A New Quantization Map

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Abstract

In this paper we find a simple rule to reproduce the algebra of quantum observables using only the commutators and operators which appear in the Koopman-von Neumann (KvN) formulation of classical mechanics. The usual Hilbert space of quantum mechanics becomes embedded in the KvN Hilbert space: in particular it turns out to be the subspace on which the quantum positions Q and momenta P act irreducibly.

1 Introduction

It is well known that in classical statistical mechanics the evolution of the probability densities in phase space \( \rho(q,p,t) \) is given by the Liouville equation

\[
i \frac{\partial}{\partial t} \rho(q,p,t) = \hat{L} \rho(q,p,t)
\]

where \( \hat{L} \) is the Liouville operator

\[
\hat{L} = -i \partial_q H(q,p) \partial_p + i \partial_p H(q,p) \partial_q
\]

and \( H(q,p) \) is the Hamiltonian in the phase space \( M \) of the system. In [1] and [2] KvN formulated classical mechanics in a Hilbert space made up of complex square integrable functions over the phase space variables \( \psi(q,p,t) \). In particular they postulated, as equation of evolution for \( \psi(q,p,t) \), the Liouville equation itself:

\[
i \frac{\partial}{\partial t} \psi(q,p,t) = \hat{L} \psi(q,p,t).
\]

Starting from [3] it is easy to prove that, since the Liouvillian \( \hat{L} \) contains only first order derivatives, the Liouville equation [1] for the probability densities \( \rho(q,p,t) \) can be derived via the postulate \( \rho(q,p,t) = |\psi(q,p,t)|^2 \). Finally KvN imposed on the states of their Hilbert space the following scalar product:

\[
\langle \psi | \tau \rangle = \int dq dp \psi^*(q,p) \tau(q,p).
\]
With this choice the Liouvillian $\hat{L}$ is a Hermitian operator. Therefore $\langle \psi | \psi \rangle = \int dqdp |\psi(q,p)|^2$ is a conserved quantity and $|\psi(q,p)|^2$ can be consistently interpreted as the probability density of finding a particle in a point of the phase space.

Now every theory formulated via states and operators in a Hilbert space can be formulated also via path integrals. It was proved in Ref. [3] and [4] that it is possible to describe the evolution (3) in the following manner:

$$\psi(\varphi, t) = \int D\varphi \langle \varphi, t | \varphi_i, t_i \rangle \psi(\varphi_i, t_i)$$

where we have indicated with $\varphi \equiv (q, p)$ all the phase space variables. The kernel of propagation $\langle \varphi, t | \varphi_i, t_i \rangle$ has the following path integral expression:

$$\langle \varphi, t | \varphi_i, t_i \rangle = \int D\varphi D\lambda \exp \left[ i \int dt (\lambda_a \dot{\varphi}^a - L) \right]$$

(5)

where the double prime in $D''\varphi$ indicates that the integration is over paths with fixed end points in $\varphi$. The $L$ in the weight of the path integral (5) is given by:

$$L = \lambda_a \omega_{ab} \partial_b H \equiv \lambda_q \partial_p H - \lambda_p \partial_q H.$$  

(6)

Having a path integral we can introduce the concept of a commutator as Feynman did in the quantum case: given two functions $O_1(\varphi, \lambda)$ and $O_2(\varphi, \lambda)$, we can evaluate the following quantity under the path integral (5): $\hat{[O_1, O_2]} = \lim_{\varepsilon \to 0} \langle O_1(t + \varepsilon)O_2(t) - O_2(t + \varepsilon)O_1(t) \rangle$. What we get is [3]:

$$\hat{[\varphi^a, \varphi^b]} = 0, \quad \hat{[\hat{\lambda}_a, \hat{\lambda}_b]} = 0, \quad \hat{[\hat{\varphi}^a, \hat{\lambda}_b]} = i\delta^a_b.$$  

(7)

The first commutator of (7) tells us that the positions $\hat{q}$ commute with the momenta $\hat{p}$, i.e. that we are doing classical and not quantum mechanics. The last commutator instead tells us the $\hat{\lambda}_a$ are something like the momenta conjugate to $\hat{\varphi}^a$. In order to satisfy (7) we can use the representation in which $\hat{\varphi}^a$ is a multiplicative operator and $\hat{\lambda}_a$ a derivative one:

$$\varphi^a = \varphi^a, \quad \hat{\lambda}_a = -i \frac{\partial}{\partial \varphi^a}.$$  

(8)

Via the previous operatorial realization, the $L$ of (6) can also be turned into an operator: $L \rightarrow \hat{L} = -i\omega_{ab} \partial_b H \partial_a = -i\partial_p H \partial_q + i\partial_q H \partial_p$. Therefore $\hat{L}$ is just the Liouville operator of Eq. (2). This confirms that the operatorial formalism lying behind the path integral (5) is nothing more than the KvN one.

The main problem we are interested in this paper is the quantization of classical mechanics, once it is formulated via operatorial or path integral techniques. We know that usually the quantization of a system is performed via the Dirac’s correspondence rules, i.e. by replacing the classical Poisson brackets $\{\cdot, \cdot\}_P$ by commutators according to the following relation:

$$\{\cdot, \cdot\}_P \rightarrow \frac{\{\cdot, \cdot\}_Q}{i\hbar}.$$  

Now, in classical mechanics formulated à la KvN, the Poisson brackets are already replaced by the KvN commutators [7]. So the quantization of the system can be performed:

1) either finding suitable rules to go from the KvN commutators to the quantum ones;

2) or finding a way to reproduce the algebra of quantum observables using KvN commutators.
and operators.

In this paper we will find a compact way to implement the second alternative, by using the fact that the set of Hermitian operators in the KvN theory includes also operators depending on \( \hat{\lambda} \) and not only the functions \( f(\hat{q}, \hat{p}) \). In particular in Sec. 2 we will introduce the following map:

\[ Q : f(\hat{q}, \hat{p}) \rightarrow f\left(\hat{q} - \frac{\hbar}{2} \hat{\lambda}_p, \hat{p} + \frac{\hbar}{2} \hat{\lambda}_q\right) \]  

(9)

where the domain of \( Q \) is made up of all the standard classical observables \( f(\hat{q}, \hat{p}) \) while the image of \( Q \) is made up of suitable Hermitian operators living in the KvN Hilbert space and depending also on \( \hat{\lambda} \). Using the KvN commutators \( [\hat{q}, \hat{p}] = \hbar \) all the functions of the form \( f\left(\hat{q} - \frac{\hbar}{2} \hat{\lambda}_p, \hat{p} + \frac{\hbar}{2} \hat{\lambda}_q\right) \) which appear on the RHS of (9) satisfy exactly the algebra of the observables of quantum mechanics. So we can say that, among all the operators of the KvN Hilbert space, there is a set whose algebra is isomorphic to the algebra of quantum observables. The most interesting feature of our approach is the split between the quantum observables of the RHS of (9) and the classical ones \( f(\hat{q}, \hat{p}) \). In particular in Sec. 3 we will show that, while the classical energy \( H(\hat{q}, \hat{p}) \) commutes with the Liouvillian \( \hat{L} \), the associated quantum energy is not conserved under the evolution generated by \( \hat{L} \). Therefore, in order to preserve the conservation of energy, we must replace also the Liouvillian \( \hat{L} \) with a more complicated object \( \hat{G} \) which is well known in the literature because it is the Moyal operator which appears in the equation of evolution of the Wigner functions [5]. This replacement is the first step necessary in order to go from the classical path integral (5) to the quantum one. The second step is a sort of polarization prescription which reduces the KvN Hilbert space to the standard Hilbert space of quantum mechanics. In Sec. 4 we will see how this polarization procedure can be justified by requiring the irreducibility of the Hilbert space of quantum mechanics under the action of the quantum observables. In the Conclusions we will outline some of the possible future applications of the content of this paper.

## 2 From Classical to Quantum Observables

Usually physicists identify the observables of classical mechanics with the functions of the phase space variables \( f(\varphi) \). In the operatorial approach to classical mechanics such observables become the functions of the commuting operators \( \hat{\varphi}^a \). Therefore the algebra of classical observables is Abelian and there is no uncertainty principle involving the operators \( \hat{\varphi}^a \) which can be diagonalized simultaneously. Every simultaneous eigenstate of \( \hat{\varphi}^a \) determines uniquely a point in the phase space \( |\varphi_{(0)}^a\rangle \) which represents the state of the system. On such a state all the observables of classical mechanics assume a well defined value. In fact if \( \hat{\varphi}|\varphi_{(0)}^a\rangle = \varphi_{(0)}^a|\varphi_{(0)}^a\rangle \) then \( f(\hat{\varphi})|\varphi_{(0)}^a\rangle = f(\varphi_{(0)}^a)|\varphi_{(0)}^a\rangle \), i.e. the eigenstate \( |\varphi_{(0)}^a\rangle \) of \( \hat{\varphi} \) is an eigenstate also for all the observables \( f(\hat{\varphi}) \) and the associated eigenvalue is just the function \( f \) evaluated on the point of the phase space \( \varphi_{(0)}^a \). In this approach the mean values and the probability distributions of the classical observables are completely independent of the phases of the wave functions \( \psi(\varphi) \) and only the modulus \( |\psi(\varphi)| \) is significant from a physical point of view [6].

Besides the operators \( \hat{\varphi}^a \) in the operatorial formulation of classical mechanics we have also the operators \( \hat{\lambda}_a \) which allow us to generate a non trivial evolution via the Liouvillian \( \hat{L} \) of Eq. (6). Such operators \( \lambda_a \) do not commute with \( \hat{\varphi}^a \) and this immediately suggests that it
might be possible to construct a non Abelian algebra of operators by considering suitable combinations of $\hat{\varphi}$ and $\hat{\lambda}$. In particular it would be very interesting to discover whether it is possible to reproduce, using only the KvN commutators $[\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{jk}$, the non Abelian algebra of quantum observables. In order to do this we can define the following operators, known in the literature on Wigner functions as Bopp operators, $[4]$

$$
\begin{align*}
\hat{Q}_j &\equiv \hat{q}_j - \frac{1}{2}\hbar \hat{\lambda}_{pj} \\
\hat{P}_j &\equiv \hat{p}_j + \frac{1}{2}\hbar \hat{\lambda}_{qj},
\end{align*}
$$

(10)

Via the KvN commutators $[7]$ it is very easy to prove that they satisfy the usual Heisenberg algebra: $[\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{jk}$. Since both $\hat{\varphi}$ and $\hat{\lambda}$ are Hermitian under the KvN scalar product $[4]$ we have that also $\hat{Q}_j$ and $\hat{P}_j$ are Hermitian operators with the same scalar product. Not only, but the classical operators $\hat{q}_j$ and $\hat{p}_j$ can be obtained from the associated quantum ones in the limit $\hbar \to 0$. We would like to stress that the quantum positions and momenta are now different “entities” than the classical ones. This may be difficult to accept but at the same time it may be the crucial element which makes quantum mechanics so counter-intuitive and different from classical mechanics.

The rules $[10]$ can be easily generalized and the quantum operator $\hat{F}$ associated to the classical observable $f(\hat{q}, \hat{p})$ will be given by the following function:

$$
\hat{F} = f\left(\hat{q}_j - \frac{1}{2}\hbar \hat{\lambda}_{pj}, \hat{p}_j + \frac{1}{2}\hbar \hat{\lambda}_{qj}\right).
$$

(11)

Since $\hat{\varphi}$ and $\hat{\lambda}$ do not commute it is clear that in $[11]$ we must specify also the ordering used. The one we choose, once we expand $[11]$ in $\hbar$, is the following:

$$
\hat{F} = f(\hat{q}, \hat{p}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\hbar}{2}\right)^n \hat{\varphi}_{a_1} \ldots \hat{\varphi}_{a_n} \omega^{a_1 b_1} \ldots \omega^{a_n b_n} \partial_{b_1} \ldots \partial_{b_n} f.
$$

(12)

The operators $\hat{F}$ are Hermitian under the KvN scalar product $[11]$ and, since the basic commutators $[\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{jk}$ are satisfied by the choice $[10]$, the Hermitian functions of $\hat{Q}_j$ and $\hat{P}_j$ given by $[11]$ or $[12]$ will satisfy the correct quantum commutators. For example the three components of the angular momentum turn out to be:

$$
\begin{align*}
\hat{M}_x &= \hat{\varphi}_z \hat{p}_y - \hat{\varphi}_y \hat{p}_z - \frac{1}{2}\hbar [\hat{\lambda}_{y z} \hat{\varphi}_z - \hat{\lambda}_{z \varphi} \hat{\varphi}_z - \hat{\lambda}_{\varphi z} \hat{p}_z] - \frac{1}{4}\hbar^2 [\hat{\lambda}_{p y} \hat{\varphi}_z - \hat{\lambda}_{y \varphi} \hat{p}_z] \\
\hat{M}_y &= \hat{\varphi}_x \hat{p}_z - \hat{\varphi}_z \hat{p}_x - \frac{1}{2}\hbar [\hat{\lambda}_{x z} \hat{\varphi}_z - \hat{\lambda}_{z \varphi} \hat{\varphi}_z - \hat{\lambda}_{\varphi z} \hat{p}_z] - \frac{1}{4}\hbar^2 [\hat{\lambda}_{p z} \hat{\varphi}_x - \hat{\lambda}_{z \varphi} \hat{p}_x] \\
\hat{M}_z &= \hat{\varphi}_x \hat{p}_y - \hat{\varphi}_y \hat{p}_x - \frac{1}{2}\hbar [\hat{\lambda}_{x y} \hat{\varphi}_y - \hat{\lambda}_{y \varphi} \hat{\varphi}_y - \hat{\lambda}_{\varphi y} \hat{p}_x] - \frac{1}{4}\hbar^2 [\hat{\lambda}_{p y} \hat{\varphi}_x - \hat{\lambda}_{y \varphi} \hat{p}_x]
\end{align*}
$$

and they obey the usual algebra $[\hat{M}_i, \hat{M}_j] = i\hbar \epsilon_{ijk} \hat{M}_k$ under the KvN commutators $[7]$.

All this confirms that it is possible to build the quantum observables $\hat{F}$ by adding to the associated classical ones $f(\hat{q}, \hat{p})$ a suitable combination of KvN operators which vanishes in the classical limit: $\lim_{\hbar \to 0} \hat{F} = f(\hat{q}, \hat{p})$. 

4
3 From Classical to Quantum Evolutions

As we have seen in the previous section, in classical mechanics the observable energy is given by the operator $H(\dot{q}, \dot{p})$. According to the map $Q$ of (12) at the quantum level the energy has instead to be identified with the operator $H(\hat{Q}, \hat{P})$ where the classical position $\dot{q}$ and momentum $\dot{p}$ are replaced by the associated Bopp operators $\hat{Q}$ and $\hat{P}$. Now at the classical level the energy $H(\dot{q}, \dot{p})$ is a conserved quantity since it commutes with the Liouvillian $\hat{L}$ which is the generator of the time evolution: $[\hat{L}, H(\dot{q}, \dot{p})] = -i \partial_q H \omega^{ab} \partial_b H = 0$. What happens at the quantum level? Does the operator $H(\hat{Q}, \hat{P})$ still commute with the Liouvillian $\hat{L}$? The answer is no. In fact, using (12), we have that the commutator of the Liouvillian with the quantum energy $H(\hat{Q}, \hat{P})$ is:

$$[\hat{L}, H(\hat{Q}, \hat{P})] = \frac{\hbar^2}{8} \omega^{\alpha \beta} \omega^{a_1 b_1} \omega^{a_2 b_2} (i \dot{\lambda}_{a_1} \dot{\lambda}_{a_2} \partial_\alpha \partial_{b_1} \partial_{b_2} H \partial_\beta H + + \dot{\lambda}_\alpha \partial_\beta \partial_{a_1} \partial_{a_2} H \partial_{b_1} \partial_{b_2} H) + O(\hbar^3).$$

(13)

For Hamiltonians more than quadratic in $q$ and $p$ the previous commutator is different from zero and the energy is not conserved under the evolution generated by the Liouvillian. Therefore if we identify the quantum energy with $H(\hat{Q}, \hat{P})$ and we require that this quantity has to be conserved in time, then we must consistently change also the operator which generates the time evolution. In particular in order to compensate the RHS of (13) we must add to the Liouvillian a term cubic in $\lambda$ and in the symplectic matrix: $\hat{G}_{(1)} = \frac{1}{24} \hbar^2 \dot{\lambda}_a \dot{\lambda}_b \dot{\lambda}_c \omega^{ad} \omega^{be} \omega^{cf} \partial_d \partial_e \partial_f H$.

In fact the commutator of such a term with $H(\hat{Q}, \hat{P})$:

$$[\hat{G}_{(1)}, H(\hat{Q}, \hat{P})] = \frac{\hbar^2}{8} \omega^{\alpha \beta} \omega^{a_1 b_1} \omega^{a_2 b_2} (i \dot{\lambda}_{a_1} \dot{\lambda}_{a_2} \partial_\alpha \partial_{b_1} \partial_{b_2} H \partial_\beta H + + \dot{\lambda}_\alpha \partial_\beta \partial_{a_1} \partial_{a_2} H \partial_{b_1} \partial_{b_2} H) + O(\hbar^3)$$

is just the opposite of the RHS of (13). If we consider also the other terms of the expansion in $\hbar$ we have that $H(\hat{Q}, \hat{P})$ is conserved if we replace the Liouvillian $\dot{\lambda}_a \omega^{ab} \partial_b H$ with the following operator:

$$\hat{G} = \sum_{j=0}^{\infty} \frac{\hbar^{2j}}{2^{2j}(2j+1)!} \dot{\lambda}_{a_1} \cdots \dot{\lambda}_{a_{2j+1}} \omega^{a_1 b_1} \cdots \omega^{a_{2j+1} b_{2j+1}} \partial_{b_1} \cdots \partial_{b_{2j+1}} H.$$

(14)

From (14) we see that $\hat{G}$ is given by the Liouvillian ($j = 0$) plus a sum of terms which are proportional to the even powers of $\hbar$ and in the limit $\hbar \rightarrow 0$ we have that $\hat{G}$ reduces to the Liouvillian $\hat{L}$ itself. At the path integral level this first step of replacing $\hat{L}$ with $\hat{G}$ implies that the kernel of propagation (3) must be replaced by (7):

$$\langle \varphi, t | \varphi_i, t_i \rangle = \int D^n \varphi D \lambda \exp \left[ i \int dt (\lambda_a \dot{\varphi}^a - \hat{G}) \right].$$

(15)

If we include, besides $\lambda$ and $\varphi$, also the forms $d\varphi$ and the vector fields then the associated path integral has been worked out in Ref. [5]. Since the kinetic term of (15) is the usual one, $\lambda_a \dot{\varphi}^a$, the KvN commutators (7) and their operatorial realization (8) are unchanged. If we
limit ourselves to the case of a 2D phase space with a Hamiltonian quadratic in the momenta, 

\[ H(q, p) = \frac{p^2}{2m} + V(q), \]

then the operatorial realization of \( G \) becomes:

\[
\hat{G} = \hat{L} - \sum_{j=1}^{\infty} \left( \frac{\hbar}{2} \right)^{2j} \frac{1}{(2j+1)!} (\hat{\lambda}_p)^{2j+1} \frac{d^{2j+1}}{dq^{2j+1}} V(q) = -\frac{\hbar}{m} \frac{\partial}{\partial q} + i \sum_{j=0}^{\infty} \left( \frac{\hbar}{2} \right)^{2j} \frac{(-1)^j}{(2j+1)!} \frac{d^{2j+1}}{dq^{2j+1}} V(q) \frac{\partial^{2j+1}}{\partial p^{2j+1}}.
\]

Before going on it is interesting to note that \( \hat{G} \) is just the operator which appears in the equation of evolution of the Wigner functions \( W(q, p, t) \), see Ref. [5]:

\[ i \frac{\partial}{\partial t} W(q, p, t) = \hat{G} W(q, p, t). \]

In any case, in order to avoid possible misunderstandings, we want to stress that in our approach the operator \( \hat{G} \) replaces the Liouvillian \( \hat{L} \) in making the evolution of the elements of the KvN Hilbert space, which are the probability amplitudes \( \psi(q, p, t) \) and not the quasi-probability distributions \( W(q, p, t) \) of the Wigner approach.

Let us now define, besides the Bopp operators of Eq. (10), the following operators:

\[
\begin{align*}
\hat{Q}_j &\equiv \hat{q}_j + \frac{1}{2} \hbar \hat{\lambda}_p, \\
\hat{P}_j &\equiv \hat{p}_j - \frac{1}{2} \hbar \hat{\lambda}_q.
\end{align*}
\]

It is then possible to prove [9] that the \( \hat{G} \) of (14) can be written in terms of the standard Hamiltonian \( H \) as:

\[
\hat{G} = \frac{1}{\hbar} H(\hat{Q}, \hat{P}) - \frac{1}{\hbar} H(\hat{\bar{Q}}, \hat{\bar{P}}) \tag{17}
\]

where the expansion in \( \hbar \) of \( H(\hat{Q}, \hat{P}) \) is given by the RHS of (12) with \( f \) replaced by \( H \) while the expansion in \( \hbar \) of \( H(\hat{\bar{Q}}, \hat{\bar{P}}) \) is:

\[
H(\hat{\bar{Q}}, \hat{\bar{P}}) = H(\hat{q}, \hat{p}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\hbar}{2} \right)^n \lambda_{a_1} \cdots \lambda_{a_n} \omega^{a_1 b_1} \cdots \omega^{a_n b_n} \partial_{b_1} \cdots \partial_{b_n} H.
\]

From (17) it is also easy to understand the reason why the energy \( H(\hat{Q}, \hat{P}) \) commutes with the operator \( \hat{G} \). In fact \( \hat{G} \) is proportional to the difference of the quantum energy itself \( H(\hat{Q}, \hat{P}) \) and an operator, \( H(\hat{\bar{Q}}, \hat{\bar{P}}) \), which commutes with the quantum energy because the unbarr ed operators \( \hat{Q} \) and \( \hat{P} \) always commute with the barred ones \( \hat{\bar{Q}} \) and \( \hat{\bar{P}} \) of (16).

Since the operator of evolution \( \hat{G} \) is now expressed via the operators (10)-(16), it is quite natural to write down also the path integral (15) in terms of \( Q, \bar{Q}, P \), and \( \bar{P} \). Since \( \hat{q} \) and \( \hat{\lambda}_p \) are coupled together in the definition of \( \hat{Q} \) and \( \hat{\bar{Q}} \), it is more convenient to use the representation in which the multiplicative operators are \( \hat{q} \) and \( \hat{\lambda}_p \) instead of \( \hat{q} \) and \( \hat{\bar{p}} \) [10]. In this representation the KvN Hilbert space is made up of functions of \( (q, \lambda_p) \) and their kernel of propagation becomes:

\[
\langle q, \lambda_p, t|q_i, \lambda_{p_i}, t_i \rangle = \int \frac{dp}{\sqrt{2\pi}} \frac{dp_i}{\sqrt{2\pi}} e^{-i\lambda_p p} \langle \varphi, t|\varphi_i, t_i \rangle e^{i\lambda_{p_i} p_i} = \int D\lambda_q Dp \lambda_p \exp \left[ i \int dt (\lambda_q \dot{q} - p \dot{\lambda}_p - G) \right]. \tag{18}
\]
The inverse of \((10)-(16)\) are given by:

\[
\begin{align*}
\hat{q} &= \frac{\hat{Q} + \hat{\bar{Q}}}{2} \\
\hat{\lambda}_p &= \frac{\hat{Q} - \hat{\bar{Q}}}{\hbar}
\end{align*}
\]

Using them we can rewrite the kernel of propagation \((18)\) as:

\[
\langle Q, \bar{Q}, t | Q_i, \bar{Q}_i, t_i \rangle = \int D^\prime Q D^\prime P \frac{\hbar}{\iota} \exp \left[ i \hbar \int dt [P \dot{Q} - H(Q, P)] \right] .
\]

This path integral generates the evolution of the square integrable functions \(\psi(Q, \bar{Q})\). From \((19)\) it is easy to note that the evolution of the variables \((Q, P)\) is completely decoupled from the evolution of the variables \((\bar{Q}, \bar{P})\). So if we consider as initial wave functions those \(\psi(Q, \bar{Q})\) which can be factorized as \(\psi(Q, \bar{Q}) = \psi(Q) \cdot \psi(\bar{Q})\), then they will remain factorized during the time evolution. In fact the variables \((Q, P)\) and \((\bar{Q}, \bar{P})\) cannot become entangled because in \((19)\) there is no interaction between them. Furthermore if we limit the KvN Hilbert space to the one whose basis is given by all the eigenstates \(|Q\rangle\) of the operator \(\hat{Q}\), then from \((19)\) we have that the wave functions \(\psi(Q)\) evolve just with the usual quantum kernel of evolution for a system described by a Hamiltonian \(H(Q, P)\):

\[
\langle Q, t | Q_i, t_i \rangle = \int D^\prime Q D^\prime P \frac{\hbar}{\iota} \exp \left[ i \hbar \int dt [P \dot{Q} - H(Q, P)] \right] .
\]

In the next section we will analyze the reasons why we have to restrict the KvN Hilbert space to the usual one in the quantum case and we will show that also this prescription is just a consequence of the choice of the observables \((9)\).

4 From Classical to Quantum States

In this section we will show how the usual Hilbert space of quantum mechanics can be embedded in the KvN Hilbert space. Let us suppose we use in quantum mechanics the coordinate representation: the positions \(\hat{Q}\) are operators of multiplication by \(Q\), the momenta \(\hat{P}\) are given by \(\hat{P} = -i\hbar \frac{\partial}{\partial Q}\), while the Hilbert space is made up of the complex and square integrable functions of \(Q\). How can we obtain all this starting from the KvN states and operators? As we have already seen in the previous sections, since \(\hat{Q}\) is a linear combination of \(\hat{q}\) and \(\hat{\lambda}_p\), it is more convenient to use, at the KvN Hilbert space level, the representation in which \(\hat{q}\) and \(\hat{\lambda}_p\) are multiplicative operators while, to satisfy \((7)\), \(\hat{\lambda}_q\) and \(\hat{p}\) must be given by \(\hat{\lambda}_q \equiv -i \frac{\partial}{\partial q}\) and \(\hat{p} \equiv i \frac{\partial}{\partial \lambda_p}\) respectively. In this representation the Hilbert space of the theory is made up of the complex wave functions \(\psi(q, \lambda_p)\) which are square integrable under the KvN scalar product \(\langle \psi | \tau \rangle = \int dq d\lambda_p \psi^*(q, \lambda_p) \tau(q, \lambda_p)\). Equivalently, if we use the variables
Also the KvN scalar product between two different states of \(H\) functions \(\psi\) function \(\chi\) square integrable function in \(\text{mechanics}\):

The Hermitian quantum observables \(f\) and of every other quantum observable \(\hat{P}\) space which is invariant under the action of the quantum positions \(\hat{H}\) words the quantum observables map every vector of standard wave functions of quantum mechanics. In fact:

\[ \hat{Q}\psi(Q,\bar{Q}) = Q\psi(Q,\bar{Q}) \]
\[ \hat{P}\psi(Q,\bar{Q}) = \left(i\frac{\partial}{\partial \lambda_p} - \frac{i}{2}\hbar\frac{\partial}{\partial q}\right)\psi(Q,\bar{Q}) = -i\hbar\frac{\partial}{\partial Q}\psi(Q,\bar{Q}). \]

In general the action of the quantum observable \(\hat{F}\) on \(\psi(Q,\bar{Q})\) is given by:

\[ \hat{F}\psi(Q,\bar{Q}) = f(Q,\bar{Q})\psi(Q,\bar{Q}) = f(Q,\bar{Q})\psi(Q,\bar{Q}). \] (20)

Clearly the space of the complex and square integrable wave functions \(\psi(Q,\bar{Q})\) is too big in order to be an irreducible representation of the algebra of quantum observables. In fact, from \([20]\) we see that every quantum observable \(\hat{F}\) modifies only the part in \(Q\) of the wave function \(\psi(Q,\bar{Q})\) while it acts as the identity operator on the part in \(\bar{Q}\). Therefore if we require that the quantum operators \(\hat{Q}\) and \(\hat{P}\) act irreducibly (see page 434 of [11]) on a space of states \(\mathbf{H}\) then we cannot identify \(\mathbf{H}\) with the full KvN Hilbert space of the square integrable wave functions \(\psi(Q,\bar{Q})\). In fact there exists at least one non-trivial subspace of the KvN Hilbert space which is invariant under the action of the quantum positions \(\bar{Q}\), the quantum momenta \(\hat{P}\) and of every other quantum observable \(\hat{F} = f(Q,\bar{Q})\). This non-trivial subspace is given by the set of all the possible square integrable functions in \(Q\) times one particular square integrable function in \(\bar{Q}\), let us call it \(\chi(\bar{Q})\):

\[ \mathbf{H}_\chi = \left\{ \psi(Q)\chi(\bar{Q}) \text{ with } \int dQd\bar{Q} |\psi(Q)|^2|\chi(\bar{Q})|^2 = 1 \right\}. \] (21)

The Hermitian quantum observables \(f(Q,\bar{Q})\) do not modify the function \(\chi(\bar{Q})\) while they map a wave function of the form \(\psi(Q)\) in another of the same type \(\psi'(Q)\). In other words the quantum observables map every vector of \(\mathbf{H}_\chi\) in another vector of \(\mathbf{H}_\chi\). Since the function \(\chi(\bar{Q})\) is fixed the space \(\mathbf{H}_\chi\) is isomorphic to the usual Hilbert space of quantum mechanics:

\[ \mathbf{H} = \left\{ \psi(Q) \text{ with } \int dQ|\psi(Q)|^2 = 1 \right\}. \] (22)

Also the KvN scalar product between two different states of \(\mathbf{H}_\chi\), let us say \(\psi(Q)\chi(\bar{Q})\) and \(\tau(Q)\chi(\bar{Q})\):

\[ \langle \psi|\tau \rangle = \int dQd\bar{Q} \psi^*(Q)\chi^*(\bar{Q})\tau(Q)\chi(\bar{Q}). \]
induces the standard scalar product of quantum mechanics \( \langle \psi | \tau \rangle = \int dQ \psi^*(Q)\tau(Q) \) if the fixed state \( \chi \) is normalized according to: \( \int d\bar{Q} \chi^*(\bar{Q})\chi(\bar{Q}) = 1. \)

Before going on let us notice that the request of irreducibility is crucial in order to guarantee a one to one correspondence between the states of the theory and the vectors of the space \( H_\chi \)\(^{13}\). In fact let us suppose we consider an arbitrary state belonging to \( H_\chi \) and we replace in it the function \( \chi(\bar{Q}) \) with a different square integrable function \( \sigma(\bar{Q}) \). What we obtain is a different state of the KvN Hilbert space which corresponds to the same expectation values for all the quantum observables \( f \left(Q, -i\hbar \frac{\partial}{\partial Q}\right)\), i.e. which corresponds to the same physical situation. To avoid this redundancy we have to restrict the KvN Hilbert space to only one of the non-trivial subspaces which are invariant under the action of \( \hat{Q} \) and \( \hat{P} \). It is clear that the choice of this invariant subspace is not unique. For example we could choose the \( H_\chi \) of \(^{21}\) in which the wave function \( \chi(\bar{Q}) \) is fixed or every other subspace of the same form. Anyway it is easy to realize that all these subspaces are isomorphic to the standard quantum Hilbert space of \(^{22}\) and that they are completely equivalent from a physical point of view.

These considerations do not depend on the particular representation that we use. For example, in order to obtain the momentum representation of quantum mechanics, we have first of all to consider the \((\lambda_q, p)\) representation of the KvN theory. In this representation \( \hat{\lambda}_q \) and \( \hat{p} \) are multiplicative operators, \( \hat{q} = i\frac{\partial}{\partial \lambda_q}, \hat{\lambda}_p = -i\frac{\partial}{\partial p} \) and the wave functions are of the form \( \psi(\lambda_q, p) \) or, using the Bopp variables, \( \psi(P, \bar{P}) \). On these wave functions the positions \( \hat{Q} \) act as derivative operators: \( \hat{Q} = i\hbar \frac{\partial}{\partial P} \) while the momenta \( \hat{P} \) act as operators of multiplication by \( P \). Also in this representation if the operators \( \hat{Q} \) and \( \hat{P} \) must act irreducibly on \( H \), then we have to identify \( H \) with the space of all the square integrable functions which depend only on the quantum momenta \( P \) times a particular \( \chi(P) \). Such a space is again isomorphic to the usual quantum Hilbert space of the square integrable wave functions in the momentum representation \( \psi(P) \).

Before concluding this section we want to stress that it is the “quantization map” itself \( Q \) of Eq. \(^{30}\) which implies that we must change the operator of evolution by replacing the Liouvillian \( \hat{L} \) with the operator \( \hat{G} \) in order to have a conserved quantum energy. Moreover we must reduce the KvN Hilbert space to the usual Hilbert space of quantum mechanics in order that the positions \( \hat{Q} \) and the momenta \( \hat{P} \) constructed via our map \(^{31}\) form an irreducible set of operators. So the role of the map \(^{32}\) in this new approach to quantization of classical mechanics is crucial.

5 Conclusions and Open Problems

In this paper we have shown how to embed quantum mechanics in the KvN Hilbert space, which was first introduced to give an operatorial formulation of classical mechanics. In
particular we have proved that the KvN Hilbert space turns out to be a suitable mathematical framework for a unifying picture of classical and quantum mechanics: in fact among all the KvN Hermitian operators there are both the classical observables $f(\hat{q},\hat{p})$ and the quantum ones $f(\hat{Q},\hat{P})$, see the figure below.

![Diagram](https://example.com/diagram.png)

**Fig. 1:** Classical and quantum observables can be seen as non-trivial subsets of all the possible Hermitian KvN operators.

As a consequence of the choice of the observables in classical mechanics only the moduli of the KvN wave functions $\psi(q,p)$ are physically significant [6] while in quantum mechanics, if we impose the condition that $\hat{Q}$ and $\hat{P}$ are an irreducible set of operators, the subspace of the physically significant KvN wave functions is isomorphic to the space of the square integrable wave functions $\psi(Q)$ or $\psi(P)$, according to the representation we use.

Some ideas of the approach contained in this paper are similar to those of geometric quantization, see for example [12]. In both cases the starting point is the KvN Hilbert space of square integrable functions on the phase space, in both cases a map is introduced to build the quantum observables and in both cases a sort of polarization is used to reduce the dimension of the quantum Hilbert space to the usual one. However in our approach the recipe to construct the observables is very simple and far from the complicated tools used in geometric quantization. The simple map (9) itself suggests in fact the two other ingredients needed to quantize a classical system. The first one is a well-motivated physical prescription, the conservation of the quantum energy, which implies that the dynamics of the theory has to be modified. The second ingredient is the mathematical prescription which says that the operators $\hat{Q}$ and $\hat{P}$ must act irreducibly and which is necessary to eliminate the redundancy in the physical description of the system. Anyway, like in every other approach to quantization also in our case we cannot bypass the Grönewald-van Hove theorem [14] and an obstruction to a full quantization [11] of the system still remains. In geometric quantization
people construct a very complicated ∧-map just to satisfy the relation
\[
\{ f, g \}_{Pb} \rightarrow \frac{1}{\hbar} [\hat{F}, \hat{G}]
\] (23)
for every function \( f \) and \( g \) but, when they impose the request of irreducibility, the number of observables which can be quantized gets limited. In our approach the irreducibility prescription reduces the KvN Hilbert space to the usual quantum one in a more natural way than in geometric quantization and again this is motivated by the a priori choice of the observables \[ \] that we made. Nevertheless our map \[ \] cannot satisfy Eq. (23) for every function \( f \) and \( g \) as a simple calculation can show. Therefore the Grönewald-van Hove theorem is not bypassed by our quantization method.

The last point we want to mention is that having a unique big Hilbert space for both classical and quantum mechanics can help in studying and appreciating the similarities and differences between the two theories. In particular, for what concerns the observables we must note, see Fig. 1, that within the KvN theory there is enough space to describe both classical and quantum mechanics but there is further space available for further observables which could describe some other possible regimes, like the ones at the border between classical and quantum mechanics \[ \] or the models of dynamical reduction \[ \]. We think that it would be interesting to investigate these issues in the future.

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