Efficient method for fractional Lévy-Feller advection-dispersion equation using Jacobi polynomials

N. H. Sweilam\textsuperscript{a}, M. M. Abou Hasan\textsuperscript{b}

\textsuperscript{a,b}Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
E-mails: nsweilam@sci.cu.edu.eg \textsuperscript{a}, muneere@live.com \textsuperscript{b}

Abstract:

In this paper, a novel formula expressing explicitly the fractional-order derivatives, in the sense of Riesz-Feller operator, of Jacobi polynomials is presented. Jacobi spectral collocation method together with trapezoidal rule are used to reduce the fractional Lévy-Feller advection-dispersion equation (LFADE) to a system of algebraic equations which greatly simplifies solving like this fractional differential equation. Numerical simulations with some comparisons are introduced to confirm the effectiveness and reliability of the proposed technique for the Lévy-Feller fractional partial differential equations.

Keywords: Lévy-Feller advection-dispersion equation; Riesz-Feller fractional derivative; Spectral method; Jacobi polynomials; Trapezoidal rule.

1 Introduction

Recently, the notion of fractional calculus and differential operators of fractional order has gained great popularity due to its engaging applications in a different fields, such as engineering, finance, system control, hydrology, viscoelasticity, and physics ([7], [13], [27], [39], [42]). The fractional models often can be described as an ordinary fractional differential equations or partial fractional differential equations. These differential equations are more appropriate than the standard integer-order ones for describing the memorial and genetic characteristic of many phenomena and materials [46].

As is well-known, analytic solutions of most fractional differential equations can not always be obtained explicitly, and therefore the employment of different numerical techniques for solving such equations is necessary. Some of the proposed numerical methods for solving such equations are: finite difference methods ([10], [12], [22], [23], [30], [34]), finite element methods [49], semi-analytic methods ([1], [48], [54], [43]), spectral methods ([3], [31], [33]), higher order numerical methods [55].

Spectral methods are a class of techniques in which the numerical solutions are expressed in terms of certain basis functions, which may be orthogonal polynomials (see, e.g. [8], [14], [20] and the references therein). Collocation method, one of the three well-known kinds of spectral methods, has become increasingly common for solving ordinary and partial differential equations. Spectral collocation method has an exponential convergence rate, so it is valuable in providing highly accurate solutions to nonlinear differential equations even using a small number of grids. Choosing of collocation points has very important role in the convergence and efficiency of the collocation method [14]. In reality, spectral collocation methods have been used for the linear and nonlinear fractional partial differential equations.
(31, 32, 33, 45) and for the fractional integro-differential equations (24, 25, 26).

The Jacobi polynomials, which we will denote by \((J_k^{\beta,\gamma}(x), k \geq 0, \beta > -1, \gamma > -1)\), have been used extensively in mathematical analysis and practical applications, and play an important role in the analysis and implementation of spectral methods. The use of general Jacobi polynomials has the advantages of obtaining the solutions in terms of the Jacobi parameters \(\beta\) and \(\gamma\). Hence to generalize and instead of developing approximation results for each particular pair of indexes, it would be very useful to carry out a systematic study on Jacobi polynomials with general indexes, which can then be directly applied to other applications.

In physics, anomalous diffusion phenomena are modeled using fractional derivatives, such that the particles sawing differently than the classical Brownian motion model (42), they follow Lévy stable motion (22). The Fokker-Planck equations, Reaction-diffusion equations, diffusion advection equations, and Kinetic equations of the diffusion can be applications of the phenomenon of anomalous diffusion. The fractional advection-dispersion equation (FADE), also known as the fractional kinetic equation (11), was shown to be an extension of the continuous-time random walk model. In groundwater hydrology FADE is utilized to model the transport of passive tracers carried by fluid flow in a porous medium (36).

Considerable numerical methods for solving FADE have been proposed. Meerschaert et al. (26) developed practical numerical methods to solve the one-dimensional space FADE with variable coefficients on a finite domain. Roop (19) proposed the finite-element method to approximate FADE in two spatial dimensions. Liu et al. (36) proposed an approximation of the Lévy-Feller advection-dispersion process by employing a random walk and finite difference methods. EI-Sayed et al. (5) used Adomian’s decomposition method to solve an advection-dispersion model with a fractional derivative in the Caputo sense. Golbabai et al. (1) used homotopy perturbation method for finding analatic solution of FADE. Shen et al. (47) used weighted Riesz fractional finite-difference approximation scheme for FADE. Recently, Bhrawy et al. presented the operational matrices for solving the two sided FADE in (2). More recently, Feng et al. (21) based on the weighted and shifted Grünwald difference (WSGD) operators they approximated the Riesz space fractional derivative to find numerical solution of Riesz space FADE.

In this paper, we consider the Lévy-Feller advection-dispersion equation (LFADE) (36), with source term, which obtains form standard advection-dispersion equation by replacing the second-order space derivative with the Riesz-Feller derivative of order \(\alpha\) and skewness \(\theta\), \((|\theta| \leq \min\{\alpha, 2 - \alpha\})\). LFADE takes the following form:

\[
\frac{\partial u(x,t)}{\partial t} = d D_\theta^{\alpha} u(x,t) - e \frac{\partial u(x,t)}{\partial x} + s(x,t), \quad d > 0, \ e \geq 0, \ t \geq 0, \ x \in \mathbb{R},
\]

with initial condition:

\[
u(x,0) = f(x),
\]

where the operator \(D_\theta^{\alpha}\) is the Riesz-Feller fractional derivative of order \(\alpha\) and skewness \(\theta\). The constants \(d, e\) represent the dispersion coefficient and the average fluid velocity respectively and \(s(x,t)\) is the source term. The fundamental solution of (1) and (2) has been derived using the Fourier transform (9) as:

\[
G_\alpha(k,t;\theta) = exp(-ta|k|^\alpha e^{sign(k)\theta \pi/2} + itbk), \quad k \in \mathbb{R}.
\]

Numerical studies of Eq. (1) have been mostly obtained by finite difference methods (FDMs) (see (36)) with their limited accuracy. That because FDMs have a local character, while fractional derivatives are essentially global differential operators. Hence, global
schemes such as spectral methods may be more appropriate for discretizing fractional operators.

Our purpose of this paper is to construct an accurate numerical technique to solve (1) and (2) using Jacobi spectral collocation (JSC) method combined with the trapezoidal rule (Crank-Nicolson method) in the one dimensional domain \( \Omega : a \leq x \leq b \) subject to the Dirichlet boundary conditions as:

\[
u(a, t) = 0, \quad u(b, t) = 0,
\]

More precisely, implementing JSC method to the spatial variable of the fractional advection-dispersion equation and using the boundary conditions reduces the problem to solving a system of ordinary differential equations with respect to the time variable. Then this system will be solved using the trapezoidal rule to reduce the problem to solve system of algebraic equations which are far easier to be solved. This is a generalization of the previous authors’ work in [33].

Indeed, there is a little work in the literature for solving numerically fractional differential equations when the Riesz-Feller operator is used to describe the fractional derivatives (see [12], [22], [23], [34] and [36], most of them used FDM). We used in the previous work [33] Chebyshev-Legendre collocation method with first-order Euler method for solving Lévy-Feller diffusion equation. To the best of our knowledge, there is no paper used spectral method for solving Lévy-Feller advection-dispersion equation, and this motivated our interest in such method.

This article is organized as follows: In the following section, we will write down some definitions on fractional calculus and give some relevant properties of Jacobi polynomials. In Section 3, we suggest and prove an explicit formula corresponding to Riesz-Feller fractional derivative of Jacobi polynomials. In Section 4 we applied JSC method to solve (1, 2, 4) on \( \Omega \) and change them to a system of ordinary differential equations which will be solved using the trapezoidal rule. Section 5, reports some numerical results to show the accuracy and the applicability of the proposed method. Finally, in Section 6 some conclusions are given.

2 Definitions and fundamentals

Here, we introduce some necessary definitions and mathematical preliminaries of the fractional derivative theory.

2.1 Some properties of fractional calculus

**Definition 2.1.** For \( 0 < \alpha < 2, \alpha \neq 1 \) and \( |\theta| \leq \min\{\alpha, 2-\alpha\} \), the Riesz-Feller fractional operator \( D_0^\alpha \) represents in the following form (see e.g. [6], [12], [22], [23])

\[
D_0^\alpha f(x) = -(c_+ D_+^\alpha + c_- D_-^\alpha) f(x),
\]

where the coefficients \( c_\pm \) are given by

\[
c_+ = c_+(\alpha, \theta) = \frac{\sin((\alpha-\theta)\pi/2)}{\sin(\alpha\pi)}, \quad c_- = c_-(\alpha, \theta) = \frac{\sin((\alpha+\theta)\pi/2)}{\sin(\alpha\pi)},
\]

and

\[
(D_+^\alpha f)(x) = (\frac{d}{dx})^n(I_+^{n-\alpha} f)(x), \quad (D_-^\alpha f)(x) = (-\frac{d}{dx})^n(I_-^{n-\alpha} f)(x),
\]

are the left-side and right-side Riemann-Liouville fractional derivatives with \( x \in \mathbb{R} \) and \( \alpha > 0, \ n-1 < \alpha \leq n, \ n = 1, 2. \)
We introduce in this section some basic properties of (shifted) Jacobi polynomials (2.2 Some properties of Jacobi polynomials).

For $\alpha > 0$,  
\begin{align*}
(I^\alpha_+ f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(\xi)}{(x - \xi)^1-\alpha} d\xi, \quad &\quad (I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \frac{f(\xi)}{(\xi - x)^1-\alpha} d\xi. \quad (8)
\end{align*}

For $\alpha = 1$, the representation (5) is not valid and has to be replaced by the formula  
\begin{align*}
D^1_0 f(x) &= [\cos(\theta \pi / 2)D^1_0 - \sin(\theta \pi / 2)D]f(x), \quad (9)
\end{align*}
where the operator $D^1_0$ is related to the Hilbert transform as first noted by Feller in 1952 in his pioneering paper [51]  
\begin{align*}
D^1_0 &= \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{f(\xi)}{x - \xi} d\xi,
\end{align*}
and $D$ refers for the first standard derivative.

For $\alpha = 2$ (surely, $\theta = 0$), $D^2_0 f(x) = \frac{d^2 f(x)}{dx^2}$.

From definition (2.1), we see the Riesz-Feller fractional derivative is a linear combination of left-side and right-side Riemann-Liouville fractional derivatives, so:  
\begin{align*}
D^\alpha_0 (\lambda f(x) + \gamma g(x)) &= \lambda D^\alpha_0 f(x) + \gamma D^\alpha_0 g(x).
\end{align*}
We also recall from (13) a useful property of the left-side and right-side Riemann-Liouville fractional derivatives. Assume that $x \in [a, b]$, $a, b \in \mathbb{R}$ such that $f(a) = f(b) = 0$ then for $0 < p, q \leq 1$ we have  
\begin{align*}
a D^p_x \cdots I^p_x f(x) = f(x), \quad &\quad x D^p_b \cdots I^p_b f(x) = f(x). \quad (10)
\end{align*}

Also, for Riesz-Feller fractional derivative we have:

2.2 Some properties of Jacobi polynomials

We introduce in this section some basic properties of (shifted) Jacobi polynomials ($J^\beta_\gamma k(x)$, $k \geq 0$, $\beta > -1$, $\gamma > -1$) that are most relevant to the proposed spectral collocation approximations ([15], [16], [17], [18]).

Jacobi polynomials may be archived from the recurrence relation:  
\begin{align*}
J_{k+1}^\beta_\gamma(x) &= (a_k^\beta_\gamma x - b_k^\beta_\gamma) J_k^\beta_\gamma(x) - c_k^\beta_\gamma J_{k-1}^\beta_\gamma(x), k = 1, 2, ..., \quad (10) \\
J_0^\beta_\gamma(x) &= 1, \quad J_1^\beta_\gamma(x) = \frac{(\beta + \gamma + 2)x + \beta - \gamma}{2},
\end{align*}
where  
\begin{align*}
a_k^\beta_\gamma &= \frac{(2k + \beta + \gamma + 1)(2k + \beta + \gamma + 2)}{2(k + 1)(k + \beta + \gamma + 1)}, \\
b_k^\beta_\gamma &= \frac{(2k + \beta + \gamma + 1)(\gamma^2 - \beta^2)}{2(k + 1)(k + \beta + \gamma + 1)(2k + \beta + \gamma)}, \\
c_k^\beta_\gamma &= \frac{(k + \beta)(k + \gamma)(2k + \beta + \gamma + 2)}{(k + 1)(k + \beta + \gamma + 1)(2k + \beta + \gamma)}.
\end{align*}
We mention here some of the most important properties of Jacobi polynomials  
\begin{align*}
J_k^\beta_\gamma(-x) &= (-1)^k J_k^\beta_\gamma(x), \quad J_k^\beta_\gamma(1) = \frac{\Gamma(k + \beta + 1)}{k! \Gamma(\beta + 1)}, \quad (11)
\end{align*}


\[
\frac{d^m}{dx^m} J_k^{\beta,\gamma}(x) = \frac{\Gamma(m + k + \beta + \gamma + 1)}{2^m \Gamma(k + \beta + \gamma + 1)} L_{k-m}^{\beta+m,\gamma+m}(x). \tag{12}
\]

A basic property of the Jacobi polynomials is that they are the eigenfunctions of the singular Sturm-Liouville problem:

\[(1 - x^2)\phi''(x) + [\gamma - \beta + (\beta + \gamma + 2)x]\phi'(x) + k(k + \beta + \gamma + 1)\phi(x) = 0.\]

In order to use these polynomials on the interval \([0, L]\), \(L > 0\), we recall here the shifted Jacobi polynomials \(J_{L,k}^{\beta,\gamma}(x) = J_k^{\beta,\gamma}\left(\frac{2x - L}{L}\right)\).

The analytic form of the shifted Jacobi polynomials \(J_{L,k}^{\beta,\gamma}(x)\) of degree \(k\) (\(k\) integer) is given by

\[
J_{L,k}^{\beta,\gamma}(x) = \sum_{i=0}^{k} (-1)^{k-i} \frac{\Gamma(k + \gamma + 1)\Gamma(k + i + \beta + \gamma + 1)}{\Gamma(i + \gamma + 1)\Gamma(k + \beta + \gamma + 1)(k - i)!} L^i x^i, \tag{13}
\]

or,

\[
J_{L,k}^{\beta,\gamma}(x) = \sum_{i=0}^{k} \frac{\Gamma(k + \gamma + 1)\Gamma(k + i + \beta + \gamma + 1)}{\Gamma(i + \beta + 1)\Gamma(k + \beta + \gamma + 1)(k - i)!} L^i x^i, \tag{14}
\]

where

\[
J_{L,k}^{\beta,\gamma}(0) = (-1)^{k} \frac{\Gamma(k + \gamma + 1)}{(\gamma + 1)k!},
\]

and

\[
J_{L,k}^{\beta,\gamma}(L) = \frac{\Gamma(k + \beta + 1)}{\Gamma(\beta + 1)k!}.
\]

The orthogonality condition of shifted Jacobi polynomials is

\[
\int_0^L J_{L,k}^{\beta,\gamma}(x) J_{L,j}^{\beta,\gamma}(x) w^\beta_L(x) dx = h_k,
\]

where

\[
w^\beta_L(x) = x^\gamma (L - x)^\beta \quad \text{and} \quad h_k = \begin{cases} \frac{\Gamma(\beta + 1)\Gamma(k + \beta + 1)}{\Gamma(k + \beta + \gamma + 1)k!}, & k = j, \\ 0, & k \neq j. \end{cases}
\]

The expansion of \(x^i\) and \((x - L)^i\) in terms of shifted Jacobi polynomials are given, respectively, by:

\[
x^i = \frac{(\gamma + 1)i}{(\beta + \gamma + 2)i} \sum_{k=0}^{i} (-1)^k L^i(-i)_k(\beta + \gamma + 2k + 1)(\beta + \gamma + 2k - 1)\frac{1}{(1 + \gamma)_k(\beta + \gamma + i + 2)_k} J_{L,k}^{\beta,\gamma}(x),
\]

\[
(x - L)^i = \frac{(\beta + 1)i}{(\beta + \gamma + 2)i} \sum_{k=0}^{i} L^i(-i)_k(\beta + \gamma + 2k + 1)(\beta + \gamma + 2k - 1)\frac{1}{(1 + \beta)_k(\beta + \gamma + i + 2)_k} J_{L,k}^{\beta,\gamma}(x),
\]

where \((\cdot)_k\) is Pochhammer’s symbol.

Assume \(f(t) \in L^2_w(x^\gamma)_{L,k}(0, L)\), then it can be expanded by means of the shifted Jacobi polynomials as the following form \([44]\):

\[
f(x) = \sum_{j=0}^{\infty} c_j J_{L,j}^{\beta,\gamma}(x), \tag{15}
\]
where
\[ c_j = \frac{1}{h_k} \int_0^L w_L^{\beta,\gamma}(x)f(x)J_{L,j}^{\beta,\gamma}(x)dx, \quad j = 0, 1, 2, \ldots. \]

If we approximate \( f(x) \) by the first \( M \) term, then we can write
\[ f(x) = \sum_{j=0}^{M} c_j J_{L,j}^{\beta,\gamma}(x). \tag{16} \]

We mention here that Chebyshev, Legendre, and ultraspherical polynomials are particular cases of the Jacobi polynomials.

### 3 Riesz-Feller fractional derivative of shifted Jacobi polynomials

For \( 1 < \alpha < 2 \), depending on definition of Riemann-Liouville fractional derivatives on \([0, L]\),
\[ 0D_x^\alpha (x)^k = \frac{\Gamma(k + 1)}{\Gamma(k - p + 1)}(x)^{k-\alpha}, \quad k > -1, \tag{17} \]
\[ xD_L^\alpha (x - L)^k = \frac{(-1)^k \Gamma(k + 1)}{\Gamma(k - \alpha + 1)}(L - x)^{k-\alpha}, \quad k > -1. \tag{18} \]

**Theorem 3.1.** The analytic form of the left-side Riemann-Liouville fractional derivative of the shifted Jacobi polynomial on \([0, L]\) is given by:
\[ 0D_x^\alpha J_L^{\beta,\gamma}(x) = \sum_{k=0}^{j} \sum_{i=0}^{k} \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\beta,\gamma} \times x^{-\alpha} \times J_L^{\beta,\gamma}(x), \tag{19} \]
where,
\[ \Theta_{i,j,k}^{\alpha,\beta,\gamma} = \frac{(-1)^{(i+j+k)}}{\Gamma(1 + \beta + \gamma + j + k)\Gamma(1 + \gamma + j)\Gamma(1 + k)\Gamma(1 + \beta + \gamma + j + k)\Gamma(1 + \gamma + j)\Gamma(1 + k - \alpha)(j - k)!k!}, \]
\[ \Upsilon_{i,k}^{\beta,\gamma} = \frac{(-k)_i(1 + \gamma)_k(2 + \beta + \gamma)i-1(1 + \beta + \gamma + 2i)}{(1 + \gamma)_i(2 + \beta + \gamma)_k(2 + k + \beta + \gamma)_i}. \tag{20} \]

**Proof.** See [4]. The proof was driven depending on linearity of Riemann-Liouville fractional operator, relation (17) and the expansion of \( x^k \) in terms of shifted Jacobi polynomials. \( \square \)

**Theorem 3.2.** The analytic form of the right-side Riemann-Liouville fractional derivative of the shifted Jacobi polynomial on \([0, L]\) is given by:
\[ xD_L^\alpha J_L^{\beta,\gamma}(x) = \sum_{k=0}^{j} \sum_{i=0}^{k} \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\beta,\gamma} \times (L - x)^{-\alpha} \times J_L^{\beta,\gamma}(x), \tag{21} \]
where,
\[ \Theta_{i,j,k}^{\alpha,\beta,\gamma} = \frac{(-1)^{(i+j+k)}}{\Gamma(1 + \beta + \gamma + j + k)\Gamma(1 + \beta + j)\Gamma(1 + k)\Gamma(1 + \beta + \gamma + j + k)\Gamma(1 + \beta + j)\Gamma(1 + k - \alpha)(j - k)!k!}, \]
\[ \Upsilon_{i,k}^{\beta,\gamma} = \frac{(-k)_i(1 + \beta + \gamma)_k(2 + \beta + \gamma)i-1(1 + \beta + \gamma + 2i)}{(1 + \beta)_i(2 + \beta + \gamma)_k(2 + k + \beta + \gamma)_i}. \tag{22} \]
Proof. See [4]. The proof was driven depending on linearity of Riemann-Liouville fractional operator, relation (18) and the expansion of \((x - L)^k\) in terms of shifted Jacobi polynomials.

**Theorem 3.3.** The analytic form of the Riesz-Feller fractional derivative of the shifted Jacobi polynomial on \([0, L]\) is given by:

\[
D_0^\alpha J_{\beta,\gamma}^{\beta,\gamma}(x) = -(c_+ D_x^\alpha J_{\beta,\gamma}^{\beta,\gamma}(x) + c_- D_L^\alpha J_{\beta,\gamma}^{\beta,\gamma}(x)),
\]

\[
= -(c_+ \sum_{k=0}^j \sum_{i=0}^k \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x) \\
+ c_- \sum_{k=0}^j \sum_{i=0}^k \Theta_{j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times (L - x)^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x)),
\]

\[
= - \sum_{k=0}^j \sum_{i=0}^k \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} + c_- \Theta_{j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times (L - x)^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x),
\]

\[
= \sum_{k=0}^j \sum_{i=0}^k \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x). \tag{23}
\]

Where

\[
\Psi_{i,j,k}^{\alpha,\beta,\gamma} = -(c_+ \Theta_{i,j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times x^{-\alpha} + c_- \Theta_{j,k}^{\alpha,\beta,\gamma} \times \Upsilon_{i,k}^{\alpha,\beta,\gamma} \times (L - x)^{-\alpha} \times J_{L,j}^{\beta,\gamma}(x). \tag{24}
\]

4 Proceedings of solution for the Lévy-Feller advection-dispersion equation

In this section, we will introduce numerical algorithm for approximating the solution of the following Lévy-Feller advection-dispersion equation:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= dD_0^\alpha u(x,t) - e \frac{\partial u(x,t)}{\partial x} + s(x,t), \quad t > 0, \quad 0 < x < L, \quad 1 < \alpha \leq 2, \\
u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\
u(x,0) &= f(x), \quad 0 \leq x \leq L,
\end{align*}
\]

assuming \(u(x,t) = 0\) for \(x \in \mathbb{R}\ \setminus [0, L]\).

In order to use Jacobi spectral collocation method, we approximate \(u(x,t)\) as:

\[
u_m(x,t) = \sum_{j=0}^m u_j(t)J_{L,j}^{\beta,\gamma}(x). \tag{26}
\]

Depending on properties of Riesz-Feller fractional derivatives we can write

\[
D_0^\alpha u_m(x,t) = \sum_{j=0}^m u_j(t)D_0^\alpha J_{L,j}^{\beta,\gamma}(x),
\]

\[
= \sum_{j=0}^m \sum_{k=0}^j \sum_{i=0}^k \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x),
\]

\[
= \sum_{j=0}^m \sum_{k=0}^j \sum_{i=0}^k u_j(t) \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x).
\]
So Eq. (25) take the following form:

\[
\sum_{j=0}^{m} \frac{du_j(t)}{dt} J_{L,j}^{\beta,\gamma}(x) = d \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{i=0}^{k} u_j(t) \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \times J_{L,j}^{\beta,\gamma}(x) - e \sum_{j=0}^{m} u_j(t) \frac{d}{dx} J_{L,j}^{\beta,\gamma}(x) + s(x,t).
\]

We collocate Eq. (27) at \((m - 1)\) points \(x_q, \ q = 1, 2, ..., m - 1\) \((a < x_q < b)\) as follows:

\[
\sum_{j=0}^{m} \frac{du_j(t)}{dt} J_{L,j}^{\beta,\gamma}(x_q) = d \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{i=0}^{k} u_j(t) \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \big|_{x=x_q} \times J_{L,j}^{\beta,\gamma}(x_q) - e \sum_{j=0}^{m} u_j(t) \frac{d}{dx} J_{L,j}^{\beta,\gamma}(x_q) \big|_{x=x_q} + s(x_q,t).
\]

Substituting Eq. (26) in the initial condition gives us the constants \(u_i\) in the initial case at \(t = 0\) and substituting by the same equation in the boundary conditions will give two equations as follows:

\[
\sum_{j=0}^{m} (-1)^j \frac{\Gamma(j + \gamma + 1)}{\Gamma(\gamma + 1)j!} u_j(t) = 0, \quad \sum_{j=0}^{m} \frac{\Gamma(j + \beta + 1)}{\Gamma(\beta + 1)j!} u_j(t) = 0,
\]

Equations (28) and (29) constitute system of \((m + 1)\) ordinary differential equations in the unknown \(u_j, \ j = 0, 1, ..., m\). This system will be solved using the trapezoidal rule (which is implicit, second-order and stable method) as described in the following:

Let \(0 < t_n < T_{final}\) and suppose \(\Delta t = \frac{T_{final}}{N}\), \(t_n = n\Delta t\), for \(n = 0, 1, 2, ..., N\), then we have the following algebraic system:

\[
\begin{align*}
\sum_{j=0}^{m} (-1)^j \frac{\Gamma(j + \gamma + 1)}{\Gamma(\gamma + 1)j!} u_j^n &= 0, \\
\sum_{j=0}^{m} \frac{u_j^n - u_j^{n-1}}{\Delta t} J_{L,j}^{\beta,\gamma}(x_q) &= \frac{1}{2} \left[ \left( d \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{i=0}^{k} u_j^n \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \big|_{x=x_q} \times J_{L,j}^{\beta,\gamma}(x_q) \right. \\
&\left. - e \sum_{j=0}^{m} u_j^n \frac{d}{dx} J_{L,j}^{\beta,\gamma}(x) \big|_{x=x_q} + s_q^n \right) \\
&\quad + \left( d \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{i=0}^{k} u_j^{n-1} \times \Psi_{i,j,k}^{\alpha,\beta,\gamma} \big|_{x=x_q} \times J_{L,j}^{\beta,\gamma}(x_q) \right. \\
&\left. - e \sum_{j=0}^{m} u_j^{n-1} \frac{d}{dx} J_{L,j}^{\beta,\gamma}(x) \big|_{x=x_q} + s_q^{n-1} \right) \right] \quad q = 1, 2, ..., m - 1, \\
\sum_{i=0}^{m} \frac{\Gamma(j + \beta + 1)}{\Gamma(\beta + 1)j!} u_j^n &= 0,
\end{align*}
\]

with the initial conditions:

\[
\sum_{j=0}^{m} u_j^n J_{L,j,q}^{\beta,\gamma} = f_q, \quad q = 0, 1, 2, ..., m,
\]

where \(u_j^n = u_j(t_n), \ J_{L,j,q}^{\beta,\gamma} = J_{L,j}^{\beta,\gamma}(x_q), \ s_q^n = s(x_q,t_n)\) and \(f_q = f(x_q)\).

System (30) can be written in a matrix form as the following:

\[
(J_1 - A)U^n = (J_0 + A)U^{n-1} + \frac{1}{2} \Delta t (S^n + S^{n-1}), \quad (31)
\]
Some test examples are introduced in this section to illustrate the accuracy of the presented convergence and efficiency of the Jacobi spectral collocation method.

Substitute the computed coefficients such that:

\[
U^n = (u_0^n, u_1^n, ..., u_m^n)^T, \
S^n = (0, s_2^n, s_3^n, ..., s_{m-1}^n, 0)^T,
\]

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
\Omega_{0,1} & \Omega_{1,1} & \Omega_{2,1} & \Omega_{3,1} & \cdots & \Omega_{m,1} \\
\Omega_{0,2} & \Omega_{1,2} & \Omega_{2,2} & \Omega_{3,2} & \cdots & \Omega_{m,2} \\
\Omega_{0,3} & \Omega_{1,3} & \Omega_{2,3} & \Omega_{3,3} & \cdots & \Omega_{m,3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Omega_{0,m-1} & \Omega_{1,m-1} & \Omega_{2,m-1} & \Omega_{3,m-1} & \cdots & \Omega_{m,m-1} \\
0 & 0 & 0 & 0 & \cdots & 0 
\end{pmatrix},
\]

\[
J_1 = \begin{pmatrix}
\Gamma(\gamma+1)/\Gamma(1+\gamma) & -\Gamma(1+\gamma+1)/\Gamma(1+\gamma+1) & \Gamma(2+\gamma+1)/\Gamma(1+\gamma+1) & \cdots & (-1)^m \Gamma(m+\gamma+1)/\Gamma(1+\gamma+m)! \\
J^{1,1}_L & J^{1,2}_L & J^{1,3}_L & \cdots & J^{1,m}_L \\
J^{2,1}_L & J^{2,2}_L & J^{2,3}_L & \cdots & J^{2,m}_L \\
J^{3,1}_L & J^{3,2}_L & J^{3,3}_L & \cdots & J^{3,m}_L \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
J^{m,1}_L & J^{m,2}_L & J^{m,3}_L & \cdots & J^{m,m}_L 
\end{pmatrix},
\]

\[
J_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
J^{1,1}_L & J^{1,1}_L & J^{1,1}_L & J^{1,1}_L & \cdots & J^{1,1}_L \\
J^{2,1}_L & J^{2,1}_L & J^{2,1}_L & J^{2,1}_L & \cdots & J^{2,1}_L \\
J^{3,1}_L & J^{3,1}_L & J^{3,1}_L & J^{3,1}_L & \cdots & J^{3,1}_L \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
J^{m,1}_L & J^{m,1}_L & J^{m,1}_L & J^{m,1}_L & \cdots & J^{m,1}_L \\
0 & 0 & 0 & 0 & \cdots & 0 
\end{pmatrix},
\]

and

\[
\Omega_{j,q} = \frac{1}{2}\langle \Delta t d \sum_{k=0}^{j} \sum_{i=0}^{k} \Psi_{i,j,k}^{\alpha,\beta,\gamma} |_{x=x_q} \times J^{\beta,\gamma}_{L,j,q} + \Delta t e \frac{d}{dx} J^{\beta,\gamma}_{L,j,q} |_{x=x_q} \rangle,
\]

\[
i = 0, 1, 2, ..., m, \quad j = 1, 2, ..., m - 1, \quad n = 1, 2, ..., N.
\]

Substitute the computed coefficients \(u_j\), \(j = 0, 1, 2, ..., m\), and the Jacobi polynomials in Eq. (20) give us the approximation solutions \(u\) of the proposed problem.

In this work we use the Jacobi Gauss-Lobatto points which are useful for the stability, convergence and efficiency of the Jacobi spectral collocation method.

### 5 Numerical simulations

Some test examples are introduced in this section to illustrate the accuracy of the presented method.
Example 1. [36] We consider the following Lévy-Feller advection-dispersion equation in a bounded domain:

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = dD_\alpha^\theta u(x,t) - e\frac{\partial u(x,t)}{\partial x}, & t > 0, \quad 0 < x < \pi, \quad 1 < \alpha \leq 2, \\
u(0,t) = 0, & t > 0, \\
u(x,0) = \sin(x), & 0 \leq x \leq \pi.
\end{cases}
\]

(32)

Let \( d = 1.5, \ e = 1, \ \alpha = 1.7, \ \theta = 0.3, \ T_{final} = 0.3 \), table (1) lists the numerical results computed by explicit finite difference approximation (EFDA) in [36] and the presented scheme in this paper JSC for problem (1).

Let \( d = 1.5, \ e = 1, \ \alpha = 1.7, \ T_{final} = 0.4 \) and \( \Delta t = 0.008 \), figure (1) shows the obtained numerical results by means of the presented scheme JSC (\( m = 5 \)) for different values of \( \theta \), which indicates the skewness.

When \( \alpha = 2, \ d = 1, \ e = 0, \) the exact solution of example (1) is \( u(x,t) = \sin(x)e^{-t}. \)

Let us consider \( T_{final} = 3 \) and \( \Delta t = 0.05. \) Figure (2) shows the exact solution and the obtained numerical results by means of the presented scheme JSC for example (1) when \( m = 7 \) and shows that the EFDA [36] is divergent.

Table 1: Comparison of the numerical results calculated by EFDA [36] when \( h = \pi/100 \) and by the presented scheme JSC when \( m = 5, \) and \( \beta = \gamma = 0 \) for example (1), where \( \alpha = 1.7, \ \theta = 0.3 \) and \( t = 0.3. \)

| \((x,0.3)\) | EFDA in [36] | Present method JSC |
|-----------|-------------|---------------------|
| 0.3142    | 0.23041     | 0.21208             |
| 0.6283    | 0.40603     | 0.38590             |
| 0.9425    | 0.54876     | 0.51814             |
| 1.2566    | 0.64661     | 0.60546             |
| 1.5708    | 0.68848     | 0.64455             |
| 1.8850    | 0.66770     | 0.63208             |
| 2.1991    | 0.58292     | 0.56471             |
| 2.5133    | 0.43764     | 0.43913             |
| 2.8274    | 0.23952     | 0.25200             |

Example 2. In this example we consider the following Lévy-Feller advection-dispersion equation:

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = dD_\alpha^\theta u(x,t) - e\frac{\partial u(x,t)}{\partial x} + s(x,t), & t > 0, \quad 0 < x < 1, \\
u(0,t) = 0, & t > 0, \\
u(x,0) = x(1-x), & 0 \leq x \leq 1,
\end{cases}
\]

(33)

where

\[
d = \Gamma(3 - \alpha), \quad e = 1,
\]

\[
s(x,t) = \left( \frac{2 - \alpha}{\sin(\alpha \pi)} \right) \left( \sin\left(\frac{(\alpha - \theta)\pi}{2}\right)x^{1-\alpha} + \sin\left(\frac{(\alpha + \theta)\pi}{2}\right)(1-x)^{1-\alpha} \right)
\]

\[
- \frac{2}{\sin(\alpha \pi)} \left( \sin\left(\frac{(\alpha - \theta)\pi}{2}\right)x^{2-\alpha} + \sin\left(\frac{(\alpha + \theta)\pi}{2}\right)(1-x)^{2-\alpha} \right) + \frac{3}{2}\frac{x^2 - 7x + 1}{2}e^{-\frac{3}{2}t},
\]

and the exact solution for this equation, when \( 1 < \alpha \leq 2, \) is:

\[
u(x,t) = x(1-x)e^{-\frac{3}{2}t}.
\]
Figure 1: Comparison between the approximate solution using the proposed method JSC for example (1) at $t = 0.4$ with $\alpha = 1.7$ and different values of $\theta$.

Figure 2: Shows the exact and numerical solution using EFDA [36] and the approximate solution using the proposed method JSC for example (1) at $t = 3$ with $\alpha = 2$.

The weighted difference scheme (WDS) which was presented in [23] is used here to study...
Table 2: Comparison of the errors calculated by JSC \((\beta = \gamma = 0.5)\) \((E_1)\) and by WDS \([23]\) \((E_2)\) with \(\sigma = 0.5\) for example \((2)\) at \(t = 0.5\).

| \(x\) | \(m = 3\) | \(m = 6\) | \(m = 12\) | \(h = 0.1\) | \(h = 0.05\) | \(h = 0.025\) |
|------|----------|----------|----------|----------|----------|----------|
| 0    | 1.9827e-05 | 1.5831e-06 | 1.2334e-06 | 1.0946e-02 | 9.5111e-03 | 8.3772e-03 |
| 0.1  | 2.9278e-05 | 6.1593e-06 | 1.5369e-06 | 5.2330e-02 | 1.4747e-02 | 1.4856e-02 |
| 0.2  | 3.0590e-05 | 9.4183e-06 | 9.5057e-07 | 2.1783e-02 | 1.7541e-02 | 1.9247e-02 |
| 0.3  | 2.6003e-05 | 1.0927e-05 | 6.1780e-06 | 5.7803e-02 | 1.7939e-02 | 2.1571e-02 |
| 0.4  | 1.7757e-05 | 1.1467e-05 | 8.1340e-06 | 2.7705e-02 | 1.6039e-02 | 2.1855e-02 |
| 0.5  | 8.0902e-06 | 3.1435e-06 | 1.2858e-06 | 3.8135e-02 | 1.2109e-02 | 2.0173e-02 |
| 0.6  | 7.5812e-06 | 9.0309e-07 | 9.0856e-07 | 1.3635e-02 | 6.6847e-03 | 1.6691e-02 |
| 0.7  | 6.5487e-06 | 2.1818e-06 | 8.6676e-07 | 9.1295e-03 | 7.4279e-04 | 1.1747e-02 |
| 0.8  | 7.0424e-06 | 2.3546e-06 | 1.7558e-06 | 3.5132e-03 | 3.7610e-03 | 6.0224e-03 |
| 1    | 0        | 0        | 0        | 0        | 0        | 0         |

numerically example 2, with weight factor \(\sigma = 0\), 0.5, 1, and the presented scheme JSC also is used. The difference between the approximate solution and the exact solution (absolute error) is given by:

\[ E_1(x, t) = |u_{exact}(x, t) - u_{JSC}(x, t)|, \quad E_2(x, t) = |u_{exact}(x, t) - u_{WDS}(x, t)|, \]

where \(E_1\) and \(E_2\) are the errors of the presented scheme JSC and of the WDS \([23]\) respectively. Moreover, the maximum absolute errors are given by

\[ M_1 = \max\{E_1(x, t) : a \leq x \leq b, \quad 0 \leq t \leq T_{final}\}, \]

\[ M_2 = \max\{E_2(x, t) : a \leq x \leq b, \quad 0 \leq t \leq T_{final}\}, \]

where \(M_1\), \(M_2\) are the maximum absolute errors of JSC and WDS \([23]\), respectively.

In order to show that JSC is more accurate than WDS \([23]\), in table (2), for \(T_{final} = 0.5\) and \(\Delta t = 0.01\), for problem (2), we compare the errors \(E_1(x, 0.5)\) with \(E_2(x, 0.5)\) when \(\alpha = 1.4\), \(\theta = -0.5\) for various values of \(m\) and \(h\).

In table (3), for \(T_{final} = 1\) and \(\Delta t = 0.005\), for problem (2), we compare the maximum errors \(M_1\) when \(m = 3\) with \(M_2\) when \(h = 0.05\) for various values of \(\alpha\) and \(\theta\).

Figure (3) shows the exact solution for Ex.(2) when \(T_{final} = 2\) and the errors of using JSC \((m = 3)\) and WDS \([23]\) where \(h = 0.05\), \(\sigma = 0\) and \(\Delta t = 0.002\).

Also, Taking \(T_{final} = 5\) and \(\Delta t = 0.02\), figure (4) shows the exact solution for Ex.(2) at \(t = 5\) and the obtained numerical results by means of the presented scheme JSC when \(m = 3\) and by the WDS \((\sigma = 0)\) \([23]\) when \(h = 0.1\).
Table 3: Comparison of the maximum errors calculated by JSC ($\beta = \gamma = 0$) ($M_1$) and by WDS [23] ($M_2$) for example (2)

| $\alpha$, $\theta$ | $M_1$ | $M_2$ |
|---------------------|-------|-------|
| $\alpha = 1.8$, $\theta = 0.1$ | 1.0995e-05 | divergent | 1.0064e-02 | 9.4012e-03 |
| $\alpha = 1.6$, $\theta = 0.1$ | 4.5779e-05 | 2.0452e-02 | 2.0031e-02 | 1.9644e-02 |
| $\alpha = 1.6$, $\theta = 0.3$ | 8.9812e-05 | 2.2966e-02 | 2.2429e-02 | 2.1961e-02 |
| $\alpha = 1.4$, $\theta = 0.3$ | 7.6254e-05 | 3.9459e-02 | 3.8952e-02 | 1.2893e-02 |
| $\alpha = 1.4$, $\theta = 0.5$ | 6.4506e-05 | 4.4378e-02 | 4.3804e-02 | 4.3232e-02 |
| $\alpha = 1.2$, $\theta = 0.5$ | 5.2149e-05 | 6.3144e-02 | 6.2514e-02 | 6.1884e-02 |
| $\alpha = 1.2$, $\theta = -0.5$ | 1.3202e-05 | divergent | divergent | divergent |
| $\alpha = 1.1$, $\theta = -0.5$ | 4.1385e-05 | divergent | divergent | divergent |

Figure 3: Comparison between the errors obtain by JSC ($\beta = \gamma = -0.5$) and by WDS [23] for example (2) when $T_{final} = 2$, $\alpha = 1.7$, and $\theta = 0.2$.

6 Conclusion

An accurate numerical method is constructed to approximate the solutions of the Lévy-Feller advection-dispersion equation. This algorithm is based on the Jacobi collocation method in combination with the trapezoidal rule to create a system of algebraic equations of the unknown coefficients of the spectral expansion. One of the main advantages of the
Figure 4: Comparison between, the numerical solution using JSC and by WDS [23] for example (2) at $t = 5$ with $\alpha = 1.9$, $\theta = 0$.

Presented algorithms is that the availability for application on fractional differential equations, inclusive the case of the fractional derivative is defined in the Riesz-(Feller) sense. Another advantage of the proposed technique is that the high accurate approximate solutions are achieved by using a few number of terms of the suggested expansion. Comparisons between our approximate solutions of the problems with its exact solutions and with the approximate solutions achieved by other methods were introduced to highlight the validity and the accuracy of the proposed scheme. Summarizing, when we used the proposed method for solving both Lévy-Feller diffusion equation and Lévy-Feller advection-dispersion equation we found that this method is more efficient than finite difference method which was used in [12] and [36] for solving these two equations respectively.

References

[1] A. Golbabai, K. Sayevand, Analytical modelling of fractional advection-dispersion equation defined in a bounded space domain, Math. Comput. Modelling, 53, 1708-1718, (2011).

[2] A. H. Bhrawy, M. A. Zaky, J. T. Machado, Efficient Legendre spectral tau algorithm for solving the two-sided space-time Caputo fractional advection-dispersion equation, Journal of Vibration and Control 1-16, (2015), doi: 10.1177/1077546314566835.

[3] A. H. Bhrawy, M. A. Zaky, Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations, Computers and Mathematics with Applications 73 (2017), 1100-1117.

[4] A. H. Bhrawy, M. A Zaky, An improved collocation method for multi-dimensional spacetime variable-order fractional Schrödinger equations, Applied Numerical Mathematics, 111, (2017), 197218, doi: http://dx.doi.org/10.1016/j.apnum.2016.09.009.
[5] A. M. A. EI-Sayed, S. H. Behiry, W. E. Raslan, Adomian’s decomposition method for solving an intermediate fractional advection-dispersion equation. Comput. Math. Appl., 59, 1759-1765, (2010).

[6] B. Al-Saqabi, L. Boyadjiev, Yu. Luchko, Comments on employing the Riesz-Feller derivative in the Schrödinger equation. The European Physical Journal Special Topics, 222, 1779-1794, (2013).

[7] D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, The fractional-order governing equation of Lévy motion. Water Resour Res, 36, 1413-1423, (2000).

[8] D. A. Kopriva, Implementing Spectral Methods for Partial Differential Equations, Algorithms for Scientists and Engineers, (2009).

[9] F. Huang, F. Liu, The fundamental solution of the space-time fractional advection-dispersion equation, J. Appl. Math. Comput. 18 (1-2), 339-350, (2005).

[10] F. Zeng, F. Liu, C. Li, K. Burrage, I. Turner, V. Anh, A Crank-Nicolson adi spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. Siam J. Numer. Anal. Vol. 52, No. 6, 2599-2622, (2014).

[11] G. Szegő, Orthogonal Polynomials, 4th ed., American Mathematical Society, Providence, R.I., American Mathematical Society, Colloquium Publications, Vol XXIII, (1975).

[12] H. Zhang, F. Liu, V. Anh, Numerical approximation of Lévy-Feller diffusion equation and its probability interpretation. Journal of Computational and Applied Mathematics, 206, 1098-1115, (2007).

[13] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, (1999).

[14] J. Shen, T. Tang, Li-Lian Wang, Spectral Methods, Algorithms, Analysis and Applications. Springer-Verlag Berlin Heidelberg (2011).

[15] T.S. Chihara, An Introduction to Orthogonal Polynomials, Math. Appl., vol. 13, Gordon and Breach Science Publishers, New York, (1978).

[16] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2010).

[17] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, vol. 98 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, (2005).

[18] A. F. Nikiforov, S. K. Suslov, V. B. Uvarov, Classical orthogonal polynomials of a discrete variable, Springer Series in Computational Physics, Springer-Verlag, Berlin, (1991).

[19] J. P. Roop, Computational aspects of FEM approximation of fractional advection-dispersion equations on bounded domains in $R^2$. J. Comput. Appl. Math., 193, 243-268, (2006).

[20] L. Trefethen, Spectral Methods in MATLAB, Software, Environments, and Tools, vol 10. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2000).
[21] L. B. Feng, P. Zhuang, F. Liu, I. Turner, J. Li, High-order numerical methods for the Riesz space fractional advection-dispersion equations, Computers and Mathematics with Applications, (2016), http://dx.doi.org/10.1016/j.camwa.2016.01.015.

[22] M. Ciesielski, J. Leszczyński, Numerical solutions to boundary value problem for anomalous diffusion equation with Riesz-Feller fractional operator. Journal of Theoretical and Applied Mechanics, 44, 2, 393-403, (2006).

[23] M. Ciesielski, J. Leszczyński, Numerical treatment of an initial-boundary value problem for fractional partial differential equations, Signal Processing, 86, 10, 2503-3094, (2006).

[24] M. M. Khader, N. H. Sweilam, On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method, Applied Mathematical Modelling, 37, 9819-9828, (2013).

[25] M. M. Khader, N. H. Sweilam and A.M.S. Mahdy, An Efficient Numerical Method for Solving the Fractional Diffusion Equation, Journal of Applied Mathematics and Bioinformatics, vol.1, no.2, 1-12, (2011).

[26] M. M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, J. Comput. Appl. Math. 172, 65-77, (2004).

[27] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high-frequency financial data, An empirical study. Physica A, 314, 749-755, (2002).

[28] M. R. Eslahchi, M. Dehghan, M. Parvizi, Application of the collocation method for solving nonlinear fractional integro-differential equations. J. Comput. Appl. Math. 257, 105-128, (2014).

[29] N. H. Sweilam, M. M. Khader, A Chebyshev pseudo-spectral method for solving fractional order integro-differential equations. ANZIAM 51, 464-475, (2010).

[30] N. H. Sweilam, M. M. Khader, H. M. Almarwn, Numerical studies for the variable order nonlinear fractional wave equation, FCAA., 15, 4, (2012).

[31] N. H. Sweilam, A. M. Nagy, Adel A. El-Sayed, Second kind shifted Chebyshev polynomials for solving space fractional order diffusion equation, Chaos. Solitons Fract., 73, 141-147, (2015).

[32] N. H. Sweilam, A. M. Nagy, Adel A. El-Sayed, On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind, Journal of King Saud University - Science (2015), http://dx.doi.org/10.1016/j.jksus.2015.05.002.

[33] N. H. Sweilam, M. M. Abou Hasan, Numerical approximation of Lévy-Feller fractional diffusion equation via Chebyshev-Legendre collocation method, Eur. Phys. J. Plus (2016) 131: 251, http://dx.doi.org/10.1140/epjp/i2016-16251-y.

[34] N. H. Sweilam, T. Assiri, M. M. Abou Hasan, Numerical solutions of nonlinear fractional Schrödinger equations using nonstandard discretizations, Numer Methods Partial Differential Equations, 1-21, (2016), http://dx.doi.org/10.1002/num.22117.

[35] N. H. Tuan, D. N. D. Hai, L. D. Long, V. T. Nguyen, M. Kirane, On a Riesz-Feller space fractional backward diffusion problem with a nonlinear source, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.003.
[36] Q. Liu, F. Liu, I. Turner, V. Anh, Approximation of the Lévy-Feller advection-dispersion process by random walk and finite difference method, Journal of Computational Physics, 222, 57-70, (2007).

[37] R. Askey, J. Fitch, Integral representations for Jacobi polynomials and some applications, Journal Of Mathematical Analysis And Applications, 26, 411-437 (1969).

[38] R. Askey, Inequalities via fractional integration. In: Fractional Calculus and Its Applications. Lecture Notes in Mathematics, 457. Berlin: Springer-Verlag, 106-115, (1975).

[39] R. C. Koeller, Application of fractional calculus to the theory of viscoelasticity. J Appl Mech, 51, 229-307, (1984).

[40] R. Herrmann, Fractional Calculus, An Introduction For Physicists. World Scientific Publishing Co. Pte. Ltd., (2011).

[41] R. Hilfer, Application of Fractional Calculus in Physics. Singapore: World Scientific, (2000).

[42] R. Metzler and J. Klafter, The random walk’s guide to anomalous diffusion, A fractional dynamics approach, Phys. Rep., 339, 177, (2000).

[43] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, Comput. Math. Appl., 57, 3, 483-487, (2009).

[44] E. Godoy, R. Ronveaux, A. Zarzo, I. Area. Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case. J. Comput. Appl. Math., 84, 257-275 (1997).

[45] S. Esmaeili, R. Garrappa, A pseudo-spectral scheme for the approximate solution of a time-fractional diffusion equation, International Journal of Computer Mathematics, Vol. 92, No. 5, 980-994, (2015), http://dx.doi.org/10.1080/00207160.2014.915962

[46] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivative: Theory and Applications. New York: Gordon and Breach, (1993).

[47] S. Shen, F. Liu, V. Anh, I. Turner, J. Chen, A novel numerical approximation for the space fractional advection-dispersion equation, IMA Journal of Applied Mathematics, 79 (2014) 431-444.

[48] V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation using Adomian decomposition, Appl. Math. Comput., 189, 1, 541-548, (2007).

[49] V.J . Ervin, J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differential Equations, 22, 558-576, (2006).

[50] W. W. Bell, Special Functions for Scientists and Engineers. Great Britain, Butler and Tanner Ltd, Frome and London, (1968).

[51] W. Feller, On a generalization of Marcel Riesz’ potentials and the semi-groups generated by them, Meddelanden Lunds Universitets Matematiska Seminarium (Comm. Sém. Mathém. Université de Lund), Tome suppl. dédié à M. Riesz, Lund, 73, (1952).

[52] W. Sun Don, D. Gottlieb, The Chebyshev-Legendre Method: Implementing Legendre Methods on Chebyshev Points, SIAM Journal on Numerical Analysis, Vol. 31, No. 6, pp. 1519-1534, (1994).
[53] X. Ma, C. Huang, Spectral collocation method for linear fractional integro-differential equations. Appl. Math. Modell., 38, 1434-1448, (2014)

[54] Y. Cenesiz, Y. Keskin, A. Kurnaz, The solution of the Bagley-Torvik equation with the generalized Taylor collocation method, J. Franklin Inst., 347, 2, 452-466, (2010).

[55] Y. Yan, K. Pal, N. J. Ford, Higher order numerical methods for solving fractional differential equations, BIT Numer. Math., 54, 2, 555-584, (2014).