GALOIS COHOMOLOGY OF SIMPLY CONNECTED REAL GROUPS
AND LABELINGS OF DYNKIN DIAGRAMS

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ABSTRACT. For each simply connected absolutely simple linear algebraic group $G$ over the field of real numbers $\mathbb{R}$, we compute the first Galois cohomology set $H^1(\mathbb{R}, G)$. The cardinalities of these sets $H^1$ were earlier computed by Jeffrey Adams. We give a functorial description of $H^1(\mathbb{R}, G)$ in terms of labelings of the Dynkin diagram of $G$, which permits to compute the set of orbits of $G(\mathbb{R})$ in $(G/H)(\mathbb{R})$, where $H$ is a simply connected $\mathbb{R}$-subgroup of $G$.

0. Introduction

Let $G$ be a linear algebraic group defined over the field of real numbers $\mathbb{R}$. We denote by $G(\mathbb{C})$ the set of $\mathbb{C}$-points of $G$, and by $G(\mathbb{R})$ the set of $\mathbb{R}$-points of $G$. The first Galois cohomology set $H^1(\mathbb{R}, G)$ is, by definition, $Z^1(\mathbb{R}, G)/\sim$, where the set of 1-cocycles is defined by $Z^1(\mathbb{R}, G) = \{z \in G(\mathbb{C}) \mid z\overline{z} = 1\}$, and two cocycles $z, z' \in Z^1(\mathbb{R}, G)$ are cohomologous (we write $z \sim z'$) if $z' = g\overline{z}g^{-1}$ for some $g \in G(\mathbb{C})$. Here the bar denotes the complex conjugation in $G(\mathbb{C})$.

In terms of Galois cohomology one can state answers to many natural questions, see e.g. Berhuy [Be]. In particular, if $X$ is a homogeneous space of $G$ defined over $\mathbb{R}$ and containing an $\mathbb{R}$-point $x \in X(\mathbb{R})$, and $H$ denotes the stabilizer of $x$ in $G$, then the set of orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ is in a canonical bijection with the kernel

$$
\ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)];
$$

see Serre [S] Section I.5.4, Corollary 1 of Prop. 36], or Berhuy [Be] Corollary II.4.5.

By a reductive (or semisimple) group we always mean a connected reductive (or semisimple) group. The Galois cohomology of classical groups and adjoint groups is well known. The Galois cohomology of compact groups was computed by Borel and Serre [BS], Theorem 6.8, see also Serre [S, Section III.4.5]. Namely, if $G$ is a connected compact (anisotropic) reductive $\mathbb{R}$-group and $T \subset G$ is a maximal torus, then $H^1(\mathbb{R}, G)$ is in a canonical bijection with $W\backslash T(\mathbb{R})_2$, where $T(\mathbb{R})_2$ is the subgroup of elements of order dividing 2 in $T(\mathbb{R})$, and $W = W(G_{\mathbb{C}}, T_{\mathbb{C}})$ is the Weyl group. See Serre [S, Section III.4.5, Example (a)]. This result was generalized in [Bo1] to arbitrary (not necessarily compact) reductive $\mathbb{R}$-groups; see also [Bo2] and Section 2 below.

In this paper we consider simply connected groups. Adams [A] computed, using in particular [Bo1], a table of the cardinalities of the Galois cohomology sets of simply connected simple real groups. In the present paper we give a functorial combinatorial description of the finite pointed sets $H^1(\mathbb{R}, G)$ for simply connected simple real groups $G$ in terms of

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labelings of Dynkin diagrams. This description permits to compute the kernel in formula (1). Examples of calculation of this kernel are given in the last section.

The description of $H^1(\mathbb{R}, G)$ is given in terms of equivalent classes of labelings of a Dynkin diagram $D$ related to the Dynkin diagram $D(G_C)$ of $G$. A labeling is a family $\mathbf{a} = (a_i)_{i=1, \ldots, n}$ of labels $a_i \in \mathbb{Z}/2\mathbb{Z}$ (i.e., $a_i = 0, 1$) for each vertex $i$ of $D$ (labelings correspond to points of order dividing 2 for a suitable compact torus of $G$). We also define elementary transformations $T_i$ of labelings (the elementary transformation $T_i$ corresponds to the reflection associated with the vertex $i$ of $D$). We say that two labelings $\mathbf{a}, \mathbf{a}' \in (\mathbb{Z}/2\mathbb{Z})^n$ are equivalent if they can be related by a sequence of elementary transformations.

Then $H^1(\mathbb{R}, G)$ is in canonical bijection with the set of equivalence classes of labelings with respect to the elementary transformations.

We describe the diagram $D$, the labelings of which we classify. If $G$ is an inner form of a compact group, then $D = D(G_C)$ is the Dynkin diagram of $G_C$. If $G$ is an outer form of a compact group, then the complex conjugation acts on $D(G_C)$ via an automorphism $\nu$ of order 2, and in this case $D = (D(G_C))^\nu$, the subdiagram of fixed points of $\nu$ (from the classification of Dynkin diagrams we know that in this case $D$ is always of type $A_l$ for some $l$).

The rest of the paper is structured as follows. In Section 1 we state well known results on Galois cohomology of real tori. In Section 2 we recall results of [Bo1]. In Sections 3 and 4 we compute the elementary transformations $T_i$ in the case when $G$ is compact, when it is a noncompact inner form of a compact group, and when it is an outer form of a compact group, respectively. In Sections 6 - 15 we describe the equivalence classes of labelings for the cases $A_1 - G_2$. In Section 16 for the reader’s convenience we give summary tables of the cardinalities of $H^1(\mathbb{R}, G)$ (which were earlier computed by Adams [A]). In the last Section 17 we give examples of computations of orbits of real algebraic groups in homogeneous spaces, using our results.

1. Galois cohomology of real tori

In this section we state some extremely well known results on Galois cohomology of real tori. For an algebraic torus $T$ over $\mathbb{R}$, we denote by $X(T)$ its character group

$$X(T) := \text{Hom}(T, \mathbb{G}_m, \mathbb{C}).$$

Any $\mathbb{R}$-torus (an algebraic torus over $\mathbb{R}$) is a direct product of the 1-dimensional and 2-dimensional tori $S$ of the following three types:

1. $S$ is the one-dimensional split $\mathbb{R}$-torus, $X(S) = \mathbb{Z}$, $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $X(S)$ trivially,

$$S = \mathbb{G}_m, \mathbb{R}, \quad (\mathbb{G}_m, \mathbb{R})(\mathbb{R}) = \mathbb{R}^\times, \quad H^1(\mathbb{R}, \mathbb{G}_m, \mathbb{R}) = 1$$

by Hilbert’s Theorem 90.

2. $X(S) = \mathbb{Z}^2$, the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$ permutes the basis vectors of $X(S) = \mathbb{Z}^2$,

$$S = R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C}, \quad (R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C})(\mathbb{R}) = \mathbb{C}^\times, \quad H^1(\mathbb{R}, R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C}) = 1$$

by Shapiro’s lemma. Here $R_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction of scalars, see e.g. [Vo] Subsection 3.12.

3. $S$ is the one-dimensional compact (anisotropic) $\mathbb{R}$-torus, $X(S) = \mathbb{Z}$, the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $X(S)$ by multiplication by $-1$,

$$S = R_{\mathbb{C}/\mathbb{R}}^{(1)} \mathbb{G}_m, \mathbb{C} := \ker[\text{Nm}_{\mathbb{C}/\mathbb{R}} : R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C} \to \mathbb{G}_m, \mathbb{R}],$$

$$R_{\mathbb{C}/\mathbb{R}}^{(1)} \mathbb{G}_m, \mathbb{C}(\mathbb{R}) = \ker[\text{Nm}_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \to \mathbb{R}^\times],$$

$$H^1(\mathbb{R}, R_{\mathbb{C}/\mathbb{R}}^{(1)} \mathbb{G}_m, \mathbb{C}) = \mathbb{R}^\times / \text{Nm}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = S(\mathbb{R})_2,$$
where $\text{Nm}_{\mathbb{C}/\mathbb{R}}$ denotes the norm map.

2. GALOIS COHOMOLOGY OF REDUCTIVE REAL GROUPS

In this section we state briefly the results of [Bo1]. For details see [Bo1] or [Bo2].

Let $G$ be a connected reductive group over $\mathbb{R}$. Let $T$ be a fundamental maximal torus of $G$, i.e., a maximal torus of $G$ containing a maximal compact torus $T_0$ of $G$. Then $T$ is the centralizer of $T_0$ in $G$; see [Bo2] Section 7]. Let $T_1$ be the largest split torus of $T$. We write $T(\mathbb{R})_2$ for the group of elements of $T(\mathbb{R})$ of order dividing 2. By [Bo1] Lemma 1.1, see also [Bo2] Lemma 3, the map $T(\mathbb{R})_2 \to H^1(\mathbb{R}, T)$ induces a canonical isomorphism $T(\mathbb{R})_2/T(\mathbb{R})_0 \cong H^1(\mathbb{R}, T)$.

We define a left action of the group $W_0(\mathbb{R})$ on the set $H^1(\mathbb{R}, T)$. Let $w \in W_0(\mathbb{R})$ be represented by $n \in N_0(\mathbb{C})$ and let $\xi \in H^1(\mathbb{R}, T)$, $\xi = [z]$, where $z \in Z^1(\mathbb{R}, T)$ is a cocycle and $[z]$ denotes the cohomology class of $z$. We set $w \ast \xi = [nzn^{-1}] = [nzn^{-1} \cdot n\bar{n}^{-1}]$,

where the bar denotes the complex conjugation in $G(\mathbb{C})$. This is a well-defined action; see [Bo2] Construction 8]. (Note that in general the action $\ast$ does not respect the group structure on $H^1(\mathbb{R}, T)$.) It is easy to see that the images of $\xi$ and $w \ast \xi$ in $H^1(\mathbb{R}, G)$ coincide. Therefore, we obtain a canonical map

$$W_0(\mathbb{R}) \backslash H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G).$$

Proposition 2.1 ([Bo1] Theorem 1], see also [Bo2] Theorem 9). The map

$$W_0(\mathbb{R}) \backslash H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

3. Weyl action for compact groups

We change our notation. From now on, $G$ is a simply connected, simple, compact (anisotropic) linear algebraic group over $\mathbb{R}$.

Let $T$ be a maximal torus of $G$, $R = R(G_{\mathbb{C}}, T_{\mathbb{C}})$ the root system, $\Pi \subset R$ a basis (a system of simple roots), $W = W(G, T) = N/T$ be the Weyl group, where $N$ is the normalizer of $T$ in $G$.

Let $X_\ast = \text{Hom}(G_{m, \mathbb{C}}, T_{\mathbb{C}})$ denote the cocharacter group of $T$. Write $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, then we have a canonical basis $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ of $X_\ast$. We are interested in the action of $W$ in $T(\mathbb{R})_2$. We identify $T(\mathbb{R})_2$ with $X_\ast/2X_\ast$. The canonical $\mathbb{Z}$-basis $\alpha_1^\vee, \ldots, \alpha_n^\vee$ of $X_\ast$ gives a $\mathbb{Z}/2\mathbb{Z}$-basis of $X_\ast/2X_\ast$, which we shall again write as $\alpha_1^\vee, \ldots, \alpha_n^\vee$.

We write $z \in T(\mathbb{R})_2$ as

$$z = \prod_{i=1}^n (\alpha_i^\vee(-1))^{a_i}, \text{ where } a = (a_i) \in (\mathbb{Z}/2\mathbb{Z})^n.$$

With $z$ we associate $y = \sum_{i} a_i \alpha_i^\vee \in X_\ast/2X_\ast$. Furthermore, we associate with $z$ a labeling $a = (a_i)$ of the Dynkin diagram $D$ of $(G, T, \Pi)$: at any vertex $i$ of $D$ we write the label $a_i \in \mathbb{Z}/2\mathbb{Z}$, $a_i = 0, 1.$
We wish to compute the orbits of $W$ in $T(\mathbb{R})_2$ with respect to the canonical left action. The Weyl group $W$ is generated by the reflections $r_i = r_{\alpha_i}$. We define the induced elementary transformations $T_i: (\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$ of the set of labelings by (2)

$$T_i a = a', \text{ where } r_i \left( \prod_{i=1}^{n} (\alpha_i^{\vee}(-1))^{a_i} \right) = \prod_{i=1}^{n} (\alpha_i^{\vee}(-1))^{a'_i} \text{ i.e., } r_i \left( \sum_{j=1}^{n} a_j \alpha_j^{\vee} \right) = \sum_{j=1}^{n} a'_j \alpha_j^{\vee}. $$

We say that two labelings $a, a' \in (\mathbb{Z}/2\mathbb{Z})^n$ are equivalent if we can relate them by a series of elementary transformations. The set of orbits of $W$ in $T(\mathbb{R})_2$ is in a canonical bijection with the set of equivalence classes of labelings of the Dynkin diagram $D$ of $(G, T, \Pi)$ with respect to the elementary transformations.

The following Lemma 3.1 gives a simple combinatorial description of the elementary transformations.

**Lemma 3.1.** Let $G$ be a simply connected simple compact $\mathbb{R}$-group of absolute rank $n$, and $D$ its Dynkin diagram, as above. Define the elementary transformations $T_i: (\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$ by (2). Then we have $a'_j = a_j$ for $j \neq i$, and $a'_i$ is given by the following formulas:

(i) if $D$ is a simply laced Dynkin diagram or $D = G_2$, then

$$a'_i = a_i + \sum_{k \text{ adj.}} a_k, $$

with the sum over all the vertices $k$ adjacent to $i$;

(ii) if $D$ is a Dynkin diagram with a double edge, then when $\alpha_i$ is a short root, $a'_i$ is given by the same formula (3), while when $\alpha_i$ is a long root, the sum in the formula for $a'_i$ is taken over the neighbors $k$ of $i$ with long roots $\alpha_k$ only:

$$a'_i = a_i + \sum_{k \text{ adj. long}} a_k. $$

In both formulas the addition is done modulo 2.

**Proof.** A reflection $r_i$ acts on $X_*$ by

$$r_i(y) = y - \langle \alpha_i, y \rangle \alpha_i^{\vee}. $$

If $y = \sum_k a_k \alpha_k^{\vee} \in X_*$, then

$$r_i(y) = y - \sum_k a_k \langle \alpha_i, \alpha_k^{\vee} \rangle \alpha_i^{\vee}, $$

and the same formula holds if $y = \sum_k a_k \alpha_k^{\vee} \in X_*/2X_*$. If we write $r_i(y) = \sum_k a'_k \alpha_k^{\vee}$, then clearly $a'_j = a_j$ for $j \neq i$, and

$$a'_i = a_i - \sum_k a_k \langle \alpha_i, \alpha_k^{\vee} \rangle, $$

so we need only to compute the sum in (5).

If $k = i$, then $\langle \alpha_i, \alpha_i^{\vee} \rangle = \langle \alpha_i, \alpha_i^{\vee} \rangle = 2 \equiv 0 \mod 2$. If two different vertices $i$ and $k$ are not connected by an edge, then $\langle \alpha_i, \alpha_k^{\vee} \rangle = 0$. Thus the sum in (5) is taken over vertices $k$ different from $i$ that are connected with $i$ by an edge. Now we consider cases.

We prove (i). If $D$ is simply laced and the vertices $i$ and $k$ are connected by an edge, then $\langle \alpha_i, \alpha_i^{\vee} \rangle = -1$, and we obtain (3). If $D = G_2$ and $k \neq i$, then either $\langle \alpha_i, \alpha_k^{\vee} \rangle = -1$ or $\langle \alpha_i, \alpha_k^{\vee} \rangle = -3 \equiv -1 \mod 2$, and we again obtain (3).

We prove (ii). If $D$ has a double edge and $\alpha_i$ is a short root, then $\langle \alpha_i, \alpha_i^{\vee} \rangle = -1$ for any vertex $k$ connected by an edge with $i$, and we again obtain (3). However, if $\alpha_i$ is a
long root, and $k$ is connected by an edge with $i$, then $\langle \alpha_i, \alpha_k^\vee \rangle = -1$ if $\alpha_k$ is long, and $\langle \alpha_i, \alpha_k^\vee \rangle = -2 \equiv 0 \mod 2$ if $\alpha_k$ is short, and we obtain (4).

\[ \square \]

4. Weyl Action for Inner Forms

As in Section 3 $G$ is a simply connected, simple, compact (anisotropic) linear algebraic group over $\mathbb{R}$. Write $G^\text{ad} = G/Z(G)$, $T^\text{ad} = T/Z(G)$, then $T^\text{ad}$ is a maximal torus in the adjoint group $G^\text{ad}$. We consider inner twisted forms of $G$. Let $t \in Z^1(\mathbb{R}, G^\text{ad})$, and let $iG$ be the corresponding twisted form. We have $iG(\mathbb{C}) = G(\mathbb{C})$, but the complex conjugation in $iG(\mathbb{C})$ is given by

$$g \mapsto \overline{g} = \text{Inn}(t)(g).$$

It is well known that we may choose $t \in T^\text{ad}(\mathbb{R})_2$ (see e.g. [S, III.4.5, Example (a)]). We lift $t$ to some $\tilde{t} \in T(\mathbb{R})$. Then the complex conjugation conjugation in $iG(\mathbb{C})$ is given by

$$\overline{g} = \tilde{g} \tilde{t}^{-1}.$$

Let $t_i \in T^\text{ad}(\mathbb{R})_2 = T(\mathbb{R})_2$ be the element such that $\alpha_i(t_i) = -1$, and $\alpha_j(t_i) = 1$ for $j \neq i$. We write $t = \prod_i (t_i)^{s_i}$, where $s_i = 0, 1$, then $(-1)^{s_i} = \alpha_i(t)$.

Since $t \in T^\text{ad}(\mathbb{R})_2$, we have $tT = T$, hence $tT$ is a compact maximal torus in $iG$. Let $W_0 := W_0(iG, iT)$ be the group $W_0$ of Section 2 then $W_0 = W(G, T) =: W$.

We consider the twisted action of $W$ on $H^1(\mathbb{R}, T) = T(\mathbb{R})_2$. Let $r_j \in W$ be a reflection. Write $r_j = n_j T(\mathbb{R})$ for some $n_j \in N(\mathbb{R})$. For $z \in T(\mathbb{R})_2$ we have

$$r_j * z = n_j z * n_j^{-1} = n_j z \tilde{t} n_j^{-1} \tilde{t}^{-1} = n_j \tilde{z} n_j^{-1} \cdot n_j \tilde{n}_j^{-1} \tilde{t}^{-1}$$

Note that

$$r_j * z = r_j(z) \cdot n_j \tilde{n}_j^{-1} \tilde{t}^{-1},$$

where $r_j(z) = n_j z \tilde{n}_j^{-1}$. In particular, we have $r_j * 1 = n_j \tilde{n}_j^{-1} \tilde{t}^{-1}$, so in general $r_j * 1 \neq 1$ and therefore, the twisted action does not preserve the group structure in $T(\mathbb{R})_2$.

We compute the elementary transformations induced by the $t$-twisted action. We define the elementary transformations by

$$\mathcal{T}_i a = a', \text{ where } r_i^* \left( \prod_{i=1}^n (\alpha_i^\vee (-1))^{a_i} \right) = \prod_{i=1}^n (\alpha_i^\vee (-1))^{a_i'} \text{ i.e., } r_i^* \left( \sum_{j=1}^n a_j \alpha_j^\vee \right) = \sum_{j=1}^n a_j' \alpha_j^\vee.$$

\[ \text{Lemma 4.1. For the $t$-twisted action of $W$, for } \mathcal{T}_i \text{ we have, as above, } a'_j = a_j \text{ for } j \neq i, \text{ while in the formulas (3) and (4) we should add } s_i, \text{ where } (-1)^{s_i} = \alpha_i^\vee(t). \text{ Namely, we have} \]

$$a'_i = a_i + s_i + \sum_{k \text{ adj.}} a_k,$$

for simply laced Dynkin diagrams and for $G_2$, and also for a vertex $i$ corresponding to a short root in a diagram with a double edge. We have

$$a'_i = a_i + s_i + \sum_{k \text{ adj. long}} a_k.$$

for a vertex $i$ corresponding to a long root in a diagram with a double edge.
Proof. By (10) it suffices to show that \( n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} = (\alpha_j^\vee (-1))^{s_j} \). We are indebted to Dmitri A. Timashev for the following proof.

Let \( \omega_i^\vee \in \text{Lie} T \) be the element such that
\[
\langle \alpha_j, \omega_i^\vee \rangle = 0 \text{ if } j \neq i \quad \text{and} \quad \langle \alpha_i, \omega_i^\vee \rangle = 1.
\]

We set \( \tilde{t} = \exp (\pi i \sum_i s_i \omega_i^\vee) \in T(\mathbb{R}) \), where \( i^2 = -1 \), then
\[
\alpha_j(\tilde{t}) = \alpha_j \left( \exp \left( \pi i \sum_i s_i \omega_i^\vee \right) \right) = \exp \left( \pi i \sum_i s_i \langle \alpha_j, \omega_i^\vee \rangle \right) = \exp (\pi i s_j) = (-1)^{s_j},
\]

hence the image of \( \tilde{t} \) in \( T^{\text{ad}}(\mathbb{R}) \) is indeed \( t \). We have
\[
n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} = r_j(\tilde{t}) \tilde{t}^{-1} = \exp \left( \pi i \sum_i s_i (\langle r_j(\omega_i^\vee) - \omega_i^\vee \rangle) \right).
\]

It remains to notice that if \( i \neq j \), then
\[
\langle r_j(\omega_i^\vee) - \omega_i^\vee \rangle = -\langle \alpha_j, \omega_i^\vee \rangle \alpha_j^\vee = 0,
\]
and if \( i = j \), then
\[
\langle r_j(\omega_j^\vee) - \omega_j^\vee \rangle = -\langle \alpha_j, \omega_j^\vee \rangle \alpha_j^\vee = -\alpha_j^\vee.
\]
Thus \( n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} = (\alpha_j^\vee (-1))^{s_j} \).

According to Victor G. Kac, up to conjugation an inner twisting can given by \( t = t_i \) for some \( i \), where \( i \) a vertex of \( D \). Namely, we consider the the Kac diagram of \( tG \), see Onishchik and Vinberg [OV1], Table 7, types I and II. This diagram contains one or two black vertices. We call the automorphism diagram of \( tG \) the Kac diagram of \( tG \) with vertex 0 removed. This automorphism diagram has exactly one black vertex, say vertex \( i \). Then we set \( t = t_i \). We say the twisted at \( i \) action of \( W \) for the twisted by \( t_i \) action of \( W \). The next lemma follows immediately from Lemma 4.1

Lemma 4.2. For the twisted at \( i \) action of \( W \), we have the same formulas (3) and (4) for \( T_j \) for \( j \neq i \) as in Lemma 3.1 above. For \( T_i \) we have, as above, \( a_j' = a_j \) for \( j \neq i \), while in formulas (3) and (4) for \( a_i' \) we should add 1. Namely, we have
\[
a_i' = a_i + 1 + \sum_{k \text{ adj.}} a_k,
\]
for simply laced Dynkin diagrams and for \( G_2 \), and also for a vertex \( i \) corresponding to a short root in a diagram with a double edge. We have
\[
a_i' = a_i + 1 + \sum_{k \text{ adj. long}} a_k.
\]
for a vertex \( i \) corresponding to a long root in a diagram with a double edge.

5. Weyl action for outer forms

Let \( G \) be again as in Section 3 i.e., a simply connected, simple, compact (anisotropic) linear algebraic group over \( \mathbb{R} \). We consider outer forms of \( G \). An outer form of \( G \) corresponds to an outer involutive automorphism \( \tau \) of \( G \).

Let \( T, \ R, \ Pi \) and \( D \) be as in Section 3. Let \( \nu \) be an automorphism of order 2 of the Dynkin diagram \( D \) of \( G \). We write \( \Pi' \) for the set of fixed points of \( \nu \) in \( \Pi \), and \( D' \) for the induced Dynkin subdiagram. Note that \( \nu \) acts on \( T, \ G \), and \( W \). We write \( W' \) for the algebraic subgroup of fixed points of \( \nu \) in \( W \). We write \( _\nu T, \ _\nu G \), and \( _\nu W \) for the corresponding
twisted algebraic groups. Let \( t \in (\nu T)_{2} = (T_{2})_{2} \). We denote by \( \text{Inn}(t) \) the corresponding inner automorphism of \( \nu G \) of order dividing 2. We set \( \tau = \text{Inn}(t) \circ \nu \). Note that \( \tau \) is an outer automorphism of order 2 of \( G \). We write \( \tau G = \text{Inn}(t)(\nu G) \) for the corresponding twisted form of \( G \). Then \( \nu G(\mathbb{C}) = G(\mathbb{C}) \), but the complex conjugation in \( \nu G(\mathbb{C}) \) is given by

\[
\tau g = \tau(\nu g) = \text{Inn}(t)(\nu(\nu g)).
\]

Note that \( \text{Inn}(t) \) acts trivially on \( \nu T \), hence also on \( \nu W \), because \( \nu W \subset \text{Aut}(\nu T) \). We see that \( \text{Inn}(t) \nu T = \nu T \) and \( \text{Inn}(t) \nu W = \nu W \).

Let \( \Pi = \{\alpha_{1}, \ldots, \alpha_{n}\} \). We write \( \alpha_{i}(t) = (-1)^{s_{i}} \), \( s_{i} \in \mathbb{Z}/2\mathbb{Z} \).

We set

\[
T(D^\nu)_{\mathbb{C}} = \prod_{\alpha_{i} \in D^\nu} \mathbb{G}_{m, \mathbb{C}},
\]

\[
T(D \setminus D^\nu)_{\mathbb{C}} = \prod_{\alpha_{i} \in D \setminus D^\nu} \mathbb{G}_{m, \mathbb{C}}.
\]

Then the subtorus \( T(D^\nu)_{\mathbb{C}} \) and \( T(D \setminus D^\nu)_{\mathbb{C}} \) of \( (\nu T)_{\mathbb{C}} \) are defined over \( \mathbb{R} \). We write \( \nu T(D^\nu) \) and \( \nu T(D \setminus D^\nu) \) for the corresponding \( \mathbb{R} \)-tori. We write also \( T(D^\nu) \) for the corresponding subtorus of \( T \), then \( \nu T(D^\nu) = T(D^\nu) \). We have

\[
\nu T = \nu T(D^\nu) \times \nu T(D \setminus D^\nu).
\]

The torus \( \nu T(D^\nu) \) is compact, and

\[
\nu T(D \setminus D^\nu) \simeq (\mathbb{R}/\mathbb{Z})_{\mathbb{C}}^{\nu},
\]

where \( \nu = \frac{1}{2} \#(D \setminus D^\nu) \). We see that

\[
H^{1}(\mathbb{R}, \nu T(D \setminus D^\nu)) = 1
\]

(cf. Section 1), and we obtain an isomorphism

\[
H^{1}(\mathbb{R}, \nu T) \simeq H^{1}(\mathbb{R}, T(D^\nu)) = T(D^\nu)(\mathbb{R})_{2} = (\mathbb{Z}/2\mathbb{Z})^{D^\nu}.
\]

In other words, \( H^{1}(\mathbb{R}, \nu T) \) is the set of labelings \( \alpha = (a_{i})_{i \in D^\nu} \) of the Dynkin diagram \( D^\nu \). We consider the group \( W_{0} = W_{0}(\nu G) \). We have \( W_{0}(\mathbb{C}) = W_{0}(\mathbb{R}) = \nu W(\mathbb{R}) \); see [Bo2 Section 7]. Clearly \( \nu W(\mathbb{R}) = W(\mathbb{C}) \). The group \( W_{0}(\mathbb{R}) \) acts on \( H^{1}(\mathbb{R}, \nu T) \) and thus on the set of labelings \( (\mathbb{Z}/2\mathbb{Z})^{D^\nu} \) via isomorphism (14). We wish to describe this action.

Note that if \( D \) is of type \( \mathbf{A}_{2n} \), then \( D^\nu = \emptyset \), \( T(D^\nu) = 1 \), \( H^{1}(\mathbb{R}, \nu T) = 1 \), \( H^{1}(\mathbb{R}, \nu G) = 1 \) (in this case \( \nu G \simeq \mathbf{SL}_{2n+1} \)). From now on we shall assume that \( D \) is not of type \( \mathbf{A}_{2n} \). Then from the classification of Dynkin diagrams we know that for any \( i \in D \setminus D^\nu \), the vertices \( i \) and \( \nu(i) \) are not connected by an edge, and therefore, the reflections \( r_{i} \) and \( r_{\nu(i)} \) commute.

**Lemma 5.1.** Assume that \( D \) is not of type \( \mathbf{A}_{2n} \). Then the group \( W_{0}(\mathbb{R}) \) is generated by the reflections \( r_{i} \) for \( i \in D^\nu \) and by the products \( r_{j} \cdot r_{\nu(j)} \) for \( j \in D \setminus D^\nu \).

**Proof.** We have

\[
W_{0}(\mathbb{R}) = \nu W(\mathbb{R}) = W(\mathbb{C}).
\]

Now the lemma follows from [Cr Proposition 13.1.2].

**Lemma 5.2.** Assume that \( D \) is not of type \( \mathbf{A}_{2n} \). Let \( j \in D \setminus D^\nu \). Then the product \( r_{j} \cdot r_{\nu(j)} \) acts trivially on \( H^{1}(\mathbb{R}, \nu T) \).
Proof. Let $z \in Z^1(\mathbb{R}, \nu T) \subset T(\mathbb{C})_2$, and let $b = (b_k) \in (\mathbb{Z}/2\mathbb{Z})^D$ be the corresponding labeling. Then $r_j \in X(\mathbb{C})$ acts on $T(\mathbb{C})_2$ by formula (9), and Lemma 4.4 says that the corresponding elementary transformation $T_j$ can change only the $j$-coordinate $b_j$ of $b$.

Now consider $r_{\nu(j)} r_j \in W_0(\mathbb{R}) \subset W(\mathbb{C})$, then we see that $T_{\nu(j)} T_j$ can change only the $j$- and the $\nu(j)$-coordinates of $b$. In particular, if we write $b' = (T_j T_{\nu(j)}) b$, then $b'_j = b_j$ for any $i \in D^\nu$.

Since $r_{\nu(j)} r_j \in W_0(\mathbb{R})$, we see that $z' := (r_{\nu(j)} r_j)(z)$ is contained in $Z^1(\mathbb{R}, \nu T)$. Since $b'_j = b_j$ for any $i \in D^\nu$, by formula (13) $z' \sim z$ in $H^1(\mathbb{R}, \nu T)$. Thus $r_{\nu(j)} r_j$ acts on $H^1(\mathbb{R}, \nu T)$ trivially. □

Lemma 5.3. Let $\xi \in H^1(\mathbb{R}, \nu T) = H^1(\mathbb{R}, T(D^\nu)) = (\mathbb{Z}/2\mathbb{Z})^D^\nu$. Let $T_i$ denote the elementary transformation of $(\mathbb{Z}/2\mathbb{Z})^D^\nu$ corresponding to $r_i \in W_0(\mathbb{R})$ for $i \in D^\nu$. Write $T_i a = a'$, where $a = (a_j)_{j \in D^\nu}$, $a' = (a'_j)_{j \in D^\nu}$. Then $a'_j = a_j$ for $j \neq i$ and

\begin{equation}
(15) \quad a'_i = a_i + s_i + \sum_{k \in D^\nu \text{ adj}} a_k,
\end{equation}

where $(-1)^{s_i} = \alpha_i(t)$ and the sum is taken over the vertices $k$ adjacent to $i$ and lying in $D^\nu$.

Proof. Let $z \in Z^1(\mathbb{R}, \nu T)$ be a cocycle. Up to equivalence, we may take $z \in T(D^\nu)(\mathbb{R})_2 \subset Z^1(\mathbb{R}, \nu T)$. Write $z = \prod_{j \in D} (\alpha_j^{\nu}(−1)) a_j$, then $a_j = 0$ for $j \in D \setminus D^\nu$. Now let $i \in D^\nu$, then the elementary transformation $T_i$ acts on $(\mathbb{Z}/2\mathbb{Z})^D$ by formula (9):

\begin{equation}
(16) \quad a'_i = a_i + s_i + \sum_{k \text{ adj.}} a_k,
\end{equation}

where the sum is taken over all $k$ adjacent to $i$. However, if $k \in D \setminus D^\nu$, then $a_k = 0$. We see that in formula (16) we may take the sum only over $k$ adjacent to $i$ such that $k \in D^\nu$. □

Set $T^\text{ad} = T/Z(G) \subset G^\text{ad}$ and $T(D^\nu)^\text{ad} = T(D^\nu)/Z(G(D^\nu)) \subset G(D^\nu)^\text{ad}$. Since the basis $D^\nu$ of $X(T(D^\nu)^\text{ad})$ is a subset of the basis $D$ of $T^\text{ad}$, the torus $T(D^\nu)^\text{ad}$ is naturally a direct factor of $\nu T^\text{ad}$, and we have a canonical projection $p: \nu T^\text{ad} \to T(D^\nu)^\text{ad}$.

Corollary 5.4 ([Bo1, Theorem 3]). Let $l = \#D^\nu$. Let $G(D^\nu) \subset G$ be the $\mathbb{R}$-subgroup of type $A_l$ corresponding to the Dynkin subdiagram $D^\nu$ of $L$. Let $t \in \nu T^\text{ad}(\mathbb{R})_2$. Then the natural embedding

\[ p(t) G(D^\nu) \hookrightarrow \nu T \]

induces a bijection

\[ H^1(\mathbb{R}, p(t) G(D^\nu)) \cong H^1(\mathbb{R}, t \nu T). \]

(17)

Proof. The embedding $T(D^\nu) \hookrightarrow \nu T$ induces the isomorphism (14)

\[ H^1(\mathbb{R}, T(D^\nu)) \cong H^1(\mathbb{R}, \nu T). \]

The group $W(D^\nu)(\mathbb{R})$ acts on the left-hand side of (17); this group is generated by $r_i$ for $i \in D^\nu$. The group $W_0(\mathbb{R})$ acts on the right-hand side; by Lemma 5.4 this group is generated by $r_i$ for $i \in D^\nu$ and $r_{\nu(j)} r_j$ for $j \in D \setminus D^\nu$. By Lemma 5.2 these products $r_{\nu(j)} r_j$ for $j \in D \setminus D^\nu$ act on the right-hand side of (17) trivially. Comparing formulas (13) and (15), we see that the actions of a reflection $r_i$ for $i \in D^\nu$ on the left-hand side and the right-hand side of (17) are compatible with the isomorphism. Thus we obtain a bijection of the quotients:

\[ H^1(\mathbb{R}, p(t) G(D^\nu)) = W(D^\nu)(\mathbb{R}) \setminus T(D^\nu)(\mathbb{R})_2 \cong W_0(\mathbb{R}) \setminus H^1(\mathbb{R}, \nu T) = H^1(\mathbb{R}, t \nu G), \]
where the left and right equalities are the bijections of Proposition 2.1.

Note that we may take \( t \in \nu T^{\text{ad}}(\mathbb{R})_2 \) from [OV1, Table 7, Type III]. Our \( t \) corresponds to the black vertex of the Kac diagram (a Kac diagram of type II has exactly one black vertex). If the black vertex has number 0, then \( t = 1 \), so \( t \nu G = \nu G \), hence \( s_i = 0 \) for all \( i \), and therefore, \( T_i \) is given by formula (3), where the sum is taken over the vertices \( k \) adjacent to \( i \) and lying in \( D^{\nu} \). If the black vertex has number \( i \neq 0 \), then \( i \in D^{\nu} \), and we take \( t = t_i \), i.e., \( s_j = 0 \) for \( j \neq i \), and \( s_i = 1 \). Thus \( T_j \) for \( j \neq i \) is given by formula (18)

\[
a'_j = a_j + \sum_{k \in D^{\nu} \text{ adj.}} a_k,
\]

while \( T_i \) is given by formula (12),

\[
a'_i = a_i + 1 + \sum_{k \in D^{\nu} \text{ adj.}} a_k,
\]

where again in both cases the sum is taken over the adjacent vertices \( k \) lying in \( D^{\nu} \).

6. Orbits: definitions

Starting this section, we describe case by case the orbits of the group \( W \) (or \( W_0 \)) generated by the elementary transformations \( T_i \) (i.e., reflections \( r_{\alpha_i} \)) on the set \( L(D) \) of labelings \( a = (a_1, \ldots, a_n) \) of a Dynkin diagram \( D \) with vertices \( i = 1, \ldots, n \), where \( a_i \in \mathbb{Z}/2\mathbb{Z} \) and each \( i \) corresponds to the simple root \( \alpha_i \). We number the vertices of \( D \) as in Onishchik and Vinberg [OV1, Table 1].

We say that two vertices \( i, j \) of a Dynkin diagram \( D \) are neighbors if they are connected by an edge (single or multiple).

By (connected) components (of 1’s) of a labeling of a Dynkin diagram we mean the connected components of the graph obtained by removing the vertices with zeros and the corresponding edges. For example, the following labeling of \( A_9 \) has 3 connected components:

\[
1--0--0--0--1---1---1.
\]

For some groups \( G \) the number of of components of a labeling is an invariant of the action of \( W \). For some others, the parity of the number of components is an invariant.

By a fixed (invariant) labeling we mean a fixed point of the action of \( W \), that is a labeling which is fixed under all the elementary transformations \( T_i \). For example for the action of Lemma 3.1 on \( A_9 \) the labelings

\[
0--0--0--0 \quad \text{and} \quad 1--0--1--0--1
\]

are fixed (invariant). Fixed labelings correspond to elements of order dividing 2 of the center \( Z(G) \) of \( G \).

We say that \( i \) is a vertex of degree \( d \) if it has exactly \( d \) neighbors. We are especially interested in vertices of degree 3. The Dynkin diagrams \( D_n \) (\( n \geq 4 \)), \( E_6, E_7 \) and \( E_8 \) have vertices of degree 3. Now let \( D \) be a Dynkin diagram with a vertex \( i \) of degree 3, at let \( a \) be a labeling of \( D \) that looks as

\[
\ldots1--1--1--\ldots
\]

The elementary transformation \( T_i \) of Lemma 3.1 splits the corresponding component to three:

\[
\ldots1--0--1--\ldots
\]
and therefore changes the number of components by 2. We call this process *splitting at* \(i\). The reverse process is called *unsplitting*.

Assume we have a Dynkin diagram with a *twisting* vertex (black vertex) \(i\):

\[
\cdots \circ i \circ \cdots
\]

By Section 4 we have formula (12) for \(T_i\):

\[
a'_i = a_i + 1 + \sum_{k \text{ adj.}} a_k,
\]

In order to get formula (23) instead, we formally add an additional vertex which we call *boxed 1*, connected by a simple edge with vertex \(i\).

\[
\cdots \circ m \circ \cdots
\]

Here 1 in the box means that we put 1 as the label at this new vertex. Now the formula for \(T_i\) becomes

\[
a'_i = a_i + \sum_{k \text{ adj.}} a_k,
\]

where the sum is taken over all the neighbors of \(i\), including the boxed 1. Thus this boxed 1 accounts for twisting at \(i\). Note that we do not add an elementary transformation corresponding to the boxed 1, so the label 1 at the boxed 1 cannot be changed under the action of the group \(W\).

Finally, we introduce some notation. Let \(G\) be a group with Dynkin diagram \(D\). We denote the set of labelings on the diagram by \(L(D)\). We denote the set of orbits under the Weyl action by \(\text{Orbs}(G) := W(D) \backslash L(D)\) and the number of orbits by \(#\text{Orbs}(G)\).

### 7. Groups of type \(A_n\)

#### 7.1. The compact group \(SU_{n+1}\) of type \(A_n\).

The Dynkin diagram is

\[
\circ \cdots \circ
\]

The Weyl group acts by the elementary transformations that are described in Lemma 3.1.

**Lemma 7.1.** For \(A_n\) we have:

(a) An elementary transformation does not change the number of components.

(b) Every component can be shrunk to length of 1, e.g.

\[
(0 - 1 - 1 - 1 - 0) \mapsto (0 - 0 - 1 - 0 - 0).
\]

(c) Components may be pushed so the space between components is of length 1, e.g.

\[
(1 - 0 - 0 - 1 - 0) \mapsto (1 - 0 - 1 - 0 - 0).
\]

**Proof.** Easy. \(\square\)

**Definition 7.2.** Labelings of \(A_n\) of the form

\[
\xi_r = \left( 1 - 0 - \cdots - 1 - 0 - \cdots \right)
\]

which have \(r\) components, defined by

\[
(\xi_r)_i = \begin{cases} 
1 & \text{for } i = 1, 3, \ldots, 2r - 1, \\
0 & \text{for } i \neq 1, 3, \ldots, 2r - 1.
\end{cases}
\]
are denoted $\xi_r$ and are called labelings with $r$ components packed maximally to the left.

**Example 7.3.** For $A_7$, $\xi_3 = (1-0-1-0-1-0)$. 

**Corollary 7.4.** Each labeling with $r$ components can be brought to the form $\xi_r$.

Thus, in $A_n$ the number of components is an invariant which fully characterizes orbits. We see that the orbit of zero consists of one labeling zero. For representatives of orbits we can take $\xi_0, \xi_1, \ldots, \xi_r$, where $r = \lceil n/2 \rceil$. We have

$$
\#H^1(\mathbb{R}, G) = \#\text{Orbs}(A_n) = r + 1 - \lfloor n/2 \rfloor + 1 = \begin{cases} 
k + 1 & \text{if } n = 2k, \\
k + 2 & \text{if } n = 2k + 1. 
\end{cases}
$$

### 7.2. The group $SU_{m,n+1-m}$ with diagram $A_n^{(m)}$.

The maximal compact subgroup is of type $A_{m-1} \times A_{n-m} \times T^1$, where $T^1$ is a 1-dimensional compact torus. For $1 \leq m < n$ the Kac diagram is

$$
\begin{array}{c}
\vdots \\
1 \quad \cdots \quad m \\
\end{array}
$$

We erase vertex 0. The twisting vertex $m$ can be accounted by adding a neighbor vertex with fixed 1 (which cannot be changed), which we denote [1] and call boxed 1. Thus we obtain the augmented diagram

$$
\begin{array}{c}
1 \quad \cdots \quad m \\
\end{array}
$$

This retains the total number of labelings and accounts for the twisting (see Sections 4 and 5).

We can shrink the the component of [1] to [1], then the label at $m$ is 0. We view the twisting vertex $m$ as a “barrier”. We then have a schematic diagram:

$$
\text{LHS} \quad 0 \quad \text{RHS}
$$

where LHS denotes the left-hand side and RHS denotes the right-hand side.

**Lemma 7.5 (The Barrier Lemma).** For the schematic diagram (26) we have:

(a) If we have a component in LHS and a component in RHS, then they can cancel each other by unsplitting at the barrier (which is a vertex of degree 3 for $1 < m < n$). Formally, if $l(a), r(a) \geq 1$ are the number of components in the left arm (LHS) and right arm (RHS) respectively then the labeling $a$ is equivalent to the labeling $a'$ with $l(a') = l(a) - 1$ and $r(a') = r(a) - 1$.

(b) A component cannot pass through the barrier from one side to the other side. Formally, a labeling $a$ with $l(a)$ and $r(a)$ cannot be changed by elementary transformations to a labeling $a'$ with $l(a') = l(a) \pm 1$ and $r(a') = r(a) \mp 1$. This means, for example, that

$$
\cdots \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad \cdots
$$

cannot be changed to

$$
\cdots \quad 0 \quad 0 \quad \boxed{1} \quad 0 \quad \cdots
$$

by a sequence of elementary transformations.
Proof. Proof of (a) is easy (see Section 6). Let us prove (b). Consider a labeling \( \mathbf{a} \) which does not lie in the orbit of 0, this means \( l(\mathbf{a}) \neq r(\mathbf{a}) \). From (a) we may assume either \( l(\mathbf{a}) > 0 \) and \( r(\mathbf{a}) = 0 \) or \( l(\mathbf{a}) = 0 \) and \( r(\mathbf{a}) > 0 \) (we do min\{\( l(\mathbf{a}), r(\mathbf{a}) \)\} unsplittings at the barrier). Without loss of generality let us assume \( l(\mathbf{a}) > 0 \) and \( r(\mathbf{a}) = 0 \). Then \( \mathbf{a} \) is of the form

\[
\mathbf{a} = \left( \begin{array}{cccc}
\cdots & 1 & 0 & 0 \\
\end{array} \right)
\]

and we see that applying \( T_m \) we get \( 1 + 0 + 1 = 0 \) (mod 2) so for \( \mathbf{a}' = T_m(\mathbf{a}) \) we get \( a'_m = a_m = 0 \). Moreover, applying \( T_i \) for \( i > m \) again yields only 0’s. The result follows.

Let \( \mathbf{a} \) be a labeling. Let \( l(\mathbf{a}) \) denote the number of components of \( \mathbf{a} \) in the LHS and \( r(\mathbf{a}) \) denote the number of components of \( \mathbf{a} \) in the RHS (in both cases excluding the component of boxed 1). Then the only invariant of a labeling \( \mathbf{a} \) is \( r(\mathbf{a}) - l(\mathbf{a}) \), and this invariant can take values between \( -\left\lceil \frac{m-1}{2} \right\rceil \) and \( \left\lceil \frac{n-m}{2} \right\rceil \).

Thus

\[
\# H^1(\mathbb{R}, G) = \# \text{Orbs}(A_n^{(m)}) = \left\lceil \frac{m-1}{2} \right\rceil + 1 + \left\lfloor \frac{n-m}{2} \right\rfloor
\]

\[
= \begin{cases}
  k+1 & \text{if } n = 2k \\
k+1 & \text{if } n = 2k+1 \text{ and } m \text{ is odd}, \\
k+2 & \text{if } n = 2k+1 \text{ and } m \text{ is even}.
\end{cases}
\]

The orbit of zero consists of is the set of labelings \( \mathbf{a} \) with \( l(\mathbf{a}) = r(\mathbf{a}) \), and for a set of representatives of orbits we can take labelings \( (26) \) where either LHS is \( \xi_i \) with \( i = 1, \ldots, \left\lceil \frac{m-1}{2} \right\rceil \) and RHS is 0, or LHS is 0 and RHS is \( \xi_i \) with \( i = 1, \ldots, \left\lceil \frac{n-m}{2} \right\rceil \), or both LHS and RHS are 0. We denote

\[
(a|b) := \left( \begin{array}{c}
\xi_0 0 \\
0 \xi_0
\end{array} \right)
\]

which means the LHS is \( \xi_0 \) and the RHS is \( \xi_0 \) (the \( | \) represents the barrier). Then we can write \((0|0), (1|0), \ldots, \left( \left\lceil \frac{m-1}{2} \right\rceil | 0 \right) \) and \((0|1), \ldots, \left( \left\lceil \frac{n-m}{2} \right\rceil | n \right)\) for the representatives.

Put it more explicitly, for representatives of orbits we can take

\[
(0|0) = \left( \begin{array}{c}
\xi_0 0 \\
0 \xi_0
\end{array} \right), \quad (i|0) = \left( \begin{array}{c}
\xi_i 0 \\
0 \xi_i
\end{array} \right), \quad (0|j) = \left( \begin{array}{c}
\xi_0 0 \\
0 \xi_j
\end{array} \right)
\]

for \( 1 \leq i \leq \left\lceil \frac{m-1}{2} \right\rceil \) and \( 1 \leq j \leq \left\lceil \frac{n-m}{2} \right\rceil \).

7.3. Outer forms of \( SU_{n+1} \). Here \( \nu \) is the nontrivial involutive automorphism of the Dynkin diagram \( A_n \).

Case \( G = SL_{n+1}, n = 2l \). Then \( D^\nu = 0 \), and it is well known that \( H^1(\mathbb{R}, SL_{n+1}) = 1 \).

Case \( G = SL_{n+1}, n = 2l + 1 \). Then \#D^\nu = 1, \( D^\nu = \circ \), \( H^1(\mathbb{R}, G) = \# \text{Orbs}(A_1) = 2 \).

The orbit of 0 consists of 0 (with \( a_{\nu(i)} = a_i \) for \( i \in D \setminus D^\nu \)). For representatives of orbits we take 0 and 1.
8. Groups of type $\mathbf{B}_n$

8.1. The compact group $\text{Spin}_{2n+1}$ of type $\mathbf{B}_n$. The Dynkin diagram is

$$\begin{array}{ccccccc}
1 & \cdots & n-1 & \equiv & n & \equiv & 0 \\
\end{array}$$

The long roots do not see the short root: i.e. they are not affected by the label at the short root, see formula (2) in Section 3. Writing every labeling $b$ as

$$b = (a \Rightarrow \kappa),$$

where $\kappa \in \{0,1\}$ is the label at the short root and $a \in L(A_{n-1})$ is the tail with $n - 1$ vertices, the elementary transformations $T^B_i$ of $L(B_n)$ acts as follows:

(a) For $i < n - 1 : T^B_i(b) = (T^A_i(a) \Rightarrow \kappa)$.
(b) For $i = n - 1 : T^B_{n-1}(b) = (T^A_{n-1}(a) \Rightarrow \kappa)$.
(c) For $i = n : (T^B_i(b))_n = \kappa + (b)_{n-1}$.

(here $T^A_i$ is the Weyl transformation on $L(A_{n-1})$).

Lemma 8.1 (The Swallowing Lemma). If $a \in L(A_n)$, $a \neq 0$, then $a \Rightarrow 0$ is equivalent to $a \Rightarrow 1$ under $W(B_n)$.

Proof. Consider the rightmost component of 1’s in the long roots. It can be expanded to the right (see Lemma 7.1), swallow the short root's 1 (by elementary transformation $T_n$) and then shrink back to the original place and length,

$$(\cdots - 1 \Rightarrow 0 \Rightarrow 1) \xrightarrow{T_{n-1}} (\cdots - 1 \Rightarrow 1) \xrightarrow{T_n} (\cdots - 1 \Rightarrow 0 \Rightarrow 0) \xrightarrow{T_{n-1}} (\cdots - 1 \Rightarrow 0 \Rightarrow 0).$$

The actions of $T_{n-1}$ and $T_n$ described in (a)–(c) above assure us that this algorithm works: property (a) allows us to do the usual manipulations on the tail $A_{n-1} = (b_i)_{i=1}^{n-1}$, property (b) implies that $T_{n-1}$ can change the label $b_{n-1}$ from 0 to 1 even when $\kappa = 1$, then (c) gives us swallowing by $T_n$ and applying $T_{n-1}$ by (b) give us shrinking back.

Lemma 8.2. Denote by $[\ell_1] \in \text{Orbs}(B_n)$ the orbit $\{\ell_1\}$ of $\ell_1$. We have a map $\varphi : L(A_{n-1}) \rightarrow L(B_n) \smallsetminus \{\ell_1\}$ defined by

$$L(A_{n-1}) \ni a \xrightarrow{\varphi} (a \Rightarrow 0) = b,$$

which induces a bijection on the orbits $\varphi_* : \text{Orbs}(A_{n-1}) \xrightarrow{\sim} \text{Orbs}(B_n) \smallsetminus \{[\ell_1]\}$ defined by $\varphi_*([a]) = [\varphi(a)]$.

Proof. We denote by $L(\Delta)$ the set of labelings of a Dynkin diagram $\Delta$. Let $T^A_i$ be the elementary transformations of $L(A_n)$, and $T^B_i$ be the elementary transformations for $L(B_n)$. Let us denote by $b = (a \Rightarrow \kappa)$ the labeling with tail $a \in L(A_{n-1})$ and short root with label $\kappa \in \{0,1\}$. Then clearly

$$(28) \quad T^B_i(a \Rightarrow \kappa) = (T^A_i(a) \Rightarrow \kappa)$$

for $i < n - 1$, and by (28) it is also true for $i = n - 1$ (where we used the assumption that $\alpha_{n-1}$ is long and $\alpha_n$ is short). We also have

$$T^B_n(a \Rightarrow \kappa) = (a \Rightarrow T_n(\kappa)) .$$

We now need to show that $a \sim a' \iff \varphi(a) \sim \varphi(a')$ where the equivalence relation is induced by the corresponding Weyl action (sequence of elementary transformations) on $A_{n-1}$ and $B_n$. That $a \sim a' \iff \varphi(a_B) \sim \varphi(a'_B)$ is clear. Conversely, if $\varphi(a) \sim \varphi(a')$ then $(a \Rightarrow 0) \sim (a' \Rightarrow 0)$ then due to (28) we can change the tail from $a$ to $a'$ by a sequence of
Let \( b \in L(B_n) \setminus \{\ell_1\} \). If \( b \neq B_1 \) then \( b \) has at least one long root with label 1. Then by Lemma 8.1 we have \( b \sim (a \Rightarrow 0) = \varphi(a) \) for some \( a \in L(A_{n-1}) \). Thus \( [b] = \varphi_*(a) \).

If \( b = B_1 \) then clearly \( b = \varphi(0_A) \). This implies \( \varphi_* \) is surjective on \( \text{Orbs}(B_n) \setminus \{[\ell_1]\} \).

Clearly, \( \ell_1 \neq \varphi(a) \) for any \( a \in L(A_{n-1}) \) since \( (\ell_1)_n = 1 \). Since \( \ell_1 \) is a fixed point then \( \ell_1 \sim_B \varphi(a) \) for any \( a \in L(A_{n-1}) \). These show that \( \text{Im} \varphi_* = \text{Orbs}(B_n) \setminus \{[\ell_1]\} \).

Now, let \( \varphi(a) \sim \varphi(a') \). Then \( (a \Rightarrow 0) \sim (a' \Rightarrow 0) \) which implies \( a \sim a' \). Thus \( \varphi_*(a) = \varphi_*(a') \Rightarrow [a] = [a'] \) and thus \( \varphi_* \) is injective.

We conclude that \( \varphi_* : \text{Orbs}(A_{n-1}) \xrightarrow{\sim} \text{Orbs}(B_n) \setminus \{[\ell_1]\} \) is bijective and the lemma follows. □

Therefore,

\[
\# H^1(\mathbb{R}, G) = \# \text{Orbs}(B_n) = \# \text{Orbs}(A_{n-1}) + 1 = \begin{cases} k + 2 & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1. \end{cases}
\]

We use the following notation: \( (\xi_j > \infty) \) means we chain a tail of \( n-1 \) vertices represented by \( \xi_j \) (see Section 7.1) to the short root with label \( \infty \in \{0, 1\} \).

The orbit of zero consists of the invariant 0 and represented by \( (\xi_0 > 0) \). For representatives of orbits we can take \( (\xi_0 > 1), (\xi_0 > 0), (\xi_1 > 0), (\xi_2 > 0), \ldots, (\xi_r > 0) \) with \( r = k = \lfloor \frac{n}{2} \rfloor \). Note that by Lemma 8.1 we can choose the short root label to be 0 whenever \( j > 0 \).

### 8.2. The diagram \( B_n^{(m)} \)

For \( 1 \leq m < n \) the Kac diagram is

```
0
\downarrow
2 \cdots m \cdots n-1 \Rightarrow n
1
```

which corresponds to the group \( \text{SO}_{2m,2(n-m)+1} \) with maximal compact subgroup \( D_m \times B_{n-m} \). We erase the vertex 0 and to account for the twisting vertex, we add a neighbor with fixed 1. We get the augmented diagram:

```
1 \cdots 2 \cdots m \cdots n-1 \Rightarrow n
```

**Lemma 8.3.**

\[
\# \text{Orbs}(B_n^{(m)}) = \begin{cases} \# \text{Orbs}(A_{n-1}^{(m)}) & \text{if } m \text{ is odd}, \\ \# \text{Orbs}(A_{n-1}^{(m)}) + 1 & \text{if } m \text{ is even.} \end{cases}
\]

**Proof.** Let \( b \in L(B_n^{(m)}) \) be a labeling of \( B_n^{(m)} \). We assume that \( b_n = 1 \) and try to change it to 0. We write \( b = (a \Rightarrow \infty) \), where \( a \in L(A_{n-1}^{(m)}), \infty \in \{0, 1\} \). Then \( a \) is equivalent (under \( W(A_{n-1}^{(m)}) \)) to a labeling of the form

LHS \( \rightarrow \) RHS
If $\text{RHS} \neq 0$, then we can expand the rightmost component so that $b_{n-1} = a_{n-1} = 1$, then change $b_n$ from 1 to 0 using $T_n$, and then shrink back. If $\text{RHS} = 0$, but $b_{m-1} = 0$ (or $m = 1$), then we can expand the component of the boxed 1 to RHS so that $b_{n-1} = 1$, then again change $b_n$ from 1 to 0 using $T_n$, and then shrink back. We see that we cannot change $b_n$ only if $a$ has $\text{RHS} = 0$ and is not equivalent (under $W(A_{n-1}^{(m)})$) to a labeling with $a_{m-1} = 0$. It is easy to see that then $m$ is even and $b = \ell_1$, where

$$
\ell_1 = \left( 1-0\cdots-1-0\cdots0\iff1 \right)
$$

is a fixed point. We denote by $[\ell_1] \in \text{Orbs}(B_n^{(m)})$ the orbit $\{\ell_1\}$ of $\ell_1$.

We have shown that any labeling $b \in L(B_n^{(m)})$, except the fixed labeling $\ell_1$ (existing for $m$ even), is equivalent to a labeling of the form $b = (a\iff0)$. Now it follows from formula (4) for $T_{n-1}$ that two labelings $\ell = (a\iff0)$ and $\ell' = (a'\iff0)$ are equivalent under $W(B_n^{(m)})$ if and only if $a$ and $a'$ are equivalent under $W(A_{n-1}^{(m)})$. We conclude that the map

$$
\ell 
\mapsto
(a\iff0)
L(A_{n-1}^{(m)}) 
\to
L(B_n^{(m)})
$$

induces a bijection

$$
\text{Orbs}(A_{n-1}^{(m)}) \to \text{Orbs}(B_n^{(m)})
$$

when $m$ is odd, and a bijection

$$
\text{Orbs}(A_{n-1}^{(m)}) \to \text{Orbs}(B_n^{(m)}) \sim \{[\ell_1]\}
$$

when $m$ is even, and the lemma follows. \(\square\)

From Lemma 8.3 we obtain that the number or orbits is

$$
#H^1(\mathbb{R}, G) = #\text{Orbs}(B_n^{(m)}) = \begin{cases} 
k & \text{if } n = 2k \text{ and } m \text{ is odd,} 
\k + 2 & \text{if } n = 2k + 2 \text{ and } m \text{ is even,} 
\k + 1 & \text{if } n = 2k + 1 \text{ and } m \text{ is odd,} 
\k + 2 & \text{if } n = 2k + 1 \text{ and } m \text{ is even.}
\end{cases}
$$

We write $b \in L(B_n^{(m)})$ as $b = (a\iff\kappa)$, where $a \in L(A_{n-1}^{(m)})$, $\kappa \in \{0, 1\}$. Then we set $l(b) := l(a)$ and $r(b) := r(a)$, where $l(a)$ and $r(a)$ were defined in Section 7.2. Namely, $l(a)$ is the number of components in LHS of $a$ and $r(a)$ is the number of components in RHS of $a$, in both cases excluding the the component of the boxed 1. Then the orbit of zero in $L(B_n^{(m)})$ is the set of all labelings $b = (a\iff\kappa)$ with $r(b) = l(b)$, i.e., $r(a) = l(a)$.

We now denoted by $(p|q > \kappa)$ the labeling with $\xi_q$ on the LHS, $\xi_q$ on the RHS (excluding the short root), 0 at the vertex $m$ and $\kappa \in \{0, 1\}$ at the short root. It follows from the proof of Lemma 8.3 that for a set of representatives of orbits in $L(B_n^{(m)})$ we can take the labelings $a\iff0$, where $a$ runs over the set of representatives of orbits in $L(A_{n-1}^{(m)})$ from Section 7.2 and when $m$ is even add the fixed labeling $\ell_1 \in L(B_n^{(m)})$. Explicitly, we obtain

- For $n = 2k + 1$ and even $m$
  $$(a|0 > 0) \text{ for } a = 0, ..., m/2 \text{ and } (0|b > 0) \text{ for } b = 1, ..., (2k - m)/2 \text{ and } \ell_1$$
- For $n = 2k + 1$ and odd $m$
  $$(a|0 > 0) \text{ for } a = 0, ..., (m - 1)/2 \text{ and } (0|b > 0) \text{ for } b = 1, ..., (2k - m + 1)/2$$
• For $n - 1 = 2k + 1$ and even $m$:

\[(a|0 > 0) \text{ for } a = 0, \ldots, \frac{m}{2} \text{ and } (0|b > 0) \text{ for } b = 1, \ldots, \frac{2k + 1 - m + 1}{2} \text{ and } \ell_1 \]

• For $n - 1 = 2k + 1$ and odd $m$:

\[(a|0 > 0) \text{ for } a = 0, \ldots, \frac{m - 1}{2} \text{ and } (0|b > 0) \text{ for } b = 1, \ldots, \frac{2k + 1 - m}{2}. \]

8.3. The diagram $B_n^{(n)}$. $G = \text{Spin}_{2n,1}$ with maximal compact subgroup isogenous to $\text{Spin}_{2n}$. The Kac diagram is:

\[
\begin{array}{c}
\circ \rightarrow \cdots \rightarrow \circ \Rightarrow \bullet \\
\end{array}
\]

The augmented diagram is

\[
\begin{array}{c}
\circ \rightarrow \cdots \rightarrow \circ \Rightarrow \circ \Rightarrow \bullet \\
\end{array}
\]

Lemma 8.4.

\[
\#\text{Orbs}(B_n^{(n)}) = \begin{cases} 
\#\text{Orbs}(A_n - 1) + 1 & \text{if } n = 2k, \\
\#\text{Orbs}(A_n - 1) & \text{if } n = 2k + 1.
\end{cases}
\]

Proof. Let us denote by $L(\Delta)$ the set of labelings of a diagrams $\Delta$. Let $T_i^A$ be the elementary transformations of $L(A_n)$, and $T_i^B$ be the elementary transformations for $L(B_n^{(n)})$. Let us denote by $b = (a \Rightarrow \kappa - [1])$ the labeling with tail $a \in L(A_n - 1)$ and short root with label $\kappa \in \{0, 1\}$. Then clearly

\[(30) \quad T_i^B(a \Rightarrow \kappa - [1]) = (T_i^A(a) \Rightarrow \kappa - [1])\]

for $i < n - 1$, and by (30) it is also true for $i = n - 1$ (where we used the assumption that $\alpha_{n-1}$ is long and $\alpha_n$ is short).

We define a map $\varphi : L(A_n - 1) \rightarrow L(B_n^{(n)})$ defined by

\[
\varphi(a) = (a \Rightarrow 0 - [1]).
\]

Clearly, if $a, a' \in L(A_n - 1)$ and $a \sim a'$ then $\varphi(a) \sim \varphi(a')$. Thus we obtain an induced map $\varphi_* : \text{Orbs}(A_n - 1) \rightarrow \text{Orbs}(B_n^{(n)})$. Furthermore, if $\varphi(a) \sim \varphi(a')$ then $a \sim a'$. Thus, $\varphi_*$ is injective.

If $n = 2k$ is even we have a fixed point (labeling whose orbit contains only the labeling itself)

\[
\ell_3 = \left(1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \Rightarrow [1] \right) = (\xi_k \Rightarrow 1 - [1]) \in L(B_n^{(n)}).
\]

Clearly, $\ell_3 \not\sim \varphi(a)$ for any $a \in L(A_n - 1)$ since $(\ell_3)_n = 1$. Since $\ell_3$ is a fixed point then $\ell_3 \not\sim \varphi(a)$ for any $a \in L(A_n - 1)$.

Now, if $b \neq \ell_3$ then there exists $a \in L(A_n - 1)$ such that $b \sim \varphi(a)$. Indeed, if $b_n = 0$ then $b = (a \Rightarrow 0 - [1])$ and we take $a \in L(A_n - 1)$, and if $b_n = 1$ then $b = (a' \Rightarrow 1 - [1])$ and since $b \neq \ell_3$ then $a' \neq \xi_k$ and it has less than $k$ components which we can push to the left such that $b_{n-1} = a_{n-1} = 0$. Then we have $(\cdots 0 \Rightarrow 1 - [1])$ which by applying $T_i^B$ we get $b_n = 0$ and $b \sim (a \Rightarrow 0 - [1])$ for some $a \in L(A_n - 1)$. Thus, $[b] = \varphi_*([a])$.

Hence, for $n = 2k$ we get $\text{Im} \varphi_* = \text{Orbs}(B_n^{(n)}) \setminus \{[\ell_3]\}$.

For $n = 2k + 1$ we do not have $\ell_3$ in $L(B_n^{(n)})$ and then $\text{Im} \varphi_* = \text{Orbs}(B_n^{(n)})$.

Then the lemma follows from the bijectivity of the above constructed maps. \qed
It follows from Lemma 8.4 that
\[
\#H^1(\mathbb{R}, G) = \#\text{Orbs}(B_n^{(n)}) = \begin{cases} 
 k + 2 & \text{if } n = 2k, \\
 k + 1 & \text{if } n = 2k + 1.
\end{cases}
\]

The orbit of zero consists of two labelings:

\[
0 \rightarrow \cdots \rightarrow 0 \Rightarrow 0 \Rightarrow 1
\]

and

\[
0 \rightarrow \cdots \rightarrow 0 \Rightarrow 1 \Rightarrow 1.
\]

It follows from the proof of Lemma 8.4 that for representatives of orbits for \(B_n^{(n)}\) we can take \(\ell_3\) (when \(n\) is even) together with the images of representatives of orbits for \(A_{n-1}\) under the map

\[
a \mapsto (a \Rightarrow 0 \Rightarrow 1) : L(A_{n-1}) \rightarrow L(B_n^{(n)}).
\]

Therefore for representatives of orbits we can take

\[
\xi_0 \Rightarrow 0 \Rightarrow 1, \quad \xi_1 \Rightarrow 0 \Rightarrow 1, \quad \ldots, \quad \xi_r \Rightarrow 0 \Rightarrow 1
\]

where \(r = \lceil \frac{n-1}{2} \rceil\); we add \(\ell_3\) when \(n\) is even.

9. Groups of type \(C_n\)

9.1. The diagram \(C_n\). The Dynkin diagram of the compact group \(Sp_n\) of type \(C_n\) is

\[
\begin{array}{ccccccc}
1 & \circ & \cdots & \circ & \Leftarrow & \circ & n
\end{array}
\]

where the most right root is the long root. The long root does not see the short roots and therefore is not affected by them (see formula (4) in Section 3).

The number of orbits is

\[
\#H^1(\mathbb{R}, G) = \#\text{Orbs}(C_n) = n + 1.
\]

Let us explain why.

**Lemma 9.1.** \(\#\text{Orbs}(C_n) = \#\text{Orbs}(A_{n-1}) + \#\text{Orbs}(A_{n-1}^{(n-1)})\).

**Proof.** The left term in the RHS, \(\#\text{Orbs}(A_{n-1})\), is the case when the long root’s label is 0 (this 0 cannot be changed by elementary transformation) and then we have diagram of \(A_{n-1}\). The right term in the RHS, \(\#\text{Orbs}(A_{n-1}^{(n-1)})\), is the case when the long root’s label is 1, this 1 cannot be changed by elementary transformation and hence acts as a boxed 1, thus we have a twisted diagram of \(A_{n-1}^{(n-1)}\).

Next we note that the label of the long root cannot be changed by elementary transformation and hence kept fixed for every orbit. It follows that the total number of orbits is the sum of the number of orbits with 0 at the long root and the number of orbits with 1 at the long root.

Let us formulate this idea.

We have:

(a) \(T_i^C(a \Leftarrow \infty) = (T_i^A(a) \Leftarrow \infty)\) for \(i < n - 1\).
(b) \(T_{n-1}^C(a \Leftarrow \infty)) = a_{n-2} + a_{n-1} + \infty\) for \(i = n - 1\).
(c) \(T_n^C(a \Leftarrow \infty) = (a \Leftarrow T_n(\infty)) = (a \Leftarrow \infty)\) for \(i = n\).
(b) and (c) are derived from (1) since $\alpha_{n-1}$ is a short root and $\alpha_n$ (with label $\kappa$) is a long root.

We construct a bijection $\varphi : L(C_n) \rightarrow L(A_{n-1}) \sqcup L(A_{n-1})^{(n-1)}$ as follows: let $c = (a \leftarrow \kappa) \in L(C_n)$, if $\kappa = 0$ we send $c$ to $a \in L(A_{n-1})$ and if $\kappa = 1$ we send $c$ to $(a - [1]) \in L(A_{n-1})^{(n-1)}$. The inverse map would be sending $a \in L(A_{n-1})$ to $a \leftarrow 0 L(C_n)$ and $(a - [1]) \in L(A_{n-1})^{(n-1)}$ to $a \leftarrow 1 L(C_n)$. We see that both maps are well-defined (as explained above). We obtain an induced map $\varphi_* : \text{Orbs}(C_n) \rightarrow \text{Orbs}(A_{n-1}) \sqcup \text{Orbs}(A_{n-1})^{(n-1)}$ defined by $\varphi_*([c]) = [\varphi(c)]$. Clearly we have $c \sim c' \iff \varphi(c) \sim \varphi(c')$. Indeed

$$c \sim c' \iff (a \leftarrow \kappa) \sim (a' \leftarrow \kappa)$$

(as the label of $\kappa$ cannot be changed by elementary transformation $T_n$ as explained above).

If $\kappa = 0$ we have

$$(a \leftarrow 0) \sim (a' \leftarrow 0) \iff a \sim a'$$

and if $\kappa = 1$ we have

$$(a \leftarrow 1) \sim (a' \leftarrow 1) \iff (a - [1]) \sim (a' - [1]) \iff a \sim a'.$$

The map $\varphi_*$ is injective: if $\varphi(c) \sim \varphi(c')$ then either $a \sim a'$ and then $c \sim c'$ for $c_n = 0$ or $(a - [1]) \sim (a' - [1])$ and $c \sim c'$ for $c_n = 1$. This map is clearly surjective: for $[a] \in \text{Orbs}(A_{n-1})$ we have $\varphi_*([a \leftarrow 0]) = [a]$ and for $[(a - [1])] \in \text{Orbs}(A_{n-1})^{(n-1)}$ we have $\varphi_*([a \leftarrow 1]) = [(a - [1])]$. Thus, $\varphi$ is a bijection and the lemma follows.

Applying the results we have for $\#\text{Orbs}(A_{n-1})$ and $\#\text{Orbs}(A_{n-1})^{(n-1)} = \#\text{Orbs}(A_{n-1})^{(1)}$, we get

\[
\#\text{Orbs}(C_n) = \#\text{Orbs}(A_{n-1}) + \#\text{Orbs}(A_{n-1})^{(n-1)} = \\
\frac{(k - 1) + 2 + (k - 1) + 1 = 2k + 1 = n + 1}{(k + 1) + (k + 1) = 2k + 2 = n + 1} \quad \text{if } n = 2k,
\]

$$\text{if } n = 2k + 1.$$

We again use the notation of chaining a tail of $n - 1$ vertices left to the long root,

$$\text{if } n < 0 \rightarrow (\text{tail } \equiv 0).$$

The orbit of zero consists of just 0. Representatives of orbits are:

$$\xi_0 < 0, \quad \xi_1 < 0, \quad \cdots, \quad \xi_r < 0$$

where $r = \lfloor \frac{n - 1}{2} \rfloor$, and

$$\xi_0 < 1, \quad \xi_1 < 1, \quad \cdots, \quad \xi_s < 1$$

where $s = \lfloor \frac{n}{2} \rfloor - 1$. When $n$ is odd, $(1010...10 \leftarrow 1)$ is a fixed point, when $n$ is even, $(1010...10 \leftarrow 0)$ is a fixed point.

9.2. The diagram $C_n^{(n)}$. The Kac diagram of the group $\text{Sp}_{2n}(\mathbb{R})$ with maximal compact subgroup $D_n$ is
where the most right root is the long root. The long root does not see the short roots and therefore not affected by them (formula (4) in Section 3). We erase the vertex 0 and write a boxed 1 as a neighbor to the twisting vertex \( n \). We obtain

\[
\begin{array}{c}
1 \circ \cdots \circ n-1 \circ \quad \Rightarrow \quad n \circ \\
\end{array}
\]

In that case there is only one orbit, \( \#H^1(\mathbb{R}, G) = \#Orbs(C_n^{(n)}) = 1 \) (Of course, it is well known that \( H^1(\mathbb{R}, \text{Sp}_{2n}) = 1 \)).

9.3. The diagram \( C_n^{(m)} \). For \( n \geq 4 \) and \( 1 < m < n-1 \) the Kac diagram of \( \text{Sp}(m, n-m) \) with maximal compact subgroup \( C_m \times C_{n-m} \) is a diagram with vertex \( m \) of degree 3.

\[
\begin{array}{c}
0 \circ \circ \cdots \circ m \circ \cdots \circ n-1 \circ \quad \Rightarrow \quad n \circ \\
\end{array}
\]

which is equivalent to

\[
\begin{array}{c}
1 \circ \cdots \circ m \circ \cdots \circ n-1 \circ \quad \Rightarrow \quad n \circ \\
\end{array}
\]

where we erased the vertex 0 and accounted for the twisting vertex by adding boxed 1 as a neighbor (see Sections 4 and 6).

It is well known that

\[
\#H^1(\mathbb{R}, G) = \#Orbs(C_n^{(m)}) = n+1.
\]

Let us denote by \( (a|b < i) \) the labeling with \( \xi_a \) on the LHS to the barrier, \( \xi_b \) on the RHS to the barrier (excluding the long root) and with label \( i \in \{0, 1\} \) of the long root (this is similar to the notation of Subsection 8.2). For example, for \( C_5^{(3)} \) we have that

\[
(1|0 < 1) = \begin{pmatrix}
1 & -0 & 0 & -0 & 1 \\
\end{pmatrix}.
\]

For \( m = 1 \) the orbit of zero consists of the representative \( [1] - 0 - ... - 0 < 0 \) and all the labelings with a single component at the short roots and 0 label at the long root. For \( m > 1 \) the orbit of zero consists of the representative \( (0|0 < 0) \) and all the labelings with \( q \) components on LHS and \( q \) components on RHS, \( l(a) = r(a) \), and label 0 at the long root. For representatives of orbits we can take \( (a|0 < 0) \) with \( a = 1, ..., \left\lceil \frac{m-1}{2} \right\rceil \) and \( (0|b < 0) \) with \( b = 1, ..., \left\lceil \frac{n-1-m}{2} \right\rceil \), and \( (a|0, 1) \) with \( a = 1, ..., \left\lfloor \frac{m}{2} \right\rfloor \) and \( (0|b, 1) \) with \( b = 1, ..., \left\lfloor \frac{n-1-m}{2} \right\rfloor \), and

\[
\ell_1 = \begin{pmatrix}
1 & -0 & -1 & 0 & 1 \\
\end{pmatrix}
\]

when \( m \) is odd.

10. Groups of type \( D_n \)

10.1. The diagram \( D_n \). The Dynkin diagram of the compact group \( \text{Spin}_{2n} \) of type \( D_n \) is

\[
\begin{array}{c}
1 \circ \cdots \circ n-3 \circ n-2 \circ n-1 \\
\end{array}
\]
This diagram has a vertex of degree 3, the vertex $n - 2$. For brevity we introduce the following short notation:

$$
\ldots \circ n^{-2} \circ n^{-1} \circ n = \begin{pmatrix}
\ldots & n^{-2} & n^{-1} \\
\ldots & 1 & 1 \\
\cdot & \cdot & \cdot 
\end{pmatrix}.
$$

To count the number of orbits we state the following lemma:

**Lemma 10.1.** Let $D_n$ with $n \geq 4$. Then:

1. The labelings $0 = (0 \ldots 0 \frac{0}{1})$ and $0 \ldots \frac{1}{1}$ are fixed points.
2. The number of components can be changed by from $r$ either by splitting to $r + 2$ or by unsplitting to $r - 2$. In particular, elementary transformations in $D_n$ do not change the parity of the number of components.
3. Denote the tail by $\cdot \ldots \cdot$, then $(\cdot \ldots - 1 - 0 \frac{0}{1})$ and $(\cdot \ldots - 1 - 0 \frac{1}{1})$ are in the same orbit.
4. For $n = 2k + 1$ the labeling with $k$ component (which can be represented by $\xi_{k-1} \frac{0}{1}$) is not equivalent to any labeling with less than $k$ components.
5. If a labeling $d$ has less than $k$ components (where $n = 2k$ or $n = 2k + 1$) and it is not a fixed point (see (a)) then $d$ can be brought to the form $(\xi_i \frac{0}{1})$ for some $i < k$.

**Proof.** It can be proven by direct calculation. The assertions (a) and (b) are easy. For (c) we write

$$(\ldots 10 \frac{0}{0}) \mapsto (\ldots 11 \frac{0}{0}) \mapsto (\ldots 11 \frac{1}{1}) \mapsto (\ldots 10 \frac{1}{1}) .$$

Part (d) can be proved by a direct check. It is enough to check for the “trimmed” diagram

$$d = \left(\xi_{k-2} 101 \frac{0}{0}\right) \quad \text{with} \quad (10 \ldots 10) = \xi_{k-2} \in L(A_{n-5}).$$

Let us show that it is impossible to reduce the number of components of $d$ to less than $k$. The only way to do so is to unsplit at vertex $n - 2$, but we have:

- Case 1:
  $$(\xi_{k-2} 101 \frac{0}{0}) \mapsto (\xi_{k-2} 101 \frac{1}{1})$$
  in which $a_{n-3}$ cannot be changed from 0 to 1.
- Case 2:
  $$(\xi_{k-2} 101 \frac{0}{0}) \mapsto (\xi_{k-2} 101 \frac{0}{1}) \mapsto (\xi_{k-2} 100 \frac{0}{1})$$
  or
  $$(\xi_{k-2} 101 \frac{0}{0}) \mapsto (\xi_{k-2} 101 \frac{1}{0}) \mapsto (\xi_{k-2} 100 \frac{1}{0})$$
  in which we cannot change the label $a_{n-1}$ (or $a_n$ respectively) from 0 to 1 without changing the entire labeling to that of Case 1.

We see that in all cases we cannot change $\xi_{k-2} 101 \frac{0}{0}$ to a labeling with three 1’s around the vertex $n - 2$. Hence, we cannot do unsplitting and reduce the number of components.

For part (e): let $d = (a_i)_{i=1}^n$ be a labeling with $i < k$ components. If $a_n = a_{n-1} = 1$ then as $d \neq (0 \ldots 0 \frac{1}{1})$ by assumption there is at least one component in the tail, and we can do unsplitting at vertex $n - 2$ and then reduce to the case of $i - 1 < k$ components with
$a_n = a_{n-1} = 0$. If $a_n = a_{n-1} = 0$ then we can push all the components left to the tail.

Section 7.4 assures that if $i < k$ we can do this: we have enough room for $i < k$ components in the tail $A_{n-3}$. Consider the case where either $a_n = 1$, $a_{n-1} = 0$ or $a_n = 0$, $a_{n-1} = 1$.

Then since $i < k$ we have in the tail $i - 1$ components and a room for at least one more component, and we push the component of $a_n$ (or $a_{n-1}$ respectively) to the tail. Finally, when all the components are in the tail $A_{n-3}$ and the labeling is $d = (a00\overline{1})$, by Corollary 7.4 it can be changed to the form $d = (\xi,0\overline{1})$ by elementary transformations.

The lemma is proved. \hfill \square

We call the vertices 1 to $n - 3$ the tail and note it corresponds to the diagram $A_{n-3}$. Therefore the total number of orbits will be $\#\text{Orbs}(A_{n-3}) + \text{additional orbits which come from the vertices } n - 2, n - 1 \text{ and } n$. Let us consider two cases to see what additional orbits come from the vertices $n - 2, n - 1$ and $n$.

- For $n = 2k$ there are 3 fixed points $0...0\overline{1}$, $101...010\overline{1}$ and $101...010\overline{0}$.
- For $n = 2k + 1$ there is one fixed point $0...0\overline{1}$ and the orbit of $0...010\overline{0}$.

Lemma 10.1 and the proof of (c) imply that there are no additional orbits other than those specified above. If the number of components $i < k$ (and the labeling is different than $0...0\overline{1}$) we can pull all the components to the tail $A_{n-3}$ to get $\xi,0\overline{0}$. For $i \geq k$ we use Lemma 10.1 for the truncated diagram $\xi_{k-1} \circ \frac{1}{2}$ to see that the list above exhaust all the possible cases.

**Lemma 10.2.** For $D_n$ with $n \geq 4$ we have

$$\#H^1(\mathbb{R}, G) = \#\text{Orbs}(D_n) = \begin{cases} \#\text{Orbs}(A_{n-3}) + 3 & \text{if } n = 2k, \\ \#\text{Orbs}(A_{n-3}) + 2 & \text{if } n = 2k + 1. \end{cases}$$

**Proof.** In the first case $n = 2k$, when there are 3 fixed points, we construct a map

$$\varphi: L(A_{n-3}) \to L(D_n) \setminus \{\xi_0, 0\overline{1}, \xi_{k-1}\overline{1}, 0\overline{1}, \xi_{k-1}0\overline{1}\}$$

and the an induced map

$$\varphi_*: \text{Orbs}(A_{n-3}) \to \text{Orbs}(D_n) \setminus \{[\xi_0, 0\overline{1}], [\xi_{k-1}\overline{1}, 0\overline{1}], [\xi_{k-1}0\overline{1}]\} = \Omega.$$ 

Clearly, if $a \sim_A a'$ then $\varphi(a) \sim_D \varphi(a')$ and thus $\varphi_*$ is well-defined.

If $\varphi(a) \sim_D \varphi(a')$ then

$$\left(\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right) \sim \left(\begin{array}{c} a_1' \\ a_2' \\ a_3' \end{array}\right)$$

which clearly implies $a \sim_A a'$. This shows that $\varphi_*$ is injective.

Clearly the fixed points are not in $\text{Im } \varphi$ (as at least one of the labels of the vertices $a_{n-1}$ and $a_n$ is not 0). Since they are fixed point, then there is no $a \in L(A_{n-3})$ such that $\varphi(a)$ is equivalent (in $D_n$) to one of the fixed points. We now show that $\varphi_*$ is surjective on $\Omega$. Let $d \in L(D_n)$ be a labeling which is not a fixed point. Then $d$ has $i < k$ components. These can be pushed to the left arm and be brought to a form $\xi,0\overline{0}$ and hence $d \sim_D \varphi(\xi_i)$.

We conclude that $\text{Im } \varphi_* = \Omega$.

From the bijectivity of $\varphi_*$ the results follows in the even case.

In the second case $n = 2k + 1$, let us define a map

$$\varphi: L(A_{n-3}) \to L(D_n) \setminus \{[\xi_0, 0\overline{1}], [\xi_{k-1}\overline{1}, 0\overline{1}], [\xi_{k-1}0\overline{1}]\}$$

$$a \mapsto a00\overline{1}$$
Let us consider \( \varphi_* : \text{Orbs}(A_{n-3}) \to \text{Orbs}(D_n) \setminus \{[\xi_{k-1}1_0^0], [\xi_00_0^1]\} \) induced by \( \psi \) as above. That is:

\[
\varphi_*([a]) = \left[ \begin{pmatrix} a0 & 0 \\ 0 & 0 \end{pmatrix} \right],
\]

and in particular

\[
\varphi_*([\xi_i]) = \left[ \begin{pmatrix} \xi_i0 & 0 \\ 0 & 0 \end{pmatrix} \right].
\]

(see Section 7.1, Corollary 7.4) and send it to the orbit (equivalence class) of \( \xi_i \) not in the image of \( \varphi \).

Moreover, if \( a \sim_A a' \) then both are equivalent to \( \xi_i \) for some \( i \). Then \( \varphi_*([a]) = \varphi_*([a']) = \varphi_*([\xi_i]) \) by Lemma 10.1(d). This shows that \( \varphi_* \) is well-defined.

The map \( \varphi_* \) is also injective. Indeed, let \( a \) and \( a' \) be non-equivalent labelings \( (a \not\sim_A a') \) with less then \( k \) components. Then \( a' \sim_A \xi_i \) and \( a' \sim_A \xi_j \) for some \( i \neq j \) with \( i, j < k \). Then

\[
\varphi(a) \sim_D \varphi(\xi_i) \not\sim_D \varphi(\xi_j) \sim_D \varphi(a')
\]

as explained above. We showed \( [a] \neq [a'] \Rightarrow \varphi_*[a] \neq \varphi_*[a'] \) which proves injectivity.

By construction \( \varphi_* \) is surjective on \( \text{Orbs}(D_n) \setminus \{[\xi_{k-1}1_0^0], [\xi_00_0^1]\} \). Now, clearly \( \xi_00_0^1 \) is not in the image of \( \varphi \) (since \( a_n = 1 \)) and since it is a fixed point \( \xi_00_0^1 \not\in \text{Im} \varphi_* \). Now, \( \xi_{k-1}1_0^0 \) is not in the image of \( \varphi \) since \( a_{n-2} = 1 \). We need to show that the orbit of \( \xi_{k-1}1_0^0 \) is not in \( \text{Im} \varphi_* \). Suppose per contra that it is. Then since \( \varphi_* \) is well-defined by representatives we can write that \( (\xi_{k-1}1_0^0) \sim_D \varphi(\xi_i) \) with \( a \sim_A \xi_i \) for some \( i < k \).

But

\[
(\xi_{k-1}1_0^0) \not\sim_D (\xi_00_0^1) = \varphi(\xi_i)
\]

by Lemma 10.1(d). Contradiction. Hence, \( [\xi_{k-1}1_0^0] \not\in \text{Im} \varphi_* \). We conclude that \( \text{Im} \varphi_* = \text{Orbs}(D_n) \setminus \{[\xi_00_0^1], [\xi_{k-1}1_0^0]\} \).

From the bijectivity of \( \varphi_* \) the results follows in the odd case.

\[\square\]

From Section 7.1 we have

\[
\#\text{Orbs}(A_{2k-3}) = \#\text{Orbs}(A_{2(k-2)+1}) = k - 2 + 2 = k
\]

and

\[
\#\text{Orbs}(A_{2k+1-3}) = \#\text{Orbs}(A_{2k-2}) = \#\text{Orbs}(A_{2(k-1)}) = k - 1 + 1 = k,
\]

and then using Lemma 10.2 above we get

\[
\#H^1(\mathbb{R}, G) = \#\text{Orbs}(D_n) = \begin{cases} 
  k + 3 & \text{if } n = 2k, \\
  k + 2 & \text{if } n = 2k + 1.
\end{cases}
\]

Representatives of orbits are:

- For \( n = 2k + 1 \) we can take the following representatives

\[
\xi_00_0^0 = 0...0_0^0, \quad \xi_10_0^0 = 10...0_0^0, \quad ..., \quad \xi_{k-1}0_0^0 = 10...1_0^0,
\]

and

\[
\xi_{k-1}1_0^0 = 101...0_0^1, \quad \xi_01_1^0 = 0...1_1^1.
\]
For $n = 2k$ we can take the following $k$ representatives

\[ \xi_0 \begin{array}{c} 0 \\ 0 \end{array} = 0...0 \begin{array}{c} 0 \\ 0 \end{array}, \quad \xi_1 \begin{array}{c} 0 \\ 0 \end{array} = 10...0 \begin{array}{c} 0 \\ 0 \end{array}, \quad ..., \quad \xi_{k-1} \begin{array}{c} 0 \\ 0 \end{array} = 10...1 \begin{array}{c} 0 \\ 0 \end{array}, \]

and the 3 fixed points:

\[ \xi_{k-1} \begin{array}{c} 1 \\ 0 \end{array} = 10...1 \begin{array}{c} 0 \\ 0 \end{array}, \quad \xi_{k-1} \begin{array}{c} 0 \\ 1 \end{array} = 10...0 \begin{array}{c} 1 \\ 0 \end{array}, \quad \xi_0 \begin{array}{c} 1 \\ 1 \end{array} = 0...0 \begin{array}{c} 1 \\ 1 \end{array}. \]

In all cases $\xi_i \in L(A_{n-3})$ (i.e. it includes $n - 3$ vertices) for all $0 \leq i \leq k - 1$.

**Example 10.3.** For $D_5$ we have representatives of orbits

\[ \begin{array}{c} 0 \\ 0 \end{array}, \quad \begin{array}{c} 0 \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ 0 \end{array}, \quad \begin{array}{c} 1 \\ 1 \end{array}. \]

For $D_6$ we have representatives of orbits

\[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}, \quad \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}, \quad \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{array}, \quad \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}, \quad \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array}, \quad \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}. \]

10.2. **The diagram $D_n^{(n)}$.** The Kac diagram of the group Spin$^*_2$ with maximal compact subgroup $A_{n-1} \times T^1$ is

\[ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \]

Erasing the vertex 0 and accounting for the twisting vertex $n$ by adding boxed 1 as a neighbor (see Section 6), we get an equivalent augmented diagram

\[ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \]

In that case there are 2 orbits:

1. **The orbit of zero** which consists by odd number of components (including the boxed 1) and we can take the representative to be

\[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}. \]

2. The other orbit which is characterized by even number of components (including the boxed 1) and we can take the representative to be

\[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array}. \]

In other words, the parity of number of components is the invariant property of each orbit. The idea is that the boxed 1 spawns components which split in the 3-junction (vertex $n - 2$ of degree 3). The splitting changes the number of components by +2 and hence preserves the parity. Then these components can be pushed to the left end, with 0’s between them. Then we can spawn 2 more components from the boxed 1. We continue until no more components can be squeezed in the arms (two sides) of the 3-junction vertex $n - 2$.

Let us write this more formally:
Lemma 10.4. For $D^{(n)}_n$ we have:

(a) Parity of number of components is preserved under elementary transformations.
(b) Every labeling with odd number of components is equivalent to (32).
(c) Every labeling with even number of components is equivalent to (33).
(d) By (a), we get that (32) and (33) are not equivalent.

Proof. For (a), if we apply $T_i$ for $i \neq n - 2$ we get by Lemma 7.1 that the number of components is preserved, and hence also the parity. If we apply $T_{n-2}$ we see that the number of components may change only by $\pm 2$ (splitting and unsplitting). Hence, parity of the number of components is preserved as well.

For (b), suppose we have odd number $r$ of components (including the boxed 1). We may assume that all the components, except at most 2 (the boxed 1 and another component) are in the left arm (left to the 3-junction, the vertex of degree 3). By Lemma 7.1 we may assume that they are packed maximally to the left. Assume that by induction we proved that the any labeling with $r - 2$ components is equivalent to the 0 labeling (the labeling with all 0’s but the boxed 1) and hence in the orbit of zero. If $r = 1$ there is nothing to prove so we can assume $r \geq 3$. If the right arm label (vertex $n - 1$) is 1 then we push a component from the left arm to vertex $n - 3$. If the right arm (vertex $n - 1$) is 0 then push one component from the left arm to this vertex. Then push one component to vertex $n - 3$ and also apply $T_n$.

We get

\[ \cdots 0 \quad 1 \quad 0 \quad 1 \]

We then do unsplitting at the 3-junction (vertex $n - 2$) and get $r - 2$ components. We recess this merged component to the boxed 1 (by applying $T_{n-1}$ then $T_{n-3}$ then $T_{n-2}$ then $T_n$). Now we have a labeling with $r - 2$ components with odd $r$. We use the induction hypothesis to conclude it is equivalent to the 0 labeling.

(c) is similar to (b). We use the above described process and swallow components by pushing, unsplitting and recessing to the boxed 1. By induction, we are then left with one “free” component and the boxed 1 component.

For (d), if (32) and (33) were equivalent, then there was in some step an elementary transformation that change the parity from odd to even, which contradicts (a).

10.3. The diagrams $D^{(m)}_n$. Here we assume that $1 \leq m < \left\lfloor \frac{n}{2} \right\rfloor$. The Kac diagram is of the group $SO_{2m,2(n-m)}$ with maximal compact subgroup $D_m \times D_{n-m}$. That is

\[ \begin{array}{c}
\begin{array}{c}
1 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
2 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
m \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
n - 2 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
n - 1 \circ
\end{array}
\end{array}
\end{array} \]

By erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1 we get an equivalent diagram:

\[ \begin{array}{c}
\begin{array}{c}
1 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
2 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
m \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
n - 2 \circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
n - 1 \circ
\end{array}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
\circ
\end{array}
\end{array} \]
Then we have:

(34) \[ \#H^1(\mathbb{R}, G) = \#\text{Orbs}(D_n^m) = \begin{cases} k + 2 & \text{if } n = 2k + 1, \\ k + 3 & \text{if } n = 2k \text{ and } m \text{ is even,} \\ k & \text{if } n = 2k \text{ and } m \text{ is odd.} \end{cases} \]

Proof of (34). The idea behind this formula is a generalization of Section 7.2. Let us call the twisted vertex the barrier. We then have a schematic diagram:

\[ \text{LHS} \quad \overset{0}{\text{---}} \quad \overset{1}{\text{---}} \quad \text{RHS} \]

As in Section 7.2 if we have a component on the LHS and a component on the RHS, then they cancel each other by unsplitting at the barrier. Moreover, a component cannot pass through the barrier to the other side. See The Barrier Lemma (Lemma 7.5) in subsection 7.2.

Let us check when do we have additional fixed points (that are not of the forms $\xi_i \cdot \cdot \cdot \xi_j$ where $\cdot$ denotes the twisted vertex, and we put $a_n = 0$):

(a) In the case $m$ is even we have the fixed points

\[ \ell_2 = \left( \begin{array}{ccccccc} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & 1 & 0 & \cdots & 1 & 0 & 0 \\ \end{array} \right) \]

and

\[ \ell_4 = \left( \begin{array}{ccccccc} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & 1 & 0 & \cdots & 1 & 0 & 1 \\ \end{array} \right). \]

The point $\ell_2$ is counted as the maximal number of components in LHS but $\ell_4$ should be added.

(b) When $n$ and $m$ have the same parity we have another fixed point we need to count,

\[ \ell_3 = \left( \begin{array}{ccccccc} \xi_0 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & \xi_0 & 0 & \cdots & 1 & 0 & 0 \\ \end{array} \right). \]

This is in addition to

\[ \left( \begin{array}{ccccccc} \xi_0 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & \xi_0 & 0 & \cdots & 1 & 0 & 1 \\ \end{array} \right), \]

which is counted by the maximal number of components in RHS.

Note that cases (a) and (b) can occur together.

There are no additional orbits as the following Lemma shows:

Lemma 10.5. In the above settings, if

\[ d = \left( \begin{array}{ccccccc} \text{LHS} & 0 & a \\ \end{array} \right) = (a_i)_{1 \leq i \leq n} \]
in not a fixed point, then $d$ can be changed by a sequence of elementary transformations to a labeling

$$d' = \begin{pmatrix} 
\text{LHS'--0--a'} \\
\text{\vdots} \\
0 
\end{pmatrix} = (a'_i)_{1 \leq i \leq n}$$

with $a'_n = 0$ and $\xi_i$ on the vertices $a'_{m+1}$ to $a'_{n-1}$, for some $0 \leq i \leq [(n - m)/2]$ (the LHS might change as well).

Proof of Lemma 10.5. The proof of this lemma is rather technical and consists of checking two cases: $n - m$ is even and $n - m$ is odd.

1. First case: $n - m$ is odd. In that case we can squeeze $r = (n - m + 1)/2$ components in RHS, with the rightmost component being on vertex $n - 2$ (that is: $a_{n-2} = 1$). If the number of components in $a$ is less than $r$ the assertion is clear: one unsplits at vertex $n - 2$ if necessary (it is possible since $d \neq \ell_4$ by assumption) to change $a_n$ from 1 to 0, and then pushes left the components. If the number of components is $r$ the labeling is one of the forms

   Case 1.1 = \begin{pmatrix} 
\text{LHS--0--1} \\
\text{\vdots} \\
0 
\end{pmatrix}

   or

   Case 1.2 = \begin{pmatrix} 
\text{LHS--0--0} \\
\text{\vdots} \\
1 
\end{pmatrix}

   or

   Case 1.3 = \begin{pmatrix} 
\text{LHS--0--0} \\
\text{\vdots} \\
1 
\end{pmatrix}

   or

   Case 1.4 = \begin{pmatrix} 
\text{LHS--0--1} \\
\text{\vdots} \\
1 
\end{pmatrix}

with $\xi_{r-1} = (1 - 0 - 1 - 0 - 1 - 0) \in L(A_{n-m-3})$. Then Case 1.1 is of the desired form $\xi_r \in L(A_{n-m-1})$. It is easily seen that Case 1.2, Case 1.3 and Case 1.4 can be brought to the form of Case 1.1 by sequences of elementary transformations.

2. Second case: $n - m$ is even. In that case we can squeeze $r = (n - m)/2$ components in RHS, with the rightmost component being on vertex $n - 1$ (that is: $a_{n-1} = 1$). If the number of components in $a$ is less than $r$ the entire labeling is different than $\ell_4$ (i.e, $d \neq \ell_4$) the assertion is clear (one unsplits at vertex $n - 2$ if necessary to change $a_n$ from 1 to 0, and then pushes left the components). If the number of components is $r$ the labeling is one of the forms

   Case 2.1 = \begin{pmatrix} 
\text{LHS--0--1} \\
\text{\vdots} \\
0 
\end{pmatrix}
Case 2.2 = 
\[
\begin{array}{c}
\text{LHS} \\
0 \\
\xi_{r-1} \\
0 \\
1 \\
\end{array}
\]
with \(\xi_{r-1} = (1 - 0 - 1 - 0 - 1) \in L(A_{n-m-3})\). If the labeling is of the form Case 2.1 we have the desired form \(\xi_r \in L(A_{n-m-1})\). If the labeling is of the form Case 2.2 and is not a fixed point (thus \(d \neq \ell_3\)) it must either have less than \(r\) components (which we assumed is not the case) or have at least 1 component at the LHS, which cancel one of the components in RHS and reduces the case to a number of components less than \(r\). The assertion follows.

This implies that the number of orbits is: the maximal number of components on LHS + the maximal number of components on RHS + zero components + the number of additional fixed points (\(\ell_3\) and \(\ell_4\) which do not belong to the cases of \(\xi_i\) at LHS or \(\xi_j\) at RHS). The maximal number of components in LHS, the maximal number of components in RHS and the number of fixed points depend on \(n\) and \(m\) and their parity. We do the calculation of

\[
\#\text{Orbs}(D_n^{(m)}) = \# \left( \text{components at LHS} \right) + \# \left( \text{components at RHS} \right) + 1 + \# \text{ fixed labelings}
\]

by dividing into cases:

- \(n = 2k + 1\) and \(m = 2s\) (here case (a) holds):
  \[
  \#\text{Orbs}(D_n^{(m)}) = \left\lceil \frac{2s - 1}{2} \right\rceil + \left\lceil \frac{2k + 1 - 2s - 1}{2} \right\rceil + 1 + 1 = s + (k - s) + 1 + 1 = k + 2
  \]

- \(n = 2k + 1\) and \(m = 2s + 1\) (here case (b) holds):
  \[
  \#\text{Orbs}(D_n^{(m)}) = \left\lceil \frac{2s + 1 - 1}{2} \right\rceil + \left\lceil \frac{2k + 1 - (2s + 1) - 1}{2} \right\rceil + 1 + 1 = s + (k - s) + 1 + 1 = k + 2
  \]

- \(n = 2k\) and \(m = 2s\) (here cases (a) and (b) hold):
  \[
  \#\text{Orbs}(D_n^{(m)}) = \left\lceil \frac{2s - 1}{2} \right\rceil + \left\lceil \frac{2k - 2s - 1}{2} \right\rceil + 1 + 2 = s + (k - s) + 1 + 2 = k + 3
  \]

- \(n = 2k\) and \(m = 2s + 1\):
  \[
  \#\text{Orbs}(D_n^{(m)}) = \left\lceil \frac{2s + 1 - 1}{2} \right\rceil + \left\lceil \frac{2k - (2s + 1) - 1}{2} \right\rceil + 1 = s + (k - s - 1) + 1 = k
  \]

Let us denote by \((a|b, i_1, i_2)\) the labeling with \(\xi_a\) left to the barrier, \(\xi_b\) right to the barrier (excluding vertices \(n - 1\) and \(n\)) and \(i_1\) the labeling of the \(n - 1\)-th vertex and \(i_2\) the labeling of the \(n\)-th vertex \((i_1, i_2 \in \{0, 1\})\). Note that for any labeling different from \(\ell_2\) and \(\ell_4\) (the fixed points) the labelings \((a|b, 0, 0)\) and \((a|b, 1, 1)\) are in the same orbit due to the process of splitting in the \(n - 2\)-th vertex.
In the case \( m = 1 \) the orbit of zero consists of all the elements in the following characterization:

(i) Start with all vertex set to 0 (except for the boxed 1).

\[
\begin{array}{ccccccc}
1 & 0 & \cdots & 0 & 0 & 0 \\
\end{array}
\]

(ii) Fill the row with 1’s, starting from the boxed 1, up to the 3-junction.

\[
\begin{array}{ccccccc}
1 & 1 & \cdots & 1 & 1 & 0 \\
\end{array}
\]

(iii) Fill in any combination of its neighbors.

\[
\begin{array}{ccccccc}
1 & 1 & \cdots & 1 & 1 & 1 \\
& & & & & 0 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
\end{array}
\]

(iv) From the all 1’s labeling, split by apply elementary transformation on the 3-junction.

\[
\begin{array}{ccccccc}
1 & 1 & \cdots & 1 & 0 & 1 \\
\end{array}
\]

(v) Recess the 1’s in the left arm of the diagram.

\[
\begin{array}{ccccccc}
0 & \cdots & 0 & 0 & 0 & 1 \\
\end{array}
\]

(vi) Each intermediate step (i.e. application of elementary transformation) is a labeling that is in the orbit of 0.

We call this characterization or process \textit{Spawning, Filling, Splitting, Recessing} or \textbf{SFSR} for short.

For any \( 1 \leq m < \lfloor \frac{n}{2} \rfloor \) the orbit of zero consists of all the following labelings:

\begin{itemize}
  \item The canonical representative \((0|0, 0, 0)\).
  \item The labeling \((0|0, 1, 1)\).
  \item All the labeling with \(q\) component on LHS and \(q\) components on RHS (excluding vertices \(n - 1\) and \(n\)), and \(a_{n-1} = a_n = 0\).
  \item All the labeling with \(q\) component on LHS and \(q\) components on RHS (excluding vertices \(n - 1\) and \(n\)), and \(a_{n-1} = a_n = 1\).
  \item Labelings with single component (including the boxed 1).
\end{itemize}

For \textbf{representatives of orbits} we can take \((0|0, 0, 0)\), \((a|0, 0, 0)\) and \((0|b, 0, 0)\) where \(1 \leq a \leq \lfloor m/2 \rfloor = \lceil (m - 1)/2 \rceil\) and \(1 \leq b \leq \lfloor (n - m)/2 \rfloor\), and the fixed points \(\ell_4\) and \(\ell_3\) when they occur.

This explanation may be cumbersome so an example will help to understand what we mean.
**Example 10.6.** Consider $D_{10}^{(5)}$.

Then all the options are:

$(0|0, 0, 0)$, $(1|0, 0, 0)$, $(2|0, 0, 0)$, $(0|1, 0, 0)$, $(0|2, 0, 0)$ \implies \#(D_{10}^{(5)}) = 5$.

Note that The orbit of 0 in this example is characterized by the SFSR component. Since there are two vertices of degree 3 there can only be two splittings and hence only 1, 3 or 5 components in the forms $(0|0, 0, 0)$, $(1|0, 0, 0)$, $(2|0, 0, 0)$, $(0|0, 1, 1)$, $(1|1, 1, 1)$ or $(2|2, 1, 1)$ (note that the boxed 1 is another component).

Let us check that $(0|0, 1, 1) = \left( \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \end{array} \right)$ is in the orbit of zero. One can see that by spawning from the boxed 1 (explicitly: applying $T_5$ then $T_6$ the $T_7$) and then unsplitting ($T_8$) one gets a single component which can be recessed back to the boxed 1 (apply $T_{10}$ then $T_6$ then $T_8$ then $T_7$ then $T_6$ and then $T_5$), thus giving us the canonical representative of the orbit of zero (i.e. all 0’s but the boxed 1). We may say that “the boxed 1 swallowed the 1’s of vertices $n - 1$ and $n$”.

Let us check that $(1|1, 1, 1) = \left( \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \end{array} \right)$ is also in the orbit of zero, as can be seen by unsplitting the vertex $n - 1$ (the rightmost vertex of degree 3), then pushing the components to the barrier and then again unsplitting, recessing and swallowing in the boxed 1. The elementary transformation which do it are: $T_8$ (unsplitting), then $T_9$ and $T_{10}$, then $T_8$ (recessing), then pushing the components by $T_6$ and $T_7$ (RHS) and $T_2$, $T_1$, $T_3$, $T_2$, $T_4$, $T_3$ (LHS),

\begin{align*}
0 &-0-1-0-1-0-0-0 \\
\end{align*}

unsplitting by $T_5$ and then recessing by $T_4$, $T_6$ and finally $T_5$ to swallow the last one.

10.4. **The diagram $D_{4}^{(2)}$.** The Kac diagram is

which is equivalent to

```plaintext
Although the results of the previous subsection are correct for this case, the considerations are different since we have a vertex of degree 4. Therefore we state the results here explicitly. It has 5 orbits:

1. The orbit of zero consists of labelings with odd number of components (including the boxed 1).

2. The fixed point

3. The fixed point

4. The fixed point

5. The fixed point

10.5. The group $\text{Spin}_{2m+1,2(n-m)+1}$. Here $0 \leq m \leq \lfloor n/2 \rfloor$

This is an outer form of the compact group $\text{Spin}_{2n+2}$ of type $D_{n+1}$, with maximal compact subgroup of type $B_m \times B_{n-m}$. The Kac diagram is:

We erase the vertex 0 and the glued together vertex $n$.

If $m = 0$, we obtain $D^\nu = A_{n-1}$ (non-twisted). By Corollary 5.4 and (25)

$$\#H^1(\mathbb{R}, G) = \#\text{Orbs}(D^\nu) = \lfloor n/2 \rfloor + 1.$$  

The orbit of 0 is 0 (in $D^\nu$). For representatives of orbits we can take $\xi_0, \xi_1, \ldots, \xi_r$, where $r = \#H^1(\mathbb{R}, G) - 1 = \lfloor n/2 \rfloor$.

If $m \neq 0$, after erasing the vertices 0 and $n$ we obtain the twisted diagram $A^{(m)}_{n-1}$. We add boxed 1 as a neighbor to vertex $m$ and obtain the augmented diagram

By Corollary 5.4 and formula (27)

$$\#H^1(\mathbb{R}, G) = \#\text{Orbs}(D^\nu) = \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{n-1-m}{2} \right\rfloor$$

The orbit of zero consists of labelings having the same number of components in LHS and in RHS (not taking in account the component of the boxed 1), see Subsection
For representatives of orbits we can take
\[(0|0) = \begin{pmatrix} \xi_0 & 0 & -\xi_0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (i|0) = \begin{pmatrix} \xi_i & 0 & -0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (0|j) = \begin{pmatrix} 0 & 0 & -\xi_j \\ 1 & 0 & 0 \end{pmatrix},\]
where \(1 \leq i \leq \lfloor (m-1)/2 \rfloor\) and \(1 \leq j \leq \lfloor (n-m-1)/2 \rfloor\).

11. Groups of type \(E_6\)

11.1. The compact group of type \(E_6\). The Dynkin diagram is

\[\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}\]

It has three orbits:

1. The orbit of zero consists of 0 which is invariant.
2. The orbit of odd number (1 or 3) of components, with representative 10000.
3. The orbit with positive even number (2) of components with representative 10001.

Remark 11.1. Note that elementary transformations on the \(E_n\), \(n = 6, 7, 8\) diagrams preserve the parity of the number of components. The proof is the same as in Lemma 10.4(a) but note that (b) and (c) may not necessarily hold since we need to consider also fixed points (invariants).

11.2. The group \(E^{(1)}_6\) with diagram \(E^{(1)}_6\). The maximal compact subgroup is isogenous to \(D_5 \times T^1\). The Kac diagram is

\[\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}\]

Erasing the vertex 0 and adding a boxed 1 to account for the twisting vertex we get the following augmented diagram:

\[\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}\]

It has three orbits:

1. The orbit of zero consists of labelings with odd number (1 or 3) of components, excluding the fixed point \(\ell_3\).
2. The orbit with positive even number (2) of components with representative [1]00001.
3. The fixed point
\[\ell_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}\]
11.3. **The group $EII$ with diagram $E_6^{(2)}$.** The maximal compact subgroup is of type $A_1 \times A_5$. The Kac diagram is

```
  1 - 2 - 3 - 4 - 5
   \   \   \   \   \6
    \   \   \   \   \
     \   \   \   \   \
      \   \   \   \   \
       \   \   \   \   \
        \   \   \   \   \
         \   \   \   \   \
          \   \   \   \   \
           \   \   \   \   \
            \   \   \   \   \
             \   \   \   \   \
              \   \   \   \   \
```

Erasing the vertex 0 and adding a boxed 1 to account for the twisting vertex we get the following equivalent diagram:

```
  1 - 2 - 3 - 4 - 5
   \   \   \   \   \6
    \   \   \   \   \
     \   \   \   \   \
      \   \   \   \   \
```

It has three orbits:

1. **The orbit of zero** consists of labelings with odd number (1 or 3) of components (including the boxed 1).
2. The orbit with positive even number (2) of components with representative

```
    0\[\boxed{1}\]0\[\boxed{1}\]
```

excluding the fixed point $\ell_2$.
3. The fixed point

```
    \ell_2 = \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0
    \end{pmatrix}
```

11.4. **The group $EIV$ of type $E_6$.** This is an outer form of the compact group of type $E_6$ with maximal compact subgroup of type $F_4$. The Kac diagram is

```
  0 - - - - -
       \   \   \   \   \
        \   \   \   \   \
```

We denote by $\nu$ the nontrivial automorphism of the Dynkin diagram $D = E_6$. We have $D^\nu = 3 - 6$ and $\# H^1(\mathbb{R}, G) = \# \text{Orbs}(D^\nu) = 2$. **The orbit of 0** consists of 0 on $D^\nu$. For representatives of orbits we can take 00 and 10.

11.5. **The group $EI$ of type $E_6$.** This is an outer form of the compact group of type $E_6$ with maximal compact subgroup of type $C_4$. The Kac diagram is

```
  0 - - - - -
       \   \   \   \   \
        \   \   \   \   \
```

We have $D^\nu = 3 - 6$. The augmented diagram is

```
  0 - - - - -
       \   \   \   \   \
```

We have $\# H^1(\mathbb{R}, G) = \# \text{Orbs}(D^\nu) = 2$. **The orbit of 0** consists of labelings with one component (including the boxed 1). For representatives of orbits we can take 00[1] and 10[1].
12. Groups of type $E_7$

12.1. The compact group of type $E_7$. The Dynkin diagram is

```
1 --- 2 --- 3 --- 4 --- 5 --- 6
    |      
    7
```

The orbits are:

1. The orbit of zero consists of the invariant zero:

```
0 --- 0 --- 0 --- 0 --- 0
    0
```

2. The fixed point $\ell_3$:

```
1 --- 0 --- 0 --- 0 --- 0
```

3. All the labelings with 1 or 3 components, excluding $\ell_3$.
4. All the labelings with 2 or 4 components.

12.2. The group $E_7^{(7)}$ with diagram $E_7^{(7)}$. This is a twisted (noncompact) form with maximal compact subgroup of type $A_7$. The Kac diagram is

```
1 --- 2 --- 3 --- 4 --- 5 --- 6 --- 0
    |      
    7
```

By erasing the vertex 0 and treating the twisting vertex by adding a boxed 1, we obtain the augmented diagram

```
1 --- 2 --- 3 --- 4 --- 5 --- 6
    |      
    7
```

```

The are two orbits, $\#H^1(\mathbb{R}, G) = \#\text{Orbs}(E_7^{(7)}) = 2$. The orbits are:

1. The orbit of zero

```
0 --- 0 --- 0 --- 0 --- 0
    0
```

which is

```
0 --- 0 --- 0 --- 0 --- 0
    0
```

is the orbit of all labelings with 1 or 3 components (including the boxed 1).

2. The orbit of

```
1 --- 0 --- 1 --- 1 --- 0
```

which is
\[
1\rightarrow 0\rightarrow 1\rightarrow 0\rightarrow 1\rightarrow 0
\]

is the orbit of all labelings with 2 or 4 components (including the boxed 1).

12.3. **The group EVII with diagram** $E_7^{(1)}$. This is a twisted (noncompact) form with maximal compact subgroup isogenous to $E_6 \times T^1$. The Kac diagram is
\[
\begin{array}{c}
\bullet \\
1 & 2 & 3 & 4 & 5 & 6 & 0 \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
7
\end{array}
\]
Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1, we get an augmented diagram:
\[
\begin{array}{c}
\bullet \\
1 & 2 & 3 & 4 & 5 & 6 \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
7
\end{array}
\]
It has 2 orbits. The orbits are:
1. The orbit of zero:
\[
0\rightarrow 0\rightarrow 0\rightarrow 0\rightarrow 0\rightarrow 0
\]
which is
\[
\begin{array}{c}
\bullet \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
7
\end{array}
\]
is the orbit of all labelings with 1 or 3 components (including the boxed 1).
2. The orbit of all labelings with 2 or 4 components (including the boxed 1).

12.4. **The group EVI with diagram** $E_7^{(2)}$. This is a twisted (noncompact) form with maximal compact subgroup isogenous to $A_1 \times D_6$. The Kac diagram is
\[
\begin{array}{c}
\circ \\
1 & 2 & 3 & 4 & 5 & 6 & 0 \\
\bullet & \circ & \circ & \circ & \circ & \circ & \circ \\
7
\end{array}
\]
Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1, we get an augmented diagram
\[
\begin{array}{c}
\circ \\
1 & 2 & 3 & 4 & 5 & 6 \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
7
\end{array}
\]
Then $\#H^1(\mathbb{R}, G) = \#\text{Orbs}(E_7^{(2)}) = 4$ and the orbits are:
1. The orbit of zero:
\[
0\rightarrow 0\rightarrow 0\rightarrow 0\rightarrow 0
\]
which is

\[
\begin{array}{cccccccc}
0 & - & 0 & - & 0 & - & 0 & - & 0 \\
\end{array}
\]

is the orbit of all labelings with 1 or 3 or 5 components (including the boxed 1) excluding \(\ell_3\) (see below).

2. The fixed point \(\ell_2\) which is

\[
\begin{array}{cccccccc}
1 & - & 0 & - & 0 & - & 0 & - & 0 \\
\end{array}
\]

3. The fixed point \(\ell_3\) which is

\[
\begin{array}{cccccccc}
0 & - & 0 & - & 1 & - & 0 & - & 0 \\
\end{array}
\]

4. The orbit of all labelings with 2 or 4 components (including the boxed 1), excluding \(\ell_2\).

### 13. Groups of type \(E_8\)

#### 13.1. The compact group with diagram \(E_8\).

The Dynkin diagram is

\[
\begin{array}{cccccccc}
1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 \\
\end{array}
\]

It has 3 orbits.

1. **The orbit of zero** which contains only 0.
2. The orbit represented by \(1 - 0 - 0 - 0 - 0 - 0 - 0 - 0\).
3. The orbit represented by \(1 - 0 - 1 - 0 - 0 - 0 - 0 - 0\).

#### 13.2. The group \(EIX\) with diagram \(E_8^{(1)}\).

The maximal compact subgroup is isogenous to \(A_1 \times E_7\). The Kac diagram is

\[
\begin{array}{cccccccc}
0 & & 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 \\
\end{array}
\]

Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1, we get an augmented diagram

\[
\begin{array}{cccccccc}
\boxed{1} & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 \\
\end{array}
\]

It has 3 orbits.

1. **The orbit of zero** consists of labelings with odd number of components (including the boxed 1).
2. The fixed point

\[
\ell_4 = \begin{pmatrix}
\boxed{1} & 0 & - & -1 & - & -1 & - & 0 & - & 0
\end{pmatrix}
\]
3. The orbit represented by $[1] - 0 - 1 - 0 - 0 - 0 - 0 - 0$ and characterized by even number of components excluding $\ell_4$.

13.3. **The group EVIII with diagram $E_8^{(7)}$.** The maximal compact subgroup is of type $D_8$. The Kac diagram is

```
         0     1     2     3     4     5     6     7
         o---o---o---o---o---o---o
            |               |
            v               v
              8
```

Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1, we get an augmented diagram

```
         1     2     3     4     5     6     7     8
         o---o---o---o---o---o---o
            |               |
            v               ↓
              8
```

It has 3 orbits.

1. **The orbit of zero** consists of labelings with odd number of components (including the boxed 1) and excluding the fixed point $\ell_3$.
2. The fixed point

$$\ell_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \rightrule\boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. The orbit represented by $1 - 0 - 0 - 0 - 0 - 0 - 0 - [1]$ and characterized by even number of components.

14. **Groups of type $F_4$**

14.1. **The compact group of type $F_4$.** The Dynkin diagram is

```
         1     2     3     4
         o---o---o---o
```

where the right roots are long and the left roots are short.

It has 3 orbits.

1. **The orbit of 0** which contains only $(0 - 0 \Leftarrow 0 - 0)$.
2. The orbit

$$\{(1 - 0 \Leftarrow 0 - 0) \mapsto (1 - 1 \Leftarrow 0 - 0) \mapsto (0 - 1 \Leftarrow 0 - 0)\}$$

3. The orbit which contains the rest, represented by $(0 - 0 \Leftarrow 0 - 1)$.

14.2. **The group $FII$ with diagram $F_4^{(1)}$.** The maximal compact subgroup is of type $B_4$. The Kac diagram is

```
         1     2     3     4     0
         o---o---o---o---o
```

Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1 we get an equivalent diagram

```
         1     2     3     4
         o---o---o---o
```

where the right roots are long and the left roots are short.

It has 3 orbits.
1. **The orbit of zero** consists of

\[ \{([1] - 0 - 0 \Leftarrow 0 - 0) \mapsto ([1] - 1 - 0 \Leftarrow 0 - 0) \mapsto ([1] - 1 - 1 \Leftarrow 0 - 0) \} . \]

2. The fixed point \( \ell_2 = ([1] - 0 - 1 \Leftarrow 0 - 0) \).

3. The orbit which contains the rest, represented by \([1] - 0 - 0 \Leftarrow 0 - 1\).

### 14.3. The group \( FI \) with diagram \( F_4^{(4)} \)

The maximal compact subgroup is of type \( C_3 \times A_1 \).

The Kac diagram is

\[
\begin{array}{cccc}
1 & 2 & \Rightarrow & 3 \\
\circ & \circ & \Rightarrow & 4 \\
& & & 0 \\
\end{array}
\]

Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1 we get the augmented diagram

\[
\begin{array}{cccc}
1 & 2 & \Rightarrow & 3 \\
\circ & \circ & \Rightarrow & 0 \Box \\
& & & 1 \\
\end{array}
\]

where the right roots are long and the left roots are short.

It has 3 orbits.

1. **The orbit of zero** consists of labeling having only 1 component in the long roots (including the boxed 1). It is the largest orbit and can also be characterized by what labelings are not in it, since there are only 4 such labelings.

2. The fixed point \( \ell_3 = (1 - 0 \Leftarrow 1 - 0 - [1]) \).

3. The orbit which contains

\[ \{([0 - 0 \Leftarrow 1 - 0 - [1]) \mapsto (0 - 1 \Leftarrow 1 - 0 - [1]) \mapsto (1 - 1 \Leftarrow 1 - 0 - [1])\} . \]

### 15. Groups of type \( G_2 \)

#### 15.1. The compact group of type \( G_2 \)

The Dynkin diagram is

\[
\begin{array}{cc}
1 & 2 \\
\circ & \circ \\
\end{array}
\]

It behaves similarly to \( A_2 \) since \( 3 \equiv 1 \) (mod 2). The 2 orbits are

\[ [\xi_0] = \{(0 - 0)\} \quad \text{and} \quad [\xi_1] = \{(1 - 0), (1 - 1), (0 - 1)\} . \]

#### 15.2. The split group of type \( G_2^{(2)} \)

The maximal compact subgroup is of type \( A_1 \times A_1 \).

The Kac diagram is

\[
\begin{array}{cc}
1 & 2 & 0 \\
\circ & \circ & \circ \\
\end{array}
\]

Erasing the vertex 0 and accounting for the twisting vertex by adding boxed 1 we get the augmented diagram

\[
\begin{array}{cc}
1 & 2 \\
\circ & \circ \Box \\
\end{array}
\]

It behaves similarly to \( A_2^{(2)} \). The 2 orbits are

\[ [0] = \{(0 - 0 - [1]), (0 - 1 - [1]), (1 - 1 - [1])\} \quad \text{and} \quad \{(1 - 0 - [1])\} . \]
16. Tables

The following tables summarize the number of orbits for each diagram. (These results have been earlier obtained by Adams [A] by other methods.)

| Type of $G$ | $|H^1(\mathbb{R}, G)| = \#\text{Orbs}(G)$ |
|-------------|------------------------------------------|
| $A_n$       | $\#\text{Orbs}(A_n) = \begin{cases} k + 1 & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1. \end{cases}$ |
| $A_n^{(m)}$ | $\#\text{Orbs}(A_n^{(m)}) = \begin{cases} k + 1 & \text{if } n = 2k, \\ k + 1 & \text{if } n = 2k + 1 \text{ and } m \text{ is odd}, \\ k + 2 & \text{if } n = 2k + 1 \text{ and } m \text{ is even}. \end{cases}$ |
| $B_n$       | $\#\text{Orbs}(B_n) = \begin{cases} k + 2 & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1. \end{cases}$ |
| $B_n^{(m)}$, $1 \leq m < n$ | $\#\text{Orbs}(B_n^{(m)}) = \begin{cases} k & \text{if } n = 2k \text{ and } m \text{ is odd}, \\ k + 2 & \text{if } n = 2k \text{ and } m \text{ is even}, \\ k + 1 & \text{if } n = 2k + 1 \text{ and } m \text{ is odd}, \\ k + 2 & \text{if } n = 2k + 1 \text{ and } m \text{ is even}. \end{cases}$ |
| $B_n^{(n)}$ | $\#\text{Orbs}(B_n^{(n)}) = \begin{cases} k + 2 & \text{if } n = 2k, \\ k + 1 & \text{if } n = 2k + 1. \end{cases}$ |
| $C_n$       | $\#\text{Orbs}(C_n) = n + 1$ |
| $C_n^{(m)}$, $1 \leq m < n$ | $\#\text{Orbs}(C_n^{(m)}) = n + 1$ |
| $C_n^{(n)}$ | $\#\text{Orbs}(C_n^{(n)}) = 1$ |
| $D_n$, $n \geq 4$ | $\#\text{Orbs}(D_n) = \begin{cases} k + 3 & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1. \end{cases}$ |
| $D_n^{(m)}$, $1 \leq m < \left\lfloor \frac{n}{2} \right\rfloor$ | $\#\text{Orbs}(D_n^{(m)}) = \begin{cases} k + 2 & \text{if } n = 2k + 1, \\ k + 3 & \text{if } n = 2k \text{ and } m \text{ is even}, \\ k & \text{if } n = 2k \text{ and } m \text{ is odd}. \end{cases}$ |
| $D_n^{(n)}$ | $\#\text{Orbs}(D_n^{(n)}) = 2$ |
| $D_4^{(2)}$ | $\#\text{Orbs}(D_4^{(2)}) = 5$ |
| Type of $G$ | $|H^1(\mathbb{R}, G)| = \#\text{Orbs}(G)$ |
|-------------|------------------------------------------|
| $E_6$       | 3                                        |
| $E_6^{(1)}$ | 3                                        |
| $E_6^{(2)}$ | 3                                        |
| $E_7$       | 4                                        |
| $E_7^{(1)}$ | 2                                        |
| $E_7^{(2)}$ | 4                                        |
| $E_7^{(3)}$ | 2                                        |
| $E_8$       | 3                                        |
| $E_8^{(1)}$ | 3                                        |
| $E_8^{(2)}$ | 3                                        |
| $F_4$       | 3                                        |
| $F_4^{(1)}$ | 3                                        |
| $F_4^{(2)}$ | 3                                        |
| $G_2$       | 2                                        |
| $G_2^{(1)}$ | 2                                        |

17. ORBITS OF ALGEBRAIC GROUPS IN HOMOGENEOUS SPACES

Let $G$ be a linear algebraic group over $\mathbb{R}$, for short an $\mathbb{R}$-group. Let $H \subset G$ be an $\mathbb{R}$-subgroup. Set $X = G/H$, it is an $\mathbb{R}$-variety with a left $\mathbb{R}$-action of $G$. Therefore, the group $G(\mathbb{R})$ of $\mathbb{R}$-points acts on the left on $X(\mathbb{R})$. By [S, I.5.1, Corollary 1 of Proposition 36], there is a canonical bijection

$$G(\mathbb{R}) \backslash X(\mathbb{R}) \cong \ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)].$$

The following lemma is well known.

**Lemma 17.1.** If $G$ is semisimple and simply connected, then any orbit of $G(\mathbb{R})$ in $(G/H)(\mathbb{R})$ is a connected component of $(G/H)(\mathbb{R})$.

**Proof.** Write $X = G/H$. Let $x \in X(\mathbb{R})$, then we have a map

$$\phi_x : G(\mathbb{R}) \to X(\mathbb{R}), \quad g \mapsto g \cdot x.$$ Since the differential of $\phi_x$ at any point of $X(\mathbb{R})$ is surjective, by the implicit function theorem the map $\phi_x$ is open, hence the orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ are open, hence they are open and closed. Now assume that $G$ is semisimple and simply connected. Then by [OV1] Theorem 5.2.3, p. 240] $G(\mathbb{R})$ is connected, hence the orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ are connected, hence they are the connected components of $X(\mathbb{R})$. $\square$

**Corollary 17.2.** If $H \subset G$ is an $\mathbb{R}$-subgroup of a simply connected semisimple $\mathbb{R}$-group $G$, $X = G/H$, then there is a canonical bijection between $\ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)]$ and the set $\pi_0(X(\mathbb{R}))$ of connected components of $X(\mathbb{R})$.

**Proof.** The corollary follows immediately from [35] and Lemma [17.1]. $\square$

**Lemma 17.3** (well-known). Let $\varphi : S \to T$ be a homomorphism of $\mathbb{C}$-tori, and let $\varphi_* : X_*(S) \to X_*(T)$ denote the induced homomorphism of the cocharacter groups. Then

$$\#\pi_0(\ker \varphi) = [(\ker \varphi) \otimes \mathbb{Q} \cap X_*(T) : \ker \varphi_*].$$
Proof. If $\dim S \leq 1$, $\dim T = 1$, then the assertion is evident. In the general case there exist bases of $X_*(S)$ and $X_*(T)$ in which $\varphi_*$ is given by a diagonal matrix, (see, e.g., [Vi Prop. 9.13]), and the assertion reduces to the one-dimensional case. □

Corollary 17.4 (well-known). Let $\varphi : H \to G$ be a homomorphism with finite kernel of connected reductive $\mathbb{C}$-groups. Let $T_H \subset H$ and $T \subset G$ be maximal tori such that $\varphi(T_H) \subset T$. Let $\varphi_* : X_*(T_H) \to X_*(T)$ denote the induced homomorphism of the cocharacter groups. Then $\# \ker[\varphi : H \to G] = [\langle \text{im } \varphi_* \rangle \otimes_{\mathbb{Z}} \mathbb{Q} \cap X_*(T) : \text{im } \varphi_*]$. Proof. We have $\ker[\varphi : H \to G] = \ker[T_H \to T]$, and the corollary follows from Lemma 17.3. □

17.5. Let $G$ be a connected semisimple $\mathbb{C}$-group, $T \subset G$ a maximal torus, $\Pi = R(G,T)$ the root system, $\Pi \subset \mathbb{R}_\ast$ a basis of $\mathbb{R}_\ast$. Let $\Pi^\vee$ denote the dual root system, $\Pi^\vee \subset \mathbb{R}^\vee \subset X_*(T)$. Recall that $G$ is simply connected if and only if $\Pi^\vee$ generates $X_*(T)$, i.e., $\Pi^\vee$ is a basis of $X_*(T)$.

We give three examples of calculations of $\pi_0((G/H)(\mathbb{R}))$.

Example 17.6. Let $G = EV$, the split simply connected simple $\mathbb{R}$-group with maximal torus $T$, of type $E_7^{(7)}$ with Dynkin diagram

```
1 -- 2 -- 3 -- 4 -- 5 -- 6
   \  \  \  \  \  \  \  \\
\  \  \  \  \  \    \  \\
7
```

and the augmented diagram

```
O -- O -- O -- O -- O -- O -- O
   \  \  \  \  \  \  \  \  \\
O
```

Let $\Pi_G = \{\alpha_1, \ldots, \alpha_7\}$ be the simple roots (with the numeration of [OV1]). We remove the vertex 3. Set $\Pi_H = \Pi_G \setminus \{\alpha_3\}$, and let $H$ be the corresponding semisimple $\mathbb{R}$-subgroup with maximal torus $T_H$ (contained in $T$) with Dynkin diagram of type $A_2 \times A_4^{(1)}$

```
O -- O
   \  \\
\  \\
```

and the augmented diagram:

```
O -- O -- O -- O
   \  \  \  \\
O
```

We show that the semisimple group $H$ is simply connected. Let $\tilde{H}$ be the universal covering of $H$, with the canonical homomorphism $\varphi : \tilde{H} \to H \to G$ and the maximal torus $T_H \subset \tilde{H}$ such that $\varphi(T_H) \subset T$. Consider the induced homomorphism $\varphi_* : X_*(T_H) \to X_*(T)$. Since the coroots $(\alpha_i^\vee)_{i \neq 3}$ constitute a basis of $X_*(T_H)$, we see that $\text{im } \varphi_* = \langle (\alpha_i^\vee)_{i \neq 3} \rangle \subset X_*(T)$, hence $\text{im } \varphi_*$ is a direct summand of $X_*(T)$, and by Corollary 17.4 the homomorphism $\varphi$ is injective, hence $\tilde{H} = H$ and $H$ is simply connected.

By [OV1] Table 3] $H$ does not contain the center of $G$. We have $H = H_1 \times H_2$, where $H_1$ is a compact groups of type $A_2$ and $H_2$ is a twisted (noncompact) group of type $A_4^{(1)}$. By Section 7.1 for $H_1$ we have $\# H_1^1(\mathbb{R}, H_1) = 2$ with representatives

$$0 - 0, \quad 1 - 0.$$
By Section 7.2 for $H_2$ we have $\#H^1(\mathbb{R}, H_2) = 3$ with representatives 

\[
[1] - 0 - 0 - 0 - 0, \quad [1] - 0 - 1 - 0 - 0, \quad [1] - 0 - 1 - 0 - 1.
\]

We have $H^1(\mathbb{R}, H) = H_1(\mathbb{R}, H_1) \times H^1(\mathbb{R}, H_2)$, hence $\#H^1(\mathbb{R}, H) = 2 \cdot 3 = 6$. The kernel $\ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)]$ is the set of labelings of the Kac diagram of $A_2 \times A_4^{(1)}$ that are in the orbit of 0 in $E_7^{(7)}$. By Section 12.2 the labelings of $E_7^{(7)}$ in the orbit of 0 are those with 1 or 3 components (including the boxed 1). Thus $\ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)]$ consists of the classes of the following labelings of $A_2 \times A_4^{(1)}$:

\[
\begin{array}{cccc}
0 & 0 & [1] - 0 & - 0 - 0 - 0 \\
0 & 0 & [1] & 0 - 1 - 0 - 1 \\
1 & 0 & [1] & 0 - 1 - 0 - 0
\end{array}
\]

We conclude that

\[
\# \pi_0((G/H)(\mathbb{R})) = \# \ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)] = 3.
\]

**Example 17.7.** Let $G = \text{Spin}_{2n}^\ast$, the simply connected “quaternionic” $\mathbb{R}$-group of type $D_4^{(n)}$ with Dynkin diagram

\[
\begin{array}{cccc}
1 & \cdots & n-3 & n-2 & n-1 \\
\circ & \cdots & \circ & \circ & \circ
\end{array}
\]

and the augmented diagram

\[
\begin{array}{cccc}
1 & \cdots & n-3 & n-2 & n & \circ \\
\circ & \cdots & \circ & \circ & \circ & \circ
\end{array}
\]

Let $\Pi_G = \{\alpha_1, \ldots, \alpha_n\}$ be the simple roots (with the numeration of [OV1]). We remove the vertex $n - 1$. Set $\Pi_H = \Pi_G \setminus \{\alpha_{n-1}\}$, and let $H$ be the corresponding semisimple $\mathbb{R}$-subgroup with diagram of type $A_4^{(n-1)}$

\[
\begin{array}{cccc}
1 & \cdots & n-3 & n-2 & n & \circ \\
\circ & \cdots & \circ & \circ & \circ & \circ
\end{array}
\]

and the augmented diagram

\[
\begin{array}{cccc}
1 & \cdots & n-3 & n-2 & n & \circ & \circ \\
\circ & \cdots & \circ & \circ & \circ & \circ & \circ
\end{array}
\]

The semisimple $\mathbb{R}$-subgroup $H$ is simply connected (one can prove this as in Example 17.6 using Corollary 17.4), and $H$ does not contain $\ker[\text{Spin}_{2n}^\ast \to \text{SO}_{2n}^\ast]$ (see [OV1], Table 3). By Section 7.2 the cohomology classes in $H^1(\mathbb{R}, H)$ correspond to the orbits of the labelings $\xi_i$ with $i$ components (including the boxed 1). We have $\lceil \frac{n}{2} \rceil$ such orbits. By Section 10.2 an orbit $\xi_i \in H^1(\mathbb{R}, H)$ goes to the orbit of 0 in $H^1(\mathbb{R}, G)$ if and only if $i$ is odd. We have $\lceil \frac{n}{2} \rceil$ odd numbers $i$ between 1 and $\lceil \frac{n}{2} \rceil$. We conclude that

\[
\# \pi_0((G/H)(\mathbb{R})) = \# \ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)] = \lceil \frac{n}{2} \rceil.
\]

For the next example we need two lemmas.

**Lemma 17.8** (well-known). Let $\Gamma$ be a finite group acting on a finite abelian group $A$. If the orders $|\Gamma|$ and $|A|$ are relatively prime, then $H^q(\Gamma, A) = 0$ for any $q \in \mathbb{Z}$, where $H^q(\Gamma, A)$ denotes the Tate modified cohomology group.
Proof. Clearly $\hat{H}^q(\Gamma, A)$ is annihilated by multiplication by $|A|$. On the other hand, by [AW Sect 6, Cor. 1 of Prop. 8] the group $\hat{H}^q(\Gamma, A)$ is annihilated by $|\Gamma|$. Since $|\Gamma|$ and $|A|$ are relatively prime, there exist integers $r$ and $s$ such that $r|\Gamma| + s|A| = 1$. We see that $\hat{H}^q(\Gamma, A)$ is annihilated by 1, hence is zero. □

Lemma 17.9 (well-known). Let

$$1 \to A \to B \xrightarrow{\psi} C \to 1$$

be a short exact sequence of algebraic $\mathbb{R}$-groups, where $A$ is finite and central in $B$. If the order $\# A(\mathbb{C})$ of $A(\mathbb{C})$ is odd, then the induced map

$$\psi_* : H^1(\mathbb{R}, B) \to H^1(\mathbb{R}, C)$$

is bijective.

Proof. Since $A$ is central, we have a cohomology exact sequence

$$C(\mathbb{R}) \to H^1(\mathbb{R}, A) \to H^1(\mathbb{R}, B) \xrightarrow{\psi_*} H^1(\mathbb{R}, C) \to H^2(\mathbb{R}, A);$$

see [S I.5.7, Prop. 43]. Since $\# \text{Gal}(\mathbb{C}/\mathbb{R}) = 2$ and $\# A(\mathbb{C})$ is odd, by Lemma 17.8 we have $H^1(\mathbb{R}, A) = 1$ and $H^2(\mathbb{R}, A) = 1$. It follows that the map $\psi_*$ is surjective and that $\ker \psi_* = 1$. We show that any fiber of $\psi_*$ contains only one element. Indeed, let $\beta \in H^1(\mathbb{R}, B)$ and let $b \in Z^1(\mathbb{R}, B)$ be a cocycle representing $\beta$. By [S I.5.5, Cor. 2 of Prop. 39], the fiber $\psi_*^{-1}(\psi_*(\beta))$ is in a bijection with the quotient of $H^1(\mathbb{R}, A)$ by an action of the group $\mathbb{Z}C(\mathbb{R})$. Since $H^1(\mathbb{R}, A) = 1$, our fiber $\psi_*^{-1}(\psi_*(\beta))$ contains only one element. Thus $\psi_*$ is bijective. □

Example 17.10. Let $G = EVIII$, the split form $E_8^{(7)}$ of $E_8$ with maximal torus $T$, with Kac diagram

and augmented diagram

We remove vertex 4 from the Kac diagram (that is, from the extended diagram). We obtain a maximal connected algebraic subgroup $H \subset G$ (see [OV2 Table 5]) with Dynkin diagram

and augmented diagram

We compute the fundamental group $\pi_1(H_{\mathbb{C}})$ of the semisimple group $H$. Let $\tilde{H}$ denote the universal covering of $H$. Consider the composite morphism

$$\varphi : \tilde{H} \to H \to G,$$
and let $\tilde{T}_H$ denote the maximal torus of $\tilde{H}$ such that $\varphi(\tilde{T}_H) = T$. We denote by $\varphi_*: X_*(\tilde{T}_H) \to X_*(T)$ the induced homomorphism of the character groups The cocharacter group $X_*(\tilde{T}_H)$ has a basis
\begin{equation}
\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee, \alpha_5, \alpha_6, \alpha_7, \alpha_8,
\end{equation}
where $\alpha_4^\vee$ means that $\alpha_4^\vee$ is of type $\frac{2}{5}$, while the subgroup $\text{im} \varphi_* \subset X_*(T)$ is generated by the cocharacters $\alpha_4^\vee$. There is a linear relation between $\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee, \alpha_5, \alpha_6, \alpha_7, \alpha_8$, in which the removed simple coroot $\alpha_4^\vee$ appears with coefficient 5, while $\alpha_0^\vee$ appears with coefficient 1: see [OVi, Table 6] or [Bon, Planche VII]. We see that $\text{im} \varphi_* \subset X_*(T)$ contains $\alpha_i^\vee$ for $i \neq 4$, and it contains $5\alpha_4^\vee$, but not $\alpha_4^\vee$. Thus $\text{im} \varphi_*$ is a subgroup of index 5 in $X_*(T)$. By Corollary 17.3 the kernel of the canonical epimorphism $\tilde{H} \to H$ is of order 5, hence $\pi_1(H_C) = (\mathbb{Z}/5\mathbb{Z})$.

Since the order 5 of $\ker \varphi$ is odd, by Lemma 17.9 the induced map $H^1(\mathbb{R}, \tilde{H}) \to H^1(\mathbb{R}, H)$ is bijective, hence
\[\# \ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)] = \# \ker[H^1(\mathbb{R}, \tilde{H}) \to H^1(\mathbb{R}, G)].\]
We compute $\ker[H^1(\mathbb{R}, \tilde{H}) \to H^1(\mathbb{R}, G)]$. We have $\tilde{H} = H_1 \times H_2$, where $H_1$ is compact of type $\mathbf{A}_4$ and $H_2$ is of type $\mathbf{A}_4^{(1)}$. By Section 7.2 for representatives of orbits of labelings of $\mathbf{A}_4$ we can take 0000, 0001 and 0101. By Section 7.2 for representatives for $\mathbf{A}_4^{(1)}$ we can take 0001, 0101 and 1010. By Section 13.3 the orbit of 0 in $\mathbf{E}_8^{(7)}$ consists of labelings with odd number of components (including the boxed 1), excluding the invariant $\ell_3$. There are 5 classes of labelings for $\tilde{H}$ with odd number of components:
\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}
\]
and one of them $(0 - 0 - 0 - 0 1 - 0 - 1 - 0 - 0 - [1])$ is the preimage of the invariant. We see that $\# \ker[H^1(\mathbb{R}, \tilde{H}) \to H^1(\mathbb{R}, G)] = 4$. Thus
\[\# \pi_0((G/H)(\mathbb{R})) = \# \ker[H^1(\mathbb{R}, H) \to H^1(\mathbb{R}, G)] = \# \ker[H^1(\mathbb{R}, \tilde{H}) \to H^1(\mathbb{R}, G)] = 4.\]

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