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Is the log-law a first principle result from Lie-group invariance analysis?

A comment on the Article by Oberlack (2001)

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Abstract

The invariance method of Lie-groups in the theory of turbulence carries the high expectation of being a first principle method for generating statistical scaling laws. The purpose of this comment is to show that this expectation has not been met so far. In particular for wall-bounded turbulent flows, the prospects for success are not promising in view of the facts we will present herein.

Although the invariance method of Lie-groups is able to generate statistical scaling laws for wall-bounded turbulent flows, like the log-law for example, these invariant results yet not only fail to fulfill the basic requirements for a first principle result, but also are strongly misleading. The reason is that not the functional structure of the log-law itself is misleading, but that its invariant Lie-group based derivation yielding this function is what is misleading. By revisiting the study of Oberlack (2001) we will demonstrate that all Lie-group generated scaling laws derived therein do not convince as first principle solutions. Instead, a rigorous derivation reveals complete arbitrariness rather than uniqueness in the construction of invariant turbulent scaling laws. Important to note here is that the key results obtained in Oberlack (2001) are based on several technical errors, which all will be revealed, discussed and corrected. The reason and motivation why we put our focus solely on Oberlack (2001) is that it still marks the core study and central reference point when applying the method of Lie-groups to turbulence theory. Hence it is necessary to shed the correct light onto that study.

Nevertheless, even if the method of Lie-groups in its full extent is applied and interpreted correctly, strong natural limits of this method within the theory of turbulence exist, which, as will be finally discussed, constitute insurmountable obstacles in the progress of achieving a significant breakthrough.

Keywords: Symmetries, Lie groups, Scaling laws, Turbulence, Law of the wall, Log-law;

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1. Introduction

Predicting the mean temporal evolution and spatial structure of turbulent fluid flow still remains the Holy Grail of systematic turbulence research for more than a century now. Statistical scaling laws play a key role in this context. To date, aside from several inequalities (Constantin & Fefferman, 1994; Constantin et al., 1999), the only exact statistical scaling equality that has been rigorously derived from first principles, i.e. by only making sole use of the Navier-Stokes equations without any major assumptions or approximations, is, so far, Kolmogorov’s 4/5-law for the third-order longitudinal structure function of the velocity field (Kolmogorov, 1941c; Frisch, 1995; Davidson, 2004). However its exact validity is only restricted to the asymptotic regime of infinite Reynolds number within the inertial range of an idealized unbounded flow, that of statistically homogeneous isotropic flow.

For wall-bounded flows, which inherently give raise to inhomogeneous turbulent boundary layers, the situation is even more worse: All attempts to derive scaling laws from first principles were doomed to failure. Every derivation done so far necessarily involves such
strong assumptions and approximations that the initially attempted linkage to the Navier-Stokes equations with its associated geometrical boundary conditions is lost. However, when withdrawing the claim of dealing with first principle derivations a vast functional range of different scaling laws exist.

Unfortunately this lack of mathematical rigor split the turbulence community into different schools each favoring their own beliefs in never-ending debates on holding the correct scaling behavior within fully developed turbulent boundary layers. In particular when regarding the issue of the functional form for the mean velocity distribution in the overlap region between the inner (near-wall) and outer regions. Even in the asymptotic limit of large Reynolds numbers, theoretical analysis does not predict a unique statistical law as how the mean velocity profile should scale within this thin layer from the wall. Two contrasting laws coexist in the literature: One is the classical log-law where its coefficients are believed to be universal, i.e. independent of the Reynolds number and first deduced by von Kármán (1930), while the second one is a power-law, where, opposed to universality, its coefficients are believed to depend weakly on the Reynolds number (Barenblatt, 1993; Barenblatt & Prostokishin, 1993; Cipra, 1996; George & Castillo, 1997; George, 2007; Barenblatt et al., 2014). Usually the log-law has been emphasized over the power-law, sometimes to the exclusion of the latter (Zagarola et al., 1997; Österlund et al., 2000; Lindgren et al., 2004; Jiménez, 2012, 2013). However in recent years it shows that maybe also the opposite is a reasonable possibility, but nevertheless, the question which scaling law ultimately applies is definitely not resolved yet (Dallas et al., 2009; Marusic et al., 2010; Gad-el-Hak & Buschmann, 2011).

This situation provokes a different perspective in deriving and interpreting statistical scaling laws for high-Reynolds-number turbulent boundary layer flows. Due to an overall lack of mathematical rigor the still-going debate would greatly benefit from a twofold change in attitude:

i) From an attitude which would regard all scaling laws as only being approximative models to the exact but unknown Navier-Stokes scaling behavior. Going with the insight of George Box that essentially “all models are wrong, but some are useful” (Box & Draper, 1987), the adequate question to ask is: How wrong do statistical scaling laws have to be to not be useful?

ii) From an attitude which shares the position of Paul Feyerabend in that “all methodologies, even the most obvious ones, have their limits” (Feyerabend, 1975). Nevertheless, in extending or in using new methodologies to construct a theoretical framework for generating turbulent scaling laws “anything goes” (Feyerabend, 1975), however only under the premiss, of course, that mathematical consistency is always ensured.

Exactly in this context we want to place our comment regarding the method of Lie-group invariance analysis when applied to the theory of turbulence. Although this method carries the high expectation of being a first principle method for generating turbulent scaling laws, we will show that it faces the same analytical problems as any other method when trying to generate such laws a priori. In other words, within the theory of turbulent flows the invariant method of Lie-groups needs to be reduced from its high expectation level of being a first principle method down to a method, which also, like any other analytical method in turbulence theory, is only capable to generate turbulent scaling laws a posteriori, i.e. on basis of physical experience by making use of assumptions and approximations. In this sense the invariant method of Lie-groups has the immanent property, too, to generate next to useful also non-useful turbulent scaling laws, which, if no physical or numerical experiment can be matched to them, need to be discarded.

The aim of our comment is twofold. At first to draw attention to the fact that the first-principle approach in Oberlack (2001), namely to generate statistical scaling laws for wall-bounded turbulent flows based on the “maximum symmetry principle”, is heavily misleading. Moreover, our analysis will show that the results as they are derived in Oberlack
A comment on the Article by Oberlack (2001) cannot be reproduced. In one of the two necessary steps to generate the Lie symmetries in Oberlack (2001) we obtain a different intermediate result, leading thus to a diverse and more general symmetry in particular for the mean velocity field. Secondly, to clarify that although the methodology of Lie-group symmetry analysis (Ovsianikov, 1982; Stephani, 1989; Fushchich et al., 1993; Olver, 1993; Ibragimov, 1994; Bluman & Kumei, 1996; Hydon, 2000; Cantwell, 2002) is a powerful and indispensable tool to generate non-trivial solutions for non-linear differential equations, it nevertheless brings along strong natural limits when applied to the theory of turbulence in general. These limits need all to be recognized in order to correctly assess the feasibility of this method in future work.

This study is organized as follows: Section 2 revisits in detail the results given in Oberlack (2001). We will show that all statistical scaling laws derived in Oberlack (2001) are based on an incorrectly calculated invariance transformation, which misleadingly lead, for example, to the log-law as a specific result for the mean streamwise velocity profile. In contrast to these published and as we believe incorrect results, our newly recalculated and cross-validated invariance transformation does not allow for such a specification. Rather, it results in complete arbitrariness in the scaling behavior for the mean streamwise velocity profile, thus showing that the method of Lie-groups is unable to generate a functionally unique set of turbulent scaling laws from first principles as it is misleadingly claimed in Oberlack (2001).

Section 3 then concludes by generally addressing the limits of the Lie-group methodology in turbulence research, in particular when generating statistical multi-point scaling laws beyond the one-point case. Next to the limits as statistical unclosedness and the presence of boundary conditions, the limits are further driven by the property of intermittency. The issue is that all multi-point functions are highly sensitive to intermittent effects, while a Lie-group based invariance analysis performed upon them, which standardly (up to some exceptions) only returns a finite dimensional Lie-group, is effectively unable to account for these effects. Yet, it should be clear that we do not criticize the method of Lie-groups itself, being a very useful mathematical tool indeed, when only applied to the right problems.

2. Phantom turbulent scaling laws as first principle solutions

As a representative example for a wall-bounded turbulent scaling law, we will for brevity herein only focus on the invariant Lie-group based derivation of the log-law for the mean streamwise velocity profile in the inertial region. In the conclusion of this section, however, we will briefly compare this log-law derivation also to its corresponding and equally valid power-law derivation.

At this point we already would like to emphasize that it is not the log-law itself to be criticized here, which, as we know, undoubtedly acts as a useful scaling law in the inertial region. In the conclusion of this section, however, we do not criticize the method of Lie-groups itself, being a very useful mathematical tool indeed, when only applied to the right problems.

Currently there are two different and independent Lie-group based derivations existing in the literature to generate the log-law. The first and older derivation is based on the deterministic fluctuating Navier-Stokes equations (Oberlack, 2001), while the second and more recent derivation is based on the infinite statistical hierarchy of the multi-point velocity correlation functions (Oberlack & Rosteck, 2010). Although both derivations claim to generate the log-law a priori from first principles, i.e. directly from the Navier-Stokes equations without any major assumptions and approximations, it is only the former derivation in Oberlack (2001) that possibly can claim this.

The reason is that the derivation in Oberlack & Rosteck (2010) is based on a statistical description in which the system of equations is obviously not closed\(^1\), that is, the invariance

\(^1\)Furthermore, in Frewer et al. (2014), which is supported by the study Frewer et al. (2015a), it is shown that the log-law derivation in Oberlack & Rosteck (2010) is essentially based on an unphysical equivalence transformation.
analysis in Oberlack & Rosteck (2010) is based on a system with more unknown functions than equations, while in Oberlack (2001) the corresponding invariance analysis at first sight is presumably based on an opposite system having more equations than unknown functions, due to augmenting the unclosed fluctuating equations by the so-called “velocity product equations” [Eq. (2.12), p. 302].

At first sight it thus seems that Oberlack (2001) has a decisive advantage over Oberlack & Rosteck (2010) when taking the perspective of an invariance analysis into which the “velocity product equations” are incorporated. Because, on the level of the fluctuating Navier-Stokes equations it seems that with this approach a natural invariant closure constraint has been achieved, since “the purpose of (2.12) [the velocity product equations] regarding the symmetry properties of plane shear flows is quite different” [p. 302] and “that (2.12) is crucial to find self-similar mean velocity profiles consistent with the second moment and all higher-order correlation equations” [p. 307], where “the major difference between the classical turbulence modelling approach and the present procedure is the treatment of equation (2.12)” [p. 309].

However, these statements are misleading. As we will demonstrate, this additional set of “velocity product equations” are completely redundant, not only from the perspective of the fluctuating equations themselves, but also from the perspective of any invariance analysis performed upon them. The immediate consequence is that, instead of the log-law, a completely indifferent result is obtained, being incapable to gain any mathematical insight into the statistical solution manifold of the underlying dynamical process. Since this is also the case for all other velocity moments, it simply shows that for wall-bounded turbulent flows a first principle result based on the invariance method of Lie-groups is yet still missing. The obvious reason is that the closure problem of turbulence cannot be bypassed, also then not when utilizing the analytical invariance method of Lie-groups.

2.1. Revisiting the results derived in Oberlack (2001)

If we restrict to non-rotating flows, the transport equations which were analyzed for invariance in Oberlack (2001) are [Eqs. (2.7b), (2.11) and (2.12), respectively]

\[ \mathcal{C} = \frac{\partial u'_k}{\partial x_k} = 0, \]  
\[ \mathcal{N}_i = \frac{\partial u'_i}{\partial t} + u_1 \frac{\partial u'_1}{\partial x_1} + \delta_{i1} u'_2 \frac{\partial \bar{u}_1}{\partial x_2} - \delta_{i1} \left( K + \nu \frac{\partial^2 u'_1}{\partial x_2^2} \right) + \delta_{i2} \frac{\partial \bar{p}^*}{\partial x_2} + \frac{\partial u'_1 u'_1}{\partial x_k} + \frac{\partial p'}{\partial x_i} - \nu \frac{\partial^2 u'_1}{\partial x_k^2} = 0, \]  
\[ \mathcal{P}_{ij} = \mathcal{N}_i u'_j + \mathcal{N}_j u'_i = 0, \]

denoting the continuity (\(\mathcal{C}\)), the momentum (\(\mathcal{N}_i\)) and the above-discussed “velocity product” (\(\mathcal{P}_{ij}\)) equations for the fluctuating fields of the Reynolds decomposed Navier-Stokes equations. All overbarred quantities denote the corresponding averaged fields. Since the analysis is restricted to stationary parallel shear flows, the above set of fluctuating transport equations are augmented by the defining restrictions for the mean fields

\[ \frac{\partial \bar{u}_1}{\partial t} = \frac{\partial \bar{u}_1}{\partial x_1} = \frac{\partial \bar{u}_1}{\partial x_3} = \frac{\partial \bar{p}^*}{\partial x_1} = \frac{\partial \bar{p}^*}{\partial x_3} = \bar{u}_2 = \bar{u}_3 = 0. \]

Note that in Oberlack (2001) (from Eqs. (2.7a-b), p. 301 onwards) not the original mean pressure field \(\bar{p} = \bar{p}(x_1, x_2)\) is used, but instead the effective (lower dimensional) mean pressure field \(\bar{p}^* = \bar{p}^*(x_2)\) is used by decomposing the original field as \(\bar{p} = - K \cdot x_1 + \bar{p}^*\), where \(K\) is the mean constant pressure gradient to drive the flow in the streamwise \(x_1\)-direction.

In order to allow for a possible Reynolds number dependence in the scaling law coefficients, the search in Oberlack (2001) to determine all invariant point-transformations for system (2.1)-(2.4) was hence extended to the class of equivalence transformations, in which the viscosity
\( \nu \sim 1/Re \) is not treated as a parameter, but, next to the space-time coordinates, as an own independent variable.

The corresponding infinitesimal invariance operator (Stephani, 1989; Olver, 1993; Ibragimov, 1994; Bluman & Kumei, 1996; Hydon, 2000; Cantwell, 2002) to generate these Lie-point equivalence transformations for the complete system (2.1)-(2.4) is thus given by the following scalar structure

\[
X = \xi_t \partial_t + \xi_x \partial_x + \xi_\nu \partial_\nu + \eta_\xi \partial_\xi + \eta_\nu \partial_\nu + \eta_{u_1} \partial_{u_1} + \eta_{\nu_1} \partial_{\nu_1} + Y,
\]

where \( Y \) is the prolongation in the infinitesimals for all derivatives appearing in system (2.1)-(2.3) including the constraints (2.4):

\[
\begin{align*}
Y = \eta_u \partial_{u'}, & \quad \eta_u \partial_{u'_i}, & \quad \eta_u \partial_{u'_{ij}}, & \quad \eta_u \partial_{u''_{i,j}}, & \quad \eta_u \partial_{u''_{i,j,k}} + \eta_{\nu'} \partial_{\nu'} \\
& \quad + \eta_{u_1} \partial_{u_1}, & \quad \eta_{u_1} \partial_{u_1,i}, & \quad \eta_{u_1} \partial_{u_1,i,j}, & \quad \eta_{\nu_1} \partial_{\nu_1}, & \quad \eta_{\nu_1} \partial_{\nu_1,i}, & \quad \eta_{\nu_1} \partial_{\nu_1,i,j}.
\end{align*}
\]

The prolongation \( Y \) appears due to the already predetermined transformation rules for all relevant derivatives, as e.g. for \( u'_{i,t} := \partial_t u'_i \) or \( u'_{i,j} := \partial_{x_j} u'_i \), etc., given then by the yet unknown and still to be determined transformation rules for the corresponding dependent variables with respect to their independent variables. In other words, all components of \( Y \) in (2.6) can be explicitly expressed as functions of both the unknown infinitesimals for the independent variables \( \xi_t, \xi_x, \xi_\nu \) and dependent variables \( \eta_\xi, \eta_\nu, \eta_u \). The general and explicit formula for a prolongation of any differential order in terms of its independent and dependent infinitesimals can be found, for example, in the above cited literature.

Now, according to Oberlack (2001), the desired equivalence transformations \( X \) for the considered system (2.1)-(2.4) are determined by solving, in respect to the constraints (2.4), the following corresponding invariance conditions [Eqs. (3.12a-c), p. 305]

\[
\begin{align*}
X \mathcal{C} \mid \phi = 0 &= 0, \\
X \mathcal{K} \mid \kappa = 0 &= 0, \\
X \mathcal{F}_{ij} \mid \phi_{ij} = 0 &= 0.
\end{align*}
\]

In order to explicitly yield these invariant transformations, the calculation in Oberlack (2001) was performed in two steps. In the first step, by solving according to the flow assumptions (2.4) the invariance conditions (2.7) and (2.8), the resulting general equivalence transformation of subsystem (2.1)-(2.2) augmented by (2.4) got calculated in infinitesimal form as [Eq. (3.14)]

\[
\begin{align*}
\xi_t &= a_1(\nu)x_1 + a_2(\nu)x_3 + f_1(t, \nu), \\
\xi_x &= a_1(\nu)x_2 + a_3(\nu), \\
\xi_\nu &= a_4(\nu)t + a_5(\nu), \\
\eta_\xi &= [2a_1(\nu) - a_4(\nu)] \nu, \\
\eta_u &= [a_1(\nu) - a_4(\nu)] u'_i + a_2(\nu)u'_i + \frac{\partial f_1}{\partial t} - g_1(x_2, \bar{u}_1, \bar{p}^*, \nu), \\
\eta_{u_1} &= [a_1(\nu) - a_4(\nu)] u'_i + a_3(\nu) [u'_i + \bar{u}_1] + \frac{\partial f_2}{\partial t}, \\
\eta_{\nu_1} &= [a_1(\nu) - a_4(\nu)] u'_i - a_2(\nu) [u'_i + \bar{u}_1] + \frac{\partial f_2}{\partial t} - x_3 \left[ \frac{\partial^2 f_2}{\partial \nu^2} - a_2(\nu) K \right] - g_2(x_2, \bar{u}_1, \bar{p}^*, \nu) + f_3(t, \nu), \\
\eta_{\nu_1} &= [a_1(\nu) - a_4(\nu)] \bar{u}_1 + g_1(x_2, \bar{u}_1, \bar{p}^*, \nu), \\
\eta_{\nu_1} &= [a_1(\nu) - a_4(\nu)] \bar{p}^* + g_2(x_2, \bar{u}_1, \bar{p}^*, \nu).
\end{align*}
\]

\(^1\)Note that Eq. (3.14) in Oberlack (2001) is the general result for the non-rotating case; the rotating case \( \Omega \neq 0 \) is only considered later in Sec. 3.4, p. 313.
This result of this first step can be obtained and confirmed e.g. by using the computer algebra system (CAS) based symmetry-software-packages GeM (Cheviakov, 2007), SADE (Filho & Figueiredo, 2011), or the latest DESOLV-II package (Vu et al., 2012). However, it should be noted that in first instance the latter DESOLV-II package of Vu et al. (2012) is not giving the invariance (2.10) in its general form. Instead it restricts the result (2.10) to
\[ a_2(\nu) = 0, \quad g_1(x_2, \ddot{u}_1, p^*, \nu) = F(x_2, \nu). \] (2.11)

In the following we will, for reasons of simplicity, only consider this special restriction (2.11), which, as a consequence, will only simplify but not change the general statement and message of this study. Hence, irrespective of this choice, whether to restrict (2.10) by (2.11) or not, the first step in Oberlack (2001) provides a reproducible result.

In clear contrast now to the second step, where the remaining invariance condition (2.9) for the system of the “velocity product equations” (2.3) got incorrectly recognized in Oberlack (2001) as a symmetry breaking condition within the already analyzed subsystem (2.7)-(2.8). This lead (amongst others) to the following wrong key restriction [Eq. (3.18), p. 308]:
\[ g_1(x_2, \ddot{u}_1, p^*, \nu) = b_1(\nu). \] (2.12)

However, our recalculation of this second step only shows complete redundancy in the invariant condition for the “velocity product equations” (2.3), rather than a symmetry-breaking mechanism in the invariances admitted by (2.1)-(2.2). That is to say, reduction (2.12) is not supported by our recalculation (see second part of Appendix B, and entire Appendix C).

The essential error in Oberlack (2001), not to recognize this redundancy, already resides in the idea to consider the invariance conditions of system (2.1)-(2.3) in the uncoupled infinitesimal form (2.7)-(2.9). This allowed to split the invariance analysis into two separate and independent steps. As was shown above, (2.7) and (2.8) were solved in a first step, followed by solving (2.9) in a second step using the results then from the first step. This reasoning, however, is incorrect in that it’s not complete. Since (2.1)-(2.3) represents a strongly coupled system, the corresponding invariance conditions may not be decoupled as expressed in (2.7)-(2.9), but must rather be solved as a coupled, or more precisely, as a combined functional vector, namely as

\[ X_{\mathcal{F}_n} |_{\mathcal{F}_n=0} = 0, \quad \text{with} \quad \mathcal{F}_n := (\mathcal{E}, \mathcal{M}_i, \mathcal{P}_{ij}), \] (2.13)

in order to not only guarantee for consistency, but to also allow for the most general solution, i.e. to allow for the minimal restricted invariant solution manifold in that system. As a result, the coupled invariance condition (2.13) immediately shows that all restrictions coming from the “velocity product equations” (2.9) are essentially redundant to (2.7) and (2.8), and thus not symmetry-breaking as incorrectly claimed in Oberlack (2001). In other words, condition (2.9) has no impact on evaluating (2.7) and (2.8). Because, splitting equation (2.13) into its relevant components
\[ X_{\mathcal{E}} |_{\mathcal{F}_n=0} = 0, \] (2.14)
\[ X_{\mathcal{M}_i} |_{\mathcal{F}_n=0} = 0, \] (2.15)
\[ X_{\mathcal{P}_{ij}} |_{\mathcal{F}_n=0} = 0, \] (2.16)
the last condition (2.16), which includes (2.9), is now satisfied identically when evaluating its left-hand side
\[ X_{\mathcal{P}_{ij}} |_{\mathcal{F}_n=0} = \left[ \mathcal{M}_X u'_j + \mathcal{M}_j X u'_i + u'_j X_{\mathcal{M}_i} + u'_i X_{\mathcal{M}_j} \right] |_{\mathcal{F}_n=0} \]
\[ \equiv 0, \] (2.17)
by using the condition (2.2) for the first two summands, since \( \mathcal{M}_i |_{\mathcal{F}_n=0} = \mathcal{M}_j |_{\mathcal{F}_n=0} \equiv 0 \), and the condition (2.15), which includes (2.8), for the last two summands.
\[ \xi_{x_1} = a_1(\nu)x_1 + b_1(\nu)t + b_2(\nu), \]
\[ \xi_{x_2} = a_1(\nu)x_2 + a_3(\nu), \]
\[ \xi_{x_3} = a_1(\nu)x_3 + b_3(\nu), \]
\[ \xi_t = a_4(\nu)t + a_5(\nu), \]
\[ \xi_\nu = [2a_1(\nu) - a_4(\nu)] \nu, \]
\[ \eta_{u_1} = [a_1(\nu) - a_4(\nu)] u_1', \]
\[ \eta_{u_2} = [a_1(\nu) - a_4(\nu)] u_2', \]
\[ \eta_{u_3} = [a_1(\nu) - a_4(\nu)] u_3', \]
\[ \eta_{p'} = 2[a_1(\nu) - a_4(\nu)] p' + g_2(x_2, \bar{u}_1, \bar{p}^*, \nu) - x_1 [a_1(\nu) - 2a_4(\nu)] K + f_3(t, \nu), \]
\[ \eta_{\bar{u}_1} = [a_1(\nu) - a_4(\nu)] \bar{u}_1 + b_1(\nu), \]
\[ \eta_{\bar{p}^*} = 2[a_1(\nu) - a_4(\nu)] \bar{p}^* - g_2(x_2, \bar{u}_1, \bar{p}^*, \nu). \]

Table 1: Comparison of the final resulting set of infinitesimal generators between Oberlack (2001) [Eq. (3.19), p. 308] (left-hand side) and our re-evaluation ((2.10) restricted by (2.11)) (right-hand side).

Hence, the correct equivalence transformation for system (2.1)-(2.4) is thus either given by the unrestricted result (2.10), or by its restriction (2.11), but not if it’s restricted by (2.12), which, according to Oberlack (2001), would also go along with two other wrongly enforced restrictions \( f_1(t, \nu) = b_1(\nu) t + b_2(\nu) \) and \( f_2(t, \nu) = b_3(\nu) \) [Eq. (3.18), p. 308]. To provide an overview, Table 1 lists the final resulting set of all infinitesimal generators. It explicitly shows and summarizes the substantial difference between the final results given in Oberlack (2001) and our on the restriction (2.11) based re-evaluation. In addition, Appendix B explicitly demonstrates that the key invariant transformation for the mean velocity field is definitely a to-the-restrictions (2.4) compatible symmetry for the equations (2.1)-(2.2), and, again, that all remaining equations (2.3) are redundant and thus obviously not symmetry breaking.

This change in the result is decisive as it now leads to a completely different picture in Oberlack (2001) when generating invariant scaling laws for the 1D mean streamwise velocity profile \( \bar{u}_1 = \bar{u}_1(x_2) \) in the limit of infinite Reynolds number (\( \nu \to 0 \)). Considering the derivation of the log-law, for which, according to Oberlack (2001), the friction velocity breaks the scaling symmetry of the mean velocity profile as \( a_1 = a_4 \neq 0 \), the corresponding invariant surface condition is then no longer given as

\[ \frac{dx_2}{a_1x_2 + a_3} = \frac{d\bar{u}_1}{b_1}, \]

which then would lead to the misleading result [Eq. (3.29), p. 312]

\[ \bar{u}_1(x_2) = d_2 \log(x_2 + d_1) + C, \] with \( d_1 = a_3/a_1, \ d_2 = b_1/a_1. \)

Instead, one obtains the more general and correct invariant surface condition

\[ \frac{dx_2}{a_1x_2 + a_3} = \frac{d\bar{u}_1}{F(x_2)}, \]

which in contrast to (2.19) leads to complete arbitrariness in the result

\[ \bar{u}_1(x_2) = \int \frac{F(x_2)}{a_1x_2 + a_3} dx_2 + C, \]
as the function \( F(x_2) \) is not closer specified, except for only being at least once integrable. Hence the derivation of the invariant log-law (2.19) is completely misleading in Oberlack (2001) as the correct equivalence transformation of the system (2.1)–(2.4) does not allow for such a unique functional specification. It rather induces the generation of any desirable function for the mean velocity profile as given in (2.21), making this result thus essentially useless. The same holds true not only for the log-law, but for all invariant scaling laws which were derived in Oberlack (2001), including, for example, the exponential scaling law in the wake region of turbulent boundary layer flow which were later extensively utilized in the papers by Khujadze & Oberlack (2004) and Guenther & Oberlack (2005).

Considering, for example, the following statements made in Oberlack (2001) such as that “In the case of the logarithmic law of the wall, the scaling with the distance from the wall arises as a result of the analysis and has not been assumed in the derivation.” [p. 299], or “... important to note that group theoretical arguments very much guide the finding ... where the mean velocity profiles are applicable.” [p. 306], or “This is an assumption in the classical derivation [von-K´arm´an-derivation] of the log law of the wall but is a result of the present analysis [Oberlack-(2001)-derivation].” [p. 312], or “The theory is fully algorithmic and no intuition is needed to find a self-similar mean velocity profile.” [p. 321],

we have to conclude, in retrospect, that all these statements are not correct and are even misleading if the correct result (2.21) is not being put forward. That is, without considering the generalized result (2.21), the study in Oberlack (2001) basically gives a wrong impression for using the method of Lie groups in turbulence theory. Because, when performing the analysis of Lie-groups for turbulent flows correctly, in the end one still has to make strong assumptions and to use more than a strong intuition in order to get from a general expression as (2.21) the correct statistical scaling law in the way as it would be proposed by a corresponding physical or numerical experiment.

For example, Oberlack (2001) claims that under certain conditions which are assumed to be valid close to the wall, either the algebraic-law “(3.27)” [Eq. (3.27), p. 312] or the log-law “(3.29)” [Eq. (3.29), p. 312] emerges as a functionally unique result when employing the method of Lie groups in turbulence theory. However, our re-evaluated and correct result for the algebraic-law conditions \( a_1 \neq a_4 \neq 0 \) and \( F(x_2) \neq 0 \)

\[
\frac{dx_2}{a_1 x_2 + a_3} = \frac{d\bar{u}_1}{(a_1 - a_4)\bar{u}_1 + F(x_2)} \quad \Leftrightarrow \quad \bar{u}_1(x_2) = (a_1 x_2 + a_3)^{1-a_4/a_1} \left( \int \frac{F(x_2)}{(a_1 x_2 + a_3)^{2-a_4/a_1}} dx_2 + C \right),
\]

and the correct result (2.20)-(2.21) for the log-law conditions \( a_1 = a_4 \neq 0 \) and \( F(x_2) \neq 0 \)

\[
\frac{dx_2}{a_1 x_2 + a_3} = \frac{d\bar{u}_1}{F(x_2)} \quad \Leftrightarrow \quad \bar{u}_1(x_2) = \int \frac{F(x_2)}{a_1 x_2 + a_3} dx_2 + C,
\]

both show that this is not the case. The reason is that (2.22) as well as (2.23) each form a completely indifferent result: Any arbitrary (integrable) function \( F(x_2) \) can be chosen to represent after its integration the law of the wall. That means, from the perspective of Lie-group theory, neither the algebraic-law nor the log-law is a special or privileged scaling law; any desirable function can be derived from this theory, e.g. choosing \( F(x_2) = \text{const.} \) in (2.22) or (2.23) gives of course the algebraic-law “(3.27)” or the log-law “(3.29)” respectively. But this choice for \( F(x_2) \) is not privileged, one can also choose e.g. a complicated hypergeometric
function, then leading to a law-of-the-wall which behaves not like a pure power-law or a log-law, but as an integrated hypergeometric function, thus ultimately having infinitely many different but equally privileged laws of the wall.

Hence, neither the power-law “(3.27)” nor the log-law “(3.29)” is a first-principle result from Lie-group theory. In Oberlack (2001) both these laws were just derived under wrong conditions, in that a mathematical correct theory has been unfortunately misapplied. In this regard the reader should note that all these problems initially already exist in the earlier publication Oberlack (1999) which only then got generalized in Oberlack (2001).

In summary, the open problem is that turbulence theory should predict a certain scaling function for the one-dimensional mean velocity profile $\bar{u}_1(x_2)$ a priori, which in our opinion, even for a particular flow regime, has not been achieved yet. For example, choosing the sophisticated method of Lie-groups to construct e.g. a scaling law in the vicinity of the wall, will only give the useless result (2.22) or (2.23), which in each case is just an alternative yet more complicated representation for the unknown one-dimensional function $\bar{u}_1(x_2)$. One thus gained nothing in using Lie-groups, one just shifted the problem from one unknown function $\bar{u}_1(x_2)$ to another unknown function $F(x_2)$. However, in Oberlack (2001) the result for the law-of-the-wall is that $F(x_2)$ must be a constant function in the wall-normal coordinate $x_2$, which definitely is an assumption, and thus forms a contradiction to all relevant statements made in Oberlack (2001), in particular to the one that “the theory is fully algorithmic and no intuition is needed to find a self-similar mean velocity profile.” [p. 321].

In this sense Lie-group theory offers no answer, nor does it give any prediction a priori in how turbulence should scale. This failure simply reflects the classical closure problem of turbulence, which, also with Lie-group theory, cannot be bypassed. However, using this method to nevertheless get such an answer would be the same as guessing it, and if one knows what to expect a posteriori then, of course, one can manually arrange everything backwards, and pretend that theory is predicting this result. But such an approach has nothing to do with science.

This concludes the section on deriving a specific turbulent scaling law when using the invariance method of Lie-groups. It showed that although this method carries the high expectation of being a first principle method in turbulence theory, a convincing result is yet still missing in the literature to date — and remains very questionable if such an analytical result from first principles is achievable at all.

While the invariance analysis just presented is set in the broader context of an analysis on Lie-groups in general, we would like to close this investigation by pointing out all natural limits this method faces when using it within the theory of turbulence.

3. The limits of Lie-group methodology in the theory of turbulence

Regarding the limits in using the invariance theory of Lie-groups for differential equations in turbulence research, the main fact should be faced that this analytical approach experiences effectively the same closure-problem issues as the corresponding statistical transport equations for the correlation moments of the flow fields. Changing even to a possible nonlocal integral and formally closed framework, e.g. to the higher statistical level of probability densities in order to allow for a formally complete and fully determined statistical description, as in the framework of the infinite Lundgren-Monin-Novikov chain of equations (Lundgren, 1967; Friedrich et al., 2012) or the functional Hopf equation (Hopf, 1952; McComb, 1990), will only render every systematic invariance analysis nearly infeasible. In particular, because the usage of a nonlocal integral framework, the accompanying existence of several internal and for wall-bounded flows also additional geometric boundary conditions, are all obstructive for an analysis based on Lie-groups, as they all favor the mechanism of symmetry breaking (Frewer et al., 2015a, 2016b; Frewer & Khujadze, 2016a). Thus the main possibilities of the Lie-group methodology are already exhausted at this point.

A further serious limit for the methodology of Lie-groups in turbulence, in particular as standardly used since the key paper by Oberlack (2001), e.g. as in Oberlack & Guenther
(2003); Khujadze & Oberlack (2004); Guenther & Oberlack (2005); Oberlack et al. (2006); Oberlack & Rosteck (2010); Oberlack & Zieleniewicz (2013); Waclawczyk & Oberlack (2013); Avsarkisov et al. (2014) and Waclawczyk et al. (2014), is in the direction to consistently aim at generating useful statistical scaling laws for all higher order moments of a certain \( n \)-point correlation function in the inertial range. Even when applying a Lie-group based invariance analysis for a complex spatiotemporal (spatially nonlocal and temporally chaotic) system correctly, the result is standardly only given by a finite dimensional Lie-group, particularly in elements as scaling or translation of flow fields, i.e. the result is in general (up to some exceptions) repeatedly only given by invariant transformations of global nature, and predominantly is due to the non-integrability of these systems. The consequence are global invariant scaling functions, which are not only incapable of locating the domain where they should apply, but also in which, in a naive way, all higher moments are iteratively connected by constant global group parameters. It is therefore obvious that an extremely complex and omnipresent spatiotemporal property as that of intermittency cannot be captured in this restrictive manner. An invariantly constructed statistical set of global scaling laws simply cannot account for the complicated intermittent behavior which is constantly observed even in the inertial range of constant energy flux.

Exceptions can be found of course in lower dimensional systems, as e.g. in the Burgers equation. However, the nonlinear Burgers equation is in itself a very specific and special case as it’s connected to the linear heat equation via the Cole-Hopf transformation (see e.g. Kevorkian & Cole (1996)). It therefore not only allows for an infinite dimensional Lie-group due to the linear superposition principle, but also to formally express the general solution of the Burgers equation in a closed form, which straightforwardly then also extends to any statistical description of the Burgers equation, as e.g. to the statistical functional Hopf approach — a result somehow or other not recognized by Waclawczyk & Oberlack (2013) as several formal particular solutions of the Burgers-Hopf equation get calculated although the general solution has been already formulated in Hosokawa & Yamamoto (1970). For more details, we refer to our comment Frewer et al. (2016b) and to its reaction Frewer et al. (2016c).

Although the origin of intermittency is still object of research, debates and conjectures there is nevertheless strong evidence that intermittent behavior in a complex system as found in all turbulent flows is closely related to the continuous dynamical breaking of symmetries induced by the system itself (Tsiomker, 2013; Saint-Michel et al., 2013; Castellani, 2003; Kurths & Pikovsky, 1995; Frisch, 1985). This breaking mechanism however acts such that when the system finally reaches a statistically fully developed state the broken symmetries are not globally restored in a statistical sense (Frisch, 1995; Biferale et al., 2003). For example in the inertial range of turbulence the results clearly show that the flow cannot be globally invariant under scaling, neither in a deterministic nor in a statistical sense. Inertial range intermittency when measured with the longitudinal multi-point structure functions show a clear lack of global statistical self-similarity. This breakdown of global scale invariance in the structure functions is expressed by an anomalous or multifractal scaling behavior. The general rule is, the higher the order of the structure functions, the larger is the departure from global statistical self-similarity, that is, inertial-range intermittency becomes more important for higher-order moments and becomes even more pronounced the larger the Reynolds number gets (Falkovich & Sreenivasan, 2006).

But in this regime intermittency not only breaks global statistical self-similarity, also global statistical isotropy may break, however then in a more weaker sense. The reason is that a noncompact group as that of scale invariance is more prone to be broken than a compact group as that of rotation invariance (Frisch, 1983). A statistical anisotropy is measured by comparing the scaling exponents between the longitudinal and the transversal structure functions, which turn out to be different even for an initially prepared homogeneous isotropic flow (Biferale & Procaccia, 2005; Biferale et al., 2008; Benzi et al., 2010). A recent theoretical investigation could show that this difference does not seem to depend on the Reynolds number, i.e. that the weak but existent breaking of isotropy in the inertial range is essentially not based on a
finite Reynolds number effect (Grauer et al., 2012). Hence there is a strong indication that next to global self-similarity also global isotropy may not be statistically restored in the limit of infinite Reynolds numbers, as was originally and contrarily postulated in both the K41-theory (return to global self-similarity and isotropy, Kolmogorov 1941) and in the later refined K62-theory (only return to global isotropy, Kolmogorov 1962).

It is important to mention here that the dynamical process of symmetry-breaking inside such an intermittent-evolving system does not remove the property of scaling per se. Quite the contrary, it rather allows for fundamentally new and more complex scaling laws, which all unfold on a local level without being necessarily connected to an underlying global symmetry or global invariance principle (Frisch, 1991; Benzi et al., 1993; Sreenivasan, 1991; L’vov & Procaccia, 2000; Fujisaka & Grossmann, 2001; Chakraborty et al., 2010).

With this knowledge at hand, it is thus highly questionable if reasonable results can be gained at all when applying the method of Lie-groups onto any set of multi-point equations in the sense of Oberlack et al., as it standardly only leads to finite dimensional transformation groups involving constant (globally valid) group parameters. A result, which, with reasonable certainty, is unable to produce the correct scaling behavior of the turbulent velocity structure functions, all the more so, as with each increase in their order the intermittent behavior consecutively becomes more pronounced. Anyhow, ignoring this insight by continuously using these finite dimensional (global) scaling Lie-groups would in fact only give the same wrong results as a direct dimensional analysis would do.

Appendix A. Complete list of all technical errors in Oberlack (2001)

This section will show all other mistakes that can be found in Oberlack (2001). They will be listed, discussed and corrected in the order as they appear in the text. To note is that all mistakes shown are independent of each other and, interestingly, also independent of the main mistake, which was discussed in Section 2.1 and which for completeness will be repeated here again briefly.

(1): The first technical mistake appearing in Oberlack (2001) is also the key mistake done in that study. As was elaborately discussed in Section 2.1, the considered set of Eqs. (3.12a)-(3.12c) are incorrect in the way that they are too restrictive in not providing the full and complete set of Lie-point symmetries which the underlying system of differential equations Eqs. (3.6a)-(3.6c) can admit. In order to obtain this complete set, the incorrect Eqs. (3.12a)-(3.12c) have to replaced by the correct ones (2.14)-(2.16). In solving these equations, we additionally could prove that Eq. (3.12c) is redundant to the information already provided by Eqs. (3.12a)-(3.12b), in particular not having the ability to break any symmetries as it is incorrectly claimed in Oberlack (2001). Our proof to this redundancy statement is given from three different perspectives: The proof (2.17) by only considering the determining equations (2.14)-(2.16), then the analytic proof in Appendix B by also including the underlying differential equations (2.1)-(2.3), and finally the proof in Appendix C via a full systematic symmetry analysis assisted by a computer algebra system (CAS).

The negative consequence of correcting the symmetry analysis in Oberlack (2001) according to the above mentioned criteria is the complete arbitrariness when constructing invariant turbulent scaling laws, as was demonstrated at the end of Section 2.1. Ultimately this just reflects the closure problem of turbulence, a problem that also the method of Lie-group symmetry analysis cannot solve or bypass.

(2): In the first line of p. 308 it is said, referring to the result obtained in Eq. (3.15), that “the last two terms do not contribute to the constraints for the infinitesimals since $M_{ij}u_j + M_{ji}u_i$ may be factored out”. This statement we cannot confirm. A correct analysis rather shows the opposite: Not the last two but rather the first two terms do not contribute. In fact, the last two terms just reduce to the already given constraints Eq. (3.12b). This finding we will now explicitly demonstrate for the diagonal term $i = j = 1$ which, of course, can be easily transferred to the two other diagonal and then extended to all off-diagonal terms.
Note that for the following proof we only make use of the information and notation as given in Oberlack (2001). Starting point is Eq. (3.15), which itself is derived from Eq. (3.12c) by carrying out the product rule of differentiation and thus then to be notated correctly as

\[
\left(\mathcal{N}_j X u_j + \mathcal{N}_j X u_i + u_j X \mathcal{N}_i + u_i X \mathcal{N}_j\right)_{|\langle \mathcal{N}_j u_j + \mathcal{N}_j u_i \rangle = 0} = 0. \quad (A.1)
\]

We will see that it is important not to suppress the preassigned evaluation constraint of this equation, as it was unfortunately done in Oberlack (2001) with the consequence that important information got lost. For example, for the diagonal term \(i = j = 1\) the above constraint reduces to

\[
\left(\mathcal{N}_1 X u_1 + u_1 X \mathcal{N}_1\right)_{|\langle \mathcal{N}_1 u_1 \rangle = 0} = 0, \quad (A.2)
\]

where we divided the equation and its evaluation constraint by 2. Looking now at the evaluation constraint \(\mathcal{N}_1 u_1 = 0\) more carefully, we see that we can only conclude that \(\mathcal{N}_1 = 0\), since obviously \(u_1 \neq 0\). Hence, equation (A.2) can be equivalently written and thus evaluated as

\[
0 = \left(\mathcal{N}_1 X u_1 + u_1 X \mathcal{N}_1\right)_{\mathcal{N}_1 u_1 = 0} = \mathcal{N}_1 \left| \mathcal{N}_1 = 0 \right| + \left(u_1 X \mathcal{N}_1\right)_{\mathcal{N}_1 = 0} = \mathcal{N}_1 \left| \mathcal{N}_1 = 0 \right| \left(X u_1\right)_{\mathcal{N}_1 = 0} + u_1 \cdot \left(X \mathcal{N}_1\right)_{\mathcal{N}_1 = 0} \quad u_1 \neq 0 \quad \left(X \mathcal{N}_1\right)_{\mathcal{N}_1 = 0}, \quad (A.3)
\]

leading to the result that the constraint for the “velocity product equation” (A.2) reduces to the given constraint \(\left.X \mathcal{N}_1\right|_{\mathcal{N}_1 = 0} = 0\) of Eq. (3.12b). Hence, since the above conclusion easily transfers to all other diagonal and then also naturally to all off-diagonal terms, this procedure shows again that the praised constraints of the “velocity product equations” Eq. (3.12c) are just redundant to the constraints Eq. (3.12b) already given by the momentum equations and thus not symmetry-breaking as incorrectly claimed through the (wrongly) induced constraint equations Eq. (3.18).

Moreover, the evaluation (A.3) also demonstrates that it is the last two terms in (A.1) that give a contribution and not the first two terms, which, oppositely as claimed in Oberlack (2001), evaluate to zero. Also the statement that “\(\mathcal{N}_j u_j + \mathcal{N}_j u_i\) may be factored out” [p. 308] is not visible at all.

(3): Let’s assume that Eq. (3.16) is a constraint equation as claimed in Oberlack (2001) and not an identity equation as proven before in (2). Then, nevertheless, it is not justified to deduce Eq. (3.17) from Eq. (3.16) when evaluating it for \(\mathcal{N}_j u_j + \mathcal{N}_j u_i = 0\) as by Eq. (2.12). For simplicity, we will show this again only for the diagonal term \(i = j = 1\); for all other diagonal and off-diagonal terms the same argument applies:

\[
0 = \left(\mathcal{N}_1 \eta_{u_1}\right)_{(3.16)} = \left(\mathcal{N}_1 \eta_{u_1}\right)_{(3.14)} = \left[a_1(\nu) - a_4(\nu)\right] \mathcal{N}_1 u_1 + \mathcal{N}_1 \left[a_2(\nu) u_3 + \frac{df_1}{dt} - g_1(x_2, \bar{u}_1, \bar{p}, \nu)\right] = \mathcal{N}_1 \left[a_2(\nu) u_3 + \frac{df_1}{dt} - g_1(x_2, \bar{u}_1, \bar{p}, \nu)\right]_{\mathcal{N}_1 u_1 = 0}, \quad (A.4)
\]

where it is now not justified to conclude that the sum of the terms in the square bracket has to be zero, as incorrectly proposed by Oberlack (2001) in putting forward Eq. (3.17) — note that the second unjustified constraint in Eq. (3.17) is obtained when considering the diagonal term \(i = j = 3\). The reason why this choice is not justified is again simply because \(\mathcal{N}_1\) itself is zero when using the evaluation constraint \(\mathcal{N}_1 u_1 = 0\), which is equivalent to \(\mathcal{N}_1 = 0\) since \(u_1 \neq 0\).
In fact, as already said and proven in (2), Eq. (3.16) and thus (A.4) is not a constraining equation but only a zero identity.

(4): The statement that “only the equations for $R_{22}$ in (B1), $\overline{p_{12}}$ in (B3) and $\overline{u_{2p}}$ in (B4) need to be examined, because these equations decouple from the other components in the tensor equations” [p. 325] is only true for the non-rotating case. For the general rotating case $\Omega_k \neq 0$ in (B1)-(B4) the equations do not decouple from the other components as claimed. Hence, the result obtained in (B8) is thus only valid for the case $\Omega_k = 0$.

(5): The result (B8), although correctly obtained from a group classification of the equations $R_{22}$ in (B1), $\overline{p_{12}}$ in (B3) and $\overline{u_{2p}}$ in (B4), and only valid in the case for $\Omega_k = 0$ (see comment (4) above), is not uniquely connected to the proposed symmetry (B7), as the conclusion in Sec. B.1 [pp. 325-327] in Oberlack (2001) misleadingly tries to suggest. Because, as can be readily seen from the following (stationary and non-rotating inviscid parallel flow) equations for $R_{22}$ in (B1),

$$ 0 = -[\tilde{u}_1(x_2 + r_2) - \tilde{u}_1(x_2)] \frac{\partial R_{22}}{\partial r_1} - \left[ \frac{\partial \overline{p_{12}}}{\partial x_2} - \frac{\partial \overline{p_{12}}}{\partial r_2} + \frac{\partial \overline{u_{2p}}}{\partial r_2} \right] - \frac{\partial R_{(22)}(2)}{\partial x_2} + \frac{\partial R_{(2k)}(2k)}{\partial r_k} - \frac{\partial R_{(2(2k))}}{\partial r_k}, $$

(A.5)

for $\overline{p_{12}}$ in (B3)

$$ \frac{\partial^2 \overline{p_{12}}}{\partial x_2^2} - 2 \frac{\partial^2 \overline{p_{12}}}{\partial x_2 \partial r_2} + \frac{\partial^2 \overline{p_{12}}}{\partial r_k \partial r_k} = 2 \frac{d\tilde{u}_1}{dx_2} \frac{\partial R_{22}}{\partial r_1} - \left[ \frac{\partial^2 R_{(22)}(2)}{\partial x_2^2} - 2 \frac{\partial^2 R_{(2k)}(2k)}{\partial x_2 \partial r_k} + \frac{\partial^2 R_{(2(2k))}}{\partial r_k \partial r_1} \right], $$

(A.6)

and for $\overline{u_{2p}}$ in (B4)

$$ \frac{\partial^2 \overline{u_{2p}}}{\partial r_k \partial r_k} = -2 \frac{d\tilde{u}_1}{dx_2} \frac{\partial R_{22}}{\partial r_1} - \frac{\partial^2 R_{(2(2k))}}{\partial r_k \partial r_1}, $$

(A.7)

the particularly considered Lie-point symmetry (B7) can be straightforwardly generalized, for example, to

$$ \begin{align*}
\xi_{r_1} &= q_1 r_1 + q_2, \quad \xi_{r_2} = q_1 r_2 + q_3, \quad \xi_{x_2} = q_1 x_2 + q_4, \\
\eta_{R_{22}} &= q_5 R_{22} + \Gamma_{22}(x_2, r_2, r_3), \quad \eta_{\overline{p_{12}}} = q_5 \overline{p_{12}} - \Lambda_2^{(1)}(x_2, r_1, r_2, r_3), \\
\eta_{\overline{u_{2p}}} &= q_6 \overline{u_{2p}} - \Lambda_2^{(2)}(x_2, r_1, r_2, r_3), \\
\eta_{R_{(2k)(2)}} &= q_6 R_{(2k)(2)} + \delta_{kl} \Lambda_2^{(1)}(x_2, r_1, r_2, r_3), \\
\eta_{R_{(2(2k))}} &= q_6 R_{(2(2k))} + \delta_{kl} \Lambda_2^{(2)}(x_2, r_1, r_2, r_3),
\end{align*} $$

(A.8)

without effecting or changing the constraint equation (B8) for the mean flow $\tilde{u}_1$. The problem with the generalized symmetry now is that we are faced with complete arbitrariness due to the unknown functions $\Gamma_{22}, \Lambda_2^{(1)}$ and $\Lambda_2^{(2)}$: For example, for the two-point velocity correlation $R_{22}$, due to the presence of the unknown and thus arbitrary function $\Gamma_{22}$, any arbitrary scaling law or scaling dependency, in particular in the inhomogeneous direction $x_2$, can be generated now by also showing full compatibility to the log-law ($q_5 = q_6$) or to a power-law ($q_5 \neq q_6$) for the mean flow field $\tilde{u}_1$ as induced by the constraint equation (B8), and not only solely by the particular scaling dependency as misleadingly proposed in (B11) and (B13).

In other words, the Lie-group symmetry method cannot provide an answer as how, for example, the higher-order moment $R_{22}$ (being the diagonal Reynolds stress $\tau_{22}$ in the one-point limit $r \to 0$) should scale if we assume a log-law or a power-law for the mean flow field according to the constraint (B8), or to its integrated form (B9). For that, modelling procedures and exogenous information from numerical simulations or physical experiments are necessarily needed to get further insights. Of course, this problem of arbitrariness in invariant scaling we not only face for $R_{22}$, but also for all other higher-order correlations as
on the right-hand side of Table the mean velocity field $\bar{u}$ result of the first part. In the first part we will demonstrate that the key transformation for

This section consists of two parts, where the result of the second part will depend on the result of the first part. In the first part we will demonstrate that the key transformation for the mean velocity field $\bar{u}_1$ (which is taken as a sub-group from the Lie-group transformation on the right-hand side of Table 1) is definitely admitted as a symmetry transformation of the equations (2.1)-(2.2) and that it’s compatible to all restrictions (2.4). Then, in the second part, we will easily see that at this stage the remaining “velocity product equations” (2.3) are just redundant to the already transformed momentum equations (2.2) and thus not symmetry breaking as contrarily claimed in Oberlack (2001). Both parts will be performed without using a computer algebra system (CAS).

Part I. The key sub-group transformation we want to consider is obtained if we put all group functions to zero (on the right-hand side of Table 1), except $F \neq 0$, which, for simplicity, we want to restrict only to a pure $x_2$-dependence. In non-infinitesimal form the transformation is thus given as the following arbitrary $x_2$-dependent translation of the mean and fluctuating streamwise velocity fields

$$
T : \quad \begin{aligned}
\bar{t} &= t, \quad \bar{x}_i = x_i, \quad \bar{\nu} = \nu, \quad \bar{u}'_i = u'_i - \delta_{11}F(x_2), \quad \bar{p}' = p', \\
\bar{u}_1 &= \bar{u}_1 + F(x_2), \quad \bar{p}^* = \bar{p}^*.
\end{aligned}
$$

Now, when inserting this transformation into the continuity equation (2.1) we obtain the invariant result

$$
0 = \mathcal{C} = \frac{\partial u'_k}{\partial x_k} = \frac{\partial \bar{u}'_k}{\partial \bar{x}_k} + \delta_{k1}\delta_{k2} \frac{dF(\bar{x}_2)}{d\bar{x}_2} = \frac{\partial \bar{u}'_k}{\partial \bar{x}_k} = \mathcal{C},
$$

(B.2)

which is also the case for the momentum equations (2.2)

$$
0 = \mathcal{N}' = \frac{\partial u'_i}{\partial t} + \bar{u}_1 \frac{\partial u'_i}{\partial x_1} + \delta_{i1}\delta_{k2} \frac{d\bar{u}_1}{d\bar{x}_2} - \delta_{i1} \left( K + \nu \frac{d^2 \bar{u}_1}{dx_2^2} \right)
+ \delta_{i2} \frac{\partial \bar{p}^*}{d\bar{x}_2} + \frac{\partial u'_i u'_k}{\partial x_k} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u'_i}{\partial x_i^2}
$$

\(^1\)Note that since the considered system of equations (2.1)-(2.4) is underdetermined (unclosed) all admitted invariances only act as equivalence transformations and not as symmetry transformations (Meleshko, 1996; Ibragimov, 2004; Bila, 2011; Chirkunov, 2012; Frewer et al., 2014). Although in this section as well as in the next section we will designate these invariant transformations as symmetries, we have to keep in mind that in a strict sense they are not symmetries but only equivalences.
\[
\frac{\partial \tilde{u}_1'}{\partial t} + \tilde{u}_1 \frac{\partial \tilde{u}_1'}{\partial \tilde{x}_1} - F(\tilde{x}_2) \frac{\partial \tilde{u}_1'}{\partial \tilde{x}_1} + \delta_{11} \left( \tilde{u}_1' \frac{d \tilde{u}_1'}{d \tilde{x}_2} - \tilde{u}_2' \frac{d F(\tilde{x}_2)}{d \tilde{x}_2} - K - \nu \frac{d^2 \tilde{u}_1'}{d \tilde{x}_2^2} + \tilde{\nu} \frac{d^2 F(\tilde{x}_2)}{d \tilde{x}_2^2} \right) \\
+ \delta_{12} \frac{d \tilde{p}^*}{d \tilde{x}_2} + \frac{\partial \tilde{u}_1' \tilde{u}_k'}{\partial \tilde{x}_k} + F(\tilde{x}_2) \frac{\partial \tilde{u}_1'}{\partial \tilde{x}_1} + \delta_{11} \tilde{u}_2' \frac{d F(\tilde{x}_2)}{d \tilde{x}_2} - \frac{\partial \tilde{p}'}{\partial \tilde{x}_1} - \tilde{\nu} \frac{d^2 \tilde{u}_1'}{d \tilde{x}_2^2} - \delta_{11} \frac{d^2 F(\tilde{x}_2)}{d \tilde{x}_2^2} = \tilde{N}_1, \\
\text{(B.3)}
\]

and for the restriction equations (2.4), which, after employing transformation (B.1), stay unchanged as well

\[
\frac{\partial \tilde{u}_1'}{\partial t} = \frac{\partial \tilde{u}_1}{\partial \tilde{x}_1} = \frac{\partial \tilde{u}_1}{\partial \tilde{x}_3} = \frac{\partial \tilde{p}^*}{\partial \tilde{x}_1} = \frac{\partial \tilde{p}^*}{\partial \tilde{x}_3} = 0.
\text{(B.4)}
\]

Hence, the (2.4)-restricted equations \( \mathcal{C} \) (2.1) and \( \mathcal{M}_i \) (2.2) admit transformation \( T \) (B.1) as a symmetry.

**Part II.** In order to complete the validation procedure of Part I, we also have to check if the “velocity product equations” \( \mathcal{P}_{ij} \) (2.3) admit \( T \) (B.1) as a symmetry transformation. But these “equations” are redundant to the momentum equations \( \mathcal{M}_i \) (2.2). They all show a degenerate behavior when trying to independently transform them from the underlying momentum equations \( \mathcal{M}_i \). Consider e.g. the first diagonal component

\[
0 = \mathcal{P}_{11} = \mathcal{M}_1 u_1' + \mathcal{M}_1 u_1' = 2 \mathcal{M}_1 u_1'.
\text{(B.5)}
\]

If we would treat (B.5) as an own and from \( \mathcal{M}_1 \) independent equation, we have to globally conclude that either \( u_1' \) must be zero or that \( \mathcal{M}_1 \) must be zero in order to satisfy equation (B.5). But, since the former choice \( u_1' = 0 \) has to be excluded, obviously, equation (B.5) can only be equivalent (necessarily and sufficiently) to the latter choice

\[
0 = \mathcal{P}_{11} = 2 \mathcal{M}_1 u_1', \text{ with } u_1' \neq 0 \iff \mathcal{M}_1 = 0,
\text{(B.6)}
\]

i.e. the “velocity product equation” \( \mathcal{P}_{11} \) is redundant and thus mathematically equivalent to the momentum equation \( \mathcal{M}_1 \). Therefore, since the latter equation stays invariant under transformation \( T \) (B.1), the former equation will stay invariant too. To be explicit, let’s apply transformation \( T \) (B.1), with the already obtained result \( \mathcal{M}_1 = \tilde{\mathcal{M}}_1 \) (B.3), to equation (B.5)

\[
0 = \mathcal{P}_{11} = 2 \mathcal{M}_1 u_1' = 2 \tilde{\mathcal{M}}_1 \left[ \tilde{u}_1' + F(\tilde{x}_2) \right].
\text{(B.7)}
\]

Now, since \( \tilde{\mathcal{M}}_1 = 0 \) (as shown in (B.3)), and since by construction \( F \neq 0 \) (the initial assumption), we obtain the equivalent relation

\[
\tilde{\mathcal{M}}_1 = 0 \iff \tilde{\mathcal{M}}_1 F(\tilde{x}_2) = 0, \text{ with } F \neq 0,
\text{(B.8)}
\]

which therefore will turn equation (B.7) into the (redundant and necessary) invariant form

\[
0 = \mathcal{P}_{11} = 2 \mathcal{M}_1 u_1' = 2 \tilde{\mathcal{M}}_1 \tilde{u}_1' + 2 \tilde{\mathcal{M}}_1 F(\tilde{x}_2) = 2 \tilde{\mathcal{M}}_1 \tilde{u}_1' = \tilde{\mathcal{P}}_{11}.
\text{(B.9)}
\]

The conclusion that \( \mathcal{P}_{11} \) admits \( T \) (B.1) as a symmetry transformation due to that \( \mathcal{M}_1 \) admits it (and vice versa), can also be readily verified when using any symmetry-determining computer algebra package (see Appendix C): The symmetry results stay completely unaffected when either including or excluding the “velocity product equation” \( \mathcal{P}_{11} \) (2.3) in a corresponding symmetry analysis next to the continuity equation \( \mathcal{C} \) (2.1), the momentum equations \( \mathcal{M}_i \) (2.2) and all constraint equations (2.4).

Hence, relation (B.9) is a fully redundant invariance of the actual system (B.2)-(B.4) and thus clearly not symmetry breaking. This conclusion, of course, also applies to all other components of \( \mathcal{P}_{ij} \) (2.3) — the procedure is exactly the same as given just before for \( \mathcal{P}_{11} \).
Appendix C. Explicit symmetry validation with CAS

By using the Maple-based symmetry-packages GeM (Cheviakov, 2007), SADE (Filho & Figueiredo, 2011) and DESOLV-II (Vu et al., 2012), we will demonstrate that irrespective of whether the “velocity product equations” $\mathcal{R}_{ij}$ (2.3) are excluded from or included into the analysis, the final result will always stay unchanged in all three symmetry-determining algorithms. In the case of GeM and SADE this result is given by (2.10), while for DESOLV-II it is further restricted by (2.11).

Hence, all three CAS results explicitly show that the “velocity product equations” $\mathcal{R}_{ij}$ (2.3) are not only redundant from the perspective of the fluctuating Navier-Stokes equations themselves (2.1)-(2.2), but also redundant from the perspective of a symmetry analysis performed upon them (2.14)-(2.15). The symmetry breaking mechanism as claimed in Oberlack (2001), which should solely arise from $\mathcal{R}_{ij}$ (2.3), is thus not supported.

In the following we will list the corresponding codes for all three packages. In each of these three sections we begin to first list the procedure without the $\mathcal{R}_{ij}$-equations (2.3) in order to first compare to the central result (2.10). Then, in a second part, we will list the procedure with the $\mathcal{R}_{ij}$-equations in order to then easily compare and to see that the central symmetry result (2.10) stays entirely unchanged. Finally note again that the DESOLV-II algorithm implicitly restricts the result (2.10) consistently to (2.11) in both cases, i.e., with or without the $\mathcal{R}_{ij}$-equations, the corresponding symmetry result simply stays unchanged too.

Ia. GeM-Package excluding the $\mathcal{R}_{ij}$-equations (2.3):

Definition:

> restart: with(linalg): with(PDEtools): with(DEtools): with(GeM):

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Definitions:

> X:=(t,x,y,z,nu):
> C:=diff(u(X),x)+diff(v(X),y)+diff(w(X),z);
> N1:=diff(u(X),t)+U(X)*diff(u(X),x)+v(X)*diff(U(X),y)-(K+nu*diff(U(X),t, x, y, z, ν)) + v(t, x, y, z, ν) (\frac{∂}{∂y} U(t, x, y, z, ν)) - K - ν (\frac{∂^2}{∂y^2} U(t, x, y, z, ν)) + 2 u(t, x, y, z, ν) (\frac{∂}{∂y} u(t, x, y, z, ν)) + 2 u(t, x, y, z, ν) w(t, x, y, z, ν)
> N2:=diff(v(X),t)+U(X)*diff(v(X),x)+diff(P(X),y)+diff(v(X)*u(X),x)+diff(v(X)*w(X),z)+diff(p(X),y)-nu*(diff(v(X),x,x)+diff(u(X),y,y)+diff(u(X),z,z));

IIa. GeM-Package excluding the $\mathcal{R}_{ij}$-equations (2.3):

Definition:

> X:=(t,x,y,z,nu):
> C:=diff(u(X),x)+diff(v(X),y)+diff(w(X),z);
> N1:=diff(u(X),t)+U(X)*diff(u(X),x)+v(X)*diff(U(X),y)-(K+nu*diff(U(X),t, x, y, z, ν)) + v(t, x, y, z, ν) (\frac{∂}{∂y} U(t, x, y, z, ν)) - K - ν (\frac{∂^2}{∂y^2} U(t, x, y, z, ν)) + 2 u(t, x, y, z, ν) (\frac{∂}{∂y} u(t, x, y, z, ν)) + 2 u(t, x, y, z, ν) w(t, x, y, z, ν)
> N2:=diff(v(X),t)+U(X)*diff(v(X),x)+diff(P(X),y)+diff(v(X)*u(X),x)+diff(v(X)*w(X),z)+diff(p(X),y)-nu*(diff(v(X),x,x)+diff(u(X),y,y)+diff(u(X),z,z));

IIIa. GeM-Package excluding the $\mathcal{R}_{ij}$-equations (2.3):

Definition:
N3 := diff(w(X),t) + U(X)*diff(w(X),x) + diff(w(X)*u(X),x) + diff(w(X)*v(X),y) + diff(w(X)*w(X),z) + diff(p(X),z) - nu*(diff(w(X),x,x) + diff(w(X),y,y) + diff(w(X),z,z));

Equations (2.1), (2.2) & (2.4):

symmetry_algorithm:

gem_init_defs([X],[u(X),v(X),w(X),p(X),U(X),P(X)],[],[K],0,[eqnC,eqnN1,eqnN2,eqnN3,eqnR1,eqnR2,eqnR3,eqnR4,eqnR5,eqnR6]);
sym\_sol := \{ eta\_P = -F12(y, \nu, U, P) - F14(\nu, U, P) + 2(-F9(\nu) + F6(\nu)) P - F16(\nu, U) + F18(\nu),
eta\_U = -F9(\nu) U + U F6(\nu) - F11(y, \nu, U, P),
eta\_p = -z F5_{t,t} - x F8_{t,t} + F12(y, \nu, U, P) + F14(\nu, U, P) + F16(\nu, U) + F17(t, \nu) + (2p - Kx) F6(\nu) + (-2p + 2Kx) F9(\nu) - z F3(\nu) K,
eta\_u = -F3(\nu) w + (-F9(\nu) + F6(\nu)) u + F8_t + F11(y, \nu, U, P),
eta\_v = v (-F9(\nu) + F6(\nu)),
eta\_w = F5_{t} + (U + u) F3(\nu) - F9(\nu) w + w F6(\nu),
x_{\mu} = \nu (2 F6(\nu) - F9(\nu)),
x\_t = F9(\nu) t + F10(\nu),
x_{,x} = F6(\nu) x - F3(\nu) z + F8(t, \nu),
x_{,y} = F6(\nu) y + F7(\nu),
x_{,z} = F3(\nu) x + F6(\nu) z + F5(t, \nu)\}

Redefinition of group functions as used in (2.10):
> _F6(\nu) := a1(\nu); _F3(\nu) := -a2(\nu); _F8(t, \nu) := f1(t, \nu);
> _F7(\nu) := a3(\nu); _F5(t, \nu) := f2(t, \nu); _F9(\nu) := a4(\nu);
> _F10(\nu) := a5(\nu); _F11(y, \nu, U, P) := g1(y, \nu, U, P);
> _F17(t, \nu) := f3(t, \nu) + F18(\nu);
> _F12(y, \nu, U, P) := -g2(y, \nu, U, P) - F14(\nu, U, P) - F16(\nu, U) + F18(\nu);
Ib. GeM-Package including the $\mathbf{P}_{ij}$-equations (2.3):

Header:

> restart: with(linalg): with(PDEtools): with(DEtools): with(GeM):

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Definitions:

> X:=(t,x,y,z,nu):
> C:=diff(u(X),x)+diff(v(X),y)+diff(w(X),z);
> eqnP11:=2*N1*u(X)=0: eqnP12:=2*N2*u(X)=0: eqnP13:=2*N3*u(X)=0:
> eqnP22:=2*N2*v(X)=0: eqnP33:=2*N3*w(X)=0:
> eqnC:=C=0: eqnN1:=N1=0: eqnN2:=N2=0: eqnN3:=N3=0:

Equations (2.1), (2.2) & (2.4) including the "velocity product equations" (2.3):

$$
\text{eqnR1} := \frac{\partial}{\partial t} \mathbf{u}(X) = 0: \quad \text{eqnR2} := \frac{\partial}{\partial x} \mathbf{u}(X) = 0: \quad \text{eqnR3} := \frac{\partial}{\partial z} \mathbf{u}(X) = 0:
$$

$$
\text{eqnR4} := \frac{\partial}{\partial t} \mathbf{v}(X) = 0: \quad \text{eqnR5} := \frac{\partial}{\partial x} \mathbf{v}(X) = 0: \quad \text{eqnR6} := \frac{\partial}{\partial z} \mathbf{v}(X) = 0:
$$

$$
\text{eqnR7} := \frac{\partial}{\partial t} \mathbf{w}(X) = 0: \quad \text{eqnR8} := \frac{\partial}{\partial x} \mathbf{w}(X) = 0: \quad \text{eqnR9} := \frac{\partial}{\partial z} \mathbf{w}(X) = 0:
$$

$$
\text{eqnR10} := \frac{\partial}{\partial t} \mathbf{u}(X) = 0: \quad \text{eqnR11} := \frac{\partial}{\partial x} \mathbf{u}(X) = 0: \quad \text{eqnR12} := \frac{\partial}{\partial z} \mathbf{u}(X) = 0:
$$

$$
\text{eqnR13} := \frac{\partial}{\partial t} \mathbf{v}(X) = 0: \quad \text{eqnR14} := \frac{\partial}{\partial x} \mathbf{v}(X) = 0: \quad \text{eqnR15} := \frac{\partial}{\partial z} \mathbf{v}(X) = 0:
$$

$$
\text{eqnR16} := \frac{\partial}{\partial t} \mathbf{w}(X) = 0: \quad \text{eqnR17} := \frac{\partial}{\partial x} \mathbf{w}(X) = 0: \quad \text{eqnR18} := \frac{\partial}{\partial z} \mathbf{w}(X) = 0:
$$

$$
\text{eqnR19} := \frac{\partial}{\partial t} \mathbf{u}(X) = 0: \quad \text{eqnR20} := \frac{\partial}{\partial x} \mathbf{u}(X) = 0: \quad \text{eqnR21} := \frac{\partial}{\partial z} \mathbf{u}(X) = 0:
$$

$$
\text{eqnR22} := \frac{\partial}{\partial t} \mathbf{v}(X) = 0: \quad \text{eqnR23} := \frac{\partial}{\partial x} \mathbf{v}(X) = 0: \quad \text{eqnR24} := \frac{\partial}{\partial z} \mathbf{v}(X) = 0:
$$

$$
\text{eqnR25} := \frac{\partial}{\partial t} \mathbf{w}(X) = 0: \quad \text{eqnR26} := \frac{\partial}{\partial x} \mathbf{w}(X) = 0: \quad \text{eqnR27} := \frac{\partial}{\partial z} \mathbf{w}(X) = 0:
$$

$$
\text{eqnR28} := \frac{\partial}{\partial t} \mathbf{u}(X) = 0: \quad \text{eqnR29} := \frac{\partial}{\partial x} \mathbf{u}(X) = 0: \quad \text{eqnR30} := \frac{\partial}{\partial z} \mathbf{u}(X) = 0:
$$

$$
\text{eqnR31} := \frac{\partial}{\partial t} \mathbf{v}(X) = 0: \quad \text{eqnR32} := \frac{\partial}{\partial x} \mathbf{v}(X) = 0: \quad \text{eqnR33} := \frac{\partial}{\partial z} \mathbf{v}(X) = 0:
$$

$$
\text{eqnR34} := \frac{\partial}{\partial t} \mathbf{w}(X) = 0: \quad \text{eqnR35} := \frac{\partial}{\partial x} \mathbf{w}(X) = 0: \quad \text{eqnR36} := \frac{\partial}{\partial z} \mathbf{w}(X) = 0:
$$

Equations (2.1), (2.2) & (2.4) including the "velocity product equations" (2.3):

> eqnC := C=0: eqnN1 := N1=0: eqnN2 := N2=0: eqnN3 := N3=0:
> eqnP11 := 2*N1*u(X)=0: eqnP22 := 2*N2*v(X)=0: eqnP33 := 2*N3*w(X)=0:
> eqnP12 := N1*v(X)+N2*u(X)=0: eqnP13 := N1*w(X)+N3*u(X)=0:
> eqnP23 := N2*w(X)+N3*v(X)=0:
> eqnR1 := diff(U(X),t)=0: eqnR2 := diff(U(X),x)=0: eqnR3 := diff(U(X),z)=0:
> eqnR4 := diff(P(X),t)=0: eqnR5 := diff(P(X),x)=0:
> eqnR6 := diff(P(X),z)=0:
Symmetry Algorithm:

Starting definitions for Point Symmetry or Conservation Law computation...

Independent variables: \( \nu, t, x, y, z \)
Dependent variables: \( P, U, p, u, v, w \)

Free functions: \( [] \)
Free constants: \( [K] \)

Variable definition successful

Computing necessary differential consequences...
29 differential consequence(s) computed.

Symmetry Algorithm:

Starting definitions for Symmetry/Conservation Law analysis successful.

Generating t.v.f. coordinates of derivatives...

Done. Generating determining equations...

Done. Splitting...

Split successful. The split system returned. Number of equations: 4865

Redefinition of group functions as used in (2.10):

\[
\begin{align*}
_F6(\nu) &= a_1(\nu); _F3(\nu) := -a_2(\nu); _F8(t, \nu) := f_1(t, \nu); \\
_F7(\nu) &= a_3(\nu); _F5(t, \nu) := f_2(t, \nu); _F9(\nu) := -a_4(\nu); \\
_F10(\nu) &= a_5(\nu); _F11(y, u, U) := -\beta(y, u, U, P); \\
_F17(t, \nu) &= f_3(t, \nu) - F18(\nu); \\
_F12(y, u, U, P) &= -g_2(y, u, U, P) - F14(\nu, U, P) - F16(\nu, U) + F18(\nu); \\
\end{align*}
\]
\[ F_6(\nu) := a_1(\nu) \]
\[ F_3(\nu) := -a_2(\nu) \]
\[ F_8(t, \nu) := f_1(t, \nu) \]
\[ F_7(\nu) := a_3(\nu) \]
\[ F_5(t, \nu) := f_2(t, \nu) \]
\[ F_9(\nu) := a_4(\nu) \]
\[ F_{10}(\nu) := a_5(\nu) \]
\[ F_{11}(y, \nu, U, P) := -g_1(y, \nu, U, P) \]
\[ F_{17}(t, \nu) := f_3(t, \nu) - F_{18}(\nu) \]
\[ F_{12}(y, \nu, U, P) := -g_2(y, \nu, U, P) - F_{14}(\nu, U, P) - F_{16}(\nu, U) + F_{18}(\nu) \]

Final Result (still identical to result (2.10)):

\[ x_t = a_1(\nu) x + a_2(\nu) z + f_1(t, \nu) \]
\[ x_y = a_1(\nu) y + a_3(\nu) \]
\[ x_z = -a_2(\nu) x + a_1(\nu) z + f_2(t, \nu) \]
\[ x_t = a_4(\nu) t + a_5(\nu) \]
\[ x_{\nu} = \nu (2a_1(\nu) - a_4(\nu)) \]
\[ \eta_u = a_2(\nu) w + (-a_4(\nu) + a_1(\nu)) u + f_1(t) - g_1(y, \nu, U, P) \]
\[ \eta_v = v (-a_4(\nu) + a_1(\nu)) \]
\[ \eta_w = f_2(t) - (u + U) a_2(\nu) - a_4(\nu) w + w a_1(\nu) \]
\[ \eta_p = -z f_2(t, t) - x f_1(t, t) - g_2(y, \nu, U, P) + f_3(t, \nu) + (2p - x K) a_1(\nu) + (-2p + 2x K) a_4(\nu) + z a_2(\nu) K \]
\[ \eta_U = g_1(y, \nu, U, P) - a_4(\nu) U + a_1(\nu) U \]
\[ \eta_P = g_2(y, \nu, U, P) + 2(-a_4(\nu) + a_1(\nu)) P \]

IIa. SADE-Package excluding the \( \mathcal{P}_{ij} \)-equations (2.3):

Header:

\[ \text{restart: with(sade):} \]

Symmetry Analysis of Differential Equations

By Tarcisio M. Rocha Filho − Annibal Figueiredo − 2010

Definitions:

\[ \text{alias(sigma=(t,x,y,z,nu,u,v,w,p,U,P)): X:=(t,x,y,z,nu):} \]
\[ C:=\text{diff}(u(X),x)+\text{diff}(v(X),y)+\text{diff}(w(X),z); \]
\[ C := \frac{\partial}{\partial x} u(t, x, y, z, \nu) + \frac{\partial}{\partial y} v(t, x, y, z, \nu) + \frac{\partial}{\partial z} w(t, x, y, z, \nu) \]
\[ M1:=\text{diff}(u(X),t)+U(X)\text{diff}(u(X),x)+v(X)\text{diff}(U(X),y)-(K+nu)\text{diff}(U(X), \]
\[ y,y)]+\text{diff}(u(X)*u(X),x)+\text{diff}(u(X)*v(X),y)+\text{diff}(u(X)*w(X),z)+\text{diff}(p(X), \]
\[ x)-nu*(\text{diff}(u(X),x,x)+\text{diff}(u(X),y,y)+\text{diff}(u(X),z,z)); \]
\[ N_1 := \left( \frac{\partial}{\partial t} u(t, x, y, z, \nu) \right) + U(t, x, y, z, \nu) \left( \frac{\partial}{\partial y} U(t, x, y, z, \nu) \right) - K - \nu \left( \frac{\partial^2}{\partial x^2} u(t, x, y, z, \nu) \right) + 2 u(t, x, y, z, \nu) \left( \frac{\partial}{\partial y} v(t, x, y, z, \nu) \right) + u(t, x, y, z, \nu) \left( \frac{\partial}{\partial z} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial y} p(t, x, y, z, \nu) \right) - \nu \left( \frac{\partial^2}{\partial x^2} u(t, x, y, z, \nu) \right) \]

\[ N_2 := \left( \frac{\partial}{\partial t} v(t, x, y, z, \nu) \right) + U(t, x, y, z, \nu) \left( \frac{\partial}{\partial x} v(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial^2}{\partial x^2} v(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial y} v(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial z} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial y} p(t, x, y, z, \nu) \right) - \nu \left( \frac{\partial^2}{\partial x^2} v(t, x, y, z, \nu) \right) \]

\[ N_3 := \left( \frac{\partial}{\partial t} w(t, x, y, z, \nu) \right) + U(t, x, y, z, \nu) \left( \frac{\partial}{\partial x} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial^2}{\partial x^2} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial y} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial z} w(t, x, y, z, \nu) \right) + \nu \left( \frac{\partial}{\partial y} p(t, x, y, z, \nu) \right) - \nu \left( \frac{\partial^2}{\partial x^2} w(t, x, y, z, \nu) \right) \]

Equations (2.1), (2.2) & (2.4):

```plaintext
> eqnC := C = 0;
> eqnN1 := N1 = 0;
> eqnN2 := N2 = 0;
> eqnN3 := N3 = 0;
> eqnR1 := diff(U(X), t) = 0;
> eqnR2 := diff(U(X), x) = 0;
> eqnR3 := diff(U(X), z) = 0;
> eqnR4 := diff(P(X), t) = 0;
> eqnR5 := diff(P(X), x) = 0;
> eqnR6 := diff(P(X), z) = 0;
> eqns := [eqnC, eqnN1, eqnN2, eqnN3, eqnR1, eqnR2, eqnR3, eqnR4, eqnR5, eqnR6];
```

Symmetry Algorithm:

a) Size of the determining system:

```plaintext
> detsys := liesymmetries(eqns, [u(X), v(X), w(X), p(X), U(X), P(X)], determining);
> nops(detsys[1]);
```

b) Solving the determining system:

```plaintext
> symsol := liesymmetries(eqns, [u(X), v(X), w(X), p(X), U(X), P(X)]);
```
Redefinition of group functions as used (2.10):

\[
\begin{align*}
_\mathcal{F}10(nu,t) & := f3(t,nu); \\
_\mathcal{F}2(U,P,nu,y) & := 2(a1(nu) - a4(nu))P + g2(y,nu,U,P); \\
_\mathcal{F}7(nu) & := a5(nu); \\
_\mathcal{F}1(U,P,nu,y) & := (a1(nu) - a4(nu))U + g1(y,nu,U,P); \\
_\mathcal{F}5(nu) & := a3(nu); \\
_\mathcal{F}4(nu,t) & := f2(t,nu); \\
_\mathcal{F}9(nu) & := -a1(nu) + a4(nu); \\
_\mathcal{F}6(nu,t) & := f1(t,nu); \\
_\mathcal{F}3(nu) & := a1(nu); \\
_\mathcal{F}8(nu) & := a2(nu); \\
_\mathcal{F}1(U,P,nu,y) & := (a1(nu) - a4(nu))U + g1(y,nu,U,P); \\
_\mathcal{F}2(U,P,nu,y) & := 2(a1(nu) - a4(nu))P + g2(y,nu,U,P); \\
_\mathcal{F}10(nu,t) & := f3(t,nu);
\end{align*}
\]

Final Result (identical to (2.10)): 

\[
\begin{align*}
_\eta[P](\sigma) & = \text{simplify(coeff(infgen,D[P]));} \\
_\eta[U](\sigma) & = \text{simplify(coeff(infgen,D[U]));} \\
_\eta[p](\sigma) & = \text{simplify(coeff(infgen,D[p]));} \\
_\eta[w](\sigma) & = \text{simplify(coeff(infgen,D[w]));} \\
_\eta[v](\sigma) & = \text{simplify(coeff(infgen,D[v]));} \\
_\xi[\nu](\sigma) & = \text{simplify(coeff(infgen,D[\nu]));} \\
_\xi[z](\sigma) & = \text{simplify(coeff(infgen,D[z]));} \\
_\xi[y](\sigma) & = \text{simplify(coeff(infgen,D[y]));} \\
_\xi[x](\sigma) & = \text{simplify(coeff(infgen,D[x]));} \\
_\eta[u](\sigma) & = -g1(y,\nu,U,P) + (\frac{\partial}{\partial \nu} f1(t,\nu)) + a1(\nu)u - a4(\nu)u + a2(\nu)w \\
_\eta[v](\sigma) & = (a1(\nu) - a4(\nu))v \\
_\eta[w](\sigma) & = (\frac{\partial}{\partial \nu} f2(t,\nu)) + wa1(\nu) - wa4(\nu) - a2(\nu)u - a2(\nu)V
\end{align*}
\]
\[ \eta_p(\sigma) = z a 2(\nu) K - x a 1(\nu) K + f 3(t, \nu) - g 2(y, \nu, U, P) + 2 a 1(\nu) p - 2 a 4(\nu) p + 2 a 4(\nu) x K - z \left( \frac{\partial^2}{\partial x^2} f 2(t, \nu) \right) - x \left( \frac{\partial^2}{\partial x^2} f 1(t, \nu) \right) \]

\[ \eta U(\sigma) = U a 1(\nu) - U a 4(\nu) + g 1(y, \nu, U, P) \]

\[ \eta_p(\sigma) = 2 P a 1(\nu) - 2 P a 4(\nu) + g 2(y, \nu, U, P) \]

**IIb. SADE-Package including the \( \mathcal{R}_ij \)-equations (2.3):**

**Definitions:**

\( \text{alias}(\sigma = (t,x,y,z,nu,u,v,w,p,U,P)) : X = (t,x,y,z,nu) ; \)

\( C := \text{diff}(u(X),x)+\text{diff}(v(X),y)+\text{diff}(w(X),z) ; \)

\( N1 := \left( \frac{\partial}{\partial t} u(t, x, y, z, \nu) \right) + \left( \frac{\partial}{\partial x} v(t, x, y, z, \nu) \right) - \left( \frac{\partial}{\partial y} u(t, x, y, z, \nu) \right) + \left( \frac{\partial}{\partial z} w(t, x, y, z, \nu) \right) \)

\( N2 := \left( \frac{\partial}{\partial x} u(t, x, y, z, \nu) \right) + \left( \frac{\partial}{\partial y} v(t, x, y, z, \nu) \right) + \left( \frac{\partial}{\partial z} w(t, x, y, z, \nu) \right) + \left( \frac{\partial}{\partial \nu} p(t, x, y, z, \nu) \right) \)

**Header:**

\( \text{restart: with(sade):} \)

\( \text{Symmetry Analysis of Differential Equations} \)

\( \text{By Torcisco M. Rocha Filho – Annibal Figueiredo – 2010} \)
Equations (2.1), (2.2) & (2.4) including the "velocity product equations" (2.3):
> eqnC := C = 0;
> eqnN1 := N1 = 0;
> eqnN2 := N2 = 0;
> eqnN3 := N3 = 0;
> eqnP11 := 2*N1*u(X) = 0;
> eqnP22 := 2*N2*v(X) = 0;
> eqnP33 := 2*N3*w(X) = 0;
> eqnP12 := N1*v(X) + N2*u(X) = 0;
> eqnP13 := N1*w(X) + N3*u(X) = 0;
> eqnP23 := N2*w(X) + N3*v(X) = 0;
> eqnR1 := diff(U(X), t) = 0;
> eqnR2 := diff(U(X), x) = 0;
> eqnR3 := diff(U(X), z) = 0;
> eqnR4 := diff(P(X), t) = 0;
> eqnR5 := diff(P(X), x) = 0;
> eqnR6 := diff(P(X), z) = 0;
> eqns := [eqnC, eqnN1, eqnN2, eqnN3, eqnP11, eqnP22, eqnP33, eqnP12, eqnP13, eqnP23, eqnR1, eqnR2, eqnR3, eqnR4, eqnR5, eqnR6];

Symmetry Algorithm:

a) Size of the determining system:
> detsys := liesymmetries(eqns, [u(X), v(X), w(X), p(X), U(X), P(X)], determining);
> nops(detsys[1]);

b) Solving the determining system:
> symsol := liesymmetries(eqns, [u(X), v(X), w(X), p(X), U(X), P(X)]);
> symsol

Redefinition of group functions as used in (2.10):
> _F6(nu, t) := f1(t, nu);
> _F3(nu) := a1(nu);
> _F8(nu) := a2(nu);
> _F5(nu) := a3(nu);
> _F4(nu, t) := f2(t, nu);
> _F9(nu) := a4(nu);
> _F7(nu) := a5(nu);
> _F1(U, P, nu, y) := (a1(nu) - a4(nu))*U*g1(y, nu, U, P);
> _F2(U, P, nu, y) := 2*(a1(nu) - a4(nu))*P*g2(y, nu, U, P);
> _F10(nu, t) := f3(t, nu);
> _F6(nu, t) := f1(t, nu);
> _F3(nu) := a1(nu);
> _F8(nu) := a2(nu);
> _F5(nu) := a3(nu);
> _F4(nu, t) := f2(t, nu);
> _F9(nu) := a4(nu);
> _F7(nu) := a5(nu);
> _F1(U, P, nu, y) := (a1(nu) - a4(nu))*U*g1(y, nu, U, P);
> _F2(U, P, nu, y) := 2*(a1(nu) - a4(nu))*P*g2(y, nu, U, P);
> _F10(nu, t) := f3(t, nu);

Final Result (still identical to (2.10)):
> n := nops(symsol[1]);
> infgen := sum(symsol[1, i], i = 1 .. n);
> xi[x](sigma)=simplify(coeff(infgen,D[x]));
> xi[y](sigma)=simplify(coeff(infgen,D[y]));
> xi[z](sigma)=simplify(coeff(infgen,D[z]));
> xi[t](sigma)=simplify(coeff(infgen,D[t]));
> xi[nu](sigma)=simplify(coeff(infgen,D[nu]));
> eta[u](sigma)=simplify(coeff(infgen,D[u]));
> eta[v](sigma)=simplify(coeff(infgen,D[v]));
> eta[w](sigma)=simplify(coeff(infgen,D[w]));
> eta[p](sigma)=simplify(coeff(infgen,D[p]));
> eta[U](sigma)=simplify(coeff(infgen,D[U]));
> eta[P](sigma)=simplify(coeff(infgen,D[P]));

\[
\begin{align*}
\xi_x(\sigma) &= f_1(t, \nu) + a_2(\nu) z + a_1(\nu) x \\
\xi_y(\sigma) &= a_3(\nu) + a_1(\nu) y \\
\xi_z(\sigma) &= -a_2(\nu) x + a_1(\nu) z + f_2(t, \nu) \\
\xi_t(\sigma) &= a_5(\nu) + t a_4(\nu) \\
\xi_{\nu}(\sigma) &= 2a_1(\nu) \nu - \nu a_4(\nu) \\
\eta_u(\sigma) &= -g_1(y, \nu, U, P) + (\frac{\partial}{\partial \nu} f_1(t, \nu)) + a_2(\nu) w + a_1(\nu) u - a_4(\nu) u \\
\eta_v(\sigma) &= -(a_1(\nu) + a_4(\nu)) v \\
\eta_w(\sigma) &= -a_2(\nu) u - a_2(\nu) U + (\frac{\partial}{\partial \nu} f_2(t, \nu)) + w a_1(\nu) - w a_4(\nu) \\
\eta_p(\sigma) &= z a_2(\nu) K - x a_1(\nu) K + \frac{\partial}{\partial \nu} f_2(t, \nu) - x (\frac{\partial^2}{\partial \nu^2} f_1(t, \nu)) + 2 a_1(\nu) p - 2 a_4(\nu) p \\
&+ 2 a_4(\nu) x K + f_3(t, \nu) - g_2(y, \nu, U, P) \\
\eta_U(\sigma) &= U a_1(\nu) - U a_4(\nu) + g_1(y, \nu, U, P) \\
\eta_P(\sigma) &= 2 a_1(\nu) P - 2 a_4(\nu) P + g_2(y, \nu, U, P)
\end{align*}
\]

IIIa. DESOLV-II-Package excluding the \( \mathcal{R}_{ij} \)-equations (2.3):

Header:

\[
\text{restart: read "Desolv-V5R5.mpl": with(desolv):}
\]

\[
\text{DESOLVII\_V5R5 (March \ - \ 2011)(c)}
\]

by Dr. K. T. Vu, Dr. J. Carminati and Miss. G. Jefferson

Definitions:

\[
\begin{align*}
\text{alias(sigma=(t,x,y,z,nu,u,v,w,p,U,P))}: \ X:=(&t,x,y,z,nu): \\
\text{C:=diff(u(X),x)+diff(v(X),y)+diff(w(X),z);} \\
\text{C:=(D_u u(t, x, y, z, \nu)) + (D_v v(t, x, y, z, \nu)) + (D_w w(t, x, y, z, \nu))} \\
\text{N1:=diff(u(X),t)+U(X)*diff(u(X),x)+v(X)*diff(U(X),y)-(K+nu*diff(U(X),y)+y,y)+diff(u(X)*v(X),y)+diff(u(X)*w(X),z)+diff(p(X),x)+2*nu*(diff(u(X),x,x)+diff(u(X),y,y)+diff(u(X),z,z))};
\end{align*}
\]
> N2:=diff(v(X),t)+U(X)*diff(v(X),x)+diff(P(X),y)+diff(v(X)*u(X),x)+diff(v(X),y)+diff(v(X),z)+diff(P(X),x)+nu*(diff(v(X),x)+diff(P(X),y)-nu*(diff(v(X),x)+diff(P(X),y)+nu*diff(v(X),z))+

> v(X),y,y)+diff(v(X),z)+diff(p(X),y)-nu*(diff(v(X),x)+diff(P(X),y)+nu*diff(v(X),z))+

Equations (2.1), (2.2) & (2.4):

\[ \text{sym} := \text{pdesolv}(\text{op(detsys)}); \]

\[ \text{eqns} := [\text{eqnC, eqnN1, eqnN2, eqnN3, eqnR1, eqnR2, eqnR3, eqnR4, eqnR5, eqnR6}]; \]

Symmetry Algorithm:

\[ a) \text{ Size of the determining system:} \]

\[ \text{detsys} := \text{gendef}([u, v, w, p, U, P], [X]); \]

\[ \text{nops(detsys[1])} \]

\[ \text{113} \]

\[ \text{b) Solving the determining system:} \]

\[ \text{sym} := \text{pdesolv}(\text{op(detsys)}); \]

\[ \xi_x (\sigma) = \frac{F_{109}(\nu)}{\nu} + \frac{t F_{51}(\nu)}{\nu} + 2 t F_{97}(\nu), \]

\[ \xi_y (\sigma) = F_{110}(\nu) + \frac{y F_{51}(\nu)}{\nu} + y F_{97}(\nu), \]

\[ \xi_z (\sigma) = -F_{107}(t, \nu) + \frac{z F_{51}(\nu)}{\nu} + z F_{97}(\nu), \]

\[ \eta_x (\sigma) = F_{81}(y, \nu) + \frac{u F_{108}(t, \nu)}{\nu} - u F_{97}(\nu), \]

\[ \eta_y (\sigma) = -\frac{\partial}{\partial t} F_{107}(t, \nu) - w F_{97}(\nu), \]

\[ \eta_z (\sigma) = -2 p F_{97}(\nu) + F_{41}(y, \nu, U, P) \]

\[ -2 p F_{97}(\nu) + \frac{K x F_{51}(\nu)}{\nu} + 3 K x F_{97}(\nu), \]

\[ \eta_U (\sigma) = -F_{81}(y, \nu) - U F_{97}(\nu), \]

\[ \eta_P (\sigma) = -F_{41}(y, \nu, U, P), [F_{41}(y, \nu, U, P), F_{114}(t, \nu), F_{81}(y, \nu), F_{108}(t, \nu), F_{107}(t, \nu), F_{97}(\nu), F_{51}(\nu), F_{110}(\nu), F_{109}(\nu)] \]
Redefinition of group functions as used in (2.10) and (2.11):

\[
\begin{align*}
F_{51}(\nu) &:= (2a_1(\nu) - a_4(\nu)) \cdot \nu \\
F_{97}(\nu) &:= -(a_1(\nu) - a_4(\nu)) \\
F_{108}(t, \nu) &:= f_1(t, \nu) \\
F_{110}(\nu) &:= a_3(\nu) \\
F_{107}(t, \nu) &:= -f_2(t, \nu) \\
F_{109}(\nu) &:= \nu \cdot a_5(\nu) \\
F_{81}(y, \nu) &:= -F(y, \nu) \\
F_{114}(t, \nu) &:= \nu \cdot f_3(t, \nu) \\
F_{41}(y, \nu, U, P) &:= -2(a_1(\nu) - a_4(\nu)) \cdot P - \frac{g_2(y, \nu, U, P)}{\rho} \\
\end{align*}
\]

Final Result (identical to (2.10) and (2.11)):

\[
\begin{align*}
\xi_x(\sigma) &:= f_1(t, \nu) + x \cdot a_1(\nu) \\
\xi_y(\sigma) &:= a_3(\nu) + y \cdot a_1(\nu) \\
\xi_z(\sigma) &:= f_2(t, \nu) + z \cdot a_1(\nu) \\
\xi_t(\sigma) &:= a_5(\nu) + t \cdot a_4(\nu) \\
\xi_\nu(\sigma) &:= (2a_1(\nu) - a_4(\nu)) \cdot \nu \\
\eta_u(\sigma) &:= -F(y, \nu) + \frac{\partial}{\partial t} f_1(t, \nu) + u \cdot a_1(\nu) - u \cdot a_4(\nu) \\
\eta_v(\sigma) &:= v \cdot (a_1(\nu) - a_4(\nu)) \\
\eta_w(\sigma) &:= (\partial^2 f_2(t, \nu)) + w \cdot a_1(\nu) - w \cdot a_4(\nu) \\
\eta_p(\sigma) &:= 2p \cdot a_1(\nu) - 2p \cdot a_4(\nu) - \frac{g_2(y, \nu, U, P)}{\rho} - \frac{\partial^2}{\partial t^2} f_2(t, \nu) \cdot z \\
&- x \cdot (\partial^2 f_1(t, \nu)) + f_3(t, \nu) - K \cdot x \cdot a_1(\nu) + 2K \cdot x \cdot a_4(\nu) \\
\eta_U(\sigma) &:= F(y, \nu) + U \cdot a_1(\nu) - U \cdot a_4(\nu) \\
\eta_P(\sigma) &:= 2P \cdot a_1(\nu) - 2P \cdot a_4(\nu) + g_2(y, \nu, U, P) \\
\end{align*}
\]

IIIb. DESOLV-II-Package including the \(\mathcal{P}_{ij}\)-equations (2.3):

Header:

\[
\begin{align*}
&> \text{restart: read "Desolv-V5R5.mpl": with(desolv):} \\
&\text{DESOLVII-V5R5 (March - 2011)(c)} \\
&\text{by Dr. K. T. Vu, Dr. J. Carminati and Miss. G. Jefferson} \\
\end{align*}
\]

Definitions:

\[
\begin{align*}
&> \text{alias(sigma=(t,x,y,z,nu,u,v,w,p,U,P)): X:=(t,x,y,z,nu):} \\
&\text{C:=diff(u(X),x)+diff(v(X),y)+diff(w(X),z);} \\
&\text{C := (}\frac{\partial}{\partial t} u(t, x, y, z, \nu)) + (\frac{\partial}{\partial y} v(t, x, y, z, \nu)) + (\frac{\partial}{\partial z} w(t, x, y, z, \nu)) \\
\end{align*}
\]
A comment on the Article by Oberlack (2001)

Equations (2.1), (2.2) & (2.4) including the "velocity product equations" (2.3):

\begin{align*}
N1 := & (\frac{\partial}{\partial t} u(t, x, y, z, \nu)) + U(t, x, y, z, \nu) (\frac{\partial}{\partial x} u(t, x, y, z, \nu)) \\
& + v(t, x, y, z, \nu) (\frac{\partial}{\partial y} u(t, x, y, z, \nu)) - K - \nu (\frac{\partial^2}{\partial y^2} U(t, x, y, z, \nu)) \\
& + 2 u(t, x, y, z, \nu) (\frac{\partial}{\partial x} u(t, x, y, z, \nu)) + \nu (\frac{\partial^2}{\partial y^2} U(t, x, y, z, \nu)) v(t, x, y, z, \nu) \\
& + u(t, x, y, z, \nu) (\frac{\partial}{\partial x} v(t, x, y, z, \nu)) + (\frac{\partial}{\partial y} u(t, x, y, z, \nu)) w(t, x, y, z, \nu) \\
& + u(t, x, y, z, \nu) (\frac{\partial}{\partial y} w(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} p(t, x, y, z, \nu)) \\
& - \nu ((\frac{\partial^2}{\partial y^2} u(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} v(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} w(t, x, y, z, \nu))) \\
N2 := & (\frac{\partial}{\partial t} v(t, x, y, z, \nu)) + U(t, x, y, z, \nu) (\frac{\partial}{\partial x} v(t, x, y, z, \nu)) + (\frac{\partial}{\partial y} p(t, x, y, z, \nu)) \\
& + (\frac{\partial}{\partial x} u(t, x, y, z, \nu)) v(t, x, y, z, \nu) + u(t, x, y, z, \nu) (\frac{\partial}{\partial x} v(t, x, y, z, \nu)) \\
& + 2 v(t, x, y, z, \nu) (\frac{\partial}{\partial x} v(t, x, y, z, \nu)) + \nu (\frac{\partial^2}{\partial y^2} U(t, x, y, z, \nu)) v(t, x, y, z, \nu) \\
& + v(t, x, y, z, \nu) (\frac{\partial}{\partial y} w(t, x, y, z, \nu)) + (\frac{\partial}{\partial y} p(t, x, y, z, \nu)) \\
& - \nu ((\frac{\partial^2}{\partial y^2} v(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} w(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} p(t, x, y, z, \nu)) \\
N3 := & (\frac{\partial}{\partial t} w(t, x, y, z, \nu)) + U(t, x, y, z, \nu) (\frac{\partial}{\partial x} w(t, x, y, z, \nu)) \\
& + (\frac{\partial}{\partial x} u(t, x, y, z, \nu)) w(t, x, y, z, \nu) + u(t, x, y, z, \nu) (\frac{\partial}{\partial x} w(t, x, y, z, \nu)) \\
& + (\frac{\partial}{\partial y} v(t, x, y, z, \nu)) w(t, x, y, z, \nu) + \nu (\frac{\partial^2}{\partial y^2} U(t, x, y, z, \nu)) w(t, x, y, z, \nu) \\
& + 2 w(t, x, y, z, \nu) (\frac{\partial}{\partial x} w(t, x, y, z, \nu)) + (\frac{\partial}{\partial y} p(t, x, y, z, \nu)) \\
& - \nu ((\frac{\partial^2}{\partial y^2} w(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} p(t, x, y, z, \nu)) + (\frac{\partial^2}{\partial y^2} p(t, x, y, z, \nu)) \\
\end{align*}

Symmetry Algorithm:

a) Size of the determining system:

\begin{align*}
& \text{detsys} := \text{gendef}(\text{eqns}, [u, v, w, p, U, P], [X]): \text{nops(detsys}[1]); \\
& 113
\end{align*}

Note that since by default the gendef-routine checks on redundancy, it automatically returns, as a consequence, the same number of determining equations as in the previous non-extended case IIIa.
b) Solving the determining system:

\[ \text{sym}:=\text{pdesolv}(\text{op}(\text{detsys})) \]

\[
\begin{align*}
\xi_x(\sigma) &= F_{108}(t, \nu) + \frac{x F_{71}(\nu)}{\nu} + x F_{97}(\nu), \\
\xi_y(\sigma) &= F_{110}(\nu) + \frac{y F_{71}(\nu)}{\nu} + y F_{97}(\nu), \\
\xi_z(\sigma) &= -F_{107}(t, \nu) + \frac{z F_{71}(\nu)}{\nu} + z F_{97}(\nu), \\
\xi_\nu(\sigma) &= F_{71}(\nu), \\
F_{51}(y, \nu) &= \frac{2}{a_1(\nu) - a_4(\nu)} P + g_2(y, \nu, U, P), \\
F_{109}(\nu) &= \frac{\nu}{a_5(\nu)} P + g_2(y, \nu, U, P), \\
F_{114}(t, \nu) &= \nu F_3(t, \nu), \\
F_{107}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{110}(\nu) &= \nu F_{114}(t, \nu), \\
F_{108}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{60}(y, \nu) &= -F(y, \nu), \\
F_{114}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{108}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{60}(y, \nu) &= -F(y, \nu), \\
F_{114}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{51}(y, \nu, U, P) &= 2 (a_1(\nu) - a_4(\nu)) P + g_2(y, \nu, U, P).
\end{align*}
\]

Redefinition of group functions as in (2.10) and (2.11):

\[
\begin{align*}
F_{71}(\nu) &= (2a_1(\nu) - a_4(\nu)) \nu, \\
F_{97}(\nu) &= -a_1(\nu) + a_4(\nu), \\
F_{108}(t, \nu) &= f_1(t, \nu), \\
F_{110}(\nu) &= a_3(\nu), \\
F_{109}(\nu) &= -f_2(t, \nu), \\
F_{60}(y, \nu) &= -F(y, \nu), \\
F_{114}(t, \nu) &= \nu F_3(t, \nu), \\
F_{108}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{60}(y, \nu) &= -F(y, \nu), \\
F_{114}(t, \nu) &= \nu F_{114}(t, \nu), \\
F_{51}(y, \nu, U, P) &= 2 (a_1(\nu) - a_4(\nu)) P + g_2(y, \nu, U, P).
\end{align*}
\]

Final Result (still identical to (2.10) and (2.11)):

\[
\begin{align*}
\text{simplify} &\text{sym}(3,2); \\
\text{simplify} &\text{sym}(3,3); \\
\text{simplify} &\text{sym}(3,4); \\
\text{simplify} &\text{sym}(3,1); \\
\text{simplify} &\text{sym}(3,5); \\
\text{simplify} &\text{sym}(3,6); \\
\text{simplify} &\text{sym}(3,7); \\
\text{simplify} &\text{sym}(3,8); \\
\text{simplify} &\text{sym}(3,9); \\
\text{simplify} &\text{sym}(3,10); \\
\text{simplify} &\text{sym}(3,11);
\end{align*}
\]

\[
\begin{align*}
\xi_x(\sigma) &= f_1(t, \nu) + x a_1(\nu), \\
\xi_y(\sigma) &= a_3(\nu) + y a_1(\nu), \\
\xi_z(\sigma) &= f_2(t, \nu) + z a_1(\nu), \\
\xi_\nu(\sigma) &= (2a_1(\nu) - a_4(\nu)) \nu, \\
\eta_a(\sigma) &= -F(y, \nu) + \frac{\partial}{\partial \nu} f_1(t, \nu) + u a_1(\nu) - u a_4(\nu), \\
\eta_\nu(\sigma) &= v (a_1(\nu) - a_4(\nu)), \\
\eta_w(\sigma) &= \frac{\partial}{\partial \nu} f_2(t, \nu) + w a_1(\nu) - w a_4(\nu)
\end{align*}
\]
\[ \eta_p(\sigma) = 2 p a_1(\nu) - 2 p a_4(\nu) - g_2(y, \nu, U, P) - \left( \frac{\partial^2}{\partial t^2} f_2(t, \nu) \right) z \]
\[- x \left( \frac{\partial^2}{\partial t^2} f_1(t, \nu) \right) + f_3(t, \nu) - K x a_1(\nu) + 2 K x a_4(\nu) \]
\[ \eta_\nu(\sigma) = F(y, \nu) + U a_1(\nu) - U a_4(\nu) \]
\[ \eta_P(\sigma) = 2 P a_1(\nu) - 2 P a_4(\nu) + g_2(y, \nu, U, P) \]

References

Avsarkisov, V., Oberlack, M. & Hoyas, S. 2014 New scaling laws for turbulent Poiseuille flow with wall transpiration. J. Fluid Mech. 746, 99–122.

Barenblatt, G. I. 1993 Scaling laws for fully developed turbulent shear flows. Part 1. Basic hypothesis and analysis. J. Fluid Mech. 248, 513–520.

Barenblatt, G. I., Chorin, A. J. & Prostokishin, V. M. 2014 Turbulent flows at very large Reynolds numbers: new lessons learned. Physics-Uspekhi 57, 250–256.

Barenblatt, G. I. & Prostokishin, V. M. 1993 Scaling laws for fully developed turbulent shear flows. Part 2. Processing of experimental data. J. Fluid Mech. 248, 521–529.

Benzi, R., Biferale, L., Fisher, R., Lamb, D. Q. & Toschi, F. 2010 Inertial range Eulerian and Lagrangian statistics from numerical simulations of isotropic turbulence. J. Fluid Mech. 653, 221–244.

Benzi, R., Ciliberto, S., Tripiccione, R., Baudet, C., Massaioli, F. & Succi, S. 1993 Extended self-similarity in turbulent flows. Phys. Rev. E 48 (1), R29–R32.

Biferale, L., Boffetta, G. & Castaing, B. 2003 Fully developed turbulence. In The Kolmogorov Legacy in Physics (ed. R. Livi & A. Vulpiani), pp. 149–172. Springer.

Biferale, L., Lanotte, A. S. & Toschi, F. 2008 Statistical behaviour of isotropic and anisotropic fluctuations in homogeneous turbulence. Physica D 237, 1969–1975.

Biferale, L. & Procaccia, I. 2005 Anisotropy in turbulent flows and in turbulent transport. Phys. Rep. 414 (2), 43–164.

Bila, N. 2011 On a new method for finding generalized equivalence transformations for differential equations involving arbitrary functions. J. Sym. Comp. 46, 659–671.

Bluman, G. W. & Kumei, S. 1996 Symmetries and Differential Equations, 2nd edn. Springer Verlag.

Box, G. E. P. & Draper, N. R. 1987 Empirical model-building and response surfaces. John Wiley & Sons.

Cantwell, B. J. 2002 Introduction to Symmetry Analysis. Cambridge University Press.

Castellani, E. 2003 On the meaning of symmetry breaking. In Symmetries in Physics: Philosophical Reflections (ed. K. Brading & E. Castellani), pp. 321–334. Cambridge University Press.

Chakraborty, S., Frisch, U. & Ray, S. S. 2010 Extended self-similarity works for the Burgers equation and why. J. Fluid. Mech. 649, 275–285.

Cheviakov, A. F. 2007 GeM software package for computation of symmetries and conservation laws of differential equations. Comp. Phys. Comm. 176 (1), 48–61.
Chirkunov, Y. A. 2012 Generalized equivalence transformations and group classification of systems of differential equations. J. App. Mech. Tech. Phys. 53 (2), 147–155.

Cipra, B. 1996 A new theory of turbulence causes a stir among experts. Science 272 (5264), p. 951.

Constantin, P. & Fefferman, C. 1994 Scaling exponents in fluid turbulence: some analytic results. Nonlinearity 7 (1), 41–57.

Constantin, P., Nie, Q. & Tanveer, S. 1999 Bounds for second order structure functions and energy spectrum in turbulence. Phys. Fluids 11, 2251–2256.

Dallas, V., Vassilicos, J. C. & Hewitt, G. F. 2009 Stagnation point von Kármán coefficient. Phys. Rev. E 80 (4), 046306.

Davidson, P. A. 2004 Turbulence: An Introduction for Scientists and Engineers. Oxford University Press.

Falkovich, G. & Sreenivasan, K. R. 2006 Lessons from hydrodynamic turbulence. Physics Today 59, 43–49.

Feferabend, P. K. 1975 Against method: Outline of an anarchistic theory of knowledge. Humanities Press: Atlantic Highlands, NJ.

Filho, T. M. R. & Figueiredo, A. 2011 [SADE] a maple package for the symmetry analysis of differential equations. Comp. Phys. Comm. 182 (2), 467–476.

Frewer, M. 2015a An example elucidating the mathematical situation in the statistical non-uniqueness problem of turbulence. arXiv:1508.06962.

Frewer, M. 2015b On a remark from John von Neumann applicable to the symmetry induced turbulent scaling laws generated by the new theory of Oberlack et al. ResearchGate, 1–3.

Frewer, M. & Khujadze, G. 2016a Comments on Janocha et al. Lie symmetry analysis of the Hopf functional-differential equation”. Symmetry 8 (4), 23.

Frewer, M. & Khujadze, G. 2016b An example of how a methodological mistake aggravates erroneous results when only correcting the results and not the method itself. ResearchGate, 1–10.

Frewer, M., Khujadze, G. & Foysi, H. 2014 On the physical inconsistency of a new statistical scaling symmetry in incompressible Navier-Stokes turbulence. arXiv:1412.3061.

Frewer, M., Khujadze, G. & Foysi, H. 2015a Comment on “Statistical symmetries of the Lundgren-Monin-Novikov hierarchy”. Phys. Rev. E 92, 067001.

Frewer, M., Khujadze, G. & Foysi, H. 2015b Objections to a Reply of Oberlack et al. ResearchGate, 1–9.

Frewer, M., Khujadze, G. & Foysi, H. 2016a A note on the notion “statistical symmetry”. arXiv:1602.08039.

Frewer, M., Khujadze, G. & Foysi, H. 2016b Comment on “Application of the extended Lie group analysis to the Hopf functional formulation of the Burgers equation”. J. Math. Phys. 57, 034102.

Frewer, M., Khujadze, G. & Foysi, H. 2016c On a new technical error in a further Reply by Oberlack et al. and its far-reaching effect on their original study. ResearchGate, 1–6.
Friedrich, R., Daitle, A., Kamps, O., Lülff, J., Vosskühle, M. & Wilczek, M. 2012 The Lundgren-Monin-Novikov hierarchy: Kinetic equations for turbulence. Comptes Rendus Physique 13, 929–953.

Frisch, U. 1983 Fully developed turbulence, singularities and intermittency. In Chaotic Behaviour in Deterministic Systems (ed. G. Iooss, H. G. Helleman & R. Stora). Amsterdam, North-Holland.

Frisch, U. 1985 Fully developed turbulence: Intermittency as a broken symmetry. In Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics (ed. M. Ghil, R. Benzi & G. Parisi), pp. 71–88. Amsterdam, North-Holland.

Frisch, U. 1991 From global scaling, a la Kolmogorov, to local multifractal scaling in fully developed turbulence. Proc. R. Soc. Lond. A 434 (1890), 89–99.

Frisch, U. 1995 Turbulence. The Legacy of A.N. Kolmogorov. Cambridge University Press.

Fujisaka, H. & Grossmann, S. 2001 Scaling hypothesis leading to extended self-similarity in turbulence. Phys. Rev. E 63 (2), 026305.

Fushchich, W. I., Shtelen, W. M. & Serov, N. I. 1993 Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics. Springer Verlag.

Gad-el-Hak, M. & Buschmann, M. H. 2011 Turbulent boundary layers: is the wall falling or merely wobbling? Acta Mech. 218 (3-4), 309–318.

George, W. K. 2007 Is there a universal log law for turbulent wall-bounded flows? Phil. Trans. R. Soc. A 365 (1852), 789–806.

George, W. K. & Castillo, L. 1997 Zero-pressure-gradient turbulent boundary layer. Appl. Mech. Rev. 50, 689–730.

Grauer, R., Homann, H. & Pinton, J.-F. 2012 Longitudinal and transverse structure functions in high Reynolds-number turbulence. New J. Phys. 14, 063016, pp. 1–10.

Guenther, S. & Oberlack, M. 2005 Incompatibility of the exponential scaling law for a zero pressure gradient boundary layer flow with Reynolds averaged turbulence models. Phys. Fluids 17, 048105–1–4.

Hopf, E. 1952 Statistical hydromechanics and functional calculus. J. Rational Mech. Anal. 1, 87–123.

Hosokawa, I. & Yamamoto, K. 1970 Numerical study of the Burgers’ model of turbulence based on the characteristic functional formalism. Phys. Fluids 13 (7), 1683–1692.

Hydon, P. E. 2000 Symmetry Methods for Differential Equations: A Beginner’s Guide. Cambridge University Press.

Ibragimov, N. H. 1994 Lie Group Analysis of Differential Equations, CRC Handbook, vol. I-III. CRC Press.

Ibragimov, N. H. 2004 Equivalence groups and invariants of linear and non-linear equations. Archives of ALGA 1, 9–69.

Jiménez, J. 2012 Cascades in wall-bounded turbulence. Annual Rev.Fluid Mech. 44, 27–45.

Jiménez, J. 2013 Near-wall turbulence. Phys. Fluids 25, 101302.

Kevorkian, J. & Cole, J. D. 1996 Multiple Scale and Singular Perturbation Methods. Springer Verlag.
Khujadze, G. & Frewer, M. 2016 Revisiting the Lie-group symmetry method for turbulent channel flow with wall transpiration. arXiv:1606.08396.

Khujadze, G. & Oberlack, M. 2004 DNS and scaling laws from new symmetries of ZPG turbulent boundary layer flow. Theor. Comp. Fluid Dyn. 18, 391–411.

Kolmogorov, A. N. 1941a Local structure of turbulence in an incompressible viscous fluid at very large Reynolds numbers. Dokl. Akad. Nauk SSSR 30, 299–301.

Kolmogorov, A. N. 1941b On the degeneration of isotropic turbulence in an incompressible viscous liquid. Dokl. Akad. Nauk SSSR 31, 319–323.

Kolmogorov, A. N. 1941c Dissipation of energy in isotropic turbulence. Dokl. Akad. Nauk SSSR 32, 325–327.

Kolmogorov, A. N. 1962 A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech. 13, 82–85.

don Kármán, T. 1930 Mechanische Ähnlichkeit und Turbulenz. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Fachgruppe 1 (Mathematik) 58-7.

Kurths, J. & Pikovsky, A. S. 1995 Symmetry breaking in distributed systems and modulational spatio-temporal intermittency. Chaos, Solitons & Fractals 5 (10), 1893–1899.

Lindgren, B., Österlund, J. M. & Johansson, A. V. 2004 Evaluation of scaling laws derived from Lie group symmetry methods in zero-pressure-gradient turbulent boundary layers. J. Fluid Mech. 502 (1), 127–152.

Lundgren, T. S. 1967 Distribution functions in the statistical theory of turbulence. Phys. Fluids 10, 969–975.

L’vov, V. S. & Procaccia, I. 2000 Analytic calculation of the anomalous exponents in turbulence. Phys. Rev. E 62, 8037–8057.

Marusic, I., McKeon, B. J., Monkewitz, P. A., Nagib, H. M., Smits, A. J. & Sreenivasan, K. R. 2010 Wall-bounded turbulent flows at high Reynolds numbers: Recent advances and key issues. Phys Fluids 22, 065103.

McComb, W. D. 1990 The Physics of Fluid Turbulence. Clarendon Press Oxford.

Meleshko, S. V. 1996 Generalization of the equivalence transformations. J. Nonlin. Math. Phys. 3 (1-2), 170–174.

Oberlack, M. 1999 Similarity in non-rotating and rotating turbulent pipe flows. J. Fluid Mech. 379, 1–22.

Oberlack, M. 2001 A unified approach for symmetries in plane parallel turbulent shear flows. J. Fluid Mech. 427, 299–328.

Oberlack, M., Cabot, W., Reif, B. A. Pettersson & Weller, T. 2006 Group analysis, direct numerical simulation and modelling of a turbulent channel flow with streamwise rotation. J. Fluid Mech. 562, 383–403.

Oberlack, M. & Guenther, S. 2003 Shear-free turbulent diffusion - classical and new scaling laws. Fluid Dyn. Res. 33, 453–476.

Oberlack, M. & Rosteck, A. 2010 New statistical symmetries of the multi-point equations and its importance for turbulent scaling laws. Discrete Continuous Dyn. Syst. Ser. S 3, 451–471.
Oberlack, M. & Zieleniewicz, A. 2013 Statistical symmetries and its impact on new decay modes and integral invariants of decaying turbulence. *Journal of Turbulence* **14** (2), 4–22.

Olver, P. J. 1993 *Applications of Lie Groups to Differential Equations*, 2nd edn. Springer Verlag.

Österlund, J. M., Johansson, A. V., Nagib, H. M. & Hites, M. H. 2000 A note on the overlap region in turbulent boundary layers. *Phys. Fluids* **12** (1), 1–4.

Ovsiannikov, L. V. 1982 *Group Analysis of Differential Equations*, 2nd edn. Academic Press.

Saint-Michel, B., Daviaud, F. & Dubrulle, B. 2013 A zero-mode mechanism for spontaneous symmetry breaking in a turbulent von Kármán flow. *Preprint, arXiv:1305.3389* pp. 1–17.

Sreenivasan, K. R. 1991 Fractals and multifractals in fluid turbulence. *Annu. Rev. Fluid Mech.* **23**, 539–600.

Stephani, H. 1989 *Differential Equations. Their solutions using symmetries*. Cambridge University Press.

Tsinober, A. 2013 *The Essence of Turbulence as a Physical Phenomenon*. Springer Verlag.

Vu, K. T., Jefferson, G. F. & Carminati, J. 2012 Finding higher symmetries of differential equations using the MAPLE package DESOLV-II. *Comp. Phys. Comm.* **183** (4), 1044–1054.

Waclawczyk, M. & Oberlack, M. 2013 Application of the extended Lie group analysis to the Hopf functional formulation of the Burgers equation. *J. Math. Phys.* **54**, 072901.

Waclawczyk, M., Staffolani, N., Oberlack, M., Rosteck, A., Wilczek, M. & Friedrich, R. 2014 Statistical symmetries of the Lundgren-Monin-Novikov hierarchy. *Phys. Rev. E* **90** (1), 013022.

Zagarola, M. V., Perry, A. E. & Smits, A. J. 1997 Log laws or power laws: The scaling in the overlap region. *Phys. Fluids* **9** (7), 2094–2100.