Transverse modes and instabilities of a bunched beam with space charge and resistive wall impedance

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Transverse instability of a bunched beam is investigated with synchrotron oscillations, space charge, and resistive wall wakefield taken into account. Boxcar model is used for all-round analysis, and Gaussian distribution is invoked for details. The beam spectrum, instability growth rate and effects of chromaticity are studied in a wide range of parameters, both with head-tail and collective bunch interactions included. Effects of the internal bunch oscillations on the of collective instabilities is investigated thoroughly. Landau damping caused by the space charge tune spread is discussed, and the instability thresholds of different modes of Gaussian bunch are estimated.

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I. INTRODUCTION

Transverse instability of a bunch in a ring accelerator has been considered independently by Pellegrini [1] and Sands [2] with synchrotron oscillations and some internal degrees of freedom of the bunch taken into account (“head-tail instability”). Later Sacherer have investigated the effect in depth including dependence of the bunch eigenmodes on amplitude of synchrotron oscillations (“radial modes” [3]).

Space charge field of the bunch was not taken into account in these pioneer works. Its role was studied first in Ref. [1] where it has been shown that the space charge tune spread results in Landau damping which suppresses many of the head-tail modes, much like other sources of the incoherent tune spread.

However, the last result has been obtained with an assumption that the space charge tune shift is significantly less than the synchrotron tune. A closer examination of the problem in Ref. [2] has led to the conclusion that almost all head-tail modes are prone to Landau damping till then the space charge tune shift is about twice less then the synchrotron tune. The damping vanishes when the shift becomes more, lower eigenmodes being free from the decay first. The lowest (rigid) mode is the only universal exception from the rule being potentially unstable with any space charge. Sometimes 1–2 next modes can demonstrate similar behavior, in dependence on the bunch shape.

However, no wakefield was included in the analysis of Ref. [2] in fact, so the results might be interpreted only as conditions which permit the bunch instability but do not determine its characteristics completely. Effects of short wakes were actually studied in Ref [3] where the instability growth rate has been found at different bunch parameters like its length, chromaticity, etc. Transverse modes coupling instability was considered in the work as well, and it has been shown that this more rigorous effect can be caused only by positive wake (for comparison, resistive wall creates a negative wake).

A single bunch was investigated in the mentioned articles because only short-range forces have been accepted in all the cases. More general theory is developed in this paper where a bunched beam with arbitrary number of the bunches is examined taking into account the space charge, intra-bunch and bunch-to-bunch interactions. Both kinds of the interactions affect the beam eigenmodes and the instability growth rate, although one of them can dominate is specific situation. Just from this standpoint the common used terms like “collective modes instability” or “head-tail instability” are treated in the paper.

The presentation is focused first on the resistive wall instability [7–9] but the results can be rather easy adapted for other known wakefields. The boxcar model is intensively used to get a general outlook of the problem in a wide range of parameters. More realistic Gaussian bunch is closely examined in a limiting case of low synchrotron frequency, although the opposite case of high frequency is also invoked to estimate thresholds of Landau damping.

Both betatron and synchrotron bare oscillations are implied to be linear in this paper. The nonlinear effects unquestionably require a special investigation, especially as an additional factors of Landau damping.

II. BUNCHED BEAM GENERAL EQUATIONS

With space charge and wakefield taken into account, equation of coherent betatron oscillations of a bunched beam in the rest frame can be written in the form [6]:

$$\left( \frac{\omega}{\Omega_0} - Q_0 \right) Y + i Q_s \frac{\partial Y}{\partial \phi} + \Delta Q \left( Y - \bar{Y} \right) = 2 \int_0^\infty \exp \left[ i \left( \theta' - \theta \right) \left( \frac{\omega}{\Omega_0} - Q_0 - \zeta \right) \right] \times q \left( \theta' - \theta \right) \bar{Y}(\theta') \rho(\theta') d\theta' \quad (1)$$
where $\theta$ is longitudinal coordinate (azimuth), $\omega$ is frequency of the coherent oscillations, $Q_0$ and $\Omega_0$ are central betatron tune and revolution frequency (in the laboratory frame), $\Delta Q(\theta)$ is the space charge tune shift averaged over transverse directions, $Q_s$ and $\phi$ are synchrotron tune and phase, $\zeta = -\xi/y$ is normalized chromaticity, and $q(\theta)$ is normalized wakefield which specific form will be defined later. The bunch shape in the longitudinal phase space $(\theta, u)$ is described by a distribution function $F(\theta, u)$ with corresponding linear density:

$$\rho(\theta) = \int_{-\infty}^{\infty} F(\theta, u) \, du \quad (2)$$

(normalization of the functions will be specified later). The function $Y(\theta, u)$ is an average transverse displacement of particles located in the point $(\theta, u)$ of the longitudinal space, multiplied by the factor $\exp[i (Q_0 + \zeta) \theta]$. Additional averaging of this function over momentum is denoted as $\bar{Y}(\theta)$ being defined by the expression:

$$\rho(\theta) \bar{Y}(t, \theta) = \int_{-\infty}^{\infty} F(\theta, u) \, Y(t, \theta, u) \, du \quad (3)$$

It is more convenient to represent the right hand part of Eq. (1) as a sum over all the bunches preceding the examined one (including all preceding turns):

$$\nu Y_n + i Q_s \frac{\partial Y_n}{\partial \phi} + \Delta Q (Y_n - \bar{Y}_n)$$

$$= 2 \int_{\theta}^{\infty} \exp[-i \lambda (\theta' - \theta)] q(\theta' - \theta) \bar{Y}(\theta') \rho(\theta') \, d\theta'$$

$$+ 2 \sum_{m=n+1}^{\infty} \int \exp[-i \lambda \left(\frac{2\pi m}{M} + \theta' - \theta\right)]$$

$$\times q \left(\frac{2\pi m}{M} + \theta' - \theta\right) \bar{Y}_{m}(\theta') \rho_{m}(\theta') \, d\theta' \quad (4)$$

where $\nu = \omega/\Omega_0 - Q_0$, $\lambda = \zeta - \nu \approx \zeta$, $M$ is number of bunches in the beam, $\rho_{m}(\theta') = \rho(2\pi m/M + \theta')$, and $Y_m(\theta') = Y(2\pi m/M + \theta')$. Any integral in this expression is taken over one of the bunches which are not presumed yet to be identical (in particular, some of them may be empty). However, periodicity conditions must be satisfied in any case:

$$\rho_{m+M}(\theta) = \rho_m(\theta),$$

$$\bar{Y}_{m+M}(\theta) = \bar{Y}_{m}(\theta) \exp[-2\pi i (Q_0 + \zeta)] \quad (5)$$

### A. Symmetric beam

In subsequent sections we will consider symmetric beams composed of identical equidistant bunches: $\rho_{m}(\theta) = \rho(\theta)$. It would be checked than all the solutions of Eq. (4) fall into $M$ groups which are known as collective modes:

$$\bar{Y}^{(k)}_{n}(\theta) = \bar{Y}(\theta) \exp\left[\frac{2\pi i n}{M}(Q_0 + \zeta - k)\right] \quad (6)$$

where $k = 1, 2, ..., M$. There are the head-tail modes inside the groups, each satisfying the equation:

$$\nu Y + i Q_s \frac{\partial Y}{\partial \phi} + \Delta Q (Y - \bar{Y})$$

$$= 2 \int_{\theta}^{\infty} \exp[-i \lambda (\theta' - \theta)] q(\theta' - \theta) \bar{Y}(\theta') \rho(\theta') \, d\theta'$$

$$+ 2 \sum_{m=1}^{\infty} \exp(-2\pi i m \kappa) \int_{\theta_0}^{\theta_0} \exp[-i \lambda (\theta' - \theta)]$$

$$\times q \left(\frac{2\pi m}{M} + \theta' - \theta\right) \bar{Y}(\theta') \rho(\theta') \, d\theta' \quad (7)$$

where $2\theta_0$ is the bunch length, and $\kappa = (k - \omega/\Omega_0)/M \simeq (k - Q_0)/M$. Note that the head-tail modes are not autonomous formations but each of them more or less depends on the collective mode which it falls.

The variable $\tau = \theta/\theta_0$ will be used further as a new longitudinal coordinate having a range $[-1,1]$, and the normalization condition will be imposed

$$\int_{-1}^{1} \rho(\tau) \, d\tau = 1 \quad (8)$$

### III. RESISTIVE WAKE

Resistive wall instability was predicted first it Ref. [7]. Corresponding wakefield function $q(\theta)$ is negative, and can be presented in the convenient form:

$$q(\theta) = -q_0 \sqrt{\frac{2\pi}{\theta}}, \quad q_0 = \frac{\alpha r_0 N_b R^2}{2\pi \gamma b Q_0 b_y \sqrt{\Omega_0}} \left(\frac{2\pi}{2\pi \sigma}\right) \quad (9)$$

where $N_b$ is number of particles per bunch, $r_0 = e^2/mc^2$ is classic radius of the particle, $R$ is the machine radius, $\sigma$ is specific conductivity of the beam pipe, $b_y$ is its semi-height, and $\alpha$ is the pipe form-factor ($\alpha = 1$ for a round pipe). Then one can rewrite Eq. (7) in the form:

$$\nu Y + i Q_s \frac{\partial Y}{\partial \phi} + \Delta Q (Y - \bar{Y}) = -2q_0 \sqrt{\frac{2\pi}{\theta_0}} \exp(i \lambda \theta_0 \tau)$$

$$\times \left[ \int_{\tau}^{1} y(\tau') \, d\tau' + \sum_{m=1}^{\infty} \exp(-2\pi i m \kappa) \int_{-1}^{1} \frac{y(\tau')}{\sqrt{1 + M + \tau - \tau}} \, d\tau' \right] \quad (10)$$

where $T = 2\pi/M \theta_0$ is the bunch spacing in terms of the variable $\tau$, and the notation is used for a shorthand: $y(\tau) = \bar{Y}(\tau) \rho(\theta_0) \exp(-i \lambda \theta_0 \tau)$. Square root in the last integral not much depends on the addition $(\tau' - \tau)$, so the expansion into Taylor series can be applied resulting in:

$$\nu Y + i Q_s \frac{\partial Y}{\partial \phi} + \Delta Q (Y - \bar{Y}) \approx q_0 \exp(i \lambda \theta_0 \tau)$$

$$\times \left[ -2 \sqrt{\frac{2\pi}{\theta_0}} \int_{\tau}^{1} y(\tau') \, d\tau' + \sqrt{M} V_1(\kappa) \int_{-1}^{1} y(\tau') \, d\tau' \right.$$

$$\left. - V_2(\kappa) \int_{-1}^{1} y(\tau')(\tau' - \tau) \, d\tau' \right] \quad (11)$$
where $B = M \theta_0 / \pi$ is bunch factor, and

\[
V_1(\kappa) = -2 \sum_{m=1}^{\infty} \frac{\exp(-2\pi im\kappa)}{\sqrt{m}}, \quad \text{(12)}
\]

\[
V_2(\kappa) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{\exp(-2\pi im\kappa)}{m\sqrt{m}} \quad \text{(13)}
\]

Both the functions are periodical, one period of $V_2(\kappa)$ being plotted in Fig. 1. Because the function $V_1(\kappa)$ has the singularities, its plot is multiplied by aperiodic factor $\sqrt{|\kappa|}$ for a convenience.

It goes without saying that form of these plots depends on the wakefield. However, in most cases it does not change a general structure of Eq. (11) as well as many subsequent inferences.

### IV. BOXCAR MODEL

The boxcar model is described by the expressions:

\[
F = \frac{1}{2\pi \sqrt{1 - \tau^2 - u^2}}, \quad \rho(\tau) = \frac{1}{2} \quad \text{at} \quad |\tau| < 1 \quad \text{(14)}
\]

where $u$ is normalized transverse momentum conjugated with the longitudinal coordinate $\tau$: $u^2 = A^2 - \tau^2$, $A = \tau_{max}$. A virtue of this model is that all solutions of Eq. (11) are exactly known at $q_0 = 0$ with any parameters $Q_s$ and $\Delta Q$, as it was shown first in Ref. 3 and developed in detail in Ref. 3. This circumstance gives a great possibility to overlook the wakefield produced effects. Required information is shortly reminded below.

All the solutions of Eq. (11) with $q = 0$ are derivable from Legendre polynomials $Y(\tau) = P_n(\tau)$, $n = 0, 1, 2, \ldots$ At any $n$, there are $n + 1$ different eigenfunctions $Y_{n,m}(\tau, u)$ with eigenvalues $\nu_{n,m}$ where $m = n, n - 2, \ldots, 2 - n, -n$. They satisfy the orthogonality conditions:

\[
\int \int \left( F \delta(\nu - \nu_{1,m}) \hat{Y}_{n,m}(\nu, \delta) \delta \nu \right) = \sum_{\nu, \delta} \int \int \left( F \delta(\nu - \nu_{1,m}) \hat{Y}_{n,m}(\nu, \delta) \delta \nu \right) = 0 \quad \text{(15)}
\]

All the eigenvalues are real numbers. Some of them are plotted in Fig. 2 where the space charge tune shift is used for a scaling. At $\mu \equiv Q_s / \Delta Q = 0$, the eigenvalues $\nu_{n,n}$ take start from the point $\nu = 0$ that is $\omega / \Omega_0 = Q_0$ (the bare tune). Other eigenvalues $\nu_{n,m<n}$ start from the point $\nu = -\Delta Q$ which corresponds the incoherent tune with the space charge included: $\omega / \Omega_0 = Q_0 - \Delta Q$. At $\mu \ll 1$, all eigenvalues acquire the additions $\delta \nu \propto Q_s^2 / \Delta Q$. In this limiting case, the eigenfunctions have about linear polarization: $Y_{n,n}$ depends mostly on longitudinal coordinate $\tau$ at any $n$, and all other functions $Y_{n,m<n}$ depend mostly on momentum $u$ (many examples are given in Ref. 3). However, these solutions merge together at $\mu \gg 1$, forming multipoles $Y_{n,m}(A) \exp(im\phi)$ with different dependence from synchrotron amplitude. These radial modes $Y_{n,m}(A)$ are born by Legendre polynomials of powers $n = |m|, |m| + 2$, etc. Note that the functions $Y_{n,n}(A)$ are the lowest (minimally oscillating) radial modes with given $n = m$. Their eigenvalues are $n(n + 1)/2 \times Q_s^2 / \Delta Q$ at small $\mu$ and $n Q_s$ at large one. The functions $Y_{n,m<n}$ are treated usually as higher radial modes.

#### A. Low wake

The study of the boxcar model is continued in this subsection with an assumption that the wake is small enough to apply the perturbation methods (applicability of this approximation will be discussed in Sec. V). Then
\( \lambda = \zeta \) in Eq. (11), and the additions to the eigentunes can be presented in the form:

\[
\Delta \nu_{n,m} = q_0 \Lambda_{n,m}(\mu) \times \left[ 2 \sqrt{\frac{2\pi}{\theta_0}} f_n(\theta_0 \zeta) + \sqrt{M} \left( g_n(\theta_0 \zeta) V_1(\kappa) + i h_n(\theta_0 \zeta) V_2(\kappa) B \right) \right]
\]

with the coefficients:

\[
\Lambda_{n,m}(\mu) = \left[ \int \int F |Y_{n,m}|^2 d\tau du \right]^{-1}
\]

\[
f_n(\theta_0 \zeta) = -\int_{-1}^{1} y^*_n(\tau) d\tau \int_{\tau}^{1} \frac{y(\tau') d\tau'}{\sqrt{\tau - \tau}}
\]

\[
g_n(\theta_0 \zeta) = \left| \int_{-1}^{1} y_n(\tau) d\tau \right|^2
\]

\[
h_n(\theta_0 \zeta) = 2 \text{Im} \int_{-1}^{1} y^*_n(\tau) \tau d\tau \int_{\tau}^{1} y_n(\tau') d\tau'
\]

Generally, these relations are applicable to a bunch of any form with low wakefield. However, the boxcar model is the only known case at present which allows to investigate the problem in depth because its eigenfunctions \( Y_{m,n}(\tau, u) \) and \( y_n(\tau) = F_n(\tau) \rho(\tau) \exp(-i \theta_0 \zeta \tau) \) are really known with any \( \mu \). The results are presented graphically in Figs. 3–7 and commented below.

The coefficients \( \Lambda_{n,m} \) depend only on the ratio \( \mu = Q_s/\Delta Q \), and does not depend on the bunch length or chromaticity (Fig. 3). They describe a general effect of synchrotron oscillations and space charge on transverse coherent motion of the bunches, including possible instability growth rate. It is seen that the space charge tune shift enhances influence of the wakefield on the lowest radial modes \( Y_{n,n} \) but depresses its influence on the higher modes \( Y_{n,m<n} \). It is very understandable result because higher radial modes are polarized mostly in \( u \)-direction at small \( \mu \), so that the global bunch displacement at given azimuth and, correspondingly, excited wakefield should be relatively small in this limiting case.

In contrast with this, the part of Eq. (16) in square brackets describes effects of the bunch length and chromaticity on different \( n \)-modes. There are three terms here which are concerned with different physical effects.

Interaction of particles inside the bunch is described by the coefficients \( f_n(\theta_0 \zeta) \) some of which are plotted in Figs. 4–5. With non-zero chromaticity, this part is capable...
able to cause the head-tail instability of different modes which basic properties were predicted in earliest works [1]–[2]. It is a single-bunch effect which is proportional to the bunch population and does not depend on number of the bunches.

Second term in Eq. (16) describes the main effect of the collective interaction of the bunches (factor \( V_1(\kappa) \)) in Fig. 1 including its dependence on chromaticity (factor \( g_n(\theta_0\zeta) \)). As a rule, the term gives a maximal contribution to the tune shift, especially if the beam consists of many bunches. Indeed, with \( M \gg 1 \) one can get \( \kappa = \sqrt{(k - Q_0)/M} \ll 1 \) that is \( V_1(\kappa) \simeq (1 + \kappa/|\kappa|)/\sqrt{|\kappa|} \) for the most unstable modes \( (k > Q_0) \). Then the total contribution of this term to the tune is:

\[
\Delta \nu_{n,m} \simeq \frac{q_0 M \Lambda_{n,m} g_n(\theta_0\zeta)}{\sqrt{|k - Q_0|}} \left( 1 + i \frac{k - Q_0}{|k - Q_0|} \right) \tag{21}
\]

With Eq. (9) for \( q_0 \) used, it gives the expression:

\[
\frac{\Delta \omega_{nm}}{\Omega_0} = \frac{\alpha r_0 N R \delta(\omega)}{2\pi \gamma Q_0 b_0^*} \left( 1 + i \frac{\omega}{|\omega|} \right) \Lambda_{n,m} g_n(\theta_0\zeta) \tag{22}
\]

where \( N = MN_b \) is the total beam intensity, \( \omega = \Omega_0(k - Q_0) \) is the coherent frequency in the laboratory frame, and \( \delta(\omega) \) is corresponding skin depth. First part of the formula coincides with well known expression for resistive wall instability of a coasting beam [7]–[8]. The factors \( \Lambda_{n,m} \) describe an impact of the bunching upon different head-tail modes, including their dependence on synchrotron frequency and space charge tune shift. Note that the coefficient of the most important rigid mode \( \Lambda_{0,0} = 1 \) independently on the mentioned parameters, that is the bunching does not affect this mode.

The last term \( g_n(\theta_0\zeta) \) in Eq. (22) describes an influence of chromaticity. In this regard, it is pertinent to dwell upon the different character of the chromatic effects in coasting and bunched beams.

In first case, chromaticity leads to a spread of incoherent betatron frequencies which phenomenon can cause Landau damping resulting in total suppressions (prevention) of instability.

In contrast with this, average momenta of all the particles are equalized in the bunch through synchrotron oscillations. In such conditions, chromaticity does not provide a systematic tune spread and cannot bring about Landau damping at once. There is no questions that additional slip of betatron phases of particles with respect to the coherent field phase affects the interaction and can change the instability growth rate. As it follows from Fig. 6, switch of the instability peaks to the higher internal modes is the most descriptive result of this.

However, it is necessary to take into account also other then chromaticity factors which can result in Landau damping. In particular, space charge tune spread itself can produce this effect in bunched beams. As it is shown in Ref. [4]–[5], the higher internal modes are more sensitive to this kind of Landau damping. Therefore, one can expect that the above mentioned shift of the instability peaks is capable to suppress all internal bunch modes except the rigid one which is not prone to this kind of the damping [8]. However, this problem cannot be solved in frames of the boxcar model which ignores this part of the tune spread at all. Therefore, we postpone its detailed analysis to Sec. VI where more realistic Gaussian distribution will be invoked.

A peculiarity of a single bunch beam is that its instability is possible in a restricted region of parameters. As it follows from Fig. 1 at \( M = 1 \), imaginary part of the coefficients \( V_1 \) and \( V_2 \) is positive at \( 0 < k - Q_0 < 0.5 \). It means that, without chromaticity, the instability is feasi-
ble only if betatron tune is located between half-integer and next integer (e.g., \( Q_0 = 0.75 \) but not 0.25). This result was first obtained in Ref. [9] for a short bunch where the chromaticity was ignored by the model. Actually, it is apparent from Eq. (16) that the chromaticity is an essential factor, mainly because it triggers the head-tail interaction. For example, without chromaticity the bunch is quite stable at \( k - Q = -0.25, \ B = 0.5 \). However, the rigid mode becomes unstable if \( \theta_0 \zeta = 1 \) obtaining the growth rate \( \text{Im} \Delta \nu_{0,0} \simeq (0.35 - 0.26) q_0 = 0.09 q_0 \) (first term in this formula is the head-tail contribution, and second one – turn by turn interaction). Of course, chromaticity of opposite sigh could prevent such a situation but higher modes instability would be enforced by this, as it follows from Fig. 6.

The last term in Eq. (16) is a part of the collective interaction which describes the field variation of preceding bunches inside the considered one. Actually, only the nearest bunch gives a noticeable contribution to this part so that the result does not depend on number of bunches, in practice. Influence of this addition looks much like to the head-tail interaction which statement can be checked by comparison of Figs. 5 and 7. However, the effect strongly depends on collective mode as it is described by the coefficient \( V_2(\kappa) \). For the example above, its contribution to the rigid mode instability is about \( -q_0 h_0(\theta_0 \zeta)/16 \) which is less then the “usual” head-tail effect in order of magnitude. Probably, this part of the resistive wake is negligible at any conditions.

**V. LOW SYNCHROTRON FREQUENCY**

The boxcar model gives a broad outlook of the problem but maybe it omits some important details being not sufficiently realistic itself. Therefore another point of view is developed in this section based on the approach \( Q_s \ll \Delta Q \) which is rather characteristic of many proton machines. As it is shown in previous section and illustrated by Fig. 3, space charge suppresses all the modes \( Y_{n,m<n}(\tau, u) \) in this limit. Therefore the following results are actually concerned only with the modes \( Y_{n,n}(\tau) \) which will be denoted later simply as \( Y_n(\tau, u) \). Following Ref. [10] with resistive wakefield added, one can show that the space part of this function satisfies the equation:

\[
U^2(\tau) \tilde{Y}''(\tau) - \left[ \tau + \frac{U^2 \Delta \nu (\tau)}{(\Delta Q + \nu) \rho} \right] \tilde{Y}'(\tau) + \frac{\nu (\Delta Q + \nu)}{Q_s^2} \tilde{Y} = \frac{q_0 \Delta Q \exp(i \lambda \theta_0 \tau)}{Q_s^2} \left[ -2 \sqrt{\frac{2 \pi}{\theta_0}} \int_{\tau}^{1} y(\tau') d\tau' - V B \int_{-1}^{1} y(\tau')(\tau' - \tau) d\tau' \right]
\]

\[ (23) \]

where

\[ U^2(\tau) = \frac{1}{\rho(\tau)} \int F(\tau, u) u^2 du \]  \[ (24) \]

As it is shown in Ref. [11] and [10], the eigentunes of Eq. (23) are \( \sim n(n + 1) Q_s^2 / \Delta Q \) in order of value. Therefore, with rather small synchrotron frequency and for not very high modes, it can be simplified using the approximation \( |\nu| \ll \Delta Q \) which results in:

\[ U^2(\tau) \tilde{Y}''(\tau) = R(\tau) \]  \[ (25) \]

with

\[ R(\tau) = \left( \tau + \frac{U^2 \rho'}{\rho} \right) \tilde{Y}'(\tau) - \frac{\nu \Delta Q}{Q_s^2} \tilde{Y} + \frac{q_0 \Delta Q \exp(i \lambda \theta_0 \tau)}{Q_s^2} \left[ -2 \sqrt{\frac{2 \pi}{\theta_0}} \int_{\tau}^{1} y(\tau') d\tau' + \sqrt{M} \left( V_1 \int_{-1}^{1} y(\tau') d\tau' - V B \int_{-1}^{1} y(\tau')(\tau' - \tau) d\tau' \right) \right] \]

\[ (26) \]

Boundary conditions of this equation are evident from the relation \( U^2(\pm 1) = 1 \) which follows from definition (24) and will be reinforced by examples in the subsequent sections. Therefore, any appropriate solution of Eq. (25) should satisfy the relations:

\[ R(\pm 1) = 0. \]  \[ (27) \]

Because Eq. (25) is linear and uniform, initial conditions \( \tilde{Y}(1) = 1, R(1) = 0 \) can be used in practice to calculate the function \( R(\tau) \) everywhere with any trial \( \nu \), and to separate thereafter the eigentunes \( \nu_n \) assuring the condition \( R(-1) = 0 \). The method is especially effective at \( q_0 = 0 \) to determine the basic modes. In particular, it confirms that Legendre polynomials are solutions of the boxcar bunch. Generally, it is easy to show that the basic solutions of any bunch are regular functions satisfying the orthogonality conditions:

\[ \int_{-1}^{1} Y_{n_1}(\tau) Y_{n_2}(\tau) \rho(\tau) d\tau = \delta_{n_1, n_2} \]  \[ (28) \]

Therefore, with enough small \( q_0 \), additions to the eigentunes can be found by standard perturbation methods which way results in expression like Eq. (16):

\[ \Delta \nu_n = q_0 \left[ \frac{\sqrt{2 \pi}}{\theta_0} f_n + \sqrt{M} \left( g_n V_1 + i h_n V_2 B \right) \right] \]  \[ (29) \]

Eqs. (18)–(20) can be used as well with appropriate eigenfunctions \( y_n(\tau) = Y_n(\tau) \rho(\tau) \exp(-i \theta_0 \tau) \) to calculate the factors \( f_n, g_n, h_n \). All the coefficients \( A_{n,n} = 1 \) in this case due to normalization condition (28).

**A. Gaussian bunch with low wake**

Truncated Gaussian bunch is considered in this subsection for a comprehensive investigation of the instability. Its distribution function is:

\[ F = \frac{C}{2 \sqrt{2 \pi}} \left( \exp \left( -\frac{A^2}{2 \sigma^2} - 1 \right) \right) \]  \[ (30) \]
where the normalizing coefficient $C$ depends on $\sigma$. Other involved functions are:

$$\rho(\tau) = C \left( \frac{\sqrt{\pi}}{2} \exp(T^2) \text{erf}(T) - T \right) \simeq \frac{2CT^3}{3} \left( 1 + \frac{2T^2}{5} \right)$$

(31)

and

$$U^2 = \sigma^2 \left( 1 - \frac{2CT^3}{3\rho} \right) \simeq \frac{2\sigma^2T^2}{5}$$

(32)

where $T^2 = (1 - \tau^2)/(2\sigma^2)$, and approximate expressions at $|\tau| \simeq 1$ are added for references.

The case $\sigma = 1/3$ (3σ truncation, $C = 0.016$) is actually considered below. Six basic normalized eigenfunctions of the bunch are shown in Fig. 8. Their eigenvalues have a form $\nu_n = \alpha_n Q_s^2/\Delta Q_c$ with the coefficients $\alpha_n$ which are also presented in Fig. 8 in the brackets (for comparison: $\alpha_n = n(n+1)/2$ for the boxcar model). Here and further, the subindex $c$ marks the bunch center.

The coefficients $f_n$, $g_n$, and $g_n$ are plotted in Figs. 9–12 which look much like Figs. 4–7 of the boxcar model. Of course, it is necessary to take into account that the Gaussian bunch has less rms length in comparison with the boxcar one (1/3 instead 1/√3), so the Gaussian plots should be proportionally wider. The absence of secondary oscillations is well explicable because of more smooth bunch shapes. With these reservations, one can assert that the boxcar model provides an adequate description of the bunch coherent instability, at least within the limit of low synchrotron frequency.

FIG. 8: Normalized eigenfunctions of truncated Gaussian bunch (0th–5th modes). Corresponding eigennumbers are given in the parentheses.

![FIG. 8: Normalized eigenfunctions of truncated Gaussian bunch (0th–5th modes). Corresponding eigennumbers are given in the parentheses.](image)

FIG. 9: Functions $\text{Re} f_n(\theta\zeta)$ of truncated Gaussian bunch. The right-hand parts of these even functions are shown.

![FIG. 9: Functions $\text{Re} f_n(\theta\zeta)$ of truncated Gaussian bunch. The right-hand parts of these even functions are shown.](image)

FIG. 10: Functions $\text{Im} f_n(\theta\zeta)$ of truncated Gaussian bunch. The right-hand parts of these odd functions are shown.

![FIG. 10: Functions $\text{Im} f_n(\theta\zeta)$ of truncated Gaussian bunch. The right-hand parts of these odd functions are shown.](image)

B. Expanded low wake approach

Formally, relation (29) is applicable if the spectrum shifts $\Delta \nu_n$ are small in comparison with distances between the basic spectrum lines which are

$$\nu_{n+1} - \nu_n = (\alpha_{n+1} - \alpha_n) \frac{Q_s^2}{\Delta Q_c} \simeq (n + 1) \frac{Q_s^2}{\Delta Q_c}$$

Therefore, condition of applicability of Eq. (29) is for the lowest mode:

$$|\Delta \nu_0| \ll \frac{Q_s^2}{\Delta Q_c}$$

(33)
The left-hand part of this expression is close to the instability growth rate which is essentially less of 1 probably in all practical cases (e.g. $|\Delta \nu_0| < 0.1$). However, the right-hand part can be still less, for example $0.05^2/0.25 = 0.01$. The example demonstrates that a violation of condition (33) is quite possible occasion, especially when collective modes instabilities of a multi-bunch beam are examined. Therefore, we consider this important case in greater detail, without the assumption that the multi-bunch contribution is small.

As a first step, we need to solve Eq. (25) at $V_1 = 0$ to know the basic modes of this case. Let us denote corresponding eigenfunctions and eigenvalues as $\Upsilon_n$ and $\nu_n$. Then solution of the total equation can be presented in the form:

$$
\bar{Y}(\tau) = q_0 V_1 (\kappa) \sqrt{M} \int_{-1}^{1} \bar{Y}(\tau') \rho(\tau') \exp(-i\zeta \theta_0 \tau') d\tau'
$$

$$
\times \sum_{n=0}^{\infty} e_n \Upsilon_n(\tau) \quad (34)
$$

where $e_n$ are coefficients of the expansion:

$$
\exp(i \lambda_0 \tau) = \sum_{n=0}^{\infty} e_n \Upsilon_n(\tau) \quad (35)
$$

It immediately results in the dispersion equation:

$$
1 = q_0 V_1 \sqrt{M} \sum_{n=0}^{\infty} \frac{e_n \Upsilon_n(\tau)}{\nu - \nu_n} \int_{-1}^{1} \Upsilon_n(\tau) \rho(\tau') \exp(-i\zeta \theta_0 \tau') d\tau \quad (36)
$$

In principle, new eigenfunctions $\Upsilon_n(\tau)$ and eigentunes $\nu_n$ could be found by the same method which was used for $\bar{Y}_n(\tau)$. However, it would be a more difficult problem to calculate the coefficients $e_n$ because the functions $\Upsilon_n(\tau)$ are not orthogonal, in contrast with $\bar{Y}_n$. Therefore we turn back to the approximation $\Upsilon = \bar{Y}$ which is certainly acceptable at low intra-bunch interaction and does not violate the overall structure of Eq. (36). It results in the dispersion equation

$$
1 = \sum_{n=0}^{\infty} \frac{q_0 V_1 \sqrt{M} g_n(\theta_0 \zeta)}{\nu - \nu_n - q_0 \left(\sqrt{8\pi/\theta_0} f_n + i h_n V_2 B \sqrt{M}\right)} \quad (37)
$$

with the same coefficients $f_n$, $g_n$, $h_n$ as before (Figs. 9–12 for Gaussian bunch). With an additional condition

$$
q_0 |V_1| \sqrt{M} \ll \frac{Q^2}{\Delta \nu_0} \quad (38)
$$

the low wakefield approximation is totally satisfied, and the equation gives the same result as the earlier Eq. (29).

Another easy case is zero chromaticity when sum (37) holds the only term $n = 0$ because $g_n(0) = \delta_{n,0}$. It means that solely the rigid internal mode can be excited without chromaticity. Because it can appear inside any collective mode, the resulting tune shift is:

$$
\frac{\Delta \omega}{\Omega_0} = q_0 \left[2 \sqrt{2\pi} f_0(0) + \sqrt{M} V_1 \left(\frac{k - Q_0}{M}\right)\right] \quad (39)
$$

This expression formally coincides with Eq. (29) at $\zeta = 0$, but can be applied at any value of the coefficient $V_1(\kappa)$.

Generally, series (37) contains restricted number of summands which conclusion follows from Figs. (6) and (11). In particular, one can see that the terms $n = 0$ and 1 give major contributions at $|\theta_0 \zeta| < \sim 3$. Eq. (37) has two actual solutions in this case, at least one of them being unstable. In the extreme case when inverse of Eq. (38)
inequality is fulfilled, one of the eigentunes is real, and another tune is:
\[ \Delta \omega = q_0 V_1(\kappa) \sqrt{M} \left[ g_0(\theta_0 \zeta) + g_1(\theta_0 \zeta) \right] \]
\[ \approx \alpha r N R \delta(\omega) \frac{2 \pi \gamma Q \omega b_0^2}{\sqrt{\left(1 + i \frac{\omega}{|\omega|}\right)}} \left[ g_0(\theta_0 \zeta) + g_1(\theta_0 \zeta) \right] \] (40)

Rather weak dependence of this expression on chromaticity engages an attention. In relative units, the addition is 1 at \( \theta_0 \zeta = 0 \) and about 0.8 at \( \theta_0 \zeta = 3 \). However, it should be mentioned again that the low synchrotron frequency limit is considered here. Role of this factor is discussed in the next section.

VI. THE INSTABILITY THRESHOLD

It could be concluded from previous analysis that the boxcar is a quite adequate model for investigation of the bunched beam instability, and only minor and almost obvious changes are needed for more realistic distributions. However, it would be a premature conclusion because the space charge tune spread is ignored at all in the boxcar model. Meanwhile, at certain situations the spread can cause Landau damping and suppress many unstable modes of a real bunch, as it has been shown in Ref. [4]–[6]. The rigid intra-bunch mode is the only occasion when this mechanism does not work and cannot prevent instability at any combination of parameters.

Unfortunately, at present the problem is adequately covered only in the limiting cases \( \mu \gg 1 \) and \( \mu \ll 1 \). In first case, this kind of Landau damping really works and suppresses almost all internal modes [3]. However, the mechanism is turned off in opposite limiting case which was just the subject of previous section. The only conclusion can be drawn from these facts: this stabilization mechanism has a threshold character and actually begin to work at \( \mu \gg 1 \). The goal of this section is to get more exact estimation of the thresholds. Because the space charge will be treated here as a promoting and dominating factor, the wakefield is excluded from the analysis.

It should be reminded preliminary that Landau damping arises when coherent frequency penetrates rather deeply in a region of incoherent tunes of the system. Then the particles which own tunes are located below or above the coherent tune could be exited in contra-phases by the coherent field, transforming its energy to the incoherent form (beam heating). Such a mechanism affects the beam decoherence and creates the instability threshold.

Well known practical recommendation follows from this statement for coasting beams: incoherent tune spread should exceed the space charge tune shift to avoid the instability. In principle, a wake field (e.g. the resistive wall contribution) affects this criterion, however, its influence is small in practice if the space charge dominates in the impedance budget. The last is just the case of our study.

An additional important property of bunched beams is that, with a coherent frequency \( \omega \), the particles undergo an influence of harmonics of frequencies \( \omega + j \Omega_s \), where \( \Omega_s \) is synchrotron frequency, and \( j \) is integer. An intense energy transfer is possible if any of these frequencies falls in the incoherent region. Fast look in Fig. 2 reveals that harmonics \( j = m \) have the most chances to do this. Therefore, more informative graph can be obtained from Fig. 2 by a transformation of each curve \( \nu_{n,m}(\mu) \) to the \( \nu_{n,m} - m \mu \). The results are presented in Fig. 13 by the solid lines of the same color as in Fig. 2.

Averaged over synchrotron phases incoherent tunes of the truncated Gaussian bunch lie in the region \( -1 < \nu/\Delta Q_s < -0.274 \). Drawing corresponding boundary line in Fig. 13 and assuming that the coherent tunes of Gaussian bunch have about the same behavior as in the boxcar model, one can make the conclusions: (i) the lowest (rigid) mode \((0, 0)\) is unstable with any \( \mu \); (ii) the higher modes \((n, n)\) can be unstable at \( \mu < -0.5 \); (iii) the more is \( n \) the less is corresponding threshold of \( \mu \); (iv) all higher radial modes like \((n, m < n)\) are stable in any case.

These conclusions are in a reasonable agreement with results of Ref. [3], according which only the rigid mode \((0, 0)\) of Gaussian bunch is unstable at \( \mu > 1 \). One can anticipate from Fig. (13) that thresholds of all other modes are located at \( \mu < 0.5 \). It is a region where the low \( \mu \) approximation could be fitted to refine the thresholds of Gaussian bunch. To accomplish this, we consider Eq. (23) with \( q_0 = 0 \) but without the additional simplification \( \nu \ll \Delta Q_s \). Correspondingly, boundary conditions (27) should be changed by the following one:

\[ Y'(\pm 1) = \frac{\nu^2}{Q_s^2} \]

FIG. 13: Transformed eigentunes: solid lines – boxcar model; dashed lines - Gaussian bunch with 3σ truncation.
TABLE I: Instability thresholds of Gaussian bunch with $3\sigma$ truncation (lower radial modes $(n,n)$).

| $n$ | 0  | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|----|
| $Q_s/\Delta Q_c < Q_{s0}$ | $\infty$ | 0.63 | 0.21 | 0.13 | 0.095 |

Obtained eigentunes are plotted in Fig. (13) by dashed lines above the Gaussian incoherent boundary. The crossing points which are the expected thresholds of the head-tail modes are presented in Table 1.

Distributions with more abrupt bunch boundaries demonstrate similar behavior but have higher thresholds. For example, three modes of parabolic bunch retain the stability at $\mu \to \infty$ [4]. The boxcar bunch is an extreme case which is unaffected by this sort of damping at all.

An important conclusion follows from this or similar table, concerning an influence of chromaticity in real conditions. As might be expected from Figs. 6 and 11, an increase of chromaticity does not affect drastically the instability growth rate because its main effect is a simple switch of the bunch oscillations from a lower internal mode to a higher one. For example, it follows from Fig. 11 that the most unstable internal modes of Gaussian bunch are: $n = 1$ at $|\theta_0\zeta| = 4$, and $n = 2$ at $|\theta_0\zeta| = 6$, in both cases the instability growth rate being about 0.45 in used relative units (the head-tail contribution is neglected in the estimations because it is reasonably small in a multibunch beam). However, now we must take into account that these results were obtained at $\mu = 0$. Let us consider the case $\mu = 0.4$ as an another example. Then the modes $n \geq 2$ cannot appear being suppressed by Landau damping. Generally, only 0th and 1st internal modes could be unstable in this case, and chromaticity $|\theta_0\zeta| \approx 10$ is sufficient to suppress both of them that is to reach a total beam stability.

Of course, Table 1 is only an estimation of the thresholds because approximate Eq. (23) lies in its foundation. It would be a good idea to solve more general Eq. (7) with arbitrary $Q_s$ and $\Delta Q_c$, and to use the results for exact instability thresholds of realistic bunches. Note that all eigennumbers of this equation are real. Therefore, the lack of regular solutions with real $\nu$ at some combination of synchrotron frequency and space charge tune shift would be a sign of Landau damping. However, the boxcar model is the only solved case at present.

VII. SUMMARY AND CONCLUSION

Transverse instability of a bunched beam is studied with synchrotron oscillations, space charge, and wakefield taken into account. Resistive wall wakefield is considered as the most universal and practically important case. However, many results have a common sense and, with small changes, can be adapted to other wakes. Boxcar model of the bunch is extensively used in the paper to get a general outline of the problem in wide range of parameters. A realistic Gaussian distribution is invoked in some cases for comparison and more detailed investigations of important problems like Landau damping.

Eigenfunctions and eigentunes of the beam are investigated with both intra-bunch and inter-bunch interactions taken into account. Contributions of the interactions to the instability growth rate are studied over a wide range of the parameters, including effects of the bunch length and chromaticity. It is shown that known head-tail and collective modes instabilities are the extreme cases when one type of the interaction certainly dominates. However, an essential influence of the intra-bunch degrees of freedom on the collective instabilities is especially marked and investigated in detail. In particular, it is shown that a variability of the internal bunch modes explains why the instability growth rate depends on the bunch parameters including space charge tune shift, synchrotron tune, bunch length, and chromaticity.

It is emphasized that the space charge tune spread can cause Landau damping and suppress the instability (other than the space charge sources of the tune spread are not considered in the paper). The phenomenon appears at rather large ratio of synchrotron frequency to the space charge tune shift, lower internal modes obtaining the stability at larger the ratio. Several modes of Gaussian bunch are considered in the paper, and their thresholds are estimated by comparison of the limiting cases.

VIII. ACKNOWLEDGMENTS

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