Brownian motion on graph-like spaces

by

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Abstract. We construct a natural diffusion on a wide class of metric spaces generalising graphs, and show that its cover time admits an upper bound depending only on the Hausdorff measure of the space.

1. Introduction. The aim of this paper is to construct an analog of Brownian motion on metric spaces that are similar to graphs in a sense made precise below, and study some of its basic properties. It turns out that, under mild conditions, there is a unique stochastic process qualifying for this. Interestingly, this process covers the whole space in finite expected time, which admits an upper bound depending only on the total length—i.e. the 1-dimensional Hausdorff measure—of the space and not its structure.

Figure 1 shows some example spaces on which our process can live; the numbers indicate the lengths of the corresponding arcs. The first one is the Hawaiian earring: an infinite sequence of circles attached to a common point \( p \), to which they converge. It might at first sight seem impossible to have a Brownian motion on this space started at \( p \), unless we impose some ad-hoc bias as to the probability with which each circle is visited first. However, there need not be a ‘first’ circle visited by a continuous path from \( p \), and indeed our process will traverse infinitely many of them before moving to any distance \( r > 0 \) from \( p \). Still, each of the finitely many points at distance exactly \( r \) from \( p \) has the same probability to be reached first. The second example is an \( \mathbb{R} \)-tree of finite total length. Our Brownian motion will reach the ‘boundary’ at the top after some finite time \( \tau \), and will continue its continuous path after this, almost surely visiting infinitely many boundary points in any interval \([\tau, \tau + \varepsilon]\). The third example is obtained from the Sierpiński gasket by...
replacing articulation points with arcs. This space contains a homeomorphic copy of the second example, and a subspace homotopy equivalent to the first example; our process on it is more complex, combining features of both the above.

In all these examples, and in much greater generality, our process behaves locally like standard Brownian motion on a real interval \( I \) on each open arc of our space isometric to \( I \), its sample paths are continuous, it has the strong Markov property, and it almost surely covers the whole space after finite time.

We call a topological space \( X \) graph-like if it contains a set \( E \) of pairwise disjoint copies of \( \mathbb{R} \), called edges, each of which is open in \( X \), such that the subspace \( X \setminus \bigcup E \) is totally disconnected. This notion was introduced by Thomassen and Vella [38], and was motivated by recent developments in graph theory; see also [18].

Recall that a continuum is a compact, connected, non-empty metric space (some authors replace ‘metric’ by Hausdorff). We will use \( \mathcal{H}(X) \) to denote the 1-dimensional Hausdorff measure of \( X \), or in other words, the total length of \( X \). Although our processes can be constructed on any graph-like continuum, the process is unique if and only if \( \mathcal{H}(X) < \infty \).

In order to construct our process, we use a result from [18] stating, roughly speaking, that every graph-like space \( X \) can be approximated by a sequence of finite graphs (i.e. 1-complexes) contained in \( X \). Such a sequence of graphs is called a graph approximation of \( X \); see Section 3 for the precise definition. For example, any sequence \((G_n)_{n \in \mathbb{N}}\) where \( G_n \) consists of finitely many of the circles of the Hawaiian earring and each circle appears in almost every \( G_n \) is a graph approximation. The main goal of this paper is to show that if \( B_n \) denotes Brownian motion on the \( n \)th member of any graph approximation of \( X \), then the \( B_n \) converge weakly—in the space of measures on continuous paths on \( X \), see Section 2.2—to a stochastic process \( B \) on \( X \) with all the desired properties, and this \( B \) does not depend on the choice of the graph approximation.
Theorem 1.1. Let $X$ be a graph-like continuum with $\mathcal{H}(X) < \infty$, and $x$ a point of $X$. Then there is a stochastic process $B$ on $X$ with continuous sample paths starting at $x$, the strong Markov property, and a stationary distribution proportional to $\mathcal{H}$.

Moreover, for every graph approximation $(G_n)_{n \in \mathbb{N}}$ of $X$, and every choice of points $x_n \in G_n$ such that $\lim x_n = x$, if $B_n$ is the standard Brownian motion on $G_n$ from $x_n$, then $B_n$ converges weakly to $B$, and $B$ is unique with this property.

It should be stressed that, unlike the above examples, our spaces do not need to satisfy any homogeneity or self-similarity properties; all we need is finiteness of the total length. Therefore, most of the techniques used so far to construct such processes on fractals cannot be employed.

It was shown in [19] that the expected time for Brownian motion on a finite, connected 1-complex $G$ to cover all of $G$ is bounded from above by a value depending only on the total length of $G$ and not on its structure. Applying this to each member of our graph approximations, and with some additional work, we prove the corresponding result for our Brownian motion on an arbitrary graph-like continuum.

Theorem 1.2. The expected cover time of the process $B$ of Theorem 1.1 is at most $20\mathcal{H}(X)^2$.

The cover time of a graph is an important notion in several respects. Apart from its applications in computer science (see [12] for references), it is of interest to probabilists due to relationships with the Gaussian Free Field [13, 37]. In particular, a lot of research is devoted to the study of the fractal structure of the uncovered set of a graph at time scales approaching the cover time (see [10, 12] and references therein). To the best of our knowledge, our $B$ is the only process constructed so far on spaces with infinitely many ramifications that has finite cover time. (A somewhat related result of Krebs [27] shows that the hitting times for Brownian motion on nested fractals are bounded.)

There have been many further constructions of Brownian motion on spaces similar to the ones considered in this paper: on finite graphs [7], on trees and their boundaries [11, 8, 9, 26], on the Sierpiński gasket [5, 20, 29] and many other fractals [23, 22, 30]. Brownian motion especially on fractals has attracted a lot of interest, with motivation coming both from pure mathematics and mathematical physics (see [29] and references therein), and has many connections with other analytic properties of fractals which also attract a lot of research [25, 35].

A general theory of diffusion processes on metric spaces was developed in [36]. An important assumption in this context is that the space satisfies a certain condition called Measure Contraction Property. Our construction
however also applies to spaces that do not satisfy this condition. One particular example is an infinite binary rooted tree whose edges at the $n$th generation have length $4^{-n}$.

The first author had asked for a construction of Brownian motion on a special type of graph-like spaces, namely metric completions of infinite graphs [16, Section 8], and the present paper gives a satisfactory answer to that question.

This paper is structured as follows. After reviewing some definitions and basic facts in Section 2, we prove the existence part of Theorem 1.1 in Section 3. The uniqueness part is then proved in Section 5. Next we prove that our process has the strong Markov property (Theorem 6.3), and the bound on the cover time is given in Section 7. Finally, in Section 8 we prove that $H$ is a stationary distribution and that our process behaves locally like standard Brownian motion inside any edge.

2. Preliminaries

2.1. Graph-like spaces. An edge of a topological space $X$ is an open subspace $I \subseteq X$ homeomorphic to the real interval $(0, 1)$ such that the closure of $I$ in $X$ is homeomorphic to $[0, 1]$. (We could allow the closure of $I$ to be a circle; it is only for convenience in certain situations that we disallow this.) Note that the frontier of an edge consists of two points, which we call its endvertices. An edge-set of a topological space $X$ is a subspace consisting of finitely many, pairwise disjoint, edges of $X$.

A topological space $X$ is graph-like if there is an edge-set $E$ of $X$ such that $X \setminus E$ is totally disconnected. In that case, we call $E$ a disconnecting edge-set.

Throughout the paper we assume that $X$ is a metric space. The following fact provides an equivalent definition of a graph-like continuum.

Lemma 2.1 ([18]). A continuum $X$ is graph-like if and only if for every $\varepsilon$ there is a finite set of edges $S_\varepsilon$ of $X$ such that the diameter of every component of $X \setminus S_\varepsilon$ is less than $\varepsilon$.

The following property of graph-like spaces is very useful to us, as it implies that Brownian motion on such a space cannot travel a long distance without traversing a long edge.

Proposition 2.2. If $X$ is a graph-like continuum, then for every $\rho > 0$ there is a finite edge-set $R_\rho$ of $X$ such that for every topological path $p : [0, 1] \rightarrow X$ in $X$, if $d(p(0), p(1)) > \rho$ then $p$ traverses an edge in $R_\rho$.

Proof. Applying Lemma 2.1 for $\varepsilon = \rho/3$, we obtain a finite set $S$ of edges such that the diameter of every path-component of $X \setminus S$ is less than $\rho/3$. 


Subdivide each edge $e \in S$ into a finite set of edges each of length at most $\rho/6$, and let $R$ be the set of edges resulting from $S$ after all these subdivisions. Now note that any topological path $p$ as in the assertion has to traverse an element of $R$; to see this, contract each path-component of $X \setminus S$ into a point to obtain a new metric space $X'$, and note that $X'$ is isometric to a finite graph whose edge-set can be identified with $R$. Moreover, after the contractions we have $d(p(0), p(1)) > \rho - 2\rho/3 = \rho/3$, and as each edge of our graph has length at least $\rho/6$, the assertion easily follows by geometric arguments. Thus we can choose $R_\rho = R$. ■

Graph-like spaces have nice bases:

**Lemma 2.3** ([18]). Let $X$ be a graph-like metric continuum. Then the topology of $X$ has a basis consisting of connected open sets $O$ such that the frontier of $O$ is a finite set of points each contained in an edge.

For further properties and characterisations of graph-like continua we refer the interested reader to [14].

### 2.2. Measures on the space of sample paths and weak convergence

Given a graph-like space $(X,d_X)$, we denote by $\mathcal{C} = C([0,T], X)$ the set of continuous functions from the real interval $[0,T]$ to $X$. We call $\mathcal{C}$ the space of sample paths; our process will be formally defined as a probability measure on $\mathcal{C}$. We endow $\mathcal{C}$ with the $L^\infty$ metric $d_{\mathcal{C}}(b,d) := \sup_{t \in [0,T]} d_X(b(t), d(t))$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{C})$ denote the space of all Borel probability measures on $\mathcal{C}$. The weak topology on $\mathcal{M}$ is the topology generated by the open sets of the form

$$O_\mu(f_1, \ldots, f_k; \varepsilon_1, \ldots, \varepsilon_k) = \left\{ \nu \in \mathcal{M} : \left| \int f_i \, d\nu - \int f_i \, d\mu \right| < \varepsilon_i, 1 \leq i \leq k \right\},$$

where $\mu$ ranges over all elements of $\mathcal{M}$, the $f_i$ range over all bounded continuous functions $f_i : \mathcal{C} \to \mathbb{R}$, and the $\varepsilon_i$ range over $\mathbb{R}_{>0}$. An immediate consequence of this definition is that a sequence of measures $\mu_i \in \mathcal{M}$ converges in this topology to $\mu \in \mathcal{M}$ if and only if $\int f \, d\mu_i$ converges to $\int f \, d\mu$ for every bounded continuous function $f : \mathcal{C} \to \mathbb{R}$. If such a sequence converges, then the limit is unique [34, Chapter II, Theorem 5.9].

Let us recall that the existence of an $X$-valued stochastic process $B = (B_t)_{0 \leq t \leq T}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ such that $B(0) = x$ and $B$ is continuous $\mathbb{P}_x$-almost surely is equivalent to the existence of a probability measure $\mu$ on $\mathcal{C}$ such that $\mu(\{p \in \mathcal{C} : p(0) = x\}) = 1$. Indeed, the measure $\mu$ defined via $\mu(A) = \mathbb{P}_x(B \in A)$ satisfies the condition above. On the other hand, if such $\mu$ exists then we can take $(\Omega, \mathcal{F}, \mathbb{P}_x) = (\mathcal{C}, \text{Bor}(\mathcal{C}), \mu)$ and $B : \Omega \to \mathcal{C}$ as the identity map. We will denote by $\mathbb{E}_x$ the expectation with respect to the measure $\mathbb{P}_x$. 
Our main tool in obtaining limits of stochastic processes is the following standard fact; see e.g. [34, Chapter VII, Lemma 2.2].

**Lemma 2.4.** Let $\Gamma \subseteq \mathcal{M}$ be a set of probability measures on $\mathcal{C}$. Then $\Gamma$ is compact in the weak topology if and only if for all $\varepsilon, \rho > 0$ there is $\eta = \eta(\varepsilon, \rho) > 0$ such that

$$\mu(\{ p : \omega_p(\eta) > \rho \}) < \varepsilon$$

for every $\mu \in \Gamma$.

where $\omega_p(\eta) := \sup_{|t - t'| \leq \eta} |p(t) - p(t')|$.

### 2.3. Metric graphs and their Brownian motion.

In this paper, by a **graph** $G$ we will mean a topological space homeomorphic to a simplicial 1-complex. We assume that any graph $G$ is endowed with a fixed homeomorphism $h : K \to G$ from a simplicial 1-complex $K$, and call the images under $h$ of the 0-simplices of $K$ the **vertices** of $G$, and the images under $h$ of the 1-simplices of $K$ the **edges** of $G$. The sets of vertices and of edges are denoted by $V(G)$ and $E(G)$ respectively. All graphs considered will be **finite**, that is, they will have finitely many vertices and edges.

A **metric graph** is a graph $G$ endowed with an assignment of lengths $\ell : E(G) \to \mathbb{R}_{>0}$ to its edges. This assignment naturally induces a metric $d_\ell$ on $G$ with the following properties. Edges are locally isometric to real intervals, their lengths (i.e. 1-dimensional Hausdorff measures) with respect to $d_\ell$ coincide with $\ell$, and for all $x, y \in V(G)$ we have $d_\ell(x, y) := \inf_{P \text{ is an } x-y \text{ arc}} \ell(P)$, where $\ell(P) := \sum_{P \supseteq e \in E(G)} \ell(e)$; see [17] for details on $d_\ell$.

The length $\ell(G)$ of a metric graph $G$ is defined as $\sum_{e \in E(G)} \ell(e)$.

An **interval** of an edge $e$ of $G$ is a connected subspace of $e$.

Brownian motion on $\mathbb{R}$ extends naturally to Brownian motion on a metric graph. The edges incident to a vertex constitute a ‘Walsh spider’ (see, e.g., [39, 6]) with equiprobable legs, and it is easily verified that in such a setting the probability of traversing a particular incident edge (or oriented loop) first is proportional to the reciprocal of the length of that edge, while inside any interval of an edge, the process behaves like standard Brownian motion on a real interval of the same length. To make this more precise, it is shown in [7] that there is a unique probability distribution on the space $C([0, t], G)$ of continuous functions from a real interval $[0, t]$ to $G$, which we will call **standard Brownian motion** on $G$, that has the following properties:

(i) the strong Markov property;

(ii) for every vertex $v$ of $G$ and any choice of points $p_i$, $1 \leq i \leq k$, one inside each edge incident to $v$, the probability to reach $p_j$ before any other $p_i$, $i \neq j$, when starting at $v$ is $(1/\ell_j)/\sum_{1 \leq i \leq k} 1/\ell_i$, where $\ell_i$

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(1) Condition (i) in [34, Chapter VII, Lemma 2.2] is void in our case because our spaces have finite diameter.
denotes the length of the interval from \( v \) to \( p_i \) \( \mathbb{Z} \) §4, Lemma 1 applied with \( \tilde{p}_i := 1/k \);

(iii) for every vertex \( v \) of \( G \), the expected time to exit the ball of radius \( r \) around \( v \) when starting at \( v \) tends to 0 as \( r \) tends to 0 \( \mathbb{Z} \) (3.1);

(iv) when starting at a point \( p \) inside an edge \( e \), the expected time till the first traversal of one of the two intervals of \( e \) of length \( \ell \) starting at \( p \) is \( \ell^2 \) \( \mathbb{Z} \) (3.4)].

2.4. Electrical network basics. An electrical network is a graph \( G \) endowed with an assignment of resistances \( r : E \to \mathbb{R}_+ \) to its edges. The set \( \vec{E} \) of directed edges of \( G \) is the set of ordered pairs \( (x,y) \) such that \( x \in N(x) \) denotes the set of vertices sharing an edge with \( x \) (\( i \) satisfies Kirchhoff’s node law outside \( p,q \));

(iii) \( \sum_{y \in N(p)} i(\vec{p}y) = 1 \) and \( \sum_{y \in N(q)} i(\vec{q}y) = -1 \) (\( i \) satisfies the boundary conditions at \( p,q \)).

The effective resistance \( R_G(p,q) \) from a vertex \( p \) to a vertex \( q \) of \( G \) is defined by

\[
R(p,q) = R_G(p,q) := \inf \{ E(i) : i \text{ is a } p-q \text{ flow of strength } 1 \},
\]

where the energy \( E(i) \) of \( i \) is defined by \( E(i) := \sum_{\vec{e} \in \vec{E}} i(\vec{e})^2 r(e) \). In fact, it is well-known that this infimum is attained by a unique \( p-q \) flow, called the corresponding electrical current.

For any \( A \subseteq G \) we denote by \( \tau_A \) the first hitting time of \( A \) for the Brownian motion, i.e.

\[
\tau_A = \inf \{ s \geq 0 : B(t) \in A \}.
\]

If \( A = \{x\} \), i.e. \( A \) is a singleton, then we denote \( \tau_x := \tau_{\{x\}} \). The expected time for Brownian motion started at a vertex \( a \) to visit a vertex \( z \) and then return to \( a \), i.e., \( \mathbb{E}_a[\tau_z] + \mathbb{E}_z[\tau_a] \), is called the commute time between \( a \) and \( z \).

**Lemma 2.5** \([\mathbb{Z}] 31\). Let \( G \) be a finite metric graph, and \( a, z \) two vertices of \( G \). The commute time between \( a \) and \( z \) equals \( 2\ell(G)R(a,z) \).

The following property of the effective resistance justifies its name.

**Lemma 2.6.** Let \( G \) be an electrical network contained in an electrical network \( H \) in such a way that there are exactly two vertices \( p,q \) of \( G \) connected to vertices of \( H \setminus G \) with edges. Then if \( H' \) is obtained from \( H \) by replacing
with a $p$–$q$ edge of effective resistance $R_{G}(p, q)$, then for every two vertices $v, w$ of $H'$ we have $R_{H'}(v, w) = R_{H}(v, w)$.

The proof of this follows easily from the definition of effective resistance. See e.g. [31] for details.

Any metric graph naturally gives rise to an electrical network by setting $r = \ell$, and we will assume this whenever talking about effective resistances in metric graphs.

The importance of effective resistances for this paper is due to the following fact, showing that they determine transition probabilities between any two points in a finite set for Brownian motion on a metric graph.

**Lemma 2.7** ([31, Exercise 2.68]). Let $G$ be a metric graph and $x, y \in G$ a finite set of points of $G$. Then for any $o \in G$,

$$P_{o}(\tau_{x} < \tau_{y}) = \frac{R_{G}(y, o) - R_{G}(x, o) + R_{G}(x, y)}{2R_{G}(x, y)}.$$

### 3. Tightness of approximating Brownian motions

In this section we prove the existence part of Theorem [1.1] in other words, the existence in $\mathcal{M}(\mathcal{C})$ of an accumulation point of every sequence $(B_{n})_{n \in \mathbb{N}}$ such that $B_{n}$ is standard Brownian motion on a graph $G_{n} \subseteq X$ and $(G_{n})_{n \in \mathbb{N}}$ is a graph approximation of $X$, that is, a sequence of finite metric graphs that are subspaces of $X$ satisfying the following two properties:

(i) for every edge $e \in E(G_{n})$ the length $\ell(e)$ of $e$ in $G_{n}$ coincides with the length of the corresponding arc of $X$;

(ii) for every finite edge-set $F$ of $X$, and every component $C$ of $X \setminus F$, there is a unique component of $G_{i} \setminus F$ meeting $C$ for almost all $i$.

The existence of graph approximations was established in [18]. In fact, we can furthermore assume that each $G_{n}$ is connected, and that $G_{n} \subseteq G_{n+1}$ for every $n$, although it will not make a formal difference for our proofs. It is also shown in [18] that (ii) implies that $\bigcup G_{n}$ contains every edge of $X$ and is dense in $X$.

So let us fix such a sequence $(G_{n})_{n \in \mathbb{N}}$. For every $n \in \mathbb{N}$ and $x \in G_{n}$, let $\mu_{n,x}$ be the measure on $\mathcal{C}$ corresponding to standard Brownian motion on $G_{n}$ starting at the point $x$. Let

$$\Gamma := \{\mu_{n,x} : n \in \mathbb{N}, x \in G_{n}\}.$$

The following result shows that this family of measures has accumulation points in $\mathcal{M}(\mathcal{C})$, which we think of as candidates for our Brownian motion on $X$. We will show in Section [5] that if the $x_{n}$ converge to a point $x$ of $X$, then $\Gamma$ has in fact a unique accumulation point.

Throughout the paper we denote by $B_{n}$ the Brownian motion on the metric graph $G_{n}$, i.e. the law of $B_{n}$ under $P_{x}$ for $x \in G_{n}$ is equal to $\mu_{n,x}$. 

**Lemma 3.1.** The family $\Gamma$ is compact (with respect to the weak topology).

**Proof.** We are going to show that our family $\Gamma$ satisfies the condition of Lemma 2.4, that is, for any $\varepsilon > 0$,

$$
\lim_{\delta \to 0} \mathbb{P}_x \left( \sup_{t, s < T, |t-s| < \delta} d(B_n(t), B_n(s)) > \varepsilon \right) = 0
$$

uniformly in $n$ and $x \in G_n$.

So fix $\varepsilon > 0$. Let $R = R_\varepsilon$ be a finite set of edges as in Proposition 2.2 and let $\varepsilon_1 = \min \{ \ell(e) : e \in R \}$. Thus we have the following bound for the probability appearing in (1):

$$
\mathbb{P}_x \left( \sup_{t, s < T, |t-s| < \delta} d(B_n(t), B_n(s)) > \varepsilon \right) \leq \mathbb{P}_x(B_n([t, t + \delta]) \text{ traverses an edge } e \in R \text{ for some } t \in [0, T - \delta]).
$$

It remains to show that the last probabilities converge to 0 uniformly in $n$ as $\delta \to 0$. For this we will use the fact that each Brownian motion $B_n$ in the interior of an edge behaves locally like standard Brownian motion $W$ on the real line. Let us make this more precise. Let $R'$ be the set of half-edges of $R$, that is, each element of $R'$ is an open subinterval of an edge of $R$ from an endpoint to the midpoint. Let us subdivide the time interval $[0, T]$ into the $[T/\delta]$ subintervals $I_0, I_1, \ldots, I_k$ of the form $I_i = [i\delta, (i+1)\delta]$; note that each $I_i$ has duration at most $\delta$. Then, if $B_n$ traverses an edge of $R$ in time $\delta$ at some point, then there is a time interval $I_i$ during which $B_n$ traverses an element of $R'$. Thus we can write

$$
\mathbb{P}_x(B_n([t, t + \delta]) \text{ traverses an edge } e \in R \text{ for some } t \in [0, T])
$$

$$
\leq \sum_i \mathbb{P}_x(B_n([i\delta, (i+1)\delta]) \text{ traverses an edge } e \in R').
$$

Now denote by $M$ the set of midpoints of elements of $R'$, and by $\tau_n^i = \inf \{ t \geq i\delta : B_n(t) \in M \}$ the associated hitting times. Then we can bound the last expression by

$$
\sum_i \mathbb{P}_x(B_n([\tau_n^i, \tau_n^i + \delta]) \text{ traverses an edge } e \in R''),
$$

where $R''$ is the set of half-edges of $R'$, in other words, ‘quarter-edges’ of $R$.

Now since inside an edge, $B_n$ behaves like standard Brownian motion $W$, the above sum is at most

$$
[T/\delta] \mathbb{P} \left( \max_{t \in [0, \delta]} |W(t)| > \varepsilon_1/4 \right) \leq 2[T/\delta] \mathbb{P}(|W(\delta)| > \varepsilon_1/4),
$$

by the reflection principle [32, Theorem 2.21]. This expression converges to 0 with $\delta$, since the second factor decays rapidly with $\delta$. Moreover, it does not depend on $n$, and so it yields (1) as desired. □
Remark. If \((\mu_{n_k}, x_{n_k})\) is a convergent sequence of elements of \(\Gamma\) with limit \(\mu\), then for every \(x \in X\),
\[
\mathbb{E}_\mu[d(b(0), x)] = \lim_{k} \mathbb{E}_{\mu_{n_k}, x_{n_k}}[d(b(0), x)].
\]
In particular, if \(x_{n_k}\) converges to \(x\), then the starting point of \(\mu\) is \(x\) a.s.

4. Occupation time of small subgraphs. A subgraph \(H\) of a graph \(G\) is a subspace of \(G\) that is a graph itself. If \(G\) is a metric graph, then we consider \(H\) to be a metric graph as well, with its edge-lengths induced from those of \(G\) in the obvious way. Note that the vertices of \(H\) need not be vertices of \(G\); an interval of an edge of \(G\) can be an edge of \(H\).

For a (finite) metric graph \(G\) and standard Brownian motion \(B\) on \(G\), the occupation time \(\text{OT}_t(H) = \text{OT}_t(H, B)\) of a subgraph \(H \subseteq G\) up to time \(t\) is defined to be \(\int_0^t \mathbf{1}_{\{B(s) \in H\}} \, ds\), the amount of time spent by \(B\) in \(H\) in the time interval \([0, t]\). We define the occupation time of \(H\) for random walk on \(G\) similarly.

In this section we show that the occupation time of a subgraph \(H\) of \(G\) is short with high probability when the length \(\ell(H)\) is small compared to \(\ell(G)\), and in fact can be bounded above by a function depending only on the proportion of the lengths but not on the structure of \(G\) and \(H\).

First we need some estimation of transition probabilities on a metric graph. We denote by \(p_t(x, y)\) the heat kernel on \(G\), i.e. the density of the distribution of Brownian motion on \(G\) with respect to the Hausdorff measure.

Lemma 4.1. Let \(G\) be a finite metric graph. Then for any \(x, y \in G\) and \(t > 0\), we have
\[
p_t(x, y) \leq 4 \max(t^{-1/2}, \ell(G)^{-1}).
\]

Proof. The proof uses a well-known idea going back to Nash [33], which we present in full in order to keep the paper self-contained.

Let \(P_t\) be the heat semigroup associated with the Brownian motion \(B\) on \(G\), i.e. \(P_t f(x) = \mathbb{E}_x[f(B(t))]\) for any bounded function \(f\). By duality, \(P_t\) acts on the space of probability measures on \(G\). Our assertion will be proved if we show that the \(P_t \delta_x\) have densities with respect to the Hausdorff measure \(\mathcal{H}\) bounded by the right side of the above inequality. Since any \(\delta_x\) is a weak limit of a probability measure with a density that is continuous on \(G\) and differentiable inside every edge, it is sufficient to obtain a uniform bound on \(\|P_t f\|_\infty\), where \(f\) is a probability density on \(G\).

The idea (cf. [33, 21]) is to prove first a Nash type inequality:
\[
\|u\|_6^6 \leq 8(c\|u\|_2^2 + \|u'\|_2^2)\|u\|_1^4 \tag{2}
\]
for every continuous function \(u\) which is differentiable inside every edge, where \(c = (2\ell(G))^{-2}\). It is enough to show (2) for \(\|u\|_1 = 1\). Since \(u\) is
continuous, there is \( x_0 \in G \) with \( |u(x_0)| = 1/\ell(G) \). Now for any \( x \in G \) there is a path \( \gamma \) connecting \( x \) to \( x_0 \), and so by the Schwarz inequality we have

\[
 u(x)^2 - u(x_0)^2 = \int_{\gamma} 2u(y)u'(y) \, dy \leq 2\|u\|_2\|u'\|_2,
\]

which implies

\[
 u(x)^2 \leq \sqrt{\ell(G)^{-2} + 2\|u\|_2\|u'\|_2} |u(x)|.
\]

Integrating the above inequality we obtain

\[
 \|u\|_2^2 \leq \sqrt{\ell(G)^{-2} + 2\|u\|_2\|u'\|_2} \leq \sqrt{\ell(G)^{-1}\|u\|_2^2 + 2\|u\|_2\|u'\|_2}.
\]

By the inequality between the quadratic and arithmetic means, this implies that

\[
 \|u\|_2^6 \leq 2\ell(G)^{-2}\|u\|_2^2 + 8\|u'\|_2^2,
\]

and so (2) is proved.

Next, following another idea due to Nash \[33\], define \( U(t) = \|e^{-ct}P_tf\|_2^2 \).

An easy observation (cf. \[28\]) gives

\[
 \frac{d}{dt} U(t) = -2c\|e^{-ct}P_tf\|_2^2 - 2\|e^{-ct}(P_tf)'\|_2^2.
\]

In view of (2) this leads to

\[
 U^3(t) \leq -4\frac{d}{dt} U(t)e^{-4\delta t},
\]

since \( \|P_tf\|_1 = 1 \). By elementary computations,

\[
 U(t) \leq \sqrt{\frac{8c}{e^{4ct} - 1}} \leq \begin{cases} 
 \sqrt{2/t} & \text{if } t \leq \ell(G)^2, \\
 \sqrt{2/\ell(G)}e^{-2ct} & \text{if } t \geq \ell(G)^2,
\end{cases}
\]

hence

\[
 \|P_tf\|_2^2 \leq 2\sqrt{e} \max(t^{-1/2}, \ell(G)^{-1})
\]

The semigroup principle gives \( P_t = P_{t/2} \circ P_{t/2} \) and by symmetry \( \|P_{t/2}\|_{1 \to 2} = \|P_{t/2}\|_{2 \to \infty} \), therefore

\[
 \|P_tf\|_{\infty} \leq 4\max(t^{-1/2}, \ell(G)^{-1}). \]

As a consequence of the above lemma we obtain a bound for the occupation time.

**Lemma 4.2.** For every finite metric graph \( G \) and every \( x \in H \subseteq G \), \( t \geq 0 \), we have \( \mathbb{E}_x[\text{OT}_t(H)] \leq \frac{16\ell(H)}{\ell(G)} \max(\ell(G)\sqrt{t}, t) \). In particular, for any \( \varepsilon > 0 \),

\[
 \mathbb{P}_x(\text{OT}_t(H) > \varepsilon) \leq \frac{16\ell(H)}{\varepsilon\ell(G)} \max(\ell(G)\sqrt{t}, t).
\]
Proof. By the definition of occupation time we have
\[ E_x[\text{OT}_t(H)] = E_x \int_0^t \mathbf{1}_{\{B(s) \in H\}} \, ds = \int_0^t p_s(x, y) \mathcal{H}(dy) \, ds, \]
which by Lemma 4.1 is bounded by
\[ 4\ell(H) \int_0^t \max(s^{-1/2}, \ell(G)^{-1}) \, ds \leq 4\ell(H)(2\sqrt{t} + t\ell(G)^{-1}) \]
and now the assertion follows easily.

The second part is just a consequence of Markov’s inequality. □

Another consequence of Lemma 4.1 is the following corollary.

Corollary 4.3. Let \( X \) be a graph-like continuum with \( \mathcal{H}(X) < \infty \), and \( (G_n)_{n \in \mathbb{N}} \) a graph approximation of \( X \). For any \( t_0 > 0 \), \( H \subseteq X \) and any \( x_n \in G_n \), we have
\[ \sup_n \mathbb{P}_{x_n}(B_n(t_0) \in H) \leq 4\mathcal{H}(H) \max(t_0^{-1/2}, \ell(G_0)^{-1}). \]

5. Uniqueness. The following fact implies that if \( \mathcal{H}(X) < \infty \), then the Brownian motion we constructed in Section 3 is uniquely determined by the metric space \((X, d)\); in particular, it does not depend on the choice of the graph approximation used.

Theorem 5.1. Let \( X \) be a graph-like space with \( \mathcal{H}(X) < \infty \). Then for every increasing graph approximation \((G_n)_{n \in \mathbb{N}}\), and any convergent sequence \((x_n)_{n \in \mathbb{N}}\) of points of \( X \) with \( x_n \in G_n \), \((\mu_n, x_n)_{n \in \mathbb{N}}\) converges weakly to an element of \( \mathcal{M} \) independent of the choice of \((G_n)_{n \in \mathbb{N}}\) and \((x_n)_{n \in \mathbb{N}}\).

This is immediate from the following lemma. The independence of the limit from \((G_n)\) follows from the fact that if \((H_n)\) is another graph approximation of \( X \), then \( G_1, H_1, G_2, H_2, \ldots \) is also a graph approximation. The lemma is the most crucial part of our uniqueness argument.

Lemma 5.2. Let \( X \) be a graph-like space with \( \mathcal{H}(X) < \infty \) and \((G_n)_{n \in \mathbb{N}}\) a graph approximation of \( X \). Let \( x_n \in G_n \) be a sequence of points that converges to a point \( x \in X \) and \((B_n(t))_{t \geq 0}\) be a Brownian motion on \( G_n \) starting at \( x_n \). Then the finite-dimensional distributions of \((B_n)_{n \geq 0}\) converge weakly, i.e. for any \( k \in \mathbb{N} \), any times \( 0 \leq t_1 < \cdots < t_k \) and Borel subsets \( A_1, \ldots, A_k \) of \( X \), the sequence \( \mathbb{P}(B_n(t_1) \in A_1, \ldots, B_n(t_k) \in A_k) \) converges.

The rest of this section is devoted to the proof of Lemma 5.2.

5.1. Useful facts about graph-like spaces. We will be using the following terminology and facts from [18].
Theorem 5.3 ([18]). Let $X$ be a graph-like space with $\mathcal{H}(X) < \infty$, and $(G_n)_{n \in \mathbb{N}}$ be a graph approximation of $X$. Then for any two sequences $(p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}}$ with $p_n, q_n \in G_n$, each converging to a point in $X$, the effective resistance $R_{G_n}(p_n, q_n)$ converges. If $p_n = p$, $q_n = q$ are constant sequences, then this convergence is from above, i.e. $\lim_n R_{G_n}(p, q) \leq R_G(p, q)$ for every $i$.

A pseudo-edge of a metric space $X$ is an open connected subspace $f$ such that $|\partial f| = 2$ and no homeomorphic copy of the interval $(0, 1)$ contained in $\overline{f}$ contains a point in $\partial f$. We denote the elements of $\partial f$ by $f^0$, $f^1$, and call them the endpoints of $f$. Note that every edge is a pseudo-edge. See [18] for further examples.

We define the discrepancy $\delta(f)$ of a pseudo-edge $f$ by $\delta(f) := \mathcal{H}(f) - d(f^0, f^1)$, which is always non-negative [18]. Similarly, for a metric graph $G \subseteq X$ and pseudo-edge $f$ of $X$ such that $G \cap \overline{f}$ contains an $f^0$-$f^1$ path, we define $\delta_G(f) := \ell(f^G) - d_{G \cap \overline{f}}(f^0, f^1)$, where $f^G$ is the connected component of $G \cap \overline{f}$ which contains $f^0$ and $f^1$.

Theorem 5.4 ([18]). For every graph-like continuum $X$ with $\mathcal{H}(X) < \infty$, and every $\varepsilon > 0$, there is a finite set $\mathcal{F}$ of pairwise disjoint pseudo-edges of $X$ with the following properties:

(i) $\sum_{f \in \mathcal{F}} \mathcal{H}(f) > \mathcal{H}(X) - \varepsilon$;
(ii) $\sum_{f \in \mathcal{F}} \delta(f) < \varepsilon$;
(iii) for every $f \in \mathcal{F}$, and every graph approximation $(G_n)_{n \in \mathbb{N}}$, $G_n \cap \overline{f}$ is connected and contains an $f^0$-$f^1$ path for almost every $n$;
(iv) for any $x \in X$ the set $\mathcal{F}$ can be chosen in such a way that $x \notin \bigcup \mathcal{F}$;
(v) for any finite edge-set $E$ the set $\mathcal{F}$ can be chosen in such a way that $E \subseteq \mathcal{F}$.

If $G \subseteq X$ is a metric graph and $\mathcal{F}$ a finite collection of pseudo-edges such that for any $f \in \mathcal{F}$, $G \cap \overline{f}$ is connected and contains an $f^0$-$f^1$ path then we can define a reduced version $G^\mathcal{F}$ of $G$ as a metric graph where each $f \cap G$ is replaced by an edge $e_f$ with length $R_{G \cap \overline{f}}(f^0, f^1)$. If $f \in \mathcal{F}$ is already an edge then $e_f$ is the same edge. We denote the corresponding Brownian motion on $G^\mathcal{F}$ by $B^\mathcal{F}$.

5.2. Proof of Lemma 5.2. We will prove the lemma using two couplings; the first will show that $B_n(t)$ is ‘close’ to $B_n^\mathcal{F}(t)$, and the second will allow us to compare $B_n^\mathcal{F}(t)$ with $B_m^\mathcal{F}(t)$. These couplings are the subject of the next two lemmas $[2]$.

(2) An alternative description of these couplings, offering some figures and a simplified version, can be found at arXiv:1405.6580.
We write $\mathbb{1}_f$ for the indicator function of the set $f \subset X$, and $e_f^c$ for the complement of $e_f$ in $G^F$.

**Lemma 5.5.** Let $t, \delta > 0$ and $0 < \beta \leq t$. Then for any metric graph $G$ and a finite collection of disjoint pseudo-edges $\mathcal{F}$ with $\bigcup \mathcal{F} \subseteq G \subseteq X$ and
\[
\sum_{f \in \mathcal{F}} \delta_G(f) < \delta,
\]
there is a coupling $(B, B^F)$ of the Brownian motions defined on $G$ and $G^F$ respectively, both starting from a given $x_0 \in X \setminus \bigcup \mathcal{F}$, such that
\[
\mathbb{P}\left( \sum_{f \in \mathcal{F}} \sup_{0 \leq s \leq t} \inf_{|r| \leq \beta} |r| \mathbb{1}_f(B(s))\mathbb{1}_{e_f^c}(B^F(s + r)) > 0 \right) 
\leq 256 \delta \max \left( \sqrt{t}, \frac{t}{\sum_{f \in \mathcal{F}} \ell(P_f) - \sum_{f \in \mathcal{F}} \ell((G \cap \bar{f}) \setminus P_f)} \right).
\]

In words, this lemma provides a coupling $(B, B^F)$ such that if $B(s)$ lies in some pseudo-edge then the trajectory $B^F(s - \beta, s + \beta)$ intersects the corresponding edge with high probability.

**Proof of Lemma 5.5.** For the purposes of the proof we need a more elaborate way of constructing $G^F$ than described above. Namely, for any $f \in \mathcal{F}$ let $P_f$ be a shortest arc in $G \cap f$ joining $f_0$ and $f_1$. We define $I_f$ to be the set of line segments of $G \cap f$, i.e.
\[
I_f := \text{Int}\{x \in P_f : (G \cap \bar{f}) \setminus \{x\} \text{ has two connected components}\},
\]
where Int denotes interior. Clearly, $I_f = (x_1, y_1) \cup \cdots \cup (x_l, y_l)$ is a finite (possible empty) union of open subarcs of $P_f$ and, since $P_f$ is the shortest path,
\[
\ell(P_f \setminus I_f) \leq \ell((G \cap \bar{f}) \setminus P_f) = \delta_G(f),
\]
which in turn gives
\[
\ell((G \cap \bar{f}) \setminus I_f) \leq 2\delta_G(f).
\]
Now the edge $e_f$ is obtained by replacing each connected component of $(G \cap \bar{f}) \setminus I_f$ by an edge (see Figure 2). More precisely, a connected component of $(G \cap \bar{f}) \setminus I_f$ joining $y_i$ to $x_{i+1}$ is replaced by an edge of length $R((G \cap \bar{f}) \setminus I_f, (y_i, x_{i+1}))$. Then the length of $e_f$ is equal to $R_G(f^0, f^1)$. Let $J_f := (G \cap \bar{f}) \setminus I_f$ and let $J_{e_f}$ be the corresponding subset of $e_f$ where each component of $J_f$ is replaced by an edge. Note that $\partial J_f = \partial J_{e_f}$ for any $f \in \mathcal{F}$. 
We choose $\delta_1 > 0$ such that the $\delta_1$-neighborhoods of any two connected components of $\bigcup_{f \in F} J_f$ remain disjoint and such that

$$\sum_{f \in F} \ell(J_{\delta_1}^f \setminus J_f) < \delta,$$

where $J_{\delta_1}^f := \{ x \in G \cap \overline{f} : d(x, J_f) < \delta_1 \}$ is the $\delta_1$-neighborhood of $J_f$ in $G \cap \overline{f}$. In particular, $\partial J_{\delta_1}^f = \partial J_{\varepsilon_0}^f$. Now we set

$$\Pi := \bigcup_{f \in F} \partial J_f \quad \text{and} \quad \Pi' := \bigcup_{f \in F} \partial J_{\delta_1}^f.$$

Observe that both sets are subsets of $G^F$ as well.

As the above construction preserves resistances, i.e.

$$R_G(x, y) = R_G^F(x, y) \quad \text{for any } x, y \in G \setminus \bigcup_{f \in F} J_f,$$

Lemma 2.7 implies that the distribution of $B(\tau_{\Pi})$ is the same as that of $B^F(\tau_{\Pi})$ under $P_x$ for any $x \in \Pi' \cup \{x_0\}$. Similarly, the random variables $B(\tau_{\Pi'})$ and $B^F(\tau_{\Pi'})$ are equidistributed under any measure from the set $\{P_x\}_{x \in \Pi}$.

On the bipartite graph $\Pi \cup \Pi'$ we consider Markov chains $Y^{(1)}, Y^{(2)}$ defined by

$$Y^{(1)}_n := B(\sigma_n^{(1)}), \quad Y^{(2)}_n := B^F(\sigma_n^{(2)}).$$
Then \( \tilde{B} \) is a Brownian motion on \( G \) starting at \( x_0 \), and \( \tilde{B}^F \) is a Brownian motion on \( G^F \) also starting from \( x_0 \). Moreover, if we denote by \( \tilde{\sigma}_k^{(1)} \) (resp. \( \tilde{\sigma}_k^{(2)} \)) the corresponding stopping time with \( B \) replaced by \( \tilde{B} \) (resp. with \( B^F \) replaced by \( \tilde{B}^F \)), then \( \tilde{B}(\tilde{\sigma}_n^{(1)}) = \tilde{B}^F(\tilde{\sigma}_n^{(2)}) = Y_n^{(1)} \).

Now we claim that if

\[
\text{(3)} \quad \text{OT}_{t+\beta} \left( \bigcup_{f \in \mathcal{F}} J_{f}^{\tilde{\delta}_1}, \tilde{B} \right) < \beta/2, \\
\text{(4)} \quad \text{OT}_{t+\beta} \left( \bigcup_{f \in \mathcal{F}} J_{e_f}^{\tilde{\delta}_1}, \tilde{B}^F \right) < \beta/2,
\]

then for any \( 0 \leq s \leq t \) and \( f \in \mathcal{F} \), we have

\[
\text{(5)} \quad \inf_{|r| \leq \beta} 1_f(B(s))1_{e_f}(\tilde{B}^F(s + r)) = 0.
\]
Indeed, as $D$’s part of the trajectory of $\tilde{B}$ is included in $\bigcup_{f \in \mathcal{F}} J_f^{\delta_1}$, i.e.

$$\tilde{B}(r) \in \bigcup_{f \in \mathcal{F}} J_f^{\delta_1} \quad \text{whenever } \tilde{\sigma}^{(1)}_{2k} \leq r \leq \tilde{\sigma}^{(1)}_{2k+1},$$

we have

$$\sum_{k=0}^{\infty} (r \wedge \tilde{\sigma}^{(1)}_{2k+1} - r \wedge \tilde{\sigma}^{(1)}_{2k}) < \beta/2$$

for any $r \leq t + \beta$. The same argument yields

$$\sum_{k=0}^{\infty} (r \wedge \tilde{\sigma}^{(2)}_{2k+1} - r \wedge \tilde{\sigma}^{(2)}_{2k}) < \beta/2.$$

As $\tilde{\sigma}^{(1)}_{2k+2} - \tilde{\sigma}^{(1)}_{2k} = \tilde{\sigma}^{(2)}_{2k+2} - \tilde{\sigma}^{(2)}_{2k+1}$, we infer $|r \wedge \tilde{\sigma}^{(1)}_{j} - r \wedge \tilde{\sigma}^{(2)}_{j}| \leq \beta/2$ for any $j \in \mathbb{N}_0$ and $r \leq t + \beta$.

Suppose now that $\tilde{\sigma}^{(1)}_{2k+1} \leq s \leq \tilde{\sigma}^{(1)}_{2k+2}$ for some $k \in \mathbb{N}_0$. Then, as $\sigma^{(1)}_{2k+1} = \sigma^{(2)}_{2k+1} + q$ with $|q| \leq \beta/2$, we conclude that $B(s) = \tilde{B}^F(s-q)$ which implies (5). If now $\tilde{\sigma}^{(1)}_{2k} \leq s \leq \tilde{\sigma}^{(1)}_{2k+1}$ for some $k \in \mathbb{N}_0$ then $\tilde{B}(s) \in f$ if and only if $\tilde{B}^{(\tilde{\sigma}^{(1)}_{2k})} \in f$. However, by the same argument as in the previous case we get $\tilde{B}^{(\tilde{\sigma}^{(1)}_{2k})} = \tilde{B}^F(\tilde{\sigma}^{(1)}_{2k} - q)$ with $|q| \leq \beta/2$. In conclusion, if $\tilde{B}(s) \in f$ then $\tilde{B}^F(s+r) \in \mathcal{E}_f$ for some $|r| \leq \beta$, showing (5).

Finally, taking into account that $\beta < t$, $\ell(\bigcup_{f \in \mathcal{F}} J_f^{\delta_1}) \leq \ell(\bigcup_{f \in \mathcal{F}} J_f^{\delta_1}) \leq 2\delta$ and $\ell(G) \geq \sum_{f \in \mathcal{F}} \ell(e_f)$, we can estimate

$$\mathbb{P}\left(\sum_{f \in \mathcal{F}} \sup_{0 \leq s \leq t} \inf_{|t| \leq \beta} \mathbb{1}_f(B(s)) \mathbb{1}_{e_f}(B^F(s+r)) > 0\right)$$

$$\leq \mathbb{P}\left(\text{OT}_{t+\beta} \left(\bigcup_{f \in \mathcal{F}} J_f^{\delta_1}, \tilde{B}\right) \geq \beta/2\right) + \mathbb{P}\left(\text{OT}_{t+\beta} \left(\bigcup_{f \in \mathcal{F}} J_f^{\delta_1}, \tilde{B}^F\right) \geq \beta/2\right)$$

$$\leq 256 \frac{\delta}{\beta} \max\left(\sqrt{t}, t/\sum_{f \in \mathcal{F}} \ell(e_f)\right).$$

Let $G$ be a metric graph and let $S \subseteq G$ be a finite set. We define $p_{x,y}^{S,G}$ as the probability that Brownian motion on $G$ starting at $x$ hits $S$ at $y$.

The next lemma is similar to the previous one. The main difference is that the set of pseudo-edges is replaced by a set of edges. On the other hand, the resistances are no longer the same and hence the corresponding hitting distributions are also different.

**Lemma 5.6.** Let $G^1, G^2$ be metric graphs and $\mathcal{E}$ be a finite set of disjoint edges in $G^1 \cap G^2$. Let $\gamma \in (0, 1/2)$ be fixed such that the $\gamma$-neighborhood of an edge from $\mathcal{E}$ is again an edge in $G^1 \cap G^2$ and set $\Pi := \bigcup_{e \in \mathcal{E}} \partial e^{\gamma}$,
\( \Pi' := \bigcup_{e \in \mathcal{E}} \partial e \). Let \( x^1 \in G^1 \setminus \bigcup_{e \in \mathcal{E}} e^\gamma \) and \( x^2 \in G^2 \setminus \bigcup_{e \in \mathcal{E}} e^\gamma \) and suppose that

\[
\sum_{y \in \Pi'} \left| p_{x^1,y}^{\Pi',G^1} - p_{x^2,y}^{\Pi',G^2} \right| \leq \gamma^3, \tag{6}
\]

\[
\sum_{x \in \Pi, y \in \Pi'} \left| p_{x,y}^{\Pi,G^1} - p_{x,y}^{\Pi,G^2} \right| \leq \gamma^3. \tag{7}
\]

Then for any \( \delta > 0 \) such that \( \ell(G^1 \setminus \mathcal{E}), \ell(G^2 \setminus \mathcal{E}) \leq \delta \) and any \( \beta \in (0, t) \) there exists a coupling \( B^1, B^2 \) of Brownian motions on \( G^1 \) and \( G^2 \) starting at \( x^1 \) and \( x^2 \) respectively, such that

\[
P \left( \sum_{e \in \mathcal{E}} \sup_{0 \leq s \leq t} \inf_{|r| \leq \beta} 1_e(B^1(s))1_e^c(B^2(s + r)) > 0 \right) \leq 256 \frac{\delta}{\beta} \max \left( \sqrt{t}, \frac{t}{\sum_{e \in \mathcal{E}} \ell(e)} \right) + 6\gamma^2 + 24\gamma t. \]

**Proof.**

Fig. 3. Intervals between blue points stand for elements of \( \mathcal{E} \) in Lemma 5.6 whereas intervals between red points are elements of the form \( e^\gamma \). The metric graphs \( G^1 \) and \( G^2 \) may differ only on the part covered by the orange balls.

Similarly to the proof of Lemma 5.5 for \( i = 1, 2 \), we define stopping times \( \sigma^{(i)} \), processes \( W, D^{(i)} \) and random variables \( Y^{(i)} \). Namely,

\[
\sigma^{(i)}_0 := \inf \{ t > 0 : B^i(t) \in \Pi' \},
\]

\[
\sigma^{(i)}_{2k+1} := \inf \{ t > \sigma^{(i)}_{2k} : B^i(t) \in \Pi \},
\]

\[
\sigma^{(i)}_{2k+2} := \inf \{ t > \sigma^{(i)}_{2k+1} : B^i(t) \in \Pi' \},
\]

\( Y^{(i)}_n := B^n(\sigma^{(i)}_n) \), and \( W = (W^{x \to y})_{x \in \Pi', y \in \Pi} \) is a collection of independent processes, where \( W^{x \to y} \) is a Brownian motion starting at \( x \) and conditioned to hit \( \Pi \) at \( y \). Similarly, \( D_i \) is a collection of \( (D^{x \to y}_i)_{x \in \Pi \cup \{x^1\}, y \in \Pi'} \), where \( D^{x \to y}_i \) is a Brownian motion starting at \( x \) and conditioned to hit \( \Pi' \) at \( y \).
First, we construct the coupling between $Y^{(1)}$ and $Y^{(2)}$. For this let us note that

$$\nu_i^i(y) := p^{(\Pi')}_{x_i, y} = \mathbb{P}(\nu_0^i = y) \quad \text{for } y \in \Pi',$$

$$p_i(x, y) := p^{(\Pi')}_{x, y} = \mathbb{P}(\nu_{2k+2}^i = y \mid \nu_{2k+1}^i = x) \quad \text{for } (x, y) \in \Pi \times \Pi',$$

$$p^i(x, y) = p^2(x, y) := p^{(\Pi')}_{x, y} = \mathbb{P}(\nu_{2k+1}^{(1)} = y \mid \nu_{2k}^{(1)} = x) \quad \text{for } (x, y) \in \Pi' \times \Pi.$$

From our assumptions we infer that

$$I_0 := \sum_{y \in \Pi'} |\nu_1^i(y) - \nu_2^i(y)| < \gamma^3,$$

and for any $x \in \Pi$,

$$I(x) := \sum_{y \in \Pi'} |p^1(x, y) - p^2(x, y)| \leq \gamma^3.$$

For $(x, y) \in \Pi \times \Pi'$ we define

$$q_0(y) := \min(\nu_1^i(y), \nu_2^i(y)) \quad \text{and} \quad q(x, y) := \min(p^1(x, y), p^2(x, y)).$$

Then $\sum_{y \in \Pi'} q_0(y) = 1 - I_0$ and $\sum_{y \in \Pi'} q(x, y) = 1 - I(x)$.

Let $(U_n)_{n \geq 0}$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$ and let $(Z_n)$ be an independent Markov chain with the following transition probabilities:

$$\mathbb{P}(Z_0 = y) := q_0(y)/(1 - I_0),$$

$$\mathbb{P}(Z_{2k+1} = y \mid Z_{2k} = x) := \mathbb{P}(\nu_{2k+1} = y \mid \nu_{2k} = x) = \mathbb{P}(\nu_{2k+1} = y \mid \nu_{2k} = x),$$

$$\mathbb{P}(Z_{2k+2} = y \mid Z_{2k+1} = x) := q(x, y)/(1 - I(x)).$$

We define $\sigma := \inf \{2k \geq 0 : U_{2k} < I(Z_{2k-1})\}$, with the convention $I(Z_{-1}) := I_0$. Let $(\tilde{Y}_1^1, \tilde{Y}_2^2)$ be a Markov chain on $G^1 \times G^2$ such that

$$\tilde{Y}_k^1 = \tilde{Y}_k^2 = Z_k \quad \text{for } k < \sigma,$$

$$\mathbb{P}(\tilde{Y}_0^1, \tilde{Y}_0^2) = (y, y') = \frac{(\nu_1^1(y) - q_0(y))}{I_0} \frac{(\nu_2^2(y') - q_0(y'))}{I_0} \quad \text{if } \sigma = 0,$$

$$\mathbb{P}(\tilde{Y}_k^1, \tilde{Y}_k^2) = (y, y') \mid Y_{k-1}^1 = Y_{k-1}^2 = y$$

$$= \frac{(p^1(x, y) - q(x, y))}{I(x)} \frac{(p^2(x, y') - q(x, y'))}{I(x)} \quad \text{for } k = \sigma,$$

$$\mathbb{P}(\tilde{Y}_k^1, \tilde{Y}_k^2) = (y, y') \mid (\tilde{Y}_{k-1}^1, \tilde{Y}_{k-1}^2) = (x, x') = p^1(x, y)p^2(x', y') \quad \text{for } \sigma > k.$$

In words, $\tilde{Y}_1$ and $\tilde{Y}_2$ coincide with $Z$ up to time $\sigma$ and then evolve independently.

Let $\tilde{B}^1$ be the concatenation of

$$D_{1,1}^{x_1 \to \tilde{Y}_0^1}, W_{1}^{\tilde{Y}_0^1 \to \tilde{Y}_1^1}, D_{1,2}^{\tilde{Y}_1^1 \to \tilde{Y}_2^1}, W_{2}^{\tilde{Y}_2^1 \to \tilde{Y}_3^1}, D_{1,3}^{\tilde{Y}_3^1 \to \tilde{Y}_4^1}, \ldots$$
and $\tilde{B}^2$ the concatenation of 
$$D_{2,1} \rightarrow \tilde{Y}_6^2, W_1 \rightarrow \tilde{Y}_1^2, D_{2,2} \rightarrow \tilde{Y}_2^2, W_2 \rightarrow \tilde{Y}_3^2, D_{2,3} \rightarrow \tilde{Y}_4^2, \ldots$$

Suppose that 
$$\max \left( \text{OT}_{t+\gamma} \left( G^1 \setminus \bigcup \mathcal{E}, \tilde{B}^1 \right), \text{OT}_{t+\gamma} \left( G^2 \setminus \bigcup \mathcal{E}, \tilde{B}^2 \right) \right) < \beta/2,$$
$$\min \left( \tilde{\sigma}^{(1)}_{2\gamma+2}, \tilde{\sigma}^{(2)}_{2\gamma+2} \right) > 2t$$

Then the same reasoning as in the proof of Lemma 5.5 leads to 
$$\inf_{|r| \leq \beta} 1_e(\tilde{B}^1(s))1_e(\tilde{B}^2(s+r)) = 0$$
for any $e \in \mathcal{E}$ and $s \leq t$.

Now let $10t/\gamma^2 \leq M \leq 12t/\gamma^2$ and observe that 
$$\mathbb{P}(\sigma < M) \leq 1 - (1 - \gamma^3)^M \leq 1 - (81/256)^\gamma^3 M \leq 2\gamma^3 M \leq 24t\gamma.$$

On the other hand, since every path between times $\tilde{\sigma}^{(i)}_{2k+1}$ and $\tilde{\sigma}^{(i)}_{2k+2}$ has to traverse an edge between points in $\Pi$ and $\Pi'$, it contains a part of the trajectory from the first hitting (after $\tilde{\sigma}^{(i)}_{2k+1}$) to a midpoint and then to the set $\Pi \cup \Pi'$. Thus, we conclude that $\tilde{\sigma}^{(i)}_{2k+2} - \tilde{\sigma}^{(i)}_{2k+1}$ stochastically dominates $T$, the first hitting time of $\{-\gamma/2, \gamma/2\}$ for the standard Brownian motion. Standard calculations (e.g. using the exponential martingales and the optional stopping theorem) show that $\mathbb{E}T = \gamma^2/4$ and $\mathbb{E}T^2 = \frac{5}{48}\gamma^4$. Hence, by Chebyshev’s inequality, 
$$\mathbb{P}(\sigma^{(i)}_{2M+2} \leq 2t) \leq \mathbb{P}(T_1 + \cdots + T_M \leq 2t) \leq \frac{5}{48}\gamma^4 M \frac{1}{t^2} < 3\gamma^2.$$ 

As a result we obtain 
$$\mathbb{P} \left( \sum_{e \in \mathcal{E}} \sup_{0 \leq s \leq t} \inf_{|r| \leq \beta} 1_e(\tilde{B}^1(s))1_e(\tilde{B}^2(s+r)) > 0 \right)$$
$$\leq \mathbb{P} \left( \text{OT}_{t+\gamma} \left( G^1 \setminus \bigcup \mathcal{E} \right) \geq \beta/2 \right) + \mathbb{P} \left( \text{OT}_{t+\gamma} \left( G^2 \setminus \bigcup \mathcal{E} \right) \geq \beta/2 \right)$$
$$+ \mathbb{P}(\sigma^{(1)}_{2M+2} \leq 2t) + \mathbb{P}(\sigma^{(2)}_{2M+2} \leq 2t) + \mathbb{P}(\sigma < M)$$
$$\leq 256\frac{\delta}{\beta} \max \left( \sqrt{t}, \frac{t}{\sum_{f \in \mathcal{E}} \ell(e)} \right) + 6\gamma^2 + 24\gamma t,$$

proving the lemma. □

In order to be able to apply Lemma 5.6, we need to justify that the corresponding hitting distributions converge. Namely, let $X_0$ be a connected component of $X \setminus \bigcup \mathcal{F}$ and set $H_n := G_n \cap X_0$. Furthermore, let $\tilde{H}_n$ be $H_n$ with attached edges $e_f$ in place of $f \in \mathcal{F}$ (the same edges for all $n$).
Lemma 5.7. Suppose that for every \( f \in \mathcal{F} \) there is a sequence \((y^f_n)\) such that \((y^f_n) \in e_f, y^f_n \to y^f\) and \(\inf_n d(y^f_n, H_n) > 0\). Then for any \( x_n \in H_n \) with \( x_n \to x \) and any \( f \in \mathcal{F} \), the sequence \( p^{S_n, \tilde{H}_n}_{x_n, y^f_n} \), where \( S_n := \{y^f_n : f \in \mathcal{F}\} \), converges.

Proof. Fix \( \varepsilon > 0 \). Since \( y^f_n \to y^f \) and \(\inf_n d(y^f_n, H_n) > 0\), we can choose points \( z^f \) between \( y^f \) and \( H_n \) such that for large enough \( n \) the Brownian motion on \( \tilde{H}_n \) starting from \( z^f \) hits \( y^f \) with probability at least \( 1 - \varepsilon \), so
\[
d(z^f, y^f) < \varepsilon d(z^f, H_n).
\]
This implies that, for large \( n \) and \( S := \{z^f : f \in \mathcal{F}\} \),
\[
|p^{S_n, \tilde{H}_n}_{x_n, y^f_n} - p^{S_n, \tilde{H}_n}_{x_n, z^f}| \leq 2\varepsilon,
\]
and hence it suffices to prove the convergence of the sequence \( p^{S_n, \tilde{H}_n}_{x_n, z^f} \).

To this end, fix \( f_0 \in \mathcal{F} \) and denote by \( \hat{H}_n \) the metric graph obtained from \( \tilde{H}_n \) by contracting all vertices \( \{z^f : f \neq f_0\} \) to a single point \( z \); in other words, all the points \( z^f \) different from \( z^{f_0} \) are being glued together. Then \( p^{S_n, \hat{H}_n}_{x_n, z^{f_0}} = p^{\{z, z^{f_0}\}, \hat{H}_n}_{x_n, z^{f_0}} \). In view of Lemma 2.7,
\[
p^{\{z, z^{f_0}\}, \hat{H}_n}_{x_n, z^{f_0}} = \frac{R_{\hat{H}_n}(z, x_n) - R_{\hat{H}_n}(z^{f_0}, x_n) + R_{\hat{H}_n}(z, z^{f_0})}{2R_{\hat{H}_n}(z, z^{f_0})}.
\]
As \( (\hat{H}_n) \) is an increasing sequence of metric graphs, with \( \sup_n \ell(\hat{H}_n) < \infty \), by Theorem 5.3 we deduce that all the resistances appearing above converge. Moreover, as \( R_{\hat{H}_n}(z, z^{f_0}) \geq d(z^{f_0}, H_n) \), we conclude that the probability \( p^{\{z, z^{f_0}\}, \hat{H}_n}_{x_n, z^{f_0}} \) converges as well, proving the lemma. \( \blacksquare \)

Now we are ready to prove the convergence result.

Proof of Lemma 5.2. Our aim is to show that for any \( k \in \mathbb{N} \), \( 0 < t_1 < \cdots < t_k \) and Borel subsets \( A_1, \ldots, A_k \subseteq X \) the sequence
\[
\mathbb{P}(B_n(t_1) \in A_1, \ldots, B_n(t_k) \in A_k)
\]
converges as \( n \) goes to infinity. Since the space \( X \) is separable, the sigma-algebra generated by a topological basis is the Borel sigma-algebra. By Lemma 2.3 we may assume that each \( A_i \) is an open set such that each point from \( \partial A_i \) is contained in an edge. We show that the sequence above is a Cauchy sequence.

For this let us fix \( 0 < \varepsilon < 1 \). There is a finite collection \( \mathcal{A} \) of edges contained in \( X \), each of length \( \delta_1 > 0 \), such that \( \bigcup_i \partial A_i \subseteq \bigcup \mathcal{A} \) and
\[
\ell\left( \bigcup \mathcal{A} \right) \leq \varepsilon.
\]
For large enough \( n_0 \), \( \bigcup \mathcal{A} \subseteq G_{n_0} \).
Let $0 < \beta < t_k \land 1$ be such that for the standard Brownian motion $W$ on $\mathbb{R}$ we have
\[ \mathbb{P}\left( \max_{0 \leq s \leq 2\beta} |W_s| \geq \delta_1/2 \right) < \varepsilon, \]
and choose $\delta$ satisfying $0 < \delta \leq \beta \varepsilon \land (\mathcal{H}(X)/4)$. Theorem 5.4 yields a finite collection $\mathcal{F}$ of pseudo-edges such that $x_n \notin \bigcup \mathcal{F}$ for $n \geq n_1$, $A \subseteq \mathcal{F}$, and
\begin{align*}
& (i) \sum_{f \in \mathcal{F}} \mathcal{H}(f) > \mathcal{H}(X) - \delta; \\
& (ii) \sum_{f \in \mathcal{F}} \delta(f) < \delta; \\
& (iii) \text{for every } f \in \mathcal{F}, \text{ a } G_n \cap \overline{f} \text{ is connected and contains an } f^0-f^1 \text{ path.}
\end{align*}
Denote by $\mathcal{F}_n$ the collection of all $f \cap G_n$ where $f \in \mathcal{F}$. The choice of $\delta$ and $n_1$ ensures that $\sum_{f \in \mathcal{F}} \ell(f \cap G_n) \geq \sum_{f \in \mathcal{F}} d(f^0, f^1) \geq \frac{1}{2} \mathcal{H}(X)$. Now for each $A_i$ and $f \in \mathcal{F} \setminus A$, the intersection $f \cap G_n$ is either contained in $A_i$ or disjoint from $A_i$. Therefore, we can define the corresponding set $A^n_i \subseteq G^n_{\mathcal{F}_n}$ by
\[ A^n_i := (A_i \cap (X \setminus \bigcup_{f \in \mathcal{F} \setminus A} f)) \cup \bigcup_{f \in \mathcal{F} \setminus A, f \subseteq A} e^n_f, \]
where $e^n_f$ is the contracted pseudo-edge $\overline{f} \cap G_n$. Next, we have
\[ |\mathbb{P}(B_n(t_1) \in A_1, \ldots, B(t_k) \in A_k) - \mathbb{P}(B^m(t_1) \in A_1, \ldots, B(t_k) \in A_k)| \]
\[ \leq |\mathbb{P}(B_n(t_i) \in A_i, 1 \leq i \leq k) - \mathbb{P}(B^m_n(t_i) \in A^n_i, 1 \leq i \leq k)| \\
+ |\mathbb{P}(B_m(t_i) \in A_i, 1 \leq i \leq k) - \mathbb{P}(B^m(t_i) \in A^n_{n_m}, 1 \leq i \leq k)| \\
+ |\mathbb{P}(B_n(t_i) \in A^n_i, 1 \leq i \leq k) - \mathbb{P}(B^m(t_i) \in A^n_{n_m}, 1 \leq i \leq k)|, \]
where $B^m_k$ is the Brownian motion on $G^m_{\mathcal{F}}$ starting at $x_k$. We estimate each of the summands separately. Let $(B_n, B^m)$ be the coupling from Lemma 5.5 with $t = t_k$:
\[ |\mathbb{P}(B_n(t_i) \in A_i, 1 \leq i \leq k) - \mathbb{P}(B^m(t_i) \in A^n_i, 1 \leq i \leq k)| \]
\[ \leq \mathbb{P}(\{B_n(t_i) \in A_i, 1 \leq i \leq k\} \triangle \{B^m(t_i) \in A^n_i, 1 \leq i \leq k\}) \]
\[ \leq \sum_{i=1}^{k} \mathbb{P}(\{B_n(t_i) \in A_i\} \triangle \{B^m(t_i) \in A^n_i\}). \]
Now observe that on the event
\[ (\{B_n(t_i) \in A_i\} \triangle \{B^m(t_i) \in A^n_i\}) \cap \left\{ B_n(t_i) \in \bigcup \mathcal{F}_n \right\} \cap \left\{ B^m(t_i) \notin \bigcup A \right\} \]
\[ \cap \left\{ \sum_{f \in \mathcal{F}} \sup_{0 \leq s \leq t} \inf_{|r| \leq \beta} 1_f(B_n(s))1_{(e^n_f)}^c(B^m(s + r)) = 0 \right\} \]
there are $f \in \mathcal{F} \setminus A$ and $-\beta \leq r \leq \beta$ such that $B_n(t_i) \in f$ and $B^m(t_i + r) \in e^n_f$. This means that the trajectory $B^m([t_i - \beta, t_i + \beta])$ crosses an edge
from $\mathcal{A}$. The latter, by the choice of $\beta$, happens with probability smaller than $\varepsilon$:

\[
\mathbb{P}(\{B_n(t_i) \in A_i\} \triangle \{B^F_n(t_i) \in A^m_i\})
\leq \varepsilon + \mathbb{P}\left(B_n(t_i) \notin \bigcup F_n\right) + \mathbb{P}\left(B_n(t_i) \in \bigcup \mathcal{A}\right) + \mathbb{P}\left(B^F_n(t_i) \in \bigcup \mathcal{A}\right)
\]
\[
+ \mathbb{P}\left(\sum_{f \in F} \sup_{0 \leq s \leq t} \inf_{|t| \leq \beta} \mathbb{I}_f(B_n(s))\mathbb{I}(\varepsilon) \sigma(B^F_n(s + r)) > 0\right)
\]
\[
\leq \varepsilon + 4\delta \max(t_i^{-1/2}, (\frac{1}{2} \mathcal{H}(X))^{-1}) + 8\varepsilon \max(t_i^{-1/2}, (\frac{1}{2} \mathcal{H}(X))^{-1})
\]
\[
+ 256\frac{\beta}{\delta} \max(\sqrt{t_i}, t_i(\frac{1}{2} \mathcal{H}(X))^{-1})
\]
\[
\leq \varepsilon(1 + t_i^{-1/2} + \mathcal{H}(X)^{-1})(25 + 256t_k).
\]

It remains to estimate the difference

\[
\mathbb{P}(B^F_n(t_i) \in A^n_i, 1 \leq i \leq k) - \mathbb{P}(B^F_m(t_i) \in A^m_i, 1 \leq i \leq k).
\]

To this end, observe that for any $f \in \mathcal{F}$ we have $\ell(e^n_f) \rightarrow \ell(e_f)$, and let $l_0 := \min_{f \in \mathcal{F}} \ell(f)$. We choose $0 < \gamma < \min(\delta/|\mathcal{F}|, l_0/4)$ and $n_2$ such that for $n \geq n_2$, $\ell(e^n_f) \geq \ell(e_f) - 3\gamma$. For any $f \in \mathcal{F}$ and large $n_2$ we define $\tilde{e}_f^n$ as a subedge of $e^n_f$ with the same center and length $\ell(e_f) - 3\gamma$. We denote by $\mathcal{E}$ the set of $\tilde{e}_f^n$ (there is a natural identification between $\tilde{e}_f^n$ and $\tilde{e}_f^m$ for $n, m \geq n_2$). Let $\Pi$ and $\Pi'$ be defined as in Lemma 5.6. If $x \in \Pi$, $y \in \Pi'$ belong to the closure of the same connected component of $G^F_n \setminus \Pi'$, then there are $f_1, f_2 \in \mathcal{F}$ and $p \in \partial f_1$, $q \in \partial f_2$ in the same component such that

\[
R_{(G^F_n \setminus \Pi') \cup \{x,y\}}(x,y) = (\ell(e^n_{f_1}) - \ell(e_{f_1}) + 3\gamma)/2 + R_{(G_n \setminus \Pi')} (p,q)
\]
\[
+ (\ell(e^n_{f_2}) - \ell(e_{f_2}) + 3\gamma)/2,
\]

which converges, by Theorem 5.3. As a result we conclude that $p_{\Pi',G^F_n}$ converges. A similar argument shows that $p_{\Pi',G^F_n}$ converges as well. Therefore, there is $n_3$ such that for $n, m \geq n_3$ and $G^1 := G^F_n$, $G^1 := G^F_m$, conditions (6) and (7) are fulfilled. An application of Lemma 5.6 with $\delta$ replaced by $2\delta$, provides a coupling between $B^F_n$ and $B^F_m$. We can now estimate

\[
|\mathbb{P}(B^F_n(t_i) \in A^n_i, 1 \leq i \leq k) - \mathbb{P}(B^F_m(t_i) \in A^m_i, 1 \leq i \leq k)|
\]
\[
\leq \sum_{i=1}^{k} \mathbb{P}(\{B^F_n(t_i) \in A^n_i\} \triangle \{B^F_m(t_i) \in A^m_i\}).
\]
The same argument as before gives

\[ P(\{B^{F_n}(t_i) \in A^n_i \} \triangle \{B^{F_m}(t_i) \in A^m_i \}) \]

\[ \leq \epsilon + P(B^{F_n}(t_i) \notin \bigcup \mathcal{E}) + P(B^{F_n}(t_i) \in \bigcup A) + P(B^{F_m}(t_i) \in \bigcup A) \]

\[ + P\left( \sum_{f \in \mathcal{E}} \inf_{0 \leq s \leq t} \mathbb{1}_e(B^{F_n}(s)) \mathbb{1}_{(e)}(B^{F_m}(s + r)) > 0 \right) \]

\[ \leq \epsilon + 8\delta \max(t_i^{-1/2}, (1/2 \mathcal{H}(X))^{-1}) + 8\epsilon \max(t_i^{-1/2}, (1/2 \mathcal{H}(X))^{-1}) \]

\[ + 512\frac{\delta}{\beta} \max(\sqrt{t_k}, t_k(1/2 \mathcal{H}(X))^{-1}) + 6\gamma^2 + 24\gamma t_k \]

\[ \leq \epsilon(1 + t_i^{-1/2} + \mathcal{H}(X)^{-1})(50 + 536t_k). \]

Summing up, for \( n, m \geq \max(n_0, n_1, n_2) \), we have

\[ |P(B_n(t_1) \in A_1, \ldots, B(t_k) \in A_k) - P(B_m(t_1) \in A_1, \ldots, B(t_k) \in A_k)| \]

\[ \leq \epsilon 2k(1 + t_i^{-1/2} + \mathcal{H}(X)^{-1})(50 + 536t_k), \]

showing that (9) is a Cauchy sequence as claimed. ■

6. Strong Markov property. By the previous section we know that for any open \( A \) in \( X \) and \( x_n \to x \) the probabilities \( P_{x_n}(B(t) \in A) \) converge to \( P_X(B(t) \in A) \). For any continuous function \( u \) on \( G \) we define \( P^n_t u(y) := E_y[u(B_n(t))] \) for \( y \in G_n \) and zero otherwise. By the Portmanteau theorem, \( P^n_t u(x_n) \) converges to \( P_t u(x) := E_x[u(B(t))] \).

The strong Markov property follows by similar methods to those in [4]. We start with an elementary lemma.

**Lemma 6.1.** Suppose \( u \) and \( u_n \) are functions on \( X \) with the property that \( u_n(x_n) \to u(x) \) for all sequences \( (x_n)_{n \in \mathbb{N}} \) such that \( x_n \in G_n \) and \( x_n \to x \). Then \( u \) is continuous and

\[ \sup_{y \in G_n} |u_n(y) - u(y)| \to 0. \]

**Proof.** In order to prove continuity observe that, since \( \bigcup G_n \) is dense in \( G \), it is enough to show that for \( x_n \in G_n, x_n \to x \) we have \( u(x_n) \to u(x) \). Since \( u_m(x_n) \to u(x_n) \) as \( m \to \infty \), we can take an increasing sequence \( m_n \) such that \( u_{m_n}(x_n) - u(x_n) \) goes to zero. Since \( u_{m_n}(x_n) \) is a subsequence of \( u_k(x'_k) \), where \( x'_k = x_n \) when \( k \in [m_n, m_{n+1}) \), we see that \( u_{m_n}(x_n) \to u(x) \).

This implies that \( u \) is continuous.

Suppose that the second part of the theorem fails. Then we have a subsequence \( n_k \) and \( x_{n_k} \to x \) with \( |u_{n_k}(x_{n_k}) - u(x_{n_k})| > \epsilon \) for some \( \epsilon > 0 \). But

\[ |u_{n_k}(x_{n_k}) - u(x_{n_k})| \leq |u_{n_k}(x_{n_k}) - u(x)| + |u(x) - u(x_{n_k})| \]

The same argument as before gives

\[ P(\{B^{F_n}(t_i) \in A^n_i \} \triangle \{B^{F_m}(t_i) \in A^m_i \}) \]

\[ \leq \epsilon + P(B^{F_n}(t_i) \notin \bigcup \mathcal{E}) + P(B^{F_n}(t_i) \in \bigcup A) + P(B^{F_m}(t_i) \in \bigcup A) \]

\[ + P\left( \sum_{f \in \mathcal{E}} \inf_{0 \leq s \leq t} \mathbb{1}_e(B^{F_n}(s)) \mathbb{1}_{(e)}(B^{F_m}(s + r)) > 0 \right) \]

\[ \leq \epsilon + 8\delta \max(t_i^{-1/2}, (1/2 \mathcal{H}(X))^{-1}) + 8\epsilon \max(t_i^{-1/2}, (1/2 \mathcal{H}(X))^{-1}) \]

\[ + 512\frac{\delta}{\beta} \max(\sqrt{t_k}, t_k(1/2 \mathcal{H}(X))^{-1}) + 6\gamma^2 + 24\gamma t_k \]

\[ \leq \epsilon(1 + t_i^{-1/2} + \mathcal{H}(X)^{-1})(50 + 536t_k). \]
goes to zero by assumption and the continuity of $u$. This contradiction proves the theorem.

**Corollary 6.2.** For $t > 0$ and every continuous function $f$ on $G$, the function $P_t f$ is also continuous and
\[
\sup_{y \in G_n} |P_t f(y) - P^n_t f(y)| \to 0.
\]

**Theorem 6.3.** The heat semigroup $P_t$ is a Feller semigroup. In particular the process $B(t)$ satisfies the strong Markov property.

**Proof.** By Corollary 6.2 we know that $P_t$ maps $C(X)$ into $C(X)$. We have to show that the family \{${P}_t$\} is a semigroup.

From the Markov property of $B_n$ we know that $P^n_{t+s} = P^n_t P^n_s$. Therefore it is enough to show that, for any $u \in C(X)$, $P^n_t P^n_s u(x_n)$ converges to $P_t P_s u(x)$ whenever $x_n \to x$. We have
\[
|P^n_t P^n_s u(x_n) - P_t P_s u(x)| \leq |P^n_t P^n_s u(x_n) - P^n_t P_s u(x_n)| + |P^n_t P_s u(x_n) - P^n_t P_s u(x_n)|.
\]
Since the first term is bounded by $\sup_{y \in G_n} |P^n_s u(y) - P^n_s u(y)|$, it goes to 0 by Corollary 6.2. Similarly, the second term converges to 0 since $P_s u$ is continuous. The last term vanishes by the continuity of $P_t P_s u$.

Since $B(t)$ is continuous and $B(0) = x$, we have $P_t u(x) \to u(x)$ as $t \to 0$ for any continuous function $u$.

7. **Cover time.** The (expected) cover time $CT_x(G)$ of a finite metric graph $G$ from a point $x \in G$ is the expected time until standard Brownian motion from $x$ on $G$ has visited every point of $G$. The cover time of $G$ is $CT(G) := \sup_{x \in G} CT_x(G)$. It is proved in [19] that there is an upper bound on $CT(G)$ depending only on the total length $\ell(G)$ of $G$ and not on its structure.

**Theorem 7.1 ([19]).** For every finite graph $G$ and $\ell : E(G) \to \mathbb{R}_{>0}$, we have $CT(G) \leq 2\ell(G)^2$.

In this section we use this fact to deduce the corresponding statement for our Brownian motion $B$ on a graph-like continuum $X$: defining $CT(X)$ as above, with standard Brownian motion replaced by our process $B$, we prove the following theorem.

**Theorem 7.2.** For every graph-like continuum $X$ with $\mathcal{H}(X) = L < \infty$, we have $CT(X) \leq 20L^2$.

Let $B$ be a Brownian motion on a metric graph $G$. We define the (random) cover time $\tau$ by
\[
\tau = \inf \{t : G = B((0, t])\}.
\]
Then $\text{CT}_x(G) = \mathbb{E}_x[\tau]$. In order to prove the above theorem we will need the following bound on the second moment of the cover time in terms of its expectation.

**Lemma 7.3.** Let $G$ be a finite metric graph. Suppose that for a constant $Q \in \mathbb{R}$ we have $\mathbb{E}_x[\tau] \leq Q$ for every $x \in G$. Then $\mathbb{E}_x[\tau^2] \leq 24Q^2$ for every $x \in G$.

**Proof.** By the Markov inequality we have

$$P_x(\tau \geq s) \leq \mathbb{E}_x[\tau]/s \leq Q/s$$

for every $s$; setting $s = 2Q$, we obtain

(10) $$P_x(\tau \geq 2Q) \leq 1/2.$$ We claim that for every $k \in \mathbb{N}$ we have

(11) $$P_x(\tau \geq 2Qk) \leq (1/2)^k.$$ To see this, we subdivide time into intervals of length $2Q$. Since inequality (10) holds for every initial point $x$, the probability that in the $i$th time interval $[(i-1)2Q, i2Q]$ the process fails to cover the whole space $G$ is at most $1/2$. Thus, if we run the process up to time $2Qk$, in which case we have $k$ such ‘trials’, the probability of not covering $G$ in any of them is at most $(1/2)^k$, proving our claim. Note that we have been generous here, as we are ignoring the part of $G$ that was covered before the $i$th interval begins.

Using this, we can bound the second moment of $\tau$ as follows:

$$\mathbb{E}_x[\tau^2] = \int_0^\infty 2tP_x(\tau \geq t) \, dt,$$

by Fubini’s theorem. Splitting time $t$ into intervals of length $2Q$, the last integral can be rewritten as

$$\sum_{k=0}^\infty \int_{k2Q}^{(k+1)2Q} 2tP_x(\tau \geq t) \, dt \leq 2 \sum_{k=0}^\infty \int_{k2Q}^{(k+1)2Q} tP_x(\tau \geq k2Q) \, dt$$

$$\leq 2 \sum_{k=0}^\infty (2Q)^2(k + 1/2)P_x(\tau \geq 2kQ)$$

$$\leq 8Q^2 \sum_{k=0}^\infty (k + 1/2)(1/2)^k = 24Q^2.$$ Using our bound for the second moment of $\tau$ from Lemma 7.3 we can prove the following lemma.

**Lemma 7.4.** Let $(G_n)_{n \in \mathbb{N}}$ be a graph approximation of a graph-like continuum $X$. Denote by $\tau_n$ the cover time for $G_n$ (by $B_n$). Suppose that for
a constant $Q \in \mathbb{R}$ we have $E_x[\tau_n] \leq Q$ for every $x \in G_n$. Then for every $x \in X$,

$$E_x[\tau] \leq 10Q.$$ 

**Proof.** We would like to use the weak convergence of the law $\mu_n$ of Brownian motion $B_n$ on $G_n$ to the law $\mu$ of our limit process $B$ (Theorem 1.1) to deduce that $E_x[\tau]$ is finite from Theorem 7.1. However, we cannot do so directly as the cover time $\tau$ is not a continuous function from $C$ to $\mathbb{R}$. To overcome this difficulty, we introduce a function $h(t) : C \to \mathbb{R}$ (parametrised by time $t$) that is continuous and is closely related to $\tau$.

Let $r > 0$ be some (small) real number. For a path $\omega \in C$, denote by $h_r(t)[\omega]$ the total length of the set $\{x \in G : d(x, \omega(s)) > r$ for every $s \leq t\}$; in other words, if we think of $\omega$ as the trajectory of a particle of ‘width’ $r$, then $h_r(t)[\omega]$ is the length of the part of $G$ that this particle has not covered by time $t$. We also define the normalised version $h_r(t)[\omega] := h_r(t)[\omega]/L$, where $L$ is again the total length of $G$. It is no loss of generality to assume that $L = 1$.

For any fixed $T, M \in \mathbb{R}$, the function

$$\omega \mapsto \left( \int_0^T (h_r(t)[\omega])^{1/M} dt \right)^2$$

as a mapping from $C$ to $\mathbb{R}$ is continuous. We can now apply the weak convergence of $\mu_{n,x}$ to $\mu_x$ to obtain

$$E_x\left[ \left( \int_0^T (h_r(t))^{1/M} dt \right)^2 \right] = \lim_{n \to \infty} E^n_x\left[ \left( \int_0^T (h_r(t))^{1/M} dt \right)^2 \right] \leq \lim_{n \to \infty} E^n_x\left[ \left( \int_0^T \mathbf{1}_{[h_r(t)>0]} dt \right)^2 \right],$$

where we have used the fact that $h_r(t) \leq 1$. Since $\ell(G) - \ell(G_n)$ converges to 0, we deduce that if a path $\omega$ covers $G_n$ at time $t$, for sufficiently large $n$ compared to $r$, then $h_r(t)[\omega] = 0$. It follows that the expression in parenthesis can be bounded from above by $\tau_n$, and so by Lemma 7.3 we conclude that

$$E_x\left[ \left( \int_0^T (h_r(t))^{1/M} dt \right)^2 \right] \leq \lim_{n \to \infty} E^n_x [\tau_n^2] \leq 24Q^2. \quad (12)$$

Now let $\varepsilon > 0$. Note that if $h_r(T) > \varepsilon$, then $h_r(t) > \varepsilon$ for every $t < T$ since $h_r(t)$ is decreasing in $t$. This easily implies

$$E_x[T^2 \varepsilon^{2/M} \mathbf{1}_{[h_r(T) > \varepsilon]}] \leq E_x\left[ \left( \int_0^T (h_r(t))^{1/M} dt \right)^2 \right],$$
which combined with (12) yields

\[ T^2 \varepsilon^2/M \mathbb{P}_x(h_r(T) > \varepsilon) \leq 24Q^2. \]

As \( M \) can be chosen arbitrarily large independently of \( \varepsilon \), we have

\[ \mathbb{P}_x(h_r(T) > \varepsilon) \leq 24Q^2/T^2. \]

Letting \( \varepsilon \) tend to 0 we deduce

\[ \mathbb{P}_x(h_r(T) > 0) \leq 24Q^2/T^2. \]

Observe that the events \( \{h_r(T) > 0\} \) decrease to \( \{h_0(T) > 0\} = \{\omega : \tau(\omega) > T\} \) as \( r \) goes to 0. Hence

\[ \mathbb{P}_x(\tau > T) \leq 24Q^2/T^2. \]

Finally, we have

\[ \mathbb{E}_x[\tau] = \int_0^\infty \mathbb{P}_x(\tau > t) \, dt \leq Q\sqrt{24} + \int_0^\infty \frac{24Q^2}{t^2} \, dt = 2\sqrt{24}Q < 10Q. \]

**Proof of Theorem 7.2** Let \((G_n)_{n \in \mathbb{N}}\) be any graph approximation of \( X \). Note that \( \ell(G_n) \leq \mathcal{H}(X) =: L \) for every \( n \) by the definition of \( \mathcal{H} \). Thus we can plug the constant \( Q = 2L^2 \) from Theorem 7.1 into Lemma 7.4 to obtain the bound \( 10Q = 20L^2 \) on the cover time of \( X \).

**Corollary 7.5.** The process \((B_t)\) is positive recurrent.

**8. Further properties.** In this section we show that the Hausdorff measure on \( X \) is stationary for our process, and that our process behaves locally like standard Brownian motion on \( \mathbb{R} \) inside any edge of \( X \).

Recall that any edge \( e \subset X \) can be viewed as an interval contained in the real line, that is, there is \( F : e \to \mathbb{R} \) which is an isometry onto its image. The next lemma shows that our process \( B \) locally coincides with the standard Brownian motion \( W \).

**Proposition 8.1.** Let \( e \) be an edge in \( X \). For any continuous function \( \phi \) with \( k - 1 \) arguments each taking values in \( \mathbb{R} \), any increasing sequence \( t_1, \ldots, t_k \), and any \( x \in \mathbb{R} \), we have

\[ \mathbb{E}_x[\phi(F(B(t_1)), \ldots, F(B(t_{k-1})))1_{t_k < \tau_{\partial e}}] = \mathbb{E}_{F(x)}[\phi(W(t_1), \ldots, W(t_{k-1}))1_{t_k < \tau_{\partial F(e)}}]. \]

**Proof.** Since the equation is true for \( B_n \), we would like to pass to the limit with \( n \) to prove that \( B \) also satisfies it, but first we have to deal with the discontinuity of the indicator under the expectation sign. For any \( \delta > 0 \)
and $n$ we have
\[
\mathbb{E}_x \left[ \phi(F(B_n(t_1)), \ldots, F(B_n(t_{k-1}))) \text{dist}(B_n[0, t_k], \partial e)^\delta \right] = \mathbb{E}_{F(x)} \left[ \phi(W(t_1), \ldots, W(t_{k-1})) \text{dist}(W[0, t_k], \partial F(\varepsilon))^\delta \right].
\]
Since the function under the expectation sign is continuous, now we can pass to the limit with $n$ and next, by Lebesgue’s theorem, with $\delta$ to 0, proving the desired equality. ■

**Proposition 8.2.** The Hausdorff measure $\mathcal{H}$ on $X$ is the unique (up to multiplicative constant) invariant measure for the process $B$.

**Proof.** Let $(G_n)_{n \in \mathbb{N}}$ be a graph approximation of $X$. Then $\mathcal{H}_n := \mathcal{H}(G_n)$ is the sum of the lengths of the edges of $G_n$, and it is proved in [18] that $\mathcal{H}(X) = \lim_n \mathcal{H}(G_n)$. Moreover, it is not hard to check that the measure $\mathcal{H}_n$ is invariant for $P_t^n$. Hence, by Lebesgue’s theorem, for any bounded continuous $u$, we have
\[
\int P_t u \, d\mathcal{H} = \lim_n \int P_t^n u \, d\mathcal{H} = \lim_n \int 1_{G_n} P_t^n u \, d\mathcal{H} = \lim_n \int u \, d\mathcal{H}_n = \int u \, d\mathcal{H}.
\]
Since by Theorem 7.2 the process is recurrent, $\mathcal{H}$ is the unique invariant measure (cf. [24]). ■

**9. Comparison with the process of [2].** In this section we show that when the space $X$ is an $\mathbb{R}$-tree $T$ with $\mathcal{H}(T) < \infty$, the process constructed in this paper coincides with the one in [2]. In order to prove this, we investigate resolvents associated with these two processes and show that they are equal.

First, let us briefly recall the definitions introduced in [2]. We assume that the $\mathbb{R}$-tree $(T, d)$ has finite Hausdorff measure $\mathcal{H}(T)$. For any $x_0 \in T$ and any absolutely continuous function $f : T \to \mathbb{R}$ there is a measurable $\nabla f \in L^1_{\text{loc}}(\mathcal{H})$ such that
\[
f(x) - f(x_0) = \int_{[x_0, x]} \nabla f(y) \, d\mathcal{H}(dy),
\]
where $[x_0, x]$ is the unique closed arc between $x_0$ and $x$. This allows us to define a bilinear form by
\[
\mathcal{E}(f, g) = \int_T \nabla f(x) \nabla g(x) \, d\mathcal{H}(dx)
\]
for any $f, g \in D(\mathcal{E})$, where the domain $D(\mathcal{E})$ is defined by
\[
D(\mathcal{E}) = \{ f \text{ absolutely continuous} : f, \nabla f \in L^2(\mathcal{H}) \}.
\]
Notice that although the sign of $\nabla f$ depends on the choice of $x_0$, the form
$\mathcal{E}$ itself is independent of $x_0$. For any $\alpha > 0$ we set

$$\mathcal{E}_\alpha(f,g) = \mathcal{E}(f,g) + \alpha \langle f, g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $L^2(\mathcal{H})$. Then $D(\mathcal{E})$ with the scalar product $\mathcal{E}_\alpha$ is a Hilbert space, i.e. $(\mathcal{E}, D(\mathcal{E}))$ is in fact a Dirichlet space.

Having defined the Dirichlet form we can now introduce the $\alpha$-capacity and resolvent. For $\alpha > 0$ we set

$$\text{cap}_\alpha, x = \inf \{ \mathcal{E}_\alpha(f,f) : f \in D(\mathcal{E}), f(x) = 1 \}.$$

By [2, Lemma 3.6] there is a unique function $h_{\alpha, x}$ for which the above infimum is attained. Next, we define the resolvent $G_\alpha$ as a bounded operator $G_\alpha : L^2(\mathcal{H}) \to D(\mathcal{E})$ such that

$$\mathcal{E}_\alpha(G_\alpha f, g) = \langle f, g \rangle$$

for any $f \in L^2(\mathcal{H})$ and $g \in D(\mathcal{E})$. Then by [2, Proposition 3.9] we infer

$$G_\alpha f(x) = \int f(y) \frac{h_{\alpha, y}(x)}{\text{cap}_{\alpha, y}} \mathcal{H}(dy).$$

On the other hand, if $W_t$ is the process associated with the Dirichlet form $\mathcal{E}$ then

$$G_\alpha f(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} f(W_t) \, dt \right]$$

for any $f \in L^2(\mathcal{H})$.

Let $T^n$ be an increasing graph approximation of $T$, i.e. $T^n$ is a metric graph with finitely many vertices such that $T^n \subset T^{n+1}$ and $\bigcup T^n = T$. By $W^n_t$, $\mathcal{E}^n_\alpha$, $\text{cap}^n_{\alpha, x}$, etc. we denote the corresponding process, Dirichlet form, capacity, etc., for the metric graph $T^n$. Since for $T^n$ the process $W^n$ coincides with the Brownian motion $B^n$, the corresponding resolvents are equal. In order to show that the process $W_t$ has the same law as the process constructed in this paper we prove that the resolvents corresponding to both processes are limits of resolvents on $T^n$.

We begin with a preparatory lemma:

**Lemma 9.1.** For every $x \in T$ and $\alpha \geq 0$, there is $c > 0$ such that $\text{cap}^n_{\alpha, x} > c$ and $\lim_n \text{cap}^n_{\alpha, x} = \text{cap}_{\alpha, x}$. Moreover, $(h^n_{\alpha, x})$ converges to $h_{\alpha, x}$ in $L^2(\mathcal{H})$.

---

(3) There is a typo in this proposition: equation (3.21) in [2] should read (in their notation)

$$g_A^*(\kappa, \cdot) := \int \kappa(dx) \frac{h_{A, x}^*(\cdot)}{\text{cap}_{A}(x)}.$$
Proof. Take any \( x \in T^n \). Then there is an arc \([x, y]\) whose length is \( d = \frac{1}{2} \text{diam}(T^1) \). For \( c = \text{cap}_{[0, d]} \), the \( \alpha \)-capacity associated with the Dirichlet form on the interval \([0, d]\), we have \( \text{cap}_{\alpha, x}^n > c \).

To simplify notation, write \( h \) instead of \( h_{\alpha, x} \) and \( \text{cap} \) instead of \( \text{cap}_{\alpha, x} \).

It is easy to see that the values of \( h^n \) and \( h_\alpha \) are bounded between 0 and 1, because otherwise we could reduce e.g. \( \mathcal{E}_\alpha(h_\alpha, h_\alpha) \) by truncating its values that lie outside \([0, 1]\), that is, by setting \( h_\alpha(x) = 0 \) whenever \( h_\alpha(x) < 0 \) and setting \( h_\alpha(x) = 1 \) whenever \( h_\alpha(x) > 1 \).

To prove the first equality of the statement, let \( g^n = g^n_x : T \to \mathbb{R} \) be the function that coincides with \( h^n \) on \( T^n \), and is extended to \( T \setminus T^n \) constantly, that is, for every component \( C \) of \( T \setminus T^n \) (which is an \( \mathbb{R} \)-tree since \( T \) is an \( \mathbb{R} \)-tree), we let \( x_C \) be the unique point of \( T^n \) in the closure of \( C \), and set \( g^n(y) = h^n(x_C) \) for every \( y \in C \).

We may assume without loss of generality that \( x \in T^n \), and so \( g^n \) is a candidate for the definition of \( \text{cap}_{\alpha} \) on \( T \). Note that \( \nabla g^n \) vanishes in \( T \setminus T^n \), and coincides with \( \nabla h^n \) on all of \( T^n \) except on a set of ramification points, which set has measure 0 because it is countable. Here, we have used the fact that \( \mathcal{H}(T) < \infty \), hence \( T \setminus T^n \) has at most countably many components because each component has positive measure. Since \( \mathcal{H}(T \setminus T^n) \to 0 \), and our functions are uniformly bounded, we deduce that \( \mathcal{E}_\alpha(g^n, g^n) \leq \mathcal{E}_\alpha(h^n, h^n) \), which means that \( \text{cap} \leq \lim \inf_n \text{cap}^n \).

To prove the converse inequality, let \( g^n : T^n \to \mathbb{R} \) be the restriction of \( h \) (i.e. the unique minimiser of \( \mathcal{E}_\alpha(g, g) \) on \( T \)) to \( T^n \). Again, \( \nabla g^n \) can differ from the restriction of \( \nabla h \) to \( T^n \) only at ramification points, which have measure 0. We deduce that \( \mathcal{E}_\alpha(g^n, g^n) \leq \mathcal{E}_\alpha(h, h) \), and therefore \( \lim \sup_n \text{cap}^n \leq \text{cap} \).

Combining these two inequalities shows that \( \lim_n \text{cap}^n \) exists and equals \( \text{cap} \).

Next, we prove that the sequence \( h^n \) converges in \( L^2 \), where we think of \( h^n \) as a function on \( T \) rather than on its subspace \( T^n \), which we can by e.g. setting \( h^n = 0 \) on \( T \setminus T^n \). If not, then for some \( \epsilon > 0 \) we can find arbitrarily large \( n < m \) such that

\[
(h^n - h^m)^2 \, d\mathcal{H} = (h^n)^2 \, d\mathcal{H} + (h^m)^2 \, d\mathcal{H} - 2h^n h^m \, d\mathcal{H} \geq \epsilon.
\]

Let \( g^n \) be the restriction of \( h^m \) to \( T^n \). Since \( \mathcal{H}(T \setminus T^n) \to 0 \), the above inequality remains true if we replace \( h^m \) by \( g^n \), and choose \( \epsilon \) slightly smaller. We will use this observation to show that, letting \( f := (h^n + g^n)/2 \), we have \( \mathcal{E}_\alpha(f, f) < \mathcal{E}_\alpha(h^n, h^n) \), which contradicts the definition of \( h^n \) as the minimiser of \( \mathcal{E}_\alpha(\cdot, \cdot) \). For this, we bound the two summands in the definition of \( \mathcal{E}_\alpha(f, f) \) separately:

\[
\int f^2 \, d\mathcal{H} = \frac{1}{4} \left( \int (h^n)^2 \, d\mathcal{H} + \int (g^n)^2 \, d\mathcal{H} \right) + \frac{1}{2} \int h^n g^n \, d\mathcal{H}.
\]
Combining this with inequality (14), with $h^n$ replaced by $g^n$, we deduce
\[ \int f^2 \, d\mathcal{H} \leq \frac{1}{2} \left( \int (h^n)^2 \, d\mathcal{H} + \int (g^n)^2 \, d\mathcal{H} \right) - \frac{\epsilon}{4}. \]
For the other summand, note first that
\[ \nabla f = \nabla h^n + \nabla g^n. \]
Thus applying the same reasoning, except that we now replace $\epsilon$ by 0, we deduce
\[ \mathcal{E}(f, f) \leq \frac{1}{2} \left( \mathcal{E}(h^n, h^n) + \mathcal{E}(g^n, g^n) \right) - \frac{\epsilon}{4}. \]
Combining these two inequalities we obtain
\[ \mathcal{E}_\alpha(f, f) \leq \frac{1}{2} \left( \mathcal{E}_\alpha(h^n, h^n) + \mathcal{E}_\alpha(g^n, g^n) \right) - \frac{\epsilon}{4}. \]
But as we have proved that $\mathcal{E}_\alpha(h^n, h^n)$ converges (to cap), by choosing $n$ large enough we can make $\mathcal{E}_\alpha(g^n, g^n)$ arbitrarily close to $\mathcal{E}_\alpha(h^n, h^n)$, whence the latter inequality implies $\mathcal{E}_\alpha(f, f) < \mathcal{E}_\alpha(h^n, h^n)$, contradicting the fact that $h^n$ is the minimiser of $\mathcal{E}_\alpha(\cdot, \cdot)$.

This contradiction proves that the sequence $(h^n)$ converges in $L^2$. The same arguments imply that $(\nabla h^n)$ also converges in $L^2$. To see that the limit $h' := \lim_n h^n$ coincides with $h$, observe that
\[ \mathcal{E}_\alpha(h', h') = \lim_n \mathcal{E}_\alpha(h^n, h^n) = \text{cap} \]
(this can be seen by using the fact that both $(h^n)$ and $(\nabla h^n)$ converge in $L^2$), i.e. $h'$ is a minimiser of $\mathcal{E}_\alpha(\cdot, \cdot)$ on $T$. Thus $h' = h$ by the uniqueness of minimisers. \Box

**Theorem 9.2.** Let $B$ be the process on $T$ constructed in Theorem 1.1. Then for any $x \in T$ under the measure $\mathbb{P}_x$ the process $B$ has the same law as $W$.

**Proof.** The law of the process is uniquely determined by the corresponding resolvent hence it suffices to prove that $\tilde{G}_\alpha = G_\alpha$, where $\tilde{G}_\alpha$ is the resolvent associated with $B$ and $G_\alpha$ with $W$. Since the resolvent is a bounded operator, it is enough to check that $\tilde{G}_\alpha f(x) = G_\alpha f(x)$ for any non-negative continuous function $f : T \to \mathbb{R}$.

Using the fact that $W^n$ converges weakly to $B$, by the dominated convergence theorem we deduce that $G^n_\alpha f(x)$ converges to $\tilde{G}_\alpha f(x)$. Our aim is to prove that $G^n_\alpha f(x)$ converges as well to $G_\alpha f(x)$. To this end, take any non-negative $\psi \in L^2(\mathcal{H})$. Then by Fubini’s theorem, Lemma 9.1 and the
dominated convergence theorem, we have
\[
\lim_{n \to \infty} \langle G_n^\alpha f, \psi \rangle = \lim_{n \to \infty} \int \int f(y) \frac{h_n^\alpha(y)}{\text{cap}_{\alpha,y}^n} \psi(x) \mathcal{H}(dy) \mathcal{H}(dx)
\]
\[
= \lim_{n \to \infty} \int f(y) \frac{h_\alpha^y(x) \psi(x) \mathcal{H}(dx)}{\text{cap}_{\alpha,y}} \mathcal{H}(dy)
\]
\[
= \int \int f(y) \frac{h_\alpha^y(x) \psi(x) \mathcal{H}(dx)}{\text{cap}_{\alpha,y}} \mathcal{H}(dy) = \langle G_\alpha f, \psi \rangle.
\]

Therefore we conclude that \( \hat{G}_\alpha = G_\alpha \).

10. Outlook. In this paper we constructed a diffusion \( B \) on graph-like spaces of finite length. The finite length condition plays an important role for the uniqueness of \( B \), and it is indeed not hard to find graph-like spaces of infinite length where the limit of the \( B_n \) as in our construction depends on the choice of the graph approximation \( (G_n)_{n \in \mathbb{N}} \).

An approach that can be used to try to avoid this situation, and hence extend our construction to spaces \( X \) of infinite length, is to endow \( X \) with a probability measure \( \mu \), and use this \( \mu \) in order to control the speed of the \( B_n \) as follows. Given any measured metric space \( (X, d, \mu) \), and a diffusion \( B : \mathbb{R}_+ \to Y \) on \( Y \), one can consider the function
\[
A_t := \int_Y L_t(x) d\mu(x),
\]
where \( L_t(x) \) denotes the local time of \( B \) at \( x \), and then reparametrise the diffusion by letting \( B'(t) = B(A_{t}^{-1}) \). This approach is standard in the study of diffusions on fractals; see e.g. [3, Chapter 4]. (We thank D. Croydon for suggesting this approach.)

A further interesting quest would be to relate our process to the theory of Dirichlet forms of [15].

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References

[1] D. Aldous and S. N. Evans, *Dirichlet forms on totally disconnected spaces and bipartite Markov chains*, J. Theoret. Probab. 12 (1999), 839–857.
[2] S. Athreya, M. Eckhoff, and A. Winter, *Brownian motion on \( \mathbb{R} \)-trees*, Trans. Amer. Math. Soc. 365 (2013), 3115–3150.

[3] M. T. Barlow, *Diffusions on fractals*, in: Lectures on Probability Theory and Statistics, Lecture Notes in Math. 1690, Springer, Berlin, 1998, 1–121.

[4] M. T. Barlow and R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet*, Ann. Inst. H. Poincaré Probab. Statist. 25 (1989), 225–257.

[5] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields 79 (1988), 543–623.

[6] M. T. Barlow, J. Pitman, and M. Yor, *On Walsh’s Brownian motions*, in: Séminaire de Probabilités XXIII, Lecture Notes in Math. 1372, Springer, Berlin, 1989, 275–293.

[7] J. R. Baxter and R. V. Chacon, *The equivalence of diffusions on networks to Brownian motion*, in: Conference in Modern Analysis and Probability, Contemp. Math. 26, Amer. Math. Soc., Providence, RI, 1984, 33–47.

[8] M. Baxter, *Markov processes on the boundary of the binary tree*, in: Séminaire de Probabilités XXVI, Lecture Notes in Math. 1526, Springer, Berlin, 1992, 210–224.

[9] A. D. Bendikov, A. A. Grigor’yan, Ch. Pittet, and W. Woess, *Isotropic Markov semigroups on ultra-metric spaces*, Russian Math. Surveys 69 (2014), 589–680.

[10] I. Benjamini and A.-S. Sznitman, *Giant component and vacant set for random walk on a discrete torus*, J. Eur. Math. Soc. 10 (2008), 133–172.

[11] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari, *The electrical resistance of a graph captures its commute and cover times (detailed abstract)*, in: Proc. 21st Annual ACM Sympos. on Theory of Computing, ACM, New York, 1989, 574–586.

[12] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni, *Cover times for Brownian motion and random walks in two dimensions*, Ann. of Math. 160 (2004), 433–464.

[13] J. Ding, J. R. Lee, and Y. Peres, *Cover times, blanket times, and majorizing measures*, Ann. of Math. 175 (2012), 1409–1471.

[14] B. Espinoza, P. Gartside, and M. Pitz, *Graph-like compacta: Characterizations and Eulerian loops*, J. Graph Theory 95 (2020), 209–239.

[15] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin, 2011.

[16] A. Georgakopoulos, *Uniqueness of electrical currents in a network of finite total resistance*, J. London Math. Soc. 82 (2010), 256–272.

[17] A. Georgakopoulos, *Graph topologies induced by edge lengths*, Discrete Math. 311 (2011), 1523–1542.

[18] A. Georgakopoulos, *On graph-like continua of finite length*, Topology Appl. 173 (2014), 188–208.

[19] A. Georgakopoulos and P. Winkler, *New bounds for edge-cover by random walk*, Combin. Probab. Comput. 23 (2014), 571–584.

[20] S. Goldstein, *Random walks and diffusions on fractals*, in: H. Kesten (ed.), Percolation Theory and Ergodic Theory of Infinite Particle Systems, IMA Vol. Math. Appl. 8, 1987, Springer, New York, 121–129.

[21] S. Haeseler, *Heat kernel estimates and related inequalities on metric graphs*, arXiv:1101.3010 (2011).

[22] B. M. Hambly, *Brownian motion on a random recursive Sierpiński gasket*, Ann. Probab. 25 (1997), 1059–1102.

[23] T. Hattori, *Asymptotically one-dimensional diffusions on scale-irregular gaskets*, J. Math. Sci. Univ. Tokyo 4 (1997), 229–278.
R. Z. Khas’minskiı, Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations, Theory Probab. Appl. 5 (1960), 179–196.

J. Kigami, Analysis on Fractals, Cambridge Univ. Press, Cambridge, 2008.

J. Kigami, Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees, Adv. Math. 225 (2010), 2674–2730.

W. B. Krebs, Hitting time bounds for Brownian motion, Proc. Amer. Math. Soc. 118 (1993), 223–232.

P. Kuchment, Quantum graphs: I. Some basic structures, Waves Random Media 14 (2004), 107–128.

T. Kumagai, Function spaces and stochastic processes on fractals, in: Fractal Geometry and Stochastics III, C. Bandt et al. (eds.), Progr. Probab. 57, Birkhäuser, Basel, 2004, 221–234.

S. Kusuoka, Lecture on diffusion processes on nested fractals, in: Statistical Mechanics and Fractals, Lecture Notes in Math. 1567, Springer, Berlin, 1993, 39–98.

R. Lyons and Y. Peres, Probability on Trees and Networks, Cambridge Ser. Statist. Probab. Math. 42, Cambridge Univ. Press, Cambridge, 2016.

P. Mörters and Y. Peres, Brownian Motion, Cambridge Ser. Statist. Probab. Math. 30, Cambridge Univ. Press, Cambridge, 2010.

J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958), 931–954.

K. R. Parthasarathy, Probability Measures on Metric Spaces, Academic Press, New York, 1967.

R. S. Strichartz, Analysis on fractals, Notices Amer. Math. Soc. 46 (1999), 1199–1208.

K. T. Sturm, Diffusion processes and heat kernels on metric spaces, Ann. Probab. 26 (1998), 1–55.

A.-S. Sznitman, Topics in Occupation Times and Gaussian Free Fields, Eur. Math. Soc., Zürich, 2012.

C. Thomassen and A. Vella, Graph-like continua, augmenting arcs, and Menger’s theorem, Combinatorica 28 (2008), 595–623.

J. B. Walsh, A diffusion with a discontinuous local time, Astérisque 52-53 (1978), 37–45.

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