Continuum limit of the spectrum of the hadronic string

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Abstract

We look at the continuum limit of the two universal predictions of the hadronic string theory namely the Lüscher term and the energy difference between successive energy states in three dimensional SU(2) lattice gauge theory using Wilson loops and Polyakov loop correlators. We also compare our simulation data with the theoretical predictions in [10].

1 Introduction

Formation of a flux tube between quarks and anti-quarks in the QCD vacuum is an appealing mechanism for quark confinement. Effective theories for these flux tubes can be written in terms of string theories [1]. An interesting feature of these theories is that they predict a universal coefficient of the large distance $1/r$ term $c = \frac{-\pi}{24}(d - 2)$; $d$ : no. of space-time dimensions) [2] also known as the Lüscher term. While ground states of such string theories have been studied for a long time [3], now with improvement of both computing power and algorithms [4], it is possible to do careful studies of other important properties of the flux tube and several such studies have been carried out recently [5].

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In a previous work we had applied the Lüscher-Weisz multilevel algorithm to look at the spectrum of the hadronic string \[6\]. Since that algorithm allows an exponential error reduction, we were able to look at physically large time extents for the Wilson loops. In this paper we apply the same algorithm to study the continuum limit of two of the string observables, the universal coefficient “c” and the energy difference between the ground state and the first excited state. Our previous experience had hinted that systematic corrections to the energies of the string states arising from the presence of higher excited states played a very crucial role in the determination of the energy differences between successive states. Recently there have been extensive studies of the string spectrum using asymmetric lattices and sophisticated wave functions \[7\]. Such studies have suggested a hyperfine structure of the spectrum. Here we use completely independent and in some sense complimentary methods to look at the string spectrum.

2 Simulation parameters

We carried out simulations of three dimensional SU(2) lattice gauge theory on three different lattices chosen so as to roughly cover the same physical volume. Let us call these lattices A, B and C. We restrict ourselves to square lattices whose parameters are summarized in table 1. The lattice spacing \(a\) was obtained by assuming \(\sqrt{\sigma} = (0.5 \text{ fm})^{-1}\). On all these lattices, we computed Polyakov loop correlators \(\langle P^*(x)P(y) \rangle\) for various values of \(r = y - x\) and Wilson loops of various space and time extents \(\langle W(r, T) \rangle\).

| lattice | \(\beta\) | \(L/a\) | \(a\sqrt{\sigma}\) | \(r_0/a\) | \(a \text{ (fm)}\) | \(\sqrt{\sigma r_0}\) |
|---------|--------|--------|-----------------|---------|----------------|----------------|
| A       | 7.5    | 36     | 0.19622 (2)     | 6.285 (4)| 0.09811 (1)   | 1.2332 (10)   |
| B       | 10.0   | 48     | 0.143691 (14)   | 8.578 (4)| 0.071846 (7)  | 1.2319 (11)   |
| C       | 12.5   | 60     | 0.113243 (5)    | 10.92 (3)| 0.056622 (3)  | 1.2366 (34)   |

To reliably extract signals of these observables which are exponentially decreasing functions of \(r\) and \(T\), we used the Lüscher-Weisz multilevel algorithm \[1\]. As is well known, this algorithm has several optimization parameters among which the number of sub-lattice updates employed seem to be
the most important one [9]. In tables 2 and 3 we tabulate the number of sub-lattice updates (iupd) used in our measurements.

| lattice | A     | B     | C     |
|---------|-------|-------|-------|
| r       | 4–8   | 7–11  | 10–14 |
| iupd    | 2000  | 3200  | 8000  |

Table 3: Sublattice updates for Wilson loops

| lattice | A     | B     | C     |
|---------|-------|-------|-------|
| T       | r     | iupd  | T     | r     | iupd  | T     | r     | iupd  |
| 4       | 4–8   | 300   | 4     | 8–12  | 300   | 6     | 8–12  | 200   |
|         | 8–12  | 400   |       | 12–16 | 400   |       | 12–16 | 250   |
|         | 6     | 450   | 8     | 8–12  | 600   | 12    | 8–12  | 400   |
|         | 8–12  | 600   |       | 12–16 | 800   |       | 12–16 | 500   |
|         | 12    | 900   | 12    | 8–12  | 900   | 20    | 8–12  | 800   |
|         | 8–12  | 1200  |       | 12–16 | 1200  |       | 12–16 | 1000  |
|         | 18    | 1350  | 16    | 8–12  | 1200  | 30    | 8–12  | 1000  |
|         | 8–12  | 1800  |       | 12–16 | 1600  |       | 12–16 | 1250  |

Another important parameter is the thickness of the time-slice over which the sub-lattice averages are carried out. We found that it was helpful to increase this thickness as one goes from stronger to weaker coupling. While a thickness of 2 was sufficient at $\beta = 5.0$ [6], for $\beta = 10$ we chose the thickness to be 4 uniformly while for the other two $\beta$ values, 7.5 and 12.5, we chose the higher value between 4 and 6 whenever possible.
Only one level of averaging was sufficient for most of the observables. However to get a stable value of the Lüscher term it was necessary to do a second level of averaging of the Polyakov loop data before taking the derivatives especially at higher values of $\beta$.

3 Polyakov loop

The Polyakov loop correlator can be expressed as

$$\langle P^*(r,T)P(0,T) \rangle = \sum_{i=0}^{\infty} b_i \exp[-V_i(r)T]$$

where $b_i$'s are integers and $T = L$ is the full time extent of the lattice. For such large values of $T$ this expectation value projects almost exclusively to the ground state and we estimate that corrections due to higher states are lower than our statistical errors. From the Polyakov loop correlators we determine the potential $V(r)$ and the force $F(r)$ as

$$V(r) = -\frac{1}{T} \log\langle P^*(r,T)P(0,T) \rangle$$

$$F(r) = \frac{\partial V(r)}{\partial r} \equiv (V(r + 1) - V(r - 1))/2.$$  

The string tension is determined from the intercept of the curve of $F(r)$ versus $1/r^2$. This will yield the correct value only for asymptotically large distances as corrections $\propto r^{-3}$ become negligible. However in our simulations at moderate values of $r$ these corrections are not yet negligible. Thus for us $F(r)$ is not really a straight line but one which approaches a straight line close to the origin. To avoid the short distance effects in our determination of the string tension as much as possible, we fit a straight line only to the final few points closest to the origin and take the intercept from that fit. These fits are shown in figure I.

To have a theoretical prediction against which we can compare the simulation results, we will consider two different forms of the potential. These are the potential assuming a free bosonic string description of the flux tube and the Arvis potential [9] which comes from assuming a Nambu-Goto type description of the flux tube. While the free bosonic string is the simplest model, recent studies [10] suggest that the only consistent description of the
Polyakov loop correlator is given by the truncated Arvis potential. At large distances, predictions from both these models coincide. Our comparison will be at moderate distances to see which form is preferred.

For the free bosonic string in three dimensions, we have $V(r)$ to be of the form

$$V(r) = \sigma r + \hat{V} - \pi/24 r$$

while the Arvis potential is given by

$$V(r) = \sigma r \left(1 - \frac{\pi}{12 \sigma r^2}\right)^{1/2}.$$  

We define the truncated Arvis potential by expanding the potential in a power series and retaining the first three terms.

The Lüscher term $c(r)$ is determined locally at different values of $r$'s by

$$c(r) = \frac{r^3}{2} \frac{\partial^2 V(r)}{\partial r^2} = \frac{r^3}{2} (V(r + 1) - 2 V(r) + V(r - 1)).$$

The theoretical predictions in the continuum limit are

$$c(r) = -\pi/24$$

free bosonic string  

$$= -\frac{\pi}{24} \left(1 + \frac{\pi}{8 \sigma r^2}\right)$$

truncated Arvis  

$$= -\frac{\pi}{24} \left(1 - \frac{\pi}{12 \sigma r^2}\right)^{-3/2}.$$ 

Arvis

To study the continuum limit of the behaviour of $c(r)$ we need to look at the variation of $c(r)$ at a fixed physical distance for various values of the bare coupling. Since the lattice spacing varies with the coupling constant, to compare the data at different values of $\beta$, we need to introduce a common scale. To that end let us define the Sommer scale $r_0$ by $r_0^2 F(r_0) = 1.65$. For each value of $\beta$, the Sommer scale has been estimated and is shown in table 1. We also see from the same table that $\sigma r_0^2$ is constant ($\approx 1.23$) to a very good approximation.

We now plot all the data together in figure 2 using $r/r_0$ as the x-axis. The data for $\beta = 5$ has been taken from 6. On this figure we also plot the prediction of $c(r)$ from 2-loop perturbation theory given by 11

$$c(r/r_0) = -\frac{3(r/r_0)r_0}{4 \pi \beta}.$$  

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Figure 1: Forces at various values of $\beta$
Figure 2: \( c(r) \) at various values of \( \beta \). ▲, ●, ▼, ■ correspond to \( \beta = 5.0, 7.5, 10.0 \) and 12.5 respectively. The 4 lines denoted by \(-\), · · ·, −·− and −·− are the corresponding 2-loop perturbation theory curves. −−−− is \( c(r) \) from free bosonic string while the continuous line and the dashed curve are the values predicted by the truncated Arvis and the Arvis potentials.

Our data approaches the theoretical values from above. For \( \beta = 5 \) and 7.5 the data points seem to cross the free string value and come closer to the predictions from the Arvis potentials. For the other two \( \beta \) values the explored regions of \( r \) unfortunately turned out to be too small to see if they too actually cross \(-\pi/24\) and again reapproach that value from below. It would therefore be important to see what happens to \( c(r) \) as one extends the range of \( r \).

It is also interesting to observe that the perturbative curve is closest to the lattice data for the largest lattice spacing. Thus it seems that the departure from the perturbation theory sets in earlier as one goes towards the continuum limit. Also we see that the data sets at \( \beta = 10.0 \) and \( \beta = 12.5 \)
are degenerate within error bars. This of course makes it pointless to do a further extrapolation to the continuum limit and we conclude that within our error bars the data at $\beta = 12.5$ is identical with the continuum limit.

4 Wilson loop

Space-time Wilson loops of extent $(r, T)$ can be interpreted in the transfer matrix formalism, as the correlation of sources between points separated by a distance $r$, propagating for time $T$. While the Polyakov loop correlators do not give us the option of having sources and project strongly onto the ground state, Wilson loops enable us to study the excited states by constructing sources which project onto some state preferentially.

In three dimensions the flux-tube has only one transverse direction to fluctuate and thus possesses only discrete symmetries according to the transformation properties under charge conjugation and parity. Classification of the low lying states according to these symmetries have been carried out in \cite{6}, and using the sources described there, we study the ground state and the first excited state of the string spectrum and take a careful look at the energy difference between the two states, comparing them with the prediction from the free bosonic string and the Arvis potential.

In our measurements, we compute correlation matrices $C(r, T)$ among different sources, each between points separated by a distance $r$, and then diagonalize the correlation matrix by taking appropriate combinations of the matrix elements. Energies of the string states can then be obtained by fitting the eigenvalues $\lambda(r, T)$ of $C(r, T)$ to the form

$$\lambda(r, T) = \alpha e^{-E(r)T}$$

using different $T$ values. The energies obtained in this way are reported under “no removal” in tables \ref{table1} and \ref{table2}. This however is a naive estimate and in practice there are corrections due to higher excited states. That is because, while sources with different symmetry properties allow us to project onto different channels, there are still infinitely many states in each channel and they all contribute to the correlator in that channel. We are interested only in the leading exponential in each of the channels, but even these are significantly contaminated by the higher states at the smaller values of $T$. To make a meaningful comparison with the prediction from the string picture, we need to remove this contamination as much as possible.
We do the removal in two ways. First we drop the smallest value of $T$ and do a fit to the rest of the points. These values are reported under the head “partial removal”. Second we assume that

$$\lambda(r, T) = a(r)e^{-E_0(r)T} + b(r)e^{-(E_0+\delta E_0)(r)T}$$

(12)

where $E_0 + \delta E_0$ is the energy of the state above $E_0$ in the same symmetry channel. This allows us to define an improved observable $\bar{E}$ by

$$-\frac{1}{T_2 - T_1} \log \frac{\lambda(T_2)}{\lambda(T_1)} = \bar{E} + \frac{1}{T_2 - T_1} \left[ \frac{b(r)}{a(r)} e^{-\delta E_0 T_1} \left(1 - e^{-\delta E_0 (T_2 - T_1)}\right) \right].$$

(13)

We estimate that further corrections coming from higher states are smaller than our statistical errors. We denote the $\bar{E}$ values in the tables under “full removal”. We plot the first two states for the different $\beta$ values in figure 6.

As seen from the figure, for the ground state the corrections due to finite time extents are small and so all the three sets of points are on top of each other. For the first excited state, the corrections are clearly visible. For $\beta = 12.5$, our data was not good enough to do the extrapolation for a “full removal”. Hence it has only two sets of points for the excited state. In table 8 the points marked with a * were also too unstable for extrapolation and the values reported there are the values extracted from the Wilson loops with the two largest time extents.

Let us now come to the energy difference between the ground state and the first excited state. A free bosonic string description of the flux tube predicts this energy difference to be $\pi/r$ for a string formed between points separated by a distance $r$. The Arvis energy states are given by

$$E_n = \sigma r \sqrt{1 + \frac{2\pi}{\sigma r^2} \left(n - \frac{d - 2}{24}\right)}$$

(14)

where $d$ is the number of space-time dimensions. It is interesting to observe that degeneracies of the Arvis states is the same as that of the free string [10] For us $d = 3$ and therefore for $E_1 - E_0$ we get

$$E_1 - E_0 = \frac{\pi}{r} \left(1 - \frac{11\pi}{24\sigma r^2}\right)$$

(truncated Arvis)

(15)

$$= \sigma r \left(\sqrt{1 + \frac{23\pi}{12\sigma r^2}} - \sqrt{1 - \frac{11\pi}{24\sigma r^2}}\right)$$

(Arvis)

(16)
In figure 7, we plot these energy differences along with the predictions from the free bosonic string and Arvis potentials. The two sets of points are the differences computed in different ways. The upper set is just the naive energy difference obtained from the column “no removal”. The lower set is computed from the ratios of the eigenvalues of the correlation matrices. To do this we need two correlators at two different times for each state. Suppressing the $r$ dependencies we denote these states by

$$c(t_1) = a e^{-E_0 t_1} + b e^{-(E_0 + \delta E_0) t_1}$$  \hspace{1cm} (17)$$

$$c(t_2) = a e^{-E_0 t_2} + b e^{-(E_0 + \delta E_0) t_2}$$  \hspace{1cm} (18)$$

$$d(t_1) = a' e^{-E_1 t_1} + b' e^{-(E_1 + \delta E_1) t_1}$$  \hspace{1cm} (19)$$

$$d(t_2) = a' e^{-E_1 t_2} + b' e^{-(E_1 + \delta E_1) t_2}$$  \hspace{1cm} (20)$$

Then it is easy to see that to leading order

$$\frac{1}{\Delta t} \log \left[ \frac{c(t_1)/c(t_2)}{d(t_1)/d(t_2)} \right] = (E_1 - E_0) + \frac{1}{\Delta t} \left[ \frac{b}{a} e^{-\delta E_0 t_1} - \frac{b'}{a'} e^{-\delta E_1 t_1} - \frac{b}{a} e^{-\delta E_0 t_2} + \frac{b'}{a'} e^{-\delta E_1 t_2} \right]$$  \hspace{1cm} (21)$$

where $\Delta t = t_1 - t_2$. Thus we see that the corrections are exponentially suppressed. Moreover to leading order $\delta E_0$ and $\delta E_1$ are identical and thus a further cancellation takes place between the coefficients. Here we use the fact that since the different correlators at the same times were measured in the same simulation we take the errors with the same sign on them. Taking the two largest time extents we get values for $E_1 - E_0$ consistent with what we would obtain from the columns “full removal”, but with lower errors. We will call the energy differences obtained in this way “Correlated energy difference”. In table 4 we tabulate the “Correlated energy difference” for $E_1 - E_0$. These values are in agreement with values obtained elsewhere using asymmetric lattices [12].

As can be seen from figure 7, the variation of the energy difference with $r$ is much better described by the Arvis potential than the free bosonic string for smaller values of $r$. In the range of $r$ we have looked at the systematic corrections are very important. It is only the data set corrected for higher energy states that stay close to the Arvis potential and keep approaching the free string prediction as one goes to larger values of $r$. Without taking these corrections into account, one can draw wrong conclusions about the validity of the string picture as our naive data actually crosses the free string value.
Table 4: Correlated energy difference for $E_1 - E_0$

| Lattice A | Lattice B |
|-----------|-----------|
| $r$ | $E_1 - E_0$ | error | $r$ | $E_1 - E_0$ | error |
| 4  | 0.432 (8)  | 8  | 0.272 (1)  |
| 5  | 0.386 (7)  | 9  | 0.255 (1)  |
| 6  | 0.349 (5)  | 10 | 0.240 (1)  |
| 7  | 0.318 (6)  | 11 | 0.2267 (7) |
| 8  | 0.300 (7)  | 12 | 0.211 (2)  |
| 9  | 0.277 (6)  | 13 | 0.200 (3)  |
| 10 | 0.257 (4)  | 14 | 0.191 (3)  |
| 11 | 0.242 (4)  | 15 | 0.183 (3)  |
| 12 | 0.228 (4)  | 16 | 0.176 (5)  |

and the difference seems to increase with $r$ instead of decreasing. We expect as one goes to larger values of $r$ even more careful treatment of the higher energy corrections are going to be necessary.

To have an idea of the magnitude of the lattice effects, we compare $r(E_1 - E_0)$ for the three different lattice spacings at $1.2r_0$, $1.3r_0$, and $1.5r_0$. The energy differences at these distances are obtained by interpolation. We plot these differences against $a^2$ in figure 3. If we assume that the continuum limit is approached as $a^2$, then we can fit a straight line through these points to obtain the continuum limit values of the energy difference. We tabulate these values in table 5.

In figure 4 we plot $r(E_1 - E_0)$ against $r/r_0$ for all the three values of $\beta$ that we have investigated along with the continuum limit values at $r/r_0 = 1.2$, $1.3$ and $1.5$. We see that the entire data set lies between the free string value and the values from the Arvis potential. Lattice effects are present but do not seem to be too large. As in the case of $c(r)$, the continuum limit values are not too far from the lattice at $\beta = 12.5$. The direction of the shift with smaller lattice spacing seems to be away from the Arvis and towards the free string description. Also as we have mentioned before, for $\beta = 12.5$ we only have a partial removal of the higher states for the first excited state and hence the estimated energy differences are higher than the true ones. Therefore we our values indicate only an upper bound for the continuum limit rather than actual values.
Figure 3: \( r(E_1 - E_0) \) against \( a^2 \). \( \triangle \), \( \square \) and \( \nabla \) correspond to \( r/r_0 = 1.2, 1.3 \) and 1.5 respectively.

Table 5: Continuum limit extrapolation of \( r(E_1 - E_0) \)

| \( a^2 \)   | \( a^2 = 0.009666 \) | \( a^2 = 0.005162 \) | \( a^2 = 0.003206 \) | \( a^2 = 0 \) |
|------------|----------------------|----------------------|----------------------|---------------|
| 1.2r_0     | 2.33 (5)             | 2.35 (4)             | 2.47 (4)             | 2.50 (9)      |
| 1.3r_0     | 2.42 (6)             | 2.44 (4)             | 2.59 (3)             | 2.66 (11)     |
| 1.5r_0     | 2.51 (5)             | 2.59 (4)             | 2.72 (2)             | 2.83 (6)      |

5 Conclusions

In this paper we have looked at the continuum limit of the Lüscher term and the energy difference between the ground state and the first excited state. Our study indicates that for the Lüscher term, the asymptotic value is approached in a non-monotonic way. This non-monotonicity is seen on our coarser lattices. On our finer lattices, the range of \( r \) looked at and big error bars prevent drawing any definite conclusion but similar tendencies are present even on our finest lattice.

The energy difference at finite lattice spacing seems to be well described by the Arvis potential. However the continuum extrapolation pushes it away towards the free string description. It is very important to take into account the corrections due to the higher energy states. Otherwise one has a large
systematic error and the resulting energy difference seems to diverge from either description rather than tend to it with increasing $r$. Another important observable is $(E_2 - E_0) - 2(E_1 - E_0)$. We plot this quantity at $\beta = 10$ in figure 5. In spite of large error bars, the behaviour of the “full removal” data set is consistent with the prediction from the Arvis potential. The “no removal” set on the other hand has a qualitatively different behaviour.

We are finally in a position to start distinguishing between different string models as the subleading effects are visible and it will be really interesting to extend the study to larger $r$ on the finer lattices.
Figure 5: $(E_2 - E_0) - 2(E_1 - E_0)$ against $r/a$ at $\beta = 10.0$. The free bosonic string predicts the value zero while the other curve is the Arvis potential. •, and ■ correspond to the sets “no removal” and “full removal” respectively.

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Figure 6: The first two energy states at various values of $\beta$. The three sets of points are “no removal”, “partial removal” and “full removal” with “no removal” being the highest and “full removal” being the lowest.
Figure 7: $E_1 - E_0$ at various values of $\beta$. The two sets of points correspond to the energy differences computed from the sets “no removal” (upper) and “Correlated difference” (lower). The dotted curve is the free bosonic string prediction. The dashed curve and the continuous curve correspond to the truncated and full Arvis potentials respectively.
Table 6: Polyakov loop data

| lattice A |   |   |
|-----------|---|---|
| \( r \)   | \( V(r) \) | \( F(r) \) | \( c(r) \) |
| 4         | 0.29620 (4) | -- | -- |
| 5         | 0.34065 (7) | 0.04353 (2) | -0.1138 (3) |
| 6         | 0.38327 (9) | 0.04207 (3) | -0.1190 (5) |
| 7         | 0.4248 (1) | 0.04116 (3) | -0.123 (1) |
| 8         | 0.4656 (2) | 0.04069 (6) | -0.125 (2) |
| 9         | 0.5067 (3) | 0.04027 (6) | -0.128 (3) |
| 10        | 0.5468 (3) | 0.03996 (7) | -0.135 (5) |
| 11        | 0.58601 (7) | 0.03965 (2) | -0.134 (4) |
| 12        | 0.62556 (8) | 0.03947 (2) | -0.138 (4) |
| 13        | 0.6649 (1) | 0.03933 (2) | -0.138 (10) |
| 14        | 0.7042 (1) | -- | -- |

| lattice B |   |   |
|-----------|---|---|
| \( r \)   | \( V(r) \) | \( F(r) \) | \( c(r) \) |
| 6         | 0.24727 (8) | -- | -- |
| 7         | 0.2707 (1) | 0.02315 (3) | -0.1097 (5) |
| 8         | 0.2935 (1) | 0.02261 (3) | -0.1146 (7) |
| 9         | 0.3158 (2) | 0.02223 (4) | -0.118 (1) |
| 10        | 0.3378 (2) | 0.02195 (4) | -0.122 (1) |
| 11        | 0.3595 (2) | 0.02174 (4) | -0.123 (3) |
| 12        | 0.3811 (3) | 0.02151 (4) | -0.125 (3) |
| 13        | 0.4027 (3) | 0.02138 (4) | -0.126 (3) |
| 14        | 0.4240 (4) | 0.02128 (2) | -0.126 (2) |
| 15        | 0.4452 (2) | 0.02120 (2) | -0.127 (4) |
| 16        | 0.4664 (2) | -- | -- |

| lattice C |   |   |
|-----------|---|---|
| \( r \)   | \( V(r) \) | \( F(r) \) | \( c(r) \) |
| 8         | 0.21098 (9) | -- | -- |
| 9         | 0.2254 (1) | 0.01428 (2) | -0.1095 (9) |
| 10        | 0.2395 (1) | 0.01401 (3) | -0.114 (1) |
| 11        | 0.2536 (1) | 0.01381 (3) | -0.117 (2) |
| 12        | 0.2673 (1) | 0.01368 (3) | -0.120 (1) |
| 13        | 0.2809 (2) | 0.013555 (8) | -0.122 (1) |
| 14        | 0.2944 (2) | 0.013454 (9) | -0.124 (2) |
| 15        | 0.3079 (2) | 0.01337 (1) | -0.126 (3) |
| 16        | 0.3212 (2) | 0.01332 (2) | -0.128 (2) |
| 17        | 0.3346 (2) | 0.01326 (2) | -0.130 (3) |
| 18        | 0.3479 (2) | 0.01322 (3) | -0.130 (4) |
| 19        | 0.3611 (2) | 0.01318 (3) | -0.132 (4) |
| 20        | 0.3742 (2) | -- | -- |
Table 7: Ground state from Wilson loops for lattices A, B & C

|     | no removal | partial removal | full removal |
|-----|------------|----------------|--------------|
| A   | r          | E      | ΔE     | χ²/d.o.f | r | E      | ΔE     | χ²/d.o.f | E      | ΔE     | χ²/d.o.f |
| 4   | 0.2966     | 0.0001 | 24.4   | 0.2964  | 0.0004 | 1.24 | 0.29631| 0.00007|
| 5   | 0.3414     | 0.0003 | 75.3   | 0.34102 | 0.00009| 4.25 | 0.3408 | 0.0001 |
| 6   | 0.3844     | 0.0005 | 114    | 0.3838  | 0.0001 | 6.85 | 0.3835 | 0.0001 |
| 7   | 0.4265     | 0.0007 | 202    | 0.4257  | 0.0002 | 12.3 | 0.4251 | 0.0002 |
| 8   | 0.468      | 0.001  | 242    | 0.4668  | 0.0004 | 27.3 | 0.4657 | 0.0004 |
| 9   | 0.509      | 0.001  | 329    | 0.5076  | 0.0006 | 37.3 | 0.5060 | 0.0004 |
| 10  | 0.550      | 0.002  | 341    | 0.5479  | 0.0007 | 41.1 | 0.5460 | 0.0006 |
| 11  | 0.590      | 0.002  | 400    | 0.5882  | 0.0009 | 50.2 | 0.5857 | 0.0007 |
| 12  | 0.631      | 0.002  | 380    | 0.6282  | 0.0011 | 51.5 | 0.6251 | 0.0008 |
| B   | r          | E      | ΔE     | χ²/d.o.f | r | E      | ΔE     | χ²/d.o.f | E      | ΔE     | χ²/d.o.f |
| 8   | 0.296      | 0.001  | 542    | 0.2941  | 0.0002 | 8.5  | 0.2934 | 0.0002 |
| 9   | 0.320      | 0.001  | 708    | 0.3167  | 0.0003 | 11.8 | 0.3158 | 0.0003 |
| 10  | 0.342      | 0.002  | 764    | 0.3390  | 0.0004 | 13.9 | 0.3378 | 0.0004 |
| 11  | 0.366      | 0.002  | 878    | 0.3611  | 0.0005 | 17.0 | 0.3595 | 0.0005 |
| 12  | 0.388      | 0.002  | 883    | 0.3830  | 0.0007 | 18.4 | 0.3811 | 0.0005 |
| 13  | 0.411      | 0.003  | 959    | 0.4050  | 0.0007 | 18.0 | 0.4029 | 0.0009 |
| 14  | 0.434      | 0.003  | 927    | 0.4268  | 0.0008 | 18.8 | 0.424  | 0.001  |
| 15  | 0.457      | 0.003  | 940    | 0.449   | 0.001  | 20.2 | 0.445  | 0.001  |
| 16  | 0.479      | 0.004  | 879    | 0.470   | 0.001  | 20.0 | 0.467  | 0.001  |
| C   | r          | E      | ΔE     | χ²/d.o.f | r | E      | ΔE     | χ²/d.o.f | E      | ΔE     | χ²/d.o.f |
| 8   | 0.2117     | 0.0003 | 220    | 0.21122 | 0.00003| 0.93 | 0.21115| 0.00007|
| 9   | 0.2265     | 0.0005 | 317    | 0.22573 | 0.00005| 1.51 | 0.22562| 0.00009|
| 10  | 0.2409     | 0.0006 | 387    | 0.23994 | 0.00006| 2.04 | 0.2398 | 0.0001 |
| 11  | 0.2552     | 0.0008 | 480    | 0.25395 | 0.00009| 2.77 | 0.2537 | 0.0001 |
| 12  | 0.2693     | 0.0010 | 631    | 0.2677  | 0.0002 | 8.28 | 0.2672 | 0.0001 |
| 13  | 0.283      | 0.001  | 715    | 0.2814  | 0.0002 | 9.75 | 0.2808 | 0.0002 |
| 14  | 0.297      | 0.001  | 751    | 0.2950  | 0.0003 | 10.9 | 0.2943 | 0.0002 |
| 15  | 0.311      | 0.002  | 804    | 0.3085  | 0.0003 | 12.3 | 0.3076 | 0.0003 |
| 16  | 0.325      | 0.002  | 808    | 0.3221  | 0.0003 | 10.2 | 0.3212 | 0.0004 |
| 17  | 0.339      | 0.002  | 838    | 0.3355  | 0.0003 | 11.0 | 0.3345 | 0.0005 |
| 18  | 0.353      | 0.002  | 827    | 0.3489  | 0.0004 | 11.5 | 0.3477 | 0.0006 |
| 19  | 0.366      | 0.002  | 832    | 0.3622  | 0.0005 | 12.3 | 0.3608 | 0.0007 |
| 20  | 0.380      | 0.003  | 805    | 0.3755  | 0.0005 | 12.9 | 0.3738 | 0.0008 |
Table 8: First excited state from Wilson loops for lattices A, B & C

|   | A               | B               | C               |
|---|-----------------|-----------------|-----------------|
|   | no removal      | partial removal | full removal    |
|   | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f |
| 4 | 0.780 | 0.015 | 552  | 0.751 | 0.001 | 1.38  | 0.723 | 0.024 |
| 5 | 0.778 | 0.014 | 446  | 0.750 | 0.002 | 2.15  | 0.719 | 0.028 |
| 6 | 0.784 | 0.015 | 498  | 0.758 | 0.002 | 4.78  | 0.725 | 0.019 |
| 7 | 0.798 | 0.016 | 498  | 0.769 | 0.003 | 5.35  | 0.737 | 0.016 |
| 8 | 0.813 | 0.016 | 516  | 0.785 | 0.002 | 4.93  | 0.763 | 0.009 |
| 9 | 0.836 | 0.019 | 545  | 0.804 | 0.003 | 5.23  | 0.781 | 0.010 |
|10 | 0.855 | 0.018 | 479  | 0.825 | 0.003 | 7.29  | 0.801 | 0.009 |
|11 | 0.885 | 0.020 | 447  | 0.850 | 0.003 | 6.33  | 0.825 | 0.010 |
|12 | 0.907 | 0.019 | 341  | 0.875 | 0.004 | 5.81  | 0.851 | 0.009 |
|   | no removal      | partial removal | full removal    |
|   | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f |
| 8 | 0.625 | 0.013 | 938  | 0.580 | 0.004 | 20.6  | 0.558 | 0.009 |
| 9 | 0.633 | 0.014 | 885  | 0.585 | 0.005 | 16.4  | 0.564 | 0.008 |
|10 | 0.640 | 0.015 | 833  | 0.592 | 0.005 | 16.8  | 0.571 | 0.008 |
|11 | 0.655 | 0.016 | 781  | 0.601 | 0.005 | 13.4  | 0.581 | 0.007 |
|12 | 0.664 | 0.016 | 690  | 0.611 | 0.005 | 13.5  | 0.589 | 0.006 |
|13 | 0.684 | 0.018 | 696  | 0.622 | 0.007 | 19.3  | 0.603 | 0.006(*)|
|14 | 0.695 | 0.018 | 587  | 0.634 | 0.007 | 15.6  | 0.616 | 0.007(*)|
|15 | 0.716 | 0.019 | 499  | 0.648 | 0.007 | 12.6  | 0.629 | 0.008(*)|
|16 | 0.729 | 0.019 | 328  | 0.662 | 0.007 | 9.1   | 0.644 | 0.009(*)|
|   | no removal      | partial removal | full removal    |
|   | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f | E   | ΔE | χ²/d.o.f |
| 8 | 0.499 | 0.005 | 58.8 | 0.457 | 0.002 | 0.45  | 0.450 |
| 9 | 0.499 | 0.005 | 57.5 | 0.455 | 0.002 | 0.38  | 0.380 |
|10 | 0.502 | 0.005 | 59.6 | 0.456 | 0.003 | 0.68  | 0.670 |
|11 | 0.505 | 0.006 | 56.9 | 0.457 | 0.003 | 0.67  | 0.670 |
|12 | 0.509 | 0.005 | 48.1 | 0.465 | 0.002 | 0.19  | 0.190 |
|13 | 0.516 | 0.006 | 39.1 | 0.471 | 0.003 | 0.34  | 0.340 |
|14 | 0.522 | 0.005 | 31.7 | 0.478 | 0.002 | 0.18  | 0.180 |
|15 | 0.532 | 0.005 | 23.5 | 0.486 | 0.001 | 0.06  | 0.060 |
|16 | 0.539 | 0.005 | 23.0 | 0.491 | 0.001 | 0.03  | 0.030 |
|17 | 0.550 | 0.005 | 17.4 | 0.5002 | 0.0090 | 0.02  | 0.020 |
|18 | 0.558 | 0.005 | 13.8 | 0.508 | 0.001 | 0.04  | 0.040 |
|19 | 0.570 | 0.005 | 10.1 | 0.519 | 0.002 | 0.08  | 0.080 |
|20 | 0.578 | 0.005 | 8.2  | 0.528 | 0.005 | 0.34  | 0.340 |