Infinite Games and Ramsey Properties of $F_{\sigma}$ Ideals

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Abstract

In this work, we investigate various combinatorial properties of Borel ideals on countable sets. We extend a theorem presented in [13] and identify an $F_{\sigma}$ tall ideal in which player II has a winning strategy in the Cut and Choose Game, thereby addressing a question posed by J. Zapletal. Additionally, we explore the Ramsey properties of ideals, demonstrating that the random graph ideal is critical for the Ramsey property when considering more than two colors. The previously known result for two colors is extended to any finite number of colors. Furthermore, we comment on the Solecki ideal and identify an $F_{\sigma}$ tall $K$-uniform ideal that is not equivalent to $\mathcal{D}_{\text{fin}}$, thereby addressing a question from Michael Hrušák's work [10].

Introduction

This article focuses on various combinatorial properties of ideals on $\omega$, particularly their Ramsey properties. We adopt standard set-theoretic notation from [2], [3], [4], [18], and [17]. An ideal on $\omega$ is a family of subsets of $\omega$ closed under subsets and finite unions, and (for this context) it always contains every finite subset of $\omega$. We only consider proper ideals, meaning we do not consider the power set $\mathcal{P}(\omega)$ as an ideal. The concept of an ideal is dual to that of a filter. A set $A \subseteq \omega$ is $I$-positive if $A \notin I$, and $I^+$ denotes the family of all $I$-positive sets.

In this work, we investigate definable ideals, primarily focusing on Borel ideals, which are viewed as subsets of the Cantor space $2^\omega$ by identifying

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subsets of \( \omega \) with their characteristic functions. The majority of the examples discussed are \( F_\sigma \) ideals. The discussion is primarily relevant to tall ideals, that is, ideals \( I \) for which every infinite subset \( A \subset \omega \) contains an infinite subset \( B \subseteq A \) with \( B \in I \). We also explore the Katětov order \( \leq_K \) on ideals \( I \) and \( J \) over \( \omega \), defined such that \( I \leq_K J \) if there exists a function \( f : \omega \to \omega \) for which \( f^{-1}[A] \in J \) for all \( A \in I \).

**Definition 0.1.** Let \( I \) be an ideal on \( \omega \), and let \( n \) and \( k \) be natural numbers. The notation \( \omega \to (I^+)^k_n \) signifies a Ramsey type property, indicating that for any coloring (function) \( c : [\omega]^k \to n \), there exists a subset \( A \subseteq \omega \), which is \( c \)-homogeneous, such that \( A \in I^+ \). A set \( A \) is deemed \( c \)-homogeneous if, for all sets \( p, q \in [A]^k \), \( c(p) = c(q) \).

In this paper, we explore this property and its extensions to other related Ramsey properties. Each section can be read independently, but later sections sometimes rely on definitions introduced earlier.

**Section 1 (Cut and Choose Game).** In this section, we introduce the Cut and Choose Game \( G_1(I) \). We demonstrate that certain tall \( F_\sigma \) ideals provide player II with a winning strategy, a characteristic that is unique among known Borel tall ideals. For such ideals previously studied, either player I has been established as having a winning strategy, or the winning player has remained undetermined. Thus, our results present the first confirmed cases where player II prevails. Additionally, some remarks are made regarding the relationship between the Cut-and-Choose game and certain Ramsey-type properties.

**Section 2 (Ramsey Properties).** We then investigate Ramsey-type phenomena. It is known that the random graph ideal \( R \) is critical for the property \( \omega \to (I^+)^2_2 \), since \( R \leq_K I \) if and only if \( \omega \not\to (I^+)^2_2 \). We generalize this result from 2-colorings to any finite number of colors. Prior work showed that \( \omega \to (I^+)^2_3 \) does not necessarily imply \( \omega \to (I^+)^3_3 \) (see [13]). We construct a family \( I_n \) such that \( \omega \to (I^+_n)^2_n \) but \( \omega \not\to (I^+_n)^2_{n+1} \), thus extending that negative answer to all finite colors.

**Section 3 (The Solecki Ideal and \( S_\omega \)).** Here we consider the Solecki ideal \( S \). It remains open whether \( \omega \to (S^+)^2_2 \) holds ([10], Question 5.5). Although we do not resolve this question, we introduce a new ideal \( S_\omega \) that contains \( S \). We investigate its Katětov relationship with other known ideals (e.g., the \( nwd \) ideal from [1], the random graph ideal, and the \( conv \) ideal...
of [10]). We also show that \( \omega \rightarrow (\omega, \mathcal{S}^+)^2 \), a weaker Ramsey-type property than \( \omega \rightarrow (\mathcal{S}^+)^2 \). Along the way, additional games akin to Cut and Choose are introduced.

Section 4 (K-Uniform Ideals). Prompted by an open question of Hrušák from [10] regarding whether \( \mathcal{E}D_{\text{fin}} \) is the only tall \( F_\sigma \) K-uniform ideal, we construct a new tall \( F_\sigma \) K-uniform ideal not Katětov-equivalent to \( \mathcal{E}D_{\text{fin}} \). This example, devised from graphs on infinite partitions of \( \omega \), enriches our comprehension of K-uniformity. Consequently, our findings confirm that \( \mathcal{E}D_{\text{fin}} \) is not uniquely representative of this class, affirmatively addressing Hrušák’s query.

1 The Cut and Choose Game

Definition 1.1. The "Cut and Choose Game" is played between two players, Player I and Player II, on a countable set \( \omega \) using an ideal \( \mathcal{I} \) on \( \omega \). The game begins with player I partitioning \( \omega \) into two pieces, \( A_0^0 \) and \( A_0^1 \). Player II then selects one of these pieces and an element within that piece, \( i_0 \in 2 \) and \( n_0 \in A_{i_0}^0 \). On the \((m+1)\)-th move, player I partitions the chosen piece \( A_{i_{m-1}}^m \) into two new pieces, \( A_{i_0}^m \) and \( A_{i_1}^m \). Player II then chooses one of these two new pieces and an element within that piece, \( i_m \in 2 \) and \( n_m \in A_{i_m}^m \). The game continues in this manner, with player I partitioning and player II choosing, until the infinite set of chosen elements \( \{n_i: i \in \omega\} \) is determined. Player I wins the game if this set is an element of the ideal \( \mathcal{I} \); otherwise, player II wins the game.

| I  | \( \omega = A_0^0 \cup A_0^1 \) | \( A_{i_0}^0 = A_{i_1}^0 \cup A_{i_1}^1 \) | ... |
|----|-------------------------------|---------------------------------|-----|
| II | \( i_0 \in 2, n_0 \in A_{i_0}^0 \) | \( i_1 \in 2, n_1 \in A_{i_1}^1 \) | ... |

It is important to note that in the Cut and Choose Game, Player I is declared the winner immediately if Player II selects an element from the ideal \( \mathcal{I} \). Furthermore, it is straightforward to prove that if Player I has a winning strategy in the game \( G_1(\mathcal{I}) \), and \( \mathcal{I} \) is Katětov below \( \mathcal{J} \), then Player I retains a winning strategy in \( G_1(\mathcal{J}) \). Conversely, if Player II holds a winning strategy in \( G_1(\mathcal{I}) \) and \( \mathcal{J} \) is Katětov below \( \mathcal{I} \), then Player II will similarly have a winning strategy in \( G_1(\mathcal{J}) \).

Proposition 1.1. If the ideal \( \mathcal{I} \) is not tall, then player II has a winning strategy in the game \( G_1(\mathcal{I}) \).
Proof. Assume $\mathcal{I}$ is not tall. Consequently, there exists an infinite set $A \subseteq \omega$ such that every infinite subset $B \subseteq A$ is in $\mathcal{I}^+$. During the game $G_1(\mathcal{I})$, player II can adopt a strategy of selecting, at each move, a part of the partition that intersects $A$ infinitely and chooses a natural number $n_i \in A$, ensuring it differs from all previously selected numbers. In the end $\{n_i : i \in \omega\} \in \mathcal{I}^+$. ■

Definition 1.2. The Random Graph Ideal. Denote the random graph on $\omega$ by $\langle \omega, E \rangle$. The random graph ideal $\mathcal{R}$ is the ideal on $\omega$ generated by all cliques and anti-cliques of this graph.

For a comprehensive discussion of the random graph ideal and the derivation of an independent family of sets, consult Definition 4.1 and Theorem 4.1 in [13]. Further, for alternative definitions of the random graph that are equivalent, refer to [5]. An additional elaboration on the random graph ideal are also available in Section 2 of this paper.

Proposition 1.2. In the Cut and Choose Game $G_1(\mathcal{R})$, player I has a winning strategy.

Proof. Consider the coloring $c : [\omega]^2 \to 2$ defined by

$$c(\{n, m\}) = \begin{cases} 0 & \text{if } \{n, m\} \in E, \\ 1 & \text{otherwise}, \end{cases}$$

where $E$ denotes the edge set of the random graph. In the first move of the game, player I partitions $\omega$ into two sets: $A_0^0 = \emptyset$ and $A_1^0 = \omega$. Since $A_0^0$ is empty, player II is compelled to choose $A_1^0$ and selects an element from it, denoted by $n_0$.

On the $k$-th move ($k \geq 1$), let $n_{k-1}$ be the element chosen by player II in the $(k - 1)$-th move. Player I partitions the remaining set, $A_{ik-1}^{k-1}$, based on the coloring $c$ with respect to $n_{k-1}$. Specifically, player I partitions $A_{ik-1}^{k-1}$ into two pieces:

$$A_0^k = \{m \in A_{ik-1}^{k-1} : c(\{n_{k-1}, m\}) = 0\}$$

and

$$A_1^k = \{m \in A_{ik-1}^{k-1} : c(\{n_{k-1}, m\}) = 1\}.$$  

Player II then chooses one of these pieces, either $A_0^k$ or $A_1^k$, denoted as $A_{ik}^k$, and selects an element from it, denoted $n_k$. 

4
This process continues, ensuring that in each step, player I partitions the chosen piece in a way that enforces c-homogeneity relative to the previously chosen element. At the end of the game, the set of selected elements, \( \{ n_k : k \in \omega \} \), will be the union of two c-homogeneous sets (one consisting of cliques and the other of anti-cliques).

Since the union of a clique and an anti-clique from the random graph is an element of the ideal \( \mathcal{R} \), player I wins the game. 

**Proposition 1.3.** The Ramsey property denoted by \( \omega \to (\mathcal{I}^+)^3 \) is satisfied if and only if \( \mathcal{R} \not\in \mathcal{K} \mathcal{I} \).

**Proof.** For a detailed proof, see [22]. 

**Proposition 1.4.** If player II possesses a winning strategy in the game \( G_1(\mathcal{I}) \), then the property \( \omega \to (\mathcal{I}^+)^3 \) necessarily holds.

**Proof.** This proof draws parallels with the proof of Proposition 1.2. Let \( c : [\omega]^2 \to 2 \) be any given coloring. Consider the gameplay in \( G_1(\mathcal{I}) \) where player I and player II employ their strategies, with player II following their winning strategy.

In the initial move, player I partitions \( \omega \) into two subsets: \( A_0^0 = \emptyset \) and \( A_1^0 = \omega \). Because \( A_0^0 \) is empty, player II is compelled to choose \( A_1^0 \), from which they select an element \( n_0 \).

For each subsequent move \( k \geq 1 \), assume \( n_k \) is the element previously chosen by player II. Player I then partitions the remaining set \( A_{i_k}^k \), based on the coloring \( c \) with respect to \( n_k \), into:

\[
A_{0}^{k+1} = \{ m \in A_{i_k}^k : c(\{n_k,m\}) = 0 \}
\]

and

\[
A_{1}^{k+1} = \{ m \in A_{i_k}^k : c(\{n_k,m\}) = 1 \}.
\]

Player II, adhering to their winning strategy, chooses one of these sets, either \( A_0^{k+1} \) or \( A_1^{k+1} \), designated as \( A_{i_{k+1}}^{k+1} \), and selects an element from it, denoted \( n_{k+1} \).

As player II is operating under a winning strategy, the set \( \{ n_k : k \in \omega \} \) forms an \( \mathcal{I} \)-positive set. This set, comprising elements selected by player II during the game, satisfies that, for each \( i \in \omega \) and for all \( k, k' > i \), \( c(\{n_i,n_k\}) = c(\{n_i,n_{k'}\}) \), so \( \{ n_k : k \in \omega \} \) can be expressed as the union of two c-homogeneous subsets, \( H_0 \cup H_1 = \{ n_k : k \in \omega \} \). Since their union forms an \( \mathcal{I} \)-positive set, at least one of these subsets, either \( H_0 \) or \( H_1 \), must not belong to \( \mathcal{I} \). Therefore, this verifies that \( \omega \to (\mathcal{I}^+)^3 \) indeed holds. 

5
The question of whether there exists a Borel tall ideal such that player II has a winning strategy in the Cut and Choose Game, previously posed by J. Zapletal, has been answered in the affirmative.

In the following, we will construct an $F_\sigma$ tall ideal $PC$ for which player II has a winning strategy in the Cut and Choose Game $G_1(I)$. The work in [10] outlines various known Borel ideals; however, it is important to note that none of these serve as suitable candidates, with the exception of the Solecki ideal (see the third section of this paper for more information on the Solecki ideal). Moreover, no ideal that is larger (in the Katetov order) than any of those Borel ideals qualifies as a potential candidate for this specific inquiry. It follows from Propositions 1.3 and 1.4 that if player II has a winning strategy in $G_1(I)$, then $I$ must be relatively small within the Katetov order.

We define a family of subsets of $\omega$ indexed by $\omega^{<\omega}$. This definition is crucial for the work presented in this paper and is considered to be the most important definition of this section.

**Definition 1.3.** We define a family of subsets of $\omega$ indexed by $\omega^{<\omega}$, denoted by $\langle A_s \subseteq \omega : s \in \omega^{<\omega} \rangle$. This family must satisfy the following properties:

- $A_\emptyset = \omega$ and $|A_s| = \omega$ for every $s \in \omega^{<\omega}$.
- $\{A_{s-n} : n \in \omega\}$ is a partition of $A_s$.
- For every $n \neq m$ natural numbers, there is $k \in \omega$ and $s \neq t \in \omega^k$ such that $n \in A_s$ and $m \in A_t$.

Furthermore, we say that $A \subseteq \omega$ is a major set if $|A| = \omega$ and $|A \cap A_s| = \omega \Rightarrow \exists n^\omega (|A \cap A_{s-n}| = \omega)$. A set $B$ is a dominant set if it contains a major set. Note that $\omega$ is a dominant set because it is a major set. Define $\text{Dominant} = \{D \subseteq \omega : A$ is a dominant set$\}$ and $\mathcal{H} = \mathcal{P}(\omega) \setminus \text{Dominant}$.

**Proposition 1.5.** $\mathcal{H}$ is an ideal.

**Proof.** We will prove that if a dominant set $B$ is partitioned into $C_0 \cup C_1$, at least one of the sets $C_0$ or $C_1$ is a dominant set. Without loss of generality, assume $B = \omega$. For each ordinal $\alpha \in \omega_1 + 1$, we will recursively define a function $\phi_\alpha : \omega^{<\omega} \rightarrow 3$ as follows:

**Base case:** For $i \in 2$, we define $\phi_0(s) = 1-i$ if $|A_s \cap C_i| < \omega$ and $\phi_0(s) = 2$ otherwise.

**Recursive case:** For $\alpha = \beta + 1$ and $i \in \{0,1\}$, we set $\phi_\alpha(s) = i$ if either $\phi_\beta(s) = i$ or $\phi_\beta(s) = 2$ and there are infinitely many $n$ such that
\( \phi_{\beta}(s \sim n) = i \). If there are infinitely many \( n \) such that \( \phi_{\beta}(s \sim n) = 0 \) and infinitely many \( n \) such that \( \phi_{\beta}(s \sim n) = 1 \), we set \( \phi_{\alpha}(s) = 0 \) (this choice does not affect the proof). If \( \phi_{\beta}(s) = 2 \) and the set \( \{ n \in \omega : \phi_{\beta}(s \sim n) \in \{0, 1\} \} \) is finite, then we set \( \phi_{\alpha}(s) = 2 \).

**Limit case:** For \( i \in \{0, 1\} \), we set \( \phi_{\alpha}(s) = i \) if there exists \( \beta < \alpha \) such that \( \phi_{\beta}(s) = i \), and \( \phi_{\alpha}(s) = 2 \) whenever for every \( \beta < \alpha \), \( \phi_{\beta}(s) = 2 \).

**Claim 1.1.** If \( \phi_{\omega_1}(\emptyset) = i \) then \( C_i \) is a dominant set.

**Proof.** To prove this, we will show that if \( \phi_{\omega_1}(s) = i \), then \( |C_i \cap A_s| = \omega \) and for infinitely many \( n \), \( \phi_{\omega_1}(s \sim n) = i \). Thus, \( C_i \) contains a major set.

Suppose that \( \phi_{\omega_1}(s) = i \), then there exists an ordinal \( \alpha \) such that \( \phi_{\alpha}(s) = i \) and \( \alpha \) is a successor ordinal or \( \alpha = 0 \).

If \( \phi_0(s) = i \), then \( \phi_0(t) = i \) for every \( t \) that extends \( s \), in particular also \( \{ n \in \omega : \phi_{\omega_1}(s \sim n) = i \} \) is infinite. If \( \alpha \) is greater than 0, then the set \( X = \{ n \in \omega : \phi_{\alpha-1}(s \sim n) = i \} \) is infinite whenever \( \phi_{\alpha}(s) = i \) by the recursive definition.

For every \( n \in X \), there exists a decreasing sequence of ordinals \( \alpha_0 > \alpha_1 > \ldots \) and natural numbers \( m_0, m_1, \ldots \) such that \( \phi_{\alpha_j}(s \sim n \sim m_0 \sim \ldots \sim m_j) = i \). Since every decreasing sequence of ordinals has a finite length, there exists a natural number \( k \in \omega \) such that \( \alpha_k = 0 \) and \( \phi_0(s \sim n \sim m_0 \sim \ldots \sim m_k) = i \). In particular, \( |C_i \cap A_s| = |A_s \sim n \cap C_i| = \omega \), because \( A_s \sim n \sim m_0 \sim \ldots \sim m_k \subset A_s \sim n \subset A_s \) and by the definition of \( \phi_0 \).

**Claim 1.2.** If \( \phi_{\omega_1}(\emptyset) = 2 \) then \( C_0 \) and \( C_1 \) are dominant sets.

**Proof.** From the definitions of \( \phi_{\alpha} \), it is clear that if \( \phi_{\omega_1}(s) = 2 \), then both \( A_s \cap C_0 \) and \( A_s \cap C_1 \) are infinite sets, and for all but finitely many \( n \in \omega \), \( \phi_{\omega_1}(s \sim n) = 2 \). Hence, both \( C_0 \) and \( C_1 \) contain a major set.

**Definition 1.4.** We say that a set \( A \) is a major set below a string \( s \in \omega^\omega \) if the following two conditions hold:

1. \( |A \cap A_s| = \omega \),
2. For each \( t \) that extends \( s \), if \( |A_t \cap A| = \omega \), then there are infinitely many \( n \) such that \( |A_t \sim n \cap A| = \omega \).

A set \( B \) is deemed a dominant set below \( s \) if it contains a major set below \( s \). Denote by \( \text{Dominant}_s \) the family of all dominant sets below \( s \) and \( \mathcal{H}_s = \mathcal{P}(\omega) \setminus \text{Dominant}_s \).
Proposition 1.6. Let $s \in \omega^{<\omega}$ be a string. $H_s$ is an ideal. Furthermore, if a set $A$ is dominant below $s$, then there are infinitely many $n$ such that $A$ is dominant below $s^{\sim n}$.

Proof. This follows the structure of the proof for Proposition 1.5. □

Definition 1.5. A tree $T \subseteq \omega^{<\omega}$ is called a boundedly branching tree if for every string $s \in T$, the set $\{n \in \omega : s^{\sim n} \in T\}$ has cardinality at most $|s| + 1$. For a given set $X \subseteq \omega$, we denote by $T_X$ the tree $\{s \in \omega^{<\omega} : X \cap A_s \neq \emptyset\}$. Using this notation, we define the following sets:

$S_0 = \{A \subseteq \omega : \exists s \in \omega^{<\omega} (A \subseteq A_s \land \forall n \in \omega (|A \cap A_s^{\sim n}| = 1))\}$,

$S_1 = \{A \subseteq \omega : T_A$ is a boundedly branching tree$\}$,

$\mathcal{PC}$ = the ideal generated by the union of $S_0$ and $S_1$.

It is worth noting that for every set $X \subseteq \omega$, the tree $T_X$ has no terminal nodes.

Theorem 1.1. Player II has a winning strategy in the game $G_1(\mathcal{PC})$.

Proof. Assume $A \subseteq \omega$ has the property that for every natural number $n$, there is a string $s \in \omega^n \cap T_A$ such that the set $\{k \in \omega : s^{\sim k} \in \omega^{n+1} \cap T_A\}$ has cardinality at least $(n + 1)^2$. Then $A$ is $\mathcal{PC}$-positive. To prove this, consider natural numbers $k$ and $l$ along with sets $X_0, X_1, \ldots, X_k \in S_0$ and $Y_0, Y_1, \ldots, Y_l \in S_1$. Define $X = \bigcup_{i \leq k} X_i$ and $Y = \bigcup_{j \leq l} Y_j$. According to the definition of $S_0$, for all $s \in T_X$, the inequality $|\{n \in \omega : s^{\sim n} \in T_X\}| \leq k(|s| + 1)$ holds. From $S_1$, there exists some $N \in \omega$ such that for all $s \in \omega^{<\omega}$ with $|s| > N$, $|\{n \in \omega : s^{\sim n} \in T_Y\}| \leq l$. Given that the function $(n + 1)^2$ grows faster than $k(n + 1) + l$, there is $M \in \omega$ where $(M + 1)^2 > k(M + 1) + l$ and for that reason $A$ can not be contained in $X \cup Y$, because there is a node in $T_A \cap \omega^M$ possessing more descendants than any node in $T_{X \cup Y} \cap \omega^M$.

Player I starts the game by presenting a partition of $\omega$ into $B_0 \cup C_0$. If $B_0$ is a dominant set, Player II chooses $D_0 = B_0$; otherwise, Player II chooses $D_0 = C_0$. There exists a natural number $n$ such that $D_0 \cap A_{(n)}$ is non-empty, and Player II selects $n_0 \in D_0 \cap A_{(n)}$.

Player I then splits $D_0$ into two pieces, $B_1$ and $C_1$. Player II chooses the dominant set among them and calls it $D_1$. Player II then selects $n_1 \in D_1 \cap A_{(m)}$ where $n \neq m$.

Player I presents the partition $B_2 \cup C_2$, and Player II chooses the dominant set among them. Player II then looks for an $i_0 \in \omega$ such that $D_2$ is a dominant set below $(i_0)$. Player II selects $n \in \omega$ such that $D_2 \cap A_{(i_0, n)}$ is non-empty, and takes $n_2 \in D_2 \cap A_{(i_0, n)}$. 

8
For the next four rounds, Player II chooses a part that is major below $(i_0)$ and selects $n_3, n_4, n_5$ and $n_6$ such that all of them are in different $A_{(i_0,n)}$.

Player I continues giving partitions, and Player II stays in level $k$ for $(k+1)^2$ rounds. After this, Player II chooses $i_k$ such that $D_m$ is major below $(i_0, i_1, \ldots , i_k)$. By construction and the initial observation, the final set $\{n_j : j \in \omega\}$ is $\mathcal{PC}$-positive.

**Proposition 1.7.** $\mathcal{PC}$ is an $F_\sigma$ tall ideal.

**Proof.** To demonstrate that $\mathcal{PC}$ qualifies as a tall ideal, consider an arbitrary set $X \subseteq [\omega]^\omega$. Let $\text{succ}_T(s)$ denote the set of successors of $s$ in $T_X$. We proceed by considering two cases:

**Case 1:** Suppose there exists $s \in T_X$ such that $\text{succ}_T(s)$ is infinite. In this scenario, let $s = \{ n \in \omega : s \cap n \in T_X \}$. For each $k \in S$, select $n_k \in X \cap m_s - k$. Observe that the collection $\{n_k : k \in S\}$ belongs to $S_0$, and consequently, it is also contained in $\mathcal{PC}$.

**Case 2:** Assume for every $s \in T_X$, the set $\text{succ}_T(s)$ is finite. We construct recursively sequences $\{n_k \in X : k \in \omega\}$ and $\langle s_k \in T_X : k \in \omega\rangle$ such that $n_i \neq n_j$ for $i \neq j$, $s_m$ extends $s_n$ if $n < m$, each $|A_{s_k} \cap X| = \omega$, and each node $t$ in the tree $T_{\{n_k \in X : k \in \omega\}}$ has at most $|t| + 1$ immediate successors. Begin with $s_0 = (n) \in \omega^1$ where $|A_n \cap X| = \omega$ and choose $n_0 \in A_n \cap X$. Assuming $n_i$ and $s_i$ have been defined, and given $A_{s_i} \cap X$ is infinite, there exists an extension $s_{i+1} \in T_X$ of $s_i$ such that $|A_{s_{i+1}} \cap X| = \omega$. Select $n_{i+1} \in A_{s_{i+1}} \cap X \setminus \{n_j : j \leq i\}$. The sequence $s_{i+1}$ has length at least $i + 1$, and the number of successors of each node in $T_{\{n_j : j \leq i+1\}}$ is limited to $i + 1$, since the set size is $i + 1$. It is notable that any choice of $n_{i+1}$ does not introduce a new branch into the tree $T_{\{n_k : k \leq i+1\}} \cap \omega^{< |s_i|}$. By construction, $\{n_k : k \in \omega\} \subseteq S_1 \cap [X]^\omega$.

Next, we aim to show that $\mathcal{PC}$ is an $F_\sigma$ set. To do so, we will prove that $\mathcal{PC}$ is generated by compact sets. We define the following subset of $\mathcal{PC}$:

$$S_2 = \{ A \subseteq \omega : \exists s \in \omega^{< \omega} (A \subseteq A_s \text{ and } \forall n \in \omega (|A \cap A_{s-n}| \leq 1)) \}.$$ 

It is clear that $S_2 \supseteq S_0$.

**Claim 1.3.** $S_2$ is compact.

**Proof.** Consider any $A \notin S_2$. Then note that there exists $n \in \omega$ and $s, t \in \omega^n$ such that $|A \cap A_s| \geq 2$ and $|A \cap A_t| \geq 1$. Let $m_0, m_1 \in A \cap A_s$ and $m_2 \in A \cap A_t$ be natural numbers. Then, the set $\{X \subseteq \omega : m_0, m_1, m_2 \in X\}$ is open and contains $A$, but does not intersect $S_2$. Therefore, every set $A$ is an interior point of $S_2$. $\blacksquare$
Claim 1.4. $S_1$ is compact.

Proof. Take $A \notin S_1$. Then there exists $n \in \omega$ such that the set \{s $\in \omega^n : A \cap A_s \neq \emptyset$\} has at least $n + 2$ elements. Fix $s_0, \ldots, s_{n+1} \in \omega^n$ and $m_0 \in A \cap A_{s_0}, \ldots, m_{n+1} \in A \cap A_{s_{n+1}}$. The set $X = \{x \subseteq \omega : m_0, \ldots, m_{n+1} \in X\}$ is an open set which contains $A$, but it does not have any subsets in $S_1$. ■

By Claims 1.3 and 1.4, $PC$ is generated by compact sets, so it is an $F_\sigma$ set.

Definition 1.6. Let $I$ and $J$ be ideals on $\omega$. For $X \in I^+$, the restriction of $I$ to $X$, denoted $I \upharpoonright X$, is defined as \{I \cap X : I \in I\}. We say that $I^+ \to (J^+)_2$ holds if, for every $A \in I^+$ and every coloring function $c : [A]^2 \to 2$, there exists a subset $B \subseteq A$ that is $J$-positive and $c$-homogeneous.

In the following part of this section, we present a family of tall ideals for which the property $I^+ \to (J^+)_2$ is satisfied.

Definition 1.7. Let $f \in \omega^\omega$ be an increasing function. A tree $T \subseteq \omega^{<\omega}$ is bounded by $f$ if for every $n \in \omega$, the number of nodes at level $n$ of $T$ does not exceed $f(n)$. In this context, define $S_3(f) = \{A \subseteq \omega : T_A$ is bounded by $f\}$ and $TC(f) = \langle S_0 \cup S_3(f)\rangle$ (the ideal generated by $S_0$ and $S_3(f)$).

Proposition 1.8. Let $f \in \omega^\omega$ be an increasing function. $TC(f)$ is a tall $F_\sigma$ ideal.

Proof. The proof follows a similar argument as the proof of Proposition 1.7. ■

Theorem 1.2. Let $f \in \omega^\omega$ be an increasing function. For any set $D \in H^+$ and any coloring function $c : [D]^2 \to 2$, there exists a subset $B \subseteq D$ that is both $TC(f)$-positive and $c$-monochromatic; that is, $H^+ \to (TC(f)^+)_2$.

Proof. It is sufficient to show that for any $D \in H^+$, Player II has a winning strategy in $G_1(H \upharpoonright D)$. This assertion is corroborated by Proposition 1.4 which ensures that any such $D$ satisfies $D \to (TC(f)^+)_2$.

Suppose $B \subseteq D$ satisfies the condition that for each $n \in \omega$, there exists a sequence $s \in \omega^n \cap T_B$ such that $|\{k \in \omega : s \upharpoonright k \in \omega^{n+1} \cap T_B\}| \geq 2n \cdot f(n) + 1$. Then $B \in TC(f)^+$. This line of reasoning aligns with the argumentation outlined in the proof of Theorem 1.1.

For the sake of contradiction, suppose $B \subseteq X \cup Y$, where $X = \bigcup_{i < j} X_i$ with each $X_i \in S_0$ and $Y = \bigcup_{i < k} Y_i$ with each $Y_i \in S_3(f)$. There exists $N \in \omega$ such that $|suc_{T_X}(s)| \leq j$ for each $s \in T_X$ if $|s| \geq N$, and $|suc_{T_Y}(s)| \leq \omega$. ■


\[ k \cdot f(|s|) \text{ for each } s \in T_Y. \] This is because the total number of nodes in \( T_Y \) at level \(|s|\) is at most \( k \cdot f(|s|) \). Let \( M = \max\{N, j, k\} \). Since \( f \) is increasing, we have \( f(M) \geq M \). For each \( s \in \omega^M \cap T_B \), it holds that \(|\text{suc}_T_B(s)| \leq \sum_{i=0}^{j} |\text{suc}_T_Y(s)| \leq j \cdot M + k \cdot f(M) \leq M \cdot f(M) + M \cdot f(M) = 2M \cdot f(M) \). Hence, \( B \) cannot satisfy the property described above.

The winning strategy for Player II is analogous to that in the proof of Theorem 1.1; the only difference is the number of nodes Player II needs to select at each level. This variation poses no issue, as whenever we partition a dominating set, there are infinitely many nodes from which to choose.

Given an increasing function \( f \in \omega^\omega \), Definition 1.5 can be adapted to define a boundedly branching tree determined by \( f \), where the number of children at each node is bounded by \( f \). It is straightforward to conceptualize an ideal \( \mathcal{PC}(f) \), for which the previously presented arguments remain valid. In short, we identify two definable families of ideals, \( \mathcal{PC}(f) \) and \( \mathcal{T}\mathcal{C}(f) \), both of minimal possible complexity \( F_\sigma \), where Player II is assured a winning strategy in the cut and choose game \( G_1(I) \).

For an increasing function \( f \in \omega^\omega \), the ideals \( \mathcal{PC}(f) \) and \( \mathcal{T}\mathcal{C}(f) \) are distinct; in particular, \( \mathcal{T}\mathcal{C}(f) \) is a proper subset of \( \mathcal{PC}(f) \). Additionally, given an increasing function \( f \), one can find a function \( g > f \) such that \( \mathcal{PC}(f) \) is contained within \( \mathcal{T}\mathcal{C}(g) \).

Several questions merit further investigation, such as whether these ideals are equivalent in the Katětov order, or what specific properties \( f \) must possess for these ideals to be distinguishable (in the Katětov order), both inter se and relative to other family associated with different increasing functions.

### 2 Ramsey Numbers

**Definition 2.1.** Let \( I \) be an ideal on \( \omega \). Suppose \( c : [\omega]^n \to k \) is a coloring function. In this context, we define

\[
\text{colors}(c, A) = \{i \in k : (\exists a \in A^n)(c(a) = i)\}.
\]

We say that \( \omega \to (I^+)^n_{k,l} \) holds if, for every coloring \( c : [\omega]^n \to k \), there exists an \( I^+ \)-positive set \( A \) such that \(|\text{colors}(c, A)| \leq l \). When \( l = 1 \), we simplify the notation to \( \omega \to (I^+)^n_k \).

In this section, we extend Theorem 5.5 from [13] by introducing a framework for Ramsey properties of ideals on \( \omega \). Essentially, the original result
established the existence of a tall $F_{\sigma}$ ideal $I$ satisfying $\omega \to (I^+)^2_2$, while $\omega \not\rightarrow (I^+)^2_3$.

To proceed, we recall that the classic Ramsey property is related to whether an ideal is above the random graph ideal. We will now present an equivalence of this kind, but first, we need to introduce some definitions based on Cameron’s work [5].

**Definition 2.2 (Star property).** A coloring $c : [\omega]^n \to k$ is said to have the $(n, k)$-star property if, for every collection of disjoint finite sets $A_1, A_2, \ldots, A_k \subseteq [\omega]^{n-1}$, there exists an $j \in \omega$ such that for every $\{j_2, j_3, \ldots, j_n\} \in A_i$, we have $c(\{j, j_2, j_3, \ldots, j_n\}) = i$.

Note that the $(2, 2)$-star property is the same as the $(*)$-property defined by Cameron. Regarding this new definition, we have the following proposition.

**Proposition 2.1.** Let $n$ and $k$ be natural numbers. There exists a function $R^n_k : [\omega]^n \to k$ with the $(n, k)$-star property. This function $R^n_k$ is called the $n$-random graph with $k$ colors.

**Proof.** The construction of $R^n_k$ will be done by recursion. Let $\{X_l : l \in \omega\}$ be an increasing sequence of finite subsets of natural numbers, with $X_0 = \{0, 1, \ldots, n-2\}$.

- **Base case:** The only possibility for $k$ subsets $(A_1, A_2, \ldots, A_k)$ of $[X_0]^{n-1}$, such that at least one of the $A_i$ is non-empty and $A_1, A_2, \ldots, A_k$ have empty pairwise intersections, is: $A_1 = X_0$ and $A_i = \emptyset$ (for $i \neq 1$), or $A_2 = X_0$ and $A_i = \emptyset$ (for $i \neq 2$), etc. In this case, define $R^n_k(0, 1, 2, \ldots, n-1) = 0$, $R^n_k(0, 1, 2, \ldots, n) = 1$, $R^n_k(0, 1, \ldots, n+k-2) = k - 1$ and let $X_1 = \{0, 1, \ldots, n+k-2\}$.

- **Successor case:** Suppose that we have defined a finite subset $X_l$ of natural numbers (which by construction is also a natural number, i.e., $|X_l| = X_l$). For every family of subsets $A_1, A_2, \ldots, A_k \subseteq [X_l]^{n-1}$ such that at least one of the $A_i$ is non-empty and the sets $A_i$ are disjoint, we take the first natural number $j$ not used in the construction and set $R^n_k(j, j_2, j_3, \ldots, j_n) = i$ for any $\{j_2, j_3, \ldots, j_n\} \in A_i$. Since there are only a finite number of such families due to $X_l$ being finite, we let $X_{l+1}$ be such that $X_l \subseteq X_{l+1}$ and $j \in X_{l+1}$ (for each family of subsets).
Note that $[\omega]^{n} = \bigcup_{l \in \omega} [X_l]^{n}$, so at the end, we have defined $R_k^n$ on the entire domain $[\omega]^{n}$, and by construction, $R_k^n$ has the $(n,k)$-star property.

The random graph is also often referred to as the universal graph, due to the following proposition:

**Proposition 2.2.** Given a coloring $c : [\omega]^{n} \to k$, there exists an injective function $f : \omega \to \omega$ such that for every selection of distinct natural numbers $a_1, a_2, \ldots, a_n$, we have

$$c\{a_1, a_2, \ldots, a_n\} = R_k^n\{f(a_1), f(a_2), \ldots, f(a_n)\}.$$

**Proof.** The proof of this proposition is done by recursion. We start by defining the function $f(i) = i$ for $i \in \{0,1,\ldots,n-1\}$. Then, we proceed with the following steps:

- **Base case:** Assume that $c\{0,1,\ldots,n-1\} = i$, where $0,1,\ldots,n-1$ are the first $n$ natural numbers. Using the $(n,k)$-star property of $R_k^n$, there exists an $x \in \omega$ such that $R_k^n\{0,1,\ldots,n-2,x\} = i$. Thus, we set $f(n-1) = x$.

- **Successor case:** Assume that we have already defined $f$ up to the natural number $l$. We define $X_i$ as the set of all $x \in [l+1]^{n} \setminus [l]^{n}$ such that $c(x) = i$, and $X_i'$ as the set of all $x \setminus \{l+1\}$ where $x \in X_i$. We also define $A_i$ as the set of all $f(j)$ such that $j \in X_i'$, for every $i < k$. Since $R_k^n$ satisfies the $(n,k)$-star property with the sets $A_i$, there exists an $x \in \omega$ such that $R_k^n\{x,j_2,\ldots,j_n\} = i$ whenever $j_2,\ldots,j_n \in A_i$ are distinct natural numbers. In this case, we assign $f(l+1) = x$.

By construction, the function $f$ satisfies the required property.

**Definition 2.3.** Let $A \subseteq \omega$ be a non-empty set and $c : [\omega]^{n} \to k$ a coloring function. The function $c$ satisfies the $(n,k)$-star property in $A$ if for all finite and pairwise disjoint $A_0, A_1, \ldots, A_{k-1} \subseteq [A]^{n-1}$ there exists $x \in A \setminus \bigcup_{i<k} A_i$ such that for each $i < k$ and every subset $\{j_2,\ldots,j_n\} \in A_i$, we have $c(x,j_2,\ldots,j_n) = i$.

In this context, we define the set $W(A,\{A_i : i < k\})$ as:

$$W(A,\{A_i : i < k\}) = \{x \in A \setminus \bigcup_{i<k} A_i : (\forall i < k)(\forall\{j_2,\ldots,j_n\} \in A_i)[c(x,j_2,\ldots,j_n) = i]\}.$$
Note that if $c$ satisfies the $(n,k)$-star property in $A$, then $A$ must be infinite. For $A = \omega$, this definition aligns precisely with Definition 2.2 which asserts that the set $W(\omega, \{A_i : i < k\})$, derived from any family of finite disjoint subsets $A_0, A_1, \ldots, A_{k-1} \subseteq [\omega]^{n-1}$, is always non-empty.

**Lemma 2.1.** If a coloring function $c$ satisfies the $(n,k)$-star property on a set $A$, and $A_0, A_1, \ldots, A_{k-1} \subseteq [A]^{n-1}$ are finite and pairwise disjoint, then $c$ also satisfies the $(n,k)$-star property on the set $W = W(A, \{A_i : i < k\})$. In particular, the set $W$ is not merely non-empty; it is necessarily infinite.

**Proof.** To prove that $c$ satisfies the $(n,k)$-star property in $W$, we must establish that for any finite disjoint sets $A'_0, A'_1, \ldots, A'_{k-1} \subseteq [W]^{n-1}$, there exists $x \in W$ such that $c(x, j_2, \ldots, j_n) = i$ for all $\{j_2, \ldots, j_n\} \in A'_i$. Let $B_i = A_i \cup A'_i$ for each $i$. Since $c$ fulfills the $(n,k)$-star property in $A$, there must exist a witness $x \in A$ effective for the sets $B_i$. Given $x \in A \bigcup_{i < k} A_i$, by definition, $x$ also resides in $W$, thus serving as the required witness for the sets $A'_i$. ■

**Lemma 2.2.** Let $c$ be a function that satisfies the $(n,k)$-star property in a set $A$. If $A$ is partitioned into $B$ and $C$, then either $c$ satisfies the $(n,k)$-star property in $B$, or $c$ satisfies the $(n,k)$-star property in $C$.

**Proof.** Suppose it is false for the partition $A = B \cup C$. Then in $B^{n-1}$, there exist finite disjoint sets $B_0, \ldots, B_{k-1}$ such that no $x \in B$ satisfies $f(x, j_2, \ldots, j_n) = i$ for any $\{j_2, \ldots, j_n\} \in B_i$ and some $i < k$. Similarly, there exist $C_0, \ldots, C_{k-1}$ subsets of $C^{n-1}$ that satisfy an equivalent property. Therefore, in $A$, there are no witnesses for the sets $A_i = B_i \cup C_i$, which contradicts the fact that $A$ satisfies the $(n,k)$-star property. ■

By the previous proposition, we have that the family of sets $X \in \mathcal{P}(\omega)$ such that $R^0_k$ satisfies the $(n,k)$-star property in $X$ is a family of positive sets for an ideal. Therefore, we can define the following:

**Definition 2.4.** Let $n, k, l$ be natural numbers such that $l < k$ and $c : [\omega]^n \to k$ a coloring function. The ideal $\mathcal{R}_{k,l}^n$ is, by definition, the ideal generated by the set $\{A \subseteq \omega : |\text{colors}(R^0_k, A)| \leq l\}$.

Note that the ideal $\mathcal{R}_{k,l}^n$ is a proper ideal, meaning $\mathcal{R}_{k,l}^n \neq \mathcal{P}(\omega)$. To prove this, it is enough to see that if we take $X_0, X_1, \ldots, X_{j-1} \in \mathcal{R}_{k,l}^n$, then $R^0_k$ does not satisfy the $(n,k)$-star property on each $X_i$. Consequently, the set $Y = \bigcup_{i \in j} X_i$ cannot have the $(n,k)$-star property either, by Lemma 2.2 in particular, $Y \neq \omega$. 

14
Proposition 2.3. Let $n, k, l$ be natural numbers such that $l < k$. Then $\mathcal{R}_{k,l}^n$ is an $F_\sigma$ tall ideal.

Proof. First, we show that $\mathcal{R}_{k,l}^n$ is generated by a closed set. The ideal $\mathcal{R}_{k,l}^n$ is generated by the set $\{A \subseteq \omega : |\text{colors}(R^n_k, A)| \leq l\}$, which can be represented as the intersection of closed sets of the form $\{A \subseteq \omega : |\text{colors}(R^n_k, A)| \neq i\}$, where $i \in \{0, 1, \ldots, l - 1\}$. Therefore, $\mathcal{R}_{k,l}^n$ is generated by a closed set.

Next, we show that $\mathcal{R}_{k,l}^n$ is tall. This follows from the classical Ramsey theorem, which states that for any positive integers $n$ and $k$, there exists a positive integer $R(n, k)$ such that for any $n$-coloring of the edges of a complete graph on $R(n, k)$ vertices, there exists a monochromatic complete subgraph on $k$ vertices. In this case, the set of vertices in the monochromatic subgraph corresponds to a set in $\mathcal{R}_{k,l}^n$, showing that it is tall.

Remark 2.1. 
- If $l < l'$, then $\mathcal{R}_{k,l}^n \subseteq_K \mathcal{R}_{k,l'}^n$.
- If $k < k'$, then $\mathcal{R}_{k,l}^n \subseteq_K \mathcal{R}_{k',l}^n$.

The first part of the remark is due to the fact that $\mathcal{R}_{k,l}^n$ is a subset of $\mathcal{R}_{k,l'}^n$. The second part is an immediate corollary of the following proposition.

Proposition 2.4. Let $\mathcal{I}$ be an ideal on $\omega$. Then $\mathcal{R}_{k,l}^n \subseteq_K \mathcal{I}$ if and only if $\omega \not\rightarrow (\mathcal{I}^+)^n_{k,l}$.

Proof. Suppose $\mathcal{R}_{k,l}^n \subseteq_K \mathcal{I}$. Let $f : \omega \rightarrow \omega$ be a Katetov function, that is, $A \in \mathcal{R}_{k,l}^n$ implies $f^{-1}[A] \in \mathcal{I}$. Define a coloring $c : [\omega]^n \rightarrow k$ by

$$c(\{a_0, \ldots, a_{n-1}\}) = R^k_\sigma(\{f(a_0), \ldots, f(a_{n-1})\})$$

when $|\{f(a_0), \ldots, f(a_{n-1})\}| = n$. If $|\{f(a_0), \ldots, f(a_{n-1})\}| \leq n - 1$, assign $c(\{a_0, \ldots, a_{n-1}\}) = 0$. We aim to show that if $A \subseteq \omega$ and $|\text{colors}(c, A)| \leq l$, then $A \in \mathcal{I}$, thus $\omega \not\rightarrow (\mathcal{I}^+)^n_{k,l}$ is false for the coloring $c$. Assume $A \subseteq \omega$ such that $|\text{colors}(c, A)| \leq l$.

Claim 2.1. $|\text{colors}(R^k, f[A])| \leq l$

Proof. By contradiction, suppose $|\text{colors}(R^k_n, f[A])| \geq l + 1$. For each $i \in n$ and $j \in l + 1$, let $a^i_j \in A$ such that for each $j < l + 1$, there exists a distinct color $N_j \in k$ where $R^k_n(\{f(a^i_0), \ldots, f(a^i_{n-1})\}) = N_j$. By the definition of $c$, we have $c(\{a^i_0, \ldots, a^i_{n-1}\}) = N_j$, thus $|\text{colors}(c, A)| \geq l + 1$, which is a contradiction.
Since \(|\text{colors}(R^n_k, f[A])| \leq l, f[A] \in R^n_{k,l}\). As \(f\) is a Katětov function, 
\(f^{-1}[f[A]] \in I, \) and \(A \subseteq f^{-1}[f[A]]\), therefore \(A \in I\).

For the converse implication, suppose \(\omega \not\rightarrow (I^+)^n_{k,l}\). There exists a 
coloring \(c : [\omega]^n \rightarrow k\) such that if \(|c([A]^n)| \leq l\), then \(A \in I\). By proposition 2.2 there is an injective function \(f : \omega \rightarrow \omega\) such that for every \(a_1, a_2, \ldots, a_n\), distinct numbers, \(c(\{a_1, a_2, \ldots, a_n\}) = R^n_k(\{f(a_1), f(a_2), \ldots, f(a_n)\})\). This function \(f\) serves as a Katětov function proving that \(R^n_{k,l} \leq K I\).  
\[\square\]

The open question of whether the properties \(\omega \rightarrow (I^+)^n_2\) and \(\omega \rightarrow (I^+)^n_3\) 
were equivalent was answered in the negative by M. Hrušák, D. Meza-Alcántara, E. Thümmel, and C. Uzcátegui in their paper [13]. They constructed an ideal called \(ED\) which satisfies \(\omega \rightarrow (I^+)^n_2\), but does not satisfy \(\omega \rightarrow (I^+)^n_3\). I then extended this idea to construct a family of ideals \((ED_m : m \in \omega)\) on \(\omega\) such that \(\omega \rightarrow (ED_m^+)^n_{m+1}\) but \(\omega \not\rightarrow (ED_m^+)^n_{m+2}\).

The ideal \(ED\) is well-documented in the literature, for instance, defined in [10] and [22], while the ideal \(\hat{ED}\) was introduced in [13]. Later in this section, we will present a generalization of these definitions.

**Definition 2.5.** Let \(X\) be any set. If \(Y \subseteq \omega \times X\), we denote the projection 
of \(Y\) onto the \(i\)-th coordinate by \((Y)_i = \{x \in X : (i, x) \in Y\}\). Let \(\Delta \subseteq \omega^2\) 
given by \(\Delta = \{(i, j) \in \omega^2 : i < j\}\). The ideals \(ED, \hat{ED}\) and \(ED_{fin}\) are defined 
over \(\omega^2, \omega^3\) and \(\Delta\) respectively as follows:

\[
ED = \{Y \subseteq \omega^2 : (\exists k \in \omega)(\forall n > k)(|Y(n)| \leq k)\},
\]

\[
\hat{ED} = \{Y \subseteq \omega^3 : (\exists k \in \omega)(i < k \Rightarrow (Y)_i \in ED) \land (i > k \Rightarrow |Y(i)| \leq k)\},
\]

\[
ED_{fin} = ED \upharpoonright \Delta = \{X \cap \Delta : X \in ED\}
\]

**Proposition 2.5.** Player I has a winning strategy in the game \(G_1(ED)\).

**Proof.** In the game \(G_1(ED)\), Player I can effectively utilize a simple strategy. 
Let \(n_0 = 0\). Player I initiates the game by choosing \(A_0 = \{n_0\} \times \omega\) and 
\(B_0 = \{n \in \omega : n > 0\} \times \omega\). If Player II selects \(A_0\), Player II immediately 
loses. Thus, Player II must choose \(B_0\) and pick a pair \((n_0, m_0) \in B_0\).

Subsequently, at each round \(i\), Player I presents \(A_i = \{n \in \omega : n < n \leq n_i - 1\} \times \omega\) and 
\(B_i = \{n \in \omega : n > n_i - 1\} \times \omega\). Player II is forced to 
consistently choose \(B_i\) and select a pair \((n_i, m_i)\). Ultimately, the set 
\(\{(n_i, m_i) : i \in \omega\}\) forms a subset with at most one element in each column 
of \(\omega^2\), which leads to Player II’s defeat.  
\[\square\]

Note that the same proof with minor modifications applies to the \(ED_{fin}\) and \(\hat{ED}\) ideals as well.
**Definition 2.6.** Let $m \in \omega$ be a natural number. The ideal $\mathcal{E}D_m$ is defined as the ideal generated by the set $\mathcal{A}_m \cup S_0$, where $\mathcal{A}_m = \{ A_s : |s| = m + 1 \}$ and $S_0$ is the family of subsets defined in Definition 1.5.

In simpler terms, the ideal $\mathcal{E}D_m$ is generated by subsets of the $m + 1$ level in the tree, as well as selectors for the partition $(A_{s-n} : n \in \omega)$ of all $A_s$ sets.

**Definition 2.7.** Let $\varphi : \mathcal{P}(\omega) \rightarrow [0, \infty]$ be a function. $\varphi$ is called a lower semicontinuous submeasure if it satisfies the following conditions:

1. **Monotonicity:** $\varphi(A) \leq \varphi(B)$ for all $A \subseteq B \subseteq \omega$.
2. **Subadditivity:** $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for all $A, B \subseteq \omega$.
3. **Continuity from below:** For any set $A \subseteq \omega$,
   \[ \varphi(A) = \lim_{n \to \infty} \varphi(A \cap n). \]

We define $\text{Fin}(\varphi)$ as the set $\{ A \subseteq \omega : \varphi(A) < \infty \}$.

It is straightforward to demonstrate that $\text{Fin}(\varphi)$ constitutes an $F_\sigma$ ideal. Mazur’s theorem [21] asserts that every $F_\sigma$ ideal can be represented as $\text{Fin}(\varphi)$ for some lower semicontinuous submeasure $\varphi$.

**Proposition 2.6.** For all natural numbers $m$, the ideal $\mathcal{E}D_m$ is an $F_\sigma$ tall ideal.

*Proof.* To demonstrate that $\mathcal{E}D_m$ is an $F_\sigma$ ideal, it suffices to construct a lower semicontinuous submeasure $\varphi_m$ such that $\text{Fin}(\varphi_m) = \mathcal{E}D_m$. Define $\varphi_m : \mathcal{P}(\omega) \rightarrow \omega + 1$ as follows:

\[ \varphi_m(E) = \min\{ \alpha + 1 : (\exists X \in [\mathcal{A}_m \cup S_0]^\alpha)(E \subseteq \bigcup X) \}. \]

It is easily verified that $\varphi_m$ is a lower semicontinuous submeasure such that $\text{Fin}(\varphi_m) = \mathcal{E}D_m$. The ideal $\mathcal{P}C$ introduced in the first section satisfies $\mathcal{P}C \subset \mathcal{E}D_m$ and is a tall ideal. Consequently, $\mathcal{E}D_m$ is also a tall ideal. 

Directly from the definition, it follows that $\mathcal{E}D_m \subseteq \mathcal{E}D_n$ whenever $m \geq n$. In particular, $\mathcal{E}D_m \leq_K \mathcal{E}D_n$.

**Definition 2.8.** Let $X$ and $Y$ be two countably infinite sets, and let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $X$ and $Y$ respectively. We say that the ideal $\mathcal{I}$ is equivalent to the ideal $\mathcal{J}$ if there exists a bijective function $f : X \rightarrow Y$ such that for every subset $A \subseteq X$, $A \in \mathcal{I}$ if and only if $f[A] \in \mathcal{J}$. In this case, we write $\mathcal{I} \approx \mathcal{J}$. 

17
Proposition 2.7. The ideals $\mathcal{D}$ (from Definition 2.5) and $\mathcal{D}_0$ (from Definition 2.6) are equivalent. Similarly, it holds that $\mathcal{D}_1 \approx \hat{\mathcal{D}}$.

Proof. Let $f_n : \omega \to \omega$ be a bijective function defined for every $n \in \omega$, mapping $A_{(n)}$ to $\omega$. Assign to each $k \in \omega$ an index $i_k \in \omega$ such that $k \in A_j$ if and only if $i_k = j$. We then define $f : \omega \to \omega \times \omega$ by $f(k) = (i_k, f_{i_k}(k))$. It’s not hard to see that $f$ meets the desired condition; this function is simply a translation between the two definitions.

Similarly, assign a bijective function $g_s : \omega \to \omega$ to each $s \in \omega^2$, mapping $A_s$ to $\omega$. For every $k \in \omega$, define $i_k \in \omega^2$ such that $k \in A_j$ if and only if $i_k = j$. The function $g : \omega \to \omega \times \omega \times \omega$ is defined as $g(k) = (i_k(0), i_k(1), g_{i_k}(k))$. Once again, it is straightforward to verify that the function $g$ demonstrates the equivalence of the two ideals. ■

In summary, the preceding proposition indicates that the definition of the family of ideals $\mathcal{D}_m$ extends the definitions of the known ideals $\mathcal{D}$ and $\hat{\mathcal{D}}$.

Proposition 2.8. It holds that $\mathcal{R}_{m+2} \leq_K \mathcal{D}_m$, which is equivalent to the statement $\omega \not\rightarrow (\mathcal{D}_m^+)_{m+2}^2$.

Proof. For each $n \in \omega$, let $x_n \in \omega^\omega$ be a sequence such that $n \in A_{x_n,i}$ for every $i \in \omega$. We define the function $\text{split} : [\omega]^2 \to \omega$ by setting $\text{split}([n,k]) = \min\{i \in \omega : x_n(i) \neq x_k(i)\}$, which is well-defined due to the third property in Definition 1.3.

Define a coloring $c : [\omega]^2 \to m+2$ as follows:

$$c([n,k]) = \begin{cases} 
\text{split}([n,k]) & \text{if } \text{split}([n,k]) \leq m, \\
 m+1 & \text{if } \text{split}([n,k]) \geq m+1.
\end{cases}$$

Note that if $A \subseteq \omega$ forms a monochromatic set for some color $i \leq m$, then there exists a set $B \supseteq A$ such that $B \in \mathcal{S}_0$. Conversely, if $Y \subseteq \omega$ forms a monochromatic set for color $m+1$, then there exists $s \in \omega^{m+1}$ such that $Y \subseteq A_s \in \mathcal{A}_m$. This coloring $c$ demonstrates that $\mathcal{D}_m$ does not satisfy the Ramsey property with $m+2$ colors. Furthermore, the set $\{A \subseteq \omega : \text{colors}(c,A) = 1\}$ forms a base for the ideal $\mathcal{D}_m$. ■

Definition 2.9. Let $\mathcal{I}$ be an ideal on $\omega$. We define the Ramsey-type property $\omega \to (\angle \omega, \ldots, \angle \omega \mid \mathcal{I}^+)_{n}^2$ to hold if, for any coloring $c : [\omega]^2 \to n$, at least one of the following two conditions is satisfied:

- **Condition 1:** There exists $i \in n$ and $A \in \mathcal{I}^+$ such that $A$ is monochromatic of color $i$, i.e., $\text{colors}(A) = \{i\}$.
• Condition 2: For each $N \in \omega$ and $i \in n$, there exists $A \in [\omega]^N$ such that $A$ is monochromatic of color $i$.

Lemma 2.3. For each pair of natural numbers $n, m \in \omega$, the ideal

$$\mathcal{E}D_{m+1} \upharpoonright A(n) = \{X \cap A(n) : X \in \mathcal{E}D_{m+1}\}$$

over the countably infinite set $A(n)$ is equivalent to the ideal $\mathcal{E}D_m$.

In simple terms, the ideal $\mathcal{E}D_{m+1}$ contains countably infinite copies of the ideal $\mathcal{E}D_m$.

Proof. Firstly, observe that for any $n, m \in \omega$, the set $A(n)$ is a positive set within the ideal $\mathcal{E}D_{m+1}$, confirming that $\mathcal{E}D_{m+1} \upharpoonright A(n)$ forms an ideal.

Consider any sequence $s$ in $\omega^{m+1}$. Let $(n) \prec s$ denote the extended sequence in $\omega^{m+2}$ where $(n) \prec s(0) = n$ and the subsequent entries are given by $(n) \prec s(i) = s(i-1)$ for $0 < i \leq m+1$. For any sequence $s \in \omega^{m+1}$, assign to $s$ an arbitrary bijective function $f_s : A(n) \rightarrow A_s$, mapping elements between these two infinite sets.

For any $k \in A(n)$, identify the unique sequence $s_k$ such that $k$ belongs to $A(n) \cap s_k$. Construct the function $f : A(n) \rightarrow \omega$ by assigning $f(k) = f_{s_k}(k)$. Note that the function $f$ is bijective and certifies that those two ideals are equivalent.

Lemma 2.4. If $M \subseteq \omega$ is a major set and $m$ is a natural number, then the ideal $\mathcal{E}D_m \upharpoonright M = \{X \cap M : X \in \mathcal{E}D_m\}$ on $M$ is equivalent to $\mathcal{E}D_m$.

Proof. Let $g : T_M \lessdot \omega < \omega$ be an order isomorphism with respect to the tree ordering (the tree $T_M$ from Definition 1.5). For each $s \in T_M$, let $f_s$ be a bijective function between the infinite set $A_s \cap M$ and $A_{g(s)}$. For each $n \in M$, there exists a unique $s_n \in T_M \cap \omega^{m+1}$ such that $n \in A_s \cap M$. We define the function $f : M \rightarrow \omega$ given by $f(n) = f_{s_n}(n)$. It is not difficult to see that $f$, as defined, demonstrates the equivalence between the ideals $\mathcal{E}D_m \upharpoonright M$ and $\mathcal{E}D_m$.

Remark 2.2. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals over the countable sets $A$ and $B$ respectively. If $\mathcal{I} \approx \mathcal{J}$ and $n \in \omega$, then the following two statements hold:
• $A \rightarrow (I^+)_n^2$ if and only if $B \rightarrow (J^+)_n^2$

• $A \rightarrow (< \omega, \ldots, < \omega | I^+)_n^2$ if and only if $B \rightarrow (< \omega, \ldots, < \omega | J^+)_n^2$

This remark is useful because we will later use induction in a proof and wish to apply the inductive hypothesis to an equivalent ideal. Its proof, being straightforward, is omitted.

**Lemma 2.5.** Let $M$ be a major set in $\omega$ and $X \subseteq M$. If $X$ is a positive set for the ideal $ED_m \upharpoonright M$, then $X$ is also a positive set for the ideal $ED_m$. Similarly, if $n, m$ are natural numbers and $X \subseteq A(n)$ is a positive set for the ideal $ED_{m+1} \upharpoonright A(n)$, then $X$ is also a positive set for the ideal $ED_{m+1}$. Finally, if $X \subseteq \omega$ and for every $n \in \omega$ there exists $N \in \omega$ such that $|X \cap A(N)| \geq n$, then $X \in ED^+$.

**Proof.** Let $I$ be an ideal on $\omega$, and suppose $X \subseteq \omega$ is a positive set for $I$. If $Y \subseteq X$ and $Y \notin I \upharpoonright X$, then $Y \notin I$. This argument substantiates the first two claims of the lemma.

For the third claim, note that the ideal $ED_m$ is generated by $A_m \cup S_0$. Proceeding by contraposition, assume $X \subseteq X_0 \cup \cdots \cup X_k \cup Y_0 \cup \cdots \cup Y_l$, with $X_i \in A_m$ and $Y_i \in S_0$. For each $0 \leq i \leq k$, there exists $n_i$ such that $X_i \subseteq A(n_i)$. For each $0 \leq j \leq l$ and each $Y_j$, consider two scenarios:

- There exists $n_j$ such that $Y_j \subseteq A(n_j)$,
- For every $n \in \omega$, $|A_n \cap Y_j| = 1$.

Let $N$ be a natural number greater than all $n_j$ and $n_i$. Then, for each $k > N$, we have $|A_k \cap X| \leq l$, thus $X$ does not satisfy the property posited by the proposition. ■

**Definition 2.10.** Let $I$ be an ideal on $\omega$. We will say that $I$ satisfies $\omega \rightarrow (\omega, I^+)_2^2$ if for every coloring $c: [\omega]^2 \rightarrow 2$, either for every $N < \omega$ there is a 0-homogeneous set $X$ of size $N$, or there is an $I$-positive 1-homogeneous set.

**Proposition 2.9.** Suppose $I$ is an ideal on $\omega$. The following properties are equivalent:

- $\omega \rightarrow (\omega, I^+)_2^2$
- $\omega \rightarrow (< \omega, < \omega | I^+)_2^2$
Proof. In both directions, consider a coloring \( c : [\omega]^2 \to 2 \). By applying the hypothesis to \( c \) and its complement \( 1 - c \), one ensures the existence of either a positive set for one of the colors or arbitrarily large monochromatic subsets for both colors. ■

**Lemma 2.6.** Let \( m \) be a natural number, and suppose \( X \subseteq \omega \) is a positive set for the ideal \( \mathcal{E} \mathcal{D}_m \). There exists a subset \( Y \subseteq X \) which is also positive, such that \( \mathcal{E} \mathcal{D}_m \upharpoonright Y \) is equivalent to \( \mathcal{E} \mathcal{D}_{\text{fin}} \).

**Proof.** We aim to find a sequence \( s \in \omega \leq m \) such that \( A_s \cap X \) belongs to \( \mathcal{E} \mathcal{D}_m^+ \) and for each \( n \in \omega \), \( A_s \upharpoonright n \cap X \) belongs to \( \mathcal{E} \mathcal{D}_m \). This is achieved recursively starting with the empty sequence \( \emptyset \). If for every \( n \in \omega \), \( A_s \upharpoonright n \cap X \) belongs to \( \mathcal{E} \mathcal{D}_m \), then we set \( s = \emptyset \). Otherwise, if there exists an \( n_0 \) such that \( A_s \upharpoonright n_0 \cap X \) is in \( \mathcal{E} \mathcal{D}_m^+ \), we initiate \( s_0 = (n_0) \).

Continuing this process at step \( k \), assuming \( n_0, n_1, \ldots, n_k \) are defined such that the finite sequence \( s_k = (n_0, n_1, \ldots, n_k) \) ensures \( A_{s_k} \cap X \) belongs to \( \mathcal{E} \mathcal{D}_m^+ \), if there exists \( n_{k+1} \) such that \( A_{s_k \upharpoonright n_{k+1}} \cap X \) remains in \( \mathcal{E} \mathcal{D}_m^+ \), the recursion continues; otherwise, we set \( s = s_k \). The recursion cannot proceed indefinitely since for each \( t \in \omega^{m+1} \), \( A_t \cap X \) must belong to \( \mathcal{E} \mathcal{D}_m \), indicating a finite conclusion within at most \( m + 1 \) steps.

Define \( X' = X \cap A_s \). We shall prove that for each \( k \in \omega \), the set \( I_k = \{ n \in \omega : |X' \cap A_{s-n}| \geq k \} \) is infinite. Assuming the contrary for some \( k \), that \( I_k \) is finite, leads to \( X' \) being contained in a finite union of \( X' \cap A_{s-n} \in \mathcal{E} \mathcal{D}_m \) and sets \( S_i \subseteq S_0 \subseteq \mathcal{E} \mathcal{D}_m \), placing \( X' \) in \( \mathcal{E} \mathcal{D}_m \).

Let \( (k_n)_{n \in \omega} \) be a sequence of distinct natural numbers, each \( k_n \) belonging to \( I_k \), and select \( y_n \in [X' \cap A_{s-n}]^n \). Define \( Y \) as \( \bigcup_{n \in \omega} y_n \). According to Lemma 2.5 \( Y \) constitutes a positive set for the ideal \( \mathcal{E} \mathcal{D}_m \). Furthermore, the function \( f : Y \to \Delta \), which bijectively maps each \( y_n \) to \( \{ (i, n) : i \in n \} \subseteq \Delta \), verifies the equivalence of \( \mathcal{E} \mathcal{D}_m \upharpoonright Y \) with \( \mathcal{E} \mathcal{D}_{\text{fin}} \).

The following proposition is the most pivotal of this section.

**Proposition 2.10.** Let \( m \) be a natural number. The property \( \omega \to (\mathcal{E} \mathcal{D}_m^+)^2_{m+1} \) holds, and additionally, \( \omega \to (\omega, \ldots, \omega \upharpoonright \mathcal{E} \mathcal{D}_m^+)_{m+2}^2 \) is also true.

**Proof.** The proof proceeds by induction on \( n \in \omega \) to demonstrate the simultaneous validity of the following two Ramsey properties:

1. \( \omega \to (\mathcal{E} \mathcal{D}_n^+)_{n+1}^2 \),

2. \( \omega \to (\omega, \ldots, \omega \upharpoonright \mathcal{E} \mathcal{D}_n^+)_{n+2}^2 \).
The base case \( n = 0 \) is straightforward, where the first property simply involves a one-color partitioning. The second property for \( n = 0 \) has been previously established by David Meza for the ideal \( \mathcal{E}D \) and it is known that \( \mathcal{E}D_0 \approx \mathcal{E}D \). It can also be demonstrated using Theorem 4.14 from [13], which states that \( \omega \rightarrow (\omega, \mathcal{E}D^+_{fin})_{2}^{2} \), hence particularly \( \omega \rightarrow (\omega, \mathcal{E}D^+_{fin})_{2}^{2} \).

By Proposition 2.9 and Remark 2.2 we have that \( \omega \rightarrow (\omega, < \omega, \mathcal{E}D^+_{0})_{2}^{2} \).

Assuming the properties hold for some \( n \in \omega \) as our induction hypothesis, consider a coloring \( c : [\omega]^2 \rightarrow n + 2 \). The objective is to identify a subset \( X \subseteq \omega \) such that \( X \in \mathcal{E}D_{n+1}^{+} \) and \( X \) is monochromatic under \( c \).

For each \( k \in \omega \), applying Lemma 2.3 and Remark 2.2 we utilize the induction hypothesis for the ideal \( \mathcal{E}D_{n+1} \upharpoonright A(k) \approx \mathcal{E}D_{n} \), ensuring that \( A(k) \rightarrow (\omega, \ldots, < \omega, (\mathcal{E}D_{n+1} \upharpoonright A(k))^{+})_{n+2}^{2} \) holds. This property suggests two scenarios:

- There exists some \( k \in \omega \) and \( X \subseteq A(k) \) such that \( X \) is monochromatic and not in \( \mathcal{E}D_{n+1} \upharpoonright A(k) \). This case is straightforward, as Lemma 2.5 confirms \( X \notin \mathcal{E}D_{n+1}^{+} \).

- For each \( i \in n + 2 \), and for all \( k, N \in \omega \), there exists \( X_{k} \subseteq A(k) \) of size \( N \) that is monochromatic in color \( i \). We will construct a positive set analogous to the one defined in the third part of Lemma 2.5. This necessitates the introduction of the following lemma.

**Lemma 2.7.** Let \( k_{0}, N \in \omega \) be natural numbers. Assuming \( D \) is a major set below \( k_{0} \), \( B \in [\omega]^{\omega} \), and for each \( m \in B \), \( D_{m} \subseteq A(m) \) is a major set below \( (m) \). There exists a set \( a \in [D]^N \), a subset \( B' \subseteq B \) with \( |B'| = \omega \), a color \( i \in n + 2 \), and for each \( m \in B' \), \( D'_{m} \subseteq D_{m} \) a major set below \( (m) \), such that \( a \) is monochromatic with color \( i \) and also for all distinct \( x \in a \), \( y \in \bigcup_{m \in B'} D'_{m} \), \( c(x, y) = i \).

**Proof.** Note that, if \( X \subseteq A(m) \) is a major set below \( (m) \), then in particular \( X \notin \mathcal{E}D_{n+1} \upharpoonright A(m) \). To prove the lemma, let \( \{x_{j} : j \in \omega \} \) be an enumeration of the set \( D \). For \( j = 0 \) and for every \( m \in B \), we partition \( D_{m} \) into \( n + 2 \) subsets, denoted by \( X_{m,i} = \{ y \in D_{m} : c(\{x_{0}, y\}) = i \} \). It is guaranteed that for every \( m \), there exists an \( i \) such that \( X_{m,i} \) is a dominant set below \( A(m) \). One color must be repeated infinitely many times, so let \( i_{0} \in n + 2 \) and \( B_{0} \in [B]^{\omega} \) be such that \( X_{m,i_{0}} \) is a dominant set for every \( m \in B_{0} \), and define \( D_{m}^{0} = X_{m,i_{0}} \). Recursively, we repeat this construction for each \( x_{j} \) and \( j \in \omega \), thus defining the following:

- \( B_{i+1} \in [B_{i}]^{\omega} \),
\begin{itemize}
\item $D_{m+1}^{l+1} \in [D_m^l]^{\omega}$ such that $D_{m+1}^{l+1}$ is a dominant set below $A_{(m)}$ for every $m \in B_{l+1}$.
\item For every $y \in \bigcup_{m \in B_l} D_m^l$ and $j \in \omega$, the color assigned by $c$ to the pair $(x_j, y)$ is the fixed number $i_j$ (again, this is possible because one color must be repeated infinitely many times).
\end{itemize}

Let $Z_i = \{x_j \in D : i_j = i\}$. As $\{Z_i : i \in n + 2\}$ is a partition (in a finite number of pieces) of $D$, it follows that there exists $i \in n + 2$ such that $Z_i$ is a dominant set below $(k_0)$. Without loss of generality, $Z_i$ is a major set below $(k_0)$, because if it were not, one would simply take $Z'_i \subseteq Z_i$ as a major set below $(k_0)$. We designate $D' = Z_i$.

Note that $\mathcal{E}D_{n+1} \upharpoonright D' \cong \mathcal{E}D_{n}$, by lemmas 2.3 and 2.4. By the Remark 2.2, we can apply the induction hypothesis to the ideal $\mathcal{E}D_{n+1} \upharpoonright D'$, thus there exists an $a \in |D'|^N$ which is $i$-monochromatic. Let $J = \max\{j : x_j \in a\}$ and let $B' = B_J$, $D'_m = D_m^J$ for every $m \in B'$. Again, we can assume, without loss of generality, that $D'_m$ is a major set below $(m)$ (if it is not, take a subset that is a major set below $(m)$). By construction, $a, B', i,$ and $\{D'_m : m \in B'\}$ are exactly as required.

Using Lemma 2.7, we construct an increasing sequence $\langle k_j : j \in \omega \rangle$ and sets $a_j, j \in \omega$ such that $k_j < k_{j+1}$ for all $j \in \omega$. For each $j \in \omega$, we choose a subset $a_j \subseteq A_{(k_j)}$ such that $|a_j| = j, c(x, y) = i_j$ for all $x, y \in a_j$ with $x \neq y$, and $c(x, y) = i_j$ for all $x, y \in a_N$ for all $N > j$. The set $\bigcup_{i_j = i_j} a_j$ is an $i$-monochromatic positive set for the ideal $\mathcal{E}D_{n+1}$ because there are only a finite number of colors and there exists $i \in n + 2$ such that the set of indices $\{j \in \omega : i_j = i\}$ is infinite. This completes the first part of the proof.

The proof that $\omega \rightarrow (\omega, \ldots, \omega | \mathcal{E}D_{n+2}^{n+2})^2$ holds is as follows. Consider a coloring $c : [\omega]^2 \rightarrow n + 3$. Define $c' : [\omega]^2 \rightarrow n + 2$ as:

$$c'(\{a, b\}) = \begin{cases} 0 & \text{if } c(\{a, b\}) = 0 \\ c(\{a, b\}) - 1 & \text{if } c(\{a, b\}) > 0 \end{cases}$$

By the previous proof, there exists $B \subseteq \omega$ which is a monochromatic positive set for the coloring $c$. If $B$ is monochromatic for some color $i > 0$, then we are done because that $i$-monochromatic set for $c'$ is also an $i + 1$-monochromatic set for $c$.

Suppose $B \in \mathcal{E}D_{n+1}^{\omega}$ is monochromatic positive for color 0 under the coloring $c$. For the coloring $c'$ restricted to $[A(0)]^2$, by the induction hypothesis applied to the ideal $\mathcal{E}D_{n+1} \downarrow A(0) \cong \mathcal{E}D_{n}$, there are two possibilities: there
exists a color \( i \in n + 2 \) and \( C \subseteq A(0) \) a monochromatic positive set for the ideal \( \mathcal{E}D_{n+1} \upharpoonright A(0) \), by Lemma 2.3, \( C \) is also positive for the ideal \( \mathcal{E}D_{n+1} \). If the color of the monochromatic is different from 0, we conclude again. Finally, suppose that is not the case, then for each \( N \in \omega \) and each \( i > 0 \), there exists an \( I_N \) monochromatic positive of size \( N \) and color \( i \), for the coloring \( c' \); that is, for the coloring \( c \) there are monochromatic sets of arbitrarily large sizes of all colors from 2 onwards; we just need to find a positive monochromatic set of color 0, of color 1 or well arbitrarily large finite monochromatic sets for both colors.

By Lemma 2.6, there exists a subset \( Y \subseteq B \) such that \( \mathcal{E}D_{n+1} \upharpoonright Y \approx \mathcal{E}D_{fin} \), by Theorem 4.14 of [13], it is established that \( \omega \rightarrow (\omega, \mathcal{E}D_{fin}^+)^2 \). By Proposition 2.9 and Remark 2.2, it is shown that \( Y \rightarrow (\omega, \omega, < \omega, (\mathcal{E}D_{n+1} \upharpoonright Y)^+)^2 \), that is, for the coloring \( c \) restricted to \( [Y]^2 \), there exists a positive monochromatic of one of the two colors 0 or 1, in which case we finish because a positive for the ideal \( \mathcal{E}D_{n+1} \upharpoonright Y \), is also a positive for the ideal \( \mathcal{E}D_{n+1} \), or there are arbitrarily large finite homogeneous sets, which also concludes the proof because we already knew that there were arbitrarily large finite monochromatic sets for all other colors (in this case).

**Theorem 2.1.** Let \( m \) be a natural number. Then \( \omega \rightarrow (\mathcal{E}D_\omega^+)^2 \) but \( \omega \not\rightarrow (\mathcal{E}D_\omega^+)^2 \).

In addition to the defined ideals, we can consider the ideal \( \mathcal{E}D_\omega = \bigcap_{n \in \omega} \mathcal{E}D_n \). It is important to note that, by definition, \( \mathcal{PC} \subseteq \mathcal{E}D_\omega \), which implies that the ideal \( \mathcal{E}D_\omega \) is tall, and it is the intersection of \( \omega \mathcal{F}_\sigma \) sets, which makes it of type \( \mathcal{F}_\sigma \delta \).

**Theorem 2.2.** Let \( n \) be a natural number. Then \( \omega \rightarrow (\mathcal{E}D_\omega^+)^2 \) holds. In particular, for any increasing function \( f \in \omega^\omega \), we have that \( \omega \rightarrow (\mathcal{PC}(f)^+)^2_n \) and \( \omega \rightarrow (\mathcal{T}\mathcal{C}(f)^+)^2_n \).

**Proof.** For any \( n \in \omega \), it holds that \( \mathcal{E}D_\omega \subseteq \mathcal{E}D_n \), \( \mathcal{PC}(f) \subseteq \mathcal{E}D_n \), and \( \mathcal{T}\mathcal{C}(f) \subseteq \mathcal{E}D_n \). This implies that any positive set for \( \mathcal{E}D_n \) is also a positive set for the other three ideals.

**Question 2.1.** Does the property \( \omega \rightarrow (\mathcal{PC}^+)^n \) hold for \( n > 2 \)?

The answer to the previous question should be affirmative; however, it was not further explored.
3 Insights into the Solecki ideal

In this section, we will explore some combinatorial properties of the Solecki ideal as well as another ideal that has not been previously defined. Let us begin with a couple of important definitions that will be used throughout this section.

Definition 3.1. Let $\Omega$ be the countable family of clopen subsets of the Cantor space $2^\omega$, each with a Haar measure of $\frac{1}{2}$. For the purposes of this discussion, we will use $\mu$ to denote the Haar measure. Here, $\Omega$ is defined as follows:

$$\Omega = \{U \subseteq 2^\omega : U \text{ is a clopen set and } \mu(U) = \frac{1}{2}\}.$$  

The Solecki ideal on $\Omega$, denoted by $S$, is defined as:

$$S = \{A \subseteq \Omega : \exists F \in [2^\omega]^{<\omega} \text{ such that } \forall U \in A, U \cap F \neq \emptyset\},$$

In this case, the letter $S$ refers to the Solecki ideal and should not be confused with the set family $S_0$ (selectors) from the first two sections of this article. It is known that the Solecki ideal $S$ is a tall $F_\sigma$ ideal.

In the first section, we sought a Borel ideal (indeed, we found a couple families of $F_\sigma$ ideals) for which player II has a winning strategy in the Cut and Choose Game related to that ideal. It is worth mentioning that we do not currently know if player II has a winning strategy in the Cut and Choose Game for Solecki’s ideal. We think it is likely that player I has a strategy (not player II). This question was also posed by J. Zapletal.

Question 3.1. Does player II have a winning strategy in $G_1(S)$?

Definition 3.2. Let $n$ be a natural number. We define the set $\Omega_n = \{U \in \text{Clopen}(2^\omega) : \mu(U) = \frac{1}{2n}\}$. For each natural number $n$ greater than 1, we define the set $S_n^+ = \{A \subseteq \Omega : (\forall V \in \Omega_n)(\exists U \in A)(U \cap V = \emptyset)\}$ and the set $S_\omega^+ = \bigcup_{n \in \omega} S_n^+$.

Note that for every $n \geq 2$, the family $S_n^+$ consists of positive sets with respect to the Solecki ideal (hence the notation with a ”+” in the definition). Before continuing, we require the following lemma:

Lemma 3.1. Let $A \in S_n^+$ and let $A = B \cup C$ be a partition of $A$. Then, either $B \in S_{n+1}^+$ or $C \in S_{n+1}^+$.

Proof. Suppose for the sake of contradiction that $B \notin S_{n+1}^+$ and $C \notin S_{n+1}^+$. Then, there exist clopen sets $V_1$ and $V_2$ of measure $\frac{1}{2n+1}$ each, such that for
every $U_1 \in B$ and $U_2 \in C$, we have $V_1 \cap U_1 \neq \emptyset$ and $V_2 \cap U_2 \neq \emptyset$. Therefore, $V = V_1 \cup V_2$ is a clopen set of measure at most $\frac{1}{2^n}$ that intersects every $U \in A$, which contradicts the assumption that $A \in S^+_n$. This completes the proof. ■

Proposition 3.1. The set $S_\omega = \mathcal{P}(\Omega) \setminus S^+\omega$ is an ideal.

Proof. By the previous lemma, if $A \in S^+\omega$ and $A$ can be written as the union of $B$ and $C$, then either $B \in S^+\omega$ or $C \in S^+\omega$. Additionally, it can be observed that $\Omega \in S^+\omega$. ■

Proposition 3.2. The ideal $S_\omega$ is an $F_{\sigma\delta}$ ideal that contains the Solecki ideal $S \subset S_\omega$. Consequently, $S_\omega$ is also a tall ideal.

Proof. To demonstrate that $S_\omega$ is an $F_{\sigma\delta}$ ideal, it suffices to show that $S^+_n$ is a $G_\delta$ set for every $n \in \omega$. Given that $\Omega_n$ is countable, for each $V \in \Omega_n$, the set

$$X_V = \{ A \subseteq \Omega : (\exists U \in A)(U \cap V = \emptyset) \}$$

is open in $2^\Omega$. Consequently, $S^+_n = \bigcap_{V \in \Omega_n} X_V$ is a $G_\delta$ set.

Moreover, for every $n > 0$ and for each $A \in S^+_n$, we have that $A \in S^+$. ■

Definition 3.3. We will say that $\omega \rightarrow (\omega, I^+)\frac{3}{2}$ holds if for every function $c : [\omega]^2 \rightarrow 2$, one of the following is true:

- There exists an infinite set $A \subseteq \omega$ such that $\text{colors}(c, A) = 0$, i.e., $A$ is homogeneous for the color 0.

- There exists a set $B \in I^+$ such that $B$ is homogeneous for the color 1 with respect to $c$.

We write $I^+ \rightarrow (\omega, I^+)\frac{3}{2}$ if, for every $A \in I^+$, the property $A \rightarrow (\omega, I \upharpoonright A^+)\frac{3}{2}$ holds.

Theorem 3.1. The ideal $S_\omega$ satisfies $S^+_\omega \rightarrow (\omega, S^+_\omega)\frac{3}{2}$.

Proof. This proof leverages the following claim, which will be applied repeatedly:

Claim: If $A \in S^+_n$ and $B \in S_\omega$, then $A \setminus B \in S^+_{n+1}$. This follows directly from Lemma 3.1.

Consider a positive set $A \in S^+_\omega$ and a coloring $c : [A]^2 \rightarrow 2$. We aim to recursively define a sequence of sets $\{A_n \in S^+_\omega : n \in \omega\}$ and elements $x_n \in A_n$ such that:
A0 = A,

xn ∈ An,

An+1 = \{ y ∈ An : c(\{xn, y\}) = 0 \}.

If the recursive construction continues indefinitely, then \{xn : n ∈ ω\} forms an infinite 0-homogeneous set, fulfilling our requirement. If the process halts, then there exists an n ∈ ω such that for every x ∈ An, the set \{ y : c(\{x, y\}) = 0 \} ⊆ S_ω. Given An ∈ S_ω^+, there is some m ∈ ω such that An ∈ S_m^+.

Let \{Vi : i ∈ ω\} enumerate Ω_{m+1}. Choose xo ∈ An such that xo ∩ V0 = \emptyset, possible because An ∈ S_m^+ and thus in S_{m+1}^+. Define

B1 = \{ y ∈ An : c(\{xo, y\}) = 1 \},

and note, by the claim, B1 ∈ S_{m+1}^+.

Recursively, for each i ∈ ω, define the following:

• Bi ∈ S_{m+1}^+,

• an element xi ∈ Bi such that xi ∩ Vi = \emptyset,

• Bi+1 = An \ (∪j<i\{ y ∈ Bj : c(\{xi, y\}) = 0 \}).

Here, An ∈ S_m^+ and ∪j<i\{ y ∈ Bj : c(\{xi, y\}) = 0 \} ∈ S_ω, hence by the claim, Bi+1 ∈ S_{m+1}^+ for each i ∈ ω (m + 1 is a fixed number). By construction, \{xi : i ∈ ω\} is a 1-homogeneous S_ω^+-positive set, concluding the proof.

A direct consequence of the previous theorem is a Ramsey-type property concerning the Solecki ideal. In particular, if there exists a witness coloring that demonstrates that Ω ↛ (S^+)^2_2, such a coloring must have infinite monochromatic sets for both colors.

**Corollary 3.1.** Ω → (ω, S^+)^2_2.

Now, we will prove that Ω ↛ (S^+)^2_2. To accomplish this, we will utilize a game closely related to the cut-and-choose game defined in the initial section.

**Definition 3.4** (See [22] and [9] as references for this game). Given an ideal I on ω, the game G_{fin}(I) is an infinite game defined as follows: player I starts the game by partitioning ω into two pieces ω = A_0^I ∪ A_1^I. Then, player
II chooses \(i_0 \in \{0, 1\}\) and selects \(a_0 \in [A_{i_0}^0]^{<\omega}\). In the \((n+1)\)-th move, player I partitions \(A_i^n\) into \(A_0^{n+1}\) and \(A_1^{n+1}\), and player II chooses \(i_{n+1} \in \{0, 1\}\) and selects \(a_{n+1} \in [A_{i_{n+1}}^{n+1}]^{<\omega}\). Player I wins the game if \(\bigcup_{n \in \omega} a_n \in \mathcal{I}\); otherwise, player II wins.

|    |  \(\omega = A_0^0 \cup A_1^0\) |  \(A_i^n = A_0^i \cup A_1^i\) |  \(i_0 \in 2, a_0 \subset A_i^0\) |  \(i_1 \in 2, a_1 \subset A_i^1\) |
|----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| II |                                 |                                 |                                 |                                 |

**Remark 3.1.** Given that player I has a winning strategy in \(G_{\text{fin}}(\mathcal{I})\) and \(\mathcal{I} \leq_K \mathcal{J}\), it follows that player I also possesses a winning strategy in \(G_{\text{fin}}(\mathcal{J})\). Conversely, if player II has a winning strategy in \(G_{\text{fin}}(\mathcal{I})\) and \(\mathcal{J} \leq_K \mathcal{I}\), then player II maintains a winning strategy in \(G_{\text{fin}}(\mathcal{J})\). The proofs of these assertions, while straightforward, are omitted for brevity.

**Proposition 3.3.** If \(\mathcal{I}\) is an ideal that can be extended to an \(F_\sigma\) ideal, that is, there exists \(\mathcal{J}\), an \(F_\sigma\) ideal such that \(\mathcal{I} \subseteq \mathcal{J}\), then player II has a winning strategy in \(G_{\text{fin}}(\mathcal{I})\).

**Proof.** Let \(\mathcal{J}\) be an \(F_\sigma\) ideal such that \(\mathcal{I} \subseteq \mathcal{J}\), and let \(\phi\) be a lower semicontinuous submeasure such that \(\text{Fin}(\phi) = \mathcal{J}\). A winning strategy for player II is as follows: in the \(n\)-th move, player I partitions a positive set of the ideal \(\mathcal{J}\). Then, player II chooses one of the partitions, which remains a positive set of the ideal \(\mathcal{J}\), and selects \(a_n\) to be a finite set such that \(\phi(a_n) = n\). At the end of the game, we have that \(\phi\left(\bigcup_{n \in \omega} a_n\right) = \infty\), thus player II wins the game. □

**Definition 3.5.** The \(\text{conv}\) ideal is defined on the countable set \(\mathbb{Q} \cap [0, 1]\) and consists of sets with at most finitely many accumulation points. This ideal is known to be a tall \(F_{\sigma\delta\sigma}\) ideal.

**Proposition 3.4.** Player I has a winning strategy in \(G_{\text{fin}}(\text{conv})\).

**Proof.** Let \(a_0 = 0\) and \(b_0 = 1\). To begin, Player I plays the partition \([0, \frac{1}{2}] \cup (\frac{1}{2}, 1]\) (note that these intervals are in the rational numbers, not the real numbers). Player II then selects one of the intervals, either \([a_0, \frac{a_0 + b_0}{2}]\) or \((\frac{a_0 + b_0}{2}, b_0]\), and provides a finite set \(F_0\) contained within the chosen interval.

Recursively, we construct two sequences of rational numbers, \(\{a_n : n \in \omega\}\) and \(\{b_n : n \in \omega\}\), such that:

- \(a_0 = 0\) and \(b_0 = 1\),
• if, in the $i$-th move, Player II chooses $[a_i, \frac{a_i+b_i}{2}]$, then $a_{i+1} = a_i$ and $b_{i+1} = \frac{a_i+b_i}{2}$,

• if, in the $i$-th move, Player II chooses $(\frac{a_i+b_i}{2}, b_i]$, then $a_{i+1} = \frac{a_i+b_i}{2}$ and $b_{i+1} = b_i$.

Player I’s winning strategy is as follows: in the $i$-th move, Player I partitions $[a_i, b_i)$ as $[a_i, \frac{a_i+b_i}{2}] \cup (\frac{a_i+b_i}{2}, b_i)$. Player II must then choose one of these two intervals and provide a finite set $F_i$ within the chosen interval. By the end of the game, the sequence of finite sets chosen by Player II will have a unique accumulation point. 

Definition 3.6. Let $U \in \Omega$ be a clopen set (of measure $\frac{1}{2}$). For a finite sequence $s \in 2^{<\omega}$, we denote by $\langle s \rangle$ the set of extensions of $s$, that is, $\langle s \rangle = \{x \in 2^\omega : x$ extends $s\}$. Using this notation, we define the sequence $x_U \in 2^\omega$ as follows:

• $x_U(0) = 0$ if $\mu(U \cap \langle (0) \rangle) \geq \mu(U \cap \langle (1) \rangle)$, otherwise $x_U(0) = 1$.

• In general, $x_U(n+1) = 0$ if $\mu(U \cap \langle x \upharpoonright n \neg 0 \rangle) \geq \mu(U \cap \langle x \upharpoonright n \neg 1 \rangle)$, otherwise $x_U(n+1) = 1$.

Note that, as $U$ is a clopen set, there exists an $n \in \omega$ such that for all $m \geq n$, $x_U(m) = 0$.

The following proposition illustrates the distinction between the ideals $S$ and $S_\omega$ in terms of the outcome of the game $G_{\text{fin}}$.

Proposition 3.5. Player I has a winning strategy in $G_{\text{fin}}(S_\omega)$, whereas Player II has a winning strategy in $G_{\text{fin}}(S)$.

Proof. The reason Player II has a winning strategy in $G_{\text{fin}}(S)$ is that $S$ is an $F_\sigma$ ideal.

Player I’s strategy in $G_{\text{fin}}(S_\omega)$ unfolds as follows:

Initially, Player I partitions $\Omega$ into two sets: $A_{(0)} = \{U \in \Omega : x_U(0) = 0\}$ and $A_{(1)} = \{U \in \Omega : x_U(0) = 1\}$. Player II then chooses $i_0 \in 2$ and a finite subset $a_0 \subseteq A_{(i_0)}$.

At the $n$-th move, having chosen the sequence $y = (i_0, i_1, \ldots, i_{n-1}) \in 2^{<\omega}$, Player II prompts Player I to partition $A_y$ into two subsets:

$A_y^{-0} = \{U \in A_y : x_U(n) = 0\}$ and $A_y^{-1} = \{U \in A_y : x_U(n) = 1\}$.

This pattern continues indefinitely.
At the game’s conclusion, Player II has selected the sequence \( x = (i_n : n \in \omega) \) and finite subsets \( a_n \in [A_{x[n]}]^{<\omega} \). Despite Player II’s choices, we assert that \( \bigcup_{n \in \omega} a_n \in S_\omega \). Choose \( n \in \omega \) such that \( \frac{1}{2^n} \leq \frac{\varepsilon}{2} \). For each \( U \) in the union \( \bigcup_{i \geq n} a_i \), the set \( U \) intersects the cone \( \langle x \upharpoonright n + 2 \rangle \), ensuring that \( \bigcup_{i \geq n} a_i \not\in S_\omega^{n+1} \). Since \( \bigcup_{i \leq n} a_i \) is finite, it is also true that \( \bigcup_{i \leq n} a_i \not\in S_\omega^{n+1} \). By Lemma 3.1 it follows that \( \bigcup_{n \in \omega} a_n \in S_\omega \). Consequently, \( \bigcup_{n \in \omega} a_n \in S_\omega \). ■

**Proposition 3.6.** \( \text{conv} \leq_K S_\omega \).

**Proof.** Define the function \( f : \Omega \to [0,1] \cap \mathbb{Q} \), given by the binary expression \( x_U \), where \( f(U) \) is a number written in binary and the \( n \)-th coordinate of its expression \( 0.d_0d_1\ldots d_n\ldots \) is given by \( x_U(n) = d_n \). Note that \( f \) is well-defined on the rationals, because for each \( U \in \Omega \), \( x_U \) ends with an infinite sequence of zeros.

For \( U, V \in \Omega \), we say that \( U \leq V \) if \( x_U \preceq x_V \), where \( \preceq \) denotes the lexicographic order on \( 2^\omega \). Observe that the function \( f \) respects this order defined on \( \Omega \) and the order on \( [0,1] \cap \mathbb{Q} \), that is, \( U \leq V \) if and only if \( f(U) \leq f(V) \).

Finally, it is not difficult to prove that the preimage under \( f \) of any convergent sequence belongs to \( S_\omega \). This is demonstrated in the same manner as the last paragraph of the proof of the previous proposition. Consequently, \( f \) serves as a witness that \( \text{conv} \leq_K S_\omega \). ■

**Corollary 3.2.** \( \Omega \not\rightarrow (S_\omega^+)^2 \).

**Proof.** This follows directly from the fact that \( R \leq_K \text{conv} \) (see [10] for a proof of this). ■

**Definition 3.7.** The nowhere dense ideal, denoted by \( \text{nwd} \), is the ideal on the set of rational numbers \( \mathbb{Q} \), consisting of all subsets of \( \mathbb{Q} \) that are nowhere dense in \( \mathbb{Q} \). It is known that this is an \( F_{\sigma\delta} \) tall ideal.

**Definition 3.8.** In the game \( G_3(\mathcal{I}) \), at each \( k \)-th move, player I selects an element \( I_k \) from the ideal \( \mathcal{I} \), and player II then picks an element \( n_k \) from the set \( \omega \setminus I_k \). Player I wins if the set \( \{n_k : k \in \omega\} \), chosen by player II, belongs to the ideal \( \mathcal{I} \).

**Proposition 3.7** (M. Hrušák and D. Meza, [22] Theorem 3.4.1). Let \( \mathcal{I} \) be an ideal on \( \omega \). If player II has a winning strategy in the game \( G_3(\mathcal{I} \upharpoonright X) \) for every \( X \in \mathcal{I}^+ \), then \( \mathcal{I} \leq_K \text{nwd} \).

**Proposition 3.8.** \( S_\omega \leq_K \text{nwd} \)
Proof. Let \( X \in S^+_\omega \) be a positive set. There exists \( n \in \omega \) such that \( X \in S^+_n \). Enumerate all clopen subsets of \( 2^\omega \) with measure \( \frac{1}{2^n} \) as \( \{ U_k : k \in \omega \} \). In the \( k \)-th move of the game \( G_3(S_\omega \restriction X) \), if player I plays \( I_k \in S_\omega \), then by Lemma 3.1, \( X \setminus I_k \in S^+_n \). Therefore, there exists \( V_k \in X \setminus I_k \) such that \( V_k \cap U_k = \emptyset \). Hence, player II can choose \( V_k \). At the end of the game, player II’s selection, \( \{ V_k : k \in \omega \} \), belongs to \( S^+_{n+1} \), establishing that player II has a winning strategy.

With that proposition, we conclude this section. In summary, we have performed a classification of the ideal \( S_\omega \) in the Katětov order, and as a corollary, we have derived a Ramsey-type property for the ideal \( S \). It is worth mentioning that the question of whether \( R \leq_K S \) remains unresolved, equivalently questioning if \( \Omega \to (S^+)^2 \).

4 Remarks on \( K \)-Uniform Ideals

Let \( \mathcal{I} \) be an ideal on \( \omega \). We say that \( \mathcal{I} \) is \( K \)-uniform if for every \( X \in \mathcal{I}^+ \), we have that \( \mathcal{I} \restriction X \leq_K \mathcal{I} \). As discussed on the second chapter, the ideal \( \mathcal{ED} \) restricted to \( \Delta = \{ (n, m) \in \omega^2 : m \leq n \} \) is known as the \( \mathcal{ED}_{\text{fin}} \) ideal.

The following are known properties of the \( \mathcal{ED}_{\text{fin}} \) ideal (see [10] and [13]):

- \( \mathcal{ED}_{\text{fin}} \) is an \( F_\sigma \) tall ideal.
- \( \mathcal{ED}_{\text{fin}} \) is a \( K \)-uniform ideal.
- If \( \mathcal{I} \) is an \( F_\sigma \) tall \( K \)-uniform ideal, then \( \mathcal{ED}_{\text{fin}} \leq_K \mathcal{I} \).

Moreover, previously, \( \mathcal{ED}_{\text{fin}} \) was the only known \( F_\sigma \) tall \( K \)-uniform ideal.

In this section, when discussing graphs on the natural numbers, we will primarily focus on the relation they represent, which can be viewed as a subset of \( [\omega]^2 \). Note that a graph is a non-empty set and an irreflexive, symmetric relation.

**Definition 4.1.** Let \( G \subseteq [\omega]^2 \) be a graph. A function \( c : \omega \to \kappa \) is a coloring of \( G \) with \( \kappa \) colors if for every pair of distinct elements \( a, b \in \omega \), \( c(a) = c(b) \) implies \( \{ a, b \} \notin G \). The chromatic number of \( G \), denoted by \( \chi(G) \), is the minimum cardinality of a set of colors such that there exists a coloring of \( G \) with that set of colors.

**Definition 4.2** (See [22] or [10]). \( G_f \) is the ideal on \( [\omega]^2 \) for which a set \( A \) belongs if and only if the chromatic number of the graph defined by \( A \) is finite. This ideal can also be described as the ideal generated by the collection of all graphs whose chromatic number is at most 2 (bipartite graphs).
It has been established that the ideal $G_{fc}$ is an $F_\sigma$ tall type and we have that $S \leq_K G_{fc}$ and $\mathcal{ED}_{fin} \leq_K G_{fc}$, refer to Chapter 1 of [22] for a proof.

**Definition 4.3.** Let $\langle I_n : n \in \omega \rangle$ be a partition of $\omega$ such that $|I_n| = n$. We denote by $K_n$ the complete graph on $I_n$, that is $K_n = [I_n]^2$, and let $K = \bigcup_{n \in \omega} K_n$.

We define the ideal $\mathcal{K}$ on $K$ as the set of subsets $X \subseteq K$ such that for some $n \in \omega$, $X$ does not contain a complete subgraph isomorphic to $K_n$.

**Remark 4.1.** $G_{fc} \leq_K G_{fc} \mid K$.

**Theorem 4.1.** The ideal $\mathcal{K}$ is an $F_\sigma$ tall $K$-uniform ideal which is not equivalent to $\mathcal{ED}_{fin}$ in the Katětov ordering.

**Proof.** Let $A \in K^+$ be a positive set. We recursively construct $K'_n$, a graph isomorphic to $K_n$, such that $K'_n$ is disjoint from $K'_i$ for all $i < n$. This is possible because $A \setminus \bigcup_{i<n} K'_i \in K^+$, allowing us to take each $K'_n$ disjoint from all previous ones. Thus, $\mathcal{K}$ is a $K$-uniform ideal. Since $G_{fc} \mid K \subseteq \mathcal{K}$, we establish that $S \leq_K \mathcal{K}$, in particular $\mathcal{K}$ is also a tall ideal. In the article [10] by M. Hrušak, a list of some known ideals and their relationships in the Katětov order is presented. The relationships provided are exhaustive; that is, $\mathcal{I} \leq_K \mathcal{J}$ if and only if this relationship is explicitly presented. In particular, it is proven that $S \not\leq_K \mathcal{ED}_{fin}$. Therefore, it follows that $\mathcal{K}$ and $\mathcal{ED}_{fin}$ are not equivalent in the Katětov ordering. Define

$$\mathcal{K}_n = \{ X \subset K : X \text{ does not contain a complete graph with } n \text{ vertices} \}.$$ 

It is clear that $\mathcal{K} = \bigcup_{n \in \omega} \mathcal{K}_n$. We aim to show that $\mathcal{K}_n$ is a closed set. Consider a set $X \notin \mathcal{K}_n$, meaning that $X$ includes a complete graph with $n$ vertices. Let $X_n \subseteq X$ be a finite subset representing a complete graph with $n$ vertices, with cardinality $[n]^2$. Note that $\{ Y \subseteq K : Y \supseteq X_n \}$ constitutes an open set that is disjoint from $\mathcal{K}_n$. Consequently, $X$ is an interior point of the complement of $\mathcal{K}_n$. This demonstrates that $\mathcal{K}_n$ is closed, thereby establishing that $\mathcal{K}$ is $F_\sigma$ and completing the proof. \(\blacksquare\)

With this theorem, a question posed by M. Hrušák is answered. The proof was quite straightforward once the correct definitions were in place. It is not difficult to imagine a family of distinct $K$-uniform, non-equivalent ideals in the Katětov order. For example, instead of using graphs where the edges are subsets of size 2, one could consider subsets of size $n \in \omega$ and ideals over $[\omega]^n$.

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