Comment on “Tricritical Behavior in Rupture Induced by Disorder”

In their letter, Andersen, Sornette, and Leung describe possible behaviors for rupture in disordered media. Their analysis is first based on the mean field-like democratic fiber bundle model (DFBM). In the DFBM, $N_0 (\rightarrow \infty)$ fibers are initially pulled with a force $F$, which is distributed uniformly. A fiber breaks if the stress it undergoes exceeds a threshold chosen from a probability distribution $p(x) = \frac{\lambda}{x} e^{-\lambda x}$. After each set of failures, the force $F$ is redistributed over the remaining fibers. Let $N(F)$ be the final number of intact fibers. What is the behavior of $N(F)$, as $F$ increases from $0$ to $\infty$? Ref. [1] claims the existence of a tricritical point, separating a “first-order” regime, characterized by a sudden global failure, from a “second-order” regime, characterized by a divergence in the breaking rate (response function) $N'(F)$.

Here, we present a graphical solution of the DFBM. Unlike an analytical solution, this enables us to consider the qualitatively different classes of disorder distribution, and to distinguish the corresponding generic behaviors of $N(F)$. We find that, for continuous distributions with finite mean, the system always undergoes a macroscopic failure, preceded by a diverging breaking rate. A “first-order” failure, with no preceding divergence, is an artifact of a (large enough) discontinuity in $p(x)$.

Suppose that a set of failures leaves the system with $N_i$ unbroken fibers. Each of these is now under a stress $F/N_i$. This leads to another set of failures, bringing the number of intact fibers to $N_{i+1} = N_0 \left\{ 1 - P \left( \frac{F}{x} \right) \right\}$. The function $N(F)$ defined above is nothing but $N_\infty$. A graphical scheme for this iteration is facilitated by setting $x_i = N_i / F$, $f = F / N_0$, and $\pi(x) = 1 - P(1/x)$, leading to

$$fx_{i+1} = \pi(x_i).$$

(1)

Since

$$\pi'(x) = \frac{1}{x^2} p \left( \frac{1}{x} \right),$$

(2)

$\pi(x)$ is a monotonic function of $x$, increasing from 0 to 1. Therefore, from iterating Eq. (1) graphically, $N(F)$ is given by the rightmost intersection of the curve $y = \pi(x)$ with the straight line $y = fx$. As the force is increased, the straight line becomes steeper, and the intersection consequently moves to the left.

We first consider continuous infinite-support distributions $p(x)$. We can distinguish three qualitatively different cases, depending on the behavior of $p(x)$ at large $x$ (see Fig. 1). (i) For $p(x) \sim x^{-r}$, with $r < 2$, intact fibers remain at any force $F$. (ii) For $p(x) = \alpha x^{-2}$, $N(F)$ goes continuously to zero at $F_c = N_0 \alpha$, with a diverging breaking rate ($N'(F_c) = \infty$). In both cases, there may or may not be jumps in $N(F)$ at smaller forces. In particular, if the slope of $\pi(x)$ is monotonically decreasing ($\pi$ convex), $N(F)$ has no discontinuity. This is the case for $e.g.$ $p(x) = \frac{\lambda}{x^r} e^{-\lambda x}$ ($1 < r \leq 2$), with any $\alpha$. Note also that both classes of distributions yield an infinite mean $\langle x \rangle = \infty$. (iii) For $p(x)$ such that $x^2 p(x) \rightarrow 0$, e.g. $p(x) = \frac{\lambda}{x^{1/2}} e^{-\lambda x}$ or $p(x) = \frac{\lambda}{x^{1/2}} e^{-\lambda x}$, with any $\lambda$, there is at least one jump in $N(F)$, leading to $N = 0$. As seen graphically, any such jump is preceded by a diverging breaking rate, i.e. the curve $N'(F)$ reaches its discontinuity vertically, generically according to $N'(F) \sim |F - F_c|^{-r}$.

![Fig. 1. Illustration of the graphical scheme for the three generic cases (i), (ii) and (iii) discussed in the text.](image)

From our graphical method, it clearly appears that a sudden jump in $N(F)$ with no divergence preceding it, is possible only if $\pi(x)$ has a non-differentiable point, which in turn, by Eq. (3), requires a discontinuity in $p(x)$. We illustrate this in the context of finite-support distributions, for which $p(x) = 0$ for $x < a$ and $x > b$. No fiber breaks up to $F_a = N_0 a$. Then, as can be shown by the graphical method, a sudden failure occurs at $F_a$ only if $\pi'(1/a) \geq a$, which is equivalent to requiring a minimal discontinuity $p(a) \geq 1/a$ at $a$. Also, note that for any finite $b$, there will be at least one jump in $N(F)$, leading to $N = 0$. Thus, for the more physical case of a continuous - albeit finite-support - distribution, the behavior of the solution is identical to that of case (iii) above.

In summary, the generic form of the DFBM’s solution depends on the disorder distribution only via its large $x$ behavior - and possible discontinuity points.

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[1] J.V. Andersen, D. Sornette and K.-T. Leung, Phys. Rev. Lett. 78, 2140 (1997)