Confining gauge fields *

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Abstract

By superposition of regular gauge instantons or merons, ensembles of gauge fields are constructed which describe the confining phase of SU(2) Yang-Mills theory. Various properties of the Wilson loops, the gluon condensate and the topological susceptibility are found to be in qualitative agreement with phenomenology or results of lattice calculations. Limitations in the application to the glueball spectrum and small size Wilson loops are discussed.

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Introduction

Despite convincing evidence for confinement in Yang-Mills theories from lattice gauge calculations and despite a host of analytical studies and conjectures, the intricacies of non-abelian gauge fields have prevented an unambiguous identification of the mechanism of confinement. In particular, the gauge dependence of many of the mechanisms proposed constitutes a severe obstacle in reaching this goal. I will choose a different path towards understanding confinement. I will describe construction and analysis of ensembles of gauge fields which exhibit confinement and account for other facets of the dynamics of SU(2) Yang Mills theory. The studies to be presented are based on the ideas that instantons [1] or merons [2], solutions of the classical Yang-Mills equations, play a prominent role also in the quantum theory [3, 4]. They are related to investigations which led to the instanton gas and liquid models [5, 6] which successfully describe a variety of phenomena of strong interaction physics [7, 8]. These models however miss confinement, the trademark of the dynamics of Yang-Mills fields. I will discuss an alternative construction of ensembles of gauge-fields obtained by superposition of “regular gauge” instantons or of merons [9]. Although irrelevant for single instanton properties, the choice of either the regular or the singular gauge in the superposition of instantons leads to two different phases, the gas or liquid phase of singular gauge instantons with a well defined low density limit and the strongly correlated confining phase of regular gauge instantons (or merons).

Gauge fields of instantons and merons

After appropriate choice of the coordinate system in color space and after regularization of the singularity, the SU(2) gauge field for a meron or an (regular gauge) instanton in Lorenz gauge with the center at the origin, is given by

\[ a_\mu(x) = \xi \frac{\eta_{\mu\nu} x_\nu \sigma^a}{x^2 + \rho^2/2}, \]

(1)

with the Pauli matrices \( \sigma^a \), the ’t Hooft tensor \( \eta_{\mu\nu} \), and with \( \xi = 1 \) denoting merons and \( \xi = 2 \) instantons. The size parameter \( \rho \) controls the short distance behavior of instanton and meron fields. For instantons, and for merons with vanishing \( \rho \), \( a_\mu(x) \) is a solution of the Euclidean classical field equations [1, 2].

Important for the following are the infrared properties of the constituents. The gauge fields of a single meron and of a single regular gauge instanton behave asymptotically as \( 1/x \) while the asymptotics of the field strengths,

\[ F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu, \]

is different for merons and instantons

\[ x \to \infty, \quad F_{\xi=1} \to x^{-2}, \quad F_{\xi=2} \to x^{-4}, \]

(2)

due to the cancellation of the abelian and non-abelian contributions to leading order for instantons. Therefore the action density

\[ s(x) = \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} \]

(3)
behaves asymptotically as $x^{-4}$ and $x^{-8}$ for merons and instantons respectively and gives rise to an infinite action for merons. The topological charge density is defined by

$$\tilde{s}(x) = \frac{1}{2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu},$$

and yields a finite value for the topological charge

$$\nu = \xi/2.$$  \hspace{1cm} (4)

The fast asymptotic decrease of the field strength of an instanton is the basis for the alternate representation of instantons in “singular gauge”

$$a^\text{sing}_\mu = \frac{2\rho^2}{\rho^2 x^2 + \rho^2} \tilde{\eta}_\mu x_\nu \sigma^a,$$  \hspace{1cm} (5)

where the gauge field decays asymptotically as $x^{-3}$ and $\tilde{\eta}$ denotes the anti-self-dual 't Hooft tensor.

**Gauge fields from superposition of merons or instantons**

The SU(2) gauge fields to be considered are superpositions of instanton or meron fields

$$A_\mu(x) = \sum_{i=1}^{N_p} h(i)a_\mu(x - z(i))h^{-1}(i).$$  \hspace{1cm} (6)

The dynamical variables of these fields are the positions of the centers $z(i)$ of the instantons or merons and their color orientations

$$h(i) = h_0(i) + ih(i) \cdot \sigma, \quad h_0^2(i) + h^2(i) = 1.$$  \hspace{1cm} (7)

In general, the action of gauge fields generated by superposition of either merons or instantons is logarithmically divergent in the infrared. By a judicious choice of the color orientation, finite values of the total action can be obtained. This is achieved, e.g., by the following superposition of 4 instanton or merons

$$A(x) = \sum_{i=1}^{3} R_i^\sigma a(x - z(i)) + a(x - z(4)),$$

with the color rotations $R_i^\sigma$ by $\pi$ around the three color directions $i$. With this choice

$$x \to \infty: \quad A \sim \frac{1}{x^2}, \quad F[a] \sim \frac{1}{x^3}, \quad s(x) \sim \frac{1}{x^6},$$

the system is “neutral” and the action is finite. The constituents, instantons or merons, are “confined”. Removal of a single constituent gives rise to an infinite action. Thus ensembles of instantons and merons exhibit similar infrared properties and in both cases strong correlations between the constituents are required to guarantee a finite action.
The ensembles of gauge fields (cf. Eq.(7)) to be constructed in the following are defined by the partition function

\[ Z = \int dz_i dh_i e^{-\frac{1}{g^2} S[A(z_i, h_i)]}, \]  

with the effective action \( S \) identified with the standard (Euclidean) action (cf. Eq. (3))

\[ S = \int_V d^4x s(x). \]

This definition of the partition function guarantees that for finite \( g^2 \) a non-zero weight is assigned to neutral configurations only, i.e. to fields with the asymptotic behavior (9). After construction of ensembles of field configurations using the Metropolis algorithm, vacuum expectation values of observables \( O \) can be computed

\[ \langle O \rangle = \frac{1}{Z} \int dz_i dh_i e^{-\frac{1}{g^2} S[A(z_i, h_i)]} O[A(z_i, h_i)]. \]

In the numerical determination of these ensembles, the location \( z(i) \) is restricted to a hypercube

\[-1 \leq z_\mu(i) \leq 1, \]

an equal number of instanton and anti-instantons (merons and antimerons) is used, and in most of the applications the values of size and coupling constant are chosen as

\[ \rho = 0.16, \quad g^2 = 32. \]  

Characteristics of instanton and meron ensembles

Action densities

In this section I will characterize qualitative properties of gauge fields in the ensembles defined by (10). In Fig.1 is shown a typical landscape of the action density displaying peaks of single merons on top of a background. Background and single meron contributions to the total action are of the same order of magnitude. Unlike in the superposition of singular instantons, the appearance of single meron or instanton structures is not automatic, it rather reflects the dynamics defined by the partition function (10). This becomes evident in the comparison (Fig.1) with the action density of a meron field of the corresponding stochastic ensemble \((g^2 = \infty)\). The values of the action density of the two ensembles

\[ g^2 = 32, \quad \langle s \rangle = 1500, \quad 200 < s(x) < 11000, \]
\[ g^2 = \infty, \quad \langle s \rangle = 110000, \quad 15000 < s(x) < 330000, \]

differ by almost two orders of magnitude. As a result of the huge background generated by the stochastic superposition of the constituents no single meron contribution \( (s_{max} = 180000) \) can be identified. The maximum close to the center of Fig.1 is a result of the stochastic superposition as can be verified in an analytical calculation. The reduction of the action from the strong coupling value is a first quantitative signature of the importance of correlations in the \((g^2 \neq \infty)\) ensembles. Another measure of the strength of the correlations
FIG. 1: Action density along a plane for a gauge field of 500 merons (left) and lines of equal action density ($g^2 = 32$, middle) and ($g^2 = \infty$ right).

FIG. 2: Response of the action of 50 instantons to changes of the color orientation along three independent geodesics on the manifold $S^3$ of the color rotations [8] (left) of the instanton closest to the center and to changes of its position (right).

between constituents is the response of the action to changes of the color orientation or of the position of a single meron or instanton [10]. As shown in Fig. 2 under variations of the color orientation of a single constituent the total action is increased by up to 60% and by up to 20% under variations of the position. In a weakly interacting system of 50 instantons (instanton gas) one would expect a 2% decrease of the action when removing a single instanton and even smaller changes under variations of the color orientation.
Confinement

In this section I will present evidence that the phase of highly correlated constituents is confining. In Yang Mills theories, the area law of Wilson loops is the signature of confinement. The Wilson loop is defined by

$$W = \frac{1}{2} \text{tr} \{ P \exp i g \oint_C A_\mu(x) dx^\mu \},$$  \hspace{1cm} (16)

with the integral ordered along the closed path C. The following results refer to rectangular paths with aspect ratio 2:1. For sufficiently large loops the results for the logarithm of the expectation value of the Wilson loop can be parametrized as a sum of constant, perimeter ($\mathcal{P}$) and area ($\mathcal{A}$) terms,

$$\ln \langle W \rangle = \omega + \tau \mathcal{P} - \sigma \mathcal{A},$$  \hspace{1cm} (17)

with $\sigma$ denoting the string tension. The parameters of the fit for ensembles of 500 merons ($\rho = 0.16$) are

$$\omega = -2.1, \quad \tau = 2.3, \quad \sigma = 22.9$$  \hspace{1cm} (18)

for the stochastic ensemble ($g^2 = \infty$). For ensembles with $g^2 = 32$ the following values

$$\omega = -0.7, \quad \tau = 1.1, \quad \sigma = 11.8$$  \hspace{1cm} (19)

are obtained if the color orientations are dynamical variables while the positions are randomly chosen but fixed, and

$$\omega = -0.7, \quad \tau = 1.1, \quad \sigma = 11.5$$  \hspace{1cm} (20)

for the ensemble with dynamical color orientations and dynamical positions. These three ensembles give rise to an area law with values of the string tension differing by a factor of 2 between stochastic and dynamically correlated ensembles. The large difference in the dimensionless ratios formed by action density and string tension,

$$\frac{g^2\langle s \rangle}{\sigma^2} \approx 500, \quad (g^2 = \infty), \quad \frac{g^2\langle s \rangle}{\sigma^2} \approx 11 \quad (g^2 = 32),$$  \hspace{1cm} (21)

reflects the different dynamics of the ensembles. As in lattice gauge theories, confinement in the strong coupling limit is a result of disorder of the system generated by the unconstrained fluctuations of the gauge fields. Suppression of the fluctuations if $g^2 \neq \infty$ affects more strongly the local action density than the non local-Wilson loop and in turn leads to the strong decrease in the ratio $g^2\langle s \rangle/\sigma$. Only minor changes of Wilson loops are obtained if, in addition to the color orientations, also the meron positions are treated as dynamical variables. This result is in accordance with the above findings concerning the response to changes in the color orientation and position (Fig.2) of single constituents. It suggests that the confining phase is a close relative of the nematic phase of liquid crystals with strong correlations in internal and comparatively weak correlations in position space [11].

Quantitative results

Scaling and confinement

Quantitative predictions for observables evaluated in the ensembles defined above require definition of a scale. As in lattice gauge calculations the phenomenological value, $\sigma =$
FIG. 3: Logarithm of a Wilson loop as a function of its area for the three meron ensembles of Eqs. (18) (green ×), (19) (pink −), and (20) (blue +).

4.4 fm$^{-2}$, of the string tension is chosen to set the scale. Rescaling the different contributions to the Wilson loop (17),

\[
\ln\langle W \rangle = \omega + \tau \sqrt{\lambda} \mathcal{P} - \sigma \lambda \mathcal{A},
\]

and fitting the scaling parameter $\lambda$ separately for each ensemble gives rise to a universal curve as shown in Fig. 4. The values of the universal parameters and of the unit of length are for the instanton ensembles

\[
\omega = -0.52, \quad \tau = 1.18, \quad \sigma = 19.0, \quad 1 \text{ u.l.} = 2.08 \sqrt{\lambda} \text{fm}.
\]

Similar results have been obtained for the meron ensembles.

As the Wilson loops, also other observables exhibit approximate independence of the number of degrees of freedom. Table I summarizes for ensembles with $N_I$ instantons the results for the vacuum expectation value of the action density $s$ and the topological susceptibility (also in physical units),

\[
\chi = \left( \frac{1}{8\pi^2} \right)^2 \int d^4x \langle \tilde{s}(x) \tilde{s}(0) \rangle,
\]

which is a measure of the strength of the fluctuations of the topological charge [5]. In comparison to the values obtained from sum rules [12, 13] and from lattice gauge calculations [14]

\[
85 \text{ fm}^{-4} \leq \langle s \rangle \leq 260 \text{ fm}^{-4},
\]

the expectation value of the action density i.e. the gluon condensate is of the correct order of magnitude. It receives about equal contributions from the background field generated by the superposition of the constituents and from the peaks of the fields of single constituents as suggested by Fig. 1. Also the value of the topological susceptibility is in reasonable agreement with the lattice result $\chi^{1/4} \sim 215 \text{ MeV}$ [14]. The topological susceptibility is
FIG. 4: Logarithm of a Wilson loop as a function of its area for instanton ensembles $N_I = 50$ (star), 100 (diamond), 200 (square), 500 (circle). The area has been rescaled with $\lambda$ given in Table I. Also shown is the curve corresponding to the parametrization (22) with the values of the parameters given in (23).

| $N_I$ | $\langle s \rangle$ | $\lambda$ | $n_I$ | $\rho$ | $\langle s \rangle$ | $\chi^{1/4}$ |
|-------|------------------|--------|------|------|------------------|-------------|
| 500   | 5430             | 1.0    | 0.33 | 2490 | 0.66             | 1.54        |
| 200   | 2490             | 0.66   | 0.27 | 200  | 0.48             | 1.35        |
| 100   | 1350             | 0.48   | 0.23 | 100  | 0.32             | 0.83        |
| 50    | 651              | 0.32   | 0.19 | 50   | 0.32             | 0.52        |

TABLE I: Properties of instanton ensembles: Vacuum expectation value of the action density $s$ defined in (3) and the topological susceptibility $\chi$ (24) (also in physical units), with the standard choice of the parameters (14) for ensembles of $N_I$ instantons and with instanton density $n_I = N_I/L^4$ and the values of the scaling parameter $\lambda$ (cf. (22)) and the size parameter $\rho$ in physical units.

dominated by the topological charge density of single constituents. The single constituent contributions in the evaluation of the integral (24) yield the result,

$$\chi^{1/4}_I = 0.83 n^{1/4}_I, \quad \chi^{1/4}_M = 0.52 n^{1/4}_M,$$

which agrees within 10% with the result of the numerical evaluation (Table I). Due to the dominance of single instanton properties similar values of this observable in the liquid phase [7] of singular gauge and in the confining phase of regular gauge instantons are obtained. Table I shows that ensembles with increasing numbers of constituents ($N_I$) and located in a hypercube of fixed volume (13) describe, after rescaling, systems of essentially the same
density $n_I$ of instantons or merons. The increase in number is accounted for by the increase in volume of the hypercube. The value of the string tension determines the density of constituents. Variations in the constituent size in the interval $0.06 \leq \rho \leq 0.25$ have been considered and found to leave the scaling properties essentially unperturbed. Origin of the scaling properties is the scale independence of the asymptotics of regular gauge instantons or merons (Eq. 1). In ensembles of singular gauge instantons on the other hand, $\rho$ controls the strength of the asymptotics of these fields (Eq. 6) and therefore the overlap and in turn the strength of the interaction of these constituents [16].

### Wilson loops and glueball masses

In addition to the area law, also more detailed properties of Wilson loops have been found to agree with those of lattice gauge calculations. I mention the Wilson loop distributions shown in Fig. 5 which, as similar lattice results [17], can be interpreted as distributions of a diffusion process on $S^3$, the group manifold of SU(2), given by

$$p(\cos \vartheta, t) = \frac{2}{\pi} \theta(t) \sum_{n=1}^{\infty} n \sin n \vartheta e^{-(n^2-1)t}, \quad (26)$$

with the diffusion time $t$ determined by the expectation value of the Wilson loop. Diffusion on $S^3$ entails “Casimir scaling” of the Wilson loop expectation values in higher representations which has been observed in lattice gauge calculations [18]. Also in agreement with lattice gauge calculations is the breakdown of Casimir scaling observed for sufficiently large loops for ensembles of gauge fields similar to the ones discussed here [19].

In a study of correlation functions of circular loops of equal radius $r$ which are parallel to each other and orthogonal to the direction of separation ($t$)

$$C_r(t) = \langle W_r(n, x_0, t)W_r(n, x_0, 0) \rangle, \quad (27)$$
indications for formation of flux tubes in connection with the area law have been found. If

gauge strings are important degrees of freedom the application of the Wilson loop operator
to the vacuum should generate such a gauge string with energy \( E \approx 2\pi \sigma r \), and the relevant
variable for describing the correlation function should be \( rt \). Indeed the three correlation
functions are described approximatively by a universal curve in terms of this variable as
shown in Fig. 6. A rough estimate of the string tension based on this calculation yields
\( \sigma = (3.7 \pm 1.3) \text{ fm}^{-2} \).

As the last topic I discuss the glueball spectrum calculated via (Euclidean) correlation
functions

\[
\langle \mathcal{O}(x)\mathcal{O}(0) \rangle \sim \sum_n \langle \Omega | \mathcal{O} | n \rangle e^{-E_n x} \langle n | \mathcal{O} | \Omega \rangle .
\]

of appropriately chosen operators carrying the quantum numbers of the glueballs. For
sufficiently large separations, the correlation function is dominated by the state of lowest
energy which can be excited by the operator \( \mathcal{O} \). For determination of the masses, local and
non-local operators have been used and various ensembles differing in the size parameter and
the number of constituents have been studied. The final result is shown in Fig. 7. Qualitative
agreement with respect to the ordering of the level has been reached. The scale however
is not compatible with the scale set by the string tension. The values of the masses have
been multiplied with a factor of 1.8 (1.5 for meron ensembles) to reproduce the \( 0^+ \) glueball
mass obtained in lattice gauge calculations. Determined by the requirement of the infrared
finiteness of the action, the degrees of freedom do not adequately account for the dynamics
on the scale of the size of the glueballs (0.2 - 0.5 fm)\cite{21}. Missing strength in the ultraviolet
also shows up in the Wilson loops of small size as is shown in the right part of Fig. 7.
While the stochastic ensemble with its unconstrained fluctuations generates a Coulomb-like
behavior at small distances, the ensembles of dynamical merons or instantons fail in the
description of Wilson loops in the perturbative regime. A remedy for the shortcoming in the
calculation of the glueball spectrum could be the inclusion of perturbative contributions in
FIG. 7: Left: Spectra of glueball states. Lattice gauge theory (left part) Instanton ensembles after rescaling (right part). Right: Enlarged portion of Fig. 3 for small size Wilson loops. Upper curve obtained for the ensemble with dynamical meron positions and color orientations, lower curve for the stochastic ensemble ($g^2 = \infty$).

addition to the non-perturbative correlation functions, as applied for ensembles of singular gauge instantons [22].

Conclusion

The central achievement of these studies is the construction of ensembles of gauge fields which exhibit confinement. These fields are obtained by superposition of regular gauge instantons or merons. It is remarkable that superposition of the gauge equivalent regular and singular instantons gives rise to two distinctively different phases, the gas or liquid phase generated by singular and the confining phase by regular gauge instantons respectively. The confining phase owes its existence to the non-trivial requirement of finite action in the presence of the slow asymptotic $1/x$ decay of the regular gauge instanton or meron fields. Unlike the disordered strong coupling limit, the confining phase is an ordered phase similar to the nematic phase of liquid crystals. The infrared behavior of the constituents is not only the source of confinement it’s scale independence also induces the scaling properties of observables under changes in the number of constituents. Besides various properties of Wilson loops, such as the area law, Casimir scaling or flux tube formation, also the action density (gluon condensate) and the topological susceptibility are reasonably well reproduced. These latter quantities are determined by the relative strength of the gauge fields close to the center of the constituents and of the background gauge field generated by the superposition of the constituents. If observables such as the topological susceptibility are dominated by single instanton properties similar results in the liquid and in the confining phase can be expected. Limits in the applicability of meron and instanton ensembles are encountered if
short distance properties are important as is the case for the glueball spectrum and for small size Wilson loops.

[1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Yu. S. Tyupkin, *Phys. Lett.* B 59, 85 (1975).
[2] V. de Alfaro, S. Fubini and G. Furlan, *Phys. Lett.* B 65, 163 (1976).
[3] C. G. Callan, R. F. Dashen and D. J. Gross, *Phys. Rev.* D 17, 2717 (1978).
[4] C. G. Callan, R. F. Dashen and D. J. Gross, *Phys. Lett.* B 66, 375 (1977).
[5] E. V. Shuryak, *Nucl. Phys.* B 203, 93 (1982).
[6] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* B 245, 259 (1984).
[7] T. Schafer and E. V. Shuryak, *Rev. Mod. Phys.* 70, 323 (1998), [arXiv:hep-ph/9610451].
[8] D. Diakonov, *Prog. Part. Nucl. Phys.* 51, 173 (2003), [arXiv:hep-ph/0212026].
[9] F. Lenz, J. W. Negele, and M. Thies, *Phys. Rev.* D 69, 074009 (2004), [arXiv:hep-th/0306105]; *Ann. Phys.* 323, 1536 (2008), [arXiv:hep-ph/0708.1687].
[10] F. Lenz and C. Szasz, to be published.
[11] F. Lenz, “Topological Concepts in Gauge Theory”, in “Topology and Geometry in Physics”, Lecture Notes in Physics, Springer Berlin (2005), [arXiv:hep-th/0403286].
[12] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, *Nucl. Phys.* B 147, 385, 448 (1979).
[13] S. Narison, *Phys. Lett.* B 387, 162 (1996), [arXiv:hep-ph/9512348].
[14] M. Camostrini, A. Di Giacomo and G. Paffuti, *Z. Phys.* C 22, 143 (1984).
[15] B. Lucini and M. Teper, *JHEP* 0106, 050 (2001), [arXiv:hep-lat/0103027].
[16] D. I. Diakonov, V. Y. Petrov and P. V. Pobylitsa, *Phys. Lett.* B 206, 372 (1989).
[17] A. M. Brzoska, F. Lenz, J. W. Negele, and M. Thies, *Phys. Rev.* D 71, 034008 (2005), [arXiv:hep-th/0412003].
[18] G. S. Bali, *Phys. Rev.* D 62, 114503 (2000), [arXiv:hep-lat/0006022].
[19] Ch. Szasz and M. Wagner, *Phys. Rev.* D 78, 036006 (2008), [arXiv:hep-ph/0806.1977].
[20] M. J. Teper, [arXiv:hep-ph/9812187].
[21] P. de Forcrand and K.-F. Liu, *Phys. Rev. Lett.* 69, 245 (1992), [arXiv:hep-lat/9211054].
[22] T. Schäfer and E. V. Shuryak, *Phys. Rev. Lett.* 75, 1707 (1995), [arXiv:hep-lat/9410372].