu/\lambda$ into the set of functions $\{1, \ldots, d-1\} \to \{1, 2, 3\}$. We claim that at most $3^{d-2}$ functions are contained in the image of this map. Indeed, if the three parts of $\lambda^{(d-3)}$ have the same parity, then $(f_T(d-2), f_T(d-1)) \in \{(1, 1), (2, 2), (3, 3)\}$, since the three parts of $\mu = \lambda^{(d-1)}$ must have the same parity. If the three parts of $\lambda^{(d-3)}$ do not have the same parity, say the first two components of $\lambda^{(d-3)}$ have the same parity, and the third part has the opposite parity, then $(f_T(d-2), f_T(d-1)) \in \{(1, 2), (2, 1)\}$. Hence $r(\lambda)$ is not greater than 3-times the number of functions from a $(d-3)$-element set to a 3-element set. Thus $r(\lambda) \leq 3^{d-2}$. \hfill \Box

### 3.1. Proof of Theorem 1.1.

For $d \geq 3$ the statement follows from Corollary 3.6 and Corollary 3.4. For the cases $d \leq 3$ see Remark 1.2 (i).

### 4. A lower bound

**Proposition 4.1.** For $d \geq 2$ we have the equality

$$\text{mult}_{(d-1,1)}(sl_2(\mathbb{C}), \rho^{(d)}) = d-1.$$

**Proof.** For $k = 0, 1, \ldots, d-2$ consider the element

$$w_k := x_1^k x_2 \alpha_1^{d-2-k} \in \mathbb{C}[x_1, x_2] = F_2.$$  

These elements are $GL_2(\mathbb{C})$-highest weight vectors with weight $(d-1, 1)$, hence each generates an irreducible $GL_2(\mathbb{C})$-submodule isomorphic to $W_{(d-1,1)}$ (see the proof of Proposition 3.4 for the notation $W_\lambda$: it is the polynomial $GL_2(\mathbb{C})$-module with
highest weight $\lambda = (\lambda_1, \lambda_2)$. Moreover, they are linearly independent modulo the ideal $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$: indeed, make the substitution $x_1 \mapsto \rho(h), x_2 \mapsto \rho(e)$. Then we get

$$w_k(\rho(h), \rho(e)) = \left( \sum_{i=1}^{d} (d + 1 - 2i)E_{i,i} \right)^k \cdot \left( \sum_{i=1}^{d-1} iE_{i,i+1} \right) \cdot \left( \sum_{i=1}^{d} (d + 1 - 2i)E_{i,i} \right)^{d-2-k}$$

$$= 2 \sum_{i=1}^{d-1} i(d + 1 - 2i)^k(d - 1 - 2i)^{d-2-k}E_{i,i+1}.$$ 

Denote by $Z = (Z_{i,j})_{i,j=1}^{d-1}$ the $(d-1) \times (d-1)$ matrix whose $(i,k+1)$ entry is the $(i,i+1)$-entry of $w_k(\rho(h), \rho(e))$ (i.e. the coefficient of $E_{i,i+1}$ on the right hand side of the above equality). If $i \neq d-1$, then

$$Z_{i,k+1} = 2(d - 1 - 2i)^{d-2} \cdot \left( \frac{d + 1 - 2i}{d - 1 - 2i} \right)^k.$$ 

Thus when $d$ is even, $Z$ is obtained from a Vandermonde matrix via multiplying each row by a non-zero integer. Since the numbers $\frac{d + 1 - 2i}{d - 1 - 2i}, \; i = 1, \ldots, d - 1$ are distinct, we conclude that $\det(Z) \neq 0$. When $d = 2f - 1$ is odd, the $(f-1)$th row of $Z$ is

$$(0, \ldots, 0, 2(f-1)^{d-2}).$$ 

Expand the determinant of $Z$ along this row: the $(d-2) \times (d-2)$ minor of $Z$ obtained by removing the $(f-1)$th row and the last column of $Z$ is again obtained from a Vandermonde matrix by multiplying each row by a non-zero integer. So $\det(Z)$ is non-zero also when $d$ is odd. This shows that the elements $w_k(\rho(h), \rho(e)), \; k = 0, 1, \ldots, d-2$ are linearly independent in $\mathbb{C}^{d \times d}$. Consequently, no non-trivial linear combination of $w_0, w_1, \ldots, w_{d-2}$ belongs to $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$. It follows that $F_2/(I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_2)$ contains the irreducible $\text{GL}_2(\mathbb{C})$-module $W_{\lambda_1}$ with multiplicity $\geq d - 1$. This multiplicity is in fact equal to $d - 1$, because $d - 1$ is the multiplicity of $W_{\lambda_1}$ as a summand in $F_2$. Recall finally that for $\lambda = (\lambda_1, \lambda_2) \vdash m$, the multiplicity of $\chi^\lambda$ in the cocharacter sequence coincides with the multiplicity of $W_{\lambda}$ in $F_2/(I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_2).$ 

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Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary
Email address: domokos.matyas@renyi.hu