ON NORMALIZATION OF QUASI-LOG CANONICAL PAIRS

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Abstract. The normalization of an irreducible quasi-log canonical pair naturally becomes a quasi-log canonical pair.

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1. Introduction

In [A], Florin Ambro introduced the notion of quasi-log varieties, which are now called quasi-log schemes, in order to establish the cone and contraction theorem for generalized log varieties. Note that a generalized log variety is a pair $(X, \Delta)$ consisting of a normal irreducible variety $X$ and an effective $\mathbb{R}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Although the main result of [A] was recovered without using the theory of quasi-log schemes in [F], it became clear that quasi-log schemes are ubiquitous in the theory of minimal models (see, for example, [F2] and [F5]). As Ambro said in [A], the definition of quasi-log schemes is motivated by Kawamata’s X-method. Therefore, it is not surprising that quasi-log schemes often appear naturally in the theory of minimal models. In this paper, we prove that the normalization of an irreducible quasi-log canonical pair (qlc pair, for short) becomes a quasi-log canonical pair. Note that the notion of quasi-log canonical pairs is one of the useful generalizations of log canonical pairs in the framework of quasi-log schemes. In general, a quasi-log canonical pair may be reducible and may not necessarily be equidimensional. We also note that the result of this paper plays a crucial role when we show that every quasi-log canonical pair has only Du Bois singularities in [FL].

Let $(X, \Delta)$ be a log canonical pair and let $W$ be a log canonical center of $(X, \Delta)$. Then $[W, \omega]$ has a natural qlc structure, where $\omega = (K_X + \Delta)|_W$. For the details, see Example 2.11 below and [F3, 6.4.1 and 6.4.2]. Let $\nu : W^\nu \to W$ be the normalization. Then we expect that there exists an effective $\mathbb{R}$-divisor $\Delta_{W^\nu}$ on $W^\nu$ such that $(W^\nu, \Delta_{W^\nu})$ is log canonical and that $K_{W^\nu} + \Delta_{W^\nu} \sim_\mathbb{R} \nu^*\omega$. However, it is still a difficult open problem to find $\Delta_{W^\nu}$ with the above properties. For a related topic, see [FG]. By Theorem 1.1 below, which is the main theorem of this paper, we see that $[W^\nu, \nu^*\omega]$ naturally becomes a qlc pair. Therefore, we can apply the theory of quasi-log schemes to $[W^\nu, \nu^*\omega]$.

Theorem 1.1 (Normalization of qlc pairs). Let $[X, \omega]$ be a qlc pair such that $X$ is irreducible. Let $\nu : Z \to X$ be the normalization. Then $[Z, \nu^*\omega]$ naturally becomes a qlc pair with the following properties:

Date: 2018/8/20, version 0.17.
2010 Mathematics Subject Classification. Primary 14E30; Secondary 14C20.
Key words and phrases. quasi-log canonical pairs, normalization, Du Bois singularities.
if \( C \) is a qlc center of \([Z; \nu^*\omega]\), then \( \nu(C) \) is a qlc center of \([X; \omega]\), and

(ii) \( \text{Nqklt}(Z, \nu^*\omega) = \nu^{-1}(\text{Nqklt}(X, \omega)) \). More precisely, the equality

\[
\nu_*\mathcal{I}_{\text{Nqklt}(Z, \nu^*\omega)} = \mathcal{I}_{\text{Nqklt}(X, \omega)}
\]

holds, where \( \mathcal{I}_{\text{Nqklt}(X, \omega)} \) and \( \mathcal{I}_{\text{Nqklt}(Z, \nu^*\omega)} \) are the defining ideal sheaves of \( \text{Nqklt}(X, \omega) \) and \( \text{Nqklt}(Z, \nu^*\omega) \) respectively.

For the definition of qlc pairs and \( \text{Nqklt}(X, \omega) \), see Definitions 2.4 and 2.7 respectively. By the theory of quasi-log schemes discussed in [F5, Chapter 6] and Theorem 1.1, the fundamental theorems of the minimal model program hold for \([Z; \nu^*\omega]\). More precisely, the cone and contraction theorem and the basepoint-free theorem of Reid–Fukuda type hold for \([Z; \nu^*\omega]\) by [F5, Theorem 6.4.7] and [F5, Theorem 6.9.1] respectively (see also [F4]). We can also apply various vanishing theorems to \([Z; \nu^*\omega]\). As a special case, we have the following vanishing theorem.

**Corollary 1.2** (Vanishing theorem for normalizations). We use the same notation as in Theorem 1.1. Let \( \pi : X \to S \) be a proper morphism onto a scheme \( S \) and let \( L \) be a Cartier divisor on \( X \) such that \( L - \omega \) is nef and log big over \( S \) with respect to \([X; \omega]\). Then

\[
R^i(\pi \circ \nu)_*\mathcal{O}_Z(\nu^*L) = 0
\]

for every \( i > 0 \).

Let us discuss some conjectures for qlc pairs. The second author poses the following conjecture on Du Bois singularities.

**Conjecture 1.3.** Let \([X; \omega]\) be a qlc pair. Then \( X \) has only Du Bois singularities.

The statement of Conjecture 1.3 is a complete generalization of [K, Corollary 6.32]. For the details of Du Bois singularities, see [F5, Section 5.3] and [K, Chapter 6]. By Theorem 1.1, we have:

**Proposition 1.4.** It is sufficient to prove Conjecture 1.3 under the extra assumption that \( X \) is normal.

Finally, we pose the following conjecture on normal qlc pairs.

**Conjecture 1.5.** Let \([X; \omega]\) be a qlc pair such that \( X \) is quasi-projective and normal. Then there exists an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( X \) such that \((X; \Delta)\) is log canonical.

We note that [F3, Theorem 1.1] strongly supports Conjecture 1.5. Of course, Conjecture 1.5 follows from Conjecture 1.3 by Proposition 1.4. This is because log canonical singularities are known to be Du Bois (see [K]).

Although Theorem 1.1 may look somewhat artificial, it plays an important role in [FLw1], [FLw2], and [FLh]. Roughly speaking, in [FL], we prove that \( X \) is generalized lc in the sense of Birkar–Zhang (see [BZ]) with some good properties when \([X; \omega]\) is a normal irreducible quasi-log canonical pair. We can see it as a weak solution of Conjecture 1.5. We note that [FL] heavily depends on the theory of variations of mixed Hodge structure on cohomology with compact support (see [FF]). Then, in [FLh], we completely confirm Conjecture 1.5, that is, we show that \( X \) has only Du Bois singularities if \([X; \omega]\) is a quasi-log canonical pair.

We will work over \( \mathbb{C} \), the complex number field, throughout this paper. A scheme means a separated scheme of finite type over \( \mathbb{C} \). A variety means a reduced scheme, that is, a reduced separated scheme of finite type over \( \mathbb{C} \). We will freely use the basic notation of the minimal model program as in [F1], [F2], and [F3]. For the details of the theory of quasi-log schemes, we recommend the reader to see [F5, Chapter 6].
Acknowledgments. The first author was partially supported by JSPS KAKENHI Grant Numbers JP16H03925, JP16H06337. The authors would like to thank Professor Wenfei Liu, whose question is one of the motivations of this paper.

2. Quick review of the theory of quasi-log schemes

In this section, we quickly review the theory of quasi-log schemes because it is not so popular yet.

Before we explain the definition of quasi-log canonical pairs, we prepare some basic definitions.

Definition 2.1 (R-divisors). Let X be an equidimensional variety, which is not necessarily regular in codimension one. Let D be an \( \mathbb{R} \)-divisor, that is, \( D \) is a finite formal sum \( \sum_i d_i D_i \), where \( D_i \) is an irreducible reduced closed subscheme of \( X \) of pure codimension one and \( d_i \) is a real number for every \( i \) such that \( D_i \neq D_j \) for \( i \neq j \). We put

\[
D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad \text{and} \quad [D] = \sum_i [d_i] D_i,
\]

where \([d_i]\) is the integer defined by \( d_i \leq [d_i] < d_i + 1\).

Let \( B_1 \) and \( B_2 \) be \( \mathbb{R} \)-Cartier divisors on \( X \). Then \( B_1 \sim_\mathbb{R} B_2 \) means that \( B_1 \) is \( \mathbb{R} \)-linearly equivalent to \( B_2 \).

We note that we can define \( \mathbb{Q} \)-divisors and \( \sim_\mathbb{Q} \) similarly.

The notion of globally embedded simple normal crossing pairs play a crucial role in the theory of quasi-log schemes described in [Fuj, Chapter 6].

Definition 2.2 (Globally embedded simple normal crossing pairs). Let \( Y \) be a simple normal crossing divisor on a smooth variety \( M \) and let \( B \) be an \( \mathbb{R} \)-divisor on \( M \) such that \( Y \) and \( B \) have no common irreducible components and that the support of \( Y + B \) is a simple normal crossing divisor on \( M \). In this situation, \( (Y, B_Y) \), where \( B_Y := B|_Y \), is called a globally embedded simple normal crossing pair.

Definition 2.3 (Strata of simple normal crossing divisors). Let \( Y \) be a simple normal crossing divisor on a smooth variety and let \( Y = \bigcup_{i \in I} Y_i \) be the irreducible decomposition of \( Y \). A stratum of \( Y \) is an irreducible component of \( Y_{i_1} \cap \cdots \cap Y_{i_k} \) for some \( \{i_1, \ldots, i_k\} \subset I \).

Let us recall the definition of quasi-log canonical pairs.

Definition 2.4 (Quasi-log canonical pairs). Let \( X \) be a scheme and let \( \omega \) be an \( \mathbb{R} \)-Cartier divisor (or an \( \mathbb{R} \)-line bundle) on \( X \). Let \( f : Y \to X \) be a proper morphism from a globally embedded simple normal crossing pair \((Y, B_Y)\). If \( B_Y \) is a subboundary \( \mathbb{R} \)-divisor, that is, \( B_Y = B_Y^{\leq 1} \), \( f^* \omega \sim_\mathbb{R} K_Y + B_Y \) holds, and the natural map

\[
\mathcal{O}_X \to f_* \mathcal{O}_Y([-(B_Y^{\leq 1})])
\]

is an isomorphism, then \((X, \omega, f : (Y, B_Y) \to X)\) is called a quasi-log canonical pair (qlc pair, for short). If there is no danger of confusion, we simply say that \([X, \omega]\) is a qlc pair.

The notion of qlc strata and qlc centers is very important. It is indispensable for inductive treatments of quasi-log canonical pairs.

Definition 2.5 (Qlc strata and qlc centers). Let \((X, \omega, f : (Y, B_Y) \to X)\) be a quasi-log canonical pair as in Definition 2.4. Let \( \nu : Y^{\nu} \to Y \) be the normalization. We put

\[
K_{Y^{\nu}} + \Theta = \nu^*(K_Y + B_Y),
\]
that is, $\Theta$ is the sum of the inverse images of $B_Y$ and the singular locus of $Y$. Then $(Y^\nu, \Theta)$ is sub log canonical in the usual sense. Let $W$ be a log canonical center of $(Y^\nu, \Theta)$ or an irreducible component of $Y^\nu$. Then $f \circ \nu(W)$ is called a qlc stratum of $(X, \omega, f : (Y, B_Y) \to X)$. If there is no danger of confusion, we simply call it a qlc stratum of $[X, \omega]$. If $C$ is a qlc stratum of $[X, \omega]$ but it is not an irreducible component of $X$, then $C$ is called a qlc center of $(X, \omega, f : (Y, B_Y) \to X)$ or simply of $[X, \omega]$.

One of the most important results in the theory of quasi-log schemes is **adjunction**.

**Theorem 2.6** (Adjunction, see [F5, Theorem 6.3.5]). Let $[X, \omega]$ be a qlc pair and let $X'$ be the union of some qlc strata of $[X, \omega]$. Then $[X', \omega|_{X'}]$ is a qlc pair such that the qlc strata of $[X', \omega|_{X'}]$ are exactly the qlc strata of $[X, \omega]$ that are contained in $X'$.

We strongly recommend the reader to see [F5, Theorem 6.3.5] and its proof for the details of Theorem 2.6. Theorem 2.6 is a special case of [F5, Theorem 6.3.5 (i)].

**Definition 2.7** (Union of all qlc centers). Let $[X, \omega]$ be a qlc pair. The union of all qlc centers of $[X, \omega]$ is denoted by $\operatorname{Nqklt}(X, \omega)$. It is very important that

$$\operatorname{Nqklt}(X, \omega), \omega|_{\operatorname{Nqklt}(X, \omega)}$$

has a quasi-log canonical structure induced from $(X, \omega, f : (Y, B_Y) \to X)$ by adjunction (see Theorem 2.6 and [F5, Theorem 6.3.5 (i)]).

The vanishing theorem is also a very important result. Theorem 2.8 is a special case of [F5, Theorem 6.3.5 (ii)].

**Theorem 2.8** (Vanishing theorem, see [F5, Theorem 6.3.5]). Let $[X, \omega]$ be a qlc pair and let $\pi : X \to S$ be a proper morphism between schemes. Let $L$ be a Cartier divisor on $X$ such that $L - \omega$ is nef and log big over $S$ with respect to $[X, \omega]$, that is, $L - \omega$ is $\pi$-nef and $(L - \omega)|_W$ is $\pi$-big for every qlc stratum $W$ of $[X, \omega]$. Then $R^i \pi_* \mathcal{O}_X(L) = 0$ for every $i > 0$.

The notion of $\mathbb{Q}$-structures is introduced in [F6].

**Definition 2.9** ($\mathbb{Q}$-structures). If $\omega$ is a $\mathbb{Q}$-Cartier divisor (or a $\mathbb{Q}$-line bundle) on $X$, $B_Y$ is a $\mathbb{Q}$-divisor on $Y$, and $f^* \omega \sim_\mathbb{Q} K_Y + B_Y$ holds in Definition 2.9, then we say that $(X, \omega, f : (Y, B_Y) \to X)$ has a $\mathbb{Q}$-structure or simply say that $[X, \omega]$ has a $\mathbb{Q}$-structure.

**Remark 2.10.** If $[X, \omega]$ has a $\mathbb{Q}$-structure, then we can easily see that for any union of qlc strata $X'$ the qlc pair $[X', \omega|_{X'}]$ naturally has a $\mathbb{Q}$-structure in Theorem 2.6. For the details, see the proof of [F5, Theorem 6.3.5].

We close this section with an important example.

**Example 2.11.** Let $(X, \Delta)$ be a log canonical pair. We put $\omega = K_X + \Delta$. Let $f : Y \to X$ be a resolution such that $K_Y + B_Y = f^*(K_X + \Delta)$. We assume that the support of $B_Y$ is a simple normal crossing divisor on $Y$. Then we can easily see that $(Y, B_Y)$ is a globally embedded simple normal crossing pair, $B_Y = B_Y^{\leq 1}$, and the natural map

$$\mathcal{O}_X \to f_* \mathcal{O}_Y([(B_Y^{\leq 1})])$$

is an isomorphism. Therefore, we can see that $(X, \omega, f : (Y, B_Y) \to X)$ is a qlc pair. In this situation, $W$ is a qlc stratum of $[X, \omega]$ if and only if $W$ is a log canonical center of $(X, \Delta)$ or $X$ itself.

Anyway, we recommend the reader to see [F5, Chapter 6] for the theory of quasi-log schemes.
3. Proof

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. Let \( f : (Y, B_Y) \to X \) be a proper surjective morphism from a globally embedded simple normal crossing pair \((Y, B_Y)\) as in Definition 2.4. By \([F5, \text{Proposition } 6.3.1]\), we may assume that the union of all strata of \((Y, B_Y)\) mapped to \(Nqklt(X, \omega)\), which is denoted by \( Y'' \), is a union of some irreducible components of \( Y \). We put \( Y' = Y - Y'' \) and \( K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'} \). Then we obtain the following commutative diagram:

\[
\begin{array}{ccc}
  Y'' & \xrightarrow{\iota} & Y \\
  f' \downarrow \quad & \quad & \downarrow f \\
  V & \xrightarrow{\nu} & X \\
\end{array}
\]

where \( \iota : Y' \to Y \) is a natural closed immersion and

\[
Y' \xrightarrow{f'} V \xrightarrow{\nu} X
\]

is the Stein factorization of \( f \circ \iota : Y' \to X \). By construction, \( \iota : Y' \to Y \) is an isomorphism over the generic point of \( X \). By construction again, the natural map \( \mathcal{O}_V \to f'_* \mathcal{O}_{Y'} \) is an isomorphism and every stratum of \( Y' \) is dominant onto \( V \). Therefore, \( p \) is birational.

Claim 1. \( V \) is normal.

Proof of Claim 1. (cf. the proof of \([F3, \text{Lemma } 6.3.9]\)). Let \( \pi : V^n \to V \) be the normalization. Since every stratum of \( Y'' \) is dominant onto \( V \), there exists a closed subset \( \Sigma \) of \( Y'' \) such that \( \text{codim}_{Y''} \Sigma \geq 2 \) and that \( \pi^{-1} \circ f' : Y' \dashrightarrow V^n \) is a morphism on \( Y' \setminus \Sigma \). Let \( \tilde{Y} \) be the graph of \( \pi^{-1} \circ f' : Y' \dashrightarrow V^n \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
  \tilde{Y} & \xrightarrow{q} & Y' \\
  \tilde{f} \downarrow \quad & \quad & \downarrow f' \\
  V^n & \xrightarrow{\pi} & V \\
\end{array}
\]

where \( q \) and \( \tilde{f} \) are natural projections. Note that \( q : \tilde{Y} \to Y' \) is an isomorphism over \( Y' \setminus \Sigma \) by construction. Since \( Y' \) is a simple normal crossing divisor on a smooth variety and \( \text{codim}_{Y'} \Sigma \geq 2 \), the natural map \( \mathcal{O}_{Y'} \to q_* \mathcal{O}_{\tilde{Y}} \) is an isomorphism. Therefore, the composition

\[
\mathcal{O}_V \to \pi_* \mathcal{O}_{V^n} \to \pi_* f_* \mathcal{O}_{\tilde{Y}} = f'_* q_* \mathcal{O}_{\tilde{Y}} \simeq \mathcal{O}_V
\]

is an isomorphism. Thus we have \( \mathcal{O}_V \simeq \pi_* \mathcal{O}_{V^n} \). This implies that \( V \) is normal. \( \square \)

Therefore, \( p : V \to X \) is nothing but the normalization \( \nu : Z \to X \). So we have the following commutative diagram:

\[
\begin{array}{ccc}
  Y'' & \xrightarrow{\iota} & Y \\
  f' \downarrow \quad & \quad & \downarrow f \\
  Z & \xrightarrow{\nu} & X \\
\end{array}
\]

Claim 2. The natural map

\[
\alpha : \mathcal{O}_Z \to f'_* \mathcal{O}_{Y'}([-B_{Y'}^1])
\]

is an isomorphism.
Proof of Claim \(\Box\). Note that \(\nu : Z \to X\) is an isomorphism over \(X \setminus \Nqklt(X, \omega)\). Therefore, \(\alpha\) is an isomorphism outside \(\nu^{-1}\)(\(\Nqklt(X, \omega)\)). Since \(Z\) is normal and \(f'_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle])\) is torsion-free, it is sufficient to see that \(\alpha\) is an isomorphism in codimension one. Let \(P\) be any prime divisor on \(Z\) such that \(P \subset \nu^{-1}(\Nqklt(X, \omega))\). Then, by construction, there exists an irreducible component of \(B^\leq_1\) which maps onto \(P\). We note that every fiber of \(f\) is connected by \(f_*\mathcal{O}_Y \cong \mathcal{O}_X\). Therefore, the effective divisor \([-\langle B^\leq_1 \rangle]\) does not contain the whole fiber of \(f'\) over the generic point of \(P\). Thus, \(\alpha\) is an isomorphism at the generic point of \(P\). This means that \(\alpha\) is an isomorphism. \(\Box\)

Therefore, by Claim \(\Box\), \(f' : (Y', B_{Y'}) \to Z\) defines a quasi-log structure on \([Z, \nu^*\omega]\). By construction, the property (i) automatically holds. Let us consider the following ideal sheaf:

\[
\mathcal{I} = f'_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} \subset f'_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) = \mathcal{O}_Z.
\]

We note that \(\mathcal{I} = \mathcal{I}_{\Nqklt(Z, \nu^*\omega)}\) since \(\Nqklt(Z, \nu^*\omega) = f'(Y''|_{Y'})\).

Claim 3. \(f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} = \mathcal{I}_{\Nqklt(X, \omega)}\) holds.

Proof of Claim \(\Box\). (cf. the proof of [K, Theorem 6.3.5 (i)]). Since \(\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} \subset \mathcal{O}_Y([-\langle B^\leq_1 \rangle])\), we get

\[
f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} \subset f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) = \mathcal{O}_X,
\]

that is, \(f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'}\) is an ideal sheaf on \(X\). By construction,

\[
f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} = \mathcal{I}_{\Nqklt(X, \omega)}
\]

holds. Here, we used the fact that every fiber of \(f\) is connected. \(\Box\)

Claim \(\Box\) implies that

\[
\nu_*\mathcal{I} = \nu_*f'_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} = f_*\mathcal{O}_Y([-\langle B^\leq_1 \rangle]) - Y''|_{Y'} = \mathcal{I}_{\Nqklt(X, \omega)}.
\]

Since \(\nu\) is finite, \(\mathcal{I} = \nu^{-1}\mathcal{I}_{\Nqklt(X, \omega)}\). Therefore, we have \(\nu^{-1}(\Nqklt(X, \omega)) = \Nqklt(Z, \nu^*\omega)\).

This means that (ii) holds. \(\Box\)

Proof of Corollary 1.2. This follows from Theorems 1.1 and 1.8 (see also [K, Theorem 6.3.5 (ii)])).

Finally, we prove Proposition 1.4.

Proof of Proposition 1.4. We prove Conjecture 1.3 under the extra assumption that Conjecture 1.3 holds true for normal qlc pairs. Let \([X, \omega]\) be a qlc pair. Let \(X_1\) be an irreducible component of \(X\) and let \(X_2\) be the union of the irreducible components of \(X\) other than \(X_1\). Then \(X_1, X_2,\) and \(X_1 \cap X_2\) are qlc pairs by adjunction (see Theorem 2.6 and [K, Theorem 6.3.5 (i)])]. In particular, they are seminormal (see [K, Remark 6.2.11]). Then we have the following short exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_1 \cap X_2} \to 0
\]

(see, for example, [K, Lemma 10.21]). By [K, Lemma 5.3.9], it is sufficient to prove Conjecture 1.3 under the extra assumption that \(X\) is irreducible by induction on \(\dim X\) and the number of the irreducible components of \(X\). Therefore, from now on, we assume that \(X\) is irreducible. Let \(\nu : Z \to X\) be the normalization. Then, by Theorem 1.1, we have

\[
(3.1) \quad R\nu_*\mathcal{I}_{\Nqklt(Z, \nu^*\omega)} = \mathcal{I}_{\Nqklt(X, \omega)}.
\]

By induction on dimension, \(\Nqklt(Z, \nu^*\omega)\) and \(\Nqklt(X, \omega)\) are Du Bois since they are qlc (see Definition 1.4). Since \(Z\) is normal and \([Z, \nu^*\omega]\) is qlc, \(Z\) is Du Bois by assumption. Therefore, by [K, Corollary 6.28] and (1.1), \(X\) is Du Bois. This is what we wanted. \(\Box\)
We close this section with a remark on $\mathbb{Q}$-structures.

**Remark 3.1.** If $[X, \omega]$ has a $\mathbb{Q}$-structure in Theorem 1.1, then we can easily see that $[Z, \nu^*\omega]$ also has a $\mathbb{Q}$-structure by the proof of Theorem 1.1.

**References**

[A] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova 240 (2003), Biratsion. Geom. Lineîn. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1(240), 214–233.

[BZ] C. Birkar, D.-Q. Zhang, Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs, Publ. Math. Inst. Hautes Études Sci. 123 (2016), 283–331.

[F1] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. 47 (2011), no. 3, 727–789.

[F2] O. Fujino, Fundamental theorems for semi log canonical pairs, Algebr. Geom. 1 (2014), no. 2, 194–228.

[F3] O. Fujino, Some remarks on the minimal model program for log canonical pairs, J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, 149–192.

[F4] O. Fujino, Basepoint-free theorem of Reid–Fukuda type for quasi-log schemes, Publ. Res. Inst. Math. Sci. 52 (2016), no. 1, 63–81.

[F5] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, 35. Mathematical Society of Japan, Tokyo, 2017.

[F6] O. Fujino, Fundamental properties of basic slc-trivial fibrations, preprint (2018). arXiv:1804.11134 [math.AG]

[FF] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, Publ. Res. Inst. Math. Sci. 50 (2014), no. 4, 589–661.

[FG] O. Fujino, Y. Gongyo, On canonical bundle formulas and subadjunctions, Michigan Math. J. 61 (2012), no. 2, 255–264.

[FLh] O. Fujino, H. Liu, Quasi-log canonical pairs are Du Bois, preprint (2018). arXiv:1804.11138 [math.AG]

[FLw1] O. Fujino, W. Liu, Simple connectedness of Fano log pairs with semi-log canonical singularities, preprint (2018). arXiv:1712.03417 [math.AG]

[FLw2] O. Fujino, W. Liu, Subadjunction for quasi-log canonical pairs and its applications, preprint (2018).

[K] J. Kollár, *Singularities of the minimal model program*. With a collaboration of Sándor Kovács, Cambridge Tracts in Mathematics, 200. Cambridge University Press, Cambridge, 2013.