Spin Foam Diagrammatics and Topological Invariance

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Abstract

We provide a simple proof of the topological invariance of the Turaev-Viro model (corresponding to simplicial 3d pure Euclidean gravity with cosmological constant) by means of a novel diagrammatic formulation of the state sum models for quantum BF-theories. Moreover, we prove the invariance under more general conditions allowing the state sum to be defined on arbitrary cellular decompositions of the underlying manifold. Invariance is governed by a set of identities corresponding to local gluing and rearrangement of cells in the complex. Due to the fully algebraic nature of these identities our results extend to a vast class of quantum groups. The techniques introduced here could be relevant for investigating the scaling properties of non-topological state sums, being proposed as models of quantum gravity in 4d, under refinement of the cellular decomposition.
**Introduction**

Since the 1980’s, topology of low dimensional manifolds is at the meeting point of many different areas of mathematics and physics. This development was initiated by Donaldson who was studying 4-manifolds by the means of Yang-Mills equations. Then Jones found his famous polynomial of links in the 3-sphere, using algebraic structures: Von Neumann algebras. This opened up a new way for algebra to come into low dimensional topology. Quantum groups play an important role through the Reshetikhin-Turaev functor \[1\]. This latter gives a diagrammatic approach to categories, and enables one to construct new topological invariants of 3-manifolds. A state sum invariant of this type was first given by Turaev and Viro \[2\] using an arbitrary triangulation of the manifold. This state sum is expressed in terms of 6j-symbols of \(U_q(sL_2)\), which are associated with tetrahedra of the triangulation. Invariance of the state sum under change of triangulation implies topological invariance. The proof of invariance in \[2\] relied heavily on properties of the \(U_q(sL_2)\) 6j-symbols (Biedenharn-Elliott property) and so was mainly algebraic. Later, the invariant was generalized to arbitrary spherical categories \[3\], again using algebraic properties of (now generalized) 6j-symbols.

State sum invariants arise in the quantization of BF-theory due to the absence of local degrees of freedom. A typical example is the case of 3d Euclidean gravity with cosmological constant represented by the Turaev-Viro model (see \[4\] and references therein). Models of quantum gravity in four dimensions have been defined by means of appropriately constraining the state sum models for 4d BF-theory \[5, 6, 7, 8\]. Constraining the state sum restores the local degrees of freedom of gravity while topological invariance is lost. Physical amplitudes are expected to be recovered by means of summing over discretizations or by an appropriate limiting procedure in which the discretization is refined. We expect that the techniques introduced here would shed light on the issues involved in studying the scaling properties of amplitudes in non-topological models as well.

A new diagrammatic approach to spin foam models has been introduced in \[9\] where generalized lattice gauge theory is considered. The aim of this article is to show that this allows for a very simple proof of topological invariance of 3d BF-theory (which is a special case of lattice gauge theory). This applies to the group case (Ponzano-Regge \[10\]), the supersymmetric case, and the quantum group case (Turaev-Viro \[2\], Barrett-Westbury \[3\]). In the group case the invariance of the partition function can be rather easily demonstrated through manipulations with delta functions. These algebraic operations have simple diagrammatic analogues (which are even more intuitive than the handling of delta functions). If one considers the 3d BF-amplitude with (any) quantum groups, the traditional proof of invariance (through algebraic manipulations) is much less trivial, and relies heavily on
properties of generalized 6j-symbols [1, 3]. We show here that the diagram-
matics defined for ordinary groups extends naturally to the quantum groups
case and therefore renders the proof of topological invariance just as easy.
Instead of complicated algebraic relations and dealing with 6j-symbols, the
proof requires only a few pictures.

What is more, the new diagrammatic approach allows the definition of
the partition function on arbitrary cellular decompositions (as opposed to
just triangulations) without any additional complications. Indeed, contrary
to what one might expect, this renders the proof of invariance even simpler.
While first performing the proof in the simplicial case employing Pachner
moves [2] we then refine it to the general cellular case. To this end we
introduce a set of moves relating cellular decompositions. As it turns out,
these correspond to diagrammatic identities of the partition function which
are in a sense elementary, rendering the invariance proof particularly simple.

In Section 1, we introduce discretized 3d BF-theory and introduce the
diagrammatic language for expressing the partition function. In Section 2,
we discuss the generalization to the quantum group case. In Section 3, we
give the proof of topological invariance first for the simplicial case and extend
it then to the general cellular case. To this end we introduce moves relating a
cellular decomposition to a simplicial one. We end with concluding remarks.
Among other things we discuss there how the presented moves are extended
to relate arbitrary cellular decompositions.

1 Diagrammatics for BF-theory

In this section we introduce BF-theory, its quantization on a discretized
manifold, and a diagrammatic formalism to represent its partition function
similar to the spin foam/network formalism.

1.1 The partition function

Let $M$ be an oriented compact piecewise-linear manifold, $G$ a compact Lie
group and $P$ a principal $G$-bundle over $M$. Let $A$ be a connection on $P,$
and $B$ an ad($P$)-valued 1-form on $M$. Here, ad($P$) is the associated vector
bundle to $P$ via the adjoint action of $G$ on its Lie algebra. Let $F$ be the
curvature 2-form of $A$. The action is then defined as

$$ S = \int_M \text{Tr}(B \wedge F), $$

Note that we do not require the bundle to be trivial. Indeed any principal $G$-bundle
will lead to the same discretized partition function as any transition function are “inte-
grated out”.

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where \( \text{Tr} \) is the trace in the adjoint representation. The partition function of the quantum field theory is formally defined to be

\[
Z(\mathcal{M}) = \int \mathcal{D} B \mathcal{D} A \ e^{i \int \text{Tr}[B \wedge F]}.
\]  

(2)

Integrating out the \( B \)-field one formally obtains

\[
Z = \int \mathcal{D} A \ \delta(F).
\]  

(3)

This last expression can be given a precise meaning if one replaces \( M \) by a cellular decomposition \( \mathcal{K} \). We recall the definition of cellular decomposition (a decomposition as a finite CW-complex, see [13]). The basic element is a “cell”. An \( n \)-cell is an open ball of dimension \( n \). A cellular decomposition \( \mathcal{K} \) of \( M \) is a presentation of \( M \) as a disjoint union of cells.

Consider now the complex \( \mathcal{K}^* \) dual to \( \mathcal{K} \). That is, the complex where every \( n \)-cell is replaced by a corresponding \( (\dim M - n) \)-cell. If in \( \mathcal{K} \) an \( n \)-cell is in the boundary of an \( m \)-cell then in \( \mathcal{K}^* \) this relationship is reversed for the corresponding cells. As in lattice gauge theory, one assigns a group element, \( g_e \), to each 1-cell (denoted edge and labeled by \( e \)) in \( \mathcal{K}^* \), representing the holonomies of the connection. Then one defines the measure over the discretized connections to be \( \prod_e dg_e \), where \( dg \) is the \textit{Haar measure} on \( G \).

The curvature \( F \) is represented by the holonomy along edges bounding 2-cells (which are denoted \textit{faces} and labelled by \( f \)), i.e., the product \( g_{e_1} \cdots g_{e_n} \), where \( e_1 \cdots e_n \) denote the edges bounding the corresponding face. We assume that each face is given an arbitrary but fixed orientation. In this way the partition function is defined as

\[
Z = \int \prod_e dg_e \prod_f \delta(g_{e_1} \cdots g_{e_n}).
\]  

(4)

Let \( \mathcal{C}_{alg}(G) \) be the algebra of complex valued representative functions on \( G \), that is, the functions arising as matrix elements of finite-dimensional complex representations of \( G \). The Peter-Weyl decomposition asserts then that

\[
\mathcal{C}_{alg}(G) = \bigoplus_{\rho} (V^*_\rho \otimes V_\rho),
\]

where \( \rho \) are the finite-dimensional irreducible representations of \( G \) and \( V_\rho \) the representation spaces. Concretely, any function \( f \) over the group \( G \) can be written as

\[
f(g) = \sum_{\rho,n,n'} C^{nn'}_{\rho} \rho_{nn'}(g) \quad \forall g \in G
\]

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with coefficients $C_{\rho}^{m'n'}$. In particular, the delta function is

$$\delta(g) = \sum_{\rho} \dim \rho \operatorname{Tr}(\rho(g)),$$

where the sum runs over irreducible representations as above. Thus, we can rewrite the partition function as

$$Z = \sum_{c: \{l\} \rightarrow \{\rho_l\}} \left( \prod_{f} \dim \rho_f \right) Z_C$$

with

$$Z_C = \int (\prod_{e} dg_e) \prod_{f} \operatorname{Tr} [\rho_f(g_{e_1} \cdots g_{e_n})],$$

where $e_1, \ldots, e_n$ are the edges bounding the face $f$. $C$ is the set of functions associating to each face $f$ an irreducible representation $\rho_f$ of $G$. In order to proceed to the integration, one can now expand the trace into matrix elements

$$\operatorname{Tr}(\rho_l(g_{e_1} \cdots g_{e_n})) = \sum_{p_1, \ldots, p_n} \rho_{l \rho_{p_1} \rho_{p_2}}(g_{e_1}) \cdots \rho_{l \rho_{p_n} \rho_{p_1}}(g_{e_n})$$

and collect for each edge the functions taking as value the corresponding group element. However, instead of proceeding formally we are going to obtain a diagrammatic representation of the partition function, or more precisely of $Z_C$.

Note that by construction, we are using in (5) the Haar measure on $G$. On the algebraic functions this is the map $\int : C_{\text{alg}}(G) \rightarrow \mathbb{C}$ given by the projection

$$\bigoplus_{\rho} V_{\rho}^* \otimes \rho \rightarrow V_1^* \otimes V_1 \cong \mathbb{C},$$

where $1$ denotes the trivial representation. Now, define a family of maps $\{T_\rho\}$ for any representation $\rho$ as

$$T_\rho : V_\rho \rightarrow V_\rho \quad \text{with} \quad v \mapsto \int dg \, \rho(g)v.$$

This is simply the projection onto the trivial subrepresentation.

**Proposition 1.1** (9). $T_\rho$ (with $\rho(g) : V_\rho \rightarrow V_\rho$) defines a family of intertwiners with the following properties:

(a) $T_1 = \text{id}_1$.

(b) $T_\rho = 0$ for $\rho$ irreducible and non-trivial.

(c) $T$ is a projector, i.e. $T_\rho^2 = T_\rho$. 

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(d) $T$ commutes with any intertwiner, i.e. for $\Phi : V_\rho \to V_{\rho'}$ an intertwiner we have $T_{\rho'} \circ \Phi = \Phi \circ T_\rho$.

(e) $T$ is self-dual, i.e. $(T_\rho)^* = T_{\rho^*}$.

(f) $T_\rho \otimes T_{\rho'} = T_{\rho \otimes \rho'} \circ (T_\rho \otimes \text{id}_{\rho'})$.

We depict representations by lines and $T_\rho$ by an unlabelled box. E.g. in the case of $T$ applied to a tensor product of three representations $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$ the diagram is

\[
v_1 \otimes v_2 \otimes v_3 \mapsto \int dg (\rho_1(g)v_1 \otimes \rho_2(g)v_2 \otimes \rho_3(g)v_3).
\]

Furthermore, every line carries an arrow and reversal of the arrow corresponds to exchanging the representation associated with that line with its dual.

The diagrammatic representation of the partition function is now obtained starting from the dual cellular complex $K^*$ as follows. Put a loop (called a wire) into each face close to its boundary. Give each wire an arrow according to an arbitrarily chosen but fixed orientation the face and the representation label of the face (given by $C$ in Eq. (5)). Each group integration, $dg_e$, over the group element associated with the corresponding edge $e$ gives rise to an intertwiner $T_{\rho_1 \otimes \cdots \otimes \rho_N}$ defined above. $\rho_1 \otimes \cdots \otimes \rho_N$ is the tensor product of the irreducible representations labeling the faces adjacent to the edge $e$, i.e., the corresponding wires adjacent to $e$. According to Eq. (7) we represent $T_{\rho_1 \otimes \cdots \otimes \rho_N}$ by a box or cylinder, denoted cable, which we put around the edge. The terminology comes from the fact that the diagram obtained in this way resembles an arrangement of cables and wires.

In Fig. 1 we give an illustration of such a diagram. The tetrahedron on the left hand side of Fig. 1 is part of a bigger cellular complex $K^*$. On the right hand side there are six internal cables (cylinders) corresponding to the six edges forming the tetrahedron plus four external cables corresponding to external edges that connect the tetrahedron with the rest of $K^*$. There are four internal faces, the four triangles bounding the tetrahedron, represented by closed wires going from one cable to the other.

We call the whole (embedded) diagram the circuit diagram. To extract the value of the partition function (or more precisely $Z_C$) one projects the
Figure 1: The circuit diagram construction for a tetrahedron.

circuit diagram (arbitrarily) onto the plane. The evaluation proceeds by using (the symmetric category version of) the Reshetikhin-Turaev functor, see \cite{9} for details. Technically, every element of the (projected) circuit diagram represents a morphism (intertwiner) in the category of representations of $G$. In particular, the circuit diagram as a whole labelled by the collection $\mathcal{C}$ represents an intertwiner $\mathbb{C} \rightarrow \mathbb{C}$ and thus a complex number which is precisely $\mathbb{Z}_c$. This is crucial for the generalization to the quantum group case (see Section 2).

Note that this construction generalizes from BF-theory to lattice gauge theory \cite{9}.

The main advantage of the diagrammatic formulation is that it allows to perform otherwise cumbersome and complicated algebraic manipulations through simple diagrammatic identities. This is crucial for our treatment in Section 3. Furthermore, this formulation is manifestly covariant as it does not require any choice of bases for the representations.

1.2 Diagrammatic identities

In this section, we introduce the diagrammatic identities that are instrumental for our new proof of topological invariance.

1.2.1 Gauge fixing

In lattice gauge theory, some group integrals in the partition function can be removed by using the gauge symmetry. The considered group variables are set (“gauge fixed”) to an arbitrary element, usually the unit element. This applies equally to discretized BF-theory which can be considered a weak coupling limit of lattice gauge theory. Diagrammatically, gauge fixing of a group variable corresponds precisely to removing the cable from the corresponding edge. Thus, the integration over the group element at this edge given by $T$ is replaced by the identity operation corresponding to the action of the identity element of the group in the relevant representations.
As is well known from lattice gauge theory, edges can be gauge fixed (i.e.,
cables removed) precisely as long as the gauge fixed edges do not form any
closed loop. In fact, this can be seen to follow purely diagrammatically from
the identity depicted in Figure 2. This identity follows from Proposition 1.1
(see [9] for details). Note that gauge fixing is not just an identity of the
partition function $Z$ as a whole, but of each summand $Z_C$ individually.

![Figure 2: Diagrammatic gauge fixing identity.](image)

1.2.2 Summation identity

In contrast to the gauge fixing the summation identity allows the removal
not only of a cable but of a wire as well. It takes the form

$$\sum_{\rho_0} \dim \rho_0 \ \begin{array}{c}
\rho_1 \\
\rho_0 \\
\rho_0
\end{array} = \begin{array}{c}
\rho_1 \\
\rho_0 \\
\rho_0
\end{array}. \quad (8)$$

The sum ranges over all finite-dimensional irreducible representations.

In order to prove (8) one uses the fact that every representation can be
decomposed as a direct sum of irreducible ones (the category of representa-
tions of the group is semisimple). For irreducible representations one can
then deduce (8) from the identity

\[ \rho_{1,2} = \delta \rho_1 \rho_2 (\dim \rho_1)^{-1}. \] (9)

See \[9\] for details.

2 Quantum group generalization in 3d

Unfortunately, BF-theory as presented in the previous section is divergent, even in its discretized form. A way to render the theory finite is to replace groups by \( q \)-deformed groups at roots of unity. This can be done in the 3-dimensional case (to which we confine ourselves in the following) and gives rise to interesting topological invariants. This was first carried out by Turaev and Viro using \( U_q(\mathfrak{sl}_2) \) \[2\]. It was generalized to the case of spherical quantum groups (or categories) by Barrett and Westbury \[3\]. We consider here the (sufficiently general) case of ribbon quantum groups (or categories).

Crucially, the formalism introduced in the previous section also permits this generalization. We only give a rough sketch here while referring the reader to \[9\] for an extensive treatment with full proofs.

The quantum group case is most conveniently described in a categorial language. In the group case the relevant category is the category of finite-dimensional representations of the group. Instead of starting with a group and then considering its representations we need to think now of abstractly given objects with certain operations and properties. More precisely, we use the notion of semisimple ribbon category (in the sense of Turaev \[11\]). As each \( q \)-deformed enveloping algebra \( U_q(\mathfrak{g}) \) gives rise to such a category (after “purification” in the root of unity case, see \[11\]) this is sufficient for our purposes. Note however, that this notion is more general than that of modular category.

The diagrammatics is modified as follows: Previously, crossings of lines were simply given by the intertwiner \( V \otimes W \rightarrow W \otimes V : v \otimes w \mapsto w \otimes v \). Now, over- and under-crossings are distinct morphisms (“intertwiners”) \( \psi_{V,W} : V \otimes W \rightarrow W \otimes V \), called the braiding. Furthermore, lines are framed, i.e., replaced by ribbons. The twist of a ribbon is a new diagrammatic element corresponding algebraically to a family of morphisms \( \nu_V : V \rightarrow V \). However, in all of the following we can omit the framings in the diagrams and implicitly use blackboard framing. That means that each line represents a ribbon which lies face-up and has no twist.
The family of morphisms \( T \) is now defined as follows. Consider the identity morphism \( \text{id}_V \) of an object \( V \) to itself. It has a decomposition as a finite sum

\[
\text{id}_V = \sum_i f_i \circ g_i
\]

(10)

where \( g_i : V \to V_i \) and \( f_i : V_i \to V \) and \( V_i \) are simple objects in the category (axiom of domination \([11]\)). Now \( T_V : V \to V \) is defined as the restricted sum \( \sum'_i f_i \circ g_i \) where we only sum over the indices \( i \) such that \( V_i \) is isomorphic to the trivial object \( 1 \cong \mathbb{C} \). The morphisms \( T \) have now precisely the properties listed in Proposition 1.1. Furthermore, they have the trivial braiding and trivial twisting property

**Proposition 2.1 (\([9]\)).**  
\((g)\) \( \psi_{V,W} \circ (T_V \otimes 1_W) = \psi_{W,V}^{-1} \circ (T_V \otimes 1_W) \).  
\((h)\) \( \nu_V \circ T_V = T_V = T_V \circ \nu_V \).

Let us now briefly discuss how the partition function is generalized to this categorial setting. The idea is to use its diagrammatic representation in terms of the circuit diagram in order to define its value. However, this is less straightforward now than in the group case. Very roughly, it is achieved by cutting the circuit diagram into pieces by cutting through each cable, then projecting each piece onto the plane and reconnecting the pieces with \( T \)-diagrams. The emerging diagram resembles the circuit diagram in a particular 2-dimensional projection. (See \([9]\) for details.) Indeed, as we can think of the identities used in the following sections as identities between the corresponding projections, the results derived there apply to the general quantum group case.

The sum over labellings with irreducible representations in (5) becomes a sum over labellings with equivalence classes of simple objects in the category. At the same time the dimensions of the representations are replaced by the quantum dimensions of the simple objects, given diagrammatically by a loop labelled by the object. If the category has only finitely many inequivalent simple objects the partition function is manifestly finite. In particular, this is true in the relevant case of \( q \)-deformed enveloping algebras \( U_q(\mathfrak{g}) \) at roots of unity giving rise to modular categories.

Importantly, the gauge fixing identity (Section 1.2.1) as well as the summation identity (Section 1.2.2) generalize to the quantum group case. For the gauge fixing this is a consequence of Propositions 1.1 and 2.1. In the summation identity, the dimension is replaced by the quantum dimension (the loop diagram) and it is proven using the domination property (10) instead of conventional semisimplicity.

Although for simplicity we use solely the language of the group context, the results of the following section apply to the quantum group context of semisimple ribbon categories in the above described way. (In fact, they even apply to semisimple spherical categories, but we do not consider this here).
3 Generalized proof of topological invariance in 3d

In this section we show, in the 3-dimensional case, that the partition function of BF-theory is independent of the chosen cellular decomposition of the manifold $M$. To be more precise, the invariant quantity corresponds to a rescaling of the partition function (5), namely

$$\Gamma = \tau^{-n(0)}Z,$$

where $n(0)$ is the number of 0-cells in $K$ (corresponding to the number of 3-cells in $K^*$) and $\tau := \sum_{\rho}(\dim \rho)^2$ which is divergent in the Lie group case. The rescaling factor is associated to the non-compact gauge symmetry $B \to B + d_A\eta$ of BF-theory [14]. Gauge equivalent configurations are summed over in (5). Gauge volume factors should be divided out of the partition function according to the standard Faddeev-Popov procedure.

In Section 1 we introduced a formulation of quantum BF-theory for an arbitrary cellular decomposition of $M$. Traditionally, one restricts to simplicial decompositions $\Delta$ of $M$. This usually simplifies the proof of topological invariance of (5) due to the particular combinatorial properties of $\Delta^*$: in this case all cables have only three wires and vertices (0-cells) are four-valent. Equivalent simplicial complexes are related by a finite set of moves (Pachner moves). In [11, 3], $6j$-symbols for $U_q(\mathfrak{sl}_2)$ as well as their main property (Biedenharn-Elliot), were generalized to any modular (or spherical) category; then the invariance of the amplitude was shown by translating the Pachner moves in terms of transformations of $6j$-symbols. Generalizing this proof to arbitrary cellular decompositions $\mathcal{K}$ of $M$, would amount to knowing an infinite number of identities of this kind relating arbitrary ‘$nj$-symbols’.

In contrast, our diagrammatic formulation allows for this generalization by utilizing the simple gauge fixing and summation identities of the Section 1.2. In turn, it should provide a way of inferring such ‘higher’ order identities for the appearing ‘$nj$-symbols’.

In this section we construct the generalized invariance proof by first concentrating on the special case of simplicial decompositions and then extending our construction to arbitrary cellular decompositions.

3.1 The simplicial case

Topologically equivalent triangulations are related by Pachner moves [12]. For showing that (5) is a topological invariant for simplicial decompositions $\Delta$ it is thus sufficient to show invariance under Pachner moves. In order to do this we first need to translate Pachner moves into the dual complex $\Delta^*$.

In three dimensions there are four Pachner moves: the (1,4) Pachner move, the (2,3) Pachner move, and their inverses. The (1,4) move creates four tetrahedra out of one in the following way: put a point $p$ in the interior of the tetrahedron whose vertices are labeled $p_i, \ i = 1, \ldots, 4$, add the four
1-simplices \((p, p_i)\), the six triangles \((p, p_i, p_j)\) (where \(i \neq j\)), and the four tetrahedra \((p, p_i, p_j, p_k)\) (where \(i \neq j \neq k \neq i\)). In the dual complex \(\Delta^*\), the original tetrahedron corresponds to a single vertex at which four edges and six faces meet. After the move is implemented, one has four vertices in \(\Delta^*\) (corresponding to the four tetrahedra in \(\Delta\)) connected by edges and surfaces to form a tetrahedron in \(\Delta^*\). In terms of circuit diagrams the move is illustrated in Fig. 3.

![Figure 3: The (1,4) Pachner move represented by circuit diagrams.](image)

The (2,3) Pachner move consists of the splitting of two tetrahedra into three tetrahedra. One replaces two tetrahedra \((u, p_1, p_2, p_3)\), and \((d, p_1, p_2, p_3)\), sharing the triangle \((p_1, p_2, p_3)\) with the three tetrahedra \((u, d, p_i, p_j)\) where \(i \neq j = 1, \ldots 3\). In terms of circuit diagrams in \(\Delta^*\) the move corresponds to the one illustrated in Fig. 4.

![Figure 4: The (2,3) Pachner move represented by circuit diagrams.](image)

The proof of topological invariance consists of showing that the Figs. 3 and 4 give rise to identical values of \(\Gamma\) in Eq. (11), as they are essentially diagrammatic identities.

Let us first consider the (1,4) move. We start with the circuit diagram on the right hand side of Fig. 3 and we eliminate three redundant cables using the gauge fixing. In this way we obtain the first diagram on the left hand side of Fig. 5. We then use the summation identity (8) to erase all the remaining internal cables as follows. The factor \(\dim \rho\) in (8) and the corresponding sum over representations is provided by the partition function (5). In the second diagram in Fig. 5 one such internal cable and corresponding closed wire are
emphasized. After the three internal cables are eliminated in this way, we obtain the circuit diagram on the right hand side of Fig. 5. This diagram corresponds precisely to the one tetrahedron diagram on the left of Fig. 3 multiplied by a closed loop evaluation. Such a closed loop corresponds to the trace of the identity in the corresponding representation, i.e., \( \dim \rho \), which together with the factor \( \dim \rho \) in (5) gives an overall factor \( \tau = \sum \rho (\dim \rho)^2 \).

In the process the number \( n^{(0)} \) of 3-cells in \( \Delta^* \) (0-cells in \( \Delta \)) has decreased by one; therefore, this extra \( \tau \) precisely compensates for this change and (11) remains invariant.

\[ \Gamma(\Delta) = \Gamma(\Delta'), \]

\[ (11) \]

Figure 5: Invariance of the partition function under the (1,4) Pachner move.

Using the same techniques we prove invariance of (11) under the (2,3) Pachner move. We start with the diagram on the right hand side of Fig. 4 and eliminate two of the internal cables by means of gauge fixing. In this way we obtain the left hand side diagram on Fig. 6, where we have emphasized the remaining internal cable and its corresponding closed wire. The latter can be eliminated using (8), see the diagram in the center of Fig. 6. In the diagram on the right hand side of Fig. 6 we have deformed the previous diagram and we have added a redundant cable (by inverse gauge fixing) to obtain the circuit diagram corresponding to the diagram on the left hand side of Fig. 4. The number of 3-cells does not change under this move.

\[ \Gamma(\Delta) = \Gamma(\Delta'), \]

\[ (11) \]

Figure 6: Invariance of the partition function under the (2,3) Pachner move.

We have proven the following result:

**Proposition 3.1.** Consider two simplicial decompositions \( \Delta \) and \( \Delta' \) of a 3-dimensional manifold \( M \). Then the following equality holds:

\[ \Gamma(\Delta) = \Gamma(\Delta'), \]

\[ (11) \]
where the value of the partition function is computed according to (11) on the corresponding complex.

3.2 General cellular decompositions

We now generalize the above results from simplicial to general cellular decompositions of a 3-dimensional manifold. We make use of the following result (see [15]).

**Proposition 3.2.** A cellular complex $\mathcal{K}$ can be subdivided into a simplicial complex $\Delta\mathcal{K}$ without introducing any new vertices (0-cells).

We start by proving the following lemma.

**Lemma 3.1.** Let $\mathcal{K}$ be a cellular decomposition of a 3-dimensional manifold $M$ and $\Delta\mathcal{K}$ an associated simplicial decomposition (Prop. 3.2). Then, the following equality holds

$$\Gamma(\mathcal{K}) = \Gamma(\Delta\mathcal{K}).$$

Note that even $Z(\mathcal{K}) = Z(\Delta\mathcal{K})$ as $n^{(0)}$ is the same for $\mathcal{K}$ and $\Delta\mathcal{K}$.

This lemma is proven using a finite set of moves relating a cellular decomposition with an associated simplicial decomposition. For definiteness we consider these moves as leading from the simplicial subdecomposition to the cellular decomposition.

For illustration let us consider first the 2-dimensional case. There is just one move, namely forming the union of two 2-cells by removing a common boundary (the inverse consisting of the subdivision of a 2-cell). That this move leaves the partition function invariant is a straightforward consequence of the gauge fixing identity. See Fig. 7 for illustration.

![Figure 7: Lemma 3.1 in two dimensions.](image)

The 3-dimensional case of interest to us here is more involved. The relevant moves are described by the following proposition.
Proposition 3.3. Let $K$ be a cellular decomposition of a 3-manifold and $\Delta_K$ an associated simplicial decomposition. Then, $K$ and $\Delta_K$ are related by a sequence of moves and their inverses from the following list:

3-cell fusion Let $\tau$ be a 2-cell which bounds two distinct 3-cells $\sigma, \sigma'$. Remove $\tau$, $\sigma$ and $\sigma'$ and insert the new 3-cell $\sigma'' := \sigma \cup \tau \cup \sigma'$.

2-cell retraction Let $\mu$ be a 1-cell that bounds only the 2-cell $\tau$ which in turn bounds only the 3-cell $\sigma$. Remove $\mu, \tau, \sigma$ and insert the new 3-cell $\sigma' := \sigma \cup \tau \cup \mu$.

Proof. We show that starting with the simplicial subdecomposition we can recover the original cellular decomposition by applying the stated moves.

We proceed in two main steps: First we show that we can remove all the subdivisions in all 3-cells of the original decomposition. Second, we show that we can remove the subdivisions on the boundaries of the 3-cells.

Thus, consider a 3-cell $\sigma$ together with its subdivision $\Delta_\sigma$. Assume that $\Delta_\sigma$ has internal 2-cells. (Otherwise it would consist only of one 3-cell and we are done.) We claim that we can always remove one such 2-cell by either applying the 3-cell fusion move or the 2-cell retraction move. It is thus sufficient to show that if we cannot apply the 3-cell fusion move, we can apply the 2-cell retraction move. The only situation that prevents us from applying the 3-cell fusion move is when every internal 2-cell is bounded on both sides by the same 3-cell. Assume this. Consider the union of all internal 2-cells and their bounding 1-cells. This forms a branched surface. Consider its boundary. If it was contained in the boundary of $\sigma$ it would divide $\sigma$ into at least two disjoint 3-cells contrary to the assumption. Thus, the boundary of the branched surface must contain 1-cells internal to $\sigma$. Pick such a 1-cell. By construction this bounds only one (internal) 2-cell. Thus, we can apply the 2-cell retraction move to it. This completes step one of the proof.

For the second step we introduce an auxiliary move as follows. Let $\mu$ be a 1-cell that bounds only the distinct 2-cells $\tau$ and $\tau'$. Remove $\mu, \tau, \tau'$ and insert the new 2-cell $\tau'' := \tau \cup \mu \cup \tau'$. We call this the 2-cell fusion move. It can be obtained as a sequence of 3-cell fusion and 2-cell retraction moves as follows. Consider the (not necessarily distinct) 3-cells $\sigma, \sigma'$ that are bounded by $\tau$. Indeed they are also bounded by $\tau'$ as $\mu$ does not bound other 2-cells than $\tau$ and $\tau'$. Consider the boundary of $\tau \cup \mu \cup \tau'$ and introduce a 2-cell $\tau''$ with this boundary subdividing $\sigma$ into two 3-cells $\sigma''$ (bounded by the 2-cells $\tau, \tau', \tau''$) and (using the previous name) $\sigma$ by the inverse 3-cell fusion move. By construction, $\sigma'$ and $\sigma''$ are distinct. Thus, we can remove $\tau$ via a 3-cell fusion move, denoting the resulting 3-cell again by $\sigma'$. As $\mu$ now only bounds $\tau'$ which in turn only bounds $\sigma'$ we can apply the 2-cell retraction move to eliminate $\mu$ and $\tau'$ to arrive at the desired configuration.

Now consider a 2-cell $\tau$ in the boundary of a 3-cell of the original cellular complex. As the subdivision of $\tau$ does not contain any internal vertices every
internal 1-cell bounds two distinct 2-cells. We can thus apply the 2-cell fusion move to remove all internal 1-cells one by one. This completes the proof.

\textit{Proof of Lemma 3.1.} It is now sufficient to show that the moves of Proposition 3.3 leave the partition function invariant. Consider the Figures 8 and 9. These show the two moves together with relevant pieces of the circuit diagram.

The 3-cell fusion move is depicted in Figure 8. The 2-cell that separates the initial 3-cells carries one cable. The removal of the cable corresponds precisely to the move. It is possible due to the gauge fixing identity (Figure 2) and leaves not only $Z$ but even $Z_C$ invariant. The dotted line in Figure 2 corresponds to the boundary of one of the initial 3-cells in Figure 8.
The 2-cell retraction move is depicted in Figure 9. Diagrammatically it corresponds to removing the cable that is carried by the 2-cell to be deleted, together with the wire that goes around the 1-cell to be deleted. This is an invariance of $Z$ due to the summation identity (8). The summation over the representations in (8) with weight the (quantum) dimension is precisely contained in the partition function (5).

Combining Proposition 3.1 with Lemma 3.1 we have the following theorem:

**Theorem 3.1.** Given two cellular decompositions $K$ and $K'$ of a 3-dimensional manifold $M$, the following equality holds:

$$\Gamma(K) = \Gamma(K'),$$

(12)

where the value of the partition function is computed according to (11).

**Concluding Remarks**

In this final section we offer some concluding remarks and consider possible developments.

Invariance of the partition function under Pachner moves was proven originally in terms of the Biedenharn-Elliott identities for 6$j$-symbols. Conversely, one could deduce these identities from the invariance of the state sum. Our diagrammatic generalization to arbitrary cellular decompositions allow us to infer, in this way, an infinite number of relations among '$nj$-symbol'. The precise structure of these ‘generalized’ Biedenharn-Elliott identities remains to be studied.

In the proof of topological invariance we have started with simplicial decompositions (related by Pachner moves) as this is the more familiar setting. Instead we could have carried out the proof directly with cellular decompositions alone. To this end we would need a complete set of moves relating cellular decompositions in 3d. One could wonder whether the moves given in Proposition 3.3 are already sufficient for this. Indeed, the (2,3) Pachner move can be obtained as a sequence of these moves. However, the (1,4) Pachner move cannot. But it can, if we add just one more move: The 1-cell retraction move. This move removes a 0-cell and a 1-cell inside a 3-cell and is defined analogous to the 2-cell retraction move. See Figure 10. In the diagrammatic picture for the (1,4) Pachner move this is reflected in the presence of the loop that is extracted from the last tetrahedron (see Figure 9). In summary, we have thus shown that any two cellular decompositions of a compact 3-manifold are related by a sequence of the following moves: 3-cell fusion, 2-cell retraction, 1-cell retraction. The generalization of these moves to higher dimensions should be investigated.
The advantage of cellular decompositions over simplicial ones might appear a purely technical one in the topological context. However, the story becomes very different in the non-topological situation, such as for realistic models of quantum gravity in 4 space-time dimensions. In particular, the diagrammatic methods considered here are still applicable for many of the recently proposed models which are modifications of 4d BF-theory \[5, 6, 7, 8\]. At first sight one might expect that cellular decompositions (being more general) just make things more complicated. But exactly the opposite is suggested by our present work. We think here in particular of the elusive question of renormalization or “coarse-graining”, see \[21\]. In that context, renormalization identities would presumably be connected to elementary moves between complexes. One could imagine these renormalization identities as “deformations” of the diagrammatic identities in the topological case. We have shown that these identities are much simpler for the cellular moves than for the Pachner moves, suggesting renormalization (rather surprisingly) to be more manageable in the general cellular case.

When one applies the state sum model of BF-theory to manifolds with boundaries this naturally defines a TQFT. In the simplicial case, boundary data are encoded in three-valent spin networks based on the graphs that are dual to the triangulation of the boundaries. As our generalization allows for general cellular decomposition of the boundaries, boundary data can be encoded in spin network states of arbitrary valence \[9\]. A simple application of this generalization is the proof of equivalence between different spin network representations of the same physical state (due to representation theory skein relations \[4\]). Since skein relations are equations between spin networks generally involving arbitrary graphs, their proof in the state-sum context can only be realized in terms of our generalization \[16\]. The relation between the state sum (spin foam) approach, the canonical approach and the issue of the continuum limit \[17\] might become more transparent in terms of arbitrary cellular decompositions.

Generalizations of the techniques introduced here to the case of non-compact groups are of great relevance. Models for Lorentzian quantum general relativity have been introduced in \[18, 19\] as constrained state sums of BF-theory on \(SL(2,\mathbb{C})\). The models have been proven to be well defined
for a given discretization using the undeformed group \([20]\). Although the generalization of our identities to infinite dimensional representations is not obvious, a modification involving “volume factors” should be valid at least for ordinary groups such as \(SL(2, \mathbb{C})\). The investigation of scaling properties of these models via our technique might then be possible.

Further issues to be investigated are the application of the diagrammatics in the context of spin foams generated by field theories \([22]\) and for defining supersymmetric spin foam models. See \([8]\) for more remarks on these points.

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