Symmetries in left-invariant optimal control problems

A. V. Podobryaev
A. K. Ailamazyan Program Systems Institute of RAS
alex@alex.botik.ru

July 25, 2018

Abstract

We consider left-invariant optimal control problems on connected exponential Lie groups (in particular connected nilpotent or compact). We introduce a construction for symmetries of the exponential map. These symmetries play a key role in investigation of optimality of extremal trajectories.

Keywords: symmetry, geometric control theory, Riemannian geometry, sub-Riemannian geometry.

AMS subject classification: 49J15, 53C17.

Introduction

Geometric control theory (see for example [1]) deals with left-invariant optimal control problems on a Lie group $G$. Consider a family of left-invariant vector fields $F_u$ that depend analytically on $u \in U \subseteq \mathbb{R}^n$. Consider also a left-invariant analytic function $\varphi : G \times U \to \mathbb{R}$, a point $q_1 \in G$, and a fixed time $t_1 > 0$. The problem is to find a control $u \in L^\infty([0, t_1], U)$ and a Lipschitz curve $q_u : [0, t_1] \to G$ such that

$$\int_0^{t_1} \varphi(q_u(t), u(t))dt \to \min, \quad \dot{q}_u(t) = F_u(q_u(t)), \quad q_u(0) = \text{id}, \quad q_u(t_1) = q_1 \in G. \quad (1)$$

Consider functions $h_u$ on the cotangent bundle $T^*G$ that depend on parameter $u \in U$:

$$h_u(\lambda) = \lambda(F_u(\pi(\lambda)) - \varphi(\pi(\lambda), u)), \quad \lambda \in T^*G,$$

where $\pi : T^*G \to G$ is the natural projection. Assume that for all $\lambda \in T^*G$ the quadratic form $\frac{\partial^2}{\partial u^2}h_u(\lambda)$ is negative definite and the function $u \mapsto h_u(\lambda)$ has maximum. Then via

---

*This work is supported by the Russian Science Foundation under grant 17-11-01387 and performed in A. K. Ailamazyan Program Systems Institute of Russian Academy of Sciences.
Pontryagin maximum principle [2, 1] we obtain a Hamiltonian differential equation on the cotangent bundle $T^\ast G$, such that its phase curves project to optimal trajectories on the group $G$:

$$\dot{\lambda} = \vec{H}(\lambda), \quad \pi(\lambda(t)) = q_u(t), \quad \lambda : [0, t_1] \to T^\ast G, \quad (2)$$

where $H(\lambda) = \max_{u \in U} h_u(\lambda)$ is the analytic maximized Hamiltonian of Pontryagin maximum principle, $\vec{H}$ is the corresponding analytic Hamiltonian vector field. The curve $\lambda(t)$ is called a normal extremal. Next we will consider only such extremals. The curve $q_u(t)$ is called a normal extremal trajectory.

If one could have explicit solution of differential equation (2), then one has a parametrization of extremal trajectories. After that it remains to study an optimality of extremal trajectories.

**Definition 1.** A Maxwell point for an optimal control problem (1) is a point where two distinct extremal trajectories meet one another with the same value of the cost functional and the time. This time is called a Maxwell time.

It is well known (see for example [7]), that an extremal trajectory can not be optimal after a Maxwell point. That is why description of Maxwell points plays an important role in investigation an optimality of extremal trajectories. In particular, the first Maxwell time is an upper bound for the time of loss of optimality (the cut time).

A natural reason of appearance of Maxwell points is a symmetry of extremal trajectories. Let us give definitions.

**Definition 2.** The exponential map of problem (1) is the map

$$\text{Exp} : g^\ast \times \mathbb{R}_+ \to G, \quad \text{Exp} (p, t) = \pi \circ e^{\tau \vec{H}}(\text{id}, p), \quad (p, t) \in g^\ast \times \mathbb{R}_+,$$

where $g$ is the Lie algebra of the Lie group $G$, and $e^{\tau \vec{H}}$ is the flow of the Hamiltonian vector field $\vec{H}$.

**Definition 3.** A symmetry of the exponential map is a pair of diffeomorphisms

$$s : g^\ast \times \mathbb{R}_+ \to g^\ast \times \mathbb{R}_+, \quad S : G \to G \quad \text{such that} \quad \text{Exp} \circ s = S \circ \text{Exp}.$$

Consider the trivialization of cotangent bundle via left shifts:

$$\tau : G \times g^\ast \to T^\ast G, \quad \tau(g, p) = dL_{g^{-1}}^* p \in T_g^\ast G, \quad g \in G, \quad p \in g^\ast,$$

where $L_g : G \to G$ is the left shift by an element $g \in G$.  

The Hamiltonian $H$ is left-invariant, so we assume that $H \in C^\infty(\mathfrak{g}^*)$. A Hamiltonian vector field is a sum of the horizontal and the vertical parts \cite{1}:

$$\vec{H}(\tau(g, p)) = \vec{H}_{\text{hor}}(\tau(g, p)) + \vec{H}_{\text{vert}}(\tau(g, p)),$$

$$\vec{H}_{\text{hor}}(\tau(g, p)) = dL_g d_p H, \quad \vec{H}_{\text{vert}}(\tau(g, p)) = dL_{g^{-1}}'(\text{ad}^* d_p H),$$

(3)

where $d_p H \in T_p \mathfrak{g}^* \simeq \mathfrak{g}$ is the differential of $H$ at a point $p$. Denote the vector field $\vec{H}_{\text{vert}}|_{T^*_p \mathfrak{g}} \in \text{Vec}(\mathfrak{g}^*)$ by the same symbol $\vec{H}_{\text{vert}}$, the meaning of this notation will be clear from a context.

The Hamiltonian system $\lambda = \vec{H}(\lambda)$ is triangular (its vertical part is independent of state variables). So, one can naturally consider symmetries of the exponential map induced by symmetries of the vertical part of the Hamiltonian system (see complete statement in Theorem \[\text{I}\]).

A plan of investigation of optimality of extremal trajectories reads as follows.

1. Parametrization of extremal trajectories.

2. Description of symmetries of the vertical part of the Hamiltonian system. Extension of these symmetries to symmetries of the exponential map.

3. Search for Maxwell points that correspond to symmetries. Search for the first Maxwell time as a function $t_{\max} : \mathfrak{g}^* \to \mathbb{R}_+ \cup \{+\infty\}$.

4. Estimation of the first conjugate time, i.e., the function $t_{\text{conj}} : \mathfrak{g}^* \to \mathbb{R}_+ \cup \{+\infty\}$ such that a pair $(p, t_{\text{conj}}(p))$ is a critical point of the exponential map.

5. Verification of the condition $t_{\max}(p) \leq t_{\text{conj}}(p)$ for almost all $p \in \mathfrak{g}$.

6. Application of the Hadamard theorem on global diffeomorphism \cite{3} to the map

$$\text{Exp}(\cdot, t_1) : \{p \in \mathfrak{g}^* \setminus 0 \mid t_1 < t_{\max}(p)\} \to G \setminus (\{\text{id}\} \cup \overline{\mathcal{M}}),$$

where $\overline{\mathcal{M}}$ is the closure of the Maxwell set. (A smooth non-degenerate proper map of connected and simply connected manifolds of equal dimensions is a diffeomorphism.)

We need items 4 and 5 to verify the non-degenerateness condition of the Hadamard theorem. If implementation of all these steps is complete, than the first Maxwell time is actually the cut time.

Notice, that implementation of this program is not guaranteed. For example, symmetries of the vertical part of the Hamiltonian system may not produce a complete description of the Maxwell set. Such situation appears in Euler elasticae problem \cite{1, 4}. However this method works in several sub-Riemannian and Riemannian problems (see references below):
1. Generalized Dido problem (Yu. L. Sachkov [6, 7, 8]).

2. Sub-Riemannian problem on Engel group (A. A. Ardentov, Yu. L. Sachkov [9, 10, 11, 12]).

3. Free nilpotent sub-Riemannian problem with growth vector (3, 6) (O. Myasnichenko [13]).

4. Sub-Riemannian problems on the Lie groups SL$_2$(R), PSL$_2$(R), SO$_3$, SU$_2$ (U. Boscain, F. Rossi [14], and independently and using another techniques by V. N. Berestovskii and I. A. Zubareva [15, 16]).

5. Riemannian problems on the Lie groups SL$_2$(R), PSL$_2$(R), SO$_3$, SU$_2$ (A. V. Podobryaev, Yu. L. Sachkov [17, 18]).

6. Sub-Riemannian problem on the Lie group SE$_2$ (Yu. L. Sachkov [19, 20, 21], the first paper in collaboration with I. Moiseev).

7. Sub-Riemannian problem on the Lie group SH$_2$ (Ya. A. Butt, Yu. L. Sachkov, A. I. Bhatti [22, 23]).

8. The problem of a rolling sphere on the plane without twisting and slipping (Yu. L. Sachkov [24]).

Here we have problems on nilpotent groups (1–3), compact groups (SO$_3$, SU$_2$), semisimple groups (SL$_2$(R), PSL$_2$(R)), semidirect product of commutative and compact groups (6, SE$_2$ = R$^2$ × SO$_2$), semidirect product of commutative and nilpotent groups (7, SH$_2$ = R$^2$ × R), direct product of compact and commutative groups (8, SO$_3$ × R$^2$).

Left-invariant optimal control problems on compact or nilpotent Lie groups are of special interest. Nilpotent ones due to existence of a nilpotent approximation [25] of control systems.

In the problems listed above an extension of symmetries of the vertical subsystem to symmetries of the exponential map were constructed by an explicit formulas for the map Exp (i.e., an explicit parametrization of extremal trajectories) or by an explicit form of the Hamiltonian system. Existence of such extension was not guaranteed a priori.

In Section [1] we introduce conditions for existence of extension of symmetries of the vertical subsystem to symmetries of the exponential map. Also there is a general construction of such symmetries and some corollaries. The proof is in Section [2]. We describe a non-trivial example in Section [3]. In addition there is Appendix for a technical definition and a lemma.
1 The main result

Let $G$ be a Lie group, $\mathfrak{g}$ be its Lie algebra. Consider the cotangent bundle $T^*G$ with the action of the group $G$ by left shifts. Let $H \in C^\infty(T^*G)$ be a left-invariant Hamiltonian, $\vec{H}$ be the corresponding Hamiltonian vector field, $\vec{H}_{\text{hor}}$ and $\vec{H}_{\text{vert}}$ be its horizontal and vertical parts, respectively, see (3).

Recall that a Lie group $G$ is called exponential, if the map $\exp : \mathfrak{g} \to G$ is surjective.

Theorem 1. Let $G$ be an exponential Lie group, $H$ be a left-invariant Hamiltonian, and an operator $\sigma : \mathfrak{g} \to \mathfrak{g}$ be such that $\sigma^*$ preserves the Hamiltonian $H$ and there hold one of two conditions:

(a) $\sigma^*(\vec{H}_{\text{vert}}) = \vec{H}_{\text{vert}}$ and $\sigma^*$ is an inner automorphism of the Lie algebra $\mathfrak{g}$;

(b) $\sigma^*(\vec{H}_{\text{vert}}) = -\vec{H}_{\text{vert}}$ and $\sigma^*$ is an inner anti-automorphism of the Lie algebra $\mathfrak{g}$.

Then the pair of diffeomorphisms $(s_\sigma, S_\sigma)$ is a symmetry of the exponential map, where

$$s_\sigma(p, t) = \begin{cases} (\sigma p, t), & \text{in case (a)}, \\ (\sigma e^{t\vec{H}_{\text{vert}}} p, t), & \text{in case (b)}, \end{cases}$$

$$S_\sigma(\exp \xi) = \exp (\sigma^* \xi), \quad \xi \in \mathfrak{g}.$$

Remark 1. The diffeomorphism $S_\sigma$ is correctly defined, because the Lie group $G$ is exponential and the (anti-)automorphism $\sigma^*$ preserves fibers of the map $\exp$. Indeed, $\sigma^* = \pm \text{Ad} f$ for some $f \in G$. Then for all $\xi \in \mathfrak{g}$, we have $\exp (\sigma^* (\xi)) = f (\exp \xi)^\pm f^{-1}$.

Everywhere below we consider symmetries $(s_\sigma, S_\sigma)$ of the exponential map such that $\sigma$ satisfies the hypotheses of Theorem 1.

Definition 4. The Maxwell sets in the pre-image and in the image of exponential map, corresponding to the symmetry $(s_\sigma, S_\sigma)$, are the sets

$$M_\sigma = \{(p, t_1) \in \mathfrak{g}^* \times \mathbb{R}_+ \mid \text{Exp}(p, t_1) = \text{Exp} \circ s_\sigma(p, t_1)\}, \quad M_\sigma = \text{Exp} M_\sigma,$$

respectively.

Corollary 1. Let $G$ be a connected compact Lie group. If a normal extremal trajectory meets a geodesic of the Killing metric, then the symmetric extremal trajectory meets the symmetric geodesic of the Killing metric at the same instant of time.

Corollary 2. Let $G$ be a connected compact Lie group. The Maxwell sets corresponding to symmetries in the image of the exponential map are subsets of the Maxwell sets corresponding to symmetries of the Riemannian problem for the Killing metric.
**Corollary 3.** Assume that there are two left-invariant optimal control problems on a connected compact Lie group. If two extremal trajectories corresponding to these problems meet one another, then the symmetric trajectories meet one another as well with the same values of time and cost functional.

**Proof of Corollaries 1–3.** For a compact Lie group $G$ one-parametric subgroups $g_\xi(t) = \exp (t\xi)$ are geodesics of the bi-invariant Riemannian metric that is defined by the Killing form $[1]$. This remark and Theorem 1 imply Corollary 1.

Corollary 2 follows directly from Theorem 1 and Corollary 1. One can get Corollary 3 by application of Corollary 1 two times. □

**2 Proof of Theorem 1**

Define a diffeomorphism $\hat{S}_\sigma = \tau \circ (S_\sigma, \sigma) \circ \tau^{-1}$, where $(S_\sigma, \sigma) : G \times g^* \to G \times g^*$ is the direct product of the maps $S_\sigma$ and $\sigma$. Denote by $\hat{H}_{\text{hor}}^\tau(\tau(g, p)) = dR_g d_p H = (\text{Ad } g)^{-1} \hat{H}_{\text{hor}}^\tau(\tau(g, p))$, where $R_g$ is the right shift by an element $g$.

**Lemma 1.** The diffeomorphism $\hat{S}_\sigma$ preserves the vector field $\vec{H}_{\text{hor}}$ in case (a), and maps it to $\vec{H}_{\text{hor}}^\tau$ in case (b).

**Proof.** The differential of the map $\hat{S}_\sigma$ at a point $g = \exp \xi$ equals

$$d_g \hat{S}_\sigma = (d_{\sigma^* \xi} \exp) \sigma^* (d_\xi \exp)^{-1}.$$ 

By the Helgason formula $[26]$ $d_\xi \exp = dR_g \frac{\exp (\text{ad } \xi)}{\text{ad } \xi}$. (See the definition of this fraction in Appendix. Also in Appendix there is a technical Lemma 4.) Substituting it to the formula of $d_g \hat{S}_\sigma$ and using the equality $\text{ad } (\sigma^* \xi) = \pm \sigma^* \text{ad } \xi$ for the (anti-)automorphism $\sigma^*$, we get

$$d_g \hat{S}_\sigma = dR_{S_\sigma(g)} \sigma^* \left( \frac{(\exp (\pm \text{ad } \xi) - 1)}{\pm \text{ad } \xi} \right) \left( \frac{(\exp \text{ad } \xi - 1)}{\text{ad } \xi} \right)^{-1} dR_g^{-1}.$$ 

In case (a) we obtain

$$d_g \hat{S}_\sigma dL_g d_p H = dR_{S_\sigma(g)} \sigma^* (\text{Ad } g) d_p H = dL_{S_\sigma(g)} (\text{Ad } S_\sigma(g))^{-1} \sigma^* (\text{Ad } g) d_p H = dL_{S_\sigma(g)} \sigma^* d_p H.$$ 

In case (b) from Lemma 4 it follows that $(\exp (-\text{ad } \xi) - 1) \left( \frac{\exp \text{ad } \xi - 1}{\text{ad } \xi} \right)^{-1} = (\exp (\text{ad } \xi))^{-1} = (\text{Ad } g)^{-1}$ and $d_g \hat{S}_\sigma dL_g d_p H = dR_{\sigma^*} d_p H$. The operator $\sigma^*$ preserves $H$, then $\sigma^* d_p H = d_{\sigma p} H$. □

Denote by $P^t = e^{t \vec{H}_{\text{vert}}}$ the flow of the vertical part of the Hamiltonian vector field, and denote by $P^t_\sigma$ its differential.
Lemma 2. The equality $\tilde{H}_{\text{hor}} = P^t \tilde{H}_{\text{hor}}$ is satisfied.

Proof. Let $\tau(g_t, p_t)$ be a trajectory of the Hamiltonian system in the left trivialization. It is well known (see for example [27]) that $p_t = (\text{Ad}^* g_t)p_0$. Next,

$$P^t \tilde{H}_{\text{hor}}(\tau(g_t, p_0)) = P^t dL_{g_t}(\text{Ad} g_t)^{-1} d_{p_0} H = d_{(\text{id}, p_0)} \tau (\text{Ad} g_t)^{-1} d_{p_0} H, 0).$$

On the other hand, $H(p_t) = H((\text{Ad}^* g_t)p_0)$, so

$$P^t \tilde{H}_{\text{hor}}(\tau(g_t, p_0)) = d_{(\text{id}, p_t)} \tau (\text{Ad} g_t)^{-1} d_{p_0} H \text{Ad}^* g_t, 0) = d_{(\text{id}, p_0)} \tau (\text{Ad} g_t)^{-1} \text{Ad} g_t d_{p_0} H, 0).$$

This is equal to $dL_{g_t} d_{p_0} H = \tilde{H}_{\text{hor}}(\tau(g_t, p_0))$. □

Lemma 3. In case (a) the equality $e^{t\hat{H}} \circ \sigma = \tilde{\sigma} \circ e^{t\hat{H}}$ is satisfied, in case (b) the equality $e^{t\hat{H}} \circ \sigma \circ P^t = P^t \circ \tilde{\sigma} \circ e^{t\hat{H}}$ is satisfied.

Proof. In case (a) the diffeomorphism $\tilde{\sigma}$ preserves the horizontal and the vertical parts of the Hamiltonian vector field $\hat{H}$. This implies that the diffeomorphism $\tilde{\sigma}$ commutes with the flow $e^{t\hat{H}}$.

In case (b) from the formula of full variation from the chronological calculus [28] and from Lemma [1] it follows that

$$e^{t\hat{H}} \circ \tilde{\sigma} \circ P^t = \exp \int_0^t (\text{Ad} P^t) \tilde{H}_{\text{hor}} d\alpha \circ P^t \circ \tilde{\sigma} \circ P^t = \tilde{\sigma} \circ \exp \int_0^t (\text{Ad} P^{-\alpha}) \tilde{H}_{\text{hor}} d\alpha \circ P^{-t} \circ P^t = \tilde{\sigma} \circ e^{td\tilde{\sigma}(\hat{H})} \circ P^t.$$

Let us prove that $e^{td\tilde{\sigma}(\hat{H})} \circ P^t = P^{-t} e^{t\hat{H}}$. From this equality the statement of Lemma follows. Consider the operator $D^t = P^{-t} e^{t\hat{H}} P^{-t} e^{-t\tilde{\sigma}(\hat{H})}$. It is necessary to show that $D^t \equiv \text{id}$. Notice that $D^0 = \text{id}$. Let us prove that $D^t = 0$.

Indeed, from

$$\dot{P}^{-t} = -\tilde{H}_{\text{vert}} P^{-t}, \quad \frac{d}{dt} e^{t\hat{H}} = e^{t\hat{H}} \tilde{H}, \quad \frac{d}{dt} e^{-t\tilde{\sigma}(\hat{H})} = -\tilde{\sigma} (\hat{H}) e^{-t\tilde{\sigma}(\hat{H})},$$

we have

$$\dot{D}^t = -\tilde{H}_{\text{vert}} P^{-t} e^{t\hat{H}} P^{-t} e^{-t\tilde{\sigma}(\hat{H})} + P^{-t} e^{t\hat{H}} \tilde{H} P^{-t} e^{-t\tilde{\sigma}(\hat{H})} -$$

$$P^{-t} e^{t\hat{H}} \tilde{H}_{\text{vert}} P^{-t} e^{-t\tilde{\sigma}(\hat{H})} - P^{-t} e^{t\hat{H}} P^{-t} \tilde{\sigma}(\hat{H}) e^{-t\tilde{\sigma}(\hat{H})}.$$

Taking the common factors out of brackets, we obtain

$$\dot{D}^t = P^{-t} e^{t\hat{H}} \left( -\text{Ad} e^{-t\hat{H}}(\text{Ad} P^t) \tilde{H}_{\text{vert}} + \tilde{H} - \tilde{H}_{\text{vert}} - (\text{Ad} P^{-t}) \tilde{\sigma}(\hat{H}) \right) P^{-t} e^{-t\tilde{\sigma}(\hat{H})}.$$
Calculate the expression in the brackets. Firstly, notice that
\[(\text{Ad} e^{-t\vec{H}})(\text{Ad} P_t^t)\vec{H}_{\text{vert}} = e^{t\vec{H}} P_t^{-1} \vec{H}_{\text{vert}} = e^{t\vec{H}} \vec{H}_{\text{vert}}.\]
Using the formula of full variation, we have
\[e^{t\vec{H}} \vec{H}_{\text{vert}} = \left(\exp \int_0^t (\text{Ad} P^\alpha) \vec{H}_{\text{hor}} d\alpha\right) \circ P_t \vec{H}_{\text{vert}},\]
where the first operator \(P_t\) preserves \(\vec{H}_{\text{vert}}\), and the second operator realizes some horizontal motion, and also preserves \(\vec{H}_{\text{vert}}\).
Secondly, \((\text{Ad} P^{-t})\tilde{S}_\sigma(\vec{H}) = P^t_s(\vec{H}_{\text{hor}} - \vec{H}_{\text{vert}}) = P^t_s \vec{H}_{\text{hor}} - \vec{H}_{\text{vert}}\). So, the expression in the brackets in the formula for \(\dot{D}_t\) equals
\[-\vec{H}_{\text{vert}} + \vec{H} - \vec{H}_{\text{vert}} - P^t_s \vec{H}_{\text{hor}} + \vec{H}_{\text{vert}} = \vec{H}_{\text{hor}} - P^t_s \vec{H}_{\text{hor}}.\]
This is equal to zero by Lemma 2. So, we have \(\dot{D}_t \equiv 0\). □

**Proof of Theorem 1.** Notice that \(\text{Exp} \circ s_\sigma(p, t)\) equals \(\pi \circ e^{t\vec{H}} \circ \tilde{S}_\sigma \tau(\text{id}, p)\) in case (a), and \(\pi \circ e^{t\vec{H}} \circ S_\sigma \circ P^t \tau(\text{id}, p)\) in case (b). Now Theorem follows from Lemma 3 and the equality \(\pi \circ P = \pi\). □

### 3 Action of symmetries on a semi-direct product

Consider a non-trivial example of action of symmetries of exponential map for a left-invariant optimal control problem on a semi-direct product \(G = V \ltimes K\), where \(\rho : K \rightarrow \text{GL}(V)\) is a representation of an exponential Lie group \(K\) on a vector space \(V\).

**Proposition 1.** A Lie group \(G = V \ltimes K\) is exponential iff for all elements \(\xi \in \mathfrak{k}\) of the Lie algebra of the Lie group \(K\) the operator \(d\rho(\xi)\) has no eigenvalues of the form \(2\pi ik\), where \(k \in \mathbb{Z} \setminus 0\).

**Proof.** From the Campbell-Hausdorf formula (derived by E. B. Dynkin [29]) we have
\[
\exp(v + \xi) = \left(\frac{\exp (d\rho(\xi)) - 1}{d\rho(\xi)}v, \exp(\xi)\right) \in V \ltimes K, \quad v \in V, \quad \xi \in \mathfrak{k}, \quad (4)
\]
this implies the statement. □

The considered method was successfully applied to sub-Riemannian problems on the Lie groups \(\text{SE}_2 = \mathbb{R}^2 \ltimes \text{SO}_2\) (Yu. L. Sachkov, I. Moiseev [19, 20, 21]), \(\text{SH}_2 = \mathbb{R}^2 \ltimes \mathbb{R}\) (Ya. A. Butt, Yu. L. Sachkov, A. I. Bhatti [22, 23]), \(\text{SO}_2 \times \mathbb{R}^2\) (the problem of a rolling sphere on the plane without twisting and slipping, Yu. L. Sachkov [24]). In these papers an action of symmetries of exponential map on endpoints of extremal trajectories were described. From Theorem 1 and formula (4) follows a general formula for action of a symmetry \(S_\sigma\) on the Lie group \(G = V \ltimes K\).
Proposition 2. Let \((v, \exp \xi) \in V \times K\), then
\[
S_\sigma(v, \exp \xi) = \begin{cases} 
(\sigma^* v, \exp (\sigma^* \xi)), & \text{in case (a)}, \\
(\text{Ad} (\exp (\sigma^* \xi))\sigma^* v, \exp (\sigma^* \xi)), & \text{in case (b)}.
\end{cases}
\] (5)

Proof. By definition
\[
S_\sigma(v, \exp \xi) = \exp (\sigma^* (A_\xi^{-1} v, \xi)) = (A_{\sigma^* \xi} \sigma^* A_\xi^{-1} v, \exp (\sigma^* \xi)),
\]
where \(A_\xi = \frac{\exp \left(\frac{d\rho}{d\xi} \right) - 1}{\text{ad}_\xi} \bigg|_{V} \).

The operator \(\sigma^*\) is an (anti-)automorphism, so the last expression equals
\[
(\sigma^* A_{\pm \xi} A_\xi^{-1} v, \exp (\sigma^* \xi)).
\]

Using Lemma 4 we obtain the statement. \(\square\)

Example 1. Formula (5) allows us to clarify a geometric sense to the Maxwell strata that were found in paper [19] for the sub-Riemannian problem on the Lie group \(\text{SE}_2\) (group of proper isometries of Euclidean plane).

There are four symmetries \(\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_6\) (we use notation from [19]) that change the sign of the vertical part of the Hamiltonian vector field. These symmetries correspond to two-dimensional Maxwell strata: two Möbius strips and two disks, respectively. Other symmetries give Maxwell strata of lower dimensions that belong to the Maxwell strata listed above.

Two disks corresponding to the symmetries \(\varepsilon_5\) and \(\varepsilon_6\) are the set of central symmetries and the set of translations, respectively.

Two Möbius strips corresponding to the symmetries \(\varepsilon_1\) and \(\varepsilon_2\) consist of transformations such that their fixed points are on the axes \(y\) and \(x\), respectively. Also there are translations along axis \(x\) and \(y\) (respectively) in these Möbius strips. Let us consider translation along axis \(x\) (respectively \(y\)) as rotation around infinite center that is on the axis \(y\) (respectively \(x\)). Let us show, that the set of rotations around centers that lies on a fixed line (including infinite center) is a Möbius strip. Indeed, consider a cylinder \([-\infty, +\infty] \times S^1\) of pairs consisting of a center of rotation and an angle of rotation. To construct the set mentioned above from this cylinder one needs to perform the following steps (see Figure 1):

1. identify the bases of the cylinder (by identifying opposite points on the base circles);

2. identify all points of the circle \(S^1\), corresponding to rotations by zero angle, because all points of this circle generate the same transformation of the plane;
Figure 1: The Möbius strip consisting of rotations around centers that are on a fixed line. Base circles are identified by the directions of arrows.

3. remove the point $T_\infty$ (on the identified bases) that corresponds to the infinite translation.

We get a Klein bottle after step 1. To make step 2 cut the bottle along a circle, we get a Möbius strip. Identify all its boundary points, i.e., glue this Möbius strip and a disk along their boundaries. We get a projective plane. It remains to cut one point (step 3), we get a Möbius strip.

It is easy to see from this geometric interpretation for Maxwell strata, that points of the Möbius strip of rotations around centers on the axis $y$ cannot be cut points. Indeed, the corresponding optimal trajectory in the plane $(x, y)$ is a circle with center on the axis $y$.

**Appendix**

**Definition 5.** Let $A$ be a linear operator. Define an operator

$$\exp A - 1\over A = \sum_{k=0}^{+\infty} \frac{A^k}{(k+1)!}.$$ 

This series converges uniformly like the exponential one.

**Lemma 4.** The equality

$$(\exp A) \left( \frac{\exp(-A) - 1}{-A} \right) = \frac{\exp A - 1}{A}$$

is satisfied.

**Proof.** The product of two series that correspond to two multipliers of the left-hand side of the equality reads as

$$\sum_{k=0}^{+\infty} \frac{A^k}{k!} \sum_{l=0}^{+\infty} \frac{(-A)^l}{(l+1)!} = \sum_{n=0}^{+\infty} C_n A^n,$$
where the coefficient is

\[ C_n = \sum_{l=0}^{n} \frac{(-1)^l}{(n-l)!(l+1)!} \]

Then

\[ (n+1)!C_n = -\sum_{l=1}^{n+1} \binom{n+1}{l} (-1)^l = -\sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^l + 1 = 1. \]

Hence, \( C_n = \frac{1}{(n+1)!} \), and so \( \sum_{n=0}^{+\infty} C_n A^n = \frac{\exp A - 1}{A} \). \( \square \)

**Conclusion**

In this paper we consider left-invariant optimal control problems on exponential Lie groups. We get general conditions for existence of extension of a symmetry \( \sigma \) of the vertical part of the Hamiltonian system to a symmetry \((s_\sigma, S_\sigma)\) of the exponential map. This allows us to find the set of fixed points \( G^{S_\sigma} \) of such symmetry on the Lie group and its equation \( F_\sigma(g) = 0 \), where \( F : G \to \mathbb{R}^k \), for some \( k \). The Maxwell set \( \mathcal{M}_\sigma \) is a subset of \( G^{S_\sigma} \), then the Maxwell time \( t_{\sigma \max}^\sigma : g^* \to \mathbb{R}_+ \cup \{+\infty\} \), corresponding to the symmetry \( \sigma \), is defined by the implicit function

\[ F_\sigma(\text{Exp}(p, t_{\sigma \max}^\sigma(p))) = 0. \]

We need to investigate these implicit functions for different symmetries to find the first Maxwell time.

Besides that, we find a formula for action of symmetries of the exponential map for semi-direct product of an exponential and a commutative Lie groups. As a consequence, we get a geometric interpretation of Maxwell strata in sub-Riemannian problem on the Lie group of proper isometries of Euclidean plane.

**References**

[1] A.A. Agrachev, Yu.L. Sachkov. Control theory from the geometric viewpoint. Springer. 2004.

[2] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko. The Mathematical Theory of Optimal Processes. Pergamon Press, Oxford. 1964.

[3] S.G. Krantz, H.R. Parks. The Implicit Function Theorem: History, Theory and Applications, Birkauser, 2001.

[4] Yu.L. Sachkov. Maxwell strata in Euler’s elastic problem // Journal of Dynamical and Control Systems. 2008. 14, 2. pp. 169–234.
[5] A. A. Ardentov. *Multiple solutions in Euler elasticae problem* // Automation and remote control. 2018. (to appear).

[6] Yu. L. Sachkov. *Discrete symmetries in the generalized Dido problem* // Sbornik: Mathematics. 2006. 197, 2, pp. 235–257.

[7] Yu. L. Sachkov. *The Maxwell set in the generalized Dido problem* // Sbornik: Mathematics. 2006. 197, 4, pp. 595–621.

[8] Yu. L. Sachkov. *Complete description of the Maxwell strata in the generalized Dido problem* // Sbornik: Mathematics. 2006. 197, 6, pp. 901–950.

[9] A. A. Ardentov, Yu. L. Sachkov. *Extremal trajectories in a nilpotent sub-Riemannian problem on the Engel group* // Sbornik: Mathematics. 2011. 202, 11, pp. 1593–1615.

[10] A. A. Ardentov, Yu. L. Sachkov. *Conjugate points in nilpotent sub-Riemannian problem on the Engel group* // Journal of Mathematical Sciences. 195, 3. 2013. pp. 369–390.

[11] A. A. Ardentov, Yu. L. Sachkov. *Cut time in sub-Riemannian problem on Engel group* // ESAIM:COCV. 21, 4. 2015. pp. 958–988.

[12] A. A. Ardentov, Yu. L. Sachkov. *Maxwell Strata and Cut Locus in Sub-Riemannian Problem on Engel group* // Regular and Chaotic Dynamics. 22, 8. 2017. pp. 909–936.

[13] O. Myasnichenko, *Nilpotent (3, 6) sub-Riemannian problem* // Journal of Dynamical and Control Systems. 8, 4. 2002. pp. 573–597.

[14] U. Boscain, F. Rossi. *Invariant Carnot-Caratheodory metrics on S^3, SO(3), SL(2) and lens spaces* // SIAM Journal on Control and Optimization. 47. 2008. pp. 1851–1878.

[15] V. N. Berestovskii. *Locally shortest arcs of special sub-Riemannian metric on the Lie group SO_{0}(2, 1)* // St. Petersburg Math. J. 27, 1. 2016. pp. 1–14.

[16] V. N. Berestovskii, I. A. Zubareva, *Geodesics and shortest arcs of a special sub-Riemannian metric on the Lie group SO(3)* // Siberian Math. J. 56, 4. 2015. pp. 601–611.

[17] A. V. Podobryaev, Yu. L. Sachkov. *Cut locus of a left invariant Riemannian metric on SO(3) in the axisymmetric case* // Journal of Geometry and Physics. 110. 2016. pp. 436–453.

[18] A. V. Podobryaev, Yu. L. Sachkov. *Symmetric Riemannian problem on the group of proper isometries of hyperbolic plane* // Journal of Dynamical and Control Systems. 24, 3. 2018. pp. 391–423.
[19] I. Moiseev, Yu. L. Sachkov. *Maxwell strata in sub-Riemannian problem on the group of motions of a plane* // ESAIM: Control, Optimisation and Calculus of Variations. 16. 2010. pp. 380–399.

[20] Yu. L. Sachkov. *Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane* // ESAIM: Control, Optimisation and Calculus of Variations. 17. 2010. pp. 1018–1039.

[21] Yu. L. Sachkov. *Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane* // ESAIM: Control, Optimisation and Calculus of Variations. 17. 2011. pp. 293–321.

[22] Ya. A. Butt, Yu. L. Sachkov, A. I. Bhatti. *Maxwell strata and conjugate points in the sub-Riemannian problem on the Lie group SH(2)* // Journal of Dynamical and Control Systems. 4, 22. 2016. pp. 747–770.

[23] Ya. A. Butt, Yu. L. Sachkov, A. I. Bhatti. *Cut Locus and Optimal Synthesis in Sub-Riemannian Problem on the Lie Group SH(2)* // Journal of Dynamical and Control Systems. 1. 2017. pp. 155–196.

[24] Yu. L. Sachkov. *Maxwell strata and symmetries in the problem of optimal rolling of a sphere over a plane* // Sbornik: Mathematics. 2010. 201, 7. pp. 1029–1051.

[25] A. A. Agrachev, A. V. Sarychev. *Filtrations of a Lie algebra of vector fields and the nilpotent approximation of controllable systems* // Soviet Math. Dokl. 36, 1. 1988. pp. 104–108.

[26] È. B. Vinberg, A. L. Onishchik. *Foundations of the theory of Lie groups* // Lie groups and Lie algebras–1, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr. 20. VINITI, Moscow. 1988. pp. 5–101. [in Russian]

[27] V. Jurdjevic. *Optimal Control, Geometry and Mechanics*. // Mathematical Control Theory, J. Bailleul, J. C. Willems (ed.), Springer. 1999. 227–267.

[28] A. A. Agrachev, R. V. Gamkrelidze. *The exponential representation of flows and the chronological calculus* // Math. USSR-Sb. 35, 6. 1979. pp. 727–785.

[29] E. B. Dynkin. *On representation of the series log (e^x e^y) of non-commutative variables x and y throw commutators* // Math. Sbornik. 25(67), 1. 1949. pp. 155–162. [in Russian]