Secret Sharing Schemes Based on Min-Entropies

Mitsugu Iwamoto
Center for Frontier Science and Engineering,
The University of Electro-Communications, Japan
mitsugu@uec.ac.jp

Junji Shikata
Graduate School of Environment and Information Sciences,
Yokohama National University, Japan
shikata@ynu.ac.jp

Abstract—Fundamental results on secret sharing schemes (SSSs) are discussed in the setting where security and share size are measured by (conditional) min-entropies. We first formalize a unified framework of SSSs based on (conditional) Rényi entropies, which includes SSSs based on Shannon and min entropies etc. as special cases. By deriving the lower bound of share sizes in terms of Rényi entropies based on the technique introduced by Iwamoto–Shikata, we obtain the lower bounds of share sizes measured by min entropies as well as by Shannon entropies in a unified manner.

As the main contributions of this paper, we show two existential results of non-perfect SSSs based on min-entropies under several important settings. We first show that there exists a non-perfect SSS for arbitrary binary secret information and arbitrary monotone access structure. In addition, for every integers \( k \) and \( n \) \((k \leq n)\), we prove that the ideal non-perfect \((k,n)\)-threshold scheme exists even if the distribution of the secret is not uniformly distributed.

I. INTRODUCTION

A secret sharing scheme (SSS) [1, 2] is one of the most fundamental primitives in cryptography. In SSSs, a secret information is encrypted into several information called shares, each of which has no information on the secret in the sense of information theoretic security. Each share is distributed to a participant, and the secret can be recovered by collecting the shares of specified set of participants called a qualified set.

In a narrow sense, information theoretic security implies so-called perfect security [3]. For instance, SSSs require that no information can be obtained by the non-qualified set of participants, called a forbidden set, even if they have unbounded computing power. Letting \( S \) and \( V_F \) be the random variables corresponding to the secret and the set of shares of a forbidden set \( F \), respectively, this requirement is mathematically formulated as \( H(S|V_F) = H(S) \) where \( H(\cdot) \) and \( H(\cdot|\cdot) \) are Shannon and conditional entropies, respectively. Namely, the random variables \( S \) and \( V_F \) are statistically independent.

On the other hand, another milder and optimistic scenario is studied to define the information theoretic security: Suppose that an adversary can guess a plaintext only once. Then, the best way to do this is guessing the plaintext with the highest probability given the ciphertext. Merhav [4] studied the exponent of this kind of success probability in guessing for symmetric-key cryptography with variable-length keys\(^1\) in asymptotic setup. Recently, Alimomeni and Safavi-Naini [5] and Dodis [6] revisited the same scenario for the symmetric-key cryptography with fixed-length keys such as Vernam cipher [7] in non-asymptotic setup. The security criteria in [5, 6] are based on min-entropy and its conditional version since the min-entropy is defined by negative logarithm of the highest probability in a probability distribution. Under such security criteria, the lower bounds of key length are discussed in [5, 6].

In this paper, we are interested in SSSs by using (conditional) min-entropies, and clarify their fundamental results. Particularly, we investigate the lower bounds of share sizes, and the constructions of SSSs based on min-entropies.

In order to derive the lower bounds of share sizes in terms of min-entropies, we take the similar strategy developed recently by Iwamoto and Shikata [8]. Namely, we first formalize a unified framework of SSSs based on (conditional) Rényi entropies, which includes SSSs based on Shannon and min entropies etc., as special cases. Then, by deriving the lower bound of share sizes in terms of Rényi entropies, we can obtain the lower bounds of share sizes measured by Shannon and min entropies in a unified manner.

Then, we show two existential results on SSSs based on min-entropies. Noticing that SSSs satisfying perfect security also satisfy the security criteria based on min-entropies, we are particularly interested in so-called non-perfect SSSs [9–11] while they are secure in the sense of min-entropies. As a result, we clarify the following two fundamental facts:

The first result is on the existence of SSSs with general access structures. In SSSs satisfying perfect security, Ito, Saito, and Nishizeki [12] proved the well known result that SSSs can be constructed if and only if the access structure satisfies a certain property called monotone. Combining this result with the one by Blundo, De Santis, and Vaccaro [13], this existential result can be extended to the case of arbitrary distribution of secret information. Inspired by these results, we will clarify that we can always construct non-perfect SSS based on min-entropies for arbitrary monotone access structures and arbitrary binary probability distribution of the secret.

The second result is on the optimality of SSSs based on min-entropies. In SSSs with perfect security, SSS is called ideal if \( H(S) = H(V_i) \) for all \( i = 1, 2, \ldots, n \). Note that the ideal SSS only exists only in the case where \( S \) is uniformly distributed [13]. In this case, \( R_\infty(S) = R_\infty(V_i) \) obviously holds since Shannon and min entropies coincide for uniform

\(^1\)In this symmetric-key cryptography, each key depends on the ciphertext, and hence, its length can be varied depending on the ciphertext.
distributions. Hence, we are interested in the existence of ideal non-perfect SSSs based on min-entropies for non-uniform probability distribution of the secret. Surprisingly, we will prove that there actually exists such a non-uniform probability distribution of the secret realizing the ideal non-perfect \((k, n)\)-threshold SSS based on min-entropies.

In Section II, we provide a formal model and security definition of SSSs based on Rényi entropies for general access structures. Under such a model and security definitions, we derive the lower bound of share sizes measured by average conditional min-entropies in Section III, by proving the extended lower bound based on Rényi entropies. Sections IV and V are devoted to construct SSSs based on min-entropies. In Section IV, we show the existence of non-perfect SSSs based on min-entropies for general access structures. Then, it is clarified in Section V that an ideal non-perfect SSS based on min-entropies for arbitrary distribution on binary secret and for arbitrary monotone access structures. Hence, we are interested in the existence of ideal SSSs based on min-entropies for non-uniform distribution of the secret. Technical proofs of Theorems 3 and 4 are provided in Appendices A and B, respectively.

II. MODEL AND SECURITY DEFINITION

Let \([n] := \{1, 2, \ldots, n\}\) be a finite set of IDs of \(n\) users. For every \(i \in [n]\), let \(V_i\) be a finite set of possible shares of the user \(i\), and denote by \(P_{V_i}\) its associated probability distribution on \(V_i\). Similarly, let \(S\) be a finite set of secret information and \(P_S\) be its associated probability distribution. In the following, for any subset \(\mathcal{U} := \{i_1, i_2, \ldots, i_u\} \subset [n]\), we use the notation \(v_\mathcal{U} := \{v_{i_1}, v_{i_2}, \ldots, v_{i_u}\}\) and \(V_\mathcal{U} := \{V_{i_1}, V_{i_2}, \ldots, V_{i_u}\}\).

For a finite set \(\mathcal{X}\), let \(\mathcal{P}(\mathcal{X})\) be the family of probability distributions defined on \(\mathcal{X}\).

A. Access Structures

In SSSs, we normally assume that each set of shares is classified into either a qualified set or a forbidden set. A qualified set is the set of shares that can recover the secret. On the other hand, the secret must be kept secret against any collusion of members of a forbidden set in the sense of information theoretic security, the meaning of which will be formally specified later. Let \(\mathcal{F} \subset 2^{[n]}\) and \(\mathcal{F} \subset 2^{[n]}\) be families of qualified and forbidden sets, respectively. Then we call \(\Gamma := (\mathcal{F}, \mathcal{F})\) an access structure. In particular, the access structure is called \((k, n)\)-threshold access structure if it satisfies that \(\mathcal{U} := \{Q : |Q| \geq k\}\) and \(\mathcal{F} := \{F : |F| \leq k - 1\}\).

In this paper, we assume that the access structure is a partition of \(2^{[n]}\), namely, \(\mathcal{F} \cup \mathcal{F} = 2^{[n]}\) and \(\mathcal{F} \cap \mathcal{F} = \emptyset\).

An access structure is said to be monotone if it satisfies that for all \(Q \in \mathcal{F}\), every \(Q' \supset Q\) satisfies \(Q' \in \mathcal{F}\) and, for all \(F \in \mathcal{F}\), every \(F' \subseteq F\) satisfies \(F' \in \mathcal{F}\). In addition, we define the maximal forbidden sets as \(\mathcal{F}^+ := \{F \in \mathcal{F} | F \cup \{i\} \in \mathcal{F}, \text{ for all } i \in [n] \setminus F\}\). Clearly, the monotone property is necessary condition for the existence of secret sharing schemes. Furthermore, it is proved that this property is actually sufficient for the existence of SSSs [12].

B. Secret Sharing Schemes for General Access Structures

Let \(\Pi = ([P_S], [\Pi_{share}], [\Pi_{comb}])\) be a secret sharing scheme for an access structure \(\Gamma\), as defined below:

- \([P_S]\) is a sampling algorithm for secret information, and it outputs a secret \(s \in S\) according to a probability distribution \(P_S\);
- \([\Pi_{share}]\) is a randomized algorithm for generating shares for all users, and it is executed by a honest entity called dealer. It takes a secret \(s \in S\) on input and outputs \((v_1, v_2, \ldots, v_n) \in \prod_{i=1}^{n} V_i\) and \(s \in S\);
- \([\Pi_{comb}]\) is an algorithm for recovering a secret. It takes a set of shares \(v_Q, Q \in \mathcal{F}\), on input and outputs a secret \(s \in S\).

In this paper, we assume that \(\Pi\) meets perfect correctness: for any possible secret \(s \in S\), and for all possible shares \((v_1, v_2, \ldots, v_n) \leftarrow \Pi_{share}(s)\), it holds that \(\Pi_{comb}(v_Q) = s\) for any subset \(Q \in \mathcal{F}\).

C. Information Measures and Security Criteria

In information theoretic cryptography, information measures play a significant role since it is used to define the security as well as to measure the share sizes. In this paper, we are interested in Rényi entropy and its conditional one since Rényi entropies include several useful entropy measures as special cases.

For a non-negative real number \(\alpha\) and a random variable \(X\) taking its values on a finite set \(\mathcal{X}\), Rényi entropy of order \(\alpha\) with respect to \(X\) is defined as [14]

\[
R_\alpha(X) := \frac{1}{1-\alpha} \log \sum_{x \in \mathcal{X}} P_X(x)^\alpha. \quad (1)
\]

It is well known several entropy measures are special cases of Rényi entropy. For instance, Shannon entropy \(H(X) := -\sum_{x \in \mathcal{X}} P_X(x) \log P_X(x)\) and min-entropy \(R_\infty(X) := -\log \max_{x \in \mathcal{X}} P_X(x)\) are derived as special cases of \(R_\alpha(X)\) by letting \(\alpha \rightarrow 1\) and \(\alpha \rightarrow \infty\), respectively.

In addition, for a non-negative real number \(\alpha\) and random variables \(X\) and \(Y\) taking values on finite sets \(\mathcal{X}\) and \(\mathcal{Y}\), respectively, conditional Rényi entropy of order \(\alpha\) with respect to \(X\) given \(Y\) is defined as [15]

\[
R_\alpha(X|Y) := \frac{\alpha}{1-\alpha} \log \sum_{y \in \mathcal{Y}} P_Y(y) \left( \sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^\alpha \right)^{1/\alpha}. \quad (2)
\]

Note that there are many definitions of conditional Rényi entropies [8, 16]. There are two reasons why we choose the definition (2) of conditional Rényi entropy. The first reason is that it is connected to the cryptographically important conditional min-entropy \(R_\infty(\cdot|\cdot)\) (e.g., see [17]) defined as

\[
R_\infty(X|Y) := -\log \sum_{y \in \mathcal{Y}} P_Y(y) \max_{x \in \mathcal{X}} P_{X|Y}(x|y) \quad (3)
\]

which plays a crucial role in this paper. We note that the relation \(R_\infty(X|Y) = \lim_{\alpha \rightarrow \infty} R_\alpha(X|Y)\) holds [8].
The second reason is that it satisfies the very useful properties similar to Shannon entropy as shown below:

**Proposition 1** ([8, 15, 18]): Let $X$, $Y$, and $Z$ be random variables taking values on finite sets $X$, $Y$, and $Z$, respectively. Then, for arbitrary $\alpha \in [0, \infty]$ we have:

(A) **Conditioned Monotonicity:**

1. $R_\alpha(X|Z) \leq R_\alpha(XY|Z)$, and
2. $R_\alpha(X|Z) = R_\alpha(XY|Z)$ holds if and only if $Y = f(X, Z)$ for some (deterministic) mapping $f$.

(B) **Conditioning Reduces Entropy:** $R_\alpha(X) \geq R_\alpha(X|Y)$ where the equality holds if $X$ and $Y$ are statistically independent.

Note that the properties (A)–(i) and (B) are proved in [8] and [15], respectively. The property (A)–(ii) is explicitly pointed out in [8].

Based on the above foundations of (conditional) Rényi entropies, we give security formalization for a secret sharing scheme for an access structure $\Gamma$ as follows.

**Definition 1 (Security based on Rényi entropies):** Let $\Pi$ be a secret sharing scheme for an access structure $\Gamma$. Then, $\Pi$ is said to meet $\epsilon$-security with respect to $R_\alpha(\cdot)$, if for any forbidden set $F \subseteq F$ it satisfies

$$R_\alpha(S) - R_\alpha(S|V_F) \leq \epsilon.$$  

In particular, $\Pi$ is said to meet perfect security with respect to $R_\alpha(\cdot)$ if $\epsilon = 0$ above.

For simplicity we abbreviate $\epsilon$-security with respect to $R_\alpha(\cdot)$ as $(R_\alpha(\cdot), \epsilon)$-security, and perfect security with respect to $R_\alpha(\cdot)$ as $R_\alpha(\cdot)$-security.

Note that Definition 1 includes the security of SSSs based on several types of entropies as special cases. In particular, we are interested in the case of $\alpha \to 1$ and $\alpha \to \infty$ which define $\epsilon$-security with respect to $H(\cdot)$ and $R_\infty(\cdot)$, i.e., the security based on Shannon and min-entropy, respectively.

Finally, we note that if a SSS is $H(\cdot)$-secure, i.e., perfectly secure, it is $R_\infty(\cdot)$-secure as well. Hence, in this paper, we are mainly interested in the SSS satisfying $R_\infty(\cdot)$-security but not satisfying $H(\cdot)$-security. Such a SSS is called non-perfect SSS [11], which is formally defined as follows:

**Definition 2 ([11]):** If there exists a set of $F \subseteq [n]$ satisfying $H_\alpha(S) > H_\alpha(S|V_F)$, it is called a non-perfect SSS.

### III. Unified Proofs for Lower Bounds of Share Sizes in SSSs

We begin with this section by reviewing the following well-known tight lower bound of SSSs for arbitrary access structure $\Gamma$ meeting $(H(\cdot), \epsilon)$-secure SSSs.

**Proposition 2** ([19, 20]): Let $\Pi$ be an $H(\cdot)$-secure secret sharing scheme for an access structure $\Gamma$. Then, for every $i \in [n]$, it holds that

$$H(V_i) \geq H(S) \text{ and } |V_i| \geq |S|.$$  

Note that (5) is proved for threshold access structures and general access structures in [19] and [20], respectively. From Proposition 2, we define that the secret sharing scheme $\Pi$ is called **ideal** if $\Pi$ satisfies (5) with equalities.

As is shown in Proposition 2, we note that the SSSs based on Shannon entropy are well studied. On the other hand, there is no study on a SSS based on min-entropy, which is the main topic of this paper. Noticing the fact that Rényi entropy is a generalization of Shannon, min, and several kinds of entropies, it is fruitful to derive the lower bounds of share sizes of $(R_\alpha(\cdot), \epsilon)$-secure SSSs, i.e., in terms of Rényi entropies, in a comprehensive way. Then, we can directly prove that the lower bounds of share sizes of $(R_\infty(\cdot), \epsilon)$-secure SSSs as a corollary. The following theorem can be considered as an extension of impossibility result with respect to key sizes measured by Rényi entropy discussed in [8] in symmetric-key cryptography to the impossibility result with respect to SSSs.

**Theorem 1:** Let $\Pi$ be a secret sharing scheme for an access structure $\Gamma$ which meets $(R_\alpha(\cdot), \epsilon)$-security. Then, it holds that, for arbitrary $\alpha \in [0, \infty]$,

$$R_\alpha(V_i) \geq R_\alpha(S) - \epsilon$$  

for every $i \in [n]$.

**Proof.** For any $i \in [n]$, there exists a for hidden set $F \subseteq F$ such that $i \notin F$ and $F \cup \{i\} \notin \mathcal{F}$. Then, we have

$$R_\alpha(S) \leq R_\alpha(S|V_F) + \epsilon \overset{(a)}{=} R_\alpha(V_i|V_F) + \epsilon \overset{(b)}{=} R_\alpha(V_i|V_F) + \epsilon \overset{(c)}{=} R_\alpha(V_i) + \epsilon,$$

where (a) follows from Proposition 1 (A)–(i), (b) follows from Proposition 1 (A)–(ii) since $\Pi$ meets perfect correctness, and (c) follows from Proposition 1 (B). In the case of $R_\alpha(\cdot)$-security, i.e., $\epsilon = 0$, the following corollary obviously holds.

**Corollary 1:** For arbitrarily fixed $\alpha \in [0, \infty]$, let $\Pi$ be a secret sharing scheme for an access structure $\Gamma$ meeting $R_\alpha(\cdot)$-security. Then, it holds that $R_\alpha(V_i) \geq R_\alpha(S)$ for every $i \in [n]$.

Hence, we can immediately obtain the following corollary with respect to the lower bound of share sizes measured by min-entropy under $R_\infty(\cdot)$-security by letting $\alpha \to \infty$ in Corollary 1.

**Corollary 2:** Let $\Pi$ be a secret sharing scheme for an access structure $\Gamma$ meeting $R_\infty(\cdot)$-security. Then, it holds that $R_\infty(V_i) \geq R_\infty(S)$ for every $i \in [n]$.

Moreover, noting that $R_\alpha(S|V_F) = R_\alpha(S)$ holds for arbitrary $\alpha \in [0, \infty]$ if the random variables $S$ and $V_F$ are statistically independent, i.e., $H(S|V_F) = H(S)$. Hence, we can prove that the lower bounds of share sizes of SSSs with $H(\cdot)$-security.

**Corollary 3:** Let $\Pi$ be a secret sharing scheme for an access structure $\Gamma$ meeting $H(\cdot)$-security. Then, it holds for arbitrary $\alpha \in [0, \infty]$ that $R_\alpha(V_i) \geq R_\alpha(S)$, $i \in [n]$.

Hence, Proposition 2 is immediately obtained from Corollary 3 as a special cases of Corollary 3 by taking the limits $\alpha \to 1$ and $\alpha \to 0$.

\[\text{However, as will be shown in Sections IV and V, we note the converse of this implication is not always true.}\]
In a similar manner, we can also obtain the following corollary with respect to the lower bound of share sizes measured by \( \text{min-entropy} \) under traditional \( H(\cdot|\cdot) \)-security by letting \( \alpha \to \infty \) in Corollary 3.

**Corollary 4:** Let \( \Pi \) be a secret sharing scheme for an access structure \( \Gamma \) meeting \( H(\cdot|\cdot) \)-security. Then, it holds that \( R_{\infty}(V_i) \geq R_{\infty}(S) \) for every \( i \in [n] \).

**Remark 1:** Recently, Alimomeni and Safavi-Naini [5] proved that the key size in symmetric-key encryption must be equal to or larger than the message size if these sizes are measured by \( \text{min-entropy} \) and the security criteria is based on \( R_{\infty}(\cdot|\cdot) \). Similarly, Dodis [6] proved that the key size in symmetric-key encryption must be equal to or larger than the message size if these sizes are measured by \( \text{min-entropy} \) and the security criteria is based on \( H(\cdot|\cdot) \). Hence, Corollaries 2 and 4 can be considered as SSS versions of [5] and [6], respectively.

**Remark 2:** All discussions in Sections II and III are valid if we replace \( R_{\alpha}(\cdot|\cdot) \) with the conditional Rényi entropy proposed by Hayashi [21]. In this case, the only difference is the security definition based on Hayashi’s conditional Rényi entropy, we can also prove Theorem 1 in the same way.

**IV. Existence of Non-Perfect SSS Based on Min-entropies for General Access Structures**

Hereafter, we are concerned with the existence of SSSs satisfying \( R_{\infty}(\cdot|\cdot) \)-security. Recalling the discussion on Definition 2, we are concerned with a \( R_{\infty}(\cdot|\cdot) \)-secure non-perfect SSS, since, if a SSS is perfectly secure, it is \( R_{\infty}(\cdot|\cdot) \)-secure as well. Note that non-perfect SSSs are meaningless if the secret if deterministic since \( H(S) = 0 \) in such a case. Hence, we assume that the secret is not deterministic in the following discussion.

In this section, we address the existence of non-perfect SSS satisfying \( R_{\infty}(\cdot|\cdot) \)-security for arbitrary monotone access structure \( \Gamma \).

**A. Existence of non-perfect SSS Based on Min-entropies**

Combining the results [12] and [13], there exist SSSs satisfying \( H(\cdot|\cdot) \)-security for arbitrary monotone access structure and for arbitrary probability distribution of secret information. However, it is not known whether this fact is still valid or not for non-perfect SSSs satisfying \( R_{\infty}(\cdot|\cdot) \)-security. We obtain a positive result for such a question, summarized as the following theorem:

**Theorem 2:** For an arbitrary binary and non-uniform probability distribution \( P_{S}(\cdot) \in \mathcal{P}\{0,1\} \) of secret information, there exists a non-perfect SSS with \( R_{\infty}(\cdot|\cdot) \)-security with an arbitrary monotone access structure \( \Gamma = (\mathcal{D}, \mathcal{F}) \).

**Proof.** Let \( P_{S}(\cdot) \in \mathcal{P}\{0,1\} \) be an arbitrarily fixed non-uniform probability distribution of the binary secret. For a given monotone access structure \( \Gamma = (\mathcal{D}, \mathcal{F}) \), let \( \mathcal{F}^+=\{F_1, F_2, \ldots, F_m\} \) where \( m := |\mathcal{F}^+| \).

Suppose that we can generate a set of shares denoted by \{\( w_1, w_2, \ldots, w_m \)\} of non-perfect \( (m,m) \)-threshold SSS for the secret \( s \in \{0,1\} \) satisfying \( R_{\infty}(\cdot|\cdot) \)-security. The construction of such \( (m,m) \)-threshold SSS based on min-entropies will be provided in Construction \( \Pi_1 \). For \( j \in [m] \), let \( W_j \) be the random variable taking its values on a finite set \( \mathcal{W}_j \), which corresponds to \( w_j \).

Consider a cumulative map [12] \( \varphi^{\mathcal{F}} : [n] \to 2^{[m]} \) given by \( \varphi^{\mathcal{F}}(i) := \{ j \mid i \notin F_j \in \mathcal{F}^+ \} \) for \( i \in [n] \), and define \( \varphi^{\mathcal{F}}(\mathcal{U}) := \bigcup_{i\in\mathcal{U}} \varphi^{\mathcal{F}}(i) \) for \( \mathcal{U} \subset [n] \). Then, it is proved in [12] that
\[
|\varphi^{\mathcal{F}}(\mathcal{U})| \geq m, \quad \text{if } \mathcal{U} \in \mathcal{D} \tag{8}
\]
\[
|\varphi^{\mathcal{F}}(\mathcal{U})| \leq m - 1, \quad \text{if } \mathcal{U} \in \mathcal{F}. \tag{9}
\]

Now, we assume that each share \( v_i \) for the SSS with the access structure \( \Gamma \) consists of a set of \( w_j \). Specifically, let \( v_i := \{ w_j \mid j \in \varphi^{\mathcal{F}}(i) \} \). Then, the secret \( s \) can be recovered from a qualified set \( \mathcal{Q} \subset [n] \) due to (8). On the other hand, we have
\[
R_{\infty}(S|\mathcal{F}) = R_{\infty}(S|\varphi^{\mathcal{F}}(\mathcal{F})) = R_{\infty}(S) \tag{10}
\]
holds for arbitrary \( \mathcal{F} \in \mathcal{F} \) where the first equality holds from the definition of \( v_i \), and the second equality is due to (9) and \( R_{\infty}(\cdot|\cdot) \)-security for the non-perfect \( (m,m) \)-threshold SSS.

**Remark 3:** Similar argument also holds by combining monotone circuit construction [22] with \( (n,n) \)-threshold non-perfect SSS with \( R_{\infty}(\cdot|\cdot) \)-security, which is omitted here.

Hence, the remaining to prove Theorem 2 is the construction of a non-perfect \( (n,n) \)-threshold SSS satisfying \( R_{\infty}(\cdot|\cdot) \)-security for arbitrary non-uniform probability distribution \( P_{S}(\cdot) \in \mathcal{P}\{0,1\} \).

**B. Construction of Non-perfect \( (n,n) \)-threshold SSS Based on Min-entropies for Arbitrary Binary Secret Information**

**Construction \( \Pi_1 \):** Let \( S \), and \( V_1, V_2, \ldots, V_n \) be binary random variables. Assume that \( S \) and \( V_1, V_2, \ldots, V_{n-1} \) are statistically independent and they satisfy \( P_{S}(0) = P_{V_1}(0) = \cdots = P_{V_{n-1}}(0) = p \), for \( 1/2 < p < 1 \). Then, we generate \( V_n \) by \( V_n := S \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1} \) where \( \oplus \) denotes the exclusive OR operation.

**Theorem 3:** The construction \( \Pi_1 \) realizes
\[
R_{\infty}(S) = R_{\infty}(V_i) = -\log p \quad \text{for } i \in [n-1], \tag{11}
\]
\[
R_{\infty}(V_n) = -\log\{p^2 + (1-p)^2\}, \tag{12}
\]
and
\[
R_{\infty}(S|\mathcal{F}) = -\log p \quad \text{for } \mathcal{F} \subset [n] \text{ s.t. } |\mathcal{F}| = n-1. \tag{13}
\]

Hence, the construction \( \Pi_1 \) is a non-perfect \( (n,n) \)-threshold SSS which meets \( R_{\infty}(\cdot|\cdot) \)-security.

**Proof.** See Appendix A.

\[\square\]
Remark 4: The above construction \( \Pi_1 \) works in the cases of \( p = 1/2, 1 \). In both cases, the random variables \( S \) and \( V_F \) in the construction \( \Pi_1 \) are statistically independent, and hence, they result in \((n, n)\)-threshold SSSs satisfying \( H(\cdot) \)-security.

On the other hand, if the random variables \( S \) and \( V_F \) in the construction \( \Pi_1 \) are not statistically independent if \( n \in \mathcal{F} \) since \( p \neq 1/2, 1 \). Therefore, \( \Pi_1 \) is a non-perfect \((n, n)\)-threshold SSS while it satisfies \( R(\cdot) \)-security.

Remark 5: As we will see in Section V, there exists a specific probability distribution of the secret that realizes a non-perfect SSS satisfying \( R_\infty(\cdot) \)-security for an arbitrary monotone access structure \( \Gamma = (2, \mathcal{F}) \) even if the set \( S \) is non-binary. However, it is an open problem to construct a monotone access structure threshold SSS while it satisfies \( R(\cdot) \)-security.

In our problem setting, we can use Corollary 2 to define the ideal SSS for \( R_\infty(\cdot) \)-security.

Definition 3 (Ideal SSS based on Min-entropies): A SSS \( \Pi \) meeting \( R_\infty(\cdot) \)-security is called ideal if \( \Pi \) satisfies \( R_\infty(V_i) = R_\infty(S) \) for arbitrary \( i \in [n] \).

Based on the above definition, the share sizes in \( \Pi_1 \) are “almost” ideal in the sense that \( R_\infty(V_i) = R_\infty(S) \) for \( i \in [n-1] \), but \( R_\infty(V_n) > R_\infty(S) \). In order to obtain (fully) ideal SSS which meets \( R_\infty(\cdot) \)-security, the parameter \( p \) must satisfy \( -\log(p^2+(1-p)^2) = -\log p \) because of \( R_\infty(S) = R_\infty(V_n) \), and we have \( p = 1/2, 1 \). However, each of \( p = 1/2, 1 \) makes \( \Pi_1 \) to be \( H(\cdot) \)-secure as pointed out in Remark 4 and hence, we assume \( p \neq 1/2, 1 \) in \( \Pi_1 \). Namely, \( \Pi_1 \) cannot realize ideal non-perfect \( R_\infty(\cdot) \)-secure SSSs while it is applicable to arbitrary probability distribution of binary secret.

Summarizing, although the protocol \( \Pi_1 \) is applicable to arbitrary binary probability distribution of the secret, it has the following problems; \( \Pi_1 \) is designed for \((n, n)\)-threshold schemes with binary secrets, and; \( \Pi_1 \) cannot realize ideal non-perfect SSSs satisfying \( R_\infty(\cdot) \)-security.

V. Existence of Ideal Non-perfect \((k, n)\)-threshold SSS based on Min-entropies

From the discussion at the end of the last section, we show that there exist an ideal non-perfect \((k, n)\)-threshold SSSs satisfying \( R_\infty(\cdot) \)-security with specific non-uniform probability distributions of a secret \( S \) over arbitrary finite field \( S \). This result implies the essential difference between \( R_\infty(\cdot) \)- and \( H(\cdot) \)-security since it is proved in [13, Theorem 7] that ideal \( H(\cdot) \)-secure SSS is realized only when \( S \) is uniform.

Construction \( \Pi_2 \): For a finite field \( \mathbb{F}_t \) with a prime power \( t \), generate a set

\[
DT(k,n) := \{ (s, v_1, v_2, \ldots, v_n) \mid v_i = s + \sum_{t=1}^{k-1} i^t r_t, \}
\]

\[
(s, r_1, r_2, \ldots, r_{k-1}) \in (\mathbb{F}_t)^k \}
\]

\( \in \mathcal{F}_t \). Therefore, \( \Pi_2 \) is called distribution table [23], where we assume that each \( i \in [n] \) in (14) is appropriately encoded so as to be regarded as \( i \in \mathbb{F}_t \). Let \( S \), and \( V_1, V_2, \ldots, V_n \) be random variables with joint probability given by

\[
P_{SV_1 V_2 \ldots V_n}(s, v_1, v_2, \ldots, v_n) = \begin{cases} p, & \text{if } (s, v_1, v_2, \ldots, v_n) = (0, 0, \ldots, 0), \\ \frac{1-p}{t^k - 1}, & \text{if } (s, v_1, v_2, \ldots, v_n) \neq (0, 0, \ldots, 0) \\
0, & \text{if } (s, v_1, v_2, \ldots, v_n) \notin DT(k,n). \end{cases}
\]

where \( p \geq 1/t^k \).

Remark 6: Let \( \varphi : (\mathbb{F}_t) \rightarrow (\mathbb{F}_t)^{n+1} \) be the mapping defined by \( \varphi((s, r_1, r_2, \ldots, r_{k-1})) = (s, v_1, v_2, \ldots, v_n) \) where \( s, r_1, r_2, \ldots, r_{k-1} \) and \( v_1, v_2, \ldots, v_n \) are specified in (14). Then, it is seen that \( \varphi \) is injective due to the Lagrange interpolation, \( \varphi((0, 0, \ldots, 0)) = (0, 0, \ldots, 0) \), and \( \text{Im} \varphi = DT(k,n) \). Hence, we have \( |DT(k,n)| = t^k \). From this fact, it is easy to see that (15) actually forms a probability distribution.

Theorem 4: In the above construction \( \Pi_2 \), it holds for arbitrary \( \mathcal{F} \subset [n] \) satisfying \( |\mathcal{F}| = k - 1 \) that

\[
R_\infty(S|V_F) = R_\infty(S) = R_\infty(V_1) = \cdots = R_\infty(V_n) = -\log \frac{pt^k + (1-p)t^{k-1} - 1}{t^k - 1},
\]

which means that, for every integers \( k \) and \( n \), and for arbitrary finite field of secret information, there exists a non-uniform distribution \( P_S(\cdot) \in \mathcal{P}(S) \) to realize an ideal non-perfect \( R_\infty(\cdot) \)-secure SSS.

Proof. See Appendix B.

Remark 7: In SSSs satisfying \( H(\cdot) \)-security, the ideal one exists only in the cases that the distribution of a secret is uniform or deterministic, and hence, \( H(S) = H(V_i) = \log |S| \) or \( H(S) = H(V_i) = 0 \) is allowed in ideal \( H(\cdot) \)-secure SSSs. On the other hand, it is interesting to note that, for every \( 0 \leq R \leq \log |S| \), there exists an ideal non-perfect SSS satisfying \( R_\infty(\cdot) \)-security that attains \( R = R_\infty(S) \).

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REFERENCES

[1] A. Shamir, “How to share a secret,” Communications of the ACM, vol. 22, no. 11, pp. 612–613, 1979.

[2] G. R. Blakley, “Safekeeping cryptographic keys,” AFIPS 1979 National Computer Conference, vol. 48, pp. 313–317, 1979.

[3] C. E. Shannon, “Communication theory of secrecy systems,” Bell Tech. J., vol. 28, pp. 656–715, Oct. 1945.

[4] N. Merhav, “A large-deviations notions of perfect secrecy,” IEEE Trans. Information Theory, vol. 30, no. 2, pp. 506–508, 2003.

[5] M. Itoh, A. Saito, and T. Nishizeki, “Multiple assignment scheme for secret sharing,” Advances in Cryptology–CRYPTO’84, LNCS 196, Berlin-Heidelberg, pp. 242–269, 1985.

[6] H. Yamamoto, “On secret sharing systems using (k, t, n) threshold scheme,” IEEE Trans., vol. 36-A, no. 9, pp. 945–952, 1985.

[7] K. Kurosawa, K. Okada, K. Sakano, W. Ogata, and T. Tsujii, “Non-Rényi conditional Rényi entropies,” Proc. of the 6th International Conference on Information Theoretic Security (ICITS 2012), LNCS7412, Springer-Verlag, pp. 1–13, August 2012.

[8] Y. Dodis, “Shannon impossibility, revisited,” Proc. of the 6th International Conference on Information Theoretic Security (ICITS 2012), LNCS7412, Springer-Verlag, pp. 100–110, August 2012. IACR Cryptology ePrint Archive (preliminary short version): http://eprint.iacr.org/2012/053.

[9] G. S. Vernam, “Cipher printing telegraph systems for secret wire and radio telegraphic communications,” J. of American Institute for Electrical Engineering, vol. 45, pp. 109–115, 1926.

[10] M. Iwamoto and J. Shikata, “Information theoretic security for encryption based on conditional Rényi entropies,” Proc. of International Conference on Information Theoretic Security (ICITS), LNCS8317, Springer-Verlag, pp. 103–121, 2013.

[11] J. Benaloh and J. Leichter, “Generalized secret sharing and monotone functions,” Advances in Cryptology–CRYPTO’88, LNCS 403, Springer-Verlag, pp. 27–35, 1990.

[12] D. R. Stinson, CRYPTOGRAPHY Theory and Practice. CRC Press, third ed., 2005.

APPENDIX

A. Proof of Theorem 3

First, we show (11) and (12). It is easy to see that
\[ R_\infty(S) = R_\infty(V_1) = \cdots = R_\infty(V_{n-1}) = -\log p. \]

Noticing that
\[ V_n = S \oplus V_1 \oplus \cdots \oplus V_{n-1}, \]

it holds that \[ S \oplus V_1 = V_n, \]
and hence, we have \[ P_{V_n}(0) = p^2 + q^2 \] where \( q := 1 - p \).

In the remaining of this proof, we check (13). Since \( R_\infty(X|Y) \) satisfies Corollary 1 (A), it is sufficient to show \( R_\infty(S|V_F) = R_\infty(S) \) only in the case of \( |F| = n-1 \).

Consider the case where \( n \not\in F \). In this case, it is easy to see that
\[ P_{S|V_1V_2\ldots V_{n-1}}(s|v_1,v_2,\ldots,v_{n-1}) = P_S(s) \]
holds since \( S \) and \( V_1, V_2, \ldots, V_{n-1} \) are independent. Hence, \( R_\infty(S|V_F) = R_\infty(S) \) obviously holds in this case.

Next, we consider the case of \( n \in F \). From the symmetry, it is sufficient to consider the case where \( F = \{2,3,\ldots,n\} \). For simplicity of notation, define \( v := v_{F(n)} \) and \( V := V_{\overline{F(n)}} \). Let \( \sigma : \{0,1\}^{n-2} \to \{0,1\} \) be the mapping that computes exclusive OR of all inputs. Due to the construction and the independency among \( S \) and \( V_1, V_2, \ldots, V_{n-1} \), the probability \( P_{V_F(v,F)} = P_{V_{n-V}}(v,\overline{v}) \) can be calculated in the following cases:

**Case 1:** \( \sigma(v) \oplus v_n = 1 \), i.e., \( (\sigma(v), v_n) = (0,1) \) or \( (\sigma(v), v_n) = (1,0) \):

\[ P_{V_{n-V}}(v, v_n) = P_{SV_1V}(1,0,v) + P_{SV_1V}(0,1,v) = 2pqP_V(v) \]

**Case 2:** \( \sigma(v) \oplus v_n = 0 \), i.e., \( (\sigma(v), v_n) = (0,0) \) or \( (\sigma(v), v_n) = (1,1) \):

\[ P_{V_{n-V}}(v, v_n) = P_{SV_1V}(0,0,v) + P_{SV_1V}(1,1,v) = (p^2 + q^2)P_V(v) \]

Furthermore, note that the following relation:

\[ P_{S|V_{n-V}}(s|v, v_n) = P_{SV_1V_{n-V}}(s,1|v, v_n) + P_{SV_1|V_{n-V}}(s,0|v, v_n). \]

Now, consider Case 1). In this case it is easy to see that
\[ P_{SV_1|V_{n-V}}(0,1|v, v_n) = P_{SV_1|V_{n-V}}(1,0|v, v_n) = \frac{1}{2} \]

and
\[ P_{SV_1|V_{n-V}}(0,0|v, v_n) = P_{SV_1|V_{n-V}}(1,1|v, v_n) = 0. \]

Hence, (19) becomes \( P_{S|V_{n-V}}(s|v, v_n) = 1/2 \), which leads to
\[ P_{V_{n-V}}(v, v_n) \max_{s \in s} P_{S|V_{n-V}}(s|v, v_n) = pq \cdot PV(v). \]

Next, consider Case 2). In this case, it is easy to see that
\[ P_{SV_1|V_{n-V}}(0,1|v, v_n) = P_{SV_1|V_{n-V}}(1,0|v, v_n) = 0 \]

and hence, (19) becomes
\[ \max_{s \in s} P_{S|V_{n-V}}(s|v, v_n) = \max \{ P_{SV_1|V_{n-V}}(0,0|v, v_n), P_{SV_1|V_{n-V}}(1,1|v, v_n) \}. \]

In the case of \( n = 2 \), we set \( v = \emptyset \) and \( V = \emptyset \).
Here, \( P_{SV|V^n}(0, 0|v, v_n) \) can be calculated as follows:

\[
P_{SV|V^n}(0, 0|v, v_n) = \frac{P_{SV|V^n}(0, 0, v, v_n)}{P_V(v, v_n)} = \frac{P_{SV}(0, 0)P_V(v)}{P_V(v, v_n)} = \frac{p^2}{p^2 + q^2} \tag{24}\]

Similarly, we have \( P_{SV|V^n}(1, 1|v, v_n) = \frac{q^2}{p^2 + q^2} \). Hence, because of \( p \geq q \), (23) becomes

\[
\sum_{v, v_n} P_{SV|V^n}(v, v_n) \max_s P_{S|V}(s|v, v_n) = \sum_{v, v_n} (pq + p^2)P_V(v) = p. \tag{26}\]

Hence, we obtain \( R_{\infty}(S|V_F) = R_{\infty}(S) = -\log p \), which completes the proof.

**B. Proof of Theorem 4**

Since the joint probability of \( (S, V_1, V_2, \ldots, V_n) \) is \( \frac{1-p}{t^k-1} \) except the case where \( S, V_1, V_2, \ldots, V_n \) are all 0, it is easy to see that

\[
P_S(0) = P_{V_1}(0) = P_{V_2}(0) = \cdots = P_{V_n}(0) = \frac{p + (t^k - 1) \frac{1-p}{t^k-1}}{1 - \frac{1-p}{t^k-1}} = \frac{pt^k + (1-p)t^k - 1}{t^k - 1}, \tag{27}\]

\[
P_S(z) = P_{V_1}(z) = P_{V_2}(z) = \cdots = P_{V_n}(z) = t^k \frac{1-p}{t^k-1}, \text{ for } z \in [t-1] \tag{28}\]

Here we note that \( p \geq 1/t^k \) implies that \( p \geq \frac{1-p}{t^k-1} \). Comparing with (27) and (28) taking \( p \geq \frac{1-p}{t^k-1} \) into account, it is easy to see that \( P_S(0) \geq P_S(z) \) as well as \( P_{V_i}(0) \geq P_{V_i}(z) \), \( i \in [n] \), for arbitrary \( z \in F_t^+ := F_t \setminus \{0\} \). Hence, we have

\[
R_{\infty}(S) = R_{\infty}(V_1) = \cdots = R_{\infty}(V_n) = -\log \frac{pt^k + (1-p)t^k - 1}{t^k - 1} \tag{29}\]

Now, we calculate \( H(S|V_F) \) where we assume that \( V_F := \{V_1, V_2, \ldots, V_{k-1}\} \) without loss of generality.

First, we consider the case where \( (v_1, v_2, \ldots, v_{k-1}) = (0, 0, \ldots, 0) \) is the condition. In the case, we have

\[
P_{V_1V_2\cdots V_{k-1}}(0, 0, \ldots, 0) = p + (t-1)\frac{1-p}{t^k-1}\]

and we denote this probability by \( R(p, t, k) \) for simplicity. Hence, we have

\[
P_{S|V_1V_2\cdots V_{k-1}}(s|0, 0, \ldots, 0) = \begin{cases} \frac{p}{R(p, t, k)} & \text{if } s = 0 \\ \frac{1-p}{R(p, t, k)} & \text{if } s = 1 \end{cases} \tag{30}\]

which results in \( \max_{s \in F_t} P_{S|V_1V_2\cdots V_{k-1}}(s|0, 0, \ldots, 0) = p/R(p, t, k) \) since we assume that \( p \geq 1/t^k \).

On the other hand, consider the case of \( (v_1, v_2, \ldots, v_{k-1}) \neq (0, 0, \ldots, 0) \). In this case, we have

\[
P_{V_1V_2\cdots V_{k-1}}(v_1, v_2, \ldots, v_{k-1}) = t \frac{1-p}{t^k - 1} \tag{31}\]

and hence, it holds that

\[
P_{S|V_1V_2\cdots V_{k-1}}(s|v_1, v_2, \ldots, v_{k-1}) = \frac{1}{t} \tag{32}\]

and hence, we obtain \( R_{\infty}(S|V_F) = 1/t \) for simplicity.

\[
H(S|V_F) = -\log \left\{ \frac{p}{R(p, t, k)} \times \frac{t}{1-p} + \frac{1}{t} \times \frac{1-p}{t^k - 1} \times (t^{k-1} - 1) \right\} \]

\[
= -\log \left\{ p + (t^{k-1} - 1) \frac{1-p}{t^k - 1} \right\} \]

\[
= -\log \frac{pt^k + (1-p)t^{k-1} - 1}{t^k - 1}. \tag{33}\]