Riesz Transform Characterizations for Multidimensional Hardy Spaces

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Received: 18 August 2021 / Accepted: 12 February 2022 / Published online: 17 March 2022
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Abstract
We study Hardy space $H^1_L(X)$ related to a self-adjoint operator $L$ defined on an Euclidean subspace $X$ of $\mathbb{R}^d$. We continue study from [27], where, under certain assumptions on the heat semigroup $\exp(-tL)$, the atomic characterization of local type for $H^1_L(X)$ was proved. In this paper we provide additional assumptions that lead to another characterization of $H^1_L(X)$ by the Riesz transforms related to $L$. As an application, we prove the Riesz transform characterization of $H^1_L(X)$ for multidimensional Bessel and Laguerre operators, and the Dirichlet Laplacian on $\mathbb{R}^d_+$.

Keywords Hardy space · Riesz transform · Bessel operator · Laguerre operator · Dirichlet Laplacian

Mathematics Subject Classification Primary 42B30 · Secondary 42B20 · 42B25 · 33C45

1 Introduction

Let $H_t = \exp(t\Delta)$ be the heat semigroup on $\mathbb{R}^d$, i.e. $H_t f(x) = \int_{\mathbb{R}^d} H_t(x-y)f(y)\,dy$ and

$$H_t(x-y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad x, y \in \mathbb{R}^d, \quad t > 0. \quad (1.1)$$

The authors are supported by the Grant No. 2017/25/B/ST1/00599 from National Science Centre (Narodowe Centrum Nauki), Poland.

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The classical Hardy space $H^1(\mathbb{R}^d)$ can be defined by the maximal operator related to the operators $H_t$ and plays an important role in harmonic analysis. We say that a function $f \in L^1(\mathbb{R}^d)$ is in $H^1(\mathbb{R}^d)$ if and only if

$$
\|f\|_{H^1(\mathbb{R}^d)} := \left\| \sup_{t > 0} |H_t f(\cdot)| \right\|_{L^1(\mathbb{R}^d)} < \infty.
$$

There are many equivalent definitions of $H^1(\mathbb{R}^d)$ related to various objects in harmonic analysis. The interested reader is referred to [37] and references therein. Let us recall that the Riesz transforms $\widetilde{R}_j = \partial_x x_j \left( -1 / \Delta^1 \right)^{-1/2}$, $j = 1, \ldots, d$, are given by

$$
\widetilde{R}_j f(x) = C_d \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \, dy,
$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. One of the classical results states that one can give equivalent definition of $H^1(\mathbb{R}^d)$ in terms of the Riesz transforms, c.f. [22]. More precisely a function $f$ belongs to $H^1(\mathbb{R}^d)$ if and only if all the functions: $f$, $\widetilde{R}_1 f$, \ldots, $\widetilde{R}_d f$ belong to $L^1(\mathbb{R}^d)$ and

$$
\|f\|_{H^1(\mathbb{R}^d)} \simeq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\widetilde{R}_j f\|_{L^1(\mathbb{R}^d)}. \quad (1.2)
$$

On the other hand, a function $f$ in $H^1(\mathbb{R}^d)$ can be decomposed as an infinite linear combination of simple functions called atoms, see [14] and [28]. More precisely, for a function $f \in H^1(\mathbb{R}^d)$ we can write

$$
f = \sum_{k=1}^\infty \lambda_k a_k, \quad (1.3)
$$

where $\sum_k |\lambda_k| < \infty$ and $a_k$ are atoms, i.e. there exist balls $B_k$ in $\mathbb{R}^d$ such that:

$$
supp a_k \subseteq B_k, \quad \|a_k\|_\infty \leq |B_k|^{-1}, \quad \int_{B_k} a_k(x) \, dx = 0. \quad (1.4)
$$

Here $|B_k|$ is the Lebesgue measure of the ball $B_k$. For more properties of $H^1(\mathbb{R}^d)$ we refer the reader to [37] and references therein.

One can consider $H^1(\mathbb{R}^d)$ as related to the classical Laplacian $\Delta$ on $\mathbb{R}^d$, since many possible definitions of $H^1(\mathbb{R}^d)$ are given in terms of $\Delta$. Since the 60’s many researchers considered the Hardy spaces $H^1_L(X)$ related to various self-adjoint operators $L$ on some metric-measure spaces $X$, see e.g. [2,6,15,16,18–23,25,28,36,38]. A natural question in this theory is the following: can we have decompositions of the type (1.3) for $f \in H^1_L(X)$? Also, whether the equivalence similar to (1.2) holds or not? It appears that now we have many general results concerning atomic decompositions for $H^1_L(X)$,
see e.g. [20,25,34,38]. However, the characterization of $H^1_L(X)$ in terms of the Riesz transforms is not known in such generality.

In the present paper, we shall continue study in the context introduced in [27]. Recall, that in [27] we consider a space $X \subseteq \mathbb{R}^d$ and a non-negative self-adjoint operator $L$ on $L^2(X)$. The semigroup $\exp(-tL)$ satisfy the upper Gaussian estimates and, roughly speaking, the kernel $T_t(x,y)$ of $\exp(-tL)$ is similar to $H_t(x-y)$ for local times and $T_t(x,y)$ decays faster for global times, where the scale of time is adjusted to some covering $\mathcal{Q} = \{Q_j\}_{j \in \mathbb{N}}$ of $X$. For a precise statement of these assumptions see [27] or Sect. 1.1 below. The main issue considered in [27] was the characterization of $H^1_L(X)$ in terms of the atomic decompositions. It was proved there that in this context one has atoms for $H^1_L(X)$ that are either classical atoms (as in (1.4)) or atoms of the form $a(x) = |Q|^{-1} \mathbb{1}_Q(x)$, $Q \in \mathcal{Q}$. The latter atoms are called "local atoms", c.f. [23].

Our goal here is to characterize $H^1_L(X)$ by the Riesz transforms $D_j L^{-1/2}$, $j = 1, \ldots, d$, where $D_j = \partial_{x_j} + V_j$ is a derivative adapted to $L$. To this end we add additional assumptions for the kernels: $\partial_{x_j} T_t(x,y), V_j(x) T_t(x,y)$. Using this we show a result similar to (1.2), i.e. the Hardy space $H^1_L(X)$ is characterized by appropriate Riesz transforms. For other results concerning this question, see e.g. [3,6,17,19,22,24,29,32,33].

Our main motivation here is to give an uniform approach that will work in different contexts to study operators such as: multidimensional Bessel and Laguerre operators, and the Dirichlet Laplacian on $\mathbb{R}^d_+$. In the last and most technical section we verify that our assumptions are indeed satisfied for these examples.

### 1.1 Assumptions

In this section, we state assumptions that will be used throughout the paper. Let $X \subseteq \mathbb{R}^d$ be a space that is a product of: finite intervals, half-lines, or lines equipped with the Lebesgue measure, i.e. $X = (a_1, b_1) \times \ldots \times (a_d, b_d)$, where $a_j \in [-\infty, \infty)$ and $b_j \in (-\infty, \infty]$. We shall study a non-negative self-adjoint operator $L$ that is densely defined on $L^2(X)$. The semigroup generated by $-L$ will be denoted by $T_t = \exp(-tL)$ and we further assume that there exists an integral kernel $T_t(x,y)$, such that for $f \in L^p(X)$, $1 \leq p \leq \infty$, we have

$$T_t f(x) = \int_X T_t(x,y) f(y) \, dy, \quad \text{a.e. } x \in X.$$ 

The Hardy space $H^1_L(X)$ related to $L$ is defined in terms of the maximal operator related to $T_t$, namely

$$H^1_L(X) = \left\{ f \in L^1(X) : \|f\|_{H^1_L(X)} := \sup_{t > 0} \|T_t f\|_{L^1(X)} < \infty \right\}.$$ 

In this paper, we shall study the spaces $H^1_L(X)$ that will be related to some coverings $\mathcal{Q} = \{Q_k : k \in \mathbb{N}\}$ of $X$, where $Q_k$ are cuboids. We assume that $\mathcal{Q}$ is an admissible...
covering in the sense of Definition 2.1 below. Let $d_Q$ be the diameter of $Q$ and denote by $Q^*$ a slight enlargement of $Q$, see the comments after Definition 2.1 below. Following [27] we assume that there exists $\gamma \in (0, 1/3)$ and $C, c > 0$, such that $T_t(x, y)$ satisfies:

$$0 \leq T_t(x, y) \leq C t^{-d/2} \exp\left(-\frac{|x - y|^2}{ct}\right), \quad x, y \in X, t > 0,$$

(A_0)

$$\sup_{y \in Q^*} \int_{(Q^*)^c} t^{\delta} T_t(x, y) \, dx \leq C d_Q^{2\delta}, \quad \delta \in [0, \gamma), Q \in Q,$$

(A_1)

$$\sup_{y \in Q^*} \int_{Q^*} t^{-\delta} |T_t(x, y) - H_t(x - y)| \, dx \leq C d_Q^{-2\delta}, \quad \delta \in [0, \gamma), Q \in Q.$$

(A_2)

In [27], the authors studied $H^1_L(X)$ for operators satisfying $(A_0)$–$(A_2)$. It was proved that $H^1_L(X)$ can be characterized by atomic decompositions with local atoms of the form $|Q|^{-1/2} Q$, where $Q \in Q$, see [27, Thm. A] and Theorem 2.4 below.

In the present paper, we shall study the Riesz transform characterization of $H^1_L(X)$, when $L$ satisfies $(A_0)$–$(A_2)$ and the following assumptions that are inspired by certain known examples like: Bessel, Laguerre, or Schrödinger operators. On $L^2(X)$ consider the operators $R_j$ formally given by:

$$R_j = (\partial_{x_j} + V_j)L^{-1/2}, \quad j = 1, \ldots, d,$$

where $\partial_{x_j}$ is the standard derivative and $V_j$ is a function that depends only on $x_j$. Suppose that $T_t(x, y)$ satisfy:

$$\sup_{y \in Q^*} \int_{T_t(x, y) Q} d_Q \left|\partial_{x_j} T_t(x, y)\right| \frac{dt}{\sqrt{t}} \, dx \leq C, \quad Q \in Q, j = 1, \ldots, d,$$

(A_3)

$$\sup_{y \in Q^*} \int_{X} \int_{d_Q^2} \left|\partial_{x_j} T_t(x, y)\right| \frac{dt}{\sqrt{t}} \, dx \leq C, \quad Q \in Q, j = 1, \ldots, d,$$

(A_4)

$$\sup_{y \in Q^*} \int_{Q^*} \int_{0}^d \left|\partial_{x_j} (T_t(x, y) - H_t(x - y))\right| \frac{dt}{\sqrt{t}} \, dx \leq C, \quad Q \in Q, j = 1, \ldots, d.$$

(A_5)

$$\sup_{y \in X} \int_{X} \int_{0}^\infty \left|V_j(x)\right| T_t(x, y) \frac{dt}{\sqrt{t}} \, dx \leq C, \quad j = 1, \ldots, d.$$

(A_6)
For \( j = 1, \ldots, d \) define the kernels
\[
R_j(x, y) := \pi^{-1/2} \int_0^\infty \left( \partial_{x_j} + V_j(x_j) \right) T_t(x, y) \frac{dt}{\sqrt{t}}.
\] (1.5)

Notice that our assumptions guarantee that the integral above exists for a.e. \((x, y)\).

The operators \( R_j \) are defined as follows:
\[
R_j f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} R_j(x, y) f(y) \, dy, \quad x \in X.
\]
We always assume that \( R_j \) are bounded on \( L^2(X) \).

1.2 Results

Our first main result is the following theorem, that describes the Hardy space \( H^1_L(X) \) in terms of the Riesz transforms.

**Theorem A** Assume that there is an operator \( L \) and an admissible covering \( Q \) as in Sect. 1.1. In particular, we assume that \((A_0)-(A_6)\) are satisfied. Then \( f \in H^1_L(X) \) if and only if \( f, R_1 f, \ldots, R_d f \in L^1(X) \). Moreover, there exists a constant \( C > 0 \) such that
\[
C^{-1} \| f \|_{H^1_L(X)} \leq \| f \|_{L^1(X)} + \sum_{j=1}^{d} \| R_j f \|_{L^1(X)} \leq C \| f \|_{H^1_L(X)}.
\]

The proof of Theorem A is given in Sect. 3.1 below and it is based on known techniques. The main idea is to compare (locally) \( R_j \) with the classical Riesz transforms \( \tilde{R}_j = \partial_{x_j}(-\Delta)^{1/2} \) and use additional decay as \( t \to \infty \).

One of our main motivations is to study product cases. Assume that for \( i = 1, \ldots, N \) we have operators \( L_i \) satisfying the assumptions as in Sect. 1.1. In particular, \( L_i \) is associated with the semigroup \( T^{[i]}_t \) that has a kernel \( T^{[i]}_t(x_i, y_i), \ x_i, y_i \in X_i \). Then we can define
\[
X = \prod_{i=1}^{N} X_i \subseteq \prod_{i=1}^{N} \mathbb{R}^{d_i} = \mathbb{R}^d
\] (1.6)
and
\[
L = L_1 + \cdots + L_N,
\] (1.7)
such that each \( L_i \) acts only on the variable \( x_i \in X_i \). For more precise description see Sect. 2.2 below. The following theorem gives the Riesz transform characterization for \( H^1_L(X) \) in the product case.
Theorem B  Let $X$ and $L$ be as in (1.6)–(1.7) and assume that for each $i = 1, \ldots, N$ the semigroup kernel $T_i^{[1]}(x_i, y_i)$ together with an admissible covering $Q_i$ of $X_i$ satisfy the conditions $(A_0)$–$(A_6)$. Then $f \in H_1^1(X)$ if and only if $f, R_1 f, \ldots, R_d f \in L_1^1(\mathbb{R}^d)$. Moreover,

$$C^{-1} \| f \|_{H_1^1(X)} \leq \| f \|_{L_1^1(\mathbb{R}^d)} + \sum_{j=1}^d \| R_j f \|_{L_1^1(\mathbb{R}^d)} \leq C \| f \|_{H_1^1(X)}.$$  

The proof of Theorem B is given in Sect. 3.2 below. We shall use [27, Thm. B], where we proved that assuming $(A_0)$–$(A_2)$ for $T_i^{[1]}(x_i, y_i)$ and $Q_i$ we can define an admissible covering $Q_1 \times \cdots \times Q_N$ that describes $H_1^1(X)$ for $L = L_1 + \cdots + L_N$, see [27, Def. 1.5].

As an example of applications of Theorem B, we study certain multidimensional Bessel and Laguerre operators. Thanks to Theorem B it is enough to verify $(A_0)$–$(A_6)$ only in the one-dimensional case. Then, the Riesz transform characterization for $H_1^1(X)$ for the multidimensional case (when $L$ is the sum of Bessel or Laguerre operators) follows from Theorem B. Furthermore, a similar argument (see Sect. 4) allows us to study the Dirichlet Laplacian on the half-space $\mathbb{R}^d_+$. Below we briefly recall the operators that we work with and state the results.

Bessel operator. Let $X = (0, \infty)^d$. For $\beta = (\beta_1, \ldots, \beta_d)$ assume $\beta_i > 0, i = 1, \ldots, d$, and consider the multidimensional Bessel operator

$$L_B^{[\beta]} = -\sum_{i=1}^d \left( \frac{d^2}{dx_i} - \frac{\beta_i^2 - \beta_0}{x_i^2} \right), \quad x_1, \ldots, x_d > 0. \quad (1.8)$$

More precisely, by $L_B^{[\beta]}$ we shall denote a proper self-adjoint operator defined on $L_2^2(X)$, see e.g. [13]. Harmonic analysis related to $L_B^{[\beta]}$ was studied in e.g. [5–8,10,13]. In [6], the authors describe the Hardy space related to $L_B^{[\beta]}$ for $d = 1$ in terms of either atomic decompositions or Riesz transforms

$$R_j = \left( \partial_{x_j} - \frac{\beta_j}{x_j} \right) \left( L_B^{[\beta]} \right)^{-1/2}, \quad j = 1, \ldots, d.$$  

Denote

$$Q_B = \left\{ 2^n, 2^{n+1} : n \in \mathbb{Z} \right\}. \quad (1.9)$$

Then $Q_B$ is an admissible covering for $(0, \infty)$ and for $d > 1$ we have the admissible coverings $Q_B \times \cdots \times Q_B$ defined in [27, Def. 1.5]. The following theorem follows directly from [27, Prop. 4.3], Theorem B and Proposition 5.5 below.

Theorem C  Let $d \geq 1$, $\beta_1, \ldots, \beta_d > 0$ and $L_B^{[\beta]}$ be the multidimensional Bessel operator, see (1.8). Then, $f \in H_1^1(L_B^{[\beta]})((0, \infty)^d)$ if and only if $f, R_1 f, \ldots, R_d f \in L_1^1(\mathbb{R}^d)$. 

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$L^1((0, \infty)^d)$. Moreover, the associated norms are comparable, i.e.
\[
\|f\|_{L^1_{\beta}(\mathbb{R}^d)} \simeq \|f\|_{L^1((0, \infty)^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1((0, \infty)^d)}.
\]

**Laguerre operator.** Let $\beta = (\beta_1, \ldots, \beta_d)$, where $\beta_i > 0$, $i = 1, \ldots, d$, and denote the multidimensional Laguerre operator
\[
L_{\beta} = -\sum_{i=1}^{d} \left( \frac{d^2}{dx_i^2} - x_i^2 - \frac{\beta_i^2 - \beta_i}{x_i^2} \right), \quad x_1, \ldots, x_d > 0. \tag{1.10}
\]

Set $X = (0, \infty)^d$. By $L_{\beta}$ we shall denote a known self-adjoint operator on $L^2(X)$, see e.g. [31]. In [3,4,9,30,31] we find some studies on harmonic analysis related to $L_{\beta}$. In particular the authors of [3] proves the atomic decomposition theorem for the Hardy space related to $L_{\beta}$ in the one-dimensional case. For $d = 1$ we have the following admissible covering of $(0, \infty)$,
\[
Q_L = \left\{ [2^n + (k - 1)2^{-n}, 2^n + k2^{-n}]: k = 1, \ldots, 2^{2n}; n \in \mathbb{N} \right\} \cup \left\{ [2^{-n}, 2^{-n+1}]: n \in \mathbb{N}_+ \right\}. \tag{1.11}
\]

and, using this covering, we produce $Q_L \boxtimes \ldots \boxtimes Q_L$ for $d > 1$, see [27, Def. 1.5]. Combining [27, Prop. 4.5], Prop. 5.11 below, and Theorem B we arrive at the following characterization of $H^1_{\beta}(0, \infty)^d)$ in terms of the Riesz transforms
\[
R_j = \left( \partial x_j + x_j - \frac{\beta_j}{x_j} \right) \left( L_{\beta} \right)^{-1/2}.
\]

**Theorem D** Let $d \geq 1$, $\beta_1, \ldots, \beta_d > 0$ and $L_{\beta}$ be the multidimensional Laguerre operator, c.f. (1.10). Then, $f \in H^1_{\beta}(0, \infty)^d)$ if and only if $f, R_1 f, \ldots, R_d f \in L^1((0, \infty)^d)$. Moreover, the associated norms are comparable, i.e.
\[
\|f\|_{H^1_{\beta}(\mathbb{R}^d)} \simeq \|f\|_{L^1((0, \infty)^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1((0, \infty)^d)}. \tag{1.12}
\]

**Dirichlet Laplacian on $\mathbb{R}^d_+$.** As a third example let us consider the Dirichlet Laplacian on the half-space $\mathbb{R}^d_+$. To be more precise for $d \geq 1$ we consider
\[
X = \mathbb{R}^d_+ = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0 \right\}.$$
and the Laplacian \( L_D = -\Delta \) with the Dirichlet boundary condition at \( x_d = 0 \). The semigroup generated by \(-L_D\) is given by 
\[
T_t f(x) = \int_X T(t)(x, y) f(y) \, dy,
\]
where \( \tilde{y} = (y_1, \ldots, y_{d-1}, -y_d) \) and \( H_t \) is as in (1.1). In this case the appropriate covering \( Q_D \) is a covering that consists of cubes such that a cube \( Q \in Q_D \) has the diameter comparable to the distance from the boundary \( x_d = 0 \) (in the case \( d = 1 \) one can just take dyadic covering of \((0, \infty)\)). The Hardy space for this operator was studied in e.g. [1,12,34]. It is not hard to check that \((A_0)–(A_5)\) are satisfied (see Sect. 5.3) and we obtain the following characterization of \( H^1_{L_D}(\mathbb{R}^d_+) \) by means of the Riesz transforms \( R_j = \partial_j x_j L_D^{-1/2} \), \( j = 1, \ldots, d \).

**Theorem E** Let \( d \geq 1 \) and \( L_D \) be the Laplacian on \( \mathbb{R}^d_+ \) with the Dirichlet boundary condition at \( x_d = 0 \). Then, \( f \in H^1_{L_D}(\mathbb{R}^d_+) \) if and only if \( f, R_1 f, \ldots, R_d f \in L^1(\mathbb{R}^d_+) \). Moreover, the associated norms are comparable, i.e.
\[
\| f \|_{H^1_{L_D}(\mathbb{R}^d_+)} \simeq \| f \|_{L^1(\mathbb{R}^d_+)} + \sum_{j=1}^d \| R_j f \|_{L^1(\mathbb{R}^d_+)} .
\]

**Organization of the paper.** In Sect. 2 we recall some known facts and prove preliminary estimates. Theorems A and B are proved in Sect. 3. In Sect. 4 we state and prove a modification of Theorem B that will be needed in the proof of Theorem E. Propositions 5.5 and 5.11, that are crucial for Theorems C and D, are stated and proved in Sect. 5. At the end of Sect. 5 we briefly discuss the Dirichlet Laplacian and prove Theorem E. We shall use a standard convention that \( C \) and \( c \) at each occurrence denote some positive constants independent of relevant quantities (depending on the context). We will write \( A \lesssim B \) for \( A \leq CB \) and \( A \simeq B \) for \( A \lesssim B \lesssim A \).

**2 Preliminaries**

**2.1 Admissible Coverings**

Let \( X \subseteq \mathbb{R}^d \) be as in Sect. 1.1. For \( z = (z_1, \ldots, z_d) \in X \) and \( r_1, \ldots, r_d > 0 \) we denote the closed cuboid 
\[
Q(z, r_1, \ldots, r_d) = \{ x \in X : |x_i - z_i| \leq r_i \text{ for } i = 1, \ldots, d \},
\]
and the cube \( Q(z, r) = Q(z, r, \ldots, r) \). The following definition will be used throughout the paper, c.f. [27, Def. 1.2].

**Definition 2.1** Let \( \mathcal{Q} \) be a set of cuboids in \( X \subseteq \mathbb{R}^d \). We call \( \mathcal{Q} \) an admissible covering if:

1. \( X = \bigcup_{Q \in \mathcal{Q}} Q \),

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2. if \( Q_1, Q_2 \in \mathcal{Q} \) and \( Q_1 \neq Q_2 \), then \( |Q_1 \cap Q_2| = 0 \),
3. if \( Q = Q(z, r_1, \ldots, r_d) \in \mathcal{Q} \), then \( r_i \cong r_j \) for \( i, j \in \{1, \ldots, d\} \),
4. if \( Q_1, Q_2 \in \mathcal{Q} \) and \( Q_1 \cap Q_2 \neq \emptyset \), then \( d_{Q_1} \simeq d_{Q_2} \),
5. if \( Q \in \mathcal{Q} \), then \( \text{dist}_{\mathbb{R}^d}(Q, \mathbb{R}^d \setminus X) \gtrsim d_Q \).

Having an admissible covering \( \mathcal{Q} \) and \( Q = (z, r_1, \ldots, r_d) \in \mathcal{Q} \), we define
\[
Q^* := Q(z, \kappa r_1, \ldots, \kappa r_d),
\]
where \( \kappa > 1 \) is chosen so that for \( Q_1, Q_2 \in \mathcal{Q} \),
\[
Q_1^{***} \cap Q_2^{***} \neq \emptyset \iff Q_1 \cap Q_2 \neq \emptyset
\]
and
\[
\text{dist}_{\mathbb{R}^d}(Q^{***}, \mathbb{R}^d \setminus X) > 0.
\]

The family \( \{Q^{***}\}_{Q \in \mathcal{Q}} \) is a finite covering of \( X \), namely
\[
\sum_{Q \in \mathcal{Q}} \mathbb{1}_{Q^{***}}(x) \leq C, \quad x \in X. \tag{2.2}
\]

Let us notice that we have a flexibility in choosing the enlargements \( Q^*, Q^{**}, Q^{***} \) etc. In particular the notation in [27] is slightly different. Recall that having admissible coverings \( \mathcal{Q}_i \) of \( X_i, i = 1, \ldots, N \), we can produce a natural admissible covering \( \mathcal{Q}_1 \bowtie \ldots \bowtie \mathcal{Q}_N \) of \( X \) as in (1.6), see [27, Def. 1.5].

### 2.2 Products

In this subsection, \( i \) will be always an index from \( \{1, \ldots, N\} \). Let \( X_i \subseteq \mathbb{R}^{d_i} \) and \( L_i \) are as in Sect. 1.1 on \( L^2(\mathbb{R}^{d_i}) \). Set \( d = d_1 + \cdots + d_N \) and let \( X \) be as (1.6). Now, we shall explain the precise meaning of (1.7). Slightly abusing the notation we keep the symbol \( L_i \) for the operator
\[
\underbrace{I \otimes \ldots \otimes I}_{i-1 \text{ times}} \otimes L_i \otimes \underbrace{I \otimes \ldots \otimes I}_{N-i \text{ times}}
\]
on \( L^2(X) \), where \( I \) denotes the identity operator on the corresponding subspace, and we define
\[
L f(x) = L_1 f(x) + \ldots + L_N f(x), \quad x = (x_1, \ldots, x_N) \in X.
\]

Since the operators \( L_i \) are self-adjoint, the operator \( L \) is well defined and essentially self-adjoint, see e.g. [35, Thm. 7.23].
Recall that the semigroups $T_t^{[i]} = \exp(-tL_i)$ on $X_i$ have the kernels $T_t^{[i]}(x_i, y_i)$, $x_i, y_i \in X_i, t > 0$, so that the semigroup $T_t = \exp(-tL)$ is related to the kernel

$$T_t(x, y) = T_t^{[1]}(x_1, y_1) \cdot \ldots \cdot T_t^{[N]}(x_N, y_N).$$

### 2.3 Local Atomic Hardy Spaces

For an admissible covering $Q$ of $X \subseteq \mathbb{R}^d$ (see Definition 2.1), we shall define the **local atomic Hardy space** $H_{at}^1(Q)$ related to $Q$ as follows.

**Definition 2.3** A function $a : X \to \mathbb{C}$ is called a $Q$-atom if either:

(i) there is $Q \in Q$ and a cube $K \subset Q^*$, such that:

$$\text{supp } a \subseteq K, \quad \|a\|_{\infty} \leq |K|^{-1}, \quad \int a(x) \, dx = 0;$$

or

(ii) there exists $Q \in Q$ such that $a(x) = |Q|^{-1} 1_Q(x)$.

Then, the atomic space $H_{at}^1(Q)$, is defined in a standard way. Namely, we say that a function $f$ is in $H_{at}^1(Q)$ if $f = \sum \lambda_k a_k$ with $Q$-atoms $a_k$ and $\sum |\lambda_k| < \infty$. Moreover, the norm of $H_{at}^1(Q)$ is given by

$$\|f\|_{H_{at}^1(Q)} = \inf \sum |\lambda_k|,$$

where the infimum is taken over all possible representations of $f = \sum \lambda_k a_k$ as above.

A standard argument shows that $H_{at}^1(Q)$ is a Banach subspace of $L^1(X)$.

Here we state the atomic decomposition result that follows from [27, Thm. A]. This will be needed later on in the proof of Theorem A.

**Theorem 2.4** Assume that for $L, T_t$, and an admissible covering $Q$ the assumptions $(A_0) – (A_2)$ are satisfied. Then $H_{L}^1(X) = H_{at}^1(Q)$ and the corresponding norms are equivalent.

### 2.4 Classical Local Hardy Spaces

In this section, we recall briefly some theory related to the classical local Hardy spaces on $\mathbb{R}^d$, c.f. [23,37]. In particular, we shall present the relation between classical local Hardy spaces and local Riesz transforms in Proposition 2.5.
Recall that the kernel of the Riesz transform $\tilde{R}_j = \partial_{x_j}(-\Delta)^{-1/2}$ can be given by

$$\tilde{R}_j(x, y) = \pi^{-1/2} \int_0^\infty \partial_{x_j} H_t(x - y) \frac{dt}{\sqrt{t}}$$

and for $\tau > 0$ denote

$$\tilde{R}_{\tau, \text{loc}}^j(x, y) = \pi^{-1/2} \int_0^{\tau^2} \partial_{x_j} H_t(x - y) \frac{dt}{\sqrt{t}}$$

$$\tilde{R}_{\tau, \text{glob}}^j(x, y) = \pi^{-1/2} \int_{\tau^2}^\infty \partial_{x_j} H_t(x - y) \frac{dt}{\sqrt{t}}.$$ .

It is well known that these kernels are related (in the principal value sense) with the operators $\tilde{R}_{\tau, \text{loc}}^j$ and $\tilde{R}_{\tau, \text{glob}}^j$ that are well defined and bounded on $L^2(\mathbb{R}^d)$ (uniformly in $\tau > 0$). In what follows we shall need the following version of the characterization of the local Hardy spaces.

**Proposition 2.5** There exist $C_1, C_2 > 0$ that does not depend on $\tau > 0$ such that:

1. If $a$ is either a classical atom or local atom of the form $a = |Q|^{-1/2} 1_Q$, where $Q = Q(z, r_1, \ldots, r_d)$, $r_1 \simeq \ldots \simeq r_d \simeq \tau$, we have

$$\|a\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \left\| \tilde{R}_{\tau, \text{loc}}^j a \right\|_{L^1(\mathbb{R}^d)} \leq C_1.$$

2. Assume that $\text{supp } f \subseteq Q^*$, where $Q = Q(z, r_1, \ldots, r_d)$, $r_1 \simeq \ldots \simeq r_d \simeq \tau$, and

$$M := \|f\|_{L^1(Q^*)} + \sum_{j=1}^d \left\| \tilde{R}_{\tau, \text{loc}}^j f \right\|_{L^1(Q^{**})} < \infty.$$

Then there exist sequences $\{\lambda_k\}_k$ and $\{a_k\}_k$, such that: $f = \sum_k \lambda_k a_k$, $\sum_k |\lambda_k| \leq C_2 M$, and $a_k$ are either the classical atoms supported in a cube $K \subseteq Q^{**}$ or $a_k = |Q|^{-1/2} 1_Q$.

**Sketch of the proof** This fact is well known and has quite standard proof. For the convenience of the reader we provide a sketch of the proof. Notice that

$$\tilde{R}_{\tau, \text{loc}}^j(x, y) = c_d \frac{x_j - y_j}{|x - y|^{d+1}} \psi\left(\frac{|x - y|}{\tau}\right),$$

where $\psi$ is smooth on $[0, \infty)$, $\psi(0) = c'_d$ and $\psi(s) \simeq e^{-s^2}$ as $s \to \infty$.

Part 1. follows by a standard Calderón–Zygmund argument. The main idea is to use the $L^2$-estimate on $Q(x_0, 2\tau)$ and the estimate $\tilde{R}_{\tau, \text{loc}}^j(x, y) \leq \tau |x - y|^{-d-1}$ for $y \in Q(x_0, \tau)$ and $x \notin Q(x_0, 2\tau)$.

In order to prove 2, define $\lambda_0 = \int f$ and let

$$g(x) = f(x) - \lambda_0 |Q|^{-1} 1_Q(x).$$
Then $a_0(x) = |Q|^{-1} 1_Q(x)$ is one of our atoms, $|\lambda_0| \leq M$, $\text{supp } g \subseteq Q^*$ and $\int g = 0$. By standard computations one may check that

$$\|g\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|\widehat{R_j g}\|_{L^1(\mathbb{R}^d)} \lesssim M.$$ 

Using the classical characterization of $H^1(\mathbb{R}^d)$ by means of the Riesz transforms, see (1.2), we obtain

$$g(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x),$$

where $a_k$ are classical atoms on $\mathbb{R}^d$ and

$$\sum_{k=1}^{\infty} |\lambda_k| \lesssim M.$$ 

Then

$$f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x), \quad \sum_{k=0}^{\infty} |\lambda_k| \lesssim M.$$ 

This may look that we are done, but notice that we also want to have atoms $a_k$ supported in $Q^{**}$ (not anywhere in $\mathbb{R}^d$). This can be done by a standard procedure, for details see e.g. [26, Thm. 2.2(b)]. Let us notice, that here we make use of the property 5. of Definition 2.1, i.e. we enlarge $Q$ in $\mathbb{R}^d$, but we want to have atoms supported in $Q^{**}$ that is still in $X$. \hfill \Box

2.5 Partition of Unity

In what follows we shall decompose functions using an admissible covering $Q$ of $X \subseteq \mathbb{R}^d$ see Definition 2.1. One can find functions $\psi_Q \in C^1(X)$ such that:

$$0 \leq \psi_Q(x) \leq 1_Q^*(x), \quad \|\psi_Q'\|_{\infty} \leq C d_Q^{-1}, \quad \sum_{Q \in \mathcal{Q}} \psi_Q(x) = 1_X(x).$$

The family $\{\psi_Q\}_{Q \in \mathcal{Q}}$ will be called a partition of unity related to $Q$.

2.6 Auxiliary Estimates

In what follows we shall use a slight generalization of $(A_2)$–$(A_5)$ that we state below for further references.
Lemma 2.6 Assume that $T_t$ together with admissible covering $Q$ satisfy $(A_0)$ and $(A_2)$–$(A_5)$. Let $\gamma$ be as in $(A_2)$. Then, for $c \geq 1$ there exists $C > 0$ such that

$$
\sup_{y \in Q^{**}} \int_{Q^{***}} \sup_{t \leq c d_Q^2} t^{-\delta} |T_t(x, y) - H_t(x - y)| \, dx \leq C d_Q^{-2\delta}, \quad \delta \in [0, \gamma), \; Q \in Q.
$$  \hspace{1cm} (A_0')

$$
\sup_{y \in Q^{**}} \int_{(Q^{***})^c} \int_0^{c d_Q^2} |\partial_{x_j} T_t(x, y)| \, dt \, dx \leq C, \quad Q \in Q, \; j = 1, \ldots, d, \hspace{1cm} (A_2')
$$

$$
\sup_{y \in Q^{**}} \int_{X} \int_{c^{-1} d_Q^2} \int_0^{\infty} |\partial_{x_j} T_t(x, y)| \, dt \, dx \leq C, \quad Q \in Q, \; j = 1, \ldots, d, \hspace{1cm} (A_3')
$$

$$
\sup_{y \in Q^{**}} \int_{Q^{***}} \int_0^{c d_Q^2} \int_0^{\infty} |\partial_{x_j} (T_t(x, y) - H_t(x - y))| \, dt \, dx \leq C, \quad Q \in Q, \; j = 1, \ldots, d. \hspace{1cm} (A_4')
$$

The proof of Lemma 2.6 is a simple exercise that follows easily from $(A_0)$ and $(A_2)$–$(A_5)$.

2.7 Riesz Transforms

For $\tau > 0$ and $j = 1, \ldots, d$ we split the kernel (1.5) as $R_j(x, y) = R_{j, loc}^j(x, y) + R_{j, glob}^j(x, y) + R_V^j(x, y)$, where

$$
R_{j, loc}^j(x, y) = \pi^{-1/2} \int_0^{\tau^2} \partial_{x_j} T_t(x, y) \frac{dt}{\sqrt{t}}, \quad x, y \in X,
$$

$$
R_{j, glob}^j(x, y) = \pi^{-1/2} \int_{\tau^2}^{\infty} \partial_{x_j} T_t(x, y) \frac{dt}{\sqrt{t}}, \quad x, y \in X,
$$

$$
R_V^j(x, y) = \pi^{-1/2} \int_0^{\infty} V_j(x) T_t(x, y) \frac{dt}{\sqrt{t}}, \quad x, y \in X.
$$  \hspace{1cm} (2.7)

Here we shall prove some preliminary estimate that will be needed later on.

Lemma 2.8 Suppose that $(A_3)$–$(A_6)$ are satisfied for $T_t$ and $Q$. Then

$$
\sup_{y \in X} \sum_{Q \in Q} \int_{Q^{**}} |R_j(x, y)| |\psi_Q(x) - \psi_Q(y)| \, dx \leq C.
$$
Proof Fix \( y \in X \) and \( Q_0 \in Q \) such that \( y \in Q_0 \). Write

\[
\sum_{Q \in \mathcal{Q}} \int_{Q^*} |R_j(x, y)| |\psi_Q(x) - \psi_Q(y)| \, dx \leq \sum_{Q \in \mathcal{Q}} \int_{Q^*} |R^I_{dQ_0, \text{glob}}(x, y)| \\
\left|\psi_Q(x) - \psi_Q(y)\right| \, dx \\
+ \sum_{Q \in \mathcal{Q}} \int_{Q^* \cap (Q^*_0)^c} |R^I_{dQ_0, \text{loc}}(x, y)| \left|\psi_Q(x) - \psi_Q(y)\right| \, dx \\
+ \sum_{Q \in \mathcal{Q}} \int_{Q^*} |R^I_{V}(x, y)| \left|\psi_Q(x) - \psi_Q(y)\right| \, dx
\]

\[= S_1 + S_2 + S_3 + S_4.\]

Using \( \|\psi_Q\|_{\infty} \leq 1 \), (2.2), (A4), (A3) and (A6) we have

\[
S_1 \lesssim \int_X \int_{d_{Q_0}^2} |\partial_{x_j} T_t(x, y)| \frac{dt}{\sqrt{t}} \, dx \lesssim 1, \\
S_2 \lesssim \int_{(Q^*_0)^c} \int_{0}^{d_{Q_0}^2} |\partial_{x_j} T_t(x, y)| \frac{dt}{\sqrt{t}} \, dx \lesssim 1, \\
S_4 \lesssim \int_X \int_{0}^{\infty} |V_j(x)| T_t(x, y) \frac{dt}{\sqrt{t}} \, dx \lesssim 1.
\]

For \( S_3 \) consider \( Q \in \mathcal{Q} \) such that \( Q^* \cap (Q^*_0)^c \neq \emptyset \). The number of such \( Q \) is bounded by an universal constant, \( d_Q \simeq d_{Q_0} \), and \( |\psi_Q(x) - \psi_Q(y)| \lesssim d_{Q_0}^{-1}|x - y| \). Applying (A5) we obtain

\[
S_3 \lesssim \int_{Q^*_0 \cap (Q^*_0)^c} \int_{0}^{d_{Q_0}^2} |\partial_{x_j} (T_t(x, y) - H_t(x - y))| \frac{dt}{\sqrt{t}} \, dx \\
+ \int_{Q^*_0 \cap (Q^*_0)^c} \int_{0}^{d_{Q_0}^2} |\partial_{x_j} H_t(x - y)| \frac{dt}{\sqrt{t}} \, dx \\
\lesssim 1 + \int_{Q^*_0 \cap (Q^*_0)^c} \frac{|x - y|}{d_{Q_0}} \int_{0}^{\infty} t^{-d/2} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dt}{t} \, dx \\
\lesssim 1 + d_{Q_0}^{-1} \int_{Q^*_0 \cap (Q^*_0)^c} |x - y|^{-d+1} \, dx \lesssim 1.
\]
3 Proofs of Theorems A and B.

3.1 Proof of Theorem A

Proof Denote
\[
\|f\|_{H^1_{L,Riesz}(X)} := \|f\|_{L^1(X)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(X)}.
\]

First inequality: \( \|f\|_{H^1_{L,Riesz}(X)} \lesssim \|f\|_{H^1_L(X)} \). We shall show that
\[
\|R_j a\|_{L^1(X)} \leq C \quad (3.1)
\]
for \( j = 1, 2, \ldots, d \) and a \( Q \)-atom \( a \) with \( C \) independent of \( a \). In general, (3.1) may not be enough to prove boundedness of an operator on \( H^1 \), see [11]. However, here Theorem 2.4, (3.1), and a standard continuity argument imply
\[
\|f\|_{H^1_{L,Riesz}(X)} \lesssim \|f\|_{H^1_L(X)}.
\]
To show (3.1), according to Definition 2.3, suppose that \( a \) is an \( Q \)-atom associated with \( Q \in Q \). Let \( R^j_{dQ,loc}, R^j_{dQ,loc} \) and \( R^j_{V} \) denote the operators with the integral kernels defined in (2.7). Applying (A6), (A4), (A3), (A5), and part I. of Proposition 2.5 we have
\[
\|R_j a\|_{L^1(X)} \leq \|R^j_{V} a\|_{L^1(X)} + \|R^j_{dQ,loc} a\|_{L^1(X)} + \|R^j_{dQ,loc} a\|_{L^1(Q^{**})}
\]
\[
+ \|R^j_{dQ,loc} - \tilde{R}^j_{dQ,loc}\|_{L^1(Q^{**})} + \|\tilde{R}^j_{dQ,loc} a\|_{L^1(Q^{**})} \leq C
\]
and (3.1) is proved. Let us notice here that since \( a \) is bounded and \( \text{supp } a \subseteq Q^{**} \) then our assumptions guarantee that all the operators appearing above are well defined.

Second inequality: \( \|f\|_{H^1_L(X)} \lesssim \|f\|_{H^1_{L,Riesz}(X)} \). Assume that \( \|f\|_{H^1_{L,Riesz}(X)} < \infty \). According to Theorem 2.4 it is enough to decompose \( f \) as \( \sum_k \lambda_k a_k \) with \( Q \)-atoms \( a_k \) and \( \sum_k |\lambda_k| \leq \|f\|_{H^1_{L,Riesz}(X)} \). Let \( \psi_Q \) be a partition of unity related to \( Q \), see Sect. 2.5. We have \( f(x) = \sum_{Q \in Q} f_Q(x) \), with \( f_Q = \psi_Q f \) and \( \text{supp } f_Q \subseteq Q^* \). Notice that
\[
\tilde{R}^j_{dQ,loc} f_Q = \left( \tilde{R}^j_{dQ,loc} - R^j_{dQ,loc} \right) f_Q + (R^j_{V} f_Q - \psi_Q R^j f)
\]
\[- R^j_{dQ,loc} f_Q - R^j_{dQ,loc} f_Q + \psi_Q R^j f.\]

We use (A5), Lemma 2.8, (A4), (A6) getting
\[
\sum_{Q \in Q} \|\tilde{R}^j_{dQ,loc} f_Q\|_{L^1(Q^{**})} \leq \sum_{Q \in Q} \left[ \left\| \left( \tilde{R}^j_{dQ,loc} - R^j_{dQ,loc} \right) f_Q \right\|_{L^1(Q^{**})} \right.
\]
\[+ \|R^j f_Q - \psi_Q R^j f\|_{L^1(Q^{**})} \]
\[
\tilde{R}^j_{dQ,loc} f_Q \leq \sum_{Q \in Q} \left[ \left\| \left( \tilde{R}^j_{dQ,loc} - R^j_{dQ,loc} \right) f_Q \right\|_{L^1(Q^{**})} \right.\]
\[+ \|R^j f_Q - \psi_Q R^j f\|_{L^1(Q^{**})} \].
for every \( j = 1, \ldots, d \). Now we use part 2. of Proposition 2.5 for each \( f_Q \), getting \( \lambda_{Q,k}, a_{Q,k} \) such that

\[
f_Q = \sum_{k} \lambda_{Q,k} a_{Q,k}, \quad \sum_{k} |\lambda_{Q,k}| \lesssim \left\| \tilde{R}^j_{d_Q,\text{loc}} f_Q \right\|_{L^1(Q^{**})}.
\]

The proof is finished by noticing that all \( a_{Q,k} \) are \( Q \)-atoms and

\[
f = \sum_{Q,k} \lambda_{Q,k} a_{Q,k}, \quad \sum_{Q,k} |\lambda_{Q,k}| \lesssim \sum_{Q \in \mathcal{Q}} \left\| \tilde{R}^j_{d_Q,\text{loc}} f_Q \right\|_{L^1(Q^{**})} \lesssim \|f\|_{H^1_{L,\text{Riesz}}(X)}.
\]

\[\square\]

### 3.2 Proof of Theorem B

**Proof** According to Theorem A, it is enough to prove \((A_0)–(A_6)\) for the kernel

\[T_i(x, y) = T_i^{[1]}(x_1, y_1) \cdots T_i^{[N]}(x_N, y_N)\]

with the covering \( Q_1 \otimes \ldots \otimes Q_N \), see [27, Def. 1.5]. It is enough to consider \( N = 2 \) and then use an inductive argument. Assume that the conditions \((A_0)–(A_6)\) are satisfied for \( T_i^{[1]}(x_1, y_1) \) and \( T_i^{[2]}(x_1, y_1) \) with \( Q_1 \) and \( Q_2 \), respectively. The estimate \((A_0)\) for \( T_i(x, y) \) follows directly. Moreover, \((A_1)–(A_2)\) were already proved in the proof of [27, Thm. B].

To deal with \((A_3)–(A_6)\) denote

\[x = (x_1, \ldots, x_{d_1}, x_{d_1+1}, \ldots, x_{d_1+d_2}) = (x_1, x_2) \in X_1 \times X_2 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.
\]

Recall that a cuboid in \( Q_1 \otimes Q_2 \) is of the form \( K = K_1 \times K_2 \), where \( K_j \subseteq Q_j \in \mathcal{Q}_j \), \( j = 1, 2 \), and \( d_K \simeq d_{K_1} \simeq d_{K_2} \simeq \min(d_{Q_1}, d_{Q_2}) \), see [27, Def. 1.5]. For the rest of the proof we fix \( y \in K^{**} = K_1^{**} \times K_2^{**} \subseteq Q_1^{**} \times Q_2^{**} \) and without loss of generality we consider \( \partial_{s_j} \) for \( j \in \{d_1+1, \ldots, d_1+d_2\} \).

**Proof of \((A_3)\).** Notice that \((K^{**})^c = (K_1^{**} \times K_2^{**})^c = S_1 \cup S_2 \cup S_3\), where

\[S_1 = X_1 \times (Q_2^{**})^c, \quad S_2 = X_1 \times (Q_2^{**} \setminus K_2^{**}), \quad S_3 = (K_1^{**})^c \times K_2^{**}.
\]
Using \((A_0)\) for \(T_t^{[1]}\) and \((A'_3)\) for \(T_t^{[2]}\) we have

\[
\int_{S_1} \int_0^{d_k} \left| \partial_{x_j} T_t(x, y) \right| \frac{dt}{\sqrt{t}} \, dx = \int_{S_1} \int_0^{d_k} \left| \partial_{x_j} T_t^{[1]}(x_1, y_1) \right| \frac{dt}{\sqrt{t}} \, dx \\
\leq \int_{(Q_2^{**})^c} \int_0^{d_k} \left| \partial_{x_j} T_t^{[2]}(x_2, y_2) \right| \frac{dt}{\sqrt{t}} \, dx_2 \lesssim 1.
\]

\(\square\)

Using \((A_0)\) for \(T_t^{[1]}\) we have

\[
\int_{S_2} \int_0^{d_k} \left| \partial_{x_j} T_t(x, y) \right| \frac{dt}{\sqrt{t}} \, dx \leq \int_{S_2} \int_0^{d_k} \left| \partial_{x_j} T_t^{[1]}(x_1, y_1) \right| \frac{dt}{\sqrt{t}} \, dx \\
\leq \int_{Q_2^{**}} \int_0^{d_k} \left| \partial_{x_j} \left( T_t^{[2]}(x_2, y_2) - H_t(x_2 - y_2) \right) \right| \frac{dt}{\sqrt{t}} \, dx_2 \\
+ \int_{Q_2^{**} \setminus K_2^{**}} \int_0^{d_k} \left| \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{\sqrt{t}} \, dx_2 \\
= A_1 + A_2.
\]

We have that \(d_K \lesssim d_Q\) and \((A'_2)\) for \(T_t^{[2]}\) implies \(A_1 \lesssim 1\). Moreover, for \(y_2 \in K_2^{**}\) and \(x_2 \notin K_2^{**}\) we have \(|x_2 - y_2| \gtrsim d_K\) and

\[
A_2 \lesssim \int_{Q_2^{**} \setminus K_2^{**}} \int_0^{d_k} t^{-d_2/2} \exp \left( -\frac{|x_2 - y_2|^2}{ct} \right) \frac{dt}{t} \, dx_2 \\
\lesssim \int_0^{d_k} t^{-d_2/2 - 1} \frac{dt}{(K_2^{**})^c} \cdot |x_2 - y_2|^{-2M} \, dx_2 \lesssim 1,
\]

where \(M\) is a fixed constant larger than \(d_2/2\). What is left is to estimate the integral on \(S_3\). Write

\[
\int_{S_3} \int_0^{d_k} \left| \partial_{x_j} T_t(x, y) \right| \frac{dt}{\sqrt{t}} \, dx \leq A_3 + A_4,
\]

where

\[
A_3 = \int_{S_1} \int_0^{d_k} \left| \partial_{x_j} T_t^{[1]}(x_1, y_1) \right| \frac{dt}{\sqrt{t}} \, dx, \\
A_4 = \int_{S_1} \int_0^{d_k} \left| \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{\sqrt{t}} \, dx.
\]
From \((A_0)\) for \(T_i^{[1]}\) and \((A'_2)\) for \(T_i^{[2]}\) we easily get \(A_3 \lesssim 1\). Let \(\delta > 0\) be fixed, then,  

\[
A_4 = \int_{(K^{**})^c} \int_K \int_0^{d_K} t^{-2\delta} T_i^{[1]}(x_1, y_1) \left| \delta^{1/2} \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{t^1-\delta} \, dx_2 \, dx_1 \]

\[
\lesssim \int_{(K^{**})^c} \sup_{s \leq d_K^2} \left( s^{-d_1/2-2\delta} \exp \left( -\frac{|x_1 - y_1|^2}{cs} \right) \right) \, dx_1 \]

\[
\times \int_{K^{**}} \sup_{r \leq d_K^2} \left( r^{-d_2/2+\delta} \exp \left( -\frac{|x_2 - y_2|^2}{cr} \right) \right) \, dx_2 \cdot \int_0^{d_K^2} t^{-1+\delta} \, dt \]

\[
\lesssim \int_{|x_1 - y_1|^2 \leq d_K} |x_1 - y_1|^{-d_1-4\delta} \, dx_1 \cdot \int_{|x_2 - y_2|^2 \leq d_K} |x_2 - y_2|^{-d_2+2\delta} \, dx_2 \cdot d_K^{2\delta} \]

\[
\lesssim d_K^{4\delta} d_K^{2\delta} \lesssim 1.
\]

**Proof of \((A_4)\).** We have that \(d_K \simeq d_Q\) or \(d_K \simeq d_Q^2\). In the latter case \(d_K \simeq d_Q^2\) the inequality \((A_4)\) for \(T_i(x, y)\) follows simply from \((A_0)\) for \(T_i^{[1]}\) and \((A'_2)\) for \(T_i^{[2]}\). Assume then that \(d_K \simeq d_Q^2 \lesssim d_Q^2\). Let \(t \geq d_K^2\) and \(y \in K^{**} \subseteq Q^{**}\). Write

\[
\int \int d_K^2 \left| \partial_{x_j} T_i(x, y) \right| \frac{dt}{\sqrt{t}} \, dx = \int \int d_Q^2 \ldots + \int \int d_Q^2 \ldots = A_5 + A_6.
\]

By \((A_0)\) for \(T_i^{[1]}\) and \((A'_4)\) for \(T_i^{[2]}\) we easily get \(A_6 \lesssim 1\). Let \(\delta \in (0, \gamma)\) be as in \((A_1)\)–\((A_2)\). For \(A_5\) write

\[
A_5 \leq \int X_1 \sup_{t \geq d_K^2} \left( t^\delta T_i^{[1]}(x_1, y_1) \right) \, dx_1 \cdot \int X_2 \int d_K^2 t^{-\delta} \left| \partial_{x_j} T_i^{[2]}(x_2, y_2) \right| \frac{dt}{\sqrt{t}} \, dx_2 = A_{5,1} \cdot A_{5,2}.
\]

By \((A_0)\) and \((A_1)\) for \(T_i^{[1]}\) we have

\[
A_{5,1} \lesssim \int Q^{***} \sup_{t \geq d_K^2} t^\delta \, dx_1
\]

\[
\quad + \int (Q^{***})^c \sup_{t > 0} t^\delta \, dx_1
\]

\[
\lesssim d_k^{d_1} d_Q^{d_1-\delta} + d_Q^{2\delta} \simeq d_k^{2\delta}.
\]
Moreover,

\[ A_{5,2} \leq \int_{Q_2^{***}} \int_0^{d_{Q_2}^2} t^{-\delta} \left| \partial_{x_j} T_t^{[2]}(x_2, y_2) \right| \frac{dt}{\sqrt{t}} dx_2 + \int_{Q_2^{***}} \int_0^{d_{Q_2}^2} \]

\[ t^{-\delta} \left| \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{\sqrt{t}} dx_2 \]

\[ + \int_{Q_2^{***}} \int_0^{d_{Q_2}^2} t^{-\delta} \left| \partial_{x_j} T_t^{[2]}(x_2, y_2) - \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{\sqrt{t}} dx_2 = A_{5,2,1} + A_{5,2,2} + A_{5,2,3}. \]

Using \((A_3)\) and \((A_5)\) for \(T_t^{[2]}\) and the estimate \(t^{-\delta} \leq d_K^{-2\delta}\) we easily get \(A_{5,2,1} + A_{5,2,3} \lesssim d_K^{-2\delta}\). Also,

\[ A_{5,2,2} \leq \int_{d_K^2}^{\infty} t^{-1-\delta} \int_{X_2} t^{-d_2/2} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) dx_2 dt \lesssim d_K^{-2\delta}. \]

Combining all the estimates above, we finish the proof of \((A_4)\) by noticing that \(A_5 + A_6 \lesssim 1\).

**Proof of \((A_5)\)** We have that \(d_K \simeq \min(d_{Q_1}, d_{Q_2})\) and \(K_j \subseteq Q_j\) for \(j = 1, 2\). Using the triangle inequality write

\[ \int_{K_j} \int_0^{d_{Q_2}^2} \left| \partial_{x_j} (T_t(x, y) - H_t(x - y)) \right| \frac{dt}{\sqrt{t}} dx \]

\[ \leq \int_{K_j} \int_0^{d_{Q_2}^2} T_t^{[1]}(x_1, y_1) \left| \partial_{x_j} \left(T_t^{[2]}(x_2, y_2) - H_t(x_2 - y_2)\right) \right| \frac{dt}{\sqrt{t}} dx \]

\[ + \int_{K_j} \int_0^{d_{Q_2}^2} \left| T_t^{[1]}(x_1, y_1) - H_t(x_1 - y_1) \right| \left| \partial_{x_j} H_t(x_2 - y_2) \right| \frac{dt}{\sqrt{t}} dx = A_7 + A_8 \]

By \((A_0)\) for \(T_t^{[1]}\) and \((A_5')\) for \(T_t^{[2]}\) we have that \(A_7 \lesssim 1\).

For \(A_8\) we use \((A_2')\) for \(T_t^{[1]}\) obtaining

\[ A_8 \lesssim \int_{Q_1^{***}} \sup_{s \leq d_{Q_2}^2} s^{-\delta} \left| T_t^{[1]}(x_1, y_1) - H_s(x_1 - y_1) \right| dx_1 \]

\[ \times \int_0^{c d_{Q_2}^2} \int_0^{d_{Q_2}^2} t^{-\delta} \left| \partial_{x_j} H_t(x_2 - y_2) \right| dx_2 \frac{dt}{\sqrt{t}} \]

\[ \lesssim d_{Q_1}^{-2\delta} \cdot \int_0^{d_{Q_1}^2} t^{-1+\delta} \int_{X_2} t^{-d_2/2} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) dx_2 dt \lesssim 1. \]
Proof of (A₆) Fix \( y \in X \). Using (A₀) for \( T_{t}^{[1]} \) and (A₆) for \( T_{t}^{[2]} \) we have

\[
\int_{X} \int_{0}^{\infty} |V_{j}(x)T_{t}(x, y)| \frac{dt}{\sqrt{t}} \, dx \lesssim \int_{X_{2}} \int_{0}^{\infty} |V_{j}(x_{2})| T_{t}^{[2]}(x_{2}, y_{2}) \, dx_{2} \int_{X_{1}} T_{t}^{[1]}(x_{1}, y_{1}) \, dx_{1} \frac{dr}{\sqrt{r}} \, dx_{2} \lesssim 1.
\]

The proof of Theorem B is finished.

4 Products of Local and Nonlocal Atomic Hardy Spaces

In this section, we present an alternative version of Theorem B. Consider the operator \( L = -\Delta + L_{2} \), where \( L_{2} \) is related to an admissible covering \( Q_{2} \) and satisfies all the assumptions of Sect. 1.1. It turns out that our methods work equally fine in this context. Notice that the Hardy space related to \( -\Delta \) does not have local nature, but the Hardy space for \( L = -\Delta + L_{2} \) will have local character as in Definition 2.3. Let us mention that this section will be needed to investigate the Dirichlet Laplacian on \( \mathbb{R}_{+}^{d} \), see Sects. 1.2 and 5.3.

More precisely, let \( X = \mathbb{R}^{d_{1}} \times X_{2} \), where \( X_{2} \subseteq \mathbb{R}^{d_{2}} \) is as in Sect. 1.1. We consider an operator \( L \) on \( L^{2}(X_{2}) \) and its semigroup \( T_{t}^{[2]} \) with the kernel \( T_{t}^{[2]}(x_{2}, y_{2}) \), \( x_{2}, y_{2} \in X_{2}, t > 0 \). Assume that \( L_{2} \) and an admissible covering \( Q_{2} \) of \( X_{2} \) satisfy (A₀)–(A₆), see Sect. 1.1 and Definition 2.1. On \( \mathbb{R}^{d_{1}} \) we consider the Laplacian \( -\Delta \) with the heat semigroup kernel \( H_{t}(x_{1} - y_{1}) \), \( x_{1}, y_{1} \in \mathbb{R}^{d_{1}}, t > 0 \), see (1.1). Following [27, Sec. 1.4.4.] we define the covering \( \mathbb{R}^{d_{1}} \otimes Q_{2} \) by splitting the strips \( \mathbb{R}^{d_{1}} \times Q_{2}, Q_{2} \in Q_{2} \), into countably many cuboids \( Q_{1,n} \times \mathbb{R}^{d_{2}} \) such that \( d_{Q_{1,n}} = d_{Q_{2}} \). Then \( L = -\Delta + L_{2} \) is understood in the sense as in Sect. 2.2.

The atomic characterization for \( H_{t}^{1}(X) \) is given in [27, Cor. 1.14], where the atoms are related to the covering \( \mathbb{R}^{d_{1}} \otimes Q_{2} \). In Theorem 4.1 below we provide a characterization by means of the Riesz transforms \( R_{j} = D_{j}L_{-1/2} \), where \( D_{j} = \partial_{x_{j}} \) for \( j = 1, \ldots, d_{1} \), and \( D_{j} = \partial_{x_{j}} + V_{j} \) for \( j = d_{1} + 1, \ldots, d_{1} + d_{2} \).

**Theorem 4.1** Let \( L = -\Delta + L_{2} \), where \( -\Delta \) is the standard Laplacian on \( \mathbb{R}^{d_{1}} \) and \( L_{2} \) with an admissible covering \( Q_{2} \) of \( X_{2} \subseteq \mathbb{R}^{d_{2}} \) satisfy (A₀)–(A₆). Then there exists \( C > 0 \) such that

\[
C^{-1} \| f \|_{H_{t}^{1}(X)} \leq \| f \|_{L^{1}(X)} + \sum_{j=1}^{d} \| R_{j} f \|_{L^{1}(X)} \leq C \| f \|_{H_{t}^{1}(X)}.
\]

The proof of Theorem 4.1 follows directly from Theorem A and the following lemma.

**Lemma 4.2** If (A₀)–(A₆) are satisfied for \( L_{2} \) with an admissible covering \( Q_{2} \) of \( X_{2} \subseteq \mathbb{R}^{d_{2}} \), then (A₀)–(A₆) are satisfied for \( L = -\Delta + L_{2} \) (see Sect. 2.2) with an admissible covering \( \mathbb{R}^{d_{1}} \otimes Q_{2} \) of \( X = \mathbb{R}^{d_{1}} \times X_{2} \subseteq \mathbb{R}^{d_{1} + d_{2}} \).
The proof of Lemma 4.2 uses the same techniques as the proof of Theorem B and will be omitted.

5 Examples

The goal of this section is to prove Theorems C, D, and E. According to Theorem B it is enough to prove \((A_0)-(A_6)\) for the one-dimensional Bessel operator \(L_B^{[\beta]}\) and the one-dimensional Laguerre operator \(L_L^{[\beta]}\).

Recall that \((A_0)-(A_2)\) were proved in [27, Prop. 4.3 and 4.5], so we shall deal only with \((A_3)-(A_6)\) in Propositions 5.5 and 5.11. By \(T_t(x, y)\) we will denote the semigroup kernel related to: \(L_B\) in Sect. 5.1, \(L_L\) in Sect. 5.2, and \(L_D\) in Sect. 5.3. Denote \(\partial_x = \frac{d}{dx}\), the partial derivative on \((0, \infty)\).

5.1 Bessel Operator

The semigroup \(T_t = \exp(-t L_B^{[\beta]})\) is given in terms of the integral kernel

\[
T_t(x, y) = \frac{(xy)^{1/2}}{2t} I_{\beta-1/2} \left( \frac{xy}{2t} \right) \exp \left( -\frac{x^2 + y^2}{4t} \right), \quad x, y \in X, t > 0, \tag{5.1}
\]

i.e. \(T_t f(x) = \int_X T_t(x, y) f(y) \, dy\). Here, \(I_\tau\) is the modified Bessel function of the first kind. For further reference recall some properties of the Bessel function \(I_\tau\):

\[
I_\tau(x) = C_\tau x^\tau + O(x^{\tau+1}), \quad \text{for } x \sim 0, \tag{5.2}
\]

\[
I_\tau(x) = (2\pi x)^{-1/2} e^x + O(x^{-3/2} e^x), \quad \text{for } x \sim \infty, \tag{5.3}
\]

\[
\partial_x (x^{-\tau} I_\tau(x)) = x^{-\tau} I_{\tau+1}(x) \quad \text{for } x > 0, \tag{5.4}
\]

see e.g. [39]. The main goal of this section is to prove the following proposition.

**Proposition 5.5** Let \(X = (0, \infty)\) and \(\beta > 0\). Then \((A_3)-(A_6)\) hold for \(L_B^{[\beta]}\) with \(Q_B\), see (1.9).

**Proof** Using (5.4) we have

\[
\partial_x T_t(x, y) = \frac{(xy)^{1/2}}{2t} \exp \left( -\frac{x^2 + y^2}{4t} \right) \left( \frac{y}{2t} I_{\beta+\frac{1}{2}} \left( \frac{xy}{2t} \right) + \frac{\beta}{x} I_{\beta-\frac{1}{2}} \left( \frac{xy}{2t} \right) \right), \tag{5.6}
\]

Denote case 1: \(xy \lesssim t\). In this case, by (5.6) and (5.2),

\[
|\partial_x T_t(x, y)| \lesssim t^{-1/2} \left( \frac{xy}{t} \right)^\beta \exp \left( -\frac{x^2 + y^2}{ct} \right) \left( \frac{1}{x} + \frac{x}{t} \right). \tag{5.7}
\]
In case 2: \( t \lesssim xy \), using (5.6) and (5.3), we have

\[
|\partial_x T_t(x, y)| \lesssim \frac{x + y}{t^{3/2}} \exp\left(-\frac{|x - y|^2}{ct}\right).
\]

(5.8)

For the rest of the proof let us fix \( I = [2^n, 2^{n+1}] \in Q_B \) and \( y \in I^{**} \). Then \( y \simeq 2^n = d_I \). Fix \( 2^{-1} < \kappa_1 < 1 < \kappa_2 < 2 \) such that \( I^{**} = [\kappa_1 2^n, \kappa_2 2^n+1] \). \( \Box \)

**Proof of (A3)** Write

\[
\int_{(A^{**}c)} \int_0^{2^n} \frac{d_1^2}{|\partial_x T_t(x, y)|} \frac{dt}{\sqrt{t}} \ dx \leq \int_0^{\kappa_1 \ast 2^n} \int_0^{xy} \frac{dx}{t^{3/2}} \exp\left(-\frac{2^n}{ct}\right) \ dx \lesssim 2^{-n} \int_0^{2^n} \left(\frac{1}{x} + \frac{x}{t}\right) \ dx \lesssim 1,
\]

\[
A_2 \lesssim \int_0^{2^n} \int_0^{xy} \frac{x^{2n}}{t} \exp\left(-\frac{2^n}{ct}\right) \ dx \lesssim 2^{-n} \int_0^{2^n} \frac{dx}{t} \lesssim 1,
\]

\[
A_3 \lesssim \int_{2^n+1}^{\infty} \int_0^{2^n} \frac{x}{t^{3/2}} \exp\left(-\frac{x^2}{ct}\right) \ dx \lesssim \int_{2^n+1}^{\infty} \frac{dt}{t} \lesssim 1,
\]

where \( N \) is arbitrarily large constant (here \( N > 1 \) is enough).

**Proof of (A4)** Let us write

\[
\int_X \int_0^{\infty} \frac{|\partial_x T_t(x, y)|}{\sqrt{t}} \ dx \ dt = \int_0^{2^n+2} \int_0^{2^n} \ dx \ dt \lesssim 1.
\]

(5.7)

For \( A_4 \) we have observe that \( x/t \lesssim 2^n/t \lesssim x^{-1} \). Using (5.7),

\[
A_4 \lesssim \int_0^{2^n+2} \int_0^{2^n} \frac{x^{2n}}{t} \exp\left(-\frac{2^n}{ct}\right) \ dx \ dt \lesssim 2^{n/2} \int_0^{2^n+2} x^{-1+\beta} \ dx \int_0^{\infty} t^{-1+\beta} \exp\left(-\frac{2^n}{ct}\right) \ dt \lesssim 1.
\]
In $A_5$ and $A_6$ we use (5.8) and (5.7), respectively. For an arbitrary large $N$ we have:

\[
A_5 \lesssim \int_0^\infty \int_{2^{2n}+1}^{2^{2n}+2^n} \frac{x}{t^{3/2}} \exp \left( -\frac{x^2}{ct} \right) \frac{dt}{\sqrt{t}} dx
\]

\[
\lesssim \int_0^\infty x^{-2N+1} \int_0^{2^n} t^{N-2} dt dx \lesssim 1,
\]

\[
A_6 \lesssim \int_0^\infty \int_0^{2^n} \left( \frac{2^n x}{t} \right)^\beta \exp \left( -\frac{x^2}{ct} \right) \frac{1}{x} \left( 1 + \frac{x^2}{t} \right) \frac{dt}{t} dx
\]

\[
\lesssim 2^n \int_0^\infty x^{-1-\beta} \int_0^{2^n} \left( \frac{x^2}{t} \right)^{\beta-1} \exp \left( -\frac{x^2}{ct} \right) \frac{dt}{t} dx
\]

\[
\lesssim 2^n \int_0^\infty x^{-1-\beta} dx \int_0^{\infty} t^{\beta-1} (1+t)e^{-t} dt \lesssim 1.
\]

**Proof of (A_5)** Observe that for $x \in I^{**}$, $y \in I^{*}$, and $t \leq d_1^2 = 2^{2n}$ we have $t \lesssim xy$. Therefore, using (5.6) and (5.3) we get

\[
\partial_x T_t(x, y) = \frac{y - x}{2t} \left( \frac{x^2}{4t} \right)^{\beta-1/2} + R(x, y)
\]

where

\[
|R(x, y)| \lesssim t^{-1/2} \exp \left( -\frac{|x-y|^2}{4t} \right) \left( \frac{x+y}{xy} + x^{-1} \right)
\]

and

\[
\lesssim x^{-1} t^{-1/2} \exp \left( -\frac{|x-y|^2}{4t} \right),
\]

since $x \simeq y \simeq d_1$. Notice that $|x-y| \lesssim 2^n$. By (5.9) and (5.10) we obtain

\[
\int_{I^{**}} \int_0^{2^n} \left| \partial_x T_t(x, y) - \partial_x H_t(x, y) \right| \frac{dt}{\sqrt{t}} dx \leq \int_{I^{**}} \int_0^{2^n} |R(x, y)| \frac{dt}{\sqrt{t}} dx
\]

\[
\lesssim \int_{I^{**}} x^{-1} \int_0^{2^n} \exp \left( -\frac{|x-y|^2}{4t} \right) \frac{dt}{t} dx
\]

\[
= C \int_{I^{**}} x^{-1} \int_{|x-y|^2/2^n}^{\infty} \exp \left( -t/4 \right) \frac{dt}{t} dx
\]

\[
\lesssim \int_0^{2^n+2} \ln \left( 2 + \frac{2^n}{|x-y|} \right) dx \lesssim \int_{-2}^{2} \ln \left( 2 + |x|^{-1} \right) dx \lesssim 1.
\]
Proof of \((A_6)\) Using \((5.1)\), \((5.2)\) and \((5.3)\), we have that

\[
\int_0^\infty T_t(x,y) \frac{dt}{\sqrt{t}} \lesssim \int_0^y \exp \left( -\frac{|x-y|^2}{4t} \right) \frac{dt}{t} + \int_0^\infty \left( \frac{xy}{t} \right)^\beta \exp \left( -\frac{x^2+y^2}{ct} \right) \frac{dt}{t} 
\]

\[
\lesssim \begin{cases} 
(x/y)^\beta & x \leq y/2, \\
\ln(y|x-y|^{-1}) & |x-y| \leq y/2, \\
(y/x)^\beta & x \geq 3y/2.
\end{cases}
\]

Hence,

\[
\int_X \int_0^\infty x^{-1} T_t(x,y) \frac{dt}{\sqrt{t}} \lesssim y^{-\beta} \int_0^{y/2} x^{-1+\beta} \, dx + \int_{|x-y| \leq y/2} \ln \left( \frac{y}{|x-y|} \right) \frac{dx}{x} \\
+ y^\beta \int_{3y/2}^\infty x^{-1-\beta} \, dx \lesssim 1.
\]

This ends the proof of Proposition 5.5.

5.2 Laguerre Operator

Recall that \(\beta > 0\) denotes the parameter related to the Laguerre operator \(L^{[\beta]}_L\), see \((1.10)\). The goal of this section is to prove we have the following proposition.

Proposition 5.11 Let \(X = (0, \infty)\) and \(\beta > 0\). Then \((A_3)-(A_6)\) hold for \(L^{[\beta]}_L\) with \(Q_L\) given in \((1.11)\).

Before going to the proof let us make some preparations. In what follows we shall use the notation \(\text{sh}(t) = \sinh(t)\), and \(\text{ch}(t) = \cosh(t)\). The semigroup \(T_t = T_{L,t} = \exp \left( -tL^{[\beta]}_L \right)\) has a kernel given by

\[
T_t(x,y) = \frac{(xy)^{1/2}}{\text{sh}(2t)} I_{\beta-1/2} \left( \frac{xy}{\text{sh}(2t)} \right) \exp \left( -\frac{\text{ch}(2t)}{2 \text{sh}(2t)} (x^2+y^2) \right), \quad x, y \in X, \quad t > 0.
\]

(5.12)

Denote

\[
U_{\beta-1/2}(x) = I_{\beta-1/2}(x) \exp(-x)\sqrt{2\pi x},
\]

so that

\[
|U_{\beta-1/2}(x) - 1| \lesssim x^{-1}, \quad |U_{\beta-1/2}(x) - U_{\beta+1/2}(x)| \lesssim x^{-1}, \quad x \sim \infty,
\]

(5.14)

c.f. \((5.3)\). Denote

\[
\Theta(t, x, y) = \exp \left( \frac{(1-\text{ch}(2t))(x^2+y^2)}{2\text{sh}(2t)} \right).
\]
In some cases, we shall use different expression for $T_t(x, y)$, namely

$$T_t(x, y) = \frac{\Theta(t, x, y)}{\sqrt{2\pi \text{sh}(2t)}} U_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) \exp \left( -\frac{|x - y|^2}{2\text{sh}(2t)} \right), \ x, y \in X, \ t > 0.$$  

(5.15)

Using (5.12), (5.15), (5.4), and (5.13) we get three expressions for $\partial_x T_t(x, y)$, i.e.

$$\partial_x T_t(x, y) = \frac{\sqrt{xy}}{\text{sh}(2t)} \exp \left( -\frac{\text{ch}(2t)}{2\text{sh}(2t)} (x^2 + y^2) \right) \cdot F_1(t, x, y)$$

(5.16)

$$= \frac{\Theta(t, x, y)}{\sqrt{2\pi \text{sh}(2t)}} \exp \left( -\frac{|x - y|^2}{2\text{sh}(2t)} \right) \cdot F_2(t, x, y)$$

(5.17)

$$= \frac{\Theta(t, x, y)}{\sqrt{2\pi \text{sh}(2t)}} \exp \left( -\frac{|x - y|^2}{2\text{sh}(2t)} \right) \cdot \left( \frac{y - x}{\text{sh}(2t)} U_{\beta + 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) + F_3(t, x, y) \right),$$

(5.18)

where

$$F_1(t, x, y) = \frac{y}{\text{sh}(2t)} I_{\beta + 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) + \frac{\beta}{x} I_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) - x \frac{\text{ch}(2t)}{\text{sh}(2t)} I_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right),$$

$$F_2(t, x, y) = \frac{y}{\text{sh}(2t)} U_{\beta + 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) + \frac{\beta}{x} U_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) - x \frac{\text{ch}(2t)}{\text{sh}(2t)} U_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right),$$

$$F_3(t, x, y) = \frac{\beta}{x} U_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) - x \left( \frac{\text{ch}(2t)}{\text{sh}(2t)} U_{\beta - 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) - U_{\beta + 1/2} \left( \frac{xy}{\text{sh}(2t)} \right) \right).$$

Observe that

$$0 < \Theta(t, x, y) \lesssim \exp \left( -ct(x^2 + y^2) \right), \quad \text{for} \quad t \lesssim 1, \ x, y \in X \quad (5.19)$$

$$0 < \Theta(t, x, y) \lesssim \exp(-ct(x^2 + y^2)), \quad \text{for} \quad t \gtrsim 1, \ x, y \in X. \quad (5.20)$$

Moreover, using (5.2) and (5.14) we get

$$|F_1(t, x, y)| \lesssim \left( \frac{xy}{\text{sh}(2t)} \right)^{\beta - 1/2} \left( \frac{1}{x} + \frac{x\text{ch}(2t)}{\text{sh}(2t)} \right), \quad xy \lesssim \text{sh}(2t),$$

(5.21)

$$|F_2(t, x, y)| \lesssim \left( \frac{y}{\text{sh}(2t)} + \frac{x\text{ch}(2t)}{\text{sh}(2t)} \right), \quad xy \gtrsim \text{sh}(2t),$$

(5.22)

$$|F_3(t, x, y)| \lesssim \left( \frac{1}{x} + xt + \frac{1}{y} \right), \quad xy \gtrsim \text{sh}(2t), \ t \leq 1. \quad (5.23)$$

Now we are almost ready to prove Proposition 5.11 but first let us make a few comments and fix some notion. The proof relies on a detailed and lengthy analysis, but essentially one uses only simple calculus and properties of $I_{\beta - 1/2}$. We shall write $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Recall that $\mathcal{Q}_L$ is the set of intervals given in
(1.11). The proof will be given in two cases. First we shall deal with the sub-intervals of \([0, 1]\) in Sect. 5.2.1. Then we shall consider sub-intervals of \([1, \infty)\) in Sect. 5.2.2. The letter \(n\) will always be a positive integer. Moreover, we shall use \(N\) as a constant that is fixed and large enough, depending on the context (most often we shall use the inequality \(\exp(-x) \lesssim x^{-N}\)).

5.2.1 Case 1: \(I \subseteq [0, 1]\)

We consider \(I = [2^{-n}, 2^{-n+1}]\), \(n \in \mathbb{N}\), and \(y \in I^{**}\). Then \(y \approx 2^{-n} = d_I\). Fix \(2^{-1} < \kappa_1 < 1 < \kappa_2 < 2\) such that \(I^{***} = [\kappa_1 2^{-n}, \kappa_2 2^{-n+1}]\).

**Proof of (A_3)** in Case 1 We deal with \(0 < t \leq 2^{-2n} \leq 1\), \(sh(t) \approx t\) and \(ch(t) \approx 1\).

Then

\[
\int_{(I^{***})^c} \int_0^{d_I^2} |\partial_x T_t(x, y)| \frac{dt}{\sqrt{t}} \, dx \leq \int_0^{\kappa_1 2^{-n}} \int_0^{2^{-2n} \wedge xy} \cdots + \int_0^{\kappa_2 2^{-n+1}} \int_0^{2^{-2n} \wedge xy} \cdots + \int_{(I^{***})^c} \int_0^{2^{-2n}} \cdots = A_1 + A_2 + A_3.
\]

For \(A_1\) we have \(xy \gtrsim t\), \(x < y\), \(|x - y| \approx y\), and \(|F_2(t, x, y)| \lesssim y/t\). Using (5.17), (5.19), and (5.22),

\[
A_1 \approx y \int_0^{2^{-n}} \int_0^{2^{-2n}} t^{-1} \exp\left(-\frac{y^2}{ct}\right) \frac{dt}{t} \, dx \lesssim y^{1-2N} \int_0^{2^{-n}} dx \cdot \int_0^{2^{-2n}} t^{N-2} \, dt \lesssim 1.
\]

For \(A_2\) we have \(xy \gtrsim t\), \(y < x\), \(|x - y| \approx x\), and \(|F_2(t, x, y)| \lesssim x/t\). Using (5.17), (5.19), and (5.22),

\[
A_2 \approx \int_{2^{-n+1}}^{\infty} \int_0^{2^{-2n}} t^{-1} \exp\left(-\frac{x^2}{ct}\right) \frac{dt}{t} \, dx \lesssim \int_{2^{-n+1}}^{\infty} x^{1-2N} \, dx \cdot \int_0^{2^{-2n}} t^{N-2} \, dt \lesssim 1.
\]

Notice that \(A_3\) appears only when \(x \leq \kappa_1 2^{-n}\). Moreover, \(x^2 \lesssim xy \lesssim t\), and \(|F_1(t, x, y)| \lesssim x^{-1}(xy/t)^{\beta - 1/2}\). Using (5.16) and (5.21),

\[
A_3 \approx \int_0^{\kappa_1 2^{-n}} x^{-1} \int_0^{2^{-2n}} \left(\frac{xy}{t}\right)^\beta \exp\left(-\frac{y^2}{ct}\right) \, dx \, dt \\
\lesssim y^{-2N+\beta} \int_0^{2^{-n}} x^{\beta-1} \, dx \cdot \int_0^{2^{-2n}} t^{N-\beta-1} \, dt \lesssim 1.
\]
Proof of (A4) in Case 1. Recall that \( y \simeq 2^{-n} \). We shall consider \( t \geq d_I^2 = 2^{-2n} \).

Write

\[
\int_0^\infty \int_{d_I^2}^\infty |\partial_x T_t(x, y)| \frac{dr}{\sqrt{t}} \, dx = \int_0^{2^{-n+3}} \int_{2^{-2n}}^1 \, dx + \int_0^{2^{-n+3}} \int_{2^{-2n}}^1 \, dx + \int_0^{2^{-n+3}} \int_{1 \wedge y}^1 \, dx ... + \int_0^{2^{-n+3}} \int_{1 \wedge y}^1 \, dx
\]

\[
+ \int_0^\infty \int_1^{1 \wedge \ln(\sqrt{xy})} \, dx + \int_0^\infty \int_1^{1 \wedge \ln(\sqrt{xy})} \, dx
\]

\[
= A_4 + A_5 + A_6 + A_7 + A_8.
\]

In the integrals \( A_4-A_6 \) we have \( t \leq 1 \), so that \( \text{sh}(2t) \simeq t \) and \( \text{ch}(2t) \simeq 1 \).

For \( A_4 \) we have \( x^2 \lesssim xy \lesssim t \), so that \( |F_1(t, x, y)| \lesssim x^{-1}(xy/t)^{\beta - 1/2} \). Using (5.16) and (5.21),

\[
A_4 \lesssim \int_0^{2^{-n+3}} x^{-1} \int_{2^{-2n}}^1 \left( \frac{xy}{t} \right)^{\beta} \frac{dr}{t} \, dx \lesssim y^{\beta} \int_0^{2^{-n+3}} x^{\beta-1} \, dx \cdot \int_0^\infty t^{-\beta-1} \, dt \lesssim 1.
\]

For \( A_5 \), we have \( xy \gtrsim t \) and \( |x - y| \simeq x \gtrsim y \), since \( x \geq 2^{-n+3} \) and \( y \leq 2^{-n+2} \). Then \( |F_2(t, x, y)| \lesssim x/t \). Using (5.17), (5.19), and (5.22),

\[
A_5 \lesssim \int_0^\infty \int_{2^{-n+3}}^{2^{-n+2}} x \int_{2^{-2n}}^{2^{-2n+2}} \exp \left( -\frac{x^2}{ct} \right) \frac{dt}{t^2} \, dx \lesssim \int_0^\infty x^{1-2N} \int_0^{2^{-n+2}} t^{N-2} \, dt \, dx \lesssim 1.
\]

For \( A_6 \), we have \( xy \lesssim t \) and \( x \gtrsim y \). Then \( |F_1(t, x, y)| \lesssim x^{-1}(xy/t)^{\beta - 1/2}(1+x^2/t) \). Using (5.16) and (5.21),

\[
A_6 \lesssim \int_0^\infty \int_{2^{-n+3}}^{2^{-n+3}} x^{-1} \int_{xy}^1 \left( \frac{xy}{t} \right)^{\beta} \exp \left( -\frac{x^2}{ct} \right) \left( 1 + \frac{x^2}{t} \right) \frac{dr}{t} \, dx
\]

\[
\lesssim y^{\beta} \int_0^\infty x^{\beta-1} \int_0^\infty t^{-\beta-1} \exp \left( -\frac{x^2}{ct} \right) \, dt \, dx
\]

\[
\lesssim y^{\beta} \int_0^\infty x^{-\beta-1} \, dx \cdot \int_0^\infty t^{-\beta-1} \exp \left( -\frac{1}{ct} \right) \, dt \lesssim 1.
\]

In the integrals \( A_7-A_8 \), we deal with \( t > 1 \), so that \( \text{sh}(2t) \simeq e^{2t} \) and \( \text{sh}(2t)/\text{ch}(2t) \simeq 1 \).

The term \( A_7 \) appears only when \( x \gtrsim 2^n \). Here \( xy \gtrsim \text{sh}(2t) \), \( x > y \), and \( |F_2(t, x, y)| \lesssim x \). Using (5.17), (5.20), and (5.22),

\[
A_7 \lesssim \int_0^\infty \int_{1/(\text{sh}(2t))^{1/2}}^\infty \frac{x}{\text{sh}(2t)^{1/2}} \exp \left( -cx^2 \right) \frac{dr}{\sqrt{t}} \, dx \lesssim 1.
\]
For $A_8$, we have $xy \lesssim \text{sh}(2t)$ and $|F_1(t, x, y)| \lesssim (xy/\text{sh}(2t))^{\beta-1/2}(x + x^{-1})$.
Using (5.16) and (5.21),

\[
A_8 \lesssim \int_0^\infty \int_1^\infty \frac{(xy)^\beta}{(\text{sh}(2t))^{\beta + 1/2}} \exp \left(-cx^2\right) \left(x + x^{-1}\right) \frac{dt}{\sqrt{t}} \, dx \\
\lesssim y^\beta \int_0^\infty x^\beta \left(x + x^{-1}\right) \exp \left(-cx^2\right) \, dx \cdot \int_1^\infty \frac{(\text{sh}(2t))^{\beta-1/2} \, dt}{\sqrt{t}} \lesssim 1,
\]

where we have used that $y \leq 2$ and $\beta > 0$.

**Proof of (A_5) in Case 1.** In (A_5), we deal with $x \simeq y \simeq 2^{-n}$ and $t \leq 2^{-2n}$, so $t \lesssim xy \lesssim 1$. Recall that $H_t(x - y)$ denotes the classical heat kernel on $\mathbb{R}$. Using (5.18),

\[
|\partial_x (T_t(x, y) - H_t(x - y))| \\
\leq \left|\partial_x H_{\frac{2t}{\text{sh}(2t)}}(x - y) - \partial_x H_t(x - y)\right| \Theta(t, x, y) U_{\beta+1/2} \left(\frac{xy}{\text{sh}(2t)}\right) \\
+ \left|\Theta(t, x, y) - 1 - \Theta(t, x, y) \left(1 - U_{\beta+1/2} \left(\frac{xy}{\text{sh}(2t)}\right)\right)\right| \cdot |\partial_x H_t(x, y)| \\
+ \frac{\Theta(t, x, y)}{(2\pi \text{sh}(2t))^{1/2}} \exp \left(-\frac{|x - y|^2}{2\text{sh}(2t)}\right) F_3(t, x, y)
\]

\[
= K_t^{[1]}(x, y) + K_t^{[2]}(x, y) + K_t^{[3]}(x, y).
\]

Recall that $t \leq 1$ and notice that $|\partial_r \partial_x H_t(x - y)| \lesssim t^{-3/2} \exp \left(-|x - y|^2/(8t)\right)$.
Using (5.14), (5.19), and the mean-value theorem we have

\[
K_t^{[1]}(x, y) \lesssim |\text{sh}(2t)/2 - t| \cdot t^{-3/2} \exp \left(-|x - y|^2/(ct)\right) \lesssim t^{3/2}.
\]

Therefore,

\[
\int_{I}^{2n} \int_0^t K_t^{[1]}(x, y) \frac{dt}{\sqrt{t}} \, dx \lesssim \int_{2^{-n-1}}^{2^{-n+2}} \, dx \cdot \int_0^t t \, dt \lesssim 1.
\]

Turning to $K_t^{[2]}$ notice that

\[
|1 - \Theta(t, x, y)| = \left|\exp(0) - \exp \left((1 - \text{ch}(2t))(x^2 + y^2)\right)/2\text{sh}(2t)\right| \lesssim ty^2.
\]

Using (5.14), (5.19) and (5.25) we get $K_t^{[2]}(x, y) \lesssim t(y^2 + (xy)^{-1})|\partial_x H_t(x - y)| \lesssim 2^{2n}$, hence

\[
\int_I^{2n} \int_0^t K_t^{[2]}(x, y) \frac{dt}{\sqrt{t}} \, dx \lesssim 2^{2n} \cdot \int_{2^{-n-1}}^{2^{-n+2}} \, dx \cdot \int_0^t \frac{dt}{\sqrt{t}} \lesssim 1.
\]
For $K^{[3]}_t$ by (5.23), we have $|F_3(t, x, y)| \lesssim y^{-1} \lesssim 2^n$. Using (5.19),

$$\int_{I^{**}} \int_0^{2^{-n}} |\partial_x T_t(x, y)| \frac{dt}{\sqrt{t}} \frac{dx}{\sqrt{t}} \lesssim 2^n \int_0^{2^{-n+2}} \int_0^{2^{-n}} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dt}{t} \frac{dx}{t}$$

$$\lesssim 2^n \int_{|x-y| \leq 2^{-n}} \frac{dx}{t} \frac{dt}{t} \frac{dx}{t} \lesssim 2^n \int_{|x-y| \leq 2^{-n}} \ln(2^n |x - y|^{-1}) \frac{dx}{t} \frac{dt}{t} \frac{dx}{t} \lesssim 1.$$

### 5.2.2 Case 2: $I \subseteq [1, \infty)$

Fix $y \in I^{**}$ and $n \in \mathbb{N}$ such that $I \subseteq [2^n, 2^{n+1}]$. We have $y \simeq 2^n = d_I^{-1}$.

**Proof of (A3) in Case 2.** Notice that we deal with $0 < t \leq 2^{-2n} \leq 1$, sh($t$) $\simeq t$ and ch($t$) $\simeq 1$. For $y \in I^{**}$ and $x \notin I^{***}$ we have $|x - y| \geq 2^{-n}$, so that

$$\int_{I^{**} \setminus I^{***}} \int_0^{2^{-2n}} |\partial_x T_t(x, y)| \frac{dt}{\sqrt{t}} \frac{dx}{\sqrt{t}} \leq \int_0^{2^{-2n}} \frac{dx}{t} \frac{dt}{t} \frac{dx}{t} \lesssim \int_{|x-y| \geq 2^{-2n}} \frac{dx}{t} \frac{dt}{t} \frac{dx}{t} = A_9 + A_{10} + A_{11}.$$

For $A_9$ we have $xy \lesssim t$ and $x \lesssim 2^{-3n}$, so that $|x - y| \simeq y$. Thus $|F_1(t, x, y)| \lesssim x^{-1}(xy/t)^{\beta-1/2}$. Using (5.16) and (5.21),

$$A_9 \lesssim \int_0^{c2^{-3n}} x^{-1} \int_0^{2^{-2n}} (\frac{xy}{t})^\beta \exp \left( -\frac{y^2}{ct} \right) \frac{dx}{t} \frac{dt}{t} \frac{dx}{t} \lesssim y^{-2N+\beta} \int_0^{c2^{-3n}} x^{\beta-1} \frac{dx}{t} \int_0^{2^{-2n}} t^{N-\beta-1} \frac{dt}{t} \lesssim 2^{-4Nn} \lesssim 1.$$

For $A_{10}$, we have $xy \gtrsim t$, $x \simeq y \simeq 2^n$, $x^{-1} \gtrsim xt$, so that $|F_3(t, x, y)| \lesssim y^{-1} \lesssim |x - y|/t$. Using (5.18), (5.23), and (5.19),

$$A_{10} \lesssim \int_{2^{-n} \lesssim |x-y| \leq 2^{-n-2}} \frac{|x - y|}{t} \int_0^{2^{-2n}} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dx}{t}$$

$$\lesssim \int_{2^{-n} \lesssim |x-y|} |x - y|^{1-2N} \frac{dx}{t} \int_0^{2^{-2n}} t^{N-2} \frac{dt}{t} \lesssim 1.$$
For $A_{11}$, we have $xy \gtrsim t$, $|x - y| \simeq x + y$, and $|F_2(t, x, y)| \lesssim (x + y)/t$. Using (5.17), (5.19), and (5.22),

$$A_{11} \lesssim \int_{|x - y| \geq 2^{n-2}} (x + y) \int_0^{2^{2n}} \exp \left( -\frac{(x + y)^2}{ct} \right) \frac{dt}{t^2} \, dx$$

$$\lesssim \int_0^\infty (x + y)^{1-2N} \, dx \cdot \int_0^{2^{2n}} t^{N-2} \, dt \lesssim 2^{4n(1-N)} \lesssim 1.$$

**Proof of (A4) in Case 2.** Write

$$\int_X |\partial_x T_t(x, y)| \frac{dt}{\sqrt{t}} \, dx = \int_0^{2^{-n}} \int_0^{2^{-n}x \sqrt{2^{-2n}}} \ldots + \int_0^{2^{-n}} \int_{x \sqrt{2^{-2n}}}^1 \ldots + \int_0^{2^{n+2}} \int_2^{2^{-n}} \ldots$$

$$+ \int_0^{(2^{-n}, 2^{n+2}) \cap \{|x - y| < 2^{-n}\}} \int_2^{2^{-n}} \ldots + \int_0^{(2^{-n}, 2^{n+2}) \cap \{|x - y| > 2^{-n}\}} \int_2^{2^{-n}} \ldots$$

$$+ \int_0^\infty \int_1^{1/\ln(\sqrt{x^2})} \ldots + \int_0^\infty \int_1^{1/\ln(\sqrt{x^2})} \ldots$$

$$= A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18}.$$

For $A_{12}$, we have $xy \gtrsim t$, $t \leq 1$ and $x < y$, so that $|F_2(t, x, y)| \lesssim y/t$. Using (5.17), (5.22), and (5.19),

$$A_{12} \lesssim y \int_0^{2^{-n}} \int_0^\infty \exp \left( -\frac{y^2}{ct} \right) \frac{dt}{t^2} \, dx$$

$$\lesssim y^{-1} \int_0^{2^{-n}} dx \cdot \int_0^\infty t^{-1} \exp \left( -\frac{1}{ct} \right) \frac{dt}{t} \lesssim 2^{-2n} \lesssim 1.$$

For $A_{13}$, we have $xy \lesssim t$, $t \leq 1$, and $x/y \lesssim x^{-1}$, so that $|F_1(t, x, y)| \lesssim x^{-1} (xy/t)^{\beta-1/2}$. Using (5.16) and (5.21),

$$A_{13} \lesssim y^\beta \cdot \int_0^{2^{-n}} x^{\beta-1} \int_0^\infty t^{-\beta} \exp \left( -\frac{y^2}{ct} \right) \frac{dt}{t} \, dx$$

$$\lesssim y^{-\beta} \cdot \int_0^{2^{-n}} x^{\beta-1} dx \cdot \int_0^\infty t^{-\beta} \exp \left( -\frac{1}{ct} \right) \frac{dt}{t}$$

$$\lesssim 2^{-2\beta n} \lesssim 1,$$

where in the last inequality we have used that $\beta > 0$.

For $A_{14}$, we have $xy \gtrsim t$, $|x - y| \simeq x$, and $x > y$, so that $|F_2(t, x, y)| \lesssim x/t$. Using (5.17), (5.19), and (5.22),

$$A_{14} \lesssim \int_0^{2^{n+2}} x \int_0^1 \exp \left( -\frac{x^2}{ct} \right) \frac{dt}{t^2} \, dx \lesssim \int_0^{2^{n+2}} x^{1-2N} \, dx \cdot \int_0^1 t^{N-2} \, dt \lesssim 1.$$
For $A_{15}$, we have that $xy \gtrsim t$, $x \simeq y \simeq 2^n$, and $|F_3(t, x, y)| \lesssim xt$. Using (5.18), (5.19), and (5.23),

\[
A_{15} \lesssim \int_{|x-y|<2^{-n}} \int_0^1 \exp\left(-cty^2\right)\left(\frac{|x-y|}{t} + xt\right) \frac{dt}{t} \, dx
\]

\[
\lesssim y^{-2N} \int_{|x-y|<2^{-n}} |x-y| \, dx \cdot \int_0^\infty t^{-N-2} \, dt
\]

\[
+ y^{-2N} \int_{|x-y|<2^{-n}} x \, dx \cdot \int_0^\infty t^{-N} \, dt \lesssim 1.
\]

For $A_{16}$, we have that $xy \gtrsim t$, $t \leq 1$, $x \lesssim y$, and $|F_3(t, x, y)| \lesssim x^{-1} + xt$. Using (5.18), (5.19), and (5.23),

\[
A_{16} \lesssim \int_{(2^{-n}, 2^{n+2}) \cap \{|x-y|>2^{-n}\}} \int_0^1 \exp\left(-cty^2\right)\left(\frac{|x-y|}{t} + x^{-1} + tx\right) \frac{dt}{t} \, dx
\]

\[= A_{16,1} + A_{16,2} + A_{16,3}, \]

where $A_{16,1}$, $A_{16,2}$, $A_{16,3}$ are the integrals with: $|x-y|t^{-1}$, $x^{-1}$, $xt$, respectively.

\[
A_{16,1} \lesssim y^{-2N} \int_{|x-y|>2^{-n}} |x-y| \int_0^\infty t^{-N-1} \exp\left(-\frac{|x-y|^2}{ct}\right) \frac{dt}{t} \, dx
\]

\[
\lesssim 2^{-2nN} \int_{|x-y|>2^{-n}} |x-y|^{-2N-1} \, dx \cdot \int_0^\infty t^{-N-2} \exp\left(-\frac{1}{ct}\right) \, dt \lesssim 1.
\]

Notice that $x^{-1} \leq 2^n$, thus

\[
A_{16,2} \lesssim y^{-2N} \int_{(2^{-n}, \infty) \cap \{|x-y|>2^{-n}\}} x^{-1} \int_0^\infty t^{-N} \exp\left(-\frac{|x-y|^2}{ct}\right) \frac{dt}{t} \, dx
\]

\[
\lesssim 2^{n(1-2N)} \int_{|x-y|>2^{-n}} |x-y|^{-2N} \, dx \cdot \int_0^\infty t^{-N} \exp\left(-\frac{1}{ct}\right) \frac{dt}{t} \lesssim 1.
\]

\[
A_{16,3} \lesssim \int_0^{2^{n+2}} x \, dx \cdot \int_0^\infty e^{-cty^2} \, dt \lesssim 2^{2n} \cdot 2^{2n} \lesssim 1.
\]

For $A_{17}$ we have that $xy \gtrsim \text{sh}(2t)$, $t \geq 1$, and $|F_2(t, x, y)| \lesssim x + y \lesssim y(x + 1)$. Using (5.17), (5.20), and (5.22),

\[
A_{17} \lesssim ye^{-cy^2} \int_0^\infty (x + 1)e^{-cx^2} \, dx \cdot \int_1^\infty (\text{sh}(2t))^{-1/2} \frac{dt}{\sqrt{t}} \lesssim 1.
\]
For $A_{18}$, we have that $xy \lesssim \text{sh}(2t)$, $t \geq 1$, and $|F_1(t, x, y)| \lesssim (xy/\text{sh}(2t))^{\beta-1/2}$. Using (5.16) and (5.21),

$$A_{18} \lesssim \int_0^\infty \int_1^\infty e^{-c(x^2+y^2)} \left( \frac{xy}{\text{sh}(2t)} \right)^\beta (x + x^{-1}) \frac{dt}{\sqrt{t} \cdot \text{sh}(2t)} dx$$

$$\lesssim y^\beta e^{-cy^2} \cdot \int_0^\infty x^\beta (x + x^{-1}) e^{-cx^2} dx \cdot \int_1^\infty (\text{sh}(2t))^{-\beta-1/2} \frac{dt}{\sqrt{t}} \lesssim 1.$$  

**Proof of (A_5) in Case 2.** In this case we have $x, y \simeq 2^n, |x - y| \lesssim 2^{-n} = d_I$. The proof follows by similar argument to those in Case 1. In particular, one uses (5.24) and estimate $K_I^{[1]} - K_I^{[3]}$ in a similar way. The details are left to the reader.

**Proof of (A_6)** Let us write

$$\int_X \left( x + x^{-1} \right) \int_0^\infty T_I(x, y) \frac{dt}{\sqrt{t}} dx = \int_0^\infty \int_0^{1 \wedge xy} \cdots + \int_0^\infty \int_1^{1 \wedge xy} \cdots + \int_0^\infty \int_1^{1 \wedge xy} \cdots$$

$$= A_{19} + A_{20} + A_{21} + A_{22}.$$  

Our goal is to prove $A_{19} + A_{20} + A_{21} + A_{22} \lesssim 1$. Observe that by using (5.15), (5.19), (5.14), for $x \leq y/2$, we have

$$\int_0^{1 \wedge xy} T_I(x, y) \frac{dt}{\sqrt{t}} \lesssim \int_0^{xy} \exp \left( -\frac{y^3}{ct} \right) \frac{dt}{t} \lesssim \int_0^\infty e^{-ct} \frac{dt}{t} \lesssim e^{-cy/x}. \quad (5.26)$$

Similarly, we get the estimates

$$\int_0^{1 \wedge xy} T_I(x, y) \frac{dt}{\sqrt{t}} \lesssim e^{-cy^2}, \quad 2x \leq y, \ y \geq 1, \quad (5.27)$$

$$\int_0^{1 \wedge xy} T_I(x, y) \frac{dt}{\sqrt{t}} \lesssim e^{-cx/y}, \quad 2x/3 \geq y, \quad (5.28)$$

$$\int_0^{1 \wedge xy} T_I(x, y) \frac{dt}{\sqrt{t}} \lesssim e^{-cx^2}, \quad 2x/3 \geq y \geq 1. \quad (5.29)$$

Moreover, by (5.15), (5.19), (5.14), for $|x - y| \leq y/2$, we have

$$\int_0^{1 \wedge xy} T_I(x, y) \frac{dt}{\sqrt{t}} \lesssim \int_0^{xy} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dt}{t} \quad \lesssim \int_{|x-y|^2/(xy)}^\infty e^{-ct} \frac{dt}{t} \lesssim \ln \left( \frac{y}{|x-y|} \right), \quad (5.30)$$

\[ Springer \]
and, for \(|x - y| \leq y/2\) and \(y \geq 1\),

\[
\int_0^{1/xy} T_t(x, y) \frac{dt}{\sqrt{t}} \lesssim \int_0^1 \exp \left( -\frac{|x - y|^2}{ct} \right) \Theta(t, x, y) \frac{dt}{t} \\
\lesssim \int_0^{y/2} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dt}{t} + \int_0^{1/xy} (ty)^{-1} \exp \left( -\frac{|x - y|^2}{ct} \right) \frac{dt}{t} \\
\lesssim \int_0^{\infty} e^{-ct} \frac{dt}{t} + |x - y|^{-2} \int_0^{y/2} y^2 e^{-ct} \frac{dt}{t} \\
\lesssim \frac{\ln(2 + (y|x - y|)^{-1})}{1 + y^2|x - y|^2}.
\tag{5.31}
\]

Consider first \(A_{19}\) in the case \(y \leq 1\). Using (5.26), (5.30), and (5.28),

\[
A_{19} \lesssim \int_0^{y/2} e^{-cy/x} \frac{dx}{x} + y^{-1} \int_{|x - y| \leq y/2} \ln \left( \frac{y}{|x - y|} \right) \frac{dx}{x} \\
+ \int_0^{\infty} x e^{-cx/y} dx + \int_0^{\infty} \frac{dx}{x} e^{-cx/y} \lesssim 1.
\]

Now consider \(A_{19}\) in the case \(y \geq 1\). Using (5.26), (5.27), (5.31), and (5.29)

\[
A_{19} \lesssim \int_0^{(2y)^{-1}} e^{-cy/x} \frac{dx}{x} + \int_0^{y/2} (x + x^{-1}) e^{-cy} dx \\
+ y \int_{|x - y| \leq y/2} \frac{\ln(2 + (y|x - y|)^{-1})}{1 + y^2|x - y|^2} \frac{dx}{x} + \int_0^{\infty} x e^{-cx^2} dx \\
\lesssim e^{-cy} + (y^2 + \ln y) e^{-cy} + \int_{-\infty}^{\infty} \frac{\ln(2 + |x|^{-1})}{1 + x^2} dx + \int_1^{\infty} x e^{-cx^2} dx \lesssim 1.
\]

Recall that \(\beta > 0\). For \(A_{20}\) we use (5.12) and (5.2) getting

\[
A_{20} \lesssim \int_0^{\infty} (x + x^{-1}) \int_0^1 \left( \frac{xy}{t} \right)^\beta \exp \left( -\frac{x^2 + y^2}{ct} \right) \frac{dt}{t} dx \\
\lesssim \int_0^{\infty} (x + x^{-1}) \left( \frac{xy}{x^2 + y^2} \right)^\beta \int_{x^2 + y^2}^{\infty} t^\beta \exp(-t/c) \frac{dt}{t} dx \\
\lesssim \int_0^{\infty} (x + x^{-1}) \left( \frac{xy}{x^2 + y^2} \right)^\beta \exp \left( -cx^2 \right) dx \\
\lesssim \int_0^{\infty} \left( \frac{xy}{x^2 + y^2} \right)^\beta \frac{dx}{x} \lesssim 1.
\]
For $A_{21}$, we have $xy \gtrsim \text{sh}(2t)$ and $x^{-1} \lesssim y$ (otherwise $A_{21} = 0$). Applying (5.15), (5.20), (5.14), we get

$$A_{21} \lesssim \int_0^\infty (x + y) \int_1^{1\sqrt{\ln(xy)}} \text{sh}(2t)^{-1/2} \Theta(t, x, y) \frac{dt}{\sqrt{t}} \, dx$$

$$\lesssim (y + 1)e^{-cy^2} \cdot \int_0^\infty (x + 1) \exp(-cx^2) \, dx \cdot \int_1^{1\sqrt{\ln(xy)}} \text{sh}(2t)^{-1/2} \frac{dt}{\sqrt{t}} \lesssim 1.$$

For $A_{22}$, we have $xy \lesssim \text{sh}(2t)$ and $\text{sh}(2t) \simeq \text{ch}(2t)$. Using (5.12) and (5.2),

$$A_{22} \lesssim \int_0^\infty (x + x^{-1}) \int_1^{1\sqrt{\ln(xy)}} \text{sh}(2t)^{-1/2} \left( \frac{xy}{\text{sh}(2t)} \right)^\beta \exp(-c(x^2 + y^2)) \frac{dt}{\sqrt{t}} \, dx$$

$$\lesssim y^\beta e^{-cy^2} \cdot \int_0^\infty (x + x^{-1}) x^\beta e^{-cx^2} \, dx \cdot \int_1^{1\sqrt{\ln(xy)}} \text{sh}(2t)^{-\beta - 1/2} \frac{dt}{\sqrt{t}} \lesssim 1.$$

We have shown that $A_{19} + A_{20} + A_{21} + A_{22} \lesssim 1$. This finishes the proofs of (A6) and Proposition 5.11.

### 5.3 Dirichlet Laplacian on $\mathbb{R}^d_+$

The goal of this section is to prove Theorem E. Recall that we consider $X = \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, \infty)$ and the Laplacian $L_D = -\Delta$ with the Dirichlet boundary condition at $x_d = 0$. The semigroup generated by $-L_D$ is associated with the kernel

$$T_t(x, y) = H_t(\tilde{x}, \tilde{y})T_{t,D}(x_d, y_d),$$

where $\tilde{x} = (x_1, \ldots, x_{d-1})$, $H_t$ is as in (1.1), and

$$T_{t,D}(x, y) = H_t(x - y) - H_t(x + y), \quad x, y, t > 0. \quad (5.32)$$

Observe that $L_D$ is of the form as in Sect. 4. Therefore, due to Theorem 4.1, it is enough to prove Theorem E in the case $d = 1$.

Assume then that $d = 1$, $X = (0, \infty)$, and $L_D$ that is related to the kernel (5.32). It is sufficient to check that for $T_{t,D}$ and the admissible covering

$$Q_D = \left\{ [2^n, 2^{n+1}] : n \in \mathbb{Z} \right\} \quad (5.33)$$

of $(0, \infty)$ the conditions $(A_0)$–$(A_5)$ are satisfied (obviously, in this case $(A_6)$ is automatically satisfied). This is stated in the following lemma.

**Lemma 5.34** Let $T_{t,D}(x, y)$ and $Q_D$ be as in (5.32) and (5.33), respectively. Then $(A_0)$–$(A_5)$ are satisfied.
Sketch of the proof  Notice that

\[ T_{t,D}(x, y) = H_t(x, y)\Omega(t, x, y), \quad x, y, t > 0, \]

where

\[ \Omega(t, x, y) = 1 - \exp(-xy/t) \simeq \min(1, xy/t). \]

Moreover,

\[ \partial_x T_{t,D}(x, y) = H_t(x, y) \left( \frac{x - y}{t} + \frac{y}{t} \exp\left(-\frac{xy}{t}\right) \right). \quad (5.35) \]

The conditions \((A_0)\)–\((A_5)\) can be proved by using standard estimates. For the convenience of the reader, we shall present the proof of \((A_4)\). Let \(I \in \mathcal{D}\) and assume that \(I = [2^n, 2^{n+1}]\) for some \(n \in \mathbb{Z}\). Fix \(y \in I^{**}\). From (5.35) we have

\[
\int_X \int_{d_i^2}^\infty \left| \partial_x T_{t,D}(x, y) \right| \frac{dt}{\sqrt{t}} dx = \int_0^{2^{n+2}} \int_{d_i^2}^\infty \left( \int_0^{2^{n+2}} \int_{d_i^2}^\infty \cdots \right) dt \cdot \int_{d_i^2}^\infty \cdots
\]

\[ = A_1 + A_2. \]

For \(A_1\) we have that \(xy \lesssim t\) and \(x \lesssim \sqrt{t}\). From (5.35) we have \(\left| \partial_x T_{t,D}(x, y) \right| \lesssim yt^{-3/2}(x/\sqrt{t} + 1) \lesssim yt^{-3/2}\) and

\[ A_1 \lesssim \int_0^{2^{n+2}} \int_{2^n}^\infty yt^{-3/2} \frac{dt}{\sqrt{t}} dx \lesssim 2^n \int_0^{2^{n+2}} dx \cdot \int_{2^n}^\infty t^{-2} dt \lesssim 1. \]

For \(A_2\) we have that \(|x| \simeq |x - y|\) and \(y \lesssim \sqrt{t}\). From (5.35),

\[ \left| \partial_x T_{t,D}(x, y) \right| \lesssim t^{-1} \exp\left(-\frac{|x - y|^2}{ct}\right) \left( \min(1, xy/t) + y/\sqrt{t} \right) \]

\[ \lesssim t^{-1} \exp\left(-\frac{x^2}{ct}\right) \cdot \left\{ \begin{array}{ll}
\frac{y}{\sqrt{t}} & \text{for } \frac{xy}{t} \lesssim 1 \\
1 & \text{for } \frac{xy}{t} \gtrsim 1.
\end{array} \right. \]

Therefore,

\[ \int_{d_i^2}^\infty \left| \partial_x T_{t,D}(x, y) \right| \frac{dt}{\sqrt{t}} \lesssim \int_{d_i^2}^{xy} \cdots + \int_{xy}^\infty \cdots \]

\[ \lesssim \int_0^{xy} \exp\left(-\frac{x^2}{ct}\right) \frac{dt}{t\sqrt{t}} + y \int_0^\infty \exp\left(-\frac{x^2}{ct}\right) \frac{dt}{t^2} \]

\[ \lesssim \int_0^{xy} \left( \frac{t}{x^2} \right)^{3/2} \frac{dt}{t\sqrt{t}} + yx^{-2} \lesssim yx^{-2}. \]
Hence,

\[ A_2 \lesssim 2^n \int_{2^n+\mathbb{R}^{n+2}} x^{-2} \, dx \lesssim 1. \]

This finishes the proof of \((A_4)\). \(\square\)

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