Estimating the Hausdorff dimensions of univoque sets for self-similar sets

Xiu Chen, Kan Jiang* and Wenxia Li

Abstract

An approach is given for estimating the Hausdorff dimension of the univoque set of a self-similar set. This sometimes allows us to get the exact Hausdorff dimensions of the univoque sets.

Key words. Hausdorff dimension; univoque set; sets of $k$-codings; self-similar sets.
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1 Introduction

Let $\{f_i\}_{i=1}^m$ be an iterated function system (IFS) of contractive similitudes on $\mathbb{R}^d$ defined as

$$f_i(x) = r_i R_i x + b_i, \quad i \in \Omega = \{1, \ldots, m\},$$

where $0 < r_i < 1$ is the contractive ratio, $R_i$ is an orthogonal transformation and $b_i \in \mathbb{R}^d$. Then there exists a unique nonempty compact set $K \subseteq \mathbb{R}^d$ satisfying (cf. [7])

$$K = \bigcup_{i=1}^m f_i(K). \quad (1)$$

The set $K$ is called the self-similar set generated by the IFS $\{f_j\}_{j=1}^m$. The IFS $\{f_j\}_{j=1}^m$ is said to satisfy the open set condition (OSC) (cf. [7]) if there exists a non-empty bounded open set $V \subseteq \mathbb{R}^d$ such that

$$V \supseteq \bigcup_{i=1}^m f_i(V) \text{ with disjoint union on the right side.}$$

Under the open set condition, the Hausdorff dimension of $K$ coincides with the similarity dimension, denoted by $\dim_S K$, which is the unique solution $s$ of the equation $\sum_{j=1}^m r_j^s = 1$. For any $x \in K$, there exists a sequence $(i_n)_{n=1}^{\infty} \in \{1, \ldots, m\}^\mathbb{N}$ such that

$$x = \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

*Corresponding author
Such sequence \((i_n)_{n=1}^\infty\) is called a coding of \(x\). The attractor \(K\) defined by (1) may equivalently be defined to be the set of points in \(\mathbb{R}^d\) which admit a coding, i.e., one can define a surjective projection map between the symbolic space \(\{1, \ldots, m\}^\mathbb{N}\) and the self-similar set \(K\) by
\[
\Pi((i_n)_{n=1}^\infty) := \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).
\]
A point \(x \in K\) may have multiple codings. \(x \in K\) is called a univoque point if it has only one coding. The set of univoque points is called the univoque set, denoted by \(U\) or \(U_1\). Generally, for \(k \in \mathbb{N}\) we set
\[
U_k = \{x \in K : x \text{ has exact } k \text{ codings}\}.
\]
The univoque set plays a pivotal role in studying the sets of multiple codings (cf. [8, 3, 4]), e.g., we have
\[
\dim H U_k \leq \dim H U \text{ for } k \geq 2,
\]
(2) since \(U_k \subseteq \bigcup_{i \in \Omega^*} f_i(U)\) where, as usual, \(\Omega^* = \bigcup_{n=1}^\infty \Omega^n\). Therefore, it is crucial to find the Hausdorff dimension of the univoque set for self-similar sets. There are many papers about the Hausdorff dimension of \(U\) when \(K\) is an interval (cf. [2, 6, 5, 10, 11, 1, 16, 9, 15]).

In the present paper, we offer an approach to estimate \(\dim H U\) for general self-similar sets. Let \(M\) be a nonempty compact subset of \(\mathbb{R}^d\) satisfying \(f_i(M) \subseteq M\) for \(1 \leq i \leq m\) (so \(K \subseteq M\)). Let
\[
S_1 = \{k \in \Omega : f_k(M) \cap f_j(M) = \emptyset \text{ for all } j \in \Omega \setminus \{k\}\} \text{ and } T_1 = \Omega \setminus S_1.
\]
For positive integer \(i\) let
\[
S_{i+1} = \{k \in (T_i \times \Omega) : f_k(M) \cap f_j(M) = \emptyset \text{ for all } j \in (T_i \times \Omega) \setminus \{k\}\},
\]
\[
T_{i+1} = (T_i \times \Omega) \setminus S_{i+1}.
\]
Note that \(S_i\) may be empty for some \(i\). Let
\[
\Gamma = \bigcup_{i \geq 1} S_i.
\]
(4)
It is clear that \(\Gamma\) becomes largest when \(M\) is taken as \(K\). An \(i \in \Omega^N\) is said to begin with \(\Gamma\) if \(i|k \in \Gamma\) for some \(k \in \mathbb{N}\). Let
\[
V = \{i \in \Omega^\mathbb{N} : i \text{ does not begin with } \Gamma\}.
\]
(5)
In this paper we obtain

**Theorem 1.1.** Let \(\Gamma\) and \(V\) be defined by (4) and (2) respectively. Then
\[
\dim H U = \max\{\dim H \Pi (\Gamma^\mathbb{N}), \dim H \Pi(V \cap \Pi^{-1}(U))\}.
\]

Let \(s\) be determined by
\[
\sum_{i \in \Gamma} r_i^s = 1.
\]
Then we have \(\dim H \Pi (\Gamma^\mathbb{N}) = s\) which will be proved in Lemma 2.1. Hence
Corollary 1.2. We have $\dim_H U \geq s$ and the equality holds if and only if $\dim_H \Pi(V \cap \Pi^{-1}(U)) \leq s$.

The OSC plays an important role in determining the Hausdorff dimension of a self-similar set. Let us recall that $K$ is generated by the IFS $\{f_i\}_{i=1}^m$ in (I). The following fact is obvious:

$$0 < H^s(U) = H^s(K) < \infty \text{ if } \{f_i\}_{i=1}^m \text{ satisfies the OSC,}$$

where $s$ is given by $\sum_{i=1}^m r_i^s = 1$. In fact, we have $U = K \setminus \bigcup_{i \in \Omega^*} f_i(K^*)$ with $K* = \bigcup_{i \neq j} (f_i(K) \cap f_j(K))$ and the OSC implies that $H^s(f_i(K) \cap f_j(K)) = 0$ for any $i \neq j$ (see [14]).

From (6) it follows that $\dim_H U = \dim_H K = \dim_S K$ if the IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition. We shall show that under some extra condition the inverse is also true. An IFS $\{f_i\}_{i=1}^m$ is said to have an exact overlap if there exist distinct $i, j \in \Omega^*$ such that $f_i = f_j$. The notion of “general finite type” appeared in the following Lemma which was posed by Lau and Ngai in [12]. We have

**Theorem 1.3.** Let $K$ be the self-similar set generated by the IFS $\{f_i\}_{i=1}^m$. Suppose that $\{f_i\}_{i=1}^m$ is of general finite type. Then $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if $\dim_H U = \dim_S K$.

This paper is organized as follows. In section 2, we give the proofs of Theorems 1.1 and 1.3. The section 3 is devoted to some examples.

## 2 Proof of Theorems 1.1 and 1.3

Denote by $ij$ the concatenation of $i, j \in \Omega^*$ and $i^k$ stands for the concatenation of $i$ with itself $k$ times. By $|i|$ we denote the length of $i \in \Omega^*$. For $i = i_1 \cdots i_k \in \Omega^*$ we denote by $[i]$ the cylinder set based on $i$, i.e., $[i] = \{(x_i) \in \Omega^N \mid x_i = i_i \text{ for } 1 \leq i \leq k\}$. For an $i = (i_k)_{k \geq 1} \in \Omega^N$ let $ip = i_1 \cdots i_p$. For $i = i_1 \cdots i_k \in \Omega^*$ denote $f_i = f_{i_1} \circ \cdots \circ f_{i_k}$ and $r_i = \prod_{i=1}^k r_i$.

**Lemma 2.1.** Let $\Gamma \subseteq \Omega^*$ be given by (7). Then $\Pi(G^N) \subseteq U$ and $\dim_H \Pi(G^N) = s$ where $s$ is determined by $\sum_{i \in \Gamma} r_i^s = 1$.

**Proof.** Note that by the definition of $\Gamma$ we have

(I) $[i], i \in \Gamma$ are pairwise disjoint;

(II) $f_i(K) \cap \Pi((\Omega^N \setminus [i]) = \emptyset$ for each $i \in \Gamma$.

First we show that $\Pi(G^N) \subseteq U$. For an $x \in \Pi(G^N)$ let $x = \Pi((x_k)_{k \geq 1})$ with $(x_k)_{k \geq 1} \in G^N$. Suppose that $(y_k)_{k \geq 1} \in G^N$ satisfies that $x = \Pi((y_k)_{k \geq 1})$. We claim that $(y_k)_{k \geq 1} = (x_k)_{k \geq 1}$. On the contrary, let $\ell$ be the smallest integer such that $y_{\ell} \neq x_{\ell}$. Let $(x_k)_{k \geq 1} = (i_k)_{k \geq 1}$ with $i_k \in \Gamma$. Let $\gamma$ be smallest integer such that $\ell \leq |i_1 \cdots i_{\gamma}|$. Then

$$\Pi((x_k)_{k \geq \delta}) = \Pi((y_k)_{k \geq \delta})$$

where $\delta = |i_1 \cdots i_{\gamma}| - |i_{\gamma}| + 1$. 

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However, $\Pi((x_k)_{k \geq \delta}) \in f_k(K)$, $i_r \in \Gamma$ and $(y_k)_{k \geq \delta} \notin [i_r]$. This leads to a contradiction to the fact $f_k(K) \cap \Pi(\Omega^N \setminus [i_r]) = \emptyset$.

In what follows we prove that $\dim_H \Pi(\Gamma^N) = s$. If $\Gamma$ is finite, then $\Pi(\Gamma^N)$ is a self-similar set generated by the IFS $\{f_i : i \in \Gamma\}$. This IFS satisfies the OSC since $K \supseteq \bigcup_{i \in \Gamma} f_i(K)$ with disjoint union. Thus $\dim_H \Pi(\Gamma^N) = s$.

In the following we assume that $\Gamma$ is infinite. Denote $\Gamma_k = \{i \in \Gamma : |i| \leq k\}$, $k \in \mathbb{N}$. Then $\Gamma_k$ is finite (we assume $k$ is big enough such that $\Gamma_k \neq \emptyset$). Thus

$$\dim_H \Pi(\Gamma^N_k) = s_k \text{ where } \sum_{i \in \Gamma_k} r_i^{s_k} = 1.$$ 

Therefore, $\dim_H \Pi(\Gamma^N) \geq \sup_k s_k = \lim_{k \to \infty} s_k = s$ where the last equality can be obtained by the equation $\sum_{i \in \Gamma} r_i^s = 1$ and $\Gamma = \bigcup_{k \geq 1} \Gamma_k$.

Arbitrarily fix a $t > s$. For any $\delta > 0$ one can take a big integer $n$ such that each set in $\{f_i(K) : i \in \Gamma^n\}$ has diameter less that $\delta$. Note that

$$\sum_{i \in \Gamma^n} |f_i(K)|^t = |K|^t \left(\sum_{i \in \Gamma} r_i^t\right)^n \leq |K|^t,$$

which implies that $\dim_H \Pi(\Gamma^N) \leq t$. □

**Proof of Theorem 1.4** Note that

$$\Omega^N = V \cup V^c = V \cup \Gamma^N \cup \{ui : u \in \Gamma^*, \ i \in V\}.$$ 

Thus

$$\Pi^{-1}(U) = \Gamma^N \cup (V \cap \Pi^{-1}(U)) \cup \{uj : u \in \Gamma^*, \ j \in V \cap \Pi^{-1}(U)\},$$

which implies the desired result. □

Now we turn to proving Theorem 1.3. We need following

**Lemma 2.2.** [18, Theorem 2.1] An IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if it is of general finite type and has no exact overlaps.

**Proof of Theorem 1.3** The necessity follows from (6). We now prove the sufficiency. Note that $\{f_i\}_{i=1}^m$ is of general finite type. Thus by Lemma 2.2 it suffices to show that the IFS $\{f_i\}_{i=1}^m$ has no exact overlaps. Otherwise, there exist distinct $i, j \in \Omega^*$ such that $f_i = f_j$. Let $K_1$ be the self-similar set generated by the IFS $\{f_k : k \in \Omega^N\}$ and $k \neq i$. Then $\dim_H K_1 \leq \dim_H K < \dim_S K$. On the other hand, for any $x \in U$ its unique coding cannot contain the block $i$ and so $x \in K_1$. Thus, $\dim_H U \leq \dim_H K_1 \leq \dim_H K < \dim_S K$, a contradiction! □

### 3 Examples

The result in the following example was obtained in [17] by giving a lexicographical characterization of the unique codings. Now we reprove it by applying Theorem 1.1 which provides a quite different way from that in [17].
Example 3.1. (see [17]) Let $K$ be the self-similar set generated by the IFS

\[\{ f_1(x) = \rho x, f_2(x) = \rho x + \rho, f_3(x) = \rho x + 1 \} \text{ where } 0 < \rho < (3 - \sqrt{5})/2.\]

Then \( \dim_H U = \frac{\log \lambda}{\log \rho} \), where \( \lambda \approx 2.3247 \) is the appropriate solution of

\[x^3 - 3x^2 + 2x - 1 = 0.\]

Proof. First one can check that \( f_1 \circ f_3 = f_2 \circ f_1 \). Take \( M = [0, (1 - \rho)^{-1}] \). Then

\[f_1(M) \cap f_2(M) = [0, \rho/(1 - \rho)] \cap [\rho, (2\rho - \rho^2)/(1 - \rho)] = [\rho, \rho/(1 - \rho)]\]

and

\[f_1(M) \cap f_3(M) = f_2(M) \cap f_3(M) = \emptyset.\]

Thus one has that \( S_1 = \{3\} \) and \( S_2 = \{23\} \). For \( k \geq 3 \) the sets \( S_k \) becomes a bit complicated. However, it is not so difficult to find out that \( |S_k| = k - 1 \) by noting that \( f_1 \circ f_3 = f_2 \circ f_1 \), where \( |A| \) denotes the cardinality of set \( A \). Let \( \Gamma = \bigcup_{k \geq 1} S_k \). Thus by Lemma 2.1

\[\dim_H \Pi(\Gamma^\mathbb{N}) = s, \text{ where } s = \frac{\log \lambda}{\log \rho} \text{ and } \lambda \approx 2.3247 \text{ is the appropriate solution of } x^3 - 3x^2 + 2x - 1 = 0.\]

It is an easy exercise to check that \( s = \frac{\log \lambda}{\log \rho} \) where \( \lambda \approx 2.3247 \) is the appropriate solution of \( x^3 - 3x^2 + 2x - 1 = 0.\)

Now we show that \( \dim_H \Pi(V) \leq s \), where \( V = \{i \in \Omega^\mathbb{N} : i \text{ does not begin with } \Gamma\} \) is as that in Theorem 1.1. By the geometric structure of \( K \) one can see that for each positive integer \( k \), the set \( \Pi(V) \) can be covered by \( 2^k \) many number of intervals of length \( \rho^k(\rho - 1)^{-1}. \) Thus

\[\mathcal{H}_{\rho^k(\rho - 1)^{-1}}(\Pi(V)) \leq (1 - \rho)^{-s}(2\rho^s)^k \to 0 \text{ as } k \to \infty\]

since \( 2\rho^s < 1 \). Thus, \( \dim_H U = \frac{\log \lambda}{\log \rho} \) by Theorem 1.1.

Example 3.2. Take \( 0 < \lambda < (3 - \sqrt{5})/2 \). Let \( K \) be the self-similar set generated by the IFS \( \{ f_1, \cdots, f_3 \} \) where

\[f_i(x, y) = (\lambda x, \lambda y) + (a_i, b_i)\]

with \( (a_1, b_1) = (0, 0), (a_2, b_2) = (1 - \lambda, 0), (a_3, b_3) = (1 - \lambda, 1 - \lambda), (a_4, b_4) = (0, 1 - \lambda) \) and

\( (a_5, b_5) = (\lambda(1 - \lambda), (1 - \lambda)^2). \) Then \( \dim_H U = s \approx \frac{\log 4.61347}{-\log \lambda} \), where \( \lambda^3 - 2\lambda^2 + 5\lambda - 1 = 0.\)
Proof. First one can check that $f_4 \circ f_2 = f_5 \circ f_4$. Among the squares $f_i([0,1]^2), 1 \leq i \leq 5$, only $f_4([0,1]^2) \cap f_5([0,1]^2) \neq \emptyset$ (see Figure 2). Thus $S_1 = \{1,2,3\}$ and $S_2 = \{41,43,51,52,53\}$. As in above example, for $k \geq 3$ the sets $S_k$ becomes a bit complicated. However, it is not so difficult to find out that $|S_k| = 3^k - 1$ by noting that $f_4 \circ f_2 = f_5 \circ f_4$. Let $\Gamma = \bigcup_{k \geq 1} S_k$. Thus by Lemma 2.1 we have $\dim H(\Gamma \cap [\Omega^N]) = s \approx \log 4.61347 - \log \lambda$ where

$$3\lambda^s + 5\lambda^{2s} + \sum_{k=3}^{\infty} (3k - 1)\lambda^{ks} = 1,$$

which is equivalent to $\lambda^{3s} - 2\lambda^{2s} + 5\lambda^s - 1 = 0$.

Now we show that $\dim H(V \cap \Pi^{-1}(U))) \leq s$, where

$$V = \{i \in \Omega^N : i \text{ does not begin with } \Gamma\}$$

is as that in Theorem 1.1. By the geometric structure of $K$ one can see that for each positive integer $k$, the set $\Pi(V \cap \Pi^{-1}(U)))$ can be covered by $2^k$ many number of squares with diameter $\sqrt{2}\lambda^k$. Thus

$$H^{s}_{\sqrt{2}\lambda^k}(K_\alpha) \leq 2^k \sqrt{2} \lambda^{sk} \to 0 \text{ as } k \to \infty$$

since $2\lambda^s < 1$. Thus, $\dim H U = s$ by Theorem 1.1. \hfill $\square$

In the above we change the map $f_5$ by letting

$$(a_5,b_5) = (\lambda - \lambda^{u+1}, 1 - 2\lambda + \lambda^{u+1}) \text{ with } u \in \mathbb{N},$$

where we require that $\lambda^{u+1} - 3\lambda + 1 > 0$. Then $\dim_H U$ can be also obtained by the same way as in Example 3.2 and so $\dim H U_k, k \geq 2$ can be obtained as well. In fact, we have

Example 3.3. Suppose that $\lambda \in (0,1), u \in \mathbb{N}$ satisfy $\lambda^{u+1} - 3\lambda + 1 > 0$. Let $K$ be the self-similar set generated by the IFS $\{f_1, \cdots, f_5\}$ where

$$f_i(x, y) = (\lambda x, \lambda y) + (a_i, b_i)$$

with $(a_1,b_1) = (0,0), (a_2,b_2) = (1 - \lambda, 0), (a_3,b_3) = (1 - \lambda, 1 - \lambda), (a_4,b_4) = (0,1 - \lambda)$ and $(a_5,b_5) = (\lambda - \lambda^{u+1}, 1 - 2\lambda + \lambda^{u+1})$. Then

$$\dim H U_{k+1} = \dim H U \text{ for any } k \in \mathbb{N}.$$
Proof. By (2) we only need to show that \( \dim H U_{k+1} \geq \dim H U \). This will be done by showing
\[
f_{42^{u+1}}(U) \subseteq U_{k+1} \text{ for each } k \geq 1.
\]
Now arbitrarily fix a point \( c \in U \) with the unique coding \( (c_i) \). We prove that \( f_{42^{u+1}}(c) \in U_{k+1}, k \geq 1 \) by induction.

Let \( k = 1 \). Note that \( x_1 = f_{42^{u+1}}(c) = f_{54^{u+1}}(c) \in f_{42^u}([0, 1]^2) = f_{54^u}([0, 1]^2) \) and
\[
f_i([0, 1]^2) \cap f_{42^u}([0, 1]^2) = \emptyset \text{ for all } i \in \{ 1, 2, 3, 4, 5 \} \cup \{ 42^u, 54^u, 54^{u-15} \}.
\]
Hence any coding \( (d_i) \) of \( x_1 \) has to begin with \( 42^u, 54^u \) or \( 54^{u-15} \). We claim that \( (d_i) \) cannot begin with \( 54^{u-15} \). Otherwise, we have \( f_{41}(c) = f_{51}(c) \in f_{41}([0, 1]^2) \cap f_{51}([0, 1]^2) = \emptyset \). On the other hand, we have \( \pi(\sigma^{u+1}(d_i)) = \pi(1(c_i)) = f_1(c) \in U \) where \( \sigma \) is the left shift on \( \Omega^\mathbb{N} \). Thus, \( (d_i) \) has to be \( 42^u 1(c_i) \) or \( 54^u 1(c_i) \), i.e., \( x_1 \in U_2 \).

Suppose that \( x_k = f_{42^{u+1}}(c) = \pi(2^{4k+1}(c_i)) \in U_{k+1} \). Let \( (d_i) \) be a coding of \( x_{k+1} := f_{42^{u+k+1}}(c) \). As before we know that \( (d_i) \) has to begin with \( 42^u, 54^u \) or \( 54^{u-15} \), and so \( (d_i) \) has to begin with \( 42^{u-1} \) or \( 54^{u-15} \). Note that
\[
x_{k+1} = f_{42^u}(\pi(2^{4k+1}(c_i))) = f_{54^u}(\pi(2^{4k+1}(c_i))) = f_{54^{u-1}}(\pi(2^{4k+1}(c_i))) = f_{54^{u-1}}(x_k).
\]
For the case that \( (d_i) \) begins with \( 42^{u-1} \) we have that \( (d_i) = 42^{u+1}(c_i) \) since \( \pi(\sigma^u(d_i)) = \pi(2^{u+1}(c_i)) \in U \). For the case that \( (d_i) \) begins with \( 54^{u-1} \) we have that \( (d_i) \) has exactly \( k+1 \) many choices since
\[
\pi((d_i)_{i \geq u}) = \pi(\sigma^u(d_i)) = \pi(2^{4k}(c_i)) = x_k \in U_{k+1}.
\]
Hence we complete the proof. \( \square \)

In the last example we try to describe \( \Gamma \) by a way which was developed in \[13, 12\].

Example 3.4. Let \( K \) be the self-similar set generated by the IFS
\[
\begin{align*}
\left\{ f_1(x) &= \frac{x}{4}, \\ f_2(x) &= \frac{x}{4} + \frac{9}{17}, \\ f_3(x) &= \frac{x + 3}{4} \right. \end{align*}
\]
Then \( \dim H U = s \), where \( s \) is the unique solution of the following equation:
\[
\frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty} (a_n + c_n) \frac{1}{4^{(n+1)s}} = 1,
\]
where \( a_n, c_n \) for \( n \geq 2 \) are determined by
\[
\begin{pmatrix}
a_n \\
b_n \\
c_n \\
d_n \\
e_n
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}^{n-2}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]
(7)

Proof. We take \( M = [0, 1] \) (one can check that \( f_i(M) \subseteq M \) for \( i \in \Omega := \{ 1, 2, 3 \} \)) and label it by \( T_1 \). Its offspring are
\[
f_1(M) = [0, 1/4], \ f_2(M) = [9/17, 53/68] \text{ and } f_3(M) = [3/4, 1].
\]
Figure 3: The location of $f_i(M), i = 1, 2, 3$. 

Then (see Figure 3)

$S_1 = \{1\}$ and $T_1 = \{2, 3\}$.

Note that the offspring of $f_1(M)$ have the same geometric location as the offspring of $M$. So $f_1(M)$ is labeled by $T_1$ as well. We label $f_2(M)$ and $f_3(M)$ by $T_2$ and $T_3$, respectively. Thus one can simply denote $M$ and its offspring as follows:

$$(M, T_1) \rightarrow (f_1(M), T_1) + (f_2(M), T_2) + (f_3(M), T_3).$$

Now let us calculate $f_i(M)$:

$$(M, T_1) \rightarrow (f_1(M), T_1) + (f_2(M), T_2) + (f_3(M), T_3).$$

Thus we have (see Figure 4)

$S_2 = \{21\}$ and $T_2 = (T_1 \times \Omega) \setminus S_2 = \{22, 23, 31, 32, 33\}$.

By the same argument as above the offspring $f_2(M)$ of $f_2(M)$ has label $T_1$, while the other two offspring $f_{22}(M), f_{23}(M)$ of $f_2(M)$ will obtain new labels $T_4, T_5$, respectively. This can be simply denoted by

$$(f_2(M), T_2) \rightarrow (f_2(M), T_1) + (f_{22}(M), T_4) + (f_{23}(M), T_5).$$

Similarly, for the $f_3(M)$ and its offspring we have

$$(f_3(M), T_3) \rightarrow (f_3(M), T_6) + (f_{32}(M), T_2) + (f_{33}(M), T_3).$$

Figure 4: The location of $f_i(M), i \in T_1 \times \Omega$. 

| $f_{21}(M)$ | $f_{22}(M)$ | $f_{31}(M)$ | $f_{32}(M)$ |
|-------------|-------------|-------------|-------------|
| $\frac{9}{17}$, $\frac{9}{17}$ + $\frac{1}{16}$ | $\frac{45}{68}$, $\frac{45}{68}$ + $\frac{1}{16}$ | $\frac{195}{272}$, $\frac{195}{272}$ + $\frac{1}{16}$ | $\frac{15}{16}$, $\frac{15}{16}$ |
Thus we have (see Figure 5)

\[ f_{221}(M) = \begin{bmatrix} 45 & 45 & 1 \ \\ 68 & 68 & 64 \end{bmatrix}, \quad f_{222}(M) = \begin{bmatrix} 189 & 189 & 1 \ \\ 272 & 272 & 64 \end{bmatrix}, \quad f_{223}(M) = \begin{bmatrix} 771 & 771 & 1 \ \\ 1088 & 1088 & 64 \end{bmatrix}, \quad f_{311}(M) = \begin{bmatrix} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ \end{bmatrix}, \quad f_{312}(M) = \begin{bmatrix} 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ 5 \ \end{bmatrix}, \quad f_{313}(M) = \begin{bmatrix} 64 \ 64 \ 64 \ 64 \ 64 \ 64 \ 64 \ 64 \ \end{bmatrix}. \]

Thus we have (see Figure 5)

\[
(f_{22}(M), T_4) \rightarrow (f_{221}(M), T_1) + (f_{222}(M), T_4) + (f_{223}(M), T_5) \\
(f_{23}(M), T_5) \rightarrow (f_{231}(M), T_6) + (f_{232}(M), T_2) + (f_{233}(M), T_3) \\
(f_{31}(M), T_6) \rightarrow (f_{311}(M), T_2) + (f_{312}(M), T_2) + (f_{313}(M), T_3)
\]

(10)

It is important to notice that no more labels occur in the above expression. Note that

\[
\begin{array}{cccc}
f_{221}(M) & f_{222}(M) & f_{223}(M) & f_{311}(M) \\
45 & 189 & 771 & 1 \\
68 & 272 & 1088 & 1 \\
\end{array}
\]

Figure 5: The location of \( f_1(M) \), \( i \in (T_2 \setminus \{32, 33\}) \times \Omega \)

we have \( f_{232} = f_{311} \). Thus \( f_{232}(M) \) and \( f_{311}(M) \) contribute nothing to \( \Gamma \). Therefore, we replace (10) by

\[
(f_{22}(M), T_4) \rightarrow (f_{221}(M), T_1) + (f_{222}(M), T_4) + (f_{223}(M), T_5) \\
(f_{23}(M), T_5) \rightarrow (f_{231}(M), T_6) + (f_{232}(M), T_2) + (f_{233}(M), T_3) \\
(f_{31}(M), T_6) \rightarrow (f_{311}(M), T_2) + (f_{312}(M), T_2) + (f_{313}(M), T_3)
\]

(11)

It follows from (8), (9) and (11) that only \( T_2 \) and \( T_4 \) have contribution to \( \Gamma \). Together with (3) one knows that the cardinality of \( S_{n+1} \) \( (n \geq 2) \) equals to the number of \( T_2 \) and \( T_4 \) occurring in the \( n \)-th generation offspring. By \( a_n, b_n, c_n, d_n, \) and \( e_n \) we denote the number of \( T_2, T_3, T_4, T_5 \) and \( T_6 \) occurring in the \( n \)-th generation offspring. By (8), (9) and (11) we have

\[
\begin{pmatrix}
a_{n+1} \\
b_{n+1} \\
c_{n+1} \\
d_{n+1} \\
e_{n+1}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a_n \\
b_n \\
c_n \\
d_n \\
e_n
\end{pmatrix}
:= A
\begin{pmatrix}
a_n \\
b_n \\
c_n \\
d_n \\
e_n
\end{pmatrix}, \quad n \geq 2.
\]

By (8) and (9) we have

\[
a_2 = b_2 = c_2 = d_2 = e_2 = 1,
\]

and so (7) is obtained. Therefore, we have \( \dim_H \Pi (\Gamma^N) = s \), where \( s \) is the unique solution of the following equation:

\[
1 = \frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty} |S_{n+1}| \frac{1}{4^{(n+1)s}} = \frac{1}{4^s} + \frac{1}{4^{2s}} + \sum_{n=2}^{\infty} (a_n + c_n) \frac{1}{4^{(n+1)s}}.
\]
In what follows we will show that $\dim_H \Pi(V \cap \Pi^{-1}(U)) \leq s$. One can check that the spectral radius of $A$ is about $\lambda \approx 2.2775$. We claim that $\lambda < 4^s$. In fact, we have $4^s > 4^t \approx 2.4693$ where $t$ is determined by

$$\frac{1}{4^t} + \frac{1}{4^{2t}} + \sum_{n=2}^{5} (a_n + c_n) \frac{1}{4^{(n+1)t}} = \frac{1}{4^t} + \frac{1}{4^{2t}} + \frac{2}{4^{3t}} + \frac{4}{4^{4t}} + \frac{9}{4^{5t}} + \frac{21}{4^{6t}} = 1.$$ 

Note that

$$\lim_{n \to \infty} \mathcal{H}_{4-n-2}^t(\Pi(V \cap \Pi^{-1}(U))) \leq \lim_{n \to \infty} (a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} + e_{n+1})4^{(-n-2)s} < \infty,$$

where the last inequality holds since all the $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}$ are bounded by $c\lambda^n$ for some $c > 0$, and the fact $\lambda < 4^s$. 

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\end{acknowledgment}

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**Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, People’s Republic of China**

*E-mail address:* chenxiu1216@163.com.

**Department of Mathematics, Ningbo University, Ningbo, Zhejiang, People’s Republic of China**

*E-mail address:* kanjiangbunnik@yahoo.com

**Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, People’s Republic of China**

*E-mail address:* wxli@math.ecnu.edu.cn.