WEAKENING IDEMPOTENCY IN $K$-THEORY

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Abstract. We show that the $K$-theory of $C^*$-algebras can be defined by pairs of matrices satisfying less strict relations than idempotency.

1. Introduction

$K$-theory of a $C^*$-algebra $A$ is patently defined by pairs (formal differences) of idempotent matrices (projections) over $A$. Regretfully, being a projection is a very strict property, and it is usually very hard to find projections in a given $C^*$-algebra. Many famous conjectures (Kadison, Novikov, Baum–Connes, Bass, etc.) are related to projections and would become more tractable if one could provide enough projections for a given $C^*$-algebra. Our aim is to show that the $K$-theory can be defined using less restrictive relations in hope that it would be easier to find elements satisfying these relations than the genuine idempotency. We show that $K$-theory is generated by pairs $a, b$ of matrices over $A$ satisfying $(a - a^2)(a - b) = (b - b^2)(a - b) = 0$, which means that $a$ and $b$ have to be “projections” only when $a \neq b$.

2. Definitions and some properties

Let $A$ be a $C^*$-algebra. For $a, b \in A$, consider the relations

\[ \|a\| \leq 1; \quad \|b\| \leq 1; \quad a, b \geq 0; \quad (a - a^2)(a - b) = 0; \quad (b - b^2)(a - b) = 0. \]  

(1)

Two pairs, $(a_0, b_0)$ and $(a_1, b_1)$ of elements in $A$, are homotopy equivalent if there are paths $a = (a_t), b = (b_t) : [0, 1] \to A$, connecting $a_0$ with $a_1$ and $b_0$ with $b_1$ respectively, such that the relations

\[ \|a_t\| \leq 1; \quad \|b_t\| \leq 1; \quad a_t, b_t \geq 0; \quad (a_t - a_t^2)(a_t - b_t) = 0; \quad (b_t - b_t^2)(a_t - b_t) = 0 \]

hold for each $t \in [0, 1]$.

A pair $(a, b)$ is homotopy trivial if it is homotopy equivalent to $(0, 0)$.

Lemma 2.1. The pair $(a, a)$ is homotopy trivial for any $a \in A$.

Proof. The linear homotopy $a_t = t \cdot a$ would do. \qed

Lemma 2.2. If $a, b$ satisfy (1) then $f(a) = f(b)$ and $f(a)(a - b) = 0$ for any $f \in C_0(0, 1)$.

Proof. It follows from $(a - a^2)(a - b) = 0$, or, equivalently, from $(a - a^2)a = (a - a^2)b$, that

\[ (a - a^2)a^2 = a(a - a^2)a = a(a - a^2)b = (a - a^2)b^2, \]

hence

\[ (a - a^2)(a - a^2) = (a - a^2)(b - b^2). \]

Similarly,

\[ (b - b^2)(b - b^2) = (a - a^2)(b - b^2), \]

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Corollary 2.3. If \((a - a^2)^2 = (b - b^2)^2\). Then (2) and positivity of \(a - a^2\) and \(b - b^2\) imply that
\[a - a^2 = b - b^2.\]
Also,
\[(a - a^2)a = (a - a^2)b = (b - b^2)b.\]
Since the two functions \(g, h, g(t) = t - t^2, h(t) = tg(t)\), generate \(C_0(0, 1)\), and \(g(a) = g(b), h(a) = h(b)\), we conclude that the same holds for any \(f \in C_0(0, 1)\). Similarly, \(g(a)(a - b) = 0\) and \(h(a)(a - b) = 0\) implies \(f(a)(a - b) = 0\) for any \(f \in C_0(0, 1)\).

Corollary 2.3. If \(||a|| < 1, ||b|| < 1\) and the pair \((a, b)\) satisfies (1) then \(a = b\), hence the pair \((a, b)\) is homotopy trivial.

Proof. Take \(f \in C_0(0, 1)\) such that \(f(t) = t \in \text{Sp}(a) \cup \text{Sp}(b)\) and \(f(1) = 0\). Then \(a = f(a), b = f(b)\), and the claim follows from Lemma 2.2.

Lemma 2.4. The pair \((f(a), f(b))\) is homotopy equivalent to \((a, b)\) for any continuous map \(f : [0, 1] \rightarrow [0, 1]\) such that \(f(0) = 0, f(1) = 1\).

Proof. As the set of all functions with the stated properties is convex, it suffices to show that for any such function \(f\), the pair \((f(a), f(b))\) satisfies the relations (1).

Set \(f_0(t) = f(t) - t\). Then \(f_0 \in C_0(0, 1)\). As \(f_0(a) = f_0(b)\) by Lemma 2.2, so
\[f(a) - f(b) = a - b.\]

Set
\[g(t) = t - t^2 + f_0(t) - f_0^2(t) - 2tf_0(t).\]
Then \(g \in C_0(0, 1)\) and
\[\begin{align*}
(f(a) - f^2(a))(f(a) - f(b)) &= g(a)(a - b) = 0; \\
(f(b) - f^2(b))(f(a) - f(b)) &= g(a)(a - b) = 0.
\end{align*}\]

Corollary 2.5. \(\text{Sp}(a) \setminus \{0, 1\} = \text{Sp}(b) \setminus \{0, 1\}\).

Proof. The inner points of \([0, 1]\) in the two spectra must coincide by Lemma 2.2.

Lemma 2.6. The addition is commutative and associative.
Proof. If \((u_t)_{t \in [0,1]}\) is a path of unitaries in \(A\), \(u_1 = 1\), \(u_0 = u\), then \(\[(u^* a u, u^* b u]\] = [(a, b)]\) for any \(a, b \in A\), as the relations (1) are not affected by unitary equivalence. The standard argument with a unitary path connecting \((1 0\ 0 1)\) and \((0 1\ 1 0)\) proves commutativity. A similar argument proves associativity.

\[\square\]

Lemma 2.7. \([(a, b)] + [(b, a)] = [(0, 0)]\) for any \(a, b\).

Proof. Set \(U_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\), \(B_t = U_t^* \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U_t\). We claim that the pair \([(a 0\ 0 b), B_t]\) satisfies the relations (1) for all \(t\).

One has

\[B_t = \begin{pmatrix} b \cos^2 t + a \sin^2 t & (a-b) \cos t \sin t \\ (a-b) \cos t \sin t & b \sin^2 t + a \cos^2 t \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + (a-b)C_t,\]

where \(C_t = \begin{pmatrix} -\cos^2 t & \cos t \sin t \\ \cos t \sin t & \cos^2 t \end{pmatrix}\).

Then

\[\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) (\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t) = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right) (\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t) = 0.\]

It remains to show that

\[A = (B_t - B_t^2)\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right) = 0.\]

Using (3) we have

\[A = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right)^2 = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - B_t\right)^2 = 0.\]

Thus, the pair \((a \oplus b, b \oplus a)\) is homotopy equivalent to the pair \((a \oplus b, a \oplus b)\), and the latter is homotopy trivial by Lemma 2.1.

\[\square\]

So we see that the equivalence classes of pairs satisfying the relations (1) in matrix algebras over \(A\) form an abelian group for any \(C^*\)-algebra \(A\). Let us denote this group by \(L(A)\).

Note that pairs of projections patently group the relations (1). If \(A\) is a unital \(C^*\)-algebra then \(K_0(A)\) consists of formal differences \([p] - [q]\) with \(p, q\) projections in matrices over \(A\). Then

\[\iota([p] - [q]) = [(p, q)]\]

gives to a morphism \(\iota : K_0(A) \to L(A)\).

In the non-unital case, \(\iota\) can be defined after unitalization. But, as we shall see later, unlike \(K_0\), there is no need to unitalize for \(L\). The following example shows the reason for that in the commutative case.

Example 2.8. Let \(X\) be a compact Hausdorff space, \(x \in X\), \(Y = X \setminus \{x\}\), \(A = C_0(Y)\), \(A^+ = C(X)\). Let \([p] - [q] \in K_0(A)\), where \(p, q \in M_n(A^+))\) are projections. Then \(p = p_0 + \alpha\), \(q = p_0 + \beta\), where \(p_0\) is constant on \(X\), and \(\alpha, \beta \in M_n(A)\). Without loss of generality we may assume that \(\alpha, \beta \neq 0\) not only at the point \(x\), but also in a small neighborhood \(U\) of \(x\). Let \(h \in C(X)\) satisfy \(0 \leq h \leq 1\), \(h(x) = 0\) and \(h(z) = 1\) for any \(z \in X \setminus U\). Set \(a = h p_0 + \alpha\), \(b = h p_0 + \beta\), then \(a, b \in M_n(A)\) and \([a, b] \in L(A)\).
Lemma 2.9. \( L(\mathbb{C}) \cong \mathbb{Z} \).

Proof. Let \( a, b \in M_n, 0 \leq a, b \leq 1 \). Let \( e_1, \ldots, e_n \) (resp. \( e'_1, \ldots, e'_n \)) be an orthonormal basis of eigenvectors for \( a \) (resp. for \( b \)) with eigenvalues \( \lambda_1, \ldots, \lambda_n \) (resp. \( \lambda'_1, \ldots, \lambda'_n \)). Let \( 0 < \lambda_i < 1 \). Then \( e_i \) is an eigenvector for \( a - a^2 \) with a non-zero eigenvalue \( \lambda_i - \lambda_i^2 \). As \( (a - a^2)(a - b) = 0 \), so \( (a - b)(a - a^2) = 0 \), hence

\[
(a - b)(a - a^2)(e_i) = (\lambda_i - \lambda_i^2)(a - b)(e_i) = 0,
\]

thus \( (a - b)(e_i) = 0 \), or, equivalently, \( a(e_i) = b(e_i) \). As \( e_i \) is an eigenvector for \( a \), so it is an eigenvector for \( b \) as well, \( b(e_i) = \lambda_i e_i \). So, eigenvectors, corresponding to the eigenvalues \( \neq 0,1 \), are the same for \( a \) and \( b \).

Re-order, if necessary, the eigenvalues so that

\[
\lambda_1, \ldots, \lambda_k \in (0,1), \quad \lambda_{k+1}, \ldots, \lambda_n \in \{0,1\},
\]

and denote the linear span of \( e_1, \ldots, e_k \) by \( L \). Similarly, assume that

\[
\lambda'_1, \ldots, \lambda'_{k'} \in (0,1), \quad \lambda'_{k'+1}, \ldots, \lambda'_{n'} \in \{0,1\},
\]

and denote the linear span of \( e'_1, \ldots, e'_{k'} \) by \( L' \). As \( e_1, \ldots, e_k \in L \) and, symmetrically, \( e'_1, \ldots, e'_{k'} \in L' \), so \( \dim L = \dim L', k = k' \), and \( \lambda_i = \lambda'_i \) for \( i = 1, \ldots, k \).

Then \( L^\perp \) is an invariant subspace for both \( a \) and \( b \), and the restrictions \( a|_{L^\perp} \) and \( b|_{L^\perp} \) are projections (as their eigenvalues equal 0 or 1). We may write \( a \) and \( b \) as matrices with respect to the decomposition \( L \oplus L^\perp \):

\[
a = \begin{pmatrix} c & 0 \\ 0 & p \end{pmatrix}; \quad b = \begin{pmatrix} c & 0 \\ 0 & q \end{pmatrix}, \tag{4}
\]

where \( p, q \) are projections. The linear homotopy

\[
a_t = \begin{pmatrix} tc & 0 \\ 0 & p \end{pmatrix}; \quad b_t = \begin{pmatrix} tc & 0 \\ 0 & q \end{pmatrix}, \quad t \in [0,1],
\]

connects the pair \( (a,b) \) with the pair \( (p,q) + (0,0) \). Therefore, \( L(\mathbb{C}) \) is a quotient of \( \mathbb{Z} \) (which is the set of homotopy classes of pairs of projections modulo stable equivalence). To see that \( L(\mathbb{C}) \) is exactly \( \mathbb{Z} \), note that (4) implies that \( \text{tr}(a-b) \in \mathbb{Z} \) for any \( a, b \) satisfying the relations (1), so this integer is homotopy invariant.

\[\square\]

Remark 2.10. One may think that the relations (1) imply that \( a, b \) are something like projections plus a common part and can be reduced to just a pair of projections by cutting out the common part. The following example shows that this is not that simple.

Example 2.11. Let \( A = C(X) \), let \( Y, Z \) be closed subsets in \( X \) with \( Y \cap Z = K \). Let \( p, q \in M_n(C(Y)) \) be projection-valued functions on \( Y \) such that \( p|_K = q|_K = r \), and let \( r \) cannot be extended to a projection-valued function on \( Z \) due to a \( K \)-theory obstruction, but can be extended to a matrix-valued function \( s \in M_n(C(Z)) \) on \( Z \) (with \( 0 \leq s \leq 1 \)). Then set \( a = \begin{cases} p & \text{on } Y; \\ s & \text{on } Z \end{cases} \) and \( b = \begin{cases} q & \text{on } Y; \\ s & \text{on } Z. \end{cases} \)

3. Universal \( C^* \)-algebra for relations (1)

Denote the \( C^* \)-algebra generated by \( a, b \) satisfying (1) by \( C^*(a, b) \). The universal \( C^* \)-algebra is the least \( C^* \)-algebra \( D \) such that for any \( a, b \) with (1) there is a surjective \( * \)-homomorphism \( \varphi : D \to C^*(a, b) \), [5]. ‘The least’ means that for any surjective \( * \)-homomorphism \( \psi : E \to C^*(a, b) \) there is a surjective \( * \)-homomorphism \( \chi : E \to D \) such that \( \psi = \varphi \circ \chi \).
Let $I \subset C^*(a, b)$ denote the ideal generated by $a - a^2$, and let $C^*(a, b)/I$ be the quotient $C^*$-algebra. Then $C^*(a, b)/I$ is generated by $\dot{a} = q(a)$ and $\dot{b} = q(b)$, where $q$ is the quotient map. But since $q(a - a^2) = q(b - b^2) = 0$, $\dot{a}$ and $\dot{b}$ are projections, and $C^*(a, b)/I$ is generated by two projections.

Then the $C^*$-algebra $C^*(a, b)$ is completely determined by the ideal $I$, by the quotient $C^*(a, b)/I$, and by the Busby invariant $\tau : C^*(a, b)/I \rightarrow Q(I)$ (we denote by $M(I)$ the multiplier algebra of $I$ and by $Q(I) = M(I)/I$ the outer multiplier algebra). The latter is defined by the two projections $\tau(\dot{a}), \tau(\dot{b}) \in Q(C_0(Y))$, where $X = \text{Sp}(a)$, $Y = X \setminus \{0, 1\}$. Let $C_b(Y)$ denote the $C^*$-algebra of bounded continuous functions on $Y$ and let

$$\pi : C_b(Y) \rightarrow C_b(Y)/C_0(Y) = Q(C_0(Y))$$

be the quotient map. Using Gelfand duality, we identify $a$ with the function id on $\text{Sp}(a)$. Let $f \in C_0(Y)$. Then

$$\tau(\dot{a})\pi(f(a)) = \tau(\dot{b})\pi(f(a)) = \pi(af(a)),$$

so we can easily calculate these two projections.

If $1 \notin X$ then $\tau(\dot{a}) = \tau(\dot{b}) = 0$; if $X = \{1\}$ then $I = 0$; if $1 \in X$ and $X$ has at least one more point $x$ then $\tau(\dot{a}) = \tau(\dot{b})$ is the class of functions $f$ on $X$ such that $f(1) = 1$ and $f(t) = 0$ for all $t \leq x$.

Let $M_1 \subset M_2$ denote the upper left corner in the 2-by-2 matrix algebra. Set

$$D = \{f \in C([-1, 1]; M_2) : f(-1) = 0, f(1) \text{ is diagonal, } f(t) \in M_1 \text{ for } t \in (-1, 0]\}.$$ 

The structure of $D$ is similar to that of $C^*(a, b)$. The ideal

$$J = \{f \in D : f(t) = 0 \text{ for } t \in [0, 1]\} \cong C_0(-1, 0)$$

is the universal $C^*$-algebra for $I$ (surjects on $I$ for any $0 \leq a \leq 1$), and the quotient is the universal (nonunital) $C^*$-algebra

$$D/J = \mathbb{C} \ast \mathbb{C} = \{m \in C([0, 1], M_2) : m(1) \text{ is diagonal, } m(0) \in M_1\}$$

generated by two projections [6]. Note that this $C^*$-algebra is an extension of $\mathbb{C}$ by the $C^*$-algebra $q\mathbb{C} = \{m \in C_0((0, 1], M_2) : m(1) \text{ is diagonal}\}$ used in the Cuntz picture of $K$-theory.

**Lemma 3.1.** The $C^*$-algebra $D$ is universal for the relations (1).

**Proof.** For any $a, b$ satisfying (1) there are standard surjective $*$-homomorphisms $\alpha : J \rightarrow I$ and $\gamma : D/J \rightarrow C^*(a, b)/I$. Since $\alpha$ is surjective, it induces $*$-homomorphisms $M(\alpha) : M(J) \rightarrow M(I)$ and $Q(\alpha) : Q(J) \rightarrow Q(I)$ in a canonical way. As

$$D \cong \{\{m, f\} : m \in M(J), f \in D/J, q_f(m) = \tau(f)\},$$

$$C^*(a, b) \cong \{(n, g) : n \in M(I), g \in C^*(a, b)/I, q_g(n) = \sigma(g)\},$$

where $q_* : M(\bullet) \rightarrow Q(\bullet)$ is the quotient map, so the map $\beta : D \rightarrow C^*(a, b)$ can be defined by $\beta(m, f) = (M(\alpha)(m), \gamma(f))$. This map is well defined if the diagram

$$\begin{array}{ccc}
D/J & \xrightarrow{\tau} & Q(J) \\
\downarrow{\gamma} & & \downarrow{Q(\alpha)} \\
C^*(a, b)/I & \xrightarrow{\sigma} & Q(I)
\end{array}$$

commutes. It does commute. The case $X = \text{Sp}(a) = \{1\}$ is trivial. For other cases, notice that the image of $\tau$ lies in $C_0(0, 1)/C_0(0, 1) \subset Q(J)$, and the image of $\sigma$ lies in $C(X)/C_0(X \setminus \{0\})$, which is either $\mathbb{C}$ or 0 (when $1 \in X$ or $1 \notin X$ respectively), and the
restriction of $Q(\alpha)$ from the image of $\tau$ to the image of $\sigma$ is induced by the inclusion $X \subset [0, 1]$.

So, for any $A$ and any $a, b \in A$ satisfying (1) there is a surjective $*$-homomorphism from $D$ to $C^*(a, b)$. To see that $D$ is universal it suffices to show that $D$ is generated by some $a, b$ satisfying (1). Set

$$a(t) = \begin{cases} \left( \frac{\cos^2 \frac{\pi}{2} t}{0} \right) & \text{for } t \in [-1, 0]; \\ \left( \frac{1}{0} \right) & \text{for } t \in [0, 1], \end{cases}$$

$$b(t) = \begin{cases} \left( \frac{\cos^2 \frac{\pi}{2} t}{0} \right) & \text{for } t \in [-1, 0]; \\ \left( \frac{\cos^2 \frac{\pi}{2} t \sin \frac{\pi}{2} t}{\sin^2 \frac{\pi}{2} t} \right) & \text{for } t \in [0, 1]. \end{cases}$$

Then $D$ is generated by these $a$ and $b$. 

The $C^*$-algebra $D$ allows one more description. Set $A_0 = \mathbb{C}^2$, $F = \mathbb{C} \oplus M_2$ and define a $*$-homomorphism $\gamma : A_0 \to F \oplus F$ by $\gamma = \gamma_0 \oplus \gamma_1$, where $\gamma_0, \gamma_1 : \mathbb{C}^2 \to \mathbb{C} \oplus M_2$ are given by

$$\gamma_0(\lambda, \mu) = \lambda \oplus \left( \frac{\lambda}{0} \right) \oplus \left( \frac{0}{\mu} \right); \quad \gamma_1(\lambda, \mu) = 0 \oplus \left( \frac{\lambda}{0} \right) \oplus \left( \frac{0}{\mu} \right); \quad \lambda, \mu \in \mathbb{C}.$$

Let $\partial : C([0, 1]; F) \to F \oplus F$ be the boundary map, $\partial(f) = f(0) \oplus f(1)$, $f \in C([0, 1]; F)$. Then $D$ can be identified with the pullback

$\begin{array}{cccc} D = A_1 \ar[r] & A_0 \ar[d] \ar[l] & \\ C([0, 1]; F) \ar[r]^\partial & F \oplus F, \ar[u] & \end{array}$

$$D = \{(f, a) : f \in C([0, 1]; F), a \in A_0, \partial(f) = \gamma(a)\}.$$

Such pullback is called a 1-dimensional noncommutative CW complex (NCCW complex) in [4]; in this terminology, $A_0$ is a 0-dimensional NCCW complex.

Recall ([1]) that a $C^*$-algebra $B$ is semiprojective if, for any $C^*$-algebra $A$ and increasing chain of ideals $I_n \subset A$, $n \in \mathbb{N}$, with $I = \bigcup_n I_n$ and for any $*$-homomorphism $\varphi : B \to A/I$ there exists $n$ and $\hat{\varphi} : B \to A/I_n$ such that $\varphi = q \circ \hat{\varphi}$, where $q : A/I_n \to A/I$ is the quotient map.

**Corollary 3.2.** The $C^*$-algebra $D$ is semiprojective.

**Proof.** Essentially, this is Theorem 6.2.2 of [4], where it is proved that all unital 1-dimensional NCCW complexes are semiprojective. The non-unital case is dealt in Theorem 3.15 of [7], where it is noted that if $A_1$ is a 1-dimensional NCCW complex then $A_1^+$ is a 1-dimensional NCCW as well, and semiprojectivity of $A_1$ is equivalent to semiprojectivity of $A_1^+$. 

One more picture of $D$ can be given in terms of amalgamated free product: $D = C(0, 1) *_{C_0(0,1)} C(0, 1)$. 
4. Identifying $L$ with $K_0$

Our definition of $L(A)$ can be reformulated in terms of the universal $C^*$-algebra $D$ as

$$L(A) = \lim_{\to \rightarrow} [D, M_n(A)],$$

where $[-,-]$ denotes the set of homotopy classes of $*$-homoorphisms. Recall that semiprojectivity is equivalent to stability of relations that determine $D$, (Theorem 14.1.4 of [5]). The latter means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $c,d \in A$ satisfy

$$\|c\| \leq 1, \quad \|d\| \leq 1, \quad c, d \geq 0, \quad \|(c - c^2)(c - d)\| < \delta, \quad \|(d - d^2)(c - d)\| < \delta,$$

there exist $a, b \in A$ such that $\|a - c\| < \varepsilon, \|b - d\| < \varepsilon$, and $a, b$ satisfy the relations (1). Stability of the relations (1) implies that

$$L(A) = [D, A \otimes \mathbb{K}] = [[D, A \otimes \mathbb{K}]],$$

where $\mathbb{K}$ denotes the $C^*$-algebra of compact operators, and $[[\cdot, \cdot]]$ is the set of homotopy classes of asymptotic homomorphisms.

**Lemma 4.1.** The functor $L$ is half-exact.

**Proof.** Let

$$0 \longrightarrow I \overset{i}{\longrightarrow} B \overset{p}{\longrightarrow} A \longrightarrow 0$$

be a short exact sequence of $C^*$-algebras. It is obvious that $p_* \circ i_* = 0$, so it remains to check that $\ker p_* \subset \mathrm{im} i_*$. Suppose that $a, b \in M_n(B)$ satisfy (1) and $(p(a), p(b)) = 0$ in $L(A)$. This means that there is a homotopy connecting $(p(a), p(b))$ to $(0,0)$ in $M_k(A)$ for some $k \geq n$ such that the whole path satisfies (1). This homotopy is given by a $*$-homomorphism $\psi : D \to C([0,1], M_k(A))$ such that $\mathrm{ev}_1 \circ \psi = 0$, where $\mathrm{ev}_t$ denotes the evaluation map at $t \in [0,1]$.

When $D$ is a semiprojective $C^*$-algebra, the homotopy lifting theorem ([2], Theorem 5.1) asserts that, given a commuting diagram

$$\begin{array}{ccc}
D & \xrightarrow{\varphi} & C([0,1]; M_k(B)) \\
\downarrow{\psi} & & \downarrow{\mathrm{ev}_0} \\
C([0,1]; M_k(A)) & \xrightarrow{\mathrm{ev}_0} & M_k(A),
\end{array}$$

where $\overline{p}_k$ and $p_k$ are the $*$-homomorphisms induced by a surjection $p$, there exists a $*$-homomorphism $\varphi$ completing the diagram. Replacing $A$ and $B$ by matrices over these $C^*$-algebras, we get a lifting $\varphi$ for the given homotopy. As $\mathrm{ev}_1 \circ \psi = 0$, so $\mathrm{ev}_1 \circ \varphi$ maps $D$ to $M_k(I)$. Thus the pair $(a,b)$ lies in the image of $i_*$. 

\[\square\]

In a standard way, set $L_n(A) = L(S^n A)$, where $SA$ denotes the suspension over $A$. Then, by Theorem 21.4.3 of [3], $L_n(A)$, being homotopy invariant and half-exact, is a homology theory. Also, by Theorem 22.3.6 of [3] and by Lemma 2.9, it coincides with the $K$-theory on the bootstrap category of $C^*$-algebras. We shall show now that it coincides with the $K$-theory for any $C^*$-algebra.

Set
where \( a, b \) are generators for \( D \) ((6), (7)), and \( f \in C_0(0,1) \) is given by \( f(t) = (t - t^2)^{1/2} \). Then \( P, Q \in M_2(D^+) \), where \( D^+ \) denotes the unitization of \( D \).

By Lemma 2.2, \( f(a) = f(b) \) and \( af(a) = bf(a) \), so \( P \) and \( Q \) are projections. One also has \( P - Q \in M_2(D) \), hence

\[
x = [P] - [Q] \in K_0(D).
\]

**Lemma 4.2.** \( K_0(D) \cong \mathbb{Z} \) with \( x \) as a generator.

**Proof.** Consider the short exact sequence

\[
0 \longrightarrow J \longrightarrow D \xrightarrow{\pi} C \ast C \longrightarrow 0,
\]

where \( C \ast C \) is the universal (nonunital) \( C^* \)-algebra (5) generated by two projections, \( p \) and \( q \) \([6]\), and \( \pi \) is given by restriction to \([0, 1] \), \( \pi(a) = p, \pi(b) = q \). We have \( \pi(P) = (1-q) \oplus p \), \( \pi(Q) = (1-q) \oplus q \), so \( \pi_*(x) = [p] - [q] \in K_0(C \ast C) \). As \( P(t) = Q(t) \) when \( t \in [-1, 0] \), so for the boundary (exponential) map \( \delta : K_0(C \ast C) \rightarrow K_1(J) \) we have \( \delta(P) = \delta(Q) \). Recall that \( J \cong C_0(-1, 0) \). Direct calculation shows that \( \delta(P) = \delta(Q) \neq 0 \). The claim follows now from the \( K \)-theory exact sequence

\[
0 = K_0(J) \longrightarrow K_0(D) \xrightarrow{\pi_*} K_0(C \ast C) \xrightarrow{\delta} K_1(J) \cong \mathbb{Z}.
\]

\( \square \)

Let us define a map \( \kappa : L(A) \rightarrow K_0(A) \). If \( l = [(a, b)] \in L(A) \) then the pair \((a, b)\) determines a \(*\)-homomorphism \( \varphi : D \rightarrow M_n(A) \) by \( \varphi(a) = a; \varphi(b) = b \). So, \( l \in L(A) \) determines a \(*\)-homomorphism \( \varphi \) up to homotopy (for some \( n \)). Put

\[
\kappa(l) = \varphi_*(x) \in K_0(A).
\]

As this definition is homotopy invariant and as direct sum of pairs corresponds to direct sum of \(*\)-homomorphisms, so the map \( \kappa \) is a well defined group homomorphism.

Recall that there is also a map \( \iota : K_0(A) \rightarrow L(A) \) given by \( \iota([p] - [q]) = ([p, q]) \), where \( [p] - [q] \in K_0(A) \).

**Lemma 4.3.** For any unital \( C^* \)-algebra \( A \), one has \( \kappa \circ \iota = \text{id}_{K_0(A)}; \iota \circ \kappa = \text{id}_{L(A)} \), hence \( L(A) = K_0(A) \).

**Proof.** To show the first identity, let \( z \in K_0(A) \) and let \( p, q \in M_n(A) \) be projections such that \( z = [p] - [q] \). Let \( \varphi : D \rightarrow M_n(A) \) be a \(*\)-homomorphism determined by the pair \((p, q)\). Then, due to universality of \( C \ast C \), \( \varphi \) factorizes through \( C \ast C \), \( \varphi = \psi \circ \pi \), where \( \pi : D \rightarrow C \ast C \) is the quotient map and \( \psi : C \ast C \rightarrow M_n(A) \) is determined by \( \psi(i_1(1)) = p \) and \( \psi(i_2(1)) = q \), where \( i_1, i_2 : C \rightarrow C \ast C \) are inclusions onto the first and the second copy of \( C \). Then

\[
\varphi(x) = \psi_*([i_1(1)] - [i_2(1)]) = [p] - [q],
\]

hence \( \kappa(\iota(z)) = z \).

Let us show the second identity. For \( [(a, b)] \in L(A) \), let \( \varphi : D \rightarrow M_n(A) \) be a \(*\)-homomorphism defined by the pair \((a, b)\) (i.e. by \( \varphi(a) = a; \varphi(b) = b \)), and let \( \varphi^+ : D^+ \rightarrow M_n(A) \) be its extension, \( \varphi^+(1) = 1 \). Then \( \iota(\kappa([(a, b)])) = [[\varphi_2^+(P), \varphi_2^+(Q)]] \), where \( \varphi_2^+ = \varphi^+ \otimes \text{id}_{M_2} \).

For \( s \in [0, 1] \), set
\[ P_s = C_s P C_s; \quad Q_s = C_s Q C_s, \quad \text{where} \quad C_s = \begin{pmatrix} s \cdot 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then
\[ P_s, Q_s \in M_2(D^+), \quad P_s - Q_s \in M_2(D), \quad 0 \leq P_s, Q_s \leq 1, \]
\[ (P_s - P_s^2)(P_s - Q_s) = 0, \quad (Q_s - Q_s^2)(P_s - Q_s) = 0 \]
for all \( s \in [0, 1]; P_0, Q_0 \in M_2(D), \) and
\[ P_1 = P, \quad Q_1 = Q; \quad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}. \]
Therefore, \((\varphi_2^+(P_s), \varphi_2^+(Q_s))\) provides a homotopy connecting \((\varphi_2^+(P), \varphi_2^+(Q))\) with \((0 \oplus a, 0 \oplus b)\), hence, the pair \((\varphi_2^+(P), \varphi_2^+(Q))\) is equivalent to the pair \((a, b)\). \(\square\)

**Theorem 4.4.** The functors \(L\) and \(K_0\) coinside for any \(C^*\)-algebra \(A\).

**Proof.** Both functors are half-exact and coinside for unital \(C^*\)-algebras, so the claim follows. \(\square\)

**Remark 4.5.** Similarly to \(D\), one can define a \(C^*\)-algebra \(D_B\) for any \(C^*\)-algebra \(B\) as an appropriate extension of \(B \ast B\) by \(CB\), where \(CB\) is the cone over \(B\) (or by \(D_B = CB \ast_{SB} CB\)). Then one gets the group \([D_B, A \otimes \mathbb{K}]\). Regretfully, \(D_B\) has no nice presentation (unlike \(D = D_C\)), so we don’t pursue here the bivariant version.

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