ADDITIVE TWISTS AND A CONJECTURE BY MAZUR, RUBIN AND STEIN

NIKOLAOS DIAMANTIS, JEFFREY HOFFSTEIN, EREN MEHMET KIRAL, AND MIN LEE

Abstract. In this paper, a conjecture of Mazur, Rubin and Stein concerning certain averages of modular symbols is proved. To cover the levels that are most important for elliptic curves, namely those that are not square-free, we establish results about $L$-functions with additive twists that are of independent interest.

1. Introduction

In this paper we prove a conjecture of Mazur, Rubin and Stein concerning certain averages of modular symbols.

Motivated by a question regarding ranks of elliptic curves defined over cyclic extensions of $\mathbb{Q}$, B. Mazur and K. Rubin [10] studied the statistical behaviour of modular symbols associated to a weight 2 cusp form corresponding to an elliptic curve. Based on both theoretical and computational arguments (the latter jointly with W. Stein) they formulated a number of precise conjectures. We state one of them in its formulation given in [11].

For a positive $q$, let $\Gamma = \Gamma_0(q)$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 with $a, b, c, d \in \mathbb{Z}$ and $q \mid c$. Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz} = \sum_{n=1}^{\infty} A(n)n^{1/2}e^{2\pi inz}$

be a newform of weight 2 for $\Gamma$. For convenience, we define $a(n)$ to be 0 when $n \leq 0$.

For each $r \in \mathbb{Q}$, we set

$$\langle r \rangle^+ = 2\pi \int_{i\infty}^{r} \Re(if(z)dz) \quad \text{and} \quad \langle r \rangle^- = 2\pi i \int_{i\infty}^{r} \Re(f(z)dz).$$

For each $x \in [0,1]$ and $M \in \mathbb{N}$, set

$$G^\pm_M(x) = \frac{1}{M} \sum_{0 \leq a \leq Mx} \langle \frac{a}{M} \rangle^\pm.$$ 

Mazur, Rubin and Stein, in [10], stated the following conjecture:

Conjecture 1.1. For each $x \in [0,1]$, we have

$$\lim_{M \to \infty} G^+_M(x) = \frac{1}{2\pi} \sum_{n \geq 1} \frac{a(n)\sin(2\pi nx)}{n^2};$$

$$\lim_{M \to \infty} G^-_M(x) = \frac{1}{2\pi i} \sum_{n \geq 1} \frac{a(n)(\cos(2\pi nx) - 1)}{n^2}.$$
The heuristic for this conjecture can be seen by the computation
\[ G_M^+(x) = \frac{1}{M} \sum_{0 \leq a \leq Mx} \langle \frac{a}{M} \rangle^+ = 2\pi \Re \left( i \int_0^\infty \frac{1}{M} \sum_{0 \leq a \leq Mx} f \left( \frac{a}{M} + iy \right) \, dy \right). \]

The inner sum is a Riemann sum for the horizontal integral \( \int_0^x \). As a heuristic let us replace the sum with the integral, even though the error is not controlled for small \( y \). Then computing the integral using the Fourier expansion of \( f \) gives us the right hand side of the above formulas.

An average version of this conjecture in the case of square-free levels was proved in \([11]\). The same paper contains the proofs of other conjectures from the original set listed in \([10]\). More recently, one of the original conjectures of \([10]\) was proved in \([4]\). The authors established a form of Conjecture 1.1 in the special case that \( x = 1 \) and \( M \) goes to infinity over the primes.

Our main theorem is as follows.

**Theorem 1.2.** For each \( x \in [0, 1] \), we have as \( M \to \infty \)

\[
G_M^+(x) = \frac{1}{2\pi} \sum_{n \geq 1} \frac{a(n) \sin(2\pi nx)}{n^2} + O \left( \left( Mq \right)^{\epsilon} M^{\frac{1}{2}} \prod_{p \mid \gcd(q,M), p^2 \mid q} p^{\frac{1}{4} \ord_p(q) + \frac{1}{2}} \right);
\]

\[
G_M^-(x) = \frac{1}{2\pi i} \sum_{n \geq 1} \frac{a(n)(\cos(2\pi nx) - 1)}{n^2} + O \left( \left( Mq \right)^{\epsilon} M^{-\frac{1}{2}} \prod_{p \mid \gcd(q,M), p^2 \mid q} p^{\frac{1}{4} \ord_p(q) + \frac{1}{2}} \right),
\]

for any \( \epsilon > 0 \).

**Remark 1.3.** The error term vanishes as \( M \to \infty \) because the product is over primes dividing \( \gcd(q, M) \) and the power of the prime never exceeds \( 3 \ord_p(q)/4 \). Hence the product is no bigger than \( q^{\frac{3}{4}} \), and \( M \) goes to infinity while \( q \) is fixed. Also note that product equals 1 if \( q \) is square free or \( \gcd(q,M) = 1 \).

See also \([8]\). H.-S. Sun in personal communication has told us that M. Kim and he are now able to prove this theorem, in the case that \( q \) is square free, with a slightly weaker exponent in \( q \).

Our method is ultimately based on Fourier coefficients of second-order modular forms which, in \([5]\), are expressed in terms of shifted convolution series. A specific second-order modular form was the main tool employed in \([11]\) too, but in this paper we have succeeded in avoiding its use. This allowed a simplification of our argument. However, the shifted convolution series itself remains a key tool, and a novelty of our approach is that we convert this to a double shifted convolution series. We prove that a certain integral transform of that double shifted convolution can be exactly computed and, in particular, that it can be analytically continued. This, in effect, resolves one of the obstacles in making heuristic arguments in support of the conjecture rigorous, namely that the series

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n}
\]

obtained by term-by-term integration of the modular symbol diverges.

As noted above, previous progress towards the Mazur, Rubin and Stein conjecture concerned only the case of square-free level (or prime \( M \)). That was a significant restriction because the questions motivating the conjecture pertain to elliptic curves, which very rarely have square free level. In this work we succeed in proving the conjecture for general levels. Extending
to non-square-free levels proved much less routine than we expected and it led to results of independent interest. We single out the functional equation of additive twists of \(L\)-functions for general levels and weights (Theorem 2.1). This theorem implies the “approximate functional equation” \((30)\), and does not seem to appear in the literature in that generality.

Of particular interest is Lemma 2.7. As mentioned in \([7\, Section\, 14.9]\), the Ramanujan-Petersson bound for Fourier coefficients of a Dirichlet twist of \(f\) holds even when the twist is not a newform, but there is an implied constant which may depend on the level badly. In Lemma 2.7 we make that dependence entirely explicit.

The bound \((29)\) should also be useful in future applications because bounds for modular symbols are often needed in this area.

In the next section we study in detail the \(L\)-functions with additive twists we will need in the sequel and prove the results for general level mentioned above. Section 3 deals with the expression of the average \(G_M(x)\) as the “limit” of certain weighted averages which, in turn, are reformulated as integrals. In Section 4 we obtain the shifted convolution series mentioned above and the two-variable version is studied. In Section 5 an integral transform of the two-variable shifted convolution is explicitly computed and thus an explicit formula for the weighted average of modular symbols is deduced in Section 6. The main term and the asymptotics of the weighted average are established in Section 7. With this preparation, the main theorem is deduced in the final section of the paper.

Acknowledgements We thank A. Cowen, D. Goldfeld, J. Louko, P. Michel, Y. Petridis, M. Radziwill, M. Risager, F. Strömberg, C. Wuthrich for helpful discussions and feedback. Part of the first author’s work was done during visits at the University of Patras and at Max-Planck-Institut für Mathematik. He is grateful for their hospitality and the excellent working conditions they provided. The third author thanks RIKEN iTHEMS for their hospitality where part of the first author’s work was done. The fourth author was supported by a Royal Society University Research Fellowship.

2. Properties of \(L\)-functions with additive twists

In this section we will establish the analytic continuation and functional equation of an \(L\)-function with additive twists. We will work more generally than in the rest of the paper because the result is of independent interest and we have not found it in this generality in the literature. In particular, all references we are aware of give the functional equation only for special combinations of the level and the denominator of the additive twist \([9]\).

2.1. Notations. In this section we closely follow \([1]\) and use similar notations. Let \(k\) be an integer. For any function \(h : \mathbb{H} \to \mathbb{C}\) and any matrix \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R})\), define

\[
(h \mid \gamma)(z) = \det(\gamma)^{\frac{k}{2}}(cz + d)^{-k}h\left(\frac{az + b}{cz + d}\right).
\]

For a positive integer \(q\) and a Dirichlet character \(\xi \pmod{q}\), let \(M(q, \xi, k)\) (resp. \(S(q, \xi, k)\)) be the space of holomorphic modular forms (resp. cusp forms) of level \(q\), weight \(k\) and central character \(\xi\). Then \(f \in S(q, \xi, k)\) has the following Fourier expansion

\[
f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}.
\]
As discussed in \[1\], we define the following operators - Hecke operators \(T_n\) for \(\gcd(n, q) = 1\), \(U_d\) and \(B_d\) for \(d \mid q\):

\[
f \mid T_n = n^{\frac{k}{2} - 1} \sum_{a+c=n} \sum_{b=0}^{c-1} \xi(a)f \mid \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},
\]

\[
f \mid U_d = d^{\frac{k}{2} - 1} \sum_{b=0}^{d-1} f \mid \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix},
\]

\[
f \mid B_d = d^{-\frac{k}{2}} f \mid \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \(N(q, \xi, k)\) denote the set of Hecke-normalized (i.e., the first Fourier coefficient is 1) cuspidal newforms of weight \(k\) and level \(q\) and central character \(\xi\). More precisely, if \(f \in N(q, \xi, k)\) then \(f \in S(q, \xi, k)\) is an eigenform of all Hecke operators \(T_n\) for \(\gcd(n, q) = 1\) and \(U_d\) for \(d \mid q\) (\([1\) p. 222]).

We now define the multiplicative twist of \(f\) for \(f \in N(q, \xi, k)\). For a primitive character \(\chi\) (mod \(r\)):

\[
f^\chi(z) := \sum_{n=1}^{\infty} a(n)\chi(n)e^{2\pi i nz}.
\]

Let \(R\) be the \(r\)-primary factor of \(q\), meaning that the prime decomposition of \(R\) only contains those primes dividing \(r\), and that \(\gcd(r, q/R) = 1\). Since \(\gcd(R, q/R) = 1\), the Dirichlet character \(\xi\) can be written as a product of Dirichlet characters \(\xi_R\) (resp. \(\xi_{q/R}\)) modulo \(R\) (resp. \(q/R\)), i.e., \(\xi = \xi_R\xi_{q/R}\). From \([1\) Proposition 3.1], we can deduce that \(f^\chi \in S(R'q/R, \xi\chi^2, k)\), where \(R' = \text{lcm}(R, \text{cond}(\xi_R)r, r^2)\).

2.2. The \(W_R\)-operator and additive twists. Assume that \(R \mid q\) and \(\gcd(R, q/R) = 1\). Put

\[
W_R = \begin{pmatrix} R x_1 & x_2 \\ q x_3 & R x_4 \end{pmatrix},
\]

where \(x_1, x_2, x_3, x_4 \in \mathbb{Z}, x_1 \equiv 1 \pmod{q/R}, x_2 \equiv 1 \pmod{R}\) and \(\det(W_R) = R(x_1x_4 - \frac{q}{R}x_2x_3) = R\).

By \([1\) Proposition 1.1], for \(f \in M(q, \xi, k)\) (resp. \(S(q, \xi, k)\)), we have \(f \mid W_R \in M(q, \overline{\xi_R\xi_{q/R}}, k)\) (resp. \(S(q, \overline{\xi_R\xi_{q/R}}, k)\)) and

\[
f \mid W_R \mid W_R = \xi_R(-1)^{\overline{\xi_{q/R}(R)}}f.
\]

For \(f \in S(q, \xi, k)\), let

\[
\tilde{f}_R = f \mid W_R \in S(q, \overline{\xi_R\xi_{q/R}}, k).
\]

For \(M_1 \in \mathbb{Z}_{\geq 1}, f \in S(q, \xi, k)\) and \(\alpha \pmod{M_1}\) with \(\gcd(\alpha, M_1) = 1\), let

\[
L(s, f, \frac{\alpha}{M_1}) = \sum_{n=1}^{\infty} \frac{a(n)e^{2\pi in\alpha/M_1}}{n^s},
\]

which is absolutely convergent for \(\Re(s) > 1 + \frac{k-1}{2}\), and

\[
\Lambda(s, f, \frac{\alpha}{M_1}) = \left(\frac{M_1}{2\pi}\right)^s \Gamma(s) L(s, f, \frac{\alpha}{M_1}) = M_1^s \int_0^{\infty} f(\frac{\alpha}{M_1} + iy)y^s \frac{dy}{y}.
\]
The aim of this section is to prove the following theorem.

**Theorem 2.1.** For \( q, M_1 \in \mathbb{Z}_{\geq 1} \) let

\[
M = \prod_{p \mid M_1, \text{ord}_p(M_1) \geq \text{ord}_p(q)} p^\text{ord}_p(M_1),
\]

\[
r = \prod_{p \mid M_1, \text{ord}_p(M_1) < \text{ord}_p(q)} p^\text{ord}_p(M_1),
\]

\[
R = \prod_{p \mid \gcd(q, r)} p^\text{ord}_p(q).
\]

Then \( M_1 = rM \) and \( r \mid q \), with \( \gcd(r, M) = 1 \). Also \( R \mid q \) is the \( r \)-primary factor of \( q \), which implies \( \gcd(R, q/R) = 1 \). Moreover, \( \frac{q}{R} \mid M \) and \( r < R \), except for when \( R = 1 \) in which case \( q \mid M \) and \( r = 1 \).

For any \( \alpha \pmod{M_1} \), set \( \alpha \equiv ar + uM \pmod{M_1} \) for \( a \pmod{M} \) and \( u \pmod{r} \). For a Hecke-normalized newform \( f \in N(q, \xi, k) \), we have

\[
\frac{\Gamma \left( s + \frac{k-1}{2} \right)}{(2\pi)^s \Gamma \left( \frac{k-1}{2} \right)} \sum_{r_0 \equiv 1 \pmod{M} \atop \gcd(r, \pi) \neq 1} \chi(u)\tau(\bar{\chi}) \mu \left( \frac{r_0}{d} \right) \varphi \left( \frac{r_0}{d} \right)
\]

\[
\times \left( \frac{\xi_q}{\xi_q} \right)^2 \left( -M \right) \left( M^2 R' \right)^{\frac{1}{2} - s} \left( \frac{r_0 a}{r_0 d} \right) ^{\frac{1}{2} - s + \frac{k-1}{2}} \left( \frac{r_0 a}{r_0 d} \right) ^{s + \frac{k-1}{2}} L(1 - s + \frac{k-1}{2}, f_{R'}), -\frac{R'a}{M}.
\]

Here \( R' = \text{lcm}(R, \text{cond}(\xi_R); r, R^2) \), \( R'a = \text{cond}(\xi_R); r, R^2) \equiv 1 \pmod{M} \), \( \tilde{f}_{R'} = f^x \mid W_{R'} \) and \( \tau(\bar{\chi}) = \sum_{\alpha \pmod{r}} \chi(\alpha) e^{2\pi i \alpha} \) is the Gauss sum for \( \chi \).

### 2.2.1. Proof of Theorem 2.1

**Lemma 2.2.** For \( q \in \mathbb{Z}_{\geq 1} \), assume that \( R \mid q \) and \( \gcd(R, q/R) = 1 \). Take \( M \in \mathbb{Z}_{\geq 1} \) such that \( \frac{q}{R} \mid M \) and \( \gcd(R, M) = 1 \). For \( a \pmod{M} \) with \( \gcd(a, M) = 1 \), set

\[
V_{q,R}^{M,a} = \left( \begin{array}{c}
M & \frac{R Ra - R a}{R} \\
q & \frac{R a}{R} \\
\end{array} \right)
\]

be an integral matrix with \( \det(V_{q,R}^{M,a}) = R \). Here \( Ra \equiv 1 \pmod{M} \).

When \( f \in S(q, \xi, k) \), we get

\[
f \left( \frac{a}{M} + iy \right) = \xi_R(-M) \xi_q(R) \xi_R \left( \frac{R a}{M} \right) \left( -\frac{R a}{M} + \frac{1}{M^2 R y} \right).
\]

**Proof.** Applying [1 Proposition 1.1],

\[
\tilde{f}_R \mid V_{q,R}^{M,a} = \xi_R(M) \xi_q(R) \xi_R(-1) \xi_q(R) f = \xi_R(-M) \xi_q(R) f.
\]

Note that

\[
V_{q,R}^{M,a} \left( \frac{a}{M} + iy \right) = -\frac{R a}{M} + i \frac{1}{M^2 R y}.
\]
So we get
\[
f\left(\frac{\alpha}{M} + iy\right) = \xi_R(-M)\xi_{q/R}(a)(\tilde{f}_R \mid \mathbb{V}_{q,R} \left(\frac{\alpha}{M} + iy\right))
= \xi_R(-M)\xi_{q/R}(a)R^{\frac{k}{2}}(-iMRy)^{-k}\tilde{f}_R(-\frac{Ra}{M} + i\frac{1}{M^2 Ry}).
\]

Now we consider more general cases. For \( r \in \mathbb{Z}_{\geq 1} \) and a Dirichlet character \( \chi \mymod r \), define
\[
c_\chi(n) = \sum_{u \mymod r} \chi(u)e^{2\pi in\frac{\alpha}{M}}.
\]
Then by orthogonality, for \( a \in \mathbb{Z} \) with \( \gcd(a, r) = 1 \), we have
\[
e^{2\pi in\frac{\alpha}{M}} = \frac{1}{\varphi(r)} \sum_{\chi \mymod r} \chi(a)c_\chi(n).
\]

**Lemma 2.3.** Assume that \( q, M_1, M, r \) and \( R \) are as given in the statement of Theorem 2.1. Since \( M_1 = Mr \) and \( \gcd(M, r) = 1 \), for any \( \alpha \in \mathbb{Z} \) with \( \gcd(\alpha, M_1) = 1 \), there exist \( \alpha \mymod M \) and \( u \mymod r \) with \( \gcd(a, M) = 1 \) and \( \gcd(u, r) = 1 \) such that \( \alpha \equiv aq + uM \mymod Mr \). We then have \( \frac{\alpha}{M_1} = \frac{\alpha}{M} + \frac{u}{r} \) and
\[
(9) \quad f\left(\frac{\alpha}{M_1} + iy\right)
= \frac{1}{\varphi(r)} \sum_{r | r_0} \frac{r}{r_*r_0} \sum_{d | r_0} \mu\left(\frac{r_0}{d}\right) \phi\left(\frac{r_0}{d}\right) \sum_{\chi \mymod r_* \text{ primitive}} \tau(\chi) \chi(ud)f(\frac{a r_0}{M} + i\frac{r}{r_*d}y).
\]

**Proof.** Since \( \frac{\alpha}{M_1} = \frac{\alpha}{M} + \frac{u}{r} \), we get
\[
(10) \quad f\left(\frac{\alpha}{M} + iy\right) = \sum_{n=1}^{\infty} a(n)e^{2\pi in\frac{\alpha}{M}}e^{2\pi in\frac{\alpha}{M} + iy} = \frac{1}{\varphi(r)} \sum_{\chi \mymod r} \chi(u) \sum_{n=1}^{\infty} a(n)c_\chi(n)e^{2\pi in\left(\frac{\alpha}{M} + iy\right)}.
\]
For a Dirichlet character \( \chi \mymod r \), assume that \( \chi \) is induced from a primitive character \( \chi_* \mymod r_* \). Let \( r_0 = \prod_{p | r, p \nmid r_*} p \) and \( r_2 = \frac{r}{r_0} \). By [3] Lemma 4.11, we have \( c_\chi(n) = 0 \) if \( r_2 \nmid n \) and for any \( n \in \mathbb{Z}_{\geq 1} \),
\[
(11) \quad c_\chi(nr_2) = r_2\chi_*\left(r_0\right)\tau(\chi_*)\bar{\chi}(n}\mu(\gcd(r_0, n))\varphi(\gcd(r_0, n)).
\]
Applying this to (10), we have
\[
(12) \quad f\left(\frac{\alpha}{M} + iy\right) = \frac{1}{\varphi(r)} \sum_{\chi \mymod r} \chi(u) \sum_{n=1}^{\infty} a(nr_2)c_\chi(nr_2)e^{2\pi inr_2\left(\frac{\alpha}{M} + iy\right)}
= \frac{1}{\varphi(r)} \sum_{\chi \mymod r} \chi(u)r_2\chi_*\left(r_0\right)\tau(\chi_*)a(r_2) \sum_{n=1}^{\infty} a(n)\bar{\chi}(n}\mu(\gcd(r_0, n))\varphi(\gcd(r_0, n))e^{2\pi inr_2\left(\frac{\alpha}{M} + iy\right)}.
\]
The last equality holds because \( f \in N(q, \xi, k) \), so \( f \mid U_p = a(p)f \) for any prime \( p \mid q \), so \( a(nr_2) = a(r_2)a(n) \). Note that \( r_2 \mid r \) and \( r \mid q \) so \( r_2 \mid q \). By definition \( r_0 \) is square-free, so we
have
\[
\sum_{n=1}^{\infty} a(n)\chi(n) \mu(\gcd(r_0, n)) \varphi(\gcd(r_0, n)) e^{2\pi i n r_2 z} = \sum_{d|r_0} \mu(d) \varphi(d) \sum_{n=1}^{\infty} a(dn) \chi_s(dn) e^{2\pi i n r_2 z}
\]
\[
= \sum_{d|r_0} \mu(d) \varphi(d) a(d) \chi_s(d) \sum_{n=1}^{\infty} a(n) \chi(n) e^{2\pi i n r_2 z} = \sum_{d|r_0} \mu(d) \varphi(d) a(d) \chi_s(d) f^{\chi}(dr_2 z).
\]
By applying (13) to (12), taking \( z = \frac{a}{M} + iy \), we get
\[
f\left( \frac{\alpha r}{M} + iy \right) = \frac{1}{\varphi(r)} \sum_{\chi \bmod r} \chi(u) r_2 \tau(\chi_s) a(r_2) \sum_{d|r_0} \mu(d) \varphi(d) a(d) \chi_s \left( \frac{r_0}{d} \right) f^{\chi_s}\left( dr_2 \left( \frac{a}{M} + iy \right) \right)
\]
\[
= \frac{1}{\varphi(r)} \sum_{r_0 | r, r_0 = \prod_{p|r, p \not| r_*} p} \frac{r}{r_0} a \left( \frac{r}{r_0} \right) \sum_{d|r_0} \mu \left( \frac{r_0}{d} \right) \varphi \left( \frac{r_0}{d} \right) \sum_{\chi \bmod r, \text{primitive}} \tau(\chi) \chi(ud) f^{\chi} \left( \frac{a r_s}{M} + i \frac{r}{r_0} dy \right).
\]

\[\Box\]

**Lemma 2.4.** Let \( \chi \) be a primitive Dirichlet character modulo \( r_* > 1 \). For \( q \in \mathbb{Z}_{\geq 1} \), let \( R_* | q \) be the \( r_* \)-primary factor of \( q \).

For \( f \in N(q, \xi, k) \), there exist \( R'_* | \lcm(R_*, r_*, \text{cond}(\xi R_\ast \chi)) \), and a Hecke-normalized newform \( F_\chi \in N(R'_* e^{\frac{\alpha r}{R_*}}, \xi \chi^2, k) \), such that
\[
f^{\chi}(z) = \sum_{\ell | R_*} \mu(\ell) (F_\chi \mid U_\ell \mid B_\ell)(z).
\]
Here we consider \( \xi \chi^2 \) as a Dirichlet character modulo \( R'_* e^{\frac{\alpha r}{R_*}} \). Moreover, let
\[
F_\chi(z) = \sum_{n=1}^{\infty} a_\chi(n) e^{2\pi inz}.
\]
By comparing the Fourier coefficients for both sides, for \( n \in \mathbb{Z}_{\geq 1} \), \( \gcd(n, r_*) = 1 \), we get
\[
a_\chi(n) = \chi(n) a(n).
\]

**Proof.** The existence of such a newform \( F_\chi \in N(q', \xi \chi^2, k) \) follows by applying [1, Theorem 3.2] for every prime \( p | r_* \). By [2] Lemma 1.4, we get
\[
q' \mid \lcm(q, \text{cond}(\chi) \text{cond}(\xi R_\ast \chi)) = q \frac{\lcm(R_*, r_*, \text{cond}(\xi R_\ast \chi))}{R_*}.
\]
Then \( R'_* := \frac{q'}{q/R_*} \mid \lcm(R_*, r_*, \text{cond}(\xi R_\ast \chi)) \).

\[\Box\]

Now we are ready to prove Theorem 2.1. Again take \( q, M_1, M, r \) and \( R \) as given in the assumption of Theorem 2.1 and let \( R' = \lcm(R, \text{cond}(\xi R)r^2) \). Then by [1] Proposition 3.1, \( f^{\chi} \in S(R'q/R, \xi \chi^2, k) \) for any primitive character \( \chi \) with \( \text{cond}(\chi) | r \).
Applying (9), we get

\begin{equation}
(14) \quad f^x(\frac{r \chi}{M} + i \frac{r}{r \chi} dy) = i^k (\xi_R \chi_2)(-M) \zeta_{q/R} \left( \frac{r}{r \chi} \right) (MR^\frac{1}{2} \frac{r}{r \chi} y)^{-k} \widetilde{f}_R(\alpha - \frac{R^a r \chi}{r \chi} y).
\end{equation}

Here \( \widetilde{f}_R = f^x | W_R' \in S \left( R^a \frac{\chi_R \xi_{q/R}}{R}, k \right) \) and we set

\begin{equation}
(15) \quad \widetilde{f}_R(x)(z) = \sum_{m \geq 1} b_{\chi, R'}(m) e^{2\pi imz}.
\end{equation}

Consider

\begin{equation}
(16) \quad \int_0^\infty f\left( \frac{\alpha}{M r} + i y \right) y^{s+\frac{k-1}{2}} \frac{dy}{y} = \frac{\Gamma \left( s + \frac{k-1}{2} \right)}{(2\pi)^{s+\frac{k-1}{2}}} \sum_{n=1}^\infty a(n) e^{2\pi in \frac{\alpha}{Mr}} \int_0^\infty e^{-2\pi ny} y^{s+\frac{k-1}{2}} \frac{dy}{y} = \frac{\Gamma \left( s + \frac{k-1}{2} \right)}{(2\pi)^{s+\frac{k-1}{2}}} L(s + \frac{k-1}{2} + 1, f, \frac{\alpha}{Mr}).
\end{equation}

Applying (9), we get

\begin{equation}
\frac{\Gamma \left( s + \frac{k-1}{2} \right)}{(2\pi)^{s+\frac{k-1}{2}}} L(s + \frac{k-1}{2} + 1, f, \frac{\alpha}{Mr}) = \int_0^\infty f\left( \frac{\alpha}{M r} + i y \right) y^{s+\frac{k-1}{2}} \frac{dy}{y} = \frac{1}{\varphi(r)} \sum_{r | r_0} \sum_{d | r_0} \sum_{\chi \bmod r, \text{primitive}} \chi(\alpha d) \tau(\chi) \mu\left( \frac{r_0}{d} \right) \varphi \left( \frac{r_0}{d} \right) a \left( \frac{r}{r \chi} \right) \times \int_0^\infty f^x(\frac{r \chi}{M} + i \frac{r}{r \chi} dy) y^{s+\frac{k-1}{2}} \frac{dy}{y}
\end{equation}

By (14),

\begin{equation}
\int_0^\infty f^x(\frac{r \chi}{M} + i \frac{r}{r \chi} dy) y^{s+\frac{k-1}{2}} \frac{dy}{y} = i^k (\xi_R \chi_2)(-M) \zeta_{q/R} \left( \frac{r}{r \chi} \right) \int_0^\infty (MR^\frac{1}{2} \frac{r}{r \chi} y)^{-k} \widetilde{f}_R(\alpha - \frac{R^a r \chi}{r \chi} y) y^{s+\frac{k-1}{2}} \frac{dy}{y} = i^k \frac{\Gamma \left( 1 - s + \frac{k-1}{2} \right)}{(2\pi)^{1-s+\frac{k-1}{2}}} \xi_{q/R} \chi_2(-M) (M^2 R')^{\frac{1}{2} - s} \left( \frac{\xi_{q/R} \left( \frac{r \chi}{R} \right)}{(r \chi)} \right)^{s+\frac{k-1}{2}} L(1 - s + \frac{k-1}{2}, \widetilde{f}_R, - \frac{R^a r \chi}{r \chi} y).
\end{equation}

This implies (8).

2.3. Decomposition of \( \widetilde{f}_R \) and its Fourier coefficients.

**Lemma 2.5.** Let \( f \) be a Hecke-normalized newform \( f \in N(q, \xi, k) \). For \( r | q \), let \( R \) be the \( r \)-primary factor of \( q \). Let \( \chi \) be a primitive Dirichlet character modulo \( r_* | r \) and \( R_* \) be the
Proof. In this set-up, by Lemma 2.4 there exists \( R' | \text{lcm}(R_*, r_*) \) since \( \text{gcd} \left( \frac{q}{R}, R' \right) \). We set \( F(z) = \sum_{n=1}^{\infty} a\chi(n) e^{2\pi i n z} \) and \( r_0 = \prod_{p \mid r_0, a\chi(p) \neq 0} p \).

Let \( R' = \text{lcm}(R, \text{cond}(\xi R), r_0, r_2) \). Then \( F(z) \in S \left( \mathcal{R}_{\xi, 0}, \mathcal{R}_{\xi, 2}, k \right) \), and we also have

\[
(17) \quad \tilde{f}_R(z) = \sum_{\ell \mid r_0} \mu(\ell)(F_\chi | U_\ell | B_\ell)(z).
\]

Here \( \tilde{f}_R(z) = F_\chi | W_{R, R'} \) and there exists a constant \( \lambda_{\frac{R'}{R_*}}(F_\chi) \) of absolute value one such that

\[
\lambda_{\frac{R'}{R_*}}(F_\chi) \tilde{f}_{\frac{R'}{R_*}} \in N \left( \mathcal{R}_{\xi, 0}, \mathcal{R}_{\xi, 2}, \mathcal{R}_{%R, 0}, \mathcal{R}_{\xi, 0}, k \right).
\]

Remark 2.6. Note that \( \lambda_{\frac{R'}{R_*}}(F_\chi) \) is a pseudo eigenvalue given in \([1]\).

Proof. In this set-up, by Lemma 2.4 there exists \( F_\chi \in N \left( \mathcal{R}_{\xi, 0}, \mathcal{R}_{\xi, 2}, k \right) \), such that

\[
f_\chi(z) = \sum_{\ell \mid r_0} \mu(\ell)(F_\chi | U_\ell | B_\ell)(z).
\]

As stated in Lemma 2.4, here we consider \( \xi \mathcal{R}_{\xi, 2} \) as a character modulo \( \frac{R' \mathcal{R}_{\xi, 2}}{R_*} \). More precisely, since \( \text{gcd} \left( \frac{q}{R_*, r_*} \right) = 1 \), we have \( \xi \mathcal{R}_{\xi, 2} = (\xi \mathcal{R}_{\xi, 2})_{R_*, R_*} \). If there exists \( p | r_* \), with \( p \nmid R' \), this implies that \( \xi_{p \mid r_*} \mathcal{R}_{\xi, 2} \) is induced from the trivial character. So this allows us to consider \( \xi_{R_*} \mathcal{R}_{\xi, 2} \) as a Dirichlet character modulo \( R_* \) and \( \xi_{R_*} \mathcal{R}_{\xi, 2} = (\xi \mathcal{R}_{\xi, 2})_{R_*} \).

For a prime \( p \nmid \frac{R' \mathcal{R}_{\xi, 2}}{R_*} \), by definition,

\[
a\chi(p)F_\chi = F_\chi | T_p = p^{k-1}((\xi \mathcal{R}_{\xi, 2})_{R_*, R_*})(p)F_\chi | B_p + F_\chi | U_p.
\]

Note that \( ((\xi \mathcal{R}_{\xi, 2})_{R_*, R_*})(p) = 0 \) if \( p \nmid \frac{R' \mathcal{R}_{\xi, 2}}{R_*} \). So for any prime \( p \), we have

\[
F_\chi | U_p = a\chi(p)F_\chi - p^{k-1}((\xi \mathcal{R}_{\xi, 2})_{R_*, R_*})(p)F_\chi | B_p.
\]

For a prime \( p \mid r_* \) when \( a\chi(p) = 0 \), i.e. \( p \nmid r_0 \), we get \( F_\chi | U_p = 0 \). Then for each square-free divisor \( \ell \mid r_* \) if \( \ell \nmid r_0 \), we have \( F_\chi | U_\ell = 0 \). Moreover for \( p \mid r_0 \), if \( p \mid R' \), we get \( F_\chi | U_p = a\chi(p)F_\chi \). Thus

\[
f_\chi(z) = \sum_{\ell \mid r_0} \mu(\ell)(F_\chi | U_\ell | B_\ell)(z) = \sum_{\ell \mid r_0} \sum_{\ell' \mid \text{gcd}(r_0, R_*), \ell' \mid \text{gcd}(r_0, R'_*)} \mu(\ell\ell')(F_\chi | U_{\ell'} | U_\ell | B_{\ell'})(z)
\]

\[
= \sum_{\ell' \mid \text{gcd}(r_0, R'_*)} \sum_{\ell \mid \text{gcd}(r_0, R'_*)} \mu(\ell\ell')a\chi(\ell')(F_\chi | U_\ell | B_{\ell'})(z).
\]
For each $\ell \mid \frac{r_0}{\gcd(r_0, R')}$, we then have
\[
F_\chi | U_\ell = F_\chi | \prod_p (T_p - p^{k-1}((\xi^2)_{R'_\xi/R_\ell})(p)B_p) \\
= \sum_{\ell_1 | \ell} \mu(\ell_1)a_\chi(\ell/\ell_1)\ell_1^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(F_\chi | B_{\ell_1}).
\]
This implies
\[
f^\chi = \sum_{\ell' | \gcd(r_0, R') \ell} \sum_{\ell_0 | \gcd(r_0, R')} \mu(\ell')a_\chi(\ell')(F_\chi | U_\ell | B_{\ell\ell'}) \\
= \sum_{\ell' | \gcd(r_0, R') \ell} \sum_{\ell_0 | \gcd(r_0, R')} \mu(\ell')a_\chi(\ell') \sum_{\ell_1 | \ell} \mu(\ell_1)a_\chi(\ell/\ell_1)\ell_1^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(F_\chi | B_{\ell_1\ell\ell'}).
\]
Set $\ell_2 = \frac{\ell}{\ell_1}$ and write $\ell = \ell_2\ell_1$. Since $r_0$ is square-free, this becomes
\[
\sum_{\ell' | \gcd(r_0, R') \ell_2} \sum_{\ell_0 | \gcd(r_0, R')} \sum_{\ell_1 | \ell} \mu(\ell_2\ell')a_\chi(\ell_2\ell')\ell_2^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_2)(F_\chi | B_{\ell_2\ell_2\ell_1}).
\]
Set $\ell\ell_2 = \ell$. Then $\ell_2 = \gcd(r_0/\gcd(r_0, R'), \ell)$ and we get
\[
(18) \quad f^\chi = \sum_{\ell | r_0} \mu(\ell)a_\chi(\ell) \sum_{\ell_1 | \gcd(r_0, \ell \gcd(r_0, R'))} \ell_1^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(F_\chi | B_{\ell\ell_1}).
\]
Set $r_{s+1} = \frac{r_0}{\gcd(r_0, R')}$. Note that $\ell\ell_2^2 | r_0r_{s+1}$. Since $F_\chi$ is a newform of level $R_\ell' \frac{R'}{R_\ell}$, from (18), we see that $f^\chi$ is a cusp form of level $R_\ell' \frac{q_{0R_\ell}}{R_\ell}$. So we conclude that $R_\ell' r_0r_{s+1} | R'$.

Our aim is to get a formula for
\[
f^\chi | W_{R'} = \sum_{\ell | r_0} \mu(\ell)(F_\chi | U_\ell | B_{\ell}) | W_{R'} \\
= \sum_{\ell | r_0} \mu(\ell)a_\chi(\ell) \sum_{\ell_1 | \gcd(r_0, \ell \gcd(r_0, R'))} \ell_1^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(F_\chi | B_{\ell\ell_1} | W_{R'}). \\
\]
Since $\ell, \ell_1 | r_0$, $\gcd(\ell_1, \ell R') = 1$, we get $\ell\ell_1^2 | R'$, and by [1] Proposition 1.5],
\[
F_\chi | B_{\ell\ell_1^2} | W_{R'} = (\ell\ell_1^2)\frac{1}{2\xi_{q/R_\ell}}(\ell\ell_1^2)(F_\chi | W_{R'} | \ell_1). \\
\]
Note that the $W_{R'}$-operator on the RHS is an operator for level $R_\ell' \frac{q}{\ell_1^2} R$. We get
\[
f^\chi | W_{R'} = \sum_{\ell | r_0} \mu(\ell)a_\chi(\ell) \sum_{\ell_1 | r_0, \gcd(\ell_1, \ell R') = 1} \ell_1^{k-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(\ell\ell_1^2)\frac{1}{2\xi_{q/R_\ell}}(\ell\ell_1^2)(F_\chi | W_{R'} | \ell_1^2) \\
= \sum_{\ell | r_0} \mu(\ell)a_\chi(\ell)\ell\frac{1}{2\xi_{q/R_\ell}}(\ell)(\ell) \sum_{\ell_1 | r_0, \gcd(\ell_1, \ell R') = 1} \ell_1^{-1}((\xi^2)_{R'_\xi/R_\ell})(\ell_1)(F_\chi | W_{R'} | \ell_1^2). \\
\]
Let
\[ W_{\frac{R'}{R_1}} = \begin{pmatrix} \frac{R'}{R_1} x_1 & x_2 \\ \frac{q}{R_1} \frac{R'}{R_1} x_3 & \frac{R'}{R_1} x_4 \end{pmatrix}, \]
where \( x_1, x_2, x_3, x_4 \in \mathbb{Z} \), \( \det(W_{\frac{R'}{R_1}}) = \frac{R'}{R_1} \), \( x_1 \equiv 1 \pmod{q/R} \) and \( x_2 \equiv 1 \pmod{R'/\ell_1^2} \), as given in Section 2.2. Since \( F_\chi \in N(R_1^{R'/R}, \xi x^2, k) \), we lower the level of \( W_{\frac{R'}{R_1}} \) to \( R'_1/R_1 \):
\[ W_{\frac{R'_1}{R_1}} = \begin{pmatrix} \frac{R'_1}{R_1} r_1 x_1 & x_2 \\ \frac{q}{R_1} \frac{R'_1}{R_1} x_3 & \frac{R'_1}{R_1} x_4 \end{pmatrix} \begin{pmatrix} \frac{R'}{R_1} R'_1 \ell_1^2 & 1 \end{pmatrix} W_{\frac{R_R'}{R_1}} \begin{pmatrix} \frac{R'}{R_1} R'_1 \ell_1^2 & 1 \end{pmatrix}. \]

Here \( W_{\frac{R_R'}{R_1}} \) is an operator for level \( R'_1q/R_1 \). Note that \( \frac{R_R'}{R_1}, \frac{R_R'}{R_1} \ell_1 \in \mathbb{Z} \) because of the construction. Thus we get
\[ (F_\chi | W_{\frac{R'}{R_1}})(z) = (F_\chi | W_{\frac{R_R'}{R_1}}) \left( \begin{pmatrix} \frac{R}{R_1} R'_1 \ell_1 & 1 \end{pmatrix} \right)(z) = \left( \begin{pmatrix} \frac{R}{R_1} R'_1 \ell_1 & 1 \end{pmatrix} \right)^{-1} F_{\frac{R_R'}{R_1}} \left( \begin{pmatrix} \frac{R}{R_1} R'_1 \ell_1 & 1 \end{pmatrix} \right) \left( \frac{R}{R_1} R'_1 \ell_1 \right)^{k} F_{\frac{R_R'}{R_1}} \left( \begin{pmatrix} \frac{R}{R_1} R'_1 \ell_1 & 1 \end{pmatrix} \right)^{-1} (F_\chi)(z). \]

Here \( \frac{R_R'}{R_1} = F_\chi | W_{\frac{R_R'}{R_1}} \).

By [1], there exists a constant \( \lambda_{\frac{R_R'}{R_1}}(F_\chi) \) of absolute value one, such that
\[ \lambda_{\frac{R_R'}{R_1}}(F_\chi) \frac{R_R'}{R_1} \in N \left( \frac{R_R'}{R_1}, (\xi x^2)_{R_1} \xi_{R/R_1} \xi_{q/R}, k \right). \]

Therefore, we finally obtain (17). \( \square \)

Applying Lemma 2.5, we set the Fourier expansion of \( \frac{R_R'}{R_1} \) to be
\[ \frac{R_R'}{R_1}(z) = \lambda_{\frac{R_R'}{R_1}}(F_\chi) \sum_{n=1}^{\infty} \tilde{a}_{\chi, \frac{R_R'}{R_1}}(n) e^{2\pi i nz}. \]

Lemma 2.7. Set
\[ \tilde{f}_{\chi, R'}(z) = \sum_{m=1}^{\infty} b_{\chi, R'}(m) e^{2\pi imz}. \]

Let \( r_{*0} = \prod p | r_{*a} \neq p \) and \( r_{*1} = \frac{r_{*a}}{\gcd(r_{*0}, R_1)} \). Then \( b_{\chi, R'}(m) = 0 \) when \( r_{*0} R_{1, r_{*0} r_{*1}} \mid m \), and otherwise, for \( n \in \mathbb{Z}_{\geq 1} \),
\[ \left| \left( \frac{R_1 R'}{R_1 R_{1, r_{*0} r_{*1}}^n} \right)^{-b_{\chi, R'}} \left( \frac{R_1 R'}{R_1 R_{1, r_{*0} r_{*1}}^n} \right) \right| \leq \left( \frac{n}{r_{*0} r_{*1}} \right)^{\epsilon} \left( \frac{R_1 R'}{R_1 R'} \right)^{\frac{1}{2}} \sigma_0(r_{*0}) \sigma_0(r_{*1}), \]
for any \( \epsilon > 0 \).
Proof. Applying \((19)\) to \((17)\), we get

\[
\tilde{f}_{R'}(z) = \sum_{m=1}^{\infty} b_{\chi,R'}(m)e^{2\pi imz} = \lambda_{\frac{R'\overline{R}}{R}}(F_{\chi}) \sum_{\ell | r_{*0}} \mu(\ell) a_{\chi}(\ell) e^{-\frac{1}{2} \overline{\xi_{q/R_{*}}} (\ell)} \sum_{\ell | r_{*0}, \gcd(\ell, \ell R'_{*}) = 1} \ell_1^{-1}((\xi_{\chi}^2)_{R_{*}' \overline{R_{*}} \xi_{q/R_{*}}})(\ell_1) \left( \frac{R_{*}' R'}{R_{*}' R' \ell_1} \right)^{\frac{k}{2}} \chi_{\overline{R_{*}'}}(n)e^{2\pi in \frac{R_{*}' R'}{R_{*}' R' \ell_1} z}.
\]

Note that for any \(\ell, \ell_1 | r_{*0}\) with \(\gcd(\ell_1, \ell R'_{*}) = 1\), we have \(\ell \ell_1^2 | r_{*0} r'_{*1}\). (We also note that by the construction from the proof of Lemma \((2.3)\) \(\frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} \in \mathbb{Z}\).) This implies that the \(m\)th Fourier coefficient of \(\tilde{f}_{R'}(z)\) is 0 when \(\frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} \not| m\). In other words, when \(\frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} \not| m\), we get \(b_{\chi,R'}(m) = 0\).

For any \(n \in \mathbb{Z}_{\geq 1}\), we have

\[
b_{\chi,R'} \left( \frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} n \right) = \lambda_{\frac{R' \overline{R}}{R_{*}}}(F_{\chi}) \sum_{\ell | r_{*0}} \mu(\ell) a_{\chi}(\ell) e^{-\frac{1}{2} \overline{\xi_{q/R_{*}}} (\ell)} \times \sum_{\ell | r_{*0}, \gcd(\ell, \ell R'_{*}) = 1} \ell_1^{-1}((\xi_{\chi}^2)_{R_{*}' \overline{R_{*}} \xi_{q/R_{*}}})(\ell_1) \left( \frac{R_{*}' R'}{R_{*}' R' \ell_1} \right)^{\frac{k}{2}} \tilde{a}_{\chi_{R_{*}}}(n) e^{2\pi in \frac{R_{*}' R'}{R_{*}' R' \ell_1} z}.
\]

Note that \(\tilde{a}_{\chi_{R_{*}}}(n) = 0\) if \(n \notin \mathbb{Z}_{\geq 1}\). For any \(m \in \mathbb{Z}_{\geq 1}\), we have \(|a_{\chi}(m)| \leq m^{\frac{k-1}{2} + \epsilon}\) and \(|\tilde{a}_{\chi_{R_{*}}}(m)| \leq m^{\frac{k-1}{2} + \epsilon}\), for any \(\epsilon > 0\). Thus we finally get

\[
\left| \left( \frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} n \right)^{\frac{k-1}{2}} b_{\chi,R'} \left( \frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} n \right) \right| \leq \left( \frac{R_{*}' R'}{R_{*}' R' r_{*0} r_{*1}} n \right)^{-\frac{k-1}{2}} \sum_{\ell | r_{*0}} \ell^{\frac{k+1}{2} + \epsilon} \ell^{-\frac{1}{2}} \sum_{\ell | r_{*0}, \gcd(\ell, \ell R'_{*}) = 1} \ell_1^{-1} \left( \frac{R_{*}' R'}{R_{*}' R' \ell_1} \right)^{\frac{k}{2}} \left( \frac{n \ell_1^2}{r_{*0} r_{*1}} \right)^{\frac{k-1}{2} + \epsilon}
\]

\[
= \left( \frac{n}{r_{*0} r_{*1}} \right)^{\epsilon} \left( \frac{R_{*}' R'}{R_{*}' R' R} \right)^{\frac{k}{2}} \sum_{\ell | r_{*0}, \gcd(\ell, \ell R'_{*}) = 1} \ell^{-1 + 2\epsilon} \prod_{p | r_{*0}} \left( 1 + p^{-2 + 2\epsilon} \right)
\]

\[
\leq \left( \frac{n}{r_{*0} r_{*1}} \right)^{\epsilon} \left( \frac{R_{*}' R'}{R_{*}' R'} \right)^{\frac{k}{2}} \prod_{p | r_{*0}} (1 + p^{-1 + 2\epsilon}) \prod_{p | r_{*1}} (1 + p^{-2 + 2\epsilon}).
\]

\[\square\]
2.4. **Additive twists in the special case applying to Theorem 1.2.** We return to the special case of interest where $f$ is a Hecke-normalized newform of weight 2 for level $q$. Then by Theorem 2.1 and Lemma 2.7 we have the following corollary.

**Corollary 2.8.** Let $f$ be a Hecke-normalized newform of weight 2 for level $q$. Let $a, d$ be coprime integers and set

$$
M_d = \prod_{p \mid d, \text{ord}_p(d) \geq \text{ord}_p(q)} p^{\text{ord}_p(d)},
$$

$$
r_d = \prod_{p \mid \gcd(q, r_d)} p^{\text{ord}_p(d)},
$$

$$
R_d = \prod_{p \mid \gcd(q, r_d)} p^{\text{ord}_p(q)}.
$$

Note that $\gcd(M_d, r_d) = 1$, $r_d \mid q$, $d = M_d r_d$, and $R_d$ is the $r_d$-primary factor of $q$. Moreover, $r_d < R_d$ unless $r_d = R_d = 1$. Further consider $a_1 \pmod{M_d}$ and $a_2 \pmod{r_d}$ such that $a \equiv a_1 r_d + a_2 M_d \pmod{d}$. Let $R_d' = \text{lcm}(R_d, r_d^2)$. Then we have

$$
(22) \quad (M_d^2 R_d')^{\frac{1}{2}} \frac{\Gamma(s + \frac{1}{2})}{(2\pi)^{s+\frac{1}{2}}} L(s + \frac{1}{2}, f, \frac{a}{d})
$$

$$
= -\frac{(M_d^2 R_d')^{\frac{1}{2}} \Gamma(1 - s + \frac{1}{2})}{(2\pi)^{1-s+\frac{1}{2}}} \frac{1}{\varphi(r_d)} \sum_{r_d | d} \sum_{e | r_d, \text{primitive}} \chi(a_2 e) \tau(\chi) \mu \left( \frac{r_d}{e} \right) \varphi \left( \frac{r_d}{e} \right)
$$

$$
\times \chi^2(M_d) a \left( \frac{r_d}{d} \right) \left( \frac{\pi}{r_d} \right)^{s+\frac{1}{2}} L(1 - s + \frac{1}{2}, \tilde{f} R_d'; \frac{R_d a_1}{R_d d e})
$$

Here $R_d a_1, \frac{R_d}{r_d e} R_d a_1, \frac{r_d}{r_d e} \equiv 1 \pmod{M_d}$.

Moreover, for a primitive Dirichlet character $\chi$ for $\text{cond}(\chi) = r_d$, set

$$
\tilde{f} R_d'(z) = \sum_{m=1}^{\infty} b_{x, R_d'}(m) e^{2\pi imz}.
$$

Let $R_{ds}$ be the $r_{ds}$-primary factor of $q$. Take $R_{ds} \mid \text{lcm}(R_{ds}, r_{ds0}^2)$, $r_{ds0}$ and $r_{ds1}$, as described in Lemma 2.5. Then $b_{x, R_d'}(m) = 0$ when $\frac{R_d R_d'}{R_{ds} R_d r_{ds0} r_{ds1}} \nmid m$ and for $n \in \mathbb{Z}_{\geq 1}$, we get

$$
(23) \quad \left| \left( \frac{R_d r_{ds0} r_{ds1}}{R_{ds} R_d r_{ds0} r_{ds1}} \right)^n \right|^\frac{1}{2} b_{x, R_d'} \left( \frac{R_d R_d'}{R_{ds} R_d r_{ds0} r_{ds1}} \right)^n \ll (nr_{ds0} r_{ds1})^\epsilon \left( \frac{R_d r_{ds0} r_{ds1}}{R_{ds} R_d} \right)^\epsilon,
$$

for any $\epsilon > 0$.

Note that

$$
(24) \quad M_d^2 R_d' = \frac{d^2 R_d}{\gcd(R_d, r_d^2)} = \text{lcm}(q, d^2).
$$

This is because

$$
M_d^2 R_d' = M_d^2 \text{lcm}(R_d, r_d^2) = M_d^2 \frac{R_d r_{ds}^2}{\gcd(R_d, r_d^2)} = d^2 \frac{R_d}{\gcd(R_d, r_d^2)}.
$$
Also,
\[
\gcd(q, d^2) = \gcd(R_d q / R_d, M_d^2 r_d^2) = \gcd(q / R_d, M_d^2) \gcd(R_d, r_d^2) = \frac{q}{R_d} \gcd(R_d, r_d^2),
\]
since \(\frac{q}{R_d} \mid M_d\). Thus we have \(\gcd(R_d, r_d^2) = \gcd(q, d^2) \frac{R_d}{q}\). Combining with the above, we get \(M_d^2 R_d = d^2 q / \gcd(q, d^2) = \text{lcm}(q, d^2)\), as claimed.

It follows from this that at \(\Re(t) = 1 + \epsilon\),
\[
(\text{lcm}(q, d^2))^{t/2} \frac{\Gamma\left(t + \frac{1}{2}\right)}{(2\pi)^{t+\frac{1}{2}}} L(t + \frac{1}{2}, f, \frac{a}{d}) \ll (\text{lcm}(q, d^2))^{1/2 + \epsilon}
\]
because of the Stirling bound for the Gamma function.

Similarly, using Corollary 2.8 we will deduce the following bound for \(t\) with \(\Re(t) = -\epsilon\):
\[
(\text{lcm}(q, d^2))^{t/2} \frac{\Gamma\left(t + \frac{1}{2}\right)}{(2\pi)^{t+\frac{1}{2}}} L(t + \frac{1}{2}, f, \frac{a}{d}) \ll (dq) dq^\frac{1}{2} \prod_{p \mid d, \frac{1}{2} \ord_p(q) < \ord_p(d) < \ord_p(q)} p^{1/2}.
\]

This analysis is more involved, and we present most of the details. For \(\Re(t) = -\epsilon\), by (22), we get
\[
(M_d^2 R_d)^{1/2} \frac{1}{\phi(r_d)} \sum_{r_d \mid d} \sum_{r_d \mid d} \sum_{\chi \mod r_d, \text{primitive}} \sqrt{r_d e \varphi}(r_d/e) \left( \frac{r_d}{r_d e} \right)^{\epsilon' + \epsilon} \sum_{m=1}^\infty \frac{|m^{-\frac{1}{2}} b_{\chi, R_d}(m)|}{m^{1+\epsilon}}
\]
for any \(\epsilon' > 0\). Note that \(b_{\chi, R_d}(m) = 0\) unless \(\frac{R_d R_d'}{R_d R_d' R_d^0 r_d^0 d_1} \mid m\). Applying the upper bound (23), for any \(0 < \epsilon' < \epsilon\), we get
\[
\sum_{m=1}^\infty \frac{|m^{-\frac{1}{2}} b_{\chi, R_d}(m)|}{m^{1+\epsilon}} \ll \left( \frac{R_d R_d'}{R_d R_d' R_d^0 r_d^0 d_1} \right)^{-\frac{1}{2}} (r_d r_d' r_d^0 r_d^0 d_1)^{\frac{1}{2} + \epsilon' - \epsilon} \sigma_0(r_d r_d') \sum_{n=1}^\infty n^{\epsilon'} \zeta(1 + \epsilon - \epsilon') \ll (r_d r_d' r_d^0 r_d^0 d_1)^{\frac{1}{2} + \frac{\epsilon'}{2}}
\]
since \(1 \leq \frac{R_d R_d'}{R_d R_d' R_d^0 r_d^0 d_1}\). Also we have
\[
r_d r_d' r_d^0 r_d^0 d_1 \leq \prod_{p \mid r_d} p \prod_{p \mid r_d' p \mid R_d} p \leq \prod_{p \mid r_d} p^2.
\]
Applying this to (27), we get

\[
(M_d^2 R_d')^{\frac{1}{2} - e'} \sum_{r_{d*} \mid r_{d*}, p} r_{d*}^{-\frac{1}{2} - e'} e_{e' - \epsilon} \sum_{e \mid r_{do}} e^{-1 - e'} \sum_{\chi \mod r_{d*}, \chi \mod p \mid r_{d*}} \prod_{p \mid r_{d*}} p^{1 + e''}.
\]

Note that \(\sum_{\chi \mod r_{d*}, \chi \mod p \mid r_{d*}} < \varphi(r_{d*}) < r_{d*}\) and \(\sum_{e \mid r_{do}} e^{-1 - e'} \ll 1\). It follows that the above is

\[
\ll (M_d^2 R_d')^{\frac{1}{2} - e'} \sum_{r_{d*} \mid r_{d*}, p} r_{d*}^{-\frac{1}{2} - e'} e_{e' - \epsilon} \sum_{\chi \mod r_{d*}, \chi \mod p \mid r_{d*}} \prod_{p \mid r_{d*}} p^{1 + e''}
\ll (M_d^2 R_d')^{\frac{1}{2} - e'} \sum_{r_{d*} \mid r_{d*}, p} r_{d*}^{-\frac{1}{2} - e'} e_{e' - \epsilon} \prod_{p \mid r_{d*}} p^{1 + e''}
\ll (M_d^2 R_d')^{\frac{1}{2} - e'} r_d \prod_{p \mid r_{d*}} p^{1 + e''} = (M_d^2 R_d')^{\frac{1}{2} - e'} r_d \prod_{p \mid r_{d*}} p^{1 + e''}.
\]

Thus

(28)

\[
(M_d^2 R_d')^{\frac{1}{2} - \frac{1}{2}} \frac{\Gamma \left( t + \frac{1}{2} \right)}{(2\pi)^{t+\frac{1}{2}}} L(t + \frac{1}{2}, f, \frac{a}{d}) \ll (\text{lcm}(q, d^2))^{\frac{1}{2}} (dq)^{\frac{1}{2}} \prod_{p \mid r_{d*}} p, \]

as \(M_d^2 R_d' = \text{lcm}(q, d^2)\). More explicitly,

\[
(\text{lcm}(q, d^2))^{\frac{1}{2}} r_d \prod_{p \mid r_{d*}} p = q^{\frac{1}{2}} \prod_{p \mid r_{d*}} p \prod_{p \mid d, \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2} \text{ord}_p(d)} \prod_{p \mid r_{d*}, \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2} \text{ord}_p(d)} \prod_{p \mid d, \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2} \text{ord}_p(d) - \min\{\text{ord}_p(q), 2 \text{ord}_p(d)\}} + 1
\]

and we have

\[
\frac{\prod_{p \mid d, \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2} \text{ord}_p(d) + 1}}{(\text{gcd}(q, d^2))^{\frac{1}{2}}} \leq \prod_{p \mid d, \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2} \text{ord}_p(d) - \min\{\text{ord}_p(q), 2 \text{ord}_p(d)\}} + 1 = \prod_{p \mid d, \text{ord}_p(q) < \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2}}.
\]

Combining the above with (28), we have, for \(\Re(t) = -\epsilon\),

\[
(M_d^2 R_d')^{\frac{1}{2} - \frac{1}{2}} \frac{\Gamma \left( t + \frac{1}{2} \right)}{(2\pi)^{t+\frac{1}{2}}} L(t + \frac{1}{2}, f, \frac{a}{d}) \ll (dq)^{e} dq \prod_{p \mid d, \frac{1}{2} \text{ord}_p(q) < \text{ord}_p(d) < \text{ord}_p(q)} p^{\frac{1}{2}}.
\]

Recall that at \(\Re(t) = 1 + \epsilon\),

\[
(\text{lcm}(q, d^2))^{\frac{1}{2}} \frac{\Gamma \left( t + \frac{1}{2} \right)}{(2\pi)^{t+\frac{1}{2}}} L(t + \frac{1}{2}, f, \frac{a}{d}) \ll (\text{lcm}(q, d^2))^{\frac{1}{2} + \epsilon},
\]
which implies that at $\Re(t) = 1 + \epsilon$,
\[
\Gamma \left( t + \frac{1}{2} \right) L \left( t + \frac{1}{2}, f, \frac{a}{d} \right) \ll (qd)^{\epsilon}.
\]

Similarly, by (26), for $\Re(t) = -\epsilon$,
\[
\Gamma \left( t + \frac{1}{2} \right) L \left( t + \frac{1}{2}, f, \frac{a}{d} \right) \ll (dq)^{\epsilon} dq^{\frac{1}{2}} \prod_{p \mid d, \frac{1}{2} \ord_p(q) < \ord_p(d) < \ord_p(q)} p^{\frac{1}{2}}.
\]

By convexity, we then have via (5), at $\Re(t) = \frac{1}{2}$,
\[
\int_{0}^{\frac{1}{2}} f(z) \, dz = \int_{0}^{\infty} f \left( \frac{a}{d} + iy \right) dy \ll d^{\frac{1}{2}} q^{\frac{1}{2}} (qd)^{\epsilon} \prod_{p \mid d, \frac{1}{2} \ord_p(q) < \ord_p(d) < \ord_p(q)} p^{\frac{1}{2}}.
\]

Note that the product over $p$ equals 1 if $q$ is square-free or $\gcd(q, d) = 1$.

The functional equation of Corollary 2.8 implies the “approximate functional equation” (see e.g. [7] Theorem 5.3). This states:

\begin{align}
    & L \left( 1, f, \frac{a}{d} \right) = \sum_{n \geq 1} a(n) e^{2\pi in \frac{a}{d}} V \left( \frac{M_d R_d^{\frac{1}{2}} X}{2\pi n} \right) \\
    & - \frac{1}{\varphi(r_d)} \sum_{r_{d,0} = 1}^{r_{d,0} \mid r_d \mid} \sum_{e \mid r_{d,0}} \sum_{\chi \mod r_{d,0} \text{ primitive}} \chi(a_2 e) \tau(\chi) \mu \left( \frac{r_{d,0}}{e} \right) \varphi \left( \frac{r_{d,0}}{e} \right) \chi^2(M_d) \\
    & \times a \left( \frac{r_d}{r_{d,0} e} \right) \sum_{n=1}^{\infty} b_X R_t \left( \frac{R_{d,0} R'_{d,0} n}{R_{d,0} R_{d,0} d + d + 1} \right) e^{-2\pi i \frac{R_{d,0}}{R_{d,0} d + d + 1} \frac{R_{d,0}}{R_{d,0} R_{d,0} d + d + 1} n} V \left( \frac{M_d R_d^{\frac{1}{2}} \frac{r_d}{r_{d,0} e} \frac{r_{d,0}}{R_{d,0} R_{d,0} d + d + 1} n X}{2\pi} \right),
\end{align}

for all $X > 0$, with
\[
    V(y) := \frac{1}{2\pi i} \int_{(2)} (2\pi y)^u G(u) \Gamma(u) \, du.
\]

Here $G(u)$ is any even function which is entire and bounded in vertical strips, of arbitrary polynomial decay as $|\Im u| \to \infty$ and such that $G(0) = 1$.

3. An expression of $G_M(x)$ as an integral

For a fixed $x \in [0, 1]$, consider the characteristic function $1_{[0,x]}$ of $[0, x]$ extended to $\mathbb{R}$ periodically with period 1. We will construct a family of smooth $h : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ approximating $1_{[0,x]}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth, non-negative function, compactly supported in $(-1/4, 1/4)$ with $\int_{-1/2}^{1/2} \phi(t) \, dt = 1$ and $\phi(0) = 1$. For each $\delta < 1$ and $t \in (-1/2, 1/2)$, set
\[
    \phi_\delta(t) = \delta^{-1} \phi(t/\delta)
\]
and extend this to \( \mathbb{R} \) periodically, with period 1. The approximating functions are \( h_\delta \) defined by

\[
h_\delta(t) := 1_{[-\delta, x+\delta]} \ast \phi_\delta(t) = \int_{-\delta}^{x+\delta} \phi_\delta(t-v) \, dv = \int_{t-x-\delta}^{t+\delta} \phi_\delta(v) \, dv,
\]
where \( \ast \) denotes the convolution. This function is smooth, and satisfies \( 0 \leq h_\delta(t) \leq 1 \). It vanishes in \((5\delta/4 + x, 1 - 5\delta/4)\) and its translates. Indeed, for \( 1 - 5\delta/4 > t > 5\delta/4 + x \), we have \( \delta/4 < t-x-\delta < t+\delta < 1-\delta/4 \). Since the support of \( \phi_\delta(v) \) is contained in \((-\delta/4, \delta/4)\) and its translations, \(\eqref{32}\) implies that \( \phi_\delta(t) \) vanishes in that range.

We further have

\[
\begin{align*}
\eqref{33} & \quad h_\delta(t) = 1 \quad \text{for } t \in [0, x] \\
\eqref{34} & \quad \hat{h}_\delta(n) = 1_{[-\delta, x+\delta]}(n) \cdot \hat{\phi}_\delta(n)
\end{align*}
\]
for the corresponding \( n \)-th Fourier coefficients. This implies that, for \( n \neq 0 \),

\[
\eqref{35} \quad \hat{h}_\delta(n) = \frac{e^{2\pi in\delta} - e^{-2\pi in(x+\delta)}}{2\pi in} \int_{-1/2}^{1/2} \phi_\delta(t) e^{-2\pi int} \, dt = \frac{e^{2\pi in\delta} - e^{-2\pi in(x+\delta)}}{2\pi in} \int_{-1/2}^{1/2} \phi(t) e^{-2\pi int} \, dt = \frac{e^{2\pi in\delta} - e^{-2\pi in(x+\delta)}}{2\pi in} \int_{-1/2}^{1/2} \phi(t) e^{-2\pi int} \, dt.
\]

The last equality follows because \( \phi \) is supported in \((-1/4, 1/4)\). With the smoothness of \( h_\delta \) we deduce that, for each \( K \geq 0 \) and \( n \neq 0 \),

\[
\eqref{36} \quad |\hat{h}_\delta(n)| \ll_K (|n| + 1)^{-1}(\delta(1 + |n|))^{-K}.
\]
This inequality combines a bound which is uniform in \( \delta \) with a stronger one which, however, is not uniform in \( \delta \). With this notation, we have

**Lemma 3.1.** For \( M > 1 \), consider any fixed \( \delta = \delta_M < 1 \). Then,

\[
G_{\delta_M}^\pm(x) = \frac{1}{M} \sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M}) + \mathcal{O}\left(\delta_M M^{2} q^{\frac{1}{4}}(qM)^{\epsilon} \prod_{p|M, \frac{\ord_p(q)}{2}<\ord_p(M)<\ord_p(q)} p^{\frac{1}{4}}\right).
\]

**Note that the product over \( p | M \) equals 1 if \( q \) is square-free or \( \gcd(q, M) = 1 \).**

**Proof.** If \( a \leq Mx \), then \( \frac{a}{M} \leq x \) and thus \( h_\delta(t) = 1 \) by \(\eqref{33}\). If \( xM < a \leq Mx + \frac{5}{4} M \delta_M \), then \( x < \frac{a}{M} \leq x + \frac{5}{4} \delta_M \).

By definition, \( \langle \frac{a}{M} \rangle^\pm \) is a linear combination of \( \int_{a/M}^{a/M} f(z) \, dz \) and its complex conjugate. With \(\eqref{29}\), we deduce that

\[
\langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M}) \ll M^{2} q^{\frac{1}{4}}(qM)^{\epsilon} \prod_{p|M, \frac{\ord_p(q)}{2}<\ord_p(M)<\ord_p(q)} p^{\frac{1}{4}},
\]
and thus
\[
\frac{1}{M} \sum_{Mx + \frac{5}{4}M \delta_M \geq a > Mx} \langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M}) \ll \frac{1}{M} M^\frac{1}{2} q^{\frac{1}{4}}(qM)^c \prod_{p|M, \frac{\text{ord}_p(q)}{2} < \text{ord}_p(M) < \text{ord}_p(q)} p^\frac{1}{4} \cdot M \delta_M \\
= M^\frac{1}{2} q^{\frac{1}{4}}(qM)^c \delta_M \prod_{p|M, \frac{\text{ord}_p(q)}{2} < \text{ord}_p(M) < \text{ord}_p(q)} p^\frac{1}{4}.
\]
Similarly,
\[
\frac{1}{M} \sum_{M \geq a > M - \frac{5}{4}M \delta_M} \langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M}) \ll M^\frac{1}{2} q^{\frac{1}{4}}(qM)^c \delta_M \prod_{p|M, \frac{\text{ord}_p(q)}{2} < \text{ord}_p(M) < \text{ord}_p(q)} p^\frac{1}{4}.
\]
If \(xM + \frac{5}{4}M \delta_M < a \leq M - \frac{5}{4}M \delta_M\), then \(x + \frac{5}{4} \delta_M < \frac{a}{M} \leq 1 - \frac{5}{4} \delta_M\) and thus, as shown above, \(h_\delta(a/M)\) decreases rapidly. Therefore
\[
\frac{1}{M} \sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M})
= \frac{1}{M} \left( \sum_{0 \leq a \leq Mx} + \sum_{xM < a \leq Mx + \frac{5}{4}M \delta_M} + \sum_{Mx + \frac{5}{4}M \delta_M \leq a \leq M - \frac{5}{4}M \delta_M} + \sum_{M - \frac{5}{4}M \delta_M < a \leq M} \right) \langle \frac{a}{M} \rangle^\pm h_\delta(\frac{a}{M})
= \frac{1}{M} \sum_{0 \leq a \leq Mx} \langle \frac{a}{M} \rangle^\pm \cdot 1 + O \left( \delta_M M^\frac{1}{2} q^{\frac{1}{4}}(qM)^c \prod_{p|M, \frac{\text{ord}_p(q)}{2} < \text{ord}_p(M) < \text{ord}_p(q)} p^\frac{1}{4} \right)
\]
as required.

In view of this lemma, we will initially study this average for an arbitrary smooth periodic \(h\). To avoid overburdening the notation we omit the \(\pm\) sign from \(\langle \cdot \rangle\) for the rest of this subsection, as the arguments clearly hold for both the plus and the minus cases.

We first see that
\[
\sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle h(\frac{a}{M}) = \sum_{0 < c \leq M} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}) - \sum_{0 < c \leq M - 1} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}).
\]
We next use the trivial decomposition:
\[
\sum_{0 < c \leq M} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}) = \frac{1}{\log(\frac{M+1}{2})} \sum_{0 < c \leq M} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}) \left( \log \left( \frac{M+1}{c} \right) - \log \left( \frac{M}{c} \right) \right)
\]
and its analogue with \(M\) replaced by \(M - 1\). Substituting these decompositions into (37) we obtain:
\[
\frac{1}{M} \sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle h(\frac{a}{M}) = \frac{X_{M+1} - X_M}{M \log \left( \frac{M+1}{M} \right)} - \frac{X_M - X_{M-1}}{M \log \left( \frac{M}{M-1} \right)}
\]
for
\[
X_m := \sum_{0 < c \leq m} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}) \log \left( \frac{m}{c} \right) = \sum_{0 < c \leq m-1} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h(\frac{a}{c}) \log \left( \frac{m}{c} \right).
\]
The last equality holds because \( \log(m/m) = 0 \).

The term \( X_m \) equals:

\[
(39) \quad \sum_{k \in \mathbb{N}} \sum_{0 < c \leq m} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h\left(\frac{a}{k}\right) \log\left(\frac{m/k}{c/k}\right) = \sum_{k \in \mathbb{N}} \sum_{0 < c \leq m/k} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h\left(\frac{a}{c}\right) \log\left(\frac{m/k}{c}\right)
\]

and, with [6, Eq. 7.1.(5)], this equals

\[
\sum_{k \in \mathbb{N}} \sum_{0 < c \leq m} \sum_{0 \leq a \leq c} \langle \frac{a}{c} \rangle h\left(\frac{a}{c}\right) \frac{1}{4\pi i} \int_{(2)} \left(\frac{m/k}{c}\right)^{2s} ds \frac{1}{s^2}.
\]

Using (29) we see that we can interchange summation and integration to deduce

\[
X_m = \frac{1}{4\pi i} \int_{(2)} \zeta(2s)m^{2s} \sum_{c > 0} \sum_{0 \leq a \leq c} \frac{c^{-2s} \langle \frac{a}{c} \rangle h\left(\frac{a}{c}\right)}{\gcd(a,c) = 1} ds \frac{1}{s^2}
\]

\[
= \frac{1}{4\pi i} \sum_{n \in \mathbb{Z}} \hat{h}(n) \int_{(2)} \zeta(2s)m^{2s} \sum_{c > 0} \sum_{0 \leq a \leq c} c^{-2s} \langle \frac{a}{c} \rangle e^{\frac{2\pi in}{c}} F_s(M) ds \frac{1}{s^2}.
\]

(Here have used (36) to justify interchanging integration and summation over \( n \).) This implies

**Proposition 3.2.** For each smooth \( h : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \) and each \( M \in \mathbb{Z} \), we have

\[
\frac{1}{M} \sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle h\left(\frac{a}{M}\right) = \frac{1}{4\pi i} \sum_{n \in \mathbb{Z}} \hat{h}(n) \int_{(2)} \zeta(2s) \sum_{c > 0} \sum_{0 \leq a \leq c} c^{-2s} \langle \frac{a}{c} \rangle e^{\frac{2\pi in}{c}} F_s(M) ds \frac{1}{s^2},
\]

where

\[
F_s(M) = \frac{(M + 1)^{2s} - M^{2s}}{M \log\left(\frac{M+1}{M}\right)} - \frac{M^{2s} - (M - 1)^{2s}}{M \log\left(\frac{M}{M-1}\right)}.
\]

4. **Shifted convolutions series**

Following the approach of [5], we will now show that the study of the integral of Proposition 3.2 can be rephrased in terms of shifted convolution series.

With the notation given in (5) and (16), we have,

\[
(40) \quad \langle \frac{a}{c} \rangle^\pm = -\pi \int_{-\infty}^{0} \left( f\left(\frac{a}{c} + ix\right) \pm f\left(-\frac{a}{c} + ix\right) \right) dx
\]

\[
= \frac{\pi}{c} \left( \Lambda(f, 1, \frac{a}{c}) \pm \Lambda(f, 1, -\frac{a}{c}) \right) = \frac{1}{2} \left( L(1, f, \frac{a}{c}) \pm L(1, f, -\frac{a}{c}) \right).
\]

Here we used \( f\left(\frac{a}{c} + ix\right) = f\left(-\frac{a}{c} + ix\right) \).

This implies

**Lemma 4.1.** For each \( s \) with \( \Re(s) = 2 \) and \( n \in \mathbb{Z} \), we have

\[
(41) \quad \sum_{c > 0} \sum_{a \mod c} \langle \frac{a}{c} \rangle^\pm e^{\frac{2\pi i n \frac{a}{c}}{c^{2s}}} = \pi \sum_{c > 0} \sum_{a \mod c} \left( \Lambda(f, 1, \frac{a}{c}) \pm \Lambda(f, 1, -\frac{a}{c}) \right) e^{\frac{2\pi i n \frac{a}{c}}{c^{2s+1}}}
\]
On the other hand, for $\Re(s) = 2$ and $\Re(t) > 3/2$,

$$
\sum_{c>0} \sum_{a \mod c \atop (a,c)=1} \Lambda(f,t,-a/c)e^{2\pi in^2/2c^{s+t}} = \frac{\Gamma(t)}{(2\pi)^t} \sum_{\ell\geq 1} a(\ell) \sum_{c>0} \sum_{a \mod c \atop (a,c)=1} c^{-2s}e^{2\pi i(n-\ell)c/2}.
$$

The last sum gives the $(n-\ell)$th Fourier coefficient $\phi(n-\ell,s)$ (or $\phi(s)$) of the standard Eisenstein series $E(z,s)$ for $SL_2(\mathbb{Z})$. Specifically,

$$
\sum_{c>0} \sum_{a \mod c \atop (a,c)=1} c^{-2s}e^{2\pi i(n-\ell)c/2} = \begin{cases} 
\phi(s) \frac{\Gamma(s)}{\sqrt{\pi} \Gamma(s-1/2)} & \text{if } \ell = n \\
\phi(n-\ell,s) \frac{\Gamma(s)}{\pi |n-\ell|^{s-1/2}} & \text{if } \ell \neq n.
\end{cases}
$$

Therefore,

$$
\sum_{c>0} \sum_{a \mod c \atop (a,c)=1} \Lambda(t,f,-a/c)e^{2\pi in^2/2c^{s+t}} = \frac{\Gamma(t)}{(2\pi)^t} D(n,s,t)
$$

with

$$
D(n,s,t) := \sum_{\ell=1}^{\infty} \frac{a(\ell)\sigma_{1-2s}(|n-\ell|)}{\ell^t},
$$

where we have set

$$
\sigma_{1-2s}(0) := \zeta(2s - 1).
$$

We thus have

**Proposition 4.2.** For $\Re(s) = 2$ and $\Re(t) > 3/2$,

$$
\zeta(2s) \sum_{c>0} \sum_{a \mod c \atop (a,c)=1} \left( \Lambda(t,f,-a/c) \pm \Lambda(t,f,a/c) \right) e^{2\pi in^2/2c^{s+t}} = \frac{\Gamma(t)}{(2\pi)^t} (D(-n,s,t) \pm D(n,s,t)).
$$

5. **An explicit formula for an integral of $D(n,s,t)$**.

If we knew that $D(n,s,t)$ could be analytically continued to $t = 1$, then, with Lemma 4.1 and Proposition 4.2, it would be possible to write the integrand in Proposition 3.2 directly as a linear combination of shifted convolution series. We will instead analytically continue the integral corresponding to $D(n,s,t)$ by evaluating it at the same time. Specifically, for each $n \in \mathbb{Z}$ and for $\Re(t) > 3/2$, we will compute the integral

$$
\int_{(2)} D(n,s,t) \frac{F_s(M)}{s^2} \, ds
$$

for each $M \in \mathbb{N}$.

We first prove

**Proposition 5.1.** For each $\Re(t) > 3/2$, $n \in \mathbb{Z}$ and $M > 1$ we have

$$
\int_{(2)} D(n,s,t) \frac{M^{2s}}{s^2} \, ds = 4\pi i \sum_{d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M}{d} \right),
$$

where we have set

$$
\sigma_{1-2s}(0) := \zeta(2s - 1).
$$
where

\[ \alpha_{n,d}(t) := \sum_{r \geq 1, \, \ell \equiv n \mod d} \frac{a(r)}{r^t}. \]

Proof. For \( s \in \mathbb{C} \) with \( \Re(s) = 2 \) we have

\[ D(n, s, t) = \sum_{\ell \geq 1} \frac{a(\ell) \sigma_{1-2s}(n - \ell)}{\ell^t} = \sum_{\ell \geq 1} \sum_{d | n - \ell} \frac{a(\ell)}{\ell^t} d^{1-2s}. \]

We are allowed to change the order of summation as we are in the region of absolute convergence. The condition \( d | (n - \ell) \) can be reinterpreted as \( \ell \equiv n \mod d \). Therefore,

\[
\int_{(2)} D(n, s, t) \frac{M^{2s}}{s^2} ds = \int_{(2)} \sum_{d \geq 1} d^{1-2s} \sum_{\ell \geq 1, \, \ell \equiv n \mod d} \frac{a(\ell) M^{2s}}{\ell^t} s^2 \frac{ds}{s^2} = \sum_{d \geq 1} d \alpha_{n,d}(t) \int_{(2)} \left( \frac{M}{d} \right)^{2s} ds = 4\pi i \sum_{d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M}{d} \right).
\]

With the definition of \( F_s(M) \) we have, for \( \Re(t) > 3/2 \),

\[
\int_{(2)} D(n, s, t) \frac{F_s(M)}{s^2} ds = \frac{1}{M} \left\{ \frac{1}{\log \left( \frac{M+1}{M} \right)} \left( \int_{(2)} D(n, s, t) \frac{(M+1)^{2s}}{s^2} ds - \int_{(2)} D(n, s, t) \frac{M^{2s}}{s^2} ds \right) \right. \\
- \left. \frac{1}{\log \left( \frac{M}{M-1} \right)} \left( \int_{(2)} D(n, s, t) \frac{M^{2s}}{s^2} ds - \int_{(2)} D(n, s, t) \frac{(M-1)^{2s}}{s^2} ds \right) \right\}.
\]

Now, with (43) we get for the first inner parentheses:

\[
\int_{(2)} D(n, s, t) \frac{(M+1)^{2s}}{s^2} ds - \int_{(2)} D(n, s, t) \frac{M^{2s}}{s^2} ds
\]

\[
= 4\pi i \left( \sum_{1 \leq d \leq M+1} d \alpha_{n,d}(t) \log \left( \frac{M+1}{d} \right) - \sum_{1 \leq d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M}{d} \right) \right)
\]

\[
= 4\pi i \left( \sum_{1 \leq d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M+1}{d} \right) - \sum_{1 \leq d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M}{d} \right) \right)
\]

\[
= 4\pi i \sum_{1 \leq d \leq M} d \alpha_{n,d}(t) \log \left( \frac{M+1}{M} \right).
\]

The passage from (46) to (47) is justified because \( \log((M+1)/(M+1)) = 0 \) and for the passage from (47) to (48) we use

\[
\log \left( \frac{M+1}{d} \right) - \log \left( \frac{M}{d} \right) = \log \left( \frac{M+1}{M} \right).
\]
Applying this with $M$ instead of $M + 1$ we get

$$\int (2) D(n, s, t) \frac{M^{2s}}{s^2} ds - \int (2) D(n, s, t) \frac{(M - 1)^{2s}}{s^2} ds = 4\pi i \sum_{d \leq M - 1} d \alpha_n,d(t) \log \left( \frac{M}{M - 1} \right).$$

Plugging (48) and (49) into (45), we obtain

$$\int (2) D(n, s, t) \frac{F_s(M)}{s^2} ds = 4\pi i \frac{1}{M} \left\{ \sum_{1 \leq d \leq M} d \alpha_n,d(t) \log \left( \frac{M + 1}{M} \right) - \frac{1}{\log \left( \frac{M}{M - 1} \right)} \sum_{1 \leq d \leq M - 1} d \alpha_n,d(t) \log \left( \frac{M}{M - 1} \right) \right\}$$

$$= \frac{4\pi i}{M} \left\{ \sum_{1 \leq d \leq M} d \alpha_n,d(t) - \sum_{1 \leq d \leq M - 1} d \alpha_n,d(t) \right\} = \frac{4\pi i}{M} \left\{ M \alpha_n,M(t) \right\}.$$

Thus we finally get

$$\int (2) D(n, s, t) \frac{F_s(M)}{s^2} ds = 4\pi i \alpha_n,M(t).$$

6. An explicit expression for the weighted average of modular symbols

Putting together Proposition 4.2 and (50) we have that, for $\Re(t) > 3/2$,

$$\int \zeta(2s) \left( \sum_{c > 0} \sum_{\text{a mod c}} \left( \Lambda(t, f \frac{a}{c}) \pm \Lambda(t, f, -\frac{a}{c}) \right) \frac{e^{2\pi i n a}}{c^{2s+t}} \right) \frac{F_s(M)}{s^2} ds$$

$$= \frac{\Gamma(t)}{(2\pi i)^t} 4\pi i \left( \alpha_{n,M}(t) \pm \alpha_{n,M}(t) \right).$$

Now we observe that $\alpha_n,d(t)$ has an analytic continuation to the entire complex plane. Indeed, for $\Re(t) > 3/2$,

$$\alpha_n,d(t) = \sum_{\ell \equiv n \mod d} \frac{a(\ell)}{\ell^t} = \frac{1}{d} \sum_{\text{a mod d}} e^{-\frac{2\pi i a t}{d}} \sum_{\ell \geq 1} \frac{a(\ell)}{\ell^t} e^{\frac{2\pi i a t}{d}} = \frac{1}{d} \sum_{\text{a mod d}} e^{-\frac{2\pi i a}{d}} L(t, f, \frac{a}{d}).$$

Since, as mentioned in Section 2 the $L$-function in the RHS has an analytic continuation to $t \in \mathbb{C}$, that is the case for $\alpha_n,d(t)$ too.

Also, for each $s \in \mathbb{C}$ with $\Re(s) = 2$,

$$\sum_{c > 0} \sum_{\text{a mod c}} \Lambda(t, f, -\frac{a}{c}) \frac{e^{2\pi i n a}}{c^{2s+t}}$$

is analytic as a function of $t \in \mathbb{C}$ for $\Re(t) > 1 - \epsilon$. 
It follows that both sides of (51) are holomorphic for $\Re(t) > 1 - \epsilon$, and hence the identity must hold for $t = 1$. With Proposition 3.2 and Lemma 4.1 this gives

$$\frac{1}{M} \sum_{0 \leq a \leq M} \langle \frac{a}{M} \rangle^\pm h(\frac{a}{M}) = \frac{1}{2} A_h^\pm(M),$$

where

$$A_h^\pm(M) := \sum_{n \in \mathbb{Z}} \hat{h}(n) (\alpha_{-n,M}(1) \pm \alpha_{n,M}(1)).$$

7. The asymptotics of $A_h^\pm(M)$ as $M \to \infty$.

To analyze the asymptotics of $A_h^\pm(M)$ we first observe that,

$$\alpha_{n,M}(t) = \frac{1}{M} \sum_{a \equiv \frac{t}{M}} e^{-\frac{2\pi i an}{M}} L(t,f,\frac{a}{M}) = \frac{1}{M} \sum_{d|M} \sum_{a \equiv \frac{t}{d}} e^{-\frac{2\pi i an}{d}} L(t,f,\frac{a}{d}).$$

For $1 \leq d \mid M$, we recall the notations $M_d$, $r_d$ and $R_d$ given in Corollary 2.8

$$d = M_d r_d, \quad \gcd(M_d,r_d) = 1 \quad \text{and} \quad R_d \text{ is the } r_d\text{-primary factor of } q.$$ 

Moreover $r_d < R_d$ unless $r_d = R_d = 1$. Also $R'_d = \text{lcm}(R_d, r_d^2)$, $M'_d R'_d = \text{lcm}(q, d^2)$. For any divisor $r_d'$ of $r_d$, we have $R'_d \mid \text{lcm}(R_d, r_d)$ and $r_d R_d | r_d$ are square-free, as described in Lemma 2.5 and Lemma 2.7. Finally, $a_1 \pmod{M_d}$ and $a_2 \pmod{r_d}$ are such that $a \equiv a_1 r_d + a_2 M_d \pmod{d}$.

We apply (30) to each $L(t,f,\frac{a}{d})$, with $X = X_d$, and substitute into (54) with $t = 1$:

$$\alpha_{n,M}(1) = \frac{1}{M} \sum_{d|M} \sum_{a \equiv \frac{t}{d}} e^{-\frac{2\pi i an}{d}} \sum_{\ell \geq 1} a(\ell) e^{2\pi i \frac{an}{d} \ell} V\left( \frac{M_d R'_d X_d}{2\pi \ell} \right)$$

$$- \frac{1}{M} \sum_{d|M} \sum_{a \equiv \frac{t}{d}} e^{-\frac{2\pi i an}{d}} \sum_{r_d \mid r_d} \sum_{r_d \mid r_d} \sum_{\ell \geq 1} \chi(a_2 \overline{e}) \tau(\overline{\chi}) \mu\left( \frac{r_d}{e} \right) \varphi\left( \frac{\tau(d)}{e} \right) \chi^2(M_d)$$

$$\times a\left( \frac{r_d}{r_d} \right) \sum_{c \geq 1} b_c X_d \left( \frac{R_d}{R'_d} \right)^{\frac{R_d}{R'_d}} \sum_{r_d \mid r_d} \sum_{r_d \mid r_d} \sum_{\ell \geq 1} \chi(a_2 \overline{e}) \tau(\overline{\chi}) \mu\left( \frac{r_d}{e} \right) \varphi\left( \frac{\tau(d)}{e} \right) \chi^2(M_d)$$

Set $X_d = \frac{X}{M_d R'_d}$, with $X$ independent of $d$. Since

$$\sum_{d|N} \sum_{a \equiv \frac{t}{d}} e^{2\pi i \frac{an}{d} \ell} = \begin{cases} M, & \text{if } n \equiv \ell \pmod{M}, \\ 0, & \text{otherwise} \end{cases}$$
and

\[
\sum_{a \mod d, \phantom{(a,d)=1}} e^{-2\pi in_a d} \chi(a_2 \overline{\ell}) \chi^2(M_d) e^{-2\pi i \frac{R_{d_1} R_{d'}}{R_{d_2} R_{d_0} r_{d_1} r_{d+1}} \frac{R_{d_0} p_1 \overline{r_d}}{M_d}} = \sum_{a_2 \mod r_d, \phantom{(a_2,r_d)=1}} e^{-2\pi i a_2 d} \chi(a_2 \overline{\ell}) \chi^2(M_d)
\]

\[
= S(n, \ell \frac{R_{d_1} R_{d'}}{R_{d_2} R_{d_0} r_{d_1} r_{d+1}} \frac{R_{d_0} p_1 \overline{r_d}}{r_{d_1} c}; M_d) \chi(-M_d \overline{\ell}) \chi_{|r_d}(n),
\]

we get

\[(55) \quad \alpha_{n,M}(1) = \sum_{\ell \equiv n \mod M} \frac{a(\ell)}{\ell} V \left( \frac{X}{2\pi \ell} \right) \]

\[-\frac{1}{M} \sum_{d | M} \frac{1}{\varphi(r_d)} \sum_{r_{d1} | r_d, \phantom{r_{d1} \mod r_d, \phantom{r_{d1} \mod r_d}}} \frac{r_{d1}}{r_{d1} r_{d0}} \sum_{e | r_{d0}} \sum_{\tau(\chi) \mu \left( \frac{r_{d0}}{e} \right) \ell(\frac{r_{d0}}{e})} \chi(-M_d \overline{\ell}) \chi_{|r_d}(n)
\]

\[\times \frac{a_{(r_{d1}, e)}}{\left( \frac{r_{d1}, e}{r_{d1}, e} \right)} \sum_{\ell \geq 1} b_{X, R}(\frac{R_{d_1} R_{d'}}{R_{d_2} R_{d_0} r_{d1} r_{d+1}} \ell) S(n, \ell \frac{R_{d_1} R_{d'}}{R_{d_2} R_{d_0} r_{d1} r_{d+1}} \frac{R_{d_0} p_1 \overline{r_d}}{r_{d1} c}; M_d) V \left( \frac{M_d^2 R_{d_1} R_{d_0} r_{d1} r_{d+1} r_{d0}}{2\pi R_{d1} r_{d0} e l X} \right).\]

Here \(\chi_{|r_d}\) is a Dirichlet character modulo \(r_d\), which is induced from the primitive character \(\chi\) (mod \(r_a\)).

For the last sum of (55) we use Weil’s bound for Kloosterman sums which implies, as \((R'_{d_1}, M_d) = 1,\)

\[(56) \quad |S(n, \ell \frac{R_{d_1} R_{d'}}{R_{d_2} R_{d_0} r_{d1} r_{d+1}} \frac{R_{d_0} p_1 \overline{r_d}}{r_{d1} c}; M_d)| \leq \gcd(n, \ell, M_d)^{\frac{1}{2}} M_d^{\frac{1}{2}} \sigma_0(M_d).\]

By applying (56) and (21), we get

\[(57) \quad \left| \sum_{\ell \geq 1} b_{X, R}(\frac{R_{d_1} R_{d'}}{R_{d_2} R_{d0} r_{d1} r_{d+1}} \ell) S(n, \ell \frac{R_{d_1} R_{d'}}{R_{d_2} R_{d0} r_{d1} r_{d+1}} \frac{R_{d0} p_1 \overline{r_d}}{r_{d1} c}; M_d) V \left( \frac{M_d^2 R_{d_1} R_{d0} r_{d1} r_{d+1} r_{d0}}{2\pi R_{d1} r_{d0} e l X} \right) \right| \leq M_d^{\frac{1}{2}} \sigma_0(M_d)(r_{d0} r_{d1})^{\frac{1}{2}} \sigma_0(r_{d0}) \sigma_0(r_{d1}) \sum_{\ell \geq 1} \frac{\gcd(n, \ell, M_d)^{\frac{1}{2}}}{(\ell)^{\frac{1}{2} - \epsilon}} V \left( \frac{M_d^2 R_{d_1} R_{d0} r_{d1} r_{d+1} r_{d0}}{2\pi R_{d1} r_{d0} e l g X} \right).\]

Set \(g = \gcd(n, M_d).\) Then \(\gcd(n, \ell, M_d) \mid g,\) so for the summation over \(\ell \geq 1,\) we get, after replacing \(\ell\) by \(g\ell,\)

\[
\sum_{\ell \geq 1} \frac{\gcd(n, \ell, M_d)^{\frac{1}{2}}}{(\ell)^{\frac{1}{2} - \epsilon}} V \left( \frac{M_d^2 R_{d_1} R_{d0} r_{d1} r_{d+1} r_{d0}}{2\pi R_{d1} r_{d0} e l g X} \right) \leq g^e \sum_{\ell \geq 1} \frac{1}{(\ell)^{\frac{1}{2} - \epsilon}} V \left( \frac{M_d^2 R_{d_1} R_{d0} r_{d1} r_{d+1} r_{d0}}{2\pi R_{d1} r_{d0} e l g X} \right).
\]
If \( y < M^{-\epsilon} \), one easily checks, by moving the line of integration in (51) to the right, that
\( V(y) \ll M^{-K_r} \), for arbitrarily large \( K_r \). Consequently,

\[
g^\epsilon \sum_{\ell \geq 1} \frac{1}{\ell^\frac{1}{2} - \epsilon} V \left( \frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e \ell gX} \right) \ll g^\epsilon \sum_{\ell < M^r \frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e \ell gX}} \ell^{-\frac{1}{2} + \epsilon} \ll g^\epsilon M^r \left( \frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e \ell gX} \right)^{\frac{1}{2} + \epsilon}.
\]

Since \( \frac{R_d R'_d}{R_d d_0 r_{d+1}} \geq 1 \) (see Lemma 2.7), we have
\[
\frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e gX} = \frac{M_d^2 R'_d d R'_d}{2\pi gX} \left( \frac{R_d R'_d}{R_d r_d d_0 r_{d+1}} \right)^{-1} \leq \frac{M_d^2 R'_d d R'_d}{2\pi gX}.
\]
Applying this to (57), we get

\[
\sum_{\ell \geq 1} b_{x, r'} \left( \frac{R_d R'_d}{R_d d_0 r_{d+1}} \ell \right) S(n, \ell, R_d R'_d d R_d d_0 r_d r_{d+1} r_d; M_d) V \left( \frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e \ell gX} \right) \leq M_d^2 \sigma_0(M_d) (r_d d_0 r_{d+1})^{\frac{1}{2} - \epsilon} \sigma_0(r_d) \sigma_0(r_{d+1}) g^\epsilon M^r \left( \frac{M_d^2 R'_d d R'_d}{2\pi gX} \right)^{\frac{1}{2} + \epsilon} \ll X^{-\frac{1}{2} - \epsilon} M^r M_d^{\frac{3}{2} + 2\epsilon} \sigma_0(M_d) R_d^{\frac{1}{2} + \epsilon} \left( \frac{r_d}{r_d e} \right)^{\frac{1}{2} + \epsilon} \prod_{p \mid r_d} p.
\]

By definition of \( R'_d \), we have
\[
R'_d = \text{lcm}(R_d, \frac{d}{r_d}) = \prod_{p \mid r_d} p^{\max\{\text{ord}_p(q), 2\text{ord}_p(d)\}} = \frac{R_d \prod p^{\max\{0, 2\text{ord}_p(d) - \text{ord}_p(q)\}} < R_d r_d.
\]
The last inequality holds since for every \( p \mid r_d, \text{ord}_p(q) > \text{ord}_p(d) \). Also recall that \( d = M_d r_d \). Therefore, referring to (58), we have

\[
\sum_{\ell \geq 1} b_{x, r'} \left( \frac{R_d R'_d}{R_d d_0 r_{d+1}} \ell \right) S(n, \ell, R_d R'_d d R_d d_0 r_d r_{d+1} r_d; M_d) V \left( \frac{M_d^2 R'_d d R_d d_0 r_d r_{d+1} r_d}{2\pi R_d r_d e \ell gX} \right) \ll X^{-\frac{1}{2} - \epsilon} M^r M_d^{\frac{3}{2} + 2\epsilon} \sigma_0(M_d) R_d^{\frac{1}{2} + \epsilon} \left( \frac{r_d}{r_d e} \right)^{\frac{1}{2} + \epsilon} \prod_{p \mid r_d} p.
\]

Note (see (11)) that \( c_{x \mid r_d}(n) = 0 \) if \( \frac{r_d}{r_d r_{d+1}} \nmid n \). When \( \frac{r_d}{r_d r_{d+1}} \mid n \),
\[
c_{x \mid r_d}(n) = \frac{r_d}{r_d r_{d+1}} \mu \left( \frac{n}{r_d r_{d+1}} \right) \varphi \left( \frac{r_d}{r_d e} \right) \varphi \left( \gcd \left( r_d, \frac{n}{r_d r_{d+1}} \right) \right) \phi \left( \frac{\gcd \left( r_d, \frac{n}{r_d r_{d+1}} \right)}{r_d} \right).
\]
Thus, when \( \frac{r_d}{r_d r_{d+1}} \mid n \)
\[
|c_{x \mid r_d}(n)| \leq \frac{r_d}{r_d r_{d+1}} \sqrt{r_d} \varphi \left( \gcd(n, r_d) \right) < \frac{r_d}{\sqrt{r_d}}.
\]
By applying (59) and (60) to the second summation in (55), we get

\[
\frac{1}{M} \sum_{d \mid M} \frac{1}{\varphi(r_d)} \sum_{r_{d^2} \mid r_d, r_{d^2} \equiv \bar{r}_{d^2} \mod {p}} \frac{r_d}{r_d x r_{d0} e} \sum_{e \mid r_{d0}} \chi(r_{d0} \frac{e}{e}) \tau(\bar{\chi}) \mu(\frac{r_{d0}}{e}) \varphi(\frac{r_{d0}}{e}) \chi(-M_2^2 \tau) c_{\chi r_d} (n)
\]

\[
\times \left( \frac{r_{d^2}}{r_{d0} e} \right) \sum_{\ell \geq 1} b_{\chi, R} \left( \frac{R_{d^2} R_{d0}}{R_{d^2} R_{d0}^2 + 1} \right) S(n, \ell) \frac{R_{d^2} R_{d0}}{R_{d^2} R_{d0} + 1} \frac{r_{d^2}}{r_{d0} e} \frac{r_{d0}}{r_{d0} x r_{d0} e} \frac{M_2^2 R_{d^2} R_{d0}^2 r_{d0} x r_{d0}}{2 \pi r_{d0} r_{d0} e} \right)
\]

\[
\leq X^{-\frac{1}{2} - \epsilon} M^\epsilon \sum_{d \mid M} d^{\frac{3}{2} + 2\epsilon} \sigma_0(M_d) R_d^{\frac{1}{2} + \epsilon} r_d^{1 - \epsilon} \frac{1}{\varphi(r_d)} \sum_{r_{d^2} \mid r_d, r_{d^2} \equiv \bar{r}_{d^2} \mod {p}} \frac{r_d}{r_d x r_{d0} e} \sum_{e \mid r_{d0}} \varphi(\frac{r_{d0}}{e}) \prod_{p \mid r_{d^2}} p
\]

\[
= X^{-\frac{1}{2} - \epsilon} M^\epsilon \sum_{d \mid M} d^{\frac{3}{2} + 2\epsilon} \sigma_0(M_d) R_d^{\frac{1}{2} + \epsilon} \frac{1}{\varphi(r_d)} \sum_{r_{d^2} \mid r_d, r_{d^2} \equiv \bar{r}_{d^2} \mod {p}} \frac{r_d}{r_d x r_{d0} e} \sum_{e \mid r_{d0}} \varphi(\frac{r_{d0}}{e}) \prod_{p \mid r_{d^2}} p
\]

\[
\leq X^{-\frac{1}{2} - \epsilon} M^\epsilon \sum_{d \mid M} d^{\frac{3}{2} + 2\epsilon} \sigma_0(M_d) R_d^{\frac{1}{2} + \epsilon} \frac{1}{\varphi(r_d)} \sum_{r_{d^2} \mid r_d, r_{d^2} \equiv \bar{r}_{d^2} \mod {p}} \frac{r_d}{r_d x r_{d0} e} \sum_{e \mid r_{d0}} \varphi(\frac{r_{d0}}{e}) e^{-2\epsilon}
\]

Again, since we have \(r_{d0}\) square-free, we get

\[
\frac{1}{r_{d0}} \sum_{e \mid r_{d0}} \varphi(\frac{r_{d0}}{e}) e^{-2\epsilon} = r_{d0}^{1 - 2\epsilon} \sum_{e \mid r_{d0}} \varphi(\frac{r_{d0}}{e}) \left( \frac{r_{d0}}{e} \right)^{2\epsilon} = r_{d0}^{1 - 2\epsilon} \prod_{p \mid r_{d0}} (1 + p^{2\epsilon}(p - 1))
\]

\[
= \prod_{p \mid r_{d0}} (1 - p^{-1} + p^{-1 - 2\epsilon}) \ll \prod_{p \mid r_{d0}} (1 - p^{-1}),
\]
and therefore we get

\[
\frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{M} \sum_{d|M} d^{\frac{3}{2}+2\epsilon} \sigma_0(Md) R_d^{\frac{1}{2}+\epsilon} \varphi(r_d) \left( \sum_{r_{ds} \neq r_d} \frac{r_d^{1+\epsilon} r_{ds}^{-2\epsilon}}{\varphi(r_d)} \prod_{p|r_{ds}} p(1-p^{-1}) \frac{1}{r_d} \sum_{e|r_d} \varphi \left( \frac{r_d e}{e} \right) e^{-2\epsilon} \right) \\
< \frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{M} \sum_{d|M} d^{\frac{3}{2}+2\epsilon} \sigma_0(Md) R_d^{\frac{1}{2}+\epsilon} \varphi(r_d) \left( \sum_{r_{ds} \neq r_d} \frac{r_d^{1+\epsilon} r_{ds}^{-2\epsilon}}{\varphi(r_d)} \prod_{p|r_{ds}} p(1-p^{-1}) \prod_{p|r_d, p|r_{ds}} (1-p^{-1}) \right) \\
= \frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{M} \sum_{d|M} d^{\frac{3}{2}+2\epsilon} \sigma_0(Md) R_d^{\frac{1}{2}+\epsilon} \varphi(r_d) \prod_{p|r_d} \sum_{r_{ds} \neq r_d} r_d^{-2\epsilon} \prod_{p|r_{ds}} p.
\]

For \( \varphi(r_d) = r_d \prod_{p|r_d} (1-p^{-1}) \), we get

\[
\frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{M} \sum_{d|M} d^{\frac{3}{2}+2\epsilon} \sigma_0(Md) R_d^{\frac{1}{2}+\epsilon} \prod_{p|r_d} (1-p^{-1}) \sum_{r_{ds} \neq r_d} r_d^{-2\epsilon} \prod_{p|r_{ds}} p \\
< \frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{M} \sum_{d|M} d^{\frac{3}{2}+2\epsilon} \sigma_0(Md) R_d^{\frac{1}{2}+\epsilon} \prod_{p|r_d} p \leq \frac{X^{-\frac{1}{2}-\epsilon}M^\epsilon}{X^{\frac{1}{2}+\epsilon}} \sigma_0(M) \sum_{d|M} \prod_{p|d, \text{ord}_p(q) > \text{ord}_p(d)} p^{(\frac{1}{2}+\epsilon) \text{ord}_p(q)+1} \\
< \frac{M^{\frac{1}{2}+\epsilon} \sigma_0(M)}{X^{\frac{1}{2}+\epsilon}} \prod_{p|\text{gcd}(q,M), p^2|q} p^{(\frac{1}{2}+\epsilon) \text{ord}_p(q)+1}.
\]

The condition of the last product implies that the term is 1 when \( q \) is square-free or \( \text{gcd}(q, M) = 1 \).

We thus have

(61)

\[
\alpha_{n,M}(1) = \sum_{\ell \equiv n (\mod M)} \frac{a(\ell)}{\ell} V \left( \frac{X}{2\pi \ell} \right) + \mathcal{O} \left( X^{-\frac{1}{2}-\epsilon} M^{\frac{1}{2}+4\epsilon+\epsilon'} \prod_{p|\text{gcd}(q,M), p^2|q} p^{(\frac{1}{2}+\epsilon) \text{ord}_p(q)+1} \right).
\]

This allows us to prove

**Lemma 7.1.** Let \( M > 1 \). For each \( X > 0 \), we have

(62) \[
\sum_{n \in \mathbb{Z}} \hat{h}(n) \alpha_{n,M}(1) = \sum_{n \in \mathbb{Z}} \hat{h}(n) \sum_{m \equiv n (\mod M)} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) \\
+ \mathcal{O} \left( X^{-\frac{1}{2}} M^{\frac{1}{2}+\epsilon} \prod_{p|\text{gcd}(q,M), p^2|q} p^{(\frac{1}{2}+\epsilon) \text{ord}_p(q)+1} \right),
\]

for any \( \epsilon > 0 \).
Proof. Replacing \( \ell \) by \( m \) in (61), we get

\[
(63) \quad \sum_{n \in \mathbb{Z}} \hat{h}(n) \alpha_{\pm n, M}(1) = \sum_{n \in \mathbb{Z}} \hat{h}(n) \sum_{m \equiv \pm n \mod M} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) + O \left( X^{-\frac{1}{2} - \epsilon} M^{\frac{1}{2} + 4\epsilon + \epsilon'} \prod_{p | \gcd(q, M), p^2 | q} p^{\frac{1}{2} + \epsilon, \ord_p(q) + 1} \left( \sum_{n \in \mathbb{Z}} \hat{h}(n) \right) \right).
\]

Now, for \( h = h_\delta \) with \( \delta = \delta_M > M^{-1+\eta} \), for some \( 0 < \eta < 1 \), (36) implies that

\[
\sum_{\substack{n \in \mathbb{Z}, \\ \delta(|n|+1) > (|n|+1)^{1-\eta} M^{1-\eta}}} \hat{h}_\delta(n) \ll K \sum_{n \in \mathbb{Z}, \delta(|n|+1) > (|n|+1)^{1-\eta} M^{1-\eta}} (1 + |n|)^{-1}(\delta(|n|+1))^{-K},
\]

for arbitrary \( K \). Choosing \( K = K'/(1-\eta) \), with \( K' \gg 1 \), we see that this portion of the sum is \( \ll M^{-K'} \), for arbitrary \( K' \).

Taking the remaining portion of the sum,

\[
\sum_{\substack{n \in \mathbb{Z}, \\ \delta(|n|+1) \leq (|n|+1)^{1-\eta} M^{1-\eta}}} \hat{h}_\delta(n) \ll M^{K'},
\]

as for \( n \neq 0 \), \( |\hat{h}_\delta(n)| \leq 1/|n| \). Thus the error term of (63) is

\[
O \left( X^{-\frac{1}{2} - \epsilon} M^{\frac{1}{2} + \epsilon} \prod_{p | \gcd(q, M), p^2 | q} p^{\frac{1}{2} + \epsilon, \ord_p(q) + 1} \right),
\]

with a new \( \epsilon > 0 \).

\[ \square \]

The next proposition gives us an estimate for the first term of the RHS of (62).

Lemma 7.2. For \( h = h_\delta \) with \( \delta = \delta_M > M^{-1+\eta} \), for some fixed \( \eta > 0 \), we have,

\[
\sum_{n \in \mathbb{Z}} \hat{h}_\delta(n) \sum_{m \equiv \pm n \mod M} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) = \sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} + O \left( X^{-\frac{1}{2} + \epsilon} \right) + O \left( X^{\frac{1}{2} + \epsilon} M^{-1+\epsilon} \right).
\]

Proof. Referring to (62), we consider

\[
\sum_{n \in \mathbb{Z}} \hat{h}(n) \sum_{m \equiv \pm n \mod M} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) = \sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} V \left( \frac{X}{2\pi n} \right) + \sum_{n \in \mathbb{Z}} \sum_{\substack{m \equiv \pm n \mod M, \\ m \neq \pm n}} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right).
\]

We first consider the terms with \( m = \pm n \) in the sum. Upon moving the line of integration, we get

\[
\sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} V \left( \frac{X}{2\pi n} \right) = \sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} + \sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} \int (-\frac{1}{2} + \epsilon) \left( \frac{X}{2\pi n} \right)^u G(u) \frac{\Gamma(u+1)}{(2\pi)^u} \frac{du}{u}.
\]
Since the second sum is $\ll \sum_{n \geq 1} |\hat{h}(\pm n)| n^{-\epsilon} X^{-1/2+\epsilon}$, inequality (60) implies that the sum converges and we have

$$(64) \quad \sum_{n \geq 1} \hat{h}(\pm n) \frac{a(n)}{n} + O \left( X^{-1/2+\epsilon} \right).$$

Now we are left with the $n \neq \pm m$ terms. Note that the length of the sum over $m$ is $X^{1+\epsilon}$ by the fast decay of $V$. We separate into two cases: $|n| \leq M/2$ and $|n| > M/2$.

For the latter, (60) implies

$$\sum_{|n| > M/2, n \equiv \pm m \mod M, m \neq \pm n} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) \ll \sum_{|n| > M/2, n \equiv \pm m \mod M, m \neq \pm n} \frac{\delta_M (1 + |n|)^{-K}}{1 + |n|} \sum_{0 < m \ll X^{1+\epsilon} \atop m \neq \pm n} \frac{1}{m^{1/2-\epsilon}} \ll \delta_M^{-K} \sum_{|n| > M/2} (1 + |n|)^{-K-1} \sum_{0 < m \ll X^{1+\epsilon} \atop m \neq \pm n} \frac{1}{m^{1/2-\epsilon}} \ll \delta_M^{-K} M^{-K} X^{1/2+\epsilon}.$$  

Since, $\delta_M > 1/M^{1-u}$, that is, $\delta_M M > M^u$, by renaming $K$, and assuming $X$ will be less than some fixed power of $Mq$, we get $O \left( (qM)^{-K} \right)$ with $K > 1$ arbitrarily large.

For the former case we note that as $m \neq \pm n$, the congruence relation modulo $M$ forces $m > M/2$. We then calculate using $K = 0$ in (60), and recalling that the contribution from $m > X^{1+\epsilon}$ is smaller than $(qM)^{-K}$ for arbitrary $K \gg 1$, we get

$$(65) \quad \sum_{n \in \mathbb{Z}, |n| \leq M/2} \hat{h}(n) \sum_{n \equiv \pm m \mod M, m \neq \pm n} \frac{a(m)}{m} V \left( \frac{X}{2\pi m} \right) \ll \sum_{|n| \leq M/2} \frac{1}{|n| + 1} \sum_{0 < |l| \ll X^{1+\epsilon}/M} \frac{a(\pm n + Ml)}{(\pm n + Ml)} \ll \sum_{|n| \leq M/2} \frac{1}{|n| + 1} \sum_{0 < |l| \ll X^{1+\epsilon}/M} \frac{1}{(\pm n + Ml)^{1/2-\epsilon}} \ll X^{1/2+\epsilon} M^{-1+\epsilon}.$$  

Combining (64) with (65) yields the proposition. \hfill \Box

We can combine Lemma 7.2 with Lemma 7.1 to get the asymptotics of $\sum \hat{h}(n)\alpha_{\pm n,M}(1)$. To this end, we will compare the error terms produced in Lemmas 7.1 and 7.2 to determine a value of $X$ that gives the optimal bound. Setting the error terms from (62) and (65) equal, we get

$$X^{-\frac{1}{2}} M^{\frac{1}{2}} \prod_{p \mid \gcd(q,M), p^2 \mid q} p^{\frac{1}{2} \ord_p(q) + 1} = X^{\frac{1}{2}} M^{-1}.$$  

This gives us

$$X = M^{\frac{3}{2}} \prod_{p \mid \gcd(q,M), p^2 \mid q} p^{\frac{1}{2} \ord_p(q) + 1}.$$  

Thus the error from these two contributions is

$$X^{\frac{1}{2}} M^{-1} (qM)^{\epsilon} = (qM)^{\epsilon} M^{-\frac{1}{2}} \prod_{p \mid \gcd(q,M), p^2 \mid q} p^{\frac{1}{2} \ord_p(q) + \frac{1}{2}}.$$
The remaining error (from (64)) is dominated by these terms since $X^{-1/2+\epsilon} \ll X^{-1/2+\epsilon} M$. From this, together with Lemmas 7.1, 7.2 we deduce

**Proposition 7.3.** Let $M > 1$. For $h = h_\delta$ with $\delta = \delta_M > M^{-1+\eta}$ for some fixed $0 < \eta < 1$, we have,

$$
\sum_{n \in \mathbb{Z}} \hat{h}_\delta(n) a(\pm n, M)(1) = \sum_{n \geq 1} \hat{h}_\delta(\pm n) a(n) + O \left( (Mq)^{\epsilon} M^{-\frac{1}{2}} \prod_{p \mid \gcd(q, M), p \geq \epsilon} \frac{1}{p^{\frac{1}{2} \ord_p(q) + \frac{1}{2}}} \right).
$$

8. **Proof of Theorem 1.2**

For fixed $x$ we consider $h = h_\delta$. Combining (52) and Proposition 7.3 we deduce

$$
\frac{1}{M} \sum_{0 \leq a \leq M} \langle a, h_\delta \rangle = \frac{1}{2} \left\{ \sum_{n \geq 1} \left( \hat{h}_\delta(-n) \pm \hat{h}_\delta(n) \right) \frac{A(n)}{n} \right\} + O \left( (Mq)^{\epsilon} M^{-\frac{1}{2}} \prod_{p \mid \gcd(q, M), p \geq \epsilon} \frac{1}{p^{\frac{1}{2} \ord_p(q) + \frac{1}{2}}} \right).
$$

**Lemma 8.1.** For $h = h_\delta$ with $\delta = \delta_M$, we have

$$
\sum_{n \geq 1} \hat{h}_\delta(n) a(n) = \sum_{n \geq 1} \frac{1 - e^{-2\pi i x n}}{2\pi i n} + O \left( \delta_M^{\frac{1}{2} - \epsilon} \right).
$$

**Proof.** We have

$$
\left| \sum_{n \geq 1} \hat{h}_\delta(n) \frac{a(n)}{n} - \sum_{n \geq 1} \frac{1 - e^{-2\pi i x n}}{2\pi i n} \frac{a(n)}{n} \right| \leq \sum_{n > \delta_M^{-1}} \left| \hat{h}_\delta(n) \frac{a(n)}{n} \right| + \sum_{n > \delta_M^{-1}} \frac{1 - e^{-2\pi i x n}}{2\pi i n} \frac{a(n)}{n} + \sum_{n = 1}^{\delta_M^{-1}} \hat{h}_\delta(n) - \frac{1 - e^{-2\pi i x n}}{2\pi i n} \frac{a(n)}{n}.
$$

Because of (61), we have

$$
\left| \sum_{n > \delta_M^{-1}} \hat{h}_\delta(n) \frac{a(n)}{n} \right| \leq \sum_{n > \delta_M^{-1}} \frac{1}{n^{2\epsilon_1}} \leq \epsilon \delta_M^{\frac{1}{2} - \epsilon}.
$$

Since $\frac{1 - e^{-2\pi i x n}}{2\pi i n}$ is likewise $\ll n^{-1}$, the same bound holds for the second sum in the RHS of (67).

For the last sum of (67), we observe that, because of (35), we have

$$
\hat{h}_\delta(n) = \frac{1 - e^{-2\pi i x n}}{2\pi i n} + \frac{1}{2\pi i n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(t) \left( (e^{2\pi i x n(1-t)} - 1) - e^{-2\pi i x n} (e^{-2\pi i x n(1+t)} - 1) \right) dt
$$

$$
= \frac{1 - e^{-2\pi i x n}}{2\pi i n} + \frac{1}{2\pi i n} O(n \delta_M).
$$
because $e^{2\pi i \delta_M n(1-t)} = 1 + \mathcal{O}(n \delta_M)$ (since $n \delta_M \leq 1$). Thus

$$\sum_{n=1}^{\delta_M^{-1}} \hat{h}_\delta(n) - 1 - e^{-2\pi i x} \frac{a(n)}{2\pi i n} \ll \delta_M \sum_{n=1}^{\delta_M^{-1}} \frac{1}{n^{1-\epsilon}} \ll \delta_M \delta_M^{-\frac{1}{2} - \epsilon} = \delta_M^{-\frac{1}{2} - \epsilon}.$$

\[ \square \]

Similarly, we can prove that

$$\sum_{n \geq 1} \hat{h}_\delta(-n) \frac{a(n)}{n} = \sum_{n \geq 1} \frac{1 - e^{2\pi i x}}{-2\pi i n} \frac{a(n)}{n} + \mathcal{O}(\delta_M^{\frac{1}{2} - \epsilon}).$$

Entering this and Lemma 8.1 into (66), we derive the main terms of Theorem 1.2.

To determine the error term we note that the error terms we have obtained from our analysis are $\delta_M^{\frac{1}{2} - \epsilon}$ from the above,

$$M^{\frac{1}{2} + \epsilon} q^{\frac{1}{4} + \epsilon} \prod_{p|M, \text{ord}_p(q) < \text{ord}_p(M) < \text{ord}_p(q)} p^{\frac{1}{2}} \delta_M$$

from Lemma 3.1, and

$$M^{-\frac{1}{4} + \epsilon} \prod_{p|\gcd(q,M), p^2 | q} p^{\frac{1}{2} \text{ord}_p(q) + \frac{1}{2}}$$

from Lemma 7.2.

Setting

$$\left( M^{\frac{1}{2} q^{\frac{1}{4}}} \prod_{p|M, \text{ord}_p(q) < \text{ord}_p(M) < \text{ord}_p(q)} p^{\frac{1}{2}} \right) \delta_M = M^{-\frac{1}{4}} \prod_{p|\gcd(q,M), p^2 | q} p^{\frac{1}{2} \text{ord}_p(q) + \frac{1}{2}}$$

gives us

$$\delta_M^{-\frac{1}{2}} M^{-\frac{1}{4}} \prod_{p|\gcd(q,M), p^2 | q} p^{\frac{1}{2} \text{ord}_p(q) + \frac{1}{2}} \prod_{p|M, \text{ord}_p(q) < \text{ord}_p(M) < \text{ord}_p(q)} p^{\frac{1}{2}} < M^{-\frac{1}{4}} \prod_{p|\gcd(q,M), p^2 | q} p^{\frac{1}{2} \text{ord}_p(q) + \frac{1}{2}}.$$ 

Thus the final error is

$$(Mq)^{\epsilon} M^{-\frac{1}{4}} \prod_{p|\gcd(q,M), p^2 | q} p^{\frac{1}{2} \text{ord}_p(q) + \frac{1}{2}}.$$

This complete the proof of Theorem 1.2.

References

[1] A.O. L. Atkin, W. Li Twists of newforms and pseudo-eigenvalues of W-operators Invent. Math., 48-3, 221–243 (1978)

[2] A. R. Booker, M. Lee, A. Strömbergsson, Twist-minimal trace formulas and the Selberg eigenvalue conjecture, arXiv:1803.06016

[3] S. Bettin, J. Bober, A. Booker, B.Conrey, M. Lee, G. Molteni, T. Oliver, D. Platt, R. Steiner, A conjectural extension of Hecke’s converse theorem, Raman. J. (2017)

[4] V. Blomer, É. Fouvry, E. Kowalski, P. Michel, D. Milíčević, W. Sawin The second moment theory of families of L-functions arXiv:1804.01450

[5] R. Bruggeman, N. Diamantis Fourier coefficients of Eisenstein series formed with modular symbols, J. Number Theory 167:317-335, 2016

[6] Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. G. Tables of integral transforms. Vol. I. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. xx+391 pp.
[7] H. Iwaniec, E. Kowalski *Analytic Number Theory* Colloquium Publications, Vol. 53, American Mathematical Society; New ed. edition (June 8, 2004)

[8] M. Kim, H.-S. Sun *Modular symbols and modular L-values with cyclotomic twists*. Preprint (2017).

[9] E. Kowalski, P. Michel, P. Vandekam *Rankin-Selberg L-functions in the level aspect* Duke Math. J. 114 (2002), no. 1, 123-191

[10] B. Mazur, K. Rubin. *The statistical behavior of modular symbols and arithmetic conjectures*. Presentation at Toronto, Nov 2016 (2016). [http://www.math.harvard.edu/~mazur/papers/heuristics.Toronto.12.pdf](http://www.math.harvard.edu/~mazur/papers/heuristics.Toronto.12.pdf)

[11] Y. Petridis, M. Risager *Arithmetic Statistics of Modular Symbols* Invent. math. (2018) 212:997-1053

E-mail address: nikolaos.diamantis@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, United Kingdom

E-mail address: jhoff@math.brown.edu

Mathematics Department, Brown University, Providence, RI 02912, USA

E-mail address: erenmehmetkiral@protonmail.com

Department of Mathematics, Keio University, Building 14, 443, 3-14-1 Kouhoku-ku, Hiyoshi, Yokohama, 223-8522 Japan

E-mail address: min.lee@bristol.ac.uk

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom