Application of engineering analysis techniques to the design of Magnetic Resonance Imaging (MRI) coils

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Abstract

In this paper, we develop a new approach to analysing and designing the gradient coils for magnetic resonance imaging (MRI) scanners for medical applications. More specifically, a novel higher-order BEM which satisfies the continuity equation for the current density is proposed. We also present solution procedures for applying this method to the inverse problem whereby the divergence-free surface current distribution in the gradient coil is deduced from knowledge of the magnetic flux density in a prescribed region of interest. The novel BEM proposed is a non-traditional one, in the sense that the collocation points are given by the vertices of the triangular elements only and not all the BEM nodes used to define the boundary elements. Furthermore, the degree of the interpolation is one degree less than that of the geometry of the triangular elements employed, so that (for example) the linear boundary elements involve constant interpolation for the surface current density. Moreover, the present method can be easily extended in order to obtain any desired degree of the interpolation for the surface current density. Within the inverse problem, care must be taken to employ the optimal value of the Tikhonov regularisation parameter. Results are presented relating to various geometries of coil, obtained using linear, quadratic and cubic variants of the boundary element formulation; those obtained using the quadratic and cubic elements agree almost precisely, while those from the linear elements exhibit small differences from those of the higher-order formulations.

Keywords: Inverse problem; Regularization; Divergence-free BEM; Magnetic resonance imaging (MRI)

1. Introduction

MRI is a non-invasive technique for imaging the human body, which has revolutionised the field of diagnostic medicine. MRI relies on the generation of highly controlled magnetic fields that are essential to the process of image production. In particular, an extremely homogeneous, strong, static field is required to polarise the sample and provide a uniform frequency of precession, while pure field gradients (which are required separately along the direction of the static field and in two perpendicular directions) are needed to encode the spatial origin of signals. The field gradients are generated by carefully arranged wire distributions generally placed on cylindrical surfaces surrounding the imaging subject, known as gradient coils [1-3].

The objective of the research reported here is to provide a computational tool to enable more complex geometries of gradient coil to be designed. This involves two stages, beginning with the development of a so-called direct boundary element method which enables the magnetic field distribution to be calculated from a known current distribution on a chosen geometry of coil. This approach is then incorporated into an inverse technique in which the desired distribution of magnetic field is used to calculate the current distribution necessary to achieve it.

The project was originated in response to an EPSRC initiative to promote collaboration between engineers and physicists, and in order to provide tools to improve the design of MRI equipment. Those involved in the project include two physicists, an electrical engineer, three mechanical engineers and a mathematician. Unusually for such a collaboration, the project involves the application of engineering techniques to the solution of a physics-based problem, rather than the use of specialist techniques from physics to solve an engineering problem.
2. Mathematical Formulation

The approach to the problem is to derive and implement a boundary element technique which enables the magnetic field distribution to be predicted from the current distribution on the gradient coil surface and vice versa. As will be seen, the eventual application of this BEM to the inverse problem requires in practice that the current be constrained within the formulation to flow tangentially to the surface of the coil and to show zero divergence, corresponding to the practical constraints that the conductors lie on the coil surface and there is conservation of current within the conductors.

In a non-magnetic material, as is the case of biological tissue, the magnetic flux density \( B = (B_x, B_y, B_z)^T \) satisfies the following system of partial differential equations [4]:

\[
\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}), \quad \nabla \cdot \mathbf{B}(\mathbf{x}) = 0, \quad \mathbf{x} = (x, y, z)^T \in \mathbb{R}^3. \tag{1}
\]

Here \( \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \) is the permeability of the free-space and \( \mathbf{J} = (J_x, J_y, J_z)^T \) is the current density which is defined as a surface current density \( \mathbf{J} = \mu_0 \mathbf{A} \), i.e.

\[
\mathbf{J}(\mathbf{x}) = \mathbf{J}^{\text{coil}}(\mathbf{x}') \delta(\mathbf{x}', \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{x}' \in \Gamma_{\text{coil}}, \tag{2}
\]

where \( \Gamma_{\text{coil}} \subset \mathbb{R}^3 \) is the coil surface and \( \delta(\mathbf{x}', \mathbf{x}) \) is the Kronecker delta function, such that

\[
\nabla \cdot \mathbf{J}^{\text{coil}}(\mathbf{x}) = 0, \quad \mathbf{J}^{\text{coil}}(\mathbf{x}) \cdot \mathbf{\nu}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_{\text{coil}}. \tag{3}
\]

with \( \mathbf{\nu} \) the outward unit vector normal to the coil surface \( \Gamma_{\text{coil}} \).

If the vector potential \( \mathbf{A} = (A_x, A_y, A_z)^T \) is introduced as:

\[
\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{4}
\]

then the system of partial differential equations (1) reduces to the following Poisson equation for the vector potential \( \mathbf{A} \):

\[
\nabla^2 \mathbf{A}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{5}
\]

In the direct problem formulation, the current density \( \mathbf{J}^{\text{coil}} \) is known on the coil surface \( \Gamma_{\text{coil}} \) and satisfies condition (3), whilst the vector potential \( \mathbf{A} \) is determined from the Poisson equation (5) by employing its integral representation, namely

\[
\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\mathbf{x}' = \frac{\mu_0}{4\pi} \int_{\Gamma_{\text{coil}}} \frac{\mathbf{J}^{\text{coil}}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\Gamma(\mathbf{x}'), \quad \mathbf{x} \in \mathbb{R}^3. \tag{6}
\]

On using eqns. (4) and (6), the magnetic flux density may be recast as

\[
\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\Gamma_{\text{coil}}} \frac{-(\mathbf{x} - \mathbf{x}') \times \mathbf{J}^{\text{coil}}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\Gamma(\mathbf{x}'), \quad \mathbf{x} \in \mathbb{R}^3. \tag{7}
\]

Motivated by the design of gradient coils used in MRI, we investigate the reconstruction of the divergence-free surface current distribution \( \mathbf{J}^{\text{coil}} \) from knowledge of one component of the magnetic flux density \( \mathbf{B} \) in a prescribed region of interest \( \Omega \subset \mathbb{R}^3 \), i.e. we focus on the following inverse problem:

Given \( \bar{B}_z(\mathbf{x}), \mathbf{x} \in \Omega \), find \( \mathbf{J}^{\text{coil}}(\mathbf{x}), \mathbf{x} \in \Gamma_{\text{coil}} \), such that:

\[
\begin{align*}
\bar{B}_z(\mathbf{x}) &= B_z(\mathbf{x}), \mathbf{x} \in \Omega, \quad \text{and} \quad \nabla \cdot \mathbf{J}^{\text{coil}}(\mathbf{x}) &= 0, \quad \mathbf{J}^{\text{coil}}(\mathbf{x}) \cdot \mathbf{\nu}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_{\text{coil}}. \tag{8}
\end{align*}
\]

3. Divergence-Free BEM

Assume that the coil surface \( \Gamma_{\text{coil}} \) is approximated as \( \Gamma_{\text{coil}} \approx \bigcup_{n=1}^{N} \Gamma_n \), where \( \Gamma_n, 1 \leq n \leq N \), are triangular boundary elements (not necessarily flat). In the following, we use the following notation:

- \( \Gamma_n := \Delta x^{n1}x^{n2}x^{n3}, 1 \leq n \leq N \), triangular boundary elements.
3.1. Geometry of the BEM

The parametrization of the triangular boundary elements is given by

\[
(x, y) \in \left\{ (\xi, \eta) \mid \xi \geq 0, \eta \geq 0, \xi + \eta \leq 1 \right\} \mapsto x(\xi, \eta) \in \Gamma_n, \quad x(\xi, \eta) = \sum_{j=1}^{N_n} N_j(\xi, \eta) x^{nj},
\]

where $N_j(\xi, \eta)$, $1 \leq j \leq N_n$, are given geometrical shape functions [5]. Consequently, the derivatives in the $\xi$– and $\eta$–directions may be recast as:

\[
\begin{cases}
\tau^{n\xi}(\xi, \eta) := \tau^{n\xi}(x(\xi, \eta)) = \frac{\partial x(\xi, \eta)}{\partial \xi} = \sum_{j=1}^{N_n} \frac{\partial N_j(\xi, \eta)}{\partial \xi} x^{nj} \\
\tau^{n\eta}(\xi, \eta) := \tau^{n\eta}(x(\xi, \eta)) = \frac{\partial x(\xi, \eta)}{\partial \eta} = \sum_{j=1}^{N_n} \frac{\partial N_j(\xi, \eta)}{\partial \eta} x^{nj}.
\end{cases}
\]

Then the surface metric (Jacobian) $J^n$ and the outward unit normal $\nu^n$ to the triangular boundary element $\Gamma_n$ are given by:

\[
J^n(\xi, \eta) := J^n(x(\xi, \eta)) = |\tau^{n\xi}(\xi, \eta) \times \tau^{n\eta}(\xi, \eta)|
\]

and

\[
\nu^n(\xi, \eta) := \nu^n(x(\xi, \eta)) = \frac{1}{J^n(\xi, \eta)} \tau^{n\xi}(\xi, \eta) \times \tau^{n\eta}(\xi, \eta).
\]

3.2. Basis Functions

On every triangular boundary element $\Gamma_n$, we define the following vectors, see also Figure 1:

\[
\begin{cases}
v^{n1}(\xi, \eta) := v^{n1}(x(\xi, \eta)) = -\frac{1}{J^n(\xi, \eta)} \tau^{n\eta}(\xi, \eta) \\
v^{n2}(\xi, \eta) := v^{n2}(x(\xi, \eta)) = \frac{1}{J^n(\xi, \eta)} \tau^{n\xi}(\xi, \eta) \\
v^{n3}(\xi, \eta) := v^{n3}(x(\xi, \eta)) = \frac{1}{J^n(\xi, \eta)} \left[-\tau^{n\xi}(\xi, \eta) + \tau^{n\eta}(\xi, \eta)\right].
\end{cases}
\]

From definition (13), it follows that the vectors $v^{ni}(\xi, \eta)$ satisfy the identity:

\[
\sum_{i=1}^{3} v^{ni}(\xi, \eta) = 0, \quad x = x(\xi, \eta) \in \Gamma_n.
\]

Next, we define the incidence function $i$ as follows:

\[
i(\cdot, \cdot) : \{1, 2, \ldots, M\} \times \{1, 2, \ldots, N\} \rightarrow \{0, 1, 2, 3\}
\]

\[
(m, n) \mapsto i(m, n) = \begin{cases}
0 & \text{if } x^m \neq x^{nj}, \forall j \in \{1, 2, 3\} \\
j & \text{if } \exists j \in \{1, 2, 3\} : x^m = x^{nj}.
\end{cases}
\]
For every global node $x^m$, $1 \leq m \leq M$, we define the set $C_m \subset \Gamma_{coil}$ of triangular boundary elements $\Gamma_n$, $1 \leq n \leq N$, adjacent to $x^m$, see also Figure 2, i.e.

$$C_m := \bigcup_{i(n,m) \neq 0}^{N} \Gamma_n, \quad 1 \leq m \leq M. \quad (16)$$

The vector basis function $f^m$ associated with the global node $x^m$ is defined by

$$f^m(\cdot) : \Gamma_{coil} \rightarrow \mathbb{R}^3, \quad f^m(x) = \begin{cases} v^{n,i(m,n)}(x) & \text{if } x \in C_m \\ 0 & \text{if } x \notin C_m \end{cases} \quad (17)$$

and, clearly, its support is a subset of $C_m$, i.e. $\{x \in \Gamma_{coil} | f^m(x) \neq 0 \} \subset C_m$. 

Figure 1: Schematic diagram of the (a) quadratic triangular boundary element $\Gamma_n$ in the physical space $\mathbb{R}^3$, and (b) the transformed quadratic triangular boundary elements $\Gamma_n(\xi, \eta)$ in the parametric space $(\xi, \eta)$.

Figure 2: The set $C_m$ of boundary elements $\Gamma_n$ adjacent to the global node $x^m$ and the corresponding vector $v^{n,i(m,n)}(x)$ in the physical space $\mathbb{R}^3$. 

$$\Gamma_{n1} \quad v^{n1}(x) \quad x \quad v^{n2}(x) \quad \Gamma_{n2} \quad \Gamma_{n3} \quad x^{n1} \quad x^{n2} \quad x^{n3} \quad x^{n4} \quad v^{n4}(x) \quad x^{n5} \quad x^{n6} \quad v^{n6}(x) \quad \Gamma_n \quad v^{n3}(x) \quad x \quad v^{n5}(x, 1/2) \quad x^{n5}(0, 1/2) \quad \Gamma_n(\xi, \eta) \quad x^{n3}(0, 0) \quad x^{n4}(1/2, 1/2) \quad x^{n6}(1/2, 0) \quad x^{n1}(1, 0) \quad \eta \quad \xi$$

$v^{n3}(\xi, \eta)$
3.3. Surface Current Density

The current density \( \mathbf{J}^{\text{coil}} \) on the coil surface \( \Gamma_{\text{coil}} \) is then approximated by

\[
\mathbf{J}^{\text{coil}}(\mathbf{x}) \approx \sum_{m=1}^{M} \mathbf{I}_m \mathbf{G}_m(\mathbf{x}) = \sum_{m=1}^{M} \mathbf{I}_m \sum_{i(m,n)}^{N} \mathbf{v}^{n,i(m,n)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_{\text{coil}},
\]

where \( \mathbf{I}_m \in \mathbb{R}, \, 1 \leq m \leq M \), are unknown coefficients that correspond to the stream function intensities. For direct problems, the stream function intensities are determined from appropriate boundary conditions, while in the case of inverse problems, they are obtained by solving a minimisation problem.

It should be noted that the degree of the approximation (18) for the surface current density \( \mathbf{J}^{\text{coil}} \) is one unit less than the degree of the triangular boundary elements \( \Gamma_n \), \( 1 \leq n \leq N \), since the vectors \( \mathbf{v}^i(\xi, \eta) \), \( 1 \leq i \leq 3 \), are related to the derivatives of the geometrical shape functions \( N_i(\xi, \eta) \), \( 1 \leq i \leq N \), associated with the triangular boundary element \( \Gamma_n \), see eqns. (9) – (13). More precisely, linear, quadratic and cubic triangular boundary elements provide constant, linear and quadratic approximations for the surface current density, respectively. From eqns. (12) and (13) it follows that for every triangular boundary element \( \Gamma_n \), the vectors \( \mathbf{v}^i(\xi, \eta) \), \( 1 \leq i \leq 3 \), and the outward unit normal vector \( \mathbf{n}^\Gamma(\xi, \eta) \) are orthogonal and hence expression (18) enforces the approximated current density \( \mathbf{J}^{\text{coil}} \) to lie in the plane tangent to the coil surface \( \Gamma_{\text{coil}} \), i.e. condition (31) is satisfied, furthermore, the interpolation given by eqn. (18) is divergence-free pointwise, i.e. condition (31) is satisfied, since

\[
\nabla \cdot \frac{\partial \mathbf{x}}{\partial \xi} = \frac{\partial}{\partial \xi} (\nabla \cdot \mathbf{x}) = 0 \quad \text{and} \quad \nabla \cdot \frac{\partial \mathbf{x}}{\partial \eta} = \frac{\partial}{\partial \eta} (\nabla \cdot \mathbf{x}) = 0.
\]

3.4. Magnetic Vector Potential and Magnetic Flux Density

According to eqns. (6), (7) and (18), the magnetic vector potential \( \mathbf{A} \) and magnetic flux density \( \mathbf{B} \) are approximated by

\[
\mathbf{A}(\mathbf{x}) \approx \frac{\mu_0}{4\pi} \sum_{m=1}^{M} \mathbf{I}_m \sum_{i(m,n)}^{N} \int_{\Gamma_n} \frac{\mathbf{v}^{n,i(m,n)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\Gamma(\mathbf{x}'),
\]

and

\[
\mathbf{B}(\mathbf{x}) \approx \frac{\mu_0}{4\pi} \sum_{m=1}^{M} \mathbf{I}_m \sum_{i(m,n)}^{N} \int_{\Gamma_n} \frac{-(\mathbf{x} - \mathbf{x}') \times \mathbf{v}^{n,i(m,n)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, d\Gamma(\mathbf{x}'),
\]

4. Description of the Algorithm

If the \( z \)-component of the magnetic flux density \( \mathbf{B} \) is known at \( L \) points in the region of interest \( \Omega \) then the BEM discretisation of the inverse problem (8) yields the following system of linear algebraic equations

\[
\mathbf{H} \mathbf{I} = \mathbf{B}_z.
\]

Here \( \mathbf{H} \in \mathbb{R}^{L \times M} \) is the BEM matrix used for computing the \( z \)-component of the magnetic flux density \( \mathbf{B} \) given by eqn. (20) calculated at \( L \) points in the region of interest \( \Omega \), \( \mathbf{B}_z = (\mathbf{B}_z^1, \ldots, \mathbf{B}_z^L)^T \in \mathbb{R}^L \) is a vector containing the \( z \)-component of the magnetic flux density at \( L \) points in the region of interest \( \Omega \) and \( \mathbf{I} \in \mathbb{R}^M \) is a vector containing the unknown values of the stream function \( \mathbf{I}_m, \, 1 \leq m \leq M \), at the global nodes.

The system of linear algebraic equations (21) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce an inaccurate and/or physically meaningless solution due to the large value of the condition number of the system matrix \( \mathbf{H} \) which increases dramatically as the BEM mesh is refined. Several regularization procedures have been developed to solve such ill-conditioned systems [6, 7]. In the following, we only consider the Tikhonov regularization method and for further details on this method, we refer the reader to [6].
4.1. Magnetic Energy and Regularization

The magnetic energy $W$ defined by

$$W = \frac{1}{2} \int_{\Gamma_{\text{coil}}} J^{\text{coil}}(x) \cdot A(x) \, d\Gamma(x)$$

(22)

is approximated, according to eqns. (18) and (19), as

$$W \approx \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} L_{mn} I_n I_m,$$

(23)

where the components of the inductance matrix $L = [L_{mn}] \in \mathbb{R}^{M \times M}$ are given by

$$L_{mn} := \frac{\mu_0}{4\pi} \sum_{i(m_i, m_j) \neq 0}^{N} \sum_{i(n_i, n_j) \neq 0}^{N} \int_{\Gamma_{m'}} \int_{\Gamma_{n'}} \frac{\mathbf{v}^{m', i(m, m')} \cdot \mathbf{v}^{n', i(n, n')}}{|x - x'|} \, d\Gamma(x') \, d\Gamma(x).$$

(24)

The approximated magnetic energy $W$ given by eqn. (23) is a quadratic and positive definite form which induces the following discrete energy norm:

$$\|I\|_{W}^2 := \|\tilde{L}I\|^2 = \sum_{m=1}^{M} \sum_{n=1}^{M} L_{mn} I_n I_m = 2W,$$

(25)

where $\tilde{L} \in \mathbb{R}^{M \times M}$ such that $\tilde{L}^T = \tilde{L}$ and $L = \tilde{L}^T \tilde{L}$.

The Tikhonov regularized solution $I_\lambda$ to the inverse problem (8) is sought as [5]

$$I_\lambda \in \mathbb{R}^M : \quad \mathcal{F}_\lambda(I_\lambda) = \min_{I \in \mathbb{R}^M} \mathcal{F}_\lambda(I),$$

(26)

where $\mathcal{F}_\lambda$ is the Tikhonov functional given by

$$\mathcal{F}_\lambda(\cdot) : \mathbb{R}^M \rightarrow [0, \infty), \quad \mathcal{F}_\lambda(I) = \frac{1}{2}\|H I - \tilde{B}_z\|^2 + \frac{1}{2}\lambda\|I\|_W^2,$$

(27)

with $\lambda > 0$ the regularization parameter to be chosen. Formally, the Tikhonov regularized solution $I_\lambda$ of the minimisation problem (26) is given by the solution of the regularized normal equation [6]

$$\left( H^T H + \lambda \tilde{L}^T \tilde{L} \right) I_\lambda = H^T \tilde{B}_z.$$

(28)

5. Numerical Results

In order to present the performance of the proposed method, we solve the inverse problem (8) for the following examples:

**Example 1.** We consider a hemispherical coil $\Gamma_{\text{coil}} = \partial B(0, R) \cap \{z \geq 0\}$, where $R = 0.175 \text{m}$. Here the region of interest is a sphere of radius $r = 0.065 \text{m}$ and centered at $x^c = (0, 0, 0.081)$, i.e. $\Omega = B(x^c, r)$.

**Example 2.** We consider a cylindrical coil $\Gamma_{\text{coil}} = \partial B(0, R) \times (-h/2, h/2)$, where $R = 0.5 \text{m}$ and $h = 2.0 \text{m}$. Here the region of interest is a sphere of radius $r = 0.2 \text{m}$ and centered centered at the origin of the coordinate system $x^c = (0, 0, 0)$, i.e. $\Omega = B(x^c, r)$.

Since the geometry of the coils considered in this paper is symmetrical with respect to the $z$–axis, it is sufficient to investigate only the design of $x$– and $z$–gradients, i.e. $\tilde{B}_z(x) = G_z x$, $x \in \Omega$ and $\tilde{B}_z(x) = G_z x$, $x \in \Omega$, where $G_x = G_z = 1.0 \text{ T m}^{-1}$.

The choice of the regularization parameter $\lambda$ in the minimisation process of the Tikhonov functional (27) is crucial for obtaining a stable, accurate and physically correct numerical solution $I_\lambda$. The optimal value $\lambda_{\text{opt}}$ of the regularization parameter $\lambda$ should be chosen such that a trade-off between the two
quantities $\|HI-\tilde{B}_z\|$ and $\|I\|_W=\|\tilde{L}I\|$ involved in the minimisation of the functional (27) is attained. To do so, we introduce a global measure for error that relates the computed and desired $z-$components of the magnetic flux density in the region of interest $\Omega$, namely the maximum relative percentage error

$$\text{Err} (B_z; \lambda) = \max_{x \in \Omega} \frac{|B_z^\lambda(x) - \tilde{B}_z(x)|}{|\tilde{B}_z(x)|} \times 100$$

(29)

where $B_z^\lambda(x)$ is the numerical $z-$component of the magnetic flux density calculated at the point $x$ in the region of interest $\Omega$, for a given regularization parameter $\lambda$, by employing the proposed BEM-based algorithm. On assuming that a deviation $\epsilon > 0$ from the desired $z-$component of the magnetic flux density $\tilde{B}_z$ is admissible in $\Omega$, such that

$$\tilde{B}_z^\epsilon(x) := \tilde{B}_z(x) (1 \pm \epsilon), \quad x \in \Omega,$$

(30)

then the choice of the optimal regularization parameter $\lambda_{\text{opt}}$ is made by employing the maximum relative percentage error given by eqn. (29) and the admissible level of noise in $B_z|\Omega$ defined by relation (30), namely

$$\lambda_{\text{opt}} = \max \{ \lambda > 0 \mid \text{Err} (B_z; \lambda) \leq \epsilon \}.$$  

(31)

The numerical solution $I_{\lambda}$ of the regularized system of normal equations (28), with $\lambda = \lambda_{\text{opt}}$ given by eqn. (31), provides only a discrete distribution of the stream function at the global nodes of the BEM mesh employed. However, these discrete values should be extended to a continuous distribution of the numerical stream function over the entire coil surface $\Gamma_{\text{coil}}$ and this is achieved by employing the contours of the stream function using its discrete distribution and the Matlab (The Mathworks, Inc., Natick, MD, USA) contouring function. Hence, in the following, the numerically retrieved solutions of the inverse problem given by eqn. (8) are presented in terms of the contours of the stream function as described above.

Figures 3(a) and (b) present the contours of the stream function in the so-called Lambert cylindrical equal-area projection, i.e. the $\theta - \cos \phi$ plane, corresponding to the hemispherical $x-$ and $z-$gradient coils, respectively, obtained using the optimal regularization parameter $\lambda_{\text{opt}}$ given by eqn. (31), $L = 351$ internal points in the region of interest and $N = 2840$ linear (---), quadratic (----) and cubic (-----) triangular boundary elements. From these figures it can be seen that, for the hemispherical $x-$ and $z-$gradient coils given by Example 1, the numerical results retrieved using linear boundary elements
are more inaccurate than those obtained by employing higher-order boundary elements, with the mention that there are no major quantitative differences between the contours of the stream function corresponding to quadratic and cubic triangular elements. Similar results have been obtained for the cylindrical x– and z–gradient coils and therefore they are not presented here.

It is also important to mention that the Tikhonov regularization method, in conjunction with the proposed divergence-free BEM, is also convergent with respect to increasing the number of boundary elements used to discretise the coil surface Γcoil. Furthermore, the finer the BEM mesh size is, the smoother are the contours of the stream function corresponding to the cylindrical and hemispherical x– and z–gradient coils. These properties are illustrated in Figures 4(a) and (b) which present the contours of the stream function in the θ–z plane corresponding to the cylindrical x– and z–gradient coils given by Example 2, respectively, obtained using the optimal regularization parameter λopt chosen according to eqn. (31), L = 351 internal points in the region of interest and various numbers of quadratic triangular boundary elements, i.e. N = 6, namely N = 1152 (—), N = 2084 (−−) and N = 3200 (⋯).

Figure 4: The contours of the stream function corresponding to the cylindrical (a) x–, and (b) z–gradient coils given by Example 2, obtained using the optimal regularization parameter λopt chosen according to eqn. (31), L = 351 internal points in the region of interest and various numbers of quadratic triangular boundary elements, i.e. N = 6, namely N = 1152 (―), N = 2084 (−−) and N = 3200 (⋯).

6. Conclusions

In this paper, we have investigated the design of MRI cylindrical and hemispherical gradient coils by considering the reconstruction of a divergence-free surface current distribution from knowledge of the magnetic flux density in a prescribed region of interest. This inverse problem was formulated in the framework of static electromagnetism using its corresponding integral representation according to potential theory. In order to retrieve an accurate and physically correct numerical solution of this inverse problem, a minimisation problem for the Tikhonov functional was solved, in conjunction with a novel higher-order BEM which satisfies the continuity equation for the current density. The numerical solutions were presented in terms of the contours of the stream function and using various types of boundary elements. For the examples analysed, it was proved the efficiency of the proposed method, as well as an improvement in the accuracy of the numerical solutions in the case of higher-order elements. However, there are no major quantitative differences between the contours of the stream function corresponding to quadratic and cubic triangular elements.

Furthermore, higher-order divergence-free interpolation formulae of any degree k > 0 for the surface current density can be easily derived by considering the appropriate geometrical shape functions.
of degree \((k + 1)\) associated with the triangular boundary elements. It is important to mention that the collocation points, i.e. global nodes, are always located at the vertices of the triangular boundary elements employed in the BEM meshing of the coil surface and, therefore, increasing the degree of the interpolation for the surface current density does not affect the number of collocation points and hence the dimension of the resulting BEM system of linear algebraic equations.

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