KILLING WILD RAMIFICATION

MANISH KUMAR

Abstract. We compute the inertia group of the compositum of wildly ramified Galois covers. It is used to show that even the p-part of the inertia group of a Galois cover of $\mathbb{P}^1$ branched only at infinity can be reduced if there is a jump in the ramification filtration at two (in the lower numbering) and certain linear disjointness statement holds.

1. Introduction

Let $k$ be a field of characteristic $p$. Let $\phi : X \to Y$ be a finite Galois $G$-cover of regular irreducible $k$-curves branched at $\tau \in Y$. Let $I$ be the inertia subgroup of $G$ at a point of $X$ above $\tau$. It is well known, $I = P \rtimes \mu_n$ where $P$ is a $p$-group, $\mu_n$ is a cyclic group of order $n$ and $(n, p) = 1$. Abhyankar’s lemma can be viewed as a tool to modify the tame part of the inertia group. For instance, suppose $k$ contains $n^{th}$-roots of unity. Let $y$ be a regular local parameter of $Y$ at $\tau$. Let $Z \to Y$ be the Kummer cover of regular curves given by the field extension $k(Y)[y^{1/n}]/k(Y)$ and $\tau' \in Z$ be the unique point lying above $\tau$. Then the pullback of the cover $X \to Y$ to $Z$ is a Galois cover of $Z$ branched at $\tau'$. But the inertia group at any point above $\tau'$ is $P$. A wild analogue of this phenomenon appears as Theorem 3.5.

Assume $k$ is also algebraically closed field and let $X \to \mathbb{P}^1$ be a Galois $G$-cover of $k$-curves branched only at $\infty$. Let $I$ be the inertia subgroup at some point above $\infty$ and $P$ be the sylow-$p$ subgroup of $I$. Then noting that the tame fundamental group of $\mathbb{A}^1$ is trivial, it can be seen that the conjugates of $P$ in $G$ generate the whole of $G$. Abhyankar’s inertia conjecture states that the converse should also be true. More precisely, any subgroup of a quasi-$p$ group $G$ of the form $P \rtimes \mu_n$ where $P$ is a $p$-group and $(n, p) = 1$ such that conjugates of $P$ generate $G$ is the inertia group of a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$.

An immediate consequence of a result of Harbater ([Ha1, Theorem 2]) shows that the inertia conjecture is true for every sylow-$p$ subgroup of $G$. In fact Harbater’s result shows that if a $p$-subgroup $P$ of $G$ occurs as the inertia group of a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ and $Q$ is a $p$-subgroup of $G$ containing $P$ then there exists a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ so that the inertia group is $Q$. Proposition 3.4 and a study of wild ramification filtration (Proposition 2.6) enables us to show that in certain cases the given $G$-cover of $\mathbb{P}^1$ can be modified to obtain a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ so that the inertia group of this new cover is smaller than the inertia group $P$ of the original cover (Theorem 3.6).

So far the inertia conjecture is only known for some explicit groups. See for instance [BP, Theorem 5] and [MP, Theorem 1.1].

The author is supported by SFB/TR-45 grant.
2. Filtration on ramification group

For a complete discrete valuation ring (DVR) \( R \), \( v_R \) will denote the valuation associated to \( R \) with the value group \( \mathbb{Z} \). Let \( S/R \) be a finite extension of complete DVRs such that \( \text{QF}(S)/\text{QF}(R) \) is a Galois extension with Galois group \( G \). Let us define a decreasing filtration on \( G \) by

\[ G_i = \{ \sigma \in G : v_S(\sigma x - x) \geq i + 1, \forall x \in S \} \]

Note that \( G_{-1} = G \) and \( G_0 \) is the inertia subgroup. This filtration is called the ramification filtration. For every \( i \), \( G_i \) is a normal subgroup of \( G \). The following are some well-known results.

**Proposition 2.1.** [Ser IV, 4, Proposition 2 and 3] Let \( S/R \) be a finite extension of complete DVRs such that \( \text{Gal}(\text{QF}(S)/\text{QF}(R)) = G \). Let \( H \) be a subgroup \( G \). Let \( K \) be the fixed subfield of \( \text{QF}(S) \) under the action of \( H \). Let \( T \) be the normalization of \( R \) in \( K \). Then \( T \) is a complete DVR, \( \text{Gal}(\text{QF}(S)/K) = H \) and the ramification filtration on \( H \) is induced from that of \( G \), i.e. \( H_i = G_i \cap H \). Moreover, if \( H = G_j \) for some \( j \geq 0 \) then \( (G/H)_i = G_i/H \) for \( i \leq j \) and \( (G/H)_{i} = \{ e \} \) for \( i \geq j \).

**Proposition 2.2.** [Ser IV, 2, Corollary 2 and 3] The quotient group \( G_0/G_1 \) is a prime-to-\( p \) cyclic group and if the residue field has characteristic \( p > 0 \) then for \( i \geq 1 \), \( G_i/G_{i+1} \) is an elementary abelian group of exponent \( p \). In particular \( G_1 \) is a \( p \)-group.

**Lemma 2.3.** Let \( S/R \) be an extension of DVRs such that \( \text{QF}(S)/\text{QF}(R) \) = \( G \). Let \( H \) be a normal subgroup of \( G \) and \( T \) be the normalization of \( R \) in \( \text{QF}(S)^H \) then

\[
\sum_{i=0}^{\infty} (|G_i| - 1) = e_{S/T} \sum_{i=0}^{\infty} (|(G/H)_i| - 1) + \sum_{i=0}^{\infty} (|H_i| - 1)
\]

**Proof.** This follows from the transitivity of the different \( D_{S/R} = D_{S/T}D_{T/R} \) [Ser III, 4, Proposition 8], Hilbert’s different formula \( d_{S/R} = \sum_{i=0}^{\infty} (|G_i| - 1) \) (Sti Theorem 3.8.7]) and \( v_S(x) = v_{S/T}v_T(x) \) for \( x \in \text{QF}(T) \).

**Lemma 2.4.** Let \( S/R \) be a totally ramified extension of complete DVRs over a perfect field \( k \) of characteristic \( p \) > 0. Suppose \( \text{QF}(S) \) is generated over \( \text{QF}(R) \) by \( \alpha \in \text{QF}(S) \) with \( \alpha^p - \alpha \in \text{QF}(R) \) and \( v_R(\alpha^p - \alpha) = -1 \). Then the degree of the different \( d_{S/R} = 2|G| - 2 \).

**Proof.** Note that since \( S/R \) is totally ramified, their residue fields are same and by [Con] the residue field is isomorphic to the field of coefficient of \( R \) and \( S \). Replacing \( k \) by this residue field we may assume that the residue fields of \( S \) and \( R \) are \( k \).

We know that \( |G| = p^l \) for some \( l \geq 0 \). We will prove the lemma by induction on \( l \). If \( l = 0 \) then the statement is trivial. Suppose \( l = 1 \). Then by hypothesis there exists \( \alpha \in \text{QF}(S) \) with \( \alpha^p - \alpha \in R \) and \( v_R(\alpha^p - \alpha) = -1 \). Let \( x = (\alpha^p - \alpha)^{-1} \) and \( y = \alpha^{-1} \) then \( v_S(x) = e_{S/R}v_R(x) = p \) and \( v_S(y) = 1 \). By Cohen structure theorem \( R = k[[x]] \) and \( S = k[[y]] \). Also we have that \( m(y) = 0 \) where \( m(T) = T^p + xT^{p-1} - x \in R[T] \). So \( m(T) \) is a minimal polynomial of \( y \) over \( \text{QF}(R) \).

Now in general assume \( l \geq 1 \). Note that \( G = (\mathbb{Z}/p\mathbb{Z})^l \), so by hypothesis there exist \( \alpha_1, \ldots, \alpha_l \in \text{QF}(S) \) such that
We define $L_n$ in third statement follows from the definition of $v$ for each $0$.

Claim. For each $0 \leq i \leq l - 1$ and $i < j \leq l$, there exist $\beta_{i,j} \in \text{QF}(S)$ such that the following holds

1. $\beta_{i,j}^p - \beta_{i,j} \in L_i$,
2. $v_i(\beta_{i,j}^p - \beta_{i,j}) = -1$,
3. $L_i(\beta_{i,j}; i < j \leq n) = L_n$ for $i < n \leq l - 1$,
4. $v_{i+1}(\beta_{i,i+1}) = -1$.

We define $\gamma_{i+1} = \beta_{i+1}^{-1}$.

Proof of the claim. We shall proof this by induction. For $i = 0$, we take $\beta_{0,j} = \alpha_j$. The first and the second statement is same as the hypothesis of the lemma. The third statement follows from the definition of $L_n$'s. For the fourth statement note that $\beta_{0,1} = \alpha_1$. Since $v_1(\alpha_1) < 0$, we have $v_1(\alpha_1^p) = v_1(\alpha_1 - \alpha_1) = v_1(x^{-1})$. So $v_1(\alpha_1) = p^{-1}v_1(x^{-1}) = p^{-1}v_0(x^{-1}) = -1$.

Suppose the claim is true for a fixed $i + 1 \geq 0$ and $i < l - 1$. Then we have $\beta_{i,j} \in \text{QF}(S)$ for $i < j \leq l$ satisfying the four properties listed in the claim. Also note that $v_i(y_i) = 1$. So $T_i = k[[y_i]]$. Hence we can write explicitly $\beta_{i,j}^p - \beta_{i,j} = c_j y_i^{-1} + d_j + f_j(y_i)$ where $c_j, d_j \in k, c_j \neq 0$ and $f_j(y_i) \in T_i$ has order at least 1. Let $g_j = f_j + f_j^p + f_j^{p^2} + \ldots \in T_i$ then $g_j - g_j^p = f_j$. Let $\gamma_{i,j} = \beta_{i,j} - g_j$. Then $\gamma_{i,j}$ also satisfies the four properties of the claim. Moreover $\gamma_{i,j}^p - \gamma_{i,j} = c_j y_i^{-1} + d_j$. Hence replacing $\beta_{i,j}$ by $\gamma_{i,j}$, we may assume

\[
\beta_{i,j}^p - \beta_{i,j} = c_j y_i^{-1} + d_j
\]

Now for any $j$ such that $i + 1 < j \leq l$. We define $\beta_{i+1,j} = \beta_{i,j} - a_j \beta_{i,i+1}$ where $a_j \in k$ is such that $a_j^p = c_j^{-1} c_{i+1}$. Note that $k$ is perfect so such an $a_j$ exists.

We shall verify that these $\beta_{i+1,j}$ satisfy the four assertions of the claim. Firstly, since $L_{i+1} = L_i(\beta_{i,i+1})$, for $i+1 < n \leq l - 1$ we have

$$L_{i+1}(\beta_{i+1,j}; i + 1 < j \leq n) = L_i(\beta_{i,j}; i < j \leq n) = L_n$$

Hence the third property is satisfied.

We Compute

\[
\beta_{i+1,j}^p - \beta_{i+1,j} = \beta_{i,j}^p - \beta_{i,j} - a_j^p \beta_{i,i+1} + a_j \beta_{i,i+1} = c_j y_i^{-1} + d_j - a_j^p (\beta_{i,i+1} + c_{i+1} y_i^{-1} + d_{i+1}) + a_j \beta_{i,i+1} = (c_j - a_j^p c_{i+1}) y_i^{-1} + d_j - a_j^p d_{i+1} + (a_j - a_j^p) \beta_{i,i+1} = (a_j - a_j^p) \beta_{i,i+1} + d_j - a_j^p d_{i+1}
\]

Hence $\beta_{i+1,j}^p - \beta_{i+1,j} \in L_{i+1}$. If $a_j = a_j^p$ then $\beta_{i+1,j}^p - \beta_{i+1,j} \in k$ but this will lead to a residue field extension for $S/R$ which contradicts the assumption that $S/R$ is totally ramified. Hence $a_j \neq a_j^p$ and

\[
\beta_{i+1,j}^p - \beta_{i+1,j} = (\text{nonzero constant}) \beta_{i,i+1} + \text{constant}
\]
So \( v_{i+1}(\beta_{i+1,j} - \beta_{i+1,j}) = v_{i+1}(\beta_{i,i+1}) = -1 \). We have now verified the first two properties of the claim too.

Finally, \( v_{i+2}(\beta_{i+1,i+2}) = v_{i+2}(\beta_{i+1,i+2} - \beta_{i+1,i+2}) = v_{i+2}(\beta_{i,i+1}) \). So we deduce that \( v_{i+2}(\beta_{i+1,i+2}) = p^{-1}v_{i+2}(\beta_{i,i+1}) = p^{-1}pv_{i+1}(\beta_{i,i+1}) = -1 \). This completes the proof of the claim. \( \square \)

The field extension \( L_{i-1}/QF(R) \) is Galois with Galois group \((\mathbb{Z}/p\mathbb{Z})^{l-1}\) and \( \text{Gal}(QF(S)/L_{i-1}) = \mathbb{Z}/p\mathbb{Z} \). Moreover, both \( T_{i-1}/R \) and \( S/T_{i-1} \) are totally ramified extension. Note that \( L_{i-1} = QF(R)(\alpha_1, \ldots, \alpha_{i-1}) \). So by induction hypothesis \( d_{T_{i-1}/R} = 2p^{l-1} - 2 \).

Since \( QF(S) = L_{i-1}(\beta_{i-1,i}) \), \( \beta_{i-1,i} \in L_{i-1} \) and \( v_{i-1}(\beta_{i-1,i}) = -1 \), we have \( d_{S/T_{i-1}} = 2p - 2 \) by “\( l = 1 \) case”.

Finally using the transitivity of different, we see that \( d_{S/R} = e_{S/T_{i-1}}d_{T_{i-1}/R} + d_{S/T_{i-1}} = p(2p^{l-1} - 2) - 2p - 2 = 2p^l - 2 \). This completes the proof of the lemma. \( \square \)

**Proposition 2.5.** Let \( i \geq 1 \) and \( S/R \) be a finite extension of complete DVRs over a perfect field \( k \) of characteristic \( p \) such that \( \text{Gal}(QF(S)/QF(R)) = G = G_i \). Let \( L \) be the subfield of \( QF(S) \) generated over \( QF(R) \) by all \( \alpha \in QF(S) \) such that \( v_R(\alpha^p - \alpha) = -i \). Then \( G_{i+1} \supset \text{Gal}(QF(S)/L) \).

**Proof.** Let \( L' = QF(S)^{G_{i+1}} \) and \( H = \text{Gal}(QF(S)/L) \leq G \). Let \( T \) and \( T' \) be the normalization of \( R \) in \( L \) and \( L' \) respectively. Since \( G_{i+1} \) is a normal subgroup of \( G \), the extension \( L'/QF(R) \) is Galois and \( \text{Gal}(L'/QF(R)) = G_{i+1} \) (say). Moreover the ramification filtration on \( \bar{G} \) is given by \( G_i = G \) and \( G_{i+1} = \{e\} \) (Proposition 2.3). If \( G_{i+1} = G \) then \( H \subset G_{i+1} \) and we are done. So we may assume \( G_{i+1} \neq G \). By Proposition 2.2 \( \bar{G} \neq \{e\} \) is isomorphic to the direct sum of copies of \( \mathbb{Z}/p\mathbb{Z} \).

Let \( L'' \subset L' \) be any \( \mathbb{Z}/p\mathbb{Z} \)-extension of \( QF(R) \). By Artin-Schrier theory there exists \( \alpha \in L'' \setminus QF(R) \) such that \( \beta = \alpha^p - \alpha \in QF(R) \). Let \( x \) be a local parameter of \( R \) then \( R = k[x] \). If \( v_R(\beta) > 0 \) then \( \alpha = c - \beta - \beta^p - \beta^p \beta^p - \ldots \in R \) for some \( c \in \mathbb{F}_p \). So \( v_R(\beta) \leq 0 \). Moreover since \( G_0 = G \), \( S/R \) is totally ramified. So \( v_R(\beta) \neq 0 \) and hence \( v_R(\beta^p) \leq 0 \). If \( v_R(\beta) \) is a multiple of \( p \) then \( \beta = c_0x^l + c_1x^{l+1} + \ldots \). for some integer \( l < 0 \). Let \( c \in k \) be such that \( c^p = 0 \) and let \( \alpha' = \alpha - cx' \). Then \( \beta' = \alpha'^p - \alpha' = -c_0x^l + c_1x^{l+1} + \ldots \). \( v_R(\beta') \) and \( L'' = QF(R)(\alpha) = QF(R)(\alpha') \). Hence by such modifications we may assume \( v_R(\alpha^p - \alpha) = -r < 0 \) is coprime to \( p \). Let \( T'' \) be the normalization of \( R \) in \( L'' \). By explicit calculation of the different and using Hilbert’s different formula, the degree of the different \( d_{T'/R} = (r+1)(p-1) \). Since \( G_{i+1} \) is trivial and \( G_i = \bar{G} \), by Hilbert’s different formula \( d_{T'/R} = (i+1)|\bar{G}| - i - 1 \). Let \( H \) be the index \( p \) subgroup of \( G \) such that \( L'' = L'H \). Then the ramification filtration on \( H \) (coming from the extension \( T'/T'' \)) is induced from \( \bar{G} \). Hence \( d_{T'/T''} = (i+1)|H| - i - 1 \). Using Lemma 2.3 and \( e_{T'/T''} = |H| \), we obtain

\[
(i+1)|\bar{G}| - i - 1 = |H|(r+1)(p-1) + (i+1)|H| - i - 1
\]

Using \( |\bar{G}| = |P| \) above and solving for \( r \), one gets \( r = i \). Hence \( L'' \subset L \). Since \( L'' \) was an arbitrary \( \mathbb{Z}/p\mathbb{Z} \)-extension of \( QF(R) \) contained in \( L' \) and \( L' \) is generated by such \( \mathbb{Z}/p\mathbb{Z} \)-extensions, we have that \( L' \subset L \). So by the fundamental theorem of Galois theory \( H \subset G_2 \). \( \square \)
Proposition 2.6. Let $S/R$ be a finite extension of complete DVRs over a perfect field $k$ of characteristic $p$ such that $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G = G_1$. Let $L$ be the subfield of $\text{QF}(S)$ generated over $\text{QF}(R)$ by all $\alpha \in \text{QF}(S)$ such that $v_R(\alpha^p - \alpha) = -1$. Then $G_2 = \text{Gal}(\text{QF}(S)/L)$.

Proof. In view of Proposition 2.5 it is enough to show $G_2 \subset H := \text{Gal}(\text{QF}(S)/L)$. Let $T$ be the normalization of $R$ in $L$. Note that $L/\text{QF}(R)$ is a Galois extension with Galois group $G/H$. By Lemma 2.4 $d_{T/R} = 2|G/H| - 2$. So using Lemma 2.3 one gets:

$$2|G| - 2 + \sum_{i=2}^{\infty}(|G_i| - 1) = |H|(2|G/H| - 2) + 2|H| - 2 + \sum_{i=2}^{\infty}(|H_i| - 1)$$

Rearranging and using $|G| = |G/H| \cdot |H|$, the above reduces to the following

$$2|G/H| - 2 + |H| - \sum_{i=2}^{\infty}(|G_i| - |H_i|) = 2|G/H| - 2$$

So $G_i = H_i$ for $i \geq 2$. Hence $G_2 = H \cap G_2$ which implies $G_2 \subset H$. \qed

Corollary 2.7. Let $S/R$ be a finite extension of complete DVRs over a perfect field $k$ of characteristic $p$ such that $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G = F^3G$. Then $F^3G \neq G$ iff there exists $\alpha \in \text{QF}(S)$ such that $\alpha^p - \alpha \in \text{QF}(R)$ and $v_R(\alpha^p - \alpha) = -1$.

3. Reducing Inertia

For a local ring $R$, let $m_R$ denote the maximal ideal of $R$. In this section we shall show how even the wild part of inertia subgroup of a Galois cover can be reduced. We begin with the following lemma.

Lemma 3.1. Let $R$ be a DVR and $K$ be the quotient field of $R$. Let $L$ and $M$ be finite separable extensions of $K$ and $\hat{\Omega} = LM$ their compositum. Let $A$ be a DVR dominating $R$ with quotient field $\hat{\Omega}$. Note that $S = A \cap L$ and $T = A \cap M$ are DVRs. Let $\hat{K}$, $\hat{L}$, $\hat{M}$ and $\hat{\Omega}$ be the quotient field of the complete DVRs $\hat{R}$, $\hat{S}$, $\hat{T}$ and $\hat{A}$ respectively. If $A/m_A = S/m_S$ then $\hat{\Omega} = \hat{LM}$. Here all fields are viewed as subfields of an algebraic closure of $\hat{K}$.

Proof. Note that $\hat{L}$ and $\hat{M}$ are contained in $\hat{\Omega}$. So $\hat{LM} \subset \hat{\Omega}$. Let $\pi_A$ denote a uniformizing parameter of $A$. Then $\pi_A \in LM \subset \hat{LM}$. So it is enough to show that $\hat{\Omega} = \hat{L}[[\pi_A]]$. Note that $\hat{S}[[\pi_A]]$ is a finite $\hat{S}$-module, hence it is a complete DVR [Coh]. Also $\hat{S} \subset S[[\pi_A]] \subset \hat{A}$ and $\pi_A$ generate the maximal ideal of $\hat{A}$, hence $\pi_A S$ is the maximal ideal of $S[[\pi_A]]$. Moreover, the residue field of $\hat{S}$ is equal to $S/m_S = A/m_A$ which is same as the residue field of $\hat{A}$. Hence the residue field of $\hat{S}[[\pi_A]]$ is also same as the residue field of $\hat{A}$. So $\hat{S}[[\pi_A]] = \hat{A}$ (by [Coh] Lemma 4). Hence the quotient field of $\hat{S}[[\pi_A]]$ is $\hat{\Omega}$. But that means $\hat{L}[[\pi_A]] = \hat{\Omega}$. \qed

Corollary 3.2. Let the notation be as in the above theorem. If $\hat{L} \subset \hat{M}$ then $A/T$ is an unramified extension.

Proof. Since $\Omega/M$ is finite extension, so is $\hat{\Omega}/\hat{M}$. Hence $\hat{A}$ is a finite $\hat{T}$-module. By the above lemma and the hypothesis $\hat{\Omega} = \hat{M}$. So $\hat{A} = \hat{T}$, i.e. $A/T$ is unramified. \qed

Let $k$ be any field.
**Theorem 3.3.** Let \( X \to Y \) and \( Z \to Y \) be Galois covers of regular \( k \)-curves branched at \( \tau \in Y \). Let \( \tau_x \) and \( \tau_z \) be closed points of \( X \) and \( Z \) respectively, lying above \( \tau \). Suppose \( k(\tau_x) = k(\tau) \). Let \( W \) be an irreducible dominating component of the normalization of \( X \times_Y Z \) containing the closed point \( (\tau_x, \tau_z) \). Then \( W \to Y \) is a Galois cover ramified at \( \tau \) and the decomposition subgroup of the cover at \( \tau \) is the Galois group of the field extension \( QF(\mathcal{O}_{X,\tau_x}) QF(\mathcal{O}_{Z,\tau_z}) / QF(\mathcal{O}_{Y,\tau}) \).

**Proof.** Let \( R = \mathcal{O}_{Y,\tau} \). Note that \( R \) is a DVR. Let \( K \) be the quotient field of \( R \). Let \( L \) and \( M \) be the function field of \( X \) and \( Z \) respectively and \( \Omega = LM \) be their compositum. By definition \( W \) is an irreducible regular curve with function field \( \Omega \) and the two projections give the covering morphisms to \( X \) and \( Y \). Let \( \tau_w \) denote the closed point \( (\tau_x, \tau_z) \in W \) and \( A = \mathcal{O}_{W,\tau_w} \). Since \( \tau_w \) lies above \( \tau_x \) under the covering \( W \to X \) and above \( \tau_z \) under the covering \( W \to Z \), we have that \( A \cap L = \mathcal{O}_{X,\tau_x} (= S \text{ say}) \) and \( A \cap M = \mathcal{O}_{Z,\tau_z} (= T \text{ say}) \). Since \( k(\tau_x) = k(\tau) \) and \( k(W) = k(X)k(Z) \) we get that \( k(\tau_w) = k(\tau_x)k(\tau_z) = k(\tau) \). But this is same as \( A/m_A = S/m_S \). So using the above lemma, we conclude that \( \hat{L}M = \hat{\Omega} \).

The decomposition group of the cover \( W \to Y \) at \( \tau_w \) is given by the Galois group of the field extension \( \hat{\Omega}/K \) ([Bou, Corollary 4, Section 8.6, Chapter 6]). This completes the proof because \( \Omega = \hat{L}M = QF(\mathcal{O}_{X,\tau_x}) QF(\mathcal{O}_{Z,\tau_z}) \) and \( \hat{K} = QF(\mathcal{O}_{Y,\tau}) \).

**Proposition 3.4.** Let \( \Phi : X \to Y \) be a \( G \)-cover of regular \( k \)-curves ramified at \( \tau_x \in X \) and let \( \tau = \Phi(\tau_x) \). Let \( G_{\tau} \) and \( I_{\tau} \) be the decomposition subgroup and the inertia subgroup respectively at \( \tau_x \). Let \( N \leq I_{\tau} \) be a normal subgroup of \( G_{\tau} \).

Suppose there exist a Galois cover \( \Psi : Z \to Y \) of regular \( k \)-curves ramified at \( \tau_z \in Z \) with \( \Psi(\tau_z) = \tau \) such that \( k(\tau_z) = k(\tau) \) and the fixed field \( QF(\mathcal{O}_{Z,\tau_z})^N \) is same as the compositum \( QF(\mathcal{O}_{Z,\tau_z})k(\tau_x) \). Let \( W \) be an irreducible dominating component of the normalization of \( X \times_Y Z \) containing \( (\tau_x, \tau_z) \). Then the natural morphism \( W \to Z \) is a Galois cover. The inertia group and the decomposition group at the point \( (\tau_x, \tau_z) \) are \( N \) and an extension of \( N \) by \( \text{Gal}(k(\tau_x)/k(\tau)) \) respectively.

**Proof.** Let \( \tau_w \in W \) be the point \( (\tau_x, \tau_z) \). Applying Theorem 3.3 we obtain that the decomposition group of the Galois cover \( W \to Y \) at \( \tau_w \) is isomorphic to \( G_{\tau_w} = \text{Gal}(QF(\mathcal{O}_{X,\tau_x}) QF(\mathcal{O}_{Z,\tau_z}) / QF(\mathcal{O}_{Y,\tau})) \). Since \( QF(\mathcal{O}_{Z,\tau_z}) \subset QF(\mathcal{O}_{X,\tau_x}) \), we have \( G_{\tau_w} = G_{\tau} = \text{Gal}(QF(\mathcal{O}_{X,\tau_x}) / QF(\mathcal{O}_{Y,\tau})) \). Since \( k(\tau_z) = k(\tau) \), the inertia group and the decomposition group of the cover \( Z \to Y \) at \( \tau_z \) are both \( \text{Gal}(QF(\mathcal{O}_{Z,\tau_z}) / QF(\mathcal{O}_{Y,\tau})) \). Since \( QF(\mathcal{O}_{Z,\tau_z})^N = QF(\mathcal{O}_{Z,\tau_z})k(\tau_x) \) we also obtain that \( \text{Gal}(QF(\mathcal{O}_{Z,\tau_z})k(\tau_x) / QF(\mathcal{O}_{Y,\tau})) = G_{\tau}/N \). Moreover, we have \( G_{\tau}/I_{\tau} = \text{Gal}(k(\tau_x)/k(\tau)) = \text{Gal}(k(\tau_x) QF(\mathcal{O}_{Y,\tau}) / QF(\mathcal{O}_{Y,\tau})) \). Since \( \mathcal{O}_{Z,\tau_x}/\mathcal{O}_{Y,\tau} \) is totally...
ramified, $\text{QF}(\hat{O}_{Z,\tau_x})$, $k(\tau_x)/\text{QF}(\hat{O}_{Y,\tau})$ are linearly disjoint over $\text{QF}(\hat{O}_{Y,\tau})$.

\[
\begin{array}{c}
\text{QF}(\hat{O}_{X,\tau_x}) \\
N \downarrow \\
\text{QF}(\hat{O}_{Z,\tau_x})k(\tau_x) \\
G_r/I_r \quad G_r/N \\
\text{QF}(\hat{O}_{Y,\tau})k(\tau_x) \\
\text{QF}(\hat{O}_{Y,\tau})
\end{array}
\]

So $\text{Gal}(\text{QF}(\hat{O}_{Z,\tau_x})k(\tau_x)/\text{QF}(\hat{O}_{Z,\tau_x})) = \text{Gal}(k(\tau_x)/k(\tau))$. So the decomposition group of $W \to Z$ is $\text{Gal}(\text{QF}(\hat{O}_{X,\tau_x})/\text{QF}(\hat{O}_{Z,\tau_x}))$ which is an extension of $N$ by $\text{Gal}(k(\tau_x)/k(\tau))$ and the inertia group is $\text{Gal}(\text{QF}(\hat{O}_{X,\tau_x})/\text{QF}(\hat{O}_{Z,\tau_x})k(\tau_x)) = N$. □

Let $k$ be an algebraically closed field of characteristic $p > 0$.

**Theorem 3.5.** Let $\Phi : X \to Y$ be a $G$-Galois cover of regular $k$-curves. Let $\tau_x \in X$ be a ramification point and $\tau = \Phi(\tau_x)$. Let $I$ be the inertia group of $\Phi$ at $\tau_x$. There exists a cover $\Psi : Z \to Y$ of degree $[I]$, such that the cover $W \to Z$ is étale over $\tau_z$ where $W$ is the normalization of $X \times_Y Z$ and $\tau_z \in \overline{Z}$ is such that $\Psi(\tau_z) = \tau$. Moreover if there are no non-trivial homomorphism from $G \to P$ where $P$ is a $p$-Sylow subgroup of $I$ then $W \to Z$ is a $G$-cover of irreducible regular $k$-curves.

**Proof.** Since $I$ is the inertia group, it is isomorphic to $P \times \mu_n$ where $(p, n) = 1$ and $\mu_n$ is a cyclic group of order $n$. Let $y$ be a local coordinate of $Y$ at $\tau$ such that $k(Y)[y^{1/n}] \cap k(X) = k(Y)$. Let $Z_1$ be the normalization of $Y$ in $k(Y)[y^{1/n}]$. Then $Z_1 \to Y$ is a $\mu_n$-cover branched at $\tau$ such that $k(Z_1)$ and $k(X)$ are linearly disjoint over $k(Y)$. Let $\tau_1 \in Z_1$ be a point lying above $\tau$. Let $X_1$ be the normalization of $X \times_Y Z_1$. Then by the above theorem $\Phi_1 : X_1 \to Z_1$ is a $G$-cover of irreducible regular $k$-curves and the inertia group at $(\tau_x, \tau_1)$ is $P$.

Let $Y_1 = Z_1$, $\tau_{x1} = (\tau_x, \tau_1)$ and $\tau_1 = \tau_{z1}$. Then $\Phi_1 : X_1 \to Y_1$ is a $G$-cover with $\Phi_1(\tau_{x1}) = \tau_1$ and the inertia group of this cover at $\tau_{x1}$ is $P$. Let $y_1$ be a regular parameter of $Y_1$ at $\tau_1$. Then $k(Y_1)/k(y_1)$ is a finite extension. Since $Y_1$ is a regular curve, we get a finite morphism $\alpha : Y_1 \to \mathbb{P}^1_{y_1}$ such that $\alpha(\tau_1)$ is the point $y_1 = 0$ and $\alpha$ is étale at $\tau_1$ (as $\hat{O}_{Y_1,\tau_1} = k[[y_1]]$).

Note that $\text{QF}(\hat{O}_{Y,\tau})/k((y_1))$ is a $P$-extension. By [Ha, Cor 2.4], there exist a $P$-cover $V \to \mathbb{P}^1_{y_1}$ branched only at $y_1 = 0$ (where it is totally ramified) such that $\text{QF}(\hat{O}_{V,\theta}) = \text{QF}(\hat{O}_{X_1,\tau_{x1}})$ as extensions of $k((y_1))$. Here $\theta$ is the unique point in $V$ lying above $y_1 = 0$. Since $V \to \mathbb{P}^1_{y_1}$ is totally ramified over $y_1 = 0$ and $Y_1 \to \mathbb{P}^1_{y_1}$ is étale over $y_1 = 0$, the two covers are linearly disjoint. Let $Z$ be the normalization of $V \times_{\mathbb{P}^1_{y_1}} Y_1$. Then the projection map $Z \to Y_1$ is a $P$-cover. Let $\tau_z \in Z$ be the closed point $(\theta, \tau_1)$. By Lemma 3.1, $\text{QF}(\hat{O}_{Z,\tau_z}) = \text{QF}(\hat{O}_{V,\theta})\text{QF}(\hat{O}_{Y_1,\tau_1}) = \text{QF}(\hat{O}_{X_1,\tau_{x1}})$. Applying Proposition 3.3 with $N = \{e\}$, we get that an irreducible dominating component $W$ of the normalization of $X_1 \times_Y Z$ is a Galois cover of $Z$ such that
the inertia group over $\tau_z$ is $\{e\}$. Hence the normalization of $X_1 \times_Y Z$ is a cover of $Z$ étale over $\tau_z$.

Moreover, there are no nontrivial homomorphism from $G$ to $P$ implies that $k(\tau)$ and $k(X_1)$ are linearly disjoint over $k(Y_1)$. Hence $W \to Z$ is a $G$-cover. We take $Z \to Y$ to be the composition $Z \to Y_1 \to Y$. Note that the morphism $X \times_Y Z \to Z$ is same as $X_1 \times_Y Z \to Z$ and the degree of the morphism $Z \to Y$ is $|P|n = |I|$. $\square$

**Theorem 3.6.** Let $\Phi : X \to \mathbb{P}^1$ be a $G$-Galois cover of regular $k$-curves. Suppose $\Phi$ is branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of $\Phi$ over $\infty$ is $I$. Let $P$ be a subgroup of $I$ such that $I_1 \supset P \supset I_2$. Suppose there are no nontrivial homomorphism from $G$ to $P$. Then there exist a $G$-cover $W \to \mathbb{P}^1$ ramified only at $\infty$ and the inertia group at $\infty$ is $P$.

**Proof.** Let $n = |I : I_1|$ be the tame ramification index of $\Phi$ at $\infty$. Let $x$ be a local coordinate on $\mathbb{P}^1$ and the point $\infty$ is $x = \infty$. Let $\mathbb{P}^1_y \to \mathbb{P}^1_x$ be the Kummer cover obtained by sending $y^n$ to $x$. Since $\Phi$ is étale at $x = 0$ and the cover $\mathbb{P}^1_y \to \mathbb{P}^1_x$ is totally ramified at $x = 0$ the two covers are linearly disjoint. So letting $W$ to be the normalization of $X \times_{\mathbb{P}^1} \mathbb{P}^1_y$, we obtain a $G$-cover $\Phi_1 : W \to \mathbb{P}^1_y$ of regular $k$-curves. Moreover by Abhyankar’s lemma $\Phi_1$ is ramified only at $y = \infty$ and the inertia group of $\Phi_1$ at $y = \infty$ is same the subgroup $I_1$ of $I$. So replacing $\Phi$ by $\Phi_1$, we may assume $I = I_1$. Also since $I_1/I_2$ is abelian, $P$ is a normal subgroup of $I$.

Let $\tau \in X$ be a point above $x = \infty$. Let $S = \hat{O}_{X,\tau}$ and $R = \hat{O}_{\mathbb{P}^1,\infty}$ then $R = k[[x^{-1}]]$ and $\text{Gal}(\overline{F}(S)/\overline{F}(R)) = I$. Let $L = \text{GF}(S)^P$. Then by Proposition 2.6 $L = \text{GF}(R)(\alpha_1, \ldots, \alpha_l)$ where $\alpha_i \in \text{GF}(S)$ is such that $v_R(\alpha_i^2 - \alpha_i) = -1$ for $1 \leq i \leq l$. Let $T$ be the normalization of $R$ in $L$. Then $\text{Spec}(T)$ is a principal $P$-cover of $\text{Spec}(R)$. By [14 Corollary 2.4], this extends to a $P$-cover $\Psi : Z \to \mathbb{P}^1$ ramified only at $x = \infty$ where it is totally ramified. Let $\tau_z \in Z$ be the point lying above $x = \infty$ then $\text{GF}(\hat{O}_{Z,\tau_z}) = L = \text{GF}(S)^P$. By Lemma 2.2 $d_{T/R} = 2|P| - 2$. So by Riemann-Hurwitz formula, the genus of $Z$ is given by

$$2g_Z - 2 = |P|(0 - 2) + d_{T/R}$$

Hence $g_Z = 0$. So $Z$ is isomorphic to $\mathbb{P}^1$.

Since there are no nontrivial homomorphism from $G$ to $P$, $\Phi$ and $\Psi$ are linearly disjoint covers of $\mathbb{P}^1_z$. Let $W$ be the normalization of $X \times_{\mathbb{P}^1} Z$. Now we are in the situation of Proposition 3.3. Hence the $G$-cover $W \to Z$ is ramified only at $\tau_z$ and the inertia group at $\tau_z$ is $P$. This completes the proof as $Z$ is isomorphic to $\mathbb{P}^1$. $\square$

**Remark 3.7.** Note that if $G$ is a simple group different from $Z/pZ$ then there are no nontrivial homomorphism from $G$ to $P$. Hence the above results apply in this scenario.

**Corollary 3.8.** Let $\Phi : X \to \mathbb{P}^1$ be a $G$-Galois cover of regular $k$-curves branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of $\Phi$ over $\infty$ is $I$. Suppose there are no nontrivial homomorphism from $G$ to $I_2$. Then conjugates of $I_2$ generate $G$.

**Proof.** Applying the above theorem with $P = I_2$, we get an étale $G$-cover of $\mathbb{A}^1$ with the inertia group $I_2$ at $\infty$. Hence the conjugates of $I_2$ generate $G$ since a nontrivial étale cover of $\mathbb{A}^1$ must be wildly ramified over $\infty$. $\square$
References

[Bou] Nicholas Bourbaki *Commutative algebra. Chapters 1–7*. Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. xxiv+625 pp.

[BP] I. I. Bouw and R. J. Pries *Rigidity, reduction, and ramification*, Math. Ann. 326:4 (2003), 803-824.

[MP] Jeremy Muskat and Rachel Pries *Title: Alternating group covers of the affine line*, http://arxiv.org/abs/0908.2140v2 (arxiv preprint).

[Coh] I. S. Cohen *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. 59, (1946). 541-106.

[Ha] David Harbater *Moduli of p-covers of curves*, Comm. Algebra 8 (1980), no. 12, 10951122.

[Ha1] David Harbater *Embedding problems and adding branch points*, in “Aspects of Galois Theory”, London Mathematical Society Lecture Note series, 256 Cambridge University Press, pages 119-143, 1999.

[Ser] Jean-Pierre Serre *Local fields*, Translated from the French by Marvin Jay Greenberg. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979. viii+241 pp.

[Sti] Henning Stichtenoth *Algebraic function fields and codes*, Second edition. Graduate Texts in Mathematics, 254. Springer-Verlag, Berlin, 2009. xiv+355 pp.

Department of Mathematics, Universität Duisburg-Essen, 45117 Essen, Germany
E-mail address: manish.kumar@uni-due.de