Affine differential geometry and smoothness maximization as tools for identifying geometric movement primitives

Felix Polyakov

Abstract Neuroscientific studies of drawing-like movements usually analyze neural representation of either geometric (e.g., direction, shape) or temporal (e.g., speed) parameters of trajectories rather than trajectory’s representation as a whole. This work is about identifying geometric building blocks of movements by unifying different empirically supported mathematical descriptions that characterize relationship between geometric and temporal aspects of biological motion. Movement primitives supposedly facilitate the efficiency of movements’ representation in the brain and comply with such criteria for biological movements as kinematic smoothness and geometric constraint. The minimum-jerk model formalizes criterion for trajectories’ maximal smoothness of order 3. I derive a class of differential equations obeyed by movement paths whose $n$th-order maximally smooth trajectories accumulate path measurement with constant rate. Constant rate of accumulating equi-affine arc complies with the 2/3 power-law model. Candidate primitive shapes identified as equations’ solutions for arcs in different geometries in plane and in space are presented. Connection between geometric invariance, motion smoothness, compositionality and performance of the compromised motor control system is proposed within single invariance-smoothness framework. The derived class of differential equations is a novel tool for discovering candidates for geometric movement primitives.

Keywords Geometric primitives · Invariance · Compact representation · Smoothness · Parabola · Logarithmic spiral

1 Introduction

Various neuroscientific studies have analyzed geometric features of primates’ and humans’ drawing-like movements and their representation in the brain. In particular, single neurons and neural populations in motor cortex were found to be tuned to movement direction (Georgopoulos et al. 1982; Schwartz 1992, 1994; Moran and Schwartz 1999a, b).

In addition to continuous representation of movements reported in studies with humans and animals, the idea of representing complex movements based on compositionality of a limited “alphabet” of primitive components is analyzed in numerous works. Existence of motor primitives was demonstrated at the levels of forces produced by muscles operating on the limbs (Bizzi et al. 1991; Giszter et al. 1993; Giszter and Hart 2013), of muscle synergies (Tresch et al. 1999; d’Avella et al. 2003; Hart and Giszter 2004; Iavnenko et al. 2004), of motion kinematics (Morasso and Mussa-Ivaldi 1982; Flash and Henis 1991; Krebs et al. 1999; Rohrer and Hogan 2003; Flash and Hochner 2005) of units of computation in the sensorimotor system (van Zuylen et al. 1988; Thoroughman and Shadmehr 2000), and as a vector cross product between a limb-segment position and a velocity or acceleration (Tanaka and Sejnowski 2015). It was proposed that movement primitive is an action of a neuromuscular system controlling automatic synergy whose elements produce stereotypical and repeatable results (Woch and Plamondon 2010; Woch et al. 2011). Decomposition of complex movements into primi-
The 2/3 power-law model was interpreted as evidence for movement segmentation as it establishes a roughly piecewise constant relationship between movement’s speed and curvature (Lacquaniti et al. 1983):

\[
K = \text{Speed} \cdot \text{Curvature}^{1/3} \\
= \text{Angular speed} \cdot \text{Curvature}^{-2/3} \approx \text{const.} 
\] (1)

The 2/3 power law was demonstrated in the studies of visual perception (Viviani and Stucchi 1992; Levit-Binnun et al. 2006; Dayan et al. 2007; Casile et al. 2010; Meirovitch et al. 2015), locomotion (Vieilledent et al. 2001; Iavnenko et al. 2002) and imagined movements (Karklinsky and Flash 2015). Segmentation of hand movements based on powers of trajectory curvature was analyzed (Endres et al. 2013), and mixture of power laws in complex trajectories was reported (Huh and Sejnowski 2015).

Hogan (1984) and Flash and Hogan (1985) reported that planar hand trajectories are smooth and quantified non-smoothness with the cost functional called jerk leading to the model named “minimum jerk”. Further studies focused on commonalities in predictions of (1) the local geometric constraint formalized by the 2/3 power law and (2) global smoothness requirement of the minimum-jerk model by comparing predictions of the two models for some geometric shapes (Viviani and Flash 1995), comparing the two models when minimum-jerk trajectory is constrained by the entire path and not only pointwise (Todorov and Jordan 1998), and analyzing relationship between the 2/3 power law and smoothness of arbitrary degrees using approximation to a number of curves (Richardson and Flash 2002).

Pioneering works (Pollick and Sapiro 1997; Handzel and Flash 1999) reported equivalence of the 2/3 power law to moving with constant equi-affine velocity and proposed relevance of equi-affine geometry1 to the mechanisms of biological movements. Later studies supported presence of features characterizing equi-affine invariants in empirical data recorded during production and perception of biological motion (Polyakov 2001; Polyakov et al. 2001; Polyakov 2006; Polyakov et al. 2009a,b; Maoz et al. 2009; Pollick et al. 2009; Maoz and Flash 2014), while Bennequin et al. (2009), Fuchs (2010) and Pham and Bennequin (2012) reported relevance of a more restricted, affine, invariance to the field of motor control.

Aiming to find paths that make local geometric constraint equivalent to smoothness maximization (Polyakov 2001, 2006; Polyakov et al. 2009b) derived differential equation satisfied by the curves along which the minimum-jerk trajectories accumulate path’s measurement with constant rate and proposed solutions of the equation as candidate movement primitives. Correspondingly, solutions of the equation parameterized with the equi-affine arc were proposed as candidates for providing identical predictions to the minimum-jerk and the 2/3 power-law models; parabolic paths satisfy this property.

Moreover, any parabolic segment can be mapped to another arbitrary parabolic segment by unique affine transformation such that segments’ initial (and final) points are matched. So, provided a direction of motion, a sequence of concatenated parabolic-like shapes can be obtained by applying a unique sequence of affine transformations to a single parabolic template with prescribed starting point and thus simplifying the representation of complex movements in the brain. Such representation implies that geometric movement primitive is a set of transformations endowed with a primitive geometric shape upon which the transformations are applied. Simulated pattern composed of parabolic segments obtained by applying a sequence of affine transformations to a single parabolic segment resembled actual movement path performed by monkey (Polyakov et al. 2009b).

Parabolic shapes were suggested as geometric building blocks of well-practiced drawing-like movements (Polyakov 2001; Polyakov et al. 2001; Polyakov 2006; Polyakov et al. 2009a,b). During practice, monkey scribbling movements became composed of relatively long parabolic-like fragments belonging to a small number of clusters and being represented in motor cortical activity synchronized to neural states (Polyakov 2006; Polyakov et al. 2009a,b). For humans, over the course of practice, sequences of nearly straight point-

---

1 The book Shirokov and Shirokov (1959) provides the most comprehensive treatise on affine differential geometry that I am aware of, see also Guggenheimer (1977). All necessary definitions and formulae are provided further in text.
to-point drawing movements got coarticulated into smooth movements approximated well with minimum-jerk trajectories passing through a single via-point (Sosnik et al. 2004). Such trajectories are parabolic-like (Polyakov 2006; Shpigelmacher 2006; Polyakov et al. 2009b). Therefore practice-induced convergence of non-smooth trajectories into smooth, parabolic-like movements that allow compact representation is natural for monkeys and humans.

Movement path enriched with temporal rule of its drawing defines movement trajectory. Earlier attempts to suggest models for duration of a movement or of its part were based on optimization principles, e.g., Hogan (1984), Flash and Hogan (1985), Uno et al. (1989), Harris and Wolpert (1998), Todorov and Jordan (1998) and Tanaka et al. (2006), Bennequin et al. (2009) and Fuchs (2010) proposed that movement timing and invariance arise from mixture of arcs in different geometries. This theory accounted well for the kinematic and temporal features of a number of repeatable drawing and locomotion movements and proposed that the equi-affine geometry was the most dominant, affine geometry second most important during drawing and Euclidian second most important during locomotion.

On the level of brain activity underlying movements with different geometric characteristics, Harpaz et al. (2014) observed scale invariance in the neural representation of handwriting movements; superposition of uniform scaling and equi-affine transformations constitutes the group of affine transformations. At the same period, Sosnik et al. (2004) found that the level of activation in different motor areas (M1, PMd, pre SMA) is related to the level of motion smoothness acquired during learning to coarticulate point-to-point segments into complex smooth trajectories.

Here I extend existing mathematical tools aiming at identifying primitive shapes by merging the properties of movements’ compositionality, smoothness, geometric invariance and compact representation. In particular, I derive a novel class of differential equations for identifying trajectories obeying criteria of both (1) maximal smoothness of arbitrary degree and (2) geometric invariance for conservation of accumulated arc and use the equations to find exact functional expression for candidate primitive curves. Trajectories along those primitive shapes would be smooth and have arc invariant under classes of geometric transformations, thus leading to compact representation of biologically plausible complex movement paths in the brain. Given ideas of employing multiple geometric arcs for representing movements (Bennequin et al. 2009; Fuchs 2010), the proposed method is applied here to derive candidate primitives corresponding to arcs invariant in different geometries. Some candidates are reported for the first time, while several other candidates were reported in earlier works that considered the level of smoothness implied by minimizing jerk. The reader can further apply this ready to use machinery on his own for arbitrary levels of smoothness and path’s measurements.

1.1 Prerequisites for the mathematical problem from the motor control studies

From now on, differentiation with respect to parameter $\sigma$ that usually stands here for a geometric measurement along a path is denoted with primes and numbers in brackets, while differentiation with respect to $t$ (that usually stands here for time or an arbitrary parameter) up to order 3 is denoted with dots:

$$f'(\sigma) \equiv \frac{df}{d\sigma}, \quad f''(\sigma) \equiv \frac{d^2f}{d\sigma^2}, \quad f'''(\sigma) \equiv \frac{d^3f}{d\sigma^3},$$

$$f^{(k)}(\sigma) \equiv \frac{d^kf}{d\sigma^k};$$

$$\dot{f}(t) \equiv \frac{df}{dt}, \quad \ddot{f}(t) \equiv \frac{d^2f}{dt^2}, \quad \dddot{f}(t) \equiv \frac{d^3f}{dt^3}.$$

Trajectory’s smoothness criterion was initially defined as minimization of the integrated squared rate of change of acceleration called also movement jerk (Hogan 1984; Flash and Hogan 1985):

$$\int_0^T \left( \dddot{x}^2 + \dddot{y}^2 \right) dt. \quad (2)$$

The information about movement’s trajectory can be split into two parts: (1) geometric specification or movement path and (2) temporal specification relating each moment of time to location on the movement path and fully determined by the speed of motion along the path. Minimum-jerk trajectory is geometrically constrained by its initial, final and via-points, so its path can be revealed simultaneously with identifying movement speed.2

According to the constrained minimum-jerk model, hand movements tend to minimize the jerk cost

$$\int_0^T \left\{ \left( \frac{d^3x}{dr^3}(\sigma_{eu}(t)) \right)^2 + \left( \frac{d^3y}{dr^3}(\sigma_{eu}(t)) \right)^2 + \left( \frac{d^3z}{dr^3}(\sigma_{eu}(t)) \right)^2 \right\} dt. \quad (3)$$

2 Coordinates of the minimum-jerk trajectories constrained by via-points are composed of fifth-order polynomials with respect to time, the third-order derivatives of $x(t), y(t)$ are continuous (Flash and Hogan 1985). Minimum-jerk trajectories with a single via-point satisfy isochrony principle stating that different movement portions have nearly the same duration independently of their extent (Viviani and Terzuolo 1982; Bennequin et al. 2009). Movement durations from the start to the via-point and from the via-point to the end point are very similar (Polyakov 2006; Shpigelmacher 2006; Polyakov et al. 2009b),
for the prescribed trajectory path \( \{x(\sigma_{ea}), y(\sigma_{ea}), z(\sigma_{ea})\} \) (Todorov and Jordan 1998). Only speed profile \( \sigma_{ea} \) remains modifiable for minimizing (3). Executed three-dimensional curve in the cost functional (3) is parameterized with Euclidian arc-length

\[
\sigma_{ea}(t) = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, d\tau. \tag{4}
\]

Euclidian speed \( \dot{\sigma}_{ea} \) of planar trajectories minimizing the cost functional with an arbitrary order of smoothness \( n \)

\[
\int_0^T \left\{ \left[ \frac{d^n x(t)}{dt^n} \right]^2 + \left[ \frac{d^n y(t)}{dt^n} \right]^2 \right\} \, dt \tag{5}
\]

was compared to the experimental data for point-to-point movements\(^3\) (Richardson and Flash 2002). Approximated predictions of movement speed that minimizes the cost functional (5) along a number of periodic paths were derived and compared to the predictions of the 2/3 power law and experimental data (Richardson and Flash 2002).

The 2/3 power law is equivalent to piecewise constancy of the equi-affine velocity of drawing movements as \( \dot{\sigma}_{ea} = K \) for \( K \) from (1) (Polllick and Sapiro 1997; Handzel and Flash 1999). Such equivalence was extrapolated to the 1/6 power law equivalent to the conservation of the spatial equi-affine velocity whose empirical validity was studied for both action and perception (Maoz 2007; Pollick et al. 2009; Maoz et al. 2009; Maoz and Flash 2014).

In case of the constrained minimum-jerk model, the problem of finding paths whose maximally smooth trajectories accumulate path’s measurement with constant rate was reduced to the following necessary condition (Polyakov 2001, 2006; Polyakov et al. 2009b)

\[
r'' - 2r'' \cdot r^{(4)} + 2r' \cdot r^{(5)} = \text{const} \tag{6}
\]

being the first integral of an equivalent but shorter equation

\[
r' \cdot r^{(6)} = 0. \tag{7}
\]

For curves in plane, Eq. (7) has the following form:

\[
x' x^{(6)} + y' y^{(6)} = 0. \tag{8}
\]

Equation (6) was derived using notation of the planar equi-affine arc for \( \sigma \) while derivation is applicable as is for an arbitrary feasible (strictly monotonic) parametrization of the path, that is, when:

\[\frac{d\sigma(t)}{dt} \neq 0, \quad 0 \leq t \leq T. \tag{9}\]

Polyakov et al. (2001), Polyakov (2006) and Polyakov et al. (2009b) used Eqs. (6) and (7) to look for candidate primitive shapes whose invariant measurement was equi-affine arc in plane and in space.

Any curve can be geometrically parameterized in infinitely many different ways. For example, a curve without inflection points can be parameterized with the integral of Euclidian speed weighted with Euclidian curvature raised to power \( \beta \):

\[
\bar{\sigma}_\beta(t) = \int_0^t \dot{\sigma}_{ea}(\tau) \cdot [\sigma_{ea}(\sigma_{ea}(\tau))]^\beta \, d\tau;
\]

\( \bar{\sigma}_\beta \) is invariant under Euclidian transformations and measures path’s equi-affine arc when \( \beta = 1/3 \) and path’s length when \( \beta = 0 \).

In order to insiu that solutions of Eq. (8) are parameterized with planar equi-affine arc, Eq. (8) was endowed with additional condition:

\[x' y'' - x'' y' = 1; \tag{10}\]

corresponding constraint was also used for three-dimensional version of (6) and spatial equi-affine arc. So necessary condition for vector functions to describe planar paths along whose maximally smooth trajectories satisfy the 2/3 power-law model was obtained in the form of system of Eqs. (7) and (10).

\[\int_0^T \left\{ \left[ \frac{d^n x(t)}{dt^n} \right]^2 + \left[ \frac{d^n y(t)}{dt^n} \right]^2 \right\} \, dt = 0, \quad 0 \leq t \leq T. \tag{11}\]

\(^3\) The model predicts straight point-to-point trajectories when boundary conditions (velocity and higher-order derivatives at start and end points) are parallel to the point-to-point straight line including the case when the derivatives are zero.
I call the order of derivative $n$ in the cost functional (11) degree of smoothness or order of smoothness.

Movement paths whose optimal trajectories accumulate measurement with constant rate. Introduce a system of two differential equations:

\[
\begin{align*}
\| \mathbf{r}^{(n)} \|^2 + 2 \sum_{i=1}^{n-1} (-1)^i \left( \mathbf{r}^{(n-i)} \cdot \mathbf{r}^{(n+i)} \right) &= \text{const.} \\
\dot{\sigma} (t)|_{\sigma=\sigma(t)} &= 1,
\end{align*}
\]

(12)

differentiation in the upper equation is implemented with respect to $\sigma$, dot between two vectors denotes their scalar product. Differentiation of the upper equation in (12) turns it into equivalent condition of orthogonality of the first- and 2nth-order derivatives:

\[
\begin{align*}
\mathbf{r}' \cdot \mathbf{r}^{(2n)} &= 0 \\
\dot{\sigma} (t)|_{\sigma=\sigma(t)} &= 1,
\end{align*}
\]

(13)

so every result related to either of the two systems (12), (13) is equivalently relevant for the other.

I propose systems (12), (13) as a tool for identifying candidates for geometric movement primitives because the systems are satisfied by the paths whose optimal trajectories with degree of smoothness $n$ accumulate measurement $\sigma(t)$ with constant rate. In other words, the systems are satisfied by the paths $\mathbf{r}(\sigma)$ allowing

\[
\text{arg min}_{\sigma(t)} J_\sigma (\mathbf{r}(\sigma), n) = \text{const.}, \ t \in [0, T].
\]

(14)

The upper equations in (12), (13) are derived in Supplementary Files 1, 2 using Euler–Poisson equation for variational problems. Particular cases of the upper differential equation for planar curves when $2 \leq n \leq 4$ and for an arbitrary $n$ are provided\textsuperscript{4} in Table 1.

Solutions of the system (13) are invariant (remain solutions) under similarity transformations (see Table 2) whenever this orthogonality-preserving transformation preserves also constancy of $\dot{\sigma}$.$^5$ Maximally smooth trajectories in general loose optimality under non-orthogonal transformations, e.g., arbitrary affine; however for some shapes, like parabolas, maximal smoothness is non-orthogonally invariant (Polyakov 2001). Particular cases of the system (13) for different geometric parameterizations and known solutions are demonstrated further in text, in Table 3 and in Supplementary File 3.$^6$

The lower equation in (12), (13) guarantees that $\dot{\sigma}(t)$ is constant for an optimal trajectory along a path satisfying parameter-independent upper equation; this implies two equivalent one to the other necessary conditions for $n$th order maximal smoothness when a trajectory is known while measurement $\sigma$ is not under consideration:

\[
\begin{align*}
\left\| \frac{d^n r}{d\tau^n} \right\|^2 + 2 \sum_{i=1}^{n-1} (-1)^i \left( \frac{d^{n-i} r}{d\tau^{n-i}} \cdot \frac{d^{n+i} r}{d\tau^{n+i}} \right) &= \text{const.}, \\
\frac{dr}{d\tau} \cdot \frac{d^2 r}{d\tau^2} &= 0.
\end{align*}
\]

(15)

(16)

The process of identifying candidates for geometric primitives based on proposed systems (12), (13) may follow either of the four approaches

1. Specify desirable parametrization $\sigma$ and a class of curves $\{r(t)\}$. Find elements of $\{r(t)\}$ that satisfy (12) (and (13)) when parameterized with desired $\sigma$. Usually curves are parameterized by polar angle, length, one of the coordinates (e.g., $y = y(x)$ for a plane curve), etc, and not necessarily by the required measurement. Assuming that (9) is satisfied, all derivatives of $r(t(\sigma))$ with respect to $\sigma$ in (13) can be computed recursively:

\[
\begin{align*}
\mathbf{r}'(\sigma)|_{\sigma=\sigma(t)} &= \frac{r(t)}{\dot{\sigma}}, \quad \mathbf{r}''(\sigma)|_{\sigma=\sigma(t)} &= \frac{\frac{d}{d\tau} \left( \frac{r(t)}{\dot{\sigma}} \right)}{\dot{\sigma}}, \ldots
\end{align*}
\]

(17)

instead of building explicit relationships $r(\sigma)$. In such a way, the constraint formalized by the lower equation of (12) can be easily plugged into the upper equation. For example, this approach is appropriate for a class of logarithmic spirals.$^7$

---

\textsuperscript{4} Different motor control studies, e.g., Hogan (1984), Flash and Hogan (1985), Viviani and Flash (1995), Todorov and Jordan (1998), Polyakov (2001), Richardson and Flash (2002), Ben-Itzkah and Karniel (2008), Polyakov et al. (2009b), used cost functionals $J_\sigma (\mathbf{r}(\sigma), n)$ for the planar ($L = 2$) and spatial ($L = 3$) curves with orders of smoothness $n$ equal to 2–4.

\textsuperscript{5} Constancy of $\dot{\sigma}$ is preserved under similarity transformations for all planar measurements $\sigma$ mentioned in Table 2 besides equi-center-affine and center-affine arcs whose speed of accumulation looses constancy under translations.

\textsuperscript{6} The last version of Polyakov (2014) contains exposition of this work in a single text.

\textsuperscript{7} Say, constancy of $\dot{\sigma}_{\text{eu}}$ is required for $n = 3$. Then the following recursive computation in Mathematica implies a specific logarithmic spiral (31) with $\beta = \pm 1/\sqrt{5}$:

\begin{align*}
\text{SetAttributes}[\beta, \text{Constant}];
\text{x[}\phi_\text{]} &= e^{\phi \beta} \cos(\phi); \\
\text{y[}\phi_\text{]} &= e^{\phi \beta} \sin(\phi); \\
\text{dx[}\phi_\text{]} &= \text{Dt}[x[\phi]]; \\
\text{dy[}\phi_\text{]} &= \text{Dt}[y[\phi]]; \\
\text{dx\_d[}\phi_\text{]} &= \text{Dt}[\text{dx}[\phi]]; \\
\text{dy\_d[}\phi_\text{]} &= \text{Dt}[\text{dy}[\phi]]; \\
\text{dx\_d\_d[}\phi_\text{]} &= \text{Dt}[\text{dx\_d}[\phi]]; \\
\text{dy\_d\_d[}\phi_\text{]} &= \text{Dt}[\text{dy\_d}[\phi]]; \\
\text{FullSimplify}[\text{dx}[\phi] \cdot \text{dy}[\phi] + \text{dx}[\phi] \cdot \text{dy}[\phi] + \text{dy}[\phi] \cdot \text{dy}[\phi]] &= \\
&= 10 \beta (1 + \beta^2) (1 + 5 \beta^2) \left( e^{2 \phi \beta} \right)^{1/2}.
\end{align*}
between two vectors denotes their scalar product. Details about planar solutions mentioned in Table 1 are provided in Table 3.

### Table 1

| Order | Equation, exemplar or general case | Derivative of the equation | Comments about equation | Known solutions when \( \sigma \) is equi-affine arc |
|-------|-----------------------------------|----------------------------|------------------------|-----------------------------------------------------|
| 2     | \( x'^2 + y'^2 - 2x'y'(3) - 2y'x'(3) = \) const | \( x'(4) + y'(4) = 0 \) | Planar “minimum-acceleration” criterion in motor control | 2D: Parabolas, circles, logarithmic spirals |
| 3     | \( x''^2 + y''^2 - 2x''y''(4) - 2y''x''(4) + 2x''x''(5) + 2y''y''(5) = \) const | \( x'(6) + y'(6) = 0 \) | Planar “minimum-jerk” criterion in motor control | |
| 4     | \( (x'^2 + y'^2) - 2x''y''(5) + 2y''x''(5) + 2x''x''(6) + 2y''y''(6) - 2x''x''(7) - 2y''y''(7) = \) const | \( x'(8) + y'(8) = 0 \) | Planar “minimum-snap” criterion in motor control | |
| \( \cdots \) | \( \cdots \) | \( r' \cdot r(2n) = 0 \) | \( L \)-dimensional space | For \( L < 2n - 1 \), \( r = [x_1, \ldots, x_L] \) s.t. \( x_k(\sigma) = \sigma^k \), \( \sigma \) is equi-affine arc in \( L \)-dimensional space |

Upper equation in the systems (12), (13). Prime and order of differentiation in the brackets correspond to the derivative with respect to \( \sigma \). Dot between two vectors denotes their scalar product. Details about planar solutions mentioned in Table 1 are provided in Table 3.

2. Specify desirable parametrization \( \sigma \) and solve (13) to identify novel candidate curves or classes of curves along which optimal trajectories conserve time derivative of prescribed measurement \( \sigma \). Here the lower equation of the system is used as is to guarantee that the solution of the parameter-independent upper equation indeed represents a curve with required parametrization meaning that conserved measurement possesses required geometric invariance. For example, parabolas satisfy the upper equation for parameterization with equi-affine arc and do not satisfy when parameterized with Euclidian arc.

3. Specify desirable parametrization \( \tilde{\sigma} \) and guess novel candidate curves among solutions of (16) for geometrically unspecified \( \sigma(t) = \text{const} \cdot t \). For example, such solutions as polynomials of degree \( \leq 2n - 1 \) with respect to \( t \) or certain periodic functions can be easily guessed. Then among the solutions determine those (if any) consistent with desired parametrization \( \tilde{\sigma} \) by testing constancy of curves’ \( \tilde{\sigma} \). For example, polynomial solution \( x(t) = 8t \), \( y(t) = t^2/2 \) of (16) describes parabola whose \( \tilde{\sigma}_{eu} = 2 \) while \( \tilde{\sigma}_{eu} \neq \text{const} \); the solution will be among desired candidate primitive curves for \( \tilde{\sigma} = \sigma_{eu} \) but not for \( \tilde{\sigma} = \sigma_{eu} \).

4. Do not request any specific parametrization of candidate curves and so neglect the geometric meaning of \( \sigma \) by setting directly \( \sigma(t) = \text{const} \cdot t \) and making the condition formalized by the lower equation of (12) unnecessary. Solve Eq. (16) and possibly look for geometric meaning of \( t \) of the solution. If, for example, \( \sqrt{x^2 + y^2} = \text{const} \), then \( t \) is proportional to Euclidian length measured along the curve defined by the solution.

In earlier studies (Polyakov 2001, 2006; Bright 2006; Polyakov et al. 2009b), known candidate curves were found using educated guess in Approaches 1–3. Approach 4 is used in the present work to identify novel candidate curves (53), (54) without prior knowledge about geometric meaning of the parameter \( t \) making these curves solutions of (16); that \( t \) appears to be polar angle.

There is actually a duality between geometric and temporal properties of trajectories satisfying optimality criterion (14). (1) One may decide about desirable geometric measurement \( \sigma \) and look for the paths along which optimal trajectories have constant \( \tilde{\sigma} \); in other words, geometry would guide temporal properties of such trajectories. (2) One may take trajectories satisfying Eq. (16) and find geometric meaning for parameterizations \( \sigma = \text{const} \cdot t \); in other words, optimal tempo for paths would guide the choice of corresponding appropriate geometric measurements and types of invariance.

**Optimal trajectories for a given path** Let some path \( r(s) \) be given. (I) Consider the problem of finding trajectory \( r(s(t)) \) that minimizes (11) along the known path. As in case of the constrained minimum-jerk model, optimal \( s(t) \) should be found, but now with degree of smoothness \( n \). (II) One may also wish to test whether a given trajectory is optimal in the sense of \( n \)-th order smoothness. Equations (15), (16) are necessarily satisfied for solutions of the problems (I), (II) because path’s parameterization \( \sigma(t) = \text{const} \cdot t : \text{const} \neq 0 \) satisfies (9) and is feasible. Concerning optimization problem (I) apply differentiation of a composite function in (16):
Geometric transformations for seven geometries and speeds of accumulating their arcs are summarized. Application of the system (13) to measurement with Fourier series (Richardson and Flash 2002) may provide a more efficient way to describing a numerical solution.

A recent study of the minimum-jerk trajectories in plane (Meirovitch 2014) used path as holonomic constraint for variational problem (van Brunt 2004) and optimized $x(t)$, $y(t)$ instead of $\dot{x}(t)$ by neglecting geometric specification of $\sigma$ and setting directly $\sigma(t) = t$; such formulation led to an alternative, shorter, proof showing that Eq. (8) is necessarily satisfied by a planar trajectory that minimizes jerk cost along its path. That elegant derivation can actually be extended to the case of curves in three-dimensional space and in space with arbitrary number of dimensions $L$.8

8 Define a curve in $L$-dimensional space as the system $F_k(x_1, \ldots, x_L) = 0, k = 1, \ldots, L - 1$ representing intersection of $L - 1$ hypersurfaces of dimension $L - 1$. Correspondingly, the system of Euler–Poisson (E–P) equations (Gelfand and Fomin 1961) (called also Euler-Lagrange) with $L - 1$ Lagrange multipliers: $(E-P) \sum_{k=1}^{L-1} \lambda_k(t) F_k = 0$ will lead to the system $(-1)^L \triangle x_{L-1} / \triangle t^2 + \sum_{k=1}^{L-1} \lambda_k(t) F_k / \triangle t x_{L-1} = -d x_{L-1} / \triangle t^2 + \sum_{k=1}^{L-1} \lambda_k(t) F_k / \triangle t x_{L-1} = 0, \triangle t = 1, \ldots, L$, implying that at every point $t(t)$ of the curve the vector $dr^6 / \triangle t^2$ belongs to the hyperplane spanned by $L - 1$ gradients (normals) to corresponding hypersurfaces $F_k$ and thus vector $dr^6 / \triangle t^2$ is orthogonal to the vector $\dot{r}$ parallel to curve’s tangent; note that the path is parameterized here by $\sigma(t) = t$. Identical derivation for $n$th-order smoothness for a trajectory in $L$-dimensional space leads to Eq. (16).
Table 3  Known solutions and their invariance for different degrees of smoothness

| Object                  | Invariance for solutions of (13) | $n$  | Equi-affine arc | Affine, similarity; center-affine when circle and log spiral are centered at the origin | Equi-center-affine arc | Euclidian arc | Polar angle |
|-------------------------|----------------------------------|------|-----------------|--------------------------------------------------------------------------------------|------------------------|---------------|-------------|
| Parabola                | Affine                           | ≥ 2  | Any             | None                                                                                  | None                   | None          | None        |
| Straight line           | Affine                           | ≥ 1  | Not relevant (zero arc) | Not relevant (arc zero or not defined)                                                      | Not relevant (zero arc) | Any           | Not relevant (zero) |
| Circle                  | Similarity and reflection        | ≥ 1  | Any             | Any                                                                                  | Any                    | Any           | Any         |
| Logarithmic spiral (31) | Similarity and reflection        | 2    | None            | $\beta = \pm \sqrt{3}$                                                                    | $\beta = \pm \sqrt{0.6}$ | None          | Same        |
|                         |                                  | 3    | $\beta = \pm 3\sqrt{3}$ | $\pm \sqrt{5 \pm 2\sqrt{5}}$                                                            | $189\beta^4 - 190\beta^2 = -5$ | $\beta = \pm \frac{1}{\sqrt{3}}$ |
|                         |                                  | 4    | $\pm \sqrt{\frac{13+4\sqrt{7}}{35}}$ | $\beta^6 - 21\beta^4 + 35\beta^2 = 7$                                                   | $19305\beta^6 - 25333\beta^4 + 1435\beta^2 = 7$ | $84\beta^4 - 35\beta^2 = -1$ as |
|                         |                                  | 5    | $\pm \sqrt{\frac{9}{17} \pm \sqrt{\frac{499+12\sqrt{707}}{215}}}$ | $\beta^6 - 33\beta^4 + 27\beta^2 = 3$                                                   | $1276275\beta^8 - 1992932\beta^6 + 169442\beta^4 - 1988\beta^2 = -3$ | $3044\beta^6 - 1869\beta^4 + 126\beta^2 = 1$ affine |
| “cosh spiral” (42), “sinh spiral” (43) | Similarity and reflection        | None | None            | None                                                                                  | None                   | None          | $\beta$ the same as for logarithmic spiral |

Known solutions of the systems (12) and (13) in plane for different orders of trajectory smoothness $n$ and geometric parameterizations invariant in affine group and five of its subgroups. Invariance of the class of curves and values of $\beta$ for logarithmic spirals are indicated. Curves parameterized with center-affine and equi-center-affine arcs and with polar angle are considered centered at the origin by parallel translation. Results for Euclidian, similarity, equi-center-affine and center-affine arcs and for the polar angle are derived in Supporting File 3.
2.2 Different parameterizations, arcs in the geometries of affine group in plane and some of its subgroups, equations and solutions

Different kinds of invariance were analyzed in the studies of action and perception of motion. For example, point-to-point hand movements are assumed to produce nearly straight paths. Straight trajectories parameterized with Euclidian arc (4) and having constant Euclidian velocity paths. Straight trajectories parameterized with Euclidian point hand movements are assumed to produce nearly straight of action and perception of motion. For example, point-to-point different kinds of invariance were analyzed in the studies (Shirokov and Shirokov 1959):

\[
\sigma_{ea}(\sigma_{ea}) = x''(\sigma_{ea})y'''(\sigma_{ea}) - x'''(\sigma_{ea})y''(\sigma_{ea})
\]

Equi-affine group Equi-affine transformations of coordinates

\[
x_1 = \alpha x + \beta y + a, \quad \gamma x + \delta y + b,
\]

involve five independent parameters. The rate of accumulating equi-affine arc called equi-affine velocity is computed as follows (Shirokov and Shirokov 1959):

\[
\dot{\sigma}_{ea} = \begin{vmatrix} \frac{\dot{x}}{x} & \frac{\dot{y}}{y} \end{vmatrix}^{1/3}
\]

System (13) for \( \sigma_{ea} \) becomes:

\[
\begin{aligned}
x''(x^{(2n)} + y^{(2n)}) &= 0 \\
x''y''' - x'''y'' &= 1.
\end{aligned}
\]

The lower equation guarantees that the solution of the upper equation is parameterized with the equi-affine arc

\[
\sigma_{ea} = \int_{0}^{t} \dot{\sigma}_{ea}(\tau) d\tau.
\]

Earlier studies used system (21) for the minimum-jerk cost functional (n = 3) (Polyakov 2001, 2006; Polyakov et al. 2009b). Here concrete curves are considered as candidate solutions; some of them are filtered out for the prescribed equi-affine parametrization (Approaches 1, 3).

The first- and the third-order derivatives of the position vector of a planar curve \( r(\sigma_{ea}) \) with respect to the equi-affine arc are parallel as \( x'y'' - x''y' = 1 \) whose differentiation implies \( x'y''' - y'x''' = 0 \). The equi-affine curvature of a curve (Shirokov and Shirokov 1959; Guggenheimer 1977; Calabi et al. 1996)

\[
\kappa_{ea}(\sigma_{ea}) = x''(\sigma_{ea})y'''(\sigma_{ea}) - x'''(\sigma_{ea})y''(\sigma_{ea})
\]

is a scaling factor between the first- and the third-order derivatives of the position vector: \( r''(\sigma_{ea}) + \kappa(\sigma_{ea})r'(\sigma_{ea}) = 0 \) making it possible to express higher-order derivatives of the vector \( r(\sigma_{ea}) \) in terms of its first- and second-order derivatives when equi-affine curvature is a known function of the equi-affine arc. In particular, system (21) with \( n = 3 \) can be rewritten. Given that

\[
\begin{aligned}
r''(\sigma_{ea}) &= -\kappa_{ea}(\sigma_{ea})r'(\sigma_{ea}), \\
r^{(4)}(\sigma_{ea}) &= r''(\sigma_{ea})(\kappa_{ea}^2(\sigma_{ea}) - 3\kappa_{ea}(\sigma_{ea})) \\
&+ r'(\sigma_{ea})(4\kappa_{ea}(\sigma_{ea})\kappa_{ea}(\sigma_{ea}) - \kappa_{ea}(\sigma_{ea})),
\end{aligned}
\]

and the upper equation of the system (21) for the case of \( n = 3 \) becomes

\[
\begin{aligned}
r'(\sigma_{ea}) \cdot r^{(6)}(\sigma_{ea}) &= (r'(\sigma_{ea}) \cdot r''(\sigma_{ea}))(\kappa_{ea}^2(\sigma_{ea}) - 3\kappa_{ea}(\sigma_{ea})) \\
&+ r'(\sigma_{ea})(4\kappa_{ea}(\sigma_{ea})\kappa_{ea}(\sigma_{ea}) - \kappa_{ea}(\sigma_{ea})) = 0.
\end{aligned}
\]

Noting that \( r'(\sigma_{ea}) \cdot r''(\sigma_{ea}) = \frac{1}{2} \left( r^2(\sigma_{ea}) \right)' \), if \( 3\kappa_{ea} \neq \kappa_{ea}^2 \), Eq. (25) implies that

\[
\frac{1}{2} \left( r^2(\sigma_{ea}) \right)' = r^2(\sigma_{ea}) \frac{4\kappa_{ea}(\sigma_{ea})\kappa_{ea}(\sigma_{ea}) - \kappa_{ea}(\sigma_{ea})}{3\kappa_{ea}^2(\sigma_{ea}) - \kappa_{ea}^2(\sigma_{ea})}.
\]

After integrating (26) the system (21) with \( n = 3 \) can be rewritten as follows:

\[
r^2(\sigma_{ea}) = r^2(0) \exp \left[ 2 \int_{0}^{\sigma_{ea}} \frac{4\kappa_{ea}(\sigma_{ea}) - \kappa_{ea}(\sigma_{ea})^2}{3\kappa_{ea}^2(\sigma_{ea}) - \kappa_{ea}^2(\sigma_{ea})} dS \right]
\]

if \( 3\kappa_{ea} \neq \kappa_{ea}^2 \).

The expression for the equi-affine curvature can be written for an arbitrary artument t (Shirokov and Shirokov 1959):
The results for candidate solutions (parabola, circle, logarithmic spiral) are as follows:

1. **Parabola** is parameterized with equi-affine arc, up to an equi-affine transformation, as follows:

   \[ x = \sigma_{ea} \]
   \[ y = \sigma_{ea}^2/2. \]

   So \( x \) and \( y \) coordinates of the parabola in equi-affine parametrization are polynomials of up to second degree with respect to \( \sigma_{ea} \). The class of parabolas constitutes obvious solution of (21) for \( n \geq 2 \); moreover, parabolas are the only solutions invariant under arbitrary affine transformations. Drawing parabolas with constant equi-affine velocity does minimize the cost functional (11) for \( n \geq 2 \) and provides zero cost. In plane only curves whose \( x \) and \( y \) coordinates are second-order polynomials of \( \sigma \) may have zero jerk cost. Corresponding results for \( n = 3 \) were reported in Polyakov (2001, 2006) and Polyakov et al. (2009b).

2. **Circle** is parameterized with equi-affine arc as follows:

   \[ x = x_0 + \kappa_{ea}^{-3/4} \cdot \cos(\sqrt{\kappa_{ea}} \sigma_{ea}) \]
   \[ y = y_0 + \kappa_{ea}^{-3/4} \cdot \sin(\sqrt{\kappa_{ea}} \sigma_{ea}). \]

   For an arbitrary \( n \geq 1 \) circle is non-invariant solution under arbitrary equi-affine transformations; however, circles are invariant solutions (remain circles) under similarity transformations (see Table 2).

3. **Logarithmic spiral** can be parameterized with polar angle:

   \[ x = x_0 + \text{const} \cdot e^{\beta \varphi} \cdot \cos \varphi \]
   \[ y = y_0 + \text{const} \cdot e^{\beta \varphi} \cdot \sin \varphi. \]

   Applying (20):

   \[ \text{d} \sigma_{ea}/\text{d} \varphi = (\text{const}^2(1 + \beta^2))^{1/3} \cdot e^{2\beta \varphi/3} \]

   and integrating with initial condition \( \sigma_{ea}(0) = 0 \) results in:

   \[ \varphi(\sigma_{ea}) = \frac{3}{2\beta} \ln \left( \frac{2\beta \sigma_{ea}}{3} \cdot (\text{const}^2(1 + \beta^2))^{-1/3} + 1 \right) \]

   that can be substituted into (31) to imply the expressions for \( x(\sigma_{ea}), y(\sigma_{ea}) \). Alternatively, one can plug the lower equation of (21) into the upper one just by applying the chain rule (17) to (31). The curve (31) is solution of (21) only for certain values of \( \beta \) depending on the degree of smoothness \( n \). The values of \( \beta \) corresponding to \( n \leq 5 \) are summarized in Table 3. For \( n = 3 \) see also Bright (2006), Polyakov et al. (2009b).

**Affine group** Planar affine transformations of coordinates

\[
\begin{align*}
    x_1 &= \alpha x + \beta y + a, \\
    y_1 &= \gamma x + \delta y + b, \\
    & \quad |\alpha \beta| \neq 0
\end{align*}
\]

involve six independent parameters. The absolute value of the speed of accumulating affine arc is computed as follows (Shirokov and Shirokov 1959):\(^{10}\)

\[
|\dot{\sigma}_a| = |\dot{\sigma}_{ea}| \sqrt{\kappa_{ea}}
\]

\[
= \sqrt{\frac{3\dot{\sigma}_{ea}^3 \cdot \ddot{x} d^4 x/dt^4}{9\kappa_{ea}^8} + 12\dot{\sigma}_a^3 \cdot \ddot{x} \ddot{y} d^4 y/dt^4 - 5 \dot{x} \dot{y} \dddot{x} \dddot{y}}
\]

where \( \kappa_{ea} \) is equi-affine curvature (23).

System (13) in plane becomes

\[
\begin{align*}
    x'x''(2n) + y'y''(2n) &= 0 \\
    3(x'y'' - x''y') &+ 12(x'y'' - x''y') \cdot \frac{|x' x''| y'' y''}{y' y''} \\
    -5 \frac{x' x''}{y' y''} &/ (9(x'y'' - x''y')^2) = 1
\end{align*}
\]

Here concrete curves are considered as candidate solutions; some of them are filtered out for the prescribed affine parametrization (Approaches 1, 3).

1. **Parabolas’** affine arc is zero, same as equi-affine arc of the straight line is zero or Euclidian length of a point is zero. Therefore testing whether parabolas are solutions of the system (36) is meaningless. Affine curvature (Shirokov and Shirokov 1959)

\[
\kappa_a = \kappa_{ea}^{-3/2} \cdot d \kappa_{ea}/d \sigma_{ea}
\]

of a parabola is not defined. Formula (37) implies that affine curvature is zero for conic sections with nonzero

\(^{10}\) Formula from Shirokov and Shirokov (1959) corresponding to (35) contains misprint corrected here.
equi-affine curvature, that is, for ellipses and hyperbolas, and only for them.

2. **Circle.** Noting from (34) that affine arc is integrated square root of equi-affine curvature (23) \( (\sigma_a = \int \sqrt{\kappa_{ea}}(\sigma_a)) \) and that equi-affine curvature of a circle is positive constant one immediately obtains for a circle: \( \sigma_{ea} = \kappa_{ea}^{-1/2} \sigma_a \). Substituting into (30) one gets:

\[
\begin{align*}
x &= x_0 + \kappa_{ea}^{-3/4} \cdot \cos(\sigma_a) \\
y &= y_0 + \kappa_{ea}^{-3/4} \cdot \sin(\sigma_a)
\end{align*}
\]  

(38)

which is solution of the system (36). Circles constitute non-invariant solutions under arbitrary affine transformation for arbitrary \( n \geq 1 \). The class of circles is invariant under similarity transformations (see Table 2 or Supplementary File 3).

3. **Logarithmic spiral.** The speed of accumulating affine arc of the logarithmic spiral (31) with respect to changing polar angle \( \varphi \) is the following constant (Bright 2006):

\[
d\varphi_a/d\varphi = \frac{\sqrt{9 + \beta^2}}{3} .
\]

After setting \( \varphi(0) = 0 \),

\[
\sigma_a(\varphi) = \frac{\sqrt{9 + \beta^2}}{3} \varphi .
\]  

(39)

Expression (31) becomes

\[
\begin{align*}
x(\sigma_a) &= x_0 + \text{const} \cdot \exp\left(\beta \frac{3}{\sqrt{9 + \beta^2}} \sigma_a\right) \\
y(\sigma_a) &= y_0 + \text{const} \cdot \exp\left(\beta \frac{3}{\sqrt{9 + \beta^2}} \sigma_a\right)
\end{align*}
\]

(40)

The values of \( \beta \) that make logarithmic spirals solutions of the system (36) depend on the degree of smoothness \( n \). Values of \( \beta(n) \) when \( 1 \leq n \leq 5 \) are summarized in Table 3. For \( n = 3 \) see also Bright (2006); Meirovitch (2014).

Represent a logarithmic spiral (31) centered at the origin\(^{11} \) in polar coordinates: \( \rho = \text{const} \cdot e^{\rho \varphi} \). Equations (31) (with \( x_0 = y_0 = 0 \)) and (32) imply that

\[
\sigma_{ea} = \frac{3/(2\beta)}{(1 + \beta^2)^{1/3}} \left( \rho^{2/3} - \text{const}^{2/3} \right) .
\]  

(41)

Without loss of generality, the arcs in (39), (41) are measured from the point corresponding to \( \varphi = 0 \). Connection between polar coordinates and equi-affine and affine arcs for logarithmic spiral can be used, for example, for approximating numerically equi-affine and affine arcs of empirical drawings if their elements can be fit well with segments of a logarithmic spiral.

**Novel spiral solutions** I call “cosh spiral”, “sinh spiral” the following curves

\[
\begin{align*}
x(t) &= x_0 + \text{const} \cdot \cos(t) \cosh(\beta t) \\
y(t) &= y_0 + \text{const} \cdot \sin(t) \cosh(\beta t) ,
\end{align*}
\]  

(42)

\[
\begin{align*}
x(t) &= x_0 + \text{const} \cdot \cos(t) \sinh(\beta t) \\
y(t) &= y_0 + \text{const} \cdot \sin(t) \sinh(\beta t) .
\end{align*}
\]  

(43)

The coordinates of both curves are equal to translated sum/difference of coordinates of two logarithmic spirals with the same const and origin and whose \( \beta \) have opposite signs. Curves (42), (43) satisfy Eq. (16) when values of \( \beta(n) \) for \( n \leq 5 \) are the same as in the case of logarithmic spiral parameterized with affine arc, as summarized in Table 3. Geometric meaning of \( t \) can be easily identified: \( t \) is polar angle.

Neither of the curves (42), (43) satisfies system (12) when parameterized by any of six planar arcs analyzed in this study and summarized in Table 2. The curves in (42), (43) are invariant solutions of (13) under similarity transformations (see Table 2). The “cosh”, “sinh” and logarithmic spirals, all three with the same value of \( \beta = \sqrt{5} - \sqrt{3} (\in \{\beta(3)\}) \) for affine parametrization of logarithmic spiral) are depicted in Fig. 1.

**Summary for known planar solutions** The results for candidate planar solutions considered above are summarized in Table 3. Parabolas constitute affinely invariant solutions of the plane version of the system (12) for all degrees of smoothness above 1 when parameterized with equi-affine arc and are not solutions for any \( n \) for any other measurement considered. Measuring equi-affine arc along parabola has an interesting property. If tangent to a parabola at some point \( G \) is parallel to the chord connecting points \( F \) and \( H \) on the parabola, then the equi-affine arc measured along parabola from \( F \) to \( G \) is equal to the equi-affine arc measured from \( G \) to \( H \).\(^{12} \) Such chord and tangent are demonstrated in Fig. 2.

The main theorem of the equi-affine theory of plane curves states that “the natural equation \( \kappa_{ea} = f(\sigma_{ea}) \) defines a plane

---

\(^{11}\) Meaning \( x_0 = y_0 = 0 \). Equivalently, tangents to a logarithmic spiral centered at the origin have constant angle with the vectors connecting corresponding points on the curve to the origin.

\(^{12}\) Proven in Appendix D of Polyakov (2006).
Fig. 1 Three spirals. “cosh spiral” (42) (solid), “sinh spiral” (43) (dash dots) and logarithmic spiral (31) (dashed), all three have \( \beta = \sqrt{5} - 2\sqrt{5} \) that makes the curves solutions of the upper equation for parameterizations with polar angle (cosh and sinh spirals) and affine/center-affine/similarity arcs/polar angle (logarithmic spiral). Drawing is implemented counterclockwise, \( 0 \leq \theta \leq 4 \). The spirals in the plot are identified with their equations in polar coordinates.

\[ \rho(\theta) = \cosh(\beta \theta) \]
\[ \rho(\theta) = \sinh(\beta \theta) \]
\[ \rho(\theta) = e^{\beta \theta} \]

Fig. 2 Tangent to a parabola and parallel chord are connected with path’s portions having equal equi-affine arc. Chord \( FH \) of the depicted parabolic segment is parallel to the tangent at point \( G \). Equi-affine arc between \( F \) and \( G \) is equal to the equi-affine arc between \( G \) and \( H \). In the canonical coordinate system parabola is described by the relationship \( x = -y^2/(2p) \), where \( p \) is the focal parameter. If tangent is parallel to the chord then, in canonical coordinate system, \( y \) coordinate of the point of touch by the tangent \( (G) \) is average of the \( y \)-coordinates of the chord \( (FH) \).

Therefore, given two arbitrary parabolic segments with prescribed initial and final points, there exists a unique affine transformation \( (33) \) mapping one segment to the other\(^{13} \) in such a way that initial and final points are matched. When direction of drawing is not prescribed, there exist two affine transformations \( (33) \) mapping one parabolic segment to the other as demonstrated in Fig. 3. The determinants of the linear part of the two transformations have opposite signs and the same absolute value, say \( |w| \). Note also that \( |w| = |\sigma_{ea,2}/\sigma_{ea,1}|^{3/2} \), \( \sigma_{ea,i} \) is equi-affine arc \( (22) \) of the \( i \)th segment.

Parabolic segments connected into a sequence always have well identified initial and final points\(^{14} \). Therefore piecwise parabolic representation of complex patterns with sequences of affine transformations applied to a single (though arbitrary) parabolic template is unambiguous.

Decomposition of affine transformation \( (44) \) or similarity transformations\(^{15} \) into composition of two transformations, one of them scaling, may also take place in processing of spatial information in the brain. In particular, Sekuler and Nash (1972) demonstrated that a pair of mental transformations, size scaling and rotation, produced additive effects on reaction time, consistent with serial processing of these.

\(^{13}\) Already without requirement for equality of their equi-affine arcs as in case of equi-affine transformation.

\(^{14}\) Sequences of parabolic-like components revealed in monkey drawing movements (Polyakov et al. 2009a,b) were usually implemented with uncharged direction of motion—either counterclockwise or clockwise.

\(^{15}\) Planar similarity transformations are uniquely decomposable into Euclidian and scaling transformations in the same manner as affine in \( (44) \).
In these two geometries I consider circles and logarithmic spirals centered at the origin and straight lines crossing the origin.

### 2.3 Solutions in space for spatial curves parameterized with 3D equi-affine arc

Equi-affine transformations of coordinates in space involve 11 independent parameters and are of the form:

\[
\begin{align*}
    x_1 &= \alpha_{11} x + \alpha_{12} y + \alpha_{13} z + a, \\
    y_1 &= \alpha_{21} x + \alpha_{22} y + \alpha_{23} z + b, \\
    z_1 &= \alpha_{31} x + \alpha_{32} y + \alpha_{33} z + c,
\end{align*}
\]

\[
\begin{bmatrix}
    \alpha_{11} & \alpha_{12} & \alpha_{13} \\
    \alpha_{21} & \alpha_{22} & \alpha_{23} \\
    \alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix}
= 1. \tag{47}
\]

The speed of accumulating spatial equi-affine arc (Shirokov and Shirokov 1959)

\[
\dot{\sigma}_{ea3} = \begin{bmatrix}
    \dot{x} & \dot{x}' & \ldots & \dot{x}^{1/6} \\
    \dot{y} & \dot{y}' & \ldots & \dot{y}^{1/6} \\
    \dot{z} & \dot{z}' & \ldots & \dot{z}^{1/6}
\end{bmatrix}
\]

is called spatial equi-affine velocity. Pollick et al. (2009) and Maoz et al. (2009) proposed that three-dimensional movements conserve spatial equi-affine velocity calling the phenomenon the “1/6 power law”. The system (13) becomes

\[
\begin{align*}
    x'x^{(2n)} + y'y^{(2n)} + z'z^{(2n)} &= 0, \tag{49}
\end{align*}
\]

1. **Parabolic screw line** (see also Polyakov et al. 2009b) can be parameterized with spatial equi-affine arc

\[
\sigma_{ea3} = \int_0^t \dot{\sigma}_{ea3}(\tau) d\tau. \tag{50}
\]

up to a spatial equi-affine transformation, as follows:

\[
\begin{align*}
    x &= \sigma_{ea3}, \\
    y &= \sigma_{ea3}^2 / 2, \\
    z &= \sigma_{ea3}^3 / 6. \tag{51}
\end{align*}
\]

Spatial affine transformations applied to parabolic screw line result in solutions of the system (49) when \( n \geq 2 \).

2. **Elliptic screw line** can be parameterized with spatial equi-affine arc up to a spatial equi-affine transformation as follows:

\[
\begin{align*}
    x &= \text{const} \cdot \cos \left( \text{const}^{-1/3} \sigma_{ea3} \right) \\
    y &= \text{const} \cdot \sin \left( \text{const}^{-1/3} \sigma_{ea3} \right)
\end{align*}
\]

---

16 Left-hand side = \( f(\beta) \cdot \exp(g(\beta) \cdot \varphi) = \text{const}, g(\beta) \neq 0 \) when \( \beta \neq 0 \).
\[ z = \text{const}^{-1/3} \sigma_{\text{ea3}}. \tag{52} \]

Elliptic screw is solution for \( n \geq 1 \). However, the cost functional with \( n = 1 \) obtains the same values for any rule \( \sigma(t) \). Therefore the case of \( n = 1 \) is not interesting. Arbitrary spatial equi-affine transformations of the elliptic screw line of the form (52) will not necessarily be solutions of the system (49).

Solutions of the system (49) are invariant under spatial similarity transformations. Both parabolic and elliptic screws have constant spatial equi-affine curvature (zero for the parabolic screw) and zero equi-affine torsion\(^{17}\) (Shirokov and Shirokov 1959). Their Euclidian curvature and torsion are not zero. Elliptic screw has constant Euclidian curvature and torsion. Bright (2006) used another method to show that trajectories along the curves with constant Euclidian curvature and torsion simultaneously satisfy the constrained minimum-jerk model and the 1/6 power law.

Parabolic screw line parameterized with Euclidian arc is not solution of the three-dimensional versions of (16) while elliptic screw is.

### 2.4 Case of arbitrary parametrization, dimension and order of smoothness

Following Approach 4, Eq. (16) can be used to easily identify some curves being solutions for parameterizations whose invariance properties are not prescribed in advance.

1. In \( L \)-dimensional space curves \( r_L(t) = (r_1(t), \ldots, r_L(t)) \) whose coordinates are polynomials of order \( \leq 2n-1 \) of an arbitrary parameter \( t \) obviously satisfy Eq. (16) as their \( 2n \)th-order derivatives with respect to \( t \) are all zero:

\[ r_1(t) = \sum_{k=0}^{2n-1} a_{1,k} t^k, \quad i = 1, \ldots, L. \tag{53} \]

Such solutions are invariant under arbitrary linear transformations and translations in \( L \)-dimensional space, including affine transformations. Parabolas constitute a particular case of (53) for \( L = 2 \) and curve’s coordinates being particular 2-nd order polynomials with respect to \( t \).

2. In spaces of dimensions \( L \geq 2 \) take any two coordinates to be described as coordinates \( x \) and \( y \) of (a) logarithmic spiral (31) or of (b) “cosh spiral” (42), or of (c) “sinh spiral” (43).

Whenever \( L \geq 3 \) describe the remaining \( L - 2 \) coordinates with arbitrary polynomials of order \( \leq 2n-1 \). The curves of the form (a)–(c) will satisfy Eq. (16) with the same values \( \beta(n) \) as in case of parametrization with polar angle in plane, see Table 3 and Supplementary File 3. So, without limitation of generality

\[ r_1(t) = x(t), \]
\[ r_2(t) = y(t), \]
\[ r_i(t) = \sum_{k=0}^{2n-1} a_{i-2,k} t^k, \quad i = 3, \ldots, L, \tag{54} \]

with \( x(t), y(t) \) being coordinates of the above-mentioned curves (a)–(c). If \( L \geq 4 \), additional (to \( r_1, r_2 \) from (54)) couples of coordinates can be described as curves (a)–(c). Parameterization of the curves (a)–(c) induced by \( t \) is invariant (still parameterizes solutions) when a curve undergoes orthogonality-preserving similarity transformations in \( L \)-dimensional space.

In general, complex trajectories in plane or in space may be constructed based on concatenating planar or spatial vector functions composed of different components proposed above. For example, a part of a logarithmic spiral being followed (in time) by a polynomial that in turn is followed by “sinh spiral”.

Without limitation of generality the four above-mentioned (in items 1, 2) trajectories in three-dimensional space are:

\[ r_1(t) = \sum_{k=0}^{2n-1} a_{1,k} t^k, \]
\[ r_2(t) = \sum_{k=0}^{2n-1} a_{2,k} t^k, \]
\[ r_3(t) = \sum_{k=0}^{2n-1} a_{3,k} t^k; \]
\[ r_1(t) = \text{const} \cdot e^{\beta t} \cdot \cos t, \]
\[ r_2(t) = \text{const} \cdot e^{\beta t} \cdot \sin t, \]
\[ r_3(t) = \sum_{k=0}^{2n-1} a_k t^k; \]
\[ r_1(t) = \text{const} \cdot \cosh \left[ \beta(n) t \right] \cdot \cos t, \]
\[ r_2(t) = \text{const} \cdot \cosh \left[ \beta(n) t \right] \cdot \sin t, \]
\[ r_3(t) = \sum_{k=0}^{2n-1} a_k t^k; \]

\(^{17}\) From Shirokov and Shirokov (1959), formula for the spatial equi-affine curvature: \( \chi(\sigma_{\text{ea3}}) = \left| \begin{array}{ccc} x' & x'' & x(4) \\ y' & y'' & y(4) \\ z' & z'' & z(4) \end{array} \right| \); for the equi-affine torsion:

\( \tau(\sigma_{\text{ea3}}) = \left| \begin{array}{ccc} x'' & x'' & x(4) \\ y'' & y'' & y(4) \\ z'' & z'' & z(4) \end{array} \right| \) The differentiation is implemented with respect to \( \sigma_{\text{ea3}} \).
\[ r_1(t) = \text{const} \cdot \sinh[\beta(n)t] \cdot \cos t, \]
\[ r_2(t) = \text{const} \cdot \sinh[\beta(n)t] \cdot \sin t, \]
\[ r_3(t) = \sum_{k=0}^{2n-1} a_k t^k. \]

As an example consider an abstract case of seven-dimensional space. Coordinates of the following trajectories are composed of the pairs of coordinates from (a)–(c) above and polynomial of \( t \); the trajectories satisfy Eq. (16):

\[ r_1(t) = \sum_{k=0}^{2n-1} a_{1,k} t^k, \]
\[ r_2(t) = \text{const}_1 \cdot e^{\tilde{\beta}(n)t} \cdot \cos t, \]
\[ r_3(t) = \text{const}_1 \cdot e^{\tilde{\beta}(n)t} \cdot \sin t, \]
\[ r_4(t) = \text{const}_2 \cdot \cosh[\beta(n)t] \cdot \cos t, \]
\[ r_5(t) = \text{const}_2 \cdot \cosh[\beta(n)t] \cdot \sin t, \]
\[ r_6(t) = \text{const}_3 \cdot \sin[\beta(n)t] \cdot \cos t, \]
\[ r_7(t) = \text{const}_3 \cdot \sin[\beta(n)t] \cdot \sin t. \]

Here values of \( \beta(n) \), \( \tilde{\beta}(n) \), \( \beta(n) \) correspond to the solutions identified with logarithmic spiral for parametrization with polar angle. Values for \( 2 \leq n \leq 5 \) are provided in Table 3.

All curves in plane and in space considered earlier in this work are particular cases of the curves from (53), (54).

### 3 Discussion

This work considers the problem of finding paths whose maximally smooth trajectories accumulate measurement along the path with constant rate. The order of smoothness is arbitrary. Class of differential equations obeyed by such paths is derived. Equations of special interest and their known solutions for different orders of smoothness in seven geometries are demonstrated. Derived class of equations constitutes a tool for finding more solutions and thus revealing new candidates for primitive shapes.

Earlier works proposed several curves whose constrained minimum-jerk \( (n = 3) \) trajectories have constant rate of accumulating equi-affine velocity (Polyakov 2001, 2006; Bright 2006; Polyakov et al. 2009b), or planar Euclidian or affine velocity (Bright 2006), see also Meirovitch (2014). Logarithmic spirals with such properties were proposed in Bright (2006). Equations with values of \( n \) different from 3 and solutions of the system of equations in arbitrary dimensional space \( L \) constitute main novelties of this work. Formulae (53), (54) describe two novel classes of solutions. Spirals (42), (43) are particular cases of (54).

#### 3.1 Empirical rationale

This work presents mathematical result of a largely predictive study. Nevertheless, there exists an empirical evidence for 1) invariance in different geometries (Lacquaniti et al. 1983; Viviani and Stucchi 1992; Vieilledent et al. 2001; Iavnenko et al. 2002; Levit-Binnun et al. 2006; Polyakov 2006; Dayan et al. 2007; Polyakov et al. 2009b; Bennequin et al. 2009; Fuchs 2010; Casile et al. 2010; Harpaz et al. 2014; Meirovitch et al. 2015; Karklinsky and Flash 2015) and 2) level of trajectory’s smoothness (Sosnik et al. 2014) being relevant for control of hand movements and in particular being represented in neural activity. Those two empirical characteristics of drawing-like movements and their mathematical properties motivated extension of Eq. (7) to arbitrary degree of smoothness and further demonstration of the method in different geometries here, including geometries defined by the similarity, center-affine and equi-center-affine groups of transformations. The derived class of differential equations merges the two above-mentioned features of movement trajectories in order to predict mathematically candidate primitive shapes. The methodology demonstrated in the manuscript may be further applied for parameterizations not mentioned in this work.

Earlier empirical works demonstrated existence of spontaneously generated hand trajectories with nearly straight (Flash and Hogan 1985) and parabolic-like (Polyakov 2001; Polyakov et al. 2009a,b) paths. Existence of non-parabolic solutions of (13) points to existence of concrete non-parabolic candidates for primitive shapes; their use in production of complex trajectories might be efficient, for example, for movement segments that presumably are not represented solely in equi-affine geometry. At the moment I point to the following additional candidates: circles, specific logarithmic spirals (31), “cosh” spirals (42) and “sinh” spirals (43) (two dimensions) and parabolic and elliptic screws (three dimensions).

Geometric invariance and smoothness of contours are also relevant for the visual system. In this respect the primary visual cortex (V1) can be viewed as the bundle of what are called 1-jets of curves\(^\text{18}\) in \( \mathbb{R} \) (Petitot 2003). “Jets are feature detectors specialized in the detection of tangents. The fact that V1 can be viewed as a jet space explains why V1 is functionally relevant for contour integration. … The Frobenius integrability condition … is an idealized mathematical

---

\(^\text{18}\) The 1-st order jet of a function \( f \) is characterized by three slots: the coordinate \( x \), the value of \( f \) at \( x \), \( y = f(x) \) and the value of its derivative \( p = f'(x) \). The latter is the slope of the tangent to the graph of \( f \) at the point \( a = (x, f(x)) \) of \( \mathbb{R} \).
version of the Gestalt principle of good continuation” (Petitot 2003).

Smooth drawings possess nice integrability properties. Edge completion as the interpolation of gaps between edge segments, which are extracted from an image, can be performed by parabolas (Handzel and Flash 2001). Smoothing may be applied by the motor system at the transitions between neighboring superimposed movement elements and thus the geometric levels of planning may precede the temporal level (see also Torres and Andersen 2006). Connection between the mechanisms of (1) action and (2) perception, and relevance of geometric invariance and smoothness for both mechanisms were observed in earlier studies. Therefore the proposed method of identifying geometric primitives may be meaningful for both motor and visual systems.

Different levels of smoothness at different movement’s stages may be employed by the motor control system even for well-practiced performance. In that case solutions of (13) with different degrees of smoothness \( n \) might be combined together. Schrader et al. (2011) and Hanuschkin et al. (2011) proposed neural network models for composing complex movements from primitives. Incorporation of movement primitives paradigm, type of geometric invariance and level of trajectory’s smoothness into methods for decoding neural data may provide additional information useful for the algorithms employed for brain–machine interfaces.

3.2 Representation of movements in different geometries, compositionality, variability and decision making

Straight point-to-point trajectory has symmetric bell-shaped tangential velocity profile (Hogan 1984; Flash and Hogan 1985). Minimum-jerk trajectories connecting two end points and passing through a via-point can be approximated based on vectorial composition of 3 point-to-point (straight) minimum-jerk movements and fit well piecewise parabolic monkey scribbling movements (Polyakov et al. 2009b). Moreover, a (straight) point-to-point minimum-jerk trajectory can itself be approximated with composition of 3 smaller and slower point-to-point minimum-jerk trajectories with each of 3 having the same direction, amplitude and duration (Polyakov 2006). So piecewise parabolic trajectory can be decomposed in a hierarchical manner into short point-to-point minimum-jerk trajectories and the algorithm is described in Supplementary File 3. Different syntactic rules of combining higher level primitives (e.g., piecewise parabolic sequences) may be developed during practice.

Straight segments form primitive shapes in Euclidian geometry, while parabolic segments are primitive shapes in equi-affine geometry and constitute the only affine invariant solution of the derived class of equations when \( \sigma = \sigma_{\text{eq}} \) for an arbitrary degree of smoothness \( n \) above 1. Unorganized monkeys’ movements converged into low-dimensional piecewise parabolic performance (Polyakov 2006; Polyakov et al. 2009a), while sequences of point-to-point trajectories by humans got coarticulated into smooth movements (Sosnik et al. 2004) that were parabolic-like. That is, primitives in Euclidian geometry got concatenated into primitives in equi-affine geometry. So construction of parabolic-like (equi-affine) primitives can be based on sequential representation of Euclidian primitives getting coarticulated! Simultaneous representation of movements in several geometries can be viewed as coarticulation of primitives in one geometry into primitives in another geometry and not only as timing being weighted mixture of different geometric arcs.

Variability of well-practiced spontaneous monkey scribbling movements was influenced by getting or not getting a reward (Polyakov 2006; Polyakov et al. 2009a). Tuning of primitives’ onset in different kinds of goal-directed movements (achieving a prescribed movement goal or implementing spontaneous search for invisible target) may be guided by decision making and/or action selection based on ongoing feedback/reinforcement signals, e.g., receiving or not receiving a reward. Therefore greater variability of non-rewarded movements was interpreted as characterizing monkey’s decision making about concatenating concurrent, already preplanned, piecewise parabolic movement sequence with another primitive element of monkey’s scribbling repertoire. Polyakov 2006 and Polyakov et al. (2009a) suggested that paradigms involving decision making might be advantageous in the studies investigating movement construction based on compositionality of movement primitives.

3.3 What happens when the motor control system is compromised

View on motor output when the system is compromised may provide an additional insight. Motor control studies of patients suffering from Parkinson’s disease (PD) observed and quantified violations of known motor regularities like the 2/3 power law, isochrony, kinematic smoothness. Compliance with the isochrony principle (Viviani and Terzuolo 1982) was impaired for the PD patients versus the control group in experiment involving point-to-point movement via an intermediate target (Flash et al. 1992). The same study also reported that patients’ velocity profiles demonstrated...
substantial abnormalities including lack of smoothness and multiple small peaks or plateaus in the velocity profile.

In the framework of geometric invariance the 2/3 power law implies proportionality of movement time to accumulated equi-affine arc. Patients with PD demonstrated impairments in how the 2/3 power law characterizes their perception of planar motion (Dayan et al. 2012). In particular, patients with PD perceived on average movements closer but not equal to a constant Euclidian velocity as more uniform than movements with constant equi-affine velocity, in contrast to choices of control subjects (Dayan et al. 2012). In compliance with other studies mentioned in Dayan et al. (2012) this result demonstrates central, e.g., visuo-motor, and not purely motor impairment of PD patients and supports again central (not purely motor) role of geometric characteristics of biological motion. Supporting the central role of geometric invariance, fMRI study of healthy humans demonstrated that basal ganglia respond preferentially to visual motion with constant rate of accumulating equi-affine arc (Dayan et al. 2007) while basal ganglia is also the main location of dysfunction in PD. In terms of the mixed-geometry approach (Bennequin et al. 2009) one can hypothesize that dominance of the equi-affine contribution to movement representation is replaced with more dominant contribution of Euclidian measurement of trajectory’s arc in case of PD patients.

Task INCIDENTAL degrees of freedom values of PD patients (while off dopaminergic medication) were abnormally variable during automated movements, while task-relevant components abnormally dominated patients’ intentional motions (Torres 2011). Moreover, patients’ transition between voluntary and automated modes of joint control was abrupt, and, unlike normal controls, the type of visual guidance differentially affected their postural trajectories. These findings provided support to the view that PD patients lack automated control that contributes to impairments in voluntary control and that basal ganglia are critical for multi-joint control (Torres 2011). A different study demonstrated that for PD patients attention induced a shift from the automatic mode to the controlled pattern within the striatum—a component of the basal ganglia—while for the control subjects attention had no apparent effect on the striatum when movement achieved the automatic stage (Wu et al. 2015, 204). Basal ganglia contributes to decision-making processes including decisions related to perception and action (e.g., see Berns and Sejnowski 1996; Cheng and Anderson 2012; Ding and Gold 2013).

Given a plausibly intimate relationship between movement variability and decision making in the framework of movement compositionality (no need for decision making during completing a preplanned well-practiced motor program would reduce variability), to my view, non-typical variability patterns demonstrate that PD may impair the decision-making process for choosing primitives composing movements. Disfunction of this decision-making process may also contribute to such PD disorders as lack of smoothness and irregularity of the velocity profiles leading in turn to (at least partial) failure to exploit more advantageous parsimonious representation provided by invariance of geometric primitives as geometric primitives provide more parsimonious representation for smooth movements. Another cause to observed movement variability features of PD patients could be disruption of their ability to coarticulate sequences of elementary submovements into complex and smooth task dependent primitives. Study with the double-step paradigm reported that the PD patients have impaired abilities to process simultaneously the motor responses to two visual stimuli which are presented in rapid succession (Plotnik et al. 1998). I hypothesize that during practice intact motor control system tends to achieve motor performance based on smooth and stereotypical (often following coartication) patterns characterized with convergence to low variability and low-dimensional representation; in other words, the motor control system tends to achieve more parsimonious control strategies through practice/learning (Polyakov 2006).

Assessment of behaviors by means of measuring stochastic properties of intra-trial variability based on special characteristic (micro-movements) was proposed recently in the framework of autism spectrum disorders (ASD) and PD studies (Torres et al. 2013; Torres 2013). The method successfully characterized behaviors of subjects. Apparently, ASD results, in part, from impaired basal ganglia function (e.g., see Aui et al. 2010; Prat and Stocco 2012). Probably, mechanisms related to geometric invariance and decision making could also be impaired to certain degree in case of ASD. Knowing the differences between judgements of observed movement speed uniformity between the PD patients (whose basal ganglia function is impaired) and control subjects (Dayan et al. 2012), it would be interesting to implement similar experiment with ASD patients and to assess their location on the axis Euclidian—equi-affine uniformity versus control and PD subjects. Another interesting approach could be to implement setup of compositionality and movement primitives studies with ASD and PD patients, for example, setups from the studies of movement coarticulation (Sosnik et al. 2004) and point of no return (Polyakov 2006; Sosnik et al. 2007), and to compare the performance of the three types of subjects: PD, ASD and controls.

Impairment of decision-making mechanisms employed in motor control may disrupt a plausible hierarchical procedure (described in Supplementary File 3) of constructing smooth complex trajectories from geometric primitives. Such impairment may also destroy higher level mechanisms of (1) binding primitives from different geometries and (2)
Bennequin’s time representation based on weighted mixture of different geometric arcs. Lack of smoothness is a typical consequence of a vast range of neurological disorders, for example, stroke, ataxia, Huntington’s disease, secondary parkinsonism. Probably, some of those disorders are characterized with certain levels of disfunction in movement compositionality.

I am not aware about studies analyzing equi-affine and affine invariants of movement trajectories produced by the patients with neurodegeneration (e.g., PD) or neurodevelopmental (e.g., ASD) disorder. Numerical computations of such quantities (Calabi et al. 1996) are highly sensitive to non-smoothness and irregularities in trajectories’ data, require data regularization and therefore would be especially challenging for motor output produced by patients characterized by non-smooth movements. Still, in the current work, geometric invariance of movement primitives is associated with simultaneous ability to successfully coarticulate basic movement elements into smooth movement blocks after practicing a motor task.

### 3.4 Afterword

To my view the following insight of a prominent mathematician of the twentieth century Andrey Kolmogorov anticipated the idea of geometric movement primitives (Kolmogorov 1988): “If we turn to the human activity—conscious, but not following the rules of formal logic, i.e., intuitive or semi-intuitive activity, for example to motor reactions, we will find out that high perfection and sharpness of the mechanism of continuous motion is based on the movements of the continuous-geometric type … One can consider, however, that this is not a radical objection against discrete mechanisms. Most likely the intuition of continuous curves in the brain is realized based on the discrete mechanism” (Translated from Russian by FP).

The way of representing the “continuous curves in the brain” as coarticulated geometric primitives might go beyond planning trajectory paths and may correspond to perception processes and geometric imagination as well. Moreover, I speculate that at certain hierarchical level of cognitive processes the “discrete mechanisms” of complex movements and language intersect. Observations of low-dimensional representation of monkey scribbling movements with parabolic primitives and reinforcement-related concatenation of parabolic segments into complex trajectories (Polyakov 2006; Polyakov et al. 2009a) support feasibility of this speculation.

---

21 Text S1 of Polyakov et al. (2009a) describes regularization procedure used in equi-affine analysis of monkey scribbling movements.

### References

Aui A, Adler M, Crocetti D, Miller M, Mostofsky S (2010) Basal ganglia shapes predict social, communication, and motor dysfunctions in boys with autism spectrum disorder. J Am Acad Child Adolesc Psychiatry 49(6):539–551

Averbeck BB, Crowe DA, Chafee MV, Georgopoulos AP (2003a) Neural activity in prefrontal cortex during copying geometrical shapes 1. Single cells encode shape, sequence, and metric parameters. Exp Brain Res 150(2):127–141

Averbeck BB, Crowe DA, Chafee MV, Georgopoulos AP (2003b) Neural activity in prefrontal cortex during copying geometrical shapes 2. Decoding shape segments from neural ensembles. Exp Brain Res 150(2):142–153

Ben-Itzhak S, Karniel A (2008) Minimum acceleration criterion with constraints implies bang-bang control as an underlying principle for optimal trajectories of arm reaching movements. Neural Comput 20(3):779–812

Bennequin D, Fuchs R, Berthoz A, Flash T (2009) Movement timing and invariance arise from several geometries. PLoS Comput Biol 5(7):e1000426. doi:10.1371/journal.pcbi.1000426

Berns G, Sejnowski TJ (1996) How the basal ganglia make decisions. In: Damasio AR et al (eds) Neurobiology of decision-making. Springer, Berlin, Heidelberg

Bizzi E, Mussa Ivaldi FA, Giszter S (1991) Computations underlying the execution of movement: a biological perspective. Science 253(5017):287–291

Bright I (2006) Motion planning through optimization. MSc Thesis, Department of Computer Science and Applied Mathematics, Weizmann Institute of Science

Calabi E, Olver PJ, Tannenbaum A (1996) Affine geometry, curve flows, and invariant numerical approximations. Adv Math 124:154–196

Casile A, Dayan E, Caggiano V, Hendler T, Flash T, Giese M (2010) Neuronal encoding of human kinematic invariants during action observation. Cereb Cortex 20(7):1647–55

Cheng J, Anderson W (2012) The role of the basal ganglia in decision making: a new fMRI study. Neurosurgery 71(4):N14–N15

d’Avella A, Saltiel P, Bizzi E (2003) Combinations of muscle synergies in the construction of a natural motor behavior. Nat Neurosci 6:300–308

Dayan E, Casile A, Levit-Binnun N, Giese M, Hendler T, Flash T (2007) Neural representations of kinematic laws of motion: evidence for action–perception coupling. Proc Natl Acad Sci USA 104:20582–20587

Dayan E, Inzelberg R, Flash T (2012) Altered perceptual sensitivity to kinematic invariants in Parkinson’s disease. PLoS ONE 7(2)

Dickey AS, Amit Y, Hatsopoulos NG (2013) Heterogeneous neural coding of corrective movements in motor cortex. Front Neural Circuits 7:Article 51

Ding L, Gold JI (2013) The basal ganglia’s contributions to perceptual decision-making. Neuron 79(4):640–649

Dipietro L, Poizner H, Krebs HI (2014) Spatiotemporal dynamics of online motor correction processing revealed by high-density electroencephalography. J Cogn Neurosci 26(9):1966–1980

Endres D, Meirovitch Y, Flash T, Giese MA (2013) Segmenting sign language into motor primitives with bayesian binning. Front Comput Neurosci 7(68). doi:10.3389/fncom.2013.00068

Flash T, Henis E (1991) Arm trajectory modification during reaching towards visual targets. J Cogn Neurosci 3(3):220–230

Flash T, Hochner B (2005) Motor primitives in vertebrates and invertebrates. Curr Opin Neurobiol 15:1–7

Flash T, Hogan N (1985) The coordination of arm movements: an experimentally confirmed mathematical model. J Neurosci 5(7):1688–1703

© Springer
Flash T, Henis E, Inzelberg R, Korczyn A (1992) Timing and sequencing of human arm trajectories: normal and abnormal motor behaviour. Hum Mov Sci 11:83–100
Fuchs R (2010) Geometry invariants and optimization. PhD thesis. http://lib-phds.l.weizmann.ac.il/Dissertations/Fuchs_Ronit_2011.pdf
Gelfand I, Fomin S (1961) Calculus of variations. Nauka, Moscow
Georgopoulos AP, Kalaska JF, Caminiti R, Massey JT (1982) On the relations between the direction of two-dimensional arm movements and cell discharge in primate motor cortex. J Neurosci 2(11):1527–1537
Giszer S, Hart C (2013) Motor primitives and synergies in the spinal cord and after injury— the current state of play. Annals of the New York Academy of Sciences, New York
Giszer SF, Mussa-Ivaldi FA, Bizzi E (1993) Convergent force fields organized in the frog’s spinal cord. J Neurosci 13(2):467–491
Guggenheimer HW (1977) Differential geometry. Dover, New York
Handzel AA, Flash T (1999) Geometric methods in the study of human motor control. Cogn Stud Bull Jpn Cogn Sci Soc 6(3):309–321
Handzel A, Flash T (2001) Affine invariant edge completion with affine geodesics. In: IEEE workshop on variational and level set methods (VLSM’01)
Hanuschkin A, Herrmann JM, Morrison A, Diesmann M (2011) Geometrically based network theory of the organization of the frog’s spinal cord and after injury—the current state of play. Annals of the New York Academy of Sciences, New York
Harpaz NK, Flash T, Dinstein I (2014) Scale-invariant movement encoding in the human motor system. Neuron 81:452–462
Harris CM, Wolpert DM (1998) Signal-dependent noise determines motor planning. Nature 394:780–784
Harr C, Giszer S (2004) Modular premotor drives and unit bursts as primitives for frog motor behaviors. J Neurosci 24:5269–5282
Hatsopoulos NG, Amit Y (2012) Synthesizing complex movement fragment representations from motor cortical ensembles. J Physiol Paris 106:112–119
Hatsopoulos NG, Xu Q, Amit Y (2007) Encoding of movement fragments in the motor cortex. J Neurosci 27(19):5105–5114
Hochemann S, Wise SP (1991) Effects of hand movement path on motor cortical activity in awake, behaving rhesus-monkeys. Exp Brain Res 83(2):285–302
Hogan N (1984) An organizing principle for a class of voluntary movements. J Neurosci 83(2):2745–2754
Huh D, Sejnowski TJ (2015) Spectrum of power laws for curved hand movements. PNAS 112(29):3950–3958
Iavnenko Y, Grasso R, Macellari V, Lacquaniti F (2002) Two-thirds power law in human locomotion: role of ground contact forces. Neuron 37(3):347–353
Iavnenko Y, Poppele R, Lacquaniti F (2004) Five basic muscle activation patterns account for muscle activity during human locomotion. J Physiol 556:267–282
Karklinsky M, Flash T (2015) Timing of continuous motor imagery: the two-thirds power law originates in trajectory planning. J Neurophysiol 113(2):2490–2499
Kolomogorov A (1988) Mathematics—science and profession. Nauka, Moscow
Krebs H, Aisen M, Volpe B, Hogan N (1999) Quantization of continuous arm movements in humans with brain injury. Proc Natl Acad Sci USA 96(8):4645–4649
Lacquaniti F, Terzuolo C, Viviani P (1983) The law relating the kinematic and figural aspects of drawing movements. Acta Psychol Paris 106:112–119
Levit-Binnun N, Schechtman E, Flash T (2006) On the similarities between the perception and production of elliptical trajectories. Exp Brain Res 172(4):533–555
Maoz U (2007) Trajectory formation and units of action, from two- to three-dimensional motion. The Hebrew University of Jerusalem, PhD Thesis
Maoz U, Flash T (2014) Spatial constant equi-affine speed and motion perception. J Neurophysiol 111(2):336–49
Maoz U, Berthoz A, Flash T (2009) Complex unconstrained three-dimensional hand movement and constant equi-affine speed. J Neurophysiol 101(2):1002–1015
Meirovitch Y (2014) Movement decomposition and compositionality based on geometric and kinematic principles. PhD thesis, Department of Computer Science and Applied Mathematics, Weizmann Institute of Science
Meirovitch Y, Harris H, Dayan E, Arieli A, Flash T (2015) Alpha and beta band event-related desynchronization reflects kinematic regularities. J Neurosci 35(4):1627–1637
Moran DW, Schwartz AB (1999a) Motor cortical activity representation of speed and direction during reaching. J Neurophysiol 82:2676–2692
Moran DW, Schwartz AB (1999b) Motor cortical activity during drawing movements: population representation during spiral tracing. J Neurophysiol 82:2693–2704
Morasso P, Mussa-Ivaldi F (1982) Trajectory formation and handwriting: a computational mode. Biol Cybern 45:131–142
Petitot J (2003) The neurogeometry of pinwheels as a sub-riemannian contact structure. J Physiol Paris 97:265–309
Pham Q-C, Bennequin D (2012) Affine invariance of human hand movements: a direct test. Quant Biol (arXiv)
Plotnik M, Flash T, Inzelberg R, Schechtman E, Korczyn AD (1998) Motor switching abilities in parkinson’s disease and old age: temporal aspects. Neurol Neurosurg Psychiatry 65:328–337
Pollick FE, Sapiro G (1997) Constant affine velocity predicts the 1/3 power law in human planar motion perception and generation. Vis Res 37(3):347–353
Pollick FE, Maoz U, Handzel A, Giblin P, Sapiro G, Flash T (2009) Three-dimensional arm movements at constant equi-affine speed. Cortex 45:325–339
Polyakov F (2001) Analysis of monkey scribbles during learning in the framework of models of planar hand motion. MSc. thesis. Department of Computer Science and Applied Mathematics, Weizmann Institute of Science. http://dl.dropboxusercontent.com/u/18260609/Texts/PolyakovThesisMSc.pdf
Polyakov F (2006) Motion primitives and invariants in monkey scribbling movements: analysis and mathematical modeling of movement kinematics and neural activities. PhD thesis. Department of Computer Science and Applied Mathematics, Weizmann Institute of Science. http://dl.dropboxusercontent.com/u/18260609/Texts/PolyakovThesisPhD.pdf
Polyakov F (2014) A class of differential equations for merging movements’ kinematic optimality with geometric invariance. arXiv:1409.0675v1
Polyakov F, Flash T, Abeles M, Ben-Shaul Y, Drori R, Nadasdy Z (2001) Analysis of motion planning and learning in monkey scribbling movements. In: Proceedings of the tenth biennial conference of the International Graphonomics Society. The University of Nijmegen, Nijmegen, The Netherlands. http://dl.dropboxusercontent.com/u/18260609/Texts/IGS2001.pdf
Polyakov F, Drori R, Ben-Shaul Y, Abeles M, Flash T (2009a) A compact representation of drawing movements with sequences of parabolic primitives. PLoS Comput Biol 5(7):e1000427. doi:10.1371/journal.pcbi.1000427
Polyakov F, Stark E, Drori R, Abeles M, Flash T (2009b) Parabolic movement primitives and cortical states: merging optimality with geometric invariance. Biol Cybern 100(2):159–184
Prat CS, Stocco A (2012) Information routing in the basal ganglia: Highways to abnormal connectivity in autism? comment on “disrupted cortical connectivity theory as an explanatory model for autism spectrum disorders” by kana et al. Phys Life Rev 9:1–2
Richardson MJE, Flash T (2002) Comparing smooth arm movements with the two-thirds power law and the related segmented-control hypothesis. J Neurosci 22(18):8201–8211
Rohrer B, Hogan N (2003) Avoiding spurious submovement decompositions: a globally optimal algorithm. Biol Cybern 89:190–199
Schrader S, Diesmann M, Morrison A (2011) A compositionality machine realized by a hierarchical architecture of synfire chains. Front Comput Neurosci 4:Article 154
Schwartz AB (1992) Motor cortical activity during drawing movements: population representation during sinusoid tracing. J Neurophysiol 70(1):28–36
Schwartz AB (1994) Direct cortical representation of drawing. Science 265:540–542
Sekuler R, Nash D (1972) Speed of size scaling in human vision. Psychon Sci 27(2):93–94
Shanenchi MM, Hu RC, Powers M, Wornell GW, Brown EN, Williams ZM (2012) Neural population partitioning and a concurrent brain-machine interface for sequential motor function. Nat Neurosci 15(2):1715–1722
Shirokov P, Shirokov A (1959) Affine differential geometry. GIFML, Moscow, 1959. German edition: affine differentialgeometrie, Teubner, 1962. English translation of relevant parts of the book can be obtained from the author of the manuscript (FP) by request for non-commercial use in research and teaching. Some parts of the book are translated into English in Appendix A of (Polyakov, 2006)
Shigelmacher M (2006) Directional-Geometrical Approach to via-point movement. MSc Thesis
Sosnik R, Hauptmann B, Karni A, Flash T (2004) When practice leads to co-articulation: the evolution of geometrically defined movement primitives. Exp Brain Res 156:422–438
Sosnik R, Shemesh M, Abeles M (2007) The point of no return in planar hand movements: an indication of the existence of high level motion primitives. Cogn Neurodyn 1(4):341–358
Sosnik R, Flash T, Sterkin A, Hauptmann B, Karni A (2014) The activity in the contralateral primary motor cortex, dorsal premotor and supplementary motor area is modulated by performance gains. Front Hum Neurosci 8(1):201
Sosnik R, Chaim E, Flash T (2015) Stopping is not an option: the evolution of unstoppable motion elements (primitives). J Neurophysiol 114(2):846–856
Tanaka H, Sejnowski TJ (2015) Motor adaptation and generalization of reaching movements using motor primitives based on spatial coordinates. J Neurophysiol 113(4):1217–1233
Tanaka H, Krakauer JW, Qian N (2006) An optimization principle for determining movement duration. J Neurophysiol 95(6):3875–3886
Thoroughman KA, Shadmehr R (2000) Learning of action through adaptive combination of motor primitives. Nat 407:742–747
Todorov E, Jordan MI (1998) Smoothness maximization along a pre-defined path accurately predicts the speed profiles of complex arm movements. J Neurophysiol 80(2):696–714
Torres EB (2011) Impaired endogenously evoked automated reaching in Parkinson’s disease. J Neurosci 31(49):17848–17863
Torres EB (2013) Atypical signatures of motor variability found in an individual with asd. Neurocase Neural Basis Cogn 19(2):150–165
Torres E, Andersen R (2006) Space-time separation during obstacle avoidance learning in monkeys. J Neurophysiol 96:162–167
Torres EB, Brincker M, Isenhower RW, Yanovich P, Stigler KA, Nurnberger JL, Metaxas DN, Jose JV (2013) Autism: the micro-movement perspective. Front Integr Neurosci 7:13–38
Tresch M, Saltiel P, Bizzi E (1999) The construction of movement by the spinal cord. Nat Neurosci 2:162–167
Uno Y, Suzuki R, Kawato M (1989) Formation and control of optimal trajectory in human multijoint arm movement. Biol Cybern 61:89–101
van Brunt B (2004) The calculus of variations. Springer, New York
van Zuylen E, Gieelen C, van der Gon Denier J (1988) Coordination and inhomogeneous activation of human arm muscles during isometric torques. J Neurophysiol 60:1523–1548
Vieilledent S, Kerlirzin Y, Dalbera S, Berthoz A (2001) Relationship between velocity and curvature of a human locomotor trajectory. Neurosci Lett 305(1):65–69
Viviani P, Flash T (1995) Minimum-jerk, two-thirds power law, and isochrony: converging approaches to movement planning. J Exp Psychol Hum Percept Perform 21(1):233–242
Viviani P, Stucchi N (1992) Biological movements look uniform: evidence of motor-perceptual interactions. J Exp Psychol Hum Percept Perform 18(3):603–623
Viviani P, Terzuolo C (1982) Trajectory determines movement dynamics. Neuroscience 7:431–437
Woch A, Plamondon R (2010) Characterization of bi-directional movement primitives and their agonist-antagonist synergy with the delta-lognormal model. Motor Control 14(1):1–25
Woch A, Plamondon R, O’Reilly C (2011) Kinematic characteristics of bidirectional delta-lognormal primitives in young and older subjects. Hum Mov Sci 30(1):1–17
Wu T, Liu J, Zhang H, Hallett M, Zheng Z, Chan P (2015) Attention to automatic movements in Parkinson’s disease: modified automatic mode in the striatum. Cereb Cortex 25:3330–3342
Zelman I, Titon M, Yekutieli Y, Hanassy S, Hochner B, Flash T (2013) Kinematic decomposition and classification of octopus arm movements. Front Comput Neurosci 7:60. doi:10.3389/fncom.2013.00060