Internal Frame Dragging and a Global Analogue of the Aharonov-Bohm Effect

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It is shown that the breakdown of a global symmetry group to a discrete subgroup can lead to analogues of the Aharonov-Bohm effect. At sufficiently low momentum, the cross-section for scattering of a particle with nontrivial $Z_2$ charge off a global vortex is almost equal to (but definitely different from) maximal Aharonov-Bohm scattering; the effect goes away at large momentum. The scattering of a spin-$1/2$ particle off a magnetic vortex provides an amusing experimentally realizable example.
The Aharonov-Bohm effect is generally thought to be inextricably connected to gauge symmetry, and to quantum mechanics. However, upon reflection there are some funny aspects to these connections. When the flux \( \Phi \) is expressed in terms of the fundamental unit \( \hbar/e \), so \( \Phi \equiv \tilde{\Phi} h/e \), and the scattered charge is measured in terms of the fundamental unit \( e \), so \( q \equiv \tilde{q} e \), then the Aharonov-Bohm phase factor \( \exp(\imath q \Phi / \hbar) = \exp(2\pi \imath \tilde{q} \tilde{\Phi}) \) is independent of \( e \) and \( \hbar \). This observation suggests that the Aharonov-Bohm effect might survive as \( e \) and \( \hbar \) approach zero, if the limit is defined in a suitable way.

We make these remarks not so much to outrage conventional wisdom concerning the Aharonov-Bohm effect, as to motivate the possibility of generalizing it. Can something like it occur for vortices of broken global symmetry, and in essentially classical contexts? We shall argue here that indeed it can, and that these generalizations have many potentially interesting incarnations.

1. Frame Dragging by Broken Symmetry

To be definite let us consider a model with global \( U(1) \) symmetry broken down to \( Z_2 \) by condensation of a scalar field \( \lambda \). Let \( \eta \) be a complex scalar field carrying half the \( U(1) \) charge of \( \lambda \). (For simplicity we shall assume that we are dealing with a relativistic theory; it will be clear that the main points do not depend on this.) Then generically one expects there to be a coupling of the type

\[
\Delta \mathcal{L} = g\lambda \eta^2 + \text{h.c.} \quad (1.1)
\]

In the homogeneous ground state where \( \langle \lambda \rangle = v \) this term generates a mass splitting between the real and imaginary components of \( \eta \equiv (\rho_1 + i\rho_2)/\sqrt{2} \):

\[
\Delta \mathcal{L} \to \frac{1}{2} \Gamma (\rho_1^2 - \rho_2^2) \quad (1.2)
\]

where \( \Gamma \equiv 2gv \).
Now in a vortex configuration for $\lambda$, where $\langle \lambda(r,\phi) \rangle \rightarrow ve^{i\phi}$ outside a core region, it will still be possible to regard the interaction (1.1) as generating a mass splitting between two real fields. However as $\phi$ varies the orientation of these fields in internal space is dragged along—in fact, rotated by $\phi/2$. In analyzing the dynamical effect of this frame dragging, it is convenient to work with fields which have a definite mass. Thus let us introduce the local mass eigenstates

$$\tilde{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi/2} & e^{-i\phi/2} \\ -ie^{i\phi/2} & ie^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \eta \\ \eta^* \end{pmatrix}. \tag{1.3}$$

(Strictly speaking this transformation will have to be regulated near the origin.) Now one can analyze the wave equation, to compute possible scattering of excitations in the $\rho_i$ fields from the vortex. Because of the transformation (1.3) this wave equation will have two unusual features:

1. Each of the fields $\rho_i$ obeys the boundary condition

$$\rho_i(\phi + 2\pi) = -\rho_i(\phi). \tag{1.4}$$

This means they must be defined with a cut, or alternatively that the allowed spectrum of partial waves includes only half-odd integers.

2. The gradient term $|\partial \eta|^2$ becomes modified, in its azimuthal component, to read

$$\frac{1}{2r^2} \left( (\partial_{\phi} + \frac{i}{2} \sigma_2) \tilde{\rho} \right)^2, \tag{1.5}$$

where $\sigma_2$ is the Pauli matrix.

What is the effect of these modifications? We claim that at small momenta ($k^2 \lesssim \Gamma$) the second modification reduces to an additional potential

$$V_{A^2} = \frac{1}{4r^2} \tag{1.6}$$

Indeed $\rho_1$ and $\rho_2$ have different effective masses, and one should expect that perturbations connecting them are suppressed at small momenta. Thus we can neglect
the terms linear in $\sigma_2$ at small momenta. (This is not quite obvious, because the $1/r^2$ interaction is potentially singular. However here the fact that the allowed partial waves are half-integral saves the day, because it means that there is always a centrifugal barrier shielding the origin.) Thus the only significant effect of the interaction with the vortex is to modify the boundary conditions, as in (1.4), and to add an additional potential (1.6). We shall compute the resulting cross-section, and justify our neglect of the off-diagonal terms, in the following section. If we neglected the additional potential (1.6) then we would have exactly the set-up which leads to maximal Aharonov-Bohm scattering. The additional term introduces a calculable modification, which is relatively small for high partial waves (or small angles).

On the other hand, clearly as $\Gamma \to 0$ the effect of the vortex must go away, apart from a possible contribution from ordinary scattering off the core (in the lowest partial wave). Thus at large momenta $k^2 \gg \Gamma$ the induced gauge field must essentially cancel the effect of the modified boundary conditions. Notice that the induced “gauge field” appearing in the gradient energy, far from being responsible for the Aharonov-Bohm-like scattering, plays a crucial role in cancelling it off!

If the $U(1)$ broken symmetry were a gauge symmetry, then the “gauge field” induced by the transformation (1.3) would be exactly cancelled by the true gauge field present in the gauge covariant derivative of $\eta$ in the vortex background. Then we would have the classic Aharonov-Bohm scattering induced by the change in boundary conditions, at all momenta. Related to this, in a broken gauge theory the scattering described here, which (since it arises from the coupling to the scalar Higgs field) might appear to be additional to the classic Aharonov-Bohm scattering, in a sense reduces to an alternative representation of it.

Thus far we have considered the case of $Z_2$ charges. For higher global charges, a more complex situation emerges. Consider for concreteness a $Z_3$ charge. The interaction corresponding to (1.1) is

$$\Delta \mathcal{L} = g \lambda \eta^3 + \text{h.c.}.$$  (1.7)
The equation of motion for $\eta$ receives a contribution of order $\eta^2$ from (1.7). Thus, for small amplitudes its effect is negligible. In particular, there is no scattering from the $\lambda$ vortex, even for small momenta, in the low amplitude limit. On the other hand for finite amplitude waves an analysis similar to the one given above applies. For small momenta (where “small” now depends on the amplitude of the wave) it will be appropriate to diagonalize (1.7), and one will find the appropriate Aharonov-Bohm-like cross-section. It is of special interest to introduce quantum considerations at this point. If $\eta$ represents a quantum field, then the additional interaction (1.7) represents a vertex where a single $\eta$ quantum breaks up into two in the presence of a vortex (or two fuse into one, or three appear from or annihilate into the vacuum). Thus, for single quanta, it appears quite different from an Aharonov-Bohm scattering effect. For coherent states of the $\eta$ field which can be treated as approximate solutions of the classical field equations, however, the preceding analysis applies. Evidently the breakup and fusion of individual $\eta$ quanta induced by the vertex, induces space-time deflection of their coherent superposition.

One might be concerned that, since the Nambu-Goldstone excitations associated with the broken symmetry field $\lambda$ are massless and therefore may be radiated with arbitrarily little energy, the effect discussed here could be washed out by Nambu-Goldstone boson emission. However since the Nambu-Goldstone field is derivatively coupled, it is clear that on general grounds its emission is an order $(k/F)^2$ correction to the elastic process for small momenta, where $F$ is the scale of symmetry breaking. Therefore it can be made arbitrarily small in regimes of interest, and clearly cannot wash out the generic effect discussed here.

An essentially geometrical cross-section associated with a global symmetry poses a potential paradox; it is noteworthy how this paradox is resolved. While gauge charges have a universal coupling strength, global charges do not, and so it is difficult at first hearing to understand how an essentially geometrical, parameter-independent form of the cross-section could emerge for them. Put another way, global charges have the character of forbidding couplings, not mandating them—
so how could they have a positive effect? What we have seen is that there is a geometrical cross-section determined by the global charge, but the *domain of validity* of the cross-section, *i.e.* the range in momenta for which it is valid, is a non-universal parameter that depends on the strength of the allowed coupling that fixes the charge assignment. As the strength of this coupling goes to zero (removing, in principle, our ability to define the charge) the form of the cross-section remains unchanged where it is valid, but its range of validity shrinks to zero.

2. Calculation

Now we shall treat the prototype problem discussed above more quantitatively. For simplicity we will treat the non-relativistic case where the momenta $k$ of the incident particles is much less than both of the perturbed masses $\mu^2_{(1,2)} = (m^2 \pm \Gamma)^{1/2}$. We will also consider the quantum mechanical scattering problem, although similar considerations would apply to the classical scattering of waves.

The substitution of eq.(1.3) into the equation for the $(\eta, \eta^*)$ modes results in a non-relativistic Schrödinger equation of the form

$$i\partial_t \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\mu_1} (\nabla^2 - 1/4r^2) & -\frac{\partial \phi}{2\mu_1 r^2} \\ \frac{\partial \phi}{2\mu_2 r^2} & -\frac{1}{2\mu_2} (\nabla^2 - 1/4r^2) \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad (2.1)$$

together with the boundary condition eq. (1.4). The off-diagonal entries (terms linear in the induced effective gauge field) connect states of different effective mass. Therefore, at low incident momenta it is reasonable to expect that their effect will be small. Our strategy will be to first solve the scattering problem ignoring the off-diagonal terms, and then take them into account perturbatively. Of course, if we send in a pure $\rho_2$ state then there can be no real $\rho_1$ production for incident momenta below the threshold enforced by energy conservation. However, even below this threshold, the off-diagonal terms can affect the elastic scattering of the $\rho_2$ modes at second order in perturbation theory. We will argue below that this effect is indeed small for incident energies much less than the mass splitting.
The solution of the diagonal scattering problem proceeds by performing a mode expansion (in two spatial dimensions—or in three at normal incidence)

\[ \rho_2(t, r, \phi) = \sum_{n \in \mathbb{Z}} e^{-i(\omega + \mu_2)t} e^{i(n+1/2)\phi} P_n^{(2)}(r), \]  

(similar for \( \rho_1 \)). Note that the partial wave expansion is shifted by one-half due to the boundary conditions. Defining \( x = k_2r \) where \( \omega = (k_2)^2/2\mu_2 \) we find that the radial eigenfunctions \( P_n(x) \) satisfy a Bessel’s equation of order \( \nu_n^2 = (n+1/2)^2 + 1/4 \). The shift of 1/4 from the usual order, \( \nu_n^2 = (n + 1/2)^2 \), expected with a mode expansion of the form eq.(2.2), is due to the on-diagonal induced potential eq.(1.6).

Thus the radial eigenfunctions that we will use for the solution of the scattering problem will be selected from the Bessel and Neumann functions \( J_{\pm \nu_n}(k_2r) \) and \( N_{\pm \nu_n}(k_2r) \). To select the appropriate set of solutions we must demand square integrability of the solution near the position of the vortex (taken to be the origin) and self-adjointness of the Hamiltonian. This results in the selection of only positive order Bessel functions in all angular momentum modes except for \( n = -1, 0 \). As discussed (in the context of gauge strings) in the appendix of [2], in both of these modes we have a one parameter family of allowed boundary conditions corresponding to a mixture of \( J_{\nu_n} \) and \( J_{-\nu_n} \). This apparent ambiguity results from the unphysical limit of zero core radius \( R \to 0 \). The correct choice of boundary condition is discovered by first performing a calculation at finite \( R \), and then taking the limit \( R \to 0 \) [2]. In our case the correct choice is that we should use only positive order Bessel functions in all modes.

Now we are ready to construct the scattering solution and calculate the elastic differential cross-section for incident \( \rho_2 \)-modes. This is most easily done if we reexpress the selected Bessel functions in terms of outgoing \( (H_{\nu_n}^{(1)}) \) and incoming \( (H_{\nu_n}^{(2)}) \) Hankel functions. If we take the incident wave to be a plane wave \( \exp(-ik_2x) \), then we must construct out of the Hankel functions a solution of the form

\[ \rho_2^{\text{sol}} = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{i(n+1/2)\phi} e^{-i\pi\alpha_n/2} \left( H_{\nu_n}^{(2)}(k_2r) + H_{\nu_n}^{(1)}(k_2r) \right), \]  

(2.3)
(note $J_n = (H_n^{(1)} + H_n^{(2)})/2$), where $\nu_n$ is given by the positive square root. The $\alpha_n$ are determined by demanding that eq. (2.3) match onto the incoming plane wave plus an outgoing scattered wave at infinity; we require

$$
\rho_2^{\text{sol}} \sim e^{i\phi/2} \left( e^{-ik_2x} + f(\phi) \frac{e^{ik_2r}}{r^{1/2}} \right),
$$

where $f(\phi)$ is the scattering amplitude. The phase $e^{i\phi/2}$ in front is necessary because of the double-valuedness of our solution, but it is harmless—if we construct narrow wave packets that travel in along the positive $x$-axis, then this phase is trivial. Using the usual expansion of the plane wave in terms of integer order Bessel functions

$$
\exp(-ikr \cos \phi) = \sum_{n \in \mathbb{Z}} e^{-i\pi|n|/2} e^{in\phi} J_{|n|}(kr),
$$

and the asymptotic behavior $H_{n}^{(1,2)}(x) \sim (2/\pi x)^{1/2} \exp[\mp i(x - \nu\pi/2 - \pi/4)]$ of the Hankel functions, the constraint of matching onto the plane wave determines $\alpha_n = \nu_n$ for all $n$.

We can now calculate the phase shifts $\delta_n(k_2)$ defined by the asymptotic relation

$$
\rho_2^{\text{sol}} \sim \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{i(n+1/2)\phi} e^{-i\pi|n|/2} \left( H_{|n|}^{(2)}(k_2r) + e^{i\delta_n(k_2)} H_{|n|}^{(1)}(k_2r) \right).$$

A simple calculation involving the asymptotic behavior of the Hankel functions leads to the result $\delta_n = \pi(|n| - \nu_n)$. From these phase shifts we can calculate the scattering amplitude $f(\phi)$. The result is

$$
f(\phi) = \frac{e^{-i\phi/2}}{(2\pi ik_2)^{1/2}} \left( \frac{1}{\cos(\phi/2)} + 2 \sum_{n=0}^{\infty} (-1)^n (e^{i\Delta_n} - 1) \cos \left( \left( n + \frac{1}{2} \right) \phi \right) \right),
$$

where

$$
\Delta_n = \pi \left( n + 1 - \sqrt{\left( n + \frac{1}{2} \right)^2 + \frac{1}{4}} \right).
$$

The first term in eq. (2.7) is the usual maximal Aharonov-Bohm amplitude. The
corrections are due to the diagonal $1/4r^2$ potential, and are most significant in low partial waves. Retaining only the first term in the sum, we obtain the differential scattering cross-section

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k_2} \left( \frac{1}{\sin^2(\theta/2)} - 8 \sin^2(\pi(1 - \sqrt{2})/4) \cos \theta + \cdots \right) , \quad (2.9)$$

where we have transformed to the correct scattering angle $\theta = \pi - \phi$. Equations (2.7)-(2.9) are the final results of our calculation. We see that the differential scattering cross-section of the mass eigenstates off a global vortex (in this $Z_2$ example) is almost, but not exactly, of maximal Aharonov-Bohm form.

A calculation in second-order perturbation theory shows that the effect of the neglected off-diagonal terms on the elastic scattering of $\rho_2$ is bounded by a constant times $(k_2)^4/(m^2\Gamma^2)$, uniformly in all partial waves [3], and so can be neglected at low incoming momenta.

3. Examples

1. Spin $1/2$ scattering by a magnetic vortex

Consider a material with a planar magnetization $\vec{M}(x)$—namely, a material described by the XY model. This model, of course, supports vortices, which indeed play an important role in its dynamics. A spin-1/2 particle, which might be an electron or a neutron for example, couples to the magnetization with an interaction

$$\Delta \mathcal{H} = g\psi^\dagger \vec{\sigma} \psi \cdot \vec{M} , \quad (3.1)$$

where $\psi$ is the spinor field representing the particle. The scattering of the spin-1/2 particle from the magnetic vortex is an instance of the general analysis above, but let us state it in fresh terms. In the presence of a vortex, the frame of the spin is dragged around. Thus if the magnetization is given by the vortex form $\vec{M}(r, \phi) \rightarrow M_0(-\delta_{11} \sin \phi + \delta_{12} \cos \phi)$, then to keep the effective mass term generated by the
interaction (3.1) diagonal, we shall need to transform to the frame-dragged variable \( \tilde{\psi} \equiv \exp(\i\phi\sigma_3/2)\psi \). Now as a spinor is rotated through \( 2\pi \), its sign changes. Thus for consistency, at low momentum where parallel transport of the spin is appropriate, the boundary condition on the spinor wave function requires it contain only half-odd-integer angular momenta. As there also occurs an induced diagonal potential (1.6), the spin will scatter off the vortex with the cross-section (2.9).

The accumulation of phase by a spin subject to a magnetic field whose direction varies as a function of an external parameter is the classic example of Berry’s phase [4]. The effect discussed here may be considered in this framework, with the angle around the vortex playing the role of external parameter. Indeed this point of view is instructive on several counts. The restriction to low momenta we found above may be considered as the adiabatic condition for applicability of Berry’s phase—at low momenta, the relevant trajectories (in the sense of a Feynman path integral) for looping around the vortex are traversed slowly. Also the special role of the vortex topology is clarified—in circling the core, we surround a point where an irremovable degeneracy between the masses of \( \rho_1 \) and \( \rho_2 \), which are the eigenvalues of the local static Hamiltonia, occurs. Finally various generalizations, such as to magnetizations tipped out of the plane by angle \( \beta \) and sweeping out a cone, suggest themselves. That particular generalization will change the calculation and cross-section as follows. Upon diagonalizing the interaction (3.1) we find that the effective mass term takes the form \( \tilde{\psi}^\dagger(\sin\beta \sigma_3 + \cos\beta \sigma_2)\tilde{\psi} \), with eigenspinors \( \tilde{\psi}_\pm = e^{-\i\beta/2}\sigma_1(1, \pm i)^T \). Now between these eigenspinors the effective gauge potential proportional to \( \sigma_3 \) in the modified gradient term

\[
|\partial_\phi \psi|^2 = |(\partial_\phi - i\sigma_3/2)\psi|^2,
\]

does have non-vanishing diagonal matrix elements, which must be included in the calculation. As a result, the quantities \( \nu_n \) are modified to become

\[
\nu_n^2 = (n + \frac{1}{2})(n + \frac{1}{2} \pm \sin \beta) + \frac{1}{4},
\]

where the \( \pm \) refers to the different eigenspinors. From these the cross-section is
readily computed, but the formula is not particularly transparent. It is noteworthy, however, that the leading correction to the canonical Aharonov-Bohm result contains terms in $\sin \phi$ as well as $\cos \phi$, giving explicit parity and time-reversal asymmetries.

2. *Polarized light*

The essential requirement for the analysis of the previous section to apply is that there should be two degrees of freedom with different dispersion relations that are rotated into one another by the variation of a material parameter, such as a magnetization, and that when the material parameter rotates through a closed cycle each degree of freedom returns to itself, with a change of phase. This general set-up can be realized in a variety of optical contexts, where the degrees of freedom are two polarizations of light of a given frequency. Realizations of frame-dragging for polarized light have already been used for interference experiments [5,6]; we are merely adapting it to a realization in scattering. Of course there is nothing special about the optical region of the electromagnetic spectrum in this regard, and an alternative macroscopic realization could be constructed for microwaves propagating through ferrites.

3. *Passport to exotica*

Quite a few remarkable phenomena involving among others Alice strings [7], Cheshire charge [8,9], and flux tube-flux tube scattering [10,11] have been studied in the context of spontaneously broken non-abelian gauge symmetries. Unfortunately, however, the list of spontaneously broken non-abelian gauge symmetries available for experimental manipulation is vanishingly small. The main point emphasized above, that frame-dragging phenomena usually associated with gauge theories also occur at low momenta for broken global symmetries, opens the strong possibility that effects closely analogous to these can be realized in suitable laboratory condensed matter systems. Particularly interesting in this regard are helium 3 [12] and liquid crystals, which are known to have complicated order parameter spaces and to support non-abelian vortices [13,14].
Assuming the existence of a global Aharonov-Bohm effect, it has been suggested in the context of black hole physics that new forms of “hair” become measurable [15]. The effects we have described are important for the interaction of matter with global and axion strings, and may affect their evolution in the early universe.

These matters are under active investigation [3].

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