NEW STEPPING-UP CONSTRUCTIONS FOR MULTICOLOURED HYPERGRAPHS

QUENTIN DUBROFF, ANTÓNIO GIRÃO, EOIN HURLEY, AND CORRINE YAP

Abstract. Generalizing the classical Ramsey numbers, \( r_k(t; q, p) \) is the smallest integer \( n \) such that in every \( q \)-colouring of the \( k \)-sets on \( n \) vertices contains a set of \( t \) vertices spanning fewer than \( p \) colours. We prove the first tower-type lower bounds on these numbers via two new stepping-up constructions, both variants of the original stepping-up lemma due to Erdős and Hajnal. We use these to resolve a problem of Conlon, Fox, and Rödl. More precisely, we construct a family of hypergraphs with arbitrarily large tower height separation between their 2-colour and \( q \)-colour Ramsey numbers.

1. Introduction

We denote by \( r_k(G; q, p) \), for \( q \geq p \), the smallest integer \( n \) such that in every \( q \)-colouring of \( K_n^{(k)} \), there is a copy of the hypergraph \( G \) whose edges span fewer than \( p \) colours. For simplicity when \( G \) is \( K_n^{(k)} \), we write \( r_k(G; q, p) = r_k(t; q, p) \). Observe that when \( q = p = 2 \), \( r_k(G; 2, 2) \) coincides with the classical Ramsey numbers \( r_k(G) \) and \( r_k(t) \), and we will denote them as such. One of the most central open problems in Ramsey theory is determining the growth rate of the 3-uniform Ramsey number \( r_3(t) \). A famous result of Erdős, Hajnal, and Rado [7] from the 60’s shows that there exist constants \( c \) and \( c' \) such that

\[
2^{c't^2} \leq r_3(t) \leq 2^{c't^2}.
\]

Note that the upper bound is essentially exponential in the lower bound, and that despite much attention, this remains the state of the art. Perhaps surprisingly, if we allow four colours instead of two, Erdős and Hajnal (see e.g. [9]) showed that the double exponential upper bound is essentially correct, i.e. there is a \( c > 0 \) such that \( r_3(t; 4, 2) \geq 2^{ct^2} \). More recently Conlon, Fox, and Sudakov [4] proved a super-exponential bound with three colours, that is, that there exists \( c > 0 \) such that \( r_3(t; 3, 2) \geq 2^{c \log t} \). Erdős conjectured that the double exponential bound should hold without using extra colours, offering $500 dollars for a proof that \( r_3(t) \geq 2^{ct^2} \) for some constant \( c > 0 \). Raising the stakes for this conjecture is the ingenious stepping-up construction of Erdős and Hajnal (see e.g. [9]), which shows that for all \( q \) and \( k \geq 3 \),

\[
r_{k+1}(2t + k - 4; q, 2) > 2^{r_k(tq, 2) - 1}.
\]

For the past 60 years, we have used (1) to stack our lower bounds for \( r_k(t; q, 2) \) upon that of \( r_3(t; q, 2) \), yielding that \( r_k(t) \geq T_k - 1(ct^2) \), where \( T_k(x) \), the tower of height \( k \) in \( x \), is defined by \( T_1(x) = x, T_{i+1}(x) = 2^{T_i(x)} \). The corresponding upper bounds of \( r_k(t) \leq T_k(O(t)) \) (see [5, 6, 7]) are once again exponential in the lower bounds, and thus a positive resolution of Erdős’s Conjecture would be the decisive step in showing that \( r_k(t) = T_k(\Theta(t)) \) for all \( k \geq 3 \).

Due to the lack of progress on this central conjecture, it is natural to try to understand just how significant a role the number of colours can play in hypergraph Ramsey numbers and whether or not there could really be such a large difference between \( r_3(t; 2, 2) \) and \( r_3(t; 4, 2) \). To this end, Conlon, Fox, and Rödl [3] asked if there exist an integer \( q \) and number \( c > 0 \) such that \( r_3(t; q, 3) \geq 2^{ct^2} \). This question gets to the heart of our ignorance of \( r_k(t; q, p) \) for any \( k, q, p \geq 3 \), which is caused by the absence of a stepping-up construction for
any \( p > 2 \). A standard application of the first moment method (see e.g. [1]) shows that for any \( k, q \in \mathbb{N} \) there exists \( c > 0 \) such that \( r_k(t; q, q) \geq 2^{t^{k-1}} \) for all \( t \in \mathbb{N} \). The only nontrivial improvement to date on this bound has been made by Mubayi and Suk [13] who proved there exists \( c > 0 \) such that for \( q \geq 9 \), \( r_3(t; q, 3) \geq 2^{t^{q-1}} \) for \( t \in \mathbb{N} \) sufficiently large; for all other values of \( k, q, p \geq 3 \), the random construction is essentially the state of the art. Our knowledge (or lack thereof) is thus summarised by the following bounds for \( k, q, p \geq 3 \) and sufficiently large \( t \in \mathbb{N} \),

\[
2^{t^C} \leq r_k(t; q, p) \leq T_k(O(t)),
\]

where \( c \geq 1 \) is allowed to depend on \( k, q \) and \( p \). Note that in this case our upper bounds are a staggering tower of height \( k - 2 \) in the lower bounds. Our first contribution is the development of two new stepping-up constructions captured by Theorems 1.1 and 1.4. Both borrow key ideas from the original, and each of them is relevant for different values of \( k, q \) and \( p \). They yield the first tower-type results of their kind.

**Theorem 1.1.** Let \( k, q, p \geq 3 \) be such that \( p \leq C_k \), the \( k \)th Catalan number. There exist \( c \geq 1 \) and \( t_0 \) such that for all \( t > t_0 \)

\[
r_{k+1}(t^C; q, p - 2) > 2^{r_k(t; q, p - 2) - 1}, \quad r_{k+1}(t^C; 2q + p, p) > 2^{r_k(t; q, p) - 1}.
\]

Unfortunately, this does not allow us to answer the question of Conlon, Fox, and Rödl on \( r_3(t; q, 3) \), since \( C_2 = 2 \), but already for \( k \geq 4 \) we have the following two corollaries:

**Corollary 1.2.** For all \( k \geq 4 \), there is \( q \in \mathbb{N} \) and \( c > 0 \) such that \( r_k(t; q, 5) \geq T_{k-1}(t^C) \).

**Corollary 1.3.** For all \( k \geq 4 \), there is \( c > 0 \) such that \( r_k(t; 3, 3) \geq T_{k-1}(t^C) \).

Observe that by the second corollary the growth rate of \( r_k(t; 3, 3) \) matches the current best lower bounds for \( r_k(t; 2, 2) \) up to a polynomial in \( t \). The reason we have an absolute constant \( c \) in the exponent is due to the use of an Erdős-Hajnal type result on sequences (see Section 2). Our second stepping-up construction yields the following.

**Theorem 1.4.** Let \( k, q, p \geq 3 \) be such that \( p \leq k! \). There exist \( c \geq 1 \) and \( t_0 \) such that for all \( t > t_0 \)

\[
r_{2k}(t^C; q, p) > 2^{r_k(t; q, p) - 1}.
\]

Note that the growth rate in \( k \) which is implied by Theorem 1.4 (approximately a tower of height \( \log_2 k \)) is much less than that of Theorem 1.1 because we can only step up at the cost of doubling the uniformity size. Despite this, it allow us to answer a question of Conlon, Fox, and Rödl which relates to the aforementioned conjecture of Erdős. Recall the conjecture claimed that the Ramsey numbers of 3-uniform hypergraphs with 2 colours should behave as a double exponential function; and that we know that this is the case when we allow four colours. One argument in favour of the conjecture is that the reliance on extra colours to prove a double exponential lower bound may be a technical limitation of the stepping-up construction. This is challenged by a stunning discovery of Conlon, Fox, and Rödl [3] who exhibited an infinite family of 3-uniform hypergraphs called *hedgehogs*, whose Ramsey numbers display strong dependence on the number of colours. Namely, they showed that 2-colour Ramsey number of hedgehogs is polynomial in their order, while the 4-colour Ramsey number is at least exponential. To understand just how significant a role the number of colours could play they asked the following:

**Question 1.5.** For any integer \( h \geq 3 \), do there exist integers \( k \) and \( q \) and a family of \( k \)-uniform hypergraphs for which the 2-colour Ramsey number grows as a polynomial in the number of vertices, while the \( q \)-colour Ramsey number grows as a tower of height \( h \)?
Our second contribution is to answer this in the affirmative. Define the \( k \)-uniform balanced hedgehog \( \hat{H}^{(k)}_i \) with body of order \( t \) to be the graph constructed as follows: take a set \( S \) of \( t \) vertices, called the body, and for each subset \( X \subset S \) of order \( \lceil \frac{k}{2} \rceil \) add a \( k \)-edge \( e \) with \( e \cap S = X \) such that for all \( e, f \in E(\hat{H}^{(k)}_i) \) we have \( e \cap f \subset S \). The hedgehog \( H^{(k)}_i \) as defined by Conlon, Fox, and Rödl differs only in that they consider every \( X \subset S \) of order \( k - 1 \) rather than \( \lceil \frac{k}{2} \rceil \). We observe that for \( k = 3 \) both definitions coincide. When the uniformity is clear from the context we shall drop the superscript.

**Theorem 1.6.** There exist \( c > 0 \) and \( q : \mathbb{N} \to \mathbb{N} \) such that for all \( k \in \mathbb{N} \) and sufficiently large \( t \), we have

(1) \( r_{2k+1}(\hat{H}^{(k)}_i; 2, 2) \leq t^{k+3} \), and

(2) \( r_{2k+1}(\hat{H}^{(k)}_i; q(k), 2) \geq T_{[c \log_2 \log_2 k]}(t) \).

The following result connects the problem of avoiding monochromatic balanced hedgehogs to that of avoiding cliques that span few colours. It is a straightforward adaptation of ideas from Conlon, Fox, and Rödl [3].

**Lemma 1.7.** Given \( k, q, t \in \mathbb{N} \), let \( p = \binom{2k+1}{k+1} \) and \( q' = \binom{q}{p} \). Then

\[ r_{2k+1}(\hat{H}^{(k)}_i; q', 2) > r_{k+1}(t; q, p + 1) - 1. \]

Using this result along with Theorem 1.4 yields the lower bound in Theorem 1.6, (b). It is natural to ask whether one can combine the growth rate in \( k \) given by Theorem 1.1 with the ability to impose as many colours as in Theorem 1.4. Unfortunately, the condition \( p \leq C_k \) prevents us from using Theorem 1.1 as the right-hand side because \( C_k = \frac{1}{k+1} \binom{2k}{k} < \binom{2k+1}{k+1} \). This is tantalisingly close, if not a little curious, as the dependence on \( C_k \) comes from our exact solution to a subsequence avoidance problem (Corollary 2.4). We show that \( C_k \) presents a natural barrier in this endeavour. This barrier is made concrete by some new and tight results on the Ramsey theory of sequences, including an Erdős-Hajnal-type result in Section 2.

The outline of the paper is as follows: in Section 2 we introduce and prove results on the Ramsey theory of sequences; in Section 3 we use these to prove our stepping up constructions, namely, Theorem 1.1 and 1.4; and in Section 4 we prove our hedgehog-related results, including Theorem 1.6 and Lemma 1.7, and provide a construction of a degenerate hypergraph that relates to the Burr-Erdős Conjecture. To finish, we pose some questions and problems highlighted by our results.

## 2. Ramsey Theory of Sequences

Let \( V = \{0, 1\}^m \) and given vectors \( v, w \in V \), define

\[ \delta(v, w) = \max \{ i : v_i \neq w_i \}. \]

We say \( v < w \) if \( v_\delta < w_\delta \). For every set of vertices \( v_1 < v_2 < \cdots < v_{k+1} \) in \( V \) there is a corresponding sequence \( (\delta_1, \delta_2, \ldots, \delta_k) \) given by \( \delta_i = \delta(v_i, v_{i+1}) \). In this section, we introduce some definitions that will be useful in order to analyse the structure of these \( \delta \)-sequences.

We say a sequence \( S = (a_1, a_2, \ldots, a_m) \) is *homogeneous* if \( a_1 \leq a_2 \leq \cdots \leq a_m \) or \( a_1 \geq a_2 \geq \cdots \geq a_m \). Two sequences \( (a_1, a_2, \ldots, a_i) \) and \( (b_1, b_2, \ldots, b_i) \) have the same pattern if the relative ordering of every pair of elements is the same, i.e. \( a_i > a_j \), \( a_i = a_j \), or \( a_i < a_j \) if and only if \( b_i > b_j \), \( b_i = b_j \), or \( b_i < b_j \) respectively for all \( 1 \leq i < j \leq t \). A *pattern* is then the equivalence class of sequences with respect to this relative ordering. Given sequences \( S \) and \( P \), we say \( S \) *contains the pattern* \( P \) if we can find a subsequence of \( S \) which has the
same pattern as $P$; otherwise, we say $S$ avoids $P$. A pattern is a permutation pattern if there is a permutation in the equivalence class, and we often use this permutation as a representative for the pattern.

With the aforementioned applications to hypergraph Ramsey theory in mind, we introduce “max-induced” subsequences. We say $(a_1, a_2, \ldots, a_n)$ with $i_1 < i_2 < \cdots < i_t$ is a max-induced subsequence of $(a_1, a_2, \ldots, a_m)$ if the maximum of $(a_{i_1}, a_{i_1+1}, \ldots, a_{i_t})$ is attained at $a_{i_1}$ or $a_{i_t}$, i.e., at the left or right extreme, for all $1 \leq j < t$. We say a sequence $S$ contains a max-induced pattern $P$ if there is a max-induced subsequence of $S$ which has pattern $P$, and a family of sequences $F$ has the max-induced Erdős-Hajnal property with exponent $c(F)$ if every sequence $S$ that avoids all members of $F$ as a max-induced subsequence has a homogeneous max-induced subsequence of order $|S|^{c(F)}$, where $|S|$ denotes the length of the sequence. We are able to characterize the families with this property.

The following fact, whose short proof can be found in Section 3, relates the max-induced property to our original motivation:

**Claim 2.1.** Suppose $v_1 < v_2 < \cdots < v_{t+1}$ are vectors in $\{0,1\}^m$ and let $\delta_i = \delta(v_i, v_{i+1})$. If $(\delta_{i_1}, \ldots, \delta_{i_t})$ is a max-induced subsequence of $(\delta_1, \ldots, \delta_k)$, then there are $v_{j_1}, \ldots, v_{j_k+1}$ such that $\delta(v_{j_s}, v_{j_{s+1}}) = \delta_i$ for each $s \in [k]$.

Define an interval in a sequence to be a subsequence of consecutive elements. We say a pattern $P = (a_1, a_2, \ldots, a_n)$ has the left property if the following holds: for each interval $I \subset P$ with maximum element $a_m$, all elements to the left of $a_m$ in $I$ — meaning the elements $a_i \in I$ such that $i < m$ — are greater than or equal to all elements to the right of $a_m$ in $I$. We similarly say a pattern has the right property if within each interval, every element to the left of the maximum is smaller than every element to the right of the maximum.

The following theorem is our core result on max-induced patterns:

**Theorem 2.2.** A finite family of patterns $F$ has the max-induced Erdős-Hajnal property if and only if $F$ contains a permutation pattern with the left property and a permutation pattern with the right property.

Observe that if a permutation has both the left and the right property, it must be decreasing up to some element and increasing for the rest of the permutation. We say such a permutation has a unique local minimum, and we immediately get the following corollary of Theorem 2.2:

**Corollary 2.3.** A permutation $P$ has the max-induced Erdős-Hajnal property (with exponent $c(P) = 16^{-|P|}$) if and only if it has a unique local minimum.

An old result of Shelah [14] states that any graph which does not have large cliques or large independent sets must contain exponentially many non-isomorphic induced subgraphs. Taking inspiration from this, we may ask: given $k \in \mathbb{N}$, what is the largest integer $f(k)$ for which there is $\varepsilon > 0$ such that every sequence of length $n$ either contains at least $f(k)$ distinct max-induced patterns on $k$ elements or a max-induced homogeneous subsequence of length $n^{\varepsilon}$? Theorem 2.2 allows us to answer this question exactly and is the origin of the Catalan number that appears in Theorem 1.1.

**Corollary 2.4.** The value of $f(k)$ is the number of permutations of $[k]$ with the right property (equivalently, the left property), which is equal to the Catalan number $C_k$.

Up to this point, our sequence results do not use a useful fact about the $\delta$-sequences which features often in stepping-up. A sequence $(a_1, a_2, \ldots, a_k)$ has the unique maximum property if for any interval $I$, $|\{i \in I : a_i = \max(a_i)_{i \in I}\}| = 1$. We capitalize on this useful property in Lemma 2.5, which studies
a second notion of subsequence that is used in our proof of Theorem 1.4. Call \((a_{i_1}, a_{i_2}, \ldots, a_{i_t})\) with 0 < \(i_1 < \cdots < i_t \leq m\) a separated subsequence of \((a_1, a_2, \ldots, a_m)\) if \(i_{j+1} > i_j + 1\) for all \(j \in [t]\). Similar to max-inducedness, the following simple fact captures the usefulness of this definition for stepping up: suppose \(v_1 < v_2 < \cdots < v_{t+1}\) are vectors in \(\{0, 1\}^m\) and let \(\delta_i = \delta(v_i, v_{i+1})\). If \((\delta_{i_1}, \ldots, \delta_{i_t})\) is a separated subsequence of \((\delta_1, \ldots, \delta_k)\), then there are \(v_{j_1}, \ldots, v_{j_{2k}}\) such that \(\delta(v_{j_{2s-1}}, v_{j_{2s}}) = \delta_i\) for each \(s \in [k]\). Given a sequence \(S = (s_1, \ldots, s_n)\), define \(\|S\| := |\{s_1, \ldots, s_n\}|\), that is, the number of distinct values in the sequence. We can now state our key result on separated subsequences.

**Lemma 2.5.** Let \(A = (a_r)_{r=1}^n\) be a sequence with the unique maximum property and \(k \in \mathbb{N}\). If \(\|A\| < n^{1/(k+1)}\) then \(A\) contains every permutation on \([k]\) as a separated subsequence.

We defer the proof of this lemma to the end of the section and first proceed to prove Theorem 2.2 and Corollary 2.4.

### 2.1. Max-induced Subsequences

Recall that a family of patterns \(\mathcal{F}\) has the **max-induced Erdős-Hajnal property** with exponent \(c = c(\mathcal{F})\) if every sequence \(S\) that avoids all patterns in \(\mathcal{F}\) as a max-induced subsequence must have a max-induced homogeneous subsequence of length \(|S|^{c(\mathcal{F})}\). To characterize the families with this property, we require the following lemma:

**Lemma 2.6.** For every \(k \geq 1\), there is a permutation \(S_k\) of length \(2^{k+1} - 1\) that contains neither a max-induced copy of \((2, 3, 1)\) nor a max-induced homogeneous subsequence of length greater than \(k + 1\).

**Proof.** We proceed by induction on \(k\). Note that for \(k = 1\), the permutation \((1, 3, 2)\) works. Let \(S_{k-1} = (a_1, a_2, \ldots, a_{2^k-1})\) be a permutation of length \(2^k - 1\) with no max-induced 231 and no max-induced homogeneous subsequence of length greater than \(k\). Define the sequence \(S_k = (b_1, \ldots, b_{2^{k+1}-1})\) by

\[
b_i = \begin{cases} a_i & \text{if } i < 2^k; \\ 2^k + 1 & \text{if } i = 2^k; \\ a_{i-2^k} + 2^k - 1 & \text{if } i > 2^k. \end{cases}
\]

If \((b_1, \ldots, b_{2^k})\) is a max-induced homogeneous subsequence of \(S_k\), then either \(i_1 \leq 2^k\) or \(i_1 \geq 2^k\). Thus, by our choice of \(S_{k-1}\), there is no max-induced homogeneous component of length greater than \(k + 1\) in \(S_k\). Similarly, every max-induced 231 pattern \((b_1, b_2, b_3)\) has either \(i_1 < 2^k\) or \(i_1 > 2^k\), which shows \(S_k\) avoids a max-induced 231. \(\Box\)

**Proof of Theorem 2.2.** For the forward direction, suppose (without loss of generality) that \(\mathcal{F}\) contains no permutation with the right property. Observe that every permutation without the right property must contain the pattern 231. Therefore, each sequence in \(\mathcal{F}\) contains either two elements which are equal or a copy of 231, so the sequence \(S_k\) defined in Lemma 2.6 is an \(\mathcal{F}\)-avoiding sequence of length at least \(2^k\) containing no max-induced homogeneous subsequence of length greater than \(k + 1\).

For the backwards direction, let \(L\) and \(R\) be two permutations such that \(L\) has the left property and \(R\) has the right property, and suppose \(S\) is a sequence of length \(n\). We prove by induction on \(t := |L| + |R|\) that \(S\) contains at least one of \(L, R\), or a max-induced homogeneous sequence of length at least \(n^t/2\), where \(\varepsilon = 4^{-t}\). If \(t \leq 4\), then one of \(L\) or \(R\) has at most two elements, so the statement is clear. We therefore assume \(t > 4\), in which case we may assume also that \(n\) is large enough for the inequalities below.
We first describe an algorithm that shows either $S$ contains a max-induced homogeneous sequence of length at least $n^{-\epsilon}$ or there is an index $i$ such that $a_i$ is the maximum in a substantial interval, i.e. $a_j \leq a_i$ for $i - n^{1-\epsilon} \leq j \leq i + n^{1-\epsilon}$. Initialize index sets $S' = [n]$ and $F_{\ell} = F_r = \emptyset$. In the $k$th iteration of the algorithm, suppose the index $j_k \in S'$ is such that $a_{j_k} = \max_{i \in S'} a_i$. Let $\ell = \max F_{\ell}$ and $r = \min F_r$. For $k = 1$, we set $\ell = 0$ and $r = n + 1$. Then

- if $j_k - \ell < n^{1-\epsilon}$, add $j_k$ to $F_{\ell}$, delete all $i \in S'$ such that $i \leq j_k$, and proceed to the next iteration;
- if $r - j_k < n^{1-\epsilon}$, add $j_k$ to $F_r$, delete all $i \in S'$ such that $i \geq j_k$, and proceed to the next iteration;
- otherwise, the algorithm terminates.

Observe that $(a_i)_{i \in F_{\ell}}$ and $(a_i)_{i \in F_r}$ are max-induced sequences in $S$, so if either $|F_{\ell}| \geq n^\epsilon/2$ or $|F_r| \geq n^\epsilon/2$, then we have constructed a max-induced homogeneous subsequence of length $n^\epsilon/2$. This must be the case if the algorithm terminates due to deleting all of $S'$. Otherwise, suppose the algorithm terminates on the $k$th iteration. Then $j_k$ satisfies $a_i \leq a_{j_k}$ for $j_k - n^{1-\epsilon} \leq i \leq j_k + n^{1-\epsilon}$ as desired. Furthermore, for $F_m := \{i \in S' : a_i = a_{j_k}\}$, the sequence $(a_i)_{i \in F_m}$ is a max-induced sequence in $S$, so we may assume $|F_m| < n^\epsilon/2$.

Now let $S'_i = \{i \in S' : i < j_k\}$ and $S'_j = \{i \in S' : i > j_k\}$. Let $M_{\ell} := M \cap S'_i$ or $M_r := M \setminus M_{\ell}$ has at least $\frac{1}{2} n^{(1-\epsilon)/2}$ elements. Suppose the former (the argument for the latter is similar). Since $|S'_i| \geq n^{1-\epsilon}$ there must be an interval $I \subseteq S'_i$ of length at least $n^{(1-\epsilon)/2}$ such that $I \cap M_{\ell} = \emptyset$. If $I_m := \{i \in I : a_i = \max_{j \in I} a_j\}$ then $(a_i)_{i \in I_m}$ is a max-induced sequence in $S$, so we may assume $|I_m| < n^\epsilon/2$.

Now $|M_{\ell}| \geq n^{(1-\epsilon)/2}/2$ and $|F_m| \leq n^\epsilon/2$ so there is an interval $J$ such that $|J \cap M_{\ell}| \geq n^{1-3\epsilon}/2$ and $J \cap F_m = \emptyset$. Similarly, there is a subinterval $I' \subseteq I$ such that $|I' \cap I_m| = \emptyset$. Let $A = J \cap M_{\ell}$ and $B = I'$ so that any max-induced subsequence in $A$ or $B$ is a max-induced subsequence of $S$. Also if $i \in A$, $j \in B$ then $a_j < a_i < a_{j_k}$. Recall that the permutation $L = (l_i)_{i=1}^{|L|}$ has the left property. Let $k$ be the index of the maximum of $L$, and define $L_{\ell} = (l_i)_{i<k}$ and $L_r = (l_i)_{i>k}$, i.e. $L_{\ell}$ is the subpermutation preceding max $L$ and $L_r$ is the subpermutation following max $L$. Observe that $L_{\ell}$ and $L_r$ both have the left property.

Apply the inductive hypothesis to $S_A := (a_i)_{i \in A}$ with the permutations $L_{\ell}$ and $R$, and to $S_B := (a_i)_{i \in B}$ with the permutations $L_r$ and $R$. If we find a max-induced copy of $R$ in $S_A$ or $S_B$ we are done. If $S_A$ contains a max-induced homogeneous subsequence of length at least

$$\frac{1}{2} |A| 4^{-(|L_{\ell}| + |R|)} \geq \frac{1}{2} (n^{1-3\epsilon}/2)^{4^{-(t-1)}} = \frac{1}{2} (n^{1-3\epsilon}/2)^{4^t} \geq \frac{n^\epsilon}{2},$$

or $S_B$ contains a max-induced homogeneous subsequence of length at least

$$\frac{1}{2} |B| 4^{-(|L_r| + |R|)} \geq \frac{1}{2} (n^{1-3\epsilon}/2)^{4^{-(t-1)}} = \frac{1}{2} (n^{1-3\epsilon}/2)^{4^t} \geq \frac{n^\epsilon}{2},$$

we are finished. Otherwise, there is a max-induced copy of $L_{\ell}$ in $S_A$ and $L_r$ in $S_B$, which along with $a_{j_k}$ forms a max-induced copy of $L$ in $S$.

Given $n, k \in \mathbb{N}$, recall $f(k)$ is the largest integer for which there exists some $\epsilon > 0$ such that every sequence of length $n$ contains either at least $f(k)$ distinct max-induced patterns on $k$ elements or a homogeneous subsequence of length $n^\epsilon$.

**Proof of Corollary 2.4.** Let $\Pi_k$ be the set of permutations of $[k]$ with the right property, and let $r_k := |\Pi_k|$. Let $S$ be a sequence of length $n$. If $S$ contains fewer than $r_k$ max-induced patterns, then $S$ must avoid at
least one left-property permutation and at least one right-property permutation, which implies \( S \) contains a long max-induced homogeneous subsequence by Theorem 2.2. So \( f(k) \geq r_k \).

To see \( f(k) \leq r_k \), observe that the only max-induced patterns in the construction \( S_k \) from Lemma 2.6 are permutations with the right property.

It remains to show \( f(k) = C_k \). Let \( \pi \in \Pi_k \) and suppose \( \pi(i) = k \). The sequence \( \{\pi(1), \ldots, \pi(i-1)\} \), must be a permutation—call it \( \pi_\ell \) of \( [i-1] \) and \( \{\pi(i+1), \ldots, \pi(k)\} \) must be a permutation \( \pi_r \) of \( [i+1, \ldots, k-1] \). Both \( \pi_\ell \) and \( \pi_r \) have the right property, so \( r_k = \sum_{i=1}^k r_{i-1} r_{k-i} \), which is the Catalan recurrence. The initial terms of \( (r_k)_{k=0}^\infty \) coincide with the Catalan numbers (take \( r_0 = 1 \), so they must define the same sequence. \( \square \)

By symmetry \( C_k \) is also the number of permutations of \( [k] \) with the left property.

### 2.2. Separated Subsequences

We now prove a result on separated subsequences. The proof relies on a simple density argument to find a very rich substructure that contains within it all possible small structures.

**Proof of Lemma 2.5.** We first find constant subsequences \( A_1, \ldots, A_k \) such that

- \( |A_i| \geq n^{1-i/(k+1)} \) for each \( i \in [k] \),
- if \( a \in A_i \) (meaning \( a \) is equal to some element of \( A_i \)) and \( a' \in A_j \) with \( i < j \), then \( |a| < |a'| \), and
- (interlacing property) if \( a_j, a_\ell \in A_i \) such that \( j < \ell \), then there is some \( a_m \in A_{i+1} \) such that \( j < m < \ell \)

As \( \|A\| < n^{k+1} \), by the pigeonhole principle there is a constant subsequence \( A_1 \) such that \( |A_1| \geq n^{1-1/(k+1)} \). Note that by the unique maximum property, \( A_1 \) must be a separated subsequence. For \( i \geq 1 \), given a constant separated subsequence \( A_i = (a_{i,j})_j \) satisfying the above properties, we construct \( A_{i+1} \) as follows:

Let \( A_i' = (a'_{i,j})_j \) be the subsequence of \( A \) consisting of the maximum values between the elements of \( A_i \), meaning if \( a_{i,j}, a_{i,j+1} \in A_i \), then \( a'_{i,j} = \max\{a_{\ell,j} : i_j \leq \ell \leq i_{j+1}\} \). By the unique maximum property of \( A \), \( a'_{i,j} \) is well-defined and is strictly greater than \( a_{i,j} \) and \( a_{i,j+1} \). Then \( |A_i'| = |A_i| - 1 \geq n^{1-1/(k+1)} - 1 \), and \( \|A_i'\| \leq n^{1/(k+1)} - 1 \) from the assumption that \( \|A\| \leq n^{1/(k+1)} \). By the pigeonhole principle, there is a constant subsequence of \( A_i' \) with length at least

\[
|A_i'| \|A_i'\| \geq n^{1-(i+1)/(k+1)}.
\]

Let \( A_{i+1} \) be this subsequence. It is clear from the construction that the three desired properties are satisfied.

Now suppose \( \sigma \) is a permutation of \( [k] \). We use the interlacing property to find a subsequence of \( A \) with the pattern \( \sigma \). Let the elements of \( A_1 \) be indexed by \( (i_1, i_2, \ldots, i_m) \). For \( j \in [k] \), the interlacing property shows there is \( a_{s_j} \in A_{\sigma(j)} \) such that \( s_j \in \{i_{j,2^k + 1}, \ldots, i_{(j+1)2^k - 1}\} \). The subsequence \( (a_{s_1}, a_{s_2}, \ldots, a_{s_k}) \) is separated by construction, and \( \sigma(i) < \sigma(j) \) if and only if \( a_{s_i} < a_{s_j} \) by the second property. Thus, \( (a_{s_i})_i \) has pattern \( \sigma \). \( \square \)

We are now ready to employ these results in our stepping-up constructions.

### 3. Imposing many colours in hypergraph Ramsey theory

In this section, we explain how the results on max-induced and separated patterns in sequences allow us to step-up colourings in which all large sets of vertices span many colours. We prove Theorems 1.1 and 1.4 as well as Corollary 1.2. Throughout, all colourings discussed will be (hyper)edge-colourings. We say a colouring of a complete hypergraph with \( q \) colours is \((t,q,p)\)-rainbow if every set of \( t \) vertices spans at least \( p \) colours.
In this language $r_k(t; q, p) - 1$ is the largest integer $n$ for which there exists a $(t; q, p)$-rainbow colouring of $K_n^{(k)}$. Let $V = \{0, 1\}^m$ and given $v, w \in V$, recall that

$$\delta(v, w) := \max\{i : v_i \neq w_i\},$$

and that $v < w$ if and only if $v_i < w_i$. To each set of vertices $v_1 < v_2 < \cdots < v_{k+1}$ in $K_n^{(r+1)}$ corresponds a sequence $(\delta_1, \delta_2, \ldots, \delta_k)$ given by $\delta_i = \delta(v_i, v_{i+1})$, which we refer to as the corresponding $\delta$-sequence. The crucial properties for our application of the results of Section 2 to hypergraph Ramsey theory are

\begin{equation}
\text{(2)} \quad \text{For an interval } I, \text{ let } \delta_I = \max_{i \in I} \delta_i. \text{ Then } |\{i \in I : \delta_i = \delta_I\}| = 1.
\end{equation}

\begin{equation}
\text{(3)} \quad \text{For any } v_1 < v_2 < \cdots < v_{k+1}, \delta(v_1, v_{k+1}) = \max\{\delta_1, \delta_2, \ldots, \delta_k\}.
\end{equation}

See e.g. [9] for proofs. A useful perspective is gained by viewing $[2^n]$ as the leaves of a binary tree of depth $n$, ordered from left to right as you would draw them. Given two leaves $v, w \in [2^n]$, the number $2\delta(v, w)$ is simply the length of the unique path between them.

Recall that Corollary 2.4 shows that a sequence $(\delta_1, \delta_2, \ldots, \delta_k)$ either contains many distinct max-induced subsequences or a long homogeneous max-induced subsequence. In order to make use of this fact, we recall Claim 2.1, which simply says the following: suppose $v_1 < v_2 < \cdots < v_{i+1}$ are vectors in $\{0, 1\}^m$ and let $\delta_i = \delta(v_i, v_{i+1})$. If $(\delta_1, \ldots, \delta_k)$ is a max-induced subsequence of $(\delta_i)$, then there are $v_{j_1}, \ldots, v_{j_k}$ such that $\delta(v_{j_s}, v_{j_{s+1}}) = \delta_i$, for each $s$. □

We are ready to use Theorem 2.2 to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $k, p, q \geq 3$ be such that $p \leq C_k$, the $k$th Catalan number, and let $c := 16^k + 1$. We first prove the inequality $r_{k+1}(t^c; 2q + p, p) > 2r_k(t^{(c,q,p)} - 1)$ by showing such that we can construct a $(t^c; 2q + p, p)$-rainbow colouring $\chi'$ of $K_n^{(k+1)}$ from any $(t; q, p)$-rainbow colouring $\chi$ of $K_n^{(k)}$. Let $\chi$ be such a rainbow colouring.

We identify $V(K_n^{(k)})$ and $V(K_n^{(k+1)})$ with $\{0, 1\}^n$ and $[n]$, respectively. Partition the set of patterns of $[k]$, excluding the two homogeneous permutations, into $p - 2$ classes $C_1, \ldots, C_{p-2}$ such that each class contains at least one permutation with the left property and one permutation with the right property. Let $C_{p-1}$ be the family of patterns containing only the strictly increasing permutation and $C_p$ the family containing the strictly decreasing permutation. Observe that such a partition is guaranteed to exist by Corollary 2.4 and the assumption that $C_k \geq p$ (note that a permutation may have both the left and right property). For a sequence $S$, we say $S \in C_i$ if the pattern of $S$ is in $C_i$. Assign to each class $C_i$ a distinct colour $c_i$. Then, for an edge $e = \{v_1, \ldots, v_{k+1}\}$ of $K_n^{(k+1)}$, with corresponding $\delta$-sequence $(\delta_1, \ldots, \delta_k)$, let

$$\chi'(e) = \begin{cases} 
\chi(\{\delta_1, \ldots, \delta_k\}) \times 1 & \text{if } \delta_1 < \delta_2 < \cdots < \delta_k; \\
\chi(\{\delta_1, \ldots, \delta_k\}) \times 2 & \text{if } \delta_1 > \delta_2 > \cdots > \delta_k; \\
c_i & \text{if } (\delta_1, \ldots, \delta_k) \in C_i, \ i \leq p - 2.
\end{cases}$$

We claim $\chi'$ is the desired colouring. Consider a set of $t^c$ vertices $\{v_1, \ldots, v_{t^c}\} \subset V(K_n^{(k+1)})$ and its corresponding $\delta$-sequence $(\delta_1, \ldots, \delta_{t^c-1})$. For each colour $c_1, \ldots, c_{p-2}$ or the set of colours $(\cdot, 1)$ and $(\cdot, 2)$, the corresponding sets of patterns contain a permutation with the left and a permutation with the right property. Theorem 2.2 thus tells us that $(\delta_1, \ldots, \delta_{t^c-1})$ either contains a max-induced pattern from each
of \(C_1, \ldots, C_p\) or a max-induced homogeneous sequence of length \(t\) (the value of \(c\) was chosen for this purpose). Claim 2.1 shows that for every max-induced sequence \(S \in (\delta_1, \ldots, \delta_{t-1})\) of length \(k\) we can find \(\{v_{i_1}, \ldots, v_{i_{t+1}}\} \subset \{v_1, \ldots, v_{t}\}\) whose corresponding sequence is \(S\). In other words, we have that \(\chi'((v_{i_1}, \ldots, v_{i_{t+1}})) = c_i\) if \(S \in C_i\). Therefore, \((\delta_1, \ldots, \delta_{t-1})\) containing a max-induced pattern from each of \(C_1, \ldots, C_p\) implies that \(\{v_1, \ldots, v_{t}\}\) spans at least \(p\) colours.

If instead \((\delta_1, \ldots, \delta_{t-1})\) contains a max-induced homogeneous subsequence, say \((\delta_1, \ldots, \delta_i)\), then by Claim 2.1 we can find \(\{v_{i_1}, \ldots, v_{i_{t+1}}\} \subset \{v_1, \ldots, v_{t}\}\) whose corresponding pattern is \((\delta_1, \ldots, \delta_i)\). We may assume without loss of generality that \((\delta_1, \ldots, \delta_i)\) is increasing, and by property (2), \(\delta_1 < \cdots < \delta_i\). Every subsequence of a strictly increasing sequence is a max-induced strictly increasing subsequence. Thus, again by Claim 2.1, for every subsequence \(S \subset (\delta_1, \ldots, \delta_i)\) of length \(k\) we can find a subsequence \(S' \subset (v_{i_1}, \ldots, v_{i_{t+1}})\) of length \(k+1\) whose corresponding \(\delta\)-sequence is \(S\). Observe that \(\chi'(S') = \chi(S, 1)\), so the set of \((k+1)\)-edges on \(\{v_{i_1}, \ldots, v_{i_{t+1}}\}\) spans at least as many colours as the set of \(k\)-edges on \((\delta_1, \ldots, \delta_i)\). By assumption, this is at least \(p\).

All that remains is to count the number of colours used by \(\chi'\). This is at most 2 colours for each of the \(q\) colours used by \(\chi\) plus an extra \(p-2\) for the \(c_i\)'s. Combined with the above arguments, this shows that \(\chi'\) is a \((t^2; 2q+p-2, p)\)-rainbow colouring and in particular a \((t^2; 2q+p, p)\)-rainbow colouring.

We now describe the construction of the \((t^2; q, p-2)\)-rainbow colouring, say \(\chi''\), which shows that \(r_{k+1}(t^2; q, p-2) > 2r_k(t^2; q, p-2)^{p-1}\). Let a \((t; q, p-2)\)-rainbow colouring \(\chi\) of \(K_n^{(k)}\) be given. As before, assign to each class \(C_i\) a distinct colour \(c_i\) for \(i \in [p-2]\), only this time identify these colours with the first \([p-2]\) colours used by \(\chi\) (trivially we have \(q \geq p-2\)). For an edge \(e = \{v_1, \ldots, v_{k+1}\}\) with corresponding \(\delta\)-sequence \((\delta_1, \ldots, \delta_k)\), let

\[
\chi''(e) = \begin{cases} 
\chi(\{\delta_1, \ldots, \delta_k\}) & \text{if } \delta_1 < \delta_2 < \cdots < \delta_k \text{ or } \delta_1 > \delta_2 > \cdots > \delta_k; \\
\c_i & \text{if } (\delta_1, \ldots, \delta_k) \in C_i.
\end{cases}
\]

We omit the proof that \(\chi''\) is in fact the desired colouring, as it is almost identical to the proof for \(\chi'\). We simply note that now we can only force \(p-2\) colours in any set of \(t^2\) vertices, as an edge with a homogeneous pattern may receive any one of the colours \(c_i\) for \(i \in [p-2]\).

In applying our stepping-up results, we will need the following result on rainbow colorings which is proven by a standard use of the first moment method (see e.g. [1]).

**Proposition 3.1.** For every \(k\) and \(q\), there exists \(\varepsilon > 0\) and \(t_0 = eq\) so that for \(t > t_0\) there is a \((t; q, q)\)-rainbow colouring of \(K_n^{(k)}\) with \(n = 2^{c_{t-1}}\).

**Proof.** Set \(\varepsilon = 1/(qk!)\) and \(t_0 = eq\). Consider a uniformly random \(q\)-colouring of \(K_n^{(k)}\) with \(n = 2^{ct^{k-1}}\). The probability a given \(t\)-set contains fewer than \(q\) colours is at most \(q(1 - 1/q)^{(\varepsilon n)} < (\varepsilon n/t^{k-1})^t < (q^{1/q}e^{(\varepsilon n)/(t^{k-1})})^t < 1\). As the expectation is strictly less than one, there must exist some \((t; q, q)\)-rainbow colouring. \(\square\)

**Proof of Corollary 1.2.** Proposition 3.1 shows that there is an \(\varepsilon > 0\) such that, with \(n = 2^{ct}\), a uniformly random colouring of the edges of \(K_n^{(3)}\) is \((t; q, q)\)-rainbow with high probability. Thus \(r_3(t; q, q) > 2^{ct}\). Since the Catalan number \(C_i \geq 5\) for \(i \geq 3\), we may apply the inequality \(r_{i+1}(t^2; 2q+5, 2q+5) > 2^{ct^{(i+5)/4}}\) from
Theorem 1.1, once for each \( i \in \{3, \ldots, k-1\} \) and with a different \( c \geq 1 \) each round. This ultimately yields a lower bound of \( r_k(t'; q', 5) \geq T_{k-1}(t) \) where \( q' = 3^{k} q \) and \( c' \geq 1 \).

The proof of Corollary 1.3 follows the exact same procedure as the proof of Corollary 1.2, simply using the bound \( r_{k+1}(t'; 3, 3) > 2^{r_k(t'; 3,3)-1} \) in place of the bound \( r_{k+1}(t'; 2q + 5, 3) > 2^{r_k(t';q,5)-1} \). We thus omit it. We now prove Theorem 1.4, which combines Lemma 2.5 with a slightly different stepping-up technique.

**Proof of Theorem 1.4.** Let \( k, p, q \geq 3 \) be such that \( p \leq k \) and let \( c := k + 2 \). We prove the inequality \( r_{2k}(t'; pq, p) > 2^{r_k(t';q,p)-1} \) by showing such that we can construct a \((t'; pq, p)\)-rainbow colouring \( \chi' \) of \( K_{2n}^{(2k)} \) from any \((t; q, p)\)-rainbow colouring \( \chi \) of \( K_{n}^{(k)} \).

We identify \( V(K_{2n}^{(2k)}) \) and \( V(K_{n}^{(k)}) \) with \( \{0, 1\}^n \) and \([n]\) respectively. Order the \( k! \) permutations of \([k]\) and let \( P_i \) be the set of sequences whose pattern is the \( i \)th permutation for \( i \in [p] \). Then, for an edge \( e = \{v_1, \ldots, v_{2k}\} \) with corresponding \( \delta \)-sequence \( (\delta_1, \ldots, \delta_{2k-1}) \) we set

\[
\chi'(e) = \chi(\{\delta_1, \delta_3, \ldots, \delta_{2k-1}\}) \times i
\]

if \( (\delta_1, \delta_3, \ldots, \delta_{2k-1}) \in P_i \) for some \( i \in [p] \) and we let \( \chi'(e) \) be arbitrary otherwise.

We now check that this is the desired colouring. It is clear that it uses at most \( pq \) colours. Suppose \( v_1 < \cdots < v_{2k} \) are vertices of \( K_{2n}^{(2k)} \) and let \( (\delta_1, \ldots, \delta_{2k-1}) \) be the corresponding \( \delta \)-sequence. If \( \|\{\delta_i\}_{i=1}^{t' - 1}\| \geq 2t \) then we can find an index set \( I \subset [t' - 1] \) such that \( (\delta_i)_{i \in I} \) is a separated subsequence with \( \|\{\delta_i\}_{i \in I}\| \geq t \). Let \( f = \{\delta_{i_1}, \ldots, \delta_{i_k}\} \subset (\delta_i)_{i \in I} \) where \( i_1 < i_2 < \cdots < i_k \). Then the edge \( e = \{v_{i_1}, v_{i_1+1}, v_{i_2}, v_{i_2+1}, \ldots, v_{i_k}, v_{i_k+1}\} \) has colour \( \chi'(e) = (\chi(f), i) \). As \( f \) was arbitrary we have that \( \{v_1, \ldots, v_{2k}\} \) spans at least \( c \) colours as \( \{\delta_i\}_{i \in I} \). But \( \|\{\delta_i\}_{i \in I}\| \geq t \) and so \( (\delta_i)_{i \in I} \) spans at least \( p \) colours by assumption.

Now suppose that \( \|\{\delta_i\}_{i=1}^{t' - 1}\| < 2t \). By (2), \( (\delta_i)_{i=1}^{t' - 1} \) satisfies the unique maximum property. Therefore, by our choice of \( c \), we can apply Lemma 2.5 with \( A = (\delta_i)_{i=1}^{t' - 1} \) to conclude that \( A \) contains every permutation of \([k]\) as a separated subsequence. Suppose we have a separated subsequence \( S \subset A \) whose pattern is the \( i \)th permutation for some \( i \in [p] \). As before we can find a \((2k)\)-edge \( e \subset \{v_1, \ldots, v_{2k}\} \) for whom the corresponding \( \delta \)-pattern is \( S \). Thus \( \chi'(e) = (\chi(f), i) \). Repeating this for the first \( p \) permutations implies that \( \{v_1, \ldots, v_{2k}\} \) spans at least \( p \) colours.

Both of our stepping-up constructions (like the original) rely on breaking the stepping-up into two parts: a lifted colouring from a lower uniformity graph and a new colouring. The constructions have the property that every large set of vertices satisfies one of the following: either it spans a colouring that is lifted from a large clique on the lower graph or it sees many distinct new colours. The number of new colours one can guarantee using this approach depends strongly on the uniformity of the hypergraphs in question. We ask a question in the concluding remarks concerning this.

4. Hedgehogs

In this section, we relate our results on many-coloured Ramsey numbers of complete hypergraphs to a Ramsey problem on a certain class of hypergraphs which we call generalized hedgehogs. We note our idea is very much inspired by the Conlon-Fox-Rödl construction described in the introduction. The main result is Theorem 1.6, (a) and we also prove Lemma 1.7 for a more general family of hedgehogs.

We define the generalized hedgehog \( H_t^{(k)}(s) \) for \( s, k, t \in \mathbb{N} \) with \( k > s \) and \( t \geq s \) to be the following \( k \)-uniform hypergraph: fix a set of \( t \) vertices called the body. The edge set of \( H_t^{(k)}(s) \) consists of the following edges: for
Theorem 4.2

Lemma 4.1. Given $k, q, p', t \in \mathbb{N}$, let $p = \binom{k}{q}$ and $q' = \binom{q}{p}$. Then

$$r_k(H_t^{(k)}(s); q', p' + 1) > r_s(t; q, p'p + 1) − 1.$$ 

Proof of Lemma 4.1. Apply Lemma 4.1 with $2k + 1, k + 1$ and $1$ playing the role of $k, s$ and $p'$ respectively. 

Proof of Lemma 4.1. We prove $r_k(H_t^{(k)}(s); q', p' + 1) > r_s(t; q, p'p + 1) − 1$ by showing that given a $(t; q, p'p + 1)$-rainbow colouring of $K_n^{(k)}$, we can construct a $q'$-colouring $\chi'$ of $K_n^{(k)}$ in which every copy of $H_t^{(k)}(s)$ spans at least $p' + 1$ colours. Let such a $\chi$ be given. Identify the vertex sets of $K_n^{(k)}$ and $K_t^{(s)}$. We colour $e' \in E(K_n^{(k)})$ by $\chi'(e') = \{\chi(e) : e \in E(K_n^{(s)}[e'])\}$, that is, the union of the colours of the $s$-edges contained in $e'$. The number of $s$-edges contained in $e'$ is exactly $p = \binom{k}{q}$, so $\chi'(e')$ will be the union of at most that many colours. If $|\chi'(e')| < p$ then we add arbitrary colours to the set until $|\chi'(e')| = p$. Thus the new colouring uses at most $q' = \binom{q}{p}$ colours.

Now consider a copy $H \subset K_n^{(k)}$ of $H_t^{(k)}(s)$. As the body contains $t$ vertices, by assumption it spans at least $p'p + 1$ colours under $\chi$. Each of these colours appears as an element of $\chi'(e')$ for some $e' \in E(H)$, i.e. $|\cup_{e' \in E(H)} \chi'(e')| \geq p'p + 1$. However, $p$ colours appear in $\chi'(e')$ for a given $e'$, so by pigeonhole there must be more than $p'$ edges $e' \in E(H)$ with distinct colours in $\chi'(H)$. 

We now derive Theorem 1.6, (b).

Proof of Theorem 1.6, (b). Throughout the proof we assume $t \in \mathbb{N}$ is sufficiently large. Let $m = \left\lfloor \frac{e(k+1)}{\log_2(k+1)} \right\rfloor$, which is chosen so that $m! > \left(\frac{m}{e}\right)^m > 2^{2k} > \left(\frac{2k + 1}{k + 1}\right)$ for $k$ sufficiently large. Recall that by Proposition 3.1, $r_m(t; m!, m!) \geq T_2(\varepsilon t)$ for some $\varepsilon > 0$. We now use the inequality $r_{2k}(t'; pq, p) \geq 2^{r_{2k}(t'; pq, p)}$ from Theorem 1.4 to obtain that there exists $q'$ and $c' \geq 1$ such that $r_{2k+1}(t'; q', m!) \geq T_{\log_2 \log_2 k + 4}(t)$ (if $k + 1 > 2^m$ with $j = \lfloor \log_2 \frac{k + 1}{m} \rfloor$ we use the simple observation $r_{i+1}(t; q, p) > r_i(t; q, p)$ for some $q' > q$). This is possible because the uniformity increases at each step while the number of colours imposed remains at $m!$. Since $m! > \left(\frac{2k + 1}{k + 1}\right)$, we can apply Lemma 1.7 to get that for some $q''$, $r_{2k+1}(\hat{H}_t; q'', 2) \geq T_{\log_2 \log_2 k - 2}(t)$, where we removed the exponent $c'$ at the cost of a tower height. We choose $c > 0$ such that if $m! \leq \left(\frac{2k + 1}{k + 1}\right)^c$ then $e \log_2 \log_2 k \leq 1$.

Thus, for all $k$, $r_{2k+1}(\hat{H}_t; q'', 2) \geq T_{e \log_2 \log_2 k}(t)$. Furthermore, it is clear that $q''$ is only a function of $k$ and of $m$, which is itself a function of $k$, as required. 

Conlon, Fox, and Rödl showed the following using a similar argument to Theorem 1.6, (a):

Theorem 4.2 ([3]). For all $k \geq 4$ there exists $c > 0$ such that

$$r_k(H_t^{(k)}; 2, 2) \leq T_{k-2}(ct).$$

For the case $k = 4$, a construction of Kostochka and Rödl [10] shows this is approximately sharp, i.e. that $r_4(H_t^{(4)}(3); 2, 2) = T_2(\Omega(t))$. We cannot prove a matching lower bound for $k = 5$ as when we attempt to apply Lemma 1.7 we need to impose 6 colours on 4-uniform graphs. When stepping up to uniformity 4,
Theorem 1.1 allows us to impose at most $C_3 = 5$ colours, so we cannot beat the random argument here. We do, however, obtain the following:

**Lemma 4.3.** For $5 \leq k \leq 13$, there exist $c > 0$ and $q \in \mathbb{N}$ such that for all $t$

$$r_k(H_t^{(k)}; q, 2) \geq T_{k-3}(t^c).$$

**Proof.** The proof mimics that of Theorem 1.6, (b), but now we start from the fact that $r_4(t; 14, 14) \geq T_2(ct)$ (using Proposition 3.1) and leverage that $C_4 = 14$. Starting from $\ell = 4$, we apply the relation $r_{\ell+1}(t^c; 2q+p, p) \geq 2^{\ell(t^c_0+p)}$ from Theorem 1.1 $k-5$ times. This gives us $r_{k-1}(t; q', 14) \geq T_{k-3}(t'^c)$ for some $q' \in \mathbb{N}$ and $c' \geq 1$. Applying Lemma 1.7 then gives the result using $q''$ colours and with $c' = c$. \hfill \Box

We now prepare to prove our upper bounds on the Ramsey numbers of balanced hedgehogs, Theorem 1.6, (a). Let $H$ be an $r$-uniform hypergraph and $A \subset V(H)$ with $|A| < r$. The **piercing number** of $A$ (denoted $\tau_H(A)$) is the size of the smallest set of vertices from $V(H) \setminus A$ that intersects every edge of $H$ containing $A$. Equivalently it is the minimum number of vertices that must be deleted from $V(H) \setminus A$ to delete all edges containing $A$.

**Proposition 4.4.** Suppose $H$ is an $r$-uniform hypergraph and $v \in V(H)$ has $\tau(v) \geq (r-1)m$. Then there exist $m$ edges whose pairwise intersections are all precisely $v$.

**Proof of Proposition 4.4.** Let $X \subset V(H)$ be a witness to $\tau(v)$. It is clear the set of edges incident to $v$ has order at least $(r-1)m$. As each such edge $e$ contains at most $r-1$ elements of $X$, and $|X|$ is minimal with respect to intersecting all edges incident to $v$, we can greedily find the desired $m$ edges. \hfill \Box

**Proof of Theorem 1.6, (a).** Let $G = K_n^{(2k+1)}$ where $n \geq t^{k+3}$ and let $\chi$ be an arbitrary red/blue colouring of the edges of $G$. We will show that there exists a monochromatic copy of $H_t^{(2k+1)}(k+1)$ in $G$, and thus that $r_{2k+1}(H_t^{(2k+1)}(k+1); 2, 2) \leq n$. We will do so by using $\chi$ to define a partial edge 2-colouring $\chi' \subseteq H_t^{(2k+1)}$ and in turn using $\chi'$ to define a 2-colouring $\chi''$ of $V(G)$. We will then find a large monochromatic set of vertices in $\chi''$, use this to find a large (red or blue) independent set in $G'$, and finally use this to find our monochromatic $H_t^{(2k+1)}(k+1)$ in $G$.

Throughout the proof let $H_t := H_t^{(2k+1)}(k+1)$. We say a set of vertices is in **red danger** if its red piercing number (its piercing number in the subgraph of red edges) is less than $t^{k+1}$ and similarly define **blue danger**. Then for $e \in G' \subseteq K_n^{(k+1)}$, let $\chi'(e)$ be red if $e$ is in red danger and blue if $e$ is in blue danger. As $\tau_G(e) > 2t^{k+1}$, we have that this partial colouring $\chi'$ is well defined.

Now say a vertex $v$ is in **red peril** if its red piercing number in $G'$ (under $\chi'$) is at most $2kt^{k+1}$ and similarly define **blue peril**. Let $\chi''(v)$ be red if $v$ is in red peril and blue if $v$ is in blue peril, with ties broken arbitrarily. We claim that $\chi''$ assigns a colour to every vertex. Indeed suppose that some vertex $v$ has red piercing number and blue piercing number at least $2kt^{k+1}$ under $\chi'$.

By Proposition 4.4, there exist $(k+1)$-edges $e_1, \ldots, e_s, f_1, \ldots, f_s \in G'$ where $s = 2t^{k+1}$ such that

- $e_i \cap e_j = f_i \cap f_j = \{v\}$ for all $i \neq j$,
- $\chi'(e_i)$ is red for all $i \in [s]$, and
- $\chi'(f_j)$ is blue for all $j \in [s]$

Let $A_1, A_2, \ldots, A_s$ be disjoint subsets each of size $k-1$ in $V(G) \setminus (\bigcup_{i=1}^s (e_i \cup f_i))$. For each $i, j \in [s]$, define a $(2k+1)$-edge $g_{ij}$ to be $e_i \cup f_j$ along with an arbitrary choice of $(2k+1) - |e_i \cup f_j|$ vertices from $A_{(i+j-1) \mod s}$. Observe that $e_i \cup f_j \neq e_{i'} \cup f_{j'}$ for $(i, j) \neq (i', j')$, so in particular, these edges are all distinct.
At least half of these edges must have been, without loss of generality, red under the colouring \( \chi \) of \( G \). Therefore, there is some \( i \in [8] \) such that \( e_i \) is contained in at least \( \frac{t^2}{2t} = \frac{t}{2} > t^{k+1} \) red \((2k+1)\)-edges under \( \chi \). But since \( g_{i,j} \cap g_{i,j'} = e_i \) for \( j \neq j' \), this contradicts the fact that \( e_i \) was a red-danger edge of \( G' \). Thus, the colouring \( \chi'' \) of \( V(G) \) colours every vertex.

We now choose a set \( X \) of red vertices (without loss of generality) which has order \( \frac{t}{2} \). By the definition of \( \chi'' \), we have that the red piercing number under \( \chi' \) of each \( v \in X \) is at most \( 2kt^{k+1} \). Therefore we can greedily find a set \( Y \) of order \( \frac{n}{2} \) which contains no red edges of \( G' \) (here we use that \( t \) is large relative to \( k \)).

All edges of \( G' \) in \( Y \) are not in red danger and thus have red piercing number at least \( t^{k+1} \) under \( \chi \). By the definition of piercing number and as \( t^{k+1} > t + k(t^{k+1}) = |V(H_1^{(2k+1)}(k+1))| \), we can build the hedgehog greedily using \( Y \) as the body.

\[ \square \]

### 4.1. Burr-Erdős Conjecture in hypergraphs.

The degeneracy of a hypergraph is the minimum \( d \) such that every induced subgraph contains a vertex incident to at most \( d \) edges; such a hypergraph is called \( d \)-degenerate.

Burr and Erdős conjectured that for every \( d \), there exists a constant \( c_d \geq 1 \) such that every \( d \)-degenerate graph \( G \) on \( n \) vertices satisfies \( r_2(G) < c_d n \) [2]. This was finally proven by Lee in [12], building on the previous work of several authors ([8], [11]). In the case of hypergraphs, however, the conjecture fails: Kostochka and Rödl [10] showed \( r_4(H_1^{(3)}) \geq 2^c \), and Conlon, Fox, and Rödl [8] proved \( r_3(H_1^{(3)}; 3) \geq \Omega(t^3/\log^6 t) \), while it is easy to see the degeneracy of hedgehogs is 1.

This shows that the Burr-Erdős Conjecture fails for \( k \)-uniform hypergraphs where \( k \geq 4 \) and for 3-uniform hypergraphs provided the number of colours is at least 3. For the sake of completeness, we show the following, in this way disproving fully the Burr-Erdős Conjecture for hypergraphs.

**Proposition 4.5.** There exists a 3-uniform hypergraph on \( C_n^2 \) vertices which is 8-degenerate and for which the 2-colour Ramsey number is at least \( Cn^3 \).

**Proof.** Let \( V \) be a set of \( n \) vertices, and with \( m = \binom{n}{3} \), let \( B \) be a set of \( m + 1 \) vertices disjoint from \( V \) with a total ordering \( x_1, x_2, \ldots, x_m, x_{m+1} \). Consider an ordering on the edges \( E \) of the complete graph on \( V \), say \( e_1, e_2, \ldots, e_m \). Let \( H \) be a 3-uniform hypergraph consisting of the following: for each edge \( e_i = (x, y) \in E \), add the two triples \( \{x, y, x_i\} \) and \( \{x, y, x_{i+1}\} \), and for every \( i \in [m-3] \), add all triples within
\[ \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}. \]
By considering the smallest element of \( B \) in each subset of vertices, we see that \( H \) is 8-degenerate.

We claim \( r_3(H) \geq n^3/8 \). Indeed, given \( K_{n^3/8} \), let \( W_i, \ldots, W_{n/4} \) be an arbitrary partition of the vertices where \( |W_i| = n^2/2 \) for all \( i \in [n/4] \). We colour blue all edges inside any \( W_i \) and all edges with vertices in distinct sets \( W_{i1}, W_{i2}, W_{i3} \). We colour red all edges with exactly two vertices in a set \( W_i \), for some \( i \in [n/4] \).

Suppose there is a red copy of \( H \). In particular, the edges induced by \( S = \{x_1, x_2, x_3, x_4, x_5\} \) are all red. If \( S \) intersects distinct parts \( W_i, W_j, \) and \( W_k \) for some \( i, j, k \in [n/4] \), then the vertices in the intersections form a blue edge. However, if \( S \subseteq W_i \cup W_j \), then \( |S \cap W_i| \geq 3 \) or \( |S \cap W_j| \geq 3 \), which again forms a blue edge. So we have a contradiction.

Suppose instead there is a blue copy of \( H \). By pigeonhole, there are two vertices \( v, w \in V \) which both lie in the same \( W_i \). Let \( e_j = \{v, w\} \). The edges \( \{v, w, x_j\} \) and \( \{v, w, x_{j+1}\} \) are both blue, so \( x_j, x_{j+1} \in W_i \). But \( \{x_j, x_{j+1}, x_{j+2}\} \) and \( \{x_{j-1}, x_j, x_{j+1}\} \) are also blue, so this implies \( x_{j-1}, x_{j+2} \in W_i \). Proceeding inductively, we conclude that all of \( B \) must lie in \( W_i \).
However, $|W_i| = n^2/2$ and $|B| = \binom{n}{2}$, so there are $n/2$ vertices of $V$ outside of $W_i$. Two of these vertices $z$ and $u$ must lie in the same $W_j$ with $j \neq i$. Letting $e_k = \{z, u\}$, we get that the edge $\{z, u, x_k\}$ is coloured red, a contradiction which finishes the proof. □

5. Concluding remarks

We have proved that for every positive integer $h$, there exist $q, k$, and an infinite family of $k$-uniform hypergraphs whose 2-colour Ramsey numbers differ by a tower of height $h$ from the $q$-colour Ramsey numbers. This reinforces the fact that the number of colours plays an important role in the behaviour of Ramsey numbers of hypergraphs and casts a shadow on Erdős’s conjecture on the 2-colour Ramsey number of a 3-uniform clique.

Observe that both of our new stepping-up constructions rely on a dichotomy: either we can find many suitable substructures within the $\delta$-sequences (which give rise to many colours) or we must have a long homogeneous sequence (which allows us to use induction). Since for every $k$-edge there are at most $k!$ distinct permutations, our methods fail to give good lower bounds for $r_k(t; q, p)$ whenever $k \ll p$. Even in the simplest case $r_3(t; q, 3)$, we were not able to prove a double exponential lower bound, leaving open the following question of Conlon, Fox, and Rödl on $r_3(t; q, 3)$.

Problem 5.1. [3, Problem 1] Is there an integer $q$, a positive constant $c$, and a $q$-colouring of the 3-uniform hypergraph on $2^{2^t}$ vertices such that every subset of order $t$ receives at least 3 colours?

We propose an even weaker problem which we were not able to resolve. A negative answer would uncover a radical new phenomenon in the Ramsey numbers of hypergraphs.

Problem 5.2. Does there exist $k \in \mathbb{N}$ such that the following holds? For all $p \in \mathbb{N}$ there exist $q \in \mathbb{N}$ and $c > 0$ such that $r_k(t; q, p) \geq 2^{2^t}$ for all $t$ sufficiently large.

Recall Proposition 4.5 shows there is an infinite family of 3-uniform hypergraphs which are 8-degenerate and for which the the 2-colour Ramsey numbers grows faster than linear in the order of the hypergraphs. It would be interesting to improve the quantitative aspects of this result.

Problem 5.3. Give an infinite family of 1-degenerate 3-uniform hypergraphs whose 2-colour Ramsey number is not polynomial in the order of the hypergraph.

Finally, we make the following conjecture regarding the Ramsey numbers of $k$-uniform hedgehogs. This would in particular demonstrate that the 2-colour and $q$-colour Ramsey numbers of these hedgehogs, unlike those of balanced hedgehogs, do not differ by arbitrarily large tower heights.

Conjecture 5.4. There is $\ell \in \mathbb{N}$ such that for every positive integer $k$, for every sufficiently large $t$,

$$r_k(H_t^{(k)}) \geq T_{k-\ell}(t).$$

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Department of Mathematics, Rutgers University, Piscataway, NJ, 08854, USA

*Email address:* qcd2@math.rutgers.edu

Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

*Email address:* antonio.girao@maths.ox.ac.uk

Institute for Computer Science, Heidelberg University, 69120 Heidelberg, Germany

*Email address:* hurley@informatik.uni-heidelberg.de

Department of Mathematics, Rutgers University, Piscataway, NJ, 08854, USA

*Email address:* corrine.yap@rutgers.edu