RELATIVE KÄHLER-RICCI FLOWS AND THEIR QUANTIZATION

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Abstract. Let \( \pi : X \to S \) be a holomorphic fibration and let \( L \) be a relatively ample line bundle over \( X \). We define relative Kähler-Ricci flows on the space of all Hermitian metrics on \( L \) with relatively positive curvature and study their convergence properties. Mainly three different settings are investigated: the case when the fibers are Calabi-Yau manifolds and the case when \( L = \pm K_{X/S} \) is the relative (anti-) canonical line bundle. The main theme studied is whether “positivity in families” is preserved under the flows and its relation to the variation of the moduli of the complex structures of the fibers. The “quantization” of this setting is also studied, where the role of the Kähler-Ricci flow is played by Donaldson’s iteration on the space of all Hermitian metrics on the finite rank vector bundle \( \pi_*L \to S \). Applications to the construction of canonical metrics on the relative canonical bundles of canonically polarized families and Weil-Petersson geometry are given. Some of the main results are a parabolic analogue of a recent elliptic equation of Schumacher and the convergence towards the Kähler-Ricci flow of Donaldson’s iteration in a certain double scaling limit.

1. Introduction

1.1. Background. On an \( n \)-dimensional Kähler manifold \((X, \omega_0)\) Hamilton’s Ricci flow [26] on the space of Riemannian metrics on \( X \) preserves the Kähler condition of the initial metric and may be written as the Kähler-Ricci flow

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}_{\omega_t},
\]

When \( X \) is a Calabi-Yau manifold (which here will mean that the canonical line bundle \( K_X \) is holomorphically trivial) it was shown by Cao [13] that the corresponding flow in the space of Kähler metrics in \([\omega_0] \in H^2(X, \mathbb{R})\) has a large time limit. The limit is thus a fixed point of the flow which coincides with the unique Ricci flat Kähler metric in \([\omega_0]\), whose existence was first established by Yau [54] in his celebrated proof of the Calabi conjecture. The non-Calabi-Yau cases when \([\omega_0]\) is the first Chern class \( c_1(L) \) of \( L = rK_X \), where \( r = \pm 1 \) have also been studied extensively (where \(-r\omega \) is added to the right hand side in equation [13]). In general the fixed points of the corresponding Kähler-Ricci flows are hence Kähler-Einstein metrics of negative \((r = 1)\) and positive \((r = -1)\) scalar curvature. The convergence towards a fixed point - when it exists - in the latter positive case (i.e. \( X \) is a Fano manifold) was only established very recently by Perelman and Tian-Zhu [43].

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A distinctive feature of Kähler geometry is that a Kähler metric $\omega$ may be locally described in terms of a local function $\phi$, such that $\omega = dd^c \phi$, where $\phi$ is determined up to an additive constant. In the integral case, i.e., when $[\omega_0] = c_1(L)$ is the first Chern class of an ample line bundle $L \to X$ this just amounts to the global fact that the space of Kähler metrics $\omega$ in $c_1(L)$ may be identified with the space $\mathcal{H}_L$ of smooth metrics $h$ on the line bundle $L$ with positive curvature form $\omega$, modulo the action of $\mathbb{R}$ on $\mathcal{H}_L$ by scalings. Locally, $h = e^{-\phi}$ and we will refer to the additive object $\phi$ as a weight on $L$ (see section 2.1). In this notation the Kähler-Einstein equations may be expressed as Monge-Ampère equations on $\mathcal{H}_L$. For example, on a Calabi-Yau manifold $\omega_\phi := dd^c \phi$ is Ricci flat precisely when

$$ (dd^c \phi)^n / n! = \mu, $$

where $\mu$ is the canonical probability measure on $X$ such that $\mu = e^{n^2} \omega \wedge \bar{\omega}$, for $\Omega$ a suitable global holomorphic $n-$form trivializing $K_X$ (to simplify the notion we will in the following always assume that the volume of the given class $[\omega_0]$ is equal to one, so that $\omega_0^n / n!$ defines a probability measure on $X$ for any $\omega \in [\omega_0]$). By letting $\mu$ depend on $\phi$ in a suitable way general Kähler-Einstein are obtained.

As emphasized by Yau 53 one can expect to obtain approximations to Kähler-Einstein metrics by using holomorphic sections of high powers of a line bundle. In this direction Donaldson recently introduced certain iterations on the “quantization” (at level $k$) of the space $\mathcal{H}_L$ of Kähler metrics in $c_1(L)$ 19. Geometrically, this quantized space, denoted by $\mathcal{H}^{(k)}$, is the space of all Hermitian metrics on the finite dimensional vector space $H^0(X, kL)$ of global holomorphic sections of $kL$, where $kL$ denotes the $k$th tensor power of $L$, in our additive notation (for the definition see section 2.4). In other words $\mathcal{H}^{(k)}$ can be identified with the symmetric space $GL(N_k, \mathbb{C})/U(N_k)$ of $N_k \times N_k$ Hermitian matrices which in turn, using projective embeddings, corresponds to the space of level $k$ Bergman metrics on $L$. The fixed points of Donaldson’s iteration are called balanced metrics at level $k$ (with respect to $\mu$) and they first appeared in the previous work of Bourguignon-Li-Yau 12. Again, in the $\pm K_X-$setting one lets $\mu$ depend on $\phi$ in a suitable way leading to different settings (see below). In the limit when $L$ is replaced by a large tensor power it has very recently been shown that balanced metrics in the different settings indeed converge to Kähler-Einstein metrics 53 28 3. It was pointed out by Donaldson in 19 that it seems likely that these iterations can be viewed as discrete approximations of the Ricci flow. This will be made precise and confirmed in the present paper (Theorem 26 and Theorem 44).

1.2. Outline of the present setting and the main results. The aim of the present paper is to study relative versions of the Kähler-Ricci flow and Donaldson’s iteration (in the various settings) and investigate whether “positivity in families” is preserved under the flows. In other words, the given geometric setting is that of a holomorphic fibration $\pi : X \to S$ of relative dimension $n$ and a relatively ample line bundle $L \to X$. The fibration will mainly be assumed to be a proper submersion over a connected base, so that all fibers are diffeomorphic (for general quasi-projective morphisms see section 4.5). Denote by $\mathcal{H}_{L/S}$ the space of all metrics on $L$ which are fiber-wise of positive curvature. In other words, $\mathcal{H}_{L/S}$ is an infinite dimensional fiber bundle over $S$ whose fibers are of the form $\mathcal{H}_L$, as in the previous section. The relative Kähler-Ricci flows are now defined as suitable flows on $\mathcal{H}_{L/S}$ such that the induced flow of curvature forms restricts to the usual Kähler-Ricci flow fiber-wise:
We will say that “positivity is preserved under the flow” if, for any initial metric with positive curvature (in all directions on $X$), the evolved metric also has positive curvature for all times, i.e. the flow induces a flow of Kähler forms on the total space $X$ of the fibration (and not only along the fibers).

**The Calabi-Yau setting.** Let us first summarize the main results in the setting when the fibers are Calabi-Yau. In this case the flow $\phi_t$ in $H_{L/S}$ is defined fiber-wise by

$$
\frac{\partial \phi_t}{\partial t} = \log \left( \frac{(dd^c \phi_t)^n}{\mu} \right),
$$

with $\mu$ a measure as in equation (1.2). Of course, adding the pull-back of a time-dependent function on the base $S$ to the right hand side of the previous equation does not alter the induced flows of the fiber-wise restricted Kähler forms $dd_X^c \phi_t$, but it certainly effects the flow of $dd^c \phi_t$ on $X$ which will typically not preserve the initial Kähler property.

One of the main results of the present paper is a parabolic evolution equation along the flow (1.3) for the function

$$
c(\phi) := c(\phi) := \frac{1}{n} (dd^c \phi)^{n+1} / (d_X^c \phi)^n \wedge ids \wedge d\bar{s}
$$
on $X$ which is well-defined when $S$ is embedded in $\mathbb{C}$. The point is that $c(\phi) > 0$ precisely when $dd^c \phi > 0$ on $X$. The evolution equation for $c(\phi_t)$ reads (Theorem 1.4)

$$
\frac{\partial}{\partial t} - \Delta_{dd^c X} c(\phi_t) = |A_{K}^X|_{\omega_{t}^X}^2 - \omega_{WP},
$$

where $\omega^X_t$ denotes the flow of the fiber-wise restricted curvature forms, $A_{\omega_t}$ is a certain representative of the Kodaira-Spencer class of the fiber $X_s$ and $\omega_{WP}$ is the pull-back to $X$ of the (generalized) Weil-Petersson form on the base $S$; by a result of Tian and Todorov that we will reprove it can be represented by the global squared $L^2-$norm of $A_{\omega_{KE}}$ for $\omega_{KE}$ the unique Ricci flat metric in $c_1(L)$. Applying the maximum principle then gives (Corollary 1.5) that the initial condition $dd^c \phi_0 > 0$ implies that

$$
dd^c \phi_t > -t \omega_{WP}
$$
(and similarly when the initial curvature is semi-positive). By its very definition $\omega_{WP}$ vanishes at $s$ precisely when the infinitesimal deformation of the complex structure on the fibers $X_s$ (i.e. the Kodaira-Spencer class) vanishes at $s$. Hence, if the fibration $\pi : X \rightarrow S$ is holomorphically trivial, then, by inequality (1.5) positivity is indeed preserved along the flow. This latter situation appears naturally in Kähler geometry. Indeed, if the base $S$ is an annulus in $\mathbb{C}$ and $\phi_s$ is rotationally invariant, then $\phi_s$ corresponds to a curve in $H_L$ and $c(\phi_s)$ is then the geodesic curvature of the curve $\phi_s$ when $H_L$ is equipped with its symmetric space Riemannian metric (see [14] and references therein). In the non-normalized $K_X -$setting (see section [4]) the equation (1.4) can be seen as a parabolic generalization of a very recent elliptic equation of Schumacher [41].

Similarly, the “quantized” version of the previous setting is studied, i.e. the relative version of Donaldson’s iteration. It gives an iteration on the space of all Hermitian metrics $H$ on the finite rank vector bundle $\pi_s : L \rightarrow S$ for any positive integer $k$ (recall that the fiber of $\pi_s : L$ over $s$ is, by definition, the space $H^0(X_s, L_s)$ of all global holomorphic sections on the fiber $X_s$ with values in $L(X_s)$). More precisely,
we will study the equivalent fiber-wise iteration $\phi_m^{(k)}$ in $\mathcal{H}_{L/S}$ obtained by applying the (scaled) Fubini-Study map to Donaldson’s iteration. It will be called the relative Bergman iteration at level $k$. When the discrete time $m$ tends to infinity it is shown (Theorem 20) that the iteration converges to a fiber-wise balanced weight:

$$\phi_m^{(k)} \to \phi^{(k)}$$

in the $C^\infty$-topology on $X_s$, uniformly with respect to $s$. It is also observed that an analogue of the inequality (1.5) holds, i.e.

$$(1.6) \quad dd^c \phi_m^{(k)} \geq -\frac{k}{m} \omega_{WP}.$$  

This turns out to be a simple consequence of a recent theorem of Berndtsson [5] about the curvature of vector bundles of the form $\pi^* (L + K_{X/S})$. We also confirm Donaldson’s expectation about the semi-classical limit when the level $k$ tends to infinity. More precisely, it is shown that, in the double scaling limit where $m/k \to t$ the (relative) Bergman iteration at level $k$ approaches the (relative) Kähler-Ricci flow

$$(1.7) \quad \phi_m^{(k)} \to \phi_t$$

uniformly on $X$. In particular, combining this convergence with (1.6) gives an alternative proof of the semi-positivity in the inequality (1.5). Moreover, by taking $m = m_k$ such that $m/k \to \infty$ this gives a dynamical construction of solutions to the inhomogeneous Monge-Ampère equation (1.2) in the setting where $\mu$ is any fixed volume form (Corollary 27).

The (anti-) canonical setting. The previous results are also shown to have analogues in the setting when the ample line bundle $L$ is either the relative canonical line bundle $K_{X/S}$ over $X$ or its dual, which we write as $L = \pm K_{X/S}$ in our additive notation. The starting point is the fact that any metric $h = e^{-\phi}$ on $\pm K_X$ induces, by the very definition of $K_X$, a volume form on $X$ which may be written suggestively as $e^{\pm \phi}$. The previous constructions, i.e. the relative Kähler-Ricci flows and the Donaldson iteration, can then be repeated word for word for these $\phi$-dependent measures $\mu = \mu(\phi)$. For example, the relative Kähler-Ricci flows are are defined by

$$(1.8) \quad \frac{\partial \phi_t}{\partial t} = \log \left( \frac{(dd^c \phi_t)^n/n!}{e^{\pm \phi_t}} \right),$$

and we obtain (Theorem 34) a corresponding parabolic equation for $c(\phi_t)$ :

$$\left( \frac{\partial}{\partial t} - (\Delta_{\omega_t} - \pm 1) \right) c(\phi_t) = |A_{\omega_t}|_{\omega_t}^2$$

and as a consequence the flows always preserve positivity (Corollary 36) in these settings. In fact, in the case of infinitesimally non-trivial fibration the flows will even improve the positivity, i.e. any initial weight which is merely semi-positively curved instantly becomes positively curved under the flows. In the $+K_X$-setting the unique fixed point of the flow (1.8) is the (fiber-wise) normalized Kähler-Einstein weight uniquely determined by

$$e^{-\phi_{KE}} = (\omega_{KE})^n/n!,$$

where $\omega_{KE}$ is the unique Kähler-Einstein metric on $X$ (Corollary 30). The corresponding elliptic equation for $c(\phi_{KE})$ was first obtained by Schumacher [11] who used it to deduce the following interesting result: $\phi_{KE}$ is always semi-positively
curved on the total space of $\mathcal{X}$ and strictly positively curved for an infinitesimally non-trivial fibration. As a consequence he obtained several applications to the geometry of moduli spaces. For example, applied to the case when $\mathcal{X} \to S$ is the universal curve over the Teichmüller space of Riemann surfaces of genus $g \geq 2$ it gives, when combined with Berndtsson’s Theorem 21 a new proof of the hyperbolicity result of Liu-Sun-Yau [32] saying that the curvature of the Weil-Petersson metric on the Teichmüller space is dual Nakano positive.

In the $-K_X$-setting the relative Kähler-Ricci flow will diverge for generic initial data. But using the convergence on the level of Kähler forms, established by Perelman and Tian-Zhu, will show that it in case the Fano manifold $X$ admits a unique positively curved Kähler-Einstein metric $\omega_{KE}$, the flow does converge to a weight for $\omega_{KE}$ in the normalized $\pm K_X$-setting. This latter setting is simple obtained by normalizing the volume forms $e^{\pm \phi}$ used above.

We will also use the relative Bergman iteration to obtained a “quantized” version of Schumacher’s result: the canonical “semi-balanced” metric at level $k$ on $K_{X/S}$, which by definition is fiber-wise normalized and balanced, is smooth with semi-positive curvature on $\mathcal{X}$ (Corollary 13) and strictly positively curved in the case of an infinitesimally non-trivial fibration. As a consequence the semi-balanced metric gives an alternative to the canonical metric on $kK_{X/S}$ introduced by Narasimhan-Simha (see [34, 27, 51, 8] for positivity properties of this latter metric). In section 4.5 some of the results concerning the setting when $K_X$ is ample are generalized to projective fibrations of varieties of general type (i.e. $K_X$ is merely big) and the corresponding canonical semi-balanced metric is shown to have a positive curvature current (Theorem 48). Relations to deformation invariance of plurigenera [13] are also briefly discussed.

1.3. Further relations to previous results. A variant of Donaldson iteration (but with a single parameter $k$) in the $K_X$-setting was introduced by Tsuji [50]. He proved convergence in the $L^1$-topology towards the normalized Kähler-Einstein weight $\phi_{KE}$ in the large $k$-limit (see [45] for a proof of uniform convergence) and deduced the semi-positivity result for $\phi_{KE}$ of Schumacher referred to above. These works of Tsuji and Schumacher provided an important motivation for the present one. Steve Zelditch has also informed the author of a joint work in progress with Jian Song, where they show that the linearization of Tsuji’s iteration at the fixed point coincides with the linearization of the Kähler-Ricci flow. It should also be pointed out that another discretization of the Kähler-Ricci flow on a Fano manifold was studied by Rubinstein [39] and Keller [28].

The $C^0$-convergence of the Bergman iteration at a fixed level $k$ in the Calabi-Yau setting (or more generally in the setting of a fixed measure $\mu$) was pointed out by Donaldson in [19] and the proof was sketched. Sano provided an explicit proof in the constant scalar curvature setting [40] (see section 4.6).

It is also interesting to compare with the very recent work of Fine [22] concerning the constant scalar curvature setting. He shows that a continuous version of Donaldson’s iteration in this latter setting, called balancing flows, converges to the Calabi flow, when the latter flow exists. Julien Keller and Xiaodong Cao have informed the author of a joint work in progress where an analogue of Fine’s balancing flows in the Calabi-Yau setting (or more generally in the setting of a fixed volume form $\mu$) is shown to converge to a flow on metrics, which however is different than the Kähler-Ricci flow.
There are also, at last tangential, relations to the work of Gross-Wilson [24], where fibrations with Calabi-Yau fibers are considered. In particular, they construct certain semi-flat Kähler metrics $\omega$ on the fibration $\mathcal{X}$, i.e. $\omega$ is fiber-wise Ricci flat. Such metrics first appeared in the string theory literature [21]. In this terminology the inequality (1.5) shows that the relative Kähler-Ricci flow deforms any given Kähler metric to a semi flat one, when there is no variation of the moduli of the complex structure of the fibers. More generally, this latter statement holds in a double scaling limit when the variation of the complex structure is very small in the sense that $\omega_{FS}(s_t)t \to 0$ as $t \to \infty$.

A Kähler-Ricci flow on compact fibrations $\mathcal{X}$ with Calabi-Yau fibers was also considered very recently by Song-Tian. But they consider the usual (i.e. non-relative) Kähler-Ricci flow (with $r = 1$) when the canonical line bundle is only semi-ample and relatively trivial (i.e. the base $S$ is the canonical model of $\mathcal{X}$). They prove that the flow collapses the fibers so that the limit is the pull-back of metric on the base $S$ solving a “twisted” Kähler-Einstein equation where the twist is described by the (generalized) Weil-Petersson form $\omega_{FS}$.

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1.4. Organization of the paper. In section 2, a general setting is introduced and the associated relative Kähler-Ricci flow and its quantization are defined. General convergence criteria for the flows are given. In the following two sections the general setting is applied to get convergence results in particular settings of geometric relevance: the Calabi-Yau setting (section 3) and the (anti-) canonical setting 4 respectively. The new feature of these convergence results for the Kähler-Ricci flows is that the convergence takes place on on the level of weights, i.e. for the potentials of the evolving Kähler metrics. Furthermore, the main question whether “positivity in families” is preserved under the flows is studied in these two sections and relations to Weil-Petersson geometry are also discussed. It is also shown that the quantized flows converge to Kähler-Ricci flows in the large tensor power limit. Applications to canonical metrics on relative canonical bundles are also given.

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2. The general setting

In this section we will consider a general setup that will subsequently be applied to particular settings in sections 3–4.

We assume given a holomorphic submersion $\pi: X \to S$ of relative dimension $n$ over a connected base and a relatively ample line bundle $L \to X$. In the absolute case when $S$ is a point we will often use the notation $L \to X$ for the corresponding ample line bundle. In this latter case we will write $\mathcal{H}_L$ for the space of all smooth Hermitian metrics on $L$ with positive curvature form. In the relative case we will denote by $\mathcal{H}_{L/S}$ the space of all metrics on $L$ which are fiber-wise of positive curvature. We will denote by $c_1(L)$ the first Chern class of $L$, normalized so that it lies in $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. To simplify the formulas to be discussed we will also assume that the relative volume of $L$ is equal to one, i.e.

$$V := \int_X c^1(L)^n/n! = 1$$

for some (and hence any) fiber $X$. The general formulas may then be obtained by trivial scalings by $V$ at appropriate places. When considering tensor powers of $L$, written as $kL$ in additive notation, we will always assume that $kL$ is very ample (which is true for $k$ sufficiently large).

2.1. The weight notation for $\mathcal{H}_L$. It will be convenient to use the “weight” representation of a metric $h$ on $L$ locally, any metric $h$ on $L$ may be represented as $h = e^{-\phi}$, where $h$ is the point-wise norm of a local trivializing section $s$ of $L$. We will call the additive object $\phi$ a “weight” on $L$. One basic feature of this formalism is that even though the functions representing $\phi$ are merely locally defined the normalized curvature form of the metric $h$ may be expressed as

$$\omega_\phi := dd^c \phi := \frac{i}{2\pi} \partial \bar{\partial} \phi$$

which is hence globally well-defined (but it does not imply that $\omega$ is exact!). The normalizations are made so that $[\omega_\phi] = c_1(L) \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. In the absolute setting we will denote by $\mathcal{H}_L$ the space of all weights such that $\omega_\phi > 0$. In other words, the map $\phi \mapsto \omega_\phi$ establishes an isomorphism between $\mathcal{H}_L/\mathbb{R}$ and the space of Kähler metrics in $c_1(L)$. In the relative setting we will denote by $\mathcal{H}_{L/S}$ the space of all smooth weights on $L$ such that the restriction to each fiber is of positive curvature.

After fixing a reference weight $\phi_0$ in $\mathcal{H}_L$ the map $\phi \mapsto u := \phi - \phi_0$ identifies the affine space of all smooth weights on $L$ with the vector space $C^\infty(X)$. Moreover, the subspace $\mathcal{H}_L$ of all positively curved smooth weights gets identified with the open convex subspace $\mathcal{H}_w := \{u: \ dd^c u + \omega_0 > 0\}$ of $C^\infty(X)$, where $\omega_0$ denotes the Kähler form $dd^c \phi_0$. The $L^1$–closure of $\mathcal{H}_w$ is usually called the space of all $\omega_0$–plurisubharmonic functions in the literature 25. In fact, all the results in the present paper whose formulation does not use that the given class $[\omega_0]$ is integral are valid in the more general setting when $\mathcal{H}_L$ is replaced by $\mathcal{H}_w$ (with essentially the same proofs). However, since the quantized setting (section 2.3) only makes sense for integral classes we will stick to the weight notation in the following.

2.2. The measure $\mu_\phi$ and associated functionals on $\mathcal{H}_L$. First consider the absolute case when $S$ is a point. In each particular setting studied in sections 3–4 we will assume given a function $\mu$ on $\mathcal{H}_L$, $\phi \mapsto \mu(\phi)$ (also denoted by $\mu_\phi$) taking
values in the space of volume forms on $X$, which is exact in the following sense. First observe that we may identify $\mu(\phi)$ with a one-form on the affine space $H_L$ by letting its action on a tangent vector $v \in C^\infty(X)$ at the point $\phi \in H_L$ be defined by

$$\langle \mu(\phi), v \rangle := \int_X v \mu(\phi).$$

The assumption on $\mu(\phi)$ is then simply that this one-form is closed and hence exact, i.e. there is a functional $I_\mu$ on $H_L$ such that $dI_\mu = \mu$:

$$dI_\mu(\phi_t) = \int_X \frac{\partial \phi_t}{\partial t} \mu_{\phi_t} \text{ for any path } \phi_t \text{ in } H_L.$$  

(2.1)

We will also assume that for any fixed $v \in C^\infty(X)$ the functional $\phi \mapsto \langle \mu(\phi), v \rangle$ is continuous with respect to the $L^\infty$-topology on $H_L$.

Two particular examples of such exact one-forms and their anti-derivatives that will be used repeatedly are as follows:

- The Monge-Ampère measure $\phi \mapsto (dd^c \phi)^n/n! := MA(\phi)$. Its anti-derivative will be denoted by $E(\phi)$, normalized so that $E(\phi_0) = 0$ for a fixed reference weight in $\phi_0$ in $H_L$. Integrating along line segments in $H_L$ gives an explicit expression for $E$, but it will not be used here.

- $\phi \mapsto \mu_0$ for a volume form $\mu_0$ on $X$, fixed once and for all with $I_{\mu_0}(\phi) := \int_X (\phi - \phi_0)\mu_0$. Since we have already fixed a reference weight $\phi_0$ it will be convenient to take $\mu_0 := (dd^c \phi_0)^n/n!$.

Given $\mu = \mu(\phi)$ we define the associated functional

$$F_\mu := E - I_\mu,$$

By construction its critical points in $H_L$ are precisely the solutions to the Monge-Ampère equation

$$(dd^c \phi)^n/n! = \mu(\phi)$$

(2.2)

We will say that $\mu(\phi)$ is normalized if it is a probability measure for all $\phi$. Equivalently, this means that $I_\mu$ is equivariant under scalings, i.e. $I_\mu(\phi + c) = I_\mu(\phi) + c$ which in turn is equivalent to $F_\mu$ being invariant under scalings.

In the relative setting we assume that $\mu_s(\phi)$ is a smooth family of measures on the fibers $X_s$ as above, parametrized by $s \in S$.

Propersness and coercivity. We first recall the definition of the well-known $J$–functional, defined with respect to a fixed reference weight $\phi_0$ (see \cite{3} for a general setting and references). It is the natural higher dimensional generalization of the (squared) Dirichlet norm on a Riemann surface and it will play the role of an exhaustion function of $H_L/\mathbb{R}$ (but without specifying any topology!). In our notation $J$ is simply given by the scale invariant function

$$J = -F_{\mu_0}.$$ 

We will then say that a functional $G$ is proper if

$$J \to \infty \implies G \to \infty.$$

and coercive if for there exists $\delta > 0$ and $C_\delta$ such that

$$J \to \infty \implies G \geq \delta J - C_\delta.$$
Note that $\delta$ may be taken arbitrarily small at the expense of increasing $C_\delta$. In many geometric applications properness (and coercivity) of suitable functionals can be thought as analytic versions of algebro-geometric stability (compare remark 29).

2.3. The relative Kähler-Ricci flow with respect to $\mu$. Given an initial weight $\phi_0 \in H_{L/S}$ the relative Kähler-Ricci flow in $\mathcal{H}_{L/S}$ is defined by the fiberwise parabolic Monge-Ampère equation

\[
\frac{\partial \phi_t}{\partial t} = \log \frac{(dd^c \phi_t)^n}{n!} \frac{\mu(\phi_t)}{\mu(\phi_t)}
\]

for $\phi_t$ smooth over $X \times [0, T]$, where $T \geq 0$. We will make the following assumptions on the flow which will all be satisfied in the particular settings studied in sections 3, 4.

Analytical assumptions on the flow:

- **Existence:** The flow exists and is smooth over $X \times [0, \infty[$
- **Uniqueness:** Any fixed point in $\mathcal{H}_L$ of the flow is unique mod $R$
- **Stability:** For any $l > 0$ and $M > 0$ there is a constant $B_{l,M}$ only depending on the upper bound on the $C^l$-norm of the initial weight $\phi_0$ (with respect to a fixed reference weight) and a lower bound on the absolute value of $dd^c \phi_0$ such that

\[
\|\phi_t - \phi_0\|_{C^l(X \times [0, M])} \leq B_{l,M} (\text{locally uniformly with respect to } s \text{ in the relative setting})
\]

It follows immediately that $\phi$ is fixed under the flow if it solves the Monge-Ampère equation 2.2. Note that since we have assume that $Vol(L) = 1$ a necessary condition to be stationary is hence that $\int_X \mu(\phi) = 1$. For any solution $\phi_t$ and fixed fiber $X = X_s$ the Kähler metrics $\omega_t$ on $X$ obtained as the restricted curvature forms of $\phi_t$ hence evolve according to

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}_{\omega_t} - \eta_{\mu},
\]

where $\text{Ric}_{\omega_t}$ in the Ricci curvature of the Kähler metric $\omega_t$ and $\eta_{\mu} = dd^c \log \mu(\phi)$.

Thanks to the following simple lemma the Kähler-Ricci flow is “gradient-like” for the functional $\mathcal{F}_\mu$. For the Fano case see [15].

**Lemma 1.** The functional $\mathcal{F}_\mu$ is increasing along the Kähler-Ricci flow on $\mathcal{H}_L$ (defined with respect to $\mu_\phi$). Moreover, it is strictly increasing at $\phi_t$ unless $\phi_t$ is stationary.

**Proof.** Differentiating along the flow gives $\frac{d\mathcal{F}(\phi_t)}{dt} = \int_X \log \frac{MA(\phi_t)}{\mu(\phi_t)}(MA(\phi_t) - \mu(\phi_t)) = \int_X \log \frac{MA(\phi_t)}{\mu(\phi_t)} \frac{MA(\phi_t)}{\mu(\phi_t)} - 1) \mu(\phi_t) \geq 0$

where the last inequality follows since both factors in the last integrand clearly have the same sign.

If moreover, $\mu(\phi)$ is normalized then both terms appearing in the definition of $\mathcal{F}_\mu$ are monotone:
Lemma 2. Assume that $\mu(\phi)$ is normalized. Then the functionals $-I_\mu$ and $E$ are both increasing along the Kähler-Ricci flow on $\mathcal{H}_L$ with respect to $\mu(\phi)$. Moreover, they are strictly increasing at $\phi_t$ unless $\phi_t$ is stationary.

Proof. Differentiating along the flow gives

$$\frac{dI(\phi_t)}{dt} = -\int_X \log \left( \frac{MA(\phi_t)}{\mu(\phi_t)} \right) \mu(\phi_t) \geq 0$$

using Jensen’s inequality applied to the concave function $f(t) = \log t$ on $\mathbb{R}_+$ in the last step (recall that $MA(\phi_t), \mu(\phi_t)$ are both probability measures). Similarly,

$$\frac{dE(\phi_t)}{dt} = \int_X \log \left( \frac{MA(\phi_t)}{\mu(\phi_t)} \right) MA(\phi_t) = -\int_X \log \left( \frac{\mu(\phi_t)}{MA(\phi_t)} \right) MA(\phi_t) \geq 0,$$

again using Jensen’s inequality, but with the roles of $MA(\phi_t), \mu(\phi_t)$ reversed. The statement about strict monotonicity also follows from Jensen’s inequality since $f(t) = \log t$ is strictly concave.

From the previous lemma we deduce the following compactness property of the flow.

Lemma 3. Assume that $\mu(\phi)$ is normalized and that the associated functional $-F_\mu$ is coercive. Then there is a constant $C$ such that $J(\phi_t) \leq C$ and $\int |\phi_t - \phi_0| \mu_0 \leq C$ along the Kähler-Ricci flow for $\phi_t$ (with respect to $\mu(\phi)$).

Proof. Combining the monotonicity of $F_\mu$ and the assumption that $F_\mu$ be coercive (and in particular proper) immediately gives the first inequality $J(\phi_t) \leq C$. Next, by the definition of coercivity there are $\delta \in [0,1]$ and $C_\delta > 0$ such that $I_\mu - E \geq \delta I_{\mu_0} - \delta E - C_\delta$ i.e.

$$\delta I_{\mu_0} \leq (-1 + \delta)E + I_\mu + C_\delta.$$

along the flow. Since by the previous lemma $-E$ and $I_\mu$ are both bounded from above along the flow it follows that there is a constant $A$ such that $I_{\mu_0} \leq A$ along the flow. Finally, by basic pluripotential theory the set $\{ \phi \in \mathcal{H}_L : J(\phi) \leq C, I_{\mu}(\phi) \leq C \}$ is relatively compact in the $L^1$-topology. This proves the last inequality in the statement of the lemma.

The next proposition shows that, under suitable assumptions, the Kähler-Ricci flow with respect to a normalized measure $\mu_\phi$ converges on the level of weights precisely when it converges on the level of Kähler metrics. In sections 3 and 4 the proposition will be applied to the usual geometric Kähler-Ricci flows, where the convergence is already known to hold on the level of Kähler metrics. To simplify the notation we will only state the result in the absolute case, the extension to the relative case being immediate.

Proposition 4. Assume that $\mu(\phi)$ is normalized and that the associated functional $-F_\mu$ is coercive. Let $\phi_t$ evolve according to the Kähler-Ricci flow defined with respect to $\mu_\phi$ and write $\omega_t = dd^c \phi_t$. Then the following is equivalent:

- The sequence of Kähler metrics $\omega_t$ is relatively compact in the $C^\infty$-topology on $X$, i.e. for any positive integer $l$ the sequence $\omega_t$ is uniformly bounded in the $C^l$-norm on $X$.
- The weights converge: $\phi_t \to \phi_\infty \in \mathcal{H}_L$ in the $C^\infty$-topology on $X$ as $t \to \infty$. 

The Kähler metrics $\omega_t \to \omega_\infty$ in the $C^\infty$–topology on $X$, where $\omega_\infty$ is a Kähler form.

Proof: Assume that the first point of the proposition holds. Then it is a basic fact that the sequence of normalized weights $\hat{\phi}_t := \phi_t - C_t$ where $C_t := I_{\mu_0}(\phi_t)$ is relatively compact in the $C^\infty$–topology on $X$ to $\hat{\phi}_\infty \in \mathcal{H}_L$ (as is seen by inverting the associated Laplacians). By the previous lemma $|C_t| \leq D$ for some positive constant $D$ and hence $\{\phi_t\}$ is also relatively compact in the $C^\infty$–topology on $X$.

In the rest of the argument we will use the $C^l$–topology on $\mathcal{H}_L$ for $l$ a large fixed integer. Let $\mathcal{K} := \overline{\{\phi_t\}}$ be the closure of $\{\phi_t\}$ which is relatively compact in $\mathcal{H}_L$ by the previous argument. Denote by $\psi_0$ an accumulation point in $\mathcal{K}$:

$$\lim_{j} \phi_{t_j} = \psi_0$$

By continuity of the “time $s$ flow map” (which follows immediately from the stability assumption on the flow) and the semi-group structure of the flow we deduce that

$$\lim_{j} \phi_{t_j+s} = \psi_s$$

for any fixed $s > 0$. In other words, $\mathcal{K}$ is in fact compact and invariant under the “time $s$ flow map”. Note also that by monotonicity

$$(2.6) \quad \lim_{t} \mathcal{E}(\phi_t) = \mathcal{E}(\psi_0) = \sup_{\mathcal{K}} \mathcal{E}$$

Assume now to get a contradiction that $\psi_s \neq \psi_0$. By the strict monotonicity in lemma 19 we have that $\mathcal{E}(\psi_s) > \mathcal{E}(\psi_0)$, contradicting (2.6) (since $\psi_s \in \mathcal{K}$ as explained above). Hence, $\psi_0$ is a fixed point of the flow and hence, by the uniqueness assumption on the flow, it is determined up to an additive constant. This means that for any two limit points $\psi_0$ and $\psi_0'$ of the flow there is a constant $C$ such that

$$\psi_0 - \psi_0' = C$$

But as explained above $\mathcal{E}(\psi_0) = \mathcal{E}(\psi_0')$ and hence, by the scaling equivariance of $\mathcal{E}$ it follows that $C = 0$. All in all this means that we have shown that the flow $\phi_t$ converges, in the $C^\infty$–topology on $X$ to a limit $\phi_\infty$ in $\mathcal{H}_L$, i.e. that the second point of the proposition holds. The rest of the implications are trivial.

Remark 5. The coercivity is used to make sure that the compactness property of the flow $\phi_t$ holds without normalizing $\phi_t$ (say, by subtracting $I_{\mu_0}(\phi_t)$). If one only assumes properness then the same proof shows that the statement still holds upon replacing $\phi_t$ by $\phi_t - I_{\mu_0}(\phi_t)$ (which, of course, does not effect the curvature forms). The same remark applies to Proposition 23 below.

2.4. Quantization: The Bergman iteration on $\mathcal{H}_L$. Proceeding fiber-wise it will be enough to consider the absolute case when $S$ is a point and we are given an ample line bundle $L \to X$. For any positive integer $k$ such that $kL$ is very ample the quantization at level $k$ of the space $\mathcal{H}_L$ is defined as the space $\mathcal{H}^{(k)}$ of all Hermitian metrics on the $N_k$–dimensional complex vector space $H^0(X, kL)$. Hence, $\mathcal{H}^{(k)}$ may be identified with the symmetric space $GL(N_k, \mathbb{C})/U(N_k)$. In the relative setting $\mathcal{H}^{(k)}$ is replaced by the space of all Hermitian metrics on the rank $N_k$–vector bundle $\pi_*(k\mathcal{L})$ over the base $S$ (compare section 3.2.2).
Fix a volume form $\mu_\phi$ on $X$ depending on $\phi$ as above. Then any given $\phi \in H_L$ induces a Hermitian metric $\text{Hilb}^{(k)}(\phi)$

$$\text{Hilb}^{(k)}(\phi)(f, f) := \int_X |f|^2 e^{-k\phi} \, d\mu_\phi,$$

giving a map

$$\text{Hilb}^{(k)} : H_L \to \mathcal{H}^{(k)},$$

There is also a natural injective map (independent of $\mu_\phi$) in the reverse direction, called the (scaled) Fubini-Study map $FS^{(k)} :$

$$FS^{(k)}(H) := \log\left(\frac{1}{N_k} \sum_{i=1}^{N_k} |f_i^H|^2\right)$$

where $f_i^H$ is any bases in $H^0(X, k\mathcal{L})$ which is orthonormal with respect to $H$.

Donaldson’s iteration (with respect to $\mu_\phi$) on the space $\mathcal{H}^{(k)}$ is then obtained by iterating the composed map

$$T^{(k)} := \text{Hilb}^{(k)} \circ FS^{(k)} : \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}$$

and its fixed points are called balanced metrics at level $k$ (with respect to $\mu$).

In order to facilitate the comparison with the Kähler-Ricci flow it will be convenient to consider the (essentially equivalent) iteration on the space $H_L$ obtained by iterating the map $FS^{(k)} \circ \text{Hilb}^{(k)}$. This latter iteration will be called the Bergman iteration at level $k$ (with respect to $\mu_\phi$) and we will denote the $m$th iterate by $\phi_m^{(k)}$ and call the parameter $m$ discrete time. Hence, the iteration immediately enters the finite dimensional submanifold $FS(\mathcal{H}^{(k)}) \subset H_L$ of Bergman metrics at level $k$ and stays there forever. By the very definition of the Bergman iteration it may be written as the difference equation

$$\phi_{m+1}^{(k)} - \phi_m^{(k)} = \frac{1}{k} \log \rho^{(k)}(\phi_m^{(k)}),$$

where $\rho^{(k)}(\phi)$ is the Bergman function at level $k$ associated to $(\mu_\phi, \phi)$, i.e.

$$\rho^{(k)}(\phi) = \frac{1}{N_k} \sum_{i=1}^{N_k} |f_i|^2 e^{-k\phi},$$

where $f_i$ is an orthonormal basis with respect to the Hermitian metric $\text{Hilb}^{(k)}(\mu_\phi, \phi)$.

Note that the Bergman measure $\rho^{(k)}(\phi)\mu_\phi$ is a probability measure on $X$ and independent of the choice of orthonormal bases. It plays the role of the Monge-Ampère measure in the quantized setting.

It will also be convenient to, following Donaldson [19], study functionals defined directly on the space $\mathcal{H}^{(k)}$. Fixing the reference metric $H_0^{(k)} := \text{Hilb}^{(k)}(\phi_0) \in \mathcal{H}^{(k)}$ we may identify $\mathcal{H}^{(k)}$ with the space of all rank $N_k$ Hermitian matrices. We let

$$\mathcal{F}_\mu^{(k)}(H) := -\frac{1}{N_k} \log \det(H) - I_\mu \circ FS^{(k)}(H)$$

whose critical points in $\mathcal{H}^{(k)}$ are precisely the balanced metrics (with respect to $\mu_\phi$); this is proved exactly as in the particular cases considered in [18, 3]. We will also consider the following functional on $H_L$

$$\mathcal{L}^{(k)}(\phi) := -\frac{1}{N_k} \log \det(\text{Hilb}^{(k)}(\mu_\phi, \phi),$$
Monotonicity. The following monotonicity properties were shown by Donaldson in the particular setting considered in [19] (where \( \mu_\phi \) is independent of \( \phi \)). See also [18] for the setting when \( \mu(\phi) = MA(\phi) \) (compare section 4.6). The main new observation here is that concavity of \( I_\mu \) implies monotonicity.

**Lemma 6.** Assume that \( \mu_\phi \) is normalized. Then the following monotonicity with respect to the discrete time \( m \) holds along Bergman iteration \( \phi_m^{(k)} \) on \( \mathcal{H}_L \) (defined with respect to \( \mu_\phi \)):

- The functional \( \mathcal{L}^{(k)} \) is increasing along the Bergman iteration and strictly increasing at \( \phi_m^{(k)} \) unless \( \phi_m^{(k)} \) is stationary. Equivalently, the functional \( -\log \det \) is strictly increasing along the Donaldson iteration in \( \mathcal{H}^{(k)} \) away from balanced metrics.
- If \( I_\mu \) is concave on the space \( \mathcal{H}_L \) with respect to the affine structure then it is decreasing along the iteration and strictly decreasing at \( \phi_m^{(k)} \) unless \( \phi_m^{(k)} \) is stationary. Equivalently, the functional \( I_\mu \circ FS^{(k)} \) is strictly decreasing along the Donaldson iteration in \( \mathcal{H}^{(k)} \) away from balanced metrics.

**Proof.** The proof of the first point is essentially the same as in Donaldson’s setting in [19], but for completeness we repeat it here. By definition

\[
\mathcal{L}^{(k)}(\phi_{m+1}) - \mathcal{L}^{(k)}(\phi_m) = -\frac{1}{N_k k} \log \det(\text{Hilb}^{(k)})(\phi_{m+1}) - \det(\text{Hilb}^{(k)})(\phi_m)
\]

By the concavity of \( \log \) and Jensen’s inequality we hence get

\[
\mathcal{L}^{(k)}(\phi_{m+1}) - \mathcal{L}^{(k)}(\phi_m) \geq -\frac{1}{k} \log \frac{1}{N_k} \sum_{i=1}^{N_k} \|f_i\|^2_{T(\text{Hilb}^{(k)})(\phi_m)},
\]

where \( f_i \) is an orthonormal basis with respect to the Hermitian metric \( \text{Hilb}^{(k)}(\phi_m) \) and where by definition \( T(\text{Hilb}^{(k)}(\phi_m)) = \text{Hilb}^{(k)}(FS(\text{Hilb}^{(k)}(\phi_m))) \). Writing out the norms explicitly shows that the rhs above may be written as

\[
-\frac{1}{k} \log \frac{1}{N_k} \left( \sum_{i=1}^{N_k} |f_i|^2 / \sum_{i=1}^{N_k} |f_i|^2 \right) \mu_{FS(\text{Hilb}^{(k)}(\phi_m))} = -\frac{1}{k} \log(1) = 0,
\]

using that \( \mu_\phi \) is normalized. This hence proves the first point.

To prove the second point we use that \( I_\mu \) is assumed concave and that, by definition, \( \mu_\phi = dI_\mu \) as a differential, to get

\[
I_\mu(\phi_{m+1}^{(k)}) - I_\mu(\phi_m^{(k)}) \leq \int (\phi_{m+1}^{(k)} - \phi_m^{(k)}) \mu_{\phi_m^{(k)}} = \frac{1}{k} \int \log \rho^{(k)}(\phi_m^{(k)}) \mu_{\phi_m^{(k)}} \leq \frac{1}{k} \log \int \rho^{(k)}(\phi_m^{(k)}) \mu_{\phi_m^{(k)}} = 0
\]

using the definition of the iteration and Jensen’s inequality in the last step (and the fact that \( \rho^{(k)}(\phi) \mu_\phi \) and \( \mu_\phi \) are both probability measures). This proves the monotonicity of \( I_\mu \). The statement about strict monotonicity follow immediately from the fact that \( \log t \) is strictly concave.

\( \square \)
**Proof.** For the benefit of the reader we repeat Donaldson’s simple proof: let $X$ and hence integrating over $X$ we get

$$\max_i \lambda_i$$

be an index such that

$$(2.8) \quad \lambda_i = \lambda_{i, \text{max}}.$$  

Next, it will be enough to prove the last point of the lemma (the second point then follows since we may by scaling invariance assume that $\det(H_\lambda) = 1$). We may assume that $\inf_i \lambda_i = \lambda_0$ and since, by assumption,

$$- \log \det(H) = \sum_i \lambda_i \geq -C$$

we get

$$- \inf \lambda_i \leq C + \sum_{i \neq 0} \lambda_i \leq C + (N - 1) \max_i \lambda_i.$$  

By the assumption $(I_{\mu_0} \circ FS)(H) \leq C$ and the first point of the lemma the rhs above is bounded from above and hence we conclude that so is $- \inf \lambda_i$. All in all

Properness and coercivity. Properness and coercivity of functionals on $\mathcal{H}^{(k)}$ are defined as in section 2.3 but with the functional $J$ replaced by its quantized version on the space $\mathcal{H}^{(k)}$:

$$J^{(k)}(H) := -\mathcal{F}^{(k)}_{\mu_0} := I_{\mu_0} \circ FS^{(k)} + \frac{1}{kN} \log \det(H)$$

The content of the following lemma is essentially contained in the proof of Proposition 3 in [19]. We will fix a metric $H_0 \in \mathcal{H}^{(k)}$. For any given $H_0$—orthonormal base $(f_i)$ we can then identify an Hermitian metric $H$ with a matrix and we will denote by $H_\lambda$ the diagonal matrix with entries $e^{-\lambda_i}$ on the diagonal.

**Lemma 7.** The following holds

- For $\lambda \in \mathbb{C}^N$ let $\phi_\lambda = FS^{(k)}(H_\lambda) := \frac{1}{k} \log(\frac{1}{N} \sum_i e^{k\lambda_i} |f_i|^2)$. Then there is a constant $C$ such that

$$\max_i \lambda_i \leq I_{\mu_0}(\phi_\lambda) + C.$$  

- The functional $J^{(k)}$ is an exhaustion function on $\mathcal{H}^{(k)}/\mathbb{R}^*$ with respect to its usual topology

- In particular, the set of all $H \in \mathcal{H}^{(k)}$ such that

$$- \log \det(H) \geq -C, \quad (I_{\mu_0} \circ FS)(H) \leq C$$

is relatively compact.

**Proof.** For the benefit of the reader we repeat Donaldson’s simple proof: let $i_{\text{max}}$ be an index such that $\max_i \lambda_i = \lambda_{i, \text{max}}$. Clearly,

$$\max_i \lambda_i + \frac{1}{k} (\log(\frac{1}{N} |f_{i, \text{max}}|^2) \leq \phi_\lambda \leq \max_i \lambda_i + \frac{1}{k} \log(\frac{1}{N} \sum_i |f_i|^2),$$

and hence integrating over $X$ and using the first inequality above gives

$$\max_i \lambda_i + \int_X (\log(|f_{i, \text{max}}|^2) - \phi_0) d\mu_0 \leq I_{\mu_0}(\phi_\lambda),$$

which proves the lemma since it is well-known that $I_{\mu_0}(\psi) > -\infty$ for any psh weight $\psi$ if $\mu_0$ is a smooth volume form (as follows from the local fact that any psh function is in $L^1$) and in particular $-C := I_{\mu}(\log(|f_{i, \text{max}}|^2)) > -\infty$. This proves the first point. As for the second and third one we first note that any Hermitian metric $H$ can be represented by a diagonal matrix (which we write in the form $H_\lambda$) after perhaps changing the base $(f_i)$ above. Moreover, by the compactness of $U(N)$ the constant $C$ in the previous point can be taken to be independent of the base $(f_i)$.

Next, it will be enough to prove the last point of the lemma (the second point then follows since we may by scaling invariance assume that $\det(H_\lambda) = 1$). We may assume that $\inf_i \lambda_i = \lambda_0$ and since, by assumption,

$$- \log \det(H) = \sum_i \lambda_i \geq -C$$

we get

$$- \inf \lambda_i \leq C + \sum_{i \neq 0} \lambda_i \leq C + (N - 1) \max_i \lambda_i.$$  

By the assumption $(I_{\mu_0} \circ FS)(H) \leq C$ and the first point of the lemma the rhs above is bounded from above and hence we conclude that so is $- \inf \lambda_i$. All in all
this means that \( \max_{i} |\lambda_i| \) is uniformly bounded from above by a constant, i.e. \( H \) stays in a relatively compact subset of \( H^{(k)} \).

**Remark 8.** The proof of the previous lemma shows that the conclusion of the lemma remains valid for any choice of a fixed reference weight \( \phi_0 \) and probability measure \( \mu_0 \) (which are used in the definition of \( J^{(k)} \)) such that \( \int_X \log(|f| - \phi_0)\mu_0 \) is finite for any section \( f \in H^0(X, kL) \).

**Criteria for convergence in the large time limit.**

**Proposition 9.** Assume that \( \mu_\phi \) is normalized, that \( I_\mu \) is decreasing along the Bergman iteration, that \( F^{(k)}_\mu \) is coercive and that there is at most one balanced metrics (modulo scaling). Then, for any given positive integer \( k \) the following holds: In the large time limit, i.e. when \( m \to \infty \) the weights \( \phi_m^{(k)} \to \phi^{(k)}_\infty \) in the \( C^\infty \)-topology on \( X \). Moreover, in the relative setting the convergence is uniform with respect to the base parameter \( s \).

**Proof.** a) uniform convergence:

We equip \( FS(H^{(k)}) \), i.e. the space of all Bergman weights at level \( k \), with the topology induced by the sup-norm. It is not hard to see that this is the same topology as the one induced from the finite dimensional symmetric space \( \mathcal{H}^{(k)} = GL(N_k, \mathbb{C})/U(N_k) \) with its usual Riemannian metric, or with respect to the operator norm on \( GL(N_k, \mathbb{C}) \). Hence, it will be enough to prove the convergence of Donaldson’s iteration in \( \mathcal{H}^{(k)} \).

Since \( \mu_\phi \) is assumed normalized Lemma 6 shows that \( -\log \det H \) is uniformly bounded from below along the Donaldson iteration in \( \mathcal{H}^{(k)} \). Moreover, by assumption \( I_\mu_\phi \circ FS^{(k)} \) is uniformly bounded from above along the Donaldson iteration. Hence, just as in the proof of Lemma 3 it follows from the coercivity assumption that \( I_\mu_\phi \circ FS^{(k)} \) is also uniformly bounded from above along the Donaldson iteration. But then it follows from Lemma 7 that the iteration \( H_m^{(k)} \) stays in a compact subset of \( \mathcal{H}^{(k)} \).

Let now \( \mathcal{K} := \{H^{(k)}_m\} \) be the closure of the orbit of \( T^{(k)} \) which is relatively compact in \( \mathcal{H}^{(k)} \) by the previous argument. Denote by \( G \) an accumulation point

\[
\lim_j H_{m_j}^{(k)} = G
\]

in \( \mathcal{H}^{(k)} \). By the continuity of \( H \mapsto T^{(k)}(H) \) on \( \mathcal{H}^{(k)} \) we deduce that

\[
\lim_j T^{(k)}(H_{m_j}^{(k)}) = T^{(k)}(G).
\]

In other words, \( \mathcal{K} \) is in fact compact and invariant under \( T^{(k)} \). Note also that by monotonicity

\[
\lim_j -\log \det(H_{m_j}^{(k)}) = -\log \det(G) = \sup_{\mathcal{K}} (-\log \det)
\]

Assume now to get a contradiction that \( T^{(k)}(G) \neq G \). By the strict monotonicity in lemma 19 we have that \( \log \det(T^{(k)}(G)) > \log \det(G) \) contradicting 2.6 (since \( T^{(k)}(G) \in \mathcal{K} \)). All in all this means that we have shown that the subsequence \( (H_{m_j}^{(k)}) \) of Donaldson iteration converges to a fixed point, i.e. a balanced metric. By the assumption on uniqueness up to scaling it follows, again using monotonicity (just like in the proof of Proposition 3) that all accumulation points coincide, i.e. the iteration converges.
b) Higher order convergence:

To simplify the notation we set \( k = 1 \) and write \( \phi^{(k)}_{m} = \phi_{m} \). First note that the \( L^{\infty} \)-estimate above is uniform over \( S \), as follows by combining the monotonicity of the functionals with the uniform boundedness of the initial weight \( \phi_{0} \). By the uniform convergence of \( \phi_{m} \), it will hence be enough to prove that

\[
\| \partial^{\alpha}_{X} (h_{0}/h_{m+1}) \|_{L^{\infty}(X)} \leq C_{\alpha} \| (h_{m}/h_{0}) \|_{L^{\infty}(X)}
\]

where \( h_{m} = e^{-\phi_{m}} \) and \( \partial^{\alpha}_{X} \) denotes a real linear differential operator on \( X \) of order \( \alpha \) (note that while \( h_{m} \) globally corresponds to a metric on \( L \) the quotient \( h_{0}/h_{m+1} \) defines a global function on \( X \)). Accepting this estimate for the moment the uniform convergence of \( (h_{m}) \) hence gives that \( \| \partial^{\alpha}_{X} (h_{0}/h_{m}) \|_{L^{\infty}(X)} \) is uniformly bounded in \( m \) and since \( h_{m}/h_{0} \rightarrow h_{\infty}/h_{0} \) it then follows that \( \| \partial^{\alpha}_{X} (\phi_{m} - \phi_{0}) \|_{L^{\infty}(X)} \) is also uniformly bounded in \( m \). Hence, standard compactness arguments show the \( C^{\infty} \)-convergence of \( (\phi_{m}) \).

Finally, the estimate \( 2.9 \) is a consequence of the following quasi-explicit integral formula for the Bergman function familiar from the theory of determinantal random point processes (see \cite{1} and references therein): \( \rho(\phi)(x) = \)

\[
\int_{y \in X^{N-1}} f(x, y)e^{-(\phi - \phi_{0})(x)}e^{-(\phi - \phi_{0})(y)}d\mu_{\phi}(y)^{\otimes N-1}/Z_{\phi}, \quad Z_{\phi} := \int_{X^{N}} f_{0}e^{-(\phi - \phi_{0})}d\mu_{\phi}^{\otimes N}
\]

where \( f(x_{1}, x_{2}, ..., x_{N}) = \det_{1 \leq i, j \leq N}(f_{i}(x_{i}))_{i,j}^{2}e^{-\phi_{0}(x_{1})}...e^{-\phi_{0}(x_{N})} \) and \( f_{i} \) is any given orthonormal base with respect to the Hermitian metric \( Hilb^{(1)}(\phi_{0}) \) on \( H^{0}(X, L) \) (note that \( Z_{\phi} \) appears as the normalizing constant). We have used the notation \( \phi(x, ...x_{m}) = \phi(x_{1}) + ...\phi(x_{m}) \). In particular,

\[
(h_{0}/h_{m+1})(x) = \int_{y \in X^{N-1}} f(x, y)e^{-(\phi_{m} - \phi)(y)}d\mu_{\phi_{m}}^{\otimes N-1}/Z_{\phi_{m}}
\]

and hence differentiating wrt \( x \) by applying \( \partial^{\alpha}_{X} \) gives

\[
| \partial^{\alpha}_{X} (h_{0}/h_{m+1})(x) | = \int \left( \partial^{\alpha}_{X} f(x, y) \right)e^{-(\phi - \phi_{0})(y)}d\mu_{\phi_{m}}^{\otimes N-1}/Z_{\phi} \leq \frac{A_{\alpha}}{Z_{\phi_{m}}} \left\| e^{-(\phi - \phi_{0})} \right\|_{L^{\infty}(X)},
\]

where \( A_{\alpha} \) is a constant independent of \( m \). Since, by the uniform convergence of \( \phi_{m} \), we have that \( Z_{\phi_{m}} \rightarrow C > 0 \) for some positive constant \( C \) this concludes the proof of the estimate 2.9.

The following basic lemma gives a natural criterion for the assumptions (a part from the monotonicity of \( f_{m} \)) in the previous theorem to be satisfied.

**Lemma 10.** Suppose that \( \mathcal{G} \) is a functional on \( \mathcal{H}^{(k)} \) which is geodesically strictly convex with respect to the symmetric Riemann structure and strictly convex modulo scaling. Then \( \mathcal{G} \) has at most one critical point (modulo scaling). Moreover, if it has some critical point then \( \mathcal{G} \) is coercive.

**Proof.** Uniqueness follows immediately from strict convexity and hence we turn to the proof of coercivity. By a simple compactness argument it will be clear that, after fixing a reference metric \( H_{0} \in \mathcal{H}^{(k)} \), which we take to be a critical point of \( \mathcal{G} \), it is enough to prove coercivity along any fixed geodesic passing through \( H_{0} \). To this end let \( H_{t} \) be a geodesic in \( \mathcal{H}^{(k)} \) starting at \( H_{0} \), i.e. the orbit of the action of a one-parameter subgroup of \( GL(N_{k}) \). In the notation of Lemma \( 7 \) this means...
that $H_t = H_{t\lambda}$ for $\lambda \in \mathbb{C}^N$ fixed. By scaling invariance we may assume that the determinant of $H_t$ vanishes along the geodesic. Integrating the upper bound in 2.8 over $X$ gives

$$J(H_t) = 0 + (I_{\mu_0} \circ FS)(H_t) \leq Ct + D$$

Now, let $f(t) = \mathcal{G}(H_t)$. Since by assumption $f$ is convex and $0$ is a critical point have that $df/dt \geq 0$ for all $t$. Hence, if we fix some number $\epsilon > 0$, then

$$f(t) \geq f(0) + \int_0^t (df/ds) ds.$$ 

But by the assumption on strict convexity the latter integrand is bounded from below by some $\delta > 0$. All in all this shows that

$$\mathcal{G}(H_t) \geq \delta t - A \geq \frac{\delta}{C} J(H_t) - A'$$

which finishes the proof. \qed

**Large $k$ asymptotics.** Next, we will recall the following proposition which is the link between the Bergman iteration and the Kähler-Ricci flow. It is essentially due to Bouche and Tian, apart from the uniformity with respect to $\phi$. In fact, a complete asymptotic expansion in powers of $k$ holds as was proved by Catlin and Zelditch and the uniformity can be obtained by tracing through the same arguments (as remarked in connection to Proposition 6 in [17]). For references see the recent survey [56].

**Proposition 11.** Assume that the volume form $\mu_\phi$ depends smoothly on $\phi$. Then the following uniform convergence for the corresponding Bergman function $\rho_{(k)}(\phi)$ holds: there is an integer $l$ such that

$$\sup_X |\rho_{(k)}(\phi) - (dd^c \phi)^n/n! \mu_\phi| \leq C/k$$

for all weights $\phi$ such that $dd^c \phi$ is uniformly bounded from above in $C^l-$norm with $dd^c \phi$ uniformly bounded from below by some fixed Kähler form.

3. The Calabi-Yau setting

First consider the absolute case where we assume given an ample line bundle $L \to X$. In this section we will the apply the general setting introduced in the previous setting to the case when the measure $\mu$ is independent of $\phi$. We will assume that it is normalized, i.e. a probability measure. We will mainly be interested in the case when $X$ is a Calabi-Yau manifold, which induces a canonical probability measure $\mu$ on $X$ defined by

$$\mu = c_n \Omega \wedge \bar{\Omega}$$

where $\Omega$ is any given holomorphic $n-$form trivializing the canonical line bundle $K_X$ and $c_n$ is a normalizing constant. In the relative Calabi-Yau setting, where each fiber is assumed to be a Calabi-Yau manifold, this hence yields a canonical smooth family of measures on the fibers.

For a fixed reference element $\phi_0 \in \mathcal{H}_L$ we set

$$I_\mu(\phi) := \int_X (\phi - \phi_0) \mu,$$

which is equivariant under the usual actions of the additive group $\mathbb{R}$: $I_\mu(\phi + c) = I_\mu(\phi) + c$. Moreover, be definition the associated functional $-\mathcal{F}_\mu$ is coercive.
3.1. The relative Kähler-Ricci flow. The convergence on the level of Kähler forms in the following theorem is due to Cao (a part from the uniqueness which was first shown by Calabi). We just observe that, since $\mu$ is normalized, the convergence of the flow also holds on the level of weights.

**Theorem 12.** The Kähler-Ricci flow on $H_L$ with respect to $\mu$ exists for all times $t \in [0, \infty]$ and the solution $\phi_t$ is smooth on $X \times [0, \infty]$. Moreover, $\phi_t \to \phi_\infty$ uniformly in the $C^\infty$- topology on $X$ when $t \to \infty$, where $\phi_\infty$ is the unique (modulo scaling) solution to the inhomogeneous Monge-Ampère equation $[16, 2]$. More precisely, all the analytical assumptions in section 2.3 are satisfied. In the Calabi-Yau case $\omega_\infty$ is Ricci flat.

**Proof.** As shown by Cao $[13]$ $\omega_t \to \omega_\infty$ in the $C^\infty$-topology. But then it follows from Proposition $[1]$ that, $\phi_t \to \phi_\infty$ uniformly in the $C^\infty$- topology on $X$. The smoothness in the relative case was not stated explicitly in $[13]$ but follows from basic maximum principle arguments. □

3.1.1. Preliminaries: Kodaira-Spencer classes and Weil-Petersson geometry. In this section we will assume that the base $S$ is one-dimensional and embedded as a domain in $\mathbb{C}$. Recall that the infinitesimal deformation of the complex structures on the smooth manifold $X_s$ as $s$ varies is captured by the the Kodaira-Spencer class $\rho(\frac{\partial}{\partial s}) \in H^{1,0}(T^{1,0}X_s)$ $[52]$. When the fibers are Calabi-Yau manifolds the "size" of the deformation is measured by the (generalized) Weil-Petersson form $[23]$ $\omega_{WP}$ on the base $S$. It was extensively studied by and Tian $[40]$ and Todorov $[19]$ when the base $S$ is a moduli space of Calabi-Yau manifolds and $X$ is the corresponding Kuranishi family. The form $\omega_{WP}$ is defined by

$$
\omega_{WP}(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) := \|A_{CY}\|^2_{\omega_{CY}},
$$

where $A_{CY}$ denotes the unique representative in the Kodaira-Spencer class $\rho(\frac{\partial}{\partial s}) \in H^{1,0}(T^{1,0}X_s)$ which is harmonic with respect to a given Ricci flat metric $\omega_{CY}$ on $X_s$ and the $L^2$-norm is computed with respect to this latter metric. Moreover, as shown in $[19]$ the following formula holds

$$
\|A_{CY}\|^2_{\omega_{CY}} = \frac{\partial^2 \psi_{\Omega}}{\partial s \partial \bar{s}}, \quad \psi_{\Omega}(s) := \log^{\|n^2\|} \int_{X_s} \Omega \wedge \bar{\Omega},
$$

where $\Omega$ now denotes any given, nowhere vanishing, global holomorphic $n$-form on $p^{-1}(U)$ and where $U$ denotes some neighborhood of a fixed point $s$ in $S$. More generally, for an arbitrary smooth base $S$ the $(1,1)$-form $\omega_{WP}$ on $S$ may be defined as the curvature of the line bundle $\pi_s(K_{X/s})$ on $S$. It is in the latter form that $\omega_{WP}$ will appear in the proof of Theorem $[14]$ below. In fact, the formula $[3, 1]$ may then be deduced from Theorem $[14]$ (see remark $[18]$).

Next we will explain how, for a fixed base parameter $s$, a weight $\phi$ on the line bundle $L \to X \to S$ induces the following two objects:

- a $(0,1)$-form $A_\phi$ with values in $T^{1,0}X_s$ representing the Kodaira-Spencer class $\rho(\frac{\partial}{\partial s})$ in $H^{1,0}(T^{1,0}X_s)$.
- A function $c(\phi)$ on $X$ measuring the positivity (or lack of positivity) of $dd^c\phi$ on $X$ in terms of the positivity of the restrictions of $dd^c\phi$ to the fibers $X_s$.

In fact $A_\phi$ will only depend on the family, parametrized by $s$, of two-forms $\omega_s$ obtained as the restrictions of the curvature form $\omega_\phi$ on $X$ to all fibers $X_s$, while $c(\phi)$ will depend on the whole form $\omega_\phi$. 

Hence, for any fixed index $i$

$$\delta_{V_\phi}(X) = \mathcal{D}_X(\partial_s \phi),$$

(3.3)

where $\delta_{V_\phi}$ denotes interior multiplication (i.e. contraction) with $V_\phi$. Now the $(0,1)$--form $A_\phi$ with values in $T^{1,0}X$ (for $s$ fixed) is simply defined by

$$A_\phi := -\mathcal{D}_X V_\phi$$

(3.4)

Denote by $\omega^X_\phi$ the curvature forms on $X$ evolving with respect to the time parameter $t$ according to the Kähler-Ricci flow (for $s$ fixed). The Laplacian on $X$ with respect to $\omega^X_\phi$ will be denoted by $\Delta_\omega^X$. Given $\phi(s, \cdot)$ we define the following function on $X$:

$$c(\phi) := \frac{1}{n}((dd^c \phi)^{n+1}/(d_X d_X^c \phi)^n \wedge ids \wedge d\bar{s})$$

(3.5)

Note that, since $\omega^X_\phi > 0$ on $X$ we have that $c(\phi) > 0$ at $(s,x) \in X$ iff $dd^c \phi > 0$ at $(s,x)$.

General submersions. Next we turn to the case of a general holomorphic submersion $\pi : X \rightarrow S$. Any given point in $X$ has a neighborhood $U$ such that the fibration $\pi : U \rightarrow S$ is holomorphically trivial and the restriction $L_U$ is isomorphic to $\pi^* L$ over $U$. Hence, the vector field $V_\phi$ defined above is locally defined, but in general not globally well-defined on $X$. However, the expression (3.4) turns out to still be globally well-defined. For completeness we will give a proof of this well-known fact [11, 23].

**Proposition 13.** The $(0,1)$--form $A_\phi$ with values in $T^{1,0}X$, locally defined by formula (3.4), is globally well-defined. It represents the Kodaira-Spencer class in $H^1(T^{1,0}X)$.

**Proof.** Step 1: the locally defined expression

$$W_\phi := \frac{\partial}{\partial s} - V_\phi$$

defines a global vector field on $X$ of type $(1,0)$.

Indeed $W_\phi$ may be characterized as the horizontal lift of $\frac{\partial}{\partial s}$ with respect to the $(1,1)$--form $dd^c \phi$ on $X$, which is non-degenerate along fibers. To see this first note that

$$i) \ d\pi(W_\phi) = \frac{\partial}{\partial s}, \ (ii) \ dd^c \phi(W_\phi, ker d\pi) = 0$$

(3.6)

The first point is trivial and the second one follows from a direct calculation: locally we may decompose

$$dd^c \phi = dz d^c \phi + \phi_s ds \wedge d\bar{s} + (\overline{\partial}_z \phi_s) \wedge ds + (\partial_z \phi_s) \wedge d\bar{s}.$$

Hence, for any fixed index $i$

$$dd^c \phi(W_\phi, \frac{\partial}{\partial z_i}) = -d_z d^c \phi(V_\phi, \frac{\partial}{\partial z_i}) + 0 + (\frac{\partial}{\partial z_i} \phi_s) + 0 = 0,$$
using the definition \[3.3\] of \(V_\phi\) in the last step. Finally, note that the properties \[3.6\] determine \(W_\phi\) uniquely: if \(W'\) is another local vector field satisfying \[3.6\] then clearly \(Z := W_\phi - W'\) satisfies
\[(i')\ d\pi(Z) = 0, \quad (ii')\ d\phi(Z, \ker d\pi) = 0\]
In particular, \(Z\) is tangential to the fibers and \(d\phi(Z, Z) = 0\). But since \(d\phi\) is assumed to be non-degenerate along the fibers it follows that \(Z = 0\).

Step 2: \(A_\phi(s) = (\partial W_\phi)_{X_s}\) and \(A_\phi(s)\) represents the Kodaira-Spencer class in \(H^{1,0}(T^{1,0}X_s)\).

The first formula above follows immediately from a local computation and the second one then follows directly from the definition of the Kodaira-Spencer class (where \(W_\phi\) may be taken as any smooth lift to \(T^{1,0}X\) of the vector field \(\frac{\partial}{\partial t}\) \[52\]). \(\square\)

As for the function \(c(\phi)\) defined by formula \[3.3\] it is still well-defined as we have fixed an embedding of \(S\) in \(\mathbb{C}\).

3.1.2. Conservation of positivity along the relative Kähler-Ricci flow. Next, we will prove the following theorem which is one of the main results of the present paper.

**Theorem 14.** Let \(\pi : X \to S\) be a proper holomorphic submersion with Calabi-Yau fibers and let \(L\) be a relatively ample line bundle over \(X\). Assume that the base \(S\) is a domain in \(\mathbb{C}\). The following equation holds along the corresponding relative Kähler-Ricci flow:

\[(\partial_t - \Delta_{\omega_X})c(\phi) = |A_\phi|^2_{\omega_X} - |A_{CY}|^2_{\omega_{CY}}.\]  \(\text{(3.7)}\)

**Proof.** Since it will be enough to prove the identity at a fixed point \(x\) in \(X\) in some local holomorphic coordinates and trivializations we may as well assume that \(\omega_\phi\) is the Euclidean metric at the point \(x\), i.e. that the complex Hessian matrix \((\partial^2 \phi/\partial z_i \partial_{\bar{z}_j})\) is the identity for \(z = 0\) (corresponding to the fixed point \(x\) in \(X\)). Moreover, we may assume that locally the holomorphic \(n\)-form \(\Omega\) may be expressed as \(\Omega = dz_1 \wedge ... \wedge dz_n\). Partial derivatives with respect to \(s\) will be indicated by a subscript \(s\) and partial derivatives with respect to \(z_i\) and \(\bar{z}_j\) by subscripts \(i\) and \(j\) respectively. If \(h = (h_{ij})\) is an Hermitian matrix we will write \((h^{ij})\) for the matrix \(h^{-1}\). The summation convention according to which repeated indices are to be summed over will be used. Next, we turn to the proof of the theorem which is based on a direct and completely elementary calculation.

**Step 1:** the following formula holds in the case of a holomorphically trivial fibration

\[\frac{\partial}{\partial t} c(\phi) = \phi_{\bar{s}i\bar{s}} + \phi_{s\bar{s}} \phi_{\bar{s}i\bar{s}} - \phi_{i\bar{j}s} \phi_{\bar{s}i\bar{j}s} - \phi_{s\bar{s}} \phi_{i\bar{j}s} \phi_{\bar{s}i\bar{j}k} - \phi_{\bar{s}i\bar{j}s} \phi_{k\bar{k}s} \phi_{\bar{s}i\bar{j}s} + 2\Re(\phi_{k\bar{k}s} \phi_{\bar{s}i\bar{s}}) + 2\Re(\phi_{k\bar{k}s} \phi_{\bar{s}i\bar{s}})\]

To see this first recall that
\[c(\phi) = \phi_{s\bar{s}} - \Re(\phi_{s\bar{s}} \phi_{i\bar{j}s} \phi_{i\bar{j}s})\]
and hence (using that \(\phi_{i\bar{j}s} = \delta_{ij}\) at \(z = 0\), so that \(\frac{\partial}{\partial t} \phi_{i\bar{j}s} = -\delta_{ij}\) at \(z = 0\))

\[\frac{\partial}{\partial t} c(\phi) = \frac{\partial}{\partial t} \phi_{s\bar{s}} - 2\Re(\phi_{\bar{s}i\bar{s}} \phi_{k\bar{k}s} \phi_{\bar{s}i\bar{s}}) + (\phi_{s\bar{s}} \phi_{i\bar{j}s}) \frac{\partial}{\partial t} \phi_{i\bar{j}s} = \text{(3.8)}\]

Using the definition of the relative Kähler-Ricci flow in the Calabi-Yau case and the simple fact that the linearization of \(\psi \mapsto \log \det(\psi_{k\bar{k}})\) at \(\psi\) is given by \(u \mapsto \Delta_{\omega_u} u\),
where $\Delta_{\omega_p} u = \psi^k l u_{kl}$ is the Laplacian with respect to the Kähler metric $\omega_p$ hence gives

$$
\frac{\partial}{\partial t} c(\phi) = (\log(\det \phi_{ij}))_{si} - 2\Re((\log(\det(\phi_{kl}))_{si})\phi_{si}) + (\phi_{si}\phi_{j})((\log(\det \phi_{kl}))_{ij} = 
$$

$$= (\phi_{ij}\phi_{ij})_{si} - 2\Re(\phi_{kl}\phi_{ij})_{si} + (\phi_{si}\phi_{j})(\phi_{kl}\phi_{ij})_{ij} = 
$$

$$\phi_{ijsi} - \phi_{ijsi} \phi_{jkl} - 2\Re[\phi_{kl} \phi_{isi} - \phi_{kls} \phi_{isi}] + (\phi_{si} \phi_{j}) (\phi_{ijkl} - \phi_{ikj} \phi_{jl})
$$

(again using $\phi_{ij} = \delta_{ij}$ at $z = 0$), finishing the proof of step 1.

**Step 2:** the following formula holds in the case of a trivial fibration:

$$
c(\phi)_{kk} = \phi_{kk}\overline{s}_{kk} + (\phi_{si}\overline{s}_{j})(\phi_{kk}\overline{s}_{j}) - 2(\phi_{si}\overline{s}_{j})\phi_{kim}\phi_{kmj} - \phi_{ksi}\overline{s}_{j} - \phi_{ksi}\overline{s}_{k} - 2\Re(\phi_{ksi}\phi_{j}\phi_{kmj})
$$

$$+ 2\Re(\phi_{ksi}\phi_{j})\phi_{kmj} - 2\Re(\phi_{ksi}\overline{s}_{j})
$$

To see this we first differentiate $c(\phi)$ with respect to $z_k$ to get

$$
c(\phi)_k = \phi_{ksi} - [(\phi_{si}\overline{s}_{j})_{k}\phi_{ij} + (\phi_{si}\overline{s}_{j})(\phi_{ij})_k] = 
$$

$$= \phi_{ksi} - (\phi_{ksi}\overline{s}_{j} + \phi_{ksi}\phi_{j})\phi_{ij} - (\phi_{ksi}\overline{s}_{j})(\phi_{ij})_k
$$

Next, note that if $h$ is a function with values in the space of Hermitian matrices and $\partial$ a derivation satisfying Leibniz rule, then

$$
\partial(h^{-1}) = -h^{-1}(\partial h)h^{-1}.
$$

In particular, if $h(0) = I$ then the following holds at 0:

$$(h^{-1})_{kk} = -\overline{h}_{kk} + (\overline{h}_k\overline{h}_k + \overline{h}_k\overline{h}_k)
$$

Applying this to $h = (\phi_{ij})$ (when expanding the term $A$ below) gives

$$
c(\phi)_{kk} = \phi_{kk}\overline{s}_{kk} - (\phi_{ksi}\overline{s}_{j})(\phi_{ijs})_{k} + (\phi_{ksi}\overline{s}_{j})(\phi_{js})_{ki} - \phi_{ksi}\overline{s}_{j} - \phi_{ksi}\phi_{j})\phi_{ki} - A = 
$$

$$\phi_{kk}\overline{s}_{kk} - (\phi_{ksi}\overline{s}_{j})_{k}(\phi_{ijs})_{j} + (\phi_{ksi}\overline{s}_{j})(\phi_{js})_{ki} = -(\phi_{ksi}\overline{s}_{j})(\phi_{ijs})_{j} + (\phi_{ksi}\overline{s}_{j})(\phi_{js})_{ki} - 2\Re(\phi_{ksi}\phi_{kmj})
$$

$$= -(\phi_{ksi}\overline{s}_{j})_{k}(\phi_{ijs})_{j} + (\phi_{ksi}\overline{s}_{j})(\phi_{js})_{ki} + 2\Re(\phi_{ksi}\phi_{kmj})
$$

Hence,

$$
c(\phi)_{kk} = \phi_{kk}\overline{s}_{kk} - (\phi_{ksi}\overline{s}_{j})_{k}(\phi_{ijs})_{j} + (\phi_{ksi}\overline{s}_{j})(\phi_{js})_{ki} - 2\Re(\phi_{ksi}\phi_{kmj})
$$

which finishes the proof of Step 2.

**Step 3:** end of proof of the theorem for a trivial fibration

Subtracting the formulas from the previous steps gives, due to cancellation of several terms, $\frac{\partial}{\partial t} c(\phi) - c(\phi)_{kk} =

$$
= \phi_{smk}\phi_{smk} + (\phi_{sm}\phi_{s})(\phi_{km})_{m} - 2\Re(\phi_{sm}\phi_{s})\phi_{km} = 
$$

$$= \sum_{m,k} \phi_{smk} - \sum_{l} \phi_{sl}\phi_{km}|^2
$$
Finally, note that
\[ \phi_{s\bar{m}k} - \sum_{l} \phi_{s\bar{l}}\phi_{\bar{m}l} = (\phi_{sm})_{\bar{k}} - (\phi_{\bar{m}l})_{k} \sum_{l} \phi_{s\bar{l}} = (\phi_{s\bar{l}}\phi_{\bar{m}l})_{k} = (V_{m})_{\bar{k}} \]

(using \( \phi_{ij} = \delta_{ij} \) at \( z = 0 \)), where \( V = (V_{1},...V_{n}) \) is the \((0,1)-vector field \text{3.3}\) expressed in local normal coordinates. This hence finishes the proof of the theorem in the case of a trivial fibration.

*Step 4:* equation \text{3.4}\ holds for a general holomorphic submersion:

Computing locally as before the only new contribution comes from the derivatives appearing the resulting local formula have been shown to be globally well-defined this hence finishes the proof of Step 4. \(\square\)

Now the maximum principle for parabolic equations \text{38}\ implies the following

**Corollary 15.** Let \( \mathcal{L} \to \mathcal{X} \to S \) be a line bundle over a fibration as in the previous theorem.

- If the fibration is holomorphically non-trivial, then the function \( c(t) := \inf_{\mathcal{X}} c(\phi) \) is, for a fixed value on \( s \), increasing along the relative Kähler-Ricci flow and hence the flow preserves (semi-) positivity of the curvature of \( \phi \).
- For a holomorphically trivial fibration \( \mathcal{X} = X \times S \), with \( \mathcal{L} \) the pull-back of an ample line bundle \( L \to X \), the flow improves the positivity of a generic initial weight in the following sense: if \( \phi_{0} \) is a semi-positively curved weight on \( \mathcal{L} \) over \( X \times S \) such that \( \partial \phi/\partial s \) does not vanish identically on \( X \times \{s\} \) for any \( s \), then \( \phi_{t} \) is strictly positively curved on \( X \times S \) for \( t > 0 \).
- In the general case the (semi-) positivity of the curvature of the weight on \( \phi - t\psi_{1} \) on the \( \mathbb{R}-line bundle \mathcal{L} - tK_{X/S} \) is preserved under the flow, i.e.

\[ dd^{c}\phi_{t} \geq -t\omega_{WP} \]

for all \( t \) (and similarly in the strict case)

*Proof.* The first and third point follow from the maximum principle exactly as in in the proof of Cor \text{36}\ below. The second point is proved as follows: if strict positivity does not hold then one concludes (see the proof of Cor \text{38}\ below) that \( -A_{\phi_{0}} = \bar{\partial}_{X}V_{\phi_{0}} \) vanishes identically on \( X \) for some \( s_{0} \), i.e. the corresponding vector field \( V_{\phi_{0}} \) defined by \text{3.5}\ is holomorphic on \( X \). But, it is a well-known fact that any such holomorphic vector field \( V_{1,0} \) vanishes identically when \( X \) is a Calabi-Yau manifold and hence \( \partial \phi/\partial s \) vanishes identically on \( X \times \{s_{0}\} \) giving a contradiction. The vanishing of \( V_{1,0} \) may be proved as follows: by a Bochner-Weitzenböck formula \( V_{1,0} \) is covariantly constant wrt any Ricci flat metric on \( X \). Moreover, the imaginary part \( V_{t} \) satisfies \( \omega_{\phi_{0}}(V_{t}, \cdot) = df \) for some real smooth function \( f \). But since \( \omega_{\phi_{0}}^{X} > 0 \) on \( X \times \{s_{0}\} \) the latter equation forces the vanishing of \( V_{t} \) at any point where \( f \)
achieves it maximum and hence \( V_t \equiv 0 \) on \( X \). Similarly, the real part \( V_R \) of \( V^{1,0} \) vanishes identically (by replacing \( df \) with \( d^c f \)).

Of course, in the case of a infinitesimally non-trivial fibration the inequality in the previous corollary is useless for the limit \( \phi_\infty \), but its interest lies in the fact that it gives a lower bound on the (possible) loss of positivity along the relative Kähler-Ricci flow, which is independent of the initial data.

**Remark 16.** Throughout the paper we assume, for simplicity, that the initial weight \( \phi_0 \) has relatively positive curvature, when restricted to the fibers of the \( \mathcal{X} \). But, as in the previous corollary, we do allow \( \phi_0 \) to have merely semi-positive curvature over the total space \( \mathcal{X} \). However, using recent developments for the Kähler-Ricci flow (see for example [44]) the relative Kähler-Ricci flows are actually well-defined for any weight \( \phi_0 \) which has merely relatively semi-positive curvature and \( \phi_t \) becomes relatively positively curved for any \( t > 0 \). Using this result the previous corollary can be seen to be valid for a general semi-positively curved initial weight \( \phi_0 \).

### 3.1.3. Evolution of the curvature of the top Deligne pairing.

For a general smooth base \( S \) (i.e. not necessarily embedded in \( \mathbb{C} \)) the weight \( \phi \) on \( L \) naturally induces a closed \((1,1)\)-form \( \Theta_\phi(s) \) on \( S \) expressed as

\[
\Theta_\phi := \pi_*((dd^c \phi)^{n+1}/(n+1)!) \]

Equivalently, for any local holomorphic curve \( C \subset S \) with tangent vector \( \frac{\partial}{\partial s} \in TS \)

\[
\Theta_\phi(\frac{\partial}{\partial s}, \frac{\partial}{\partial \pi}) := \int_{\mathcal{X}} c(\phi)\omega^*_\phi/n! \]

where \( s \in C \) and \( \pi \) is the induced map \( \pi : \mathcal{X} \to C \). Geometrically, the form \( \Theta_\phi \) on \( S \) may be described as the curvature of the Hermitian holomorphic line bundle \((\mathcal{L}, \phi)^{n+1} \over S \) defined as the top Deligne pairing of the Hermitian holomorphic line bundle \((\mathcal{L}, \phi) \to \mathcal{X} \to S \) (see [16]; the relevance of Deligne pairings for Kähler geometry has been emphasized by Phong-Sturm [35]). The form \( \Theta_\phi \) also appears as a multiple of the curvature of the Quillen metric on the determinant of the direct image of a certain virtual vector bundle over \( \mathcal{X} \) (see [23] and references therein).

Similarly, one can define a \((1,1)\)-form \( \omega_{WP} \) on \( S \) depending on \( \phi \) by letting

\[
\omega_{WP}(\frac{\partial}{\partial s}, \frac{\partial}{\partial \pi}) := \int_{\mathcal{X}} |A_\phi(s)|^2\omega^*_\phi/n! \]

It can be checked that this yields a well-defined \((1,1)\)-form on \( \mathcal{X} \). Anyhow this latter fact is also a consequence of the following corollary of the previous theorem.

**Corollary 17.** (same assumptions as in the previous theorem). Let \( \Theta_{\phi_t} \) be the curvature form on \( S \) of the top Deligne pairing of \((\mathcal{L}, \phi) \to \mathcal{X} \to S \), where \( \phi_t \) evolves according to the relative Kähler-Ricci flow in the Calabi-Yau case. Then

\[
\frac{\partial}{\partial t} \Theta_\phi(s) = -\pi_*((dd^c \phi)^{n+1}/(n+1)! + \omega_{WP} - \omega_{WP}) \]

where \( R_{\omega^*_\phi} \) denotes the fiber-wise scalar curvature of the metric \( \omega_\phi \).

**Proof.** We may without loss of generality assume that \( S \) is embedded in \( \mathbb{C} \). Then

\[
\frac{\partial}{\partial t} \int_{\mathcal{X}} c(\phi)\omega^*_\phi/n! = \int_{\mathcal{X}} \frac{\partial}{\partial t} c(\phi)\omega^*_\phi/n! + \int c(\phi)\frac{\omega^*_\phi}{(n-1)!} \wedge dd^c \frac{\partial}{\partial t} \phi. \]
Now by the definition of the Kähler-Ricci flow in the Calabi-Yau case
\[ \frac{\omega^2}{(n-1)!} \wedge \frac{\partial}{\partial t} \phi = \frac{\omega^2}{(n-1)!} \wedge (-Ric(\omega_X)) =: -R_{\omega_X} \omega^2/\phi \]
where we have used the definition of the (normalized) scalar curvature $R_{\omega_X}$ of the Kähler metric $\omega_X$ in the last step. Finally, integrating the formula in the previous theorem finishes the proof of the corollary. \(\square\)

Remark 18. Note that if the initial weight $\phi$ for the Kähler-Ricci flow is taken so that $\omega_\phi$ restricts to a Ricci flat metric on all fibers of $X$, then $\phi$ is stationary for the Kähler-Ricci flow and hence the previous corollary (and the proof of the previous theorem) shows that $\hat{\chi}_s|_{A(\phi(s))}|^2/\omega^n = \partial^2 \psi_{\Omega}/\partial \bar{s} \partial s$.

3.2. Quantization: The Bergman iteration on $\mathcal{H}_L$. In this section we will specialize and develop the general results in section 2.4 to the present setting where we have fixed a family of probability measure $\mu_s$ (independent of $\phi$) on the fibers $\chi_s$.

3.2.1. Convergence and positivity of the Bergman iteration at a fixed level $k$. The following monotonicity properties were shown by Donaldson [19] in the present setting.

**Lemma 19.** The functionals $-I_\mu$ and $\mathcal{L}^{(k)}$ are both increasing along the Bergman iteration on $\mathcal{H}_L$ with respect to $\mu$. Moreover, they are strictly increasing at $\phi_m^{(k)}$ unless $\phi_m^{(k)}$ is stationary.

**Proof.** Since $I_\mu$ is affine and in particular concave on the affine space of all smooth weights the lemma follows immediately from Lemma [6]. \(\square\)

We can now prove the convergence of the Bergman iteration at a fixed level $k$ in the present setting.

**Theorem 20.** Let $L \to X$ be an ample line bundle and $\mu$ a fixed volume form on $X$ giving unit volume to $X$. Assume given a smooth initial weight $\phi_0$. For any given positive integer $k$ the following holds: In the large time limit, i.e. when $m \to \infty$, the weights $\phi_m^{(k)} \to \phi_{\infty}^{(k)}$ in the $C^\infty$-topology on $X$. Moreover, in the relative setting the convergence is locally uniform with respect to the base parameter $s$.

**Proof.** By the previous lemma $-I_\mu$ is increasing and by definition $-\mathcal{F}_\mu^{(k)}$ is coercive. Moreover, as shown in [3] balanced weights are unique modulo scaling and hence all the convergence criteria in Proposition [9] are satisfied. \(\square\)
3.2.2. Conservation of positivity. Recall that, given a relatively ample line bundle \( \mathcal{L} \) over a fibration \( \pi : \mathcal{X} \to S \) as above the corresponding direct image bundle \( \pi_* (\mathcal{L} + K_{\mathcal{X}/S}) \to S \) is the vector bundle such that the fiber over \( s \) is naturally identified with the space \( H^0(X, L + K_X) \) of all holomorphic \( n \)-forms \( f \) on \( X := \mathcal{X}_s \) with values in \( L := \mathcal{L}_X \) (as is well-known this is indeed a vector bundle, which is shown using vanishing theorems). Moreover, any given weight \( \phi \) on \( \mathcal{L} \) induces an Hermitian metric on \( \pi_* (k\mathcal{L} + K_{\mathcal{X}/S}) \) whose fiber-wise restriction will be denoted by \( \text{Hilb}_{L+K_X}(\phi) \):

\[
\text{Hilb}_{L+K_X}(\phi)(f, f) := i^n^2 \int_X f \wedge \overline{f} e^{-\phi}
\]

The point is that there is no need to specify an integration measure \( \mu \) thanks to the twist by the relative canonical line bundle \( K_{\mathcal{X}/S} \). We will have great use for the following recent results of Berndtsson.

**Theorem 21.** (Berndtsson) Let \( \pi : \mathcal{X} \to S \) be a proper holomorphic submersion and let \( \mathcal{L} \) be a relatively ample line bundle over \( \mathcal{X} \) equipped with a smooth weight with semi-positive curvature. Then

- [5] the curvature of the Hermitian vector bundle over \( S \) defined as the direct image bundle \( \pi_* (\mathcal{L} + K_{\mathcal{X}/S}) \) is semi-positive in the sense of Nakano (and in particular in the sense of Griffiths).
- (see Theorem 1.2 in [9] and the subsequent discussion) The vector bundle \( \pi_* (\mathcal{L} + K_{\mathcal{X}/S}) \) has strictly positive curvature in the sense of Griffiths if either the curvature form of \( \phi \) is strictly positive over all of \( \mathcal{X} \) or strictly positive along the fibers of \( \pi : \mathcal{X} \to S \) and the fibration is infinitesimally non-trivial (i.e. the Kodaira-Spencer classes are non-trivial for all \( s \in S \)).

We will only use the following simple consequence of Berndtsson's theorem (compare [5] [7]):

**Corollary 22.** Under the assumptions in the first point of the previous theorem we have

\[
\dd^c (FS(k) \circ \text{Hilb}_{k\mathcal{L}+K_{\mathcal{X}/S}})(\phi) \geq 0
\]

and the inequality is strict under the assumptions in the second point of the theorem.

**Proof.** We will denote the line bundle \( k\mathcal{L} + K_{\mathcal{X}/S} \) over \( \mathcal{X} \) by \( \mathcal{F} \) and the vector bundle \( \pi_* (\mathcal{F}) \) over \( S \) by \( E \) (and its dual by \( E^* \)). First note that the weight on \( \mathcal{F} \) that we are interested in may be written as

\[
(FS \circ \text{Hilb}_{\mathcal{F}})(s, x_s) = \log \sup_{f_s \in E_s} \frac{|f_s(x_s)|^2}{\|f_s(x_s)\|^2} = \log |\Lambda_{(s, x_s)}|^2
\]

where \( \Lambda_{(s, x_s)} \) is the element in \( E^*_s \otimes \mathcal{F}_s \) defined by

\[
(\Lambda_{(s, x_s)} f_s) := f_s(x_s)
\]

Let now \( t \mapsto (s_t, x_{s_t}) \) be a local holomorphic curve in \( \mathcal{X} \) with \( t \in \Delta \) (the unit-disc). Trivializing \( \mathcal{F} \) in a neighborhood of the previous curve we may pull back \( \Lambda_{(s, x_s)} \) to a holomorphic section \( \Lambda_t \) of \( E^* \) over the unit-disc and identify the weight defined by (3.10) with a function \( \log |\Lambda_t|^2 \) on \( \Delta \). We have to prove that this latter function is (strictly) psh. But this follows from the following well-known fact: a vector bundle \( E \to \Delta \) is (strictly) positive in the sense of Griffiths iff \( \log(\|\Lambda_t\|^2) \) is...
(strictly) subharmonic on $\Delta$ where $\Lambda$ is any non-trivial holomorphic section of the dual vector bundle $E^*$. For example, to get the required (strict) subharmonicity one just notes that, after a standard computation,

$$
\frac{\partial^2}{\partial t \partial \bar{t}} \log(\|\Lambda_t\|^2)_{t=0} \geq -\Theta_{E^*}(\Lambda_0, \Lambda_0) \|\Lambda_0\|^2,
$$

where $\Theta_{E^*}$ at $t$ is the Hermitian endomorphism of $E^*$ representing the curvature of $E$. By the previous theorem $\Theta_{E^*}$ is (strictly) positive which is equivalent to $\Theta_{E^*}$ being (strictly) negative and the corollary hence follows from the previous inequality.

We next obtain a “quantized” version of Corollary 15.

**Corollary 23.** Let $\pi : X \to S$ be a proper holomorphic submersion with Calabi-Yau fibers and let $L$ be a relatively ample line bundle over $X$.

- When $\pi$ is holomorphically trivial the relative Bergman iteration preserves semi-positivity of the curvature of $\phi$.
- In the case of a general submersion with Calabi-Yau fibers,

$$
dd^c \phi^{(k)}_{(m)} \geq -m \frac{\omega_{WP}}{k}
$$

for all $m$.

**Proof.** For simplicity first consider the case of a trivial fibration. Fix a holomorphic $n$-form $\Omega$ on $X := X_0$ trivializing $K_X$. Under the assumption that $X \to S$ is holomorphically trivial $\Omega$ extends to a holomorphic $n$-form on all of $X$ such that $\psi_\Omega := \log \int_{X_s} e^{\phi(s, \cdot)} \Omega$ is independent of $s$. In this notation

$$
\text{Hilb}^{(k)}((\phi(s, \cdot))(f, f) := \int_{X_s} |f|^2 e^{-(k\phi(s, \cdot) - \psi_\Omega(s))} \omega^2 \Omega \wedge \bar{\Omega}
$$

Now consider the fiber-wise isomorphism

$$
j : H^0(X, kL) \to H^0(X, kL + K_X), \quad j(f) = f \otimes \Omega,
$$

which clearly satisfies $\text{Hilb}^{(k)}((\phi(s, \cdot))(f) = e^{\psi_\Omega} j^* \text{Hilb}^{(k)}_{L + K_X}(\phi(s, \cdot)))$. This means that, up to a multiplicative constant independent of $s$, the map $j$ is an isometry when $H^0(X_s, kL + K_{X_s}) = H^0(X, kL + K_{X/S})_{X_s}$ is equipped with its natural Hermitian product. In particular, by $3.9$

$$
dd^c \phi \geq 0 \implies dd^c(FS^{(k)} \circ \text{Hilb}^{(k)})(\phi) \geq 0
$$

Iterating hence proves the first point in the statement of the corollary. Finally, for a general submersion the same argument gives, but now taking into account the fact that $\psi_\Omega$ depends on $s$ that

$$
dd^c \phi \geq 0 \implies dd^c(FS^{(k)} \circ \text{Hilb}^{(k)})(\phi) \geq -dd^c \psi_\Omega(s)/k := -\omega_{FS}(s)/k
$$

using formula $3.2$ in the last equality. Replacing $\phi$ with $FS^{(k)} \circ \text{Hilb}^{(k)} - \psi_\Omega(s)$ and iterating hence finishes the proof of the corollary. □
3.2.3. Convergence towards the Kähler-Ricci flow. The following very simple proposition will turn out to be very useful:

**Proposition 24.** The following monotonicity holds for the Bergman iteration at level $k$ (with respect to $\mu$). Assume that $\phi^{(k)}_m \leq \psi^{(k)}_m$. Then $\phi^{(k)}_{m+1} \leq \psi^{(k)}_{m+1}$. In particular, the Bergman iteration decreases the distance in $H_L$ defined with respect to the sup-norm: $d(\phi, \psi) := \sup_X |\phi - \psi|$.

*Proof.* By definition we have

$$\phi^{(k)}_{m+1} = \phi^{(k)}_m + \frac{1}{k} \log \rho^{(k)}(k\phi^{(k)}_m) = \frac{1}{N_k} \sum_i |f_i|^2.$$

By a well-known identity for Bergman kernels

$$\sum_{i=1}^N |f_i|^2(x) = \sup_{f \in H^0(X,kL)} (|f(x)|^2 / \int_X |f|^2 e^{-k\phi_m} d\mu).$$

But this latter expression is clearly monotone in $\phi_m$ proving the first statement of the proposition. As for the last statement just let $C := \sup_X |\phi^{(k)}_m - \psi^{(k)}_m|$ so that

$$\phi^{(k)}_m \leq \psi^{(k)}_m + C, \quad \psi^{(k)}_m \leq \phi^{(k)}_m + C.$$ 

Applying the first statement of the proposition hence finishes the proof. $\square$

**Remark 25.** The previous proposition can be seen as a “quantum” analog of the corresponding result for the Kähler-Ricci flow [123], which follows directly from the maximum principle for the Monge-Ampère operator and its parabolic analogue.

Now we can prove the following Theorem which is one of the main results in this paper.

**Theorem 26.** Let $L \to X$ be an ample line bundle and $\mu$ a volume form on $X$ giving unit volume to $X$. Fix a smooth weight $\phi_0$ on $L$, whose curvature form is fiber-wise strictly positive and consider the corresponding Bergman iteration $\phi^{(k)}_m$ at level $k$ and discrete time $m$, as well as the Kähler Ricci flow $\phi_t$ - both defined with respect to $\mu$. Then there is a constant $C$ such that

$$\sup_X |\phi^{(k)}_m - \phi_m/k| \leq Cm/k^2$$

In particular, if $m_k$ is a sequence such that $m_k/k \to t$, then

$$\phi^{(k)}_{m_k} \to \phi_t$$

uniformly on $X$. Moreover, in the relative setting $C$ is locally bounded in the base parameter $s$ if $\mu$ depends smoothly on $s$.

*Proof.* Write $\psi_{k,m} = \phi_{m/k}$ and $F^{(k)}(\psi) = \frac{1}{k} \log \rho^{(k)}(\psi)$.

*Step 1:* The following holds for all $(k,m)$

$$\psi_{k,m+1} - \psi_{k,m} = F^{(k)}(\psi_{k,m}) + O(1/k^2),$$

where the error term is uniform in $(k,m)$ (and we will in the following take that as a definition of $O(1/k)$ etc).

To prove this we write the lhs as

$$\frac{1}{k} \left( \phi_{m,k+1} - \phi_{m,k} \right) = \frac{1}{k} \left( \frac{\partial \phi_t}{\partial t} \right)_{t=m/k} + O(1/k)$$
using that \( |\frac{\partial^2 \phi}{\partial t^2}| \leq C \) on \( X \times [0, T] \) by Theorem 14. More precisely, by the mean value theorem the error term \( O(1/k) \) may be written as \( \frac{1}{k} \frac{\partial^2 \phi}{\partial t^2}(\zeta)/2 \) for some \( \zeta \in [0, 1/k] \).

Since \( \phi_t \) evolves according to the Kähler-Ricci flow this means that

\[
\psi_{k,m+1} - \psi_{k,m} = \frac{1}{k} \log\left( \frac{(dd^c \phi_{m/k})^n}{\mu} \right) + O(1/k^2).
\]

But by Proposition 11 we have that \( F(k)(\phi_{m/k}) = \frac{1}{k} \log\left( \frac{(dd^c \phi_{m/k})^n}{\mu} \right) + O(1/k^2) \),

where the error term is uniformly bounded in \((m,k)\) for \( m/k \leq T \) by Theorem 14. In fact, as is well-known the uniform estimates 2.4 on the “space-derivatives” of \( \phi_t \) in Theorem 14 also hold for all time-derivatives \( d^r \phi_t/d^r t \) (and in particular for \( r = 1 \) and \( r = 2 \) used above). This is well-known and shown by differentiating the flow equation with respect to time and applying the maximum principle repeatedly.

Hence, \( T \) may be taken to be equal to infinity, which finishes the proof of Step 1.

**Step 2:** Given step 1 and the fact that the Bergman iteration decreases the sup norm, we have

\[
\sup_X |\phi_{m/k}^{(k)} - \psi_{k,m}| \leq C m/k^2
\]

We will prove this by induction over \( m \) (for \( k \) fix) the statement being trivially true for \( m = 0 \). By Step 1 there is a uniform constant \( C \) such that

\[
\sup_X |\psi_{k,m+1} - (\psi_{k,m} + F(k)(\psi_{k,m}))| \leq C(1/k^2).
\]

for all \((m,k)\). Now we fix the integer \( k \) and assume as an induction hypothesis that \( 3.11 \) holds for \( m \) with \( C \) the constant in the previous inequality. By Proposition 24

\[
\sup_X |(\psi_{k,m} + F(k)(\psi_{k,m})) - (\phi_{m}^{(k)} + F(k)(\phi_{m}^{(k)}))| \leq \sup_X |(\psi_{k,m} - \phi_{m}^{(k)})| \leq C m/k^2
\]

with the same constant \( C \) as above, using the induction hypothesis in the last step. Combining this estimate with the previous inequality gives

\[
\sup_X |\psi_{k,m+1} - \phi_{m+1}^{(k)}| \leq C m/k^2 + C/k^2
\]

proving the induction step and hence Step 2.

Of course, it seems natural to expect that \( C^\infty - \)convergence holds but we leave this problem for the future.

Combining the previous corollary with Theorem 28 and the variational principle in 13 (the \( C^\infty - \)convergence rather uses 28, 53) now gives the following

**Corollary 27.** The conservation of semi-positivity of the curvature of \( \phi_t \) in Corollary 14 holds. Moreover, for a fixed initial data \( \phi_0 = \phi_0^{(k)} \in \mathcal{H}_L \) the following convergence results hold for the Bergman iteration \( \phi_{m/k}^{(k)} \):

- For any sequence \( m_k \) such that \( m_k/k \to \infty \) the convergence \( \phi_{m_k}^{(k)} \to \phi_\infty \) holds in the \( L^1 - \)topology on \( X \). Moreover, if it is also assumed that \( m_k/k^2 \to 0 \) then the convergence holds in the \( C^0 - \)topology.
The balanced weights \( \phi^{(k)}_\infty := \lim_{m \to \infty} \phi^{(k)}_m \) at level \( k \) converge, when \( k \to \infty \), in the \( C^\infty \) topology, to the weight \( \phi_\infty \) which is the large time limit of the corresponding Kähler-Ricci flow (and in particular a solution to the corresponding inhomogeneous Monge-Ampère equation).

In the relative case the convergence holds fiberwise locally uniformly with respect to the base parameter \( s \).

**Proof.** The first statement follows immediately by combining Theorem 26 and the previous corollary, since semi-positivity is preserved under uniform limits of weights. Hence, we turn to the proof of the first point. It is based on the following inequalities:

\[
\limsup_{k \to \infty} I_\mu(\phi^{(k)}_{m_k}) \leq I_\mu(\phi_\infty), \quad \liminf_{k \to \infty} E(\phi^{(k)}_{m_k}) \geq E(\phi_\infty).
\]

To prove these inequalities take a sequence \( m'_k \) such that \( m'_k / k \to t \) and \( m'_k \leq m_k \).

By monotonicity (Lemma 19)

\[
I_\mu(\phi^{(k)}_{m_k}) \leq I_\mu(\phi^{(k)}_{m'_k}).
\]

Hence, letting \( k \to \infty \) and using that \( \phi^{(k)}_{m'_k} \to \phi_t \) uniformly (by Theorem 26) gives

\[
\limsup_{k \to \infty} I_\mu(\phi^{(k)}_{m_k}) \leq I_\mu(\phi_t).
\]

Finally, letting \( t \to \infty \) and using Theorem 12 proves the first inequality in (3.12). As for the second inequality in (3.12), it is similarly proved by noting that, by monotonicity,

\[
\mathcal{L}^{(k)}(\phi^{(k)}_{m'_k}) \leq \mathcal{L}^{(k)}(\phi^{(k)}_{m_k}).
\]

To proceed we will use that \( \psi_k \to \psi \) uniformly in \( H_L \) implies that

\[
\mathcal{L}^{(k)}(\psi_k) \to \mathcal{E}(\psi)
\]

To see this recall that this is well-known when \( \psi_k = \psi \) for all \( k \) (as follows for example from Proposition 11 saying that the convergence holds for the differentials \( d\mathcal{L}^{(k)} \) and \( d\mathcal{E} \); for more general convergence results see [2]). But then the general case follows easily from the fact that \( \mathcal{L}^{(k)} \) is monotone in the argument \( \psi \) and scaling equivariant. Hence, letting \( k \to \infty \) gives, since \( \psi_k := \phi^{(k)}_{m'_k} \to \phi_t \) uniformly, that

\[
\mathcal{E}(\phi_t) \leq \liminf_{k \to \infty} \mathcal{L}^{(k)}(\phi^{(k)}_{m'_k}).
\]

Finally, the proof of the second inequality in (3.12) is finished by using that (as shown in [3]) for any sequence \( (\psi_k) \) in \( H_L \)

\[
\limsup_{k \to \infty} \mathcal{L}^{(k)}(\psi_k) \leq \liminf_{k \to \infty} \mathcal{E}(\psi_k)
\]

Now, adding up the two inequalities in (3.12) gives

\[
\liminf_{k \to \infty} \mathcal{F}(\phi^{(k)}_{m_k}) \geq \mathcal{F}(\phi_\infty).
\]

But then it follows from the variational results in [3] that

\[
\lim_{k \to \infty} \mathcal{F}(\phi^{(k)}_m) = \mathcal{F}(\phi_\infty)
\]

and

\[
dd^c \phi^{(k)}_{m_k} \to \dd^c \phi_\infty
\]
in the weak topology of currents. Next, note that by the inequalities 3.12 the sequence $\phi^{(k)}_{m_k}$ is contained in a compact subset of $\mathcal{H}_L$ equipped with the $L^1$-topology (compare the proof of Lemma 29 and hence we may assume (perhaps after passing to a subsequence) that $\phi^{(k)}_{m_k} \to \psi$ in the $L^1$-topology. But then the convergence in 3.14 forces $\psi = \phi_\infty + C$ for some constant $C$. Finally, it will hence be enough to prove that $C = 0$. To this end note that combining 3.14 and the inequalities 3.12 shows that the latter inequalities are in fact equalities. In particular,

$$\lim_{k \to \infty} E(\phi^{(k)}_{m_k}) = E(\phi_\infty)$$

By the scaling equivariance of $E$ it hence follows that $C = 0$, which finishes the proof of the first point. If one assumes that $m_k / k^2 \to 0$ then it follows immediately from combining Theorem 12 and Theorem 26 that the convergence holds uniformly on $X$, i.e. in the $C^0$-topology.

To prove the second point in the statement of the corollary note that replacing $\phi^{(k)}_{m_k}$ by $\phi_\infty^{(k)}$ in the previous argument gives, just as before, that $\phi_\infty^{(k)} \to \phi_\infty$ in the $L^1$-topology. Moreover, since it was shown in [28, 53] that the convergence of the corresponding curvature forms holds in the $C^\infty$-topology this proves the second point.

4. The (anti-) canonical setting

In this section we will consider another particular case of the general setting in section 2 arising when the line bundle $L := rK_X$ is ample, where $r = 1$ or $r = -1$ (for any fiber $X$ of the fibration). Hence, $X$ is necessarily of general type in the former “positive” case and a Fano manifold in the latter “negative” setting. We will also refer to these two different settings as the $\pm K_X$-settings.

By the very definition of the canonical line bundle any weight $\phi$ on $\pm K_X$ determines a canonical scale invariant probability measure $\mu_\pm(\phi)$ on $X$, where

$$\mu_\pm(\phi) := e^{\pm \phi} / \int_X e^{\pm \phi}$$

(with a slight abuse of notation), so that $\mu_\pm(\phi + c) = \mu_\pm(\phi)$. Equivalently, $\mu_\pm(\phi)$ may be identified with the one-form on $\mathcal{H}_{\pm K_X}$ obtained as the differential of the following functional $I_\pm(\phi)$ on $\mathcal{H}_{\pm K_X}$:

$$I_\pm(\phi) := \pm \log \int_X e^{\pm \phi}, \quad \mu_\pm(\phi) = dI_\pm.$$

A characteristic feature of the $\pm K_X$-setting is that the anti-derivative $I_\pm$ is canonically defined (i.e. not only up to scaling). As a consequence there is a canonical normalization condition for weights that will occasionally be used below, namely the condition that $I_\pm(\phi) = 0$.

We will also have use, as before, for the equivariant functional

$$F_\pm := E - I_\pm,$$

where $E$ is the functional defined in section 2.2 (with respect to a fixed reference weight in $\pm K_X$). Note that the critical points of $F_\pm$ on $\mathcal{H}_{\pm K_X}$ are the Kähler-Einstein weights $\phi$, i.e. the weights such that $\omega_\phi$ is a Kähler-Einstein metric on $X$ (compare Theorem 28 below).

1Note that $F_\pm$ is minus the functional introduced by Ding-Tian [47].
It will also be important to consider a non-normalized variant of \( \mu_\pm(\phi) \) defined by

\[
\mu'_\pm(\phi) := e^{\pm\phi}
\]

(which is the differential of the non-equivariant functional \( \phi \mapsto \int e^{\pm\phi} \)). In the sequel we will refer to the two different settings defined by \( \mu_\pm(\phi) \) and \( \mu'_\pm(\phi) \) as the normalized \( \pm K_X \)-setting and the non-normalized \( \pm K_X \)-setting, respectively. It should be pointed out that it is the latter one which usually appears in the literature on the Kähler-Ricci flow (see for example [13, 48, 37]).

4.1. The relative Kähler-Ricci flow. According to the general construction in section 2 each particular setting introduced above comes with an associated relative Kähler-Ricci flow. For future reference we will write out the fiber-wise flow in the non-normalized \( \pm K_X \)-setting in local holomorphic coordinates:

\[
\frac{\partial \phi}{\partial t} = \text{log det}\left( \frac{1}{\pi} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right)/n! - (\pm \phi)
\]

The normalized and the non-normalized setting induce the same evolution of the fiber-wise curvature forms \( \omega_t \):

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric} \omega_t - \pm \omega_t,
\]

in \( c_1(\pm K_X) \).

In particular, if \( \omega_t \) converges to \( \omega_\infty \) in the large time limit, then \( \omega_\infty \) is necessarily a Kähler-Einstein metric, which is of negative scalar curvature in the \( K_X \)-setting and positive scalar curvature in the \( - K_X \)-setting.

The main virtue of the Kähler-Ricci flow in the normalized setting as compared with the non-normalized one is that the first one is convergent precisely when the flow of curvature forms \( \omega_t \) is. On the other hand, as will be seen later the flow in the non-normalized setting (and its quantized version) has better monotonicity and positivity properties.

**Theorem 28.** The Kähler-Ricci flow in the \( \pm K_X \)-settings always exists and is smooth on \( X \times [0, \infty[ \). More precisely, all the analytical assumptions in section 2.3 are satisfied. In the normalized \( K_X \)-setting it converges to a Kähler-Einstein metric of negative scalar curvature. In the \( - K_X \)-setting the flow converges to a Kähler-Einstein metric of positive scalar curvature under the assumptions that \( H^0(TX) = 0 \) and \( X \) a priori admits a Kähler-Einstein metric. Furthermore, in the relative case the convergence is locally uniform with respect to the base parameter.

A part from the uniqueness statement, the first part of the previous theorem is due to Cao [13]. The convergence on the level of Kähler metrics in the Fano case, i.e. when \( - K_X \) is ample, was proved by Perelman (unpublished) and Tian-Zhu [48]. The convergence on the level of weights then follows directly from Proposition 4 and the known coercivity of the functionals \( - F_\pm \); the coercivity of \( - F_+ \) follows immediately from Jensen’s inequality, while the coercivity of \( - F_- \) was shown in

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2In the literature this latter flow of Kähler forms is sometimes referred to as the normalized Kähler-Ricci flow as opposed to Hamilton’s original flow, but our use of the term normalized is different and only applies on the level of weights on \( L \).
[36], confirming a conjecture of Tian. The uniqueness in the difficult case of $-K_X$ is due to Bando-Mabuchi (for a comparatively simple proof see [3]).

**Remark 29.** The first key analytical ingredient in the proof of the convergence of the flow of Kähler metric $\omega_t$ in the case (i.e the $-K_X$-setting) is an estimate of Perelman saying that the Ricci potential $h_t$ of $\omega_t$, when suitably normalized, is always bounded along the Kähler-Ricci flow for $\omega_t$ (see [48, 37]). In fact, in the present notation $h_t$ coincides (modulo signs) with the time derivative of $\phi_t$ evolving according to the normalized Kähler-Ricci flow in the $-K_X$-setting. The second key ingredient is the fact that the existence of a Kähler-Einstein metric implies that $-F +$ is proper (and conversely [47, 36]). As is well-known there are, in general, obstructions to existence of Kähler-Einstein metrics in the $-K_X$-setting. According to a conjecture of Yau the existence of a Kähler-Einstein metric should be equivalent to a suitable notion of algebraic stability (in the sense of Geometric Invariant Theory). From this point of view the properness (or coercivity) assumption on the functional $-F_+$ can be considered as an analytic stability [47].

It will be convenient to make the following

**Definition.** A weight $\phi_{KE}$ on $\pm K_X$ will be called a normalized Kähler-Einstein weight if $I_{\pm}(\phi_{KE}) = 0$, or equivalently if $e^{\pm \phi_{KE}} = \omega_{KE}^n/n!$.

Hence, there is precisely one normalized Kähler-Einstein weight on $+K_X$ when it is ample. The following simple corollary of the previous theorem and the subsequent remark illustrates the difference between the normalized and non-normalized settings.

**Corollary 30.** In the $+K_X$-setting the non-normalized flow [4.1] always converges to the normalized Kähler-Einstein weight.

**Proof.** Write $\phi'_t$ for the evolution under the Kähler-Ricci flow in the non-normalized $K_X$-setting so that

$$\phi'_t = \phi_t + C_t,$$

where $C_t$ is a constant for each $t$. Since $\phi \mapsto (dd^c \phi)^n$ is invariant under scalings, comparing the two flow equations gives

$$(4.3) \quad \frac{\partial C_t}{\partial t} = -C_t - I_+(\phi_t).$$

Let $D_t := C_t - I(\phi_t)$. Then we get

$$\frac{\partial D_t}{\partial t} = -D_t + \epsilon_t,$$

where $\epsilon_t := \frac{\partial I_+(\phi_t)}{\partial t}$. In the $+K_X$-setting Theorem [28] implies that $\epsilon_t \to 0$ and hence it follows for elementary reasons that $D_t \to 0$. Indeed, assume for a contradiction that $D_t$ does not converge to 0. Then $\frac{\partial \log |D_t|}{\partial t} \to -1$ i.e. $|D_t| \leq C_\delta e^{-t(1-\delta)} \to 0$ for $0 < \delta <<< 1$, giving a contradiction. Finally, in the non-normalized $-K_X$-setting it was shown in [47] (building on [15]) that there is a constant $c_0$ such that $\phi'_t$ converges. But then it follows immediately from combining the scaling invariance of $\phi \mapsto (dd^c \phi)^n$ and the scaling equivariance of $\mu_-$ that the flow diverges exponentially for any other choice of constant $c_0$. $\square$

As for the non-normalized $-K_X$-setting we make the following
Remark 31. In the non-normalized $-K_X$-setting (under the assumptions in the previous theorem) it was shown in [37] (building on [15]) that the flow converges when the initial weight $\phi_0$ is replaced by $\phi_0 + c_0$ for a unique constant $c_0$. The argument in the proof of the previous corollary then gives that for a generic initial weight the flow is divergent.

4.2. Weil-Petersson geometry. As before we may in the following assume that the base $S$ is embedded in $\mathbb{C}$. In the relative $\pm K_X$-setting the (generalized) Weil-Petersson form $\omega_{WP}$ on $S$ was introduced in [30] (see also [23] for generalizations):

\[(4.4) \quad \omega_{WP}(\frac{\partial}{\partial s}, \frac{\partial}{\partial \bar{s}}) := \|A_{KE}\|^2_{\omega_{KE}},\]

where $A_{KE}$ denotes the unique representative in the Kodaira-Spencer class $\rho(\frac{\partial}{\partial s}) \in H^{1,0}(T^{1,0}X_s)$ which is harmonic with respect to the Kähler-Einstein metric on $X_s$ and the $L^2$-norm is computed with respect to this latter metric. In fact, as showed in [23] (Proposition 4.12) $A_{KE} = -\partial V_{\omega_s}$, where $V_{\omega_s}$ is the local vector field defined by formula 3.3. This is a consequence of the following proposition proved in [23].

Proposition 32. Let $\pi : X \to S$ be a proper holomorphic submersion and $\omega_s$ a smooth family of 2-forms on the fibers $X_s$ such that $\omega_s$ is Kähler-Einstein on $X_s$. Then $A_{\omega_s}$ is the unique element in $H^{0,1}(T^1\!\!\!\!\!\,_X^\star s)$ which is harmonic with respect to $\omega_s$.

Note that “harmonic” lifts of vector fields were previously used by Siu [42] in the context of Weil-Petersson geometry.

Remark 33. When the relative dimension is one the space $H^{0,1}(T^1\!\!\!\!\!\,_X^\star s)$ is isomorphic to $H^{1,0}(T^1\!\!\!\!\!\,_X^\star s) = H^{0,2}(K_X)$ under Serre duality. Hence, the Weil-Petersson form as defined in terms of harmonic representatives then coincides with the metric on $X_S$ introduced by Weil in the case when $X$ is the universal family over Teichmüller space. As conjectured by Weil and subsequently proved by Ahlfors this latter $(1,1)$-form is closed and hence Kähler. In the higher dimensional case, it was observed in [23] that the Kähler property of $\omega_{WP}$ as defined by (4.4) follows immediately from Corollary 37 below.

By an application of the implicit function theorem (in appropriate Banach spaces) the smoothness of the family $\omega_s$ (and of the associated normalized weight) in the previous proposition is automatic in the $+K_X$-case case, as well as in the $-K_X$ case if there are no non-trivial holomorphic vector fields tangential to the fibers of the fibration (see Theorem 6.3 in [23]).

Now we can prove the following variant of Theorem 14.

Theorem 34. Let $\pi : X \to S$ be a proper holomorphic submersion. Assume that $\pm K_X|_S$ is relatively ample and that $\phi_t$ evolves according to the Kähler-Ricci flow in the non-normalized setting. Then

\[(4.5) \quad \frac{\partial c(\phi_t)}{\partial t} = \Delta_{\omega_X} c(\phi_t) - \pm c(\phi_t) + |A_{\phi_t}|^2_{\omega_X}.\]

In particular, if $\phi_{KE}$ is a fiber-wise normalized Kähler-Einstein weight, then

$$\Delta_{\omega_{KE}} c(\phi_{KE}) - \pm c(\phi_{KE}) + |A_{\omega_{KE}}|^2_{\omega_{KE}} = 0.$$
Proof. To simplify the notation we will only consider the $+K_X$-setting, but the proof in the $-K_X$ setting is essentially the same. We will just indicate the simple modifications of the proof of Theorem 3.1 which arise in the present setting.

Let us first consider the modifications to the calculation of the $t$-derivative of $c(\phi)$ that arise from the additional term $-\partial_t \phi$ appearing in the calculation of the time derivative $\phi_t$, since now

$$\frac{\partial}{\partial t} \phi_t = \log \det(\phi_{k\ell}) - \phi$$

in local coordinates. To this end we assume to simplify the notation that $X$ is one-dimensional (but the general argument is essentially the same). First recall that according to formula 3.3

$$\frac{\partial}{\partial t} c(\phi) = \frac{\partial}{\partial t} \phi_{s\bar{s}} - (|s|_X^2 \phi_{zz})_{t} - (\phi_{s\bar{s}})_{t} \phi_{zz} - \frac{\partial}{\partial t} \phi_{zz}\phi_{zz}.$$

Hence, the additional contribution referred to above is of the form

$$ B := (-\phi)_{s\bar{s}} - 2\Re(-\phi_{s\bar{s}}\phi_{zz}) + |s|_X^2 \phi_{zz}^2 (\phi_{zz}^2 - \phi_{zz}) = (-\phi_{s\bar{s}}) + |s|_X^2 \phi_{zz}^2 - \phi_{s\bar{s}} \phi_{zz} = -c(\phi).$$

Hence, the local calculations in the Calabi-Yau case give that

$$\frac{\partial}{\partial t} c(\phi) = \Delta_{s\bar{s}} c(\phi) - c(\phi) + |A_{s\bar{s}}|_{X}^2.$$

Finally, since a normalized Kähler-Einstein weight is stationary for the non-normalized Kähler-Ricci flow this finishes the proof of the theorem. □

The last fiber-wise elliptic equation in the previous corollary (in the $K_X$-setting) was first obtained very recently by Schumacher [41], who used the maximum principle to deduce the following interesting

Corollary 35. Let $\pi : X \to S$ be a fibration as in the previous theorem and assume that $K_{X/S}$ is relatively ample. Then the canonical fiber-wise Kähler-Einstein weight $\phi_{KE}$ on $K_{X/S}$ is smooth with semi-positive curvature form on $X$. Moreover, if the Kodaira-Spencer classes of the fibration are non-trivial for all $s$, then the curvature form of $\phi_{KE}$ is strictly positive on $X$.

The first part of the corollary was also shown by Tsuchi [50, 51] using his iteration. Similarly, by a simple application of the parabolic maximum principle we deduce the following corollary from the parabolic equation in the previous theorem.

Corollary 36. (same assumptions as in the previous theorem). Let $\phi_t$ evolve according to the Kähler-Ricci flow in the non-normalized $\pm K_X$-setting. If the initial weight has (semi-) positive curvature form on $X$ then so has $\phi_t$ for all $t$. More precisely, $(dd^c \phi_t)_{x_s} > 0$ (in all $n + 1$ directions) at any point $x_s$ in the fiber $X_s$ unless $(dd^c \phi_0)_{x_s}^{n+1}$ and $A_{x_s}$ vanish identically on $X_s$.

Proof. As usual we may assume that $S$ is embedded in $\mathbb{C}$. Let us start with the semi-positive case where the conclusion follows from the weak maximum principle. Indeed, assume to get a contradiction that $c(\phi_t) \leq 0$ on $X$ for $t = 0$ but that there is $(t, s, x)$ such that at $(t, s, x)$ we have $c(\phi_t)(s, x) < 0$. By optimizing over $(x, t)$ we may also assume that $\frac{\partial}{\partial t}(c(\phi_t)) \leq 0$, $\Delta_{s\bar{s}} c(\phi_t) \geq 0$. Then equation 3.3 gives

$$0 \geq e^{at}(ac(\phi_t) + \frac{\partial c(\phi_t)}{\partial t}) = e^{at}([\Delta_{s\bar{s}} c(\phi_t)] + (a \pm 1)c(\phi_t) + |A_{s\bar{s}}|_{X}^2).$$
But if $a$ is chosen so that $a \pm 1 > 0$, then the rhs above is strictly positive, giving a contradiction. To handle the remaining cases we invoke the following well-known strong maximum principle for the heat operator (which by standard argument can be reduced to the corresponding local statement in [38]): let $h_t \geq 0$ satisfy
\[
\frac{\partial h_t}{\partial t} \geq \Delta_{g_t} h_t \quad \text{on } [0, T] \times X
\]
for any smooth family $g_t$ of Riemannian metrics. Then either $h_t > 0$ for all $t > 0$ or $h_0 \equiv 0$. In our case we set $h_t = e^{at}c(\phi_t)$ with $a = -1$ and conclude that if it is not the case that $c(\phi_t) > 0$ for all $t > 0$ then $c(\phi_0) \equiv 0$ and hence
\[
\frac{\partial}{\partial t} c(\phi_t)_{t=0} = |A_{\phi_0}|^2 \omega^n_g
\]
If we now assume, to get a contradiction, that the rhs above is strictly positive at $x_0$ then it follows that there is an $\epsilon > 0$ such that $c(\phi_t)(x_0) > 0$ for $t \in [0, \epsilon]$ i.e. for such $t$ it is not the case that $c(\phi_t) \equiv 0$ on $X$. Hence, as explained above $c(\phi_t) > 0$ on all of $[0, \infty] \times X$, which yields the desired contradiction. $\square$

In particular, the previous corollary says that if the fibration $\mathcal{X}$ is infinitesimally non-trivial then the non-normalized Kähler-Ricci flows instantly makes any semi-positively curved initial weight strictly positive.

Next we note that integrating the last formula in the previous theorem immediately gives the following corollary first shown by Fujiki-Schumacher [23] (Theorem 7.9).

**Corollary 37.** (same assumptions as in the previous theorem). Let $\phi_{KE}$ be the weight of a smooth metric on $\pm K_{\mathcal{X}/S}$ which restricts to a normalized Kähler-Einstein weight on each fiber, then
\[
\pi_*(((dd^c \phi_{KE})^n+1)/(n+1)!)) = \pm \omega_{WP}
\]
on $S$, where $\pi_*$ denotes the fiber integral. In particular, if $S$ is effectively parametrized (i.e. all Kodaira-Spencer classes are non-trivial) then $\pm \pi_*(((dd^c \phi_\pm)^n+1$ and hence the Weil-Petersson metric $\omega_{WP}$ is a Kähler form on the base $S$.

**Remark 38.** It follows immediately from the previous corollary that, in the $-K_X$-setting, the normalized Kähler-Einstein weight $\phi_{KE}$ never has semi-positive curvature on all of $\mathcal{X}$ if the family is effectively parametrized. Combining this fact with Corollary [38] shows that the relative Kähler-Ricci flow in the non-normalized $K_X$-setting never converges in the $L^1(\mathcal{X})$—topology for an initial weight $\phi_0$ with semi-positive curvature form on an effectively parametrized fibration $\mathcal{X}$.

### 4.3. Quantization: the Bergman iteration

The (normalized) Bergman iteration in the $\pm K_X$—setting on $\mathcal{H}_{\pm K_X}$ is defined precisely as in section 3.2 but using the probability measure $\mu_{\pm}(\phi)$ in the definition of $\text{Hilb}^{(k)}(\phi, \mu_{\pm}(\phi))$. Similarly, the non-normalized Bergman iteration is defined in terms of the measure $\mu'_{\pm}$.

The virtue of the non-normalized setting is that the corresponding Hilbert norms correspond to the “adjoint” norms appearing in Berndtsson’s Theorem [21].

\[
(4.6) \quad \text{Hilb}^{(k)}(\phi, \mu'_{\pm})(f, f) := i^n \int_X f \wedge \bar{f} e^{-(k\pm 1)\phi} = \text{Hilb}_{(k-1)L+K_X}(\phi)
\]
for $L = \pm K_X$. Moreover, they are clearly decreasing in $\phi$ (for $k \geq 1$) and hence the analogue of Proposition [24] of the corresponding Bergman iteration holds:
Proposition 39. Consider the Bergman iteration $\phi^{(k)}_{m}$ in the non-normalized $\pm K_X$-setting and assume that $\phi^{(k)}_{m} \leq \psi^{(k)}_{m}$. Then $\phi^{(k)}_{m+1} \leq \psi^{(k)}_{m+1}$. Moreover, if $d(\phi, \psi)$ denotes the sup norm of $\phi - \psi$ then

$$d(\psi_{m+1}, \phi_{m+1}) \leq d(\psi_{m+1}, \phi_{m+1})(1 \pm \frac{1}{k})$$

In particular, the Bergman iteration decreases the distance $d(\phi, \psi)$ in the non-normalized $K_X$-setting.

Proof. Given the discussion preceding the proposition we just have to prove the claimed property of the distance $d$. But this follows directly from the monotonicity in the first part combined with the fact that $\log \rho^{(k)}(\phi_{m} + c)/k = \log \rho^{(k)}(\phi_{m})/k - \pm \frac{c}{k}$, which in turn follows from $\mu^{\prime}_{\pm}(\phi + c) := e^{\pm (\phi+c)} = \mu^{\prime}_{\pm}(\phi)e^{\pm c}$.

On the other hand, the following monotonicity of functionals holds in the normalized setting:

Lemma 40. The functionals $-I_{\mu_{\pm}}$ and $L^{(k)}$ are increasing along the normalized Bergman iteration on $H_{\pm K_X}$. Moreover, they are strictly increasing at $\phi^{(k)}_{m}$ unless $\phi^{(k)}_{m}$ is stationary (when $k > 1$ in the case of $I_{\mu_{\pm}}$).

Proof. By the general Lemma 48 $L^{(k)}$ is increasing and $I_{\pm}$ is decreasing under the iteration, since $\phi \mapsto I_{\pm}(\phi)$ is concave with respect to the affine structure by Jensen’s inequality. To show that $I_{\pm}^{(k)}$ is increasing in the $K_X$-setting just observe that,

$$I_{\pm}^{(k)}(\phi^{(k)}_{m+1}) - I_{\pm}^{(k)}(\phi^{(k)}_{m}) := \log \int e^{(\phi^{(k)}_{m+1} - \phi^{(k)}_{m})} e^{\phi^{(k)}_{m+1}} = \log \int (\rho^{(k)})^{1/k} \mu(\phi^{(k)}_{m}) \leq \log((\int \rho^{(k)}(\phi^{(k)}_{m}) \mu)^{1/k}) = 0,$$

using Jensen’s inequality applied to the concave function $t \mapsto t^{1/k}$, which is strictly concave for $k > 1$.

4.3.1. Convergence of the Bergman iteration at a fixed level $k$.

Theorem 41. The Bergman iteration $\phi^{(k)}_{m}$ at level $k$ converges, when the discrete time $m \to \infty$, to a balanced weight $\phi^{(\infty)}_{k}$ in the following settings:

- The normalized $K_X$-setting
- The normalized $-K_X$-setting if it is a priori assumed that there exists some balanced metric at level $k$ and $H^0(TX) = 0$.
- The normalized $-K_X$-setting for $k$ sufficiently large under the assumption that $X$ admits a Kähler-Einstein metric and $H^0(TX) = 0$.
- The non-normalized $+K_X$-setting, where the limiting balanced weight is the unique normalized one.

Proof. Proof of the first point: By the previous lemma $-I_\mu$ is increasing and as shown in [8] $-F^{(k)}_\mu$ is coercive (as follows immediately from Jensen’s inequality). Moreover, as shown in [8] balanced weights are unique modulo scaling and hence all the convergence criteria in Proposition [9] are hence satisfied.

Proof of the second point: By the previous lemma $-I_\mu$ is increasing and as shown in [8] it follows immediately from Berndtsson’s theorem [21] applied to $L = -K_X$
that $-\mathcal{F}_\mu^{(k)}$ is strictly convex modulo scaling. Hence, the convergence follows by combining Proposition 9 and Lemma 10.

Proof of the third point: The fact that $-\mathcal{F}_\mu^{(k)}$ is coercive was shown in 3 (using the corresponding coercivity of $-\mathcal{F}_\mu$ on $\mathcal{H}_L$). Given this coercivity the convergence follows as in the previous point.

Proof of the fourth point: let $(\phi')_m^{(k)} = \phi_m^{(k)} + C_m^{(k)}$ denote the non-normalized Bergman iteration in the $K_X$-setting. By the definition of the Bergman iteration (compare equation 4.10 below):

$$(C_m^{(k)} - C_{m+1}^{(k)}) = -C_m^{(k)}/k - I(\phi_m^{(k)})/k$$

where by the first point above $I(\phi_m^{(k)}) \to I_\infty$, when $m \to \infty$. Set $D_m := C_m^{(k)} + I(\phi_m^{(k)})$. Then

$$D_{m+1} = (1 - 1/k)D_m + \epsilon_m,$$

where $\epsilon_m = (I(\phi_{m+1}^{(k)}) - I(\phi_m^{(k)})) \to 0$ as $m \to \infty$. But then it follows for elementary reasons that $D_m \to 0$, i.e. $C_m^{(k)} \to -I_\infty$ showing that $(\phi')_m^{(k)}$ indeed converges and $I_\infty((\phi')_m^{(k)}) \to 0$, proving the second point. For completeness we finally show that $D_m \to 0$. Assume for a contradiction that this is not the case. But then $D_{m+1}/D_m \to (1 - 1/k)$ and hence $D_m \leq C_3(1 - 1/k + \delta)^m \to 0$ for $\delta$ sufficiently small, giving a contradiction. \qed

The convergence in the fourth point above also follows immediately from the contracting property of the corresponding iteration (compare the proof of Theorem 47 below). We also note the following direct consequence of Berndtsson’s Theorem 21 using formula 4.6 in the non-normalized setting.

Corollary 42. The Bergman iteration in the non-normalized $\pm K_X$-setting preserves the (semi-)positivity of the curvature of the initial weight. Moreover, if the fibration $\mathcal{X}$ is assumed infinitesimally non-trivial then any initial weight on $\pm K_{X/S}$ on which is semi-positively curved and strictly positively curved along the fibers of $\mathcal{X}$ becomes strictly positively curved under the iteration.

Combining the previous corollary and Theorem 41 now gives the following:

Corollary 43. Let $\pi : \mathcal{X} \to S$ be a proper holomorphic submersion with $K_{X/S}$ relatively ample. Let $\phi^{(k)}$ be the weight on $K_{X/S}$ obtained by requiring that its restriction to any fiber is the unique normalized balanced weight at level $k$, i.e. $\int_{\mathcal{X}_s} e^{\phi^{(k)}} = 1$. Then $\phi^{(k)}$ is smooth with semi-positive curvature form. Moreover, if the fibration $\mathcal{X}$ is assumed infinitesimally non-trivial then $\phi^{(k)}$ is strictly positively curved.

Proof. Since positivity and smoothness are local notions it is enough to prove the corollary when when $S$ is embedded in $\mathbb{C}$.

Smoothness: By definition $\phi^{(k)} = FS^{(k)}(H^{(k)})$ where $H^{(k)}$ is an element in the finite-dimensional smooth manifold $\mathcal{H}^{(k)}$ uniquely determined by $G^{(k)}(H^{(k)}, s) = 0$, 3 where $G^{(k)}$ is the smooth map defined by

$$G^{(k)}(H^{(k)}, s) := (T^{(k)} - I, I_+ \circ FS^{(k)}) \in \mathcal{H}^{(k)} \times \mathbb{R}.$$ 

Moreover, as shown in 3 the linearization of $T^{(k)} - I$ is invertible modulo scaling (since it represents the differential of a functional on $H^{(k)}$ which is strictly convex.
modulo scaling). Hence, the claimed smoothness follows from the implicit function theorem.

**Positivity:** Since $K_{X/S}$ is assumed relatively ample it admits a smooth weight $\phi_0$, which has fiber-wise positive curvature form. After adding a sufficiently large multiple of the pull-back from the base of $|s|^2$ we may assume that $\phi_0$ has positive curvature over $X$. By the last point of the previous theorem the Bergman iteration $\phi_m^{(k)}$ in the non-normalized $K_X$—setting with initial weight $\phi_0$ yields a sequence of weights on $K_{X/S}$ converging, when $m \to \infty$, uniformly to the unique normalized balanced weight $\phi^{(k)}$ at level $k$. As a consequence $d^P\phi^{(k)} \geq 0$ on $X$. Moreover, if the fibration $X$ is assumed infinitesimally non-trivial the previous corollary shows that applying the Bergman iteration to $\phi^{(k)}$ yields a strictly positively curved metric. But since $\phi^{(k)}$ is fixed under the iteration this finishes the proof of the corollary. □

**Corollary 44.** Let $\pi : X \to S$ be the universal curve of the Teichmuller space of complex curves of a genus $g \geq 2$. Fix a positive integer $k$ (for $g = 2$ we assume that $k \geq 2$). Under the natural isomorphism

$$(T^{1,0}S)^* = \pi_*(2K_{X/S})$$

the fiber-wise normalized balanced weight $\phi^{(k)}$ on $K_{X/S}$ at level $k$ (appearing in the previous corollary) induces an Hermitian metric $\omega^{(k)}$ on $S$ with a curvature which is dually Nakano positive. Moreover, when $k \to \infty$ the metric $\omega^{(k)}$ converges towards the Weil-Petersson metric $\omega_{WP}$ point-wise on $S$.

**Proof.** As is classical the assumptions on $k$ ensure that $K_{X/S}$ is very ample. By the previous corollary $\phi^{(k)}$ is a smooth weight on $K_{X/S} \to X$ with strictly positive curvature and hence the $L^2$—metric on the direct image bundle $\pi_*(L + K_{X/S})$ (with $L = K_{X/S}$) induced by $\phi^{(k)}$ has, according to the first point in Theorem 21 a curvature which is positive in the sense of Nakano. Since, $T^{1,0}S|_s = H^1(T^{1,0}X_s) \cong H^0(2K_{X_s})^*$ this proves the first statement. To prove the point-wise convergence on $S$ of $\omega^{(k)}$ towards $\omega_{WP}$ it is enough to prove that

$$e^{-\phi^{(k)}} \to e^{-\phi_{KE}}$$

in $L^1_{loc}(X)$ for $X = X_s$ (since, by definition, it implies the point-wise convergence of the corresponding Hermitian metrics on $\pi_*(L + K_{X/S})$). But this convergence follows from the $L^1$ convergence of $\phi^{(k)}$ towards $\phi_{KE}$ (Theorem 11) combined with the fact that $J(\phi^{(k)})$ is uniformly bounded, as shown in [3] (see Lemma 6.4 therein). Alternatively, it follows immediately from the uniform convergence in Theorem 17 below.

The convergence in the previous corollary should be compared with the approximation results for the Weil-Petersson metric for moduli spaces of higher dimensional manifolds recently obtained by Keller-Lukic [29]. The approximating Kähler metrics $\omega^t$ in [29] are related to different balanced metrics, namely those defined wrt Donaldson’s original setting in [17] (where $\mu(\phi) = MA(\phi)$).

4.3.2. **Convergence towards the Kähler-Ricci flow.**

**Theorem 45.** The following convergence results hold in all settings introduced in the beginning of section 4 (i.e. in the (non-) normalized $\pm K_X$—settings). Fix a
smooth and strictly psh weight initial weight \( \phi_0 \) on \( \pm K_X \) and consider the corresponding Bergman iteration \( \phi_m^{(k)} \) at level \( k \) and discrete time \( m \), as well as the corresponding Kähler Ricci flow \( \phi_t \). Then there is a constant \( A \) such that

\[
\sup_X |\phi_m^{(k)} - \phi_{m/k}| \leq Am/k^2
\]

uniformly in \( (m,k) \) satisfying \( m/k \leq T \) (in the \( K_X \)–setting \( A \) is independent of \( T \)).

In particular, if \( m_k \) is a sequence such that \( m_k/k \to t \), then \( \phi_m^{(k)} \to \phi(t) \) uniformly on \( X \) and

\[
ddf \phi_m^{(k)} \to \omega_t
\]
on \( X \) in the sense of currents, where \( \omega_t \) evolves according to the corresponding Kähler-Ricci flow \( 4.2 \). The corresponding result also holds for the corresponding non-normalized flows and in the relative setting, where the convergence is locally uniform with respect to the base parameter \( s \).

Proof. In the case of the non-normalized \( K_X \)–setting (denoted by primed objects) the proof of Theorem \( 26 \) carries over essentially verbatim, thanks to the last statement in Proposition \( 39 \) and Corollary \( 30 \) which gives the uniformity wrt \( T \in [0, \infty) \). To handle the non-normalized \( -K_X \)–setting we need to modify the previous argument slightly. More precisely, we will prove that

\[
\sup_X |\psi_{k,m+1}^{(k)} - \phi_{m/k}\rangle| \leq A(1 + \frac{1}{k})^{-m/m/k^2}
\]

(4.8)

for all \((m,k)\) such that \( m/k \leq T \). Now we fix the integer \( k \) and assume as an induction hypothesis that (4.8) holds for \( m \) with \( A \) the constant in the previous inequality. By Proposition \( 24 \)

\[
\sup_X |(\psi_{k,m} + F^{(k)}(\psi_{k,m}) - (\phi_m + F^{(k)}(\phi_m))| \leq \sup_X |(\psi_{k,m} - \phi_m)(1 + \frac{1}{k})| \leq \\
\leq (A(1 + \frac{1}{k})^{-m/m/k^2})(1 + \frac{1}{k})
\]

with the same constant \( A \) as above, using the induction hypothesis in the last step. Combining this estimate with the previous inequality gives

\[
\sup_X |(\psi_{k,m+1} - \phi_m^{(k)}| \leq A(1 + \frac{1}{k})^{-m+1/m/k^2} + A/k^2.
\]

But using that \( 1 \leq (1 + \frac{1}{k})^{-m+1} \) in the last term above proves the induction step and hence finishes the proof of the estimate \( 26 \).

To treat the Kähler-Ricci flows \( \phi_t \) in the normalized settings we write

\[
\phi_t = \phi_t + C_t,
\]
where $C_t$ is a constant for each $t$. Then
\begin{equation}
\frac{\partial C_t}{\partial t} = -(I_\phi'(t))
\end{equation}
Indeed, by the definition of the flow $\phi'_t$ and $\phi_t$ we have
\begin{align*}
\frac{\partial \phi'_t}{\partial t} &= \log(MA(\phi'_t)) - \varepsilon \phi'_t \\
\frac{\partial \phi_t}{\partial t} &= \log(MA(\phi_t)) - \varepsilon \phi_t \pm (I_\varepsilon(\phi_t))
\end{align*}
By scale invariance we may as well replace $\phi_t$ with $\phi'_t$ in the rhs of the second equation above and hence subtracting the second equation from the first one proves 4.9.

Similarly, writing $(\phi'_t(k))^{(k)}_m = \phi'_m + C_m^{(k)}$
we obtain the the following difference equation, using that $\phi \mapsto \rho^{(k)}(\phi)$, defined with respect to $\mu_\pm$ is scale invariant:
\begin{equation}
C_{m+1}^{(k)} - C_m^{(k)} = -\frac{1}{k} I_\varepsilon((\phi')^{(k)}_m)
\end{equation}
Now, as explained above the estimate 4.8, holds for the “primed” objects and hence by the scaling equivariance of $I_\varepsilon$
\begin{equation}
|I_\varepsilon((\phi'_m/k)) - I_\varepsilon((\phi')^{(k)}_m)| \leq Am/k^2
\end{equation}
A simple version of the argument given in the proof of Theorem 45 now shows, by comparing the differential equation 4.9 with the difference equation 4.10 and using 4.11, that
\begin{equation}
|C_m^{(k)} - C_{m/k}| \leq Bm/k^2
\end{equation}
for a uniform constant $B$. All in all this hence finishes the proof of the theorem. \(\square\)

We also have the following analogue of Cor 27:

**Corollary 46.** For a fixed initial data $\phi_0 = \phi_0^{(k)} \in H_{\pm X}$ the following convergence results hold for the Bergman iteration $\phi^{(k)}_m$ in the normalized $\pm X -$setting (in the $-X -$setting it is assumed that $H^0(TX) = 0$ and $X$ a priori admits a Kähler-Einstein metric):

- For any sequence $m_k$ such that $m_k/k \to \infty$ the convergence $\phi^{(k)}_{m_k} \to \phi_\infty$ holds in the $L^1 -$topology on $X$.
- The balanced weights $\phi_\infty^{(k)} := \lim_{m \to \infty} \phi^{(k)}_m$ at level $k$ converge, when $k \to \infty$, in the $C^\infty -$topology, to the weight $\phi_\infty$ which is the large time limit of the corresponding Kähler-Ricci flow.

Moreover, the convergence in the second point also holds in the non-normalized $X -$setting, where the limit $\phi_\infty$ coincides with the canonical Kähler-Einstein weight $\phi_{KE}$. In the relative case all convergence results hold fiberwise locally uniformly with respect to the base parameter $s$.

**Proof.** The proof of the first two points proceeds exactly as in the previous setting (again using the variational characterization in [3]). As for the claimed convergence in the non-normalized setting it is obtained by noting that the large $m$ limit $(\phi')^{(k)}_m$ in the non-normalized setting is the unique balanced weight such that $I_\varepsilon((\phi')^{(k)}_m) = 0$. In other words, $(\phi')^{(k)}_\infty = \phi_\infty^{(k)} - I_\varepsilon(\phi_\infty^{(k)})$, where $\phi_\infty^{(k)}$ is the large $m$ limit of the iteration in the normalized setting. But by the second point above this means
that \((\phi')_{\infty} \to \phi_{\infty} - I_{\pm}(\phi_{\infty})\) in \(L^1\) (also using the continuity with respect to the \(L^1\)–topology of the functional \(I_{\pm}\) on compacts; compare [3]). By uniqueness, this means that the limit must be \(\phi_{KE}\).

4.4. Uniform convergence of the balanced weights in the \(K_X\)–setting.

Next we point out that in the \(K_X\)–setting the convergence of the balanced weights is actually uniform (the proof is independent of the variational one given in [3]). The proof simply uses that \(\phi^{(k)}\) is close to \(\phi_t\), where \(\phi_t\) is the corresponding Kähler-Ricci flow and \(t_k\) is a suitable sequence tending to infinity.

**Theorem 47.** Let \(\phi^{(k)}\) be the the balanced weight at level \(k\) on the canonical line bundle \(K_X\) (in the non-normalized setting). When \(k \to \infty\), the weights \(\phi^{(k)}\) converge uniformly towards the normalized Kähler-Einstein weight \(\phi_{KE}\).

**Proof.** Fix a smooth and positively curved weight \(\phi_0\) on \(K_X\) and denote by \(\phi^{(k)}_m\) the Bergman iteration at level \(k\) with initial data \(\phi^{(k)}_0 = \phi_0\). By Proposition the map whose iterations define the Bergman iterations is a contraction mapping with contacting constant \(q = (1 - \frac{1}{k}) < 1\) and hence it follows from the Banach fix point theorem that

\[
\|\phi^{(k)} - \phi^{(k)}_m\|_{L^\infty} \leq \frac{q^m}{(1 - q)} \|\phi^{(k)}_1 - \phi_0\|_{L^\infty}
\]

By definition we have \(\phi^{(1)}_1 - \phi_0 = \frac{1}{k} \log \rho(k\phi)\) which, according to Prop [11] is uniformly bounded by a constant times \(\frac{1}{k} \log k\) and hence

\[
\|\phi^{(k)} - \phi^{(k)}_m\|_{L^\infty} \leq C(1 - \frac{1}{k})^m k \log k
\]

Next we take the sequence \(m = m_k := [k^{3/2}]\) where \([c]\) denotes the smallest integer which is larger than \(c\). Then \(t_k := m_k/k = k^{1/2} \to \infty\) as \(k \to \infty\) and since \((1 - \frac{1}{k})^k \to e^{-1} < 1\) we conclude that

\[
\|\phi^{(k)}_m - \phi_0\|_{L^\infty} \to 0
\]

as \(k \to \infty\). If now \(\phi_t\) denotes the Kähler-Ricci flow in the non-normalized \(K_X\)–setting we have, according to Theorem [15] that

\[
\|\phi^{(k)}_m - \phi^{(k)}\|_{L^\infty} \to 0
\]

using that \(m_k/k^2 \to \infty\). Finally, since \(\phi_{t_k} \to \phi_{KE}\) uniformly as \(t_k \to \infty\) this proves the theorem. Of course, the last convergence is not really needed for the proof as we may as well start with \(\phi_0 = \phi_{KE}\) which is trivially fixed under the Kähler-Ricci flow.

It should be pointed out that the uniform convergence in the previous theorem has been previously obtained by Berndtsson (who also related it to Tsuji’s iteration [50]), using a different approach - see the announcement in [10]. But hopefully the relation to the convergence of the Kähler-Ricci flow above may shed some new light on the convergence.
4.5. **Families of varieties of general type and comparison with the NS-metric.** The quantized setting concerning the case when $K_X$ is ample admits a straightforward generalization to the case where $K_X$ is merely $\mathbb{Q}$-effective [31]. For simplicity we will only discuss the case when $K_X$ is big, i.e. $X$ is a non-singular variety of general type. Moreover, we will no longer assume that the map $\pi$ is a submersion. More precisely, we are given a surjective quasi-projective morphism $\pi : X \to S$ between non-singular varieties such that the generic fiber is a variety of general type. We denote by $S^0$ the maximal Zariski open subset of $S$ such that $\pi$ restricted to $\mathcal{X}^0 := \pi^{-1}(S^0)$ is a submersion, i.e. a smooth morphism (and hence the fibers of $S^0$ are non-singular varieties of general type).

Let us first consider the general absolute case, where we are given a line bundle $L \to X$ and an integer $k$ such that $kL$ is effective, i.e. $H^0(X, kL) \neq \{0\}$. The main new feature in this more general setting is that any Bergman weight $\psi_k$ at level $k$, i.e. $\psi_k \in FS^{(k)}(kH)$, will usually have singularities, i.e. it defines a singular metric on $L$ with positive curvature form. More precisely, the weight $k\psi_k$ on $kL$ is singular precisely along the base locus $Bs(kL)$ of $kL$, i.e. the intersection of the zero sets of all elements in $H^0(kL)$. Anyway, the difference of any two Bergman metrics is clearly bounded. Moreover, when $L = K_X$ the measure $\mu_{\psi_k} := e^{\psi_k}$ has a smooth density which vanishes precisely along $Bs(kL)$. As a consequence, we may fix such a reference (singular) weight $\phi_0 := \psi_k$ and the reference measure $\mu_0 := e^{\psi_k}$. Then Lemma 7 still applies (as explained in the remark following the lemma). As a consequence the proof of the convergence of the Bergman iteration to a balanced weight at level $k$ in the non-normalized $K_X$-setting (Theorem 11) is still valid as long as $kK_X$ is effective. Combining this latter convergence with the generalizations [7, 8] of Berndtsson’s Theorem 21 and the invariance of plurigenera 43 then gives the following generalization of Corollary 13.

**Theorem 48.** Let $\pi : \mathcal{X} \to S$ be a surjective quasi-projective morphism such that the generic fiber is a variety of general type. Then, for $k$ sufficiently large there is a unique singular weight $\phi^{(k)}$ on the relative canonical line bundle $K_{\mathcal{X}/S} \to \mathcal{X}$ with positive curvature current, such that the restriction of $\phi^{(k)}$ to any fiber over $S^0$ is a normalized and balanced weight at level $k$. Moreover, the weight $\phi^{(k)}$ is smooth on the Zariski open set defined as the complement in $\mathcal{X}$ of $\bigcup_{\ell \in S^0} Bs(kK_{\mathcal{X}^\ell}) \cup \pi^{-1}(S - S^0)$.

**Proof.** Let us first prove the positivity statement. As before we may assume that $S$ is a domain in $\mathbb{C}$. First we consider the behavior over the set $S^0$, i.e. where the fibration is a submersion. Fix $s_0 \in S_0$ and write $X = \mathcal{X}_{s_0}$. Let $(f_i)$ be a bases in $H^0(X, kK_X)$. By the invariance of plurigenera 43 $s_0$ has a neighborhood $U \subset S_0$ with holomorphic sections $F_i$ of $kK_{\mathcal{X}/S} \to U$ such that $F_i$ restricts to $f_i$ on $X$. After perhaps shrinking $U$ we may hence assume that the restrictions of $F_i$ to any fiber give a base in $H^0(\mathcal{X}_s, kK_{\mathcal{X}_s})$. Let now $\phi_0 := \frac{1}{k} \log(\sum |F_i|^2)$ so that $\phi_0$ is a singular weight on $K_{\mathcal{X}/S}$ over $U$ with positive curvature and such that $\phi_0$ restricts to a Bergman weight at level $k$ on each fiber. In particular,

\begin{equation}
\int_{\mathcal{X}_s} |f|^2 e^{-(k-1)\phi_0} < \infty
\end{equation}

for any $f \in H^0(\mathcal{X}_s, kK_{\mathcal{X}_s})$. Decomposing, as before, $kK_X = (k-1)L + K_X$ with $L = K_X$, but now using Theorem 3.5 in [7] shows that the curvature current of the weight $\phi_1^{(k)} := FS^{(k)} \circ Hilb^{(k)}(\phi_0)$ on $K_{\mathcal{X}/S}$ is positive over $U$. Since, by definition,
$\phi^{(k)}_1$ is still fiber-wise a Bergman weight at level $k$ we may iterate the same argument and conclude that $\phi^{(k)}_m$ has a positive curvature current for any $m$. Now, as explained in the discussion before the statement of the theorem,

$$m \to \infty \implies \sup_{\mathcal{X}_s} |\phi^{(k)}_m - \phi^{(k)}| \to 0,$$

locally uniformly with respect to $s$, where $\phi^{(k)}$ is the unique normalized fiber-wise balanced weight at level $k$. In particular, it follows that $\phi^{(k)}$ has a curvature current which is positive over $S^0$.

Finally, to prove the claimed extension property of $\phi^{(k)}$ over $S - S_0$ first note that, writing $X = \mathcal{X}_s$ for a fixed fiber,

$$\phi^{(k)} \leq \phi^{(k)}_{NS} := \log(\sup_{f \in H^0(X, kK_X)} (|f|^{2/k} / \int_X (f \wedge \overline{f})^{1/k}))$$

where $k\phi^{(k)}_{NS}$ is the weight of the Narasimhan-Simha (NS-) metric on $kK_X/S$.\[34, 25, 31, 3\]. Accepting this for the moment we can use the result in [3] saying that $\phi^{(k)}_{NS}$ is locally bounded from above, with a constant which does not blow-up as $s$ converges to a point in $S - S^0$ (this is proved by an $L^{2/k}$ variant of the local Ohsawa-Takegoshi $L^2$-extension theorem). By the inequality 4.13 it hence follows that $\phi^{(k)}$ is also locally bounded from above by the same constant and then the claimed extension property follows from basic pluripotential theory.

Finally, to prove the inequality 4.13 fix a point $x \in X$. By the extremal definition of Bergman kernels there are sections $f_i$ (depending on $x$) such that $\phi^{(k)}(x) = \frac{1}{k} \log(\frac{1}{N_k} |f_i|^2(x))$ and $\phi^{(k)} = \frac{1}{k} \log(\frac{1}{N_k} \sum_i |f_i|^2)$ on $X$. Since, $\int_X e^{\phi^{(k)}} = 1$ it hence follows that $\int_X (\frac{1}{N_k} f_1 \wedge \overline{f_1})^{1/k} \leq 1$ which finishes the proof of the inequality 4.13 since $f_1/(N_k)^{1/2}$ is a candidate for the sup defining $\phi^{(k)}_{NS}$.

As for the last smoothness statement in the theorem it is proved exactly in Corollary 4.13 using that $\pi_* (kK_{X/S})$ is a locally trivial vector bundle over $S^0$. Indeed, it follows as before that the fiber-wise normalized balanced metrics $H^{(k)}_s$, which by the local freeness may be identified with a family in $GL(N_k)$, form a smooth family. Applying the Fubini-Study map to get $\phi^{(k)}$ then introduces the singular locus described in the statement of the theorem. \hfill $\square$

Remark 49. If one does not invoke the invariance of plurigenera in the proof of the previous theorem then the same argument gives the slightly weaker statement where $S^0$ is replaced by the intersection of $S^0$ with a Zariski open set where $\pi_* (kK_{X/S})$ is a locally trivial vector bundle. If one could then prove that the extension of $\phi^{(k)}$ is such that the integrability condition 4.12 holds over all of $S$, then the invariance of plurigenera would follow from a well-known version of the Ohsawa-Takegoshi extension theorem. It would be interesting to see if this approach is fruitful in the non-projective Kähler case where the invariance of plurigenera is still open. When $\phi^{(k)}$ is replaced by the weight of the $NS$-metric $\phi^{(k)}_{NS}$ (see formula 3.13) this approach was recently used by Tsuji [31] to give a new proof of the invariance of plurigenera (in the projective case).

It should also be pointed out that (singular) Kähler-Einstein metrics and Kähler-Ricci flows have been studied recently for $K_X$ big. For example, using the deep finite generation of the canonical ring there is a unique Kähler-Einstein weight with
minimal singularities which satisfies the Monge-Ampère equation
\[(dd^c \phi_{KE})^n / n! = e^{\phi_{KE}}\]
on a Zariski open set in \(X\). It seems likely that the positivity result in Corollary 35 can be extended to families of such singular weights \(\phi_{KE}\). But there are several regularity issues which need to be dealt with. Moreover, it also seems likely that the canonical balanced weights \(\phi^{(k)}\) converge to \(\phi_{KE}\), when \(K_X\) is big, but this would require a generalization of the convergence results in [3] (which only concern ample line bundles). This latter conjectural convergence should be compared with the convergence of the weight of the NS–metrics proved in [4], saying that \(\phi^{(k)}_{NS}\) converges in \(L^1\) (and uniformly on compacts of an Zariski open set) to
\[\phi_{\text{can}} := \sup\{\psi : \int_X e^\psi = 1\},\]
where the sup is taken over all singular weights \(\psi\) on \(K_X\) with positive curvature current. In particular, \(\phi_{KE} \leq \phi_{\text{can}}\), which is consistent with the inequality 4.13.

4.6. **Comparison with the constant scalar curvature and other settings.**

Given an ample line bundle \(L \to X\) the absolute setting when \(\mu(\phi) := (dd^c \phi)^n / n!\) was studied in depth by Donaldson in [17, 18]. Of course, in this setting the Kähler-Ricci flow is trivial, but the corresponding quantized setting and the study of its large \(k\) limit is highly non-trivial. In fact, it was shown by Donaldson in [17] that, if it is a priori assumed that \(c_1(L)\) contains a Kähler metric \(\omega\) with constant scalar curvature and if \(H^0(TX) = \{0\}\), then the curvature forms of any sequence of balanced weights converge in the \(C^\infty\)–topology to \(\omega\). Moreover, Donaldson showed that such balanced weights do exist for \(k\) sufficiently large. As earlier shown by Zhang this latter fact is equivalent to the polarized variety \((X, kL)\) being stable in the sense of Chow-Mumford (with respect to a certain action of the group \(SL(N_k)\)). An explicit proof of the convergence of the Bergman iteration in this setting was given by Sano [40] (see also [18]).

Note that in this setting the functional \(I_\mu\) is precisely the functional \(E\) (compare the beginning of section 2). Since, \(E\) is well-known to be concave on \(H_L\) with respect to the affine structure and \(E \circ F^{(k)}\) is geodesically convex on \(H^{(k)}\) the convergence of the corresponding Bergman iteration is also a consequence of Proposition 9.

It should also be pointed out that the role of the Kähler-Ricci flow of Kähler metrics in this setting is played by the Calabi flow. Indeed, as shown by Fine [22], the balancing flow, which is a continuous version of Donaldson’s iteration, converges, at this level of Kähler metrics, in the large \(k\) limit to the Calabi flow. More precisely, the balancing flow \(H^{(k)}\) is simply the scaled gradient flow on the symmetric space \(H^{(k)}\) of the functional \(F^{(k)}_\mu\) in this setting and the convergence holds for the curvature forms of the weights \(F^{(k)}_\mu(H_{l})\) in \(H_L\).

**Remark 50.** Another, less studied, setting of geometric relevance (see [6, ?]) appears when we let
\[\mu(\phi) := \frac{1}{N_l} \sum_{i=1}^{N_l} f_i \wedge \bar{f}_i e^{-l\phi}\]
for a fixed integer \(l\) where \(f_i\) is an orthonormal base for \(H^0(lL + K_X)\) equipped with the Hermitian metric induced by \(\phi\). When \(L = -K_X\) and \(l = 1\) this is precisely the normalized \(-K_X\)–setting. In the general case \(I_\mu(\phi)\) is essentially the induced...
metric on the top exterior power of the Hilbert space $H^0(lL + K_X)$. Moreover, as soon as the corresponding functional $F^{(k)}_\mu$ has a critical point and $H^0(TX) = \{0\}$ the assumptions for convergence in Proposition 9 are satisfied (see [6]).

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