Pinned QMA: The power of fixing a few qubits in proofs

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What could happen if we pinned a single qubit of a system and fixed it in a particular state? First, we show that this can greatly increase the complexity of static questions – ground state properties of local Hamiltonian problems with restricted types of terms. In particular, we show that the Pinned commuting and Pinned Stoquastic Local Hamiltonian problems are QMA complete. Second, we show that pinning a single qubit via often repeated measurements also results in universal quantum computation already with commuting and stoquastic Hamiltonians. Finally, we discuss variants of the Ground State Connectivity (GSCON) problem in light of pinning, and show that Stoquastic GSCON is QCMA complete. We hence identify a comprehensive picture of the computational power of pinning, reminiscent of the power of the one clean qubit model.

I. INTRODUCTION

The goal of quantum Hamiltonian complexity [44, 65] is to study the computational power of physical models described by local Hamiltonians, the intricate properties of their dynamics and their eigenstates, as well as to understand the computational complexity of determining these properties. Many Hamiltonians are known to be universal for quantum computation [41], while others are thought to be much simpler, but still hard to investigate classically [37] or even efficiently simulable by classical computation [53]. There is a long history of searching for the simplest possible, closest to realistically and efficiently implementable, and robustly controllable interaction with universal dynamics for quantum computation with local Hamiltonians. Restrictions on the type and strength of interactions, locality, and geometrical restrictions have been investigated, e.g., in Refs. [41, 47, 52, 60, 62, 63]. Thinking about universality for computation often comes hand in hand with asking complexity questions such as identifying the hardness of determining the properties of the eigenstates of these Hamiltonians.

Looking at this from a quantum control theory viewpoint provides us with an interesting observation. An extra level of control over a subsystem can result in a surprising boost in possible universality properties, state generation power, or the difficulty of complexity questions. We have seen this with the DQC1 ("one clean qubit") model [54, 59], whose single clean qubit gives rise to unexpected computing power. Similarly, if one is allowed to use magic states, computing with a restricted set of universal gates such as Clifford gates [38] becomes universal for quantum computation. Effectively fixing parts of the system to a particular state using perturbation gadgets allowed us to build complex effective Hamiltonians from simpler ones [50]. It has also been shown that a Zeno-effect measurement of a small subsystem can grant universal power to a non-universal set of commuting gates [39].

In this work, we investigate the computational potential offered by controlling a small subsystem, with the goal of finding efficient constructions for quantum computation with restricted control and answering complexity questions about such systems. We ask: What happens when a part of the system, e.g., a single qubit, is forced to live in a particular state – either as a written down condition, or by the Zeno-effect from often repeated measurements? As often in Hamiltonian complexity, there are two views of this, a static and a dynamic one. In both approaches, a seemingly innocent pinning condition brings quite a lot to the table.

Statically, asking about the difficulty of determining state properties (e.g., the ground state energy) of such pinned Hamiltonians, we show that these questions can become QMA complete even for permutation-matrix Hamiltonians. Several of our results on determining ground state energies complement the conclusions of Ref. [51]. We pin a qubit by an external prescription and show that the ground state question becomes QMA-complete for commuting and stoquastic problems. Next, going beyond Markov matrices, we show QMA-completeness of the Pinned Local Hamiltonian problem for permutation matrices.

Dynamically, asking about the preparation power of evolution with restricted time-independent Hamiltonians combined with Zeno pinning of a qubit, we find connections to previous work on Hamiltonian purification [39, 64], showing that the quantum Zeno-effect can, in fact, drive efficient universal quantum computations in several restricted settings. This includes, in particular, commuting and stoquastic Hamiltonians. We find another application of the pinning technique in the context of the Ground State Connectivity (GSCON) problem and its variants with restricted types of terms, e.g., the Commuting GSCON problem [46] — where, in fact, our investigation originated. Specifically, we prove that GSCON with stoquastic Hamiltonians is QCMA-complete, complementing a similar recent result [46] on Commuting GSCON.

On a formal level, “pinning” basically means “projecting” down into a selected subspace. Our goal is to uncover in which situations this can result in a large increase in complexity, or state preparation power, generating complex effective terms – weighted sums of the original restricted terms. Orsucci et al. [64] have formulated the related question of Hamiltonian purification, investigating the possible universal dynamics for a set of commuting Hamiltonians, when projected onto a particular subspace. We investigate several other

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scenarios beyond commuting Hamiltonians, in particular, stoquastic, stochastic, and permutation Hamiltonians. In our study we put emphasis on complexity questions, as well as efficient constructions, compared to only provable universality (that the dynamics are rich enough to cover the whole algebra, but without easily calculable guarantees on runtimes).

We conclude that there are strong limits to the pinning technique. First, dimensionality arguments from Ref. [64] mean a necessary increase in the size of the purified system in which interactions are restricted. Second, we encounter questions regarding locality of the required terms. Note that pinning does not allow us to create multiplicative effective terms, as perturbative gadgets do – creating effective 3-local terms from 2-local ones. We do not know if it is possible to build gadgets for effective \( k + 1 \) local interactions from \( k \)-local Hamiltonian terms with the help of pinning, for our commuting or stoquastic settings. Many such questions with low locality thus remain open.

Doing something special on one extra qubit is not new. Besides [46], where it is used to show that the \( GSCON \) problem is QCMA-complete already for commuting Hamiltonians, Jordan, Gosset & Love [51] have used techniques tracing back to Ref. [48] to get rid of varying signs of matrix elements by increasing the system size and replacing positive 1’s by 2x2-identity matrices and negative 1’s by the Pauli X matrices. They prove universality of adiabatic quantum computation in an excited state of a \( \text{Stoquastic Local Hamiltonian} \), instead of the usual ground state computation, by splitting the Hilbert space into two, depending on the state of an ancilla qubit. Moreover, adding a stoquastic term effectively pinning this ancilla into a state that results in a high energy, they showed \( QMA \)-completeness of understanding energy bounds for the highest excited energy of a \( \text{Stoquastic Local Hamiltonian} \). Next, they also show \( QMA \) hardness of bounding the lowest energy of doubly stochastic (Markov) matrices, and \( QMA \) hardness of the \( \text{Stochastic 6-SAT problem} \) (deciding whether a sum of stochastic matrices is frustration-free or not).

This work is structured as follows: First, in Section II, we show that several restricted versions of the \( \text{Pinned Local Hamiltonian} \) problem are \( QMA \)-complete, in particular, commuting, stoquastic and permutation Hamiltonians. In Section III we then turn to the dynamical problem of universal time evolution, showing that the Zeno-pinned time evolution under both commuting and stoquastic Hamiltonians is complete for universal quantum computations. Finally, in Section IV, we prove that the stoquastic \( GSCON \) problem is QCMA-complete and discuss the free fermionic \( GSCON \) problem.

II. PINNED LOCAL HAMILTONIANS: A COMPLEXITY VIEWPOINT

A. Local Hamiltonians and states with fixed qubits

In \( QMA \), a verifier asks for a witness of the form \( |\psi\rangle \), to which she adds a few ancillas and verifies it with a quantum circuit \( V \). Does anything change, if she demands that the witness must have a few qubits that are \textit{pinned} to some fixed state? No, as the verifier can ask for all but the pinned qubits of the witness, supply those pinned qubits on her own, and verify the whole state as before.

Rather straightforwardly, we can show that problems in the class \( QMA \) can be verified using \( \text{Pinned QMA} \) and vice versa, so that \( \text{Pinned QMA} = QMA \). If we ask for a \textit{pinned} proof of the form \( |\psi'\rangle = |\psi\rangle |0\rangle \), with one pinned qubit, the extra demand does not increase the complexity of the problem. If the verifier that asks for \( |\psi'\rangle = |V\rangle \), the same thing can be verified in \( QMA \) with a modified circuit \( V \) which adds one more ancilla that stores a check of whether the pinned qubit is really \( |0\rangle \), and then does the verification \( V' \), accepting only if both are accepted. Thus, \( \text{Pinned QMA} \) can be verified in \( QMA \). On the other hand, for any \( QMA \) verifier circuit \( W' \) that demands a witness \( |\phi'\rangle \), there exists a pinned version, which demands a witness \( |\phi\rangle = |\phi'\rangle |0\rangle \) with one extra qubit, and whose verifier circuit \( W \) simply disregards the pinned qubit and verifies only the \( |\phi'\rangle \) part with \( V' \).

However, things are not quite as straightforward when instead of \( QMA \) witnesses we start pinning qubits of low energy states for the \textit{Local Hamiltonian problem}. Let us consider the \( \text{QMA-complete} \) problem \textit{Local Hamiltonian (LH)}, and investigate the pinning requirement. Imagine we look at a Hamiltonian \( H' \), and ask if there exists a low energy state of the form \( |\psi\rangle = |\psi\rangle |0\rangle \). We call this problem \( \text{Pinned LH} \).

Definition 1 (The \( p \)-Pinned \( k \)-Local Hamiltonian Problem). Consider a \( k \)-local Hamiltonian \( H \) for a system of size \( n \), a \( p \)-qubit state vector \( |\phi\rangle \), with \( p = \text{poly}(n) \) and two energy bounds \( b, a \), such that \( b - a \geq 1/\text{poly}(n) \). You are promised that either:

YES There exists an \( n-1 \) qubit state vector \( |\psi\rangle \), such that the energy of the \( n \)-qubit state vector \( |\psi\rangle |\phi\rangle \) with respect to \( H \) is at most \( a \), or

NO for any state vector \( |\psi\rangle \), the energy of the \( n \)-qubit state vector \( |\psi\rangle |\phi\rangle \) with respect to \( H \) is at least \( b \).

Decide, which is the case.

We will prove the following theorem:

Theorem 2 (\( QMA \) complete of the Pinned \( k \)-Local Hamiltonian Problem). \textit{The Pinned \( k \)-Local Hamiltonian Problem is \( QMA \) complete.}

\textbf{Proof.} First, on the one hand, \( \text{Pinned LH} \) is no easier than \( LH \), because for any local Hamiltonian \( H \), we can choose choose \( |\phi\rangle = |0\rangle \) and set up

\[ H' = H \otimes 1. \]  

(1)

There exists a low-energy state of \( H' \) of the form \( |\psi\rangle |0\rangle \) if and only if there exists a low-energy state vector \( |\psi\rangle \) of \( H \). Thus, \( \text{Pinned LH} \) is \( QMA \)-hard as solving it allows one to solve the \( LH \) Problem. On the other hand, observe that \( \text{Pinned LH} \) belongs to \( QMA \). We can set up a quantum verifier that receives the witness \( |\psi\rangle \), adds its own single-qubit state vector \( |\phi\rangle \),
and then tests whether the state vector $|\psi\rangle\langle\phi|$ has low enough energy for the Pinned Local Hamiltonian $G'$. In summary, Pinned LH is QMA-complete.

This could be the end of the proof. However, one might desire more details in order to understand how to translate the energy bounds between these problems. We can explicitly set up the LH problem to contain Pinned LH for example as follows. Let us construct a Local Hamiltonian $G$, which has a low-energy state if and only if a Pinned LH $G'$ has a low-energy state vector of the form $|\psi\rangle|\phi\rangle$. Without loss of generality, we can again take $|\phi\rangle = |0\rangle$, by a local basis transformation on the operators acting on the last qubit.

Let us then set up a Local Hamiltonian $G$ retaining the properties of a pinned $G'$ by penalizing the additional qubit with energy $\Delta > 0$ if it is not in the desired pinned state vector $|0\rangle$,

$$G = G' + \Delta \mathbb{1} \otimes |1\rangle\langle 1|.$$

(2)

If there exists a state vector of the form $|\psi\rangle|0\rangle$ for the Pinned Local Hamiltonian $G'$ with energy $E_{G',0} \leq a$, the same state will also have a “low” energy for the local Hamiltonian $G$,

$$\langle 0|\langle \psi|G|\psi\rangle|0\rangle \leq a.$$

(3)

On the other hand, if it is the case that any state vector of the form $|\psi\rangle|0\rangle$ has energy at least $E_{G',0} \geq b$, then taking a general state vector,

$$|S\rangle = (\cos \varphi)|\psi_0\rangle|0\rangle + (\sin \varphi)|\psi_1\rangle|1\rangle,$$

we can show that the ground state energy of the local Hamiltonian $G$ obeys

$$E_S = \langle S|G|S\rangle$$

$$= (\cos^2 \varphi)\langle 0|\langle \psi_0|G'|\psi_0\rangle|0\rangle$$

$$+ (\sin^2 \varphi) \left[ (1|\langle \psi_1|G'|\psi_1\rangle|1\rangle + \Delta) \right]$$

$$+ (\cos \varphi \sin \varphi) \left[ (1|\langle \psi_1|G'|\psi_0\rangle|0\rangle + c.c.) \right]$$

$$\geq b \cos^2 \varphi + \Delta \sin^2 \varphi$$

$$+ (\sin^2 \varphi) (1|\langle \psi_1|G'|\psi_1\rangle|1\rangle$$

$$+ (\sin 2\varphi) \text{Re} \left[ (1|\langle \psi_1|G'|\psi_0\rangle|0\rangle \right]$$

$$\geq b \cos^2 \varphi + (\Delta - \|G'\|) \sin^2 \varphi - \sin 2\varphi \|G'\|.$$

(7)

Let us label $c = \Delta - \|G'\|$ and $d = \|G'\|$ to write

$$E_S \geq b \left( 1 + \cos 2\varphi \right) + \frac{c}{2} \left( 1 - \cos 2\varphi \right) - d \sin 2\varphi.$$

(8)

Assuming $c - b > d > 0$, it is easy to find that the extrema of this expression appear at

$$\tan 2\varphi = \frac{2d}{c - b},$$

producing

$$E_S \geq \frac{1}{2} \left( c + b - \sqrt{(c - b)^2 + (2d)^2} \right).$$

(10)

Let us now set

$$c := \frac{1}{2} \left( b + a + \frac{(2d)^2}{b - a} \right),$$

(11)

i.e.

$$\Delta = c + d = \frac{b + a}{2} + d \left( \frac{2d}{b - a} + 1 \right) = \text{poly}(n).$$

(12)

With basic algebra, recalling $b > a$, we can show that this satisfies $c - \sqrt{(c - b)^2 + (2d)^2} \geq a$, and thus

$$E_S \geq \frac{a + b}{2},$$

which means in the NO instances, the ground state energy will be at least $(a + b)/2$, which is at least an inverse polynomial above the lower bound $a$ in the YES instances. Together with (3), this means we have translated the original problem’s energy bounds to $a' = a$ and $b' = (a + b)/2$, halving the promise gap of the original Pinned LH.

Therefore, we have not really changed the complexity of the general local Hamiltonian problem by the pinning requirement. However, the situation surprisingly changes when we start thinking about Hamiltonians whose terms come from a restricted class, as we will show in the following sections.

**B. Pinned Commuting Local Hamiltonian**

Pinning a qubit effectively projects into a subspace of the entire Hilbert space. When the original Hamiltonian comes with some restrictions, these may be lifted after this projection. Here and in the following sections, we investigate such cases. First, we claim that pinning a qubit for a Commuting Local Hamiltonian and asking about the lowest possible energy of such a state is as difficult as asking about the ground state energy of an generic local Hamiltonian.

**Theorem 3** (QMA-completeness of the Pinned Commuting 3-local Hamiltonian problem). The Pinned Commuting 3-local Hamiltonian problem is QMA-complete.

The Pinned Commuting $k$-local Hamiltonian problem is defined analogously to Definition 1, with an additional condition: the Hamiltonian’s terms commute with each other. Let us prove it is QMA-complete.

**Proof.** First, note that the Pinned Commuting $k$-local Hamiltonian problem is in QMA, just as Pinned LH is. The harder direction is to show that commuting terms plus pinning can result in complexity equal to the case of unrestricted local Hamiltonians. Thanks to Ref. [35], we know that the 2-local Hamiltonian problem made from $Z$, $X$, $ZZ$, and $XX$ terms is QMA-complete. Let us take such a Hamiltonian and split it into two groups, one made from $ZZ$ and $Z$ terms, and the other made from $XX$ and $X$ terms. The terms within each group commute with each other. Let $H = \sum_i A_i + \sum_j B_j$ be such a non-commuting $k$-local Hamiltonian, where in the
group $A = \sum_i A_i$, all the $A_i$ commute with each other, and in $B = \sum_j B_j$, all the terms $B_j$ commute with each other. Assume the Local Hamiltonian promise problem for this $H$ has energy bounds $b$ and $a$. Let us now add another qubit to the system, and modify the terms to

$$A'_i = A_i \otimes \frac{1}{2} (|+\rangle \langle +|)_{n+1} = A_i \otimes |+\rangle \langle +|_{n+1}, \quad (14)$$

$$B'_j = B_j \otimes \frac{1}{2} (|-\rangle \langle -|)_{n+1} = B_j \otimes |-\rangle \langle -|_{n+1},$$

similarly to the approach taken in Ref. [46]. These terms form a fully commuting, $(k + 1)$-local Hamiltonian $H' = \sum_i A'_i + \sum_j B'_j$. How much power would we have if we could figure out whether $H'$ has a low-energy state vector of the form $|\psi\rangle|0\rangle$? Notice on the one hand that when we pin the last qubit to the state vector $|0\rangle$, the expectation values of the $A'_i$’s and $B'_j$’s become

$$\langle 0|\langle \psi| A'_i |\psi\rangle|0\rangle = \frac{1}{2} \langle \psi | A_i |\psi\rangle, \quad (15)$$

$$\langle 0|\langle \psi| B'_j |\psi\rangle|0\rangle = \frac{1}{2} \langle \psi | B_j |\psi\rangle. \quad (16)$$

Thus, if the original $k$-local $H$ has a ground state vector $|\psi\rangle$ with energy $a$, the state vector $|\psi\rangle|0\rangle$ will have energy $a/2$ for the new commuting Hamiltonian $H'$, as $\langle 0|\langle \psi| H'|\psi\rangle|0\rangle = \frac{1}{2} \langle \psi | H |\psi\rangle$. On the other hand, if the energy of any state vector $|\psi\rangle$ for the Hamiltonian $H$ is at least $b$, the energy of any state vector $|\psi\rangle|0\rangle$ for the new commuting Hamiltonian $H'$ is at least $b/2$.

Therefore, if one could solve a Pinned Commuting $(k + 1)$-Local Hamiltonian problem on $n + 1$ qubits, with promise $\frac{b}{2} \leq \frac{a}{2}$, one could use this to solve a $k$-Local Hamiltonian problem (made from two commuting groups of terms) on an $n$ qubit systems, with promise bounds $b, a$. As the original problem is QMA-hard for $k = 2$, we have thus proven that 3-local Pinned Commuting Local Hamiltonian is QMA-complete.

### C. Pinned Stoquastic Local Hamiltonian

Let us look at another restricted class – stoquastic Hamiltonians with non-positive off-diagonal terms. For such Hamiltonians an important obstacle to classical simulation via Quantum Monte Carlo – the sign problem – does not arise [58]. The local Hamiltonian for stoquastic Hamiltonians defines the complexity class StoqMA [36], which is believed to be strictly smaller than QMA for the above reason. In particular, stoquastic Hamiltonians are not thought to be universal for quantum computing. What happens when we pin some of the qubits of such Hamiltonians? We show the following.

**Theorem 4** (QMA-completeness of the Pinned Stoquastic 3-Local Hamiltonian problem). The Pinned Stoquastic 3-Local Hamiltonian problem is QMA-complete.

A different viewpoint of this problem is given in Ref. [51], where the authors show universality of adiabatic evolution in the highest excited state of a stoquastic Hamiltonian, and the QMA hardness of lower bounding the highest energy of such a Hamiltonian.

**Proof.** As in the proof of Theorem 3, we start with observing that Pinned stoquastic $k$-local Hamiltonian is in QMA, because Pinned $LH$ is in $QMA$. We will now show that looking at the ground state energy of a Hamiltonian with stoquastic terms with pinning a qubit results is as hard as for a general local Hamiltonian.

Let us start with an instance of the QMA complete problem Local Hamiltonian. For each such Hamiltonian $H$, we can write another using only stoquastic terms, in order to deal with possible positive off-diagonal elements in $H$. For this, we will divide $H = \hat{O} + \hat{P}$ into local terms $\hat{O}$ which are diagonal or have negative off-diagonal elements, and local terms $\hat{P}$ with positive off-diagonal elements. Let us replace the latter with stoquastic terms as follows. First, add an extra qubit $q$ in a state vector $|\phi\rangle = (\langle 0| - |1\rangle)/\sqrt{2}$ to the system. Second, modify each term $\hat{P}$ by attaching the operator $X_q$ and change its sign, generating a new, stoquastic Hamiltonian $H' = \hat{O} \otimes I - \hat{P} \otimes X_q$. When we then look at state vectors of the form $|\phi\rangle|\rangle$, the expectation values of the modified Hamiltonian will be

$$\langle \langle \langle \phi| H'|\phi\rangle|\rangle = \langle \langle \langle \phi| \hat{O} \otimes I - \hat{P} \otimes X_q |\phi\rangle|\rangle = \langle \langle \langle \phi| H |\phi\rangle|\rangle. \quad (17)$$

The expectation value of a pinned state vector $|\phi\rangle|\rangle$ for the stoquastic $H'$ is the same as for the state vector $|\phi\rangle$ and the original Hamiltonian $H$.

In more detail, let us start with the QMA-complete 2-local Hamiltonian made from terms $X, Z, X \otimes X, Z \otimes Z,$ and $X \otimes Z$ [35]. First, we will change each term of the $X$ type with a positive prefactor $x_{\alpha} > 0$ into

$$x_{\alpha}X_a \mapsto -x_{\alpha}X_a \otimes X_q, \quad (19)$$

which is stoquastic. When we pin the qubit $q$ in the state vector $|\rangle\rangle$, the expectation value of the new term in the state vector $|\langle \phi\rangle\rangle$ will be simply $x_{\alpha} \langle \langle \langle \phi| X_a |\phi\rangle|\rangle$, thanks to $\langle \langle \langle X_q \rangle|\rangle = -1$. We can deal with the terms of the type $XX$ with a positive prefactor just as easily. Next, we will look at the terms $X \otimes Z$ in $H$, whose off-diagonal terms have a varying sign. Because we can rewrite $X \otimes Z = X \otimes |0\rangle\langle 0| - X \otimes |1\rangle\langle 1|$, assuming $x_{\alpha, b} > 0$, the corresponding terms in $H'$ will be

$$x_{\alpha, b}X_a \otimes Z_b \mapsto -x_{\alpha, b}X_a \otimes (|0\rangle\langle 0|b \otimes X_q + |1\rangle\langle 1| \otimes I_q), \quad (20)$$

$$-x_{\alpha, b}X_a \otimes Z_b \mapsto -x_{\alpha, b}X_a \otimes (|0\rangle\langle 0|b \otimes I_q + |1\rangle\langle 1| \otimes X_q). \quad (21)$$

Observe that the modified terms are stoquastic, with only negative off-diagonal elements.

Consider now the new stoquastic 3-local Hamiltonian $H'$. What if we can figure out if there exists a low-energy state vector of $H'$ of the form $|\langle \phi\rangle\rangle$? If the original $H$ has a ground
state vector $|\phi\rangle$ with energy $a$, on the one hand, the state vector $|\phi\rangle|\rangle$ will have energy $a$ for the new stoquastic Hamiltonian $H'$. On the other hand, if the energy of any state vector $|\phi\rangle$ for the Hamiltonian $H$ is at least $b$, the energy of any state vector of the form $|\phi\rangle|\rangle$ is at least $b$ for the new commuting Hamiltonian $H'$. Therefore, we have turned a Local Hamiltonian problem with promise $b, a$, into a Pinned Stoquastic Local Hamiltonian with the same promise, with a doubled Hilbert space (adding a qubit), and stoquastic terms that have a local-Hamiltonian structure increased by 1. Solving Pinned Stoquastic Hamiltonian space (adding a qubit), and stoquastic terms that have a local-Hamiltonian structure is thus at least as hard as $LH$, and thus QMA complete. □

Note that in the proof we provided, the type of the terms in $H'$ is different from $H$, as we were only interested in making them stoquastic, not keeping their form. It remains open to analyze what is the hardness of Pinned Stoquastic Hamiltonian with restricted form (e.g., only XXX, ZZZ) or locality below 3. After showcasing the pinning technique in two examples, we will continue exploring how far it takes us, applying it to simpler and simpler original Hamiltonians.

D. Pinned Permutation Hamiltonians

The possibilities opened in the previous sections motivate us to go further and design a classically looking problem about 0/1 permutation matrices that will still be QMA-complete. We claim the following.

**Theorem 5** (QMA-completeness of the Pinned Local Permutation Hamiltonian). Pinned Local Permutation Hamiltonian is QMA-complete.

Note that universality for quantum computation with 0/1 matrices has been previously demonstrated for example in the PromiseBQP string-rewriting problem of Wocjan and Janzing [49], or the universal computation by quantum walk construction of Childs et al. [40].

**Proof.** One direction of Theorem 5 is easy – pinned local permutation Hamiltonian is obviously in QMA. The more difficult part is again to construct QMA-hard instances of pinned 0/1 Hamiltonian. First, we will take a target Hamiltonian made from Pauli matrices, and replace them by 0/1 matrices on a larger Hilbert space, with a technique similar to those of Ref. [51], where it has been used to build QMA-hard instances of stochastic matrices. Second, we will utilize pinning to generate the desired real-valued prefactors for the permutation, and thus also the effective original Pauli terms.

Consider an instance of the QMA-complete, 2-local Hamiltonian problem with a Hamiltonian $H$ made from $X, Z, XX$ and $ZZ$ terms, as in Section II B, with real-valued prefactors. Let us deal with Pauli terms first, and consider the prefactors later. The $X$ and $XX$ terms already are permutation matrices. For the $Z$ and $ZZ$ terms, we will add an ancilla qubit $z$, and transform the interactions as

$$Z \mapsto |0\rangle\langle 0| \otimes Z_z + |1\rangle\langle 1| \otimes X_z,$$  

$$Z \otimes Z \mapsto (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes Z_z + (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes X_z,$$  

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most 2. In step 1, we built an \( n + 1 \) qubit permutation Hamiltonian \( H' \) with locality at most 3, which did not yet include the desired real prefactors. In step 2, we constructed the final permutation Hamiltonian \( H'' \) which works on \( n + Q + 2 \) qubits, and has at most \( 2M \times (Q + 1) \) terms, with locality at most 5. We pinned the ancilla qubits \( z \) and \( q_0 \) into the state vector \(|-\rangle\), and the ancilla qubits \( q_1, \ldots, q_Q \) into the states (25). Determining the lowest energy of the pinned 5-local permutation Hamiltonian \( H'' \), with \( Q + 2 \) pinned qubits, is thus \( QMA \)-hard, as it implies determining the ground state energy of the target local Hamiltonian \( H \), an instance of the \( QMA \)-complete problem Local Hamiltonian. Therefore, the Pinned permutation Hamiltonian problem is \( QMA \)-hard, as well as \( QMA \)-complete.

### III. A DYNAMICAL VIEWPOINT ON PINNING

In the previous section, we looked at how pinning can contribute to the complexity of determining the static properties of local Hamiltonians – the bounds on the energies of states from the pinned subspace. We now turn to a dynamical question, asking what pinning can contribute when applied to an evolution with a local, time-independent Hamiltonian. Imagine we can evolve with a restricted set of interactions (or unitaries), while constantly measuring one qubit in a particular basis, pinning it via the Zeno effect to a particular state. In the circuit model, we know that having access to specific states can greatly enhance the power of a restricted model. For example, a source of magic states is enough to turn computation with Clifford gates into a universal quantum computation [38].

Could pinning have the same type of implication, when used in an evolution with a fixed, restricted type, local Hamiltonian in continuous time? We will now show that building Hamiltonians following a similar strategy as in the previous section together with pinning via the quantum Zeno effect allows us to achieve non-stoquastic evolution with only stoquastic Hamiltonians, as well as evolution with non-commuting Hamiltonians with only commuting terms in the Hamiltonian. We thus get universal quantum computation out of evolution with a restricted set of Hamiltonians together with a fixed Pauli basis measurement of a single qubit.

#### A. Pinned evolution with stoquastic Hamiltonians is computationally universal

Let us delve into the details of applying pinned evolution. Assume we can set up our system to interact with a stoquastic Hamiltonian

\[
H' = A \otimes \mathbb{I}_q + B \otimes X_q,
\]

made from two groups of local, stoquastic terms \( A \) and \( B \), with no positive off-diagonal entries. Furthermore, we demand \( B \) to be entirely off-diagonal. The terms \( B \otimes X_q \) also interact with an ancilla qubit \( q \), similarly to (20). Based on this construction, we will now show:

**Theorem 6 (Universality of Pinned Evolution).** Pinned Evolution with time-independent, local stoquastic Hamiltonians is universal for BQP.

**Proof.** Let us investigate what happens when we initialize the ancilla qubit \( q \) as \(|-\rangle\), and measure it in the \( X \) basis often enough. This likely pins the ancilla qubit to the state vector \(|-\rangle\). Meanwhile, the system wants to evolve with \( H' \), but effectively, we get something different. Let us cut the time evolution into small steps of size \( \delta \to 0 \). The evolution can be approximated as alternating the evolution \( e^{-i\delta H} \) with a projection of the last qubit onto the state vector \(|-\rangle\). It will be helpful to express

\[
\langle -| e^{-i\delta H} |\psi\rangle |q\rangle \approx \langle -| e^{-i\delta A} e^{-i\delta B \otimes X_q} |\psi\rangle |q\rangle
\]

\[
\approx \langle -| (1 - i\delta A - i\delta B \otimes X_q) |\psi\rangle |q\rangle
\]

\[
= 1 - i\delta (A - B) \langle \psi \rangle \approx e^{-i\delta (A - B)} \langle \psi \rangle,
\]

valid up to first order in \( \delta \). This allows us to effectively evolve the state vector \(|\psi\rangle\) with the general, non-stoquastic Hamiltonian \( H = A - B \).

Moreover, because the last qubit is in an eigenstate of \( X_q \), it never gets flipped into the state vector \(|+\rangle\). Thus, taking \( \delta = t/N \), with \( N \to \infty \), we can confidently say that

\[
|\psi(t)\rangle_{PE} = \prod_{j=1}^{N} P_{-} \left( e^{-it(A+B \otimes X_q)} \right) |\psi\rangle
\]

\[
= e^{-it(A-B)} |\psi\rangle + |\delta\rangle,
\]

where \(|\delta\rangle\) is an error state vector with norm of order at most \( \delta = t/N \), i.e., going to zero as \( N \to \infty \). Therefore, we can simulate evolution with unrestricted, time-independent, local Hamiltonians using evolution according to stoquastic Hamiltonians and pinning. This is universal for quantum computation (BQP), when we recall various constructions for universal quantum computation by evolution with a time-independent, local Hamiltonian, see, e.g., Ref. [60].

#### B. Pinned evolution with commuting Hamiltonians

Pinning can induce new types of evolution also for commuting local Hamiltonians. We will show how one can simulate evolution with a non-commuting Hamiltonian \( H = A + B \), made from two groups of terms that commute within the group. For this, we will construct a Hamiltonian

\[
H' = 2A \otimes |+\rangle\langle +|_q + 2B \otimes |-\rangle\langle -|_q
\]

all of whose terms commute, by adding an ancilla qubit \( q \) as in (14). Let us now analyze what happens when we alternate computational basis measurements on the last qubit, initialized as \(|0\rangle\), with evolution according to \( H' \). We will prove the following.

**Theorem 7 (Universality of commuting Pinned Evolution).** Pinned evolution with time-independent, local commuting Hamiltonians is universal for BQP.
Proof. Let us look at a short time interval $\delta = t/N$, with $N \to \infty$. The pinned evolution of the system will be well approximated by the evolution $e^{-i\delta H'}$ according to $H'$ for time $\delta$, and then a measurement in the computational basis. This repeated measurement should on the one hand effectively pin the ancilla qubit $q$ in the state vector $|0\rangle$, as in Vaidman’s bomb-testing procedure [43] in its circuit setting [55]. This is the Zeno effect, explained in detail e.g., in Ref. [61], where we also find that the probability of a “bad” projection (a flip of the $|0\rangle$ to $|1\rangle$ scales as $O(\delta^2)$, and can be made arbitrarily small even after $O(\delta^{-1})$ repetitions. On the other hand, what is the effective evolution of the rest of the system? Let us calculate

\[
e^{-i\delta H'} |\psi\rangle |0\rangle = e^{-i2\delta A \otimes |+\rangle} |+\rangle e^{-i2\delta B \otimes |\rangle} |\rangle |0\rangle |0\rangle (34) \\
\approx (1 - i2\delta A \otimes |+\rangle) |+\rangle - i2\delta B \otimes |\rangle |\rangle |0\rangle |0\rangle (35) \\
\approx |\psi\rangle |0\rangle - i2\delta (A|\psi\rangle |+\rangle + B|\psi\rangle |\rangle) (36) \\
= (1 - i\delta (A + B)) |\psi\rangle |0\rangle - i\delta (A - B)|\psi\rangle |1\rangle (37) \\
\approx e^{-i\delta(A + B)} |\psi\rangle |0\rangle - i\delta (A - B)|\psi\rangle |1\rangle, (38)
\]

correct up to order $\delta$. Therefore, when we now measure the ancilla qubit, we will get the result $0$ and obtain the state $e^{-i\delta(A + B)} |\psi\rangle$, with probability $1 - O((\delta \|A - B\|)^2)$. Moreover, the state can also contain an error vector with norm $O(\delta \|A + B\|^2)$, as the evolution (34) with commuting terms cannot produce mixed terms such as $AB$, while (38) does include them in its series expansion.

What happens when we repeat this evolve-measure procedure $N = t/\delta$ times? We end up with the state vector $e^{-i\delta(A + B)} |\psi\rangle$, with an error vector of norm $O(t\delta \|A + B\|^2)$, while the probability that all the $N$ measurements of the pinned ancilla qubit result in $|0\rangle$ is lower bounded by $1 - O(t\delta \|A - B\|^2)$. Therefore, we can simulate evolution with unrestricted (non-commuting) Hamiltonians using commuting Hamiltonians and pinning. Starting with a universal local Hamiltonian built from two groups of commuting terms as in Section II.B, this directly translates into the statement of the theorem: pinned evolution with commuting local Hamiltonians is universal for quantum computation.

IV. GROUND STATE CONNECTIVITY

Our original motivation for exploring pinning was to understand better the variants of Gharibian and Sikora’s Ground State Connectivity (GSCON) problem [45]. It asks about the possibility of traversing the low-energy subspace of a local Hamiltonian from one specific ground state to another, using local unitary transformations. Gosset, Mehta and Vidick [46] have shown that the problem remains QCMA complete even if only commuting Hamiltonians are used. In their proof, they use a trick similar to pinning – combining the original Hamiltonian’s terms with projections on ancilla qubits to make the terms commute. Then they demand that the initial and final ground state have a few qubits in a specific state – which means that the original non-commuting Hamiltonian’s terms are effectively applied. Moreover, this has to be combined with the impossibility of a simple flip of this state without a computation being verified first. Nevertheless, it helped us realize that the GSCON formulation allows one to essentially fix some part of the ground state, adding extra power to restricted forms of Hamiltonians.

Therefore, using techniques similar to Ref. [46], hardness results for pinned local Hamiltonians should be translatable to hardness of GSCON for similarly restricted Hamiltonians. For example, we will be able to show QCMA-hardness of GSCON for stoquastic Hamiltonians thanks to Theorem 6 and Ref. [46]. Moreover, in this context we will also provide some evidence into the free-fermionic variant of GSCON, to be further developed in future work.

A. Stoquastic GSCON

First, we will show how to build on the proof that the Ground State Connectivity (GSCON) problem is QCMA-complete for commuting Hamiltonians, as well as on universality of pinned stoquastic LH, and prove that Stoquastic GSCON is QCMA-complete. The statement of the problem is identical to the Commuting GSCON problem in Ref. [46], the only difference being the replacement of the word “commuting” by “stoquastic”. We thus have:

Definition 8 (Stoquastic Ground State Connectivity \((H, k, \eta_1, \eta_2, \eta_3, \Delta, I, m, U_\psi, U_\phi)\)).

Input:

1. $k$-local Hamiltonian $H = \sum_i H_i$ with stoquastic terms (i.e. with no positive off-diagonal elements), satisfying $\|H_i\| \leq 1$.
2. $\eta_1, \eta_2, \eta_3, \Delta \in \mathbb{R}$, and integer $m \geq 0$, such that $\eta_2 - \eta_1 \geq \Delta$ and $\eta_4 - \eta_3 \geq \Delta$.
3. Polynomial size quantum circuits $U_\psi$ and $U_\phi$ generating “starting” and “target” state vectors $|\psi\rangle$ and $|\phi\rangle$ starting from the $|0\rangle^\otimes n$ state, respectively, satisfying $\langle \psi| H |\psi\rangle \leq \eta_1$ and $\langle \phi| H |\phi\rangle \leq \eta_1$.

Output:

1. If there exists a sequence of $l$-local unitaries $(U_i)_{i=1}^m$ in $U$ such that

   (a) (Intermediate states remain in low energy space) For all $i \in [m]$ and intermediate states $|\psi_i\rangle := U_i \cdots U_2 U_1 |\psi\rangle$, one has $\langle \psi_i| H |\psi_i\rangle \leq \eta_1$, and

   (b) (Final state close to target state) $\|U_m \cdots U_1 |\psi\rangle - |\phi\rangle\| \leq \eta_3$,

   then output YES.

2. If for all $l$-local sequences of unitaries $(U_i)_{i=1}^m$, either:

   (a) (Intermediate state obtains high energy) There exists $i \in [m]$ and an intermediate state vector $|\psi_i\rangle := U_i \cdots U_2 U_1 |\psi\rangle$, such that $\langle \psi_i| H |\psi_i\rangle \geq \eta_2$, or
(b) (Final state far from target state) 
\[ \| U_m \cdots U_1 |\psi\rangle - |\phi\rangle \| \geq \eta_s, \]
then output NO.

There is not that much that we need change in the proof of Theorem 6 in Ref. [46], when we want to build a generic effective Hamiltonian from stoquastic instead of commuting terms, using “pinning” thanks to a restriction on the initial and final states, as well as the form of the Hamiltonian that we construct.

**Theorem 9** (QCMA-completeness of the Stoquastic Ground State Connectivity Problem). The Stoquastic Ground State Connectivity Problem is QCMA-complete.

**Proof.** It is straightforward to see that the Stoquastic GCON is in QCMA, with a witness encoding the sequence of unitaries, verifiable by a quantum computation. For the other direction, we are directly inspired by the proof of QCMA-completeness of Commuting GCON [46]. There, the authors split a target generic (non-commuting) local Hamiltonian \( G = A + B \) into two groups of local commuting terms, add two 3-qubit ancilla registers, and set up the commuting Hamiltonian

\[ A \otimes \Pi_S \otimes \Pi_+ + B \otimes \Pi_S \otimes \Pi_+ + I \otimes I \otimes \Pi_S, \tag{39} \]

where \( \Pi_S \) projects onto \( S = \text{span}\{ |0,0,0\rangle, |1,1,1\rangle \} \), and \( \Pi_+ \) are projectors onto \( \{ |0,0,0\rangle \pm |1,1,1\rangle \} / \sqrt{2} \). The QCMA-hard GCON question concerns the possible low-energy traversal from the state vector \( |0\rangle^{\otimes n} |1\rangle^3 |0\rangle^3 \) to the state vector \( |0\rangle^{\otimes n} |0\rangle^3 |0\rangle^3 \) by 2-local operations. This is possible by using the first \( n \)-qubit register to prepare a low-energy witness for the Hamiltonian \( G = A + B \). This effectively “turns off” the first two terms in (39), allowing one to flip the middle register to \( |1,1,1\rangle \) by 2-local operations without a high energy cost. Finally, one uncomputes the first register. Meanwhile, the last register stays “pinned” in \( |0,0,0\rangle \), making sure both groups of terms \( A \) and \( B \) are in play and contribute significantly to the energy of the intermediate states. For more details, see the proof of Theorem 6 of Ref. [46].

Let us then work out the stoquastic version of this. We start with an \( n \)-qubit register, and the target generic, non-stoquastic, 2-local, \( n \)-qubit Hamiltonian \( H \) made from \( ZZ, ZX, XX, Z \) and \( X \) terms. The Local Hamiltonian problem for this variant of \( H \) is QMA-complete. The GCON problem based on \( H \) is thus QCMA-complete.

We will construct a stoquastic GCON Hamiltonian \( H'' \) similarly to (39), with a few important differences. First, let us define two operators

\[ Q = \frac{1}{3} \left( X_{q_1} + X_{q_2} + X_{q_3} \right), \tag{40} \]

an analogue of \( X_q \) from Section II C, effectively flipping the sign when the ancilla register is in the state vector \( |\cdot\rangle^3 \), and

\[ R_3 = \frac{3}{4} I - \frac{1}{4} \left( X_{q_1} X_{q_2} + X_{q_2} X_{q_2} + X_{q_3} X_{q_3} \right), \tag{41} \]

a 2-local, stoquastic operator equivalent to the projector onto the space orthogonal to the span of \( |\cdot\rangle^3 \) and \( |+\rangle^3 \).

Second, let us add a 3-qubit ancilla register and combine the original Hamiltonian \( H \) with the operator \( R_3 \) as \( H' = H \otimes R_3 \). Similarly to Section II C, we can split this local Hamiltonian \( H' \) acting on \( n + 3 \) qubits into groups of local terms \( H' = \hat{O}' + \hat{P}' \), with non-positive off-diagonal terms \( \hat{O}' \) and a group of strictly off-diagonal local terms with positive elements \( \hat{P}' \).

Finally, we combine the group \( \hat{P}' \) with the operator \( Q \) on the final ancilla register, in order to ensure that \( -\hat{P}' \otimes Q \) is stoquastic, with strictly negative off-diagonal elements, as \( \hat{P}' \otimes Q \) is a tensor product of two operators which each have strictly off-diagonal elements and no diagonal elements. Altogether, we arrive at the local, stoquastic Hamiltonian

\[ H'' = \hat{O}' \otimes I - \hat{P}' \otimes Q + I \otimes R_3. \tag{42} \]

Observe that for the state vectors of the form \(|\psi\rangle|\cdot\rangle^3 |\cdot\rangle^3 \) and \(|\psi\rangle|+\rangle^3 |\cdot\rangle^3 \), the expectation value of \( H'' \) is zero. Meanwhile, when the middle register is in an \( X \)-basis state vector \( |x_1, x_2, x_3\rangle \) other than \( |\cdot\rangle^3 \) or \( |+\rangle^3 \), and the last register remains in \( |\cdot\rangle^3 \), the expectation value

\[ \langle \psi | (x_1, x_2, x_3) (-|\cdot\rangle^3 H'' |\psi\rangle) |x_1, x_2, x_3\rangle |\cdot\rangle^3 \]

\[ = \langle \psi | (x_1, x_2, x_3) (\hat{O}' + \hat{P}') |\psi\rangle |x_1, x_2, x_3\rangle \]

\[ = \langle \psi | (x_1, x_2, x_3) H \otimes R_3 |\psi\rangle |x_1, x_2, x_3\rangle = \langle \psi | H |\psi\rangle \]

is equivalent to the expectation value of the original non-stoquastic Hamiltonian \( H \) acting on \(|\psi\rangle\), thanks to

\[ (x_1, x_2, x_3) R_3 (x_1, x_2, x_3) = 1. \tag{45} \]

The hard ground space traversal question we ask is then: Decide, if starting in the state vector \(|0\rangle^{\otimes n} |\cdot\rangle^3 |\cdot\rangle^3 \), one can traverse the low-energy subspace of \( H'' \) without energy above \( \alpha \) (where this bound comes from the QCMA-complete LH problem with energy bounds \( \alpha \) and \( \beta \) and at most \( \eta_3 \) far from the state \(|0\rangle^{\otimes n} |+\rangle^3 |\cdot\rangle^3 \)), using a sequence of 2-local unitaries of length polynomial in \( n \), or whether one must exit at least \( \eta_4 \) far from the final state, or some of the intermediate states have energy at least \( \eta_2 \)?

Showing completeness is straightforward with the following sequence of transformations. Note the third register stays in \( |\cdot\rangle^3 \) throughout the process. First, we prepare the low-energy witness for \( H \) in the first register. The energy is zero during this process. Second, we flip the second register from \(|\cdot\rangle^3 \) to \( |+\rangle^3 \), qubit by qubit. In this process, the energy of the states is at most \( \alpha \), thanks to (44). Finally, we uncompute the first register, keeping the energy zero.

For soundness, one can directly follow [46] to show that no sequence of 2-local unitaries will satisfy well enough the two conditions – end near enough the final state and stay low enough in energy throughout the sequence. The lower bound on the energy of the intermediate states if one is to end up close to the final state is in this case \( \eta_2 = \Omega \left( \beta^2 / m^4 \right) \), just as in the proof of Soundness of Theorem 10 in Ref. [46], where \( \langle \psi | H |\psi\rangle \geq \beta \) is the bound in the NO case of the original LH.
problem and $m$ is the number of unitaries in the sequence. One has only to replace
\begin{align}
P_0 = |0, 0, 0\rangle\langle 0, 0, 0| &\quad \rightarrow \quad |\rangle\langle -|^{\otimes 3}, \quad (46) \\
P_1 = |1, 1, 1\rangle\langle 1, 1, 1| &\quad \rightarrow \quad |+\rangle\langle +|^{\otimes 3}, \quad (47) 
\end{align}
and follow the proof. \hfill \square

Observe that in the NO case, to obtain soundness, an efficient (poly-length) sequence of 2-local transformations keeping the energy of intermediate states low enough simply could not exist, and this was guaranteed by the lower bound from the Small Projection Lemma 8 [46]. Would this be also true in other settings besides history state preparation connected to QCMA-complete problems? We ask this question about quantum memories, e.g., based on the toric code, in forthcoming work.

### B. Ground state connectivity for free fermions

In the context of studies of Majorana fermionic quantum memories, variants of GSCON for free fermions are particularly interesting [34, 57]. Here we provide insights that we expect to be helpful in tackling this version of the problem relevant when assessing Majorana fermionic quantum memories: we provide evidence that between any pair of low-energy free-fermionic states, there exists a local free-fermionic circuit that interpolates between them within the low-energy subspace. Before we get there, let us define the Free Fermionic Ground State Connectivity Problem, though. Note also that our discussion of the free-fermionic problem does not rely on pinning, but complements our understanding of GSCON in a practically relevant setting.

**Definition 10** (Free Fermionic Ground State Connectivity $(H, k, \eta_1, \eta_2, \eta_3, \eta_4, \Delta, l, m, U_\psi, U_\phi)$).

1. Input parameters:
   - (a) $k$-local free fermionic Hamiltonian $H = \sum_i H_i$ acting on $n$ fermionic modes with each $H_i$ being supported on no more than $k$ modes, satisfying $\|H_i\| \leq 1$.
   - (b) $\eta_1, \eta_2, \eta_3, \eta_4, \Delta \in \mathbb{R}$, and integer $m \geq 0$, such that $\eta_2 - \eta_1 \geq \Delta$ and $\eta_4 - \eta_3 \geq \Delta$.
   - (c) Polynomial size fermionic Gaussian quantum circuits $U_\psi$ and $U_\phi$ generating “starting” and “target” fermionic Gaussian state vectors $|\psi\rangle$ and $|\phi\rangle$ (starting from the fermionic vacuum), respectively, satisfying $\langle \psi | H | \psi \rangle \leq \eta_1$ and $\langle \phi | H | \phi \rangle \leq \eta_1$.

2. Output:
   - (a) If there exists a sequence of $l$-local unitaries $(U_i)_{i=1}^m \in U$ supported on $m$ modes each such that

   i. (Intermediate states remain in low energy space) For all $i \in [m]$ and intermediate states $+ |\psi_i\rangle := U_i \cdots U_2 U_1 |\psi\rangle$, one has $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$, and
   
   ii. (Final state close to target state) $\|U_m \cdots U_1 |\psi\rangle - |\phi\rangle\| \leq \eta_3$, then output YES.

   (b) If for all $l$-local sequences of unitaries $(U_i)_{i=1}^m$, either:

   i. (Intermediate state obtains high energy) There exists $i \in [m]$ and an intermediate state vector $|\psi_i\rangle := U_i \cdots U_2 U_1 |\psi\rangle$, such that $\langle \psi_i | H | \psi_i \rangle \geq \eta_2$, or
   
   ii. (Final state far from target state) $\|U_m \cdots U_1 |\psi\rangle - |\phi\rangle\| \geq \eta_4$, then output NO.

Here, we do not assess the hardness of the Free Fermionic GSCON problem. We conjecture that in contrast to the general case, in free fermions there will always exist a local low-energy path between any pair of low-energy quantum states.

**Conjecture 1** (Free Fermionic Ground State Connectivity). For any free fermionic Hamiltonian $H$ and any pair of low-energy Gaussian fermionic states $|\psi\rangle, |\phi\rangle$ there exists a 2-local finite Gaussian fermionic circuit interpolating between them such that all intermediate states satisfy the energy constraint.

We here provide evidence in favour of this conjecture. Let us denote the fermionic covariance matrix of the initial state vector $|\psi\rangle$ with $\gamma$ (in the conventions of Ref. [42]), and with $\omega$ the covariance matrix of the final state vector $|\phi\rangle$. For $n$ modes, this is a real $2n \times 2n$ matrix satisfying $\gamma = -\gamma^T$ (as is the case for any covariance matrix) and $\gamma^T \gamma = 0$ (reflecting purity). The application of Gaussian fermionic gates to achieve $|\psi_i\rangle = U_i \cdots U_2 U_1 |\psi\rangle$ corresponds to a transformation

\[
\gamma_i := O_i \cdots O_2 O_1 \gamma \gamma^T O_2^T \cdots O_i^T
\]

with $O_i \in SO(2n)$ for all $i$, on the level of covariance matrices. In the Free Fermionic Ground State Connectivity Problem, the initial covariance matrix can be written as

\[
\gamma = O \gamma_0 O^T
\]

with $O \in SO(2n)$ and either

\[
\gamma_0 = \bigoplus_{j=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

or

\[
\gamma_0 = \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bigoplus \bigoplus_{j=1}^{n-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)
\]

depending on having even or odd parity. Turning to Hamiltonians, energy expectation values are computed as

\[
\langle \psi | H | \psi \rangle = \text{tr}(\gamma \mathbf{h}),
\]
with $h = -h^T$. For a local Hamiltonian $H = \sum_i H_i$, each of the terms $H_i$ will correspond to a matrix $h_i = -h_i^T$ with $||h_i|| \leq 1$ that is a zero matrix except a $2k \times 2k$ block, since each $h_i$ acts on $k$ modes only. The Hamiltonian matrix $h$ can without loss of generality be assumed to be $2 \times 2$ block diagonal, as any special orthogonal transformation to bring it into this form can be absorbed in the $O$ of the initial covariance matrix. The attainable energy expectations can be computed from the reachable set

$$\{P(O\gamma_0 O^T) : O \in SO(2n)\},$$

where $P$ is the projection onto $2 \times 2$ block diagonal form. By virtue of the analog of the Schur-Horn theorem for skew-symmetric matrices [56], it becomes clear that within both the even and the odd parity sectors, the reachable set are all $2 \times 2$ skew-symmetric real block diagonal matrices for even and odd parity, respectively. As a consequence of that, there is a parametrized curve $t \mapsto O(t)$ for $t \in [0, 1]$ with $O(t) \in SO(2n)$ for all $t$ so that

$$\gamma = O(0)\gamma_0 O(0)^T$$

and

$$\omega = O(1)\gamma_0 O(1)^T$$

so that

$$\text{tr}(O(t)\gamma_0 O(t)^T h) = (1 - t)\text{tr}(\gamma h) + t\text{tr}(\omega h).$$

That is to say, one can linearly interpolate between the initial and final energy values. One can then chop the linear interpolation into a finite number $N$ steps, each of which is characterized by an orthogonal matrix in $SO(2n)$ close in operator norm to the identity. What is more, following the special orthogonal fermionic analog of the decomposition of Ref. [66], this transformation can be exactly decomposed into a an $O(n^2)$ sized circuit of 2-local fermionic Gaussian quantum gates that are also close to the identity. The so obtained discrete local fermionic circuit $\prod_{t=1}^{N(t)} O_t$, therefore remains close to the continuous curve $O(t)$ for all $t \in [0, 1]$. This implies that the energy along this circuit cannot deviate too much from the initial and final value. By increasing the value of $N$ we can push this deviation down arbitrarily far so as to satisfy the energy constraint throughout the path, providing evidence for our conjecture. We leave the details of this interesting problem relevant for practical quantum memories with Majorana fermions for future work.

V. DISCUSSION

Although pinning might seem simple technically, it has a variety of contexts where it applies naturally. It exemplifies the mathematical question of Hamiltonian purification [64], which we generalize here to a variety of contexts (not only to commuting Hamiltonians, but also stoquastic ones, permutation Hamiltonians and other restricted Hamiltonians). Here, we used it to demonstrate several results in Hamiltonian complexity, while raising questions about the static (complexity) and dynamical (evolution and universality) implications of a special type of control on a small subsystem. Let us now discuss a few observations that we made during our investigation of pinning.

First, quantum perturbation gadgets that have been used in Hamiltonian complexity for a long time ever since [52], are also based on a form of pinning – effectively fixing part of a system into a subspace by providing a large energy penalty to the orthogonal subspace. They can result in an effective Hamiltonian with multiplicatively combined, higher-locality terms, thanks to the form of the perturbative expansion of the Hamiltonian’s self-energy. On the other hand, pinning as we view it here is a geometrical restriction on a part of a system. First of all, it is not perturbative, and second, it can effectively generate only linear and not multiplicative combinations of operators. Therefore, it does not allow one to combine operators to increase the effective locality of terms, which perturbative gadgets are designed to do. On the contrary, we need $k + 1$ local terms in a pinned Hamiltonian to get an effective $k$-local Hamiltonian. In particular, to show that Pinned Commuting 3-LH is QMA complete in Section II B, we have turned a 2-Local Hamiltonian problem with promise $b, a$, into a pinned version with a doubled Hilbert space by adding a qubit. Moreover, the newly formed up to 3-local and commuting terms have the form $Z, X, ZX, XX, ZZ, ZZX, XXX$. However, is the increase in locality essential? The complexity of Pinned 2-Local Commuting Hamiltonian remains open. Straightforward attempts mimicking perturbation gadgets to generate effective interactions with higher locality do not work. Similarly, we have shown in Section II C that the Pinned Stoquastic 3-LH is QMA complete. However, it remains open to figure out how hard the Pinned Stoquastic 2-LH problem is. One way to go could be to show that 2-LH with $\pm ZZ, \pm XX, \pm X, \pm Z$ terms is QMA-complete.

Second, our reason for investigating pinning was its application to Hamiltonians with a restricted form. Could pinning be “forced” with such restricted terms? Sometimes, as in the application to GSCON, there exist operators with the desired form, which energetically penalize a subspace. For example, in Section IV A, we wrote down the stoquastic operator (41) that works as a projector onto the complement of $|\pm\rangle^3$ and $|+\rangle^3$, or in Ref. [46], where a 3-local projector has the required form commuting with the rest of the Hamiltonian. However, in other situations we can not do this. For example, we can not energetically prefer the state vector $|\rangle$ of a qubit by stoquastic terms, as that would imply QMA-completeness of the Stoquastic LH problem, which is considered unlikely. Thus, we require pinning as an external condition in the Pinned Stoquastic LH problem. Similarly, we added dynamical pinning based on repeated measurements in Section III as an external resource, and not directly as a part of the Hamiltonian. Third, it would be interesting to see whether pinning for some restricted models could result in intermediate complexity (e.g., completeness for transverse Ising models), as classified in Ref. [41].

Fourth, we hope that dynamical pinning based on extra con-
trol (repeated measurements) of a single qubit, described in Section III, with a fixed interaction Hamiltonian of a restricted form, could be readily implemented in today’s experimental settings. Finally, it is also our hope that the present work can substantially contribute to the growing body of solutions to problems in Hamiltonian complexity beyond assessing the computational complexity of approximating ground state energies, signifying the richness of the field.

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[1] R. Barends, J. Kelly, A. Megrant, A. Veitia, Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. O’Malley, P. Roushan, A. Vainsencher, J. Wenner, A. N. Korotkov, A. N. Cleland, and J. M. Martinis. Superconducting quantum circuits at the surface code threshold for fault tolerance. Nature, 508:500–5003, 2014.

[2] J. D. Biamonte and P. J. Love. Realizable Hamiltonians for universal adiabatic quantum computers. Physical Review A, 78:012352, 2008.

[3] S. Bravyi, D. P. DiVincenzo, R. I. Oliveira, and B. M. Terhal. The complexity of stoquastic local Hamiltonian problems. June 2006. arXiv: quant-ph/0606140.

[4] S. Bravyi and M. Hastings. On Complexity of the Quantum Ising Model. Communications in Mathematical Physics, 349:1–45, 2017.

[5] S. Bravyi and A. Kitaev. Universal quantum computation with ideal Clifford gates and noisy ancillas. Physical Review A, 71, 2005. arXiv: quant-ph/0403025.

[6] D. K. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio, and K. Yuasa. Exponential rise of dynamical complexity in quantum computing through projections. Nature Communications, 5:5173, 2014.

[7] A. M. Childs, D. Gosset, and Z. Webb. Universal computation by multiparticle quantum walk. Science, 339:791–794, 2013.

[8] T. S. Cubitt, A. Montanaro, and S. Piddock. Universal quantum Hamiltonians. Proceedings of the National Academy of Sciences, page 201804949, 2018.

[9] J. Eisert, V. Eisler, and Z. Zimborás. Entanglement negativity bounds for fermionic gaussian states. Phys. Rev. B, 97:165123, 2018.

[10] A. C. Elitzur and L. Vaidman. Quantum mechanical interaction-free measurements. Foundations of Physics, 23:987–997, 1993.

[11] S. Ghiribian, Y. Huang, Z. Landau, and S. W. Shin. Quantum Hamiltonian complexity. Foundations and Trends in Theoretical Computer Science, 10:159–282, 2015.

[12] S. Ghiribian and J. Sikora. Ground State Connectivity of Local Hamiltonians. In M. M. Halldórsson, K. Iwama, N. Kobayashi, and B. Speckmann, editors, Automata, Languages, and Programming, pages 617–628, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[13] D. Gosset, J. C. Mehta, and T. Vidick. QCMA hardness of ground space connectivity for commuting Hamiltonians. Quantum, 1:16, 2017.

[14] D. Gosset, B. M. Terhal, and A. Vershynina. Universal adiabatic quantum computation via the space-time circuit-to-Hamiltonian construction. Physical Review Letters, 114, 2015. arXiv: 1409.7745.

[15] D. Janzing and P. Wocjan. BQP-complete problems concerning mixing properties of classical random walks on sparse graphs. 2006. arXiv: quant-ph/0610235.

[16] D. Janzing and P. Wocjan. A promiseBQP-complete String Rewriting Problem. Quantum Info. Comput., 10:234–257, 2010.

[17] S. P. Jordan and E. Farhi. Perturbative gadgets at arbitrary orders. Phys. Rev. A, 77:062329, 2008.

[18] S. P. Jordan, D. Gosset, and P. J. Love. Quantum-Merlin-Arthur–complete problems for stoquastic Hamiltonians and Markov matrices. Physical Review A, 81:032331, 2010.

[19] J. Kempe, A. Kitaev, and O. Regev. The complexity of the local Hamiltonian problem. SIAM Journal on Computing, 35:1070–1097, 2006.

[20] J. Kempe and K. B. Whaley. Exact gate sequences for universal quantum computation using the xy interaction alone. Physical Review A, 65:052330, 2002.

[21] E. Knill and R. Laflamme. Power of one bit of quantum information. Physical Review Letters, 81:5672–5675, 1998.

[22] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. A. Kasevich. Interaction-free measurement. Physical Review Letters, 74:4763, 1995.

[23] R. S. Leite, T. R. W. Richa, and C. Tomei. Geometric proofs of some theorems of schur-horn type. 286(1):149–173, 1999.

[24] D. Litinski, M. S. Kesselring, J. Eisert, and F. v. Oppen. Combining topological hardware and topological software: Color-code quantum computing with topological superconductor networks. Phys. Rev. X, 7:031048, 2017.

[25] E. Y. Loh, J. E. Gubernatis, R. T. Scalettar, S. R. White, D. J. Scalapino, and R. L. Sugar. Sign problem in the numerical simulation of many-electron systems. Phys. Rev. B, 41:9301–9307, 1990.

[26] T. Morimae. Hardness of classically sampling one clean qubit model with constant total variation distance error. Physical Review A, 96:040302(R), 2017. arXiv: 1704.03640.

[27] D. Nagaj. Universal two-body-Hamiltonian quantum computing. Physical Review A, 85:032330, 2012.

[28] D. Nagaj, O. Sattath, A. Brodutch, and D. Unruh. An adaptive attack on Wiesner’s quantum money. Quantum Information & Computation, 16:1048–1070, 2016.

[29] D. Nagaj and P. Wocjan. Hamiltonian quantum cellular automata in 1d. Physical Review A, 78, 2008. arXiv: 0802.0886.

[30] R. I. Oliveira and B. M. Terhal. The complexity of quantum spin systems on a two-dimensional square lattice. Quantum
