Single Scale Tadpoles and $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ Corrections to the $\rho$ Parameter

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Abstract

We present a new set of high precision numerical values of four-loop single-scale vacuum integrals, which we subsequently use to obtain the non-singlet corrections to the $\rho$ parameter at $\mathcal{O}(G_F m_t^2 \alpha_s^3)$. Our result for $\Delta \rho$ is in agreement with the recent calculation [1].

1 Introduction

Single-scale four-loop vacuum integrals have attracted a lot of attention in recent years. Even though the first applications were connected to the calculation of anomalous dimensions [2,3], the class of solved problems counts by now such topics as the pressure in hot QCD [4], coupling constant and mass decoupling relations [5,6], moments of the hadronic production cross section [7,8], and corrections to the $\rho$ parameter [9,1]. Studies of four-loop tadpoles have also led to new ideas in computational techniques, such as the introduction of special integral bases [10].

This impressive progress has been made possible to a large extent by the Laporta algorithm for the reduction of integrals to masters described in [11], and by the difference equation method for the numerical evaluation of the masters proposed in the same publication. The first sets of integrals have been
evaluated precisely using these principles [12,13]. At present, other methods are also available, see [14] and [1].

One of the applications of four-loop tadpoles mentioned at the beginning concerns a quantity of primary importance in the area of electroweak physics, namely the $\rho$ parameter introduced by Veltman [15]. Defined as the ratio of the charged and neutral current strengths, it differs from its leading-order value of one, by a shift which can be expressed through the transverse parts of $W$ and $Z$ boson self-energies

$$\Delta \rho = \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2}. \quad (1)$$

This shift occurs as a universal correction in all electroweak observables and is thus related to the indirect prediction of the Higgs boson mass from the experimental data, and in particular from the $W$ boson mass [16] and the effective weak mixing angle [17].

In view of the importance of $\Delta \rho$, several corrections have been computed. In particular, the two- [18] and three-loop [19] QCD effects, and various electroweak effects in the limit of a large top quark mass [20] have been accounted for. At the three-loop level, the leading behavior in the limit of a large Higgs boson mass is also available [21]. From now on, we will denote the QCD corrections to $\Delta \rho$ in leading order in the electroweak interaction by $\delta \rho$.

At the four-loop level, the singlet QCD corrections, i.e. corrections where the external gauge bosons couple to different fermion loops have been evaluated in [9]. Motivated by that publication, we started the calculation of the non-singlet contributions, which are obtained by attaching gluons (with possible fermion loop insertions) to the leading one-loop diagrams. The major obstacle to overcome is the calculation of the many new master integrals. It is the purpose of the present work to present our results for those integrals and apply them to the calculation of the four-loop non-singlet QCD corrections to $\Delta \rho$.

Recently, Ref. [1] containing a result for the very same corrections appeared. Anticipating the content of Section 3, we can state that we agree with this calculation.

This paper is organized as follows. In the next section we present high precision numeric expansions of the master integrals. We then give our on-shell result and conclusions. An appendix contains the corrections expressed in the $\overline{\text{MS}}$ scheme.
2 Master Integrals

Upon reducing the complete set of scalar integrals occurring in the diagrams contributing to $\Delta \rho$ at the four-loop level, we are left with 65 masters. However, the latter number is only correct if we consider the integration-by-parts identities for a given prototype and not those of its parents\(^1\). It has been noticed in the case of two \([22]\) and three-loop \([23]\) on-shell propagators that the identities of the parents can further reduce the set of master integrals. Interestingly, the same happens also in the present calculation. In fact, we found the following two relations

\[
\begin{align*}
\text{(continuous)} & = \frac{3 - 4\varepsilon}{2(\varepsilon - 1)} & + \frac{1}{2} \\
\text{(dashed)} & = \frac{2 - 3\varepsilon}{3} & + \frac{\varepsilon - 1}{6}
\end{align*}
\]

where the continuous and dashed lines denote massive and massless lines respectively. The final number of master integrals is, therefore, 63. Of these 23 have either been already presented in the literature \([13]\), or are products of vacuum integrals with a lower number of loops, or can be trivially expressed in terms of gamma functions. Ultimately, we need only to calculate 40 new integrals. At this point, a few words are due to explain our choice of masters. Since the algorithm we use for numerical evaluation can provide high precision numbers and deep expansions in the dimensional regularization parameter $\varepsilon^2$, we let the reduction software chose the masters automatically, keeping, however, integrals with dots (higher powers of denominators) instead of irreducible numerators.

As mentioned in the Introduction, the method of difference equations \([11]\) provides a particularly efficient approach for the evaluation of vacuum integrals. Its main steps are

1. introduction of a symbolic power, $x$, on a chosen massive line of a given

\(^1\)integrals with a larger number of lines, such that removing some lines and merging the remaining vertices leads to the original integral.

\(^2\)we assume the dimension of space-time to be $d = 4 - 2\varepsilon$. 

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vacuum integral, which is then a function $T(x, \varepsilon)$;

(2) determination of a difference equation in $x$,

$$
\sum_{i=0}^{n} c_i(x, \varepsilon) T(x + i, \varepsilon) = U(x, \varepsilon),
$$

where $c_i(x, \varepsilon)$ are rational functions and $U(x, \varepsilon)$ is given by vacuum integrals with less lines;

(3) determination of the boundary condition at $x \to \infty$, which is equivalent to finding the low momentum expansion of the propagator subloop, when the line carrying power $x$ is cut;

(4) solution of the difference equation after expansion in $\varepsilon$, assuming that every term of the expansion of $T(x, \varepsilon)$ takes the form of a factorial series

$$
\mu^x \sum_{s=0}^{\infty} \frac{a_s \Gamma(x + 1)}{\Gamma(x - K + s + 1)},
$$

where $\mu$, $K$ and $a_s$ remain to be determined from the difference equation and the boundary condition.

Further details can be found in the original work [11]. We note here only that the difference equations are derived from integration-by-parts identities, which are now more complicated, because they involve two variables and not just one, as those of the reduction to masters. The procedure we sketched above is recursive and performed numerically. The recursion leads to the loss of many terms in the expansion in $\varepsilon$. Even if we start with a three-loop input up to $\varepsilon^{10}$, in some cases we are only left with terms up to $\varepsilon^2$ in the four-loop results. Apart from these exceptions, all the 5-, 6- and 7-liners are provided up to $\varepsilon^4$. The remaining 8-, and 9-liners are given up to $\varepsilon^3$. The depth of the expansion of our four-loop masters would be sufficient for five-loop calculations. As far as the number of digits is concerned, our goal was to obtain at least 30 digits for all the masters. This has turned out to be difficult for the last three 9-liners shown below, because of weak convergence. In the worst case, we have still obtained 18 digits.

Before we list our results, we have to define the normalization of the integrals. We simply assume that the integration measure is $(e^{\varepsilon \gamma} / i \pi^{d/2})^4 \int d^d k_1 \ldots d^d k_4$ and that the propagators in the subsequent expressions are $1/(k^2 - m^2)$ and we set everywhere the mass to unity. The values of the computed integrals are

$$
\begin{align*}
= & 0.25000000000000000000000000000000 \varepsilon^{-4} \\
+ & 1.00000000000000000000000000000000 \varepsilon^{-3}
\end{align*}
$$
\[
\sum_{n=2}^{4} \frac{\pi^2}{12} \left( \frac{\pi^2}{6} \right)^n = 0.4166666666666666666667 \varepsilon^{-3}
\]

\[
\sum_{n=2}^{4} \frac{\pi^2}{12} \left( \frac{\pi^2}{6} \right)^n = 0.4166666666666666666667 \varepsilon^{-3}
\]
\[-110899.174077030571795400409124 \varepsilon^4 \quad (5)\]

\[-\frac{1}{250000000000000000000000000000 \varepsilon^4 + 0.9166666666666666666666666 \varepsilon^{-3} + 1.90580036675744655156954091666 \varepsilon^{-2} + 2.52073453954145394473436269644 \varepsilon^{-1} - 62.9263953267520790182858402690 - 258.692941139129800870656513172 \varepsilon - 2245.88328900424600912128374855 \varepsilon^2 - 7643.28441760863244533368790482 \varepsilon^3 - 48359.2665176408023251619751028 \varepsilon^4 \quad (6)\]

\[-\frac{1}{5.83333333333333333333333333333 \varepsilon^4 + 3.58333333333333333333333333333 \varepsilon^{-3} + 13.6690897446562641758844843611 \varepsilon^{-2} + 38.3870766266324744689703482793 \varepsilon^{-1} - 51.22522128011891524118819912275 - 477.94746099519461927728562192 \varepsilon - 6127.93433249128299350151222877 \varepsilon^2 - 23752.7464620411276168084881607 \varepsilon^3 - 161921.400195144895866566378264 \varepsilon^4 \quad (7)\]

\[-\frac{1}{3.33333333333333333333333333333 \varepsilon^4 + 1.33333333333333333333333333333 \varepsilon^{-3} + 3.42995604456548429098161011110 \varepsilon^{-2} + 5.55063171514827193110902146145 \varepsilon^{-1} - 57.5851192392905729834630220283 - 226.906688638637087111211732760 \varepsilon - 2357.42884823258106273364123809 \varepsilon^2 - 6885.09836097771127634712790473 \varepsilon^3 - 51888.4446361973078707935677603 \varepsilon^4 \quad (8)\]

\[-\frac{1}{1.00000000000000000000000000000000 \varepsilon^{-4} + 6.250000000000000000000000000000 \varepsilon^{-3}} \]
\[ -59651.1029754541134802977172410 \varepsilon^4 \quad (12) \]

\[ = + 0.87500000000000000000000000000000 \varepsilon^{-4} \]
\[ + 5.312500000000000000000000000000 \varepsilon^{-3} \]
\[ + 24.127285712567359185353492916 \varepsilon^{-2} \]
\[ + 71.586987470921357789204087564 \varepsilon^{-1} \]
\[ + 197.55837193513443514418819015 \varepsilon \]
\[ + 6.41187248372530806306191468163 \varepsilon^2 \]
\[ - 1028.3312883387533383445867353 \varepsilon^3 \]
\[ - 16249.8777552641957747492917792 \varepsilon^4 \]
\[ - 65696.8989733875039196986140933 \varepsilon^5 \quad (13) \]

\[ = + 0.62500000000000000000000000000000 \varepsilon^{-4} \]
\[ + 4.12500000000000000000000000000000 \varepsilon^{-3} \]
\[ + 20.6694020171659560336324750416 \varepsilon^{-2} \]
\[ + 74.2503544994774685337366210644 \varepsilon^{-1} \]
\[ + 147.807870353419457006020027881 \varepsilon \]
\[ + 467.918626785158952385003318864 \varepsilon^2 \]
\[ - 1988.53979380180044626102894892 \varepsilon^3 \]
\[ - 4066.0500495929160439936393130 \varepsilon^4 \]
\[ - 84210.5386674689601672799852884 \varepsilon^5 \quad (14) \]

\[ = + 0.08333333333333333333333333333333 \varepsilon^{-4} \]
\[ + 0.66666666666666666666666666666667 \varepsilon^{-3} \]
\[ + 5.1690897446562417588448436109 \varepsilon^{-2} \]
\[ + 27.8538606812276657878535072006 \varepsilon^{-1} \]
\[ + 146.237985312171144034228066867 \varepsilon \]
\[ + 662.260137198260174836138241719 \varepsilon^2 \]
\[ + 2974.21982572225816025721879182 \varepsilon^3 \]
\[ + 12445.5794442004943383914555146 \varepsilon^4 \]
\[ + 52217.5702319691751782305982084 \varepsilon^5 \quad (15) \]

\[ = + 0.12500000000000000000000000000000 \varepsilon^{-4} \]
\[ + 1.0833333333333333333333333333333 \varepsilon^{-3} \]
\[= + 0.166666666666666666666666666667 \varepsilon^{-4} + 1.50000000000000000000000000000 \varepsilon^{-3} + 9.214978022827421454908050555 \varepsilon^{-2} + 41.2601364660351459773071526496 \varepsilon^{-1} + 209.510736830949219399940244688 + 736.116832299866041378628875270 \varepsilon + 374.4598433116742262482544777 \varepsilon^{2} + 11749.430216934050683438761905 \varepsilon^{3} + 62286.5836001930375873157912315 \varepsilon^{4} (17)\]

\[= + 0.083333333333333333333333333333 \varepsilon^{-4} + 0.833333333333333333333333333333 \varepsilon^{-3} + 7.93678486290272798525102010851 \varepsilon^{-2} + 36.646066943862068010045914737 \varepsilon^{-1} + 194.097392655331631239559043545 + 770.17891682309688733987658212 \varepsilon + 3399.99482439741716780023304896 \varepsilon^{2} + 13282.1053505362611944372969993 \varepsilon^{3} + 54753.4641354130075585240962863 \varepsilon^{4} (18)\]

\[= - 0.12500000000000000000000000000000 \varepsilon^{-4} - 0.75000000000000000000000000000000 \varepsilon^{-3} - 2.71376648328794339088189620834 \varepsilon^{-2} - 7.09547330357733318621286446542 \varepsilon^{-1} - 22.2503497804982861883215445262 - 17.7874425940476975226672333232 \varepsilon - 175.317651447398940503032885592 \varepsilon^{2} + 313.983105353518264983361554717 \varepsilon^{3}\]
\[-2142.79466723532874375769322841 \varepsilon^4 \quad (19)\]

\[= + 0.0833333333333333333333333333333 \varepsilon^{-4} + 0.8333333333333333333333333333333 \varepsilon^{-3} + 8.53781331448252512795088918927 \varepsilon^{-2} + 41.6650082807281404435406778114 \varepsilon^{-1} + 179.973949309303080544355066651 + 915.99674271556240782158232880 \varepsilon + 2767.39379170194861041563852821 \varepsilon^2 + 16538.6792505904580566239289602 \varepsilon^3 + 40456.0909713394095410013351009 \varepsilon^4 \quad (20)\]

\[= + 0.125000000000000000000000000000 \varepsilon^{-4} + 1.0833333333333333333333333333333 \varepsilon^{-3} + 8.39400978662909744608738286983 \varepsilon^{-2} + 47.441872716323093889524659287 \varepsilon^{-1} + 158.265894985614754765609093664 + 1040.51973958438238064321813619 \varepsilon + 2220.5264983098662027273311300 \varepsilon^2 + 19009.3819367933781054485906380 \varepsilon^3 + 29872.2890554813361688482086054 \varepsilon^4 \quad (21)\]

\[= + 0.16666666666666666666666666666 \varepsilon^{-4} + 1.500000000000000000000000000000 \varepsilon^{-3} + 9.81600647386253928819067413630 \varepsilon^{-2} + 46.2790780523770796198432389873 \varepsilon^{-1} + 192.14394998518979278313357259 + 895.583232707441960151372330064 \varepsilon + 3015.506091588685899428979534043 \varepsilon^2 + 15431.714565054206165629974866 \varepsilon^3 + 45950.3851577110455998246982000 \varepsilon^4 \quad (22)\]

\[= + 0.125000000000000000000000000000 \varepsilon^{-4} + 1.250000000000000000000000000000 \varepsilon^{-3}\]
\[ + 9.95972900171596697005418045574 \varepsilon^{-2} \\
+ 41.216020334804518858225510288 \varepsilon^{-1} \\
+ 210.851234758305984612090112605 \\
+ 790.910352868677999468381955470 \varepsilon \\
+ 3471.90742617992105936864664248 \varepsilon^2 \\
+ 13388.2008481237206516463529138 \varepsilon^3 \\
+ 54666.0539850126748720361226803 \varepsilon^4 \] \hfill (23)

\[ = + 0.125000000000000000000000000000 \varepsilon^{-4} \\
+ 1.250000000000000000000000000000 \varepsilon^{-3} \\
+ 10.560754532957641127540495365 \varepsilon^{-2} \\
+ 46.2349619211464755283586373665 \varepsilon^{-1} \\
+ 188.248493815420722512481676436 \\
+ 965.746295254204695422785736581 \varepsilon \\
+ 2625.62618545640179663793065741 \varepsilon^2 \\
+ 17532.058598152815334470784400 \varepsilon^3 \\
+ 36105.6708001198638661491047454 \varepsilon^4 \] \hfill (24)

\[ = + 0.08333333333333333333333333333333 \varepsilon^{-4} \\
+ 0.6666666666666666666666666666667 \varepsilon^{-3} \\
+ 6.37114664781585928445076208433 \varepsilon^{-2} \\
+ 37.8917438539115415433555091730 \varepsilon^{-1} \\
+ 154.604631544617018499470855434 \\
+ 894.807380418881266203622406485 \varepsilon \\
+ 2607.13652650580177242514674361 \varepsilon^2 \] \hfill (25)

\[ = + 0.08333333333333333333333333333333 \varepsilon^{-4} \\
+ 0.33333333333333333333333333333333 \varepsilon^{-3} \\
+ 2.50242307798959750921781769442 \varepsilon^{-2} \\
+ 7.17750170260260867273229975754 \varepsilon^{-1} \\
+ 10.0033662615671539152667896043 \\
+ 121.389726250269120381127113931 \varepsilon \\
- 338.169587547191526007167931728 \varepsilon^2 \] \hfill (26)
\[
\begin{align*}
&= + 1.80308535473939142809960724227 \varepsilon^{-2} \\
&\quad - 2.03566656453161423532377023190 \varepsilon^{-1} \\
&\quad + 40.4773901150829247806609147362 \varepsilon \\
&\quad - 100.628977840966481693832556086 \varepsilon \\
&\quad + 679.056339067891403282049708894 \varepsilon^2 \\
&\quad - 2387.60452082758635632061839710 \varepsilon^3 \quad (27) \\

&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&\quad + 0.374844191926851274910805165871 \varepsilon \\
&\quad + 141.683133328263640451220557622 \varepsilon \\
&\quad - 146.684014785607112711629455843 \varepsilon^2 \\
&\quad + 2448.77867872444787260224256255 \varepsilon^3 \quad (28) \\

&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&\quad - 13.7206302625920162070487342659 \varepsilon \\
&\quad + 135.92896108952872038638377087 \varepsilon \\
&\quad - 497.764644552772376477970102393 \varepsilon^2 \\
&\quad + 2695.35125395194610632909729485 \varepsilon^3 \quad (29) \\

&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&\quad - 7.85876279518922242360822900783 \varepsilon \\
&\quad + 128.53138667643130344223252317 \varepsilon \\
&\quad - 349.879176215675436895102723351 \varepsilon^2 \\
&\quad + 2332.83399433790756728143973285 \varepsilon^3 \quad (30) \\

&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&\quad - 20.8585976739540823192137169924 \varepsilon \\
&\quad + 148.814327289184790880960752284 \varepsilon \\
&\quad - 657.418920712168196526528768117 \varepsilon^2 \\
&\quad + 3182.47221782449086620274231596 \varepsilon^3 \quad (31)
\end{align*}
\]
\[
\begin{align*}
&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&- 18.7699726985410818615637417243 \\
&+ 142.191941064595949485071666672 \varepsilon \\
&- 600.255511127362803430335175869 \varepsilon^2 \\
&+ 2957.18558813594868083060620764 \varepsilon^3 \quad (32)
\end{align*}
\]

\[
\begin{align*}
&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&- 19.0793750927079960314257405876 \\
&+ 141.252248186107747164092632797 \varepsilon \\
&- 605.029621201870258904662671455 \varepsilon^2 \\
&+ 2946.48740435117339409606564510 \varepsilon^3 \quad (33)
\end{align*}
\]

\[
\begin{align*}
&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&- 19.1118657059490732311103013898 \\
&+ 141.144175505094858012998872298 \varepsilon \\
&- 605.431190575217055408942399967 \varepsilon^2 \\
&+ 2945.58041293149149762720437022 \varepsilon^3 \quad (34)
\end{align*}
\]

\[
\begin{align*}
&= + 5.18463877571684963165682743229 \varepsilon^{-1} \\
&+ 7.30445968508659934705130294982 \\
&+ 141.443417856878944115040221346 \varepsilon \\
&+ 40.4773793831579631350040651322 \varepsilon^2 \quad (35)
\end{align*}
\]

\[
\begin{align*}
&= - 6.72847056008568105547188977521 \\
&- 26.0876465999666155389659770717 \varepsilon \\
&- 214.647717912411362028052727052 \varepsilon^2 \\
&- 613.715203096626075654874908838 \varepsilon^3 \quad (36)
\end{align*}
\]

\[
\begin{align*}
&= - 3.71140264536682392682628965373 \\
&- 2.11520599545877264756135261804 \varepsilon
\end{align*}
\]
\[-71.9899451389829491830674629550 \varepsilon^2 + 41.1881294174309244417864419468 \varepsilon^3 \]  
\[(37)\]

\[= + 5.18463877571684963165682743229 \varepsilon^{-1} + 52.8346981753279451179346027590 + 447.323275771878352225930613955 \varepsilon \]  
\[(38)\]

\[= -3.4497511317390349922288 + 6.3127694885459812824115 \varepsilon - 63.668771344187502234181 \varepsilon^2 + 196.34402612627359322923 \varepsilon^3 \]  
\[(39)\]

\[= -2.42695639537700735 + 4.01554669961524192 \varepsilon - 43.6962533603324647 \varepsilon^2 + 128.936157875347890 \varepsilon^3 \]  
\[(40)\]

\[= + 0.473611472272364450 + 1.09585342206826990 \varepsilon + 5.37764333252884269 \varepsilon^2 + 8.82896457590640998 \varepsilon^3 \]  
\[(41)\]

Let us finally note that the prototypes in Eqs. 15 and 38 contain only one massive line, and are therefore trivially amenable to massless three-loop propagators. In order to evaluate these, we have used the results from [24,25,26]. For example the result in Eq. 38 can be obtained from

\[-\varepsilon^{4 \gamma} \frac{\Gamma(-4 \varepsilon)\Gamma^3(1 + \varepsilon)\Gamma^6(1 - \varepsilon)\Gamma(1 + 4 \varepsilon)}{\Gamma^3(2 - 2 \varepsilon)\Gamma(2 - \varepsilon)} \left(20\zeta_5 + (68\zeta_3^2 - 80\zeta_5 + 50\zeta_6) \varepsilon + (-272\zeta_3^2 + 204\zeta_3\zeta_4 + 80\zeta_5 - 200\zeta_6 + 450\zeta_7) \varepsilon^2 + O(\varepsilon^3)\right) \]  
\[(42)\]

Although not all of the above numerical values could be checked with independent techniques, we were able to verify many of them, either with the
sector decomposition method [27], or with Mellin-Barnes techniques [28,29] implemented in the MB package [30] (see also [31]).

3 On-shell Results

Once the master integrals are calculated, it is just a matter of substituting their values into the reduced diagrams and performing \( \overline{\text{MS}} \) renormalization to obtain the corrections to \( \Delta \rho \) in the \( \overline{\text{MS}} \) scheme. The latter can be found in the Appendix. As far as applications to electroweak physics are concerned, the on-shell scheme is more relevant. We used the relation between the \( \overline{\text{MS}} \) and on-shell masses of a heavy quark from [32], in order to perform the translation between the schemes.

In our result below, we keep the explicit dependence on the logarithms of the ratio of the on-shell top quark mass and the dimensional regularization mass unit. These can be checked, or recovered if not given, by using the simple renormalization group equation satisfied by \( \delta \rho \)

\[
0 = \frac{d}{d \log(\mu^2)} \delta \rho ,
\]

which in the on-shell scheme does only require the knowledge of the QCD \( \beta \)-function. Clearly, in the \( \overline{\text{MS}} \) scheme the anomalous dimension of the mass would also be needed.

If we introduce the following notation

\[
X_t = \frac{G_F m_t^2}{8\sqrt{2}\pi^2}, \quad L_t \equiv \log \left( \frac{m_t^2}{\mu^2} \right),
\]

the QCD corrections up to \( \mathcal{O}(\alpha_s^3) \) to the leading-order \( \Delta \rho \) result in the on-shell scheme are

\[
\delta \rho^{\text{OS (non-singlet)}} = 3X_t \left\{ 1 + \left( \frac{\alpha_s}{\pi} \right) C_F \left( -2.144934067 \right) \\
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ C_F^2 \left( 3.228455773 \right) \\
+ C_F \, C_A \left( -6.288851333 + 1.966189561 \, L_t \right) \\
+ C_F \, T_F \left( 0.1470525995 - 0.7149780223 \, L_t \right) \right]\right\}.
\]
\[
+ n_t \left( 2.679319209 - 0.7149780223 L_t \right)
\]
\[
+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ C_F^3 \left( -0.7845479837 \right) \\
+ C_F^2 C_A \left( 17.20096563 - 5.918835584 L_t \right) \\
+ C_F^2 T_F \left( -0.4393186129 + 1.616070332 L_t \\
+ n_t \left( -8.740003239 + 1.616070332 L_t \right) \right) \\
+ C_F C_A^2 \left( -30.95679757 + 13.0488891 L_t - 1.802340431 L_t^2 \right) \\
+ C_F C_A T_F \left( -0.5400590182 - 5.355886515 L_t + 1.310793041 L_t^2 \\
+ n_t \left( 24.8274162 - 9.998375300 L_t + 1.310793041 L_t^2 \right) \right) \\
+ C_F T_F^2 \left( 0.3035659457 + 0.09803506636 L_t - 0.2383260074 L_t^2 \\
+ n_t \left( 0.7160711769 + 1.884247873 L_t - 0.4766520149 L_t^2 \right) \\
+ n_t^2 \left( -3.448039206 + 1.786212806 L_t - 0.2383260074 L_t^2 \right) \right) \right] \}
\]

After setting \( \mu = m_t \), which is equivalent to \( L_t = 0 \), our result is in agreement with [1].

### 4 Conclusions

In this work, we have computed a further subset of four-loop single scale vacuum integrals with enough terms in the \( \varepsilon \) expansion to even allow for five-loop calculations. Our results can be applied whenever the physical process shows large scale differences and the large mass procedure can be applied, leading naturally to expansion (Wilson) coefficients expressed through tadpoles.

The immediate motivation for this computation has been the calculation of the non-singlet corrections to the \( \rho \) parameter at the four-loop level. After completing this task, we have obtained a result in agreement with the one recently presented in [1]. Thus, the complete four-loop corrections are now available and checked by two independent calculations for both the singlet ([9] and [1]) and non-singlet ([1] and the present work) parts. We can, of course, only confirm the numerical smallness of the final correction.
5 Acknowledgments

Parts of the presented calculations were performed on the DESY Zeuthen Grid Engine computer cluster. This work was supported by the Sofja Kovalevskaja Award of the Alexander von Humboldt Foundation sponsored by the German Federal Ministry of Education and Research.

A The ρ Parameter in the MS Scheme

The substitution of the results of Section 2 into the expression for the sum of the four-loop diagrams after reduction to masters leads to the bare contribution. Together with lower order corrections it can be written as follows

\[ \delta \rho = \frac{G_F m_0^2}{8\sqrt{2}\pi^2} \left( \frac{m_0}{\mu} \right)^{-2\varepsilon} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^0}{\pi} m_0^{-2\varepsilon} \right)^i c_i(\varepsilon), \]  

(A.1)

where \( c_i(\varepsilon) \) are given as Laurent expansions in \( \varepsilon \), with expansion coefficients being pure numbers. We perform the MS renormalization with

\[ \alpha_s^0 = \mu^{2\varepsilon} Z_{\alpha_s} \alpha_s(\mu^2), \]  

(A.2)

\[ m_0 = Z_m \mu_t(\mu^2), \]  

(A.3)

where we only need the two-loop strong coupling renormalization constant and the three-loop mass renormalization constant. With the following definitions

\[ x_t(\mu^2) = \frac{G_F \mu_t^2(\mu^2)}{8\sqrt{2}\pi^2}, \quad l_t \equiv \log \left( \frac{\mu_t^2(\mu^2)}{\mu^2} \right), \]  

(A.4)

the contribution to \( \Delta \rho \) of the QCD dressed one-loop diagrams up to \( O(\alpha_s^3) \) can be written as

\[ \delta \rho^{\text{MS}(\text{non-singlet})} = 3x_t \left\{ 1 + \left( \frac{\alpha_s}{\pi} \right) C_F \left( -0.1449340668 - 1.5 l_t \right) \right. \]  

\[ + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ C_F^2 \left( -0.04028771897 + 2.27901100 l_t + 1.125 l_t^2 \right) \right. \]  

\[ + C_F C_A \left( 0.3716744884 - 1.887977105 l_t + 0.6875 l_t^2 \right) \]  

(A.5)

\[ ^3 \text{dressing the one-loop diagrams cannot generate singlet contributions.} \]
\[ + C_F T_F \left( 0.4577540666 + 0.3683553111 l_t - 0.25 l_t^2 \right) \\
+ n_l \left( -0.447815241 + 0.3683553111 l_t - 0.25 l_t^2 \right) \] \\
+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ C_F^3 \left( 1.521138276 - 5.066619934 l_t - 3.256800825 l_t^2 - 0.5625 l_t^3 \right) \right] \\
+ C_F^2 C_A \left( 1.236336499 + 6.680264837 l_t - 0.1895515129 l_t^2 - 1.031245 l_t^3 \right) \\
+ C_F^2 T_F \left( -4.596159345 - 2.35323612 l_t + 0.3587005501 l_t^2 + 0.375 l_t^3 \right) \\
+ n_l \left( -2.313187377 - 0.9994272257 l_t + 0.3587005501 l_t^2 + 0.375 l_t^3 \right) \] \\
+ C_F C_A^2 \left( 0.7437683464 - 3.881114283 l_t + 2.26189568 l_t^2 - 0.4201388889 l_t^3 \right) \\
+ C_F C_A T_F \left( 2.503654176 + 1.794781882 l_t - 1.279484737 l_t^2 + 0.3055555556 l_t^3 \right) \\
+ n_l \left( -1.370511922 + 3.449430465 l_t - 1.279484737 l_t^2 + 0.3055555556 l_t^3 \right) \] \\
+ C_F T_F^2 \left( 0.6880486468 + 0.4672064147 l_t + 0.1227851037 l_t^2 - 0.0555555556 l_t^3 \right) \\
+ n_l \left( 0.8495320699 + 0.3327224357 l_t + 0.2455702074 l_t^2 - 0.111111111 l_t^3 \right) \\
+ n_l^2 \left( 0.4681052884 - 0.1344839790 l_t + 0.1227851037 l_t^2 - 0.0555555556 l_t^3 \right) \right) \] 

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