World quantum gravity: An approach based on Synge’s world function

Ding Jia (贾丁)\textsuperscript{1,2,*}

\textsuperscript{1}Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
\textsuperscript{2}Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada

Abstract

A new approach of quantum gravity based on the world function (invariant distance) is presented. The approach takes a relational scalar quantity as a basic variable, conveniently incorporates matter, and facilitates the study of quantum causal structure of spacetime. The core of the approach is an application of Parker’s observation that under a Feynman sum, a gravitational phase can be traded into the Van Vleck-Morette determinant – a functional of the world function. A formula for quantum amplitudes of processes on quantum spacetime is obtained. Quantum gravity not only modifies the form of the matter propagators, but also break them into smaller pieces.

* ding.jia@uwaterloo.ca
I. INTRODUCTION

A goal of quantum gravity is to describe quantum matter processes over quantum spacetime. The superposition principle and the path integral approach suggest a sum over spacetime and matter configurations for the quantum amplitude of processes:

\[ A = \sum_g A_{QG}[g] \sum_\gamma A_M[\gamma, g], \]  

where \( A_{QG}[g] \) is the quantum gravity amplitude for the spacetime configuration \( g \), and \( A_M[\gamma, g] \) is the matter amplitude for the matter configuration \( \gamma \) on the spacetime configuration \( g \).

For example, in the ordinary functional integral QFT approach,

\[ \int Dg_{ab} e^{iS_{EH}[g_{ab}]} \int D\phi e^{iS_M[\phi, g_{ab}]}. \]  

The prescriptions are \( g \rightarrow g_{ab} \) the metric field, \( \gamma \rightarrow \phi \) the matter field, \( A_{QG}[g] \rightarrow e^{iS_{EH}[g_{ab}]} \) based on the Einstein-Hilbert action, and \( A_M[\gamma, g] \rightarrow e^{iS_M[\phi, g_{ab}]} \) based on the matter action. There are conceptual and technical difficulties associated with this approach in a perturbative treatment (e.g., background independence and non-renormalizability). Alternative approaches based on other variables include Quantum Regge Calculus [1–3], Euclidean Quantum Gravity [4], Spin-foam Models [5, 6], Dynamical Triangulation [7, 8], and Causal Set [9, 10].

We present a new "World Quantum Gravity" approach that takes the world function \( \sigma(x, y) \) [11] as a basic variable. The world function is one half the squared geodesic distance (appendix A), and there are several motivations for choosing it as a basic variable:

- \( \sigma \) is simple. In particular, it is invariant under changes of coordinates.
- \( \sigma \) is matter-friendly. The summations over gravity and matter can be unified in the summation of \( \sigma \).
- \( \sigma \) indicates causal structure. \( \sigma <,=,> 0 \) manifestly correspond to time-, light-, and space-separations, which facilitates the study of quantum causal structure [12–14].

That it is possible to change variable from \( g_{ab} \) to \( \sigma \) is suggested by the formula [11]

\[ g_{ab}(x) = -\lim_{y \to x} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b} \sigma(x, y) \]  

(3)
showing that \( g_{ab} \) is can be recovered from \( \sigma \). In principle, one can plug the above expression in (2) to change \( g_{ab} \) for \( \sigma \), but the result can be unmanageably complicated.

Our strategy to obtain a practical version of (1) in terms of \( \sigma \) is as follows. To capture the notion of locality, the variables \( \sigma \) are located to a skeleton that encodes topological information of spacetime (section II). \( A_M \) is obtained from the standard worldline formalism for matter QFT (section III). \( A_{QG} \) is prescribed in terms of the Van Vleck-Morette determinant based on a crucial observation by Parker (section IV). The sums \( \sum_g \) and \( \sum_\gamma \) are unified into a single one by using geometric diffeomorphisms to generate different matter configurations \( \gamma \) (section V). The pieces are put together to obtain (22) as a formula for quantum amplitudes of processes on quantum spacetime in section VI.

II. TOPOLOGICAL STRUCTURE

In contrast to \( g_{ab}(x) \) as a field variable, \( \sigma(x, y) \) is a relational variable defined for pairs of locations. As seen from the definition (A1) of \( \sigma \) as an integral, \( \sigma \) is also a local quantity in the sense that local contributions add up for global values. Whereas the natural arena for the pointwise defined \( g_{ab} \) is a manifold, that for the local and pairwise defined \( \sigma \) is a skeleton (a mathematical graph), with locations modelled as points, and neighboring points connected by edges. Specifying a spacetime configuration \( g \) amounts to assigning a world function value \( \sigma_k \) for each edges \( k \).

Using a skeleton is also a common strategy for evaluating path integrals for particles, fields, and gravity. In approaches to quantum gravity different skeleton structures have been used, e.g., simplicial triangulations, dual triangulations, hypercubes etc. It turns out convenient to use hypercubes for the present model, although a generalization to a broader class of skeletons is possible. For concreteness we also focus on \( 3 + 1 \) spacetime dimensions.

The use of a discrete skeleton is a form of “algorithmic discreteness” and in no way assumes spacetime to be fundamentally discrete. The situation is similar to using discrete time steps as part of an algorithm to approximate the path integral for a point particle [15]. This in no way assumes that time is fundamentally discrete. In both cases, the exact result is approached by fine-graining the discrete skeleton.

If the \( \sum_g \) in (1) contains a sum over spacetime topology, skeletons with different topolo-
gies are summed over. This sum can be separated out and performed in the end. In the rest of the paper we focus on $\sum_g$ on an individual skeleton.

III. MATTER AMPLITUDE

Consider a scalar field $\phi$ on curved 3+1-dimension spacetime with the coupling constant $\xi$ and governed by $(\Box + m^2 + \xi R)\phi(x) = 0$. In the Schwinger proper time representation [16] the Feynman propagator can be expressed as a path integral [17, 18]

$$G(x, y) = i \int_0^\infty \langle x, l | y, 0 \rangle e^{-im^2l} dl,$$

$$\langle x, l | y, 0 \rangle = \int d[x(l')] \exp\left\{ i \int_0^l dl' \left[ \frac{1}{4} g_{ab} \frac{dx^a}{dl'} \frac{dx^b}{dl'} - (\xi - \frac{1}{3}) R(l') \right] \right\},$$

$$\lim_{N \to \infty} \left[ \frac{1}{l (\frac{1}{4\pi i\epsilon})^2} \right]^{N+1} \int \prod_{n=1}^N d^4x_n [-g(x_n)]^{1/2} \exp \left\{ \sum_{m=0}^N i \int_{m\epsilon}^{(m+1)\epsilon} \left[ \frac{1}{4} g_{ab} \frac{dx^a}{dl'} \frac{dx^b}{dl'} - (\xi - \frac{1}{3}) R(l') \right] dl' \right\},$$

with fixed starting point $x_0 = x$ and ending point $x_{N+1} = y$. $m^2$ has an infinitesimal negative imaginary part in accordance with the Feynman prescription. The $l'$ integrals are evaluated along geodesics connecting $x_m$ and $x_{m+1}$. If there are multiple geodesics the shortest is used. $l$ is partitioned equally to the $N + 1$ segments so that $\epsilon := l/(N + 1)$. The technique of applying particle path integral to re-express propagators and deal with quantum field theory problems is systematically applied in the so-called worldline formalism [19–23], and has been applied to quantum gravity to incorporate matter, notably by Freidel and collaborators [24, 25]. In the worldline formalism, additional internal degrees of freedom can be introduced within points of the worldlines to incorporate higher spin fields [26]. In the current context of quantum gravity we focus on the external, spacetime degrees of freedom of matter, encoded already in (5).

Graphically, the above re-expression amounts to replacing the sum over Feynman diagrams by a sum over correlation diagrams. A correlation diagram is simply a Feynman diagram whose edges are replaced by curves localized in spacetime. A curve bounded by $x$ and $y$ has as the amplitude the integrand of (5). The sum over curves sharing the same vertices amounts to the path integral of (5), supplemented by the residue integral of (4), eliminating the dependence on the artificial parameter $l$. The correlation diagrams
represent ways matter correlations are mediated in spacetime, and provides a “basis” of matter configurations $\gamma$ in (1).

On a skeleton graph, a correlation diagram becomes a subgraph. Each edge $k \in \gamma$ is assigned the infinitesimal matter amplitude $[i(4\pi il_k)^2]^{-1}\exp\left\{i\frac{\sigma_k}{2l_k} - i(\xi - \frac{1}{3})R_kl_k\right\}$, an analogue (cf. (A1)) of the segment amplitude in (6). The $R_k$ term is eventually replaced by a $\sigma_k$ term, so only $\sigma_k$ and $l_k$ are free variables. $l_k$ is an unphysical parameter that the physical amplitude should not depend on. On a manifold, $l$ is eliminated by an integral in (4). The $l$-integral is from vertex to vertex, but does not extend beyond vertices. The fundamental reason is that the reparametrization invariance of the curves do not extend beyond vertices that connect three or more curves (shifting such vertices along the curve changes the physical configuration). The skeleton we start with may have nodes with only two edges. In this case we group the two edges into one and eliminate the node, so that all nodes connect three or more edges. Then an analogue $l$-integral should be performed for each edge, yielding

$$A_M[k \in \gamma, g] = \int \frac{dl_k}{(4\pi il_k)^2}\exp\left\{i\frac{\sigma_k}{2l_k} - i(\xi - \frac{1}{3})R_kl_k - im^2l_k\right\},$$

(7)

$$A_M[\gamma, g] = \prod_{k \in \gamma} A_M[k \in \gamma, g]V[\gamma],$$

(8)

where $V[\gamma]$ is the coupling constants factor associated to the vertices of the Feynman diagram that $\gamma$ is associated with.

IV. GRAVITY AMPLITUDE

The proposal for $A_{QG}[\sigma]$ in terms of the world function is inspired by Parker’s remarkable observation [17] in the context of QFT on curved spacetime. The following exchange is possible (appendix C) under a sum over paths for arbitrary constants $a, b, c$,

$$\Delta^a \exp\left\{i[\frac{\sigma}{2l} - \left(\frac{a}{3} - c\right)Rl]\right\} \xrightarrow{\text{path}} \Delta^b \exp\left\{i[\frac{\sigma}{2l} - \left(\frac{b}{3} - c\right)Rl]\right\},$$

(9)

where $\Delta$ is the Van Vleck-Morette determinant, a functional of $\sigma$ and its second order derivative (appendix B). In particular, for $a = 0$ and $b = 3c$, $\exp\left\{i(\frac{\sigma}{2l} + clR)\right\} \xrightarrow{\text{path}} \Delta^{3c} \exp\left\{i\frac{\sigma}{2l}\right\}$. This means the $R$ term in (6) can be traded into a term $\Delta^{3c}$.

The plan (implemented in section VI) is to use this correspondence to trade away a $g_{ab}$-dependent Einstein-Hilbert action term for a $\sigma$-dependent $\Delta^{3c}$ term in the skeleton.
setup. Initially an edge $j$ is formally assigned an infinitesimal gravitational amplitude
\[ \exp\{i\bar{\alpha}d^4x_j\sqrt{-g_j}R_j\}, \]
with $\bar{\alpha}$ as the coupling constant. In a Riemann normal coordinate system around $x$, $\sqrt{-g(y)} \approx \Delta^{-1}(x,y)$ [27]. This motivates the prescription $\sqrt{-g_j} \to \Delta_j^{-1}$ on each edge $j$. Let $s_j = |2\sigma_j|^{1/2}$ be the physical proper distance along each edge $j$, and define

\[ \alpha_j = \alpha \sum_{\{k,m,n\}} s ks ms n, \tag{10} \]

where the sum is over all sets of three edges sharing a same vertex with $j$ such that together with $j$, the four edges bound an elementary 4-cube. Assign to the edge $j$ the gravitational amplitude

\[ \exp\{i\alpha_j s_j \Delta_j^{-1}R_j\}. \tag{11} \]

$\alpha_j s_j = \alpha \sum_{\{k,m,n\}} s ks ms n s_j$ is thought of as corresponding to $\bar{\alpha}d^4x_j$, so that the amplitude corresponds to $\exp\{i\bar{\alpha}d^4x_j\sqrt{-g_j}R_j\}$. The sum enumerates over elementary 4-cubes containing $j$. Note that the elementary 4-cubes of the skeleton are counted multiple times in this sum, since two different three-element sets in this sum associated with $j$ may correspond to the same 4-cube. Moreover, the same elementary 4-cube appear in the amplitude of different edges $j$. Therefore $\alpha \neq \bar{\alpha}$ is used in (10) to compensate the multiplicities. Ultimately, the value of this new coupling constant is supposed to be determined by matching with experimental data.

The assignment (11) should be viewed as one prescription out of other possibilities. For instance, we may alternatively reduce the multiplicity of elementary 4-cubes by systematically reducing the sets included in the sum of (10). For concreteness we work with (11) in the following, keeping in mind that different prescriptions may be useful for different contexts.

V. UNIFYING MATTER AND SPACETIME SUMS

In pure quantum gravity, spacetime diffeomorphism is a complete redundancy for the spacetime sum. In quantum gravity with matter this is no longer so. Spacetime diffeomorphisms can generate physically distinct matter configurations, and be used to perform the matter sum [24].
Consider the situation illustrated in fig. 1. A matter configuration encoded in correlation diagram $\gamma$ is defined by coordinate values on some spacetime $g$. Under an active diffeomorphism to some geometry $g'$ physically equivalent to $g$, $\gamma$ will stay at the same coordinate locations but (generically) move to new physical locations. Alternatively, under a passive diffeomorphism to some other coordinate system $x'$, $\gamma$ will move to different coordinate locations and (generically) move to new physical locations. Note that a diffeomorphism does not always generate a distinct matter configuration, since it may simply reparametrize $\gamma$.

In a Feynman sum we want to sum over physically distinct configurations. Namely, we want to perform $\sum_{[g]} \sum_{[\gamma]}$, where $[\cdot]$ denotes physical equivalence class of configurations. The above observation suggests that starting with a fixed $\gamma \in \Gamma$,

$$\sum_{g} f[\gamma, g] = \sum_{[g]} \sum_{g \in [g]} f[\gamma, g] = \sum_{[g]} \sum_{[\gamma] \in \Gamma} \sum_{\gamma \in [\gamma]} f[\gamma, g].$$

(12)

The first equality just uses the definition of $[g]$ as equivalence classes. The second equation assumes that all $[\gamma] \in \Gamma$ can be reached by summing over $g \in [g]$.

Let $N[\gamma, g]$ count cardinality of the redundancy in $\sum_{\gamma \in [\gamma]}$. Then using $f/N$ in the above formula, we get

$$\sum_{g} \frac{f[\gamma, g]}{N[\gamma, g]} = \sum_{[g]} \sum_{\gamma \in \Gamma} f[\gamma, g].$$

(13)

The wanted double sum over spacetime and matter configurations is thus unified into a single sum over spacetimes. The sum over topological classes can be performed by picking a representative $\gamma_{\Gamma}$ from each $\Gamma$:

$$\sum_{[g]} \sum_{[\gamma]} f[\gamma, g] = \sum_{\Gamma} \sum_{g} \frac{f[\gamma_{\Gamma}, g]}{N[\gamma_{\Gamma}, g]}.$$  

(14)
VI. MATTER ON QUANTUM SPACETIME

By the previous formula,

\[ A = \sum_{[\gamma]} \sum_{[\sigma]} A_{QG}[g]A_M[\gamma, g] = \sum_{\Gamma} \sum_g A_{QG}[g] A_M[\gamma_T, g] \frac{N[\gamma_T, g]}{N[\gamma_T, g]} \tag{15} \]

Using the prescriptions (8) for \(A_M\) and (11) for \(A_{QG}\) (for simplicity suppress \(\sum_{\Gamma}, N[\gamma_T, g], V[\gamma_T]\), and \(\exp^{-imz_k}\) for the moment as they carry through the equations), we get

\[
\sum_{[\sigma]} \prod_j \exp\left\{ i\sigma_j \Delta^{-1} R_j \right\} \prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \exp\left\{ i\frac{\sigma_k}{2l_k} - i(\xi - \frac{1}{3})R_k l_k \right\} = \sum_{[\sigma]} \prod_j \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} \prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} \exp\left\{ i\sigma_j \Delta^{-1} R_j \right\} \tag{16} \]

\[
\prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} \sum_{[\sigma]} \prod_{h \in \gamma_T} \exp\left\{ i\frac{\sigma_h}{2s_h} + i\alpha_h s_h \Delta^{-1} R_h \right\} \tag{17} \]

\[
\prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \sum_{[\sigma]} \prod_{h \in \gamma_T} \Delta^{3\alpha_h} \Delta^{-1} \exp\left\{ i\frac{\sigma_h}{2s_h} \right\} \tag{18} \]

\[
\prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \Delta^{3\alpha_h} \Delta^{-1} \exp\left\{ i\frac{\sigma_h}{2s_h} \right\} \tag{19} \]

\[
\sum_{[\sigma]} \prod_{h \in \gamma_T} \Delta^{3\alpha_h} \Delta^{-1} \prod_{k \in \gamma_T} \int \frac{dl_k}{(4\pi i l_k)^2} \Delta^{3\alpha_h} \Delta^{-1} \exp\left\{ i\frac{\sigma_h}{2s_h} \right\} \tag{20} \]

where

\[
C_k = \left( \frac{1}{s_k} + \frac{1}{l_k} \right) \alpha_k s_k \Delta^{-1} - (\xi - \frac{1}{3})l_k \right]. \tag{21} \]

The formal sum \(\sum_{[\sigma]}\) is over all the spacetime configurations in terms of \(\sigma\) and its derivative on the edges. In the first step we introduced a factor \(1 = \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} \exp\left\{ i\frac{\sigma_j}{2s_j} \right\} \). In the second step we distributed a factor under \(j\) to \(h\) and \(k\). We also used the fact that for \(\sigma, \sigma' \in [\sigma]\), \(\prod_j \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} = \prod_j \exp\left\{ -i\frac{\sigma_j}{2s_j} \right\} \) to push \(\sum_{\sigma \in [\sigma]} \text{in} \sum_{[\sigma]} = \sum_{[\sigma]} \sum_{\sigma \in [\sigma]}\) rightwards. In the third step we applied (9). The condition that there is a path sum is fulfilled by realizing that the paths in a Feynman sum has a fractal structure, so each
FIG. 2. Traditional Feynman diagram vs. correlation diagram from WQG

segment should be viewed as arising from a finer path sum [18] [28]. In the fourth step we pulled $\sum_{\sigma \in [\sigma]}$ back and distributed the $\exp\left\{-i \sigma_j^2 s_j\right\}$ factors to $h$ and $k$.

Inserting the suppressed factors, we obtain the final formula for the amplitude

$$
\sum_{\Gamma} \sum_{\sigma} \frac{V[\gamma_{\Gamma}]}{N[\gamma_{\Gamma}, \sigma]} \prod_{h \in \gamma_{\Gamma}} \Delta_h^{3\alpha_h} \Delta_h^{-1} \prod_{k \in \gamma_{\Gamma}} \int \frac{dl_k}{(4\pi i l_k)^2} \Delta_k^{3C_k} \exp \left\{ i \sigma_k \frac{2 l_k}{l_k} - i \alpha^2 l_k \right\}
$$

(22)

with $C_k$ defined in (21) and $\alpha_h$ in (10). $N[\gamma_{\Gamma}, \sigma]$ is a generalization of the Feynman diagrams symmetry factor. It counts the number of graph relabellings that preserve the matter subgraph $\gamma$ and the physical configuration ($\sigma$ on the edges and $V[\gamma_{\Gamma}]$ on the vertices). On a particular skeleton, Equation (22) gives an approximation. The exact result is to be approached by fine-graining the skeletons.

VII. CONCLUSION

Using the world function as a basic variable for gravity and the worldline formalism for matter physics, we obtained formula (22) for the amplitude of quantum physics on quantum spacetime.

The picture (fig. 2) emerging from (22) is quite interesting. The vacuum quantum gravity amplitude is encoded in the dashed edges $H_h$, which brings the Van Vleck-Morette determinant $\Delta$ to the center stage in this approach to quantum gravity. A pure matter Feynman propagator is broken into multiple propagators $D_k$, receiving modification from gravity through the $\Delta_k^{3C_k}$ factor and the sum $\sum_{\sigma}$. That quantum gravitational path integral modifies matter propagators has long been suggested and investigated [29]. Here the modification is obtained from a non-perturbative, background independent treatment of quantum gravity. This modification is expected to regularize the UV divergence of ordinary
QFT [30, 31], but this needs to be checked. We leave it for future work to explore the consequences of the new picture.

ACKNOWLEDGEMENT

I am very grateful to Lucien Hardy, Achim Kempf, and Laurent Freidel for valuable discussions.

Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade. This publication was made possible through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

Appendix A: World function

We introduce the world function and collect some relevant formulas in this section. See [11, 27] for a more comprehensive treatment of the formalism and formulas quoted below.

Consider two points $x, y$ on a spacetime manifold such that $y$ lies in the convex normal neighborhood of $x$. Parametrize the unique geodesic $z^a(l)$ connecting $x$ to $y$ by an affine parameter $l$. Define $\sigma(x, y)$ by

$$\sigma(x, y) = \frac{1}{2}(l_y - l_x) \int_{l_x}^{l_y} g_{ab}(z) \frac{dz^a}{dl} \frac{dz^b}{dl} dl.$$  \hspace{1cm} (A1)

This is nothing but one half the squared geodesic distance. For example, on flat spacetime (A1) reduces to the familiar expression $\sigma(x, y) = \frac{1}{2} \eta_{ab}(y - x)^a(y - x)^b$. On a general curved spacetime, first let us introduce

$$t^a = \frac{dz^a}{dl}. \hspace{1cm} (A2)$$

Then $\frac{D}{dl} t^a = 0$, since $z^a(l)$ is a geodesic. Hence $c = g_{ab}(l) \frac{dz^a}{dl} \frac{dz^b}{dl} = g_{ab} t^a t^b$ is constant along $z^a(l)$, and $\sigma(x, y) = \frac{c}{2}(l_y - l_x)^2$. Picking $l$ to be the proper distance $s$ implies $c = \pm 1$ for spacelike and timelike separations, whence $\sigma(x, y) = \pm s^2/2$. In the null case $\sigma(x, y) = 0 =$
Therefore the formula
\[ \sigma(x,y) = \pm s^2/2 \] (A3)

can be used for all three cases. This equation expresses \( \sigma \) in terms of \( s \). Conversely, the proper distance \( s = |2\sigma|^{1/2} \) can be expressed in terms of \( \sigma \).

Synge calls \( \sigma(x,y) \) the world function, because “it determines the curved world of spacetime” \[11\]. Indeed one can show that \( \sigma \) knows all about \( g_{ab} \):
\[ g_{ab}(x) = -\lim_{y \to x} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b} \sigma(x,y). \] (A4)

Higher order covariant derivatives of \( \sigma \) are useful quantities. We use subscripts to express derivatives, with actions on \( x \) and \( y \) distinguished by a prime. For instance,
\[ \nabla_{y^b} \nabla_{x^d} \nabla_{x^c} \sigma(x,y) = \sigma_{a'b'c'd'}(x,y). \] In this notation (A4) becomes
\[ g_{ab}(x) = -\lim_{y \to x} \sigma_{ab}(x,y). \] A similar equation is
\[ g_{ab}(x) = \lim_{y \to x} \sigma_{ab}(x,y). \] (A5)

Indices can be raised and lowered by the metric. For instance, \( \sigma^{a}_{b}(x,y) = \nabla_{x^b} [g^{ac}(x) \sigma_c(x,y)] \).

Differentiating (A1) leads to
\[ \sigma_a(x,y) = -(\lambda_y - \lambda_x)g_{ab}(x)t^b(x). \] (A6)

Up to \(- (\lambda_y - \lambda_x)\), \( \sigma_a \) agrees with the tangent vector along the geodesic at \( x \). This implies
\[ g_{ab} \sigma^a \sigma^b = (\lambda_y - \lambda_x)^2 g_{ab} t^a t^b = 2\sigma. \] (A7)

The norm of \( \sigma^a \) equals \( |2\sigma|^{1/2} = s \), the proper distance.

**Appendix B: Van Vleck-Morette determinant**

The Van Vleck-Morette arose early on in studying the classical limit of quantum mechanics \[32\] and in studying the path integral transition amplitudes \[33\]. Later on it found applications to several other subjects such as the heat kernel expansion, geometrical optics, Riemannian geometry etc. \[34\].
Consider a Lorentzian $d + 1$-dimensional spacetime with the metric $g_{ab}$ and world function $\sigma(x,y)$. The Van Vleck-Morette determinant is usually defined as

$$\Delta(x, y) = (-1)^d \frac{\det|\sigma_{a\nu}(x, y)|}{\sqrt{-g(x)} \sqrt{-g(y)}},$$  \hspace{1cm} (B1)

which is manifestly symmetric in $x$ and $y$. A near coincidence expansion of $\Delta$ establishes the connection between $\Delta$ and the Ricci tensor:

$$\Delta(x, y) = 1 + \frac{1}{6} R_{ab}(x) \sigma_a(x, y) \sigma_b(x, y) + O(\epsilon^3),$$  \hspace{1cm} (B2)

where $\epsilon$ measures the size of of a typical component of $\sigma^a$. A proof can be found in [27].

$\Delta(x, y)$ can be re-expressed in terms of the world function and its second order derivative without reference to the metric. When $x$ and $y$ are timelike or spacelike with proper distance $s = |2\sigma(x,y)|^{1/2}$, $\Delta(x,y)$ obeys the differential equation

$$\frac{d\Delta(s)}{ds} = (\frac{d}{s} - \theta)\Delta(s),$$  \hspace{1cm} (B3)

with the boundary condition $\Delta(0) = 1$ [27, 34]. This differential equation can be derived starting with differentiating (A7) twice (Section 7.2 of [27]). Here $\theta(x) = \nabla_a u^a$ is the expansion of the geodesic congruence originating from $y$ with the normalized tangent vector $u^a = \sigma^a/\|\sigma^a\| = \sigma^a/s$. One can check that

$$\theta(x) = \frac{\sigma_a(x, y)}{s} - \frac{1}{s}.$$  \hspace{1cm} (B4)

The solution to the differential equation can be obtained by integration

$$\Delta(s) = C s^d \exp\left\{-\int \theta ds'\right\},$$  \hspace{1cm} (B5)

where the integral is along the geodesic from $y$ to $x$, and $C$ is a constant. As both $s$ and $\theta$ can be written in terms of $\sigma$, (B5) expresses $\Delta$ as a functional of $\sigma$ (and its second order derivative).

The lightlike case is complicated by the fact that the proper distance vanishes and is no longer an affine parameter. Fortunately in the integral over spacetime geometries $\sigma = 0$ has measure zero, so we do not need to delve into the lightlike case (although there is an expression similar to (B5) in this case [34]).

Equation (B5) offers the intuition that the Van Vleck-Morette determinant measures the curvature of spacetime in terms of the amount of focusing/defocusing of geodesic sprays.
Without any curvature, the transverse density of geodesics would fall by \(s^{-d}\) as they reach out a proper distance \(s\). With curvature, the density falls instead by \(\exp\{ - \int \theta ds' \}\). The Van Vleck-Morette determinant quantifies the ratio.

The Van Vleck-Morette determinant in the form of (B5) interestingly resembles the Wilson line expression \(\mathcal{P} \exp\{ i \int A_a dx^a \}\). The connection to gauge theories is worth exploring.

**Appendix C: Parker’s magic**

The path integral representation of the Feynman propagator of equations (4) to (6) is reproduced here.

\[
G(x, y) = i \int_0^\infty \langle x, l | y, 0 \rangle e^{-im^2 l} dl, \quad (C1)
\]

\[
\langle x, l | y, 0 \rangle = \int d[l(x')] \exp\left\{ i \int_0^l dl' \left[ \frac{1}{4} g_{ab} \frac{dx^a}{dl'} \frac{dx^b}{dl'} - (\xi - \frac{1}{3}) R(l') \right] \right\} \quad (C2)
\]

\[
= \lim_{N \to \infty} \left[ i \frac{1}{4 \pi i \epsilon} \right]^2 \int_0^N d^4 x_n [-g(x_n)]^{1/2} \exp \left\{ \sum_{m=0}^N i \int_{m\epsilon}^{(m+1)\epsilon} \left[ \frac{1}{4} g_{ab} \frac{dx^a}{dl'} \frac{dx^b}{dl'} - (\xi - \frac{1}{3}) R(l') \right] dl' \right\}, \quad (C3)
\]

Parker [17] found that using

\[
\langle x, l | y, 0 \rangle = \lim_{N \to \infty} \left[ i \frac{1}{4 \pi i \epsilon} \right]^2 \int_0^N d^4 x_n [-g(x_n)]^{1/2} \exp \left\{ \sum_{m=0}^N i \int_{m\epsilon}^{(m+1)\epsilon} \left[ \frac{1}{4} g_{ab} \frac{dx^a}{dl'} \frac{dx^b}{dl'} - [\xi - \frac{1}{3} (1 - p)] R(l') + p \ln \Delta(x_m, x_{m+1}) \right] dl' \right\}, \quad (C4)
\]

instead does not affect \(G(x, y)\). The magic is that for an *arbitrary* constant \(p\), a multiplication by \(\Delta^p(x_m, x_{m+1})\) can be compensated by \(\exp\left\{ -i \int_{m\epsilon}^{(m+1)\epsilon} (R(l') p/3) dl' \right\}\). Equation (9) is an another way to express the arbitrariness of \(p\), with \(p = a\) on one side and \(p = b\) on the other. This magical freedom in \(p\) allows us to trade between an \(R\) term and a \(\Delta\) term.

A proof of this freedom by carrying out the path integral is given in Parker’s original paper [17]. Later Bekenstein and Parker offered a short explanation (Appendix A of [18]), which we review here. Consider carrying out all the integrals in (C4) except the last

\[
\langle x, l | y, 0 \rangle = \int \langle x, l | x_N, l - \epsilon \rangle \langle x_N, l - \epsilon | y, 0 \rangle \sqrt{-g} d^4 x_N, \quad (C5)
\]
\[ \langle x, l| x_N, l - \epsilon \rangle = \Delta^p(x, x_N) \exp \left\{ \frac{i \sigma(x, x_N)}{2 \epsilon} - i \epsilon [\xi - \frac{1}{3} (1 - p) ] R \right\}, \tag{C6} \]

where (A1) is used to introduce \( \sigma \). From (B2),

\[ \Delta^p = 1 + \frac{1}{6} p R_{ab} \sigma_a \sigma_b + \mathcal{O}(\epsilon^3). \tag{C7} \]

For simplicity denote the exponential in (C6) by \( E \). Then

\[ \langle x, l| x_N, l - \epsilon \rangle = \{ 1 - \frac{2}{3} p R_{ab} \epsilon^2 [\nabla_a \nabla_b - \frac{i \sigma_{ab}}{2 \epsilon} - \frac{1}{2} (\xi - \frac{1}{3} + \frac{1}{3} p) (\sigma_a R_{,b} + \sigma_b R_{,a})] \} E + \mathcal{O}(\epsilon^3). \tag{C8} \]

If \( \langle x, l| x_N, l - \epsilon \rangle \) was alone, \( \nabla_a \nabla_b \) acts on \( E \) to create a dominating \( \mathcal{O}(\epsilon^{-2}) \) term in the square bracket. Yet crucially, \( \langle x, l| x_N, l - \epsilon \rangle \) is composed with \( \langle x_N, l - \epsilon| y, 0 \rangle \) in (C5). \( E \) multiplied by \( \langle x_N, l - \epsilon| y, 0 \rangle \) generates the transition amplitude from \( y \) to \( x \) through \( x_N \). \( \nabla_a \nabla_b \) acts on this amplitude to create a \( \mathcal{O}(\epsilon^0) \) term, dominated by the \( \epsilon^{-1} \) term in the square bracket. Hence as \( \epsilon \to 0 \),

\[ \langle x, l| y, 0 \rangle = \int \left[ 1 + \frac{i}{3} \epsilon p R_{ab} \sigma_{ab} + \mathcal{O}(\epsilon^2) \right] E \langle x_N, l - \epsilon| y, 0 \rangle \sqrt{-gd^4 x_N} \tag{C9} \]

As \( N \to \infty, x \to x_N \), and by (A5) \( \sigma_{ab} \to g_{ab} \), so \( R_{ab} \sigma_{ab} \to R \). Thus

\[ \langle x, l| y, 0 \rangle = \int \exp \left[ \frac{i \sigma(x, x_N)}{2 \epsilon} - i \epsilon [\xi - \frac{1}{3}] R \right] \langle x_N, l - \epsilon| y, 0 \rangle \sqrt{-gd^4 x_N}. \tag{C10} \]

The \( p \)-dependence has dropped out, implying that \( p \) is arbitrary. The reasoning can be applied to other segments, the whole path integral is independent of \( p \).

Appendix D: A summation scheme

The focus of this work is on deriving (22), but not evaluating it. The latter task is rich enough to deserve an independent treatment. Nevertheless, we discuss a scheme for evaluating the expression, as this outlines a direction for future work.

\( \Delta \) as given in (B5) depends on both \( \sigma \) itself (through \( s = |2\sigma|^{1/2} \)) and its second order derivative \( \sigma_a^a \) (through \( \theta = [\sigma_a^a - 1] / s \)). One strategy for evaluating \( \sum \sigma \) is to introduce an auxiliary variable \( \rho \) independent of \( \sigma \) to replace the second order derivative, and then integrate over both \( \sigma \) and \( \rho \) to implement the sum over spacetime configurations. This is
analogous to the strategy of the first order formulation of general relativity [35] that eliminates higher than first order derivatives of the metric $g_{ab}$ by treating the affine connection as an independent variable.

We want to use the Raychaudhuri equation to introduce the auxiliary variable and obtain an alternative expression for $\theta(s)$ to be used in (B5). Recall the Raychaudhuri equation for timelike [36] and spacelike [37] geodesic congruences (the lightlike case is of measure zero in the integral and hence unimportant):

$$\frac{d\theta}{ds} = -\frac{1}{3} \theta^2 - \hat{\sigma}^2 + \omega^2 - R_{ab}u^a u^b. \quad (D1)$$

Here $s$ is an affine parameter along the geodesics, $\theta$ is the expansion, $\hat{\sigma}^2$ is the squared shear, $\omega^2$ is the squared rotation, $R_{ab}$ is the Ricci tensor, and $u^a$ is the unit tangent vector along the geodesics.

As a first approximation, assume that $\hat{\sigma}^2 = \omega^2 = 0$, and that $R_{ab}u^a u^b = \rho$ is constant on each edge. This approximation is similar in spirit to the standard prescription of piecewise linear (vanishing acceleration and constant velocity) trajectories in evaluating the path integral for a point particle [15]. Then the equation becomes $\frac{d\theta}{ds} = -\frac{1}{3} \theta^2 - \rho$. From the coincidence limit of $\sigma_{ab}(x, y)$, one can derive the boundary condition $\theta(0) = \infty$ [27]. The solution to the differential equation is then

$$\theta(s) = \sqrt{3\rho} \cot\left(s\sqrt{\frac{\rho}{3}}\right). \quad (D2)$$

One can check that in the flat spacetime limit $\rho \rightarrow 0$, the familiar expression $\theta(s) \rightarrow \frac{3}{s}$ is recovered.

Plugging (D2) in (B5) and using the boundary condition $\Delta(0) = 1$, we obtain

$$\Delta(s, \rho) = \left[s \sqrt{\frac{\rho}{3}} \csc\left(s \sqrt{\frac{\rho}{3}}\right)\right]^3. \quad (D3)$$

When this expression is used, the formula (22) no longer depends on the second order derivative $\sigma^a_a$. All the spacetime degrees of freedom are now encoded in the values of $\sigma_j$ and $\rho_j$ on the edge $j$ for all edges. The sum $\sum_{\sigma}$ in (22) is implemented as $\int dp_j \int d\sigma_j$ on each edge $j$.

The remaining ambiguity of this evaluation scheme is the integration range. At this stage it is still unclear what the integration range for $\rho = R_{ab} u^a u^b$ should be. This is a question regarding what spacetime configurations to include in the summation, and appears to
be related to the question of fundamental energy conditions on spacetime configurations. We leave it to future work to address this ambiguity, and focus on the $\sigma$ integral below.

We illustrate the integral on a single edge $h$ in pure quantum gravity without matter. We single out the terms in (22) depending on $\sigma_h$ and $\rho_h$ as

$$\int d\rho_h \int d\sigma_h \Delta_h^{3\alpha_h \Delta_h^{-1}} B^{s_h}. \quad \text{(D4)}$$

The first factor comes from the amplitude $H_h$ on the edge $h$. The second factor in which $B$ is a constant comes from $G_j = \Delta_j^{3\alpha_j \Delta_j^{-1}}$ for the edges $j$ sharing vertices with $h$. $\Delta_j$ is independent of $\sigma_h$ and $\rho_h$, but $\alpha_j$ defined in (10) depends on $s_h$, and hence $\sigma_h$. Treating only $\sigma_h$ and $\rho_h$ as variables, $H_j$ is of the form $B_j^{s_h}$ for some constant $B_j$. The product of such $G_j$’s for different $j$ can be written as $B^{s_h}$ for some constant $B$ as in (D4).

We want to conduct the $\sigma_h$ integral by changing variable from $\sigma = \pm s^2/2$ to $s$. Note from (D2) that as $s \to \pi \sqrt{3/\rho}$, $\theta \to -\infty$. At this caustic point, $\sigma$ stops being well-defined as there are multiple geodesics connecting the two points. This suggests the introduction of a new edge at this point. Practically this amounts to integrating $s$ only up to

$$a = \pi (3/\rho)^{1/2} \quad \text{(D5)}$$

(equivalently, integrating $\sigma$ from $-a^2/2$ to $a^2/2$). Suppressing the subscript $h$ for simplicity, we have

$$\int_{-a^2/2}^{a^2/2} d\sigma \Delta^{3\alpha \Delta^{-1}} B^s = \int_0^0 d\sigma \Delta^{3\alpha \Delta^{-1}} B^s + \int_{-a^2/2}^{a^2/2} d\sigma \Delta^{3\alpha \Delta^{-1}} B^s \quad \text{(D6)}$$

$$= \int_0^a ds (-s) \Delta^{3\alpha \Delta^{-1}} B^s + \int_0^a ds s \Delta^{3\alpha \Delta^{-1}} B^s \quad \text{(D7)}$$

$$= 2 \int_0^a ds s \Delta^{3\alpha \Delta^{-1}} B^s, \quad \text{(D8)}$$

with $\Delta(s, \rho)$ given in (D3).

Here $\alpha$ and $B$ are constants with respect to $s = s_h$, but $\alpha$ depends on $s_j$, and $B$ depends on $s_j$ and $\rho_j$ for other edges $j$. In addition, $\Delta$ depends on $\rho_h$. Hence the integral is a function of $\alpha = \alpha_h, B = B_h$, and $\rho = \rho_h$, which implicitly depend on $s_j$ and $\rho_j$ for the adjacent edges. Figure 3 shows some plots of (D8) with varying $\alpha, \rho$, and $B$, obtained from numerical integration. For larger $B$, as $\rho$ gets smaller the integral grows fast. This is because $s$ is integrated from 0 to $\pi \sqrt{3/\rho}$, which becomes unbounded in the flat spacetime limit $\rho \to 0$. This suggests that we introduce an IR regularization in further studies. To
FIG. 3. Numerical integration of (D8) for different $\alpha, \rho$, and $B$.

FIG. 4. Numerical integration of (D8) with bound ($\rho_0 = 2$) on $s$ for different $\alpha, \rho$, and $B$.

illustrate, consider imposing an upper bound on the $s$-integral for small $\rho$. This can be achieved, for instance, by integrating $s$ from 0 to $\min(\pi \sqrt{3/\rho}, \pi \sqrt{3/\rho_0})$ for some constant $\rho_0$. Then for $\rho < \rho_0$, the bound $\pi \sqrt{3/\rho_0}$ is in effect. An example with $\rho_0 = 2$ is shown in fig. 4. In contrast to fig. 3, the values of the integral for smaller $\rho$ are now bounded. Incidentally, one can go one step further to integrate $\rho$ from 0 to $\infty$ numerically, and the results are shown in fig. 5. These are not meant as final values for (D4), since as mentioned
FIG. 5. Numerical integration of (D8) with bound ($\rho_0 = 2$) on $s$ followed by $\int_0^\infty d\rho$ for different $\alpha$ and $B$.

Further studies are needed to determine the integration range for $\rho$.

In summary, we introduced an auxiliary variable $\rho$ taking the values $\rho_j$ at each edge $j$ to eliminate the second derivative $\sigma^a_a$ from the amplitude (22). The formal sum over spacetime configurations $\sum_\sigma$ can then be specified as a double integral $\int d\rho_j \int d\sigma_j$ on all the edges. As illustrated above, the $\sigma$-integral can already be carried out. However, further work is needed to fully evaluate the double integral. In particular, we need to study:

- The integration range of $\rho = R_{ab} u^a u^b$. This appears to be related to the question of what energy conditions are imposed on allowed spacetimes.
- The applicability of the approximation of $\sigma^2 = \omega^2 = 0$. Does this approximation need to be relaxed on certain ranges of the integrals?
- The flat spacetime limit $\rho \to 0$. Here the $s$-integration range of 0 to $\pi \sqrt{3/\rho}$ becomes unbounded. Is directly imposing an upper-bound on the $s$-integral a good regularization?

Appendix E: Related approaches

We point out some connections to (and differences from) some related approaches of quantum gravity.

Regge Calculus. The world function is a form of invariant spacetime distance. Regge Calculus [1] can also be viewed as taking an invariant distance as a basic variable for spacetime configurations. Yet the way to capture spacetime curvature differs. Regge Calculus takes as a starting point the geometric setup of a piecewise flat triangulation. Cur-
vature is concentrated at the hinges – places where simplices meet. The WQG model in contrast does not assume a piecewise flat geometric setup. Although the Einstein-Hilbert action term involving curvature is assigned to the skeleton edges, this is only a way of book-keeping. The curvature should not be thought of as concentrated within the edges. Rather, the curvature should be thought of as from the region close to the edge, as an average over elementary 4-cubes containing the edge is taken in assigning the gravitational amplitude (11).

Quantum Regge Calculus [3] applies the path integral to the classical Regge spacetime configurations. This follows from first working at the classical level by assigning a gravitational action corresponding to the Einstein-Hilbert action to each classical configuration. The WQG model differs by working directly at the quantum level in the presence of a Feynman sum to trade the gravitational phase into a functional of $\sigma$, as (9) is possible only under a sum over paths. Here there is no expression in terms of the invariant distance for the Einstein-Hilbert action at the classical level in the absence of the Feynman path sum to start with. In this restricted sense, the WQG model does not arise from “quantizing” a classical theory of gravity.

**Spin-foam and Dynamical Triangulation.** Spin-foam [5, 6] and Dynamical Triangulation [7, 8] models are also path integral models of quantum gravity. The use of skeleton structures for evaluating the path integral is shared by these models with the Quantum Regge Calculus models and the WQG model.

A major difference among the approaches is the choice of variables to capture gravitational degrees of freedom. Spin-foam models use elements of some gauge group and Lie algebra such as $SU(2)$ and $su(2)$ as the basic variables. At the end of appendix B we noted that the resemblance between the Van Vleck-Morette determinant and the Wilson line. It would be interesting to study if there is a gauge theory aspect of the WQG approach.

Dynamical Triangulation models use the number of simplices of spacetime triangulation as the basic variable. In the Lorentzian signature models of Causal Dynamical Triangulation [8], a global time foliation is implemented. The WQG model presented does not assume a global time foliation, although a model with this structure is conceivable within the WQG approach, should this structure turn out useful within the approach.

**Causal Set.** In the Causal Set approach [9, 10], spacetime causal structure is used as a basic variable to capture spacetime configurations. As such the spacetime causal structure
is manifest in this approach. This is also the case in the WQG approach, with the sign of $\sigma$ telling the spacetime causal relations. As noted in the introduction section, one of the major motivations for the WQG approach is to use this feature to study the quantum causal structures of quantum spacetimes.

The Causal Set approach assumes that spacetime is fundamental discrete. This assumption is needed to capture the conformal factor of spacetime configurations through the number of points in a region. In the WQG approach the information associated with the conformal factor is to be found in the world function as an invariant distance. Fundamental discreteness of spacetime is not assumed, although the WQG model can be applied to a fundamentally discrete spacetime that provides a fundamental skeleton.

[1] T. Regge, Il Nuovo Cimento 19, 558 (1961).
[2] G. Ponzano and T. E. Regge, in Spectroscopic and Group Theoretical Methods in Physics, edited by F. Bloch (North-Holland Publishing Company, Amsterdam, 1968) pp. 1–58.
[3] H. W. Hamber, Quantum Gravitation: the Feynman Path Integral Approach (Springer, Berlin, 2009).
[4] S. W. Hawking, in General Relativity - An Einstein Centenary Survey, edited by W. Hawking, S. W.; Israel (Cambridge University Press, Cambridge, 1979) pp. 746–789.
[5] J. C. Baez, in Knots and Quantum Gravity, edited by J. C. Baez (Clarendon Press, Oxford, UK, 1994) p. 133.
[6] A. Perez, Living Reviews in Relativity 16, 3 (2013).
[7] D. Weingarten, Nuclear Physics B 210, 229 (1982).
[8] J. Ambjørn, J. Jurkiewicz, and R. Loll, Nuclear Physics B 610, 347 (2001).
[9] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Physical Review Letters 59, 521 (1987).
[10] D. M. T. Benincasa and F. Dowker, Physical Review Letters 104, 181301 (2010).
[11] J. L. Synge, Relativity: the General Theory (North-Holland Publishing Company, Amsterdam, 1971).
[12] L. Hardy, arXiv:gr-qc/0509120 ; Journal of Physics A: Mathematical and Theoretical 40, 3081 (2007).
[13] G. Chiribella, G. M. D’Ariano, P. Perinotti, and B. Valiron, Physical Review A 88, 22318
(2013); arXiv:0912.0195.

[14] O. Oreshkov, F. Costa, and Č. Brukner, Nature Communications 3, 1092 (2012).

[15] R. P. R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).

[16] J. Schwinger, Physical Review 82, 664 (1951).

[17] L. Parker, Physical Review D 19, 438 (1979).

[18] J. D. Bekenstein and L. Parker, Physical Review D 23, 2850 (1981).

[19] R. P. Feynman, Physical Review 80, 440 (1950); Physical Review 84, 108 (1951).

[20] A. M. Polyakov, Gauge Fields and Strings (Harwood Academic Publishers, Chur, Switzerland, 1987).

[21] Z. Bern and D. A. Kosower, Physical Review Letters 66, 1669 (1991).

[22] M. J. Strassler, Nuclear Physics B 385, 145 (1992).

[23] M. G. Schmidt and C. Schubert, arXiv:hep-ph/9412358.

[24] L. Freidel and D. Louapre, Classical and Quantum Gravity 21, 5685 (2004); arXiv:gr-qc/0410141; L. Freidel and E. R. Livine, Classical and Quantum Gravity 23, 2021 (2006).

[25] A. Baratin and L. Freidel, Classical and Quantum Gravity 24, 1993 (2007); Classical and Quantum Gravity 24, 2027 (2007).

[26] O. Corradini and C. Schubert, arXiv:1512.08694.

[27] E. Poisson, A. Pound, and I. Vega, Living Reviews in Relativity 14, 7 (2011).

[28] One can also view the edge factor as a 1-segment evaluation of the path integral (a special case of the general $N+1$-segment evaluation in (6) before taking the limit). Due to the fractal structure of the Feynman paths, this evaluation is actually fairly accurate when the variation in the curvature is small [18].

[29] S. Deser, Reviews of Modern Physics 29, 417 (1957).

[30] H. C. Ohanian, Physical Review D 55, 5140 (1997).

[31] H. C. Ohanian, Physical Review D 60, 104051 (1999).

[32] J. H. Van Vleck, Proceedings of the National Academy of Sciences of the United States of America 14, 178 (1928).

[33] C. Moretto, Physical Review 81, 848 (1951).

[34] M. Visser, Physical Review D 47, 2395 (1993).

[35] A. Palatini, Rendiconti del Circolo Matematico di Palermo 43, 203 (1919).
[36] E. Poisson, *A Relativist’s Toolkit* (Cambridge University Press, Cambridge, 2004).

[37] G. Abreu and M. Visser, *Physical Review D* 83, 104016 (2011).