High-gradient operators in perturbed Wess-Zumino-Witten field theories in two dimensions

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(Dated: January 31, 2010)

Many classes of non-linear sigma models (NLσMs) are known to contain composite operators with an arbitrary number $2s$ of derivatives (“high-gradient operators”) which appear to become strongly relevant within renormalization group (RG) calculations at one (or fixed higher) loop order, when the number $2s$ of derivatives becomes large. This occurs at many conventional fixed points of NLσMs which are perturbatively accessible within the usual $\epsilon$-expansion in $d = 2 + \epsilon$ dimensions. Since such operators are not prohibited from occurring in the action, they appear to threaten the very existence of such fixed points. At the same time, for NLσMs describing metal-insulator transitions of Anderson localization in electronic conductors, the strong RG-relevance of these operators has been previously related to statistical properties of the conductance of samples of large finite size (“conductance fluctuations”). In this paper, we analyze this question, not for perturbative RG treatments of NLσMs, but for two-dimensional Wess-Zumino-Witten (WZW) models at level $k$, perturbatively in the current-current interaction of the Noether currents (”non-Abelian Thirring/Gross-Neveu models”). WZW models are special (“Principal Chiral”) NLσMs on a Lie Group $G$ with a WZW term at level $k$. In these models the role of high-gradient operators is played by homogeneous polynomials of order $2s$ in the Noether currents, whose scaling dimensions we analyze. For the Lie Supergroup $G = GL(2N|2N)$ and $k = 1$, this corresponds to time-reversal invariant problems of Anderson localization in the so-called chiral symmetry classes, and the strength of the current-current interaction, a measure of the strength of disorder, is known to be completely marginal (for any $k$). We find that all high-gradient (polynomial) operators are, to one loop order, irrelevant or relevant depending on the sign of that interaction.

I. INTRODUCTION

Fluctuations are known to play a key role in sufficiently low-dimensional systems, whether classical or quantum, as they can preempt spontaneous symmetry breaking. When the symmetry is both global and continuous, the tool of choice to address the role of fluctuations in low-dimensional systems is the non-linear sigma model (NLσM). However, the usefulness of NLσMs has come to transcend situations in which a pattern of symmetry breaking is immediately obvious. For example, NLσMs have been used with success in the context of Anderson localization (see Ref. 4 for a review) to access the transition from a metallic to an insulating phase induced by weak disorder or to compute probability distributions of spectral wavefunctions and transport characteristics in chaotic metallic grains and disordered electronic systems.

Quite generally, the construction of a generic NLσM on a connected Riemannian manifold $\mathcal{M}$ of finite dimension $n$, the “target manifold”, can proceed in the following way. One assigns to any point from Euclidean space in $d$ dimensions, specified by coordinates $x^\mu$ ($\mu = 1, \cdots, d$), a point in the manifold $\mathcal{M}$ with the coordinates $\phi^i(x)$ ($i = 1, \cdots, n$). The simplest action $S$, which is made of two derivatives of the coordinates $\phi^i$, and is invariant under both the rotations of Euclidean space and reparametrization of the target manifold, is

$$S = \frac{1}{4\pi \ell} \int \frac{d^d x}{a^{d-2}} G_{ij} [\phi(x)] \partial_\mu \phi^i(x) \partial_\mu \phi^j(x)$$ \hspace{1cm} (1.1)$$

where $G_{ij} [\phi]$ is a component of the metric tensor on $\mathcal{M}$, $\ell$ is the coupling constant, and $a$ is the short-distance cutoff.

The target manifold can be either compact or non-compact. An example of a NLσM on a compact target manifold is the $O(N)/O(N-1)$ NLσM with $2 < N = 3, 4, 5, \cdots$ when the target manifold is the unit sphere $S^{N-1}$ in $N$-dimensional Euclidean space. When $N = 3$ it describes spontaneous symmetry breaking in a classical ferromagnet. Non-compact target manifolds are of relevance to the problem of Anderson localization in the bosonic “replica limit” $N \to \infty$ or when the manifold is generalized to a supermanifold in Anderson localization the coupling constant $t$ has the meaning of the inverse of the mean dimensionless conductance.

The implicit assumption made in the construction (1.1) is that all the invariant scalars that contain $2s$ ($1 < s = 2, 3, \cdots$) derivatives of the field can be ignored. The standard justification for this assumption is that their “engineering dimension” $2s$ is much larger than the spatial dimension $d = 2 + \epsilon$, i.e., they are irrelevant in the renormalization group (RG) sense, and this is expected to remain so after renormalization in $d = 2 + \epsilon$ dimensions for small $\epsilon$, and thus small $t$. 
This assumption was called into question in Refs. 3–16 for which the main results can be illustrated most simply by the example of the $O(N)/O(N-1)$ NLSM. We recall that the $O(N)/O(N-1)$ NLSM has an infrared unstable fixed point located, to one loop order, at $t^* = \epsilon/(N-2)$, from which emerges a renormalization group (RG) flow to strong and weak coupling. In Ref. 3, a family of perturbations of the $O(N)/O(N-1)$ NLSM action (1.1), which we shall call high-gradient operators, was considered. A high-gradient operator of order $s$ is a homogeneous polynomial of order $2s$ in the derivatives of the fields (all located at the same point) which is a scalar with respect to both the symmetry group of the NLSM [i.e., $O(N)$] and the rotation group of Euclidean space. The minimum (i.e., dominant, or “leading”) value of the one-loop scaling dimension $x^{(s)}$ of the high-gradient operators of order $s$ at the fixed point $t^*$ is found to be

$$x^{(s)} = 2s - s(s-1)t^* + O(\epsilon^2). \quad (1.2)$$

Although strongly irrelevant by power counting (i.e., in the absence of fluctuation corrections, $t^* \to 0$), high-gradient operators of order $s$ thus acquire a one-loop scaling dimension smaller than two when the order $2s$ of derivatives is large enough so that $st^* \approx sc/(N-2) \sim 2$, and thus would appear to become relevant, based on the one-loop result. In $d = 2$ dimensions, the lowest one-loop scaling dimension $x^{(s)}$ for all high-gradient operators of order $s$ is

$$x^{(s)} = 2s - s(s-1)t + O(t^2) \quad (1.3)$$

along the trajectory to strong coupling away from the infra-red unstable fixed point $t = 0$. Two-loop counterparts to Eqs. (1.2) and (1.3) yield the same conclusion. High-gradient operators of sufficiently high-order $s$ appear to be relevant for any given dimension $d = 2 + \epsilon$ at the non-trivial fixed point.

Similar results hold for the NLSMs defined on the compact target manifolds $(M$ and $N$ are positive integers) $Sp(M+N)/Sp(M) \times Sp(N)/U(M)$, $U(M+N)/U(M) \times O(M+N)/O(M) \times O(N)$ and on families of compact Kähler (and super) manifolds [9]. Generalizations to the non-compact target manifolds $Sp(M,N)/Sp(M) \times Sp(N)$, $U(M,N)/U(M) \times U(N)$, and $O(M,N)/O(M) \times O(N)$ follow from the rule that the coupling $t$ of the compact NLSMs entering in one-loop anomalous dimensions must be replaced by $-t$ in the corresponding non-compact NLSM. In Anderson localization, compact target manifolds arise when using fermionic replicas for disorder averaging, whereas non-compact target manifolds arise when using the bosonic replicas for disorder averaging. If one uses supersymmetric disorder averaging, the resulting NLSM has both compact and non-compact sectors. The high-gradient operators in the NLSM defined on $AdS_5 \times S^5$ (AdS$_5$ is non-compact whereas $S^5$ is compact) have also been discussed in the context of the AdS/CFT correspondence. (See, for example, Refs. 14–21.)

The substitution $t \to -t$ does not affect the value of the minimal (i.e., dominant, or “leading”) one-loop scaling dimension $x^{(s)}$ of all high gradient operators of order $s$ is distributed symmetrically about zero. This turns out to be the case whenever $m, n > 1$ in the above examples. On the other hand, there are some target manifolds, the simplest examples being $S^{N-1} = O(N)/O(N-1)$ and $CP^{N-1} = U(N)/U(N-1) \times U(1)$, for which the full spectrum of one-loop anomalous dimensions of order $s$ turns out to be not symmetric about zero, in which case the substitution $t \to -t$ matters. For example, high-gradient operators are made more irrelevant by one-loop renormalization effects in the non-compact NLSM on $U(N-1,1)/U(N-1) \times U(1)$. (We refer the reader to Appendix A for a more detailed discussion of “one-sided” versus “two-sided” spectra of one-loop anomalous scaling dimensions for high-gradient operators in NLSMs.)

Of course, one can only conclude that high-gradient operators become relevant for sufficiently large values of $s$, if the strong relevance seen in the one-loop expressions for their scaling dimensions persists when all higher loop contributions (not computed here or in other works on this subject) have been taken into account. For example, the one-loop expressions may not be characteristic in the large-$s$ limit, if the actual expansion parameter is not $\epsilon$ but $s \epsilon$. As any insight for resolving the nature of the $\epsilon$ expansion for high-gradient operators in NLSMs must come from outside the $\epsilon$ expansion itself, progress has stalled since the early 1990’s.

The aim of this paper is to study the operators that play the role of the high-gradient operators in field theories which are two-dimensional Wess-Zumino-Witten (WZW) theories [22,23] on a Lie group $G$, perturbed by an interaction quadratic in the Noether currents (“current-current interaction”). Such theories are often referred to as “two-dimensional non-Abelian Thirring (or Gross-Neveu) models”. Any WZW theory, which is a Principal-Chiral-Non-Linear-sigma model supplemented by a WZW term at level $k$, gives a prescription to construct high-gradient operators in terms of powers of Noether currents. Because it is possible to represent the Noether currents in WZW theories in terms of free fermions, one might be inclined to think that such operators are perhaps not capable of displaying a “pathological” spectrum of scaling dimension as in Eq. (1.2). However, as we demonstrate in this paper, the situation is more interesting. Indeed, we will see that under conditions specified below, the one-loop spectra of the form (1.3) and (1.4) can be realized by perturbing a WZW critical point by a current-current perturbation.

We also want to investigate if there is a difference between the properties of high-gradient operators in unitary and non-unitary non-Abelian Thirring models. This is important because NLSMs describing the physics of Anderson localization are non-unitary field theories. Moreover, high-gradient operators in these theories have been previously related to the statistical fluctuations of the
conductance of a disordered metal. In this context, an appealing physical interpretation of the spectra has been proposed, attributing them to a broad tail in the probability distribution of the conductance. However, given that this interpretation depends crucially on the ability to invert the \( s \to \infty \) and \( \epsilon \to 0 \) limits in spectra which are analogous to those in Eqs. (1.3) and (1.2), it would be useful to have an example of a critical field theory describing a problem of Anderson localization for which one can study the RG-relevance of high-gradient operators without resorting to the \( \epsilon \)-expansion, and for which one can reasonably expect a broad distribution of the conductance.

We now provide an outline of the article and a summary of our results.

It is shown in Sec. 11 that high-gradient operators in the (unitary) \( \hat{su}(2)_k \) Thirring model with strength \( g \) of the “current-current interaction”, are made more irrelevant by the presence of these interactions when the latter are (marginally) irrelevant \((g < 0, \text{in our conventions})\). On the other hand, along the renormalization group (RG) flow driven by a (marginally) relevant current-current interaction \((g > 0, \text{in our conventions})\), a one-loop spectrum of the form \( (1.3) \) is recovered in the “classical” limit \( 1/k \to 0 \). The inverse level \( 1/k \) plays here the role of a “quantum” parameter. Indeed, for any finite \( k \), we find that the quadratic growth in \( s \) in the unbounded one-loop spectrum \( (1.3) \) does not persist for values of \( s \) larger than \( k \). In effect, \( 1/k \) determines the efficiency in “taming” the strong RG-relevance of high-gradient operators seen at one-loop order, which is related to the fact that there exists a representation of the current algebra of the level-\( k \) WZW theory in terms of free fermions.

Section II is devoted to high-gradient operators in what we will call the \( \hat{gl}(M|M)_k \) Thirring (or Gross-Neveu) model which was discussed in Ref. [33]. This is the \( \hat{gl}(M|M)_k \) WZW theory on the Lie Supergroup \( GL(M|M) \), perturbed by two current-current perturbations, one of which we call \( g_A \), which is exactly marginal, and another which we call \( g_M \) which flows logarithmically under the RG at a rate dependent on \( g_M \). In spite of the presence of an RG flow of the coupling \( g_A \) there exists a sector of the theory, the so-called \( PSL(M|M) \) sector, which is scale (conformally) invariant throughout. The high-gradient operators turn out to reside in this conformally invariant sector, and are unaffected by the presence of the coupling \( g_A \). We will show that, for \( k = 1 \), the spectrum of one-loop anomalous dimensions of high-gradient operators is fundamentally different for positive and negative values of the coupling constant \( g_M \).

In particular, when \( g_M > 0 \) all high-gradient operators are made more irrelevant by the current-current perturbations, whereas they are made more relevant when \( g_M < 0 \). We close Sec. II by comparing the anomalous scaling dimensions of high-gradient operators in the \( \hat{gl}(M|M)_k \) Thirring (or Gross-Neveu) models and those in the \( GL(2N|2N)/OSp(2N|2N) \) NL\( \sigma \)Ms, observing that they behave in the same way.

The result that the spectrum of one-loop anomalous dimensions of high-gradient operators is strongly dependent on the sign of \( g_M \) has important implications in the context of Anderson localization because, as discussed in Ref. [33], the \( \hat{gl}(M|M)_k \) Thirring (or Gross-Neveu) model at \( k = 1 \) describes a disordered electronic system, where \( g_M > 0 \) and \( g_M < 0 \) correspond to the strengths of disorder potentials. The theory with \( g_M > 0 \) thus offers an example of a critical theory for Anderson localization with no relevant high-gradient operator. For example, this field theory describes a tight-binding model of electrons on the honeycomb lattice with (real-valued) random hopping matrix elements which are non-vanishing only between the two sublattices of the bipartite honeycomb lattice (see also Ref. [34]). Versions of the honeycomb tight-binding model provide the basic electronic structure of graphene. In the classification scheme of Zirnbauer and Altland and Zirnbauer, this model belongs to the “chiral-orthogonal” symmetry class (class BDI). Another example of a problem of Anderson localization in the same symmetry class is provided by a random tight-binding model on a square lattice with \( \pi \)-flux through every plaquette.

By contrast, when \( g_M < 0 \), the spectrum of one-loop scaling dimensions is unbounded from below for any \( k \) as is the case in Eq. (1.2). The full spectrum of one-loop anomalous scaling dimensions of high gradient operators as it appears, e.g., in the Grassmanian NL\( \sigma \)Ms with target manifolds \( Sp(M+N)/Sp(M) \times Sp(N), U(M+N)/U(M) \times U(N), O(M+N)/O(M) \times O(N) \), which is symmetric about zero, is only recovered in the extreme “classical” limit \( k \to \infty \). In the context of Anderson localization, the case with \( g_M < 0 \) describes the surface state of a three-dimensional topological insulator in the chiral-symplectic class (symmetry class CII) of Anderson localization.

After concluding in Sec. IV, we review in Appendix B the realization of the \( \hat{gl}(2N|2N)_{k=1} \) Thirring (or Gross-Neveu) model as a problem of Anderson localization in two dimensions in symmetry class BDI, which was established in Ref. [33].

II. HIGH-GRADIENT OPERATORS AND \( \hat{su}(2)_k \) WZW THEORIES

The \( O(3)/O(2) \) NL\( \sigma \)M with coupling constant \( t \) is the simplest example of a NL\( \sigma \)M containing infinitely many high-gradient operators all of which would appear to become relevant based on one-loop results. This happens at the infra-red unstable fixed point \( t^* = \epsilon \) in \( d = 2 + \epsilon > 2 \) dimensions within the one-loop approximation as long as the order \( s \) of these high-gradient operators is large enough. A precursor to this perturbative property also occurs in \( d = 2 \) dimensions close to the infra-red unstable fixed point \( t = 0 \) as the NL\( \sigma \)M flows to strong coupling. Along this flow, the spectrum of one-loop dimensions for the high-gradient operators is unbounded from below.
In two dimensions, the O(3)/O(2) NLSM, supplemented by a topological theta-term at \( \theta = \pi \), flows to a critical field theory, the SU(2) WZW theory with \( \tilde{\text{su}}(2)_{k=1} \) current algebra, \( \tilde{\text{su}}(2)_{k=1} \) WZW theory. The strongly relevant high-gradient operators near the infra-red unstable fixed point \( t = 0 \) must become irrelevant at the WZW critical point, because the full operator content of the \( \tilde{\text{su}}(2)_{k=1} \) WZW theory is known to contain only a finite number of relevant fields (with scaling dimensions bounded from below and above by zero and two, respectively). The purpose of this section is to perturb the \( \tilde{\text{su}}(2)_{k} \) WZW theory with a current-current perturbation and to examine the fate of those operators in the \( \tilde{\text{su}}(2)_{k} \) WZW theory which correspond to the high-gradient operators in the O(3)/O(2) NLSM. We will refer to these operators still as "high-gradient operators".

We are going to argue that the spectrum of one-loop scaling dimensions associated with all high-gradient operators is bounded from below by the lowest one-loop scaling dimension corresponding to high-gradient operators of order \( k \). This result is very different from the unbounded spectrum \[1.3\] of one-loop scaling dimensions associated with high-gradient operators in the two-dimensional O(3)/O(2) NLSM.

In the following sections, we first review the \( \tilde{\text{su}}(2)_{k} \) WZW theory perturbed by a current-current interaction. Second, we identify high-gradient operators of order \( s \). Finally, we compute the leading one-loop dimensions of high-gradient operators of order \( s \) up to one loop.

### A. Definitions

The most fundamental property of the \( \tilde{\text{su}}(2)_{k} \) WZW theory is the existence of a pair of holomorphic and antiholomorphic SU(2) Noether currents, \( J_{1}, J_{2}, J_{3}, \) and \( J_{\tilde{1}}, J_{\tilde{2}}, J_{\tilde{3}} \), respectively, which satisfy the affine (Kac-Moody) current algebra

\[
J_{\alpha}(z)J_{\beta}(0) = \frac{kC_{\alpha\beta}}{z} + \frac{i}{z} f_{\alpha\beta\gamma} J_{\gamma}(0) + \cdots, \\
\tilde{J}_{\bar{\alpha}}(\bar{z})\tilde{J}_{\bar{\beta}}(0) = \frac{k\tilde{C}_{\bar{\alpha}\bar{\beta}}}{\bar{z}} + \frac{i}{\bar{z}} f_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \tilde{J}_{\bar{\gamma}}(0) + \cdots, \\
J_{\alpha}(z)\tilde{J}_{\bar{\beta}}(0) = 0,
\]

(2.1a)

at level \( k = 1, 2, 3, \ldots \), where the invariant (Casimir) tensor of rank 2 in \( \text{su}(2) \) has the contravariant and covariant representations (in our conventions)

\[
C_{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta}, \quad C^{\alpha\beta} = 2\delta_{\alpha\beta},
\]

(2.1b)

respectively, while the structure constant of \( \text{su}(2) \) is the fully antisymmetric Levi-Civita tensor of rank 3,

\[
f_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}, \quad \alpha, \beta, \gamma = 1, 2, 3.
\]

(2.1c)

The dots in Eq. (2.1a) are terms of order zero and higher in powers of \( z (\bar{z}) \) when \( z = x + iy \) (\( \bar{z} = x - iy \)) the holomorphic (antiholomorphic) coordinates of the Euclidean plane. We shall also refer to the (anti) holomorphic sector of the theory as the (right-) left-moving sector.

The \( \tilde{\text{su}}(2)_{k} \) current algebra \[2.1\] has a representation in terms of free-fermions. More precisely, it is obtained from the action

\[
S_{s} := \sum_{i=1}^{k} \int \frac{dz d\bar{z}}{2\pi i} (\psi_{i}^{\dagger} \partial \psi_{i} + \bar{\psi}_{i}^{\dagger} \partial \bar{\psi}_{i})
\]

constructed from \( k \)-independent flavors of left \( (\psi) \) and right \( (\bar{\psi}) \) moving Dirac fermions, whereby each one transforms in the fundamental representation of \( \text{SU}(2) \times \text{SU}(k) \), with the partition function

\[
Z_{s} := \int D[\psi^{\dagger}, \psi, \bar{\psi}^{\dagger}, \bar{\psi}] \exp (-S_{s}).
\]

One has the operator product expansions (OPE)

\[
\psi_{\alpha}(z)\psi_{\beta}^{\dagger}(0) = \delta_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{z} + \frac{i}{z} f_{\alpha\beta\gamma} \psi_{\gamma}(0) + \cdots, \\
\psi_{\alpha}(z)\bar{\psi}_{\bar{\beta}}^{\dagger}(0) = \delta_{\alpha\bar{\beta}} + \frac{\delta_{\alpha\bar{\beta}}}{\bar{z}} + \frac{i}{\bar{z}} f_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \bar{\psi}_{\bar{\gamma}}(0) + \cdots, \\
\bar{\psi}_{\bar{\alpha}}(\bar{z})\bar{\psi}_{\bar{\beta}}^{\dagger}(0) = \delta_{\bar{\alpha}\bar{\beta}} + \frac{\delta_{\bar{\alpha}\bar{\beta}}}{\bar{z}} + \frac{i}{\bar{z}} f_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \bar{\psi}_{\bar{\gamma}}(0) + \cdots
\]

for \( i, i' = 1, \ldots, k \) and \( c, d = 1, 2 \). In turn, the OPE \[2.3\] imply that the left and right Noether currents

\[
J_{\alpha} := \sum_{i=1}^{k} \psi_{i}^{\dagger} (\sigma_{\alpha})_{i}^{d} \psi_{d}, \quad \tilde{J}_{\bar{\alpha}} := \sum_{i=1}^{k} \bar{\psi}_{i}^{\dagger} (\sigma_{\bar{\alpha}})_{i}^{d} \bar{\psi}_{d},
\]

(2.4)

with \( \alpha = 1, 2, 3 \) obey the \( \tilde{\text{su}}(2)_{k} \) current algebra \[2.1\].

The field theory defined by Eq. \[2.3\] is a free-fermion field theory. The content of local operators is thus known. It contains a finite number of fields whose scaling dimensions are bounded between 0 and 2 and are thus relevant, as it should be for a field theory defined on a Hilbert space with a positive definite inner product and with a spectrum bounded from below which is built on the Dirac-Fermi sea, in short a unitary field theory. Clearly, within the set of powers of the Noether currents \[2.4\] there is thus no room for an infinite family of relevant operators.

We perturb the free-fermion field theory by a current-current interaction \( O_{I} \) of the \( \tilde{\text{su}}(2)_{k} \) currents \[2.4\].

\[
Z := \int D[\psi^{\dagger}, \psi, \bar{\psi}^{\dagger}, \bar{\psi}] \exp (-S), \\
S := S_{s} + g \int \frac{dz d\bar{z}}{2\pi i} O_{I}(z, \bar{z}), \\
O_{I}(z, \bar{z}) := C^{\alpha\beta} J_{\alpha}(z)\tilde{J}_{\bar{\beta}}(\bar{z}) = 2J_{\alpha}(z)\tilde{J}_{\bar{\alpha}}(\bar{z}).
\]

We take the coupling constant \( g \) to be real. The (unitary) field theory \[2.4\] is often referred to as a non-Abelian Thirring (Gross-Neveu) model. Suitable non-unitary generalizations of the field theory \[2.3\] compute (disorder average) moments of Green’s functions in a
class of problems of Anderson localization in $d = 2$ dimensions that we will investigate later on in this paper.

The one-loop beta function,

$$\beta_g = \frac{dg}{dl} = 4g^2, \quad (2.6)$$

encodes the change in the coupling constant caused by the infinitesimal rescaling $a \to (1 + dl)a$ of the short-distance cutoff $a$. Thus, the current-current interaction is (marginally) irrelevant (relevant) for $g < 0$ ($g > 0$) with the free-fermion fixed point at $g = 0$.

The SU(2)$_k$ Noether currents (2.4) are those appearing at the non-trivial fixed point of the Principal Chiral NL$\sigma$M on the SU(2) group manifold with a Wess-Zumino term. This has, the well-known (Euclidean) action

$$S = \frac{k}{16\pi} \int d^2x \, \text{tr} \left( \partial_\mu G^{-1} \partial^\mu G \right) + k\Gamma[G], \quad (2.7a)$$

where $G \in \text{SU(2)}$ is a group element, and the integral $\Gamma[G]$ over a three-dimensional ball $B$ with coordinates $r_\mu$ and whose boundary $\partial B$ is $d = 2$-dimensional Euclidean space,

$$\Gamma[g] := \frac{1}{24\pi} \int_B d^3r \, \epsilon_{\mu\nu\lambda} \text{tr} \left( G^{-1} \partial_\mu G G^{-1} \partial_\nu G G^{-1} \partial_\lambda G \right) \quad (2.7b)$$

is the Wess-Zumino term. The Noether currents which generate the SU(2)$_{\text{left}} \times$ SU(2)$_{\text{right}}$ symmetry at the critical point of the WZW theory can be fully represented by the fermionic expressions in Eq. (2.4). In the bosonic (i.e., NL$\sigma$M) representation, these currents are built out of first-order derivatives of the bosonic fields,

$$J_\alpha \propto k \partial_\mu \left( [G^{-1} \partial G]^{-1}_{\sigma\alpha} \right), \quad \bar{J}_\alpha \propto k \partial_\mu \left( G^{-1} [\partial G]^{-1}_{\sigma\alpha} \right). \quad (2.8)$$

The relationship (2.8) suggests that composite operators built out of monomials in the currents (2.4) in the WZW theory are the counterparts of the high-gradient operators in NL$\sigma$M. For this reason, we shall still call the former family of composite operators high-gradient operators. The “classical” counterparts of the high-gradient operators of order $s$ in the NL$\sigma$M are thus the homogeneous polynomials

$$T^{\alpha_1 \cdots \alpha_s \tilde{\alpha}_1 \cdots \tilde{\alpha}_s} \sigma_{\alpha_1} \cdots \sigma_{\tilde{\alpha}_s} \tilde{J}_{\alpha_1} \cdots \tilde{J}_{\tilde{\alpha}_s} \quad (2.9)$$

of the left and right currents that are invariant under the diagonal SU(2) symmetry group of the interacting theory. The generating set of classical high-gradient operators of order $s$ is specified once all the linearly independent rank $2s$ tensors $T^{\alpha\beta \cdots \gamma \delta \cdots}$ in the adjoint representation of SU(2) that are invariant under SU(2) transformations can be fully enumerated. In turn, the most general SU(2) invariant tensor of even rank in the adjoint representation is the product of the Casimir tensor of rank $2$.

The high-gradient operators in Eq. (2.9) are classical in the sense that quantum fluctuations encoded through the Pauli principle (or, equivalently, through the underlying Dirac-Fermi sea) in the free-fermion representation (2.4) of the current algebra, have not yet been accounted for. To account for these quantum fluctuations, one needs to introduce a point-splitting procedure that allows for the proper normal ordering, i.e., the correct subtraction of all short-distance singularities:

$$\lim_{z_i \to z} \left[ T^{\alpha_1 \cdots \tilde{\alpha}_s} J_{\alpha_1} \cdots \tilde{J}_{\tilde{\alpha}_s}(z) - (\text{all short-distance singularities}) \right]. \quad (2.10)$$

Two objects of the form (2.3) that are linearly independent classically might not survive as a pair of distinct quantum operators of the form (2.10) after normal ordering has been implemented. More precisely, one might anticipate that the underlying free-fermion representation of the current algebra must manifest itself as soon as the order $s$ becomes larger than the number $k$ of fermionic flavors by changing the book-keeping relating classical expressions labeled by SU(2) tensors of rank $2s$ and quantum operators.

Indeed, we are going to show that this is the mechanism that prevents high-gradient operators of order $s > k$ from acquiring one-loop scaling dimensions smaller than the smallest one-loop scaling dimensions associated with the set of all high-gradient operators of order $s \leq k$. In other words, the smallest one-loop dimension associated with the set of all high-gradient operators is reached within the set of all high-gradient operators of order $s < k$ when $g > 0$. It is thus bounded from below when $g > 0$.

Had we ignored the underlying free-fermion representation of the current algebra altogether, we would have wrongly predicted that, when $g > 0$, the one-loop dimensions associated with the classical objects (2.3) are of a form similar to the ones in Eq. (1.2) i.e., that the set of one-loop dimensions of high-gradient operators is unbounded from below. On the other hand, this classical prediction is recovered in the limit $k \to \infty$ with $s/k \to 0$. For this reason we shall separate the computation of the most relevant one-loop dimension associated with high-gradient operators of order $s$ into the case when $s < k$ and the case when $k < s$.

In this context, we would like to remind the reader that the su(2)$_k$ WZW theories are known to describe quantum critical points in the parameter space of quantum spin-$S$ antiferromagnetic chains when $k = 2S$. Here, we observe that both, the number of relevant perturbations and the number of independent local composite operators built out of the generators of SU(2) which are SU(2) singlets, grows with $S$. [For $S = 1/2$ the algebra obeyed by the Pauli matrices only allows one invariant SU(2) tensor of rank 2, the $2 \times 2$ unit matrix.] On the other hand, the
strength of quantum fluctuations in \( \hat{s}(2) \) WZW theories decreases with increasing \( k = 2s \) for the same reason as the role of quantum fluctuations decreases with increasing \( S \) in quantum spin chains.

B. Anomalous dimensions of high-gradient operators

As the most general SU(2) invariant tensor of even rank in the adjoint representation is the product of the Casimir tensor of rank 2, we define the three diagonal SU(2) invariants out of the three current bilinears

\[
H := C^\alpha\beta J_\alpha\bar{J}_\beta, \quad A := C^\alpha\beta J_\alpha\bar{J}_\beta, \quad B := C^\alpha\beta J_\alpha\bar{J}_\beta,
\]

(2.11a)

together with the SU(2) invariant

\[
C := AB.
\]

(2.11b)

The space of the high-gradient operators is then spanned by the family

\[
\left\{ H^s, H^{s-2}C, \ldots, H^2C^{[s/2]-1}, C^{[s/2]} \right\}
\]

(2.12)

made of \( [s/2] + 1 \) “classical” operators. We call these operators “classical” because we have not yet taken into account the short distance singularities associated with the definition of composite operators (i.e., the “Pauli principle” discussed above). As announced below Eq. (2.10), these singularities need to be subtracted from the “classical” expressions (2.12) upon normal ordering. We shall nevertheless ignore the issue of normal ordering at first and compute the one-loop RG equation for these “un-regularized” (or un-normal-ordered) operators, a step of no consequence in the (“classical”) limit \( s/k \to 0 \). We shall then contrast this un-regularized calculation with the full quantum calculation for the special case of \( k = 1 \), i.e., when the proper normal ordering procedure has been accounted for.

We shall see that the calculation without normal ordering gives an infinite tower of high-gradient operators that are all relevant to one-loop order for sufficiently large \( s \) and for \( g > 0 \). The one-loop spectrum of anomalous dimensions is identical to the one in the \( O(N)/O(N-1) \) NL\( \sigma \)M when \( N = 3 \). Indeed, once the normal ordering procedure is ignored, the high-gradient operators (2.12) are analogous to those discussed in Ref. 13, and the calculations of the anomalous dimensions in the \( \hat{s}(2)_k \) WZW model and in the \( O(N)/O(N-1) \) NL\( \sigma \)M run along parallel tracks.

The effect of normal ordering is weaker the smaller \( s/k \) is, i.e., the closer proximity to the semi-classical limit of the WZW theory. To see this, consider the case when \( k > s \). The “classical” expression \( T^{\alpha_{1} \ldots \alpha_{s}} J_{\alpha_{1}} \cdots \bar{J}_{\alpha_{s}}(z, \bar{z}) \) for the composite operator made of a local product of holomorphic and antiholomorphic currents is modified upon normal ordering. To leading order in a short-distance expansion, this classical expression is replaced by

\[
\sum_{s_{1} \neq \cdots \neq s_{k} \neq \cdots \neq s_{k}=1}^{k} T^{\alpha_{1} \ldots \alpha_{s_{k}}} J_{\alpha_{1}} \cdots \bar{J}_{\alpha_{s_{k}}}(z) \times \bar{J}_{\alpha_{k+1}} \cdots \bar{J}_{\alpha_{s}}(\bar{z}) + \cdots.
\]

(2.13)

Here, the terms included in the \( \cdots \) arise from the OPE for the product \( J_{\alpha_{1}}(z)J_{\alpha_{2}}(0) \) when any two flavor indices \( \alpha_{i} \) and \( \alpha_{j} \) are identical. Evidently, normal ordering (or the Pauli principle) has a much more potent effect when \( k < s \), for the condition \( s_{1} \neq \cdots \neq s_{k} \neq \cdots \neq s_{k} \) can then never be met so that the leading order term above is absent. The operator contents with and without normal ordering thus look very different. When \( k < s \), some operators in the set (2.12) completely disappear to leading order because of Fermi statistics. This will be demonstrated explicitly for the case of \( k = 1 \) [see Eqs. (2.28) and (2.29) below], for which we will show, after correctly taking into account normal ordering, that all high-gradient operators which would be relevant classically (when \( g > 0 \)) disappear from the operator content.

1. RG equation for un-regularized high-gradient operators

To compute the leading one-loop scaling dimensions for the high-gradient operators (2.12), we start from the field theory (2.5) in which we substitute the action by

\[
S := S_0 + g \int \frac{dzd\bar{z}}{2\pi i} O_I - \sum_{m,n=0}^{2m+n=s} Z_{m,n}^{(s)} 2^{s-2} \int \frac{dzd\bar{z}}{2\pi i} C^m H^n.
\]

(2.14)

To determine the one-loop dimensions of the couplings \( \{ Z_{m,n}^{(s)} | 2m+n = s \} \), we do not need the full one-loop RG flows, i.e., the RG equations for the coupling constants up to and including order \( Z_{m,n}^{(s)} Z_{p,q}^{(s)} \), but only the linear in \( Z_{m,n}^{(s)} \) contributions to the one-loop RG flows. Thus, all we need are the OPE of \( C^m H^n(z, \bar{z}) \) with \( O_I(0) \), where the integers \( m \) and \( n \) satisfy \( 1 \leq 2m+n = s \leq k \). Furthermore, we shall introduce the short-hand notation

\[
A \times B = C \iff A(z, \bar{z})B(0) = \frac{1}{2\pi i} C(z, \bar{z}) + \cdots
\]

(2.15)

for the OPE relating the operators \( A, B, \) and \( C \). Here, the dots are meant to contain not only regular terms of zeroth and higher order in \( z \) or \( \bar{z} \) but also second and higher order poles in \( z \) or \( \bar{z} \).

As an intermediary step, one verifies that the OPE (2.13) between the building blocks \( H \) and \( C \) with \( O_I \) (observe that \( O_I = H \)) are

\[
\overset{\text{\( H \)}}{\text{\( x \)}} O_I = -4H, \quad (C \times O_I) = 0.
\]

(2.16)
Here, we introduced yet another short-hand notation \( A \cdots B \) or \( A \cdots B \), by which we mean that one current in \( A \) and one current in \( B \) are contracted with the rule
\[
J_{\alpha}(z)J_{\beta}(0) = \delta_{\alpha\beta} \left( \frac{C_{\alpha\beta}}{z^2} + i \frac{f_{\alpha\beta}}{z} J_{\gamma}(0) + \cdots \right),
\]
\[
\bar{J}_{\alpha}(z)\bar{J}_{\beta}(0) = \delta_{\alpha\beta} \left( \frac{C_{\alpha\beta}}{z^2} + i \frac{f_{\alpha\beta}}{z} \bar{J}_{\gamma}(0) + \cdots \right),
\]
\[
J_{\alpha}(z)\bar{J}_{\beta}(0) = 0,
\]
for any \( \alpha, \beta = 1, 2, 3 \) and \( \iota, \iota' = 1, \ldots, k \) at the free-fermion fixed point \( g = 0 \), where
\[
J_{\alpha} := \bar{\psi}_{\iota}^{a}(\sigma_{\alpha})_a \frac{b}{2} \psi_{b}, \quad \bar{J}_{\alpha} := \bar{\psi}_{\iota}^{a}(\sigma_{\alpha})_a \frac{\bar{b}}{2} \bar{\psi}_{b}. \tag{2.18}
\]

When \( A \) and \( B \) consist of more than one \( J_{\alpha} \) or \( \bar{J}_{\alpha} \), and when there are many possible Wick contractions between \( A \) and \( B \), the short-hand notations \( A \cdots B \), \( A \cdots B \) and \( \cdots \cdots B \) mean the resulting operator obtained by taking all possible such Wick contractions. One also verifies that the OPE (2.15) between the building blocks \( HH, CH \), and \( CC \) with \( O_I \) are
\[
HH \times O_I = -4H^2 + 4C,
\]
\[
CH \times O_I = CC \times O_I = 0. \tag{2.19}
\]

We then infer that, for any pair \((m, n)\) of positive integer that satisfies \( 1 < 2m + n = s \leq k \),
\[
C^m H^n \times O_I = C \times O_I \times m C^{m-1} H^n + \bar{H} \times O_I \times n C^{n-1} H^n + CH \times O_I \times mn C^{m-1} H^{n-1} + CC \times O_I \times \frac{m(m-1)}{2} C^{m-2} H^n + HH \times O_I \times n(n-1) C^{n-2} H^{n-2} = -2n(n+1) C^{m} H^{n} + 2n(n-1) C^{m+1} H^{n-2}.
\]

The contributions to the RG equations obeyed by the couplings \( Z_{m,n}^{(s)} \), where \( 1 < 2m + n = s \leq k \) needed to extract the spectrum of one-loop dimensions are
\[
\frac{dZ_{m,n}^{(s)}}{dl} = (2 - 2s) Z_{m,n}^{(s)} + 4gn(n+1) Z_{m,n}^{(s)} - 4g(n+2)(n+1) Z_{m-1,n+2}^{(s)} + \cdots. \tag{2.21}
\]

Here, the dots include non-linear contributions of second order in \( g \) or \( Z_{m,n}^{(s)} \).

The linearized RG flows (2.21) are closed. This is a justification a posteriori for neglecting the RG effects of current monomials with repeating flavor indices. The linearized RG flows (2.21) have a lower triangular structure, i.e., there is no feedback effect on the flow of a high-gradient operator of order \( s \) from lower-order high-gradient operators. Thus, we conclude that the leading \( [s/2] + 1 \) one-loop scaling dimensions associated with the family of high-gradient operators (2.12) when \( \geq s = 2m + n \) are given by
\[
x_{m,n}^{(s)} = 2(2m + n) - 4gn(n+1). \tag{2.22}
\]

Observe that the spectrum of anomalous dimensions
\[
\gamma_{m,n}^{(s)} := -4gn(n+1), \quad 2m + n = s \tag{2.23}
\]
is one sided with respect to 0. When \( g \leq 0 \) these anomalous dimensions are positive, i.e., the scaling dimensions are larger than their engineering value. The opposite happens when \( g \geq 0 \), i.e., when the current-current perturbation is (marginally) relevant. When \( g > 0 \) and for a given \( 1 < s \leq k \), the smallest one-loop anomalous dimension occurs for the pair \((m, n) = (0, s)\),
\[
\gamma_{m,n}^{(s)} := \min_{2m+n=s} \gamma_{m,n}^{(s)} = -4gs(s+1). \tag{2.24}
\]

For \( g > 0 \), the quadratic dependence on \( s \) can overcome the linear dependence on \( s \) in the one-loop dimension \( x_{m,n}^{(s)} := 2s + \gamma_{m,n}^{(s)} \). If the order \( s \) \( 1 < s \leq k \) is allowed to be sufficiently large, the one-loop dimension \( x_{m,n}^{(s)} \) decreases past the value 2 and eventually becomes negative. The quadratic dependence on \( s \) is reminiscent of that for the one-loop dimensions (2.1) in the \( O(N)/O(N-1) \) \( NL \sigma M \). However, in contrast to the \((2 + \epsilon)\)-dimensional \( O(N)/O(N-1) \) \( NL \sigma M \) at its non-trivial fixed point \( t^* \), a value smaller than 2 for the one-loop dimensions \( x_{m,n}^{(s)} \) is not a threat to the internal stability of the WZW fixed point \( g = 0 \) since it occurs along a flow to strong coupling. Moreover, it is known that in \( d = 2 \) dimensions the \( O(3)/O(2) \) \( NL \sigma M \) with theta term at \( \theta = \pi \) flows in the infrared into the level \( k = 1 \) SU(2) WZW fixed point. While the spectrum of one-loop dimensions of high-gradient operators at the WZW fixed point is bounded from below (as we will recall below), the spectrum of these operators is unbounded from below in the weakly coupled 2-dimensional \( O(3)/O(2) \) \( NL \sigma M \) (the presence of the theta term does not affect this result).

2. Normal ordering revisited

We shall illustrate the effects of the Fermi statistics for the family of high-gradient operators (2.12) when \( s = 2 \) for the case of \( g \) (marginally) relevant \((g > 0)\) current-current interaction. We shall then show for the special case of \( k = 1 \) and \( s = 2 \) that the two one-loop dimensions associated with the family of high-gradient operators (2.12) are unchanged, to one loop order, i.e.,
\[
x_{0,2}^{(s)} = x_{1,0}^{(s)} = 4. \tag{2.25}
\]
We start from the family of high-gradient operators \( \{\mathcal{O}_2\} \) with \( s = 2 \). For clarity of presentation, we rename the two members of this family,

\[
\mathcal{O}_1 \equiv C^{\alpha \beta} J_{\alpha} J_{\beta}, \quad \mathcal{O}_2 \equiv C^{\alpha \beta} J_{\alpha} J_{\beta} C^{\gamma \delta} J_{\gamma} J_{\delta}, \quad (2.26)
\]

As implied by Eq. (2.14) these are two classical expressions. The two quantum expressions involve point splitting and normal ordering as in Eq. (2.10).

Without loss of generality, we consider only the left current sector. Normal ordering of

\[
J_\alpha(z) J_\beta(0) = \frac{k C_{\alpha \beta}}{z^2} + \frac{i}{z} \epsilon_{\alpha \beta \gamma} J_\gamma(0) + \frac{i}{2} \epsilon_{\alpha \beta \gamma} \partial J_\gamma(0) + \frac{\delta_{\alpha \beta}}{4} \sum \frac{k}{i} : \left( \psi_{i}^a \partial \psi_{i}^a - \psi_{i}^a \partial \psi_{i}^a \right) : (0) + \frac{k}{i} : \psi_{i}^a \psi_{i}^b \psi_{i}^c \psi_{j}^d \partial \psi_{j}^e : (0) + \cdots \quad (2.27)
\]

amounts to the subtraction from Eq. (2.22) of the terms singular in the limit \( z \rightarrow 0 \),

\[
: J_\alpha J_\beta : (0) = \sum_{i \neq j}^k J_\alpha J_\beta (0) + \frac{i}{2} \epsilon_{\alpha \beta \gamma} \partial J_\gamma (0) + \frac{\delta_{\alpha \beta}}{4} \sum \frac{k}{i} : \left( \partial \psi_{i}^a \psi_{i}^a - \psi_{i}^a \partial \psi_{i}^a \right) : (0) \quad (2.28)
\]

for \( \alpha, \beta = 1, 2, 3 \). The proper quantum interpretation of the classical currents (2.28) is then

\[
: \mathcal{O}_1 : (\bar{z}, z) = 4 \sum_{\alpha, \beta = 1}^3 : J_\alpha J_\beta : (\bar{z}), \quad (2.29)
\]

\[
: \mathcal{O}_2 : (\bar{z}, z) = 4 \sum_{\alpha, \beta = 1}^3 : J_\alpha J_\beta : (\bar{z}), \quad (2.30)
\]

3. **High-gradient operators when \( k = 1 \)**

When \( k = 1 \), the summation over unequal flavors disappears in Eq. (2.22). (Observe in passing that : \( \mathcal{O}_1 : \) is then proportional to one component of the energy-momentum stress tensor.) One then verifies the OPE

\[
: \mathcal{O}_1 : \times \mathcal{O}_1 = 3 : \mathcal{O}_1 : - 9 : \mathcal{O}_2 :, \quad : \mathcal{O}_2 : \times \mathcal{O}_1 = : \mathcal{O}_1 : - 3 : \mathcal{O}_2 :: (2.30)
\]

If we diagonalize the linearized one-loop RG flows for the coupling \( \widetilde{Z}^{(2)}_{1,0} \) associated with : \( \mathcal{O}_1 : \) and the coupling \( \widetilde{Z}^{(2)}_{0,2} \) associated with : \( \mathcal{O}_2 : \), we find that their one-loop dimensions remain equal to their engineering dimensions,

\[
\tilde{Z}^{(2)}_{1,0} = \tilde{Z}^{(2)}_{0,2} = 4. \quad (2.31)
\]

The lesson that we draw from the example \( s = 2 \) and \( k = 1 \) is that it is necessary to use normal ordering to properly define composite operators. Had we not used normal ordering, we would have incorrectly predicted that there are infinitely many high-gradient operators which become relevant, at one-loop order, for large enough \( s \) and for \( g > 0 \). We believe that for a generic value of \( k \), there is no infinity of one-loop relevant high-gradient operators. Only a finite number of high-gradient operators become relevant, at one-loop order, for large enough \( s \) and for \( g > 0 \) when \( k > 1 \).

In the next section, we turn attention to a non-unitary WZW model of relevance to the problem of Anderson localization to investigate whether the loss of unitarity opens the door to an infinity of relevant high-gradient operators.

**III. HIGH-GRADIENT OPERATORS AND \( \mathfrak{g}l(M|M)_k \) WZW THEORIES**

An interesting example of a problem of Anderson localization in two dimensions which possesses a special so-called sublattice (or chiral) symmetry (SLS) and TRS (thus belonging to the “chiral-orthogonal” symmetry class BDI in the classification scheme of Zirnbauer, and Altland and Zirnbauer to be described by a WZW model on the supergroup GL(2 \( | \) 2 \( | \) k) is as follows. Consider a tight-binding model of fermions on a honeycomb lattice with random real-valued hopping matrix elements of non-vanishing mean, which do not connect the same sublattice (so that SLS is preserved). A related realization of the same problem of Anderson localization is provided by a random tight-binding model on a square lattice with flux-\( \pi \) through every plaquette, introduced in Ref. [33]. In the absence of disorder this band structure is known to exhibit the energy-momentum dispersion law of two species of (relativistic) Dirac fermions at two points in the Brillouin zone at low energy near the Fermi level (at zero energy). It was shown in Ref. [39] that the SLS-preserving disorder discussed above leads to a theory for the disorder averages which, in the supersymmetric formulation, is a GL(2N|2N) Thirring (Gross-Neveu) model. In other words, the problem of two-dimensional Anderson localization on the honeycomb lattice preserving SLS and TRS, is described by a set of Dirac fermions (and SUSY boson partners) perturbed by a current-current interaction of the Noether currents of its underlying GL(2N|2N) (super) symmetry. The interaction strength corresponds to the strength of the disorder. The system of free Dirac fermions (and SUSY boson partners) is well known to be described by a WZW model on the supergroup GL(2N|2N) with \( \mathfrak{g}l(2N|2N)_k \) conformal Kac-Moody current algebra symmetry at level \( k = 1 \).
This section is devoted to the one-loop RG analysis of high-gradient operators in the perturbed $\mathfrak{gl}(M|M)_k$ WZW theory. The main result of this section and of this article applies to the case of level $k = 1$ of relevance for the random tight-binding models discussed above. In order to state this result, we first need to recall from Ref. [3] that the $\mathfrak{gl}(2N|2N)$ Thirring (Gross-Neveu) models possess two coupling constants; one, $g_M$, which does not flow under the renormalization group (RG) and another, $g_A$, which flows logarithmically under the RG and a rate dependent on $g_M$. Our main result then suggests that all higher-order gradient operators are more irrelevant in the presence of the current-current interaction with $g_M > 0$ than at zero coupling $g_M = 0$. A positive $g_M$ can be interpreted as the variance of the disorder strength in the random tight-binding model in symmetry class BDI. For the opposite sign of the coupling constant $g_M < 0$, on the other hand, higher-order gradient operators have a spectrum of one-loop dimensions that is unbounded from below very much as in Eq. (1.2). In the context of Anderson localization, the case with $g_M < 0$ describes the surface state of a three-dimensional topological insulator in the chiral-symmetric class (symmetry class CII) of Anderson localization.

As in Sec. II, we are going to distinguish two limits. In the first (classical) limit,

$$M \to \infty, \quad k \to \infty,$$

(3.1)

OPEs between the high-gradient operators can be obtained without any reference to the composite nature of the currents. One then recovers a spectrum of one-loop scaling dimensions for high-gradient operators that mimics closely that of the NLσMs discussed above. The second limit,

$$M = 1, 2, 3, \cdots, \quad k = 1,$$

(3.2)

is the opposite extreme to the first one in that the normal ordering of the currents and thus of the high-gradient operators is essential and changes dramatically the spectrum of one-loop scaling dimensions from the “classical” limit [3].

### A. Definitions

Our starting point is a two-dimensional conformal field theory characterized by the current algebra [4]

$$J_A^B(z)J_C^D(0) = \frac{k\epsilon_{AC}^B}{z^2} + \frac{1}{z} \left[ \delta_{AC}^B J_A^D(0) + \epsilon_{AC}^B J_C^D(0) \right] + \cdots,$$

$$J_A^B(z)\bar{J}_C^D(0) = \frac{k\epsilon_{AC}^B}{z^2} + \frac{1}{z} \left[ \delta_{AC}^B J_A^D(0) + \epsilon_{AC}^B J_C^D(0) \right] + \cdots,$$

$$J_A^B(z)\bar{J}_C^D(0) = 0,$$

(3.3a)

where

$$\epsilon_{AC}^B := (-)^B + 1 \delta_{AC}^B \delta_D^B,$$

(3.3b)

and

$$\delta_{AC}^B = -(-)^{BC} \delta_{BC}^D, \quad \epsilon_{AC}^B = (-)^{BC + D(B+C)} \delta_{AC}^D,$$

(3.3c)

with the indices $A, B, C, D = 1, \cdots, M+N$, where $\delta_{BC}$ denotes the Kronecker delta. The capitalized indices $A, B, C$, and $D$ also carry a grade which is either $0$ for $M$ out of the $M+N$ values that they take or $1$ for the remaining $N$ values. It is the grade of the indices $A$ and $B$ that enters expressions such as $(-)^A$ or $(-)^{AB}$. The grade $0$ (1) will shortly be associated with bosons (fermions). The positive integer $k$ is the level of the current algebra [3]. The current algebra [3] is associated with the Lie superalgebra $\mathfrak{gl}(M|N)$ defined by the structure constants Eq. (3.3c) for $A, B, C, D = 1, \cdots, M + N$. When $N = 0$, the structure constants [3] reduce to

$$\delta_{AC}^B = -\delta_{BC}^A, \quad \epsilon_{AC}^B = +\delta_{AC}^D,$$

(3.4)

for $A, B, C, D = 1, \cdots, M$. These define the Lie algebra $\mathfrak{gl}(M)$ of the non-compact Lie group GL(M). When $M = 0$, the structure constants [3] reduce to

$$\delta_{AC}^B = +\delta_{BC}^A, \quad \epsilon_{AC}^B = -\delta_{AC}^D,$$

(3.5)

for $A, B, C, D = 1, \cdots, N$. These define the Lie algebra $\mathfrak{u}(N)$ of the compact Lie group U(N).

There exists a free-fermion and free-boson realization of the current algebra [3] defined by the action

$$S_* := \frac{1}{2} \int \frac{dz}{2\pi i} \left( \psi_A^\dagger \partial \psi_A + \bar{\psi}_A^\dagger \partial \bar{\psi}_A \right),$$

(3.6a)

with the partition function

$$Z_* := \int \mathcal{D}[\psi^\dagger, \psi, \bar{\psi}, \bar{\psi}] \exp(-S_*),$$

(3.6b)

where it is understood that $\psi_A$ and $\bar{\psi}_A$ are complex-valued integration variables for the $M$ values of $A$ with grade 0 while $\bar{\psi}_A$ and $\bar{\psi}_A$ are Grassmann-valued integration variables for the $N$ values of $A$ with grade 1, regardless of the value taken by the flavor index $i = 1, \cdots, k$. The current algebra [3] is then realized by the representation

$$J_A^B := \sum_{i=1}^k \psi_A^i \psi_A^i, \quad \bar{J}_A^B := \sum_{i=1}^k \bar{\psi}_A^i \bar{\psi}_A^i,$$

(3.7)

as follows from the OPE

$$\psi_A(z)\psi_A^i(0) = (-1)^{AB + 1} \psi_A^i(z)\psi_A(0) = \frac{\delta_{\psi A A}^i}{z},$$

$$\bar{\psi}_A(z)\bar{\psi}_A^i(0) = (-1)^{AB + 1} \bar{\psi}_A^i(z)\bar{\psi}_A(0) = \frac{\delta_{\psi A A}^i}{z},$$

$$\psi_A(z)\bar{\psi}_A^i(0) = 0,$$

(3.8)
with $A, B = 1, \cdots, M + N$ and $\ell = 1, \cdots, k$.

The expressions in Eq. (3.7) form a representation of the $\mathfrak{gl}(M|N)_k$ current algebra in terms of free fermions. There are two Casimir invariants of rank 2 in $\mathfrak{gl}(M|N)$ that we use to perturb the free field theory (3.6) with two types of current-current interactions, both of which are invariant under the global GL($M|N$) symmetry. \[ Z := \int \mathcal{D}[\psi, \bar{\psi}] \exp(-S), \]

\[ S := S_0 + \int \frac{d^2z}{2\pi i} \left( \frac{g_A}{2\pi} O_A + \frac{g_M}{2\pi} O_M \right), \tag{3.9} \]

$O_A := -J_A^A \left( -1 \right)^A \bar{J}_B^B \left( -1 \right)^B \equiv -\text{str} J \text{ str} \bar{J},$

$O_M := -J_B^B \bar{J}_A^A \left( -1 \right)^A \equiv -\text{str} (J \bar{J}).$\]

Formally, one may allow the coupling constants $g_A$ and $g_M$ to take on any real (i.e., positive or negative) values. However, to make connection with the above mentioned two-dimensional tight-binding models in symmetry class BDI of Anderson localization, we must demand that $g_A$ and $g_M$ be positive. (See Appendix B.)

The “classical” counterparts to the high-gradient operators of order $s$ in Eq. (2.9) are the homogeneous polynomials

\[ T_{A_1A_2\cdots A_s}^{B_1B_2\cdots B_s} J_{A_1} \cdots J_{A_s} \bar{J}_{B_1} \cdots \bar{J}_{B_s} \] \tag{3.10} \]

of the left and right currents that are invariant under the diagonal GL($M|N$) symmetry group of the interacting theory. The set of “classical” high-gradient operators of order $s$ is specified once all the linearly independent rank $2s$ invariant tensors $T_{A_1\cdots A_s}^{A_1\cdots A_s}$ in the adjoint representation of GL($M|N$) which are invariant under GL($M|N$) transformations have been enumerated. At the quantum level, normal ordering defines the quantum high-gradient operators of order $s$ as in Eq. (2.11).

We are now going to specialize to the case $M = N$ where the beta function for the coupling constant $g_M$ vanishes identically, an exact result. (As already mentioned, the other coupling constant $g_A$ flows logarithmically at a rate set by $g_M$.) The sector which we loosely denote by

\[ \text{PSL}(M|M) \sim \text{GL}(M|M)/U(1) \times U(1) \] \tag{3.11} \]

remains scale (conformally) invariant for any value of $g_M$. More specifically, PSL($M|M$) is obtained by first factoring out the U(1) subgroup thereby obtaining the subgroup SL($M|M$) of GL($M|M$), followed in a second step by the “gauging away” of the states carrying the U(1) charges under $j := J_A^A$ and $\bar{j} := J_A^A$. This turns out to realize a line of RG fixed points (and conformal field theories) labeled by the coupling constant $g_M$.

\section{High-gradient operators when $M, k \to \infty$}

We are going to show that, when $M$ and $k$ are very large, the spectrum for the one-loop scaling dimensions of high-gradient operators shares the same structure as that in Eq. (2.4). It will become clear by comparison to the case of $k = 1$ that the limit $M, k \to \infty$ is the extreme “classical” limit whereas the limit $k = 1$ is the extreme “quantum” limit.

We restrict the family of “classical” high-gradient operators to objects of the form

\[ \text{str} (J\bar{J}J\bar{J}) \text{ str} (J\bar{J}) \cdots, \tag{3.12} \]

e.g., to diagonal GL($M|M$)-invariant monomials of order $s$ in both the holomorphic and antiholomorphic currents. For any given order $s$, the engineering dimensions are all equal and given by $2s$. This degeneracy is lifted to first order in the coupling constant $g_M$. The task of enumerating all linearly-independent high-gradient operators (3.12) of order $s$ is greatly simplified by the assumption $M, k \to \infty$. We can rule out the scenario by which it is a finite set of independent Casimir operators of GL($M|M$) that fixes all the linearly independent classical high-gradient operators of order $s$ once the limit $M \to \infty$ has been taken. We can also rule out the scenario by which normal ordering changes the book-keeping between classical and quantum high-gradient operators of order $s$ once the limit $k \to \infty$ has been taken.

For high-gradient operators of type Eq. (3.11) or (3.12), the coupling $g_A$ does not renormalize their scaling dimensions, since $g_A$ (or $O_A$) can be removed from the action (3.3) by chiral transformation. All that therefore is needed to compute their one-loop scaling dimensions are their OPE with the quadratic Casimir operator $O_M$. We will write the following expressions for the general case of GL($M|N$), and will set $M = N$ (i.e., the case of interest) only in Eqs. (3.19), (3.20), and (3.21). The required OPEs follow from (a) the intra-trace formula

\[ \text{str} \left[ JMJN \right] \times \text{str} \left[ J\bar{J} \right] = \text{str} (J\bar{J}N) \text{ str} (JM) - \text{str} (JM) \text{ str} (J\bar{J}N) + \text{str} (JM) \text{ str} (J\bar{J}) \] \tag{3.13a} \]

and (b) the inter-trace formula

\[ \text{str} \left[ JM \right] \text{ str} \left[ J\bar{J} \right] = \text{str} (J\bar{J}N) \text{ str} (JM) - \text{str} (JM) \text{ str} (J\bar{J}N) + \text{str} (J\bar{J}JM) \] \tag{3.13b} \]

with $M$ and $N$ arbitrary operators. Here we have used the short-hand notation of Eq. (2.13).

To proceed we also need to distinguish linearly independent high-gradient operators of order $s$. To this end, a “quantum number”, the number of switches, is introduced. The number of switches of type $n_+ i$ and of type $n_- i$ in a single trace are defined as follows. Consider the trace

\[ \text{str} (J_{\mu_1} J_{\mu_2} J_{\mu_3} \cdots J_{\mu_{2n}}) \] \tag{3.14a} \]
where $\mu_1, \cdots, \mu_{2n} = \pm$ while $J_\pm = J$ and $J_+ = \tilde{J}$. Write the sequence of “conformal” indices

$$
\mu_1, \cdots, \mu_{2n}, \mu_{2n+1}
$$

(3.14b)

where $\mu_{2n+1} = \mu_1$ by cyclicity of the trace. The number $n_\lambda$ of switches of type $\uparrow$ is the number of sign changes from $+ \rightarrow -$ in two consecutive conformal indices when reading the sequence $\mu_1, \cdots, \mu_{2n}, \mu_{2n+1}$ from left to right. The number $n_\lambda$ of switches of type $\downarrow$ is the number of sign changes from $-$ $\rightarrow$ $+$ in two consecutive conformal indices when reading the sequence $\mu_1, \cdots, \mu_{2n}, \mu_{2n+1}$ from left to right.

These quantum numbers are useful as it can be shown that there is no contribution in the one-loop RG of supertraces made out of $2n$ currents as in Eq. (3.14a) from the subspace with $n_\lambda$ and $n_\lambda$ to the one with at least $n_\lambda + 1$ and $n_\lambda + 1$. This implies a lower triangular structure for the linearized RG equations obeyed by all supertraces of order $2n$ as in Eq. (3.14a) which allows to treat separately each sector defined by a given number of switches. We shall assume that the strongest renormalization of the engineering scaling dimensions occurs within the sector made of the maximum number of switches.

Within the subspace of maximal switches it is sufficient to introduce

$$
\omega : = J\tilde{J} \equiv J_- J_+,
$$

$$
\Omega_m := \text{str} (\omega^m),
$$

(3.15)

for any $m = 1, 2, 3, \cdots$. With the help of the OPE (3.13a) and (3.13b) one verifies the OPE

$$
\text{str} (\omega^m \omega^n) \times \mathcal{O}_M = -\Omega_{m+1} \Omega_n - \Omega_{n+1} \Omega_m
$$

(3.16a)

$$
\text{str} (\omega^m) \text{str} (\omega^n) \times \mathcal{O}_M = -4 \Omega_{m+n+2},
$$

$$
\text{str} (\omega^m) \times \mathcal{O}_M = -\Omega_1 \Omega_m - (N - M) \Omega_{m+1},
$$

and

$$
\Omega_m \times \mathcal{O}_M = -2m \sum_{k,l=1}^{k+l=m} \Omega_k \Omega_l - 2m(N - M) \Omega_m,
$$

$$
\Omega^r_m \Omega^r_n \times \mathcal{O}_M = -4r_m r_n m n \Omega_{m+n} \Omega_{m+n}^{-1} \Omega_n^{-1},
$$

(3.16b)

for any $m, n, r_m, r_n = 1, 2, 3, \cdots$.

The action of the linearized one-loop RG flow on the space of composite operators in the subspace of maximal switches spanned by

$$
\Omega_1^r \Omega_2^r \cdots \Omega_L^r, \quad \sum_{p=1}^{L} p r_p = 2s,
$$

(3.17)

is encoded by the operator

$$
\hat{R} : = -2(N - M) \sum_k k \Omega_k \frac{\partial}{\partial \Omega_k} + \frac{(l+n) \Omega_n \partial}{\partial \Omega_{l+n}} + \ln \Omega_{l+n} \frac{\partial}{\partial \Omega_l} \frac{\partial}{\partial \Omega_n}.
$$

(3.18)

It is instructive to compare the OPE (3.16) and the RG equation (3.18) with the corresponding result in the weakly coupled NL$\sigma$M on the symmetric space $U(P + Q)/U(P) \times U(Q)$ with $P, Q > 1$. They are essentially identical to the corresponding result for the $U(P + Q)/U(P) \times U(Q)$ NL$\sigma$M.

Now we return to the case $M = N$. The diagonalization of $\hat{R}$ gives the largest and smallest eigenvalues

$$
\lambda_{\text{max}}^{(s)} = 2s(s - 1) - \lambda_{\text{min}}^{(s)}.
$$

(3.19)

Thus, both largest and smallest eigenvalues depend quadratically on $s$. In turn, one obtains a spectrum of one-loop scaling dimensions with the upper and lower bounds

$$
x_{\text{max}}^{(s)} = 2s + \frac{g_m}{2\pi} s(s - 1),
$$

$$
x_{\text{min}}^{(s)} = 2s - \frac{g_m}{2\pi} s(s - 1),
$$

for any given $1 < s = 2, 3, \cdots$. Observe that these bounds are interchanged when $g_m \to -g_m$.

C. High-gradient operators when $k = 1$

Having dealt with the extreme “classical” limit, we turn our attention to the extreme “quantum” limit $M = 1, 2, 3, \cdots$ and $k = 1$ for which the interacting field theory (3.10) describes a problem of Anderson localization in $d = 2$ dimensions reviewed in Appendix [4].

The classification of all independent high-gradient operators in GL($M$|$M$) or in PSL($M$|$M$) is more involved than in SU(2) because the problem of listing all invariants is more complex. [4] An increase of complexity can already be seen at the level of SU($N$) for which the invariant tensors of rank 2s are obtained from all possible products of one rank 2 tensor and two rank 3 tensors. Instead of considering the most generic family of “classical” high-gradient operators (3.10), we consider the GL($M$|$M$) invariant family of “classical” objects

$$
\{ \mathcal{O}_M^m \mathcal{O}_A^n | m, n = 0, 1, 2, 3, \cdots, m + n = s \},
$$

(3.22)

which must then be normal ordered. We are going to prove that the coupling constant $Z_{m,n}^{(s)}$ of the high-gradient operator $\mathcal{O}_M^m \mathcal{O}_A^n$ in the action

$$
S = S_* + \int \frac{d^2z}{2\pi i} \left( \frac{g_m}{2\pi} \mathcal{O}_A^m + \frac{g_m}{2\pi} \mathcal{O}_M^n \right) - \sum_{m,n=0}^{m+n=s} Z_{m,n}^{(s)} \mathcal{O}_M^m \mathcal{O}_A^n \int \frac{d^2z}{2\pi i} \mathcal{O}_M^m \mathcal{O}_A^n
$$

(3.23)
obeys the linearized one-loop RG equation

$$\frac{dZ_{m,n}^{(s)}}{dl} = (2 - 2s) Z_{m,n}^{(s)} - \frac{4g_M^2}{2\pi} m(m - 1) Z_{m,n}^{(s)} + \frac{4g_M^2}{2\pi} (m + 1)^2 Z_{m+1,n-1}^{(s)}$$  \hspace{1cm} (3.24)$$

for any \(m, n = 0, 1, 2, 3, \cdots\) with \(m + n = s > 1\). For the \(PSL(M|M)\) theory the operators \(O^m_A\) are all absent.

The RG equation (3.24) shows that there is no feedback from high-gradient operators containing a factor \(O^m_A\) to those containing a factor \(O^n_A\) provided \(n' < n\). Diagonalization of the RG equation gives the set of one-loop scaling dimensions

$$x_{m,n}^{(s)} = 2s + \frac{2g_M}{\pi} m(m - 1)$$  \hspace{1cm} (3.25)$$

for all \(m, n = 0, 1, 2, 3, \cdots\) such that \(m + n = s\). For a positive \(g_M\) we get the lower and upper bounds

$$x_{\min}^{(s)} = 2s, \hspace{1cm} x_{\max}^{(s)} = 2s + \frac{2g_M}{\pi} s(s - 1)$$  \hspace{1cm} (3.26)$$

respectively, i.e., \(x_{m,n}^{(s)}\) with \(m + n = s\) is always much larger than the engineering dimension \(2s\) so that the high-gradient operator \(O^n_M\) is irrelevant. For a negative \(g_M\), the spectrum of lower bounds on \(x_{m,n}^{(s)}\) with \(m + n = s\) is unbounded from below when \(s \to \infty\), i.e.,

$$x_{\min}^{(s)} = 2s - \frac{2|g_M|}{\pi} s(s - 1), \hspace{1cm} x_{\max}^{(s)} = 2s$$  \hspace{1cm} (3.27)$$

Proof: Having made the simplification \(g_A = 0\) we only need to compute the OPE \(O^n_M O^m_A \times O_M\), where \(1 \leq m + n = s\), to justify Eqs. (3.24) and (3.25). Each operator in Eq. (3.24) contains terms with 4s bosons, 4s - 2 bosons and 2 different fermions, 4s - 4 bosons and 4 different fermions, ..., \(4s - 2M\) bosons and \(2M\) different fermions, and so on. The terms that contain identical fermions have short-distance singularities and hence they should be interpreted as operators that involve gradients over fermion fields after normal ordering. It is understood from now on that the OPE \(O^n_M O^m_A \times O_M\) is only over the terms in the expansion \(O^n_M O^m_A\) involving different fermions, i.e., the OPE we present are “accurate” up to terms involving gradients over fermionic spinors. Neglecting the OPE between derivatives of the fermionic spinors and \(O_M\) is harmless insofar as these OPE cannot feedback into the RG flows of those contributions that we keep.

Let

$$\chi(\xi) := \sum_{A=1}^{2M} A^A \xi_A - \sum_{A=1}^{2M} (-)^A \xi_A A^A$$  \hspace{1cm} (3.28)$$

and remember that \(O_{M} = - (\psi^\dagger \psi) \bar{\psi} \psi\) while \(O_{A} = - (\psi^\dagger \bar{\psi}) \bar{\psi} \psi\). The OPE that involve \((\psi^\dagger \psi)\) and \((\bar{\psi} \psi)\) are

$$\begin{align*}
(\psi^\dagger \psi) \times (\bar{\psi} \psi) &= 0, \\
(\psi^\dagger \psi)(\bar{\psi} \psi)(\psi^\dagger \psi) &= - (\psi^\dagger \psi), \\
(\psi^\dagger \psi)(\bar{\psi} \psi)(\psi^\dagger \psi) &= - O_A.
\end{align*}$$  \hspace{1cm} (3.29)$$

On the other hand, the OPE that involve \((\psi^\dagger \psi)\), \((\bar{\psi} \psi)\), \((\psi^\dagger \bar{\psi})\), and \((\bar{\psi} \psi)\) are given by

$$\begin{align*}
(\psi^\dagger \psi)(\bar{\psi} \psi)(\psi^\dagger \psi) &= (\psi^\dagger \psi)(\psi^\dagger \bar{\psi})(\psi^\dagger \psi) = - O_M, \\
(\psi^\dagger \psi)(\psi^\dagger \bar{\psi})(\bar{\psi} \psi) &= (\bar{\psi} \psi)(\bar{\psi} \psi)(\psi^\dagger \psi) = (\psi^\dagger \psi)(\bar{\psi} \psi)(\psi^\dagger \psi) = (\psi^\dagger \psi)(\bar{\psi} \psi)(\psi^\dagger \psi).
\end{align*}$$  \hspace{1cm} (3.30)$$

Both \(O_M\) and \(O_A\) are generated through the OPE (3.29) and (3.30), respectively. However, two OPE in Eq. (3.30) always appear in a pairwise fashion and cancel each other,

$$\begin{align*}
(\psi^\dagger \psi)(\bar{\psi} \psi) A \times O_M + (\psi^\dagger \psi)(\bar{\psi} \psi) A \times O_M = 0.
\end{align*}$$  \hspace{1cm} (3.31)$$

where \(A\) is some operator. Hence, the total number of \(O_M\) contained in a high-gradient operator never increases under the linearized RG flow.

From the OPE \((3.29)\) and \((3.30)\) one deduces the OPE

$$\begin{align*}
O^n_M O^m_A \times O_M &= mO^{m-1}_M O^m_A O_M \times O_M + nO^m_A O^{n-1}_M O_A \times O_M \\
&= mO^{m-1}_M O^m_A O_M \times O_M + nO^m_A O^{n-1}_M O_A \times O_M \\
&= mO^{m-1}_M O^m_A O_M \times O_M + nO^m_A O^{n-1}_M O_A \times O_M \\
&= mO^{m-1}_M O^m_A O_M \times O_M + nO^m_A O^{n-1}_M O_A \times O_M.
\end{align*}$$  \hspace{1cm} (3.32)$$

(When \(m = 0\), the term with \(O^{n-1}_M\) is absent from the last line.) The linearized one-loop RG equation (3.24) thus follows from the OPE (3.32). \(\Box\)

Had we assumed the level \(k\) to be larger than \(k = 1\), the family (3.22) would not have been closed under the OPE with \(O_M\). For example, in the extreme classical limit \(M, k \to \infty\) the family of high-gradient operators is given by the much larger family (3.12).

We close by pointing out that we could have reached the same conclusions on the spectrum of one-loop scaling dimensions of high-gradient operators had we used, instead of the effective action with diagonal GL(M|M) symmetry, an action built out of fermionic replicas or an action built out of bosonic replicas and taken the number of replicas to zero at the end of the day. Using bosonic
replicas mimics very closely the line of argument presented here. Using fermionic replicas singles out high-gradient operators made of fermionic spinors that are all distinct through their replica index and then taking this replica index to zero, very much in the same way as replicated vortices in certain classes of classical random two-dimensional Coulomb gases.

We would like to stress that our results depend crucially on the continuous symmetry GL(2N|2N) of the \( \mathfrak{gl}(2N|2N) \) Thirring model. (From the point of view of Anderson localization, it is the existence of a continuous symmetry not the symmetry group per se that matters since the symmetry group changes depending on the choice made to represent single-particle Green’s functions, say a supersymmetric, bosonic replicas, fermionic replicas, or Keldysh path integral.) If we consider local perturbations (local operators) that break the GL(2N|2N) symmetry, an infinite set of local operators with relevant (negative) scaling dimensions can appear. This alternative set of local operators may be related to the situation recently considered by Le Doussal and Schehr. The microscopic starting point of Ref. 2 is a class of classical random XY models in two dimensions. These models can also be viewed as interacting models of Dirac fermions subjected to disorder, by the magic of the boson-fermion duality in \( d = (1 + 1) \) dimensions. The difference with our paper is that their model is not invariant under a continuous symmetry group, but only under the discrete symmetry group which permutes the replica indices. It is then necessary to use the full machinery of functional RG to account for the one-loop relevance of high-gradient operators.

D. Comparison with the \( \mathbb{C}P^{1|2} \) NLσM

The perturbed \( \tilde{\mathfrak{g}}(2N|2N)_{k=1} \) WZW model with \( g_M > 0 \) (Thirring model) describes a problem of Anderson localization in two dimensions. As briefly reviewed in Appendix 3, this problem of Anderson localization arises as the long-wavelength description of a tight-binding model on a two-dimensional bipartite lattice with a form of disorder that preserves sublattice and time-reversal symmetries. The long-wavelength theory is a \( (2+1) \)-dimensional Dirac equation subject to disorder potentials consistent with these symmetries. In terms of the symmetry-based classification of Anderson localization, the relevant symmetry class is the class BDI (chiral-orthogonal symmetry class).

It is possible to use a different representation of this Anderson localization problem, in terms of a non-compact supermanifold

\[
\text{GL}(2N|2N)/\text{OSp}(2N|2N).
\]  

(A suitable analytical continuation in the boson-boson sector is needed to implement the non-compactness.) These two descriptions, one in terms of the Thirring model and the other in terms of the NLσM, are complementary to each other in that when one of the models is strongly coupled, the other is weakly coupled. A reflection of this appears in the conductivity. The coupling constant of the NLσM is inversely proportional to the conductivity. In the clean limit \( g_M = 0 \) of the Thirring model the conductivity is of order unity (in units of \( e^2/h \)), consistent with the strongly coupled regime of the NLσM. The conductivity decreases with \( g_M > 0 \) as seen in perturbation theory. Furthermore, both \( g_M \) and the conductivity are exactly marginal. This suggests a deeper relationship between the Thirring model and the NLσM, and indeed (following Ref. 41), one can turn the Thirring model into the NLσM continuously by tuning \( g_M \) (or equivalently the conductivity) continuously.

We consider the case \( N = 1 \) for which the non-compact target supermanifold \( U(1) \times U(1) \times \mathbb{C}P^{1|2} \), where again a suitable analytical continuation is obtained for \( \mathbb{C}P^{1|2} \), i.e., we need to consider the non-compact counterpart to \( \mathbb{C}P^{1|2} \) as defined in Appendix 4. Obtaining the non-compact \( \mathbb{C}P^{1|2} \) target supermanifold of the NLσM from \( U(1) \times U(1) \times \mathbb{C}P^{1|2} \) corresponds in the Thirring model to the reduction of the GL(2|2) to the PSL(2|2) current algebra in Eq. (3.11).

It is explicitly shown in Appendix 5 that all high-gradient operators are made more irrelevant at one-loop order by fluctuations in any non-compact \( \mathbb{C}P^{N+M-1|N} \) NLσM labeled by the non-negative integers \( M \) and \( N \). To be more precise, we find that the largest and smallest one-loop scaling dimensions for the high-gradient operators of type \( A_{14} \), for a given \( s \), are

\[
\begin{align*}
x^{(s)}_{\text{max}} &= 2s + 2|s(s - 1)|, \\
x^{(s)}_{\text{min}} &= 2s + 2|s| \times 0,
\end{align*}
\]  

where \( |s| > 0 \) is the coupling constant of the non-compact \( \mathbb{C}P^{N-1|N} \) NLσM. This is fully consistent with our finding [28] in the Thirring model. We conclude that, in symmetry class BDI, high-gradient operators in the Thirring model with \( g_M > 0 \) behave in the same way as in the corresponding NLσM (i.e., the one that belongs to the symmetry class BDI).

The sign of \( g_M \) in the perturbed \( \tilde{\mathfrak{g}}(2N|2N)_{k=1} \) WZW model can be chosen to be negative, \( g_M < 0 \). If so, this field theory does not represent anymore the moments of the single-particle Green’s function in a problem of Anderson localization in (bulk) two dimensions. Nevertheless, this field theory does describe a problem of Anderson localization which, however, now belongs to the different symmetry class CII (chiral-symplectic symmetry class) describing the effect of disorder on the Dirac fermions which are known to appear at the two-dimensional boundary of a three-dimensional topological band insulator in the same symmetry class. Equation (3.29) implies that high-gradient operators are now made more relevant by the current-current perturbation \( g_M < 0 \) to one-loop order. As for the case with \( g_M > 0 \), a problem of Anderson localization in the symmetry class
CII is characterized by a NLσM with a corresponding target manifold. As before, the beta function of the coupling constant $g_M < 0$ of the Thirring model as well as that of the coupling constant of the corresponding NLσM vanish, and one can interpolate between the weak coupling limit of the Thirring model and the strong coupling limit of the NLσM and conversely, by tuning $g_M$ continuously. The target supermanifold in symmetry class CII is the compact supermanifold $\mathbb{C}P^{|\hat{N}|}$, from which one extracts when $N = 1$ the NLσM on the compact supermanifold $\mathbb{C}P^{|\hat{N}1|}$. It is explicitly shown in Appendix [4] that all high-gradient operators are made more relevant at one-loop order by fluctuations in any compact $\mathbb{C}P^{N+M-1|N}$ NLσM labeled by the non-negative integers $M$ and $N$. In particular, for $M = 0$, we find that the largest and smallest one-loop scaling dimensions for the high-gradient operators of type $A(s)$, for a given $s$, are

$$x^{(s)}_{\text{max}} = 2s - 2ts(s - 1),$$

$$x^{(s)}_{\text{min}} = 2s + 2t \times 0,$$

where $t > 0$ is the coupling constant of the compact $\mathbb{C}P^{N-1|N}$ NLσM. Once again, we conclude that high-gradient operators behave in the same way in the Thirring model with $g_M < 0$ and in the corresponding NLσM that belongs to the symmetry class CII.

**IV. CONCLUSIONS**

More than twenty years after their discovery, the role of high-gradient operators, which appear to be highly relevant in one-loop computations of anomalous dimensions in a great variety of NLσMs, still remains a puzzle. Indeed, this perturbative property is rather general as it can apply to both compact and non-compact target manifolds. In the absence of an exact calculation of observables that would be sensitive to high-gradient operators, it is still an outstanding question whether the extreme RG-relevance of these operators is an artifact of the one-loop calculation (e.g., in the $2 + \epsilon$-expansion), or is a feature that is generally valid. (For an attempt to compare the $\epsilon$ expansion in $d = 2 - \epsilon$ dimensions with exact results obtained for $d = 1$, see Ref. [16].)

In order to shed some light on these issues we have asked in this paper the following question. Can high-gradient operators become relevant in the family of two-dimensional $\mathcal{gl}(M|M)_k$ Thirring models with $M$ and $k$ positive integers? The strategy that we followed has three steps. The first step consists of identifying all the independent “classical” high-gradient operators of order $s$. This is a problem of group theory that involves the enumeration of all distinct $GL(M|M)$ singlets in the direct product of $2s$ adjoint representations of $GL(M|M)$. The second step consists of normal-ordering all independent classical high-gradient operators of order $s$. This step depends crucially on the level $k$ of the non-Abelian Thirring model. The inverse level $1/k$ plays the role of a quantum parameter that vanishes in the limit $k \rightarrow \infty$. The level $k = 1$ is thus the most “quantum”. The computation of the linearized RG flows for the high-gradient operators is the final step.

We could not solve the first step in its full generality. We were nevertheless able to construct two sets of high-gradient operators in the extreme “classical” limit $\mathcal{gl}(M|M)_k$ with $M, k \rightarrow \infty$ and the extreme “quantum” case $\mathcal{gl}(M|M)_k$ with $M$ a positive integer and $k = 1$, respectively, and carry out the second and third steps consistently, i.e., show that each family of normal-ordered high-gradient operators is closed under the linearized RG flow equations.

The set of high-gradient operators that we considered in the extreme “quantum” limit is much smaller than the set of high-gradient operators for the extreme “classical” case. This is to be expected as normal ordering is extremely sensitive to the free-field fermionic representation of the $\mathcal{gl}(M|M)_k$ current algebra at the unperturbed WZW critical point. This difference has dramatic consequences for the spectrum of one-loop anomalous scaling dimensions in the extreme “classical” and “quantum” cases.

In the extreme “classical” case, anomalous one-loop scaling dimensions for high-gradient operators of order $s$ are distributed in a symmetric fashion about zero with the minimum and the maximum both depending quadratically on the order $s$, very much like for the family of NLσMs on the target spaces $U(M+N)/U(M) \times U(N)$ with $M$ and $N$ positive integers. Hence, high-gradient operators must become (one-loop) relevant for both signs of the current-current interaction with increasing order $s$ very much in the same way as their cousins do in both the compact family $U(M+N)/U(M) \times U(N)$ and the non-compact family $U(M,N)/U(M) \times U(N)$ with $M, N > 1$.

In the extreme quantum case $k = 1$, the spectrum of anomalous one-loop scaling dimensions of order $s$ is always one-sided, i.e., positive for one sign of the current-current interaction. For $\mathcal{gl}(M|M)_{k=1}$ with $M$ a positive integer the sign of the current-current interaction for which high-gradient operators are always irrelevant corresponds to the interpretation of the $\mathcal{gl}(2N|2N)_{k=1}$ Thirring model as a problem of Anderson localization in random tight-binding models on two-dimensional bipartite lattices (symmetry class BDI). We have shown in this paper that the high-gradient operators in these random tight-binding models are irrelevant at one-loop order.

High-gradient operators in those NLσMs of relevance to the physics of Anderson localization are related to the moments of the dc conductance. Their perturbative one-loop relevance has been interpreted as the signature of broad tails in the probability distribution of the conductance in Refs. [28]. (One should, however, bear in mind that the current-current correlation function entering the Kubo formula for the conductance looks rather different from a simple $GL(2N|2N)$ current-current cor-
relation function. It would thus be very interesting to study the probability distribution of the dc conductance in the relevant random tight-binding model using nonperturbative techniques (this may include, e.g., also numerical approaches) in order to establish if it is broad or not.

Acknowledgments

CM would like to thank Eduardo Fradkin for important comments. This research was supported in part by the National Science Foundation under Grant No. PHY05-51164 and under Grant No. DMR-0706140 (AWWL). SR thanks the Center for Condensed Matter Theory at University of California, Berkeley for its support.

Appendix A: High-gradient operators in NLσMs on the complex projective superspaces CP^{N+M−1|N}

Whether or not the spectra of anomalous one-loop scaling dimensions of high-gradient operators in NLσMs are symmetric about zero or not can be very important when the analytical continuation of the coupling constant $t$ in the NLσM from positive to negative values is meaningful from a physical point of view. We shall call spectra which are fully symmetric about zero two-sided spectra. Spectra of anomalous one-loop scaling dimensions that are strictly positive (negative) will be called one-sided. NLσMs with the target manifolds $S^{N−1} = O(N)/O(N − 1)$ are already known to be one-sided. We are going to show that this is also the case for the NLσMs with the target manifolds $\sigma$.

$$C P^{N+M−1|N} \simeq U(N + M|N)/[U(1) \times U(N + M − 1|N)]. \quad (A1)$$

The complex projective superspaces (A1) are generalizations of the compact complex projective spaces

$$C P^{M−1} \simeq U(M)/[U(1) \times U(M − 1)]. \quad (A2)$$

We shall also study on their own right the high-gradient operators in NLσMs with the non-compact complex projective target spaces

$$U(1, N + M − 1|N)/[U(1) \times U(N + M − 1|N)]. \quad (A3)$$

These non-compact manifolds follow from the compact complex projective spaces (A2) upon analytical continuation of some real coordinates to imaginary ones. The complex projective superspaces (A1) are special cases of the Kähler supermanifolds whose high-gradient operators were studied in Ref. [16]. We refer the reader for notations, conventions, and the relevant intermediary results to Ref. [14].

Appendix A is organized as follows. We first define NLσMs on the projective superspaces (A1) using a geometrical approach. We then present the main result of Appendix A on the one-loop scaling dimensions of high-gradient operators and show that they are one sided. We close by briefly outlining how the one-loop scaling dimensions of high-gradient operators are computed.

1. Geometry of the $CP^{N+M−1|N}$ NLσMs

A NLσM on a Hermitian supermanifold can be defined with the help of the partition function

$$Z := \int \mathcal{D} [\varphi^A, \chi] e^{-S[\varphi^A, \chi]},$$

$$S := \frac{1}{2\pi t} \int d^D r \left( \partial_a \varphi^a \right) a^* g_b (\varphi^1, \varphi) \left( \partial_b \varphi^b \right). \quad (A4)$$

Here, (ψ^1, ψ) are the coordinates on the Hermitian supermanifold, $a^* g_b (\varphi^1, \varphi)$ is the metric on the Hermitian supermanifold, and $t$ is the NLσM’s coupling constant.

We are going to restrict our analysis to Hermitian supermanifolds such that their metric can be derived from a Kähler potential. Furthermore, we shall choose the Kähler potential so that the target supermanifold is none of the compact complex projective spaces (A2) upon analytical continuation of some real coordinates to imaginary ones. The complex projective superspaces (A1) are special cases of the compact complex projective manifolds (A2).

The derivative of the Kähler potential (A5a) gives the metric on the Hermitian supermanifold:

$$\partial[\varphi^a \varphi^b], \partial[\varphi^a \varphi^b] \equiv Z_{a^* b}. \quad (A5a)$$

The bilinear form

$$\varphi^a \varphi^b := \varphi^a \varphi^b \quad (A5b)$$

is presented in terms of the diagonal tensor $\xi$ with the components

$$\partial[\varphi^a \varphi^b], \partial[\varphi^a \varphi^b] \equiv Z_{a^* b}. \quad (A5c)$$

that do not depend on the coordinates $(\varphi^1, \varphi)$. Hence,

$$1 + \varphi^1 \varphi = 1 + \sum_{i=1}^{N+M-1} \varphi^i \varphi^i + \sum_{i=1}^{N} \psi^2 \psi^i \quad (A5d)$$

where $(\varphi^1, \varphi^i)$ with $i = 1, \cdots, N + M − 1$ and $(\psi^1, \psi^i)$ with $i = 1, \cdots, N$ are the bosonic and fermionic coordinates of $(\varphi^1, \varphi)$, respectively. We observe that $(\varphi^1, \varphi^i)$ has $N + M − 1$ bosonic and $N$ fermionic coordinates. Equations (A4) and (A5) define the $CP^{N+M−1|N}$ NLσMs. Setting $N = 0$ in Eqs. (A4) and (A5) defines the NLσMs on the compact complex projective manifold (A2). The analytical continuation $\varphi^i \rightarrow i \varphi^i$ and $\varphi^1 \rightarrow i \varphi^1$ in Eqs. (A4) and (A5) defines the NLσMs on the non-compact complex projective manifold (A1). The derivative of the Kähler potential (A5a) gives the metric (a superanalogue of the Fubini-Study metric) for $CP^{N+M−1|N}$ through

$$a^* g_b = \frac{\partial}{\partial[\varphi^a]^{a^*}} K \frac{\partial}{\partial[\varphi^b]} \equiv Z_{a^* b}. \quad (A6a)$$
We have introduced the scalar
\[ Z := \frac{1}{1 + \varphi^2 \xi \varphi} \]  
(A6b)
and the tensor
\[ a^* Y_b := a^* \xi_b - a^* \xi_c \varphi^a Z \varphi^d d^e \xi_b. \]  
(A6c)

Following the usage for graded indices from Ref. [16], we also define the tensors
\[ a^b Y^b := a^c \varphi^b + \varphi^a \varphi^{ab}, \]
\[ a^* Y^b := ( -1)^{a+b+ab} b^a Y_a, \]
\[ a^b \varphi^b := ( -1)^{ab} b^a \varphi^{ab}. \]  
(A7)

It then follows that the metric indices can be raised, lowered, or shifted according to
\[ a^* g_b = Z_a Y_b, \]
\[ a g^{\nu} = Z_a Y_{\nu}, \]
\[ a^* g^b = Z^{-1} a^* \varphi^b + \varphi^a \varphi^{ab} \]
\[ a g^{\nu} = Z^{-1} a^* \varphi^b = Z^{-1} a^* \varphi^b. \]  
(A8)

The metric tensor (A6) can be expanded about any point of the manifold at which it is finite (flat geometry). The lowest order in this expansion defines the “kinetic” contribution to the Lagrangian of the NL$\sigma$M, whereas the higher-order contributions define the “interactions”. The bosonic contribution to the kinetic energy must be positive definite for the path integral to be well defined. This condition fixes the sign of the coupling constant $t$. For the compact complex projective manifolds (A3), $t > 0$ must be chosen. For the non-compact complex projective manifolds (A3), $t < 0$ must be chosen (see, for example, Ref. [7]). A consequence of the analytical continuation $t \rightarrow -t$ for the one-loop beta function of $t$, if it is proportional to $t^2$ as it is in $d = 2$ dimensions, is that it changes by a sign. Similarly, the one-loop corrections to the scaling dimensions of high-gradient operators also change by a sign under the analytical continuation $t \rightarrow -t$. It then matters greatly whether the anomalous one-loop scaling dimensions are two-sided or one-sided.

According to Friedan, the one-loop beta function for the coupling constant $t$ of a NL$\sigma$M on a Riemannian manifold is given by the curvature of the manifold. The curvature follows from the Ricci tensor, which we now compute for $\mathbb{C}P^{N+M-1|N}$. Needed are the coefficients of the connection. They are
\[ \Gamma^a_{bc} := a^b g^d g_{bc}, \]
\[ = - Z \left[ a^b \varphi^d \varphi^e d^f \xi_c + ( -1)^{bc} a^b \varphi^d \varphi^e d^f \xi_b \right]. \]  
(A9a)

and
\[ \Gamma^a_{b^* c^*} := a^* g^b g_{b^* c^*}, \]
\[ = - Z \left[ ( -1)^{a} a^b \varphi^d \varphi^e d^f \xi_c + ( -1)^{b+c} a^b \varphi^d \varphi^e d^f \xi_b \right]. \]  
(A9b)

The curvature tensor field on $\mathbb{C}P^{N+M-1|N}$ can then be expressed solely in terms of the metric tensor field,
\[ R^a_{bc \gamma} = - \delta^a_{bc} \partial_{\gamma} + \delta^b_{c \gamma} \partial_{c \gamma} + \delta^c_{b \gamma} \partial_{c \gamma} \]  
(A10a)
and
\[ R^a_{b^* c^* d} = - \delta^a_{b^* c^* d} - \delta^b_{c^* d} + \delta^c_{b^* d} - \delta^d_{b^* c^*}. \]  
(A10b)

For $\mathbb{C}P^{N+M-1|N}$, the Ricci tensor field is proportional to the metric with $M$ the proportionality constant,
\[ R_{bd} = Mg_{bd}. \]  
(A11)

For $\mathbb{C}P^{N+M-1|N}$, it follows that the Ricci tensors vanishes when $M = 0$, and so does the one-loop beta function according to Friedan. The beta function vanishes to all orders in the loop expansion. The special case of $\mathbb{C}P^{1|2}$ \((M, N) = (0, 2)) has also been discussed in Refs. [8 and 9].

2. High-gradient operators for the $\mathbb{C}P^{N+M-1|N}$ NL$\sigma$Ms

From the property (A10), i.e., that the curvature tensor field of the supermanifold $\mathbb{C}P^{N+M-1|N}$ depends solely on its metric, follows that the RG equations among the infinite set of operators made of local polynomials in
\[ G_{\mu \nu} := \partial_{\mu} \varphi^a g^a g^b g_{\nu}, \]  
(A12)
are closed.

Near two dimensions $(d = 2 + \epsilon)$, it is convenient to use the conformal coordinates,
\[ \partial_{\pm} = \partial_x \pm i \partial_y, \quad \mu = \pm, \nu = \pm, \]  
(A13)
i.e., we use $G_{++}$, $G_{+-}$, $G_{-+}$, and $G_{--}$ as the building blocks for the high-gradient operators. It can be shown that the one-loop RG equations are closed within the family
\[ \left\{ G^p_{++} + G^q_{--} \right\}_{p=q+2r=s} \]  
(A14)
of high-gradient operators for any given number of gradients $2s$, where $p$, $q$, and $r$ are any non-negative integer satisfying $p+q+2r=s$. Furthermore, for any given $s$, $r$, and $r'$ the family (A14) obeys one-loop RG equations with an upper triangular structure in the sense that all high-gradient operators of the form (A14) with $r' > r$ do not enter the one-loop RG equations for those high-gradient operators with $r$ fixed. The task of diagonalizing the closed one-loop RG equations obeyed by the family (A14) thus simplifies greatly. It is indeed sufficient to fix $s$ and $r$ and to diagonalize the one-loop RG equations obeyed by the family (A14) labeled by the non-negative...
integers $p$ and $q$. For any finite order $s$, diagonalization of the one-loop RG flows obeyed by the family \((A14)\) of high-gradient operators yields the one-loop RG eigenvalues

$$\alpha^{(s)}_{p,q,r} = -2Mr + 2\left(-pq + p(p-1) + q(q-1)\right), \quad (A15a)$$

here labeled by the non-negative integers $q$, $p$, and $r$ that satisfy

$$p + q + 2r = s. \quad (A15b)$$

Combining Eq. \((A15)\) with the engineering scaling dimension $2s$ yields the one-loop scaling dimensions

$$x^{(s)}_{p,q,r} = 2s - t\alpha^{(s)}_{p,q,r}$$

$$= 2s - 2t\left(-Mr - pq + p(p-1) + q(q-1)\right)$$

\((A16)\)

for the family \((A14)\) of high-gradient operators. Equations \((A15)\) and \((A16)\) are the main result of this Appendix. Observe that this result is independent of the integer $N$ in $\mathbb{C}P^{N+M-1}$. Hence, it applies to the case $N = 0$, both in its compact and non-compact incarnations \((A2)\) and \((A3)\), respectively.

We now take a closer look at the spectrum when $M = 0$. In this case, the projective supermanifold is Ricci flat, i.e., $R_{\sigma \tau} = 0$ according to Eq. \((A11)\), and hence the one-loop beta function of the NL$\sigma$M coupling constant $t$ vanishes. (These are in fact lines of critical points labeled by the coupling constant $t$ of the $\mathbb{C}P^{N-1}$ NL$\sigma$Ms.) We also distinguish the compact case from the non-compact case by demanding that $t > 0$ in the former case and that $t < 0$ in the latter case.

The compact case corresponds to $t > 0$. For any given order $s$, we seek the largest and smallest one-loop RG eigenvalues that govern the RG flow of the high-gradient operators \((A14)\) in the NL$\sigma$Ms $\mathbb{C}P^{N-1}$. Needed are the extremal values of $\alpha^{(s)}_{p,q,r}$ while holding $p + q + 2r = s$ fixed. We find that the most and least dominant one-loop scaling dimensions in two dimensions and for a fixed $s$ are

$$x^{(s)}_{\min} = 2s - 2ts(s-1),$$

$$x^{(s)}_{\max} = 2s - 2t \times 0. \quad (A17)$$

We conclude that the spectrum of one-loop anomalous scaling dimensions \((A13)\) for any “compact” $\mathbb{C}P^{N-1}$ NL$\sigma$M is one-sided in the sense that it is not symmetrically distributed about zero: While $x^{(s)}_{\min}$ is not bounded as a function of $s$, $x^{(s)}_{\max} = 2s$ irrespective of $s$. The result for the most dominant scaling dimension $x^{(s)}_{\min}$ is the same as that for the NL$\sigma$Ms with $P, Q > 1$ is two-sided: The one-loop anomalous scaling dimensions are symmetrically distributed about zero.

The non-compact case corresponds to $t < 0$. For any given order $s$, we seek the largest and smallest one-loop RG eigenvalues that govern the RG flow of the high-gradient operators \((A14)\) when $M = 0$. These follow from Eq. \((A17)\) with the substitution $t \rightarrow -t$,

$$x^{(s)}_{\max} = 2s + 2t|s(s-1)|,$$

$$x^{(s)}_{\min} = 2s + 2t|0| \times 0. \quad (A19)$$

So, there is no relevant high-gradient operator in this non-compact case. This is the consequence of the one-sided property of the spectrum \((A13)\) when $M = 0$. On the other hand, in the case of the non-compact

$$U(P, Q)/U(P) \times U(Q) \quad (A20)$$

NL$\sigma$Ms with $P, Q > 1$, there are always relevant high-gradient operators. We note that the one-loop scaling dimensions \((A17)\) and \((A19)\) turn into the corresponding scaling dimensions \((3.24)\) and \((3.26)\) for the $gl(2N\mid 2N)_{k=1}$ WZW model, if we identify $t$ with $-9\eta_{M}/\pi$.

We now relax the condition for criticality $M = 0$ of the $\mathbb{C}P^{N-1}$ NL$\sigma$M target manifold. It can then also be shown that the spectra \((A13)\) labeled by $s$ and $M$ are one-sided. Since this result is, as required, independent of $N$, it applies to the $\mathbb{C}P^{M-1}$ NL$\sigma$Ms as well. In turn, $\mathbb{C}P^{M-1} \sim U(M)/U(1) \times U(M - 1)$ is obtained from $U(P + Q)/U(P) \times U(Q)$ by specializing to $(P, Q) = (M - 1, 1)$ or $(1, M - 1)$. The reason why the spectrum of one-loop anomalous scaling dimensions in $U(P + Q)/U(P) \times U(Q)$ with $P, Q > 1$ looks so different from the cases with either $P$ or $Q$ being unity is the following. The $U(P + Q)/U(P) \times U(Q)$ NL$\sigma$Ms with $P, Q > 1$ have a larger set of high-gradient operators than in the projective (super) spaces. This can be seen by comparing the set of high-gradient operators \((A14)\) against their counterparts when the target manifold is $U(P + Q)/U(P) \times U(Q)$ with $P, Q > 1$, which can be found in Eqs. \((2.12)\) and \((2.16b)\) from Ref. \[10].

High-gradient operators for $U(P + Q)/U(P) \times U(Q)$ NL$\sigma$Ms with $P, Q > 1$ can be expressed as a product of traces of matrix fields, while in the complex projective space, there is no such trace. Here, note that $Z$ defined in Eq. \((A6a)\) is a scalar while the corresponding object in $U(P + Q)/U(P) \times U(Q)$ with $P, Q > 1$ is a matrix, Eq. \((2.9b)\) from Ref. \[10\]. Similarly, the set of high-gradient operators in the $gl_{k=1}(M|M)$ WZW theory \((3.12)\) is larger than the set of high-gradient operators in the $gl_{k=1}(M|M)$ WZW theory.

3. Sketch of the one-loop RG computation

We now outline the calculations leading to the main results \((A13)\) and \((A16)\).
We choose the covariant background field method to renormalize the NLσMs. The merit of the background field method is that there is no need to worry about the appearance of redundant operators. This is very convenient when considering the mixing of a large set of operators under the RG, that cannot be distinguished by the symmetries of the NLσM. The background field method consists in resolving the coordinates \( \varphi^a = \varphi^a_{(1)} + \zeta^a \) of a NLσM into slow (mean-field) modes \( \varphi^a_{(1)} \) that satisfy the classical equations of motion and fast (fluctuating) modes \( \zeta^a \) in terms of which the Taylor expansion of the action transforms covariantly under reparametrization of the target manifold, i.e., in terms of which only the metric, the Riemann tensor, the covariant derivative, etc, of the target manifold appear in the action. For Kähler manifolds, this is achieved by choosing \( \zeta^a \) to be (either Riemannian or Kählerian) normal coordinates. The very same expansion of the action is also applied to the building blocks (A12) to the high-gradient operators, i.e.,

\[
G_{\mu \nu} = [G_{\mu \nu}]_{\zeta^0} + [G_{\mu \nu}]_{\zeta^1} + \cdots, \quad (A21)
\]

where \([G_{\mu \nu}]_{\zeta^p}\) represents a \(p\)-th term in this expansion.

To compute the anomalous scaling dimensions of high-gradient operators, they are first expanded in terms of the fast mode \( \zeta^a \), and are then pairwise Wick contracted. For example, the relevant formula for calculating \( \langle [G_{\mu \nu}]^2 \rangle \) and \( \langle [G_{\mu \nu}]^2, [G_{\rho \sigma}]^2 \rangle \), whereby the angular bracket \( \langle \cdots \rangle \) denotes pairwise Wick contraction of the fast modes \( \zeta^a \), can be found in Ref. 10, e.g., Eq. (C.40). When applied to the \( \mathbb{C}P^{N+M-1} \) NLσM, we obtain

\[
\langle [G_{\mu \nu}]^2 \rangle = -IM\delta_{\mu,\nu}G_{\mu \nu},
\]

\[
\langle [G_{\mu \nu}]^2, [G_{\rho \sigma}]^2 \rangle = Iq\left( \delta_{\rho,\nu} - \delta_{\rho,\mu} - \delta_{\sigma,\nu} + \delta_{\sigma,\mu} \right) \times \left( G_{\mu \nu}G_{\rho \sigma} + G_{\mu \rho}G_{\nu \sigma} \right), \quad (A22)
\]

where \( I = \int d^d k/(2\pi)^d(1/k^2) \). After substituting \( \mu = \pm, \nu = \pm \), this gives

\[
\langle [G_{++}]^2 \rangle = -IMG_{++},
\]

\[
\langle [G_{--}]^2 \rangle = -IMG_{--},
\]

\[
\langle [G_{++}]^2, [G_{++}]^2 \rangle = \langle [G_{--}]^2, [G_{--}]^2 \rangle = 0,
\]

\[
\langle [G_{++}]^2, [G_{--}]^2 \rangle = +4IG_{++}G_{--},
\]

\[
\langle [G_{++}]^2[G_{++}]^2 \rangle = -2I(G_{++}G_{--} + G_{++}G_{--}),
\]

\[
\langle [G_{--}]^2[G_{--}]^2 \rangle = -2I(G_{--}G_{++} + G_{--}G_{--}),
\]

\[
\langle [G_{--}]^2[G_{++}]^2 \rangle = +4IG_{--}G_{--}. \quad (A23)
\]

Furthermore, if \( p, q, \) and \( r \) are non-negative integers, we find

\[
\langle [G_{++}]^p[G_{--}]^q \rangle = +p\langle [G_{++}]^p \rangle [G_{++}]^{q-1}G_{--}^2
\]

\[
+ q\langle [G_{++}]^p \rangle [G_{--}]^{q-1}G_{++}^2
\]

\[
+ pq\langle [G_{++}]^p \rangle [G_{--}]^{q-1} \langle [G_{--}]^q \rangle [G_{++}]^r \]

\[
+ \frac{p(p-1)}{2}G_{++}^{p-2}G_{--}^q \langle [G_{--}]^q \rangle [G_{++}]^r \]

\[
+ \frac{q(q-1)}{2}G_{--}^{q-2}G_{++}^p \langle [G_{++}]^p \rangle [G_{--}]^r \]

\[
= +2I(pq + p(p-1) + q(q-1))G_{++}^pG_{--}^q - 2IpqG_{++}^{p-1}G_{--}^{q-1}G_{++}G_{--} \quad (A24)
\]

and

\[
\langle [G_{++}]^p[G_{--}]^q(G_{++}G_{--})^r \rangle = +r\langle [G_{++}]^p \rangle [G_{--}]^{q+1}G_{++}^{r-1}[G_{++}]_{\zeta^2}G_{--}^r
\]

\[
+ r\langle [G_{++}]^p \rangle [G_{--}]^{q+1}G_{++}^{r-1}[G_{--}]_{\zeta^2}G_{++}^r
\]

\[
+ \langle [G_{++}]^p \rangle [G_{--}]^{q+1}G_{++}^{r-1}(G_{++}G_{--})^r \quad (A25)
\]

\[
= -2IMrG_{++}^{p-1}G_{--}^{q+1}(G_{++}G_{--})^r
\]

\[
- 2I[pq - p(p-1) - q(q-1)]G_{++}^{p-1}G_{--}^q(G_{++}G_{--})^r
\]

\[
- 2IpqG_{++}^{p-1}G_{--}^{q-1}(G_{++}G_{--})^{r+1}. \quad (A25)
\]

Equation (A25) justifies the claim that the family (A14) of high-gradient operators is closed under one-loop RG
and yields Eqs. (A15) and (A16).

Appendix B: Relationship between the \( \hat{g}(2N|2N)_{k=1} \) Thirring model and Anderson localization with “sublattice” symmetry — a review

The \( \hat{g}(2N|2N)_{k=1} \) Thirring model represents the physics of observables in a class of problems of Anderson localization in symmetry classes BDI (see Ref. 33) and CII (see Ref. 11) within the classification scheme of Refs. 35–38. The fundamental physical observables are disorder averages of (products) of Green’s functions. Here, we review some basic steps of this connection for the example of symmetry class BDI, whose simplest representative is a two-dimensional random tight-binding model on the two-dimensional honeycomb lattice. Only sites on the different sublattices are connected, the model inherits a special symmetry called sublattice (or chiral) symmetry. It turns out to imply the presence of an operator that anti-commutes with the Hamiltonian, which thus relates the spectrum at positive and negative energies, and makes the zero of energy, \( E = 0 \) (often called the “band center”), special. Taking the low-energy limit near zero energy one obtains a random Dirac equation. (See, e.g., Ref. 54 for details.) We now start from the continuum limit of the so-obtained Hamiltonian, which reads

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{V}(r),
\]

where the kinetic energy is (we set \( \hbar \) and the Fermi velocity \( v_F \) to be one)

\[
\mathcal{H}_0 = -\sum_{\mu=1}^{2} i(\sigma_\mu \otimes \tau_1) \partial_\mu,
\]

and the static disorder is

\[
\mathcal{V}(r) = \sum_{\mu=1}^{2} (\sigma_\mu \otimes \tau_2) A_\mu(r) - (\sigma_0 \otimes \tau_2) V(r) + (\sigma_3 \otimes \tau_1) M(r).
\]

Here, \( \sigma_{1,2,3} \) and \( \tau_{1,2,3} \) are two independent sets of Pauli matrices together with another two independent \( 2 \times 2 \) identity matrix \( \sigma_0 \equiv I_2 \) and \( \tau_0 \equiv I_2 \). The \( 2 \times 2 \) matrix space associated with the \( \tau \) Pauli matrices originates from the bipartite symmetry of the underlying lattice model. The real-valued functions (potentials) \( A_\mu(r) \), \( V(r) \), and \( M(r) \), which do not vary appreciably on the scale of the lattice spacing, represent four independent sources of (static) randomness.

The above potentials are random variables. We will assume first that they are white-noise distributed according to a Gaussian probability distribution with vanishing mean,

\[
\overline{A_\mu(r)A_\nu(r')} = g_A \delta_{\mu\nu}\delta^{(2)}(r - r'), \quad \mu, \nu = 1, 2;
\]

\[
\overline{V(r)V(r')} = \overline{M(r)M(r')} = g_M \delta^{(2)}(r - r'),
\]

where \( \delta^{(2)}(r - r') \) is the two-dimensional delta function, \( -\delta \) represents disorder averaging, and we assume that the variances of \( V(r) \) and \( M(r) \) are identical. Of course, the disorder strengths \( g_A \) and \( g_M \) are positive.

The tight-binding Hamiltonian is invariant under time-reversal and so is its continuum limit

\[
T(\mathcal{H})^* T = \mathcal{H}, \quad T := \sigma_1 \otimes \tau_3.
\]

Since the tight-binding Hamiltonian preserves the bipartite nature of the underlying lattice for any realization of the disorder, so does its continuum limit through the chiral symmetry

\[
C \mathcal{H} C = -\mathcal{H}, \quad C := \sigma_0 \otimes \tau_3.
\]

As already mentioned, because of its chiral and time reversal symmetries the Hamiltonian belongs to the so-called BDI symmetry class within the classification scheme of Refs. 33–38.

2. Path integral representation of the single-particle Green’s function

In problems of Anderson localization, physical quantities are expressed by disorder averages of (products of) the retarded and advanced Green’s functions

\[
\hat{G}^{R/A}(E) := (E \pm i\eta - \mathcal{H})^{-1}.
\]

In the present model, the retarded and advanced Green’s functions are related, at the band center \( E = 0 \), by the chiral symmetry through

\[
C \hat{G}^R(E = 0) C = -\hat{G}^A(E = 0).
\]

Hence, any arbitrary product of retarded or advanced Green’s function at the band center equates, up to a sign, a product of retarded Green’s functions at the band center. From now on we will omit the energy argument of the Green’s function having in mind that it is always fixed to the band center \( E = 0 \).
Since the two kinds of Green’s functions are related by the chiral symmetry, it suffices to introduce functional integrals for the retarded Green’s function defined by the supersymmetric partition function $Z \equiv Z_F \times Z_B$ with

$$Z_F := \int \mathcal{D}[\bar{\chi}, \chi] \exp \left( i \int_r \bar{\chi} (i \eta - \mathcal{H}) \chi \right),$$

$$Z_B := \int \mathcal{D}[\bar{\xi}, \xi] \exp \left( i \int_r \bar{\xi} (i \eta - \mathcal{H}) \xi \right).$$  \hspace{1cm} (B7)

Here, $\int_r = \int d^2 r = \int d \tilde{z} d \bar{z} / (2i)$, $(\bar{\chi}, \chi)$ is a pair of two independent four-component fermionic fields, and $(\bar{\xi}, \xi)$ is a pair of four-component bosonic fields related by complex conjugation.

The matrix elements of the retarded Green’s function can be represented as

$$i \hat{G}^R (r, r') = \langle \chi (r) \bar{\chi} (r') \rangle = \langle \xi (r) \bar{\xi} (r') \rangle$$  \hspace{1cm} (B8)

with $\langle \cdots \rangle$ denoting the expectation value taken with the partition function $Z$. With the help of the property $T = T^\dagger$ of time-reversal and the property in Eq. (B3),

$$\int_r \bar{\chi} (i \eta - \mathcal{H}) \chi = - \int_r \chi^T T (i \eta - \mathcal{H}) T \chi^T,$$

$$\int_r \bar{\xi} (i \eta - \mathcal{H}) \xi = + \int_r \xi^T T (i \eta - \mathcal{H}) T \xi^T,$$  \hspace{1cm} (B9)

Eq. (B8) is also given by

$$i \hat{G}^R (r, r') = - \langle (T \bar{\chi}^T (r) (\chi^T T) (r')) \rangle$$

$$= + \langle (T \bar{\xi}^T (r) (\xi^T T) (r')) \rangle.$$  \hspace{1cm} (B10)

We now perform the change of integration variables $\bar{\chi}, \chi \rightarrow \psi^\dagger, \psi$ and $\bar{\xi}, \xi \rightarrow \beta^\dagger, \beta$ where, when the matrix space on which the $\tau$ Pauli matrices acts is made explicit,

$$\bar{\chi} \rightarrow \sqrt{\frac{1}{2 \pi}} \left( \psi^{\dagger} \sigma_x - i \psi_2 \sigma_x \right), \quad \chi \rightarrow \sqrt{\frac{1}{2 \pi}} \left( \psi \psi^{\dagger} \right),$$

$$\bar{\xi} \rightarrow \sqrt{\frac{1}{2 \pi}} \left( \beta^{\dagger} \sigma_x - i \beta_2 \sigma_x \right), \quad \xi \rightarrow \sqrt{\frac{1}{2 \pi}} \left( \beta \beta^{\dagger} \right).$$  \hspace{1cm} (B11)

We also define

$$\bar{A} := A_x + i A_y, \quad A := A_x - i A_y, \quad \bar{m} := V - i M, \quad m := V + i M,$$  \hspace{1cm} (B12)

With these changes of variables, the partition function at $E = 0$ can be written as

$$Z_F = \int \mathcal{D}[^\dagger, \psi] e^{-S^F - S^F_{\eta}}, \quad Z_B = \int \mathcal{D}[\beta^\dagger , \beta] e^{-S^B - S^B_{\eta}},$$  \hspace{1cm} (B13a)

with the effective action for the fermionic part given by

$$S^F = \int_r \frac{1}{2 \pi} \sum_{a=1}^2 \left[ \psi^{\dagger} (\partial \bar{\psi} + \bar{A}) \psi_a + \bar{\psi}^{\dagger} (\partial \psi + A) \psi_a \right.$$  \hspace{1cm} (B13b)

$$+ m \psi^{\dagger} \bar{\psi}_a + m \psi^{\dagger} \psi_a \right],$$

and the bosonic part of the effective action given by

$$S^B = \left( \psi, \bar{\psi} \rightarrow \beta, \bar{\beta}, \quad \psi^\dagger, \bar{\psi}^\dagger \rightarrow \beta^\dagger, \bar{\beta}^\dagger \text{ in } S^B \right),$$  \hspace{1cm} (B13c)

$$S^B_{\eta} = \int_r \frac{i \eta}{2 \pi} \left( \psi^{\dagger} \psi^{\dagger} + \psi^\dagger \psi^\dagger - \psi^\dagger \psi_1 - \psi_1 \psi^\dagger \right),$$

and

$$S^F_{\eta} = \int_r \frac{i \eta}{2 \pi} \left( \bar{\psi}^{\dagger} \bar{\psi}^{\dagger} + \bar{\psi}^\dagger \bar{\psi}^\dagger - \bar{\psi}^\dagger \bar{\psi}_1 - \bar{\psi}_1 \bar{\psi}^\dagger \right),$$

Observe the non-Hermitian appearance and asymmetry between fermions and bosons in $S^F_{\eta}$, which are necessary to maintain supersymmetry.

The time-reversal invariance (B5) and (B13) in terms of the new basis implies invariance under

$$\psi_2 \rightarrow \psi_1, \quad \psi_1 \rightarrow - \psi_2, \quad \beta_2 \rightarrow - i \beta_1, \quad \beta_1 \rightarrow + i \beta_2.$$  \hspace{1cm} (B14)

The finite level-broadening term $S^F_{\eta} / \eta$ is necessary for the computation of certain physical observables, including for example the Kubo conductivity, the Einstein conductivity, and the local density of states. However, when we compute the conductance from the Landauer formula by attaching ideal leads to the disordered region described by the Hamiltonian $H_{b}$ in Sec. III, we can set $\eta = 0$ in the disordered region (while still keeping $\eta \neq 0$ in the leads).

The last step consists in averaging the partition function $Z = Z_F \times Z_B$ over the probability distribution for the white-noise and Gaussian distributed random potentials. In this way, one finds a generating function for the averages of Green’s functions which is nothing but the $\mathfrak{gl}(2|2)_{k=1}$ Thirring model. Specially, integration over the vector potential yields the term proportional to

$$\psi^\dagger \psi_A \times \bar{\psi}^\dagger \bar{\psi}_B = (-)^A \psi_A \psi^\dagger \times (-)^B \bar{\psi}_B \bar{\psi}^\dagger,$$  \hspace{1cm} (B15a)

while integration over the complex-valued mass yields the term proportional to

$$\psi^{\dagger} \psi_B \times \bar{\psi}^\dagger \bar{\psi}_A = (-)^A \psi_A \psi^{\dagger} \times (-)^B \bar{\psi}_B \bar{\psi}^{\dagger},$$  \hspace{1cm} (B15b)

where we have combined bosonic $\beta_\eta, \bar{\beta}_\eta$ and fermionic $\psi_a, \bar{\psi}_a (a = 1, 2)$ spinors into the supersymmetric vector $\psi_A, \bar{\psi}_A (A = 1, \cdots, 4)$ as in Sec. II. The $N$-th moment of the retarded Green’s function evaluated at the band center is obtained by allowing the index $a$ to run from 1 to $2N$ in Eq. (B13) or, equivalently, by allowing the indices $A$ and $B$ to run from 1 to $2N + 2N$ in Eq. (B15), thereby obtaining the $\mathfrak{gl}(2N|2N)_{k=1}$ Thirring model.
We have assumed so far that the random imaginary vector potential $A, \bar{A}$ and the complex random mass $\tilde{m}, m$ possess a Gaussian probability distribution. If we assume instead that their distributions have non-vanishing higher cumulants (but still assuming that they have no spatial correlations), the quenched disorder averaging necessarily yields high-gradient operators of the form $\tilde{\mathcal{Q}}$. 

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$$T_{1,\ldots,p} = U_{1,\ldots,j_1} \cdots U_{p,\ldots,j_p} T_{j_1,\ldots,j_p}$$
for any $U$ in the adjoint representation, i.e., $U_{jk} = \exp (-t f_{ijk})$ ($t \in \mathbb{R}$), is called an invariant tensor for the adjoint representation. The scalar constructed from the invariant tensor $T_{1,\ldots,p}$ and the Lie algebra generators $\{X_i\}$ according to
$$C^{(p)} = T_{1,\ldots,p} X_{i_1} \cdots X_{i_p}$$

is a $p$-th order Casimir, i.e., $[C^{(p)}, X_i] = 0$ for all $X_i$. For example, if $V(X_i)$ is a representation of $X_i$, then
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