On semigroups of matrices with nonnegative diagonals✩

Grega Cigler, Roman Drnovšek∗

Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

Abstract

We give a short proof of a recent result by Bernik, Mastnak, and Radjavi, stating that an irreducible group of complex matrices with nonnegative diagonal entries is diagonally similar to a group of nonnegative monomial matrices. We also explore the problem when an irreducible matrix semigroup in which each member is diagonally similar to a nonnegative matrix is diagonally similar to a semigroup of nonnegative matrices.

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1. Introduction

Multiplicative semigroups of matrices with nonnegative diagonal entries have been studied in the papers [2] and [4]. Their authors considered the general question under which additional assumptions such a semigroup is simultaneously similar to a semigroup of nonnegative matrices. The main result of [2] is that every irreducible group of complex matrices with nonneg-
ative diagonal entries is diagonally similar to a group of nonnegative monomial matrices. In Section 2 we give a short proof of this result. Our proof is more geometric and less group-theoretic than the proof in [2]. Multiple authors of the paper [4] provided several examples showing that it is impossible to extend this result from groups to semigroups. So, to obtain similarity to a semigroup of nonnegative matrices, stronger assumptions on a given semigroup must be imposed. In Section 3 we explore the problem when an irreducible matrix semigroup in which each member is diagonally similar to a nonnegative matrix is necessarily diagonally similar to a semigroup of nonnegative matrices.

We now recall some definitions and basic facts. The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$. A convex set $K \subseteq \mathbb{R}^n$ is said to be a cone if $rK \subseteq K$ for all $r \in \mathbb{R}_+$. A cone $K \subseteq \mathbb{R}^n$ is proper if it is closed, pointed ($K \cap (-K) = \{0\}$), and solid (the interior of $K$ is nonempty). The most natural example of a proper cone is the nonnegative orthant $\mathbb{R}^+_n$. A cone $K \subseteq \mathbb{R}^n$ is reproducing if $K - K = \mathbb{R}^n$. It is well-known that a closed cone is solid if and only if it is reproducing.

Let $K$ be a closed cone in $\mathbb{R}^n$. A vector $x \in K$ is an extremal vector of $K$ if $y \in K$ and $x - y \in K$ imply that $y$ is a nonnegative multiple of $x$. By Ext $(K)$ we denote the set of all extremal vectors of $K$. By the Krein-Milman theorem, $K$ is the convex hull of Ext $(K)$. The angle $\phi \in [0, \pi]$ between non-zero vectors $x, y \in \mathbb{R}^n$ is determined by the equality $x^T y = \|x\| \|y\| \cos \phi$.

If $F$ is a subset of complex numbers, then $M_n(F)$ denotes the set of all $n \times n$ matrices with entries in $F$. If $C \subseteq M_n(\mathbb{C})$ is a collection of complex matrices, then $\overline{C}$ denotes its closure in the Euclidean topology, and $\mathbb{R}_+ C$ denotes its homogenization, i.e., $\mathbb{R}_+ C = \{rC : r \in \mathbb{R}_+, C \in C\}$. We say that a matrix has a nonnegative diagonal if all of its diagonal entries are nonnegative. A matrix is called monomial if it has the same nonzero pattern.
as a permutation matrix, i.e., there is exactly one nonzero entry in each row and in each column.

A collection \( \mathcal{C} \subseteq M_n(\mathbb{C}) \) (where \( n \geq 2 \)) is reducible if there exists a common invariant subspace other than the trivial ones \( \{0\} \) and \( \mathbb{C}^n \), or equivalently, there exists an invertible matrix \( S \in M_n(\mathbb{C}) \) such that the collection \( S \mathcal{C} S^{-1} \) has a block upper-triangular form; otherwise, the collection \( \mathcal{C} \) is said to be irreducible. If the matrix \( S \) can be chosen to be a permutation matrix, then the collection \( \mathcal{C} \) is said to be decomposable; otherwise, it is called indecomposable (or ideal-irreducible).

2. Groups of matrices with nonnegative diagonals

The study of semigroups of matrices having nonnegative diagonals was initiated by the authors of [2]. They started their discussion by the following result (see [2, Theorem 4.1]).

**Theorem 2.1.** Let \( S \subseteq M_n(\mathbb{C}) \) be an irreducible semigroup of matrices of rank at most one having nonnegative diagonals. If \( \mathbb{R}_+ S = S \), then, after a diagonal similarity, \( S = XY^T \) for some subsets \( X \) and \( Y \) of \( \mathbb{R}_+^n \) each of which spans \( \mathbb{C}^n \).

Using the Haar measure one can prove the following assertion (see [2, Proposition 4.3]).

**Proposition 2.2.** Let \( S \subseteq M_n(\mathbb{C}) \) be an irreducible semigroup of matrices. Suppose that \( \mathbb{R}_+ S = S \) and that there exists a non-zero functional \( \varphi : M_n(\mathbb{C}) \to \mathbb{C} \) such that \( \varphi(S) \in \mathbb{R}_+ \) for all \( S \in S \). Then \( S \) has members of rank one.

The following theorem is the main result of [2, Theorem 5.5]. We provide a short proof that is more geometric and less group-theoretic than the original one.
Theorem 2.3. If \( G \subset M_n(\mathbb{C}) \) is an irreducible group of matrices with non-negative diagonals, then, up to a diagonal similarity, \( G \) is a group in \( M_n(\mathbb{R}_+) \). Therefore, each member of the group \( G \) is a nonnegative monomial matrix.

Proof. With no loss of generality we may assume that \( tG \in G \) for all \( t > 0 \) and \( G \in G \). Let \( S = \overline{G} \). Applying Proposition 2.2 for the trace functional, we conclude that \( S \) contains elements of rank one. The semigroup ideal \( S_1 \) of all elements of rank at most one in \( S \) is irreducible (see [3]). By Theorem 2.1, we can assume that, after a diagonal similarity, \( S_1 = XY^T \) for some subsets \( X \) and \( Y \) of \( \mathbb{R}_+^n \) each of which spans \( \mathbb{C}^n \). We can also assume that \( \mathbb{R}_+X = X \) and \( \mathbb{R}_+Y = Y \). The cone \( \hat{X} \) generated by \( X \) is closed, and it is invariant under any \( S \in S \), since \( (Sx)y^T = S(xy^T) \in S_1 \) for every \( x \in X \) and \( y \in Y \). Similarly, it follows from \( x(S^Ty)^T = (xy^T)S \in S_1 \) that \( Y \) is invariant under \( S^T \). The dual cone

\[
Y^d = \{ z \in \mathbb{R}^n : z^Ty \geq 0 \text{ for all } y \in Y \}
\]

of the set \( Y \) obviously contains \( \mathbb{R}_+^n \), and it is invariant under any \( S \in S \), as \( (Sz)^T y = z(S^Ty) \geq 0 \) for all \( y \in Y \) and \( z \in Y^d \). It follows that every \( G \in G \) is a bijective mapping on both \( \hat{X} \) and \( Y^d \), implying that every \( G \in G \) maps \( \text{Ext}(\hat{X}) \) to itself, and the same holds for the cone \( Y^d \). We want to show that the inclusions \( \hat{X} \subseteq \mathbb{R}_+^n \subseteq Y^d \) are in fact equalities.

Assume, if possible, that \( \hat{X} \neq Y^d \). Then there exists a unit vector \( x \in X \setminus Y^d \) which is extremal for the cone \( \hat{X} \). Since the cone \( Y^d \) is closed, the distance between \( x \) and \( Y^d \) is strictly positive. It follows that there is a number \( \phi \in (0, \pi/2) \) such that, for each \( z \in \text{Ext}(Y^d) \), the angle between \( z \) and \( x \) is at least \( \phi \). Since \( x \in X \) and the set \( Y \) is spanning, there is a vector \( y \in Y \) such that \( P = xy^T \in S \) with \( y^Tx > 0 \). We can assume that \( y^Tx = 1 \), so that \( Px = x \). Choose any \( \epsilon > 0 \). Since \( S = G \), there is a matrix \( G \in G \).
such that \(|G - P| < \epsilon\). Now, for any \(z \in \text{Ext}(Y^d)\) with norm 1, we have
\[
\epsilon^2 > \|Gz - Pz\|^2 = \|Gz - (y^T z)x\|^2 = \|Gz\|^2 + (y^T z)^2 - 2(y^T z)\|Gz\| \cos \phi_z,
\]
where \(\phi_z\) is the angle between the vector \(x\) and the vector \(Gz \in \text{Ext}(Y^d)\).
Since \(y^T z \in \mathbb{R}_+\) and \(\phi_z \geq \phi\), we conclude that
\[
\epsilon^2 > \|Gz\|^2 + (y^T z)^2 - 2(y^T z)\|Gz\| \cos \phi = (y^T z - \|Gz\| \cos \phi)^2 + \|Gz\|^2 \sin^2 \phi.
\]
It follows that
\[
\|Gz\| \sin \phi < \epsilon \quad \text{and} \quad |y^T z - \|Gz\| \cos \phi| < \epsilon,
\]
and so
\[
0 \leq y^T z < \epsilon + \|Gz\| \cos \phi < \epsilon + \frac{\epsilon}{\sin \phi} \cos \phi.
\]
Since \(\epsilon > 0\) is arbitrary, we obtain that \(y^T z = 0\) for all vectors \(z \in \text{Ext}(Y^d)\), implying that \(y = 0\). This contradiction completes the proof of the equality \(\hat{X} = Y^d = \mathbb{R}_+^n\). Consequently, the inclusion \(\mathcal{G} \subset M_n(\mathbb{R}_+)\) holds, as asserted.

Since the map associated to any matrix \(G \in \mathcal{G}\) maps \(\text{Ext}(\mathbb{R}_+^n)\) to itself and it is invertible, the matrix \(G\) must be monomial, and so the proof is complete.

3. Semigroups of matrices diagonally similar to nonnegative ones

Let \(\mathcal{S} \subseteq M_n(\mathbb{C})\) be a semigroup in which each member \(A \in \mathcal{S}\) is diagonally similar to a nonnegative matrix. In this section we are looking for additional assumptions under which the whole semigroup \(\mathcal{S}\) is diagonally similar to a semigroup of nonnegative matrices. We first show that it does not suffice to assume that the semigroup \(\mathcal{S}\) is indecomposable.

**Example 3.1.** Define \(n \times n\) matrices \(A = aa^T\) and \(B = bb^T\), where \(n \geq 2\), \(a = [1, 1, \ldots, 1]^T\) and \(b = [1, 1, \ldots, 1, 1 - n]^T\). Then every nonzero member
of the semigroup $S$ generated by $A$ and $B$ is an indecomposable matrix of rank one that is diagonally similar to a nonnegative matrix. However, the whole semigroup $S$ is not diagonally similar to a semigroup of nonnegative matrices.

**Proof.** Note that $A^k = n^{k-1}A$ and $B^k = (n(n-1))^{k-1}B$ for all $k \in \mathbb{N}$, while $AB = BA = 0$. Therefore, $S$ is contained in the semigroup $\mathbb{R}_+A \cup \mathbb{R}_+B$. If $D$ is the diagonal matrix with diagonal $(1,1,\ldots,1, -1)$, then the matrix $DBD^{-1}$ is nonnegative, and therefore each matrix from $S$ is diagonally similar to a nonnegative matrix. Since the matrices $A$ and $B$ are indecomposable, every nonzero member of $S$ is indecomposable as well. It is easy to verify that the whole semigroup $S$ is not diagonally similar to a semigroup of nonnegative matrices. □

In the rest of the paper we explore the case when the semigroup $S$ is irreducible. We first show that, with no loss of generality, we may assume that $S$ is a closed set.

**Lemma 3.2.** Let $C \subset M_n(\mathbb{C})$ be a collection in which each member $A \in C$ is diagonally similar to a nonnegative matrix. Then the closure $\overline{\mathbb{R}_+C}$ also consists of matrices which are diagonally similar to nonnegative matrices.

**Proof.** Clearly, we may assume that $\mathbb{R}_+C = C$. If $A \in \overline{C}$, then there is a sequence $\{A_k\}_{k \in \mathbb{N}}$ in $C$ converging to the matrix $A$. For each $k \in \mathbb{N}$, let $D_k$ be a diagonal matrix such that $D_kA_kD_k^{-1}$ is a nonnegative matrix. We may assume that each diagonal entry of $D_k$ has absolute value one. Since the sequence $\{D_k\}_{k \in \mathbb{N}}$ is bounded, it has a convergent subsequence $\{D_{k_m}\}_{m \in \mathbb{N}}$ converging to some diagonal matrix $D$. Since $DAD^{-1} = \lim_{m \to \infty} D_{k_m}A_{k_m}D_{k_m}^{-1}$, the matrix $DAD^{-1}$ is nonnegative, and so $A$ is also diagonally similar to a nonnegative matrix. This completes the proof. □
We continue with a reduction of the problem to the real setting.

Lemma 3.3. Let $\mathcal{S} = \mathbb{R}_+ \mathcal{S} \subseteq M_n(\mathbb{C})$ be an irreducible semigroup such that each member $A \in \mathcal{S}$ is diagonally similar to a nonnegative matrix. Then there exists an invertible diagonal matrix $D \in M_n(\mathbb{C})$ such that the semigroup $DSD^{-1}$ consists of real matrices, and there exist two sets $X, Y \subseteq \mathbb{R}^n_+$, each of which spans $\mathbb{C}^n$, such that

$$DS_1D^{-1} = (DSD^{-1})_1 = XY^T,$$

where $S_1$ is the ideal of $\mathcal{S}$ consisting of members of rank at most one. Furthermore, the subcone of $\mathbb{R}^n_+$ generated by $X$ is a proper cone invariant under every member of $\mathcal{S}$.

PROOF. Our assumption implies in particular that all diagonal elements of any member of $\mathcal{S}$ must be nonnegative. By Proposition 2.2, the ideal $S_1$ of all members of $\mathcal{S}$ with rank at most one is nonzero. Since $\mathcal{S}$ is an irreducible semigroup, it is also necessarily irreducible (see [3]). Then by Theorem 2.1, we can find an invertible diagonal matrix $D$ and two sets $X, Y \subset \mathbb{R}^n_+$, each of which spans $\mathbb{C}^n$, such that $DS_1D^{-1} = XY^T$. As we are interested in diagonal similarities, we can assume that $D$ is the identity, so that $S_1 = XY^T$. To prove the inclusion $\mathcal{S} \subset M_n(\mathbb{R})$, pick any $A \in \mathcal{S}$ and $x \in X$. Since for any nonzero vector $y \in Y$ the matrix $A(xy^T) = (Ax)y^T$ belongs to $S_1$, we conclude that $Ax \in X \subseteq \mathbb{R}^n_+$. It follows that the cone of $\mathbb{R}^n_+$ generated by $X$ is a proper cone invariant under $A$. Since the set $X$ spans $\mathbb{C}^n$, it follows that $A(\mathbb{R}^n) \subseteq \mathbb{R}^n$, and therefore $A \in M_n(\mathbb{R})$. This completes the proof. \hfill $\Box$

From now on we consider real matrices. If a real matrix $A$ is diagonally similar to a nonnegative matrix via diagonal matrix $D$, we clearly may assume that each diagonal entry of $D$ is either 1 or $-1$. In this case we say that $D$ is a $\pm 1$-diagonal matrix.
**Lemma 3.4.** Let $A \in M_n(\mathbb{R})$ be an indecomposable matrix and $D$ a $\pm 1$-diagonal matrix such that $A' = DAD$ is a nonnegative matrix. If there exists a proper cone $K$ such that $A(K) \subseteq K$ and $K \subseteq \mathbb{R}_+^n$, then $D = \pm I$ and $A$ itself is a nonnegative matrix.

**Proof.** By the Perron-Frobenius Theorem, the spectral radius $\rho(A') = \rho(A)$ of the indecomposable matrix $A'$ is a simple eigenvalue having exactly one (up to a scalar multiplication) strictly positive eigenvector $e$. On the other hand, since the proper cone $K$ is invariant under $A$, the extension of the Perron-Frobenius Theorem (see [1, Theorem 3.2]) ensures that there is a non-zero vector $x \in K$ such that $Ax = \rho(A)x$. However, $A'Dx = DAx = \rho(A)Dx$, and so the vectors $Dx$ and $e$ are collinear. It follows that either $De$ or $-De$ belongs to $K \subseteq \mathbb{R}^n_+$, and this implies that $D = \pm I$ and $A$ itself is a nonnegative matrix. \[\square\]

The following simple example shows that in Lemma 3.4 we cannot omit the assumption that the cone $K$ is proper.

**Example 3.5.** Let $n \geq 2$, $a = [1, 1, \ldots, 1, 1 - n]^T$ and $K = \mathbb{R}_+[1, 1, \ldots, 1]^T$. The matrix $A = aa^T$ is indecomposable, and the cone $K$ is invariant under $A$, while $DAD$ is a nonnegative matrix for the diagonal matrix $D$ with diagonal $(1, 1, \ldots, 1, -1)$. \[\square\]

For $n \geq 2$ we say that a matrix $A \in M_n(\mathbb{R})$ is **1-decomposable** if there is a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where each of $A_1$ and $A_2$ is either an indecomposable (square) matrix or a $1 \times 1$ block.

The following assertion is crucial for the proof of the main result.
Proposition 3.6. Let \( A \in M_n(\mathbb{R}) \) be a 1-decomposable matrix that is diagonally similar to a nonnegative matrix. Let \( K \) and \( L \) be proper cones of \( \mathbb{R}_n^+ \) that are invariant under \( A \) and \( A^T \), respectively. Then \( A \) is a nonnegative matrix.

Proof. Let \( P \) be a permutation matrix such that the matrix \( PAP^T \) has the block form

\[
PAP^T = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}
\]

with respect to the decomposition \( \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l \), where \( 1 \leq k < n, l = n - k \), and each of \( A_1 \) and \( A_2 \) is either an indecomposable (square) matrix or a \( 1 \times 1 \) block. We first prove that the diagonal blocks \( A_1 \) and \( A_2 \) are nonnegative matrices. If \( DAD \) is a nonnegative matrix for a suitable \( \pm 1 \)-diagonal matrix \( D \), then \( E = PDP^T \) is a \( \pm 1 \)-diagonal matrix such that \( E(PAP^T)E \) is a nonnegative matrix. It follows that matrix \( PAP^T \) satisfies our assumptions provided that the cones \( K \) and \( L \) are replaced by the cones \( P(K) \) and \( P(L) \). We can therefore assume that \( A \) itself is of the block form

\[
A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}.
\]

Let \( \Pi_1 : \mathbb{R}^n \to \mathbb{R}^k \) and \( \Pi_2 : \mathbb{R}^n \to \mathbb{R}^l \) be the corresponding projections, and let \( C \subseteq \mathbb{R}_+^n \) be a proper cone. As \( C \subseteq \Pi_1(C) + \Pi_2(C) \) and \( \Pi_1(C) \) contains at most \( k \) linearly independent vectors, it follows that \( \Pi_2(C) \) contains at least \( n - k = l \) linearly independent vectors. Consequently, \( \Pi_2(C) \) contains exactly \( l \) linearly independent vectors, so that \( \Pi_2(C) \) is a generating cone of \( \mathbb{R}^l \). Similarly, \( \Pi_1(C) \) is a generating cone of \( \mathbb{R}^k \). Since \( C \subseteq \mathbb{R}_+^n \), both \( \Pi_1(C) \) and \( \Pi_2(C) \) are pointed and therefore proper cones. Assume now that the cone \( C \) is invariant under \( A \). If \( x_2 \in \Pi_2(C) \), then \( x_2 = \Pi_2(x) \) for some \( x \in C \), and so \( A_2(x_2) = A_2(\Pi_2(x)) = \Pi_2(Ax) \in \Pi_2(C) \), since \( A(C) \subseteq C \). Therefore, the cone \( \Pi_2(C) \) is invariant under \( A_2 \). This means that \( \Pi_2(K) \subseteq \mathbb{R}_+^l \) is a proper
cone invariant under $A_2$. Since the indecomposable matrix $A_2$ is diagonally similar to a nonnegative matrix, we can apply Lemma 3.4 to conclude that $A_2$ is a nonnegative matrix.

In order to show that $A_1$ is also a nonnegative matrix, we consider the transposed matrix $A^T$. The proper cone $L \subseteq \mathbb{R}^n_+$ is invariant under $A^T$. Then the cone $\Pi_1(L)$ is a proper cone invariant under $A_1^T$. Since $A_1$ is indecomposable, $A_1^T$ is indecomposable and again by Lemma 3.4 we conclude that $A_1$ must be a nonnegative matrix.

It remains to prove that the block $B$ is nonnegative. Suppose to the contrary that $B$ has some strictly negative entries. If $D = D_1 \oplus D_2$ is a $\pm 1$-diagonal matrix such that $DAD$ is a nonnegative matrix, then $D_i A_i D_i$ for $i = 1, 2$ and $D_1 BD_2$ are nonnegative matrices. Using Lemma 3.4 we conclude that $D_i = \pm I$ for $i = 1, 2$ and $D_1 BD_2 = \pm B$. Since $B$ contains some strictly negative entries, the matrix $-B$ must be nonnegative. Since we can add the identity matrix to the matrix $A$, without loss of generality we can assume that the matrices $A_1$ and $A_2$ are both primitive, i.e., the spectral radius $\rho(A_i)$ is the only point in the peripheral spectrum of $A_i$, $i = 1, 2$. For $k \in \mathbb{N}$ we have

$$A^k = \begin{bmatrix} A_1^k & B_k \\ 0 & A_2^k \end{bmatrix},$$

where

$$B_k = \sum_{l=0}^{k-1} A_1^{k-1-l} BA_2^l.$$ 

If we multiply the matrix $A$ by a suitable positive scalar, we can assume that $\rho(A) = \max\{\rho(A_1), \rho(A_2)\} = 1$. We must consider the following three cases:

1. $\rho(A_1) = \rho(A_2) = 1$: By Perron-Frobenius theory, the limits

$$\lim_{k \to \infty} A_1^k = E_1 \text{ and } \lim_{k \to \infty} A_2^k = E_2$$

are strictly positive idempotents of rank 1. In particular, there is a constant $C > 0$ such that $\|A_1^k\|, \|A_2^k\| \leq C$ for all $k \in \mathbb{N}$. Then we have, for any
\( m \in \mathbb{N} \),

\[
\|B_{4m}\| = \left\| \sum_{l=0}^{4m-1} A_{4m-1-l} B A_2^l \right\| \leq \sum_{l=0}^{4m-1} \|A_{4m-1-l}\| \|B\| \|A_2\| \leq 4m C^2 \|B\|,
\]

and so the sequence \( \left\{ \frac{1}{4m} B_{4m} \right\}_{m \in \mathbb{N}} \) is bounded. It follows that some subsequence \( \left\{ \frac{1}{4m_k} A_{4m_k} \right\}_{k \in \mathbb{N}} \) of the sequence \( \left\{ \frac{1}{4m} A_{4m} \right\}_{m \in \mathbb{N}} \) converges to the matrix of the form

\[
A_{\infty} = \lim_{k \to \infty} \frac{1}{4m_k} A_{4m_k} = \begin{bmatrix} 0 & B_{\infty} \\ 0 & 0 \end{bmatrix}.
\]

Choose \( m \in \mathbb{N} \) such that \( \frac{1}{4} E_i \leq A_{1}^l \) for \( i = 1, 2 \) and all \( l \geq m \). As \(-B\) is a nonnegative matrix, we obtain that \( A_{1}^{4m-1-l} B A_2^l \leq \frac{1}{4} E_1 B E_2 \) for all \( l = m, m + 1, m + 2, \ldots, 3m - 1 \). Since the matrices \(-A_{1}^{4m-1-l} B A_2^l\) are nonnegative, we have

\[
B_{4m} = \sum_{l=0}^{4m-1} A_{1}^{4m-1-l} B A_2^l \leq \sum_{l=m}^{3m-1} A_{1}^{4m-1-l} B A_2^l \leq \frac{1}{4} \sum_{l=m}^{3m-1} E_1 B E_2.
\]

It follows that

\[
B_{\infty} \leq \lim_{m \to \infty} \frac{1}{4m} \left( \frac{1}{4} \sum_{l=m}^{3m-1} E_1 B E_2 \right) = \frac{1}{8} E_1 B E_2,
\]

and so \( B_{\infty} \) is a matrix with some strictly negative entries. Therefore, there is a strictly positive vector \( e \in K \) such that the vector \( A_{\infty} e \) is not in \( \mathbb{R}_+^n \). As the cone \( K \) is closed and invariant under all powers of \( A \), it has to be invariant under \( A_{\infty} \), so that \( A_{\infty} e \in K \subseteq \mathbb{R}_+^n \). This contradiction completes the proof in this case.

(2) \( 1 = \rho(A_1) > \rho(A_2) \): As before, the limit \( \lim_{k \to \infty} A_1^k = E_1 \) is a strictly positive idempotent of rank 1. Since \( L \subseteq \mathbb{R}_+^n \) is a proper cone invariant under \( A^T \), we can find a strictly positive vector \( e \in L \) such that for all \( k \in \mathbb{N} \) we have \( (A^T)^k e \in L \subseteq \mathbb{R}_+^n \). If \( k \) is large enough, we have \( A_1^{k-1} \geq \frac{1}{2} E_1 \) and therefore \( B_k \leq A_1^{k-1} B \leq \frac{1}{2} E_1 B \). Writing \( e = e_1 \oplus e_2 \) with respect to the
given decomposition, we get $B_k^T e_1 \leq \frac{1}{2} (E_1 B)^T e_1 = \frac{1}{2} B^T E_1^T e_1$. Since the vector $B^T E_1^T e_1$ has at least one strictly negative component, the same holds for $B_k^T e_1$. Since $\lim_{k \to \infty} A_2^k = 0$, there is some power $k$ such that the vector $(A^T)^k e = ((A_1^T)^k e_1) \oplus (B_k^T e_1 + (A_2^T)^k e_2)$ has at least one strictly negative component. This is a contradiction with $(A^T)^k e \in L \subseteq \mathbb{R}^n_+$. 

(3) $\rho(A_1) < \rho(A_2) = 1$: This case can be handled in a way similar to the case (2); we get the contradiction with the assumption that $K$ is a proper cone invariant under $A$. □

The next example shows that in Proposition 3.6 none of the cones $K$ and $L$ can be omitted.

**Example 3.7.** The proper cone $K = \{(x, y) \mid x \geq y \geq 0\} \subset \mathbb{R}^2_+$ is invariant under the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

which is diagonally similar to a nonnegative matrix, but it is not nonnegative itself. Therefore, the cone $L$ cannot be omitted in Proposition 3.6. By duality, the cone $K$ cannot be omitted as well. □

The following is the main result of the paper.

**Theorem 3.8.** Let $S \subset M_n(\mathbb{C})$ be an irreducible semigroup such that each member of $S$ is diagonally similar to a nonnegative matrix. Suppose that every member of rank at least 2 is either indecomposable or 1-decomposable. Then $S$ is (simultaneously) diagonally similar to a semigroup of nonnegative matrices.

**Proof.** By Lemma 3.2, we can assume that $S = \mathbb{R}_+ S$. Then, by Lemma 3.3, we can assume that $S \subset M_n(\mathbb{R})$ and that there are spanning sets $X, Y \subseteq \mathbb{R}^n_+$ such that $S_1 = XY^T$. We can also assume that $X = \mathbb{R}_+ X$ and $Y = \mathbb{R}_+ Y$.
Denote by \( \hat{X} \) and \( \hat{Y} \) the cones generated by \( X \) and \( Y \), respectively. Since \( X \) and \( Y \) are spanning sets, the cones \( \hat{X}, \hat{Y} \subseteq \mathbb{R}^n_+ \) are proper. Choose any member \( A \in S \) of rank at least 2. Then, for all \( x \in X \) and \( y \in Y \), the matrices \( Axy^T = (Ax)y^T \) and \( xy^TA = x(A^Ty)^T \) belong to \( S_1 = XY^T \). It follows that \( Ax \in X \) and \( A^Ty \in Y \), and therefore the proper cone \( \hat{X} \) is invariant under \( A \), while the proper cone \( \hat{Y} \) is invariant under \( A^T \). Since the matrix \( A \) is either indecomposable or 1-decomposable, we now apply either Lemma 3.4 or Proposition 3.6 to conclude that \( A \) is nonnegative. This completes the proof. \( \square \)

**Corollary 3.9.** Let \( S \subset M_2(\mathbb{C}) \) be an irreducible semigroup such that each member of \( S \) is diagonally similar to a nonnegative matrix. Then \( S \) is (simultaneously) diagonally similar to a semigroup of nonnegative matrices.

We conclude the paper with the following example showing that the (in)decomposability assumptions in Proposition 3.6 and Theorem 3.8 cannot be omitted.

**Example 3.10.** Define the matrix

\[
A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\]

and the proper cones \( K_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq z \geq 0 \} \subset \mathbb{R}^3_+ \) and \( L_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq y \geq 0, z \geq 0 \} \subset \mathbb{R}^3_+ \). It is easy to see that \( K_3 \) is invariant under \( A_3 \), while \( L_3 \) is invariant under \( A_3^T \). For \( n \geq 3 \) we define the proper cones \( K_n = K_3 \oplus \mathbb{R}^{n-3}_+ \) and \( L_n = L_3 \oplus \mathbb{R}^{n-3}_+ \). Now we define an irreducible semigroup \( S_1 = K_nL_n^T \), consisting of matrices of rank at most 1. We extend the matrix \( A_3 \) with a zero block to get a matrix \( A_n = A_3 \oplus 0 \in M_n(\mathbb{R}) \). As \( K_3 \) is invariant under \( A_3 \) and \( L_3 \) is invariant under \( A_3^T \), it is clear that the cones \( K_n \) and \( L_n \) are invariant under \( A_n \) and
$A_n^T$, respectively. Since $A_n^2 = A_n$, $\mathcal{S} = \mathcal{S}_1 \cup \{A_n\}$ is an irreducible semigroup in which each member is diagonally similar to a nonnegative matrix, while the whole semigroup is not diagonally similar to a semigroup of nonnegative matrices.

\[\square\]

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