Purely nonlinear disorder-induced localizations and their parametric amplification

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We investigate spatial localization in a quadratic nonlinear medium in the presence of randomness. By means of numerical simulations and theoretical analyses we show that, in the down conversion regime, the transverse random modulation of the nonlinear susceptibility generates localizations of the fundamental wave that grow exponentially in propagation. The localization length is optically controlled by the pump intensity which determines the amplification rate. The results also apply to cubic nonlinearities. © 2013 Optical Society of America

Over the past few years, the interplay between nonlinearity and randomness generated relevant interest, specifically in the fields of light-matter interaction [1–4] and Bose-Einstein condensation [5]. In all these studies, disorder is given by a randomly varying linear response. In the presence of a disordered potential, multiple scattering may yield localized states, the Anderson localizations (ALs) [6–9], which decay exponentially over a characteristic length l. Nonlinearity is always introduced as a process affecting observables like the degree of localization, or destroying the disorder-induced Anderson states. This kind of analysis has been reported by several authors, for example in non-resonant systems by the propagation of solitons in local and nonlocal media [10–14] or shock-waves [15] and in resonant systems by the propagation of Self-Induced Transparency (SIT) pulses [16]. Related phenomena are absence of equilibrium, glassy regimes and complexity [17,18].

In this Letter, we analyze the dynamics of the existence of a kind of light localization with a purely nonlinear origin [19]. We observe that unstable localized states arise in a linearly homogeneous medium with a random modulation of the nonlinear response. The predicted effect is general and occurs for various kinds of nonlinearity. We consider a quadratic nonlinearity $\chi^{(2)}$ [20], for which experiments can be envisaged by parametric down-conversion in nonlinear lattices [21] with disorder quasi-phase matching (QPM) [22] and the case of cubic nonlinearity $\chi^{(3)}$ [23] attainable, for example, in Bose-Einstein condensation by modulating the nonlinear interaction through Feshbach resonance [24]. We stress these are localizations only due to a random modulation of the nonlinear response; albeit they are unstable, they can be observed for short propagation distances (of the order of the diffraction length). We also show that random nonlinearity produces a parametric amplification that enhances the localized states during their evolution. In addition, at variance with the linear case, the nonlinear localization length is determined by the optical fluence, i.e. by the input pump beam for $\chi^{(2)}$ and $\chi^{(3)}$. The analysis presented in what follows concerns frequency degenerate down-conversion via $\chi^{(2)}$ nonlinearities, hence the optical pump will be denoted as the second harmonic (SH), while the degenerate outputs (signal=idler) as the fundamental frequency (FF). The main conclusions of the analysis remain valid in the non-degenerate case. Furthermore they can also be extended to $\chi^{(3)}$ processes.

We start from the normalized $\chi^{(2)}$ coupled-mode system [20,25,26] for slowly varying envelopes of continuous-wave light beams, propagating in a randomly distributed quadratic nonlinearity

$$
\begin{align*}
i \partial_x A_1 + \partial_x^2 A_1 + d(x) A_1^* A_2 & = 0, \\
i \partial_x A_2 + \frac{1}{2} \partial_x^2 A_2 + \delta k A_2 + \frac{1}{2} d(x) A_1^2 & = 0,
\end{align*}
$$

(1)

where $\delta k = (2k_1 - k_2)L_D$ is the wave-vector mismatch and $L_D = 2k_1 w_0^2$ is the diffraction length ($w_0$ is the beam waist); $d(x)$ is the normalized effective scaled second-order nonlinear coefficient, $x$ is the transverse coordinate in units of $w_0$ and $z$ is the propagation coordinate in units of $L_D$. $A_1$ and $A_2$ are respectively the amplitudes of the normalized fields at the fundamental frequency (FF) and second harmonic (SH); details can be found in [25].

Randomness is introduced either by a Gaussian random nonlinear coefficient $d(x)$ or, in the considered undepleted pump approximation ($|A_2| >> |A_1|$), by using a spatially modulated $SH$ pump beam $A_2(x,0)$. In the former case, we consider a QPM profile $d(x)$ such that the correlation function of the disordered potential is $\langle d(x)d(x') \rangle = d_0^2 \delta(x-x')$ and the brackets denote average over disorder realizations. Various quasi-phase matching (QPM) profiles can be considered and provide analogue results. We anticipate that, in both cases, the disorder strength will be proportional to $d_0 A_2(x,0)$,
hence the disorder degree can be controlled by acting on the nonlinear coefficient or on the pump intensity.

Writing the \( SH \) and \( FF \) fields respectively as \( A_2(x, z) \rightarrow A_2(x, z)e^{i\delta kz} \) and \( A_1(x, z) \rightarrow A_1(x, z)e^{i\delta k z/2} \) and assuming \( A_2(x, z) \) to be real and slowly depending on \( z \), from Eq. (1) we have for \( FF \):

\[
i\partial_x A_1 + \partial_z^2 A_1 - V(x)A_1 - \frac{\delta k}{2} A_1 = 0,
\]

where \( V(x) = -d(z)A_2(x, 0) \) represents the disordered potential. For the sake of simplicity we assume \( A_2(x, 0) \)

The relevant disorder induced localizations \( \varphi_n(x) \) are given by \( (n = 0, 1, 2, \ldots) \):

\[
-\partial_x^2 \varphi_n + V(x)\varphi_n(x) = E_n\varphi_n.
\]

The states for \( E_n < 0 \) are exponentially localized, such that \( \varphi_n \approx e^{-|x|/l_n}/\sqrt{l_n} \) with \( l_n \approx |E_n|^{-1/2} \) [27]. We study their evolution in (2) by projecting over the state \( \varphi_m(x) \) and obtaining the equation

\[
\frac{d e_n}{dz} - e_n \left( E_n + \frac{\delta k}{2} \right) + \sum_n V_{mn} (c_n - c_n^*) = 0
\]

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\]

by using the averaged state, \( \varphi(x) \approx e^{-|x|/\sqrt{l}} \), we have

\[
V_{mm} = E_m - \frac{1}{l_m^2} \simeq 2E_m < 0,
\]

so that the instability growth-rate directly gives the localization eigenvalue and the localization length.

We point out the relation between the localization length and the rate of growth with the \( SH \) amplitude. Being \( \langle V(x)V(x') \rangle = V_0^2 \delta(x-x') \) with \( V_0 = d_0A_{20} \), and letting \( x = \tilde{x}/V_0^2/3 \) and \( E_n = E_0V_0^2/3 \), we obtain the scaled equation \( -\varphi_{xx} + r(\tilde{x})\varphi = \tilde{E}\varphi \), with \( r(\tilde{x})r(\tilde{x}') = \delta(\tilde{x} - \tilde{x}') \). The localization length for a state \( \varphi \) can be hence written as

\[
l = \tilde{l}/V_0^{2/3} = \tilde{l}/(d_0A_{20})^{2/3},
\]

with \( \tilde{l} \) the value for \( V_0 = 1 \). The scaling law for the maximum growth-rate:

\[
\lambda_R^m = 2V_{mm} = 2E_m = -2E_0(d_0A_{20})^{4/3}.
\]

By varying the \( SH \) amplitude, we expect an increase of the growth-rate and an enhanced localization, as numerically verified below.

Numerically, we use a split-step beam propagation method (BPM) to integrate Eqs. (1), with the input condition \( |A_1(x, z = 0)| \ll |A_2(x, z = 0)| \). We solve the eigenvalue problem (4) by a pseudo-spectral method for the \( FF \) modes with potential \( V(x) = d_0A_{20}r(x) \), and \( r(x) \) a normalized Gaussian distribution, \( (r(x)r(x')) = \delta(x - x') \) with periodic boundary condition. The degree of localization depends on the strength of disorder \( V_0 = d_0A_{20} \). In Fig. 1 we show the \( FF \) during propagation for two values of \( A_{20} \). The evolution reveals the role of the \( SH \) field as a disorder potential that induces localizations for the \( FF \), a process that is more evident when increasing the amount of \( SH \), i.e., the strength of randomness. In order to test the previous theoretical analysis, we use the ground-state of (4) as initial condition for the \( FF \) and measure the growth-rate \( \lambda_R \) for the localization during the evolution for several wave-vector mismatches. We fit by an exponential the evolution of the \( FF \) peak at a fixed disorder realization. In Fig. 2, we show \( \lambda_R \) as a function of \( \delta k \). The negative values of \( \lambda_R \) in figure 2 are due to the unsuitability of the fit outside the instability \( \delta k \) region. The found instability regions are in agreement with Eq. (7). In Fig. 3, panel (a), we show the localization length of the ground state at \( z = 1 \) versus \( A_{20} \) and for \( \delta k = 0.1 \) (in proximity of the maximum expected growth-rate). As the \( SH \) amplitude increases, the strength of disorder grows and the states get more localized. Correspondingly, the localization becomes more unstable and the gain \( \lambda_R \) increases (see Fig. 3, panel (b)).

The curves follow the scaling laws, Eqs. (10) and (11). To investigate the dynamics of the localized states, we consider an initial wave function \( A_1(x, 0) = A_{10}e^{-\beta x^2} \). For a flat input \( \beta = 0 \), \( A_{10} = 0.01 \), we compare in Fig. 4, panel (a), the profile of the \( FF \) after propagation.
In conclusion, the parametric down-conversion of a light beam in a quadratic medium in conjunction with a random transverse modulation of the nonlinear susceptibility (or, equivalently, of the pump beam) brings about the generation of localized regions that grow exponentially with the intensity of the SH field. In addition, we show that the localization length of these states can be controlled by the optical fluence, which also determines the coefficients $c_m(z)$, found numerically and shown in figure 4, panel (b), confirm the predicted exponential trend. The FF state evolves towards the ground-state and the FF growth-rate (continuous thick line) is closer to $c_0(z)$ (dashed line). Similar dynamics occurs by taking $A_1(x, 0)$ as a Gaussian state, $\beta = 1$. Statistically, the state that has the nearest growth-rate to $A_1(x, z)$ is the mostly localized one, i.e., the ground state as shown by the histograms in Fig. 5.

We have obtained similar results in the case of Kerr media with a disordered cubic nonlinear term, $\chi(x)$, resulting in a randomly modulated nonlinear Schrödinger equation, $i\partial_t A + \partial_x^2 A + \chi(x)|A|^2 A = 0$, where $A(x, z)$ is the scaled field amplitude. Repeating the analysis above, this system admits exponentially unstable localizations with a growth-rate of $\lambda_R^2 = -E_m^2 + \chi_{mm}$, where $E_m$ is the $m$-th eigenvalue and $\chi_{mm} = -\int dx \phi_m^2(x) \chi(x)|A_0(x)|^2$, being $A_0(x)$ a slowly varying field envelope. The numerical investigations for Kerr media will be reported elsewhere.

We stress that the predicted localizations are unstable, because subject to parametric $\chi^{(2)}$ or hyper-parametric $\chi^{(3)}$ amplification, however they can be observed for finite propagation. As an example, in the down-conversion regime, assuming $d_0 \simeq 1 \text{ pm}/V$ with $w_0 \sim 10 \mu m$ and an effective transverse waveguide length of the order of $10 \mu m$, a peak power of $100 \text{ kW}$ at $\lambda = 1 \mu m$ results in a localization length of the order of $50 \mu m$.

In conclusion, the parametric down-conversion of a light beam in a quadratic medium in conjunction with a random transverse modulation of the nonlinear susceptibility (or, equivalently, of the pump beam) brings about the generation of localized regions that grow exponentially with the intensity of the SH field. In addition, we show that the localization length of these states can be controlled by the optical fluence, which also determines...
Fig. 4. (Color online) (a) $|A_1(x,z)|$ at $z = 1$ (continuous line) and the ground state arbitrarily scaled for: $A_{01} = 0.01$, $\delta k = 0.1$, $A_{20} = 10$. (b) Evolution of the peak value of the FF wave (continuous thick line) and coefficients $c_m(z)$, in Eq. 12, for: $m = 0$ (dashed line) and $m > 0$ (continuous thin lines).

Fig. 5. (Color online) Histogram of the minimum difference between the growth rate of $A_1(x,z)$ and $\varphi_k$, with 50 realizations of disorder, $A_{01} = 0.01$ and $\delta k = 0.1$. (a) $A_{20} = 1$, (b) $A_{20} = 2$, (c) $A_{20} = 3$, (d) $A_{20} = 4$.

the strength of disorder. Our results also apply to other kind of nonlinearities, such as a randomly modulated Kerr nonlinearity, and can be extended to 2D and 3D cases and have implication in classical frequency down-conversion devices and for quantum optical applications, such as parametric sources of entangled photon pairs. We remark that nonlinear amplification can be used to analyze the states induced by a given distribution of disorder and hence retrieve the properties of the latter, thus providing a characterization tool for the disorder that can associated to e.g. periodic poling in ferroelectric crystals or related approaches.

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