Interference Alignment with Partial CSI Feedback in MIMO Cellular Networks

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Abstract

Interference alignment (IA) is a linear precoding strategy that can achieve optimal capacity scaling at high SNR in interference networks. However, most existing IA designs require full channel state information (CSI) at the transmitters, which would lead to significant CSI signaling overhead. There are two techniques, namely CSI quantization and CSI feedback filtering, to reduce the CSI feedback overhead. In this paper, we consider IA processing with CSI feedback filtering in MIMO cellular networks. We introduce a novel metric, namely the feedback dimension, to quantify the first order CSI feedback cost associated with the CSI feedback filtering. The CSI feedback filtering poses several important challenges in IA processing. First, there is a hidden partial CSI knowledge constraint in IA precoder design which cannot be handled using conventional IA design methodology. Furthermore, existing results on the feasibility conditions of IA cannot be applied due to the partial CSI knowledge. Finally, it is very challenging to find out how much CSI feedback is actually needed to support IA processing. We shall address the above challenges and propose a new IA feasibility condition under partial CSIT knowledge in MIMO cellular networks. Based on this, we consider the CSI feedback profile design subject to the degrees of freedom requirements, and we derive closed-form trade-off results between the CSI feedback cost and IA performance in MIMO cellular networks.

Index Terms

MIMO cellular networks, interference alignment (IA), partial CSI feedback, CSI feedback dimension, IA feasibility condition.

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I. INTRODUCTION

It is well known that inter-cell interference is one of the most important performance bottlenecks in wireless networks. There are many works on interference mitigation techniques and conventional approaches either treat interference as noise or rely on *interference avoidance* by means of channel orthogonalization [1]. However, these schemes are far from optimal [2]. Recently, interference alignment (IA) was proposed as an effective means to mitigate interference in $K$-user interference channels [3], [4]. By aligning the interference from different transmitters (Txs) into a lower dimensional subspace at each receiver (Rx), IA can achieve the optimal capacity scaling with respect to (w.r.t.) SNR. As such, there is a surge in the research interest of IA and it has been extended to other topologies such as MIMO cellular networks in [5], [6].

Despite the fact the IA can achieve substantial throughput gain, conventional IA designs [3]–[6] require full channel state information at the Tx side (CSIT). Such full CSIT requirement is quite difficult to achieve in practice due to limited CSI feedback capacity in the reverse link in practice. As such, naive IA design will be very sensitive to CSIT errors [7], [8] and it is important to take into account the CSI feedback constraint in the IA design. There are, in general, two ways to reduce the CSI feedback overhead, namely *CSI quantization* and *CSI filtering*. In [7], [8], the authors considered using Grassmannian codebooks to quantize and feedback the channel direction information (CDI) for IA processing. In [9], [10], some adaptive quantization schemes are proposed to exploit the channel statistics so as to enhance the limited CSI feedback efficiency. However, these schemes considered CSI quantization of the full CDI in the interference networks only.

In fact, full CDI may not always be needed to achieve IA processing at the Txs. We illustrate two examples in which substantially reduced CSI is fed back to achieve IA processing. Furthermore, the *CSI quantization* and the *CSI filtering* techniques are complementary to each other and in some situations, the CSI filtering will be a first order contributor towards enhancing the CSI feedback efficiency in MIMO cellular networks. The CSI filtering techniques to reduce feedback overhead are relatively less explored. In [11], a CSI filtering scheme by CSI truncation is proposed to reduce the CSI feedback in MIMO interference network. In [12], a CSI filtering scheme with zero-forcing IA is proposed to eliminate the intercell CSI feedback in MIMO cellular networks. However, a more systematic understanding is still needed to determine how much CSI feedback is required for IA processing. In this paper, we propose a systematic framework of CSI filtering and analyze the associated tradeoff between CSI feedback cost and IA degrees of freedom (DoF) performance in MIMO cellular networks. There are several unique
challenges that need to be tackled.

- **How to quantify the CSI Feedback Cost?** It may be natural to measure the CSI feedback cost in MIMO cellular networks in terms of the total number of the feedback bits. However, this metric mixes the *CSI filtering* and *CSI quantization* together. To obtain some key design insights, it is desirable to have a metric that can solely focus on the CSI filtering aspect because the CSI quantization is complementary and can always be considered on top of the CSI filtering as in Figure 1.

- **IA Feasibility Conditions under Partial CSI Feedback:** It is well known that the IA scheme is not always feasible and the feasibility conditions are topology specific. The IA feasibility condition is studied for MIMO interference channels in [13]–[16], and for MIMO cellular networks in [17]. However, these works have assumed full CSIT and hence the precoders can be designed as a function of the full CSI. While in MIMO cellular networks with CSI feedback filtering, the precoders can only be designed based on the *partial CSI knowledge* from CSI feedback filtering and hence the IA feasibility conditions are different.

- **CSI Feedback Design:** Further, it remains a question what is the CSI filtering scheme with the least amount of CSI feedback overhead to support the required IA DoFs for a given antenna configuration. Such a question involves minimization of the CSI feedback cost subject to IA feasibility constraint. However, this problem is highly non-trivial because of the combinatorial nature of CSI filtering scheme design.

In this paper, we will address the challenges above as follows. We first define a novel CSI feedback cost metric, namely the *CSI feedback dimension*. The CSI feedback dimension enables us to isolate the CSI quantization effects from the CSI filtering design so as to obtain tractable and first order design insights. Based on the proposed metric, we propose the idea of IA processing under partial CSI feedback in MIMO cellular networks. After that, we investigate the feasibility conditions and derive the associated precoder / decorrelator solutions for IA under a given partial CSI feedback scheme. Based on these results, we attempt to find out the least amount of CSI feedback overhead by formulating the problem of minimizing CSI feedback dimension subject to IA constraints with a given IA DoFs in the network for a given antenna configuration. Using specific insights from the problem, we derive a low complexity asymptotically optimal solution and obtain closed-form tradeoff results between the number of DoFs and the CSI feedback dimension. Finally, we compare the proposed IA design with various state-of-the-art

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1For instance, in conventional IA formulation [13]–[16], the IA precoders / decorrelators \{V_i, U_i : \forall i\} are found to be a function of the *entire* CSIs \{H_{ji} : \forall j, i\} such that rank \( U_j^\dagger H_{ji} V_i \) = d, \( U_j^\dagger H_{ji} V_i = 0, \forall j, i \neq j. \)
baselines and illustrate that the proposed solution achieves significant CSI feedback cost reduction in MIMO cellular networks.

Notation: Uppercase and lowercase boldface letters denote matrices and vectors respectively. The operators (·)\(^T\), (·)\(^\dagger\), rank(·), |·|, tr(·), dim(·), \(\otimes\), \(\lfloor\cdot\rfloor\), \(\lceil\cdot\rceil\), \(||\cdot||\) and vec(·) are the transpose, conjugate transpose, rank, cardinality, trace, dimension of subspace, Kronecker product, integer floor, integer ceiling, Frobenius norm and vectorization respectively; \(I_d\), \(\mathbb{Z}\) and \(\mathbb{U}(A, B) = \{U \in \mathbb{C}^{A \times B} : U^\dagger U = I\}\) denote the identity matrix, the set of non-negative integers, and the set of \(A \times B\) semi-unitary matrices respectively; \(P(A) = \{aA : a \in \mathbb{C}\}\) and span\(\{A_i\}\) denotes the vector space spanned by all the column vectors of the matrices in \(\{A_i\}\), and \(d \mid M\) denotes that integer \(M\) is divisible by integer \(d\).

II. System Model

A. MIMO Cellular Networks

Consider a MIMO cellular network with \(G\) base stations (BSs) and each BS serves \(K\) mobile stations (MSs) as illustrated in Figure 2. Consider that each BS and MS are equipped with \(N\) and \(M\) antennas respectively, and \(d\) data streams are transmitted to each MS from its serving BS. We focus on the case when \(M \leq (G - 1)Kd + d\) because otherwise, i.e., \(M > (G - 1)Kd + d\), the number of antennas at the MS is over-sufficient to cancel all the inter-cell interference using pure zero forcing at the MS.

Denote the transmit SNR at each BS as \(P\), the \(k\)-th MS of BS \(j\) as the \((j, k)\)-th MS, the channel matrix from the \(i\)-th BS to the \((j, k)\)-th MS as \(H_{jk,i} \in \mathbb{C}^{M \times N}\). The received signal at the \((j, k)\)-th MS is given by:

\[
y_{jk} = U_{jk}^\dagger \left( H_{jk,j} V_{jk} x_{jk} + \sum_{p=1, p \neq j}^{K} H_{jk,j} V_{jp} x_{jp} + \sum_{i=1}^{G} \sum_{p=1}^{K} H_{jk,i} V_{ip} x_{ip} + n_{jk} \right), \forall j, k
\]

where \(x_{jk} \sim \mathcal{CN}(0, \frac{P}{Kd} I_d)\) is the encoded information symbol sent from the \(j\)-th BS to the \((j, k)\)-th MS, \(V_{jk} \in \mathbb{C}^{N \times d}\) and \(U_{jk} \in \mathbb{C}^{M \times d}\) are the corresponding precoder and decorrelator matrix for the \((j, k)\)-th MS, \(n_{jk} \sim \mathcal{CN}(0, I_M)\) is the white Gaussian noise.

Assumption 1 (Channel Matrices): Assume the elements of \(H_{jk,i}\) are i.i.d. complex Gaussian random variables with zero mean and unit variance. The CSIs are observable at the MSs and the CSI feedback from the \((j, k)\)-th MS will be received error-free by BS \(j\). Furthermore, we assume the BSs \(\{1, \cdots, G\}\) have backhaul connections such that the feedback CSI can be shared among them.
B. CSI Feedback Filtering and Feedback Cost

The CSI feedback reduction in MIMO cellular networks contains two processes in general, namely the CSI filtering and the CSI quantization as illustrated in Figure 1. To simplify the analysis, we shall consider these two factors separately. We consider CSI filtering only in Sections II-IV (no quantization is performed) and then analyze the effects of CSI quantization (block (b)) in Section V. Since IA processing aims at nulling off interferences at the MS, only the CDI \( P(H) = \{aH : a \in \mathbb{C}\} \), which is contained in \( G(1, MN) \) [18]. For example, consider two CSI

\[ P(H) = \{aH : a \in \mathbb{C}\}, \forall a \in \mathbb{C}. \]

\[ U^\dagger HV = 0, \text{ then we have } U^\dagger(aH)V = 0, \forall a \in \mathbb{C}. \]

For example, in IA designs, if \( U^\dagger HV = 0 \), then we have \( U^\dagger(aH)V = 0, \forall a \in \mathbb{C} \). Hence, it is sufficient to feeding back the CDI for IA, i.e., \( P(H) = \{aH : a \in \mathbb{C}\} \), which is contained in \( G(1, MN) \) [18].

\[ P(H) = \{aH : a \in \mathbb{C}\}, \forall a \in \mathbb{C}. \]
matrices $H_1, H_2 \in \mathbb{C}^{2 \times 3}$. If we feedback $P(H_1) = \{aH_1 : a \in \mathbb{C}\}, P(H_2) = \{aH_2 : a \in \mathbb{C}\}$, then this corresponds to the feedback filtering function $F = \left( P(H_1), P(H_2) \right) \in \mathbb{G}(1, 6) \times \mathbb{G}(1, 6)$. Note that under given feedback filtering functions $\{F_{j,k}\}$, the partial CSI $\{F_{j,k}(H_{j,k})\}$ will be fed back to the BSs for the IA precoder designs $\{V_{j,k} : \forall j, k\}$. To highlight the role of feedback cost reduction due to CSI filtering at the MS, we define the notion of feedback dimension below.

**Definition 2 (CSI Feedback Dimension):** Define the feedback dimension $D$ as the sum of the dimension of the Grassmannian manifolds \[ \sum_{i=1}^{G} \sum_{j=1}^{K} \sum_{k=1}^{l_{jk}} A_{j,k}(B_{j,k}^i - A_{j,k}^i), \] where $A_{j,k}(B_{j,k}^i - A_{j,k}^i)$ represents the difference between the Grassmannian manifolds $\{G(A_{j,k}^i, B_{j,k}^i) : i = 1, \cdots, l_{jk}; j = 1, \cdots, G; k = 1, \cdots, K\}$.

**Remark 1 (Interpretation of CSI Feedback Dimension):** The feedback dimension in Def. 2 is a first order measure of CSI feedback cost in MIMO cellular networks because it isolates the contribution of CSI feedback reduction due to CSI feedback filtering from CSI quantization. First, a Grassmannian manifold of dimension $D$ is locally homeomorphic to $\mathbb{C}^{D \times 1}$. Intuitively, this means that a Grassmannian manifold of dimension $D$ locally looks like the $D$-dimensional Euclidean space and a feedback dimension $D$ means that $D$ scalars are required to feedback to the BS side. Second, the feedback dimension is also directly proportional to the total number of bits allocated for CSI feedback in MIMO cellular networks. As in Theorem 5 in Section V, we demonstrate that with a total number of CSI feedback bits $D \log \text{SNR}$, it is sufficient to support certain DoF in MIMO cellular networks.

C. Interference Alignment under Partial CSI Feedback

One commonly adopted IA formulation in MIMO cellular networks is to find out the precoder and decorrelator solutions $\{U_{j,k}, V_{j,k}\}$ based on the full CSIT knowledge, such that the following set of conditions can be satisfied:

\begin{align*}
\text{rank}(U_{j,k}^\dagger H_{j,k,j} V_{j,k}) &= d, \forall j, k; \\
U_{j,k}^\dagger H_{j,k,j} V_{j,p} &= 0, \forall j, k \neq p; \quad \text{(intracell interference nulling)} \\
U_{j,k}^\dagger H_{j,k,i} \begin{bmatrix} V_{i1} & \cdots & V_{iK} \end{bmatrix} &= 0, \forall j, k, i \neq j. \quad \text{(intercell interference nulling)}
\end{align*}

\[ \text{The locally homeomorphic relationship between a Grassmannian } \mathbb{G} \text{ with dimension } D \text{ and } \mathbb{C}^{D \times 1} \text{ means: there exist a mapping } f : \mathbb{G} \rightarrow \mathbb{C}^{D \times 1}, \text{ such that for any point } x \in \mathbb{G}, \text{ there exists an open set } U \subseteq \mathbb{G} \text{ containing } x \text{ and the image } f(U) \text{ is open in } \mathbb{C}^{D \times 1}. \]
However, in the above formulation of IA constraints (3)-(5), the precoders \( \{V_{jk} : \forall j, k\} \) serve to null both the intracell interference in (4) and intercell interference in (5). As such, this formulation makes it hard to find out the CSI dependencies of the precoders \( \{V_{jk} : \forall j, k\} \). Consequently, it is difficult to know which part of CSI can be filtered out while still achieving the IA (3)-(5). To simplify the interference nulling structure, we consider using a two-stage precoding structure for the precoders \( \{V_{jk} \} \).

**Definition 3 (Two Stage Precoding at the BS):** Two stage precoding is applied at each of the BSs \( \{1, \cdots, G\} \), i.e., the precoder \( V_{jk} \) is given by \( V_{jk} = T_j V_{jk}^s \), where the semi-unitary matrix \( T_j \in U(N, Kd) \), \( N \geq Kd \), is the outer precoder for intercell interference nulling and \( V_{jk}^s \in U(Kd, d) \) is the inner precoder for intracell interference nulling between the MSs.

With two stage precoding, the IA constraints (3)-(5) can be reformulated as: Find out the outer precoders \( \{T_i \in U(N, Kd) : \forall i\} \), inner precoders \( \{V_{jk}^s \in C^{Kd \times d} : \forall j, k\} \) and decorrelators \( \{U_{jk} : \forall j, k\} \) based on the full CSIT knowledge such that:

\[
\begin{align*}
\text{rank} \left( U_{jk}^\dagger H_{jk,j} T_j V_{jk}^s \right) &= d, \forall j, k; \\
U_{jk}^\dagger H_{jk,j} T_j V_{jp}^s &= 0, \forall j, k \neq p; \quad \text{(intracell interference nulling)} \\
U_{jk}^\dagger H_{jk,i} T_i &= 0, \forall j, k, i \neq j. \quad \text{(intercell interference nulling)}
\end{align*}
\]

As can be seen above, the outer precoders \( \{T_i\} \) serve to null the intercell interference only (as in (8)), and based on the outer precoders \( \{T_i\} \), the inner precoders \( \{V_{ip}^s\} \) serves to null the intracell interference only (as in (7)). This decoupled interference nulling structure enables us to find how the precoders adapt to the CSI and may guide us to design efficient CSI feedback reduction schemes. Note that the two formulations of IA constraints, i.e., (3)-(5) and (6)-(8), are in fact equivalent.

**Lemma 1 (Equivalent IA Formulation):** With full CSIT, there exist \( \{U_{jk}, V_{jk}\} \) satisfying constraints (3)-(5) iff there exist \( \{T_i\}, \{V_{jk}^s\}, \{U_{jk}\} \) satisfying (6)-(8).

Based on the new IA constraints (6)-(8), we then investigate how the CSI can be filtered to reduce the CSI feedback dimension. In the literature, there are some CSI feedback designs [7]–[9] that feedback the full CDI, i.e., \( F_{jk} = \left( \cdots, \text{P}(H_{jk,i}), \cdots \right)_{\forall i}, \forall j, k \), which correspond to a CSI feedback dimension of \( G^2 K(MN - 1) \). By using the two stage precoding structure, we show in Example 1 and 2 below that the IA constraints (6)-(8) can still be achieved with substantially reduced feedback cost.

**Example 1 (Two Stage Precoding with Fixed Outer Precoders):** Consider a MIMO cellular network as illustrated in Figure 2. Suppose BS 1, 2 use fixed outer precoder \( T_1, T_2 \in U(3, 2) \). The intercell interference space at the (2,1)-th MS is given by \( \text{span}(H_{21,1} T_1) \). This can be cancelled by choosing a
Intercell interference
Intracell interference
fixed
fixed

Figure 2. Toy Example 1: Two-stage precoding with fixed outer precoder at the BSs can help to reduce the CSI feedback dimension for IA. BS 1 has fixed outer precoder $T_1 \in U(3, 2)$ and the $(2, 1)$-th, $(2, 2)$-th MSs can cancel the intercell interference by designing the decorrelator $U_{21} = R_{21}, U_{22} = R_{22} \in U(3, 1)$ to be orthogonal to span$(H_{21,1}T_1)$, span$(H_{22,1}T_1)$ respectively (similarly for BS 2).

decorrelator at the $(2, 1)$-th MS as: $U_{21} = R_{21} \in U(3, 1)$, where $R_{21}$ is orthogonal to the intercell interference, i.e., $(R_{21})^\dagger H_{21,1}T_1 = 0$. The remaining freedom at BS 2 are the inner precoders $\{V_{s1}, V_{s2}\}$ which are designed to cancel the intracell interference, i.e. $(R_{22})^\dagger H_{22,1}T_2 V_{s1} = 0, (R_{21})^\dagger H_{21,2}T_2 V_{s2} = 0$. As such, the BS 2 only needs to know $F_{21} = P((R_{21})^\dagger H_{21,2}T_2), F_{22} = P((R_{22})^\dagger H_{22,2}T_2)$ to compute the inner precoders (similarly for BS 1). Hence, using a feedback function $F_{1k} = P((R_{1k})^\dagger H_{1k,1}T_1), F_{2k} = P((R_{2k})^\dagger H_{2k,2}T_2), \forall k = 1, 2$, the IA conditions in (6)-(8) can be achieved by using the first 2 antennas at the MSs only with conventional IA design [17]. As a result, the feedback dimension is only $6 \times 2 \times (2 \times 5 - 1) = 108$ compared with $6 \times 2 \times (3 \times 5 - 1) = 168$ under full CDI feedback.

Example 2 (CSI Submatrix Feedback): Consider a MIMO cellular network with $G = 2$ BSs and $K = 3$ MSs for each BS. The BS and MS have $N = 5, M = 3$ antennas respectively and $d = 1$ data stream is transmitted for each MS. Suppose the CSI filtering functions at the MSs are given by: $F_{jk} = \left(P\left(H_{jk,1}\right), P\left(H_{jk,2}\right)\right), \forall j, k$, where $H_{jk,i}^s$ is the $2 \times 5$ upper submatrix of $H_{jk,i} \in \mathbb{C}^{3 \times 5}$, i.e., $H_{jk,i}^s = \begin{bmatrix} I_2 & 0_{2 \times 1} \end{bmatrix} H_{jk,i} \forall j, k, i$. Based on this CSI feedback $\{F_{jk}\}$, the IA conditions (6)-(8) can be achieved by using the first 2 antennas at the MSs only with conventional IA design [17]. As a result, the feedback dimension is only $6 \times 2 \times (2 \times 5 - 1) = 108$ compared with $6 \times 2 \times (3 \times 5 - 1) = 168$ under full CDI feedback.

Note that the strategy described in Example 1 is first mentioned in [12] and it can be generalized to MIMO cellular networks with a subset of BSs to have fixed outer precoders.

\[\text{Note that a } G = 2, K = 3, N = 5, M = 2, d = 1 \text{ MIMO cellular network with full CSIT is IA feasible [17].}\]
Remark 2: Examples 1 and 2 are only simple toy examples to illustrate two effective CSI feedback filtering policy (two-stage precoding with fixed outer precoders and CSI submatrix feedback respectively) to reduce the CSI feedback dimension. While these are trivial in these simple toy examples, the challenge is to have a CSI feedback filtering solution that embrace both strategies to minimize the CSI feedback dimension for general topology under DoF and IA feasibility constraints.

In the following, we shall formally give the structural form for the CSI filtering function $F_{jk}$ that embraces the above two policies. We first partition the BSs into two sets and define CSI submatrix feedback as follows.

Definition 4 (Partitioning of BSs): The group of BSs $\{1, \ldots, G\}$ are partitioned into two subsets, namely the type-I BSs, $\mathbb{B}_g^I = \{1, \ldots, g\}$ and the type-II BSs, $\mathbb{B}_g^I = \{g + 1, \ldots, G\}$. The type-II BSs use fixed outer precoder $T_{II}^{jg} \in \mathbb{U}(N, Kd), i \in \mathbb{B}_g^I$.

Definition 5 (CSI Submatrix Feedback): The CSI submatrices $\{H_{jk,i}^s\}$ are considered for CSI filtering feedback, where $\{H_{jk,i}^s\}$ correspond to the CSI on the first $m_{jk} \leq M$ antennas at the $(j, k)$-th MS, $\forall j, k$, the first $n_i \leq N$ antennas at the $i$-th BS, $i \in \mathbb{B}_g^I$, and degenerated $n_i = N$ antennas at the $i$-th BS, $i \in \mathbb{B}_g^I$. That is:

$$H_{jk,i}^s = \begin{cases} I_{m_{jk}} & H_{jk,i} \ I_{n_i} & \end{cases}^T, \forall j, k, i \in \mathbb{B}_g^I \ (9)$$

Note that $\{m_{jk} : \forall j, k\}, \{n_i : \forall i \in \mathbb{B}_g^I\}$ characterizes the size of the CSI submatrices $\{H_{jk,i}^s\}$. Denote $\mathbb{N}^r(\cdot)$ as the left null space, i.e., $\mathbb{N}^r(A) = \{u \mid u^\dagger A = 0\}$. Based on the above two definitions, we have the following definition on the CSI filtering functions $\{F_{jk}\}$.

Definition 6 (Structural Form of $F_{jk}$): The CSI filtering functions $F_{jk}(H_{jk})$ in (1) are given by

$$F_{jk}(H_{jk}) = (\cdots, \mathbb{P}(H_{jk,i}^c), \cdots)_{i \in \mathbb{B}_g^I \cup \{j\}}, \forall j, k; \ (10)$$

$$H_{jk,i}^c = \begin{cases} (R_{jk})^\dagger H_{jk,i}^c \in \mathbb{C}^{A_{jk} \times n_i}, & \forall j, k, i \in \mathbb{B}_g^I \\ (R_{jk})^\dagger H_{jk,i}^c T_{II}^{jg} \in \mathbb{C}^{A_{jk} \times Kd}, & \forall k, j \in \mathbb{B}_g^I \end{cases} \ (11)$$

Note that when $m_{jk} \neq M$ or $n_i \neq N$, $\forall j, k \in \mathbb{L}$, $\mathbb{L}$ is instead of directly selecting the upper left $m_{jk} \times n_i$ submatrix as in (9), there is in fact extra space of carefully selecting the $m_{jk}$ or $n_i$ effective antennas to improve the direct link power gain. However, in this paper, we are more interested in the tradeoff between the first-order DoF performance and the CSI feedback cost, and note the possible power gain mentioned will not affect the performance in the DoF sense [3]. Therefore, to better illustrate the insights, we consider a simple effective antenna reduction scheme as in [9] and focus on the feedback dimension reduction of $\mathbb{L}$. 

March 18, 2014 DRAFT
where $H_{jk,i}^e$ denotes the effective CSI, $R_{jk} \in \mathbb{U}(m_{jk}, A_{jk})$ is a semi-unitary matrix that defines the left null space of the intercell interference from all type-II BSs at the $(j,k)$-th MS:

$$
\text{span}(R_{jk}) = \mathbb{N}^r \left( \left[ \cdots H_{jk,i}^s T_i^I \cdots \right]_{i \in \mathbb{B}_g^I \setminus \{j\}} \right), \forall j, k; \quad (12)
$$

$$
A_{jk} = m_{jk} - \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} Kd, \; \forall j, k. \quad (13)
$$

Note that there is no need to feedback the intercell cross link CSIs $\{H_{jk,i}^s : \forall j, k, i \in \mathbb{B}_g^I \setminus \{j\}\}$ because the intercell interference from type-II BSs can be canceled by setting the decorrelator $U_{jk}$ to be in the subspace spanned by $R_{jk}$. The above feedback structure in Def. 6 corresponds to the tuple $H_{jk}^{\text{fed}} = F_{jk}(H_{jk}) \in \mathbb{G}(1, B_{jk,1}) \times \cdots \times \mathbb{G}(1, B_{jk,l_j})$, where the length $l_j = |\mathbb{B}_g^I \cup \{j\}|$ and

$$
B_{jk,i} = \begin{cases} 
KdA_{jk}, & i = l_j, j \in \mathbb{B}_g^I \\
n_iA_{jk}, & j \in \mathbb{B}_g^I \text{ or } i < l_j 
\end{cases} \quad (14)
$$

Note that $m_{jk}$ and $n_i$ in $L$ control the size of the CSI submatrices to feedback and $g = |\mathbb{B}_g^I|$ is the number of the type-I BS. In fact, there is a one-to-one correspondence between the feedback profile $L$ and the feedback function in (10). Note that the proposed feedback profile $L$ embraces Example 1, 2 with the corresponding $L = \{\{m_{jk} = 3, \forall j, k, g = 0\}\}$ in Example 1 and $L = \{\{m_{jk} = 2 : \forall j, k, g = 2, \{n_i = 5 : i \leq 2\}\}$ in Example 2. Furthermore, it also includes some existing works as special cases:

- **Special Case I (Full CDI Feedback):** When $L = \{\{m_{jk} = M, \forall j, k, g = G, \{n_i = N : \forall i\}\}$, $L$ will be reduced to conventional full CDI feedback in [7]–[9].
- **Special Case II (Zero-forcing IA Feedback):** When $G = 2, M = N = K + 1, d = 1$ and $L = \{\{m_{jk} = M, \forall j, k, g = 0\} \text{ (all BSs are type-II BSs)}, L$ will be reduced to the feedback scheme in [12] (Example 1 corresponds to one such example).

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*We define $R_{jk} = I$ when $\mathbb{B}_g^I \setminus \{j\} = \emptyset$. 

March 18, 2014 DRAFT
For a given feedback profile $L$, the total feedback dimension is given by,

$$D(L) = \sum_{j=1}^{G} \sum_{k=1}^{K} \sum_{i=1}^{g} (n_i A_{jk} - 1) + \sum_{j=g+1}^{G} \sum_{k=1}^{K} (KdA_{jk} - 1).$$  \hfill (16)

Next, we discuss IA constraints under the proposed CSI filtering $L$, to achieve $d$ data streams for each MS in the following.

**Constraints 1 (IA under $L$):** Given the CSI feedback profile $L$ and the outer precoders $\{T^H_i \in \mathbb{U}(N, Kd) : i \in \mathbb{B}^I_g\}$ for the type-II BSs, find the outer precoders $\{T^I_j \in \mathbb{U}(N, Kd) : j \in \mathbb{B}^I_g\}$ for type-I BSs, the inner precoders $\{V_{jk} \in \mathbb{U}(Kd, d) : \forall j, k\}$ for all BSs and decorrelators $\{U_{jk}\}$ for all MSs, such that:

$$\text{rank}(U_{jk}^\dagger H_{jk,j} T_j V_{jk}^s) = d, \forall j, k;$$ \hfill (17)

$$U_{jk}^\dagger H_{jk,j} T_j V_{jp}^s = 0, \forall j, k \neq p; \quad \text{intracell IA constraints} \hfill (18)$$

$$U_{jk}^\dagger H_{jk,i} T_i = 0, \forall j, k, i \neq j; \quad \text{intercell IA constraints} \hfill (19)$$

$$\{T^I_j : j \in \mathbb{B}^I_g\}, \{V_{jk}^s : \forall j, k\} \text{ can only be adaptive to } \{F_{jk}(H_{jk}) : \forall j, k\} \text{ according to } L. \quad \text{CSI knowledge constraint} \hfill (20)$$

where $T_j = T^I_j$, $j \in \mathbb{B}^I_g$ and $T_j = T^H_j$, $j \in \mathbb{B}^H_g$ for notation convenience.

Note that (17)-(19) refers to the IA constraints and (20) refers to the CSI knowledge constraint. Specifically, compared with conventional IA with full CSIT in (6)-(8), there are two unique challenges, namely the CSI knowledge and feasibility, associated with Constraints 1 under partial CSIT knowledge. First, the CSI knowledge constraint is an implicit constraint which is difficult to handle. Second, adjusting the feedback profile $L$ may reduce the CSI feedback dimension $D(L)$ in (16) but the IA constraints may no longer be feasible. The following summarizes the challenges we face.

**Challenge 1:** Adjust the feedback profile $L$ so as to minimize the feedback dimension $D(L)$ subject to Constraints 1.

### III. IA Feasibility Conditions Under a Given Feedback Profile $L$

In this section, we shall investigate Constraints 1 and find out the requirements on $L$ to make the IA problem in Constraint [I] feasible, i.e., for what kind of CSI feedback profile $L$, Constraints [I] can have feasible solutions $\{T^I_j\}$, $\{V_{jk}^s, U_{jk}\}$. We further derive the corresponding IA transceiver solutions $\{T^I_j\}$, $\{V_{jk}^s, U_{jk}\}$ to satisfy the conditions in Constraints [I] for a given feedback profile $L$. 

March 18, 2014 DRAFT
A. IA Constraints Transformation

To investigate (20) in Constraints 1, we shall first have a better understanding on how to utilize the partial CSI knowledge \( \{ F_{jk} \} \). Specifically, the information available at BS \( j \) from the feedback CSI \( \{ F_{jk} \} : \forall k \) is denoted by the set of matrices \( \mathbb{H}_j \),

\[
\mathbb{H}_j = \left\{ \tilde{H}_{jk,i} = a_{jk,i} H_{jk,i} : \forall k, i \in B_I \cup \{ j \} \right\}
\]  

(21)

where \( \{ a_{jk,i} \} \) are some \(^7\) non-zero scalars. Based on \( \{ \mathbb{H}_j \} \), we study Constraints 1.

**Challenge 2:** Constraints 1 is difficult because 1) the conditions (17) and (18) are coupled as \( \{ H_{jk,j} \} \) act as both the direct link in (17) and the cross link in (18); 2) the CSI knowledge constraint (20) requires that the precoders can only be designed based on the partial CSI knowledge \( \{ \mathbb{H}_j \} \).

We first introduce an equivalent IA constraint transformation, which can explicitly handle the CSI knowledge constraint and the coupling issues.

**Constraints 2 (IA Constraint Transformation under \( \mathcal{L} \)):** Find \( \tilde{T}_{Ii}^f \in \mathbb{U}(n_i, Kd), n_i \geq Kd, i \in \mathbb{B}_g^f \), and \( \tilde{U}_{jk} \in \mathbb{U}(A_{jk}, d), A_{jk} \geq d, \forall j, k \), satisfying the following equations:

\[
(\tilde{U}_{jk})^\dagger \tilde{H}_{jk,i} \tilde{T}_{Ii}^f = 0, \forall j, k, i \in \mathbb{B}_g^f \setminus \{ j \}.
\]  

(22)

Note that constraint in (22) involves intercell IA constraints from the type-I BSs only. This is because the intercell interference from the type-II BSs has already been cancelled at the MSs via designing the decorrelator \( U_{jk} \) in the subspace spanned by \( R_{jk} \). Furthermore, Constraint 2 contains no intracell IA constraints because of the two stage precoding structures in (17)–(19). The equivalent relationship between Constraints 1 and Constraints 2 is established in the lemma below.

**Lemma 2 (Equivalence of Constraints 1 and Constraints 2):** Given the CSI feedback profile \( \mathcal{L} \) and the outer precoders \( \{ T_{IIi}^f \in \mathbb{U}(N, Kd) : i \in \mathbb{B}_g^{II} \} \) for type-II BSs:

(a): Constraints 2 is feasible iff Constraints 1 is feasible.

(b): If \( \{ \tilde{T}_{Ii}^f \} \) and \( \{ \tilde{U}_{jk} \} \) are solutions of Constraints 2 then \( \{ T_{Ii}^f \}, \{ V_{jk}^g, U_{jk} \} \) given by

\[
T_{Ii}^f = \begin{bmatrix} \tilde{T}_{Ii}^f \\ 0 \end{bmatrix}, i \in \mathbb{B}_g^f, \quad U_{jk} = \begin{bmatrix} R_{jk} \tilde{U}_{jk} \\ 0 \end{bmatrix}, \forall j, k; \]

(23)

\(^7\) As an example, one common approach \[18\] to feedback \( \mathbb{P}(H) = \{ aH : a \in \mathbb{C} \}, H \in \mathbb{C}^{A \times B} \) is to feedback the unitary vector \( \frac{1}{||H||} \text{vec}(H) \). In this case, the scalar \( a = \frac{1}{||H||} \).
\[ V_{jk}^d = v_d \left( \sum_{p=1}^{K} \left( (\hat{U}_{jp})^\dagger \hat{H}_{jp}^e T_i \right) \right), \forall k, j \in \mathbb{B}_g^I; \] 
\[ V_{jk}^s = v_d \left( \sum_{p=1}^{K} \left( (\hat{U}_{jp})^\dagger \hat{H}_{jp}^e \right) \right), \forall k, j \in \mathbb{B}_g^I \] 

are the solutions for Constraints 1 almost surely, where \( v_d(A) \) is the matrix of eigenvectors corresponding to the \( d \) least eigenvalues of a Hermitian matrix \( A \).

Note that the precoder solutions in (23)-(25) automatically satisfy the CSI knowledge constraint (20). Furthermore, the IA Constraints (2) contain the intercell constraints from type-I BSs only as in (22). Consequently, the aforementioned Challenge 2 is tackled by using Constraint (2) and Lemma (2).

B. Feasibility Conditions on \( L \)

Based on Lemma (2) and Constraints 2, we obtain the following necessary feasibility conditions for Constraint 1.

**Theorem 1 (Necessary Conditions for IA Feasible on \( L \))**: If Constraints (1) is feasible, the CSI feedback profile \( L \) should satisfy: 1) \( m_{jk} - \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} Kd - d \geq 0, \forall j, k, 2) N \geq Kd, n_i \geq Kd, i \in \mathbb{B}_g^I, 3) \forall J^{[\ell]}_{sub} \subseteq \{ (j, k) : \forall j, k \}, J^{[\ell]}_{sub} \subseteq \mathbb{B}_g^I, \)
\[ \sum_{(j, k) \in J^{[\ell]}_{sub}} \left( m_{jk} - \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} Kd - d \right) + \sum_{i \in J^{[\ell]}_{sub}} K(n_i - Kd) \geq \sum_{j \in J^{[\ell]}_{sub} i \in J^{[\ell]}_{sub} \setminus \{j\}} Kd. \]

For instance, if we have 0 type-I BS \( (g = 0) \) and \( m_{jk} = m \), \( \forall j, k \), in \( L \), then Theorem 1 requires \( N \geq Kd, m \geq (G - 1)Kd + d \) for \( L \) to be IA feasible (see Example 1); if we have 0 type-II BS \( (g = G) \) and \( m_{jk} = m, \forall j, k, n_i = n, \forall i \), in \( L \), then Theorem 1 requires \( m \geq d, n \geq Kd, m + n \geq (GK + 1)d \) for \( L \) to be IA feasible (see Example 2). Using the max-flow theory [20], [21], Theorem 1 can be expressed in an alternative way.

**Corollary 1 (Equivalent Condition)**: \( L \) satisfies the three conditions in Theorem 1 iff \( N \geq Kd \) and there exist non-negative variables \( \{ f'_{jk,i}, f''_{jk,i} \}, f'_{jk,i} \geq 0, f''_{jk,i} \geq 0, \forall j, k, i \in \mathbb{B}_g^I \setminus \{j\} \), that satisfy
\[ f'_{jk,i} + f''_{jk,i} \geq Kd, \forall j, k, i \in \mathbb{B}_g^I \setminus \{j\}, \]
\[ \left( m_{jk} - \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} Kd - d \right) \geq \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} G f''_{jk,i}, \forall j, k, \]
\[ (n_i - Kd) \geq \sum_{j \neq i}^K \sum_{k=1}^G f'_{jk,i}, \forall i \in \mathbb{B}_g^I. \]
Similar to conventional IA \[13\], \[16\], checking condition (26) in Theorem 1 involves an exponential number of comparisons (i.e., \(O(2^{KG})\)). By Corollary \[1\] this exponential complexity can be reduced to a polynomial number. On the other hand, Corollary \[1\] also provides a constructive approach to verify the IA feasibility conditions (i.e., construct \(f^t_{jk,i}, f^t_{jk,i}\) in terms of the parameters in \(L\) and check the conditions (27)-(29)).

We also have that the conditions in Theorem \[1\] are sufficient in the divisible cases.

**Theorem 2 (Sufficient IA Feasibility Conditions):** Suppose \(L\) satisfies the three conditions in Theorem \[1\]. If \(L\) further satisfies \(d \mid n_i, \forall i \in \mathbb{B}_g^I\), or \(Kd \mid (m_{jk} - d), \forall j, k,\), Constraints \[1\] is feasible.

**Remark 3 (Backward Compatibility with Previous Results):** Suppose \(g = G, K = 1, m_{jk} = M, n_i = N\) in \(L\) (full CDI feedback). Then the required conditions on parameter \(G, M, N, d\) from Theorem 2 in this paper, are the same as from Corollary 3.4 of \[16\]. Suppose \(g = G, m_{jk} = M, n_i = N, \forall j, k, i,\) in \(L\) (full CDI feedback) and \(d \mid N, d \mid M\). Then the required conditions on parameter \(G, K, M, N, d\) from Theorem 2 in this paper, are the same as from Theorem 2 in \[17\].

**C. Transceiver Design under \(L\)**

In this section, we derive the IA solutions \(\{T_j^I\}, \{V_{jk}^g, U_{jk}\}\) to Constraints 1. Note conventional IA designs \[4\], \[5\] require full CSIT and hence can not be directly applied to Constraints \[1\] which have the CSI knowledge constraint (20). Specifically, we adopt the alternating interference leakage minimization (AILM) techniques \[4\] and solve the equivalent Constraints \[2\]. Similar to \[4\], we propose the following problem to find the solutions to satisfy Constraints \[2\].

**Problem 1 (Interference Leakage Minimization):**

\[
\min_{\{\tilde{T}_i^I, \tilde{U}_{jk}\}} \quad I \triangleq \sum_{(j,k) \in \mathbb{B}_g^I \setminus \{j\}} \sum_{i \in \mathbb{B}_g^I} \text{tr} \left( (\tilde{U}_{jk})^\dagger \tilde{H}_{jk,i}^e \tilde{T}_i^I \left( (\tilde{U}_{jk})^\dagger \tilde{H}_{jk,i}^e \tilde{T}_i^I \right)^\dagger \right) \\
\text{s.t.} \quad \tilde{T}_i^I \in \mathbb{U}(n_i, Kd), i \in \mathbb{B}_g^I; \quad \tilde{U}_{jk} \in \mathbb{U}(A_{jk}, d), \forall j, k.
\]

(30)

Problem 1 has closed-form optimal \(\{\tilde{T}_i^I\}\) for fixed \(\{\tilde{U}_{jk}\}\) and closed-form optimal \(\{\tilde{U}_{jk}\}\) for fixed \(\{\tilde{T}_i^I\}\), and hence we shall apply alternating optimization techniques \[4\] to derive solutions.

**Algorithm 1 (Iterative Solution to Constraints 2 under \(L\)):**

- **Step 1 (Initialization):** Randomly initialize \(\tilde{T}_i^I \in \mathbb{U}(n_i, Kd), \forall i \in \mathbb{B}_g^I, \tilde{U}_{jk} \in \mathbb{U}(A_{jk}, d), \forall j, k.\)
- **Step 2 (Update \(\tilde{U}_{jk}\)):** Update \(\tilde{U}_{jk} = v_d \left( \sum_{i \in \mathbb{B}_g^I \setminus \{j\}} (\tilde{H}_{jk,i}^e \tilde{T}_i^I) (\tilde{H}_{jk,i}^e \tilde{T}_i^I)^\dagger \right), \forall j, k.\)
- **Step 3 (Update \(\tilde{T}_i^I\)):** Update \(\tilde{T}_i^I = v(Kd) \left( \sum_{(j,k)} (\tilde{H}_{jk,i}^e \tilde{U}_{jk}) (\tilde{H}_{jk,i}^e \tilde{U}_{jk})^\dagger \right), \forall i \in \mathbb{B}_g^I.\)
Repeat Step 2 and Step 3 until the value of $I$ in (30) converges.

Note that based on the converged solution of \{\hat{T}_i\} and \{\hat{U}_{jk}\} from Algorithm 1, we can obtain the overall solutions \{T_{ij}\} \{V_{jk}, U_{jk}\} to Constraints 1 by using Lemma 2.

Remark 4 (Characterization of Algorithm 1): Note Algorithm 1 can automatically adapt to the partial CSI knowledge constraint (20). On the other hand, the value of $I$ converges in Algorithm 1 because: 1) the total interference leakage $I$ in (30) is monotonically decreasing in the alternating updates of Step 2 and Step 3; 2) $I$ is non-negative so that $I$ is bounded below. However, the convergence to global optimality is not guaranteed due to the nonconvexity of Problem 1 [4]. Note that if the total interference leakage $I$ at the converged point is 0, then the converged solution is a feasible solution to Constraints 2. Furthermore, from extensive simulations, it is observed that the converged value of $I$ is always 0 when Constraints 2 is feasible (similar to conventional AILM works [4], [5], [14]).

D. Implementation Consideration

In this section, we give a summary on how to implement the proposed IA scheme with partial CSI feedback $L$ in MIMO cellular networks.

Algorithm 2 (Implementation of Proposed IA Scheme under $L$):

- **Step 1 (CSI Observation):** The $(j, k)$-th MS observes the local CSI $H_{jk} = (H_{jk,1}, H_{jk,2}, \ldots H_{jk,G})$, $\forall j, k$.

- **Step 2 (Partial CSI Feedback under $L$):** The $(j, k)$-th MS feedbacks the filtered CSI generated by $H_{jk}^{fed} = F_{jk}(H_{jk})$ to BS $j$, where $F_{jk}$ is the CSI filtering function as in Definition 6 according to feedback profile $L$.

- **Step 3 (Transceiver Computation):** BS $j$ obtains $H_{j}$ in (21) from the feedback $\{H_{jk}^{fed} : \forall k\}$. One BS collects the $\{H_{j} : \forall j\}$ from other BSs through the backhaul and computes $\{\hat{T}_{i} : i \in B_{I}\}$, $\{\hat{U}_{jk} : \forall j, k\}$ according to Algorithm 1 in a centralized manner.

- **Step 4 (Transceiver Distribution):** The BS mentioned in Step 3 distributes the obtained $\hat{T}_{j}$, $\hat{U}_{jk} : \forall j, k$ to BS $j$ for $j \in B_{I}$ and $\{\hat{U}_{jk} : \forall k\}$ to BS $j$ for $j \in B_{II}$. BS $j$ forward $\hat{U}_{jk}$ to the $(j, k)$-th MS, $\forall j, k, k$.

  - BS $j$ uses $T_{j}$ as the outer precoder for type-I BSs, $V_{jk}$ as the inner precoder for the $(j, k)$-th MS designed via equations (23)-(25) in Lemma 2.
  - The $(j, k)$-th MS uses $U_{jk}$ as the decorrelator designed via equation (23) in Lemma 2.
IV. FEEDBACK DIMENSION MINIMIZATION AND ASYMPTOTIC OPTIMAL FEEDBACK PROFILE

A. Problem Formulation

In this section, we solve Challenge 1 by solving the following problem of CSI feedback dimension minimization subject to the requirement of IA DoFs (Constraints 1) under partial CSI feedback $L$ in MIMO cellular networks.

Problem 2 (Feedback Dimension Minimization):

$$\min_{\mathcal{L}} D(\mathcal{L})$$

s.t.  
$$n_i \leq N, \forall i \in \mathbb{B}_g^I, m_{jk} \leq M, \forall j, k;$$

$$0 \leq g \leq G, \quad g, n_i, m_{jk} \in \mathbb{Z};$$

Constraints 1 under $\mathcal{L}$.  

Note that Problem 2 is an offline optimization where we try to find the optimal feedback profile $\mathcal{L}^*$ to minimize the feedback dimension $D(\mathcal{L})$ so that the BS can still deliver $d$ data streams to each MS in the MIMO cellular network with the given antenna configurations. Note that the Constraints 1 in (34) is an implicit constraint on $\mathcal{L}$ and the feasibility conditions are specified in Theorem 1 and 2. Figure 3 summarizes the relationship between Problem 2 and Theorem 1, 2. By using the necessary conditions in Theorem 1, we first have the following property for any feasible $\mathcal{L}$ to Problem 2. Denote $N_1 = \min(GKd, N)$, $g_1 = \left\lfloor \frac{G(G-1)Kd-M+d}{N_1-Kd} \right\rfloor$.

Lemma 3 (Number of Type-II BSs): Suppose $\mathcal{L} = \{m_{jk} : \forall j, k, \{n_i : i \in \mathbb{B}_g^I\}\}$ is a feasible solution to Problem 2, then $\mathcal{L}$ has no more than $(G-g_1)$ type-II BSs, i.e., $g \geq g_1$.

Lemma 3 indicates that we may only allow a finite number of type-II BSs to satisfy the required IA DoF in the network.

Challenge 3: Design a low-complexity solution to Problem 2 despite the implicit constraint (34) on $\mathcal{L}$ and the combinatorial nature of the optimization variable ($\mathcal{L}$).

B. Proposed Greedy Algorithm of Feedback Profile Design

To tackle the challenges, we obtain an achievable upper bound of feedback dimension by (a) restricting constraint (34) with its sufficient conditions in Theorem 2 and (b) find a low complexity greedy algorithm that gives a feedback profile $\mathcal{L}_0$ satisfying the sufficient condition. Specifically, the greedy feedback profile solution $\mathcal{L}_0$ is designed to aggressively select the largest number of type-II BSs. While the solution is
a suboptimal upper bound of the minimum feedback dimension \( D(L^*) \), we will show later that it is asymptotically optimal as \( G \to \infty \). Denote \( N_0 = \min \left( G K d, \left\lfloor \frac{N}{d} \right\rfloor d \right) \). The details of the greedy algorithm are as follows:

**Algorithm 3 (Greedy Solution \( L_0 \) to Problem [2])**

- **Step 1 (Initialization):** Initialize \( L_0 = \left\{ \{ m_{jk} = M : \forall j, k \} \right\} \), where
  \[
g_0 = \left\lceil \frac{G ((G - 1) K d - M + d)}{N_0 - K d} \right\rceil. \tag{35}\]

- **Step 2 (Antenna Pruning Preparation):** Construct the max flow graph \( N' = (V, E) \):  
  1) The vertices are given by \( V = \{ a, b, u_{jk}, v_i, c_{ji,k} \} \), \( \forall j, k, i \in \mathbb{B}_{g_0} I \), where \( a, b \) are the source, destination node respectively and \( u_{jk}, v_i, c_{ji,k} \) are the intermediate nodes in \( N \).
  2) The edges are given by \( E = \{ (a, u_{jk}), (a, v_i), (u_{jk}, c_{ji,i}), (v_i, c_{ji,i}), (c_{ji,i}, b) : \forall j, k, i \in \mathbb{B}_{g_0} I \} \), where \( (u, v) \) denotes the edge from node \( u \) to node \( v \).
  3) The edge capacities are given by \( c(a, u_{jk}) = c(u_{jk}, c_{ji,i}) = (m_{jk} - \sum_{i \in \mathbb{B}_{g_0} I \setminus \{ j \}} K d - d), c(a, v_i) = c(v_i, c_{ji,i}) = K (n_i - K d), c(c_{ji,i}, t) = K d, \forall j, k, i \in \mathbb{B}_{g_0} I \), where \( c(u, v) \) denotes the edge capacity on the edge \( (u, v) \).
  4) Find the max flow solutions \( \{ f(a, b) : (a, b) \in E \} \) for \( N \).

- **Step 3 (Antenna Pruning):** Based on the max-flow \( \{ f(a, b) : (a, b) \in E \} \) obtained in **Step 2**, perform antenna reduction as
  \[
n_i = N_0 - \left\lceil \frac{c(a, v_i) - f(a, v_i)}{K d} \right\rceil d, i \in \mathbb{B}_{g_0} I, \]
  \[
m_{jk} = M - \left\lceil c(a, u_{jk}) - f(a, u_{jk}) \right\rceil, \forall j, k. \]

**Remark 5 (Interpretation of Algorithm 3):** The feedback profile \( L_0 \) design in Algorithm 3 contains two stages and in the first stage (Step 1), we design \( g_0 \) in \( L \) by choosing the largest number of type-II BSs, in the second stage (Step 2, 3), we further reduce the feedback antennas via max-flow techniques.
As the computation mainly comes from finding the max flow solutions, the overall worst case complexity of Algorithm 3 is $O(G^4K^2)$ [20].

By deploying Corollary 1 and by using the max-flow graph in Algorithm 3, we derive that $L_0$ satisfies the conditions in Theorem 2, and is therefore a feasible solution to Problem 2.

**Theorem 3 (Feasibility of $L_0$):** The obtained feedback profile $L_0$ from Algorithm 3 is a feasible solution to the feedback dimension optimization problem (Problem 2).

### C. Asymptotic Optimality of the Proposed Greedy Solution

In this section, we further show that $L_0$ is in fact asymptotically optimal. To do this, we relax the constraint (34) in Problem 2 with its necessary conditions in Theorem 1, and find a strict lower bound on the minimum feedback dimension under the necessary conditions (through algebraic manipulations). Specifically, we have the following bounds on the optimal feedback dimension.

**Theorem 4 (Bounds on the Optimal Feedback Dimension):** Suppose $L^*$ is the optimal solution of Problem 2, then

$$D_{low} \leq D(L^*) \leq D(L_0) \quad (36)$$

where $D(L_0)$ is the feedback dimension induced by feedback profile $L_0$ and $D_{low}$ is given by:

$$D_{low} = KGN_1g_1(M - (G - g_1)Kd) - KG^2. \quad (37)$$

From Theorem 4, we derive that $L_0$ can achieve the asymptotic optimality of Problem 2.

**Corollary 2 (Asymptotic Optimality of $L_0$):** Suppose the number of antennas $N, M$ are given by $N = \lfloor C_1KG \rfloor, M = \lfloor C_2KG \rfloor$, where $0 < C_1, C_2 < d, d < C_1 + C_2$. As $G \to \infty$, we have

$$\lim_{G \to \infty} \frac{D(L^*)}{G^4K^3} = \lim_{G \to \infty} \frac{D(L_0)}{G^4K^3} = \frac{(d - C_1)(d - C_2)^2}{C_1}. \quad (38)$$

**Proof:** As $G \to \infty$, we have $g_0 = \frac{d-C_2}{C_1} G + \mathcal{O}(1)$ and $g_1 = \frac{d-C_2}{C_1} G + \mathcal{O}(1)$. Substituting $g_0$ and $g_1$ into $D(L_0)$ and $D_{low}$, we obtain

$$\lim_{G \to \infty} \frac{D(L_0)}{G^4K^3} = \lim_{G \to \infty} \frac{D_{low}}{G^4K^3} = \frac{(d - C_1)(d - C_2)^2}{C_1}.$$ 

From this and (36), the corollary is proved.

**Remark 6 (Interpretation of Corollary 2):** Corollary 2 depicts the scaling law of the optimal feedback dimension w.r.t. the size of the network $G$ and (38) indicates that the proposed greedy solution $L_0$ is an asymptotically optimal solution to Problem 2. Furthermore, using Lemma 3 and Corollary 2 we can infer that the asymptotic optimal $L_0$ has the largest number of type-II BSs.
From (38), the value of $\lim_{G \to \infty} \frac{D(L^*)}{G^2K^2}$ gets larger as $d$ increases ($0 < C_1, C_2 < d$). This agrees with our intuition that we should pay a larger CSI feedback overhead as the required IA DoF increases in the network for a given number of antennas.

**Corollary 3 (Performance Comparison):** Under the same setup as in Corollary 2, the ratio between the feedback dimension of $L_0$ and the full CDI feedback scheme (sum feedback dimension $D_{full} = G^2K(MN - 1)$) is given by

$$\Upsilon \triangleq \lim_{G \to \infty} \frac{D(L_0)}{D_{full}} = \frac{(d - C_1)(d - C_2)^2}{(C_1)^2C_2} < 1.$$  \hspace{1cm} (39)

Note that $(x)$ comes from $\forall i$, $0 < C_1, C_2 < d$, $d < C_1 + C_2$ as in Corollary 2. (39) further implies that larger values of $C_1, C_2$ with $0 < C_1, C_2 < d$, $d < C_1 + C_2$ tends to have smaller $\Upsilon$ and hence the proposed scheme achieves larger CSI feedback reduction gain. This is because a larger number of antennas at the BSs and MSs (larger $C_1, C_2$) leads to a larger design space for CSI filtering and hence better schemes may be obtained.

V. RELATIONSHIP BETWEEN CSI FEEDBACK DIMENSION AND FEEDBACK BITS

Recall that in Section II, we propose a novel metric (feedback dimension $D$) to quantify the effectiveness of CSI feedback filtering. In this section, we justify the physical meaning of $D$ in MIMO cellular networks by deriving the scaling relationship between the CSI feedback bits $B_{tot}$ and the CSI feedback dimension $D(L)$. Specifically, we show that when $B_{tot}$ scales with $D$ and SNR as $B_{tot} = D(L) \log \text{SNR}$, the sum DoF of $KGd$ can be achieved in the MIMO cellular network. This result indicates that the proposed feedback dimension can serve as a first-order measurement of the CSI feedback overhead, and highlights the importance of feedback dimension optimization in MIMO cellular networks.

A. MIMO Cellular Networks with Limited CSI Feedback Bits

Suppose that we deploy a feasible feedback profile $L$ (feasible solution to Problem 2) in the MIMO cellular network with a total of $B_{tot}$ CSI feedback bits to quantize and feedback the partial CSI $\{F_{jk} : \forall j,k\}$ generated at the MSs (block (b) in Figure 1). Assume $b$ bits per each feedback dimension and then $B_{tot} = bD(L)$.

To begin with, we illustrate how the elements in $F_{jk}$ are quantized using the Grassmannian codebook. We quantize the direction information $P(H)$ of the matrix $H$ by first stacking $H$ into a long vector $\text{vec}(H)$, and then quantizing the normalized vector $h = \frac{1}{||H||} \text{vec}(H)$ to be $\hat{h}$, $||\hat{h}|| = 1$, with the Grassmannian
vector codebooks \cite{18}. We recover the quantized version of $H$ (denoted as $\hat{H}$) by reverse-stacking $\hat{h}$. Based on this quantization approach, we denote the quantized version of $H^e_{jk,i}$ in $\{F_{jk}\}$ as $\hat{H}^e_{jk,i}$. The relationship between $H^e_{jk,i}$ and $\hat{H}^e_{jk,i}$ can be expressed as

$$H^e_{jk,i} = C_{jk,i} \hat{H}^e_{jk,i} + \Delta_{jk,i}, \quad \forall j, k, i$$

where $\{C_{jk,i}\}$ are certain scalars, $\Delta_{jk,i}$ is the quantization distortion part and $\text{vec}(\Delta_{jk,i})$ lies in the orthogonal complement space of $\text{vec}(\hat{H}^e_{jk,i})$ \cite{18}.

**Lemma 4 (CSI Quantization Distortion):** Denote $\mathbb{E}(||\Delta_{jk,i}||^2)$ as the average quantization distortion of $H^e_{jk,i}$, we have

$$\mathbb{E}(||\Delta_{jk,i}||^2) = (B_{jk,i} - 1)2^{-b}, \quad \forall j, k, i$$

where $B_{jk,i}$ is given in (14).

Denote $\{\hat{T}_j \in \mathbb{U}(N, Kd) : j \in \mathbb{B}_g^f\}$, $\{\hat{V}^s_{jk} \in \mathbb{U}(Kd, d) : \forall j, k\}$, $\{\hat{U}_{jk} \in \mathbb{U}(N, d) : \forall j, k\}$ as the designed outer precoders for type-I BSs, inner precoders for all BSs, decorrelators for all MSs respectively, based on the quantized CSI $\{\hat{H}^e_{jk,i} : \forall j, k, i \in \mathbb{B}_g^f \cup \{j\}\}$. Due to the quantization of the feedback CSI, IA cannot be perfectly achieved and there will be residual interference leakage. Denote the residual interference covariance matrix at the $(j, k)$-th MS as $\Phi_{jk}$, then

$$\Phi_{jk} = \frac{P}{Kd} \sum_{(i,p)\neq(j,k)} \left( \left( \hat{U}^+_{jk} H_{jk,i} \hat{V}_{ip} \right) \left( \hat{U}^+_{jk} H_{jk,i} \hat{V}_{ip} \right)^\dagger \right)$$

where $\hat{V}_{ip} = \hat{T}_i^f \hat{V}^s_{ip}$, $i \in \mathbb{B}_g^f$, $\hat{V}_{ip} = \hat{T}_i^{II} \hat{V}^s_{ip}$, $i \in \mathbb{B}_g^{II}$. We have the following lemma on the average residual interference leakage.

**Lemma 5 (Residual Interference Bound):** Denote $\mathbb{E}(\text{tr}(\Phi_{jk}))$ as the average interference leakage, then

$$\mathbb{E}(\text{tr}(\Phi_{jk})) \leq \frac{P}{d} c_{jk} \cdot 2^{-b}$$

where $c_{jk} = \sum_{i \in \mathbb{B}_g^f \cup \{j\}} (B_{jk,i} - 1)$.

**B. Throughput Analysis under Limited CSI Feedback Bits**

Denote $\{T_j\}$, $\{V^s_{jk} : \forall j, k\}$, $\{U_{jk} : \forall j, k\}$ as the perfect CSIT IA transceivers. Then the network throughput under perfect CSIT can be expressed as \cite{4}.

$$R_{\text{per}} = \sum_{j=1}^{G} \sum_{k=1}^{K} \mathbb{E}\left\{ \text{log det} \left( I_d + \frac{P}{Kd} (U^\dagger_{jk} H_{jk,j} T_j V^s_{jk}) (U^\dagger_{jk} H_{jk,j} T_j V^s_{jk})^\dagger \right) \right\}.$$

\cite{March 18, 2014 DRAFT}
Following the above definition and treating the residual interference due to CSI quantization as noise, the network throughput under limited feedback can be expressed as
\[
R_{\text{lim}} = \sum_{j=1}^{G} \sum_{k=1}^{K} \mathbb{E} \left\{ \log \det \left( I_d + \frac{P}{Kd} (\hat{U}_{jk}^\dagger H_{jk,j} \hat{V}_{jk}) (\hat{U}_{jk}^\dagger H_{jk,j} \hat{V}_{jk})^\dagger (I_d + \Phi_{jk})^{-1} \right) \right\}.
\] (42)

We have the following throughput bounds regarding \( R_{\text{lim}} \).

**Lemma 6 (Throughput Bounds):** \( R_{\text{lim}} \) is bounded by
\[
GKd \int_0^\infty \log \left( 1 + \frac{P}{Kd} \cdot v \right) \cdot f(v) dv = R_{\text{per}} \geq R_{\text{lim}} \geq R_{\text{lb}} = R_{\text{per}} - \sum_{j=1}^{G} \sum_{k=1}^{K} d \cdot \log \left( 1 + \frac{P}{d^2 c_{jk}} \cdot 2^{-b} \right)
\] (43)
where \( f(v) \) is the marginal probability density function (p.d.f.) of the unordered eigenvalues of the \((d \times d)\) central Wishart matrix with \(d\) degrees of freedom and covariance matrix \( I \). (For more details see pp 32-33).

Based on Lemma 6 we have the following Theorem.

**Theorem 5 (Scaling Law Between CSI Feedback Bits and Feedback Dimension):** When the total number of CSI feedback bits \( B_{\text{tot}} \) is given by:
\[
B_{\text{tot}} = D(\mathcal{L}) \log P
\] (44)
the MIMO cellular network can achieve the sum DoF of \( GKd \) data streams, i.e.,
\[
\lim_{P \rightarrow \infty} \frac{R_{\text{lim}}}{\log P} = GKd.
\]

**Proof:** From (44), we obtain \( b = \log P \). Hence
\[
R_{\text{lb}} = R_{\text{per}} - \sum_{j=1}^{G} \sum_{k=1}^{K} d \cdot \log \left( 1 + \frac{1}{d^2 c_{jk}} \right).
\]
Note \( \sum_{j=1}^{G} \sum_{k=1}^{K} d \cdot \log \left( 1 + \frac{1}{d^2 c_{jk}} \right) \) is bounded. Therefore,
\[
\lim_{P \rightarrow \infty} \frac{R_{\text{lb}}}{\log P} = \lim_{P \rightarrow \infty} \frac{R_{\text{per}}}{\log P} = GKd.
\]
From this and (43), Theorem 5 is proved.

**Remark 7 (Interpretation of Theorem 5):** Theorem 5 demonstrates a linear scaling relationship between the CSI feedback bits and feedback dimension in MIMO cellular networks. This result indicates that the proposed metric of CSI feedback dimension can separate the CSI filtering and CSI quantization in MIMO cellular networks, and can serve as a first-order measurement of the feedback overhead.
VI. Numerical Results

In this section, we verify the performance of the proposed feedback scheme in MIMO cellular networks through simulation. We consider limited feedback with Grassmannian codebooks \([18]\) to quantize the partial CSI \(\{F_{jk}\}\) at each MS. The precoders/decorrelators are designed using the Algorithm 1 developed in Section III-C. We consider \(10^4\) i.i.d. Rayleigh fading channel realizations and compare the performance of the proposed feedback scheme with the following 3 baselines.

- **Baseline 1 (Feedback Full CDI As in [7]–[9]):** Each MS quantizes and feedbacks the full CDI using Grassmannian codebooks, i.e., \(F_{jk} = (\cdots, P(H_{jk,i}), \cdots)_{\forall i}, \forall j, k\).

- **Baseline 2 (Feedback Truncated CDI As in [11]):** Each MS quantizes and feedbacks the CDI of the smallest CSI submatrices, i.e., \(F_{jk} = (\cdots, P(H^s_{jk,i}), \cdots)_{\forall i}, \forall j, k\), where \(H^s_{jk,i} = \begin{bmatrix} I_m & 0 \end{bmatrix} H_{ji,k}\), and \(m\) are chosen to make the network tightly feasible by \(m = GKd + d - N\) \([17]\).

- **Baseline 3 (Random Beamforming):** The BS, MS randomly choose the transceivers \(\{T_j, V^s_{jk}\}, \{U_{jk}: \forall j, k\}\).

Consider a MIMO cellular network with \(G = 3, K = 2, N = M = 4, d = 1\) for simulation tests. We obtain the following feedback profile for the proposed scheme via Algorithm 3, \(L = \{\{m_{jk} = 4 : \forall j, k\}, g = 2, \{n_1 = 4, n_2 = 3\}\}\). Note the sum feedback dimension for the proposed scheme, baseline 1 and baseline 2 are 114, 198, and 270 respectively under the considered network topology.

A. Throughput Comparison w.r.t. Transmit SNR

Figure 4 illustrates the network throughput versus the transmit SNR \(P\) under a sum feedback bits of \(B_{\text{tot}} = 800\). The proposed scheme achieves substantial throughput gain over the baselines. This is because the proposed scheme significantly reduces the CSI feedback dimension while preserving the IA feasibility. Under the same number of feedback bits, more CSI feedback bits can be utilized to reduce the quantization error per dimension. The dramatic performance gain highlights the importance of reducing the feedback dimension in MIMO cellular networks. Furthermore, we observe that the gain is larger at high SNR because residual interference, which is the major performance bottleneck in high SNR regimes, is significantly reduced by the proposed scheme. On the other hand, we observe that the throughputs of all the schemes saturate at high SNR. This is because under fixed number of CSI feedback bits, the leakage interference power due to CSI quantization scales with the transmit SNR.
B. Relationship between CSI Feedback Dimension and Feedback Bits

Figure 5 illustrates the network throughput versus the transmit SNR when the number of CSI feedback bits scales as $B_{tot} = D \log \text{SNR}$ as in Theorem 5. Note $D = 114$ as derived for the proposed feedback scheme. As we can see, the throughput of the proposed scheme achieves the same slope as that of the perfect CSIT throughput, which justifies that the sum DoFs of the network are maintained under the given CSI feedback bits scaling condition as in Theorem 5. However, the baseline 1, 2 can not achieve the same slope because they have larger CSI feedback dimension and hence require more feedback bits.

VII. CONCLUSIONS

In this paper, we consider IA processing with CSI feedback filtering in MIMO cellular networks. We characterize the feedback cost by the feedback dimension and demonstrate that it can serve as a first order metric of the CSI feedback overhead. Based on these, we formulate the problem of feedback dimension minimization subject to the required IA DoF for a given antenna configuration and we further propose an asymptotic optimal solution. Both analytical and simulation results show that the proposed scheme can significantly reduce the CSI feedback cost of IA in MIMO cellular networks.
Figure 5. Throughput scaling versus transmit SNR under $B_{\text{tot}} = D \log \text{SNR}$ in a $G = 3$, $K = 2$, $N = M = 4$, $d = 1$ network.

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