On various types of density of numerical radius attaining operators

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ABSTRACT
In this paper, we are interested in studying Bishop–Phelps–Bollobás type properties related to the denseness of the operators which attain their numerical radius. We prove that every Banach space with a micro-transitive norm and the second numerical index strictly positive satisfies the Bishop–Phelps–Bollobás point property, and we see that the one-dimensional space is the only one with both the numerical index 1 and the Bishop–Phelps–Bollobás point property. We also consider two weaker properties $L_p^\eta$-nu and $L_0^\eta$-nu, the local versions of Bishop–Phelps–Bollobás point and operator properties respectively, where the $\eta$ which appears in their definition does not depend just on $\varepsilon > 0$ but also on a state $(x, x^*)$ or on a numerical radius one operator $T$. We address the relation between the $L_p^\eta$-nu and the strong subdifferentiability of the norm of the space $X$. We show that finite dimensional spaces and $c_0$ are examples of Banach spaces satisfying the $L_p^\eta$-nu, and we exhibit an example of a Banach space with a strongly subdifferentiable norm failing it. We finish the paper by showing that finite dimensional spaces satisfy the $L_0^\eta$-nu and that, if $X$ has a strictly positive numerical index and has the approximation property, this property is equivalent to finite dimensionality.

1. Introduction

E. Bishop and R. Phelps asked if it was possible to extend their result on denseness of norm attaining functionals to bounded linear operators (see [1]). J. Lindenstrauss, in [2], was the one who gave a negative answer for this question opening possibilities to develop a whole new theory with very elegant and deep results in connection to the geometry of the involved Banach spaces. Parallel to this, I. D. Berg and B. Sims initiated in [3] the study of the numerical radius attaining operators. Recall that an operator $T$ on $X$ attains the numerical radius if there are $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$ and...
$|x^*_0(Tx_0)| = \sup |x^*(Tx)|$, the supremum being taken over all $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Although for the numerical radius one has to deal with the extra condition $x^*(x) = 1$, and the techniques seem to require much more ingenuity and ability, it attracted the attention of many authors to see when the set of all numerical radius attaining operators is norm dense in the set of bounded operators. For instance, C.S. Cardassi proved that such denseness holds for classical Banach spaces as $\ell_1$, $c_0$, and $L_1(\mu)$ as well as for uniformly smooth Banach spaces (see [4–6]). We emphasize the fact that the denseness holds for every Banach space with the Radon-Nikodým property (see [7, Theorem 2.4]) but it does not hold in general (see [8, Section 2]).

After the Bishop-Phelps theorem had been shown, B. Bollobás in [9] improved the theorem in the following sense: for given norm one elements $x$ and $x^*$ such that $x^*(x) \approx 1$, it is possible to get new elements $y$ and $y^*$ such that $y^*$ attains the norm at $y$, $y \approx x$, and $y^* \approx x^*$. That is, one can control the distances between the points and the functionals simultaneously. Since Bollobás’ theorem is no longer true for operators due to Lindenstrauss’ results, M. Acosta, R. Aron, D. García, and M. Maestre introduced a new property, which opened even more possibilities to develop the theory, called the Bishop–Phelps–Bollobás property (see [10]). A pair of Banach spaces $(X, Y)$ satisfies the Bishop–Phelps–Bollobás property if it is possible to get a Bollobás’ theorem for bounded linear operators from $X$ into $Y$, that is, for given norm one bounded linear operator $T$ and element $x$ such that $\|Tx\| \approx 1$, there are a new norm one bounded linear operator $S$ and $x_0$ such that $S$ attains the norm at $x_0$, with $x_0 \approx x$ and $S \approx T$. It is clear that this property implies the denseness of all operators that attain the norm. Again, parallel to the study of the Bishop–Phelps–Bollobás property, the study of this property for numerical radius was initiated (see [11–13]) and it brings us to the main topic of this paper.

Let $X$ be a Banach space over the scalar field $\mathbb{K}$, which can be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Unless otherwise stated, all the results are valid for both real and complex Banach spaces, but we give their proofs only for complex spaces since the others can be proved analogously with simpler proofs. We denote by $S_X$ the unit sphere of $X$. The Banach space of all bounded linear operators from $X$ into itself is denoted by $\mathcal{L}(X)$ with the operator norm $\|T\| := \sup\{\|T(x)\| : x \in S_X\}$ for each $T \in \mathcal{L}(X)$. Especially, the dual space of $X$ is written as $X^*$. We define the set of states of $X$ by

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}.$$  

The numerical radius of $T \in \mathcal{L}(X)$ and the numerical index of $X$ are defined, respectively, by

$$v(T) = \sup\{|x^*(Tx)| : (x, x^*) \in \Pi(X)\} \quad \text{and}$$

$$n(X) = \inf\{v(T) : T \in \mathcal{L}(X), \quad \|T\| = 1\}.$$  

It is clear that $0 \leq n(X) \leq 1$ and $n(X)\|T\| \leq v(T) \leq \|T\|$ for all $T \in \mathcal{L}(X)$. So, if $n(X) = 1$, then $\|T\| = v(T)$ for every operator $T \in \mathcal{L}(X)$ and we are using this fact throughout the paper without any explicit reference. Moreover, $v(\cdot)$ is a seminorm in $\mathcal{L}(X)$ and if $n(X) > 0$, then it becomes an equivalent norm to the usual one on $\mathcal{L}(X)$. With this notation in mind, we say that $T \in \mathcal{L}(X)$ attains the numerical radius if there is $(x_0, x_0^*) \in \Pi(X)$ such
that $|x_0^*(T(x_0))| = v(T)$. We denote by NRA($X$) the set of all numerical radius attaining operators on $X$. We refer the interested reader in this topic to the classical books [14,15].

We say that $X$ satisfies the Bishop–Phelps–Bollobás property for numerical radius (BPBp-nu, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X)$ with $v(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy

$$|x^*(T(x))| > 1 - \eta(\varepsilon),$$

there are $S \in \mathcal{L}(X)$ with $v(S) = 1$ and $(y, y^*) \in \Pi(X)$ such that

$$|y^*(S(y))| = 1, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$ 

It is immediate to see that if $X$ satisfies the BPBp-nu, then the set of all operators that attain the numerical radius is norm dense in $\mathcal{L}(X)$. As an overview of the known results, a Banach space $X$ satisfies the BPBp-nu when

- $X$ is finite dimensional (see [13, Proposition 2]);
- $X$ is uniformly convex and uniformly smooth, $n(X) > 0$ (see [13, Proposition 4 and 6]);
- $X$ is a Hilbert space (see [16, Corollary 3.3]);
- $X$ is $c_0$ or $\ell_1$ (see [12]);
- $X$ is $L_1(\mu)$ (see [13, Theorem 9] and also [11, Theorem 9]);
- $X$ is the subspace of all
  - finite-rank operators on $L_1(\mu)$;
  - compact operators on $L_1(\mu)$;
  - weakly compact operators on $L_1(\mu)$, where $\mu$ is $\sigma$-finite measure space (see [17, Theorem 2.1 and Corollary 2.1]);
- $X$ is $C(K)$ with compact metrizable $K$ (real) (see [18, Theorem 2.2]).

Very recently, a stronger property than the BPBp-nu was considered in [19, Theorem 2.5]. Stronger in the sense that if we have $T \in \mathcal{L}(X)$ with $v(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfying $|x^*(T(x))| \approx 1$, then the new operator $S \in \mathcal{L}(X)$ with $v(S) = 1$ will satisfy $|x^*(S(x))| = 1$ and $S \approx T$. That is, we do not change the initial state $(x, x^*)$ where $T$ almost attains the numerical radius. As it occurs with the BPBp-nu, which is a numerical radius version of the Bishop–Phelps–Bollobás property for operators defined in [10], this new property is the corresponding of the Bishop–Phelps–Bollobás point property (see [20] and the references therein) for the numerical radius. This is one of the properties we are focussing on in this paper. We give the precise definition in the next section. We comment that analogously the operator property can be considered by fixing the operator $T$ instead of fixing the state $(x, x^*)$ which is the corresponding version of the Bishop–Phelps–Bollobás operator property (see [21] and the references therein). However, it is not difficult to see that this property implies the domain space is one-dimensional.

Let us now describe the content of this paper. In the next section, we consider the Bishop–Phelps–Bollobás point property for numerical radius. In Theorem 2.2, we prove that a space with a micro-transitive norm and the second numerical index strictly positive satisfies the point property for the numerical radius. In particular, real Hilbert spaces satisfy this property, a result that generalizes [19, Theorem 2.5]. We also show that, for Banach spaces with numerical index 1, the point property for the numerical radius is too strong, in the sense that just one-dimensional spaces enjoy it (see Proposition 2.5).
In Sections 3 and 4, we consider the corresponding local versions of the already mentioned point and operator properties for the numerical radius, meaning that the \( \eta \) that appears in their definitions depends not only on \( \varepsilon \), but also on a state \((x,x^*)\) \( \in \Pi(X) \) or on a numerical radius one operator \( T \). In Proposition 3.2 we prove that the local point property for the numerical radius implies that the Banach space must have a strongly subdifferentiable norm, whenever its numerical index is 1. We also prove that finite dimensional spaces with \( n(X) > 0 \) satisfy it. In particular, every complex finite dimensional Banach space satisfies this property. Moreover, in Theorem 3.7 we prove that the Banach space \( c_0 \) has the local point property for the numerical radius (whereas, on the other hand, \( \ell_1 \) fails it). As the strong subdifferentiability of the norm is very related to this property, we also ask if it is a sufficient condition for the space to have the local point property: in Theorem 3.14 we see that this is not the case, by exhibiting a counterexample. We finish the paper by considering the local operator property for the numerical radius. In Theorem 4.2 and Proposition 4.3 we prove that every finite dimensional space enjoys this property and that, for spaces \( X \) with the approximation property and \( n(X) > 0 \), the local operator property is, indeed, equivalent to finite dimensionality of the space.

2. The point property for numerical radius

In this section, we study the (uniform) Bishop–Phelps–Bollobás point property for numerical radius. By uniform, we mean that the \( \eta \) that appears in their definitions depends just on a given \( \varepsilon > 0 \) (in contrast with the local properties defined in Section 3, \( \eta \) depends on \( \varepsilon \) and a state, or \( \varepsilon \) and an operator). It is worth noting that the Bishop–Phelps–Bollobás point property for numerical radius was already introduced by Choi et al. [19] in the context of complex Hilbert spaces.

**Definition 2.1:** Let \( X \) be a Banach space. We say that \( X \) has the Bishop–Phelps–Bollobás point property for numerical radius (BPBpp-nu, for short) if given \( \varepsilon > 0 \), there is \( \eta(\varepsilon) > 0 \) such that whenever \( T \in \mathcal{L}(X) \) with \( \nu(T) = 1 \) and \((x,x^*) \in \Pi(X)\) satisfy \(|x^*(T(x))| > 1 - \eta(\varepsilon)\), there exists \( S \in \mathcal{L}(X) \) with \( \nu(S) = 1 \) such that \(|x^*(S(x))| = 1\) and \( \|S - T\| < \varepsilon\).

We prove that every Banach space with a micro-transitive norm and the second numerical index strictly positive satisfies the BPBpp-nu and, also, that this property is too strict when we consider Banach spaces with numerical index 1. In order to do this, we need some background.

Given a Banach space \( X \), an operator \( T \in \mathcal{L}(X) \) is said to be skew-hermitian if \( \nu(T) = 0 \). We denote by \( \mathcal{Z}(X) \) the set of all skew-hermitian operators on \( X \), which is a closed subspace of \( \mathcal{L}(X) \). In the quotient space \( \mathcal{L}(X)/\mathcal{Z}(X) \), we define \( \|T + \mathcal{Z}(X)\| := \inf\{\|T - S\| : S \in \mathcal{Z}(X)\} \). Then, we have \( \nu(T) \leq \|T + \mathcal{Z}(X)\| \) for every \( T \in \mathcal{L}(X) \). The second numerical index of \( X \) is defined by

\[
n'(X) = \max\{k \geq 0 : k\|T + \mathcal{Z}(X)\| \leq \nu(T) \quad \forall \ T \in \mathcal{L}(X)\}
\]

\[
= \inf \left\{ \frac{\nu(T)}{\|T + \mathcal{Z}(X)\|} : T \in \mathcal{L}(X) \setminus \mathcal{Z}(X) \right\}.
\]

We refer the interested reader on this topic to [16], where many properties of the second numerical index were obtained and the condition \( n'(X) = 1 \) was intensively studied.
Let $G$ be a Hausdorff topological group with the identity element $e$ and $T$ be a Hausdorff topological space. An action $(\cdot \cdot)$ of $(G, T)$ is a continuous function from $G \times T$ to $T$ such that $(e, t) = t$ and $(g_1, (g_2, t)) = (g_1g_2, t)$ for every $g_1, g_2 \in G$ and $t \in T$. The action is said to be transitive if $T = \{(g, t) : g \in G\}$ for every $t \in T$, and said to be micro-transitive if $\{(g, t) : g \in U\}$ is a neighbourhood of $t$ in $T$ for every $t \in T$, whenever $U$ is a neighbourhood of $e$ in $G$. Given a Banach space $X$, we may take the group of surjective isometries on $X$ as the group $G$ and $S_X$ as the topological space $T$. We then say that $X$ (or the norm of $X$) is micro-transitive (respectively, transitive) if the canonical action is micro-transitive (respectively, transitive). It is known that micro-transitivity of a norm implies transitivity. The famous open problem, known as the Mazur rotation problem, asks whether transitive separable Banach spaces are isometrically isomorphic to Hilbert spaces (see, for instance, [22]). It is worth remarking that the non-separable version of the Mazur rotation problem had been solved negatively by S. Rolewicz (see [23]) and Hilbert spaces are the only known spaces with micro-transitive norms. We kindly send the interested reader on this topic to [24,25] and the references therein.

Our main result in this section is the following.

**Theorem 2.2:** Let $X$ be a Banach space. Suppose that the norm of $X$ is micro-transitive and that $n'(X) > 0$. Then, $X$ satisfies the BPBp-nu.

Let us notice that if $X$ has a micro-transitive norm, then there is a function $\beta : (0, 2) \rightarrow \mathbb{R}^+$ such that whenever $x, y \in S_X$ satisfy $\|x - y\| < \beta(\varepsilon)$, there is a surjective isometry $T \in \mathcal{L}(X)$ satisfying $T(x) = y$ and $\|T - \text{Id}_X\| < \varepsilon$, where $\text{Id}_X$ is the identity operator on $X$ (see [26, Proposition 2.1]). It is worth remarking that we may take $\beta$ so that $\beta(\varepsilon) < \varepsilon$ for any $\varepsilon \in (0, 2)$.

**Proof of Theorem 2.2:** Suppose that $X$ is micro-transitive with some function $\varepsilon \mapsto \beta(\varepsilon)$. Then, by [26, Corollary 2.13], we see that $X$ is both uniformly convex and uniformly smooth. Moreover since $n'(X) > 0$, $X$ satisfies the BPBp-nu with some function $\varepsilon \mapsto \eta(\varepsilon)$ (see [16, Theorem 3.2] and [13, Proposition 4]).

Let $\varepsilon \in (0, 1)$ be given and set

$$
\varepsilon' := \frac{n'(X)}{2 + 5n'(X)} \varepsilon > 0 \quad \text{and} \quad \eta'(\varepsilon') := \eta(\beta(\varepsilon')) > 0.
$$

Let $(x, x^*) \in \Pi(X)$ and $T \in \mathcal{L}(X)$ with $\nu(T) = 1$ be such that $|x^*(T(x))| > 1 - \eta'(\varepsilon')$. From the definition of the second numerical index, there exists $G \in Z(X)$ so that

$$
\|T + G\| = \frac{\nu(T)}{n'(X)} + \varepsilon' = \frac{1}{n'(X)} + \varepsilon'.
$$

Since $\nu(G) = 0$, we also have that $\nu(T + G) = 1$ and $|x^*((T + G)(x))| = |x^*(T(x))| > 1 - \eta'(\varepsilon')$. Therefore, there are $(y, y^*) \in \Pi(X)$ and $S_1 \in \mathcal{L}(X)$ with $\nu(S_1) = 1$ such that

$$
|y^*(S_1(y))| = 1 \quad \text{and} \quad \max \left\{|y^* - x^*|, |y - x|, |S_1 - (T + G)|\right\} < \beta(\varepsilon') < \varepsilon'.
$$

In particular $|y - x| < \beta(\varepsilon')$ and, since $X$ is micro-transitive with the function $\beta$, there is a linear surjective isometry $U \in \mathcal{L}(X)$ such that $U(x) = y$ and $\|U - \text{Id}_X\| < \varepsilon'$. Now, notice
that $1 = y^*(y) = y^*(U(x)) = (U^*(y^*))(x)$. Since $X$ is uniformly smooth and $x^*(x) = 1$, we have that $U^*(y^*) = x^*$. Analogously, we get that $(U^{-1})^*(x^*) = y^*$.

Define $S_2 := U^{-1} \circ S_1 \circ U \in \mathcal{L}(X)$. Since $((U^{-1})^*(x^*))(U(z)) = z^*(U^{-1}U(z)) = z^*(z) = 1$, whenever $(z, z^*) \in \Pi(X)$, we have that $\nu(S_2) \leq \nu(S_1) = 1$. Moreover, the equality

$$|x^*(S_2(x))| = |(x^*(U^{-1}S_1U)(x))| = |((U^{-1})^*(x^*))(S_1U(x))| = |y^*(S_1(y))| = 1$$

shows that $S_2$ has numerical radius 1 and it is attained at $(x, x^*) \in \Pi(X)$. Consequently, the operator $S := S_2 - G$ also has numerical radius 1 and it is attained at $(x, x^*)$. Finally,

$$\|S_2 - G - T\| \leq \|S_2 - S_1\| + \|S_1 - (T + G)\|$$

$$= \|U^{-1}S_1U - S_1U\| + \|S_1U - S_1\| + \epsilon'$$

$$< 2\epsilon'\|S_1\| + \epsilon'$$

$$< 2\epsilon'\left(\frac{1}{n'(X)} + 2\epsilon'\right) + \epsilon'$$

$$= \left(2 + (4\epsilon' + 1)n'(X)\right)\epsilon' \leq \epsilon.$$

Since real Hilbert spaces have second numerical index 1 (see [16, Theorem 2.3]), real Hilbert spaces satisfy the BPBpp-nu. On the other hand, since the numerical index of a complex Banach space $X$ is always greater than or equal to $1/e$ (and, hence, strictly positive), we have $\mathcal{Z}(X) = \{0\}$ and, consequently, $n'(X) = n(X) > 0$. This means that we have the same result for complex Hilbert spaces. We state these results in the next corollary, which should be compared with [19, Theorem 4.1.(a)].

**Corollary 2.3:** Let $H$ be a (real or complex) Hilbert space. Then, $H$ has the BPBpp-nu.

In what follows, we focus on spaces with numerical index 1. Examples of such spaces include $C(K)$-spaces, $L_1(\mu)$-spaces, isometric preduals of $L_1(\mu)$, and all function algebras, such as the disk algebra $A(\mathbb{D})$ and $H^\infty$. We prove that these spaces don’t have the BPBpp-nu. To prove this, we need the following characterizations of uniformly smooth and uniformly convex Banach spaces. On the one hand, we have that $X$ is uniformly smooth if and only if given $\epsilon > 0$, there is $\eta(\epsilon) > 0$ such that whenever $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|x^*(x)| > 1 - \eta(\epsilon)$, there is $x_0^* \in S_{X^*}$ such that $|x_0^*(x)| = 1$ and $\|x^* - x_0^*\| < \epsilon$ (see [20, Proposition 2.1]). On the other hand, it is proved in [27, Theorem 2.1] that $X$ is uniformly convex if and only if given $\epsilon > 0$, there is $\eta(\epsilon) > 0$ such that whenever $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|x^*(x)| > 1 - \eta(\epsilon)$, there is $x_0 \in S_X$ such that $|x^*(x_0)| = 1$ and $\|x_0 - x\| < \epsilon$.

**Lemma 2.4:** If a Banach space $X$ with $n(X) = 1$ has the BPBpp-nu, then $X$ is uniformly smooth.
**Theorem 2.1**, \[ 29, 30 \]. Here, we keep a similar notation.

Motivated by the very restrictive behaviour of the Bishop–Phelps–Bollobás point property for the numerical radius (as we have seen in the previous section), we are now dealing with a weaker property. By weaker, we mean that the \( \eta \) that appears in the definition of the BPBp-nu does not depend just on a given \( \varepsilon > 0 \) but also on the state \( (x, x^*) \in \Pi(X) \).

For the Bishop–Phelps–Bollobás properties, this local variant was already considered in \[ 29, 30 \]. Here, we keep a similar notation.

**Definition 3.1:** Let \( X \) be a Banach space. We say that \( X \) has the \( L_{p,p}-nu \) if given \( \varepsilon > 0 \) and \( (x, x^*) \in \Pi(X) \), there is \( \eta(\varepsilon, (x, x^*)) > 0 \) such that whenever \( T \in \mathcal{L}(X) \) with \( \nu(T) = 1 \) satisfies \( |x^*(T(x))| > 1 - \eta(\varepsilon, (x, x^*)) \), there is \( S \in \mathcal{L}(X) \) with \( \nu(S) = 1 \) such that \( |x^*(S(x))| = 1 \) and \( \|S - T\| < \varepsilon \).
It is immediate to notice that the BPBpp-nu implies the \( L_{p,p} \)-nu. We start by proving that all spaces with numerical index 1 which satisfy the \( L_{p,p} \)-nu must be strongly subdifferentiable. Let us recall that a norm in a Banach space \( X \) is strongly subdifferentiable (SSD, for short) at \( x \in X \) whenever the limit \( \lim_{t \to 0^+} \frac{1}{t} (\|x + th\| - \|x\|) \) exists uniformly for \( h \in B_X \). If this happens for every \( x \in S_X \), we say that the norm of \( X \) is SSD. The norm of any finite dimensional space and the sup-norm on \( c_0 \) are examples of SSD norms. Moreover, the \( \ell_1 \)-norm is SSD only at points in the sphere of \( \ell_1 \) that are sequences with finitely many nonzero terms. For a background on this topic, we refer the reader to [31] and the references therein.

**Proposition 3.2:** Let \( X \) be a Banach space with \( n(X) = 1 \). If \( X \) has the \( L_{p,p} \)-nu, then the norm of \( X \) is SSD.

**Proof:** In order to prove that the norm of \( X \) is SSD, we use a characterization given in [29, Theorem 2.3.(a)], which says that the norm of \( X \) is SSD at \( x \in S_X \) if and only if given \( \varepsilon > 0 \), there is \( \eta(\varepsilon, x) > 0 \) such that whenever \( x^* \in S_{X^*} \) satisfies \( |x^*(x)| > 1 - \eta(\varepsilon) \), there is \( z^* \in S_{X^*} \) such that \( |z^*(x)| = 1 \) and \( \|z^* - x^*\| < \varepsilon \). Let \( \varepsilon > 0 \) and \( x_0 \in S_X \) be given.

Let \( x_0^* \in S_{X^*} \) be such that \( \|x_0^*(x_0)\| = 1 \), which implies \( (x_0^*, x_0^0) \in \Pi(X) \). Since \( X \) has the \( L_{p,p} \)-nu, we may consider \( \eta(\varepsilon, x_0) := \eta(\varepsilon, (x_0^*, x_0^0)) > 0 \). Let \( x^* \in S_{X^*} \) be a functional such that \( |x^*(x_0)| > 1 - \eta(\varepsilon, x_0) \) and define \( T := x^* \otimes x_0 \). Then \( \nu(T) = \|T\| = \|x^*\| = 1 \) and \( |x^*_0(T(x_0))| = |x^*(x_0)| > 1 - \eta(\varepsilon, x_0) \). By hypothesis, there is \( S \in \mathcal{L}(X) \) with \( \nu(S) = \|S\| = 1 \) such that \( |x_0^*(S(x_0))| = 1 \) and \( \|S - T\| < \varepsilon \). Setting \( z^* := S^*x_0^* \in B_{X^*}, \) we have that \( |z^*(x_0)| = |(S^*(x_0^*))(x_0)| = |x_0^*(S(x_0))| = 1 \). Moreover, we get that \( \|z^* - x^*\| < \varepsilon \). Indeed, for arbitrary \( x \in S_X \),

\[
|z^*(x) - x^*(x)| = |(S^*(x_0^*))x_0^*(x) - x^*(x)| \\
= |x_0^*(S(x)) - x^*(x_0)| \\
= |x_0^*(S(x)) - x_0^*(x_0) - x_0^*(T(x))| \\
\leq \|S(x) - T(x)\|.
\]

This implies that \( \|z^* - x^*\| \leq \|S - T\| < \varepsilon \) which shows the norm of \( X \) is SSD at \( x_0 \in S_X \). Since \( x_0 \) is arbitrary, the norm of \( X \) is SSD.

On the other hand, we do not know what happens in general.

**Question 3.3:** Let \( X \) be a Banach space having the \( L_{p,p} \)-nu such that \( n(X) \neq 1 \). Does \( X \) have SSD norm?

Let us show now some positive results regarding the validity of the \( L_{p,p} \)-nu. Thanks to Proposition 3.2, if we are looking for spaces satisfying the \( L_{p,p} \)-nu, it is natural to look at those with strong subdifferentiable norm, even in case that \( 0 < n(X) < 1 \). Since the norm of finite dimensional spaces are SSD, we analyse the \( L_{p,p} \)-nu for these spaces. Suppose that \( X \) is finite dimensional and does not have the property \( L_{p,p} \)-nu. Then, by definition, there
are $\varepsilon_0 > 0$ and $(x_0, x_0^*) \in \Pi(X)$ such that there exists a sequence of operators $(T_n) \subset \mathcal{L}(X)$ so that

$$v(T_n) = 1 \geq |x_0^*(T_n(x_0))| \geq 1 - \frac{1}{n}$$

for all $n \in \mathbb{N}$ and, whenever $S \in \mathcal{L}(X)$ satisfies $v(S) = 1$ and $\|T - S\| < \varepsilon$, the number $|x_0^*(S(x_0))|$ is strictly smaller than 1. If the set of operators with numerical radius 1 is compact, then a subsequence of $(T_n)$ converges to an operator $T_0$, also with numerical radius 1, and, in this case, we have $|x_0^*(T_0(x_0))| = 1$, which is a contradiction. It is clear that if $n(X) > 0$, then $v(\cdot)$ is an equivalent norm on $\mathcal{L}(X)$. Hence, if $X$ is finite dimensional and $n(X) > 0$, the closed unit ball of $(\mathcal{L}(X), v(\cdot))$ is compact. So, we have the following result.

**Proposition 3.4:** Let $X$ be a finite dimensional Banach space with $n(X) > 0$. Then, $X$ satisfies the property $L_{p,p}$-nu.

Arbitrary complex Banach spaces always have strictly positive numerical index. So, we have the following consequence from Theorem 3.4.

**Corollary 3.5:** Every finite dimensional complex Banach space has the $L_{p,p}$-nu.

We do not know if the same statement holds for real spaces. However, we can conclude that it is true for 2-dimensional real Banach spaces. Indeed, if $X$ is 2-dimensional and $n(X) = 0$, then $X$ is isometrically isomorphic to the 2-dimensional Hilbert space (see [32, Theorem 3.1]) and, consequently, it has the BPBpp-nu.

**Question 3.6:** Do finite dimensional spaces with $n(X) = 0$ have the $L_{p,p}$-nu?

Next, we consider the property on $c_0$ which has a SSD norm and has the numerical index 1. It is known that the pairs $(c_0, c_0)$ and $(c_0, X)$, where $X$ is $\mathbb{C}$-uniformly convex, satisfy the $L_{p,p}$ (see [29, Proposition 2.8 and Theorem 2.12], respectively). In particular, the pair $(c_0, L_p(\mu))$ satisfies it for a positive measure $\mu$ and $1 \leq p < \infty$. Here, we have the following result.

**Theorem 3.7:** $c_0$ satisfies the $L_{p,p}$-nu.

In order to prove this theorem, we need two auxiliary results. The first one concerns a characterization of the strong subdifferentiability of the norm of a Banach space in terms of finite convex sums, which was proved in [30]. The second is a straightforward fact about functionals on $c_0$ which attain the norm.

**Lemma 3.8 ([30, Proposition 3.2]):** For every Banach space $Y$, the following are equivalent.

(a) The norm of $Y$ is SSD.
(b) For each $\varepsilon > 0$, $y \in S_Y$ and a finite sequence $(\alpha_j)_{j \in A}$ of positive numbers with $\sum_{j \in A} \alpha_j = 1$, there exists $\eta = \eta(\varepsilon, (\alpha_j)_{j \in A}, y) > 0$ such that whenever a sequence of functionals
\[(y_j^*)_{j \in A} \subset By^*\) satisfies
\[
\text{Re} \sum_{j \in A} \alpha_j y_j^*(y) > 1 - \eta,
\]
there is \((z_j^*)_{j \in A} \subset S_{Y^*}\) such that
\[
z_j^*(y) = 1 \quad \text{and} \quad \|z_j^* - y_j^*\| < \varepsilon \quad \text{for all} \ j \in A.
\]

**Lemma 3.9:** Let \(x^* = (x_1^*)_{i=1}^{\infty} \in S_{c^*}\) be a linear functional on \(c_0\) and suppose that it attains the norm at \(x_0 = (x_0(i))_{i=1}^{\infty} \in S_{c_0}\). If there is \(j \in \mathbb{N}\) such that \(0 < |x_0(j)| < 1\), then \(x^*(e_j) = 0\), where \((e_i)_{i=1}^{\infty}\) is the canonical basis of \(c_0\).

**Proof of Theorem 3.7:** Denote the canonical basis of \(c_0\) and \(\ell_1\) by \((e_i)\) and \((e_i^*)\), respectively. Let \((x_0, x_0^*) \in \Pi(c_0)\) be given. We write \(x_0 = (x_0(i))_{i=1}^{\infty} = \sum_{i=1}^{\infty} x_0(i) e_i \in S_{c_0}\). Since \(x_0 \in S_{c_0}\), there is a finite collection \(A = \{n_1, \ldots, n_m\} \subset \mathbb{N}\) such that \(|x_0(n_i)| = 1\) for \(i = 1, \ldots, m\). By Lemma 3.9, we have
\[
x_0^* = \alpha_1 e_{n_1}^* + \cdots + \alpha_m e_{n_m}^* \quad \text{with} \quad \|x_0^*\| = \sum_{i=1}^{m} |\alpha_i| = 1.
\]
We may suppose that all \(\alpha_i \neq 0\) and notice that
\[
|\alpha_1| |x_0(n_1)| + \cdots + |\alpha_m| |x_0(n_m)| = |\alpha_1| + \cdots + |\alpha_m| = 1 = x_0^*(x_0) = \alpha_1 x_0(n_1) + \cdots + \alpha_m x_0(n_m)
\]
This implies that \(\alpha_i x_0(n_i) = |\alpha_i x_0(n_i)|\) for every \(i = 1, \ldots, m\). Let \(\varepsilon > 0\) and \(0 < \xi < \frac{\varepsilon}{4}\) be given. Since the norm of \(c_0\) is SSD, using \(\eta\) of Lemma 3.8, we consider
\[
\eta := \eta(\xi, (|\alpha_i|)_{i=1}^{m}, x_0) > 0.
\]
Let \(T \in \mathcal{L}(c_0)\) with \(\nu(T) = \|T\| = 1\) be such that \(|x_0^*(T(x_0))| > 1 - \eta\). For a suitable modulus 1 scalar \(r\) and \(y_{n_i}^* = \frac{\alpha_i}{|\alpha_i|} e_{n_i}^*\) for \(i = 1, \ldots, m\), we have
\[
|\alpha_1 y_{n_1}^*(T(x_0)) + \cdots + |\alpha_m y_{n_m}^*(T(x_0)) = |x_0^*(T(x_0))| > 1 - \eta.
\]
Then, by Lemma 3.8, there is \((z_{n_i}^*)_{i=1}^{m} \subset S_{c^*}\) such that \(z_{n_i}^*(x_0) = 1\) and \(\|z_{n_i}^* - y_{n_i}^*\| < \xi\) for \(i = 1, \ldots, m\). Now, define \(S \in \mathcal{L}(c_0)\) by
\[
S(x) := T(x) + \sum_{i=1}^{m} \left(1 + \frac{\varepsilon}{4}\right) z_{n_i}^*(x) - y_{n_i}^*(T(x)) y_{n_i}
\]
where \(y_{n_i} := \frac{\alpha_i}{|\alpha_i|} e_{n_i}\) for \(i = 1, \ldots, m\) and \(x \in c_0\). Then, it is clear that
\[
S^*(y^*) = T^*(y^*) + \sum_{i=1}^{m} y^*(y_{n_i}) \left(1 + \frac{\varepsilon}{4}\right) z_{n_i}^* - T^*(y_{n_i}^*) \quad (y^* \in c_0^*).
\]
We have that if \( n \neq n_1, \ldots, n_m, \) then \( \|S^*(e^*_n)\| = \|T^*(e^*_n)\| \leq 1. \) On the other hand, if \( n = n_j \) for some \( j = 1, \ldots, m, \) then
\[
\|S^*(e^*_n)\| = \|S^*(\gamma^*_n)\| = \left\| T^*(\gamma^*_n) + \left(1 + \frac{\varepsilon}{4}\right) z^*_n - T^*(\gamma^*_n) \right\| = 1 + \frac{\varepsilon}{4}.
\]

Since \( v(S) = \|S\| = \|S^*\| = \sup_n \|S^*(e^*_n)\|, \) we have that \( v(S) = 1 + \frac{\varepsilon}{4}. \) Moreover, we have that
\[
|\tilde{x}^*_0(S(x_0))| = \left| x^*_0(T(x_0)) + \sum_{i=1}^{m} \left[ \left(1 + \frac{\varepsilon}{4}\right) z^*_n(x_0) - y^*_n(T(x_0)) \right] x^*_0(y_n) \right|
\]
\[
= x^*_0(T(x_0)) + \sum_{i=1}^{m} |\alpha_i| \left(1 + \frac{\varepsilon}{4}\right) - \sum_{i=1}^{m} |\alpha_i| y^*_n(T(x_0))
\]
\[
= \sum_{i=1}^{m} \alpha_i e^*_i(T(x_0)) + \sum_{i=1}^{m} |\alpha_i| \left(1 + \frac{\varepsilon}{4}\right) - \sum_{i=1}^{m} \alpha_i e^*_i(T(x_0))
\]
\[
= 1 + \frac{\varepsilon}{4}.
\]

Then, \( v(S) = \|S\| = |\tilde{x}^*_0(S(x_0))| = 1 + \frac{\varepsilon}{4}. \) Hence, the inequality
\[
\frac{S}{1 + \frac{\varepsilon}{4}} - T \leq \frac{S}{1 + \frac{\varepsilon}{4}} - S + \|S - T\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon
\]
shows that \( \frac{S}{1 + \frac{\varepsilon}{4}} \in \mathcal{L}(X) \) is the desired operator. \( \blacksquare \)

**Remark 3.10:** Theorem 3.7 shows that the property \( L_{p,p} \)-nu is strictly weaker than the BPBp-nu since \( c_0 \) fails to have the BPBpp-nu (see Proposition 2.5). Concerning \( \ell_1 \), since the numerical index of \( \ell_1 \) is 1 and the norm of \( \ell_1 \) is not SSD, it cannot have the property \( L_{p,p} \)-nu by Proposition 3.2. Moreover, we can also notice that the denseness of numerical radius attaining operators does not imply the property \( L_{p,p} \)-nu since \( \overline{\text{NRA}}(\ell_1) = \mathcal{L}(\ell_1) \) (see [5]) but \( \ell_1 \) fails the property \( L_{p,p} \)-nu. On the other hand, we do not know whether the \( L_{p,p} \)-nu implies the denseness of numerical radius attaining operators or not.

**Question 3.11:** Let \( X \) be a Banach space with the \( L_{p,p} \)-nu. Is it true that \( \overline{\text{NRA}}(X) = \mathcal{L}(X) \)?

In view of Theorems 3.4 and 3.7, it is natural to ask whether the strong subdifferentiability of the norm is a sufficient condition for the validity of the property \( L_{p,p} \)-nu. We show that this is not the case by exhibiting the following counterexample. Let \( Z \) be the space \( c_0 \) equipped with the equivalent strictly convex norm
\[
\|x\|_Z = \|x\|_\infty + \left( \sum_{i=1}^{\infty} \frac{|x(i)|^2}{2^i} \right)^{1/2}.
\]
This space appears in classical counterexamples for norm attaining results. In [2] it is proved that the set \( \text{NA}(c_0, Z) \) is not dense in \( \mathcal{L}(c_0, Z) \), where \( \text{NA}(c_0, Z) \) is the set of all
norm attaining operators from \( c_0 \) to \( Z \). Indeed, if \( T \in \text{NA}(c_0, Z) \) then there is \( n_0 \in \mathbb{N} \) such that \( T(e_n) = 0 \) for all \( n \geq n_0 \). Hence, the formal identity \( \text{id} \colon c_0 \to Z \) cannot be approximated by norm attaining operators. As a consequence, it is also shown in [2] that, if we take \( X = c_0 \oplus_\infty Z \), then \( \text{NA}(X, X) \) is not dense in \( \mathcal{L}(X, X) \). When looking at the denseness of numerical radius attaining operators, the space \( X = c_0 \oplus_\infty Z \) appears as a natural candidate to show that \( \text{NRA}(X, X) \) is not dense in \( \mathcal{L}(X, X) \) and, indeed, this is the main result in [8]. Before showing the desired counterexample, we observe two facts.

**Remark 3.12:** The norm of the Banach space \( c_0 \oplus_\infty Z \) is strongly subdifferentiable.

**Proof:** Since the norm of \( c_0 \) is SSD, by [31, Proposition 2.2] it suffices to show that the norm of \( Z \) is SSD. We want to prove that, for each \( z \in S_Z \), the one-sided limit \( \lim_{t \to 0^+} \| z + th \|_Z - 1 \) exists uniformly for \( h \in B_Z \). Consider \( \tilde{z} = (\frac{z(i)}{\sqrt{2}}) \) and \( \tilde{h} = (\frac{h(i)}{\sqrt{2}}) \) and note that

\[
\frac{\| z + th \|_Z - 1}{t} = \frac{\| z + th \|_Z - \| z \|_Z}{t} + \frac{\| z + th \|_2 - \| \tilde{z} \|_2}{t}.
\]

On the one hand, since \( z \in c_0 \) and \( (c_0, \| \cdot \|_Z) \) is SSD, we have that \( \lim_{t \to 0^+}(I_t) \) exists uniformly for \( h \in B_Z \) (note that if \( h \in B_Z \), then \( h \in B_{c_0} \)). On the other hand, since \( \tilde{z} \in \ell_2 \) and \( (\ell_2, \| \cdot \|_2) \) is SSD, then \( \lim_{t \to 0^+}(I_{i_t}) \) exists uniformly for \( h \in B_Z \) (again, note that if \( h \in B_Z \), then \( \tilde{h} \in B_{\ell_2} \)). Therefore, \( \lim_{t \to 0^+} \| z + th \|_Z - 1 \) exists uniformly for \( h \in B_Z \).

**Remark 3.13:** Suppose that a Banach space \( X \) has the \( L_{p,p} \)-nu. Then, given \( \varepsilon > 0 \) and \( (x, x^*) \in \Pi(X) \), there is \( \tilde{\eta}(\varepsilon, (x, x^*)) > 0 \) such that, whenever \( T \in \mathcal{L}(X) \) with \( \nu(T) \leq \| T \| \leq 1 \) satisfies

\[
|x^*(T(x))| > 1 - \tilde{\eta}(\varepsilon, (x, x^*)),
\]

there is \( S \in \mathcal{L}(X) \) with \( \nu(S) = 1 \) such that \( |x^*(S(x))| = 1 \) and \( \| S - T \| \leq \varepsilon \). Notice that the difference with the property \( L_{p,p} \)-nu is that we are considering the initial operator \( T \) in the ball of the space, instead of considering \( T \) such that \( \nu(T) = 1 \). Indeed, let \( 0 \leq 0 < \varepsilon < 1 \) and \( 0 < \eta(\varepsilon, (x, x^*)) < \varepsilon/2 \) as in the definition of the \( L_{p,p} \)-nu. Put \( \tilde{\eta}(\varepsilon, (x, x^*)) = \frac{\eta(\varepsilon, (x, x^*))}{\sqrt{2}} \) and suppose \( T \in \mathcal{L}(X) \) is such that \( \nu(T) \leq \| T \| \leq 1 \) and \( |x^*(T(x))| > 1 - \tilde{\eta}(\varepsilon, (x, x^*)) \). Then, if we consider \( T_1 = \frac{T}{\nu(T)} \), we have \( \nu(T_1) = 1 \) and

\[
|x^*(T_1(x))| = \frac{1}{\nu(T)}|x^*(T(x))| > \frac{1 - \tilde{\eta}(\varepsilon, (x, x^*))}{\nu(T)} \geq 1 - \tilde{\eta}(\varepsilon, (x, x^*)).
\]

By hypothesis, there is \( S \in \mathcal{L}(X) \) with \( \nu(S) = 1 \) such that \( |x^*(S(x))| = 1 \) and \( \| S - T_1 \| < \varepsilon/2 \). Then,

\[
\| S - T_1 \| \leq \| S - T_1 \| + \| T_1 - T \| < \frac{\varepsilon}{2} + \frac{\| T \|}{\nu(T)} (1 - \nu(T))
\]

\[
< \frac{\varepsilon}{2} + \frac{\| T \|}{\nu(T)} \tilde{\eta}(\varepsilon, (x, x^*)) < \frac{\varepsilon}{2} + \frac{\| T \|}{\nu(T)} \frac{\varepsilon}{4}.
\]

Since \( \| T \| \leq 1 \) and \( \nu(T)^{-1} \leq 2 \) we deduce \( \| S - T \| < \varepsilon \), which is the desired statement.
Theorem 3.14: The norm of the space $X = c_0 \oplus \infty \mathcal{Z}$ is SSD and fails the $L_{p,p}$-nu.

Proof: Assume that $X$ has the $L_{p,p}$-nu and fix $z_0 \in S_\mathcal{Z}$ such that $z_0(1) \geq z_0(2) \geq \cdots > 0$. Note that $z_0(1) \geq \frac{1}{2}$, otherwise, we would have

$$\|z_0\|_\mathcal{Z} = \|z_0\|_\infty + \left(\sum_{i=1}^{\infty} \frac{|z_0(i)|^2}{2^i}\right)^{1/2} < \frac{1}{2} + \frac{1}{2} \left(\sum_{i=1}^{\infty} \frac{1}{2^i}\right)^{1/2} = 1.$$ 

For $z_0^* \in S_\mathcal{Z}^*$ so that $z_0^*(z_0) = 1$, let $x_0 = (e_1, z_0) \in S_{c_0 \oplus \infty \mathcal{Z}}$ where $e_i$ is the canonical basis of $c_0$ and $x_0^* \in S_{(c_0 \oplus \infty \mathcal{Z})^*}$ be a functional such that $x_0^*(y, z) = z_0^*(z)$ for arbitrary $(y, z) \in c_0 \oplus \infty \mathcal{Z}$.

It is clear that $(x_0, x_0^*) \in \Pi(X)$ and, by hypothesis, given $0 < \varepsilon < \frac{1}{2}$ we have $\eta(\varepsilon, (x_0, x_0^*))$ which is written in Remark 3.13 as $\tilde{\eta}(\varepsilon, (x_0, x_0^*))$. Then, we can take $N$ such that

$$z_0^*((z_0(1), \ldots, z_0(N), 0, \ldots)) > 1 - \eta(\varepsilon, (x_0, x_0^*)),$$

and (once we fix $N$) we can also choose a sequence $N < m_1 < m_2 < \cdots$ such that

$$\frac{1}{2^{m_1+2}} \leq \frac{|z_0(N+1)|^2}{2^{N+1}}, \quad \frac{1}{2^{m_2+2}} \leq \frac{|z_0(N+2)|^2}{2^{N+2}}, \ldots. \quad (1)$$

Let $T: X \rightarrow X$ be the operator defined by for $y \in c_0$ and $z \in \mathcal{Z}$

$$T(y, z) = \begin{cases} 0, \\ y(1)z_0(1), y(1)z_0(2), \ldots, y(1)z_0(N), 0, \ldots, 0, \frac{y(m_1)}{2}, \\ \times 0, \ldots, 0, \frac{y(m_2)}{2}, \ldots \\ \text{m_1-thcoord.} \quad \text{m_2-thcoord.} \end{cases}$$

and let us show that $\|T\| \leq 1$. Note that for each $y \in B_{c_0}$ and $z \in B_{\mathcal{Z}}$

$$\|T(y, z)\|_X = \left\| \left( y(1)z_0(1), y(1)z_0(2), \ldots, y(1)z_0(N), 0, \ldots, 0, \right. \right.$$ 

$$\left. \times \frac{y(m_1)}{2}, 0, \ldots, 0, \frac{y(m_2)}{2}, 0, \ldots \right\|_{\mathcal{Z}} \leq \left\| \left( z_0(1), \ldots, z_0(N), 0, \ldots, 0, \frac{y(m_1)}{2}, 0, \ldots \right) \right\|_{\infty} + \left( \sum_{i=1}^{N} \frac{|z_0(i)|^2}{2^i} + \sum_{j=1}^{\infty} \frac{|y(m_j)|^2}{2^{m_j+2}} \right)^{1/2},$$

$$\times 0, \ldots, 0, \frac{y(m_2)}{2}, \ldots \right\|_{\mathcal{Z}} \leq \left\| \left( z_0(1), \ldots, z_0(N), 0, \ldots, 0, \frac{y(m_1)}{2}, 0, \ldots \right) \right\|_{\infty} + \left( \sum_{i=1}^{N} \frac{|z_0(i)|^2}{2^i} + \sum_{j=1}^{\infty} \frac{|y(m_j)|^2}{2^{m_j+2}} \right)^{1/2},$$
where the inequality is due to the fact that $|y(1)| \leq 1$. Since $z_0(1) \geq \frac{1}{2}$ and $|y(m_j)| \leq 1$, it is clear that

$$
\|T(y,z)\| \leq \|z_0\|_\infty + \left( \sum_{i=1}^{N} \frac{|z_0(i)|^2}{2^i} + \sum_{j=1}^{\infty} \frac{1}{2^{mj+2}} \right)^{1/2}
$$

and, by (1), we deduce that

$$
\|T(y,z)\| \leq \|z_0\|_\infty + \left( \sum_{i=1}^{\infty} \frac{|z_0(i)|^2}{2^i} \right)^{1/2} = \|z_0\| \leq 1.
$$

Since $x_0^*(T(x_0)) > 1 - \eta(\varepsilon, (x_0, x_0^*))$, by hypothesis there is an operator $S: X \to X$, $v(S) = 1$, such that

$$
x_0^*(S(x_0)) = 1 \quad \text{and} \quad \|T - S\| < \varepsilon.
$$

Let $P$ and $Q$ be the projections from $X$ onto $Z$ and $c_0$, respectively. It is clear that $PT = T$ and $\|PT - PS\| < \varepsilon$. Also, by [8, Lemma 1.2] we have that $v(S) = \max\{v(PS), v(QS)\} = 1$ and, since $v(QS) \leq \|QS\| < \varepsilon$, we deduce that $PS$ attains its numerical radius. Hence, we may consider $U = PS$. Following the ideas in [8] (see Equation (4) in there) we have

$$
U(y,z) = Az + By
$$

with $A \in \mathcal{L}(Z)$ and $B \in \mathcal{L}(c_0, Z)$ and $\|A\| < \|B\|$ (this last inequality is due to the fact that $S$ is close to $T$). Then, by [8, Proposition 2.4] we have $\lim_{n \to \infty} e_n^*(B(e_n)) = 0$ for the canonical basis $(e_n^*)$ of $\ell_1$. Naming $\tilde{T}: c_0 \to Z$ to the operator

$$
\tilde{T}(y) = \left( y(1)z_0(1), y(1)z_0(2), \ldots, y(1)z_0(N), 0, \ldots, 0, \frac{y(m_1)}{2}, 0, \ldots, 0, \frac{y(m_2)}{2}, 0, \ldots \right),
$$

we have $\|\tilde{T} - B\| < \varepsilon$ and $e_{m_j}^*(\tilde{T}(e_{m_j})) = \frac{1}{2}$ for all $j \in \mathbb{N}$. Consequently,

$$
\frac{1}{2} = \lim_{j \to \infty} |e_{m_j}^*(\tilde{T} - B)(e_{m_j})| \leq \varepsilon,
$$

which is the desired contradiction. 

We finish the study of the property $L_{p,p}$-nu by pointing out the immediate fact that a reflexive Banach space $X$ satisfies the property $L_{p,p}$-nu if and only if its dual $X^*$ satisfies it. This is deduced from the fact that $x^*(Tx) = \hat{x}(T^*x^*)$ and $(T^*)^* = T$ and $(x, x^*) \in \Pi(X, X^*)$ if $(x, x^*) \in \Pi(X)$, where $^\ast: X \to X^{**}$ is the canonical isometric inclusion (see [13, Proposition 3]). Notice that this is no longer true if we remove the hypothesis of $X$ being reflexive. Indeed, $c_0$ satisfies the $L_{p,p}$-nu by Theorem 3.7 but $\ell_1$ does not by Remark 3.10.

**Proposition 3.15:** A reflexive Banach space $X$ has the $L_{p,p}$-nu if and only if $X^*$ has the $L_{p,p}$-nu.
4. The local property $L_{o,o}$-nu

We now study the property $L_{o,o}$-nu which is a weaker version of the Bishop–Phelps–Bollobás operator property for the numerical radius (a trivial property satisfied only in the one-dimensional case, as we already mentioned in the introduction), in which the $\eta$ depends on a numerical radius one operator $T \in \mathcal{L}(X)$. We will see this is much more restrictive than the property $L_{p,p}$-nu. We note that this concept for norm was also considered in [29,30].

Definition 4.1: Let $X$ be a Banach space. We say that $X$ has the $L_{o,o}$-nu if given $\varepsilon > 0$ and $T \in \mathcal{L}(X)$ with $v(T) = 1$, there is $\eta(\varepsilon, T) > 0$ such that whenever $(x, x^*) \in \Pi(X)$ satisfies $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, there is $(y, y^*) \in \Pi(X)$ such that $|y^*(T(y))| = 1$, $\|y - x\| < \varepsilon$, and $\|y^* - x^*\| < \varepsilon$.

It is immediate that if $X$ satisfies the $L_{o,o}$-nu, then every operator attains its numerical radius. We are starting with finite dimensional Banach spaces. Let us notice that if $\dim(X) < \infty$ and $X$ fails to have the $L_{o,o}$-nu, we get a contradiction by using the compactness of the unit balls of $X$ and $X^*$. Thus, we have the following result.

Proposition 4.2: Let $X$ be a finite dimensional Banach space. Then, $X$ has the $L_{o,o}$-nu.

In view of the previous result, it is natural to ask whether there are infinite dimensional Banach spaces satisfying the $L_{o,o}$-nu. Under some general assumptions on the space, we prove that this is not the case.

Proposition 4.3: Let $X$ be a Banach space with the approximation property and $n(X) > 0$. If $X$ has the $L_{o,o}$-nu, then $X$ is finite dimensional.

Proof: Note that, since $X$ has the $L_{o,o}$-nu, then every $T \in \mathcal{L}(X)$ attains its numerical radius and, hence, $X$ is reflexive by [33, Theorem 1]. Now, in the space $Bil(X \times X^*)$ of bilinear forms from $X \times X^*$ to $\mathbb{K}$, we consider the norm

$$
\|\varphi\|_\Pi = \sup_{(x, x^*) \in \Pi(X)} |\varphi(x, x^*)|.
$$

We note that this value becomes a norm from the assumption $n(X) > 0$. Since $X$ is reflexive, the mapping

$$
(\mathcal{L}(X), v(\cdot)) \rightarrow (Bil(X \times X^*), \|\cdot\|_\Pi)
$$

$$
T \mapsto \varphi_T, \quad \text{with } \varphi_T(x, x^*) = x^*(Tx)
$$

is an isometric isomorphism.

We now claim that $(Bil(X \times X^*), \|\cdot\|_\Pi)$ is a reflexive space. In that case, $(\mathcal{L}(X), v(\cdot))$ is reflexive and, since $v(\cdot)$ and $\|\cdot\|$ are equivalent norms in $\mathcal{L}(X)$, $(\mathcal{L}(X), \|\cdot\|)$ is reflexive. Then, by [34] all the operators in $\mathcal{L}(X)$ are compact, and, consequently, $X$ is a finite dimensional space.
To prove that \((\text{Bil}(X \times X^*), \| \cdot \|_\Pi)\) is a reflexive space, consider the (algebraic) subspace of \(X \otimes X^*\) given by

\[
Z = \left\{ \sum_{i=1}^{n} \lambda_i x_i \otimes x_i^* : n \in \mathbb{N}, \lambda_i \in \mathbb{K}, (x_i, x_i^*) \in \Pi(X) \right\}
\]

endowed with the norm

\[
\pi_\Pi(u) = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \|x_i\| \|x_i^*\| : u = \sum_{i=1}^{n} \lambda_i x_i \otimes x_i^* \right\}
\]

where the infimum is taken over all the representations of \(u\) of the form \(\sum_{i=1}^{n} \lambda_i x_i \otimes x_i^*\) with \((x_i, x_i^*) \in \Pi(X)\). Let \(Y\) be the completion of \(Z\) (with the norm \(\pi_\Pi(\cdot)\)) and let us note that \((\text{Bil}(X \times X^*), \| \cdot \|_\Pi)\) and \(Y^* = (Y, \pi_\Pi(\cdot))^*\) are isometrically isomorphic. Consider the (linear) mapping

\[
(\text{Bil}(X \times X^*), \| \cdot \|_\Pi) \rightarrow Y^*
\]

\[
\varphi \mapsto L_{\varphi |_Y},
\]

where \(L_{\varphi |_Y}\) is the restriction to \(Y\) of the functional \(L_{\varphi} \in (X \hat{\otimes}_\pi X^*)^*\) associated to \(\varphi\). Noting that

\[
\|L_{\varphi |_Y}\|_{Y^*} = \sup_{\pi_\Pi(u) = 1} |L_{\varphi}(u)|,
\]

and it is easy to check that \(\|L_{\varphi |_Y}\|_{Y^*} = \|\varphi\|_\Pi\). Hence, the mapping in (3) is an isometry. It remains to prove that it is surjective. Given \(L \in Y^*\) it is clear that \(L|_Z \in Z^*\) (the algebraic dual of \(Z\)), and we can consider \(\tilde{L}\) an algebraic extension of \(L\) to the vector space \(X \otimes X^*\), and \(\varphi_{\tilde{L}}\) the (non-necessarily bounded) bilinear form associated to \(\tilde{L}\). Now, since

\[
\|\varphi_{\tilde{L}}\|_\Pi = \sup_{(x, x^*) \in \Pi(X)} |\tilde{L}(x \otimes x^*)| = \sup_{(x, x^*) \in \Pi(X)} |L(x \otimes x^*)| \leq \|L\|_{Y^*},
\]

we see that \(\varphi_{\tilde{L}} \in (\text{Bil}(X \times X^*), \| \cdot \|_\Pi)\) and that \(L_{\varphi_{\tilde{L}} |_Z} = \tilde{L}|_Z = L|_Z\). Then, by a continuity argument (note that \(\tilde{L}|_Z\) and \(L|_Z\) are bounded) we deduce that \(L_{\varphi_{\tilde{L}} |_Y} = L|_Y\), which proves that the mapping in (3) is surjective.

Since every \(T \in \mathcal{L}(X)\) attains its numerical radius, every \(\psi \in \text{Bil}(X \times X^*)\) attains the \(\| \cdot \|_\Pi\)-norm and, consequently, every functional in \(Y^*\) is norm-attaining. Then, by James’ theorem, \(Y\) is reflexive and, hence, \((\text{Bil}(X \times X^*), \| \cdot \|_\Pi)\) is reflexive.

We do not know what happens in the general case. We finish the paper by highlighting this open question.

**Question 4.4:** Let \(X\) be any Banach space. If \(X\) has the \(L_{o,o}\)-nu, then \(X\) is finite dimensional?

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References

[1] Bishop E, Phelps R. A proof that every Banach space is subreflexive. Bull Amer Math Soc. 1961;67(1):97–98.
[2] Lindenstrauss J. On operators which attain their norm. Israel J Math. 1963;1(3):139–148.
[3] Berg ID, Sims B. Denseness of operators which attain their numerical radius. J Aust Math Soc Ser A. 1984;36(1):130–133.
[4] Cardassi CS. Density of numerical radius attaining operators on some reflexive spaces. Bull Aust Math Soc. 1985;31(1):1–3.
[5] Cardassi CS. Numerical radius attaining operators, Banach spaces (Columbia, Mo, 1984). (Lecture Notes in Math; 1166). Berlin: Springer; 1985. p. 11–14.
[6] Cardassi CS. Numerical radius-attaining operators on C(K). Proc Amer Math Soc. 1985;95(4):537–543.
[7] Acosta MD, Paña R. Numerical radius attaining operators and the Radon-Nikodým property. Bull Lond Math Soc. 1993;25(1):67–73.
[8] Payá R. A counterexample on numerical radius attaining operators. Israel J Math. 1992;79(1):83–101.
[9] Bollobás B. An extension to the theorem of bishop and phelps. Bull Lond Math Soc. 1970;2(2):181–182.
[10] Acosta MD, Aron RM, García D, et al. The Bishop–Phelps–Bollobás theorem for operators. J Funct Anal. 2008;254(11):2780–2799.
[11] Falcó J. The Bishop-Phelps-Bollobás property for numerical radius on L1. J Math Anal Appl. 2014;414(1):125–133.
[12] Guirao AJ, Kozhushkina O. The Bishop-Phelps-Bollobás property for numerical radius in ℓ1 (C). Stud Math. 2013;218(1):41–54.
[13] Kim SK, Lee HJ, Martín M. On the Bishop–Phelps–Bollobás property for numerical radius. Abs Appl Anal. 2014:Article ID 479208.
[14] Bonsall FF, Duncan J. Numerical ranges of operators on normed spaces and of elements of normed algebras. Cambridge University Press; 1971. (London Math. Soc. Lecture Note Series; Vol. 2).
[15] Bonsall FF, Duncan J. Numerical ranges II. Cambridge University Press; 1973. (London Math. Soc. Lecture Note Series; Vol. 10).
[16] Kim SK, Lee HJ, Martín M, et al. On a second numerical index for Banach spaces. Proc R Soc Edinb Sect. 2020;150(2):1003–1051.
[17] Acosta MD, Fakhar M, Soleimani-Mourchehkhorti M. The Bishop–Phelps–Bollobás property for numerical radius of operators on L1(μ). J Math Anal Appl. 2018;458(2):925–936.
[18] Avilés A, Guirao AJ, Rodríguez J. On the Bishop-Phelps-Bollobás property for numerical radius in $C(K)$ spaces. J Math Anal Appl. 2014;419(1):395–421.
[19] Choi YC, Dantas S, Jung M. The Bishop–Phelps–Bollobás properties in complex Hilbert spaces. Math Nachr. 2021;294(11):2105–2120.
[20] Dantas S, Kim SK, Lee HJ. The Bishop–Phelps–Bollobás point property. J Math Anal Appl. 2016;444(2):1739–1751.
[21] Dantas S, Kadets V, Kim SK, et al. There is no operatorwise version of the Bishop–Phelps–Bollobás property. Linear Multilinear Algebra. 2020;68(9):1767–1778.
[22] Banach S. Theory of linear operations. Amsterdam: North-Holland; 1987. (North-Holland Mathematical Library; vol. 38).
[23] Rolewicz S. Metric linear spaces. Warsaw: PWN-Polish Scientific Publishers; 1985. (Mathematics and its Applications; Vol. 20).
[24] Becerra-Guerrero J, Rodríguez-Palacios A. Transitivity of the norm on Banach spaces. Extr Math. 2002;17(1):1–58.
[25] Effros EG. Transformation groups and $C^*$-algebras. Ann Math. 1965;81(1):38–55.
[26] Cabello-Sánchez F, Dantas S, Kadets V, et al. On Banach spaces whose group of isometries acts micro-transitively on the unit sphere. J Math Anal Appl. 2020;488(1):124046.
[27] Kim SK, Lee HJ. Uniform convexity and the Bishop–Phelps–Bollobás property. Can J Math. 2014;66(2):373–386.
[28] Kadets V, Martín M, Merí J, et al. Convexity and smoothness of Banach spaces with numerical index one. Illinois J Math. 2009;53(1):163–182.
[29] Dantas S, Kim SK, Lee HJ, et al. Local Bishop–Phelps–Bollobás properties. J Math Anal Appl. 2018;468(1):304–323.
[30] Dantas S, Kim SK, Lee HJ, et al. Strong subdifferentiability and local Bishop–Phelps–Bollobás properties. Rev Real Acad Cienc Exactas Fis Nat A Mat. 2020;114(2):16. Article ID: 47.
[31] Franchetti C, Payá R. Banach spaces with strongly subdifferentiable norm. Boll Uni Mat Ital. 1993;7(1):45–70.
[32] Rosenthal H. The lie algebra of a Banach space, Banach spaces (Columbia, Mo, 1984). Berlin: Springer; 1985. p. 129–157. (Lecture Notes in Math; 1166).
[33] Acosta M, Ruiz Galán M. A version of James’ theorem for numerical radius. Bull Lond Math Soc. 1999;31(1):67–74.
[34] Holub JR. Reflexivity of $L(E, F)$. Proc Amer Math Soc. 1973;39(1):175–177.