General-relativistic perturbation equations for the dynamics of elastic deformable astronomical bodies expanded in terms of generalized spherical harmonics

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In our previous paper, based on the Carter & Quintana framework and the Damour-Soffel-Xu scheme, we deduced a complete and closed set of post-Newtonian dynamical equations for elastically deformable astronomical bodies. In this paper, we expand the general relativistic perturbation equations of elastic deformable bodies (field equations, stress-strain relation, Euler equation) in terms of Generalized Spherical Harmonics. This turns the set of complicated partial differential equations into a set of ordinary differential equations. This will be useful for numerical applications that mainly deal with the global dynamics of the Earth.

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I. INTRODUCTION

In a series of papers [1, 2] the dynamical equations for elastic deformable astronomical bodies in the framework of Einstein’s theory of gravity (GRT) were derived. Fields of application range from problems of seismology of astronomical bodies including the problem of normal modes to the global motion of such bodies in space. Actually one of our main interests is the problem of global geodynamics where the accuracy of modern geodetic space techniques such as VLBI has long reached the level where effects from relativity have to be taken into account. Unfortunately the field of relativistic global geodynamics has not yet reached a satisfactory level despite the fact that published amplitudes of nutation series are of $\mu$as (microarcsec) accuracies.

This paper presents another necessary step to improve this situation. It extends our previous work mentioned above that was based upon two frameworks: (1) the Carter-Quintana formalism [3, 4] for the description of elastic deformable bodies in GRT by means of a displacement field and (2) the Damour-Soffel-Xu one (the DSX scheme [5, 6, 7, 8]) on relativistic celestial mechanics in the first post-Newtonian approximation to GRT.

Here we shall expand the general relativistic perturbation equations for the dynamics of elastic deformable bodies in terms of Generalized Spherical Harmonics (GSH). This turns the set of partial differential equations for scalars, vectors and tensors into a set of ordinary differential equations. Actually in the Newtonian approximation numerical programs that have been written mainly to deal with the global dynamics of the Earth (e.g., Wahr [9], Dehant [10], Schastok [11]) by considering its elastic properties in a local framework usually integrate such (Newtonian) equations. In general relativity, the general expansion for vectors and tensors by means of GSH has been discussed in several papers [12, 13]. Here, we apply these methods and expand the general relativistic dynamical equations for elastic deformable, nonrotating astronomical bodies by means of GSH.

In the following we consider the dynamics of one of these bodies in its own local coordinate system $(cT, X)$ restricting ourselves to a spherically symmetric non-rotating relaxed ground state. The extension to some rotating axially symmetric ground state with nonvanishing dynamical ellipticity so that problems of precession and nutation can be treated will be the subject of another paper. In Sec. II, we briefly review the main results in Ref.[1]. We also rewrite the equations and introduce suitable notation in order to facilitate the comparison
with the corresponding Newtonian equations (Ref. [9]).

In Sec. III, Generalized Spherical Harmonics are introduced. Our main new results are presented in Sec. IV, where the post Newtonian perturbation equations (PDE) are expanded so that they reduce to a set of ordinary differential equations (ODE). In Sec. V some conclusion can be found.

II. PERTURBED FIELD EQUATION AND PN EULER EQUATION

Let us first recall some relations that will be relevant for this paper. In the DSX-formalism, the Einstein field equations to first post-Newtonian order can be written as [5]

\[ \nabla^2 W - \frac{1}{c^2} \frac{\partial^2 W}{\partial T^2} = -4\pi G \Sigma + O(4), \]
\[ \nabla^2 W^a = -4\pi G \Sigma^a + O(2), \]

where \( W \) and \( W^a \) are the scalar and vector potentials that describe the gravitational interaction, and the gravitational mass-density and mass-current-density, \( \Sigma \) and \( \Sigma^a \), are related with the energy momentum tensor by \( \Sigma = (T^{00} + T^{aa})/c^2 \), \( \Sigma^a = T^{0a}/c \). We shall often abbreviate the order symbol \( O(c^{-n}) \) simply by \( O(n) \).

For a non-rotating static spherically symmetric body (unperturbed state), the energy-momentum tensor takes the form

\[ T^{00} = \rho c^2 \left( 1 + \frac{2W}{c^2} \right) + O(2), \]
\[ T^{aa} = 3p \left( 1 - \frac{2W}{c^2} \right) + O(4), \]
\[ T^{0a} = O(3), \]

where \( \rho c^2 \) is the energy density and \( p \) is the pressure. The field equations reduce to

\[ \nabla^2 W = -4\pi G \left[ \rho^* \left( 1 + \frac{2W}{c^2} \right) + \frac{2p}{c^2} \right] + O(4), \]
\[ \nabla^2 W^a = O(2), \]

where \( \rho^* = \rho + \frac{p}{c^2} \) is the chemical potential per unit volume. For a static equilibrium configuration the scalar potential \( W \) is a function of the radial coordinate \( r \) only, and the gravito-magnetic potential \( W^a \) vanishes.
Now we consider a perturbed state. The perturbed field equations for the Eulerian variations of gravitational potentials, $\delta W$ and $\delta W^a$ read

$$\nabla^2 \delta W - \frac{1}{c^2} \frac{\partial^2 \delta W}{\partial T^2} = -4\pi G \delta \Sigma + O(4), \quad (2.8)$$
$$\nabla^2 \delta W^a = -4\pi G \delta \Sigma^a + O(2), \quad (2.9)$$

where, by using Eqs.(4.27), (4.31), (4.40) and (4.41) of our previous paper (Ref.[1]),

$$\delta \Sigma = \delta \rho + \frac{1}{c^2} (2\rho \delta W + 2W \delta \rho + 3\delta p)$$
$$= -\nabla \cdot (\rho s) - \frac{1}{c^2} [\nabla \cdot (\rho s) + 3\rho s \cdot \nabla W + \rho \delta W + 2\nabla \cdot (\rho W s) + 3\kappa \nabla \cdot s] + O(4), \quad (2.10)$$

$$\delta \Sigma^a = \rho s^a + O(2) \quad (2.11)$$

and $\kappa$ is the compression modulus. Then the field equations can be expressed as

$$\nabla^2 \delta W - \frac{1}{c^2} \frac{\partial^2 \delta W}{\partial T^2} = 4\pi G \{ \nabla \cdot (\rho s) + \frac{1}{c^2} [\nabla \cdot (\rho s) + 3\rho s \cdot \nabla W + \rho \delta W$$
$$+ 2\nabla \cdot (\rho W s) + 3\kappa \nabla \cdot s] \} + O(4), \quad (2.12)$$

$$\nabla^2 \delta W^a = -4\pi G \rho s^a + O(2). \quad (2.13)$$

The perturbed PN Euler equation (for non-rotating, static and spherically symmetric ground state) reads (Eq.(4.32) of Ref. [1] for $\Omega = 0$)

$$0 = \rho^* s_a \left(1 + \frac{2W}{c^2}\right) + \rho^* \Theta W_a - \rho^* s^b_a W_{,b} - \rho^* \delta W_{,a} - \rho^* s^b_{ba}$$
$$- (\kappa \Theta)_{,a} - (2\mu s^\beta_{a,\beta}) - \frac{1}{c^2} \left[-4\rho^*(\delta W_{,a})_{,T} + \kappa \Theta W_{,a} + 4(\kappa W \Theta)_{,a} \right] + O(4), \quad (2.14)$$

where $\mu$ is the shear modulus, and

$$(2\mu s^\beta_{a,\beta}) = (2\mu s_{ba})_{,b} + \frac{1}{c^2} \left[-(4\mu W s_{ba})_{,b} + 4\mu W_{,c} s_{ac} \right] + O(4). \quad (2.15)$$

The shear modulus $\mu$ and the compression modulus $\kappa$ are related with the Lamé parameter $\lambda$ by the relation

$$\kappa = \lambda + 2\mu/3. \quad (2.16)$$

The shear tensor $s_{ab}$ and the volume dilatation $\Theta$ are given by

$$s_{ab} = \left(1 + \frac{2W}{c^2}\right) \left[s^{(a, b)} - \frac{1}{3} s^{k, k} s_{ab}\right] + O(4), \quad (2.17)$$
$$\Theta = s^k_{,k} + \frac{1}{c^2} \left[3W_{,k} s^k + 3\delta W\right] = \nabla \cdot s + \frac{3}{c^2} (\nabla W \cdot s + \delta W) + O(4). \quad (2.18)$$

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The gravitational scalar potential $W$ and the vector potential $W^a$ can be decomposed into a sum of two contributions (see paper [5])

$$W = W^+ + \mathbf{W}, \quad W_a = W_a^+ + \mathbf{W}_a,$$  \hspace{1cm} (2.19)

where $W^+$ and $W_a^+$ are the self parts (resulting from the gravitational action of the body under consideration) and $\mathbf{W}$ and $\mathbf{W}_a$ are the external parts (describing tidal and inertial forces) of the metric potentials. Since we take the elastic mechanical ground state as an isolated body the external potentials vanish for this state

$$\mathbf{W} = \mathbf{W}_a = 0, \quad W = W^+, \quad W_a = W_a^+.$$ \hspace{1cm} (2.20)

Similarly for the perturbed state we write

$$\delta W = \delta W^+ + \delta \mathbf{W}, \quad \delta W_a = \delta W_a^+ + \delta \mathbf{W}_a.$$ \hspace{1cm} (2.21)

The external parts of metric potentials, $(\delta \mathbf{W}, \delta \mathbf{W}_a)$, that result from the ephemerides of external bodies are assumed to be known, so only the Eulerian variations of the internal potentials have to be determined self-consistently by partial differential equations. The perturbed Euler equation can be rewritten as

$$\rho^* f^a = \rho^* s^a \left(1 + \frac{4W^+}{c^2}\right) + \rho^* \Theta W_a^+ - \rho^* s^b a W_b^+ - \rho^* s^b W_a^+$$

$$-(\kappa \Theta)_a - (2\mu s_{ba})_b + \frac{1}{c^2} \left[ (4\mu W^+ s_{ba})_b - 4\mu s_{ac} W_a^+ - 4\rho^* (\delta W_a^+)_T ight]$$

$$+ \kappa \Theta W_a^+ + 4(\kappa W^+ \Theta)_a,$$ \hspace{1cm} (2.22)

where

$$f^a = \delta \mathbf{W}_a + \frac{4}{c^2} (\delta \mathbf{W}_a)_T$$ \hspace{1cm} (2.23)

is the external tidal force density.

To facilitate the comparison with well known results from Newtonian theory we write the equations in a form that was employed in Ref. [9]. To this end we first introduce the post-Newtonian Cauchy elastic stress tensor $T^{ab}$:

$$T^{ab} = \kappa \Theta \delta_{ab} + 2\mu s_{ab} - \frac{4W^+}{c^2} \left( \mu s_{ab} + \kappa \Theta \delta_{ab} \right) + O(4).$$ \hspace{1cm} (2.24)

Then Eq.(2.22) takes a form that can easily be compared with the results from Wahr’s paper (Ref.[9])

$$\left(1 + \frac{4W^+}{c^2}\right) \rho^* \ddot{s} = - \left[ \rho^* \Theta \nabla W^+ - \rho^* \nabla W^+ \cdot (\nabla s)^T - \rho^* \nabla \delta W^+ - \rho^* s \cdot \nabla (\nabla W^+) \right]$$
\[+\nabla \cdot \vec{T} + \frac{1}{c^2} \left\{ -\kappa \Phi \nabla W^+ + 2\mu \nabla W^+ \cdot \left[ (\nabla s + (\nabla s)^T) - \frac{2}{3} \nabla \cdot \vec{s} \right] \right\} + \frac{4}{c^2} \rho^* (\delta W^+).T + \rho^* f + O(4), \]  

where \( \vec{I} \) is the second rank identity tensor and the superscript \( T \) denotes the transpose (not to be confused with the local time variable).

Let us now introduce the following notation generalizing the one from [9]:

\[
\begin{align*}
W^+(r) &= -\Phi(r) \quad \text{the potential of the ground state} \\
\delta W^+ &= -\phi_1^E \quad \text{the incremental Eulerian gravitational potential energy} \\
\delta \overrightarrow{W} &= -\phi_T \quad \text{the tidal potential} \\
(\delta W^+).T &= \dot{\vec{A}}^a \quad \text{the time derivative of the self part of vector potential} \\
(\delta \overrightarrow{W}).T &= \dot{\overrightarrow{A}}^a \quad \text{the time derivative of the external part of vector potential} \\
f &= \nabla \delta \overrightarrow{W} + \frac{1}{c^2} (\delta \overrightarrow{W}).T = -\nabla \phi_T + \frac{4}{c^2} \overrightarrow{A} \\
& \quad \text{the tidal force}
\end{align*}
\]  

(2.26)

Going into Fourier space with local time variable \( T \) and replacing \( \partial/\partial T \) by \( i\omega \) the first perturbed field equation (Eq.(2.12)) becomes

\[
\left( \nabla^2 + \frac{\omega^2}{c^2} \right) \delta W^+ = 4\pi G \left\{ \nabla \cdot (\rho s) + \frac{1}{c^2} \left[ \nabla \cdot (\rho s) + 3\rho s \cdot \nabla W^+ + \rho (\delta W^+ + \delta \overrightarrow{W}) \right. \right. \\
\left. \left. + 2\nabla \cdot (\rho W^+ s) + 3\kappa \nabla \cdot s \right] \right\} + O(4),
\]

(2.27)

since the tidal potential \( \delta \overrightarrow{W} \) satisfies D’Alembert’s equation.

The second perturbed field equation (Eq.(2.13)), for the same reason, takes the form

\[
\nabla^2 (\delta W^+).T \equiv \nabla^2 \dot{\vec{A}}^a = 4\pi G \omega^2 \rho s^a + O(2).
\]

(2.28)

By using the definition above (Eqs.(2.26)), the Eq.(2.27) can be rewritten as

\[
- \left( \nabla^2 + \frac{\omega^2}{c^2} \right) \phi_1^E = 4\pi G \left\{ \nabla \cdot (\rho^* s) + \frac{1}{c^2} \left[ -5\rho^* s \cdot \nabla \Phi - \rho^* \phi_1^E - \rho^* \phi_T \\
- 2\Phi \nabla \cdot (\rho^* s) + 3\kappa \nabla \cdot s \right] \right\} + O(4).
\]

(2.29)

The perturbed Euler equation (Eq.(2.25)) takes the form

\[
- \left( 1 - \frac{4\Phi}{c^2} \right) \rho^* \omega^2 s = - \left[ \rho^* \nabla \phi_1^E + \rho^* \nabla \Phi \cdot (\nabla s)^T + \rho^* s \cdot \nabla (\nabla \Phi) \right. \\
\left. \quad - \rho^* \Theta \nabla \Phi \right] + \nabla \cdot \vec{T} + \frac{1}{c^2} \left\{ \kappa \Theta \nabla \Phi - 2\mu \nabla \Phi \cdot \left[ (\nabla s + (\nabla s)^T) - \frac{2}{3} \nabla \cdot \vec{s} \right] \right\} + \frac{4}{c^2} \rho^* (\dot{\vec{A}} + \overrightarrow{A}) - \rho^* \nabla \phi_T + O(4),
\]

(2.30)
where
\[ T^{ab} = \kappa \Theta \delta_{ab} + 2 \mu s_{ab} + \frac{4 \Phi}{c^2} (\mu s_{ab} + \kappa \Theta \delta_{ab}) + O(4), \] (2.31)
and
\[ \Theta = \nabla \cdot s - \frac{3}{c^2} (\nabla \Phi \cdot s + \phi^E_1 + \phi_T) + O(4). \] (2.32)

Eqs.(2.28), (2.29), (2.30) and (2.31) are the ones that will now be expanded in terms of
generalized spherical harmonics (scalar-, vector- and tensor spherical harmonics).

III. GENERALIZED SPHERICAL HARMONICS

Eqs.(2.28)—(2.31) are complicated partial differential equations (PDE) that will be
turned into a set of ordinary differential equations (ODE) by means of expansions in terms
of scalar-, vector- and tensor spherical harmonics or, briefly, in terms of generalized spherical
harmonics that are described e.g., by Phinney & Burridge [14], Smith [15] and Wahr [9]. Here tensors of arbitrary rank are represented by their components along the complex basis
\[ e_-, e_0, e_+ \], where
\[ e_- = \frac{1}{\sqrt{2}}(e_\theta - i e_\phi), \quad e_0 = e_r, \quad e_+ = -\frac{1}{\sqrt{2}}(e_\theta + i e_\phi) \] (3.1)
and \( e_r, e_\theta \) and \( e_\phi \) are (Euclidean) unit vectors in \( r, \theta \) and \( \phi \) direction. Components of a
tensor field along the complex basis are called canonical components and the basis itself is
called canonical basis. Generalized spherical harmonics (GSH) are denoted by \( D^l_{mn}(\theta, \phi) \)
and defined by
\[ D^l_{mn}(\theta, \phi) = (-1)^{m+n} P^m_l(\cos \theta) \exp(i m \phi), \] (3.2)
where \( P^m_l \) is a generalized associated Legendre function
\[ P^m_l(x) = \frac{(-1)^{l-n}}{2^l(l-n)!} \left[ \frac{(l-n)!(l+m)!}{(l+n)!(l-m)!} \right]^{1/2} (1-x)^{(n-m)/2} (1+x)^{-(m+n)/2} \times \left( \frac{d}{dx} \right)^{l-m} \left( (1-x)^{-n}(1+x)^{l+n} \right). \] (3.3)
\( D^l_{mn} \) are defined only for \( l \geq 0 \) and \( |m| \leq l \) (see, e.g., [14] and [15] for a complete discussion).

By using GSH, a scalar, vector and 2nd rank tensor can be expanded as
\[ \phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi^m_l(r) D^l_{mn}(\theta, \phi), \] (3.4)
\[ \mathbf{u}(r, \theta, \phi) = \sum_{n=-1}^{+1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{mn}^{l}(r) D_{mn}^{l}(\theta, \phi) \mathbf{e}_{n}, \quad (3.5) \]

\[ \Tilde{T}(r, \theta, \phi) = \sum_{a,b=-1}^{+1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_{l}^{mab}(r) D_{m(a+b)}^{l}(\theta, \phi) \mathbf{e}_{a} \mathbf{e}_{b}. \quad (3.6) \]

The \( D_{m0}^{l} \) are proportional to the usual spherical harmonics (\( D_{m0}^{l} = (-1)^{m} \sqrt{4\pi/(2l+1)} Y_{m}^{l} \)) and the \( D_{mn}^{l}(\theta, \phi) \) are similar to the \( Y_{mn}^{l}(\theta, \phi) \) of paper [14] (\( D_{mn}^{l} = (-1)^{m+n} Y_{mn}^{l} \)). Some useful formulas for objects involving GSH can be found in the Appendix.

IV. POST-NEWTONIAN PERTURBATION EQUATIONS

In this section we will rewrite Eqs.(2.28), (2.29), (2.30) and (2.31) in a compact form by using GSH. First we expand the scalars, vectors and tensors appearing in these equations in the following way

\[ \phi_{E}^{l}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{m}^{l}(r) D_{m0}^{l}(\theta, \phi), \quad (4.1) \]

\[ \phi_{T}(r, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} T_{\phi_{m}}^{l}(r) D_{m0}^{l}(\theta, \phi) \quad (4.2) \]

\[ s_{a}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} s_{ma}^{l}(r) D_{ma}^{l}(\theta, \phi), \quad (4.3) \]

\[ A_{a}^{l}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{ma}^{l}(r) D_{ma}^{l}(\theta, \phi), \quad (4.4) \]

\[ \mathbf{T}_{ab}^{l}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_{l}^{mab}(r) D_{m(a+b)}^{l}(\theta, \phi). \quad (4.5) \]

Here \( s_{a} \) denotes the displacement field, \( \mathbf{T}_{ab}^{l} \) the incremental Cauchy elastic stress tensor \((a, b = -1, 0, +1)\). The gravitational potential \( \Phi \) of the ground (equilibrium) state should not be expanded, since it is a known function of the radial coordinate \( r \).

A. Field equations

By using the expansions from Eqs.(4.1)—(4.3), the formulas given in the Appendix and the orthogonality properties of \( D_{mn}^{l} \) for each appropriate value of \( l, m \) and \( n \), the Eq.(2.29)
reduces to

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{\omega^2}{c^2} \right) \phi_i^m \\
= -4\pi G \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) \left( \rho^* s_i^{m0} \right) + \frac{\rho^*}{r} \sqrt{\frac{l(l+1)}{2}} \left( s_i^{m+} + s_i^{m-} \right) \right] \\
+ \frac{4\pi G}{c^2} \left[ 5\rho^* \left( \frac{d}{dr} \Phi \right) s_i^{m0} + \rho^* \left( \phi_i^m + T \phi_i^m \right) + (2\rho^* \Phi - 3\kappa) \left( \frac{d}{dr} + \frac{2}{r} \right) s_i^{m0} \right] \\
+ (2\rho^* \Phi - 3\kappa) \frac{1}{r} \sqrt{\frac{l(l+1)}{2}} \left( s_i^{m+} + s_i^{m-} \right) + 2(\Phi \frac{d}{dr} \rho^*) s_i^{m0}. \tag{4.7}
\]

To write this equation in a more compact form it is useful to define new scalar functions ([9])

\[
\begin{align*}
U_i^m &= s_i^{m0} \\
V_i^m &= s_i^{m+} + s_i^{m-} \\
W_i^m &= s_i^{m+} - s_i^{m-} \\
P_i^m &= T_i^{m00} \\
Q_i^m &= T_i^{m0+} + T_i^{m0-} \\
R_i^m &= T_i^{m0+} - T_i^{m0-}
\end{align*}
\]

Let

\[
\frac{d}{dr} g_i^m = \frac{d}{dr} \phi_i^m + 4\pi G \rho^* U_i^m, \tag{4.9}
\]

then

\[
\frac{d^2}{dr^2} \phi_i^m = \frac{d}{dr} g_i^m - 4\pi G \frac{d}{dr} \left( \rho^* U_i^m \right). \tag{4.10}
\]

Substituting Eqs.(4.8), (4.9) and (4.10) into Eq.(4.7), we get

\[
\frac{d}{dr} g_i^m = \left( \frac{l(l+1)}{r^2} - \frac{\omega^2}{c^2} \right) \phi_i^m - 2 g_i^m - 4\pi G \frac{d}{dr} \rho^* L_1 V_i^m + 4\pi G \left( 5\rho^* g_0 U_i^m + \rho^* \left( \phi_i^m + T \phi_i^m \right) \right) \\
+ (2\rho^* \Phi - 3\kappa) \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) U_i^m + \frac{L_1}{r} V_i^m \right] + 2\Phi \left( \frac{d}{dr} \rho^* \right) U_i^m, \tag{4.11}
\]

where

\[
g_0 \equiv \frac{d}{dr} \Phi, \tag{4.12}
\]

\[
L_1 \equiv \sqrt{\frac{l(l+1)}{2}}. \tag{4.13}
\]

In the Newtonian limit Eq.(4.9) and Eq.(4.11) are correspond to Eq.(III.33) and Eq.(III.34) of [9] respectively.
Next we consider the second perturbed field equation Eq.(2.28). Expand \( \dot{A}^{(-/0/+)} \) according to Eq.(4.4), i.e.,

\[
\dot{A}^{(-/0/+)}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \dot{A}_l^{m(-/0/+)}(r) D_{m-1}^l(\theta, \phi). \tag{4.14}
\]

For the explicit calculation we have

\[
\left( \nabla^2 \dot{A} \right)^a = \left( \nabla \cdot \nabla \dot{A} \right)^a \equiv \xi^a. \tag{4.15}
\]

Since \( \nabla \dot{A} \) is not a symmetric second rank tensor so that we cannot use Phinney’s formulas directly. We obtain (for more details see the Appendix)

\[
\xi^0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_1^2) \right] \dot{A}_l^m + \frac{2L_1}{r^2} (\dot{A}_l^m + \dot{A}_l^m) \right\} D_{m0}^l, \tag{4.16}
\]

\[
\xi^- = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_2^2) \right] \dot{A}_l^m + \frac{2L_1}{r^2} \dot{A}_l^m \right\} D_{m-1}^l, \tag{4.17}
\]

\[
\xi^+ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_2^2) \right] \dot{A}_l^m + \frac{2L_1}{r^2} \dot{A}_l^m \right\} D_{m+1}^l, \tag{4.18}
\]

where \( L_1 \) is defined in Eq.(4.13), and

\[
L_2 = \sqrt{\frac{(l-1)(l+2)}{2}}. \tag{4.19}
\]

The expansion of \( s^a \) takes the form

\[
s^{(-/0/+)}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} s_l^{m(-/0/+)}(r) D_{m(1/0/+)}^l(\theta, \phi). \tag{4.20}
\]

Using this expansion in the field equations (2.28) and the orthogonality relations of \( D_{mn}^l \) one finds

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_1^2) \right] \dot{B}_l^m = \frac{2L_1}{r^2} \dot{E}_l^m = 4\pi G \omega^2 \rho U_l^m, \tag{4.21}
\]

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_2^2) \right] \dot{E}_l^m - \frac{4L_1}{r^2} \dot{B}_l^m = 4\pi G \omega^2 \rho V_l^m, \tag{4.22}
\]

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^2}(1 + L_2^2) \right] \dot{H}_l^m = 4\pi G \omega^2 \rho W_l^m. \tag{4.23}
\]
with the definition
\begin{align}
\dot{A}_m^0 &= \dot{B}_1^m \\
\dot{A}_m^0 + \dot{A}_m^- &= \dot{E}_1^m \\
\dot{A}_m^0 - \dot{A}_m^- &= \dot{H}_1^m \\
\mathcal{A}_m^0 &= \mathcal{E}_1^m \\
\mathcal{A}_m^0 + \mathcal{A}_m^- &= \mathcal{E}_1^m \\
\mathcal{A}_m^0 - \mathcal{A}_m^- &= \mathcal{T}_1^m \\
\end{align}
\tag{4.24}

(The definitions for \(\mathcal{E}_1^m\), \(\mathcal{H}_1^m\) and \(\mathcal{T}_1^m\) will be used in Section IV.C.) For numerical applications it is useful to reduce the set of second order differential equations (Eq.(4.21)—(4.23)) to an equivalent set of first order. With
\begin{align}
\frac{d}{dr} b_i^m &= b_i^m, \\
\frac{d}{dr} e_i^m &= e_i^m, \\
\frac{d}{dr} h_i^m &= h_i^m, \\
\end{align}
\tag{4.25-4.27}

the post-Newtonian field equations, Eqs.(4.21) - (4.23), finally take the form
\begin{align}
\frac{d}{dr} b_i^m &= -\frac{2}{r} b_i^m + \frac{2}{r^2} (1 + L_1^2) \dot{b}_i^m + \frac{2L_1}{r^2} \dot{E}_1^m + 4\pi G \omega^2 \rho U_i^m, \\
\frac{d}{dr} e_i^m &= -\frac{2}{r} e_i^m + \frac{2}{r^2} (1 + L_2^2) \dot{e}_i^m + \frac{4L_1}{r^2} \dot{B}_1^m + 4\pi G \omega^2 \rho V_i^m, \\
\frac{d}{dr} h_i^m &= -\frac{2}{r} h_i^m + \frac{2}{r^2} (1 + L_2^2) \dot{h}_i^m + 4\pi G \omega^2 \rho W_i^m. \\
\end{align}
\tag{4.28-4.30}

Compared with the Newtonian case we have six additional equations (Eqs.(4.25—4.30)) and six additional unknown functions \(b_i^m\), \(e_i^m\), \(h_i^m\), \(\dot{B}_1^m\), \(\dot{E}_1^m\), \(\dot{H}_1^m\) that only appear in the post-Newtonian formalism.

\section*{B. Stress-strain relation}

According to Eqs.(2.31), (2.32) and (2.17), and the stress tensor takes the form
\begin{equation}
\mathbf{T} = \lambda \nabla \mathbf{s} \mathbf{I} + \mu \left[ \nabla \mathbf{s} + (\nabla \mathbf{s})^T \right] + \frac{\kappa}{c^2} \left[ 4\Phi \nabla \mathbf{s} - 3(\nabla \Phi \cdot \mathbf{s} + \phi_E + \phi_T) \right] \mathbf{I}. 
\tag{4.31}
\end{equation}

Using the GSH expansion for the Cauchy elastic stress-tensor one finds that
\begin{align}
\mathbf{T}^{ab}(r, \theta, \phi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_l^{mab}(r) D_m(a+b)(\theta, \phi) \\
&= \lambda \nabla \cdot \mathbf{s} e^{ab} + \mu (s^{a,b} + s^{b,a}) + \frac{\kappa}{c^2} \left[ 4\Phi \nabla \mathbf{s} - 3(\nabla \Phi \cdot \mathbf{s} + \phi_E^a + \phi_T^a) \right] e^{ab}, 
\tag{4.32}
\end{align}

\begin{align}
\mathbf{T}^{ab}(r, \theta, \phi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_l^{mab}(r) D_m(a+b)(\theta, \phi) \\
&= \lambda \nabla \cdot \mathbf{s} e^{ab} + \mu (s^{a,b} + s^{b,a}) + \frac{\kappa}{c^2} \left[ 4\Phi \nabla \mathbf{s} - 3(\nabla \Phi \cdot \mathbf{s} + \phi_E^a + \phi_T^a) \right] e^{ab}, 
\tag{4.32}
\end{align}
where $e_{ab}$ and $e^{ab}$ are the canonical components of the identity tensor and are equal to

$$
(e_{ab}) = (e^{ab}) = \begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix}
$$

$a, b = -, 0, +$  \hspace{1cm} (4.33)

Using

$$
\nabla \mathbf{s} = s^{c,d}e_{cd} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{d}{dr} \left( \frac{2}{r} \right) s_{lm}^{m} + \frac{1}{r} \sqrt{\frac{l(l+1)}{2}} (s_{l}^{m+} + s_{l}^{m-}) \right] D_{i}^{0m},
$$

$$
\nabla \Phi \cdot \mathbf{s} = \Phi^{c} s^{d} e_{cd} = \Phi^{0} s^{0},
$$

and the orthogonality properties of $D_{mn}^{l}$ we get

$$
T_{lm}^{m-} = -\frac{2\mu}{r} L_{2} s_{l}^{m-},
$$

$$
T_{lm}^{m0} = T_{l}^{m0} = \mu \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) s_{l}^{m-} - \frac{1}{r} L_{1} s_{l}^{m0} \right],
$$

$$
T_{lm}^{00} = \left( \lambda + \frac{4\kappa}{c^{2}} \Phi \right) \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) s_{l}^{m0} + \frac{L_{1}}{r} (s_{l}^{m+} + s_{l}^{m-}) \right]
+ 2\mu \left( \frac{d}{dr} s_{l}^{m0} \right) - \frac{3\kappa}{c^{2}} \left( \frac{d}{dr} \Phi \right) s_{l}^{m0} + \phi_{l}^{m} + T \phi_{l}^{m},
$$

$$
T_{lm}^{m+} = T_{l}^{m+} = -\left( \lambda + \frac{4\kappa}{c^{2}} \Phi \right) \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) s_{l}^{m0} + \frac{L_{1}}{r} (s_{l}^{m+} + s_{l}^{m-}) \right]
- \mu \left[ \frac{L_{1}}{r} (s_{l}^{m+} + s_{l}^{m-}) + \frac{2}{r} s_{l}^{m0} \right] + \frac{3\kappa}{c^{2}} \left( \frac{d}{dr} \Phi \right) s_{l}^{m0} + \phi_{l}^{m} + T \phi_{l}^{m},
$$

$$
T_{lm}^{0+} = T_{l}^{0+} = \mu \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) s_{l}^{m+} - \frac{L_{1}}{r} s_{l}^{m0} \right],
$$

$$
T_{lm}^{++} = -\frac{2\mu}{r} L_{2} s_{l}^{m+}.
$$

The relation for $T_{lm}^{00} \equiv P_{l}^{m}$ can be written in the form

$$
\left[ (\lambda + 2\mu) + \frac{4\kappa}{c^{2}} \Phi \right] \frac{d}{dr} U_{l}^{m} = P_{l}^{m} - \left( \lambda + \frac{4\kappa}{c^{2}} \Phi \right) \left( \frac{2}{r} U_{l}^{m} + \frac{L_{1}}{r} V_{l}^{m} \right) + \frac{3\kappa}{c^{2}} (g_{0} U_{l}^{m} + \phi_{l}^{m} + T \phi_{l}^{m}),
$$

\hspace{1cm} (4.42)

which is the post-Newtonian version of Eq.(III.35) from Wahr (1982).

By means of Eqs.(4.37) and (4.40) we get

$$
\frac{d}{dr} V_{l}^{m} = Q_{l}^{m} + \frac{\mu}{r} (V_{l}^{m} + 2L_{1} U_{l}^{m}),
$$

$$
\frac{d}{dr} W_{l}^{m} = R_{l}^{m} + \frac{\mu}{r} W_{l}^{m},
$$

\hspace{1cm} (4.43)

that generalizes Wahr’s Eqs.(III.36) and (III.37) to the post-Newtonian level. Eqs.\hspace{1cm}(4.42),

\hspace{1cm}(4.43) and (4.44) are the post-Newtonian version of the stress-strain relation.
C. Euler equation (Conservation of momentum)

According to Eqs.(2.30) and (2.32), in the complex basis \( \mathbf{e}_- \), \( \mathbf{e}_0 \), \( \mathbf{e}_+ \) the perturbed Euler equation can be expressed as

\[
- \left( 1 - \frac{4\Phi}{c^2} \right) \rho^* \omega^2 s^a = - \left\{ \rho^* \phi^E_{i,a} + \rho^* \Phi^b s^c d e_{bc} + \rho^* s^b \phi^E_{i,a} e_{bc} \right. \\
\left. - \rho^* \left[ (\nabla \cdot s) - \frac{3}{c^2} \left( \Phi^c s^d e_{cd} + \phi^E_{i,T} + \phi^E_{j,T} \right) \Phi^a \right] + \zeta^a - \rho^* \phi^a_{i,T} + \frac{4}{c^2} \rho^* (A^a + \overline{A}^a) \right. \\
\left. + \frac{1}{c^2} \left\{ \kappa \Phi^a (\nabla \cdot s) - 2\mu \left[ \phi^g (s^{a,b} + s^{b,a}) e_{gb} - \frac{2}{3} \Phi^a (\nabla \cdot s) \right] \right\},
\]

(4.45)

where \( \zeta^a = \left( \nabla \cdot \overline{T}^a \right)^a = T^{ab,c} e_{bc} \). This relation comprises the three equations for \( a = (-, 0, +) \).

By means of Eqs.(4.36) and (4.39), the \( a = - \) equation can be written in the form

\[
\frac{d}{dr} T^{m0-}_l = - \frac{3}{r} T^{m0-}_l + \frac{2\mu}{r^2} L^2 s^m_l + \frac{L_1}{r} \left\{ \frac{\mu + \lambda}{r} L_1 (s^m_l + s^m_{i,T}) + 2s^m_l + \lambda \frac{d}{dr} s^m_l \right\} \\
- \frac{L_1}{r} \rho^* (g_0 s^m_l + \phi^m_i + T \phi^m_i) - \rho^* \omega^2 s^m_l \\
+ \frac{1}{c^2} \left\{ \frac{4\kappa}{r} \Phi L_1 \left[ \frac{L_1}{r} (s^m_l + s^m_{i,T}) + \left( \frac{d}{dr} + \frac{2}{r} \right) s^m_l \right] \\
- 3\kappa \left[ \left( g_0 s^m_l + \phi^m_i + T \phi^m_i \right) + 4\Phi \rho^* \omega^2 s^m_l \right. \\
\left. + 2\mu g_0 \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) s^m_l - \frac{1}{r} L_1 s^m_l \right] - 4\rho^* (A^m_l + \overline{A}^m_l) \right\}.
\]

(4.46)

The \( a = 0 \) equation, by using Eq.(4.39), can be written as

\[
\frac{d}{dr} T^{m00}_l = - \frac{2}{r} T^{m00}_l - \frac{L_1}{r} (T^{m0+}_l + T^{m0-}_l) + \frac{2}{r} \left\{ \frac{\mu + \lambda}{r} L_1 (s^m_l + s^m_{i,T}) + 2s^m_l + \lambda \frac{d}{dr} s^m_l \right\} \\
- \rho^* \omega^2 s^m_l + \rho^* \left( \frac{d}{dr} \phi^m_i + 4\pi G \rho^* s^m_l \right) - \frac{4}{r} \rho^* g_0 s^m_l - \frac{L_1}{r} \rho^* g_0 (s^m_l + s^m_{i,T}) + \rho \frac{d}{dr} (T \phi^m_i) \\
+ \frac{1}{c^2} \left\{ \frac{8\kappa}{r} \Phi - (\lambda + 2\mu) g_0 \left[ \frac{d}{dr} + \frac{2}{r} \right] s^m_l + \left[ \frac{8\kappa}{r} \Phi - (\lambda + 2\mu) g_0 \right] \frac{L_1}{r} (s^m_l + s^m_{i,T}) \\
+ 4\mu g_0 \frac{d}{dr} s^m_l - 4\rho^* (A^m_l + \overline{A}^m_l) + 3 \left( \rho^* g_0 - \frac{2\kappa}{r} \right) (g_0 s^m_l + \phi^m_i + T \phi^m_i) \right. \\
\left. + 4\Phi \rho^* \omega^2 s^m_l + 8\pi G (p - \rho^* \Phi) \right\},
\]

(4.47)
where we have used Eq.(2.6). The pressure $p$ is determined from the equation of state $p = p(\rho)$ that is assumed to be a known function inside the reference body.

Finally, the $a = +$ equation can be brought into the form

$$
\frac{d}{dr} T_i^{m0+} = -3 \frac{r}{r} T_i^{m0+} + 2 \frac{\mu}{r^2} L_i^{2s_m+} + \frac{L_1}{r} \left( \frac{\mu + \lambda}{r} \left[ L_i^{s_m+} + s_i^{m-} \right] + 2 s_i^{m0} \right) + \mu \frac{d}{dr} s_i^{m0} \right) \\
- \frac{L_1}{r} \rho^* (g_0 s_i^{m0} + \phi_i^{m} + T \phi_i^{m}) - \rho^* \omega^2 s_i^{m+} \\
+ \frac{1}{c^2} \left\{ \frac{4 \kappa}{r} \Phi L_1 \left( \frac{L_i}{r} (s_i^{m+} + s_i^{m-}) + \left( \frac{d}{dr} + \frac{2}{r} \right) s_i^{m0} \right) \\
- \frac{3 \kappa}{r} L_1 (g_0 s_i^{m0} + \phi_i^{m} + T \phi_i^{m}) + 4 \Phi \rho^* \omega^2 s_i^{m+} \\
+ 2 \mu g_0 \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) s_i^{m+} - \frac{1}{r} L_i s_i^{m0} \right] - 4 \rho^* (A_i^{m+} + \overline{A}_i^{m+}) \right\}. \tag{4.48}
$$

We will also rewrite these PN Euler equations to facilitate the comparison with the corresponding Newtonian equations. With the definitions from Eqs.(4.8), (4.9) and (4.24), the $a = 0$ Equation (4.47) can be rewritten as

$$
\frac{d}{dr} P_i^m = -2 \frac{r}{r} P_i^m - \frac{L_1}{r} Q_i^m + \frac{2}{r} \left[ \frac{\mu + \lambda}{r} \left( L_i V_i^m + 2 U_i^m \right) + \lambda \frac{d}{dr} U_i^m \right] \\
- \rho^* \omega^2 U_i^m + \rho^* g_i^m - \frac{4}{r} \rho^* g_0 U_i^m - \frac{L_1}{r} \rho^* g_0 V_i^m + \rho^* \frac{d}{dr} T \phi_i^{m} \\
+ \frac{1}{c^2} \left\{ \frac{8 \kappa}{r} \Phi - (\lambda + 2 \mu) g_0 \right\} \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) U_i^m + \frac{L_1}{r} V_i^m \right] + 4 \mu g_0 \frac{d}{dr} U_i^m \\
+ 3 \left( \rho^* g_0 - \frac{2 \kappa}{r} \right) (g_0 U_i^m + \phi_i^m + T \phi_i^{m}) + 4 \Phi \rho^* \omega^2 U_i^m + 8 \pi G (p - \rho^* \Phi) \\
- 4 \rho^* \left( \overline{B}_i^m + \overline{B}_i^{m} \right) \right\}, \tag{4.49}
$$

generalizing Wahr’s Equation (III.30).

Similarly, combining Eq.(4.48) with (4.46) we get

$$
\frac{d}{dr} Q_i^m = -3 \frac{r}{r} Q_i^m + 2 \frac{\mu}{r^2} L_i^{2V_i^m} + \frac{2 L_i}{r} \left[ \frac{\mu + \lambda}{r} \left( L_i V_i^m + 2 U_i^m \right) + \lambda \frac{d}{dr} U_i^m \right] \\
- \rho^* \omega^2 V_i^m - \frac{2 L_1}{r} \rho^* (g_0 U_i^m + \phi_i^m + T \phi_i^{m}) \\
+ \frac{1}{c^2} \left\{ \frac{8 \kappa}{r} \Phi L_1 \left[ \frac{L_i}{r} V_i^m + \left( \frac{d}{dr} + \frac{2}{r} \right) U_i^m \right] - \frac{6 \kappa}{r} L_1 (g_0 U_i^m + \phi_i^m + T \phi_i^{m}) \\
+ 2 \mu g_0 \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) V_i^m - \frac{2 L_i}{r} U_i^m \right] + 4 \Phi \rho^* \omega^2 V_i^m - 4 \rho^* \left( \overline{B}_i^m + \overline{E}_i^m \right) \right\}. \tag{4.50}
$$
which is the post-Newtonian version of Wahr’s Eq.(III.31).

Wahr’s Eq.(III.32) is generalized if we combine Eq.(4.48) with (4.46) (using Eq.(4.44))

\[
\frac{d}{dr} R_{l}^{m} = - \left( \frac{3}{r} - \frac{2g_0}{c^2} \right) R_{l}^{m} + \frac{2\mu L^2}{r^2} W_{l}^{m} - \left( 1 - \frac{4\Phi}{c^2} \right) \rho^* \omega^2 W_{l}^{m} - \frac{4\rho^*}{c^2} (H_{l}^{m} + \bar{H}_{l}^{m}).
\]

(4.51)

Up to now, we have fourteen equations (Eqs.(4.9), (4.11), (4.25), (4.26), (4.27), (4.28), (4.29), (4.30), (4.42), (4.43), (4.44), (4.49), (4.50) and (4.51)) for the determination of fourteen unknown functions \( (U_{l}^{m}, V_{l}^{m}, W_{l}^{m}, g_{l}^{m}, \phi_{l}^{m}, P_{l}^{m}, Q_{l}^{m}, R_{l}^{m}, B_{l}^{m}, E_{l}^{m}, H_{l}^{m}, b_{l}^{m}, e_{l}^{m}, h_{l}^{m}) \) in contrast to eight equations and eight unknown functions in the Newtonian limit (Ref.[9]).

V. CONCLUSION

In this paper we have successfully derived the post-Newtonian equations for the dynamics of some elastic deformable astronomical body expanded in terms of Generalized Spherical Harmonics(GSH). We have shown that in the Newtonian limit they reduce to the corresponding Newtonian equations that can be found in Wahr’s paper[9]. The importance to separate a set of partial differential equation (PDE) into a set of ordinary differential equation (ODE) is obvious for numerical calculations. In that form the equations of motion can be directly used in numerical computer programs so that the orders of magnitude of the various relativistic terms for concrete problems can be assessed.

In this paper we considered a non-rotating body as reference state for our perturbation theory. Since the precessional and nutational motion of some astronomical body depends upon its dynamical ellipticity that is related with its rotational motion due to the inertial forces, this formalism cannot be applied directly to problems of precession and nutation. The derivation of corresponding equations and junction conditions represented in terms of GSH for a rotating elastomechanical ground state will be the subject of a forthcoming paper.

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Based on the results of Phinney’s paper [14], we derive here some useful formulas for computing the perturbed PN equations. The expansion of scalars, vectors and tensors is given in Eqs. (4.1)–(4.6). For simplicity, in the following we will omit the summation symbols $\sum_{l=0}^{\infty} \sum_{m=-l}^{l}$.

1. **Gradient and Laplacian of a scalar $\Phi(r, \theta, \phi)$**

By means of GSH, a scalar can be expanded as

$$\Phi(r, \theta, \phi) = \phi_i^m(r) D_{m0}^l(\theta, \phi)$$

and the gradient is given by

$$\nabla \Phi = \Phi^\alpha$$

where

$$\Phi^\pm = - \frac{L_1}{r} \phi_i^m D_{m\pm 1}^l \quad (A.1)$$

$$\Phi^0 = \frac{d}{dr} \phi_i^m D_{m0}^l \quad (A.2)$$

The Laplacian of $\Phi$ can be written as

$$\nabla^2 \Phi = \Phi^{\alpha\beta} \epsilon_{\alpha\beta} = \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] \phi_i^m D_{m0}^l \quad (A.3)$$

where $\epsilon_{\alpha\beta}$ are the canonical components of the identity tensor, its value is given by Eq.(4.33). For simplicity, we represent $[\nabla(\nabla \Phi)]^{\alpha\beta}$ as $\Phi^{\alpha\beta}$, then the second partial derivatives of $\Phi$ are given by

$$\Phi^{--} = \frac{1}{r^2} L_1 L_2 \phi_i^m D_{m-2}^l \quad (A.4)$$

$$\Phi^{-0} = \phi^{0-} = \left( \frac{L_1}{r^2} \phi_i^m - \frac{L_1}{r} \frac{d}{dr} \phi_i^m \right) D_{m-1}^l \quad (A.5)$$

$$\Phi^{--} = \phi^{+0} = \left( \frac{L_2}{r^2} \phi_i^m - \frac{1}{r} \frac{d}{dr} \phi_i^m \right) D_{m0}^l \quad (A.6)$$

$$\Phi^{00} = \frac{d^2}{dr^2} \phi_i^m D_{m0}^l \quad (A.7)$$

$$\Phi^{0+} = \phi^{+0} = \left( \frac{L_1}{r^2} \phi_i^m - \frac{L_1}{r} \frac{d}{dr} \phi_i^m \right) D_{m+1}^l \quad (A.8)$$

$$\Phi^{++} = \frac{L_1 L_2}{r^2} \phi_i^m D_{m+2}^l \quad (A.9)$$
where
\[ L_1 \equiv \sqrt{\frac{l(l+1)}{2}}, \quad (A.10) \]
\[ L_2 \equiv \sqrt{\frac{(l-1)(l+2)}{2}}. \quad (A.11) \]

2. Operations involving a scalar \( \Phi(r) \)

If a scalar is a function of the radial coordinate \( r \) only, i.e., \( \Phi = \Phi(r) \), we have

\[ \Phi^\pm = 0, \quad (A.12) \]
\[ \Phi^0 = \frac{d}{dr} \Phi, \quad (A.13) \]
\[ \nabla^2 \Phi = \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \Phi, \quad (A.14) \]
\[ \Phi^{--} = \Phi^{++} = \Phi^{-0} = \Phi^{0-} = \Phi^{+0} = \Phi^{0+} = 0, \quad (A.15) \]
\[ \Phi^{-+} = \phi^{+-} = -\frac{1}{r} \frac{d}{dr} \Phi, \quad (A.16) \]
\[ \Phi^{00} = \frac{d^2}{dr^2} \Phi. \quad (A.17) \]

3. Gradient and divergence of a vector-field

The expansion of a vector is given by Eq.(4.3), i.e.,

\[ u^\alpha(r, \theta, \phi) = U^{m\alpha}_l(r) D^{l}_{m\alpha}(\theta, \phi). \quad (\alpha = -1, 0, +1) \quad (A.18) \]

We write the vector gradient as

\[ u^{\alpha\beta}(r, \theta, \phi) = U^{m\alpha\beta}_l(r) D^{l}_{m(\alpha+\beta)}(\theta, \phi), \quad (A.19) \]

where

\[ U^{m\alpha\beta}_l(r) = \begin{cases} -\frac{1}{r} \left( L_{\alpha} U^{m\alpha}_l + \epsilon_{\alpha-} (-1)^\alpha U^{m0}_l + \delta_{0\alpha} (-1)^\alpha U^{m-}_l \right), & \beta = -1 \\ \frac{d}{dr} U^{m\alpha}_l, & \beta = 0 \quad (A.20) \\ -\frac{1}{r} \left( L_{\alpha+1} U^{m\alpha}_l + \epsilon_{\alpha+} (-1)^\alpha U^{m0}_l + \delta_{0\alpha} (-1)^\alpha U^{m+}_l \right), & \beta = +1 \end{cases} \]

where

\[ L_0 = L_1, \quad L_{-1} = L_2. \quad (A.21) \]
The symmetric vector gradient can be written as
\[
\frac{1}{2} (u^{+,+} + u^{-,-}) = -\frac{1}{2r} \left[ L_1 (U_i^{m+} + U_i^{m-}) + 2U_i^{m0} \right] D_{m0}^l, \tag{A.22}
\]
\[
\frac{1}{2} (u^{0,0} + u^{0,a}) = \frac{1}{2} \left[ \left( \frac{d}{dr} - \frac{1}{r} \right) U_i^{ma} - \frac{L_1}{r} U_i^{m0} \right] D_{ma}^l. \quad (\alpha = \pm 1) \tag{A.23}
\]

The divergence of a vector-fields is expanded in the form
\[
\nabla \cdot \mathbf{u} = u^{\alpha,\beta} e_{\alpha\beta} = \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) U_i^{m0} + \frac{1}{r} \sqrt{\frac{l(l+1)}{2}} \left( U_i^{m+} + U_i^{m-} \right) \right] D_{m0}^l. \tag{A.24}
\]

4. Divergence of a symmetric, second order tensor-field

For a symmetric tensor-field \( T^{\alpha\beta} \), the divergence, \( (\nabla \cdot \mathbf{T})^\alpha = T^{\alpha\beta,\gamma} e_{\beta\gamma} = \xi^\alpha \), is given by
\[
\xi^0 = \left[ \left( \frac{d}{dr} + \frac{2}{r} \right) T_i^{m00} + \frac{2}{r} T_i^{m+-} + \frac{1}{r} \sqrt{\frac{l(l+1)}{2}} \left( T_i^{m0+} + T_i^{m0-} \right) \right] D_{m0}^l, \tag{A.25}
\]
\[
\xi^\alpha = \left[ \left( \frac{d}{dr} + \frac{3}{r} \right) T_i^{m0\alpha} + \frac{1}{r} \sqrt{\frac{l(l+1)}{2}} T_i^{m+-} + \frac{1}{r} \sqrt{\frac{(l-1)(l+2)}{2}} T_i^{m0\alpha} \right] D_{ma}^l. \quad (\alpha = \pm 1) \tag{A.26}
\]

5. Laplacian of a vector-field

The Laplacian of a vector-field can be written in the form
\[
(\nabla^2 A)^\alpha = (A^{\alpha,\beta,\gamma} + A^{\beta,\alpha,\gamma} - A^{\beta,\gamma,\alpha}) e_{\beta\gamma} \equiv \xi^\alpha, \quad (\alpha = -1, 0, +1) \tag{A.27}
\]
where \( A^{\alpha,\beta} \) is an unsymmetric second order tensor. In terms of GSH, a tensor-field of second rank can be expanded as
\[
m^{\alpha\beta}(r, \theta, \phi) = M_i^{m\alpha\beta}(r) D_{m(\alpha+\beta)}(\theta, \phi). \quad (\alpha, \beta = -1, 0, +1)
\]
Differentiation yields
\[
m^{\alpha\beta,\gamma}(r, \theta, \phi) = M_i^{m\alpha\beta\gamma}(r) D_{m(\alpha+\beta+\gamma)}(\theta, \phi),
\]
where the calculation of the quantity \( M_i^{m\alpha\beta\gamma} \) is similar to Eq.(2.13) of Ref.[14], note that \( D_{mn}^l = (-1)^{m+n} Y_i^{mn} \). Some terms, used in this paper, read
\[
M_i^{m000} = \frac{d}{dr} M_i^{m00}, \tag{A.28}
\]

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\[ M_{l}^{m_{0}+} = -\frac{1}{r} \left( L_{1}M_{l}^{m_{0}-} + M_{l}^{m+} + M_{l}^{m00} \right), \quad (A.29) \]
\[ M_{l}^{m_{0}-} = -\frac{1}{r} \left( L_{1}M_{l}^{m0+} + M_{l}^{m-} + M_{l}^{m00} \right), \quad (A.30) \]
\[ M_{l}^{m_{0}0} = \frac{d}{dr} M_{l}^{m_{0}0}, \quad (A.31) \]
\[ M_{l}^{m_{0}-} = -\frac{1}{r} \left( L_{2}M_{l}^{m_{0}-} + M_{l}^{m_{0}0} \right), \quad (A.32) \]
\[ M_{l}^{m_{0}+} = -\frac{1}{r} \left( L_{1}M_{l}^{m_{0}+} + M_{l}^{m_{0}0} \right), \quad (A.33) \]
\[ M_{l}^{m_{0}+0} = \frac{d}{dr} M_{l}^{m_{0}+0}, \quad (A.34) \]
\[ M_{l}^{m_{0}+} = -\frac{1}{r} \left( L_{1}M_{l}^{m_{0}+} + M_{l}^{m_{0}+0} \right), \quad (A.35) \]
\[ M_{l}^{m_{0}+} = -\frac{1}{r} \left( L_{2}M_{l}^{m_{0}+} + M_{l}^{m_{0}+0} + M_{l}^{m_{0}0} \right). \quad (A.36) \]

Now we have

\[ A^{\alpha}(r, \theta, \phi) = A_{l}^{m_{\alpha}}(r)D_{m_{\alpha}}^{l}(\theta, \phi), \]
\[ A^{\alpha,\beta}(r, \theta, \phi) = A_{l}^{m_{\alpha}\beta}(r)D_{m_{\alpha+\beta}}^{l}(\theta, \phi). \]

Set

\[ m_{\alpha,\beta}(r, \theta, \phi) = A^{\alpha,\beta} = M_{l}^{m_{\alpha}\beta}(r)D_{m_{\alpha+\beta}}^{l}(\theta, \phi), \quad (\alpha, \beta = -1, 0, +1) \quad (A.37) \]

then

\[ (A^{\alpha,\beta})_{\gamma}(r, \theta, \phi) = m_{\alpha,\beta,\gamma}(r, \theta, \phi) = M_{l}^{m_{\alpha}\beta\gamma}(r)D_{m_{\alpha+\beta+\gamma}}^{l}(\theta, \phi). \]

The quantities \( M_{l}^{m_{\alpha}\beta\gamma}(r) \) are derived from Eqs.\( (A.28), (A.29), (A.30), (A.31), (A.32), (A.33), (A.34), (A.35) \) and \( (A.36) \), and \( M_{l}^{m_{\alpha}\beta}(r) = A_{l}^{m_{\alpha}\beta}(r) \) can be calculated by Eq.\( (A.20) \), then we can deduce

\[ \xi^{0} = \left\{ \frac{d}{dr} \right\}^{2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^{2}} \left( 1 + L_{2}^{2} \right) \frac{A_{l}^{m0}}{A_{l}^{m_{0}0}} - \frac{2L_{1}}{r^{2}} \left( A_{l}^{m+} + A_{l}^{m_{0}} \right) \frac{D_{m0}^{l}}{D_{m0}^{l}}, \quad (A.38) \]
\[ \xi^{-} = \left\{ \frac{d}{dr} \right\}^{2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^{2}} \left( 1 + L_{2}^{2} \right) \frac{A_{l}^{m_{0}0}}{A_{l}^{m_{0}0}} - \frac{2L_{1}}{r^{2}} \frac{A_{l}^{m_{0}0}}{A_{l}^{m_{0}0}} \frac{D_{m-1}^{l}}{D_{m-1}^{l}}, \quad (A.39) \]
\[ \xi^{+} = \left\{ \frac{d}{dr} \right\}^{2} + \frac{2}{r} \frac{d}{dr} - \frac{2}{r^{2}} \left( 1 + L_{2}^{2} \right) \frac{A_{l}^{m_{0}+}}{A_{l}^{m_{0}+}} - \frac{2L_{1}}{r^{2}} \frac{A_{l}^{m_{0}+}}{A_{l}^{m_{0}+}} \frac{D_{m+1}^{l}}{D_{m+1}^{l}}, \quad (A.40) \]

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