THE MINIMAL DISPLACEMENT AND EXTREMAL SPACES

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Abstract. We show that both separable preduals of $L_1$ and non-type I $C^*$-algebras are strictly extremal with respect to the minimal displacement of $k$-Lipschitz mappings acting on the unit ball of a Banach space. In particular, every separable $C(K)$ space is strictly extremal.

1. Introduction

Throughout the paper $(X, \|\cdot\|)$ denotes a real or complex infinite-dimensional Banach space. The notion of the minimal displacement was introduced by K. Goebel in [2]. Let $C$ be a bounded closed and convex subset of $X$ and $T : C \to C$ a mapping. The minimal displacement of $T$ is the number
\[ d_T = \inf \{ \|x - Tx\| : x \in C \}. \]

Goebel showed that if $T$ is $k$-Lipschitz then
\[ d_T \leq \left( 1 - \frac{1}{k} \right) r(C) \quad \text{for } k \geq 1, \]
where $r(C) = \inf_{x \in X} \sup_{y \in C} \|x - y\|$ denotes the Chebyshev radius of $C$. There are some spaces and sets with $d_T = \left( 1 - \frac{1}{k} \right) r(C)$. The minimal displacement characteristic of $X$ is a function
\[ \psi_X(k) = \sup \{ d_T : T : B_X \to B_X, T \in L(k) \}, \quad k \geq 1, \]
where $B_X$ denotes the closed unit ball of $X$ and $L(k)$ is the class of $k$-Lipschitz mappings. It is known that
\[ \psi_X(k) \leq 1 - \frac{1}{k} \]
for any space $X$ and the spaces with $\psi_X(k) = 1 - \frac{1}{k}$ are said to be extremal. Among extremal spaces are some sequence and function spaces such as $c_0, c, C[0, 1], BC(\mathbb{R}), BC_0(\mathbb{R})$ (see [2, 9]).

Recently Bolibok [1] proved that in $\ell_\infty$,
\[ \psi_{\ell_\infty}(k) \geq \begin{cases} (3 - 2\sqrt{2}) (k - 1), & \text{if } 1 \leq k \leq 2 + \sqrt{2}, \\ 1 - \frac{2}{k}, & \text{if } k > 2 + \sqrt{2}. \end{cases} \]
It is still an open problem whether the space $\ell_\infty$ is extremal with respect to the minimal displacement. Recall that $\ell_\infty$ is isometric to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$

2010 Mathematics Subject Classification. Primary 47H09, 47H10; Secondary 46B20, 46L35.

Key words and phrases. Minimal displacement, Lipschitz mapping, Optimal retraction.
is the Stone-Čech compactification of \( \mathbb{N} \). In this note we show that every separable \( C(K) \) space, where \( K \) is compact Hausdorff, is strictly extremal (see Definition 2.1). This is a consequence of a more general Theorem 2.5 which states that all separable preduals of \( L_1 \) are (strictly) extremal. An analogous result holds in the non-commutative case of separable non-type I \( C^* \)-algebras.

2. Results

We begin by recalling the arguments which show that (a real or complex) \( c_0 \) is extremal (see [3.5]). Fix \( k \geq 1 \) and define a mapping \( T : B_{c_0} \to B_{c_0} \) by

\[
Tx = T(x_1, x_2, x_3, ...) = (1, k |x_1| \land 1, k |x_2| \land 1, ...).
\]

It is clear that \( T \in L(k) \) and for any \( x = (x_1, x_2, x_3, ...) \in B_{c_0} \), \( \|Tx - x\| > 1 - \frac{1}{k} \), since the reverse inequality implies \( |x_1| \geq \frac{1}{k} \), \( k |x_1| \land 1 = 1 \) and, consequently, \( |x_i| \geq \frac{1}{k} \) for \( i = 1, 2, 3, ..., \), which contradicts \( x \in c_0 \). Notice that the minimal displacement \( d_T = 1 - \frac{1}{k} \) is not achieved by \( T \) at any point of \( B_{c_0} \). This suggests the following definition.

Definition 2.1. A Banach space \( X \) is said to be strictly extremal if for every \( k > 1 \), there exists a mapping \( T : B_X \to B_X, T \in L(k) \), such that \( \|Tx - x\| > 1 - \frac{1}{k} \) for every \( x \in B_X \).

It follows from the above that \( c_0 \) is strictly extremal. On the other hand we have the following result.

Proposition 2.2. Suppose that \( B_X \) has the fixed point property for nonexpansive mappings (i.e., every nonexpansive mapping \( S : B_X \to B_X \) has a fixed point). Then for every \( k \)-Lipschitz mapping \( T : B_X \to B_X, k \geq 1 \), there exists \( x \in B_X \) such that \( \|Tx - x\| \leq 1 - \frac{1}{k} \). In particular, \( X \) is not strictly extremal.

Proof. Let \( T : B_X \to B_X \) be \( k \)-Lipschitz. Then \( \frac{1}{k} T \) is nonexpansive and consequently there exists \( \|x\| \leq \frac{1}{k} \) such that \( Tx = kx \). Hence \( \|Tx - x\| = (k - 1) \|x\| \leq 1 - \frac{1}{k} \).

Proposition 2.2 applies to all uniformly nonsquare Banach spaces, uniformly noncreasy spaces as well as to \( \ell_\infty \).

In what follows we need the following observation.

Lemma 2.3. Suppose that \( Y \) is a subspace of a Banach space \( X \) and there exists an \( m \)-Lipschitz retraction \( R : B_X \to B_Y \). Then

\[
\psi_X (k) \geq \frac{1}{m} \psi_Y \left( \frac{k}{m} \right)
\]

for every \( k \geq 1 \).

Proof. Fix \( \varepsilon > 0 \) and select a \( k \)-Lipschitz mapping \( T : B_Y \to B_Y \) such that \( \|Ty - y\| > \psi_Y (k) - \varepsilon \) for every \( y \in B_Y \). Define \( \tilde{T} : B_X \to B_X \) by putting \( \tilde{T}x = (T \circ R)x, x \in B_X \). Then

\[
\psi_Y (k) - \varepsilon < \|TRx - Rx\| = \|RTRx - Rx\| \leq m \|\tilde{T}x - x\|
\]

for every \( k \geq 1 \).
for every $x \in B_X$. Notice that $\tilde{T}$ is $km$-Lipschitz and hence

$$\psi_X(km) \geq \frac{1}{m}(\psi_Y(k) - \varepsilon).$$

This completes the proof since $\varepsilon$ is arbitrary. \qed

Recall that $Y$ is said to be a $k$-complemented subspace of a Banach space $X$ if there exists a (linear) projection $P : X \to Y$ with $\|P\| \leq k$. The well known Sobczyk theorem asserts that $c_0$ is 2-complemented in any separable Banach space $X$ containing it.

**Proposition 2.4.** Let $X$ be a separable space which contains $c_0$. Then $$\psi_X(k) \geq 1 - \frac{1}{k}, \quad k \geq 1.$$  

**Proof.** Let $P : X \to c_0$ be a projection with $\|P\| \leq 2$ and define a retraction $R : X \to c_0$ by

$$(Rx)(i) = \begin{cases} (Px)(i), & \text{if } |(Px)(i)| \leq 1, \\ \frac{(P_x)(i)}{|(P_x)(i)|}, & \text{if } |(Px)(i)| > 1. \end{cases}$$

Then $R$ is 2-Lipschitz and $R(B_X) \subset R(B_{c_0})$. It is enough to apply Lemma 2.3 since $c_0$ is extremal. \qed

The most interesting case is if $X$ contains a 1-complemented copy of $c_0$.

**Theorem 2.5.** Let $X$ be a separable infinite-dimensional Banach space whose dual is an $L_1(\mu)$ space over some measure space $(\Omega, \Sigma, \mu)$. Then $\psi_X(k) = 1 - \frac{1}{k}$, i.e., $X$ is an extremal space.

**Proof.** It follows from the Zippin theorem (see [11, Theorem 1]) that if $X$ is a separable infinite-dimensional real Banach space whose dual is an $L_1(\mu)$, then $X$ contains a 1-complemented copy of $c_0$. If $X$ is complex, consider its real part to obtain a 1-Lipschitz retraction $R : X \to c_0$. It is now enough to apply Lemma 2.3. \qed

By examining the proof of Lemma 2.3 we conclude that every separable predual of $L_1(\mu)$ is in fact strictly extremal. It is well known that $C(K)$, the Banach space of scalar-valued continuous functions on the Hausdorff compact space $K$, is a predual of some $L_1(\mu)$ (see, e.g., [6]). Hence we obtain the following corollary.

**Corollary 2.6.** Every separable infinite-dimensional $C(K)$ space, for some compact Hausdorff space $K$, is strictly extremal.

There exists an extensive literature regarding complemented subspaces of Banach spaces. Our next simple observation is concerned with the connection between the existence of a nonexpansive retraction and the fixed point property.

**Proposition 2.7.** Suppose that $Y$ is a subspace of a Banach space $X$ and there exists a nonexpansive retraction $R : X \to Y$ such that $R(B_X) \subset B_Y$. If $B_X$ has the fixed point property for nonexpansive mappings, then $B_Y$ has the fixed point property, too.
Proof. Suppose, contrary to our claim, that there exists a nonexpansive mapping $T : B_Y \to B_Y$ without a fixed point. Then the mapping $T \circ R : B_X \to B_X$ is nonexpansive and fixed point free which contradicts our assumption.

It is well-known that $B_{\ell_\infty}$ has the fixed point property, whereas $B_{c_0}$ does not have the fixed point property for nonexpansive mappings (see, e.g., [4]). Hence we obtain another proof of the well-known result that there is no nonexpansive retraction from $B_{\ell_\infty}$ into $B_{c_0}$. In the same way, we conclude that there is no nonexpansive retraction from $B_{\ell_\infty}$ into $\hat{c}_0$, where $\hat{c}_0$ denotes the space of scalar-valued sequences converging to 0 with respect to a Banach limit. Notice that $\hat{c}_0$ is a strictly extremal subspace of $\ell_\infty$ of codimension one.

The space $C(K)$ of complex-valued continuous functions on $K$ forms a commutative $C^*$-algebra under addition, pointwise multiplication and conjugation. The remainder of this paper deals with a special class of $C^*$-algebras. Recall (see, e.g., [10, Definition 1.5]) that a $C^*$-algebra $\mathcal{A}$ is called type I if every irreducible $*$-representation $\varphi : \mathcal{A} \to B(H)$ on a Hilbert space $H$ satisfies $K(H) \subset \varphi(\mathcal{A})$ ($\varphi$ is called irreducible if $\varphi(\mathcal{A})$ has no invariant (closed linear) subspaces other than $\{0\}$ and $H$). A fundamental example of non-type I $C^*$-algebra is the CAR (canonical anti-commutation relations) algebra, defined as follows. Identify $B(\ell_2)$ with infinite matrices and define $\text{CAR}_d$ to be all $T \in B(\ell_2)$ so that there exist $n \geq 0$ and $A \in M_{2^n}$ (the space of all $2^n \times 2^n$ matrices over $\mathbb{C}$) such that

$$T = \begin{bmatrix} A & \cdots \\ A & A \\ \vdots \end{bmatrix}.$$ 

The CAR algebra is the norm-closure of $\text{CAR}_d$. 

Theorem 2.8 (W. Lusky [8]). Every separable $L_1$-predual space $X$ (over $\mathbb{C}$) is isometrically isomorphic to a 1-complemented subspace of the CAR algebra.

Combining a remark after Theorem 2.5 with Theorem 2.8 we deduce that the CAR algebra is strictly extremal. It turns out that the same is true for all separable non-type I $C^*$-algebras.

Theorem 2.9. Every separable non-type I $C^*$-algebra is strictly extremal.

Proof. It follows from [10] Corollary 1.7 that if $\mathcal{A}$ is a separable non-type I $C^*$-algebra then the CAR algebra is isometric to a 1-complemented subspace of $\mathcal{A}$. It suffices to follow the arguments of Lemma 2.3 since the CAR algebra is strictly extremal. \qed

References

[1] K. Bolibok, The minimal displacement problem in the space $\ell_\infty$, in: Topics in Nonlinear Analysis: dedicated to Kazimierz Goebel and Lech Górniewicz on the occasion
of their 70th birthday, W. Krysiewski and R. Skiba (eds.), Cent. Eur. J. Math. 12 (2012), 2211–2214.

[2] K. Goebel, On the minimal displacement of points under lipschitzian mappings, Pacific J. Math. 48 (1973), 151-163.

[3] K. Goebel, Concise Course on Fixed Point Theorems, Yokohama Publishers, Yokohama, 2002.

[4] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[5] K. Goebel, G. Marino, L. Muglia, R. Volpe, The retraction constant and the minimal displacement characteristic of some Banach spaces. Nonlinear Anal. 67 (2007), 735–744.

[6] H. E. Lacey, The isometric theory of classical Banach spaces, Springer-Verlag, New York-Heidelberg, 1974.

[7] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces. II. Function spaces, Springer-Verlag, Berlin-New York, 1979.

[8] W. Lusky, Every separable $L_1$-predual is complemented in a $C^*$-algebra, Studia Math. 160 (2004), 103–116.

[9] L. Piasecki, Retracting ball onto a sphere in some Banach spaces. Nonlinear Anal. 74 (2011), 396-399.

[10] H. Rosenthal, Banach and operator space structure of $C^*$-algebras, in: Trends in Banach spaces and operator theory, A. Kamińska (ed.), Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003, 275–294.

[11] M. Zippin, On some subspaces of Banach spaces whose duals are $L_1$ spaces, Proc. Amer. Math. Soc. 23 (1969), 378–385.

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