Complex projective scheme approach to the geometrical structures of multipartite quantum systems

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Abstract

We investigate the geometrical structures of multipartite quantum systems using the language of complex projective schemes, which are fundamental objects in algebraic geometry. In particular, we will explicitly construct multi-qubit states in terms of these schemes and also discuss separability and entanglement of bipartite and multipartite quantum states.

1 Introduction

Multipartite quantum systems are very interesting complex composite systems which are defined on the projective Hilbert spaces. The geometrical structures of these complex projective spaces which are directly related to separability and entanglement of quantum states are very important in the field of quantum information and quantum computing. There are many approaches to consider the geometrical structure multipartite quantum systems, e.g., based on complex projective varieties [1, 2]. In this paper, we will investigate the geometrical structure of multipartite states based on the geometry of the complex projective schemes which generalize the concept of the complex projective varieties. Projective scheme are very important class of scheme in algebraic geometry [3, 4, 5, 6]. The scheme-theoretic construction of algebraic geometry formulated by A. Grothendieck and his coworkers. This construction is a solid ground for a grand unification of number theory and algebraic geometry. Grothendieck realized that an affine variety corresponds to a finitely generated integral domain over a field. Then, he generalized this construction by considered any commutative ring $A$, defining a topological space Spec($A$), and a sheaf of rings on Spec($A$) and he called it an affine scheme. Moreover, an arbitrary scheme can be obtained by gluing together affine scheme. Scheme-theoretic approach has advantage over the classical algebraic geometry by allowing geometric arguments about infinitesimals and limits in such way that are superior to the classical theory. This approach can be regarded as abstract. However, schemes make things simpler and theory behind the idea are directly related to differential and complex geometry and algebraic topology. Thus the basic definition of scheme theory appear convincing way of dealing with many ordinary geometric phenomena. In many branch of mathematic such as differential geometry the compact objects can be embedded in affine space, however it is not the case.
in algebraic geometry. Thus one needs to construct projective schemes which forms the most important family of schemes. The subject of this paper is to apply these scheme-theoretic tools to geometry of multipartite quantum system. In particular, in section 2 we will give a short introduction to sheaves which are necessarily for definition of schemes. Then, we define and discuss the basic properties of schemes. Finally, in section 3 we apply these mathematical objects to geometrical structures of single, bipartite, and multipartite quantum systems. We assume that the reader have some familiarity with algebraic varieties. However we will give a short introduction to sheaf theory and schemes. For those who want to known more about scheme we recommend following books [3, 4, 5, 6] which are also our main references.

2 Geometry of scheme

In differential and topological geometry, manifolds are made by gluing together open balls from Euclidean space. In analogy, schemes are made by gluing together simple open sets which are called affine schemes. However, there is an important difference between these two constructions. In manifold one point looks locally just like another. But schemes admit much more local variation e.g., by allowing the smallest open sets to be so large that many interesting geometry can happens within each one of them. Affine scheme is an object that is constructed from commutative ring. This is a generalization of the relationship between an affine variety and its coordinate ring. For example in the classical algebraic geometry we have correspondence between the set of affine variety and the finitely generated nilpotent-free rings over an algebraically closed field such as complex number field $K = \mathbb{C}$. In scheme-theoretic approach there is a correspondence between the set of affine scheme and the set of commutative rings with identity. Thus the ring and the corresponding affine scheme are equivalent objects. To give a general definition of scheme we need the concept of sheaf which we will introduced next.

2.1 Sheaf theory

Sheaves are standard objects in many branches of mathematics such as algebraic topology and geometry. The concept of sheaf gives a systematic way of keeping track of local data on topological space. But it was the Serre who introduced sheaf theory into algebraic geometry. Then, Grothendieck generalized Serre’s works to establish the theory of scheme. This include heavily use of homological algebra in algebraic geometry. Now, we will review the basic definition of sheaves. Let $X$ be a topological space. Then, a presheaf $\mathcal{E}$ on $X$ consists of a set $\mathcal{E}(U)$, where $U \subset X$ and a restriction map $\text{res}_{U_2, U_1} : \mathcal{E}(U_2) \rightarrow \mathcal{E}(U_1)$, where $U_1 \subset U_2$ such that the following axioms are satisfied:

1. $\text{res}_{U, U} = I_{\mathcal{E}(U)}$ and

2. If $U_1 \subset U_2 \subset U_3 \subset X$, then $\text{res}_{U_2, U_1} \circ \text{res}_{U_3, U_2} = \text{res}_{U_3, U_1}$.

The elements of $\mathcal{E}(U)$ are called the sections of $\mathcal{E}$ over $U$ and elements of $\mathcal{E}(X)$ are called global sections. Note also that a presheaf can be seen as a contravariant functor from the category of open sets in $X$ to the category of sets. Now, let
\[ \mathcal{E}_1, \mathcal{E}_2 \] be presheaves on \( X \) and \( \phi : \mathcal{E}_1 \to \mathcal{E}_2 \) be a collection of maps \( \phi(U) : \mathcal{E}_1(U) \to \mathcal{E}_2(U) \) for each open \( U \) such that if \( U \subset V \). Then \( \psi(U) \circ \text{res}_{V,U} = \text{res}_{V,U} \circ \phi(V) \). We call a presheaf \( \mathcal{E} \) a sheaf if for every collection \( \{U_i\} \) of open sets in \( X \) with \( U = \bigcup U_i \), we have an exact diagram
\[
\mathcal{E}(U) \longrightarrow \prod_{i,j} \mathcal{E}(U_i) \longrightarrow \prod_{i,j} \mathcal{E}(U_i \cap U_j), \tag{2.1.1}
\]
For example, the map \( \prod \text{res}_{U,U_i} : \mathcal{E}(U) \to \prod_i \mathcal{E}(U_i) \) is injective and its image is the set on which \( \prod_i \text{res}_{U,U_i} \circ \text{res}_{U_i \cap U_j} : \prod_i \mathcal{E}(U_i) \to \prod_{i,j} \mathcal{E}(U_i \cap U_j) \) and \( \prod \text{res}_{U_i \cap U_j} \circ \mathcal{E}(U_j) \to \prod_{i,j} \mathcal{E}(U_i \cap U_j) \) agree. The meaning of these diagrams are that if \( x_1, x_2 \in \mathcal{E}(U) \) and \( \text{res}_{U,U_1} x_1 = \text{res}_{U,U_2} x_2 \), then \( x_1 = x_2 \) for all \( i \) which also means that elements are uniquely determined by local data. Moreover, if for local collection of elements \( x_i \in \mathcal{E}(U_i) \) such that \( \text{res}_{U_i \cap U_j} x_i = \text{res}_{U_j \cap U_i} x_j \) for all \( i,j \), then there is an \( x \in \mathcal{E}(U) \) such that \( \text{res}_{U,U_i} x = x_i \) for all \( i \). This means that if we have local data which are compatible. Then, this local data actually patch together to form something in \( \mathcal{E}(U) \). Let \( \mathcal{E} \) be a sheaf on \( X \) and \( x \in X \). Then for \( U \) open containing \( x \), the collection of \( \mathcal{E}(U) \) is an inverse system and we have \( \mathcal{E}_x = \lim_{\xrightarrow{\text{local}} U} \mathcal{E}(U) \), that is, the disjoint union of \( \mathcal{E}(U) \) over all open sets \( U \) containing \( x \), modulo the equivalence relation \( a \sim b \) if \( a \in \mathcal{E}(U), b \in \mathcal{E}(V) \) and there is an open neighborhood \( W \) of \( x \) contained in \( U \cap V \) such that the restrictions of \( a \) and \( b \) to \( W \) are equal. \( \mathcal{E}_x \) is called the stalk of \( \mathcal{E} \) at \( x \). Usually, one writes \( \Gamma(U, \mathcal{E}) \) for \( \mathcal{E}(U) \) and calls it the set sections of \( \mathcal{E} \) over \( U \). Moreover, \( \Gamma(X, \mathcal{E}) \) is the set of global sections of \( \mathcal{E} \).

In following sections we also need the definition of the direct image of a sheaf. Let \( f : X \to Y \) be a continuous map of topological spaces. Then for a sheaf \( \mathcal{E} \) of additive groups over \( X \) we have the sheaf \( f_* \mathcal{E} \) over \( Y \) associated with the presheaf \( \mathcal{E}(f^{-1}(U)) \) which is called the direct image of \( \mathcal{E} \).

### 2.2 Scheme theory

In this section we will define presheaf, affine scheme and scheme. Note that the category of schemes is an enlargement of the category of varieties. We associate a geometric object to an arbitrary commutative ring \( R \) which called \( \text{Spec}(R) \), the spectrum of \( R \). If \( R \) is a finitely generated integral domain over an algebraic closed field, then \( \text{Spec}(R) \) will be almost the same as an affine variety associated to the ring \( R \). Moreover, we define \( \text{Spec}(R) \) to be the set of prime ideals \( P \not\subseteq R \). where \( R \) itself is not counted as a prime ideal but \( (0) \), if, prime, is counted. To distinguish a prime ideal \( P \) of \( R \) from a point \( P \) of \( \text{Spec}(R) \), we will write \( \{P\} \) for element of \( \text{Spec}(R) \). Next, we define a topology on \( \text{Spec}(R) \) which is called Zariski topology: Closed set is a sets of the form
\[
V(A) = \{\{P\} : P \supseteq A \text{ for some ideal } A \subseteq R\}. \tag{2.2.1}
\]
Now, we define the distinguishing open subset of \( \text{Spec}(R) \) by
\[
\text{Spec}(R)_f = \{\{P\} : f \text{ is not an element of } P\} = \text{Spec}(R) - V(f). \tag{2.2.2}
\]
Let \( Z \) be an irreducible closed subset of \( \text{Spec}(R) \). Then, a point \( z \in Z \) is called a generic point of \( Z \) if \( Z \) is the closure of \( z \). Now, we equip the topological
space \( \text{Spec}(R) \) with a sheaf of rings \( \mathcal{O}_{\text{Spec}(R)} \), which is called the structure sheaf of \( \text{Spec}(R) \). Moreover, let \( R_f \) be the localization of the ring \( R \) with respect to the multiplicative system \( \{ f, f^2, f^3, \ldots \} \) to the open set \( \text{Spec}(R)_f \). Furthermore, if \( [P] \in \text{Spec}(R)_f \), then \( f \) is not an element of \( P \) and there exists a natural map \( R_f \to R_f \), because the multiplicative system \( R - P \) contains the system \( \{ f, f^2, f^3, \ldots \} \). We have also

\[
R_P = \lim_{f \in R - P} R_f = \lim_{[P] \in \text{Spec}(R)_f} R_f. \tag{2.2.3}
\]

Next, let \( U \subset \text{Spec}(R) \) and \( \Gamma(U, \mathcal{O}_{\text{Spec}(R)}) \) be the set of elements \( \{ r_P \} \in \prod_{[P] \in U} R_P \) for which there exists a covering of \( U \) by distinguished open sets \( \text{Spec}(R)_f \) together with elements \( r_\beta \in R_\beta \) in such a way that \( r_P \) is equal to the image of \( r_\beta \in R_P \) whenever \( [P] \in \text{Spec}(R)_f \). Now, if \( V \subset U \), then we have the coordinate projection \( \prod_{[P] \in U} R_P \to \prod_{[P] \in V} R_P \) that gives \( \Gamma(U, \mathcal{O}_{\text{Spec}(R)}) \to \Gamma(V, \mathcal{O}_{\text{Spec}(R)}) \) and by taking the restriction map we get a presheaf \( \mathcal{O}_{\text{Spec}(R)} \) which is also a sheaf. Moreover, we have \( \Gamma(\text{Spec}(R)_f, \mathcal{O}_{\text{Spec}(R)}) = R_f \) and the stalk of \( \mathcal{O}_{\text{Spec}(R)} \) at \( [P] \) is \( R_P \), that is

\[
(\mathcal{O}_{\text{Spec}(R)})_P = \lim_{[P] \in \text{Spec}(R)_f} R_f = R_P. \tag{2.2.4}
\]

Let \( R \) be a commutative ring. Then, a prescheme is a topological space \( X \) with a sheaf of rings \( \mathcal{O}_X \) on \( X \) if there exists an open covering \( \{ U_\beta \} \) of \( X \) such that \( (U_\beta, \mathcal{O}_X|_{U_\beta}) \) is isomorphic to \( (\text{Spec}(R)_\beta, \mathcal{O}_{\text{Spec}(R)_\beta}) \). Moreover, an affine scheme is a prescheme \( (X, \mathcal{O}_X) \) that is isomorphic to \( (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) \). Furthermore, if \( (X, \mathcal{O}_X) \) and \( (Y, \mathcal{O}_Y) \) are two preschemes, then a morphism from \( X \) to \( Y \) is a continuous map \( f : X \to Y \) and a collection of homomorphism \( f^*_V : \Gamma(V, \mathcal{O}_Y) \to \Gamma(f^{-1}(V), \mathcal{O}_X) \) for open set \( V \subset Y \), such that

1. for open sets \( V_1 \subset V_2 \subset Y \), \( \text{res}_{V_2, V_1} : \Gamma(V_2, \mathcal{O}_Y) \to \Gamma(V_1, \mathcal{O}_Y) \), and \( \text{res}_{f^{-1}(V_2), f^{-1}(V_1)} \), we have \( \text{res}_{f^{-1}(V_2), f^{-1}(V_1)} f^*_V = f^*_V \circ \text{res}_{V_2, V_1} \).

2. for open set \( V \subset Y \) and \( x \in f^{-1}(V) \), and \( \mu \in \Gamma(V, \mathcal{O}_Y) \), we have \( \mu(f(x)) = 0 \implies f^*_V(\mu)(x) = 0 \).

Note that the category of affine schemes is isomorphic to the category of commutative rings with unit if we reverse the arrows. Moreover, a pair \( (X, \mathcal{O}_X) \) consisting of a topological space \( X \) and \( \mathcal{O}_X \) of commutative rings is also called a ringed space. Let \( \phi : R \to S \) be a ring homomorphism, \( \phi^c : \text{Spec}(S) \to \text{Spec}(R) \) be a continuous map defined by \( P \mapsto \phi^{-1}(P) \), and \( \phi^* : \mathcal{O}_{\text{Spec}(R)} \to \mathcal{O}_{\text{Spec}(S)} \) induced by homomorphism \( \phi_f : R_f \to S_{\phi(f)} \) for an open set of \( \text{Spec}(R) \), and \( f \in R \). Then, the ring homomorphism \( \phi \) determines a morphism of ringed spaces between the induced affine schemes

\[
(\phi^c, \phi^*) : (\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)}) \to (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}). \tag{2.2.5}
\]

Now, if there exists an open covering \( \{ U_i \}_{i \in I} \) of \( X \) such that \( (U_i, \mathcal{O}_{U_i}) \) isomorphic to an affine scheme as a local ringed space, then \( (X, \mathcal{O}_X) \) is said to be a scheme. The topological space \( X \) is called the underlying space and \( \mathcal{O}_X \) is called the structure sheaf.

A sheaf \( \mathcal{E} \) over a scheme \( (X, \mathcal{O}_X) \) is said to be an \( \mathcal{O}_X \)-module whenever the following condition is satisfied. For each open set \( U \), \( \mathcal{E}(U) \) is an \( \mathcal{O}_X(U) \)-module such that for open sets \( V \subset U \), \( p_V : \mathcal{O}_X(U) \times \mathcal{E}(U) \to \mathcal{E}(U) \), \( p_V : \)
\( \mathcal{O}_X(V) \times \mathcal{E}(V) \rightarrow \mathcal{E}(V) \), and \( \theta : \mathcal{O}_X(U) \times \mathcal{E}(U) \rightarrow \mathcal{O}_X(V) \times \mathcal{E}(V) \) we have \( \text{res}_{U,V} \circ p_U = p_V \circ \theta \), where \( p_U \) and \( p_V \) indicate that \( \mathcal{O}_X(U) \) and \( \mathcal{O}_X(V) \) are module structures on \( \mathcal{E}(U) \) and \( \mathcal{E}(V) \). If \( \mathcal{E} \) and \( \mathcal{F} \) are additive groups, then the direct sum \( \mathcal{E}(U) \oplus \mathcal{F}(U) \) is also a sheaf, where \( U \) is an open set and we will denote it by \( \mathcal{E} \oplus \mathcal{F} \). Moreover, If \( \mathcal{E} \) and \( \mathcal{F} \) are sheaves of \( \mathcal{O}_X \)-modules, then \( \mathcal{E} \oplus \mathcal{F} \) is also \( \mathcal{O}_X(U) \). In general one writes \( \mathcal{O}^{\oplus n}_X = \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X \). Furthermore, if an \( \mathcal{O}_X \)-module \( \mathcal{E} \) is isomorphic to \( \mathcal{O}^{\oplus n}_X \) as an \( \mathcal{O}_X \)-modules, then \( \mathcal{E} \) is said to be a free module of rank \( n \) and is said to be locally free \( \mathcal{O}_X \)-module of rank \( n \) if there exists an open covering \( \{ U_i \}_{i \in I} \) of \( X \) such that the map \( \mathcal{E}|_{U_i} \rightarrow \mathcal{E} \) to \( U_i \) is a free module of rank \( n \) over \( \mathcal{O}_{U_i} = \mathcal{O}_X|_{U_i} \). Finally a locally free \( \mathcal{O}_X \)-module of rank \( n \) is called a locally free sheaf of rank \( n \) and in particular, if \( n = 1 \) is said to be an invertible sheaf over \( X \). Invertible sheaf are important in construction of multipartite systems which we will encounter in following section. Let \( \mathcal{E} \) be an \( \mathcal{O}_X \)-module and also let \( \mathcal{O}_U \) denotes the sheaf \( \mathcal{O}_X \) restricted to an open set \( U \). Then, for each point \( x \in X \) there exists an open neighborhood \( U \) of \( x \) such that the sequence of \( \mathcal{O}_X \)-modules

\[
\mathcal{O}^{\oplus I}_X \rightarrow \mathcal{O}^{\oplus J}_X \rightarrow \mathcal{E}|_U \rightarrow 0 \quad (2.2.6)
\]

is exact and \( \mathcal{E} \) is called a quasicoherent sheaf, where \( I \) and \( J \) are two indexing sets which does not need to be finite.

We will discuss the construction of complex projective scheme in the next section in relation with the space of a general quantum system. However, there are still some new terms such as closed immersion and tensor algebras which we need to define in this section. A closed immersion is a morphism \( f : X \rightarrow Y \) of schemes such that \( f \) induces a homeomorphism of underlying topological space of \( X \) to underlying topological space of \( Y \). Moreover, the induced map \( f^* : \mathcal{O}_X \rightarrow f_* \mathcal{O}_Y \) of sheaves on \( X \) is surjective. Note that one can define an abstract complex variety to be an integral separated scheme of finite type over algebraically closed complex field \( \mathbb{C} \). Let \( (X, \mathcal{O}_X) \) be a ringed space, and let \( \mathcal{E} \) be a sheaf of \( \mathcal{O}_X \)-modules. Then we define the tensor algebra \( T(M) = \bigoplus_{m \geq 0} T^m(M) = \bigoplus_{m \geq 0} (M \otimes \cdots \otimes M) \), symmetric algebra \( S(M) = \bigoplus_{m \geq 0} S^m(M) \), and exterior algebra \( \wedge(M) = \bigoplus_{m \geq 0} \wedge^m(M) \) of \( \mathcal{E} \) by taking the sheaves associated to presheaf. This operation assigns to each open set \( U \) corresponding tensor operation applied to \( \mathcal{E}(U) \) as an \( \mathcal{O}_X \)-module. Thus the results are \( \mathcal{O}_X \)-algebras and their components in each degree are \( \mathcal{O}_X \)-modules.

## 3 Quantum systems and schemes

In this section we develop the geometrical construction of multipartite quantum system based on the complex projective schemes. We begin our construction with the simplest quantum system namely a general single quantum state. Then, we in detail construct multi-qubit quantum system. Our construction is abstract but it may gives some new insight to the geometrical structure of quantum systems. A general, composite quantum system with \( m \) subsystems is denoted by \( \mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_m \), consisting of the pure states

\[
|\Psi\rangle = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} \alpha_{i_1 i_2 \cdots i_m} |i_1 i_2 \cdots i_m\rangle \in \mathcal{H} \mathcal{Q} = \mathcal{H} \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{H} \mathcal{Q}_m, \quad (3.0.7)
\]
where \( N_j + 1 = \dim(\mathcal{H}_{Q_j}) \) is the dimension of the \( j \)th Hilbert space. In the following text, we will construct the Hilbert space \( \mathcal{H}_Q \) of a quantum system \(|\Psi\rangle\) as a complex projective scheme. Then, we will discuss the properties of this state based on our construction.

3.1 Construction of a single quantum system based on complex projective scheme

First we will discuss the geometrical structure of a general single quantum system \(|\Psi\rangle = \sum_{i=0}^{N_i} \alpha_i |i\rangle = \sum_{i=0}^{N} \alpha_i |i\rangle \). For an \((N+1)\)-dimensional vector space \( E \) over a complex number field \( \mathbb{C} \), the spectrum \( \mathbb{P}(E) \) of complex-valued points of \( \mathbb{P}(E) \) over \( \mathbb{C} \) is isomorphic to \((\mathbb{C}^{N+1}) \setminus \{(0,0,\ldots,0)\})/\sim\) as sets, where the equivalence relation \( \sim \) is defined by \((x_0, \ldots, x_N) \sim (y_0, \ldots, y_N) \Leftrightarrow \exists \lambda \in \mathbb{C} - \{0\}, \) such that \( x_i = y_i \forall \frac{1}{N} \leq i \leq N \).

Let \( R = \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_N] \) be the ring of polynomial in \( N + 1 \) variables over complex number field \( \mathbb{C} \), \( R_d \) be the set of all homogeneous polynomials of degree \( d \), and \( I_d \) be the set of polynomials in \( R_d \) and in an ideal \( I \) of \( R \), that is \( I_d = I \cap R_d \). Then, \( I = \bigoplus_{d=0}^{\infty} I_d \) is called the homogeneous ideal of \( R \). If a prime ideal \( P \) is homogeneous, the \( P \) is called homogeneous prime ideal. Now, we define

\[
P^N_{\mathbb{C}} = \{ P : P \text{ is a homogeneous prime ideal of } R \text{ and } P \neq \{\alpha_0, \alpha_1, \ldots, \alpha_N\} \}
\]

and by defining \( V(I) = \{ P \in P^N_{\mathbb{C}} : I \in P \} \) as a closed set of \( P^N_{\mathbb{C}} \) we define a topology on \( P^N_{\mathbb{C}} \). We can also form an open set for \( P^N_{\mathbb{C}} \) by \( D(I) = \{ P \in P^N_{\mathbb{C}} : f \not\in P \} \) for some homogeneous polynomial \( f \in R \) and for this open set we also define

\[
\Gamma(D(I), \mathcal{O}_{P^N_{\mathbb{C}}}) = \{ \frac{g}{f^n} : g \in R, \text{ } g \text{ is homogeneous with } \deg g = n \deg f, \text{ } n \geq 1 \}.
\]

and we also obtain the ring space \( (\mathcal{O}_{P^N_{\mathbb{C}}}, P^N_{\mathbb{C}}) \), where the structure sheaf \( \mathcal{O}_{P^N_{\mathbb{C}}} \) of \( P^N_{\mathbb{C}} \) is defined in similar manner as the affine scheme. Then lengthy construction one can see that this ringed space has a complex projective scheme structure. For example we define \( U_i = D(\alpha_i), \) \( P^N_{\mathbb{C}} = \bigcup_{i=0}^{\infty} U_i, \) and \( R_i = \mathbb{C}[\alpha_0, \alpha_1, \ldots, \frac{\alpha_i}{\alpha_0}, \ldots, \frac{\alpha_{i-1}}{\alpha_0}, \ldots, \frac{\alpha_1}{\alpha_0}] \). Then one needs to show that \( U_i \) is homeomorphic to \( \text{Spec}(R_i) \). Moreover, we have \( \Gamma(U_i, \mathcal{O}_{P^N_{\mathbb{C}}}) = R_i \) and \( \Gamma(U_i \cap \text{Spec}(F), \mathcal{O}_{P^N_{\mathbb{C}}}) = (R_i)_{f} \), where \( F = \alpha_i^{\deg f} \). Thus we can conclude that \( \Gamma(U_i, \mathcal{O}_{P^N_{\mathbb{C}}}|U_i) \) is isomorphic to \( (\text{Spec}(R_i), \mathcal{O}_{\text{Spec}(R_i)}) \).

Now we will illustrate this construction by explicitly discuss the space of a quantum bit (qubit) \(|\Psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \). Let \( U_0 = (\text{Spec}\mathbb{C}[\alpha_0], \mathcal{O}_{\text{Spec}(\alpha_0)}) \) and \( U_1 = (\text{Spec}\mathbb{C}[\alpha_1], \mathcal{O}_{\text{Spec}(\alpha_1)}) \). Then we can define an affine scheme structure on an open set \( X_{\alpha_0} = D(\alpha_0) \) of \( X = U_0 = \text{Spec}\mathbb{C}[\alpha_0] \) as

\[
U_{01} = (\text{Spec}\mathbb{C}[\alpha_0, 1/\alpha_0], \mathcal{O}_{\text{Spec}(\alpha_0, 1/\alpha_0)}), \quad (3.1.3)
\]

where \( \mathcal{O}_{\text{Spec}(\alpha_0, 1/\alpha_0)} = \mathcal{O}_{X_{\alpha_0}} \). Moreover, we define \( D(\alpha_1) \) in similar manner. Furthermore, the isomorphic \( \phi : \mathbb{C}[\alpha_1, 1/\alpha_1] \rightarrow \mathbb{C}[\alpha_0, 1/\alpha_0] \) defined by \( f(\alpha_1, 1/\alpha_1) \mapsto f(\alpha_0, 1/\alpha_0) \) induces an isomorphism of affine scheme \( (\phi^\circ, \phi_0) : U_{01} \rightarrow U_{10} \). Then by gluing \( U_0 \) and \( U_1 \) through this isomorphism gives the scheme \( \mathbb{P}^1_{\mathbb{C}} = (Z, \mathcal{O}_Z) \), where \( Z = X \sqcup_{\phi^\circ} Y \) is obtained by gluing \( X \) and \( Y \) by
identifying the open sets $X_\alpha$ and $Y_\alpha$ by the map $\phi^\ast$. We have also $O_Z|X = O_X$ and $O_Z|Y = O_Y$. Thus the structure sheaf $O_Z$ is obtained by identifying $O_X|X_\alpha$ and $O_Y|Y_\alpha$ by $\phi^\ast$.

We can also generalize the above construction as follows. A commutative ring $S = \bigoplus_{d=0}^\infty S_d$ that satisfies $S_d S_e \subset S_{d+e}$ is called graded ring. For example let $S = \mathbb{C}[\alpha_0, \alpha_1, \ldots, \alpha_N]$ be the ring of polynomial in $N+1$ variables over complex number field $\mathbb{C}$. Moreover, let

$$S_d = \{ F \in S : \text{ F be a homogeneous polynomials of degree } d \}. \quad (3.1.4)$$

Then $S = \bigoplus_{d=0}^\infty S_d$ is a graded ring. Now, an ideal $I = \bigoplus_{d=0}^\infty I_d$ of $S$, where $I_d = I \cap S_d$ is called a homogeneous ideal. Note that, an ideal $I$ is homogeneous if and only if for an arbitrary element $F = \bigoplus_{d=0}^\infty F_d$, we have $F_d \in I$ for all $j = 1, 2, \ldots, l$. If $I$ is prime, then is called the homogeneous prime ideal of $S$.

For example, $S_+ = \bigoplus_{d=1}^\infty S_d$ is a homogeneous ideal of the graded ring $S$. Next, we define

$$\text{Proj} S = \{ P : P \text{ is a homogeneous prime ideal of } S \text{ and } P \not\supset S_+ \}, \quad (3.1.5)$$

to be the homogeneous prime spectrum of $S$. Now, the Zariski topology can be defined on $\text{Proj} S$ by taking $V(Q) = \{ P \in \text{Proj} S : Q \supset P \}$ as closed set. Moreover, we define an open set in $\text{Proj} S$ by

$$D(f) = \{ P \in \text{Proj} S : f \text{ is not an element of } P, \text{ for } f \in S_d \}. \quad (3.1.6)$$

Now, we will define the structure sheaf $O_{\text{Proj} S}$ by defining

$$\Gamma(D(f), O_{\text{Proj} S}) = \{ g/f^n : g \in S_{nd}, \ n \geq 1 \}. \quad (3.1.7)$$

An open set in $\text{Proj} S$ has a covering by the open set of type $D(f)$. There, we can define the the sheaf $O_{\text{Proj} S}$ of commutative rings over $\text{Proj} S$ and we obtain a local ringed space $(\text{Proj} S, O_{\text{Proj} S})$. Moreover, for the construction of the sheaf $O_{\text{Proj} S}$ we conclude that $(D(f), O_{\text{Proj} S})/D(f)$ is isomorphic to affine scheme $(\text{Spec} S_0, O_{\text{Spec} S_0})$, where $S_0 \subset S_1$ consists of elements in $\{ g/f^n : g \in S_{nd}, \ n \geq 1 \}$ is a commutative ring. We call $(\text{Proj} S, O_{\text{Proj} S})$ a projective scheme determined by the graded ring $S$. Now, based on this construction, the space of a single qubit is the one dimensional complex projective scheme $\mathbb{P}_C^1 = \text{Proj} \mathbb{C}[\alpha_0, \alpha_1]$. For this scheme we have $\Gamma(\mathbb{P}_C^1, O_{\mathbb{P}_C^1}) = O_{\mathbb{P}_C^1}(\mathbb{P}_C^1) = \mathbb{C}$.

### 3.2 Scheme-theoretic construction of multi-qubit states

In this section, we will investigate the geometrical structure of multi-qubit quantum states $|\Psi\rangle = \sum_{i_0, i_1, \ldots, i_m=0}^1 \sum_{j_0, j_1, \ldots, j_m=0}^1 \alpha_{i_0, i_1, \ldots, i_m} |i_1 j_2 \cdots i_m \rangle$ based complex projective scheme. Our construction is based on a map called the Segre morphism. Let us consider the following map

$$\mathbb{P}_C^1 \times \mathbb{P}_C^1 \times \cdots \times \mathbb{P}_C^1 \quad \rightarrow \quad \mathbb{P}_{C}^{2^m - 1} = \text{Proj} \mathbb{C}[\alpha_{00 \cdots 0}, \alpha_{00 \cdots 1}, \ldots, \alpha_{11 \cdots 1}]$$

$$\left((\alpha^0_0 : \alpha^0_1), \ldots, (\alpha^{m-1}_0 : \alpha^{m-1}_1)\right) \quad \mapsto \quad \left(\ldots, \alpha^0_{i_1} \alpha^1_{i_2} \cdots \alpha^{m-1}_{i_m}, \ldots\right). \quad (3.2.1)$$

where $\alpha_{i_1 i_2 \cdots i_m}, 0 \leq i_j \leq 1$ are homogeneous coordinate-functions on $\mathbb{P}_{C}^{2^m - 1}$. Moreover, let $X_1 = \text{Proj} \mathbb{C}[\beta^1_0 : \beta^1_1] = \mathbb{P}_C^1, \ldots, X_m = \text{Proj} \mathbb{C}[\beta^{m-1}_0 : \beta^{m-1}_1] = \mathbb{P}_C^1.$
and $f_j : X_j \rightarrow \text{Spec}\mathbb{C}$ for all $j = 1, 2, \ldots, m$. Furthermore, let $g : Z = X_1 \times_{\text{Spec}\mathbb{C}} X_2 \rightarrow \text{Spec}\mathbb{C}$ be the structure morphism and

$$p_j : Z = X_1 \times_{\text{Spec}\mathbb{C}} X_2 \times_{\text{Spec}\mathbb{C}} \cdots \times_{\text{Spec}\mathbb{C}} X_2 \rightarrow X_j$$

be the projection for $j = 1, 2, \ldots, m$. So, we have $\Gamma(X_j, \mathcal{O}_{X_j}(1)) = \mathbb{C}\beta_0^j \oplus \mathbb{C}\beta_0^j = V_j$. Now, the natural $\mathcal{O}_X$-homomorphism

$$\gamma_j : f_j^*V_j = \mathcal{O}_{X_j},\beta_0^j \oplus \mathcal{O}_{X_j},\beta_0^j \rightarrow \mathcal{O}_{X_j}(1)$$

is surjective and $\phi(\mathcal{O}_{X_j}(1),\gamma_j) : X_j \rightarrow \mathbb{P}(V_j)$ is an isomorphism over $\mathbb{C}$. Next we construct the invertible sheaf $\mathcal{L}$ over $Z$ by

$$\mathcal{L} = p_1^*\mathcal{O}_{X_1}(1) \otimes p_2^*\mathcal{O}_{X_2}(1) \otimes \cdots \otimes p_m^*\mathcal{O}_{X_m}(1).$$

Then, we have also the following surjective homomorphism

$$\gamma : g^*(V_1 \otimes V_2 \otimes \cdots \otimes V_m) = p_1^*f_1^*(V_1) \otimes p_2^*f_2^*(V_2) \otimes \cdots \otimes p_m^*f_m^*(V_m) \rightarrow \mathcal{L}.$$  

This construction gives us a scheme morphism

$$\phi(\mathcal{L},\gamma) : Z \rightarrow \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_m) \simeq \mathbb{P}^{2m-1},$$

since $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ is isomorphic to $\mathbb{C}^{2m}$. Then one can show that the map $Z(\mathbb{C}) = \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_m)(\mathbb{C}) = \mathbb{P}^{2m}(\mathbb{C})$. Let us consider the map $\psi_j : V_j \rightarrow \mathbb{C}$. Then the $\mathbb{C}$-value point of the space $X_j$ are in one-to-one correspondence with the equivalence classes of non-zero map $\psi_j$ which is uniquely determined by $\psi_j(\beta_j^{m-1}) = \alpha_0^{j-1}$. Thus $((\alpha_0^0 : \alpha_0^1), \ldots, (\alpha_0^{m-1} : \alpha_1^{m-1}))$ corresponds to an element of $X_1(\mathbb{C}) \times X_2(\mathbb{C}) \times \cdots \times X_2(\mathbb{C})$ in one-to-one way. That is, the image of $((\alpha_0^0 : \alpha_0^1), \ldots, (\alpha_0^{m-1} : \alpha_1^{m-1}))$ under the morphism

$$\phi(\mathcal{L},\gamma) : Z \rightarrow \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_m)$$

corresponds to the equivalent class of

$$\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_m : V_1 \otimes V_2 \otimes \cdots \otimes V_m \rightarrow \mathbb{C}.$$  

This map is uniquely determined by

$$\psi(\beta_j^0 \otimes \beta_j^1 \otimes \cdots \otimes \beta_j^{m-1}) = \alpha_0^0 \alpha_0^1 \cdots \alpha_0^{m-1} \equiv \alpha_{00}\ldots\alpha_{01} \equiv \alpha_{10}\ldots\alpha_{11} \equiv \alpha_{11}\ldots\alpha_{10},$$

which is the equivalent class of $\psi$ corresponding to

$$(\alpha_0^0 \cdots \alpha_0^{m-1}, \alpha_0^1 \cdots \alpha_0^{m-1}, \cdots, \alpha_1^0 \cdots \alpha_1^{m-1}) \equiv (\alpha_{00}\ldots\alpha_{01}, \alpha_{01}\ldots\alpha_{10}, \cdots, \alpha_{11}\ldots\alpha_{10}).$$

Thus the map $\phi(\mathcal{L},\gamma)$ is exactly the Segre map (3.2.1) and the space of separable state of a multi-qubit quantum state is give by $\text{Im}\phi(\mathcal{L},\gamma)$ the image of $\phi(\mathcal{L},\gamma)$

$$\text{Im}\phi(\mathcal{L},\gamma) = \{\alpha_{i_1i_2\ldots i_m} \in \mathbb{P}_C^{2m-1} : \alpha_{[k_1k_2\ldots k_m\alpha_{i_1i_2\ldots i_m}] = 0, 0 \leq i_j \leq 1, i = k, l}\}.$$
The measure of entanglement for multi-qubit states can also be constructed based on quadratic polynomial defining the Segre morphism which also coincides with well-known concurrence for two-qubit and three-qubit states [1]. There is also another method that generalizes the above map based on quasicoherent sheaf which we would like to discuss next. Let \( E_j \) for all \( j = 1, 2, \ldots, m \) be quasicoherent sheaves over a scheme \( X \) and \( P(E_j) \) be the projective schemes obtained from \( E_j \). Moreover, let \( \pi_j : P(E_j) \to X \) be the structure morphisms. Furthermore consider the projection

\[
p_j : P(E_1) \times_X \cdots \times_X P(E_m) \to P(E_j), \quad j = 1, 2, \ldots, m.
\]

Then we have \( \pi = \pi_j \circ p_j \). Now, let also

\[
\mathcal{L} = p_1^* \mathcal{O}_P(E_1)(1) \otimes p_2^* \mathcal{O}_P(E_1)(1) \otimes \cdots \otimes p_m^* \mathcal{O}_P(E_1)(1). \tag{3.2.12}
\]

Then, a surjective homomorphism \( \pi_j^* E_j \to \mathcal{O}_P(E_1)(1) \) gives the surjective homomorphism

\[
\gamma : \pi^* (E_1 \otimes_X \cdots \otimes_X E_m) = p_1^* (\pi_1^* E_1) \otimes \cdots \otimes p_m^* (\pi_m^* E_m) \to \mathcal{L}. \tag{3.2.13}
\]

Thus, we can define the Segre morphism by

\[
\phi_{(\mathcal{L}, \gamma)} : P(E_1) \times_X \cdots \times_X P(E_m) \to P(E_1 \otimes_X \cdots \otimes_X E_m). \tag{3.2.14}
\]

One can also see that the Segre morphism is a closed immersion. This construction may seems abstract but is a very natural generalization of the Segre variety which is very important in the construction of the geometrical structures of multipartite quantum systems.

### 3.3 Geometrical structure of a two-qubit state based on complex projective scheme

Now, we illustrate our construction by working out in detail a non-trivial example of quantum system, namely a two qubit state. The image of the map \( P^1 \times P^1 \to P^3 = \text{Proj}[a_{00}, a_{01}, a_{10}, a_{11}] \) defined by

\[
((a_0^0 : a_1^0, a_0^1 : a_1^1)) \mapsto (a_0^0 a_1^0, a_0^0 a_1^1, a_0^1 a_1^0, a_0^1 a_1^1) \tag{3.3.1}
\]

is a quadratic surface in \( P^3 \). Now, we want to construct this surface in terms of schemes. Let \( X_1 = \text{Proj}[a_0^0 : a_1^0] = P^1 \) and \( X_2 = \text{Proj}[a_0^0 : a_1^1] = P^1 \). Moreover, let \( f_1 : X_1 \to \text{Spec} \mathbb{C} \) and \( f_1 : X_1 \to \text{Spec} \mathbb{C} \). Furthermore, let \( g : Z = X_1 \times_{\text{Spec} \mathbb{C}} X_2 \to \text{Spec} \mathbb{C} \) be the structure morphism and \( p_j : Z = X_1 \times_{\text{Spec} \mathbb{C}} X_2 \to X_j \) be the projection for \( j = 1, 2 \).

So, we have \( \Gamma(X_1, \mathcal{O}_{X_1}(1)) = \mathbb{C}a_0 \oplus \mathbb{C}a_1 = V_1 \) and \( \Gamma(X_2, \mathcal{O}_{X_2}(1)) = \mathbb{C}a_0 \oplus \mathbb{C}a_1 = V_2 \). Now, the natural \( \mathcal{O}_X \)-homomorphism

\[
\gamma_1 : f_1^* V_1 = \mathcal{O}_{X_1,a_0^0} \oplus \mathcal{O}_{X_1,a_0^1} \to \mathcal{O}_{X_1}(1) \tag{3.3.2}
\]

is surjective and \( \phi_{(\mathcal{O}_{X_1}(1), \gamma_1)} : X_1 \to \text{Proj}(V_1) \) is an isomorphism over \( \mathbb{C} \). Next we construct the invertible sheaf over \( Z \) by \( \mathcal{L} = p_1^* \mathcal{O}_{X_1}(1) \otimes p_2^* \mathcal{O}_{X_2}(1) \). Then we have also following surjective homomorphism

\[
\gamma : g^*(V_1 \otimes V_2) = p_1^* f_1^*(V_1) \otimes p_2^* f_2^*(V_2) \to \mathcal{L}. \tag{3.3.3}
\]
This construction gives us a scheme morphism \( \phi : Z \to \mathbb{P}(V_1 \otimes V_2) \simeq \mathbb{P}^3 \), since \( V_1 \otimes V_2 \) is isomorphic to \( \mathbb{C}^4 \). Then one can show that the map \( Z(\mathbb{C}) = \mathbb{P}(V_1 \otimes V_2) = \mathbb{P}^3(\mathbb{C}) \), see also ref. [5].

Let us consider the map \( \psi_1 : V_1 \to \mathbb{C} \). Then the \( \mathbb{C} \)-value point of the space \( X_1 \) is in one-to-one correspondence with the equivalence classes of non-zero map \( \psi_1 \) which is uniquely determined by \( \psi_1(\alpha_0^0) = \alpha_0^0 \) and \( \psi_1(\alpha_1^0) = \alpha_1^0 \). We can also get a similar construction for the \( \psi \) map. This map is uniquely determined by

\[
\psi(\beta_0^0 \otimes \beta_0^0) = \alpha_0^0 \alpha_1^0 \equiv \alpha_{00} \\
\psi(\beta_0^0 \otimes \beta_0^1) = \alpha_0^0 \alpha_1^1 \equiv \alpha_{01} \\
\psi(\beta_0^1 \otimes \beta_0^0) = \alpha_0^1 \alpha_1^0 \equiv \alpha_{10} \\
\psi(\beta_0^1 \otimes \beta_0^1) = \alpha_0^1 \alpha_1^1 \equiv \alpha_{11}
\]

which is the equivalent class of \( \psi \) corresponding to \( (\alpha_0^0, \alpha_0^1, \alpha_0^1, \alpha_0^1) = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}) \). Thus the map \( \phi(\mathcal{L}, \gamma) \) is exactly the Segre map and the image of \( \phi(\mathcal{L}, \gamma) \) is the space of separable two-qubit quantum states \( \Psi = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \), that is

\[
\text{Im} \phi(\mathcal{L}, \gamma) = \{ \alpha_{k_1k_2} \in \mathbb{P}_C^3 : \alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10} \text{ for } k_1, k_2 = 1, 2 \} \quad (3.3.4)
\]

The measure of entanglement for such state can also be constructed by using the definition of the Segre morphism which also coincides with well-known concurrence. Now, let \( Q \) be the nonsingular quadric surface \( \alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10} \) in \( \mathbb{P}_C^3 \). Then the Picard group of \( Q \) is given by \( \text{Pic}Q \cong \mathbb{Z} \oplus \mathbb{Z} \) and we have \((a, b)\) type of an invertible sheaf, where \( a, b \in \mathbb{Z} \). For \( Q \cong \mathbb{P}_C^1 \times \mathbb{P}_C^1 \), if \( a, b > 0 \), then we consider an \( a \)-uple and \( b \)-uple embedding of \( \mathbb{P}_C^1 \to \mathbb{P}_C^{N_1} \) and \( \mathbb{P}_C^1 \to \mathbb{P}_C^{N_2} \) respectively. Now, we obtain the following closed immersion

\[
Q = \mathbb{P}_C^1 \times \mathbb{P}_C^1 \to \mathbb{P}_C^{N_1} \times \mathbb{P}_C^{N_2} \to \mathbb{P}_C^N. \quad (3.3.5)
\]

This map corresponds to an invertible sheaf of type \((a, b)\) on \( Q \). This also means that for any \( a, b, > \), the corresponding invertible sheaf is very ample and ample. In general we have a one-to-one correspondence between linear equivalence classes of divisors and isomorphism classes of invertible sheaves.

### 3.4 Grassmannian scheme construction of multipartite quantum systems

In this section we will discuss the construction complex Grassmannian scheme and its application to problem of quantifying multipartite quantum systems. The construction of Grassmannian scheme is parallel with the classical construction of Grassmannian variety which we have discussed in our paper [2].

Let \( A \) be a commutative ring, \( S = \text{Spec}(A) \) be a scheme and \( N \) be a positive number and \( k < N \). Then we can define the Grassmannian scheme \( \mathcal{G}_S(k, N) \) over \( S \) as follows. For any morphism \( T \to S \) of affine scheme we have \( \mathcal{G}_T(k, N) = \mathcal{G}_S(k, N) \times_S T \). Thus, we can construct Grassmannians over arbitrary schemes \( S \) by gluing together the Grassmannians \( \mathcal{G}_{U_i}(k, N) \) over collection of affine open subsets \( U_i \subset S \). The classical complex Grassmannian can be constructed either as an union of open sets which are isomorphic to
affine space $\mathbb{C}^{k(N-k)}$ or by embedding in the complex projective space $\mathbb{P}_C^M$ defined by Plücker equations. These two constructions have a direct extension to the category of schemes. Moreover, in the scheme-theoretic case, we have also a third construction, namely by characterizing Grassmannian as Hilbert schemes. This means we can represent the Grassmannian schemes as functors of families of linear subspaces of a fixed complex vector space. Here, we will discuss the construction of complex Grassmannian scheme based on Plücker equations as a subscheme of complex projective space $\mathbb{P}_C^M$, where $M = \binom{N}{k} - 1$.

Let $\mathbb{C}[\ldots, P_I, \ldots]$ be a polynomial ring in $M + 1$ variables labeled by subsets $I = (i_1 < \cdots < i_k) \subset \{1, 2, \ldots, N\}$. The variables $P_I$ corresponds to the maximal minors of a $k \times N$ matrix $G$. Let $\phi : \mathbb{C}[\ldots, P_I, \ldots] \longrightarrow \mathbb{C}[p_1,1, \ldots, p_k,N]$ be defined by

$$P_I \longmapsto \begin{vmatrix} p_{i_1,i_1} & \cdots & p_{i_1,i_k} \\ \vdots & \ddots & \vdots \\ p_{i_k,i_1} & \cdots & p_{i_k,i_k} \end{vmatrix}$$

(3.4.1)

which sends each generator $P_I$ to corresponding minor of the matrix $(p_{i,j})$. Then we define the complex projective Grassmannian scheme by

$$\mathcal{G}(k,N) = \text{Proj}\mathbb{C}[\ldots, P_I, \ldots]/\text{Ker}\phi \subset \text{Proj}\mathbb{C}[\ldots, P_I, \ldots] = \mathbb{P}_C^M.$$  

(3.4.2)

We can also define the Grassmannian $\mathcal{G}(k,\mathcal{E})$ of $k$-dimensional subspaces of a locally free sheaf $\mathcal{E}$ over scheme $S$ as follows. Let the map

$$\mathcal{E}^{\otimes k} = \mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E} \longrightarrow \bigwedge \mathcal{E}$$

be defined by $w \otimes w \otimes \cdots \otimes w \longmapsto P_{i_1,i_2,\ldots,i_k} = w \wedge w \wedge \cdots \wedge w$. Moreover, let

$$\phi : \text{Sym}(\bigwedge \mathcal{E})^* \longrightarrow \text{Sym}(\bigwedge \mathcal{E})^*$$

(3.4.4)

be the induced map on the symmetric algebras. Then, we define the Grassmannian $\mathcal{G}(k,\mathcal{E})$ to be the subscheme of $\mathbb{P}(\mathcal{E}^*) = \text{Proj}(\text{Sym}(\bigwedge^k \mathcal{E})^*)$ given by the ideal sheaf $\text{Ker}\phi$ which is also generated by the quadratic polynomials

$$\mathcal{P}_{I,J} = \sum_{i=1}^{k+1} (-1)^i P_{i_1,\ldots,i_{k-1},j_i} P_{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{k+1}},$$

(3.4.5)

where $I = (i_1, \ldots, i_{k-1}), 1 \leq i_1 < \cdots < i_{k-1} < j_i$, for $i = 1, \ldots, k + 1$, and $J = (j_1, \ldots, j_{k+1}), 1 \leq j_1 < \cdots < j_{k+1} \leq N$ are two increasing sequences of numbers from the set $\{1, 2, \ldots, N\}$. Now, based on this Plücker coordinates we can define a measure of entanglement for multi-qubit states. The measure constructed by taking the square root of product of the Plücker coordinates with its conjugate of a matrix which is constructed by elements of $(\alpha_{00,0}, \alpha_{00,01}, \ldots, \alpha_{11,11})$. For classical construction of Grassmannian variety and a measure of entanglement for multipartite quantum systems see the ref. [2].

In this paper we have presented a new geometrical construction of multipartite quantum systems based on a very important class of abstract schemes, namely the complex projective schemes. We have reviewed the construction...
of sheaves and based on these abstract mathematical objects we have defined
prescheme, affine scheme, and scheme. Then, we have investigated the geomet-
rical structures of multipartite quantum systems based on complex projective
schemes. We have also discussed the separability and entanglement of bipartite
and multipartite quantum systems based on complex multi-projective schemes
and complex Grassmannian scheme.

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