On the Brane Configuration of $N = (4, 4)$ 2D Supersymmetric Gauge Theories

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Abstract

We study two dimensional $N = (4, 4)$ supersymmetric gauge theories with various gauge groups and various hypermultiplets in the fundamental as well as bi-fundamental and adjoint representations. They have ”mirror theories” which become equivalent to them at strong coupling. The theory with one fundamental and one adjoint has a Higgs branch which is parametrized by the adjoint matter. We also consider theories which involve an orientifold plane. The brane realization of the Matrix theory formulation of NS 5-branes in Type II string theories is also considered.
1 Introduction

Recently, supersymmetric gauge theory in various dimensions with various supercharges have been studied in the context of brane theory. There are two approaches for this purpose. One is considering wrapped D-branes on the Calabi-Yau cycles in Type II A and B string theories compactified on K3-fibred Calabi-Yau 3-fold \cite{1} (For review see \cite{2}). This approach was generalized in \cite{3} to explain mirror symmetry in $N = 4$ three dimensional gauge theory as well as Seiberg-Witten models both with simple groups and product of simple gauge groups.

Another approach is to consider the configuration of intersecting D-branes and NS 5-branes in Type II A and B string theories. At first, Hanany and Witten \cite{4} used a particular brane configuration in Type II B string theory in order to describe mirror symmetry in three dimensional supersymmetric gauge theories with 8 supercharges \cite{5}. Applying string duality to the configuration of branes in Type II B string theory, one can provide an explanation for the mirror symmetry.

It has been shown \cite{6} by a particular brane configuration of type II A string theory that one can obtain the supersymmetric $U(n)$ gauge theory in four dimension with four supercharges. Then by making a certain deformation in the brane configuration, Seiberg’s duality can be realized. Introducing an orientifold plane in the brane configuration helps us to generalize this work to other classical Lie gauge groups. \cite{7}

$N = 2$ four dimensional gauge theory with gauge group $SU(n)$ was also obtained from the brane configuration in type II A string theory \cite{8}. By lifting from type II A to M-theory, the exact solution of $N = 2 \ D = 4$ SYM theory was obtained. More precisely, from the M-theory point of view, we have a five brane with worldvolume $R^{3,1} \times \Sigma$; where the theory on the $R^{3,1}$ is an $N = 2 \ D = 4$ SYM theory; and $\Sigma$ is the Seiberg-Witten curve corresponding to it. This work was generalized to other classical Lie gauge groups in \cite{9}.

In this context, chiral gauge theories in four dimensions with 4 supercharges have been obtained by studying the brane configurations in the non-flat spacetime backgrounds, specially in the orbifold background \cite{10}. These theories have been also studied by introducing an orientifold six plane.

Recently, two dimensional $N = (4, 4)$ supersymmetric gauge theories have been studied by considering a particular brane configuration in Type II A string theory \cite{11}. The same configuration was used in \cite{12} to show the relation between the Higgs branch of the theory and the moduli space of the instantons \cite{13}. Similar configuration is considered in \cite{14}. Two dimensional $N = (2, 2)$ supersymmetric gauge theories as well as five and six dimensional theories have been also studied in \cite{15}, \cite{16}, \cite{17}. (For review of this approach see \cite{18}.)

In this article we consider a class of brane configurations of Type II A string theory to study several two dimensional gauge theories which have 8 or 16 supercharges. These theories have been appeared in the context of Matrix theory description of NS 5-brane in Type II A and B \cite{19}. It has been shown that coincident NS 5-branes
in Type II A and B lead to non-trivial gauge theory in the limit of \( g_{A,B} \to 0 \) [20]. In [21] Witten obtained new gauge theories which can be described by \( (p,q) \) 5-branes in Type II B or equivalently by considering Type II A/ M-theory on the non-flat background. He was also shown that the Matrix theory formulation of these theories are two dimensional supersymmetric gauge theory with 8 supercharges.

In section two we introduce our brane configuration in Type II A string theory. We will consider the theory with one fundamental and one adjoint matter. This model has a Higgs branch, but there is no smooth transition from the Coulomb branch to the Higgs branch in the week coupling limit. We also consider the brane configuration in the presence of an O 4-plane. So we can study the theory with \( SP(N) \) gauge group and \( D_{N_f} \) singularity. In section three we lift this brane configuration to M-theory. We will see that there are two theories which become equivalent in the strong coupling limit. In section four we consider the brane realization of the Matrix theory formulation of NS 5-branes in Type II string theory. It seems that the mirror symmetry in two dimensions which we will study in section three corresponds to the duality between Type II A on an \( A_{k-1} \) singularity and \( k \) NS 5-branes in Type II B string theory. Note that in [22] the correspondence between M-theory 5-branes and ALE backgrounds in Type II B string theory is realized as three dimensional mirror symmetry. This correspondence is studied by the Matrix theory description of them.

2 Brane Configuration

The Type II A brane configuration which we will use, involve three kinds of branes. 1) An NS 5-brane with worldvolume \((x^0, x^1, x^2, x^3, x^4, x^5)\) which lives at a point in the \((x^6, x^7, x^8, x^9)\) (and \(x^{10}\) in M-theory point of view) directions. Let \(Q_L\) and \(Q_R\) be the left and right moving supercharges in Type II A. The supercharges obey:

\[
\Gamma^0 \ldots \Gamma^9 Q_L = Q_L, \quad \Gamma^0 \ldots \Gamma^9 Q_R = -Q_R
\]

(1)

NS 5-brane is invariant under half of the supersymmetries \(\epsilon_L Q_l + \epsilon_R Q_R\) with:

\[
\Gamma^0 \ldots \Gamma^5 \epsilon_L = \epsilon_L, \quad \Gamma^0 \ldots \Gamma^5 \epsilon_R = \epsilon_R
\]

(2)

2) A D 2-brane with worldvolume \((x^0, x^1, x^6)\) at the point \((x^2, x^3, x^4, x^5, x^7, x^8, x^9)\).
It is invariant under half of the supersymmetries

\[
\Gamma^0 \Gamma^1 \Gamma^6 \epsilon_R = \epsilon_L
\]

(3)

3) A D 4-brane with worldvolume \((x^0, x^1, x^7, x^8, x^9)\) living at the point \((x^2, x^3, x^4, x^5, x^6)\) and invariant under half of the supersymmetries with

\[
\Gamma^0 \Gamma^1 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R = \epsilon_L
\]

(4)
Each relation (2), (3), (4) by itself breaks half of the supersymmetries, but it is easy to see that they are not independent, so altogether they break $\frac{1}{4}$ of the original supersymmetries of Type II A string theory.

The presence of all these branes break the ten dimensional Lorentz group to 1+1 dimensional Lorentz group with $SO(4) \times SU(2)_R$ global symmetry. $SO(4)$ corresponds to rotation in the $x^2, x^3, x^4, x^5$ directions and $SU(2)_R$ to the rotation in the $x^7, x^8, x^9$ directions. Brane configuration which we will consider, consists of NS 5-branes at the points $r_i = (x^7_i, x^8_i, x^9_i)$ and $x^6_i$; the D 2-branes which are suspended between these NS 5-branes; so that they are finite in $x^6$ direction. Also we will have some D 4-branes at $m_i = (x^2_i, x^3_i, x^4_i, x^5_i)$ in between the NS 5-branes. This configuration of branes preserves 8 supercharges in intersection worldvolume $(x^0, x^1)$. So we have an $N = (4, 4)$ gauge theory in two dimensions.

For example consider two NS 5-branes at $r_1, x^6_1 = 0$ and $r_2, x^6_2 = L$, and N D 2-branes suspended between them, and $N_f$ D 4-branes in between these two NS 5-branes at points $m_i, x^6_i$.

\[ x^2, x^3, x^4, x^5 \]
\[ x^6 \]
\[ x^7, x^8, x^9 \]

Notation

\[ r = r_2 - r_1 \] is the Fayet-Iliopoules terms and is in $(1, 1, 3)$ representation of $SO(4) \times SU(2)_R$, where the representation of $SO(4)$ are labelled by $SU(2) \times SU(2)$. When $r = 0$, the D 2-branes can suspend between two NS 5-branes and preserve 8 supercharges in 1+1 dimension. In this case we have a $U(N)$ gauge theory which is in the Coulomb phase and can be parametrized by scalars in vector multiplet. In brane language, these scalars are fluctuations of D 2-branes in $x^2, x^3, x^4, x^5$ directions; we set $u = x^2 + ix^3, v = x^4 + ix^5$. They transform as $(2, 2, 1)$ of $SO(4) \times SU(2)_R$.

The presence of $N_f$ D 4-branes between the two NS 5-branes correspond to $N_f$ matter in the fundamental representation of the gauge groups. In fact, strings stretched between D 2-branes and D 4-branes give us hypermultiplets in the fundamental representation. The position of these D 4-branes, $m_i$, are bare masses of the hypermultiplets which transform as $(2, 2, 1)$ under global symmetry $SO(4) \times SU(2)_R$. If two of $m_i$'s become equal, the D 2-brane can break and suspends between these two D 4-branes. In this case the scalars in the hypermultiplets correspond to the fluctuations of D 2-branes in the directions $x^7, x^8, x^9$ and also one should include the component $A_6$ from the gauge field which can survive from boundary conditions. These scalars parametrize the Higgs branch of the theory. So, along the
Higgs branch there are scalars transforming in $3 + 1$ of $SU(2)_R$ \[23\]. If we look at this configuration from the point of view of M-theory the last one which is singlet under $SU(2)_R$, becomes more manifest; in fact it is the fluctuations in $x^{10}$ (compact direction in M-theory). The position of NS 5-branes in $x^{10}$ can be interpreted as $\theta$ angle ($\theta = x_{2}^{10} - x_{1}^{10}$)

The distance between two NS 5-branes determines the gauge coupling constant of the two dimensional theory; more precisely

$$\frac{1}{g^2} = \frac{L}{g} \quad \quad (5)$$

where $g$ is the string coupling. The quantum Coulomb branch can occur when $r = 0$ and $\theta = 0$ \[24\]. If $r \neq 0$ (or $\theta \neq 0$) the theory can not have the Coulomb branch, which means that if we want to have a supersymmetric configuration, the D 2-branes must be broken into D 2-branes between NS 5-branes and D 4-branes. Note that in this case one should also consider that the $s$-configuration is not supersymmetric \[13\]. If the D 4-branes have the same position in $x^2, x^3, x^4, x^5$ (equal mass $m_i = m_j$), the theory can be in the Higgs branch. The complete Higgsing is only possible for $2N \leq N_f$.

If we have several NS 5-branes one can also have another hypermultiplet. Strings between D 2-branes which end from left and right to a NS 5-brane give us bi-fundamental matter. In particular if we compact the $x^6$ direction we can also have hypermultiplets in the adjoint representation. This can give us a two dimensional gauge theory with 16 supercharges \[12\].

An interesting model is $U(1)$ gauge theory with $N_f = 1$ and one hypermultiplet in the adjoint representation as $U(1)$ is Abelian is simply free. This theory has been recently studied and argued that it can have a Higgs branch \[13\]. In brane language it is easy to see the occurrence of the Higgs branch. In this case we have one NS 5-brane and one D 2-brane which wraps around $x^6$ direction and both ends are in the NS 5-brane (in fact it intersects the NS 5-brane). We also have a D 4-brane. The D 2-brane can end on the NS 5-brane and move in $x^2, x^3, x^4, x^5$, so the theory is in the Coulomb branch. When D 2-brane intersects D 4-brane it can break into two D 2-branes between D 4-brane and NS 5-brane. One can also imagine that the position of these two D 2-branes in the directions of $x^2, x^3, x^4, x^5$ become equal, so they can connect and leave the NS 5-brane. This means that the adjoint hypermultiplet (free hypermultiplet) gets expectation value and the theory is in the Higgs branch, which is parametrized by the position of the D 2-brane in the directions $x^7, x^8, x^9$ and $x^6$ ($A_6$ component of gauge field). So the model has a Higgs phase as well as a Coulomb phase, but in this case the Higgs branch is obtained from the expectation value of matter in the adjoint representation as studied in \[13\].

For gauge group $U(N)$ the story is the same as $U(1)$. The Higgs branch is parametrized by motion of ends of D 2-branes in the directions $x^7, x^8, x^9$. This can happen if D 2-branes leave NS 5-brane. In this case the Higgs branch is also parametrized by the adjoint hypermultiplets. In the Higgs branch we have N D
2-branes that end on the D 4-brane which can be interpreted by N points in the \( R^4 \). Of course one should mod the Weyl group action on them. So the Higgs branch of the theory is the symmetric product of the Higgs branch of N=1, i.e. \( S^N R^4 \). 

The Coulomb branch can occur when D 2-branes end on the NS 5-brane and move in the directions \( x^2, x^3, x^4, x^5 \). In this case the gauge group is broken to \( U(1)^N \). Note that although it is difficult to see the transition from Coulomb branch to Higgs branch in the context of brane configuration, it is easy to see that the final configuration can occur; which means that there is Higgs branch for the case \( N_f = 1 \). In the next section we will see that this transition can happen by going to strong coupling limit. Note also that as we will see, in the strong coupling limit of the theory, these two branch become equivalent.

One can also add an orientifold plane parallel to D 4-branes in the Type II A brane configuration. Consider an O 4-plane parallel to D 4-branes with world-volume \((x^0, x^1, x^7, x^8, x^9)\). The orientifold plane can be introduced by moding out \((x^2, x^3, x^4, x^5, x^6) \rightarrow (-x^2, -x^3, -x^4, -x^5, -x^6)\) together with gauging of the world sheet parity. There are two type of O 4-plane in the Type II A String theory, classified with the RR charges \([25]\). One type has -1 D 4-brane charge and the other has +1. When \( 2N \) D 4-branes are close to the fixed point, the former leads to \( SO(2N) \) gauge enhancement and the latter to \( SP(2N) \). Since we are interested in two dimensional theory, this enhancement corresponds to global symmetry.

Now consider the following brane configuration. 2 NS 5-branes, 2 D 2-branes (in fact only one half of them is physical the other is its image with respect to O 4-plane), \( 2N_f \) D 4-branes and an O 4-plane parallel to them. We choose O 4-plane charge to be negative of that of the D 4-brane, so the global symmetry is
$SO(2N_f) \times SO(4) \times SU(2)_R$. In the presence of the O 4-plane the Coulomb branch is $R^4/Z_2$ and the gauge group is $SP(1) \simeq SU(2)$. This theory is the $D_{N_f}$ model discussed in [23].

In the spirit of [4], presence of $N_f$ D 4-branes induce magnetic charge in the $(u, v)$ space. Note that the O 4-plane also induces twice magnetic charge with opposite sign. They can affect the metric on the Coulomb branch. Minimizing the total NS 5-brane worldvolume, we find the four dimensional Laplace equation; therefore $SU(2)$ gauge symmetry with $2N_f$ flavours has the following metric on the Coulomb branch.

$$ds^2 = \left( \frac{1}{g^2} + \frac{2N_f - 2}{X^2} \right) d^2 X$$

(6)

where $X^2 = u\bar{u} + v\bar{v}$. In the case of $m_i \neq 0$ we find

$$ds^2 = \left( \frac{1}{g^2} + \sum_{i=1}^{N_f} \frac{1}{|X-m_i|^2} + \frac{1}{|X+m_i|^2} - \frac{2}{X^2} \right) d^2 X$$

(7)

which is the one-loop correction to the metric in the Coulomb branch[23]. Note that the last term is the effect of O 4-plane. As in [11], the torsion on the Coulomb branch can be interpreted as the anti-symmetric $B_{\mu\nu}$ living on the NS 5-brane of Type II A string theory. In our case the B-field charge is $2N_f - 2$. Note that in the case $N_f = 1$ the effect of O 4-plane can cancel by these two D 4-branes, so the metric will be well defined over all the range of the moduli space. For $N_f = 0$ there is only O 4-plane, so there is no global symmetry as well as the Higgs branch. Also metric is not flat and not Riemannian and it has a singularity at finite distance $X = \sqrt{2g}$.

From the above discussion, it seems that for N D 2-branes and $N_f$ D 4-branes (and one should also consider their images) the theory is $SP(N)$ with $D_{N_f}$ singularity and the Coulomb branch is $(R^4)^N/W$, where W is Weyl group of $SP(N)$.

### 3 M-theory description and mirror symmetry

The brane configuration in M-theory consists of M 5-branes and M 2-branes. By lifting to M-theory the D 4-branes become M 5-branes. In this case the Lorentz group of 11-dimensional M-theory is broken to $SO(1,1) \times SO(4) \times SO(4)$ by our brane configuration. These two $SO(4)$’s can be interpreted as R-symmetry. The first $SO(4)$ which acts on the $x^2, x^3, x^4, x^5$ directions, corresponds to R-symmetry of the Higgs branch. The second $SO(4)$ which acts on the $x^7, x^8, x^9, x^{10}$ directions, is the R-symmetry of the Coulomb branch. Note that the R-symmetry of the Coulomb branch of the theory is enhanced from $SU(2)_R$ to $SO(4)$ at strong coupling [11], as conjectured in [13]. So in the strong coupling, the $R$-symmetry of the Higgs and Coulomb branches are the same. It is very similar to the three dimensional $N = 4$ gauge theory, where we have mirror symmetry, which exchanges Coulomb and Higgs branches. Therefore we expect to see a similar symmetry in two dimensional $N = (4,4)$ gauge theory in strong coupling. In fact there are two different theories
that become equivalent at strong coupling. More precisely, the Coulomb branch of one theory is equivalent to the Higgs branch of the other theory at strong coupling. When we are in strong coupling limit - M-theory - there is no difference between these theories. In brane language it means that, they are the same brane configurations in M-theory. If we have a brane configuration in M-theory, the theory which we will find in Type II A limit, depends on the direction compactified.

Consider an operator \( U \) acting as \( x^i \to x^{i+5} \) and \( x^{i+5} \to -x^i \) for \( i = 2, 3, 4, 5 \). \( U \) is a reflection operator which changes the theory to its mirror. Assume we have a particular brane configuration in M-theory. It can be obtained from a brane configuration of Type II A by opening up the 11th direction. Now by applying the operator \( U \) on this configuration; then going back to the Type II A, we will find the mirror theory which is equivalent to the first one we began with in the strong coupling limit. Note that under \( U \), the number of degrees of freedom of the theories do not change, so the superconformal field theories which we will obtain from these two, one from the Higgs branch and another from the Coulomb branch, will have the same central charge.

If we obtain our brane configuration from T-dualizing the brane configuration of Type II B, which describes the mirror symmetry of \( N = 4, D = 3 \) supersymmetric gauge theory [4], then the \( U \)-transformation will be equivalent to \( RS \) transformation introduced in [4]. In fact exchanging of \( x^2, x^3, x^4 \) with \( x^7, x^8, x^9 \) corresponds to the \( R \) transformation and exchanging \( x^5 \) with \( x^{10} \) corresponds to \( S \)-duality.

As an example, consider the theory with gauge group \( U(N) \) and \( N_f \) hypermultiplets in the fundamental representation [4]. It has the following brane configuration. (for simplicity the case \( N=3, N_f = 7 \) is indicated)

Since the Higgs branch of the theory is equivalent to the Coulomb branch of the other theory in the strong coupling, the first theory should be in the complete Higgs branch, so \( 2N \leq N_f \). To finding the mirror theory, one should first Higgs the theory, then go to the M-theory and after \( U \)-transforming go backing to the Type II A. Doing so, we will find the theory with gauge group \( U(1) \times U(2) \times \cdots U(N - 1) \times U(N)^{N_f - 2N + 1} \times U(N - 1) \cdots \times U(1) \) with hypermultiplets transforming as \( (1, 2) \oplus (2, 3) \oplus \cdots (N - 1, N) \oplus (N, N) \oplus \cdots (N, N) \oplus N \oplus (N, N - 1) \oplus \cdots (2, 1) \). In finding

\[ \text{Brane realization of this theory helps us to write the metric of the moduli space of the Coulomb branch for gauge groups with arbitrary rank}[3]\]
the matter content of the mirror theory we used the fact that the $x^6$ coordinate of D 4-branes (matter) appear to be irrelevant as was noted in [4]. Although $x^6$ direction of matter has no physical meaning in the first theory, it becomes physical in the mirror theory. In fact, it becomes the gauge coupling in the mirror theory. As this is a strong coupling phenomena, it would be difficult to explain it in the context of field theory.

Note that under this symmetry, the number of bosonic degrees of freedom of the Higgs branch of the first theory (scalars in hypermultiplets) is equal to that of the number of bosonic degrees of freedom of the Coulomb branch of the second one (scalars in vector multiplets); so the supercoformal theories corresponding to them have the same central charges. One can also start with the Coulomb branch of the first theory. In this case after $U$-transformation we will find the Higgs branch of the second one.

Let us return to the case $N_f = 1$ with one adjoint hypermultiplet. If we begin with the Coulomb branch of the theory and apply the $U$-transformation, we will end up with the theory in the Higgs branch. So as we said, the transition from Coulomb branch to Higgs branch can occur when we go to the strong coupling and apply $U$-transformation. Note that in the M-theory limit, there is no difference between these two brane configuration, which means the Coulomb branch and the Higgs branch of the theory become equivalent at strong coupling.

By introducing the $U$-transformation it is possible to study a large class of theories and their mirrors. In principal we can start by an arbitrary brane configuration in the Type II A string theory (in Higgs branch or Coulomb branch or a mixed branch) then lift to the M-theory and apply $U$-transformation, then back to the Type II A. In this case we can find a different theory which becomes equivalent in strong coupling limit, to the original theory; Although it may not have a Lagrangian formalism.

Consider the model (A-model) which has $U(N)$ gauge group, $N_f$ hypermultiplets in the fundamental representation of the gauge group, and one hypermultiplet in the adjoint representation. In the brane language, this model consists of a NS 5-brane, $N$ D 2-branes which wrap on the $x^6$ direction and end on the NS 5-brane (in fact it can be viewed as intersecting), so there is a hypermultiplet in the adjoint representation. We should also add $N_f$ D 4-branes at points $m_i$ and $x^6$ as hypermultiplets in the
fundamental representation. In this model $x^6$ component of the matter appears to be irrelevant, but $m_i$’s ($\eta_i = m_i - m_{i+1}$) correspond to the bare masses of the hypermultiplets.

Assume the theory is in the Higgs branch, then the mirror model (B-model) will be the Coulomb branch of theory with gauge group $\prod_{i=0}^{N_f-1} U(N)_i$ and its hypermultiplets consist of one fundamental matter charged under $U(N)_1$ and $N_f$ bifundamental which have charge respectively under $U(N)_i \times U(N)_{i+1}$ in the representation $(N, \bar{N})$ with the cyclic identification $i \sim i + N$.

The above information may be completely encoded in the "quiver" diagram [26].
For A and B models the "quiver" diagrams are as follows

In the mirror theory $x^6$ corresponds to the gauge coupling and $\eta_i$’s are the Fayet-Iliopoules terms. In order to have Higgs branch in the A-model, the mass of adjoint matter should be zero, in the B-model it means that $\sum \eta_i = 0$. Note that we can have a theory as the same of B-model but without the fundamental hypermultiplet, it can do if we do not use NS 5-brane in the A-model or assume that we are far from it, in this case the fundamental matter decoupled from the theory.

By the same method as above we can find more complicated models which become equivalent in the strong coupling limit. These models can be indicated by their "quiver" diagrams. Note that two theories which become equivalent in strong coupling, correspond to mirror pairs in three dimensions. These mirror pairs of the three dimensional theories have been studied in the case of Sp gauge groups as we do not understand the action of S-duality on the O-planes completely.

4 Relation with Matrix theory

In this section we will consider some particular two dimensional supersymmetric gauge theories and construct their brane configurations. These theories have recently been studied in the context of Matrix theory formulation of NS 5-brane in Type II string theories.

First consider $N = (4,4) \ D = 2$ supersymmetric gauge theory with gauge group $F = U(1)^k$ and $k$ hypermultiplets in the bi-fundamental representation of the gauge group ($(1,1)$ representation of $U(1)_i \times U(1)_{i+1}$ with identification $i \sim i + k$). It has the following brane configuration:
$k$ NS 5-branes at the same point in the directions $x^7, x^8, x^9$ and at $k$ points in the $x^6$ direction. We assume that the $x^6$ direction is compact with radius $R_6$ and $k$ NS 5-branes positioned in the $x^6$ direction in an $Z_k$ invariant way. There is also a D 2-brane wrapped around the $x^6$ direction. This D 2-brane is broken into segments stretched between each pair of adjacent NS 5-branes.

The freedom on adding an arbitrary constant phase to the position of the branes on the circle, leads to the decoupling of the $U(1)$ from the gauge group, so that the non-trivial gauge group is actually $F/U(1)$. The ends of the D 2-branes can move inside the NS 5-branes. The position of the D 2-brane on the NS 5-brane can be interpreted as the scalars in the vector multiplets. So they parametrize the Coulomb branch of the theory. It has $4k$ dimensions. Note that since the gauge group is Abelian there is no symmetry breaking in the Coulomb branch.

If D 2-branes meet from two side of one NS 5-brane, they can reconnect, so we can have closed D 2-branes which is wrapped around the $x^6$ direction. In this case the D 2-brane can leave the NS 5-branes and move far from them along the $x^7, x^8, x^9$ directions. This is a transition from the Coulomb branch to the Higgs branch. Note that, the Higgs branch occurs via expectation value of the bi-fundamental hypermultiplets.

This model is exactly the theory of D 1-brane probe of Type II B string theory on the ALE space. To see this, consider the limit of $R \to 0$; in this case, T-duality on the $x^6$ directions maps us to the Type II B theory on a circle with radius $R_B = \frac{1}{M_2 R}$.

The $k$ NS 5-branes of Type II A theory, which are located on the transverse circle $R$, are mapped into $k$ Kaluza-Klein monopoles in the Type II B string theory. The $k$ Kaluza-Klein monopoles can be constructed from four dimensional multi-Taub-NUT metric tensored with flat space. The non-trivial metric on the $R^3 \times S^1$ is

$$ds^2 = V(x)d\vec{x}^2 + V(x)^{-1}(d\theta + \vec{A} \cdot d\vec{x})^2,$$

where

$$\nabla V = \nabla \times \vec{A}, \quad V = 1 + R_B \sum_{i=1}^{k} \frac{1}{|\vec{r} - \vec{x}_i|}$$

The position of the $k$ branes are given by the $\vec{x}_i$ and the angular variable $\theta$ which has the period proportional to $R_B$. When all the branes are separated the space is smooth. For the $k \geq 2$ coalescing branes the multi-Taub-NUT has an $A_{k-1}$ singularity at the position of the branes. In the limit $R_B \to \infty$ the space becomes $R^4/Z_k$. So in our case we end up with the Type II B on the $A_{k-1}$ ALE space. Also the D 2-brane maps to D 1-brane and then one can consider, after T-duality, motion of the D 1-brane in the ALE space. So the theory corresponds to D 1-brane probe of Type II B string theory on the $A_{k-1}$ singularity which is discussed in [28], where the same theory was discussed. The Higgs branch of the theory corresponds to motion of D 1-branes in the ALE space as a background of the Type II B string theory. One can also consider the theory with gauge groups $U(N)^k$. In this case we can study the NS 5-branes in the Type II A or Type II B on an $A_{k-1}$ singularity.
Now consider the theory with gauge group $U(N)^k$ and $k$ bi-fundamental hypermultiplets. This is exactly the B-model discussed in the previous section, in the limit that the fundamental hypermultiplet decouples. Moreover we assume that the theory is compactified over a circle with radius $R_1$. This theory corresponds to the matrix theory description of the Type II A string theory on an $A_{k-1}$ singularity. In our brane realization, it corresponds to $k$ wrapped NS 5-branes (wrap around $x^1$) at the same points in the directions $x^7, x^8, x^9$ and at $k$ points on the $x^6$ direction in an $Z_k$ invariant way. There are also $N$ D 2-branes wrapped around $x^1$ and $x^6$. In fact we can also have one D 4-brane, but we are far from it; so it decoupled from the theory. So we have exactly the same gauge group and matter content as above. These $N$ D 2-branes can break into segments stretched between each pair of adjacent NS 5-branes.

This theory has a Coulomb branch as well as a Higgs branch. The Higgs branch can be obtained from the expectation value of the bi-fundamental hypermultiplets and describes the bulk theory. The Coulomb branch describes the decoupled six dimensional theory. It has $4kN$ dimensions with tube-like metric. In fact this theory corresponds to the Type II A Matrix theory on an ADE singularity as studied in [19][29].

Let us apply the $U$-transformation to this configuration as before. Doing so, we will have $N$ D 2-branes wrapped around $x^1, x^2$, and $k$ wrapped D 4-branes at points on $x^6$. Since both directions of wrapped D 2-branes are compactified, and in general they can be in the same order, so in fact we have 2+1 dimensional gauge theory with gauge group $U(N)$ compactified on the torus. This gauge theory has 16 supercharges and is the bulk theory; but presence of the $k$ D 4-branes which can be interpreted as impurities in the 2+1 dimensional gauge theory, break half of the supercharges. These impurities correspond to the $k$ hypermultiplets in the fundamental representation.

The bulk theory which is described by the D 2-branes, has seven scalars in the adjoint representation corresponding to the seven directions transverse to D 2-branes and has $SO(7)$ global symmetry. But presence of the D 4-branes break this symmetry to $SO(4) \times SU(2)$ and the scalars break in to two parts. One, the scalars in the vector multiplet which parametrize the Coulomb branch (fluctuation of D 2-branes in the directions $x^2, x^3, x^4, x^5$) and another, the scalars in the hypermultiplets which parametrize the Higgs branch (fluctuation of D 2-branes in the directions $x^7, x^8, x^9$). The $SO(4)$ acts on the scalars in the vector multiplet and $SU(2)$ on the scalars in the hypermultiplets.

Here, the bulk theory, which is the space-time physics with background $k$ Type II B wrapped NS 5-branes, can be described by the Coulomb branch and the Higgs branch describes the decoupled theory. In the limit of $R_6 \to 0$, this theory is 1+1 di-

\footnote{Note that the decoupled fundamental matter maps to NS 5-brane wrapped around $x^1$ direction and in general can intersect the D 2-branes and from point of view of two dimensional theory, it gives us the matter in the adjoint representation.}
mensional $N = (4, 4)$ theory with gauge group $U(N)$ and one adjoint hypermultiplet and $k$ fundamental hypermultiplets.

The decoupled theory is described by the Higgs branch of the theory. In this case the $k$ hypermultiplets get expectation values and $N$ D 2-branes break into segments stretched between D 4-branes. The Higgs branch is parametrized by the fluctuation of the D 2-branes in the directions $x^7, x^8, x^9$ and the last scalar corresponds to non-zero component of the gauge field $A_6$. This branch has $4kN$ dimensions, as we expect from the mirror theory. If we go to the strong coupling limit, the last scalar will be the fluctuation of D 2-branes in the $11^{th}$ direction and as we said, in this limit, it leads to R-symmetry enhancement from $SU(2)$ to $SO(4)$ [19][11]. As we said there is a Higgs branch for the case $k = 1$, it means that the theory for one NS 5-brane should also be non-trivial at the decoupled limit!

Note that it seems that, the two dimensional mirror symmetry which we considered here, corresponds to the duality between Type II A on an $A_{k-1}$ singularity and $k$ NS 5-branes in Type II B and also the duality between Type II B on an $A_{k-1}$ singularity and $k$ NS 5-branes in Type II A. Note also that, This mirror symmetry may shed light on the dual realization of the 1+1 dimensional $N = (4, 4)$ gauge theory in the such way that the tube metric is absent, but the dual theory has ADE singularity as conjectured in [24][23] and argued in [30] and recently discussed in the second reference in [19].

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