On the signed graphs with two distinct eigenvalues

F. Ramezani $^1$

Department of Mathematics,
K.N.Toosi University of Technology, Tehran, Iran
P.O. Box 16315-1618

School of Mathematics,
Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

Abstract. We consider signed graphs, i.e, graphs with positive or negative signs on their edges. We construct some families of bipartite signed graphs with only two distinct eigenvalues. This leads to constructing infinite families of regular Ramanujan graphs.

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1 Introduction

We consider only simple graphs, i.e., graphs with out loops and multiple edges. The vertex set and edge set of the graph $G$ will be denoted by $V(G)$ and $E(G)$, respectively. If there is no doubt about $G$ we simply write $V$ and $E$. A signature on a graph $G$ is a function $s : E \rightarrow \{1, -1\}$. A graph $G$ provided with a signature $s$ is called a signed graph, and will be denoted by $(G, s)$. We call the graph $G$ the ground graph of the signed graph $(G, s)$. The adjacency matrix, $A^s$ of the signed graph $(G, s)$ on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$,

$^1$Email: ramezani@kntu.ac.ir.
is an $n \times n$ matrix whose entries are

$$A^s(i,j) = \begin{cases} 
1, & \text{if } v_i \text{ is adjacent to } v_j \text{ and } s(\{v_i, v_j\}) = 1; \\
-1, & \text{if } v_i \text{ is adjacent to } v_j \text{ and } s(\{v_i, v_j\}) = -1; \\
0, & \text{otherwise.} 
\end{cases}$$

The $n \times n$ matrix $A$, whose entries are the absolute values of the entries of $A^s$, is called the \textit{ordinary adjacency matrix} of the graph $G$ and will be denoted by $A(G) = |A^s|$. Note that any symmetric $(0, \pm 1)$--matrix $A$ with zero entries on diagonal is corresponding to a signature of the graph whose adjacency matrix is $|A|$. Then we sometimes use the symmetric matrix $A$ rather than the signature which is in its correspondence. The \textit{spectrum} of a signed graph is the eigenvalues of its adjacency matrix. By $O_n$, $J_n$, and $I_n$ we mean the all zero matrix, all one matrix, and identity matrix of size $n$, respectively. If there is no doubt about size of the matrices we simply write $O, J, I$. We say the $n \times n$ matrices $A$ and $B$ commute if $AB = BA$. An $n \times n$ matrix $A$ is called anti-symmetric if $A^t = -A$, where $A^t$ is the transpose of the matrix $A$. For an $n \times n$ signed matrix $A$, by $A^*$ we mean the matrix \begin{pmatrix} O_n & A \\ A^t & O_n \end{pmatrix}. An $n \times n$ matrix $C$ is called \textit{orthogonal signed matrix} if the entries of $C$ belong to the set $\{0, 1, -1\}$ and $CC^t = C^tC = \alpha I_n$, where $\alpha$ is a positive integer. A signed graph is called \textit{orthogonal} if its adjacency matrix is an orthogonal signed matrix. Recently some problems on the spectrum of signed adjacency matrices have attracted many studies. In \cite{2}, the authors have considered the lollipop graph and proved its signed graphs are determined by their spectrum. Energy of signed matrices has been considered in \cite{5}. The spectrum of signed graphs can be used to find the spectrum of 2-lifts of graphs, which leads to some results on the existence of infinite families of regular Ramanujan graphs with a fixed degree (cf. \cite{8}). It is known that the only graphs with two distinct eigenvalues of the ordinary adjacency matrix are the complete graphs. In this article we consider bipartite graphs and find some signatures of them which lead to signed graphs with only two distinct eigenvalues. A Ramanujan graph, is a $d$--regular graph whose the second largest eigenvalue is less than or equal to $2 \sqrt{d-1}$. These graphs have several applications to complexity theory, design of robust computer networks, and the theory of error-correcting codes, see \cite{6}. In \cite{9} the authors have proved the existence of infinite families of regular bipartite Ramanujan graphs of every degree bigger than $2$. Here, using the Hadamard matrices, we construct some regular bipartite Ramanujan graphs. A \textit{Conference matrix} $C_{0n}$ is an $n \times n$ $(0, \pm 1)$--matrix with $0$ on the
diagonal and ±1 elsewhere such that $Co_n^tCo_n = (n - 1)I_n$. During the preparation of this paper we encounter interesting results from other combinatorial areas such as regular two-graphs, Hadamard matrices, Conference matrices and Orthogonal codes.

2 preliminaries

In this section we present some results which will be used to construct signed graphs with only two distinct eigenvalues.

In a signed graph $(G, s)$ by resigning at a vertex $v \in V(G)$ we mean multiplying the signs of all the edges incident to $v$ by $-1$. Two signed graphs $(G, s)$ and $(G, s')$ are called equivalent if one is obtained from the other by a sequence of resigning around prescribed vertices. Otherwise we call them distinct. Most properties of signed graphs specially the spectrum are the same in equivalent signed graphs. In the following proposition from [10] the number of distinct signed graphs on the labeled graph $G$ is enumerated.

Proposition 1 [10] If the labeled graph $G$ has $m$ edges, $n$ vertices and $c$ components, then there are $2^{(m-n+c)}$ distinct signed graphs on it.

It is well-known that any $n \times n$ symmetric matrix has $n$ real eigenvalues. If $\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_k^{m_k}$ are the eigenvalues of a symmetric matrix considering their multiplicity then the minimal polynomial of it will be equal to $M(A, x) = \prod_{i=1}^{k} (x - \lambda_i)$. Hence any symmetric matrix with only two distinct eigenvalues has a minimal polynomial of the following form.

$$x^2 + ax + b = 0.$$

In the following lemmas we mention the basic properties of possible signed graphs with just two distinct eigenvalues.

Lemma 1 Suppose the signed graph $(G, s)$ has just two distinct eigenvalues, then the graph $G$ is regular.

Proof. The diagonal entries of the matrix $(A^s)^2$ are the vertex degrees of $G$. On the other hand if $\lambda, \mu$ are the distinct eigenvalues of $(G, s)$, then $(A^s)^2 - (\lambda + \mu)A^s + \lambda\mu I = O$. The
diagonal entries of the matrix $A^s$ are zero hence the diagonal entries of $(A^s)^2$ (therefore the vertex degrees of $G$) should be equal to $-\lambda \mu$, this implies that $G$ is regular. $\square$

2.1 Complete signed graphs

In this short section we review the existing results on the signed graph $(K_n, s)$ with only two distinct eigenvalues, where $K_n$ is the complete graph with $n$ vertices.

A two-graph $\Gamma$ is a set consisting of 3-subsets of a finite set $X$, say set of vertices, such that every 4-subset of $X$ contains an even number of 3-subsets of $\Gamma$. A two-graph is called regular if every pair of vertices lies in the same number of 3-subsets of the two-graph. Let $\Gamma$ be a two-graph on the set $X$. For any $x \in X$, we define a graph with vertex set $X$ where vertices $y$ and $z$ are adjacent if and only if $\{x, y, z\}$ is in $\Gamma$. For a graph $G$, a signed complete graph $\Sigma$ has been corresponded on the same vertex set, whose edges have negative sign if it is an edge of $G$ and positive sign otherwise. Actually the graph $G$ consists of all vertices and all negative edges of $\Sigma$. The signed adjacency matrix of a two-graph is the adjacency matrix of the corresponding signed complete graph.

In [11] it is proved that a two-graph is regular if and only if its signed adjacency matrix has just two distinct eigenvalues. Hence any regular two graph gives a complete signed graph with only two distinct eigenvalues. For more details about two-graphs see [4].

Example. The following set of 3-subsets is a regular two-graph on the set of vertices $\{1, 2, \ldots, 6\}$.

$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}$.

For the vertex 1 of the two-graph, the corresponding graph $G$ is on the vertex set $\{1, \ldots, 6\}$ and the edges are 23, 24, 35, 46, 56. Hence the corresponding signed complete graph have the following adjacency matrix.

$$ A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & 1 & 1 \\
1 & -1 & 0 & 1 & -1 & 1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & 1 & -1 & 1 & 0 & -1 \\
1 & 1 & 1 & -1 & -1 & 0 \\
\end{pmatrix}. $$
The spectrum of $A$ is $[2.2361^3, -2.2361^3]$.

2.2 Bipartite signed graphs

In this section we focus on the signed graphs with only two distinct eigenvalues where the ground graph is bipartite.

**Lemma 2** Let $G$ be a bipartite graph on $2n$ vertices, then the signed graph $(G, s)$ has only two distinct eigenvalues if and only if its adjacency matrix is of the form

\[
\begin{pmatrix}
O_n & C \\
C^t & O_n
\end{pmatrix},
\]

where $C$ is an orthogonal signed matrix of size $n$.

**Proof.** If $G$ is a bipartite graph on $2n$ vertices, then the adjacency matrix of $(G, s)$ is of the block form

\[
\begin{pmatrix}
O_n & C \\
C^t & O_n
\end{pmatrix},
\]

on the other hand we have $(A^s)^2 + \alpha A^s + \beta I_{2n} = O_{2n}$, for some integers $\alpha, \beta$. It implies

\[
\begin{pmatrix}
O_n & C \\
C^t & O_n
\end{pmatrix}^2 + \alpha \begin{pmatrix}
O_n & C \\
C^t & O_n
\end{pmatrix} + \beta I_{2n} = O_{2n}.
\]

Therefore,

\[
\begin{pmatrix}
CC^t & O_n \\
O_n & C^t C
\end{pmatrix} + \alpha \begin{pmatrix}
O_n & C \\
C^t & O_n
\end{pmatrix} + \beta I_{2n} = O_{2n}.
\]

The above equality yields

\[
CC^t = C^t C = -\beta I_n, \quad \alpha C = O_n.
\]

Hence $\alpha = 0$ and $C$ and $A^s$ are orthogonal signed matrices. On the other hand suppose that $C$ is an orthogonal signed matrix of size $n$, with $CC^t = C^t C = \gamma I$. Then for the symmetric signed matrix $A^s = C^s$, we have $(A^s)^2 = \gamma I_{2n}$ and hence the corresponding signed graph has just two distinct eigenvalues $\pm \sqrt{\gamma}.$

The above lemma implies that any bipartite signed graph with just two distinct eigenvalues is in correspondence with an orthogonal signed matrix $C$. Therefore from now on we will refer to orthogonal signed matrices instead of bipartite signed graphs with just two distinct
eigenvalues. We will denote the corresponding orthogonal signed matrix of a bipartite signed graph \((G, s)\) by \(C_s(G)\).

**Corollary 1** If \(G\) is a complete bipartite graph with \(2n\) vertices, and \((G, s)\) has only two distinct eigenvalues, then the matrix \(C_s(G)\) is an Hadamard matrix of size \(n\).

**Corollary 2** If \(G\) is a bipartite \((n-1)\)-regular graph on \(2n\) vertices, and \((G, s)\) has only two distinct eigenvalues, then the matrix \(C_s(G)\) is a Conference matrix of size \(n\).

### 3 Constructing bipartite signed graphs with two distinct eigenvalues

In this section we first recall some well known methods of constructing the Hadamard matrices, then apply them to find orthogonal signed matrices and hence bipartite signed graphs with just two distinct eigenvalues.

**Definition.** If \(A\) is an \(m \times n\) matrix and \(B\) is a \(p \times q\) matrix, then the Kronecker product \(A \otimes B\) is the \(mp \times nq\) block matrix,

\[
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{pmatrix},
\]

It is proved that if the spectrum of an \(n \times n\) matrix \(A\) and an \(m \times m\) matrix \(B\) are \(\lambda_1, \lambda_2, \ldots, \lambda_n\) and \(\mu_1, \mu_2, \ldots, \mu_m\), respectively, then the spectrum of \(A \otimes B\) is \(\lambda_i \mu_j\), for \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\), cf. [7].

The following lemma from [7] will be used for constructing orthogonal signed matrices from small examples.

**Lemma 3** Let \(A\) and \(B\) are orthogonal matrices, then the matrix \(A \otimes B\), which is the Kronecker product of \(A, B\), is orthogonal.

The following corollaries are immediate consequences of the above lemma.
**Corollary 3** If there exist bipartite signed graphs on respectively $2n, 2m$ vertices, with only two distinct eigenvalues, then there exists a bipartite signed graph on $4mn$ vertices, which has only two distinct eigenvalues.

**Corollary 4** For any positive integer $n$, there is an orthogonal signed matrix of size $2n$, and hence a bipartite signed graph on $4n$ vertices and only two distinct eigenvalues

**Proof.** Consider the matrix $A = I_n \otimes H_2$, where $H_2$ is the Hadamard matrix of size 2, i.e.

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

The matrix $A$ is an orthogonal matrix with spectrum $\pm \sqrt{2^n}$, hence the symmetric matrix $A^*$ will be the adjacency matrix of a signed bipartite graph with only two distinct eigenvalues. □

**Lemma 4** Let $C$ be a symmetric $n \times n$ orthogonal signed matrix with zero entries on the diagonal. Then the matrix defined below is a symmetric orthogonal signed matrix,

$$B = \begin{pmatrix} C + I & C - I \\ C - I & -C - I \end{pmatrix}.$$ 

**Proof.** Since $C$ is symmetric so is $B$. Suppose that $C^2 = \alpha I$, we have

$$B^2 = \begin{pmatrix} (C + I)^2 + (C - I)^2 & (C + I)(C - I) + (C - I)(-C - I) \\ (C - I)(C + I) + (-C - I)(C - I) & (C - I)^2 + (-C - I)^2 \end{pmatrix},$$

$$= \begin{pmatrix} C^2 + 2I & O \\ O & C^2 + 2I \end{pmatrix} = \begin{pmatrix} (\alpha + 2)I & O \\ O & (\alpha + 2)I \end{pmatrix},$$

so the assertion follows. □

**Lemma 5** Let $C$ be an antisymmetric orthogonal matrix with zero entries on the diagonal. Then the matrix $C + I$ is an orthogonal matrix.
Proof. Suppose that $CC^t = \alpha I$ for some positive integer $\alpha$. We have
\[
(C + I)(C^t + I) = CC^t + C + C^t + I = (\alpha + 1)I,
\]since $C$ is an antisymmetric orthogonal signed matrix. Hence the assertion follows. □

In the following theorem, we establish a method for constructing orthogonal matrices, based on the Williamson method (cf. [7]).

**Theorem 1** Let the matrices $A_i, 1 \leq i \leq 4$, be symmetric of order $n$, with entries $0, \pm 1$, having constant number of zeros in each row and assume that they are mutually commuting with each other. Consider the matrix $H$ defined bellow,
\[
H = \begin{pmatrix}
A_1 & A_2 & A_3 & A_4 \\
-A_2 & A_1 & -A_4 & A_3 \\
-A_3 & A_4 & A_1 & -A_2 \\
-A_4 & -A_3 & A_2 & A_1
\end{pmatrix},
\]then $H$ is an orthogonal matrix if and only if
\[
\sum_{i=1}^{4} A_i^2 = (\sum_{i=1}^{4} k_i)I.
\]Where $k_i$ is the number of non-zero entries in each row of $A_i$, for $i = 1, \ldots, 4$.

Proof. The assertion follows by simply calculating $HH^t$. □

**Proposition 2** Let $C$ be a symmetric orthogonal signed matrix of order $n$. Then the matrix $H$, in the previous theorem, obtained by substituting any of the following matrices, is an orthogonal signed matrix.

- $A_1 = A_2 = A_3 = A_4 = C$, or
- $A_1 = A_2 = C$, $A_3 = C - I$ and $A_4 = C + I$, or
- $A_1 = A_2 = C + I$, $A_3 = A_4 = C - I$,
providing the matrix $C$ has zero diagonal in the second and last cases.

**Proposition 3** If the matrices $A_i$ in Theorem 1 are equal to a given orthogonal matrix $C$, then the matrix $H$ will be orthogonal.

**Proof.** In general the matrices $A_i$, defined in Theorem 1 do not need to be symmetric. By calculation we see that the only requirement for the orthogonality of $H$ is that the matrices $A_iA_j^t$ for $i, j = 1, \ldots, 4$ need to be symmetric. Since we have $CC^t = \alpha I$, then the assertion follows. □

One may consider the matrix $C$ in the previous propositions to be a symmetric conference matrix. An infinite family of symmetric conference matrices exist, for instance with Paley method we may construct symmetric conference matrices of sizes $q + 1$, where $q$ is a prime power and $q \equiv 1 \pmod{4}$. (cf. [7], pp. 175–176)

4 Some Ramanujan graphs

Bilu and Linial conjectured that every regular Ramanujan graph of degree $d$ has a signature $s$ on the edge set with the property $\lambda_1(A^s) \leq 2\sqrt{d - 1}$, see [3]. We will call the mentioned signature a good signature. In this section for some families of known Ramanujan graphs, we verify the conjecture of Bilu and Linial and introduce new families of Ramanujan graphs. To state our main result we need some lemmas.

It is well-known that if $n \times n$ symmetric matrices $A$ and $B$ commute then the eigenvalues of $A + B$ are the sum of eigenvalues of $A$ and $B$, i.e

$$\lambda_i(A + B) = \lambda_i(A) + \lambda_i(B), \ i = 1, 2, \ldots, n.$$ 

**Lemma 6** If $G$ is a $k$–regular graph on $n$ vertices and $\frac{1}{4}(k - 1)^2 + k + 2 \leq n$ then the complement of $G$, $G^c$ is a Ramanujan graph.

**Proof.** The graph $G^c$ is a $(n - k - 1)$–regular graph, hence we need to prove that

$$\lambda_2(G) \leq 2\sqrt{n - k - 2}.$$
It is well-known that the eigenvalues of the complement of $G$ can be given in decreasing order as in the following list,

$$ n - k - 1, -1 - \lambda_n(G), -1 - \lambda_{n-1}(G), \ldots, -1 - \lambda_2(G). $$

By Perron-Frobenius Theorem we know that $\lambda_n(G) \geq -k$ then $\lambda_2(G^c) \leq k - 1$, and the assertion follows by the following chain of inequalities,

$$ \lambda_2(G^c) \leq k - 1 \iff \frac{1}{4} \lambda_2^2(G^c) \leq \frac{1}{4}(k - 1)^2 \leq n - (k + 2), $$

hence

$$ \lambda_2(G^c) \leq 2\sqrt{n - k - 2} $$

thus $G^c$ is a Ramanujan graph.

\[ \square \]

**Proposition 4** Let $C$ be a symmetric orthogonal signed matrix of order $n$, with $CC^t = \alpha I$, $\alpha \geq 2$ and $k = n - 1 - \alpha$, holds in the inequality $\frac{1}{4}(k - 1)^2 + k + 2 \leq n$ then the ground graph $G$ corresponding to the matrix $C$ is a Ramanujan graph and $C$ is a good signature of $G$.

**Proof.** The graph $G$ is $\alpha$–regular, on the other hand $C$ is a symmetric orthogonal signed matrix hence $\lambda_1(C) = \sqrt{\alpha} \leq 2\sqrt{\alpha - 1}$. The complement of $G$ is a regular graph of degree $k = n - 1 - \alpha$. By the assumption and Lemma 6 the assertion follows.

In the case that the orthogonal signed matrix $C$ is not symmetric we can use it to construct bipartite Ramanujan graphs. But before that we need some preliminaries. For a bipartite graph $G$ with bipartition $X, Y$ of vertices, the *bipartite complement* of $G$ which is denoted by $G_b^c$ is a bipartite graph on the vertex set $V = X \cup Y$, where a vertex $x \in X$ is adjacent to a vertex $y \in Y$ if and only if $x$ was not adjacent to $y$ in $G$.

**Lemma 7** If $G$ is a $k$–regular bipartite Ramanujan graph on the disjoint sets $X, Y$ of size $n$, and $k \leq \frac{4}{k}$ then the bipartite complement of $G$ is also a Ramanujan graph.

**Proof.** We have $A(G) + A(G^c_b) = A(K_n,n)$. The graph $G$ is regular and bipartite, hence the adjacency matrix of $G$ is of the following form.

$$ A(G) = \begin{pmatrix} O & B \\ B^t & O \end{pmatrix}, $$

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where \( B \) is a \((0, 1)\)-matrix with a fix number of 1’s on each row and column. Therefore the matrices \( A(G) \) and \( A(K_{n,n}) \) commute and hence the eigenvalues of \( G_b^c \) follows

\[
\lambda_i(G_b^c) = \lambda_i(K_n, n) - \lambda_i(G) = \begin{cases} 
  n - k & i = 1, \\
  \lambda_i(G) & 1 < i < 2n, \\
  k - n & i = 2n.
\end{cases}
\]

By the assumption \( \lambda_2(G) \leq 2\sqrt{k - 1} \), and \( k \leq n - k \) hence

\[
\lambda_2(G_b^c) = \lambda_2(G) \leq 2\sqrt{k - 1} \leq 2\sqrt{n - k - 1},
\]

therefore \( G_b^c \) is a Ramanujan graph.

\[\square\]

**Lemma 8** The bipartite complement of the graph \( kC_4 \), \((k \geq 2)\) that is the \( k \) disjoint copy of \( C_4 \), is a Ramanujan graph.

**Proof.** As discussed in the proof of the above lemma, the eigenvalues of \((kC_4)_b^c\) can be calculated by the eigenvalues of the graph \( K_{2k,2k} \) and the graph \( kC_4 \). Thus the spectrum of \((kC_4)_b^c\) is \( \pm(2k - 2), \pm2^{k-1}, 0^{2k} \), hence the assertion follows.

\[\square\]

**Lemma 9** If \( C \) is an orthogonal signed matrix of size \( n \) and the ground graph \( G \) corresponding to \( C^* \) is a Ramanujan graph, then \( C^* \) is a good signature of \( G \).

For a signature \( s \) on the edges of the graph \( G \) the 2–lift \( \tilde{G}_s \) of the graph \( G \) is a graph on the vertex set \( V(G) \times \{1, 2\} \), where there is an edge between \((u, i) \) and \((v, j) \) if and only if

\[
\{u, v\} \in E(G) \text{ and } \begin{cases} 
  i = j & \text{if } s(uv) = 1, \\
  i = \bar{j} & \text{otherwise},
\end{cases}
\]

where \( \bar{1} = 2 \) and \( \bar{2} = 1 \). In Figure 1, we illustrate the definition. Note that in the figure the negative edge is denoted by dot line. In the graph \( \tilde{G}_s \), the vertices of \((G, s)\) are duplicated and the positive edges are replaced by two parallel lines and the negative edge is replaced by two crossed edges.
The spectrum of the graph $\tilde{G}_s$ is determined in [8].

**Lemma 10** [8] The spectrum of $A(\tilde{G}_s)$ is the multi set union of spectrum of $A$ and $A^s$.

By the above Lemma the authors of the paper [9] conclude that for any Ramanujan graph $G$ and a good signature $s$ of it, the graph $\tilde{G}_s$ is a Ramanujan graph. As a final result, in the following table we list some Ramanujan graphs and a good signature of them.

Suppose that $H_n$ is a Hadamard matrix and $Co_n$ is a conference matrix of order $n$.

| $G$          | a good signature $A^s$          | eigenvalues of $\tilde{G}_s$         |
|--------------|----------------------------------|--------------------------------------|
| $K_{n,n}$    | $H^*_n$                          | $\pm n, \pm \sqrt{n^n}, 0^{2n-2}$   |
| $K_{n,n} \setminus M$ | $Co^*_n$                         | $\pm (n-1), \pm \sqrt{n-1}^n, \pm 1^{n-1}$ |
| $(nC_4)_b$  | $(-Co_n, Co_n)$                  | $\pm (2n-2), (\pm 2)^{n-1}, 0^{2n}$  |

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