GENERAL ABELIAN ORIENTIFOLD MODELS
AND ONE LOOP AMPLITUDES

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Abstract

We construct a one loop amplitude for any Abelian orientifold point group for arbitrary complex dimensions. From this we show several results for orientifolds in this general class of models as well as for low dimensional compactifications. We also discuss the importance and structure of the contribution of orientifold planes to the dynamics of D-branes, and give a physical explanation for the inconsistency of certain $Z_4$ models as discovered by Zwart.
1 Introduction

Though there has been much excitement about $D$–branes in the past few years, it has only been more recently that the importance of orientifolds in these models has been appreciated. An orientifold can stand for two related things, a compactification model or a plane. A natural way of looking at the latter is as the unorientated counterparts of the $D$–branes. As they are unoriented they have somewhat different properties and are undynamical, but they can not be discounted as they can couple to $D$–branes via open and unoriented closed strings. Indeed the open type I string can be considered as a twisted sector of the closed string under these conditions.

Orientifold models arise when we combine the worldsheet parity symmetry with an orbifold point group. The fixed points in the target space then become the fixed orientifold plane. These planes carry RR charges which in turn require the presence of $D$–branes to cancel. The $D$–branes and $o$–planes interact via open strings which have Chan–Paton factors associated with them, and hence also Wilson lines. In the T–dual models which produce the $D$–branes from string theory the Wilson lines describe the position of the $D$–branes in the compact space, and thus play a significant roll in their dynamics by giving the gauge groups associated with the $D$–branes. When this is combined with an discrete modding to produce a orbifold space the $D$–branes become identified under the point group so that not all of the are dynamical. This is reflected in the Wilson lines. In order to understand how the Wilson lines behave under the action of the point group we require the action of the point group on the Chan–Paton factors. This we do by investigating the open string one loop amplitudes in the T–dual model.

The aim of this paper is to find a general solution to the action of the point group on the Chan–Paton factors for the set of Abelian point groups defined on complex compact coordinates. Though these are not the most general orbifold models, they do form the most important class. Much work has been performed in this area already and this paper hopes to finish part of that project. Some general results, independant of the number of dimensions being compacted, are produced and we discuss some of the problems that have arisen in the consistancy of these models. We also show how strongly the dynamics of the $D$–branes are in fact controlled by the presence of the orientifold planes. They do not simply increase the number of possible gauge groups that $D$–branes can have, but tightly control the way they can move.
together by defining with unexpected rigidity the action of the point group on the Chan–Paton factors and Wilson lines.

In section 2 we outline the theory and features we wish to employ as well as our notation. Some of the notation will change for different parts of the theory but this is to emphasise certain patterns in the amplitudes. In section 3 we calculate the one loop open string amplitudes we will need \([4, 16]\), giving them in the most general form possible; while in section 4 we extract the divergences from them. These divergences, in fact tadpole equations, are the charge cancellation constraints in the dual model and are what allow us to solve for the action of the point group on the Chan–Paton factors. Surprisingly enough, we are able to find a very general formula for this, which is what gives us the ability to discuss the strong influence of the orientifold planes. Also in section 4 we take a look at orientifold models which have been compacted down to two dimensions. In section 5 we use some of the results of section 4 to give a physical explanation for the strange inconsistancy of certain \(Z_4\) point group models as recently discovered by Zwart. Finally we take a quick look at how the orientifold groups can be extended.

## 2 The Amplitudes I: Features

In this section we start to write down an scheme for constructing the Klein bottle, Möbius strip and cylinder amplitudes for the open string. We follow the conventions of Gimon and Polchinski \([8]\), and use the work of Gimon and Johnson \([6]\) as our basis. Most of the background work and theory can be found in refs \([5, 6, 7, 8, 9, 10, 11, 12]\) and we will leave the reader to consult those papers for the relevant material.

Here we shall just give the basic tools and definitions of what we are calculating. Our ultimate aim is to produce a set of equations that will allow us to write down the conditions on the projective representation of the orientifold point group acting on the Chan Paton factors for any given model.
2.1 General Formulae

The complete amplitude we wish to calculate can be written as

\[ \int_{0}^{\infty} \frac{dt}{t} \left[ Tr_{c}(P(-1)^F e^{-2\pi t(L_0+\bar{L}_0)} + Tr_{o}(P(-1)^F e^{-2\pi tL_0}) \right] \]  \hspace{1cm} (1)

We use $P$ here to denote the generalized GSO projector extended to include the point group of the orientifold model and $F$ is the space-time fermion number. $t$ is the loop modulus for the cylinder; it can be related to the cylinder length, to give a common length scale, by $t = 1/2\ell, 1/8\ell, 1/4\ell$ for the cylinder, Möbius strip and Klein bottle diagrams respectively. $L_0$ represents the hamiltonian of the string in the various twisted and untwisted sectors.

2.2 General Point Groups

At this point we are going to relax the constraints of supersymmetry and modular invariance to allow a larger range of abelian point group models. Since we desire a degree of arbitrariness for the number of compact dimensions in our analysis this extends in to the lattices used for compactification and hence the allowed automorphisms. Though for the most part the actual group structure of the lattice is not as important as the point group we have increasing number of possibilities as the number of compact dimensions increase. However, for this paper we will restrict our discussion to the purely Abelian case on complex coordinates as these can be related in many cases to Calabi-Yau spaces. We will denote the $k$th element of the Abelian groups as $\alpha^k_N$ where $\alpha_N$ is the generator. Thus the elements of the orientifold group can be written as

\[ \{1, \Omega, \alpha^k_N, \Omega \cdot \alpha^k_N \} \]  \hspace{1cm} (2)

The extension to point groups with more than one generator is obvious.

Without loss of generality we can divide our compact space up using complex coordinates, $z_i$. The reason for this is that the $T^2$ has two possible Lie algebra valued lattices corresponding to the root diagrams for $SU(3)$ and $SU(2) \times SU(2)$ and allows the incorporation of all Abelian point groups of interest. We divide our spacetime into three sections:

- The compact coordinates $I = 1, ..., 2d$.
- The uncompact coordinates $\mu = 1, ..., 2\nu$. 

3
• The ghost coordinates which swallow two dimensions (corresponding
to the light cone quantization).

If the total number of dimensions in the theory is $D$ then we must have

$$2\nu + 2d + 2 = D$$

where $D$ in the case of superstring theory is 10.

Because we are complexifying our space we can define the action of each
element of our point group on a torus independently of the rest to find the
amplitudes, though the final overall results will take into account all sectors
and the overlap between the action of the point groups.

Take a general cyclic group $\mathbb{Z}_N$ with element $\alpha_N^k$; following the literature
we define its action on a particular complex coordinate, say $\rho_1 = x^8 + ix^9$ as:

$$\alpha_N^k: \rho_1 \to e^{\pm \frac{2\pi i k}{N}} \rho_1$$

for the bosonic and NS sectors; while for the R sector the point groups acts
on the appropriate fermions with the action:

$$\alpha_N^k: \frac{e^{\pm 2\pi i k J_{89}}}{N}$$

Note that when the point group is of the form $\mathbb{Z}_N \times \mathbb{Z}_M$, which we will call
a cross group as opposed to a pure group otherwise, we can define the action
of elements of both groups, generated by $\alpha_N^n$ and $\alpha_M^m$ on a single complex
coordinate in a combined form, i.e.

$$\rho_1 \to e^{\frac{2\pi i n}{N}} e^{\pm \frac{2\pi i m}{M}} \rho_1$$

In all cases we must choose $\pm$ a priori. Models with point groups of the
form $\mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_P$ act on a single complex coordinate are not allowed to
occur, though more complex actions where each group acts on a separate but
overlapping pair of coordinates are allowed.

### 2.3 Dp-Branes

Since we are dealing with T-dual models we have the presence of $D$-branes in
the theory corresponding to open strings with Dirichlet boundary conditions.
These make the open string one loop amplitudes much more interesting but
unfortunately reduce the generality of our expressions. In Type I string theory, taken as a orientifold of type IIB theory the eigenvalues of the parity operator $\Omega$ satisfy

$$\Omega^2 = (\pm i)^{(9-p)/2} \tag{7}$$

This has values of $\pm 1$ for $p = 1, 5, 9$ which is consistent with what is known about Type I theory. Other values of $p$ get projected out. As a result of this we can not use arbitrary $D$ above but must set it equal to 10 for superstrings. It is not obvious to us how to generalize this expression with this technique but as there is no need to we shall leave it as it stands. Later we shall see that the amplitudes do seem to see this value for the total dimension.

A side consequence of this form for the $\Omega^2$ is that the group nature of the projection representation of the point group and thus the Chan Paton factors associated with each $D$-brane via the open string interactions is fixed. The $D9$-brane and the $D1$-brane have orthogonal factors, $SO(n)$, while the $D5$-brane symplectic factors, $Sp(n)$. See [5, 13, 14] for further discussion on this point.

The $D9$-Brane is always present as it corresponds to the action of the worldsheet parity, $\Omega$ with the identity of the point group. It is essentially the open string with all Neumann boundary conditions and is required to cancel the presence of the $o9$-plane induced in the theory by $\Omega$. Other members of the point group determine which other $o$-planes can exist. These latter objects carry RR charge which needs to be cancelled by the opposite charge appearing on the $Dp$-branes. However, depending on the action of the point group and hence the conserved charge not all these will be present in the theory. For example, the $D1$-brane only appears in low dimensional compactification, while $D5$-branes are not present in $T^8/Z_2$ but are present in $T^8/Z_n^2$ for $n = 2, 3$ [11].

A point to note is that for five branes there are different configurations possible under the various point groups so that there are different types giving rise to sectors of interaction between $5_i, 5_j, 9_5, 5, 9, 15_i, 5_j, 1$ according to the model. This is a feature of the way in which we choose the models to act on the complex coordinates.

\footnote{we are grateful to Stefan Förste for help with this point}
2.4 Compact Zero Modes

In dealing with a variation of standard orbifold models there are twisted and untwisted sectors in the theories. In the former the bosonic zero modes, corresponding to discrete momentum and winding factors vanish, but they must be accounted for in the untwisted sector. There are various types of contribution to consider depending on the $D$-branes involved. Following Berkooz and Leigh we will write their contributions in the form

$$M_j = \left( \sum_{n \in \mathbb{Z}} e^{-\pi tn^2/v_j} \right)^2$$

$$W_j = \left( \sum_{\omega \in \mathbb{Z}} e^{-\pi t\omega^2 v_j} \right)^2$$

where the momentum sum is on the root lattice $\Gamma$; and the winding modes are summed over the dual lattice $\Gamma^\star$. $v_j$ is the volume of the particular torus.

2.5 Chan Paton Factors

Chan Paton factor are intrinsically features of the open string, and as such will not appear in the Klein bottle expressions. Traditionally they have provided gauge groups for the open string but in the dual models they play a very fundamental role and are responsible for a lot of the interesting features of the theory.

The action of the orientifold group on the Chan Paton factors, $\lambda$, should form a projective representation of the group up to a phase controlled by the various $D$-brane sectors they are being taken in. If $P$ represents an element of the point group and $p$ the $D$-brane sector, then the projective representation of the orientifold group is given by the matrices $\gamma$ satisfying:

$$\lambda \propto \gamma_{P,p} \lambda \gamma_{P,p}^{-1}$$

$$\lambda \propto \gamma_{\Omega,p} \lambda^T \gamma_{\Omega,p}^{-1}$$

What appears in the one loop amplitudes for the Möbius strip and cylinder are the trace over the different $\gamma$'s according to sector. Thus by demanding cancellation of tadpoles there are constraints on the $\gamma$’s which in turn will allow specific representations to be formed which are unique upto unitary transformations on the Chan–Paton factors.
3 Amplitudes II: The Equations

3.1 General Features of the Solutions

The amplitudes we are specifically calculating are

\[ KB : T_{NSNS+RR}^{U+T} \frac{\Omega}{2} P \frac{1 + (-1)^F}{2} e^{-2\pi t L_0} \]  
(12)

\[ MS : T_{NS-R}^{\lambda \lambda} \frac{\Omega}{2} P \frac{1 + (-1)^F}{2} e^{-2\pi t L_0} \]  
(13)

\[ C : T_{NS-R}^{\lambda \lambda} \frac{1}{2} P \frac{1 + (-1)^F}{2} e^{-2\pi t L_0}, \]  
(14)

where the point group projection operator \( P \) is given by

\[ \frac{1}{N} \sum_{k=0}^{N_i-1} \alpha^k_{N_i} \]  
(15)

\( N \) is the order of \( P \) and \( N_i \) is the order of the subgroups where applicable.

In each amplitude there is a sum over three sets of theta functions. In order these will represent the contributions from the Neveu–Schwarz and Ramond sectors. The theta functions appearing with negative index are the bosonic contributions with the factors of \( f(t) \) (and in some of the expressions trigonometric contributions depending on \( z_i \)) being present to allow the rewriting in terms of the theta functions. There also appears in each amplitude a factor proportional to \( t^{-\left(D-2d-2\ell\right)} \) which is the uncompactified momentum contribution. This is rewritten in simpler form as \( t^{-(\nu+1)} \) or equivalently \( t^{(d-5)} \) using \( D = 10 \).

The types of \( D \)-brane the open string ends on is given by the labels \( \lambda \) and \( \lambda' \). For the Möbius strip we have \( \lambda = \lambda' \), that is 99, 5, 5, 11 sectors. For the cylinder we must take into account all possible combinations of \( D \)-branes that can appear in the particular model under construction eg

99, 95, 91, 5, 5j(5), 5, 1, 11

The Klein bottle ends only on \( o \)-planes so does not receive these contributions.

Though so far we have been very general in our approach we have yet to incorporate the Gimon–Polchinski consistancy conditions. They affect the presence of the Klein bottle and Möbius strip in the different twisted sectors.
3.2 Conventions

We will express the amplitudes using the Jacobi $\vartheta$–functions for the twisted sectors and the $f_i$ functions for the untwisted sectors as this allows us to easily see the relevant patterns for a general point group. For the open string amplitudes, the $D9$-branes correspond to the string with Neumann boundary conditions on both ends. These will give a contribution to the traces according to

$$\text{Tr}(e^{\pm\frac{2\pi i k}{N_i}}) = (4\sin^2\frac{\pi k}{N_i})^{-d'}$$

(16)

where $d'$ means we take into account every complex dimension the group $\mathbb{Z}_{N_i}$ is acting upon.

For our theta functions we use the definitions as

$$\vartheta_1(z|t) = 2^{q_1/4}\sin \pi z \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 - q^2 e^{2\pi i z}) \prod_{n=1}^{\infty} (1 - q^2 e^{-2\pi i z})$$

$$\vartheta_2(z|t) = 2^{q_1/4} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 + q^2 e^{2\pi i z}) \prod_{n=1}^{\infty} (1 + q^2 e^{-2\pi i z})$$

$$\vartheta_3(z|t) = \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 + q^{2n-1} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 + q^{2n-1} e^{-2\pi i z})$$

$$\vartheta_4(z|t) = \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i z}) \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{-2\pi i z})$$

(17)

with $q = e^{-\pi t}$ and $z_i = \frac{k}{N_i}$.

Also required is

$$f_1(q) = q^{1/12} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right), \quad f_2(q) = q^{1/12} \sqrt{2} \prod_{n=1}^{\infty} \left(1 + q^{2n}\right)$$

$$f_3(q) = q^{-1/24} \prod_{n=1}^{\infty} \left(1 + q^{2n-1}\right), \quad f_4(q) = q^{-1/24} \prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right).$$

(18)

3.3 Klein Bottle

(a) Untwisted sector

$$\int_0^{\infty} \frac{dt}{t} (4\pi^2 \alpha' t)^{d-5} \frac{f_1^8(2t)}{f_1^2(2t)} \times F(M_i, W_j)$$

(19)

Here $F(M_i, W_j)$ stands for the zero mode contribution which is
• $\prod_{j=1}^d M_j + \prod_{j}^d W_j$ if the point group consists of a single $Z_N$ acting on all the compact coordinates simultaneously.

• $\prod_{j=1}^d M_j + \sum_5 \prod_{i}^d M_i \prod_{j \neq i}^d W_j$ when there are $D9$- and $D5$-branes in a cross theory.

• $\prod_{j=1}^d M_j + \sum_5 \prod_{i}^d M_i \prod_{j \neq i}^d W_j + \prod_j^d W_j$ when all possible branes, including $D1$-branes are present in the theory.

We use $\sum_5$ to denote the fact that we must sum over the possible ways we can include $D5$-branes in the theory which will be dependant on the action of the point group on the compact coordinates.\(^2\)

(b) Twisted sector

In the twisted sectors we need to define the action on the modulus of the torus $t_i$. We use the notation

$$t_i = t + \zeta t, \quad \zeta = \zeta(i)$$

(20)

where the twist $\zeta = (m - n)/N_i$ in the closed string channel. For cross models we have the same only now the denominator is the larger of the Abelian point groups where applicable and the numerator contributions are appropriately normalized. That is if the relevant point group is $Z_N \times Z_M$ with $M = anN$ such that $a \in \mathbb{Z}^+$, then

$$\zeta = \pm \left(\frac{m}{M} - \frac{n}{N}\right) = \pm \frac{m - an}{M}$$

(21)

In the $T^4/Z_N$ model of Gimon and Johnson \([6]\) then this simplifies to setting $t_i$ to $t^+ = t + \zeta t$, $t^- = t - \zeta t$ appropriately for the separate tori. The amplitude is:

$$\frac{N-1}{\sum_{n=1}^{N-1}} \int_0^\infty \frac{dt}{t} (4\pi^2 \alpha' t)^{d-5} f_{1}^{-3\nu}(2t) \times$$

$$\left\{ -\vartheta_4'(0|2t) \prod_{i=1}^{d} \vartheta_4(z_i|2t_i) \vartheta_1^{-1}(z_i|2t_i) \right\} \frac{(2\sin 2\pi(z_i - \zeta t))}{(4\sin^2 \pi z_i)}$$

\(^2\)A quick rule of thumb, though not entirely correct, is that $\prod_{j=1}^d M_j$ is from the $D9$-branes, $\prod_{i}^d M_i \prod_{j \neq i}^d W_j$ is from the presence of $D5$-branes, while $\prod_j^d W_j$ is normally present if there are $D1$-branes in the theory.
For the

(a) Untwisted Sector

3.4 Möbius Strip

(b) Twisted Sector

Thus depending on our compacting scheme only now we must take into account traces over Chan Paton factors in the amplitude. Thus depending on our compacting scheme \( \mathcal{F}(M_i, W_j) \) equals

- \( \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d M_j + \sum_{P(i)} \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d W_j \) if the point group consists of a single \( Z_N \) acting on all the compact coordinates simultaneously. \( B \) stands for a \( D1 \)-brane when \( d = 4 \) and a \( D5 \)-brane otherwise.

- \( \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d M_j + \sum_{P(i)} \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d M_i \prod_{j \neq i}^d W_j \) when there are \( D9 \)- and \( D5 \)-branes for the cross theory.

- \( \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d M_j + \sum_{P(i)} \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \prod_{j=1}^d M_i \prod_{j \neq i}^d W_j \) when \( d = 4 \).

(b) Twisted Sector

The twisted Möbius strip amplitudes are\(^3\):

For the \( D9 \)-branes:

\[
- \sum_{P(i)} \text{Tr}(\gamma_{0,9}^{d-1} \gamma_{0,9}^{*}) \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha')^{d-5} f_1^{3d-5} f_3^{3d-5} \times \mathcal{F}(M_i, W_j) \times \\
\left\{ \begin{array}{l}
\vartheta_3^{d}(i) \prod_{i=1}^d \vartheta_3(i, z_i \vartheta_1^{-1}(i, z_i) \frac{(2 \sin \pi z_i)}{(4 \sin^2 \pi z_i)} \\
- \vartheta_4^{d}(i) \prod_{i=1}^d \vartheta_4(i, z_i \vartheta_1^{-1}(i, z_i) \frac{(2 \sin \pi z_i)}{(4 \sin^2 \pi z_i)} \\
\end{array} \right. 
\]

\(^3\)note that for the Möbius strip we have \( q = \exp(-2\pi t) \)
For the $D5$–branes, we have to be very careful where we place the brane in the compact and non–compact space time. We shall assume that the $D5$–branes shall fill the uncompact spacetime before the compact dimensions. In the case that the number of uncompact dimensions is less than that of the five brane we can organize the permitted $D5$–branes in the theory in the compact dimensions according to the action of the point group. This naturally splits the mode expansion, and thus the amplitudes, into two: those with DD boundary conditions and those with NN boundary conditions. We will use a double prime to denote those compact dimensions with a $D5$–brane in them and a double prime those without, the product being symbolic over the compact dimensions.

\[
- \vartheta_2'(iq, 0) \prod_{i=1}^{d} \vartheta_2(iq, z_i) \vartheta_1^{-1}(iq, z_i) \frac{(2 \sin \pi z_i)}{(4 \sin^2 \pi z_i)} \left\{ \begin{array}{l}
\vartheta_3'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_4'(iq, z_i) \vartheta_3''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \\
- \vartheta_4'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_2'(iq, z_i) \vartheta_4''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \\
- \vartheta_2'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_1'(iq, z_i) \vartheta_2''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \end{array} \right\} (24)
\]

For the $D1$–branes present when $d = 4$, the equation is essentially the same as for the $D9$–branes only the change of boundary conditions from NN to DD means a shift in the theta functions and a loss of the trace contribution that arose out of the NN sector:

\[
- \sum_{P(i)} Tr(\gamma_{\Omega P,5_i}^{-1} \gamma_{\Omega P,5_i}^T) \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{d-5} f_1^{-3\nu}(iq) \times \left\{ \begin{array}{l}
\vartheta_3'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_4'(iq, z_i) \vartheta_3''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \\
- \vartheta_4'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_2'(iq, z_i) \vartheta_4''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \\
- \vartheta_2'(iq, 0) \prod_{i=1}^{d} \frac{\vartheta_1'(iq, z_i) \vartheta_2''(iq, z_i) (2 \cos \pi z_i)}{\vartheta_2'(iq, z_i) \vartheta_1'(iq, z_i)} \end{array} \right\} (25)
\]

For the $D1$-branes present when $d = 4$, the equation is essentially the same as for the $D9$–branes only the change of boundary conditions from NN to DD means a shift in the theta functions and a loss of the trace contribution that arose out of the NN sector:

\[
- \sum_{P(i)} Tr(\gamma_{\Omega P,1}^{-1} \gamma_{\Omega P,1}^T) \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{d-5} \times \left\{ \begin{array}{l}
\prod_{i=1}^{d} \vartheta_4(iq, z_i) \vartheta_3^{-1}(iq, z_i) (2 \cos \pi z_i) \\
- \prod_{i=1}^{d} \vartheta_3(iq, z_i) \vartheta_2^{-1}(iq, z_i) (2 \cos \pi z_i) \end{array} \right\}
\]
\[- \prod_{i=1}^{i=d} \vartheta_1(iq, z_i) \vartheta_2^{-1}(i q, z_i)(2 \cos \pi z_i) \}\}

(26)

3.5 Cylinder

(a) Untwisted Sector

The contribution from the 99 cylinders are

\[(Tr(\gamma_{1,9}))^2 \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{d-5} \frac{f_5^8(t)}{f_5^1(t)} \prod_j M_j\]

From the 5,5 \text{i cylinders:}

\[
\int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{d-5} \frac{f_5^8(t)}{f_5^1(t)} \times
\sum_i M_i \sum_{a, b \in 5_i} (\gamma_{1,5_i})_{aa} (\gamma_{1,5_i})_{bb} \prod_{m_j \neq i} \sum_\omega e^{-t(2\pi \omega r_j + X_{m_j}^{m_j} - X_b^{m_j})^2/2\pi \alpha'}
\]

That is there is a momentum contribution from the compact dimensions where the $D_{5_i}$-branes live and winding contributions otherwise. For the 11 cylinders there will only be winding contributions so we have:

\[
\int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{d-5} \frac{f_5^8(t)}{f_5^1(t)} \times
\sum_{a, b \in 1} (\gamma_{1,1})_{aa} (\gamma_{1,1})_{bb} \prod_{m_j \neq 1} \sum_\omega e^{-t(2\pi \omega r_j + X_{m_j}^{m_j} - X_b^{m_j})^2/2\pi \alpha'}
\]

(b) Twisted Sector

For the twisted cylinder we have to be careful as there are a variety of boundary conditions that will give different results according the action of $P$. The boundary conditions can be grouped as NN, DD and ND according to the particular branes being used; while the action of $P$ is labelled by $z_i$. Note that $P = 1$ can also label the uncompact dimensions as well. Thus with the given prefactors listed below there is a sum over three sets of products of four $\vartheta$ functions. Following the same set of conventions as above we have the following theta functions for each contribution from the three fermion and boson sectors respectively:

1. NN, $P = 1 : \vartheta_3(0|t)$, $\vartheta_4(0|t)$, $\vartheta_2(0|t)$, $f_1^{-1}(t)$
2. \( \text{NN, } P = z_i : \vartheta_3(z_i|t), \vartheta_4(z_i|t), \vartheta_2(z_i|t), \vartheta_1^{-1}(z_i|t)(2 \sin \pi z_i) \)

3. \( \text{ND, } P = 1 : \vartheta_2(0|t), \vartheta_1(0|t), \vartheta_3(0|t), f_1^{-1}(t) \)

4. \( \text{ND, } P = z_i : \vartheta_2(z_i|t), \vartheta_1(z_i|t), \vartheta_3(z_i|t), \vartheta_1^{-1}(z_i|t) \)

5. \( \text{DD, } P = 1 : \vartheta_3(0|t), \vartheta_4(0|t), \vartheta_2(0|t), f_1^{-1}(t) \)

6. \( \text{DD, } P = z_i : \vartheta_3(z_i|t), \vartheta_4(z_i|t), \vartheta_2(z_i|t), \vartheta_1^{-1}(z_i|t)(2 \sin \pi z_i) \)

Note that for NN with \( P = z_i \) we must also account for the contribution of \((4 \sin^2 \pi z_i)\) from the trace over the representation of \( P \) as noted earlier. In the equation below we will have this as being implicit. In the \( P \neq 1 \) sectors all the amplitudes will contain a factor of \( f_1^{-3\nu}(t) \) to correctly normalize the theta functions. Naturally in the case \( d = 4 \) we have \( \nu = 0 \) and this contribution becomes 1.

A general cylinder amplitude will be of the form

\[
\sum_{P(i)} Tr(\gamma_{P,B})Tr(\gamma_{P,C}) \int_0^\infty dt \left( 8\pi^2 \alpha' t \right)^{-1} f_1^{-3\nu}(t) \times \left\{ \prod_{i=1}^4 \frac{\vartheta_{F_i}(z_i|t)}{\vartheta_{B_i}(z_i|t)} - \prod_{i=1}^4 \frac{\vartheta'_{F_i}(z_i|t)}{\vartheta'_{B_i}(z_i|t)} - \prod_{i=1}^4 \frac{\vartheta''_{F_i}(z_i|t)}{\vartheta''_{B_i}(z_i|t)} \right\} \]

(30)

Where \( B, C \) can be a 1, 5, or 9 D–brane, \( z_i \) can be zero and \( F_i \) labels the relevant boundary conditions according to \( B \) and \( C \) for the fermions, while \( B_i \) labels the corresponding bosons for the same conditions. The primes simply label the fermion sector. When \( B, C \) or both are D5–branes or D1–branes then there is a sum over the fixed points of the orbifold group as well.

4 Divergences

Here we will give a general set of rules that can be derived for arbitrary compactification (up to complexification) then concentrate on developing the more general case in \( d = 4 \). For convention we adopt

\[
v_L = V_L(4\pi^2 \alpha t)^{L/2} \]

(31)

to represent the appropriate volume expressions associated with each compactification scheme.
Taking the limit $t \to 0$ and using the relevant correspondances between $t$ and $l$ we can calculate the tadpole divergences from each diagram in each sector. In the untwisted sector all dependance on $\nu$ and $d$ vanishes in this limit giving us a set of relations for each type of brane that holds in all cases. For the $D5_i$-branes and $D1$-brane the presence of $P(i)$ represents the set of transformations that defines the branes. In fact the $P(i)$ in question will be an action of a single point group element of order two (if there is no such element then the brane will not exist in the model as can also be seen by consideration of charge cancellation), so we can use the Gimon–Polchinski condition for an element $g$ of the point group

$$\gamma^T_{\Omega g} = \pm \gamma_{\Omega g}$$ (32)

These relations, dropping common factors, are:

(a) for the $D9$-brane,

$$Tr(\gamma_{0,9})^2 - 64Tr(\gamma^{-1}_{\Omega,9}\gamma^T_{\Omega,9}) + 32^2$$ (33)

(b) for the $D5_i$-branes,

$$Tr(\gamma_{0,5,i})^2 - 64Tr(\gamma^{-1}_{\Omega P(i),5}\gamma^T_{\Omega P(i),5}) + 32^2$$ (34)

(c) for the $D1$-brane,

$$Tr(\gamma_{0,1})^2 - 64Tr(\gamma^{-1}_{\Omega P(i),1}\gamma^T_{\Omega P(i),1}) + 32^2$$ (35)

Now using that the number of branes of a particular type is given by the trace of $\gamma_{1,p}$ we can solve these three equations to find that the number of each type of brane is always 32 provided they exist in the relevant model to start with. This is a general result and is consistent with what we know about $D$–branes [13].

For the twisted sectors we have the results,

(a) from the Klein bottle

$$\left[2^{5-d} \sum_{P(i)} \prod_i \frac{2\cos \pi z_i}{2\sin \pi z_i} \right]^2$$ (36)

which for $\Omega_{k+N/2}$ becomes

$$\left[2^{5-d} \sum_{P(i)} \prod_i \frac{-\sin \pi z_i}{\sin \pi z_i} \right]^2$$ (37)

14
(b) from the Möbius strip

\[-2^{6-d} \sum_{P(i)}' \left\{ Tr(\gamma_{\Omega,9}^{-1} \gamma_{\Omega,9}^T) \prod_{i=1}^{d} \frac{1}{2 \sin \pi z_i} \right. \]

\[+ Tr(\gamma_{\Omega P(i),5}^{-1} \gamma_{\Omega P(i),5}^T) \prod_{i=1}^{d} (2 \cos \pi z_i) \]

\[+ Tr(\gamma_{\Omega P(i),1}^{-1} \gamma_{\Omega P(i),1}^T) \prod_{i=1}^{d} (2 \cos \pi z_i) \right\} \]

(38)

(c) for the cylinder we must sum over all possible sectors. This can cause problems with the NN trace contributions but the expression can be written, after factorising into perfect squares, as:

\[\sum_{P(i)}' \left\{ \left( Tr(\gamma_{P(i),9}) \prod_{i=1}^{d} (2 \sin \pi z_i)^{-1} - Tr(\gamma_{P(i),5}) \prod_{i=1}^{d} (2 \sin \pi z_i) \right)^2 \right. \]

\[+ \left( Tr(\gamma_{P(i),9}) \prod_{i=1}^{d} (2 \sin \pi z_i)^{-1} - Tr(\gamma_{P(i),1}) \prod_{i=1}^{d} (2 \sin \pi z_i) \right)^2 \]

\[+ \left( Tr(\gamma_{P(i),5}) \prod_{i=1}^{d} (2 \sin \pi z_i) \prod_{i=1}^{d} (2 \sin \pi z_i) \right)^2 \]

\[+ \left( Tr(\gamma_{P(i),5}) \prod_{i=1}^{d} (2 \sin \pi z_i) - Tr(\gamma_{P(i),1}) \prod_{i=1}^{d} (2 \sin \pi z_i) \right)^2 \}

(39)

where the tilde means that the contribution is from the coordinates where the branes do not overlap in the compact dimensions (ie ND coordinates) and the prime indicates we do not sum over the order two element. Note also that there are sums over the fixed points in the case of the 51 and 1 branes.

Before we start solving for the individual sectors and extracting the divergences from these tadpoles let us note some general properties of $\gamma_{\Omega P(i)}$ that hold in all cases. First we note that the Abelian group elements $\alpha_{kN}$ and $\alpha_{k+N/2}$ both square to the same element $\alpha_{N}^{2k}$. Also in solving the contributions from the untwisted sector we made the choice $\gamma_{N/2} = -1$. Finally we note that the values of $\Omega^2$ in the various sectors are, following [5], 1 for the 99, 91 and 11 sectors, $-1$ for the 51, 59 and 51 sectors. Using this we can
define a set of relations allowing us to simplify the above equations.

\[ \text{Tr}(\gamma_{\Omega k}^{-1}) = \text{Tr}(\gamma_{\Omega 2k}) \quad (40) \]

\[ \text{Tr}(\gamma_{\Omega k + N/2}^{-1}) = -\text{Tr}(\gamma_{\Omega 2k}) \quad (41) \]

when \( \Omega^2 = 1 \); and

\[ \text{Tr}(\gamma_{\Omega k}^{-1}) = -\text{Tr}(\gamma_{\Omega 2k}) \quad (42) \]

\[ \text{Tr}(\gamma_{\Omega k + N/2}) = \text{Tr}(\gamma_{\Omega 2k}) \quad (43) \]

when \( \Omega^2 = -1 \) for the various brane combinations.

Going back to purely Abelian case and looking ahead somewhat, we know that depending on the particular elements of the point group, the cylinder equation above is what we get for some constraints while for others we have to take into account the Möbius strip and Klein bottle contributions. Assuming that we have to do the latter notice that if we take this charge squared expression to be of the following form and do a little algebra:

\[ (a - b - c)^2 = (a - b)^2 + c^2 - 2c(a - b) \quad (44) \]

we can immediately take \((a - b)^2\) as the contribution from the cylinder, \(c^2\) as the contribution from the Klein bottle and the Möbius strip must be twice the square root of the Klein bottle times the square root of the cylinder contribution \((a - b)\). An examination of the amplitudes shows that this indeed holds generally. We can do this due to the Gimon–Polchinski conditions which mean that the Klein bottle and Möbius strip will not contribute in every twisted sector so that the cylinder expression must form a perfect square on its own. So whereas before the Gimon–Polchinski conditions seemed to imply more work they now give the remarkable result of simplifying the expressions for the purely Abelian case.

Apply this we get the set of charge contraints for arbitrary \(d\),

\[
\sum_{P(i)} \left( \text{Tr}(\gamma_{P(i)}, g)^d \prod_{i=1}^{d}(2\sin\pi z_i)^{-1} - \text{Tr}(\gamma_{P(i), 5i}) \prod_{i=1}^{d}(2\sin\pi z_i) - K \right)^2
\]

\[
\sum_{P(i)} \left( \text{Tr}(\gamma_{P(i)}, g)^d \prod_{i=1}^{d}(2\sin\pi z_i)^{-1} - \text{Tr}(\gamma_{P(i), 1}) \prod_{i=1}^{d}(2\sin\pi z_i) - K \right)^2
\]

(45) (46)
\[
\sum_{\mathbf{P}(i)}' \left( \text{Tr}(\gamma_{\mathbf{P}(i),5}) \prod_{i=1}^d \frac{(2\sin\pi z_i)}{(2\sin\pi z_i)} - \text{Tr}(\gamma_{\mathbf{P}(i),1}) \prod_{i=1}^d (2\sin\pi z_i) - K \right)^2
\]

(47)

\[
\sum_{\mathbf{P}(i)}' \left( \text{Tr}(\gamma_{\mathbf{P}(i),5}) \prod_{i=1}^d (2\sin\pi z_i) - \text{Tr}(\gamma_{\mathbf{P}(i),5}) \prod_{i=1}^d (2\sin\pi z_i) - K \right)^2
\]

(48)

where we have defined \(K\) as

\[
2^{5-d} \left\{ \prod_{i=1}^d \frac{2\cos\pi z_i}{2\sin\pi z_i} + \prod_{i=1}^d \frac{-2\sin\pi z_i}{2\sin\pi z_i} \right\}
\]

(49)

corresponding to the \(\Omega_k\) and \(\Omega_{k+N/2}\) Klein bottle amplitudes respectively. Each expression is individually set to zero. What we are looking for is that the total \(D\)-brane plus orientifold charge squared in each twisted sector is zero. Remarkably this can be solved as it stands if we make the substitutions

\[
\begin{align*}
2\cos\pi z_i &= e^{\pi iz_i} + e^{-\pi iz_i} \\
2i\sin\pi z_i &= e^{\pi iz_i} + e^{-\pi (z_i+1)}
\end{align*}
\]

(50)

First look at the \(D9\)-branes, pulling out the common factor of \((2\sin\pi z_i)^{-1}\). As a result the factor in front of the \(D5\)-branes and \(D1\)-branes is related to the Lefschetz fixed point formula for the appropriate element of \(\mathbf{P}(i)\). In the cases we will be interested in this is in fact an integer. We now let

\[
\text{Tr}(\gamma_{\mathbf{P}(i),9}) = \text{Tr}(\gamma_{\mathbf{P}(i),1})
\]

(51)

which implies that the trace of \(\gamma_{\mathbf{P}(i),9}\) is equal to the contribution from the Klein bottle up to an integer. It is worth looking at the Klein bottle contribution a little closer. For clarity let us change the notation so that we have instead \(\pi z_1 = a, \pi z_2 = b, \pi z_3 = c, \pi z_4 = d, \) and assume we have a hypothetical \(\pi z_5 = f\). Calculating \((2\cos\pi z_i)^d\) for \(d = 2, 3, 4\) we get the corresponding pattern:

\[
\begin{align*}
e^{i(\pm a \pm b)} &= 4 \text{ terms} \\
e^{i(\pm a \pm b \pm c)} &= 8 \text{ terms} \\
e^{i(\pm a \pm b \pm c \pm d)} &= 16 \text{ terms} \\
e^{i(\pm a \pm b \pm c \pm d \pm f)} &= 32 \text{ terms}
\end{align*}
\]

(52)
Similarly for \((2\sin \pi z_i)^d\).

Each of these terms can be interpreted as the diagonal components of the \(\gamma\) matrix associated to the \(D9\)-brane operating in the various sectors. They encode all the group properties. If we multiply the number of terms by the remaining coefficient of the Klein bottle contribution then we recover the number 32 in all cases as required to make up a trace of a matrix corresponding to 32 \(D\)-branes. If we did not have 32 terms then this would imply that we would have less than 32 \(D\)-branes which from the untwisted sector is not possible. This also essentially implies the uniqueness of the solution.

Thus we can write the solution for the \(D9\)-brane in an arbitrary sector as

\[
\gamma_{1,9} = \text{diag}\{e^{i\pi(\xi)}\} \otimes I_{2^k}
\]

where \((\xi)\) is all possible combinations of \(\pm z_i\) and then all possible combinations of \(\pm(z_i + 1)\) when they apply, i.e. even \(N\), with \(k\) being the appropriate value need to make up the number of terms to 32. What is more is that with this notation we do not need to worry about how the \(\gamma\)'s will look in a cross model when there are two or more point groups of the same type, as they will have different values for \(a, b, c, d\) which give rise to the same terms but in a different order. We simply are required to choose the initial order for the \(\pm\) signs. Notice that setting any one of \(a, b, c, d\) to zero is equivalent to switching off the compactification in the associated dimensions, or equivalently moving into the untwisted sector for that dimension.

As simple as making this choice appears it actually has quite far reaching consequences. The expansion shows that each value on the diagonal of the \(\gamma\) and hence every \(D\)-brane feels the action of each point group in the model. The value of the diagonal terms will also determine whether a particular \(D\)-brane is dynamical or not, and also its relation to the other \(D\)-branes. Once set we cannot arbitrarily change the order as that would amount to changing the action of the point group on the coordinates. This is the heart of the control the orientifold planes have on the \(D\)-branes. The charge cancellation equations also give us a way of solving the \(\gamma\) for the 1 and 5, sectors by using the solution for \(\gamma_9\) but these need to be considered for the individual models.

The purpose of including the hypothetical \(z_5\) which would be the contribution if we compactified the \(x^0, x^1\) coordinates reserved for the lightcone quantization is to show that to compactify further is not possible. This is as the number of terms would be greater than 32 so the equations are unsolvable,
besides violating what we know from the completely untwisted sector. What this tell us is that the scheme sees the total number of dimensions and hence the superconformal anomaly which fixes it as would be expected, but that we are unable to use this to place any constraints on how the compactification works.

Before moving onto specific examples, we should note something about the solution for $\gamma$ when $N$ is even. We can consider the $\sin \pi/N$ term as infact being $\cos \pi(N - 1)/N$. In fact we can use this as an alternative method for constructing the same solutions building on the fact that the trace of $\gamma_{1,9}$ is zero while $\gamma_{2,9}$ is not. However, more importantly, that we have to include both the $\Omega_k$ and $\Omega_{k+N/2}$ gives rise to the fact that the solution will always factorise to something with a $Z_2$ structure.

Since the cases for $d \leq 3$ have already been dealt with in the literature we complete the tale by constructing the $\gamma$’s for $d = 4$, that is compactification down to two dimensions. The work on the cases on containing $N_i = 2$, $i = 1, 2, 3$ has been done in [11] and as it’s solutions are somewhat different in structure than for $N_i \neq 2$ we will not cover them here.

### 4.1 The $T^8/Z_N$ Orientifold

Here we have the action of the point group acting on all the coordinates at once so we can write $z_i = z$ for all $i$, with $z = n/N$. In this case we have no $D5$-branes, only $D1$-branes and $D9$-branes. We need to take care as to whether $N$ is odd or even and in the case of $N$ even whether $n$ is odd or even. Putting all this into our general solution we get the following results for even $N$

$$Tr(\gamma_{2k-1,9}) - (2\sin\frac{(2k - 1)\pi}{N})^8 Tr(\gamma_{2n-1,1}) = 0 \quad (54)$$

for $n = 2k - 1$; and for $n = 2k$

$$Tr(\gamma_{2k,9}) - (2\sin\frac{2k\pi}{N})^8 Tr(\gamma_{2k,1}) = 2. \left(2\cos\frac{2k\pi}{N}\right)^4 + \left(2\sin\frac{2k\pi}{N}\right)^4 \quad (55)$$

While for odd $N$

$$Tr(\gamma_{n,9}) - (2\sin\frac{n\pi}{N})^8 Tr(\gamma_{n,1}) - 2. \left(2\cos\frac{n\pi}{N}\right)^4 = 0 \quad (56)$$
These can now be solved to find $\gamma$. First solve for the case when $N$ is odd. Since we have only one constraint per sector we set

$$Tr(\gamma_{n,9}) = Tr(\gamma_{n,1})$$

which gives us

$$\gamma_{n,9} = \text{diag}\{e^{4\pi iz}, e^{-4\pi iz}, e^{2\pi iz}(4), e^{-2\pi iz}(4), 1(6)\} \otimes I_2$$

The number in parenthesis is the multiplicity of that particular value, we are not going to overly concern ourselves with the order as we are not going to evaluate the Chan–Paton factors or Wilson lines specifically. For the case $N = 3$, we get:

$$\gamma_{1,9} = \text{diag}\{e^{\pi i/3}, e^{-\pi i/3}, e^{2\pi i/3}(4), e^{-2\pi i/3}(4), 1(6)\} \otimes I_2$$

When $N$ is even we have the solutions:

$$Tr(\gamma_{2k-1,9}) = Tr(\gamma_{2k-1,1}) = 0$$

which tells us that we can set $\gamma_{n,9} = e^{i\pi m/N}$ $\gamma_{n,1}$ where $m$ is an odd integer, while the solution for $n = 2k$ gives the result for the diagonal of $\gamma_{1,9}$

$$\{e^{2\pi i/N}(2), e^{-2\pi i/N}(2), e^{\pi i/N}(4), e^{-\pi i/N}(4), -e^{\pi i/N}(4), -e^{-\pi i/N}(4), 1(12)\}$$

which for $N = 2M$ has the factorisation $Z_N = Z_M \otimes Z_2$ as expected for $Z_4$ and $Z_6$.

### 4.2 The $T^8/(Z_N \times Z_M)$ Orientifold

We define the action of the point group in this case as being

$$z : (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (x^{2,3}, x^{4,5}, e^{-2\pi iz} x^{6,7}, e^{2\pi iz} x^{8,9})$$ $$y : (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (e^{-2\pi iy} x^{2,3}, e^{2\pi iy} x^{4,5}, x^{6,7}, x^{8,9})$$

where $z = n/N$ and $y = m/M$ labels the action of $Z_N$ and $Z_M$, respectively.

We can see that this divides the compact space up into $T^4/Z_N \otimes T^4/Z_M$ but now due to the existence of two $D5$-branes, one for each invariant space left under the action of $Z_N$ or $Z_M$, it is not possible to treat them completely independently except perhaps for those point groups without an element of
order two. However, we can use this fact to see that the tadpole divergences are evenly split between the two sectors with \( z_1 = z_2 = z \), \( z_3 = z_4 = y \). We should also consider the cases when \( n = 0, m \neq 0 \) and \( m = 0, n \neq 0 \) respectively though following the prescription given earlier they can be obtained simply enough from the following equations; \( n = 0, m = 0 \) is just the untwisted sector. Define \( 5_1 \) to be the \( D_5 \)-brane left invariant under the action of \( z \) and \( 5_2 \) to be the \( D_5 \)-brane left invariant under the action of \( y \).

The tadpole contributions are for \( n \neq 0, m \neq 0 \):

(i) from the Klein bottle

\[
4 \left[ \frac{(2\cos \pi z)^2(2\cos \pi y)^2}{(2\sin \pi z)^2(2\sin \pi y)^2} \right]^2 \tag{63}
\]

and a similar contribution for the \( \Omega_{(k+N/2)} \) diagram where as before the relevant topline \( (2\cos \pi z_i) \) goes to \((-2\sin \pi z_i)\).

(ii) from the Möbius strip, \(-4\times\)

\[
Tr(\gamma^{-1}_{\Omega,z,y,9} \gamma^{T}_{\Omega,z,y,9}) \frac{1}{(2\sin \pi z)^2(2\sin \pi y)^2} + Tr(\gamma^{-1}_{\Omega,z,y,5_1} \gamma^{T}_{\Omega,z,y,5_1})(2\cos \pi z)^2(2\cos \pi y)^2 + Tr(\gamma^{-1}_{\Omega,z,y,5_2} \gamma^{T}_{\Omega,z,y,5_2})(2\cos \pi z)^2(2\cos \pi y)^2 + Tr(\gamma^{-1}_{\Omega,z,y,1} \gamma^{T}_{\Omega,z,y,1})(2\cos \pi z)^2(2\cos \pi y)^2 \tag{64}
\]

(iii) from the cylinder

\[
\sum_{n,m}^{N-1} \sum_{r,s}^{M-1} \left\{ \left( Tr(\gamma_{z,y,9})(2\sin \pi z)^{-2}(2\sin \pi y)^{-2} - Tr(\gamma^{(G)}_{z,y,1}) \right)^2 
+ \left( Tr(\gamma_{z,y,9})(2\sin \pi z)^{-2}(2\sin \pi y)^{-2} - Tr(\gamma^{(G)}_{z,y,5_1})(2\sin \pi z)^2 \right)^2 
+ \left( Tr(\gamma_{z,y,9})(2\sin \pi z)^{-2}(2\sin \pi y)^{-2} - Tr(\gamma^{(G)}_{z,y,5_2})(2\sin \pi y)^2 \right)^2 
+ \left( Tr(\gamma_{z,y,9})(2\sin \pi z)^2 - Tr(\gamma^{(G)}_{z,y,5_1}) \right)^2 
+ \left( Tr(\gamma_{z,y,9})(2\sin \pi y)^2 - Tr(\gamma^{(G)}_{z,y,5_2}) \right)^2 
+ \left( Tr(\gamma_{z,y,9})(2\sin \pi y)^2 - Tr(\gamma^{(G)}_{z,y,1}) \right)^2 \right\} \tag{65}
\]
where the \((G)\) is the sum over fixed points. This can be solved as it stands using the method outlined already, or we can switch a \(z\) or \(y\) off and see that we simply have the solutions for \(T^4/Z_N\) which are already presented in [6] which can be multiplied together as desired to obtain the relative sectors. All we have to be aware of is the phases between the 5_i and 1 branes which will be the same as given in [11].

4.3 The \(T^8/(Z_N \times Z_M \times Z_P)\) Orientifold

We define the action of the point group as being

\[
\begin{align*}
  z &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (x^{2,3}, x^{4,5}, e^{2\pi i z} x^{6,7}, e^{-2\pi i z} x^{8,9}) \\
  y &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (e^{2\pi i y} x^{2,3}, e^{-2\pi i y} x^{4,5}, x^{6,7}, x^{8,9}) \\
  x &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (e^{-2\pi i x} x^{2,3}, x^{4,5}, e^{2\pi i x} x^{6,7}, x^{8,9})
\end{align*}
\]

with \(z = n/N, y = m/M\) and \(x = p/P\) labels the action of \(Z_N\), \(Z_M\) and \(Z_P\) respectively. This case will have all brane types, but requires that

\[
\frac{N}{M} \in \mathbb{Z}, \quad \frac{M}{P} \in \mathbb{Z}
\]

so restricting the number of possible combinations of point groups somewhat, eg. we can have 2,2,2 or 3,3,6 but not 2,3,6 etc. for \(N, M, P\) respectively. As in the previous case most of the work has already been done for us in [10] with the phases again from [11]. All we have to do is switch off all point groups except for one, which effectively leaves us with a \(T^4/Z_N\) to solve again. What makes this one different is the ordering of terms in the solutions for \(\gamma\) but this is done by comparison with the general solution presented earlier. As will be discussed in the next section models containing one or more \(Z_4\)’s will be inconsistent.

Other types of \(d = 4\) models can be constructed by playing around with the basic constructions presented here, such as a \(T^8/Z_{N_i}\), where

\[
\begin{align*}
  z &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (x^{2,3}, x^{4,5}, e^{2\pi i z} x^{6,7}, e^{-2\pi i z} x^{8,9}) \\
  y &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (e^{2\pi i y} x^{2,3}, e^{-2\pi i y} x^{4,5}, x^{6,7}, x^{8,9}) \\
  x &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (e^{-2\pi i x} x^{2,3}, x^{4,5}, e^{2\pi i x} x^{6,7}, x^{8,9}) \\
  w &: (x^{2,3}, x^{4,5}, x^{6,7}, x^{8,9}) \rightarrow (x^{2,3}, x^{4,5}, e^{-2\pi i w} x^{6,7}, e^{-2\pi i w} x^{8,9})
\end{align*}
\]
which works for $Z_N$, $Z_M$, $Z_P$ and $Z_Q$ and

$$\frac{N}{M} \in \mathbb{Z}, \frac{M}{P} \in \mathbb{Z}, \frac{P}{Q} \in \mathbb{Z}$$  \hspace{1cm} (69)

and so on. However, the solutions will be simply those already known with reordering of the terms as explained earlier.

5 Zwart Inconsistencies

In [10] Zwart noticed that the $T^6/(Z_2 \times Z_4)$ and $T^6/(Z_4 \times Z_4)$ orientifold models are inconsistent as the relevant $\gamma$ matrices do not satisfy the appropriate group properties required. We are now in a position to give an explanation of why this anomalous result occurs and how it might occur in other models. It can also be seen that part of the problem lies in the $Z_4$ orbifold model. However, the pure $T^6/Z_4$ model is consistent so another part of the problem lies in the fact that it is a cross model. From Berkooz and Leigh [3], we know that the representation for $Z_2 \times Z_2$ is unique.

The key to this lies one of the results in the previous section, which was that we can decompose the $\gamma$ matrices corresponding to $Z_N$ with $N = 2M$ in to the representations for $Z_M \otimes Z_2$. Going back to the set of automorphisms for the two dimensional (complex) lattice from which we get the available Abelian point groups we notice that the maximum point group for the $SU(2) \times SU(2)$ and $SU(3)$ lattices are both even. We can immediately see that the $Z_2$ in our decompostion corresponds to the general result for Lie algebra roots that every root has a negative value that is also a root and the $Z_2$ is simply the transformation between them:

$$Z_2 : \alpha \rightarrow -\alpha$$  \hspace{1cm} (70)

This is a feature of all even point groups and thus we can take it as given. The problem must therefore lie in the remaining part of the point group.

Let us stay with the $T^6$ models of Zwart. We see that models with $Z_6$ work fine. $Z_6$ splits up into $Z_3 \times Z_2$. Divide the root diagram into the six component triangles, the Weyl chambers, delineated by the various root vectors. The lattice we start with are the first two triangles between the two simple roots. The action of $Z_3$ is to rotate the root vectors to the
third and fourth triangles and then the fifth and sixth triangles in turn. In each case we have the same lattice structure so thus the same physics, but the lattice has a different orientation relative to the original lattice. This difference in orientation means we can not identify the lattices under the lattice property. Lattice rotations under the $Z_2$ rotation of the simple roots is trivially identified with the original lattice.

Doing the same for $Z_4$ which in terms of the $\gamma$’s has a $Z_2 \times Z_2$ decomposition. As before one $Z_2$ corresponds to the interchange of the positive roots with their negative counterparts and the new lattice formed is trivially identified with the original one. The second $Z_2$, call it $Z'_2$, corresponds to a reflection in the axes, but as this is now the $SU(2) \times SU(2)$ root system these are orthogonal so they divide the space up into four identical squares that under the lattice identification can all be mapped onto each other. Also the model must contain the $Z_2 \times Z_2$ solution in it. Acting with the $\gamma$ matrix corresponding to $Z'_2$ on the Berkooz–Leigh solution swaps the components of the Berkooz–Leigh solution around. However, we must be able to identify the $\gamma$ matrices of the Berkooz–Leigh solution with the new matrices due to the fact that they are identified under the lattice. An examination of the solutions quickly shows that it is not possible without making the identification $\epsilon = -\tilde{\epsilon}$ for at least one of the sectors.

Thus the origin of the inconsistancy can be taken back directly to the uniqueness of the Berkooz–Leigh solution for the $Z_2 \times Z_2$ model. That this solution is unique stems from the fact that the $Z_2$’s involved correspond to the interchanging of the positive and negative simple roots in each $T^2$ the complex coordinate system divides up the compact space into. This with the fact it is a cross model is what makes the solution unique so the same analysis is not directly applicable to the pure $Z_2$ models.

We must also note that this does not appear to be a feature of simple orbifold models but arise out of the fact we are taking T–dual models and enhancing the point group to contain a worldsheet parity, which we can assosiate the $Z_2$ with \[7\] \[8\].

Any orientifold model whose component lattices shows similiar behaviour to the $Z_4$ cross models will suffer the same fate of having a inconsistent representation of the action for the point group on the Chan Paton factors. Thus we can also expect similar behaviour to be observed in the solutions of orientifold compactification on $T^8$. 

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6 Discussion

In this paper we have examined the general orientifold model on the $SU(N)$ subgroups of the $SO(N)$ space groups in compactifying the type IIB superstring down to 6, 4 and 2 dimensions. What we have discovered is that in all these models there must always be 32 $D$–branes in them. We can also see that the dynamics of the $D$–branes is much more strongly controlled by the Klein bottle contributions to the amplitudes, and thus by the orientifold planes, than was suspected. Rather than being present to ensure proper charge cancellation, they impose stringent conditions on the form of the $\gamma$’s representing how the point group act on the $D$–branes.

It is possible to generalise these models even further. So far all work in the area has been limited to the $SU$ group lattices and Abelian point groups. This has been because they have supersymmetry preserving properties and can be related to special points in Calabi–Yau moduli spaces. However, it is possible to consider more general models that have features such as discrete torsion, asymmetry or non–Abelian point groups. While the former two can be considered for the models examined in this paper the non–Abelian point groups require a fresh approach. The \( \vartheta \) notation used in this paper simplified the amplitudes considerable but they are only suitable for a fermion on a complex manifold. For non–Abelian group we would want to allow for real manifold so have to retreat to a more basic form utilising the $f$ functions modified to take a $exp(2\pi iz)$ term \[15\].

Another approach that needs to be explored more fully is the use of boundary operators, which have been used to describe $D$–branes. Analogous to them are crosscaps \[16, 17\] which in the T–dual model may give a description of the orientifold planes. A test for them would be that they should be able to reproduce all the amplitudes presented here. This would be useful as they are easier to construct for non–standard point groups.

After this paper was finished refs \[18, 19\] were brought to our attention. In particular the treatment of the tadpoles and the construction of the $\gamma$’s \[18\] overlaps somewhat with ours, and corroborates many of the results which we have obtained. Indeed they have considered Abelian groups not explicitly presented in this paper, though it is obvious they can be readily accounted for by our results.

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