Log-Lindley generated family of distributions

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Abstract

A new generator of univariate continuous distributions, with two additional parameters, called the Log-Lindley generated family is introduced. Some special distributions in the new family are presented. Some mathematical properties of the new family are studied. The maximum likelihood method to estimate model parameters is employed. The potentiality of the new generator is illustrated using a real data set.

Keywords: Log-lindley distribution; generated family; moments; maximum likelihood; order statistics.

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1 Introduction

In recent years, several authors have proposed various methods for generating new families of univariate continuous distributions, by adding one or more shape parameters to the baseline distribution, in order to obtain more flexibility for modeling different types of data sets. For example, Eugene et al. (2002) proposed the beta generated (G) family of distributions, Cordeiro and de Castro (2011) proposed the Kumaraswamy-G family, Alexander et al. (2012) proposed the McDonald-G, transformed-transformer (T-X) family by Alzaatreh et al. (2013), Weibull-G by Bourguignon et al. (2014), Lomax-G by Cordeiro et al. (2014), log-gamma-G by Amini et al. (2014), type-1 half-logistic family of distributions by Cordeiro et al. (2015), Lindley family of distributions by Cakmakyapan and Ozel (2016), a new Weibull-G by Tahir et
al. (2016), Topp-Leone-G by Rezaei et al. (2017), Gompertz-G by Alizadeh et al. (2017), odd Lindley-G by Gomes-Silva et al. (2017).

In this paper, we propose a new family of distributions based on the Log-Lindley distribution. This distribution, with bounded domain, introduced by Gomez-Deniz et al. (2014) as an alternative to the beta distribution. They studied the most important properties and gave some nice applications of this model to insurance and inventory management. The log-lindley distribution is a flexible and simple model obtained by a logarithmic transformation of the generalized Lindley random variable defined in Zakerzadeh and Dolati (2009).

The probability density function (pdf) and cumulative distribution function (cdf) are respectively given by

\[ f(x) = \frac{a^2}{1 + ab} x^{a-1} (b - \log x) \]

and

\[ F(x) = \frac{x^a}{1 + ab} (1 + ab - a \log x), \]

for \( 0 < x < 1, \ a > 0 \) and \( b \geq 0 \).

Motivated by the pioneering work of Cordeiro and de Castro (2011), we define the Log-Lindley (“LL for short”) generator with two extra positive parameters \( a \) and \( b \) by the following cdf

\[ F(x; a, b, \theta) = \frac{1}{1 + ab} \left[ 1 + ab - a \log G(x; \theta) \right] G^a(x; \theta), \tag{1} \]

where \( G(x; \theta) \) is the continuous baseline cdf depending on a parameter vector \( \theta \). Then, for an arbitrary parent cdf \( G(x) \), the LL-G family is defined by the cdf (1). The LL generator has the same parameters of the baseline \( G \) distribution plus two additional parameters \( a \) and \( b \). The density function corresponding to LL-G distribution is

\[ f(x; a, b, \theta) = \frac{a^2 g(x; \theta)}{1 + ab} \left[ b - \log G(x; \theta) \right] G^{a-1}(x; \theta), \tag{2} \]

where \( g(x; \theta) \) is the baseline pdf. Hereafter, a random variable \( X \) with pdf (2) will be denoted by \( X \sim LL-G(a, b, \theta) \). Further, we can omit the dependence on the vector \( \theta \) of the parameters and write simply \( G(x) = G(x; \theta) \). The hazard rate function of \( X \) is given by

\[ h(x) = \frac{a^2 g(x; \theta) [b - \log G(x; \theta)] G^{a-1}(x; \theta)}{1 + ab - (1 + ab - a \log G(x; \theta)) G^a(x; \theta)}. \]
The quantile function of the LL-G distribution is obtained, by inverting (1), as
\[ Q(u) = Q_G \left( \exp \left\{ \frac{1 + ab}{a} \right\} \exp \left\{ \frac{1}{a} W_{-1} \left( -\frac{(1 + ab) u}{\exp (1 + ab)} \right) \right\} \right), \]
where \( Q_G \) is the quantile function of the baseline \( G \) distribution and \( W_{-1} \) is the negative branch of the Lambert \( W \) function (i.e., the solution of the equation \( W(z)e^{W(z)} = z \)). For more details on Lambert \( W \) function, see Section 2 of Jodrá (2010). If \( U \) is a uniform random variable on the unit interval \((0, 1)\), then
\[ X = Q_G \left( \exp \left\{ \frac{1 + ab}{a} \right\} \exp \left\{ \frac{1}{a} W_{-1} \left( -\frac{(1 + ab) U}{\exp (1 + ab)} \right) \right\} \right) \]
will be a LL-G random variable.

The rest of the paper is organized as follows. In Section 2, we present some new generated distributions. Shapes of the pdf and hazard rate function are given in Section 3. Some mathematical properties of the LL-G family is derived in Sections 4. Maximum likelihood estimation of the model parameters and the observed information matrix are presented in Section 5. In Section 6, an application of the LL-G family is illustrated using a real data set.. Finally, conclusions are given in Section 7.

## 2 Special LL-G distributions

In this section, we give some special models of the LL-G family of distributions.

### 2.1 The Log-lindley-normal (LL-N) distribution

If \( G(x) = \Phi \left( \frac{(x - \mu)}{\sigma} \right) \) and \( g(x) = \phi \left( \frac{(x - \mu)}{\sigma} \right) \) are the cdf and pdf of the normal \( N(\mu, \sigma^2) \) distribution with parameters \( \mu \) and \( \sigma^2 \), respectively, where \( \phi \) and \( \Phi \) are the pdf and cdf of the standard normal distribution, respectively. Then, the LL-N density function is given by
\[
f(x) = \frac{a^2 \phi \left( \frac{(x - \mu)}{\sigma} \right)}{1 + ab} \left[ b - \log \left\{ \Phi \left( \frac{(x - \mu)}{\sigma} \right) \right\} \right] \left[ \Phi \left( \frac{(x - \mu)}{\sigma} \right) \right]^{a-1}.
\]

### 2.2 The Log-lindley-Weibull (LL-W) distribution

The pdf of the LL-W distribution is defined from (2) by taking \( G(x) = 1 - e^{-ax^\beta} \) and \( g(x) = \alpha \beta x^{\beta-1}e^{-ax^\beta} \) to be the cdf and pdf of the Weibull
distribution respectively. Then, the LL-W density function is given
for \( x > 0 \), by
\[
f(x) = \frac{a^2 \alpha \beta x^{\alpha - 1} e^{-\alpha x^\beta}}{1 + ab} \left[ b - \log \left( 1 - e^{-\alpha x^\beta} \right) \right] \left[ 1 - e^{-\alpha x^\beta} \right]^{\alpha - 1}.
\] (3)

3 Shapes

We can describe analytically the shapes of the density and hazard rate functions. The critical points of the pdf of the LL-G distribution are the roots of the following equation:
\[
\frac{g'(x)}{g(x)} - \frac{g(x)}{G(x) \left[ b - \log \{G(x)\} \right]} + (a - 1) \frac{g(x)}{G(x)} = 0.
\] (4)

If \( x = x_0 \) is a root of (4) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether \( \lambda(x_0) < 0 \), \( \lambda(x_0) > 0 \) or \( \lambda(x_0) = 0 \), where
\[
\lambda(x) = \frac{g(x) g''(x) - [g'(x)]^2}{g^2(x)} + (a - 1) \frac{g'(x) G(x) - g^2(x)}{G^2(x)} - \frac{g'(x) G(x) \left[ b - \log \{G(x)\} \right]^2 - g^2(x) \left( [b - \log \{G(x)\}]^2 - 1 \right)}{G^2(x) \left[ b - \log \{G(x)\} \right]^3}.
\]

The critical points of the hazard rate function of the LL-G distribution are the roots of the following equation:
\[
\frac{g'(x)}{g(x)} - \frac{g(x)}{G(x) \left[ b - \log \{G(x)\} \right]} + (a - 1) \frac{g(x)}{G(x)}
- \frac{g(x) G^{a-1}(x) + a G^{a-1}(x) \left[ 1 + a \left( b - \log \{G(x)\} \right) \right]^2}{(1 + ab - [1 + a \left( b - \log \{G(x)\} \right)] G^a(x) (1 + a \left( b - \log \{G(x)\} \right)))} = 0.
\] (5)

If \( x = x_0 \) is a root of (5) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether \( \kappa(x_0) < 0 \), \( \kappa(x_0) > 0 \).
or $\kappa(x_0) = 0$, where

$$
\kappa(x) = \frac{g''(x) g(x) - [g'(x)]^2}{g^2(x)} + (a - 1) \frac{g'(x) G(x) - g^2(x)}{G^2(x)} \\
- \frac{g'(x) G(x) [b - \log \{G(x)\}] - g(x) \{G(x) [b - \log \{G(x)\}]\}'}{G^2(x) [b - \log \{G(x)\}]^2} \\
- \frac{g'(x) G^{a-1}(x) + (a - 1) g^2(x) G^{a-2}(x)}{(1 + ab - [1 + ab - a \log \{G(x)\}] G^a(x)) (1 + a (b - \log \{G(x)\}))} \\
- \frac{(1 + ab - [1 + ab - a \log \{G(x)\}] G^a(x))^2 (1 + ab - a \log \{G(x)\})}{a^2 g^2(x) G^{2a-2}(x) (b - \log G(x))} \\
+ \frac{a^3 g(x) G^{2a-2} (1 + ab - a \log G(x)) (b - \log G(x))}{(1 + ab - [1 + ab - a \log \{G(x)\}] G^a(x))^2}.
$$

### 4 Mathematical properties

In this section, we present some properties of the LL-G distribution.

If $|z| < 1$ and $\delta > 0$ is a real non-integer, we have the following expansions:

$$
(1 - z)^\delta = \sum_{j=0}^\infty (-1)^j \binom{\delta}{j} z^j,
$$

and

$$
\log (1 - z) = - \sum_{j=0}^\infty \frac{z^{j+1}}{j + 1}.
$$

By making use the previous expansions, we can obtain

$$
f(x) = \sum_{i=0}^\infty w_i h_{i+1}(x),
$$

where

$$
w_i = \sum_{k=l}^\infty \frac{a^2 (-1)^{k+i}}{(1 + ab) (i + 1)} \binom{k}{i} \left[ b \binom{a - 1}{k} + \sum_{j=0}^j \sum_{l=0}^{j+1} (-1)^l \binom{j + 1}{l} \binom{a + l - 1}{k} \right],
$$

5
and \( h_{i+1}(x) = (i + 1)g(x)G^i(x) \) is the pdf of the exponentiated-G (exp-G) distribution with power parameter \( i + 1 \). Equation (6) indicates that the LL-G density function is a linear combination of exp-G density functions. Therefore, some mathematical properties of the new model can be derived from those exp-G properties. For example, the ordinary and incomplete moments and moment-generating function.

The \( r \)th moment of \( X \sim\text{LL-G}(a, b, \theta) \), is

\[
E(X^r) = \sum_{i=0}^{\infty} w_i E(Y_{i+1}^r).
\]

where \( Y_{i+1} \sim\text{exp-G}(i + 1) \). Nadarajah and Kotz (2006) gave the closed-form expressions for moments of several exp-G distributions.

The moment generating function of \( X \), say \( M(t) = E(e^{tX}) \), is given by

\[
M(t) = \sum_{i=0}^{\infty} w_i E(e^{tY_{i+1}}).
\]

The order statistics play an important role in reliability and life testing. Let \( X_{1:n}, \ldots, X_{n:n} \) denote the order statistics obtained from this sample. The pdf of \( X_{k:n} \) is given by

\[
f_{k:n}(x) = \frac{n!}{(n-k)!(k-1)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}, \text{ for } k = 1, \ldots, n.
\]

Since \( 0 < F(x) < 1 \) for \( x > 0 \), by using the expansion (9) of \([1 - F(x)]^{n-k}\), we obtain

\[
f_{k:n}(x) = \frac{n!}{(n-k)!(k-1)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j f(x) [F(x)]^{k+j-1}.
\]

5 Maximum likelihood estimation

Let \( x_1, \ldots, x_n \) be a random sample of size \( n \) from LL-G distributions with unknown \( r \times 1 \) parameter vector \( \xi = (a, b, \theta)^T \). The likelihood function is

\[
L(\xi) = \left( \frac{a^2}{1 + ab} \right)^n \prod_{i=1}^{n} g(x_i) [b - \log G(x_i)] G^{a-1}(x_i).
\]
Then, we obtain the log-likelihood function \( \ell (\xi) \)

\[
\ell (\xi) = 2n \log (a) + n \log (1 + ab) + \sum_{i=1}^{n} \log g (x_i; \theta) \\
+ (a - 1) \sum_{i=1}^{n} \log G (x_i; \theta) + \sum_{i=1}^{n} \log \left[ b - \log G(x_i; \theta) \right].
\]

The components of the score vector are given as follows

\[
U_a (\xi) = 2n a + nb \frac{1}{1 + ab} + \sum_{i=1}^{n} \log G (x_i; \theta),
\]

\[
U_b (\xi) = na \frac{1}{1 + ab} + \sum_{i=1}^{n} \frac{1}{b - \log G(x_i; \theta)},
\]

and

\[
U_\theta (\xi) = \frac{n}{g (x_i; \theta)} \left( \frac{\hat{g} (x_i; \theta)}{g (x_i; \theta)} \right) + (a - 1) \sum_{i=1}^{n} \frac{\hat{G} (x_i; \theta)}{G (x_i; \theta)} + \sum_{i=1}^{n} \frac{\hat{G} (x_i; \theta)}{G(x_i; \theta) (b - \log G(x_i; \theta))},
\]

where \( \hat{g} (x_i; \theta) = \frac{\partial g (x_i; \theta)}{\partial \theta} \) and \( \hat{G} (x_i; \theta) = \frac{\partial G (x_i; \theta)}{\partial \theta} \).

The MLE \( \hat{\xi} = \left( \hat{a}, \hat{b}, \hat{\theta} \right)^T \) of \( \xi = (a, b, \theta)^T \) is obtained by solving the non-linear likelihood equations simultaneously \( U_a (\xi) = 0, U_b (\xi) = 0 \) and \( U_\theta (\xi) = 0 \).

The observed information matrix, for making interval inference, is given by

\[
\begin{pmatrix}
U_{11} & U_{12} & U_{13} \\
U_{21} & U_{22} & U_{23} \\
U_{31} & U_{32} & U_{33}
\end{pmatrix},
\]

where

\[
U_{11} = \frac{2n a^2 - nb^2}{(1 + ab)^2},
\]

\[
U_{12} = \frac{n (1 + ab) - nab}{(1 + ab)^2},
\]

\[
U_{13} = \sum_{i=1}^{n} \frac{G (x_i; \theta)}{G(x_i; \theta)},
\]
$$U_{22} = - \frac{na^2}{(1 + ab)^2} - \sum_{i=1}^{n} \frac{\log G(x_i; \theta)}{(b - \log G(x_i; \theta))^2},$$

$$U_{23} = \sum_{i=1}^{n} \frac{\hat{G}(x_i; \theta)}{G(x_i; \theta)} \left[ b - \log G(x_i; \theta) \right],$$

$$U_{33} = \sum_{i=1}^{n} \frac{\hat{g}(x_i; \theta) g(x_i; \theta) - [\hat{g}(x_i; \theta)]^2}{g^2(x_i; \theta)}$$

$$+ (a - 1) \sum_{i=1}^{n} \frac{\hat{G}(x_i; \theta) G(x_i; \theta) - \left[ \hat{G}(x_i; \theta) \right]^2}{G^2(x_i; \theta)}$$

$$+ \sum_{i=1}^{n} \frac{\hat{G}(x_i; \theta) G(x_i; \theta) (b - \log G(x_i; \theta))}{[G(x_i; \theta) (b - \log G(x_i; \theta))]^2},$$

$$- \sum_{i=1}^{n} \frac{\left[ \hat{G}(x_i; \theta) \right]^2 [b - 1 - \log G(x_i; \theta)]}{[G(x_i; \theta) (b - \log G(x_i; \theta))]^2},$$

where \( \hat{g}(x_i; \theta) = \frac{\partial^2 g(x_i; \theta)}{\partial \theta^2} \) and \( \hat{G}(x_i; \theta) = \frac{\partial^2 G(x_i; \theta)}{\partial \theta^2} \).

### 6 A real data application

In this section, the performance of the proposed family is illustrated by considering one real data set. This data set is given by Bjerkedal (1960) which represents the survival time in days of 72 guinea pigs infected with virulent tubercle bacilli. The data set is: 12, 15, 22, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376. We compare the fit of the LL-W distribution defined in (6), with the beta W (BW), Kumaraswamy-W (KW), McDonald-W (McW), Marshall-Olkin-W (MOW), Weibull-W (WW), Lomax-W (LoW), Lindley-W (LiW), Topp-Leone-W (TW), Gompertz-W (GW), odd Lindley-W (OLW),
where their pdfs are

\[ f_{TW}(x) = 2ab\alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 - e^{-\alpha x^\beta} \right)^{ab-1} \]
\[ \times \left[ 1 - \left( 1 - e^{-\alpha x^\beta} \right)^b \right] \left[ 2 - \left( 1 - e^{-\alpha x^\beta} \right)^b \right]^{a-1}, \]

\[ f_{GW}(x) = a\alpha \beta x^{\beta-1} e^{(1+b)\alpha x^\beta} e^{\frac{b}{2} \left( 1 - e^{-\alpha x^\beta} \right)}, \]

\[ f_{LoW}(x) = ab\alpha \beta \frac{x^{\beta-1}}{b + \alpha x^\beta}, \]

\[ f_{LiW}(x) = ab\alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 + \alpha x^\beta \right), \]

\[ f_{OLW}(x) = \frac{a^2}{(1 + a)^2} \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 + \alpha x^\beta \right), \]

\[ f_{WW}(x) = \frac{a^2}{(1 + a)^2} \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 + \alpha x^\beta \right), \]

\[ f_{MOW}(x) = \frac{a\beta \alpha x^{\beta-1} e^{-\alpha x^\beta}}{\left[ 1 - (1 - a) e^{-\alpha x^\beta} \right]^2}, \]

\[ f_{McW}(x) = \frac{c}{B(a/c, b)} \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 - e^{-\alpha x^\beta} \right)^a \left[ 1 - \left( 1 - e^{-\alpha x^\beta} \right)^b \right]^{b-1}, \]

\[ f_{KW}(x) = ab\alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \left( 1 - e^{-\alpha x^\beta} \right)^a \left[ 1 - \left( 1 - e^{-\alpha x^\beta} \right)^b \right]^{b-1}, \]

\[ f_{BW}(x) = \frac{\alpha \beta x^{\beta-1} e^{-\alpha x^\beta}}{B(a, b)} \left( 1 - e^{-\alpha x^\beta} \right)^{a-1}, \]

We estimate the model parameters of the distributions by the method of maximum likelihood. Table 1 lists the MLEs and the corresponding standard errors in parentheses of the model parameters.

To choose the best model, we use the maximized loglikelihood \((-2 \log(L))\), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC):

\[ \text{AIC} = -2 \log(L) + 2k, \quad \text{BIC} = -2 \log(L) + k \log(n), \]

\[ \text{CAIC} = -2 \log(L) + \frac{2kn}{n - k - 1} \quad \text{and} \quad \text{HQIC} = -2 \log(L) + 2k \log(\log(n)), \]
where \( k \) is the number of the model parameters and \( n \) is the sample size. The better distribution to fit the data corresponds to smaller values of these criteria. Therefore, from Table 2, we conclude that the LL-W distribution gives an excellent fit than the other models.

7 Conclusions

We have introduced a new generated family of distributions, called the LL-G family. We have presented some special cases. We have proved that the LL-G density function is a linear combination of exponentiated distributions which allow us to obtain its properties such as ordinary and incomplete moments, moment-generating function, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and order statistics. The LL-G family parameters are estimated by maximum likelihood. An example to real data illustrate the performance of the proposed family.

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Table 1: MLEs of the model parameters and the corresponding standard errors given in parentheses.

| model   | $\alpha$     | $\beta$     | $a$            | $b$            | $c$            |
|---------|--------------|--------------|----------------|----------------|----------------|
| LLW     | 0.230493     | 0.547514     | 14.024860      | 0.047958       |
|         | (0.312457)   | (0.211727)   | (15.529796)    | (0.078301)     |
| TW      | 0.49220      | 0.37141      | 6.00952        | 4.46868        |
|         | (1.88508)    | (0.46321)    | (35.48929)     | (35.73443)     |
| GW      | 0.0030390    | 0.8408393    | 2.4757193      | 6.9645592      |
|         | (0.0020311)  | (0.1001659)  | (0.9023052)    | (0.0907703)    |
| LoW     | 0.0030609    | 2.0150751    | 1.2104142      | 23.8855555     |
|         | (0.0044717)  | (0.4139031)  | (0.4725888)    | (0.0090636)    |
| LiW     | 1.187295     | 0.988639     | 0.017651       |
|         | (1.309358)   | (0.080269)   | (0.018779)     |
| OLW     | 2.6443973    | 0.1650443    | 0.0068032      |
|         | (1.2154923)  | (0.0426117)  | (0.0103276)    |
| WW      | 0.177315     | 0.535990     | 9.582126       | 1.527191       |
|         | (0.036533)   | (0.103519)   | (0.050421)     | (0.862084)     |
| MOW     | 0.563061     | 0.473979     | 80.404218      |
|         | (0.266324)   | (0.076261)   | (54.900698)    |
| McW     | 0.25188      | 0.63229      | 12.93719       | 0.53561        | 1.44729        |
|         | (0.39718)    | (0.39417)    | (19.05401)     | (0.65902)      | (29.69634)     |
| KW      | 0.951877     | 0.326516     | 49.594511      | 1.518053       |
|         | (0.268110)   | (0.107911)   | (0.025434)     | (1.428337)     |
| BW      | 0.23649      | 0.64716      | 12.15966       | 0.51710        |
|         | (0.34335)    | (0.36361)    | (14.91623)     | (0.58922)      |
| Weibull | 0.0028431    | 1.2587947    |
|         | (0.0020601)  | (0.1406885)  |                |                |
Table 2: The statistics: \(-2\log(L), \text{AIC}, \text{CAIC}, \text{BIC} \text{ and HQIC}\).

| model | \(-2\log(L)\) | AIC  | CAIC | BIC   | HQIC  |
|-------|----------------|------|------|-------|-------|
| LLW   | **779.7472**   | **787.7472** | **788.3442** | **796.8539** | **791.3726** |
| TW    | 780.1929       | 788.1929  | 788.79  | 797.2996  | 791.8183 |
| GW    | 798.7376       | 806.7376  | 807.3346 | 815.8442  | 810.363  |
| LoW   | 783.3026       | 791.3026  | 791.8997 | 800.4093  | 794.928  |
| LiW   | 788.9608       | 794.9608  | 795.3138 | 801.7908  | 797.6799 |
| OLW   | 796.4631       | 802.4631  | 802.5624 | 809.0394  | 804.9285 |
| WW    | 780.3174       | 788.3174  | 788.9145 | 797.4241  | 791.9428 |
| MOW   | 792.0679       | 798.0679  | 798.4209 | 804.8979  | 800.787  |
| McW   | 780.0641       | 790.0641  | 790.9732 | 801.4474  | 794.5958 |
| KW    | 780.2858       | 788.2858  | 788.8829 | 797.3925  | 791.9112 |
| BW    | 780.0632       | 788.0632  | 788.6602 | 797.1698  | 791.6886 |
| Weibull| **795.6583**   | **799.6583** | **799.8322** | **804.2116** | **801.471** |