Kolmogorov Complexity, Cosmic Microwave Background Maps and the Curvature of the Universe

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Abstract. The information theory approach is suggested to the Cosmic Microwave Background (CMB) problem for negatively curved homogeneous and isotropic Universe. Namely, the Kolmogorov complexity of anisotropy of spots in CMB sky maps is proposed as a new descriptor for revealing crucial cosmological information, particularly on the curvature of the Universe. Such profound descriptor can be especially valuable while analyzing the data of forthcoming space and ground based experiments MAP, Planck surveyor, CAT, etc.

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The Cosmic Microwave Background radiation data and particularly the sky maps are an essential source of cosmological information. Various descriptors have been proposed to extract that information, including the hot spot number density, genus, correlation function of local maxima, Euler-Poincare characteristic, etc. (for review see [1]). Most of these descriptors have been already applied for the analysis of COBE sky maps [2]. However, in view of forthcoming new generation high precision experiments, the importance of involving of more refined descriptors, is evident.

In the present paper we turn attention to a new descriptor, namely to the Kolmogorov complexity of the CMB anisotropy spots, which can carry important information on the geometry of the Universe. The idea is based on the effect of geodesic mixing [3] occurring in Friedmann-Robertson-Walker (FRW) Universe with negative curvature. Several observable consequences of this effect for CMB properties have been predicted [4],[5], including: (a) decrease of the amplitude of the anisotropy after the last scattering epoch, (b) flattening of the angular autocorrelation function, (c) distortion of the sky maps.

 Particularly, the distortion of anisotropy spots is a result of strong sta-
tical properties - exponential instability of geodesic flows in space (locally if the space is non-compact) with constant negative curvature - Anosov systems [3]. Namely, an exponential elongation (stretching) of phase space density occurs uniformly for the dimensions of the phase space, including the spatial coordinates and, in accord to Liouville theorem it contracts in equal number of dimensions.

The signature of this predicted effect - a threshold independent elongation of spots has been discovered [7] at the refined analysis of COBE 4 year (complete) dataset. Given the absence of other reasonable physical mechanisms, this can be a direct indication of the negative curvature of the Universe.

The essence of the effect of geodesic mixing includes the projection of geodesics from d=(3+1) pseudo-Riemannian space to d=3 Riemannian space and the study of the behavior of time correlation functions of geodesic flows in 3-spaces.

The problem of the geodesics' projection for homogeneous metrics was studied by Lockhart, Misra and Prigogine [8]; in that case the reparametrization of the time of the geodesic is as follows

\[ \lambda(t) = \int_{t_0}^{t} a^{-1}(s) ds = \eta(t) - \eta(t_0). \]

At d=3 the time correlation function of geodesic flow \( f^t \) is decreasing by exponential law, i.e. \( \exists c > 0 \) such that for all \( A_1, A_2 \in L^2(SM) \) [14]

\[ \bar{b}(t) = | \int A_1(f^tu)A_2(u)d\mu - \int A_1(u)d\mu(u) \int A_2(u)d\mu(u) | = \]

\[ c \| A_1 \| \| A_2 \| (1 + t)e^{-h(f)t} + O(e^{-ht}). \]  \( (1) \)

where \( \mu(SM) = 1 \) is the Liouville measure and

\[ \| A \| = [ \int A(u)^2 d\mu(u) ]^{1/2}, \]

and \( h \) is the Kolmogorov-Sinai (KS) entropy.

This exponential law is determining the quantitative efficiency of geodesic mixing, i.e. of the 3 observable effects mentioned above, including the specific elongated distortion of CMB maps [2]. However, such elongation is a

\[ ^2 \text{Note, that the map distortion due to geodesic mixing has no direct relation with the effects discussed in [9], and exists whatever are the initial shapes of spots at the last scattering surface.} \]
simplification of the complexity of the anisotropy spots, as mentioned already in [5]. To describe quantitatively the latter here we suggest to use the invariant definition of complexity - Kolmogorov complexity - introduced by Kolmogorov in 1965 [10]. The fundamental consequences of the concepts of complexity and random sequences were revealed in studies by Solomonoff, Martin-Lof and especially, by Chaitin (see [11]) and concerning the basics of physics, e.g. second law of thermodynamics, by Zurek [12].

Kolmogorov complexity $K_u$ is defined as the minimal length of the binary coded (in bits) program which is required to describe the system completely [13], i.e. it will enable to recover the initial system via a given computer. The Kolmogorov complexity of object $y$ at given object $x$ is defined as

$$K_{\phi(p)}(x) = \min_{p: \phi = x} l(p),$$

(2)

where $l(p)$ is the length of the program $p$ with respect to computer $\phi(p)$ describing the object completely, i.e. at 0–1 representation: $l(\emptyset) = 0$. The fundamental point of the Kolmogorov’s formulation is the independence of complexity on the computer. Therefore the Turing machine can be considered as a universal computer while computing the complexity.

Thus, the Kolmogorov complexity is the amount of information which is required to determine uniquely the object $x$. Kolmogorov had proved that the amount of information of $x$ with respect to $y$ is given by the formula [10]

$$I(y : x) = K(x) - K(x | y),$$

where the conditional complexity

$$K(x | y) = \min l(p),$$

is the minimal length of the program required to describe the object $x$ when the shortest program for $y$ is known. Note, that $K(x | x) = 0$ and $I(x : x) = K(x)$ and $I(x : x) \geq 0$; where $A \geq B$ denotes $A \leq B + \text{const}$.

Kolmogorov complexity measured in bits is related to KS-entropy via the relation [15]

$$\Delta I = \log_2(2^{h(f^t)(t-t_0)}) = h(f^t)(t-t_0),$$

(3)

where the loss of information $\Delta I$ during the time interval $t - t_0$ is

$$\Delta I = K_u(t) - K_u(t_0),$$

(4)
i.e. the information corresponding to the distortion of the pattern from the state initial state $t_0$ (the last scattering epoch) up to the observer at $t$ i.e. at $z = 0$. This is guaranteed by Shannon-McMillan-Breiman theorem stating the uniform exponential rate of loss of information.

Return again to the cosmological problem. In $k = -1$ FRW Universe any initial CMB structure observed at redshift $z_{obs}$ should have more complex (amoebae-like) shape than at $z < z_{obs}$

$$K(CMB\text{spot} \mid z = z_{obs}) > K(circle \mid z < z_{obs});$$

here 'circle' denotes a circle with Gaussian or other fluctuations. The complexity estimated for the anisotropy spot observed now should exceed the complexity of the primordial spot (besides the scale expansion $1 + z$ times) by an amount depending on the KS-entropy and, hence, on the curvature and the distance of the last scattering surface. Indeed, for the last scattering epoch at redshift $z$ for the Universe with the present density parameter $\Omega_0$ the exponential factor in (1) yields

$$e^{ht} = (1 + z)^2[1 + \sqrt{1 - \Omega_0}]/(\sqrt{1 + \Omega_0} + \sqrt{1 - \Omega_0})]^4 \quad (5)$$

For FRW Universe KS-entropy is determined only by the scale factor of the Universe (this is natural, since no other scale factor exists in FRW spaces) and is equal

$$h = 2/a, \quad (6)$$

Then, from Eqs.(3) and (4) we come to a simple equation linking the geometry with the complexity

$$\Delta K = (2/a)\Delta T. \quad (7)$$

In other words, the relative complexity of the observed spot with respect to a circular spot (which one can expect for flat or positively curved spaces) will determine the curvature of the hyperbolic Universe, where $\Delta T$ is the time elapsed since photons started to move freely and thus tracing how curved the 3D space is.

Kolmogorov complexity $K$, therefore, enables one to reduce the properties of CMB and the geometry of the Universe to a simple pattern recognition analysis of the shapes of anisotropy spots. Namely, the numerical value of $K$ estimated by means of the analysis of CMB sky maps will carry information on the curvature, density parameter $\Omega$, the redshift of the last scattering epoch and the law of expansion of the Universe in post-scattering epoch.
The cosmological information should be extracted from the CMB maps in the following way. First, certain criterion should define the spots represented via given configurations of pixels (e.g. [7]). Then, the computation of the Kolmogorov complexity should be performed by means of the length of a special compressed code (string) completely defining the spots. Since the basic program will be the same, the only changes will be due to different data files, i.e. the coordinates of the pixels of various spots. This problem technically has been solved in [7] and the Kolmogorov complexity of given test configurations has been computed along with their Hausdorff dimension. Note, that COBE-DMR data are still not efficient for calculation of such a refined descriptor as the complexity, since have the ratio signal/noise=2/1; the latter enables to determine certain genuine spots from the noise [18][19] but not the reliable shape of the spots. However, the forthcoming experiments, such as MAP and Planck Surveyor, can well enable to define the anisotropies with sufficient accuracy, and hence, to elaborate the technique developed here.

Thus, the heuristic content of this approach is clear:

(a) CMB properties are reduced to a problem of random sequences and information theory known by its fundamental achievements [11];

(b) Kolmogorov complexity of CMB anisotropies is a computable descriptor containing direct information on the geometry of Universe.

Moreover, the CMB properties discussed above, can be only one of manifestations of a much deeper link - negatively curved space - mixing - complexity with the fundamental physical laws, namely, implying mixing - second law of thermodynamics - arrow of time. Thus, we observe the second law and the time asymmetry because we live in a Universe with negative curvature, and those laws may not be the same in a flat or positively curved spaces. We plan to discuss curvature/second-law conjecture, including the relations between thermodynamic, cosmological and other arrows of time, in a separate paper.

The complexity and information way of thinking can be valuable also in other key cosmological problems.

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