EXPANDERS WITH SUPERQUADRATIC GROWTH

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Abstract. We will prove several expanders with exponent strictly greater than 2. For any finite set $A \subset \mathbb{R}$, we prove the following six-variable expander results

$$| (A - A)(A - A)(A - A) | \gg \frac{|A|^{2+\frac{1}{8}}}{\log \frac{17}{16} |A|}$$

$$| A + A + A | \gg \frac{|A|^{2+\frac{1}{8}}}{\log \frac{17}{16} |A|}$$

$$| AA + AA | \gg \frac{|A|^{2+\frac{1}{8}}}{\log |A|}$$

$$| AA + A | \gg \frac{|A|^{2+\frac{1}{8}}}{\log |A|}$$

1. Introduction

Let $A$ be a finite set of real numbers. The sum set of $A$ is the set $A + A = \{a + b : a, b \in A\}$ and the product set $AA$ is defined analogously. The Erdős-Szemerédi sum-product conjecture\footnote{In fact, the conjecture was originally stated for all $A \subset \mathbb{Z}$, but it is also widely believed to be true for all $A \subset \mathbb{R}$.} states that, for any such $A$ and all $\epsilon > 0$ there exists an absolute constant $c_\epsilon > 0$ such that

$$\max\{|A + A|, |AA|\} \geq c_\epsilon |A|^{2-\epsilon}.$$  
In other words, it is believed that at least one of the sum set and product set will always be close to the maximum possible size $|A|^2$, suggesting that sets with additive structure do not have multiplicative structure, and vice versa.

A familiar variation of the sum-product problem is that of showing that sets defined by a combination of additive and multiplicative operations are large. A classical and beautiful result of this type, due to Ungar \cite{21}, is the result that for any finite set $A \subset \mathbb{R}$

$$\left| \frac{A - A}{A - A} \right| \geq |A|^2 - 2,$$

where

$$\frac{A - A}{A - A} = \left\{ \frac{a - b}{c - d} : a, b, c, d \in A, c \neq d \right\}.$$
This notation will be used with flexibility to describe sets formed by a combination of additive and multiplicative operations on different sets. For example, if $A, B$ and $C$ are sets of real numbers, then $AB + C := \{ab + c : a \in A, b \in B, c \in C\}$. We use the shorthand $kA$ for the $k$-fold sum set; that is $kA := \{a_1 + a_2 + \cdots + a_k : a_1, \ldots, a_k \in A\}$. Similarly, the $k$-fold product set is denoted $A^{(k)}$; that is $A^{(k)} := \{a_1a_2\cdots a_k : a_1, \ldots, a_k \in A\}$.

We refer to sets such as $A - A$, which are known to be large, as expanders. To be more precise, we may specify the number of variables defining the set; for example, we refer to $A - A$ as a four variable expander.

Recent years have seen new lower bounds for expanders. For example, Roche-Newton and Rudnev [16] proved that for any $A \subset \mathbb{R}$

$$|(A - A)(A - A)| \gg \frac{|A|^2}{\log |A|},$$

and Balog and Roche-Newton [2] proved that for any set $A$ of strictly positive real numbers

$$|A + A| \geq 2|A|^2 - 1.$$  

Note that equations (1.1), (1.2) and (1.3) are optimal up to constant (and in the case of (1.2), logarithmic) factors, as can be seen by taking $A = \{1, 2, \ldots, N\}$. More generally, any set $A$ with $|A + A| \ll |A|$ is extremal for equations (1.1), (1.2) and (1.3).

With these results, along with others in [5], [6], [11] and [14], we have a growing collection of near-optimal expander results with a lower bound $\Omega(|A|^2)$ or $\Omega(|A|^2/\log |A|)$. All of the near-optimal expanders that are known have at least 3 variables. The aim of this paper is to move beyond this quadratic threshold and give expander results with relatively few variables and with lower bounds of the form $\Omega(|A|^{2+c})$ for some absolute constant $c > 0$.

1.1. Statement of results. It was conjectured in [2] that for any $A \subset \mathbb{R}$ and any $\epsilon > 0$, $|(A - A)(A - A)(A - A)| \gg |A|^{3-\epsilon}$. In this paper, a small step towards this conjecture is made in the form of the following result.

**Theorem 1.1.** Let $A \subset \mathbb{R}$. Then

$$|(A - A)(A - A)(A - A)| \gg \frac{|A|^{2+\frac{\epsilon}{2}}}{\log^{17/16} |A|}.$$  

This result is the first improvement on the bound $|(A - A)(A - A)(A - A)| \gg |A|^2/\log |A|$ which follows trivially from (1.2). The proof uses some beautiful ideas of Shkredov [18].

The following theorem gives partial support for the aforementioned conjecture from a slightly different perspective.

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3Throughout the paper, this standard notation $\ll, \gg$ and respectively $O(\cdot), \Omega(\cdot)$ is applied to positive quantities in the usual way. Saying $X \gg Y$ or $X = \Omega(Y)$ means that $X \geq cY$, for some absolute constant $c > 0$. All logarithms in this paper are base 2.
Theorem 1.2. Let $A \subset \mathbb{R}$. Then for any $\epsilon > 0$ there is an integer $k > 0$ such that
$$|(A - A)^{(k)}| \gg_{\epsilon} |A|^{3-\epsilon}.$$ 

We also prove the following six variables expanders have superquadratic growth.

Theorem 1.3. Let $A \subset \mathbb{R}$. Then
$$|A + A + A| \gg |A|^{2+2/17} \log^{16/17} |A|.$$ 

Theorem 1.4. Let $A \subset \mathbb{R}$. Then
$$|AA + AA| \gg |A|^{11/8} |AA|^{3/4} \log |A|.$$ 

In particular, since $|AA| \geq |A|$,
$$|AA + AA| \gg |A|^{2+\frac{3}{8}} \log |A|.$$ 

Theorem 1.5. Let $A \subset \mathbb{R}$. Then
$$|AA + A| \gg |A|^{2+\frac{3}{8}} \log |A|.$$ 

The proofs of these three results make use of the results and ideas of Lund [10].

In fact, a closer inspection of the proof of Theorem 1.5 reveals that we obtain the inequality
$$\left| \left\{ \frac{ab + c}{ad + e} : a, b, c, d, e \in A \right\} \right| \gg |A|^{2+\frac{3}{8}} \log |A|.$$ 

Therefore, Theorem 1.5 actually gives a superquadratic five variable expander.

2. Preliminary Results

For the proof of Theorem 1.1 we will require the Ruzsa Triangle Inequality. See Lemma 2.6 in Tao-Vu [20].

Lemma 2.1. Let $A, B$ and $C$ be subsets of an abelian group $(G, +)$. Then
$$|A - B||C| \leq |A - C||B - C|.$$ 

A closely related result is the Plünnecke-Ruzsa inequality. A simple proof of the following formulation of the Plünnecke-Ruzsa inequality can be found in [13].

Lemma 2.2. Let $A$ be a subset of an abelian group $(G, +)$. Then
$$|kA - lA| \leq \frac{|A + A|^{k+l}}{|A|^{k+l-1}}.$$
We will also use the following variant, which is Corollary 1.5 in Katz-Shen [9]. The result was originally stated for subsets of the additive group $\mathbb{F}_p$, but the proof is valid for any abelian group.

**Lemma 2.3.** Let $X, B_1, \ldots, B_k$ be subsets of an abelian group $(G, +)$. Then there exists $X' \subset X$ such that $|X'| \geq |X|/2$ and

$$|X + B_1 + \cdots + B_k| \ll \frac{|X + B_1||X + B_2| \cdots |X + B_k|}{|X|^{k-1}}.$$

We will need various existing results for expanders. The first is due to Garaev and Shen [4].

**Lemma 2.4.** Let $X, Y, Z \subset \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then

$$|XY||(X + \alpha)Z| \gg |X|^{3/2}|Y|^{1/2}|Z|^{1/2}.$$

In particular,

(2.1) $$|X(X + \alpha)| \gg |X|^{5/4}$$

and

(2.2) $$\max\{|XY|, |(X + \alpha)Y|\} \gg |X|^{3/4}|Y|^{1/2}.$$

Note that Lemma 2.4 was originally stated only for $\alpha = 1$, but the proof extends without alteration to hold for an arbitrary non-zero real number $\alpha$. A similar and earlier result of Elekes, Nathanson and Ruzsa [3] will also be used.

**Lemma 2.5.** Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex or concave function and let $X, Y, Z \subset \mathbb{R}$. Then

$$|f(X) + Y||X + Z| \gg |X|^{3/2}|Y|^{1/2}|Z|^{1/2}.$$

Define

$$R[A] := \left\{ \frac{a-b}{a-c} : a, b, c \in A \right\}.$$

The following result is due to Jones [6]. An alternative proof can be found in [15].

**Lemma 2.6.** Let $A \subset \mathbb{R}$. Then

$$|R[A]| \gg \frac{|A|^2}{\log |A|}.$$

Each of the three latter results come from simple applications of the Szemerédi-Trotter Theorem.

Note that the proof of Lemma 2.6 also implies that there exists $a, b \in A$ such that

(2.3) $$|(A - a)(A - b)| \gg \frac{|A|^2}{\log |A|}.$$
See [15] for details. In particular, this gives a shorter proof of inequality (1.2), requiring only a simple application of the Szemerédi-Trotter Theorem. The inequality (1.2) will also be used in the proof of Theorem 1.1.

An important tool in this paper is the following result of Lund [10], which gives an improvement on (1.3) unless the ratio set $A/A$ is very large.

**Lemma 2.7.** Let $A \subset \mathbb{R}$. Then

$$|A + A| \gg \frac{|A|^2}{\log |A|} \left( \frac{|A|^2}{|A/A|} \right)^{1/8}.$$  

In fact, a closer examination of the proof of Lemma 2.7 reveals that it can be generalised without making any meaningful changes to give the following statement.

**Lemma 2.8.** Let $A, B \subset \mathbb{R}$. Then

$$|A + A| \gg \frac{|A||B|}{\log |A| + \log |B|} \left( \frac{|A||B|}{|A/B|} \right)^{1/8}.$$  

The proofs of Theorems 1.3 and 1.4 use Lemma 2.8 as a black box. However, for the proof of Theorem 1.5 we need to dissect the methods from [10] in more detail and reconstruct a variant of the argument for our problem. To do this, we will also need the following tools which were used in [10]. The first is a generalisation of the Szemerédi-Trotter Theorem to certain well-behaved families of curves. A more general version of this result can be found in Pach-Sharir [12].

**Lemma 2.9.** Let $P$ be an arbitrary point set in $\mathbb{R}^2$. Let $\mathcal{L}$ be a family of curves in $\mathbb{R}^2$ such that

- any two distinct curves from $\mathcal{L}$ intersect in at most two points and
- for any two distinct points $p, q \in P$, there exist at most two curves from $\mathcal{L}$ which pass through both $p$ and $q$.

Let $K \geq 2$ be some parameter and define $\mathcal{L}_K := \{ l \in \mathcal{L} : |l \cap P| \geq K \}$. Then

$$|\mathcal{L}_K| \ll \frac{|P|^2}{K^3} + \frac{|P|}{K}.$$  

We will need the following version of the Lovász Local Lemma. This precise statement is Corollary 5.1.2 in [1].

**Lemma 2.10.** Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent from all but at most $d$ of the events $A_j$ with $j \neq i$. Suppose also that the probability of the event $A_i$ occurring is at most $p$ for all $1 \leq i \leq n$. Finally, suppose that

$$ep(d + 1) \leq 1.$$  

Then, with positive probability, none of the events $A_1, \ldots, A_n$ occur.
3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Write $D = A - A$ and apply Lemma 2.3 in the multiplicative setting with $k = 2$, $X = DD$ and $B_1 = B_2 = D$. We obtain a subset $X' \subseteq DD$ such that $|X'| \gg |DD|$ and

\[
|X' DD| \ll \frac{|DDD|^2}{|DD|}.
\]

Then apply Lemma 2.1, again in the multiplicative setting, with $A = B = DD$ and $C = (X')^{-1}$. This bounds the left hand side of (3.1) from below, giving

\[
|DD/DD|^{1/2}|X'|^{1/2} \leq |X' DD| \ll \frac{|DDD|^2}{|DD|}.
\]

Recall the observation of Shkredov [18] that $R[A] - 1 = -R[A]$. Indeed, for any $a, b, c \in A$

\[
\frac{a - b}{a - c} - 1 = \frac{a - b - (a - c)}{a - c} = -\frac{c - b}{c - a}.
\]

Therefore, by Lemmas 2.4 and 2.6,

\[
|DD/DD| \geq |R[A] \cdot R[A]| = |R[A] \cdot (R[A] - 1)| \gg |R[A]|^{5/4} \gg \frac{|A|^{5/2}}{\log^{5/4} |A|}.
\]

Putting this bound into (3.2) yields

\[
\frac{|A|^{5/4}}{\log^{5/8} |A|} |X'|^{1/2} \ll \frac{|DDD|^2}{|DD|}.
\]

Finally, since $|X'| \gg |DD| \gg \frac{|A|^2}{\log |A|}$ by (1.2), it follows that

\[
|DDD|^2 \gg \frac{|A|^{5/4}}{\log^{5/8} |A|} |DD|^{3/2} \gg \frac{|A|^{5/4}}{\log^{5/8} |A|} \left( \frac{|A|^2}{\log |A|} \right)^{3/2} = \frac{|A|^{17/4}}{\log^{17/8} |A|}.
\]

and thus

\[
|DDD| \gg \frac{|A|^{2 + \frac{1}{8}}}{\log^{17/16} |A|}
\]

as claimed. \hfill \Box

We now turn to the proof of Theorem 1.2 which exploits similar ideas to the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $R := R[A]$ and $D = A - A$. Further, define $X_0 = D/D$ and recursively $X_i$ to be either $X_{i-1} R$ or $X_{i-1} (R - 1)$ such that

\[
|X_i| = \max\{|X_{i-1} R|, |X_{i-1} (R - 1)|\}.
\]
We are going to prove by induction on $k$ that

$$|X_k| \gg_k \frac{|A|^{3 - \frac{1}{2k}}}{\log^{\frac{3}{2}} |A|}.$$ 

Indeed, the base case $k = 0$ follows from (1.1). Now, let $k \geq 1$. Then applying inequality (2.2) in Lemma 2.4 Lemma 2.6 and the inductive hypothesis

$$|X_{k+1}| \gg |X_k|^{1/2}R^{3/4} \gg_k \left( \frac{|A|^{3 - \frac{1}{2k}}}{\log^{\frac{3}{2}} |A|} \right)^{1/2} \left( \frac{|A|^2}{\log |A|} \right)^{3/4} = \frac{|A|^{3 - \frac{1}{2k+1}}}{\log^{\frac{3}{2}} |A|}.$$

Now fix $\epsilon > 0$ and choose $k$ sufficiently large so that $\frac{1}{2k} < \epsilon$. It was already noted earlier, $R \subseteq D/D$ and $R - 1 \subseteq -D/D$, and so

$$|A|^{3-\epsilon} \leq \frac{|A|^{3 - \frac{1}{2k}}}{\log^{\frac{3}{2}} |A|} \ll_k |X_k| \leq \frac{|D^{(k+1)}|}{|D^{(k+1)}|}.$$

Applying Lemma 2.1 multiplicatively with $A = B = D^{(k+1)}$ and $C = 1/D^{(k+1)}$ we obtain that

$$|D^{(k+1)}||A|^{3-\epsilon} \ll \epsilon |D^{(2k+2)}|^2,$$

so $|D^{(2k+2)}| \gg \epsilon |A|^{3-\epsilon}$. Since $k$ depends on $\epsilon$ only, it completes the proof. \qed

3.1. Remarks, improvements and conjectures. An improvement to Lemma 2.4 was given in [7], in the form of the bound

$$|A(A + \alpha)| \gg \frac{|A|^{24/19}}{\log^{2/19} |A|}.$$

Inserting this into the previous argument, we obtain the following small improvement:

$$|DDD| \gg \frac{|A|^{2+\frac{7}{36}}}{\log^{\frac{85}{76}} |A|}.$$

Furthermore, a small modification of the previous arguments can also give the bound

$$|DD/D| \gg \frac{|A|^{2+\frac{7}{36}}}{\log^{\frac{85}{76}} |A|}.$$

In the spirit of Theorem 1.2 it is reasonable to conjecture the following.

**Conjecture 3.1.** For any $l > 0$ there exists $k > 0$ such that

$$|(A - A)^{(k)}| \gg_{k,l} |A|^l$$

uniformly for all sets $A \subset \mathbb{R}$.

Even the case $l = 3$ is of interest as it is seemingly beyond the limit of the methods of the present paper. An alternative form of Conjecture 3.1 is as follows.
Conjecture 3.2. For any $\epsilon > 0$ there exists $\delta > 0$ such that for any real set $X$ with

$$|XX| \leq |X|^{1+\delta}$$

the following holds: if $A \subset \mathbb{R}$ is such that

$$A - A \subset X,$$

then

$$|A| \ll_{\delta} |X|^\epsilon.$$

For comparison with Conjecture 3.1, we note that a similar sum-product estimate with many variables was proven in [2], in the form of the inequality

$$|4^{k-1}A^{(k)}| \gg |A|^k.$$

We also note that Corollary 4 in [19] verifies Conjecture 3.2 for any $\epsilon > 1/2 - c$, where $c > 0$ is some unspecified (but effectively computable) absolute constant.

It is not hard to see that Conjecture 3.2 is indeed equivalent to Conjecture 3.1. Assume that Conjecture 3.1 is true and fix $\epsilon > 0$. Next, take $l = \lceil 1/\epsilon \rceil + 3$. Assuming that Conjecture 3.1 holds, there is $k(\epsilon)$ such that

$$|(A - A)^{(k)}| \gg_{k, l} |A|^l$$

holds for real sets $A$.

Now, in order to deduce Conjecture 3.2, take $\delta = \epsilon/10k$ and assume that there are sets $X, A$ such that $|XX| \leq |X|^{1+\delta}$ and $A - A \subset X$. If we now also assume for contradiction that $|A| \geq |X|^{\epsilon}$, then by the Plünnecke-Ruzsa inequality (2.2)

$$|(A - A)^{(k)}| \leq |X^{(k)}| \leq |X|^{1+\delta k} \leq |A|^{\frac{1+\delta k}{\epsilon}} \leq |A|^l - 1,$$

which contradicts (3.5) if $|A|$ is large enough (depending on $\epsilon$), which we can safely assume.

Now let us assume that Conjecture 3.2 holds true. Let $l > 0$ be fixed and $\epsilon = \frac{1}{l+1}$. Let $A$ be an arbitrary real set. Consider the set $X_0 = (A - A)$ and define recursively

$$X_{i+1} = X_i X_i.$$

Note that by construction

$$X_i = (A - A)^{(2^i)}.$$

Let $c$ be an arbitrary non-zero element in $A - A$. Observe that

$$c^{2^i-1} \cdot A - c^{2^i-1} \cdot A = c^{2^i-1} \cdot (A - A) \subset (A - A)^{(2^i)} = X_i,$$

and so $A_i \subset X_i$ where $A_i := c^{2^i-1} \cdot A$. Thus, we are in position to apply the assumption that Conjecture 3.2 holds true. In particular, there is $\delta(\epsilon) > 0$ such that $|A| \ll_{\delta} |X|^{\epsilon}$ whenever $A - A \subset X$ and $|XX| \leq |X|^{1+\delta}$.

Now consider $X_i$ for $i = 1, \ldots, \lceil l/\delta \rceil + 1 := j$. For each $i$, if $|X_{i+1}| \leq |X_i|^{1+\delta}$ it follows from Conjecture 3.2 that $|A| = |A_i| \ll_{\delta} |X_i|^{\epsilon}$, so

$$|(A - A)^{(2^i)}| = |X_i| \gg_{\delta} |A|^{1/\epsilon} \geq |A|^l.$$
and we are done. Otherwise, if for each $1 \leq i \leq j$ holds $|X_{i+1}| \geq |X_i|^{1+\delta}$, one has

$|(A - A)^{(2j)}| = |X_j| \geq |X_0|^{1+j\delta} \geq |A|^l$.

Thus, Conjecture 3.1 holds uniformly in $A$ with

$k(l) := 2^j = 2^{|l/\delta(l)|+1}$.

For a further support, let us remark that Conjecture 3.2 holds true if one replaces the condition $|XX| \leq |X|^{1+\delta}$ with the more restrictive one $|XX| \leq K|X|$ where $K > 0$ is an arbitrary but fixed absolute constant. In this setting Conjecture 3.2 can be proved by combining the Freiman Theorem and the Subspace Theorem and then applying almost verbatim the arguments of [17]. We leave the details to the interested reader.

4. Proofs of Theorems 1.3 and 1.4

4.1. Proof of Theorem 1.3. We will first prove the following lemma.

**Lemma 4.1.** Let $A \subset \mathbb{R}$. Then

$$|A + A + A| \gg \frac{|A|^{54/32}|A/A|^{13/32}}{\log^{3/4} |A|}.$$}

**Proof.** Apply Lemma 2.5 with $f(x) = 1/x$, $X = (A + A)/(A + A)$ and $Y = Z = A/A$. Note that $f(X) = X$ and so

$$|A + A + A| \gg \frac{|A + A|^{3/4}}{A + A} |A/A|^{1/2}.$$}

Then applying Lemma 2.7 it follows that

$$|A + A + A| \gg \frac{|A|^{3/2}}{\log^{3/4} |A|} \left( \frac{|A|^2}{|A/A|} \right)^{3/2} \frac{|A/A|^{1/2}}{\log^{3/4} |A|} = \frac{|A|^{54/32}|A/A|^{13/32}}{\log^{3/4} |A|}.$$}

□

This immediately implies that

$$|A + A + A| \gg |A|^{2+\frac{1}{17} - \epsilon}.$$}

However, by optimising between Lemma 4.1 and Lemma 2.7 we can get a slight improvement in the form of Theorem 1.3.

**Proof of Theorem 1.3** Let $|A/A| = K|A|$. If $K \geq \frac{|A|^{1+\epsilon}}{\log^{17/17} |A|}$ then Lemma 4.1 implies that

$$|A + A + A| \gg \frac{|A|^{67/32}K^{13/32}}{\log^{3/4} |A|} \gg \frac{|A|^{2+2/17}}{\log^{16/17} |A|}.$$}
On the other hand, if \( K \leq \frac{|A|^{1/8}}{\log^{1/17} |A|} \) then Lemma 2.7 implies that
\[
\left| \frac{A + A}{A + A} + \frac{A}{A} \right| \geq \left| \frac{A + A}{A} \right| \gg \frac{|A|^2}{\log |A|} \left( \frac{|A|}{K} \right)^{1/8} \gg \frac{|A|^{2+2/17}}{\log^{16/17} |A|}.
\]

4.2. **Proof of Theorem 1.4.** Apply Lemma 2.8 with \( B = AA \). This yields
\[
\left| \frac{AA + AA}{A + A} \right| \gg \left| AA \right| \left( \frac{|AA|}{|A|} \right)^{1/8}.
\]
By applying Lemma 2.2 in the multiplicative setting, we have
\[
|AA/A| \leq \left| AA \right|^{3/2},
\]
and so
\[
\left| \frac{AA + AA}{A + A} \right| \gg \left| AA \right| \left( \frac{|AA|}{|A/A|} \right)^{1/8} \geq \left| AA \right| \left( \frac{|AA|^3}{|AA|^2} \right)^{1/8} = \left| AA \right|^{11/8} \left| AA \right|^{3/4} \log |A|^{-1/8}.
\]
as required.

5. **Proof of Theorem 1.5**

Consider the point set \( A \times A \) in the plane. Without loss of generality, we may assume that \( A \) consists of strictly positive reals, and so this point set lies exclusively in the positive quadrant. We also assume that \( |A| \geq C \) for some sufficiently large absolute constant \( C \). For smaller sets, the theorem holds by adjusting the implied multiplicative constant accordingly.

For \( \lambda \in A/A \), let \( A_\lambda \) denote the set of points from \( A \times A \) on the line through the origin with slope \( \lambda \) and let \( A_\lambda \) denote the projection of this set onto the horizontal axis. That is,
\[
A_\lambda := \{ (x, y) \in A \times A : y = \lambda x \}, \quad A_\lambda := \{ x : (x, y) \in A_\lambda \}.
\]
Note that \( |A_\lambda| = |A_\lambda| \) and
\[
\sum \lambda |A_\lambda| = |A|^2.
\]

We begin by dyadically decomposing this sum and applying the pigeonhole principle in order to find a large subset of \( A \times A \) consisting of points which lie on lines of similar richness. Note that
\[
\sum_{|A_\lambda| \leq \frac{|A|^2}{2|A/A|}} |A_\lambda| \leq \frac{|A|^2}{2},
\]
and so
\[
\sum_{|A_\lambda| \geq \frac{|A|^2}{2|A/A|}} |A_\lambda| \geq \frac{|A|^2}{2}.
\]
Therefore, there exists some \( \tau \geq \frac{|A|^2}{2|A|} \) such that

\[
|S_\tau| \gg \sum_{\lambda \in S_\tau} |A_\lambda| \gg \frac{|A|^2}{\log |A|},
\]

where \( S_\tau := \{ \lambda : \tau \leq |A_\lambda| < 2\tau \} \). Using the trivial bound \( \tau \leq |A| \), it also follows that

\[
|S_\tau| \gg \frac{|A|}{\log |A|}.
\]

For a point \( p = (x, y) \) in the plane with \( x \neq 0 \), let \( r(p) := y/x \) denote the slope of the line through the origin and \( p \). For a point set \( P \subseteq \mathbb{R}^2 \) let \( r(P) := \{ r(p) : p \in P \} \). The aim is to prove that

\[
|r((AA + A) \times (AA + A))| = |r((A \times A) + (AA \times AA))| \gg \frac{|A|^{2+\frac{8}{\log |A|}}}{|A|^{2+\frac{8}{\log |A|}}}
\]

\[
|S_\tau| \geq \frac{|A|^{9/8}}{\log |A|}
\]

for any absolute constant \( c > 0 \) then we are done. Therefore, we may assume for the remainder of the proof that this is not the case. In particular, by \( (5.1) \), we may assume that

\[
\tau \geq C|A|^{7/8}
\]

holds for any absolute constant \( C \).
Next, the basic lower bound (5.4) will be enhanced by looking at larger clusters of lines, a technique introduced by Konyagin and Shkredov [9] and utilised again by Lund [10]. We will largely adopt the notation from [10].

Let $2 \leq M \leq \frac{|S|}{2}$ be an integer parameter, to be determined later. We partition $S$ into clusters of size $2M$, with each cluster split into two subclusters of size $M$, as follows. For each $1 \leq t \leq \left\lceil \frac{|S|}{2M} \right\rceil$, let

$$f_t = 2M(t - 1)$$

$$T_t = \{\lambda f_{t+1}, \lambda f_{t+2}, \ldots, \lambda f_{t+M}\}$$

$$U_t = \{\lambda f_{t+M+1}, \lambda f_{t+M+2}, \ldots, \lambda f_{t+2M}\}.$$

For the remainder of the proof we consider the first cluster with $t = 1$, but the same arguments work for any $1 \leq t \leq \left\lceil \frac{|S|}{2M} \right\rceil$. We simplify the notation by writing $T_1 = T$ and $U_1 = U$

Let $1 \leq i, k \leq M$ and $M + 1 \leq j, l \leq 2M$ with at least one of $i \neq k$ or $j \neq l$ holding. For $a_i \in A_{\lambda i}$ and $a_k \in A_{\lambda k}$. Define

$$\mathcal{E}(a_i, j, a_k, l) = |\{(x, y) \in A \times A : r((a_i, \lambda_i a_i) + (\alpha_j x, \lambda_j \alpha_j x)) = r((a_k, \lambda_k a_k) + (\alpha_l y, \lambda_l \alpha_l y))\}|.$$

**Lemma 5.1.** Let $i, j, k, l$ satisfy the above conditions and let $K \geq 2$. Then there are $O(|A|^4/K^3 + |A|^2/K)$ pairs $(a_i, a_k) \in A_{\lambda_i} \times A_{\lambda_k}$ such that

$$\mathcal{E}(a_i, j, a_k, l) \geq K.$$

**Proof.** We essentially copy the proof of Lemma 2 in [10], and so some details are omitted. Let $l_{a,b}$ be the curve with equation

$$(\lambda_i a + \lambda_j \alpha_j x)(b + \alpha_l y) = (\lambda_k b + \lambda_l \alpha_l y)(a + \alpha_j x).$$

Let $\mathcal{L}$ be the set of curves

$$\mathcal{L} = \{l_{a,b} : a \in A_{\lambda_i}, b \in A_{\lambda_k}\}$$

and let $\mathcal{P} = A \times A$. Note that $(x, y) \in l_{a_i,a_k}$ if and only if

$$r((a_i, \lambda_i a_i) + (\alpha_j x, \lambda_j \alpha_j x)) = r((a_k, \lambda_k a_k) + (\alpha_l y, \lambda_l \alpha_l y)).$$

Hence $\mathcal{E}(a_i, j, a_k, l) \geq K$ if and only if $|l_{a_i,a_k} \cap \mathcal{P}| \geq K$.

We can verify that the set of curves $\mathcal{L}$ satisfies the conditions of Lemma 2. One can copy this verbatim from the corresponding part of of the proof of Lemma 2 in [10]. Therefore, there are most

$$O\left(\frac{|\mathcal{P}|^2}{K^3} + \frac{|\mathcal{P}|}{K}\right) = O\left(\frac{|A|^4}{K^3} + \frac{|A|^2}{K}\right)$$

curves $l \in \mathcal{L}$ such that $|l \cap \mathcal{P}| \geq K$. The lemma follows. □
Now, for each \((i, j)\) such that \(1 \leq i \leq M\) and \(M + 1 \leq j \leq 2M\) choose an element \(a_{ij} \in A_{\lambda_i}\) uniformly at random. Then, for any \(1 \leq i, k \leq M\) and \(M + 1 \leq j, l \leq 2M\), define \(X(i, j, k, l)\) to be the event that
\[
\mathcal{E}(a_{ij}, j, a_{kl}, l) \geq B,
\]
where \(B\) is a parameter to be specified later. By Lemma 5.1, the probability that the event \(X(i, j, k, l)\) occurs is at most
\[
\frac{C}{\bar{	au}^2} \left( \frac{|A|^4}{B^3} + \frac{|A|^2}{B} \right),
\]
where \(C > 0\) is an absolute constant.

Furthermore, note that the event \(X(i, j, k, l)\) is independent of the event \(X(i', j', k', l')\) unless \((i, j) = (i', j')\) or \((k, l) = (k', l')\). Therefore, the event \(X(i, j, k, l)\) is independent of all but at most \(2M^2\) of the other events \(X(i', j', k', l')\). With this information, we can apply Lemma 2.10 with
\[
n = M^4 - M^2, \quad d = 2M^2, \quad p = \frac{C}{\bar{	au}^2} \left( \frac{|A|^4}{B^3} + \frac{|A|^2}{B} \right).
\]
It follows that there is a positive probability that none of the the events \(X(i, j, k, l)\) occur, provided that
\[
(5.6) \quad \frac{eC}{\bar{	au}^2} \left( \frac{|A|^4}{B^3} + \frac{|A|^2}{B} \right) (2M^2 + 1) \leq 1.
\]
The validity of (5.6) is dependent on our subsequent choice of the value of \(B\). For now we proceed under the assumption that this condition is satisfied.

Let
\[
Q = \bigcup_{1 \leq i \leq M, M+1 \leq j \leq 2M} \{(a_{ij}, \lambda_ia_{ij}) + (\alpha_ja, \lambda_j\alpha_ja) : a \in A\}.
\]
Crucially,
\[
(5.7) \quad r(Q) \geq M^2|A| - \sum_{1 \leq i, k \leq M, M+1 \leq j, l \leq 2M: \{i, j\} \neq \{k, l\}} \mathcal{E}(a_{ij}, j, a_{kl}, l).
\]
In (5.7), the first term is obtained by counting the \(|A|\) slopes in \(Q\) coming from all pairs of lines in \(U \times T\). The second error term covers the overcounting of slopes that are counted more than once in the first term.

Since \(\mathcal{E}(a_{ij}, j, a_{kl}, k) \leq B\) for all quadruples \((i, j, k, l)\) satisfying the aforementioned conditions, it follows that
\[
(5.8) \quad r(Q) \geq M^2|A| - M^4B.
\]
Choosing \(B = \frac{|A|}{2M^2}\), it follows that
\[
(5.9) \quad r(Q) \geq \frac{M^2|A|}{2}.
\]
This choice of $B$ is valid as long as
\begin{equation}
\frac{eC}{\tau^2} (8M^6|A| + 2M^2|A|)(2M^2 + 1) \leq 1.
\end{equation}
This will certainly hold if
\begin{equation}
\frac{30eC}{\tau^2} M^8|A| \leq 1
\end{equation}
and so we choose
\[ M = \left\lfloor \left( \frac{\tau^2}{30eC|A|} \right)^{1/8} \right\rfloor. \]
In particular, by (5.5) we have $M \geq 2$ and so
\begin{equation}
M \gg \frac{\tau^{1/4}}{|A|^{1/8}}.
\end{equation}
It is also true that $M \leq \frac{|S_\tau|}{2}$. This is true for all sufficiently large $A$ since
\[ |S_\tau| \geq \frac{c|A|}{\log |A|} \geq |A|^{1/8} \geq 2M. \]
Therefore
\begin{equation}
\left\lfloor \frac{|S_\tau|}{2M} \right\rfloor \gg \frac{|S_\tau|}{M}.
\end{equation}
Next, note that $r(Q)$ is a subset of the interval $(\lambda_1, \lambda_2M)$. We can repeat this argument for the next cluster to find at least $M^2|A|/2$ elements of $r((AA + A) \times (AA + A))$ in the interval $(\lambda_{2M+1}, \lambda_{4M})$ and then so on for each of the $\left\lfloor \frac{|S_\tau|}{2M} \right\rfloor$ clusters of size $2M$. It then follows from (5.12) and (5.11) that
\[ \left| \frac{AA + A}{AA + A} \right| = |r((AA + A) \times (AA + A))| \geq \sum_{j=1}^{\left\lfloor \frac{|S_\tau|}{2M} \right\rfloor} \frac{M^2|A|}{2} \gg |S_\tau|M|A| \gg (|S_\tau|\tau)^{1/4}|A|^{7/8}|S_\tau|^{3/4}. \]
Applying (5.1) and (5.2), we conclude that
\[ \left| \frac{AA + A}{AA + A} \right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log |A|} \]
as required.
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