Virtual Photon Correction to the $K^+ \rightarrow \pi^+\pi^0\pi^0$ Decay

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Abstract

We consider electromagnetic corrections to the non-leptonic kaon decay, $K^+ \rightarrow \pi^+\pi^0\pi^0$, due to explicit virtual photons only. The decay amplitude is calculated at one-loop level in the framework of Chiral Perturbation Theory. The interest in this process is twofold: It is actually measured by the NA48 collaboration from one side, and, the value of the amplitude at the $\pi\pi$ threshold gives access to $\pi\pi$ scattering lengths from the other side. We found that the present correction is about 5 to 6% the value of the Born amplitude squared. Combined with another piece published recently, this fixes the size of isospin breaking correction to the amplitude squared to 7% its one-loop level value in the absence of isospin breaking and at the center of Dalitz plot.

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I. INTRODUCTION

Starting May, Nicola Cabibbo submitted to the arxiv a letter proposing a “potentially accurate” method for the determination of $\pi\pi$ scattering length $a_0^0 - a_0^2$ from the $\pi^0\pi^0$ spectrum in $K^+ \to \pi^+\pi^0\pi^0$ near the $\pi^+\pi^-$ threshold [1]. Next day, Hans Bijnens and Fredrik Borg submitted a paper treating isospin breaking in $K \to 3\pi$ decays in the framework of chiral perturbation theory [2]. In their study, the authors of Ref. [2] took into account strong isospin breaking and local electromagnetic corrections. The former is proportional to the up and down quark mass difference $m_u - m_d$. With respect to local electromagnetic correction, it includes the effects of mass square difference between charged and neutral pions $M_{\pi^\pm}^2 - M_{\pi^0}^2$ and electroweak counterterms.

Let us justify the necessity of the present work. The NA48 collaboration investigates actually the $CP$ violating asymmetry in $K^\pm \to \pi^\pm\pi^0\pi^0$ decays. So, there is no need in principle to upgrade the experimental setup in order to study the $\pi^0\pi^0$ spectrum near the $\pi^+\pi^-$ threshold. From the theoretical point of view, the $\pi\pi$ scattering length $a_0^0 - a_0^2$ is very sensitive to the value of the quark condensate. Hence, an accurate measurement of this quantity would allow a best understanding of the quantum chromodynamics vacuum structure.

The most convenient theoretical framework to study non-leptonic weak kaon decays is chiral perturbation theory extended to include non-leptonic weak interactions [3]. The effective Lagrangian constructed in Ref. [3] has been used in [4, 5, 6] to calculate $K \to 3\pi$ decays to next-to-leading chiral order. The theoretical prediction did not match the experimental data. Naturally, the need to push the chiral expansion up to higher orders is imperative. In a parallel direction, one has to improve the theoretical prediction by evaluating the size of isospin breaking corrections. To do so, electromagnetism should be incorporated appropriately in the effective Lagrangian formalism [7].

We shall distinguish between two types of virtual photons: hard and soft. Hard photons are integrated out from the effective theory leaving local electromagnetic interactions weighted by order parameters. The latter are counterterms in the sense that they absorb ultraviolet divergences produced by soft virtual photon loops. The isospin breaking corrections calculated in Ref. [2] are due to $m_u - m_d$ and to hard virtual photons. Obviously, they are ultraviolet divergent. Moreover, their size is about 10% the value of the one-loop level
squared amplitude in the absence of isospin breaking. The authors of Ref. [2] concluded that the correction they obtained is not “quite enough to solve the discrepancies” between theory and experiment and that the effects of soft virtual photons should be included. The aim of the present work is to evaluate these effects.

**II. KINEMATICS**

The decay process

\[ K^+(k) \rightarrow \pi^0(p_1)\pi^0(p_2)\pi^+(p_3) \]  \hspace{1cm} (1)

is studied in terms of three scalars

\[ s_i \doteq (k - p_i)^2, \quad i = 1, 2, 3, \]  \hspace{1cm} (2)

satisfying the on-shell condition

\[ \sum_i s_i = M_{K^\pm}^2 + M_{\pi^\pm}^2 + 2M_{\pi^0}^2. \]  \hspace{1cm} (3)

The differential decay rate is defined by

\[ d\Gamma \doteq \frac{1}{4M_{K^\pm}} |\mathcal{M}|^2 d\Phi, \]  \hspace{1cm} (4)

with the differential phase space

\[ d\Phi \doteq (2\pi)^4\delta^4\left(\sum_i p_i - k\right) \prod_i \frac{d^3p_i}{(2\pi)^32p_i^0}, \]  \hspace{1cm} (5)

and the decay amplitude \( \mathcal{M} \). The spectrum of the phase space with respect to \( s_3 \) reads

\[ \frac{d\Phi}{ds_3} = \frac{1}{2^7\pi^3} \frac{\lambda^{1/2}(s_3,M_{\pi^0}^2,M_{\pi^0}^2)}{s_3} \lambda^{1/2}(s_3,M_{\pi^\pm}^2,M_{K^\pm}^2) \frac{M_{K^\pm}^2}{M_{\pi^\pm}^2}, \]  \hspace{1cm} (6)

where \( \lambda \) stands for the Källén function

\[ \lambda(x,y,z) \doteq x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \]  \hspace{1cm} (7)

The physical region is bounded by

\[ (M_{\pi^\pm} + M_{\pi^0})^2 \leq s_1, s_2 \leq (M_{K^\pm} - M_{\pi^0})^2, \quad 4M_{\pi^0}^2 \leq s_3 \leq (M_{K^\pm} - M_{\pi^\pm})^2. \]  \hspace{1cm} (8)

In what follows, we set \( M_{K^\pm} = M_K \) and \( M_{\pi^\pm} = M_{\pi^0} = M_\pi \). This is safe since \( e^2M_{\pi^\pm}^2 = e^2M_\pi^2 + O(e^4) \).
III. THE DECAY AMPLITUDE

The decay $^1$ can proceed locally. It can also be mediated by pion or kaon multi-poles. Let us then write the decay amplitude at one-loop level as $^9$:

\[
\mathcal{M} = \mathcal{T} + \frac{\hat{Z}}{M_K^2 - M_\pi^2} + \frac{\hat{Z}}{(M_K^2 - M_\pi^2)^2}.
\]  

(9)

We follow notations of Ref. $^2$ for the strangness-changing Lagrangian parameters. The decay amplitude takes then the following form:

\[
X = CG_8X^{(8)} + CG_8'X^{(8')}, \quad X = \mathcal{M}, \mathcal{T}, \hat{Z}, \hat{Z}.
\]  

(10)

Moreover, the subscripts 0 and 1 will be assigned to Born amplitudes and one-loop explicit virtual photon corrections, respectively.

A. Born amplitude

The lowest order in the perturbative expansion of the amplitude comprises five diagrams as sketched in Fig. 1. We present the result in terms of scalar products avoiding any ambiguity

\[
\mathcal{T}_0^{(8)} = \frac{1}{3}(k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - p_1 \cdot p_3 - p_2 \cdot p_3),
\]  

(11)

\[
\mathcal{T}_0^{(8')} = \frac{1}{3}M_K^2,
\]  

(12)

\[
\mathcal{T}_0^{(27)} = \frac{1}{9}(2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3 - 30p_1 \cdot p_2 + 13p_1 \cdot p_3 + 13p_2 \cdot p_3),
\]  

(13)

\[
\hat{Z}_0^{(8)} = -\frac{1}{3}(M_K^2 - M_\pi^2) \times (M_\pi^2 + 2k \cdot p_1 + 2k \cdot p_2 - 4k \cdot p_3 + 4p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_2 \cdot p_3),
\]  

(14)

\[
\hat{Z}_0^{(8')} = -\frac{1}{3}(M_K^2 - M_\pi^2)M_K^2,
\]  

(15)

\[
\hat{Z}_0^{(27)} = -\frac{1}{18}[4M_\pi^2(M_K^2 - M_\pi^2)
\]  

\[+(8M_K^2 + 7M_\pi^2)(k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)].
\]  

(16)

Note that $\mathcal{M}_0^{(8')}$ vanishes if we add all contributions.
B. Virtual photons

In this section, we shall calculate one-loop diagrams with a virtual photon exchanged between two meson legs, one meson leg and a vertex, and two vertices. The various topologies are depicted in Figs. (2) - (9). The loop functions figuring in the expressions are tabulated in the appendix. A fictitious mass \( m_\gamma \) is attributed to the photon in order to regularize infrared divergent integrals. For completeness, we calculate the amplitudes proportional to \( G_8 \). Only those proportional to \( G_8 \) and \( G_{27} \) will be taken into account numerically. We shall give the contribution of each figure separately.

Figure 2

We have

\[
\mathcal{T}_{1}^{(8)} = \frac{e^2}{3} \left\{ 2A_0(M_\pi) + 2A_0(M_K) \right. \\
& -2(4\pi)^{-2}(k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - p_1 \cdot p_3 - p_2 \cdot p_3)[1 + \ln(m_\gamma^2) - \ln(M_\pi M_K)] \\
& + (8M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-p_3, 0, M_\pi) \\
& + 2(4M_\pi^2 - k \cdot p_1 - k \cdot p_2 + 4k \cdot p_3 + p_1 \cdot p_3 + p_2 \cdot p_3)B_1(-p_3, 0, M_\pi) \\
& + (8M_K^2 + k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-k, 0, M_K) \\
& + 2(4M_K^2 - k \cdot p_1 - k \cdot p_2 + 4k \cdot p_3 + p_1 \cdot p_3 + p_2 \cdot p_3)B_1(-k, 0, M_K) \\
& - (k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(p_3 - k, M_\pi, M_K) \\
& + 4(k \cdot p_3)(k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 \cdot p_3)C_0(-p_3, -k, m_\gamma, M_\pi, M_K) \right\}, \\
\mathcal{T}_{1}^{(27)} = \frac{e^2}{9} \left\{ 4A_0(M_\pi) + 4A_0(M_K) \right. \\
& -2(4\pi)^{-2}(2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3 \cdot p_3) \right. \\
& + 2(4\pi)^{-2}(2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3) \right. \\
& + (16M_\pi^2 + 2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3) \right. \\
& - (30p_1 \cdot p_2 + 13p_1 \cdot p_3 + 13p_2 \cdot p_3)[1 + \ln(m_\gamma^2) - \ln(M_\pi M_K)] \\
& + (16M_K^2 + 2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3) \right. \\
& - 30p_1 \cdot p_2 - 17p_1 \cdot p_3 - 17p_2 \cdot p_3)B_0(-p_3, 0, M_\pi) \\
& + 4(k \cdot p_3)(k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 \cdot p_3)C_0(-p_3, -k, m_\gamma, M_\pi, M_K) \right\},
\]

(17)
\[+2(8M_\pi^2 - 2k \cdot p_1 - 2k \cdot p_2 + 8k \cdot p_3 + 30p_1 \cdot p_2 - 13p_1 \cdot p_3 - 13p_2 \cdot p_3)B_1(-p_3, 0, M_\pi)\]
\[+(16M_K^2 - 28k \cdot p_1 - 28k \cdot p_2 - 8k \cdot p_3)\]
\[-30p_1 \cdot p_2 + 13p_1 \cdot p_3 + 13p_2 \cdot p_3)B_0(-k, 0, M_K)\]
\[+2(8M_K^2 - 2k \cdot p_1 - 2k \cdot p_2 + 8k \cdot p_3)\]
\[+30p_1 \cdot p_2 - 13p_1 \cdot p_3 - 13p_2 \cdot p_3)B_1(-k, 0, M_K)\]
\[-2(k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 - 15p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(p_3 - k, M_\pi, M_K)\]
\[-15(k \cdot p_1 + k \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_1(p_3 - k, M_\pi, M_K)\]
\[+4(k \cdot p_3)(2k \cdot p_1 + 2k \cdot p_2 - 8k \cdot p_3)\]
\[-30p_1 \cdot p_2 + 13p_1 \cdot p_3 + 13p_2 \cdot p_3)C_0(-p_3, -k, m_\gamma, M_\pi, M_K)\]
\[+60(k \cdot p_3)(p_1 \cdot p_3 + p_2 \cdot p_3)C_1(-p_3, -k, 0, M_\pi, M_K)\]
\[+60(k \cdot p_3)(k \cdot p_1 + k \cdot p_2)C_2(-p_3, -k, 0, M_\pi, M_K)\}

(19)

**Figure 3**

We have

\[\mathcal{T}_1^{(8)} = \frac{3}{2} \mathcal{T}_1^{(27)}\]

\[= -\frac{2e^2}{3} \{A_0(M_\pi) - A_0(M_K) - M_K^2 B_0(-p_3, 0, M_\pi)\]
\[-M_\pi^2 B_0(0, M_\pi, M_K) + M_\pi^2 B_0(k - p_3, M_\pi, M_K)\]
\[+(4M_K^2 + 3M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)\]
\[+2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-k, 0, M_K)\]
\[-(3M_K^2 + 3M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)\]
\[+2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-k, 0, M_\pi)\]
\[-(M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)\]
\[+2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(p_3 - k, M_\pi, M_K)\]
\[+(M_K^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)\]
\[+2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(p_3 - k, M_\pi, M_K)\]
\[+(M_\pi^2 + 4k \cdot p_3)(M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)\]
\[ T_1^{(s')} = \frac{2e^2}{3} M_K^2 \{ B_0(k - p_3, M_\pi, M_\pi) - B_0(0, M_\pi, M_K) \\
- B_0(-k, 0, M_\pi) + 2B_0(-k, 0, M_K) - B_0(-p_3, 0, M_\pi) \\
+ (M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3) \\
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(-p_3, -k, m_\gamma, M_\pi, M_K) \\
+ (M_K^2 - M_\pi^2 + 3M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3) \\
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(-p_3, -k, 0, M_\pi, M_\pi) \\
- M_\pi^2(M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3) \\
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(0, k - p_3, M_K, M_\pi, M_\pi) \\
- M_\pi^2(M_K^2 - M_\pi^2 + 2k \cdot p_3)(M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3) \\
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)D_0(-k, -p_3, -k, m_\gamma, M_K, M_\pi, M_\pi) \} . \]  

(22)

**Figure 4**

We have

\[ \hat{Z}_1^{(s')} = -\hat{Z}_1^{(s')} = \frac{3}{2} \hat{Z}_1^{(27)} \]

\[ = -\frac{2e^2}{3} M_K^2 \{-2A_0(M_\pi) \\
+ (M_K^2 + 4M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3)B_0(-p_3, 0, M_\pi) \} \]  

(23)
\[ + (3M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \]
\[ + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) B_0 (-k, 0, M_\pi) \]
\[ - (M_K^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \]
\[ + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) B_0 (p_3 - k, M_\pi, M_\pi) \]
\[ - (M_K^2 - M_\pi^2 - 4k \cdot p_3 ) (M_K^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \]
\[ + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) C_0 (-p_3, -k, 0, M_\pi, M_\pi) \} . \tag{24} \]

**Figure 5**

We have

\[ \tilde{Z}_1^{(8)} = \frac{3}{2} \tilde{Z}_1^{(27)} \tag{25} \]
\[ = - \frac{2e^2}{3} (M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) \times \]
\[ \{ - 2M_K^2 (4\pi)^{-2} [1 + \ln(m_\pi^2) - \ln(M_\pi M_K)] \]
\[ + A_0 (M_\pi) - M_\pi^2 B_0 (0, M_\pi, M_K) \]
\[ + M_\pi^2 B_0 (-k, 0, M_K) - 2M_K^2 B_1 (-k, 0, M_K) \]
\[ - 2M_K^2 B_1 (-p_3, 0, M_\pi) - (2M_K^2 + M_\pi^2 ) B_0 (-k, 0, M_\pi) \]
\[ + M_\pi^2 (3M_K^2 + M_\pi^2 ) C_0 (-k, -k, 0, M_\pi, M_K) \} , \tag{26} \]

\[ \tilde{Z}_1^{(8')} = \frac{2e^2}{3} M_K^2 (M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) \times \]
\[ \{ - 2(4\pi)^{-2} [1 + \ln(m_\pi^2) - \ln(M_\pi M_K)] \]
\[ - 2B_1 (-p_3, 0, M_\pi) - B_0 (0, M_\pi, M_K) + B_0 (-k, 0, M_\pi) \]
\[ + B_0 (-k, 0, M_K) - 2B_1 (-k, 0, M_K) \]
\[ + (3M_K^2 + M_\pi^2 ) C_0 (-k, -k, 0, M_\pi, M_K) \} , \tag{27} \]

\[ \tilde{Z}_1^{(8)} = - \tilde{Z}_1^{(8')} = \frac{3}{2} \tilde{Z}_1^{(27)} \tag{28} \]
\[ = - \frac{2e^2}{3} M_K^2 (M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3 ) \times \]
\[ [ - A_0 (M_\pi) + 2(M_K^2 + M_\pi^2 ) B_0 (-k, 0, M_\pi) ] \]. \tag{29} \]
We have

\[ \mathcal{T}_1^{(8)} = \frac{3}{2} \mathcal{T}_1^{(27)} \]

\[ = -\frac{e^2}{6} \left\{ -A_0(M_\pi) + A_0(M_K) - M_\pi^2 B_0(-k, 0, M_K) 
+ (M_K^2 + 6M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(-p_3, 0, M_\pi) 
- (M_K^2 + 5M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(-p_3, 0, M_K) 
+ M_K^2 B_0(p_3 - k, M_K, M_K) - M_K^2 B_0(0, M_\pi, M_K) 
- (2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(p_3 - k, M_\pi, M_K) 
+ (M_K^2 + 3M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(k - p_3, M_K, M_K) 
+ (M_K^2 + 4k \cdot p_3)(2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) C_0(-p_3, -k, M_\gamma, M_\pi, M_K) 
+ M_K^2 (M_K^2 + 7M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) C_0(-p_3, -p_3, 0, M_\pi, M_K) 
- (M_K^2 - M_\pi^2 + 4k \cdot p_3)(2M_K^2 + 3M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) C_0(-p_3, -k, 0, M_K, M_K) 
- M_K^2 (2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) C_0(0, p_3 - k, M_\pi, M_K, M_K) 
+ M_K^2 (M_K^2 - M_\pi^2 + 4k \cdot p_3)(2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) D_0(-p_3, -k, -p_3, m_\gamma, M_\pi, M_K, M_K) \right\} , \]

\[ \mathcal{T}_1^{(8')} = \frac{e^2}{6} M_K^2 \left\{ -B_0(-k, 0, M_K) - B_0(-p_3, 0, M_\pi) 
+ B_0(p_3 - k, M_K, M_K) + 2B_0(-p_3, 0, M_K) - B_0(0, M_\pi, M_K) 
+ (2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 
+ 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) C_0(-p_3, -k, M_\pi, M_K) \right\} . \]
\begin{align*}
+ & (M_K^2 + 7M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(-p_3, -p_3, 0, M_{\pi}, M_K) \\
- & (2M_K^2 + 2M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(0, p_3 - k, M_{\pi}, M_K, M_K) \\
- & (M_K^2 - M_{\pi}^2 + 4k \cdot p_3)C_0(-p_3, -k, 0, M_K, M_K) \\
+ & (M_K^2 - M_{\pi}^2 + 4k \cdot p_3)(2M_K^2 + 2M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)D_0(-p_3, -k, -p_3, M_K, M_K) \} . 
\end{align*}

\textit{Figure 7}

We have

\begin{align*}
\hat{Z}_1^{(8)} &= -\frac{M_{\pi}^2}{M_K^2} \hat{Z}_1^{(8')} = \frac{3}{2} \hat{Z}_1^{(27)} \\
&= \frac{e^2}{6} \left\{ -2A_0(M_K) \\
+ & (5M_K^2 + 3M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-k, 0, M_K) \\
+ & (3M_K^2 + 5M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(-p_3, 0, M_K) \\
- & (M_K^2 + 3M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)B_0(p_3 - k, M_K, M_K) \\
+ & (M_K^2 - M_{\pi}^2 + 4k \cdot p_3)(M_K^2 + 3M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \\
+ & 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3)C_0(-p_3, -k, 0, M_K, M_K) \} . 
\end{align*}

\textit{Figure 8}

We have

\begin{align*}
\hat{Z}_1^{(8)} &= \frac{3}{2} \hat{Z}_1^{(27)} \\
&= \frac{e^2}{6} (2M_K^2 + 2M_{\pi}^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) \times \\
& \{ -2M_{\pi}^2 (4\pi)^{-2} \left[ 1 + \ln(m_{\pi}^2) - \ln(M_{\pi}M_K) \right] \}
\end{align*}
\[\begin{align*}
+ A_0(M_K) - 2M_\pi^2 B_0(-p_3, 0, M_K) - M_\pi^2 B_1(0, M_\pi, M_K) \\
- 2M_\pi^2 B_1(-p_3, 0, M_\pi) - 2M_\pi^2 B_1(-k, 0, M_K) \\
+ 4M_\pi^2 M_K^2 C_0(-p_3, -p_3, 0, M_\pi, M_K) \right) , \\
\end{align*}\]

\[\begin{align*}
\dot{Z}_1^{(s')} &= -\frac{e^2}{6} M_K^2 (2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) \\
&\quad \times \left\{ -2(4\pi)^{-2} \left[ 1 + \ln(m_\gamma^2) - \ln(M_\pi M_K) \right] \\
&\quad + 2B_0(-p_3, 0, M_K) - B_0(0, M_\pi, M_K) - 2B_1(-p_3, 0, M_\pi) \\
&\quad - 2B_1(-k, 0, M_K) + 4M_\pi^2 C_0(-p_3, -p_3, 0, M_\pi, M_K) \right\} , \\
\end{align*}\]

\[\begin{align*}
\ddot{Z}_1^{(s)} &= -\frac{M_\pi^2}{M_K^2} \frac{\dot{Z}_1^{(s')}}{3} = \frac{3}{2} \ddot{Z}_1^{(27)} \\
&= -\frac{e^2}{6} M_\pi^2 (2M_K^2 + 2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) \\
&\quad \times \left[ -A_0(M_K) + 2(M_K^2 + M_\pi^2) B_0(-p_3, 0, M_K) \right] .
\end{align*}\]

Figure 9

We have

\[\begin{align*}
\dot{Z}_1^{(s)} &= -\frac{M_\pi^2}{M_K^2} \frac{\dot{Z}_1^{(s')}}{3} = -\dot{Z}_1^{(27)} \\
&= \frac{e^2}{2} M_\pi^2 \left\{ -2A_0(M_K) \\
&\quad -2(k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) \times \\
&\quad (4\pi)^{-2} \left[ 1 + \ln(m_\gamma^2) - \ln(M_\pi M_K) \right] \\
&\quad + (8M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 10k \cdot p_3 \\
&\quad + 2p_1 \cdot p_2 + 3p_1 \cdot p_3 + 3p_2 \cdot p_3) B_0(-p_3, 0, M_\pi) \\
&\quad -2(k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_1(-p_3, 0, M_\pi) \\
&\quad + (8M_K^2 - 3k \cdot p_1 - 3k \cdot p_2 - 2k \cdot p_3 \\
&\quad + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(-k, 0, M_K) \\
&\quad - 2(k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_1(-k, 0, M_K) \\
&\quad - (2M_\pi^2 + k \cdot p_1 + k \cdot p_2 - 4k \cdot p_3 \\
&\quad + 2p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) B_0(p_3 - k, M_\pi, M_K) \\
&\quad + 2(-k \cdot p_1 - k \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3) B_1(p_3 - k, M_\pi, M_K) \\
&\quad + 4(k \cdot p_3)(k \cdot p_1 + k \cdot p_2 - 2k \cdot p_3 \right\} .
\end{align*}\]
We have
\[ T^{(8)}_1 = \frac{3}{2} T^{(27)}_1 = -\frac{3}{2} (M_K^2 - M_\pi^2)^{-1} \tau_1^{(8)} = -\frac{3}{2} (M_K^2 - M_\pi^2)^{-1} \tau^{(27)}_1 = 0. \] (42)

This result is somewhat expected since the neutral pion is electrically neutral.

Our amplitude is ultraviolet divergent. So is the amplitude calculated in Ref. [2]. Only the sum of these is ultraviolet finite. Our amplitude is also infrared divergent. We shall show in the following section that the infrared divergence cancels the one coming from real soft photon emission. The cancelation occurs at the level of differential decay rates. Since we are interested in evaluating the amplitude, we have to subtract infrared divergence. The subtraction scheme is not unique. We will adopt a minimal subtraction scheme consisting on dropping out only \( \ln(m_\gamma^2) \) terms from the expression of the amplitude.

### IV. INFRARED DIVERGENCE

The infrared divergence of the amplitude reads:
\[ M_{1IR} = -\frac{e^2}{8\pi^2} M_0 \tau (-p_3, -k, M_\pi, M_K) \ln(m_\gamma^2). \] (43)

The divergence is canceled by the one generated from the emission of a real soft photon. The energy \( q \) of the latter being raised by the detector resolution \( \omega \). The cancelation takes place as usually at the differential decay rate level. Let \( M_\gamma \) denotes the decay amplitude corresponding to the emission of one real photon. We shall focus only on the infrared divergent part of \( M_\gamma \). The Feynman diagrams one needs are drawn in Fig. 11. The result is standard:
\[ M_\gamma = -e M_0 \left( \frac{k \cdot \varepsilon^*}{k \cdot q} - \frac{p_3 \cdot \varepsilon^*}{p_3 \cdot q} \right). \] (44)

One obtains after squaring the amplitude and summing over photon polarizations
\[ |M_\gamma|^2 = -e^2 |M_0|^2 \left[ \frac{M_K^2}{(k \cdot q)^2} - \frac{2(k \cdot p_3)}{(k \cdot q)(p_3 \cdot q)} + \frac{M_\pi^2}{(p_3 \cdot q)^2} \right]. \] (45)
We shall now explicitly show the infrared divergence cancelation. The infrared divergence coming from virtual photons is deduced from Eqs. (4) and (43)

\[ d\Gamma_{IR} = -\frac{e^2}{4\pi^2} d\Gamma_0 \left[ 1 - (k \cdot p_3) \tau (-p_3, -k, M_\pi, M_K) \right] \ln(m_\gamma^2) . \] (46)

From Eq. (45), the differential decay rate corresponding to the emission of one real photon reads in the soft photon approximation

\[ d\Gamma_\gamma = -e^2 d\Gamma_0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q} \left[ \frac{M_K^2}{(k \cdot q)^2} - \frac{2(k \cdot p_3)}{(k \cdot q)(p_3 \cdot q)} + \frac{M_\pi^2}{(p_3 \cdot q)^2} \right] . \] (47)

One can then easily check that

\[ d\Gamma_{IR} + d\Gamma_\gamma = 0 . \] (48)

V. RESULTS

In this section we shall evaluate isospin breaking correction due to explicit virtual photons. This correction depends only on \( s_3 \) thanks to the on-shell relation (3). It is natural then to study the variation of the correction with respect to \( s_3 \) in the physical region defined by Eq. (8). In order to evaluate the size of the correction it is convenient to compare it to the tree-level value or to the one-loop level correction in the absence of isospin breaking. Since the latter depends on two kinematical variables it is easier (but not less instructive) to compare it to the former. This is done in Fig. (12) where we plotted the Born amplitude squared \( |M|^2 \) (dashed curve) and the one-loop level amplitude squared \( |M_0|^2 + 2 R (M_0 M_1) \) (plain curve). We recall that \( M_1 \) stands for the explicit virtual photon correction. In order to draw Fig. (12) we used the following numerical input:

\[ e^2 = (4\pi)/(137.036) , \quad M_\pi = 0.134977 \text{ GeV} , \quad M_K = 0.495042 \text{ GeV} , \]
\[ C = -1.07 \times 10^{-6} \text{ GeV}^{-2} , \quad G_8 = 5.45 , \quad G_{27} = 0.392 . \] (49)

Note also that the ultraviolet divergent terms (\( \lambda \) terms) have been dropped out and the scale \( \mu \) was taken to be 0.770 GeV. Let us comment the content of Fig. (12). To this end, we shall write the amplitude squared as

\[ |M|^2 = |M_0|^2 [1 + \delta(s_3)] . \] (50)

The ratio \( \delta \) is given in Tab. (I) for different values of \( s_3 \). As can be seen from Tab. (I), the explicit virtual photon correction is about 5 to 6\% the value of the Born amplitude.
VI. CONCLUSION

In this work we calculated isospin breaking corrections to the process $K^+ \rightarrow \pi^+ \pi^0 \pi^0$ due to virtual soft photons at one-loop level and in the framework of chiral perturbation theory. The corrections generated by $m_u - m_d$ and virtual hard photons have been calculated in Ref. [2]. The corrections are individually ultraviolet divergent but jointly ultraviolet finite. They are also infrared divergent. We showed that this divergence is canceled at the differential decay level by the one coming from real soft photons. We follow Ref. [2] and denote by $A_{00+}(0, 0)$ the decay amplitude for the process in question evaluated at the center of Dalitz plot. Then the amplitude squared reads:

$$|A_{00+}(0, 0)|^2 = 2.49 \times 10^{-13} + \frac{6.84 \times 10^{-13}}{O(\rho^2)} + \frac{7.7 \times 10^{-14} + -1.3 \times 10^{-14}}{O(\rho^4)},$$

(51)

In the first line of the equation we reported the value of the amplitude squared at leading and next-to-leading chiral orders and in the absence of isospin breaking [5]. The first term in the second line represents isospin breaking corrections due to $m_u - m_d$ and to hard virtual photons [2]. The second term has been calculated in the present work and corresponds to the isospin breaking correction due to soft virtual photons. Adding all this together, isospin breaking correction to the amplitude squared represents 7% its one-loop value in the absence of isospin breaking and at the center of Dalitz plot.

APPENDIX A: LOOP INTEGRALS

We use dimensional regularization and adopt the \( \overline{\text{MS}} \) subtraction scheme

$$\overline{\chi} = -\frac{1}{32\pi^2} \left[ \frac{2}{4-n} + 1 - \gamma + \ln(4\pi) \right],$$

(A1)

where $n$ is space-time dimension and $\gamma$ the Euler constant. All the technical material necessary for the calculation of one-loop integrals is given in the appendix of Ref. [8].

It is convenient to take the following notations:

$$\sigma_P = \sqrt{1 - \frac{4M_P^2}{s_3}},$$

(A2)
\[ \sigma_{PP} = \frac{\sigma_P - 1}{\sigma_P + 1}, \]  
\( A \) integrals

The one-point function reads:
\[ A_0(m) = m^2 \left[ -2\lambda - \frac{1}{16\pi^2} \ln \left( \frac{m^2}{\mu^2} \right) \right], \]  
where \( \mu \) an arbitrary scale with mass dimension.

\[ B \] integrals

We need the following two-point functions:
\[ B_0(-p_3, 0, M_\pi) = \frac{A_0(M_\pi)}{M_\pi^2} + \frac{1}{16\pi^2}, \]  
\[ B_1(-p_3, 0, M_\pi) = -\frac{1}{2} \frac{A_0(M_\pi)}{M_\pi^2}, \]  
\[ B_0(-k, 0, M_K) = \frac{A_0(M_K)}{M_K^2} + \frac{1}{16\pi^2}, \]  
\[ B_1(-k, 0, M_K) = -\frac{1}{2} \frac{A_0(M_K)}{M_K^2}, \]  
\[ B_0(0, M_\pi, M_K) = \frac{A_0(M_\pi)}{M_\pi^2} + \frac{1}{16\pi^2} \frac{M_K^2}{M_K^2 - M_\pi^2} \ln \left( \frac{M_K^2}{M_\pi^2} \right), \]  
\[ B_0(-p_3, 0, M_K) = \frac{A_0(M_K)}{M_K^2} + \frac{1}{16\pi^2} \left[ 1 - \left( 1 - \frac{M_K^2}{M_\pi^2} \right) \ln \left( 1 - \frac{M_\pi^2}{M_K^2} \right) \right], \]  
\[ \Re B_0(-k, 0, M_\pi) = \frac{A_0(M_\pi)}{M_\pi^2} + \frac{1}{16\pi^2} \left[ 1 - \left( 1 - \frac{M_\pi^2}{M_K^2} \right) \ln \left( \frac{M_K^2}{M_\pi^2} - 1 \right) \right], \]  
\[ \Im B_0(-k, 0, M_\pi) = \frac{1}{16\pi} \left( 1 - \frac{M_\pi^2}{M_K^2} \right), \]  
\[ \Re B_0(k - p_3, M_\pi, M_\pi) = \frac{A_0(M_\pi)}{M_\pi^2} + \frac{1}{16\pi^2} [1 + \sigma_\pi \ln(-\sigma_{\pi\pi})], \]  
\[ \Im B_0(k - p_3, M_\pi, M_\pi) = \frac{\sigma_\pi}{16\pi}, \]  
\[ B_0(p_3 - k, M_K, M_K) = \frac{A_0(M_K)}{M_K^2}. \]
\[
B_0(p_3 - k, M_\pi, M_K) = \frac{1}{2} A_0(M_K) + \frac{1}{2} A_0(M_\pi) + \frac{1}{16\pi^2} \left[ 1 + \frac{M_K^2 - M_\pi^2}{s_3} \ln \left( \frac{M_\pi}{M_K} \right) \right] - \frac{1}{s_3} \lambda^{1/2}(s_3, M_\pi^2, M_K^2) \ln(\sigma_{K\pi}) \], \quad (A18)
\]

\[
B_1(p_3 - k, M_\pi, M_K) = \frac{1}{2} s_3 \left[ A_0(M_\pi) - A_0(M_K) - (s_3 - M_K^2 + M_\pi^2) B_0(p_3 - k, M_\pi, M_K) \right]. \quad (A19)
\]

### \(\tau\) integrals

The definition and expression of \(\tau\) integrals can be found in Ref. \[8\]. Note the particular expression

\[
\tau(-p_3, -k, M_\pi, M_K) = -2\lambda^{-1/2}(s_3, M_\pi^2, M_K^2) \ln(\sigma_{K\pi}). \quad (A20)
\]

### \(C\) integrals

The three-point functions needed for our purposes are:

\[
C_0(-p_3, -p_3, 0, M_\pi, M_K) = \\
\frac{1}{16\pi^2} \left[ \frac{1}{M_\pi^2} \ln \left( 1 - \frac{M_\pi^2}{M_K^2} \right) + \frac{1}{M_K^2 - M_\pi^2} \ln \left( \frac{M_\pi^2}{M_K^2} \right) \right], \quad (A21)
\]

\[
\Re C_0(-k, -k, 0, M_\pi, M_K) = \\
-\frac{1}{16\pi^2 M_K^2} \left[ \ln \left( \frac{M_\pi^2}{M_K^2 - M_\pi^2} \right) - \frac{M_K^2}{M_K^2 - M_\pi^2} \ln \left( \frac{M_\pi^2}{M_K^2} \right) \right], \quad (A22)
\]

\[
\Im C_0(-k, -k, 0, M_\pi, M_K) = -\frac{1}{16\pi M_K^2}, \quad (A23)
\]

\[
C_0(0, p_3 - k, M_K, M_\pi, M_K) = \\
\frac{1}{16\pi^2} \frac{1}{M_K^2 - M_\pi^2} \left[ \left( 1 - \frac{M_K^2 - M_\pi^2}{s_3} \right) \ln \left( \frac{M_\pi}{M_K} \right) - 2 \sqrt{\frac{4M_K^2}{s_3} - 1} \arctan \sqrt{\frac{s_3 - 1}{4M_K^2 - s_3}} + \frac{1}{s_3} \lambda^{1/2}(s_3, M_\pi^2, M_K^2) \ln(\sigma_{K\pi}) \right], \quad (A24)
\]

\[
\Re C_0(0, k - p_3, M_K, M_\pi, M_\pi) = \\
\frac{1}{16\pi^2} \frac{1}{M_K^2 - M_\pi^2} \left[ \left( 1 + \frac{M_K^2 - M_\pi^2}{s_3} \right) \ln \left( \frac{M_\pi}{M_K} \right) \right]
\]
\(-\sigma_\pi \ln(-\sigma_\pi) - \frac{1}{s_3} \lambda^{1/2}(s_3, M_\pi^2, M_K^2) \ln(\sigma_{K\pi}) \),  \hspace{1cm} (A25)

\[\Im C_0(0, k - p_3, M_K, M_\pi, M_\pi) = -\frac{1}{16\pi} \frac{\sigma_\pi}{M_\pi^2 - M_K^2}, \hspace{1cm} (A26)\]

\[C_0(-p_3, -k, 0, M_K, M_K) = \frac{1}{16\pi^2} \frac{1}{M_K M_\pi} \frac{\sigma_{K\pi}}{1 - \sigma_{K\pi}^2} \times \]

\[-\ln^2(\sigma_{K\pi}) - \frac{1}{2} \ln^2(\sigma_{\pi\pi}) - \frac{1}{2} \ln^2\left(\frac{M_\pi}{M_K}\right)\]

\[+ \ln(\sigma_{K\pi}) \left[ \ln\left(\frac{M_K M_\pi}{M_K^2 - M_\pi^2}\right) + \ln\left(\frac{M_K^2}{M_\pi^2 - M_K^2}\right) \right] \]

\[+ \text{Li}_2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\right) + \text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\right)\]

\[-\text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\right) - \text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\sigma_{\pi\pi}\right)\]

\[-\text{Li}_2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\right) - \text{Li}_2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\sigma_{\pi\pi}\right) \], \hspace{1cm} (A27)

\[\Re C_0(-p_3, -k, 0, M_\pi, M_\pi) = \frac{1}{16\pi^2} \frac{1}{M_K M_\pi} \frac{\sigma_{K\pi}}{1 - \sigma_{K\pi}^2} \times \]

\[-\frac{\pi^2}{3} - \frac{1}{2} \ln^2(\sigma_{K\pi}) - \ln^2(\sigma_{\pi\pi}) - \frac{1}{2} \ln^2\left(\frac{M_\pi}{M_K}\right)\]

\[+ \ln(\sigma_{K\pi}) \left[ \ln\left(\frac{M_K^2}{M_\pi^2 - M_K^2}\right) + \ln\left(\frac{M_K^2}{M_\pi^2 - M_K^2}\right) \right] \]

\[+ \frac{1}{2} \ln^2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\right) + \frac{1}{2} \ln^2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\sigma_{\pi\pi}\right) \]

\[+ \frac{1}{2} \ln^2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\right) + \frac{1}{2} \ln^2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\sigma_{\pi\pi}\right) \]

\[+ \text{Li}_2\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\right) + \text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\right)\]

\[+ \text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\sigma_{\pi\pi}\right) + \text{Li}_2\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\sigma_{\pi\pi}\right) \], \hspace{1cm} (A28)

\[\Im C_0(-p_3, -k, 0, M_\pi, M_\pi) = -\frac{1}{16\pi} \frac{1}{M_K M_\pi} \frac{\sigma_{K\pi}}{1 - \sigma_{K\pi}^2} \times \]

\[-2 \ln(\sigma_{K\pi}) + 2 \ln(-\sigma_{\pi\pi}) + \ln\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\right)\]

\[-\ln\left(1 - \frac{M_K}{M_\pi} \sigma_{K\pi}\sigma_{\pi\pi}\right) + \ln\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\sigma_{\pi\pi}\right) - \ln\left(1 - \frac{M_\pi}{M_K} \sigma_{K\pi}\sigma_{\pi\pi}\right) \], \hspace{1cm} (A29)

\[C_0(-p_3, -k, m_\gamma, M_\pi, M_K) = \frac{1}{16\pi^2} \frac{1}{M_K M_\pi} \frac{\sigma_{K\pi}}{1 - \sigma_{K\pi}^2} \times \]
\[
\{ \ln(\sigma_{\pi\pi}) \left[ 2 \ln(1 - \sigma_{\pi\pi}^2) - \frac{1}{2} \ln(\sigma_{\pi\pi}) - \ln \left( \frac{m_{\pi}^2}{M_K M_{\pi}} \right) \right] \\
-\frac{\pi^2}{6} + \frac{1}{2} \ln^2 \left( \frac{M_{\pi}}{M_K} \right) + \text{Li}_2(\sigma_{\pi\pi}^2) \\
+ \text{Li}_2 \left( 1 - \frac{M_{K}}{M_{\pi}} \sigma_{\pi\pi} \right) + \text{Li}_2 \left( 1 - \frac{M_{K}}{M_{\pi}} \sigma_{\pi\pi} \right) \} , \qquad (A30)
\]

\[ C_1(-p_3, -k, 0, M_{\pi}, M_K) = \]
\[ \lambda^{-1}(s_3, M_{\pi}^2, M_K^2) \left[ (M_K^2 + M_{\pi}^2 - s_3) B_0(-p_3, 0, M_{\pi}) \\
-2M_{K}^2 B_0(-k, 0, M_{\pi}) + (M_K^2 - M_{\pi}^2 + s_3) B_0(p_3 - k, M_{\pi}, M_K) \right] , \quad (A31) \]

\[ C_2(-p_3, -k, 0, M_{\pi}, M_K) = \]
\[ \lambda^{-1}(s_3, M_{\pi}^2, M_K^2) \left[ -2M_{\pi}^2 B_0(-p_3, 0, M_{\pi}) \\
+(M_K^2 + M_{\pi}^2 - s_3) B_0(-k, 0, M_{\pi}) - (M_K^2 - M_{\pi}^2 - s_3) B_0(p_3 - k, M_{\pi}, M_K) \right] . \quad (A32) \]

\textbf{D integrals}

We have one complex and one real four-point function in the physical region. The complex one reads:

\[ \Re D_0(-k, -p_3, -k, m_{\gamma}, M_K, M_{\pi}, M_{\pi}) = \]
\[ \frac{1}{16\pi^2} \frac{1}{M_K M_{\pi}} \frac{1}{M_K^2 - M_{\pi}^2} \frac{1}{1 - \sigma_{\pi\pi}^2} \times \]
\[ \left\{ \frac{\pi^2}{6} + 2 \ln(\sigma_{\pi\pi}) \left[ \ln(1 - \sigma_{\pi\pi}^2) - \ln \left( \frac{M_{\pi} m_{\gamma}}{M_K^2 - M_{\pi}^2} \right) \right] \\
-\frac{1}{2} \ln^2 \left( 1 - \frac{M_K}{M_{\pi}} \sigma_{\pi\pi} \right) + \frac{1}{2} \ln^2 \left( 1 - \frac{M_K}{M_{\pi}} \sigma_{\pi\pi} \sigma_{\pi\pi} \right) \\
-\frac{1}{2} \ln^2 \left( 1 - \frac{M_K}{M_{\pi}} \sigma_{\pi\pi} \right) + \frac{1}{2} \ln^2 \left( 1 - \frac{M_K}{M_{\pi}} \sigma_{\pi\pi} \right) \right\} , \quad (A33) \]

\[ \Im D_0(-k, -p_3, -k, m_{\gamma}, M_K, M_{\pi}, M_{\pi}) = \]
\[ \frac{1}{16\pi^2} \frac{1}{M_K M_{\pi}} \frac{1}{M_K^2 - M_{\pi}^2} \frac{1}{1 - \sigma_{\pi\pi}^2} \times \]
\[ \left\{ -2 \ln(\sigma_{\pi\pi}) + 2 \ln(-\sigma_{\pi\pi}) \right\} \]
\[ \begin{align*}
&+ \ln \left( 1 - \frac{M_K}{M_\pi} \frac{\sigma_{K\pi}}{\sigma_{\pi\pi}} \right) - \ln \left( 1 - \frac{M_K}{M_\pi} \sigma_{K\pi} \sigma_{\pi\pi} \right) \\
&+ \ln \left( 1 - \frac{M_\pi}{M_K} \frac{\sigma_{K\pi}}{\sigma_{\pi\pi}} \right) - \ln \left( 1 - \frac{M_\pi}{M_K} \sigma_{K\pi} \sigma_{\pi\pi} \right) \right) .
\end{align*} \]

(A34)

The real one is given by:

\[ D_0(-p_3, -k, -p_3, m_\gamma, M_\pi, M_K) = \]

\[ \begin{align*}
&- \frac{1}{16\pi^2} \frac{1}{M_K M_\pi} \frac{1}{M_K^2 - M_\pi^2} \left[ \sigma_{K\pi} \right] \\
&\left\{ -\frac{\pi^2}{6} + 2 \ln(\sigma_{K\pi}) \left[ \ln(1 - \sigma_{K\pi}^2) - \ln \left( \frac{M_K m_\gamma}{M_K^2 - M_\pi^2} \right) \right] \\
&+ \ln^2 \left( \frac{M_\pi}{M_K} \right) + \ln^2(\sigma_{KK}) + \text{Li}_2(\sigma_{K\pi}^2) \\
&+ \text{Li}_2 \left( 1 - \frac{M_K}{M_\pi} \frac{\sigma_{K\pi}}{\sigma_{KK}} \right) + \text{Li}_2 \left( 1 - \frac{M_K}{M_\pi} \sigma_{KK} \sigma_{K\pi} \right) \\
&+ \text{Li}_2 \left( 1 - \frac{M_\pi}{M_K} \frac{\sigma_{K\pi}}{\sigma_{KK}} \right) + \text{Li}_2 \left( 1 - \frac{M_\pi}{M_K} \sigma_{KK} \sigma_{K\pi} \right) \right) .
\end{align*} \]

(A35)

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TABLE I: The size of explicit virtual photon correction compared to the Born amplitude. We used the notations $r$ and $s_0$ for the ratio $s/(4M^2_\pi)$ and the Dalitz plot center $(s_1+s_2+s_3 = M_K^2+3M_\pi^2)/3$, respectively.

| $r$ | 1   | 1.12 | 1.24 | $s_0/(4M_\pi^2)$ | 1.48 | 1.60 | $(M_K - M_\pi)^2/(4M_\pi^2)$ |
|-----|-----|------|------|-------------------|------|------|-------------------------------|
| $10^2 \delta$ | −5.78 | −5.59 | −5.43 | −5.26 | −5.13 | −4.99 | −5.44 |
Figure 1: Born diagrams. The strangeness-changing vertex is represented by a full square.
Figure 2: The various topologies with a virtual photon (wavy line) attached to the $K\pi\pi\pi$ strangeness-changing vertex.
Figure 3: Box diagrams with a $\pi\pi$ scattering vertex.

Figure 4: Diagrams with a pion pole and a photon attached to a $\pi\pi$ scattering vertex.
Figure 5: Diagrams with a pion pole, a $\pi\pi$ scattering vertex, and a photon attached to the $K\pi$ strangeness-changing vertex.
Figure 6: Box diagrams with a $K\pi$ scattering vertex.

Figure 7: Diagrams with a kaon pole and a photon attached to a $K\pi$ scattering vertex.
Figure 8: Diagrams with a kaon pole, a $K\pi$ scattering vertex and a photon attached to a $K\pi$ strangeness-changing vertex.
Figure 9: Diagrams with a neutral kaon pole.
Figure 10: Diagrams with electromagnetic pion form factor.

Figure 11: Emission of one real soft photon diagrams. Are shown only diagrams generating infrared divergence. The filled circle represents tree vertices from Fig. 1.
Figure 12: The amplitude squared as function of $s_3$. The dashed curve represents Born amplitude squared $|\mathcal{M}_0|^2$, the plain one includes explicit virtual photon correction $|\mathcal{M}_0|^2 + 2 \Re(\mathcal{M}_0\mathcal{M}_1)$. 