TOWARDS A NONPERTURBATIVE COVARIANT
REGULARIZATION IN 4D QUANTUM FIELD THEORY

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Abstract. We give a noncommutative version of the complex projective space \( \mathbb{C}P^2 \)
and show that scalar QFT on this space is free of UV divergencies. The tools necessary
to investigate Quantum fields on this fuzzy \( \mathbb{C}P^2 \) are developed and several possibilities
to introduce spinors and Dirac operators are discussed.

Keywords: Regularization, Noncommutative Geometry, Geometric Quantization, \( \mathbb{C}P^2 \)

1. Introduction

Recently methods of noncommutative geometry ([3]) were used to introduce a covariant
regularization procedure for Quantum fields on the 2 sphere ([8, 9]). There is hope that
with these methods there will be a nonperturbative understanding of quantum effects.
For a treatment of the chiral anomaly in the Schwinger model see [11].

The idea is to approximate the algebra of functions on the space by matrix algebras
and to encode the geometrical information of the space in these algebras. In this way it
becomes possible to construct models of QFT on the virtual spaces, the matrix algebras
are thought to be functions on. Since the theory has only finite degrees of freedom
the problems one usually deals with in QFT are absent. For examples of these matrix
geometries see [17, 18, 7].

We will show that the \( \mathbb{C}P^2 \), as a 4 dimensional Riemannian manifold on which the
group \( SU(3) \) acts isometrically, can by treated in this way. We construct the Laplace
operator on the fuzzy \( \mathbb{C}P^2 \) and show that the spectrum is the same as in the classical
case, except for a truncation at higher eigenvalues, which can be interpreted as a UV
cutoff. Since \( SU(3) \) acts on the matrix algebras by automorphisms we did not lose the
\( SU(3) \) covariance by this procedure.

We discuss the quantization of several vector bundles, among them a family of \( \text{Spin}^C \)-
bundles, and compare the structures with the classical case. As well as for the functions

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we find that in the quantized sections higher order representations of $G$ are missing. A straightforward generalization of the classical Spin$^C$-Dirac operators is introduced and the spectra are calculated. It turns out, that zero modes and index are different from the commutative case. A possible solution to this problem is presented.

2. THE CLASSICAL CASE

We start with a group theoretic construction of the classical structures on the $\mathbb{C}P^2$. Let $G^C$ be the Lie group $SL_3(\mathbb{C})$ and $g^C$ its Lie algebra. We choose a Cartan subalgebra and a corresponding system of roots $\Delta$, positive roots $\Delta^+$, fundamental roots $\Delta^f$, and use the standard Cartan-Weyl basis $\{H_\delta, X_\delta, X_{-\delta}\}_{\delta \in \Delta^+}$.

![Root Diagram of the $A_2$](image)

**Figure 1.** Root Diagram of the $A_2$

and $\beta$ (see Figure 1). Hence we have

$$g^C = \text{span}_{\mathbb{C}}\{H_\alpha, H_\beta, X_\alpha, X_\beta, X_{-\alpha}, X_{-\beta}, X_{\alpha+\beta}, X_{-(\alpha+\beta)}\}$$

The compact real form $g \subset g^C$ is

$$g = \text{span}_{\mathbb{R}}\{iH_\alpha, iH_\beta, X_\alpha + X_{-\alpha}, X_\beta + X_{-\beta}, X_{\alpha+\beta} + X_{-(\alpha+\beta)}, i(X_\alpha - X_{-\alpha}), i(X_\beta - X_{-\beta}), i(X_{\alpha+\beta} - X_{-(\alpha+\beta)})\}$$

and the corresponding connected subgroup $G \subset G^C$ is the group $SU(3)$. The two fundamental weights are

$$\lambda_1 = \frac{1}{3}(2\alpha + \beta) \quad \lambda_2 = \frac{1}{3}(\alpha + 2\beta)$$

We denote the Verma module with highest weight $\lambda = n_1\lambda_1 + n_2\lambda_2$ by $D(n_1, n_2)$ the fundamental roots chosen such that $D(1,0)$ corresponds to the fundamental representation
of the $SU(3)$. The dimensions of these representations are given by
\[ \dim(D(n_1, n_2)) = \frac{(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2)}{2} \]
and the image of the quadratic Casimir element
\[ C = \sum_{\delta \in \Delta^+} (X_\delta X_{-\delta} + X_{-\delta} X_\delta) + \sum_{\delta \in \Delta^f} \frac{1}{2} H^2_\delta \]
is the multiplication operator by the number
\[ C(n_1, n_2) = \frac{2}{3}(n_1^2 + n_2^2 + n_1 n_2 + 3(n_1 + n_2)) \]
To each Verma module there corresponds an (up to equivalence unique) unitary irreducible representation of $G$ and a holomorphic irreducible representation of $G^C$, which we denote by the same symbol.

Let $p \subset g^C$ now be the parabolic subalgebra
\[ p = \text{span}_C\{H_\alpha, H_\beta, X_\alpha, X_{-\alpha}, X_\beta, X_{-(\alpha+\beta)}\} \]
and $P$ the analytic subgroup of $G^C$ with Lie algebra $p$. The quotient space $M := G^C/P$ is a complex flag manifold (see e.g. [22, 20, 21]) and has a compact realization $M = G/K$ where $K$ is the subgroup of $G$ with Lie algebra $k := g \cap p$. In our case this is the group $S(U(2) \times U(1)) \subset SU(3)$. This is just the subgroup that leaves the lowest weight subspace of $D(1,0)$ invariant, and our space $M$ is just the complex projective space $\mathbb{CP}^2$, the space of complex lines in $\mathbb{C}^3$.

$M$ is in a canonical way a Kähler manifold. The action of $G$ on $M$ is holomorphic and leaves the Kähler structure invariant. There exists a unique normalized $G$-left invariant measure on $M$ which we denote by $\mu$.

### 2.1. The quasi-left regular representation

In this section we give the decomposition of the unitary representation of $G$ on the space $L^2(M, \mu)$, given by the action
\[ g f(x) = f(g^{-1}x) \]
For the decomposition we use the fact that $M$ is a Riemannian symmetric space of compact type ([14, 15]). This means that there is a real form $G^R \subset G^C$ with Iwasawa decomposition $G^R = KAN$, where $A$ is the abelian and $N$ the nilpotent part. We denote by $g^R, a, n \subset g^C$ the corresponding real Lie algebras. $a$ can be extended to a maximal abelian Lie subalgebra $h$ of $g^R$ such that its complexification $h^C$ is a Cartan subalgebra of $g^C$. The weights are elements of the dual of the vectorspace $h^C = i(h \cap k) + a$. We choose again fundamental roots and denote by $\Sigma^+$ the positive roots that do not vanish on $a$ and interpret them as linear forms on $a$. The Killing form induces a scalar product $\langle \cdot | \cdot \rangle$ on the dual $a^*$. We have
Theorem 2.1 (Helgason). A representation with highest weight $\lambda$ contains a $K$-invariant vector if and only if

1. $\lambda(i(h \cap k)) = 0$
2. $\langle \lambda|\delta \rangle \langle \delta|\delta \rangle \in \mathbb{N}_0 \quad \forall \delta \in \Sigma^+$

(see [13])

In our case a short calculation shows that one can choose another Cartan subalgebra such that

$$k = g \cap \text{span}_C \{H_\beta - H_\alpha, X_\beta + X_{-\alpha}, X_\alpha, X_{\alpha + \beta} + X_{-(\alpha + \beta)}\}$$

and with

$$a = \text{span}_R \{H_\alpha + H_\beta\}$$
$$n = \text{span}_R \{(X_\alpha - X_{-\alpha}), (X_\beta - X_{-\beta}), (X_{\alpha + \beta} - X_{-(\alpha + \beta)})\}$$

$g^R = k + a + n$ is a real form of $g^C$. One obtains that the representations of $G$ that contain a $K$-invariant vector are just the representations with highest weight

$$\lambda = n(\lambda_1 + \lambda_2) \quad n \in \mathbb{N}_0$$

(9)

The space of $K$-invariant vectors in such a representation is furthermore one-dimensional and by Frobenius theorem exactly those representations occur with multiplicity one in the quasi-left regular representation $\mathbb{1}$.

Hence we get for our case the following result

Theorem 2.2. Let $M = \mathbb{C}\mathbb{P}^2$ with the canonical $SU(3)$ action. As $SU(3)$-modules we have

$$L^2(M, \mu) \cong \bigoplus_{n=0}^{\infty} D(n, n)$$

and the subspace of $L^2(M, \mu)$ which is, as an $SU(3)$-module, isomorphic to $D(n, n)$ is just the space of functions of the form

$$M \to \mathbb{C}, \quad uK \to \langle ue_n|v \rangle, \quad v \in D(n, n)$$

where $e_n \in D(n, n)$ is an (up to a scalar unique) $K$-invariant vector.

---

1the quasi-left regular representation on $G/K$ is just the representation of $G$ induced by the trivial representation of $K$
2.2. The Laplace operator. Identifying the Lie algebra $g^\mathbb{C}$ with the complex left invariant vector fields on the group $G$ and those with differential operators on $C^\infty(G)$, we get a natural action of the universal enveloping algebra $\mathcal{U}(g^\mathbb{C})$ on $C^\infty(G)$. The quadratic Casimir element $C$ becomes a biinvariant Differential operator $\Delta_G$ of second order and is equal to the Laplace-Beltrami operator of the (up to a scalar unique) biinvariant metric on $G$. Thus it can be restricted to the $K$-right invariant functions on $G$ and we get a differential operator $\Delta_M : C^\infty(M) \to C^\infty(M)$. This operator coincides with the metric Laplacian on $M$. We get

**Theorem 2.3.** The Laplace operator $\Delta_M$ on the $\mathbb{CP}^2$ is essentially self-adjoint on $C^\infty(\mathbb{CP}^2)$. The spectrum of its closure is purely discrete and the eigenvalues $\lambda$ and multiplicities $\text{mult}_\lambda$ are

$$
\lambda_n = 2n(n+1), \quad n \in \mathbb{N}_0
$$

$$
\text{mult}_\lambda = (n+1)^3
$$

Proof: We restrict the domain to the finite linear span of vectors in the irreducible subspaces of the group action, on each of which $\Delta_M$ acts by multiplication with $2n(n+1)$. Hence the domain contains a total set of analytic vectors and we found the operator in its spectral decomposition. \hfill \square

2.3. Homogeneous vector bundles over $M$. The homogeneous vector bundles over $M$ are in one to one correspondence with the finite dimensional representations of the group $K$. If $V$ is a homogeneous vector bundle over $M$ and $\tau$ the representation of $K$ at the fibre $V_{eK}$ at the identity class $eK$, then $V$ is isomorphic as a $G$-vector bundle to the associated bundle $G \times_\tau V_{eK}$. The sections of this bundle can be identified with the equivariant $V_{eK}$-valued functions on $G$, that is the functions $f$ that satisfy the equivariance condition $f(xk) = \tau(k^{-1})f(x) \quad \forall x \in G, k \in K$.

The homogeneous vector bundles over $M$ have a canonical holomorphic structure since $\tau$ can be extended to a holomorphic representation $\tilde{\tau}$ of $P$ and the vector bundle can be constructed as $G^\mathbb{C} \times_{\tilde{\tau}} V_{eK}$.

We define several useful bundles on $M$. Let $\tau(n)$ for $n \in \mathbb{Z}$ be the irreducible (one dimensional) representation of $K$ with highest weight $-n\lambda_2$. We denote the homogeneous line bundle $G \times_{\tau(n)} \mathbb{C}$ by $L^n$. For example $L$ is the hyperplane bundle and $L^{-1}$ the tautological line bundle over $M$.

Let $\chi$ be the representation of $K$ with highest weight $\lambda_1$ and $\chi^*$ the dual representation, we define $H$ and $H^{-1}$ to be the bundles $G \times_\chi \mathbb{C}^2$ and $G \times_{\chi^*} \mathbb{C}^2$ respectively.
Since the irreducible representations of $K$ are unitary these vector bundles have natural hermitian structures, and we denote by $\Gamma^2(V)$ the Hilbert space of square integrable sections and by $\Gamma_{hol}(V)$ the finite dimensional Hilbert space of holomorphic sections of the bundle $V$. The representation of $G$ on $\Gamma^2(G \times \tau V)$ is just the representation induced by $\tau$.

**Theorem 2.4.** For the bundles $L^n$ we have the following isomorphisms of $G$-modules

1. for $n \geq 0$: $\Gamma^2(L^n) \cong \bigoplus_{m=0}^{\infty} D(m, m + n)$
2. for $n < 0$: $\Gamma^2(L^n) \cong \bigoplus_{m=0}^{\infty} D(m - n, m)$

Sketch of Proof: We specialize to Theorem 2.4. For the bundles $\tau$ is dense in $\Gamma^2(L^n)$ where $h_n$ is a nonzero highest weight vector and $\{v_i\}_{i=1}^{\dim(D(0,n))}$ an orthonormal basis in $D(0,n)$. To see this simply choose $f_i(x) = f(x)\langle v_i | x h_n \rangle$. Hence using theorem 2.2 the subspace

$$W := \text{span}_C \{ \langle \cdot | e_m | u^{(m)} \rangle \langle \cdot | h_n | v \rangle : u^{(m)} \in D(m, m), v \in D(0, n), m \in \mathbb{N}_0 \}$$

is dense in $\Gamma^2(L^n)$. The subspaces

$$W^N := \text{span}_C \{ \langle \cdot | e_m | u^{(m)} \rangle \langle \cdot | h_n | v \rangle : u^{(m)} \in D(m, m), v \in D(0, n), m \leq N \}$$

define a filtration of $W$. A short calculation using $D(0, m) \otimes D(m, 0) = \bigoplus_{i=0}^{m} D(i, i)$ gives

$$W^N = \text{span}_C \{ \langle \cdot | h_{n+m} | u^{(m+n)} \rangle \langle \cdot | l_m | v^{(m)} \rangle : u^{(m)} \in D(0, n + m), v^{(m)} \in D(m, 0), m \leq N \}$$

where $l_m$ is a nonzero lowest weight vector in $D(m, 0)$. Since $D(0, m + n) \otimes D(m, 0) = \bigoplus_{i=0}^{m} D(i, i + n)$ and the vector $h_{n+m} \otimes l_m \in D(0, n + m) \otimes D(m, 0)$ is cyclic, we can choose a family of nonzero vectors $e^m \in D(m, m + n)$ such that

$$W^N = \text{span}_C \{ \langle \cdot | e^m | v^{(m)} \rangle : v^{(m)} \in D(m, m + n), m \leq N \}$$

and therefore

$$W = \text{span}_C \{ \langle \cdot | e^m | v^{(m)} \rangle : v^{(m)} \in D(m, m + n), m \in \mathbb{N}_0 \}$$

This establishes the desired isomorphism of modules.

The same method allows us to decompose the representation on the space of sections of another class of bundles.
Theorem 2.5. For the bundles $L^n \otimes H^{-1}$ we have the following isomorphisms of $G$-modules

1. for $n \geq 0$: $\Gamma^2(L^n \otimes H^{-1}) \cong \bigoplus_{m=0}^{\infty} (D(m, m + n + 2) \oplus D(m + 1, m + n))$
2. for $n < 0$: $\Gamma^2(L^n \otimes H^{-1}) \cong \bigoplus_{m=0}^{\infty} (D(m - n - 1, m + 1) \oplus D(m - n + 1, m))$

Since $G$ acts on $M$ by holomorphic automorphisms the bundles of antiholomorphic $k$-forms $\Lambda^{0,k}$ are homogeneous vector bundles. The vector space of antiholomorphic tangent vectors at $eK$ can be identified with the nilpotent radical $u$ of the parabolic subalgebra $p \subset g^C$, which is just span$_C\{X_{-(\alpha+\beta)}, X_{-\beta}\}$. The isotropy representation is the adjoint action of $K$ on $u$. Taking the dual representations this leads to the following isomorphisms of $G$-bundles.

$$
\Lambda^{0,0} \cong L^0, \\
\Lambda^{0,1} \cong L \otimes H^{-1}, \\
\Lambda^{0,2} \cong L^3
$$

(10)

2.4. Spin$^C$-bundles. It is well known that the $\mathbb{CP}^2$ does not admit a Spin-structure. It does however admit a Spin$^C$-structure and we give here a family of Spin$^C$-bundles. Since $M$ is a Kähler manifold the bundle of antiholomorphic forms $\Lambda^{0,*}$ with the natural $\mathbb{Z}_2$-grading is a canonical Spin$^C$-bundle and the Dolbeault-Dirac operator $D = \sqrt{2}(\partial + \overline{\partial})$ is a Spin$^C$-Dirac operator (see e.g. [13]). Tensoring this bundle with some line bundle we obtain other Spin$^C$-bundles.

Tensoring the canonical Spin$^C$-bundle $S$ with the line bundle $L^m$ and using the above described isomorphisms of $G$-bundles we get the Spin$^C$-bundles $S_m$:

$$
S_m^+ \cong L^m \oplus L^{m+3} \ldots \text{even part} \\
S_m^- \cong L^{m+1} \otimes H^{-1} \ldots \text{odd part} \\
S_m = S_m^+ \oplus S_m^-
$$

(11)

The sections of $S_m$ are identified with the $\mathbb{C}^4$-valued functions $\phi$ on $G$ which satisfy the equivariance condition

$$
\begin{pmatrix}
\phi_1(gk) \\
\phi_2(gk) \\
\phi_3(gk) \\
\phi_4(gk)
\end{pmatrix} = \tau(-m)
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \tau(-3)(k) & 0 & 0 \\
0 & 0 & \tau(-1)(k)\chi_{11}(k) & \tau(-1)(k)\chi_{12}(k) \\
0 & 0 & \tau(-1)(k)\chi_{21}(k) & \tau(-1)(k)\chi_{22}(k)
\end{pmatrix}
\begin{pmatrix}
\phi_1(g) \\
\phi_2(g) \\
\phi_3(g) \\
\phi_4(g)
\end{pmatrix}
$$

for all $k \in K, g \in G$. Here the $\chi_{ik}$ are the matrix elements of the representation $\chi$. The basis in the representation space of $\chi$ can be chosen in such a way that the
Dolbeault-Dirac operator $D_m$ takes the form

$$D_m = \sqrt{2} \begin{pmatrix}
0 & 0 & \tilde{X}_{\alpha+\beta} & \tilde{X}_\beta \\
0 & 0 & -\tilde{X}_{\beta} & \tilde{X}_{-(\alpha+\beta)} \\
\tilde{X}_{-(\alpha+\beta)} & -\tilde{X}_\beta & 0 & 0 \\
\tilde{X}_\beta & \tilde{X}_{\alpha+\beta} & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix} 0 & \tilde{D}^* \\
\tilde{D} & 0 \\
\end{pmatrix}$$

where $\tilde{X}_\delta$ is the differential operator corresponding to the left invariant vector field $X_\delta$ on $G$. This is an example of an abstract supersymmetric Dirac operator. We have

**Theorem 2.6.** The Spin$^C$-Dirac operator $D_m$ is essentially self-adjoint on the space of smooth sections of $S_m$. The spectrum of its closure is purely discrete. The eigenvalues $\lambda$ and multiplicities $\text{mult}_\lambda$ are:

1. $\lambda_n^\pm = \pm \sqrt{2(n+2) + 2|m|(n+1) - 2m}$, $n \in \mathbb{N}_0^+$
   $$\text{mult}_{\lambda_n^\pm} = \frac{(n + \frac{|m|+2}{2} + 1)(n+1)}{2}$$
2. $\lambda_m^\pm = \pm \sqrt{2(n+2) + 2|m+3|(n+1) + 2(m+3)}$, $n \in \mathbb{N}_0^+$
   $$\text{mult}_{\lambda_m^\pm} = \frac{(n + \frac{|m+3|+2}{2} + 1)(n+1)}{2}$$

The index is $\text{index}(D_m) = \text{mult}_{\lambda=0} = \frac{(m+1)(m+2)}{2}$

Sketch of Proof: The essential self-adjointness follows from Nelsons trick. Using the transformation properties on the group an easy calculation shows

$$\tilde{D}^* \tilde{D}_m = \begin{pmatrix} \Delta_G - 2m - \frac{2}{3}m^2 & 0 \\
0 & \Delta_G - 2m - \frac{2}{3}m^2 \\
\end{pmatrix}$$

The abstract Foldy-Wouthuysen transformation and the decompositions of the $G$-modules $\Gamma(L^m)$, $\Gamma(L^{m+3})$, and $\Gamma(L^{m+1} \otimes H^{-1})$ gives the desired result.

3. Quantization of the $\mathbb{C}P^2$

Since $M$ is a compact Kähler manifold and $L$ a Quantum line bundle we can use the Berezin-Toeplitz quantization procedure (see e.g. [2]). We fix an $N \in \mathbb{N}_0$ and let $\Pi_N$ be the orthogonal projection onto the subspace $\mathcal{H}^N := \Gamma_{hol}(L^N) \subset \Gamma^2(L^N)$. For each $f \in C^\infty(M)$ we get an operator $T^N(f) \in \text{End}(\mathcal{H}^N)$, defined by

$$T^N(f) := \Pi_N M_f \Pi_N$$

where $M_f$ is the multiplication-operator associated with $f$. The corresponding surjective map $C^\infty(M) \rightarrow \text{End}(\mathcal{H}^N)$ is called Toeplitz quantization map. The algebra $\mathcal{A}_N := \text{End}(\mathcal{H}^N)$ is the algebra of quantized functions, which for high values of $N$ approximates the algebra $C^\infty(M)$ in a certain sense.
By the Bott-Borel-Weyl theorem (see [20, 21]) we have the isomorphism of $G$-modules
\[ \mathcal{H}^N \cong D(0, N). \]
Identifying these two spaces we obtain for the quantized algebra of functions the matrix algebra
\[ \mathcal{A}_N = \text{Mat}(\frac{(N+1)(N+2)}{2}, \mathbb{C}) \]
on which $G$ acts by inner automorphisms
\[ G \times \mathcal{A}_N \to \mathcal{A}_N : (g, a) \to \pi(g) a \pi^{-1}(g) \]
where $\pi$ is the representation which corresponds to the $G$-module $D(0, N)$. If $\tilde{P}_N$ is the orthogonal projection onto the highest weight subspace of $\mathcal{H}^N$ we define the coherent state function on $M$ by
\[ P_N : M \to \text{End}(\mathcal{H}^N), \quad gK \to \pi(g) \tilde{P}_N \pi^{-1}(g) \]
and we get for the Toeplitz quantization map
\[ T^N_f = \frac{(N+1)(N+2)}{2} \int_M f(x) P_N(x) d\mu(x) \]
which is equivariant, real, positive, and norm decreasing.

Moreover the following results hold

**Theorem 3.1.**

1. The restriction $t^N := T^N|_{\mathcal{E}^N}$ is bijective.
2. $\lim_{N \to \infty} (T^N)^{-1} T^N(f) = f$ in the uniform topology on $C^\infty(M)$
3. for $f_1 \ldots f_n \in C^\infty(M)$ we have with the $C^*$-norms on the $\mathcal{A}_N$:
\[ \|T^N_{f_1} \cdots T^N_{f_n} - T^N_{f_1 \cdots f_n}\| = O\left(\frac{1}{N}\right) \quad \text{for} \quad N \to \infty \]
4. If $\text{tr}_1(\cdot)$ is the normalized trace on $\mathcal{A}_N$, we have:
\[ \int_M f(x) d\mu(x) = \text{tr}_1(T^N_f) \quad \forall f \in C^\infty(M), N \in \mathbb{N}_0 \]
5. for $f_1 \ldots f_n \in C^\infty(M)$ holds:
\[ \int_M f_1(x) \cdots f_n(x) d\mu(x) = \text{tr}_1(T^N_{f_1} \cdots T^N_{f_n}) + O\left(\frac{1}{N}\right) \quad \text{for} \quad N \to \infty \]
6. $\|T^N_f\| = \|f\|_{\text{sup}} + O\left(\frac{1}{N}\right)$ \quad for \quad $N \to \infty$

Here $\mathcal{E}^N \subset C^\infty(M)$ is the space of truncated functions, that is the subspace that contains only the $G$-representations $D(n, n)$ for $n \leq N$.

It follows that as $G$-modules $\mathcal{A}_N$ and $\oplus_{n=0}^N D(n, n)$ are isomorphic. This is a kind of cut-off automated by the quantization. One introduces now the linear inclusions
\[ i_N : \mathcal{A}_N \hookrightarrow \mathcal{A}_{N+1}, \quad a \to t^{N+1}(t^N)^{-1}a \]
and the sequence

\[(A_s, i_s) := A_0 \overset{i_1}{\rightarrow} \ldots \overset{i_{N-1}}{\rightarrow} A_N \overset{i_N}{\rightarrow} A_{N+1} \overset{i_{N+1}}{\rightarrow} \ldots \]

The vector space limit of this sequence is in a canonical way a normed space and its closure has the structure of a \(C^*\)-algebra, the product being the limit \(N \rightarrow \infty\) of the products in the single algebras. This algebra can be shown to be isomorphic to the algebra of continuous functions on \(M\). Moreover the isomorphism can be chosen to be equivariant. In this sense the sequence approximates the \(\mathbb{CP}^2\). We therefore understand the algebras \(A_N\) as the algebra of functions on the "virtual" fuzzy \(\mathbb{CP}^2\) of order \(N\). This requires however to have as a "correspondence principle" the Toeplitz quantization maps or equivalently the above sequence in mind.

4. The Laplace Operator on the Fuzzy \(\mathbb{CP}^2\)

Since \(G\) acts on \(A_N\) by inner automorphisms we get by differentiation an action of the Lie algebra \(g^C = \text{sl}_3(\mathbb{C})\) by inner derivations. If \(\pi\) is the representation of \(g^C\) which corresponds to the \(g^C\)-module \(D(0, N)\) this action reads as follows

\[xa = [\pi(x), a]\quad x \in g^C, a \in A_N\]

This gives a representation of the universal enveloping algebra \(U(g^C)\) and we define in analogy to the commutative case the Laplace operator

\[\Delta_N : A_N \rightarrow A_N\]

to be the image of the quadratic Casimir element, or more explicitly:

\[\Delta_N a := \left( \sum_{\delta \in \Delta^+} (X_\delta X_{-\delta} + X_{-\delta} X_\delta) + \sum_{\delta \in \Delta^f} \frac{1}{2} H_\delta \right) a\]

Since the quantization map is equivariant we have

\[\Delta_N (a) = t^N (\Delta_M (t^N)^{-1}(a))\]

Using the decomposition of the \(g^C\)-module \(A_N\) one gets easily

**Theorem 4.1.** The eigenvalues \(\lambda\) of the Laplacian \(\Delta_N\) and their multiplicities \(\text{mult}_\lambda\) are:

\[\lambda_n = 2n(n + 2),\quad \text{mult}_{\lambda_n} = (n + 1)^3\quad n \in \{0, \ldots, N\}\]

The spectrum shows the expected UV cutoff.

The above action of \(g^C\) on \(A_N\) can be thought of as a Lie algebra homomorphism from \(g^C\) to the Lie algebra of derivations \(\text{Der}(A_N)\) of the algebra. The image, as a subspace of \(\text{Der}(A_N)\), gives rise to a derivation based differential calculus on \(A_N\) (see [4]).

With
the natural scalar product on the one-forms the Laplace operator has the natural form \( \Delta_N = d^*d \), as in the commutative case.

5. Quantization of homogeneous vector bundles over \( \mathbb{C}P^2 \)

Vector bundles in the noncommutative case should be finitly generated projective modules over the algebra. In [13] the concept of Toeplitz quantization was generalized to the quantization of bundles.

Let \( E \) be a holomorphic hermitian vector bundle with base \( M \) and \( E^* \) be the dual bundle. Let furthermore \( \mathcal{H}_E^N \) be the Hilbert space of holomorphic sections of the bundle \( E^* \otimes L^N \), that is \( \mathcal{H}_E^N := \Gamma_{hol}(E^* \otimes L^N) \). Using the orthogonal projection \( \Pi_N^E \in B(L^2(M, E^* \otimes L^N)) \)

\[ (23) \quad \Pi_N^E \in B(L^2(M, E^* \otimes L^N)) \]

onto \( \mathcal{H}_E^N \), we can define the surjective map

\[ (24) \quad T_N^E : \Gamma(M, E) \to \text{Hom}_\mathbb{C}(\mathcal{H}_E^N, \mathcal{H}^N), \quad \nu \to \Pi_N \nu \Pi_N^E \]

According to [12] this gives the quantization of the bundle \( E \) in the following sense

\[ (25) \quad \lim_{N \to \infty} \|T_N^E(f)T_N^E(\nu) - T_N^E(f\nu)\| = 0 \]

Clearly \( \mathcal{M}_N^E := \text{Hom}_\mathbb{C}(\mathcal{H}_E^N, \mathcal{H}^N) \) is a finitely generated projective left \( A_N \)-module.

Let \( \pi \) be an irreducible representation of \( K \) and \( E \) the associated homogeneous vector bundle over \( M \). If \( \pi \) has highest weight \( n_1 \lambda_1 + n_2 \lambda_2 \), the isotropy representation of \( E^* \otimes L^N \) has highest weight

\[ (26) \quad \gamma = n_1 \lambda_1 + (N - n_1 - n_2) \lambda_2 \]

Then by the Bott-Borel-Weyl theorem

\[ (27) \quad \Gamma_{hol}(E^* \otimes L^N) \cong D(n_1, N - n_1 - n_2) \]

as \( G \)-modules, if we set by definition \( D(n_1, n_2) = \{0\} \) if \( n_1 < 0 \) or \( n_2 < 0 \).

5.1. The Bundles \( E = L^n \). The isotropy representation of \( L^n \) was \( \tau(n) \), which had highest weight \( n \lambda_2 \), and one obtains

\[ (28) \quad \Gamma_{hol}(E^* \otimes L^N) \cong D(0, N - n) \]

The quantization of the bundles \( L^n \) is therefore given by the \( A_N \)-modules \( M_{L^n}^N \)

\[ (29) \quad M_{L^n}^N = \text{Hom}_\mathbb{C}(D(0, N - n), D(0, N)) \]
An easy calculation shows that as $G$-modules we have

\begin{equation}
M^N_{L^n} \cong \bigoplus_{k=0}^{N-n} D(k, k + n) \quad \text{for } n \geq 0
\end{equation}

and

\begin{equation}
M^N_{L^n} \cong \bigoplus_{k=0}^{N} D(k - n, k) \quad \text{for } n < 0
\end{equation}

5.2. **The Bundles** $E = L^n \otimes H^{-1}$. The isotropy representation of $L^n \otimes H^{-1}$ is $\tau(n) \otimes \chi^*$ with highest weight $\lambda_1 + n \lambda_2$. This yields

\begin{equation}
\Gamma_{hol}(E^* \otimes L^N) \cong D(1, N - n - 1).
\end{equation}

This gives as a quantization of $L^n \otimes H^{-1}$ the modules

\begin{equation}
M^N_{L^n \otimes H^{-1}} = \text{Hom}_\mathbb{C}(D(1, N - n - 1), D(0, N)).
\end{equation}

Again as $G$-modules

\begin{equation}
M^N_{L^n \otimes H^{-1}} \cong \bigoplus_{k=0}^{N-n-1} (D(k, k + n + 2) \oplus D(k + 1, k + n)) \quad \text{for } n \geq 0
\end{equation}

and

\begin{equation}
M^N_{L^n \otimes H^{-1}} \cong \bigoplus_{k=0}^{N} D(k - n - 1, k + 1) \oplus \bigoplus_{k=0}^{N-1} D(k - n + 1, k) \quad \text{for } n < 0
\end{equation}

6. **Quantized Spin$^C$-Bundles and a Dirac Operator**

In the classical case we had a family of Spin$^C$-bundles

\begin{equation}
S_m = S^+_m \oplus S^-_m
\end{equation}

over $M$, where

\begin{equation}
S^+_m = L^m \oplus L^{m+3}
\end{equation}

\begin{equation}
S^-_m = L^{m+1} \otimes H^{-1}.
\end{equation}

Furthermore we had the twisted Dolbeault complex

\begin{equation}
0 \longrightarrow \Gamma(L^m) \overset{\partial}{\longrightarrow} \Gamma(L^{m+1} \otimes H^{-1}) \overset{\partial}{\longrightarrow} \Gamma(L^{m+3}) \longrightarrow 0
\end{equation}

which we used to define the Dirac operator

\begin{equation}
D_m = \sqrt{2}(\partial + \partial^*) = \begin{pmatrix} 0 & \tilde{D}^*_m \\ \tilde{D}_m & 0 \end{pmatrix}
\end{equation}
where $\tilde{D}_m : \Gamma(S^+_m) \to \Gamma(S^-_m)$ was a $G$-equivariant map fulfilling $\tilde{D}_m^* \tilde{D}_m = \Delta_G - 2m - \frac{2}{3}m^2$. Let $T$ be the quantization maps defined earlier. We define the Spin$^C$-bundle in the quantum case to be the module

$$
\begin{align*}
\tilde{M}_N^S & = \tilde{M}_N^{S_m} \oplus \tilde{M}_N^{S_m} \\
\tilde{M}_N^{S_m} & = M_{L_m}^N \oplus M_{L_{m+3}}^N \\
\tilde{M}_N^S & = M_{L_{m+1} \otimes H^{-1}}^N 
\end{align*}
$$

(40)

In the following we consider the case $N \geq \max(2, m + 3)$.

**Theorem 6.1.** There is a unique linear map

$$
\overline{\partial}_N : \ M_N^S \to M_N^S
$$

such that the following diagram commutes

$$
\begin{array}{cccccc}
0 & \overset{\overline{\partial}}{\longrightarrow} & \Gamma(L^m) & \overset{\overline{\partial}}{\longrightarrow} & \Gamma(L^{m+1} \otimes H^{-1}) & \overset{\overline{\partial}}{\longrightarrow} & \Gamma(L^{m+3}) & \overset{\overline{\partial}}{\longrightarrow} & 0 \\
& & \downarrow T_{L_m}^N & & \downarrow T_{L_{m+1} \otimes H^{-1}}^N & & \downarrow T_{L_{m+3}}^N & & \\
0 & \overset{\overline{\partial}_N}{\longrightarrow} & M_{L_m}^N & \overset{\overline{\partial}_N}{\longrightarrow} & M_{L_{m+1} \otimes H^{-1}}^N & \overset{\overline{\partial}_N}{\longrightarrow} & M_{L_{m+3}}^N & \overset{\overline{\partial}_N}{\longrightarrow} & 0
\end{array}
$$

$\overline{\partial}_N$ is equivariant and satisfies $\overline{\partial}_N^2 = 0$.

Proof: Uniqueness is guaranteed by the surjectivity of the maps $T$. For existence it is enough to show that

$$
\begin{align*}
\overline{\partial}(\ker(T_{L_m}^N)) & \subset \ker(T_{L_{m+1} \otimes H^{-1}}^N) \\
\overline{\partial}(\ker(T_{L_{m+1} \otimes H^{-1}}^N)) & \subset \ker(T_{L_{m+3}}^N)
\end{align*}
$$

This is easily derived from the decomposition of the modules and the fact that $\overline{\partial}$ and the quantization maps are equivariant.

Explicitely the map is given by $\overline{\partial}_N f = T^N \overline{\partial}(T^N)^{-1}(f)$, where $(T^N)^{-1}(f)$ is an arbitrary element of the set $(T^N)^{-1}(\{f\})$.

**Definition 6.2.** The Spin$^C$-Dirac operator $D_m : M_N^S \to M_N^S$ is the self-adjoint operator, given by

$$
D_m = \sqrt{2}(\overline{\partial}_N + \overline{\partial}_N^*)
$$

As in the classical case we have

$$
D_m = \begin{pmatrix} 0 & \tilde{D}_m^* \\ \tilde{D}_m & 0 \end{pmatrix}
$$

and the map $\tilde{D}_m : M_N^{S_m} \to M_N^{S_m}$ is equivariant. We clearly have
Lemma 6.3.

\[ \tilde{D}_m^* \tilde{D}_m|_{\ker(D_m)^\perp} = (\Delta - 2m - \frac{2}{3}m^2)|_{\ker(D_m)^\perp} \]

which gives, using the decomposition of the G-modules \( M^N_{S_m} \) and \( M^N_{S_m} \):

Theorem 6.4. The eigenvalues \( \lambda \) of \( D_m \) and their multiplicities \( \text{mult}_\lambda \) are:

1. \( \lambda^\pm_m = \pm \sqrt{2n(n + 2) + 2|m|(n + 1) - 2m} \)
   \[ \text{mult}_{\lambda^\pm_m} = (n + \frac{|m| + 2}{2})(n + |m| + 1)(n + 1) \]
   \( n \in \{1, \ldots, N - m - 1\} \) if \( m \geq 0 \)
   \( n \in \{0, \ldots, N - 1\} \) if \( m < 0 \)

2. \( \lambda^\pm_m = \pm \sqrt{2n(n + 2) + 2|m + 3|(n + 1) + 2(m + 3)} \)
   \[ \text{mult}_{\lambda^\pm_m} = (n + \frac{|m + 3| + 2}{2})(n + |m + 3| + 1)(n + 1) \]
   \( n \in \{0, \ldots, N - m - 3\} \) if \( m \geq -2 \)
   \( n \in \{1, \ldots, N\} \) if \( m < -2 \)

3. \( \lambda = 0 \)
   \[ \text{mult}_\lambda = \frac{N+2}{2}(m^2 - 3mN + 2N^2 + 2N + 2) \]

index\((D_m)\) = dim(\(\ker(D_m)\)) − dim(\(\ker(D^*_m)\)) = \( \frac{3}{2}(N^2 + 3N + 2) \).

Remark 6.5. The index and the number of zero modes are different from the commutative case. That is because the maps \( \overline{\partial}_N \) vanish also on higher representations. Apart from this the spectrum is truncated, as we expected.

The Dirac operator defines for each \( m \in \mathbb{N} \) and \( N \geq \max(2, m + 3) \) an even spectral triple \((\mathcal{A}, \mathcal{H}, D)\) with

\[ \mathcal{A} = \mathcal{A}_N \]
\[ \mathcal{H} = M^N_{S_m} \]
\[ D = D_m \]

which gives a differential calculus over the algebra \( \mathcal{A}_N \).

6.1. A different choice of bundle. Taking for the space of spinor fields instead of \[(40)\]

\[ M^N_{S_m} = M^N_{S_m} \oplus M^N_{S_m} \]
\[ M^N_{S_m} = M^N_{L_m} \oplus M^{N+2}_{L^{m+3}} \]
\[ M^N_{S_m} = M^{N+1}_{L^{m+1} \otimes H^{-1}} \]

we can use the same procedure to define a Dirac operator, which except for a spectral cutoff has the same spectrum as the classical Dirac operator. In particular the index
and zero modes are classical. In this case we lose the $\mathcal{A}_N$-module property, the space is however a module over the algebra $\mathcal{A}_N \oplus \mathcal{A}_{N+2} \oplus \mathcal{A}_{N+1}$. This will be investigated elsewhere.

7. Conclusions and Outlook

The tools provided in this paper make it possible to investigate scalar QFT on the fuzzy $\mathbb{C}P^2$, which is free of any divergencies, since the algebra of functions is finite dimensional and allows to construct a well defined functional integral. The attempt to construct spinors led to similar difficulties as for the known case of the fuzzy sphere. There the use of supersymmetry provided a solution (\cite{9, 10}). It seems promising to use a similar extension for the $\mathbb{C}P^2$. Noncommutative generalizations of the Dolbeault operators might as well lead to structures proposed in \cite{6}.

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