Abstract. In 2009, Sagan and Savage introduced a combinatorial model for the Fibonomial numbers, integer numbers that are obtained from the binomial coefficients by replacing each term by its corresponding Fibonacci number. In this paper, we present a combinatorial description for the $q$-analog and elliptic analog of the Fibonomial numbers. This is achieved by introducing some $q$-weights and elliptic weights to a slight modification of the combinatorial model of Sagan and Savage.

1. Introduction

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is one of the most important and beautiful sequences in mathematics. It starts with the numbers $F_0 = 0$ and $F_1 = 1$, and is recursively defined by the formula $F_n = F_{n-1} + F_{n-2}$.

Fibonacci analogs of famous numbers, such as the binomial coefficients and Catalan numbers:

$$\binom{m+n}{n} = \frac{(m+n)!}{m! \cdot n!} \quad \text{and} \quad 1 + \frac{2n}{n+1} \binom{2n}{n},$$

have intrigued some mathematicians over the last few years [ACMS14, BR14, BCMS18, CS14, SS10, TS19]. The Fibonomial and Fibo-Catalan numbers are defined, respectively, as:

$$\binom{m+n}{n}_F := \frac{F_{m+n}}{F_m \cdot F_n} \quad \text{and} \quad 1 + \frac{2n}{n+1} \binom{2n}{n}_F,$$

where $F_n := \prod_{k=1}^n F_k$ is the Fibonacci analog of the $n!$ number. These rational expressions turn out to be positive integers. In [SS10], Sagan and Savage introduced a combinatorial model to interpret the Fibonomial numbers in terms of certain tilings of an $m \times n$ rectangle. A path-domino tiling of an $m \times n$ rectangle is a tiling with monominos and dominos where one lattice path from $(0, 0)$ to $(m, n)$ is specified, and such that:

- all tiles above the path are either monominos or horizontal dominos;
- all tiles below the path are either monominos or vertical dominos; and
- all tiles that touch the path from below are vertical dominos.

We call these last tiles touching the path from below special vertical dominos, and denote by $T_{m,n}$ the collection of all path-domino tilings of an $m \times n$ rectangle. An example is illustrated on the left of Figure 1. The following result is a special case of [SS10, Theorem 3].

**Theorem 1.1 ([SS10]).** The Fibonomial number $\binom{m+n}{n}_F$ counts the number of path-domino tilings of an $m \times n$ rectangle.

The main objective of this paper is to present a $q$-analog and an elliptic analog generalization of this result. The resulting $q$-Fibonomial and elliptic Fibonomial numbers count the number of path-domino tilings of an $m \times n$ rectangle according to their $q$-weights and elliptic weights, respectively.
2. $q$-analog of the Fibonomial numbers

We denote by $\mathbb{N} := \{1, 2, 3, \ldots \}$ the set of natural numbers. The $q$-analog of $n \in \mathbb{N}$ is defined as
\[ [n]_q := 1 + q + q^2 + \cdots + q^{n-1}. \]
The evaluation of this polynomial at $q = 1$ recovers the number $n$. Sometimes we omit the subindex $q$ when it is clear from the context to simplify notation. Before studying the $q$-analog of the Fibonomial numbers, let us recall some useful and known straightforward lemmas.

**Lemma 2.1.** For $m, n \in \mathbb{N}$, the following identities hold:
\[
\begin{align*}
(m + n) \cdot [m]_q &= [m]_q + q^m [n]_q \\
(m \cdot n) \cdot [m]_q &= [m]_q [n]_q^m.
\end{align*}
\]

It is well known that the Fibonacci number $F_n$ counts the number of tilings of an $(n-1)$-strip (a rectangle from $(0, 0)$ to $(n-1, 1)$) using dominos and monominos. Given such a tiling $T$, we define the weight $\omega(T)$ of $T$ as the product of the weights of its tiles, where a monomino has weight 1 and a domino whose top-right coordinate is $(i, 1)$ has weight $q^i$. The weight of a 0-strip is by definition equal to 1.

**Lemma 2.2** (cf. [SS10]). For $n \in \mathbb{N}$, the $q$-analog of the Fibonacci numbers\(^2\) can be computed as
\[ [F_n]_q = \sum_T \omega(T), \]
where the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

**Proof.** The result is clearly true for $n = 1, 2$. Let $n > 2$, applying Equation (1) from Lemma 2.1 we get:
\[ [F_n]_q = [F_{n-1} + F_{n-2}]_q = [F_{n-1}]_q + q^{F_{n-1}}[F_{n-2}]_q \]
By induction, the first term of this sum corresponds to the tilings of an $(n-1)$-strip that finish with a monomino, while the second term to the tilings of an $(n-1)$-strip that finish with a domino. □

**Lemma 2.3.** For $m, n \in \mathbb{N}$, the following identities hold:
\[
\begin{align*}
F_{m+n} &= F_m F_{m+1} + F_n F_{n-1} \\
[F_{m+n}]_q &= [F_m]_q [F_{m+1}]_q [F_n]_q [F_{n-1}]_q.
\end{align*}
\]

**Proof.** Equation (3) is a well known identity for Fibonacci numbers, see for instance [SS10, Lemma 1]. The Fibonacci number $F_{m+n}$ counts the number of tilings of an $(m+n-1)$-strip with monominos and dominos. These tilings can be subdivided into two types: those containing a domino which is cut in two by the line $x = m$, and all other tilings. The first kind is counted by $F_m F_{n-1}$ (the number of tilings of an $(n-1)$-strip times the number of tilings of an $(m-2)$-strip), while the second kind is counted by $F_n F_{m+1}$ (the number of tilings of an $(m-1)$-strip times the number of tilings of an $(n-1)$-strip). Therefore, Equation (3) follows.

Applying Lemma 2.1 to (3) leads to Equation (4). □

For $m, n \in \mathbb{N}$, the $q$-analog of the Fibonomial number is defined as
\[ \left[ \begin{array}{c} m+n \\ n \end{array} \right]_q = \frac{[F_{m+n}]_q}{[F_m]_q [F_n]_q}, \]
where $[F_n]_q := \prod_{k=1}^n [F_k]_q$ is the $q$-Fibonomial analog of the $n!$ number. Surprisingly, this rational expression turns out to be a polynomial. Our objective is to present a combinatorial model to describe it. In order to achieve this, we will introduce some $q$-weights associated to path-domino tilings of an $m \times n$ rectangle.

Let $T \in T_{m,n}$ be a path-domino tiling of an $m \times n$ rectangle. The $q$-weights of the possible tiles in $T$ are defined as follows:

\[^2\]The elliptic and $q$-analogs of the Fibonacci numbers we use are different to the analogs considered e.g. in [SY18].
to the following equation we obtain: 

\[ q^\omega(T) = q^{1+2+3+10+24+5} = q^{51}. \]

where \((i, j)\) denotes the coordinate of the top-right corner of the tile, and the shaded vertical domino represents a special vertical domino touching the path from below. The \(q\)-weight of \(T\) is defined as the product of the weights of its tiles; see an example in Figure 1. The following theorem is one of our main results.

**Theorem 2.4.** For \(m, n \in \mathbb{N}\), the \(q\)-analog of the Fibonomial numbers is a polynomial in \(q\) with non-negative integer coefficients. It can be computed as

\[
\left[ \frac{m+n}{n} \right]_F = \sum_{T \in \mathcal{T}_{m,n}} \omega(T).
\]

**Proof.** Let us start proving the result for the initial cases \(m = 1\) or \(n = 1\).

For \(n = 1\), we have \([\frac{m+1}{n}]_F = [F_{m+1}]_q\). The collection \(\mathcal{T}_{m,1}\) coincides with the tilings of an \(m\)-strip with dominos and monominos, since only the last step of the specified lattice path can be a north step because of the special vertical domino condition. The weight of a domino in a tiling, whose top-right corner has coordinate \((i, 1)\), is \(q^{F_i} = q^{F_i}\). Therefore, the result follows from Lemma 2.2.

For \(m = 1\), we have \([\frac{1+n}{n}]_F = [F_{n+1}]_q\). The collection \(\mathcal{T}_{1,n}\) can be identified with the collection of tilings of a vertical \(n\)-strip with dominos and monominos, where the last domino has a special weight. The weight of a usual vertical domino, whose top-right corner has coordinate \((1, j)\), is \(q^{F_i} = q^{F_i}\), while the weight of a special vertical domino located at the same place is \(q^{F_i+1} = q^{F_i}\). Therefore, the result also follows from Lemma 2.2.

Now assume the result holds when \(m = 1\) or \(n = 1\). Let \(m, n > 1\), replacing \([F_{m+n}]_q\) from Lemma 2.3 to the following equation we obtain:

\[
\left[ \frac{m+n}{n} \right]_F = \frac{[F_{m+n}]_q[F_{m+n-1}]_q}{[F_m]_q[F_n]_q} + \frac{[F_{m+n}]_q[F_{m+n-1}]_q}{[F_m]_q[F_n]_q}
\]

By induction (and using again Lemma 2.2), the first term of the sum is the weighted counting of the path-domino tilings of the \(m \times n\) rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step.
Indeed, the path-domino tilings whose path ends with a north step have an extra contribution \([F_{m+1}]_{q^{F_{m}}}\) corresponding to the tilings of the last row with horizontal dominos and monominoes. The path-domino tilings whose path ends with an east step have an extra contribution \(q^{F_{m}F_{m+1}}[F_{n-1}]_{q^{F_{m}}}\); this corresponds to the weight of the forced special vertical domino \((q^{F_{m}F_{m+1}})\) and the tilings of the remaining \((n-2)\)-strip in the last column \([F_{n-1}]_{q^{F_{m}}}\).

\[ \begin{align*}
\text{Example 2.5.} & \quad \text{The polynomials } \binom{m+n}{n}_q \text{ are unimodal.} \\
\text{Example 2.6.} & \quad \text{Figure 2 illustrates an example of the } q\text{-Fibonomial for } m = n = 2. \\
\text{Example 2.7} \quad (n = 2). & \quad \text{Let } m \in \mathbb{N} \text{ and } n = 2. \text{ Theorem 2.4 leads to the identity:} \\
\text{(6) } & \quad [F_{m+2}][F_{m+1}] = \sum_{k=1}^{m+1} q^{F_{m+k}}[F_k]^2, \\
\text{where } c_k^m & = \sum_{i=k}^{m} F_{i+1}. \\
\text{The left hand side comes from the equality } & \quad \binom{m+2}{2}_q = [F_{m+2}][F_{m+1}]. \text{ The right hand side is the sum of the weights of all path-domino tilings of an } m \times 2 \text{ rectangle. In fact, the term } q^{F_{m+k}}[F_k]^2 \text{ indicates the sum of the weights of the path-domino tilings whose specified path is } E^{k-1}N^2E^{m-(k-1)}: q^{F_{m+k}} \text{ is the product of the weights of the special vertical dominos, and } [F_k]^2 \text{ is the weight of the two horizontal rows above the path. Since there are no more possibilities for the specified path due to the special vertical domino condition, the identity (6) follows. The evaluation at } q = 1 \text{ recovers} \\
\text{(7) } & \quad F_{m+2}F_{m+1} = \sum_{k=1}^{m+1} F_k^2. \\
\text{Remark 2.8.} & \quad \text{Equation (7) is a well known identity due to its relation with the golden ratio and golden spirals in nature, see for instance [Dun97, PL07]. The left hand side of the equation is the area of a } F_{m+2} \times F_{m+1} \text{ rectangle, which can be subdivided into a sequence of squares, with side lengths } F_1, F_2, \ldots , F_{m+1}, \text{ forming a spiral as illustrated in Figure 3 (left). This Fibonacci spiral is an approximation of the golden spiral, a special case of logarithmic spirals which describe the shape of various natural phenomena such as galaxies, nautilus shells and hurricanes. On the other hand, Equation (6) also has a natural geometric interpretation. The left hand side represents the weighted area of a } F_{m+2} \times F_{m+1} \text{ rectangle, where a unit square whose bottom-left corner is located at } (i,j) \text{ has weight } q^{i+j}. \text{ This rectangle can be subdivided into a sequence of squares, with side lengths } F_1, F_2, \ldots , F_{m+1}, \text{ in the north-east direction as illustrated in Figure 3 (right). The sum of their weighted areas is exactly the right hand side of Equation (6). This sum can also be interpreted as the “mass” of the rectangle, where the } F_k\text{-square has density } d(F_k) = q^{F_k}. \text{ This density increases according to the ratio } \frac{d(F_k)}{d(F_{k+1})} = q^{F_k+1}, \text{ satisfying the initial condition } d(F_{m+1}) = 1. \text{ It would be interesting to assign these densities to the squares giving rise to the Fibonacci spiral on Figure 3 (left), and see if the resulting equation has some physical meaning. Or even more interesting, to have a continuous version of the equation representing the mass of} \]
the Fibonacci spiral (or golden spiral), in order to describe some physical phenomenon in nature. For instance, it is quite natural to think that the density of galaxies grows exponentially as it approaches the center of the spiral. In Example 3.6, we additionally provide a generalization of Equation (6) using elliptic weight functions.

Figure 3. Geometric interpretation of Equations (6) and (7).

3. Elliptic analog of the Fibonomial numbers

Similarly as in the previous section, the elliptic analog of the Fibonomial number is obtained by replacing each term in the binomial coefficient by its corresponding Fibonacci “elliptic number”. The elliptic number is an elliptic function that generalizes the $q$-analog of a number and plays an important role in the theory of hypergeometric series and special functions. The elliptic number used here is an elliptic function that generalizes the $q$-analog of a number; it was introduced by Schlosser and Yoo in [SY17], motivated by work of Schlosser on elliptic binomial coefficients [Schar], and is of interest in the theory of hypergeometric series and special functions. The elliptic analog of the Fibonacci numbers we use is different to the one considered in [SY18].

An elliptic function is a function defined over the complex numbers that is meromorphic and doubly periodic. It is well known (cf. e.g. [Web91]) that elliptic functions can be obtained as quotients of modified Jacobi theta functions. These are defined as

$$\theta(x;p):= \prod_{j\geq 0} \left(1 - p^j x \right) \left(1 - \frac{p^{j+1}}{x} \right) \quad , \quad \theta(x_1,\ldots,x_\ell;p) = \prod_{k=1}^\ell \theta(x_k;p),$$

where $x,x_1,\ldots,x_\ell \neq 0$ and $|p| < 1$. The elliptic analog of a natural number $n \in \mathbb{N}$ (or simply elliptic number [SY17]) is defined as:

$$[n]_{a,b,q,p} := \frac{\theta(q^n, aq^n, bq^2; p)}{\theta(q, aq, bq^{1+n}, \frac{bq^{n-1}}{p}; p)}.$$ (8)

Here, $a,b$ are two additional parameters. The elliptic number is indeed an elliptic function [SY17, Remark 4]. Taking the limit $p \to 0$, then $a \to 0$ and then $b \to 0$, one recovers the $q$-analog $[n]_q$.

We simply define the elliptic analog of the Fibonacci number $F_n$ by $[F_n]_{a,b,q,p}$.

For $m,n \in \mathbb{N}$, the elliptic analog of the Fibonomial number is defined as

$$\left[ \begin{array}{c} m+n \\ n \end{array} \right]_{a,b,q,p} := \frac{[F_{m+n}]_{a,b,q,p}}{[F_m]_{a,b,q,p} \cdot [F_n]_{a,b,q,p}},$$ (9)

where $[F_n]_{a,b,q,p} := \prod_{k=1}^n [F_k]_{a,b,q,p}$ is the elliptic Fibonacci analog of the $n!$ number.

Similarly as before, the elliptic Fibonomial number counts path-domino tilings of an $m \times n$ rectangle according to certain elliptic weights. For $T \in \mathcal{T}_{m,n}$, the elliptic weights of the possible tiles in $T$ are defined as follows:

$$\bar{\omega} \left( \begin{smallmatrix} \\ \end{smallmatrix} \right) = 1 \quad \bar{\omega} \left( \begin{smallmatrix} * \\ \end{smallmatrix} \right) = \omega_1(i,j) \quad \bar{\omega} \left( \begin{smallmatrix} * \star \\ \end{smallmatrix} \right) = \omega_2(j,i)$$
where \((i, j)\) denotes the coordinate of the top-right corner of the tile, the shaded vertical domino represents a special vertical domino touching the path from below, and

\[
\omega_1(i, j) := v_{a, b; q, p}^1(F_i, F_{i-1}), \\
\omega_2(i, j) := v_{a, b; q, p}^2(F_{i+1} F_j, F_i F_{j-1}),
\]

are defined in terms of the following expression:

\[
v_{a, b; q, p}(m, n) := \theta(q^{2m+n} - aq^{n+1}, bq^n, bq^{n+1}, \frac{aq^{n-1}}{q^m}; p) q^n - \frac{aq^{n-1}}{q^m}; p)\]

Note that the weight of a “regular” vertical domino is evaluated at \((j, i)\) instead of \((i, j)\). This transposition does not make any difference for the \(q\)-analog of the Fibonomial numbers, but it does for the elliptic case. The elliptic weight \(\tilde{\omega}(T)\) of \(T\) is defined as the product of the weights of its tiles; see an example in Figure 4. The elliptic weight is a generalization of the \(q\)-weight, since we obtain the \(q\)-weight by taking the limit \(p \to 0, a \to 0\) and \(b \to 0\) in this order.

**Theorem 3.1.** For \(m, n \in \mathbb{N}\), the elliptic analog of the Fibonomial numbers can be computed as

\[
\left[\begin{array}{c}
m + n \\
n
\end{array}\right]_{a, b; q, p} = \sum_{T \in \mathcal{T}_{m, n}} \tilde{\omega}(T).
\]

The proof of this theorem follows the same steps as the proof of Theorem 2.4. The proofs of the technical lemmas and examples use some basic properties of theta functions summarized in the following proposition, which are essential in the theory of elliptic hypergeometric series.

**Proposition 3.2 (cf. [Web91, p. 451, Example 5]).** The theta function satisfies the following basic properties:

\[
\theta(x; 0) = 1 - x, \\
\theta(xy, u z, v x, \frac{u z}{x}; p) = \theta(uy, \frac{u z}{y}, xz, \frac{u x}{z}; p) + \frac{u}{z} \theta(zy, \frac{u x}{z}, ux, \frac{u x}{z}; p).
\]

Before proving Theorem 3.1, let us again prove some straightforward lemmas:

**Lemma 3.3.** For \(m, n \in \mathbb{N}\), the following identities hold:

\[
[m + n]_{a, b; q, p} = [m]_{a, b; q, p} + v_{a, b; q, p}(m, n)[n]_{a, b; q, p} \\
[m \cdot n]_{a, b; q, p} = [m]_{a, b; q, p}[n]_{a, b; q, p} \cdot q^m \cdot p
\]
Proof. Rearranging the left hand side of Equation (15) and applying Equation (14) (by taking $x = a \frac{q^m}{2}, y = a \frac{q^{-1}}{2}b^{-1}q^{-1}$, $u = a \frac{q^{m+n}}{2}$ and $z = a \frac{2}{2}$) yields:

$$[m + n]_{a,b,q,p} = \frac{\theta(bq^2, \frac{z}{2}; q^m)}{\theta(q, aq, bq^{m+n+1}, \frac{a}{b}q^{m+n+1}; q^m)} \cdot \frac{\theta(\frac{z}{2}q^{m-1}, bq^{m+1}, aq^{m+n}, q^{m+n}; q^m)}{\theta(\frac{z}{2}q^{m-1}, bq^{m+1}; q^m)}$$

$$= \frac{\theta(bq^2, \frac{z}{2}; q^m)}{\theta(q, aq, bq^{m+n+1}, \frac{a}{b}q^{m+n+1}; q^m)} \cdot \frac{\theta(\frac{z}{2}q^{m-1}, bq^{m+1}, aq^{m+n+1}, q^{m+n}; q^m) + q^n \theta(\frac{z}{2}, bq, aq^{2m+n}, q^n; q^m)}{\theta(\frac{z}{2}q^{m-1}, bq^{m+1}; q^m)}$$

$$= [m]_{a,b,q,p} + \frac{\theta(aq^{m+n}, bq, bq^{m+n+1}, \frac{a}{b}q^{m+n+1}; q^m) \cdot q^n [n]_{a,b,q,p}}{\theta(q, aq, bq^{m+n+1}, \frac{a}{b}q^{m+n+1}; q^m)}$$

Equation (16) follows from simple cancellations.

Using the same arguments as in the $q$-case and replacing the weight of a domino whose top-right coordinate is $(i,1)$ by $\omega_1(i,1)$, we obtain the following lemmas.

Lemma 3.4. For $n \in \mathbb{N}$, the elliptic analog of the Fibonacci numbers can be computed as

$$[F_n]_{a,b,q,p} = \sum_T \tilde{\omega}(T),$$

where the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

Proof. The result is clearly true for $n = 1, 2$. Let $n > 2$, applying Equation (15) from Lemma 3.3 we obtain:

$$[F_n]_{a,b,q,p} = [F_{n-1} + F_{n-2}]_{a,b,q,p}$$

$$= [F_{n-1}]_{a,b,q,p} + v_{a,b,q,p}(F_{n-1}, F_{n-2})[F_{n-2}]_{a,b,q,p}$$

$$= [F_{n-1}]_{a,b,q,p} + \omega_1(n-1,1)[F_{n-2}]_{a,b,q,p}.$$
Now assume the result holds when \( m = 1 \) or \( n = 1 \). Let \( m, n > 1 \), applying Lemma 3.5 we obtain:

\[
\begin{bmatrix} m + n \\ n \end{bmatrix}_{\mathcal{F}_{\omega, k, q, p}} = \frac{[F_{m+n}]_{a, b, k, q, p} [F_{m+n-1}]_{a, b, k, q, p}^1}{[F_m]_{a, b, k, q, p}^1 [F_{m-1}]_{a, b, k, q, p}^1}
\]

\[
= [F_{m+1}]_{a, b, q^1 ; q^F, n, p} [m + n - 1]_{\mathcal{F}_{\omega, k, q, p}}^{m - n - 1} + \omega_2(m, n) [F_{n-1}]_{a, b, q^1 ; q^F, m, n, p} [m - 1 + n]_{\mathcal{F}_{\omega, k, q, p}}^{m - n - 1}.
\]

By induction (and using again Lemma 3.4), the first term of the sum is the weighted counting of the path-domino tilings of the \( m \times n \) rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step.

Indeed, the path-domino tilings whose path ends with a north step have an extra contribution \([F_{m+1}]_{a, b, q^1 ; q^F, n, p}\). This corresponds to the weighted enumeration of the tilings of the last row with horizontal dominos and monominos. This follows from the fact that both quantities satisfy the same initial conditions and recurrence relation, which is obtained by applying Equation (15) to \( F_m + F_{m-1} \):

\[
[F_{m+1}]_{a, b, q^1 ; q^F, n, p} = [F_m]_{a, b, q^1 ; q^F, n, p} + \omega_2(m, n) [F_{m-1}]_{a, b, q^1 ; q^F, n, p}.
\]

The path-domino tilings whose path ends with an east step have an extra contribution \(\omega_2(m, n) [F_{n-1}]_{a, b, q^1 ; q^F, m, n, p}\).

This corresponds to the weight of the forced special vertical domino \(\omega_2(m, n)\) and the tilings of the remaining \((n-2)\)-strip in the last column \([F_{n-1}]_{a, b, q^1 ; q^F, m, n, p}\).

**Example 3.6** \((n = 2)\). The identity (6) in Example 2.7 for \(m \in \mathbb{N}\) and \(n = 2\) generalizes in the elliptic case to

\[
[F_{m+2}]_{a, b, q, p} [F_{m+n}]_{a, b, q, p} = \sum_{k=1}^{m+1} \Omega_k^m \left[ F_k \right]_{a, b, q, p},
\]

where \(\Omega_k^m = \prod_{i=k}^{m} \omega_2(i, 2)\) is the product of the weights of the special vertical dominos, and \([F_k]_{a, b, q, p} = [F_k]_{a, b, q^1 ; q^F, 2, p}\) is the weight of the two horizontal rows above the path.

**Example 3.7** \((a, b; p \to 0)\). By computing the limits \(p \to 0, a \to 0\) and \(b \to 0\) (in this order) of \(\bar{c}(T)\) and \(\omega_2(T)\), we obtain Theorem 2.4.

4. The \(q\)-Fibonacci analog of the rational Catalan numbers

Given a pair of relatively prime numbers \(m, n \in \mathbb{N}\) (that is, such that their greatest common divisor is \((m, n) := \text{gcd}(m, n) = 1\)), the \(m, n\)-Catalan number is defined as

\[
\text{Cat}_{m,n} := \frac{1}{m+n} \begin{bmatrix} m + n \\ n \end{bmatrix}.
\]

This number is equal to the number of lattice paths from \((0,0)\) to \((m,n)\) that stay above the main diagonal of the \(m \times n\) rectangle. The study of these numbers (also in the non-coprime case), and their \(q\)-analog and \(q,t\)-analog generalizations, has produced a substantial amount of research related to topics including rectangular diagonal harmonics, Shi hyperplane arrangements, affine Weyl groups, affine Hecke algebras, knot theory, and representation theory of Cherednik algebras.

The \(q\)-Fibonacci analog of the \(m,n\)-Catalan number is defined as

\[
[F\text{Cat}_{m,n}] := \frac{1}{[F_{m+n}]_{a, b, k, q, p}} \begin{bmatrix} m + n \\ n \end{bmatrix}_{\mathcal{F}} = \left[ F_{m+n-1} \right]_{a, b, k, q, p}^1.
\]

Surprisingly, this rational expression also turns out to be a polynomial when \((m, n)\) is equal to 1 or 2. Before proving it, we need the following lemma.
Lemma 4.1 ([HL74]). For $m, n \in \mathbb{N}$, we have $(F_m, F_n) = F_{(m,n)}$.

Proposition 4.2. If $(m, n) \in \{1, 2\}$, then $[\text{FCat}_{m,n}]$ is a polynomial in $q$ with integer coefficients.

Proof. First note that

\[
[F_n]^{m+n\choose n}_q = [F_m]^{m+n\choose n-1}_q .
\]

Since all the terms involved in this identity are polynomials, then $[F_{m+n}]$ divides $[F_n]^{m+n\choose n}_q$. But

\[
(F_{m+n}, F_n) = F_{(m+n,n)} = F_{(m,n)} = 1
\]

whenever $(m, n)$ is equal to 1 or 2. Therefore, the polynomials $[F_{m+n}]$ and $[F_n]$ have no roots in common. Thus, $[F_{m+n}]$ divides $[m+n\choose n]$. □

Computational experimentation suggests that the coefficients of these polynomials are nonnegative integers. However, we do not have a proof nor a combinatorial description to describe them.

Remark 4.3. The model of Sagan and Savage in [SS10] gives a combinatorial interpretation of the Lucas analog of the binomial coefficients. The Lucas numbers generalize the Fibonacci numbers, and are defined by the initial conditions $\{0\} = 0, \{1\} = 1$ and the recurrence $\{n\} = s\{n-1\} + t\{n-2\}$ for some variables $s, t$. It was proven in [BCMS18, Section 6.2], that the Lucas analog of the rational $m, n$-Catalan numbers is also polynomial in $s, t$ with integer coefficients. However, to the best of our knowledge, no $q$-analog of the Lucas-binomial coefficients have been studied in the literature. The proof of Proposition 4.2 was originally found by our colleague Shu Xiao Li, during discussions about this topic in the Algebraic Combinatorics Seminar at the Fields Institute in 2015. At this Seminar, our colleague Farid Aliniaeifard has shown that Conjecture 2.5 implies the positivity of the coefficients.

Remark 4.4. Given a crystallographic Coxeter group $W$ with Coxeter exponents $e_1 < e_2 < \ldots < e_n$, the rational $W$-Catalan number is defined as $C_W(a) = \prod_{i=1}^{n} \frac{a^{e_i}}{e_i}$, and this is an integer when $a$ is relatively prime to $e_n + 1$. The Coxeter exponents for the crystallographic Coxeter groups are:

| type of $W$ | $e_1, e_2, \ldots, e_n$ |
|-------------|-------------------------|
| $A_n$       | 1, 2, 4, \ldots, $n$    |
| $B_n$       | 1, 3, 5, \ldots, $2n - 1$ |
| $D_n$       | $n - 1$, 1, 3, 5, \ldots, $2n - 3$ |
| $E_6$       | 1, 4, 5, 7, 8, 11         |
| $E_7$       | 1, 5, 7, 9, 11, 13, 17     |
| $E_8$       | 1, 7, 11, 13, 17, 19, 23, 29 |
| $F_4$       | 1, 5, 7, 11                |
| $G_2$       | 1, 5                       |

The classical Catalan number corresponds to type $A_n$. We can now define a $q$-Fibonacci analog as follows:

\[
C_{W, a}^{q}(a) = \prod_{i=1}^{n} \frac{[F_{a+e_i}]}{[F_{e_i}+1]} .
\]

We have computationally checked that this is a polynomial with positive integer coefficients when $a$ and $e_n + 1$ are relatively prime, for each type and various values of $a$. It is interesting to note that although in type $A_n$ we have shown that it is a polynomial as long as $(F_a, F_{e_n+1}) = 1$, for other types we must have the stronger condition $(a, e_n + 1) = 1$. For example $C_{F_4, q}(2)$ is not a polynomial.

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\[3\]Their proof is reproduced from ours with our permission.
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(N. Bergeron) Department of Mathematics and Statistics, York University, Toronto

E-mail address: bergeron@mathstat.yorku.ca
URL: http://www.mathstat.yorku.ca/bergeron/

(C. Ceballos) Faculty of Mathematics, University of Vienna, Vienna, Austria

E-mail address: cesar.ceballos@univie.ac.at
URL: http://www.mat.univie.ac.at/~cesar/

(J. Küstner) Faculty of Mathematics, University of Vienna, Vienna, Austria

E-mail address: josef.kuestner@univie.ac.at