ON THE QFT RELATION BETWEEN DONALDSON-WITTEN INVARIANTS AND FLOER HOMOLOGY THEORY

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ABSTRACT

A TQFT in terms of general gauge fixing functions is discussed. In a covariant gauge it yields the Donaldson-Witten TQFT. The theory is formulated on a generalized phase space where a simplectic structure is introduced. The Hamiltonian is expressed as the anticommutator of off-shell nilpotent BRST and anti-BRST charges. Following original ideas of Witten a time reversal operation is introduced and an inner product is defined in terms of it. A non-covariant gauge fixing is presented giving rise to a manifestly time reversal invariant Lagrangean and a positive definite Hamiltonian, with the inner product previously introduced. As a consequence, the indefiniteness problem of some of the kinetic terms of the Witten’s action is resolved. The construction allows then a consistent interpretation of Floer groups in terms of the cohomology of the BRST charge which is explicitly independent of the background metric. The relation between the BRST cohomology and the ground states of the Hamiltonian is then completely stablished. The topological theories arising from the covariant, Donaldson-Witten, and non-covariant gauge fixing are shown to be quantum equivalent by using the operatorial approach.
1 Introduction

Topological Field Theories were introduced by Witten in [1] and have had since then a great development from a physical and mathematical point of view [2]-[8]. Recently an interesting relation between the ghost sector of field theories with propagating local degrees of freedom and TQFT has been also found [9]-[12].

In [1] Witten, inspired on previous work of Atiyah, gave a description of the relation of Donaldson and Floer theory. To define Donaldson’s invariants of a four manifold with boundary, $X$, one must specify a state in the Floer homology of its boundary $\Sigma$ [13]. To do so, on $X$ which Witten assumed to be $\Sigma \times R$, one constructs the topological quantum field theory with effective action $S_{\text{eff}}$. One may start with the classical action given in [14] and construct a canonical Hamiltonian formulation of the effective action. If it is possible to construct a Hilbert space $H$ of quantum states on $\Sigma$ consistent with a positive Hamiltonian of the TQFT, one may then pick a state on $\Sigma$, to construct topological invariants of $X$. These are obtained from path integrals with boundary conditions determined by $\psi$:

$$Z = \int \mu \exp(-S_{\text{eff}})O\psi, \quad (1.1)$$

where $O$ is an observable constructed from a product of local fields with the property of being BRST invariant:

$$QO = 0, \quad (1.2)$$

where $Q$ is the off-shell nilpotent BRST operator. The quantum states $\psi$ must satisfy the physical condition

$$Q\psi = 0. \quad (1.3)$$

If $\psi$ is $Q$-exact (1.3) becomes zero, hence (1.1) only depends on the BRST cohomology class, that is on Q-closed modulo Q-exact fields, which are in fact the Floer cohomology classes. This very relevant relation was found in [1], where the interesting observables $O$ were also constructed and related to Donaldson’s invariants:

$$O = \prod_{i=1}^{N} \int_{\gamma_i} W_{k_i}, \quad (1.4)$$
where $W_{k_i}$ were constructed from a set of descendent equations starting from $W_0 = \frac{1}{2} Tr \phi^2$, $\gamma_i$ is a $k_i$ dimensional homology cycle. $W_k, k = 0, \ldots, 4$, has ghost number $4 - k$. The overall ghost number of $O$ must equal the dimension of fermion zero mode space in order to have a non-trivial path integral. $\phi$ is the spin zero, BRST invariant field in Witten’s TQFT. Following Witten, we are then interested in obtaining a canonical Hamiltonian formulation of the TQFT. In particular of the off-shell nilpotent BRST operator $Q$. The cohomology groups associated to $Q$ are precisely the Floer groups. An explicit construction of the metric independent $Q$ should provide also a direct proof of the independence on the metric of Floer groups. A canonical symplectic formulation of the theory also will help to construct a positive definite inner product on the Hilbert space of the quantum field theory. In [1] Witten suggested to use a time reversal operation together with a canonical construction in order to consider this problem. The use of a time reversal operation is a natural one in any euclidean formulation in quantum field theory [15]-[18]. In fact, it is essential since it is related to the Wick rotation $t \to it$ because under conjugation $i t \to -it$ while $t \to t$. In principle, the TQFT does not need any Wick rotated Minkowskian formulation so one may think the time reversal is not necessarily relevant in this case. However as suggested by Witten it turns out to be essential in the definition of the inner product.

In general it may seem not natural to introduce a Hamiltonian formulation for a theory over a locally euclidean, orientable manifold $X$, since there isn’t any natural evolving direction on it. Moreover one usually considers in QFT different boundary conditions on time $\pm \infty$ than in any spatial boundary. In particular a classical analysis of a canonical action $\langle \dot{q} p - H(q, p) \rangle$ shows that in order to have a gauge invariant canonical action, boundary conditions at $t = \pm \infty$ are needed [19]. If we consider an euclidean formulation, for example over a compact closed manifold, these boundary conditions have to be imposed at some points (corresponding to $t = \pm \infty$). But since there isn’t any prefer “time” direction on an euclidean manifold it seems that the boundary restrictions will appear everywhere. This point was first analysed in [20] for the Seiberg-Witten topological quantum field theory. We may argue in the following way. We consider the embedding of $X$ into a euclidean manifold $R^n$ of high enough dimensions $n$, this is always possible. We then consider a direction in $R^n$ as the euclidean “time” $\tau$. It defines a height function over $X$. One may consider then the Hamiltonian analysis of the
covariant Lagrangean in terms of $\tau$. Let us call A and B the points of lowest and highest height over $X$. The only requirement on the BRST hamiltonian construction with "time" $\tau$ was found in \cite{21}-\cite{22}, it is the quantum analogue of the classical conditions on the gauge parameters discussed above:

$$\left[ Q - \sum_p \left< p \frac{\delta Q}{\delta p} \right> \right]^{\tau_f}_{\tau_i} = 0. \quad (1.5)$$

This condition is satisfied identically by the Seiberg-Witten BRST operator \cite{20}. This is a particular property of some topological theories. It will come out that it is also satisfied identically by Witten’s TQFT. The canonical hamiltonian construction is then completely consistent. The structure of the hamiltonian of Witten’s TQFT was obtained in \cite{1}, although a canonical symplectic structure was not presented. It is of the form $\{Q, \bar{Q}\}$, where $Q$ and $\bar{Q}$ were shown to be nilpotent on-shell. This particular form of the hamiltonian was then used, under the assumption of the existence of a positive inner product, to show that the BRST cohomology consists only of the ground states. The contribution of this paper is to introduce a symplectic structure in the formulation of Witten’s TQFT, to obtain the metric independent off-shell nilpotent BRST charge, to construct the time reversal operation generalizing the one introduced by Witten and to define an internal product in terms of it. We will show then that for the gauge fixing functions we will introduced in this work, the Hamiltonian is positive definite resolving the problems of indefiniteness of the kinetic terms of Witten’s action, which are related to the existence of a positive definite inner product on the space of quantum states. The explicit form of the BRST charge which is independent of the metric will give a direct proof of the topological invariant property of the Floer groups and show that consistency condition (1.5) is identically satisfied. The BRST charge that we will obtain is an extension of the previously found in \cite{14}. In that paper the BRST charge was constructed from the minimal sector of the extended phase space. The BRST transformations of that sector were there obtained from the Poisson bracket structure. The BRST transformations for the non-minimal sector of phase space were obtained directly from the general formalism in \cite{21}-\cite{22} and were non-canonical transformations. In this paper we obtain a symplectic structure which includes the minimal and non-minimal sectors. It allows the canonical construction of $H = \{Q, \bar{Q}\}$ with $Q$ and $\bar{Q}$ related by the time reversal operation and off-shell nilpotents.
This structure together with the positive definite inner product in the space of states defined in terms of the time reversal operation, completes the argument in [1] to show that the BRST cohomology consists only on ground states.

2 The gauge invariant action

The gauge invariant action introduced in [14] is

\[ S = \frac{1}{4} \int Tr (B + F) \wedge (B + F) = \frac{1}{4} \int d^4x \, \varepsilon^{\mu\nu\sigma\rho} (F_{\mu\nu}^{a} + B_{\mu\nu}^{a}) (F_{\sigma\rho}^{a} + B_{\sigma\rho}^{a}). \]  

(2.1)

where \( F \) is the curvature two form of the gauge connection one form \( A \), and \( B \) is an independent two form.

The action (2.1) is invariant under the finite gauge transformation on the principal bundle

\[ A \rightarrow \Lambda^{-1} A \Lambda + \Lambda^{-1} d \Lambda, \]
\[ B \rightarrow \Lambda^{-1} B \Lambda, \]  

(2.2)

and under the infinitesimal gauge transformations with parameter \( \varepsilon_{\mu}^{a} \):

\[ \delta A_{\mu}^{a} = (D_{\mu}\omega)^{a} + \varepsilon_{\mu}^{a} = \partial_{\mu}\omega^{a} + (A_{\mu} \times \omega)^{a} + \varepsilon_{\mu}^{a}, \]
\[ \delta B_{\mu\nu}^{a} = -(D_{\mu}\varepsilon_{\nu})^{a} + (D_{\nu}\varepsilon_{\mu})^{a} + (B_{\mu\nu} \times \omega)^{a}, \]  

(2.3)

where we have also included the infinitesimal transformations corresponding to (2.2). \( \varepsilon_{\mu}^{a} \) are the gauge parameters which eliminate the local degrees of freedom of the theory leaving only the topological excitations. We denote \((A_{\mu} \times \omega)^{a} = f_{abc} A_{\mu}^{b}\omega^{c} \). The gauge parameter \( \varepsilon_{\mu}^{a} \) is restricted by the condition that \( A + \delta A \) must also be a connection on the same principal bundle. That is, under a finite gauge transformation \( \Lambda \) on the principal bundle we have

\[ A + \varepsilon \rightarrow \Lambda^{-1}(A + \varepsilon)\Lambda + \Lambda^{-1} d \Lambda \]

Consequently

\[ \int Tr F(A + \varepsilon) \wedge F(A + \varepsilon) = \int Tr F(A) \wedge F(A), \]
that is the gauge transformations with parameter $\varepsilon$, under the restriction that $A + \delta A$ is still a connection on the principal bundle, do not change the Chern Class of $F$. It can change $A$ within the set of connections on the same principal bundle only.

From now on we suppress the group index in order to simplify the expressions. The gauge invariances (2.3) allow to make a partial gauge fixing

$$B^+_{\mu\nu} = 0$$

(2.4)

where

$$B^\pm_{\mu\nu} \equiv \frac{1}{2}(B_{\mu\nu} \pm \overline{B}_{\mu\nu}),$$

$$\overline{B}_{\mu\nu} \equiv \sqrt{g} \varepsilon_{\mu\nu\sigma\rho}B^{\sigma\rho}. \quad (2.5)$$

In this gauge the field equations reduce to:

$$F^+_{\mu\nu} = 0,$$

$$F^-_{\mu\nu} + B^-_{\mu\nu} = 0. \quad (2.6)$$

This field equations (2.6) are identical to the ones obtained in [1] by Witten.

It is interesting to notice that the action (2.1) give rise by dimensional reduction to two dimensions, to a theory describing as field equations the “self dual” Hitchin equations over Riemann surfaces [23]-[25].

We now follow the same steps as in [14] but we shall consider a different reduction procedure.

The action (2.1) may be reformulated in a canonical form. We obtain by direct expansion of the Lagrangean density

$$S = \int d^4x \sqrt{g}[\dot{A}_i \varepsilon^{ijk}(F_{jk} + B_{jk}) + A_0 D_i[\varepsilon^{ijk}(F_{jk} + B_{jk})] + B_0 \varepsilon^{ijk}(F_{jk} + B_{jk})].$$

(2.7)

Here we recognize a theory with vanishing canonical Hamiltonian and with canonical conjugate momenta to $A_i$ given by

$$\pi^i = \varepsilon^{ijk}(F_{jk} + B_{jk}), \quad (2.8)$$

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where $\varepsilon^{ijk} \equiv \varepsilon^{0ijk}$, and with Lagrange multipliers $A_0$ and $B_{0i}$ associated respectively to the following constraints

$$
\phi = D_i \pi^i = \partial_i \pi^i + (A_i \times \pi^i) = 0,
\phi^i = \pi^i = 0.
$$

(2.9)

The algebra of the constraints is given by:

$$
\{ \phi^{ia}(x), \phi^{jb}(x') \} = 0,
\{ \phi^{a}(x), \phi^{b}(x') \} = f^{abc} \phi^{c}(x) \delta^3(x - x'),
\{ \phi^{a}(x), \phi^{ib}(x') \} = f^{abc} \phi^{ic}(x) \delta^3(x - x'),
$$

(2.10)

which show that all constraints are first class. Nevertheless they are not linearly independent since they satisfy the following identity

$$
(D_i \phi^i) = \phi,
$$

(2.11)

and thus we have to deal with a reducible theory [21]-[22] with one level of reducibility. The corresponding matrix of reducibility is given by:

$$
a = (D_i, -1).
$$

(2.12)

To construct the BRST charge we follow [21]-[22] and introduce the minimal sector of the extended phase space expanded by the conjugate pairs:

$$(A_i, \pi^i); (C_1, \mu^1), (C_{i1}, \mu^{i1}); (C_{11}, \mu^{11}),$$

(2.13)

where $(A_i, \pi^i)$ are the original canonical coordinates and $(C, \mu)$ are the canonical ghost and antighost associated to the constraints (2.9).

The off-shell nilpotent BRST charge associated with (2.9) is then given by:

$$
\Omega = \langle C_1 (D_i \pi^i) + C_{1i} \pi^i + C_{11} [(D_i \mu^{i1}) - \mu^1],
- \frac{1}{2} C_1 (C_1 \times \mu^1) - C_1 (C_{i1} \times \mu^{i1}),
-C_{11} (C_1 \times \mu^{11}) \rangle,
$$

(2.14)

where $\langle \cdots \rangle$ stands for integration on the three dimensional continuous index.
We now define the non minimal sector of the extended phase space \([21]-[22]\). It contains extra ghosts, antighosts and Lagrange multipliers. First we introduce the C-fields

\[
C_m, C_i^m; \quad C_{mn}, C_{im}; \quad m, n = 1, 2, 3
\]

(2.15)

where at least one of the indices \(m, n\) take the values 2 or 3. In addition to these ghost, antighost and Lagrange multiplier fields we introduce the \(\lambda\) and \(\theta\) fields (Lagrange multipliers), also in the non minimal sector,

\[
\lambda^0_1, \lambda^0_1; \quad \lambda^0_1; m = 1, 2, 3,
\]

\(
\lambda^1_{11};
\)

\[
\theta^0_1, \theta^0_1; \quad \theta^0_{1m}; m = 1, 2, 3,
\]

\[
\theta^1_{11}.
\]

(2.16)

In this notation the 1 subscripts denote ghost associated to a gauge symmetry of the action, the 2 subscripts denote antighost associated to a gauge fixing condition in the effective action and the 3 subscripts denote Lagrange multipliers associated to a gauge fixing condition. The effective action is then given by:

\[
S_{eff} = \int d^4x \sqrt{g} \left[ \pi^i \dot{A}_i + \mu^i \dot{C}_i + \mu^{1i} \dot{C}_{1i} + \mu^{11} \dot{C}_{11}
\]

\[
+ \hat{\delta}(\lambda^0_1 \mu^1 + \lambda^0_1 \mu^{1i} + \lambda^1_{11} \mu^{11})
\]

\[
+ \hat{\delta}(C_2 \chi_2 + C_2^i \chi_{2i}) + \hat{\delta}(C_1 \chi_{12}) \right],
\]

(2.17)

In eq. (2.17) \(\chi_2, \chi_{2i}\) are the primary gauge fixing functions associated to the constraints (2.9), while \(\chi_{12}\) is the gauge fixing functions which must fix the longitudinal part of the \(C_{1i}\) field. The BRST transformation for the canonical variables is given by

\[
\hat{\delta}Z = (-1)^{\varepsilon_z} \{Z, \Omega\},
\]

(2.18)

where \(\varepsilon_z\) is the grassmanian parity of \(Z\). The BRST transformation of the variables of the non minimal sector are determined by imposing the closure of the charge as in \([21]-[22]\).
After the integration on $\mu^1, \mu^{1i}$ and $\mu^{11}$ we obtain:
\[
\hat{\delta} \lambda^0_1 = D_0 C_1 + \lambda^1_{11},
\hat{\delta} \lambda^0_{1i} = D_0 C_{1i} + D_i \lambda^1_{11} - \lambda^0_{1i} \times C_1 + \frac{1}{2} (C_{2i} \times C_{11}),
\hat{\delta} \lambda^1_{11} = -D_0 C_{11} + (\lambda^1_{11} \times C_1),
\tag{2.19}
\]
where $D_0 C = \partial_0 C + (A_0 \times C)$ with $A_0 = -\lambda^0_1$. We introduce $C_{1\mu} = (C_{10}, C_{11})$ after we have recognized $C_{10} = -\lambda^1_{11} = -\lambda^0_{11}$.

After the introduction of the a self-dual field $C_{2\mu
u}$, we finally choose gauge fixing functions that may be written in a covariant form as
\[
\chi_2 = \partial_\mu A^\mu - \frac{\alpha}{2} C_3,
\chi^\mu_{2\nu} = \frac{\alpha}{2} B^{+\mu\nu}, \quad a \neq 0,
\chi_{12} = D^\mu C_{1\mu} + \frac{1}{2} C_{11} \times C_{11} + \frac{1}{2} ((C_{12} \times C_1) \times C_{11}),
\tag{2.20}
\]
After elimination of all conjugate momenta the BRST transformation rules of all the remaining objects take the form
\[
\hat{\delta} A_\mu = -D_\mu C_1 + C_{1\mu},
\hat{\delta} B^{+\mu\nu} = -D_{[\mu} C_{1\nu]} - \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} D^\rho C_1^\sigma + (C_1 \times B^{+\mu\nu}) - \frac{1}{2} (C_{2\mu\nu} \times C_{11}),
\hat{\delta} C_1 = C_{11} + \frac{1}{2} (C_1 \times C_1),
\hat{\delta} C_{1\mu} = D_\mu C_{11} + (C_1 \times C_{1\mu}),
\hat{\delta} C_{11} = -(C_{11} \times C_1),
\hat{\delta} C_2 = C_3,
\hat{\delta} C_3 = 0,
\hat{\delta} C_{2\mu\nu} = C_{3\mu\nu} = (C_{2\mu\nu} \times C_1) - 2 (F^+ + B^+)_{\mu\nu},
\hat{\delta} C_{3\mu\nu} = 0,
\hat{\delta} C_{12} = C_{13},
\hat{\delta} C_{13} = 0.
\tag{2.21}
\]
The explicit expression for $C_{3\mu\nu}$ has been obtained from functional integration.

It turns out that the algebra (2.21) closes off-shell.

The covariant, BRST invariant action, is given by:
\[ S_{\text{eff}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} (F^+ + B^+)^2 + \tilde{\delta} (C_2 \chi_2) + \tilde{\delta} (C_{12} \chi_{12}) \
\right. \\
\left. + \tilde{\delta} (C_{2 \mu \nu} \chi_{2 \mu \nu}^{\mu \nu}) - \frac{1}{8} C_{2 \mu \nu} (C_{2}^{\mu \nu} \times C_{11}) \right], \quad (2.22) \]

and it can be rewritten using equations (2.20) and (2.21) as follows

\[ S_{\text{eff}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} F_{\mu \nu}^{\mu \nu} + \left( \frac{1}{2} - a \right) B_{\mu \nu}^{+ \mu \nu} + \left( 1 - a \right) F_{\mu \nu}^{+ \mu \nu} \right. \\
\left. + C_{2 \mu \nu} D_{\mu} C_{1 \nu} + \frac{1}{8} C_{11} (C_{2}^{\mu \nu} \times C_{2 \mu \nu}) + \overline{C}_{13} D_{\mu} C_{1}^{\mu} \right. \\
\left. + C_{12} (C_{1}^{\mu} \times C_{1 \mu}) + C_{12} D_{\mu} D_{\mu} C_{11} + \frac{1}{2} C_{11} (\overline{C}_{13} \times \overline{C}_{13}) \right. \\
\left. - \frac{1}{2} (C_{12} \times C_{11}) (C_{12} \times C_{11}) + C_{3} (\partial_{\mu} A^{\mu} - \frac{\alpha}{\lambda}) C_{3} \right. \\
\left. + C_{2} \partial_{\mu} D_{\mu} C_{1} - C_{2} \partial_{\mu} C_{1}^{\mu} \right], \quad (2.23) \]

with \( \overline{C}_{13} = C_{13} + (C_{12} \times C_{1}) \). If we relax (2.22) from the gauge fixing \( \chi_2 \) and eliminate the auxiliary fields \( B_{\mu \nu} \) we exactly obtain the Witten’s TQFT with the following identifications

\[ C_{1 \mu} = i \psi_\mu, \quad C_{11} = -i \phi, \quad \overline{C}_{13} = -\eta, \]
\[ C_{12} = \frac{1}{2} i \lambda, \quad C_{2 \mu \nu} = -\chi_{\mu \nu}. \]

We notice that the sign of the term \( B^+ B^+ \), for any value of \( a \) does not correspond to the one in a Gaussian functional. However the \( B^+ B^+ \) term decouples from the action, hence the problem is harmless. In fact we may perform a change of variable \( B^+ \rightarrow iB^+ \) in the functional integral and integrate it. There is also a problem with the sign of the quartic term \( (C_{12} \times C_{11}) (C_{12} \times C_{11}) \), since its corresponding kinetic term is indefinite. This point was first noticed by Witten in [1]. The functional integral may be correctly defined by eliminating the last two terms of the gauge fixing \( \chi_{12} \) in eq.(2.22). In [1] \( \phi \) and \( \lambda \) are taken to be complex conjugates. However this implies from (2.21) that \( \tilde{\delta} C_1 \) becomes complex valued and hence inconsistent with the original assumption that the gauge parameters \( \Lambda \) are real valued. We will resolve this problem by following a different approach.
3 Off-Shell Supersymmetry

In order to obtain a gauge supersymmetric action with off-shell closure of the SUSY algebra, we consider the action

\[
S_{\text{eff}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} (F_+^+ F_+^{+\mu} - B_+^+ B_+^{+\mu}) + C_2^{\mu\nu} \mathcal{D}_\mu C_1 \right. \\
+ \mathcal{C}_{13} \mathcal{D}_\mu C_1^\mu + C_{12} (C_1^\mu \times C_1) + C_{12} \mathcal{D}_\mu \mathcal{D}_\mu C_1 \\
\left. + \frac{1}{2} C_1 (\mathcal{C}_{15} \times \mathcal{C}_{13}) - \frac{1}{2} (C_2 \times C_1) (C_2 \times C_1) \right],
\]

(3.1)

it arises from (2.22) by supressing the gauge fixing \( \chi_2 \) term. The action (3.1) is invariant under the BRST algebra (2.21). It is also invariant under the gauge transformations of the \( SU(2) \) principal bundle, which are generated by the first class constraint \( \mathcal{D}_i \pi^i \), since the corresponding gauge fixing condition has not been imposed. Finally (3.1) is invariant under general coordinate transformations.

If the auxiliary field \( B^+ \) is eliminated from (3.1) by Gaussian integration, we obtain the action proposed by Witten [1] for the TQFT describing Donaldson’s invariants as “topological” observables.

We define the following SUSY transformation \( \delta \)

\[
\delta \equiv \hat{\delta} \mid_{C_1 = 0},
\]

when acting on any of the fields describing the topological theory.

We then have

\[
\delta A_\mu = C_1 \mu,
\]

\[
\delta A_\mu = \delta A \mid_{C_1 = 0} = \hat{\delta} A \mid_{C_1 = 0} = \mathcal{D}_\mu C_1,
\]

\[
\delta C_1 = \mathcal{D}_\mu C_1,
\]

\[
\delta C_2 = -2 (F_+ + B_+)^{\mu\nu},
\]

\[
\delta B_+^{\mu\nu} = -\mathcal{D}_[\mu C_1^\nu] - \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \mathcal{D}_\rho C_1^\sigma - \frac{1}{2} (C_2^{\mu\nu} \times C_1),
\]

\[
\delta B_+^{\mu\nu} = B_+^{\mu\nu} \times C_1,
\]

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\[ \delta C_{11} = 0, \]
\[ \delta C_{12} = C_{13}, \]
\[ \delta C_{13} = 0. \]  

This algebra closes modulo field dependent gauge transformations (with parameter \( C_{11} \)).

The SUSY transformations defined by Witten are obtained from this algebra by eliminating the auxiliary field \( B \). After this reduction the SUSY algebra closes on-shell and modulo field dependent gauge transformations. On-shell because the auxiliary field \( B \) has been eliminated, and modulo field dependent gauge transformations because the Wess-Zumino field, \( C_{1} \) in this case, has been eliminated.

In our action \( (2.23) \) we also include the Wess-Zumino field \( C_{1} \) allowing the complete off-shell closure of the SUSY algebra. It is interesting to notice that \( C_{1} \) is not the usual Wess-Zumino field that is introduced in the superfield formulation of the SUSY gauge multiplet. This is allowed in the topological theory because we are only considering the nilpotent subalgebra of the full twisted SUSY algebra.

\section{BRST Charge and Hamiltonian of the TQFT on the extended phase space}

We now introduce an extended phase space in order to define a simplectic structure over it. The canonical conjugate pairs we define are:

\[ (A_{i}, \pi_{i})_{+}, \]
\[ (C_{1}, \mu_{1})_{-}, (C_{2}, P_{2})_{-}, (C_{3}, \lambda_{1}^{0})_{+}, \]
\[ (C_{1i}, \mu_{1i})_{-}, (C_{2i}, P_{2i})_{-}, (C_{3i}, \lambda_{1i}^{0})_{+}, \]
\[ (C_{11}, \mu_{11})_{+}, (C_{12}, P_{12})_{+}, (C_{13}, \lambda_{11}^{1})_{-}, \]

where \( C_{1} \) and \( C_{1i} \) are the ghost fields associated to the first class constraints of the theory. \( C_{11} \) arises from the reducibility of the first class constraints. Their ghost numbers are:

\[ C_{1} \quad +1 \]
\[ C_{1i} \quad +1 \]
The $C_2, C_i^2$ and $C_{12}$ fields are associated to the gauge fixing conditions and appear in the action as

$$\bar{\delta}(C_2 \chi_2 + C_i^2 \chi_i + C_{12} \chi_{12})$$

where $\bar{\delta}$ is the BRST transformation. $\chi_2, \chi_i$ are the gauge fixing conditions associated to the first class constraints $\phi$ and $\phi^i$ respectively, and have ghost number 0, while $\chi_{12}$, is related to the reducibility of $C_{1i}$ and $C_1$. It has ghost number +1. Consequently the ghost numbers of $C_2, C_i^2$ and $C_{12}$ are:

$$C_2 - 1, C_i^2 - 1, C_{12} - 2$$

The $C_3, C_i^3$ and $C_{13}$ fields are the Lagrange multipliers associated to the gauge fixing conditions. Their ghost numbers are then:

$$C_3 0, C_i^3 0, C_{13} - 1$$

The sum of the ghost numbers of the canonical conjugate pairs must be zero, hence the ghost numbers of the other variables in (4.1) are determined. The BRST charge in the extended phase space is

$$Q = \langle C_1(D_i \pi^i) + C_{1i} \pi^i + C_{11}[(D_i \mu^{1i}) - \mu^1]$$

$$-\frac{1}{2} C_1(C_1 \times \mu^1) - C_{1i}(C_{1i} \times \mu^{1i})$$

$$-C_{11}(C_1 \times \mu^{11}) - C_3 P_2 - C_i^3 P_{2i} + C_{13} P_{12} \rangle$$

$$= \Omega + \langle -C_3 P_2 - C_i^3 P_{2i} + C_{13} P_{12} \rangle.$$ (4.2)

The charge $Q$ is off-shell nilpotent

$$\{Q, Q\} = 0.$$ (4.3)
The BRST transformation for the variables is given by
\[ \tilde{\delta} Z = (-1)^{\varepsilon_z} \{ Z, Q \}, \] (4.4)
where \( \varepsilon_z \) is the grassmanian parity of \( Z \).

In order to express the Hamiltonian of section 3, which reproduces Witten’s effective action after functional integration of the auxilliary fields, as a \( Q \)-anticommutator we introduce the following functional \( \tilde{Q} \) which will turn out to be the time reversal of the BRST charge when reduced to the minimal sector of phase space. We define \( \tilde{Q} \) in a very general way in terms of general gauge fixing functions, and step by step we impose some consistency conditions over them. At the end of the process we give the general structure of the gauge fixing functions solving all consistency restrictions. The \( \tilde{Q} \) is given by:
\[ \tilde{Q} = \langle C_2 \tilde{\chi}_2 + C_2^i \tilde{\chi}_{2i} + C_{12} \tilde{\chi}_{12} + \lambda_0^0 \mu^1 + \lambda_{i_1}^0 \mu^{i_1} + \lambda_{i_1}^1 \mu^{1i} \rangle. \] (4.5)

In order to have off-shell nilpotency of \( Q^\dagger \)
\[ \{ \tilde{Q}, Q \} = 0, \] (4.6)
we impose the following restrictions on the gauge fixing functions:
\[ \{ \langle C_2 \tilde{\chi}_2 + C_2^i \tilde{\chi}_{2i} + C_{12} \tilde{\chi}_{12} \rangle, \langle \lambda_0^0 \mu^1 + \lambda_{i_1}^0 \mu^{i_1} + \lambda_{i_1}^1 \mu^{1i} \rangle \} = 0, \] (4.7)
and
\[ \{ \langle C_2 \tilde{\chi}_2 + C_2^i \tilde{\chi}_{2i} + C_{12} \tilde{\chi}_{12} \rangle, \langle C_2 \tilde{\chi}_2 + C_2^i \tilde{\chi}_{2i} + C_{12} \tilde{\chi}_{12} \rangle \} = 0. \] (4.8)

Later on we will discuss these restrictions, they turn out to be mild conditions on the gauge fixing functions. We then obtain
\[ \{ Q, \tilde{Q} \} = \langle \lambda_0^0 (D_i \pi^i - C_1 \times \mu^1 - C_{i_1} \times \mu^{i_1} + C_{11} \times \mu^{1i}) + \lambda_{11}^0 (\pi^2 - C_1 \times \mu^{1i} - \lambda_{11}^1 (D_i \mu^{i_1} - \mu^1 - C_1 \times \mu^{1i}) - C_{3i} \tilde{\chi}_2 - C_{3i} \tilde{\chi}_{2i} - C_{13} \tilde{\chi}_{12} - P_2 \mu^1 - P_2 \mu^{i_1} + P_{12} \mu^{1i} + C_2 \{ \langle P_2 C_3 \rangle, \tilde{\chi}_2 \} + C_2 \{ \langle P_2 C_3^i \rangle, \tilde{\chi}_{2i} \} + C_{12} \{ \langle P_{12} C_{13} \rangle, \tilde{\chi}_{12} \} - C_2 \{ \Omega, \tilde{\chi}_2 \} - C_2 \{ \Omega, \tilde{\chi}_{2i} \} + C_{12} \{ \Omega, \tilde{\chi}_{12} \} \rangle. \] (4.9)

The right hand member of eq.(4.9) is exactly the expansion of the terms
\[ \langle -\tilde{\delta} (\lambda_0^0 \mu^1 + \lambda_{i_1}^0 \mu^{i_1} + \lambda_{i_1}^1 \mu^{1i}) - \tilde{\delta} (C_2 \tilde{\chi}_2 + C_2^i \tilde{\chi}_{2i} + C_{12} \tilde{\chi}_{12}) \rangle, \] (4.10)
in eq.(2.17). We distinguish $\chi$ from $\hat{\chi}$ since it is more interesting for our formulation to consider

$$\begin{align*}
\chi_2 &= \hat{\chi}_2 + \dot{\lambda}^0_i, \\
\chi_{2i} &= \hat{\chi}_{2i} + \dot{\lambda}^0_{1i}, \\
\chi_{12} &= \hat{\chi}_{12} - \dot{\lambda}^1_{11},
\end{align*}$$

(4.11)

We consider in the action (2.17) the gauge fixing conditions (4.11). The time derivative terms contribute to the kinetic part of the action which now becomes

$$\int d^4x \sqrt{g} \left[ \pi^i \dot{A}_i + \mu^1 \dot{C}_1 + \mu^{1i} \dot{C}_{1i} + \mu^{11} \dot{C}_{11} + P_2 \dot{C}_2 + P_{2i} \dot{C}_{2i} + P_{12} \dot{C}_{12} \\
+ \lambda^0_{1i} \dot{C}_{1i} + \lambda^0_{1} \dot{C}_{1} + \lambda^1_{11} \dot{C}_{11} \right],$$

(4.12)

while the $\hat{\chi}$ terms contribute to the Hamiltonian of the system which has then the expression (4.9). We have thus expressed the action (2.17) completely in terms of canonical conjugate pairs with the Hamiltonian having the form

$$H = \{Q, \hat{Q}\},$$

(4.13)

with $Q$ and $\hat{Q}$ off-shell nilpotent charges. From (4.13) one obtains

$$\begin{align*}
\{Q, H\} &= 0, \\
\{\hat{Q}, H\} &= 0.
\end{align*}$$

(4.14)

The important remark with respect to the BRST charge (4.2) is that it is independent of the metric over the base manifold $X$. This is a great advantage with respect to the approach in [1], where the BRST charge is obtained from the covariant effective action, already metric dependent, and hence the charge is metric dependent. Our canonical approach instead ensures that the BRST charge (4.2) is metric independent. $\hat{Q}$ may depend on the metric through the gauge fixing functions. Usually in any field theory the gauge fixing conditions leading to a covariant formulation of the effective action are metric dependent. The BRST cohomology, obtained from (4.2), which may be identified to the Floer cohomology is then a topological invariant of the boundary of $X$. In [1] it is argued, using (4.13), how to relate the BRST cohomology with the ground states of the Hamiltonian. Let $\psi$ be a state of the quantum theory. It must satisfy

$$Q \psi = 0.$$ 

(4.15)
If $\psi$ is an eigenstate of the Hamiltonian

$$H\psi = \lambda \psi,$$  \hspace{1cm} (4.16)

with $\lambda \neq 0$. Then

$$\psi = Q \left( \lambda^{-1} \tilde{Q} \psi \right),$$  \hspace{1cm} (4.17)

and this means that $\psi$ is in the trivial cohomology, it is $Q$–exact. Then the cohomology classes of $Q$ correspond to zero eigenstate of $H$. Conversely, if there exists a positive inner product and

$$H\psi = 0,$$  \hspace{1cm} (4.18)

then

$$\langle \psi | H | \psi \rangle = |Q|\psi\rangle|^2 + |\tilde{Q}|\psi\rangle|^2,$$  \hspace{1cm} (4.19)

and consequently

$$Q|\psi\rangle = 0,$$  \hspace{1cm} (4.20)

$$\tilde{Q}|\psi\rangle = 0.$$  \hspace{1cm} (4.21)

In (4.19) it is explicitly used the positivity of the inner product. However the Hilbert space in [1] as stated by Witten, is indefinite because of the terms $\eta D_0 \psi_0$ and $(D_0 \psi_i) \chi_i$ in the Hamiltonian. Moreover the Hamiltonian in [1] is not bounded from below because of the term $\phi \Delta \lambda$ when $\phi$ and $\lambda$ are real. We will discuss these problems in the next section.

### 5 Time reversal operation and positivity of the Hamiltonian

In order to discuss the problems related to the positivity of the inner product in the space of quantum states, we will be interested in analysing the quadratic part of the Hamiltonian, considered as a polynomial on the fields. We will denote it $H_2$. It may be expressed as

$$H_2 = \{ Q_2, \tilde{Q}_2 \},$$  \hspace{1cm} (5.1)
where $Q_2$ and $\tilde{Q}_2$ are the quadratic parts, when considered as a polynomial on the fields, of the BRST and anti-BRST charges:

\begin{align}
\{Q_2, Q_2\} &= 0, \quad (5.2) \\
\{\tilde{Q}_2, \tilde{Q}_2\} &= 0. \quad (5.3)
\end{align}

We may now introduce the time reversal operation $T$:

\[
\begin{align*}
P_2 & \stackrel{T}{\rightarrow} C_2 \\
P_{2i} & \stackrel{T}{\rightarrow} C_{2i} \\
P_{12} & \stackrel{T}{\rightarrow} C_{12} \\
C_3 & \stackrel{T}{\rightarrow} -\tilde{\chi}_2 \\
C^i_3 & \stackrel{T}{\rightarrow} -\tilde{\chi}_{2i} \\
C_{13} & \stackrel{T}{\rightarrow} \tilde{\chi}_{12} \\
C_{11} & \stackrel{T}{\rightarrow} \mu^{11} \\
(\partial_i \mu^{1i} - \mu^1) & \stackrel{T}{\rightarrow} \lambda^{11}_1 \\
\tau^i & \stackrel{T}{\rightarrow} \lambda^0_{1i} \\
C_1 & \stackrel{T}{\rightarrow} \mu^1 \\
C_{1i} & \stackrel{T}{\rightarrow} \mu^{1i}
\end{align*}
\]  

(5.4)

where $\partial_i$ denotes the covariant derivative with respect to the background metric. The time reversal operation $T$ by definition satisfies $T^2 = 1$. This implies that

\[ \lambda^{11}_1 = \partial^i C_{1i} - C_1, \]  

(5.5)

which will arise from the gauge fixing procedure. For the BRST charge we have

\[
Q_2 \stackrel{T}{\rightarrow} \tilde{Q}_2 \stackrel{T}{\rightarrow} Q_2,
\]

(5.6)

We will denote $\phi^i \equiv T\phi$. Later on we will interpret $\dagger$ as the adjoint under the internal product we will introduce.

We will consider now several canonical reductions of the topological action. All of them will be performed on the full action and then discuss
properties of the quadratic part of the Hamiltonian. We first assume

\[
\begin{align*}
\{\langle P_2 C_3 \rangle, \chi_2 \} &= a P_2, \\
\{\langle P_2 C_3^i \rangle, \chi_{2i} \} &= b P_{2i}, \\
\{\langle P_{12} C_{13} \rangle, \chi_{12} \} &= d P_{12},
\end{align*}
\]

(5.7)

where \(a, b, d\) are real numbers to be determined. We then functionally integrate on \(P_2, P_{2i}, P_{12}\). We obtain

\[
\delta (\mu^1 + a C_2) \delta (\mu^{1i} + b C_2^i) \delta (\mu^{11} + d C_{12}),
\]

(5.8)

in the functional measure. We finally integrate on \(C_2, C_2^i, C_{12}\) and obtain the Hamiltonian in the reduced phase space:

\[
H = \langle \lambda_1^0 (\mathcal{D}_i \pi^i - C_1 \times \mu^1 - C_{1i} \times \mu^{1i} + C_{11} \times \mu^{11}) \\
+ \lambda_{1i}^0 (\pi^i - C_1 \times \mu^{1i}) - \lambda_{11}^1 (\mathcal{D}_i \mu^{1i} - \mu^1 - C_1 \times \mu^{11}) \\
- C_3 \hat{\chi}_2 - C_3^i \hat{\chi}_{2i} - C_{13} \hat{\chi}_{12} + \frac{1}{a} \mu^1 \{\Omega, \hat{\chi}_2\} \\
+ \frac{1}{b} \mu^{1i} \{\Omega, \hat{\chi}_{2i}\} - \frac{1}{d} \mu^{11} \{\Omega, \hat{\chi}_{12}\} \rangle.
\]

(5.9)

We will impose further conditions on the gauge fixing functions. They are

\[
\begin{align*}
\{\Omega, \chi_2 \} &= -\alpha (\Delta C_1 - \partial^i C_{1i}), \\
\{\Omega, \chi_{2i} \} &= -\eta (C_{1i} - \partial_i C_1), \\
\{\Omega, \chi_{12} \} &= \gamma (\Delta C_{11} - C_{11}),
\end{align*}
\]

(5.10)

where

\[
\begin{align*}
\frac{\gamma}{d} &> 0, \\
\frac{\alpha}{a} &= -\frac{\eta}{b} > 0.
\end{align*}
\]

(5.11)

We have then imposed several restrictions to the gauge fixing functions. It turns out that a solution to all of them is given by

\[
\begin{align*}
\chi_2 &= a \lambda_1^0 - \alpha \partial^i A_i, \\
\chi_{2i} &= b \lambda_{1i}^0 + \eta A_i, \\
\chi_{12} &= d \lambda_{11}^1 - \gamma (\partial^i C_{1i} - C_1).
\end{align*}
\]

(5.12)
These gauge fixing functions may be continuously deformed with linear and nonlinear terms. They are all admissible gauge fixing functions. The QFT should be independent of the element in the admissible set. This is though if there isn’t a gauge anomaly in the theory. We will consider this point in the next section.

We may now perform a further canonical reduction. We will eliminate the pair of conjugate variables \( (C_3, \lambda^0_1) \), \( (C_3^i, \lambda^0_{1i}) \) and \( (C_{13}, \lambda^1_{11}) \) which appear linearly in the full action. In order to perform the reduction we integrate on \( C_3 \), \( C_3^i \), \( C_{13} \). We obtain in the functional measure

\[
\delta(\chi_2) \delta(\chi_{2i}) \delta(\chi_{12}).
\]

We may now integrate on \( \lambda^0_1, \lambda^0_{1i} \) and \( \lambda^1_{11} \). After this final canonical reduction we have returned to the minimal sector of phase space. In the process, however, we have been able to find gauge fixing functions which yield a canonical action whose quadratic part is invariant under \( T \) and with a positive definite quadratic part of the Hamiltonian. The explicit expressions for the quadratic parts are

\[
S_2 = \int d\tau \left[ \langle \pi^i \dot{A}_i + \mu^1 \dot{C}_1 + \mu^{11} \dot{C}_{11} \rangle - H_2 \right],
\]

\[
H_2 = \langle \pi^i \pi^i + \frac{\alpha}{d} \langle \partial^i \pi^i \rangle \partial_i \pi^i + \frac{\gamma}{d} \langle \partial^i \pi^i \rangle \partial^i \pi^i + \frac{\gamma}{d} \langle \partial^i \pi^i \rangle \partial^i \pi^i \rangle \right] - H_2.
\]

After the elimination of \( \lambda^0_{1i} \), we obtain from (5.14)

\[
T \pi^i = \lambda^0_{1i} = A_i,
\]

and since \( T^2 = 1 \)

\[
TA_i = \pi^i,
\]

we have completed the table (5.4) for all canonical conjugate pairs. It is important to remark several points with respect to the structure of \( H_2 \) in (5.15). In [1] the fields \( \phi \) and \( \lambda \) were taken to be complex and satisfying

\[
\phi = -\lambda^*.
\]
in order to ensure the positivity of the \((\phi, \lambda)\) kinetic terms in the action of the TQFT. The drawback of this approach, as explained by Witten, is that the Lagrangian is then not real. Consequently the quantum field correlation function giving rise to the Donaldson invariants are not manifestly real. In our approach, \(C_{11}\) and \(\mu^{11}\) are real fields related by the time reversal operation. In the reduction procedure we obtain from (5.8) \(C_{12}\) in terms of \(\mu^{11}\). The corresponding quadratic term is then manifestly positive definite when one defines the inner product on the Hilbert space of physical states \(\mathcal{H}\) as

\[
(\psi, \varphi)_{+} \equiv (T\psi, \varphi),
\]

where \((\ ,\ )\) is the \(L^2\) inner product. The advantage of the approach we have followed as suggested in [1] is that the adjoint in the sense of \((\ ,\ )_{+}\) of \(\phi\) is \(\lambda\) and viceversa, while in terms of the \((\ ,\ )\) inner product they are self-adjoints. The same happens to \(Q\) and \(Q^\dagger\), which are adjoint under the \((\ ,\ )_{+}\) inner product and this is precisely the property needed in the argument used to show that all ground states of the Hamiltonian are physical states. The next point to emphasize is with respect to the indefinite terms \(\eta D\psi\) and \((D\psi_i)\chi_i\) appearing in (5.8). They automatically yield, for self-adjoint fields, an indefinite Hilbert space inner product. In fact,

\[
\{\psi(\sigma), \eta(\sigma')\} = \delta_{\sigma\sigma'},
\]

\[
\{\psi^i(\sigma), \chi_j(\sigma')\} = \delta^i_j\delta_{\sigma\sigma'},
\]

and all the others anticommutators are zero.

We then applied a bra and a ket from the right and left respectively to all anticommutators and use the self-adjoint property of the fields to obtain from the left member a positive definite matrix while from the right an indefinite matrix, that is an inconsistency unless the inner product is not positive definite. In our approach, the definition (5.19) , allows to obtain positive definite results from the right and left members of (5.20). This is through because under the time reversal operation:

\[
\begin{align*}
\psi & \overset{T}{\rightarrow} \eta \overset{T}{\rightarrow} \psi \\
\chi_i & \overset{T}{\rightarrow} \psi_i \overset{T}{\rightarrow} \chi_i.
\end{align*}
\]

We then have

\[
(\varphi, \psi \eta \varphi)_{+} = (T\varphi, \psi \eta \varphi) = (\psi T\varphi, \eta \varphi) =
\]

\[
(T T\psi T \varphi, \eta \varphi) = (T \eta \varphi, \eta \varphi) = (\eta \varphi, \eta \varphi)_{+},
\]

20
that is, the matrix from the right hand side in the previous argument is now also positive definite.

We have thus obtained a canonical version of Witten’s TQFT with a different set of gauge fixing functions. Starting from the classical action \( (2.1) \) we have performed a canonical construction of the effective action on general gauge fixing functions. In the covariant gauge (section 2) we recover Witten’s TQFT effective action with manifest covariance. In the new set of gauge fixing functions defined in this paper \( (5.12) \) we obtain an effective action whose quadratic part is invariant under time reversal, with a positive definite Hamiltonian \( H_2 \) in terms of the inner product introduced in \( (5.19) \). The necessary conditions to have a Hamiltonian consistent with a positive inner product, raised in \( [1] \), are then satisfied. The interpretation of the Floer’s theory in terms of a quantum field theory may then be performed with the gauge fixing functions introduced in this section.

6 Effective Action of the TQFT

We will present in this section an admissible deformation of \( (5.12) \) with non-linear terms which allows to rewrite the Hamiltonian \( (5.9) \) in a manifestly positive form. It can be deduced from \( (5.9) \) that a convenient non-linear admissible deformation of \( (5.12) \) is given by

\[
\begin{align*}
\chi_2 &= a\lambda^0 - \alpha (D_i\pi^i - \frac{1}{2} C_1 \times \mu^1)^\dagger, \\
\chi_{2i} &= b\lambda_{1i}^0 + \eta (\pi^i - C_1 \times \mu^{1i})^\dagger, \\
\chi_{12} &= d\lambda_{11}^1 - \gamma (D_i\mu^{1i} - \mu^1 - C_1 \times \mu^{11})^\dagger.
\end{align*}
\]

(6.1)

We consider in what follows

\[
\begin{align*}
\alpha &= a = 1 \\
\eta &= -b = -1 \\
\gamma &= d = 1
\end{align*}
\]

which satisfy the restriction \( (5.11) \). Using \( (5.4) \) and \( (5.16-5.17) \) we obtain

\[
\begin{align*}
\chi_2 &= \lambda^0 - (\partial^i A_i - A_i \times \pi^i + \frac{1}{2} C_1 \times \mu^1), \\
\chi_{2i} &= \lambda_{1i}^0 - (A_i + C_1 \times \mu^i), \\
\chi_{12} &= \lambda_{11}^1 - (\partial^i C_1i + \pi^i \times C_{1i} - C_1 + C_{11} \times \mu^1),
\end{align*}
\]

(6.2)
where we have used the convention

\[(AB)^\dagger = B^\dagger A^\dagger.\]

We may now insert (6.2) into (5.9) and perform the same canonical reduction as before. We eliminate then the canonical conjugate pairs \((C_3, \lambda^0_1), (C_3^i, \lambda^0_{1i})\) and \((C_{13}, \lambda^1_{11})\).

After this canonical reduction we end with a description of the theory in terms of the minimal sector of phase space as in section 5 but now for the complete effective action. After several calculations we obtain

\[
S = \int d\tau \left[ \langle \pi^i \dot{A}_i + \mu^1 \dot{C}_1 + \mu^{1i} \dot{C}_{1i} + \mu^{11} \dot{C}_{11} \rangle - H \right]
\]

\[
H = \langle \pi^i - C_1 \times \mu^{1i} \rangle \langle \pi^i - C_1 \times \mu^{11} \rangle + (\mathcal{D}_i \pi^i - C_1 \times \mu^{11})(\mathcal{D}_i \pi^i - C_1 \times \mu^{11}) - C_{1i} \times \mu^{1i} + C_{11} \times \mu^{11} + (C_{1i} - \mathcal{D}_i C_1)(C_{1i} - \mathcal{D}_i C_1)
\]

\[
+ (\partial^i C_{1i} - C_1 + \pi^i \times C_{1i} - \mu^1 \times C_{11})(\partial^i C_{1i} - C_1 + \pi^i \times C_{1i} - \mu^1 \times C_{11}) + (\mu^{11} \times C_{1i} + \mu^1 \times A_i)^\dagger (\mu^{11} \times C_{1i} + \mu^1 \times A_i)
\]

\[
+ (\mathcal{D}_i C_{1i} + C_1 \times C_{1i})^\dagger (\mathcal{D}_i C_{1i} + C_1 \times C_{1i}) + (C_{11} + \frac{1}{2} C_1 \times C_1)^\dagger (C_{11} + \frac{1}{2} C_1 \times C_1)
\]

\[
+ (C_1 \times C_{1i})^\dagger (C_1 \times C_{1i}) + (C_{11} \times C_1)^\dagger (C_{11} \times C_1).
\]

The Hamiltonian (6.3) may be expressed as the Poisson bracket of the off-shell nilpotent BRST charge \(\Omega\) (2.14), and anti-BRST charge \(\Omega^\dagger\):

\[
\Omega^\dagger = T\Omega
\]

\[
\{\Omega, \Omega\} = 0,
\]

\[
\{\Omega^\dagger, \Omega\} = 0,
\]

\[
H = \{\Omega, \Omega^\dagger\}.
\]

We notice that the complete effective action (6.3) is invariant under the time reversal operation. We have constructed then gauge fixing functions (6.1), which are admissible deformations of (5.12), yielding an effective BRST invariant action, consistent with a positive inner product in the space of Hilbert states. The quantum equivalence between the effective action (6.3) and Witten’s effective action arises from the independence of the functional
integral on the gauge fixing functions. This latest point is based on the BRST invariance of the canonical action \cite{21}-\cite{22}. In order to ensure this point, the requirement of nilpotency of $Q$ has to be raised, to the quantum level. That is

$$\{Q,Q\} = 0,$$  

(6.5)
as an operatorial condition.

Given an ordering for the BRST charge $Q$ then the ordering for $H$ is automatically determined from

$$H = \{Q,Q^\dagger\}.$$  

(6.6)
The expression (4.2) of the BRST charge is given in the $qp$ order and as already said is classically nilpotent off-shell. We consider now the situation when $Q$ is an operator constructed from the canonical quantization approach. It has the property that is linear in the conjugate momenta variable. Consequently for any commutator of the form

$$\left[\prod_L q_j \hat{p}_i, \prod_K \hat{q}_j \hat{p}_i\right]$$  

(6.7)
where $\prod q$ denotes a product of $q_l$ operators for $l \in L$ and $\prod \hat{q}$ another product of $\hat{q}$ operators at a different point, and the conjugate pairs being \((q_i,p_i)\) $i = 1,\ldots,n$, we expand (6.7) and obtain

$$\prod_K \hat{q} \left[\prod_L q, \hat{p}_i\right] p_j + \prod_L q \left[p_j, \prod_K \hat{q}\right] \hat{p}_i$$  

(6.8)
which is again linear in the momenta and also $qp$ ordered. Any cancellation that was valid classically using Poisson brackets is then valid at the operatorial level. In fact the only difference between the two evaluations is in the ordering of the resulting terms in (6.8). Two polinomic terms which can cancel classically may not do it as operators because of possible different orderings of non commuting operators in the polinomy. However in the TQFT under consideration the resulting expression (6.8) has a determined $qp$ ordering and consequently any classical cancellation ensures an operatorial one. It can then be shown that the expression for the BRST Charge $Q$ in (4.2) satisfies (6.5) as an operator. This important property ensures that there are no gauge anomalies in this QFT and hence the quantum equivalence of Witten’s
TQFT effective action and the one presented in this paper is assured. Having defined $H = \{Q, Q^\dagger\}$, the nilpotency of $Q$ ensures the operatorial relation

$$[H, Q] = 0 \quad \text{(6.9)}$$

The property of linearity of the BRST charge $Q$ in the momenta may also be used as in [26] to show that the TQFT is independent of the coupling constant. This property is directly related to the independence of the partition function on the background metric as explained in [1]. Finally the linearity of $Q$ also ensures the consistency of the Hamiltonian approach over a compact euclidean orientable manifold $M$ [20]. The point is that one may always embed $M$ over $\mathbb{R}^N$ for $N$ large enough. One can then consider a height function over $M$ by taking a direction on $\mathbb{R}^N$. This defines an euclidean time over $M$, and allows to follow a canonical approach as we have done. The consistency requirement arises from the BRST invariance of the effective action. It requires boundary conditions at the points of heighest and lowest $\tau$. Since we may choose almost any direction on $\mathbb{R}^N$ to define $\tau$, we end up with consistency conditions almost every where over $M$. Fortunately when $Q$ is linear in the momenta, the boundary condition is satisfied identically giving rise to a consistent Hamiltonian approach. The boundary conditions on other fields theories with locally propagating degrees of freedom are in general non trivial and with a relevant physical intpretation [21]-[22].

7 Conclusions

We obtained the canonical structure for Witten’s TQFT allowing the description of the Floer theory in terms of a Hamiltonian consistent with a positive inner product. We started from a gauge action introduced in [14] and by considering a covariant gauge fixing BRST procedure we obtained Witten’s effective action including auxiliarly fields. We then found an extension of the phase space, where the explicit expression of the off-shell nilpotent charge is obtained. The Hamiltonian is expressed in the form $\{Q, Q^\dagger\}$ where $Q$ and $Q^\dagger$ are the BRST and anti-BRST nilpotent charge. The $Q^\dagger$ is expressed in a general form in terms of gauge fixing functions satisfying necessary requirements to obtain nilpotency of the anti-BRST charge. The explicit expression of $Q$ is metric independent, the resulting BRST cohomology is then identified, following Witten, to the Floer groups. This gives a direct proof that Floer’s
groups are topological invariants depending on the boundary of the base manifold \( X \). The Hamiltonian approach is shown to be consistently defined by checking that the boundary conditions arising from the BRST construction are identically satisfied. The time reversal operation introduced by Witten in [1] is generalized to the extended phase space. It is shown that the necessary conditions raised in [1] to have a positive definite inner product in the Hilbert space of states are satisfied provided that suitable admissible gauge fixing functions are chosen. The resulting TQFT with those gauge fixing functions is not manifestly covariant but satisfies the positivity requirement. Finally by going to the operatorial formulation it is shown that the nilpotency condition on the BRST operator is satisfied as a quantum operator. This property ensures that Witten’s TQFT, manifestly covariant, and the one obtained from the gauge fixing functions introduced in this paper are quantum equivalent field theories. The two properties, covariance and positivity are then obtained by considering different gauge fixing functions of the same TQFT, arising from the classical gauge action introduced in [14].

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References

[1] E. Witten, *Comm. Math. Phys.* 117(1988)353.

[2] E. Witten, *Math. Res. Lett.* 1(1994)769.

[3] N. Seiberg and E. Witten, *Nucl. Phys.* B426(1994)19.

[4] L. Baulieu and I.M. Singer, *Nucl. Phys.* (Proc. Suppl.) 5B(1988)12.

[5] C. Aragão and L. Baulieu, *Phys. Lett.* B275(1992)315.

[6] D. Birmingham, M. Blau, M. Rakowski and G. Thompson *Phys. Rep.* 209(1991)129.

[7] Y. Igarashi, H. Imai, S. Kitakado and H. So, *Phys. Lett.* B227(1989)239.

[8] L. Baulieu and B. Grossman, *Phys. Lett.* B214B(1988)223.

[9] L. Baulieu, M. Green and E. Rabinovici, *Nucl. Phys.* B498(1997).

[10] L. Baulieu and P. West, *hep-th/9805200*.

[11] L. Baulieu and E. Rabinovici, *hep-th/9805122*.

[12] R. Szabo, *hep-th/9804150*.

[13] M.F. Atiyah, *The Symposium on the Mathematical Heritage of Hermann Weyl*, Wells R. et al (eds.) (Univ. of North Carolina, May, 1987).

[14] R. Gianvittorio, A. Restuccia and J. Stephany, *Phys. Lett.* B347(1995)279.

[15] K. Osterwalder and R. Schrader, *Comm. Math. Phys.* 31(1973)83.

[16] J. Glimm and A. Jaffe, *Quantum Physics*, Springer Verlag, 1987.
[17] I. Martin and J.G. Taylor, *Phys. Lett.* B185(1987)99.

[18] I. Martin, *Euclidean Supersymmetries*, Ph.D. Thesis, London University, King’s College 1986.

[19] C. Teitelboim, *Phys. Rev.* D426(1982)3159.

[20] R. Gianvittorio, I. Martin and A. Restuccia, *Class. Quan. Grav.* 13(1996)2887.

[21] I.A. Batalin and E. Fradkin, *Phys. Lett.* B122(1983)157; *Phys. Lett.* B128(1983)307.

[22] M. I. Caicedo and A. Restuccia, *Class. Quan. Grav.* 10(1993)833; *Phys. Lett.* B307(1993)77.

[23] N. Hitchin, *Proc. London Math. Soc.* 55(1987)59.

[24] N. Hitchin, *Lectures at LASSF 89*, U.S.B., Caracas, Venezuela.

[25] I. Martin, A. Mendoza and A. Restuccia, *Lett. Math. Phys.* 21(1991)221.

[26] M. Caicedo, R. Gianvittorio, A. Restuccia and J. Stephany, *Phys. Lett.* B354(1995)292.