Global rigidity of complete bipartite graphs

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1. Introduction

In this note, we prove the following.

**Theorem 1.1.** Let $d \in \mathbb{N}$ and $m, n \geq d + 1$, with $m + n \geq \binom{d+2}{2} + 1$. Then the complete bipartite graph $K_{m,n}$ is generically globally rigid in dimension $d$.

This statement has appeared in [5, Theorem 63.2.2], but a proof hasn’t yet been circulated.

2. Setup and background

We start by introducing the necessary concepts and definitions.

2.1. Rigidity

**Frameworks** A framework $(G, p)$ a graph $G$ with $n$ vertices and a configuration $p : V \to \mathbb{E}^d$, mapping the vertex set $V$ of $G$ to a $d$-dimensional point set in Euclidean space.

By picking an origin arbitrarily, we identify points $x \in \mathbb{E}^d$ with affine coordinates of the form $\hat{x} := (\cdots, 1) \in \mathbb{R}^{d+1}$. Thus, we may identify a configuration with a vector in $\left(\mathbb{R}^d\right)^n$ or its affine counterpart $\hat{p} \in \left(\mathbb{R}^{d+1}\right)^n$. We can also write this as an $n \times (d+1)$ configuration matrix $\hat{P}$.

Fix a dimension $d$ and a graph $G$. Two frameworks $(G, p)$ and $(G, q)$ are **equivalent** if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \quad (\text{all edges } \{i, j\} \text{ of } G)$$

They are **congruent** if there is a Euclidean motion $T$ of $\mathbb{E}^d$ so that

$$q(i) = T(p(i)) \quad (\text{all verts. } i \text{ of } G)$$

A framework $(G, p)$ is **rigid** if there is a neighborhood $U \ni p$ so that if $q \in U$ and $(G, q)$ is equivalent to $(G, p)$, then $q$ is congruent to $p$. A framework $(G, p)$ is **globally rigid** if any $(G, q)$ equivalent to $(G, p)$ is congruent to it.

Rigidity [1] and global rigidity [6] are **generic properties**. A configuration is **generic** if its coordinates are algebraically independent over $\mathbb{Q}$.

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Theorem 2.1 ([1, 6]). Let $d$ be a dimension and $G$ a graph. Then either every generic framework $(G, p)$ in dimension $d$ is (globally) rigid or no generic framework is (globally) rigid.

If every generic framework $(G, p)$ in dimension $d$ is globally rigid, we say that $G$ is **generically globally rigid (GGR)** in dimension $d$.

**Infinitesimal rigidity** The rigidity matrix $R(p)$ of a framework $(G, p)$ is the matrix of the linear system

$$\langle p(j) - p(i), p'(j) - p'(i) \rangle = 0 \quad \text{(all edges } \{i, j\} \text{ of } G)$$

where the vector configuration $p'$ is variable. The kernel of $R(p)$ comprises the **infinitesimal flexes** of $(G, p)$. When $G$ is a graph with $n \geq d$ vertices, a $d$-dimensional framework $(G, p)$ is called **infinitesimally rigid** when $R(p)$ has rank $dn - \binom{d+1}{2}$. Infinitesimal rigidity implies rigidity [1].

**Generic global rigidity** The main tool we will use in this paper to prove that a graphs are GGR is the following:

Theorem 2.2 ([6]). Let $G$ be a graph and $d$ a dimension. Suppose that there is a framework $(G, p)$ that is infinitesimally rigid and globally rigid in dimension $d$. Then $G$ is GGR in dimension $d$.

To construct frameworks that are globally rigid, we use the stronger property of universal rigidity. A framework $(G, p)$ is **universally rigid** if any equivalent framework in any dimension is congruent. One important way to certify that a constructed framework is universally rigid is via the still stronger property of super stability. To define this we need a bit more terminology.

**Equilibrium stresses** For a graph $G$, define the space $S(G)$ of **graph supported matrices** to be the symmetric $n \times n$ matrices that have zeros in the off-diagonal entries indexed by non-edges $\{i, j\}$. A matrix $\Omega \in S(G)$ is a **stress matrix** if it has the vector of all ones in its kernel. A stress matrix $\Omega$ is an **equilibrium stress matrix** of a framework $(G, p)$ if $\Omega \hat{P} = 0$. A computation shows that for each vertex $i$

$$\sum_{j \neq i} \Omega_{ij} [p(j) - p(i)] = 0$$

if and only if a stress matrix $\Omega$ is an equilibrium stress matrix for $(G, p)$. Thus, equilibrium stress matrices are obtained by re-arranging **equilibrium stresses** of $(G, p)$, which are vectors $\omega$ in the cokernel of $R(p)$.

Suppose that $n \geq d$, let $(G, p)$ be a $d$-dimensional framework and denote by $m$ the number of edges, $r$ the rank of the rigidity matrix, $s$ the dimension of the space of equilibrium stresses and $f$ the dimension of infinitesimal flexes. Linear algebra duality gives us the Maxwell index theorem:

$$m - r = s - f + \binom{d + 1}{2}$$  \hspace{1cm} (1)

We then see that $(G, p)$ is infinitesimally rigid if and only if $s = m - dn + \binom{d+1}{2}$. 


Super stability. The edge directions of a framework $(G, p)$ is the configuration $e$ of $|E|$ points at infinity $e(i, j) := p(j) - p(i)$. A framework has its edge directions on a conic at infinity if there is a quadric surface $\Omega$ at infinity containing all of $e$. A framework with $d$-dimensional affine span is super stable if it has a positive semidefinite (PSD) equilibrium stress matrix $\Omega$ of rank $n - d - 1$ and its edges directions are not on a conic at infinity.

The main connection between these concepts is due to Connelly.

Theorem 2.3 (\cite{Connell}). If $(G, p)$ is super stable, then it is universally rigid.

2.2. Bipartite graphs and partitioned point sets

We are interested in graphs $G$ that are simple and bipartite, with vertex partition $U$, $V$ and edge set $E$. We denote by $u$ and $v$ the size of $U$ and $V$, respectively, and the total number of vertices by $n := u + v$. The number of edges is $m := |E|$.

For notational convenience, we denote a configuration of the vertices of a bipartite graph by a pair of mappings $p : U \to E^d$ and $q : V \to E^d$, and a framework on a bipartite graph by $(G, p, q)$. All the other definitions discussed in the previous section for point configurations extend naturally to partitioned point configurations $(p, q)$.

2.3. General position and quadric separability

We say that a point set $p$ of at least $d + 1$ points in dimension $d$ is in (affine) general position if any $d + 1$ of the points are affinely independent. This is equivalent to any $d + 1$ of the vectors in $\hat{p}$ being linearly independent.

Fix a dimension $d \in \mathbb{N}$. Let $D = \binom{d+2}{2} - 1$. Let $\mathcal{V} : \mathbb{R}^{d+1} \to \mathbb{R}^{D+1}$ denote the degree 2 homogeneous Veronese map, which is defined by $x \mapsto xx^T$. This is well-defined, since the image is a subset of symmetric $(d + 1) \times (d + 1)$ matrices, which are naturally identified with $\mathbb{R}^{D+1}$ by a suitable choice of coordinates. We give this $\mathbb{R}^{D+1}$ the trace inner product

$$\langle X, Y \rangle = \text{Tr}(XY)$$

We define the action of $\mathcal{V}$ on $x \in E^d$ by $\mathcal{V}(x) = \hat{x} \hat{x}^T$. This extends to an action $\mathcal{V}(p)$ on point configurations including partitioned point configurations. For $x \in E^d$, $\mathcal{V}(x)$ is a matrix with a 1 in the bottom right corner. Thus, we can view $\mathcal{V}$ as mapping $E^d$ to a $D$-dimensional affine space, which we denote by $\mathbb{A}^D$.

The inner product described above identifies $(\mathbb{R}^{D+1})^*$ with quadratic polynomials on $E^d$, since, if $Q$ is a $(d + 1) \times (d + 1)$ symmetric matrix,

$$\hat{x}^T Q \hat{x} = \text{Tr}((\hat{x} \hat{x}^T)Q) = \langle \mathcal{V}(x), Q \rangle$$

We note that the identification implies the following, which we need later:

Lemma 2.1. Let $d \in \mathbb{N}$ be a dimension. Then $\mathcal{V}(E^d)$ affinely spans $\mathbb{A}^D$.

Proof. If $\mathcal{V}(E^d)$ has defective affine span, then there is a non-zero quadratic polynomial vanishing on all of $E^d$, which is impossible. \qed
A partitioned point configuration \((p, q)\) is *strictly quadratically separable* if \(V(p)\) and \(V(q)\) are strictly separable by an (affine) hyperplane in \(\mathbb{A}^D\). This is equivalent to there being a quadric surface \(\Omega\) in \(\mathbb{E}^d\) strictly separating the points of \(p\) from those of \(q\).

We recall that \(V(p)\) and \(V(q)\) are not strictly separable by a hyperplane in \(\mathbb{A}^D\), if and only if their convex hulls \(\text{conv}(V(p))\) and \(\text{conv}(V(q))\) have non-empty intersection.

### 3. Super-stable realizations of \(K_{d+1,d+1}\)

In this section we construct super stable realizations of \(K_{d+1,d+1}\) in dimension \(d\) with some additional properties.

**Proposition 3.1.** For each \(d\), there are \((p, q)\) such that: (a) \(p\) and \(q\) both have \(d\)-dimensional affine span in \(\mathbb{E}^d\); (b) \(V(p, q)\) has \(2d + 1\)-dimensional linear span in \(\mathbb{R}^{d+1}\); (c) \((K_{d+1,d+1}, p, q)\) is super stable.

The key rigidity theoretic tool we need is a result of Connelly and Gortler.

**Theorem 3.1** ([4]). Let \(u \geq v \geq 1\). If \(\text{conv}(V(p))\) and \(\text{conv}(V(q))\) intersect in their relative interiors, then \((K_{u,v}, p, q)\) is super stable. If \(\text{conv}(V(p))\) and \(\text{conv}(V(q))\) are disjoint, then \((K_{u,v}, p, q)\) is not universally rigid.

The rest of this section builds the proof of Proposition 3.1 in stages.

**Not quadratically separable** Let \(C\) be a curve in \(\mathbb{E}^d\). We assume there is a parameterization \(f_C(t)\). Suppose that \((p, q)\) is on \(C\), and define \((p, q)\) to be alternating on \(C\) if there are \(s_i\) and \(t_j\) such that \(s_1 < t_1 < s_2 < \cdots\), with \(p(i) = f_C(s_i)\) and \(q(j) = f_C(t_j)\).

**Lemma 3.2.** If \(u \geq v \geq d + 1\), and \((p, q)\) is alternating on a degree \(d\) curve \(C\), then \((p, q)\) is not strictly quadratically separable.

**Proof.** The alternating property implies that any separating quadric \(\Omega\) would have to intersect \(C\) transversely in at least \(2d + 1\) points. Since \(C\) is of degree \(d\), Bezout's Theorem implies that, for any quadric \(\Omega\), either \(|C \cap \Omega| \leq 2d\) or \(\Omega\) contains a component of \(C\). Either case contradicts strict separation.

**General position** We will place \((p, q)\) so that it alternates along the rational normal curve \(C_d\), which is the projective counterpart to the moment curve. For \(s, t\) real,

\[
[s : t] \mapsto [t^d : t^{d-1}s : \cdots : ts^{d-1} : s^d].
\]

The part of \(C_d\) in \(\mathbb{E}^d\) is the moment curve, obtained by setting \(s = 1\).

The rational normal curve \(C_d\) is characterized by the property that any \(d + 1\) points on it are affinely independent. We need a similar statement for the re-embedded curve \(V(C_d)\), which has the parameterization

\[
[s : t] \mapsto V([t^d : t^{d-1}s : \cdots : ts^{d-1} : s^d]).
\]
It is immediate that the degree of $\mathcal{V}(\mathcal{C}_d)$ is $2d$. Looking more closely, we see that, in fact, $\mathcal{V}(\mathcal{C}_d)$ is a rational normal curve in its affine span. The $ij$th entry of $\mathcal{V}([t^d : t^{d-1}s : \cdots : ts^{d-1} : s^d])$ is

$$t^{d-i+1}s^{i-1}t^{d-j+1}s^{j-1} = t^{2d-i-j+2}s^{i+j-2}$$

Since every entry is determined by $i + j$, there are $2d + 1$ distinct entries, and any set of these parameterizes a rational normal curve of degree $2d$. From this, we see that any $2d + 1$ points on $\mathcal{V}(\mathcal{C}_d) \cap A^D$ are affinely independent.

**Lemma 3.3.** Suppose that that $(p, q)$ is alternating along the $d$-dimensional rational normal curve and each of $p$ and $q$ have $d + 1$ points in $\mathbb{E}^d$. Then $p$ and $q$ are in affine general position in $\mathbb{E}^d$ and, under the Veronese map, any $2d + 1$ points of $(p, q)$ have $2d$-dimensional affine span in $A^D$.

**Proof of Proposition 3.1** Parts (a) and (b) are Lemma 3.3. For part (c), we know from Lemma 3.2 and the alternating pattern, that $P = \text{conv}(\mathcal{V}(p))$ and $Q = \text{conv}(\mathcal{V}(q))$ have non-empty intersection. To conclude super-stability from Theorem 3.1 we need that $P$ and $Q$ intersect in their relative interiors.

Suppose the contrary for a contradiction. Let $P'$ and $Q'$ be maximal faces of $P'$ and $Q'$ that meet in their relative interiors. Necessarily, $P'$ and $Q'$ are proper, so they span at most $2d$ points in total. Writing any $x \in P' \cap Q'$ as a convex combination of vertices of $P'$ and $Q'$, respectively, shows that these points are affinely dependent in $A^D$. This contradicts Lemma 3.3. Hence we conclude that $P$ and $Q$ intersect in their relative interiors, as desired. \qed

4. The proof

For the rest of this section let $d$, $m$ and $n$ be as in the statement. We refer to the subgraph induced by the first $d + 1$ vertices in each part as the core of $K_{m,n}$, and denote by $U_1$ and $V_1$ the vertices of the core. For notational convenience, we also denote by $p(U')$ and $q(V')$ the sub-configurations of $p$ and $q$ indexed by $U' \subseteq U$ and $V' \subseteq V$.

The proof strategy is to start from Proposition 3.1 to produce a configuration $(p, q)$ such that $(G, p, q)$ is infinitesimally rigid and globally rigid. The desired statement then follows from Theorem 2.2.

4.1. The Bolker-Roth stress decomposition

We will use results of Bolker and Roth [2] on the stresses of complete bipartite graphs. Denote by $p^\vee$ the Gale dual (see, e.g., [8, Section 6.3]) of the vector configuration $\hat{p}$. The rank of $p^\vee$ is equal to the dimension of the cokernel of the configuration matrix $\hat{P}$. We similarly define the rank of $(\mathcal{V}(p, q))^\vee$ in terms of the images in $\mathbb{R}^{D+1}$ of $p$ and $q$ under $\mathcal{V}$.

**Theorem 4.1** ([2]). Suppose that $p$ and $q$ both have full affine span in $\mathbb{E}^d$. The the dimension of the space of equilibrium-stresses of $(K_{u,v}, p, q)$ is given by $\text{rank}(p^\vee) \text{rank}(q^\vee) + \text{rank}((\mathcal{V}(p, q))^\vee)$.

\[^{1}\text{We thank Jessica Sidman for suggesting this approach.}\]
4.2. Trilateration and global rigidity

A graph $H$ is a trilateration in dimension $d$ of a graph $G$, if $H$ is obtained from $G$ by adding a new vertex to $G$ and connecting it to at least $d + 1$ neighbors.

Global and universal rigidity are very well-behaved with respect to trilateration: it is preserved no matter where the new vertex is placed, so long as the neighbors affinely span $\mathbb{R}^d$. This seems to be a folklore result.

**Lemma 4.1.** Let $(G, p)$ be a globally (resp universally) rigid framework in dimension $d$. Let $H$ be a graph obtained by trilaterating $G$ and $U$ the set of at least $d + 1$ neighbors. If $p(U)$ affinely spans $\mathbb{R}^d$, then for any placement $p(v_0)$ of the new vertex $v_0$, the resulting framework on $H$ is globally (resp universally) rigid.

**Sketch.** This follows from the fact that $K_{d+2}$ is universally rigid in dimension $d$, and gluing two globally (or universally) rigid frameworks along $d + 1$ affinely independent vertices in dimension $d$ preserves global (or universal) rigidity.

We note that an immediate consequence is that trilaterating a generically globally rigid graph yields another generically globally rigid graph.

4.3. Proof of Theorem 1.1

By Lemma 4.1, it is sufficient to prove that $K_{m,n}$ is generically globally rigid when $m, n \geq d + 1$ and $m + n = \binom{d+2}{2} + 1 = D + 2$. From now on, we make this assumption.

**The construction** Now we describe the construction of $(p, q)$. Realize the core of $K_{m,n}$ using Proposition 3.1. Then, at each trilateration step, place the new vertex generically.

**Infinitesimally rigid** Using $m + n = \binom{d+2}{2} + 1$, we have, by direct computation

$$mn - (m - d - 1)(n - d - 1) - 1 = d(m + n) - \binom{d+1}{2}$$

The Maxwell index theorem (Equation 11) then tells us that if the dimension of the space of equilibrium stresses of $(K_{m,n}, p, q)$ has dimension

$$(m - d - 1)(n - d - 1) + 1$$

then $(K_{m,n}, p, q)$ is infinitesimally rigid.

The desired equilibrium stress space dimension follows from Theorem 4.1 and two observations:

- Since the core is realized in general position and the rest of the points are placed generically, $(p, q)$ is in general position. Hence rank$(p^V)$ rank$(q^V) = (m - d - 1)(n - d - 1)$.

- The core is realized (non-generically) so that rank$((\mathcal{V}(p(U_1), q(V_1)))^V) = 1$. This is because the $2d + 2$ points of $\mathcal{V}(p(U_1), q(V_1))$ have $2d$-dimensional affine span in $\mathbb{A}^D$ by Proposition 3.1. (And so have one non-trivial affine dependency in $\mathbb{A}^D$.)

We add $D + 2 - 2(d + 1)$ additional generic points in $\mathbb{E}^d$ during the trilateration phase. The images of these under $\mathcal{V}$ are generic in $\mathcal{V}(\mathbb{E}^d)$, which has $D$-dimensional affine span in $\mathbb{A}^D$ (Lemma 2.1), so no new affine dependencies appear during the trilateration phase. Hence, the $D + 2$ points of $\mathcal{V}(p, q))$ affinely span $\mathbb{A}^D$, which implies that rank$((\mathcal{V}(p, q))^V) = 1$. 


Globally rigid Proposition 3.1 implies that the core is super stable and thus universally rigid. Lemma 4.1 then implies that \((G, p, q)\) is universally rigid and thus globally rigid.

5. Concluding remarks

The hypotheses of Theorem 1.1 are also necessary. We need \(m, n \geq d + 1\) to avoid contradicting Hendrickson's necessary conditions \([7]\) for a graph to be GGR. The classification of equilibrium stresses in complete bipartite graphs by Bolker and Roth \([2]\) implies that, if \(m + n < \left(\frac{d+2}{2}\right) + 1\) any equilibrium stress matrix of a generic framework \((K_{m,n}, p, q)\) has all zeros on its diagonal and is thus necessarily of deficient rank.

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