ERGODIC OPTIMIZATION THEORY FOR NON-DEGENERATE AXIOM A FLOWS

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Abstract. In this article, we consider the weighted ergodic optimization problem for non-degenerate Axiom A attractors of a $C^2$ flow on a compact smooth manifold. The main result obtained in this paper is that for generic observables from function spaces $C^{0,\alpha}$ ($\alpha \in (0,1]$) and $C^1$ the minimizing measures are uniquely supported on a periodic orbit.

1. Introduction

Context and motivation. Ergodic optimization theory focuses on the ergodic measures on which a given observable taking a extreme ergodic average (maximum or minimum), which gives expression to the principle of least action in dynamical systems, and has strong connection with other fields, such as Aubry-Mather theory [Co2, Ma, CIPP] in Lagrangian Mechanics; ground state theory [BLL] in thermodynamics formalism and multifractal analysis; and controlling chaos [HO1, OGY, SGOY] in control theory.

In this paper, we study the typical optimization problem in weighted ergodic optimization theory for a non-degenerate Axiom A attractor of a $C^2$ flow on a compact smooth manifold. For discrete time case, ergodic optimization theory has been developed broadly. Among them, Yuan and Hunt proposed an open problem in [YH, Conjecture 1.1] on 1999, which provides a mathematical mechanism on Hunt and Ott’s experimental and heuristic results in [HO2, HO3] and becomes one of the fundamental questions raised in the field of ergodic optimization theory. Yuan and Hunt’s conjecture has attracted sustained attentions and yielded considerable results, for instances [BZ, Bo1, Bo2, Bo4, Co1, CLT, Mo, QS]. For a more comprehensive survey for the classical ergodic optimization theory, we refer the reader to Jenkinson [Jc1, Jc2], to Bochi [B], to Baraviera, Leplaideur, Lopes [BLL], and to Garibaldi [Ga] for a historical perspective of the development in this area. In our recent paper [HLMXZ], we extend the applicability of the theory both to a broader class of systems including Axiom A maps, Anosov diffeomorphisms and uniformly expanding maps and to a broader class of observables including both Hölder continuous functions and $C^1$ functions when it is

Huang is partially supported by NSF of China (11431012,11731003). Lian is partially supported by NSF of China (11725105,11671279). Xu is partially supported by NSF of China (11801538, 11871188). Zhang is partially supported by NSF of China (11701020,11871262).
well defined, which leads to a solution to Yuan and Hunt’s conjecture for $C^1$ smooth case. To our knowledge, because of difficulties appears on both conceptual level and technical level, there is no existing result of ergodic optimization theory for flows so far, which make the results obtained in the present paper the first achievement on flows towards ergodic optimization theory.

On the other hand, as mentioned in [HLMXZ], the reason of adding the nonconstant weight $\psi$ mainly lies in the studies on the zero temperature limit (or ground state) of the $(u, \psi)$-weighted equilibrium state for thermodynamics formalism (for more details, we refer readers to works [BF, BCW, FH]).

**Summary of the results.** To avoid unnecessary complexity, we only introduce the result in the framework of standard ergodic optimization theory. Consider a $C^2$ flow $\Phi$ on a compact smooth manifold $M$. Let $\Lambda$ be a non-degenerate Axiom A attractor of $\Phi$. By non-degenerating, we mean that there is no rest point of $\Phi$ falls in $\Lambda$. For a given observable $u : M \to \mathbb{R}$, the ergodic averages of $u$ on $\Lambda$ which is given by the integration of $u$ with respect to $\Phi|_\Lambda$-ergodic measures carry intrinsical information of the dynamics, in particular, among which the extreme values of the ergodic averages and the corresponding support measures is of most interest. As a consequence of the main result (Theorem 2.2) of the present paper, we have that for generic observables in function space $C^0,\alpha(M)$ and $C^1(M)$, the maximizing or minimizing measures are uniquely supported on a periodic orbit.

**Remarks on techniques of the proof.** It seems that the proof given in [HLMXZ] provides a more general mechanism in the study on ergodic optimization problems, which also shed a light on the case of flows for sure. However, the results of the present paper depend crucially on the continuous time nature of the system; that is to say, they do not follow from the properties of their time-1 maps. Therefore, we must build certain theoretical base and formalize certain basic techniques directly to address issues raised in the case of flows.

We mention three differences of note between our setting and the existing literatures at both conceptual level and technical level which pervade the arguments in this paper: (1) At conceptual level, the most significant issue is that the gap function of a discrete time periodic orbit, i.e. the minimum separation of finite isolated points, is not well defined for continuous periodic orbit. Such a gap function plays a key role in the proof of [HLMXZ]. (2) The presence of shear, i.e. the sliding of some orbits past other nearby orbits due to the slightly different speed at which they travel. This continuous time phenomenon causes tremendous amount of ”tail estimates” throughout this paper. (3) Several main fundamental theoretical tools are not existing and need to be rebuilt from the base, such as Anosov closing Lemma, Mañé-Conze-Guivarc’h-Bousch’s Lemma and
Periodic Approximation Lemma.

**Structure of this paper.** In Section 2, we set up the theoretic model and state the main results; In Section 3, we state some well known properties of non-degenerate Axiom A attractors, and some theoretical tools including Anosov closing Lemma, Mañé-Conze-Guivarc’h-Bousch’s Lemma and Periodic Approximation Lemma preparing for the proof the main results without proofs; In Section 4, we give the proof of Theorem 2.2, of which proving Part I) of the Theorem costs most efforts; As follows, we leave the proofs of all the technical Lemmas to Section 5. On one hand, readers may go through the main proof by assuming the validity of these technical Lemmas without extra interruptions; on the other hand, these technical Lemmas with their proofs may be of independent interest. Finally, we discuss the case when observables have higher regularity in Section 6 in which only some partial results are presented.

## 2. MAIN SETTING AND RESULTS

Let $M$ be a compact smooth Riemannian manifold with Riemannian metric $d$ and $\Phi = \{\phi_t : M \to M\}_{t \in \mathbb{R}}$ be a $C^2$ flow on $M$.

**Definition 2.1.** For $\Lambda \subset M$, $(\Lambda, \Phi)$ is called a non-degenerate Axiom A attractor if the following conditions hold:

1. $\Lambda$ is a nonempty $\Phi$-invariant compact set.
2. There exists an $\epsilon_0 > 0$ such that for any $x \in M$ with $d(x, \Lambda) < \epsilon_0$
   \[ \lim_{t \to \infty} d(\phi_t(x), \Lambda) = 0. \]
3. There exist $\lambda_0 > 0, C_0 > 1$ and a continuous splitting of tangential spaces of $M$ restricted on $\Lambda, T_x M = E^u_x \oplus E^c_x \oplus E^s_x \forall x \in \Lambda$, such that the following hold
   \[ (D_M \phi_t)_x (E^\tau_x) = E^\tau_{\phi_t(x)}, \quad \tau = u, c, s, \quad \forall t \in \mathbb{R} \text{ and } x \in \Lambda, \]
   \[ \max \left\{ \|D_M \phi_{-t} x\|_{E^c_x}, \|D_M \phi_t_{x}\|_{E^s_x} \right\} \leq C_0 e^{-\lambda_0 t}, \forall t \in \mathbb{R}^+, \]
   where $(D_M \phi_t)_x$ is the derivative of the time-$t$ map $\phi_t$ on $x$ with respect to space variables.
4. $\inf_{x \in \Lambda} \left\| \frac{d\phi_t(x)}{dt} \right\| > 0$ and $E^c_x = \text{span} \left\{ \frac{d\phi_t(x)}{dt} \right\}, \forall x \in \Lambda$.

Denote by $\mathcal{M}(\Lambda, \Phi)$ the set of all $\Phi$-invariant Borel probability measures on $\Lambda$, which is a non-empty convex and compact topological space with respect to weak* topology. Denote by $\mathcal{M}^e(\Lambda, \Phi) \subset \mathcal{M}(\Lambda, \Phi)$ the set of ergodic measures, which is the set of the extremal points of $\mathcal{M}(\Lambda, \Phi)$. Let $u : M \to \mathbb{R}$ and $\psi : M \to \mathbb{R}^+$ be continuous functions.
The quantity \( \beta(u; \psi, \Lambda, \Phi) \) defined by
\[
\beta(u; \psi, \Lambda, \Phi) := \min_{\nu \in \mathcal{M}(\Lambda, \Phi)} \frac{\int u \, d\nu}{\int \psi \, d\nu},
\] (2.1)
is called the ratio minimum ergodic average, and any \( \nu \in \mathcal{M}(\Lambda, \Phi) \) satisfying
\[
\frac{\int u \, d\nu}{\int \psi \, d\nu} = \beta(u; \psi, \Lambda, \Phi)
\]
is called a \((u, \psi)\)-minimizing measure. Denote that
\[
\mathcal{M}_{\text{min}}(u; \psi, \Lambda, \Phi) := \left\{ \nu \in \mathcal{M}(\Lambda, \Phi) : \frac{\int u \, d\nu}{\int \psi \, d\nu} = \beta(u; \psi, \Lambda, \Phi) \right\}.
\]
By compactness of \( \mathcal{M}(M, \Phi) \), and the continuity of the operator \( \frac{\int u \, d\nu}{\int \psi \, d\nu} \), it directly follows that \( \mathcal{M}_{\text{min}}(u; \psi, M, \Phi) \neq \emptyset \), which contains at least one ergodic \((u, \psi)\)-minimizing measure by ergodic decomposition.

For \( \alpha \in (0, 1] \), let \( \mathcal{C}^{0, \alpha}(M) \) be the space of \( \alpha \)-Hölder continuous real-valued function on \( M \) endowed with the \( \alpha \)-Hölder norm \( \|u\|_\alpha := \|u\|_0 + [u]_\alpha \), where \( \|u\|_0 := \sup_{x \in M} |u(x)| \) is the super norm, and \( [u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x, y)^\alpha} \). Also note that when \( \alpha = 1 \), \( \mathcal{C}^{0, 1}(X) \) becomes the collection of all real valued Lipschitz continuous functions, and \( [u]_1 \) becomes the minimum Lipschitz constant of \( u \). Additionally, denote by \( \mathcal{C}^{1, 0}(M) \) the Banach space of continuous differentiable functions on \( M \) endowed with the standard \( \mathcal{C}^1 \)-norm.

In this paper, we consider the weighted ergodic optimization problem and derive the following result.

**Theorem 2.2.** Let \( M \) be a compact smooth Riemannian manifold with Riemannian metric \( d \) and \( \Phi \) be a \( C^2 \) flow on \( M \). Suppose that \((\Lambda, \Phi)\) is an non-degenerate Axiom A attractor, then the following hold:

I) For \( \alpha \in (0, 1] \), if \( \psi \in \mathcal{C}^{0, \alpha}(M) \) with \( \inf_{x \in M} \psi(x) > 0 \), then there exists an open and dense set \( \mathfrak{P} \subset \mathcal{C}^{0, \alpha}(M) \) such that for any \( u \in \mathfrak{P} \), the \((u|_\Lambda, \psi|_\Lambda)\)-minimizing measure of \((\Lambda, \Phi)\) is uniquely supported on a periodic orbit of \( \Phi \).

II) For \( \psi \in \mathcal{C}^{0, 1}(M) \) with \( \inf_{x \in M} \psi(x) > 0 \), there exists an open and dense set \( \mathfrak{P} \subset \mathcal{C}^{1, 0}(M) \) such that for any \( u \in \mathfrak{P} \), the \((u|_\Lambda, \psi|_\Lambda)\)-minimizing measure of \((\Lambda, \Phi)\) is uniquely supported on a periodic orbit of \( \Phi \).

We remark here that \( M, \Lambda, \Phi \) are assumed to satisfy conditions in Theorem 2.2 throughout the rest of this paper.
3. Properties of non-degenerate Axiom A attractors

This section devotes to building theoretic tools as preparations for the proof of Theorem 2.2.

3.1. Invariant Manifolds. For a point \( x \in \Lambda \) and \( \epsilon > 0 \) the local stable and unstable sets are defined by

\[
W^s_\epsilon(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \forall t \geq 0, \; d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to +\infty \},
\]

\[
W^u_\epsilon(x) = \{ y \in M : d(\phi^{-t}(x), \phi^{-t}(y)) \leq \epsilon \forall t \geq 0, \; d(\phi^{-t}(x), \phi^{-t}(y)) \to 0 \text{ as } t \to +\infty \}.
\]

The following Lemma is a standard result of invariant manifold in existing literature, of which the proof is omitted.

**Lemma 3.1.** For any \( \lambda_1 \in (0, \lambda_0) \), there exists \( \epsilon_1 > 0 \) and \( C_1 \geq 1 \) such that for any \( \epsilon \in (0, \epsilon_1] \), the following hold:

i) \( W^s_\epsilon(x), W^u_\epsilon(x) \) are \( C^2 \) embedded discs for all \( x \in \Lambda \) with \( T_xW^s_\epsilon(x) = E^s_\tau \), \( \tau = u, s \);

ii) \( d(\phi^t(x), \phi^t(y)) \leq C_1 e^{-t\lambda_1}d(x,y) \) for \( y \in W^s_\epsilon(x), \; t \geq 0 \), and

\[
d(\phi^{-t}(x), \phi^{-t}(y)) \leq C_1 e^{-t\lambda_1}d(x,y) \text{ for } y \in W^u_\epsilon(x), \; t \geq 0;
\]

iii) \( W^s_\epsilon(x), W^u_\epsilon(x) \) vary continuously on \( x \) (in \( C^1 \) topology).

By choosing the Riemannian metric, the non-degenerate Axiom A flows in Theorem 2.2 meets the following basic canonical setting: There are positive constants \( \delta, \epsilon, \beta, \lambda, C \) with \( C \geq 1 \) and \( \delta \ll \epsilon \ll \min\{\epsilon_0, \epsilon_1\} \), where \( \epsilon_0 \) is as in A2) of the definition of Axiom A and \( \epsilon_1 \) is as in Lemma 3.1 such that:

1. For \( x, y \in M \) with \( d(x,y) \leq \delta \), there is unique time \( v = v(x,y) \) with \( |v| \leq \epsilon d(x,y) \) such that
   a) \( W^s_\epsilon(\phi_v(x)) \cap W^u_\epsilon(y) \) is not empty and contains only one element. We use \( w = w(x,y) \) to represent the only element.
   b) \( d(x,y) \geq C^{-1} \max\{d(\phi_v(x), w), d(y, w), d(\phi_v(x), x), d(w, x)\} \).

2. For \( x \in M \), \( y \in W^u_\epsilon(x) \) and \( t \geq 0 \), \( d(\phi^{-t}x, \phi^{-t}y) \leq Ce^{-\lambda t}d(x,y) \);
   For \( x \in M \), \( y \in W^s_\epsilon(x) \) and \( t \geq 0 \), \( d(\phi_t x, \phi_t y) \leq Ce^{-\lambda t}d(x,y) \);

3. For \( x, y \in M \), \( d(\phi_t x, \phi_t y) \leq Ce^{\beta t}d(x,y) \) for all \( t \in \mathbb{R} \).

**Remark 3.2.** In our following text, \( \delta, \epsilon, \lambda, \beta, C \) are the positive constants as above. Additionally, for convenience, we assume \( C \gg 1, 0 < \delta \ll \epsilon \ll 1 \). Otherwise, we set a positive constant \( \epsilon' \) such that \( \epsilon' \ll \frac{\epsilon}{C^{10^{10}+10^9}} \). We set another positive constant \( \delta' \) with \( \delta' \ll \delta \) such that for any \( x, y \in M \) with \( d(x,y) \leq \frac{C^{10^9}e^{10^9+10^9}}{e^{10^9-1}} \), there is unique time \( v = v(x,y) \) with \( |v| \leq C d(x,y) \) such that \( W^s_{\epsilon'}(\phi_v(x)) \cap W^u_{\epsilon'}(y) \) is not empty and contains only one element.
Remark 3.3. For proofs and more details of Lemma 3.1 and the basic canonical setting, we refer readers to \[PSh\], \[Bowen\], and \[BR\]. The only property which is not appearing in the above references is the following inequality
\[
|v(x,y)| \leq Cd(x,y) \quad (3.1)
\]
appearing in (1) of basic canonical setting. We remark here that this inequality holds when \( \Phi \) is \( C^2 \). When \( \Phi \) is \( C^{1+\alpha} \) for some \( \alpha \in (0,1] \), the above inequality will be replaced by \( |v(x,y)| \leq Cd^\alpha(x,y) \) which is still sufficient for the proof of this paper (although necessary modifications are needed). This concludes that the Theorem 2.2 still holds for \( C^{1+\alpha} \) flows.

Finally, to our knowledge, there is no explicit statement equivalent to (3.1) in existing literature. Nevertheless, (3.1) can be proved by combining Lemma 6., Proposition 8., Proposition 9. and Lemma 13. from \[LY\]. Since (3.1) is intuitively natural but at the same time the proof involves considerable technical complexity, we decide not to put the detailed proof in this paper for the sake of simplicity.

3.2. Anosov Closing Lemma. Let \( \delta', \epsilon', \delta, \epsilon, \lambda, \beta, C \) be the constants as in Remark 3.2. Then we have the following Lemma.

Lemma 3.4. Given \( \eta \leq \frac{C^{10}103+103}{\epsilon^{-1}} \delta' \) and \( T > 0 \), if \( x, y \in \Lambda \) and \( s : \mathbb{R} \to \mathbb{R} \) continuous with \( s(0) = 0 \) satisfy
\[
d(\phi_{t+s(t)}(y), \phi_t(x)) \leq \eta \text{ for } t \in [0,T],
\]
then for all \( t \in [0, T] \), the following hold:

ASH1) \( |s(t)| \leq 2C\eta \);
ASH2) \( d(\phi_{t+s(t)}(y), \phi_t(x)) \leq C^2e^{-\lambda\min(t,T-t)} (d(y,x) + d(\phi_T(y), \phi_T(x))) \), where \( v(y,x) \) is as in Remark 3.2 satisfying \( |v(y,x)| \leq Cd(x,y) \).

Especially, one has that

(1) If \( d(\phi_{t+s(t)}(y), \phi_t(x)) \leq \eta \text{ for all } t \geq 0 \), then
\[
d(\phi_{t+s(t)}(y), \phi_t(x)) \to 0 \text{ as } t \to +\infty.
\]

(2) If \( d(\phi_{t+s(t)}(y), \phi_t(x)) \leq \eta \text{ for all } t \in \mathbb{R} \), then \( \phi_{v(y,x)}(y) = x \).

A segment of \( \Phi \) is a curve \( S : [a,b] \to M : t \to \phi_t(x) \) for some \( x \in M \) and real numbers \( a \leq b \). We note the left endpoint of \( S \) by \( S^L = \phi_a(x) \), the right endpoint of \( S \) by \( S^R = \phi_b(x) \) and the length of \( S \) by \( |S| = b - a \). By a segment \( S \), if \( S^L = S^R \), we say \( S \) is a periodic segment. We have the following version of Anosov Closing Lemma.

Lemma 3.5 (Anosov Closing Lemma). There are positive constants \( L \) and \( K \) depending on the system constants only such that if segment \( S \) of \( \Phi|\Lambda \) satisfy
(a) $|S| \geq K$;
(b) $d(S^L, S^R) \leq \delta'$.

Then, there is a periodic segment $O$ such that

$$||S| - |O|| \leq Ld(S^L, S^R)$$

and

$$d(\phi_t(O^L), \phi_t(S^L)) \leq Ld(S^L, S^R) \text{ for all } 0 \leq t \leq \max(|S|, |O|).$$

Remark 3.6. In the following text, we also use $S$ (so is $O$ and $Q$) to represent the collection of points $\phi_t(S^L), 0 \leq t \leq |S|$ as no confusion being caused. By Lemma 3.4 and the choices of $\epsilon$ and $\delta$, $O$ is clearly belonging to $\Lambda$.

3.3. Mañé-Conze-Guivarc’h-Bousch’s Lemma. For $\gamma \in \mathbb{R} \setminus \{0\}$ and continuous function $u : M \to \mathbb{R}$, define that

$$u_\gamma(x) = \frac{1}{\gamma} \int_0^\gamma u(\phi_t(x)) dt. \quad (3.2)$$

Lemma 3.7 (Mañé-Conze-Guivarc’h-Bousch’s Lemma). For $0 < \alpha \leq 1$ and $N_0 > 0$, there exists a positive constant $\gamma = \gamma(\alpha) > N_0$ such that if $u \in C^{0,\alpha}(M)$ satisfies $\beta(u; 1, \Lambda, \Phi|\Lambda) \geq 0$, then there is $v \in C^{0,\alpha}(\Lambda)$ such that $u_\gamma|\Lambda \geq v \circ \phi_\gamma|\Lambda - v$.

Remark 3.8. We remark that the key point of Lemma 3.7 lies in the fact that $v$ is chosen with the same Hölder exponent as $u$. Indeed, there were a number of weak versions of Lemma 3.7 in the setting of smooth Anosov flows without fixed points, or certain expansive non-Anosov geodesic flows, where $v$ is still Hölder, but the Hölder exponent is less than $\alpha$.\cite{LRR, LT, PR}.

By using Lemma 3.7 we have the following Lemma.

Lemma 3.9. For $0 < \alpha \leq 1$, there exists large $\gamma = \gamma(\alpha)$ such that, for $u \in C^{0,\alpha}(M)$ and strictly positive $\psi \in C^{0,\alpha}(M)$, there is $v \in C^{0,\alpha}(\Lambda)$ such that

1. $u_\gamma|\Lambda - v \circ \phi_\gamma|\Lambda + v - \beta(u; \psi, \Lambda, \Phi)|\Lambda \geq 0$;
2. $Z_{u,\psi} \subset \{ x \in \Lambda : (u_\gamma|\Lambda + v \circ \phi_\gamma|\Lambda - v - \beta(u; \psi, \Lambda, \Phi)|\Lambda)(x) = 0 \}$,

where $Z_{u,\psi} = \bigcup_{\mu \in \mathcal{M}_{\min}(u; \psi, \Lambda, \Phi)} \text{supp}(\mu)$.

Remark 3.10. For convenience, in the following text, if we need to use Lemma 3.9, we use $\bar{u}$ to represent $u_\gamma|\Lambda + v \circ \phi_\gamma|\Lambda - v - \beta(u; \psi, \Lambda, \Phi)|\Lambda$ for short. Then, $\bar{u} \geq 0$ and $Z_{u,\psi} \subset \{ x \in \Lambda : \bar{u}(x) = 0 \}$.
3.4. **Periodic Approximation.** For $\alpha \in (0, 1]$, $Z \subset M$ and a segment $S$ of $\Phi$, we define the $\alpha$-deviation of $S$ with respect to $Z$ by

$$d_{\alpha, Z}(S) = \int_0^{|S|} d^\alpha (\phi_t (S^L), Z) \, dt.$$  

By a segment $S$, if $S^L = S^R$, we say $S$ is a periodic segment. For $P \geq 0$, using $O^P$ denote the collection of periodic segments in $\Lambda$ with length not larger than $P$. Now we have the following version of Quas and Bressaud’s periodic approximation Lemma.

**Lemma 3.11.** Let $Z \subset \Lambda$ be a $\Phi$-invariant compact subset of $\Lambda$. Then, for all $\alpha \in (0, 1], k \geq 0$,

$$\lim_{P \to +\infty} P^k \min_{S \in O^P} d_{\alpha, Z}(S) = 0.$$  

4. **Proof of Theorem 2.2**

Throughout the whole section, let $\delta', \epsilon', \delta, \epsilon, \lambda, \beta, C$ be the constants as in Remark 3.2.

4.1. **Proof of Part I) of Theorem 2.2**

4.1.1. **Locking Property of Periodic Segments.** In this subsection, we show that periodic segments have locking property in some sense.

For $0 < \eta \leq \delta$, the $\eta$-disk of $x$ is defined by

$$D(x, \eta) = \{ y \in \Lambda : d(x, y) \leq \eta, W^s_\epsilon(x) \cap W^u_\epsilon(y) \neq \emptyset \}. \tag{4.1}$$

$D(x, \eta)$ has the following properties:

(a) $W^s_\eta(x) \subset D(x, \eta)$ and $\phi_t(W^s_\eta(x)) \subset D(\phi_t(x), Ce^{-\lambda t} \eta)$ for $t \geq 0$;

(b) $W^u_\eta(x) \subset D(x, \eta)$ and $\phi_t(W^u_\eta(x)) \subset D(\phi_t(x), Ce^{\lambda t} \eta)$ for $t \leq 0$;

(c) $\phi_t(D(x, \eta)) \subset D(\phi_t(x), Ce^{\beta |t|} \eta)$ for $t \in \mathbb{R}$ satisfying $Ce^{\beta |t|} \eta < \delta$.

(d) for $\eta \leq \frac{\delta}{C}$ and $x, y \in \Lambda$ with $d(x, y) \leq \eta$, there exists a unique time $v = v(x, y)$ with $|v| \leq Cd(x, y)$ such that $y \in D(\phi_v(x), \delta)$. In fact, $v$ is the one given by the basic canonical setting.

Now we define $D : \Lambda \times \Lambda \to [0, +\infty)$ by

$$D(x, y) = \left\{ \begin{array}{ll} \delta', & \text{if } y \notin D(x, \delta'), \\ d(x, y), & \text{if } y \in D(x, \delta'). \end{array} \right. \tag{4.2}$$

By a periodic segment $O$ of $\Phi|_{\Lambda}$, we define the gap of $O$ by

$$D(O) = \min_{x \in O, 0 < t < |O|_{\min}} D(x, \phi_t(x)), \tag{4.3}$$
where $|\mathcal{O}|_{\min} = \min\{t > 0|\phi_t(x) = x, x \in \mathcal{O}\}$. Note that the assumptions of Theorem 2.2 imply that $D(\mathcal{O}) > 0$ automatically, therefore in the rest of this section we keep in mind that $D(\mathcal{O}) > 0$ without extra explanation.

Firstly, we present several technical Lemmas.

**Lemma 4.1.** Let $\mathcal{O}$ be a periodic segment of $\Phi|_\Lambda$. If $x, y \in \mathcal{O}$ satisfy $d(x, y) < \frac{D(\mathcal{O})}{C}$, then $\phi_v(x) = y$ where $v = v(x, y)$.

**Proof.** Let $\delta', \varepsilon', \delta, \varepsilon, \lambda, \beta, C$ be the constants as in Remark 3.2. Then, $d(x, y) \leq \frac{D(\mathcal{O})}{C} < \delta' \ll \delta$. Hence, by the basic canonical setting, there is a constant $v = v(x, y)$ such that $y \in D(\phi_v(x), Cd(x, y)) \subset D(\phi_v(x), \delta)$.

If $\phi_v(x) \neq y$, then $D(\mathcal{O}) \leq d(\phi_v(x), y) \leq Cd(x, y) < D(\mathcal{O})$, which is impossible. Thus, $\phi_v(x) = y$. This ends the proof. □

By a periodic segment $\mathcal{O}$ of $\Phi$, the periodic measure $\mu_\mathcal{O}$ is defined by

$$\mu_\mathcal{O} = \frac{1}{|\mathcal{O}|} \int_0^{|\mathcal{O}|} \delta_{\phi_t(\mathcal{O})} dt.$$ 

By an ergodic measure $\mu \in \mathcal{M}^e(\Lambda, \Phi|_\Lambda)$, a point $x \in M$ is called a generic point of $\mu$ if the following holds

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\phi_t(x)) dt = \int f d\mu \text{ for all } f \in C(M).$$

The following Lemma shows that periodic segments have locking property in some sense.

**Lemma 4.2.** Let $\mathcal{O}$ be a periodic segment of $\Phi|_\Lambda$ and $x \in M$. If

$$d(\phi_t(x), \mathcal{O}) \leq \frac{D(\mathcal{O})}{4C^2e^\beta} \text{ for all } 0 \leq t \leq T,$$  

(4.4)

then there is a $y \in \mathcal{O}$ such that $d(\phi_t(x), \phi_t(y)) \leq Cd(\phi_t(x), \mathcal{O})$ for all $0 \leq t \leq T$.

Especially, if $d(\phi_t(x), \mathcal{O}) \leq \frac{D(\mathcal{O})}{4C^2e^\beta}$ for all $t \geq 0$, then $x$ is a generic point of $\mu_\mathcal{O}$. 


Proof. Let $\delta', \epsilon', \delta, \epsilon, \lambda, \beta, C$ be the constants as in Remark 3.2. Take a positive constant $\theta \ll \delta$ such that $d(\phi_t(z), z) \leq \frac{D(O)}{4\epsilon}$ for all $|t| \leq \theta$ and $z \in M$. By assumption (4.4), there are $y'_t \in \mathcal{O}$ such that
\[
d(\phi_t(x), y'_t) = d(\phi_t(x), \mathcal{O}) \leq \frac{D(O)}{4C^2e^\beta} \ll \delta \quad \text{for all } 0 \leq t \leq T.
\]
Note $y_t = \phi_v(y'_t, \phi_t(x))(y'_t)$ for $0 \leq t \leq T$. Then
\[
\phi_t(x) \in \mathcal{D}(y_t, Cd(\phi_t(x), \mathcal{O})) \subset \mathcal{D}\left(y_t, \frac{D(O)}{4Ce^\beta}\right) \quad \text{for all } 0 \leq t \leq T. \quad (4.5)
\]
Fix $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leq \theta$. Then, by (4.5),
\[
d(y_{t_1}, y_{t_2}) \leq d(y_{t_1}, \phi_{t_1}(x)) + d(\phi_{t_1}(x), \phi_{t_2}(x)) + d(\phi_{t_2}(x), y_{t_2})
\leq \frac{D(O)}{4Ce^\beta} + \frac{D(O)}{4C} + \frac{D(O)}{4Ce^\beta} < \frac{D(O)}{C}
\]
Thus, by Lemma 4.1 there is a constant $\tau$ satisfying $|\tau| \leq Cd(y_{t_1}, y_{t_2}) \ll \delta$ such that
\[
\phi_\tau(y_{t_1}) = y_{t_2}.
\]
By the uniqueness of $v$ given in the basic canonical setting and the smallness of both $\theta$ and $|\tau|$, one has $\tau = t_2 - t_1$. Hence,
\[
y_{t_2} = \phi_{t_2-t_1}(y_{t_1}) \quad \text{for all } t_1, t_2 \in [0, T] \text{ with } |t_1 - t_2| \leq \theta.
\]
Therefore, $y_t = \phi_v(y_0)$ for all $t \in [0, T]$ by induction, which implies that
\[
d(\phi_t(y_0), \phi_t(x)) = d(y_t, \phi_t(x)) \leq Cd(y'_t, \phi_t(x)) = Cd(\phi_t(x), \mathcal{O}) \quad \text{for all } 0 \leq t \leq T.
\]
Thus, $y = y_0$ is the point as required.

Now we assume that $d(\phi_t(x), \mathcal{O}) \leq \frac{D(O)}{4Ce^\beta}$ for all $t \geq 0$. Then by the arguments above, there is $y \in \mathcal{O}$ such that
\[
d(\phi_t(x), \phi_t(y)) \leq \frac{D(O)}{4Ce^\beta} \leq \delta' \quad \text{for all } t \geq 0.
\]
Then by Lemma 3.4 we have
\[
d(\phi_t(x), \phi_t(y)) \to 0 \quad \text{as } t \to +\infty,
\]
where $v = v(x, y)$. Then $x$ must be a generic point of $\mu_\mathcal{O}$. This ends the proof. \hfill \Box

**Lemma 4.3.** There is positive constant $\tau_0$ such that for any $x \in \Lambda$
\[
\phi_t(x) \notin \mathcal{D}(x, \delta) \quad \text{for all } 0 < |t| \leq \tau_0.
\]

**Proof.** Let $\delta', \epsilon', \delta, \epsilon, \lambda, \beta, C$ be the constants as in Remark 3.2. Take $\tau_0 > 0$ small enough such that $d(\phi_t(z), z) \leq \frac{\delta'}{\epsilon'}$ for all $|t| \leq \tau_0$ and $z \in M$. Suppose that there is an $x \in \Lambda$ and $0 < |\tau| \leq \tau_0$ such that $\phi_\tau(x) \in \mathcal{D}(x, \delta)$. Note
\[
w = W^s_\epsilon(x) \cap W^u_\epsilon(\phi_\tau(x)).
\]
Then, by (1)(b) of the basic canonical setting, one has that

\[
\max\{d(w,x), d(w, \phi_r(x))\} \leq Cd(x, \phi_r(x)) \leq \frac{\delta'}{C}.
\]

Then for \(t \geq 0\),

\[
d(\phi_t(w), \phi_t(x)) \leq Ce^{-\lambda t}d(w,x) \leq Cd(w,x) \leq \frac{\delta'}{C},
\]

and for \(t < 0\),

\[
d(\phi_t(w), \phi_t(x)) \leq d(\phi_t(w), \phi_t(\phi_r(x)) + d(\phi_r \phi_t(x), \phi_t(x)) \leq \frac{\delta'}{C} + \frac{\delta'}{C^2} \leq \delta',
\]

where we used \(w = W^s(\phi_t(x)) \cap W^u(\phi_r(x))\) and the selection of \(\tau\). Then by Lemma 3.4, there is a constant \(s\) with \(|l| \ll \delta\) such that \(w = \phi_l(x) = \phi_{l-\tau}(x)\). It is clear that at least one of \(l\) and \(l-\tau\) is not zero since \(\tau \neq 0\). Without loss of any generality, we assume that \(l \neq 0\), then \(\{x, \phi_l(x)\} \subset W^s(\phi_t(x))\) (otherwise \(\{\phi_l(x), \phi_l(x)\} \subset W^u(\phi_r(x))\)). Thus

\[
W^s(\phi_t(x)) \cap W^u(\phi_l(x)) \neq \emptyset \text{ and } W^s(\phi_t(x)) \cap W^u(\phi_l(x)) \neq \emptyset,
\]

which is impossible by the uniqueness of function \(v\) given in the basic canonical setting. This ends the proof. \(\square\)

**Remark 4.4.** Lemma 4.3 provide a lower bound \(\tau_0\) of the periods of periodic segments.

**Remark 4.5.** We say a periodic segment \(\mathcal{O}\) is pure if \(\phi_t(y) \neq y\) for all \(y \in \mathcal{O}\) and \(0 < t < |\mathcal{O}|\). By Lemma 4.3, a periodic segment \(\mathcal{O}\) is pure if and only if \(|\mathcal{O}| = |\mathcal{O}|_{\text{min}}\).

**4.1.2. Good periodic orbits.** This section is devoted to looking for good periodic orbits which are the candidates to support certain minimizing measures.

**Proposition 4.6.** For any \(\alpha \in (0,1]\), a given \(\bar{L} > 0\) and \(\Phi\)-forward-invariant non-empty subset \(Z \subset \Lambda\) (i.e. \(\phi_t(Z) \subset Z, \forall t \geq 0\)), there exists a periodic segment \(\mathcal{O}\) of \(\Phi|_{\Lambda}\) such that

\[
\frac{D\alpha(\mathcal{O})}{d_{\alpha,Z}(\mathcal{O})} > \bar{L}.
\]

**Proof.** Fix \(0 < \alpha \leq 1\), and recall that \(\delta', \ell, \delta, \epsilon, \lambda, \beta, C\) are as in Remark 3.2, and \(K, L\) are as in Lemma 3.5. For the sake of convenience, we additionally assume that \(K \geq L\). By Lemma 3.11, for any \(k \in \mathbb{N}\), there exists a periodic segment \(\mathcal{O}_0\) of \(\Phi|_{\Lambda}\) with period \(P_0\) large enough such that

\[
d_{\alpha,Z}(\mathcal{O}_0) < P_0^{-k} < \delta'.
\]

We remark here that the period of a periodic segment is always assumed to be the minimum period, which will avoid unnecessary complicity without harming the argument.
If $D^\alpha(O_0) > \tilde{L}d_{\alpha,Z}(O_0)$, the proof is done. Otherwise, one has that
\[
D^\alpha(O_0) \leq \tilde{L}d_{\alpha,Z}(O_0) < \tilde{L}P_0^{-k}. \tag{4.8}
\]
Since $P_0, k$ can be chosen as large as needed, one can request that $D(O_0) < \delta'$. Therefore, by definition of $D(O_0)$ (see (4.3)), there is a $y \in O_0$ and a time $t_0 \in (0, P_0)$ such that
\[
\phi_{t_0}(y) \in D(y, D(O_0)).
\]
Split the periodic segment $O_0$ into two segments which are noted by $Q^L_0: [0, t_0] \to \Lambda: t \to \phi_t(y)$; $Q^R_0: [t_0, P_0] \to \Lambda: t \to \phi_t(y)$.

We choose the segment with smaller length and note it by $Q_0$. Then either $Q^L_0 \in D(Q^R, \delta')$ or $Q^R_0 \in D(Q^L, \delta')$. It is clear that in either case
\[
d(Q^L_0, Q^R_0) \leq \delta' \quad \text{and} \quad d_{\alpha,Z}(Q_0) \leq d_{\alpha,Z}(O_0).
\]
We will employ different discussions for two different situations according to the length of the segment for which we set $3K$ as a landmark.

If the following condition holds
\[
|Q_0| > 3K, \tag{4.9}
\]
also note that $d(Q^L_0, Q^R_0) \leq \delta'$, then Lemma 3.5 is applicable here, by which one has that there exists a periodic segment $O_1$ such that
\[
||Q_0| - |O_1|| \leq Ld(Q^L_0, Q^R_0) \leq LD(O_0) < L\tilde{L}P_0^{-k}, \tag{4.10}
\]
\[
d\left(\phi_t(Q^L_0), \phi_t(Q^L_1)\right) \leq Ld(Q^L_0, Q^R_0) \quad \forall 0 < t < \max\{|Q_0|, |O_1|\}. \tag{4.11}
\]
Since $K \geq L$ and $\delta' < 1$, (4.10) together with the assumption $|Q_0| > 3K$ implies that
\[
|O_1| \leq \frac{4}{3}|Q_0| \leq \frac{2}{3}|Q_0|. \tag{4.12}
\]
By definition, one has that

\[ d_{\alpha, Z}(O_1) = \int_0^{|O_1|} d^\alpha \left( \phi_t(O_1^L), Z \right) dt \]

\[ = \int_0^{|O_1|} d^\alpha \left( \phi_t \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

\[ \leq \int_0^{|Q_0|} d^\alpha \left( \phi_1 \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

\[ + \int_{|Q_0| - |O_1|}^{Q_1 + |Q_0|} d^\alpha \left( \phi_1 \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

\[ \leq \int_0^{|Q_0|} d^\alpha \left( \phi_1 \phi_v(O_1^L, Q_0^L)(O_1^L), \phi_1(Q_0^L) \right) dt \]

\[ + \int_0^{|Q_0|} d^\alpha \left( \phi_1(Q_0^L), Z \right) dt \]

\[ + \int_0^{|Q_0|} d^\alpha \left( \phi_{||Q_0|| - ||O_1||} \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

By applying Ash3) of Lemma 3.4, one has that

\[ \text{Int(a)} \]

\[ \text{Int(b)} = d_{\alpha, Z}(Q_0). \] (4.14)

By definition, one has that

\[ \text{Int(c)} = \int_0^{|Q_0|} d^\alpha \left( \phi_{||Q_0|| - ||O_1||} \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

\[ \leq \left( C e^{\beta \|Q_0\| - \|O_1\|} \right)^{\alpha} \int_0^{|Q_0|} d^\alpha \left( \phi_1 \phi_v(O_1^L, Q_0^L)(O_1^L), Z \right) dt \]

\[ \leq \left( C e^{\beta \|Q_0\| - \|O_1\|} \right)^{\alpha} (\text{Int(a)} + \text{Int(b)}). \]

By taking \( P_0 \) and \( k \) large, it is able to make \( |Q_0| - |O_1| < 1 \). Therefore, one has the following simplified estimate

\[ d_{\alpha, Z}(O_1) \leq L_1 d^\alpha(Q_1^L, Q_0^L) + L_2 d_{\alpha, Z}(Q_0) \leq \tilde{L} d_{\alpha, Z}(O_0), \]

where

\[ L_1 = \left( C^\alpha e^{\alpha \beta} + 1 \right) \frac{2(2C^2 L)^\alpha}{\lambda \alpha}; \]

\[ L_2 = C^\alpha e^{\alpha \beta} + 1; \]

\[ \tilde{L} = L_1 \tilde{L} + L_2 = \left( C^\alpha e^{\alpha \beta} + 1 \right) \left( \frac{2(2C^2 L)^\alpha}{\lambda \alpha} \tilde{L} + 1 \right). \]
Once \( d_{\alpha,Z}(\mathcal{O}_1) > \hat{L}d_{\alpha,Z}(\mathcal{O}_1) \), \( \mathcal{O}_1 \) is the periodic segment required. Otherwise, repeat the operation above once it is doable. Note that such a process will stop at a finite time, since the operation above will reduce the period of periodic segment at least \( \frac{1}{3} \). Therefore, under the assumption that the operation above is always doable, the process will end on an periodic segment \( \mathcal{O}_m \) for some \( m \in \mathbb{N} \cup \{0\} \), which either satisfies the requirement of this Proposition or \( |\mathcal{O}_m| \geq 3K \) while \( |\mathcal{Q}_m| < 3K \). In either cases, one has that

\[
m \leq \frac{\log P_0 - \log(3K)}{\log 1.5} + 1,
\]

and

\[
d_{\alpha,Z}(\mathcal{O}_i) \leq \hat{L}^id_{\alpha,Z}(\mathcal{O}_0) \quad \forall 1 \leq i \leq m.
\]

In order to make each operation above doable, one need that

\[
D(\mathcal{O}_i) < \delta' \quad \forall 1 \leq i \leq m - 1,
\]

which can be done by assuming the largeness of \( P_0 \) and \( k \) in advance. To be precise, one can take

\[
k > \frac{\log \hat{L}}{\log 1.5} \quad \text{and} \quad P_0^{\log \hat{L}^{-1} - k} < \frac{(\delta')^\alpha}{\hat{L}L}, \quad (4.17)
\]

where the second inequality above implies that for all \( 0 \leq i \leq m - 1 \)

\[
\hat{L}d_{\alpha,Z}(\mathcal{O}_i) \leq P_0^{-k} \hat{L}L^m < (\delta')^\alpha,
\]

which ensures the existence of \( \mathcal{O}_{i+1} \) and \( \hat{L}d_{\alpha,Z}(\mathcal{O}_m) < (\delta')^\alpha \).

Next, we will deal with the case that \( |\mathcal{Q}_m| \leq 3K \), which is the counterpart of the case when \( 4.9 \) holds. We will show that by rearranging extra largeness of \( P_0 \) and \( k \), one can make \( \mathcal{O}_m \) satisfy the requirement of Proposition 4.6. We will prove this by contradiction.

Before going to further discussion, we note that the union of all periodic orbits of \( \Phi|_{\Lambda} \) with period \( \leq 4K \) is a nonempty compact subset of \( \Lambda \), which is denoted by \( \text{Per}_{4K} \). Once \( Z \cap \text{Per}_{4K} \neq \emptyset \) Proposition 4.6 holds automatically; otherwise, there exists \( \sigma > 0 \) such that

\[
d(x,Z) > \sigma \quad \forall x \in \text{Per}_{4K}. \quad (4.18)
\]

Suppose that

\[
D^\alpha(\mathcal{O}_m) \leq \hat{L}d_{\alpha,Z}(\mathcal{O}_m) < (\delta')^\alpha. \quad (4.19)
\]

When

\[
K \leq |\mathcal{Q}_m| < 3K, \quad (4.20)
\]

by the exactly same argument as on \( \mathcal{Q}_0 \), one has that there exists a periodic segment \( \mathcal{O}_{m+1} \) such that

\[
|\mathcal{O}_{m+1}| \leq 4K \quad \text{and} \quad d_{\alpha,Z}(\mathcal{O}_{m+1}) \leq \hat{L}d_{\alpha,Z}(\mathcal{O}_m) \leq \hat{L}m^1d_{\alpha,Z}(\mathcal{O}_0) < \hat{L}m^1 P_0^{-k}.
\]
By choosing $P_0$ and $k$ large enough one can make $d_{\alpha,Z}(O_{m+1}) < \sigma$ which implies contradiction with (4.13). Therefore (4.19) and (4.20) can not hold simultaneously for large enough $P_0$ and $k$.

When  

\[ |Q_m| < K, \]

(4.21) Lemma 3.5 is not applicable directly. For sake of convenience, note $l = |Q_m|$. By (4.19), $Q_m^L \in D(Q_m^R, \delta')$ or $Q_m^R \in D(Q_m^L, \delta')$. Then by Lemma 4.3 $l > \tau_0$. Let $q$ be the integer such that

\[ K \leq ql \leq 2K \]  

and then $2 \leq q \leq \left\lfloor \frac{2K}{\tau_0} \right\rfloor$.

Note

\[ S_i : [0, l] \rightarrow M : t \rightarrow \phi_{l+i}(Q_m^L) \text{ for } i = 0, 1, 2, \cdots, q - 1, \]

and

\[ S : [0, ql] \rightarrow M : t \rightarrow \phi_t(Q_m^L). \]

Then,

\[ d(S_0^L, S_0^R) = d(Q_m^L, Q_m^R) \]

\[ d(S_1^L, S_1^R) = d(\phi_t(S_0^L), \phi_t(Q_m^R)) \leq Ce^{\beta t} d(S_0^L, S_0^R) \leq Ce^{\beta K} d(Q_m^L, Q_m^R), \]

\[ \cdots \]

\[ d(S_{q-1}^L, S_{q-1}^R) \leq (Ce^{\beta K})^{q-1} d(Q_m^L, Q_m^R). \]

Therefore,

\[ d(S_i^L, S_i^R) \leq \sum_{i=0}^{q-1} (Ce^{\beta K})^i d(Q_m^L, Q_m^R) \leq \frac{(Ce^{\beta K})^{\left\lfloor \frac{2K}{\tau_0} \right\rfloor} - 1}{Ce^{\beta K} - 1} D(O_m), \]

which together with (4.19) implies that

\[ d(S_i^L, S_i^R) \leq \frac{(Ce^{\beta K})^{\left\lfloor \frac{2K}{\tau_0} \right\rfloor} - 1}{Ce^{\beta K} - 1} \left( \tilde{L} d_{\alpha,Z}(O_m) \right)^{\frac{1}{\alpha}} \leq \frac{(Ce^{\beta K})^{\left\lfloor \frac{2K}{\tau_0} \right\rfloor} - 1}{Ce^{\beta K} - 1} \left( \tilde{L} L^m P_0^{-k} \right)^{\frac{1}{\alpha}}. \]

(4.22)

By taking $P_0$ and $k$ large enough, one can make $d(S_i^L, S_i^R) < \sigma$. Also note that $|S| \geq K$, then Lemma 3.5 is applicable to $S$. Therefore, there exists a periodic segment $O_s$ such that $|O_s| \leq |S| + Ld(S_i^L, S_i^R) \leq 3K$ and

\[ d(\phi_t(O_{l,S}(O_s^L), \phi_t(S_i^L))) \leq L \left( \frac{Ce^{\beta K} \left\lfloor \frac{2K}{\tau_0} \right\rfloor - 1}{Ce^{\beta K} - 1} \left( \tilde{L} L^m P_0^{-k} \right)^{\frac{1}{\alpha}} \right)^{\alpha} 0 \leq t \leq |S|, \]

(4.23)

where the right hand side of the above inequality can be make smaller than $\frac{1}{3} \sigma$ by taking $P_0$ and $k$ large enough. On the other hand,

\[ d_{\alpha,Z}(S_0) = d_{\alpha,Z}(Q_m) \]

\[ d_{\alpha,Z}(S_1) = \int_0^l d^\alpha(\phi_{l+t}(S_0^L), Z) \leq (Ce^{\beta l})^\alpha d_{\alpha,Z}(S_0) \leq (Ce^{\beta K})^\alpha d_{\alpha,Z}(Q_m), \]

\[ \cdots \]
\[ d_{\alpha,Z}(S_{q-1}) \leq (Ce^{\beta K})^{(q-1)\alpha} d_{\alpha,Z}(Q_m). \]

Thus,
\[
d_{\alpha,Z}(S) = \sum_{i=0}^{q-1} d_{\alpha,Z}(S_i) \leq \sum_{i=0}^{q-1} (Ce^{\beta K})^{i\alpha} d_{\alpha,Z}(Q_m)
\]
\[
\leq \sum_{i=0}^{[\frac{2K}{\tau_0}] - 1} C_i \alpha e^{i\beta K \alpha} d_{\alpha,Z}(O_m)
\]
\[
\leq \sum_{i=0}^{[\frac{2K}{\tau_0}] - 1} C_i \alpha e^{i\beta K \alpha} \hat{L}^{-m} d_{\alpha,Z}(O_m)
\]
\[
\leq \left( \frac{(Ce^{\beta K})^{[\frac{2K}{\tau_0}] - 1}}{(Ce^{\beta K})^{\alpha} - 1} \right) \alpha \hat{L}^{-m} P_0^{-k},
\]
which can be made smaller than \( \frac{1}{3} \sigma \) by taking \( P_0 \) and \( k \) large enough. Since \( |S| \geq K > 1 \), there is a point \( t^* \in [0, |S|] \) such that
\[
d(\phi_{t^*}(S^L), Z) \leq \frac{\sigma}{3}.
\]

Therefore, by (4.23), by taking \( P_0 \) and \( k \) large enough, we have
\[
d(\phi_{t^*}(O^L_\ast), Z) \leq d(\phi_{t^*}(O^L_\ast, S^L_\ast), \phi_{t^*}(S^L_\ast)) + d(\phi_{t^*}(S^L_\ast), Z) \leq \frac{2}{3} \sigma < \sigma
\]
which contradicts with (4.18) as \( K \leq |S| \leq 2K \).
Hence (4.19) can not hold for large enough \( P_0 \) and \( k \). This ends the proof. \( \square \)

4.1.3. Main Proposition. In this subsection, we state and prove our main Proposition. For a continuous function \( u \) and a segment \( S \) of \( \Phi \), define the integration of \( u \) along \( S \) with time interval \([a, b]\) and starting point \( x \) by the following
\[
\langle S, u \rangle = \int_a^b u(\phi_t(x)) dt.
\]

Now we have the following Proposition.

**Proposition 4.7.** Given \( 0 < \varepsilon \leq 1, \) \( 0 < \alpha \leq 1, \) a strictly positive function \( \psi \in C^{0,\alpha}(M) \) and \( u \in C^{0,\alpha}(M) \), if a periodic segment \( O \) of \( \Phi|_\Lambda \) satisfies the following comparison condition
\[
\frac{D^\alpha(O)}{d_{\alpha,Z(u,\psi)}(O)} > \left( 4C^3(\|\bar{u}\|_\alpha + 10\varepsilon + \frac{\|\tilde{u}\|_\alpha + 1}{\psi_{\min}}\|\psi_\gamma\|_\alpha) + 1 + \frac{1}{\tau_0} \right) \frac{\|\bar{u}\|_\alpha\|\psi\|_0}{\psi_{\min}} \frac{100(4C^3e^{2\beta})\alpha}{\varepsilon},
\]
(4.26)
where \( \bar{u} \) is defined in Remark 3.9 and \( \tau_0 \) is the constant in Lemma 4.3, then the periodic measure

\[
\mu_O \in M_{\min}(u + \varepsilon d^\alpha(\cdot, O) + h; \psi, \Lambda, \Phi),
\]

where \( h \in C^{0,\alpha}(M) \) satisfying \( \|h\|_\alpha < 10\varepsilon \) and

\[
\|h\|_0 < \min \left\{ \frac{\varepsilon}{2} \left( \frac{D(O)}{4C^3 \varepsilon^2} \right)^\alpha, \frac{1}{2} \right\}.
\]

**Proof.** Fix \( \varepsilon, \alpha, O, \psi, u, h \) as in the Proposition, \( \delta', \varepsilon', \delta, \beta, C \) as in Remark 3.2, \( \bar{u}, Z_{u,\psi}, \gamma, \psi_\gamma \) as in Remark 3.10, \( \tau_0 \) as in Lemma 4.3. Note

\[
G = \bar{u} + \varepsilon d^\alpha(\cdot, O) + h - a_O \psi_\gamma,
\]

where

\[
a_O = \frac{\langle O, \bar{u} + \varepsilon d^\alpha(\cdot, O) + h \rangle}{\langle O, \psi_\gamma \rangle} = \frac{\langle O, \bar{u} + h \rangle}{\langle O, \psi_\gamma \rangle}.
\]

By straightforward computation, one has that

\[
|a_O| \leq \frac{\langle O, \|\bar{u}\| d^\alpha(\cdot, Z_{u,\psi}) + h \rangle}{\langle O, \psi_\gamma \rangle} = \frac{\|\bar{u}\| d^\alpha(Z_{u,\psi})}{\|\psi\|_{\min}} + \|h\|_0, \tag{4.27}
\]

where we used \( \bar{u}|_{Z_{u,\psi}} = 0 \). Notice that for all \( \mu \in M(\Lambda, \Phi) \)

\[
\int u + \varepsilon d^\alpha(\cdot, O) + h d\mu = \int u + \varepsilon d^\alpha(\cdot, O) + h d\mu \tag{4.27}
\]

\[
= \int \bar{u} + \varepsilon d^\alpha(\cdot, O) + h d\mu = \int G d\mu + a_O + \beta(u; \psi, \Lambda, \Phi).
\]

Then, in order to show that \( \mu_O \in M_{\min}(u + \varepsilon d^\alpha(\cdot, O) + h; \psi, \Lambda, \Phi) \), it is enough to show that \( \mu_O \in M_{\min}(G; \psi_\gamma, \Lambda, \Phi) \). Since \( \psi \) is strictly positive and \( \int G d\mu = 0 \), it is enough to show that

\[
\int G d\mu \geq 0 \text{ for all } \mu \in M^e(\Lambda, \Phi). \tag{4.28}
\]

Define a compact set \( R \subset M \) by

\[
R = \left\{ y \in M : d(y, O) \leq \left( \frac{a_O \|\psi\|_0 + \|h\|_0}{\varepsilon} \right)^\frac{1}{\alpha} \right\}.
\]

We have the following Claim.

**Claim 1.** \( R \) contains all \( x \in M \) with \( G(x) \leq 0 \).
Proof of Claim 1. Given $x \in M \setminus R$, we are to show that $G(x) > 0$. Note that
\[ \bar{u} + h - a_{O}\psi_{\gamma} \geq -|a_{O}||\psi_{\gamma}|_{0} - \|h\|_{0}. \] (4.29)
where we used $\bar{u} \geq 0$ and $\|\psi_{\gamma}\|_{0} \leq \|\psi\|_{0}$. Then
\[ G(x) = \bar{u}(x) + \varepsilon a_{O}(x, O) + h(x) - a_{O}\psi_{\gamma} \geq \varepsilon a_{O}(x, O) - |a_{O}||\psi_{\gamma}|_{0} - \|h\|_{0} \]
\[ > \varepsilon \cdot \left(\frac{|a_{O}||\psi|_{0} + \|h\|_{0}}{\varepsilon}\right)^{\frac{1}{2}} - |a_{O}||\psi|_{0} - \|h\|_{0} \]
\[ = 0. \]
This ends the proof of Claim 1. \qed

Define a compact set $R' \subset M$ by
\[ R' = \left\{ y \in M : d(y, O) \leq \left(\frac{2(|a_{O}||\psi|_{0} + \|h\|_{0})}{\varepsilon}\right)^{\frac{1}{2}} \right\}. \]

It is easy to see that $R$ is in the interior of $R'$ and the following holds because of (4.26), (4.27) and the choices of $\|h\|_{0}$
\[ d(y, O) \leq \left(\frac{2(|a_{O}||\psi|_{0} + \|h\|_{0})}{\varepsilon}\right)^{\frac{1}{2}} \leq \frac{D(O)}{4C^{3}e^{2\beta}}, \forall y \in R'. \] (4.30)

By Claim 1, there is a constant $\tau$ with $0 < \tau < 1$ such that $G(\phi_{t}(x)) > 0$ for all $x \in M \setminus R'$ and $|t| \leq \tau$. Now we claim the following assertion:

Claim 2. If $z \in M$ is not a generic point of $\mu_{O}$, then there is $m \geq \tau$ such that
\[ \int_{0}^{m} G(\phi_{t}(z))dt > 0. \]

Next we prove the Proposition by assuming the validity of Claim 2, while the proof of Claim 2 is left to the end of this section. For a given ergodic measure $\mu \in M^{e}(\Lambda, \Phi)$, if $\mu = \mu_{O}$, (4.28) obviously holds. Otherwise, let $z$ be a generic point of $\mu$, thus $z$ is not a generic point of $\mu_{O}$. Therefore, by Claim 2, there is a $t_{1} \geq \tau$ such that
\[ \int_{0}^{t_{1}} G(\phi_{t}(z))dt > 0. \]

Since $\phi_{t}(z)$ is still not a generic point of $\mu_{O}$, by using claim 2 again, one has a $t_{2} \geq t_{1} + \tau$ such that
\[ \int_{t_{1}}^{t_{2}} G(\phi_{t}(z))dt > 0. \]
By repeating the above process, one has $0 \leq t_1 < t_2 < t_3 < \cdots$ with all gaps not less than $\tau$ such that
\[ \int_{t_i}^{t_{i+1}} G(\phi_t(z)) dt > 0 \text{ for } i = 0, 1, 2, 3, \cdots, \]
where we assign $t_0 = 0$. Therefore
\[ \int Gd\mu = \lim_{l \to +\infty} \frac{1}{l} \int_0^l G(\phi_t(z)) dt \\
= \lim_{i \to +\infty} \frac{1}{t_i} \left( \int_{t_0}^{t_1} G(\phi_t(z)) dt + \int_{t_1}^{t_2} G(\phi_t(z)) dt + \cdots + \int_{t_{i-1}}^{t_i} G(\phi_t(z)) dt \right) \\
\geq 0. \]
Thus, $\mu_\mathcal{O} \in \mathcal{M}_{\min}(u + \varepsilon d^\alpha(\cdot, \mathcal{O}) + h; \psi, \Lambda, \Phi)$. This ends the proof. \hfill \Box

**Remark 4.8.** It is not difficult to see that for any $\varepsilon' > \varepsilon$, $\mu_\mathcal{O}$ is the unique measure in $\mathcal{M}_{\min}(u + \varepsilon d^\alpha(\cdot, \mathcal{O}) + h; \psi, \Lambda, \Phi)$ whenever $\|h\|_\alpha < 10\varepsilon$ and $\|h\|_0$ is sufficiently small. The Proposition shows that there is an open set of $C_{0,\alpha}(\mathcal{M})$ near $u$ such that these $\alpha$-Hölder functions in the open set have the same unique minimizing measure with respect to $\psi$ being supported on a periodic orbit.

**Proof of Claim 2.** If $z \notin \mathcal{R}'$, just take $m = \tau$, we have nothing to prove since $G(\phi_t(z)) > 0$ for all $|t| \leq \tau$.

One need only to consider the case that $z \in \mathcal{R}'$. Also note that, since $z$ is not a generic point of $\mu_\mathcal{O}$, Lemma 4.2 implies that the following inequality
\[ d(\phi_t(z), \mathcal{O}) \leq \frac{D(\mathcal{O})}{4C^2e\beta} \]
**CANNOT** hold for all $t \geq 0$. Thus there is an $m_1 > 0$ such that
\[ d(\phi_{m_1}(z), \mathcal{O}) > \frac{D(\mathcal{O})}{4C^2e\beta}. \]
Let $m_2 > 0$ be the smallest time such that
\[ d(\phi_{m_2}(z), \mathcal{O}) = \frac{D(\mathcal{O})}{4C^2e\beta}, \tag{4.31} \]
where the existence of such $m_2$ is ensured by (4.30) and the continuity of the flow. Then, by (3) of the **basic canonical setting**, one has the following
\[ d(\phi_{m_2-t}(z), \mathcal{O}) > \frac{D(\mathcal{O})}{4C^3e\beta}, \forall 0 < t \leq 1, \]
which together with (4.30) implies that
\[ \phi_{m_2-t}(z) \notin \mathcal{R}' \text{ for all } 0 < t \leq 1. \tag{4.32} \]
Thus
\[ \int_{m_2-1}^{m_2} G(\phi_t(z)) dt = \int_{m_2-1}^{m_2} \bar{u}(\phi_t(z)) + \varepsilon d^\alpha(\phi_t(z), \mathcal{O}) + h(\phi_t(z)) - a_\mathcal{O} \psi_\gamma(\phi_t(z)) dt \]
\[ \geq \int_{m_2-1}^{m_2} \varepsilon d^\alpha(\phi_t(z), \mathcal{O}) - |a_\mathcal{O}||\psi||_0 - ||h||_0 dt \] (4.33)
\[ \geq \varepsilon \cdot \left( \frac{D(\mathcal{O})}{4C^3 e^{2\beta}} \right)^\alpha - |a_\mathcal{O}||\psi||_0 - ||h||_0, \]
where we used (4.29).

Now since \( \mathcal{R}' \) is compact, there is an \( m_3 \) which is the largest time such that \( 0 \leq m_3 \leq m_2 \) and \( \phi_t(z) \in \mathcal{R}' \). By (4.32), it is clear that \( m_3 \leq m_2 - 1 \). Then by Claim 1, for all \( m_3 < t < m_2 - 1 \)
\[ G(\phi_t(z)) > 0, \] (4.34)
where we used the fact \( \mathcal{R} \subset \mathcal{R}' \). On the other hand, since \( m_3 < m_2 \), by the choice of \( m_2 \) (note on (4.31)), one has that
\[ d(\phi_t(z), \mathcal{O}) \leq \left( \frac{2(a_\mathcal{O} ||\psi||_0 + ||h||_0)}{\varepsilon} \right)^{\frac{1}{\alpha}} < \frac{D(\mathcal{O})}{4C^2 e^{2\beta}} \]
for all \( 0 \leq t \leq m_3 \).

Therefore, by Lemma 4.2, there is \( y_0 \in \mathcal{O} \) such that
\[ d(\phi_t(z), \phi_t(y_0)) \leq C \left( \frac{2(a_\mathcal{O} ||\psi||_0 + ||h||_0)}{\varepsilon} \right)^{\frac{1}{\alpha}} \leq \delta' \text{ for all } t \in [0, m_3]. \]

By using Lemma 3.3, we have for all \( 0 \leq t \leq m_3 \),
\[ d(\phi_t\phi_v(y_0, z), \phi_t(z)) \leq 2C^2 e^{-\lambda \min(t, m_3-t)} C \left( \frac{2(a_\mathcal{O} ||\psi||_0 + ||h||_0)}{\varepsilon} \right)^{\frac{1}{\alpha}}. \]

Hence,
\[ \int_0^{m_3} d^\alpha(\phi_t(z), \phi_t\phi_v(y_0, z)(y_0)) dt \leq \int_0^{m_3} \left( 2C^3 \left( \frac{2(a_\mathcal{O} ||\psi||_0 + ||h||_0)}{\varepsilon} \right)^{\frac{1}{\alpha}} (e^{-\lambda t} + e^{-\lambda(m_3-t)}) \right)^{\alpha} dt \]
\[ \leq \frac{4C^3 \alpha}{\lambda \alpha} \cdot \left( \frac{a_\mathcal{O} ||\psi||_0 + ||h||_0}{\varepsilon} \right). \]
Therefore,
\[\int_{0}^{m_3} G(\phi_t(z)) - G(\phi_t(\phi_v(y_0))) dt\]
\[= \int_{0}^{m_3} \bar{u}(\phi_t(z)) + \epsilon d^\alpha(\phi_t(z), O) + h(\phi_t(z)) - \bar{u}(\phi_{t+v}(y_0)) - h(\phi_{t+v}(y_0))\]
\[\geq \int_{0}^{m_3} \bar{u}(\phi_t(z)) - \bar{u}(\phi_{t+v}(y_0)) + h(\phi_t(z)) - h(\phi_{t+v}(y_0))\]
\[\geq - (\|\bar{u}\|_\alpha + \|h\|_\alpha + |a_O| \|\psi_\gamma\|_\alpha) \int_{0}^{m_3} d^\alpha(\phi_t(z), \phi_{t+v}(y_0)) dt\]
\[\geq - (\|\bar{u}\|_\alpha + \|h\|_\alpha + |a_O| \|\psi_\gamma\|_\alpha) \cdot \frac{4C^3 \alpha}{\lambda \alpha} \cdot \frac{|a_O| \|\psi\|_0 + \|h\|_0}{\epsilon},\]
where we write \(v\) short for \(v(y_0, z)\). Also note that
\[|a_O| = \left|\frac{\langle O, \bar{u} + h \rangle}{\langle O, \psi_\alpha \rangle}\right| \leq \frac{\|\bar{u}\|_0 + \|h\|_0}{\psi_{min}} \leq \frac{\|\bar{u}\|_0 + 1}{\psi_{min}}.\]

Thus, one has that
\[\int_{0}^{m_3} G(\phi_t(z)) - G(\phi_t(\phi_v(y_0))) dt\]
\[\geq - (\|\bar{u}\|_\alpha + \|h\|_\alpha + \|\bar{u}\|_0 + \|h\|_0 + 1) \cdot \frac{4C^3 \alpha}{\lambda \alpha} \cdot \frac{|a_O| \|\psi\|_0 + \|h\|_0}{\epsilon},\] 
(4.35)

Rewrite \(m_3 = p|O| + r\) for some nonnegative integer \(p\) and real number \(0 \leq r \leq |O|\). By applying (4.29), one has that
\[\int_{0}^{m_3} G(\phi_t(\phi_v(y_0))) dt = \int_{m_3-r}^{m_3} G(\phi_t(\phi_v(y_0))) dt \geq -|O| \cdot (|a_O| \|\psi\|_0 + \|h\|_0),\] 
(4.36)
where we used \(\int Gd\mu_O = 0\). Combining (4.27), (4.33), (4.34), (4.35) and (4.36), we have
\[\int_{0}^{m_2} G(\phi_t(z)) dt \geq \int_{0}^{m_3} G(\phi_t(z)) dt + \int_{m_2}^{m_3} G(\phi_t(z)) dt\]
\[= \int_{0}^{m_3} G(\phi_t(z)) - G(\phi_{t+v}(y_0)) dt + \int_{0}^{m_3} G(\phi_{t+v}(y_0)) dt + \int_{m_2}^{m_3} G(\phi_t(z)) dt\]
\[\geq - (\|\bar{u}\|_\alpha + \|h\|_\alpha + \|\bar{u}\|_0 + 1) \cdot \frac{4C^3 \alpha}{\lambda \alpha} \cdot \frac{|a_O| \|\psi\|_0 + \|h\|_0}{\epsilon} \] 
(4.35)
\[- |O| \cdot (|a_O| \|\psi\|_0 + \|h\|_0) \] 
(4.36)
\[+ \epsilon \left(\frac{D(O)}{4C^3 \epsilon^2}\right)^\alpha - |a_O| \|\psi\|_0 - \|h\|_0\] 
(4.38)
is Lip-dense in $C$.

Theorem 4.9

In this theorem, Remark 4.10.

Proposition 4.11

Proof of Part II) of Theorem 2.2.

Lemma 4.11.

Proof of Claim 2.

where we used Remark [14] and condition (1.26). Therefore, $m = m_2$ is the time as required since $m_2 \geq 1 \geq \tau$. This ends the proof of Claim 2. \hfill \Box

4.2. Proof of Part II) of Theorem 2.2

First we state a technical result on function approximation, which plays a key role in proving Proposition 4.11. Proposition 4.11 can be viewed as a $C^1$-version of Proposition 4.7, which implies the part II) of Theorem 2.2.

Theorem 4.9 ($GW$). Let $M$ be a smooth compact manifold. Then $C^\infty(M) \cap C^{0,1}(M)$ is Lip-dense in $C^{0,1}(M)$.

Remark 4.10. In this theorem, $C^\infty(M) \cap C^{0,1}(M)$ is Lip-dense in $C^{0,1}(M)$ means that for any $g_1 \in C^{0,1}(M)$ and $\varepsilon > 0$ there is corresponding $g_2 \in C^\infty(M)$ such that $\|g_1 - g_2\|_0 < \varepsilon$ and $\|g_2\|_1 < \varepsilon + \|g_1\|_1$. Especially, $\|D_M g_2\|_0 < \varepsilon + \|h\|_1$, where $D_M g$ is the derivative of function $g$ with respect to space variables.

Proposition 4.11. Given $0 < \varepsilon \leq 1$, a strictly positive $\psi \in C^{1,0}(M)$ and $u \in C^{1,0}(M)$, if a periodic segment $\mathcal{O}$ of $\Phi|_\Lambda$ satisfies the following comparison condition

$$D(\mathcal{O}) > \left( \frac{4C^3(\|\tilde{u}\|_1 + 10\varepsilon + \frac{\|\tilde{u}\|_{\psi,\min}+1}{\psi_{\min}} \|\psi_1\|_1)}{\lambda_\varepsilon \tau_0} + 1 + \frac{1}{\tau_0}\right) \left( \frac{\|\tilde{u}\|_1 \|\psi_0\|_0}{\psi_{\min}} \cdot \frac{400C^3 e^{2\beta}}{\varepsilon} \cdot d_{\alpha,\psi,\varepsilon}(\mathcal{O}), \right.$$}

where $\tilde{u}$ is defined in Remark 3.3 and $\tau_0$ is the constant in Lemma 4.3.

Then there is a $w \in C^\infty(M)$ with $\|w\|_0 < 2\varepsilon \cdot \text{diam}(M)$ and $\|D_M w\|_0 < 2\varepsilon$ such that the probability measure

$$\mu_\mathcal{O} \in \mathcal{M}_{\text{min}}(u + w + h; \psi, \Lambda, \Phi),$$

where we used Remark 4.3 and condition (1.26). Therefore, $m = m_2$ is the time as required since $m_2 \geq 1 \geq \tau$. This ends the proof of Claim 2. \hfill \Box
where \( h \) is any \( C^1 \) function with \( \|D_M h\|_0 < 5\varepsilon \) and
\[
\|h\|_0 < \frac{1}{2} \cdot \min \left\{ \frac{\varepsilon}{2} \left( \frac{D(O)}{4C_3 e^2\delta} \right) + \|\psi\|_0 + 1, 1 \right\}.
\]

**Proof.** By Theorem 4.9, there exists a function \( w \in C^\infty \) such that
\[
\|D_M w\|_0 < \|\varepsilon d(\cdot, O)\|_1 + \varepsilon \leq 2\varepsilon
\]
and
\[
\|w - \varepsilon d(\cdot, O)\|_0 < \min \left( \frac{H}{2}, \varepsilon \cdot \text{diam}(M) \right),
\]
where
\[
H = \min \left\{ \frac{\varepsilon}{2} \left( \frac{D(O)}{4C_3 e^2\delta} \right) + \|\psi\|_0 + 1, 1 \right\}.
\]
Next we show that \( w \) is the function as required. Note that
\[
u + w + h = u + \varepsilon d(\cdot, O) + (w - \varepsilon d(\cdot, O) + h).
\]
Notice that,
\[
\|w - \varepsilon d(\cdot, O) + h\|_1 \leq \|D_M w\|_0 + \|\varepsilon d(\cdot, O)\|_1 + \|h\|_1 \leq 2\varepsilon + \varepsilon + 5\varepsilon < 10\varepsilon,
\]
and
\[
\|w - \varepsilon d(\cdot, O) + h\|_0 \leq \|w - \varepsilon d(\cdot, O)\|_0 + \|h\|_0 < \frac{H}{2} + \frac{H}{2} = H.
\]
Then by Proposition 4.7, we have that \( \mu_O \in \mathcal{M}_{min}(u + w + h; \psi, \Lambda, \Phi) \). Additionally,
\[
\|w\|_0 < \|\varepsilon d(\cdot, O)\|_0 + \varepsilon \cdot \text{diam}(M) \leq 2\varepsilon \cdot \text{diam}(M).
\]
This ends the proof. \( \square \)

**Remark 4.12.** Let \( \tilde{w} \in C^{1,0}(M) \) be such that \( \|\tilde{w}\|_{1,0} < \varepsilon, \tilde{w}|_O = 0 \) and \( \tilde{w}|_{M\setminus O} > 0 \). Then \( \mu_O \) is the unique measure in \( \mathcal{M}_{min}(u + \tilde{w} + w + h; \psi, \Lambda, \Phi) \) whenever \( \|h\|_1 < 5\varepsilon \) and \( \|h\|_0 \) is sufficiently small. The Proposition shows that there is an open set of \( C^{1,0}(M) \) near \( u \) such that functions in the open set have the same unique minimizing measure with respect to \( \psi \) and the measure supports on a periodic orbit.

5. Proofs of Technical Lemmas

We note that throughout this the whole section, \( \delta, \epsilon, \lambda, \beta, C, \epsilon', \delta' \) are same as the ones in Remark 3.2.
5.1. Proof of Lemma [3.4]

Proof. We put a small positive constant $\tau$ with $\tau \ll 1$ such that $|s(t_1) - s(t_2)| \leq \eta$ for all $|t_1 - t_2| \leq \tau$ and $t_1, t_2 \in [0, T]$. Since $\eta \leq \frac{C^{10^{10^3+10^3}}}{\lambda_{-1}} \delta'$, for all $0 \leq t \leq T$, there exists $r(t)$ with $|r(t)| < C\eta$ such that

$$w(\phi_{t+s(t)+r(t)}(y), \phi_t(x)) = W_{\epsilon'}^s(\phi_{t+s(t)+r(t)}(y)) \cap W_{\epsilon'}^u(\phi_t(x)). \quad (5.1)$$

Then for $t' \in [-\tau, \tau]$ and $t \in [\tau, T - \tau]$, one has that

$$\phi_{t'}(w(\phi_{t+s(t)+r(t)}(y), \phi_t(x))) \in W_{\epsilon'}^s(\phi_{t+s(t)+r(t)+t'}(y)) \cap W_{\epsilon'}^u(\phi_{t+t'}(x)).$$

On the other hand, one has that

$$w(\phi_{t+t'+s(t+t')+(r(t)+t')}(y), \phi_{t+t'}(x)) = W_{\epsilon'}^s(\phi_{t+t'+s(t+t')+(r(t)+t')}(y)) \cap W_{\epsilon'}^u(\phi_{t+t'}(x)).$$

Since

$$\left| (t + t' + s(t + t') + r(t + t')) - (t + s(t) + r(t) + t') \right| \leq (2C + 1)\eta \ll \delta,$$

by the uniqueness of $v(\phi_{t+s(t+r(t)+t'}y, \phi_{t+t'}x)$ where the function $v$ is given by (1) of the basic canonical setting, one has that

$$t + t' + s(t + t') + r(t + t') = t + s(t) + r(t) + t',$$

and

$$w(\phi_{t+t'+s(t+t')+(r(t)+t')}(y), \phi_{t+t'}(x)) = \phi_{t'}(w(\phi_{t+s(t)+r(t)}(y), \phi_t(x))),$$

for all $t' \in [-\tau, \tau]$ and $t \in [\tau, T - \tau]$. Since $\tau$ can be taken arbitrarily small, one has the following by induction

$$s(t) + r(t) = s(\tau) + r(\tau) = s(0) + r(0) = r(0) = v(y, x), \quad \forall t \in [0, T]. \quad (5.2)$$

Thus

$$|s(t)| \leq |r(t)| + |r(0)| \leq 2C\eta,$$

and for all $t \in [\tau, T - \tau]$ and $t' \in [-\tau, \tau]$

$$w(\phi_{t+t'+v(y,x)}(y), \phi_{t+t'}(x)) = \phi_{t'}(w(\phi_{t+v(y,x)}(y), \phi_t(x))),$$

which implies that

$$w(\phi_{t+v(y,x)}(y), \phi_t(x)) = \phi_t(w(\phi_{v(y,x)}(y), x)) \forall t \in [0, T]. \quad (5.3)$$

Now, we prove the rest part. Note $w = w(\phi_{v(y,x)}(y), x) = W_{\epsilon'}^s(\phi_{v(y,x)}(y)) \cap W_{\epsilon'}^u(x)$. Then by (5.1), (5.2) and (5.3), one has

$$\phi_t(w) = W_{\epsilon'}^s(\phi_{t+\phi_{v(y,x)}(y)}) \cap W_{\epsilon'}^u(\phi_t(x)) \text{ for all } t \in [0, T].$$

Thus, for all $t \in [0, T]$,

$$d(\phi_t(w), \phi_{t+\phi_{v(y,x)}(y)}) < C'd(\phi_t(x), \phi_t(y)) \text{ and } d(\phi_t(w), \phi_t(x)) < C'd(\phi_t(x), \phi_t(y)).$$

Therefore

$$d(\phi_t(w), \phi_t(x)) \leq Ce^{-\lambda(T-t)}d(\phi_T(w), \phi_T(x)) \leq C^2e^{-\lambda(T-t)}d(\phi_T(x), \phi_T(y)),$$
where we used \( w \in W_{\epsilon}^{u}(x) \) and
\[
d(\phi_{t}(w), \phi_{t} \varphi_{v(y,x)}(y)) \leq Ce^{-\lambda(t)}d(w, \phi_{v(y,x)}(y)) \leq C^{2}e^{-\lambda(t)}d(x, y).
\]
where we used \( w \in W_{\epsilon}^{s}(\phi_{y,x}v(y)) \). By summing up, we have
\[
d(\phi_{t} \varphi_{v(y,x)}(y), \phi_{t}(x)) \leq C^{2}e^{-\lambda \min(t, T-t)}(d(x, y) + d(\phi_{T}(x), \phi_{T}(y))) \text{ for } 0 \leq t \leq T.
\]

Now we assume that \( d(\phi_{t+s(t)}(y), \phi_{t}(x)) \leq \eta \) for all \( t \geq 0 \), then by the arguments above. We have for all \( t \geq 0 \),
\[
d(\phi_{t} \varphi_{v(y,x)}(y), \phi_{t}(x)) \leq C^{2}e^{-\lambda \min(t, 2t-t)}(d(x, y) + d(\phi_{2t}(x), \phi_{2t}(y)))
\]
\[
\leq 2C^{2}\eta e^{-\lambda \min(t, 2t-t)} \to 0 \text{ as } t \to +\infty.
\]

This ends the proof. \( \square \)

5.2. Proof of Lemma 3.5

Proof. We partially follow Bowen’s arguments in \cite{Bowen}. Firstly we fix a constant \( K \gg C \) with \( 2C^{2}e^{-\lambda K} \ll 1 \) and a segment \( S \) as in Lemma 3.5. We let \( \tau = |S| \) and \( \eta = d(S^{L}, S^{R}) \). Then \( \eta < \delta' \) and \( 2C^{2}e^{-\lambda \tau} \ll 1 \). Otherwise, we have the following Claim.

Claim A. There is a \( y \in \Lambda \) and a continuous function \( \hat{s}: \mathbb{R} \to \mathbb{R} \) with \( \hat{s}(0) = 0 \) and \( \text{Lip}(\hat{s}) \leq \frac{2C\eta}{\tau} \) such that \( d(\phi_{i \tau+\tau+t}, \phi_{t}(y)) \leq L_{1} \eta \) for all \( t_{1} \in [0, \tau] \) and \( i \in \mathbb{Z} \) where \( L_{1} = 4C^{2} + 2C^{2}e^{\beta} + C^{2}e^{2\lambda} \).

Since the proof of Claim A is long, the readers find the proof in the following subsection. We let \( y \in M \) and \( \hat{s}: \mathbb{R} \to \mathbb{R} \) as in Claim A. We divide the following proof into two steps.

Step 1. At first, we show that \( y \) is a periodic point.

By Claim A,
\[
d(\phi_{t+\hat{s}(t)}(y), \phi_{t+\tau+\hat{s}(\tau+t)}(y)) \leq 2L_{1}\eta \text{ for } t \in \mathbb{R}.
\]

Since \( \text{Lip}(\hat{s}) \ll 1 \) and \( \hat{s}(0) = 0, g(t) = t + \hat{s}(t) \) is a homomorphism of \( \mathbb{R} \) onto itself, the above inequality can be rewritten as the following
\[
d(\phi_{t}(y), \phi_{g^{-1}(t)+\tau+\hat{s}(g^{-1}(t)+\tau)}(y)) \leq 2L_{1}\eta \text{ for } t \in \mathbb{R}.
\]

We note
\[
y' = \phi_{g^{-1}(0)+\tau+\hat{s}(g^{-1}(0)+\tau)}(y)
\]
and
\[
s(t) = g^{-1}(t) + \hat{s}(g^{-1}(t) + \tau) - g^{-1}(0) - \hat{s}(g^{-1}(0) + \tau) - t.
\]
Then
\[
d(\phi_{t}(y), \phi_{t+s(t)}(y')) \leq 2L_{1}\eta \text{ for } t \in \mathbb{R} \text{ and } s(0) = 0.
\]
Therefore, by Lemma 3.4, one has 
\[ \phi_{v_2}(y') = y \text{ and } |v_2| \leq 2CL_1 \eta, \]
where \( v_2 = v(y', y) \). Thus 
\[ \phi_{g^{-1}(0)+\hat{s}(g^{-1}(0)+\tau)+v_2}(y) = y. \]
Notice that \( g^{-1}(0) = 0 \) since \( g(0) = 0 \). Thus,
\[ |g^{-1}(0) + \hat{s}(g^{-1}(0) + \tau) + v_2| \leq |\hat{s}(\tau)| + |v_2| \leq (2C + 2CL_1)\eta \ll \tau. \]
Therefore, \( y \) is a periodic point.

**Step 2.** There is a periodic segment \( O \) such that 
\[ ||S| - |O|| \leq Ld(S^L, S^R) \]
and 
\[ d(\phi_t(O^L), \phi_t(S^L)) \leq Ld(S^L, S^R) \text{ for all } 0 \leq t \leq \max(|S|, |O|), \]
where \( L = 2CL_2 + L_2 \) and \( L_2 = 2C^3L_1 + C^4L_1 \).

By Claim A,
\[ d(\phi_{t+\hat{s}(t)}(y), \phi_t(S^L)) \leq L_1 \eta \text{ for } t \in [0, \tau]. \]
By Lemma 3.4 for \( t \in [0, \tau] \), \( |\hat{s}(t)| \leq 2CL_1 \eta \) and
\[ d(\phi_{t,v_1}(y), \phi_t(S^L)) \leq C^2 e^{-\lambda \min(t, \tau-t)} (d(y, S^L) + d(\phi_{\tau}(y), \phi_{\tau}(S^L))) \]
\[ \leq C^2 (d(y, S^L) + d(\phi_{\tau+\hat{s}(\tau)}(y), \phi_{\tau}(S^L)) + d(\phi_{\tau+\hat{s}(\tau)}(y), \phi_{\tau}(y))) \]
\[ \leq L_2 \eta, \]  
(5.4)
where \( v_1 = v(y, S^L) \). Now we put \( y^* = \phi_{v_1} y \) and we have a periodic segment,
\[ O : [0, \tau + g^{-1}(0) + \hat{s}(g^{-1}(0) + \tau) + v_2] \rightarrow M : t \rightarrow \phi_t(y^*), \]
where \( v_2 \) is as in **Step 1**.

It is clear that 
\[ ||S| - |O|| \leq |g^{-1}(0) + \hat{s}(g^{-1}(0) + \tau) + v_2| \leq L_2 \eta. \]
If \( |O| \leq |S| \), by (5.4),
\[ d(\phi_t(y^*), \phi_t(S^L)) \leq L_2 \eta, \text{ for } t \in [0, \max(|S|, |O|)], \]
where we used \( \tau = \max(|S|, |O|) \). 
If \( |O| > |S| \), by (5.4),
\[ d(\phi_t(y^*), \phi_t(S^L)) \leq L_2 \eta, \text{ for } t \in [0, |S|], \]
and for \( t \in (|S|, |O|] \),
\[ d(\phi_t(y^*), \phi_t(S^L)) \leq d(\phi_t(y^*), \phi_{\tau}(y^*)) + d(\phi_{\tau}(y^*), \phi_{\tau}(S^L)) + d(\phi_{\tau}(S^L), \phi_t(S^L)) \]
\[ L \leq L \eta, \]
where \( L = 2CL_2 + L_2 \). This ends the proof since \( L_2 \leq L \). \( \square \)

5.2.1. Proof of Claim A.

Proof. Remind that \( S \) is a segment of \( \Phi|_\Lambda \) with \( |S| = \tau \geq K \) and \( d(S^L, S^R) = \eta < \delta' \) where \( K \) satisfies \( 2C^2 e^{-\lambda K} \ll 1 \). We define \( x_{-k}, \zeta_{-k} \) recursively for \( k \geq 0 \) by
\[ x_0 = S^L, \xi_0 = 0 \]
and
\[ \zeta_{-k-1} = v(\phi_{-\tau}(x_{-k}), S^R), x_{-k-1} = W^\xi(\phi_{-\tau+\zeta_{-k-1}}(x_{-k})) \cap W^\xi(S^R) \text{ for } k = 1, 2, \ldots. \]
We have the following assertions.

Assertion 1. \( x_{-k} \) and \( \zeta_{-k} \) are well defined and \( d(x_{-k}, S^R) \leq 2C\eta \) for \( k \geq 0 \).

Proof. In the case \( k = 0 \), it is obviously true. Now assume that we have \( \zeta_{-k}, x_{-k} \) and \( d(x_{-k}, S^R) \leq 2C\eta \). Then
\[ d(\phi_{-\tau}(x_{-k}), S^R) \leq d(\phi_{-\tau}(x_{-k}), \phi_{-\tau}(S^R)) + d(S^R, \phi_{-\tau}(S^R)) \]
\[ \leq Ce^{-\lambda \tau} \cdot 2C\eta + \eta \]
\[ \leq 2\eta, \] \( \quad (5.5) \)
where we used \( x_k \in W^\xi(S^R) \). Since \( 2\eta \leq 2\delta' \), \( x_{-k-1} \) is well defined as well as \( \zeta_{-k-1} \), and moreover one has that
\[ d(x_{-k-1}, S^R) \leq 2C\eta. \]
This ends the proof. \( \square \)

By (5.5), we have that
\[ |\zeta_k| \leq 2C\eta \ll 1. \] \( \quad (5.6) \)
Next we note \( x^{(-k)} = \phi_{k\tau - \sum_{i=0}^k \zeta_i}(x_{-k}) \) for \( k \geq 0 \). For \( k \in \mathbb{N} \), we define \( s_{-k}^*: \mathbb{R} \to \mathbb{R} \) by
\[ s_{-k}^*(t) = \begin{cases} \zeta_0, & \text{if } t \geq 0, \\ \sum_{i=0}^l \zeta_i, & \text{if } -l\tau \leq t < -(l-1)\tau, l \in \{1, 2, \ldots, k\}, \\ \sum_{i=0}^k \zeta_i, & \text{if } t < -k\tau. \end{cases} \]

Assertion 2. There exists a constant \( L_1 \) such that for \( t = -j\tau - t_0 \) satisfying \( t_0 \in [0, \tau) \) and \( j \in \{0, 1, \ldots, k - 1\} \), the following holds
\[ d\left( \phi_{t+s_{-k}^*(t)}(x^{(-k)}), \phi_{-t_0}(S^R) \right) \leq L_1\eta. \]
Proof. We fix $t = -j \tau - t_0$ for some $j \in \{0, 1, \cdots, k-1\}$ and $t_0 \in [0, \tau)$. We have
\[
d\left(\phi_{-t_0}(x_{-j}), \phi_{-t_0}(S^R)\right) \leq C e^{-\lambda t_0} d(x_{-j}, S^R) \leq 2C^2 \eta. \tag{5.7}
\]
Notice that $\tau - \zeta_{-j} - t_0 \geq -1$, we have
\[
d\left(\phi_{\tau - \zeta_{-j+1} - t_0}(x_{-j+1}), \phi_{t_0}(x_{-j})\right)
= d\left(\phi_{\tau - \zeta_{-j+1} - t_0}(x_{-j+1}), \phi_{\tau - \zeta_{-j} - t_0}(x_{-j})\right)
\leq e^{\beta} d\left(x_{-j+1}, \phi_{\tau - \zeta_{-j} - t_0}(x_{-j})\right)
\leq 2C^2 e^{\beta} \eta, \tag{5.8}
\]
where $\beta$ is the one as in Remark 3.2.

On the other hand, one has that
\[
\sum_{l=j}^{k-1} d\left(\phi_{(l+1-j)\tau - \sum_{i=j+1}^{l+1} \zeta_{-i} - t_0}(x_{-l-1}), \phi_{(l-j)\tau - \sum_{i=j+1}^{l} \zeta_{-i} - t_0}(x_{-l})\right)
\leq \sum_{l=j}^{k-1} C e^{-\lambda \left((l+1-j)\tau - \sum_{i=j+1}^{l+1} \zeta_{-i} - t_0\right)} \eta \tag{5.9}
\]
\[
\leq C e^{2\lambda} e^{\lambda - 1} \eta,
\]
where we used $|\zeta| \ll 1, \tau \gg 1$ and $t_0 \leq \tau$. Combining (5.7), (5.8) and (5.9), we have that for $t = -j \tau - t_0$
\[
d\left(\phi_{t+s^{-k}_n(t)}(x^{(-k)}), \phi_{-t_0}(S^R)\right)
= d\left(\phi_{-j \tau + \sum_{i=j+1}^{k} \zeta_{-i} - t_0}(x^{(-k)}), \phi_{-t_0}(S^R)\right)
= d\left(\phi_{(k-j)\tau - \sum_{i=j+1}^{k} \zeta_{-i} - t_0}(x_{-k}), \phi_{-t_0}(S^R)\right)
\leq \sum_{l=j+1}^{k-1} d\left(\phi_{(l+1-j)\tau - \sum_{i=j+1}^{l+1} \zeta_{-i} - t_0}(x_{-l-1}), \phi_{(l-j)\tau - \sum_{i=j+1}^{l} \zeta_{-i} - t_0}(x_{-l})\right)
+ d\left(\phi_{\tau - \zeta_{-j+1} - t_0}(x_{-j+1}), \phi_{-t_0}(x_{-j})\right)
+ d\left(\phi_{-t_0}(x_{-j}), \phi_{-t_0}(S^R)\right)
\leq L_0 \eta,
\]
where $L_0 = 2C^2 + 2C^2 e^{\beta} + C e^{2\lambda} e^{\lambda - 1}$. \qed

Now for $k \in \mathbb{N}$, we define $\bar{s}_{-k} : \mathbb{R} \to \mathbb{R}$ by
\[
\bar{s}_{-k}(t) = \begin{cases} 
\zeta_0, & \text{if } t \geq 0, \\
\sum_{i=0}^{t} \zeta_{-i} - \frac{t+1}{\tau} \zeta_{-t}, & \text{if } -l \tau \leq t < -(l-1)\tau, l \in \{1, 2, \cdots, k\}, \\
\sum_{i=0}^{k} \zeta_{-i}, & \text{if } t < -k \tau.
\end{cases}
\]
It is clear that $s_{-k}$ is Lipschitz continuous with

$$Lip(s_{-k}) \leq \frac{\max_{i \in \{0,1,\ldots,k\}} |\zeta_i|}{\tau} \leq \frac{2C\eta}{\tau},$$

and

$$|\bar{s}_{-k}(t) - s_{-k}^\ast(t)| \leq \max_{i \in \{0,1,\ldots,k\}} |\zeta_i| \leq 2C\eta.$$

Therefore, by Assertion 2., when $t = -j\tau - t_0$ for some $j \in \{0,1,2,\ldots,k - 1\}$ and $t_0 \in [0,\tau)$, one has that

$$d\left(\phi_{t+\bar{s}_{-k}(t)}(x^{(-k)}), \phi_{-t_0}(S^R)\right) \leq \left|\phi_{t+\bar{s}_{-k}(t)}(x^{(-k)})\right| + d\left(\phi_{t+s_{-k}^\ast(t)}(x^{(-k)}), \phi_{-t_0}(S^R)\right) \leq L_1 \eta,$$

where $L_1 = L_0 + 2C^2$. Now for $k \in \mathbb{N}$, we define $s_{-k} : \mathbb{R} \to \mathbb{R}$ by

$$s_{-k}(t) = \bar{s}_{-2k}(t - k\tau) - \sum_{i=0}^{k} \zeta_{-i}.$$

It is clear that $s_{-k}(0) = 0$. On the other hand, we note $y_k = \phi_{-\tau k + \sum_{i=0}^{k} \zeta_{-i}}(x^{(-2k)})$. Thus, when $t = -j\tau - t_0$ for some $j \in \{-k,-k+1,\ldots,k-1\}$ and $t_0 \in [0,\tau)$, (5.10) implies that

$$d\left(\phi_{t+s_{-k}(t)}(y_k), \phi_{-t_0}(S^R)\right) = d\left(\phi_{t-\tau k + \bar{s}_{-2k}(t-\tau k)}(x^{(-2k)}), \phi_{-t_0}(S^R)\right) \leq L_1 \eta.$$

Notice that $s_{-k}$ are Lipschitz with $Lip(s_{-k}) \leq \frac{2C\eta}{\tau} \ll \eta$ for all $k \in \mathbb{N}$. Applying the Ascoli-Azelá theorem, there exists a subsequence $(s_{-k_i})_{i=1}^{+\infty}$ that converges to a Lipschitz continuous function $\bar{s} : \mathbb{R} \to \mathbb{R}$ with $Lip(\bar{s}) \leq \frac{2C\eta}{\tau} \ll \eta$ and $s(0) = 0$. Without losing any generality, we assume that $y_{k_i} \to y$ as $i \to +\infty$. By the continuity, if $t = -j\tau - t_0$ for some $j \in \mathbb{Z}$ and $t_0 \in [0,\tau)$, then

$$d(\phi_{t+\bar{s}(t)}(y), \phi_{-t_0}(S^R)) \leq L_1 \eta.$$

In other words, if $t = -j\tau + t_0$ for some $j \in \mathbb{Z}$ and $t_0 \in [0,\tau)$, then

$$d\left(\phi_{t+\bar{s}(t)}(y), \phi_{t_0}(S^L)\right) \leq L_1 \eta.$$

Note that $y_{k_i} \in \Lambda$ for each $i \in \mathbb{N}$, thus $y \in \Lambda$. This ends the proof of Claim A.. \qed

5.3. Proof of Lemma 3.7 In this section, we mainly prove a version of the so called Mañè-Conze-Guivarc’h-Bousch’s Lemma. The proof partially follows Bousch’s arguments in [Bo3].
5.3.1. Integration Along Segment. Recall that, for a continuous function \( u \) and a segment \( S \) of \( \Phi \), the integration of \( u \) along \( S \) is defined by

\[
\langle S, u \rangle = \int_a^b u(\phi_t(x))dt.
\]

**Lemma 5.1.** Let \( u : M \to \mathbb{R} \) be an \( \alpha \)-Hölder function with \( \beta(u; 1, \Lambda, \Phi) \geq 0 \). Then for a segment \( S \) of \( \Phi |_{\Lambda} \) satisfying \(|S| \geq K \) and \( d(S^L, S^R) \leq \delta' \), the following holds

\[
\langle S, u \rangle \geq -K_1 d^\alpha(S^L, S^R),
\]

where \( K_1 = \frac{(CL)^\alpha}{\lambda \alpha} \|u\|_\alpha + L \|u\|_0 \).

**Proof.** Since \( d(S^L, S^R) < \delta' \), by Anosov Closing Lemma, there exists a periodic segment \( O \) of \( \Phi |_{\Lambda} \) such that

\[
||S| - |O|| \leq Ld(S^L, S^R)
\]

and

\[
d(\phi_t(O^L), \phi_t(S^L)) \leq Ld(S^L, S^R) \text{ for all } 0 \leq t \leq \max(|S|, |O|).
\]

Therefore, by noting \( v = v(O^L, S^L) \) as in Lemma 3.4,

\[
\langle S, u \rangle - \langle O, u \rangle = \int_0^{|O|} u(\phi_t(S^L)) - u(\phi_t(O^L))dt + \int_0^{|S|} u(\phi_t(S^L))dt
\]

\[
\geq -\|u\|_\alpha \int_0^{|O|} d^\alpha(u(\phi_t(S^L)), u(\phi_t(O^L)))dt - \|u\|_0 |S| - |O|
\]

\[
\geq -\|u\|_\alpha \int_0^{|O|} \left(Ce^{-\lambda \min(t,T-t)}Ld(S^L, S^R)^\alpha dt - \|u\|_0 Ld(S^L, S^R)\right)
\]

\[
\geq - \left(\frac{(CL)^\alpha}{\lambda \alpha} \|u\|_\alpha + L \|u\|_0 \right) d^\alpha(S^L, S^R),
\]

where we used the assumption \( 0 < \alpha \leq 1 \) and \( 0 < d(S^L, S^R) < \delta' \ll 1 \). Then the Lemma is immediately from the fact \( \langle O, u \rangle \geq 0 \) since \( \beta(u; 1, \Lambda, \Phi) \geq 0 \). This ends the proof. \( \square \)

**Lemma 5.2.** Let \( P \) be a finite partition of \( M \) with diameter small than \( \delta' \) and \( u : M \to \mathbb{R} \) be an \( \alpha \)-Hölder function with \( \beta(u; 1, \Lambda, \Phi) \geq 0 \). Then for a given segment \( S \) of \( \Phi |_{\Lambda} \), the following holds

\[
\langle S, u \rangle \geq -K_2 \delta^\alpha,
\]

where \( K_2 = \sum_{P} \left(\frac{\|w\|_0}{\delta} \right) + \left(\frac{(CL)^\alpha}{\lambda \alpha} \|w\|_\alpha + L \|w\|_0 \right) \) and \( K \) is as in Lemma 3.5.

**Proof.** For \( x \in M \), note \( P(x) \) be the element in \( P \) which contains \( x \). Assume \( |S| = (n-1)K + r \) for some \( n \geq 1 \) and \( 0 \leq r < K \). Note \( t_i = iK \) for \( 0 \leq i \leq n - 1 \) and \( t_n = |S| \). We define the function \( w : \mathbb{N} \to [0, n] \cap \mathbb{N} \) inductively by letting

\[
w(0) = 0
\]
where \( \eta : [0, n-1] \cap \mathbb{N} \to [0, n-1] \cap \mathbb{N} \) is the function that maps each \( i \) to the largest \( j \in [0, n-1] \cap \mathbb{N} \) such that \( \mathcal{P}(\varphi_i(S^L)) = \mathcal{P}(\varphi_j(S^L)) \). Let \( s \geq 0 \) be the smallest integer for which \( \eta(w(s)) = n-1 \). Then \( \mathcal{P}(\varphi_{t(w(s))}(S^L)) \neq \mathcal{P}(\varphi_{t(w(s))}(S^L)) \) for \( 0 \leq i < j \leq s \) which implies \( s \leq \#\mathcal{P} \). For \( 0 \leq j \leq s \), we have two cases: If \( \eta(w(j)) = w(j) \)

\[
\int_{t_{\eta(w(j))}}^{t_{\eta(w(j))}+1} u(\varphi_t(x)) dt = 0 \quad \text{and} \quad \int_{t_{\eta(w(j))}}^{t_{\eta(w(j))}+1} u(\varphi_t(x)) dt \geq -K\|u\|_0. \tag{5.11}
\]

If \( \eta(w(j)) > w(j) \), by using Lemma 5.1

\[
\int_{t_{\eta(w(j))}}^{t_{\eta(w(j))}+1} u(\varphi_t(x)) dt \geq -\left(\frac{(CL)^{\alpha}}{\lambda \alpha}\|u\|_\alpha + L\|u\|_0\right) \delta^{\alpha}, \tag{5.12}
\]

where we use the fact \( d(\varphi_i(S^L), \varphi_j(S^L)) < \delta \) since \( \mathcal{P}(\varphi_i(S^L)) = \mathcal{P}(\varphi_j(S^L)) \). On the other hand, as in (5.11),

\[
\int_{t_{\eta(w(j))}}^{t_{\eta(w(j))}+1} u(\varphi_t(S^L)) dt \geq -K\|u\|_0. \tag{5.13}
\]

Combining (5.11), (5.12) and (5.13), one has

\[
\langle S, u \rangle = \sum_{j=0}^{s-1} \int_{t_{\eta(w(j))}}^{t_{\eta(w(j))}+1} u(\varphi_t(S^L)) dt
\]

\[
\geq -s \left( K\|u\|_0 + \left(\frac{(CL)^{\alpha}}{\lambda \alpha}\|u\|_\alpha + L\|u\|_0\right) \delta^{\alpha} \right)
\]

\[
\geq -\#\mathcal{P} \cdot \left( K\|u\|_0 + \left(\frac{(CL)^{\alpha}}{\lambda \alpha}\|u\|_\alpha + L\|u\|_0\right) \delta^{\alpha} \right),
\]

which completes the proof.

In the following, we deal with the so called shadowing property for two finite time segments, which will allow one to use one segment to shadow two segments of which the ending point of one segment is close to the beginning point of the other. Let \( S_1 \) and \( S_2 \) be two segments of \( \Phi \), suppose that

\[
d(S^R_1, S^L_2) < \delta'.
\]

Then there exist \( v(S^L_2, S^R_1) \) and \( w(S^L_1, S^R_1) = W^v(\phi_v(S^L_2, S^R_1), (S^L_2)) \cap W^w(S^R_1) \). Define a new segment \( S_1 \ast S_2 : [-|S_1|, |S_2| - v(S^L_2, S^R_1)] \) by letting

\[
S_1 \ast S_2(t) = \phi_v \left( w(S^L_2, S^R_1) \right) \quad \forall t \in [-|S_1|, |S_2| - v(S^L_2, S^R_1)]. \tag{5.14}
\]

We remark here that the definition of \( S_1 \ast S_2 \) above is not the unique way for describing the shadowing property. Nevertheless, it is the most convenient way for the rest of the proof.
Lemma 5.3. Given $0 < \alpha \leq 1$ and a large constant $\gamma = \gamma(\alpha) \gg 1$ satisfying that $2C^{2\alpha}e^{-\frac{2\alpha}{\lambda}} \ll 1$, when two segments $S_1$ and $S_2$ of $\Phi|_\Delta$ satisfy the following
\[
d(S_1^R, S_2^L) \leq \delta' \quad \text{and} \quad \min\{|S_1|, |S_2|\} \geq \gamma,
\]
then for all $u \in C^{0,\alpha}(M)$,
\[
\frac{|\langle S_1 \ast S_2, u \rangle - \langle S_1, u \rangle - \langle S_2, u \rangle|}{d^\alpha(S_1^R, S_2^L) - d^\alpha((S_1 \ast S_2)^R, S_2^R) - d^\alpha((S_1 \ast S_2)^L, S_1^L)} \leq K_3,
\]
where $K_3 = \frac{C\|u\|_0 + 2C^{2\alpha}\|u\|_0}{1 - 2C^{2\alpha}e^{-(\gamma - 1)\alpha}}$ and the denominator of the left side of the above inequality is always positive by the choice of $\gamma$.

Proof. Fix $\alpha, \gamma, u, S_1, S_2$ as in this Lemma. Note $v = v(S_2^L, S_1^R)$, $w = w(S_2^L, S_1^R)$ and $\tilde{S}_2 : [0, |S_2| - v] : t \rightarrow \phi_{t+v}(S_2^L)$.

Thus, we have
\[
\langle S_1 \ast S_2, u \rangle - \langle \tilde{S}_2, u \rangle - \langle S_1, u \rangle = \left| \int_0^{|S_2| - v} u(\phi_t(w)) - u(\phi_t(S_2^L)) dt + \int_0^{|S_1|} u(\phi_{-t}(w)) - u(\phi_{-t}(S_1^R)) dt \right|
\]
\[
\leq \int_0^{|S_2| - v} \|u\|_\alpha d^\alpha(\phi_t(w), \phi_t(S_2^L)) dt + \int_0^{|S_1|} \|u\|_\alpha d^\alpha(\phi_{-t}(w), \phi_{-t}(S_1^R)) dt
\]
\[
\leq \int_0^{|S_2| - v} \|u\|_\alpha (Ce^{-\lambda t})^\alpha d^\alpha(w, S_2^L) dt + \int_0^{|S_1|} \|u\|_\alpha (Ce^{-\lambda t})^\alpha d^\alpha(w, S_1^R) dt
\]
\[
\leq 2\|u\|_\alpha \frac{C^{2\alpha}}{\lambda \alpha} d^\alpha(S_1^R, S_2^L),
\]
and
\[
\langle \tilde{S}_2, u \rangle - \langle S_1, u \rangle \leq \|u\|_0 |v| \leq \|u\|_0 Cd(S_1^R, S_2^L) \leq \|u\|_0 Cd^\alpha(S_1^R, S_2^L).
\]
Therefore
\[
|\langle S_1 \ast S_2, u \rangle - \langle S_1, u \rangle - \langle S_2, u \rangle| \leq \left( C\|u\|_0 + \frac{2C^{2\alpha}\|u\|_0}{\lambda \alpha} \right) d^\alpha(S_1^R, S_2^L) \quad (5.15)
\]
On the other hand, one has that
\[
d^\alpha((S_1 \ast S_2)^R, S_2^L) \leq C^\alpha e^{-\alpha \lambda (\gamma - v)} d^\alpha(w, S_2^L) \leq C^{2\alpha} e^{-\alpha \lambda (\gamma - 1)} d^\alpha(S_1^R, S_2^L),
\]
and
\[
d^\alpha((S_1 \ast S_2)^L, S_1^R) \leq C^\alpha e^{-\alpha \lambda \gamma} d^\alpha (w, S_1^R) \leq C^{2\alpha} e^{-\alpha \lambda \gamma} d^\alpha (S_1^R, S_2^L),
\]
which combining with (5.15) and the choice of $\gamma$ implies what needed, thus accomplish the proof. \qed
5.3.2. Proof of Lemma 3.7 Before the main proof, we first state a technical Lemma which can be deduced from the Lemma 1.1 of [Bo3].

**Lemma 5.4.** Given $0 < \alpha \leq 1, A > 0, \gamma \in \mathbb{R}$ and a continuous function $u : M \to \mathbb{R}$, the following are equivalent

1. For all $n \geq 1$ and $x_i \in M, i \in \mathbb{Z}/n\mathbb{Z}$,
   \[ \sum_{i \in \mathbb{Z}/n\mathbb{Z}} u(x_i) + A \sum_{i \in \mathbb{Z}/n\mathbb{Z}} d^\alpha(\phi_\gamma x_i, x_{i+1}) \geq 0. \]  \hspace{1cm} (5.16)

2. There exists an $\alpha$-Hölder function $v : M \to \mathbb{R}$ with $\|v\|_\alpha \leq A$ such that $u \geq v \circ \phi_\gamma - v$.

Now we prove Lemma 3.7

**Proof.** Let $K_1, K_2, K_3$ be the constants as in Lemmas 5.1, 5.2 and 5.3. We fix a $\gamma > N_0$ satisfying the condition in Lemma 5.3 and a large number $Q$ such that $Q > \max\{K_1, K_2, K_3\}$.

For $n \geq 1$, we note $i^{(n)} = i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ for $i \in [0, n - 1] \cap \mathbb{Z}$. Now we fix an integer $n \geq 1$ and points $x_{i^{(n)}} \in \Lambda, i^{(n)} \in \mathbb{Z}/n\mathbb{Z}$. Note

\[ S_{i^{(n)}} : [0, \gamma] \to \Lambda : t \to \phi_t(x_{i^{(n)}}) \text{ for } i^{(n)} \in \mathbb{Z}/n\mathbb{Z}, \]

\[ \mathcal{L}^{(n)} = \{S_{i^{(n)}}, i^{(n)} \in \mathbb{Z}/n\mathbb{Z}\} \]

and

\[ \Sigma^{(n)} = \sum_{i^{(n)} \in \mathbb{Z}/n\mathbb{Z}} \langle S_{i^{(n)}}, u \rangle + Q \sum_{i^{(n)} \in \mathbb{Z}/n\mathbb{Z}} d^\alpha(S_{i^{(n)}}^R, S_{i^{(n)}}^L) \]

If there is some $j^{(n)} \in \mathbb{Z}/n\mathbb{Z}$ such that $d(S_{j^{(n)}}^R, S_{j^{(n)+1}}^L) < \delta'$, just take

\[ S_{1(n-1)} = S_{j^{(n)}} * S_{j^{(n)+1}(n)} \text{ and } S_{i^{(n-1)}} = S_{j^{(n)+1}} \text{ for } i = 2, 3, \ldots n-1 \]

\[ \mathcal{L}^{(n-1)} = \{S_{i^{(n-1)}}, i^{(n-1)} \in \mathbb{Z}/(n-1)\mathbb{Z}\} \]

and

\[ \Sigma^{(n-1)} = \sum_{i^{(n-1)} \in \mathbb{Z}/(n-1)\mathbb{Z}} \langle S_{i^{(n-1)}}, u \rangle + Q \sum_{i^{(n-1)} \in \mathbb{Z}/n\mathbb{Z}} d^\alpha(S_{i^{(n-1)}}^R, S_{i^{(n-1)+1}(n-1)}) \]

Note that by Lemma 5.3

\[ \Sigma^{(n)} - \Sigma^{(n-1)} \geq - \left| \langle S_{j^{(n)}} * S_{j^{(n)+1}(n)}^L, u \rangle - \langle S_{j^{(n)}}, u \rangle - \langle S_{j^{(n)+1}(n)}, u \rangle \right| \]

\[ + Qd^\alpha(S_{j^{(n)}}^R, S_{j^{(n)+1}(n)}) \]
\[-Q \left( d^\alpha \left( S_{j(n)}^L, (S_{j(n)}^* S_{(j+1)(n)})^L \right) - d^\alpha \left( S_{(j+1)(n)}^R, (S_{j(n)}^* S_{(j+1)(n)})^R \right) \right) \geq 0. \]

That is
\[\Sigma^{(n)} \geq \Sigma^{(n-1)}. \quad (5.17)\]

Repeat the above process until \(L^{(1)}\) with \(d(S_{1(1)}^R, S_{1(1)}^L) < \delta'\) OR some \(m \in [1, n] \cap \mathbb{N}\) with
\[d \left( S_{j(m)}^R, S_{(j+1)(m)}^L \right) \geq \delta' \text{ for all } j \in \mathbb{Z}/m\mathbb{Z}. \]

In the case the process end at \(L^{(1)}\) with \(d(S_{1(1)}^R, S_{1(1)}^L) < \delta'\). We have by Lemma 5.1 that
\[\Sigma^{(1)} = \langle S_{1(1)}, u \rangle + Q d^\alpha \left( S_{1(1)}^R, S_{1(1)}^L \right) \geq -K_1 d^\alpha \left( S_{1(1)}^R, S_{1(1)}^L \right) + Q d^\alpha \left( S_{1(1)}^R, S_{1(1)}^L \right) \geq 0. \quad (5.18)\]

In the case the process end at some \(m \in [1, n] \cap \mathbb{N}\) with
\[d \left( S_{j(m)}^R, S_{(j+1)(m)}^L \right) \geq \delta' \text{ for all } j^{(m)} \in \mathbb{Z}/m\mathbb{Z}. \]

We have by Lemma 5.2 that
\[\Sigma^{(m)} = \sum_{i^{(m)} \in \mathbb{Z}/m\mathbb{Z}} \langle S_{i^{(m)}}, u \rangle + Q \sum_{i^{(m)} \in \mathbb{Z}/m\mathbb{Z}} d^\alpha \left( S_{i^{(m)}}^R, S_{i+1^{(m)}}^L \right) \geq -mK_2 \delta^\alpha + mQ \delta^\alpha \geq 0. \quad (5.19)\]

Combining the inequality (5.18), (5.19) and the fact \(\Sigma^{(n)} \geq \Sigma^{(n-1)} \geq \Sigma^{(n-2)} \geq \ldots \) by (5.17), one has
\[\Sigma^{(n)} \geq 0. \]

Then
\[\sum_{i^{(n)} \in \mathbb{Z}/n\mathbb{Z}} u_\gamma(x_{i^{(n)}}) + \frac{Q}{\gamma} \sum_{i^{(n)} \in \mathbb{Z}/n\mathbb{Z}} d^\alpha \left( S_{i^{(n)}}^R, S_{i+1^{(n)}}^L \right) = \frac{\Sigma^{(n)}}{\gamma} \geq 0. \]

By Lemma 5.4 there is \(\alpha\) Hölder function \(v\) on \(\Lambda\) with \(\|v\|_\alpha \leq \frac{Q}{\gamma} \) such that
\[u_\gamma|_\Lambda \geq v \circ \phi_\gamma|_\Lambda - v. \]

This ends the proof. \(\square\)

Finally, we give the proof of Lemma 3.9.
Proof of Lemma 3.9. (1) By Lemma 3.7, we only need to show that
\[ \int u - \beta(u; \psi, \Lambda, \Phi) \psi d\mu \geq 0 \text{ for all } \mu \in \mathcal{M} (\Phi|\Lambda). \]
It is immediately from the fact since \( \psi \) is strictly positive.

(2). Given a probability measure \( \mu \in \mathcal{M}_{\min}(u, \psi, \Lambda, \Phi) \), one has
\[ \int u_\gamma + v \circ \phi_\gamma - v - \beta(u; \psi, \Lambda, \Phi) \psi_\gamma d\mu = \int u - \beta(u; \psi, \Lambda, \Phi) \psi d\mu = 0. \]
Combining (1) and the fact \( u_\gamma|_\Lambda + v \circ \phi_\gamma|_\Lambda - v - \beta(u; \psi, \Lambda, \Phi) \psi_\gamma|_\Lambda \) is continuous on \( \Lambda \), one has
\[ \text{supp}(\mu) \subseteq \{ x \in \Lambda : (u_\gamma + v \circ \phi_\gamma - v - \beta(u; \psi, \Lambda, \Phi) \psi_\gamma)|_\Lambda(x) = 0 \}. \]
Therefore,
\[ Z_{u, \psi} \subseteq \{ x \in \Lambda : (u_\gamma|_\Lambda + v \circ \phi_\gamma|_\Lambda - v - \beta(u; \psi, \Lambda, \Phi) \psi_\gamma|_\Lambda)(x) = 0 \}. \]
This ends the proof. \( \square \)

5.4. Proof of Lemma 3.11. In this section, we mainly prove the periodic approximation. The proof partially follows the arguments in [BQ].

5.4.1. Joining of segments. In this section, we give some properties of jointed segments.

Lemma 5.5. If two segments \( S_1 \) and \( S_2 \) satisfy \( |S_1| \geq 1 \) and \( d(S_1^R, S_2^L) \leq \delta' \), then

1. \( \max_{x \in S_1 \cup S_2} d(x, S_1 \cup S_2) \leq C \delta d(S_1^R, S_2^L); \)
2. \( |S_1| + |S_2| - 1 \leq |S_1 \ast S_2| \leq |S_1| + |S_2| + 1. \)

Proof. (1). Note \( v = v(S_2^L, S_1^R), w = w(S_2^L, S_2^R) \) and
\[ \tilde{S}_2 : [v, |S_2|] : t \to \phi_t(S_2^L). \]
Then for \( t \in [-|S_1|, 0], \)
\[ d(\phi_t(w), S_1) \leq d(\phi_t(w), \phi_t(S_1^R)) \leq C e^{\lambda t} d(w, S_1^R) \leq C^2 e^{\lambda t} d(S_1^R, S_2^L) \leq C^2 d(S_1^R, S_2^L), \]
where we used \( w \in W^u(\phi(S_1^R)). \) For \( t \in [0, |S_2| - v], \)
\[ d(\phi_t(w), \tilde{S}_2) \leq d(\phi_t(w), \phi_t(S_2^L)) \leq C e^{-\lambda t} d(w, \phi_t(S_2^L)) \leq C e^{-\lambda t} d(S_1^R, S_2^L). \]
where we used \( w \in W^s(\phi(S_2^L)). \) Thus,
\[ d(\phi_t(w), S_1) \leq d(\phi_t(w), \tilde{S}_1) + \max_{x \in \tilde{S}_1} d(S_1, \tilde{S}_1) \]
First we fix a large constant $P_0 > 1$ such that $C^2 e^{\lambda(P_0 - 1)} + C^2 e^{-\lambda P_0} < \frac{1}{2}$. Fix two segments $S_1$ and $S_2$ as in Lemma. Note $v = v(S_2^L, S_1^R)$ and $w = w(S_2^L, S_1^R)$. Then

$$d(S_1^L, (S_1 \ast S_2)^L) = d(\phi_{-|S_1|}(S_1^R), \phi_{-|S_1|}(w))$$

$$\leq C e^{-\lambda|S_1|} d(S_1^R, w)$$

$$\leq C^2 e^{-\lambda P_0} d(S_1^R, S_2^L),$$

where we used $w \in W^u(S_1^R)$. On the other hand,

$$d(S_2^R, (S_1 \ast S_2)^R) = d(\phi_{|S_2|-v}(S_2^L), \phi_{|S_2|-v}(w))$$

$$\leq C e^{-\lambda(|S_2|-v)} d(\phi_v(S_2^L), w)$$

$$\leq C^2 e^{-\lambda(P_0 - 1)} d(S_1^R, S_2^L),$$

where we used $w \in W^u(\phi_v(S_2^L))$. By assumption, we have

$$d(S_1^L, (S_1 \ast S_2)^L) + d(S_2^R, (S_1 \ast S_2)^R) \leq C^2 e^{-\lambda(P_0 - 1)} d(S_1^R, S_1^L) + C^2 e^{-\lambda P_0} d(S_1^R, S_1^L)$$

$$\leq \frac{1}{2} d(S_1^R, S_1^L).$$

This ends the proof.
5.4.2. Periodic Approximation. For integer \( n \geq 1 \), let \( \Sigma_n = \{0, 1, 2, \cdots, n-1\}^N \) and \( \sigma \) be a shift on \( \Sigma_n \). Assume \( F \) is a subset of \( \bigcup_{i \geq 1} \{0, 1, 2, \cdots, n-1\}^i \), then the subshift with forbidden \( F \) is noted by \((Y_F, \sigma)\) where

\[
Y_F = \{ x \in \{0, 1, 2, \cdots, n-1\}^N, w \text{ does not appear in } x \text{ for all } w \in F \}.
\]

The following Lemma is Lemma 5 of [BQ], which will be used later.

**Lemma 5.7 ([BQ]).** Suppose that \((Y, \sigma)\) is a shift of finite type (with forbidden words of length 2) with \( M \) symbols and entropy \( h \). Then \((Y, \sigma)\) contains a periodic point of period at most \( 1 + Me^{(1-h)} \).

Now we are ready to prove Lemma 3.11, which partially follow the argument in [BQ].

**Proof of Lemma 3.11.** Fix a positive constant \( \delta'' \ll \frac{e}{\delta''^2} \). Let \( \mathcal{P} = \{B_1, B_2, \cdots, B_m\} \) be a finite partition of \( \Lambda \) with diameter small than \( \delta'' \). For \( x \in \Lambda \), \( \tilde{x} \in \{1, 2, 3, \cdots, m\}^N \) is defined by

\[
\tilde{x}(n) = j \text{ if } \phi_n(x) \in B_j, \text{ for } n = 0, 1, 2, \cdots .
\]

Note \( \tilde{Z} = \{ \tilde{x} : x \in Z \} \) and \( W_n \) is the collection of length \( n \) string that appears in \( \tilde{Z} \). One has

\[
\mathbb{Z}^{W_n} = Kne^{nh}
\]

where \( h = h_{top}(Z, \sigma) \) and \( K_n \) grows at a subexponential rate. Note

\[
Y_n = \{y_0y_1y_2\cdots \in W_n^\mathbb{Z} : y_iy_{i+1} \in W_{2n} \text{ for all } i \in \mathbb{N} \}
\]

and \((Y_n, \sigma_n)\) is the 1-step shift of finite type \( W_n \). From Lemma 5.7, the shortest periodic orbit in \( Y_n \) is at most \( 1+Ke^{nh}e^{1-nh} = 1+Ke^n \). Note one of the shortest periodic orbit in \( Y_n \) by \( z_0z_1z_2\cdots z_{p_n-1}z_0z_1z_2\cdots \) for some \( p_n \leq 1+Ke^n \) and \( z_i \in W_{n}, i = 0, 1, 2, \cdots, p_n-1 \). For \( i = 0, 1, 2, \cdots, p_n-1 \), there is \( x_i \in Z \) such that the leading \( 2n \) string of \( \tilde{x}_i \) is \( z_i z_{i+1} \) (we note \( z_{p_n} = z_0, x_{p_n} = x_0, S_{p_n} = S_0, \cdots \)). Choose segments \( S_i \) by

\[
S_i : \left[ \frac{n}{2}, \frac{3n}{2} + v_i \right] \rightarrow M : t \rightarrow \phi_t(x_i) \text{ for } i = 0, 1, 2, \cdots, p_n - 1,
\]

where \( v_i = v(\phi_n(x_i), x_{i+1}) \). We have the following Claim.

**Claim Q1.** \( \|S_i^R, S_{i+1}^L\| \leq 2C^2e^{-\frac{nh}{2}} \delta'' \) for \( i = 0, 1, 2, \cdots, p_n - 1 \).

**Proof of Claim Q1.** Note that the leading \( 2n \) string of \( \tilde{x}_i \) is \( z_{i} z_{i+1} \) and leading \( n \) string of \( \tilde{x}_{i+1} \) is \( z_{i+1} \). Thus,

\[
\mathcal{P}(\phi_{n+j}(x_i)) = \mathcal{P}(\phi_j(x_{i+1})) \text{ for } j = 0, 1, \cdots, n - 1.
\]

Therefore,

\[
d(\phi_{n+j}(x_i), \phi_j(x_{i+1})) < \delta'' \text{ for } j = 0, 1, \cdots, n - 1.
\]
Thus
\[ d(\phi_{n+t}(x_i), \phi_t(x_{i+1})) < C e^\beta \delta'' < \delta' \text{ for } t \in [0, n]. \]
Then by Lemma 3.3, we have
\[ d\left( \frac{\phi_{n+t+1}(x_i)}{\chi}, \frac{\phi_{n+1}(x_{i+1})}{\chi} \right) \leq C^2 e^{-\frac{n\lambda}{2}} \left( d(\phi_n(x_i), x_{i+1}) + d(\phi_{2n}(x_i), \phi_n(x_{i+1})) \right) \]
\[ \leq 2C^2 e^{-\frac{n\lambda}{2}} \delta''. \]
This ends the proof of Claim Q1. \( \square \)

Now we define segments \( \overline{S}_i \) recursively for \( 0 \leq i \leq n-2 \) by \( \overline{S}_0 = S_0 \) and
\[ \overline{S}_i = \overline{S}_{i-1} * S_i \text{ for } 1 \leq i \leq n-2. \]
By using Claim Q1, we have the following Claim.

**Claim Q2.** There is positive integer \( N \) such that for any \( n \geq N \), one has

1. \( \overline{S}_i \) is well defined for \( 0 \leq i \leq n-2 \);
2. \( d(\overline{S}_i^R, S_{i+1}^L) \leq 4C^2 p_n e^{-\frac{n\lambda}{2}} \delta'' < \delta' \text{ for } 0 \leq i \leq n-3 \);
3. \( d(\overline{S}_{p_n-2}^R, \overline{S}_{p_n-2}^L) \leq 2C^2 p_n e^{-\frac{n\lambda}{2}} \delta'' < \min\{\delta', \frac{1}{L}\} \), where \( L \) is as in Lemma 3.3;
4. \( (n-1)p_n \leq |\overline{S}_{p_n-2}| \leq (n+1)p_n \);
5. \( \max_{x \in \overline{S}_{p_n-2}} d(x, Z) \leq C^4 p_n^2 e^{-\frac{n\lambda}{2}} \delta'' \).

**Proof of Claim Q2.** Since \( p_n \) grows at a subexponential rate, we can take \( N \) large such that
\[ N > P_0 \text{ and } 4p_n C^2 e^{-\frac{n\lambda}{2}} \delta'' < \min\left\{\delta', \frac{1}{L}\right\} \text{ for all } n \geq N, \]
where \( P_0 \) is the constant as in Lemma 5.6. For \( 0 \leq i \leq n-2 \), we define
\[ \chi(i) = d(\overline{S}_i^R, S_{i+1}^L) + d(S_{i+1}^R, S_i^L) + \cdots + d(S_{p_n-2}^R, S_{p_n-1}^L) + d(S_{p_n-1}^R, \overline{S}_i^L). \]
By Claim Q1,
\[ \chi(0) \leq 2C^2 p_n e^{-\frac{n\lambda}{2}} \delta''. \]
Now we are to show that \( \chi(i) \) and \( \overline{S}_i \) are well defined, which satisfy that
\[ \chi(i) \leq \delta' \text{ and } |\overline{S}_i| > P_0 \text{ for } i = 0, 1, 2, \ldots, n-2. \]
These are clearly true for \( i = 0 \). Now we assume that these are true for some \( i \in \{0, 1, 2 \cdots, p_n-3\} \). Then for \( i+1 \), since \( \chi(i) \leq \delta' \), one has \( d(\overline{S}_i^R, S_{i+1}^L) \leq \delta' \). Thus, we can join \( \overline{S}_i \) and \( S_{i+1} \) by letting
\[ \overline{S}_{i+1} = \overline{S}_i * S_{i+1}. \]
It is clear that \( |\overline{S}_{i+1}| > P_0 \) by Lemma 5.3 (2).
On the other hand, by triangle inequality and Lemma 3.6, one has that
\[ \chi(i) - \chi(i + 1) \]
\[ = d(S_i^R, S_{i+1}^L) + d(S_{i+1}^R, S_{i+2}^L) - d(S_{i+1}^R, S_{i+2}^L) + d(S_{i+1}^R, S_{i+2}^R) - d(S_{i+1}^R, S_{i+2}^R) \]
\[ \geq d(S_i^R, S_{i+1}^L) - d(S_i^L, S_{i+1}^R) - d(S_{i+1}^R, S_{i+1}^L) \]
\[ \geq \frac{1}{2} d(S_i^R, S_{i+1}^L). \]
Therefore, \( \chi(i + 1) \leq \chi(i) \leq \delta' \). By induction, we ends the proof of (1).

By (5.21), one has
\[ d(S_i^R, S_{i+1}^L) \leq 2\chi(i) \leq \cdots \leq 2\chi(0) \leq 4C^2p_e^{-\frac{n\lambda}{2}}\delta'' \text{ for } i = 0, 1, 2 \ldots, p_n - 3, \]
and
\[ d(S_{p_n-2}^R, S_{p_n-2}^L) \leq \chi(p_n - 2) \leq \cdots \chi(0) \leq 2C^2p_e^{-\frac{n\lambda}{2}}\delta''. \]
This ends the proof of (2) as well as (1), (3), and (4) follows Lemma 5.5 (2).

Now we note \( D_i = \overline{S}_i \cup \bigcup_{j=i+1}^{p_n-1} S_j \) for \( i = 0, 1, 2, \ldots, p_n - 1 \). Then by Lemma 5.3 (1),
\[ \max_{x \in D_{i+1}} d(x, D_i) \leq \max_{x \in S_i \cup S_{i+1}} d(x, S_i \cup S_{i+1}) \leq C^3 d(S_i^L, S_{i+1}^R) \leq 2C^5p_e^{-\frac{n\lambda}{2}}\delta''. \]
Therefore, by triangle inequality and noting that \( D_0 \subset Z \),
\[ \max_{x \in S_{p_n-2}} d(x, Z) \leq \max_{x \in D_0} d(x, Z) + \sum_{i=0}^{p_n-3} \max_{x \in D_{i+1}} d(x, D_i) \leq 2C^5p_e^{-\frac{n\lambda}{2}}\delta''. \]
This ends the proof of Claim Q2.

Recall that \( P_0 \) is the constant as in Lemma 3.6, \( K, L \) are the constants as in Lemma 3.5 and \( N \) is the constant as in Claim Q2. We fix an integer \( n > \max(P_0, K, N) + 1 \) and let \( \overline{S}_{p_n-2} \) be the segment as in Claim Q2. Then by Claim Q2 (3),
\[ |\overline{S}_{p_n-2}| > K \text{ and } d(\overline{S}_{p_n-2}^R, \overline{S}_{p_n-2}^L) \leq \delta'. \]
Applying the Anosov Closing Lemma, we have a periodic segment \( \mathcal{O}_n \) such that
\[ |\overline{S}_{p_n-2} - |\mathcal{O}_n| \leq Ld(\overline{S}_{p_n-2}^L, \overline{S}_{p_n-2}^R) \]
and
\[ d(\phi_t(\mathcal{O}_n^L), \phi_t(\overline{S}_{p_n-2}^L)) \leq Ld(\overline{S}_{p_n-2}^L, \overline{S}_{p_n-2}^R) \forall t \in [0, \max(|\overline{S}_{p_n-2}|, |\mathcal{O}_n|)]. \]

We claim the following:

Claim Q3. \( \max_{x \in \mathcal{O}_n} d(x, Z) \leq (2C^3Lp_0 + C^4p_0^2)e^{-\frac{n\lambda}{2}}\delta'' \).
Proof of Claim Q3. If $|O| \leq |\mathcal{S}_{p_n-2}|$, by (5.23),
\[
\max_{x \in O} d(x, \mathcal{S}_{p_n-2}) \leq L d\left(\mathcal{S}_{p_n-2}^L, \mathcal{S}_{p_n-2}^R\right).
\]
If $|O| > |\mathcal{S}_{p_n-2}|$, note $t_* = \min(t, |\mathcal{S}_{p_n-2}|)$ for $t \in [0, |O|]$. Then, by (5.22) and (5.23),
\[
d \left(\phi_t(O^L), \mathcal{S}_{p_n-2}^L\right) \leq C L d\left(\mathcal{S}_{p_n-2}^L, \mathcal{S}_{p_n-2}^R\right) + L d\left(\mathcal{S}_{p_n-2}^L, \mathcal{S}_{p_n-2}^R\right).
\]
Therefore,
\[
\max_{x \in O} d(x, \mathcal{S}_{p_n-2}) = \max_{t \in [0, |O|]} d \left(\phi_t(O^L), \mathcal{S}_{p_n-2}^L\right) \leq C^2 L d\left(\mathcal{S}_{p_n-2}^L, \mathcal{S}_{p_n-2}^R\right).
\]
Combining with Claim Q2. (3) (5), we have
\[
\max_{x \in O} d(x, Z) \leq \max_{x \in O} d \left(x, \mathcal{S}_{p_n-2}\right) + \max_{x \in \mathcal{S}_{p_n-2}} d(x, Z)
\]
\[
\leq C^2 L \cdot 2C^2 p_n e^{-\frac{n\lambda}{2} \delta''} + C^5 p_n^2 e^{-\frac{n\lambda}{2} \delta''}
\]
\[
= (2C^4 L p_n + C^5 p_n^2) e^{-\frac{n\lambda}{2} \delta''}.
\]
This ends the proof of Claim Q3. \qed

By Claim Q3. and (5.22), we have
\[
d_{\alpha,Z}(O_n) \leq |O_n| \left(\max_{x \in O} d(x, Z)\right)^\alpha
\]
\[
\leq \left(|\mathcal{S}_{p_n-2}| + L d\left(\mathcal{S}_{p_n-2}^L, \mathcal{S}_{p_n-2}^R\right)\right) \cdot \left((2C^4 L p_n + C^5 p_n^2) e^{-\frac{n\lambda}{2} \delta''}\right)^\alpha
\]
\[
\leq H_n e^{-\frac{n\lambda}{2}},
\]
where $H_n = ((2C^4 L p_n + C^5 p_n^2) \delta'')^\alpha \cdot ((n + 1)p_n + 1)$. Note that $H_n$ grows at a subexponential rate as $n$ increase as $p_n$ does. Hence
\[
\limsup_{p_n \to +\infty} P^k \min_{O \in O_n^p} d_{\alpha,Z}(O) \leq \limsup_{n \to +\infty} ((n + 1)p_n + 1)^k \cdot H_n e^{-\frac{n\lambda}{2}} = 0,
\]
where we used the fact that $p_n, H_n$ grow at a subexponential rate as $n$ increase and $|O_n| \leq np_n + 1$. The proof of Lemma 3.11 is accomplished. \qed
6. Further discussions on the case of $C^{s,\alpha}$-observables

For $s \in \mathbb{N}$, $0 \leq \alpha \leq 1$ and a strictly positive function $\psi$ on $M$, $\text{Per}^*_{s,\alpha}(M, \psi)$ is defined by the collection of $C^{s,\alpha}$-continuous function $u$ on $M$ that $M_{\min}(u; \psi, \Lambda, \Phi)$ contains at least one periodic measure. And $\text{Loc}_{s,\alpha}(M, \psi)$ is defined by

$$\text{Loc}_{s,\alpha}(M, \psi) = \{ u \in \text{Per}^*_{s,\alpha}(M, \psi) : \text{there is } \varepsilon > 0 \text{ such that } M_{\min}(u + h; \psi, \Lambda, \Phi) = M_{\min}(u, \psi, \Lambda, \Phi) \text{ for all } \|h\|_{r,\alpha} < \varepsilon \}.$$

In the case $s \geq 1$ and $\alpha > 0$ or $s \geq 2$, we do not have result like Proposition 4.7. But, we have the following weak version.

**Proposition 6.1.** Let $O$ be a periodic segment of $\Phi|\Lambda$ with $D(O) > 0$ and $u \in C(M)$ with $u|O = 0$ and $u|_{M/O} > 0$. Then there exists a constant $\varrho > 0$ such that the probability measure $\mu_O \in M_{\min}(u + h; \psi, \Lambda, \Phi)$, where $h$ is any $C^{0,1}(M)$ function with $\|h\|_1 < \varrho$.

**Remark 6.2.** As in Remark 4.12 for $s \in \mathbb{N}$ and $0 \leq \alpha \leq 1$, we let $\tilde{w} \in C^{s,\alpha}(M)$ such that $\|\tilde{w}\|_{s,\alpha} < \varepsilon$, $\tilde{w}|_O = 0$ and $\tilde{w}|_{M/O} > 0$. Then $\mu_O$ is the unique measure in $M_{\min}(u + \tilde{w} + h; \psi, \Lambda, \Phi)$ whenever $\|h\|_{s,\alpha} < \varrho$. The Proposition shows that there is an open set of $C^{s,\alpha}(M)$ near $u$ such that functions in the open set have the same unique minimizing measure with respect to $\psi$ and the probability measure supports on a periodic orbit.

By using Remark 6.2, we have the following result.

**Theorem 6.3.** $\text{Loc}_{s,\alpha}(M, \psi)$ is an open dense subset of $\text{Per}^*_{s,\alpha}(M, \psi)$ w.r.t. $\| \cdot \|_{s,\alpha}$ for integer $s \geq 1$ and real number $0 \leq \alpha \leq 1$.

**Proof.** Given $s \geq 1$ and $0 \leq \alpha \leq 1$. The openness is clearly true. We prove $\text{Loc}_{s,\alpha}(M, \psi)$ is dense in $\text{Per}^*_{s,\alpha}(M, \psi)$ w.r.t. $\| \cdot \|_{s,\alpha}$. Since

$$\int ud\mu = \int \tilde{u}d\mu \text{ for all } \mu \in M(\Phi|\Lambda),$$

we have $M_{\min}(u; \psi, \Lambda, \Phi) = M_{\min}(\tilde{u}; \psi, \Lambda, \Phi)$. Then the theorem is immediately from Remark 6.2. \qed

6.1. Proof of Proposition 6.1. Now we finish the proof of Proposition 6.1.

**Proof of Proposition 6.1.** Let $O$ be a periodic segment of $\Phi|\Lambda$ and $u \in C(M)$ with $u|O = 0$ and $u|_{M/O} > 0$. For $0 \leq \rho \leq D(O)$, we note $\theta(\rho) = \min\{u(x) : d(x, O) \geq \rho, x \in M\}$.
It is clear that $\theta(0) = 0$, $\theta(\rho) > 0$ for $\rho \neq 0$ and $\theta$ is non-decreasing. Since $D(\mathcal{O}) > 0$ by assumption, there are two constants $\rho_1, \rho_2$ satisfy

$$0 < \rho_1 < \rho_2 < \frac{D(\mathcal{O})}{4C^3e^{2\beta}}.$$  \hfill (6.1)

Next we will show that $\mu_\mathcal{O} \in \mathcal{M}_{\min}(u + h; \psi, \Lambda, \Phi)$ for all $h \in C^{0,1}(M)$ with $\|h\|_1 < \varrho$ where the constant $\varrho$ is positive and

$$\varrho < \frac{1}{2} \min \left\{ \frac{\psi_{\min} \theta(\rho_1)}{1 + \psi_{\min}}, \frac{\theta\left(\frac{D(\mathcal{O})}{4C^3e^{2\beta}}\right)}{1 + \left| \mathcal{O} \right| \cdot \frac{1}{\psi_{\min}} + \frac{1}{\psi_{\min}}} \right\}.$$  \hfill (6.2)

Now we fix a function $h$ as above. Note $G = u + h - a_\mathcal{O}\psi$ where

$$a_\mathcal{O} = \frac{\langle \mathcal{O}, u + h \rangle}{\langle \mathcal{O}, \psi \rangle} \leq \frac{\|h\|_0}{\psi_{\min}}.$$  \hfill (6.3)

Then $\int Gd\mu_{\mathcal{O}} = \int \frac{u + hd\mu}{\psi d\mu} - a_\mathcal{O}$. Therefore, to show that $\mu_\mathcal{O} \in \mathcal{M}_{\min}(u + h; \psi, \Lambda, \Phi)$, it is enough to show that

$$\int Gd\mu \geq 0$$

for all $\mu \in \mathcal{M}^e(\Phi|\Lambda)$,

where we used the assumption $\psi$ is strictly positive and the fact $\int Gd\mu_\mathcal{O} = 0$. Now we let $\text{Area}_1 = \{y \in M : d(y, \mathcal{O}) \leq \rho_1\}$. We have the following Claim.

**Claim 1.** $\text{Area}_1$ contains all $x \in M$ with $G(x) \leq 0$.

**Proof of Claim 1.** For $x \notin \text{Area}_1$, we have

$$G(x) = u(x) + h(x) - a_\mathcal{O}\psi \geq \theta(\rho_1) - a_\mathcal{O}\psi - \|h\|_0 \geq \theta(\rho_1) - \frac{\|h\|_0}{\psi_{\min}} > 0.$$  \hfill (6.2)

where we used (6.2). This ends the proof of Claim 1. \hfill $\square$

Note $\text{Area}_2 = \{y \in M : d(y, \mathcal{O}) \leq \rho_2\}$. It is clear that $\text{Area}_1$ is in the interior of $\text{Area}_2$. Thus, $d(\text{Area}_1, M \setminus \text{Area}_2) > 0$. Therefore, by Claim 1, we can fix a constant $0 < \tau < 1$ such that $G(\phi_t(x)) > 0$ for all $x \in M \setminus \text{Area}_2$ and $|t| \leq \tau$.

**Claim 2.** If $z \in \Lambda$ is not a generic point of $\mu_\mathcal{O}$, then there is $m \geq \tau$ such that

$$\int_0^m G(\phi_t(z)) dt > 0.$$  \hfill (6.1)

Next we prove Proposition 6.1 by assuming the validity of Claim 2. proof of which is left to the next subsection. Same as the argument at the beginning of the proof, it is enough to show that for all $\mu \in \mathcal{M}^e(\Phi|\Lambda)$

$$\int Gd\mu \geq 0.$$
Given $\mu \in \mathcal{M}^e(\Phi|\Lambda)$, in the case $\mu = \mu_\mathcal{O}$, it is obviously true. In the case $\mu \neq \mu_\mathcal{O}$, just let $z$ be a generic point of $\mu$. Note that $z$ is not a generic point of $\mu_\mathcal{O}$. By Claim 2., we have $t_1 \geq \tau$ such that

$$\int_0^{t_1} G(\phi_t(z)) dt > 0.$$  

Note that $\phi_{t_1}(z)$ is also not a generic point of $\mu_\mathcal{O}$. By claim 2, we have $t_2 \geq t_1 + \tau$ such that

$$\int_{t_1}^{t_2} G(\phi_t(z)) dt > 0.$$  

By repeating the above process, we have $0 \leq t_1 < t_2 < t_3 < \cdots$ with gap not less than $\tau$ such that

$$\int_{t_i}^{t_{i+1}} G(\phi_t(z)) dt > 0 \text{ for } i = 0, 1, 2, 3, \cdots,$$

where $t_0 = 0$. Therefore,

$$\int Gd\mu = \lim_{m \to +\infty} \frac{1}{m} \int_0^m G(\phi_t(z)) dt = \lim_{i \to +\infty} \frac{1}{t_i} \left( \int_{t_0}^{t_1} G(\phi_t(z)) dt + \int_{t_1}^{t_2} G(\phi_t(z)) dt + \cdots + \int_{t_{i-1}}^{t_i} G(\phi_t(z)) dt \right) \geq 0.$$  

That is, $\mu_\mathcal{O} \in \mathcal{M}_{\min}(u + h; \psi, \Lambda, \Phi)$. This ends the proof of Proposition.  

6.2. Proof of Claim 2. We assume that $z$ is not a generic point of $\mu_\mathcal{O}$. If $z \notin \text{Area}_2$, just note $m = \tau$, we have nothing to prove since $G(\phi_t(z)) > 0$ for all $|t| \leq \tau$. Now we assume that $z \in \text{Area}_2$. In the case $d(\phi_t(z), \mathcal{O}) < \frac{D(\mathcal{O})}{4C^2e^\beta}$ for all $t \geq 0$, by Lemma 4.2, $z$ is a generic point of $\mu_\mathcal{O}$ which is impossible by our assumption. Hence, there must be some $m_1 > 0$ such that

$$d(\phi_{m_1}(z), \mathcal{O}) \geq \frac{D(\mathcal{O})}{4C^2e^\beta}.$$  

We can assume $m_2 > 0$ the smallest time such that

$$d(\phi_{m_2}(z), \mathcal{O}) \geq \frac{D(\mathcal{O})}{4C^2e^\beta}.$$  

The existence of $m_2$ is ensured by (6.1). Then for $0 \leq t \leq 1$,

$$d(\phi_{m_2-t}(z), \mathcal{O}) \geq \frac{D(\mathcal{O})}{4C^3e^{2\beta}}.$$  

Then
\[
\int_{m_2}^{m_2} G(\phi_t(z)) dt = \int_{m_2}^{m_2} u(\phi_t(z)) + h(\phi_t(z)) - a_\mathcal{O} \psi(\phi_t(z)) dt
\]
\[
\geq \int_{m_2}^{m_2} \theta \left( \frac{D(\mathcal{O})}{4C^3 e^{2\beta}} \right) - \|h\|_0 - a_\mathcal{O} \|\psi\|_0 dt
\]
\[
\geq \theta \left( \frac{D(\mathcal{O})}{4C^3 e^{2\beta}} \right) - \frac{1 + \psi_{\min}}{\psi_{\min}} \|h\|_0.
\]  
(6.5)

where we used (6.3) and the definition of \(\theta(\cdot)\). On the other hand, one has that
\[
\frac{D(\mathcal{O})}{4C^3 e^{2\beta}} > \rho_2
\]
which implies that
\[
\phi_{m_2-t}(z) \notin \text{Area}_2 \quad \text{for all} \quad 0 \leq t \leq 1.
\]  
(6.6)

Since \(\text{Area}_2\) is compact, we can take \(m_3\) the largest time with \(0 \leq m_3 \leq m_2\) such that
\[
\phi_{m_3}(z) \in \text{Area}_2,
\]
where we use the assumption \(z \in \text{Area}_2\). By (6.6), it is clear that \(m_3 < m_2 - 1\). Then by Claim 1 and the fact \(\text{Area}_1 \subset \text{Area}_2\),
\[
G(\phi_t(z)) > 0 \quad \text{for all} \quad m_3 < t < m_2 - 1.
\]  
(6.7)

Since \(m_3 < m_2\), one has by (6.4) that
\[
d(\phi_t(z), \mathcal{O}) < \frac{D(\mathcal{O})}{4C^2 e^{\beta}} < \delta' \quad \text{for all} \quad 0 \leq t \leq m_3.
\]

Therefore, by Lemma 4.2 there is \(y_0 \in \mathcal{O}\) such that
\[
d(\phi_t(z), \phi_t(y_0)) \leq C d(\phi_t(z), \mathcal{O}) \leq \frac{D(\mathcal{O})}{4C e^{\beta}} \quad \text{for all} \quad t \in [0, m_3].
\]

Also notice that
\[
d(z, y_0) \leq C \rho_2 \quad \text{and} \quad d(\phi_{m_3}(z), \phi_{m_3}(y_0)) \leq C \rho_2,
\]
where we used \(z, \phi_{m_3}(z) \in \text{Area}_2\). By using Lemma 3.4, we have for all \(0 \leq t \leq m_3\),
\[
d(\phi_t(z), \phi_t \phi_v(y_0)) \leq C^2 e^{-\lambda \min(t, m_3 - t)} (d(z, y_0) + d(\phi_{m_3}(z), \phi_{m_3}(y_0))) \leq 2C^3 e^{-\lambda \min(t, m_3 - t)} \rho_2,
\]
where \(v = v(y_0, z)\). Hence,
\[
\int_0^{m_3} d(\phi_t(z), \phi_t \phi_v(y_0)) dt \leq \int_0^{m_3} 2C^3 \rho_2 (e^{-\lambda t} + e^{-\lambda (m_3 - t)}) dt \leq \frac{4C^3 \rho_2}{\lambda}.
\]
Since \( u(\phi_t(y_0)) = 0 \) for all \( t \in \mathbb{R} \) and \( u \geq 0 \), one has

\[
\int_0^{m_3} G(\phi_t(z)) - G(\phi_{t+v}(y_0)) \, dt
\]

\[
= \int_0^{m_3} u(\phi_t(z)) + h(\phi_t(z)) - u(\phi_{t+v}(y_0)) - h(\phi_{t+v}(y_0)) - a_\mathcal{O}(\psi(\phi_t(z)) - \psi(\phi_{t+v}(y_0))) \, dt
\]

\[
\geq \int_0^{m_3} h(\phi_t(z)) - h(\phi_{t+v}(y_0)) - a_\mathcal{O}(\psi(\phi_t(z)) - \psi(\phi_{t+v}(y_0))) \, dt
\]

\[
\geq - \|h\|_1 + |a_\mathcal{O}||\psi||_1 \int_0^{m_3} d(\phi_t(z), \phi_{t+v}(y_0)) \, dt
\]

\[
\geq - \|h\|_1 + \frac{\|h\|_0\|\psi||_1}{\psi_{\text{min}}} \cdot \frac{4C^3\rho_2}{\lambda}
\]

\[
\geq - \|h\|_1(1 + \frac{\|\psi||_1}{\psi_{\text{min}}}) \cdot \frac{4C^3\rho_2}{\lambda}.
\]

(6.8)

By assuming that \( m_3 = p|\mathcal{O}| + q \) for some nonnegative integer \( p \) and real number \( 0 \leq q \leq |\mathcal{O}| \), one has by (6.4) that

\[
\int_0^{m_3} G(\phi_{t+v}(y_0)) \, dt = \int_0^{m_3} G(\phi_{t+v}(y_0)) \, dt \geq -|\mathcal{O}| \cdot \frac{1 + \psi_{\text{min}}}{\psi_{\text{min}}} \|h\|_0,
\]

(6.9)

where we used \( \int Gd\mu_{\mathcal{O}} = 0 \). Combining (6.3), (6.5), (6.7), (6.8) and (6.9), we have

\[
\int_0^{m_2} G(\phi_t(z)) \, dt
\]

\[
\geq \int_0^{m_3} G(\phi_t(z)) \, dt + \int_0^{m_2} G(\phi_t(z)) \, dt
\]

\[
= \int_0^{m_3} G(\phi_t(z)) - G(\phi_{t+v}(y_0)) \, dt + \int_0^{m_3} G(\phi_{t+v}(y_0)) \, dt + \int_0^{m_2} G(\phi_t(z)) \, dt
\]

\[
\geq - \|h\|_1(1 + \frac{\|\psi||_1}{\psi_{\text{min}}}) \cdot \frac{4C^3\rho_2}{\lambda} - |\mathcal{O}| \cdot \frac{1 + \psi_{\text{min}}}{\psi_{\text{min}}} \|h\|_0 + \theta \left( \frac{D(\mathcal{O})}{4C^3\epsilon^2\beta} \right) - \frac{1 + \psi_{\text{min}}}{\psi_{\text{min}}} \|h\|_0
\]

\[
= \theta \left( \frac{D(\mathcal{O})}{4C^3\epsilon^2\beta} \right) - \left( 1 + \frac{\|\psi||_1}{\psi_{\text{min}}} \cdot \frac{4C^3\rho_2}{\lambda} + |\mathcal{O}| \cdot \frac{1 + \psi_{\text{min}}}{\psi_{\text{min}}} + \frac{1 + \psi_{\text{min}}}{\psi_{\text{min}}} \right) \|h\|_1
\]

\[
> 0,
\]

where we used assumption (6.2). Therefore, \( m = m_2 \) is the time as required since \( m_2 \geq 1 > \tau \) by (6.7). This completes the proof of Claim 2.
Acknowledgement

At the end, we would like to express our gratitude to Tianyuan Mathematical Center in Southwest China, Sichuan University and Southwest Jiaotong University for their support and hospitality.

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