SOLITON METRICS FOR TWO-LOOP RENORMALIZATION GROUP FLOW ON 3D UNIMODULAR LIE GROUPS

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Abstract. The two-loop renormalization group flow is studied via the induced bracket flow on 3D unimodular Lie groups. A number of steady solitons are found. Some of these steady solitons come from maximally symmetric metrics that are steady, shrinking, or expanding solitons under Ricci flow, while others are not obviously related to Ricci flow solitons.

1. Introduction

In this note we look into perturbation of the Ricci flow on three-dimensional unimodular Lie groups. The Ricci flow on three-dimensional Lie groups has been well-studied, both in forward and backward time \cite{[13] [12] [3] [10] [2]}, as well as other geometric flows such as cross curvature flow \cite{[3] [5]} and other flows, e.g., \cite{[11] [1] [16]}. We will use the notations $E(2)$ for the universal covering of the group of isometries of the Euclidean plane, $H(3)$ for the three-dimensional Heisenberg group, $E(1,1)$ for the group of isometries of the Minkowski plane, $SU(2)$ for the special unitary group, and $SL(2,\mathbb{R})$ for the universal covering of the special linear group (this is nonstandard notation but simplifies the text). Many of these groups admit maximally symmetric Thurston geometries that are special left-invariant metrics (actually, one- and two-parameter families of metrics) with additional symmetries: the flat metric on $E(2)$, Nil on $H(3)$, Sol on $E(1,1)$, the round sphere $S^3$ metrics on $SU(2)$, and the maximally symmetric metrics on $SL(2,\mathbb{R})$. It is well-known that many of these manifolds admit steady Ricci solitons (the flat metrics), shrinking Ricci solitons (the round sphere metrics), and expanding Ricci solitons (Nil, Sol). It is notable, however, that not all of Thurston’s geometries are represented among these generalized fixed points; in particular, there is no soliton on $SL(2,\mathbb{R})$. Also note that the case of expanding solitons is a bit misleading, as the solitons do not exist on compact quotients, but instead lead to collapsing of closed manifolds with these geometries.

One may ask if we can deform the Ricci flow to another flow that has additional fixed points, steady solitons, or other special solutions. One candidate for such a perturbation is the two-loop renormalization group flow (RG-2 Flow), introduced by physicists and studied more recently in works such as \cite{[8] [6] [9]}. The flow is compactly written as

\begin{equation}
\frac{\partial}{\partial t} g = -2 \text{Rc} - \frac{\alpha}{2} \text{Rm}^2
\end{equation}

where $\alpha$ is a constant.

\begin{center}
\begin{tabular}{l}
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In this note, we describe several steady soliton solutions to RG-2 flow depending on the value of the parameter $\alpha$. It was recently brought to the authors’ attention that a similar classification of solitons for RG-2 flow was found concurrently by T. Wears in [17] by finding derivations on the Lie algebras. Our approach is instead to look at the flow of Lie bracket coefficients on an orthonormal frame, which has the advantage of making it easier (and quicker) to find soliton metrics but the disadvantage of not having an explicit family of automorphisms.

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2. RG-2 BRACKET FLOW

Instead of looking at the RG-2 flow on the metric coefficients, we prefer to work on a left-invariant orthonormal frame. It is well known [15] that three-dimensional unimodular Lie algebras can be written in terms of an orthonormal frame with at most three nonzero structure constants:

$$[e_2, e_3] = a_1 e_1, \quad [e_3, e_1] = a_2 e_2, \quad \text{and} \quad [e_1, e_2] = a_3 e_3,$$

for some constants $a_1, a_2, a_3 \in \mathbb{R}$. The structure constants determine the left-invariant metric (see e.g., [14]), and so any flow of metrics induces a flow of structure constants. This induced flow is called the corresponding bracket flow. The bracket flow for Ricci flow has been studied in a number of contexts. Note that the sign of the structure constants $a_1, a_2, a_3$ determine which unimodular Lie group the algebra corresponds to among $\mathbb{R}^3$, $H(3)$, $E(3)$, $E(1,1)$, $SL(2,\mathbb{R})$, $SU(2)$.

A calculation along the lines of that in [10] leads to the following system on the structure constants corresponding to the RG-2 flow:

$$\frac{da_1}{dt} = \frac{a_1}{2} \left[ (a_2 - a_3)^2 - 3a_1^2 + 2a_1 a_2 + 2a_1 a_3 \right] \left[ 1 + \frac{\alpha}{8} \left[ (a_2 - a_3)^2 - 3a_1^2 + 2a_1 a_2 + 2a_1 a_3 \right] \right],$$

$$\frac{da_2}{dt} = \frac{a_2}{2} \left[ (a_1 - a_3)^2 - 3a_2^2 + 2a_1 a_2 + 2a_2 a_3 \right] \left[ 1 + \frac{\alpha}{8} \left[ (a_1 - a_3)^2 - 3a_2^2 + 2a_1 a_2 + 2a_2 a_3 \right] \right],$$

$$\frac{da_3}{dt} = \frac{a_3}{2} \left[ (a_1 - a_2)^2 - 3a_3^2 + 2a_1 a_3 + 2a_2 a_3 \right] \left[ 1 + \frac{\alpha}{8} \left[ (a_1 - a_2)^2 - 3a_3^2 + 2a_1 a_3 + 2a_2 a_3 \right] \right].$$

Note that when $\alpha = 0$, the bracket formulation of the Ricci flow is recovered.

An important property of the bracket flow is that steady algebraic solitons are precisely the metrics that are fixed under the bracket flow (see, e.g., [14]). A steady algebraic soliton is a left invariant metric $g(t)$ such that

$$g(t) = \phi^*_t g_0$$

where $\phi_t$ is a one-parameter family of automorphisms of the group (the use of “algebraic” refers to the assumption that $\phi_t$ are automorphisms, not just diffeomorphisms). RG-2 flow [11] is not scale invariant in the same way that Ricci flow is scale invariant due to the fact that the constant $\alpha$ is not dimensionless; usually we specify a value of $\alpha$ and that fixes a scale. For this reason, we are mainly concerned with steady solitons.
3. Solitons for RG-2 flow

The main result is the following.

**Theorem 3.1.** The following are all steady solitons for the RG-2 flow, characterized by their structure constants \((a_1, a_2, a_3)\), the sign of \(\alpha\), their sectional curvatures (given as the three eigenvalues of the Einstein tensors), and their Ricci curvatures. The maximally symmetric metrics are described by their geometries.

| Group | \((a_1, a_2, a_3)\) | \(\alpha\) | Sectional curvatures | Ricci curvatures | Geometry       |
|-------|-------------------|---------|---------------------|-----------------|---------------|
| \(E(2)\) | \((a, a, 0), a \in \mathbb{R}\) | any     | \((0, 0, 0)\)     | \((0, 0, 0)\)   | flat          |
| \(H(3)\) | \(\pm \sqrt{\frac{3}{8\alpha}}, 0, 0\) | \(\alpha > 0\) | \(\left(\frac{3}{32\alpha}, \frac{3}{32\alpha}, -\frac{9}{32\alpha}\right)\) | \(\left(\frac{1}{16\alpha}, -\frac{3}{16\alpha}, -\frac{3}{16\alpha}\right)\) | Nil            |
| \(E(1,1)\) | \(\pm \sqrt{\frac{2}{3\alpha}}, 0, \pm \sqrt{\frac{2}{3\alpha}}\) | \(\alpha < 0\) | \(\left(\frac{2}{3\alpha}, \frac{2}{3\alpha}, -\frac{2}{3\alpha}\right)\) | \(\left(0, 0, \frac{4}{3}\right)\) | Sol            |
| \(SU(2)\) | \(\pm \sqrt{\frac{2}{3\alpha}}, \pm \sqrt{\frac{2}{3\alpha}} \pm \sqrt{\frac{2}{3\alpha}}\) | \(\alpha > 0\) | \(\left(\frac{2}{3\alpha}, \frac{2}{3\alpha}, \frac{2}{3\alpha}\right)\) | \(\left(\frac{1}{3\alpha}, \frac{1}{3\alpha}, \frac{1}{3\alpha}\right)\) | \(S^3\left(\sqrt{\frac{2}{3\alpha}}\right)\) |
| \(E(1,1)\) | \(\frac{3}{\sqrt{2\alpha}}, 0, \frac{3}{\sqrt{2\alpha}}\) | \(\alpha > 0\) | \(\left(\frac{2}{3\alpha}, \frac{2}{3\alpha}, -\frac{2}{3\alpha}\right)\) | \(\left(\frac{1}{3\alpha}, -\frac{2}{3\alpha}, -\frac{1}{3\alpha}\right)\) |               |
| \(SU(2)\) | \(\frac{3}{\sqrt{2\alpha}}, -\frac{3}{\sqrt{2\alpha}}\) | \(\alpha < 0\) | \(\left(\frac{2}{3\alpha}, \frac{2}{3\alpha}, -\frac{2}{3\alpha}\right)\) | \(\left(-\frac{1}{3\alpha}, -\frac{1}{3\alpha}, \frac{1}{3\alpha}\right)\) |               |

**Remark 3.2.** It is important to note that the first four metrics are solitons for Ricci flow, correspond to a metric in the one-parameter family of maximally symmetric metrics (which are Thurston geometries). Because they are maximally symmetric, any metric starting in the one-parameter family remains in it. These particular choices of initial metric for certain \(\alpha\) are actually fixed under RG-2, whereas they may be expanding or shrinking under Ricci flow. In addition to the fixed points listed here, it is easy to use the bracket flow to describe the dynamics under the system \((2.1)\) of any maximally symmetric metric of these forms even if they are not fixed; they correspond to self-similar expanding or shrinking solutions.

**Remark 3.3.** According to \([5]\) and \([9]\), the RG-2 flow is parabolic if \(1 + \alpha K > 0\) for all sectional curvatures \(K\). The only solitons in the table that satisfy this requirement are the first two, the fourth, and the last.

The last three metrics are new compared to the work in \([7]\), especially the metric on \(E(1,1)\), which does not satisfy the symmetry assumption in that paper; quite possibly \([7]\) were aware of the last two but did not consider these solutions since they require \(\alpha < 0\), which is not physical. As we are only considering the system as a perturbation of Ricci flow, we have no problem considering \(\alpha < 0\).

4. Sketch of proof and discussion

The main idea is to start with the ansatz that there is a solution of one of the following forms: \((a(t), 0, c(t) a(t))\) or \((a(t), a(t), c(t) a(t))\) and derive the equations to maintain these forms. One can then derive the equations for \(c\) to be constant. Any such solution will be self-similar, and this can be used to analyze self-similar solutions similar to those in the first four cases, where we get shrinkers or expanders; in particular, we can find when such solutions are fixed, which depends on the initial condition \(a(0)\). For the last three solutions, we cannot derive a solution where \(c\) is constant without \(a\) being constant as well. The solutions where \(a\) and \(c\) are both constant correspond to algebraic steady soliton metrics.
This method was motivated by studying a related problem. In [10], the Ricci flow was considered up to scaling by looking at the ratio of structure constants $m_2 = \frac{a_2}{a_1}$ and $m_3 = \frac{a_3}{a_1}$. This allowed a complete phase portrait of the system described up to scaling, and hence in the $(m_2, m_3)$-plane, expanding and shrinking solitons are also fixed points. As previously mentioned, RG-2 flow does not allow this analysis since it is not scale invariant, but one could still make an attempt to consider the flow in the $(m_2, m_3)$-plane, getting the equations

\[
\begin{align*}
\frac{dm_2}{dt} &= m_2 (1 - m_2) (1 + m_2 - m_3) \left[1 - \beta (1 + m_3 - m_2) (1 - m_2 - m_3)\right], \\
\frac{dm_3}{dt} &= m_3 (1 - m_3) (1 + m_3 - m_2) \left[1 - \beta (1 + m_2 - m_3) (1 - m_2 - m_3)\right],
\end{align*}
\]

where $\beta = \frac{\alpha a_2^2}{a_1^2}$. Note that, in general, $\beta$ is not constant since $\alpha$ is constant and $a_1$ is not. However, for steady solitons of RG-2 flow, $\beta$ is constant and so this system gives corresponding fixed points.

The system (4.1) can be solved completely in the sense that we can find all the fixed points and analyze the phase plane analogously to [10]. The fixed points are

\[
(0, 0), \ (0, \pm 1), \ (\pm 1, 0), \ (1, 1), \ \left(0, 1 \pm \sqrt{\frac{1 - \beta}{\beta}}\right), \ \left(1 \pm \sqrt{\frac{1 - \beta}{\beta}}, 0\right), \ \left(1, 1 \pm \sqrt{1 + \frac{1}{\beta}}\right),
\]

\[
\left(1 \pm \sqrt{1 + \frac{1}{\beta}}, 1\right), \ \text{and} \ \frac{1}{2} \left(1 - \frac{1}{\beta}, 1 - \frac{1}{\beta}\right).
\]

Notice that this system has fixed points (depending on the value of $\beta$) in a number of regimes including SL $(2, \mathbb{R})$, which does not have a fixed point for the bracket flow of either the Ricci flow or RG-2 flow. However, it is not clear how this flow corresponds to any geometric flow, since it would require a time dependent $\alpha$, determined in a way that is not calculated from a curvature.

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