The construction of partner potential from the general potential anharmonic in $D$-dimensional Schrödinger system

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Abstract. The Schrödinger equation is the fundamental equation in quantum physics. The characteristic of the particle in physics potential field can be explained by using the Schrödinger equation. In this study, the solution of 4 dimensional Schrödinger equation for the anharmonic potential and the anharmonic partner potential have done. The method that used to solve the Schrödinger equation was the ansatz wave method, while to construction the partner potential was the supersymmetric method. The construction of partner potential used to explain the experiment result that cannot be explained by the original potential. The eigenvalue for anharmonic potential and the anharmonic partner potential have the same characteristic. Every increase of quantum orbital number the eigenvalue getting smaller. This result corresponds to Bohr’s atomic theory that the eigenvalue is inversely proportional to the atomic shell. But the eigenvalue for the anharmonic partner potential higher than the eigenvalue for the anharmonic original potential.

1. Introduction
The Schrödinger equation is the equation that representation of electron or particle. The solution of Schrödinger equation is eigenfunction and eigenvalue a particle that used to describe the characteristic of a particle in-universe. The Schrödinger equation was developed into $D$-dimensional Schrödinger system to solve the complex system in quantum mechanic. So the Schrödinger equation can explain more about a system. There are many methods that used to solve the Schrödinger equation. For example the ansatz wave function method [1-4], the Nikiforov-Uvarov method[5-7], the the asymptotic iteration method (AIM)[8-10], supersymmetric method [11-13] and other [14-15]. In this study, we used the wave function ansatz method and the supersymmetric method. The ansatz wave function method was used to solve the Schrödinger equation and the supersymmetric method was used to construction partner potential.

The partner potential was used to explain the experiment result that cannot be explained by the original potential. To construction the partner potential we using the ground state eigenfunction, the original potential and the supersymmetric operators. Some study about the construction of partner potential has been done. For example the construction partner potential from the Hylleraas potential [16], the construction partner potential from the Hulthen potential [17-18] and the construction partner potential from the Manning Rosen and Rosen-Morse potential [19]. In this study, we choose potential anharmonic as the original potential. The general form of anharmonic potential that we used is

$$V(r) = ar^2 + br + cr^{-1} + dr^{-2}$$

where $a, b, c$ and $d$ are potential parameters.
The first section of this paper we will review the supersymmetric method and the wave function ansatz method. The next section we will explain the Schrodinger equation in the D-dimensional system. We will present the solution of D-dimensional Schrodinger equation with anharmonic potential using the ansatz wave function method at three section. The next section we will present about construction partner potential from origin anharmonic potential in D-dimensional Schrodinger system and then we will present the solution of D-dimensional Schrodinger equation with partner potential anharmonic.

2. Method

2.1. The Supersymmetric method

Definition of supersymmetric Hamiltonian $H_{ss}$ partner for changing operator that commute with $H_{ss}$ by Witten as follows [20]

$$H_{ss} = \begin{pmatrix} -\frac{d^2}{dx^2} + \frac{d\phi(x)}{dx} + \phi^2(x) & 0 \\ 0 & -\frac{d^2}{dx^2} - \frac{d\phi(x)}{dx} + \phi^2(x) \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$  \hspace{1cm} (2)

The Hamiltonian partner $H_-=H_1$ $H_+=H_2$ and partner potential $V_-=V_1$ and $V_+=V_2$ are

$$V_-(x) = V_1 = \phi^2(x) - \phi'(x) \quad \text{and} \quad V_+(x) = V_2 = \phi^2(x) + \phi'(x)$$  \hspace{1cm} (3)

where $\phi$ is superpotential supersymmetric.

The relation between effective potential $V_{ef}$ and first potential $V_1$ as follow

$$V_{ef}(x) = V_-(x; a_0) + E_0 = V_1(x; a_0)$$  \hspace{1cm} (4)

with $E_0$ is ground state energy for effective potential. By setting the new supersymmetric operator are the lowering operator $A$ and the rising operator $A^\dagger$ as

$$A^\dagger = -\frac{d}{dx} + \phi(x),$$

$$A = \frac{d}{dx} + \phi(x)$$  \hspace{1cm} (5)

so the Hamiltonian partner in Eq. (2) can be rewritten as

$$H_-(x) = H_1 = A^\dagger A,$$  \hspace{1cm} (6)

and also

$$A\psi_0^{(-)} = A\psi_0 = 0$$  \hspace{1cm} (7)

The new partner potential $V_2$ from Eq. (4) and Eq. (7) we get the relation [16]

$$V_2(x) = V_1 - 2 \frac{d^2}{dx^2} \ln \psi_0 = V_{ef} - E_0 - 2 \frac{d^2}{dx^2} \ln \psi_0$$  \hspace{1cm} (8)

This equation will be used to construct the partner potential from the original potential in Eq. (1).

2.2. The Wave Function Ansatz
The wave function ansatz was used to solve the radial part Schrödinger equation. The first step of ansatz wave function is we must determine the form of wave function ansatz that corresponds with potential that we used. From work [4] the wave function ansatz that suitable for Eq. (1) is

\[ R_n(r) = h_n(r) \exp(P(r)) \]  

(9)

with

\[ h_n(r) = \begin{cases} 1 & \text{when } n = 0 \\ \prod_{j=1}^{n} (r - \sigma_j^n) & \text{when } n = 1, 2, 3, \ldots \end{cases} \]  

(10)

The parameter \( P(r) \) in Eq. (9) was determined by superpotential supersymmetric. So we have

\[ P(r) = \frac{1}{2} Ar^2 + Br + C \ln r \]  

(11)

The Eq. (9) with \( P(r) \) in Eq. (11) is the wave function ansatz that we used to solve the Schrödinger equation with potential anharmonic.

3. Results and discussion

3.1. The Schrödinger equation in D-dimensional system

The general Schrödinger equation in \( D \)-dimensional system [21]

\[ -\frac{\hbar^2}{2m} \nabla_D^2 \Psi(r, \Omega) + V(r, \Omega) \Psi(r, \Omega) = E \Psi(r, \Omega) \]  

(12)

with \( D \) is dimensional and the Laplacian operator as [22]

\[ \nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \sum_{j=1}^{m} \frac{1}{\sin^2 \theta_{j+1} \sin^2 \theta_{j+2} \cdots \sin^2 \theta_{D-1}} \times \left\{ \frac{1}{\sin^{j-1} \theta_j} \left( \frac{\partial}{\partial \theta_j} \sin^{j-1} \theta_j \frac{\partial}{\partial \theta_j} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta_D} \right)^2 \right\} \]  

(13)

To simplify calculation we set \( \hbar = 1, 2m = 1 \). By using variable separation method from Eq. (12) and (13) we get the \( D \)-dimensional Schrödinger equation radial part and polar part. The radial part of \( D \)-dimensional Schrödinger equation as follows

\[ \frac{\partial^2 R(r)}{\partial r^2} - \left[ \frac{3}{4} \frac{\lambda_{D-1}}{r^2} \right] R(r) = -2 \left( E - V(r) \right) R(r) \]  

(14)

where by Dong [21] \( \lambda_{D-1} \) is describe as

\[ \lambda_{D-1} = l_{D-1} (l_{D-1} + 2) \]  

(15)

with \( l_{D-1} \) is angular momentum.

3.2. The Solution of \( D \)-dimensional Schrödinger equation with anharmonic potential

First calculation, we solve the radial part of Schrödinger equation with anharmonic potential by using the wave function ansatz. The anharmonic potential in Eq. (1) substituting into the radial part of Schrödinger equation in Eq. (14) so we get
The Eq. (16) is radial part of Schrodinger equation with anharmonic potential. To solve Eq. (16), substituting the wave function ansatz in Eq. (19) into Eq. (16), so we have

\[ P^\sigma(r) + \left( P'(r) \right)^2 + \frac{h_n''(r) + 2h_n'(r) P'(r)}{h_n(r)} = -2E + 2ar^2 + 2br + 2\frac{c}{r} + \frac{\left(2d + \frac{3}{4} - \lambda_{D-1}\right)}{r^2} \]  

(17)

From Eq. (11), we obtained

\[ P'(r) = Ar + B + \frac{C}{r} \]  
\[ P^\sigma(r) = A - \frac{C}{r^2} \]  

(18)

For \( n = 0 \) from Eq. (10) \( h_1 = 1 \), and by substituting Eq.(18) into Eq. (17), we have

\[ A^2r^2 + 2ABr + \frac{2BC}{r} + \frac{C^2-C}{r^2} + A + 2AC + B^2 = -2E + 2ar^2 + 2br + 2\frac{c}{r} + \frac{\left(2d + \frac{3}{4} - \lambda_{D-1}\right)}{r^2} \]  

(19)

And then, by using the separation variable method, from Eq. (19) we obtained the equations as follow

\[ A^2 = 2a \]  
\[ AB = b \]  
\[ BC = c \]  
\[ C^2 - C = 2d + \frac{3}{4} - \lambda_{D-1} \]  
\[ A + 2AC + B^2 = -2E \]  

(20a)  
(20b)  
(20c)  
(20d)  
(20e)

From Eq. (20) the parameters \( A, B, C \) can be determined, so we can get the eigenvalue and the eigenfunction for \( n = 0 \). From Eq. (20) we have

\[ A = -\sqrt{2a} \ , \ B = -\frac{b}{\sqrt{2a}} \text{ and } C = \frac{1}{2} \left(1 + \sqrt{4 + 8d - 4\lambda_{D-1}}\right) \]  

(21)

Then, by using Eq. (21) and Eq. (20e) we obtained the eigenvalue as follow

\[ E_0 = \frac{\sqrt{2a}}{2} \left(2 + \sqrt{4 + 8d - 4\lambda_{D-1}}\right) - \frac{b^2}{4a} \]  

(22)

And the eigenfunction as

\[ R_0(r) = N_0r^2\left(1 + \sqrt{4 + 8d - 4\lambda_{D-1}}\right) \exp\left[-\sqrt{2ar - \frac{b}{\sqrt{2a}}}\right] \]  

(23)

For \( n = 1 \) from Eq. (10) we have \( h_1 = \left(r - \sigma_i^{(1)}\right) \), by substituting Eq. (18) into Eq. (17), we get
\[ A - \frac{C}{r^2} + A^2 r^2 + B^2 + C^2 + 2ABr + 2AC + \frac{2BC}{r} + \frac{2 \left( Ar + B + \frac{C}{r} \right)}{r - \sigma_1^{(i)}} = \]  
\[ -2E + 2 \left( ar^2 + br + cr^{-1} + dr^{-2} \right) + \frac{3}{4} - \frac{\lambda_{D+1}}{r^2} \]

By following the steps that we used in our calculation when \( n = 0 \) we have the parameters \( A, B, C \) as
\[ A = -\sqrt{2a}, \quad B = -\frac{b}{\sqrt{2a}}, \quad C = \frac{1}{2} \left( 1 + \sqrt{4 + 8d - 4\lambda_{D+1}} \right) \]  
and
\[ A \left[ 1 + 2 \left( C + 1 \right) \right] + B^2 = -2E \]

And then, by substituting Eq. (25) into Eq. (26), we obtain the eigenvalue for \( n = 1 \) as
\[ E_1 = \frac{\sqrt{2a}}{2} \left( 4 + \sqrt{4 + 8d - 4\lambda_{D+1}} \right) - \frac{b^2}{4a} \]
and the eigenfunction as follows
\[ R_1 (r) = \left\{ N_1 (r - \sigma_1^{(i)}) \right\} r^{1/2} \left( \sqrt{4 + 8d - 4\lambda_{D+1}} \right) \exp \left[ -\frac{1}{2} \sqrt{2a} r - \frac{b}{2\sqrt{2a}} r \right] \]

By using the same steps, we can determine the eigenvalue and the eigenfunction for \( n = 2 \) and more. So we can generalisation the eigenvalue and the eigenfunction for \( n \) as follow
\[ E_n = \frac{\sqrt{2a}}{2} \left( 2 + 2n + \sqrt{4 + 8d - 4\lambda_{D+1}} \right) - \frac{b^2}{4a} \]
and
\[ R_n (r) = \left\{ N_n \prod_{i=1}^{n} (r - \sigma_i^{(i)}) \right\} r^{1/2} \left( \sqrt{4 + 8d - 4\lambda_{D+1}} \right) \exp \left[ -\frac{1}{2} \sqrt{2a} r - \frac{b}{2\sqrt{2a}} r \right] \]

with \( n = 0, 1, 2, 3, \ldots \), \( N_n \) is normalization constant.

3.3. The construction of partner potential from origin potential anharmonic in Schrodinger 4 dimensional system

From Eq. (16) the effective potential for \( D \)-dimensional Schrodinger equation with anharmonic potential is
\[ V_{\text{eff}} = 2ar^2 + 2br + \frac{2c}{r} \left( \frac{2d + 3}{4} - \lambda_{D+1} \right) \]
To construction the partner potential we use Eq. (8). So we obtained the new partner potential as follow
\[ V_2 (r) = 2ar^2 + 2br + \frac{2c}{r} \left( \frac{2d + 3}{4} - \lambda_{D+1} \right) + 2C - E_0 - 2A \]
where \( E_0 \) in Eq. (22), \( A \) in Eq (21) dan \( C \) in Eq (21).
The Eq. (31) is partner potential anharmonic that will be used as the new potential. The new partner potential solved in the $D$-dimensional Schrodinger system by using the wave function ansatz method. Substituting Eq. (31) into Eq. (14) we have the Schrodinger equation with partner potential anharmonic in the $D$-dimensional system as following

$$\frac{\partial^2 R(r)}{\partial r^2} - \left( \frac{3 - \lambda_{D-1}^2}{2r^2} \right) R(r) = (-2E - 2E_0 - 4A)R(r) + 4 \left[ ar^2 + br + \frac{c}{r^2} + \left( d + C + \frac{1}{2} \left( \frac{3}{4} - \lambda_{D-1}^2 \right) \right) \right] R(r) \tag{33}$$

By using supersymmetric approximation we determine the wave function ansatz that corresponds with partner potential anharmonic as

$$R_n(r) = h_n(r) \exp(P(r)) \tag{34}$$

with

$$P(r) = \frac{1}{2} A'r^2 + B'r + C' \ln r \tag{35}$$

And then, substituting the wave function ansatz in Eq. (33) into Eq. (32) and by using the same steps that we were used to solving the Schrodinger equation with anharmonic potential, we have the solution of Schrodinger equation with the new partner potential anharmonic as follow

$$E_n = \left[ \frac{\sqrt{4a}}{2} \left( 2 + 2n + \sqrt{4 \left( 4d + 4C + \frac{9}{4} - \lambda_{D-1}^2 - \lambda_{D-1}' \right) } \right) - E_0 - 2A - \frac{b^2}{a} \right] \tag{36}$$

and

$$R_n(r) = \left[ N_{nl} \prod_{i=1}^{n} \left( r - \sigma_i^{(m)} \right)^{1/2} \sqrt{4d+4C+\frac{9}{4} - \lambda_{D-1}^2 - \lambda_{D-1}'} \right] \exp \left[ - \frac{1}{2} \sqrt{4a} r^2 - 2 \frac{b}{\sqrt{4a}} r \right] \tag{37}$$

with $n = 0, 1, 2, 3, \ldots, E_0$ in Eq (22), $A$ and $C$ in Eq. (21).

To see the characteristic eigenvalue of the anharmonic potential and the new partner potential anharmonic, we calculate the numeric eigenvalue by using MATLAB program. We present the numeric eigenvalue in Table 1.

**Table 1.** The eigenvalue of anharmonic potential and the anharmonic partner potential with the value of parameters potential are $a = 1/32, b = 1, c = 4, d = 0$ and the angular momentum $l = 0$.

| $n$ | Eigenvalue anharmonic potential (1/fm) | Eigenvalue anharmonic partner potential (1/fm) |
|-----|--------------------------------------|-----------------------------------------------|
| 0   | 15.750                               | 30.2819                                       |
| 1   | 15.500                               | 30.0769                                       |
| 2   | 15.250                               | 29.8750                                       |
| 3   | 15.000                               | 29.6772                                       |
| 4   | 14.750                               | 29.4846                                       |

In Table 1 we can see that the eigenvalue of the original anharmonic potential and the partner potentials anharmonic have the same characteristic. Every increase of quantum orbital number the eigenvalue getting smaller. This result corresponds to Bohrn’s atomic theory that the eigenvalue is inversely proportional to the atomic shell. But the eigenvalue for the anharmonic partner potential higher than the eigenvalue for the anharmonic original potential. This phenomenon happens because the parameters of the origin potential in Eq (1) and the parameters of the partner potential in Eq (31) are different.
4. Conclusion
We have constructed the partner potential from the original anharmonic potential in Schrodinger $D$-dimensional system using the supersymmetric method. The partner potentials that we obtained is solvable. By using the wave function ansatz method we were obtained the eigenvalue and the eigenfunction of Schrodinger equation with anharmonic potential and the anharmonic partner potential. The eigenvalue of the original anharmonic potential and the partner potentials anharmonic have the same characteristic. Every increase of quantum orbital number the eigenvalue getting smaller. This result corresponds to Bohrn’s atomic theory that the eigenvalue is inversely proportional to the atomic shell. But the eigenvalue for the anharmonic partner potential higher than the eigenvalue for the anharmonic original potential.

Acknowledgments
This work partly supported by PUT-MRG of LPPM Sebelas Maret University.

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