Extended field theories are local

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Abstract

We show that all extended functorial field theories, both topological and nontopological, are local. Precisely, we show that the functor that sends a target geometry $S$ to the smooth $(\infty, d)$-category of field theories with geometric structure $S$ satisfies descent with respect to covers of $S$. We introduce a precise formulation of the smooth $(\infty, d)$-category of bordisms with geometric data, such as Riemannian metrics or geometric string structures, and prove that it satisfies codescent with respect to the target $S$. We apply this result to construct a classifying space for concordance classes of functorial field theories with geometric data, solving a conjecture of Stolz and Teichner about the existence of such a space. We also develop a geometric theory of power operations, following the recent work of Barthel, Berwick-Evans, and Stapleton.

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1 Introduction

A key feature of quantum field theory (QFT) that is usually expected, or demanded, is that it is local in the sense that large scale phenomena are entirely determined by phenomena on smaller scales. All influences should propagate through space (or spacetime) at some finite speed and there is no ‘action at a distance’.

Although the axiomatization of QFTs via functorial field theories (see the original paper by Segal [2004.a] or Atiyah’s paper [1988.b] for the topological case) provides a beautiful framework that captures many necessary ingredients, it has the unfortunate side effect of allowing field theories that are not local. This is almost evident from the axioms, for suppose we are given a bordism Σ (i.e., a d-dimensional smooth manifold with boundary) between two (d − 1)-manifolds Σ₀ and Σ₁ and suppose we have a smooth map f : Σ → X to some target manifold X. To such data, a functorial field theory associates a linear map between Hilbert spaces

\[ Z(Σ,f) : H(Σ₀,f|Σ₀) → H(Σ₁,f|Σ₁) \].

Now in this setting ‘local’ means local with respect to the target manifold X. In applications, Σ should represent the ‘worldline’ of a particle, or ‘worldvolume’ of a string and X is the ambient space in which it propagates. If this field theory were local, we should be able to restrict it to some small region U ⊂ X. Moreover, we should be able to reconstruct all of the field theory from such restrictions, to elements of an open cover of X. Here we have a problem. If the bounding manifolds Σ₀ and Σ₁ have images in X (under f) that cover a large region in X, we have no hope for being able to assemble the field theory from its restriction to a cover \( \{ U_α \} \) of X. Indeed, there is no reason to expect that the maps \( f_i : Σ_i → X, i = 0,1 \), factor through some \( U_α \) and we are unable to cut Σᵢ into smaller pieces so that it factors.

Fortunately, all hope is not lost. This defect of functorial QFTs can be resolved by allowing ourselves to further cut Σᵢ into smaller pieces by codimension 1 submanifolds. This naturally leads to the fully-extended setting of Freed [1992], Baez–Dolan [1995], and Lurie [2009.a], where we have a d-category of bordisms, starting with points as objects, 1-manifolds as morphisms, 2-manifolds as morphisms between morphisms etc. It has long been expected that fully-extended field theories ought to be local, but no actual proof has emerged in the literature. One part of the issue for this is that a rigorous definition (in the topological case) of the fully-extended bordism category has not appeared until Lurie [2009.a] and Calaque–Scheimbauer [2015.b]. Moreover, to talk about the nontopological case, one really needs to parametrize the bordism category over smooth manifolds, and this adds another layer of complexity that has not yet been explored in the literature in the fully extended case (see Stolz and Teichner [2004.b, 2011.a] for the nonextended case).

The main goal of this paper is to provide a rigorous proof that all fully-extended field theories are local. To incorporate a variety of geometric structures on smooth manifolds, such as Riemannian metrics, orientations, spin and string structures, as well as more complicated higher geometric structures such as geometric string structures, we generalize the usual definition of tangential structures, i.e., the space of lifts

\[ M \xrightarrow{τ_M} BGL(d), \]

where \( τ_M \) denotes the tangent bundle and \( Y \) is a space encoding the tangential structure. Our motivation is that, in the geometric setting, one is interested in a variety of structures that do not fit into the above formalism. For example, we can consider Riemannian metrics with constraints on the sectional curvature (e.g., positive, negative, nonpositive, nonnegative).
We adopt an alternative perspective. Tangential structures can be pulled back along open embeddings. They are also local in that structures can be glued from local data. Thus, a tangential structure naturally gives rise to an \( \infty \)-sheaf on the site of \( d \)-dimensional manifolds, with open embeddings between them, and we might as well consider any \( \infty \)-sheaf on this site as a legitimate geometric structure. This perspective is explained in detail in Section 3. The usual notion of a target in physics, for example, in the sigma model, can be encoded as a geometric structure of this type by taking \( S(M) = C^\infty(M, X) \), where \( X \) is the target manifold.

For a geometric structure \( S \), a \( T \)-valued, fully-extended field theory \( F \) with \( S \)-structure is a symmetric monoidal functor \( F : \text{Bord}_d^S \to T \), where \( T \) is some smooth symmetric monoidal \((\infty,d)\)-category. To say that \( F \) is local roughly means that for any open cover of the target \( S \) we can reconstruct \( F \) from its restriction to bordisms \( M \to S \) that factor through some element of the open cover.

Thus, a natural formulation of locality involves descent on a general smooth stack \( S \in \text{Sh}_\infty(O\text{Emb}_d^\infty) \). This leads us to the main result of the paper.

**Theorem 1.0.1.** The smooth \((\infty,d)\)-category of symmetric monoidal functors \( \text{Fun}^\circ(\text{Bord}_d^S, T) \) satisfies descent with respect to the target \( S \), for all smooth symmetric monoidal \((\infty,d)\)-categories \( T \). That is to say, the functor

\[
\text{FFT}_{d,T} : \text{PSh}_\Delta(\text{OEmb}_d) \to \text{Cat}_{\infty,d}, \quad S \mapsto \text{FFT}_{d,T}(S) := \text{Fun}^\circ(\text{Bord}_d^S, T)
\]

is an \( \infty \)-sheaf.

The results of Berwick-Evans, Boavida de Brito, and the second author [2019], see also [2020.d], can be used to construct a classifying space for field theories. More precisely, we can construct (rather explicitly) a space \( B_f \text{FFT}_{d,T}(S) \) such that the following holds.

**Corollary 1.0.2.** For all geometric structures \( S \), we have a bijective correspondence

\[
\begin{align*}
\text{Concordance classes of} & \\
\text{\( d \)-dimensional, \( T \)-valued} & \\
\text{field theories with \( S \)-structure} & \cong [Y, B_f \text{FFT}_{d,T}(S)],
\end{align*}
\]

where on the right we take homotopy classes of maps.

This resolves a long-standing conjecture of Stolz and Teichner [2011.a] that postulates the existence of such a classifying space for extended functorial field theories with any geometry.

In Section 2 we introduce the notion of a smooth, symmetric monoidal \((\infty,d)\)-category. In Section 3, we define geometric structures. In Section 4, we introduce two smooth variants of the bordism category, analogous to the distinction between bicategories and double categories. In Section 5, we develop necessary tools from homotopy theory that will be used in the proof of Theorem 1.0.1. In Section 6, we prove that the various bordism categories we have introduced satisfy codescent, which is the main theorem of the paper. Finally, in Section 7 we establish several applications. In Section 7.2, we construct an explicit classifying space of field theories (in the sense of Corollary 1.0.2). In Section 7.3, we construct a geometric model for power operations, following Barthel, Berwick-Evans, and Stapleton [2020.1].

## 2 Symmetric monoidal smooth \((\infty,n)\)-categories

### 2.1 Homotopy theory and higher categories

Recall the simplex category \( \Delta \), whose objects are nonempty, totally ordered sets \([n] = \{0, \ldots, n\}\), and whose morphisms are order-preserving maps. Recall also Segal’s category \( \Gamma \), whose opposite category has objects which are finite pointed sets \((\ell) = \{*, 1, \ldots, \ell\}\) and basepoint-preserving functions as morphisms. Finally, consider the category \( \text{CartSp} \) whose objects are Cartesian spaces \( \mathbb{R}^n \) and morphisms are smooth functions between them. The basic building blocks for symmetric monoidal, smooth \((\infty,n)\)-categories lie in the threefold product category

\[
\Delta \times \Gamma \times \text{CartSp}.
\]

One should think of each category as capturing an independent nature of the objects we are constructing. For the sake of transparency, we provide some indications of how to think of each piece conceptually.
The category \( \Delta^{\times n} \): Here an object is a multisimplex \( k = ([k_1], [k_2], \ldots, [k_n]) \in \Delta^{\times n} \), which we can think of as indexing a composable chain of morphisms, with \( k_i \) composed morphisms in the \( i \)th direction. We will generally think of such composable chains as a grid. For example, for \( n = 2 \), we have two composition directions. If \( k_1 = k_2 = 4 \), this gives a grid

The face of a single square represents a 2-morphism. The edges represent 1-morphisms and vertices represent objects. These 2-morphisms can be composed in two directions, giving rise to a grid. One can construct a model structure on presheaves on \( \Delta^{\times n} \), whose fibrant objects are \( n \)-fold Segal spaces. We will provide the details of this construction later. For an expository account of \( n \)-fold Segal spaces, see Barwick [2005], Lurie [2009], Barwick–Schommer-Pries [2011], Calaque–Scheimbauer [2015].

The category \( \Gamma \): This category captures the symmetric monoidal structure. An object \( \langle \ell \rangle \in \Gamma \) is simply a finite set \( \{1, 2, \ldots, \ell\} \) and a morphism is just a function in the opposite direction satisfying \( f(\ast) = \ast \). This category encodes the symmetric monoidal structure as follows. Consider a \( \Gamma \)-object \( X : \Gamma^{\text{op}} \to \text{sSet} \). The function \( \phi_{\ell} : \langle \ell \rangle \to \langle 1 \rangle \) sending \( i \) to \( 1 \) for \( 1 \leq i \leq \ell \) gives, by functoriality, a map

\[
X\langle \ell \rangle \to X\langle 1 \rangle.
\]

We also have maps \( \delta_i : \langle \ell \rangle \to \langle 1 \rangle \) that send \( i \) to \( 1 \) and \( j \) to \( 0 \) for \( j \neq i \). They induce a map

\[
\delta_i : X\langle \ell \rangle \to X\langle 1 \rangle^{\times \ell}.
\]

The multiplicative structure itself is given by the following zigzag, where the left leg is a weak equivalence, so can be (formally) inverted:

\[
X\langle 1 \rangle^{\times \ell} \leftarrow X\langle \ell \rangle \xrightarrow{\phi_{\ell}} X\langle 1 \rangle.
\] (2.1.1)

One can think of \( X\langle \ell \rangle \) as the space of \( \ell \)-tuples that can be multiplied. The map \( \phi_{\ell} \) then performs the multiplication. The map \( \delta_i \) extracts the individual components of an \( \ell \)-tuple and is a weak equivalence because any \( \ell \)-tuple can be deformed to an \( \ell \)-tuple that can be multiplied. The elegance of Segal’s method is that it circumvents the need for explicitly keeping track of coherent homotopies in the symmetric monoidal structure, although these are of course still hidden in the above equivalence. For more information, see Segal [1973].

The category \( \text{CartSp} \): This category captures the smooth structure. An object of \( \text{CartSp} \) is an open subset of \( \mathbb{R}^n \) that is diffeomorphic to \( \mathbb{R}^n \) and morphisms are smooth maps. This category is turned into a site by equipping it with the coverage of good open covers, i.e., open covers for which every finite intersection is either empty or diffeomorphic to \( \mathbb{R}^n \). Objects in the functor category \( \text{Fun}(\text{CartSp}^{\text{op}}, \text{D}) \) can be thought of as the category of smoothly parametrized objects of \( \text{D} \) over cartesian spaces. As a basic example, one can consider a smooth manifold \( M \) as a functor \( M : \text{CartSp}^{\text{op}} \to \text{Set} \), by assigning to each cartesian space \( U \) the set of smooth maps \( C^\infty(U, M) \). For us, this category will allow us to form smooth families of symmetric monoidal (\( \infty, n \))-categories. For more information about sheaves on this site, see Fiorenza–Schreiber–Stasheff [2010] or Sati–Schreiber [2020].

Notation 2.1.2. Throughout the remainder of the paper, we will denote simplicial presheaves on some category \( \mathcal{C} \) as \( \mathcal{PSh}_\Delta(\mathcal{C}) \), i.e.,

\[
\mathcal{PSh}_\Delta(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{sSet}).
\]

Ordinary presheaves with values in another category \( \mathcal{D} \) will be denoted

\[
\mathcal{PSh}(\mathcal{C}, \mathcal{D}) := \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D}).
\]
2.2 Smooth \((\infty, n)\)-categories

The target category for our functors will be the category of simplicial sets \(sSet\) and we will present an \((\infty, 1)\)-category of sheaves of symmetric monoidal \((\infty, n)\)-categories by a particular model structure (following Barwick [2005] in the case of \(\Delta^\times n\)). We begin by taking the injective model structure on \(\mathcal{P}Sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp})\). To encode the Segal conditions for \(\Delta\) and \(\Gamma\), completeness conditions for \(\Delta\), and descent conditions for \(\text{CartSp}\) on these objects, we perform a left Bousfield localization of this model category.

**Notation 2.2.1.** Fix some \(n \geq 0\), which corresponds to \(n\) in \((\infty, n)\)-categories. The functor

\[ j_\Gamma : \Gamma \to \mathcal{P}sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp}) \]

is defined as the composition

\[ \Gamma \to \mathcal{P}sh(\Gamma) \to \mathcal{P}sh_\Delta(\Gamma) \to \mathcal{P}sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp}), \]

where the first functor is the Yoneda embedding, the second functor is the inclusion of presheaves of sets into simplicial presheaves, and the third functor is induced by the projection functor \(\Delta^\times n \times \Gamma \times \text{CartSp} \to \Gamma\). Analogously, we define functors

\[ j_{\text{CartSp}} : \text{CartSp} \to \mathcal{P}sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp}) \]

and

\[ j_{\Delta,k} : \Delta \to \mathcal{P}sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp}), \]

where the latter uses the projection \(\Delta^\times n \times \Gamma \times \text{CartSp} \to \Delta\) onto the \(k\)th copy of \(\Delta\), where \(1 \leq k \leq n\).

**Definition 2.2.2.** Fix \(n \geq 0\). We define the following maps, with respect to which we will perform a left Bousfield localization.

(i) (Segal’s special \(\Delta\)-condition.) For \([a], [b] \in \Delta\) and \(1 \leq k \leq n\), the maps

\[ \phi^{a,b,k} : j_{\Delta,k}[a] \sqcup_{j_{\Delta,k}[0]} j_{\Delta,k}[b] \to j_{\Delta,k}[a + b]. \]  \tag{2.2.3} \]

Local objects are precisely Segal’s special \(\Delta\)-objects \(X\), defined by the condition that the map \(X_{a+b} \to X_a \times_{X_0} X_b\) is a weak equivalence.

(ii) (Completeness condition.) For \(1 \leq k \leq n\), the map

\[ x : E \to j_{\Delta,k}[0] \]  \tag{2.2.4} \]

where \(E\) is obtained by evaluating the composition of functors

\[ sSet = \mathcal{P}sh(\Delta) \to \mathcal{P}sh_\Delta(\Delta) \to \mathcal{P}sh_\Delta(\Delta^\times n \times \Gamma \times \text{CartSp}) \]

on the nerve of the groupoid with two objects \(x\) and \(y\) and two nonidentity morphisms \(x \to y\) and \(y \to x\), where the first functor converts presheaves of sets into presheaves of discrete simplicial sets and the second functor is induced by the projection \(\Delta^\times n \times \Gamma \times \text{CartSp} \to \Delta\) onto the \(k\)th copy of \(\Delta\). Local objects with respect to (i) and (ii) are Rezk’s complete Segal objects, defined by Segal’s special \(\Delta\)-condition and the completeness condition, which says that the map \(X_0 \to X_1\) that sends objects to their identity morphisms is a weak equivalence onto the subobject of invertible 1-morphisms.

(iii) (Segal’s special \(\Gamma\)-condition.) For \(\langle k \rangle, \langle \ell \rangle \in \Gamma\), the maps

\[ t^{k,\ell} : j_\Gamma(\langle k \rangle) \sqcup_{j_\Gamma(\langle 0 \rangle)} j_\Gamma(\langle \ell \rangle) \to j_\Gamma(\langle k + \ell \rangle). \]  \tag{2.2.5} \]

Also, the map

\[ \tau : \emptyset \to j_\Gamma(\langle 0 \rangle), \]  \tag{2.2.6} \]

which plays a role roughly analogous to the completion map (it forces the 0th space to be contractible).

Local objects are precisely Segal’s special \(\Gamma\)-objects, for which the maps \(X_{k+\ell} \to X_k \times X_\ell\) and \(X_0 \to *\) are weak equivalences, with the subscripts referring to \(\Gamma\), and all other arguments fixed.
(iv) (Sheaf condition.) For \( \{ U_a : a \in I \} \) a good open cover of \( V \) in \( \mathcal{C}_{\text{artSp}} \), the Čech nerve \( c_U \in \mathcal{P} \mathcal{S} \mathcal{h}_\Delta(\mathcal{C}_{\text{artSp}}) \)
has as its \( m \)-simplices the presheaf of sets

\[
\prod_{\alpha_0, \ldots, \alpha_m : U_{\alpha_0} \cap \cdots \cap U_{\alpha_m} \neq \emptyset} j_{\mathcal{C}_{\text{artSp}}}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_m}).
\]

The \( k \)th face map deletes \( U_{\alpha_k} \), the \( k \)th degeneracy map duplicates \( U_{\alpha_k} \). We have a canonical map

\[
eq \longrightarrow j_{\mathcal{C}_{\text{artSp}}}(V)
\]

(2.2.7)

that in degree \( m \) is induced by the inclusions \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_m} \to V \). The local objects are \( \infty \)-sheaves (alias \( \infty \)-stacks), for which the restriction map

\[
X(V) \to \operatorname{holim}_\alpha X(U_{\alpha_0} \cap \cdots \cap U_{\alpha_m})
\]

is a weak equivalence, where we keep all arguments but \( \mathcal{C}_{\text{artSp}} \) fixed and arbitrary.

**Remark 2.2.8.** Although it may seem as if the nature of the two maps in (iii) are different, both of them arise from inverting maps of the form

\[
j_T(\langle k_1 \rangle) \sqcup j_T(\langle k_2 \rangle) \sqcup \cdots \sqcup j_T(\langle k_m \rangle) \to j_T(\langle k_1 + \cdots + k_m \rangle).
\]

Taking \( m = 0 \) produces the map \( \tau \). Taking \( m = 2 \) produces a map weakly equivalent to the map \( t^k,\ell \) (note that \( j_T(0) \) in the pushout was already forced to be contractible by the map \( \tau \)). In fact, it is clear that we can alternatively localize at maps of the above form, which gives a somewhat more uniform description.

The above conditions should be satisfied for an arbitrary fixed choice of all objects in factors other than the one under consideration. For instance, fixing an arbitrary object in \( \Delta^{\times n} \times \Gamma \) should yield an \( \infty \)-sheaf of simplicial sets on \( \mathcal{C}_{\text{artSp}} \). This idea can be precisely expressed using the notion of a cartesian left Bousfield localization, which can be defined using the same universal property but requiring the left derived functor of the localization functor to preserve finite homotopy products. Even more concretely, we can reduce to the case of ordinary Bousfield localization as in the following definition.

**Definition 2.2.9** (The multiple model structure). We define a model structure

\[
\mathcal{C}_{\infty \text{Cat}^{\otimes, \text{uple}}} := \mathcal{P} \mathcal{S} \mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma \times \mathcal{C}_{\text{artSp}})^{\text{uple}}
\]

by performing the ordinary left Bousfield localization of the model category \( \mathcal{P} \mathcal{S} \mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma \times \mathcal{C}_{\text{artSp}})^{\text{incl}} \) at the morphisms given by taking the product of one of the maps (i), (ii), (iii), or (iv) of Definition 2.2.3 with the identity map of an object in the image of the composition

\[
R \to \mathcal{P} \mathcal{S} \mathcal{h}(R) \to \mathcal{P} \mathcal{S} \mathcal{h}_\Delta(R) \to \mathcal{P} \mathcal{S} \mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma \times \mathcal{C}_{\text{artSp}}),
\]

where \( R \) denotes the product \( \Delta^{\times n} \times \Gamma \times \mathcal{C}_{\text{artSp}} \) with the corresponding factor omitted. For example, one such local morphism for \( n = 1 \) is

\[
\phi^{a,b} \times \text{id}_{\ell} \times \text{id}_U : (j_\Delta(a) \sqcup j_\Delta(b)) \times j_T(\ell) \times j_{\mathcal{C}_{\text{artSp}}}(U) \to j_\Delta[a \times b] \times j_T(\ell) \times j_{\mathcal{C}_{\text{artSp}}}(U).
\]

Locality with respect to such a map forces the simplicial object obtained by fixing the factors corresponding to \( \Gamma \) and \( \mathcal{C}_{\text{artSp}} \) to be a special Segal object.

The existence of this localization follows from the Smith recognition theorem (see Barwick [2007, Theorem 4.7]).

**Remark 2.2.10.** A somewhat more elegant way to phrase the above left Bousfield localization is to perform the enriched left Bousfield localization (Barwick [2007, Theorem 4.46]) with respect to the maps (2.2.3), (2.2.4), (2.2.5), (2.2.6), (2.2.7), (2.2.8). The enrichment in this case is given by the cartesian product and corresponding internal hom. This automatically takes care of fixing all the other objects, and it also guarantees right away that the resulting model structure is cartesian.
2.3 Globular \((\infty, n)\)-categories

The previous section gives the correct smooth variant of \(n\)-fold complete Segal spaces. However, these are not quite a model for \((\infty, n)\)-categories until we impose the globular condition of Barwick–Schommer-Pries [2011.b, Notation 12.1]. For \(n = 2\) this amounts to passing from double categories to bicategories. Recall that double categories have two distinct notions of 1-morphisms: horizontal and vertical. Both can be composed, and 2-cells are squares involving two vertical and two horizontal morphisms. In other words, our model so far describes the \(n\)-fold analog of double categories. In order to get rid of the extra 1-morphisms, we further localize the functor category \(\mathcal{P}sh_{\Delta}(\Delta^{\times n} \times \Gamma \times \mathcal{C}art\mathcal{S}p)_{\text{uple}}\) at the morphisms given by taking the product of a map \((v)\) with the identity map of an object in the image of the composition

\[
\Gamma \times \mathcal{C}art\mathcal{S}p \rightarrow \mathcal{P}sh(\Gamma \times \mathcal{C}art\mathcal{S}p) \rightarrow \mathcal{P}sh_{\Delta}(\Gamma \times \mathcal{C}art\mathcal{S}p) \rightarrow \mathcal{P}sh_{\Delta}(\Delta^{\times n} \times \Gamma \times \mathcal{C}art\mathcal{S}p).
\]

**Definition 2.3.1.** Fix \(n \geq 0\). We define the following maps, which we include in our left Bousfield localization for the globular model structure.

\(v\) (Globular maps.) For an object \(k = ([k_1], \ldots, [k_n]) \in \Delta^{\times n}\), let \(\hat{k}\) be the object with \(j\)th component given by

\[
[\hat{k}_j] = \begin{cases} [0], & \text{if there is } i < j \text{ with } k_i = 0, \\ [k_j], & \text{otherwise}. \end{cases}
\]

There is a canonical map from \(k\) to \(\hat{k}\) given by identities or unique maps to \([0]\) for each index \(j\). The globular maps are defined as

\[
j_{\Delta^{\times n}}(k) \rightarrow j_{\Delta^{\times n}}(\hat{k}). \tag{2.3.2}
\]

Here \(j_{\Delta^{\times n}}\) is the composition

\[
\Delta^{\times n} \rightarrow \mathcal{P}sh(\Delta^{\times n}) \rightarrow \mathcal{P}sh_{\Delta}(\Delta^{\times n}) \rightarrow \mathcal{P}sh_{\Delta}(\Delta^{\times n} \times \Gamma \times \mathcal{C}art\mathcal{S}p).
\]

The local objects are multisimplicial spaces \(X\) such that \(X_0\), which we interpret as an object in \((n-1)\)-fold simplicial spaces, is constant and \(X_k\) is a local object in \((n-1)\)-fold simplicial spaces. For \(n = 2\), the locality condition boils down to forcing the maps \(X_{0,b} \rightarrow X_{0,0}\) to be equivalences. This makes all vertical morphisms identities.

**Definition 2.3.3** (The globular model structure). We define the model structure

\[
C^\infty \mathcal{C}at_{\infty, n}^{\otimes, \text{glob}} := \mathcal{P}sh_{\Delta}(\Delta^{\times n} \times \Gamma \times \mathcal{C}art\mathcal{S}p)_{\text{glob}}
\]

in the same way as in Definition 2.2.9, but with the maps (2.3.2) thrown into the list, with \(R = \Gamma \times \mathcal{C}art\mathcal{S}p\).

The existence of the localization is established in Barwick [2007, Theorem 4.7].

**Remark 2.3.4.** Our results apply to bordism categories which are fibrant objects in the multiple or globular model structure. We use the shorthand notation \(C^\infty \mathcal{C}at_{\infty, n}^{\otimes, \text{glob}}\) to refer to the model category \(C^\infty \mathcal{C}at_{\infty, n}^{\otimes, \text{glob}}\) or \(C^\infty \mathcal{C}at_{\infty, n}^{\otimes, \text{uple}}\), depending on the context.

**Remark 2.3.5.** Without the globular condition (2.3.2), we get a cartesian model structure, so internal homs in the underlying \(\infty\)-category can be computed by deriving the internal hom. One can easily see that including the globular condition yields a noncartesian model structure, so internal homs in the underlying \(\infty\)-category can no longer be computed by deriving the internal hom. However, we can still compute internal homs in the underlying \(\infty\)-category by passing to the Quillen equivalent model category of \(\Theta_n\)-spaces, see Barwick–Schommer-Pries [2011.b, Corollary 14.7].

3 Geometric structures

We now describe how to encode geometric structures on bordisms. As pointed out in the introduction, our treatment generalizes the traditional treatment of tangential structures and is more general. Roughly,
the passage from the traditional approach to our approach is given by taking the sheaf of sections of the homotopy pullback

\[
\begin{array}{ccc}
Y \times_{\text{BGL}(d)} M & \rightarrow & Y \\
\downarrow & & \downarrow \\
M & \rightarrow & \text{BGL}(d),
\end{array}
\]

where \( \tau : M \to \text{BGL}(d) \) is a classifying map for the tangent bundle. There are two subtleties which need to be accounted for. First, we need to encode geometric structures. Hence, we need to replace the classifying space \( \text{BGL}(d) \) with the classifying stack of vector bundles or rank \( d \). Second, the sheaf must live on a larger site than the open subsets of \( M \), since we will need to pullback the structure along all open embeddings. Moreover, we will need all geometric structures to vary in families, parametrized over cartesian spaces. We resolve these issues by working with sheaves on the site of submersions (with \( d \)-dimensional fibers), with fiberwise open embeddings between them.

**Remark 3.0.1.** To make all sites small and to make presheaves valued in sets, rather than proper classes, we require that the underlying set of any manifold is a subset of \( \mathbb{R} \). We do not require any compatibility with the smooth structure on \( \mathbb{R} \).

### 3.1 Sheaves on \( \text{OEmb}_d \)

Our definition resembles that of Ayala–Francis [2012.a], except that we do everything in smooth families.

**Definition 3.1.1.** Let \( \text{OEmb}_d \) be the site with objects submersions \( p : M \to U \), with \( d \)-dimensional fibers and \( U \) an object in \( \text{CartSp} \). Morphisms are smooth bundle maps \( f : M \to N \) that restrict to open embeddings fiberwise. Covering families are given by a collection of morphisms

\[
\begin{array}{ccc}
M_\alpha & \rightarrow & M \\
\downarrow & & \downarrow \\
U_\alpha & \rightarrow & U
\end{array}
\]

such that \( \{M_\alpha\} \) is an open cover of \( M \). A fiberwise \( d \)-dimensional geometric structure is a simplicial presheaf on \( \text{OEmb}_d \).

**Remark 3.1.2.** Alternatively, we could consider sheaves on manifolds and étale maps. The canonical inclusion \( \text{OEmb}_d \hookrightarrow \text{Etale}_d \) exhibits \( \text{OEmb}_d \) as a dense subsite in the 1-categorical sense, i.e., it induces an equivalence of 1-categories of sheaves. This continues to holds at the level of simplicial presheaves. Hence, we are free to use either site.

**Remark 3.1.3.** The special case of geometric structures valued in groupoids instead of simplicial sets is considered by Ludewig–Stoffel [2020.a], where sheaves on the site of submersions with fiberwise étale maps are considered, with a slightly modified Grothendieck topology where empty covers are excluded. By the above remarks, our geometric structures are a natural generalization of those considered in [2020.a].

Our definition of geometric structures is extremely versatile and captures all significant geometric structures we can think of, including metrics, topological structures, tangential structures, etc. We begin with some basic examples that are non-tangential in nature.

**Example 3.1.4 (manifolds).** Let \( X \) be a smooth manifold of any dimension. We can regard \( X \) as a geometric structure via the sheaf which assigns

\[
(M \to U) \mapsto C^\infty(M,X)
\]

to all submersion \( M \to U \). This is clearly a sheaf on \( \text{OEmb}_d \) since the total space functor \( \text{OEmb}_d \to \text{Man} \), which maps \( M \to U \) to \( M \), sends covering families to covering families. This sheaf is not representable.
Example 3.1.5. The previous example can be generalized easily to all simplicial presheaves on the site of smooth manifolds. Indeed, the total space functor $T : \mathcal{OEmb}_d^f \to \text{Man}$ induces a restriction

$$T^* : \mathcal{PSh}_\Delta(\text{Man}) \to \mathcal{PSh}_\Delta(\mathcal{OEmb}_d^f),$$

which manifestly preserves the homotopy descent property.

Example 3.1.6. An example of a geometric structure that does not come from a simplicial presheaf on smooth manifolds and smooth maps is given by the presheaf of Riemannian metrics. Riemannian metrics restrict along immersions (in particular, étale maps), but not along arbitrary smooth maps. This presheaf allows us to encode Riemannian metrics as nontangential structures. We can also consider Riemannian metrics with restrictions on sectional curvature (e.g., positive, negative, nonpositive, nonnegative), since these properties are preserved by pullbacks along étale maps.

3.2 Tangential structures

We explain how to treat traditional tangential structures in this setting (see, e.g., Lurie [2009a]). Since we have chosen to work with model categories, we will need a strict presentation of the fiberwise tangent bundle functor on $\mathcal{OEmb}_d^f$. We start with some preparatory definitions and lemmas. The following definition describes the stack of vector bundles on the site $\mathcal{OEmb}_d^f$.

Definition 3.2.1. Given an integer $d \geq 0$, the pseudofunctor

$$\text{Vect}(d) : (\mathcal{OEmb}_d^f)^{op} \to \mathcal{G}_{\mathfrak{rp}d}$$

is defined as follows. An object $(M \to U) \in \mathcal{OEmb}_d^f$ is sent to the groupoid of dualizable (equivalently, finitely generated and projective) $d$-dimensional (meaning $\Lambda^{d+1}_{C^\infty(M)}$ vanishes) modules over the real algebra $C^\infty(M)$ and their isomorphisms. A fiberwise open embedding $M \to M'$ in the category $\mathcal{OEmb}_d^f$ is sent to the functor of groupoids

$$\text{Vect}(d)(M') \to \text{Vect}(d)(M)$$

that sends a $C^\infty(M')$-module $X'$ to $X' \otimes_{C^\infty(M')} C^\infty(M)$. The remaining structure of a pseudofunctor is given by the natural isomorphisms

$$X \to X \otimes_{C^\infty(M)} C^\infty(M)$$

and

$$(X'' \otimes_{C^\infty(M'')} C^\infty(M')) \otimes_{C^\infty(M')} C^\infty(M) \to X'' \otimes_{C^\infty(M'')} C^\infty(M).$$

The definition of $\text{Vect}(d)$ as a pseudofunctor is necessary since pullbacks (implemented as tensor products here) do not compose strictly. Since we will need a strict simplicial presheaf in order to describe tangential structures, we strictify this pseudofunctor as follows.

Definition 3.2.2. Recall the adjunction $F \dashv i$, where

$$i : \mathcal{PSh}(\mathcal{OEmb}_d^f, \mathcal{G}_{\mathfrak{rp}d}) \to \Psi \mathcal{PSh}(\mathcal{OEmb}_d^f, \mathcal{G}_{\mathfrak{rp}d})$$

denotes the inclusion of strict functors into pseudofunctors, denoted by $\Psi \mathcal{PSh}$. The left adjoint $F$ formally adds pullbacks. We define the strict presheaf of groupoids $\text{BGL}(d) := F\text{Vect}(d)$. We have a canonical pseudonatural transformation

$$\text{Vect}(d) \to \text{BGL}(d),$$

given by the unit of the adjunction.

We now define the category of vector bundles over base spaces that are objects in $\mathcal{OEmb}_d^f$. More precisely, the vector bundles live over the total space of the object.
**Definition 3.2.3.** Given an integer $d \geq 0$, denote by $\mathcal{O}\text{Emb}_d^f/\text{BGL}(d)$ the comma category. Objects are pairs $(M \rightarrow U, V)$, where $(M \rightarrow U) \in \mathcal{O}\text{Emb}_d^f$ and $V \in \text{BGL}(d)(M \rightarrow U)$. Morphisms $(M \rightarrow U, V) \rightarrow (M' \rightarrow U', V')$ are pairs

$$(f : M \rightarrow M', g : V \rightarrow f^*V'),$$

where $f$ is a morphism in $\mathcal{O}\text{Emb}_d^f$ and $f^* = \text{BGL}(d)(f)$.

We now define the fiberwise tangent bundle functor on $\mathcal{O}\text{Emb}_d^f$.

**Definition 3.2.4.** Given an integer $d \geq 0$, denote by

$$\tau : \mathcal{O}\text{Emb}_d^f \rightarrow \mathcal{O}\text{Emb}_d^f/\text{BGL}(d)$$

the following model for the fiberwise tangent bundle functor. The functor $\tau$ sends an object $M \rightarrow U$ to the pair $(M \rightarrow U, V)$ with $V$ the module $\text{Der}_{M/U}$ of derivations $D : C^\infty(M) \rightarrow C^\infty(M)$ over the ring $C^\infty(U)$, where $C^\infty(M)$ is an algebra over $C^\infty(U)$ via pullback along $p : M \rightarrow U$. These derivations are sections of the fiberwise tangent bundle. A fiberwise open embedding $f : M \rightarrow M'$ is sent to the pair $(f : M \rightarrow M', g : V \rightarrow f^*V')$, where

$$g : \text{Der}_{M/U} \rightarrow \text{Der}_{M'/U'} \otimes_{C^\infty(M')} C^\infty(M) \cong f^*\text{Der}_{M'/U'}$$

denotes the canonical isomorphism.

The following construction takes as an input a vector bundle over a stack $Y$, specified by a map $Y \rightarrow \text{BGL}(d)$, and sends it to the sheaf, on the site $\mathcal{O}\text{Emb}_d^f$, whose value on $M \rightarrow U$ is the simplicial set of lifts of the fiberwise tangent bundle map $M \rightarrow \text{BGL}(d)$ through the map $Y \rightarrow \text{BGL}(d)$. The simplicial set of lifts is modeled by the simplicial set of sections of the base change of $Y \rightarrow \text{BGL}(d)$ along $M \rightarrow \text{BGL}(d)$.

**Definition 3.2.5.** Given an integer $d \geq 0$, we define a functor

$$\tau^* : \mathcal{P}\text{Sh}_\Delta(\mathcal{O}\text{Emb}_d^f)/\text{BGL}(d) \rightarrow \mathcal{P}\text{Sh}_\Delta(\mathcal{O}\text{Emb}_d^f)$$

as follows. Given an object $Y \rightarrow \text{BGL}(d)$ in $\mathcal{P}\text{Sh}_\Delta(\mathcal{O}\text{Emb}_d^f)/\text{BGL}(d)$, we construct a simplicial presheaf $(\mathcal{O}\text{Emb}_d^f)^{\text{op}} \rightarrow \text{sSet}$ as follows. An object $M \rightarrow U$ in $\mathcal{O}\text{Emb}_d^f$ is sent to the simplicial set of sections of the base change $Y \times_{\text{BGL}(d)} M \rightarrow M$ of the morphism $Y \rightarrow \text{BGL}(d)$ along the morphism

$$\tau(M \rightarrow U) : (M \rightarrow U) \rightarrow \text{BGL}(d).$$

This amounts to pulling back the bundle $Y \rightarrow \text{BGL}(d)$ to $M$ along the fiberwise tangent bundle map $M \rightarrow \text{BGL}(d)$ and taking the simplicial set of sections.

This construction defined a (strict) presheaf, since the resulting simplicial set of sections can be expressed as the simplicial subset of $Y(M \rightarrow U)$ comprising those $n$-simplices whose image in $\text{BGL}(d)(M \rightarrow U)_n$ is the degeneration of the given element in $\text{BGL}(d)(M \rightarrow U)_0$, which comes from the morphism $(M \rightarrow U) \rightarrow \text{BGL}(d)$.

In other words, a morphism $M \rightarrow M'$ in $\mathcal{O}\text{Emb}_d^f$ is sent to the simplicial map

$$\text{Map}_{/M'}(M', Y \times_{\text{BGL}(d)} M') \rightarrow \text{Map}_{/M}(M, Y \times_{\text{BGL}(d)} M \times U)$$

induced by the map $\tau(M \rightarrow M') : (M \rightarrow U, V) \rightarrow (M' \rightarrow U', V')$ in $\mathcal{O}\text{Emb}_d^f/\text{BGL}(d)$. This defines a (strict) functor because $\tau$ is a strict functor.

**Proposition 3.2.6.** The functor $\tau^*$ is a right Quillen functor, where both sides are equipped with the local injective model structure.

**Proof.** (Acyclic) cofibrations in the slice model structure are created by the forgetful functor

$$\mathcal{P}\text{Sh}_\Delta(\mathcal{O}\text{Emb}_d^f)/\text{BGL}(d) \rightarrow \mathcal{P}\text{Sh}_\Delta(\mathcal{O}\text{Emb}_d^f),$$

and (acyclic) cofibrations in the injective model structure are defined objectwise.
By construction, the functor \( \tau^* \) preserves small limits and filtered colimits. The domain and codomain of \( \tau^* \) are locally presentable categories, so \( \tau^* \) is a right adjoint functor.

Its left adjoint functor is the unique cocontinuous functor

\[
\mathcal{PSh}(\mathcal{OEmb}_d) \to \mathcal{PSh}(\mathcal{OEmb}_d)/\text{BGL}(d)
\]

that sends \( \Delta^n \otimes (M \to U) \) to the object \( \Delta^n \otimes (M \to U) \to (M \to U) \to \text{BGL}(d) \), where the first map projects \( \Delta^n \to \Delta^0 \) and the second map is the tangent bundle map \( \tau(M \to U) \). This left adjoint functor preserves injective (acyclic) cofibrations, so it is a left Quillen functor for the injective model structure and its right adjoint \( \tau^* \) is a right Quillen functor. Čech covers are sent to Čech covers, equipped with a map to \( \text{BGL}(d) \). Hence, this Quillen adjunction descends to the local injective model structure. \( \square \)

Thus, to convert a tangential structure given by a map \( Y \to \text{BGL}(d) \) into a geometric structure in the sense of Definition 3.1.1, we apply the right derived functor of \( \tau^* \).

### 3.3 Examples of geometric structures

We now discuss some examples of geometric structures.

**Example 3.3.1** (families of differential string structures). Let \( \text{BSpin}(d)_\nabla \) denote the simplicial presheaf on \( \mathcal{OEmb}_d \) which is the homotopy sheafification of the simplicial presheaf

\[
(M \to U) \mapsto N(\Omega^1_U(M; \mathfrak{so}(d))/C^\infty(M, \text{Spin}(d))),
\]

where \( \Omega^1_U(M; \mathfrak{so}(d)) \) denotes fiberwise Lie-algebra valued 1-forms. This simplicial presheaf assigns to a smooth manifold \( M \) the nerve of the action groupoid given by the action of \( C^\infty(M, \text{Spin}(d)) \) on \( \Omega^1_U(M; \mathfrak{so}(d)) \) by gauge transformations \( A \mapsto g^{-1}Ag + g^{-1}dg \) (where \( gAg^{-1} \) denotes the adjoint action and \( g^{-1}dg \) denotes the Maurer-Cartan form). This sheaf classifies smooth principal \( \text{Spin}(d) \)-bundles with connection, which live on the total space \( M \). Then we have a morphism of simplicial presheaves

\[
\text{BSpin}(d)_\nabla \to \text{BGL}(d) \tag{3.3.2}
\]

which forgets the connection (projects out forms) and maps via the composition \( \text{Spin}(d) \to \text{SO}(n) \to \text{GL}(d) \), where the first map is the double cover and the second is the inclusion. Now consider the simplicial presheaf arising as the pullback in \( \mathcal{PSh}(\mathcal{OEmb}_{d\text{an}}) \):

\[
\begin{array}{ccc}
\text{BString}(d)_\nabla & \to & \Omega^3 \times_{a, \text{B}^3\text{U}(1)_\nabla} \text{B}^3\text{U}(1)_\nabla^{\Delta[1]} \\varepsilon_1
\end{array}
\]

\[
\begin{array}{ccc}
\text{BSpin}(d)_\nabla & \xrightarrow{\psi_1} & \text{B}^3\text{U}(1)_\nabla \\varepsilon_1
\end{array}
\]

The maps in the diagram are defined as follows. The map \( a : \Omega^3 \to \text{B}^3\text{U}(1)_\nabla \) is the canonical map that includes globally defined 3-connections into circle 3-bundles with connections. It can be obtained (for example) via the morphism of sheaves of positively graded chain complexes

\[
\begin{array}{cccc}
0 & \to & C^\infty(-; \text{U}(1)) & \\xrightarrow{d \log} \\
0 & \to & \Omega^1 & \\xrightarrow{d} \\
0 & \to & \Omega^2 & \\xrightarrow{d} \\
0 & \to & \Omega^3 & \\xrightarrow{\text{id}} \\
\Omega^3 & \xrightarrow{id} & \Omega^3
\end{array}
\]
via the Dold-Kan correspondence. All forms appearing in the diagram are fiberwise differential forms with respect to the projection $M \times U \to U$. The maps $e_0, e_1 : B^3 U(1)_{\nabla} \to BU(1)_{\nabla}$ are evaluations at the endpoints. The map $\frac{1}{2}\hat{p}_1$ is the fractional differential refinement of the first Pontryagin class, which can be obtained via Chern-Weil theory (see, e.g., Bunke [2012.1]) from the invariant polynomial $p_1$ in the expansion

$$\det \left(1 - \frac{x}{2\pi}\right) = 1 + p_1(x) + p_2(x) + \cdots + p_n(x).$$

where $x \in \mathfrak{so}(n)$. The map $\text{BString}(d)_{\nabla} \to \text{BGL}(d)$ exhibits a geometric structure.

Let us unwind the space of differential string structures. We fix a fiberwise Riemannian metric and a fiberwise orientation. Since $\text{BString}(d)_{\nabla} \to \text{BSO}(d)$ is a fibration, the space of sections $\text{Map}/_M(M, \text{BString}(d)_{\nabla} \times_{\text{BSO}(d)} \tau M)$ is a Kan complex (this can also be computed as the fiber of the map $\text{BString}(d)_{\nabla}(M) \to \text{BSO}(d)(M)$ at $\tau : * \to \text{BSO}(d)(M)$). Using the universal property of the pullback describing $\text{BString}(d)_{\nabla}$, a vertex in this Kan complex is uniquely determined by the following data:

- $S_1$ A fiberwise spin structure on the fiberwise oriented frame bundle and a spin connection $\nabla$ on the resulting spin principal bundle.
- $S_2$ A fiberwise differential 3-form $\omega$ corresponding to a connection 3-form on the trivial circle 3-bundle.
- $S_3$ A fiberwise isomorphism of circle 3-bundles with connection $\lambda : \frac{1}{2}\hat{p}_1(\nabla) \simeq a(\omega)$.

The sheaf of sections sending

$$(M \to U) \mapsto \text{Map}/_M(M, \text{BString}(d)_{\nabla} \times_{\text{BGL}(d)} \tau M)$$

thus has vertices which are precisely fiberwise differential string structures (including a choice of a fiberwise oriented Riemannian metric) in the sense of Fiorenza–Schreiber–Stasheff [2010] (an equivalent definition was first given by Waldorf [2009.1]). The superscript $h$ denotes the homotopy pullback presented by the path space construction like the one used to define $\text{BString}(d)_{\nabla}$.

**Example 3.3.3.** The geometric spin and spin-c structures are encoded in exactly the same manner as differential string structures, but with the homotopy pullback diagrams

$$\begin{array}{ccc}
\text{BSpin}(d) & \longrightarrow & *\\
\downarrow & & \downarrow \\
\text{BSO}(d) & \xrightarrow{w_2} & \text{B}^2\mathbb{Z}/2
\end{array}$$

and

$$\begin{array}{ccc}
\text{BSpin}^c(d) & \xrightarrow{\tilde{c}_1} & \text{BU}(1)_{\nabla} \\
\downarrow & & \downarrow \rho_2 I \\
\text{BSO}(d) & \xrightarrow{w_2} & \text{B}^2\mathbb{Z}/2.
\end{array}$$

Here $I : \text{BU}(1)_{\nabla} \to \text{B}^2\mathbb{Z}$ is the canonical map which is part of the differential refinement (see Bunke [2012.1], Schreiber [2017.1]) and $\rho_2$ is the mod 2 reduction. These homotopy pullbacks are again presented by the path space construction.

**Example 3.3.4.** The rigid geometries of Stolz–Teichner [2011.1] (see also Ludewig–Stoffel [2020.1]) provide an example of geometric structures.

**Remark 3.3.5.** The following table provides a list of a few examples of geometric structures which can be encoded using our machinery. In the table, the functor $\delta : s\text{Set} \to \mathcal{P}\text{sh}_{\Delta}(\mathcal{O}\text{Emb}^d)$ is the locally constant sheaf functor.
| Degree | Tangential? | Stack                                      | Description               |
|--------|-------------|--------------------------------------------|---------------------------|
| 1      | Yes         | $\tau^*\text{BGL}(d)\nabla$              | connections               |
| 2      | Yes         | $\tau^*\text{BString}(d)$                | smooth string structure   |
| $\infty$ | Yes       | $\tau^*\delta(\text{BString}(d))$       | topological string structure |
| 2      | Yes         | $\tau^*\text{BString}(d)\nabla$          | differential string structure |
| 0      | No          | $\mathcal{X}$                             | target manifold           |
| 1      | Yes         | $\tau^*\text{BO}(d)$                     | Riemannian structure      |
| 0      | No          | $\mathcal{Riem}$                         | Riemannian metrics        |
| 0      | No          | $\mathcal{Riem}_{+}$                      | metrics with positive sectional curvature |
| 0      | No          | $\mathcal{M}//\mathcal{G}$               | rigid geometry            |
| Any    | No          | $\mathcal{M}//\mathcal{G}$               | higher rigid geometry     |

Remark 3.3.6. The geometric structure $\mathcal{Riem}$ is equivalent to the geometric structure induced by the tangential structure $\text{BO}(d)$ in the table, since the sheaf of sections of the pullback of $\text{BO}(d) \to \text{BGL}(d)$ along the fiberwise tangent bundle map $\tau : M \to \text{BGL}(d)$ yields precisely fiberwise Riemannian metrics.

4 Smooth bordism categories

In this section, we give a precise definition of the smooth bordism category as a symmetric monoidal smooth $(\infty,d)$-category.

4.1 The smooth $d$-uple bordism category

In this section we define the $d$-uple and $d$-globular bordism categories, following the terminology of Calaque and Scheimbauer [2015.1]. The $d$-uple bordism category is not local with respect to the globular morphisms (2.3.3) and hence there are distinguished composition directions, which need to be accounted for, just as a double category has distinguished composition directions that are put on equal footing.

Definition 4.1.1. A cut of an an object $p : M \to U$ in $\mathcal{OEmb}_d$ is a triple $(C_\preceq, C_=, C_\succeq)$ of subsets of $M$ such that there is a smooth map $h : M \to \mathcal{R}$ and the fiberwise-regular values of the map $(h,p)$ form an open neighborhood of $\{0\} \times U$ in $\mathcal{R} \times U$. Moreover, $h^{-1}(-\infty,0) = C_\preceq$, $h^{-1}(0) = C_=$ and $h^{-1}(0,\infty) = C_\succeq$. We set

$$C_\preceq = C_\prec \cup C_=$$  
$$C_\succeq = C_\succ \cup C_=$$.

There is a functor $\text{Cut} : (\mathcal{OEmb}_d)^{\text{op}} \to \mathcal{S}et$ that associates to an object $p : M \to U$ its set of cuts, and to a morphism the induced map of sets that takes preimages of cuts. We equip the set of cuts with a natural ordering $\preceq$, with $C \preceq C'$ if and only if $C_\preceq \subset C'_\preceq$.

Definition 4.1.2. Fix $d \geq 0$, a simplex $[m] \in \Delta$, and an object $p : M \to U$ in $\mathcal{OEmb}_d$. A cut $[m]$-tuple $C$ for $p : M \to U$ in $\mathcal{OEmb}_d$ is a collection of cuts $C_j = (C_{\preceq j}, C_= j, C_{\succeq j})$ of $p : M \to U$ indexed by vertices $j \in [m]$ such that

$$C_0 \preceq C_1 \preceq \cdots \preceq C_m.$$  

We set

$$C_{(j,j')} = C_{\succeq j} \cap C_{\preceq j'}, \quad C_{[j,j']} = C_{\succeq j} \cap C_{\preceq j'}.$$
There is a functor \( \text{CutTuple} : \Delta^{op} \times (\mathsf{OEmb}_0)^{op} \to \mathsf{Set} \) that associates to an object \(([m], p : M \to U)\) the set of cut \([m]\)-tuples of \(p\). The functor associates to a morphism the induced map of sets that takes preimages of the cuts and reindexes them according to the map of simplices. Thus, a face map removes a cut and a degeneracy map duplicates a cut.

We remark that the above definition implies that

\[
C_{>0} \supset C_{>1} \supset \cdots \supset C_{>j} \supset \cdots \supset C_{>m},
\]

as well as the analogous chains for \(C_{\leq}^m\) and \(C_{\geq}^m\).

**Definition 4.1.3** (\(d\)-uple bordisms). Given \(d \geq 0\), we specify an object \(\text{Bord}_{d, \text{uple}}\) in the category

\[
\mathcal{C}^\infty \mathcal{C} \setminus_{\Delta^{op}}_{\infty,d} \mathcal{C} = \mathcal{P} \mathcal{S} \mathcal{h}_D(\Delta^{d} \times \Gamma \times \mathcal{C} \mathcal{S} \mathcal{p})_{\text{uple}}
\]
as follows. For an object \((\mathbf{m}, (\ell), U) \in \Delta^{d} \times \Gamma \times \mathcal{C} \mathcal{S} \mathcal{p}\), the simplicial set \(\text{Bord}_{d, \text{uple}}(\mathbf{m}, (\ell), U)\) is the nerve of the following groupoid, which is small by Remark 3.0.1.

**Objects:** An object of the groupoid is a bordism given by the following data.

1. A \(d\)-dimensional smooth manifold \(M\).
2. For each \(1 \leq i \leq d\), a cut \([m_i]\)-tuple \(C^i\) for the projection \(p : M \times U \to U\).
3. A choice of map \(P : M \times U \to \langle \ell \rangle\), which gives a partition of the set of connected components of \(M \times U\) into \(\ell\) disjoint subsets and another subset corresponding to the basepoint (the “trash bin”),

which satisfy the transversality property:

\[
\cap. \text{ For every subset } S \subset \{1, \ldots, d\} \text{ and for any } j : S \to \mathbb{Z} \text{ such that } 0 \leq j_i \leq k_i \text{ for all } i \in S, \text{ there is a smooth map } h_S : M \times U \to \mathbb{R}^S \text{ such that for any } i \in S, \text{ the map }
\]

\[
\pi_i \circ h_S : M \times U \to \mathbb{R},
\]

where \(\pi_i : \mathbb{R}^S \to \mathbb{R}\) is the \(i\)th projection, yields the \(j_i\)-th cut \(C^i_{j_i}\) in the cut tuple \(C^i\), as in Definition 4.1.1. We require that the fiberwise-regular values of \((h_S, \hat{p})\) form an open neighborhood of \(\{0\} \times U \subset \mathbb{R}^S \times U\).

To simplify notation, we define

\[
C_{[j, j']} := \bigcap_{i \in S} C^{i}_{[j_i, j'_i]}
\]

and

\[
C_{(j, j')} := \bigcap_{i \in S} C^{i}_{(j_i, j'_i)},
\]

where \(j, j' : S \to \mathbb{Z}\) and \(0 \leq j_i \leq j'_i \leq m_i\) for all \(i \in S\). We also set

\[
\text{core}(\{j, j'\}, P) = C_{[j, j']} \setminus P^{-1}\{\ast\},
\]

and we require it to be compact, for all choices of \(j, j'\).

We will omit \(P\) in the notation when it is clear from the context.

**Morphisms:** A morphism of the groupoid is a cut-respecting map given by an equivalence class whose representatives are morphisms \(\psi : V \to \hat{V}\) in \(\mathsf{OEmb}_0\) that commute with the maps to \(\langle \ell \rangle\), where \(V\) is an open subset of \(M \times U\) that contains core\((\{0, \mathbf{m}\}, P)\), and likewise for \(\hat{V}\). Moreover, \(\psi\) satisfies the property:

\[
\square. \text{ For all } 0 \leq j_i \leq j'_i \leq m_i, \text{ there is an open neighborhood } Y_{j, j'} \text{ of core}\((\{j, j'\}, P)\) such that for any open } W_{j, j'} \subset Y_{j, j'} \text{ containing core}\((\{j, j'\}, P)\), \text{ the map } \psi \text{ restricts to a fiberwise diffeomorphism }
\]

\[
\varphi : W_{j, j'} \to \hat{W}_{j, j'},
\]

where \(\hat{W}_{j, j'} \supset \text{core}(\{j, j'\}, \hat{P})\) is open. That is, \(\psi\) restricts to a diffeomorphism on germs of all grid faces of all codimensions, throwing away the trash bin.
Two such maps $\psi : V \to \tilde{V}$ and $\psi' : V' \to \tilde{V}'$ are equivalent if they agree on an open neighborhood $W$ of $\text{core}([0, m], P)$ with $W \subset V \cap V'$. That is, we take germs of such maps around the grid.

**Presheaf structure maps:** The structure maps corresponding to morphisms in $\Delta$, $\Gamma$, and $\text{CartSp}$ are given by nerves of functors of groupoids specified as follows. For $\Delta$, the structure maps are induced by the maps in Definition $4.1.2$. A face map removes the corresponding cut and the outer face maps shrink the core and restrict germs appropriately. A degeneracy map duplicates a cut. The properties defining $\psi$ are preserved by these operations and hence the above data is natural with respect to maps in $\Delta$. For $\Gamma$, a map $\langle \ell \rangle \to \langle \ell' \rangle$ is simply composed with the given map $P : M \times U \to \langle \ell \rangle$, possibly shrinking the core and restricting germs. For a given map $U' \to U$ in $\text{CartSp}$, we pull back cut tuples and $P$ via this map. Likewise for $\psi$.

**Remark 4.1.5.** We have chosen to work exclusively with trivial bundles $M \times U \to U$ in Definition $4.1.3$, in order to ensure functoriality of pullbacks of bundles. Since we are working over the cartesian site, all bundles are trivializable and our presheaf of $(\infty, d)$-categories satisfies descent on cartesian spaces. Alternatively, we could include all bundles, but then we would need to work with Grothendieck fibrations instead of presheaves.

**Remark 4.1.6.** The multisimplicial presheaf, given by taking the nerve of the groupoid, is already local with respect to the Segal morphisms (2.2.3). Indeed, we can glue two bordisms together along a given diffeomorphism of tubular neighborhoods of the respective boundaries.

$$C^2_{\equiv 0}$$

Figure 1: A bordism defined by a cut tuples $C^1$ and $C^2$. Everything is parametrized by elements $x \in U$ in Definition $4.1.3$ and we can regard this picture as the fiber of a point $x \in U$.

### 4.2 Bordisms with geometric structures

In the above definition, the bordisms are equipped with no additional structure. We would like to equip bordisms with geometric structure, or tangential structure, in the sense of Section 3. Recall that these are given simply by simplicial presheaves on the site $\mathcal{O}_{\text{Emb}}^f_d$. Although it is relatively clear how to modify Definition $4.1.3$ so that $M$ is equipped with a geometric structure, we still need to say what happens when we pass to isomorphisms.

Let $S$ be a simplicial presheaf on $\mathcal{O}_{\text{Emb}}^f_d$. Consider a bordism $M \times U$ (Definition $4.1.3$). We have a well-defined germ of geometric structures

$$S\Box(M \times U) := \colim_{V \supset \text{core}([0, m], P)} S(V),$$

where the colimit is taken over the poset of open subsets $V \subset M \times U$ containing the subset $\text{core}([0, m]) \subset M \times U$. Since $S$ is by definition functorial with respect to fiberwise open embeddings, we can pull back along fiberwise cut-respecting maps (Definition $4.1.3$) $\varphi : V \to M \times U$, defined on a neighborhood of the core. Passing to germs, we have an induced map

$$\varphi^* : S\Box(M \times U) = \colim_{\bar{V} \supset \text{core}([0, m], \bar{P})} S(\bar{V}) \to \colim_{V \supset \text{core}([0, m], P)} S(V) = S\Box(M \times U).$$

Thus, the map $\varphi^*$ fiberwise pulls back geometric data on a germ of the core.
Example 4.2.3. Let $X$ be a smooth manifold, viewed as a sheaf on $\mathcal{O}\text{Emb}^d$ via
\[ X(N \rightarrow U) = C^\infty(N, X). \]
For $N = M \times U \rightarrow U$, the corresponding geometric structure on bordisms has elements
\[ [f] \in C^\infty(M \times U, X) := \text{colim}_{V > \text{core}(\emptyset, m)} C^\infty(V, X) \]
given by the germ of a smooth function $f : M \rightarrow X$ around the core. The pullback $\varphi^*[f]$ is the germ $[f \varphi]$.

Definition 4.2.4 ($d$-uple bordisms with structure). Fix $d \geq 0$ and $\mathcal{S}$ a simplicial presheaf on $\mathcal{O}\text{Emb}^d$. We specify an object $\text{Bord}^\mathcal{S}_d$ in the category
\[ C^\infty\text{Cat}_{\infty, d} = \mathcal{P}\text{Sh}_\Delta(\Delta \times d \times \Gamma \times \text{cartSp}) \]
by taking the diagonal of the nerve of the following presheaf of simplicial groupoids.

(GO) The simplicial set of objects is given by
\[ \text{Ob} := \coprod_{(M, C, P)} \mathcal{S}_\square(M \times U), \]
where the coproduct ranges over the objects of Definition 4.2.3 and the subscript $\square$ denotes the germ in $(4.2.1)$. This groupoid is small because of our convention on manifolds (see Remark 3.0.1).

(GM) The simplicial set of morphisms is given by
\[ \text{Mor} := \coprod_{\varphi : (M, C, P) \rightarrow (\tilde{M}, \tilde{C}, \tilde{P})} \mathcal{S}_\square(\tilde{M} \times \tilde{U}), \]
where the coproduct is taken over the morphisms in Definition 4.2.3. The target map of the groupoid structure sends the component indexed by a germ $\varphi : (M, C, P) \rightarrow (\tilde{M}, \tilde{C}, \tilde{P})$ to itself by identity. The source map pulls back the structure by $\varphi$ as defined in Eq. (4.2.2). Composition is given by functoriality of $\mathcal{S}$.

Remark 4.2.5. The (pre)bordism category defined in Definition 4.1.3 is substantially different from Lurie’s (pre)bordism category [2009.a], Calaque–Scheimbauer [2015.b], and Schommer-Pries [2017]. The role of the space of embedded bordisms is taken over by the sheaf of groupoids, whose shape recovers this space. This is necessary because we cannot allow cuts to vary continuously in a space of bordisms, due to the presence of rigid geometric structures that require smooth data.

4.3 The globular bordism category

We now extract a globular subcategory from the $d$-uple bordism category.

Definition 4.3.1 (globular bordism category). Fix $d \geq 0$ and $\mathcal{S}$ a simplicial presheaf on $\mathcal{O}\text{Emb}^f_d$. We specify an object $\text{Bord}^\mathcal{S}_d\text{glob}$ in the category
\[ C^\infty\text{Cat}_{\infty, d} = \mathcal{P}\text{Sh}_\Delta(\Delta \times d \times \Gamma \times \text{cartSp}) \]
by taking the subpresheaf of $\text{Bord}^\mathcal{S}_d\text{uple}$ whose value at $(m, (\ell), U)$ with $m = ([m_1], \ldots, [m_d])$ is defined as follows. Let $1 \leq i \leq d + 1$ be the smallest integer such that $m_i = 0$, or $d + 1$ if all $m_i$ are nonzero. Then take only the components in Definition 4.2.4 indexed by the objects $(M, C, P)$ that are isomorphic to the simplicial degeneration of an object in
\[ \text{Bord}^\mathcal{S}_d\text{uple}(([m_1], \ldots, [m_{i-1}], [0], \ldots, [0]), (\ell), U), \]
in the simplicial directions $i + 1, i + 2, \ldots, d$. 

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Example 4.3.2. Set $d = 2$. The following images depict objects in the 2-uple and 2-glob bordism categories. The two images on the left represent a 2-uple bordism in bidegree $([0],[1])$ and $([1],[1])$ respectively. The image on the right depicts a cut tuple in the globular bordism category in bidegree $([1],[1])$.

\[
C^2_{=0} \quad C^2_{=0} \quad C^2_{=0}
\]

Example 4.3.3. Again, take $d = 2$. The following picture illustrates isomorphic and non-isomorphic objects in the groupoid in bidegree $([0],[1])$ and $([1],[1])$ (respectively). The dashed lines represent germs around the cores of the simplicial faces.

\[
\mathcal{C}^2_{=0} \quad \mathcal{C}^2_{=0} = C^2_{=1}
\]

\[
\mathcal{C}^2_{=0} \quad \mathcal{C}^2_{=0} = C^2_{=1}
\]

Remark 4.3.4. The categories $\text{Bord}^S_{d,\text{uple}}$ and $\text{Bord}^S_{d,\text{glob}}$ are local objects in the uple and glob model structures (respectively). We will not provide more details, since this is not needed for the main theorem. In particular, field theories are defined using derived homs from the bordism category, which does not require any fibrancy or cofibrancy properties.

Remark 4.3.5. The bordism category $\text{Bord}^S_{d,\text{uple}}$ is not globular. We have natural monomorphisms

\[
\text{Bord}^S_{d,\text{glob}} \hookrightarrow \text{Bord}^S_{d,\text{uple}} \hookrightarrow \mathcal{R}(\text{Bord}^S_{d,\text{uple}})
\]

with $\mathcal{R}$ denoting localization in the globular model structure. Both morphisms and their composition are not equivalences in the uple model structure.

Remark 4.3.6. Our results work equally well for both the variants $\text{Bord}^S_{d,\text{uple}}$ and $\text{Bord}^S_{d,\text{glob}}$. Henceforth, we will denote both bordism categories ambiguously as $\text{Bord}^S_d$. When distinctions need to be made between the two, we will make it explicit.
5 Criteria for weak equivalences of smooth symmetric monoidal $(\infty,n)$-categories

Our goal in this section is to establish sufficient criteria for weak equivalences in $\mathcal{C}^{\infty}\text{Cat}^\otimes_{\infty,n}$ that will be used in the next section to prove the main theorem.

5.1 Cartesian spaces and stalks

First, we show how to reduce the problem of showing that a morphism in $\mathcal{C}^{\infty}\text{Cat}^\otimes_{\infty,n} = \mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma \times \text{Cart}^\mathcal{Sp})_{\text{local}}$ is a weak equivalence to the same problem in the model category $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}}$.

Proposition 5.1.1. Weak equivalences in the model category $\mathcal{C}^{\infty}\text{Cat}^\otimes_{\infty,n}$ can be detected stalkwise. Specifically, on the site $\text{Cart}^\mathcal{Sp}$, points are indexed by natural numbers $p \in \mathbb{N}$ corresponding to the dimension of the cartesian space. For each point $p \in \mathbb{N}$, the associated stalk is given by the functor

$$p^*: \mathcal{C}^{\infty}\text{Cat}^\otimes_{\infty,d} \to \text{Cat}_{\infty,d}$$

that assigns to a smooth symmetric monoidal $(\infty,d)$-category $X$ the stalk

$$X_p := \text{colim}_{B^p \subset \mathbb{R}^d} X(B^p) \in \mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma),$$

where $B^p$ is an open ball centered at the origin in $\mathbb{R}^p$ and the (homotopy) colimit is taken over the inclusions of such open balls. Thus, to show that a morphism of smooth symmetric monoidal $(\infty,d)$-categories $X \to Y$ is an equivalence, it suffices to show that the induced maps on the stalks (5.1.2) are weak equivalences in $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}}$.

Proof. See, e.g., Bunke–Nikolaus–Völkl [2013, Lemma 7.2].

5.2 Reduction of multiple to single

Next, we explain how the problem of showing that a morphism in $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}}$ is a weak equivalence can be reduced to the same problem in the model categories $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Gamma)_{\text{local}}$ or $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Gamma)_{\text{local}}$.

The model structure on $\Gamma$-spaces is given by taking the injective model structure on $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Gamma)$ and performing left Bousfield localization at the morphisms Eq. (2.2.5) and Eq. (2.2.6). This model structure has more cofibrations than the Bousfield–Friedlander model structure [1978], which, in turn, has more cofibrations than Schwede’s Q-model structure [1999]. All three model structures have the same weak equivalences. Recall also the Rezk model structure [1998] on $\Delta$-spaces, given by taking the Reedy model structure on $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta)$, which coincides with the injective model structure, and performing left Bousfield localization at the morphisms Eq. (2.2.3) and Eq. (2.2.4).

Proposition 5.2.1. If $f$ is a morphism in $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}}$ such that for each $m \in \Delta^{\times n}$ the functor

$$\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}} \to \mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Gamma)_{\text{local}}$$

induced by

$$\Gamma \to \Delta^{\times n} \times \Gamma, \quad (\ell) \mapsto (m, (\ell))$$

sends $f$ to a weak equivalence, then $f$ is a weak equivalence.

Likewise, if $f$ is a morphism in $\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}}$ such that for each $i \in 1, \ldots, n$, $(\ell) \in \Gamma$, and $m \in \Delta^{n-1}$ the functor

$$\mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta^{\times n} \times \Gamma)_{\text{local}} \to \mathcal{P}\mathcal{S}\mathcal{h}_\Delta(\Delta)_{\text{local}}$$

induced by

$$\Delta \to \Delta^{\times n} \times \Gamma, \quad [\omega] \mapsto (m_1, \ldots, m_{i-1}, \omega, m_i, \ldots, m_{n-1}, (\ell))$$

sends $f$ to a weak equivalence, then $f$ is a weak equivalence.
Proof. We will prove the second claim, the first claim is analogous. If \( f : F \to G \) is a morphism with indicated properties, then the induced natural transformation in \( \mathcal{P}sh(\Delta^{n-1} \times \Gamma, \mathcal{P}sh_{\Delta}(\Delta)_{\text{local}}) \) with components \( f_m(\ell) : F_m(\ell) \to G_m(\ell) \) is objectwise a local weak equivalence in \( \mathcal{P}sh_{\Delta}(\Delta)_{\text{local}} \). By the co-Yoneda lemma, the weighted colimit of \( f_m(\ell) \), weighted by the Yoneda embedding, is canonically isomorphic to \( f \). The weighted colimit computes the weighted homotopy colimit of the same pair. This weighted homotopy colimit can be rewritten as a homotopy coend of a natural transformation (of bifunctors) whose components are

\[
j(m, (\ell)) \boxtimes f_m'(\ell').
\]

These maps are local weak equivalences in simplicial presheaves on \( \Delta^n \times \Gamma \), by definition of the multiple model structure in Definition 2.2.9. Since local weak equivalences are closed under homotopy colimits in the injective model structure, \( f \) is a local weak equivalence. \( \square \)

5.3 Reduction of simplicial presheaves to presheaves of sets

We now explain how the problem of showing that a morphism in \( \mathcal{P}sh_{\Delta}(\Delta)_{\text{local}} \) or \( \mathcal{P}sh_{\Delta}(\Gamma)_{\text{local}} \) is a weak equivalence can be reduced to the same problem in the categories \( \mathcal{P}sh(\Delta) \) respectively \( \mathcal{P}sh(\Gamma) \), where weak equivalences of presheaves of sets are created by the inclusion functor that turns them into presheaves of discrete simplicial sets.

**Proposition 5.3.1.** If \( f \) is a morphism in \( \mathcal{P}sh_{\Delta}(\Delta)_{\text{local}} \) such that for every \([l] \in \Delta\) the functor

\[
\mathcal{P}sh_{\Delta}(\Delta)_{\text{local}} \to \mathcal{P}sh(\Delta)_{\text{local}}, \quad F \mapsto F_l = ([n] \mapsto F([n])_{[l]})
\]

sends \( f \) to a weak equivalence (here presheaves of sets are promoted to presheaves of discrete simplicial sets), then \( f \) is a weak equivalence.

Likewise for \( \Gamma \) instead of \( \Delta \).

**Proof.** This proof is analogous to the proof of Proposition 5.2.1. If \( f : F \to G \) is a morphism with indicated properties, then the induced natural transformation of simplicial diagrams in \( \mathcal{P}sh_{\Delta}(\Delta)_{\text{local}} \) with components \( f_l : F_l \to G_l \) is an objectwise local weak equivalence. The objectwise diagonal of this natural transformation is precisely \( f \). The objectwise diagonal functor computes the homotopy colimit of simplicial diagrams in the injective model structure. Since local weak equivalences are closed under homotopy colimits in the injective model structure, \( f \) is a local weak equivalence. The case of \( \Gamma \) is analogous. \( \square \)

5.4 The simplicial Whitehead theorem

First, we recall the Whitehead theorems for simplicial maps between Kan complexes. See, for example, Dugger–Isaksen [2002.a, Proposition 4.1].

**Lemma 5.4.1.** A map \( f : X \to Y \) of Kan complexes is a simplicial homotopy equivalence if and only if for any commutative diagram

\[
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{a} & X \\
\downarrow \iota & & \downarrow f \\
\Delta^n & \xrightarrow{b} & Y
\end{array}
\]

there is a diagonal arrow \( \delta : \Delta^n \to X \) and a homotopy \( H : \Delta^1 \times \Delta^n \to Y \) from \( f \delta \) to \( b \), relative boundary.

Next, using Kan’s \( \text{Ex}^\infty \) functor (see, for example, Goerss–Jardine [1999.a, §III.4]) we can formulate the simplicial Whitehead theorem for arbitrary simplicial sets.

**Lemma 5.4.2.** A map \( f : X \to Y \) of simplicial sets is a simplicial weak equivalence if and only if for any commutative diagram

\[
\begin{array}{ccc}
\text{Sd}^k \partial \Delta^n & \xrightarrow{a} & X \\
\downarrow \iota & & \downarrow f \\
\text{Sd}^k \Delta^n & \xrightarrow{b} & Y
\end{array}
\]

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there is \( l \geq k \) such that after precomposing both horizontal maps with the natural transformation \( \text{Sd}^l \to \text{Sd}^k \) given by the iterated last vertex map, the resulting commutative square with newly constructed horizontal maps \( a' \) and \( b' \)

\[
\begin{array}{ccc}
\text{Sd}^l \partial \Delta^n & \xrightarrow{a'} & X \\
\downarrow & & \downarrow f \\
\text{Sd}^l \Delta^n & \xrightarrow{b'} & Y
\end{array}
\]

admits a diagonal arrow \( \delta : \text{Sd}^l \Delta^n \to X \) and a homotopy \( H : \text{Sd}^l(\Delta^1 \times \Delta^n) \to Y \) from \( f \delta \) to \( b' \), relative boundary.

We will need the following special case of the above criterion, which establishes weak contractibility of simplicial sets.

**Lemma 5.4.3.** Suppose \( X \) is a simplicial set such that for any \( n \geq 0 \), \( k \geq 0 \), and any simplicial map \( f : \text{Sd}^k \partial \Delta^n \to X \), there is a finite chain of simplicial homotopies in either direction along \( \Delta^1 \) of the form

\[
H \times \text{Sd}^k \partial \Delta^n \to X
\]

that connects \( f \) to a simplicial map that factors through \( \Delta^0 \), where \( H \) is an arbitrary zigzag of 1-simplices. Then \( X \) is a weakly contractible simplicial set, i.e., the map \( X \to \Delta^0 \) is a simplicial weak equivalence.

**Proof.** We prove that \( \text{Ex}^\infty X \) is a contractible Kan complex. To this end, we show that any map \( g : \partial \Delta^n \to \text{Ex}^\infty X \) extends to \( \text{Ex}^\infty X \) extends to a map \( \Delta^n \to \text{Ex}^\infty X \). The map \( g \) is adjoint to a map for the form \( f : \text{Sd}^k \partial \Delta^n \to X \) for some \( k \geq 0 \).

By assumption, there is a simplicial homotopy \( h_0 : H \times \text{Sd}^k \partial \Delta^n \to X \) from \( f \) to a constant simplicial map \( \text{Sd}^k \partial \Delta^n \to \Delta^0 \to X \), where \( H \) is an arbitrary zigzag of 1-simplices. The adjoint map to \( h_0 \) has the form \( H \to \text{Hom}(\text{Sd}^k \partial \Delta^n, X) = \text{Hom}(\partial \Delta^n, \text{Ex}^k X) \), and we postcompose it with the map \( \text{Hom}(\partial \Delta^n, \text{Ex}^k X) \to \text{Hom}(\partial \Delta^n, \text{Ex}^\infty X) \) induced by the inclusion \( \text{Ex}^k X \to \text{Ex}^\infty X \). The latter composition is adjoint to the map \( h_1 : C = H \times \partial \Delta^n \to \text{Ex}^\infty X \), which is a simplicial homotopy from \( g \) to a constant simplicial map \( p : \partial \Delta^n \to \Delta^0 \to \text{Ex}^\infty X \). We extend \( h_1 \) to a map \( h_1' : A \to X \), where \( A = C \cup_{\partial \Delta^n} \Delta^n \), and the map \( h_1' \) is defined on \( \Delta^n \) by extending \( p \) to \( \Delta^n \) as a constant map.

The inclusion \( A \to H \times \Delta^n \) is an acyclic cofibration, thus the map \( A \to \text{Ex}^\infty X \) can be extended to a map \( H \times \Delta^n \to \text{Ex}^\infty X \). The restriction of the latter map to \( \Delta^0 \times \Delta^n = \Delta^n \) provides the desired extension of the map \( g : \partial \Delta^n \to \text{Ex}^\infty X \) to \( \Delta^n \).

\[\square\]

### 5.5 Dugger and Spivak’s rigidification of quasi-categories

The final tool needed for the proof of the main theorem is a rigidification of quasi-categories. The reason we will need this rigidification is the following. In Section 4, we will have a monomorphism of simplicial sets \( f : X \to Y \) which we claim is a weak equivalence in the Joyal model structure. Thus, \( f \) admits a presentation by an inner anodyne map. However, it is not at all clear how to express our given \( f \) as a retract of transfinite composition of cobase changes. On the other hand, if work with simplicial categories, then the induced map on hom spaces turns out to be much easier to understand.

By definition, a morphism of simplicial sets \( X \to Y \) is an equivalence in the Joyal model structure if it induces a Dwyer-Kan equivalence on corresponding simplicial categories \( \mathcal{C}(X) \to \mathcal{C}(Y) \). Here \( \mathcal{C} \) is left adjoint to the homotopy coherent nerve functor. Dugger and Spivak have a concrete model \([2009.c]\) for \( \mathcal{C} \), where the mapping simplicial sets are described using necklaces. We now review this description.

**Definition 5.5.1.** A necklace is a simplicial set of the form \( \Delta^n_{i_0} \cup \Delta^n_{i_1} \cup \ldots \cup \Delta^n_{i_k} \), where each wedge is obtained by gluing the final vertex of \( \Delta^n_{i_k} \) to the initial vertex of \( \Delta^n_{i_{k+1}} \). Each \( \Delta^n_{i_k} \) is called a bead and each the initial/final vertex of each \( \Delta^n_{i_k} \) is called a joint.

Morphisms of necklaces are simply morphisms of simplicial sets. Such morphisms are a composition of morphisms induced by the canonical inclusions \( \Delta^n_{i_k} \cup \Delta^n_{i_{k+1}} \hookrightarrow \Delta^{n_{i_k}+n_{i_{k+1}}} \), and the face and degeneracy maps of each bead. Now let \( X \) be a simplicial set. By Dugger–Spivak \([2009.c,\text{Theorem 5.2}]\), the simplicial category \( \mathcal{C}(X) \) is connected by a zig-zag of equivalences to the simplicial category whose objects are vertices in \( X \) and whose simplicial set of morphisms are given by the nerve of the following category
1. The objects morphisms of necklaces $\Delta^n \vee \ldots \vee \Delta^n \rightarrow X$ sending the initial vertex of the first bead to $x$ and the final vertex of the last bead to $y$.

2. The morphisms are morphisms of necklaces covering $X$.

Following Dugger–Spivak [2009.c], we call this simplicial category the *necklace category* and denote it $\mathcal{C}^{\text{neck}}(X)$. Thus, we have the following.

**Proposition 5.5.2.** Let $f : X \rightarrow Y$ be a morphism of simplicial sets which is bijective on vertices. Then $f$ is an equivalence in the Joyal model structure if for all vertices $x, y \in X$, the induced map

$$\mathcal{C}^{\text{neck}}(f) : \mathcal{C}^{\text{neck}}(X)(x, y) \rightarrow \mathcal{C}^{\text{neck}}(Y)(f(x), f(y))$$

is an equivalence in the classical model structure on simplicial sets.

**Proof.** Since $f$ is bijective on vertices, $\mathcal{C}^{\text{neck}}(f)$ is essentially surjective. If $\mathcal{C}^{\text{neck}}(f)$ is an equivalence on mapping spaces, then it is a Dwyer-Kan equivalence of simplicial categories. By Dugger–Spivak [2009.c, Theorem 5.2], $\mathcal{C}^{\text{neck}}(X)$ is connected to $\mathcal{C}(X)$ by a zig-zag of weak equivalences. Therefore, $\mathcal{C}(f)$ is a weak equivalence (by 2 out of 3).

## 5.6 $\Gamma$-spaces

We now modify the concept of Moss structure in order to detect acyclic cofibrations in the local injective model structure on $\Gamma$-spaces. Since we already reduced the problem from the case of simplicial presheaves to presheaves of sets, we work in this simplified setting.

**Definition 5.6.1.** Let $j : \Gamma \rightarrow \mathcal{PSh}(\Gamma)$ denote the Yoneda embedding. We define the boundary of a representable $j(\ell) \in \mathcal{PSh}(\Gamma)$ as the subpresheaf $\partial(\ell) \rightarrow j(\ell)$, whose value at $\langle m \rangle$ consists of all maps of finite pointed sets $\langle \ell \rangle \rightarrow \langle m \rangle$ such that at least one element $k \in \langle \ell \rangle$ with $k \neq \ast$ is sent to $\ast \in \langle m \rangle$.

Elements of $j(\ell)$ that are not in $\partial(\ell)$ are maps of finite pointed sets $f : \ell \rightarrow \langle m \rangle$ such that $f^{-1}\{\ast\} = \{\ast\}$. Such maps are also known as *active morphisms*, see Lurie [2017.c, Definition 2.1.2.1].

**Proposition 5.6.2.** The inclusion $\partial(\ell) \rightarrow j(\ell)$, viewed as a morphism of discrete simplicial presheaves, is an acyclic cofibration in the local injective model structure on $\Gamma$-spaces.

**Proof.** We prove by induction on $\ell$ that $\partial(\ell) \rightarrow j(\ell)$ is a weak equivalence. The map in $\mathcal{PSh}(\Gamma)$,

$$S_\ell := j(1) \sqcup j(0) \sqcup j(0) \sqcup \cdots \sqcup j(0) \rightarrow j(\ell),$$

is an acyclic cofibration for any $\langle \ell \rangle$, which follows by repeatedly composing cobase changes of maps of the form Eq. (2.2.5). The resulting subpresheaf picks out those maps of pointed sets $\langle \ell \rangle \rightarrow \langle m \rangle$ that send all elements to $\ast$ except for some $k \in \langle \ell \rangle$, $k \neq \ast$.

For $\ell = 2$, the map $j(1) \sqcup j(0) \rightarrow j(2)$ coincides with $\partial(2) \rightarrow j(2)$, which establishes the base of induction.

Suppose $\partial(m) \rightarrow j(\ell)$ is a weak equivalence for all $m < \ell$. To show that $\partial(\ell) \rightarrow j(\ell)$ is a weak equivalence, we present the inclusion $S_\ell \rightarrow \partial(\ell)$ as a composition of cobase changes of coproducts of maps of the form $\partial(m) \rightarrow j(m)$, and then use the 2-out-of-3 property for the maps $S_\ell \rightarrow \partial(\ell)$ and $S_\ell \rightarrow j(\ell)$ to conclude that $\partial(\ell) \rightarrow j(\ell)$ is a weak equivalence.

We introduce the following filtration on $S_\ell$:

$$S_\ell \rightarrow \partial(\ell) : W_\ell \rightarrow W_{\ell-1} \rightarrow \cdots \rightarrow W_1 = \partial(\ell).$$

Here $W_\ell$ is a subpresheaf of $j(\ell)$ comprising precisely those maps of pointed finite sets $\langle \ell \rangle \rightarrow \langle m \rangle$ such that the preimage of $\langle m \rangle \setminus \{\ast\}$ has $k$ or fewer elements. By construction, $S_\ell = W_1$ and $W_{\ell-1} = \partial(\ell)$. It remains to present the inclusion $W_{k-1} \rightarrow W_k$ for any $1 < k < \ell$ as a cobase change of a coproduct of maps of the form $\partial(k) \rightarrow j(k)$. The indexing set of the coproduct is the set of all maps $f$ of pointed finite sets $\langle \ell \rangle \rightarrow \langle k \rangle$ such that the preimage of $p \in \langle k \rangle$, $p \neq \ast$ has exactly one element (such morphisms are...
also known as inert, see Lurie \cite[Definition 2.1.1.8]{Lurie09}) and \( f \) is (strictly) increasing when restricted to this preimage. (Here we use the natural order on \( \{\ell\} = \{\ast, 1, \ldots, \ell\} \).) The map \( f : \{\ell\} \to \{k\} \) corresponds to a map \( j(k) \to W_k \) under the (contravariant) Yoneda embedding, and the attaching map \( \partial(k) \to W_{k-1} \) is the composition \( \partial(k) \to j(k) \to W_k \), which factors through \( W_{k-1} \) by construction.

To show that the resulting commutative square is a pushout square, pick an arbitrary element of \( W_k \) that is not in \( W_{k-1} \), i.e., a map of pointed finite sets \( g : \{\ell\} \to \{m\} \) such that the preimage \( A \) of \( \{m\} \setminus \{\ast\} \) has exactly \( k \) elements. We have to show that this element comes from a unique element in the coproduct, i.e., there is a unique pair \((f, \sigma)\), where \( f : \{\ell\} \to \{k\} \) is an element in the indexing set of the coproduct and \( \sigma : \{k\} \to \{m\} \) is a map of pointed finite sets such that \( \sigma f = g \). Indeed, \( f \) must be the unique map \( f : \{\ell\} \to \{k\} \) of pointed finite sets such that the preimage of \( \{k\} \setminus \{\ast\} \) equals \( A \) and \( f \) is strictly increasing. For such an \( f \) there is a unique \( \sigma : \{k\} \to \{m\} \) for which \( \sigma f = g \) because \( f \) is injective away from the preimage of \{\ast\}.

\[ \square \]

6 Codescent for bordism categories

In this section, we prove the main theorem for bordism categories. Throughout this section, we will use multi-index notation. Greek letters \( \alpha \) and \( \beta \) will denote multi-indices, which are maps of sets \( \alpha : [n] = \{0 < \cdots < n\} \to A \), where \( A \) is the indexing set for some open cover \( \{U_a\}_{a \in A} \). We set \( U_\alpha = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \).

The simplicial maps induce maps between corresponding intersections, by removing or duplicating an open set in the intersection.

Since bordisms and their cut tuples will be used frequently, we will often denote bordisms simply by \( M \), leaving the cut tuples and map \( P : M \to \{\ell\} \) implicit.

**Theorem 6.0.1.** Fix \( d \geq 0 \). Denote by \( \mathcal{P}Sh_\Delta(\mathcal{OEmb}_d)_{\text{local}} \) the local injective model structure, where the localization is taken at the Čech nerve maps corresponding to covers in \( \mathcal{OEmb}_d \), defined as in Eq. \( (2.2.7) \).

The functors

\[
\text{Bord}_d, \text{uple} : \mathcal{P}Sh_\Delta(\mathcal{OEmb}_d)_{\text{local}} \to C^\infty \mathbf{Cat}_{\infty,d}^{\text{,uple}}, \quad S \mapsto \text{Bord}^S_d, \text{uple}
\]

and

\[
\text{Bord}_d, \text{glob} : \mathcal{P}Sh_\Delta(\mathcal{OEmb}_d)_{\text{local}} \to C^\infty \mathbf{Cat}_{\infty,d}^{\text{,glob}}, \quad S \mapsto \text{Bord}^S_d, \text{glob}
\]

are left Quillen functors that preserve all weak equivalences. In particular, they are homotopy cocontinuous.

**Proof.** Using the notation of Remark \( 4.3.9 \), we denote both bordism categories by \( \text{Bord}^S_d \). By the universal property of left Bousfield localizations, it suffices to define a left Quillen functor in the injective model structure on \( \mathcal{P}Sh_\Delta(\mathcal{OEmb}_d) \), show that it preserves all weak equivalences, and sends localizing maps to weak equivalences.

By the definition of \( \text{Bord}^S_d \) (see Definition \( 4.2.4 \) and Definition \( 4.3.1 \)), both functors send monomorphisms to monomorphisms and objectwise weak equivalences to weak equivalences. Both functors are left adjoints, with right adjoints given by the functors that send \( X \in C^\infty \mathbf{Cat}_{\infty,d}^{\text{,uple}} \) to field theories with values in \( X \):

\[
X \mapsto \left( (W \to U) \mapsto \text{Map}(\text{Bord}^W_d \to U, X) \right).
\]

It remains to show that the localizing maps Eq. \( (2.2.7) \) are sent to weak equivalences. That is, for a fixed cover \( \{W_a \to U_a\}_{a \in A} \) of \( W \to U \), we must show that the canonical map

\[
\text{hocolim}_{\{n\} \in \Delta^{op}} \prod_{\alpha : [n] \to A} \text{Bord}^{W_a \to U_a}_d \leftrightarrow \text{Bord}^W_d \to U
\]

(6.0.2)

is an equivalence in \( C^\infty \mathbf{Cat}_{\infty,d}^{\text{,uple}} \). We establish this as follows.

First, by Proposition \( 5.1.2 \), the canonical map from the homotopy colimit in Eq. \( (6.0.2) \) to the corresponding strict colimit is an equivalence. Then we show that the map out of the strict colimit is a stalkwise weak equivalence (see Proposition \( 5.1.3 \)) by factoring each stalk as a composition of monomorphisms (see Definition \( 6.3.1 \))

\[
\text{colim}_{\{n\} \in \Delta^{op}} \prod_{\alpha : [n] \to A} p^* \text{Bord}^{W_a \to U_a}_d \xrightarrow{1} B_0 \xrightarrow{2} B_d = p^* \text{Bord}^W_d \to U.
\]
The map 1 is a weak equivalence by Proposition 6.4.1. The map 2 is a weak equivalence by Proposition 6.5.1. The last equality holds by Lemma 6.6.4. We remark that our constructions respect the globularity of the existing bordisms. Therefore, our proof works in both cases: globular and multiple.

6.1 Reduction of homotopy colimits to strict colimits

Recall the following fact, which will be used twice below.

Lemma 6.1.1. If I is a direct category (meaning it contains no infinite descending chains of nonidentity morphisms \( \cdots \to X_2 \to X_1 \to X_0 \)), and \( D : I \to C \) is a diagram valued in a model category \( C \) such that the canonical map \( \text{colim}_{k<i} D(k) \to D(i) \) is a cofibration for any \( i \in I \), then the canonical map \( \text{hocolim} D \to \text{colim} D \) is a weak equivalence. The cofibrancy condition holds if \( C \) is a category of simplicial presheaves, cofibrations in \( C \) are monomorphisms of simplicial presheaves, and all transition maps \( D(k) \to D(i) \) are monomorphisms of presheaves.

Proof. This follows immediately if we equip \( I \) with a Reedy structure whose arrows are all positive and observe that the cofibrancy condition expresses cofibrancy in the Reedy structure, which coincides with the projective structure. For the case of simplicial presheaves, observe that \( \text{colim}_{k<i} D(k) \to D(i) \) is a monomorphism because the left side is a union of subobjects. □

Proposition 6.1.2. Let \( \{ W_a \to U_a \} \) be a covering of \( W \to U \), with \( W \) a \( d \)-dimensional manifold. Then the canonical comparison map

\[
\text{hocolim}_{[n] \in \Delta^{op}} \left( \prod_{\alpha : [n] \to A} \text{Bord}^{W_a \to U_a}_d \right) \to \text{colim}_{[n] \in \Delta^{op}} \left( \prod_{\alpha : [n] \to A} \text{Bord}^{W_a \to U_a}_d \right)
\]

is an equivalence.

Proof. To prove the claim, we will rewrite our simplicial diagram as a filtered homotopy colimit over homotopy colimits of cubical diagrams. We then use Lemma 6.1.1 to compute these homotopy colimits as strict colimits, proving the claim.

Observe that if \( A \) is the indexing set of the cover \( \{ U_a \} \), commutativity of (homotopy) colimits and (homotopy) coproducts ensures that we can rewrite the (homotopy) colimit as a diagram over \( \Delta^{op}_A \), with objects morphisms of sets \( [n] \to A \) and morphisms commutative triangles. Explicitly, we assign

\[
(\alpha : [n] \to A) \mapsto \text{Bord}^{W_a \to U_a}_d : = \mathcal{D}(\alpha).
\]

Now let \( \Delta_a \subset \Delta \) be the subcategory with the same objects, but only the injective order preserving maps. This inclusion is cofinal (see, for example, Lurie [2017, a, Lemma 6.5.3.7]). Fix any subset \( J \subset A \) and consider the subcategory \( (\Delta^{op}_A)_{/J} \) whose objects are injective maps \( j : [n] \to J \) and morphisms are injective order preserving maps \( [n] \hookrightarrow [m] \) commuting with the \( j \)'s. Then we can rewrite the homotopy colimit as

\[
\text{hocolim}_{[n] \in \Delta^{op}} \left( \prod_{\alpha : [n] \to A} \text{Bord}^{W_a \to U_a}_d \right) \simeq \text{hocolim}_{J \subset A} \text{hocolim}_{j \in (\Delta^{op}_A)_{/J}} \mathcal{D}(j).
\]

where the first homotopy colimit appearing on the right is taken over inclusions of finite subsets \( J \subset A \). Observe that the poset of all \( J \) is a direct category, with transition maps

\[
\text{colim}_{j \in (\Delta^{op}_A)_{/J}} \mathcal{D}(j) \to \text{colim}_{k \in (\Delta^{op}_A)_{/K}} \mathcal{D}(k)
\]

being monomorphisms for all inclusions \( J \subset K \), so by Lemma 6.1.1 the strict colimit over all \( J \subset A \) computes the homotopy colimit. The second homotopy colimit, indexed by \( (\Delta^{op}_A)_{/J} \), is also represented by its corresponding strict colimit. This follows again from Lemma 6.1.1 because all transition maps are manifestly monomorphisms of simplicial presheaves. □
6.2 Reduction to representable geometric structures

As explained in the proof of Theorem 5.0.1, we reduce to the case of representable presheaves on $\text{OEmb}^f_d$. Hence, we will be interested in the bordism category $\text{Bord}^W_d \to U$. Because a representable presheaf is a presheaf of discrete simplicial sets, we obtain the simplicial presheaf whose value on $(\{m\}, \langle \ell \rangle, V) \in \Delta^{\times d} \times \Gamma \times \text{CartSp}$ is the nerve of the groupoid $\text{Ob} = \coprod_{(M,C,P), (\{m\}, \langle \ell \rangle)} \text{OEmb}^f_d(M \times V \to V, W \to U)$, $\text{Mor} = \coprod_{\varphi: (M,C,P) \to (\tilde{M}, \tilde{C}, \tilde{P})} \text{OEmb}^f_d(\tilde{M} \times V \to V, W \to U)$.

The source map sends a component indexed by $\varphi$ to the component indexed by the target of $\varphi$, via the identity map. The target map precomposes with $\varphi: (M, C, P) \to (\tilde{M}, \tilde{C}, \tilde{P})$ and maps to the the componentet indexed $(M, C, P)$, via identity. For a point $p \in \mathbb{N}$, the corresponding stalk at $p$ takes germs around $0 \in V = \mathbb{R}^p$.

**Remark 6.2.1.** Any parallel pair of morphisms in the groupoid $\text{Bord}^W_d \to U((\{m\}, \langle \ell \rangle, V)$ necessarily coincides. Indeed, automorphisms are identities, since they have to commute with an open embedding. Hence this groupoid is discrete, i.e., it is equivalent to its connected components. In fact, each connected component can be identified with a trivial bundle whose fiber over a point $v \in V$ is an open subset of the corresponding fiber over $f(v)$, where $f: V \to U$ is the smooth map on base spaces.

**Notation 6.2.2.** Given $d \geq 0$, a manifold $M \in \text{Man}$, $V \in \text{CartSp}$, and $(W \to U) \in \text{OEmb}^f_d$, we will denote germs of fiberwise open embeddings $f: M \times V \to W$ (over some $V \to U$) around the origin $0 \in V = \mathbb{R}^p$ as $f: M \Rightarrow W$.

6.3 Filtration on the bordism category

We now define the filtration on $p^*\text{Bord}^W_d \to U$ that we used in Theorem 5.0.3. The idea will be to show that all but the first layer in the filtration come from attaching simplices by inner horns, and the corresponding transition map is an acyclic cofibration. The first layer of the filtration will use the $\Gamma$-boundary inclusions (Proposition 5.6.2) and we will exhibit it as a transfinite composition of coface changes of these maps.

**Definition 6.3.1.** Fix an open cover $\{W_a \to U_a\}$ of $W \to U$. We define a filtration

$$\text{colim}_{[n] \in \Delta^{\times p}} \coprod_{\alpha: [n] \to A} p^*\text{Bord}^W_d \to U_a \hookrightarrow B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_d \subset p^*\text{Bord}^W_d \to U$$

inductively as follows. Objectify on $\Delta^{\times d} \times \Gamma$, we define $B_i(\{m\}, \langle \ell \rangle)$ as the simplicial subset of $p^*\text{Bord}^W_d \to U(\{m\}, \langle \ell \rangle)$ consisting of connected components whose vertices we now specify. A vertex $x \in B_0(\{m\}, \langle \ell \rangle)$ is given by a germ $f: M \Rightarrow W$ around $\text{core}[0, \{m\}]$ such that $f$ maps each connected component of the germ into some $W_a \subset W$. We define the vertices of $B_i(\{m\}, \langle \ell \rangle)$ to be the subset of vertices $x \in p^*\text{Bord}^W_d \to U(\{m\}, \langle \ell \rangle)$ that admit a cut tuple $\tilde{C}$ with the following properties:

- $\tilde{C}$ contains the cut tuple of $x$ in the $i$-th direction.
- For each $0 \leq j_i \leq m_i - 1$, the bordism $\Upsilon_{j_i}$ with the same data as $x$, but with cut tuple in the $i$-th direction given by two successive cuts $\tilde{C}_{j_i}$ and $\tilde{C}_{j_i+1}$, is in $B_{i-1}$.

6.4 $\Gamma$-direction

**Proposition 6.4.1 ($\Gamma$-direction).** Let $B_0$ be as in Definition 6.3.1. For each point $p \in \mathbb{N}$, the monomorphism $p: \text{colim}_{[n] \in \Delta^{\times p}} \coprod_{\alpha: [n] \to A} p^*\text{Bord}^W_d \to U_a \hookrightarrow B_0$ is an acyclic cofibration.
Proof. Fix \( m \in \Delta^{\times d} \). We evaluate \( \rho \) on \( m \), obtaining a morphism in \( \mathcal{P} \mathcal{S} \mathcal{H} \Delta(\Gamma) \). By Proposition 5.2.1, it suffices to show that this is an acyclic cofibration in \( \mathcal{P} \mathcal{S} \mathcal{H} \Delta(\Gamma)_{\text{local}} \). We will not include \( m \) in the notation.

**Step 1** (A filtration on \( B_0 \)): Let \( B_0^k \subset B_0 \) denote the simplicial subsheaf that is objectwise given by taking the connected components that contain a vertex in the strict colimit, or the germ of \( \text{core}[0, m] \) has at most \( k \) connected components. Hence, we have a filtration

\[
\colim_{[n] \in \Delta^{\infty}} \prod_{\alpha \in [n]} p^* \text{Bord}_d \alpha \rightarrow \sum \rightarrow \sum \rightarrow \cdots \rightarrow B_0.
\]

The first equality holds by definition of \( B_0 \) and \( B_1 \). We now show that each inclusion \( B_0^{k-1} \rightarrow B_0^k \) is a weak equivalence, for \( k \geq 2 \). By Proposition 5.3.1, it suffices to show that for each \( [l] \in \Delta \), the map \( (B_0^{k-1})_l \rightarrow (B_0^k)_l \) is a weak equivalence of presheaves of sets on \( \Gamma \). We claim that \( (B_0^{k-1})_l \rightarrow (B_0^k)_l \) is a cobase change of a coproduct of boundary inclusions \( \partial(k) \rightarrow j(k) \), which are weak equivalences by Proposition 5.6.2. Fix \( [l] \in \Delta \), which we will omit from the notation below. Recall that \( l \)-simplices are \( l \)-tuples of isomorphisms of bordisms \((M, C, P)\). We will work with the starting object of this chain, while implicitly extending the constructions to the entire \( l \)-simplex via the given isomorphisms. This is possible because isomorphisms of bordisms induce isomorphisms of germs of cores.

**Step 2** (The proposed pushout): Consider the set of maps

\[
f : j(k) \rightarrow B_0^k,
\]

where \( j \) is the Yoneda embedding, that do not factor through \( B_0^{k-1} \rightarrow B_0^k \), and the bordism \((M, C, P)\) corresponding to \( f \) under the Yoneda lemma has the following property: the map \( P : M \rightarrow \langle k \rangle \), restricted to the germ of \( \text{core}[0, m] \), is surjective away from the basepoint in \( \langle k \rangle \) and maps all other connected components of \( M \) to the basepoint in \( \langle k \rangle \). The symmetric group \( \Sigma_k \) acts on this set, by permuting the elements in \( \langle k \rangle \). This action is free, since the restriction of \( P : M \rightarrow \langle k \rangle \) is required to be surjective, away from the basepoint. In fact, the restriction of \( P \) is also injective on connected components because the germ of \( \text{core}[0, m] \) has at most \( k \) connected components. Choose a single representative from each \( \Sigma_k \)-orbit and denote the resulting set by \( S \).

For \( f \in S \), the composition \( \partial(k) \rightarrow j(k) \rightarrow B_0^k \) factors through \( B_0^{k-1} \), since elements of \( \partial(k) \) are maps of finite pointed sets \( \langle k \rangle \rightarrow \langle \ell \rangle \) that have at most \( k - 1 \) elements of \( \langle k \rangle \) map to non-basepoint elements of \( \langle \ell \rangle \). We have a commutative diagram

\[
\begin{array}{ccc}
\prod_{f \in S} \partial(k) & \longrightarrow & B_0^{k-1} \\
\downarrow & & \downarrow \\
\prod_{f \in S} j(k) & \longrightarrow & B_0^k.
\end{array}
\] (6.4.2)

We show that this is an objectwise pushout square.

**Step 3**: Fix \( \langle \ell \rangle \in \Gamma \) and evaluate at this object. We show that the bottom map in Eq. (6.4.2) induces a bijection of sets

\[
\prod_{f \in S} j(k)(\langle \ell \rangle) \setminus \partial(k)(\langle \ell \rangle) \rightarrow B_0^k(\langle \ell \rangle) \setminus B_0^{k-1}(\langle \ell \rangle).
\] (6.4.3)

Elements in the left side are precisely pairs \((f, \sigma)\), where \( f \in S \) and \( \sigma : \langle k \rangle \rightarrow \langle \ell \rangle \) satisfies \( \sigma^{-1}\{\ast\} = \{\ast\} \). The displayed map sends \((f, \sigma) \mapsto f(\sigma)\), i.e., we apply the \( T \)-structure map \( \sigma \) to \( f \). Observe that the core of \( f(\sigma) \) is the same as the core of \( f \). In particular, the core of \( f(\sigma) \) has \( k \) connected components and does not factor through any \( W_a \), hence \( f(\sigma) \not\in B_0^{k-1}(\langle \ell \rangle) \).

It remains to show that any element \( x \in B_0^k(\langle \ell \rangle) \setminus B_0^{k-1}(\langle \ell \rangle) \) comes from a unique pair \((f, \sigma) \in \prod_{f \in S} j(k)(\langle \ell \rangle) \setminus \partial(k)(\langle \ell \rangle) \). If \( x = f(\sigma) \), then \( x \) and \( f \) must have exactly the same bordism data (including the core) except for \( P \), for which we have \( P_x = \sigma \circ P_f \). By definition, the restriction of \( P_f \) to the germ of the core is bijective on \( \pi_0 \), after removing the basepoint in \( \langle k \rangle \). Hence the equation \( P_x = \sigma \circ P_f \) has exactly one solution in terms of \( \sigma \), which establishes uniqueness of \((f, \sigma)\).
To show the existence of \((f, \sigma)\), we construct \(f\) from \(x\) by changing the \(P\)-component of the bordism data of \(x\) to an arbitrary surjection \(P \ell\) from the germ of core\([0, m], P_2\) to \(\langle k \rangle \setminus \{\ast\} \subset \langle k \rangle\), which is bijective on \(\pi_0\). Among all such surjections, there is a unique one with the corresponding map \(f \in S\), since any two surjections have corresponding \(f\)’s in the same \(\Sigma_k\)-orbit.

\[\square\]

6.5 \(\Delta\)-direction

We now turn to the most technical part of the proof. Supporting lemmas will be proved subsequently.

**Proposition 6.5.1.** The monomorphism \(\rho_i : B_i \to B_i\) in Definition 6.3.1 is an equivalence.

**Proof.** Consider the monomorphism

\[\rho_i(m, \langle \ell \rangle) : B_{i-1}(m, \langle \ell \rangle) \to B_i(m, \langle \ell \rangle),\]

obtained by evaluating \(\rho_i\) on an arbitrary multisimplex \(m \in \Delta \times d^{-1}\) and \(\langle \ell \rangle \in \Gamma\), via restriction along the functor

\[\Delta \to \Delta \times d \times \Gamma, \quad \omega \mapsto (m_1, \ldots, m_{i-1}, \omega, m_i, \ldots, m_d, \langle \ell \rangle).\]

By Proposition 5.2.1, it suffices to show that \(\rho_i(m, \langle \ell \rangle)\) is an equivalence in the Rezk model structure on simplicial objects in simplicial sets. In fact, recall that both \(B_{i-1}\) and \(B_i\) are discrete in the target simplicial direction \([l] \in \Delta\), i.e., the canonical maps \(B_i \to \pi_0(B_i)\) and \(B_{i-1} \to \pi_0(B_{i-1})\) are objectwise equivalences (see Remark 6.2.1). This amounts to working with the images of the open embeddings of bordisms into \(W\). Hence, we have further reduced the claim to showing the \(\pi_0(\rho_i(m, \langle \ell \rangle))\) is a weak equivalence in the Joyal model structure on simplicial sets. We will redefine \(\rho_i\) to be the map \(\pi_0(\rho_i(m, \langle \ell \rangle))\), i.e., we will omit \(m, \langle \ell \rangle\) and \(\pi_0\) in the notation below.

By Proposition 5.5.2, it suffices to show that for each pair of vertices \(x\) and \(y\), the induced map

\[\mathcal{N}(\mathcal{C}^{nec}(\rho_i)) : \mathcal{N}(\mathcal{C}^{nec}(B_{i-1})(x, y)) \to \mathcal{N}(\mathcal{C}^{nec}(B_i)(x, y))\]

is a weak equivalence. The map \(\mathcal{N}(\mathcal{C}^{nec}(\rho_i))\) splits as a coproduct, indexed by the ambient bordisms with endcuts \(x\) and \(y\). By Proposition 6.6.4, each component of \(\mathcal{N}(\mathcal{C}^{nec}(B_{i-1})(x, y))\) and \(\mathcal{N}(\mathcal{C}^{nec}(B_{i-1})(x, y))\), indexed by a bordism with endcuts \(x\) and \(y\), is contractible. Hence \(\mathcal{N}(\mathcal{C}^{nec}(\rho_i))\) is an equivalence.

\[\square\]

The claim about the contractibility of \(\mathcal{N}(\mathcal{C}^{nec}(B_{i-1})(x, y))\) in Proposition 5.5.1 will occupy the remainder of this section. We begin with an explicit description of the Dugger–Spivak necklace categories. These have the following objects and morphisms.

- A necklace in \(\mathcal{C}^{nec}(B_{i-1})(x, y)\) is a composable collection of bordisms, as defined in Definition 4.1.3.
  Each bead of the necklace specifies a bordism in \(B_{i-1}\). Bordisms must coincide on the germ of each joint cut and can therefore be glued (see Milnor 1969) along each joint. The resulting bordism is not equipped with an open embedding into \(W\), but only a local diffeomorphism. Hence, we obtain a single bordism with a different geometric structure (i.e., a local diffeomorphism to \(W\)). The collection of cut tuples together forms a cut tuple of the entire bordism, with the joints forming a subset of this cut tuple. We emphasize that this resulting bordism is not a vertex in any level of the filtration of Definition 5.3.1, since the map to \(W\) is not an open embedding in general.

- A morphism only exists between necklaces whose underlying bordisms (as constructed in the previous paragraph) become isomorphic after throwing away the inner cuts. A morphism is a composition of the following elementary operations.
  1. The joint cut between \(C^j\) and \(C^{j+1}\) is converted to a cut in the union \(C^j \cup C^{j+1}\), provided that the resulting bead is in \(B_{i-1}\).
  2. New compatible cuts are added to an existing cut tuple \(C^j\).

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The necklace category associated to $B_i$ has the same description, but we require that each bead is in $B_i$.

### 6.6 Bordism categories for representable geometric structures are 1-truncated

Observe that the contractibility claim of Proposition 6.5.1 implies that the fibrant replacements (in the Rezk model structure) of $B_{i-1}$ and $B_i$ are weakly equivalent to nerves of ordinary categories. This is intuitively clear, since the operation of fibrant replacements adds formal compositions and homotopies between them. However, there is a natural candidate for a composition of bordisms, given by gluing, which produces a bordism with a local diffeomorphism to $W$.

To prove contractibility, we will invoke the simplicial Whitehead theorem in the form of Lemma 5.4.3. However, to construct the homotopy there, we need a cutting and gluing operation between cut manifolds. Moreover, we will need our surgered manifolds to have specific properties that allow us to define a homotopy. We begin with our cutting and gluing lemmas.

**Lemma 6.6.1.** Let $C$ and $C'$ be cuts on an open subset $N$ whose underlying cut manifolds $C_{\leq 0}$ and $C'_{\leq 0}$ intersect transversally. Then for any open neighborhood $U$ of the intersection $C_{\leq 0} \cap C'_{\leq 0}$, there is a cut $C''$ having the following properties:

1. Outside of $U$,
   
   $C''_{\leq 0} = (C'_{\leq 0} \cap C'_{>0}) \cup (C_{\leq 0} \cap C'_{>0})$.

2. $C''$ is compatible with $C$ and $C''$ is compatible with $C'$, in the sense of Definition 4.1.2.

**Proof.** Fix maps $h, k : N \to \mathbb{R}$ defining the cuts $C$ and $C'$. Consider the map

$$(h, k) : N \to \mathbb{R}^2.$$ 

Then $(h, k)^{-1}(0) = C_{\leq 0} \cap C'_{\leq 0}$, and the preimages of the two axes are $C_{\leq 0}$ and $C'_{\leq 0}$. Choose an open ball of radius $\delta$ around $0 \in \mathbb{R}^2$. Since transversality is a generic property, there exists a smooth embedding $\iota : \mathbb{R} \to \mathbb{R}^2$ which is transversal to $(h, k)$ and which satisfies the following:

1. $\iota(t) = (0, -t)$ for $t < -1$
2. $\iota(t) = (t, 0)$ for $t > 1$
3. $\iota(-1, 1) \subset B_\delta$, where $B_\delta$ is the open ball of radius $\delta$. 

$\hat{\mathbb{R}}$
Then \( C^m_{\delta_0} := (h,k)^{-1}(\mu(R)) \) gives rise to a cut with the desired properties. Since the tubular neighborhood of \( h^{-1}(0) \cap k^{-1}(0) \) is given by \((h,k)^{-1}(B_{\delta})\), we can get an arbitrarily small radius by letting \( \delta \to 0 \).

From the definition of the filtration in Definition 3.1, every bordism in \( B_{i}([1]) \) can be cut in the first \( i \)-directions so that each cell is contained in some \( W_a \). However, it will be useful for us to find an explicit such cut tuple.

**Definition 6.6.2.** Two cuts \( C \) and \( C' \) on a bordism \( M \) are called *transversally compatible* if locally, with respect to some open cover of \( M \), \( C \) intersects \( C' \) transversally or \( C \) is compatible with \( C' \), i.e., for the given element \( U \) in the cover, \( C \subset C' \cap U \) or \( C' \cap U \subset C \cap U \).

**Lemma 6.6.3.** Let \( M \) be a bordism in \( B_{i-1} \). There is an open neighborhood \( U \supset \text{core}(M) \) with the following property. Suppose \( M' \) is a bordism with the same bordism data as \( M \), except the cuts in the \( i \)-direction. If \( \text{core}(M') \subset U \), then \( M' \) is in \( B_{i-1} \).

**Proof.** By definition, there are cut tuples in the first \( i-1 \) directions which decomposition of \( M \) into cells \( M_{[j,j']} \) contained in some \( W_a \). Since there are only finitely many such cells, and each cell is compact, there is an \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood \( N_{a} \) of each \( M_{[j,j']} \) is still contained in \( W_a \). Since there are only finitely many cells in the decomposition, there is a \( \delta \)-neighborhood \( U \) of the core of \( M \) so that for any bordism \( M' \subset U \), the cuts decomposing \( M \) can be perturbed so that they are transversal to the cuts of \( M' \) and \( M_{[j,j']} \subset N_{a} \).

**Lemma 6.6.4.** Fix a bordism \( M \) in \( B_{i}([1]) \). Fix an arbitrary open cover \( \{W_a\} \) of \( M \). Then there is a cut tuple \( \Psi \) which satisfies the following properties:

- The endcuts of \( \Psi \) coincide with the endcuts of \( M \).
- For each \( j \) indexing a cut in the cut tuple \( \Psi \), the closure of \( \Psi_{(j,j+1)} \) is contained in some \( W_a \).

Moreover, for any finite collection of cut tuples \( \{C_k\}_{k \in K} \) on \( M \), we can find such a \( \Psi \) so that each \( \Psi_j \) is transversally compatible with each \( C_{k,j'} \in C_k \).

**Proof.** To construct the \( \Psi \), we will need to use a lemma from Morse theory for manifolds with corners (see Budney [2002, I, Section 3.1] for a one page survey, Vakhrameev [1998, A], Handran [2002, a, 2004, d] for proofs). We construct \( \Psi \) by induction on the number of critical points, for some Morse function \( f \) on the core of \( M \) (which is a manifold with corners).

For the base case, we will show that any bordism \( M \) that admits a Morse function with no critical points has such a decomposition. Fix a Morse function \( f : M \to \mathbb{R} \) with \( f \) giving the initial cut (see Definition 1.1.1). Let \( y \) denote the terminal cut of \( M \) and let \( s = \sup_{x \in \text{core}(M)} f(x) \). The gradient of \( f \) may flow outside the core, but a modification of the gradient flow (see Vakhrameev [1998, A, Theorem 2.1]) produces a flow that respects the boundary. We denote this flow by \( \varphi_t \).

Partition \([0,s]\) into intervals \( I_k = [t_{k-1}, t_k], k = 1, \ldots, N \), where \( 0 = t_0 \leq t_1 \leq \cdots \leq t_N = s \) so that for any \( x \in f^{-1}(t_{k-1}) \) the flow line \( \varphi_t(x), t \in [t_{k-1}, t_k] \), is in some \( W_a' \). We cut \( f^{-1}[0,s] \) into thin strips by the cuts \( f^{-1}(t_k) \).

Fix a strip \( f^{-1}(t_k) \). For any \( a \in A \), define \( W_a'' = \{ x \in f^{-1}(t_{k-1}) \mid \varphi_t(x) \in W_{a'}, t \in [t_{k-1}, t_k]\} \). Choose a finite subcover \( \{X_{b}\}_{1 \leq b \leq B} \) of \( \{W_a''\}_{a \in A} \). Choose a partition of unity \( \{\psi_b\} \) subordinate to the cover \( \{X_{b}\} \). Rescale \( \varphi_{\delta} \) so that their sum is \( t_k - t_{k-1} \). Define \( \Psi_{k,b} \) to be the the diffeomorphic image of \( f^{-1}(t_{k-1}) \) under the map

\[
x \mapsto \varphi_{(t_{k-1} + \psi_1(x) + \cdots + \psi_b(x))}(x).
\]

We have \( \Psi_{k,0} = f^{-1}(t_{k-1}) \) and \( \Psi_{k,B} = f^{-1}(t_k) \). The cuts \( \Psi_{k,b} \) are the desired cuts.

We now prove the inductive step. Consider an isolated critical point \( x \). If \( x \) lies in the top-dimensional stratum, then by Milnor [1969, Theorem 3.2], we can modify \( f \) to a Morse function \( F \) that coincides with \( f \) everywhere, except for an arbitrarily small neighborhood of \( x \). By construction, this \( F \) satisfies the following property. Let \( y \) denote the critical value corresponding to \( x \). There is a sufficiently small \( \epsilon \) such that the cut tuple with endcuts \( \Psi_0 = f^{-1}(y-\epsilon) \) and \( \Psi_1 = F^{-1}(y-\epsilon) \) satisfies \( \Psi_{(0,1)} \subset W_{a}' \), for some \( a \in A \). The bordism
The connected components of both Proposition 6.6.6. satisfies the condition of Lemma 6.6.3 by assumption. □

Proof. Lemma 6.6.1 to cut and glue and processing cuts in this order. Equal cuts are processed simultaneously. At each stage, we invoke where \( f \) we seek to construct a zig-zag of simplicial homotopies from \( x \) to \( y \). We denote the necklaces picked out by \( f \) we seek to construct a zig-zag of simplicial homotopies from \( x \) to \( y \).

\[ f : Sd^k \partial \Delta[n] \to \mathcal{N}(\mathcal{C}^{\text{nc}}(B_0)(x,y)_M), \]

we seek to construct a zig-zag of simplicial homotopies from \( f \) to

\[ f' : Sd^k \partial \Delta[n] \to \mathcal{N}(\mathcal{C}^{\text{nc}}(B_0)(x,y)_M), \]

where \( f' \) factors through \( \mathcal{N}(\mathcal{C}^{\text{nc}}(B_0)(x,y)_M) \). This proves the proposition, since \( \mathcal{C}^{\text{nc}}(B_0)(x,y)_M \) has a terminal object.

We denote the necklaces picked out by \( f \) by \( \{C_q\}_{q \in Q} \), where \( Q \) is the set of vertices of \( Sd^k \partial \Delta[n] \). Lemma 6.6.3 provides an \( \epsilon \) such that the \( 2\epsilon \)-neighborhood of each bead in each necklace \( C_q \) satisfies the
conditions of Lemma 6.6.3. Lemma 6.6.4 gives a cut tuple $\Psi$ such that each cut $\Psi_j$ is transversally compatible with each cut in the necklaces picked out by $f$. Set $\delta = \epsilon / |\Psi|$.

Introduce an equivalence relation on the set of pairs $(q, h)$, which is generated by $(q, h) \sim (q', h')$ if $C_{q, h} = C_{q', h'}$ and $q$ and $q'$ are vertices in the same edge in $\text{Sd}^k \partial [n]$. The equivalence classes have a natural partial order, induced by the partial order on cuts (see Definition 4.1.1). Extend this partial order to some total order.

We now prove by induction on $j$, in decreasing order, that there is a zig-zag of homotopies between $f$ and $f_j$, where

$$f_j : \text{Sd}^k \partial [n] \to \mathcal{N}(\mathcal{C}^{\text{nce}}(B_{i-1})(x, y)_M)$$

is a map factoring through $\mathcal{N}(\mathcal{C}^{\text{nce}}(B_{i-1})(x, \Psi_j)_M)$. The induction starts at $f|_{\Psi_1} = f$. Simplices in the nerve of a poset are uniquely determined by their vertices. Hence, it suffices to construct the zig-zag of homotopies on vertices and verify that the relevant inequalities are satisfied.

We apply Lemma 6.6.5 to the necklaces $\{C_q\}_{q \in Q}$, with $\epsilon$ and $\delta$ chosen above and the cut $\Psi_j$, to obtain new necklaces $\{C'_q\}_{q \in Q}$. The properties 1–3 in Lemma 6.6.5 ensure that $\{C'_q\}_{q \in Q}$ give rise to a map

$$f_j : \text{Sd}^k \partial [n] \to \mathcal{N}(\mathcal{C}^{\text{nce}}(B_{i-1})(x, y)_M).$$

It remains to connect this map to $f_{j+1}$ by a zig-zag of homotopies. This zig-zag will be a composition of zig-zags, one for each equivalence class $[(q, h)]$ such that the corresponding cut $C_{q, h}$ satisfies $C_{q, h} \not\subseteq \Psi_j$. We process these equivalence classes in increasing order by induction, extending an existing zig-zag of homotopies. Fix such an equivalence class. We extend the zig-zag by inserting $C'_{q, h}$ immediately before $C_{q, h}$, for each $(q, h)$ in the equivalence class. If $C_{q, h}$ has duplicates, i.e., the corresponding cut tuple is simplicially degenerate, we insert as many duplicates of $C'_{q, h}$. This defines the zig homotopy in the zig-zag. We then remove the old cuts corresponding to $(q, h)$ in the equivalence class, obtaining the zag homotopy in the zig-zag. Properties 4–6 in Lemma 6.6.5 ensure that this is produces a zig-zag of simplicial homotopies through maps to $\mathcal{N}(\mathcal{C}^{\text{nce}}(B_{i-1})(x, y))$. □
7 Applications

7.1 Smooth field theories

In this section, we describe smooth field theories and locality. Roughly speaking a smooth field theory is a symmetric monoidal functor \( \mathcal{F} : \text{Bord}_d^S \rightarrow T \), where \( T \) is some sheaf of symmetric monoidal \((\infty, d)\)-categories. This would be a precise definition, but we have not yet said explicitly what a symmetric monoidal functor is, nor have we ensured that the functor is appropriately derived. As explained in Remark 2.3.3, we have an \( \infty \)-internal hom available in \( C^{\infty}\text{Cat}_{\infty,d}^\otimes \), which is simply given by deriving the internal hom \( \text{Fun}^\otimes(-,-) \) in the underlying 1-category. We therefore have a corresponding smooth symmetric monoidal \((\infty, d)\)-category of functors \( \mathcal{R}(\text{Fun}^\otimes)(A, X) \), between two smooth symmetric monoidal \((\infty, d)\)-categories \( A \) and \( X \).

Remark 7.1.1. To recover the underlying \((\infty, d)\)-category of symmetric monoidal functors, one can take global sections over \( \text{CartSp} \), i.e., evaluate at the point.

We can recover the underlying space of field theories by deriving the mapping simplicial set functor in \( C^{\infty}\text{Cat}_{\infty,d}^\otimes \). Alternatively, one can evaluate the derived internal hom at \( (([0],[0],\ldots,[0]),(1),\mathbb{R}^0) \in \Delta^{\times d} \times \Gamma \times \text{CartSp} \).

The derived internal hom is given by cofibrantly replacing \( A \) and the global sections functor is a right Quillen functor, so that when \( A \) is cofibrant and \( X \) is fibrant, \( \text{Fun}^\otimes(A, X) \) is fibrant. There is an obvious comparison map \( \text{Fun}^\otimes(A, X) \rightarrow \text{Map}(A, X) \), which takes the limit over \( \Delta^{\times d} \times \Gamma \times \text{CartSp} \) (equivalently evaluates at \( (0, (1), *) \)). Under this comparison, an object in the symmetric monoidal \((\infty, d)\)-category \( \text{Fun}^\otimes(A, X) \) is identified with a vertex in the simplicial set \( \text{Map}(A, X) \).

Definition 7.1.2. Let \( \mathcal{S} \) be a sheaf on \( \text{OEmb}^d \) and let \( T \) be a smooth symmetric monoidal \((\infty, d)\)-category, i.e., a fibrant object in \( C^{\infty}\text{Cat}_{\infty,d}^\otimes \). A (fully-extended) smooth, \( d \)-dimensional field theory with \( \mathcal{S} \)-structure is a natural transformation
\[
\mathcal{F} : \text{Bord}_d^S \rightarrow T.
\]
Equivalently, it is an object in \( \text{Fun}^\otimes(\text{Bord}_d^S, T) \). A smooth family of (fully-extended) field theories is natural transformation
\[
\mathcal{F} : \text{Bord}_d^S \times U \rightarrow T.
\]
where \( U \) is embedded by the composite
\[
j_{\text{CartSp}} : \text{CartSp} \hookrightarrow \mathcal{PSh}_{\Delta}(\text{CartSp}) \hookrightarrow \mathcal{PSh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \text{CartSp}).
\]
The smooth symmetric monoidal \((\infty, d)\)-category of functorial field theories with geometric structure \( \mathcal{S} \) is defined as the internal hom \( \text{Fun}^\otimes(\text{Bord}_d^S, T) \).

7.2 Classifying spaces of field theories

In this section, we prove Corollary 1.0.2. This provides an affirmative answer to a long-standing conjecture of Stolz and Teichner. In order to prove the statement about concordance, we will need to recall the notion a cohesive \( \infty \)-topos of Schreiber 2017.1. These are certain toposes which participate in a quadruple adjunction \( (| \cdot | \dashv \delta \dashv \Gamma \dashv \delta^\dagger) \), with \( \delta \) and \( \delta^\dagger \) fully faithful and \( | \cdot | \) preserving products. In our context the adjunction takes the form
\[
| \cdot |, \Gamma : \text{Sh}_{\infty}(\text{Man}) \rightleftarrows \infty\text{-Grpd} : \delta, \delta^\dagger \tag{7.2.1}
\]
with left adjoints depicted above their corresponding right adjoints. These functors give rise to idempotent monads
\[
j := \delta | \cdot | \dashv \nu := \delta \Gamma \dashv \varpi := \delta^\dagger \Gamma, \tag{7.2.2}
\]
each of which reflects a different nature of a smooth stack (see Schreiber 2017.1 for detailed discussion). We will focus on one of these idempotent functors, namely the left most adjoint \( j := \delta | \cdot | \), which we call the shape functor.
Remark 7.2.3. The previous adjunction goes through just as well for sheaves with values in spectra \cite{Bunke-Nikolaus-Voelk}. In particular, for a sheaf of spectra \( X \in \mathcal{S}_\infty(\text{Man}; \mathcal{S}p) \), we can take the geometric realization \( |X| \), which gives a spectrum. We can also reflect back into sheaves of spectra, by taking the shape
\[
\int(X) = \delta|X| \in \mathcal{S}_\infty(\text{Man}; \mathcal{S}p).
\]

Remark 7.2.4. Recall that the \( \infty \)-category of smooth stacks is a Cartesian closed \( \infty \)-category. In particular, it admits an internal hom (i.e., a mapping stack), which we denote by \( \mathcal{M}ap(X,Y) \). If \( X \) and \( Y \) are stacks (i.e., satisfy descent), then this stack is given simply by computing the corresponding internal hom in presheaves, which itself satisfies descent (since sheafification \( L \) preserves products). The global sections of \( \mathcal{M}ap(X,Y) \) produce the usual mapping space between sheaves.

The \( \infty \)-category of sheaves of spectra is also closed monoidal, with the smash product given by the sheafification of the pointwise smash product and internal hom given by the sheaf of function spectra, which we denote \( F(X,Y) \). The global sections give an enrichment over ordinary spectra.

Remark 7.2.5. Berwick-Evans, Boavida de Brito, and the second author \cite{Berwick-Evans} (see also \cite{Bunke-Nikolaus-Voelk} for the structured case) proved an explicit formula was proved for the shape functor \( \int \) appearing in \eqref{eq:shape-functor}. More precisely, \cite[Theorem 1.1]{Berwick-Evans} proves that if \( X \) is a smooth sheaf of \( \infty \)-groupoids (i.e., \( X \) satisfies descent), then we have an equivalence of smooth sheaves
\[
\int Y \simeq \colim_{[n]\in \Delta^op} \mathcal{M}ap(\Delta^n, Y),
\]
where \( \Delta^n \) is the smooth \( n \)-simplex, viewed as a smooth stack via its diffeological space of plots. Although it is fairly easy to see that the colimit on the right is invariant under concordance, it is highly nontrivial to show that it satisfies descent.

By Theorem \ref{thm:conc-maps}, the functor
\[
\mathcal{S} \mapsto \mathcal{M}ap(\text{Bord}_d^{\mathcal{S}}, T)
\]
preserves local weak equivalences in \( \mathcal{S}_\infty(\text{OEmb}^f_d) \). If we regard a manifold as a sheaf on \( \text{OEmb}^f_d \) via its functor of smooth points, then Čech nerves of covers of manifolds are local weak equivalences in \( \mathcal{S}_\infty(\text{OEmb}^f_d) \). Hence if we restrict the above functor along the embedding \( \text{Man} \hookrightarrow \mathcal{S}_\infty(\text{OEmb}^f_d) \) we get a well defined sheaf on the site of smooth manifolds.

Formula \eqref{eq:shape-functor} has some striking consequences. For example, if \( \sim_{\text{con}} \) denotes the equivalence relation given by concordance, then for any smooth manifold \( M \), we have the formula
\[
Y(M)/\sim_{\text{con}} \cong [M, \colim_{[n]\in \Delta^op} Y(\Delta^n)]
\]
where on the right we take homotopy classes of maps and on the left we mod out by the relation of concordance. This immediately gives a classifying space construction for \( Y \). We illustrate with the following example.

Example 7.2.8. Let \( Y = \text{Vect} \) be the sheaf of groupoids given by vector bundles with isomorphisms between them. Because every vector bundle is locally trivial, we have an equivalence \( \text{Vect}(\Delta^n) \simeq \text{BGL}(\Delta^n) \) on every smooth \( n \)-simplex, where \( \text{BGL} = \coprod_{n \in \mathbb{N}} \text{BGL}(n) \) is the delooping of the stable, smooth general linear group. Then its classifying space can be computed as
\[
|\text{Vect}| \simeq \colim_{[n]\in \Delta^op} |\text{Vect}(\Delta^n)| \simeq \colim_{[n]\in \Delta^op} |\text{BGL}(\Delta^n)|
\]
\[
\simeq \colim_{[n]\in \Delta^op} \text{hocolim}_{[m]\in \Delta^op} C^\infty(\Delta^n, \text{GL}^m)
\]
\[
\simeq \text{hocolim}_{[m]\in \Delta^op} \colim_{[n]\in \Delta^op} C^\infty(\Delta^n, \text{GL}^m)
\]
\[
\simeq \text{hocolim}_{[m]\in \Delta^op} (\text{sing}(\text{GL}^m))
\]
If we further apply the geometric realization \( |\cdot| : \text{sSet} \to \text{Top} \), and use the natural equivalence \( |\text{sing}(\text{GL}^m)| \simeq \text{GL}^m \), we recover the classifying space construction
\[
\text{Vect}(M)/\sim_{\text{con}} \cong [M, \text{BGL}].
\]
We can apply this construction to field theories.

**Definition 7.2.9.** We define the *classifying space* of field theories as

\[ \mathcal{B} \mathcal{F} \mathcal{T}(d, T) := \text{colim} \text{Map}(\text{Bord}^d_{\Delta^n}, T). \]

Then by \cite{2019}, we immediately deduce Corollary 1.0.2. □

### 7.3 Power operations in the Stolz–Teichner program

Building upon the work of Barthel–Berwick-Evans–Stapleton \cite{2020b}, we develop a theory of geometric power operations for fully extended geometric field theories. Fix \( d \geq 0 \) and a target \( T \) for the field theories.

**Definition 7.3.1.** The \( n \)-th power cooperation \((n \geq 0)\) is a morphism

\[ \text{Bord}(X^{\times n}/\Sigma_n) \to \text{Bord}(X), \]

where \( X \) is a simplicial presheaf on \( \text{OEmb}^f_d \). The notation \(-/\Sigma_n\) denotes the homotopy quotient of simplicial presheaves presented as a bar construction, where \( \Sigma_n \) acts on \( X^{\times n} \) by permutations. Fixing an object in \( \Delta^{\times n} \times \Gamma \times \text{CartSp} \), the cooperation on individual bordisms is defined using a pull-push construction along

\[ X^{\times n}/\Sigma_n \leftarrow (X^{\times n} \times n)/\Sigma_n \to X, \]

where the left map is induced by the projection \( n \to 1 \) and the right map is given by evaluation. Observe that the left map is an \( n \)-fold covering, so pulling back a map \( M \to X^{\times n}/\Sigma_n \) produces an \( n \)-fold covering \( M \to M \) together with a map \( M \to (X^{\times n} \times n)/\Sigma_n \) defining the geometric structure. Postcomposing the latter map with the evaluation map \( (X^{\times n} \times n)/\Sigma_n \to X \) yields the desired bordism.

**Definition 7.3.2.** The \( n \)-th power operation \((n \geq 0)\)

\[ \mathcal{F} \mathcal{T}(X) \to \mathcal{F} \mathcal{T}(X^{\times n}/\Sigma_n) \]

is the morphism of stacks defined via precomposition with the power cooperation \( \text{Bord}(X^{\times n}/\Sigma_n) \to \text{Bord}(X) \).

Let \( T \) be a symmetric monoidal \((\infty, d)\)-category with all objects and \( k \)-morphisms for \( k \geq 1 \) invertible. Then we can regard \( \mathcal{F} \mathcal{T} \) as a stack on the site of smooth manifolds via the composite functor

\[ \text{Man} \xrightarrow{j_{\text{Man}}} \mathcal{P} \text{Sh}_\Delta(\text{Man}) \xrightarrow{i^*} \mathcal{P} \text{Sh}_\Delta(\text{OEmb}^f_d) \xrightarrow{\mathcal{F} \mathcal{T}} \mathcal{S} \text{et}, \]

where \( j_{\text{Man}} \) denotes the Yoneda embedding, \( i^* \) is the restriction along the inclusion \( \text{OEmb}^f_d \hookrightarrow \text{Man} \) and \( \mathcal{F} \mathcal{T} \) is the functor sending a geometric structure \( \mathcal{S} \) to \( \text{Fun}(\text{Bord}^S_d, T) \).

**Proposition 7.3.3.** If the target symmetric monoidal \((\infty, n)\)-category has all objects and \( k \)-morphisms for \( k \geq 1 \) invertible, then applying the shape functor to the \( n \)-th power operation

\[ \mathcal{F} \mathcal{T}(X) \to \mathcal{F} \mathcal{T}(X^{\times n}/\Sigma_n) \]

produces the \( n \)-th power operation for the associated \( E_\infty \)-spectrum \( E \):

\[ \text{Hom}(f X, E) \to \text{Hom}((f X)^{\times n}/\Sigma_n, E), \]

where \( f X \) is the space given by the shape of the stack \( X \) and \( \text{Map}(-, E) = \text{Hom}(\Sigma^\infty-, E) \) denotes the derived powering of spectra over spaces.
Proof. Observe that \( f((X^{\times n}/\Sigma_n) = (f X)^{\times n}/\Sigma_n \) because \( f \) is homotopy cocontinuous and commutes with finite homotopy products. Furthermore, applying \( f \) to the pull-push zigzag

\[
X^{\times n}/\Sigma_n \leftarrow (X^{\times n} \times n)/\Sigma_n \rightarrow X,
\]

produces the pull-push zigzag for the space \( f X \):

\[
(f X)^{\times n}/\Sigma_n \leftarrow ((f X)^{\times n} \times n)/\Sigma_n \rightarrow f X.
\]

In the previous section we proved that

\[
f(\text{FFT}(X)) = \text{Hom}(f X, E),
\]

where \( E = f \text{FFT} \) is the classifying spectrum. Applying this result to both sides, we obtain the desired claim. \( \square \)

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