PICK MATRICES AND QUATERNIONIC POWER SERIES

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Abstract. It is well known that a non-constant complex-valued function $f$ defined on the open unit disk $D$ is an analytic self-mapping of $D$ if and only if Pick matrices $\left[\frac{1-f(z_i)f(z_j)}{1-z_i\bar{z}_j}\right]_{i,j=1}^{n}$ are positive semidefinite for all choices of finitely many points $z_i \in D$. A stronger version of the “if” part was established by A. Hindmarsh [13]: if all $3 \times 3$ Pick matrices are positive semidefinite, then $f$ is an analytic self-mapping of $D$. In this paper, we extend this result to the non-commutative setting of power series over quaternions.

1. Introduction

The Schur class $\mathcal{S}$ of all analytic complex-valued functions mapping the unit disk $D$ into its closure has played a prominent role in function theory and its applications beginning with the work of I. Schur [17]. Among several alternative characterizations of the Schur class is one in terms of positive kernels: the function $f: D \to \mathbb{C}$ is in the class $\mathcal{S}$ if and only if the associated kernel $K_f(z, \zeta) = \frac{1-f(z)f(\zeta)}{1-z\bar{\zeta}}$ is positive on $D \times D$ or equivalently, if and only if the Pick matrix

$$P_f(z_1, \ldots, z_n) = \left[\frac{1-f(z_i)f(z_j)}{1-z_i\bar{z}_j}\right]_{i,j=1}^{n}$$

is positive semidefinite for any choice of finitely many points $z_1, \ldots, z_n \in D$. The “only if” part is the classical result of Pick and Nevanlinna [16, 15]. For the “if” part, let us observe that positivity of $1 \times 1$ matrices $P_f(z)$ already guarantees $|f(z)| \leq 1$ ($z \in D$). Thus, larger Pick matrices are needed in the “if” direction to guarantee the analyticity of $f$. The latter can be done via constructing the coisometric de Branges-Rovnyak realization [6] for $f$ or using a more recent lurking isometry argument [4]. A remarkable fact established by Hindmarsh [13] (see also [9]) is that analyticity is implied by positivity of all $3 \times 3$ Pick matrices. The objective of this paper is to extend this result to regular functions in quaternionic variable (Theorem 2.1 below).

Let $\mathbb{H}$ denote the skew field of real quaternions $\alpha = x_0 + ix_1 + jx_2 + kx_3$ where $x_\ell \in \mathbb{R}$ and $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = ijk = -1$. 

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−1. The real and the imaginary parts, the conjugate and the absolute value of a quaternion $\alpha$ are defined by
\[ \text{Re}(\alpha) = x_0, \quad \text{Im}(\alpha) = ix_1 + jx_2 + kx_3, \]
\[ \bar{\alpha} = \text{Re}(\alpha) - \text{Im}(\alpha) \quad \text{and} \quad |\alpha|^2 = \alpha\bar{\alpha} = |\text{Re}(\alpha)|^2 + |\text{Im}(\alpha)|^2, \]
respectively. By
\[ B = \{ \alpha \in \mathbb{H} : |\alpha| < 1 \} \]
we denote the unit ball in $\mathbb{H}$.

Since multiplication in $\mathbb{H}$ is not commutative, function theory over quaternions is somewhat different from that over the field $\mathbb{C}$. There have been several notions of regularity (or analyticity) for $\mathbb{H}$-valued functions, most notable of which are due to Moisil [14], Fueter [10, 11], and Cullen [7]. More recently, upon refining and developing Cullen’s approach, Gentili and Struppa introduced in [8] the notion of regularity which, being restricted to functions on a quaternionic ball around the origin, turns out to be the feature of power series with quaternionic coefficients on one side; we refer to the recent book [12] for a detailed exposition of the subject. Here we accept the following definition of regularity on the quaternionic unit ball.

**Definition 1.1.** A function $f : B \to \mathbb{H}$ is called left-regular on $B$ if it admits the power series expansion with quaternionic coefficients on the right which converges absolutely on $B$:
\[ f(z) = \sum_{k=0}^{\infty} z^k f_k \quad \text{with} \quad f_k \in \mathbb{H} \quad \text{such that} \quad \lim_{k \to \infty} \sqrt[k]{|f_k|} \leq 1. \quad (1.2) \]
If in addition, $f(\alpha) := \sum_{k=0}^{\infty} \alpha^k f_k \in B$ for all $\alpha \in B$, we say that $f$ belongs to the left Schur class $QS_L$. Right regular functions and the right Schur class $QS_R$ can be defined similarly.

Quaternionic Schur classes have become an object of intensive study quite recently. A number of related results (e.g., Möbius transformations, Schwarz Lemma, Bohr’s inequality) are presented in [12, Chapter 9]. Among other results, we mention realizations for slice regular functions [2], Schwarz-Pick Lemma [5], Blaschke products [3], Nevanlinna-Pick interpolation [1].

2. **Pick matrices and Hindmarsh’s theorem**

A straightforward entry-wise verification shows that the complex matrix \[ P_f(z_1, \ldots, z_n) = TP_f(z_1, \ldots, z_n)T^* = EE^* - NN^*, \quad (2.1) \]
where
\[ T = \begin{bmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_n \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad N = \begin{bmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{bmatrix}. \quad (2.2) \]

Since $|z_i| < 1$, the latter matrix is the unique matrix subject to identity \[ (2.1) \]. In case $z_i \in B$ and $f(z_i) \in \mathbb{H}$, the Stein equation \[ (2.1) \] still has a unique solution $P_f(z_1, \ldots, z_n)$ (still called the Pick matrix of $f$). Solving
this equation gives the explicit formula for the entries of \( P_f(z_1, \ldots, z_n) \) in terms of series

\[
P_f(z_1, \ldots, z_n) = \left[ \sum_{k=0}^{\infty} z_i^k (1 - f(z_i) \overline{f(z_j)}) z_j^k \right]_{i,j=1}^n
\]  \hspace{1cm} (2.3)

which converge due to the following estimate:

\[
\left| \sum_{k=0}^{\infty} z_i^k (1 - f(z_i) \overline{f(z_j)}) z_j^k \right| \leq 2 \sum_{k=0}^{\infty} |z_i|^k |z_j|^k = \frac{2}{1 - |z_i||z_j|}.
\]

According to a result from [1], for any function \( f \in QS_L \), the associated Pick matrix \( (2.3) \) is positive semidefinite for any choice of finitely many points \( z_1, \ldots, z_n \in \mathbb{B} \). The notions of adjoint matrices, of Hermitian matrices and positive semidefinite matrices over \( \mathbb{H} \) are similar to those over \( \mathbb{C} \) (we refer to a very nice survey [18] on this subject).

The following quaternionic analog of the Hindmarsh theorem [13] is the main result of the present paper.

**Theorem 2.1.** Let \( f : \mathbb{B} \to \mathbb{H} \) be given and let us assume that \( 3 \times 3 \) Pick matrices \( P_f(z_1, z_2, z_3) \) are positive semidefinite for all \( (z_1, z_2, z_3) \in \mathbb{B}^3 \). Then \( f \) belongs to \( QS_L \).

Before starting the proof we make several observations.

**Remark 2.2.** For any \( z_1, z_2 \in \mathbb{C} \), the quaternion \( z_1 j z_2 \) belongs to \( \mathbb{C} j \).

The statement follows from the multiplication table for imaginary units in \( \mathbb{H} \). We also remark that any quaternion \( \alpha = x_0 + i x_1 + j x_2 + k x_3 \) admits a unique representation \( \alpha = \alpha_1 + \alpha_2 j \) with \( \alpha_1, \alpha_2 \in \mathbb{C} \). Consequently, any quaternionic matrix \( A \) admits a unique representation \( A = A_1 + A_2 j \) with complex matrices \( A_1 \) and \( A_2 \).

**Remark 2.3.** The matrix \( A = A_1 + A_2 j \in \mathbb{H}^{n \times n} \) \( (A_1, A_2 \in \mathbb{C}^{n \times n}) \) is positive semidefinite if and only if the complex matrix \( \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \) is positive semidefinite (see [18]).

Two quaternions \( \alpha \) and \( \beta \) are called equivalent (conjugate to each other) if \( \alpha = h^{-1} \beta h \) for some nonzero \( h \in \mathbb{H} \). It follows (see e.g., [18]) that

\[
\alpha \sim \beta \quad \text{if and only if} \quad \text{Re}(\alpha) = \text{Re}(\beta) \quad \text{and} \quad |\alpha| = |\beta|.
\]  \hspace{1cm} (2.4)

Therefore, the conjugacy class of a given \( \alpha \in \mathbb{H} \) form a 2-sphere (of radius \( |\text{Im}(\alpha)| \) around \( \text{Re}(\alpha) \)).

**Remark 2.4.** If \( \alpha, \beta, \gamma \) are three distinct equivalent quaternions, then

\[
\gamma^k = (\gamma - \beta)(\alpha - \beta)^{-1} \alpha^k + (\alpha - \gamma)(\alpha - \beta)^{-1} \beta^k \quad \text{for all} \quad k = 0, 1, \ldots
\]  \hspace{1cm} (2.5)

Indeed, since \( \alpha \sim \beta \), it follows from (2.4) that \( (\alpha - \beta)\alpha(\alpha - \beta)^{-1} = \overline{\beta} \) and subsequently,

\[
(\alpha - \gamma)\alpha^k(\alpha - \gamma)^{-1} = \overline{\beta}^k = (\gamma - \beta)^{-1} \gamma^k(\gamma - \beta),
\]  \hspace{1cm} (2.6)
Proof of Theorem 2.1:
We first observe that for $t$ tending it by right linearity to power series (1.2) complete s the proof.

Indeed, for each fixed $g$ ad admits a (unique) representation (2.9) with $k$ for all integers $k \geq 0$. Then we get (2.5) from (2.6) and (2.7) as follows:

$$(\gamma - \beta)(\alpha - \beta)^{-1}\alpha^k + (\alpha - \gamma)(\alpha - \beta)^{-1}\beta^k = \gamma^k(\gamma - \beta)(\alpha - \beta)^{-1} + \gamma^k(\alpha - \gamma)(\alpha - \beta)^{-1} = \gamma^k.$$  

It turns out that the values of a regular function $f$ at two points from the same conjugacy class uniquely determine $f$ at any point from this class; the formula (2.8) below was established in [8] in a more general setting.

**Remark 2.5.** Let $f$ be left-regular on $\mathbb{B}$ and let $\alpha, \beta, \gamma \in \mathbb{B}$ be distinct equivalent points. Then

$$f(\gamma) = (\gamma - \beta)(\alpha - \beta)^{-1}f(\alpha) + (\alpha - \gamma)(\alpha - \beta)^{-1}f(\beta).$$  

Indeed, equality (2.5) verifies formula (2.8) for monomials $f(z) = z^k$. Extending it by right linearity to power series (1.2) completes the proof.

**Proof of Theorem 2.1.** We first observe that for $n = 1$, the formula (2.3) amounts to

$$P_f(z_1) = \sum_{k=0}^{\infty} z_1^k(1 - |f(z_1)|^2)z_1^k = \frac{1 - |f(z_1)|^2}{1 - |z_1|^2}$$

for each $z_1 \in \mathbb{B}$. Therefore, condition $P_f(z_1) \geq 0$ implies $|f(z_1)| \leq 1$.

It remains to show that $f$ is left-regular. Toward this end, we first show that there exist complex Schur-class functions $g$ and $h$ such that

$$f(\zeta) = g(\zeta) + h(\zeta)j \quad \text{for all} \quad \zeta \in \mathbb{D}. \quad (2.9)$$

Indeed, for each fixed complex point $\zeta \in \mathbb{C} \cap \mathbb{B} = \mathbb{D}$, the quaternion $f(\zeta) \in \mathbb{H}$ admits a (unique) representation (2.9) with $g(\zeta) \in \mathbb{C}$ and $h(\zeta) \in \mathbb{C}$. For any two points $\zeta_1, \zeta_2 \in \mathbb{D}$ and any $k \geq 0$, we then compute

$$\zeta_1^k(1 - f(\zeta_1)f(\zeta_2))\zeta_2^k = \zeta_1^k \zeta_2^k - \zeta_1^k \left[ g(\zeta_1)g(\zeta_2) - h(\zeta_1)h(\zeta_2) \right] \zeta_2^k$$

$$= \zeta_1^k \left[ 1 - g(\zeta_1)g(\zeta_2) - h(\zeta_1)h(\zeta_2) \right] \zeta_2^k + \zeta_1^k \left[ g(\zeta_1)h(\zeta_2) - h(\zeta_1)g(\zeta_2) \right] \zeta_2^k.$$

Summing up the latter equalities over all $k \geq 0$ gives

$$\sum_{k=0}^{\infty} \zeta_1^k(1 - f(\zeta_1)f(\zeta_2))\zeta_2^k = \frac{1 - g(\zeta_1)g(\zeta_2) - h(\zeta_1)h(\zeta_2)}{1 - \zeta_1\zeta_2}$$

$$+ \sum_{k=0}^{\infty} \zeta_1^k \left[ g(\zeta_1)h(\zeta_2) - h(\zeta_1)g(\zeta_2) \right] \zeta_2^k. \quad (2.10)$$
The first term on the right is complex whereas the second term belongs to $\mathbb{C}j$ by Remark 2.2. Let us consider the Pick matrix $P_f(\zeta_1, \zeta_2, \zeta_3)$ based on arbitrary points $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$. According to (2.3) and (2.10),

$$P_f(\zeta_1, \zeta_2, \zeta_3) = P_{f,1}(\zeta_1, \zeta_2, \zeta_3) + P_{f,2}(\zeta_1, \zeta_2, \zeta_3)$$

where

$$P_{f,1}(\zeta_1, \zeta_2, \zeta_3) = \left[\frac{1 - g(\zeta_i)g(\zeta_j) - h(\zeta_i)h(\zeta_j)}{1 - \zeta_i\zeta_j}\right]_{i,j=1}^3 \in \mathbb{C}^{3 \times 3} \quad (2.11)$$

and where $P_{f,2}(\zeta_1, \zeta_2, \zeta_3)$ is a matrix from $\mathbb{C}^{3 \times 3}j$. By the assumption in Theorem 2.1, the matrix $P_f(\zeta_1, \zeta_2, \zeta_3)$ is positive semidefinite. Then, the complex matrix $P_{f,1}(\zeta_1, \zeta_2, \zeta_3)$ is positive semidefinite by Remark 2.3. The matrix (2.11) can be written as

$$P_{f,1}(\zeta_1, \zeta_2, \zeta_3) = \Lambda - G\Lambda G^* - H\Lambda H^* \geq 0 \quad (2.12)$$

where

$$\Lambda = \left[\frac{1}{1 - \zeta_i\zeta_j}\right]_{i,j=1}^3, \quad G = \begin{bmatrix} g(\zeta_1) & 0 & 0 \\ 0 & g(\zeta_2) & 0 \\ 0 & 0 & g(\zeta_3) \end{bmatrix}, \quad H = \begin{bmatrix} h(\zeta_1) & 0 & 0 \\ 0 & h(\zeta_2) & 0 \\ 0 & 0 & h(\zeta_3) \end{bmatrix},$$

and it is well known that the matrix $\Lambda$ is positive semidefinite. Then we conclude from (2.12) that $\Lambda - G\Lambda G^* \geq 0$ and $\Lambda - H\Lambda H^* \geq 0$, i.e., that the $3 \times 3$ matrices

$$\left[\frac{1 - g(\zeta_i)g(\zeta_j)}{1 - \zeta_i\zeta_j}\right]_{i,j=1}^3 \quad \text{and} \quad \left[\frac{1 - h(\zeta_i)h(\zeta_j)}{1 - \zeta_i\zeta_j}\right]_{i,j=1}^3$$

are positive semidefinite for all choices of $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{D}$. Then it follows from the complex Hindmarsh theorem that the functions $g, h : \mathbb{D} \to \mathbb{C}$ are complex-analytic and belong to the classical Schur class $\mathcal{S}$. Substituting their power series expansions $g(\zeta) = \sum_{k=0}^{\infty} \zeta^k g_k$ and $h(\zeta) = \sum_{k=0}^{\infty} \zeta^k h_k$ into (2.9) leads us to the power series expansion for $f$ on $\mathbb{D}$:

$$f(\zeta) = g(\zeta) + h(\zeta)j = \sum_{k=0}^{\infty} \zeta^k f_k \quad \text{with} \quad f_k = g_k + h_k j. \quad (2.13)$$

We now extend the latter power series to the whole $\mathbb{B}$ by simply letting

$$F(z) = \sum_{k=0}^{\infty} z^k f_k \quad (z \in \mathbb{B}). \quad (2.14)$$

The resulting power series converges absolutely on $\mathbb{B}$ (as the sum of two converging series $g(z)$ and $h(z)j$) and agrees with $f$ on $\mathbb{D}$. We next show that $F$ agrees with $f$ throughout $\mathbb{B}$.
Let \( \gamma \) be any point in \( \mathbb{B} \setminus \mathbb{D} \). The points \( \alpha := \text{Re}(\gamma) + |\text{Re}(\gamma)|i \) and \( \overline{\alpha} \) belong to \( \mathbb{D} \) and are equivalent to \( \gamma \). Observe that

\[
(\gamma - \overline{\alpha})(\alpha - \overline{\alpha})^{-1} = (\gamma - \overline{\alpha})^{-1}(\gamma - \overline{\alpha}), \\
(\alpha - \gamma)(\alpha - \overline{\alpha})^{-1} = (\gamma - \overline{\alpha})^{-1}(\gamma - \alpha). 
\tag{2.15}
\]

Since \( F \) is left-regular by construction, we apply formula (2.15) to get

\[
F(\gamma) = (\gamma - \overline{\alpha})(a - \overline{\alpha})^{-1}F(\alpha) + (\alpha - \gamma)(\alpha - \overline{\alpha})^{-1}F(\overline{\alpha}) \\
= (\gamma - \overline{\alpha})^{-1}(\gamma - \overline{\alpha})f(\alpha) + (\gamma - \overline{\alpha})^{-1}(\gamma - \alpha)f(\overline{\alpha}), 
\tag{2.16}
\]

where the second equality follows due to (2.15) and since \( F \) agrees with \( f \) on \( \mathbb{D} \). On the other hand, we know that the Pick matrix \( P_f(\alpha, \overline{\alpha}, c) \) is positive semidefinite and satisfies the Stein identity (2.1):

\[
P_f(\alpha, \overline{\alpha}, \gamma) - TP_f(\alpha, \overline{\alpha}, \gamma)T^* = EE^* - NN^* 
\tag{2.17}
\]

where

\[
T = \begin{bmatrix} \alpha & 0 & 0 \\
0 & \overline{\alpha} & 0 \\
0 & 0 & \gamma \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\
1 \\
1 \end{bmatrix}, \quad N = \begin{bmatrix} f(\alpha) \\
f(\overline{\alpha}) \\
f(\gamma) \end{bmatrix}.
\]

Let us introduce the row-vector \( V = [\gamma - \overline{\alpha} \quad \gamma - \alpha \quad \overline{\gamma} - \overline{\alpha}] \). Since \( \text{Re}(\alpha) = \text{Re}(\gamma) \) and \( |\alpha| = |\gamma| \) by definition of \( \alpha \), we have

\[
VT = [(\gamma - \overline{\alpha})\alpha \quad (\gamma - \alpha)\overline{\alpha} \quad (\overline{\gamma} - \overline{\alpha})\gamma] = \gamma V, \\
VE = \gamma - \overline{\alpha} + \gamma - \alpha + \overline{\gamma} - \gamma = 2\text{Re}(\gamma) - 2\text{Re}(\alpha) = 0.
\]

Multiplying both parts of (2.17) by \( V \) on the left and by \( V^* \) on the right we get, on account of two last equalities,

\[
VP_f(\alpha, \overline{\alpha}, \gamma)V^* - \gamma VP_f(\alpha, \overline{\alpha}, \gamma)V^*\overline{\gamma} = -VN^*V^*.
\]

Since \( P_f(\alpha, \overline{\alpha}, \gamma) \) is positive semidefinite, we have \( VP_f(\alpha, \overline{\alpha}, \gamma)V^* \geq 0 \) and hence we can write the last equality as

\[
(1 - |c|^2)VP_f(\alpha, \overline{\alpha}, \gamma)V^* = -|VN|^2.
\]

The latter may occur only if \( VP_f(\alpha, \overline{\alpha}, \gamma)V^* = VN = 0 \). Thus,

\[
VN = (c - \overline{\alpha})f(\alpha) + (c - \alpha)f(\overline{\alpha}) + (\overline{\gamma} - \gamma)f(\gamma) = 0,
\]

which implies

\[
f(\gamma) = (\gamma - \overline{\gamma})^{-1}(\gamma - \overline{\alpha})f(\alpha) + (\gamma - \overline{\gamma})^{-1}(\gamma - \alpha)f(\overline{\alpha}). \tag{2.18}
\]

Comparing (2.16) and (2.18) we conclude that \( F(\gamma) = f(\gamma) \). Since \( \gamma \) was chosen arbitrarily in \( \mathbb{B} \setminus \mathbb{D} \), it follows that \( F = f \) on \( \mathbb{B} \). Since \( F \) is left-regular on \( \mathbb{B} \) by construction (2.14), it follows that \( f \) is left-regular on \( \mathbb{B} \) as well. \( \square \)
3. Schur-class of quaternionic formal power series

It turns out that the quaternionic Schur class can be defined without distinguishing the left and the right settings. Let us consider formal power series in one formal variable $z$ which commutes with quaternionic coefficients (which in turn, satisfy the same growth condition as in (1.2)):

$$g(z) = \sum_{k=0}^{\infty} z^k g_k = \sum_{k=0}^{\infty} g_k z^k \quad \text{with} \quad g_k \in \mathbb{H} \quad \text{such that} \quad \lim_{k \to \infty} \sqrt[k]{|g_k|} \leq 1. \quad (3.1)$$

For each $g \in \mathbb{H}[[z]]$ as in (3.1), we define its conjugate by $g^\sharp(z) = \sum_{k=0}^{\infty} z^k \overline{g}_k$.

The anti-linear involution $g \mapsto g^\sharp$ can be viewed as an extension of the quaternionic conjugation $\alpha \mapsto \overline{\alpha}$ from $\mathbb{H}$ to $\mathbb{H}[[z]]$. We next define $g^{e\ell}(\alpha)$ and $g^{e\ell}(\alpha)$ (left and right evaluations of $g$ at $\alpha$) by

$$g^{e\ell}(\alpha) = \sum_{k=0}^{\infty} \alpha^k g_k, \quad g^{e\ell}(\alpha) = \sum_{k=0}^{\infty} g_k \alpha^k, \quad \text{if} \quad g(z) = \sum_{k=0}^{n} z^k g_k. \quad (3.2)$$

Observe that condition $\lim_{k \to \infty} \sqrt[k]{|g_k|} \leq 1$ imposed on the coefficients guarantees the absolute convergence of the series in (3.2) for all $\alpha \in \mathbb{H}$. Since multiplication in $\mathbb{H}$ is not commutative, left and right evaluations produce different results; however, equality $g^{e\ell}(\alpha) = g^{e\ell}(\overline{\alpha})$ holds for any $\alpha \in \mathbb{H}$ as can be seen from (3.2) and the definition of $g^\sharp$.

In accordance with Definition (1.1) we define the left and the right Schur classes $QS_L$ and $QS_R$ as the sets of power series $f \in \mathbb{H}[[z]]$ such that $|g^{e\ell}(\alpha)| \leq 1$ (respectively, $|g^{e\ell}(\alpha)| \leq 1$) for all $\alpha \in \mathbb{B}$. But as was shown in (1) (in slightly different terms), the classes $QS_L$ and $QS_R$ coincide. We now recall several results from (1) in terms of the present setting. With a power series $g \in \mathbb{H}[[z]]$ as in (3.1), we associate lower triangular Toeplitz matrices

$$T_n(g) = \begin{bmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g_{n-1} & \cdots & g_1 & g_0 \end{bmatrix} \quad \text{for} \quad n = 1, 2, \ldots \quad (3.3)$$

**Theorem 3.1.** Let $g \in \mathbb{H}[[z]]$ be as in (3.1). The following are equivalent:

1. $|g^{e\ell}(\alpha)| \leq 1$ for all $\alpha \in \mathbb{B}$.
2. $|g^{e\ell}(\alpha)| \leq 1$ for all $\alpha \in \mathbb{B}$.
3. The matrix $T_n(g)$ is contractive for all $n \geq 1$.

We thus may talk about the Schur class $QS \subset \mathbb{H}[[z]]$ of formal power series $g$ such that the matrix $T_n(g)$ is contractive for all $n \geq 0$. In the latter power series setting, Theorem 2.1 can be formulated as follows: if the function $f : \mathbb{B} \to \mathbb{H}$ is such that $3 \times 3$ Pick matrices $P_f(z_1, z_2, z_3)$ are positive semidefinite for all $(z_1, z_2, z_3) \in \mathbb{B}^3$, then there is (a unique) $g \in QS$.
such that \( f(\alpha) = g^{e_\ell}(\alpha) \) for all \( \alpha \in \mathbb{B} \). The “right” version of this theorem is based on dual Pick matrices

\[
\hat{P}_f(z_1, \ldots, z_n) = \left[ \sum_{k=0}^{\infty} z_i^k (1 - f(z_i) f(z_j)) z_j^k \right]_{i,j=1}^n.
\]

**Theorem 3.2.** Let \( f : \mathbb{B} \to \mathbb{B} \) be given and let us assume that \( 3 \times 3 \) dual Pick matrices \( \hat{P}_f(z_1, z_2, z_3) \) are positive semidefinite for all \( (z_1, z_2, z_3) \in \mathbb{B}^3 \). Then there is (a unique) \( g \in QS \) such that \( f(\alpha) = g^{e_\ell}(\alpha) \) for all \( \alpha \in \mathbb{B} \).

The proof is immediate: by Theorem 2.1 there is an \( h \in QS \) such that \( f(\alpha) = h^{e_\ell}(\alpha) \) for all \( \alpha \in \mathbb{B} \). Therefore, \( f(\alpha) = h^{e_\ell}(\alpha) \) and it remains to choose \( g = h^\dagger \) which belongs to \( QS \) by Theorem 3.1.

In the proof of Theorem 2.1, we actually showed that for any \( g \in QS \), there exist (unique) Schur-class functions \( s, h \in \mathbb{D} \) so that

\[
g(\zeta) = s(\zeta) + h(\zeta)j \quad \text{for all} \quad \zeta \in \mathbb{D},
\]

and the latter equality determines \( g \) uniquely in the whole \( \mathbb{B} \). The last question we address here is how to characterize the pairs \( (s, h) \) of complex Schur functions producing via formula (3.4) a quaternionic Schur-class power series.

**Theorem 3.3.** Let \( s \) and \( h \) be Schur-class functions. The function \( g \) be given by (3.4) belongs to \( QS \) if and only if the following matrix is positive semidefinite

\[
\begin{bmatrix}
I_n - T_n(s)T_n(s)^* - T_n(h)T_n(h)^* & T_n(s)T_n(h)^\top - T_n(h)T_n(s)^\top \\
T_n(h)T_n(s)^* - T_n(s)T_n(h)^* & I_n - T_n(s)T_n(s)^\top - T_n(h)T_n(h)^\top
\end{bmatrix} \geq 0
\]

for all \( n \geq 0 \) where \( I_n \) stands for the \( n \times n \) identity matrix and \( T_n \) is defined via formula (3.3).

**Proof:** By Theorem 3.1, \( g \) belongs to \( QS \) if and only if \( I_n - T_n(g)T_n(g)^* \) is positive semidefinite for all \( n \geq 1 \). It follows from (3.4) that

\[
T_n(g)T_n(g)^* = (T_n(s) + T_n(h)j)(T_n(s)^* - jT_n(h)^*)
\]

\[
= T_n(s)T_n(s)^* + T_n(h)jT_n(s)^* - T_n(s)jT_n(h)^* + T_n(h)T_n(h)^*
\]

\[
= T_n(s)T_n(s)^* + T_n(h)T_n(h)^* + (T_n(h)T_n(s)^\top - T_n(s)T_n(h)^\top)j.
\]

Therefore, \( I_n - T_n(g)T_n(g)^* = A_1 + A_2j \) where

\[
A_1 = I_n - T_n(s)T_n(s)^* - T_n(h)T_n(h)^*, \quad A_2 = T_n(s)T_n(h)^\top - T_n(h)T_n(s)^\top
\]

and the statement follows immediately, by Remark 2.3. \( \square \)

Note that positive semidefiniteness of diagonal blocks in (3.5) is equivalent to the inequality \( |s(\zeta)|^2 + |s(\zeta)|^2 \leq 1 \) holding for all \( \zeta \in \mathbb{D} \) which is necessary (since \( |g(\zeta)|^2 = |s(\zeta)|^2 + |s(\zeta)|^2 \) for \( g \) to be in \( QS \) but not sufficient.
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