LOCAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL
DIRAC-KLEIN-GORDON EQUATIONS IN FOURIER-LEBESGUE
SPACES

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Abstract. The local well-posedness problem is considered for the Dirac-Klein-Gordon system in two space dimensions for data in Fourier-Lebesgue spaces \( \hat{H}^{s,r} \), where \( \| f \|_{\hat{H}^{s,r}} = \| \langle \xi \rangle^s \hat{f} \|_{L^{r'}} \) and \( r \) and \( r' \) denote dual exponents. We lower the regularity assumptions on the data with respect to scaling improving the results of d’Ancona, Foschi and Selberg in the classical case \( r = 2 \). Crucial is the fact that the nonlinearities fulfill a null condition as detected by these authors.

1. Introduction and main results

Consider the Cauchy problem for the Dirac-Klein-Gordon equations in two space dimensions
\[
\begin{align*}
i(\partial_t + \alpha \cdot \nabla)\psi + M\beta \psi &= -\phi \beta \psi \quad (1) \\
(-\partial_t^2 + \Delta)\phi + m\phi &= -\langle \beta \psi, \psi \rangle \quad (2)
\end{align*}
\]
with (large) initial data
\[
\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1. 
\]
Here \( \psi \) is a two-spinor field, i.e. \( \psi : \mathbb{R}^{1+2} \to \mathbb{C}^2 \), and \( \phi \) is a real-valued function, i.e. \( \phi : \mathbb{R}^{1+2} \to \mathbb{R}, \quad m,M \in \mathbb{R} \) and \( \nabla = (\partial_{x_1}, \partial_{x_2}), \quad \alpha \cdot \nabla = \alpha^1 \partial_{x_1} + \alpha^2 \partial_{x_2} \), \( \alpha \), \( \beta \) are hermitian \((2 \times 2)\)-matrices satisfying \( \beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I, \quad \alpha^3 \beta + \beta \alpha^3 = 0 \),
\( \langle \cdot, \cdot \rangle \) denotes the \( \mathbb{C}^2 \)-scalar product. A particular representation is given by \( \alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

The Cauchy data are assumed to belong to Fourier-Lebesgue spaces: \( \psi_0 \in \hat{H}^{s,r}, \quad \phi_0 \in \hat{H}^{l,r}, \quad \phi_1 \in \hat{H}^{l-1,r} \). Here \( \hat{H}^{s,r} \), \( 1 \leq r < \infty \), denotes the completion of \( S(\mathbb{R}^2) \) with respect to the norm \( \| f \|_{\hat{H}^{s,r}} = \| \langle \xi \rangle^s \hat{f} \|_{L^{r'}} \), where \( r \) and \( r' \) denote dual exponents and \( \hat{f} \) is the Fourier transform of \( f \).

Following [2] it is possible to simplify the system (1),(2),(3) by considering the projections onto the one-dimensional eigenspaces of the operator \( -i\alpha \cdot \nabla \) belonging to the eigenvalues \( \pm |\xi| \). These projections are given by \( \Pi_\pm(D) \), where \( D = \sum_{\ell} \gamma_{\ell}^2 \) and \( \Pi_\pm(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \alpha) \). Then \( -i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D) + \Pi_\pm(\xi) \beta = \beta \Pi_\pm(\xi) \). Defining \( \psi_\pm := \Pi_\pm(D)\psi \) and splitting the function \( \phi \) into the sum \( \phi = 

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\( \frac{1}{2}(\phi_+ + \phi_-) \), where \( \phi_{\pm} := \phi \pm iA^{-1/2}\partial_t \phi \), \( A := -\Delta + 1 \), the Dirac-Klein-Gordon system can be rewritten as

\[
(-i\partial_t + |D|)\psi_{\pm} = -M\beta\psi_{\pm} + \Pi_{\pm}(\phi\beta(\psi_+ + \psi_-)) \tag{4}
\]

\[
(i\partial_t + A^{1/2})\phi_{\pm} = \mp A^{-1/2}(\beta(\psi_+ + \psi_-), \psi_+ + \psi_-) \mp A^{-1/2}(m+1)(\phi_+ + \phi_-). \tag{5}
\]

The initial conditions are transformed into

\[
\psi_{\pm}(0) = \Pi_{\pm}(D)\psi_0, \quad \phi_{\pm}(0) = \phi_0 \pm iA^{-1/2}\phi_1 \tag{6}
\]

The aim is to minimize the regularity of the data so that local well-posedness holds. Persistence of higher regularity is then a consequence of the fact that the results are obtained by a Picard iteration.

The decisive detection by d’Ancona, Foschi and Selberg [2] was that both nonlinearities satisfy a null condition. This implies that the Cauchy problem in three space dimensions is locally well-posed in the classical case \( r = 2 \) for data \((\psi_0, \phi_0, \phi_1) \in H^{s} \times H^{l} \times H^{l-1} \), where \( s > 0 \), \( l = s + \frac{1}{2} \). This is almost optimal with respect to scaling.

In the case \( m = M = 0 \) the Dirac-Klein-Gordon system is invariant under the rescaling

\[
\psi_\lambda(t, x) = \lambda^2 \psi(\lambda t, \lambda x), \quad \phi_\lambda(t, x) = \lambda \psi(\lambda t, \lambda x)
\]

Because in \( N \) space dimensions

\[
\|\psi_\lambda(0, \cdot)\|_{H_{\lambda}^{s, r}} = \lambda^2 \|\psi(0, \lambda x)\|_{H_{\lambda}^{s, r}} \sim \lambda^{2s-2} \|\psi(0, \cdot)\|_{H_{\lambda}^{s, r}},
\]

\[
\|\psi_\lambda(0, \cdot)\|_{L^{2, r}_{\lambda}} = \lambda^2 \|\phi(0, \lambda x)\|_{L^{2, r}_{\lambda}} \sim \lambda^{2s-2} \|\psi(0, \cdot)\|_{L^{2, r}_{\lambda}}
\]

the scale-invariant space is

\[
(\psi_0, \phi_0, \phi_1) \in \tilde{H}^{\frac{s}{2} - \frac{1}{2}, r} \times \tilde{H}^{\frac{s}{2} - 1, r} \times \tilde{H}^{\frac{s}{2} - 2, r},
\]

where \( \|f\|_{\tilde{H}^{s, r}} = \|\xi^{s} f\|_{L^{r}_{\lambda}} \). Thus in the two-dimensional case the critical spaces are

\[
(\psi_0, \phi_0, \phi_1) \in H^{-\frac{1}{2}} \times L^{2} \times H^{-1} \quad \text{for} \ r = 2
\]

and

\[
(\psi_0, \phi_0, \phi_1) \in \tilde{H}^{\frac{s}{2} - 1 +} \times \tilde{H}^{1, +1} \times \tilde{H}^{0, +1} \quad \text{for} \ r = 1 + .
\]

We remark that \( \tilde{H}^{s, r} \sim H^{s, 2} \), where \( s = n + N(\frac{1}{2} - \frac{1}{r}) \) in terms of scaling, because \( \|\psi_\lambda(\lambda x)\|_{\tilde{H}^{s, r}} \sim \lambda^{\frac{n}{2} - \frac{2}{r}} \|\psi_0\|_{\tilde{H}^{s, r}} \).

In two space dimensions local well-posedness in the classical case \( r = 2 \) was proven by d’Ancona, Foschi and Selberg [1] for \( s > -\frac{1}{4} \) and \( \max(\frac{1}{4} - \frac{s}{2}, \frac{1}{2} + \frac{s}{2}, 1 + s) < l < \min(\frac{1}{2} + 2s, \frac{3}{2} + \frac{s}{2}, 1 + s) \), especially for \((s, l) = (-\frac{1}{2} +, \frac{7}{6}) \) and \((s, l) = (0, \frac{1}{2} +) \). Global well-posedness was obtained by Grünrock and the author [3] for \( r = 2 \) and \( s \geq 0, \ l = s + \frac{1}{2} \), using the charge conservation law \( \|\psi(t)\|_{L^{2}} = \text{const} \). This means that there is still a gap concerning LWP between the known results and the minimal regularity predicted by scaling, namely \((s, l) = (\frac{1}{2}, 0) \) leaving open the problem what happens for \(-\frac{1}{2} < s < -\frac{1}{4} \) and \( 0 < l < \frac{1}{6} \) or else \(-\frac{1}{2} < s < 0 \) and \( 0 < l \leq \frac{1}{6} \).

We want to approach this problem by leaving the \( L^{2} \)-based data and study the local well-posedness problem for data in \( \tilde{H}^{s, r} \)-spaces for \( 1 < r < 2 \), especially for \( r = 1 + \). The critical spaces are \((\psi_0, \phi_0, \phi_1) \in \tilde{H}^{\frac{s}{2} - \frac{1}{2}, r} \times \tilde{H}^{\frac{s}{2} - 1, r} \times \tilde{H}^{\frac{s}{2} - 2, r} \), i.e. \((\psi_0, \phi_0, \phi_1) \in \tilde{H}^{\frac{s}{2} - r} \times \tilde{H}^{\frac{s}{2} + r} \times \tilde{H}^{\frac{s}{2} + r} \), for \( r = 1 + \).

Our main Theorem [11] shows that especially for \( r = 1 + \) we may assume \((\psi_0, \phi_0, \phi_1) \in \tilde{H}^{\frac{s}{2} + r} \times \tilde{H}^{\frac{s}{2} + r} \times \tilde{H}^{\frac{s}{2} + r} \) leaving open the interval \( \frac{1}{2} < s < \frac{5}{6} \) for the spinor and \( 1 < l \leq \frac{1}{2} \). As remarked above in terms of scaling \( H^{\frac{s}{2} + r} \sim H^{\frac{s}{2} + r} \)
and $H^{s,r,1} \sim H^{s,r}$. Thus in this sense the gap for the spinor significantly shrinks to $-\frac{1}{2} < s \leq -\frac{3}{8}$ from $-\frac{1}{2} < s \leq -\frac{1}{4}$ in the pure $L^2$-case.

This gap phenomenon especially for the low dimensional case $N = 2$ also appears for other types of nonlinear wave equations with quadratic nonlinearities. In the three-dimensional case Grünerkøk [7] proved for derivative nonlinear wave equations like $\Box u = (\partial u)^2$ an almost optimal well-posedness result in the sense of scaling as $r \rightarrow 1$. This problem was considered in the two-dimensional case by Grigoryan-Tanguay [5]. For $r = 2$ the critical exponent is $s = 1$. The authors prove by use of Strichartz type estimates that $s > \frac{1}{2}$ is sufficient for LWP. For $1 < r < 2$ these authors proved LWP for $s > 1 + \frac{1}{r}$, thus $s > \frac{5}{2}$ for $r = 1 +$, which scales like $H^{s,r}$, half a derivative away from the critical exponent.

If however a null condition is satisfied for a system of the form $\Box u = Q(u,u)$, where $Q$ is one of the null forms of Klainerman, then this gap could be closed by Grigoryan-Nahmod [3], who established LWP for $s > \frac{1}{2} + \frac{1}{2r}$, which for $r = 1 +$ scales like $H^{s,r}$, as desired.

In the classical case $r = 2$ it is by now standard to reduce LWP for semilinear wave equations to estimates for the nonlinearities in Bourgain-Klainerman-Machedon spaces $X^{s,b}$. Grünerkøk [6] proved that a similar method also works for $1 < r < 2$. He also obtained the necessary bilinear estimates for the derivative wave equation by use of the calculations of Foschi-Klainerman [3]. Later this approach was also used by [4] and by the author [9] for the Chern-Simons-Higgs and the Chern-Simons-Dirac equations. Using the fact that the nonlinear terms in the Dirac-Klein-Gordon system fulfill a null condition, as was shown by [1], we now combine the estimates in [3] and a bilinear estimate by [5].

We now formulate the main result for the DKG system.

**Theorem 1.1.** Let $1 < r \leq 2$, $\delta > 0$ and $s = s_0 + \delta$, $l = l_0 + \delta$. Here $(s_0,l_0) = (\frac{9}{16} - \frac{3}{16}, \frac{9}{16} - \frac{3}{16})$ (minimal s) and $(s_0,l_0) = (\frac{1}{2}, \frac{1}{2} - \frac{5}{16})$ (minimal l) are admissible. Assume

$$
\psi_0 \in \tilde{H}^{s,r}(\mathbb{R}^2), \phi_0 \in \tilde{H}^{l,r}(\mathbb{R}^2), \phi_1 \in \tilde{H}^{-1,r}(\mathbb{R}^2).
$$

Then there exists $T > 0$, $T = T(\|\psi_0\|_{\tilde{H}^{s,r}}, \|\phi_0\|_{\tilde{H}^{l,r}}, \|\phi_1\|_{\tilde{H}^{-1,r}})$ such that the DKG system (1), (2), (3) has a unique solution

$$
\psi \in X_{s,b,+}^{T,r}[0,T] + X_{l,b,-}^{T,r}[0,T], \phi \in X_{l,b,+}^{T,r}[0,T] + X_{l,b,-}^{T,r}[0,T],
$$

$$
\partial_t \phi \in X_{l-1,b,+}^{T,r}[0,T] + X_{l-1,b,-}^{T,r}[0,T],
$$

where $b = \frac{1}{r}$. This solution satisfies

$$
\psi \in C^0([0,T], \tilde{H}^{s,r}), \phi \in C^0([0,T], \tilde{H}^{l,r}), \partial_t \phi \in C^0([0,T], \tilde{H}^{l-1,r}).
$$

The spaces $X_{s,b,\pm}^{T,r}$ are generalizations of the Bourgain-Klainerman-Machedon spaces $X^{s,b}$ (for $r = 2$). We define $X_{s,b,\pm}^{T,r}$ as the completion of $S^{(1+2)}$ with respect to the norm

$$
\|\phi\|_{X_{s,b,\pm}^{T,r}} := \|\langle \xi \rangle \hat{\phi}(\tau \pm |\xi|)\|_{L_{\tau,t}^{1,r}}
$$

for $1 \leq r \leq 2$, $\frac{1}{r} + \frac{1}{r} = 1$, where $\hat{\cdot}$ denotes the Fourier transform with respect to space and time.

**Remark 1:** By Theorem 1.2 the solution depends continuously on the data.

**Remark 2:** We recover the case $r = 2$ with $(s_0,l_0) = (\frac{1}{4} + \frac{3}{2r}, \frac{3}{2r})$ or $(s_0,l_0) = (0, \frac{1}{2})$ from [1] and the pair $(s_0,l_0) = (\frac{1}{2} + \frac{3}{16}, \frac{5}{16})$ for $r = 1 +$.

**Remark 3:** By interpolation of the case $r = 1 +$ with the whole range of pairs $(s,l)$ for $r = 2$ from [1] (cf. Prop. 2.3 below) one obtains further admissible pairs $(s_0,l_0)$ for $1 < r < 2$. We omit the details.
Using the following general local well-posedness theorem (cf. [4], Theorem 1) we reduce the proof of Theorem 1.1 to bilinear estimates for the nonlinearities.

**Theorem 1.2.** Let $N(u)$ be a nonlinear function of degree $\alpha > 0$. Assume that for given $s \in \mathbb{R}$, $1 < r < \infty$ there exist $b > \frac{1}{r}$ and $b' \in (b - 1, 0)$ such that the estimates

$$\|N(u)\|_{X^{r,s,b}} \leq c\|u\|_{X^{r,s,b}}^\alpha$$

and

$$\|N(u) - N(v)\|_{X^{r,s,b}} \leq c\|u\|_{X^{r,s,b}}^{\alpha - 1} \|v\|^\alpha_{X^{r,s,b}}$$

are valid. Then there exist $T = T(\|u_0\|_{H^{s,r}}) > 0$ and a unique solution $u \in X^{r,s,b}_{0,T}$ of the Cauchy problem

$$\partial_t u + i Du = N(u), \quad u(0) = u_0 \in \dot{H}^{s,r},$$

where $D$ is the operator with Fourier symbol $|\xi|$. This solution is persistent and the mapping data upon solution $u_0 \mapsto u$, $\dot{H}^{s,r} \to X^{r,s,b}_{0,T}$ is locally Lipschitz continuous for any $T_0 < T$.

## 2. Bilinear estimates

We start by collecting some fundamental properties of the solution spaces. We rely on [4]. The spaces $X^{r,s,b}_{0,T}$ with norm

$$\|\phi\|_{X^{r,s,b}_{0,T}} := \|\langle\xi\rangle^s \langle r + |\xi|\rangle^b \phi(r,\xi)\|_{L^r_T}$$

for $1 < r < \infty$ are Banach spaces with $S$ as a dense subspace. The dual space is $X^{r,s,-b}_{-b,-b}$, where $\frac{1}{r} + \frac{1}{q} = 1$. The complex interpolation space is given by

$$(X^{s_0,s_0\pm,\pm}_{0,b_0\pm}, X^{s_1,s_1\pm,\pm}_{b_1\pm}, \mathbb{T}) = X^{s,b}_{\mathbb{T}}$$

where $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{s} = \frac{1}{s_0} + \frac{\theta}{s_1}$, $b = (1 - \theta)b_0 + \theta b_1$. Similar properties has the space $X^{s,b}_{0,T}$ defined by its norm

$$\|\phi\|_{X^{s,b}_{0,T}} := \|\langle\xi\rangle^s \langle |\tau| - |\xi|\rangle^b \phi(r,\xi)\|_{L^r_T}.$$  

We also define

$$X^{r,s,b}_{0,T} = \{u = U_{0,T} \times \mathbb{R}^2 : U \in X^{r,s,b}_{0,T}\}$$

with

$$\|U\|_{X^{r,s,b}_{0,T}} := \inf\{\|U\|_{X^{r,s,b}_{0,T}} : U_{0,T} \times \mathbb{R}^2 = u\}$$

and similarly $X^{r,s,b}_{0,T}$.

If $u = u_+ + u_-$, where $u_{\pm} \in X^{r,s,b}_{0,T}$, then $u \in C^0(0,T,\dot{H}^{s,r})$, if $b > \frac{1}{r}$.

The ”transfer principle” in the following proposition, which is well-known in the case $r = 2$, also holds for general $1 < r < \infty$ (cf. [4], Prop. A.2 or [6], Lemma 1). We denote $\|u\|_{\tilde{L}^r_T(\tilde{L}^r)} := \|\tilde{u}\|_{L^r(\tilde{L}_r^r)}$.

**Proposition 2.1.** Let $1 \leq p, q \leq \infty$. Assume that $T$ is a bilinear operator which fulfills

$$\|T(e^{\pm iH_D} f_1, e^{\pm iH_D} f_2)\|_{\tilde{L}^r_T(\tilde{L}^q)} \lesssim \|f_1\|_{\dot{H}^{s_1,r}} \|f_2\|_{\dot{H}^{s_2,r}}.$$

Then for $b > \frac{1}{r}$ the following estimate holds:

$$\|T(u_1, u_2)\|_{\tilde{L}^r_T(\tilde{L}^q)} \lesssim \|u_1\|_{X^{r,s,b,\pm,\pm}_{0,T}} \|u_2\|_{X^{r,s,b,\pm,\pm}_{0,T}}.$$
At first we are primarily interested in the case $r = 1$. Thereafter we obtain the general case $1 < r \leq 2$ by bilinear interpolation with the known results for the case $r = 2$.

**Proposition 2.2.** Let $r = 1$, $t \geq s \geq \frac{2}{3}$, $\frac{1}{2} + \frac{1}{3r} < l \leq 1 + \frac{1}{3}$ and $b > \frac{2}{3}$.

The following estimates apply:

\begin{align}
\|\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle \|_{X^{s,b}_{l-1,\pm_1+}} &\lesssim \|\psi\|_{X^{s,b}_{l,\pm_2}} \|\psi'\|_{X^{s,b}_{l,\pm_2}}, \\
\langle \Pi_{\pm_2}(D)\langle \beta \Pi_{\pm_1}(D) \psi \rangle \|_{X^{s,b}_{l-1,\pm_2}} &\lesssim \|\phi\|_{X^{1,b}_{l,\pm_2}} \|\psi\|_{X^{s,b}_{l,\pm_2}}.
\end{align}

By duality (8) is equivalent to

\[
\int \int \langle \Pi_{\pm_2}(D)(\phi \beta \Pi_{\pm_1}(D) \psi), \psi' \rangle \ dt \ dx \lesssim \|\phi\|_{X^{1,b}_{l,\pm_2}} \|\psi\|_{X^{s,b}_{l,\pm_2}} \|\psi'\|_{X^{s,b}_{l-1,\pm_2}}.
\]

The left hand side equals

\[
\int \int \phi(\beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi') \ dt \ dx \lesssim \|\phi\|_{X^{1,b}_{l,\pm_2}} \|\beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi'\|_{X^{s,b}_{l-1,\pm_2}},
\]

so that (8) reduces to

\[
\|\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle \|_{X^{s,b}_{l-1,\pm_2}} \lesssim \|\psi\|_{X^{s,b}_{l,\pm_2}} \|\psi'\|_{X^{s,b}_{l-1,\pm_2}}.
\]

The null structure rests on the following property of the Fourier symbol which is given by the following lemma.

**Lemma 2.1.** (cf. [1], Lemma 2)

\[\Pi_{\pm_1}(\eta - \xi) \beta \Pi_{\pm_2}(\eta - \xi) \Pi_{\pm_1}(\eta) = O(\langle \pm_1 \eta, \pm_2 (\eta - \xi) \rangle),\]

where $\langle \eta, \xi \rangle$ denotes the angle between the vectors $\eta$ and $\xi$.

This has the following consequence:

\[
|\mathcal{F}(\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle)(\tau, \xi)| \lesssim \int |\langle \beta \Pi_{\pm_1}(\eta) \tilde{\psi}(\lambda, \eta), \Pi_{\pm_2}(\eta - \xi) \tilde{\psi}'(\lambda - \tau, \eta - \xi) \rangle| \ d\lambda \ d\eta
\]

\[
= \int |\langle \Pi_{\pm_2}(\eta - \xi) \beta \Pi_{\pm_1}(\eta) \tilde{\psi}(\lambda, \eta), \tilde{\psi}'(\lambda - \tau, \eta - \xi) \rangle| \ d\lambda \ d\eta
\]

\[
\lesssim \int \langle \pm_1 \eta, \pm_2 (\eta - \xi) \rangle |\tilde{\psi}(\lambda, \eta)| |\tilde{\psi}'(\lambda - \tau, \eta - \xi)\rangle | \ d\lambda \ d\eta.
\]

For the angle between two vectors the following elementary estimates apply.

**Lemma 2.2.** (cf. [1])

\[\langle \eta, \xi \rangle \sim \frac{||\xi||^2}{|\eta|^2 |\eta - \xi|^2} \langle |\xi| - |\eta| - |\eta - \xi| \rangle\]

\[
\langle \eta, \xi \rangle \sim \frac{||\eta||^2}{|\eta|^2 |\eta - \xi|^2} \langle |\eta| + |\eta - \xi| - |\xi| \rangle\]

\[
\langle \pm_1 \eta, \pm_2 (\eta - \xi) \rangle \lesssim \left( \frac{\langle \tau \rangle^2 + \langle \lambda \pm_2 |\eta| \rangle + \langle \lambda - \tau \pm_2 |\eta - \xi| \rangle}{\min(\langle \xi \rangle, \langle \eta - \xi \rangle)} \right)^\frac{1}{2}.
\]

**Proof of [2].** By the fractional Leibniz rule the estimate (17) follows from

\[
\|\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle \|_{X^{s,b}_{l-1,\pm_1+}} \lesssim \|\psi\|_{X^{s,b}_{l,\pm_2}} \|\psi'\|_{X^{s,b}_{l,\pm_2}}
\]

and the similar estimate

\[
\|\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle \|_{X^{s,b}_{l-1,\pm_2}} \lesssim \|\psi\|_{X^{s,b}_{l,\pm_2}} \|\psi'\|_{X^{s,b}_{l,\pm_2}}.
\]
We only prove the first one, because the last one is handled in exactly the same way. It is equivalent to

$$\|\langle \beta \Pi_{\pm_1}(D) \psi, \Pi_{\pm_2}(D) \psi' \rangle \|_{X_{L,T}^{b-1,+}} \lesssim \| \psi \|_{X^{b}_{L,T}} \| \psi' \|_{X^{b}_{L,T}} .$$

(15)

The left hand side is bounded by

$$\| \mathcal{F}(\langle \beta \Pi_{\pm_1} \psi, \Pi_{\pm_2} \psi' \rangle) \|_{L^\prime_{\xi}}$$

$$= \| \int \langle \beta \Pi_{\pm_1}(\eta) \tilde{\psi}(\lambda, \eta), \Pi_{\pm_2}(\eta - \xi) \tilde{\psi}'(\tau - \lambda, \xi - \eta) \rangle d\lambda d\eta \|_{L^\prime_{\xi}}.$$  

(16)

Let now $\psi(t, x) = e^{\pm i t D} \psi_0(x)$ and $\psi' = e^{\mp i t D} \psi_0(x)$, so that we obtain $\tilde{\psi}(\tau, \xi) = c \delta(\tau \mp 1 | \xi |) \psi_0(\xi)$ and $\tilde{\psi}'(\tau, \xi) = c \delta(\tau \pm 2 | \xi |) \psi_0'(\xi)$. Then we obtain by Lemma [2.1]

$$\| \mathcal{F}(\langle \beta \Pi_{\pm_1} \psi, \Pi_{\pm_2} \psi' \rangle) \|_{L^\prime_{\xi}}$$

$$= c^2 \| \langle \Pi_{\pm_2}(\eta - \xi) \beta \Pi_{\pm_1}(\eta) \delta(\lambda \mp 1 | \eta |) \tilde{\psi}_0(\eta), \delta(\tau - \lambda \pm 2 | \xi - \eta |) \tilde{\psi}_0'(\xi - \eta) \rangle d\eta d\lambda \|_{L^\prime_{\xi}}$$

$$\lesssim \| \int \triangle(\pm_1 \eta, \pm_2(\eta - \xi)) \delta(\tau \mp 1 | \eta | \pm 2 | \xi - \eta |) \tilde{\psi}_0(\eta) \tilde{\psi}_0'(\xi - \eta) \|_{L^\prime_{\xi}} .$$

(17)

We now distinguish between the different signs. It suffices to consider the cases $\pm_1 = \pm_2 = +$ (hyperbolic case) and $\pm_1 = +, \pm_2 = -$ (elliptic case).

**Case** $\pm_1 = \pm_2 = +$. Then we obtain from (1) and H"older’s inequality:

$$\| \mathcal{F}(\langle \beta \Pi_{\pm_1} \psi, \Pi_{\pm_2} \psi' \rangle) \|_{L^\prime_{\xi}}$$

$$\lesssim \| \int |\xi|^{\frac{3}{2}} \| |\tau| - |\xi| \|^{\frac{3}{2}} \| \delta(\tau - |\eta| + |\xi - \eta|) |\tilde{\psi}_0(\eta)| \| \tilde{\psi}_0'(\xi - \eta) \| d\eta d\lambda \|_{L^\prime_{\xi}}$$

$$\lesssim \sup_{\tau, \xi} I \| D^{\frac{3}{4}} \psi_0'' \|_{L^\prime} \| D^{\frac{3}{4}} \psi_0' \|_{L^\prime} ,$$

where

$$I = |\xi|^{\frac{3}{2}} \| |\tau| - |\xi| \|^{\frac{3}{2}} \left( \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{3}{2} - \frac{3}{2}} |\eta - |\xi|^{-\frac{3}{2}} d\eta \right) .$$

We want to show $\sup_{\tau, \xi} I \lesssim 1$.

Subcase $|\eta| + |\xi - \eta| \leq 2|\xi |$. By [3], Prop. 4.5 we obtain

$$\int_{|\eta| + |\xi - \eta| \leq 2|\xi |} \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{3}{2} - \frac{3}{2}} |\eta - |\xi|^{-\frac{3}{2}} d\eta \sim \| \xi \| \| |\tau| - |\xi| \| A \| \tau| - |\xi| \| B ,$$

with $A = \max(\frac{2}{3}, \frac{5}{3}, \frac{3}{2}) - 1 - r = \frac{1}{2} - r$ and $B = 1 - \max(\frac{2}{3}, \frac{5}{3}, \frac{3}{2}) = -\frac{1}{2}$ for $r = 1 +$. This implies

$$I' \lesssim \| |\xi| |\tau| - |\xi| |\xi|^{\frac{3}{2} - r} |\tau| - |\xi|^{-\frac{3}{2}} \| |\tau| - |\xi| \| A \| |\tau| - |\xi| \| B ,$$

because $|\tau| \leq |\xi |$.

Subcase $|\eta| + |\xi - \eta| \geq 2|\xi |$. We apply [3], Lemma 4.4, and obtain

$$\int_{|\eta| + |\xi - \eta| \geq 2|\xi |} \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{3}{2} - \frac{3}{2}} |\eta - |\xi|^{-\frac{3}{2}} d\eta$$

$$\sim \left( |\xi|^{2} - r^{2} \right)^{-\frac{1}{2}} \int_{2}^{\infty} (|\xi| x + r)^{-\frac{3}{2} + \frac{3}{2}} (|\xi| x - r)^{-\frac{3}{2} + \frac{3}{2}} (x^{2} - 1)^{-\frac{1}{2}} dx$$

$$\sim \left( |\xi|^{2} - r^{2} \right)^{-\frac{1}{2}} \int_{2}^{\infty} \left( x + \frac{r}{|\xi|} \right)^{-\frac{3}{2} + \frac{3}{2}} \left( x - \frac{r}{|\xi|} \right)^{-\frac{3}{2} + \frac{3}{2}} (x^{2} - 1)^{-\frac{1}{2}} dx |\xi|^{1-r} .$$
The lower limit of the integral can be chosen as 2 by inspection of the proof of [3]. Because $|\tau| \leq |\xi|$ the integral is bounded and we obtain

$$I^r \lesssim |\xi|^r |\tau| - |\xi||\tau| \frac{|\xi|^{1-r}}{|\tau| - |\xi||\tau| + |\xi|} \lesssim |\tau| - |\xi||\tau| \frac{|\xi|^{1-r}}{|\tau| - |\xi||\tau| + |\xi|} \lesssim 1.$$  

**Case $\pm_1 = +, \pm_2 = -$.** We use (12) and Hölder and obtain in the case $|\eta| \geq |\xi - \eta|$:

$$\sum_{\pm} \lesssim \| \delta(\tau - |\eta| - |\xi - \eta|) \psi_0^+(\eta) \psi_0^-(\xi - \eta) \|_{L^r_{\tau,\xi}} \lesssim \| \psi_0^+ \|_{L^r_{\tau}} \| \psi_0^- \|_{L^r_{\tau}},$$

where

$$I = |\tau| - |\xi||\tau| \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-\frac{d}{2}} |\eta - \xi|^{-\frac{d}{2}} d\eta.$$  

By [3], Lemma 4.3 we obtain

$$\int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-\frac{d}{2}} |\eta - \xi|^{-\frac{d}{2}} d\eta \sim \tau^A |\tau| - |\xi|^{B},$$

with $A = \max(\frac{d}{2} + \frac{d}{2}, \frac{d}{2}) - (\frac{d}{2} + 1) = \frac{d}{2} - \frac{d}{2}$ and $B = 1 - \max(\frac{d}{2} + \frac{d}{2}, \frac{d}{2}) = -\frac{d}{2}$ for $r = 1$. Using $|\xi| \leq \tau$ this implies

$$I^r \lesssim |\tau| - |\xi||\tau|^{\frac{d}{2}} |\tau| - |\xi|^{-\frac{d}{2}} \lesssim 1.$$  

We omit the case $|\eta| \leq |\xi - \eta|$ because it can be treated similarly.

In any case we arrive at the estimate

$$\| F((\beta \Pi_{\pm_1} \psi, \Pi_{\pm_2} \psi)) \|_{L^r_{\tau,\xi}} \lesssim \| D\psi_0^\pm \|_{L^r_{\tau}} \| D\psi_0^\pm \|_{L^r_{\tau}}.$$  

By the transfer principle Prop. 2.4 we obtain (14), which completes the proof.

For the proof of (17) we need the following propositions, where we refer to the authors’ paper [5] and the Grigoryan-Tanguay paper [3].

**Proposition 2.3.** Assume $1 < r \leq 2$, $\alpha_0 > \frac{1}{2} - \gamma$, $\alpha_1 + \alpha_2 > \frac{d}{2}$, $0 \leq \alpha_0 \leq \alpha_1, \alpha_2$, \textit{max}$(\alpha_1, \alpha_2) \neq \frac{3}{2} b$, $b \geq \gamma$, and either $\alpha_1 + \alpha_2 - \alpha_0 > \gamma + \frac{1}{r}$ and $\gamma \geq \frac{1}{r}$, or $\alpha_1 + \alpha_2 - \alpha_0 \geq \gamma + \frac{1}{r}$ and $\gamma \geq \frac{1}{r}$. Moreover $\gamma \geq \text{max}(\alpha_1 - \frac{d}{2}, \alpha_2 - \frac{d}{2})$, $b > \frac{1}{2}$.

Then the following estimate holds:

$$\| uv \|_{X_{\alpha_0, \gamma}} \lesssim \| u \|_{X_{\alpha_1, b}} \| v \|_{X_{\alpha_2, b}}.$$  

**Proof.** [3], Proposition 2.6.

In the case $\gamma = 0$ we need the following non-trivial result.

**Proposition 2.4.** Let $1 \leq r \leq 2$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > \frac{d}{2}$, $b_1 + b_2 > \frac{1}{2}$, and $b_1, b_2 > \frac{1}{2}$. Then the following estimate holds

$$\| uv \|_{X_{\alpha_0, \alpha_1, b_1}} \lesssim \| u \|_{X_{\alpha_1, b_1}} \| v \|_{X_{\alpha_2, b_2}}.$$  

**Proof.** Selberg [10] proved this in the case $r = 2$. The general case $1 < r \leq 2$ was given by Grigoryan-Tanguay [5], Prop. 3.1, but in fact the case $r = 1$ is also admissible. More precisely the result follows from [5] after summation over dyadic pieces in a standard way.
Proof of (14). We apply Lemma [2.4] and estimate the angle by (13), where we replace the power $\frac{1}{2}$ by $\frac{1}{2^+}$, which is certainly possible. This allows to reduce (13) by the following estimates:

$$
\|u\|_{X^{r, l} \pm \frac{1}{2^+}, b} \leq \|u\|_{X^{r, l} \pm \frac{1}{2^+}, b} \|v\|_{X^{r, l} \pm \frac{1}{2^+}, b},
$$

(18)

By duality it suffices to prove

$$
\|uw\|_{X^{r, l} \pm \frac{1}{2^+}, b} \leq \|u\|_{X^{r, l} \pm \frac{1}{2^+}, b} \|w\|_{X^{r, l} \pm \frac{1}{2^+}, b},
$$

(19)

(18) follows from the fractional Leibniz rule and Prop. [2.4], which is fulfilled for $l + \frac{1}{2^+} > \frac{1}{2^+}$, $l > \frac{1}{2}$ and $2b - \frac{1}{2^+} > \frac{1}{2^+}$. (19), (20), and (21) follow similarly.

Next we prove (23). We use Prop. [2.3] with parameters $\gamma = b - 1 + \frac{1}{2^+} = \frac{3}{2^+} - 1 + \frac{1}{2^+}$, $\alpha_0 = s > \frac{5}{2^+} > \frac{1}{2} \gamma$, $\alpha_1 = s + \frac{1}{2^+}$, $\alpha_2 = l + \frac{1}{2}$, so that $\alpha_1 + \alpha_2 = 2l + \frac{1}{2} > \frac{5}{2^+} + 1 + \frac{1}{2}$, because by assumption $l > \frac{5}{2}$ and $l > \frac{1}{2} + \frac{1}{2^+}$. Moreover $\alpha_1 + \alpha_2 = s + \frac{1}{2} + l > \frac{5}{2}$, because by assumption $s > \frac{5}{2}$ and $l > \frac{1}{2} + \frac{1}{2^+}$. Also we need $\gamma = \frac{3}{2^+} - 1 + \frac{1}{2^+} = \max(\alpha_1 - \frac{1}{2}, \alpha_2 - \frac{1}{2}) = \max(s - \frac{1}{2^+}, l - \frac{1}{2})$, because we may assume without loss of generality $l \leq \frac{5}{2} - 1$ and $s \leq \frac{5}{2} - 1$.

Finally we have to prove (22), where it suffices to consider the case $l = \frac{1}{2} + \frac{1}{2^+}$. By the fractional Leibniz rule we reduce to the estimates

$$
\|uw\|_{X^{r, l} \pm \frac{1}{2^+}, b} \leq \|u\|_{X^{r, l} \pm \frac{1}{2^+}, b} \|w\|_{X^{r, l} \pm \frac{1}{2^+}, b},
$$

(24)

Concerning (23) we apply Prop. [2.3] with $\gamma = 1$, $\alpha_0 = 0$, $\alpha_1 = \frac{1}{2^+}$, $\alpha_2 = 1 + \frac{1}{2^+}$, so that $\alpha_1 + \alpha_2 = 1 + \frac{1}{2^+} + \frac{1}{2^+}$ and $\alpha_1 + \alpha_2 - \alpha_0 = 1 + \frac{1}{2^+} + \frac{1}{2} + \frac{1}{2}$. Thus

$$
\|uw\|_{X^{r, 1}_{\frac{1}{2^+} + 1}} \leq \|u\|_{X^{r, 1}_{\frac{1}{2^+} + 1}} \|w\|_{X^{r, 1}_{\frac{1}{2^+} + 1}}.
$$

Moreover we apply Prop. [2.4] with $\alpha_1 = \frac{1}{2^+}$, $\alpha_2 = \frac{1}{2^+}$, $b_1 = b_2 = b$, thus $\alpha_1 + \alpha_2 > \frac{1}{2^+}$ and $b_1 + b_2 > \frac{1}{2^+}$. Thus

$$
\|uw\|_{X^{r, b}_{\frac{1}{2^+} + b}} \leq \|u\|_{X^{r, b}_{\frac{1}{2^+} + b}} \|w\|_{X^{r, b}_{\frac{1}{2^+} + b}}.
$$

Interpolation between these estimates implies

$$
\|uw\|_{X^{r, l}_{\frac{1}{2^+} + l}} \leq \|u\|_{X^{r, l}_{\frac{1}{2^+} + l}} \|w\|_{X^{r, l}_{\frac{1}{2^+} + l}},
$$
which proves (24).

Concerning (25) we argue similarly. We obtain
\[ \|uw\|_{X^{s,l}_{0,0}} \lesssim \|u\|_{X^{s,l}_{r,0}} \|w\|_{X^{s,l}_{r,-s,l}}, \]
and
\[ \|uw\|_{X^{s,l}_{0,0}} \lesssim \|u\|_{X^{s,l}_{r,0}} \|w\|_{X^{s,l}_{r,-s,l}}, \]
so that interpolation implies
\[ \|uw\|_{X^{s,l}_{0,0}} \lesssim \|u\|_{X^{s,l}_{r,0}} \|w\|_{X^{s,l}_{r,-s,l}}, \]
which proves (25) and completes the proof of (9).

\[ \Box \]

Remark: It is (25) which prevents the optimal choice \( s = \frac{1}{r} + \), \( l = 1^+ \) in the case \( r = 1^+ \). All the other estimates which are necessary for the proof of our main theorem are valid for this choice.

The bilinear estimates in the case \( r = 2 \) by [1], Theorem 1 are given by the following proposition.

Proposition 2.5. Let \( r = 2 \). The estimates (7) and (8) are fulfilled in the region
\[ s > -\frac{1}{5}, \quad \max(\frac{1}{2} - \frac{s}{4}, \frac{1}{2} + \frac{s}{4}) < l < \min(\frac{3}{4}, \frac{3}{4} + 3s, \frac{3}{4} + 2s, 1 + s). \]

The admissible pairs \((s, l)\) in the general case \( 1 < r \leq 2 \) are now obtained by bilinear interpolation between the estimates in Prop. 2.2 and Prop. 2.5. Because we are mainly interested in the minimal possible choice of \( s \) and \( l \) we concentrate on the following result for simplicity.

Proposition 2.6. Let \( 1 < r \leq 2 \), \( b = \frac{1}{r^+} \) and \( \delta > 0 \). The estimates (7) and (8) are fulfilled in the cases \((s, l) = (\frac{5}{8} + \frac{2}{r^+} + \delta, \frac{3}{4} + \frac{3}{4} + \delta) \) (minimal \( s \)) and \((s, l) = (\frac{5}{8} + \frac{2}{r^+} + \delta, \frac{3}{4} + \frac{3}{4} + \delta) \) (minimal \( l \)).

Proof. We interpolate between the pair \((s, l) = (\frac{5}{8} + \frac{2}{r^+} + \delta, \frac{3}{4} + \frac{3}{4} + \delta) \) in the case \( r = 1^+ \) on the one hand and the pairs \((s, l) = (\frac{5}{8} + \frac{2}{r^+} + \delta, \frac{3}{4} + \frac{3}{4} + \delta) \) and \((s, l) = (0, \frac{1}{r^+}) \) in the case \( r = 2 \) on the other hand to obtain the first and second claimed pair \((s, l)\), respectively. We concentrate on the second pair. Let \( \delta > 0 \) be given and \( s = \frac{5}{8} + \frac{2}{r^+} + \delta, \ l = \frac{3}{4} + \frac{3}{4} + \delta \). If \( r > 1 \) is sufficiently close to 1 we have \( \delta > \frac{3}{4} - \frac{1}{4} \), so that \( \delta = \frac{1}{2} - \frac{1}{4} + \omega \), where \( \omega > 0 \). For \( \omega = 0^+ \) we obtain \( s = \frac{5}{8} + \delta \) and \( l = \frac{3}{4} + \frac{3}{4} + \delta \). In this case the estimates (7) and (8) are satisfied. By the fractional Leibniz rule this is also true for every \( \omega > 0 \), thus for the given \( \delta \) and \( r \) close enough to 1. Bilinear interpolation with the case \( s = \delta \) and \( l = \frac{1}{r^+} + \delta \) in the case \( r = 2 \) implies the estimates (7) and (8) for the given pair \((s, l)\) in the whole range \( 1 < r \leq 2 \).

References

[1] P. d’Ancona, D. Foschi, and S. Selberg: Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions. J. Hyperbolic Diff. Equns. 4 (2007), no. 2, 295330
[2] P. d’Ancona, D. Foschi, and S. Selberg, Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system, J. Eur. Math. Soc. (2007), no. 4, 877-899
[3] D. Foschi and S. Klainerman: Bilinear space-time estimates for homogeneous wave equations. Ann. Sc. ENS. 4. serie, 33 (2000), 211-274
[4] V. Grigoryan and A. Nahmod: Almost critical well-posedness for nonlinear wave equation with Q_{\mu \nu} null forms in 2D. Math. Res. Letters 21 (2014), 313-332
[5] V. Grigoryan and A. Tanguay: Improved well-posedness for the quadratic derivative nonlinear wave equation in 2D. J. Math. Analysis Appl. 475 (2019), 1578-1595
[6] A. Grünrock: An improved local well-posedness result for the modified KdV equation. Int. Math. Res. Not. (2004), no.61, 3287-3308
A. Grünrock: On the wave equation with quadratic nonlinearities in three space dimensions. J. Hyperbolic Diff. Equ. 8 (2011), 1-8

A. Grünrock and H. Pecher: Global solutions for the Dirac-Klein-Gordon system in two space dimensions. Comm. Part. Diff. Equus. 35 (2009), 89-112

H. Pecher: The Chern-Simons-Higgs and the Chern-Simons-Dirac equations in Fourier-Lebesgue spaces. Discrete Contin. Dyn. Syst. 39 (2019), 4875-4893.

S. Selberg: Bilinear Fourier restriction estimates related to the 2D wave equation. Adv. Diff. Equ. 16 (2011), 667-690