Torsional chiral magnetic effect in Weyl semimetal with topological defect

Hiroaki Sumiyoshi$^1$ and Satoshi Fujimoto$^2$

$^1$Department of Physics, Kyoto University, Kyoto 606-8502, Japan and
$^2$Department of Materials Engineering Science, Osaka University, Toyonaka 560-8531, Japan

(Dated: April 1, 2016)

We propose a torsional response raised by lattice dislocation in Weyl semimetals akin to chiral magnetic effect; i.e. a fictitious magnetic field arising from screw or edge dislocation induces charge current. We demonstrate that, in sharp contrast to the usual chiral magnetic effect which vanishes in real solid state materials, the torsional chiral magnetic effect exists even for realistic lattice models, which implies the experimental detection of the effect via SQUID or nonlocal resistivity measurements in Weyl semimetal materials.

PACS numbers: 72.80.-r, 72.15.-v, 11.30.Rd, 11.15.Yc

Recently, many candidate materials for Dirac semimetals and Weyl semimetals (WSMs) [1–4], have been discovered [5–23]. These topological semimetals are intriguing because of exotic transport phenomena associated with the chiral anomaly in quantum field theory [24], such as the anomalous Hall effect [25, 26], chiral magnetic effect (CME) [27], negative longitudinal magnetoresistance [5, 12, 19–21, 28, 29], and chiral gauge field [30].

Among them, the CME has been discussed in broad areas of quantum many-body physics, including nuclear and nonequilibrium physics as well as condensed matter physics. It is the generation of charge current parallel to an applied magnetic field even in the absence of electric fields. In nuclear physics, together with the chiral vortical effect [31], it is expected to play an important role in heavy ion collisions experiments [32, 33]. The CME also caused a stir in nonequilibrium statistical physics, since it leads to the existence of the ground state which, recently, attracts a renewed interest in connection with the realization of quantum time crystal [34], and then the CME has been studied from this point of view [35, 36]. However, unfortunately, their results are negative for its realization: the macroscopic ground state current in realistic WSMs is always absent.

In this letter, we propose a chiral response in WSMs, named “torsional chiral magnetic effect (TCME)”, in which the ground state charge current is caused by the effective magnetic field induced by lattice dislocation as shown in FIG.1. By using the Cartan formalism of the differential geometry, we can describe the lattice strain and dislocation in terms of vielbein and torsion [37]. From the viewpoint of the quantum field theory in curved space-time, the TCME is raised by the mixed action of electromagnetic and torsional fields that is prohibited in four-dimensional spacetime with the Lorentz symmetry, but made possible in non-relativistic band electrons in solid state systems. Furthermore, we demonstrate that the TCME is possible in realistic lattice models by carrying out numerical calculations. Our results imply the existence of experimentally observable current induced by the TCME in real WSM materials. We also resolve the relation between our results and the no-go theorem that the CME is absent in equilibrium states [35, 36]. First of all, we clarify the notations. The indices $i,j, \cdots = x, y, z$ and $a, b, \cdots = \bar{x}, \bar{y}, \bar{z}$ represent the coordinates in the laboratory and local orthogonal (or Lorentz) frames, respectively. In the following, we use the Einstein summation convention.

Linear response theory for torsional response—Here, we briefly introduce the Cartan formalism, which can be applied to description of crystal systems with lattice strain as follows. It is an approach to curved space and based on the local orthonormal frame form $e^a = e^a_i(r)dr^i$, where the coefficient fields $e^a_i(r)$ are referred to as the vielbein [38]. We introduce the coordinate measured by an observer on the deformed lattice $R^a$ and the laboratory coordinate $r^i$, and identify its exterior derivative as the local orthonormal frame $e^a = dR^a$. Then, to the first order in the displacement field $\tilde{u}$, the vielbein is written as $e^a_i = \delta^a_i - \partial u^a_i/\partial r^i$. For this observer, the lattice is not deformed, and then the Hamiltonian of the system is given by $H(-\partial R^a, -\partial R^a, -\partial R^a) = H,-ie^a_{\bar{x}}\partial x^a_{\bar{r}}, -ie^a_{\bar{y}}\partial y^a_{\bar{r}}, -ie^a_{\bar{z}}\partial z^a_{\bar{r}}$, where $H(p_x, p_y, p_z)$ is the Hamiltonian without lattice deformation and $e^a_i$ is the inverse of $e^a_{\bar{r}}$. In this way, the emergent vielbein appears, and therefore we can describe the elastic response by using the Cartan formalism. The coupling between the vielbein and electrons is similar to the minimal coupling of the $U(1)$ gauge field, $p_{\bar{r}} \rightarrow p_{\bar{r}} - eA_{\bar{r}}$. Then, we can define the analog of the field strength by $T^a_{\bar{r}} = \partial_i e^a_{\bar{r}} - \partial_j e^a_{\bar{r}}$, which is referred to as the torsion, or “torsional magnetic field” (TMF) [39–41], where the spin connection is dropped for simplicity. Using the displacement vector, the torsion is rewritten as $T^a_{\bar{r}} = (\partial_i \partial_j - \partial_j \partial_i)u^a$. The point is that, if $u^a(r)$ is a well-defined function, the torsion is always zero, and the multivaluedness of $u^a(r)$ is necessary for nonzero torsion. Indeed, the edge dislocation along $z$-axis with Burgers vector $b_\gamma \hat{y}$ causes the TMF, $T^y_{\bar{x}} = -b_\gamma \delta^{(2)}(x, y)$, and the screw one with $b_\gamma \hat{z}$, $T^z_{\bar{x}} = -b_\gamma \delta^{(2)}(x, y)$, as shown in FIG.1. For more details about the lattice strain and differential geometry, see, for example, Refs. [42–44].
Now, using the linear response theory with the Cartan formalism, we investigate the TCME of WSMs due to dislocation. We calculate the current density in the presence of TMF and magnetic field up to the linear order. We use the model of a pair of Weyl fermions with the opposite chirality, whose Weyl points are at $k = \lambda^L$ and $\lambda^R$ in the momentum space, and Fermi energies are given by $E = \nu_F \lambda^L_0$ and $\nu_F \lambda^R_0$, respectively. Therefore the $4 \times 4$ Hamiltonian is given by

$$H(k) := \begin{pmatrix} H_L(k) & 0 \\ 0 & H_R(k) \end{pmatrix}$$

with $H_s(k) := \nu_F [\chi_s(k - \lambda^s \cdot \sigma - \lambda^0_0)$, where $s = L$ or $R$ is the index of the chirality and $\chi_{L(R)} = +1(-1)$, and $\sigma^i$ is the Pauli matrix. Well, we calculate the current density in the presence of the external fields. The calculation is performed by the variation of the effective action, $S_{\text{eff}}[A_i, e_i^a]$, with respect to the gauge field, as $j^a(r) := -\langle e^a_i(r)/|e_i(r)| \rangle (\delta S_{\text{eff}}/\delta A_i(r))$. The effective action is defined as

$$e^{-S_{\text{eff}}[A_i, e_i^a]} := \int D\psi D\bar\psi \psi\bar\psi \text{exp} \left(-S[\psi \bar\psi, A_i, e_i^a]\right),$$

$$S[\psi, \bar\psi, A_i, e_i^a] := \frac{1}{2} \int d^3r \left[ \psi^\dagger(\tau, r) \tilde{\mathcal{L}} \psi(\tau, r) + \text{c.c.} \right],$$

$$\tilde{\mathcal{L}} := |e_i(r)| |\partial/\partial \tau - H(-i\nabla_a)|,$$

where $\psi$ is the fermionic field, $\tau$ and $r$ denote the imaginary time and spatial coordinate, respectively, and $A_i$ is the vector potential. Here c.c. represents the complex conjugate combined with the change of the sign of the derivative operator $\partial/\partial \tau$. Also, in Eq.(2), the Jacobian is given by $|e_i(r)| := \text{det} e_i^a(r)$, and the covariant derivative is $-i\nabla_a := e_i^a(r)(-id_i - e_{A_i}(r))$ with $a = \bar{i}$. Using Eq.(2), we obtain that the current density up to the first order of the magnetic field and the TMF is given by

$$j(r) = \left[ \frac{\nu_F (\lambda^R - \lambda^L_0)}{4\pi^2} B + \frac{\nu_F (\lambda^L - \lambda^R_0)}{4\pi^2} T^a \right].$$

at zero temperature and up to the linear order in $\lambda^{L(R)}$, where the details of the calculations are described in Ref.[45]. Here, The vector representation of the TMF, $T^a$, is defined by $(T^a)_i := (1/2)e^{ijk}T^a_k$. For the derivation of Eq.(3), we introduced a momentum cutoff scheme $|k - \lambda^s| < \Lambda$ for the Weyl node of the chirality $s$. Physically, $\Lambda$ corresponds to the momentum range from the Weyl points in which the cone structures of the band of the lattice system is approved.

The first term represents the CME in the presence of the chiral chemical potential (i.e. $\lambda^L_0 \neq \lambda^R_0$), and then reproduces the previous result for the CME [46]. On the other hand, the second term in Eq.(3) is a new one, which raises the TCME; i.e. the current is generated by the TMF for the pair of Weyl points which are shifted in the momentum space due to broken time-reversal symmetry (TRS). This point is in sharp contrast to the usual CME, which requires breaking inversion symmetry.

We comment on the relation between the TCME and the chiral anomaly. One may expect that when $\lambda^L_0 = -\lambda^R_0$, the TCME is described by the topological $\theta$-term, which is the consequence of the chiral anomaly like the CME and anomalous Hall effect. However, there is no mixed chiral anomaly term of $U(1)$ field strength and torsion in four-dimensional spacetime [40, 47]. This point is resolved by the observation that the Lorentz symmetry, which is postulated in the calculation scheme in Refs. [40, 47], is broken in the cutoff scheme used for the derivation of the second term of Eq.(3), which is correctly applicable to realistic condensed matter systems.

Now we discuss the consequences and physical pictures of the TCME. The TCME is realized in two types of lattice dislocations. (a) case of edge dislocation: $j^z \propto \Delta \lambda_2 T^z_2$, and (b) case of screw dislocation: $j^z \propto \Delta \lambda_3 T^z_3$, with $\Delta \lambda_\alpha := \lambda^L_\alpha - \lambda^R_\alpha$. Their schematic pictures are shown in FIG.1. These responses can be understood with the following semiclassical picture: Case (a): Edge dislocation is regarded as the (0,1,0)-“surface” of the extra lattice plane made up of the blue and green atoms in FIG.1-(a), which harbors a chiral Fermi arc, when two Weyl points are shifted in the $k_z$-direction. The electrons in the Fermi arc state are the very origin of the current induced by the edge dislocation. Case (b): There is a chiral Fermi arc mode on the dislocation line. The electrons in the mode rotate around the screw dislocation line, and due to the screw dislocation the rotating motion causes the current along the Burgers vector.

The situation is similar to that of the three-dimensional integer quantum Hall state (3D IQHS) [48] with dislocation which is the staking of quantum Hall state layers characterized by the vector $G_e = (2\pi n_c/a)\hat{n}$, where $n_c$, $a$, and $\hat{n}$ are the first Chern number, the lattice constant, and the unit vector along the staking direction. In the 3D IQHS, there is one dimensional $n$ chiral modes along the dislocation line, when the topological number, $n = b_y \cdot G_e/2\pi$, is nonzero [49]. This condition for the chiral modes is similar to that for the TCME, $b_y \cdot (\lambda^L - \lambda^R) \neq 0$. 

![FIG. 1. (Color Online) Ground state current $j$ induced by (a) edge and (b) screw dislocation with the Burgers vector $b$.](image-url)
However, there are the following significant differences. The chiral modes of the 3D IQHS are exponentially localized at the dislocation, and separated from the bulk higher-energy states, while, as will be shown below, the chiral modes of the WSMs exhibit power-law decay. We call them quasi-localized modes. Moreover their spectrum is not isolated from the bulk spectrum but appears as its envelope as shown in FIG.2a, and therefore they can be easily mixed with the bulk modes.

**Spectral asymmetry and ground state current**— We confirm the TCME due to dislocation by using an alternative approach other than the linear response theory based on (3). Our approach here is to calculate the expression obtained directly from the linear response theory [45]. The schematic picture of the density-of-state compared with that of the opposite energy density-of-state for the WSMs [35] generalized to the case with dislocation, $E = \pm m_{k_z}^s$, represents the relatively higher (lower) density-of-state compared with that of the opposite energy $E = \mp m_{k_z}^s$.

![FIG. 2. (Color Online) (a) Schematic picture of the spectrum of the WSM with the screw dislocation. The black (white) bands along $E = \pm m_{k_z}^s$ represent the relatively higher (lower) density-of-state compared with that of the opposite energy $E = \mp m_{k_z}^s$. (b) Lattice with a pair of screw dislocations with opposite Burgers vectors. (c) Numerical result for the spectrum of the WSMs with a pair of screw dislocations. The blue curves are the envelope of the bulk spectrum. The opacity of the dots represents the expectation value of $|\rho - l^d|/L_z L_y$ (see the gray scale bar). In the figures (c1-3), the modes with this value smaller than 0.1, 0.15, and 0.2 are plotted. (d) Current density along z-direction, $j_z(\mathbf{x}, y)$.](image)
When $x = 0$, $-l_{\text{dis}}^x < y < l_{\text{dis}}^x$, while $\Theta(r) = 0$ for other regions. We numerically diagonalized this model and obtained the spectrums and current. Here the material parameters are set as $t = r = 1$ and $d = 3.6$. The lattice constant is 1 and the amplitudes of the Burgers vectors is set as $b_g = 1$. For the calculation, we imposed the open boundary condition along the $x-$ and $y-$directions and periodic boundary condition along the $z-$direction, and set $L_x = L_y = 4l_{\text{dis}}^x = 38$ and $L_z = 100$.

As shown in FIG.2c-3, we obtained the asymmetric spectrum in agreement with the analytic calculation. The asymmetric modes are localized at the dislocation line. The quasi-localized chiral modes are not isolated from the bulk but easily mixed with the bulk modes (FIG.2c-3). The current density at zero temperature is shown in FIG.2d. We obtained the upward current along the screw dislocation and downward current along the anti-screw dislocation due to the TCME. The total current per the unit length toward $z-$direction due to one dislocation line is $J_z^i/L_z = 0.087$, which is calculated by the summation of the current density in the $x > 0$ half-plane, and this value is in the same order as that estimated from the linear response theory (3), $J_z^i/L_z \sim 0.1$. For the estimation, we set the cutoff as $\Lambda \sim 1/$(lattice constant) = 1.

No-go theorem of CME — The existence of the TCME in the realistic lattice system may seem to contradict with the no-go theorems of the ground state current [35, 36]. However, they prohibit the total current, but not the local current density. Therefore, the current along the dislocation line can exist, as we found [45].

Experimental implication — Here we present two experimental setups to observe the TCME in TRS-broken WSMs, for which Eu$_2$Ir$_2$O$_7$ [22] and YbMnBi$_2$[23] are candidate materials. The first one is a scanning SQUID measurement, which can detect weak inhomogeneous magnetic fields [56, 57]. If there is a pair of dislocation, the circulating current occurs. The magnitudes of the current and the induced magnetic field are estimated as $I \sim 10^{-6}$A and $B \sim 10^{-7}$T, respectively, for both Eu$_2$Ir$_2$O$_7$ and YbMnBi$_2$. Here we used Eq.(3) and the material parameters, $v_F \sim 10^3$m/s and $a \sim 10$Å and set $b_g = a$ and $\lambda \sim \Lambda \sim 1/a$, where $a$ is the lattice constant. Also, for the estimation of the magnitude of the magnetic field, we used a typical value of inter-distance between dislocations, $10^3$Å [58]. It is feasible to detect $B \sim 10^{-7}$T via the scanning SQUID.

The second one is a nonlocal transport phenomenon, which was observed in quantum Hall materials [59, 60]. The experimental setup is shown in FIG.3. If the bulk contributions are completely negligible and there are only the chiral modes at the dislocation lines, $V_{34} := V_3 - V_4 = 0$ despite $I_{12} \neq 0$, then, the nonlocal resistivity $R_{12,34} := V_{34}/I_{12}$ is equal to zero [59]. On the other hand, if the nonlocal transport is negligible, $V_{34} > V_{3'4'}$ holds for $L_1L_3 < L_1L_3'$ when $I_{12} \neq 0$. Therefore, if $R_{12,34} < R_{12,3'4'}$ is observed, it is the fingerprint of the chiral current due to the TCME. The effect can be discriminated from any previously reported conventional transport induced by dislocation[58, 61–69].

We also comment on effects of impurities and disorientation of the dislocation. First, the current due to the TCME is expected to be robust against weak disorder. It is because that Eq.(3) is independent of the scattering time, like the intrinsic contribution to the anomalous Hall effect [70]. More precisely, the current is due to the edge modes in the Fermi arc on the surface of WSMs, and these modes are supported by the Weyl points, which are protected by the Chern number, and hence, robust against weak disorder.

Next, in real experimental setups, it is difficult to align the dislocation line orthogonal (parallel) to the line connecting the Weyl nodes exactly in the case (a) (case (b)). Even when they are not orthogonal (parallel), as long as they are not parallel (orthogonal), the current parallel to the dislocation line still exists. Supposing that the dislocation line is parallel to the $z-$axis, the current is given by $J_z^i = e\nu_FL_z(\lambda^{\mu} - \lambda^{\nu}) \cdot b_g/4\pi^2$ in the both cases (a) and (b).

Summary — In this letter, we have discussed the TCME in WSMs caused by dislocation. We have confirmed that it is possible to occur and experimentally observable in realistic materials, and argued that the Lorentz symmetry breaking is important for it.

Acknowledgement — We thank H. Fujita, Y. Kikuchi, T. Kimura, K. Ohmori, D. Schmeltzer, K. Shiozaki, and A. Shitade for fruitful discussions. We also thank one of the referees for suggesting scanning SQUID measurements. This work is supported by the Grant-in-Aids for Scientific Research from MEXT of Japan [Grant No. 23540406, No. 25220711, and No. 25103714 (KAKENHI on Innovative Areas Topological Quantum Phenomena”), No. 15H05852 (KAKENHI on Innovative Areas Topological Materials Science”)]. HS is supported by a JSPS
Supplemental Material

Derivation of Eq.(3)

In this section, we derive the expression for the current in the presence of the magnetic and torsional magnetic responses of the current density, Eq.(3). The derivation consists of two steps: first we derive the expression for the Green function in the presence of the gauge field and vielbein using the gradient expansion Eq.(S-4), and next we calculate the current density by using Eq.(S-4) and obtain Eq.(S-18), which is equivalent to Eq.(3).

First, we calculate the single-electron Green function. The Green function in the presence of the gauge field and vielbein which is defined by

\[ G(\tau_1, \tau_2, \tau_3, \tau_4) := \int \frac{D[\phi_1\phi_2]}{D[\phi_1\phi_2]} \exp \left( -S[\phi_1, \phi_2, A_n^\dagger, A_n] \right) \]

Then, the following differential equation holds:

\[ \frac{1}{2} \left[ \hat{L}(\varepsilon, \tau_1) - i \partial_\tau \right] G(\varepsilon, \tau_1, \tau_2) + G(\varepsilon, \tau_1, \tau_2) \hat{L}^\dagger(\varepsilon, \tau_1, \tau_2) = \delta(\tau_1 - \tau_2), \]

with \( \hat{L}(\varepsilon, \tau, -i \partial_\tau) = |e(\tau)||e(\tau) - H(\tau) - eA_0| \), and \( |e(\tau)| = \det e^{\dagger}(\tau) \) Here \( \varepsilon = (2N + 1)\pi T \) is the Fermionic Matsubara frequency with the temperature \( T \), and \( G(\varepsilon, \tau_1, \tau_2) := \int_0^\beta G(\varepsilon, \tau_1, \tau_2; \tau, 0) e^{-i\beta \varepsilon \tau} d\tau \) is the Fourier component of the Green function. Now, using the spatial Wigner transformation defined as \( \tilde{f}(\mathbf{R}, \mathbf{p}) := \int d^3r e^{-i\mathbf{p}\cdot \mathbf{r}} f(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2) \), Eq. (S-2) is rewritten into

\[ \frac{1}{2} \left[ \tilde{\hat{L}}(\varepsilon, \mathbf{R}, \mathbf{p}, \imath \partial_\mathbf{R}) \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) + \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) \tilde{\hat{L}}^\dagger(\varepsilon, \mathbf{R}, \mathbf{p}) \right] = 1. \]

In the gradient expansion up to the first order in \( \partial_i A_j \) or \( \partial_i e_j^a \), the Green function becomes

\[ \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) = \tilde{G}^{(0)}(\varepsilon, \mathbf{R}, \mathbf{p}) + \tilde{G}^{(1)}(\varepsilon, \mathbf{R}, \mathbf{p}) + \cdots, \]

\[ \tilde{G}^{(0)}(\varepsilon, \mathbf{R}, \mathbf{p}) = \frac{1}{|e(\mathbf{R})|} \left[ \mathcal{L}_0(\varepsilon, \pi) \right]^{-1}_{\pi_n = e_n^a(\mathbf{R})} \]

\[ \tilde{G}^{(1)}(\varepsilon, \mathbf{R}, \mathbf{p}) = \frac{i}{2|e(\mathbf{R})|} \partial \mathcal{L}_0(\varepsilon, \pi) \left[ \mathcal{L}_0(\varepsilon, \pi) \right]^{-1}_{\pi_n = e_n^a(\mathbf{R})} \]

\[ \mathcal{L}_0^{-1}(\varepsilon, \pi) \left[ \mathcal{F}_{ab}(\mathbf{R}) + \mathcal{T}_{ab}(\mathbf{R}) \pi_n \right] \]

Here \( \pi_n := e_n^a(\mathbf{R}) (\rho_i - eA_i(\mathbf{R})) \) is the gauge-invariant mechanical momentum, while \( \rho_i \) is the canonical momentum, defined as \( F_{ab} := e_a^b e^c F_{ij} \), and \( \mathcal{T}_{ab} := e_a^b e^c (\mathbf{T})_{ij} \). The free Lagrangian is defined as \( \mathcal{L}_0(\varepsilon, \pi) := i \varepsilon - H(\pi) \).

Next, using Eq. (S-4), we calculate the current density and derive Eq.(2). The current density are defined by \( j^a(\mathbf{r}) := -\langle e^a(\mathbf{r})/|e(\mathbf{r})| \rangle \delta S_{eh}/\delta A_i(\mathbf{r}) \). Therefore

\[ j^a(\mathbf{R}) = \frac{e}{|e(\mathbf{R})|} \int \frac{D[\phi_1\phi_2]}{D[\phi_1\phi_2]} \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_n} \bigg|_{\pi_n = e_n^a(\mathbf{R})} + \text{c.c.} \]

\[ = \frac{eT}{2} \sum_N \text{Tr} \left[ \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_1} \right]_{\pi_n = e_n^a(\mathbf{R})} \]

\[ G(\varepsilon, \mathbf{R}, \mathbf{p}) \bigg|_{\mathbf{r}_1 \to \mathbf{R}, \mathbf{r}_2 \to \mathbf{R} + \mathbf{r}} + \text{c.c.} \]

\[ = \frac{eT}{2} \sum_N \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_1} \right]_{\pi_n = e_n^a(\mathbf{R})} \]

\[ e^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} \text{Tr} \left[ \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_1} \right]_{\pi_n = e_n^a(\mathbf{R})} \]

\[ \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) \bigg|_{\mathbf{r}_1 \to \mathbf{R}, \mathbf{r}_2 \to \mathbf{R} + \mathbf{r}} + \text{c.c.} \]

Here Tr means the trace over the band indices, we used that \( \delta \mathcal{L}/\delta A_i = \langle e(\mathbf{r})/|e(\mathbf{r})| \rangle \partial \mathcal{L}_0/\partial \pi_n \) and the second line does not depend on \( \tau \) due to imaginary time-translation symmetry. Note that \( \pi \) of the third line is the operator though that of the fourth one is the c-number. Using Eq.(S-4), up to the first order in \( \partial_i A_j \) or \( \partial_i e_j^a \), the expression of the current density becomes

\[ j^a(\mathbf{R}) = j^{(0)}(\mathbf{R}) + j^{(1)}(\mathbf{R}), \]

with the zeroth-order terms,

\[ j^{(0)}(\mathbf{R}) \]

\[ = \frac{eT}{2} \sum_N \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_1} \right]_{\pi_n = e_n^a(\mathbf{R})} \]

\[ \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) \bigg|_{\mathbf{r}_1 \to \mathbf{R}, \mathbf{r}_2 \to \mathbf{R} + \mathbf{r}} + \text{c.c.} \]

and the first-order terms,

\[ j^{(1)}(\mathbf{R}) \]

\[ = \frac{eT}{2} \sum_N \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \frac{\partial \mathcal{L}_0(\varepsilon, \pi)}{\partial \pi_1} \right]_{\pi_n = e_n^a(\mathbf{R})} \]

\[ \tilde{G}(\varepsilon, \mathbf{R}, \mathbf{p}) \bigg|_{\mathbf{r}_1 \to \mathbf{R}, \mathbf{r}_2 \to \mathbf{R} + \mathbf{r}} + \text{c.c.} \]

The zeroth-order terms (S-7) can be rewritten as

\[ j^{(0)}(\mathbf{R}) = e \sum_n \int \frac{d^3\pi}{(2\pi)^3} u_n^a(\pi) n_F(\varepsilon_n, \pi). \]
For the derivation, we inserted the identity, \( 1_\pi = \sum_n |u^\pi_n \rangle \langle u^\pi_n | \), between \( \partial L_0 / \partial \pi \) and \( \hat{G}^{(0)} \) in Eq.(S-7), and used the formula, \( \sum_{N=-\infty}^{\infty} \frac{1}{[i\varepsilon_N - t]} + \frac{1}{[-i\varepsilon_N - t]} = (1 - 2n_F(t)) / T \), for the summation over the Matsubara frequency, and \( \int d^3 p = |e(R)| \int d^3 \pi \). Here, \( n \) is the band index, \( \varepsilon_{n,\pi} \) is the energy, \( v^{\pi,a}(\pi) := \partial \varepsilon_{n,\pi} / \partial \pi_n \) is the group velocity, \( n_F(\varepsilon) := 1 / (e^{\varepsilon / (T + 1)} + 1) \) is the Fermi distribution function, and \( |u^\pi_n \rangle \) is the Bloch state. This term corresponds to the summation of all contributions to the current from the electrons in the occupied states in the absence of magnetic and torsional magnetic field.

Now, we move on to the calculation of Eq.(S-8). The sum of the second term of Eq.(S-8) and its complex conjugate is zero, since \( [\cdots]^* = \cdots |_{\varepsilon_N \rightarrow -\varepsilon_N} \). Then, by using Eq.(S-4), we obtain

\[
J^{(1)}(R) = \frac{i e T}{4} \sum_{N} \int \frac{d^3 \pi}{(2\pi)^3} \mathcal{L}_0^{-1}(\varepsilon_N, \pi) \frac{\partial L_0(\varepsilon_N, \pi)}{\partial \pi_n} \mathcal{L}_0^{-1}(\varepsilon_N, \pi) \frac{\partial L_0(\varepsilon_N, \pi)}{\partial \pi_a} |c_e(R) + T^{(a)}_{ab}(R)\rangle \langle \pi_c| + c.c. \tag{S-10}
\]

Moreover, inserting the identities, \( 1_\pi = \sum_n |u^\pi_n \rangle \langle u^\pi_n | \), we obtain

\[
J^{(1)}(R) = \frac{-ieT}{4} \sum_{N, n, m, l} \int \frac{d^3 \pi}{(2\pi)^3} \langle n | \frac{\partial \mathcal{H}}{\partial \pi_l} | m \rangle \langle m | \frac{\partial \mathcal{H}}{\partial \pi_a} | l \rangle \times \langle l | \frac{1}{(i\varepsilon_N - \varepsilon_n)(i\varepsilon_N - \varepsilon_l)(i\varepsilon_N - \varepsilon_l)} \times [c_F_{ab}(R) + T^{(a)}_{ab}(R)\rangle \langle \pi_c| + c.c., \tag{S-11}
\]

where the indices \( \pi \) are omitted like \( \varepsilon_n := \varepsilon_{n,\pi} \) and \( |n \rangle := |u^\pi_n \rangle \). There are three types of contributions to the summation over the band indices \( n, m, l \): (a) all the three are the same, (b) two of them are the same and the other is different, and (c) each one is different respectively. However the contribution (a) is found to be zero because of the antisymmetry of \( [c_F_{ab}(R) + T^{(a)}_{ab}(R)\rangle \langle \pi_c| \) under \( a \leftrightarrow b \). Moreover, the contribution (c) is also zero, since our model of the WSM, Eq.(1), consists of two two-band Hamiltonians independent of each other, and then the overlap of three or more bands is zero. Therefore, we have only to consider the contribution (b), and then obtain

\[
J^{(1)}(R) = \frac{-ieT}{4} \sum_{N} \int \frac{d^3 \pi}{(2\pi)^3} [c_F_{ab}(R) + T^{(a)}_{ab}(R)\rangle \langle \pi_c| (M_{iab} + M_{ab\bar{i}} + M_{\bar{i}i\bar{a}}) + c.c., \tag{S-12}
\]

with

\[
M_{abc} := \langle v^a_N | n_F(\varepsilon_{n}) \langle n, b | (\varepsilon_n - H) | n, c \rangle + v^a_n \langle n, b | n_F(H) | n, c \rangle - v^a_N n_F(\varepsilon_{n}) \langle n, b | n, c \rangle, \tag{S-13}
\]

where we used the abridged notation, \( |n, a \rangle := | \partial \varepsilon_n / \partial \pi_a \rangle \).

For the derivation of Eqs.(S-12,S-13) we used the formulae \( \sum_{N=-\infty}^{\infty} \frac{1}{[i(\varepsilon_N - t)]^2} = \frac{n_F(\varepsilon_{n}) + n_F(-\varepsilon_{n})}{4t^2} \), \( \langle n | \frac{\partial \mathcal{H}/\partial \pi_n}{\partial \pi_a} | m \rangle = (\varepsilon_m - \varepsilon_n) \langle n, m, \pi \rangle \) for \( n \neq m \), and \( \sum_n f(\varepsilon_m) \langle n, b | n, c \rangle = (n, b) f(H) | n, c \rangle \) for any function \( f \). Moreover, using the relationship \( (M_{abc})^* = M_{acb} \), we obtain

\[
J^{(1)}(R) = -\frac{ieT}{2} \sum_{n} \int \frac{d^3 \pi}{(2\pi)^3} \langle c_F^{23} + T^{(d)}_{23} \rangle \varepsilon_{abc} M_{abc}, \tag{S-14}
\]

where \( \varepsilon_{abc} \) is the antisymmetrical symbol. Furthermore, since \( v^a_N n_F(\varepsilon_{n}) = \partial n_F(\varepsilon_{n}) / \partial \pi_a \) and only antisymmetric parts of \( M_{abc} \) contribute, using integration by parts, we find

\[
J^{(1)}(R) = \frac{ie}{2} \sum_{n} \int \frac{d^3 \pi}{(2\pi)^3} \langle c_F^{23} + T^{(d)}_{23} \rangle \varepsilon_{abc} v^a_n n_F(\varepsilon_{n}) \langle n, b | n, c \rangle + \frac{ie}{2} \sum_{n} \int \frac{d^3 \pi}{(2\pi)^3} T^{(d)}_{23} \varepsilon_{abc} n_F(\varepsilon_{n}) \langle n, b | n, c \rangle. \tag{S-15}
\]

Using the Berry curvature is defined by \( \Omega^a_{\pi} := -ic_{abc} \langle n, b | n, c \rangle \), and the vector representation of the TMF, \( T^a_{\pi} \), is defined by \( T^a_{\pi} := (1/2) \varepsilon^{ijk} T^{jk}_{\pi} \), it can be rewritten as

\[
J^{(1)}(R) = -e \sum_{n} \int \frac{d^3 \pi}{(2\pi)^3} \langle v^a_{\pi} \cdot \Omega^a_{\pi} \rangle (eB_{\pi} + T^{(a)}_{\pi} n_F(\varepsilon_{n})) - \frac{e}{2} \sum_{n} \int \frac{d^3 \pi}{(2\pi)^3} \langle \Omega^a_{\pi} n_F(\varepsilon_{n}) \rangle. \tag{S-16}
\]

It is noted that the term containing \( B_{\pi} \) is equal to the expression for the CME derived by Son and Yamamoto [46], and the others are new terms that represent the current induced by the torsion. Neglecting the last term, which is, as we will discuss later, less important than the others in the case of WSMs, Eq.(S-16) can also shortly derived from the substitution of the magnetic field or the field strength in the absence of the vielbein, \(-i[(\bar{\iota}_{\partial} - eA_2), (\bar{\iota}_{\partial} - eA_1)] = eB_{\pi} \) with the field strength in the presence of the vielbein, \(-i[\bar{\nabla}_g, -\bar{\nabla}_d] = eB_{\pi} + T^{(a)}_{\pi} (i\nabla_a), \) where \( [U, V] := UV - VU \) is the commutator. This justifies the analogy between the TMF and the magnetic field.
Finally, we substitute the energy, group velocity, and Berry curvature of the model of the WSM, (1), into Eq.(S-16) and derive Eq. (3). We characterize the four bands of the Hamiltonian (1) as $n = (s, \pm)$, with $s = L$ or $R$, where $s$ is the index of the chirality and $+(-)$ means the higher (lower) band of the Weyl cone. Then, their energy, group velocity, and Berry curvature are given by

$$\varepsilon^{s, \pm}(k) = v_F [\pm |k - \lambda^s| - \lambda_0^s]$$

$$v^{s, \pm}(k) = \pm \frac{v_F (k - \lambda^s)}{|k - \lambda^s|}$$

$$\Omega^{s, \pm}(k) = \pm \chi_s \frac{k - \lambda^s}{2|k - \lambda^s|}$$

(S-17)

Using Eqs.(S-16, S-17), we obtain

$$j^{(1)}_R = \left[ \frac{e^2 v_F (\lambda^R_0 - \lambda^L_0)}{4\pi^2} B_1(R) + \frac{e v_F (\lambda^R_0 - \lambda^L_0)\Delta}{4\pi^2} T^\|_R(R) \right]$$

(S-18)

at zero temperature and up to the linear order in $\lambda^{L(R)}$. For the derivation of Eq.(S-18), we introduced a momentum cutoff scheme $|k - \lambda^s| < \Lambda$ for the Weyl node of the chirality $s$. Physically, $A$ corresponds to the momentum range from the Weyl points in which the cone structures of the band of the lattice system is approved. Note that the last term of Eq.(S-16) yields second(or more)-order contributions in $\lambda^{L(R)}$, and then less important as mentioned before. Eq.(S-18) is the correction of current due to the TMF and magnetic field and is equivalent to Eq.(3), then the derivation of Eq.(3) has been completed.

**Ground state current in the presence of screw dislocation: analytical calculation**

In this section, we calculate the ground state current raised by the TCME in the case of screw dislocation, by calculating directly the eigenstates of the Hamiltonian (Eq.(4) in the main text),

$$H_{s,k_z}^{\text{screw}} = \chi_s [H^{\perp}_{k_z} + m^L_{k_z} \sigma^z]$$

where $m^L_{k_z} = k_z - \chi_s \lambda$, $\Phi_{k_z} = k_z b_y$, $\rho = \sqrt{x^2 + y^2}$, $\chi_{L(R)} = \pm 1(-1)$, and $\sigma^i$ is the Pauli matrix.

For the calculation of the spectrum, it is useful to clarify the symmetry of the eigenstates of $H^{\perp}_{k_z}$. Suppose $|\kappa\rangle_{k_z}$ the eigenstate of $H^{\perp}_{k_z}$ with eigenvalue $\kappa$. Since $\{H^{\perp}_{k_z}, \sigma^z\} = 0$, where $\{U,V\} := UV + VU$ is the anticommutator, the state $\sigma^z |\kappa\rangle_{k_z}$ is also the eigenstate with eigenvalue $-\kappa$. Therefore, we can choose the eigenfunctions to preserve the doublet structure, $|\pm \kappa\rangle_{k_z} = \sigma^z |\kappa\rangle_{k_z}$, for $\kappa \neq 0$. On the other hand, there is no double structure in the zero eigenstates. Since the hermitian operator $\sigma^z$ maps zero eigenstates of $H^{\perp}_{k_z}$ to zero eigenstates of $H^L_{k_z}$, then we can choose the zero eigenstates also as eigenstates of $\sigma^z$, denoted by $|0,\pm \kappa\rangle_{k_z}$ with $\sigma^z |0,\pm \kappa\rangle_{k_z} = \sigma_t |0,\pm \kappa\rangle_{k_z}$ and $\sigma_t = \pm 1$. There is another symmetrical property between the eigenstates of $H^L_{k_z}$ with different $k_z$. Since the transformation $k_z \rightarrow -k_z$ corresponds to the flip of the direction of the effective magnetic field, $\Theta H^{\perp}_{k_z} \Theta^{-1} = H^{\perp}_{-k_z}$ holds, where $\Theta = i\sigma^y K$ is the time-reversal operator for spin-1/2 fermions and $K$ is the complex conjugation operator [71]. Therefore we can impose $|\kappa\rangle_{k_z} = \Theta |\kappa\rangle_{k_z}$ and $|0,-\kappa\rangle_{k_z} = \Theta |0,\kappa\rangle_{k_z}$, because of $\{\sigma_t, \Theta\} = 0$.

The eigenstates of $H_{s,k_z}^{\text{screw}}$ can be constructed from $|\kappa\rangle_{k_z}$ and $|0,\kappa\rangle_{k_z}$. Indeed, $|\psi^{L,\pm}_{k_z}(\kappa)\rangle := (c^{L,1}_{k_z}(\kappa)|\kappa\rangle_{k_z} + c^{L,2}_{k_z}(\kappa)|-\kappa\rangle_{k_z})/\sqrt{\kappa^2 + (m^L_{k_z})^2}$ and $\sigma_t m^L_{k_z}$, respectively. Here the coefficients are given by $c^{L,1}_{k_z}(\kappa) = (4\kappa^2 + (m^L_{k_z})^2)^{-1/4} \pm \frac{m^L_{k_z}}{4\kappa^2 + (m^L_{k_z})^2}$. Moreover, $|\psi^{\perp}_{k_z}(\kappa)\rangle := \Theta |\psi^{L,\mp}_{k_z}(\kappa)\rangle$ and $|0,\kappa\rangle_{k_z}$ are the eigenstates of $H_{R,k_z}^{\text{screw}}$, with eigenvalues $\pm \sqrt{\kappa^2 + (m^R_{k_z})^2}$ and $-\sigma_t m^R_{k_z}$, respectively.

Now, we calculate the ground state current in the presence of the screw dislocation. As yet, we have not distinguished discrete and continuum states. From now on, we use $\kappa$ to express the discrete eigenvalues of $H^{\perp}_{k_z}$ and $(\kappa, l)$ to label the continuum states, where $\kappa$ is the continuum energy eigenvalue, and $l$ is a discrete quantum number, e.g., the angular momentum. The current operator is defined by $\partial H_{s,k_z}^{\text{screw}} / \partial k_z = \chi_s \sigma^x + \chi_s\{-(b y / 2\pi \rho^2) \sigma^x + (b y / 2\pi \rho^2) \sigma^y\}$. At least up to the first order in $b$, the correction to the current operator due to the dislocation, i.e., the second and third term above, does not contribute to the expectation value because these terms are odd under the transformation $x \rightarrow -x$ or $y \rightarrow -y$. Therefore, the current is the sum of the expectation values of $\chi_s \sigma^x$ for the occupied states which consist of discrete nonzero,
where \( L_z \) is the size of the system. Here we introduce the momentum cutoff scheme, \( |m_{k_z}^s| < \Lambda \), i.e. the domain of the integration is the same as that used in the calculation of Eq. (4). The first term in the square braket is equal to zero, since \( \sigma^z |\kappa_i\rangle_{k_z} = -|\kappa_i\rangle_{k_z} \) is orthogonal to \( |\kappa_i\rangle_{k_z} \).

The second term is the index of the Dirac operator, \( H_{k_z} \), which is an integer and the difference in the number of its normalizable zero modes with \( \sigma^3 = +1 \) and \( \sigma^3 = -1 \). The index is given by \( N_{k_z} := -\text{sgn}(\Phi_{k_z}) |\Phi_{k_z}|/2\pi \) [53, 54]. The normalizable zero modes exhibit power-law decay for large distance from the dislocation; i.e. they behave like \( |0_{i-1}\rangle \propto (0, \rho^{-\Phi_{k_z}/2\pi}(x - iy)^{i-1}) \) for \( \Phi_{k_z} > 0 \), and \( |0_{i+1}\rangle \propto (\rho^{\Phi_{k_z}/2\pi}(x + iy)^{i-1}) \) for \( \Phi_{k_z} < 0 \), where \( i = 1, 2, \ldots, |N_{k_z}| \) [53, 54]. Now, we move on to the third term of Eq. (S-20). One may expect that it is equal to zero, since \( \sigma^z |\kappa, l\rangle = -|\kappa, l\rangle \) is orthogonal to \( |\kappa, l\rangle \) for almost all values of \( \kappa \). However, the scattering states near \( \kappa = 0 \) (their amplitudes \( \propto \cos(\kappa \rho + \delta_1) \sqrt{\rho} \) with \( \delta_1 \) the phase shift) cause a delta function peak of \( |\kappa, l\rangle \sigma^z |\kappa, l\rangle \) at \( \kappa = 0 \). Indeed, from an explicit calculation [53], it has been shown that \( \sum_l |\kappa, l\rangle \sigma^z |\kappa, l\rangle = c_{k_z} \delta(\kappa) \), with \( c_{k_z} = \Phi_{k_z}/2\pi - N_{k_z} \), and then the third term is equal to \( c_{k_z} \). Substituting them into Eq. (S-20), we obtain

\[
J^z = \int_{-\Lambda - \lambda}^{\lambda - \lambda} \frac{L_z \varsigma k_z \Phi_{k_z}}{2\pi} = -\frac{L_z \omega g \lambda \lambda}{2\pi^2},
\]

which is coincident with the expression obtained directly from Eq. (3) by the following reason. In the presence of the screw dislocation with the Burgers vector \(-bg \vec{z}\) the torsion is given by \( T^z = T_{xy} = b g \delta^{(2)}(x, y) \). Therefore, the total current derived from Eq. (3) is \( J^z = -L_z \omega g F(\lambda^R - \lambda^L)/4\pi^2 \). In this section we have set \( \lambda^L = -\lambda^R = \lambda \) and \( e = v_F = 1 \), and therefore we obtain \( J^z = -L_z \omega g \lambda /2\pi^2 \), which reproduces Eq. (S-21).

### Absence of total current and possibility of local current

In this section, we show that, if the system is periodic in a certain direction, the total current is always zero, even in the presence of a magnetic field or lattice strain and dislocations, while the local current is not. The argument is the extension of that presented in Ref. [35]. We start with the general Hamiltonian of electrons in solids (set \( e = 1 \) in this section):

\[
H = \int d^3 r \frac{1}{2m} (-i\nabla_i - A_i(\rho, z))^2 + V(\rho, z),
\]

where \( V \) is the potential term, in which the effect of the dislocation is included. Here \( i = x, y, z, r = (\rho, z) \), and \( \rho = (x, y) \), and we impose the periodicity in the \( z \)-direction:

\[
A_i(\rho, z) = A_i(\rho, z + a), V(\rho, z) = V(\rho, z + a).
\]

Suppose \( \psi \) is one eigenfunction of the Hamiltonian and define the Bloch wave function \( \psi_{n, k_z}(\rho, z) = e^{ik_z z} u_{n, k_z}(\rho, z) \), whose energy is \( \varepsilon_{n, k_z} \). The total current along the \( z \)-direction is given by

\[
J_z = \sum_n \int_{BZ} \frac{dk_z}{2\pi} \int d^3 r \psi_{n, k_z}^*(\rho, z) \frac{\delta H}{\delta A_z} n_F(\varepsilon_{n, k_z})
\]

\[
= -\sum_n \int_{BZ} \frac{dk_z}{2\pi} \int d^3 r u_{n, k_z}^*(\rho, z) H_{k_z} u_{n, k_z}(\rho, z) n_F(\varepsilon_{n, k_z}),
\]

where \( H_{k_z} = e^{-ik_z z} H e^{ik_z z} \) and \( n_F \) is the Fermi distribution function. Here we use the identity :

\[
\int d^3 r u_{n, k_z}^*(\rho, z) \frac{\partial H_{k_z}}{\partial k_z} u_{n, k_z}(\rho, z)
\]

\[
= \frac{\partial}{\partial k_z} \int d^3 r u_{n, k_z}^*(\rho, z) H_{k_z} u_{n, k_z}(\rho, z)
\]

\[
= \frac{\partial \varepsilon_{n, k_z}}{\partial k_z},
\]

which follows from

\[
\frac{\partial}{\partial k_z} \left[ \int d^3 r t_{n, k_z}(\rho, z) c_{n, k_z}(\rho, z) \right] = \frac{\partial}{\partial k_z} 1 = 0,
\]

and then we can rewrite Eq. (S-24) into

\[
J_z = -\sum_n \int_{BZ} \frac{dk_z}{2\pi} \frac{\partial \varepsilon_{n, k_z}}{\partial k_z} n_F(\varepsilon_{n, k_z})
\]

\[
= -\frac{1}{2\pi} \sum_n \sum_{i=1}^{n} \int \varepsilon_{n, k_{i+1}}(z)^n dz n_F(\varepsilon)
\]

Here for each region \( k \in (k_{i-1}^{(n)}, k_{i}^{(n)}) \), \( \varepsilon_{n, k} \) monotonically increases or decreases, and \( k_{0}^{(n)} = 0 \) and \( k_{i+1}^{(n)} = 2\pi/a \). We
Then, in the case of local current, the above argument in the case of the total current does not hold. Hence, the local current is not always zero, unlike the total current.

\[
\frac{\partial}{\partial k_z} \left[ u^*_{n,k_z}(\mathbf{p},z) u_{n,k_z}(\mathbf{p},z) \right] \neq 0. \quad (S-28)
\]

find that Eq. (S-27) is always equal to zero owing to the periodicity of the dispersion in the wave number space, $\varepsilon_{n,k=0} = \varepsilon_{n,k=2\pi/a}$. Therefore, we found that the total current along the $z$-direction is zero. In the above derivation, it is essential that the integrand with respect to $k_z$ can be rewritten into the total derivative with respect to $k_z$, and this key factor follows from the fact that the integral over the real space of $|u_{n,k_z}(\mathbf{p},z)|^2$ is equal to 1, which results in Eq. (S-26). Instead, without the integration over the real space,

\[
\frac{\partial}{\partial k_z} \left[ u^*_{n,k_z}(\mathbf{p},z) u_{n,k_z}(\mathbf{p},z) \right] \neq 0. \quad (S-28)
\]

Then, in the case of local current, the above argument in the case of the total current does not hold. Hence, the local current is not always zero, unlike the total current.
