Letter

Tighter uncertainty relations based on \((\alpha, \beta, \gamma)\) modified weighted Wigner–Yanase–Dyson skew information of quantum channels

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Abstract

We use a novel formation to illustrate the \((\alpha, \beta, \gamma)\) modified weighted Wigner–Yanase–Dyson ((\(\alpha, \beta, \gamma\) MWWYD) skew information of quantum channels. By using operator norm inequalities, we explore the sum uncertainty relations for arbitrary \(N\) quantum channels and for unitary channels. These uncertainty inequalities are shown to be tighter than the existing ones by a detailed example. Our results are also applicable to the MWWYD skew information and the \((\alpha, \gamma)\) modified weighted Wigner–Yanase–Dyson ((\(\alpha, \gamma\) MWWYD) skew information of quantum channels as special cases.

Keywords: uncertainty relation, \((\alpha, \beta, \gamma)\) MWWYD skew information, quantum channel

1. Introduction

As an extremely important issue in quantum physics, the uncertainty principle has been widespread concerned since Heisenberg [1] proposed the notions of uncertainties in measuring non-commuting observables. Based on the variance of measurement outcomes the well-known Heisenberg–Robertson uncertainty relation [2] says that for arbitrary two observables \(A\) and \(B\) with respect to a quantum state \(|\psi\rangle\), one has:

\[
\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|,
\]

where \([A, B] = AB - BA\) and \(\Delta M = \sqrt{\langle \psi | M^2 | \psi \rangle - \langle \psi | M | \psi \rangle^2}\) is the standard deviation of an observable \(M\).

There are also many ways to describe uncertainty relations, such as entropy [3–7], variance [8–11] and majorization techniques [12–15]. In particular, the quantum uncertainty can also be characterized by skew information. The skew information has been initially proposed by Wigner and Yanase [16], termed as Wigner–Yanase (WY) skew information. Then a more general quantity has been suggested by Dyson, called the Wigner–Yanase–Dyson (WYD) skew information [16]. This quantity has been further generalized in [17] and termed as generalized Wigner–Yanase–Dyson (GWYD) skew information. The uncertainty relations based on WY skew information, WYD skew information and GWYD skew information have been studied extensively [18–23].
For a quantum state $\rho$ and an observable $A$, Furuichi et al. [24] defined another generalized Wigner–Yanase skew information,

$$K_\rho^\alpha(A) = -\frac{1}{2} \text{Tr} \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A \right)^2,$$

which, called as the weighted Wigner–Yanase–Dyson (WWYD) skew information in [23], is different from WYD skew information. Chen et al. [25] proposed then a generalized Wigner–Yanase skew information for arbitrary operator $E$ (not necessarily Hermitian):

$$K_\rho^\alpha(E) = -\frac{1}{2} \text{Tr} \left( \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, E \right)^2,$$

which is termed as the modified weighted Wigner–Yanase–Dyson (MWWYD) skew information in [26], By replacing the arithmetic mean of $\rho^\alpha$ and $\rho^{1-\alpha}$ with their convex combination, the two-parameter extension of the Wigner–Yanase skew information is introduced in [26],

$$K_\rho^{\alpha,\beta}(A) = -\frac{1}{2} \text{Tr} \left((1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, A\right)^2,$$

which is called the $(\alpha, \gamma)$ weighted Wigner–Yanase–Dyson ((\(\alpha, \gamma\)) WWYD) skew information in [27]. Note that equation (4) reduces to equation (2) when $\gamma = \frac{1}{2}$.

We defined the $(\alpha, \beta, \gamma)$ weighted Wigner–Yanase–Dyson ((\(\alpha, \beta, \gamma\)) WWYD) skew information as [27],

$$K_\rho^{\alpha,\beta}(A) = -\frac{1}{2} \text{Tr} \left( (1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, A \right)^2 \rho^{1-\alpha-\beta},$$

which reduces to equation (4) when $\beta = 1-\alpha$. We also defined the $(\alpha, \beta, \gamma)$ modified weighted Wigner–Yanase–Dyson ((\(\alpha, \beta, \gamma\) MWWYD)) skew information with respect to a quantum state $\rho$ and an arbitrary operator $E$ (not necessarily Hermitian) in [27] as:

$$K_\rho^{\alpha,\beta}(E) = -\frac{1}{2} \text{Tr} \left( (1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, E \right)^2 \rho^{1-\alpha-\beta},$$

which is the non-Hermitian extension of the $(\alpha, \beta, \gamma)$ WWYD skew information. Equation (6) reduces to equation (10) in [23] when $\gamma = \frac{1}{2}$. When $\beta = 1-\alpha$, we obtain the $(\alpha, \gamma)$ modified weighted Wigner–Yanase–Dyson ((\(\alpha, \gamma\)) MWWYD) skew information,

$$K_\rho^{\alpha,\gamma}(E) = -\frac{1}{2} \text{Tr} \left( (1-\gamma)\rho^\alpha + \gamma \rho^{1-\alpha}, E \right)^2 \rho^{1-\alpha-\gamma},$$

which is the non-Hermitian extension of the $(\alpha, \gamma)$ WWYD skew information. It reduces to equation (3) when $\gamma = \frac{1}{2}$.

Quantum channels characterize the general evolutions of quantum systems [28, 29], which play an essential role in quantum information processing. The uncertainty relations for quantum channels have been investigated from both the variance-based and entropic-based uncertainty measure [30, 31]. Specifically, the unitary channels are useful and commonly encountered in both quantum information theory and quantum computation [28]. Uncertainty relations for general unitary channels have been investigated both theoretically and experimentally [32–34]. Recently, the sum uncertainty relations for quantum channels have attracted considerable attention [27, 35–38]. Fu, Sun and Luo [35] investigated the uncertainty relations for two quantum channels based on WY skew information for arbitrary operators. Afterwards, Zhang et al. [36] generalized the uncertainty relations for two quantum channels to arbitrary $N$ quantum channels and proposed tighter lower bounds than the ones in [35] for two quantum channels. Zhang et al. [37] proposed new bounds which are tighter than the results in [36]. Cai [38] confirmed that the results in [35] also hold for all metric-adjusted skew information. By employing the norm inequalities proposed in [37], we have established sum uncertainty relations for arbitrary $N$ quantum channels based on $(\alpha, \beta, \gamma)$ MWWYD skew information [27].

Following the idea in [39], the $(\alpha, \beta, \gamma)$ MWWYD skew information of a state $\rho$ with respect to a channel $\Phi$ has been defined as [27],

$$K_\rho^{\alpha,\beta}(\Phi) = \sum_{i=1}^{n} K_\rho^{\alpha,\beta}(E_i),$$

where $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, $0 \leq \gamma \leq 1$, and $E_i(i = 1, 2, \ldots, n)$ are Kraus operators of the channel $\Phi$, i.e. $\Phi(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger$. Very recently, we provided the following uncertainty relations for arbitrary $N$ quantum channels $\{\Phi_i\}_{i=1}^{N}$ with $\Phi_i(\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger$, $i = 1, 2, \ldots, N (N > 2)$ [27].
where \( \alpha, \beta \geq 0 \), \( \alpha + \beta \leq 1 \), \( 0 \leq \gamma \leq 1 \), \( S_n \) is the \( n \)-element permutation group and \( \pi_i, \pi_s \in S_n \) are arbitrary \( n \)-element permutations.

The remainder of this paper is structured as follows. In section 2, we explore the \((\alpha, \beta, \gamma)\) MWWYD skew information-based sum uncertainty relations for arbitrary \( N \) quantum channels. Especially, we show that when \( \beta = 1 - \alpha \), i.e. when the \((\alpha, \beta, \gamma)\) MWWYD skew information becomes the \((\alpha, \gamma)\) MWWYD skew information, our new bounds are tighter than the existing ones by a detailed example. The uncertainty relations based on the \((\alpha, \beta, \gamma)\) MWWYD skew information for unitary channels are discussed in section 3. We conclude with a summary in section 4.

2. Sum uncertainty relations for arbitrary \( N \) quantum channels in terms of \((\alpha, \beta, \gamma)\) MWWYD skew information

In this section, by using a new formation we explore the uncertainty relations for arbitrary \( N \) quantum channels in terms of the \((\alpha, \beta, \gamma)\) MWWYD skew information \( K_{\rho, \gamma}^{\alpha, \beta} (\Phi) \).

Let \( \Phi \) be a quantum channel with Kraus representation, \( \Phi (\rho) = \sum_{i=1}^{n} E_i \rho E_i^\dagger \). Following the idea in [37], we define the \((\alpha, \beta, \gamma)\) MWWYD skew information of the channel as,

\[
\sum_{i=1}^{N} K_{\rho, \gamma}^{\alpha, \beta} (\Phi_i) \geq \max_{\pi, \pi_s \in S_n} \frac{1}{N - 2} \left\{ \sum_{1 \leq i < k \leq N} \sum_{i=1}^{n} K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)}) \right. \\
- \frac{1}{(N - 1)^2} \left\{ \sum_{1 \leq i < k \leq N} \left( \sum_{i=1}^{n} \sqrt{K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)})} \right)^2 \right\}, 
\]

(9)

\[
\sum_{i=1}^{N} K_{\rho, \gamma}^{\alpha, \beta} (\Phi_i) \geq \max_{\pi, \pi_s \in S_n} \frac{1}{2(N - 1)} \left\{ \frac{2}{N(N - 1)} \sum_{i=1}^{n} \left( \sum_{1 \leq i < k \leq N} \sqrt{K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)})} \right)^2 \right\} \\
+ \sum_{1 \leq i < k \leq N} \sum_{i=1}^{n} K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)}) \right\}, 
\]

(10)

\[
\sum_{i=1}^{N} K_{\rho, \gamma}^{\alpha, \beta} (\Phi_i) \geq \max_{\pi, \pi_s \in S_n} \frac{1}{2} \left\{ \sum_{1 \leq i < k \leq N} \sum_{i=1}^{n} K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)}) \right. \\
- \frac{1}{(N - 1)^2} \left\{ \sum_{1 \leq i < k \leq N} \left( \sum_{i=1}^{n} \sqrt{K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)})} \right)^2 \right\}, 
\]

(11)

where \( \alpha, \beta \geq 0 \), \( \alpha + \beta \leq 1 \), \( 0 \leq \gamma \leq 1 \), and \( \pi_i, \pi_s \in S_n \) are arbitrary \( n \)-element permutations.

Theorem 1. Let \( \Phi_1, \ldots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_i (\rho) = \sum_{i=1}^{n} E_i \rho (E_i^\dagger) \), \( i = 1, 2, \ldots, N \) \((N > 2)\). We have

\[
\sum_{i=1}^{N} K_{\rho, \gamma}^{\alpha, \beta} (\Phi_i) \geq \max \{ LB1, LB2, LB3 \}, 
\]

(13)

where

\[
LB1 = \max_{\pi, \pi_s \in S_n} \frac{1}{N - 2} \left\{ \sum_{1 \leq i < k \leq N} \sum_{i=1}^{n} K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)}) \right. \\
- \frac{1}{(N - 1)^2} \left\{ \sum_{1 \leq i < k \leq N} \left( \sum_{i=1}^{n} \sqrt{K_{\rho, \gamma}^{\alpha, \beta} (E_{\pi(i)}^k + E_{\pi_s(i)})} \right)^2 \right\}, 
\]

(14)
LB2 = \max_{\pi_1, \pi_2 \in S_n} \left\{ \frac{1}{N} \sum_{i=1}^{n} K_{\pi_1, \gamma}^{\alpha, \beta} \left( \sum_{i=1}^{n} E_{\pi_1}(i) \right) \right. \\
+ \frac{2}{N^2(N-1)} \left[ \sum_{1 \leq i < j \leq N} \left( \sum_{i=1}^{n} K_{\pi_1, \gamma}^{\alpha, \beta} (E_{\pi_1}(i) - E_{\pi_1}(j)) \right)^2 \right] \right\},
(15)

LB3 = \max_{\pi_1, \pi_2 \in S_n} \left\{ \frac{1}{2(N-1)} \sum_{1 \leq i < j \leq N} \left( \sum_{i=1}^{n} K_{\pi_1, \gamma}^{\alpha, \beta} (E_{\pi_1}(i) \pm E_{\pi_1}(j)) \right)^2 \right\},
(16)

\alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, \quad 0 \leq \gamma \leq 1, \quad S_n \text{ is the n-element permutation group and } \pi_1, \pi_2 \in S_n \text{ are arbitrary n-element permutations.}

Proof. The proof is completed directly by using the following inequalities [36, 37, 40],

\[ \sum_{i=1}^{N} |u_i|^2 \geq \frac{1}{N} \left[ \sum_{1 \leq i < j \leq N} |u_i + u_j|^2 \right] \]
\[ \geq \frac{1}{(N-1)^2} \left( \sum_{1 \leq i < j \leq N} |u_i + u_j|^2 \right) \],
\[ \sum_{i=1}^{N} |u_i|^2 \geq \frac{1}{N} \left[ \sum_{i=1}^{N} |u_i|^2 \right] + \frac{2}{N^2(N-1)} \left( \sum_{1 \leq i < j \leq N} |u_i - u_j|^2 \right) \],
\[ \sum_{i=1}^{N} |u_i|^2 \geq \frac{1}{2(N-1)} \left[ \sum_{1 \leq i < j \leq N} |u_i \pm u_j|^2 \right] + \frac{2}{N^2(N-1)} \left( \sum_{1 \leq i < j \leq N} |u_i \mp u_j|^2 \right) \],

with \[ |u_i|^2 = 2K_{\pi_1, \gamma}^{\alpha, \beta}(\Phi_\rho), \quad |u_i + u_j|^2 = 2\sum_{i=1}^{n} K_{\pi_1, \gamma}^{\alpha, \beta}(E_{\pi_1}(i) + E_{\pi_1}(j)) \text{ and } |u_i \pm u_j|^2 = 2\sum_{i=1}^{n} K_{\pi_1, \gamma}^{\alpha, \beta}(E_{\pi_1}(i) \mp E_{\pi_1}(j)). \] \]

Note that when \( \alpha = \beta = \frac{1}{2} \), theorem 1 reduce to theorem 1 in [37]. As a special case, we use the \((\alpha, \gamma)\) MWYWD skew information to compare our lower bounds with the existing ones. For convenience, we denote by \( \mathcal{L}_B1, \mathcal{L}_B2, \mathcal{L}_B3 \) the right hand sides of (9), (10) and (11), respectively. The following example shows that our results give tighter lower bounds than \( \mathcal{L}_B1, \mathcal{L}_B2 \) and \( \mathcal{L}_B3 \), see figure 1.

Example 1. Given a qubit state \( \rho = \frac{1}{2}(1 + \mathbf{r} \cdot \sigma) \), where \( \mathbf{1} \) is the \( 2 \times 2 \) identity matrix, \( \mathbf{r} = (\frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta, 0) \), \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) with \( \sigma_j (j = 1, 2, 3) \) the Pauli matrices, and \( \mathbf{r} \cdot \sigma = \sum_{j=1}^{3} r_j \sigma_j \). We consider the following three quantum channels:

(i) the amplitude damping channel \( \Phi_{AD} \),
\[ \Phi_{AD}(\rho) = \sum_{i=1}^{2} A_i \rho A_i^\dagger, \quad A_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \]
\[ A_2 = \sqrt{q}|1\rangle\langle 1|; \]

(ii) the phase damping channel \( \Phi_{PD} \),
\[ \Phi_{PD}(\rho) = \sum_{i=1}^{2} B_i \rho B_i^\dagger, \quad B_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \]
\[ B_2 = \sqrt{q}|0\rangle\langle 0|; \]

(iii) the bit flip channel \( \Phi_{BF} \),
\[ \Phi_{BF}(\rho) = \sum_{i=1}^{2} C_i \rho C_i^\dagger, \quad C_1 = \sqrt{q}|0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \]
\[ C_2 = \sqrt{1-q}|0\rangle\langle 0| + |1\rangle\langle 1| \]

with \( 0 \leq q < 1 \), respectively.

For the case \( \alpha = \gamma = \frac{1}{2}, q = 0.2 \) and \( \theta = \frac{\pi}{6} \), we have \( K_{\pi_1, \gamma}^{\frac{1}{2}}(\Phi_{AD}) + K_{\pi_1, \gamma}^{\frac{1}{2}}(\Phi_{PD}) + K_{\pi_1, \gamma}^{\frac{1}{2}}(\Phi_{BF}) = 0.283955 \). The lower bounds \( \mathcal{L}_B1, \mathcal{L}_B2, \mathcal{L}_B3 \) are 0.275596, 0.2644 and 0.256419, respectively, and the lower bounds \( LB1, LB2, LB3 \) are 0.260707, 0.26726 and 0.265758, respectively. Obviously, \( LB2 \) is tightest among \( LB1, LB2 \) and \( LB3 \), which is also greater than \( \mathcal{L}_B2 \) and \( \mathcal{L}_B3 \) given in [27].

We also consider the case \( \alpha = \gamma = \frac{1}{2} \). For \( q = 0.4 \) and \( q = 0.9 \), the sum and the lower bounds \( \mathcal{L}_B1, \mathcal{L}_B2, \mathcal{L}_B3 \), \( LB1, LB2, LB3 \) are shown in figure 1, respectively. Especially, for \( q = 0.4 \), the sum and the lower bounds are calculated for some special \( \theta \), as listed in table 1. It can be seen that for \( q = 0.4 \), our lower bounds \( LB2 \) and \( LB3 \) are tighter than \( \mathcal{L}_B1, \mathcal{L}_B2 \) and \( \mathcal{L}_B3 \). While for \( q = 0.9 \), our lower bounds \( LB2 \) and \( LB3 \) are tighter than \( \mathcal{L}_B1, \mathcal{L}_B2 \) and \( \mathcal{L}_B3 \).

The above results show that theorem 1 in this paper improve the existing results ones given in [27].

3. Sum uncertainty relations for \( N \) unitary channels in terms of \((\alpha, \beta, \gamma)\) MWYWD skew information

In this section, we consider sum uncertainty relations for arbitrary \( N \) unitary channels. For a unitary channel \( \Phi_U \), we have \( \Phi_U(\rho) = U\rho U^\dagger \). According to equation (6), the \((\alpha, \beta, \gamma)\) MWYWD skew information of an unitary operator \( U \) is given by:

\[ K_{\pi_1, \gamma}^{\alpha, \beta}(U) = -\frac{1}{2} \text{Tr}[(|1-\rho^\alpha + \gamma \rho^\beta, U|)] \]
\[ [|1-\rho^\alpha + \gamma \rho^\beta, U|]^{1-\alpha-\beta} \]
\[ = \frac{1}{2} \left( [[|1-\rho^\alpha + \gamma \rho^\beta, U|]^{1-\alpha-\beta}]^2, \right) \alpha, \beta \geq 0, \quad \alpha + \beta \leq 0, \quad 0 \leq \gamma \leq 1. \]

(17)

The \((\alpha, \beta, \gamma)\) MWYWD skew information of a unitary channel \( \Phi_U \) is defined as \( K_{\pi_1, \gamma}^{\alpha, \beta}(\Phi_U) = K_{\pi_1, \gamma}^{\alpha, \beta}(U) \). For simplicity, in
the following, we use \( K_{\rho,\gamma}^{\alpha,\beta}(U) \) to denote the quantity of the unitary channel \( \Phi_{U} \) determined by \( U \). Similar to the proof of theorem 1, we can prove the following theorem.

**Theorem 2.** Let \( U_1, \ldots, U_N \) be arbitrary \( N \) unitary operators. Then we have:

\[
\sum_{i=1}^{N} K_{\rho,\gamma}^{\alpha,\beta}(U_i) \geq \max\{Lb_1, Lb_2, Lb_3\},
\]

where

\[
Lb_1 = \frac{1}{N-2} \left\{ \sum_{1 \leq i < k \leq N} K_{\rho,\gamma}^{\alpha,\beta}(U_i + U_k) \right. \\
- \frac{1}{(N-1)^2} \left\{ \sum_{1 \leq i < k \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(U_i + U_k)} \right\}^2 
\]

(19)

\[
Lb_2 = \frac{1}{N} \left[ \sum_{i=1}^{N} U_i \right] \\
+ \frac{2}{N^2(N-1)} \left\{ \sum_{1 \leq i < k \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(U_i - U_k)} \right\}^2 
\]

(20)

\[
Lb_3 = \max_{x \in \{0,1\}} \frac{1}{2(N-1)} \left\{ \sum_{1 \leq i < k \leq N} K_{\rho,\gamma}^{\alpha,\beta}(U_i + (-1)^x U_k) \right. \\
+ \frac{2}{N(N-1)} \left\{ \sum_{1 \leq i < k \leq N} \sqrt{K_{\rho,\gamma}^{\alpha,\beta}(U_i + (-1)^x U_k)} \right\}^2 
\]

(21)

and \( x \in \{0,1\}, \alpha, \beta \geq 0, \alpha + \beta \leq 1, 0 \leq \gamma \leq 1 \).

Figure 1. The solid black line represents the sum = \( K_{\rho,\gamma}^{\frac{1}{2}}(\Phi_{AD}) + K_{\rho,\gamma}^{\frac{1}{2}}(\Phi_{PD}) + K_{\rho,\gamma}^{\frac{1}{2}}(\Phi_{AR}) \). The solid blue, green and the red lines represent the lower bounds \( Lb_1, Lb_2 \) and \( Lb_3 \) in theorem 1, respectively. The dotted magenta, dashed blue and green lines are for the lower bounds \( Lb_1, TB_2 \) and \( TB_3 \), respectively. (a) \( q = 0.4 \); (b) \( q = 0.9 \).

**Example 2.** Given a qubit state \( \rho = \frac{1}{2}(1 + r \cdot \sigma) \) with \( r = (\sqrt{2} \cos \theta, \sqrt{2} \sin \theta, 0) \). Consider the following three unitary operators,

\[
U_1 = e^{i \frac{\pi}{4} x_1} = \left( \begin{array}{cc} \cos \frac{\pi}{4} & i \sin \frac{\pi}{4} \\ i \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{array} \right),
U_2 = e^{i \frac{\pi}{4} x_2} = \left( \begin{array}{cc} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{array} \right),
U_3 = e^{i \frac{\pi}{4} x_3} = \left( \begin{array}{cc} e^{i \frac{\pi}{4}} & 0 \\ 0 & e^{-i \frac{\pi}{4}} \end{array} \right),
\]

which correspond to the rotations around the \( z \)-axis of the Bloch sphere. When \( \beta = 1 - \alpha \), i.e., when the \( (\alpha, \beta, \gamma) \) MWWYD skew information reduces to the \( (\alpha, \gamma) \) MWWYD skew information, the comparison among the lower bounds of theorem 2 is presented in figure 2, from which one sees...
that the lower bound \( Lb_3 \) is tighter than \( Lb_2 \) and \( Lb_1 \) in this case.

4. Conclusions

We have studied the sum uncertainty relations for \( N \) quantum channels based on the \((\alpha, \beta, \gamma)\) MWWYD skew information. By detailed example it has been shown that our uncertainty inequalities are tighter than the existing ones. Since the MWWYD skew information and \((\alpha, \gamma)\) MWWYD skew information are two special cases of the \((\alpha, \beta, \gamma)\) MWWYD skew information, our results are also valid for the MWWYD skew information and the \((\alpha, \gamma)\) MWWYD skew information. Finally, we have also explored sum uncertainty relations for unitary channels. These results may shed some new light on the study of skew information-based sum uncertainty relations for quantum channels.

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Conflict of interest

The authors declare that they have no conflict of interest.

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