The symplectic analysis for the four dimensional Pontryagin and Euler invariants is performed within the Faddeev-Jackiw context. The Faddeev-Jackiw constraints and the generalized Faddeev-Jackiw brackets are reported; we show that in spite of the Pontryagin and Euler classes give rise the same equations of motion, their respective symplectic structures are different to each other. In addition, a quantum state that solves the Faddeev-Jackiw constraints is found, and we show that the quantum states for these invariants are different to each other. Finally, we present some remarks and conclusions.

PACS numbers: 98.80.-k,98.80.Cq

I. INTRODUCTION

Nowadays, the study of topological theories is an interesting topic to perform. In fact, the relevance for studying topological theories has been motived in several contexts of theoretical physics because of they provide an interesting relation between mathematics and physics, just like that existing between geometry and General Relativity [GR]. From the classical point of view, topological theories are devoid of physical degrees of freedom, background independent and diffeomorphisms covariant, because of these symmetries, the topological theories are considered as good laboratories for testing ideas about the construction of a background independent quantum theory, and these ideas could be applied for the construction of a desired quantum version of GR [1]. From a global point of view, topology and quantum mechanics has an interesting overlap just like that discovered by E. Witten in [2] and extended by M. Atiyah in [3], where concepts of geometry, supersymmetry and quantum field theory where unified, giving origin to the so-called topological quantum field theory [4]. Moreover, we find in the literature that the topological theories are also important in the canonical approach of GR; In fact, when GR is considered with the addition of topological terms, namely the Pontryagin, Euler and Nieh Yan invariants, it is well-known that these topological invariants have no effect on the equations of motion of gravity, however, they give an important contribution in the symplectic structure of the theory [5]. Within the classical field theory context, either the Euler or Pontryagin classes are fundamental blocks for constructing the noncommutative form of topolog-
ical gravity. Furthermore, these topological invariants have been studied in several works due to they are expected to be related to physical observables, as for instance, in the case of anomalies. It is important to comment, that the topological invariants cited above are not the only ones with an interesting relation with physics. In fact, there are also the so-called BF theories.

In general, a BF theory is a topological theory, it is diffeomorphisms covariant, and if some extra constraints are imposed on the $B$ field, then the topological structure of the theory is broken down and theories just as complex GR, real GR or a Yang-Mills theory arise in a natural form. In addition, the BF formulations of gravity are interesting for the community, because they have allowed us to understand the spin foam formulation developed in the Loop Quantum Gravity (LQG) program. In this respect, both the Euler and Pontryagin invariants can be written as a BF-like theory, and this important fact will be used along this paper. On the other hand, the Euler and Pontryagin invariants are fundamental in the characterization of the topological structure of a manifold. In fact, they label topologically distinct four-geometries; the Pontryagin invariant gives the relation between the number of selfdual and anti-selfdual harmonic connections on the manifold. The Euler invariant, on the contrary, gives a relation between the number of harmonic $p$-forms on the manifold.

From the Hamiltonian point of view, the Euler and Pontryagin invariants treated as field theories give rise the same equations of motion, are devoid of physical degrees of freedom, background independent, diffeomorphisms covariant and there exist reducibility conditions between the constraints. Because of these symmetries, either the Pontryagin or Euler invariants are good toy models for studying the classical and quantum structure of a background independent theory.

With these antecedents, the purpose of this paper is to develop the symplectic analysis of the Euler and Pontryagin invariants. As far as we know the symplectic analysis of these invariants has not been carried out. In this respect, we have commented previously that these invariants has been analyzed only within the Dirac context; however, in these works the complete structure of the constraints and the Dirac brackets, useful for the quantization of the theory, were not constructed.

In this manner, in order to know a complete canonical description of the theories under study, we apply the symplectic formulation of Faddeev-Jackiw (FJ) which is a powerful alternative framework for studying singular systems. In fact, the FJ method is a symplectic approach where all relevant information of the theory can be obtained through an invertible symplectic matrix, which is constructed by means of the symplectic variables that are identified as the dynamical variables from the Lagrangian of the theory. Since the theory is singular, there will be constraints and FJ scheme has the advantage that all constraints are considered at the same footing. In fact, in FJ method it is not necessary to perform the classification of the constraints in primary, secondary, first class or second class, as it is done in Dirac's method. Furthermore, from the components of the symplectic tensor it is possible to identify the FJ generalized brackets; one goal of this approach is that, at the end of the calculations, the Dirac and the FJ generalized brackets are equivalents. In this manner, the cornerstone of this paper is to develop the symplectic analysis of the Pontryagin and Euler invariants. In our results we will find that in spite of the Pontryagin and Euler classes giving rise the
same equations of motion, their respective symplectic structures are different and this fact will be important in the quantization. In fact, once we have found the complete set of FJ constraints, we will find a quantum state that solves the constraints and we will see that the quantum states for these invariants are different to each other.

The paper is organized as follows. In Section I, the symplectic formalism for the Pontryagin invariant is performed. For our study, we will write the invariant as a $BF$-like theory, this fact is necessary because the FJ formalism is applicable to linear Lagrangians in the velocities. We report the complete set of FJ constraints and we will report the reducibility conditions among these constraints. Then, a symplectic tensor is constructed and the FJ generalized brackets are identified. With these results at hand, we will find a quantum state that solves the quantum FJ constraints. In Section II the symplectic analysis for the Euler invariant is carried out. For our analysis, we will use the same symplectic variables than those used in the Pontryagin invariant and we will show that despite this fact the symplectic structures are different to each other. Because of the symplectic structures are different, we will find a different quantum state that solves exactly the Euler’s constraints. Finally, we add some remarks and conclusions.

II. SYMPELTIC FORMALISM FOR THE PONTRYAGIN INVARIANT

The four-dimensional Pontryagin invariant is described by the action

$$S[A_{\mu}^{IJ}] = \Xi \int_{\mathcal{M}} \left[ F^{IJ} \wedge F_{IJ} \right],$$

(1)

here, $\Xi$ is a constant, $\mathcal{M}$ is the space-time manifold and $F$ is straight field of the Lorentz connection $A_{\beta}^{IJ}$ given by $F^{IJ}_{\alpha\beta} = \partial_{\alpha} A_{\beta}^{IJ} - \partial_{\beta} A_{\alpha}^{IJ} + A_{\alpha}^{I} K A_{K}^{J} K - A_{\alpha}^{J} K A_{K}^{I} K$, the capital letters are internal $SO(3,1)$ Lorentz indices and run from $I, J, K = 0, 1, 2, 3$ that can be raised and lowered by the internal metric $\eta^{IJ} = (-1, 1, 1, 1)$, and $\alpha, \beta, \mu = 0, 1, 2, 3$ are space-time indices.

We introduce auxiliary fields, namely $B^{IJ}$, corresponding to a set of six two-forms; thus, the action (1) takes a different fashion, a $BF$-like theory

$$S[A_{\mu}^{IJ}, B^{KL}] = \Xi \int_{\mathcal{M}} \left[ F^{IJ} \wedge B_{IJ} - \frac{1}{2} B^{IJ} \wedge B_{IJ} \right].$$

(2)

We can see that the action (1) and (2) are the same modulo equations of motion. On the other hand, with the introduction of the $B'$s variables, the action (2) is now linear in the velocities, then the FJ formalism can be carried out \[22\]. Furthermore, we will work with real variables without involve either self-dual or anti-self-dual variables; it is easy to observe that in the self-dual (anti-self-dual) scenario, the actions are reduced to the Pontryagin characteristic based on the self-dual (anti-self-dual) connection and this case is trivial.

By performing the 3+1 decomposition and breaking down the Lorentz covariance we obtain the
from the symplectic Lagrangian (6) we identify the following symplectic variables \(a, b, c\) here lowered with the Euclidean metric.

where the Lagrangian density takes the following form

\[
L = \int \mathcal{A}^{abc} \left[ B_{ij}^0 \dot{A}_{aij} + \frac{1}{2} B_{ij}^0 B_{kl}^0 \dot{A}_{ijkl} + \frac{1}{2} A_{ijkl} \left( \partial_i B_{jk}^0 \dot{A}_{ab}^j + 2 B_{kl}^0 B_{bc}^j \dot{A}_{aik}^j + 2 B_{ij}^0 B_{a0}^j \right) + A_{00i} \left( \partial_i B_{0j}^0 + B_{ij}^0 A_{00j} + B_{ij}^0 A_{0ij} \right) + B_{ij}^0 \partial_0 A_{0ij} - \partial_i A_{00j} + A_{0ij} A_{b}^i + A_{0ij} A_{0j}^i - A_{a0}^i \right] + \frac{1}{2} B_{ij}^0 \partial_0 A_{00j} - \frac{1}{2} B_{ij}^0 B_{00j} - \frac{1}{2} B_{ij}^0 B_{ij}^0 B_{0i}^0 \right] d^3x, \tag{3}
\]

here \(a, b, c = 1, 2, 3\), \(\epsilon^{abc} \equiv \eta^{abc}\) and \(i, j, k = 1, 2, 3\) are the internal indices that can be raised or lowered with the Euclidean metric \(\eta^{ij} = (1, 1, 1)\). By introducing the following variables

\[
A_{aij} \equiv -\epsilon_{ijk} A_{a}^i, \\
A_{0ij} \equiv -\epsilon_{ijk} A_{0}^i, \\
B_{abi} \equiv -\epsilon_{ijk} B_{ab}^k, \\
B_{b0i} \equiv -\epsilon_{ijk} B_{b0}^k, \\
A_{ai} \equiv \dot{Y}_{ai},
\]

the Lagrangian density takes the following form

\[
\mathcal{L} = \Xi_{abc} \left[ B_{ab}^0 \dot{A}_{cd0i} + B_{abi} \dot{Y}_{c}^i - \mathcal{V}^{(0)} \right],
\]

where \(\mathcal{V}^{(0)}\) corresponds to the symplectic potential expressed by

\[
\mathcal{V}^{(0)} = -A_{a0i} \left[ \partial_i \left( \Xi_{0a} \dot{B}_{abc} B_{ab}^c \right) + \Xi_{abc} \epsilon_{ijk} B_{ab}^j \dot{Y}_{c}^k - \Xi_{abc} \epsilon_{ijk} B_{ab}^{0j} A_{c0}^k \right] - A_{b0i} \left[ \partial_i \left( \Xi_{0b} \dot{B}_{abc} B_{ab}^c \right) - \Xi_{abc} \epsilon_{ijk} B_{ab}^{0j} \dot{Y}_{c}^k - \Xi_{abc} \epsilon_{ijk} B_{ab}^{0j} A_{c0}^k \right] - \Xi_{abc} B_{00i} \left[ \partial_i A_{00i} - \partial_i A_{00j} + \epsilon_i^{jk} A_{00j} \dot{Y}_{0k} - \epsilon_{ijk} A_{00j} \dot{Y}_{0k} \right] - \Xi_{abc} B_{0ai} \left[ \partial_i \dot{Y}_{c}^i - \partial_i \dot{Y}_{a}^i + \epsilon_i^{jk} \dot{Y}_{b}^j \dot{Y}_{c}^k - \epsilon_{ijk} A_{00j} \dot{Y}_{0k} \right] + \Xi_{abc} B_{0ai} \left[ B_{0ai} B_{c0i} + B_{bc} \dot{B}_{0ai} \right].
\]

From the symplectic Lagrangian (6) we identify the following symplectic variables

\[
\dot{\xi}^{(0)} = (A_{00i}, B_{ab}^0, \dot{Y}_a^i, B_{abi}, A_{i}^0, A_{00i}, B_{a0}^0, B_{0ai}),
\]
and the following 1-forms

\[ a^{(0)} = (\Xi \eta^{abc} B_{ab}^{0i}, 0, \Xi \eta^{abc} B_{ab0}, 0, 0, 0, 0). \]  

(9)

In this manner, the symplectic matrix defined as \( f_{ij}(x, y) = \frac{\delta a_j(y)}{\delta \xi^i(x)} - \frac{\delta a_i(x)}{\delta \xi^j(y)} \), is given by

\[
 f_{ij}^{(0)} = \begin{pmatrix}
 0 & \Xi \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \Xi \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -\Xi \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \Xi \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \end{pmatrix} \delta^3(x - y).
\]

(10)

We can observe that \( f_{ij}^{(0)} \) is singular, and therefore, there are constraints. In order to identify the constraints, we calculate the zero-modes of \( f_{ij}^{(0)} \) and they are given by the following 4 vectors

\[
 \nu_{1}^{(0)} = (0, 0, 0, 0, V A^a, 0, 0, 0),
\]

(11)

\[
 \nu_{2}^{(0)} = (0, 0, 0, 0, 0, V_{A^a0}, 0, 0),
\]

(12)

\[
 \nu_{3}^{(0)} = (0, 0, 0, 0, 0, 0, V_{B^a0}, 0),
\]

(13)

\[
 \nu_{4}^{(0)} = (0, 0, 0, 0, 0, 0, 0, V_{B^a0}),
\]

(14)

where \( V A^a, V_{A^a0}, V_{B^a0} \) and \( V_{B^a0} \) are arbitrary functions. Hence, by using these modes we find the following FJ constraints

\[
 \Omega_{1}^{(0)} = \int d^3x \frac{i}{\delta \xi^{(0)}} \int d^3y \nu^{(0)}(\xi) = \int d^3x V^{A^a} \frac{\delta}{\delta A^a} \int d^3y \nu^{(0)}(\xi)
\]

\[
 = \partial_x (\Xi \eta^{abc} B_{ab}^{0i} + \Xi \eta^{abc} \epsilon^{ijk} B_{ab0j} \eta^{k} - \Xi \eta^{abc} \epsilon^{ijk} B_{ab0j}) A_c^{0k},
\]

(15)

\[
 \Omega_{00i}^{(0)} = \int d^3x \frac{i}{\delta \xi^{(0)}} \int d^3y \nu^{(0)}(\xi) = \int d^3x V^{A^a0} \frac{\delta}{\delta A^a0} \int d^3y \nu^{(0)}(\xi)
\]

\[
 = \partial_x (\Xi \eta^{abc} B_{ab0i} - \Xi \eta^{abc} \epsilon^{ijk} B_{ab0j} \eta^{k} - \Xi \eta^{abc} \epsilon^{ijk} B_{ab0j}) A_{a0}^{0i},
\]

(16)

\[
 \Omega_{0a0i}^{(0)} = \int d^3x \frac{i}{\delta \xi^{(0)}} \int d^3y \nu^{(0)}(\xi) = \int d^3x V^{B^a0} \frac{\delta}{\delta B^a0} \int d^3y \nu^{(0)}(\xi)
\]

\[
 = \Xi \eta^{abc} \left[ \partial_x A_{a0i} - \partial_x A_{a0i} + \epsilon^{ijk} A_{a0j} \gamma^{k} - \epsilon^{ijk} A_{a0j} \gamma^{k} \right] - \Xi \eta^{abc} B_{a0i},
\]

(17)

\[
 \Omega_{a0i}^{(0)} = \int d^3x \frac{i}{\delta \xi^{(0)}} \int d^3y \nu^{(0)}(\xi) = \int d^3x V^{B^a0} \frac{\delta}{\delta B^a0} \int d^3y \nu^{(0)}(\xi)
\]

\[
 = \Xi \eta^{abc} \left[ \partial_x \gamma_{a0}^{i} - \partial_x \gamma_{a0}^{i} + \epsilon^{ijk} \gamma_{a0j} \gamma^{k} - \epsilon^{ijk} \gamma_{a0j} \gamma^{k} \right] - \Xi \eta^{abc} B_{ab}^{i},
\]

(18)

from these constraints, we observe that there exist the following 6 reducibility conditions

\[
 \partial_x \Omega_{a0}^{(0)} = \epsilon^{ijk} A_{a0k} \Omega_{a0}^{(0)} a_0^i + \epsilon^{ijk} A_{a0k} \Omega_{a0}^{(0)} a_0^j + \frac{1}{2} \Omega_{a0}^{(0)} a_0^i,
\]

(19)

\[
 \partial_x \Omega_{a0}^{(0)} a_0^i = -\epsilon^{ijk} A_{a0k} \Omega_{a0}^{(0)} a_0^i + \epsilon^{ijk} A_{a0k} \Omega_{a0}^{(0)} a_0^j + \frac{1}{2} \Omega_{a0}^{(0)} a_0^i,
\]
and this fact will be considered in the counting of physical degrees of freedom. Now we shall observe if emerge more constraints, for this aim, we calculate the following system

$$\bar{f}_{ij} \hat{\xi}^{(0)j} = Z_i(\xi),$$

where

$$\bar{f}_{ij} = \left( \begin{array}{c} f^{(0)}_{ij} \\ \frac{\delta \xi^{(0)j}}{\delta \xi^{(0)i}} \end{array} \right) \quad \text{and} \quad Z_k = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right).$$

Thus, the symplectic matrix $\bar{f}_{ij}$ is given by

$$\bar{f}_{ij} = \begin{pmatrix} 0 & \Xi \eta^{abc} \delta^i_k & 0 \\
\Xi \eta^{abc} \delta^i_k & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
2\Xi \eta^{abc} \left( \delta^k_l \partial_c - \epsilon^{ijk} Y_{bk} \right) & -\Xi \eta^{abc} \delta^i_k & 2\Xi \eta^{abc} \epsilon_{ijk} A_{b0j} \\
-\Xi \eta^{abc} \left( \delta^k_l \partial_c - \epsilon^{ijk} Y_{bk} \right) & 0 & 2\Xi \eta^{abc} \epsilon_{ijk} A_{b0j} \\
0 & -\Xi \eta^{abc} \epsilon_{ijk} A_{b0j} & \delta^3(x - y) \end{pmatrix}.$$

The matrix $\bar{f}_{ij}$ is not a square matrix as expected, however it has null vectors. The null vectors are given by

$$\bar{v}_1 = (-\epsilon_{ijl} A_{ij} V^l, \epsilon_{ijl} B_{ij}^{0j} V^l, \partial_i V_i + \epsilon_{ijkl} Y^{i} V^j, -\epsilon_{ijl} B_{abj}^{0j} V^l, 0, 0, 0, 0, V^i, 0, 0, 0),$$
$$\bar{v}_2 = (\partial_i V_i + \epsilon_{ijl} Y^{i} V^j, -\epsilon_{ijl} B_{ij}^{0j} V^l, \partial_i V_i + \epsilon_{ijkl} Y^{i} V^j, -\epsilon_{ijl} B_{abj}^{0j} V^l, 0, 0, 0, 0, V^i, 0, 0, 0),$$
$$\bar{v}_3 = (V_i - 2(\partial_i V_i - \epsilon_{ijl} Y^{i} V^j), -2(\partial_i V_i + \epsilon_{ijkl} Y^{i} V^j), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\bar{v}_4 = (0, 2\epsilon_{ijl} A_{ijl} V^i, -V_i, -2(\partial_i V_i + \epsilon_{ijkl} Y^{i} V^j), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$
where $V^i$’s are arbitrary functions. On the other hand, $Z_k(\xi)$ is given by

$$Z_k(\xi) = \begin{pmatrix} \frac{\delta \gamma^{(1)}}{\delta \xi^{(1)}_{ij}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ (24)

$$= \begin{pmatrix} \Xi \eta^{abc} \epsilon^{j_i} A_0^j B_{ab}^{0j} + \Xi \eta^{abc} \epsilon^{ijk} A_{00i} B_{abj} + 2\Xi \eta^{abc} \partial_k B_{0a}^{0k} \\ -2\Xi \eta^{abc} \epsilon^{j_i} B_{0a}^{0i} \Omega^{(0)}_{ij} + 2\Xi \eta^{abc} \epsilon^{ijk} B_{0ai} A_{0bj} \\ \Xi \eta^{abc} \epsilon_{ikj} A_0^j A_0^k + \Xi \eta^{abc} \partial_k A_{00i} + \Xi \eta^{abc} \epsilon^{j_i} A_{00j} \Upsilon^{(0)} \xi^k \\ -\Xi \eta^{abc} \epsilon^{j_i} A_0^j B_{abj} + \Xi \eta^{abc} \epsilon^{ijk} A_{00i} B_{abj}^{0j} - 2\Xi \eta^{abc} \epsilon^{ijk} A_{00j} B_{0ai}^{0i} \\ -2\Xi \eta^{abc} \partial_k B_{0ak} - 2\epsilon_j^k \Upsilon^i B_{0ai} \\ \Xi \eta^{abc} \partial_k A_0^j - \Xi \eta^{abc} \epsilon^{ijk} A_0^j \Upsilon^{(0)} \xi^k + \Xi \eta^{abc} \epsilon^{jki} A_{00j} A_{c0k} \end{pmatrix} \Omega^{(0)}_{ij}$$ (25)

In this manner, the contraction of the null vectors with $Z_k$, namely, $\tilde{N}_k Z_k(\xi) = 0$, gives identities because the result is a linear combination of constraints. Hence, there are no more FJ constraints.

Furthermore, we will add the constraints given in (15) to the symplectic Lagrangian using the following Lagrange multipliers, namely, $A^i_0 = \dot{T}^i, A_{00i} = \dot{\Lambda}_i, B_{0a}^{0i} = \dot{\lambda}^i, B_{0ai} = \dot{\lambda}_{0i},$ thus the symplectic Lagrangian reads

$$\mathcal{L}^{(1)} = \Xi \eta^{abc} B_{ab}^{0i} \dot{A}_{c0i} + \Xi \eta^{abc} B_{abj}^{0j} \dot{T}^i - \dot{T}^i \Omega^{(0)}_{ij} - \dot{\Lambda}_i \Omega^{(0)}_{00i} - \frac{\dot{\lambda}^i}{2} \Omega^{(0)}_{0ai} - \frac{\dot{\lambda}_{0i}}{2} \Omega^{(0)}_{0ai} - \gamma^{(1)}$$ (26)
where $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}(\Omega_0, \Omega_0^{a0}, \Omega_0^{ac}, \Omega_0^{0a}, \Omega_0^{00}) = 0$. This result is expected because of the general covariance of the theory such as it is present in GR.

From the symplectic Lagrangian (26) we identify the following symplectic variables

$$\xi^{(0)} = (A_{a0i}, B_{ab} \delta^i_0, \gamma_i^a, B_{abi}, A_0^i, A_{0ai}, B_{0ai}, T^i, \Lambda_i, \chi_i^a, \chi_{ai}), \quad (27)$$

and the 1-forms

$$a^{(0)} = \left( \Xi^{abc} B_{ab} \delta^i_0, 0, \Xi^{abc} B_{abi}, 0, -\Omega^{(0)}_i \delta^i_0, -\Omega^{(0)}_i 00i, -\frac{\Omega^{(0)}_i \delta^i_0}{2}, -\frac{\Omega^{(0)}_i 0ai}{2} \right). \quad (28)$$

Hence, the symplectic matrix has the following form

$$f_{ij}^{(1)} = \left( \begin{array}{cccccccc} 0 & \Xi^{abc} \delta^i_k & 0 & 0 \\ \Xi^{abc} \delta^i_k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Xi^{abc} \delta^i_k \\ -\Xi^{abc} \epsilon^j_k B_{ab} \delta^{0j} & -\Xi^{abc} \epsilon^j_k A_{00j} & -\Xi^{abc} \delta^i_k & 0 \\ -\Xi^{abc} \epsilon^j_k A_{00j} & -\Xi^{abc} \delta^i_k & 0 & 0 \\ -\Xi^{abc} \epsilon^j_k A_{abi} & -\Xi^{abc} \delta^i_k & 0 & 0 \\ -\Xi^{abc} \epsilon^j_k A_{abi} & -\Xi^{abc} \delta^i_k & 0 & 0 \\ -\Xi^{abc} \epsilon^j_k A_{abi} & -\Xi^{abc} \delta^i_k & 0 & 0 \end{array} \right) \delta^3(x - y), \quad (29)$$

where the notation $D^i_{kc} = \delta^i_k \partial_c + \epsilon^i_k \gamma_c$ and $d^i_{kc} = \delta^i_k \partial_c - \epsilon^i_k \gamma_c$ was introduced. We can observe that this symplectic matrix is still singular, however we have showed that there are not more constraints, therefore this theory is a gauge theory. In order to obtain a symplectic tensor, we need to fixing the gauge, we will use the following temporal gauge

$$A_{0i} = 0, \quad (30)$$

$$A_{00i} = 0,$$

$$B_{0a} \delta^i_0 = 0,$$

$$B_{0ai} = 0,$$

this means that $\hat{T}^i = 0, \hat{\Lambda}_i = 0, \hat{\zeta}_a^i = 0$ and $\hat{\chi}_{ai} = 0$. In this manner, we introduce more Lagrange multipliers enforcing the gauge fixing as constraints. The Lagrange multipliers introduced
are $\beta_i, \alpha^i, \rho_i^a, \sigma^a_i$ thus, the symplectic Lagrangian reads

$$L^{(2)} = \Xi \eta^{abc} B_{ab}^{0i} \dot{A}_{a0i} + \Xi \eta^{abc} B_{abi} \dot{Y}_c^i - \dot{T}^i \left[ \Omega^{(0)}_{ai} - \beta_i \right] - \dot{\Lambda}_i \left[ \Omega^{(0)}_{0ai} - \alpha^i \right]$$

$$- \dot{\chi}_a \left[ \frac{\Omega^{(0)}_{0ai}}{2} - \rho_i^a \right] - \dot{\lambda}_{ai} \left[ \frac{\Omega^{(0)}_{0ai}}{2} - \sigma^a_i \right].$$

(31)

From the symplectic Lagrangian (31) we identify the following symplectic variables

$$\xi^{(0)} = (A_{a0i}, B_{ab}^{0i}, Y_a^i, B_{abi}, A_{a0i}, B_{abi}, B_{0ai}, T^i, A_i, \zeta_a^i, \chi_{ai}, \beta_i, \alpha^i, \rho_i^a, \sigma^a_i),$$

(32)

and the 1-forms

$$\alpha^{(0)} = \left( \Xi \eta^{abc} B_{ab}^{0i} 0, 0, \Xi \eta^{abc} B_{abi}, 0, - \left[ \Omega^{(0)}_{ai} - \beta_i \right], - \left[ \Omega^{(0)}_{0ai} - \alpha^i \right] \right),$$

(33)

$$- \left[ \frac{\Omega^{(0)}_{0ai}}{2} - \rho_i^a \right], - \left[ \frac{\Omega^{(0)}_{0ai}}{2} - \sigma^a_i \right] \right).$$

(34)

Thus, the symplectic matrix is given by

$$f^{(2)}_{ij} = \begin{pmatrix}
0 & \Xi \eta^{abc} \delta^i_k & 0 & 0 & \Xi \eta^{abc} \epsilon^i_k B_{ab}^{0j} & \Xi \eta^{abc} \epsilon^j_k A_{0ij} \\
\Xi \eta^{abc} \delta^i_k & 0 & 0 & 0 & \Xi \eta^{abc} \delta^j_i & \Xi \eta^{abc} \epsilon^i_k B_{ab}^{0j} \\
0 & 0 & 0 & \Xi \eta^{abc} \delta^i_k & 0 & -\Xi \eta^{abc} \delta^j_i \\
0 & 0 & \Xi \eta^{abc} \delta^i_k & 0 & \Xi \eta^{abc} \delta^j_i & 0 \\
-\Xi \eta^{abc} \epsilon^i_k B_{ab}^{0j} & \Xi \eta^{abc} \delta^i_k & 0 & 0 & 0 & 0 \\
-\Xi \eta^{abc} \epsilon^j_k A_{0ij} & \Xi \eta^{abc} \delta^i_k & 0 & 0 & 0 & 0 \\
\Xi \eta^{abc} \delta^i_k & 0 & \Xi \eta^{abc} \delta^j_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \delta^i(x - y).$$

(35)
We can observe that this matrix is not singular, after a long calculation, the inverse of \( f_{ij}^{(2)} \) is given by

\[
\begin{pmatrix}
0 & \frac{1}{2} \eta_{ab} \delta^k_l & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} \eta_{ab} \delta^k_l & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} \eta_{ab} \delta^k_l & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} \eta_{ab} \delta^k_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[ f_{ij}^{(2)} \] is given by

\[
\begin{pmatrix}
\epsilon_{ij}^m A_{g0m} & -\epsilon_{ij}^m B_{de}^{0m} & -D_{gl}^j & \epsilon_{ij}^m B_{bgm} & -\delta^k_l & 0 & 0 & 0 \\
-\delta^k_l & 0 & 0 & 0 & 0 & \delta^k_l & 0 & 0 \\
0 & -\delta^k_l & 0 & 0 & 0 & 0 & \delta^k_l & 0 \\
0 & 0 & -\delta^k_l & 0 & 0 & 0 & 0 & \delta^k_l \\
0 & 0 & 0 & -\delta^k_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta^k_l & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta^k_l & 0 & 0 \\
\end{pmatrix}
\]

where the following definitions have been used

\[
C_{kl} = \sum \eta^{abc} \left( \epsilon_{jik} d^i_c B_{ab}^{0j} - \epsilon_{ij}^l D_{kc}^{l} B_{ab} \right),
\]

\[
G_{kl} = \sum \eta^{abc} \left( -d^i_k B_{cil} + \epsilon_{ik}^j \epsilon_{ij}^m A_{g0j} A_{cm} - \frac{1}{2} \epsilon_{ij}^k B_{ab} \right),
\]

\[
E_{kl} = \sum \eta^{abc} \left( -\epsilon_{ij}^k A_{g0j} D_{ci}^i + \epsilon_{ij}^m A_{cm} d_{ikc} - \frac{1}{2} \epsilon_{ij}^k B_{ab} \right),
\]

\[
J_{kl} = \frac{\eta^{abc} \epsilon_{jkl} B_{ab}^{0j}}{2},
\]

\[
I_{kl} = \sum \eta^{abc} \left( d_{kcd} B_{cd} - \epsilon_{ij}^k d^i_k A_{0j} + \frac{1}{2} \epsilon_{ij}^k B_{ab} \right).
\]

Therefore, from the symplectic tensor \( \Xi \) we can identify the generalized FJ brackets by means of

\[
\{ \xi_i^{(2)} (x), \xi_j^{(2)} (y) \} = \left[ f_{ij}^{(2)} (x, y) \right]^{-1},
\]
thus, the following generalized brackets arise

$$\{B_{a0i}(x), A_{d0l}(y)\}_{FJ} = \frac{1}{2\Xi} \eta_{abcd} \eta_{d1} \delta^3(x - y),$$  \hspace{1cm} (43) $$

$$\{B_{ab}(x), \Upsilon_{dl}(y)\}_{FJ} = -\frac{1}{2\Xi} \eta_{abcd} \eta_{d1} \delta^3(x - y),$$  \hspace{1cm} (44) $$

where we can observe that the FJ brackets depend on the parameter $\Xi$. It is important to remark, that these brackets were not reported in $[19]$, and these brackets will be useful in the quantization of theory. As it was commented previously, in the FJ formalism it is not necessary the classification of the constraints in first class, second class, etc, such as in Dirac’s method it is done; in FJ approach all constraints are treated at the same footing. In this manner, the counting of physical degrees of freedom is carried out as follows [DF=dynamical variables-independent constraints], thus, for the theory under study, there are 18 canonical variables given by $(A_{c0i}, \Upsilon^i_c)$ and 18 independent constraints $(\Omega^{(0)}_i, \Omega^{(0)}_{00i}, \Omega^{(0)}_{0ai}, \Omega^{(0)}_{00a, \delta i})$, then the theory is devoid of physical degrees of freedom. This result is expected because Pontryagin class is a topological theory. It is important to comment that all these results are not reported in the literature.

**A quantum state**

It is well-known from the quantum point of view, that the Dirac constraints of the Pontryagin class are solved by means the so-called Chern-Simons state. Hence, in this section we will solve the quantum FJ constraints by using the results reported above. We will observe that in spite of either Pontryagin or Euler classes sharing the same classical equations of motion, their respective quantum states will be different to each other. In order to prove this claim we need to rewrite the Chen-Simons action given by

$$S[A] = \frac{\Xi}{2} \left[ \int A^{IJ} \wedge A_{IJ} + \frac{2}{3} A^{IK} \wedge A_{KL} \wedge A_{IL} \right],$$  \hspace{1cm} (45) $$

in terms of the variables $[\mathcal{H}]$, hence, the action takes the following form

$$S[A_{a0i}^I, \Upsilon_a^i] = \int \left\{ \Xi \eta^{abc} [A_{a0i}^I \partial_b A_{c0i} + \Upsilon_a^k \partial_b \Upsilon_c^k - \Xi \epsilon_{ijk} \eta^{abc} A_{a0i}^I A_{c0j}^I \Upsilon_b^k + \frac{\Xi}{3} \epsilon_{ijk} \eta^{abc} \Upsilon_a^l \Upsilon_b^i \Upsilon_c^j] \right\} dx^3. \hspace{1cm} (46)$$

On the other hand, the generalized FJ brackets will be useful for the quantization. In fact, the dynamical variables will be promoved to operators and the brackets will be promoved to commutators. Hence, the generalized brackets are given by

$$\{B_{a0i}(x), A_{d0l}(y)\}_{FJ} = \frac{1}{2\Xi} \eta_{abcd} \eta_{d1} \delta^3(x - y),$$  \hspace{1cm} (47) $$

$$\{B_{ab}(x), \Upsilon_{dl}(y)\}_{FJ} = -\frac{1}{2\Xi} \eta_{abcd} \eta_{d1} \delta^3(x - y),$$  \hspace{1cm} (48) $$

its classical-quantum correspondence is given by

$$\{A_{a0i}(x), \Xi \eta^{abc} \tilde{B}_{ab0j}(y)\}_{FJ} = -\eta_{ij} \delta^3(x - y),$$  \hspace{1cm} (49) $$

$$\{\Upsilon_{dl}(x), \Xi \eta^{abc} \tilde{B}_{ab0j}(y)\}_{FJ} = \eta_{ij} \delta^3(x - y),$$  \hspace{1cm} (50) $$

therefore, we can identify the classical-quantum correspondence $\Xi \eta^{abc} \tilde{B}_{ab0j} \rightarrow -i \frac{\delta}{\delta A_{0j}}$ and $\Xi \eta^{abc} \tilde{B}_{abi} \rightarrow i \frac{\delta}{\delta A_{bi}}$. It is well-known, that in theories with a Hamiltonian described as a linear
combination of constraints as in our case, it is not possible to use the Schrödinger equation for quantization, because the action of the Hamiltonian on physical states is annihilation, in this manner, at quantum level we can not talk about the eigenstates of energy for the Hamiltonian \[19\]. In canonical (symplectic) quantization we have that the restriction of our physical states is archived by demanding that

\[
\{ \frac{i}{\delta A_\mu} \frac{\delta}{\delta A^\mu}, \frac{\delta}{\delta A_\mu} \right\} = 0,
\]

where the solution is given by

\[
\Psi_P(A_\mu^0, Y_a^i) = e^{-iS[A_\mu^0, Y_a^i],}
\]

here \(S[A_\mu^0, Y_a^i]\) is the action given in \[19\]. We can observe that the constraints are solved exactly and the Bianchi identities are not involved; this is a difference between our results and those reported in \[19\]. In this manner, by using the new variables, the Chern-Simons state is a quantum state of the Pontryagin class.

### III. SYMPLECTIC ANALYSIS FOR THE EULER INVARIANT

The Euler invariant is described by the following action expressed as a BF-like theory

\[
S[A^K_{IJ}, B^K_{\alpha j}] = \Omega \int_M \left[ *F_{IJ} \wedge B_{1J} - \frac{1}{2} *B_{1J} \wedge B_{1J} \right],
\]

where \(* = \epsilon^{KL}_{IJ}\) is the dual of \(SO(3, 1)\) and \(\Omega\) is a constant. Both actions \[2\] and \[53\] give rise the same equations of motion.

By performing the 3+1 decomposition, breaking down the Lorentz covariance and using the variables \[11\] we obtain the following Lagrangian density

\[
\mathcal{L} = -\Omega \eta^{abc} \dot{\gamma}_a^j B_{0jbc} + \Omega \eta^{abc} \dot{A}_a^0 B_{bc}^1 - A_0^i \left[ \partial_a \left( \Omega \eta^{abc} B_{b0ci} \right) + \Omega \eta^{abc} \epsilon^j_{in} Y_n^a B_{0jbc} - \Omega \eta^{abc} \epsilon_{kli} A_{a0}^i B_{bc}^k \right] - A_{00}^i \left[ \partial_a \Omega \eta^{abc} B_{bc}^1 \right] - \Omega \eta^{abc} \epsilon^i_{jn} Y_n^a B_{bc}^i + \Omega \eta^{abc} \epsilon^j_{ki} A_{a0}^i B_{bc}^k + \Omega \eta^{abc} B_{0a}^{ij} \left[ \partial_a Y_{ji} - \partial_b Y_{aj} - \epsilon_{jki} A_{a0}^j B_{0b0j} + \epsilon_{jin} Y_n^a T_n^b \right] + \Omega \eta^{abc} B_{0a}^{i0} \left[ \partial_a A_{b0i}^j - \partial_b A_{a0i}^j - \epsilon_{mi} A_{a0m}^0 Y_n^a + \epsilon_{mn} A_{b0m} B_{0n0} \right] + \Omega \eta^{abc} \left[ B_{ab}^i B_{0c0i} + B_{0a}^i B_{bc0i} \right],
\]

where \(\epsilon^{0ijk} = \epsilon^{ijk}\) and \(i, j, k = 1, 2, 3\) are raised and lowered with the Euclidean metric \(\eta_{ij} = (1, 1, 1)\). We can see that either Euler or Pontryagin theories share the same configuration variables, however,
We observe that the canonical partner’s of the dynamical variables have changed, this fact will be reflected in the generalized FJ brackets. In this respect, we will compare the results obtained from both actions (2) and (53), now they are at the same level written in terms of real variables, this is a different scenario to that reported in [19] where it was considered only the auto-self-dual case.

In this manner, from (54) we can identify the following symplectic Lagrangian given by

\[
\mathcal{L}^{(0)} = \Omega \eta^{abc} B_{ab}^{0i} \dot{\Upsilon}_{ci} - \Omega \eta^{abc} B_{ab} A_{c0}^{0i} - \mathcal{V}^{(0)},
\]

(55)

where \(\mathcal{V}^{(0)}\) is the symplectic potential

\[
\mathcal{V}^{(0)} = A_{00i} \left[ \partial_c (\Omega \eta^{abc} B_{ab}) + \Omega \eta^{abc} \epsilon^{ijk} B_{abj} \Upsilon_c^k - \Omega \eta^{abc} \epsilon^{ijk} B_{ab}^0 A_{c0}^0 \right] \\
- A_0^i \left[ \partial_c (\Omega \eta^{abc} B_{ab}^{0i}) - \Omega \eta^{abc} \epsilon^{ijk} B_{ab}^0 \Upsilon_c^k - \Omega \eta^{abc} \epsilon^{ijk} B_{ab} A_{c0}^0 \right] \\
+ \Omega \eta^{abc} B_{0ai} \left[ \partial_b A_{c0}^i - \partial_c A_{b0}^i + \epsilon^{ijk} A_{b0j} \Upsilon_c^k - \epsilon^{ijk} A_{b0j} A_{c0k} + B_{bc}^0 \right] \\
- \Omega \eta^{abc} B_{0a0i} \left[ \partial_b \Upsilon_a^c - \partial_c \Upsilon_a^b + \epsilon_{ijk} \Upsilon_b^j \Upsilon_c^k - \epsilon_{ijk} A_{b0j} A_{c0k} - B_{bcj} \right].
\]

(56)

Now, from the symplectic Lagrangian (55) we identify the following symplectic variables

\[
\xi^{(0)} = (A_{a0}^i, B_{ab}^{0i}, \Upsilon_{ai}, B_{abi}, A_0^i, A_{00i}, B_{a0}^{0i}, B_{0ai}),
\]

(57)

and the 1-form

\[
\alpha^{(0)} = (-\Omega \eta^{abc} B_{abi}, 0, \Omega \eta^{abc} B_{ab}^{0i}, 0, 0, 0, 0, 0).
\]

(58)

Thus, the symplectic matrix for the Euler class takes the form

\[
f_{ij}^{(0)} = \begin{pmatrix}
0 & 0 & 0 & \Omega \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 \\
0 & 0 & \Omega \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 \\
0 & -\Omega \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Omega \eta^{dec} \delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \delta^3(x - y).
\]

(59)

We observe that \(f_{ij}^{(0)}\) is singular as expected because there are constraints. The zero-modes of \(f_{ij}^{(0)}\) are given by the following 4 vectors

\[
\nu_1^{(0)} = (0, 0, 0, 0, V^{A_a^0}, 0, 0, 0),
\]

(60)

\[
\nu_2^{(0)} = (0, 0, 0, 0, V^{A_{a0}^0}, 0, 0),
\]

(61)

\[
\nu_3^{(0)} = (0, 0, 0, 0, 0, V^{B_{a0}^{0i}}, 0),
\]

(62)

\[
\nu_4^{(0)} = (0, 0, 0, 0, 0, 0, V^{B_{a0i}}),
\]

(63)
where $V^{A_0i}$, $V^{A_{00i}}$, $V^{B_{00i}}$, and $V^{B_{000}}$ are arbitrary functions. Hence, by using these modes we find the following FJ constraints

\[
\Omega_i^{(0)} = \int \mathbf{d}^3 x \mathcal{V}_{i}^{(0)} \frac{\delta}{\delta \xi_i^{(0)}} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi) = \int \mathbf{d}^3 x V^A_{0i} \frac{\delta}{\delta A_0^i} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi)
\]

\[
= \partial_c (\Omega^{abc} B_{ab}^{0i} - \Omega^{abc} \epsilon_{ijk} B_{ab}^{0j} \mathcal{Y}_{c}^{k} - \Omega^{abc} \epsilon_{i} \epsilon_{j} B_{abbk} A_{c0j}),
\]

\[
\Omega^{(0) 00i} = \int \mathbf{d}^3 x \mathcal{V}_{i}^{(0)} \frac{\delta}{\delta \xi_i^{(0)}} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi) = \int \mathbf{d}^3 x V^A_{00i} \frac{\delta}{\delta A_{00i}} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi)
\]

\[
= - \left[ \partial_c (\Omega^{abc} B_{ab}^{0i} + \Omega^{abc} \epsilon_{ijk} B_{ab}^{0j} \mathcal{Y}_{c}^{k} - \Omega^{abc} \epsilon_{jki} B_{ab}^{0j} A_{c0k}) \right],
\]

\[
\Omega^{(0) a0} = \int \mathbf{d}^3 x \mathcal{V}_{i}^{(0)} \frac{\delta}{\delta \xi_i^{(0)}} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi) = \int \mathbf{d}^3 x V^B_{a0i} \frac{\delta}{\delta B_{a0i}} \int \mathbf{d}^3 y \mathcal{V}^{(0)}(\xi)
\]

\[
= - \Omega^{abc} \left[ \partial_c A_{c0i} - \partial_c A_{b0i} + \epsilon_{ijk} A_{b0j} \mathcal{Y}_{c}^{k} - \epsilon_{jki} A_{abk} A_{c0k} - B_{bc} \right],
\]

and we observe that there are the following 6 reducibility condition between the constraints

\[
\partial_a \Omega^{(0) a0} = \epsilon_{i} \epsilon_{j} Y^{a0k} \Omega^{(0) a0} - \epsilon_{ij} A_{a0k} \Omega^{(0) a0} + \frac{1}{2} \Omega^{(0) a0},
\]

\[
\partial_a \Omega^{(0) a0} = \epsilon_{i} \epsilon_{j} A_{a0k} \Omega^{(0) a0} + \epsilon_{i} \epsilon_{j} Y^{a0k} \Omega^{(0) a0} + \frac{1}{2} \Omega^{(0) a0}.
\]

We can observe that either Pontryagin or Euler class share the same FJ constraints, however this fact does not guarantee that these actions will be equivalent at the quantum level, as we will see this point below. Now, we shall observe if there are more constraints. For this aim, we use the expression \([20]\), where the symplectic matrix $\tilde{f}_{ij}$ for the Euler theory is given by

\[
\tilde{f}_{ij} =
\begin{pmatrix}
0 & 0 & 0 & \Omega^{abc} \delta_{i}^{j} \\
0 & -\Omega^{abc} \delta_{i}^{j} & 0 & 0 \\
-\Omega^{abc} \delta_{i}^{j} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\Omega^{abc} \epsilon_{ik}^{j} B_{abj} & \Omega^{abc} (\delta_{ik} \partial_c - \epsilon_{ikj} \mathcal{Y}_{c}^{j}) & -\Omega^{abc} \epsilon_{ikj} B_{abj} & 0 \\
-\Omega^{abc} \epsilon_{jki} B_{abj}^{0j} & -\Omega^{abc} \epsilon_{jki} A_{c0j}^{j} & \Omega^{abc} \epsilon_{jki} B_{abj} & 0 \\
-2\Omega^{abc} \epsilon_{ikj} A_{b0j} & 0 & 2\Omega^{abc} (\delta_{ik} B_{ab} + \epsilon_{ikj} \mathcal{Y}_{b}^{j}) & 0 \\
2\Omega^{abc} (\delta_{ik} B_{ab} - \epsilon_{ikj} \mathcal{Y}_{b}^{j}) & \Omega^{abc} \delta_{k}^{i} & 2\Omega^{abc} \epsilon_{ijk} A_{b0j} & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
\Omega \eta^{abc} \delta^i_j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\Omega \eta^{abc} \epsilon^{jk}_i A_{i\alpha j} & 0 & 0 & 0 & 0 \\
\Omega \eta^{abc} (\delta^{ik}_j \partial_b + \epsilon^{k}_i \gamma^{j}_c) & 0 & 0 & 0 & 0 \\
-\Omega \eta^{abc} \delta^i_k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \delta^3(x-y).
\]  

The matrix \( \bar{f}_{ij} \) has the following null vectors

\[
\vec{V}_1 = (\epsilon_{ij} A_{c0}^j V^i , \epsilon_{ij} B_{ab}^{0j} V^i , \partial_i V_i - \epsilon_{ij} \gamma^i_c V^j , \epsilon_{ij} B_{ab} V^i , 0, 0, 0, 0, V^i, 0, 0, 0),
\]

\[
\vec{V}_2 = (-[\partial_i V_i + \epsilon_{ij} \gamma^i_c V^j] , -\epsilon_{ij} B_{ab} V^i , -\epsilon_{ij} A_{ab} V_i , -\epsilon_{ij} B_{ab}^{0j} V^i, 0, 0, 0, 0, V^i, 0, 0),
\]

\[
\vec{V}_3 = (V_i , -2(\partial_b V_i + \epsilon_{ij} \gamma^i_b V^j) , 0, -2\epsilon_{ij} A_{b0j} V^i, 0, 0, 0, 0, 0, V^i, 0),
\]

\[
\vec{V}_4 = (0, 2\epsilon_{ij} A_{b0j} V^i, V_i , 2(\partial_b V_i - \epsilon_{ij} \gamma^i_b V^j), 0, 0, 0, 0, 0, 0, V^i).
\]  

(69)
where $V^i$'s are arbitrary functions. On the other hand, $Z_k(\xi)$ is given by

$$
Z_k(\xi) = \begin{pmatrix}
\frac{\delta \mathcal{L}^{(i)}}{\delta (\xi_j)} \\
0 \\
0 \\
0
\end{pmatrix}
$$

(71)

\[
\begin{pmatrix}
\Omega_i^{j0} \epsilon^{kj} A_{0 i} B_{0 j} - \Omega_i^{abc} \epsilon_{jk i} A_{00 i} B_{a b j} - 2 \Omega_i^{abc} \epsilon_{jki} A_{00 i} B_{0 a k} \\
+ 2 \Omega_i^{abc} \epsilon_{jki} B_{0 a i} \Upsilon_{c j} + 2 \Omega_i^{abc} \epsilon^{jki} B_{0 a 0} A_{00 i}
- \Omega_i^{abc} \epsilon_{jki} A_{0 i}^{0} A_{c 0}^k + \Omega_i^{abc} \partial_c A_{0 i} + \Omega_i^{abc} \epsilon_{ijk} A_{0 i}^0 \Upsilon_c^k
\Omega_i^{abc} \epsilon_{ijk} A_{0 i}^{0} B_{a b j} + \Omega_i^{abc} \epsilon_{jki} A_{0 i}^{0} B_{0 a j}
+ 2 \Omega_i^{abc} \partial_b B_{0 a 0}^i - 2 \Omega_i^{abc} \epsilon_{jki} \Upsilon_{b 0}^i B_{0 a i}
- \Omega_i^{abc} \partial_c A_{00 i} + \Omega_i^{abc} \epsilon_{jki} A_{00 i} \Upsilon_{c k} + \Omega_i^{abc} \epsilon^{jki} A_{00 i} A_{0 a j}
\end{pmatrix}
\]

(72)

Hence, the contraction $\tilde{V}_i^\mu Z_\mu = 0$ gives identities because this contraction is a linear combination of constraints. Therefore, there are no more FJ constraints.

Furthermore, we will add the constraints (61) to the symplectic Lagrangian using the following Lagrange multipliers, namely $A_0^i = \dot{T}_i, A_{00 i} = \dot{\Lambda}_i, B_{0 a i} = \frac{\dot{\xi}_i}{2}, B_{0 a i} = \frac{\dot{\xi}_i}{2}$. Thus, the symplectic Lagrangian takes the form

$$
\mathcal{L}^{(1)} = \Omega_i^{abc} B_{ab}^{00 i} \Upsilon_{c i} - \Omega_i^{abc} B_{abi} A_{0 c 0}^i - \dot{T}_i \Omega_i^{(0)} 00 i + \dot{\Lambda}_i \Omega_i^{(0)} 00 i - \frac{\dot{\xi}_i}{2} \Omega_i^{(0)} 0 a i
$$

(73)
where \( \mathcal{V}^{(1)} = \mathcal{V}^{(0)}|_{\Omega^{(0)} = \Omega^{(0)}_{\alpha \beta \gamma} \equiv 0} = 0 \), this result is expected because Euler class is diffeomorphism covariant just like GR.

From the symplectic Lagrangian (73) we identify the following symplectic variables
\[
\xi^{(1)} = (A_{a0}^{i}, B_{ab}^{qi}, \Gamma_{ai}, B_{ab}, A_{0}^{i}, A_{00i}, B_{a0}^{0i}, B_{0ai}, T^{i}, \Lambda, \epsilon_{a}^{i}, \chi_{ai}),
\]
and the 1-forms
\[
a^{(1)} = \left( -\Omega^{0}_{abc} B_{abi}, 0, \Omega^{0}_{abc} B_{abi}^{0i}, 0, -\Omega_{i}^{(0)} + \Omega^{(0)}_{00}, \Omega_{i}^{(0)} 0_{ai}, + \frac{\Omega^{(0)}_{0a}}{2}, + \frac{\Omega^{(0)}_{0ai}}{2} \right).
\]

Hence, the symplectic matrix has the following form
\[
f^{(1)}_{ij} = \begin{pmatrix}
0 & 0 & 0 & \Omega^{0}_{abc} \delta^{k}_{i} \\
0 & 0 & 0 & 0 \\
-\Omega^{0}_{abc} \delta^{k}_{i} & 0 & 0 & 0 \\
\Omega^{0}_{abc} \epsilon^{i}_{k} B_{abi} & \Omega^{0}_{abc} d_{abc} & -\Omega^{0}_{abc} \epsilon_{ijk} B_{abi}^{0j} & -\Omega^{0}_{abc} \epsilon^{i}_{k} A_{00j} \\
-\Omega^{0}_{abc} \epsilon^{i}_{k} d_{a0i} & \Omega^{0}_{abc} \epsilon^{i}_{k} B_{abi}^{0j} & 0 & -\Omega^{0}_{abc} d_{abc} \\
\Omega^{0}_{abc} \epsilon^{i}_{k} A_{00j} & -\Omega^{0}_{abc} d_{abc} & 0 & \Omega^{0}_{abc} \delta^{k}_{i} \\
-\Omega^{0}_{abc} \epsilon^{i}_{k} A_{00j} & -\Omega^{0}_{abc} \epsilon^{i}_{k} d_{a0i} & \Omega^{0}_{abc} \delta^{k}_{i} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \delta^{k}(x - y),
\]
where we have defined \( D^{i}_{k} = \delta^{i}_{k} \partial_{c} + \epsilon^{ij}_{k} \partial_{c} \) and \( d^{i}_{a} = \delta^{i}_{a} \partial_{c} - \epsilon^{ij}_{k} \partial_{j} \). This matrix is singular and we have proved that there are no more constraints, thus, this theory has a gauge symmetry. In order to obtain a symplectic tensor, we fixing the temporal gauge just like was done for the Pontryagin invariant (51). In this manner, we introduce more Lagrange multipliers enforcing the gauge fixing as constraints. The Lagrange multipliers are given by \( \beta_{i}, \alpha^{i}, \rho_{i}^{a}, \sigma_{i}^{a} \), thus, the symplectic Lagrangian takes the form
\[
\mathcal{L}^{(2)} = \Omega^{0}_{abc} B_{abi}^{0j} \hat{\chi}_{ai} - \Omega^{0}_{abc} B_{abi} \hat{A}_{ai}^{j} - \hat{T}^{i} \left[ \Omega^{(0)}_{i} - \beta_{i} \right] + \hat{\Lambda}_{i} \left[ \Omega^{(0)}_{0ai} + \alpha^{i} \right] \\
- \epsilon_{a}^{i} \left[ \Omega^{(0)}_{0ai}^{0a} - \rho_{i}^{a} \right] + \hat{\chi}_{ai} \left[ \Omega^{(0)}_{0ai}^{0a} + \sigma^{a} \right].
\]
From the symplectic Lagrangian, we identify the following new set of symplectic variables
\[
\xi^{(2)} = (A_{a0}^{i}, B_{ab}^{qi}, \Gamma_{ai}, B_{ab}, A_{0}^{i}, A_{00i}, B_{a0}^{0i}, B_{0ai}, T^{i}, \Lambda_{i}, \epsilon_{a}^{i}, \chi_{ai}, \beta_{i}, \alpha^{i}, \rho_{i}^{a}, \sigma_{i}^{a}),
\]
and the 1-forms
\[
a^{(2)} = \left( -\Omega^{0}_{abc} B_{abi}, 0, \Omega^{0}_{abc} B_{abi}^{0i}, 0, -\Omega_{i}^{(0)}, + \Omega^{(0)}_{00}, \Omega_{i}^{(0)} 0_{ai}, + \frac{\Omega^{(0)}_{0a}}{2}, + \frac{\Omega^{(0)}_{0ai}}{2} \right).
\]
and the 1-forms

\[ a^{(2)} = \left( -\Omega^a_{~~b} B_{ab}, 0, \Omega^{\alpha a}_{~~b} B_{ab}, 0, \right. \]

\[ \left. - \left[ \Omega^{(0)}_{0a \rho} \delta^a_i \right], \left[ \Omega^{(0)}_{0a \rho} \alpha^i \right] \right) . \]

Thus, the symplectic matrix is given by

\[ f^{(2)}_{ij} = \left( \begin{array}{cccccccc}
0 & 0 & 0 & 0 & \Omega \eta^{abc} \delta^k_i & -\Omega \eta^{abc} \epsilon^{kj} B_{abj} \\
0 & 0 & 0 & 0 & \Omega \eta^{abc} \delta^k_i & -\Omega \eta^{abc} d_{i j c} \\
0 & 0 & 0 & 0 & -\Omega \eta^{abc} \delta^k_i & 0 & 0 & -\Omega \eta^{abc} \epsilon^{kj} B_{abj} \\
\Omega \eta^{abc} \epsilon_{ik} B_{abj} & \Omega \eta^{abc} d_{i j c} & -\Omega \eta^{abc} \epsilon_{ik} B_{abj} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\Omega \eta^{abc} \delta^k_i & 0 & 0 & -\Omega \eta^{abc} d_{i j c} \\
\Omega \eta^{abc} \epsilon_{ik} B_{abj} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Omega \eta^{abc} \epsilon_{ik} A_{b0j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Omega \eta^{abc} d_{i j b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) . \]

\[ \delta^3 (x - y). \]
We can observe that this matrix is not singular; after a long calculation, the inverse of \( f_{ij}^{(2)} \) is given by

\[
f_{ij}^{(2)\, -1} = \frac{1}{2\Omega} \eta_{abg} \delta^l_k \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{2\Omega} \eta_{abg} \delta^l_k & 0 \\
0 & 0 & -\frac{1}{2\Omega} \eta_{abg} \delta^l_k & 0 & 0 \\
0 & -\frac{1}{2\Omega} \eta_{abg} \delta^l_k & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2\Omega} \eta_{abg} \delta^l_k & 0 & 0 \\
-\epsilon_j^k A_{g0k} & -\epsilon_j^l B_{bg} & 0 & -\epsilon_j^k A_{g0k} & -\epsilon_j^l B_{bg} \\
D^j_l & -\epsilon_j^m B_{bg} & 0 & D^j_l & -\epsilon_j^m B_{bg} \\
\frac{1}{2} \delta_{ij}^g \delta_{ab}^h & \frac{1}{2} \delta_{ij}^g D^j_l & 0 & \frac{1}{2} \delta_{ij}^g \delta_{ab}^h & A_{f0m} \\
0 & \frac{1}{2} \delta_{ij}^g \delta_{ab}^h & -\frac{1}{2} \delta_{ij}^g \delta_{ab}^h & -\frac{1}{2} \delta_{ij}^g \delta_{ab}^h & 0
\end{pmatrix}
\]

\( \delta^k(x - y) \), \( 82 \)

where we have defined

\[
C_{kl} = \Omega \eta^{abc} \left( \epsilon^i_k B_{abj} d_{ilc} + \epsilon^i_j B_{abj} D^i_{kc} \right),
\]

\( 83 \)

\[
E_{kl} = \Omega \eta^{abc} \left( d_{ilc} D_{kb} - \epsilon_i^j k \epsilon^m_i A_{b0j} A_{c0m} - \frac{1}{2} \epsilon_i^j B_{abj} \right),
\]

\( 84 \)

\[
G_{kl} = \Omega \eta^{abc} \left( \epsilon^i_j A_{b0j} d_{ilc} + \epsilon_i^j A_{c0j} d_i^k + \frac{1}{2} \epsilon_i^j B_{abj} \right),
\]

\( 85 \)

\[
I_{kl} = \frac{\Omega}{2} \eta^{abc} \epsilon_{jkl} B_{ab}^{0j},
\]

\( 86 \)

\[
K_{kl}^f = \Omega \eta^{fbc} (d_k^{bc} - D_k^{bc}).
\]

\( 87 \)

Therefore, from the symplectic tensor we can identify the following FJ brackets

\[
\{ B_{ab0i}(x), Y_{di}(y) \}_{FJ} = \frac{1}{2\Omega} \eta_{abg} \eta_{lid} \delta^k(x - y),
\]

\( 88 \)

\[
\{ B_{abi}(x), A_{dpi}(y) \}_{FJ} = \frac{1}{2\Omega} \eta_{abg} \eta_{lid} \delta^k(x - y),
\]

\( 89 \)
which have been not reported in the literature. It is important to observe that in spite of we have used in both theories the same configuration variables and the same gauge fixing, the generalized brackets are different to each other. As it is showed below this fact will be important in the quantization.

Now, the counting of physical degrees of freedom is performed in the following way [DF=dynamical variables-independent constraints], thus, there are 18 canonical variables given by \((A_{\alpha i}, \Gamma^i_a)\) and 18 independent constraints \((\Omega^{(0)}_{\alpha i}, \Omega^{(0)}_{0i}, \Omega^{(0)}_{0\alpha i}, \Omega^{(0)}_{\alpha 0i})\). In this manner, the theory lacks of physical degrees of freedom.

A quantum state

We have seen in previous sections that the FJ constraints for the Pontryagin theory are exactly solved by means the so-called Chern-Simons theory. In this section we will solve the quantum FJ constraints for the Euler class. First we observe that the state given in (92) does not solve the Euler constraints, in this manner we need to find a new state. We propose the following Chen-Simons action

\[
S[A] = \frac{\Omega}{2} \left[ \int \epsilon^{ijk} A_{ij} \wedge A_{KL} + \frac{2}{3} \epsilon^{ijkl} A_i F^j + A_{ij} \wedge A_{KL} \right],
\]

and we write it in terms of the variables \(A^{0i}, \Gamma^i_a\), then it takes the following form

\[
S[A^{0i}, \Gamma^i_a] = \int \left\{ \Omega \eta^{abc} \left[ -A^{0i} \partial_b \Upsilon^i_c + A^{0i}_a \partial_c \Upsilon^i_b \right] - \Omega \eta^{abc} \epsilon_{ijk} A^{0i}_a \Upsilon^j_b \Upsilon^k_c + \frac{2}{3} \eta^{abc} \epsilon_{ijk} A^{0i}_a A^{0j}_b A^{0k}_c \right\} dx^3.
\]

On the other hand, the generalized FJ brackets for the Euler invariant will be useful for the quantization. In fact, the dynamical variables will be promoted to operators and the brackets will be promoted to commutators. Hence, the generalized brackets are given by

\[
\{ B_{\alpha 0i}(x), \Upsilon_d(y) \}_{FJ} = \frac{1}{2\Omega} \eta_{abcd} \delta^3(x - y),
\]

\[
\{ B_{a0i}(x), A_{d0i}(y) \}_{FJ} = \frac{1}{2\Omega} \eta_{abcd} \delta^3(x - y),
\]

its classical-quantum correspondence is given by

\[
\{ \Upsilon_d(x), \Omega \eta^{abc} \hat{B}_{bc0i}(y) \}_{FJ} = -\eta_{ij} \delta^a d \delta^3(x - y),
\]

\[
\{ A_{d0i}(x), \Omega \eta^{abc} \hat{B}_{c0j}(y) \}_{FJ} = -\eta_{ij} \delta^a d \delta^3(x - y),
\]

hence, we can identify the classical-quantum correspondence \(\Omega \eta^{abc} \hat{B}_{bc0i} \rightarrow -i \frac{\delta}{\delta A^{0i}_a}\) and \(\Omega \eta^{abc} \hat{B}_{bci} \rightarrow -i \frac{\delta}{\delta \Upsilon^i_a}\); this election has been used because both \(A^{0i}_a\) and \(\Upsilon^i_a\) are now the dynamical variables.

Moreover, just like in previous sections, we will demand that the restriction for the Euler physical states, namely \(\Psi_E(A^{0i}_a, \Upsilon^i_a)\), will be archived by

\[
\left\{ i \frac{\delta}{\delta A^{0i}_a} - \Omega \eta^{abc} \left[ \partial_b \Gamma^i_c - \partial_c \Gamma^i_b + \epsilon_{ijk} \Gamma^j_b \Gamma^k_c - \epsilon_{ijk} A^{0j}_b A^{0k}_c \right]\right\} \Psi_E(A^{0i}_a, \Upsilon^i_a) = 0,
\]

\[
\left\{ i \frac{\delta}{\delta \Upsilon^i_a} - \Omega \eta^{abc} \left[ \partial_b A^{0i}_a \partial_c A^{0j}_b + \epsilon_{ijk} A^{0j}_b \Upsilon^k_c - \epsilon_{ijk} A^{0j}_a \Upsilon^k_b \right]\right\} \Psi_E(A^{0i}_a, \Upsilon^i_a) = 0,
\]

\[
\left\{ -i \partial_a \frac{\delta}{\delta A^{0i}_a} + \Omega \epsilon_{ijk} \Gamma^j_a \frac{\delta}{\delta \Upsilon^i_a} + i \Omega \epsilon_{ijk} A^{0j}_a \frac{\delta}{\delta A^{0k}_a}\right\} \Psi_E(A^{0i}_a, \Upsilon^i_a) = 0,
\]

\[
\left\{ -i \partial_a \frac{\delta}{\delta \Upsilon^i_a} + \Omega \epsilon_{ijk} \Upsilon^j_a \frac{\delta}{\delta A^{0i}_a} - i \Omega \epsilon_{ijk} A^{0i}_a \frac{\delta}{\delta \Upsilon^i_a}\right\} \Psi_E(A^{0i}_a, \Upsilon^i_a) = 0,
\]

(96)
where the solution is given by
\[
\Psi_E(A^a_0, \Upsilon^a) = e^{iS[A^a_0, \Upsilon^a]},
\]
(97)
now \(S[A^a_0, \Upsilon^a]\) is given in (91). Again, the constraints (96) are solved exactly by (97), thus, a new quantum state is reported in this work. In this manner, in spite of the Euler and Pontryagin sharing the same equations of motion, its corresponding quantum states are different.

IV. CONCLUSIONS

In this paper, a complete symplectic analysis for the Euler and Pontryaging invariants has been performed. We carry out our analysis for both invariants by using the same symplectic variables and the same gauge fixing, we have observed that in spite of the Euler and Pontryagin invariants sharing the same FJ constraints, its corresponding generalized FJ brackets are different. This fact, allowed us to observe that the solution to the quantum FJ constraints are not the same. It is worth to comment, that we have found only mathematical solutions for the constraints; in order to observe if these solutions are physical (we need to remember that we have worked with real variables) then it is necessary to construct a measure for the quantization via mechanical path integral. In fact, there is an important connection between FJ quantization and path integral as that reported in [23]. In this respect, the measure acquires a factor related with the determinant of the symplectic tensors given in (37) and (82), thus, in this paper we have all tools for exploring these subjects. Finally, we have seen that the FJ formalism demands to work with less constraints than Dirac’s formalism, this fact allowed us to construct the fundamental brackets with relative simplicity. Moreover, it is possible to analice the addition of the topological invariants to theories with degrees of freedom just like bi-gravity models [24], these problems are already in progress and will be the subject of forthcoming works [25].

V. REFERENCES

[1] C. Rovelli. Quantum Gravity. Cambridge University Press, Cambridge (2004), T. Thiemann, Modern Canonical Quantum General Relativity. Cambridge University Press, Cambridge (2007).
[2] E. Witten, J. Diff. Geom., 17 (4), 661-6692, (1982).
[3] M. F. Atiyah, Publications Mathmatiques de l’IHES, Volume 68, 175-186, (1988).
[4] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121, No. 3, 351-399, (1989); E. Witten, Topological quantum field theory, Comm. Math. Phys., 117, 353-386, (1988); E. Witten, Topological sigma models, Comm. Math. Phys., 118 (1988), 411-449.
[5] D. J. Rezende and A. Perez, Phys. Rev. D 79: 064026, (2009).
[6] H. G. Compéan, O. Obregón, C. Ramírez and M. Sabido, Journal of Physics. Conference Series 24, 203-212, (2005).
[7] A. Mardones, J. Zanelli, Class. Quantum Grav. 8 (1991) 1545.
[8] T. Kimura, Prog. Theor. Phys. 42, 1191, (1969).
[9] R. Delbourgo, A. Salam, Phys. Lett. B 40 381, (1972).
[10] T. Eguchi, P. Freund, Phys. Rev. Lett. 37 1251, (1976).
[11] L. Alvarez-Gaum, E. Witten, Nucl. Phys. B 234, 269, (1984).
[12] O. Chandia, J. Zanelli, Phys. Rev. D 55, 7580, (1997).
[13] G.T. Horowitz, Commun. Math. Phys. 125, 417, (1989).
[14] G.T. Horowitz, M. Srednicki, Commun. Math. Phys. 130, 83, (1990).
[15] J. F. Plebanski, J. Math. Phys. 18, 2511, (1977).
[16] M. Celada, M. Montesinos, J. Romero, Class. Quant. Grav. 33, No 11, 115014, (2016); D. K. Wise, Class. Quant. Grav. 27, 155010, (2010);
[17] J.A. Nieto, J. Socorro, Phys. Rev. D59: 041501, (1999).
[18] J. C. Baez, Lect. Notes Phys. 543, 25-94, (2000); A. Perez, Living Rev. Relativ. (2013) 16: 3. https://doi.org/10.12942/lrr-2013-3.
[19] A. Escalante, Phys. Lett. B 676, 105-111, (2009); I. Oda, arXiv:hep-th/0311149
[20] O. Chandia and J. Zanelli, AIP Conference Proceedings 419, 251 (1998); doi: http://dx.doi.org/10.1063/1.54694
[21] A. Escalante and L. Carbajal, Annals Phys. 326, 3237339, (2011).
[22] L.D. Faddeev, R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692.; E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, F.I. Takakura, L.M.V. Xavier, Modern Phys. Lett. A 23 (2008) 829; E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, F.I. Takakura, Internat. J. Modern Phys. A 22 (2007) 3605; E.M.C. Abreu, C. Neves, W. Oliveira, Internat. J. Modern Phys. A 21 (2008) 5329; C. Neves, W. Oliveira, D.C. Rodrigues, C. Wotzasek, Phys. Rev. D 69 (2004) 045016; J. Phys. A 3 (2004) 9303; C. Neves, C. Wotzasek, Internat. J. Modern Phys. A 17 (2002) 4025; C. Neves, W. Oliveira, Phys. Lett. A 321 (2004) 267; J.A. Garcia, J.M. Pons, Internat. J. Modern Phys. A 12 (1997) 451; E.M.C. Abreu, A.C.R. Mendes, C. Neves, W. Oliveira, R.C.N. Silva, C. Wotzasek, Phys. Lett. A 374 (2010) 360373607; A. Escalante, M. Zárate, Annals Phys. 353 (2015) 163-178.
[23] D. J. Toms, Phys. Rev. D, 92, 105026, (2015).
[24] C. Deffayet, J. Mourad and G. Zahariade, JHEP, 03, 086, (2013); Tuan Q. Do. Phys. Rev. D 94, 044022 (2016).
[25] A. Escalante, work in progress.