A sharp isoperimetric inequality in metric measure spaces
with non-negative Ricci curvature

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November 5, 2021

Abstract

We prove a sharp dimension-free isoperimetric inequality, involving the
volume entropy, in non-compact metric measure spaces with non-negative
synthetic Ricci curvature.

Keywords: isoperimetric inequality, curvature-dimension condition, metric mea-
sure space, optimal transport, volume entropy.

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1 Introduction

The aim of this note is to present a sharp isoperimetric inequality for a class of metric
measure spaces with non-negative Ricci curvature in the sense of Lott-Sturm-Villani.

Let \((X, d, m)\) be a metric measure space. Recall that the Minkowski content of
a Borel set \(Z \subset X\) with \(m(Z) < +\infty\) is defined by

\[
m^{+}(Z) := \liminf_{\epsilon \to 0} \frac{m(Z^\epsilon) - m(Z)}{\epsilon}
\]

where \(Z^\epsilon \subset X\) is the \(\epsilon\)-neighbourhood of \(Z\) defined by
\(Z^\epsilon := \{x : d(x, Z) < \epsilon\}\).

Isoperimetric inequalities for a class \(M\) of metric measure spaces relate the size of
the boundary of sets to their measure. Precisely, there is a function \(I_M(v)\), called
isoperimetric profile, such that

\[
m^{+}(\Omega) \geq I_M(v)
\]

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for all \((X, d, m) \in \mathcal{M}\) and \(\Omega \subset X\) with \(m(\Omega) = v\).

Recently, isoperimetric inequalities in the class of non-compact metric measure spaces with synthetic non-negative Ricci curvature is studied in [1] and [3]. In these isoperimetric inequalities, a key number in the isoperimetric profile is a constant called asymptotic volume ratio of a metric measure space \((X, d, m)\), defined as

\[
\text{AVR}(X, d, m) := \lim_{r \to +\infty} \frac{m(B_r(x_0))}{r^N} \in [0, +\infty].
\]

When \(\text{AVR}(X, d, m) > 0\), we call that the space \((X, d, m)\) has Euclidean volume growth.

Since the dimension upper bound \(N\) appear in the asymptotic volume ratio, the isoperimetric inequalities obtained in [1] and [3] are all dimension-dependent. So it is natural to find a dimension-free version of the isoperimetric inequality for metric measure spaces with non-negative Ricci curvature. Firstly, recall that a space has non-negative Ricci curvature (without dimension restriction) in the sense of Lott-Sturm-Villani means:

**Definition 1.1** (Lott-Sturm-Villani’s curvature-dimension condition, cf. [5, 7]). We say that a metric measure space \((X, d, m)\) has non-negative Ricci curvature, or satisfies CD\((0, \infty)\) condition, if the entropy functional \(\text{Ent}_m\) is displacement convex on the \(L^2\)-Wasserstein space \((\mathcal{P}_2(X), W_2)\). This means, for any two probability measures \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\) with \(\mu_0, \mu_1 \ll m\), there is a \(L^2\)-Wasserstein geodesic \((\mu_t)_{t \in [0,1]}\) such that

\[
\text{Ent}_m(\mu_t) \leq t\text{Ent}_m(\mu_1) + (1 - t)\text{Ent}_m(\mu_0)
\]

where \(\text{Ent}_m(\mu_t)\) is defined as \(\int \rho_t \ln \rho_t \, dm\) if \(\mu_t = \rho_t \, m\), otherwise \(\text{Ent}_m(\mu_t) = +\infty\).

In the study of metric (Riemannian) geometry, there are some important spaces where the asymptotic volume ratio \(\text{AVR} = +\infty\). In these cases, we often consider instead the volume entropy, which is an important concept in both Riemannian geometry (cf. [2]) and dynamical system (cf. [6]).

**Definition 1.2** (Volume entropy). A metric measure space \((X, d, m)\) admits the volume entropy at \(x_0 \in X\), denoted by \(h(X, d, m)(x_0)\), provided

\[
h(X, d, m)(x_0) := \lim_{r \to +\infty} \frac{\ln m(B_r(x_0))}{r} \in [0, \infty].
\]

It is not hard to prove that a metric measure space with non-negative Ricci curvature in the sense of Definition 1.1, surely admits the volume entropy which is independent of the choice of \(x_0\).

The main result of this note is the following sharp isoperimetric inequality involving volume entropy:

**Theorem 1.3** (Sharp isoperimetric inequality). Let \((X, d, m)\) be a metric measure space with non-negative synthetic Ricci curvature. Then for any \(\Omega \subset X\) with finite measure, it holds the following isoperimetric inequality

\[
m^+(\Omega) \geq m(\Omega)h(X, d, m).
\]

Moreover, the constant \(h(X, d, m)\) in (1.2) can not be replaced by any larger ones.
2 Main results

In the following two theorems we will prove a sharp isoperimetric inequality for the class of metric measure spaces with non-negative synthetic Ricci curvature.

**Theorem 2.1** (Dimension-free isoperimetric inequality). Let \((X, d, m)\) be a metric measure space with non-negative Ricci curvature. Then for any \(\Omega \subset X\) with finite measure, it holds the following isoperimetric inequality

\[
m^+(\Omega) \geq m(\Omega) h(X, d, m).
\]  

(2.1)

**Proof.** Given \(x_0 \in X\) and \(R > 0\). Define \(\mu_0 = \frac{1}{m(\Omega)}m|_\Omega\) and \(\mu_1 = \frac{1}{m(B_R(x_0))}m|_{B_R(x_0)}\). By 1.1 there is an \(L^2\)-Wasserstein geodesic \((\mu_t)\) connecting \(\mu_0, \mu_1\) such that

\[
\text{Ent}_m(\mu_t) \leq t\text{Ent}_m(\mu_1) + (1 - t)\text{Ent}_m(\mu_0) \tag{2.2}
\]

Denote the support of \(\mu_t\) by \(\text{supp} \mu_t\) and the set of \(t\)-intermediate points

\[
Z_t := \{ z : \exists x \in \Omega, y \in B_R(x_0), \text{such that } \frac{d(z, x)}{t} = \frac{d(z, y)}{1 - t} = d(x, y) \}\).

It can be seen that \(\text{supp} \mu_t \subset Z_t\). Then by (2.2) and Jensen's inequality, we have

\[
-\ln \left( m(Z_t) \right) \leq -\ln \left( m(\text{supp} \mu_t) \right) \leq -t \ln \left( m(B_R(x_0)) \right) - (1 - t) \ln \left( m(\Omega) \right) \tag{2.3}
\]

Let \(\epsilon := t(d(\Omega) + R)\). It is not hard to see that \(Z_t \subset \Omega^c\). If \(m^+(\Omega) = +\infty\), there is nothing to prove. Otherwise, we have \(\lim_{\epsilon \to 0} m(\Omega^c) = m(\Omega)\). Then we have

\[
\frac{m^+(\Omega)}{m(\Omega)} = \lim_{\epsilon \to 0} \frac{1}{m(\Omega)} \frac{m(\Omega^c) - m(\Omega)}{\epsilon} \geq \lim_{t \to 0} \frac{\ln \left( m(Z_t) \right) - \ln \left( m(\Omega) \right)}{t(d(\Omega) + R)} \geq \lim_{t \to 0} \frac{\ln \left( m(B_R(x_0)) \right) + (1 - t) \ln \left( m(\Omega) \right) - \ln \left( m(\Omega) \right)}{t(d(\Omega) + R)} \geq \lim_{t \to 0} \frac{\ln \left( m(B_R(x_0)) \right) - \ln \left( m(\Omega) \right)}{d(\Omega) + R}.
\]

By (2.3)

\[
\frac{m^+(\Omega)}{m(\Omega)} \geq \frac{\ln \left( m(B_R(x_0)) \right) - \ln \left( m(\Omega) \right)}{d(\Omega) + R}.
\]

Letting \(R \to \infty\), by Definition 1.2, we get

\[
\frac{m^+(\Omega)}{m(\Omega)} \geq h(X, d, m)
\]

which is the thesis.

**Theorem 2.2** (Sharp inequality). The inequality (2.1) in Theorem 2.1 is sharp. This means, for any \(C > h(X, d, m)\), the inequality \(m^+(\Omega) \geq Cm(\Omega)\) does not hold for any Borel set \(\Omega \subset X\).
Proof. We will prove the theorem by contradiction. Assume there is a constant \( C > h(X, d, m) \), such that

\[
m^+(\Omega) \geq C m(\Omega)
\]

(2.4)

for any Borel set \( \Omega \subset X \).

Apply (2.3) with \( \Omega = B_{r+\delta}(x_0) \) and \( R = \epsilon \). We get the following interpolation inequality

\[
\ln \left( m(Z_t) \right) \geq t \ln \left( m(B_{\epsilon}(x_0)) \right) + (1 - t) \ln \left( m(B_{r+\delta}(x_0)) \right) \quad \forall t \in [0, 1].
\]

(2.5)

For any \( z \in Z_t \), we have \( d(z, x_0) \leq (1 - t)(r + \delta) + \epsilon \). Choosing \( t = \frac{\delta + \epsilon}{r + \delta} \), we have \( Z_t \subset B_r(x_0) \). Thus (2.5) implies

\[
\ln \left( m(B_r(x_0)) \right) \geq \frac{\delta + \epsilon}{r + \delta} \ln \left( m(B_{\epsilon}(x_0)) \right) + \frac{r - \epsilon}{r + \delta} \ln \left( m(B_{r+\delta}(x_0)) \right).
\]

Then

\[
\frac{r - \epsilon}{r + \delta} \left( \frac{\ln \left( m(B_{r+\delta}(x_0)) \right) - \ln \left( m(B_r(x_0)) \right)}{\delta} \right) \leq \frac{\delta + \epsilon}{\delta(r + \delta)} \left( \ln \left( m(B_{r}(x_0)) \right) - \ln \left( m(B_{\epsilon}(x_0)) \right) \right).
\]

From the proof of Theorem 2.1, we can see that \( \frac{\ln \left( m(B_{r+\delta}(x_0)) \right) - \ln \left( m(B_r(x_0)) \right)}{\delta} \geq C \), so

\[
\frac{r - \epsilon}{r + \delta} C \leq \frac{\delta + \epsilon}{\delta(r + \delta)} \left( \ln \left( m(B_{r}(x_0)) \right) - \ln \left( m(B_{\epsilon}(x_0)) \right) \right).
\]

Letting \( r \to \infty \), we get

\[
C \leq \frac{\delta + \epsilon}{\delta} h(X, d, m).
\]

Letting \( \epsilon \to 0 \) we get the contradiction. \( \square \)

For metric measure spaces satisfying a generalized Bishop-Gromov inequality, such as CD(0, N) spaces or MCP(0, N) spaces with \( N < +\infty \), whose volume entropy vanish, it is known that there is no isometric inequality in the form of (2.1) with a positive \( C \). More generally, as a direct consequence of Theorem 2.1 and Theorem 2.2, we have the following result.

**Corollary 2.3.** Let \( (X, d, m) \) be a metric measure space with non-negative Ricci curvature \( h(X, d, m) = 0 \), then there is no isoperimetric inequality in the form of

\[
m^+(\Omega) \geq C m(\Omega) \quad \forall \Omega \subset X
\]

from some \( C > 0 \).
Example 2.4. Consider the 1-dimensional metric measure space \((\mathbb{R}, |\cdot|, e^t \mathcal{L}^1)\). It can be seen that this is a RCD\((0,\infty)\) space and its volume entropy is 1. Take \(\Omega = (-\infty, 0]\), we have \((e^t \mathcal{L}^1)(\Omega) = 1\) and \((e^t \mathcal{L}^1)^+(\Omega) = 1\), so that the equality holds in (2.1).

Inspired by the example above, we propose the following conjecture, which will be studied in a forthcoming paper.

**Conjecture 2.5 (Rigidity).** Let \((X, d, m)\) be a RCD\((0,\infty)\) metric measure space with \(h(X, d, m) > 0\). Assume there is \(\Omega \subset X\) with \(m(\Omega) < +\infty\) such that

\[m^+(\Omega) = h(X, d, m)m(\Omega) > 0.\]

Then

\[(X, d, m) \cong \left(\mathbb{R}, |\cdot|, e^t dt\right) \times (Y, d_Y, m_Y)\]

for some RCD\((0,\infty)\) space \((Y, d_Y, m_Y)\) with \(m_Y(Y) < +\infty\), and up to change of variables

\[\Omega = (-\infty, e] \times Y \subset \mathbb{R} \times Y\]

where \(e \in \mathbb{R}\) satisfies \(\int_{-\infty}^{-e} e^s ds = m(\Omega)\).

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