Improved variational principle for bounds on energy dissipation in turbulent shear flow

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Abstract

We extend the Doering-Constantin approach to upper bounds on energy dissipation in turbulent flows by introducing a balance parameter into the variational principle. This parameter governs the relative weight of different contributions to the dissipation rate. Its optimization leads to improved bounds without entailing additional technical difficulties. For plane Couette flow, the high-Re-bounds obtainable with one-dimensional background flows are methodically lowered by a factor of $27/32$.

Keywords: Turbulent shear flows, energy dissipation, background flow method, rigorous estimates.

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I. INTRODUCTION

While it is not feasible to obtain exact solutions to the equations of motion for turbulent flows, it is possible to derive mathematically rigorous upper bounds on certain quantities characterizing turbulent flow fields, such as the rate of energy dissipation in shear flows or the rate of heat transport in turbulent convection. A detailed theory of upper bounds has been developed by Howard [1] and Busse [2–4]. The bounds obtained by their method, although still considerably higher than experimentally observed values, have until now resisted any attempt of further improvement (see, e.g., [5]).

Recently a quite different approach has sparked renewed interest in the theory of upper bounds. Instead of starting from the usual Reynolds decomposition, Doering and Constantin [6,7] utilized an idea put forward already by Hopf [8] and decomposed a turbulent flow field into a stationary “background flow” and a fluctuating component. The background flow should be regarded as an arbitrary mathematical auxiliary field that merely has to carry the boundary conditions of the physical flow field, rather than as a time average of the actual flow. This method turns out to be fairly versatile [7,9–12]. A given background flow immediately yields an upper bound, provided it satisfies a certain spectral constraint. The best bounds obtainable in this way can therefore be computed from a variational principle for the background flow [7]. A formal connection between this approach and the Howard-Busse theory has been elucidated by Kerswell [13].

The objective of the present paper is to point out that the variational principle suggested by Doering and Constantin can still be improved, without introducing additional technical complications. Although our arguments are of a more general kind, we restrict the discussion to the concrete example of energy dissipation in plane Couette flow. In this case the benefits of our formulation of the variational principle are particularly obvious: the laminar flow now becomes an admissible background flow for Reynolds numbers up to the energy stability border, which is just what should be required on intuitive grounds. This eliminates a shortcoming of the original Doering-Constantin approach. For asymptotically high Reynolds numbers, and employing one-dimensional background flows only, the improved principle yields upper bounds on the rate of energy dissipation that are systematically by a factor of $27/32$ lower than those that can be computed from the original principle.

II. FORMULATION OF THE VARIATIONAL PRINCIPLE

We consider an incompressible fluid confined between two infinitely extended rigid plates. The lower plate at $z = 0$ is at rest, whereas the upper one at $z = h$ moves with constant velocity $U$ in the positive $x$-direction. The velocity field $u(x, t)$ satisfies the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \tag{1}$$

with

$$\nabla \cdot u = 0 , \tag{2}$$

where $p$ is the kinematic pressure and $\nu$ the kinematic viscosity. We impose no-slip boundary conditions (b.c.) on $u$ at $z = 0$ and $z = h$,
\( \mathbf{u}(x, y, 0, t) = \mathbf{0}, \ \mathbf{u}(x, y, h, t) = U\hat{x} \) 

(\( \hat{x} \) is the unit vector in x-direction), and periodic b.c. on \( \mathbf{u} \) and \( p \) in x- and y-direction; the periodicity lengths are \( L_x \) and \( L_y \). The time-averaged energy dissipation rate per mass is given by

\[
\varepsilon_T \equiv \left \langle \frac{\langle \nu |\nabla \mathbf{u}|^2 \rangle}{T} \right \rangle = \frac{1}{T} \int_0^T dt \left[ \nu \int_{\Omega} d^3x \ u_{ij} v_{ij} \right].
\]

We employ the notation \( \langle \cdot \rangle = \frac{1}{\Omega} \int_{\Omega} d^3x (\cdot) \) for the volume average and \( \langle \cdot \rangle_T = \frac{1}{T} \int_0^T dt (\cdot) \) for the time average, \( \Omega = L_x L_y h \) denotes the periodicity volume, and \( u_{ij} \) symbolizes \( \partial_j u_i(x, y, z, t) \) with \( i, j = x, y, z \); summation over repeated indices is implied. The goal is to formulate a variational principle for rigorous upper bounds on the long time limit of \( \varepsilon_T \),

\[
\varepsilon \equiv \limsup_{T \to \infty} \varepsilon_T,
\]

or on the non-dimensionalized quantity

\[
c_\varepsilon(Re) \equiv \frac{\varepsilon}{U^3 h^{-1}},
\]

where \( Re = U h / \nu \) is the Reynolds number.

The approach by Doering and Constantin [6,7] rests on a decomposition of the velocity field \( \mathbf{u}(x, t) \) into a background flow and fluctuations,

\[
\mathbf{u}(x, t) = \mathbf{U}(x) + \mathbf{v}(x, t).
\]

The background flow \( \mathbf{U} \) is a stationary and divergence-free vector field satisfying the physical b.c.: \( \mathbf{U}(x, y, 0) = \mathbf{0}, \ \mathbf{U}(x, y, h) = U\hat{x}, \) and \( \mathbf{U} \) is assumed to be periodic in x- and y-direction; otherwise it is completely arbitrary. For the divergence-free fluctuations \( \mathbf{v} \) we then have homogeneous b.c. for all instants \( t \geq 0 \), i.e., \( \mathbf{v}(x, y, 0, t) = \mathbf{v}(x, y, h, t) = \mathbf{0} \), and \( \mathbf{v} \) is periodic in x- and y-direction. With the usual manipulations one arrives at the following equations for \( \mathbf{u} \) and \( \mathbf{v} \):

\[
\nu \left \langle |\nabla \mathbf{u}|^2 \right \rangle = \nu \left \langle |\nabla \mathbf{U}|^2 \right \rangle + 2 \nu \left \langle U_{ij} v_{ij} \right \rangle + \nu \left \langle |\nabla \mathbf{v}|^2 \right \rangle,
\]

\[
\frac{d}{dt} \left \langle \frac{1}{2} \mathbf{v}^2 \right \rangle + \left \langle \mathbf{v} \cdot (\mathbf{U} \cdot \nabla \mathbf{U}) \right \rangle + \left \langle \mathbf{v} \cdot (\nabla \mathbf{U})_{sym} \cdot \mathbf{v} \right \rangle = -\nu \left \langle U_{ij} v_{ij} \right \rangle - \nu \left \langle |\nabla \mathbf{v}|^2 \right \rangle,
\]

where \( [\nabla \mathbf{U}]_{ij} \equiv (U_{j|i} + U_{i|j})/2 \). At this point, Doering and Constantin use eq. (8) to eliminate the cross background-fluctuation term in eq. (9), namely, they form \( [\text{eq. (8)}] + 2 \cdot [\text{eq. (9)}] \). It is crucial to note that this implies putting a certain fixed weight on the different contributions to the resulting expression for \( \varepsilon_T \). In order to avoid such a weighting we introduce a new degree of freedom and consider

\[
[\text{eq. (8)}] + a \cdot [\text{eq. (9)}] \quad a > 1
\]
with the *balance parameter* $a$. Apart from this modification, we now follow closely the spirit of the original Doering-Constantin approach, which can always be recovered by setting $a = 2$. Equation (10) leads to the *energy balance equation*

$$
\varepsilon_T = \nu \langle |\nabla U|^2 \rangle - \frac{a}{T} \left\langle \frac{1}{2} \mathbf{v}(-, T)^2 \right\rangle + \frac{a}{T} \left\langle \frac{1}{2} \mathbf{v}(\cdot, 0)^2 \right\rangle
$$

$$
- a \left\langle \frac{1}{\Omega} \int_\Omega d^3x \left[ \frac{a-1}{a} \nu |\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla U)_{\text{sym}} \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v} \right] \right\rangle_T
$$

with

$$
f \equiv \nabla \cdot \nabla U - \frac{a-2}{a} \nu \Delta U ,
$$

which becomes the inequality (see (5))

$$
\varepsilon \leq \nu \langle |\nabla U|^2 \rangle - a \lim \inf_{T \to \infty} \left\langle \frac{1}{\Omega} \int_\Omega d^3x \left[ \frac{a-1}{a} \nu |\nabla \mathbf{v}|^2 + \mathbf{v} \cdot (\nabla U)_{\text{sym}} \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v} \right] \right\rangle_T .
$$

For the evaluation of (13) we have to distinguish two different cases.

**Case a) $f$ is a gradient**

In this case the term linear in $\mathbf{v}$ on the rhs. of (13) vanishes and the resulting bound on the energy dissipation rate is given by

$$
\varepsilon \leq \inf_{U,a>1} \left\{ \nu \langle |\nabla U|^2 \rangle \right\} ,
$$

provided $U$ complies with the following constraints:

i) $U$ is a divergence-free vector field satisfying the physical b.c. and $f$ is a gradient,

ii) the functional

$$
H_{U,a}\{\mathbf{w}\} \equiv \frac{1}{\Omega} \int_\Omega d^3x \left[ \frac{a-1}{a} \nu |\nabla \mathbf{w}|^2 + \mathbf{w} \cdot (\nabla U)_{\text{sym}} \cdot \mathbf{w} \right]
$$

is positive semi-definite, $H_{U,a}\{\mathbf{w}\} \geq 0$, for all stationary divergence-free vector fields $\mathbf{w}$ satisfying the homogeneous b.c.

The condition ii is equivalent to the statement that all eigenvalues $\lambda$ of the hermitian eigenvalue problem

$$
\lambda \mathbf{V} = -2 \frac{a-1}{a} \nu \Delta \mathbf{V} + 2 (\nabla U)_{\text{sym}} \cdot \mathbf{V} + \nabla P ,
$$

$$
0 = \nabla \cdot \mathbf{V} , \quad \mathbf{V} \text{ satisfies the homogeneous b.c. ,}
$$

are non-negative; $P$ is a Lagrange multiplier for the divergence condition. Following Doering and Constantin [7], who consider the eigenvalue problem without an adjustable parameter
(a = 2), we denote the requirement that all eigenvalues of the eigenvalue problem (16) be non-negative as

spectral constraint: all \( \lambda \geq 0 \).

(17)

Case b) f is no gradient

In this case we bound the second term on the rhs. of (13) by

\[
\langle \frac{1}{\Omega} \int_{\Omega} d^3x \left[ \frac{a-1}{a} \nu |\nabla v|^2 + v \cdot (\nabla U)_{\text{sym}} \cdot v + f \cdot v \right] \rangle_T \\
\geq \inf_w \left\{ \frac{1}{\Omega} \int_{\Omega} d^3x \left[ \frac{a-1}{a} \nu |\nabla w|^2 + w \cdot (\nabla U)_{\text{sym}} \cdot w + f \cdot w \right] \right\},
\]

(18)

where we seek the infimum in the space of all stationary, divergence-free vector fields \( w \) satisfying the homogeneous b.c. If \( V \) is the minimizing vector field, it solves the Euler-Lagrange equations

\[
0 = -2 \frac{a-1}{a} \nu \Delta V + 2 (\nabla U)_{\text{sym}} \cdot V + \nabla P + f, \\
0 = \nabla \cdot V, \quad V \text{ satisfies the homogeneous b.c. },
\]

(19)

so that the inequality (18) becomes

\[
\langle \frac{1}{\Omega} \int_{\Omega} d^3x \left[ \frac{a-1}{a} \nu |\nabla v|^2 + v \cdot (\nabla U)_{\text{sym}} \cdot v + f \cdot v \right] \rangle_T \geq \frac{1}{2 \Omega} \int_{\Omega} d^3x f \cdot V.
\]

(20)

The requirement

\[
H_{U,a} \{ w \} > 0 \text{ for all divergence-free } w \neq 0 \text{ satisfying the homogeneous b.c.}
\]

(21)

guarantees the uniqueness of the solution \( V \) to the Euler-Lagrange equations (19) and ensures that this solution indeed minimizes the rhs. of (18). The uniqueness can be shown as follows: if there were any other solution \( W \neq V \), then the difference \( W - V \) would be a nonvanishing eigensolution to the eigenvalue problem (16) with eigenvalue \( \lambda = 0 \). This contradicts the requirement (21), which implies that all eigenvalues of (16) are strictly positive. The minimizing character of \( V \) stems from the convexity of the expression in curly brackets on the rhs. of (18).

Putting all things together, we see that in this case b) the resulting bound on the energy dissipation rate is given by

\[
\varepsilon \leq \inf_{U,a \geq 1} \left\{ \nu \langle |\nabla U|^2 \rangle - \frac{a}{2} (v \cdot (U \cdot \nabla U)) + \frac{a-2}{2} \nu \langle V \cdot \Delta U \rangle \right\},
\]

(22)

provided

i) \( U \) is a divergence-free vector field satisfying the physical b.c. and \( f \) is no gradient,

ii) \( V \) is a solution to the Euler-Lagrange equations (19),
iii) the functional \( H_{U,a}\{w\} > 0 \) for all divergence-free vector fields \( w \neq 0 \) satisfying the homogeneous b.c. Equivalently, all eigenvalues \( \lambda \) of the eigenvalue problem (16) must be positive, so that now we have the

\[
\text{spectral constraint: all } \lambda > 0 .
\]  

(23)

The main technical problem encountered in the original Doering-Constantin approach is to verify their spectral constraint for a given background flow, i.e., to show that all eigenvalues of the eigenvalue problem (16) with \( a = 2 \) are non-negative (case a) or positive (case b). In this respect our formulation generates no additional complications because our constraining eigenvalue problem is formally identical with the original one, the only difference being that the kinematic viscosity is rescaled by a factor \( 2(a - 1)/a \). Since \( -\Delta \) is a positive definite, unbounded operator, the spectral constraint enforces the positivity of this scaling factor, as anticipated in eq. (10) by the condition \( a > 1 \).

It should be noted that our formulation differs from the original one in two major points. Firstly, if \( \Delta U \) is no gradient then the minimizing field \( V \) must be determined by solving the Euler-Lagrange equations (19) even if \( U \cdot \nabla U \) is a gradient, since the contribution to \( f \) that is proportional to \( \Delta U \) vanishes only if \( a = 2 \), see eq. (12). Secondly, our variational principle concerns not only the background flow \( U \) but also the balance parameter \( a \). We will see that in the case of one-dimensional background flows both issues can be dealt with easily; solving the Euler-Lagrange equations and minimizing over \( a \) will be independent from the intricate problem of verifying the spectral constraint. Hence, the additional freedom gained by the balance parameter \( a \) entails no additional difficulties, but will result in an improved bound on \( \varepsilon \).

III. PLANE COUETTE FLOW WITH 1D BACKGROUND FLOWS

To become more specific we restrict the following discussion to one-dimensional background flows, i.e., to flows which can be described by a profile function \( \phi \),

\[
U = U\phi(\zeta) \hat{x} ; \quad \phi(0) = 0 , \quad \phi(1) = 1 ,
\]  

(24)

depending on the dimensionless variable \( \zeta \equiv z/h \). Clearly, such a \( U \) is a divergence-free vector field satisfying the physical b.c. Additionally we impose \( \phi(\zeta) = 1 - \phi(1 - \zeta) \), so that the profile is adapted to the symmetry of the physical problem.

a) Laminar profile, \( \phi(\zeta) = \zeta \)

In this case \( U \) is the laminar solution to the equations of motion (1) and (2), and we have \( f = 0 \). Hence, case a) of our variational principle applies. Provided all eigenvalues of the eigenvalue problem (16) are non-negative, the estimate (14) yields

\[
\varepsilon = \nu \frac{U^2}{h^2} \quad \text{or} \quad c_\varepsilon = Re^{-1} ,
\]  

(25)

respectively. We have strict equalities here since upper and lower bounds on the long time limit of \( \varepsilon_T \) coincide, as long as the laminar profile fulfills the spectral constraint [4]. We are
thus led to the following question: what is the maximal Reynolds number \( Re \) (or inverse
kinematic viscosity \( \nu^{-1} \)) up to which the laminar profile is admitted as a valid testprofile for the variational principle?

Since the viscosity enters into the eigenvalue problem (16) only in rescaled form, we can tune the Reynolds number by suitably adjusting the value of \( a \). In order to find the maximal “critical” Reynolds number for the laminar profile we have to set \( a = \infty \), so that the scaling factor \( 2 (a - 1)/a \) of the viscosity takes on the highest value possible, namely 2. The eigenvalue problem then becomes

\[
\lambda \mathbf{V} = -2 \nu \Delta \mathbf{V} + \frac{U}{h} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \mathbf{V} + \nabla P ,
\]

\[ 0 = \nabla \cdot \mathbf{V} , \quad \mathbf{V} \text{ satisfies the homogeneous b.c.} \] (26)

This is exactly the eigenvalue problem appearing in energy stability theory for the plane Couette flow [14][15]. Hence, the laminar profile is an admissible testprofile up to the Reynolds number \( Re_{ES} \) characterizing the energy stability border [14][15],

\[
Re_{ES} \simeq 2 \sqrt{1707.77} \simeq 82.65 ,
\] (27)

and consequently (27) is valid for all \( Re \leq Re_{ES} \).

In the original Doering-Constantin approach \((a = 2)\) one had to discard the laminar profile already for Reynolds numbers \( Re > Re_{ES}/2 \), which necessarily led to non-optimal upper bounds on \( c_\varepsilon \) for Reynolds numbers \( Re_{ES}/2 < Re \leq Re_{ES} \). The introduction of the balance parameter cures this obvious shortcoming and guarantees that \( c_\varepsilon \) as obtained from the variational principle has the known \( 1/Re \)-behaviour up to \( Re = Re_{ES} \).

b) Non-laminar profile, \( \phi(\zeta) \neq \zeta \)

In this case we have

\[
f = -\frac{a - 2}{a} \nu \frac{U}{h^2} \phi''(\zeta) \mathbf{x} \quad \text{with} \quad \phi''(\zeta) \neq 0 .
\] (28)

The boundary conditions (24) together with the symmetry condition ensure that \( f \) cannot be a gradient if \( a \neq 2 \), so that now we have to resort to case b) of our variational principle. Because of the special form that the inhomogeneous term \( f \) acquires for non-laminar profiles, one can find an analytical solution to the Euler-Lagrange equations (19):

\[
\mathbf{V} = \frac{1}{2} \frac{a - 2}{a - 1} U [\zeta - \phi(\zeta)] \mathbf{x} ,
\] (29)

\[
P = P_0 + \frac{1}{2} \frac{a - 2}{a - 1} U^2 \left[ \frac{1}{2} \phi(\zeta)^2 - \int_0^\zeta d\xi \xi \phi'(\xi) \right] .
\] (30)

Note that \( \mathbf{V} \) vanishes if \( a = 2 \), so that we do not need to distinguish between the cases \( a \neq 2 \) and \( a = 2 \). Moreover, the rhs. of (29) is proportional to the difference between \( \phi(\zeta) \) and the the laminar profile \( \zeta \), which leads us back to the laminar case a) in the limit \( \phi(\zeta) \to \zeta \). Provided all eigenvalues of (16) are positive, the estimate (24) yields
\[ \varepsilon \leq \inf_{\phi, a > 1} \left\{ 1 + \frac{a^2}{4(a-1)} D\{\phi\} \nu \frac{U^2}{h^2} \right\}; \quad (31) \]

the inequality for \( c_\varepsilon \) reads
\[
 c_\varepsilon \leq \inf_{\phi, a > 1} \left\{ 1 + \frac{a^2}{4(a-1)} D\{\phi\} Re^{-1} \right\}. \quad (32)
\]

Here we have employed the abbreviation \( D\{\phi\} \) for the functional
\[
 D\{\phi\} \equiv \int_0^1 d\zeta |\phi'(\zeta)|^2 - 1. \quad (33)
\]

This functional is strictly positive for \( \phi(\zeta) \neq \zeta \), which means that for each non-laminar profile and each \( a > 1 \) the factor \( 1 + \frac{a^2}{4(a-1)} D\{\phi\} \) appearing in (31) and (32) exceeds one. Therefore, each non-laminar profile produces a bound that is strictly higher than the laminar bound (25).

In a manner analogous to the procedure for the laminar profile one has to investigate on the basis of the eigenvalue problem (16) up to which Reynolds number a given profile \( \phi \) is admissible as a test profile. Because of the positive definiteness of \(-\Delta\), the spectrum of the hermitian operator defined by the rhs. of (16) is lowered when the rescaled kinematic viscosity is decreased. Thus, for a given \( \phi \) we define the critical Reynolds number \( R_c\{\phi\} \) as that Reynolds number where the lowest eigenvalue \( \lambda \) of (16) with fixed balance parameter \( a = \infty \) (i.e., \( 2(a-1)/a = 2 \) is maximal) passes through zero. For \( Re \geq R_c\{\phi\} \) one has to discard \( \phi \) as a test profile for the variation. In addition, if \( a > 1 \) is finite, one finds the constraint
\[
 Re < \frac{a - 1}{a} R_c\{\phi\}. \quad (34)
\]

The factor \( a^2/(a-1) \) in (31) and (32) has a local minimum for \( a = 2 \) and increases monotonically with \( a \) for \( a > 2 \). A given \( \phi \) thus produces the following upper bound on \( c_\varepsilon \):
\[
 c_\varepsilon \leq \left[ 1 + \frac{a_{\min}^2}{4(a_{\min} - 1)} D\{\phi\} \right] Re^{-1} \quad \text{for} \quad 0 \leq Re < R_c\{\phi\} \quad (35)
\]

with
\[
a_{\min} = \begin{cases} 
 2 & \text{for} \quad 0 \leq Re < \frac{1}{2} R_c\{\phi\} \\
 \frac{R_c\{\phi\} - Re}{R_c\{\phi\} - \frac{1}{2} R_c\{\phi\}} & \text{for} \quad \frac{1}{2} R_c\{\phi\} \leq Re < R_c\{\phi\}
\end{cases}; \quad (36)
\]

hence
\[
 c_\varepsilon \leq \begin{cases} 
 [1 + D\{\phi\} Re^{-1}] & \text{for} \quad 0 \leq Re < \frac{1}{2} R_c\{\phi\} \\
 \left[ 1 + \frac{D\{\phi\} R_c\{\phi\}^2}{4[R_c\{\phi\} - Re] Re} \right] Re^{-1} & \text{for} \quad \frac{1}{2} R_c\{\phi\} \leq Re < R_c\{\phi\}
\end{cases}. \quad (37)
\]

In this way we have accomplished the optimization of the balance parameter and are left with the task of varying the profile function. But it is possible to deduce some general statements even without solving the variational principle for \( \phi \). To this end, we denote the
expression on the rhs. of (37) as $\bar{c}_e(Re)$. This is a continuous function of the Reynolds number, and even continuously differentiable at $Re = R_c\{\phi\}/2$. For every given profile, $\bar{c}_e(Re)$ has exactly one local minimum in the whole interval $0 \leq Re < R_c\{\phi\}$; this minimum appears in the upper half interval $R_c\{\phi\}/2 \leq Re < R_c\{\phi\}$. Because the variational principle tests all profile functions $\phi$ satisfying the required conditions, the minimum point determined by a particular $\phi$ is the only point that this profile could possibly contribute to the resulting upper bound on $c_e(Re)$. Thus, each $\phi$ leads to a point in the $(Re, c_e)$-plane,

$$\phi \rightarrow (R_{\min\{\phi\}}, \bar{c}_e(R_{\min\{\phi\}})) \; .$$

The Reynolds number $R_{\min\{\phi\}}$ of the minimum point can be expressed as

$$R_{\min\{\phi\}} = x_0\{\phi\} R_c\{\phi\} \; ,$$

where $x_0\{\phi\}$ is the unique (real) zero $x_0$ with $1/2 \leq x_0 < 1$ of the cubic polynomial

$$x^3 - 2x^2 + \left( 1 - \frac{3}{4} D\{\phi\} \right) x + \frac{1}{2} D\{\phi\} = 0 \; .$$

This follows directly by minimizing $\bar{c}_e(Re)$ with respect to $Re$. Although an analytical expression for the desired zero of (40) is available for all $D\{\phi\} > 0$, we will discuss only two limiting cases. In the laminar limit $\phi(\zeta) \rightarrow \zeta$ the functional $D\{\phi\}$ vanishes and the zero $x_0\{\phi\}$ tends to 1. Inserting $R_{\min\{\phi\}}$ into (36) shows that $a_{\min}$ tends to infinity in this limit,

$$\lim_{Re \downarrow Re_{ES}} a_{\min} = \infty \; .$$

Therefore, the optimal bounds on $c_e$ resulting from the variational principle will be continuous at the energy stability border. On the other hand, for sufficiently high Reynolds numbers the spectral constraint (23) singles out those profile functions as admissible for the variational principle that have large slopes within thin boundary layers and remain almost constant in the interior [7]. Hence the functional $D\{\phi\}$ tends to infinity in the asymptotic limit $Re \rightarrow \infty$, which implies that the zero $x_0\{\phi\}$ tends to $2/3$ and $a_{\min}$ approaches its asymptotic value 3,

$$\lim_{Re \rightarrow \infty} a_{\min} = 3 \; .$$

The optimal value of the balance parameter $a$, considered as a function of the Reynolds number, is monotonically decreasing from infinity at the energy stability border $Re_{ES}$ to its asymptotic limit $a_{\infty} = 3$ and never reaches the value $a = 2$ that has tacitly been taken for granted before.

In the original approach [7] a given profile $\phi$ yields an upper bound on $c_e$ in a similar way, but the range of Reynolds numbers within which this profile is admissible is only half as large as in our case. Since, as remarked before, the factor $a^2/(a - 1)$ appearing in (32) has a minimum for $a = 2$, in the common interval this bound agrees with ours:

$$c_e \leq \bar{c}_e(Re) \quad \text{for} \quad 0 \leq Re \leq \frac{1}{2} R_c\{\phi\} \; .$$

(43)
Here the minimizing point is \((R_c \{\phi\}/2, \bar{c}_\varepsilon(R_c \{\phi\}/2))\), which has to be compared with \((38)\). Therefore, in our formulation a given profile function \(\phi\) leads to a **lowered** bound on \(c_\varepsilon\) at a **higher** Reynolds number \(Re\). The improvement due to the balance parameter \(a\) relies on the fact that each \(\phi\) becomes admissible within an enlarged \(Re\)-interval. Note that in both cases exactly the same eigenvalue problem must be solved to check the spectral constraint.

Under the assumption that in both formulations the upper bounds on \(c_\varepsilon\) are approaching constant values in the asymptotic limit \(Re \to \infty\), dubbed \(\bar{c}_{\varepsilon, \infty}\) and \(\bar{c}_{\varepsilon, \infty}^{DC}\), respectively, we can calculate the relative factor

\[
g_\infty \equiv \frac{\bar{c}_{\varepsilon, \infty}}{\bar{c}_{\varepsilon, \infty}^{DC}} = \lim_{Re \to \infty} \frac{\bar{c}_\varepsilon(x_0(\phi) R_c \{\phi\})}{\bar{c}_\varepsilon(R_c \{\phi\}/2)} = \frac{27}{32}. \tag{44}
\]

Considering the same class of one-dimensional background flows, our formulation yields asymptotic upper bounds on the rate of energy dissipation that are systematically lowered by a factor of \(27/32\) compared to those that can be computed from the original Doering-Constantin approach.

### IV. ILLUSTRATION

In this section we wish to illustrate the general statements derived in the previous section. Our aim is not to calculate the best bounds that the improved variational principle has to offer, but rather to elucidate the new aspects in a simple way. Instead of verifying the spectral constraint for a given \(\phi\) on the basis of the eigenvalue problem \((16)\), one can check the positivity of the functional \(H_{U,a}\{w\}\),

\[
H_{U,a}\{w\} = \frac{1}{\Omega} \int_{\Omega} d^3 x \left[ \frac{a - 1}{a} \nu |\nabla w|^2 + \frac{U h}{h} \phi' w_x w_z \right], \tag{45}
\]

by means of the inequality

\[
\left| \frac{1}{\Omega} \int_{\Omega} d^3 x \frac{U}{h} \phi' w_x w_z \right| \leq \frac{\langle |\nabla w|^2 \rangle}{2 \sqrt{2}} \int_0^{h/2} dz \frac{U}{h} |\phi'|. \tag{46}
\]

This estimate can be shown by repeatedly using Schwarz’ inequality and utilizing the symmetry property of the profile function \(\phi(\zeta)\) (see also refs. \([10, \ldots, 14, 15]\)). Thus,

\[
H_{U,a}\{w\} \geq \left[ \frac{a - 1}{a} \nu - \frac{U h}{2 \sqrt{2}} \int_0^{1/2} d\zeta |\phi'| \right] \langle |\nabla w|^2 \rangle. \tag{47}
\]

If the profile function \(\phi(\zeta)\) satisfies the condition

\[
\int_0^{1/2} d\zeta |\phi'(\zeta)| \leq 2 \sqrt{2} \frac{a - 1}{a} Re^{-1} \quad \text{or} \quad \int_0^{1/2} d\zeta |\phi'(\zeta)| < 2 \sqrt{2} \frac{a - 1}{a} Re^{-1}, \tag{48}
\]

then the non-negativity (or the positivity) of the functional \(H_{U,a}\{w\}\) is guaranteed. It should be realized that \((18)\) is a sufficient but not a necessary condition; it is more restrictive than the spectral constraint. In addition to the b.c. \((24)\) and the symmetry requirement we assume \(\phi'(\zeta) \geq 0\), so that the modulus signs \(\|\) in \((18)\) can be skipped.
After the replacement of the variational principle’s spectral constraint (condition ii in case a) and condition iii in case b), respectively) by the sharpened profile constraint ([48], a certain \( \phi \) can, in general, no longer be admitted as a testprofile for Reynolds numbers up to \( R_c\{\phi\} \) as defined in the previous section. Rather, ([48]) leads to the border

\[
R^S_c\{\phi\} \equiv \frac{2\sqrt{2}}{\int_0^{1/2} d\zeta \, \zeta \, \phi'(\zeta)} .
\]  

(49)

For example, the laminar profile \( \phi(\zeta) = \zeta \) yields

\[
R^S_c\{\zeta\} = 16\sqrt{2} \approx 22.63,
\]  

(50)

which has to be contrasted to (27), i.e., \( R_c\{\zeta\} = Re_{ES} \approx 82.65 \). Thus \( R^S_c\{\zeta\} \) is by about a factor 4 smaller than \( R_c\{\zeta\} \). Nevertheless, it is instructive to consider the variational principle with the stronger constraint ([48]) since then even the variation of the profile \( \phi \) can be done analytically. For finite \( a > 1 \) we now have the constraint

\[
Re < \frac{a-1}{a} R^S_c\{\phi\}
\]  

(51)

instead of (34), which is exactly ([48]).

Without going into the cumbersome technical details (see also ref. [10] for comparison) we summarize our results. We have to distinguish three \( Re \)-ranges,

\[
\begin{align*}
\text{(I)} \quad 0 & \leq Re \leq 16\sqrt{2}, \\
\text{(II)} \quad 16\sqrt{2} & < Re \leq 20\sqrt{2}, \\
\text{(III)} \quad 20\sqrt{2} & < Re < \infty.
\end{align*}
\]  

(52)

In these ranges the minimizing profile function behaves as sketched in Fig. 1: in (I) \( \phi \) is equal to the laminar profile, \( \phi(\zeta) = \zeta \). In (II) the profile develops parabolic boundary layers which reach the middle of the domain \( \Omega \) (i.e., \( \zeta = 1/2 \)), while the slope at \( \zeta = 1/2 \) decreases from one to zero with increasing \( Re \). When \( Re \) is increased beyond \( 20\sqrt{2} \), the thickness \( \delta \) of these parabolic boundary layers is getting smaller and smaller; in the interior \( \phi \) remains constant, \( \phi(\zeta) = 1/2 \) for \( \delta \leq \zeta \leq 1-\delta \). Asymptotically, \( \delta \) vanishes as \( \sim 1/Re \). The analytic expression for \( \delta \) in (III) reads

\[
\delta = \frac{4\sqrt{2}}{Re} \left( \frac{3\sqrt{Re-2\sqrt{2}} + \sqrt{Re-18\sqrt{2}}}{\sqrt{Re-2\sqrt{2}} + \sqrt{Re-18\sqrt{2}}} \right) \rightarrow_{Re \to \infty} \frac{8\sqrt{2}}{Re} .
\]  

(53)

Correspondingly, the functional \( D\{\phi\} \) defined by eq. (33) diverges for \( Re \to \infty \), as anticipated in section 3.

The minimizing balance parameter \( a \) turns out to be

\[
a = \begin{cases} 
\infty & \text{(i.e., } 2 \frac{a-1}{a} = 2) \\
2 + \frac{16\sqrt{2}}{Re-16\sqrt{2}} & \text{(II)} \\
\frac{3}{2} \left( 1 + \sqrt{\frac{Re-2\sqrt{2}}{Re-18\sqrt{2}}} \right) & \text{(III)}
\end{cases}
\]  

(54)
For $Re > 16\sqrt{2}$ this parameter decreases monotonically from $\infty$ to its asymptotic value $a_\infty = 3$, as depicted in Fig. 2.

Finally, the resulting bounds on $c_\varepsilon$ are given by

$$\frac{1}{Re} \leq c_\varepsilon(Re) \leq \begin{cases} \frac{1}{Re} & (I) \\ \frac{1}{Re} - \frac{12\sqrt{2}}{Re} & (II) \\ \frac{1}{Re} - \frac{12\sqrt{2}}{Re} + \frac{96}{Re} & (III) \end{cases}$$

(55)

Note that the upper bound is continuous but not continuously differentiable at $Re = 16\sqrt{2}$, see Fig. 3. The limit $Re \to \infty$ yields

$$c_\varepsilon(\infty) \leq \lim_{Re \to \infty} \bar{c}_\varepsilon(Re) = \frac{3}{32\sqrt{2}}.$$  

(56)

As before, we denote the upper bound on $c_\varepsilon$ by $\bar{c}_\varepsilon(Re)$.

These results have to be compared with those derived in ref. [10], where the sharpened profile constraint (48) was used to calculate bounds on $c_\varepsilon$ analytically within the framework established by Doering and Constantin ($a = 2$ fixed). The analogous three $Re$-ranges found there are

$$(I') \quad 0 \leq Re \leq 8\sqrt{2}$$

$$(II') \quad 8\sqrt{2} < Re \leq 12\sqrt{2}$$

$$(III') \quad 12\sqrt{2} < Re < \infty$$

(57)

in particular, the first range $(I')$ is only half as wide as in our case. The corresponding bounds on $c_\varepsilon$ are given by

$$\frac{1}{Re} \leq c_\varepsilon(Re) \leq \begin{cases} \frac{1}{Re} & (I') \\ \frac{1}{Re} - \frac{12\sqrt{2}}{Re} + \frac{96}{Re} & (II') \\ \frac{1}{Re} - \frac{12\sqrt{2}}{Re} + \frac{96}{Re} - \frac{1}{9\sqrt{2}} & (III') \end{cases}$$

(58)

The improvement due to the parameter $a$ is measured by the ratio of the rhs. of (55) and the rhs. of (58),

$$\bar{g}(Re) \equiv \frac{\bar{c}_\varepsilon(Re)}{\bar{c}_\varepsilon DC(Re)}.$$  

(59)

This ratio has a minimum at $Re = 16\sqrt{2}$,

$$\bar{g}(16\sqrt{2}) = \frac{9\sqrt{2}}{16\sqrt{2}} = \frac{9}{16} \simeq 0.56,$$

(60)

and increases monotonically with increasing $Re$ to its asymptotic value $27/32$,

$$\lim_{Re \to \infty} \bar{g}(Re) = \frac{3 \cdot 9\sqrt{2}}{32\sqrt{2}} = \frac{27}{32} \simeq 0.84.$$  

(61)

A graphical comparison of the bounds (55) and (58) is shown in Fig. 3.
V. CONCLUSION: BACKGROUND FLOW AND BALANCE PARAMETER BELONG TOGETHER

In this paper we have been concerned with an improved formulation of the Doering-Constantin variational principle for bounds on turbulent energy dissipation, rather than with its solution. The necessity of devoting the utmost care to an optimal formulation is obvious: what has been lost by a non-optimal formulation of a variational principle cannot be regained by even the most sophisticated techniques for solving it.

The feasibility of improving the background flow method rests on two key observations. The first of these is that if one refrains from what appears to be straightforward, namely, if one does not eliminate the unwanted cross background-fluctuation term from eq. (9) but rather keeps this term with a certain weight quantified by the balance parameter $a$, one gains a new freedom that can be exploited to improve the bounds. The second observation is that the required solution to the Euler-Lagrange equations (19) can easily be written down analytically if one restricts oneself to background flows which only have a height-dependent profile but no spanwise structure. Then the technical difficulties encountered in our formulation are the same as those of the original approach — in both cases one has to solve the same eigenvalue problem in order to check whether a given profile is an admissible candidate for the computation of the bound, albeit in our case the kinematic viscosity is rescaled —, but the improved formulation leads to systematically lowered bounds.

Finally, we wish to stress that the usefulness of introducing the balance parameter is by no means restricted to the case of plane Couette flow. It should be obvious by now that the arguments employed in sections 2 and 3 can easily be adapted to other problems. When computing upper bounds on any property of a turbulent flow that is amenable to the background flow method, background flow and balance parameter should henceforth be regarded as belonging together.

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REFERENCES

[1] L.N. Howard, Ann. Rev. Fluid Mech. 4 (1972) 473.
[2] F.H. Busse, J. Fluid Mech. 41 (1970) 219.
[3] F.H. Busse, Adv. Appl. Mech. 18 (1978) 77.
[4] F.H. Busse, in: Nonlinear Physics of Complex Systems — Current States and Future Trends, J. Parisi, S.C. Müller, and W. Zimmermann, eds. (Springer, Berlin, 1996).
[5] R.R. Kerswell and A.M. Soward, J. Fluid Mech. (1996) in press.
[6] C.R. Doering and P. Constantin, Phys. Rev. Lett. 69 (1992) 1648.
[7] C.R. Doering and P. Constantin, Phys. Rev. E 49 (1994) 4087.
[8] E. Hopf, Math. Annalen 117 (1941) 764.
[9] C. Marchioro, Physica D 74 (1994) 395.
[10] T. Gebhardt, S. Grossmann, M. Holthaus, and M. Löhden, Phys. Rev. E 51 (1995) 360.
[11] P. Constantin and C.R. Doering, Phys. Rev. E 51 (1995) 3192.
[12] C.R. Doering and P. Constantin, Phys. Rev. E 53 (1996) 5957.
[13] R.R. Kerswell, to be published (1996).
[14] D.D. Joseph, Stability of Fluid Motions I, II (Springer, Berlin, 1976).
[15] P.G. Drazin and W.H. Reid, Hydrodynamic Stability (Cambridge University Press, Cambridge, 1981).
FIGURES

FIG. 1. Background flow profiles resulting from the variational principle with the sharpened profile constraint (48). Solid line: $0 \leq Re \leq 16\sqrt{2}$; short dashes: $Re = 20\sqrt{2}$, long dashes: $Re = 64\sqrt{2}$.

FIG. 2. Balance parameter $a$ resulting from the variational principle with the sharpened profile constraint (48).

FIG. 3. Bounds on the dimensionless energy dissipation rate $c_\varepsilon(Re)$ derived from the variational principle with the sharpened profile constraint (48). Solid line: lower bound, $1/Re$; long dashes: upper bound with optimized parameter $a$ (see inequality (55)); short dashes: upper bound obtained in ref. [10] for $a = 2$ fixed (see inequality (58)).
