Orbifolds and Flows from Gauged Supergravity

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Abstract

We examine orbifolds of the IIB string via gauged supergravity. For the gravity duals of the $A_{n-1}$ quiver gauge theories, we extract the massless degrees of freedom and assemble them into multiplets of $N = 4$ gauged supergravity in five dimensions. We examine the embedding of the gauge group into the isometry group of the scalar manifold, as well as the symmetries of the scalar potential. From this we find that there is a large $SU(1,n)$ symmetry group which relates different RG flows in the dual quiver gauge theory. We find that this symmetry implies an extension of the usual duality between ten-dimensional IIB solutions which involves exchanging geometric moduli with background fluxes.

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1. Introduction

Gauged supergravity has proven to be a remarkably effective tool in the construction and study of holographic RG flows. In this general approach, one uses a five-dimensional supergravity theory to capture and simplify the details of what is usually a far more complicated ten-dimensional supergravity theory. This has been most extensively employed in the study of flows of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and their holographic duals in IIB supergravity. In this instance, gauged $\mathcal{N} = 8$ supergravity in five dimensions \cite{1,2} captures (as a consistent truncation) essentially all the perturbations of $\mathcal{N} = 4$ Yang-Mills that involve gauge invariant bilinear chiral operators. There has also been work on using more general five-dimensional $\mathcal{N} = 2$ gauged supergravity theories to examine flow solutions (see, for example, \cite{3,4}) and, while this is interesting, there is often an unresolved problem in determining the dual theory on the brane and then establishing the duality precisely. This is not an issue for gauged $\mathcal{N} = 8$ supergravity precisely because of its well-established connection with the $S^5$ compactification of IIB supergravity \cite{5,6} and hence with $\mathcal{N} = 4$ supersymmetric Yang-Mills on D3-branes.

In this paper we will examine flow solutions in five-dimensional, $\mathcal{N} = 4$ gauged supergravity theories \cite{7,8,9}. Our purpose is to study perturbations of, and flows in, the $\mathcal{N} = 2$ supersymmetric quiver gauge theories in four dimensions. The quiver theories are, of course, conformal and satisfy the relationship, $c = a$, of central charges that is essential to a holographic theory \cite{10}. These theories are related to the $\mathcal{N} = 4$ theory via their construction as the world-volume theory on D3-branes at an orbifold singularity \cite{11,12}. It is thus straightforward to identify the holographic duals of quiver theories in terms of the IIB superstring on a background of the form $AdS_5 \times S^5/\Gamma$ \cite{13}, where $\Gamma$ is an appropriately chosen discrete subgroup of $SU(2)_L \subset SU(2)_L \times SU(2)_R \times U(1) \subset SO(6)$, and where $SO(6)$ is the isometry group of $S^5$.

Our first task will be to identify the correct gauging and matter content for the $\mathcal{N} = 4$ gauged supergravity theory so as to obtain a subsector of the holographic dual of the quiver theories. We will restrict our attention to the $A_{n-1}$ quivers that are obtained by taking $\mathbb{Z}_n$ orbifolds. From the supergravity perspective, the problem is to find the proper number of massless vector and tensor multiplets, along with their charges. This problem will be resolved here in exactly the same manner that it was in the 1980’s when similar ambiguities had to be resolved in the proper gauging of maximal supergravities (see, for example, \cite{14}): We will study the linearization of the corresponding compactification of the IIB theory, and this will give us precisely the proper field content and charges.
An $\mathcal{N} = 2$ superconformal Yang-Mills theory has an $SU(2)_R \times U(1)$ R-symmetry, and so this must be the gauge group of the supergravity. One can then get further insight into the gauging by looking at the untwisted sector of the orbifold. We will examine this in greater detail in section 3, but here we note that the untwisted sector must include the $SU(2)_L$ invariant sector of the $S^5$ compactification of the IIB theory. In truncating to $\mathcal{N} = 8$ gauged supergravity we see that this $SU(2)_L$ invariant sector must reduce to the $SU(2)$ invariant sector considered in [15]. This fact was noted in [15] and indeed was part of the motivation for the use of the truncation to $SU(2)$ singlets. The result is $\mathcal{N} = 4$ gauged supergravity coupled to two tensor multiplets, and the gauge group is, of course, $SU(2)_R \times U(1)$. In [15] the scalar coset was shown to be

$$\frac{SO(5,2)}{SO(5) \times SO(2)} \times SO(1,1). \quad (1.1)$$

The group $SO(5,2)$ contains an obvious $SO(3) \times SO(2) \times SO(2)$ and the $SU(2)_R$ gauge symmetry of the supergravity is to be identified with the $SO(3)$, while the $U(1)$ gauge symmetry is the diagonal subgroup of $SO(2) \times SO(2)$.

It is also worth remembering that this $SU(2)_R \times U(1)$ commutes with an $SU(1,1)$ in $SO(5,2)$, and that this $SU(1,1)$ is naturally identified with the $SU(1,1)$ of the dilaton and axion in the original IIB theory. The supergravity potential is invariant under the action of this $SU(1,1)$, and since the dilaton and axion are dual to the gauge coupling and $\theta$-angle, this invariance reflects the fact that the complex gauge coupling is a freely chooseable parameter at the UV fixed point.

We will show in section 4 that the foregoing generalizes very naturally to the quiver theories. First, we will show that the supergravity theory we seek is $\mathcal{N} = 4$ gauged supergravity coupled to $2n$ tensor multiplets and either one or three vector multiplets. The corresponding scalar manifold is of the form

$$\mathcal{S} = \frac{SO(5,2n+q)}{SO(5) \times SO(2n+q)} \times SO(1,1), \quad (1.2)$$

where $2n$ is the number of tensor multiplets and $q$ is the number of vector multiplets. The $SO(1,1)$ factor is parameterized by the “dilaton” in the five-dimensional gravity supermultiplet. There is an obvious $SO(3) \times (SO(2))^{n+1}$ subgroup of $SO(5,2n)$, and $SU(2)_R$ is still to be identified with the $SO(3)$ and the $U(1)$ is the diagonal embedding in the product of $SO(2)$’s. The supergravity gauge symmetry, $SU(2)_R \times U(1)$, thus commutes with $SU(1,n)$ in $SO(5,2n)$. We will show in section 4 that this $SU(1,n)$ is an invariance
of the potential in the gauged $\mathcal{N} = 4$ theory. Moreover we will argue that the scalars that parameterize the coset

$$\mathcal{T} = \frac{SU(1,n)}{U(1) \times SU(n)},$$

are in fact the duals of the $n$ distinct complex coupling constants of the $n$ distinct $SU(N)$ gauge group factors associated with the nodes of the quiver diagram. The $SU(1,n)$ invariance thus represents the fact that these couplings are freely chooseable parameters.

Having identified the correct matter content and gauging we will go on in section 5 to analyze part of the corresponding supergravity potential. We first describe how to reconstruct the results of $[15]$, and we then describe how the flows of $[16,17]$ must fit within the $\mathcal{N} = 4$ gauged supergravity. Having done this, we discover a rather surprising result: there is an $SU(n)$ symmetry that must map all of the flows of $[16,17]$ onto the flow of $[18]$. To be precise, the $SU(n) \subset SU(1,n)$ is a symmetry of the supergravity action that maps the supergravity scalars that describe the flow of $[18]$ onto any and all of the flows of $[16,17]$. In particular, we find that the critical point of $[18]$ is extended to a complex $(n-1)$-dimensional surface in the $\mathcal{N} = 4$ supergravity theory corresponding to the $A_{n-1}$ quiver theory. Indeed, if $g_i$ and $m_j$ are the gauge coupling constants and masses associated with the $i$th node of the quiver, then the critical surface is a $\mathbb{P}^{n-1}$ parameterized by homogeneous coordinates $g_i^2/m_i$. In particular the flows of $[16,17]$ and $[18]$ are represented by distinct points on this surface.

This is a rather remarkable claim. As we will see, it is almost trivial from the five-dimensional perspective, but from the ten-dimensional IIB perspective, it means that there must be a continuous symmetry that trades the tensor gauge field fluxes obtained in $[19]$ on a topologically trivial manifold for Kähler moduli of blow-ups on the resolved orbifold $[1]$. Such a symmetry is not unprecedented: It is certainly not the first time that one has discovered that a very simple, manifest symmetry in low dimensions could have remarkable, and unexpected geometric consequences in higher dimensions. In this case, the IIB string theory enjoys a discrete duality symmetry which acts on the metric moduli and fluxes and it is promoted to a continuous symmetry of the supergravity sector. We shall have more to say about this in our discussion in section 6.

There were also very indirect hints that there might be such a symmetry. The flows are similar in their field theory description, and have the same symmetry and supersymmetry.\footnote{More precisely, these are complex structure moduli of the deformation of the orbifold singularity in the 3-fold $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$.}
Moreover all these flows give rise to infra-red fixed points with the same central charge:

\[ \frac{c_{IR}}{c_{UV}} = \frac{27}{32}. \]

For the resolved conifolds, this result about the central charges was established for \( n = 2 \) in [20], and rather more recently for general \( n \) in [21]. For the flows involving only fluxes on the unresolved \( S^5/\mathbb{Z}_n \), the ratio of central charges follows from the results of [15] and the observation that this particular flow lies entirely in the untwisted sector. Our results here will show, amongst other things, how this generalizes to a continuous family of flows that spans smoothly across all twisted sectors.

Since we are using a five-dimensional theory to describe a solution that lives in IIB supergravity in ten-dimensions, we are once again haunted by an old ghost of gauged supergravity: Consistent Truncation. The issue is whether a solution of the five-dimensional theory really does “lift” to an exact solution in ten dimensions. The fact that one can do this is fairly well established for the maximal gauged supergravity theories in four and seven dimensions [22]. While there is no formal proof in five dimensions, the result is very plausible, and many lifts have been explicitly constructed (see, for example, [23, 24, 19, 25]). However, our work here goes beyond the “established” results of consistent truncation: we are using half-maximal supersymmetry, with added tensor multiplets. There is thus a legitimate concern that we may be working with only an effective five-dimensional theory, and our results may not have exact ten-dimensional lifts. Based on our experience of consistent truncation we know that if it fails then one typically finds a solution in lower dimensions that has no analog in higher dimensions, or vice versa. Moreover, there may be a symmetry mismatch. In the instance we consider here, we have a family of solutions in both five and ten dimensions, with the same symmetry and supersymmetry, and the families are generated from flows with the same initial data (for which there is an exact perturbative correspondence). Moreover, we will see that one can understand the result from the field theory on the brane, and the recent work in [21] shows that the ten-dimensional solutions have the correct cosmological constant, or central charge for the brane theory. We therefore feel that we are on very solid ground, and furthermore the results presented here strongly suggest that the half-maximal gauged supergravities also represent a class of consistent truncations of the IIB theory.

\[ ^2 \text{In making this statement we are thinking of the flow of [18] as being embedded in the } \mathcal{N} = 2 \text{ quiver theory and not in the original setting of } \mathcal{N} = 4 \text{ Yang-Mills theory, and so the flows start with the same value of } c_{UV}. \]
2. A Class of RG Flows in $\mathcal{N} = 2$ Quiver Gauge Theories

We will be primarily concerned with D3-branes at $A_{n-1}$ singularities $\mathbb{C}^2/\mathbb{Z}_n$. If the pullback of the orbifold projection to the brane world-volume is taken to be the regular representation, the gauge theory on the $N$ branes is a four-dimensional $\mathcal{N} = 2$ $U(N)^n$ gauge theory with $n$ adjoint vector multiplets and $n$ hypermultiplets transforming in the bifundamental representations of adjacent groups $[11]$. This field content can be associated to the extended Dynkin diagram of $A_{n-1}$ (called the “quiver”). As is usual in brane world-volume theories, the scalar degrees of freedom parameterize the space transverse to the branes. Here the scalars in the hypermultiplets describe the $\mathbb{C}^2/\mathbb{Z}_2$ directions, while those in the vector multiplets describe the remaining transverse $\mathbb{R}^2 = \mathbb{C}$.

We can represent the $\mathcal{N} = 2$ supersymmetric hypermultiplets in terms of $\mathcal{N} = 1$ chiral fields $(A_i, B_i)$ where the $A_i$ are in the $(N_{i-1}, \bar{N}_i)$ representation of $U(N_{i-1}) \times U(N_i)$ and the $B_i$ are in the $(\bar{N}_{i-1}, N_i)$. The $\mathcal{N} = 1$ chiral fields in the $\mathcal{N} = 2$ vector multiplets will be denoted by $\Phi_i$ and are in the adjoints of $U(N_i)$. The $\mathcal{N} = 1$ superpotential is then

$$W = \sum_i \lambda_i \text{Tr} \int d^2 \theta (B_{i+1} \Phi_i A_{i+1} - A_i \Phi_i B_i).$$

(2.1)

For $\mathcal{N} = 2$ supersymmetry, the Yukawa couplings, $\lambda_j$, have to be equal to the gauge couplings $g_j = \sqrt{4\pi/\text{Im} \tau_j}$ associated with each node of the quiver theory. In the orbifold gauge theory, all these gauge couplings are equal, $g_j = g$. (This is because they are all equal to $\sqrt{n}$ times the gauge coupling of the original $\mathcal{N} = 4$ Yang-Mills theory before the orbifoldization.) These $\mathcal{N} = 2$ theories are superconformal, and have the R-symmetry group $SU(2)_R \times U(1)$. Under the Cartan generator $J_3$ of $SU(2)_R$, the hypermultiplets $(A_i, B_i)$ have charge $J_3 = 1$, while the $\Phi_i$ are neutral. Under the generator, $Y$, of the $U(1)$ factor, the fields $\Phi_i$ have charge $Y = 2$, while the $(A_i, B_i)$ are neutral. The superconformal, $\mathcal{N} = 1$, R-charge is given by the expression

$$R_{sc} = \frac{Y + 2J_3}{3}.$$  

(2.2)

The mass dimension of a field with this R-charge is given by $d = 3R_{sc}/2 + \gamma$, where $\gamma$ is the anomalous dimension.

In principle one can also add D and F-terms for the $U(1)$ factors. In the usual brane-probe analysis, these terms allow the resolution of the orbifold singularity in the moduli

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3 Specifically, we mean the point in the string moduli space where the worldsheet orbifold CFT is valid.
space of the probe. In the infrared these $U(1)$'s decouple and so we cannot use D and F-terms to resolve the singularity in the solution of the IIB theory, and so we must look at relevant deformations \[13\].

Klebanov and Witten \[16\] considered a deformation to an $\mathcal{N} = 1$ theory by a twisted mass term for the adjoint chiral fields

$$\tilde{W}_2 = \frac{m}{2} \text{Tr} \int d^2\theta (\Phi_1^2 - \Phi_2^2). \tag{2.3}$$

Geometrically, this deformation corresponds to a complex structure deformation of the manifold $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ and removes the orbifold singularity, leaving a conical singularity at the origin. This deformation leads to a theory in the infra-red that can be studied by integrating out the massive $\Phi_{1,2}$ multiplets. Doing this in (2.1) leads to a quartic superpotential for the chiral fields $A_i, B_i$,

$$\tilde{W}_2 = \frac{g^2}{m} \text{Tr} \int d^2\theta (A_1B_1A_2B_2 - A_1A_2B_2B_1). \tag{2.4}$$

In \[16\], it was explained that the $\mathcal{N} = 1$ theory in the IR is the theory that describes D3-branes at the conifold singularity.

Another obvious $\mathcal{N} = 1$ deformation of the $A_1$ quiver theory is by the untwisted mass term:

$$W_2 = \frac{m}{2} \text{Tr} \int d^2\theta (\Phi_1^2 + \Phi_2^2). \tag{2.5}$$

This deformation is analogous to the deformation that gives a mass to one of the $\mathcal{N} = 1$ chiral fields in the $\mathcal{N} = 4$ super-Yang–Mills multiplet \[26,27,18\]. Now we obtain an $\mathcal{N} = 1$ theory with quartic superpotential

$$W_2 = \frac{g^2}{m} \text{Tr} \int d^2\theta ((A_1B_1)(A_1B_1 - B_2A_2) + (A_2B_2)(A_2B_2 - B_1A_1)). \tag{2.6}$$

The geometry dual to the flow generated by (2.5) can be obtained as a $\mathbb{Z}_2$ orbifold of the ten-dimensional lift \[19\] of the flow found in \[18\].

More generally, we can consider the deformations \[17,28,21,29\]:

$$W_n = \sum_i \frac{m_i}{2} \text{Tr} \int d^2\theta \Phi_i^2. \tag{2.7}$$

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4 The definition of $(A_2, B_2)$ in \[16\] is opposite to ours. Here $A_2 = B_2^{KW}, B_2 = A_2^{KW}$. 

6
(By a change of basis, we could rewrite this in terms of \(n - 1\) twisted superfield operators \(\Phi_i^2 - \Phi_{i+1}^2\) and the untwisted operator \(\sum_{i=1}^{n} \Phi_i^2\), but we will not need its explicit form.) We note that in the IR, the overall mass scale of the complex parameters \(m_i\) will decouple. For the perturbation to be non-trivial, at least one of the \(m_i\), say \(m_j\), must be non-zero and can therefore be used to set the overall scale. The ratios \(m_i/m_j\) yield inhomogeneous coordinates on \(\mathbb{P}^{n-1}\). This complex projective space is the moduli space of couplings of the deformations \((2.7)\). If all the \(m_i\) are non-zero then we expect a class of flows, generalizing those of Leigh and Strassler, to a family of fixed points parameterized by \(m_i/m_j\). These fixed point theories will all have \(c_{IR}/c_{UV} = \frac{27}{32}\).

If one, or more, of the \(m_i\) vanish then it is, a priori, less clear what will happen: there might not be a fixed point, or, if there is, the central charge may be higher than that obtained when the \(m_i \neq 0\). Thus we expect a surface of fixed points with \(c_{IR}/c_{UV} = \frac{27}{32}\) described by \(\mathbb{P}^{n-1}\), with all the hypersurfaces \(\mathbb{P}^{n-2}\) defined by \(m_i = 0\) excised. On these hypersurfaces, and their intersections, there are potentially different fixed-point theories with \(c_{IR}/c_{UV} \geq \frac{27}{32}\). Our analysis below will more properly elucidate the structure of the manifold of fixed points.

It is relatively straightforward to use the methods of [26] to see how the foregoing emerges from the conditions for conformal invariance. We begin with the quiver theory and its a priori independent collection of gauge couplings, \(\tau_i\), Yukawa couplings, \(\lambda_i\), and adjoint masses \(m_i\). Whenever one of the \(m_i\) is non-zero, we integrate out the corresponding \(\Phi_i\). This leads to a superpotential

\[
\sum_{i|m_i \neq 0} c_i \text{ Tr } \int d^2 \theta \left( A_{i+1} B_{i+1} - B_i A_i \right)^2 \\
+ \sum_{i|m_i = 0} \lambda_i \text{ Tr } \int d^2 \theta \left( B_{i+1} \Phi_i A_{i+1} - A_i \Phi_i B_i \right),
\]

where \(c_i = \lambda_i^2/(2m_i)\) and \(i|m_i = 0\) denotes the set of indices, \(i\), such that \(m_i = 0\). Let the number of \(\Phi_i\) that we have given mass to be \(\mu\). Then there are \(\mu\) couplings \(c_i\) and \(n - \mu\) couplings \(\lambda_i\) left. Vanishing of the \(\beta\)-function for the gauge coupling requires that

\[
\gamma A_i + \gamma B_i + \gamma A_{i+1} + \gamma B_{i+1} + \delta_{m_i,0} (2\gamma \Phi_i - 1) + 1 = 0, \; \forall \; i,
\]

Alternatively, we can leave the field \(\Phi_i\) in the theory. Then the vanishing of the beta function for \(m_i\) fixes \(\gamma_{\Phi_i} = \frac{1}{2}\) for all \(i\) such that \(m_i \neq 0\). Applying this to the beta functions for the Yukawa couplings will leave us with the equations (2.10).
where $\gamma_X$ is the anomalous dimension of the field $X$. The vanishing of the $\beta_{\lambda_i}$ and the $\beta_{c_i}$ lead to the equations

$$
2\gamma_{\Phi_i} + \gamma_{A_i} + \gamma_{B_i} + \gamma_{A_{i+1}} + \gamma_{B_{i+1}} = 0, \quad \forall \ i | m_i = 0,
$$

$$
\gamma_{A_i} + \gamma_{B_i} + \gamma_{A_{i+1}} + \gamma_{B_{i+1}} + 1 = 0, \quad \forall \ i | m_i \neq 0.
$$

These equations are equivalent to \((2.9)\).

Consider the unperturbed quiver theory (with all $m_i = 0$). This theory has $2n$ complex couplings $(\tau_i, \lambda_i)$ and \((2.10)\) represent $n$ constraints on them. Recalling the analysis of the $\mathcal{N} = 4$ theory in [26], we can imagine solving these constraints to give the $\lambda_i$ as functions of the $\tau_i$. Thus we recover a moduli space, $\mathcal{M}_\tau^{(n)}$, whose global structure has been described by [30,29]. It is the moduli space of an elliptic curve with modulus $\tau = \sum \tau_i$ and $n - 1$ marked points corresponding to the remaining independent couplings $\tau_i - \tau_{i+1}$.

We now add $\mu$ mass perturbations. There are now $2n + \mu$ complex couplings $(\tau_i, \lambda_i, m_i)$, but \((2.10)\) still represent only $n$ constraints. We can again use the constraints to eliminate the $\lambda_i$ in favor of the $(\tau_i, m_i)$. We recover a space of fixed points with the structure $\mathcal{M}_\tau^{(n)} \times \mathbb{P}^{\mu-1}$, where the $\mathbb{P}^{\mu-1}$ is parameterized by the ratios $m_i/m_j$, as discussed above. We find superconformal fixed points at each point of the $\mathbb{P}^{\mu-1}$. It is easy to see that the $\mathbb{P}^{\mu-1} \subset \mathbb{P}^{n-1}$ is the vanishing locus of $n - \mu$ of the $m_i$ in the theory with all $n$ masses turned on.

In [21], a field theoretic computation of the central charges was made when all $m_i \neq 0$. In general, the relations \((2.10)\) on the vanishing loci of the $m_i$ do not lend a general result for the central charge. However, the most obvious solution of \((2.10)\), given by $\gamma_{\Phi_i} = 1/2$ and $\gamma_{A_i} = \gamma_{B_i} = -1/4$, leads to the result $\frac{c_{IR}}{c_{UV}} = \frac{27}{32}$. We conjecture that this is the central charge over the whole $\mathbb{P}^{n-1}$, including the vanishing loci, but we do not have a field-theoretic proof of this.

### 3. The Massless Spectrum of IIB Strings on $\text{AdS}_5 \times S^5 / \mathbb{Z}_n$

We now turn to the gravity dual to the quiver gauge theory on D3-branes at $\mathbb{C}^2 / \mathbb{Z}_n$. Our ultimate goal is to identify the duals of the flows described in the previous section. To accomplish this we first need to identify the duals of the operators in the perturbations \((2.7)\) that drive the flow, and this is done by studying the linearized perturbations about the UV fixed point theory.
As explained in [13], the near horizon geometry of D3-branes at $\mathbb{C}^2/\mathbb{Z}_n$ is $\text{AdS}_5 \times S^5/\mathbb{Z}_n$, where the $\mathbb{Z}_n$ acts as a subgroup of $SU(2)_L$. Specifically, if the $S^5$ is embedded in $\mathbb{C}^3$ given by $(z_1, z_2, z_3)$, then the $\mathbb{Z}_n$ action is

$$(z_1, z_2, z_3) \mapsto (\alpha z_1, \alpha^{-1} z_2, z_3),$$

where $\alpha = e^{2\pi i/n}$ is a root of unity. We are interested in perturbations of the quiver gauge theory by relevant operators. In the gravity dual, these operators correspond to certain “negative-mass” modes on $\text{AdS}_5$. The spectrum of the orbifold theories was given in [31,32] (see also [33] for additional discussion), and we now review those results starting with the simplest part: the untwisted sector.

### 3.1. Untwisted Sector

The untwisted sector will come from $\mathbb{Z}_n$-invariant harmonics of the reduction of IIB supergravity on $S^5$ [31]. Relevant operators correspond to modes with mass $m^2 \leq 0$. These can be obtained from the reduction of the $\mathcal{N} = 8$ graviton supermultiplet, as summarized in Table 3.1.

| Field     | $SU(4)$ irrep | $SU(2)_L \times SU(2)_R \times U(1)$ decomposition                                      |
|-----------|---------------|--------------------------------------------------------------------------------------|
| scalars   | $20'$         | $(1, 1)_0 \oplus (1, 1)_4 \oplus (1, 1)_{-4} \oplus (2, 2)_2 \oplus (2, 2)_{-2} \oplus (3, 3)_0$ |
|           | $10_c$        | $2(2, 2)_0 \oplus (3, 1)_{2} \oplus (3, 1)_{-2} \oplus (1, 3)_2 \oplus (1, 3)_{-2}$  |
|           | $1_c$         | $2(1, 1)_0$                                                                           |
| fermions  | $20$          | $(2, 1)_{-1} \oplus (2, 1)_3 \oplus (1, 2)_{-3} \oplus (1, 2)_{1} \oplus (3, 2)_{1} \oplus (2, 3)_{-1}$ |
|           | $20^*$        | $(2, 1)_1 \oplus (2, 1)_{-3} \oplus (1, 2)_{3} \oplus (1, 2)_{-1} \oplus (3, 2)_{-1} \oplus (2, 3)_1$ |
|           | $4$           | $(2, 1)_1 \oplus (1, 2)_{-1}$                                                          |
|           | $4^*$         | $(2, 1)_{-1} \oplus (1, 2)_1$                                                          |
| vectors   | $15$          | $(1, 1)_0 \oplus (2, 2)_2 \oplus (2, 2)_{-2} \oplus (3, 1)_0 \oplus (1, 3)_0$          |
| 2-forms   | $6_c$         | $2(1, 1)_2 \oplus 2(1, 1)_{-2} \oplus 2(2, 2)_0$                                      |
| gravitini | $4$           | $(2, 1)_1 \oplus (1, 2)_{-1}$                                                          |
|           | $4^*$         | $(2, 1)_{-1} \oplus (1, 2)_1$                                                          |
| graviton  | $1$           | $(1, 1)_0$                                                                            |

**Table 3.1:** The massless fields of IIB supergravity on $\text{AdS}_5 \times S^5$ and the relevant branching rules.

Despite their $m^2 < 0$ Kaluza-Klein eigenvalues, these modes propagate on the light-cone in $\text{AdS}_5$ and so one properly calls them massless.
The isometry group of $S^5$ is $SO(6) \approx SU(4)$. The $\mathbb{Z}_n$-action of (3.1) acts as the $SU(2)_L$ component of the subgroup $SU(2)_L \times SU(2)_R \times U(1) \subset SU(4)$. The relevant branching rules of $SU(4) \rightarrow SU(2)_L \times SU(2)_R \times U(1)$ are obtained from:

$$4 \rightarrow (2, 1)_1 \oplus (1, 2)_{-1}, \quad 4^* \rightarrow (2, 1)_{-1} \oplus (1, 2)_1. \quad (3.2)$$

For the $6 = (4 \otimes 4)_a$, one then has:

$$6 \rightarrow (1, 1)_2 \oplus (1, 1)_{-2} \oplus (2, 2)_0. \quad (3.3)$$

Continuing this, the $20^*$ appears in the product $4^* \otimes 6 = 4 \oplus 20^*$, and so one obtains:

$$20^* \rightarrow (2, 1)_1 \oplus (2, 1)_{-3} \oplus (1, 2)_3 \oplus (1, 2)_{-1} \oplus (3, 2)_{-1} \oplus (2, 3)_1. \quad (3.4)$$

It is then easy to read off the states from Table 3.1 which survive the $\mathbb{Z}_n$ projection. The only subtlety is that the $\mathbb{Z}_2$ projection, unlike the $\mathbb{Z}_{n>2}$ projection, leaves invariant all the vector-like $SU(2)$ representations. For the gravitini we thus find

$$-(2, 1)_1 \oplus (2, 1)_{-1} \oplus (1, 2)_{-1} \oplus (1, 2)_1 \rightarrow 2_{-1} \oplus 2_{1},$$

where the labels on the right-hand side are those of $SU(2)_R \times U(1)$. For the vectors we find

$$(1, 1)_0 \oplus (2, 2)_2 \oplus (2, 2)_{-2} \oplus (3, 1)_0 \oplus (1, 3)_0 \rightarrow 1_0 \oplus (1 + 2\delta_{n2})(1)_0 \oplus 3_0.$$  

We see that there are two states which are $\mathbb{Z}_2$-invariant, but not $\mathbb{Z}_{n>2}$-invariant. This leads to a pair of extra vector multiplets for $\Gamma = \mathbb{Z}_2$ because $\mathbb{Z}_2$ is, of course, the center of $SU(2)_L$.

The set of fields which survive the $\mathbb{Z}_n$ projection are displayed in Table 3.2. For completeness we will also consider the $SU(2)_L$ projection used in [13]. The stronger $SU(2)_L$ projection of [13] further truncates the states, since triplets of $SU(2)_L$ will give rise to a singlet of $\mathbb{Z}_n$, leading to a single vector multiplet when $n > 2$. In addition, all of the $SU(2)_L$ triplets are $\mathbb{Z}_2$ invariant, so there are three vector multiplets when $n = 2$. Terms in brackets in Table 3.2 are $\mathbb{Z}_{n>2}$ singlets that arise from $SU(2)_L$ triplets; the additional states that are present for $\Gamma = \mathbb{Z}_2$ are listed in the last column.
Table 3.2: The massless fields in the untwisted sector of IIB string theory on $AdS_5 \times S^5 / \mathbb{Z}_n$.

From [1] we know that the gauged $\mathcal{N} = 4$ supersymmetry multiplets have the content displayed in Table 3.3.

| Field   | $SU(2)_R \times U(1)_R$ irrep | Additional states for $\Gamma = \mathbb{Z}_2$ |
|---------|-------------------------------|---------------------------------------------|
| scalars | $1_0 \oplus 4 \oplus 1_{-4} \oplus 3_0$ | $\oplus 2(3_0)$                                |
|         | $3_2 \oplus 3_{-2} \oplus 1_2 \oplus 1_{-2}$ | $\oplus 2(1_2 \oplus 1_{-2})$                |
|         | $2(1_0)$                                                   |
| fermions| $2_3 \oplus 2_{-3} \oplus 2_1 \oplus 2_{-1} \oplus 2_1 \oplus 2_{-1}$ | $\oplus 2(2_1 \oplus 2_{-1})$                |
|         | $2_1 \oplus 2_{-1}$                                        |
| vectors | $1_0 \oplus 3_0 \oplus 1_0$                               | $\oplus 2(1_0)$                                |
|         | $2(1_2) \oplus 2(1_{-2})$                                  |
| gravitini| $2_{-1} \oplus 2_1$                                     |
| graviton| $1_0$                                                      |

Table 3.3: Field content of the supermultiplets of the $\mathcal{N} = 4$ $AdS_5$ superalgebra $SU(2, 2|2)$ with the R-symmetry group $SU(2)_R \times U(1)_R \subset USp(4)$. The subscripts denote the $U(1)_R$ charges and differ from the gauged $U(1)$ charges, as given for example in Table 3.2. The gauged $U(1)$ is the diagonal subgroup of $U(1)_R$ and another $SO(2)$ symmetry of the supergravity theory.

From this we can read off the field content of the various invariant sectors of the $\mathcal{N} = 8$ theory:

- $SU(2)_L$ invariant sector: A graviton supermultiplet and two charged tensor multiplets, of charges $\pm 2$ with respect to the gauged $U(1)$. The scalar coset is $SO(5,2) / SO(5) \times SO(2) \times \mathbb{Z}_2$.

- $U(1)_L$ or $\mathbb{Z}_n > 2$ invariant sector: A graviton supermultiplet, two charged tensor multiplets, of charges $\pm 2$, and a neutral vector multiplet. The scalar coset is $SO(5,2) / SO(5) \times SO(2) \times \mathbb{Z}_2$. 
\[ \frac{SO(5,3)}{SO(5) \times SO(3)} \times SO(1,1) \]. No fields are charged under the vector field in the additional vector multiplet.

- \( \mathbb{Z}_2 \) invariant sector: A graviton supermultiplet, two charged tensor multiplets, of charges \( \pm2 \), and three vector multiplets. The scalar coset is \[ \frac{SO(5,5)}{SO(5) \times SO(5)} \times SO(1,1) \]. The three vector multiplets actually gauge \( SU(2)_L \), and this sector represents \( N = 4 \) gauged supergravity with an \( SU(2)_L \times SU(2)_R \times U(1) \) gauge group.

### 3.2. The Twisted Sector

The twisted sector states were computed in \[ \text{32} \]. They are localized at the fixed circle of the \( \mathbb{Z}_n \) action (3.1) (they propagate on \( \text{AdS}_5 \times S^1 \)), so they can be computed via KK-reduction of the IIB string theory on \( S^1 \times \mathcal{M} \). Here \( \mathcal{M} \) is a compact Einstein manifold which looks locally like \( \mathbb{C}^2/\mathbb{Z}_n \), but inherits curvature and 5-form flux (such that \( \Lambda = 4 \) in units of the \( \text{AdS}_5 \) radius) from its relation to \( S^5/\mathbb{Z}_n \).

The twisted sector states of IIB string theory on \( \mathbb{C}^2/\mathbb{Z}_n \) consist of \( (n - 1) \) six-dimensional \( (0,2) \) tensor multiplets. The field content of the tensor multiplets is a tensor in the singlet of the \( USp(4) \) \( (0,2) \) R-symmetry, a set of chiral fermions in the \( 4 \), and real scalars in the \( 5 \). The conformal field theory on this background enjoys a \( \mathbb{Z}_n \) quantum symmetry [35], which acts by phases on the twisted sector states. In the theory on branes at the singularity, the quantum symmetry acts by clock shifts on the quiver diagram [36,37].

The spectrum on \( \mathcal{M} \) will similarly consist of tensor multiplets, but there will be corrections to the masses of these states. Reducing on the \( S^1 \) to \( \text{AdS}_5 \) we thus get tensor multiplets, which, as we will discuss in section 4, are not equivalent to vector multiplets in \( \text{AdS}_5 \). Moreover, due to the mass corrections, it turns out that the massless states in \( \text{AdS}_5 \) are tensors corresponding to non-trivial KK harmonics on the \( S^1 \). This means that the massless tensor multiplets in five dimensions come in charge-conjugate pairs under the \( U(1) \) associated to the Hopf fiber. In terms of physical degrees of freedom, the scalars in the \( 3_2 \) correspond to the hyper-Kähler deformations of \( \mathbb{C}^2/\mathbb{Z}_n \). The two singlets correspond to the periods of the NS-NS and RR B-fields. We have summarized the twisted sector fields in Table 3.4.

One should note that these twisted sector tensor multiplets have precisely the same charge assignments as the pair of tensor multiplets that come from the untwisted sector. As we will see, this is an essential consequence of the \( \mathbb{Z}_n \) cyclic quantum symmetry of the quiver theory.

---

\[ \text{A completely stringy computation of the spectrum was made in } \text{34}. \]
Field & $SU(2)_R \times U(1)$ irrep &  \\
--- & --- &  \\
scalars & $1_4 \oplus 3_2 \oplus 1_0$ &  \\
 & $1_{-4} \oplus 3_{-2} \oplus 1_0$ &  \\
fermions & $2_3 \oplus 2_1 \oplus 2_{-1} \oplus 2_{-3}$ &  \\
2-forms & $1_2 \oplus 1_{-2}$ &  \\

**Table 3.4:** The massless fields in the twisted sector of IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_n$ consist of $n-1$ copies of these fields.

### 3.3. Dual Operators in the Quiver Gauge Theory

To identify the states above with operators in the field theory, it is important to recall the discussion of the $N = 2$ superconformal R-charge in section 2. We denote the scalar components of the bifundamental chiral multiplets as $(a_i, b_i)$ and those in the adjoint multiplets by $\phi_i$. The pairs $(a_i, \bar{b}_i)$ and $(b_i, \bar{a}_i)$ form doublets under $SU(2)_R$, and are neutral under the $U(1)$; the fields, $\phi_i$, are $SU(2)_R$ singlets with $U(1)$-charge +2.

It is thus elementary to identify the duals of $\phi_i^2$ and $(\bar{\phi}_i)^2$ with the $1_{+4}$ and $1_{-4}$ of the tensor multiplets.\(^8\) In the untwisted sector these tensor multiplet scalars are dual to the surviving residue of the corresponding operators in the original $N = 4$ theory. That is, they are dual to $\sum_i^n \phi_i^2$ and $\sum_i^n (\bar{\phi}_i)^2$, which are, of course, singlets under the quantum $\mathbb{Z}_n$ symmetry. The scalars in the twisted sector are, of course, dual to operators that are charged under the quantum $\mathbb{Z}_n$ symmetry, and which are orthogonal to diagonal sums like $\sum_i^n \phi_i^2$. It is therefore convenient to use a cyclic basis $\phi_i^2 - \phi_{i-1}^2$. Similarly, one should recall that the $1_0$ scalars in the tensor multiplet of the twisted sector come from the periods of the NS-NS and RR B-fields over the blown-down $\mathbb{P}^1$ cycles of the ALE space. They are dual to *differences* between the complexified gauge couplings of the quiver gauge theory\(^[13,38]\). The sum over all the gauge couplings is dual to the original IIB dilaton, which is now in the tensor multiplet of the untwisted sector.

The $SU(2)_R$ triplets are easily identified by using the superconformal algebra. They are fermion bilinear operators for the two fermions in the $N = 2$ vector multiplet: $O_i^{ab} \equiv \chi_i^a \chi_i^b$. As above, the scalars in the untwisted sectors are dual to the sums, $\sum_i^n O_i^{ab}$, while the scalars in the twisted sectors are dual to the differences $O_i^{ab} - O_{i-1}^{ab}$.

---

\(^8\) Throughout this discussion we have suppressed explicit traces on products of operators: One should remember that all the operator expressions contain gauge invariant traces.
We have thus identified the duals of operator bilinears of the $\mathcal{N} = 2$ vector multiplets on the brane with charged tensor multiplets in supergravity. One should note that having identified the dual of any scalar in the tensor multiplet (such as the gauge coupling) the entire holographic assignment follows from supersymmetry. One can indeed check that this is consistent with the foregoing identifications. One can also arrive at the same result far more generally: The gauge couplings must be $1_6$’s of the R-symmetry, and from Table 3.3 we see that this is only possible if we start from vector or tensor multiplets of charges $\pm 2$. It was shown in [9] that in a gauged $\mathcal{N} = 4$ supergravity theory only tensor multiplets can be charged under a $U(1)$ gauge symmetry. Hence we see that the bilinear operators in the vector multiplets on the brane must be dual to scalars in tensor multiplets of charge $\pm 2$.

Finally, we come to the rest of the untwisted sector of the bulk theory, which contains the graviton supermultiplet and either one or three vector multiplets. The duality is straightforward to establish: One simply projects $\mathcal{N} = 4$ Yang-Mills dual operators onto the fields of the quiver theory. The dilaton of the supergravity multiplet belongs to the $SO(1,1)$ factor of the scalar manifold (1.2) and comes from the $20'$ of $SU(4)$. Its dual is easily identified [15,18] as

$$
\sum_i \left[ |\phi_i|^2 - \frac{1}{2} (|a_i|^2 + |b_i|^2) \right].
$$

The relative normalization reflects the traceless condition on the original $20'$ of $SU(4)$, ensuring that (3.5) is chiral.

There is also a $\mathcal{N} = 4$ vector multiplet common to all $\mathbb{Z}_n$ projections. We identify the $3_0$ state with the triplet components of the product of the two $SU(2)_R$ doublets we discussed above. These are the operators $\sum_i^n a_i b_i$, $\sum_i^n \bar{a}_i \bar{b}_i$, and the $J_3$-component $\sum_i^n (|a_i|^2 - |b_i|^2)$. These are the same expressions which appear in the D and F-flatness conditions and so they are frozen to constant values in the $\mathcal{N} = 2$ vacuum.

For $\Gamma = \mathbb{Z}_2$ we obtain two more vector multiplets, giving three vector multiplets that in fact gauge an additional $SU(2)$. (In fact it is $SU(2)_L$.) The new vector multiplets are dual to the multiplets built on the $\mathbb{Z}_2$-invariant chiral primaries $a_1 a_2$ and $b_1 b_2$ and their $SU(2)_R$ images. More generally, to form gauge invariant operators like these one must take products over all the nodes of the quiver: $\prod_i^n a_i$ and $\prod_i^n b_i$, and these have dimension $n$. They are thus only relevant supersymmetric perturbations for $n = 2$. These operators could be useful to probe the relative anomalous dimensions acquired by the $A_i$ and $B_i$ multiplets.
In the 1980's gauged supergravity theories were extensively studied, but primarily in four dimensions. The five-dimensional theories were also studied, but less thoroughly. The maximal gauged, $\mathcal{N} = 8$ theory was constructed \[1,2\], and some $\mathcal{N} = 4$ theories coupled to matter were investigated (see, for example, \[7,8\]). However, it was only rather recently that the most general gauged $\mathcal{N} = 4$ supergravity with matter coupling was constructed \[9\]. As we have discussed, the most general massless matter multiplets that are consistent with $\mathcal{N} = 4$ supersymmetry are vector and tensor multiplets, and these are not equivalent in the AdS backgrounds of gauged supergravity theories.

We are interested in these theories here because we wish to find simple descriptions of RG flows in quiver gauge theories. This approach has already proven very successful in $\mathcal{N} = 4$ Yang-Mills using $\mathcal{N} = 8$ supergravity, and the successes rested heavily on the five-dimensional theory being a consistent truncation of IIB supergravity. Consistent truncation in $\mathcal{N} = 4$ supergravity is largely \textit{terra incognita}, but we will start the discussion with results that are well established. To go beyond these, it is worth remembering that one of the reasons, and perhaps the primary reason, why it works is the complete rigidity of the structure. The complete $\mathcal{N} = 8$ theory is determined by the perturbative spectrum and the choice of gauge group. In terms of the brane theory this means that the large $N$ operator product structure within the energy-momentum tensor supermultiplet is rigidly determined by supersymmetry and $R$-symmetry. It was shown in \[9\] that the entire structure of the $\mathcal{N} = 4$ supergravity theory is fixed once one has decided on the gauge group and how the tensor multiplets are charged under this gauge group. This rigidity of structure should therefore be reflected in the large $N$ operator product structure of the $\mathcal{N} = 2$ theory on the brane, and thus determine the RG flows whether one uses the five-dimensional, or the ten-dimensional descriptions.

The bottom line is, as we described in the introduction, we find a family of flow solutions in the five-dimensional theory that exactly match onto the known ten-dimensional fixed-point solutions.
4.1. Terra Cognita: Truncations of the $\mathcal{N}=8$ theory

It is an often used fact that it is always consistent to truncate any field theory to the sector of singlets under any symmetry, discrete or continuous. One may thus generate several gauged $\mathcal{N}=4$ theories by applying this technique to the $\mathcal{N}=8$ theory, and the simplest way to accomplish this is to look for singlets under some subgroup, $\Gamma \subset SU(2)_L$ in the embedding $SU(2)_L \times SU(2)_R \times U(1) \subset SO(6)$. The whole point is that $\Gamma$ singlets contain four of the eight original supersymmetries.

The obvious choices for $\Gamma$ are $SU(2)_L$, $U(1)_L \subset SU(2)_L$ and $\mathbb{Z}_n$, and as we have discussed, they lead to $\mathcal{N}=4$ supergravity coupled to two charged tensor multiplets, and either zero, one or three vector multiplets. The resulting theories are perfectly consistent closed subsectors of the $\mathcal{N}=8$ supergravity theory, and any result derived therein can be, at least in principle, unambiguously lifted to ten dimensions.

In terms of the quiver theories on the brane, these $\mathcal{N}=4$ theories represent the untwisted sector of the quiver theory, and flows within these $\mathcal{N}=4$ theories may be interpreted as flows lying entirely within the untwisted sector of the corresponding quiver. That this is consistent may also be seen within the field theory as a consequence of the $\mathbb{Z}_n$ quantum symmetry. It follows that the $\mathcal{N}=1$ supersymmetric flows of [18] must be a part of any quiver theory, and indeed this observation was made in [18].

We now wish to go beyond these results, and add in precisely $(n-1)$ pairs of tensor multiplets with charges $\pm 2$ under the $U(1)$ factor of the $SU(2)_R \times U(1)$ gauge group.

4.2. $\mathcal{N}=4$ gauged supergravity theories: Some general facts

The starting point for constructing a gauged supergravity theory is usually to start with the ungauged theory in Minkowski space. In this setting the vector and tensor fields are equivalent, and so we may start with the theory described in [7], consisting of the $\mathcal{N}=4$ Poincaré supergravity multiplet coupled to an arbitrary number, $p$, of vector multiplets.

The five-dimensional $\mathcal{N}=4$ Poincaré supergravity multiplet consists of a graviton, four gravitini, six vector fields, four “spin-$\frac{1}{2}$” fields and a single scalar (the dilaton). There is an $SO(5) \cong USp(4)$, R-symmetry with the spinors transforming as a $4$ and the vectors transforming as a $5 + 1$ under it. In the ungauged Poincaré supergravity theory the massless “matter multiplets” can only be vector multiplets, and such a multiplet contains one vector field, four spin-$\frac{1}{2}$ fields and five real scalars.

9 Additional gravitino multiplets would require more supersymmetry.
The addition of the $p$ vector multiplets to supergravity thus results in the bosonic field content:

$$\{e^m_\mu, a_\mu, A_{\tilde{I}}^\mu, \sigma, \phi^x\}, \quad (4.1)$$

where $e^m_\mu$ denotes the fünfbein, $a_\mu$ is the $SO(5)$ singlet vector field of the supergravity multiplet, the $A_{\tilde{I}}^\mu$ ($\tilde{I} = 1, \ldots, (5+p)$) comprise the $SO(5)$ five-plet of vector fields from the supergravity multiplet as well as the $p$ vector fields from the $p$ vector multiplets, $\sigma$ is the supergravity scalar (the “dilaton”), and $\phi^x$ ($x = 1, \ldots, 5p$) collectively denotes the scalar fields of the vector multiplets.

The scalar fields $(\phi^x, \sigma)$ parameterize the scalar manifold

$$S = \frac{SO(5,p)}{SO(5) \times SO(p)} \times SO(1,1), \quad (4.2)$$

where the coset part and the $SO(1,1)$ factor are due to $\phi^x$ and $\sigma$, respectively.

The isometry group $G = SO(5,p) \times SO(1,1)$ of $S$ extends to a rigid symmetry of the entire ungauged supergravity Lagrangian. Under this symmetry group, the vector fields $A_{\tilde{I}}^\mu$ transform in the $(5+p)$ of $SO(5,p)$ and have $SO(1,1)$ charge $-1$, while $a_\mu$ is $SO(5,p)$ inert and has $SO(1,1)$ charge $+2$.

A gauged version of this theory is obtained by “gauging” appropriate subgroups $K \subset G$. For the gauged $\mathcal{N} = 4$ supergravity with $\mathcal{N} = 4$ supersymmetric AdS ground states, one must consider the corresponding supermultiplets of the $\mathcal{N} = 4$ anti-de Sitter $AdS_5$ superalgebra $SU(2,2|2)$ with an R-symmetry group $SU(2)_R \times U(1)_R \subset USp(4)$. The $\mathcal{N} = 4$, $AdS_5$ graviton supermultiplet consists of the graviton, four gravitini, $(3+1)$ vector fields, two tensor fields and one scalar (the dilaton). Technically, the appearance of the two tensor fields can be traced back to the gauging of the $U(1)_R$ factor in the supergravity theory [8], which is a generic phenomenon in five-dimensions [1,2,39,39] that also carries over to the matter multiplets: vector and tensor multiplets are no longer equivalent in gauged five-dimensional supergravity. Indeed, in the gauged theory, tensor fields satisfy a first order system of self-duality equations and these equations require that there be an even number of tensor multiplets. Thus a generic gauged theory will contain an even number, $2n$, of tensor multiplets and an arbitrary number, $q$, of vector multiplets. The fermion and scalar content of vector and tensor multiplets is the same, and the generic gauged supergravity theory has a scalar coset of the form (4.2) with $p = 2n + q$.

Supersymmetry imposes severe constraints on the possible gauge groups $K$. For our purposes, the most important results of the general analysis of [8] are:
• The $SO(1,1)$ factor in $G$ cannot be gauged, i.e., any gauge group $K$ has to be a subgroup of $SO(5,p)$.

• When the gauge group, $K$, is Abelian, it has to be one-dimensional with the corresponding gauge field given by the $SO(5,p)$ singlet $a_\mu$. Furthermore, in the gauged theory there can be no vector fields charged under this Abelian gauge group: If one wants charged matter fields then they must be tensor multiplets. In the traditional supergravity description one takes $K$ to be a subgroup of $SO(5,p)$, and any vector field that is charged under $K$ must be dualized to a tensor field prior to gauging. At the linearized level, the field equations of such self-dual tensor fields are of the form

$$dB = im(\ast B),$$

where $\ast$ denotes the Hodge dual, and $m$ is a mass parameter proportional to the coupling constant for $K$. Note that five-dimensional self-duality requires the factor $i$ on the right hand side. This implies that the tensor fields have to be complex and hence of even number when split in real and imaginary parts.

Note that the converse is also true: In five-dimensional $\mathcal{N} = 4$ gauged supergravity any self-dual tensor field has to be charged with respect to a one-dimensional Abelian group $K \subset SO(5,p)$.

• When the gauge group $K$ is semi-simple, no such self-dual tensor fields can exist, and the $(5 + p)$ of $SO(5,p)$ has to decompose with respect to $K \subset SO(5,p)$ as

$$(5 + p) \rightarrow \text{adjoint}(K) \oplus \text{possible } K-\text{singlets}. \quad (4.3)$$

In particular, there must not be any non-singlets of $K$ in addition to the adjoint. Otherwise, the gauging would be inconsistent with supersymmetry.

When the gauge group $K \subset SO(5,p)$ is a direct product of a semi-simple and an Abelian group (as will be the case in all of the examples considered here), a combination of the previous two items applies: the Abelian factor has to be one-dimensional, its gauge field is $a_\mu$, and the vector fields that would transform non-trivially under this Abelian factor have to be converted to self-dual tensor fields. Conversely, all self-dual tensor fields have to be charged with respect to the Abelian factor, and they must be neutral with respect to the semi-simple part of the gauge group $K$. With respect to this semi-simple part, the $(5 + p)$ of $SO(5,p)$ again has to decompose as in $(4.3)$.

• After $K$ has been gauged, the former global symmetry group $G = SO(5,p) \times SO(1,1)$ is broken to $[SO(1,1) \times C]_{\text{rigid}} \times K_{\text{gauged}}$, where $C$ denotes the commutant of $K$ in
When the gauge group $K$ is the direct product of a semi-simple and an Abelian factor, the global $SO(1, 1)$ symmetry can be used to rescale the ratio of the two coupling constants, so that only their relative signs are physically meaningful. When the gauge group $K$ is the direct product of a semi-simple and an Abelian factor, the global $SO(1, 1)$ symmetry can be used to rescale the ratio of the two coupling constants, so that only their relative signs are physically meaningful. 

### 4.3. The scalar potential

As we commented earlier, the structure of the $\mathcal{N} = 4$ supergravity theories is almost completely rigid. Indeed, specifying how an allowed gauge group $K$ is embedded in $SO(5, p)$ fixes the complete supergravity Lagrangian including the scalar potential. The scalar structure, and most particularly the scalar potential is of central importance to the study of the holographic flows, and so we now focus upon this sector in more detail.

We are interested in the case where the gauge group $K$ is a direct product of a semi-simple and an Abelian group. Denoting by $g_S$ and $g_A$ the gauge couplings of, respectively, the semi-simple and the Abelian factor, the general form of the scalar potential reads:

$$V = \frac{1}{2} \left[ g_A^2 V_{ij} V^{aij} - 36 g_A g_S U_{ij} S^{ij} + g_S^2 \left( T_{ij} T^{aij} - 9 S_{ij} S^{ij} \right) \right].$$  \hspace{1cm} (4.4)

Here, the tensors $V_{ij}^a$, $U_{ij}$, $T_{ij}^a$, and $S_{ij}$ are appropriately contracted products of two or three coset representatives of $SO(5, p)/[SO(5) \times SO(p)]$. Denoting such a coset representative by $L_{\tilde I}^a = (L_{\tilde I}^{ij}, L_{\tilde I}^a)$ (with inverse $L_{\tilde I}^{\tilde a}$), where $\tilde I$ is an $SO(5, p)$ index, $a$ is an $SO(p)$ index, and $ij$ denotes the 5 of $SO(5)$ written in terms of $USp(4)$ indices $i, j, \ldots = 1, \ldots, 4$ (i.e., $L_{\tilde I}^{ij}$ is antisymmetric and symplectic traceless in $i$ and $j$), these quantities are

$$U_{ij} = -\frac{\sqrt{2}}{6} e^{6} \Lambda_{NM}^{N} M_{Nij} L_{MK}^{M} L_{ij}^{M} \Lambda_{NM}^{N} M_{Nij} L_{ij}^{M} \Lambda_{NM}^{N} M_{Nij} L_{ij}^{M},$$

$$V_{ij}^a = \frac{1}{\sqrt{2}} e^{6} \Lambda_{NM}^{N} M_{Nij} L_{ij}^{M} \Lambda_{NM}^{N} M_{Nij} L_{ij}^{M},$$

$$S_{ij} = -\frac{2}{9} e^{-6} L_{ij}^{\tilde I} f_{\tilde I K}^{\tilde J} L_{ij}^{\tilde K} L_{ij}^{\tilde L},$$

$$T_{ij}^a = -e^{-6} L_{ij}^{\tilde I} L_{ij}^{\tilde J} L_{ij}^{\tilde K} f_{\tilde I K}^{\tilde J} L_{ij}^{\tilde I} L_{ij}^{\tilde L}.$$

Here, $\Lambda_{NM}^{N} M_{Nij} L_{ij}^{M}$ defines the action of the Abelian factor of $K$ upon the tensor fields, and $f_{\tilde I K}^{\tilde J}$ are the structure constants of the semi-simple part of the gauge group (that is, we have split the index $\tilde I$ into $M$ and $I$, corresponding to the splitting of the vector fields $A_\mu^I$ of the ungauged theory into tensor $B_{\mu \nu}^M$ and vector fields $A_\mu^I$ in the gauged version).
As mentioned earlier, the global $SO(1,1)$ symmetry, which acts as a constant shift on $\sigma$, can be used to rescale the coupling constants $g_A$ and $g_S$ such that only their relative sign matters.

Finally, by examining the contractions in the potential (4.14) and in the definitions of the tensors in (4.3), one sees that the potential is invariant under two symmetry operations:

(i) The left multiplication of the matrix $L$ by the elements of $SO(5,p)$ that commute with $\Lambda^N_M$ and that leave $f_{IK}^J$ invariant. This is precisely the group $C_{\text{rigid}} \times K_{\text{gauged}}$ mentioned earlier.

(ii) Right multiplication by any matrix in $SO(5) \times SO(p)$, which is the composite local symmetry of the theory.

One can use the second of these symmetries to put $L$ in the “symmetric gauge,” in which $L$ is the exponential of a symmetric matrix with $5p$ purely non-compact generators:

$$L = \exp \left( \begin{array}{cc} 0_{5 \times 5} & X \\ X^T & 0_{p \times p} \end{array} \right),$$

(4.6)

Having gone to the symmetric gauge one still has the symmetry (i), but to preserve the symmetric gauge one may have to combine it with a compensating transformation using symmetry (ii).

4.4. The relevant gauged supergravity theory

In holographic duality the R-symmetry on the brane becomes the gauge symmetry of the supergravity, and so we need to gauge $SU(2)_R \times U(1)$. The perturbative compactification analysis of section 3 gives us the spectrum and charges of the additional matter multiplets: we have the $\mathcal{N} = 4$ supergravity multiplet coupled to $n$ pairs of tensor multiplets of charge $\pm2$, and we have either one ($n > 2$) or three ($n = 2$) additional vector multiplets which gauge an additional $SU(2)_L$.

We will focus initially on the theory with one additional vector multiplet, corresponding to the $\mathbb{Z}_{n>2}$ orbifolds. The scalar fields parameterize the coset space$^{10}$

$$SO(1,1) \times \frac{SO(5, 2n + 1)}{SO(5) \times SO(2n + 1)}.$$  

(4.7)

$^{10}$ This should be compared with the scalar coset for the six-dimensional effective theory of the IIB string on an $A_{n-1}$ space, which is $SO(5,n+1)/[SO(5) \times SO(n+1)]$. 

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The $SU(2)_R$ factor is embedded into $SO(5) \subset SO(5, 2n + 1)$ as $SO(3) \subset SO(3) \times SO(2) \subset SO(5)$, with $f_{IK}^J = \epsilon_{IJK}$ in the first three indices of the $SO(5)$ vector representation. The embedding of the $U(1)$ factor of $SU(2)_R \times U(1)$ is a little less obvious, but is the natural extension of that found in [15]. For the untwisted sector the coset has the numerator $SO(5, 3)$, and this contains $SO(5, 2)$ in an obvious manner. This contains $SO(3) \times SO(2, 2) = SO(3) \times SU(1, 1) \times SU(1, 1)$. One $SU(1, 1)$ represents the IIB dilaton, while the other $SU(1, 1)$ represents geometric scalars coming from the $S^5$ metric. The $U(1)$ subgroup of this second $SU(1, 1)$ is a geometric symmetry on the $S^5$ that becomes a gauge symmetry in five dimensions. This $U(1)$ sits inside $SO(2, 2)$ as the diagonal $SO(2)$ in $SO(2) \times SO(2) \subset SO(2, 2)$. Adding the twisted sector tensor multiplets extends this to be the diagonal $SO(2)$ in $(SO(2))^{n+1} \subset SO(2, 2n)$, and $SO(3) \times SO(2, 2n) \subset SO(5, 2n) \subset SO(5, 2n + 1)$. Thus we have:

$$
\Lambda^N_M = \begin{pmatrix}
0_{3 \times 3} & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & \varepsilon & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & \varepsilon & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & \ldots & 0 \\
0 & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \ldots & \ddots & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 & \varepsilon \\
0 & 0 & 0 & \ldots & \ldots & 0 & 0
\end{pmatrix}, \quad \varepsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
$$

(4.8)

and where $\varepsilon$ is repeated $(n + 1)$ times.

It is important to examine the symmetry of the corresponding potential (4.4). The symmetry from left-multiplication is $SU(2) \times U(1, n)$. The $SU(2)$ is the invariance of $f_{IK}^J$, while $U(1, n)$ is the subgroup of $SO(2, 2n)$ that commutes with $\Lambda^N_M$. The $U(1)$ center of $U(1, n)$ is, of course, generated by $\Lambda$ itself, and is part of the gauge symmetry. If one passes to symmetric gauge, then the potential in this gauge has the symmetries:

(i) Conjugation by $SU(2) \times U(1) \times U(n)$, which is the compact subgroup of the symmetry group identified above.

(ii) Left multiplication by $SU(1, n)$. If one wishes to preserve symmetric gauge (4.6) then this must be combined with a compensating right multiplication by $U(1) \times U(n)$.

These symmetries are a very natural generalization of the symmetries of the potential in the $\mathcal{N} = 8$ theory, and the truncation considered in [13]. In particular that truncation had an $SU(2) \times U(1) \times U(1)$, along with a non-compact $SU(1, 1)$ symmetry that corresponds to the IIB dilaton/axion coset.
We finish by noting that the model with three extra vector multiplets in the case \( \Gamma = \mathbb{Z}_2 \) is a simple generalization of the foregoing. First the supergravity gauge group is now \( SU(2)_L \times SU(2)_R \times U(1) \), and so the structure constants \( f_{IKJ} \) can be taken to be \( \epsilon_{IJK} \) in both the first three indices and the last three indices of the \( SO(5,7) \) vector representation. With this choice, the matrix \( \Lambda \) is as in (4.8), but extended by two extra rows and columns of zeroes. The potential still has the non-compact \( SU(1,2) \) symmetry, but has the compact conjugation symmetry of \( SU(2) \times SU(2) \times U(1) \times U(2) \).

4.5. The five-dimensional scalars and the IIB fields

Thus far we have identified holographic duals of operators on the brane in terms of the supergravity scalars in \( \mathcal{N} = 4 \) multiplets. Here we wish to make that identification a little more explicit in terms of the five-dimensional scalar manifold. It is well known that at the non-linear level the correspondence of fields can be quite subtle, but there is also a minor issue at linearized level involving choices of basis.

We will focus on two particular components of the five-dimensional tensor multiplets, namely the \( 1_0 \) and the \( U(1) \subset SU(2)_R \) singlet part of the \( 3_{\pm 2} \) scalars. We will denote these complex degrees of freedom as \( \phi_i \) and \( \chi_i \), respectively. In the supergravity theory it is natural to use an orthonormal basis in terms of the Cartan-Killing form on \( SO(5,2n+q) \), and so take a linearized kinetic term of the form:

\[
\sum_{j=1}^{n} (\partial_{\mu} \phi_j)^2 + \sum_{j=1}^{n} (\partial_{\mu} \chi_j)^2.
\]

In the linearized IIB theory the counterparts of the \( \phi_j \) and \( \chi_j \), which we will call \( \hat{\phi}_j \) and \( \hat{\chi}_i \), have very diverse origins in terms of untwisted and twisted sectors. Consider the \( \hat{\phi}_j \): One of them, say \( \hat{\phi}_1 \), comes from the untwisted sector and is the complexified axion-dilaton, which corresponds to the sum of the complexified gauge couplings, \( \sum_i \tau_i \), in the quiver theory. In the twisted sector, the \( \hat{\phi}_i \) are proportional to the periods \( B_{NS} + i B_{RR} \) over the homology 2-spheres of the ALE space. They correspond to differences, \( \tau_i - \tau_{i+1} \), between the complexified gauge couplings of the quiver theory. The intersection form on the 2-cycles is given by the Cartan matrix, \( C_{ij} \), of the \( A-D-E \) group and so \( C_{ij} \) appears as a mixing matrix in the effective action for the \( \hat{\phi}_i \). So at the linearized level, there is a natural field re-definition between \( \phi_i \) and \( \hat{\phi}_i \) to get a canonical kinetic term for the fields \( \phi_i \):

\[
\sum_{i,j=1}^{n} \delta_{ij} \partial_{\mu} \phi_i \partial^{\mu} \phi_j = (\partial_{\mu} \hat{\phi}_1)^2 + \sum_{i,j=2}^{n} C_{ij} \partial_{\mu} \hat{\phi}_i \partial^{\mu} \hat{\phi}_j.
\]
In other words, the twisted sector fields \( \hat{\phi}_i \) are in one-to-one correspondence with the simple roots \( \alpha_i \) of the \( A-D-E \) group. We choose a basis for the fields \( \phi_i \) which behaves like the fundamental weights \( e_i \), which satisfy \( e_i \cdot e_j = \delta_{ij} \), in order to obtain canonical kinetic terms (4.10). The twisted sector linearized fields \( \hat{\phi}_i \) are proportional to differences of the \( \phi_i \) through the expression \( \alpha_i = e_i - e_{i+1} \) for the simple roots in terms of the weights. We further make the choice that the untwisted sector field \( \hat{\phi}_1 = c \sum_{i=1}^{n} \phi_i \) for some normalization \( c \). With this choice of basis, the \( \phi_i \) satisfy the right properties to be dual to the coupling constants \( \tau_i \) at each node of the quiver. Similarly, the \( \chi_i \) are dual to the mass perturbations \( m_i \) in (2.7).

It is also useful to describe how these scalars sit in the coset (4.7). One must examine \( SO(2,2) \) subgroups of \( SO(5,2n) \) to determine the \( U(1) \) R-charges of the various generators [15]. One finds that the general representative with all fields except \( \phi_i \) and \( \chi_i \) turned off takes the form

\[
L = \exp \begin{pmatrix}
0_{5 \times 5} & v^{(1)}_{5 \times 2} & \cdots & v^{(n)}_{5 \times 2} & 0_{5 \times 1} \\
(v^{(1)}_{5 \times 2})^T & \vdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & \cdots & 0_{2n+1 \times 2n+1} & \vdots \\
(v^{(n)}_{5 \times 2})^T & \cdots & \cdots & \cdots & 0_{1 \times 5}
\end{pmatrix},
\]

where the \( 5 \times 2 \) matrices have the form

\[
v^{(i)}_{5 \times 2} = \begin{pmatrix}
\text{Re} \chi_i & \text{Im} \chi_i \\
-\text{Im} \chi_i & \text{Re} \chi_i \\
0 & 0 \\
\text{Re} \phi_i & \text{Im} \phi_i \\
-\text{Im} \phi_i & \text{Re} \phi_i
\end{pmatrix}.
\]

5. Holographic renormalization group flows

5.1. Some simple tests

There are several obvious consistency checks upon our putative dual of a subsector of the quiver theories. First is the fact that the \( SU(1, n) \) symmetry discovered above is essential from the field theory perspective. Recall that the \( SU(1, n) \) appeared as the commutant of the \( SU(2) \times U(1) \) gauge symmetry, which means that all the supergravity scalars in the coset:

\[
\mathcal{T} = \frac{SU(1, n)}{U(1) \times SU(n)},
\]

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are precisely the $1_0$ of the R-symmetry. As we noted earlier, these are the duals of the complex gauge coupling constants, $\tau_i$, on the nodes of the quiver theory. The fact that the supergravity potential has an $SU(1, n)$ invariance means that these coupling constants are freely chooseable parameters in the theory, as they must be from the brane perspective. This fact is the generalization to the quiver theory of the $SU(1, 1)$ symmetry of the IIB theory and its relationship to the complex gauge coupling in the $\mathcal{N} = 4$ Yang-Mills theory.

One can also check that the results of [9] and of the previous sections reproduce the solutions of [15,18]. To be more precise, in [15] a general $SO(5, 2)$ matrix was constructed (up to symmetries of the potential), and the $\mathcal{N} = 8$ supergravity potential was then constructed for this matrix. We have taken the same $SO(5, 2)$ matrix and have explicitly verified that (4.4) and (4.5), with the choices of $\Lambda_{N_M}$ and $f_{IK}^J$ in section 4, exactly reproduces the scalar potential of [15].

The flow studied in [18] used two parameters. The first, denoted $\rho = e^\alpha$, was the $SO(1, 1)$ group element, while the second was a particularly simple $SO(5, 2)$ matrix:

$$L = \exp(\mathcal{X}) \text{ where } \mathcal{X}_{1,6} = \mathcal{X}_{6,1} = \chi, \quad \mathcal{X}_{2,7} = \mathcal{X}_{7,2} = \chi,$$

with all other matrix elements equal to zero (this corresponds to $\text{Re } \chi_1 = \chi$ and all other modes turned off in (1.12)). The $\mathcal{N} = 1$ supersymmetric flow is characterized by a superpotential, $W$, that is related to the scalar potential via:

$$\mathcal{V} = \frac{g^2}{8} \sum_{j=1}^2 \left| \frac{\partial W}{\partial \varphi_j} \right|^2 - \frac{g^2}{3} \left| W \right|^2,$$

with $\varphi_1 = \chi$ and $\varphi_2 = \sqrt{6} \alpha$. The scalars, $\varphi_j$, are defined so as to have canonically normalized kinetic terms. The superpotential was computed to be:

$$W = \frac{1}{4\rho^2} \left[ \cosh(2\varphi_1) \left( \rho^6 - 2 \right) - (3\rho^6 + 2) \right],$$

and the equations of motion for the flow are:

$$\frac{d\varphi_j}{dr} = \frac{g}{2} \frac{\partial W}{\partial \varphi_j}, \quad \frac{dA}{dr} = -\frac{g}{3} W,$$

where $A(r)$ is the cosmological function in the five-dimensional metric:

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu.$$
The superpotential gives rise to two supersymmetric ground states: The maximally symmetric, and maximally supersymmetric one with $\phi_j = 0$, and the other with:

$$
\varphi_1 = \pm \frac{1}{2} \log(3) \quad \varphi_2 \equiv \sqrt{6} \alpha = \frac{1}{\sqrt{6}} \log(2). \quad (5.7)
$$

This second point has $\mathcal{N} = 2$ supersymmetry in the bulk, and on the brane it corresponds to the $\mathcal{N} = 1$ superconformal point of Leigh and Strassler \[26,18\]. The flow of interest is the steepest descent from the $\varphi_j = 0$ critical point to either of the non-trivial points (5.7). The cosmological constants at the two critical points are, respectively $\frac{-3}{4}g^2$ and $\frac{-1}{3}2^{4/3}g^2$, which leads to the result that $\frac{c_{IR}}{c_{UV}} = \frac{27}{32}$.

As we described earlier, the scalars $\chi$ and $\alpha$ are respectively dual to the following operators in the $\mathcal{N} = 4$ Yang-Mills theory:

$$
O_f = \text{Tr}(\lambda^3 \lambda^3), \quad O_b = -\sum_{j=1}^{4} \text{Tr}(X^j X^j) + 2 \sum_{j=5}^{6} \text{Tr}(X^j X^j), \quad (5.8)
$$

and the flow corresponds to turning on a mass for a single chiral multiplet.

In the $\mathcal{N} = 8$ supergravity theory the superpotential can be read off from the eigenvalues of a scalar matrix in the gravitino variation. We have indeed confirmed that the same results are obtained from the gravitino variation in \[9\] using the scalars $\alpha$ and $\chi$ defined above.

5.2. New flows from old

The unbroken global symmetry $SU(1, n)$ in the massless sector of the $\mathbb{Z}_n$ orbifolds has profound consequences for the holographic descriptions of renormalization group flows in the corresponding quiver gauge theories.

The $SO(5, 2)$ matrix defined in (5.2) can be embedded directly into $SO(5, 2n + 1)$ or $SO(5, 2n + 3)$ resulting in a particular flow solution for the quiver gauge theory. This flow corresponds to the deformation of the quiver theory by a single adjoint mass term which breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry on the brane. This embedding is guaranteed to be consistent because the other matter multiplets can be consistently truncated out, using an $SU(n - 1) \times \mathbb{Z}_2$ symmetry for $\mathbb{Z}_{n>2}$ or a $U(1) \times SU(2)_L$ symmetry for $\mathbb{Z}_2$. Now consider the action of the $SU(n)$ “flavor symmetry.” This can map the matrix elements of $\mathcal{X}$ onto any matrix with

$$
\mathcal{X}_{1,2j+4} = \mathcal{X}_{2j+4,1} = \chi_j, \quad \mathcal{X}_{2,2j+5} = \mathcal{X}_{2j+5,2} = \chi_j, \quad j = 1, \ldots, n. \quad (5.9)
$$
The parameters $\chi_j$ are the supergravity scalars that represent the couplings $m_j$ in (2.7).

The potential and superpotential are invariant under the $SU(n)$ and so we conclude that the critical point described above extends to a critical surface swept out by the $SU(n)$ action. Similarly, the flow of \[ \text{(18)} \] becomes a continuous family of flows to this critical surface, with the end-point determined by the initial velocities $\chi_j$, or $m_j$. We therefore conclude that, at least for large $N$, the quiver theories have a continuous family of $\mathcal{N} = 1$ supersymmetric fixed points with $c_{\text{IR}}/c_{\text{UV}} = \frac{27}{32}$, and these are all swept out by the continuous action of $SU(n)$. In particular, the solutions of \[ \text{(16)} \], and its generalizations in \[ \text{(17)} \], and the solution of \[ \text{(18)} \] are simply $SU(n)$ images of one another, and merely represent isolated solutions in a continuum family. To be precise, for $n = 2$, the deformation by \[ \text{(2.3)} \] corresponds to the flow with $\text{Re} \, \chi_1 = -\text{Re} \, \chi_2 = \chi/2$, while that of \[ \text{(2.5)} \] has $\text{Re} \, \chi_1 = \text{Re} \, \chi_2 = \chi/2$.

It is interesting to note that the gauged $\mathcal{N} = 4$ supergravity theory contains scalars, $\chi_j$, that are dual to fermion masses in each node of the quiver theory. On the other hand, the only scalar mass term in the $\mathcal{N} = 2$ vector multiplet on the brane that is “resolvable” within this gauged supergravity is \[ \text{(3.5)} \]: the sum over all the nodes of the quiver. In any supersymmetric flow, the scalar masses are determined by the fermion masses, and so the scalar mass data is inessential to the study of the generalizations of the “Leigh-Strassler” flows. It was however noted in \[ \text{(18)} \] that one could apparently give independent initial velocities to the supergravity scalars that are dual to the fermion and scalar bilinears on the brane, and still preserve supersymmetry. These more general flows could then be interpreted as turning on a mass for a chiral multiplet, and flowing out along the Higgs branch of the remaining massless multiplets. The gauged supergravity considered here still captures these flows for the “sum over nodes” on the quiver, but apparently not within each individual node of the quiver separately.

6. Discussion

Motivated by the many successes of gauged supergravity theory in studying holographic RG flows, we have made some progress in extending the toolkit to spacetime orbifolds and their field theory duals. The applicability of our approach to the massless string modes in the untwisted sector is guaranteed by virtue of the consistent truncation to $\Gamma$–invariant subsectors of the maximal gauged supergravity theory in five dimensions.

\[ \text{11} \] More generally, the untwisted deformation corresponds to $\text{Re} \, \chi_i = \frac{\chi}{n}$ for all $i$. 

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However, we have also described the way in which one can incorporate the twisted sector degrees of freedom. Here we have passed beyond the safe haven of consistent truncations and we cannot, \textit{a priori}, guarantee that the solutions we obtain in our five-dimensional theory always correspond to ten-dimensional solutions of the full IIB theory.

\textbf{6.1. Features of the gauged supergravity approach}

We have presented several encouraging facets of our approach which motivate our trust in it. First of all, the structure of the $\mathcal{N}=4$, five-dimensional gauged supergravity theories found in \cite{9} is completely determined by the embedding of the gauge group in the scalar coset. We found that the IIB massless modes had a particular charge structure under the $SU(2)_{R} \times U(1)$ group that was to be gauged, and so the gauged supergravity theory that is dual to quiver gauge theories is unique up to field re-definitions.

We have also used our approach to examine the family of RG flows generated by perturbations of the quiver theory by masses for the adjoint scalar fields. This yielded a picture of a $\mathbb{P}^{n-1}$ critical surface of superconformal fixed points, which was precisely in line with our field theory analysis in section 2. The gauged supergravity results also indicate that the value of the central charge on these surfaces is $\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{27}{32}$, which is consistent with the calculations of \cite{20,21}, as well as those of section 2.

In several of these RG flows, ten-dimensional solutions are known. In \cite{16}, it was argued that IIB on the conifold was the theory that described the IR fixed point of the flow generated by the deformation of the $n=2$ quiver theory by the twisted operator (2.3). From the orbifold connection between $\mathcal{N}=4$ Yang-Mills and the $\mathcal{N}=2$ quiver theories, it is easy to see that the ten-dimensional lift of the flow generated by the untwisted operator

\begin{equation}
\mathcal{W}_u = \frac{m}{2} \sum_{i=1}^{n} \int d^2 \theta \Phi_i^2,
\end{equation}

is given by a $\mathbb{Z}_n$–orbifold of the “Leigh–Strassler” solution found in \cite{23,19}.\footnote{To be precise, this $\mathbb{Z}_n$ acts as an identification on the Hopf angle, $\alpha_3$, of the $SU(2)$ sub-manifold defined in equation (3.7) of \cite{19}. This means that $SU(2)$ is replaced by the Lens space $S^3/\mathbb{Z}_n$. The rest of the ten-dimensional solution, including the 2-form and 5-form backgrounds, is invariant under the $\mathbb{Z}_n$ action, and so is unmodified.} Manifolds which provide a basis for the ten-dimensional description of the general flows were described in \cite{17} and these were used in the computation of the central charges \cite{21} that our results agree with.
The fact that there are known ten-dimensional solutions that coincide with individual points in the families of five-dimensional solutions, and that the results of [17] suggest that these ten-dimensional solutions also form families with the same symmetries and supersymmetry, is consistent with our conclusions here.

It would certainly be more satisfying (as well as extremely useful in practice) if we could directly extend the technology of uplifts of five-dimensional solutions to ten-dimensional IIB solutions (as in, for example, [23,19,25]) to the solutions of $\mathcal{N} = 4$ gauged supergravity theories. An important part of this program involves the correct approach to include the geometric data of topology change due to the partial resolutions of the orbifold singularity in the general case. This is currently under investigation.

We have reproduced the branch of moduli space which corresponds to the adjoint masses, but we have had less success in reproducing the moduli space, $\mathcal{M}_g^{(n)}$ of gauge couplings found in [30,29]. The duality group can be expressed as a certain extension of $SL(2,\mathbb{Z})$ by a braid group$^{13}$. The resulting discrete group appears to be too large to be embedded in the $SU(1, n)$ symmetry group found here. Presumably the difference lies in the large $N$ limit that is inherent in the supergravity approach. This limit may, on the one hand, collapse or trivialize some of the quantum duality symmetry, while on the other hand promote another part of it to a continuum symmetry. It is worth recalling that for $n = 1$ the $SU(1, 1)$ symmetry of the IIB supergravity is reduced to $SL(2,\mathbb{Z})$ in the string theory. It would be interesting to determine what happens to the $SU(1, n)$ symmetry of the supergravity theory considered here. A physically natural guess might involve a symmetry that leaves the hyperplanes, $m_i = 0$ fixed, and so might involve a combination of the Weyl group of $SU(n)$ and the $SL(2,\mathbb{Z})$ action inherited from the IIB string.

6.2. Additional Interesting Flows

We have studied one class of flows, but there are other interesting deformations that can be studied within the $\mathcal{N} = 4$ gauged supergravity. For example, one component of the vector multiplet in the untwisted sector is dual to a $\mathbb{Z}_n$-invariant hypermultiplet mass term, $\sum a_i b_i$, which preserves $\mathcal{N} = 2$ supersymmetry on the D3-brane. This could be used to probe the vacuum of the quiver theory, as well as of other fixed points with less supersymmetry.

More interesting flows can be generated by other initial conditions for the scalars $\phi_i$ and $\chi_i$. Since the $\phi_i$ that are dual to coupling constants in the field theory, the coupling

$^{13}$ We thank E. Witten for discussions on these points.
constants can run along these flows. It would be interesting to see if there are flows, perhaps incorporating additional domain wall configurations in five dimensions, which result in $H_{RR}$ flux in ten-dimensions that could be interpreted as fractional brane configurations.

There are some flows which might be outside of the scope of the holographic approach within IIB supergravity alone. For example, there are non-$\mathbb{Z}_n$-invariant chiral primary operators $a_i b_i$ (no sums on $i$) that do not have dual fields in the twisted sector discussed in section 3. This is consistent with the notion that the non-trivial RR flux has frozen out some moduli (like the FI-terms for the gauge theory).

We also note some features of the operator spectrum of the $\mathcal{N} = 2$ algebra in four-dimensions. We are able to find the duals of individual mass terms for the fermions in the vector and hypermultiplets. Indeed the fermion masses in the vector multiplets are what we used to generate the $\mathcal{N} = 1$ flows that we studied. On the other hand, the individual boson mass terms like $|a_i|^2$, $|b_i|^2$, and $|\phi_i|^2$ are apparently non-chiral. This can be most directly seen as a consequence of the fact that, in the projection from the $\mathcal{N} = 4$ SYM theory, they correspond to components of bilinear operators which are not chiral. Instead we see only linear combinations of the operators (3.5) and $\sum_i (|a_i|^2 - |b_i|^2)$, which are related to components of chiral $\mathcal{N} = 4$ operators.

6.3. Surprising Implications of Gauged Supergravity Symmetries

Perhaps the most intriguing result of our analysis is the $SU(1,n)$ symmetry on the tensor multiplet sector that we found in section 4. One may think of this symmetry group as in terms of the “bonus symmetries” that arise from enhancements of discrete symmetries into continuous ones in passing from a string theory to its supergravity limit. One example is the promotion of the $SL(2,\mathbb{Z})$ S-duality group to $SU(1,1)$. In the orbifolds we have studied here, an additional discrete symmetry is the $\mathbb{Z}_n$ quantum symmetry, which leads to a $SL(2,\mathbb{Z}) \times \mathbb{Z}_n$ group which is being promoted to $SU(1,n)$. In the near horizon limit, we saw that this space is enlarged to form the coset of five-dimensional scalars. Apparently the $SU(1,n)$ symmetry is non-trivially related to the residual IIB duality group in the near-horizon background. This would suggest another obvious guess of $SU(1,n;\mathbb{Z})$ for the quantum duality symmetry.
This discussion illuminates the extension of our results to the $D$ and $E$ series. In six dimensions, we will find scalar cosets $SO(5, r + 2)/[SO(5) \times SO(r + 2)]$ and duality group $SO(2, r + 2; \mathbb{Z})$, where $r$ is the order of the orbifold group. The KK-reduction of the $r$ twisted-sector tensor multiplets will again lead to $2r$ charged five-dimensional tensor multiplets with the same $SU(2)_R \times U(1)$ charges of $1_{\pm 2}$ as we find in the untwisted sector. The linearized fields will have kinetic terms which are weighted by the relevant Cartan/intersection matrix, as in (4.10), so we will make the same type of basis change in going to the five-dimensional fields as we described in section 4.5. Accounting for the single neutral vector multiplet we obtain in the untwisted sector, we find a scalar coset $SO(5, 2(r + 1) + 1)/[SO(5) \times SO(2(r + 1) + 1)]$. The gauging proceeds as in the $A$-series case and we find an analogous $SU(1, r + 1)$ symmetry.

From the perspective of supergravity we might have anticipated the $SU(1, n)$ symmetry from the outset. The fact that the quiver theories are superconformal with $n$ complex couplings, $\tau_j$ that can be chosen freely at the UV fixed point means that the corresponding supergravity must have $n$ complex scalars that represent flat directions in the supergravity potential. The presence of a $\mathbb{Z}_n$ quantum symmetry that cycles the nodes of the quiver, and hence the gauge couplings, combined with the original $SU(1,1)/U(1)$ coset of the IIB theory points rather directly at $SU(1, n)$.

There are also consequences of the $SU(n) \subset U(1, n)$ symmetry for the structure of ten-dimensional solutions. This symmetry relates fields in the untwisted sector of the IIB orbifold theory with those in the twisted sector. In fact, in five-dimensional terms, it puts these fields on the same footing. On the other hand, in terms of the ten-dimensional geometry, turning on these fields does very different things. For example, we have already described the fixed point geometry dual to the fixed point generated by the untwisted deformations (6.1) in terms of an orbifold of the ten-dimensional solution in [19]. This involves D3-branes transverse to a space with a $\mathbb{Z}_n$ orbifold singularity and certain backgrounds for the self-dual 5-form and the 2-form $B$-fields. We argued that, for $n = 2$, this fixed point is related by $SU(2)$ to the fixed point which is generated by the twisted deformation (2.3). The ten-dimensional solution which governs the fixed point in this case [10] involves D3-branes transverse to the conifold (the original $\mathbb{Z}_2$ orbifold singularity has been removed), with an appropriate 5-form field strength turned on, but no $B$-fields. For $n > 2$, the analogous fixed points are dual to cones over the level surfaces of the generalized conifolds constructed in [20]. It is also expected that the IIB solution in these cases does not involve $B$-fields.
Finally, we believe that this $SU(1,n)$ symmetry can also be used to make further statements about these fixed point theories. For example, we expect that the coupling constant dependence of correlation functions at large $N$ should be constrained to $SU(n)$ covariant expressions.

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Appendix A. Analysis of the scalar manifold

In this appendix we will determine the total number of independent scalar fields of the potential of gauged $\mathcal{N} = 4$ supergravity, with the gauge group $SU(2)_R \times U(1)$, coupled to $2n$ tensor multiplets with or without an extra spectator vector multiplet that could drive renormalization group flows. The scalar fields of these $\mathcal{N} = 4$ supergravity theories parameterize the coset space

$$S = SO(1,1) \times \frac{SO(5,p)}{SO(5) \times SO(p)},$$

where $p = 2n$ for the coupling of $2n$ tensor multiplets and $p = 2n + 1$ when we have an additional spectator vector multiplet.

For $p = 2n$ or $p = 2n + 1$ the relevant gauged supergravity has the residual symmetry $SU(2)_R \times U(1) \times SU(1,n) \times SO(1)$ \[14\]. Let us first consider the case $p = 2n$. Then the $10n$ scalar fields of the coset $SO(5,2n)/[SO(5) \times SO(2n)]$ decompose as $(5,2n)$ under the subgroup $SO(5) \times SO(2n)$. With respect to the $U(1)$ subgroup of $SU(1,n) \subset SO(2,2n)$ each of the five sets of $2n$ scalars in the fundamental representation of $SO(2n)$ decomposes as $n \oplus \bar{n}$. We shall label them as

$$Z_A^i \simeq (5, n) \in SO(5) \times U(n),$$

$$Z_A^{iA} \simeq (5, \bar{n}) \in SO(5) \times U(n),$$

where $i,j, \ldots = 1, \ldots, 5$ and $A = 1, \ldots, n$. Consider now the general case when $n \geq 4$. By using the noncompact symmetries of the residual $SU(1,n)$ symmetry we can gauge away one set of $n + \bar{n}$ scalars, namely a linear combination of $Z_A^1$ and $Z_A^5$, leaving us with four sets of $n + \bar{n}$ scalars. Now by the local $U(n)$ symmetry we can rotate the first set $Z_A^1$ to the vector $(R_1^1,0,...,0)$ where $R_1^1$ is real. Using the little group $U(n-1)$ of this vector we can rotate the second vector $Z_A^2$ into a vector of the form $(C_1^2, R_2^2, 0, ..., 0)$ where $C_1^2(R_2^2)$ is complex (real). Similarly $Z_A^3$ can be brought to the form $(C_1^3, C_2^3, R_3^3, 0, ..., 0)$ by a $U(n-2)$ rotation. The remaining linear combination of $Z_A^4$ and $Z_A^5$ can similarly be brought to the form $(C_1^4, C_2^4, C_3^4, R_4^4, 0, ..., 0)$. All this leaves us with 16 real scalars. Using the $SU(2)_R \times U(1)$ gauge symmetry we can gauge away 4 of the 16 scalars leaving us with 12 scalars plus the $SO(1,1)$ dilaton.

For $n < 4$ the above analysis needs to be slightly modified. For $n = 1$ we are left with 1 real and 3 complex scalars from the coset $SO(5,2)/[SO(5) \times SO(2)]$ after gauge fixing

\[14\] As discussed earlier we are considering the case when all the pairs of tensor fields carry equal and opposite charges with respect to $U(1)$.
the $SU(1,1)$ symmetry fully. Further fixing of the $SU(2)_R \times U(1)$ gauge symmetry leaves us with 3 real scalars plus the $SO(1,1)$ dilaton, thus agreeing with the analysis of [15]. For $n = 2$ we get $5 \times 2 + 2 = 12$ scalars surviving from the coset $SO(5,4)/[SO(5) \times SO(4)]$. After utilizing the $SU(1,2)$ symmetry. Then fully gauge-fixing the $SU(2)_R \times U(1)$ symmetry we are left with 8 scalars plus the $SO(1,1)$ dilaton. Similar analysis yields 11 real scalars plus the $SO(1,1)$ dilaton for $n = 3$ after we fix all the gauges.

The above results generalize trivially to the cases when there is an additional spectator vector multiplet present and the scalar manifold is $SO(5,2n+1)/SO(5) \times SO(2n+1)$ since the extra five scalars sitting in the coset $SO(5,2n+1)/SO(5) \times SO(2n+1)$ are all singlets of the residual $SU(1,n)$ symmetry of the gauged supergravity describing the coupling of $2n$ tensor multiplets and spectator vector multiplet to $\mathcal{N} = 4$ supergravity. Thus the net number of scalar fields left after gauge fixing all the symmetries is 5 more than the corresponding cases without the extra vector multiplet.

The fact that the number of independent moduli does not grow with increasing $n > 4$ may at first sight appear surprising. However we should keep in mind that the holographic theories we are trying to understand all arise from orbifolding of the five sphere and the total moduli available for the renormalization group flows should not increase by the orbifolding procedure. The scalar manifold of the $\mathcal{N} = 8$ supergravity in $d = 5$ is $E_6(6)/USp(8)$ corresponding to 42 scalars. The $S^5$ compactification of IIB superstring leads to gauged $\mathcal{N} = 8$ supergravity with the gauge group $SU(4)$ and an additional $SU(1,1)$ symmetry [1]. Gauge fixing of all the symmetries of this theory leaves us with 42-18=24 scalar fields that could drive renormalization group flows.
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