ON UNIFORMLY DISCONNECTED JULIA SETS

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Abstract. It is well-known that the Julia set of a hyperbolic rational map is quasisymmetrically equivalent to the standard Cantor set. Using the uniformization theorem of David and Semmes, this result comes down to the fact that such a Julia set is both uniformly perfect and uniformly disconnected. We study the analogous question for Julia sets of UQR maps in $S^n$, for $n \geq 2$. Introducing hyperbolic UQR maps, we show that the Julia set of such a map is uniformly disconnected if it is totally disconnected. Moreover, we show that if $E$ is a compact, uniformly perfect and uniformly disconnected set in $S^n$, then it is the Julia set of a hyperbolic UQR map $f : S^N \to S^N$ where $N = n$ if $n = 2$ and $N = n + 1$ otherwise.

1. Introduction

In [DS97], David and Semmes introduced a scale-invariant version of total disconnectedness towards a uniformization of all metric spaces that are quasisymmetric to the standard middle-third Cantor set $C$: A metric space is quasisymmetrically homeomorphic to $C$ if and only if it is compact, doubling, uniformly disconnected and uniformly perfect.

A rich source of Cantor set constructions in $S^n$, for $n \geq 2$, arises from dynamics. As observed in [HP10], if $f$ is a hyperbolic rational map for which the Julia set is totally disconnected, then $J(f)$ is quasisymmetrically equivalent to $C$. Comparing with the uniformization result of David and Semmes, it is clear that $J(f)$ is compact and doubling. Moreover, it is well-known that $J(f)$ is uniformly perfect, see for example [MnDR92]. Hence the important property here is that for a hyperbolic rational map, if $J(f)$ is totally disconnected, then it is uniformly disconnected. Informally, this means that on all scales, the points of $J(f)$ don’t cluster together too much, and is in some sense the opposite notion to uniform perfectness.

The condition that $f$ is hyperbolic cannot be dropped here. Every uniformly disconnected set $X \subset \mathbb{R}^n$ is porous. Then by a result of Luukainen [Luu98, Theorem 5.2], the Assouad dimension, and so also the Hausdorff dimension, is strictly less than $n$. However, Yang [Yan18] exhibited cubic polynomials with totally disconnected Julia set and Hausdorff dimension equal to 2. In these examples, $J(f)$ contains a critical point.

In this paper, we explore the analogous situation in the context of uniformly quasiregular mappings in $S^n$, for $n \geq 2$. For brevity we will call them UQR maps. This class of mappings is the correct generalization of complex dynamics to higher
real dimensions, with a well-developed theory. See Bergweiler’s survey [Ber10] for an introduction to the subject. Again it is clear that if \( f \) is UQR, then \( J(f) \) is compact and doubling. Moreover, \( J(f) \) is uniformly perfect [FN11]. So again the question comes down to the property of uniform disconnectedness.

Our first result shows that for hyperbolic UQR maps, totally disconnected implies uniformly disconnected. We make the definition for hyperbolic UQR maps in the preliminaries below, but it is the same as for rational maps: the Julia set must not intersect the post-branch set. This definition is new in the context of UQR maps, but as we note in section 2 the class is non-empty.

**Theorem 1.1.** Let \( n \geq 2 \). If \( f : \mathbb{S}^n \to \mathbb{S}^n \) is a hyperbolic UQR map and the Julia set is totally disconnected, then it is uniformly disconnected.

Therefore, by the uniformization result of David and Semmes, if \( f \) is a hyperbolic UQR map, then \( J(f) \) is quasisymmetrically equivalent to \( \mathbb{C} \). Note, however, that this does not mean that \( J(f) \) is ambiently homeomorphic to \( \mathbb{C} \) since there do exist hyperbolic UQR maps for which \( J(f) \) is a wild Cantor set, see [FW15].

The next result addresses the converse question of when a uniformizable totally disconnected subset of \( \mathbb{S}^n \) is a Julia set of a hyperbolic UQR map.

**Theorem 1.2.** Let \( n \geq 2 \). If \( E \subset \mathbb{S}^n \) is a compact, uniformly perfect and uniformly disconnected set, then it is the Julia set of a hyperbolic UQR map \( f : \mathbb{S}^N \to \mathbb{S}^N \), where \( N = 2 \) if \( n = 2 \) and \( N = n + 1 \) if \( n \geq 3 \).

One of the tools used in the proof of this result is the conformal trap method. This yields a hyperbolic UQR map \( G : \mathbb{S}^n \to \mathbb{S}^n \) with \( J(G) \) equal to the standard Cantor set \( \mathbb{C} \). Consequently, if there is a quasiconformal map \( F : \mathbb{S}^n \to \mathbb{S}^n \), then \( F(\mathbb{C}) \) is a Cantor set that also arises as a Julia set of a UQR map. This UQR map is just a conjugate of \( G \).

This idea gives one way of improving Theorem 1.2. We call a Cantor set in \( \mathbb{R}^3 \) self-similar if it is the attractor set of an iterated function system (IFS) generated by \( \mathcal{F} = \{ \phi_1, \ldots, \phi_n \} \), where each generator is a contracting similarity of \( \mathbb{R}^n \), and \( \mathcal{F} \) satisfies the open set condition. Evidently such attractors are both uniformly perfect and uniformly disconnected.

**Theorem 1.3.** Let \( X \) be a tame self-similar Cantor set in \( \mathbb{R}^3 \). Then there exists a quasiconformal map \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( F(X) = \mathbb{C} \).

The discussion above immediately yields the following consequence.

**Corollary 1.4.** Let \( X \) be a tame self-similar Cantor set in \( \mathbb{R}^3 \). Then there exists a UQR map \( f : \mathbb{S}^3 \to \mathbb{S}^3 \) such that \( J(f) = X \).

This result extends [Fle19, Theorem 1.3] from two to three dimensions. As will be clear from the proof of Theorem 1.3 the key obstruction to increasing this result to higher dimensions is the lack of results approximating orientation-preserving homeomorphisms by orientation-preserving diffeomorphisms.

We remark that Theorem 1.3 is not true if self-similar is replaced by quasi-self-similar.

**Proposition 1.5.** There exists a compact, uniformly perfect and uniformly disconnected set \( X \subset \mathbb{R}^3 \) such that \( X \) is ambiently homeomorphic to \( \mathbb{C} \) but not ambiently quasiconformal to \( \mathbb{C} \).
We leave it as an open question as to whether Theorem \ref{thm:open_question} is true in general for \( N = n \). Further work in this direction could ask for a classification of the geometry of totally disconnected Julia sets for UQR maps which are not hyperbolic, or even if there are non-hyperbolic UQR maps for which the Julia set is totally disconnected. It may be worth pointing out here that while \( z \mapsto z^d \) is a hyperbolic rational map, the UQR analogues of these constructed in \cite{May97} are not hyperbolic. This is because the branch set consists of rays from 0 to infinity, but the Julia set is the unit sphere in \( \mathbb{R}^n \).

The paper is organized as follows. In section 2, we recall the basics of UQR maps and introduce hyperbolic UQR maps. We also recall some of the geometric notions we will need. In section 3, we prove Theorem \ref{thm:main_result} in section 4, we prove Theorem \ref{thm:fatou_result} and in section 5, we prove Theorem \ref{thm:chordal_result} and Proposition \ref{prop:quasiconformal_equivalence}.

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2. Preliminaries

We denote by \( B(x, r) \) the (open) ball in a metric space \( X \) centered at \( x \in X \) and of radius \( r \). For \( n \geq 2 \) we identify \( \mathbb{R}^n \cup \{ \infty \} \) with \( \mathbb{S}^n \) and use the chordal metric. If \( X = \mathbb{S}^n \) and we want to emphasize the dimension, we write \( B^n(x, r) \).

2.1. Quasiregular maps. A continuous map \( f : E \to \mathbb{R}^n \) defined on a set \( E \subset \mathbb{R}^n \) is called quasiregular if \( f \) belongs to the Sobolev space \( W^{1,n}_\text{loc}(E) \) and if there exists some \( K \geq 1 \) such that

\[
|f'(x)|^n \leq KJ_f(x), \quad \text{for a.e. } x \in E.
\]

Here \( J_f \) denotes the Jacobian of \( f \) at \( x \in E \) and \( |f'(x)| \) the operator norm. The smallest such \( K \) for which this inequality holds is called the outer dilatation and denoted \( K_0(f) \). If \( f \) is quasiregular, then we also have

\[
J_f(x) \leq K' \min_{|h|=1} |f'(x)(h)|, \quad \text{for a.e. } x \in E.
\]

The smallest \( K' \) for which this inequality holds is called the inner dilatation and denoted \( K_I(f) \). Then the maximal dilatation of a quasiregular map \( f \) is \( K(f) = \max\{K_0(f), K_I(f)\} \). We then say that \( f \) is \( K(f) \)-quasiregular. The branch set \( B(f) \) of a quasiregular map \( f : E \to \mathbb{R}^n \) is the the closed set of points in \( E \) where \( f \) does not define a local homeomorphism. See Rickman's monograph \cite{Ric93} for an exposition on quasiregular mappings.

Quasiregular mappings can be defined at infinity and also take on the value infinity. To do this, if \( A : \mathbb{S}^n \to \mathbb{S}^n \) is a Möbius map with \( A(\infty) = 0 \), then we require \( f \circ A^{-1} \) or \( A \circ f \) respectively to be quasiregular via the definition above.

If \( f \) is quasiregular and a homeomorphism, then we say that \( f \) is quasiconformal. Quasiconformality is a generalization of conformality, while quasiregularity is a generalization of holomorphicity. A notion stronger than that of quasiconformality (and better adapted to a general metric space setting) is that of quasisymmetry. A homeomorphism \( f : (X, d) \to (Y, d') \) between metric spaces is quasisymmetric if there exists a homeomorphism \( \eta : [0, +\infty) \to [0, +\infty) \) such that

\[
\frac{d'(f(x), f(a))}{d'(f(x), f(b))} \leq \eta \left( \frac{d(x, a)}{d(x, b)} \right), \quad \text{for all } x, a, b \in X \text{ with } x \neq b.
\]

If we want to emphasize the distortion function \( \eta \), we say that \( f \) is \( \eta \)-quasisymmetric.
2.2. UQR mappings. Note that the composition of two quasiregular mappings is always quasiregular but the maximal dilatation typically increases. A quasiregular map $f$ is uniformly quasiregular (abbr. UQR) if there exists $K \geq 1$ such that for every $m \in \mathbb{N}$, the $m$-th iterate $f^m = f \circ \cdots \circ f$ is $K$-quasiregular.

If $f : \mathbb{S}^n \to \mathbb{S}^n$ is UQR, then the Fatou set of $f$

$$F(f) = \{x \in \mathbb{S}^n : (f^m|_U)_{m=1}^{\infty} \text{ is a normal family for some open set } U \ni x\}$$

and the Julia set of $f$ $J(f) = \mathbb{S}^n \setminus F(f)$.

In the following proposition, we record some properties of Julia sets of UQR mappings on $\mathbb{S}^n$ that we will need for our proofs. For a map $f : \mathbb{S}^n \to \mathbb{S}^n$ and a point $x \in \mathbb{S}^n$, recall the backward orbit $O^-(x) = \{y : f^m(y) = x, m \in \mathbb{N}\}$ and the forward orbit $O^+(x) = \{f^m(x) : m \geq 0\}$.

**Proposition 2.1.** Let $n \geq 2$ and suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is UQR. Then:

(i) $J(f)$ is closed.

(ii) If $g = f^m$, then $J(g) = J(f)$.

(iii) $J(f)$ and its complement $F(f)$ are completely invariant under $f$.

(iv) The exceptional set $E(f)$ (the set consisting of all points with finite backward orbit) is a finite set. Moreover, if $U$ is any open set intersecting $J(f)$, the forward orbit $O^+(U) = \bigcup_{x \in U} O^+(x)$ contains $\mathbb{S}^n \setminus E(f)$.

(v) For any $x \in \mathbb{S}^n$, the closure of the backward orbit $\overline{O}^-(x)$ contains $J(f)$. If $x \in J(f)$, then it equals $J(f)$.

(vi) $J(f)$ is uniformly perfect.

The proof of the first five of these properties can be found in [Ber10]. The final property is from [FN11].

We now introduce the notion of a hyperbolic UQR map.

**Definition 2.2.** Let $n \geq 2$ and let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a non-injective UQR map.

(i) The post-branch set of $f$ is

$$\mathcal{P}(f) = \overline{\{f^m(B(f)) : m \geq 0\}}.$$  

(ii) The map $f$ is called hyperbolic if $J(f) \cap \mathcal{P}(f)$ is empty.

This definition is the obvious analogue of the usual one for rational maps, but here it is a little more restrictive since the branch set of a quasiregular map in $\mathbb{S}^n$, for $n \geq 3$, cannot have isolated points. As noted in the introduction, this means that the UQR power maps are not hyperbolic and neither are the UQR analogues of Chebyshev polynomials. However, there do exist hyperbolic UQR maps. The UQR map constructed in [FW15] is in fact conformal and expanding on a neighbourhood of its Julia set. It follows that the branch set is in the escaping set and hence its orbit cannot approach $J(f)$. Moreover, the conformal trap construction from [Mar97, MP10] give hyperbolic UQR maps. Note that all these examples have a totally disconnected Julia set.

2.3. Quasi-self-similarity. A metric space $(X, d)$ is $c$-uniformly perfect if there exists $c \geq 1$ such that for any $x \in X$ and any $r \in (0, \text{diam } X)$, the set $B(x, r) \setminus B(x, r/c)$ is nonempty. A metric space $(X, d)$ is $c$-uniformly disconnected if there exists $c \geq 1$ such that for any $r \in (0, \text{diam } X)$ there exists a set $E \subset B(x, r)$ containing $x$ such that $\text{diam } E \leq r$ and $\text{dist}(E, X \setminus E) \geq r/c$. 

Following Piaggio [CP11], given a constant \( r_0 > 1 \) and a homeomorphism \( \eta : [0, +\infty) \to [0, +\infty) \), we say that a metric space \((X, d)\) is \((\eta, r_0)\)-quasi-self-similar if for every \( x \in X \) and \( r \in (0, \text{diam } X) \) there exists an \( \eta\)-quasisymmetric \( \phi : B(x, r) \to X \) such that

\[
B(\phi(x), r_0) \subset \phi(B(x, r)).
\]

Note that our definition of quasi-self-similarity is slightly weaker of that of Piaggio as we make no assumption on the size of the ball \( B(\phi(x), r_0) \). However, if \( X \) is \( c \)-uniformly perfect, then it is easy to see that \( \text{diam } B(\phi(x), r_0) \geq r_0/c \). By Proposition 2.1(vi), we can use this definition of quasi-self-similarity when discussing Julia sets of UQR maps.

3. Proof of Theorem 1.1

The aim of this section is to prove that Julia sets of hyperbolic UQR mappings are quasi-self-similar. The latter result coupled with the next lemma yields Theorem 1.1.

**Lemma 3.1.** Suppose that \( X \) is compact, uniformly perfect, quasi-self-similar and totally disconnected. Then \( X \) is uniformly disconnected.

**Proof.** Suppose that \( X \) is \( c \)-uniformly perfect and \((r_0, \eta)\)-quasi-self-similar. Rescaling, we may assume that \( \text{diam } X = 1 \). Since \( X \) is compact and totally disconnected, there exists a homeomorphism \( f : \mathcal{C} \to X \) where \( \mathcal{C} \) is the standard Cantor set. Recall that \( \mathcal{C} \) is the attractor of the IFS \((\mathbb{R}, \{\phi_1, \phi_2\})\) where

\[
\phi_i(x) = \frac{x}{3} + \frac{2(i-1)}{3}, \quad i = 1, 2.
\]

For each \( k \in \mathbb{N} \) and \( w = i_1 \cdots i_k \in \{1, 2\}^k \), we set \( X_w = f(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(\mathcal{C})) \). By the uniform continuity of \( f \), there exists \( \delta_0 \in \mathbb{N} \) such that for any \( w \in \{1, 2\}^{k_0} \),

\[
\text{diam } X_w < \delta_0 := r_0 \min \left\{ (2c)^{-1}, (2c)^{-1} \eta^{-1} ((4c)^{-3}) \right\},
\]

where \( \theta : [0, +\infty) \to [0, +\infty) \) is defined by \( \theta(t) = (\eta^{-1}(t^{-(1)}))^{-1} \). Recall that the inverse of an \( \eta\)-quasisymmetric map is \( \theta\)-quasisymmetric [Hei01, Proposition 10.6]. Define also

\[
d_0 := \min_{w \in \{1, 2\}^{k_0}} \text{dist}(X_w, X \setminus X_w).
\]

Fix \( x \in X \) and \( r > 0 \). Then, there exists an \( \eta\)-quasisymmetric map \( \phi : B(x, r) \to X \) such that

\[
B(\phi(x), r_0) \subset \phi(B(x, r)).
\]

Let \( w \in \{1, 2\}^{k_0} \) such that \( \phi(x) \in X_w \). Then, by the choice of \( \delta_0 \) we have that \( X_w \subset B(\phi(x), (2c)^{-1} r_0) \). Set \( E = \phi^{-1}(X_w) \). We show that \( \text{diam } E \) is less or comparable to \( r \), while its distance from \( X \setminus E \) is at least comparable to \( r \).

Firstly, by the uniform perfectness of \( X \), we know that

\[
\text{diam } \phi(B(x, r)) \geq \text{diam } B(\phi(x), r_0) \geq c^{-1} r_0.
\]

Therefore, by Proposition 10.8 in [Hei01] and the choice of \( \delta_0 \),

\[
(3.1) \quad \text{diam } E \leq \theta \left( \frac{2 \text{diam } X_w}{\text{diam } \phi(B(x, r))} \right) \text{diam } B(x, r) \leq 2 \theta (2c \delta_0 r_0^{-1}) r < (2c)^{-3} r.
\]

By the uniform perfectness of \( X \), there exist a point \( y_1 \in B(x, r) \setminus B(x, r/c) \) and a point \( y_2 \in B(x, (2c)^{-3} r) \setminus B(x, (2c)^{-3} r) \). Therefore,

\[
\text{diam}(B(x, r) \setminus E) \geq |y_1 - y_2| \geq r(c^{-1} - \frac{1}{8} c^{-3}).
\]
We now estimate \( \text{dist}(E, X \setminus E) \). By the choice of \( \delta_0 \), we have that \( X_w \subset B(\phi(x), (2c)^{-1}r_0) \) and, by uniform perfectness of \( X \), \( \text{diam} B(\phi(x), r_0) \geq r_0/c \). Hence,

\[
\text{diam}(\phi(B(x, r) \setminus E)) \geq \text{diam}(B(\phi(x), r_0) \setminus X_w) \geq (2c)^{-1}r_0.
\]

Now, by \([\text{Lyub88}] \) p. 532, setting \( \psi(t) = (\theta(t^{-1}))^{-1} \), we have

\[
\text{dist}(E, X \setminus E) = \text{dist}(E, B(x, r) \setminus E)
\]

\[
\geq \frac{1}{2} \psi \left( \frac{\text{dist}(X_w, \phi(B(x, r) \setminus E))}{\text{diam}(\phi(B(x, r) \setminus E))} \right) \text{diam}(B(x, r) \setminus E)
\]

\[
\geq \frac{1}{2} \psi \left( \frac{d_0}{\text{diam} X} \right) c^{-1}r.
\]

\[
\geq \frac{1}{2} \psi (d_0) c^{-1}r.
\]

\[\square\]

**Remark 3.2.** The uniform disconnectedness constant can not depend only on \( r_0, \eta \) and \( c \). Indeed, for any \( \epsilon \in (0, 1/2) \) let \( X_\epsilon \) be the Cantor set which is the attractor of the IFS \((\mathbb{R}, \{\phi_1, \phi_2\})\) with

\[
\phi_i(x) = (1 - \epsilon)x/2 + (i - 1)(1 + \epsilon)/2, \quad i = 1, 2.
\]

Since \( \epsilon \) is bounded away from 1, it is easy to see that \( X_\epsilon \) is \( c \)-uniformly perfect for some universal \( c \). Moreover, since \( X_\epsilon \) is self-similar, it is also \((\eta, 1)\)-quasi-self-similar with \( \eta(t) = t \). However, for any \( C > 1 \) there exists \( \epsilon \in (0, 1/2) \) such that \( X_\epsilon \) is not \( C \)-uniformly disconnected.

For the rest of this section we will use the chordal metric \( \sigma \) on \( S^\alpha \). If \( E, F \) are closed sets in \( \mathbb{R}^\alpha \cup \{\infty\} \), then \( \sigma(E, F) \) denotes the chordal distance between them. Moreover, denote by \( L_f(x, r) \) the quantity

\[
L_f(x, r) = \max_{\sigma(y, x)=r} \sigma(f(y), f(x)).
\]

**Lemma 3.3.** Suppose that \( f \) is a hyperbolic UQR map. There exists \( r_1 > 0 \) such that if \( x \in J(f) \), then \( f \) is injective on \( B(x, r_1) \).

**Proof.** For each \( x \in J(f) \), let \( r_x \) denote the supremum of radii \( r \) so that \( f \) is injective on \( B(x, r) \). Since \( f \) is hyperbolic, \( r_x > 0 \) for each \( x \in J(f) \).

Now suppose the result was false. Then there would exist a sequence \( x_n \in J(f) \) with \( r_{x_n} \rightarrow 0 \). Passing to a subsequence if necessary, and recalling that \( J(f) \) is compact, we may assume by relabelling that \( x_n \rightarrow x_0 \). Since \( J(f) \) is closed, \( x_0 \in J(f) \). Then there is no neighbourhood of \( x_0 \) on which \( f \) is injective. To see this, if \( \epsilon > 0 \), we can find \( N \) large enough so that \( B(x_N, r_{x_N}) \subset B(x_0, \epsilon/2) \).

This means that \( x_0 \in B(f) \). However, since \( f \) is hyperbolic, we arrive at a contradiction. \( \square \)

**Theorem 3.4.** If \( f \) is a hyperbolic UQR map, then \( J(f) \) is quasi-self-similar.

**Proof.** Recalling \( r_1 \) from Lemma 3.3 let \( r_2 = \min\{r_1, \sigma(J(f), \mathcal{P}(f))\} \). Then let \( U \) be an \( r_2 \)-neighbourhood of \( J(f) \). Note that \( U \) cannot be all of \( S^\alpha \) since \( B(f) \) is non-empty. By construction, \( \partial U \subset F(f) \).

Since the backward orbit of any non-exceptional point accumulates on \( J(f) \) by Proposition 2.11(v), we can find \( N \in \mathbb{N} \) such that \( f^{-N}(U) \subset U \).

Set \( g = f^N \). Then \( J(g) = J(f) \) by Proposition 2.11(ii) and we have \( \overline{g^{-1}(U)} \subset U \). In particular, \( \partial g^{-1}(U) \) is contained in \( U \), is compact and is in \( F(f) \). Hence
σ(∂g⁻¹(U), J(g)) := δ > 0. Moreover, g⁻¹(U) ∩ B(g) = ∅ since f, and hence g, is hyperbolic. The point is that if x ∈ J(g) and 0 < t < δ, then g is quasiconformal on B(x, t) and g(B(x, t)) ⊂ U.

Now suppose r < δ and x ∈ J(g). Let B = B(x, r) and find C > 1 so that B' = B(x, Cr) ⊂ U. Since B' is an open set meeting J(g) and the forward orbit of B' under g omits at most finitely many points, then for some m ∈ N, gᵐ(B') contains B(gᵐ(x), r₂). As long as gᵐ is K-quasiconformal on B', the egg-yolk principle [Hei01] Theorem 11.14 implies that gᵐ is η-quasisymmetric on B. It follows that gᵐ(B) contains the ball

\[ B \left( gᵐ(x), \frac{L_gᵐ(x, r)}{η(C)} \right). \]

Find M minimal so that \( L_g%M(x, Cr) \geq δ \). By construction, we must also have \( L_g%M(x, Cr) < r₂ \). Hence \( g%M(B) \) contains the ball

\[ B \left( g%M(x), \frac{δ}{η(C)} \right). \]

We therefore have obtained the condition for quasi-self-similarity of \( J(f) \) with \( r₀ = δ/η(C) \) and \( φ = g%M|_{J(g)} = f^N|M|_{J(f)} \). If \( r ≥ δ/η(C) \), then we may just take \( φ \) to be the identity map. Combining these cases, we conclude that \( J(f) \) is quasi-self-similar.

4. Proof of Theorem 1.2

Recall that David and Semmes proved that a metric space is quasisymmetrically homeomorphic to \( C \) if and only if it is compact, doubling, uniformly disconnected and uniformly perfect. Later, MacManus improved that result for sets in \( \mathbb{R}² \) by showing that a set \( E ⊂ \mathbb{R}² \) is quasisymmetric homeomorphic to \( C \) if and only if it is the image of \( C \) under a quasiconformal homeomorphism of \( \mathbb{R}² \). MacManus’ result is false in \( \mathbb{R}³ \) due to the existence of self-similar wild Cantor sets in \( \mathbb{R}³ \) [Dav07] pp. 70–75, but by increasing the dimension by 1, MacManus’ result generalizes to dimensions \( n ≥ 3 \).

**Theorem 4.1** ([Mac99] [Vel16]). Given \( c, C > 1 \) and integer \( n ≥ 2 \), there exists \( K ≥ 1 \) depending on \( c, C, n \) such that if a set \( E ⊂ \mathbb{R}ⁿ \) is compact, \( c \)-uniformly perfect, and \( C \)-uniformly disconnected, then there exists a \( K \)-quasiconformal mapping \( F : \mathbb{R}^N → \mathbb{R}^N \) with \( F(C) = E \), where \( N = 2 \) if \( n = 2 \), and \( N = n + 1 \) if \( n ≥ 3 \).

For the proof of Theorem 1.2 we require the following well-known lemma which says that the standard Cantor set is the Julia set of a hyperbolic UQR map. We include a proof for completeness; see also [MP10] and [Mar97]. The main novelty is that we check the constructed map is hyperbolic.

**Lemma 4.2.** Let \( n ≥ 2 \). There exists a hyperbolic UQR map \( G : \mathbb{S}ⁿ → \mathbb{S}ⁿ \) whose Julia set is the standard Cantor set \( C \).

**Proof.** Let \( p₀ = (-1, 0, 0, \ldots, 0) \), \( p₁ = (0, 1, 0, \ldots, 0) \) and \( p₂ = (0, -1, 0, \ldots, 0) \). Let \( g : \mathbb{R}ⁿ → \mathbb{R}ⁿ \) with

\[ g(r, \theta, x₁, \ldots, xₙ) = (r, 2\theta, x₃, \ldots, xₙ) \]

where the first two coordinates of \( \mathbb{R}ⁿ \) are in polar coordinates. It is easy to see that \( g \) is a bounded length distortion map with branch set the hyperplane \{ (0, 0) \} × \mathbb{R}ⁿ⁻².
and that

\[
g^{-1}(p_0) = \{p_1, p_2\}, \quad g(p_0) = (1,0,\ldots,0).\]

Let \(r_0 > 0\) so that \(g^{-1}(B(p_0, r_0))\) has exactly two components, one containing \(p_1\) and another containing \(p_2\). Choose also positive constants \(a, b\) so that \(b < a/2\) and

(i) \(B(p_i, a) \subset g^{-1}(B(p_0, r_0))\) for \(i = 1, 2\);
(ii) \(B(p_0, b) \subset g(B(p_i, a))\) for \(i = 1, 2\);
(iii) \(g(B(p_0, b)) \subset B(g(p_0), a) \subset g(B(p_0, r_0))\).

Now we define \(\tilde{g} : \mathbb{R}^n \to \mathbb{R}^n\) with the following rules

(i) \(\tilde{g}|_{\mathbb{R}^n \setminus \bigcup_{i=0,1,2} B(p_i, a)} = g|_{\mathbb{R}^n \setminus \bigcup_{i=0,1,2} B(p_i, a)}\);
(ii) for each \(i = 0, 1, 2\), \(\tilde{g}|_{B(p_i, b)}\) is a translation of \(B(p_i, b)\) onto \(B(g(p_i), b)\);
(iii) on each annulus \(B(p_i, a) \setminus B(p_i, b)\), \(\tilde{g}\) is defined as the quasiconformal extension of \(\tilde{g} : \partial B(p_i, a) \cup \partial B(p_i, b) \to g(\partial B(p_i, a)) \cup \partial B(g(p_i), b)\) given by Sullivan’s Annulus Theorem \([TV81,\, \text{Theorem 3.17}]\).

Clearly \(\tilde{g}\) extends to a quasiregular map \(S^n \to S^n\) that, by slight abuse of notation, we still call \(\tilde{g}\). Finally, define \(G : S^n \to S^n\) by \(G = \Phi \circ \tilde{g}\) where \(\Phi : S^n \to \mathbb{R}^n\) is the conformal inversion that maps \(\partial B(p_0, b)\) onto itself.

By construction, \(f|_{B(p_0, b)}\) is conformal and hence if an orbit ever ends up in \(B(p_0, b)\) it stays there. This is called a conformal trap. It turns out that the only way an orbit does not end up in \(B(p_0, b)\) is if it stays in \(B(p_1, b) \cup B(p_2, b)\). However, \(f\) is also conformal on this set. Hence any orbit is obtained by

(i) either always applying a conformal map,
(ii) or applying finitely many conformal maps, then a map with distortion and then conformal maps from there on.

It follows that \(G\) is UQR, the Julia set of \(G\) is a tame Cantor set contained in \(B(p_1, b) \cup B(p_2, b)\) (see \([MP10]\)) and that \(B(G) = B(g) = (\{(0,0)\} \times \mathbb{R}^{n-2}) \cup \{\infty\}\).

Finally, if \(x \in B(G)\), then \(\tilde{g}(x) = x\) and \(G(x) \in B(p_0, b)\). On the other hand, for any \(x \in B(p_0, b)\), we have \(G(x) \in B(p_0, b)\). Therefore,

\[
\mathcal{P}(G) \subset B(p_0, b) \cup (\{(0,0)\} \times \mathbb{R}^{n-2}) \cup \{\infty\}
\]

and \(G\) is hyperbolic. \(\Box\)

**Proof of Theorem 4.2** Let \(F : \mathbb{R}^N \to \mathbb{R}^N\) be the quasiconformal map from Theorem 4.1 Clearly \(F\) extends to a quasiconformal map \(S^N \to S^N\) that, again by slight abuse of notation, we still call \(F\).

By Lemma 4.2 there exists a non-injective UQR map \(G : S^N \to S^N\) such that \(J(G) = C\). Define now \(f : \mathbb{R}^N \to \mathbb{R}^N\) with \(f = F \circ G \circ F^{-1}\). Since \(f^k = F \circ G^k \circ F^{-1}\), it is clear that \(f\) is non-injective and UQR. It is immediate that \(J(f) = F(J(G)) = F(C) = E\).

Moreover, \(B(f) = F(B(G))\) and it follows that \(\mathcal{P}(f) = F(\mathcal{P}(G))\). Therefore, since \(\mathcal{P}(G) \cap B(G) = \emptyset\), it follows that \(\mathcal{P}(f) \cap B(f) = \emptyset\) and \(f\) is hyperbolic. \(\Box\)

5. **Self-similar tame Cantor sets in dimension three**

In this section, we discuss when self-similar tame Cantor sets in \(\mathbb{R}^3\) are ambiently quasiconformal to \(C\), or not, as the case may be.
Proof of Theorem 1.3. Suppose that \( X \) is the attractor of an IFS \((\mathbb{R}^3, \mathcal{F})\) where \( \mathcal{F} = \{\phi_1, \ldots, \phi_n\} \) are contracting similarities with the open set condition. Let \( C_n \) be the Cantor set which is the attractor of the IFS \((\mathbb{R}^3, \{\psi_1, \ldots, \psi_n\})\) where

\[
\psi_i(x, y, z) = \frac{1}{2n-1}(x + 2i - 2, y, z).
\]

Clearly \( C_n \) is ambiently homeomorphic to \( C \). We construct a quasiconformal homeomorphism \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( F(X) = C_n \).

Let \( f : X \to C_n \) be an orientation preserving homeomorphism. Let \( B' \) be a topological ball such that the balls \( \psi_i(B') \) have mutually disjoint closures contained in \( B' \). Since orientation preserving homeomorphisms of \( \mathbb{R}^3 \) can be approximated by orientation preserving diffeomorphisms \([\text{Mum60}]\), there exists a topological ball \( B \) with smooth boundary such that the balls \( \phi_i(B) \) have mutually disjoint closures contained in \( B \).

Define \( f : \partial B \cup \bigcup_{i=1}^n \partial \phi_i(B) \to \partial B' \cup \bigcup_{i=1}^n \partial \psi_i(B') \) so that \( f|_{\partial B} : \partial B \to \partial B' \) is an orientation preserving diffeomorphism and for each \( i = 1, \ldots, n \),

\[
f|_{\partial \phi_i(B)} = \psi_i \circ f|_{\partial B} \circ \phi_i^{-1}|_{\partial \phi_i(B)}.
\]

We claim that there exists a quasiconformal extension

\[
F : \overline{B} \setminus \bigcup_{i=1}^n \phi_i(B) \to \overline{B'} \setminus \bigcup_{i=1}^n \psi_i(B'),
\]

Assuming the claim, we can extend \( F \) quasiconformally to \( \overline{B} \setminus X \) by setting

\[
F|_{\phi_w(\overline{B}) \cup \bigcup_{i=1}^n \phi_i(B)} = \psi_w \circ F|_{\overline{B} \cup \bigcup_{i=1}^n \phi_i(B)} \circ \phi_w^{-1}, \quad \text{for} \ w \in \{1, \ldots, n\}^k.
\]

Moreover, we can extend \( F \) quasiconformally to \( \mathbb{R}^3 \setminus B \) by Ahlfors extension theorem \([\text{Ahf63}]\). Now, by a theorem of Väisälä for removable singularities \([\text{Väi71}]\), \( F \) extends quasiconformally to \( \mathbb{R}^3 \) and maps \( X \) onto \( C_n \).

To prove the claim, let \( Q_1, \Delta_1, Q_1', \Delta_1', Q_1, \ldots, Q_n, \Delta_1, \ldots, \Delta_n, Q_1', \ldots, Q_n', \Delta_1', \ldots, \Delta_n' \) be open cubes in \( \mathbb{R}^3 \) with the following properties:

(i) \( \overline{B} \cup \overline{\Delta} \subset Q \) and \( \overline{B'} \cup \overline{\Delta'} \subset Q' \);

(ii) for each \( i \in \{1, \ldots, n\} \) we have \( \overline{\Delta}_i \cap \Delta_i \subset \overline{\Delta}_i \) and \( \overline{\Delta}_i' \subset \Delta_i' \subset \Delta_i \).

We now construct two quasiconformal maps

\[
G : \overline{B} \setminus \bigcup_{i=1}^n B_i \to \overline{Q} \setminus \bigcup_{i=1}^n Q_i, \quad G' : \overline{B'} \setminus \bigcup_{i=1}^n B_i' \to \overline{Q'} \setminus \bigcup_{i=1}^n Q_i'.
\]

Assuming we have these maps, we set

\[
h : \partial Q \cup \bigcup_{i=1}^n \partial Q_i \to \partial Q' \cup \bigcup_{i=1}^n \partial Q_i' \quad \text{with} \ h = G' \circ f \circ G^{-1}.
\]

Applying Sullivan’s Annulus Theorem, we can extend \( h \) to

\[
h : (\overline{Q} \setminus \Delta) \cup \bigcup_{i=1}^n (\overline{\Delta}_i \setminus Q_i) \to (\overline{Q'} \setminus \Delta') \cup \bigcup_{i=1}^n (\overline{\Delta}_i' \setminus Q_i')
\]

so that \( h|_{\partial \Delta} \) is a similarity mapping \( \partial \Delta \) onto \( \partial \Delta' \) and for each \( i \in \{1, \ldots, n\} \), \( h|_{\partial \Delta_i} \) is a similarity mapping \( \partial \Delta_i \) onto \( \partial \Delta_i' \). By Proposition 4.8 in \([\text{Veli16}]\), there exists a
quasiconformal extension of $h$

$$H : \overline{Q} \setminus \bigcup_{i=1}^{n} Q_i \to \overline{Q} \setminus \bigcup_{i=1}^{n} Q'_i$$

and we can set $F = (G')^{-1} \circ H \circ G$.

It remains to construct the maps $G, G'$. We only work for $G$; the construction of $G'$ is similar. Let $D, D_1, \ldots, D_n$ be balls with smooth boundary such that $\overline{D} \subset B$, and for every $i = 1, \ldots, n$, $\overline{D}_i \subset D_i \subset D$. Define now $G$ as follows:

(i) $G|_{\partial B} : \partial B \to \partial Q$ is an orientation preserving diffeomorphism, $G|_{\partial D} : \partial D \to \partial D$ is the identity and $G|_{\overline{B} \setminus \partial B} : \overline{B} \setminus \partial B \to \overline{Q} \setminus B$ is the quasiconformal extension of the latter two diffeomorphisms given by Sullivan’s Annulus Theorem;

(ii) $G|_{\partial D_i} : \partial D_i \to \partial Q_i$ is an orientation preserving diffeomorphism, $G|_{\partial D_i} : \partial D_i \to \partial D_i$ is the identity and $G|_{\overline{D}_i \setminus \partial D_i} : \overline{D}_i \setminus \partial D_i \to \overline{Q_i} \setminus Q_i$ is the quasiconformal extension of the latter two diffeomorphisms given by Sullivan’s Annulus Theorem;

(iii) $G|_{\bigcup_{i=1}^{n} D_i} : \bigcup_{i=1}^{n} D_i \to \overline{Q} \setminus \bigcup_{i=1}^{n} D_i$ is the identity. \hfill $\Box$

**Proof of Proposition 1.3** Let $(T_0, \mathcal{F})$ be the IFS generating the Antoine necklace, where $T_0$ is a closed solid torus with smooth boundary and $\mathcal{F} = \{ \phi_1, \ldots, \phi_n \}$ are contracting similarities mapping $T_0$ into $T_0$; see [FW15, §3.1] for a precise description. Let also $\{ \phi'_i : i = 1, \ldots, n \}$ be contracting similarities of $\mathbb{R}^3$ such that the tori $\phi'_i(T_0)$ are contained in the interior of $T_0$ and are mutually disjoint and unlinked.

Let $\varepsilon$ be the empty word and let $\psi : T_0 \to T_0$ be the identity map. Inductively, suppose that for some $k \in \mathbb{N}$ and for some word $w$ in $\{1, \ldots, n\}^{k(k-1)/2}$ we have defined $\psi_w$.

- If $w \neq n^{k(k-1)/2} := nn \cdots n$, then for any word $u \in \{1, \ldots, n\}^{k+1}$ set $\psi_{wu} = \psi_w \circ \phi'_u$.
- If $w = n^{k(k-1)/2} := nn \cdots n$, then for any word $u \in \{1, \ldots, n\}^{k+1}$ set $\psi_{wu} = \psi_w \circ \phi_u$.

Let $X$ be the Cantor set obtained via defining sequence $(X_i)_{i=1}^{\infty}$, where $X_i$ is the collection of all $\psi_w(T_0)$ with the word length of $w$ equal to $i$. It is straightforward to check that $X$ is compact, uniformly perfect, uniformly disconnected and ambiently homeomorphic to $C$.

For a contradiction, assume that there exists a $K$-quasiconformal map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(X) = C$. Let $k \in \mathbb{N}$, let $w_1 = n^{k(k-1)/2}$, $w_2 = n^{k(k-1)/2}$ and let $A_1 = \psi_{w_1}(X)$ and $A_2 = \psi_{w_2}(X)$. By the quasisymmetry of $f$, there exists $C > 1$ such that

$$C^{-1} \text{diam } f(A_1) \leq \text{dist}(f(A_1), f(A_2)) \leq C \text{ diam } f(A_1)$$

$$C^{-1} \text{diam } f(A_1) \leq \text{diam } f(A_2) \leq C \text{ diam } f(A_1)$$

Then there exist topological balls $B_1, B_2 \subset \mathbb{R}^3$ that contain $f(A_1), f(A_2)$, respectively, such that

$$(2C)^{-1} \text{diam } f(A_1) \leq \text{dist}(B_1, B_2) \leq 2C \text{ diam } f(A_1)$$

and for $i = 1, 2$ and for all $x \in f(A_i)$

$$(2C)^{-1} \text{diam } f(A_1) \leq \text{dist}(x, \partial B_i) \leq (2C) \text{ diam } f(A_1).$$
By the quasiconformality of $f^{-1}$, there exists $C' > 1$ depending only on $C$ and $K$, and there exist two mutually disjoint topological balls $B'_i = f^{-1}(B_i)$ such that for $i = 1, 2$, $A_i \subset B'_i$, and for all $x \in f(A_i)$

$$(C')^{-1} \text{diam } A_i \leq \text{dist}(x, \partial B'_i) \leq C' \text{diam } A_i.$$ 

However, if we choose $k$ sufficiently large, there exist two 3-dimensional manifolds with boundary $M_1, M_2$ such that $A_i \subset M_i \subset B'_i$ and $M_1, M_2$ are linked to each other. This leads to a contradiction. 

□

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