COMPACT SURFACES WITH NO BONNET MATE

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Abstract. This note gives sufficient conditions (isothermic or totally nonisothermic) for an immersion of a compact surface to have no Bonnet mate.

1. Introduction

Consider a smooth immersion \( x : M \to \mathbb{R}^3 \) of a connected, orientable surface \( M \), with unit normal vector field \( e_3 \). Its induced metric \( I = dx \cdot dx \) and the orientation of \( M \) induced by \( e_3 \) from the standard orientation of \( \mathbb{R}^3 \) induce a complex structure on \( M \), which provides a decomposition into bidegrees of the second fundamental form \( II \) of \( x \) relative to \( e_3 \),

\[
- de_3 \cdot dx = II = II^{2,0} + HI + II^{0,2}.
\]

Here \( H \) is the mean curvature of \( x \) relative to \( e_3 \) and \( II^{2,0} = II^{0,2} \) is the Hopf quadratic differential of \( x \). Relative to a complex chart \((U, z)\) in \( M \),

\[
I = e^{2u} dz d\bar{z}, \quad II^{2,0} = \frac{1}{2} h e^{2u} d\bar{z},
\]

where the conformal factor \( e^u \), the Hopf invariant \( h \), and the mean curvature \( H \) satisfy the structure equations on \( U \) relative to \( z \),

\[
-4 e^{-2u} u_{zz} = H^2 - |h|^2 \quad \text{Gauss equation}
\]

\[
(e^{2u} h)_{\bar{z}} = e^{2u} H \quad \text{Codazzi equation}
\]

and the Gauss curvature is \( K = H^2 - |h|^2 \). See [JMN16, page 212].

In 1867 Bonnet [Bon67] began an investigation into the problem of whether there exist noncongruent immersions \( x, \tilde{x} : M \to \mathbb{R}^3 \) with the same induced metric, \( I = \tilde{I} \), and the same mean curvature, \( H = \tilde{H} \). This Bonnet Problem has been studied by Bianchi [Bia09], Graustein [Gra24], Cartan [Car42], Lawson–Tribuzy [LT81], Chern [Che85], Kamberov–Pedit–Pinkall [KPP98], Bobenko–Eitner [BE98, BE00], Roussos–Hernandez [RH90], Sabitov [Sab12], the present authors [JMN16], and many others cited in these references.

Definition 1. An immersion \( x : M \to \mathbb{R}^3 \) is Bonnet if there is a noncongruent immersion \( \tilde{x} : M \to \mathbb{R}^3 \) such that \( \tilde{I} = I \) and \( \tilde{H} = H \). Then \( \tilde{x} \) is called a Bonnet mate of \( x \) and \((x, \tilde{x})\) form a Bonnet pair.

A constant mean curvature (CMC) immersion \( x : M \to \mathbb{R}^3 \), for which \( M \) is simply connected and \( x \) is not totally umbilic, admits a 1-parameter family of Bonnet mates, which are known as the associates of \( x \) [JMN16, Example 10.11, page

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The local problem is thus to determine if an immersion \( \mathbf{x} \) with nonconstant mean curvature has a Bonnet mate. By nonconstant mean curvature \( H \) we mean that \( dH \neq 0 \) on a dense, open subset of \( M \).

**Definition 2.** A Bonnet immersion \( \mathbf{x} : M \to \mathbb{R}^3 \) is proper if its mean curvature is nonconstant and there exist at least two noncongruent Bonnet mates.

It is known [JMN16, page 211] that the umbilics of \( \mathbf{x} \) are precisely the zeros of its Hopf quadratic differential \( II^{2,0} \). For the following definitions we assume that \( \mathbf{x} \) has no umbilics in the domain \( U \). If \( (U,z) \) is a complex coordinate chart in \( M \), then the local coefficient \( e^{2u}h \) of \( 2II^{2,0} \) in \( U \) has the polar representation

\[
e^{2u}h = e^{G+ig},
\]

for a smooth function \( G : U \to \mathbb{R} \) and a smooth map \( e^{ig} : U \to S^1 \). The function \( g : U \to \mathbb{R} \) is defined only locally, up to an additive integral multiple of \( 2\pi \). If \( w = w(z) \) is another complex coordinate in \( U \), and if the invariants relative to it are denoted by \( \hat{u} \) and \( \hat{h} \), then

\[
e^{2u}h = e^{2\hat{u}\hat{h}(w')}^2,
\]

where \( w' = \frac{dw}{dz} \) is a nowhere zero holomorphic function of \( z \). Setting \( e^{2u}\hat{h} = e^{G+ig} \) on \( U \), we find by an elementary calculation

\[ g_{zz} = \hat{g}_{zz} \]

on \( U \). The Laplace-Beltrami operator of \((M,I)\) is given in the local chart \((U,z)\) by

\[
\Delta = 4e^{-2u} \frac{\partial^2}{\partial z^2}.
\]

We conclude from (2) that \( \Delta g = \Delta \hat{g} \) on \( U \), and therefore that \( \Delta g \) is a globally defined smooth function on \( M \) away from the umbilic points of \( \mathbf{x} \).

**Definition 3.** A surface immersion \( \mathbf{x} : M \to \mathbb{R}^3 \) is called isothermic if it has an atlas of charts \((U,(x,y))\) each of which satisfies

\[
I = e^{2u}(dx^2 + dy^2) \quad \text{and} \quad II = e^u(adx^2 + cdy^2) \quad [\text{JMN16, Definition 9.5, page 277}].
\]

Definition 3 is equivalent to the first statement of the following definition if there are no umbilics [JMN16, Corollary 9.14, page 280].

**Definition 4.** An umbilic free immersion \( \mathbf{x} : M \to \mathbb{R}^3 \) of an oriented connected surface is isothermic if \( \Delta g = 0 \) identically on \( M \). It is totally nonisothermic if \( \Delta g \neq 0 \) on a connected, open, dense subset of \( M \).

The following is known about the local situation. Suppose that \( \mathbf{x} : M \to \mathbb{R}^3 \) is an umbilic free immersion for which \( M \) is simply connected and possesses a complex coordinate \( z : M \to \mathbb{C} \). Cartan [Car42] proved that if \( \mathbf{x} \) is proper Bonnet, then it has a 1-parameter family of distinct mates [JMN16, Theorem 10.42, pages 340-342]. Graustein [Gra24] proved that if \( \mathbf{x} \) is isothermic and Bonnet, then it is proper Bonnet. The present authors [JMN16, Theorem 10.13, pages 303-304] proved that if \( \mathbf{x} \) is totally nonisothermic, then it has a unique Bonnet mate. This contrasts emphatically with the case when \( M \) is compact, as stated in item (2) of the following Theorem.

What is the global situation? Lawson–Tribuzy [LT81] proved that \( \mathbf{x} : M \to \mathbb{R}^3 \) cannot be proper Bonnet if \( M \) is compact. Since then the question whether there exist Bonnet pairs for a compact surface \( M \) of genus \( g > 0 \) has been open.” Roussos–Hernandez [RH90] proved that \( \mathbf{x} : M \to \mathbb{R}^3 \) has no Bonnet mate if \( M \) is compact and \( \mathbf{x} \) is a surface of revolution with nonconstant mean curvature. Sabitov [Sab12,
Theorem 13, page 144] gives a sufficient condition preventing the existence of a Bonnet mate when the mean curvature is nonconstant and \( M \) is compact. He gives no geometric interpretation of his condition. It is known, and proved in the next section, that a necessary condition that \( \mathbf{x} \) be Bonnet is that its set of umbilics is a discrete subset of \( M \).

The goal of this paper is to prove the following result. It generalizes the Roussos–Hernandez result, since a surface of revolution is isothermic [JMN16, Example 9.7, page 277]. It also gives a geometrical clarification of the Sabitov result.

**Theorem.** Let \( \mathbf{x} : M \to \mathbb{R}^3 \) be a smooth immersion with nonconstant mean curvature \( H \) of a compact, connected surface, and suppose that \( \mathcal{D} \), the set of umbilics of \( \mathbf{x} \), is a discrete subset of \( M \).

1. If \( \mathbf{x} : M \setminus \mathcal{D} \to \mathbb{R}^3 \) is isothermic, then \( \mathbf{x} : M \to \mathbb{R}^3 \) has no Bonnet mate.
2. If \( \mathbf{x} : M \setminus \mathcal{D} \to \mathbb{R}^3 \) is totally nonisothermic, then \( \mathbf{x} : M \to \mathbb{R}^3 \) has no Bonnet mate.

2. **The deformation quadratic differential**

From the Gauss equation above, the Hopf invariants \( h \) and \( \tilde{h} \) relative to a complex coordinate \( z \) of two immersions with the same induced metric and the same mean curvatures must satisfy

\[
|\tilde{h}| = |h|,
\]

since \( \tilde{u} = u \). Hence, the only possible difference in the invariants of two such immersions must be in the arguments of the complex valued functions \( h \) and \( \tilde{h} \). Moreover, taking the difference of their Codazzi equations, we get

\[
(e^{2u}\tilde{h} - e^{2u}h)\bar{z} = e^{2u}(H_z - \bar{H}_z) = 0,
\]
at every point of the domain \( U \) of the complex coordinate \( z \). This means that the function

\[
F = e^{2u}(\tilde{h} - h) : U \to \mathbb{C}
\]
is holomorphic.

**Definition 5.** If \( \mathbf{x}, \tilde{\mathbf{x}} : M \to \mathbb{R}^3 \) are immersions that induce the same complex structure on \( M \), then their deformation quadratic differential is

\[
Q = \widetilde{\mathcal{H}}^{2,0} - \mathcal{H}^{2,0}.
\]

If \( \mathbf{x} \) and \( \tilde{\mathbf{x}} \) have the same induced metric and mean curvature, then the expression for \( Q \) relative to a complex coordinate \( z \) is

\[
Q = \frac{1}{2}e^{2u}(\tilde{h} - h)dz\bar{dz} = \frac{1}{2}Fdz\bar{dz},
\]

which shows that \( Q \) is a holomorphic quadratic differential on \( M \), and

\[
|F + e^{2u}\tilde{h}| = |e^{2u}h| = |e^{2u}h|
\]
on \( U \), since \( |\tilde{h}| = |h| \). \( Q \) is identically zero on \( M \) if and only if \( \tilde{h} = h \) in any complex coordinate system. Therefore, by Bonnet’s Congruence Theorem, \( Q = 0 \) if and only if the immersions \( \mathbf{x} \) and \( \tilde{\mathbf{x}} \) are congruent in the sense that there exists a rigid motion \((\mathbf{y}, A) \in \mathbb{E}(3)\) such that \( \tilde{\mathbf{x}} = \mathbf{y} + Ax \to M \to \mathbb{R}^3 \). Thus, an immersion \( \tilde{\mathbf{x}} : M \to \mathbb{R}^3 \) is a Bonnet mate of \( \mathbf{x} : M \to \mathbb{R}^3 \) if it induces the same metric and mean curvature and the deformation quadratic differential is not identically zero.
Proposition 6. If an immersion $\mathbf{x} : M \to \mathbb{R}^3$ possesses a Bonnet mate $\tilde{\mathbf{x}} : M \to \mathbb{R}^3$, then the umbilics of $\mathbf{x}$ must be isolated and coincide with those of $\tilde{\mathbf{x}}$.

Proof. Under the given assumptions, the holomorphic quadratic differential $Q$ is not identically zero. Therefore, in any complex coordinate chart $(U, z)$, we have $Q = \frac{1}{2} F dz \wedge d\bar{z}$, where $F$ is a nonzero holomorphic function of $z$. Its zeros must be isolated. A point $m \in U$ is an umbilic of $\mathbf{x}$ if and only if $h(m) = 0$ if and only if $\tilde{h}(m) = 0$, by (4). In either case $F(m) = 0$ by (4). Therefore, the set of umbilic points is a subset of the set of zeros of $Q$, which is a discrete subset of $M$. \hfill \Box

Let $\mathbf{x} : M \to \mathbb{R}^3$ be an immersion with a Bonnet mate $\tilde{\mathbf{x}} : M \to \mathbb{R}^3$. Let $(U, z)$ be a complex coordinate chart in $M$ and let $h$ and $\tilde{h}$ be the Hopf invariants of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively, relative to $z$ on $U$. Let $D$ be the set of umbilics of $\mathbf{x}$, necessarily a discrete subset of $M$. On $U \setminus D$ we have $h$ never zero and

$$\tilde{h} = hA,$$

for a smooth function $A : U \setminus D \to S^1$, where $S^1 \subset \mathbb{C}$ is the unit circle. On $U \setminus D$ then, the difference of the Hopf differentials is the holomorphic quadratic differential

$$Q = \frac{1}{2} \tilde{H}^{2,0} - H^{2,0} = H^{2,0}(A - 1).$$

This shows that $A : M \setminus D \to S^1$ is a well-defined smooth map on all of $M \setminus D$.

Remark 7. Under our assumption of nonconstant $H$, the map $A$ cannot be constant, for otherwise $H^{2,0}$ would then be holomorphic and thus $H$ would be constant by the Codazzi equation.

Proposition 8 (Sabitov [Sab12]). If an immersion $\mathbf{x} : M \to \mathbb{R}^3$ possesses a Bonnet mate $\tilde{\mathbf{x}} : M \to \mathbb{R}^3$, then the deformation quadratic differential $Q$ of $\mathbf{x}$ is zero only at the umbilics of $\mathbf{x}$. Therefore, $A : M \setminus D \to S^1$ never takes the value $1 \in S^1$.

Proof. This is Theorem 1, pages 113ff of [Sab12]. He says the result is stated in [Bob08], but he believes the proof there is inadequate. Sabitov’s proof uses results from the Hilbert boundary-value problem. The following proof is essentially the same as Sabitov’s, but avoids use of the Hilbert boundary-value problem.

Seeking a contradiction, suppose $Q(m_0) = 0$ for some point $m_0 \in M \setminus D$. Since $Q$ is holomorphic, and not identically zero, its zeros are isolated. Let $(U, z)$ be a complex coordinate chart of $M \setminus D$ centered at $m_0$, containing no other zeros of $Q$, and such that $z(U)$ is an open disk of $\mathbb{C}$. Now $A(m_0) = 1$ and $A$ is continuous, so we may assume $U$ chosen small enough that $A$ never takes the value $-1$ on $U$. Then there exists a smooth map $v : U \to \mathbb{R}$ such that $-\pi < v < \pi$ and $A = e^{iv}$ on $U$. Since $A = 1$ on $U$ only at $m_0$, it follows that

$$(5) \quad v(U \setminus \{m_0\}) \subset (-\pi, 0) \text{ or } v(U \setminus \{m_0\}) \subset (0, \pi).$$

Let $e^{2u}$ and $h$ be the conformal factor and Hopf invariant of $\mathbf{x}$ relative to $z$. Then $h$ never zero on $U$ implies it has a polar representation $h = e^{f+ig}$, for some smooth functions $f, g : U \to \mathbb{R}$. Now $Q = \frac{1}{2} F dz \wedge d\bar{z}$, where

$$F = e^{2u} e^{i(f+g)} (e^{iv} - 1) = e^{2u + f}(e^{i(g+v)} - e^{ig}) : U \to \mathbb{C}$$

is holomorphic. Using the identity

$$e^{i(g+v)} - e^{ig} = e^{i(2g+v)/2}(e^{iv/2} - e^{-iv/2}) = 2i e^{i(g+v/2)} \sin(v/2),$$

we obtain

$$Q = \frac{1}{2} \frac{1}{2} H^{2,0}(A - 1) = \frac{1}{2} H^{2,0}([e^{i(g+v/2)} - 1] / e^{i(g+v)/2} : U \to \mathbb{C}).$$

Therefore, $Q(m_0) = 0$ for some point $m_0 \in M \setminus D$. This contradicts Proposition 6 and completes the proof.
we get
\[ F = 2ie^{2u+f+(g+v/2)} \sin(v/2) \]
on \( U \). The contour integral of \( d\log F \) about any circle in \( U \) centered at \( m_0 \) is \( 2\pi i \) times the number of zeros of \( F \) inside the circle. By assumption, this integral is not zero. But,
\[ d\log F = d(2u + f + i(g + v/2)) + d\log(|\sin(v/2)|), \]
and the contour integral of the right hand side is zero, since these are exact differentials on \( U \setminus \{m_0\} \). In fact, the values of \( v/2 \) on \( U \setminus \{m_0\} \) lie entirely in \((0, \pi/2)\) or entirely in \((-\pi/2, 0)\), so \( \sin(v/2) \) is never zero. This is the desired contradiction to our assumption that \( Q \) has a zero in \( M \setminus D \).

As a consequence of this Proposition, the smooth map \( A : M \setminus D \to S^1 \) never takes the value \( 1 \in S^1 \), so there exists a smooth map
\[ r : M \setminus D \to (0, 2\pi) \subset \mathbb{R}, \]
such that \( A = e^{ir} \) on \( M \setminus D \).

3. PROOF OF THE THEOREM

Proof. Seeking a contradiction, we suppose that \( x \) possesses a Bonnet mate \( \tilde{x} : M \to \mathbb{R}^3 \). Let \( II^{2,0} \) and \( \tilde{II}^{2,0} \) be the Hopf quadratic differentials of \( x \) and \( \tilde{x} \), respectively. By the preceding propositions, the quadratic differential \( \tilde{II}^{2,0} - II^{2,0} \) is holomorphic on \( M \), and on \( M \setminus D \)
\[ \tilde{II}^{2,0} - II^{2,0} = II^{2,0}(e^{ir} - 1), \]
where the function \( r : M \setminus D \to (0, 2\pi) \) is smooth. Let \((U, z)\) be a complex coordinate chart in \( M \setminus D \). Let \( h \) and \( e^u \) be the Hopf invariant and conformal factor of \( x \) relative to \( z \). Then \( h = e^{f+ig} \) on \( U \), for some smooth functions \( f : U \to \mathbb{R} \) and \( e^{ig} : U \to S^1 \).

1). If \( x \) is isothermic, then \( g_{\bar{z}z} = 0 \) identically on \( U \). Let \( G = f + 2u : U \to \mathbb{R} \). Then \( (e^{G + ig(e^{ir} - 1)})_\bar{z} = 0 \) implies
\[ (6) \quad r_{\bar{z}z} = i(G + ig)_\bar{z}(1 - e^{-ir}) \]
on \( U \). Applying \( \partial_z \) to this, and using that \( r_{\bar{z}} \) is the complex conjugate of \( r_{\bar{z}} \), we find
\[ (7) \quad r_{\bar{z}z} = 0 \]
on \( U \). Hence, \( r : M \setminus D \to (0, 2\pi) \) is a bounded harmonic function. Since the points of \( D \) are isolated and \( r \) is bounded, we know that \( r \) extends to a harmonic function on all of \( M \). But then \( r \) must be constant, since \( M \) is compact. This contradicts our assumption of nonconstant \( H \), by Remark 7.

2). If \( x \) is totally nonisothermic, we have either \( \Delta g \leq 0 \) or \( \Delta g \geq 0 \) on \( M \setminus D \). To be specific, let us suppose that \( \Delta g \leq 0 \) on \( M \setminus D \). Now (6) holds and by the proof of Theorem 10.13 on pages 303-304 of [JMN16], we have
\[ (8) \quad e^{ir} = 1 + \frac{-2g_{\bar{z}z}(g_{\bar{z}z} + iL)}{D}, \]
on \( U \), where \( L = |G_{\bar{z}z} + ig_{\bar{z}z}|^2 - G_{\bar{z}z} \) and \( D = g_{\bar{z}z}^2 + L^2 \). Applying \( \partial_z \) to (6) and using (8), we find
\[ (9) \quad r_{\bar{z}z} = -2g_{\bar{z}z}, \]
on $U$. Therefore, $\Delta r = -2\Delta g \geq 0$ on $M \setminus D$.

Recall [HK76, Def. §2.1, pages 40-41] that a function $v : V \to \mathbb{R} \cup \{ -\infty \}$ on a domain $V \subset \mathbb{C}$ is subharmonic if

1. $-\infty \leq v(z) < +\infty$ in $V$.
2. $v$ is upper semi-continuous in $V$. (This means that for any $c \in \mathbb{R}$, the set \{ $z \in U : v(z) < c$ \} is open in $V$.)
3. If $z_0$ is any point of $V$ then there exist arbitrarily small positive values of $R$ such that

\[
v(z_0) \leq \frac{1}{2\pi R} \int_0^{2\pi} v(z_0 + Re^{it})dt.
\]

If $v$ is of class $C^2$ in $V$, then $v$ is subharmonic in $V$ if and only if $v_{zz} \geq 0$ in $V$ [HK76, Example 3, page 41].

If $M$ is a connected Riemann surface, we define a function $v : M \to \mathbb{R} \cup \{ -\infty \}$ to be subharmonic if for any complex coordinate chart $(U, z)$ of $M$, the local representative $v \circ z^{-1} : z(U) \to \mathbb{R}$ is subharmonic. This is well-defined by the Corollary to Theorem 2.8 on page 53 of [HK76].

We conclude from (9) that $r$ is subharmonic on $M \setminus D$. In the event that $\Delta g \geq 0$ on $M \setminus D$, we conclude that $-r$ is subharmonic and continue as below with $-r$.

Suppose $(U, z)$ is a complex coordinate chart centered at a point $m_0 \in D$, and small enough that no other point of $D$ lies in it. Then $r \circ z^{-1}$ is subharmonic on the open set $z(U) \setminus \{ 0 \}$, so it extends uniquely to a subharmonic function on $z(U)$, by Theorem 5.8 on page 237 of [HK76]. It follows that $r$ extends uniquely to a subharmonic function on $M$.

By Theorem 1.2 on page 4 of [HK76], if $v : V \to \mathbb{R} \cup \{ -\infty \}$ is upper semi-continuous on a nonempty compact domain $V \subset \mathbb{C}$, then $v$ attains its maximum on $V$; i.e., there exists $z_0 \in V$ such that $v(z) \leq v(z_0)$ for all $z \in V$. The same proof shows that this is true for an upper semi-continuous function on a compact Riemann surface. Thus, the subharmonic function $r : M \to \mathbb{R} \cup \{ -\infty \}$ attains its maximum at some point $m_0 \in M$. Let $(U, z)$ be a complex coordinate chart centered at $m_0$. Choose $R > 0$ such that the disk $D(0, R) = \{ z \in \mathbb{C} : |z| \leq R \}$ is contained in $z(U)$. By the maximum principle for subharmonic functions [HK76, Theorem 2.3, page 47], $r \circ z^{-1}$ must be constantly equal to $r(m_0)$ on $D(0, R)$. It follows that

\[
E = \{ m \in M : r(m) = r(m_0) \}
\]

is an open subset of $M$. But

\[
E = M \setminus \{ m \in M : r(m) < r(m_0) \}
\]

is closed, since $r$ is upper semi-continuous. We conclude that $r$ is constant on $M$, which is our sought for contradiction, by Remark 7.

\[\square\]

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