Ramanujan type $1/\pi$ Approximation Formulas

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Abstract

In this article we use theoretical and numerical methods to evaluate in a closed-exact form the parameters of Ramanujan type $1/\pi$ formulas.

keywords $\pi$-formulas; Ramanujan; elliptic functions; singular modulus; alternative modular bases; approximations; numerical methods

1 Introduction

We give the definitions of the Elliptic Integrals of the first and second kind respectively (see [9],[4]):

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2(t)}} \quad \text{and} \quad E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} dt. \quad (1)$$

In the notation of Mathematica we have

$$K(x) = \text{EllipticK}[x^2] \quad \text{and} \quad E(x) = \text{EllipticE}[x^2]. \quad (2)$$

Also we have (see [9],[7]):

$$\dot{K}(k) = \frac{dK(k)}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}. \quad (3)$$

The elliptic singular moduli is defined to be the solution of the equation:

$$\frac{K(\sqrt{1 - w^2})}{K(w)} = \sqrt{r}. \quad (4)$$

In Mathematica is stated as

$$w = k = k_r = k[r] = \text{InverseEllipticNomeQ}[e^{-\pi \sqrt{r}}]^{1/2}. \quad (5)$$

The complementary modulus is given by $k_r^2 = 1 - k_r^2$.

Also we will need the following relation of the elliptic alpha function (see [7]):

$$a(r) = \frac{\pi}{4K(k_r)^2} - \sqrt{r} \left( \frac{E(k_r)}{K(k_r)} - 1 \right). \quad (6)$$
The Hypergeometric functions are defined by

\[ m+1 \text{F}_m (a_1, a_2, \ldots, a_{m+1}; b_1, b_2, \ldots, b_m; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_{m+1})_n}{(b_1)_n(b_2)_n \ldots (b_m)_n} \frac{z^n}{n!}, \text{ for } |z| < 1, \]

and \((a)_0 := 1, (a)_n := a(a+1)(a+2) \ldots (a+n-1), \) for each positive integer \(n\).

## 2 The construction of some \(1/\pi\) and \(1/\pi^2\) formulas

It holds

\[ \phi_1(z) = 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; z \right) = \frac{4K^2 \left( \frac{1}{2}(1 - \sqrt{1 - z}) \right)}{\pi^2}, \] (8)

Consider the following equation with respect to the function \(\phi_1(z)\):

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{3}\right)_n}{(n!)^3} z^n (an + b) = g \frac{9}{\pi} \iff b\phi_1(z) + az\phi'_1(z) = \frac{g}{\pi}, \]

Set \(w = 1/2 \left( 1 - \sqrt{1 - k^2} \right), 1 - 2w = \sqrt{1 - z} = k_r\).

But

\[ b\phi_1(z) + az\phi'_1(z) = \frac{g}{\pi} \iff g = \frac{4K(w)(aE(w) + (b + a(w - 1) - 2bw)k(w))}{\pi(1 - 2w)}, \]

hence

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{3}\right)_n}{(n!)^3} 4^n (w - w^2)^n (an + b) = \]

\[ = \frac{4K(\sqrt{w}) (aE(\sqrt{w}) + (b - a + aw - 2bw - 2bw)K(\sqrt{w}))}{\pi^2(1 - 2w)}. \]

For \(w = k_r\) we get

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{3}\right)_n}{(n!)^3} 4^n (k_r k'_r)^n (an + b) = \]

\[ = \frac{4K(k_r) (aE(k_r) + (b - a + ak_r^2 - 2bw - 2bk_r^2)K(k_r))}{\pi^2(1 - 2k_r^2)}. \] (9)

Now using the formula for \(a(r)\), in the sense that

\[ E(k_r) = K(k_r) - \frac{a(r) K(k_r)}{\sqrt{r}} + \frac{\pi}{4K(k_r)\sqrt{r}}, \]

(10)
for suitable values for $a, b, c$ we get the following theorem:

**Theorem 2.1**

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^3}{(n!)^3} n(k_r k'_r)^{2n} (\sqrt{r}(1 - 2k_r^2)n + a(r) - \sqrt{r}k_r^2) = \frac{1}{\pi}$$  \hspace{1cm} (11)

**Example.**

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^3}{(n!)^3} (40\sqrt{2} - 56)^n (an + b) = \frac{4a}{7\pi} + \frac{5a}{7\sqrt{2}\pi} + 4(-4a + \sqrt{2a} + 14b) \frac{\Gamma^2 \left( \frac{3}{2} \right)}{7\pi \Gamma^2 \left( \frac{5}{2} \right)}.$$

From which a special case is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^3}{(n!)^3} (40\sqrt{2} - 56)^n (n + \frac{2}{7}) - \frac{1}{7\sqrt{2}} = \frac{8 + 5\sqrt{2}}{14\pi}.$$

**Theorem 2.2**

$$\sum_{n=0}^{\infty} \frac{B^{(2)}_n}{(n!)^2} (k_r)^{2n} (\sqrt{r}k_r^2n + a(r) - \sqrt{r}k_r^2) = \frac{1}{\pi}.$$  \hspace{1cm} (12)

**Proof.**

We use the function

$$\phi_2(z) = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; z \right) = \frac{2K(\sqrt{z})}{\pi}.$$  \hspace{1cm} (13)

Then if

$$B^{(2)}_n := \sum_{j=0}^{n} \left( \binom{n}{j} \left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_{n-j} \right)^2$$

$$\phi_2^2(z) = \cdots = \sum_{n=1}^{\infty} \frac{z^n}{(n!)^2} \sum_{j=0}^{n} \left( \binom{n}{j} \left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_{n-j} \right)^2,$$  \hspace{1cm} (14)

where

$$c\phi_2(z) + bz\phi'_2(z) + az^2\phi''_2(z) = \sum_{n=0}^{\infty} \frac{B^{(2)}_n}{(n!)^2} z^n (an^2 + (b-a)n + c)$$

Hence we get

$$\sum_{n=0}^{\infty} \frac{B^{(2)}_n}{(n!)^2} k_r^{2n} (an^2 + (b-a)n + c) = \frac{2(aE(k_r) + (2b - 2bk_r^2 - 4a + 6ak_r^2)E(k_r)K(k_r))}{\pi^2(1 - k_r^2)^2} + \cdots$$
\[ + \frac{2(3a - 2b + 2c + (-4a + 2b - 2c)k_r^2)K(k_r)}{\pi^2(1 - k_r^2)}. \]

For \( a = 0, b = 1, c = (-k_r^2 + a(r)r^{-1/2})k_r^{-2} \), we get

**Theorem 2.3** Set

\[ B_n^{(3)} := \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_{n-j} \right]^3, \quad (15) \]

then an \( 1/\pi^2 \) formula is the following

\[ \sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} (2k_r k_r')^{2n}(n^2 + (b(r) - 1)n + c(r)) = \frac{3}{(1 - 2k_r^2)^2 r \pi^2} \quad (16) \]

where

\[ b(r) = \frac{3a_r + \sqrt{r} - 6a(r)k_r^2 - 9\sqrt{r}k_r^2 + 12\sqrt{r}k_r^2}{\sqrt{r}(1 - 2k_r^2)^2} \]

and

\[ c(r) = \frac{3a(r)^2 - 6a(r)\sqrt{r}k_r^2 - r k_r^2 + 4r k_r^4}{r(1 - 2k_r^2)^2} \]

**Proof.**

Set

\[ \phi_3(z) = {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; z \right)^2 = \left( \frac{16K^2 \left( \frac{1}{2}(1 - \sqrt{1 - z}) \right)}{\pi^2} \right)^2, \]

then

\[ c\phi_3(z) + bz\phi_3'(z) + az^2\phi_3''(z) = \sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} z^n(an^2 + (b - a)n + c) \]

The left hand of the above equation is a function of \( E(x), K(x) \), and can evaluated when we set certain values to the parameters \( a, b, c \).

**Examples.**

1) \[ \frac{1}{1200(161\sqrt{5} - 360)\pi^2} = \]

\[ \sum_{n=0}^{\infty} \frac{B_n^{(3)}}{(n!)^3} \left( 51841 - 23184\sqrt{5} \right)^n \left( n^2 + \left( 1 - \frac{521}{288\sqrt{5}} \right)n + \frac{5}{12} - \frac{521}{576\sqrt{5}} \right) \quad (17) \]

2)
\[ b(163) = \frac{191211325848427}{151931373056001} - \frac{101078496262538371350772720 \cdot 2^{2/3}}{151931373056001} - \frac{4 \cdot 2^{1/3}(B_1 - \sqrt{489}B_2)^{1/3}}{151931373056001} \]

\[ B_1 = 5680848001702137216093843898647314524189 \]

\[ B_2 = 7689698960589381643149203281167 \]

\[-5839006481108705728 + 95296270719550411072 \cdot b(163) - 4530513053635162884 \cdot b(163)^2 + 668649972819460401 \cdot b(163)^3 = 0 \]

\[ c(163) = \frac{141786798298969760}{24764813808128163} - \frac{4 \left(C_1 - \sqrt{489}C_2\right)^{1/3}}{24764813808128163} - \frac{6241484569597616793758999818952 \cdot 2^{2/3}}{24764813808128163} \]

\[ C_1 = 5512985602111283751597893407219881834715037026 \]

\[ C_2 = 101526256966667546381077303112958296550 \]

\[ C_3 = 2756492801055641875798946703600940917357518513 \]

\[ C_4 = 507631284833377390538651556479148275 \]

\[-24380823840878077184 + 1313102088959368594752 \cdot c(163) - 3051378096384581928640 \cdot c(163)^2 + 17765361127840243394169 \cdot c(163)^3 = 0 \]

\[ \sum_{n=0}^{\infty} \frac{4^n B_n^{(3)}}{(n!)^3} (k_{163} k_{163}')^{2n}(n^2 + (b(163) - 1)n + c(163)) = \frac{A}{\pi^2} \]

\[ A = \frac{4 \left(12660947754667 + 26680 \left(A_1 - \sqrt{489}A_2\right)^{1/3} + 26680 \left(A_1 + \sqrt{489}A_2\right)^{1/3}\right)}{8254937936042721} \]

\[ A_1 = 10686639869761339845357037 \]

\[ A_2 = 3136555671686449089 \]

\[ y_{163} = (k_{163} k_{163}')^2 = \frac{1}{16} - \frac{266933400}{\left(-1 + 557403 \sqrt{489}\right)^{1/3}} + \frac{10005}{2} \left(-1 + 557403 \sqrt{489}\right)^{1/3} \]

\[ -1 + 164085588290048048 \cdot y_{163} - 768 \cdot y_{163}^2 + 4096 \cdot y_{163}^3 = 0 \]

Formula (18) gives about 17 digits per term and is a formula for $1/\pi^2$. For $r = 253$ we have another such formula which gives 21 digits per term constructed in the same way as (18).
3 The study of a non usual $1/\pi$ formula

The $j$ invariant is given by (see [17]):

$$j(z) = \left( \frac{\eta(z/2)}{\eta(z)} \right)^{16} + 16 \left( \frac{\eta(z)}{\eta(z/2)} \right)^8 ,$$  \hspace{1cm} (19)

where $z = \sqrt{-r}$, $r$-positive real and

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi inz} \right)$$

is the Dedekind eta function.

Also

$$\frac{\eta(z)}{\eta(z/2)} = \frac{k_r^{1/12}}{2^{1/6} k_r^{1/6}} .$$  \hspace{1cm} (20)

From [24] section 7, Theorem 7.4 and from [11] formula (5.8), when $q = e^{2\pi iz}$, $z = \sqrt{-r}$, $r$ positive real, the modular $j$-invariant is also given by

$$j(z) = 1728 \frac{Q^3(q)}{Q^3(q) - R^2(q)} .$$  \hspace{1cm} (21)

where

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} , \quad Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}$$

and

$$R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} .$$

The function $t_r$ is given from

$$t_r = \frac{Q_r}{R_r} \left( P_r - \frac{6}{\pi \sqrt{r}} \right) ,$$  \hspace{1cm} (22)

where

$$P_r = P(-e^{-\pi \sqrt{r}}) , \quad Q_r = Q(-e^{-\pi \sqrt{r}}) \text{ and } R_r = R(-e^{-\pi \sqrt{r}}) .$$

i) Using Theorems 3 and 4 of [25], relation (21) equivalently can be transformed to

$$j(z) = \frac{432}{\beta_r (1 - \beta_r)} .$$  \hspace{1cm} (23)

Also note that we have

$$j(\sqrt{-r}) = j_r = \frac{256 (1 - k_r^2 + k'_r)^3}{(k_r k'_r)^4} = \frac{432}{\beta_r (1 - \beta_r)} .$$  \hspace{1cm} (24)
Hence with our method in [25] we can simplify the known results of [24] and [11] using the function $\beta_r$, which defined as the root of the equation:

$$\frac{2F_1 \left(\frac{1}{6}, \frac{5}{6}; 1; 1-w \right)}{2F_1 \left(\frac{1}{6}, \frac{5}{6}; 1; w \right)} = \sqrt{r}. \quad (25)$$

ii) Set now $m_r := k_r^2$ and let $a(r)$, $E(x)$ be the elliptic alpha function and the complete elliptic integral of the second kind respectively (see [7],[4]), then:

$$t_r = \frac{1}{(1 - 2\beta_r)u_r^2} \left( P(q) - \frac{6}{\sqrt{r\pi}} \right) =$$

$$= \frac{1}{(1 - 2\beta_r)u_r^2} \left( \frac{E(m_r/4)}{K(m_r/4)} - 2 + m_r/4 - \frac{3\pi}{4\sqrt{r/4K(m_r/4)^2}} \right) F_{r/4}^2$$

or

$$t_r = \frac{1 + m_r/4 - \frac{6}{\sqrt{r}} a \left( \frac{r}{4} \right)}{\sqrt{1 - m_r/4 + m_r^2(1 - 2\beta_r/4)}} \quad (26)$$

Hence from the above evaluations and the $1/\pi$ series in [6] and [11] we get the next reformulation:

**Theorem 3.1** If we define

$$J_r := 1728 j_{-1}^{-1} = 4\beta_r (1 - \beta_r) \quad (27)$$

$$T_r := \frac{1 + k_r^2 - 3\pi a(r)}{\sqrt{1 - k_r^2 + k_r^2(1 - 2\beta_r)}} = \frac{2j_{1/3}^3 \sigma(r) G_r^3}{\sqrt{r} \sqrt{J_r - 1728}} \quad (28)$$

then

$$\frac{3}{\pi \sqrt{r} \sqrt{1 - J_r}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{5}{6})_n (\frac{1}{2})_n a(J_r)^n (6n + 1 - T_r)}{(n!)^3} \quad (29)$$

**Note.** The function $G_r$ is the Weber invariant and

$$\sigma(r) = 2\sqrt{r} (1 + k_r^2) - 6a(r)$$

(see [7],[5] chapter 5).

The above formulas (27), (28) and (29) can be used for numerical and theoretical evaluations.

**Similarities of formula (29) and a fifth order base formula**
From the identity
\[ 2F1 \left( \frac{1}{6}, \frac{5}{6}; 1; \frac{1 - \sqrt{1-z}}{2} \right)^2 = 3F2 \left( \frac{1}{6}, \frac{5}{6}; 1, 1; z \right), \] (30)
and using the following relations found in [7]:
\[ K_s(x) = \frac{\pi}{2} F1 \left( \frac{1}{2} - s, \frac{1}{2} + s; 1; x^2 \right) \]
and
\[ E_s(x) = \frac{\pi}{2} F1 \left( -\frac{1}{2} - s, \frac{1}{2} + s; 1; x^2 \right) \]
(31)
\[ a_s(x_r) := \frac{\pi}{4} K_s(x_r) \cos(\pi s) \]
where \( s = \frac{1}{3} \) one can get, (working as in Theorem 2.1) the following Ramanujan-type \( 1/\pi \) formula:
\[ \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)_n \left( \frac{5}{6} \right)_n \left( \frac{1}{2} \right)_n \left( a_3(x_r) - a_1/6 \right)_n (n-b) = \frac{3}{2\sqrt{3} \pi (1-2\alpha_3(r))} \]
(34)
where the function \( a_5(r) = a_{1/3}(\sqrt{\beta_r}) \) is algebraic for \( r \in \mathbb{Q}^+ \).
The parameters and the corresponding function \( a_5(r) \) of (34) are those of fifth singular moduli base theory. Also (34) in comparison with (29) gives the following theorem.

**Theorem 3.2**
\[ 10a_5(r)r^{-1/2} = 10a_{1/3}(\sqrt{\beta_r})r^{-1/2} = 1 + 8\beta_r - \frac{1 + k_r^2 - 3a(r)r^{-1/2}}{\sqrt{1-k_r^2 + k_r^2}} \] (35)
The above formula is for general evaluation of elliptic alpha function in the fifth elliptic base.

Also from the cubic theory as in fifth, we have
\[ 3F2 \left( \frac{1}{3}, \frac{2}{3}; 1; 1; w \right)^2 = 2F1 \left( \frac{1}{3}, \frac{2}{3}; 1; \frac{1 - \sqrt{1-w}}{2} \right)^2 \] (36)
we get
\[ \sum_{n=0}^{\infty} \left( \frac{1}{7} \right)_n \left( \frac{2}{7} \right)_n \left( \frac{1}{2} \right)_n [4a_3(r) - 4a_2^2(r)]^n (n-b) = \frac{\sqrt{3}}{2\pi \sqrt{r(1-2a_3(r))}} \] (37)
\[ b = \frac{4 \left( a_3(r) - a_{1/6} \right) (r^{-1/2})}{3(1-2a_3(r))} \] (38)
4 Examples and Evaluations

1) For \( r = 2 \)

\[
J_2 = \frac{27}{125} \\
T_2 = \frac{5}{14}
\]

and

\[
\frac{15\sqrt{5}}{14\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{\sqrt{5}}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{27}{125}\right)^n \left(6n + \frac{9}{14}\right) \tag{39}
\]

2) For \( r = 4 \) we have

\[
\frac{11\sqrt{11}}{14\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{\sqrt{11}}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{8}{1331}\right)^n \left(6n + \frac{10}{21}\right) \tag{40}
\]

3) For \( r = 5 \) we have

\[
T_5 = \frac{1}{418} \left(139 + 45\sqrt{5}\right) \\
J_5 = \frac{27 (-1975 + 884\sqrt{5})}{33275}
\]

Hence

\[
\sqrt{21650 + 5967\sqrt{5}} = \pi \\
= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{\sqrt{5}}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(-\frac{53325 + 23868\sqrt{5}}{33275}\right)^n \left(836n + 93 - 15\sqrt{5}\right) \tag{41}
\]

4) For \( r = 8 \) we have

\[
k_8^2 = 113 + 80\sqrt{2} - 4\sqrt{2} \left(799 + 565\sqrt{2}\right)
\]

\[
a(8) = 2 \left(10 + 7\sqrt{2}\right) \left(1 - \sqrt{-2 + 2\sqrt{2}}\right)^2
\]

Then

\[
\frac{15\sqrt{\frac{5}{2} (84125 + 81432\sqrt{2})}}{9982\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{\sqrt{2}}{6}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^3} \left(\frac{5643000 - 3990168\sqrt{2}}{1520875}\right)^n \left(\frac{3276 - 1125\sqrt{2} + 29946n}{4991}\right) \tag{42}
\]

5) For \( r = 18 \) we have

\[
k_{18} = (-7 + 5\sqrt{2})(7 - 4\sqrt{3})
\]
\[ a(18) = -3057 + 2163\sqrt{2} + 1764\sqrt{3} - 1248\sqrt{6} \]
\[ \alpha_6 = \frac{1}{500}(68 - 27\sqrt{6}) \]
\[ \beta_{18} = \frac{1}{2} \cdot \frac{7 (49982 + 4077\sqrt{6})}{10\sqrt{5} (989 + 54\sqrt{6})^{3/2}} \]
\[ J_{18} = \frac{637326171 - 260186472\sqrt{6}}{453870144125} \]
\[ T_{18} = \frac{712075 + 49230\sqrt{6}}{1074514} \]

Hence we get the formula giving 8 digits per term:
(Note that the number of digits per term is determined by the value of \( J_r \), approximately.)
\[ \frac{5\sqrt{23124123365} - 13274820\sqrt{6}}{1074514\pi} = \]
\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{5}{6})_n (\frac{1}{2})_n}{(n!)^3} \left( \frac{637326171 - 260186472\sqrt{6}}{453870144125} \right)^n \times \]
\[ \left( 6n + \frac{9 (40271 - 5470\sqrt{6})}{1074514} \right) \]

(45)

6) For \( r = 27 \)
\[ k_{27} = \frac{1}{2} \sqrt{\frac{1 + 100 \cdot 2^{1/3} - 80 \cdot 2^{2/3}}{2 + \sqrt{3} - 100 \cdot 2^{1/3} + 80 \cdot 2^{2/3}}} \]
\[ a(27) = 3 \left[ \frac{1}{2} \left( \sqrt{3} + 1 \right) - 2^{1/3} \right] \]

\( a(27) \) is obtained from [7] page 172.
\[ J_{27} = \frac{56143116 + 157058640 \cdot 2^{1/3} - 160025472 \cdot 2^{2/3}}{817400375} \]
\[ T_{27} = \frac{58871825 + 22512960 \cdot 2^{1/3} + 13208820 \cdot 2^{2/3}}{132566687} \]

Hence we get the 11 digits per term formula:
\[ \frac{935}{\pi} \sqrt{\frac{935}{3 (761257259 - 157058640\sqrt{2} + 160025472\sqrt{4})}} = \]
\[ = \sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n (\frac{5}{6})_n (\frac{1}{2})_n}{(n!)^3} \left( \frac{56143116 + 157058640\sqrt{2} - 160025472\sqrt{4}}{817400375} \right)^n \times \]
7) From the Wolfram pages 'Elliptic Lambda Function' and 'Elliptic Singular Value' we have:

\[ k_{58} = \left( -1 + \sqrt{2} \right)^6 \left( -99 + 13 \sqrt{58} \right) \]

and

\[ a(58) = \frac{1}{64} \left( -70 + 99 \sqrt{2} - 13 \sqrt{29} \right) \left( 5 + \sqrt{29} \right)^6 \left( -444 + 99 \sqrt{29} \right) \]

Also using the cubic theta identities, (see [25] relations (2),(3),(4),(30)) we evaluate \( \alpha_{174} \) numerically to 1500 digits and then \( \beta_{58} \) to 1500 digits accuracy. We then apply the 'Recognize' routine of Mathematica. The result is the minimum polynomial of \( \beta_{58} \) (this can be done also from (19) and (23)):

\[
1 - 1399837865393267000x + 79684665286353732299517000x^2 - \\
-159369327773031733812500000x^3 + 79684663886515866906250000x^4 = 0.
\]

Solving this equation with respect to \( x \) we get the value of \( \beta_{58} \) in radicals. Thus

\[ J_{58} = \frac{1399837865393267 - 259943365786104\sqrt{29}}{39842331943257933453125} \] \hspace{1cm} (46)

\[ T_{58} = \frac{5 \left( 1684967251 + 24160612\sqrt{29} \right)}{10376469642} \] \hspace{1cm} (47)

The result is the formula

\[
5\sqrt{\frac{\pi}{87}} \left( 1382696980921017 - 90211316\sqrt{29} \right) \left( \frac{357809298}{1382696980921017 - 90211316\sqrt{29}} \right) = \\
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_n}{(n!)^3} \left( 1399837865393267 - 259943365786104\sqrt{29} \right) \left( \frac{39842331943257933453125}{1399837865393267 - 259943365786104\sqrt{29}} \right)^n \\
\times \left( \frac{6117973}{32528118} - \frac{8628790}{25557807\sqrt{29}} + 6n \right) \] \hspace{1cm} (48)

which gives 18 digits per term.

8) For \( r = 93 \) (see [7] pg.158), we have

\[ \sigma(93) = 6G_{93}^{-6} \left( \frac{\sqrt{3} + 1}{2} \right)^3 \left( 15\sqrt{93} + 13\sqrt{31} + 201\sqrt{3} + 217 \right). \]
From [5] chapter 34 we have

\[
G_{93} = \left(3\sqrt{3} + \sqrt{31}\right)^{1/4} \left(39 + 7\sqrt{31}\right)^{1/6}
\]
also

\[
a(r) = \sqrt{\frac{1 + k_r^2}{3}} - \frac{\sigma(r)}{6}
\]

Hence

\[
G_{93}^{-24} = 4k_{93}^2 (1 - k_{93}^2)
\]

This result is a very flexible formula that gives about 24 digits per term.

\section{5 Neat Examples with Mathematica and Simplicity}

The class number \(h(-d), d \in \mathbb{N}\) of the equivalent quadratic forms is given by

\[
h(-d) = \frac{-w(d)}{2d} \sum_{n=1}^{d-1} \left(\frac{-d}{n}\right) n,
\]
where \(w(3) = 6, w(4) = 4\) else \(w(d) = 2\). \(\left(\frac{d}{n}\right)\), is the Jacobi symbol. Observe that \(h(-163) = 1\) (see [17]). For small values of \(h(-d)\) we have greater possibility to evaluate \(J_d\) and \(T_d\) in radicals.

The simplest way to evaluate the parameters \(J_{163}\) and \(T_{163}\) is again with Mathematica.

The general algorithm is:

i) Set \(r = d\) and \(k[r] = \text{InverseEllipticNomeQ}[e^{-\pi\sqrt{r}}]^{1/2}\), then we can evaluate \(\beta_r\) and \(j_r\) from relations (19) and (23). Hence we get the value of \(J_r\) as in section 4 example 7.

ii) For the evaluation of \(T_r\) we will need the value of \(a(r)\) which is given from (see [7]):

\[
a(r) = \frac{\pi}{4K^2} - \sqrt{r} \left(\frac{E}{K} - 1\right).
\]
This in Mathematica is given from

\[
a(r) = \frac{\pi}{4 \text{EllipticK}[k[r]^2]} - \sqrt{T \left( \frac{\text{EllipticE}[k[r]^2]}{\text{EllipticK}[k[r]^2]} - 1 \right) - 1}
\]  

(51)

Hence taking the package

\[<\!\!<\text{NumberTheory'Recognize}\!>\]

and

\[
\text{Recognize}[N[J_{163}, 1500], 16, x] \\
\text{Recognize}[N[T_{163}, 1500], 16, x]
\]

we get two equations. After solving them we get if \( r \in \mathbb{N} \) (here \( r = 163 \)), the values of the parameters \( J_r \) and \( T_r \) in algebraic-closed forms. The results are the \( \pi \) formulas.

1) We have that \( J_{163} \) is root of

\[-64 + 25528108531892325888558727380998000x - 21982537902460417233779433360187500x^2 + +224514224985741154735390775022822688001953125x^3 = 0\]

hence

\[
J_{163} = 4 \frac{C_1 - C_2 \left(-A_1 + \sqrt{489} B_1\right)^{1/3} + 30591288 \left(-A_1 + \sqrt{489} B_1\right)^{1/3}}{1079255251621895869488211571345343375}
\]

\[
A_1 = 12737965652562547164590026038483232428161827096523072256574968383 \\
B_1 = 2290380731820668253780064859649503945583497227761749294205546402325349 \\
C_1 = 8888429913332498766352891 \\
C_2 = 902206261147132595923169636910570558029813352485594880
\]

From \( J_r = 4\beta_r(1 - \beta_r) \), we get the value of \( \beta_r \) and hence

\[
T_{163} = 5 \frac{12948195754365757115 + 8 \left(A_2 - B_2 \sqrt{489}\right)^{1/3} + 8 \left(A_2 + B_2 \sqrt{489}\right)^{1/3}}{83470787671093501833}
\]

where

\[
A_2 = 3802386862487392962897493239274992371253057854289262 \\
B_2 = 3865464212119923579732688315287754932200919450
\]

The above parameters give 32 digits per term

2) Another evaluation is taking \( d = r = 253 \).
\[ J_{253} = \frac{A_1 - A_2 \sqrt{11} + 31990140 \sqrt{A_3 - A_4 \sqrt{11}}}{A_5} \]

\[ \begin{align*}
A_1 &= 280436578925995909441757692179285740357087269234369 \\
A_2 &= 8455480998076515696277133493195558464492321957799872 \\
A_3 &= 14334626424019721999734311051748172965440271797713951 \\
A_4 &= 43220524871261259540733172862370537466134334936322822 \\
A_5 &= 1066755353338783886372226117351012749877681799897625
\end{align*} \]

\[ T_{253} = \frac{1875 \sqrt{B_1 - B_2 \sqrt{11} + 3847208393012364625 + 752271279708923520 \sqrt{11}}}{6969874104047710086} \]

\[ \begin{align*}
B_1 &= 21321689952816786600672118125 \\
B_2 &= 60533150139616794053500831192
\end{align*} \]

The above parameters give 41 digits per term.

**Conclusion**

We have given a way of how we can construct a very large number of Ramanujan's type 1/π formulas. It is true that in most cases, from \( r = 1 \) to 100 (or higher), using Mathematica program, such formulas are very simple, as long as \( h(-d) \) remains small and the parameters are solutions of solvable polynomial equations.

**References**

[1]: M.Abramowitz and I.A.Stegun: Handbook of Mathematical Functions. Dover Publications. (1972).

[2]: B.C.Berndt: Ramanujan's Notebooks Part I. Springer Verlag, New York. (1985).

[3]: B.C.Berndt: Ramanujan's Notebooks Part II. Springer Verlag, New York. (1989).

[4]: B.C.Berndt: Ramanujan's Notebooks Part III. Springer Verlag, New York. (1991).

[5]: B.C. Berndt: Ramanujan's Notebooks Part V. Springer Verlag, New York, Inc. (1998)

[6]: Bruce C. Berndt and Heng Huat Chan: Ramanujan and the Modular j-Invariant. Canad. Math. Bull. Vol.42(4), (1999). pp.427-440.

[7]: J.M. Borwein and P.B. Borwein: Pi and the AGM. John Wiley and Sons, Inc. New York, Chichester, Brisbane, Toronto, Singapore. (1987).
[8]: I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series and Products. Academic Press. (1980).
[9]: E.T. Whittaker and G.N. Watson: A course on Modern Analysis. Cambridge U.P. (1927)
[10]: J.J. Zucker: The summation of series of hyperbolic functions. SIAM J. Math. Ana.10.192. (1979)
[11]: Bruce.C. Berndt and Heng Huat Chan: Eisenstein Series and Approximations to $\pi$. Page stored in the Web.
[12]: S. Ramanujan: Modular equations and approximations to $\pi$. Quart. J. Math.(Oxford). 45, 350-372. (1914).
[13]: S. Chowla: Series for $1/K$ and $1/K^2$. J. Lond. Math. Soc. 3, 9-12. (1928)
[14]: N.D. Baruah, B.C. Berndt and H.H. Chan: Ramanujan’s series for $1/\pi$: A survey. American Mathematical Monthly 116, 567-587. (2009)
[15]: T. Apostol: Modular Functions and Dirichlet Series in Number Theory. Springer
[16]: Bruce.C. Berndt, S. Bhargava and F.G. Garvan: Ramanujan’s Theories of Elliptic Functions to Alternative Bases. Transactions of the American Mathematical Society. 347, 4163-4244. (1995)
[17]: D. Broadhurst: Solutions by radicals at Singular Values $k_N$ from New Class Invariants for $N \equiv 3 \mod 8$. arXiv:0807.2976 (math-ph).
[18]: J.V. Armitage W.F. Eberlein: Elliptic Functions. Cambridge University Press. (2006)
[19]: N.D. Baruah, B.C. Berndt: Eisenstein series and Ramanujan-type series for $1/\pi$. Ramanujan J.23. (2010) 17-44
[20]: N.D. Baruah, B.C. Berndt: Ramanujan series for $1/\pi$ arising from his cubic and quartic theories of elliptic functions. J. Math. Anal. Appl. 341. (2008) 357-371
[21]: B.C. Berndt: Ramanujan’s theory of Theta-functions. In Theta functions: from the classical to the modern Editor: Maruti Ram Murty, American Mathematical Society. 1993
[22]: J.M. Borwein and P.B. Borwein: A cubic counterpart of Jacobi’s identity and the AGM. Transactions of the American Mathematical Society, 323, No.2, (Feb 1991), 691-701
[23]: Habib Muzaffar and Kenneth S. Williams: Evaluation of Complete Elliptic Integrals of the first kind at Singular Moduli. Taiwanese Journal of Mathematics, Vol. 10, No. 6, pp 1633-1660, December 2006
[24]: Bruce C. Berndt and Aa Ja Yee: Ramanujans Contributions to Eisenstein Series, Especially in his Lost Notebook. (page stored in the Web).
[25]: Nikos Bagis: Eisenstein Series, Alternative Modular Bases and Approximations of $1/\pi$. arXiv:1011.3496 (2010)