Self-Inverse Functions and Palindromic Circuits

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Abstract—We investigate the subclass of reversible functions that are self-inverse and relate them to reversible circuits that are equal to their reverse circuit, which are called palindromic circuits. We precisely determine which self-inverse functions can be realized as a palindromic circuit. For those functions that cannot be realized as a palindromic circuit, we find alternative circuits are equal to their reverse circuit, which are called that are self-inverse and relate them to reversible circuits that are self-inverse and related to them to reversible circuits that are equal to their reverse circuit, which are called palindromic circuits. We precisely determine which self-inverse functions can be realized as a palindromic circuit.

I. INTRODUCTION

While the reversible circuit model has seen many practical applications (e.g., logic designs [1], [2], [3], reversible logic synthesis [4], [5], [6]), the theoretical aspects of the logic circuit model have received much less attention. This is, in itself, not a hindrance to the usage of the logic model in the aforementioned applications, but it does limit our understanding and therefore the possibility to implement the applications most efficiently.

In this paper we investigate the relationship between (reversible) self-inverse functions (involutions) and reversible palindromic circuits. By a palindromic circuit we mean a reversible circuit generated from gates and serial circuit composition (no parallel composition) that is identical when reading it from left and right.

Looking at reversible circuit as permutations is not a novel idea. This duality has been used for reversible logic synthesis [7], [8] but also as theoretical foundation for reversible logic analysis [9], [10]. Though the many results have shown these to be interesting approaches, we will take a different approach for this work. To get a deep understanding of palindromic circuits, we define which permutations (defined as transpositions in the cycle notation) are equivalent to mixed-polarity multiply-controlled Toffoli gates (MPMCT). For this purpose we exploit general theorems about permutations.

The authors in [11] have coined the term palindromic circuits and also related them to self-inverse functions. They have shown that there are some self-inverse functions that can be realized as a palindromic circuit and argued that for some no such realization can be found. In this paper we precisely determine which self-inverse functions can be realized as a palindromic circuit. For those functions that cannot be realized as a palindromic circuit, we find alternative palindromic representations that require an extra circuit line or quantum gates in their construction. In [12] palindromic circuits have been used in an optimization technique for quantum circuits.

The paper is organized as follows. Basic notations and definitions for permutations and reversible circuits are described in the next section. Section III discusses properties of self-inverse reversible functions and shows how MPMCT gates can be derived from transpositions. Section IV introduces palindromic circuits and determines the subclass of self-inverse functions that can be realized as a palindromic circuit. Section V illustrates alternative constructions for palindromic circuits that can realize all self-inverse function and Section VI concludes the paper.

II. PRELIMINARIES

A. Basic Notation and Definitions

Applying the bit-wise operations ‘&’, ‘|’, and ‘⊕’ to non-negative numbers is interpreted as applying them to their unsigned bit-wise expansion. The operation ‘ν’ is the sideways sum and counts the number of ones in a bit-string or in the bit-wise expansion of a non-negative number. The double factorial $n!! = \prod_{i=0}^{[n/2]-1} (n-2i)$ is the product of all integers from 1 to $n$ that have the same polarity as $n$. For a non-negative number $n$, an integer partition $n$ is a sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ and $\mu_1 + \mu_2 + \cdots + \mu_k = n$.

B. Permutations

Permutations are elements from the symmetric group $S_n$ i.e. bijections over the set $\{0, 1, \ldots, n-1\}$. We chose to have 0 as the lowest permutation index, in contrast to the conventional definition, as this makes computation with respect to reversible functions and gates easier. Several notations are used for permutations. Given a permutation $\pi \in S_n$ its two-line form representation is

$$
\begin{pmatrix}
i_1 & i_2 & \cdots & i_n \\
\pi(i_1) & \pi(i_2) & \cdots & \pi(i_n)
\end{pmatrix}
$$

(1)

in which all indexes are written in the first line and its function values with respect to $\pi$ in the second line. The order of indexes in the first line is arbitrary, however, if we have $i_1 < i_2 < \cdots < i_n$ we can omit the first line and have the one-line form representation

$$
\begin{pmatrix}
\pi(i_1) & \pi(i_2) & \cdots & \pi(i_n)
\end{pmatrix}
$$

(2)
A permutation can be partitioned into cycles \((i_1, i_2, \ldots, i_k)\) such that \(\pi(i_j) = i_{j+1}\) for \(j < k\) and \(\pi(i_k) = i_1\). The order of cycles and the starting value inside a cycle do not change the permutation. A cycle of length 1 is called a fixpoint and a cycle of length 2 is called a transposition. Fixpoints are usually omitted in the cyclic representation. Given a permutation \(\pi \in S_n\) in cyclic notation, we refer to the number of cycles (including fixpoints) as \(\text{cyc}(\pi)\). Also let \(\text{type}(\pi)\) be the list of sizes of these cycles, including repetitions, written in decreasing order, i.e., \(\text{type}(\pi)\) is an integer partition of \(n\). The permutation that represents the identity is denoted \(\sigma_{id}\).

**Example 1:** Let \(\pi \in S_8\) be a permutation with two-line form \((0 7 2 4 6 3 5 1 0 2)\). The two-line form in which the first line is ordered is \((0 2 4 3 1 6 5 7)\) from which we can immediately extract the one-line form \((4 2 6 0 3 1 5 7)\). The cyclic representation of \(\pi\) is \((0, 4, 3)(1, 2, 6, 5)(7)\). We have \(\text{cyc}(\pi) = 3\) and type(\(\pi\)) = \((4, 3, 1)\). There are no transpositions in the cyclic representation and the only fixpoint is 7.

The notion of type can be used to partition permutations into conjugacy classes. For this purpose, we review two well-known lemmas.

**Lemma 1:** For all permutations \(\pi, \sigma \in S_n\) we have \(\text{type}(\sigma \circ \pi \circ \sigma^{-1}) = \text{type}(\pi)\).

**Proof:** We show that if
\[
\pi = (i_1, i_2, \ldots, j_1, j_2, \ldots,)
\]
then
\[
\sigma \pi \sigma^{-1} = (\sigma(i_1), \sigma(i_2), \ldots, \sigma(j_1), \sigma(j_2), \ldots).
\]

We first assume that \(\text{cyc}(\pi) = 1\), i.e., \(\pi = (i_1, i_2, \ldots, i_k)\) and show that \(\sigma \pi \sigma^{-1}\) and \(\pi' = (\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))\) are equal by proving that both have the same effect on \(x \in \{1, 2, \ldots, n\}\). First assume that \(x = \sigma(i_s)\) for some \(1 \leq s \leq k\). Then
\[
\sigma \pi \sigma^{-1}(x) = \sigma \pi \sigma^{-1}(i_s) = \sigma \pi(i_s) = \sigma(i_{s+1})\% k = \sigma(x).
\]
If \(x \neq \sigma(i_s)\) for any \(s\), then \(\sigma(x) = \sigma^{-1}(x) = x\) and \(\pi\) fixes \(\sigma^{-1}(x)\). The general form for multiple cycles follows from conjugation being a homomorphism. See also [13].

**Lemma 2:** Let \(\pi, \pi' \in S_n\) such that \(\text{type}(\pi) = \text{type}(\pi')\). Then there exists a permutation \(\sigma\) such that \(\pi = \sigma \circ \pi' \circ \sigma^{-1}\).

**Proof:** When writing \(\pi\) atop \(\pi'\) such that the size of cycles match one obtains \(\sigma\) in two-line form. Due to ordering of same sized cycles and elements in cycles several permutations for \(\sigma\) can be obtained, unless \(\pi = \sigma_{id}\).

The inverse \(\pi^{-1}\) of a permutation \(\pi\) is found by swapping the first and second line in its two-line form. A permutation \(\pi\) is called an involution if \(\pi = \pi^{-1}\). (Sometimes, \(\pi\) is also called self-inverse or self-conjugate.)

**Lemma 3:** Let \(\pi\) be an involution. Then, the cycle representation of \(\pi\) consists only of transpositions and fixpoints.

**Proof:** The cycle representation is unique when disregarding order of cycles and order of elements within cycles. Assume that the cycle representation of \(\pi\) consists of a cycle \((i_1, i_2, \ldots, i_k)\) with \(k > 2\). Then \(\pi^{-1}\) consists of the cycle \((i_k, \ldots, i_2, i_1)\) and hence \(\pi \neq \pi^{-1}\).

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Given an involution \(\pi \in S_n\), let \(\text{size}(\pi)\) be the number of transpositions in \(\pi\). Further, let \(\text{trans}(\pi)\) be the set of transpositions in \(\pi\). We have \(|\text{trans}(\pi)| = \text{size}(\pi)\) and \(\text{cyc}(\pi) = n - \text{size}(\pi)\). Given a set of permutations \(\Pi\), we define
\[
\mathcal{P}_\sigma(\Pi) = \{\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k | \{\pi_1, \pi_2, \ldots, \pi_k\} \subseteq \Pi\},
\]
referred to as the power set of permutations.

**C. Reversible Circuits**

Reversible functions can be realized by reversible circuits that consist of at least \(n\) lines and are constructed as cascades of reversible gates that belong to a certain universal gate library. The most common gate library consists of Toffoli gates or single-target gates.

Given a set of variables \(X = \{x_1, \ldots, x_n\}\), a reversible single-target gate \(T_g(t)\) realizes a reversible functions on \(n\) lines that inverts the variable on the target line \(t \in X\) if and only if the control function \(g\) evaluates to true, where \(g\) is a Boolean function with input variables \(X \setminus \{t\}\). Only line \(t\) is updated. The domain of \(g\) can be smaller than \(X \setminus \{t\}\).

**Example 2:** Fig. 1(a) shows the graphical notation of a single-target gate \(T_{x_1 \lor x_2}(x_3)\) with control function \(x_1 \lor x_2\) and target line \(x_3\).

There exist \(n \cdot 2^{n-1}\) different single-target gates on \(n\) lines, since for each target line one can choose from \(2^{n-1}\) Boolean functions over \(n - 1\) variables. If the control function is \(\perp\) (false), the target line is never inverted and is therefore omitted from the circuit representation.

**Mixed-polarity multiple-control Toffoli (MPMCT) gates** are a subset of the single-target gates in which the control function \(g\) is \(\top\) (true) or can be represented as one product term consisting of positive and negative literals over \(X \setminus \{t\}\). As notation we use \(T(C, t)\) where \(C\) is the set of literals in the product term. If \(g = \top\), \(C\) is empty and the gate is a Not gate on line \(t\). The affected lines in \(C\) are referred to as control lines and a line \(x_i\) is called positive if \(x_i \in C\) and negative if \(\overline{x_i} \in C\). Multiple-control Toffoli gates (MCT) are a subset of MPMCT gates in which the product terms can only consist of positive literals.

**Example 3:** Figs. 1(b) and (c) show circuits consisting of MPMCT and MCT gates, respectively. The gates in Fig. 1(b)
are $T(\{x_1, x_2\}, x_3)$ and $T(0, x_3)$. The gates in Fig. 1(c) are $T(\{x_1, x_2\}, x_3)$, $T(\{x_1\}, x_3)$, and $T(\{x_2\}, x_3)$.

III. SELF-INVERSE REVERSIBLE FUNCTIONS

A reversible function $f$ on $n$ variables is called self-inverse if $f(f(x)) = x$ for all input assignments $x$, or in other words if $f = f^{-1}$. To better understand these functions, it helps a lot to investigate the respective permutations that are represented by the reversible functions, i.e., elements from the symmetric group $S_{2^n}$. Then, self-inverse functions correspond to involutions. The permutation matrix of an involution is symmetric.

A. Reversible Gates

The reversible gates that have been introduced in the previous section are obviously self-inverse. We are interested in transpositions that occur in permutation representations of reversible gates that act on $n$ circuit lines. Involutions whose number of transpositions is a power of 2 are playing a central role when describing such gates. For this purpose, we define

$$\Sigma_k = \{ \pi \in S_{2^n} | \pi = \pi^{-1} \text{ and } \text{size}(\pi) = 2^{k-1} \}$$

(4)

to be the set of all involutions over $2^n$ elements of size $2^{k-1}$ for $1 \leq k \leq n$. We also define

$$I_n = \bigcup_{k=1}^{n} \Sigma_k$$

(5)

to be the set of all involutions which size is a power of 2.

Since the introduced reversible gates only change at most one bit at a time, the occurring transpositions must be of the form $(a, b)$ such that the hamming distance of the binary expansions of $a = a_n \ldots a_2a_1$ and $b = b_n \ldots b_2b_1$ is 1. Let us refer to all of these transpositions as the set $H_n$, i.e.,

$$H_n = \{(a, b) | \nu(a \oplus b) = 1 \}$$

(6)

First note that each transposition $(a, b) \in H_n$ corresponds to one fully controlled MPMCT gate. It acts on line $i$ where $i$ is the single index for which $a_i \neq b_i$. The polarity of the controls is chosen according to the other bits. We have $|H_n| = 2^{n-1}$, because one has $2^n$ choices for $a$ and then $n$ choices for $b$ remain. Since transposition is commutative, the product needs to be halved. Note that this number corresponds to the number of fully controlled MPMCT gates $n \cdot 2^{n-1}$, i.e., one has $n$ choices for the target and then each remaining line can be either positively or negatively controlled.

Based on this observation we partition the set $H_n$ into $n$ sets $H_{n,1}, H_{n,2}, \ldots, H_{n,n}$ such that

$$H_{n,i} = \{(a, b) \in H_n | a \oplus b = 2^{i-1} \}$$

(7)

contains all transpositions in which the components differ in their $i$-th bit. Let $g$ be a single-target gate that acts on the $i$-th line and $\pi_g$ its permutation representation, then $\text{trans}(\pi_g) \subseteq H_{n,i}$. But also the reverse holds, i.e. by selecting a subset of $H_{n,i}$ one finds a set of transpositions that corresponds to a single target gate that acts on the $i$-th line. This can be easily found by counting as $|H_{n,i}| = 2^{n-1}$ and thus there exist $2^{2^{n-1}}$ subsets which equals the number of Boolean functions on $n-1$ variables.

Example 4: For $n = 3$, the following 12 transpositions can be used to form gates that act on three circuit lines (brackets and commas for the sets have been removed for clarity):

$$H_{3,1} = (0, 1)(2, 3)(4, 5)(6, 7)$$
$$H_{3,2} = (0, 2)(1, 3)(4, 6)(5, 7)$$
$$H_{3,3} = (0, 4)(1, 5)(2, 6)(3, 7)$$

From all the subsets in $H_{n,i}$, there are $3^{n-1}$ subsets that represent an MPMCT gate, since $3^{n-1}$ is the number of product terms over $n-1$ variables. The question is how these subsets are characterized. One can easily see that a MPMCT gate is represented by $2^{k-1}$ transpositions, where $n-k$ is the number of control lines, i.e., there are $k-1$ empty lines. But by simply counting we see that not all subsets which size is a power 2 can represent an MPMCT gate. We need to select $2^{k-1}$ transpositions such that the number of positions in which the overall bits of the binary expansions differ is $k$, in other words, $\pi \in I_k$ represents an MPMCT gate, if and only if $\nu_{\pi} = k$ with

$$p = \bigoplus\{a \oplus b | (a, b) \in \text{trans}(\pi)\}$$

(8)

Example 5: As an example, an MPMCT gate with one control line in a circuit of 3 lines, i.e. $k = 2$, can be characterized by two transpositions from $H_{3,i}$, for some $i$. The two transpositions $(4, 5)(6, 7)$ are a valid choice since their binary expansions $101, 111, 101$ differ in 2 positions (last two bits). The two transpositions $(2, 3)(4, 5)$, however, do not form an MPMCT gate since their binary expansions $010, 011, 100$ differ in 3 positions.

With all these observations, we finally define the set $G_n \subseteq I_n$ as the set of all permutations that represent MPMCT gates over $n$ lines according to (8), based on which

$$G_{n,i} = G_n \cap \mathcal{P}_o(H_{n,i})$$

(9)

is the set of MPMCT gates acting on line $i$ and

$$G_k^n = G^n \cap I_k^n$$

(10)

is the set of all MPMCT gates with $n-k$ control lines. From these sets one can derive

$$G_{n,i}^k = G_{n,i} \cap G_k^n$$

(11)

as the set of all MPMCT gates with $n-k$ controls acting on line $i$.

B. Counting Self-Inverse Functions

In this section we are counting self-inverse functions and subclasses of them. All results are summarized in Table I which also has a row for all reversible functions as a baseline for comparison. There are $2^n!$ reversible functions over $n$ variables due to the one-to-one correspondence with elements in $S_{2^n}$.
Self-inverse functions over \( n \) variables are characterized by their type which is an integer partition of \( 2^n \). In order to count self-inverse functions we exploit properties from integer partitions. Let \( \mu \) be an integer partition that contains \( a_1 \) ones, \( a_2 \) twos, and so on. Then we define

\[
z_\mu = \prod_{i=1}^{n} i^{a_i}. \tag{12}
\]

Lemma 4 ([13]): For a given integer partition \( \mu \) of \( n \), the number of permutations \( \pi \in S_n \) for which \( \text{type}(\pi) = \mu \) is \( \frac{N!}{z_\mu} \).

Based on this lemma, we can count self-inverse functions.

**Theorem 1:** There are

\[
2^{n-1} \left( \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} \right) \tag{13}
\]

self-inverse reversible function on \( n \) variables.

**Proof:** Let \( N = 2^n \) and \( \pi \in S_N \) be an involution, i.e., \( \mu = \text{type}(\pi) \) is an integer partition with \( k \) size(\( \pi \)) occurrences of 2 and \( N - 2k \) occurrences of 1. According to Lemma 4 we know that there exist \( \frac{N!}{z_\mu} \) such involutions, i.e.,

\[
\frac{N!}{z_\mu} = \frac{N!}{1^{N-2k} 2^k (N-2k)! k!} = \frac{N!}{2^k (N-2k)! k! (2k)!} = \frac{(2k)!}{2^k (N-2k)! (2k)!} = \frac{N!}{2^k (N-2k)! (2k)!} = \left( \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} \right)
\]

The value of \( k \) is bounded by \( 0 \) and \( 2^{n-1} \).

From (13) we can deduce

\[
|I_n| = \left( \sum_{k=1}^{n} \binom{n}{k} \right) (2^{n-1} \binom{n}{2k})
\]

which we call palindromic in Table I. The next section determines them as the exact set of involutions that can be realized as palindromic circuit.

We are now considering the subset of self-inverse functions that are represented by one single-target gate. As described above, there are \( n \cdot 2^{n-1} \) single-target gates. Single-target gates are a redundant gate representation since \( n \) gates represent the identity function, i.e., whenever the control function is \( \perp \), independent of the target line position. Hence, the number of functions represented by a single-target gate is

\[
n \cdot 2^{n-1} - n + 1 = n(2^{n-1} - 1) + 1 \tag{14}
\]

MPMCT gates are not redundant and there exist \( n \cdot 3^{n-1} \) such gates for \( n \) variables.

Single transpositions are also a subclass of self-inverse functions and there exist \( 2^{n-1} (2^n - 1) \) transpositions \((a, b)\) over \( n \) variables. One can choose from \( 2^n \) values for \( a \) and from \( 2^n - 1 \) values for \( b \). Since \((a, b) = (b, a)\), the product needs to be halved.

There are some subset relations worth to mention:

\[
\text{reversible} \supset \text{self-inverse} \supset |I_n| \supset \text{single-target gate} \supset \text{MPMCT gate}
\]

**IV. PALINDROMIC CIRCUITS**

A reversible circuit \( C = g_1 g_2 \ldots g_k \) that consists of mixed-polarity multiple-controlled Toffoli gates \( g_i \) is called palindromic if \( g_i = g_{k+1-i} \) for all \( i \in \{1, \ldots, k\} \). The circuit is called even if \( k \) is even and odd otherwise.

**Lemma 5:** A palindromic circuit is even if and only if it realizes the identity function.

**Proof:** Let \( C = g_1 g_2 \ldots g_{2k} \) be an even palindromic circuit. From the definition of a palindromic circuit we have \( g_1 g_2 \ldots g_k g_{k+1} \ldots g_{2k} = g_{k+1} \ldots g_k g_{2k} \ldots g_{k+1} \). Let \( f \) be the function represented by these two subcircuits. Then, \( C \) represents the function \( f \circ f^{-1} = \text{id} \).

Now let \( C = g_1 g_2 \ldots g_k g_{k+1} g_{k+2} \ldots g_{2k+1} \) be an odd palindromic circuit. Let \( f \) be the function represented by \( C \), \( g \) be the function represented by \( g_{k+1} \), and \( \pi_f \) and \( \pi_g \) their permutation representations. According to Lemma 1, we have \( \text{type}(\pi_f) = \text{type}(\pi_g) \) for all \( i \in \{1, \ldots, k\} \). Since \( g \) has the functionality of a single gate we have \( \text{type}(\pi_g) \neq \text{type}(\pi_{id}) \) and therefore \( f \neq \text{id} \).

**Theorem 2:** Let \( f \) be a self-inverse function on \( n \) variables and \( \pi_f \) its permutation representation. Then \( \pi_f \in I_n \) if and only if \( f \) can be realized by an odd palindromic circuit with \( n \) lines.

**Proof:** Direction ‘\( \Rightarrow \)’: Let \( C \) be an odd palindromic circuit that realizes the function \( f \) with middle gate \( g \). Let \( \pi_f \) and \( \pi_g \) their permutation representations. We have \( \pi_g \in G_n \subseteq I_n \). According to Lemma 1 we can imply that \( \pi_f \in I_n \).

Direction ‘\( \Leftarrow \)’: Let \( f \) be a self-inverse function with permutation representation \( \pi_f \) such that \( \pi_f \in I_n^k \). Choose an arbitrary gate \( g \) with permutation representation \( \pi_g \in C_n^k \). According to Lemma 2 we can always find a permutation \( \sigma \) such that \( \pi_f = \sigma \circ \pi_g \circ \sigma^{-1} \). Obviously, \( \pi_f \) can be represented by a palindromic circuit.

**V. ALTERNATIVE CONSTRUCTIONS**

Theorem 2 works only for those self-inverse functions that are in \( I_n \). We will now show two circuit constructions that...
allow to give palindromic circuits for any self-inverse function. The first construction requires an additional line and the second construction requires semi-classical quantum gates.

Both constructions are based on the same idea. Let \( f \) be a self-inverse function with permutation representation \( \pi_f \notin I_n \) such that there exists a \( k \) with \( 2^{k-1} < \text{size}(\pi_f) < 2^k \). Let \( \pi_h \) be some permutation with \( \text{type}(\pi_h) = \text{type}(\pi_f) \) such that there exists a permutation \( \pi_g \in G_n \) with \( \text{size}(\pi_g) = 2^k \) and \( \text{trans}(\pi_h) \subset \text{trans}(\pi_g) \).

**Example 6:** For \( n = 3 \) and \( \pi_f = (0,1)(3,5)(2,7) \) we can choose \( \pi_g = (0,4)(1,5)(2,6)(3,7) \) (i.e., \( T(\emptyset,x_3) \)) and \( \pi_h = (1,5)(2,6)(3,7) \) (i.e., the circuit in Fig. 1).

According to Lemma 2 we can always find a permutation \( \sigma \) such that \( \pi_f = \sigma \circ \pi_h \circ \sigma^{-1} \), however, this cannot be represented as a palindromic circuit because \( \pi_h \notin I_n \). The permutation \( \sigma \circ \pi_g \circ \sigma^{-1} \) can instead be represented as a palindromic circuit, however, it does not represent the same function. Let \( \pi_r = \pi_g \circ \pi_h \). Since \( \text{trans}(\pi_h) \subset \text{trans}(\pi_g) \) we have \( \text{trans}(\pi_r) = \text{trans}(\pi_g) \setminus \text{trans}(\pi_h) \). Note also that we have \( \pi_h = \pi_g \circ \pi_r = \pi_r \circ \pi_g \). In order to represent the same function we need to cancel the transpositions in \( \text{trans}(\pi_r) \) in the circuit computation.

**Example 7:** In the previous example we have \( \pi_r = (0,4) \).

### A. Construction Using An Additional Line

The construction using one additional line is depicted in Fig. 2. The permutation \( \pi_h \) can be realized by \( \pi_r \circ \pi_g \) as described above, where \( \pi_g \) is realized by a single gate and \( \pi_r \) can be realized by \( \text{size}(\pi_r) \) fully controlled Toffoli gates. Storing the value of that construction on a zero-initialized ancilla line in fact computes the result of applying \( \pi_h \). The value can be used to update the intended target line using a single controlled NOT gate. Since all gates in the realization of \( \pi_h \) act on the same target line, they can be arranged arbitrarily, and particularly in reverse order. This restores the zero value on the ancilla line.

### B. Construction Using Quantum Gates

Instead of using an ancilla line one can also use the semi-classical \( V \) gate that performs the so-called square-root of NOT, i.e., two consecutive applications of a \( V \) perform a NOT operation. The circuit construction is depicted in Fig. 3. Every assignment that triggers a transposition in \( \pi_h \) also triggers a transposition in \( \pi_g \) but not in \( \pi_r \). Hence, in that case only \( \pi_g \) is performed and the target line is updated as intended. However, an assignment that triggers \( \pi_g \) but is not in \( \pi_h \) must also trigger a transposition in \( \pi_r \). Since each of the \( V \) gates are fully controlled, two of them are executed which together cancel the update of \( \pi_g \). Due to the construction of \( \pi_r \), there is no such case in which a transposition in \( \pi_r \) is triggered but not \( \pi_g \).

### VI. Conclusions

In this paper we have defined palindromic circuits, a subset of the reversible circuits, and shown the exact subclass of the self-inverse functions that can be realized with such circuits. We have also shown how the complement (still restricted to the self-inverse functions) to this can be constructed with either a reversible circuit and an extra ancilla line or using quantum gates.

To achieve the results, we investigated involutions in the symmetric group \( S_{2^n} \) that are isomorphic to self-inverse reversible functions on \( n \) variables. Specifically, we define the transposition that exactly define a reversible gate and define the rest of the reversible gates using permutation product.

Our results provide a better understanding of the relationship between reversible circuits and invertible functions. The understanding of this relationship is still limited; although we only touched a subset of both areas in this paper, we believe that this paper gives a valuable step forward.

### Acknowledgement

This work was partly funded by the European Commission under the 7th Framework Programme.

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