We formulate and explore the physical implications of a new translation gauge theory of gravity in flat space-time with a new Yang-Mills action, which involves quadratic gauge curvature and fermions. The theory shows that the presence of an "effective Riemann metric tensor" for the motions of classical particles and light rays is probably the manifestation of the translation gauge symmetry. In the post-Newtonian approximation of the tensor gauge field produced by the energy-momentum tensor, the results are shown to be consistent with classical tests of gravity and with the quadrupole radiations of binary pulsars.

1 Introduction

Since the creation of the Yang-Mills theory in 1954, quantum field theory with gauge symmetry based on a flat space-time have been successfully applied to all fundamental physical theories of fields, except for the gravitational field. Dyson stressed that the most glaring incompatibility of concepts in contemporary physics is that between the principle of general coordinate invariance and a quantum-mechanical description of all of nature. [1] Quantum gravity appears to be the last challenge to the powerful gauge symmetry of the Yang-Mills theory.

In contrast to the field-theoretic approach, Einstein’s approach involving the geometrization of physics appears to be successful only in the formulation of classical gravity. If the electromagnetic force, which is velocity-dependent, is treated in the same way, it is natural to employ the Finsler geometry rather than the Riemannian geometry. [2, 3, 4] The fundamental metric tensors of the Finsler geometry depend on both position and velocity (i.e., the differential of the coordinates). So far, however, all attempts to geometrize classical electrodynamics have been unsuccessful in the sense that they have not even begun to approach the usefulness of quantum electrodynamics. Thus, it is interesting to investigate the possibility of understanding gravity within
the framework of the usual physical field theory, especially a framework with gauge symmetry in a flat space-time. An obvious challenge is to construct a gauge-invariant action (which involves quadratic gauge-curvature) in flat space-time that can produce the same good agreement with experimental results as those in general relativity (whose action involves linear space-time curvature [5, 6, 7, 8, 9, 10, 11]).

In this paper, we do just that. We generalize the usual Yang-Mills theory with internal gauge groups to a gauge theory with the external space-time translation group in which the generators of the group do not have constant matrix representations. This generalization appears to be essential for tensor gauge fields to be generated by the well-defined and conserved energy-momentum tensor. We investigate a physical system involving fermion matter which generates a gravitational tensor field. We find an interesting action with a quadratic gauge-curvature in flat space-time and with space-time translation gauge symmetry. This action leads to a linearized gauge field equation that is exactly the same as the linearized Einstein equation, and is in agreement with experimental tests of general relativity such as the red shift. To at least the second order approximation, Yang-Mills gravity also predicts correctly the perihelion shift of Mercury and the gravitational quadrupole radiation of binary pulsars. This theory of Yang-Mills gravity with translation gauge symmetry is interesting because the gauge symmetry in flat space-time appears to be crucial for a quantum field theory to be unitary and renormalizable.

2 Translation Gauge Transformations and Accelerated Frames

The formulations of electromagnetic and Yang-Mills theories associated with internal gauge groups are all based on the replacement, \( \partial_\mu \rightarrow \partial_\mu + igB_\mu \). The field \( B_\mu = B_a^\mu t_a \) involves constant matrix representations of the generators \( t_a \) of the gauge groups. However, the generators of the external space-time translation group \( T(4) \) are the displacement operators, \( p_\mu = i\partial_\mu \) (c=\( \hbar = 1 \)); thus, the replacement takes a different form,

\[
\partial_\mu \rightarrow \partial_\mu - ig\phi^{\mu\nu}p_\nu \equiv J^{\mu\nu}\partial_\nu, \quad J^{\mu\nu} = \eta^{\mu\nu} + g\phi^{\mu\nu}, \tag{1}
\]

in such a gauge theory in inertial frames, where \( \eta^{\mu\nu} = (1, -1, -1, -1) \). Since the generators of this external translation group \( T(4) \) is \( p_\mu = i\partial_\mu \), we have a symmetric tensor gauge field \( \phi^{\mu\nu} = \phi^{\nu\mu} \) (i.e., a spin-2 field) rather than a 4-vector field (i.e., a spin-1 field) in the gauge covariant derivative, \( \Delta^\mu = J^{\mu\nu}\partial_\nu \) (in inertial frames). Nevertheless, we follow the Yang-Mills approach to formulate a theory of gravity based on \( T(4) \) in flat space-time. It is precisely this unique property (1) due to displacement operators of the \( T(4) \) group that naturally leads to an effective Riemannian metric tensor,\(^1\) and a universal repulsive force for all matter and anti-matter, where the force is characterized

\(^1\)The speculation that Einstein’s theory of gravity may be an effective field theory has been around for decades among theorists. The idea of an effective Riemannian space due to the presence of the gravitational field in Minkowski space-time was extensively discussed by Logunov, Mestvirishvili and others. Their theory is not based on the space-time translation gauge group. Moreover, they postulated a different gravitational action, which involves a linear ‘scalar curvature of effective Riemannian space.’ See, for example, Ref. 12.
by a coupling constant $g$ with the dimension of length in natural units. This is in sharp contrast with all coupling constants, which are dimensionless, in the usual Yang-Mills theories. For external gauge groups related to space-time, e.g., the de Sitter group or the Poincaré group, the gauge invariant Lagrangian involving fermions turns out to be richer in content.\footnote{For example, in order for the fermion Lagrangian to be de Sitter gauge invariant, apart from the presence of the usual Yang-Mills (‘phase’) fields, there must also be distinct “scale fields” to compensate the non-commutativity between Dirac’s matrices and the generators of the de Sitter groups. It suggests the existence of an additional gravitational spin force. See Ref. 13.} If the space-time translation group $T(4)$ is replaced by the Poincaré group, we expect that a new gravitational spin force generated by the fermion spin density will appear in the theory. \cite{13} However, in this paper, we concentrate on the external gauge group of space-time translations $T(4)$, which is the Abelian subgroup of the Poincaré group and is non-compact. This group $T(4)$ is particularly interesting because it is the minimal group related to the conserved energy-momentum tensor, which couples to a tensor (or spin-2) field $\phi_{\mu\nu}$ and is the source of gravity.

Translation gauge symmetry is based on the local space-time translation with an arbitrary infinitesimal vector gauge-function $\Lambda^\mu(x)$,

$$x^\mu \rightarrow x'^\mu = x^\mu + \Lambda^\mu(x), \quad x^\mu = (w, x, y, z).$$

(2)

It is interesting that this transformation has a dual interpretation: (i) a shift of the space-time coordinates by an infinitesimal vector gauge-function $\Lambda^\mu(x)$, and (ii) an arbitrary infinitesimal transformation. For the interpretation (ii) to be consistent in the theory, we must formulate Yang-Mills gravity for both inertial and non-inertial frames (i.e., general frames). Fortunately, we can accommodate these two mathematical implications of the transformation (2) by defining a gauge transformation of space-time translations for physical quantities $Q^{\mu_1 \cdots \mu_n}_\alpha(x)$ in the Lagrangian of fields:

$$Q^{\mu_1 \cdots \mu_n}_{\alpha_1 \cdots \alpha_n}(x) \rightarrow (Q^{\mu_1 \cdots \mu_n}_{\alpha_1 \cdots \alpha_n}(x))^{\mathcal{G}} = \left( Q^{\mu_1 \cdots \mu_n}_{\beta_1 \cdots \beta_n}(x) - \Lambda^\lambda(x) \partial_\lambda Q^{\mu_1 \cdots \mu_n}_{\beta_1 \cdots \beta_n}(x) \right) \frac{\partial x'^{\mu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\mu_n}} \frac{\partial x^{\beta_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x^{\alpha_n}},$$

(3)

where $\mu_1, \nu_1, \alpha_1, \beta_1$, etc. are space-time indices. As usual, both (Lorentz) spinor field $\psi$ and (Lorentz) scalar field $\Phi$ are treated as ‘coordinate scalars’ and have the same transformation:

$$\psi \rightarrow \psi^\mathcal{G} = \psi - \Lambda^\lambda \partial_\lambda \psi, \quad \Phi \rightarrow \Phi^\mathcal{G} = \Phi - \Lambda^\lambda \partial_\lambda \Phi.$$ 

In general, the gauge transformations for scalar, vector and tensor fields are given by

$$Q(x) \rightarrow (Q(x))^{\mathcal{G}} = Q(x) - \Lambda^\lambda \partial_\lambda Q(x), \quad \Gamma^\mu \rightarrow (\Gamma^\mu)^{\mathcal{G}} = \Gamma^\mu - \Lambda^\lambda \partial_\lambda \Gamma^\mu + \Gamma^\lambda \partial_\lambda \Lambda^\mu,$$

$$T^{\mu\nu} \rightarrow (T^{\mu\nu})^{\mathcal{G}} = T^{\mu\nu} - \Lambda^\lambda \partial_\lambda T^{\mu\nu} - T^{\mu\alpha} \partial_\alpha \Lambda^\nu + T^{\nu\alpha} \partial_\alpha \Lambda^\mu, \quad T^{\mu\nu} = J^{\mu\nu}, \quad P^{\mu\nu},$$

$$Q^{\mu\nu} \rightarrow (Q^{\mu\nu})^{\mathcal{G}} = Q^{\mu\nu} - \Lambda^\lambda \partial_\lambda Q^{\mu\nu} + \Lambda^\beta \partial_\beta \Lambda^\nu + Q^{\mu\lambda} \partial_\lambda \Lambda^\nu.$$ 

Here $D_\mu$ denotes the partial covariant derivative associated with a metric tensor $\gamma_{\mu\nu}(x)$ in a general reference frame (inertial or non-inertial). Note that the functions $D_\mu Q$
and $D_{\mu} D_{\nu} Q$ transform, by definition, as a covariant vector $Q_{\mu}(x)$ and a covariant tensor $Q_{\mu\nu}(x)$ respectively under the translational gauge transformation. The change of variables in the translation gauge transformation (4) in flat space-time is formally similar to the Lie variations in the coordinate transformations in Riemannian geometry.

Since the theory of Yang-Mills gravity should be formulated in a general frame of reference (inertial or non-inertial) characterized by a certain metric tensor $P_{\mu\nu}$, let us consider specific examples of $P_{\mu\nu}(x)$ for general frames. To substantiate the existence of such a metric tensor $P_{\mu\nu}$, let us consider a general-linear-acceleration transformation between an inertial frame $F_I(x, y, z)$ and a non-inertial frame $F(x, y, z)$.

Suppose $F_I$ is at rest and the frame $F$ moves in the x-direction with an initial velocity $\beta_0$ and an arbitrary linear acceleration $\alpha(w)$. The general-linear-acceleration transformations [14, 15] between an inertial frame $F_I$ and a general frame $F$ (which moves with an arbitrary velocity $\beta(w)$ in the +x-direction) are given by

$$w_I = \gamma(\beta(w)) U - \frac{\beta_0}{\alpha_0 \gamma_0}, \quad x_I = \gamma U - \frac{1}{\alpha_0 \gamma_0}, \quad y_I = y, \quad z_I = z; \quad (5)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2(w)}}, \quad \gamma_0 = \frac{1}{\sqrt{1 - \beta_0^2}}, \quad U = x + \frac{1}{\alpha(w) \gamma_0^2}, \quad \alpha(w) = \frac{d\beta(w)}{dw}. $$

They reduce to the Wu transformation in the limit of constant-linear-acceleration, i.e., $\alpha(w) \to \alpha_o$, [16, 17, 18]

$$w_I = \gamma(\beta_o + \alpha_o w)(x + \frac{1}{\alpha_o \gamma_o^2}) - \frac{\beta_0}{\alpha_o \gamma_o}, \quad x_I = \gamma(x + \frac{1}{\alpha_0 \gamma^2}) - \frac{1}{\alpha_0 \gamma_o}, \quad y_I = y, \quad z_I = z; \quad (6)$$

In the special case $\beta_o = 0$, the Wu transformation (6) becomes the Møller transformation, [19, 20, 21, 22]

$$w_I = (x + \frac{1}{\alpha_o}) sinh(\alpha_o w^*), \quad x_I = (x + \frac{1}{\alpha_o}) cosh(\alpha_o w^*) - \frac{1}{\alpha_o}, \quad y_I = y, \quad z_I = z,$$

provided one makes a change of time variable, $w = (1/\alpha_o) tanh(\alpha_o w^*)$. Furthermore, the Wu transformation (6) reduces to the Lorentz transformation in the limit of zero acceleration, $\alpha_o \to 0$,

$$w_I = \gamma_o(w + \beta_0 x), \quad x_I = \gamma_o(x + \beta_0 w), \quad y_I = y, \quad z_I = z; \quad (7)$$

One can verify that the general-linear-acceleration transformation (5) preserves the space-time interval $ds^2$:

$$ds^2 = dw_I^2 - dt_I^2 = W^2 dw^2 + 2U_I dw dx - dx^2 - dy^2 - dz^2 = P_{\mu\nu} dx^\mu dx^\nu, \quad (8)$$

$$P_{00} = W^2, \quad P_{01} = P_{10} = U_J, \quad P_{11} = P_{22} = P_{33} = -1, \quad W^2 = W_c^2 - U_J^2, \quad W_c = \gamma^2 \left( \frac{1}{\gamma_o^2} + \alpha(w)x \right), \quad U_J = \frac{d\alpha}{dw} \frac{1}{\alpha^2(w) \gamma_o^2},$$
where \( dx^\mu = (dw, dx, dy, dz) \) and the non-vanishing components of \( P_{\mu\nu} \) are \( P_{00} = W^2, P_{01} = P_{10} = U J \), etc. for general-linear-acceleration frames. It is interesting to see that the metric tensors \( P_{0\alpha} \) involves the ‘jerk’, i.e., the time derivative of the acceleration \( d\alpha(w)/dw \), which appears very rarely in physics. Moreover, equations (6) and (8) indicate that all constant-linear-acceleration frames have the metric tensor of the form \( P_{\mu\nu} = (W^2, -1, -1, -1) \) because \( d\alpha(w)/dw = 0 \) and \( \alpha(w) = \alpha_0 \). [18] The existence of the finite space-time transformations between inertial frames and general-linear-acceleration frames implies that the space-time associated with the general-linear-acceleration (and constant-linear-acceleration) frames is flat. [20] Thus, the physical space-time of all these general frames is characterized by zero Riemann-Christoffel curvature tensor, \( R^\lambda_{\mu\nu\alpha} = 0 \).

It turns out that the group properties of the space-time transformations for these accelerated frames differ drastically from those of the Lorentz group which is defined in the Minkowski space-time with \( \eta_{\mu\nu} = (1, -1, -1, -1) \). The reason is that there is simply no equivalence between inertial and non-inertial frames. Therefore, the physics of quantum fields in these non-inertial frames are much more involved [18] and does not have the elegant Lorentz and Poincaré invariance. In view of these differences between inertial and non-inertial frames, the metric tensor \( P_{\mu\nu} \) for such a class of general non-inertial frames may be called the Poincaré metric tensor. In the limit of zero acceleration, we have \( W \to +1 \) and \( U \to 0 \), so that the Poincaré metric tensor \( P_{\mu\nu} \) in (8) for non-inertial frames reduces to the Minkowski metric tensor \( \eta_{\mu\nu} \) for inertial frames.

3 Translation Gauge Symmetry and the Field-Theoretic Origin of Effective Metric Tensors

To see the field-theoretic origin of effective Riemannian metric tensors, let us consider a fermion field \( \psi \). The kinetic term in a fermion Lagrangian in a general frame is given by

\[
iv\bar{\psi}\Gamma_\alpha D^\alpha \psi - m \bar{\psi}\psi = iv\bar{\psi}\partial^\alpha \psi - m \bar{\psi}\psi, \]

\[
\{\Gamma_\mu, \Gamma_\nu\} = 2P_{\mu\nu}(x), \quad \Gamma_\mu = \gamma_\alpha e^\alpha_\mu, \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad \eta_{ab}e^a_\mu e^b_\nu = P_{\mu\nu}. \quad (9)
\]

Here we have the usual relation \( D_\mu \psi = \partial_\mu \psi \) because the ‘Lorentz spinor’ \( \psi \) transforms as a ‘coordinate scalar’ and \( D_\mu \) is the partial covariant derivative defined in terms of the Poincaré metric tensor \( P_{\mu\nu} \) in a general frame. In the presence of the gauge field \( \phi_{\mu\nu} \), the translation gauge symmetry dictates the replacement in a general frame,

\[
D^\alpha \to D^\alpha + g\phi^{\alpha\beta}D_\beta \equiv J^{\alpha\beta}D_\beta, \quad J^{\alpha\beta} = P^{\alpha\beta} + g\phi^{\alpha\beta},
\]

\[
iv\bar{\psi}\Gamma_\alpha D^\alpha \psi \to iv\bar{\psi}\Gamma_\alpha \Delta^\alpha \psi = iv\bar{\psi}\Gamma_\alpha J^{\alpha\beta}D_\beta \psi = iv\bar{\psi}\gamma_\alpha E^{\alpha\beta}D_\beta \psi \quad (10)
\]

\[
\Delta^\alpha = J^{\mu\alpha}D_\alpha, \quad E^{\alpha\beta} = e^\alpha_\mu J^{\mu\alpha}.
\]
If one considers $E^{\alpha\alpha}$ as an ‘effective tetrad’, one has the following relation for an ‘effective metric tensor’,

$$\eta_{ab}E^{\alpha\alpha}E^{\beta\beta} = \eta_{\alpha\beta}\epsilon_\mu^a J^{\mu\alpha} J^{\nu\beta} = P_{\mu\nu} J^{\mu\alpha} J^{\nu\beta} = G^{\alpha\beta}. \quad (11)$$

Such an ‘effective metric tensor’ $G^{\alpha\beta}$ also shows up if we consider the Lagrangian of a scalar field $\Phi$ with the same replacement as that in (10),

$$\frac{1}{2} [P_{\mu\nu}(D^\mu \Phi)(D^\nu \Phi) - m^2 \Phi^2] \rightarrow \frac{1}{2} [G^{\alpha\beta}(D_\alpha \Phi)(D_\beta \Phi) - m^2 \Phi^2].$$

Therefore, it may appear as if the geometry of the physical space-time is changed from pseudo-Euclidean space-time to non-Euclidean space-time due to the presence of the tensor gauge field (or spin-2 field) $\phi_{\mu\nu}$. However, based on the Yang-Mills approach, the presence of $E^{\alpha\alpha}$ in (10) and $G^{\alpha\beta}$ is simply the manifestation of the translation symmetry in flat physical space-time.

In the literature, when one arrives at this crucial step (10), [7, 8, 9, 10] one usually gives up the Yang-Mills approach for a truly gauge invariant theory with a quadratic gauge curvature in a flat space-time, and follows Einstein’s approach to discuss gravity by postulating Riemannian space-time due to the presence of $\phi_{\mu\nu}$ or $J_{\mu\nu}$ in (11). In other words, one postulates $E^{\alpha\alpha}$ and $G^{\alpha\beta}$ in (11) as a real tetrad and a real metric tensor of physical space-time. Such an approach leads to ‘the most glaring incompatibility of concepts in contemporary physics’ as observed by Dyson. [1]

We stress that, from the viewpoint of Yang-Mills theory, the real physical space-time in (10) is still flat and the fundamental metric tensor is still $P_{\mu\nu}$ in general frames of reference. We shall take this viewpoint throughout the discussion.

All the observable effects of gravity are directly related to the motion of classical objects and light rays. Thus, it is important to understand the relation between the wave equations of fields and the corresponding classical equation of a particle and light ray. In Yang-Mills gravity, we show that the equation of motion for a classical object is essentially the limit of geometrical optics of wave (or field) equations, as demonstrated in the Appendix. It suffices to say here that if we postulate the following action $S_p$ for classical particles,

$$S_p = - \int m ds_{ci}, \quad ds_{ci}^2 = I_{\mu\nu} dx^\mu dx^\nu, \quad I_{\mu\nu} G^{\alpha\alpha} = \delta^\alpha_{\mu}, \quad (12)$$

one can derive the classical equation of motion (i.e., the Hamilton-Jacobi equation) which is the same as that obtained from the classical limit (or the limit of geometric optics) of wave equations in the Appendix.

We consistently treat $G^{\mu\nu}$ in (12) as merely an ‘effective metric tensor’ for the motion of a classical object in flat space-time and in the presence of the tensor gauge field. One can verify that the action $S_p$ (12) is not invariant under the gauge transformation (3). This is not surprising because $S_p$ is only an effective action for classical objects rather than the action for basic tensor and fermion fields. However, one can show that
the effective interval $ds^2_{\xi}$ and, hence, the action $S_p$ are invariant under the following transformation

$$Q_{\alpha_1...\alpha_m}(x) \rightarrow Q^{*\mu_1...\mu_m}(x) = Q^{\nu_1...\nu_m}(x) \frac{\partial x^{\mu_1}}{\partial x^{\nu_1}}... \frac{\partial x^{\mu_m}}{\partial x^{\nu_m}} \frac{\partial x^{\beta_1}}{\partial x^{\alpha_1}}... \frac{\partial x^{\beta_n}}{\partial x^{\alpha_n}},$$

(13)

which corresponds to the transformation (2) with the interpretation (ii), namely, an arbitrary infinitesimal transformation. This invariant property of $S_p$ leads to invariant equation of motion, e.g., the Hamilton-Jacobi equation, for classical objects and light rays. Furthermore, we are able to show the agreement between experiments and the action $S_p$ in (12) when the tensor field $\phi_{\mu \nu}$ is solved from the gauge field equation. This agreement will be discussed in sections 6 and 7 below.

4 Gauge Invariant Action with Fermions, Tensor Fields and Quadratic Gauge-Curvature

Yang-Mills' theory with internal gauge group is generalized to a theory with the external gauge group of space-time translation. In the generalized Yang-Mills theory with the external translation gauge symmetry, we have the gauge curvature

$$C^{\mu \nu \alpha} = J^{\lambda \nu}(D_{\lambda}J^{\nu \alpha}) - J^{\nu \lambda}(D_{\lambda}J^{\mu \alpha}),$$

(14)

which is given by the commutation relation of the gauge covariant derivative $\Delta^\mu = J^{\mu \nu}D_\nu$, $[\Delta^\mu, \Delta^\nu] = C^{\mu \nu \alpha}D_\alpha$.

The translation gauge curvature $C_{\mu \alpha \beta}$ involves the symmetric tensor gauge field $\phi^{\mu \nu} = \phi^{\nu \mu}$. Thus, it differs from the usual Yang-Mills gauge curvature $f_{\mu \nu}^k = \partial_\nu b^k_\mu - \partial_\mu b^k_\nu - c_{ij}^k b^i_\mu b^j_\nu$, where $c_{ij}^k$ is the structure constant of the gauge group whose generators have constant matrix representations. In view of this difference, one cannot take any property of the usual Yang-Mills theory for granted in the present theory of gravity. For example, although the Yang-Mills theory with internal gauge groups has a corresponding fiber bundle, the present Yang-Mills gravity with an external space-time group does not. It can be directly verified that the gauge-curvature $C^{\mu \nu \alpha}$ given by (14) satisfies the following identities,

$$C^{\nu \mu \alpha} = -C^{\mu \nu \alpha}, \quad C^{\mu \nu \alpha} + C^{\nu \mu \alpha} + C^{\alpha \mu \nu} = 0,$$

(15)

because $J^{\mu \nu} = J^{\nu \mu}$. One can also obtain the Bianchi identity which is complicated for the translation gauge curvature $C_{\mu \alpha \beta}$. It turns out that there are only two independent quadratic gauge-curvature scalars: namely $C_{\mu \alpha \beta}C^{\alpha \beta \gamma}$ and $C_{\mu \alpha \beta}C^{\alpha \gamma \beta}$. Other quadratic gauge-curvature scalars can be expressed in terms of them because of the identities (15). We postulate that, in a general frame, the action $S_{\phi \psi}$ for fermion matter and spin-2 fields involves the linear combination of the two independent quadratic terms of the gauge-curvature and the symmetrized fermion Lagrangian:

$$S_{\phi \psi} = \int L_{\phi \psi} \sqrt{-P}d^4x, \quad P = \text{det } P_{\mu \nu},$$

(16)
$L_{\phi \psi} = \frac{1}{2g^2} \left( C_{\mu \alpha \beta} C^{\mu \beta \alpha} - C_{\mu \alpha} C^{\mu \beta} \right) + \frac{i}{2} \bar{\psi} \Gamma_{\mu} \Delta^\mu \psi - (\Delta^\mu \bar{\psi}) \Gamma_{\mu} \psi - m \bar{\psi} \psi, \quad (17)$

$\Delta^\mu \psi = J_{\mu \nu} D_{\nu} \psi, \quad J_{\mu \nu} = P_{\mu \nu} + g_{\phi \mu \nu} = J_{\nu \mu}, \quad D_{\lambda} P_{\mu \nu} = 0. \quad (18)$

Note that the quadratic gauge-curvature term in (17) can also be expressed as

$L_{\phi \psi} = \frac{1}{2g^2} \left( \frac{1}{2} C_{\mu \alpha \beta} C^{\mu \beta \alpha} - C_{\mu \alpha} C^{\mu \beta} \right) + \frac{i}{2} \bar{\psi} \Gamma_{\mu} \Delta^\mu \psi - (\Delta^\mu \bar{\psi}) \Gamma_{\mu} \psi - m \bar{\psi} \psi, \quad (19)$

because $C_{\mu \alpha \beta} C^{\mu \alpha \beta} = 2C_{\mu \alpha \beta} C^{\mu \beta \alpha}$. The different relative sign in the two quadratic gauge curvature in the Lagrangian (17) leads to a simple linearized equation which is formally the same as that in general relativity.

Based on the translation gauge transformation (4), we can show that

$L_{\phi \psi} \rightarrow (L_{\phi \psi})^g = L_{\phi \psi} - \Lambda^\lambda (\partial_\lambda L_{\phi \psi}). \quad (20)$

Since $\Lambda^\mu$ is an infinitesimal gauge vector function, the gauge transformation of $P_{\mu \nu}$ can be written in the form,

$$(P_{\mu \nu})^g = P_{\mu \nu} - \Lambda^\lambda \partial_\lambda P_{\mu \nu} - P_{\mu \beta} \partial_\nu \Lambda^\beta - P_{\alpha \nu} \partial_\mu \Lambda^\alpha$$

$$= [(1 - \Lambda^\sigma \partial_\sigma) P_{\alpha \beta}] (\delta^\alpha_{\mu} - \partial_\mu \Lambda^\alpha) (\delta^\beta_{\sigma} - \partial_\nu \Lambda^\beta). \quad (21)$$

It follows from (21) that

$\sqrt{-P} \rightarrow \sqrt{-P^g} = [(1 - \Lambda^\sigma \partial_\sigma) \sqrt{-P}] (1 - \partial_\lambda \Lambda^\lambda), \quad P = \text{det}P_{\mu \nu}. \quad (22)$

Thus, the Lagrangian $\sqrt{-P} L_{\phi \psi}$ changes only by a divergence under the gauge transformation,

$$\int \sqrt{-P} L_{\phi \psi} d^4x \rightarrow \int \left[ \sqrt{-P} L_{\phi \psi} - \partial_\lambda (\Lambda^\lambda L_{\phi \psi} \sqrt{-P}) \right] d^4x = \int \sqrt{-P} L_{\phi \psi} d^4x. \quad (23)$$

The divergence term in (23) does not contribute to field equations because one can transform an integral over a 4-dimensional volume into the integral of a vector over a hypersurface on the boundaries of the volume of integration where fields and their variations vanish. Thus, we have shown that the action $S_{\phi \psi}$ is invariant under the gauge transformations (4).

## 5 Tensor and Fermion Equations in General Frames

In general, field equations with gauge symmetry are not well defined. One usually includes a suitable gauge-fixing term in the Lagrangian to make the solutions of gauge field equation well-defined. Yang-Mills gravity in a general frame is based on the total Lagrangian $L_{\text{tot}} \sqrt{-P}$, which is the original Lagrangian with an additional gauge-fixing term $L_{\text{gf}} \sqrt{-P}$ involving the gauge parameter $\xi$:

$L_{\text{tot}} \sqrt{-P} = (L_{\phi \psi} + L_{\text{gf}}) \sqrt{-P}, \quad (24)$
Here, the gauge-fixing term corresponds to the usual gauge condition \( \partial_\mu \phi^{\mu\nu} - (1/2) \partial^\nu \phi_\lambda = 0 \) for tensor fields. The Lagrange equations for the gravitational tensor field \( \phi^{\mu\nu} \) in a general frame can be derived from the action \( \int L_{\text{tot}} \sqrt{-P} d^4x \), we obtain

\[
H^{\mu\nu} + \xi A^{\mu\nu} = g^2 T^{\mu\nu},
\]

where \( H^{\mu\nu} \equiv D_\lambda (J_\rho^{\cdots \nu} - J_\rho^{\cdots \nu} + C^{\cdots \nu} P^{\mu \rho} + C^{\cdots \rho} J^{\nu \lambda}) - C^{\cdots \beta} D^{\nu} J_{\alpha \beta} + C^{\cdots \beta} D^{\nu} J_{\alpha \beta} - C^{\cdots \beta} D^{\nu} J_{\lambda \beta}, \]

\[
A^{\mu\nu} = D^{\mu} \left( D^{\lambda} J^{\nu \lambda} - \frac{1}{2} D^{\nu} J^{\lambda} \right) - \frac{1}{2} P^{\mu\nu} \left( D^{\alpha} D^{\lambda} J_{\alpha \lambda} - \frac{1}{2} D^{\alpha} D_{\alpha} J^{\lambda} \right),
\]

where \( \mu \) and \( \nu \) should be made symmetric. We have used the identity (15) in the derivation of (26). The energy-momentum tensor \( T^{\mu\nu} \) of fermion matter and the partial covariant derivative \( D_\lambda \) associated with the Poincaré metric tensor \( P_{\mu\nu} \) are given by

\[
T^{\mu\nu} = \frac{1}{2} \left[ \bar{\psi} i \Gamma^{\mu} D^{\nu} \psi - i (D^{\nu} \bar{\psi}) \Gamma^{\nu} \psi \right], \quad D^{\nu} \psi = \partial^{\nu} \psi,
\]

\[
D_\lambda J^{\mu\nu} = \partial_\lambda J^{\mu\nu} + \Gamma^{\mu}_{\lambda \rho} J^{\rho \nu} + \Gamma^{\nu}_{\lambda \rho} J^{\mu \rho},
\]

etc.

where the Christoffel symbol is given by \( \Gamma^{\mu}_{\lambda \rho} = \frac{1}{2} P^{\mu\sigma} (\partial_\lambda P_{\sigma \rho} + \partial_\rho P_{\sigma \lambda} - \partial_\sigma P_{\lambda \rho}) \).

The Dirac equation for a fermion interacting with the tensor fields \( \phi^{\mu\nu} \) in a general frame can also be derived from (24):

\[
i \Gamma_\mu (P^{\mu\nu} + g \phi^{\mu\nu}) D_\nu \psi - m \psi + \frac{i}{2} [D_\nu (J^{\mu\nu} \Gamma_\mu)] \psi = 0,
\]

\[
i (P^{\mu\nu} + g \phi^{\mu\nu}) (D_\nu \bar{\psi}) \Gamma_\mu + m \bar{\psi} + \frac{i}{2} \bar{\psi} [D_\nu (J^{\mu\nu} \Gamma_\mu)] = 0,
\]

where we have used the relation \( (1/\sqrt{-p}) \partial_\mu [Q^\nu \sqrt{-P}] = D_\nu Q^\nu \). If one compares the fermion equation (31) with the Dirac equation in quantum electrodynamics [i.e., \( (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - m) \psi = 0 \)] in inertial frames, one can see a distinct difference: Namely, the kinematic term \( i \gamma^\mu \partial_\mu \) and the electromagnetic coupling term \( e \gamma^\mu A_\mu \) have a different relative sign, if one takes the complex conjugate of the Dirac equations. This implies the presence of both repulsive and attractive forces between two charges. However, if one takes the complex conjugate of the fermion equation for \( \psi \) in (31), there is no change in the relative sign of the kinematical term and the spin-2 coupling term. Thus, the translation gauge symmetry of gravity naturally explains the universal attractive force of gravity for fermion matter and anti-fermion matter.

In inertial frames with \( P_{\mu\nu} = \eta_{\mu\nu} \), the gauge-field equation (27) can be linearized as follows:

\[
\partial_\lambda \partial_\lambda \phi^{\mu\nu} - \partial^\mu \partial_\lambda \phi^{\lambda \nu} - \eta^{\mu\nu} \partial_\lambda \partial_\lambda \phi + \eta^{\mu\nu} \partial_\alpha \partial_\beta \phi^{\alpha \beta} + \partial^\mu \partial^\nu \phi - \partial^\nu \partial_\lambda \phi^{\lambda \mu} - g T^{\mu\nu} = 0,
\]
for weak fields. This equation can also be written in the form:

\[ \partial_\lambda \partial^\lambda \phi^{\mu\nu} - \partial^\mu \partial_\lambda \phi^{\lambda\nu} + \partial^\nu \partial_\lambda \phi^{\lambda\mu} - \partial^\nu \partial_\lambda \phi^{\lambda\mu} = g(T^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} T^\lambda_\lambda), \]  

(33)

where we have set \( \xi = 0 \) and used \( J^{\mu\nu} = \eta^{\mu\nu} + g \phi^{\mu\nu} \). It is interesting to see that the linearized gauge-field equation (32) is formally the same as the corresponding equation in general relativity. This property may be related to the fact that the transformation (2) is formally the same as that in general relativity.

### 6 The Perihelion Shift with a New Correction Term

In order to show possible differences between Yang-Mills gravity and general relativity, let us consider the perihelion shift to the second order approximation. The perihelion shift can be seen from the solution of the Hamilton-Jacobi equation for a classical particle. The momentum \( p_\mu \) of a classical particle can be derived from the action \( S_p \) (12), we have

\[ p_\nu = -\frac{\partial S_p}{\partial x^\nu} = -m \frac{dx^\mu}{ds_{ei}} I_\mu\nu, \]

(34)

Since \( ds_{ei}^2 = I_{\mu\nu} dx^\mu dx^\nu \), we have

\[ G^{\mu\nu} p_\mu p_\nu - m^2 = 0, \quad I^{\mu\nu} G_{\nu\lambda} = \delta_\lambda^\nu. \]

(35)

The result (34) is obtained on the basis of the variation of the particle action (12)

\[ \delta S_p = -m I_{\mu\nu} \left( \frac{dx^\mu}{ds_{ei}} \right) \delta x^\nu, \]

(36)

which can be derived if we consider only the actual path with one of its end point variables. From equations (34) and (35), we obtain the following Hamilton-Jacobi equation for a particle with mass \( m \),

\[ G^{\mu\nu} (\partial_\mu S)(\partial_\nu S) - m^2 = 0, \quad S \equiv S_p. \]

(37)

This equation can also be obtained from the Dirac equation in the presence of the gravitational tensor field \( \phi^{\mu\nu} \) by considering the classical limit, which resembles the limit of geometric optics. (See Appendix.) Equations (35) and (37) are formally the same as the corresponding equations in general relativity.

Equation (32) or (33) with the only non-vanishing component, \( T^{00} = m \delta^3(\mathbf{r}) \), leads to \( g \phi^{00} = g^2 m / (8 \pi r) \) in the Newtonian limit. Also, \( G^{00} \) in the Hamilton-Jacobi equation (37) should have the usual result \( G^{00} = 1 + 2GM/r \) in this limit, where \( G \) is the gravitational constant. Based on these results, together with \( G^{\mu\nu} \) in (37) and \( I_{\mu\nu} \) in (12), we obtain the first order approximation in an inertial frame,

\[ g = \sqrt{8 \pi G}, \quad \text{and} \quad g \phi^{00} = g \phi^{11} = \frac{Gm}{r}, \quad \text{etc.} \]

(38)

These results can be obtained by solving (33) with the spherical coordinate, \( x^\mu = (w, r, \theta, \phi) \).
Let us consider the perihelion shift of Mercury, which is sensitive to the coefficient appearing in the second-order term of $G^{00}$ or $I_{00}$ and in the first-order term of $G^{11}, G^{22}$ and $G^{33}$. However, we shall calculate the second-order terms of all components for the effective metric tensors $G^\mu_\nu$ to show that the observable result is gauge invariant, i.e., independent of the gauge parameter $\xi$. We solve the non-linear gauge field equations by the method of successive approximation and carry out the related post-Newtonian approximation to a second order. For gauge field equations to be well defined to the second order, it is convenient to use gauge-field equation (26) with the gauge parameter $\xi$.

For simplicity, we consider an inertial frame and a static and spherically symmetric system, in which tensor gauge fields are produced by a spherical object at rest with mass $m$. Based on symmetry considerations, [23] the non-vanishing components of the exterior solutions $\phi^{\mu\nu}(r)$ are $\phi^{00}(r), \phi^{11}(r), \phi^{22}(r)$ and $\phi^{33}(r) = \phi^{22}/\sin^2\theta$, where $x^\mu = (r, \theta, \phi)$. To solve the static gauge field, let us write $J^{00} = J_0^0 = S, -J^{11} = J_1^1 = R$, and $-r^2 J^{22} = J_2^2 = -r^2 \sin^2 \theta J_{33} = J_3^3 = T$. The metric tensor is given by $P_{\mu\nu} = (1, -1, -r^2, -r^2 \sin^2 \theta)$. In this coordinate system, the non-vanishing components of the Christoffel symbol $\Gamma^\alpha_{\mu\nu}$ are given by $\Gamma^1_{22} = -r, \Gamma^1_{33} = -r \sin^2 \theta, \Gamma^2_{12} = 1/r, \Gamma^3_{33} = -\sin \theta \cos \theta, \Gamma^3_{13} = 1/r, \Gamma^3_{23} = \cot \theta$.

After some tedious but straightforward calculations, the gauge field equation (26) with $(\mu, \nu) = (0, 0), (1, 1), (2, 2), (3, 3)$, can be written respectively as

$$\frac{d}{dr} \left( R^2 \frac{dS}{dr} \right) + \frac{2}{r} R^2 \frac{dS}{dr} + \left( R \frac{d}{dr} + \frac{dR}{dr} + \frac{2}{r} R \right) \left( -R \left( \frac{dS}{dr} + 2 \frac{dT}{dr} \right) - \frac{2}{r} T^2 + \frac{2}{r} TR \right)$$

$$+ \xi \left[ \frac{1}{4} \frac{d^2}{dr^2} (S - R + 2T) + \frac{1}{2r} \frac{d}{dr} (S - 3R + 4T) \right] = 0, \quad (39)$$

$$R \left( \frac{dS}{dr} + 2 \frac{dT}{dr} \right) - \frac{2}{r} T^2 + \frac{2}{r} TR$$

$$+ \xi \left[ \frac{1}{4} \frac{d^2}{dr^2} (S - R + 2T) - \frac{1}{2r} \frac{d}{dr} (S + R) + \frac{3}{r^2} (R - T) \right] = 0, \quad (40)$$

$$\left( R \frac{d}{dr} + \frac{dR}{dr} + \frac{5R}{r} - \frac{2T}{r} \right) \left[ R \frac{d(T/r)}{dr} + \frac{T^2}{r^3} \right]$$

$$+ \left[ \frac{1}{r^2} \left( R \frac{d}{dr} + \frac{dR}{dr} + \frac{3R}{r^3} - \frac{2T}{r^3} \right) \right] \left[ -R \left( \frac{dS}{dr} + 2 \frac{dT}{dr} \right) - \frac{2}{r} T^2 + \frac{2}{r} TR \right]$$

$$+ \xi \left[ \frac{1}{4r^2} \frac{d^2}{dr^2} (S - R + 2T) - \frac{1}{r^3} \frac{d}{dr} (R - T) + \frac{1}{r^4} (R - T) \right] = 0. \quad (41)$$

The equation for $(\mu, \nu) = (3, 3)$ is the same as that in (41).
We can solve the gauge field equations (39)-(41) to a second order approximation by setting
\[ S = 1 + a_0/r + a/r^2, \quad R = 1 + b_0/r + b/r^2, \] etc. We obtain the second-order approximation of the tensor field which satisfies the gauge field equation (26),
\[ g\phi^{00} = \frac{Gm}{r} + \frac{G^2m^2}{2r^2}, \quad g\phi^{11} = \frac{Gm}{r} + \frac{K_1}{r^2}, \]
\[ g\phi^{22} = -\frac{1}{r^2} \left[ -\frac{Gm}{r} + \frac{K_2}{r^2} \right], \quad g\phi^{33} = g\phi^{22}/\sin^2\theta; \quad (42) \]

\[ K_1 = \left( \frac{2}{\xi} + \frac{1}{2} \right) G^2m^2, \quad K_2 = 2G^2m^2 \left( \frac{1}{\xi} - 1 \right), \]

Note that the first order approximation is independent of the gauge parameter \( \xi \). We have seen that only the second order terms in \( \phi^{11}, \phi^{22} \) and \( \phi^{33} \) depend on the gauge parameter \( \xi \). However, all the first order terms and the second order term in \( \phi^{00} \) do not depend on the gauge parameter \( \xi \), and these are the only crucial terms for the observable results of the perihelion shift.

From the result (42) and \( G^{\mu\nu}(r) \) given in (12) with \( P_{\mu\nu} = (1, -1, -r^2, -r^2\sin^2\theta) \), we obtain the effective metric tensor,
\[ G^{00}(r) = 1 + 2\frac{Gm}{r} + 2\frac{G^2m^2}{r^2}, \quad G^{11}(r) = -\left[ 1 - \frac{2Gm}{r} + \frac{L_1}{r^2} \right], \]
\[ G^{22}(r) = -\frac{1}{r^2} \left( 1 - \frac{2Gm}{r} + \frac{L_2}{r^2} \right), \quad G^{33}(r) = G^{22}(r)/\sin^2\theta; \quad (43) \]
\[ L_1 = -\frac{4}{\xi}G^2m^2, \quad L_2 = \left( \frac{4}{\xi} - 3 \right) G^2m^2. \]

These results for effective metric tensors are well defined in the limit \( \xi \to \infty \). This particular gauge may be called `static gravity gauge.' If one chooses the static gravity gauge, the effective metric tensors are given by
\[ G^{00}(r) = 1 + 2\frac{Gm}{r} + 2\frac{G^2m^2}{r^2}, \quad G^{11}(r) = -\left[ 1 - \frac{2Gm}{r} \right], \quad (44) \]
\[ G^{22}(r) = -\frac{1}{r^2} \left( 1 - \frac{2Gm}{r} - 3\frac{G^2m^2}{r^2} \right), \quad G^{33}(r) = G^{22}(r)/\sin^2\theta. \]

Let us carry out the calculation of the perihelion shift to the second order for all components of \( G^{\mu\nu}(r) \) in (43) in terms of the spherical coordinates \( x^\mu = (w, \rho, \theta, \phi) \). This can be accomplished by a change of variable \( \rho^2 = r^2/(1 - 2Gm/r + L_2/r^2) = -G^{22}(r) \), where \( G^{22}(r) \) is given in (43). We obtain
\[ r = \rho B, \quad B \equiv \left[ 1 - \frac{Gm}{\rho} + \frac{G^2m^2}{2\rho^2} \left( \frac{4}{\xi} - 6 \right) \right], \quad (45) \]
\[ dr = d\rho \left[ 1 - \frac{G^2m^2}{2\rho^2} \left( \frac{4}{\xi} - 6 \right) \right]. \]
The effective metric tensor $G_{\mu\nu}(\rho)$ in the Hamilton-Jacobi equation (37) (with $r = \rho$) is obtained in the spherical coordinate $x^\mu = (w, \rho, \theta, \phi)$ as follows:

\[
G^{00}(\rho) = G^{00}(r)\big|_{r=\rho B} = 1 + \frac{2Gm}{\rho} + \frac{4G^2m^2}{\rho^2},
\]

\[
G^{11}(\rho) = G^{11}(r)(\frac{d\rho}{dr})^2\big|_{r=\rho B} = -\left[1 - \frac{2Gm}{\rho} - \frac{8G^2m^2}{\rho^2}\right], \quad (46)
\]

\[
G_{22}(\rho) = -\rho^2, \quad G_{33}(\rho) = -\rho^2\sin^2\theta.
\]

Note that the gauge parameter $\xi$ in $G^{11}(r)$ and $(d\rho/dr)^2$ cancel each other so that $G^{11}(\rho)$ is $\xi$-independent, in agreement with gauge invariance. Therefore, all components of $G_{\mu\nu}(\rho)$ for the spherical coordinates are independent of the gauge parameter to the second-order approximation. As far as experiment is concerned, the result (46) is effectively equivalent to that of general relativity. [23] The second-order term in $G^{11}(\rho)$ differs from that in general relativity and leads to a slightly different prediction for the perihelion shift, as we shall see below.

To see the physical implications of (46), we choose $\theta = \pi/2$ so that the Hamilton-Jacobi equation (37) for a planet with mass $m_p$ has the following form:

\[
G^{00}(\rho) \left(\frac{\partial S}{\partial w}\right)^2 + G^{11}(\rho) \left(\frac{\partial S}{\partial \rho}\right)^2 + G^{33}(\rho) \left(\frac{\partial S}{\partial \phi}\right)^2 - m_p^2 = 0.
\]

According to the general procedure for solving the Hamilton-Jacobi equation, we write the solution of $S$ in the form $S = -E_0 w + M \phi + f(\rho)$. [24] We solve for $f(\rho)$, and obtain

\[
S = -E_0 w + M \phi + \int \frac{1}{\sqrt{|G^{11}|(\rho)}} \sqrt{E_0^2 G^{00}(\rho) - m_p^2 - \frac{M^2}{\rho^2}} d\rho, \quad (48)
\]

where $E_0$ and $M$ are respectively constant energy and angular momentum of the planet. The trajectory is determined by $\partial S/\partial M = constant$, so that we have

\[
\phi = \int \frac{(M/\rho^2) d\rho}{\sqrt{E_0^2 G^{00}G^{11} - m_p^2 G^{11} - M^2 G^{11}}/\rho^2}. \quad (49)
\]

To find the trajectory, it is convenient to write (49) as a differential equation of $\sigma = 1/\rho$. We obtain the following equation

\[
\frac{d^2 \sigma}{d\phi^2} = \frac{1}{P} - \sigma(1 + Q) + 3Gm\sigma^2, \quad (50)
\]

by differentiating the equation with respect to $\phi$. Thus we see that the equation for the trajectory (50) in Yang-mills gravity differs slightly from the corresponding equation in general relativity by a new correction term $Q$. This correction term $Q$ is of the order of
which is undetectable because of the velocity $\beta$ of the planet is very small in comparison with the speed of light, $\beta \ll 1$.

By the usual successive approximation, [24] we obtain the solution

$$\sigma = \frac{1}{P(1 + Q)} \left[ 1 + e \cos \left( \phi \left( 1 - \frac{3Gm}{P} + \frac{Q}{2} \right) \right) \right]. \quad (51)$$

The advance of the perihelion for one revolution of the planet is given by

$$\delta \phi = \frac{6\pi Gm}{P} \left( 1 - \frac{3(E_o^2 - m_p^2)}{4m_p^2} \right). \quad (52)$$

We note that the second term in the bracket of (52) shows the difference between the present Yang-Mills gravity and Einstein’s theory. This result shows that the observable perihelion shift is independent of the gauge parameter $\xi$ which appears in the second order approximation of the solution of $g^{\mu\nu}$. Since the observational accuracy of the perihelion shift of the Mercury is about 1 %, the prediction (52) of Yang-Mills gravity can be tested only if the Mercury were to move with a tenth of the speed of light such that $(E_o^2 - m_p^2)/m_p^2 \approx \beta^2 \approx 0.01$. It is highly unlikely for a macroscopic planet to have such a speed. Thus, the result (52) of Yang-Mills gravity is consistent with existing data for the perihelion shift. [25]

### 7 Bending of Light and Other Experiments

The bending of light can be derived from the propagation of a light ray in geometrical optics in an inertial frame. Suppose the light ray propagates in the presence of the tensor gauge fields, its path is determined by the eikonal equation,

$$G^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi = 0. \quad (53)$$

This eikonal equation can be directly derived from the Maxwell’s equation in the presence of the gravitational tensor field $\phi^{\mu\nu}$ in the limit of geometrical optics. (See Appendix.) It can also be obtained from the Hamilton-Jacobi equation (37) with $m \rightarrow 0$ and $\partial_\mu S \rightarrow \partial_\mu \Psi$, where $\Psi$ is the eikonal. As usual, we assume that the motion of the light ray is in a plane passing through the origin and having the angle $\theta = \pi/2$. Using $G^{\mu\nu}(\rho)$ given in (46) and $x^\mu = (w, \rho, \theta, \phi)$, the eikonal equation (53) can be written as

$$I^{00} \left( \frac{\partial \Psi}{\partial w} \right)^2 + I^{11} \left( \frac{\partial \Psi}{\partial \rho} \right)^2 - \frac{1}{\rho^2} \left( \frac{\partial \Psi}{\partial \phi} \right)^2 = 0. \quad (54)$$

By the general procedure of solving (54) in a spherical symmetric tensor field, we look for the eikonal $\Psi$ in the form [24]

$$\Psi = -E_0 w + M\phi + f(\rho). \quad (55)$$

One can determine $f(\rho)$ and solve for the trajectory of the ray, which is the same as (50) with $m_p \rightarrow 0$ and $E_0$ replaced by $\omega_o = -\partial \Psi/\partial w$ (c=1). We have

$$\frac{d^2 \sigma}{d\phi^2} = -\sigma (1 + Q_o) + 3Gm\sigma^2, \quad \sigma = \frac{1}{\rho}. \quad (56)$$
\[ Q_o = \frac{8G^2m^2}{M^2}, \quad (57) \]

where the new correction term \( Q_o \) is extremely small. Following the usual procedure [24], we find the following result for the deflection of a light ray,

\[ \Delta \phi \approx \frac{4Gm\omega_o}{M} \left( 1 - \frac{18G^2m^2\omega_o^2}{M^2} \right), \quad (58) \]

We note that the additional correction term in the bracket differs from that in general relativity and is negligible for the bending of light by the Sun. A ray of light passing through a spherical symmetry tensor field at a distance \( R_o \) from the center of the sun will have a deflection \( \Delta \phi \approx 4Gm/R_o \approx 1.75'' \), to the first order approximation. This result is consistent with experiment and is also the same as that obtained in general relativity, as one would expect based on the results (46) for the effective metric tensor.

Historically, red shift and time dilatation due to the gravitational effect were originally derived by using the principles of equivalence. [26] Nevertheless, these experiments can also be discussed within the present framework of Yang-Mills gravity in flat space-time. Both the red shift and the time dilatation caused by gravity (or the tensor gauge fields) can be considered as physical results of the invariance of the effective action (12) or the ‘proper time’ \( \tau = \int ds_{ei} \) under the transformation (13), without assuming the usual principle of equivalence.

With the help of geometrical optics, [24] the red shift can also be derived from the eikonal equation (53). This eikonal equation is fundamental in geometrical optics and is invariant under the transformation (13). For static tensor field, \( G^{\mu\nu} \) does not contain time \( w = x^0 \), so that the frequency \( \kappa_0 = -\partial \Psi / \partial w \) is constant during the propagation of the light ray. [24] On the other hand, the frequency \( \kappa_0 = -\partial \Psi / \partial \tau \) measured in terms of the ‘proper time’ depends on positions in space. Thus, we have

\[ \kappa_0 = -\frac{\partial \Psi \partial w}{\partial w \partial \tau} = \frac{\kappa_0}{\sqrt{I_{00}}}, \quad I_{00} = \frac{1}{G_{00}} \approx 1 - 2g\phi_{00}. \quad (59) \]

In fact, this relation with \( G_{00} \) given in (46) for the spherical coordinate is the same as that in general relativity and is consistent with the experiment of red-shift. [23, 24]

The experiment of the time delay of radar echoes passing the sun can be explained by the gauge field equation (26) under the simplifying assumption of isotropy and time independence. Specifically, this experiment can be explained by the result \( G^{11}(\rho) \) in (46) to the first order in \( Gm/\rho \). [23] The effective metric tensor in (46) is the same as that obtained in general relativity to the first order approximation. Thus, if one follows the usual procedure of calculations, [23] one can verify that Yang-Mills gravity is also consistent with the experiment of radar echoes.

In Yang-Mills gravity, the gravitational quadrupole radiations of binary pulsars can be calculated to the second-order in \( g\phi^{\mu\nu} \) in inertial frames. The energy-momentum tensor of gravitation \( t_{\mu\nu} \) is defined by the field equation (26) (with \( \xi = 0 \)) written in the following form, \( D^\lambda D_\lambda \phi^{\mu\nu} = -g(T^{\mu\nu} + t^{\mu\nu}) \), in a general frame. Using the usual approximations and gauge condition \( \partial_\mu \phi^{\mu\nu} = \partial^\nu \phi^\lambda / 2 \), we can calculate the average energy-momentum of a gravitational plane wave and the power by the usual
method. [23] For example, the power \( P_o \) emitted per unit solid angle in the direction \( \mathbf{x}/|\mathbf{x}| \) can be written as

\[
\frac{dP_o}{d\Omega} = \frac{G\omega^2}{\pi} \left( T^{\lambda\rho}(\mathbf{k}, \omega)T^{\star}_{\lambda\rho}(\mathbf{k}, \omega) - \frac{1}{2} T(\mathbf{k}, \omega)T^{\star}(\mathbf{k}, \omega) \right),
\]

(60)

where \( T(\mathbf{k}, \omega) \) is defined as follows: [23] Suppose one observes this radiation in the wave zone, one can write the polarization tensor in terms of the Fourier transform of \( T^{\mu\nu}(\mathbf{k}, \omega) \):

\[
e^{\mu\nu}(\mathbf{x}, \omega) = \frac{-g}{4\pi r} [T^{\mu\nu}(\mathbf{k}, \omega) - \frac{1}{2} \eta^{\mu\nu}T(\mathbf{k}, \omega)], \quad T = T^{\lambda\lambda},
\]

(61)

\[
T^{\mu\nu}(\mathbf{k}, \omega) \equiv \int d^3x' T^{\mu\nu}(\mathbf{x}', \omega) \exp(-i \mathbf{k} \cdot \mathbf{x}'),
\]

(62)

where the polarization tensor \( e^{\mu\nu}(\mathbf{x}, \omega) \) is defined by the relation:

\[
\phi^{\mu\nu}(\mathbf{x}, \mathbf{t}) \approx \left[ e^{\mu\nu}(\mathbf{x}, \omega) \exp(-i \mathbf{k} \cdot \mathbf{x}^\lambda) + c.c. \right].
\]

(63)

To the second-order approximation, the result (60) for the power emitted per solid angle in Yang-Mills gravity turns out to be the same as that obtained in general relativity and consistent with the data of the binary pulsar PSR 1913+16. [23, 27, 28]

8 Remarks and Discussions

Although the invariant action of Yang-Mills gravity is dictated by the space-time translation gauge symmetry, the gauge-fixing term in the action is not. We observe that the relation between the effective metric tensor \( G^{\alpha\beta} \) in Hamilton-Jacobi equation and the tensor field \( \phi^{\mu\nu} \) appears to be dependent on the choice of the specific form of the gauge-fixing term. For example, suppose one chooses

\[
L_{gf} \sqrt{-P} = \left( \frac{\eta}{2g^2} (D_{\mu}J^{\mu\alpha})D^{\nu}J_{\nu\alpha} \right) \sqrt{-P},
\]

(64)

where \( \eta \) is a gauge parameter. The gauge-dependent terms in the static field equation (40) will be modified as follows:

\[
R\left( \frac{dS}{dr} \right)^2 + 2r^3 \frac{dT}{dr} \left[ \frac{R}{r} \frac{dt}{dr} + \frac{T^2}{r^3} \right] + \left( \frac{dS}{dr} + 2\frac{dT}{dr} - \frac{2T}{r} \right) \times \]

\[
\left( -R\left( \frac{dS}{dr} + 2\frac{dT}{dr} \right) - \frac{2T^2}{r} + \frac{2T}{r} \right) - \eta \frac{d}{dr} \left( \frac{dR}{dr} + \frac{2R}{r} - \frac{2T}{r} \right) = 0,
\]

(65)

Naturally, the solution will be different from those given in (42). Thus, if one wants to preserve the result (46), the relation between the effective metric tensor \( G^{\alpha\beta} \) and the tensor field \( \phi^{\mu\nu} \) has to be modified accordingly. This property suggests that the types of gauge-fixing terms that can be used for external space-time translation symmetry are more restricted than those in the Yang-Mills theory with internal gauge groups.
So far, there is no observable difference between Yang-Mills gravity and Einstein’s theory in known experiments within the solar system, it is possible that the difference between the two theories can be tested by observations of phenomena outside the solar system. For example, the binary pulsar PSR 1913+16 provides an interesting and unique test of gravitational theories. Both the pulsar and its silent companion are about 1.4 times the mass of the Sun. They travel with a speed that range up to $4 \times 10^5$ meters per second in a tight orbit with a minimum separation roughly equal to the radius of the Sun. As a result, the binary pulsar has a very large advancing of periastron, 4.2 degrees per year. We have examine this case and we find that the data is not accurate enough to test the difference between Yang-Mills gravity and general relativity. The quadrupole radiation and experiments related to the binary pulsar will be discussed in detail in a separate paper.

The theory of Yang-Mills gravity in flat space-time has a well-defined conservation law for the energy-momentum tensor. The space-time translation symmetry plays an essential role in connecting the tensor Yang-Mills field to its source, i.e., the conserved energy-momentum tensor (through the Noether theorem). Furthermore, it is gratifying that the Hamilton-Jacobi equation (37) for a classical particle can also be derived from the corresponding fermion wave equation, as shown in Appendix. Thus, Yang-Mills gravity in flat space-time reveals a more well-defined theory and a more coherent relation between its quantum and classical aspects than conventional formalisms in curved space-time.

In previous attempts to formulate a gauge theory of gravity, one usually followed Einstein’s approach based on Riemannian space-time and obtained Einstein’s equation or closely related field equations for the metric tensor. As a result, the gauge symmetry, however powerful it may be, was unable to simplify the complicated interaction terms and, hence, the resultant theory of gravity had serious ultraviolet divergences and was not renormalizable. In view of this difficulty, we follow closely the Yang-Mills approach with a quadratic gauge-curvature and formulate the theory of Yang-Mills gravity on the basis of the translation gauge symmetry and flat space-time.

9 Conclusions

The Yang-Mills approach to gravity reveals an interesting property. Namely, the action (12) for the motion of a classical particle with the effective metric $ds^2 = I_{\mu \nu} dx^\mu dx^\nu$ (or the Hamilton-Jacobi equation (A.7) with $G^{\mu \nu}$ in the Appendix) shows that the underlying basis for gravity is probably the translation gauge symmetry in a flat space-time rather than the general coordinate invariance in curved space-time.

In general, the basic Lagrangians for vector and tensor fields in Yang-Mills gravity do not explicitly and unambiguously involve the effective metric tensor $G^{\mu \nu}$. Yang-Mills gravity reveals the field-theoretic origin of an ‘effective Riemannian metric tensor’ only in the limit of geometrical optics or classical limit of the wave equations, as shown in (A.3) and (A.7) in the Appendix. Therefore, the effective metric tensor $G^{\mu \nu}$ does not play any basic role in the quantum aspect of Yang-Mills gravity.

Based on previous discussions, we conclude that Yang-Mills gravity is viable because the gravitational gauge equation (26) is consistent with all known experiments.
of gravity.

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Appendix. Derivations of Eikonal and Hamiltonian-Jacobi Equations

In Yang-Mills gravity, the fundamental equation (53) of geometrical optics can be derived as follows: We postulate the translation gauge invariant Lagrangian \( L_{\text{em}} \) for the electromagnetic potential \( A^\mu \),

\[
L_{\text{em}} = -\frac{1}{4} P^{\mu \alpha} P^{\nu \beta} F_{\mu \nu} F_{\alpha \beta}, \quad F_{\mu \nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu, \quad \Delta_\mu = J_{\mu \lambda} \partial^\lambda,
\]

where we have used the same replacement in (10). For simplicity, let us consider an inertial frame with \( P_{\mu \nu} = \eta_{\mu \nu} \) and \( \Delta_\mu = J_{\mu \lambda} \partial^\lambda \), one can obtain the wave equation,

\[
\Delta_\mu (\Delta_\mu A^\lambda - \Delta^\lambda A_\mu) + (\partial_\alpha J_\alpha^\mu)(\Delta_\mu A^\lambda - \Delta^\lambda A_\mu) = 0.
\]

Using the electromagnetic gauge condition, \( \partial_\mu A^\mu = 0 \), and the expression for the field \( A^\lambda = a^\lambda \exp(i\Psi) \), we can derive the eikonal equation (53),

\[
G^{\mu \nu} \partial_\mu \Psi \partial_\nu \Psi = 0, \quad G_{\mu \nu} = P_{\alpha \beta} J_{\alpha \mu} J_{\beta \nu},
\]

in the limit of geometrical optics. That is, both the eikonal \( \Psi \) and the wave 4-vector \( \partial_\mu \Psi \) are very large. [30] We stress that the Lagrangian (A.1) and the wave equation (A.2) do not imply an effective metric tensor \( G^{\mu \nu} \). Only in the limit of geometrical optics of the wave equation (A.2), an effective metric tensor \( G^{\mu \nu} \) emerges.

Next, let us consider the relation between the Hamilton-Jacobi equation (37) and the massive fermion wave equation. The fermion wave equation (31) can be derived from the Lagrangian (24), i.e.,

\[
i \Gamma_\mu \Delta^\mu \psi - m\psi + \frac{i}{2} \gamma_a [D_\nu (J^{\mu \nu} e^a_\mu)]\psi = 0.
\]

Using the expression for the field \( \psi = \psi_0 \exp(iS) \), we can derive the equation

\[
\gamma_a E^{a \mu} \partial_\mu S - m - \frac{i}{2} \gamma_a [D_\nu (J^{\mu \nu} e^a_\mu)] = 0.
\]

In the classical limit, the momentum \( \partial_\mu S \) and mass \( m \) are large quantities, and one can neglect the small gravitational interacting term involving \( e^a_\mu \). To eliminate the spin variables, we multiply a factor \( (\gamma_a E^{a \mu} \partial_\mu S - m) \) to the large terms in (A.5), the resultant equation can be written in the form

\[
\frac{1}{2} (\gamma_b \gamma_a + \gamma_a \gamma_b) E^{a \mu} E^{b \nu} (\partial_\mu S)(\partial_\nu S) - m^2 = 0.
\]
With the help of the anti-commutation relation for $\gamma_a$ in (9) and the effective metric tensor (11), (A.6) leads to the Hamilton-Jacobi equation,

$$G^{\mu\nu}(\partial_\mu S)(\partial_\nu S) - m^2 = 0,$$

for the motion of a classical particle in the presence of the gravitational tensor field $\phi^{\mu\nu}$. It is important that the result (A.7) is consistent with equation (37) obtained from the particle action $S_p$ in (12).$^3$

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