Integral Operators Basic in Random Fields
Estimation Theory

Alexander Kozhevnikov
Department of Mathematics,
University of Haifa, Haifa, 31905, ISRAEL
kogevn@math.haifa.ac.il

Alexander G. Ramm
Department of Mathematics,
Kansas State University
Manhattan, KS 66502, USA
ramm@math.ksu.edu

Abstract. 1 The paper deals with the basic integral equation of random field estimation theory by the criterion of minimum of variance of the error estimate. This integral equation is of the first kind. The corresponding integral operator over a bounded domain $\Omega$ in $\mathbb{R}^n$ is weakly singular. This operator is an isomorphism between appropriate Sobolev spaces. This is proved by a reduction of the integral equation to an elliptic boundary value problem in the domain exterior to $\Omega$. Extra difficulties arise due to the fact that the exterior boundary value problem should be solved in the Sobolev spaces of negative order.

1 Introduction

For convenience of the reader, all standard notations, such as $\mathbb{N}_+$, $\mathbb{R}$, $\mathbb{C}$, the definitions of the Fourier transform, of the Sobolev spaces, etc, are placed in the Appendix.

Let $P$ be a differential operator in $\mathbb{R}^n$ of order $\mu$, 

$$P := P(x, D) := \sum_{|\alpha| \leq \mu} a_{\alpha}(x) D^\alpha,$$

where $a_{\alpha}(x) \in C^\infty(\mathbb{R}^n)$.

The polynomials

$$p(x, \xi) := \sum_{|\alpha| \leq \mu} a_{\alpha}(x) \xi^\alpha \quad \text{and} \quad p_0(x, \xi) := \sum_{|\alpha| = \mu} a_{\alpha}(x) \xi^\alpha$$

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1
are called respectively a symbol and a principal symbol of $P$.

Suppose that the symbol $p(x, \xi)$ belongs to the class $SG^{(\mu, 0)}(\mathbb{R}^n)$ consisting of all $C^\infty$ functions $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ that for any multiindices $\alpha, \beta$ there exists a constant $C_{\alpha, \beta}$ such that

$$|D_x D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta} (\xi)^{\mu-|\beta|} (x)^{-|\alpha|} \quad (x, \xi \in \mathbb{R}^n, \quad \langle \xi \rangle:=(1 + |\xi|^2)^{1/2}) \quad (1)$$

It is known (cf. [13, Prop. 7.2]) that the map $P(x, D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, where $\mathcal{S}(\mathbb{R}^n)$ is the space of smooth rapidly decaying functions (see Appendix). Let $H^s(\mathbb{R}^n) \ (s \in \mathbb{R})$ be the usual Sobolev space (see Appendix). It is known that $P(x, D)$ naturally acts on the Sobolev spaces, i.e. the operator $P(x, D)$ is (cf. [13, Sec. 7.6]) a bounded operator: $H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

$P(x, D)$ is called elliptic, if $p_0(x, \xi) \neq 0$ for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

Let $P(x, D)$ and $Q(x, D)$ be both elliptic differential operators of even orders $\mu$ and $\nu$ respectively, $0 \leq \mu < \nu$, with symbols satisfying (1) (for $Q(x, D)$ we replace $p$ and $\mu$ in (1) respectively by $q$ and $\nu$). The case $\mu \geq \nu$ is a simpler case which leads to an elliptic operator perturbed by a compact integral operator in a bounded domain.

We assume also that $P(x, D)$ and $Q(x, D)$ are invertible operators, i.e. there exist the inverse bounded operators $P^{-1}(x, D) : H^{s-\mu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ and $Q^{-1}(x, D) : H^{s-\nu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Let $R := Q^{-1}(x, D)P(x, D)$. The ellipticity and invertibility of $P(x, D)$ and $Q(x, D)$ imply that $R$ is an elliptic invertible pseudodifferential operator of negative order $\mu - \nu$ acting from $H^s(\mathbb{R}^n)$ onto $H^{s+\nu-\mu}(\mathbb{R}^n) \ (s \in \mathbb{R})$.

Since $P$ and $Q$ are elliptic, their orders $\mu$ and $\nu$ are even for $n > 2$. If $n = 2$, we assume that $\mu$ and $\nu$ are even numbers. Therefore, the number $a := (\nu - \mu)/2 > 0$ is an integer.

Let $\Omega$ denote a bounded connected open set in $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$ ($C^\infty$-class surface) and $\overline{\Omega}$ its closure in $L^2(\Omega)$, i.e. $\overline{\Omega} = \Omega \cup \partial \Omega$. The smoothness restriction on the domain can be weakened, but we do not go into detail.

The restriction $R_\Omega$ of the operator $R$ to the domain $\Omega \subset \mathbb{R}^n$ is defined as

$$R_\Omega := r_\Omega R e_{\Omega_-}, \quad (2)$$

where $e_{\Omega_-}$ is the extension by zero to $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ and $r_\Omega$ is the restriction to $\Omega$.

It is known (cf. [2, Th. 3.11, p. 312]) that the operator $R_\Omega$ defines a continuous mapping

$$R_\Omega : H^s(\Omega) \to H^{s+\nu-\mu}(\Omega) \quad (s > -1/2),$$

where $H^s(\Omega)$ is the spaces of restrictions of elements of $H^s(\mathbb{R}^n)$ to $\Omega$ with the usual infimum norm (see Appendix).

The pseudodifferential operator $R$ of negative order $\mu - \nu$ as well as its restriction $R_\Omega$ can be represented as integral operators with kernel $R(x, y) :

$$Rh = \int_{\mathbb{R}^n} R(x, y) h(y) \, dy, \quad R_\Omega h = \int_{\Omega} R(x, y) h(y) \, dy \ (x \in \Omega),$$

2
where \( R(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \text{Diag}) \), Diag is the diagonal in \( \mathbb{R}^n \times \mathbb{R}^n \), Moreover, \( R(x, y) \) has a weak singularity:
\[
|R(x, y)| \leq C |x - y|^{-\sigma} \quad n + \mu - \nu < \sigma < n.
\]
For \( n + \mu - \nu < 0 \), \( R(x, y) \) is continuous.

Let \( \gamma := n + \mu - \nu \) and \( r_{xy} := |x - y| \to 0 \). Then \( R(x, y) = O(r_{xy}^{-\gamma}) \) if \( n \) is odd or if \( n \) is even and \( \nu < n \), and \( R(x, y) = O(r_{xy}^{-\gamma} \log r_{xy}) \) if \( n \) is even and \( \nu > n \).

In [6], the equation
\[
\mathcal{R}_\Omega h = f \in H^a(\Omega), \quad h \in H^{-a}_0(\Omega)
\]
is derived as a necessary condition for the optimal estimate of random fields by the criterion of minimum of variance of the error of the estimate. The kernel \( R(x, y) \) is a known covariance function, and \( h(x, y) \) is the distributional kernel of the operator of optimal filter. The kernel \( h(x, y) \) should be of minimal order of singularity ([6]). In [6], the case was considered when \( P \) and \( Q \) are polynomial functions of a selfadjoint elliptic operator defined in the whole space. In [7] and [8], some generalizations of this theory are obtained. This paper presents an extension of some results from [7].

The purpose of this paper is to prove that, under some natural assumptions, the restriction operator \( \mathcal{R}_\Omega \) is an isomorphism of the space \( H^{-a}_0(\Omega) \) onto \( H^a(\Omega) \) where \( a = (\nu - \mu) / 2 \geq 0 \) and \( H^s(\Omega) \) denotes the subspace of \( H^s(\mathbb{R}^n) \) that consists of the elements supported in \( \Omega \) (see Appendix).

To prove the isomorphism property, we reduce the integral equation (3) to an equivalent elliptic exterior boundary-value problem. Since we look for a solution \( u \) belonging to the space \( H^a(\Omega_-) = H^{(\nu - \mu)/2}(\Omega_-) \) and the differential operator \( Q \) is of order \( \nu \), then \( Qu \) should belong to some Sobolev space of negative order. This means that we need results on the solvability of (23) in Sobolev spaces of negative order. Such spaces as well as solvability in them of elliptic differential boundary value problems in bounded domains have been investigated by Ya. Roitberg [9] and later by V. Kozlov, V. Maz’ya, J. Rossmann [5]. The case of pseudodifferential boundary value problems has been studied in [4]. A. Erkip and E. Schrohe [1] and E. Schrohe [11] have proved the solvability of elliptic differential and pseudodifferential boundary value problems for unbounded manifolds and in particular for exterior domains. These solvability results have been obtained in so-called weighted Sobolev spaces and in particular in the Sobolev spaces \( H^s \) of positive order \( s \). To obtain the isomorphism property, the solvability results by A. Erkip and E. Schrohe for exterior domain should be extended to the weighted Sobolev spaces of negative order. One can find in Appendix the definition of these spaces which is similar to [9] and [5].

2 Reduction of the basic integral equation to a boundary-value problem

**Theorem 1.** The integral equation (3) is equivalent to the following system (4), (5),
\[ \begin{cases} Qu = 0 & \text{in } \Omega_- \\ D_n^j u = D_n^j f & \text{on } \partial \Omega, \quad 0 \leq j \leq a - 1 \end{cases} \quad (4) \]

\[ \text{where } u \in H^a(\Omega_-) \text{ is an extension-function for } f, \text{i.e.} \]

\[ F \in H^a(\Omega^n), \quad F := \begin{cases} f \in H^a(\Omega) & \text{in } \Omega, \\ u \in H^a(\Omega_-) & \text{in } \Omega_- \end{cases} \quad (6) \]

**Proof.** Let \( h \in H_0^a(\Omega) \) be a solution to (3), i.e. \( R_0 h = f \in H^a(\Omega) \). Let us define \( F := Q^{-1} Ph \). Since \( h \in H_0^a(\Omega) \), then \( Ph \in H^{-a-\mu}(\Omega^n) \) and \( F = Q^{-1} Ph \in H^{-a+\mu}(\Omega^n) = H^a(\Omega^n) \). We have \( f = R_0 h = r_\Omega Q^{-1} Ph = r_\Omega F \), i.e. \( F \) is an extension of \( f \). Therefore, \( F \) can be represented in the form (6). Further, since \( F = Q^{-1} Ph \), then \( Ph = QF, \text{i.e. } h \) is a solution to (5). Since \( h \in H_0^a(\Omega) \), then \( QF = Ph \in H_0^{-a-\nu}(\Omega) \). It follows, that \( Qu = 0 \) in \( \Omega_- \). Since \( F \in H^a(\Omega^n) \), we get \( D_n^j u = D_n^j f \) on \( \partial \Omega_- \), \( 0 \leq j \leq a - 1 \). This means that \( u \in H^a(\Omega_-) \) is a solution to the boundary value problem (4). Thus, it was proved that any solution to (3) is also a solution to the system (4), (5).

In the opposite direction, let a pair \((u, h) \in H^a(\Omega_-) \times H_0^{-a}(\Omega)\) be a solution to the system (4), (5), (6). Since \( Ph = QF \), then \( Rh = Q^{-1} Ph = F \). It follows from (6) that \( R_\Omega h = Rh|_\Omega = F|_\Omega = f \), i.e. \( h \) is a solution to (3). \( \blacksquare \)

**Remark.** If \( \mu > 0 \), the boundary value problem (4) is underdetermined because \( Q \) is an elliptic operator of order \( \nu \) and needs therefore \( \nu/2 \) boundary conditions, but we have only a (\( a < \nu/2 \)) conditions in (4). Therefore, the next step is a transformation of the equation (5) into \( \mu/2 \) extra boundary conditions to the boundary value problem (4). After some preparations it will be done in Theorem 2.

**Notation.** Let \( \Xi_{t+\lambda} \) denote a family \((\lambda \in \mathbb{R}, \ t \in \mathbb{Z})\) of order-reducing pseudodifferential operators \( \Xi_{t+\lambda} : F^{-1} \chi_+ (\xi, \lambda) F \), where \( \chi_+ (\xi, \lambda) := \left(1 + |\xi|^2 + \lambda^2\right)^{1/2} + i\xi_n \) \( t \) is the symbol. It is known that the operator \( \Xi_{t+\lambda} \) maps the space \( \mathcal{S}_0(\mathbb{R}^n) := \{ u \in \mathcal{S}(\mathbb{R}^n) : \text{supp } u \subset \mathbb{R}^n_+ \} \) onto itself and has the following isomorphism properties for \( s \in \mathbb{R} \):

\[ \Xi_{t+\lambda} : H^s(\mathbb{R}^n) \simeq H^{s-t}(\mathbb{R}^n), \quad (7) \]

\[ \Xi_{t+\lambda} : H^s_0(\mathbb{R}^n_+) \simeq H^{s-t}_0(\mathbb{R}^n_+). \quad (8) \]

Suppose that there exists a real-valued \( C^\infty \) function \( \omega \) with nowhere vanishing differential such that \( \partial \Omega = \{ x \in \mathbb{R}^n : \omega (x) = 0 \} \) and \( \Omega_- = \{ x \in \mathbb{R}^n : \omega (x) > 0 \} \). This
means that $\Omega_-$ is a particular case of a SG-compatible manifold with boundary $\partial \Omega$ (cf. [1, Sect. 1.2]). Let $\bigcup_{j=1}^J \Omega_j$ be a cover of $\mathbb{R}^n$ by finitely many coordinate charts. Let \{\varphi_1, ..., \varphi_J\} be a partition of unity and \{\psi_1, ..., \psi_J\} be a set of cut-off functions such that (i) $\text{supp} \varphi_j$, $\text{supp} \psi_j \subseteq \Omega_j$, (ii) $\varphi_j \psi_j = \varphi_j$, (iii) $D^\alpha \varphi_j(x) = O \left( \langle x \rangle^{-|\alpha|} \right)$, and (iv) $D^\alpha \psi_j(x) = O \left( \langle x \rangle^{-|\alpha|} \right)$.

Let us define a family of pseudodifferential operators depending on a parameter $\lambda \geq 0$

$$\Lambda^\lambda_+ := \sum_{j=1}^J \psi_j \Xi^\lambda_{+, \lambda} \varphi_j$$

It is known ([3], [11]) that for large enough $\lambda$, the operator $\Lambda^\lambda_+$ is an isomorphism:

$$\Lambda^\lambda_+ : H^s(\mathbb{R}^n) \simeq H^{s-t}(\mathbb{R}^n), \quad s \in \mathbb{R},$$

as well as an isomorphism:

$$\Lambda^\lambda_+ : H^s_0(\Omega) \simeq H^{s-t}_0(\Omega), \quad s \in \mathbb{R}. \quad (9)$$

**Lemma 1.** Let $P(x, D)$ be an invertible differential operator of order $\mu$, i.e. there exists the inverse operator $P^{-1}(x, D)$ which is bounded: $H^{s-\mu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Then a solution $h$ to the equation

$$P(x, D) h = g \in H_0^{-a-\mu}(\Omega)$$

belongs to the space $H_0^{-a}(\Omega)$ if and only if $g$ satisfies the following $\mu/2$ boundary conditions:

$$r_{\partial \Omega} D^j_n \Lambda^{\alpha-\mu/2}_+ P^{-1}(x, D) g = 0 \quad (j = 0, ..., \mu/2 - 1).$$

**Proof. Necessity.** Let $h = P^{-1}(x, D) g \in H_0^{-a}(\Omega)$ be a solution to $P(x, D) h = g \in H_0^{-a-\mu}(\Omega)$. By (9), we have $\Lambda^{\alpha-\mu/2}_+ h \in H_0^{\mu/2}(\Omega)$. Therefore, $r_{\partial \Omega} D^j_n \Lambda^{\alpha-\mu/2}_+ h = 0 \quad (j = 0, ..., \mu/2 - 1)$.

**Sufficiency.** Assume that the equalities $r_{\partial \Omega} D^j_n \Lambda^{\alpha-\mu/2}_+ h = 0 \quad (j = 0, ..., \mu/2 - 1)$ hold. Since $g \in H_0^{-a-\mu}(\Omega) \subset H^{-a-\mu}(\mathbb{R}^n)$, we have $h = P^{-1}(x, D) g \in H^{-a}(\mathbb{R}^n)$. Therefore, $\Psi := \Lambda^{\alpha-\mu/2}_+ h \in H^{\mu/2}(\mathbb{R}^n)$. Since $r_{\partial \Omega} D^j_n \Psi = 0 \quad (j = 0, ..., \mu/2 - 1)$, we have $\Psi = \Psi_+ + \Psi_-$, where $\Psi_+ := e_\Omega \tau^\Omega \Psi \in H_0^{\mu/2}(\Omega)$ and $\Psi_- := e_{\Omega_+} \tau_{\Omega_+} \Psi \in H_0^{\mu/2}(\Omega_-)$. Since $\Lambda^{\alpha-\mu/2}_+ : H_0^{\mu/2}(\Omega) \simeq H_0^{-a}(\Omega)$, $\Lambda^{\alpha-\mu/2}_+ \Psi_+ \in H_0^{-a}(\Omega)$. Moreover, $\Lambda^{\alpha-\mu/2}_+$ is a differential operator with respect to the variable $x_n$, hence $\text{supp} \Psi_- \subset \overline{\Omega}_-$ implies $\text{supp} \Lambda^{\alpha-\mu/2}_+ \Psi_+ \subset \overline{\Omega}_-$. Since $P$ is a differential operator,

$$\text{supp} \left( P \Lambda^{\alpha-\mu/2}_+ \right) \Psi_- \subset \text{supp} \Lambda^{\alpha-\mu/2}_+ \Psi_- \subset \overline{\Omega}_-.$$
On the other hand, we have
\[ \Phi := \left( P \Lambda_{+}^{\nu/2} \right) \Psi = \left( P \Lambda_{+}^{\nu/2} \right) (\Psi_{+} + \Psi_{-}) = \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{+} + \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-}. \]

For any \( \phi \in C_{0}^{\infty} (\Omega_{-}) \)
\[ 0 = \langle \Phi, \phi \rangle = \left( \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{+}, \phi \right) + \left( \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-}, \phi \right) = \left( \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-}, \phi \right) \]

This means \( \text{supp} \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-} \subset \overline{\Omega} \). It follows that \( \text{supp} \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-} \subset \partial \Omega \). For any \( \Psi_{-} \in C_{0}^{\infty} (\Omega_{-}) \), we have \( \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-} \in C^{\infty} (\mathbb{R}^{n}) \) and \( \text{supp} \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-} \subset \partial \Omega \).

Therefore, \( \left( P \Lambda_{+}^{\nu/2} \right) \Psi_{-} = 0 \) and, moreover, \( \Lambda_{+}^{\nu/2} \Psi_{-} = 0 \) (\( \forall \Psi_{-} \in C_{0}^{\infty} (\Omega_{-}) \)). Since \( C_{0}^{\infty} (\Omega_{-}) \) is dense in \( H_{0}^{\mu/2} (\Omega_{-}) \), one gets \( \Lambda_{+}^{\nu/2} \Psi_{-} = 0 \) (\( \forall \Psi_{-} \in H_{0}^{\mu/2} (\Omega_{-}) \)). It follows, that
\[ h = \Lambda_{+}^{\nu/2} \Psi = \Lambda_{+}^{\nu/2} \Psi_{+} + \Lambda_{+}^{\nu/2} \Psi_{-} = \Lambda_{+}^{\nu/2} \Psi_{+} = H_{0}^{-\alpha} (\Omega). \]

\[ \blacktriangle \]

**Notation.** Let \( F \in C^{\infty} (\Omega) \cap \mathcal{S} \left( \overline{\Omega}_{-} \right) \). Assume that \( F \) has finite jumps \( F_{k} \) of the normal derivative of order \( k \) \( (k = 0, 1, ...) \) on \( \partial \Omega \). For \( x' \in \partial \Omega \), we will use the following notation:
\[ F_{0} (x') := [F]_{\partial \Omega} (x') := \lim_{\varepsilon \to +0} (F (x' + \varepsilon \mathbf{n}) - F (x' - \varepsilon \mathbf{n})), \]
\[ F_{k} (x') := \left[ D_{n}^{k} F \right]_{\partial \Omega} (x'). \]

Let \( f \in C^{\infty} (\overline{\Omega}) \) and \( u \in \mathcal{S} (\overline{\Omega}_{-}) \), then we set \( \gamma_{k} f (x') := r_{\partial \Omega} D_{n}^{k} f (x') \), \( \gamma_{k} u (x') := r_{\partial \Omega} D_{n}^{k} u (x') \).

Let \( \delta_{\partial \Omega} \) denote the Dirac measure supported on \( \partial \Omega \), i.e. a distribution acting as
\[ (\delta_{\partial \Omega}, \varphi) := \int_{\partial \Omega} \varphi (x) dS, \quad \varphi (x) \in C_{0}^{\infty} (\mathbb{R}^{n}). \]

It is known that for any differential operator \( Q \) of order \( \nu \) there exists a representation \( Q = \sum_{j=0}^{\nu} Q_{j} D_{n}^{j} \), where \( Q_{j} \) is a tangential differential operator of order \( \nu - j \) (cf Appendix). We denote by \( \{ D^{\alpha} F (x) \} \) the classical derivative at the points where it exists.

**Lemma 2.** Under the previous notation the following equality holds for the distribution \( Q F \):
\[ Q F = \{ Q F \} - i \sum_{j=0}^{\nu} Q_{j} \sum_{k=0}^{j-1} D_{n}^{k} (F_{j-1-k} \delta_{\partial \Omega}). \] (10)

**Proof.** Let \( \cos (\mathbf{n} x_{j}) \) denote cosine of the angle between the exterior unit normal vector \( \mathbf{n} \) to the boundary \( \partial \Omega \) of \( \Omega \) and the \( x_{j} \)-axis.
We use the known formulas
\[
\int_{\Omega} \frac{\partial u}{\partial x_j} \, dx = \int_{\partial \Omega} u(x) \cos(n x_j) \, d\sigma \quad u(x) \in C^\infty(\overline{\Omega}), \quad j = 1, \ldots, n,
\]
and
\[
\int_{\Omega_-} \frac{\partial v}{\partial x_j} \, dx = -\int_{\partial \Omega} v(x) \cos(n x_j) \, d\sigma \quad v(x) \in C^\infty_0(\overline{\Omega_-}) \quad j = 1, \ldots, n.
\]
where \(d\sigma\) is the surface measure on \(\partial \Omega\). Applying these formulas to the products \(u(x) \varphi(x)\) and \(v(x) \varphi(x)\) where \(\varphi(x) \in C^\infty_0(\mathbb{R}^n), u(x) \in C^\infty(\overline{\Omega}), v(x) \in C^\infty_0(\overline{\Omega_-})\), we get
\[
\int_{\Omega} \frac{\partial u}{\partial x_j} \varphi(x) \, dx = -\int_{\Omega} u(x) \frac{\partial \varphi}{\partial x_j} \, dx + \int_{\partial \Omega} u(x) \varphi(x) \cos(n x_j) \, d\sigma \quad j = 1, \ldots, n,
\]
and
\[
\int_{\Omega_-} \frac{\partial v}{\partial x_j} \varphi(x) \, dx = -\int_{\Omega_-} v(x) \frac{\partial \varphi}{\partial x_j} \, dx - \int_{\partial \Omega} v(x) \varphi(x) \cos(n x_j) \, d\sigma \quad j = 1, \ldots, n.
\]
By (11), (12), we have
\[
\left( \frac{\partial F}{\partial x_j}, \varphi \right) = -\left( F, \frac{\partial \varphi}{\partial x_j} \right) = -\int_{\mathbb{R}^n} F(x) \frac{\partial \varphi}{\partial x_j} \, dx =
\]
\[
= \int_{\mathbb{R}^n} \left\{ \frac{\partial F(x)}{\partial x_j} \right\} \varphi(x) \, dx + \int_{\partial \Omega} [F]_{\partial \Omega}(x) \cos(n x_j) \varphi(x) \, dS =
\]
\[
= \left( \left\{ \frac{\partial F}{\partial x_j} \right\} + [F]_{\partial \Omega} \cos(n x_j) \varphi(x) \right), \quad \varphi(x) \in C^\infty(\mathbb{R}^n).
\]
This means,
\[
\frac{\partial F}{\partial x_j} = \left\{ \frac{\partial F}{\partial x_j} \right\} + [F]_{\partial \Omega} \cos(n x_j) \delta_{\partial \Omega}, \quad j = 1, \ldots, n.
\]
It follows, \(D_n F = \{D_n F\} - i F_0 \delta_{\partial \Omega}\). Further, using the last formula we have \(D^2_n F = D_n \{D_n F\} - i D_n (F_0 \delta_{\partial \Omega}) = \{D^2_n F\} - i F_1 \delta_{\partial \Omega} - i D_n (F_0 \delta_{\partial \Omega})\) and so on. By induction one gets:
\[
D^j_n F = \{D^j_n F\} - i \sum_{k=0}^{j-1} D^k_n (F_{j-k-1} \delta_{\partial \Omega}) \quad (j = 1, 2, \ldots).
\]
Substituting this formula for \(D^j_n F\) into the representation \(Q = \sum_{j=0}^{\nu} Q_j D^j_n\), we get (10).
Denoting by $f^0$ and $u^0$ the extensions by zero to $\mathbb{R}^n$ of functions $f(x) \in C^\infty(\overline{\Omega})$ and $u(x) \in C^\infty_0(\overline{\Omega}_-)$, and using Lemma 2, we obtain the following formulas:

\[
(Qf)^0 = Q(f^0) - i \sum_{j=1}^{\nu} \sum_{k=0}^{j-1} Q_j D_k^n ((D_n^{j-1-k}f) |_{\partial\Omega} \cdot \delta_{\partial\Omega}) \quad (f \in C^\infty(\overline{\Omega})) ,
\]

\[
(Qu)^0 = Q(u^0) + i \sum_{j=1}^{\nu} \sum_{k=0}^{j-1} Q_j D_k^n ((D_n^{j-1-k}u) |_{\partial\Omega} \cdot \delta_{\partial\Omega}) \quad (u \in C^\infty_0(\overline{\Omega}_-)) .
\]

Using these formulas we can define the action of the operator $Q$ upon the elements of the spaces $\mathcal{F}^{s,\nu}(\Omega)$ and $\mathcal{F}^{s,\nu}(\Omega_-)$ ($s \in \mathbb{R}$) (defined in Appendix) as follows (cf. [5, Sect. 1.3.2], [9, Sect. 2.4]):

\[
(Q(f, \psi))^0 := Q(f^0) - i \sum_{j=1}^{\nu} \sum_{k=0}^{j-1} D_k^n (\psi_{j-k} \cdot \delta_{\partial\Omega}) \quad ((f, \psi) \in \mathcal{F}^{s,\nu}(\Omega)) ,
\]

\[
(Q(u, \phi))^0 := Q(u^0) + i \sum_{j=1}^{\nu} \sum_{k=0}^{j-1} D_k^n (\phi_{j-k} \cdot \delta_{\partial\Omega}) , \quad ((u, \phi) \in \mathcal{F}^{s,\nu}(\Omega_-)) .
\]

It is known ([9]) that $Q$ defined respectively in (15) and (16) is a bounded mapping $Q : \mathcal{F}^{s,\nu}(\Omega) \to \mathcal{H}^{s,\nu}(\Omega)$ and $Q : \mathcal{F}^{s,\nu}(\Omega_-) \to \mathcal{H}^{s,\nu}(\Omega_-)$.

Moreover, $Q$ is respectively the closure of the mapping $f \to Q(x, D)f$ ($f \in C^\infty(\overline{\Omega})$) or $u \to Q(x, D)u$ ($u \in \mathcal{S}(\overline{\Omega}_-)$) between the corresponding spaces.

**Notation.** Let $W_{m\ell}$ ($m = 1, \ldots, \mu/2$, $\ell = a + 1, \ldots, \nu$) be the operator acting on $\partial\Omega$ as follows:

\[
W_{m\ell}(\phi) := i \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{\ell=0}^{a+1} \sum_{j=\ell}^{\nu} Q_j D_n^{-j-\ell} (\phi \cdot \delta_{\partial\Omega}) , \quad \phi \in C^\infty(\partial\Omega).
\]

where $\gamma_k$ is the restriction to $\partial\Omega$ of the $D_n^k$ (cf. Appendix).

The mapping $W_{m\ell}$ is a pseudodifferential operator of order $m - \mu + \nu/2 - 1 - \ell$ acting on $\partial\Omega$. Therefore, for any real $s$, it is a bounded operator:

$W_{m\ell} : H^s(\partial\Omega) \to H^{s-m+\mu-\nu/2+1+\ell}(\partial\Omega)$.

For $(f, \psi) \in \mathcal{F}^{a,\nu}(\Omega)$, we have $g := Q(f, \psi) \in H^0_{a,\nu}(\Omega)$, and we set

\[
w_{a+m} := -\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} g^0 \quad (m = 1, \ldots, \mu/2) ,
\]

(18)
where the operator $\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} (x, D)$ is a trace operator of order $m - 1 - a - 3\mu/2$.

It follows that $w_{a+m} \in H^{\mu/2-m+1/2} (\partial \Omega)$.

**Theorem 2.** The integral equation (3)

$$R_\Omega h = f \in H^a (\Omega), \ h \in H_0^{-a} (\Omega)$$

is equivalent to the following boundary value problem:

$$\begin{cases}
Qu = 0 & \text{in } \Omega_-

\partial \Omega, \ 0 \leq j \leq a - 1

D_n^j u = D_n^j f

\sum_{\ell=a+1}^{\mu} W_{m\ell} (\gamma_{\ell-1} u) = w_{a+m} & \text{on } \partial \Omega, \ 1 \leq m \leq \mu/2
\end{cases}$$

(19)

where the functions $u$, $f$ and $h$ are related by the formulas

$$h = P^{-1} Q F, \ F \in H^a (\mathbb{R}^n), \ F := \begin{cases}
f \in H^a (\Omega) & \text{in } \Omega, \\
u \in H^a (\Omega_-) & \text{in } \Omega_-
\end{cases}$$

**Proof.** Our starting point is Theorem 1. Consider the equation $Ph = Q F, \ h \in H_0^{-a} (\Omega)$. Since $F \in H^a (\mathbb{R}^n)$ and $Qu = 0$ in $\Omega_-$ by (4), then $Q F \in H_0^{a-\mu} (\mathbb{R}^n) = H_0^{-a-\mu} (\mathbb{R}^n)$. By Lemma 1, a solution $h$ to the equation $Ph = Q F \in H_0^{-a-\mu} (\Omega)$ belongs to the space $H_0^{-a} (\Omega)$ if and only if $Q F$ satisfies the following $\mu/2$ boundary conditions:

$$r_{\partial \Omega} D_n^{m-1} \Lambda_+^{-a-\mu/2} P^{-1} Q F = 0 \quad (m = 1, ..., \mu/2).$$

(20)

Since $F = f^0 + u^0$, $Q F = Q (f^0) + Q (u^0)$. Substituting the last expression into (20) we have

$$\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} Q (u^0) = -\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} Q (f^0) \quad (m = 1, ..., \mu/2).$$

In view of (15) and (16), one gets:

$$i \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{j=1}^{\mu} Q_j \sum_{k=0}^{j-1} D_n^k (\phi_{j-k} \cdot \delta_{\partial \Omega}) =$$

$$= \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \left(Q (f, \psi)\right)^0 + i \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{j=1}^{\mu} Q_j \sum_{k=0}^{j-1} D_n^k (\psi_{j-k} \cdot \delta_{\partial \Omega})$$

(21)

Since

$$F := \begin{cases}
f \in H^a (\Omega) & \text{in } \Omega, \\
u \in H^a (\Omega_-) & \text{in } \Omega_-
\end{cases},$$

then $\gamma_{j-1} u = \gamma_{j-1} f \ (j = 1, ..., a)$. Therefore, $\phi_j = \gamma_{j-1} u = \gamma_{j-1} f = \psi_j \ (j = 1, ..., a)$.

We identify the space $H^a (\Omega)$ with the subspace of $\mathfrak{H}_{a,(\nu)} (\Omega)$ of all $(f, \psi) = (f, \psi_1, ..., \psi_{\nu})$ such that $\psi_{\nu+1} = ... = \psi_{\nu} = 0$. Let $(f, \psi)$ belong to this subspace and $(u, \phi) = (u, \phi_1, ..., \phi_{\nu}) \in \mathfrak{H}_{a,(\nu)} (\Omega_-)$. Then we can rewrite (21) as

$$\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \left(Q (f, \psi)\right)^0 + i \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{j=1}^{\mu} Q_j \sum_{\ell=a+1}^{j} D_n^{j-\ell} (\phi_{\ell} \cdot \delta_{\partial \Omega}) = 0.$$
Changing the order of summation

\[ \sum_{j=1}^{\nu} \sum_{\ell=a+1}^{j} = \sum_{\ell=a+1}^{\nu} \sum_{j=\ell}^{\nu}, \]

we have

\[ i \gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{\ell=a+1}^{\nu} \sum_{j=\ell}^{\nu} Q_j D_n^{j-\ell} (\phi_{\ell} \cdot \delta \partial \Omega) = -\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} (Q f, \psi_0) \]

(22)

where \( m = 1, \ldots, \mu/2 \). In view of the notation (17) and (18), the equalities (22) can be rewritten in the form of \( \mu/2 \) equations

\[ \sum_{\ell=a+1}^{\nu} W_{m \ell} (\phi_{\ell}) = w_{a+m} \quad \text{on } \partial \Omega, \quad m = 1, \ldots, \mu/2. \]

Since \( \phi_j = \gamma_{j-1} u \) for \( u \in S (\overline{\Omega}_-) \), we rewrite these equalities as

\[ \sum_{\ell=a+1}^{\nu} W_{m \ell} (\gamma_{\ell-1} u) = w_{a+m} \quad \text{on } \partial \Omega, \quad m = 1, \ldots, \mu/2. \]

These equalities define \( \mu/2 \) extra boundary conditions to the boundary-value problem

\[ \begin{cases} Qu = 0 & \text{in } \Omega_-, \\ D_j u = D_n f & \text{on } \partial \Omega, \quad 0 \leq j \leq a - 1. \end{cases} \]

\[ \blacksquare \]

3 Isomorphism property

We look for a solution \( u \in H^a (\Omega_-) \) to the boundary value problem (19). Let us consider the following non-homogeneous boundary value problem associated with (19):

\[ \begin{cases} Qu = w & \text{in } \Omega_-, \\ \gamma_0 B_j u := \gamma_0 D_n^{j-1} u = w_j & \text{on } \partial \Omega, \quad 1 \leq j \leq a \\ \gamma_0 B_{a+m} u := \sum_{\ell=a+1}^{\nu} W_{m \ell} (u_{\ell-1}) = w_{a+m} & \text{on } \partial \Omega, \quad 1 \leq m \leq \mu/2 \end{cases} \]

(23)

where \( w, \ w_j \ (j = 1, \ldots, \nu/2) \) are arbitrary elements from the corresponding Sobolev spaces (see below Theorem 3 and 4)

**Assumption 1.** We assume that the differential operator \( Q \) is \( md \)-elliptic (cf. [1, p. 23]), i.e. its symbol \( q (x, \xi) \in SG^{(\nu,0)} (\mathbb{R}^n) \) is invertible for large \( |x| + |\xi| \) and

\[ [q (x, \xi)]^{-1} = O \left( \langle \xi \rangle^{-\mu} \right). \]
Note that the $md$-ellipticity differs from the usual ellipticity which says that the principal symbol $q_0 (x, \xi) \neq 0$ for any $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

For the formulation of the Shapiro-Lopatinskii condition we need some notation.

**Notation.** Let $\varepsilon > 0$ be a sufficiently small number. Denote by $U$ ($\varepsilon$-conic neighborhood) the union of all balls $B(x, \varepsilon \langle x \rangle)$, with the center $x \in \partial \Omega$ and radius $\varepsilon \langle x \rangle$. Let $y = (y', y_n) = (y_1, \ldots, y_{n-1}, y_n)$ be normal coordinates in an $\varepsilon$-conic neighborhood $U$ of $\partial \Omega$, i.e., $\partial \Omega$ may be identified with $\{y_n = 0\}$, $y_n$ is the normal coordinate, and the normal derivative $D_n$ is $D_{y_n}$ near $\partial \Omega$. Each differential operator on $\mathbb{R}^n$ with $SG$-symbol can be written in $U$ as a differential operator with respect to $D_{y'}$ and $D_{y_n}$ within the $SG$-calculus [1, p. 40], i.e.

$$Q = \sum_{j=0}^\nu Q_j (y, D_{y'}) D_{y_n}^j$$

where $Q_j (y, D_{y'})$ are differential operators with symbols belonging to $SG^{(\nu, 0)} (\mathbb{R}^n)$. Let

$$q(y, \xi) = q(y, \xi', \xi_n) = \sum_{j=0}^\nu q_j (y, \xi') \xi_n^j$$

be the symbol of $Q$, where $\xi'$ and $\xi_n$ are cotangent variables associated with $y'$ and $y_n$.

**Assumption 2.** We assume that the operator $Q$ is $md$-properly elliptic (cf. [1, Assumption 1, p. 40]), i.e. for all large $|y| + |\xi'|$ the polynomial $q(y, \xi', z)$ in the complex variable $z$ has exactly $\nu/2$ zeroes with positive imaginary part $\tau_1 (y', \xi'), \ldots, \tau_{\nu/2} (y', \xi')$.

We conclude from Assumptions 1 and 2 that the polynomial $q(y, \xi', z)$ has no real zeroes and it has exactly $\nu/2$ zeroes with negative imaginary part for all large $|y| + |\xi'|$.

In particular, the Laplacian $\Delta$ in the space $\mathbb{R}^n (n \geq 2)$ is elliptic in the usual sense but not $md$-elliptic, while the operator $I - \Delta$ is $md$-elliptic as well as $md$-properly elliptic.

Let

$$\chi (y', \xi') := \left( 1 + \sum_{i,j=1}^{n-1} \xi_i (g(y)^{-1})_{ij} \xi_j \right)^{1/2}$$

where $g = (g_{ij})$ is a Riemannian metric on $\partial \Omega$. We denote

$$q^+ (y', \xi') := \prod_{j=1}^{\nu/2} \left( z - \chi (y', \xi')^{-1} \tau_j (y', \xi') \right).$$

Consider the operators $B_m$ ($m = 1, \ldots, \nu/2$) from (23). Each of them is of the form

$$B_m = \sum_{j=0}^{\nu-1} B_{mj} (y', D_{y'}) D_{y_n}^j$$

in the normal coordinates $y = (y', y_n) = (y_1, \ldots, y_{n-1}, y_n)$ in an $\varepsilon$-conic neighborhood of $\partial \Omega$. Here $B_{mj} (y', D_{y'})$ is a pseudodifferential operator of order $\rho_m - j$ ($\rho_m \in \mathbb{N}$) acting
on $\partial \Omega$. Let $b_{mj} (y', \xi')$ denote the principal symbol of $B_{mj} (y', D_y')$. Let us note that the operators $B_m$ from the boundary value problem (23) are operators of this type. Scaling down the coefficient operators $B_{mj}$ to order zero, we set

$$b_m (y', \xi', z) := \sum_{j=0}^{\nu-1} b_{mj} (y', \xi') \chi (y', \xi')^{-\rho_m + j} z^j$$

Define polynomials in $z$

$$r_m (y', \xi', z) = \sum_{j=0}^{\nu/2} r_{mj} (y', \xi') z^j$$

as the residues of $b_m (y', \xi', z)$ modulo $q^+ (y', \xi')$.

**Assumption 3.** (Shapiro-Lopatinskii condition) The determinant

$$\det \left( (r_{mj} (y', \xi'))_{m,j} \right)$$

is bounded and bounded away from zero, i.e. there exist two positive constants $c$ and $C$ such that

$$0 < c \leq \det \left( (r_{mj} (y', \xi'))_{m,j} \right) \leq C.$$

The following Theorem 3 has been proved by K. Erkip and E. Schrohe [1, Th. 3.1] in essentially more general case of the SG-manifold. The latter includes the exterior of bounded domains which is the easiest case of the SG-manifolds. This case was chosen here just for a simplicity of the exposition.

**Theorem 3.** (a particular case of [1, Th. 3.1], [11]). If the differential operator $Q$ of even order $\nu$ satisfies Assumptions 1, 2 and 3, i.e. $Q$ is $md$-elliptic, $md$-properly elliptic and such that the Shapiro-Lopatinskii condition holds for the operator $(Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2})$, then the mapping

$$(Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2}) : H^s (\Omega_-) \to H^{s-\nu} (\Omega_-) \times \prod_{j=1}^{\nu/2} H^{s-\rho_j - 1/2} (\partial \Omega) \quad (s \geq \nu)$$

is a Fredholm operator.

The following Theorem 4 is an extension of Theorem 3 to the case of the Sobolev spaces of negative order.

**Theorem 4.** Under conditions of Theorem 3, the mapping $(Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2})$ can be extended to a Fredholm operator

$$(Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2}) : \mathcal{H}^{s, \nu} (\Omega_-) \to \mathcal{H}^{s-\nu} (\Omega_-) \times \prod_{j=1}^{\nu/2} H^{s-\rho_j - 1/2} (\partial \Omega) \quad (s \in \mathbb{R}).$$

The proof can be obtained just as it was done in [4] for the case of a bounded domain. AK hopes to publish the proof separately for a more general case.
Assumption 4. Suppose that the Fredholm operator \((Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2})\) has the trivial kernel and cokernel, i.e. it is an invertible operator.

For example, if the kernel \(R(x,y)\) has the property \((Rh,h) \geq c||h||^2_{H_0^{-a}}\) for all \(h \in H_0^{-a}\), where \(c = \text{const} > 0\) does not depend on \(h\), then the operator in Assumption 4 is invertible (see [6]).

Theorem 5. Under conditions of Theorem 3 and in addition under Assumption 4, the mapping \(R_\Omega\) which is defined in Introduction is an isomorphism: \(H_0^{-a}(\Omega) \rightarrow H^a(\Omega)\).

Proof. Let us consider the operator \((Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2})\) generated by the boundary value problem (23). Taking into account that \(\rho_j = \text{order } B_j = j - 1\) \((j = 1, ..., a)\) and \(\rho_j = \text{order } B_j = j - \mu + \nu/2 - 2\) \((j = a + 1, ..., \nu/2)\), then the operator is as follows:

\[
(u, \phi) \mapsto (Q(u, \phi), \gamma_0 B_1(u, \phi), ..., \gamma_0 B_{\nu/2}(u, \phi)) = (w, w_1, ..., w_{\nu/2})
\]

It maps the space \(\mathcal{H}^{s,\nu}(\Omega_-)\) to the space

\[
\mathcal{H}^{s-\nu}(\Omega_-) \times \prod_{j=1}^{a} H^{s-j+1/2}(\partial \Omega) \times \prod_{j=a+1}^{\nu/2} H^{s-j+\mu-\nu/2+3/2}(\partial \Omega) \quad (s \in \mathbb{R}).
\]

By Theorem 4, the mapping is a Fredholm operator and, in view of Assumption 4, it is an isomorphism. Considering the isomorphism for \(s = a\) we obtain, in view of Theorem 2, that the operator \(R_\Omega\) is an isomorphism of the space \(H_0^{-a}(\Omega)\) onto \(H^a(\Omega)\).

Example. Let \(P = I\) be the identity operator (its order \(\mu = 0\)) and \(Q = I - \Delta\) \((\nu = \text{ord } Q = 2)\). Then, by Theorem 4, the corresponding operator \(R_\Omega\) is an isomorphism: \(H_0^{-1}(\Omega) \rightarrow H^1(\Omega)\).

Solution of the integral equation (3). Under conditions of Theorem 5 there exists a unique solution to the integral equation (3). The question is how to find this solution.

Examples of analytic formulas for the solution to the integral equation (3) can be found in [6]. The analytical formulas for the solution can be obtained only for domains \(\Omega\) of special shape, for example, when \(\Omega\) is a ball, and for special operators \(Q\) and \(P\), for example, for operators with constant coefficients.

We give such a formula for the solution of equation (3) assuming \(P = I\) and \(Q = -\Delta + a^2 I\). Consider the equation

\[
\int_{\Omega} \frac{\exp(-a|x-y|)}{4\pi|x-y|} h(y)dy = f(x), \quad x \in \overline{\Omega} \subset \mathbb{R}^3, \quad a > 0,
\]

(24)

with kernel \(R(x,y) := \frac{\exp(-a|x-y|)}{4\pi|x-y|}\), \(P = I\), and \(Q = -\Delta + a^2 I\). By Theorem 1, one obtains the unique solution to equation (24) in \(H_0^{-1}(\Omega)\):

\[
h(x) = (-\Delta + a^2)f + \left(\frac{\partial f}{\partial n} - \frac{\partial u}{\partial n}\right) \delta_{\partial \Omega}
\]

(25)
where \( u \) is the unique solution to the Dirichlet problem in the exterior domain \( \Omega_\epsilon \)
\[
(-\Delta + a^2)u = 0 \quad \text{in} \quad \Omega_\epsilon, \quad u|_{\partial \Omega} = f|_{\partial \Omega}, \quad u(\infty) = 0,
\]
\( \partial \Omega \) is the boundary of \( \Omega \), and \( \delta_{\partial \Omega} \) is the delta function with support \( \partial \Omega \).

For any \( \phi \in C_0^\infty(\mathbb{R}^n) \) one has:
\[
((-\Delta + a^2)R, \phi) = (R, (-\Delta + a^2)\phi) = \int_\Omega f(-\Delta + a^2)\phi dx + \int_{\Omega_\epsilon} u(-\Delta + a^2)\phi dx
\]
\[
= \int_\Omega (-\Delta + a^2)f \phi dx + \int_{\Omega_\epsilon} (-\Delta + a^2)u \phi dx - \int_{\partial \Omega} \left( f \frac{\partial \phi}{\partial n} - u \phi \right) ds + \int_{\partial \Omega} \left( \frac{\partial f}{\partial n} - \frac{\partial u}{\partial n} \right) \phi ds,
\]
where the condition \( u = f \) on \( \partial \Omega \) was used. Thus, we have checked that formula (25) gives the (unique in \( H_0^{-1}(\Omega) \)) solution to equation (24).

4 Appendix

Notation. We denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{C} \) the set of complex numbers. Let \( \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \), \( \mathbb{N} := \{0, 1, \ldots\} \), \( \mathbb{N}_+ := \{1, 2, \ldots\} \), \( \mathbb{R}^n := \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, \ i = 1, \ldots, n\} \).

Let \( \alpha \) be a multi-index, i.e. \( \alpha := (\alpha_1, \ldots, \alpha_n) \), \( \alpha_j \in \mathbb{N} \), \( |\alpha| := \alpha_1 + \cdots + \alpha_n \), \( i := \sqrt{-1} \); \( D_j := i^{-1} \partial/\partial x_j; D^\alpha := D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n} \).

Let \( C^\infty(\overline{\Omega}) \) be the space of infinitely differentiable up to the boundary functions in \( \overline{\Omega} \). Near \( \partial \Omega \) there is defined \( A \) normal vector field \( \mathbf{n}(x) = (n_1(x), \ldots, n_n(x)) \), is defined in a neighborhood of the boundary \( \partial \Omega \) as follows: for \( x_0 \in \partial \Omega \), \( \mathbf{n}(x_0) \) is the unit normal to \( \partial \Omega \), pointing into the exterior of \( \Omega \). We set
\[
\mathbf{n}(x) := \mathbf{n}(x_0) \quad \text{for} \quad x \quad \text{of the form} \quad x = x_0 + s\mathbf{n}(x_0) =: \zeta(x_0, s)
\]
where \( x_0 \in \partial \Omega \), \( s \in (-\delta, \delta) \). Here \( \delta > 0 \) is taken so small that the representation of \( x \) in terms of \( x_0 \in \partial \Omega \) and \( s \in (-\delta, \delta) \) is unique and smooth, i.e., \( \zeta \) is bijective and \( C^\infty \) with \( C^\infty \) inverse, from \( \partial \Omega \times (-\delta, \delta) \) to the set \( \zeta(\partial \Omega \times (-\delta, \delta)) \subset \mathbb{R}^n \).

We call differential operators tangential when, for \( x \in \zeta(\partial \Omega \times (-\delta, \delta)) \), they are either of the form
\[
Af = \sum_{j=1}^n a_j(x) \frac{\partial f}{\partial x_j}(x) + a_0(x) f \quad \text{with} \quad \sum_{j=1}^n a_j(x) n_j(x) = 0,
\]
or they are products of such operators. The derivative along \( \mathbf{n} \) is denoted \( \partial_{\mathbf{n}} : \)
\[
\partial_{\mathbf{n}} f := \sum_{j=1}^n n_j(x) \frac{\partial f}{\partial x_j}(x)
\]
for \( x \in \zeta (\partial \Omega \times (-\delta, \delta)) \). Let \( D_n := i^{-1} \partial_n \).

Let \( \Omega_- := \mathbb{R}^n \setminus \overline{\Omega} \) denote the exterior of the domain \( \Omega \), \( r_{\partial \Omega}, r_\Omega \) be respectively the restriction operators to \( \partial \Omega \), \( \Omega : r_{\partial \Omega} f := f|_{\partial \Omega} \), \( r_\Omega f := f|_{\Omega} \).

Let \( S(\mathbb{R}^n) \) be the space of rapidly decreasing functions, that is the space of all \( u \in C^\infty (\mathbb{R}^n) \) such that

\[
\sup \sup_{|\alpha| \leq k} \left| (1 + |x|^2)^m D^\alpha u(x) \right| < \infty \quad \text{for all } k, m \in \mathbb{N}.
\]

Let \( S(\overline{\Omega}_-) \) be the space of restrictions of the elements \( u \in S(\mathbb{R}^n) \) to \( \overline{\Omega}_- \) (this space is equipped with the factor topology).

Let \( u \in C^\infty (\overline{\Omega}) \) and \( v \in S(\overline{\Omega}_-) \), then we set \( \gamma_k u := r_{\partial \Omega} D_n^k u = (D_n^k u)|_{\partial \Omega} \), \( \gamma_k v := r_{\partial \Omega} D_n^k v = (D_n^k v)|_{\partial \Omega} \).

**Definition of the Sobolev spaces.** Let \( H^s(\mathbb{R}^n) \) \((s \in \mathbb{R})\) be the usual Sobolev space:

\[
H^s(\mathbb{R}^n) := \left\{ f \in S' \mid F^{-1}(1 + |\xi|^2)^{s/2} F f \in L_2(\mathbb{R}^n) \right\},
\]

\[
\|f\|_{H^s(\mathbb{R}^n)} := \left\| F^{-1}(1 + |\xi|^2)^{s/2} F f \right\|_{L_2(\mathbb{R}^n)},
\]

where \( F \) denotes the Fourier transform \( f \mapsto F_{x \to \xi} f(x) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx \), \( F^{-1} \) its inverse and \( S' = S' (\mathbb{R}^n) \) denote the space of tempered distributions which is dual to the space \( S(\mathbb{R}^n) \).

Let \( H^s(\Omega) \) and \( H^s(\Omega_-) \) \((0 \leq s \in \mathbb{R})\) be respectively the spaces of restrictions of elements of \( H^s(\mathbb{R}^n) \) to \( \Omega \) and \( \Omega_- \). The norms in the spaces \( H^s(\Omega) \) and \( H^s(\Omega_-) \) are defined by the relations

\[
\|f\|_{H^s(\Omega)} := \inf \|g\|_{H^s(\mathbb{R}^n)} \quad (s \geq 0),
\]

\[
\|f\|_{H^s(\Omega_-)} := \inf \|g\|_{H^s(\mathbb{R}^n)} \quad (s \geq 0),
\]

where infimum is taken over all elements \( g \in H^s(\mathbb{R}^n) \) which are equal to \( f \) in \( \Omega \) respectively in \( \Omega_- \).

By \( H^s_0(\Omega) \) \((s \in \mathbb{R})\) and \( H^s_0(\Omega_-) \), we denote the closed subspaces of the space \( H^s(\mathbb{R}^n) \) which consist of the elements with supports respectively in \( \overline{\Omega} \) or in \( \overline{\Omega}_- \) i.e.

\[
H^s_0(\Omega) := \{ f \in H^s(\mathbb{R}^n) : \text{supp } f \subseteq \overline{\Omega} \} \subset H^s(\mathbb{R}^n) \quad (s \in \mathbb{R}),
\]

\[
H^s_0(\Omega_-) := \{ f \in H^s(\mathbb{R}^n) : \text{supp } f \subseteq \overline{\Omega}_- \} \subset H^s(\mathbb{R}^n) \quad (s \in \mathbb{R}).
\]

We define the spaces

\[
\mathcal{H}^s(\Omega) := \left\{ \begin{array}{ll}
H^s(\Omega) & s > 0 \\
H^s_0(\Omega) & s \leq 0,
\end{array} \right.
\]

\[
\mathcal{H}^s(\Omega_-) := \left\{ \begin{array}{ll}
H^s(\Omega_-) & s > 0, \\
H^s_0(\Omega_-) & s \leq 0.
\end{array} \right.
\]
For $s \neq k + 1/2$ ($k = 0, 1, \ldots, \ell - 1$), we define the spaces $\mathcal{Y}^{s,\ell}(\Omega)$ and $\mathcal{Y}^{s,\ell}(\Omega_-)$ respectively as the sets of all

$$(u, \phi) = (u, \phi_1, \ldots, \phi_{\ell}) \text{ and } (v, \psi) = (v, \psi_1, \ldots, \psi_{\ell})$$

where $u \in \mathcal{H}^s(\Omega)$, $v \in \mathcal{H}^s(\Omega_-)$, $\phi = (\phi_1, \ldots, \phi_{\ell})$ and $\psi = (\psi_1, \ldots, \psi_{\ell})$ are vectors in $\prod_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega)$ satisfying the condition

$$\phi_j = D_{n}^{j-1}u|_{\partial\Omega}, \quad \psi_j = D_{n}^{j-1}v|_{\partial\Omega} \quad \text{for } j < \min(s, \ell).$$

The norms in $\mathcal{Y}^{s,\ell}(\Omega)$ and $\mathcal{Y}^{s,\ell}(\Omega_-)$ can be defined as

$$\| (u, \phi) \|^2_{\mathcal{Y}^{s,\ell}(\Omega)} = \| u \|^2_{\mathcal{H}^s(\Omega)} + \sum_{j=1}^{\ell} \| \phi_j \|^2_{H^{s-j+1/2}(\partial\Omega)};$$

$$\| (v, \psi) \|^2_{\mathcal{Y}^{s,\ell}(\Omega_-)} = \| v \|^2_{\mathcal{H}^s(\Omega_-)} + \sum_{j=0}^{\ell} \| \psi_j \|^2_{H^{s-j+1/2}(\partial\Omega)}.$$ 

Since only the components $\phi_j$ and $\psi_j$ with index $j < s$ can be chosen independently of $u$, we can identify $\mathcal{Y}^{s,\ell}(\Omega)$ and $\mathcal{Y}^{s,\ell}(\Omega_-)$ with the following spaces.

For $s, s_1 \neq k + 1/2$ ($k = 0, 1, \ldots, \ell - 1$),

$$\mathcal{Y}^{s,\ell}(\Omega) = \begin{cases} \mathcal{H}^s(\Omega), & \ell = 0, \\
\mathcal{H}^s(\Omega), & 1 \leq \ell < s + 1/2, \\
\mathcal{H}^s(\Omega) \times \prod_{j=[s+1/2]+1}^{\ell} H^{s-j+1/2}(\partial\Omega), & 0 < [s + \frac{1}{2}] < \ell, \\
\mathcal{H}^s(\Omega) \times \prod_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega), & s < \frac{1}{2} \end{cases}$$

$$\mathcal{Y}^{s,\ell}(\Omega_-) = \begin{cases} \mathcal{H}^s(\Omega_-), & \ell = 0, \\
\mathcal{H}^s(\Omega_-), & 1 \leq \ell < s + 1/2, \\
\mathcal{H}^s(\Omega_-) \times \prod_{j=[s+1/2]+1}^{\ell} H^{s-j+1/2}(\partial\Omega), & 0 < [s + \frac{1}{2}] < \ell, \\
\mathcal{H}^s(\Omega_-) \times \prod_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega), & s < \frac{1}{2} \end{cases}$$

Finally, for $s = k + 1/2$ ($k = 0, 1, \ldots, \ell - 1$), we define the spaces $\mathcal{Y}^{s,\ell}(\Omega)$, $\mathcal{Y}^{s,\ell}(\Omega_-)$ by the method of complex interpolation.

Let us note that for $s \neq k + 1/2$ ($k = 0, 1, \ldots, \ell - 1$), the spaces $\mathcal{Y}^{s,\ell}(\Omega)$, $\mathcal{Y}^{s,\ell}(\Omega_-)$ are completion of $C^\infty(\overline{\Omega})$, $\mathcal{S}(\overline{\Omega_-})$ respectively in the norms

$$\| (u, \gamma_0 u, \ldots, \gamma_{\ell-1} u) \|^2_{\mathcal{Y}^{s,\ell}(\Omega)} = \| u \|^2_{\mathcal{H}^s(\Omega)} + \sum_{j=0}^{\ell-1} \| \gamma_j u \|^2_{H^{s-j+1/2}(\partial\Omega)},$$

16
\[ \|v, \gamma_0 v, \ldots, \gamma_{\ell-1} v\|^2_{H^s(\Omega)} = \|v\|^2_{H^s(\Omega)} + \sum_{j=0}^{\ell-1} \|\gamma_j v\|^2_{H^{s-j-1/2}(\partial \Omega)}. \]

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