The $p$-spin spherical spin glass model

A. Barrat

International Center for Theoretical Physics, Strada Costiera 11, 34100 Trieste, Italy

This review presents various aspects of a mean-field spin glass model known as the $p$-spin spherical spin glass model, which has raised a lot of interest in the study of spin glasses, and also for its possible links with a mean-field theory of structural glasses.

This preprint contains no new results and is therefore not intended to be published, but its aim is to present a collection of results and formulas concerning this very rich model.

It is in fact the English translation of one of the chapters of my PhD thesis (“Quelques aspects de la dynamique hors d’équilibre des verres de spin”, “Some aspects of the out of equilibrium dynamics of spin glasses”). A postscript version (in French) of this PhD thesis will soon be available at [http://www.lpt.ens.fr](http://www.lpt.ens.fr).

The recent developments in the theory of spin glass dynamics have made clearer the similarity of behaviour in spin glasses and in glasses [1,2]. In this context it seems at the moment that a certain category of spin glasses, those which are described by a so called one step replica symmetry breaking (RSB) transition [3], are good candidate models for a mean field description of the glass phase [4–6]. In these systems the presence of metastable states generates a purely dynamical transition (which is supposed to be rounded in finite dimensional systems [4–6]) at a temperature $T_d$ higher than the one obtained within a theory of static equilibrium, $T_S$.

The spherical $p$-spin spin glass introduced in [7,8] is an interesting example of this category. It is a simple enough system in which the metastable states can be defined and studied by the TAP method [9]. Besides, its Relaxational Langevin dynamics was shown to display the interesting behaviour known as aging [10].

The model is defined by the Hamiltonian

$$H = - \sum_{1 \leq i_1 < i_2 \ldots < i_p \leq N} J_{i_1 i_2 \ldots i_p} s_{i_1} s_{i_2} \cdots s_{i_p}$$

with $p \geq 3$, where the couplings are gaussian, with zero mean and variance $p!/(2N^{p-1})$. The spins, instead of being restricted to the values $+1$ or $-1$ (Ising spins), are real variables, with the global constraint $\sum_{i=1}^N s_i^2 = N$: the system is less frustrated, but this simplification allows for a more complete analytical treatment, and the model still displays a very interesting behaviour.

In this small review are presented statical and dynamical aspects of this model. Numerous formulas are displayed in appendices.

I. STATICS

A. Replica method

This study, made by Crisanti and Sommers [7], shows a transition at a temperature $T_s$, between a high temperature replica symmetric phase, and a low temperature phase with one step of replica symmetry breaking (see appendix A.1): at low temperature, the Boltzmann measure is dominated by a small number of pure states.

The static transition temperature is given by [11]:

$$T_S = y \sqrt{\frac{p}{2y}} \left(1 - y\right)^{-\frac{p}{2}}$$

where

$$\frac{2}{p} = -2y \frac{1 - y + \ln y}{(1 - y)^2}$$

[1] e-mail: barrat@ictp.trieste.it
B. TAP equations

The TAP (Thouless-Anderson-Palmer, [3]) equations are equations on the local magnetizations \( m_i = \langle s_i \rangle \). They were derived by Rieger [12] for the p-spin model with Ising spins; they can be derived through a variational principle on the \( m_i \), from a free energy \( f (\{ m_i \}) \). In the spherical case, this free energy was obtained by various authors [11, 13]; in appendix A.2 we propose another derivation by the cavity method [6].

The free energy \( f (\{ m_i \}) \) is best written in terms of radial and angular variables, \( q \) and \( \hat{s}_i \) (with \( m_i = \sqrt{q} \hat{s}_i \)), in the form [11]:

\[
f(\{ m_i \}) = q \frac{\alpha}{2} E^0 (\{ \hat{s}_i \}) - \frac{T}{2} \ln (1 - q) - \frac{1}{4T} [(p - 1)q^p - pq^{p-1} + 1] ;
\]

where the angular energy is:

\[
E^0 (\{ \hat{s}_i \}) = \frac{1}{N} \sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \hat{s}_{i_1} \cdots \hat{s}_{i_p} .
\]

At zero temperature the TAP states are just unit vectors which minimize the angular energy \( E^0 \). There actually exist such states for \( E^0 \in [E_{min}, E_c = -\sqrt{2(p - 1)/p}] \). The states with \( E^0 = E_{min} \) correspond to the RSB solution.

We denote by \( \hat{s}_i^0 \) one zero temperature state, of energy \( E^0_{\alpha} \). The free energy per spin \( f (\{ m_i \}) \) depends on the \( \hat{s}_i \) only through \( E^0_{\alpha} \). We see therefore that each state \( \hat{s}_i^0 \), gives rise at finite temperature \( T \) to one TAP state \( \alpha \) given by:

\[
m_i^{\alpha} = \sqrt{q(E^0_{\alpha}, T)} \hat{s}_i^0 ,
\]

where \( q(E^0_{\alpha}, T) \) is given in appendix, equation (A3).

When changing the temperature, one can follow the metastable states which keep the same angular direction [3]; their order in free energy or energy, at fixed \( T \), is the same as their order in \( E^0 \); if \( E_{\alpha}^0 > E_{\beta}^0 \), \( E_{\alpha}(T) > E_{\beta}(T) \).

When raising \( T \), a state disappears at a temperature \( T_{max}(E^0) \) (where the equation defining \( q(E^0_{\alpha}, T) \) has no more solutions); \( T_{max}(E^0) \) is a decreasing function of \( E^0 \); the most excited states, with \( E^0 = E_c \), disappear first at \( T_{max}(E_c) \), and the lowest at \( T_{max}(E_{min}) \equiv T_{TAP} \). Above \( T_{TAP} \), the only remaining state is the paramagnetic one with \( q = 0 \) and free energy \( F_{para} = -1/(4T) \).

To complete the description of metastable states at any temperature, one only needs the density of states \( \rho(E^0) \) with an angular energy \( E^0 \). This has been computed in [13]; the multiplicity is exponentially large, giving a finite complexity density \( s_c^0(E^0) \), defined as:

\[
s_c^0(E^0) = \lim_{N \to \infty} \frac{\log \rho(E^0)}{N} .
\]

At finite temperature, for a free energy \( F \) between the free energy of the lowest (with \( E^0 = E_{min} \)) and the highest (with \( E^0 = E_c \)) TAP states, we have therefore an extensive value for \( S_c(F, T) \), the logarithm of the number of TAP states with free energy \( f \) at temperature \( T \).

The equation (4) gives the free energy of a TAP state with a given zero-temperature energy; to obtain the full partition function, we need to sum over the possible energies, including the complexity term. After changing variables we obtain an integral over the free energies of the TAP states [13]:

\[
Z = \int dF \exp \left( -\frac{(F - TS_c(F, T))}{T} \right) .
\]

For large \( N \) we evaluate this integral by a saddle point method. For \( T > T^* \), with \( T^* = \sqrt{\rho(p - 2)p - 2(p - 1) - 7/2} \), we find that the Boltzmann measure is dominated by the paramagnetic state \( q = 0 \), with \( F = N F_{para} \). On the contrary, below \( T^* \), this measure is dominated by TAP states with free energy \( F_{eq} \); because of their complexity, their total free energy is:

\[
F_{tot} \equiv -T \log(Z) = F_{eq}(T) - TS_c(F_{eq}(T), T) .
\]

For \( T < T_S \), these are the lowest TAP states, with \( E^0 = E_{min} \), and, for \( T_S < T < T^* \), intermediate TAP states with a parameter \( q \) given by:

\[
2
\[ \frac{p}{2T^2} q^{p-2} (1 - q) = 1. \] 

In this region \( T_S < T < T^* \), \( F_{\text{tot}} \) is equal to the paramagnetic free energy. The global situation is pictured in figure (1).

![Free energy versus temperature](image)

**FIG. 1.** Free energy versus temperature; (1) : free energy of the paramagnetic solution for \( T > T^* \), \( F_{\text{tot}} \) for \( T < T^* \); (2) : free energy of the lowest TAP states, with zero temperature energy \( E_{\text{min}} \); (3) : free energy of the highest TAP states, corresponding to \( E_c \); (4) : an intermediate value of \( E_0 \) leads to an intermediate value of \( f \) at any temperature; (5) : \( f_{\text{eq}}(T) \); the difference between curves (5) and (1) gives the complexity \( T S_c(f_{\text{eq}}(T), T) \).

Between the two transition temperatures \( T_s \) and \( T_d \), the situation is unclear: the total equilibrium free energy seems to get two equal contributions, from the paramagnetic state and from a bunch of TAP solutions with non-zero \( q \). One can wonder if there is a phase coexistence, or simply a problem of double counting in the TAP approach. This issue, which is an important one if one aims at understanding the finite dimensional behaviour of this type of systems, can in fact be clarified within a dynamical approach as. Let us also mention that some purely static approaches also carry relevant information on related issues [14, 15].

Before explaining this particular dynamical approach, we will recall the previously known results of the equilibrium and out of equilibrium dynamics.

### II. HIGH TEMPERATURE DYNAMICS

Following the study of the statics [7], Crisanti, Horner and Sommers studied the Langevin relaxation dynamics of the model [8]

\[ \frac{ds_i(t)}{dt} = -\frac{\partial H}{\partial s_i} - \mu(t) s_i(t) + \eta_i(t), \] 

where \( \mu(t) \) has to be computed self consistently in order to implement the spherical constraint. Using the same formalism as Sompolinsky and Zippelius for the SK model, they wrote coupled equations for the correlation and response functions, with random initial conditions corresponding to a quench at initial time.

If the validity of the fluctuation-dissipation theorem (FDT)

\[ \frac{\partial}{\partial t} C(t, t') = -Tr(t, t'), \] 

and invariance by time translation (TTI) are assumed, the equations reduce to one single equation for the correlation \( C(t, t') = C_{\text{eq}}(t - t') \) :
\[
\frac{\partial C_{eq}(\tau)}{\partial \tau} = -TC_{eq}(\tau) - \frac{p}{2T} \int_0^\tau du \ C_{eq}^{-1}(\tau - u) \ \frac{\partial C_{eq}(u)}{\partial u},
\]

(13)

with the initial condition \( C_{eq}(0) = 1 \) (see appendix B.3.b). These equation can be integrated numerically, with the following result (figure (2)) : at high temperature, \( \lim_{\tau \to \infty} C_{eq}(\tau) = 0 \); if the temperature is lowered, a plateau appears in the curve giving \( C_{eq}(\tau) \) as a function of \( \log(\tau) \), which length diverges at a certain temperature \( T_d \), given in appendix B.1.

Moreover, for \( T > T_d \), with \( T - T_d \ll T_d \), it can be shown analytically \[8\] that the plateau is approached with a power law, \( q + \text{constant } t^{-a} \), and the departure from the plateau is in \( q - \text{constant} \ t^b \); this behaviour is identical to the one of the density correlation function in the mode coupling theories concerning the glass transition (the analysis of Crisanti, Horner and Sommers \[8\] follows the lines of the discussion by Götze on the glass transition \[16\]).

![Figure 2](image)

**FIG. 2.** \( C_{eq}(\tau) \) as a function of \( \tau \) : result of the numerical integration of equation equation (13) for \( p = 3, T = 0.7, 0.65, 0.62, 0.615, 0.613 \); here \( T_d = 0.612372 \). The horizontal line represents the limit \( \lim_{\tau \to \infty} C_{eq}(\tau) \) for \( T = T_d \), here \( q \approx 0.509034 \).

The value of \( T_d \) is precisely equal to the temperature \( T^* \) where appears the problem of double counting mentioned earlier. Besides, the fact that \( T_d \) is higher than the static transition temperature shows that the transition at \( T_d \) is purely dynamical. Below \( T_d \) the dynamics is no longer ergodic. and the validity of the equilibrium dynamics properties should not be assumed.

### III. OUT OF EQUILIBRIUM DYNAMICS

The dynamics below \( T_d \) was studied in 1993 by Cugliandolo and Kurchan \[10\], taking into account the finite initial time. The dynamical equations for the correlation and response functions are as follow (with \( t > t' \), and random initial conditions ; a possible derivation of this equations is detailed in B.2) :

\[
\mu(t) = \int_0^t ds \ \frac{p}{2} C^{p-1}(t, s) r(t, s) + T
\]

\[
\frac{\partial r(t, t')}{\partial t} = -\mu(t) r(t, t') + \frac{p(p-1)}{2} \int_{t'}^t ds \ C^{p-2}(t, s) r(t, s) r(s, t')
\]

\[
\frac{\partial C(t, t')}{\partial t} = -\mu(t) C(t, t') + \frac{p}{2} \int_{t'}^t ds \ C^{p-1}(t, s) r(t', s)
\]

\[
+ \frac{p(p-1)}{2} \int_0^{t'} ds \ C^{p-2}(t, s) r(t, s) C(s, t').
\]
At high temperature, the system (14) reduces, via TTI and FDT, to the dynamics studied by Crisanti, Horner and Sommers, equation (13), but, below $T_d$, $C(t,t')$ and $r(t,t')$ depend on the two times. The dynamical equations can be partially solved by the separation of two time regimes (10) (see figure (3)):

- **FDT regime**: for finite time separations $\tau = t - t'$, i.e. for $\tau/t$ going to zero, the equilibrium dynamics properties are valid, and we obtain two functions $C_{FDT}(\tau)$ and $r_{FDT}(\tau)$ related by FDT. The limit $\tau \to \infty$ yields $\lim_{\tau \to \infty} C_{FDT}(\tau) = q$, with the value of the parameter of the threshold TAP states for $q$ (corresponding to $E_0 = E_c$).

- **aging regime**: for $t$ and $t'$ going to infinity, without $(t-t')/t \to 0$, i.e. for widely separated times, one can neglect the time derivatives $\partial r(t,t')/\partial t$ and $\partial C(t,t')/\partial t$, and this yields equations invariant by time reparametrization. This way, an Ansatz can be found to solve the equations: $C(t,t') = C(h(t')/h(t))$, and $r(t,t') = G(h(t')/h(t))h'(t')/h(t)$, with $C(1) = q$, and $h$ a monotonously increasing function. Moreover, $G(\lambda) = xC'(\lambda)$, with constant $x$. In this regime the correlation decreases from $q$ to zero.

![Figure 3](image-url). p-spin model with $p = 3$: $C(t_w + t, t_w)$ as a function of $t$ at temperature $T = 0.5$ ($T_d \approx 0.612$) for $t_w = 0, 10, 20, 30, 40, 50$, in logarithmic scale; this curves were obtained by numerical integration of the dynamical equations (13), with the algorithm used by [15]; the dotted curve is $C_{FDT}(t)$, the horizontal line corresponds to $q$ at $T = 0.5$, $q \approx 0.639$.

The energy density can be shown to approach asymptotically the energy density of the highest TAP states. As was noticed in [10], this dynamics does not explore the TAP states which dominate the Boltzmann distribution, and in fact stays above all TAP states, with energy density going towards the one of the threshold states, but staying at all times $O(1)$ above.

### IV. INFLUENCE OF INITIAL CONDITIONS

Using the same field theoretical techniques as for the out of equilibrium dynamics, it is possible to derive dynamical equations for a system thermalized at temperature $T'$ at the initial time, and then brought at temperature $T$. However, to implement the Boltzmann measure at $T'$ for the initial conditions, one has to reintroduce replicas (see appendix B.3, and [10 21 23]). The obtained equations describe the evolution of two times overlaps between replicas $C^{ab}(t,t') = \langle s^a(t)s^b(t') \rangle$, where $a$ and $b$ are replica indices. These equations respect the initial structure (replica symmetric or not) of the $C^{ab}$, i.e. the static replica structure describing the equilibrium at temperature $T'$. 
For the p-spin model, the initial situation is replica symmetric if $T'$ is higher than the static transition temperature, with $C^{ab}(0,0) = \delta_{ab}$. Therefore, at all times we can write $C^{ab}(t,t') = C(t,t')\delta_{ab}$, and the equations are now (for $T' > T_s$, $t > t'$):

$$
\begin{align*}
\mu(t) &= \int_0^t ds \frac{p^2}{2} C_p^{-1}(t,s)r(t,s) + T + \frac{p}{2T} C_p(t,0) \\
\frac{\partial r(t,t')}{\partial t} &= -\mu(t) r(t,t') + \frac{p(p-1)}{2} \int_0^{t'} ds C_p^{-2}(t,s)r(t,s)r(t',s) \\
\frac{\partial C(t,t')}{\partial t} &= -\mu(t) C(t,t') + \frac{p}{2} \int_0^{t'} ds C_p^{-1}(t,s)r(t',s) \\
&\quad\quad\quad\quad\quad + \frac{p(p-1)}{2} \int_0^t ds C_p^{-2}(t,s)r(t,s)C(s,t') \\
&\quad\quad\quad\quad\quad + \frac{p}{2T} C_p^{-1}(t,0) C(t',0).
\end{align*}
$$

Equations (15) yield then equation (13) of equilibrium dynamics (see appendix B.3).

We see that the only difference with the usual dynamical equations lie in terms of coupling to the initial condition. Of course, these terms vanish in the limit $T' \to \infty$.

a. $T = T'$. If the temperature does not change at the initial time, $T = T'$, we start with a thermalized system, and do not modify it. Therefore the definition of the initial time is arbitrary and the system stays in equilibrium: the dynamics is equilibrium dynamics, with $C(t,t') = C_{eq}(t-t'), r(t,t') = r_{eq}(t-t')$ and $r_{eq}(\tau) = -\frac{1}{12} \frac{\partial C_{eq}}{\partial \mu}$. The equations (13) yield then equation (13) of equilibrium dynamics (see appendix B.3).

For $T > T_d$ the system is in the paramagnetic state, with $\lim_{\tau \to -\infty} C_{eq}(\tau) = 0$. On the other hand, below $T_d$, we know that $C_{eq}(\tau)$ can not go to zero. We obtain $\lim_{\tau \to -\infty} C_{eq}(\tau) = C_{\infty}$, where $C_{\infty}$ is the highest solution of

$$
\frac{p}{2T} C_p^{-2}(1 - C_{\infty}) = 1
$$

obtained taking $\tau \to \infty$ with $\frac{\partial C_{eq}(\tau)}{\partial \mu} \to 0$ in the equation of equilibrium dynamics (13). This equation is identical to (14), which defines the Edwards Anderson parameter of a TAP solution.

Therefore, below $T_d$, the thermalized system is in a TAP state, and no more in a paramagnetic state: we would then have $C_{\infty} = 0$. The Boltzmann measure is neither a superposition of TAP states and a paramagnetic state, since the average on initial composante would yield an intermediate value for $C_{\infty}$.

This shows that, for temperatures lower than $T_d$, the Boltzmann measure is given by the bunch of TAP states with free energy $F_{eq}(T)$ such that

$$
F_{eq}(T) - T S_c(F_{eq}(T), T) = F_{RS} = -\frac{1}{4T}.
$$

b. $T$ different from $T'$. In order to push further the dynamical exploration of TAP states, we can now study the dynamics at a temperature $T$ different from the thermalization temperature $T'$. Two different scenarios are a priori possible: equations (13) can yield an explicit dependence on $t$ and $t'$ for $C(t,t')$ and $r(t,t')$, or, after a transient regime, equilibrium dynamics, with functions of $t-t'$. A numerical integration of the equations (using the algorithm of (14) (15)) leads to eliminate the first possibility (see figure (3)).
FIG. 4. $p = 3$ model, with $T_S \approx 0.586$, $T_d \approx 0.612$; numerical integration of equations (15) for $T' = 0.605$, $T = 0.6$; $C(t,0)$ is shown as a function of $t$ in full lines, $C(t,t')$ as a function of $t - t'$ for $t' = 6, 12, 18, 24$ (symbols); the dotted curve is the result of the numerical integration of (18), and the horizontal line represents the value of $C_\infty$ obtained by (27, 28).

To study analytically the behaviour of the system, it is therefore convenient to introduce as previously $C_{eq}(\tau)$, $r_{eq}(\tau)$, related by FDT, as well as $C_\infty = \lim_{\tau \to \infty} C_{eq}(\tau)$, $\mu_\infty = \lim_{t \to \infty} \mu(t)$, and $l = \lim_{t \to \infty} C(t,0)$. $l$ can differ from $C_\infty$ because of the transient regime. We obtain:

\[
\frac{\partial C_{eq}(\tau)}{\partial \tau} = - \left( \mu_\infty - \frac{p}{2T} \right) C_{eq}(\tau) + \frac{p}{2} \int_0^T du \ C_{eq}^{-1}(u) \ r_{eq}(\tau - u) - \frac{p}{2T} C_\infty + \frac{p}{2T'} l_p. \tag{18}
\]

It is possible to extract the value of $C_\infty$; the energy of the system can also be computed (10) (appendix B.3.b). For $T \neq T'$, we obtain that the asymptotic energy $E_\infty$, and $C_\infty$ are precisely the energy and the parameter $q$ of TAP states at temperature $T$: those are the TAP states obtained following the TAP states that are at equilibrium at temperature $T'$ to the new temperature $T$: these states correspond to a certain angular energy $E_0^{T'}$, and $C_\infty$ is equal to $q(E_0^{T'}, T)$ (see figure (5)).
FIG. 5. (1) : TAP states giving the statics at $T'$, with angular energy $E_{T'}^0$, and $q(E_{T'}^0, T')$ ; (2) : TAP states (1) followed at temperature $T$, with $q(E_{T'}^0, T)$ ; (3) : TAP states giving the statics at $T$, with angular energy $E_T^0$, and $q(E_T^0, T)$.

The obtained dynamics is therefore a relaxation in a TAP state, in which the system was thermalized, followed at the new temperature $T$. It differs from the TAP states dominating the statics at $T$. Moreover, taking the Laplace transform of (18), it is possible to show that the relaxation of $C_{eq}(\tau)$ for large $\tau$ is of the form $\tau^{-3/2} \exp(-\tau/\tau_0)$. The relaxation time $\tau_0$, which can be written as a function of $T$, $T'$ and $C_\infty$ (the exact expression is complicated, and I do not reproduce it here), diverges for the threshold TAP states ($E_\infty = E_c$), and this relaxation occurs only for $T$ lower than the temperature where the followed TAP state disappears. If $T$ is further increased, we observe a fast relaxation in the paramagnetic state.

V. CONCLUSIONS

In this paper have been shown various aspects of the spherical $p$-spin model, which, despite its relative simplicity (it is a mean-field model, with a global spherical constraint instead of a constraint for each spin) that allows for an analytical treatment, displays a complex phase space and interesting dynamical behaviours.

The phase space landscape is made of many metastable states whose characteristics are given by the “TAP equations”, therefore called “TAP states”.

The use of particular initial conditions for the dynamics has shown that these solutions of the TAP equations correspond to real states, with a real ergodicity breaking. Below $T_d$, the equilibrium measure is given by TAP states, the paramagnetic state vanishing at this dynamical transition. The way in which this state disappears is still an open question, whose answer would certainly help to understand the essence of the aging behaviour obtained with a quench below $T_d$.

Moreover, if the system is prepared in a TAP state, by a thermalization between $T_S$ and $T_d$, it stays trapped within, even if its temperature is changed : these states can be followed by the dynamics described in paragraph IV, as well for lower temperatures as for higher ones, and even for temperatures higher than $T_d$, until they disappear. They present a true ergodicity breaking even at these temperatures, but their complexity is not high enough to allow them to contribute to the Boltzmann measure. The structure of these metastable states is still under investigation : in [24], for example, it is shown that they are not randomly distributed in the phase space.

On the other hand, the usual dynamics at a temperature below $T_d$, starting from a random configuration, only leads to a “weak ergodicity breaking” [22,10], where the self overlap vanishes at very large time differences (much larger than the waiting time). This is explained [10,23] by the fact that the system, which was initially in the (infinite temperature) paramagnetic state, does not find any TAP state in a finite time, but stays at energy density $O(1)$ (going to zero as $t$ goes to infinity) above the threshold.

It is likely that the impossibility for the system to find the states in a finite time results from the mean-field approximation, and that, in finite dimensions, the time to find these states would be finite, the dynamics being
obtained afterwards by jumps between states. Recent numerical simulations of a model with 4 spins-interactions in three dimensions [24] tend to support this scenario.

Acknowledgments

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APPENDIX A: THE STATICS

1. Replicas

The study of the p-spin model with the replica method yields the following results:

- at high temperature, the system is replica symmetric, with \( Q_E A = 0 \) (paramagnetic region);
- below the static transition temperature

\[
T_S = y \sqrt{\frac{p}{2y}} (1 - y)^{\frac{p}{2}} - 1 , \quad \frac{2}{p} = -2y \frac{1 - y + \ln y}{(1 - y)^2},
\]

(A1)

there is a one-step replica symmetry breaking, with parameters \( q_0 = 0, q_1, x \), and free energy density

\[
F_{RSB} = - \frac{1}{4T} \left( 1 - (1 - x)q_1^p \right) + \frac{T}{2x} \ln \left( \frac{1 - q_1}{1 - (1 - x)q_1} \right) - \frac{T}{2} \ln(1 - q_1)
\]

(A2)

and

\[
q_1^{\frac{p}{2}} (1 - q_1) = \sqrt{\frac{2y}{p}} T, \quad x = T q_1^{\frac{p}{2}} \sqrt{\frac{2y}{p}} \left( \frac{1}{y} - 1 \right).
\]

(A3)

2. TAP solutions

a. Derivation of the TAP equations

We derive here the TAP equations on the local magnetizations, using the cavity method [3].

The Hamiltonian is

\[
H = - \sum_{1 \leq i_1 < i_2 \cdots < i_p \leq N} J_{i_1 i_2 \cdots i_p} s_{i_1} s_{i_2} \cdots s_{i_p} - \frac{r}{2} \sum_{i=1}^{N} s_i^2
\]

(A4)

where \( r \) is adjusted in a self consistent way to implement the spherical constraint.

To the Hamiltonian for \( N \) spins \( i = 1, \cdots, N \) we add a spin \( s_0 \), with coupling constants \( J_{0i_1 \cdots i_p} \). The field acting on \( s_0 \) is

\[
h = \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} s_{i_2} \cdots s_{i_p}.
\]

(A5)

In the absence of \( s_0 \), to each configuration of the \( s_i \) correspond a value of \( h \). The probability distribution of this field in the absence of \( s_0 \) will allow us to obtain the joint distribution of \( s_0 \) and \( h \). The mean value of \( h \) is:

\[
\langle h \rangle_N = \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} \langle s_{i_2} \rangle_N \cdots \langle s_{i_p} \rangle_N
\]

(A6)

since the \( J_{0i_2 \cdots i_p} \) and the \( s_i \) are independent variables; we have also used the hypothesis \( \langle s_{i_2} \cdots s_{i_p} \rangle_N \approx \langle s_{i_2} \rangle_N \cdot \cdots \cdot \langle s_{i_p} \rangle_N \), so called “clustering” hypothesis [3], more precisely

\[
\lim_{N \to \infty} \frac{1}{N^{p-1}} \sum_{i_2 \cdots i_p} \left( \langle s_{i_2} \cdots s_{i_p} \rangle_N - \langle s_{i_2} \rangle_N \cdots \langle s_{i_p} \rangle_N \right)^2 = 0.
\]

(A7)

Besides, in

\[
\langle (h - \langle h \rangle_N)^2 \rangle_N = \sum_{1 \leq i_2 < \cdots < i_p \leq N, 1 \leq k_2 < \cdots < k_p \leq N} J_{0i_2 \cdots i_p} J_{0k_2 \cdots k_p} \left( \langle s_{i_2} \cdots s_{i_p} - \langle s_{i_2} \rangle_N \cdots \langle s_{i_p} \rangle_N \rangle \langle s_{k_2} \cdots s_{k_p} - \langle s_{k_2} \rangle_N \cdots \langle s_{k_p} \rangle_N \rangle \right)
\]

(A8)
it is possible to show (calculating higher moments) to show that the distribution of \( h \) in the absence of \( s_0 \) is gaussian. In the presence of \( s_0 \), we add an interaction \( h s_0 \), so the joint distribution of \( h \) and \( s_0 \) is of the form

\[
P_{N+1}(h, s_0) \propto \exp \left( -\frac{\beta r}{2}s_0^2 + \beta h s_0 - \frac{(h - \langle h \rangle_N)^2}{p(1 - q^{-1})} \right).
\] (A10)

We deduce

\[
\langle s_0 \rangle_{N+1} = \frac{\langle h \rangle_N}{r - \frac{p\beta^2}{2}(1 - q^{-1})}
\] (A11)

and

\[
\langle (s_0 - \langle s_0 \rangle_{N+1})^2 \rangle_{N+1} = \frac{1}{\beta(r - \frac{p\beta^2}{2}(1 - q^{-1}))}
\] (A12)

Because of the spherical constraint, \( \langle (s_0 - \langle s_0 \rangle_{N+1})^2 \rangle_{N+1} = 1 - q \), so that:

\[
\beta r = \frac{1}{1 - q} + \frac{p\beta^2}{2}(1 - q^{-1}).
\] (A13)

We need equations between the \( \langle s_i \rangle_{N+1} \); therefore we must obtain \( \langle h \rangle_N \) as a function of these magnetizations. In order to do this, we look at the influence on \( \langle s_i \rangle \) of the new spin \( s_0 \), for a given \( i \). We write

\[
h = s_i h_i + h_i'
\] (A14)

where

\[
h_i = \sum_{1 \leq i_3 < \cdots < i_p \leq N} J_{0i_3 \cdots i_p} s_{i_3} \cdots s_{i_p},
\] (A15)

the sum being on indices non equal to \( i \). Then \( h_i \) is of order \( N^{-1/2} \), and to leading order

\[
\langle (h_i - \langle h_i \rangle_N)^2 \rangle_N = \frac{p(p - 1)}{2N}(1 - q^{-2})
\]
\[
\langle (h_i' - \langle h_i' \rangle_N)^2 \rangle_N = \frac{p}{2}(1 - q^{-1})
\] (A16)

so the joint probability distribution of \( s_0, s_i, h_i, h_i' \) is

\[
P_{N+1}(s_0, s_i, h_i, h_i') \propto \exp \left( -N \frac{(h_i - \langle h_i \rangle_N)^2}{p(p - 1)(1 - q^{-1})} - \frac{(h_i' - \langle h_i' \rangle_N)^2}{p(1 - q^{-1})} - \frac{\beta r}{2}s_0^2 + \beta s_0(h_i + h_i') - \frac{(s_i - \langle s_i \rangle_N)^2}{2(1 - q)} \right).
\] (A17)

The term \( \frac{(s_i - \langle s_i \rangle_N)^2}{2(1 - q)} \) comes from the fact that, in the absence of \( s_0 \), the spin \( s_i \) has average \( \langle s_i \rangle_N \) and variance \( 1 - q \). Integrating out \( h_i, h_i' \) yields

\[
P_{N+1}(s_0, s_i) \propto \exp \left( -\frac{\beta r}{2}s_0^2 + \frac{p\beta^2}{4}(1 - q^{-1})s_0^2 + \beta s_0(s_i \langle h_i \rangle_N + \langle h_i' \rangle_N) - \frac{(s_i - \langle s_i \rangle_N)^2}{2(1 - q)} \right),
\] (A18)

and finally with (A13):

\[
\langle s_i \rangle_{N+1} = \langle s_i \rangle_N + \beta^2(1 - q^2)\langle h_i \rangle_N \langle h_i' \rangle_N
\] (A19)
or
\[
\langle s_i \rangle_{N+1} = \langle s_i \rangle_N + \beta(1 - q) \langle s_i \rangle_{N-1} \sum_{1 \leq i_3 < \cdots < i_p \leq N} J_{0i_3 \cdots i_p} \langle s_{i_3} \rangle_N \cdots \langle s_{i_p} \rangle_N. \tag{A20}
\]

The technique used by Rieger [12] allows to write the TAP equations on \( m_i = \langle s_i \rangle_{N+1}, i = 0, \cdots, N \) : as
\[
\frac{m_0}{\beta(1 - q)} = \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} \langle s_{i_2} \rangle_N \cdots \langle s_{i_p} \rangle_N, \tag{A21}
\]
we obtain
\[
\frac{m_0}{\beta(1 - q)} = \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} m_{i_2} \cdots m_{i_p} - A, \tag{A22}
\]
where
\[
A = (p - 1)\beta(1 - q)m_0 \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} m_{i_2} \cdots m_{i_p} \sum_{1 \leq k_3 < \cdots < k_p \leq N} J_{0i_2 k_3 \cdots k_p} m_{k_3} \cdots m_{k_p}, \tag{A23}
\]
and, using the coupling symmetry under permutations:
\[
A = (p - 1)\beta(1 - q)m_0 \frac{1}{(p - 1)!} \sum_{i_2 \cdots i_p} J_{0i_2 \cdots i_p} m_{i_2} \cdots m_{i_p} \frac{1}{(p - 2)!} \sum_{k_3 \cdots k_p} J_{0i_2 k_3 \cdots k_p} m_{k_3} \cdots m_{k_p}
\]
\[
= \beta(1 - q)m_0 \frac{1}{(p - 2)!} \sum_{i_2} \left( \sum_{k_3 \cdots k_p} J_{0i_2 k_3 \cdots k_p} m_{k_3} \cdots m_{k_p} \right)^2. \tag{A24}
\]
The terms \( J_{0i_2 k_3 \cdots k_p} m_{k_3} \cdots m_{k_p} \) are dominant and yield:
\[
A = \beta(1 - q)m_0 \frac{1}{(p - 2)!} \sum_{i_2} (p - 2)! \frac{p!}{2N^{p-1}} (Nq)^{p-2} \tag{A25}
\]
The equations are finally:
\[
\frac{m_0}{\beta(1 - q)} = \sum_{1 \leq i_2 < \cdots < i_p \leq N} J_{0i_2 \cdots i_p} m_{i_2} \cdots m_{i_p} - \beta(1 - q) \frac{p(p - 1)}{2} q^{p-2} m_0. \tag{A26}
\]

b. Study of the TAP states

The states \( \alpha \) are given by
\[
m_i^\alpha = \sqrt{q(E^\alpha_i, T)} s_i^\alpha, \tag{A27}
\]
where \( q(E, T) \) is obtained by minimizing the free energy
\[
q^\beta E - \frac{T}{2} \ln(1 - q) - \frac{1}{4T} [(p - 1)q^p - pq^{p-1} + 1] ; \tag{A28}
\]
if we take
\[
z = \frac{1}{T}(1 - q)q^{\beta - 1} \tag{A29}
\]
we obtain the equation
\[
\frac{p(p-1)}{2} z^2 + pzE + 1 = 0,
\]
which yields that \( q \) is the largest solution of
\[
(1 - q)q^{\frac{p}{2}-1} = T \left( \frac{-E - \sqrt{E^2 - E_c^2}}{p - 1} \right),
\]
with \( E_c = -\sqrt{2(p-1)/p} \) (the other solution is not a minimum of the free energy \( A28 \)).

The maximum of \((1 - q)q^{p/2-1}\) occurs for \( q = 1 - \frac{2}{p} \), so that the equation \( A31 \) has no more solutions when
\[
\frac{2}{p} \left( 1 - \frac{2}{p} \right)^{\frac{p}{2}-1} = T \left( \frac{-E - \sqrt{E^2 - E_c^2}}{p - 1} \right).
\]

Therefore, a TAP state of energy \( E^0 \) disappears at
\[
T_{\text{max}}(E^0) = \left( \frac{2}{p} \right) \left( \frac{p-1}{-E^0 - \sqrt{(E^0)^2 - E_c^2}} \right) \left( 1 - \frac{2}{p} \right)^{\frac{p}{2}-1}.
\]

The energy per spin of a state \( \alpha \), given by \( \frac{\partial}{\partial T} \left( \beta f \right) \), is:
\[
E_\alpha = q^\frac{p}{2} \alpha - \frac{T}{2} \ln(1 - q) - \frac{1}{4T}[(p-1)q^p - pq^{p-1} + 1]
\]
with \( z = \left( E_0 - \sqrt{(E_0)^2 - E_c^2} \right) / (p - 1). \) This complexity goes to zero for
\[
z = \sqrt{\frac{2y}{p}},
\]
which corresponds to \( E^0 = E_{\text{min}} = E_{RSB}(T = 0) \). Therefore there exist TAP states for zero temperature energies between \( E_{\text{min}} \) and \( E_c \).

The lowest states disappear at
\[
T_{TAP} = \frac{2}{p} \sqrt{\frac{p}{2y}} \left( 1 - \frac{2}{p} \right)^{\frac{p}{2}-1}.
\]

Let us now determine the parameters of the TAP states dominating the partition function. They minimize
\[
F_{\text{tot}} = -T \ln(Z) = F_{eq}(T) - TS_c(F_{eq}(T), T),
\]
so that
\[
q^\frac{p}{2} E^0 - \frac{T}{2} \ln(1 - q) - \frac{1}{4T}[(p-1)q^p - pq^{p-1} + 1]
\]
\[
- \frac{T}{2} \left( - \ln \frac{pz^2}{2} + \frac{p-1}{2} z^2 - \frac{2}{p^2 z^2} + \frac{2 - p}{p} \right).
\]

Taking the derivative with respect to \( q \), using the relation \( \frac{p(p-1)}{2} z^2 + pzE^0 + 1 = 0 \), substituting \( z \) with its expression as a function of \( q \), and taking
we obtain

\[(1 - x)((p - 1)(1 - q)x - 1) = 0,\]  \hspace{1cm} (A41)

which yields:

\[\frac{p}{2T^2}(1 - q)q^{p-2} = 1\]  \hspace{1cm} (A42)

or

\[\frac{p(p - 1)}{2T^2}(1 - q)^2q^{p-2} = 1.\]  \hspace{1cm} (A43)

The first solution yields a free energy density equal to \(-1/(4T)\), i.e., equal to the free energy density of the paramagnetic solution. Moreover:

\[z = \sqrt{\frac{2(1 - q)}{p}}\]

\[E^0 = \sqrt{\frac{p}{2(1 - q)}} \left( \frac{p - 1}{p} q - 1 \right).\]  \hspace{1cm} (A44)

Given that \((1 - q)q^{p-2}\) is maximum for \(q = (p - 2)/(p - 1)\), this solution is only possible for temperatures lower than \(T^*\), with

\[T^* = \sqrt{\frac{p(p - 2)p^{p-2}}{2(p - 1)^{p-1}}}.\]  \hspace{1cm} (A45)

Above \(T^*\), the paramagnetic solution with \(q = 0\) is therefore dominant. For \(T = T_S\), \(q\) goes to the value \(q_1\) of the lowest TAP states. Below \(T_S\) there is no more any solution, and these states, with the lowest total free energy, dominate.

The second solution corresponds to the threshold states, and is valid only for \(T = T^*\); for any other value of \(T\) it does not yield the lowest free energy.
APPENDIX B: DYNAMICS OF THE P-SPIN SPHERICAL MODEL

1. Dynamical transition

The dynamical transition temperature can be obtained by the dynamical stability criterion: the correlation function is a decreasing function of the time $\frac{\partial C_{eq}(\tau)}{\partial \tau} \leq 0$. This criterion, together with the evolution equation of $C_{eq}$ \((B3)\), commands that, at all times,

$$-TC_{eq}(\tau) - \frac{p}{2T} C_{eq}^{p-1}(\tau) (C_{eq}(\tau) - 1) \leq 0. \quad (B1)$$

It is now easy to check that this inequality cannot be satisfied for all times, if $\lim_{\tau \to \infty} C_{eq}(\tau) = 0$, below $T_d$, with

$$T_d = \sqrt{\frac{p(p - 2)p-2}{2(p - 1)p-1}}. \quad (B2)$$

2. Dynamical equations

The starting point of the formalism is the equality

$$1 = \int Ds_1 \cdots Ds_N \prod_{t=1}^{N} \delta \left( dt \left( \dot{s}_i(t) + \frac{\partial H}{\partial s_i} + \mu(t)s_i(t) - \eta_i(t) \right) \right), \quad (B3)$$

which is simply a rewriting of the Langevin equation (the corresponding Jacobian is 1 if the Itô discretization scheme is used for the Langevin equation, i.e. if $s_i(t + \delta t) = s_i(t) - \delta t \frac{\partial H}{\partial s_i}(t) - \delta t \mu(t)s_i(t) + \int_{t}^{t+\delta t} \eta_i(t') dt'$).

The notation $D_s$ stands for $\prod_t ds_i(t)$: at each time $t$ the integration is performed over all possible values of the spins.

Writing $D_s D\dot{s}$ for $Ds_1 \cdots Ds_N D\dot{s}_1 \cdots D\dot{s}_N$ we obtain:

$$1 = \int Ds D\dot{s} \exp \left( \sum_{i=1}^{N} \int_{0}^{\infty} dt \left( \dot{s}_i(t) + \frac{\partial H}{\partial s_i} + \mu(t)s_i(t) - \eta_i(t) \right) \right), \quad (B4)$$

This equality can be averaged over the noise:

$$1 = \int Ds D\dot{s} \exp \left( \sum_{i=1}^{N} \int_{0}^{\infty} dt \left( T\dot{s}_i^2(t) + \dot{s}_i(t)s_i(t) + \dot{s}_i(t)\frac{\partial H}{\partial s_i} + \mu(t)s_i(t)s_i(t) \right) \right), \quad (B5)$$

and then over the quenched disorder, using

$$\frac{\partial H}{\partial s_i} = -\left( \sum_{i_1 < i_2 \cdots < i_p} J_{i_1 i_2 \cdots i_p} + \sum_{i_2 < i_3 \cdots < i_p} J_{i_2 i_3 \cdots i_p} + \cdots + \sum_{i_2 < \cdots < i_p < i} J_{i_2 i_3 \cdots i_p} \right) s_{i_2} \cdots s_{i_p}; \quad (B6)$$

we note

$$\sum_{i} s_i = \sum_{i_1 < i_2 \cdots < i_p} + \sum_{i_2 < i_3 \cdots < i_p} + \cdots + \sum_{i_2 < \cdots < i_p < i} \sum_{i_1}, \quad (B7)$$

and similarly $\sum_{ijk}$ the sum $\sum_{i_1 < i_2 \cdots < i_p}$ with indices $i_k$ different from $i$ and from $j$. 

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1 = \int \mathcal{D}s\mathcal{D}\dot{s} \exp S

S = \sum_{i=1}^{N} \int_{0}^{\infty} dt \left( T \dot{s}_{i}^{2}(t) + \dot{s}_{i}(t)\dot{s}_{i}(t) + \mu(t)s_{i}(t) \right)

+ \frac{p!}{4N^{p-1}} \left( \sum_{i=1}^{N} \sum_{i}^{*} \int_{0}^{\infty} dt \int_{0}^{\infty} dt' \dot{s}_{i}(t)\dot{s}_{i}(t')s_{i2}(t)s_{i2}(t') \cdots s_{ip}(t)s_{ip}(t') \right)

+ \sum_{i,j=1;i\neq j}^{N} \sum_{ij}^{*} \int_{0}^{\infty} dt \int_{0}^{\infty} dt' \dot{s}_{i}(t)s_{i}(t')s_{j}(t')s_{j}(t')s_{i3}(t)s_{i3}(t') \cdots s_{ip}(t)s_{ip}(t') \right). \quad (B8)

By definition of the correlation and response functions, these quantities are given by the average of $s_{i}(t)s_{i}(t')$ and $s_{i}(t)\dot{s}_{i}(t')$ with the action $S$. Taking the limit $N \to \infty$, we will therefore be able to write in a self consistent way :

$$\frac{1}{N} \sum_{i} s_{i}(t)s_{i}(t') = C(t,t') \quad \text{and} \quad -\frac{1}{N} \sum_{i} s_{i}(t)\dot{s}_{i}(t') = r(t,t'), \quad (B9)$$

which is equivalent to using the saddle point method. Besides, for $N \to \infty$, $\frac{1}{N} \sum_{i} \dot{s}_{i}(t)\dot{s}_{i}(t') = 0$ (it is a double derivative of 1 with respect to $\eta_{i}(t)$ and $\eta_{i}(t')$).

We then use :

$$\langle A \frac{\partial S}{\partial s_{i}(t)} \rangle = \frac{\partial A}{\partial s_{i}(t)} \quad (B10)$$

with $A = s_{i}(t')$ and $A = \dot{s}_{i}(t')$:

$$\frac{\partial S}{\partial s_{i}(t)} = 2T \dot{s}_{i}(t) + \dot{s}_{i}(t) + \mu(t)s_{i}(t)

+ \frac{p!}{2N^{p-1}} \int_{0}^{\infty} dt' \dot{s}_{i}(t') \sum_{i}^{*} s_{i2}(t)s_{i2}(t') \cdots s_{ip}(t)s_{ip}(t')

+ \frac{p!}{2N^{p-1}} \int_{0}^{\infty} dt' \dot{s}_{i}(t') \sum_{j=1;j \neq i}^{N} s_{j}(t)\dot{s}_{j}(t') \sum_{ij}^{*} s_{i3}(t)s_{i3}(t') \cdots s_{ip}(t)s_{ip}(t'). \quad (B11)$$

We multiply by $s_{i}(t')$, and sum over $i$; in the limit $N \to \infty$ we obtain :

$$0 = -2Tr(t',t) + \frac{\partial C(t,t')}{\partial t} + \mu(t)C(t,t')

- \frac{p}{2} \int_{0}^{t'} dt''r(t',t'')C(t,t'')^{p-1}

- \frac{p(p-1)}{2} \int_{0}^{t} dt''C(t',t'')r(t,t'')C(t,t'')^{p-2} \quad (B12)$$

(with $r(t',t) = 0$ for $t > t'$).

In the same way, multiplying by $\dot{s}_{i}(t')$ and summing over $i$ yields the equation for $r$. Taking the limit $t' \to t$, with $C(t,t) = 1$, yields then the self consistency equation for $\mu(t)$.

The energy is obtained multiplying (11) by $s_{i}(t')$, averaging over the noise $\eta_{i}$ and the couplings, and finally taking the limit $t' \to t$ :

$$\sum_{i} s_{i}(t')\dot{s}_{i}(t) = \sum_{i} s_{i}(t') \frac{\partial H}{\partial s_{i}}(t) - \mu(t) \sum_{i} s_{i}(t')s_{i}(t) + \sum_{i} s_{i}(t')\eta_{i}(t) \quad (B13)$$

and, with (B6) :

$$\frac{1}{N} \sum_{i} s_{i}(t') \frac{\partial H}{\partial s_{i}}(t) = pE(t) \quad (B14)$$
so, with \( \lim_{t' \to t} \frac{\partial C(t, t')}{\partial t} = -T \):

\[
E(t) = \frac{T - \mu(t)}{p}.
\]

(B15)

3. Thermalized initial conditions

\( a. \) Dynamical equations

In the previous derivation, the average over the initial conditions \( s_i(0) \) was not performed, since the system was taken disordered at \( t = 0 \). This average is therefore uniform.

If on the other hand the initial system is thermalized at a given temperature \( T' \), this average is

\[
\int Ds(0) \cdots \frac{e^{-H[s(0)]/T'}}{Z(T')},
\]

(B16)

where \( Z(T') \) is the partition function at \( T' \). As we want to average over the couplings, we have to reintroduce replicas:

\[
\frac{1}{Z(T')} = \lim_{n \rightarrow 0} \left( \int Ds^1 \cdots Ds^{n-1} \exp \left( -\frac{1}{T'} \sum_{a=1}^{n-1} H[s^a] \right) \right).
\]

(B17)

We introduce at all times \( n \) replicas, with the same disorder realization. The spins are now \( s^a_i(t) \) with \( i = 1, \cdots, N \), \( a = 1, \cdots, n \). We obtain after averaging over the noise

\[
1 = \int Ds^a D\tilde{s}^a \exp \sum_{a=1}^{n} S^a_i,
\]

\[
S^a_i = \sum_{i=1}^{N} \int_{0}^{\infty} dt \tilde{s}_i^a(t) \left( T \tilde{s}_i^a(t) + \frac{\partial H^a}{\partial \tilde{s}_i^a} + \mu(t) s_i^a(t) \right) - \frac{1}{T'} H[s_i^a(0)].
\]

(B18)

After the average over the couplings, we have the same terms as in (B8), with a sum over replica indices, and the following new terms:

\[
\frac{1}{T'} \frac{p^l}{4N^{p-1}} \sum_{a \leq b} \sum_{i=1}^{a} \sum_{i=1}^{b} \int_{0}^{\infty} dt \tilde{s}_i^a(t)s_i^b(0)s_i^a(t)s_i^b(0) \cdots s_i^a(t)s_i^b(0).
\]

(B19)

Taking the derivative with respect to \( \tilde{s}_i^a(t') \), multiplying by \( s_i^b(t') \) or \( \tilde{s}_i^b(t') \), and summing over \( i \), we obtain for \( N \rightarrow \infty \) equations for \( C^{ab}(t, t') \) and \( r^{ab}(t, t') \), with \( r^{ab}(t, t') = \delta_{ab} r(t, t') \) since the replicas are not coupled. The terms (B18) give rise to a coupling to the initial conditions \( C^{ab}(0, 0) \) and \( C^{ab}((t', 0) \). We thus obtain equations which, in the RS case, reduce to (B15).

\( b. \) Case \( T = T' \)

The system is FDT and TTI : \( C(t, t') = C_{eq}(t - t') \), \( r(t, t') = r_{eq}(t - t') \); we note \( \tau = t - t' \); then \( r_{eq}(\tau) = -\frac{1}{T} \frac{\partial C_{eq}}{\partial \tau} \).

The integral appearing in the equation for \( \mu(t) \) is:

\[
\frac{p^2}{2} \int_{0}^{t} ds C_{eq}^{-1}(s) r(t, s) = -\frac{1}{T} \frac{p^2}{2} \int_{0}^{t} du C_{eq}^{-1}(u) C_{eq}'(u) = -\frac{p}{2T} (C_{eq}'(t) - 1)
\]

(B20)

so that

\[
\mu(t) = T - \frac{p}{2T} (C_{eq}'(t) - 1) + \frac{p}{2T} C_{eq}'(t) = T + \frac{p}{2T}.
\]

(B21)
For the equation for the correlation, the first integral is
\[ \int_0^t ds \ C_p^{-1}(t, s)r(t', s) = \int_0^{t'} ds \ C_p^{-1}(t - s)r_{eq}(t' - s) = -\frac{1}{T} \int_0^t du \ C_p^{-1}(u)C_{eq}(u - \tau), \] (B22)
and the second one yields
\[ (p - 1) \int_0^t ds \ C_p^{-2}(t, s)r(t, s)C(s, t') = (p - 1) \int_0^{t'} ds \ C_p^{-2}(t - s)r_{eq}(t - s)C_{eq}(t' - s) + (p - 1) \int_t^{t'} ds \ C_p^{-2}(t - s)r_{eq}(t - s)C_{eq}(s - t') \]
\[ = -\frac{p - 1}{T} \int_\tau^t du \ C_p^{-2}(u)C_{eq}'(u)C_{eq}(u - \tau) - \frac{p - 1}{T} \int_0^\tau du \ C_p^{-2}(u)C_{eq}'(u)C_{eq}(\tau - u) \]
and, after integrating by parts:
\[ -\frac{1}{T} \left(C_{eq}(t')C_p^{-1}(t) - C_{eq}(\tau)\right) + \frac{1}{T} \int_\tau^t du \ C_p^{-1}(u)C_{eq}'(u - \tau) - \frac{1}{T} \int_0^\tau du \ C_p^{-1}(\tau - u)C_{eq}'(u). \] (B23)
We finally obtain
\[ \frac{\partial C_{eq}}{\partial \tau} = -\left(T + \frac{p}{2T}\right)C_{eq}(\tau) - \frac{p}{2T} \left(C_{eq}(t')C_p^{-1}(t) - C_{eq}(\tau)\right) - \frac{1}{T} \int_0^\tau du \ C_p^{-1}(\tau - u)C_{eq}'(u) + \frac{p}{2T}C_p^{-1}(t)C_{eq}'(t'), \] (B24)
which is equation [13].
Remark : it is possible to treat in the same way the equation for the response function; this yields the derivative of [13] with respect to \( \tau \).

c. \( T \) different from \( T' \)

For \( T \) different from \( T' \), there is a transient regime. We therefore take \( t \) and \( t' \) large, i.e. the one-time quantities have reached their limiting values:
\[ C(t, 0) = C(t', 0) = l \]
\[ \mu(t) = \mu_\infty = T + \frac{p}{2T} \left(1 - C_p^\infty\right) + \frac{p}{2T}l^p. \] (B25)
The equation [18] is obtained in the same way as for \( T = T' \). The limit \( \tau \to \infty \) yields
\[ 0 = -\left(\mu_\infty - \frac{p}{2T}\right)C_\infty - \frac{p}{2T}C_\infty^{-1}(C_\infty - 1) - \frac{p}{2T}C_p^\infty + \frac{p}{2T}l^p, \] (B26)
or
\[ TC_\infty = \frac{p}{2T}l^p(1 - C_\infty) + \frac{p}{2T}C_\infty^{-1}(1 - C_\infty)^2. \] (B27)
Taking the limit of \( t \) going to infinity in the equation for \( C(t, t') \) with \( t' = 0 \) [13], we obtain:
\[ 0 = -\mu_\infty l + \frac{p(p - 1)}{2} \lim_{t \to \infty} \int_0^t C_p^{-2}(t - s)r_{eq}(t - s)C(s, 0) + \frac{p}{2T}l^p - 1 \]
\[ = -\mu_\infty l - \frac{p}{2T}(C_\infty^{-1} - 1) + \frac{p}{2T}l^p \]
and finally:

\[ l^p - 2 = \frac{2TT'}{p(1 - C_\infty)}. \]  

(B28)

Let us consider the TAP states that dominate the partition function at \( T' \); we note

\[ q = q(E^0_T, T) \quad \text{et} \quad q_1 = q(E^0_T', T'). \]  

(B29)

Equations (A42) and (A31) show that \( q \) and \( q_1 \) verify:

\[
\frac{p}{2T'^2} q_1^{p-2} (1 - q_1) = 1, \quad \frac{1 - q}{T} q^{p-1} = \frac{1 - q_1}{T'} q_1^{p-1},
\]  

(B30)

so that

\[ q_1 = 1 - \frac{p}{2T^2} (1 - q)^2 q^{p-2} \]  

(B31)

If we now take \( X = 1 - \frac{p}{2T^2} (1 - C_\infty)^2 C_\infty^{p-2} \), the equation for \( C_\infty \) (coming from (B27) and (B28)) can be written:

\[ X - \left( \frac{p}{2} (1 - X) \right)^{-1/(p-2)} T'^{2/(p-2)} = 0, \]  

(B32)

therefore \( X = q_1 \) and:

\[ C_\infty = q(E^0_{T'}, T). \]  

(B33)

The energy is (B17):

\[ E_\infty = \frac{T - \mu_\infty}{p} = \frac{1}{2T} (C_{\infty} - 1) - \frac{l^p}{2T'} \]  

(B34)

(remark that for \( T = T' \), we get back, with \( l = C_\infty, E_\infty = -\frac{1}{2T} \)), i.e.:

\[ E_\infty = -\frac{TC_\infty}{p(1 - C_\infty)} + \frac{C_{\infty}^{p-1}}{2T} - \frac{1}{2T}, \]  

(B35)

the energy at \( T \) of the TAP states dominating the statics at \( T' \) is:

\[ E(E^0_{T'}, T) = q^{p-1} E^0_{T'} - \frac{1}{2T} ((p - 1)q^p - pq^{p-1} + 1) \]  

(B36)

which is identical to (B33) (this identity is obtained using (B33) and (A44)):

\[ E_\infty = E(E^0_{T'}, T). \]  

(B37)

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