From Vertex Operators to Calogero-Sutherland Models

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Abstract

The correlation function of the product of N generalized vertex operators satisfies an infinite set of Ward identities, related to a $W_\infty$ algebra, whose extension out of the mass shell gives rise to equations which can be considered as a generalization of the compactified Calogero-Sutherland (CS) hamiltonians. In particular the wave function of the ground state of the compactified CS model is shown to be given by the value of the product of N vertex operators between the vacuum and an excited state and the hamiltonian is identified with $W_0^2$ generator. The role of the vertex algebra as underlying unifying structure is pointed out.

Keyword: Vertex operator, $W_\infty$ algebra, Calogero-Sutherland model
PACS: 02.20.Tw, 03.65.Fd

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1 Introduction

Recently there has been a renewal of interest in the one dimensional integrable models, in particular in Calogero-Sutherland (CS) models \[1, 2\]. These models describe quantum mechanical systems of N one dimensional particles interacting through specific two-body potentials. Although they have been proposed and completely solved in the beginning of the seventy, they are at present object of intensive study for their unexpected connections with other fashionable models as the matrix models where the eigenvalues of the matrix are identified with the momenta of the particles \[3\] and the \(c = 1\) conformal field theory (CFT) models \[4\] and for their interesting and hidden infinite symmetry structure. The connection of these models with the Lie algebras root systems has been clarified just a few years after they were proposed and their complete integrability was proved \[5\]. An alternative proof of the integrability as well as a new expressions of the quantum integrals of the Sutherland model is given in \[6\] using representation theory of Lie algebra \(gl(N)\) and of its affine extension. Only recently the underlying infinite Kac-Moody symmetry has been understood and their invariance for a \(W_\infty\) discovered. In \[7\] it has been proven that the Calogero model is invariant for \(U(1)\) Kac-Moody, the \(U(1)\) current algebra appearing as the generating function of the quantum integrals of the model, and for a \(W_{1+\infty}\) algebra. In the case in which the interacting particles transform as the fundamental representation of a \(su(N)\) the symmetry is extended by an affine \(su(N)\) and the \(W_{1+\infty}\) algebra becomes a coloured one. This symmetry has been further clarified in \[8\], where it is also shown that the Sutherland model is invariant only for the Yangian \(Y(su(N))\). In \[9\] it has been proven that the ground state of the models satisfies the Knizhnik-Zamolodchikov equation. Moreover using the \(S_N\)-extended Heisenberg algebra \[10\] an universal Hamiltonian for the CS model has been written as the anticommutator of one particle operators whose commutation relations reduce to the standard bosonic annihilation-creation operators in the limit of vanishing coupling constant \[11\]. In this approach therefore the potential is connected with the statistics of the particles.

In this work we show that the very deep structure of these models, more exactly of their compactified version, i.e. the Sutherland model, is the vertex algebra structure, which constitutes the mathematically rigorous formulation of the algebraic origin of conformal field theory. We refer for an exhaustive discussion of the subject to \[13\] where references to the original papers can be found. For a pedagogical introduction for physicist see \[14\].

The generalized vertex operators provide a well known explicit construction of the vertex algebra and they are connected with the structure of Lorentzian algebras.
which includes the indefinite Kac-Moody algebras \[18\] and the Borcherds algebras \[19\], the affine algebras being only a particular and peculiar subset of these algebras.

Even if some of the topics we discuss here can be found scattered in the literature, we believe useful to present here in a unified matter. In fact we believe that our approach allows an unified discussion of the algebraic structure, the connection with conformal field theory and the statistics of the particles. The paper is organized as it follows: in Sec. 2 we write the product of \(N\) generalized vertex operators (GVO) in terms of an ordered product of the GVO times derivative of Jastrow like functions and introduce a set of differential operators in terms of which we can get all the relevant results of the vertex algebra.

In Sec. 3 we define the amplitude or correlation function for the GVOs product and show that it can be obtained by the action of differential operators applied to the amplitude of \(N\) standard vertex operators (VO).

In Sec. 4 an (infinite) set of Ward identities are then derived for the amplitudes, it is shown that the generators spanning a \(W_\infty\) algebra are a subset of the set of GVO symmetries.

In Sec. 5 we consider explicitly the particular case of the correlation functions for VOs and we show that the wave functions of the ground state of the CS model are obtained as the matrix element of the product of \(N\) VO between the vacuum and a suitable excited state, the Hamiltonian of the model appearing as a combination of differential operators appearing in the analogous of the Ward identities for off-shell amplitudes. Finally in Sec. 6 we briefly discuss further developments of the formalism previously developed which in the most general case gives rise to more general systems than the CS model. Let us remark that the vertex operators for the CS models already introduced in the literature \[20\], can be obtained by expansion of the GVOs.

## 2 Differential operators for GVOs

In this section we introduce a set of differential operators that realize the vertex algebra in a complementary way to the usual vertex operators construction \[13\].

Firstly we briefly recall the GVOs construction, which provides a well known realization of vertex algebra \[21\] and then we show that it is possible to build up a set of differential operators by means of which to carry out the relevant operations on GVOs.

Let us recall the particular choice of basis that we use in our construction that
simplifies greatly the formal calculations.

The standard (tachyonic) vertex operator (VO) is defined by

\[ U^r(z) = e^{ir\cdot Q(z)} : \]

where \( Q(z) \) are standard Fubini-Veneziano fields

\[ Q^\mu(z) = q^\mu - ip^\mu \ln z + i \sum_{n \neq 0} \alpha^\mu_n z^{-n} \quad (1) \]

on d-dimensional Minkowskian torus with periodical boundary conditions given by a vector \( r \) in a lattice \( \Lambda \).

A general GVO is a product of Schur polynomials in the derivatives of Fubini fields times a standard vertex operator

\[ U^{\{r,r_i\}}_{\{n_i\}}(z) = \prod_i P^{n_i}_i(z) U^r(z) := \Delta^{\{n_i\}}_{\{z_i\}} \prod_i U^{r_i}(z_i) U^{r'}(z) : \quad (2) \]

where \( r' = r - \sum_i r_i \) and:

\[ \Delta^{\{n_i\}}_{\{z_i\}} = \prod_i \lim_{z_i \rightarrow z} \frac{\partial^{n_i}}{\partial z_i^{n_i}} \quad (3) \]

At this point a few words of warning are necessary: in the above equation the label \( z_i \) stands for a finite set of variables; to be rigorous we should put it in curl brackets. However in order not to overweight the notation we use only the round brackets. We shall use this convention hereafter. Only for the GVO or VO the functional dependence is only from one variable. Moreover in the above definition, we should also write the explicit dependence on the \( z \) point where the limit is done at the end. For the same reason we shall omit here and in similar equations the dependence on the point(s) where the limit is (are) performed.

Formal Laurent expansion of GVO is denoted as:

\[ U^{\{r,r_i\}}_{\{n_i\}}(z) = \sum_m A^{\{r,r_i\}}_{\{n_i\}} m z^{-m-h} \quad (4) \]

where \( h = r^2 + \sum_i n_i \) is the conformal weight.

The product of two VO is simply:

\[ U^r(z) U^s(\xi) =: U^r(z) U^s(\xi) : (z - \xi)^{r \cdot s} \quad \text{for} \quad |\xi| < |z| \quad (5) \]

by means of locality properties the product is analytically extended to whole \( C^2 \) space except the poles \( z = \xi \) and \( z, \xi = 0, \infty \).
We can easily generalize this relation to \( N \) VO product:

\[
\prod_{q=1}^{N} U^{r_q}(z_q) = \prod_{q=1}^{N} U^{r_q}(z_q) : \mathcal{F}_N^{0,0,r;r'}(z_l, z_{l'})
\]  

where we have introduced a Jastrow like function:

\[
\mathcal{F}_N^{0,0,r;r'}(z_l, z_{l'}) = \prod_{l>l'} (z_l - z_{l'})^{r_l r_{l'}}
\]  

Using this result and the differential operators:

\[
\Delta^{(n_i, n_j)}(z_i, \xi_j) = \lim_{i,j} \lim_{z_i \to \xi_i} \lim_{\xi_j \to z_j} \frac{n_i! \partial^{n_i}}{n_i! \partial z_i^{n_i}} \frac{n_j! \partial^{n_j}}{n_j! \partial \xi_j^{n_j}}
\]

we can compute any two GVOs product and we get:

\[
U^{(r,r_i)}(z) U^{(s,s_j)}(\xi) = \Delta^{(n_i, n_j)}(z_i, \xi_j) : U^{(r,r_i)}(z, z_i)U^{(s,s_j)}(\xi, \xi_j) : \mathcal{F}^{(a_{ij}, a_{ij}, a_{ij}, a_{ij})}(z_i, \xi_j)
\]  

where

\[
U^{(r,r_i)}(z, z_i) =: \prod_i U^{r_i}(z_i)U^{r'}(z) : \quad r' = r - \sum_i r_i
\]

\[
U^{(s,s_j)}(\xi, \xi_j) =: \prod_j U^{s_j}(\xi_j)U^{s'}(\xi) : \quad s' = s - \sum_j s_j
\]

and

\[
\mathcal{F}^{(a_{ij}, a_{ij}, a_{ij}, a_{ij})}(z_i, \xi_j) = \prod_{ij} (z_i - \xi_j)^{a_{ij}}(z_i - \xi)^{a_{ij}}(z - \xi)^{a_{ij}}(z - \xi)^{a_{ij}}
\]

with \( a_{ij} = r_i \cdot s_j, a_{ij} = r' \cdot s_j, a_{ij} = r_i \cdot s' e a_{ij} = r' \cdot s' \).

After some algebraic calculation we obtain:

\[
U^{(r,r_1)}(z) U^{(s,s_1)}(\xi) = \sum_{\{n_{i,j}\}} \chi^{\{r,s,r_1,s_1\}}_{\{\{k_{i,j}\}\}} \frac{\lambda^{\{r,s,r_1,s_1\}}_{\{\{k_{i,j}\}\}}}{(z - \xi)^{-r-s+\sum_i k_i + \sum_j k_j}} : U^{(r,r_1)}(z)U^{(s,s_1)}(\xi) : \mathcal{F}^{(a_{ij}, a_{ij}, a_{ij}, a_{ij})}(z_i, \xi_j)
\]

with

\[
\chi^{\{r,s,r_1,s_1\}}_{\{\{k_{i,j}\}\}} = \sum_{\sum_i p_i = k_i, \sum_j p_j = k_j} (-1)^{\sum l_j} \prod_{i,j} \left( a_{i0} \frac{a_{ij}}{p_i - l_i} \right) \left( a_{ij} \frac{a_{ij}}{l_i + l_j} \right) \left( a_{ij} \frac{a_{ij}}{p_j - l_j} \right)
\]

(14)
Remark that this relation is obtained in a non-local way without OPE expansion so it holds also in the large distances limit and can be used to give non-perturbative results.

The set of \( a_{il} \) can be expressed in terms of upper triangular matrices in \( gl(N) \) so relating our approach to the approach of ref.\[6\].

To extend eq.(9) we remark that only Jastrow like functions appear in the GVOs product so we can define:

\[
\prod_q U^{\{r_q, r_{vq}\}}(z_q) = \Delta^{\{n_{vq}\}}(z_{vq}) : \prod_q U^{\{r_q, r_{vq}\}}(z_q, z_{vq}) : F_N^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'})
\]

where:

\[
\Delta^{\{n_{vq}\}}(z_{vq}) = \prod_{\{q, vq\}} \lim_{z_{vq} \to z_q} \frac{\partial^{n_{vq}}}{\partial z_{vq}}
\]

is the generalization of operators in eq.(8) and

\[
F_N^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'}) := \prod_q U^{\{r_q, r_{vq}\}}(z_q, z_{vq}) := \prod_q U^{r_{vq}}(z_{vq}) U^{r_{vq}'}(z_q) :
\]

From these definitions we can introduce differential operators that acting only on normal ordered products of tachyonic vertex give expression for general GVOs:

\[
\mathcal{D}_{\{n_{vq}\}}^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{vq}) = \Delta^{\{n_{vq}\}}(z_{vq}) F_N^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'})
\]

If we expand these operators in terms of \( \Delta^{\{n_{vq}\}}(z_{vq}) \) we obtain:

\[
\mathcal{D}_{\{n_{vq}\}}^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{vq}) = \sum_{\{k_{vq}\}} S_{\{k_{vq}\}}^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'}) \Delta^{\{n_{vq} - k_{vq}\}}(z_{vq})
\]

where

\[
S_{\{k_{vq}\}}^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'}) = \left( \Delta^{\{k_{vq}\}}(z_{vq}) F_N^{\{a_{il}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}, a_{il'}^{l'}\}}(z_{il}, z_{jl'}) \right)
\]
These functions can be explicitly computed, see the Appendix, and have the following expression:

\[
S_{\{k_{vq}\}}^{\{a_{ij}, a_{ii'}, a_{ij'}, a_{ii'}\}}(z_{\ell}, z_{\ell'}) = \sum_{\{\sum_{l>q} k_{lq} + \sum_{l' < q} k_{l'q} = k_{vq}\}} \prod_{l>q} \frac{\chi_{\{k_{ij}^\ell, (k_{ij'})^{l'}\}}^{\{r_l, r_{l'}, r_q, r_{q'}\}}}{(z_{\ell} - z_{\ell'})^{-r_l - r_{l'} + \sum_i k_{il}^\ell + \sum_{i'} k_{i'l'}^{l'}}}
\]

(22)

The explicit expansion of general \(N\) GVOs product becomes:

\[
\prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{vq}, z_{q}) = \tilde{D}_{\{\{n_{vq}\}\}}^{\{a_{ii'}, a_{ii'}, a_{ij'}, a_{ii'}\}}(z_{vq}) : \prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{vq}, z_{q}) : \quad (23)
\]

\[
= \sum_{\{\sum_{l>q} k_{lq} + \sum_{l' < q} k_{l'q} = k_{vq}\}} \prod_{l>q} \frac{\chi_{\{k_{ij}^\ell, (k_{ij'})^{l'}\}}^{\{r_l, r_{l'}, r_q, r_{q'}\}}}{(z_{\ell} - z_{\ell'})^{-r_l - r_{l'} + \sum_i k_{il}^\ell + \sum_{i'} k_{i'l'}^{l'}}} : \prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{vq}, z_{q}) : \quad (23)
\]

Locality properties for GVOs can be evaluated by means of permutations group (notice that \(\Delta^{\{n_{vq}\}}(z_{vq})\) and ordered products are \(S_N\)-symmetric so we do not indicate explicitly the symmetrization):

\[
\Pi_{qq'}^{N} \prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{q}) \Pi_{qq'} = \Delta^{\{n_{vq}\}}(z_{vq}) : \prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{q}, z_{q'}) : \quad (24)
\]

\[
= \Delta^{\{n_{vq}\}}(z_{vq}) : \prod_{q}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{q}, z_{q'}) : K_{qq'} F_N^{\{a_{ii'}, a_{ii'}, a_{ij'}, a_{ii'}\}}(z_{i}, z_{i'}) \quad (25)
\]

where \(\Pi_{qq'}\) is an operator that exchanges the order of two GVOs while \(K_{qq'}\) exchanges the sets of indices \(\{q, v_q\}\) and \(\{q', u_{q'}\}\) in the functions:

\[
K_{qq'} F_N^{\{a_{ii'}, a_{ii'}, a_{ij'}, a_{ii'}\}}(z_{i}, z_{i'}) = \epsilon \sum_{l=0}^{q'-q-1} r_{q+l} : r_{q'} F_N^{\{a_{ii'}, a_{ii'}, a_{ij'}, a_{ii'}\}}(z_{i}, z_{i'}) \quad (25)
\]

where \(\epsilon = e^{i\pi}\) so we generalize this construction also to non-local cases (in particular rational values of \(r_q, r_{q'}\) give RCFT) where vertex algebras can be extended following [22].

3 GVOs correlators

By means of this formalism we calculate explicitly also correlation functions for GVOs using the property of normal ordered VOs:

\[
\mathcal{A}^{\{r_q, r_{q'}\}}(z_{vq}, z_{q}) = \langle \prod_{q} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{q'}\}}(z_{q}, z_{vq}) : \delta_{\sum q r_q, 0} \rangle \quad (26)
\]
so by using of eq. (23) we have simply:

$$G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) = \langle \prod_q U_{\{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) \rangle = \mathcal{D}_{\{n_{eq} \}}^{\{a_{ijr}, a_{ilr}, a_{ilr}, a_{ilr} \}}(z_{eq}) \mathcal{A}_{\{r_q, r_{eq} \}}(z_{eq}, z_q)$$

$$= \mathcal{S}_{\{n_{eq} \}}^{\{a_{ijr}, a_{ilr}, a_{ilr}, a_{ilr} \}}(z_l, z_{l'}) \delta_{\sum_q r_q, 0}$$ \hspace{1cm} (27)

We can also verify the duality invariance by using of action of symmetric group:

$$\prod_{l \neq i} K_{il} G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) = \prod_{l \neq i} \eta_{il}(r_i \cdot r_l) G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) = e^{-r_i^2} G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) \forall i$$ \hspace{1cm} (28)

where $\eta_{il}(r_i \cdot r_l) = e^{r_i r_l}$ and neutrality implies $\sum_{l \neq i} r_i \cdot r_l = -r_i^2$.

A $G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q)$ amplitude can be obtained also by means of the action of the $\Delta \{n_{eq} \}$ operators on a tachyonic amplitude:

$$G_{N \{n_{eq} \}}^{\{r_q, r_{eq} \}}(z_q) = \nabla_{\{n_{eq} \}}^{[a_{il}, a_{il}]}(z_{eq}) G_{N + \sum_{eq}}^{\{r_q, r_{eq} \}}(z_{eq}, z_q)$$ \hspace{1cm} (29)

where

$$\nabla_{\{n_{eq} \}}^{[a_{il}, a_{il}]}(z_{eq}) = \Delta^{\{n_{eq} \}}(z_{eq}) \mathcal{F}_{N}^{[a_{il}, a_{il}]-1}(z_i, z_l)$$ \hspace{1cm} (30)

$$\mathcal{F}_{N}^{[a_{il}, a_{il}]}(z_i, z_l) = \prod_{l \neq i} \left( z_l - z_i \right)^{a_{il}}(z_l - z_i)^{a_{il}}$$ \hspace{1cm} (31)

in fact applying this operator we obtain:

$$\nabla_{\{n_{eq} \}}^{[a_{il}, a_{il}]}(z_{eq}) G_{N + \sum_{eq}}^{\{r_q, r_{eq} \}}(z_{eq}, z_q) = \mathcal{S}_{\{n_{eq} \}}^{\{a_{ijr}, a_{ilr}, a_{ilr}, a_{ilr} \}}(z_l, z_{l'}) \delta_{\sum_q r_q, 0}$$ \hspace{1cm} (32)

So all “massive” amplitudes properties can be deduced simply by tachyonic ones.

This set of operators gives relations between $N$ and $N + \sum n_{eq}$ vertex amplitudes so they appear as non-linear realization of a larger algebra as pointed out by Witten in the case of 2D string theory.

By using of a diverse factorization of functions:

$$\mathcal{F}_{N}^{\{a_{ijr}, a_{ilr}, a_{ilr}, a_{ilr} \}}(z_i, z_{l'}) = \mathcal{F}_{N}^{\{a_{ijr}, a_{ilr}, a_{ilr}, a_{ilr} \}}(z_i, z_{l'}) \mathcal{F}_{N}^{\{0,0,0, r_l r_{l'} \}}(z_l, z_{l'})$$ \hspace{1cm} (33)
we obtain also a relation between higher spin “excitations” and tachyonic amplitudes with the same momentum:

\[
G_{\{n_{vq}\}}^{\{r_{q},r_{vq}\}}(z_q) = \nabla_{\{n_{vq}\}}^{[a_{iljl},a_{iljl}]}(z_{vq}) G_{\{n_{vq}\}}^{\{r_{q},r_{vq}\}}(z_{vq}, z_q)
\]

\[
= \nabla_{\{n_{vq}\}}^{\{a_{iijr},a_{iijr},a_{iijr}\}}(z_{vq}) \mathcal{F}_N^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}(z_{i}, z_{j}) \mathcal{F}_N^{\{0,0,r_{l},r_{l}\}}(z_{l}, z_{l}) \delta \sum_{\gamma q} 0
\]

\[
= \mathcal{S}_{\{n_{vq}\}}^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}(z_{vq}) G_{\{n_{vq}\}}^{\{r_{q}\}}(z_q)
\]

An important point in this approach is that we can built all amplitudes by means of the action of \(\Delta^{[a_{iljl},a_{iljl}]}\) set of differential operators on the tachyonic correlation functions.

Moreover, all informations on vertex algebra are contained in the contact \(\mathcal{S}\) functions so we can remove completely the GVOs from our construction and make use only of differential operators and \(\mathcal{F}_N^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}(z_{i}, z_{j})\) functions to study any amplitudes.

### 4 Ward identity for correlation functions

At this point we can study the whole set of amplitudes in order to recover their symmetries. To do this we note that if the sum of roots vanishes (mass shell condition) ordered amplitudes \(\mathcal{A}^{[r_{q},r_{vq}]}(z_{vq}, z_q)\) are constant or in a more general case are symmetric functions, so they satisfy an infinity of differential equations:

\[
\partial_{\{n_{q}\}}^{\{n_{q}\}}(z_q) \mathcal{A}^{[r_{q},r_{vq}]}(z_{vq}, z_q) = \left( \prod_q z_q^{m_q+n_q} \frac{\partial^{m_q}}{n_q! \partial z_{q}^{n_q}} \right) \mathcal{A}^{[r_{q},r_{vq}]}(z_{vq}, z_q) = 0
\]

(35)

In the language of CFT these relations correspond to the insertion of quasi-primary fields [7], in fact:

\[
\partial_{\{n_{q}\}}^{\{n_{q}\}}(z_q) \mathcal{A}^{[r_{q},r_{vq}]}(z_{vq}, z_q) = \langle : \prod_q z_q^{m_q+n_q} \frac{\partial^{m_q}}{n_q! \partial z_{q}^{n_q}} \mathcal{A}^{[r_{q},r_{vq}]}(z_{vq}) : \rangle
\]

(36)

is the ordered amplitude for descendent fields relatives to \(\text{SL}(2,\mathbb{R})\) subalgebra.

To these ordered amplitudes symmetries we associate Ward identities for generic GVOs amplitudes, which, defining the operators:

\[
\mathcal{L}_{\{n_{vq}\}}^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}(z_q) = \mathcal{S}_{\{n_{vq}\}}^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}(z_l, z_l') \partial_{\{n_{q}\}}^{\{n_{q}\}}(z_q) \mathcal{S}_{\{n_{vq}\}}^{\{a_{iijr},a_{iijr},a_{iijr},a_{iijr}\}}^{-1}(z_l, z_l')
\]

(37)
can be written in the following form:
\[
\mathcal{L}_{\{n_q\},\{m_q\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q) G_{N_{\{n_{vq}\}}^{\{vq\}}}(z_q) = 0 \quad \text{for} \quad \{n_q \neq 0\} \quad (38)
\]

The above introduced operators satisfy the same commutation relations of free differential algebra \(\hat{\partial}_{\{m_q\}}^{\{n_q\}}(z_q)\):
\[
[\mathcal{L}_{\{n_q\},\{m_q\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q), \mathcal{L}_{\{n_q'\},\{m_q'\}}^{\{a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q) ] = \mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}^{-1}}(z_l, z_{l'}) (39)
\]
and the Jacobi identity.

Remark that the function \(\mathcal{S}_{\{k_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}^{-1}}(z_l, z_{l'})\) is a sum of Jastrow functions and is well defined everywhere except in the points \(z_l = z_{l'}\) but in these points the singularity of GVOs are regularized with normal ordering.

Using the factorization given in eq. (33) it is possible also to write a set of effective equations for the “excitations”, which, introducing the following operators:
\[
\mathcal{L}_{\{n_q\},\{m_q\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q) = \mathcal{F}_N^{\{0,0,r_l,r_{l'}\}^{-1}}(z_l, z_{l'}) \mathcal{L}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q) \mathcal{F}_N^{\{0,0,r_l,r_{l'}\}^{-1}}(z_l, z_{l'}) \quad (40)
\]
can be written as:
\[
\mathcal{L}_{\{n_q\},\{m_q\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\},\{n_{vq}\}}(z_q) \mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}}(z_{vq}) = 0 \quad \text{for} \quad \{n_q \neq 0\} \quad (41)
\]
The above equation are the analogous of eq. (38) for \(S\) functions and satisfies the same algebraic relations.

The set of differential operators introduced in eq. (38) can be considered as a generalization of Dunkl operators whose role is fundamental in the theory of integrable systems.

In fact from the property:
\[
\mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}}(z_l, z_{l'}) \quad \hat{\partial}_{\{m_{vq}\}}^{\{n_{vq}\}}(z_q) \quad \mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}^{-1}}(z_l, z_{l'}) = \prod_q \frac{z_{mq}^n q^n}{n_q!} \left( \mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}}(z_l, z_{l'}) \quad \hat{\partial}_{z_q} \mathcal{S}_{\{n_{vq}\}}^{\{a_{ij},a_{ij'},a_{ij''},a_{ij'''}\}^{-1}}(z_l, z_{l'}) \right)^{n_q} \quad (42)
\]
we can write a general differential operator in a more simple way:

\[ \mathcal{L}^{\{a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}\}}_{\{n_q\}, \{m_q\}}(z_q) = (-1)^{\sum_q n_q} \prod_q \frac{z_q^{n_q+n_q}}{n_q!} \left( \hat{L}^{\{a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}\}}_{\{1_q\}}(z_q) \right)^{n_q} \]

The generalized Dunkl operator is:

\[ \hat{L}^{\{a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}\}}_{\{1_q\}}(z_q) = -\mathcal{S}^{\{a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}\}}_{\{n_q\}}^{-1}(z_l, z_l') \frac{\partial}{\partial z_l} \mathcal{S}^{\{a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}, a_{ij\ell'}\}}_{\{n_q\}}(z_l, z_l') \]

which in fact reduces to the usual Dunkl operator in the case of VOs:

\[ \hat{L}^{\{0,0,0,a_{ij\ell'}\}}_{\{1_q\}}(z_q) = -\mathcal{F}^{\{0,0,0,r_{ij\ell'}\}}_N(z_l, z_l') \frac{\partial}{\partial z_l} \mathcal{F}^{\{0,0,0,r_{ij\ell'}\}}_N(z_l, z_l') \]

Moreover, in this case we have:

\[ \hat{L}^{\{0,0,0,a_{ij\ell'}\}}_{\{1_q\}}^{-1}(z_q) G^\{r_{ij\ell'}\}_N(z_l) = < \frac{\partial}{\partial z_l} - r_q \cdot Q^{(1)}(z_q) > \prod_i U^{r_{ij\ell'}}(z_l) > \]

where the singular product is understood as limit \( z \to z_q \).

So in this case we can identify the action of the Dunkl operator on the VOs amplitudes with the action of the operator:

\[ \frac{\partial}{\partial z_q} - r_q \cdot Q^{(1)}(z_q) \]

on the product of VOs that is just a Miura transformation generating an explicit realization of \( W_\infty \) in terms of fields.

In the general case of \( N \) GVOs, however, a Miura transformation cannot exist.
4.1 \( W_\infty \) symmetries of amplitudes

In this section we describe some interesting subalgebras of the GVOs symmetries by using the correspondence between ordered amplitudes \( A^{(r_q,r_{vq})}(z_{vq}, z_q) \) and \( G_N^{(r_q,r_{vq})}(z_q) \) which allows us to single out the relevant symmetries.

It is obvious that the subalgebra of \( \partial_{\{m_q\}}(z_q) \) algebra spanned by the generators:

\[
\omega^n_m = \sum_{q=1}^{N} z^{m+n} \frac{\partial^n}{\partial z^n_q}
\]

satisfies a \( W_\infty \) algebra.

The corresponding differential operators \( W^n_m \) obtained replacing \( \partial_{\{m_q\}}(z_q) \) with the operators of eq.(49) are a realization of the same algebra.

An important consequence of this is that all GVOs amplitudes are \( W_\infty \) invariant, as indeed the \( W^n_m \) generators satisfy quantum commutation relations without central extensions, this is an obvious consequence of zero genus of Riemann surface on which we take correlators, while anomalies arise on genus one surface:

\[
[W^n_m, W^{'n'}_{m''}] = (n'm - nm')W^{n+n'-1}_{m+m'} + \ldots = \sum_{l=1} q^{l-1} C(l)_{n,n'}^m m_{m'}^{n+n'-l}
\]

where \( q \) is a quantum deformation that is \( q = 1 \) in this case, so:

\[
C(l)_{n,n'}^m = \left( \begin{array}{c} n \\ l \end{array} \right) \left[ m' + n' \right]_l - \left( \begin{array}{c} n' \\ l \end{array} \right) \left[ m + n \right]_l
\]

and \([a]_b\) is the Pochhammer symbol.

It should be possible to realize these generators by means of a particular combination of GVOs.

In particular we compute the Virasoro (Witt) subalgebra generators:

\[
[W^1_m, W^2_{m'}] = (m - m')W^1_{m+m'}
\]

In the case of Virasoro algebra these vertices can be simply identified. In fact in this case, it is well know that the tensor field \( T(\xi) \) is the generator of transformations that act on VOs in the following way:

\[
\oint_{C_{\xi,z}} \frac{d\xi}{2\pi i} \xi^{m+2} T(\xi) U^r(z) = \left( z^{m+1} \frac{\partial}{\partial z} + \frac{r^2}{2}(m + 1) z^m \right) U^r(z)
\]
where the circuit \( C_{\xi,z} \) includes the pole \( \xi = z \).

If we specialize these operators to the projective subalgebra we obtain:

\[
\sum_{l \neq q} \frac{1}{2} \left( \sum_{q} \lambda_{lq} \right) F_{N}^{(0,0,0,a_{l})}(z_{l}, z_{l'}) \quad (53)
\]

\[
\sum_{l \neq q} \frac{1}{2} \left( \sum_{q} \lambda_{lq} \right) F_{N}^{(0,0,0,a_{l})}(z_{l}, z_{l'}) \quad (54)
\]

\[
\sum_{l \neq q} \frac{1}{2} \left( \sum_{q} \lambda_{lq} \right) F_{N}^{(0,0,0,a_{l})}(z_{l}, z_{l'}) \quad (55)
\]

where \( \sum_{l \neq q} a_{lq} = -r_{q}^{2} = -2h_{q} \).

So the invariance is satisfied with the trivial comultiplication and we can use these relations to fix three variables in \( S_{(a_{l})}^{(a_{l})} \) \((z_{l}, z_{l'})\) functions.

In this case it is simple to realize these transformations by means of \( T(\xi) \):

\[
W_{m}^{1} G_{N}^{(r)}(z_{q}) = \sum_{q} \oint_{C_{\xi,q}} \frac{d\xi}{2\pi i} \xi^{m+2} < U^{r_{1}}(z_{1}) \cdots T(\xi) U^{r_{q}}(z_{q}) \cdots U^{r_{N}}(z_{N}) > = 0 \quad (56)
\]

In the general case additional terms are needed:

\[
w_{m}^{1} F_{N}^{(0,0,0,a_{l'})}(z_{l}, z_{l'}) = \sum_{l \neq l'} \frac{a_{l'}(z_{l}^{m+1} - z_{l'}^{m+1})}{z_{l} - z_{l'}} F_{N}^{(0,0,0,a_{l'})}(z_{l}, z_{l'}) \quad (57)
\]

In fact as the vacuum state \( |0> \) is invariant only for projective transformations, in this case it is necessary to consider also the contribution of the action of \( T(\xi) \) on the vacuum:

\[
W_{m}^{1} G_{N}^{(r)}(z_{q}) = \sum_{q} \oint_{C_{\xi,q}} \frac{d\xi}{2\pi i} \xi^{m+2} < U^{r_{1}}(z_{1}) \cdots T(\xi) U^{r_{q}}(z_{q}) \cdots U^{r_{N}}(z_{N}) > + \oint_{C_{\xi,0}} \frac{d\xi}{2\pi i} \xi^{m+2} < U^{r_{1}}(z_{1}) \cdots U^{r_{N}}(z_{N}) T(\xi) > + \oint_{C_{\xi,\infty}} \frac{d\xi}{2\pi i} \xi^{m+2} < T(\xi) U^{r_{1}}(z_{1}) \cdots U^{r_{N}}(z_{N}) > = 0 \quad (58)
\]

The interpretation of this result is very simple: the \( W_{\infty} \) symmetry of GVOs in quantum case is realized taking in to account the anomalous transformations of the vacuum, so the symmetry is restored also in non-critical dimensions.
4.2 Additional equations

In the previous section it is shown that a $W_\infty$ algebras exists for any GVOs correlator and the explicit realization can be given in terms of generalized Dunkl operators. This is only a subalgebra of the full symmetry algebra that can be realized, in this section we want to understand the role of remaining differential operators.

As recently is pointed out [24], canonical quantization of two dimensional identical particles give unusual interesting results.

We consider GVOs correlator as multiparticle form factors and the generalized Dunkl operators as one-particles operators, in this framework the $W_\infty$ generators become the completely symmetric single-particles operators that give the observables of $N$ particles system.

We can also construct operators not $S_N$ invariant that must be related to the differential equations not in $W_\infty$ algebra.

Two cases can arise: in the first the particles are distinguishable then these operators give observable results; in the second case the particles are identical then they cannot be observable to avoid the possibility to identify the particle.

In this framework the existence of this additional symmetry have a very simply physical interpretation in terms of breaking the Hilbert space of the particles in $N!$ sectors that cannot be mixed by hamiltonian evolution of the system so all not-symmetric operators must be unobservable.

This aspect is very interesting because it implies that must exist many null states for $W_\infty$ representations corresponding to these unobservable operators; this is just the case of quasi-finite representations that are well studied in [25].

Notice that degenerate representations of W algebras arise also in the classification of hierarchies in quantum Hall effect [26] without any explicit request of generalized exclusion principle.

In the following section we explore in more detail this aspect in the case of CS model.

These further equations relate amplitudes in which there are quasi-secondary fields to pure quasi-primary ones. Therefore, by means of these equations we reduce the vertex operators space to quasi-primary states only. In this way we remove not only the dependence on quasi-secondary null states but also on all quasi-secondary as a signal of enhanced symmetry. This invariance has a simple interpretation in terms of universal enveloping algebra of projective symmetry and does not depend on full Virasoro constraints.

As pointed out in [17] this is very important for the consistence of GVOs construction of Lorentzian algebras.
Besides the considered linear symmetry algebra of GVOs which commutes with the $N$ operator (number of GVOs), we remark that it does exist another set of symmetry obtained applying the $\nabla$ operator of Sec.(3) which relate product of GVOs with different value of $N$, see the end of Sec.(5).

5 Relationship with Sutherland model

In the first part of this paper we have constructed the differential operators that generate the symmetries for all GVOs amplitudes, in what follows we describe the connection with the CS one dimensional integrable systems. The compactification of external space implies that periodic boundary conditions for the momentum of particles must be imposed, so the model that we describe is of Sutherland form that describes a system of non-relativistic particles on a circle interacting with an inverse square potential.

This model is completely integrable and gives eigenfunctions expressed in terms of Jack polynomials $J_{\{t\}}^{(r^2)}(z_q)$ that are indexed by the Young diagrams that can be interpreted as the distribution of the momentum of pseudo-particles (holes).

The Young diagram is parameterized by the $N$ numbers $\{t\} = (t_1, \ldots, t_N)$, $t_1 \geq t_2 \cdots \geq t_N \geq 0$ with total number of boxes denoted by $|t| = \sum_{q=1}^{N} t_q$.

To make a reduction of GVOs space we have to impose that the number of particles is a constant so we consider only the set of $G_{\{\{n_q\}\}}^{\{r_q, r_{vq}\}}(z_q)$ with a fixed value of $N$. The correspondence with the CS model is obtained by the identification of the states and of the hamiltonian, which can be done using the projective invariance to fix two states corresponding to in-vacuum and out-excited state:

$$\psi_{\{(n_j)(n_{vq})(n_i)\}}^{\{s, s_j, r_q, r_{vq}, r_i\}}(z_q) = \langle s, s_j(n_j) | \prod_{q=1}^{N} U_{\{\{n_{vq}\}\}}^{\{r_q, r_{vq}\}}(z_q) | r, r_i(n_i) \rangle$$

(59)

Specializing these wave functions to Sutherland model we consider the ground state ($r_q = r, r_{vq} = 0 \ \forall \ q$):

$$\psi_{N}^{\{r, \lambda\}}(z_q) = \langle Nr + \lambda | \prod_{q=1}^{N} U^{r}(z_q) | \lambda \rangle = \prod_{l>l'}(z_l - z_{l'})^{r^2} \prod_{q=1}^{N} z_q^{r \lambda}$$

(60)

The wave function vanishes for $z_l - z_{l'} \to 0$ if $r^2 > 0$, which indeed is the condition ensuring the absence of poles in the product of two VOs and assure the
normalizability of the wave functions, it is very interesting to notice that we could consider also the case \( r^2 < 0 \) where bound-states should exist.

When expand VOs in terms of symmetric Jack polynomials we obtain the wave functions of excited states \([14]\):

\[
\begin{aligned}
&
: \prod_{q=1}^{N} U^r(z_q) : | \lambda > = \prod_{q=1}^{N} z_q^{\lambda^2} \sum_{\{t\}} (-1)^{|t|} J^{(r^2)}_{\{t\}}(z_q) \left( j_t^{r^2} \right)^{-1/2} | \{ t \}, N r + \lambda >
\end{aligned}
\]

where \( \left( j_t^{r^2} \right)^{-1/2} \) is a normalization factor.

All informations on symmetry of states are encoded in the ground state wave functions, this allows to understand the importance of GVOs in 1-dimensional integrable models. GVOs can be factorized in a Jastrow like function and a symmetric term that depends only on ordered products. Statistical properties are related only to \( F_N^{(a_{i_1j_{i_1}}, a_{i_2j_{i_2}}, a_{i_3j_{i_3}}, a_{i_4j_{i_4}})}(z_{i_1}, z_{j_{i_1}}) \) functions.

By orthogonality properties of Jack polynomials we obtain:

\[
< \{ t \}, N r + \lambda | \prod_{q=1}^{N} U^r(z_q) | \lambda > = \prod_{l > l'} (z_l - z_{l'})^2 \prod_{q=1}^{N} z_q^{r^2 \lambda + \lambda^2} (-1)^{|t|} J^{(r^2)}_{\{t\}}(z_q) \left( j_t^{r^2} \right)^{-1/2}
\]

The completeness of Jack polynomials in the space of \( c = 1 \) CFT at \( R = \sqrt{r^2} \) implies that exist an out-state for each state of Sutherland model, so \( W_\infty \) symmetry project the functions defined in \( R^N \) into single-particle space, in this way full set of wave functions can be obtained acting only on out-state (collective state).

In this case the first two generators of \( W_\infty \) algebra are:

\[
W_0^1 = \prod_{l > l'} (z_l - z_{l'})^2 \sum_q z_q \frac{\partial}{\partial z_q} \prod_{l > l'} (z_l - z_{l'})^{-r^2} = \sum_q z_q \frac{\partial}{\partial z_q} - \frac{r^2}{2} N(N - 1)
\]

and

\[
W_0^2 = \prod_{l > l'} (z_l - z_{l'})^2 \sum_q z_q^2 \frac{\partial^2}{\partial z_q^2} \prod_{l > l'} (z_l - z_{l'})^{-r^2}
\]

\[
= \sum_q z_q^2 \frac{\partial^2}{\partial z_q^2} - 2 r^2 \sum_{l > l'} \frac{z_l^2 \frac{\partial}{\partial z_l} - z_{l'}^2 \frac{\partial}{\partial z_{l'}}}{z_l - z_{l'}} + 2 r^2 (r^2 + 1) \sum_{l > l'} \frac{z_l z_{l'}}{(z_l - z_{l'})^2} + r^4 \sum_{q,l,l',q' \neq q} \frac{z_q^2}{(z_l - z_q)(z_q - z_{l'})} + \frac{r^2(r^2 + 1)}{2} N(N - 1)
\]
The first generator corresponds, up to a constant, to total momentum $P$:

$$P = \sum_{q} z_q \frac{\partial}{\partial z_q}$$  \hspace{1cm} (66)

while the first two terms of the second generator are the differential operators whose eigenvectors are, according to Stanley’s theorem [30], the Jack polynomials, while the fourth term is a constant. This suggests that there is a close relationship between this generator and the CS hamiltonian which in our notation is written as:

$$H = \sum_{q} \frac{1}{2} \left( z_q \frac{\partial}{\partial z_q} \right)^2 - r^2(r^2 - 1) \sum_{l \neq l'} \frac{z_l z_{l'}}{(z_l - z_{l'})^2}$$  \hspace{1cm} (67)

Moreover, by construction on the ground state we have:

$$W_0^4 \psi_N^{(r, \lambda)}(z_q) = \left[ P - \frac{r^2}{2} N(N - 1) \right] \psi_N^{(r, \lambda)}(z_q) = p \psi_N^{(r, \lambda)}(z_q)$$ \hspace{1cm} (68)

$$W_0^2 \psi_N^{(r, \lambda)}(z_q) = \left[ 2H + \frac{r^4}{6} N(N^2 - 1) \right] \psi_N^{(r, \lambda)}(z_q) = 2E \psi_N^{(r, \lambda)}(z_q)$$ \hspace{1cm} (69)

where $p = Nr \cdot \lambda$ and $E = \frac{1}{2} N(r \cdot \lambda)^2$ are the eigenvalues of relative operators.

In order to be able to really identify the CS hamiltonian we have to modify our approach introducing the symmetric hermitian exchange operators $K_{ll'}$, so we can discuss the more general case in which the exchange operator does appear in the CS hamiltonian.

In the following we give more general results that it is possible to have by the VOs construction if $r_l \neq r_{l'}$.

The CS wave function $\Psi(z_q)$ can be written in a factorized form as the product of the ground state wave-function $\psi_N^{(r_q, \lambda)}(z_q)$ times the wave function of the collective CS hamiltonian i.e. for $a_{ll'} = r^2$ a Jack polynomial $\phi(z_q)$. If the particles are identical (indistinguishable) the wave function has to be invariant under the action of the operator $K_{ll'}$, so we must impose, for any couple $ll'$

$$(1 - \eta_{ll'} K_{ll'}) \Psi(z_q) = \psi_N^{(r_q, \lambda)}(z_q)(1 - K_{ll'})\phi(z_q) = 0$$  \hspace{1cm} (70)

where $\eta_{ll'}$ is the phase, computed in Sec.(2), produced by the action of $K_{ll'}$ on $\psi_N^{(r_q, \lambda)}(z_q)$.

The phase $\eta_{ll'}$ takes in account of the statistics that depends on length of the $r_q$ roots.
Let us remark that the invariance of the wave function should require identical particles, i.e. all \( q \) equal or the introduction of mutual exclusion statistics \([32]\), whose connection with CS model has been discussed in \([33]\) and \([34]\).

The VOs approach naturally leads to models with this type of statistics. We make use of results in Sec.(4.2) that allow us to replace the differential operator of eq.(47) by the Dunkl operator

\[
d_q = \frac{\partial}{\partial z_q} + \sum_{l \neq q} \frac{a_{ql}}{z_q - z_l} (1 - K_{ql})
\]

When \( a_{ql} = r^2 \) the \( d_q \) satisfy the relations of an affine Hecke algebra, but now we have:

\[
[d_q, d_q'] = \sum_{l \neq q, q'} \left( \frac{a_{qq'}a_{ql}}{(z_q - z_{q'})(z_q' - z_l)} - \frac{a_{ql}a_{ll'}}{(z_q - z_l)(z_{q'} - z_l)} \right) K_{qq'}(K_{ql} - K_{ql'})
\]

\[
[z_q, z_{q'}] = 0
\]

\[
[d_q, z_{q'}] = \delta_{qq'} \left( 1 + \sum_{l} a_{ql} K_{ql} \right) - a_{qq'} K_{qq'}
\]

so the \( W_\infty \) algebra structure is now lost. In the order to see if and how this symmetry can be recovered we note that the first two \( W_\infty \) generators are:

\[
W^1_m(d) = \sum_q z_q^m \frac{\partial}{\partial z_q} + \sum_{l > l'} a_{ll'} z_l^m z_l'^{-1} (1 - K_{ll'})
\]

\[
W^2_m(d) = \sum_q z_q^{m+2} \frac{\partial^2}{\partial z_q^2} + 2 \sum_{l > l'} a_{ll'} z_l^{m+2} \frac{\partial}{\partial z_l} - z_l'^{m+2} \frac{\partial}{\partial z_{l'}} z_l^m (1 - K_{ll'})
\]

where we have used

\[
d_q^2 = \frac{\partial^2}{\partial z_q^2} + \sum_{l \neq q} \frac{a_{ql}}{(z_q - z_l)^2} \left[ \frac{\partial}{\partial z_q} - \frac{\partial}{\partial z_l} + \frac{\partial}{\partial z_q} + \frac{\partial}{\partial z_l} \right] (1 - K_{ql})
\]

\[
- \sum_{l \neq q} \frac{a_{ql}}{(z_q - z_l)^2} (1 - K_{ql}) + \sum_{q, l \neq l' \neq q} \frac{a_{ql}a_{ll'}}{(z_q - z_l)(z_{q'} - z_{l'})} (1 - K_{lq})(1 - K_{ql'})
\]
If all $a_{l^l} = r^2$ they still close a $W_\infty$ algebra on the states that we have used to realize them on which the exchange operators $K_{qq'}$ becomes a c-number

$$\left[ W_m^n(d), W_{m'}^{n'}(d) \right] \phi(z_q) = \left[ W_m^n, W_{m'}^{n'} \right] \phi(z_q)$$

(78)

where $W_m^n$ are the $W_m^n(d)$ with $K_{ll'} = 1$.

In particular $W_0^2(d)$ becomes the effective CS hamiltonian when it is applied on the symmetric functions.

If we use the freedom of adding any antisymmetric term that cannot give observable results, we restore the $W_\infty$ structure that is realized only modulo the transformations generated by the operators of Sec.(4.2).

If we define the operator $L_q$ as

$$L_q = \mathcal{F}_N^{(0,0,b_{ll'})}(z_{it}, z_{j_{lt}})d_q \mathcal{F}_N^{(0,0,b_{ll'})} - 1(z_{it}, z_{j_{lt}})$$

$$= \frac{\partial}{\partial z_q} - \sum_{l \neq q} \frac{1}{z_q - z_l}\left[b_{ql} - a_{ql}(1 - \eta(b_{ql}K_{ql})]\right]$$

(79)

the $L_q$ satisfy the same algebra than the Dunkl operator $d_q$ with $K_{ll'} \rightarrow \eta_{ll'}(b_{ll'})K_{ll'}$.

Now we have

$$W_1^n = \mathcal{F}_N^{(0,0,b_{ll'})}(z_{it}, z_{j_{lt}})W_1^m(d)\mathcal{F}_N^{(0,0,b_{ll'})} - 1(z_{it}, z_{j_{lt}})$$

$$= \sum_q z_m^{q+1} \frac{\partial}{\partial z_q} - \sum_{l \neq q} \frac{z_l^{m+1} - z_{ll'}^{m+1}}{z_l - z_{ll'}}[b_{ql} - a_{ql}(1 - \eta(b_{ql}K_{ql})])$$

(80)

and

$$W_2^n = \mathcal{F}_N^{(0,0,b_{ll'})}(z_{it}, z_{j_{lt}})W_2^m(d)\mathcal{F}_N^{(0,0,b_{ll'})} - 1(z_{it}, z_{j_{lt}})$$

$$= \sum_q z_m^{q+2} \frac{\partial^2}{\partial z_q^2} - \sum_{l \neq q} \frac{z_l^{m+2} - z_{ll'}^{m+2}}{(z_l - z_{ll'})^2}[(b_{ql} - a_{ql})b_{ll'} - b_{ll'}(a_{ll'} - 1) - a_{ll'}(1 - \eta(b_{ll'}K_{ll'}))]$$

$$+ \sum_{l \neq q} \frac{z_l^{m+2}}{z_l - z_{ll'}}(b_{ql} - a_{ql})b_{ll'}(b_{ll'} - a_{ll'}(1 - \eta(b_{ll'}K_{ll'}))$$

(81)

$$+ \sum_{q, l \neq l' \neq q} \frac{z_l^{m+2}}{z_l - z_{ll'}}(b_{ql} - a_{ql})b_{ll'}(b_{ll'} - a_{ll'}(1 - \eta(b_{ll'}K_{ll'}))$$

(82)

When $a_{ll'} = a$ and $b_{ll'} = b$ it is possible to prove on the line of the proof of Ref. 29 that the commutators of any symmetrized operators do not contain any extra

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terms depending on $K_{ll'}$, i.e. the abstract algebra does not depend on the statistics of the particles. Moreover by a translation of the $q$-th Dunkl operator by $z_q$ we can introduce an harmonic oscillator potential in the Hamiltonian.

Notice that if we consider the case $b_{ll'} = a_{ll'}$ the generator $\mathcal{W}_0^2$ becomes the multispecies Hamiltonian of CS model with a three-body potential plus terms that contain the projectors $(1 - \eta(b_{ql}K_{ll})q(1 - \eta(b_{ql'}K_{ll'}))$ vanishing on the relative wave functions.

Now we are in position to discuss the CS model with spin. We introduce an internal space (spin space of dimension $2s + 1$) so that the true wave function is the tensor product of the $\Psi(z_q)$ with a spin wave function $\chi$. We can simulate the action of the spin by introducing a $(2s + 1)$-dim “vacuum” such that

$$P_{ll'}|\chi> = \eta_{ll'}(s)|\chi>$$

where now we write the exchange operator as the product of an operator $K_{ll'}$ acting on the labels $ll'$ of the excited states and the operator $P_{ll'}$ acting on the spin vacuum. Now the equation, which the eigenstates of the collective Hamiltonian have to satisfy, becomes

$$(P_{ll'} - K_{ll'})\phi'(z_q)|\chi> = 0$$

where $\phi'(z_q)$ can be related to the symmetric function $\phi(z_q)$ by a Jastrow like function that gives the same phases of spin vacuum:

$$\phi'(z_q) = \phi(z_q)F_s(z_l, z_{l'})$$

On this space the Dunkl operator is written as

$$d_q = \frac{\partial}{\partial z_q} - \sum_{l \neq q} \frac{a_{ql}}{z_q - z_l}(P_{ll'} - K_{ll'})$$

The above formula is a particular case ($\mu = \nu$) of the general case considered in [31]. The general case can be reproduced in our approach by a translation of any root $r_q(r_q^2 = \nu)$ by a quantity $t_q$ such that $\mu = (r_q + t_q)^2$ and by defining the generator $L_q$ with the Jastrow function which depends only on $\mu - \nu = t_q(t_q + r_q)$, that is the generators of the differential algebra are written with a factor which does not take in to account completely the statistics (case $b_{ll'} \neq a_{ll'}$).

It is interesting to look more in detail at this interpretation of the statistics in terms of shift of the roots. The free case (no interaction) is obtained when $r^2 = 0$, i.e. when $r$ is a light-like vector. So the lattice $\Lambda$ can be considered in a Minkowskian space.
If we take $\lambda$ in the dual lattice:

$$\lambda = \alpha + n_+ K_+ + n_- K_- \quad n \in \mathbb{Z}_+$$

(88)

where $K_+, K_-$ are the two lightlike vector such that $K_+ \cdot K_- = 1$ and $\alpha$ is a vector orthogonal to $K_+$ and $K_-$. The interaction with statistics $\nu$ can be introduced by translating the lightlike roots (let us say $r_q = r = nK_+$ for any $q$) by $t$ a vector which requiring the invariance of the total momentum ($\sum_q r_q \cdot \lambda$) can be written as:

$$t \cdot \lambda = 0$$

(89)

The introduction of the spin requires the extension of the algebra of the observables to a charged $W$-algebra.

Notice that the ground state function has an expression very similar to the measure of the Selberg correlation integrals, whose relevance for the CS models have been discussed in [27]. Many relations between CS model and other topics, for instance KdV equations, can by simply understood in this framework.

It is an interesting point to study the action of the operators of Sec.(3), that give amplitudes of GVOs, in the case of CS model.

We restrict the discussion to the case of VOs and apply the operator to obtain the CS ground state of $N - 1$ particles from $N$ one:

$$\psi_{N-1}^{\{r'_q, \lambda\}}(z_q) = \nabla^{\{r_N, r_N-1\}}(z_N, z_{N-1})\psi_N^{\{r_q, \lambda\}}(z_q)$$

$$= \lim_{z_N \to z_{N-1}} (z_N - z_{N-1})^{-r_N - r_{N-1}} \psi_N^{\{r_q, \lambda\}}(z_q)$$

(90)

where

$$r'_q = r_q \quad \forall \quad q = 1, \ldots, N - 2$$

$$r'_{N-1} = r_N + r_{N-1}$$

(91)

We hope that this formulation can be applied on second quantization of CS model where $\nabla$ operator can be identified with annihilator operator.

6 Conclusions

We have shown that by a similarity transformation (”dressing”) of the free differential operators generating the $W_\infty$, we still preserve the algebraic structure and that
with a particular choice of the "dress" we can identify the CS hamiltonian with the
generator $W_0^2$.

The choice of "dressing" depends on two aspect, the first is related to the ex-
istence of non-unitary similarity transformations that give the $W_\infty$ algebra for any
GVO that is discussed in Sec.(2)-(3).

The second gives a sector independent basis by using the permutations operator
$K_{ll'}$ that takes in account the multiply connected configuration space for CS model.
In term of vertex algebra this is just the duality invariance property of amplitudes.

The dressing function is given by the correlation function of the product of N
VOs, computed out of mass shell between an arbitrary in state and an out state
fixed by the choice of the in state and by the value of the roots. The $W$ algebra is so
related with the Ward identities for VOs amplitudes, identities always satisfied also
by correlation functions for the product of generic GVOs. Although we have not
yet explicitly computed the correlation function for product of any GVO, we have
presented here the whole formalism to emphasize the role, we believe fundamental,
of the vertex algebra. In the case of the product of N VOs with the same root,
corresponding to the CS model, we find all the results of [28], but our formalism
allows to get more general equations with potential whose coupling constant is not
necessarily equal for all the particles. The system of N VOs appears as a system of
N particles, of equal mass, with internal quantum numbers specified by the roots $r_i$.
The particles are identical if their roots are equal. The observables of a system of
identical particles must be symmetric operators, so operators not belonging to the
diagonal $W_\infty$, must be not observable. For instance we have ($\mathcal{L}_i$ are the operators
given by eq:(47))

$$ (z_1 \mathcal{L}_1(z_1) - z_2 \mathcal{L}_1(z_2) \psi_N(z_q)) = (r_1 - r_2) \cdot \lambda \psi_N(z_q) \quad (92) $$

and the r.h.s. is vanishing if $r_1 = r_2$. More generally as the momenta defined on
the circle have to be quantized, the product $r_q \cdot \lambda$ must be an integer, that implies
that $\lambda$ has to belong to the dual lattice of the lattice $\Lambda$ of the roots. So translating $\lambda$
by an element of the lattice does not change the value of the relative momenta (up
to an integer), so the translation of $\lambda$ can be interpreted as a Galilean boost of the
system. So the algebra of observables is the whole algebra of differential operators
One may expect that a classification of all the possible vertex operators may give a
classification of all the possible integrable models. Let us emphasize once more that
we have shown that the invariance for a $W_\infty$ algebra, which in the literature has
been established only for particular models, is a general feature, consequence of the
algebra of differential operators for correlation functions. Moreover the connection
between integrable 1+1 models as CS and vertex algebra is given by the intrinsic
structure of GVOs that are an explicit realization of the vertex algebra.

We conjecture that “integrability conditions” as Yang-Baxter equation, can be deduced by properties of vertex algebra as it happens in the simplest case of CS model.

A very interesting question to understand is if there are physical models corresponding to correlation functions of the product of arbitrary GVOs. Independently of any physical interpretation or interest it is natural to raise the question if such models are integrable. We believe that the answer is negative, at least if integrability is meant in the usual sense. In fact integrability requires that $r^2$ in eq.(56) be non negative. This condition is not guaranteed either for the roots or for the weight for a generic (not affine) Kac-Moody algebra.

It is interesting to discuss the connections and the differences between the structure of the integrable non relativistic models and the structure of string theory, where the Lorentzian algebras play an essential role. For instance many of the above considerations still hold replacing the term particle with the term string. However a thorough discussion of this topic requires further analysis and it will be eventually carried out elsewhere.

Finally let us remark that $W_\infty$ algebras are related to area preserving diffeomorphisms and these structures arise as a property of vertex algebra independently of any physical models.
APPENDIX

To give an explicit expression of amplitudes and GVOs products we need to compute any arbitrary \( S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) \) function. This can be done in the following way.

By definition:
\[
S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) = \lim_{z_{eq} \to z_q} S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'})
\] (93)

and
\[
S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) = (\partial_{\{k_{vq}\}}(z_{vq}) F_{N}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) \bigm|_{q,vq})
\] (94)

Factorizing \( F \) in terms of \( q \)-variable:
\[
F_{N}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_{i_l}, z_{i_{l'}}) = \prod_{l > l', l \neq q} F_{l'l'}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_{i_l}, z_{i_{l'}})
\] (95)

\[
\times \prod_{l > q} F_{l'q}^{(ai_{i',q},ai_{i''},ai_{q},a_{vq})}(z_{i_l}, z_{vq}) \prod_{l < q} F_{l'q}^{(ai_{i',q},ai_{i''},ai_{q},a_{vq})}(z_{vq}, z_{i_{l'}})
\]

we indicate with \( k_{vq}^{l} \) and \( k_{vq}^{l'} \) the number of derivatives that act, respectively, on terms \( F_{l'q} \) and \( F_{l'l'} \), these numbers must satisfy the identity:
\[
\sum_{l > q} k_{vq}^{l} + \sum_{l' < q} k_{vq}^{l'} = k_{vq} \quad \forall \{q, vq\}
\] (96)

At this point it is possible to give a factorization of \( S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) \) in terms of two-point functions:
\[
S_{\{k_{vq}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'}) = \sum_{\{\sum l > q k_{vq}^{l} + \sum l' < q k_{vq}^{l'} = k_{vq}\} \bigm|_{l > l'}} \prod_{l > l'} S_{\{k'_{vq}^{l'}\}}^{(ai_{i'},ai_{i''},ai_{j'},ai_{j''})}(z_l, z_{l'})
\] (97)

where indices \( l, l' \) include also the renamed \( q \), and, by means of substitutions:
\[
r_{i_l} \to r_i, \quad r_{l} \to r, \quad k_{vq}^{l'} \to k_i \quad \forall \ l, l'
\] (98)
\[
r_{j_{i'}} \to s_j, \quad r_{v} \to s, \quad k_{vq}^{l'} \to k_j \quad \forall \ l, l'
\] (99)
a general $S$ is:

$$
S^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}_{k_{i},k_{j}}(z_{i},\xi_{j}) = \left(\partial^{(k_{i},k_{j})}_{(z_{i},\xi_{j})}F^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}(z_{i},\xi_{j})\right)
$$

(101)

In this way it is necessary to compute only this class of functions. By the factorization of $F^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}(z_{i},\xi_{j})$ it is possible give an explicit expression for its derivatives:

$$
\partial^{k_{i}}(z_{i})\partial^{k_{j}}(\xi_{j})F^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}(z_{i},\xi_{j}) = \sum_{\{p_{i},p_{j}\}}\sum_{\{l_{i},l_{j}\}}\sum_{i,j}(\partial^{p_{i}}(z_{i})(z_{i}-\xi_{j})^{a_{ij}}\left[\partial^{p_{j}-l_{j}}(\xi_{j})(z_{i}-\xi_{j})^{a_{0j}}\right])
$$

(102)

that can be written in a more compact form introducing the coefficients:

$$
\chi^{(a_{i,j},a_{0,i},a_{0,j})}_{\{l_{i}+l_{j},(p_{i}-l_{i}),(p_{j}-l_{j})}\}} = (-1)^{\sum_{i,j}l_{j}}\prod_{i,j}\left(\begin{array}{c}
a_{i0} \\
p_{i}-l_{i}
\end{array}\right)\left(\begin{array}{c}
a_{ij} \\
l_{i}+l_{j}
\end{array}\right)\left(\begin{array}{c}
a_{0j} \\
p_{j}-l_{j}
\end{array}\right)
$$

(103)

$$
S^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}_{k_{i},k_{j}}(z_{i},\xi_{j}) = F^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}(z_{i},\xi_{j})
$$

(104)

$$
\times\sum_{\{s_{i},s_{j}\}}\chi^{(a_{i,j},a_{0,i},a_{0,j})}_{\{l_{i}+l_{j},(p_{i}-l_{i}),(p_{j}-l_{j})\}}\frac{\chi^{(r_{i},r_{j},s_{i})}_{\{k_{i},(k_{j})\}}}{(z_{i}-\xi_{j})^{l_{i}+l_{j}}(z_{i}-\xi_{j})^{l_{i}}(z_{i}-\xi_{j})^{l_{j}}}
$$

To give a final formula for $S^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}_{k_{i},k_{j}}(z_{i},\xi_{j})$ it is necessary to take the limits the $z_{i} \rightarrow z$ and $\xi_{j} \rightarrow \xi$ for all $i, j$:

$$
S^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}_{k_{i},k_{j}}(z_{i},\xi_{j}) = \lim_{\{z_{i} \rightarrow z\}}\lim_{\{\xi_{j} \rightarrow \xi\}}S^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}_{k_{i},k_{j}}(z_{i},\xi_{j})
$$

(105)

$$
= (\Delta^{(k_{i},k_{j})}(z_{i},\xi_{j}))F^{(a_{i,j},a_{0,i},a_{0,j},a_{00})}(z_{i},\xi_{j}) = \frac{\chi^{(r_{i},r_{j},s_{i})}_{\{k_{i},(k_{j})\}}}{(z_{i}-\xi)^{-r_{s}+\sum_{i}l_{i}+\sum_{j}k_{j}}}
$$

where

$$
\chi^{(r_{i},r_{j},s_{i})}_{\{k_{i},(k_{j})\}} = \sum_{\{p_{i},p_{j}\}}\sum_{\{l_{i},l_{j}\}}\chi^{(a_{i,j},a_{0,i},a_{0,j})}_{\{l_{i}+l_{j},(p_{i}-l_{i}),(p_{j}-l_{j})\}}
$$

(106)
Replacing this formula in the eq: (96) with the obvious renaming of indices:

\[ r_i \rightarrow r_i, \quad r \rightarrow r_l, \quad k_i \rightarrow k_{il}^l \quad \forall \ l, l' \] (107)

\[ s_j \rightarrow r_{jl}, \quad s \rightarrow r_{jl}, \quad k_j \rightarrow k_{jl}^l \quad \forall \ l, l' \] (108)

we obtain the final formula:

\[
S_{\{k_{vq}\}}^{\{a_{i_l,j_l},a_{i_l',j_{l'}},a_{i_{l''}},a_{i_{l'''}}\}}(z_l, z_{l'}) = \sum_{\{\sum_{l>q} k_{il}^l + \sum_{l'<q} k_{il}^l = k_{vq}\} \ l>q} \prod_{l>q} \chi_{\{(k_{il}^l),(k_{jl}^l)\}^l}^{\{r_{i_l},r_{i_l'},r_{i_{l''}},r_{i_{l'''}}\}} (z_l - z_{l'})^{-r_i} \cdot r_{i_l} + \sum_{i_l} k_{il}^l + \sum_{j_l} \chi_{\{(k_{jl}^l),(k_{jl'}^l)\}^l}^{\{r_{j_l},r_{j_{l'}},r_{j_{l''}},r_{j_{l'''}}\}} (z_l - z_{l'})^{-r_j} \cdot r_{j_l} + \sum_{j_l} k_{jl}^l
\] (109)

Q.E.D.
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