A DISC MAXIMIZES LAPLACE EIGENVALUES AMONG ISOPERIMETRIC SURFACES OF REVOLUTION

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Abstract. The Dirichlet eigenvalues of the Laplace-Beltrami operator are larger on a flat disc than on any other surface of revolution immersed in Euclidean space with the same boundary.

1. Introduction

Let \( \Sigma \) be a compact connected immersed surface of revolution in \( \mathbb{R}^3 \) with one smooth boundary component. The Euclidean metric on \( \mathbb{R}^3 \) induces a Riemannian metric on \( \Sigma \). Let \( \Delta_\Sigma \) be the corresponding Laplace-Beltrami operator on \( \Sigma \). Denote the Dirichlet eigenvalues of \( -\Delta_\Sigma \) by

\[
0 < \lambda_1(\Sigma) < \lambda_2(\Sigma) \leq \lambda_3(\Sigma) \leq \ldots
\]

Let \( R \) be the radius of the boundary of \( \Sigma \), and let \( D \) be a disc in \( \mathbb{R}^2 \) of radius \( R \). Let \( \Delta \) be the Laplace operator on \( \mathbb{R}^2 \), and denote the Dirichlet eigenvalues of \( -\Delta \) on \( D \) by

\[
0 < \lambda_1(D) < \lambda_2(D) \leq \lambda_3(D) \leq \ldots
\]

**Theorem.** If \( \Sigma \) is not equal to \( D \), then for \( j = 1, 2, 3, \ldots \),

\[
\lambda_j(\Sigma) < \lambda_j(D)
\]

We remark that there are compact connected surfaces, which are not surfaces of revolution, embedded in \( \mathbb{R}^3 \) whose boundary is a circle of radius \( R \) and have first Dirichlet eigenvalue larger than \( \lambda_1(D) \). This can be proven with Berger’s variational formulas [Be].

This problem resonates with the Rayleigh-Faber-Krahn inequality, which states that the flat disc has smaller first Dirichlet eigenvalue than any other domain in \( \mathbb{R}^2 \) with the same area [E] [K]. Hersch proved that the canonical metric on \( S^2 \) maximizes the first non-zero eigenvalue among metrics with the same area [H]. Li and Yau showed the canonical metric on \( \mathbb{R}P^2 \) maximizes the first non-zero eigenvalue among metrics with the same area [LY]. Nadirashvili proved the same is true for the flat equilateral torus, whose fundamental parallelogram is comprised of two equilateral triangles [NI]. It is not known if there is such a maximal metric on the Klein bottle, but Jakobson, Nadirashvili, and Polterovich showed there is a critical metric
El Soufi, Giacomini, and Jazar proved this is the only critical metric on the Klein bottle [EGJ].

As for the second eigenvalue, the Krahn-Szegő inequality states that the union of two discs with the same radius has smaller second Dirichlet eigenvalue than any other domain in $\mathbb{R}^2$ with the same area [K]. Nadirashvili proved that the union of two round spheres of the same radius has larger second non-zero eigenvalue than any metric on $S^2$ with the same area [N2]. It is conjectured that a disc has smaller third Dirichlet eigenvalue than any other planar domain with the same area. Bucur and Henrot established the existence of a quasi-open set in $\mathbb{R}^2$ which minimizes for the third eigenvalue among sets of prescribed Lebesgue measure [BH]. This was extended to higher eigenvalues by Bucur [Bu].

On a compact orientable surface, Yang and Yau obtained upper bounds, depending on the genus, for the first non-zero eigenvalue among metrics of the same area [YY]. Li and Yau extended these bounds to compact non-orientable surfaces [LY]. However, Urakawa showed that there are metrics on $S^3$ with volume one and arbitrarily large first non-zero eigenvalue [U]. Colbois and Dodziuk extended this to any manifold of dimension three or higher [CD].

For a closed compact hypersurface in $\mathbb{R}^{n+1}$, Chavel and Reilly obtained upper bounds for the first non-zero eigenvalue in terms of the surface area and the volume of the enclosed domain [C, R]. This was extended to higher eigenvalues by Colbois, El Soufi, and Girouard [CEG]. Abreu and Freitas proved that for a metric on $S^2$ which can be isometrically embedded in $\mathbb{R}^3$ as a surface of revolution, the first $S^1$-invariant eigenvalue is less than the first Dirichlet eigenvalue on a flat disc with half the area [AF]. Colbois, Dryden, and El Soufi extended this to $O(n)$-invariant metrics on $S^n$ which can be isometrically embedded in $\mathbb{R}^{n+1}$ as hypersurfaces of revolution [CDE].

We conclude this section by reformulating the theorem. Fix a plane in $\mathbb{R}^3$ containing the axis of symmetry of $\Sigma$. Identify $\mathbb{R}^2$ with this plane isometrically in such a way that the axis of symmetry is identified with

$$\{(x, y) \in \mathbb{R}^2 : x = 0\}$$

Define

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$$

We may assume $\partial \Sigma$ intersects $\mathbb{R}^2_+$ at the point $(R, 0)$. Let $L$ be the length of the meridian $\Sigma \cap \mathbb{R}^2_+$. Let $\alpha : [0, L] \to \mathbb{R}^2_+$ be a regular, arc-length parametrization of $\Sigma \cap \mathbb{R}^2_+$ with $\alpha(0) = (R, 0)$. Write $\alpha = (F_\alpha, G_\alpha)$. Note that $F_\alpha(L) = 0$ and $F_\alpha$ is positive over $[0, L)$.

Let $C^0_0(0, L)$ be the set of functions $w : [0, L] \to \mathbb{R}$ which are continuously differentiable and vanish at zero. For a non-negative integer $k$ and a positive
integer $n$, define
\[
\lambda_{k,n}(\alpha) = \min_{W} \max_{w \in W} \frac{\int_0^L |w'|^2 F_\alpha + \frac{k^2 w^2}{F_\alpha} \, dt}{\int_0^L w^2 F_\alpha \, dt}
\]
Here the minimum is taken over all $n$-dimensional subspaces $W$ of $C^1_0(0, L)$. We remark that
\[
\left\{ \lambda_j(\Sigma) \right\} = \left\{ \lambda_{k,n}(\alpha) \right\}
\]
Moreover, if we count $\lambda_{k,n}(\alpha)$ twice for $k \neq 0$, then the values occur with the same multiplicity. Define $\omega : [0, R] \to \mathbb{R}^2_+$ by
\[
\omega(t) = (R - t, 0)
\]
Define $\lambda_{k,n}(\omega)$ similarly to $\lambda_{k,n}(\alpha)$. Then
\[
\left\{ \lambda_j(D) \right\} = \left\{ \lambda_{k,n}(\omega) \right\}
\]
Again, if we count $\lambda_{k,n}(\omega)$ twice for $k \neq 0$, then the values occur with the same multiplicity. Now to prove the theorem, it suffices to prove the following lemma.

**Lemma 1.** If $\alpha$ does not equal $\omega$, then for any non-negative integer $k$ and any positive integer $n$,
\[
\lambda_{k,n}(\alpha) < \lambda_{k,n}(\omega)
\]
To prove this, we define a neighborhood of the boundary $\partial \mathbb{R}^2_+$ and treat the segments of the curve outside and inside of this neighborhood separately. For the exterior segment, we simply project $\alpha$ orthogonally onto $\omega$ and observe that this increases the eigenvalue. For the interior segment, we unroll the curve to $\omega$ and see that this increases the eigenvalue as well.

2. **Proof**

We first extend the definition of the functionals $\lambda_{k,n}$ to Lipschitz curves. Let $[a, b]$ be a finite, closed interval and let $\psi : [a, b] \to \mathbb{R}^2_+$ be a Lipschitz curve. Write $\psi = (F_\psi, G_\psi)$. Assume that $F_\psi$ is positive over $[a, b]$. Let $\text{Lip}_0(a, b)$ be the set of continuous functions $w : [a, b] \to \mathbb{R}$ which vanish at $a$ and are Lipschitz over $[a, c]$ for every $c$ in $(a, b)$. For a non-negative integer $k$ and a positive integer $n$, define
\[
\lambda_{k,n}(\psi) = \inf_{W} \max_{w \in W} \frac{\int_a^b \frac{|w'|^2 F_\psi}{|\psi'|} + \frac{k^2 w^2 |\psi'|}{F_\psi} \, dt}{\int_a^b w^2 F_\psi |\psi'| \, dt}
\]
Here the infimum is taken over all \( n \)-dimensional subspaces \( W \) of \( \text{Lip}_0(a,b) \). Let \( H^1_0(\psi, k) \) be the set of continuous functions \( w : [a,b) \to \mathbb{R} \) which vanish at \( a \) and have a weak derivative such that
\[
\int_a^b \frac{|w'|^2 F_\psi}{|\psi'|} + \frac{k^2 w^2 |\psi'|}{F_\psi} \, dt < \infty
\]

In the following lemma, we note that if \( \psi \) is a regular piecewise continuously differentiable curve which meets the axis transversally, then the infimum in the definition of the functionals \( \lambda_{k,n} \) is attained.

**Lemma 2.** Let \( \psi : [a,b] \to \mathbb{R}^2_+ \) be a piecewise continuously differentiable curve. Assume there is a positive constant \( c \) such that for all \( t \) in \( [a,b] \),
\[
|\psi'(t)| \geq c
\]
Write \( \psi = (F_\psi, G_\psi) \). Assume that \( F_\psi \) is positive over \([a,b)\). Assume that \( F_\psi(b) = 0 \) and \( F_\psi'(b) < 0 \). Let \( k \) be a non-negative integer. Then there are functions
\[
\varphi_{k,1}, \varphi_{k,2}, \varphi_{k,3}, \ldots
\]
which form an orthonormal basis of \( H^1_0(\psi, k) \) such that, for any positive integer \( n \),
\[
\lambda_{k,n}(\psi) = \int_a^b \frac{\varphi_{k,n}'^2 F_\psi}{|\psi'|} + \frac{k^2 \varphi_{k,n}^2 |\psi'|}{F_\psi} \, dt
\]
Each function \( \varphi_{k,n} \) has exactly \( n-1 \) roots in \((a,b)\) and satisfies the following equation weakly:
\[
\left( \frac{F_\psi \varphi_{k,n}'}{|\psi'|} \right)' = k^2 |\psi'| \varphi_{k,n} \left( \frac{F_\psi}{F_\psi} \right) - \lambda_{k,n}(\psi) F_\psi |\psi'| \varphi_{k,n}
\]
Also,
\[
\lambda_{k,1}(\psi) < \lambda_{k,2}(\psi) < \lambda_{k,3}(\psi) < \ldots
\]
We omit the proof which is standard and refer to Gilbarg and Trudinger [GT] and Zettl [Z].

Now fix a non-negative integer \( K \) and a positive integer \( N \), for the remainder of the article. Let
\[
\mu = \frac{K}{\sqrt{\lambda_{K,N}(\omega)}}
\]
The inequality \( \mu < R \) is a basic fact about Bessel functions [W]. Let \( \alpha \) be as defined in the introduction, and let
\[
A = \min \left\{ t \in [0, L] : F_\alpha(t) = \mu \right\}
\]
Define $\beta : [0, L] \to \mathbb{R}^2_+$ to be a piecewise continuously differentiable function such that $\beta(0) = (R, 0)$ and

$$\beta'(t) = \begin{cases} (F'_\alpha(t), 0) & t \in [0, A) \\ (F'_\alpha(t), G'_\alpha(t)) & t \in (A, L] \end{cases}$$

**Lemma 3.** Assume $\alpha$ is not equal to $\beta$ and $\lambda_{K,N}(\alpha) \geq \lambda_{K,N}(\omega)$. Then

$$\lambda_{K,N}(\alpha) < \lambda_{K,N}(\beta)$$

**Proof.** Fix a number $p$ in $(0, 1)$. Define $\alpha_p : [0, L] \to \mathbb{R}^2_+$ to be a regular piecewise continuously differentiable curve such that $\alpha_p(0) = (R, 0)$ and

$$\alpha'_p(t) = \begin{cases} (F'_\alpha(t), pG'_\alpha(t)) & t \in [0, A) \\ (F'_\alpha(t), G'_\alpha(t)) & t \in (A, L] \end{cases}$$

We first show that

$$\lambda_{K,N}(\alpha) < \lambda_{K,N}(\alpha_p)$$

By Lemma 2 there is a $N$-dimensional subspace $\Phi$ of $H_0^1(\alpha_p, K)$ such that

$$\lambda_{K,N}(\alpha_p) = \max_{w \in \Phi} \frac{\int_0^L \frac{w'^2F_\alpha}{|\alpha'|} + \frac{K^2w^2|\alpha'|}{F_\alpha} dt}{\int_0^L w^2F_\alpha |\alpha'| dt}$$

Moreover $\Phi$ is contained in $\text{Lip}_0(0, L)$ and the maximum over $\Phi$ is only attained by scalar multiples of a function $\varphi_{K,N}$ which has exactly $N - 1$ roots in $(0, L)$. Let $v$ be a function in $\Phi$ such that

$$\frac{\int_0^L \frac{v'^2F_\alpha}{|\alpha'|} + \frac{K^2v^2|\alpha'|}{F_\alpha} dt}{\int_0^L v^2F_\alpha |\alpha'| dt} = \max_{w \in \Phi} \frac{\int_0^L \frac{w'^2F_\alpha}{|\alpha'|} + \frac{K^2w^2|\alpha'|}{F_\alpha} dt}{\int_0^L w^2F_\alpha |\alpha'| dt}$$

Note this quantity is at least $\lambda_{K,N}(\alpha)$, which is at least $\lambda_{K,N}(\omega)$. It follows that

$$\frac{\int_0^L \frac{|v'|^2F_\alpha}{|\alpha'|} + \frac{K^2v^2|\alpha'|}{F_\alpha} dt}{\int_0^L v^2F_\alpha |\alpha'| dt} \leq \frac{\int_0^L \frac{|v'|^2F_\alpha}{|\alpha'|} + \frac{K^2v^2|\alpha'|}{F_\alpha} dt}{\int_0^L v^2F_\alpha |\alpha'| dt}$$

If equality holds, then $v$ must vanish on a set of positive measure. In either case, we obtain

$$\lambda_{K,N}(\alpha) \leq \frac{\int_0^L \frac{|v'|^2F_\alpha}{|\alpha'|} + \frac{K^2v^2|\alpha'|}{F_\alpha} dt}{\int_0^L v^2F_\alpha |\alpha'| dt} < \lambda_{K,N}(\alpha_p)$$

Now we repeat the argument to obtain

$$\lambda_{K,N}(\alpha) \leq \lambda_{K,N}(\beta)$$
Let $\varepsilon > 0$. There is an $N$-dimensional subspace $W$ of $\text{Lip}_0(0, L)$ such that

$$\max_{w \in W} \frac{\int_0^L |w|^2 F_\alpha}{|\beta'|} + \frac{K^2 w^2 |\beta'|}{F_\alpha} dt < \lambda_{K,N}(\beta) + \varepsilon$$

Let $u$ be a function in $W$ such that

$$\frac{\int_0^L |u|^2 F_\alpha}{|\alpha'|} + \frac{K^2 u^2 |\alpha'|}{F_\alpha} dt = \max_{w \in W} \frac{\int_0^L |w|^2 F_\alpha}{|\alpha'|} + \frac{K^2 w^2 |\alpha'|}{F_\alpha} dt$$

Note this quantity is at least $\lambda_{K,N}(\alpha_p)$, which is at least $\lambda_{K,N}(\omega)$. It follows that

$$\frac{\int_0^L |u|^2 F_\alpha}{|\alpha'|} + \frac{K^2 u^2 |\alpha'|}{F_\alpha} dt \leq \frac{\int_0^L |\alpha|^2 F_\alpha}{|\beta'|} + \frac{K^2 |\beta'|}{F_\alpha} dt$$

Now we obtain

$$\lambda_{K,N}(\alpha_p) \leq \lambda_{K,N}(\beta) + \varepsilon$$

Therefore,

$$\lambda_{K,N}(\alpha) < \lambda_{K,N}(\beta)$$

$\square$

Write $\beta = (F_\beta, G_\beta)$. Define $F_\gamma : [0, L] \to \mathbb{R}$ by

$$F_\gamma(t) = \begin{cases} 
\min \{F_\beta(s) : s \in [0, t]\} & t \in [0, A] \\
F_\beta & t \in [A, L]
\end{cases}$$

Let $G_\gamma = G_\beta$. Let $\gamma = (F_\gamma, G_\gamma)$. Note that $\gamma : [0, L] \to \mathbb{R}_+^2$ is Lipschitz.

**Lemma 4.** Assume $\lambda_{K,N}(\beta) \geq \lambda_{K,N}(\omega)$. Then

$$\lambda_{K,N}(\beta) \leq \lambda_{K,N}(\gamma)$$

**Proof.** Define

$$V = \left\{ t \in [0, A] : F_\beta(t) \neq F_\gamma(t) \right\}$$

By the Riesz sunrise lemma, there are disjoint open intervals $(a_i, b_i)$ such that

$$V = \bigcup_i (a_i, b_i)$$

and $F_\gamma$ is constant over each interval. Suppose $\lambda_{K,N}(\beta) > \lambda_{K,N}(\gamma)$. Then there is a $N$-dimensional subspace $W$ of $\text{Lip}_0(0, L)$ such that

$$\max_{w \in W} \frac{\int_0^L |w|^2 F_\alpha}{|\gamma'|} + \frac{K^2 w^2 |\gamma'|}{F_\gamma} dt < \lambda_{K,N}(\beta)$$

Note that over each interval $(a_i, b_i)$, the function $|\gamma'|$ is zero, so each $w$ in $W$ is constant. Let $J = [0, L] \setminus V$. The isolated points of $J$ are countable,
Lemma 5. This reparametrization satisfies

\[ \lambda_{K,N}(\gamma) \leq \lambda_{K,N}(\zeta) \]

Proof. Write \( \gamma = (F_\gamma, G_\gamma) \) and \( \zeta = (F_\zeta, G_\zeta) \). Let \( w \) be a function in \( \text{Lip}_0(0, L^*) \) such that

\[ \frac{\int_0^{L^*} |w|^2 F_\xi}{\int_0^{L^*} |\zeta'|^2 F_\zeta} < \infty \]

Define \( v = w \circ \ell \). Then \( v \) is in \( \text{Lip}_0(0, L) \), and changing variables yields

\[ \frac{\int_0^{L^*} |w|^2 F_\xi}{\int_0^{L^*} |\zeta'|^2 F_\zeta} = \frac{\int_0^L |w|^2 F_\gamma}{\int_0^L |\gamma'|^2 F_\zeta} \]

It follows that \( \lambda_{K,N}(\gamma) \leq \lambda_{K,N}(\zeta) \).

We can now prove Lemma 1 for the case \( K = 0 \).
Proof of Lemma 1 for the case $K = 0$. Suppose $\alpha$ is not equal to $\omega$ and
$$\lambda_{K,N}(\alpha) \geq \lambda_{K,N}(\omega).$$
Then $\alpha$ is not equal to $\beta$, so by Lemmas 3, 4, and 5
$$\lambda_{K,N}(\alpha) < \lambda_{K,N}(\beta) \leq \lambda_{K,N}(\gamma) \leq \lambda_{K,N}(\zeta).$$
But in this case, $\zeta = \omega$, so the proof is complete. □

For the remainder of the article, we assume that $K$ is positive. Write $\zeta = (F_\zeta, G_\zeta)$. Let $\chi : [0, L^*] \to \mathbb{R}^2_+$ be a piecewise continuously differentiable function such that $\chi(0) = (R, 0)$ and for $t$ in $[0, L^*]$ with $t \neq P$,
$$\chi'(t) = \left( F'_\zeta(t), |G'_\zeta(t)| \right)$$
Then $\lambda_{K,N}(\zeta) = \lambda_{K,N}(\chi)$, trivially. Write $\chi = (F_\chi, G_\chi)$. Note that, for $t$ in $[0, P]$,
$$\chi(t) = R - t$$
Also, for every $t$ in $[0, L^*]$ with $t \neq P$,
$$|\chi'| = 1$$

Let $\Phi_{K,1}, \Phi_{K,2}, \ldots$ be the functions given by Lemma 2 associated to $\omega$. Let $z_0$ be the largest root of $\Phi_{K,N}$ in $(0, R)$. It follows from basic facts about Bessel functions that $z_0 < P$ and that $\Phi_{K,N}$ has no critical points in $[P, R)$. There is a unique number $\Lambda$ such that there exists a function $u : [z_0, P] \to \mathbb{R}$ which is non-vanishing over $(z_0, P)$ and satisfies
$$\begin{cases}
(\omega u')' + (\Lambda \omega - \frac{K^2}{\omega}) u = 0 \\
u(z_0) = 0 \\
u'(P) = 0
\end{cases}$$
Moreover,
$$\Lambda < \lambda_{K,N}(\omega)$$
To compare $\lambda_{K,N}(\chi)$ and $\lambda_{K,N}(\omega)$, we need the following lemma.

**Lemma 6.** Let $Q$ and $z$ be real numbers with $z < z_0$ and $Q > P$. Let $\psi : [z, Q] \to \mathbb{R}^2_+$ be continuously differentiable over $[P, Q]$. Assume that, for $t$ in $[z, P]$,
$$\psi(t) = (R - t, 0)$$
Write $\psi = (F_\psi, G_\psi)$. Assume that $F_\psi(Q) = 0$ and $F_\psi$ is positive over $[z, Q)$. Assume that $|\psi'| = 1$ over $(P, Q)$ and that $F'_\psi(Q) < 0$. Let $\varphi$ be a function in $\text{Lip}_0(z, Q)$ such that
$$\lambda_{K,1}(\psi) = \frac{\int_z^Q |\varphi'|^2 F_\psi + \frac{K^2}{F_\zeta} \varphi^2 dt}{\int_z^Q \varphi^2 F_\psi dt}$$
Assume that $\lambda_{K,1}(\psi) > \Lambda$. Then

$$\lim_{t \to Q} \varphi(t) = 0$$

Also $\varphi$ is differentiable over $[z, Q]$, and over $[P, Q]$,

$$|\varphi'|^2 - \frac{K^2 \varphi^2}{|F_\psi|^2} \leq 0$$

Furthermore $\varphi'$ and $\frac{\varphi}{F_\psi}$ are bounded over $[z, Q]$.

**Proof.** Since $|\varphi'|^2 F_\psi$ and $\varphi^2 / F_\psi$ are integrable, the function $\varphi^2$ is absolutely continuous. Moreover $\varphi^2 / F_\psi$ is integrable, but $1 / F_\psi$ is not integrable over $(c, Q)$ for any $c$ in $(z, Q)$. It follows that

$$\lim_{t \to Q} \varphi(t) = 0$$

By Lemma 2, the function $\varphi$ is continuously differentiable over $[z, Q]$, and twice continuously differentiable over $[z, P]$ and $(P, Q)$, with

$$(F_\psi \varphi')' = \frac{K^2 \varphi}{F_\psi} - \lambda_{K,N}(\psi) F_\psi \varphi$$

It is also non-vanishing over $(z, Q)$. We may assume that $\varphi$ is positive over $(z, Q)$. Furthermore, the Picone identity (see, e.g. Zettl [Z]) implies that

$$\varphi'(P) < 0$$

The function

$$F_\psi^2 |\varphi'|^2 - K^2 \varphi^2$$

is differentiable over $(P, Q)$, and its derivative is

$$-2 \lambda_{K,N}(\psi) F_\psi^2 \varphi \varphi'$$

Therefore, we can prove the inequality by showing that

$$\lim_{t \to Q} F_\psi^2 |\varphi'|^2 = 0$$

Note that

$$((F_\psi^2 |\varphi'|^2)' = 2K^2 \varphi \varphi' - 2 \lambda_{K,N}(\psi) F_\psi^2 \varphi \varphi'$$

Since $|\varphi'|^2 F_\psi$ and $\varphi^2 / F_\psi$ are integrable, it follows that $F_\psi^2 |\varphi'|^2$ is absolutely continuous. Moreover, the limit as $t$ tends to $Q$ must be zero, because $F_\psi^2 |\varphi'|^2$ is integrable and $1 / F_\psi$ is not integrable over $(c, Q)$ for any $c$ in $(z, Q)$.

It remains to show that $\varphi'$ and $\frac{\varphi}{F_\psi}$ are bounded over $[z, Q]$. Let $z_*$ be a point in $[P, Q]$ such that over $[z_*, Q)$,

$$\frac{K^2}{F_\psi^2} - \lambda_{K,N}(\psi) F_\psi > 0$$
Then $\varphi'$ cannot vanish in $[z_*, Q)$. That is $\varphi'$ is negative over $[z_*, Q)$. We have seen that over $(z_*, Q), \quad K\varphi \geq -F_\psi \varphi'$

Now over $(z_*, Q), \quad \varphi'' \geq -\frac{F_\psi \varphi'}{F_\psi} - K\varphi' - \lambda_{K,N}(\psi)\varphi$

In particular, since $K \geq 1$, \[ \lim_{t \to Q} \varphi'' \geq 0 \]

Therefore $\varphi'$ is bounded. Since $F_\psi'(Q) < 0$, it follows from Cauchy’s mean value theorem that $\frac{\varphi'}{F_\psi}$ is bounded. \[ \square \]

To compare $\lambda_{K,N}(\chi)$ and $\lambda_{K,N}(\omega)$ we will unroll $\chi$ to $\omega$. The following lemma describes the homotopy more precisely.

**Lemma 7.** Let $\chi_0 : [P, L^*] \to \mathbb{R}^2$ be a continuously differentiable curve, parametrized by arc length. Assume $\chi_0(P) = (\mu, 0)$. Write $\chi_0 = (F_0, G_0)$, and assume that $F_0(L^*) = 0$ and $F_0'(L^*) = -1$. Also assume that $F_0$ is positive over $[P, L^*)$ and $G_0'$ is non-negative over $[P, L^*)$. Define a curve $\chi_1 : [P, L^*] \to \mathbb{R}^2$ by $\chi_1(t) = (R - t, 0)$

Then there is a $C^1$ homotopy $\chi_s : [P, L^*] \to \mathbb{R}^2$ for $s$ in $[0, 1]$ with the following properties. The homotopy fixes $P$, that is $\chi_s(P) = (\mu, 0)$ for all $s$ in $[0, 1]$. Each curve in the homotopy is parametrized by arc length, so for all $t$ in $[P, L^*$] and for all $s$ in $[0, 1]$, $|\chi_s'(t)| = 1$

If we write $\chi_s = (F_s, G_s)$, then for all $t$ in $[P, L^*)$ and for all $s$ in $[0, 1]$, $F_s'(t) \leq 0$

Finally, if $L_s^*$ is defined by $L_s^* = \min \left\{ t \in [P, L^*] : F_s(t) = 0 \right\}$

then $F_s'(L_s^*) < 0$, for all $s$ in $[0, 1]$.

**Proof.** Let $h : [0, 1] \to \mathbb{R}$ be a continuously differentiable function such that $h(0) = 0$, $h'(0) = 0$, $h(1) = 1$, $h'(1) = 0$ and $h'(s) > 0$ for all $s$ in $(0, 1)$. For functions $f_0 : [P, L^*) \to \mathbb{R}$ and $f_1 : [P, L^*) \to \mathbb{R}$, with $f_0 \geq f_1$, we define a homotopy by $f_s = (1 - h(s))f_0 + h(s)f_1$

We refer to this homotopy as the monotonic homotopy from $f_0$ to $f_1$ via $h$. 

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There is a continuous function \( \theta_0 : [P, L^*] \to [0, \pi] \) such that, for all \( t \) in \([P, L^*] \)

\[
\chi_0'(t) = \left( -\cos \theta_0(t), \sin \theta_0(t) \right)
\]

Let \( \varepsilon > 0 \) be small. There is a continuous function \( \theta_1 : [P, L^*] \to [0, \pi] \), which has the following three properties. First for all \( t \) in \([P, L^*] \)

\[
\theta_0(t) - \varepsilon \leq \theta_1(t) \leq \theta_0(t)
\]

Second \( \theta_1 \) is continuously differentiable over the set

\[
\left\{ t \in [P, L^*] : \theta_1(t) \in (\pi/4, \pi) \right\}
\]

and \( \theta_1 \) has finitely many critical points in this set. Third \( \pi/2 \) is a regular value of \( \theta_1 \). We take the monotonic homotopy from \( \theta_0 \) to \( \theta_1 \) via \( h \). The set

\[
\left\{ t \in [P, L^*] : \theta_1(t) \geq \pi/2 \right\}
\]

consists of finitely many closed intervals \([a_1, b_1], [a_2, b_2], \ldots\), indexed so that \( a_i > b_{i+1} \) for all \( i \). Let \( U_1 \) be a small neighborhood of \([a_1, b_1]\). Let \( \delta_1 > 0 \) be small, and define \( \theta_2 : [P, L^*] \to \mathbb{R} \) by

\[
\theta_2(t) = \begin{cases} 
\theta_1(t) & \text{if } t \notin U_1 \\
\min(\theta_1(t), \pi/2 - \delta_1) & \text{if } t \in U_1
\end{cases}
\]

If \( U_1 \) is sufficiently small, then for sufficiently small \( \delta_1 \), this function is continuous. Take the monotonic homotopy from \( \theta_1 \) to \( \theta_2 \) via \( h \). Repeat this for each of the closed intervals, letting \( U_2, U_3, \ldots \) be small neighborhoods of each of the intervals, and letting \( \delta_2, \delta_3, \ldots \) be small positive numbers. This yields finitely many homotopies. Finally, take the monotonic homotopy from the last function to the constant zero function via \( h \). Let \( \tilde{\theta}_s : [P, L^*] \to [0, \pi] \), for \( s \) in \([0, 1]\) be the composition of all of these homotopies. Then define \( \chi_s : [P, L^*] \to \mathbb{R}^2 \) for \( s \) in \([0, 1]\) to be the \( C^1 \) homotopy with \( \chi_s(P) = (\mu, 0) \) and for all \( t \) in \([P, L^*] \),

\[
\chi_s'(t) = \left( -\cos \tilde{\theta}_s(t), \sin \tilde{\theta}_s(t) \right)
\]

If the parameters are sufficiently small, then this homotopy satisfies the properties.

Now we can compare \( \lambda_{K,N}(\chi) \) and \( \lambda_{K,N}(\omega) \).

**Lemma 8.** If \( \chi \) is not equal to \( \omega \), then

\[
\lambda_{K,N}(\chi) < \lambda_{K,N}(\omega)
\]
Proof. Suppose $\lambda_{K,N}(\chi) \geq \lambda_{K,N}(\omega)$. Let $\varphi_{K,1}, \varphi_{K,2}, \varphi_{K,3}, \ldots$ be the functions given by Lemma 2 associated to the curve $\chi$. Let $z$ be the largest root of $\varphi_{K,N}$. Define $\chi_0 : [z, L^*] \to \mathbb{R}_+^2$ by

$$\chi_0 = \chi \big|_{[z, L^*]}$$

It follows from Lemma 2 that

$$\lambda_{K,N}(\chi) = \lambda_{K,1}(\chi_0)$$

Define $\omega_1 : [z, R^*] \to \mathbb{R}_+^2$ by

$$\omega_1(t) = (R - t, 0)$$

It follows from the Picone identity that $z < z_0$ and

$$\lambda_{K,N}(\omega) \geq \lambda_{K,1}(\omega_1)$$

Let $\chi_s : [P, L^*] \to \mathbb{R}_+^2$ be the homotopy discussed in Lemma 7. Extend the domain of each curve $\chi_s$ to $[z, L^*]$, by defining, for all $s$ in $[0, 1]$ and for all $t$ in $[z, P]$,

$$\chi_s(t) = \chi_0(t) = (R - t, 0)$$

For $s$ in $[0, 1]$, write $\chi_s = (F_s, G_s)$ and define

$$L^*_s = \min \left\{ t \in [z, L^*] : F_s(t) = 0 \right\}$$

Then define

$$\omega_s = \chi_s \big|_{[z, L^*_s]}$$

These functions map into $\mathbb{R}_+^2$. Note $\omega_1$ agrees with the previous defintion and $\omega_0 = \chi_0$. We will show that the function

$$s \mapsto \lambda_{K,1}(\omega_s)$$

is monotonically increasing over $[0, 1]$. We will do this by showing it is continuous and has non-negative lower left Dini derivative at points $\sigma$ in $(0, 1]$ where $\lambda_{K,1}(\omega_\sigma) > \Lambda$.

We first show the function

$$s \mapsto \lambda_{K,1}(\omega_s)$$

is lower semicontinuous. Fix a point $\sigma$ in $[0, 1]$ such that

$$\liminf_{\sigma \to \sigma} \lambda_{K,1}(\omega_s) < \infty$$

Let $\{s_k\}$ be a sequence in $[0, 1]$ converging to $\sigma$ such that

$$\lim_{k \to \infty} \lambda_{K,1}(\omega_{s_k}) = \liminf_{s \to \sigma} \lambda_{K,1}(\omega_s)$$
By Lemma 2, for each $s$ in $[0, 1]$, there is a function $\varphi_s$ in $\text{Lip}_0(z, L_s^*)$ such that

$$\lambda_{K, 1}(\omega_s) = \frac{\int_z^{L_s^*} |\varphi_s'|^2 F_s + K^2 \varphi_s^2 \, dt}{\int_z^{L_s^*} \varphi_s^2 F_s \, dt}$$

We may assume that each function $\varphi_s$ is normalized so that

$$\int_z^{L_s^*} |\varphi_s|^2 F_s \, dt = 1$$

For $s$ in $[0, 1]$, let $\ell_s : [z, L_s^*] \to [z, L_s^*]$ be a linear function with $\ell_s(z) = z$ and $\ell_s(L_s^*) = L_s^*$. Define $W_s = \varphi_s \circ \ell_s$, for $s$ in $[0, 1]$. Then define $\tau : [0, 1] \to \mathbb{R}$ by

$$\tau(s) = \frac{\int_z^{L_s^*} |W_s'|^2 F_s + K^2 W_s^2 \, dt}{\int_z^{L_s^*} W_s^2 F_s \, dt}$$

Changing variables yields

$$\tau(s) = \frac{\int_z^{L_s^*} |\ell_s'|^2 |\varphi_s'|^2 (F_s \circ \ell_s^{-1}) + \frac{K^2 \varphi_s^2}{(F_s \circ \ell_s^{-1})} \, dt}{\int_z^{L_s^*} \varphi_s^2 (F_s \circ \ell_s^{-1}) \, dt}$$

For $s$ in $[0, 1]$, define $\Psi_s : [0, L_s^*] \to \mathbb{R}$ by

$$\Psi_s(t) = \begin{cases} \frac{F_s \circ \ell_s^{-1}(t)}{F_s(t)} & t \in [0, L_s^*) \\ 1 & t \in [L_s^*, L^*] \end{cases}$$

Note that

$$\lim_{s \to \sigma} \Psi_s = 1$$

and the convergence is uniform. This follows from the fact that the functions

$$(s, t) \mapsto F_s \circ \ell_s^{-1}(t)$$

and

$$(s, t) \mapsto F_s(t)$$

are both differentiable at the point $(\sigma, L_s^*)$ and their derivatives at this point are equal. Now we see that

$$\lim_{s \to \sigma} \int_z^{L_s^*} \varphi_s^2 F_s \, dt - \int_z^{L_s^*} \varphi_s^2 (F_s \circ \ell_s^{-1}) \, dt = 0$$

Similarly,

$$\lim_{k \to \infty} \int_z^{L_{sk}^*} |\varphi_{sk}'|^2 F_{sk} \, dt - \int_z^{L_{sk}^*} |\varphi_{sk}'|^2 (F_{sk} \circ \ell_{sk}^{-1}) \, dt = 0$$

Also,

$$\lim_{k \to \infty} \int_z^{L_{sk}^*} \frac{K^2 \varphi_{sk}}{F_{sk}} \, dt - \int_z^{L_{sk}^*} \frac{K^2 \varphi_{sk}}{(F_{sk} \circ \ell_{sk}^{-1})} \, dt = 0$$
It follows that
\[ \lim_{k \to \infty} \left( \lambda_{K,1}(\omega_{s_k}) - \tau(s_k) \right) = 0 \]
Moreover \( \tau(s) \geq \lambda_{K,1}(\omega_s) \) for all \( s \) in \([\sigma, 1]\). Therefore,
\[ \liminf_{s \to \sigma} \lambda_{K,1}(\omega_s) \geq \lambda_{K,1}(\omega_\sigma) \]
This proves that the function
\[ s \mapsto \lambda_{K,1}(\omega_s) \]
is lower semicontinuous.

Next we show the function
\[ s \mapsto \lambda_{K,1}(\omega_s) \]
is upper semicontinuous. Fix a point \( \sigma \) in \([0, 1]\). By Lemma 2, there is a function \( \varphi_\sigma \in \text{Lip}_0(z, L_\sigma^*) \) such that
\[ \lambda_{K,1}(\omega_\sigma) = \frac{\int_z^{L_\sigma^*} |\varphi'_\sigma|^2 F_\sigma + \frac{K^2 \varphi_\sigma^2}{F_\sigma} dt}{\int_z^{L_\sigma^*} \varphi_\sigma^2 F_\sigma dt} \]
For \( s \) in \([0, 1]\), let \( \ell_s : [z, L_\sigma^*] \to [z, L_s^*] \) be a linear function with \( \ell_s(z) = z \) and \( \ell_s(L_\sigma^*) = L_s^* \). Define \( V_s = \varphi_\sigma \circ \ell_s^{-1} \), for \( s \) in \([0, 1]\). Changing variables yields
\[ \lambda_{K,1}(\omega_\sigma) = \frac{\int_z^{L_s^*} |\ell'_s(t)|^2 |F_\sigma| (F_\sigma \circ \ell_s^{-1}) + \frac{K^2 V_s^2}{(F_\sigma \circ \ell_s^{-1})} dt}{\int_z^{L_s^*} V_s^2 (F_\sigma \circ \ell_s^{-1}) dt} \]
Then define \( \Upsilon : [0, 1] \to \mathbb{R} \) by
\[ \Upsilon(s) = \frac{\int_z^{L_s^*} |V_s'|^2 F_s + \frac{K^2 V_s^2}{F_s} dt}{\int_z^{L_s^*} V_s^2 F_s dt} \]
For \( s \) in \([0, 1]\), define \( \Psi_s : [0, L_s^*] \to \mathbb{R} \) by
\[ \Psi_s(t) = \begin{cases} \frac{F_\sigma \circ \ell_s^{-1}(t)}{F_s(t)} & t \in [0, L_s^*) \\ 1 & t \in [L_s^*, L^*] \end{cases} \]
As before,
\[ \lim_{s \to \sigma} \Psi_s = 1 \]
and the convergence is uniform. Now we see that
\[ \lim_{s \to \sigma} \int_z^{L_s^*} V_s^2 F_s dt - \int_z^{L_s^*} V_s^2 (F_\sigma \circ \ell_s^{-1}) dt = 0 \]
Similarly,
\[ \lim_{s \to \sigma} \int_z^{L_s^*} |V_s'|^2 F_s dt - \int_z^{L_s^*} |V_s'|^2 (F_\sigma \circ \ell_s^{-1}) dt = 0 \]
Also,
\[
\lim_{s \to \sigma} \int_{L_s^*} K^2 V_s \frac{F_s}{L_s^*} dt - \int_{L_s^*} \frac{K^2 V_s}{(F_s \circ \ell_s^{-1})} dt = 0
\]
It follows that
\[
\lim_{s \to \sigma} \Upsilon(s) = \lambda_{K,1}(\omega_\sigma)
\]
Moreover \( \Upsilon(s) \geq \lambda_{K,1}(\omega_s) \) for all \( s \in [0, \sigma] \). Therefore,
\[
\lim \sup_{s \to \sigma} \lambda_{K,1}(\omega_s) \leq \lambda_{K,1}(\omega_\sigma)
\]
This proves that the function
\[
s \mapsto \lambda_{K,1}(\omega_s)
\]
is upper semicontinuous, hence continuous. We remark that Cheeger and Colding [CC] proved a general theorem regarding continuity of eigenvalues.

Now we show the left lower Dini derivative of the function
\[
s \mapsto \lambda_{K,1}(\omega_s)
\]
is non-negative at every point \( \sigma \) in \( (0, 1] \) such that \( \lambda_{K,1}(\omega_\sigma) > \Lambda \). Fix \( \sigma \) in \( (0, 1] \) and assume that
\[
\lambda_{K,1}(\omega_\sigma) > \Lambda
\]
By Lemma 2, there is a function \( \varphi_\sigma \) in \( \text{Lip}_0((0, L_\sigma^*)) \) such that
\[
\lambda_{K,1}(\omega_\sigma) = \frac{\int_{L_\sigma^*} |\varphi_\sigma'|^2 F_\sigma + K^2 \varphi_\sigma^2}{\int_{L_\sigma^*} \varphi_\sigma^2 F_\sigma} dt
\]
By Lemma 6,
\[
\lim_{t \to L_\sigma^*} \varphi_\sigma(t) = 0
\]
Also \( \varphi' \) and \( \frac{\varphi_\sigma^2}{\varphi_\sigma} \) are bounded over \([z, L^*] \). Over \([P, L_\sigma^*] \),
\[
|\varphi_\sigma'|^2 - \frac{K^2 \varphi_\sigma^2}{|F_\sigma|^2} \leq 0
\]
Note that, for \( s \) in \([0, \sigma] \),
\[
L_s^* \geq L_\sigma^*
\]
Define a function \( \xi : [0, \sigma] \to \mathbb{R} \) by
\[
\xi(s) = \frac{\int_z^{L_s^*} |\varphi_\sigma'|^2 F_s + K^2 \varphi_\sigma^2}{\int_z^{L_s^*} \varphi_\sigma^2 F_s} dt
\]
Now \( \lambda_{K,1}(\omega_s) \leq \xi(s) \) for \( s \) in \([0, \sigma] \), and \( \lambda_{K,1}(\omega_\sigma) = \xi(\sigma) \). Also \( \xi \) is left differentiable at \( \sigma \) with
\[
\partial_- \xi(\sigma) = \frac{\int_P^{L_\sigma^*} (|\varphi_\sigma'|^2 - \frac{K^2 \varphi_\sigma^2}{|F_\sigma|^2} - \lambda_{K,1}(\omega_\sigma) \varphi_\sigma^2) \dot{F_\sigma}}{\int_z^{L_\sigma^*} |\varphi_\sigma|^2 F_\sigma} dt
\]
The function $\dot{F}_\sigma$ is non-positive. That is, $\partial_- \xi(\sigma) \geq 0$. This implies that the lower left Dini derivative of the function

$$s \mapsto \lambda_{K,1}(\omega_s)$$

is non-negative at $\sigma$. That is, the lower left Dini derivative is non-negative at every point $\sigma$ in $(0, 1]$ such that $\lambda_{K,1}(\omega_0) > \Lambda$. Since the function is also continuous and $\lambda_{K,1}(\omega_0) > \Lambda$, it follows that the function is monotonically increasing. Moreover, if $\chi$ is not equal to $\omega$, then for some $\sigma$, the function $\dot{F}_\sigma$ is not identically zero, which yields $\partial_+ \xi(\sigma) < 0$. This implies that the lower left Dini derivative of the function

$$s \mapsto \lambda_{K,1}(\omega_s)$$

is negative at some point in $[0, 1]$. In particular, the function is not constant. Now

$$\lambda_{K,1}(\chi_0) = \lambda_{K,1}(\omega_0) < \lambda_{K,1}(\omega_1)$$

This yields $\lambda_{K,N}(\chi) < \lambda_{K,N}(\omega)$. □

**Proof of Lemma 1.** Suppose $\alpha$ is not equal to $\omega$ and $\lambda_{K,N}(\alpha) \geq \lambda_{K,N}(\omega)$. Then by Lemmas 3, 4, 5, and 8

$$\lambda_{K,N}(\alpha) \leq \lambda_{K,N}(\beta) \leq \lambda_{K,N}(\gamma) \leq \lambda_{K,N}(\zeta) = \lambda_{K,N}(\chi) \leq \lambda_{K,N}(\omega)$$

Since $\alpha$ is not equal to $\omega$, it must either be the case that $\alpha$ is not equal to $\beta$ or $\chi$ is not equal to $\omega$. In the first case, the first inequality is strict by Lemma 8. In the second case, the last inequality is strict by Lemma 3. □

**References**

[AF] M. Abreu, P. Freitas, *On the invariant spectrum of $S^1$-invariant metrics on $S^2$*, Ann. Global Anal. Geom 33 (2008), no. 4, 373-395.

[Be] M. Berger, *Sur les premières valeurs propres des variétés riemanniennes*, Compositio Math. 26 (1973), 129-149.

[Bu] D. Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Rational Mech. Anal. 206 (3) (2012), 1073-1083.

[BH] D. Bucur, A. Henrot, *Minimization of the third eigenvalue of the Dirichlet Laplacian*, Proc. R. Soc. Lond. 456, 985-996 (2000).

[CC] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below*, III, J. Differential Geom. 54 (2000), 37-74.

[C] I. Chavel, *On A. Hurwitz’ method in isoperimetric inequalities*, Proc. Amer. Math. Soc. 71 (1978) no. 2, 275-279.

[CD] B. Colbois, J. Dodziuk, *Riemannian metrics with large $\lambda_1$*, Proc. Amer. Math. Soc. 122 (1994), no. 3, 905-906.

[CDE] B. Colbois, E. Dryden, A. El Soufi, *Extremal $G$-invariant eigenvalues of the Laplacian of $G$-invariant metrics*, Math. Z. 258 (2008) no. 1, 29-41.

[CEG] B. Colbois, A. El Soufi, A. Girouard, *Isoperimetric control of the spectrum of a compact hypersurface*, J. Reine Angew. Math. 683 (2013), 49-65.
A unique extremal metric for the least eigenvalue of the Laplacian on the Klein bottle, Duke Math. J. 135 (2006), no. 1, 181-202.

G. Faber, Beweiss, dass unter allen homogenen Membrane von gleicher Fläche und gleicher Spannung die kreisförmige die tiefsten Grundton gibt. Sitzungsber.-Bayer Akad. Wiss., München, Math.-Phys. Munich. (1923) 169-172.

D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

J. Hersch, Quatre propriétés isopérimétriques de membranes sphériques homogènes, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1645-A1648.

D. Jakobson, N. Nadirashvili, I. Polterovich, Extremal metric for the first eigenvalue on a Klein bottle, Canad. J. Math. 58 (2006), no. 2, 381-400.

E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, Math. Ann. 94 (1924) 97-100.

P. Li, S.-T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces, Invent. Math. 69 (1982), no. 2, 269-291.

N. Nadirashvili, Berger’s isoperimetric problem and minimal immersions of surfaces, Geom. Funct. Anal. 6 (1996), no. 5, 877-897.

N. Nadirashvili, Isoperimetric inequality for the second eigenvalue of a sphere, J. Differential Geom. 61 (2002), no. 2, 335-340.

R. C. Reilly, On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, Comment. Math. Helv. 52 (1977), no. 4, 525-533.

H. Urakawa, On the least positive eigenvalue of the Laplacian for compact group manifolds, J. Math. Soc. Japan 31 (1979) no. 1, 209-226.

G. N. Watson, Theory of Bessel functions, 2nd edition, Cambridge University Press, 1944.

P. C. Yang, S.-T. Yau, Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) 7 (1980), no. 1, 55-63.

A. Zettl, Sturm-Liouville Theory, Amer. Math. Soc., Rhode Island, 2005.