Generalized Estimators, Slope, Efficiency, and Fisher Information Bounds

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Abstract

Point estimators may not exist, need not be unique, and their distributions are not parameter invariant. Generalized estimators provide distributions that are parameter invariant, unique, and exist when point estimates do not. Comparing point estimators using variance is less useful when estimators are biased. A squared slope $\Lambda$ is defined that can be used to compare both point and generalized estimators and is unaffected by bias. Fisher information $I$ and variance are fundamentally different quantities: the latter is defined at a distribution that need not belong to a family, while the former cannot be defined without a family of distributions, $M$. Fisher information and $\Lambda$ are similar quantities as both are defined on the tangent bundle $TM$ and $I$ provides an upper bound, $\Lambda \leq I$, that holds for all sample sizes – asymptotics are not required. Comparing estimators using $\Lambda$ rather than variance supports Fisher’s claim that $I$ provides a bound even in small samples. $\Lambda$-efficiency is defined that extends the efficiency of unbiased estimators based on variance. While defined by the slope, $\Lambda$-efficiency is simply $\rho^2$, the square of the correlation between estimator and score function.

Key words: nonasymptotics, generalized estimators, slope, Fisher information, efficiency

1 Introduction

Differential geometry has been used extensively in the study of higher order asymptotic inference. While there are many references that could be cited here, Amari [1990] stands out for its clear description of the role that geometry plays in understanding and simplifying many areas of statistical inference. This paper’s finite sample results are a departure from asymptotics, but we hope it continues geometry’s role in providing simplicity and intuition for inferential methods and concepts.
2 Estimation

2.1 Two Views of Estimation

One approach to estimation is that of choosing a point estimator with good properties and reporting this estimate along with its standard error. We label this the point-estimate-plus-se approach. The other approach is based on hypothesis testing where there is not just one null hypothesis but a continuum. This is the continuum-of-hypotheses approach and it is the view expressed by Fisher (1955).

It may be added that in the theory of estimation we consider a continuum of hypotheses each eligible as null hypothesis, and it is the aggregate of frequencies calculated from each possibility in turn as true . . . which supply the likelihood function, . . . and other indications of the amount of information available.

In the second approach, the point estimate, a real number, is replaced with a function on the parameter space, a generalized estimate, that is obtained from the continuum of hypotheses and estimation is described in terms of properties of this function.

2.2 The Estimation Problem

The estimation problem considered in this paper is described as follows. There is a finite population having a distribution on values \( \{b_1, b_2, \ldots, b_I\} \subset \mathbb{R} \), that, without loss of generality (wlog), we can order so that \( b_1 < b_2 < \cdots < b_I \). This population can be an actual population, such as all adults living in North Carolina on January 1, 2022, or it can be hypothetical, such as the collection of all pills that could be produced from a specific production line. Whether actual or hypothetical, we call this the real world population. The multivariate hypergeometric distribution for a sample of size one along with the numeric labels is an exact model for the data obtained from the real world population that we denote as \( m \). There is a one-dimensional family of distributions, \( M \), where each distribution in \( M \) is dominated by a measure having support \( \mathcal{X} \subset \mathbb{R} \) such that \( \{b_1, b_2, \ldots, b_I\} \subset \mathcal{X} \). The assumption is that there is at least one model \( m \in M \) so that \( m \) is a suitable approximation to \( m \). What constitutes a suitable approximation will depend on the application. It may be that the area histogram for \( m \) is well approximated by the area histogram of \( m \) if \( \mathcal{X} \) is countable or by the graph of the density function if \( \mathcal{X} \) is uncountable.

With this assumption the estimation problem becomes that of using points \( y = (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n = \mathcal{Y} \) to distinguish models in \( M \). To accommodate reduction to a sufficient statistic and inference conditional on an ancillary statistic, the space \( \mathcal{Y} \) is adjusted accordingly. The real world population and the exact model \( m \) motivate the estimation problem but neither play a role in defining or assessing estimators. Pointedly, we will not take hypothetical repeated samples from either the real world population or from \( m \). We assume that the family of
distributions $M$ has the structure that makes $M$ a Riemannian manifold and there is a global coordinate chart $\theta : M \to \Theta$ where $\Theta$ is an open interval in $\mathbb{R}$.

2.3 A Continuum of Hypotheses

Fisher’s continuum-of-hypotheses approach to estimation is less well-known so we provide a few details. We are given a sample $y$ and a family of models $M$ and the estimation problem is that of assessing, for each model, the extent to which $y$ is an extreme observation. For a fixed model, say the model with $\theta = \theta'$, the notion of ‘extreme’ is made precise by a test statistic which defines an ordering $\preceq$ on $\mathcal{Y}$ given by

$$y_1 \preceq y_2 \text{ iff } T(y_1) \leq T(y_2).$$

(1)

This allows us to talk about a particular $y \in \mathcal{Y}$ as being extreme, i.e., the value $T(y)$ is extreme. Under mild regularity conditions, $T^{-1}(t)$, for $t \in T(\mathcal{Y})$, provides a continuous partition of $\mathcal{Y}$ into ordered subsets of co-dimension $1$. Since the ordering does not change if $T$ is replaced with $aT + b$ for $a, b \in \mathbb{R}$ and $a > 0$ we can, wlog, define $T$ such that $ET = 0$ and $VT = 1$. With $T$ standardized, $T(y)$ is the number standard deviations $y$ is from $0$, the mean of the sampling distribution.

Since we are considering a continuum of hypotheses we have a continuum of test statistics $T(\cdot, \theta)$ where $T(y, \theta)$ indicates how far $y$ is in the tails of the sampling distribution for each model indexed by $\theta \in \Theta$. In comparing two tests $T_1$ and $T_2$, there would be a preference for $T_1$ if $T_1(y, \theta)$, as a function of $\theta$, changed more rapidly than $T_2(y, \theta)$. The derivative, which is the slope of the graph of $T(y, \theta)$, measures this change at $\theta$. This discussion motivates the following heuristic: ‘$T_1$ has more information than $T_2$ for distinguishing models near $\theta$’. The slope of the test can be a function of $y$ so we use the mean slope, $E_\theta T'(Y, \theta)$. Even though the heuristic does not invoke a formal definition of ‘information’, it is worth noting that when $T$ is the score, the mean slope is the Fisher information and when $T$ is the standardized score the mean slope is the square root of the Fisher information.

The notion of a continuum of tests is formalized below using a generalized estimator and this approach to estimation can be summarized in the following desideratum:

Generalized estimators should have large\(^1\) mean slope.

This is related to, but distinct from, the desideratum for point estimation:

Estimators should have small standard deviation.

Estimators are often compared in terms of variance, rather than standard deviation, and squared mean slope is used to show the connection between these two approaches to estimation. However, the geometric intuition for understanding the relationship between estimators, both point and generalized estimators, is best described using the mean slope rather than its square.

\(^1\)Here, ‘large’ means in absolute value.
3 Generalized Estimators

Estimators, both point and generalized, can be described by their aggregate or
distributional properties on $\mathcal{Y}$. Such properties can be used to compare and
select estimators before any specific data have been obtained. We begin by
considering aggregate properties, that is estimators. Section 3 describes the
inferential role of the generalized estimate obtained from a single sample $y \in \mathcal{Y}$.

3.1 Space of Generalized Estimators $\mathcal{G}$

A point estimator for $\theta$ is a function $u : \mathcal{Y} \rightarrow \Theta$. A generalized estimator
for $\theta$ is a function $g : \mathcal{Y} \rightarrow C^k(\Theta)$, the space of functions on $\Theta$ whose $k^{th}$
derivative is continuous. We let $C(\Theta)$ be the space of functions of required
smoothness. For fixed $y$, point estimation $y \mapsto u(y) = \hat{\theta} \in \Theta$ while generalized
estimation $y \mapsto g(y)(\theta) \in C(\Theta)$. When $y$ is fixed we simplify the notation by
dropping the subscript, $g(\theta) = g_y(\theta)$. The context will indicate whether $g =
g(\theta)$ is an estimate (fixed $y$) or an estimator (distribution of $y$ values). This
follows the notational simplification for point estimation where $\hat{\theta}$ may refer to
an estimate or to an estimator and expressions such as $E\hat{\theta}$ indicate that $\hat{\theta}$ is an
estimator. Likewise, $Eg$ indicates that $g$ is an estimator and if the expectation
were defined using an integral we may use the full notation $g_y(\theta)$ or even $g(\theta; y)$.
The generalized estimate needs to be a function on $\Theta$ for it to represent the
continuum-of-hypotheses approach since each hypothesis specifies a point in $\Theta$.
For the null hypothesis $H_0 : \theta = \theta_0$, $g(\theta_0)$ is the value of a test statistic for the
sample $y$. The function $g(\theta)$ shows how the test statistic for fixed $y$ changes
over the continuum of hypotheses indexed by $\theta$.

For another parameterization $\xi$, $y \mapsto g(\xi) \in C(\Xi)$ and while $g(\theta)$ and $g(\xi)$
are different functions, they take the same value when $\theta$ and $\xi$ label the same
point $m \in M$. Using more precise notation, a generalized estimate for $y$ is a function
$g_M : M \rightarrow \mathbb{R}$ that when expressed in the $\theta$ parameter is $g_\theta : \Theta \rightarrow \mathbb{R}$
and $g_\Xi : \Xi \rightarrow \mathbb{R}$ in the $\xi$ parameter. For any point $m_\circ \in M$, $g_M(m_\circ)$, $g_\theta(\theta(m_\circ))$, and
$g_\xi(\xi(m_\circ))$ all take the same value. Furthermore, since $y$ was any point in $\mathcal{Y}$
and $m_\circ$ specifies a distribution on $\mathcal{Y}$, the estimators $g_M(m_\circ)$, $g_\theta(\theta_\circ)$, and
$g_\xi(\xi_\circ)$ all have the same distribution. It is this property that makes generalized
estimators parameter invariant.

For point estimation the function $u$ must be measurable and we generally require
$E_\theta u^2 < \infty$ for all $\theta$. For the generalized estimator we require that
$g(\theta)$ is measurable for all $\theta$. The moment assumptions are that $E_\theta g(\theta) = 0$,
$V_\theta g(\theta) < \infty$, and $E_\theta (g(\theta) \ell'(\theta)) \geq 0$ where $\ell(\theta)$ is the log likelihood for $M$.
The smoothness assumptions are that $\partial_\theta V_\theta g(\theta)$ is finite and $g(\theta)$ can be differentiated
under the integral so that $\partial_\theta E_\theta g(\theta) = 0$ yields the identity

$$- E_\theta g'(\theta) = E_\theta (g(\theta) \ell'(\theta))$$

where the subscript $\theta$ on $E$ has been dropped since the expectations will always
be taken with respect to the same value for the parameter as appears in the
functions. Further notational simplification emphasizes that we are interested in a continuum of values

\[-Eg' = E(g\ell')\] (3)

where for a.e. \( y, g', g, \ell' \in C(\Theta) \) and the expectations also are in \( C(\Theta) \). Any function \( g \) satisfying these conditions is a generalized estimator and we denote the space of all generalized estimators as \( \mathcal{G}(M, \mathcal{Y}) \) which we abbreviate to \( \mathcal{G} \) when the context is clear regarding the model space and sampling distribution. Since \(-g(\theta)\) simply reverses the ordering of \( \mathcal{Y} \) specified by \( g(\theta) \) we don’t want both of these functions in \( \mathcal{G} \); the restriction \( E(g\ell') \geq 0 \) means \( \ell' \in \mathcal{G} \).

The relation \( g_1 \sim g_2 \) if and only if \( g_1(\theta) = k(\theta)g_2(\theta) \) for some \( k \in C(\Theta) \) with \( k(\theta) > 0 \) is an equivalence relation. The equivalence of \( g_1 \) and \( g_2 \) is motivated by the fact that for each \( \theta_0 \), \( g_1(\theta_0) \) and \( g_2(\theta_0) \) provide the same ordering of \( \mathcal{Y} \) so that the tail area obtained for each \( y \in \mathcal{Y} \) is the same for both \( g_1 \) and \( g_2 \).

Inferential properties of equivalence class \([g]\) are conveniently represented by the standardized estimator

\[ \bar{g} = g/\sqrt{V(g)}. \]

The standardized score \( \ell' \) will be denoted as

\[ \hat{s} = \ell'/\sqrt{V(\ell')} \in [\ell']. \]

A generalized estimator \( g(\theta) \) defines a point estimator \( \hat{\theta}_g \) by the equation \( g(\hat{\theta}_g) = 0 \). Note that \( \hat{\theta}_g \) need not be unique and it need not exist.

### 3.2 Mapping Point Estimators to \( \mathcal{H} \subset \mathcal{G} \)

If \( u : \mathcal{Y} \to \Theta \) is a point estimator and \( v(\theta) = Eu \), then

\[ h(\theta) = u(y) - v(\theta) \] (4)

is a generalized estimator. If \( \hat{\theta} = u(y) \) is unbiased for \( \theta \), then \( \hat{\theta}_h = \hat{\theta} \). We use the notation \( h \) for generalized estimators that are the difference between two functions, one having domain \( \mathcal{Y} \) and the other \( \Theta \). All graphs for \( h(\theta) \) have the same shape being the function \( -v(\theta) \) shifted by the amount \( u(y) \). For \( \hat{\theta} = u(y) \) unbiased \( v(\theta) = \theta \) so the graphs of \( h(\theta) \) are straight lines with slope \(-1\); in general, the graphs of \( h(\theta) \) will not be straight lines but the shapes will be the same. The set of generalized estimators of this form is \( \mathcal{H} \subset \mathcal{G} \). Note that \( h \) is parameter-invariant since

\[ h(\xi) = u(y) - v \circ \theta_\xi(\xi) \]

where \( \theta_\xi : \Xi \to \Theta \). Also, for \( k_1, k_2 \in \mathbb{R} \) with \( k_1 > 0 \), \( k_1u + k_2 \in [u] = [u - Eu] \). Equivalence of statistic \( u \) under affine transformations is required for inference to be invariant with respect to units on \( \mathcal{Y} \) (e.g., degrees Fahrenheit or Centigrade). The group of transformation defining equivalence for generalized estimator \( g \) provides invariance with respect to parameterizations on \( M \).
The point estimator $\hat{\theta} = u(y)$ is not parameter-invariant since its distribution need not be the same as that of $\hat{\xi} = \theta^{-1}(\hat{\theta})$. Point estimators for different parameterizations produce different generalized estimators even when the point estimates are parameter-invariant. An example appears in Section 7.1.1.

### 3.3 Squared Slope $\Lambda(g)$

The squared slope of $g \in G$ is a function $\Lambda(g) \in C(\Theta)$ defined by

$$\Lambda(g) = (Eg')^2 \quad = (Eg')^2 / V(g).$$

Clearly, $\Lambda(g_1) = \Lambda(g)$ whenever $g_1 \sim g$. The slope for a point estimator is defined as the slope of its corresponding generalized estimator. The squared slope for $\hat{\theta} = u(y)$ is that of $h = \hat{\theta} - v(\theta)$

$$\Lambda(\hat{\theta}) = (v')^2 / V(\hat{\theta}).$$

If $\hat{\theta}$ is unbiased, then $v(\theta) = \theta$ and

$$\Lambda(\hat{\theta}) = 1 / V(\hat{\theta}). \quad (5)$$

A caution regarding terminology: the slope of a generalized estimator is a mean since there is a distribution of curves of varying slopes; the slope of $\hat{\theta} - v(\theta)$ is $-v'(\theta)$ since each curve has the same slope.

### 4 Fisher Information Bounds

Fisher information provides an upper bound for squared slope $\Lambda$ and a lower bound for the variance, $V$, of an estimator. In the next section the bounds are calculated and efficiencies defined by $\Lambda$ and $V$ are compared. Section 4.2 shows that $V$ and Fisher information $I$ are fundamentally different; variance of a point estimator is not geometric while $\Lambda$ and $I$ are tensors.

#### 4.1 Bounds for $\Lambda$ and $V$

The Fisher information for $M$ is $I_1 = -E\ell''_1(\theta, x)$ where $\ell_1$ is the log likelihood for models in $M$; $I_1$ represents the average amount of information in a single observation $x$. The Fisher information in $y$ is $I = I_y = -E\ell''(\theta, y)$ where $\ell$ is the log likelihood for $y$. For $Y = X^n$, $\ell = \sum \ell_1(\theta, x_i)$ and $I = n I_1$. The Fisher information upper bound for $\Lambda$ is obtained from $-Eg' = E(g\ell')$. Expressing the covariance $E(g\ell')$ in terms of the correlation yields

$$-Eg' = \rho(g, \ell') \sqrt{V(g)V(\ell')} \quad (6)$$

Squaring this last equation and making substitutions $\Lambda(g) = (Eg')^2 / V(g)$ and $I = V(\ell')$ shows

$$\Lambda(g) = \rho^2(g, \ell') I \quad (7)$$
so that the Fisher information $I$ is the upper bound for $\Lambda(g)$ since $\rho \leq 1$. Clearly, $\Lambda(\ell') = I$ so that the score achieves the squared slope upper bound. Since $\Lambda(\hat{\theta}) = \Lambda(\hat{\theta} - v(\theta))$, equation (7) provides a bound for point estimator $\hat{\theta}$,

$$\Lambda(\hat{\theta}) = \rho^2(\hat{\theta}, \ell')I. \quad (8)$$

For the variance bound, we substitute $\hat{\theta} - v(\theta)$ for $g$ in (6) and solving for variance yields

$$V(\hat{\theta}) = \frac{(v')^2}{\rho^2(\hat{\theta}, \ell')I} \geq (v')^2I^{-1} \quad (9)$$

where the equality in (9) holds provided $\rho \neq 0$. The bound on $\Lambda(\hat{\theta})$ has no restriction on $\rho$. When $\hat{\theta}$ is unbiased $v(\theta) = \theta$ so that (9) gives the Cramér-Rao lower bound $V(\hat{\theta}) \geq I^{-1}$. The bound on $\Lambda(\hat{\theta})$ is the same whether $\hat{\theta}$ is biased or unbiased.

The $V$-efficiency of an unbiased point estimator $\hat{\theta}$ is defined in terms of the ratio $V(\hat{\theta})$ and the variance lower bound

$$\text{Eff}^V(\hat{\theta}) = I^{-1}/V(\hat{\theta}). \quad (10)$$

We define the $\Lambda$-efficiency of any generalized estimator $g$ using the squared slope and its upper bound

$$\text{Eff}^\Lambda(g) = \Lambda(g)/I = \rho^2(g, \ell'). \quad (11)$$

Setting $g = \hat{\theta} - v(\theta)$, the $\Lambda$-efficiency for a point estimate is, like that of generalized estimator $g$, simply the square of the correlation between the estimator and the score

$$\text{Eff}^\Lambda(\hat{\theta}) = \rho^2(\hat{\theta}, \ell').$$

For unbiased $\hat{\theta}$ and $\rho \neq 0$, the equality in (9) shows $\text{Eff}^V(\hat{\theta}) = \text{Eff}^\Lambda(\hat{\theta})$ so that $\Lambda$-efficiency is an extension of $V$-efficiency to unbiased point estimators. In fact, $\Lambda$-efficiency extends $V$-efficiency to all generalized estimators.

Since $I = nI_1$ when $\mathcal{Y}$ is $\mathcal{X}^n$, $\Lambda(g) = \rho^2 n I_1$ and $\Lambda$-efficiency can be interpreted in terms of an effective sample size

$$n(g) = \rho^2 n$$

where $\rho$ is the correlation between $g$ and $\ell'$. Or, the $\Lambda$-efficiency loss of using $g$ rather than $\ell'$ is a loss of $(1 - \rho^2)n$ observations.

4.2 Variance is Defined Point-wise

For any statistic $u : \mathcal{Y} \to \mathbb{R}$, the variance of $u$ for model $m$ depends only on $m$; it does not depend on the family $M$ that contains $m$. Fisher information is defined on the tangent bundle $TM$ and so cannot be defined for an isolated point outside of $M$. Fisher information and variance are fundamentally different quantities. Like Fisher information, $\Lambda$ is defined on $TM$ and the Fisher information bound,
\( \Lambda = \rho^2 I \), provides interpretations for finite samples. We illustrate the difference between the variance and squared slope \( \Lambda \) of an estimator using an example where a model \( m \) is not isolated but belongs to two different manifolds.

Let \( \mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2\} \), \( M \) be the two dimensional manifold of bivariate normal distributions with zero correlation and variance one, and \( \mu \) be the mean parameterization. Reduction of the sample space to the minimal sufficient statistic provides the two dimensional sample space \( \mathcal{Y} = \mathbb{R}^2 \). We consider submanifolds \( M_1 \) and \( M_2 \) that intersect in a point \( m_0 \). We start with \( M_1 \) consisting of all points having \( \mu = (\theta, 0) \) and \( M_2 \) all points with \( \mu = (0, \theta) \) so that \( m_0 \) is the point with \( \mu = (0, 0) \). In the expectation parameter space, \( M_1 \) is the horizontal axis and \( M_2 \) is the vertical axis. We compare two estimators: \( \bar{x}_j = n^{-1} \sum x_{ji} \), for \( j \in \{1, 2\} \). Both \( \bar{x}_1 \) and \( \bar{x}_2 \) have variance \( 1/n \) that equals the variance lower bound for all models in \( M_1 \) and \( M_2 \). Both estimators are unbiased for \( \theta \) at \( m_0 \).

In terms of bias and variance the estimators are indistinguishable at \( m_0 \). For inference in the family \( M_1 \), \( \bar{x}_1 \) has \( \rho(\bar{x}_1, \ell') = 1 \) while \( \bar{x}_2 \) has \( \rho(\bar{x}_2, \ell') = 0 \). The variance of an estimator at a point does not distinguish between an optimal estimator and one that is worthless (its distribution is the same for all models in the manifold).

In contrast, \( \Lambda_{M_1}(\bar{x}_1) = n \) attaining the slope upper bound while \( \Lambda_{M_1}(\bar{x}_2) = 0 \). The squared slope correctly distinguishes these estimators in terms of their inferential properties. The difference is that \( \Lambda_{M_1}(u) = \rho^2(u, \ell')n \) is a function of \( \ell' \in TM_1 \).

This problem disappears when sufficiency arguments are used to replace \( \mathcal{Y} \) with \( \mathcal{Y}_j = \{y : y = n^{-1} \sum x_{ji}\} \) for \( M_j \). However, by replacing \( M_1 \) and \( M_2 \) with manifolds whose expectation parameters are circles of large radii intersecting at \((0, 0)\) with centers on the vertical and horizontal axes the minimal sufficient statistic is two dimensional and the sample distributions for models in \( M_1 \) and \( M_2 \) are defined on \( \mathcal{Y} \). The expectation and variance of \( \bar{x}_1 \) and \( \bar{x}_2 \) at \( \mu = (0, 0) \) are unaffected by changing the manifolds; \( \Lambda_{M_1}(\bar{x}_j) \) will be a function of \( \theta \) that for \( \theta = 0 \) approaches the upper bound \( n \) when \( j = 1 \) and zero when \( j = 2 \).

The problem with using variance to assess an estimator is lessened when the variance is considered a function on the manifold. The argument that estimators should be assessed at all distributions is cogent, if not obvious, but begs the question of why estimators should not be allowed to depend on the distribution.

## 5 \( \mathcal{G} \) and Confidence Intervals

After selecting a generalized estimator \( g \) based on its aggregate properties, we use the resulting estimate \( g(\theta) \in C(\Theta) \) corresponding to a particular value \( y \in \mathcal{Y} \). We allow \( g(\theta) \) to be any generalized estimate although \( \Lambda \)-optimality would suggest using \( \ell'(\theta) \).

For \( g(\theta) \) monotone on \( \Theta \), the interval

\[
(\theta_{lo}, \theta_{hi}) = \{\theta : -k < \bar{g}(\theta) < k\}
\]

is the \( g \)-confidence interval for \( \theta \) at level \( k \). With \( g = \ell' \) and \( k = 2 \), \( (\theta_{lo}, \theta_{hi}) = \)
\[ \hat{s}^{-1}(\pm 2) \text{ consists of all models } m \text{ such that } y \text{ is within 2 standard deviations of the most likely sample value when } \mathcal{Y} \text{ is ordered by the score. The justification for ordering } \mathcal{Y} \text{ by } \ell' \text{ is that it has the greatest slope among all estimators in } \mathcal{G}. \text{ That is, } \ell' \text{ is } \Lambda\text{-optimal. This interpretation for } \hat{s}^{-1}(\pm 2) \text{ holds for any sample size } n; \text{ the interpretation describing the percentile that } y \text{ is in } \mathcal{Y} \text{ will depend on the extent to which the distribution of } \hat{s}(\theta) \text{ is approximated by the standard normal for all } \theta \in (\theta_{lo}, \theta_{hi}). \]

A linear approximation to the estimate \( \bar{g}(\theta) \) is of the form
\[ b(\theta - \hat{\theta}) \]
where \( b \) may depend on \( y \) but is constant in \( \theta \) and \( \hat{\theta} \) satisfies \( \bar{g}(\hat{\theta}) = 0 \). Function \( b(\theta - \hat{\theta}) \) is considered a good approximation to \( \bar{g}(\theta) \) when confidence intervals obtained from the linear function are a suitable approximation to those obtained from \( \bar{g}(\theta) \) over a subset \( \Theta_1 \subset \Theta \). That is, for \( k \in \mathbb{R} \),
\[ \{ \theta : -k < \bar{g}(\theta) < k \} = \{ \theta : \hat{\theta} + k/b < \theta < \hat{\theta} - k/b \} = \hat{\theta} \pm k|b|^{-1}. \tag{12} \]

Comparing \( \text{(12)} \) to the estimate-plus-standard-error approach, \( \hat{\theta} \pm k\hat{se} \), shows that, for unbiased \( \hat{\theta} \), the standard error \( \hat{se} = \sqrt{V(\hat{\theta})} \) is the reciprocal of the absolute value of the slope \( b \) of the linear approximation to \( \bar{g}(\theta) \). The linear approximation will not be parameter invariant.

\section{A Related Class of Estimators \( \mathcal{G}_{\text{lrt}} \)}

We consider the class of all functions \( G : \mathcal{Y} \to C(\Theta) \) such that \( G' \in \mathcal{G} \) and \( K = \sup_{\theta \in \Theta} G \) is finite and measurable. We denote the set of functions \( G - K \) as \( \mathcal{G}_{\text{lrt}} \) since the log likelihood ratio statistic in this class plays the role of the score in the class of estimators \( \mathcal{G} \). Following the same logic, the space \( \mathcal{G} \) defined in Section 3.1 would be denoted \( \mathcal{G}_{\text{score}} \). Note that \( \sup_{\theta} G = 0 \) for all \( G \in \mathcal{G}_{\text{lrt}} \). When \( G_{\theta} = G \) is expressed in parameter \( \xi \), \( \partial \xi G_{\Xi} = (\partial \xi \theta_{\Xi}) \partial_{\theta} G_{\theta} \) so that \( G_{\Xi} \sim G'_{\theta} \). Consequently, the functions in \( \mathcal{G}_{\text{lrt}} \), like those in \( \mathcal{G}_{\text{score}} \), are parameter-invariant and so each parameterized version is obtained from a function \( G_M \) having domain \( M \). Usually the context will indicate the domain of the function so we drop the subscript notation on \( G \).

The squared slope can be extended to \( \mathcal{G}_{\text{lrt}} \) by defining \( \Lambda(G) = \Lambda(G') \) where the interpretation is now in terms of the Laplacian of \( G \) rather than the slope of \( G' \). Since \( \hat{s} \) is \( \Lambda \)-optimal so is
\[ \hat{S}(\theta) = 2 \left( \ell(\theta, y) - \sup_{\theta \in \Theta} \ell(\theta, y) \right) \tag{13} \]
where the constant 2 is chosen for convenience as the distribution of $-\hat{S}(\theta)$ is approximately Chi-squared with 1 degree of freedom.

The choice between using $\hat{s}$ or $\hat{S}$ must depend on considerations other than $\Lambda$-optimality. Advantages of $\hat{s}$ are that the ordering of $Y$ accounts for the direction of departures from the model and confidence intervals can be interpreted in terms of standard deviations. When each model in $M$ specifies a symmetric distribution on $Y$, the tails specified by $\hat{s}$ are identical (mirror images) so that $\hat{S}$, or any $G$, can be used while still accounting for the direction of the departure. When $M$ is the family of normal location models, confidence intervals from $\hat{s}$ and $\hat{S}$ are identical. An advantage of $\hat{S}$ is that it can be used for inference for two or more parameters considered jointly.

7 Examples

7.1 Bernoulli Family

Consider a dichotomous attribute for individuals in a population. We use the sample space $\mathcal{X} = \{0, 1\}$ where 1 indicates the attribute is present and the statistical manifold is

$$M = \{m : \mathcal{X} \to \mathbb{R} : 0 < m(1) < 1, \ m(0) + m(1) = 1\}.$$ 

The sample space $\mathcal{X}^n$ can be reduced to $Y = \{0, 1, \ldots, n\}$ using the sufficient statistic $y = \sum x_i$. The log likelihood function for parameter $p(m) = m(1)$ is

$$\ell_P(p) = y \log p + (n - y) \log(1 - p) + k(n, y)$$

with score

$$\hat{s}_P(p) = n^{-1/2} \frac{y - np}{\sqrt{p(1-p)}}$$

For the log-odds parameterization $\theta = \theta(m) = \log(m(1)/m(0))$, the score is

$$\hat{s}_\Theta(\theta) = n^{-1/2} \frac{y - np_\Theta(\theta)}{\sqrt{p_\Theta(\theta)(1 - p_\Theta(\theta))}}$$

where $p_\Theta(\theta) = e^\theta/(e^\theta + 1)$.

7.1.1 Maximum Likelihood Estimator

For $y \notin \{0, n\}$ the maximum likelihood (ml) point estimates $\hat{p} = y/n$ and $\hat{\theta} = \theta_P(\hat{p})$ identify the same point in $M$, namely, the distribution for which $m(1) = y/n$. For $y \in \{0, n\}$ the ml point estimators are not defined so bias and variance cannot be used to compare these estimators. We extend the definition of maximum likelihood estimate in the next section where the variance is finite for the parameter $p$ but infinite for $\theta$. Figure 1 displays the eleven standardized score functions for both parameters when $Y = \{0, 1, \ldots, 10\}$. The location of the circles along the horizontal axis are the nine ml point estimates that exist.
Figure 1: Maximum likelihood (ml) point estimate for parameters $p$ and $\theta$. The nine open circles are the ml point estimates; the models in $M$ corresponding to the estimates in the $P$ are the same as those in $\Theta$. Maximum likelihood provides parameter invariant estimates.
Figure 2: Generalized estimator \( \hat{s} \). The two sets of eleven curves are the generalized estimator \( \hat{s} \) expressed in the \( P \) and \( \Theta \) parameter spaces. The distribution of \( \hat{s} \) is shown for four models in \( M \) as vertical slices of the curves. The distribution of \( \hat{s}(p)|_{p=1/2} \) is identical to that of \( \hat{s}(\theta)|_{\theta=0} \). This is true for all \( p \) and \( \theta \) when both label the same model in \( M \). Generalized estimators provide parameter invariant estimators.
For all \( y \in Y \) the generalized ml estimates \( \hat{s}(p) \) and \( \hat{s}(\theta) \) exist and for every \( m_\circ \in M \) \( \hat{s}(p_\circ) = \hat{s}(\theta_\circ) \) where \( p_\circ = p(m_\circ) \) and \( \theta_\circ = \theta(m_\circ) \) so the distribution of these estimators are parameter invariant. The curves in the left panel of Figure 2 are the eleven generalized estimates expressed using parameter \( p \); the right panel shows these generalized estimates using parameter \( \theta \). These are the same curves that appear in Figure 1 but we are now interested in the entire curve, not just where nine of these cross the parameter axis.

While the distribution for point estimators is restricted to the intersection of these curves with the horizontal axis, generalized estimators describe a continuum-of-hypotheses where each hypothesis considered in turn is represented by where these curves intersect the vertical line specified by a null value \( p_\circ \) or \( \theta_\circ \). Figure 2 displays the distribution for four hypotheses. Notice in each case the corresponding vertical distributions are the same for both parameterizations.

For any parameterization, these vertical distributions have the same mean and same standard deviation for all parameter values, and so provide little information for estimation. Estimation using this continuum of vertical distributions is described by the mean slope. For example, the mean of the eleven slopes at \( \theta = 0 \) has absolute value \( \sqrt{\Lambda(\hat{s}(\theta))|_{\theta=0}} \).

The Fisher information bound expressed in terms of the desideratum that generalized estimators should have large mean slopes is described as follows. For each \( g \in G \) there is a family of (at most) eleven curves that are the graphs over a parameter space, say \( \Theta \). We seek an estimator \( g \) such that the mean slope is large and the Fisher information bound says its absolute value is at most \( \sqrt{I(\theta)} \) for all \( \theta \in \Theta \). Since mean slope and Fisher information are tensors, Fisher information provides a bound for all parameterizations.

7.1.2 Uniformly Minimum Variance Unbiased Estimators

The estimator \( \tilde{p} = y/n \) for \( y \in Y \) equals the ml estimator for \( y \notin \{0, n\} \) and so is called the extended ml estimator. The extended ml estimator is the unique uniformly minimum variance unbiased estimator (UMVUE) for parameter \( p \).

The parameter \( \xi = p^2 \) taking values in \( \Xi = (0, 1) \) has UMVUE estimator \( \tilde{\xi} = y(y - 1)/(n(n - 1)) \) (Barnett 1999, page 151). There are two UMVU estimators, \( \tilde{p} \) and \( \tilde{\xi} \), and since they are not both unbiased in the same parameter, we cannot compare them directly in terms of their variances. We consider a third estimator \( (y/n)^2 \) that is unbiased for \( \eta = E(y/n)^2 \). The parameter \( \eta \) is a function of the sample size so its estimator is not likely to be selected based on its unbiasedness.

The generalized estimators associated with these point estimators

\[ h_j(\theta) = u_j - Eu_j \]

are defined by the statistics \( u_1 = y, u_2 = y(y - 1), \) and \( u_3 = y^2 \), respectively. Note \( \bar{h}_1 = \bar{s} \). Figure 3 shows the \( \Lambda \)-efficiencies for these statistics. The parameter \( \theta \) can be any parameterization, parameter \( p \) is used in Figure 3. The
curves will look different for other parameterizations but the vertical slices giving the efficiencies for corresponding points in the manifold are the same; that is, \( \Lambda \)-efficiency can be defined directly on \( M \).

In terms of \( \Lambda \)-efficiency, the estimate for \( p (u_1 = y) \) is fully efficient, the estimate for \( \xi (u_2 = y(y-1)) \) has the lowest efficiency approaching zero for small \( p \), and the estimate based on \( u_3 = y^2 \) has efficiency between the other statistics. The fact that \( u_2 = y(y-1) \) is zero for both \( y = 0 \) and \( y = 1 \) is consistent with efficiency approaching zero for models with \( p \) near zero. When \( p < 0.17 \) using \( \xi \) rather than \( \tilde{p} \) results in an efficiency loss of at least 2 observations. Note that \( u_1 \) and \( u_3 \) are both sufficient statistics but they are not both \( \Lambda \)-efficient.

Figure 3 shows an asymmetry in the estimator \( \tilde{\xi} \) whose efficiency is close to one for \( p \) near one but close to zero for \( p \) near zero. For a dichotomous population where one of the categories is rare, the efficiency of \( \tilde{\xi} \) will depend on which category is labeled 1. In other words, statistics \( y(y-1) \) and \( (n-y)(n-y-1) \) have very different efficiencies, even though they describe the same inference problem with labels for “success” and “failure” interchanged. To avoid this asymmetry, \( u \) must satisfy \( \rho^2(u(y), \ell') = \rho^2(u(n-y), \ell') \) which \( u = y \) does.
7.2 Normal Location Family

For a function \( f : \mathcal{X} \to \mathbb{R} \) such that \( f > 0 \) a.e. and \( \int f \, dx = 1 \), the set \( \{ f(x - \theta) : \theta \in \mathbb{R} \} \) defines a location family indexed by \( \theta \). All members of this family have the same shape, so without loss of generality, we choose the function \( f \) having \( \int_{-\infty}^{0} f \, dx = \frac{1}{2} \) so that \( \theta \) is the median of \( f(x - \theta) \). With suitable smoothness on \( f \) this set becomes our manifold \( M = \{ f(x - \theta) : \theta \in \mathbb{R} \} \).

For the normal location family with \( \sigma > 0 \) known, \( f = (2\pi)^{-1/2} \exp(-x^2/2) \) and the manifold is
\[
M_{\sigma} = \{ f((x - \theta)/\sigma)/\sigma : \theta \in \mathbb{R} \}.
\]

For this family \( \theta \) is also the mean and \( y = \bar{x} = n^{-1} \sum x_i \) is sufficient so that \( Y = \mathbb{R} \). The standardized score is \( \hat{s}(\theta) = \sqrt{n}/\sigma(\bar{x} - \theta) \), a line with slope \( -\sqrt{n}/\sigma \) intersecting the horizontal axis at \( \bar{x} \).

Figures for the normal family corresponding to those for the Bernoulli, Figures 1 and 2, would have a continuum of parallel lines in which case all vertical distributions are identical, each being \( \mathcal{N}(0, 1) \). For the normal manifold, the horizontal distribution of the point estimator can capture the structure of the vertical distributions that describe the continuum of hypotheses: the horizontal distribution for ml point estimator at the normal model with \( \theta = \theta_0 \) is the affine transformation \( (\sigma/\sqrt{n}) \mathcal{N}(0, 1) + \theta_0 = \mathcal{N}(\theta_0, \sigma^2/n) \).

Outside the normal model, asymptotics provide the continuum of parallel lines described above. Asymptotic calculations for a point estimator \( \hat{\theta} \) indicate the conditions under which, for large \( n \), the vertical distributions are normal and the corresponding generalized estimators are linear so that confidence intervals obtained from \( \hat{s} \) can be approximated by \( \hat{\theta} \pm z_{\alpha} \hat{s} \).

7.3 Cauchy Location Family

The manifold for the Cauchy family is \( M = \{ f_C(x - \theta) : \theta \in \mathbb{R} \} \) where \( f_C(x) = \pi^{-1}(x^2 + 1)^{-1} \) and \( Y = \mathcal{X}^n \) since there is no sufficient statistic of lower dimension. The log likelihood is
\[
\ell(\theta; y) = \sum \log((x_i - \theta)^2 + 1) - n \log \pi
\]
and the standardized score estimate for \( y \) is
\[
\hat{s}(\theta) = \left( \frac{2}{n} \right)^{1/2} \sum \frac{2(x_i - \theta)}{(x_i - \theta)^2 + 1}
\]
since the Fisher information is \( n/2 \).

Fisher (1925) considers estimates based on the sample median \( \tilde{\theta} \) whose density for a sample of size \( 2k + 1 \) is
\[
f_{\text{med}}(z; \theta) = \frac{(2k + 1)!}{(k!)^2 \pi} \left[ \frac{1}{4} - \frac{1}{\pi^2} \arctan^2(z - \theta) \right]^k \left[ 1 + (z - \theta)^2 \right]^{-1}.
\]
We consider two generalized estimators associated with the sample median,

\[
\hat{h}_{med} = \frac{\hat{\theta} - \theta}{\sqrt{V(\hat{\theta})}}
\]

\[
\hat{s} = \ell'_{med} / \sqrt{V(\ell'_{med})}
\]

where \(\ell'_{med} = \log f_{med}\) and \(Y_{med} = \{y : y = x_{(k+1)}\}\). Both \(h_{med}\) and \(\hat{s}\) are functions of the data reduced to the sample median while \(\hat{s}\) is a function of the full data \(y \in \mathbb{R}^n\).

Table 1 shows the squared slope and efficiency (Eff \(\times 100\%\)) for the sample median for odd sample sizes from 1 to 31. The values for \(\Lambda(\hat{s})\) and \(\Lambda(\hat{\theta})\) were obtained from Rider (1960) who reports \(\Lambda(\hat{s})\) and the variance of the median \((1/\Lambda(\hat{\theta}))\). Since the variance of \(\hat{\theta}\) does not exist for samples of size \(n = 1\) and \(n = 3\), \(\Lambda(\hat{\theta})\) is zero but note \(\Lambda(\hat{s}) > 0\). The loss of efficiency is consistent with the sample median not being a sufficient statistic. However, the loss of efficiency is also due to the fact that \(\hat{s}\) is a point estimate; \(\Lambda(\hat{s}) > \Lambda(\hat{\theta})\) even though both are a function of the data only through the sample median \(\hat{\theta}\).

The adjusted sample size \(n(\hat{s}) = n\eff(\hat{s})\) shows the information loss in terms of a reduction in sample size. For example, \(\hat{s}\) for a sample of size \(n = 15\) has smaller slope than \(\hat{s}\) for a sample of size \(n = 11\) \((n(\hat{s}) = 10.8)\) indicating that reducing the 15 observations to the sample median results in a loss of more than 4 observations. Using point estimator \(\hat{\theta}\) rather than generalized estimator \(\hat{s}\) (both having domain \(Y_{med}\)) results in the loss of an additional observation \((n(\hat{\theta}) = 9.8)\).

Table 1 does not include the ml point estimator \(\hat{\theta}\). To calculate \(\Lambda(\hat{\theta})\) we need the variance of \(\hat{\theta}\) and the derivative of its expectation. Instead of these calculations, we describe \(\hat{\theta}\) using a simulation of 100,000 samples of size \(n = 15\). Efron & Hinkley (1978) consider \(\hat{\theta}\) and \(\hat{\theta}\) conditional on the ancillary statistic

\[
a = (x_{(2)} - x_{(1)}, x_{(3)} - x_{(2)}, \ldots, x_{(n)} - x_{(n-1)})
\]

where \(x_{(i)}\) is the \(i\)th order statistic. To simplify notation we assume the observations are ordered so that \(x_i = x_{(i)}\).

Conditioning on \(a \in \mathbb{R}^{n-1}\) results in the sample space

\[
Y_a = \{y \in Y : a = (x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})\}
\]

which is a one dimensional subspace of \(Y \subset \mathbb{R}^n\).

The point estimators \(\hat{\theta}\) and \(\hat{\theta}_{\alpha}\) and generalized estimator \(\hat{S}\) are compared using the intervals

\[
\hat{\theta} \pm z_\alpha \hat{f}^{-1/2} = \left\{ \theta : -z_\alpha < \hat{f}^{1/2}(\theta - \hat{\theta}) < z_\alpha \right\}
\]

\[
\hat{\theta} \pm z_\alpha \ell_{obs}^{-1/2} = \left\{ \theta : -z_\alpha < \ell_{obs}^{1/2}(\theta - \hat{\theta}) < z_\alpha \right\}
\]

\[
\hat{S}^{-1}(z_\alpha^2) = \left\{ \theta : \hat{S}(\theta) > -z_\alpha^2 \right\}
\]

where \(\ell_{obs} = -\ell'(\theta = \hat{\theta})\) and \(\hat{f} = \ell'_{\theta = \hat{\theta}}\) are constants. The expressions after the equality show that the estimate-plus-se approach are linear approximations to
generalized estimators using slope $I_{\text{obs}}$ and $\hat{I}$, respectively. The choice of the latter slope is based on Fisher’s argument that $V(\hat{\theta}|a)$ is $1/I_{\text{obs}}$. This interval does not represent inference based on $(\hat{\theta}, I_{\text{obs}})$ or conditionally on $I_{\text{obs}}$. The test associated $\hat{\theta}|a$ orders points in $Y_{a}$. Barndorff-Nielsen (1978) comments on this confusion regarding the role of observed information.

We use confidence intervals $\hat{S}^{-1}(-z_{\alpha}^{2})$ rather than $\hat{s}^{-1}(\pm z_{\alpha})$ since the latter do not always exist, especially for $\alpha$ small. The score does not provide a sensible ordering on the entire sample space $Y$: for each $\theta$, $\lim_{y \to \pm \infty} \hat{s}(\theta)(y) = 0$. A reasonable ordering would have these limits tend to infinity or at least be bounded away from zero. McLeish & Small (2012, page 37) describe this difficulty as “the score function is $E$-ancillary ‘at infinity’ for a Cauchy model”. Their approach is to modify the score. We find it simpler to use $\hat{S}$.

Coverage error for confidence intervals having nominal 5% error are 7.5% ($\hat{\theta}$), 6.9% ($\hat{\theta}|a$), and 5.6% ($\hat{S}$) indicating the normal distribution is a better approximation to $\hat{S}$ than to the point estimators. McLeish & Small (2012, pages 2-3) note the normal distribution is often a better approximation to $\hat{s}$ than to its root $\hat{\theta}$. Quantile-quantile plots for 100,000 samples show the normal distribution to be a very close approximation to the distribution of $\text{sign}(\hat{\theta} - \theta)\hat{S}^{1/2}$ while the distributions of $\hat{\theta}$ and $\hat{\theta}|a$ show noticeable departures from normality. This relationship to the normal approximation holds for the other vertical and horizontal distributions: $\hat{s}$ is closely approximated the standard

| n  | $\Lambda(\hat{\theta})$ | $\Lambda(\hat{s})$ | Eff($\hat{\theta}$) | Eff($\hat{s}$) | n($\hat{\theta}$) | n($\hat{s}$) |
|----|--------------------------|--------------------|---------------------|----------------|-----------------|-------------|
| 1  | 0                        | .50000             | 0                   | 100            | 0               | 1           |
| 3  | 0                        | 1.09064            | 1.5                 | 0              | 72.71           | 0           | 2.2         |
| 5  | 0.81883                  | 1.74552            | 2.5                 | 32.75          | 69.82           | 4.9         | 6.3         |
| 7  | 1.63377                  | 2.44042            | 3.5                 | 46.68          | 69.73           | 5.5         | 7.8         |
| 9  | 2.44703                  | 3.90109            | 5.5                 | 59.26          | 69.73           | 6.5         | 7.8         |
| 11 | 3.25942                  | 4.65369            | 6.5                 | 62.64          | 71.60           | 8.1         | 9.3         |
| 13 | 4.07130                  | 5.41608            | 7.5                 | 65.10          | 72.21           | 9.8         | 10.8        |
| 15 | 4.88286                  | 6.18574            | 8.5                 | 68.48          | 73.28           | 11.4        | 12.4        |
| 17 | 5.69418                  | 6.96171            | 9.5                 | 71.64          | 74.14           | 13.0        | 13.9        |
| 19 | 6.50538                  | 7.74214            | 10.5                | 74.14          | 75.42           | 14.6        | 15.5        |
| 21 | 7.31647                  | 8.52636            | 11.5                | 77.42          | 76.73           | 16.3        | 17.1        |
| 23 | 8.12744                  | 9.31370            | 12.5                | 77.42          | 77.73           | 17.9        | 18.6        |
| 25 | 8.93839                  | 10.10363           | 13.5                | 77.42          | 77.73           | 19.5        | 20.2        |
| 27 | 9.74925                  | 10.89574           | 14.5                | 77.42          | 77.73           | 21.1        | 21.8        |
| 29 | 10.56011                 | 11.68970           | 15.5                | 77.42          | 77.73           | 22.7        | 23.4        |
| 31 | 11.37087                 | 12.47363           | 16.5                | 77.42          | 77.73           | 24.3        | 24.9        |

Table 1: The three columns to the right of the sample size $n$ are the squared slope for the median point estimate $\hat{\theta}$, the score based on the median $\hat{s}$, and the score based on the full sample $\hat{S}$. The next two columns give the percent efficiency and the final two columns give the effective sample sizes.
normal distribution while the median \( \tilde{\theta} \) is not.

Figure 4 shows that coverage error of \( \hat{\theta} \) intervals depend on \( I_{\text{obs}} \). When \( I_{\text{obs}} \) is small the coverage error is significantly larger than 5% and when \( I_{\text{obs}} \) is large the error significantly smaller than 5%. Conditioning on \( a \) addresses this problem. So does \( \hat{S} \). The generalized estimator has the advantage of avoiding other problems that can arise with ancillary statistics: the role of approximate ancillary statistics and choosing between multiple ancillary statistics.

To compare the lengths we adjust the margin of errors so all intervals have the same coverage error: \( z_{0.25} \). \( z_{0.25} \) is replaced with \( 1.08555 z_{0.25} (\hat{\theta}) \) and with \( 1.05518 z_{0.25} (\hat{\theta}|a) \). In keeping with our emphasis on parameter-invariance the length will be defined using Kullback-Leibler balls (disks in this case) contained in \( M \). For \( m_o \in M, r > 0 \), and \( D(m_1, m_2) = E_{m_1}(\log(m_1/m_2)) \) define

\[
B(m_o, r) = \{ m \in M : D(m, m_o) < r \}.
\]

The ball \( B(m_o, r) \) covers the interval \( (\theta_{lo}, \theta_{hi}) \) if \( m \in B(m_o, r) \) for all \( m \) such that \( \theta(m) \in (\theta_{lo}, \theta_{hi}) \). The KL length of \( (\theta_{lo}, \theta_{hi}) \) is the radius of the smallest KL ball that covers \( (\theta_{lo}, \theta_{hi}) \)

\[
\text{Len}_{\text{KL}}(\theta_{lo}, \theta_{hi}) = \inf \{ r : \text{for some } m_o \in M, B(m_o, r) \text{ covers } (\theta_{lo}, \theta_{hi}) \}.
\]

For the Cauchy family the KL divergence is

\[
D(m_1, m_2) = \log \left( (\theta(m_1) - \theta(m_2))^2 + 4 \right) - \log 4
\]

so that the KL length of the interval \( (\theta_{lo}, \theta_{hi}) \) is a function of \( |\theta_{hi} - \theta_{lo}| \). The location-scale Cauchy family also has a closed form for the KL divergence (see Chyzak & Nielsen (2019)).

The mean KL lengths are 0.141 (\( \hat{\theta} \)), 0.140 (\( \hat{\theta}|a \)), and 0.135 (\( \hat{S} \)). Conditioning on \( a \) provides little improvement in terms of mean KL length. It might be surprising that the mean length for \( \hat{\theta}|a \) is not closer to that of \( \hat{S} \) since conditioning on \( a \) recovers all the information as measured by Fisher information. Note that information calculations for \( \hat{\theta}|a \) use the score for statistic \( \hat{\theta} \) which differs from the linear approximation to the score used for the \( \hat{\theta}|a \) confidence intervals.

8 Discussion

Point estimators can be made parameter invariant by defining estimators as functions from \( Y \) to \( M \) so that models are compared directly to data without using any parameterization. Bias and variance that would be expressed in a particular parameter are replaced with KL bias and variance. Details appear in Wu & Vos (2012) and Vos & Wu (2015).

The class \( G_{\text{score}} \) is closely related to that of Godambe (1960). One difference is that Godambe uses \( g \) to obtain a point estimator while we consider the inferential properties of \( g \) as a function on \( M \). McLeish & Small (2012) emphasize the inferential value of \( g \) over its roots but still use \( g \) to find point estimators.
Figure 4: Coverage Errors and Observed Fisher Information. Simulations from 100,000 samples of size $n = 15$ were ordered by the observed Fisher information.
The role of $G_{\text{score}}$ and $G_{\text{lrt}}$ is to provide finite sample justification for inference based on the score and likelihood ratio using $\Lambda$-optimality. The purpose is not to introduce new estimators. For example, the $\hat{s}$ confidence interval (using the exact distribution) is the Clopper-Pearson interval. Many papers report expected length for binomial confidence intervals (e.g., [Brown et al. (2001)]) but we have found none that use mean KL length or other parameter invariant measures.

The squared slope $\Lambda$ applies to a wider class of estimators and provides simpler comparisons than does bias and variance. If one accepts that Fisher information describes important aspects of inference, then the fact that $\Lambda$ and $I$ are both tensors while the variance is not, strongly suggests that squared slope $\Lambda$ should be used to compare estimators, thereby supporting Fisher’s claim regarding the role of $I$ in finite samples.

Our justification for this claim relies on geometry, Fisher’s view on estimation, and his view on “repeated sampling from the same population.” [Fisher (1955)] said this phrase is foreign to his point of view. According to [Cox (2016)]

[Fisher] strongly emphasized that when probability was used to describe what underlay a set of data, he did not have in mind probability as a limiting frequency over a large number of repetitions. Rather, by probability Fisher meant a proportion in a hypothetical infinite population, the data being regarded as a random sample from that hypothetical population.

Each vertical distribution for a generalized estimator, represented in Figure 2, but common to all generalized estimators, corresponds to Fisher’s “hypothetical infinite population”.

Measure theory provides a rigorous treatment of Fisher’s idea of a proportion in an infinite population. This is in contrast to the notion of repeated sampling from the same population which is not part of mathematics. Rather, it is one interpretation for a measure theoretic probability. [Vos & Holbert (2022)] argue that making repeated sampling one of several possible interpretations rather than a defining aspect of frequentist inference can address the confusion regarding p-values. It seems the same is true for comparison of estimators: “repeated sampling from the same population” encourages one to think of a single distribution, $m$ perhaps, where bias and variance are easily understood, but hinders the understanding of geometric quantities such as $I$ and $\Lambda$ that are not defined for an isolated distribution. The full utility, simplicity, and beauty of geometric quantities, such as Fisher information and slope, are revealed once the notion of ‘hypothetical repeated sampling’ is no longer a defining property of frequentist inference.

References

Amari, S.-I. (1990). *Differential-geometrical methods in statistics*. Lecture notes in statistics. New York, NY: Springer.
Barndorff-Nielsen, O. (1978). Comments on paper by b. efron and d. v. hinkley. *Biometrika*, 65(3), 482.

Barnett, V. (1999). *Comparative Statistical Inference*. Wiley Series in Probability and Statistics. Chichester, England: John Wiley & Sons, 3 edition.

Brown, L. D., Cai, T. T., & DasGupta, A. (2001). Interval estimation for a binomial proportion. *Statistical Science*, 16(2).

Chyzak, F. & Nielsen, F. (2019). A closed-form formula for the Kullback-Leibler divergence between Cauchy distributions. *arXiv e-prints*, (pp. arXiv:1905.10965).

Cox, D. R. (2016). Some pioneers of modern statistical theory: a personal reflection. *Biometrika*, 103(4), 747–759.

Efron, B. & Hinkley, D. V. (1978). Assessing the accuracy of the maximum likelihood estimator: Observed versus expected fisher information. *Biometrika*, 65(3), 457–483.

Fisher, R. (1955). Statistical methods and scientific induction. *Journal of the Royal Statistical Society. Series B (Methodological)*, 17(1), 69–78.

Fisher, R. A. (1925). Theory of statistical estimation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 22(5), 700–725.

Godambe, V. P. (1960). An optimum property of regular maximum likelihood estimation. *The Annals of Mathematical Statistics*, 31(4), 1208–1211.

McLeish, D. L. & Small, C. G. (2012). *The theory and applications of statistical inference functions*. Lecture Notes in Statistics. New York, NY: Springer, 1988 edition.

Rider, P. R. (1960). Variance of the median of samples from a cauchy distribution. *Journal of the American Statistical Association*, 55(290), 322–323.

Vos, P. & Holbert, D. (2022). Frequentist statistical inference without repeated sampling. *Synthese*, 200(2).

Vos, P. & Wu, Q. (2015). Maximum likelihood estimators uniformly minimize distribution variance among distribution unbiased estimators in exponential families. *Bernoulli*, 21(4).

Wu, Q. & Vos, P. (2012). Decomposition of kullback–leibler risk and unbiasedness for parameter-free estimators. *Journal of Statistical Planning and Inference*, 142(6), 1525–1536.