LEAST ENERGY SOLUTIONS FOR COUPLED HARTREE SYSTEM WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENTS

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Abstract. In this paper we are interested in the following critical coupled Hartree system

\[
\begin{aligned}
(-\Delta)^s \tilde{u} + \lambda_1 \tilde{u} &= \alpha_1 \int_{\Omega} \frac{|\tilde{u}(z)|^{2^*_\mu}}{|x-z|^\mu} dz |\tilde{u}|^{2^*_\mu-2} \tilde{u} + \beta \int_{\Omega} \frac{|\tilde{v}(z)|^{2^*_\mu}}{|x-z|^\mu} dz |\tilde{u}|^{2^*_\mu-2} \tilde{u}, \quad \text{in } \Omega, \\
(-\Delta)^s \tilde{v} + \lambda_2 \tilde{v} &= \alpha_2 \int_{\Omega} \frac{|\tilde{v}(z)|^{2^*_\mu}}{|x-z|^\mu} dz |\tilde{v}|^{2^*_\mu-2} \tilde{v} + \beta \int_{\Omega} \frac{|\tilde{u}(z)|^{2^*_\mu}}{|x-z|^\mu} dz |\tilde{v}|^{2^*_\mu-2} \tilde{v}, \quad \text{in } \Omega, \\
\tilde{u} = \tilde{v} = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(0 < s < 1, \alpha_1, \alpha_2 > 0, \beta \neq 0, 4s < \mu < N, 2^*_\mu = (2N - \mu)/(N - 2s), \Omega \subset \mathbb{R}^N (N \geq 3)\) is a smooth bounded domain, \(-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0\) with \(\lambda_1(\Omega)\) the first eigenvalue of \((-\Delta)^s\) under the Dirichlet boundary condition. Assume that the nonlinearity and the coupling terms are both of the upper critical growth due to the Hardy–Littlewood–Sobolev inequality, by applying the Dirichlet-to-Neumann map, we are able to obtain the existence of the ground state solution of the critical coupled Hartree system.

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1. **Introduction.** The following two-component system with Hartree type nonlinearities describes the boson stars in mean-field theory \([20, 27]\), which has attracted a great deal of attention in theoretical and numerical astrophysics over the past years.

\[
\begin{aligned}
    i\partial_t \Psi_1 &= -\frac{\hbar^2}{2m} \Delta \Psi_1 + W_1(x)\Psi_1 - (K(x) \ast |\Psi_1|^2) \Psi_1 - \beta(K(x) \ast |\Psi_2|^2) \Psi_1, \\
    i\partial_t \Psi_2 &= -\frac{\hbar^2}{2m} \Delta \Psi_2 + W_2(x)\Psi_2 - (K(x) \ast |\Psi_2|^2) \Psi_2 - \beta(K(x) \ast |\Psi_1|^2) \Psi_2,
\end{aligned}
\]

(1)

where \(\Psi_i : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}, i = 1, 2\), \(m\) is the mass of the particles, \(\hbar\) is the planck constant, the real positive constant \(\beta\) is the scattering length, \(W_i(x), i = 1, 2\) are the external potentials and \(K(x)\) is the response function which possesses information on the mutual interaction between the particles. If the response function is a delta function \(K(x) = \delta(x)\), the nonlinear response is local in fact and has been widely considered in recent years. In this case system (1) appears especially in nonlinear optics \([37, 38]\). Physically, the solution \(\Psi_i\) denotes the \(i\)-th component of the beam in Kerr-like photorefractive media \([2]\). System (1) also arises in the Hartree-Fock theory to describe a binary mixture of Bose-Einstein condensates in two different hyperfine states. Set \(\Psi_1(x,t) = u(x)e^{-iE_1t}\) and \(\Psi_2(x, t) = v(x)e^{-iE_2t}\), then system (1) can be transformed into a coupled elliptic system given by

\[
\begin{aligned}
    -\varepsilon^2 \Delta u + V_1(x)u &= |u|^2 u + \beta |v|^2 u, \\
    -\varepsilon^2 \Delta v + V_2(x)v &= |v|^2 v + \beta |u|^2 v,
\end{aligned}
\]

(2)

where \(V_i(x) = W_i(x) - E_i\), \(\varepsilon^2 = \frac{h^2}{2m}\). The existence of solitary waves of the coupled Schrödinger system (2) has been investigated recently by many authors, see \([5, 8, 19, 24, 29, 30, 31, 32, 34, 36, 39, 44, 47, 51]\). The critical growth case was studied in \([15, 16, 17, 23, 42, 43]\). Among them, Chen and Zou studied the nonlinear Schrödinger system

\[
\begin{aligned}
    -\Delta u + \lambda_1 u &= \mu_1 u^{2p-1} + \beta u^{p-1}v^p, \quad \text{in } \Omega, \\
    -\Delta v + \lambda_2 v &= \mu_2 v^{2p-1} + \beta v^{p-1}u^p, \quad \text{in } \Omega, \\
    u, v &\geq 0, \quad \text{in } \Omega, \\
    u = v = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^N\), \(\mu_1, \mu_2 > 0\) and \(\beta \neq 0\) is a coupling constant. For the special critical case \(p = 2\) and \(N = 4\), the authors also investigated the existence and properties of least energy solutions in the higher dimensional case \(2p = \frac{2N}{N-2}\) and \(N \geq 5\) in \([17]\).

But nonlocal nonlinearities have attracted considerable interest as a means of eliminating collapse and stabilizing multidimensional solitary waves. It appears naturally in optical systems \([33]\) and is known to influence the propagation of electromagnetic waves in plasmas \([9]\). Non-locality also plays an important role in the theory of Bose-Einstein condensation, where it accounts for the finite-range many-body interaction \([18]\). In \([53]\) the authors applied the variational methods to study a Schrödinger system with non-local coupling nonlinearities of Hartree type

\[
\begin{aligned}
    -\varepsilon^2 \Delta u + V_1(x)u &= (\int_{\mathbb{R}^3} \frac{u^2}{|x-y|}dy)u + \beta(\int_{\mathbb{R}^3} \frac{v^2}{|x-y|}dy)u, \\
    -\varepsilon^2 \Delta v + V_2(x)v &= (\int_{\mathbb{R}^3} \frac{v^2}{|x-y|}dy)v + \beta(\int_{\mathbb{R}^3} \frac{u^2}{|x-y|}dy)v.
\end{aligned}
\]
Under suitable assumptions on the potential, the authors proved the existence of purely vector ground state solutions for the Schrödinger system if the parameter $\varepsilon$ is small and $\beta$ is large. In [10] the author considered a class of fractional Schrödinger systems of Choquard type, by using the concentration compactness techniques they were able to prove existence and stability results of the standing waves. The nonlocal Hartree system was also studied in [50], there the authors obtained the existence of standing waves. The nonlocal systems of Choquard type, by using the concentration compactness techniques they were able to prove existence and stability results of the standing waves. The nonlocal system with Hardy-Littlewood-Sobolev critical exponents that is standing waves.

The aim of the present paper is to consider the following critical coupled Hartree system with Hardy-Littlewood-Sobolev critical exponents that is

$$
\begin{cases}
(-\Delta)^s u + \lambda_1 \tilde{u} = \alpha_1 \int_{\Omega} \frac{\tilde{u}(z)}{|x-z|^\mu} dz |\tilde{u}|^{2^*_s-2} \tilde{u}, & \text{in } \Omega, \\
(-\Delta)^s \tilde{v} + \lambda_2 \tilde{v} = \alpha_2 \int_{\Omega} \frac{\tilde{v}(z)}{|x-z|^\mu} dz |\tilde{v}|^{2^*_s-2} \tilde{v}, & \text{in } \Omega,
\end{cases}
$$

where $0 < s < 1$, $\alpha_1, \alpha_2 > 0$, $\beta \neq 0$, $4s < \mu < N$, $2^*_\mu = (2N - \mu)/(N - 2s)$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$ with $\lambda_1(\Omega)$ the first eigenvalue of $(-\Delta)^s$ under the Dirichlet boundary condition.

Before stating the main results, we need to recall some basic results about the fractional Laplacian operator $(-\Delta)^s$. Consider the nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^N$ is defined on the Schwartz class of functions $g \in \mathcal{S}$ through the Fourier transform

$$
|(-\hat{\Delta})^s g|_\xi = (2\pi|\xi|)^{2s} \hat{g}(\xi),
$$

or via the Riesz potential (see, for example, [25, 48]). There is another way of defining this operator. In the remarkable paper [14] by Caffarelli and Silvestre, the authors introduced the $s$–harmonic extension technique. If $\tilde{u}$ is a regular function in $\mathbb{R}^N$, we say that $u = E_s(\tilde{u})$ is its $s$-harmonic extension to the upper half-space, $\mathbb{R}^{N+1}_+$, if $u$ the solution to the following problem

$$
\begin{cases}
\text{div}(y^{1-2s} \nabla u) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\
u(x,0) = \tilde{u}, & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+.
\end{cases}
$$

As shown in [14], $(-\Delta)^s$ can also be characterized by

$$
(-\Delta)^s \hat{u}(x) = -\frac{1}{\kappa_s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial u}{\partial y}(x,y),
$$

where $\kappa_s = (2^{1-2s} \Gamma(1-s)/\Gamma(s))$. Thus the appropriate function spaces to work with are $X^{2s}(\mathbb{R}^{N+1}_+)$ and $H^s(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ and $C_0^\infty(\mathbb{R}^N)$ respectively, under the norms $\|u\|_{X^{2s}} = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u(x,y)|^2 dx dy$ and $\|\hat{u}\|_{H^s}^2 = \int_{\mathbb{R}^N} |2\pi \xi|^{2s} |\hat{u}(\xi)|^2 d\xi$. The extension operator is well defined for smooth functions through a Poisson kernel, whose explicit expression is given in [12]. It can also be defined in the space $\tilde{H}^s(\mathbb{R}^N)$, and, in fact,

$$
\|u\|_{X^{2s}} = C \|\tilde{u}\|_{\tilde{H}^s}, \quad \text{for all } \tilde{u} \in \tilde{H}^s(\mathbb{R}^N),
$$

where $C = \sqrt{\kappa_s}$ (see for [6]). On the other hand, for a function $u \in X^{2s}(\mathbb{R}^{N+1}_+)$ we shall denote its trace on $\mathbb{R}^N$ as $tr_{\mathbb{R}^N}(u)$. This trace operator is also well defined and it satisfies

$$
\|tr_{\mathbb{R}^N}(u)\|_{H^s} \leq C \|u\|_{X^{2s}},
$$

where $C = \sqrt{\kappa_s}$.
The Sobolev trace embedding implies that the trace also belongs to $L^{2^*_s}(\mathbb{R}^N)$, where $2^*_s = \frac{2N}{N-2s}$. Thus, for all $u \in X^{2^*_s}(\mathbb{R}^{N+1}_+)$, we have

$$\left(\int_{\mathbb{R}^N} |u(x,0)|^{2N/(N-2s)} \, dx\right)^{(N-2s)/2N} \leq C \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u(x,y)|^2 \, dx \, dy\right)^{1/2}. \quad (8)$$

To treat the nonlocal nonlinearities, we need to recall the Hardy-Littlewood-Sobolev inequality, see for example [28].

**Proposition 1** (Hardy–Littlewood–Sobolev inequality). Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of $f, h$, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \, dx \, dy \leq C(t, N, \mu, r) \|f\|_t \|h\|_r, \quad (9)$$

where $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$. If $t = r = 2N/(2N - \mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^2 \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(\frac{N}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{-1 + \frac{\mu}{N}}. \quad (10)$$

In this case there is equality in (9) if and only if $f \equiv (\text{const.})h$ and

$$h(x) = C(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $C \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Denote by $D^{1,2}(\mathbb{R}^{N+1}_+)$ the closure of the set of smooth functions compactly supported in $\mathbb{R}^{N+1}_+$ with respect to the norm of $\|u\|_{D^{1,2}(\mathbb{R}^{N+1}_+)} = \left(\int_{\mathbb{R}^{N+1}_+} \|\nabla u\|^2 \, dx \, dy\right)^{1/2}$. From the Hardy-Littlewood-Sobolev inequality, we know

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,0)|^{2^*_s} |u(z,0)|^{2^*_s}}{|x-z|^\mu} \, dx \, dz\right)^{\frac{N-2s}{2N}} \leq C(N, \mu) \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \left(\int_{\mathbb{R}^N} |u(x,0)|^{2^*_s} \, dx\right)^{\frac{2^*_s}{2}}, \quad (11)$$

where $C(N, \mu)$ is defined as in the Proposition 1 and $2^*_s = \frac{2N}{N-2s}$. Consequently,

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,0)|^{2^*_s} |u(z,0)|^{2^*_s}}{|x-z|^\mu} \, dx \, dz\right)^{\frac{N-2s}{2N}} \leq C \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u|^2 \, dx \, dy. \quad (12)$$

Thus, we can define the best constant $S_C$ by

$$S_C := \inf_{u \in D^{1,2}(\mathbb{R}^{N+1}_+), \{0\}} \frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u(x,y)|^2 \, dx \, dy}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,0)|^{2^*_s} |u(z,0)|^{2^*_s}}{|x-z|^\mu} \, dx \, dz\right)^{\frac{N-2s}{2N-\mu}}}. \quad (13)$$

Let $S_0 = \inf_{u \in D^{1,2}(\mathbb{R}^{N+1}_+), \{0\}} \frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u(x,y)|^2 \, dx \, dy}{\left(\int_{\mathbb{R}^N} |u(x,0)|^{2^*_s} \, dx\right)^{2/2^*_s}}$. By [7] we know that $S_0$ is achieved by the extremal functions

$$\tilde{U}_\varepsilon(x,y) = \frac{\varepsilon^{(N-2s)/2}}{|(x,y + \varepsilon)|^{N-2s}}, \quad (14)$$

where $\varepsilon > 0$ is arbitrary. Follow [21, 54], by the Hardy-Littlewood-Sobolev inequality and the definition of $S_C$, we have $S_C = \frac{S_0}{C(N, \mu)^{\frac{2N-\mu}{2N}}}$, while

$$U_\varepsilon(x,y) = S_0^{\frac{(N-\mu)(2s-N)}{2(N-\mu-2s)}} C(N, \mu)^{\frac{2s-N}{2N-\mu+2s}} \tilde{U}_\varepsilon(x,y). \quad (15)$$
is the unique minimizer for $S_C$ and satisfies
\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla u) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\
\frac{\partial u}{\partial \nu} = \int_{\mathbb{R}^N} \frac{|u(z,0)|^{2^*_n}}{|x-z|^\mu} dz |u|^{2^*_n-2}u, & \text{in } \mathbb{R}^N, \\
u = 0, & \text{on } \partial \mathbb{R}_+^{N+1}.
\end{cases}
\]

Moreover,
\[
\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla U_{\epsilon}|^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_{\epsilon}(x,0)|^{2^*_n} |U_{\epsilon}(z,0)|^{2^*_n}}{|x-z|^\mu} dx dz = S_C^{-\frac{2N-\mu}{2N-2^*_n}}. \tag{14}
\]

Firstly we are interested in the existence of the least energy solution of the problem in the whole space
\[
\begin{cases}
(-\Delta)^s \tilde{u} = \alpha_1 \int_{\mathbb{R}^N} \frac{|\tilde{u}(z)|^{2^*_n}}{|x-z|^\mu} dz |\tilde{u}|^{2^*_n-2} \tilde{u} + \beta \int_{\mathbb{R}^N} \frac{|\tilde{v}(z)|^{2^*_n}}{|x-z|^\mu} dz |\tilde{v}|^{2^*_n-2} \tilde{u}, & \text{in } \mathbb{R}^N, \\
(-\Delta)^s \tilde{v} = \alpha_2 \int_{\mathbb{R}^N} \frac{|\tilde{v}(z)|^{2^*_n}}{|x-z|^\mu} dz |\tilde{v}|^{2^*_n-2} \tilde{v} + \beta \int_{\mathbb{R}^N} \frac{|\tilde{u}(z)|^{2^*_n}}{|x-z|^\mu} dz |\tilde{u}|^{2^*_n-2} \tilde{v}, & \text{in } \mathbb{R}^N.
\end{cases} \tag{15}
\]

By using the harmonic extension technique, we will study the following boundary value problem in a half space:
\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla u) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\
-\text{div}(y^{1-2s} \nabla v) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\
\frac{\partial u}{\partial \nu} = \alpha_1 \int_{\mathbb{R}^N} \frac{|u(z,0)|^{2^*_n}}{|x-z|^\mu} dz |u|^{2^*_n-2}u + \beta \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_n}}{|x-z|^\mu} dz |v|^{2^*_n-2}u, & \text{in } \mathbb{R}^N, \\
\frac{\partial v}{\partial \nu} = \alpha_2 \int_{\mathbb{R}^N} \frac{|v(z,0)|^{2^*_n}}{|x-z|^\mu} dz |v|^{2^*_n-2}v + \beta \int_{\mathbb{R}^N} \frac{|u(z,0)|^{2^*_n}}{|x-z|^\mu} dz |u|^{2^*_n-2}v, & \text{in } \mathbb{R}^N, \\
u = v = 0, & \text{on } \partial \mathbb{R}_+^{N+1}. \tag{16}
\end{cases}
\]

Thus if $(u, v)$ satisfies (16), then the trace $(\tilde{u}, \tilde{v})$ on $\mathbb{R}^N$ of the function $(u, v)$ will be a solution of problem (15). Clearly (16) has semi-trivial solutions $(a_1^{\frac{2N-\mu}{2N-2^*_n}} U_{\epsilon}(x, y), 0)$ and $(0, a_2^{\frac{2N-\mu}{2N-2^*_n}} U_{\epsilon}(x, y))$. Here, we are only interested in nontrivial solutions of (16).

Define $D := X^{2s}(\mathbb{R}_+^{N+1}) \times X^{2s}(\mathbb{R}_+^{N+1})$ and introduce the functional $I$ by
\[
I(u, v) = \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \left( |\nabla u|^2 + |\nabla v|^2 \right) dx dy - \frac{1}{2 \cdot 2^*_n} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2\beta |u(x,0)|^{2^*_n} |v(z,0)|^{2^*_n}}{|x-z|^\mu} dx dz \right) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\alpha_1 |u(x,0)|^{2^*_n} |u(z,0)|^{2^*_n} + \alpha_2 |v(x,0)|^{2^*_n} |v(z,0)|^{2^*_n}}{|x-z|^\mu} dx dz.
\]

We also introduce the set
\[
\mathcal{N} = \{(u, v) \in D : u \neq 0, v \neq 0, I'(u, v)(u, 0) = I'(u, v)(0, v) = 0\}
\]
and define the least energy
\[
A := \inf_{(u, v) \in \mathcal{N}} I(u, v) = \inf_{(u, v) \in \mathcal{N}} \frac{N - \mu + 2s}{4N - 2\mu} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \left( |\nabla u|^2 + |\nabla v|^2 \right) dx dy. \tag{17}
\]
Similar to [15], we introduce the nonlinear system (2)

\[
\begin{align*}
\alpha_1 k_{2r-1}^2 + \beta k_{2r-1}^2 l l_1^2 & = 1, \\
\beta k_{2r-1}^2 l l_1^2 & = 1, \\
\end{align*}
\] (18)

Then there exists \((k_0, l_0)\) such that \((k_0, l_0)\) satisfies (18) and

\[
k_0 = \min \{ (k, l) : (k, l) \text{ is a solution of (18)} \}.
\] (19)

We have the following theorem,

**Theorem 1.1.** (1) If \(\beta < 0\), then \(A\) is not attained.
(2) If \(\beta > 0\), (16) has a least energy solution \((U, V)\) with \(I(U, V) = A\), which is radially symmetric and decreasing. Moreover,
(i) If \(\beta \geq \frac{N-k_0}{N+2s}\) max\{\(\alpha_1, \alpha_2\)\}, then \(A\) is attained by \(\sqrt{k_0}U_\varepsilon, \sqrt{l_0}U_\varepsilon\), where \(k_0, l_0\) is defined in (18). That is, \((\sqrt{k_0}U_\varepsilon, \sqrt{l_0}U_\varepsilon)\) is a least energy solution of (16).

(ii) there exists \(0 < \beta_1 \leq \frac{N-k_0}{N+2s}\) \(\geq 2\) \(\lambda_{\alpha_1,\alpha_2}\), and for any \(0 < \beta < \beta_1\), there exists a solution \((k(\beta), l(\beta))\) of (18), such that

\[
I(\sqrt{k(\beta)}U_\varepsilon, \sqrt{l(\beta)}U_\varepsilon) > A = I(U, V).
\]

That is, \((\sqrt{k(\beta)}U_\varepsilon, \sqrt{l(\beta)}U_\varepsilon)\) is a different solution of (16) with respect to \((U, V)\).

The fractional Laplacian \((-\Delta)^s\) on bounded domain was defined in [7]. Let \((\varphi, \lambda_j)\) be the eigenfunctions and eigenvectors of \(-\Delta\) in \(\Omega\) with Dirichlet boundary data, Then \((\varphi, \lambda_j)\) are the eigenfunctions and eigenvectors of \((-\Delta)^s\). The first eigenvalue \(\lambda_1(\Omega)\) of \((-\Delta)^s\) on bounded is defined by

\[
\lambda_1(\Omega) := \inf_{M_1} \int_C y^{1-2s}|\nabla u|^2dxdy,
\] (20)

where \(M_1 := \{u(x, y) \in X_0^2(\mathcal{C}) : \int_\Omega u(x, 0)^2dx = 1\}\). Define the space of functions defined in our domain \(\Omega\) as \(H_0^s(\Omega) = \{\tilde{u} = \Sigma \alpha_j \varphi_j \in L^2(\Omega) : \|\tilde{u}\|_{H_0^s(\Omega)} = (\Sigma \alpha_j^2 \lambda_j^s)^{1/2} < \infty\}\), then \((-\Delta)^su = \sum \alpha_j \lambda_j \varphi_j\). We have \(\|\tilde{u}\|_{H_0^s(\Omega)} = \|(-\Delta)^{s/2}\tilde{u}\|_{L^2(\Omega)}\).

Denote the half cylinder by \(\mathcal{C} = \Omega \times (0, \infty)\) and its lateral boundary by \(\partial \mathcal{C} = \partial \Omega \times \{0\}\). We define

\[
X_0^{2s}(\mathcal{C}) = \{u \in X^{2s}(\mathcal{C}) : u = 0 \ a.e. \ on \ \partial \mathcal{C}\},
\] (21)

the Sobolev space of functions with traces vanish on \(\partial \mathcal{C}\) and the norm

\[
\|u\|_{X_0^{2s}(\mathcal{C})} = \left( \int_C y^{1-2s}|\nabla u|^2dxdy \right)^{1/2}.
\]

We denote by \(tr_\Omega\) the trace operator on \(\Omega\) for functions in \(X_0^{2s}(\mathcal{C})\):

\[
tr_\Omega u := u(x, 0), \quad u \in X_0^{2s}(\mathcal{C}).
\]

We have that \(tr_\Omega u \in H_0^s(\Omega)\), since it is well known that traces of \(X_0^{2s}(\mathcal{C})\) functions are \(H_0^s(\Omega)\) functions on the boundary. Note that the closure \(H_0^s(\Omega)\) of smooth functions with compact support \(C_0^\infty(\Omega)\), that is, \(C_0^\infty(\Omega)\) is dense in \(H_0^s(\Omega)\). Denote \(\mathcal{V}_0(\Omega)\) the space of traces on \(\Omega \times \{0\}\) of functions in \(X_0^{2s}(\mathcal{C})\):

\[
\mathcal{V}_0(\Omega) := \{\tilde{u} = tr_\Omega u | u \in X_0^{2s}(\mathcal{C}) \} \subset H_0^s(\Omega),
\] (22)

endowed with the norm of \(H_0^s(\Omega)\). The dual space of \(\mathcal{V}_0(\Omega)\) is denoted by \(\mathcal{V}_0^*(\Omega)\). For a function \(\tilde{u} \in \mathcal{V}_0(\Omega)\) on \(\Omega \subset \mathbb{R}^N\), consider the minimizing problem:

\[
\inf \left\{ \int_C y^{1-2s}|\nabla u|^2dxdy, |u \in X_0^{2s}(\mathcal{C}), u(x, 0) = \tilde{u} \ on \ \Omega \right\},
\]
follow the arguments in [13], we see that there is a unique minimizer $u \in X_0^{2s}(\Omega)$ of $J(u) = \int_{\Omega} y^{1-2s}|\nabla u|^2 \, dx \, dy$ satisfies

$$
\begin{cases}
\text{div}(y^{1-2s}\nabla u) = 0, & \text{in } \mathcal{C}, \\
u = 0, & \text{on } \partial \mathcal{C}, \\
u(x, 0) = \tilde{u}, & \text{on } \Omega.
\end{cases}
$$

The function $u \in X_0^{2s}(\mathcal{C})$ is then called the harmonic extension of $\tilde{u}$ in $\mathcal{C}$ which vanishes on $\partial \mathcal{C}$. In the following the harmonic extension of $u$ will be denoted by $u := \text{s-ext}(\tilde{u})$. We have that the extension operator is an isometry between $H_0^s(\Omega)$ and $X_0^{2s}(\mathcal{C})$. That is $\|s\text{-ext}(\tilde{u})\|_{X_0^{2s}(\mathcal{C})} = \|\tilde{u}\|_{H_0^s(\Omega)}$. It is easy to see that for every $\eta \in C^\infty$ and $\eta \equiv 0$ on $\partial \mathcal{C}$, we have $\int_\mathcal{C} \nabla u \nabla \eta \, dx \, dy = \int_\Omega \frac{\partial u}{\partial \nu} \eta \, dx$. Now one can define the operator $(-\Delta)^s u$: $\mathcal{V}_0(\Omega) \to \mathcal{V}_0^s(\Omega)$ by $(-\Delta)^s u := \frac{\partial u}{\partial \nu}|\Omega$, where $u = s\text{-ext}(\tilde{u}) \in X_0^{2s}(\mathcal{C})$ and $\nu$ is the unit outer normal to $\Omega$. We also introduce the inverse operator $(-\Delta)^{-s} u$: $\mathcal{V}_0^s(\Omega) \to \mathcal{V}_0(\Omega)$ by $f \mapsto \text{tr}_\Omega u$, where $u$ is found by solving the problem:

$$
\begin{cases}
\text{div}(y^{1-2s}\nabla u) = 0, & \text{in } \mathcal{C}, \\
u = 0, & \text{on } \partial \mathcal{C}, \\
u(x, 0) = f(u(x, 0)), & \text{in } \Omega.
\end{cases}
$$

By using the harmonic extension, we can reformulate (3) as

$$
\begin{cases}
-\text{div}(y^{1-2s}\nabla u) = 0, & \text{in } \mathcal{C}, \\
-\text{div}(y^{1-2s}\nabla v) = 0, & \text{in } \mathcal{C}, \\
\frac{\partial u}{\partial \nu^s} = \alpha_1 \int_\Omega \frac{|u(z, 0)|^{2s}}{|x - z|^\mu} \, dz |u|^{2s-2} u + \beta \int_\Omega \frac{|v(z, 0)|^{2s}}{|x - z|^\mu} \, dz |u|^{2s-2} u - \lambda_1 u, & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu^s} = \alpha_2 \int_\Omega \frac{|v(z, 0)|^{2s}}{|x - z|^\mu} \, dz |v|^{2s-2} v + \beta \int_\Omega \frac{|u(z, 0)|^{2s}}{|x - z|^\mu} \, dz |v|^{2s-2} v - \lambda_2 v, & \text{in } \Omega, \\
u = u = 0, & \text{on } \partial \mathcal{C}.
\end{cases}
$$

The critical problem with critical exponent and the the nonlocal Hartree equation driven by fractional Laplacian were studied in [7, 26, 41, 45, 54]. In [54], the authors studied the existence of least energy solution for the nonlocal critical problem

$$
\begin{cases}
(-\Delta)^s \tilde{u} + \lambda_1 \tilde{u} = \left(\frac{1}{|x - z|^\mu} * |\tilde{u}|^{2s}\right)|\tilde{u}|^{2s-2} \tilde{u}, & \text{in } \Omega, \\
\tilde{u} \in H_0^s(\Omega),
\end{cases}
$$

where $-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0$. Consequently, (3) has semi-trivial solutions $(\tilde{u}_\alpha, 0)$ and $(0, \tilde{u}_\alpha)$. We are concerned with the existence of real nontrivial solutions of (25). Define $H := X_0^{2s}(\mathcal{C}) \times X_0^{2s}(\mathcal{C})$ and introduce the energy functional

$$
E(u, v) = \frac{1}{2} \left( \int_{\mathcal{C}} y^{1-2s}(|\nabla u|^2 + |\nabla v|^2) \, dx \, dy + \int_\Omega (\lambda_1 u(x, 0)^2 + \lambda_2 v(x, 0)^2) \, dx \right) \\
- \frac{1}{2} \sum_{i=1}^2 \int_{\mathcal{C}} \alpha_i |u(x, 0)|^{2s} |u(z, 0)|^{2s} + \alpha_2 |v(x, 0)|^{2s} |v(z, 0)|^{2s} \, dx \, dz \\
+ \int_{\mathcal{C}} \int_{\mathcal{C}} 2\beta |u(x, 0)|^{2s} |v(z, 0)|^{2s} \, dx \, dz.
$$

We see that $(u, v)$ is a weak solution of (25) if and only if $(u, v)$ is a critical point of the functional $E$. 
We define a Nehari type manifold
\[ M = \{ (u, v) \in H : u \neq 0, v \neq 0, E'(u, v)(u, 0) = E'(u, v)(0, v) = 0 \}. \]

Then any nontrivial solutions of \((25)\) belong to \(M\). Define the least energy
\[
B := \inf_{(u, v) \in M} E(u, v) = \inf_{(u, v) \in M} \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y^{1-2s}(|\nabla u|^2 + |\nabla v|^2)dx dy \right.
\]
\[
+ \int (\lambda_1 u(x, 0)^2 + \lambda_2 v(x, 0)^2) dx \bigg) \tag{28}
\]

First we will study the symmetric case \(-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0\). Recall the Brezis-Nirenberg problem
\[
\begin{cases}
\text{div}(y^{1-2s}\nabla u) = 0, & \text{in } C, \\
u = 0, & \text{on } \partial C, \\
\frac{\partial u}{\partial \nu^s} = \left( \int_{\Omega} \frac{|u(z, 0)|^{2^*_s}}{|x-z|^{\mu}} \, dz \right) |u|^{2^*_s-2}u - \lambda u, & \text{in } \Omega, 
\end{cases} \tag{29}
\]

which has a least energy solution \(w\) with energy
\[
B_1 := \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y^{1-2s}|\nabla w|^2 dx dy + \lambda \int_{\Omega} w(x, 0)^2 dx \right)
\]
\[
= \frac{N - \mu + 2s}{4N - 2\mu} \int_{\Omega} \int_{\Omega} \frac{|w(x, 0)|^{2^*_s}|w(z, 0)|^{2^*_s}}{|x-z|^{\mu}} \, dxdz. \tag{30}
\]

Moreover, for any solution \(u\) of \((29)\), we know
\[
\int_C y^{1-2s} |\nabla u|^2 dx dy + \lambda \int_{\Omega} u(x, 0)^2 dx 
\]
\[
\geq \left( \frac{4N - 2\mu}{N - \mu + 2s} B_1 \right)^{\frac{N-2s}{N-2s+\mu}} \left( \int_{\Omega} \int_{\Omega} \frac{|u(x, 0)|^{2^*_s}|u(z, 0)|^{2^*_s}}{|x-z|^{\mu}} \, dxdz \right)^{\frac{N-2s}{2s}} \tag{31}
\]

The existence results for the the symmetric case can be stated in the following theorems.

**Theorem 1.2.** Suppose that \(-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0\), \((k_0, l_0)\) satisfy \((19)\). Then for any \(\beta > 0\), \((\sqrt{k_0} w, \sqrt{l_0} w)\) is a solution of \((25)\). Moreover, if \(\beta \geq \frac{N - \mu + 2s}{N - 2s} \max\{\alpha_1, \alpha_2\}\), then \(B\) is attained by \((\sqrt{k_0} w, \sqrt{l_0} w)\), that is, \((\sqrt{k_0} w, \sqrt{l_0} w)\) is a least energy solution of \((25)\).

**Theorem 1.3.** Suppose that \(-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0\), \((k_0, l_0)\) satisfy \((19)\). Then there exists \(\beta_0 \geq \frac{N - \mu + 2s}{N - 2s} \max\{\alpha_1, \alpha_2\}\) such that, if \(\beta > \beta_0\) and \((u, v)\) is any a least energy solution of \((25)\), then \((u, v) = (\sqrt{k_0} w, \sqrt{l_0} w)\), where \(w\) is a least energy solution of the Brezis-Nirenberg problem \((29)\).

We also consider the general case \(-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0\).

**Theorem 1.4.** Suppose that \(-\lambda_1(\Omega) < \lambda_1, \lambda_2 < 0\), then system \((25)\) has a least energy solution \(E(u, v) = B\) for any \(\beta \neq 0\).
case $\beta < 0$, and use mountain pass argument to prove Theorem 1.4 for the case $\beta > 0$.

2. **Proof of Theorem 1.1.** We begin this section with the theorem about the existence of solution for problem (16).

Define functions

$$\varphi_1(k,l) := \alpha_1 k^{2^*_\mu - 1} + \beta k^{\frac{2^*_\mu}{2} - 1} l^{\frac{2^*_\mu}{2}} - 1, \quad k > 0, \quad l \geq 0;$$

$$\varphi_2(k,l) := \alpha_2 k^{2^*_\mu - 1} + \beta k^{\frac{2^*_\mu}{2} - 1} l^{\frac{2^*_\mu}{2}} - 1, \quad l > 0, \quad k \geq 0;$$

$$f_1(k) := \beta^{\frac{2}{2^*_\mu}} \left( k^{1 - \frac{2^*_\mu}{2}} - \alpha_1 k^{\frac{2^*_\mu}{2}} \right)^{\frac{2^*_\mu}{2}}, \quad 0 < k \leq \alpha_1 \frac{-1}{1 - \frac{2^*_\mu}{2}};$$

$$f_2(l) := \beta^{\frac{2}{2^*_\mu}} \left( l^{1 - \frac{2^*_\mu}{2}} - \alpha_2 l^{\frac{2^*_\mu}{2}} \right)^{\frac{2^*_\mu}{2}}, \quad 0 < l \leq \alpha_2 \frac{-1}{1 - \frac{2^*_\mu}{2}}.\tag{35}$$

Then $\varphi_1(k,f_1(k)) \equiv 0$ and $\varphi_2(f_2(l),l) \equiv 0$.

In order to prove Theorem 1.1, we shall make use of the following four lemmas taken from [17].

**Lemma 2.1.** Suppose that $\beta > 0$, then equation

$$\varphi_1(k,l) = 0, \quad \varphi_2(k,l) = 0, \quad k, \quad l > 0 \tag{36}$$

has a solution $(k_0,l_0)$, which satisfies $\varphi_2(k,f_1(k)) < 0, \forall 0 < k < k_0$, that is, $(k_0,l_0)$ satisfies (19). Similarly, (36) has a solution $(k_1,l_1)$ such that $\varphi_1(f_2(l),l) < 0, \forall 0 < l < l_1$.

**Lemma 2.2.** Suppose that $\beta > (\frac{N - \mu + 2\sigma}{2N - 2\mu}) \max \{\alpha_1, \alpha_2\}$, then $f_1(k) + k$ is strictly increasing for $k \in [0, \alpha_1 \frac{-N + 2\sigma}{N - \mu}]$ and $f_2(l) + l$ is strictly increasing for $l \in [0, \alpha_2 \frac{-N + 2\sigma}{N - \mu}]$.

**Lemma 2.3.** Suppose that $\beta > (\frac{N - \mu + 2\sigma}{2N - 2\mu}) \max \{\alpha_1, \alpha_2\}$, let $(k_0,l_0)$ be in Lemma 2.1. Then

$$\max \{\alpha_1 (k_0 + l_0) \frac{-N + 2\sigma}{N - \mu}, \alpha_2 (k_0 + l_0) \frac{-N + 2\sigma}{N - \mu}\} < 1$$

and

$$\varphi_2(k,f_1(k)) < 0, \quad \forall 0 < k < k_0; \quad \varphi_1(f_2(l),l) < 0, \quad \forall 0 < l < l_0.\tag{37}$$

**Lemma 2.4.** Suppose that $\beta > (\frac{N - \mu + 2\sigma}{N - 2\mu}) \max \{\alpha_1, \alpha_2\}$. Then

$$\begin{cases}
    k + l \leq k_0 + l_0, \\
    \varphi_1(k,l) \geq 0, \quad \varphi_2(k,l) \geq 0, \\
    k, l \geq 0, \quad (k,l) \neq (0,0) \tag{38}
\end{cases}$$

has a unique solution $(k_0,l_0)$.

By the semigroup property of the Riesz potential and the Hölder inequality, we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu} \frac{dx}{|x-z|^\mu} dz$$

$$\leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x,0)|^{2^*_\mu} |u(z,0)|^{2^*_\mu} \frac{dx}{|x-z|^\mu} dz \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu} \frac{dx}{|x-z|^\mu} dz \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x,0)|^{2^*_\mu} |u(z,0)|^{2^*_\mu} \frac{dx}{|x-z|^\mu} dz + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x,0)|^{2^*_\mu} |v(z,0)|^{2^*_\mu} \frac{dx}{|x-z|^\mu} dz. \tag{39}$$
For any \((u,v) \in \mathcal{N}\), by (10) and (39), we have \(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla u^2 + |v|^2) dx dy \leq C\), which implies that
\[
A = \inf_{(u,v) \in \mathcal{N}} \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla u|^2 dx dy + \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v|^2 dx dy \right) > 0.
\] (40)

**Lemma 2.5.** If \(A\) is attained by a couple \((u,v) \in \mathcal{N}\), then this couple is a critical point of \(I\), provided \(-\infty < \beta < 0\).

**Proof.** Let \(\beta < 0\). Assume that \((u,v) \in \mathcal{N}\) such that \(A = I(u,v)\). Define
\[
I'(u,v)(u,0) = \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla u|^2 dx dy - \alpha_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,0)|^{2s}_\alpha |u(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz
\]
\[
- \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha dx dz,
\]
\[
I'(u,v)(0,v) = \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v|^2 dx dy - \alpha_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz
\]
\[
- \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x,0)|^{2s}_\alpha |u(z,0)|^{2s}_\alpha dx dz.
\]
Then there exists two Lagrange multipliers \(L_1, L_2 \in \mathbb{R}\) such that
\[
I'(u,v) + L_1 I'(u,v)(u,0) + L_2 I'(u,v)(0,v) = 0.
\]
Testing this equation with \((u,0)\) and \((0,v)\) respectively, we obtain
\[
L_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |u(z,0)|^{2s}_\alpha - (2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz + L_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |v(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz = 0,
\]
\[
L_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |v(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha - (2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz + L_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz = 0.
\]
Since \(\beta < 0\), we deduce from \(I'(u,v)(u,0) = I'(u,v)(0,v) = 0\) that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |u(z,0)|^{2s}_\alpha - (2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz
\]
\[
\times \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(2 - 2s)_\mu |v(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha - (2 - 2s)_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz
\]
\[
> \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2s_\mu |u(x,0)|^{2s}_\alpha |v(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz \right)^2
\]
a contradiction. From this we have \(L_1 = L_2 = 0\) and \(I'(u,v) = 0\). \(\square\)

**Proof of (1) in Theorem 1.1.** By (13) we see that \(w_{\alpha_i} := \alpha_i^{\frac{N-2s}{4(N-\mu-2s)}} U_1(x,y)\) satisfies equation \(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla u|^2 dx dy = \alpha_i \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,0)|^{2s}_\alpha |u(z,0)|^{2s}_\alpha}{|x-z|^\mu} dx dz\) in \(\mathbb{R}^{N+1}_+\).
Let \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N \) and \( (u_R(x, y), v_R(x, y)) = (w_{\alpha_1}(x, y), w_{\alpha_2}(x + Re_1, y)). \)

Then \( v_R(x, y) \to 0 \) weakly in \( X^{2s}(\mathbb{R}^N) \), so \( v_R(x, y) \to 0 \) weakly in \( L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \) as \( R \to +\infty \), then \( |v_R|^{2^*_s} \to 0 \) in \( L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \) as \( R \to +\infty \). By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \) to \( L^{\frac{N}{N-2s}}(\mathbb{R}^N) \), we know that \( |x|^{-\mu} \ast |v_R(z, 0)|^{2^*_s} \to 0 \) in \( L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \) as \( R \to +\infty \). We have \( (|x|^{-\mu} \ast |v_R(z, 0)|^{2^*_s})u_R|^{2^*_s}u_R \to 0 \) in \( L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \) as \( R \to +\infty \). That is, \( \lim_{R \to +\infty} \int_\mathbb{R}^N \frac{|u_R(x, 0)|^{2^*_s}|v_R(z, 0)|^{2^*_s}}{|x-z|^\mu}dxdz \to 0 \). Note that \( \beta < 0 \). Then for \( R > 0 \) sufficiently large, the following proof is inspired by [15]. We see that

\[
\left\{ \begin{array}{l}
t^2 \int_{\mathbb{R}^N} y^{1-2s} |\nabla u_R|^2 dxdy = t^{2-2s} \alpha_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x, 0)|^{2^*_s}|u_R(z, 0)|^{2^*_s}}{|x-z|^\mu}dxdz \\
+m^2 \int_{\mathbb{R}^N} y^{1-2s} |\nabla v_R|^2 dxdy = m^{2-2s} \alpha_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_R(x, 0)|^{2^*_s}|v_R(z, 0)|^{2^*_s}}{|x-z|^\mu}dxdz \\
+ t^{2-2s} m^{2-2s} \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x, 0)|^{2^*_s}|v_R(z, 0)|^{2^*_s}}{|x-z|^\mu}dxdz,
\end{array} \right.
\]

have a solution \((t_R, m_R)\) with \( t_R > 1 \) and \( m_R > 1 \), then we can get \( \lim_{R \to +\infty} (|t_R - 1| + |m_R - 1|) = 0 \). Note that \((t_Ru_R, m_Rv_R) \in \mathcal{N}\), we see from (14) that

\[
A \leq I(t_Ru_R, m_Rv_R)
\]

\[
= \frac{N - \mu + 2s}{4N - 2\mu} \left( t^2 \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla u_R|^2 dxdy + m^2 \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla v_R|^2 dxdy \right)
\]

\[
= \frac{N - \mu + 2s}{4N - 2\mu} \left( t^2 \alpha_1 \frac{N-2s}{N-2\mu} + m^2 \alpha_2 \frac{N-2s}{N-2\mu} \right) S_c^{2N-\mu}. \]

Letting \( R \to +\infty \), we get that \( A \leq \frac{N-\mu+2s}{4N-2\mu} \left( \alpha_1 \frac{N-2s}{N-2\mu} + \alpha_2 \frac{N-2s}{N-2\mu} \right) S_c^{2N-\mu} \).

On the other hand, for any \((u, v) \in \mathcal{N}\), we see from (11) and \( \beta < 0 \) that

\[
\int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla u|^2 dxdy \leq \alpha_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x, 0)|^{2^*_s}|v(z, 0)|^{2^*_s}}{|x-z|^\mu}dxdz
\]

\[
\leq \alpha_1 S_c^{2N-\mu} \left( \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla u|^2 dxdy \right)^{\frac{2N-\mu}{N-2s}},
\]

so we can get \( \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla u|^2 dxdy \geq \alpha_1 \frac{N-2s}{N-2\mu} S_c^{2N-\mu} \). Similarly, we also have \( \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla v|^2 dxdy \geq \alpha_2 \frac{N-2s}{N-2\mu} S_c^{2N-\mu} \). Combining these with (17), then we have \( A \geq \frac{N-\mu+2s}{4N-2\mu} \left( \alpha_1 \frac{N-2s}{N-2\mu} + \alpha_2 \frac{N-2s}{N-2\mu} \right) S_c^{2N-\mu} \). Hence,

\[
A = \frac{N - \mu + 2s}{4N - 2\mu} \left( \alpha_1 \frac{N-2s}{N-2\mu} + \alpha_2 \frac{N-2s}{N-2\mu} \right) S_c^{2N-\mu}.
\]
Now, assume that $A$ is attained by some $(u,v) \in N$ and $I(u,v) = A$. By Lemma 2.5, we get that $(u,v)$ is a nontrivial solution of (16) and $\beta < 0$. We have

$$
\int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla u|^2 \, dx \, dy \leq \alpha_1 S_C^{-\frac{2N-\mu}{N-\mu}} \left( \int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla u|^2 \, dx \, dy \right)^{\frac{2N-\mu}{2}}.
$$

Therefore, it is easy to see that

$$
A = I(u,v) = \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla u|^2 \, dx \, dy + \int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla v|^2 \, dx \, dy \right) > \frac{N - \mu + 2s}{4N - 2\mu} \left( \alpha_1 - \frac{2N-\mu}{N-\mu} \alpha_2 \right) S_C^{-\frac{2N-\mu}{N-\mu}},
$$

which is a contradiction. This completes the proof.

**Proof of (2) in Theorem 1.1.** Since $\beta > 0$, $(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon)$ is a nontrivial solution of (16) and

$$
A \leq I(\sqrt{k_0} U_\varepsilon, \sqrt{l_0} U_\varepsilon) = \frac{N - \mu + 2s}{4N - 2\mu} (k_0 + l_0) S_C^{-\frac{2N-\mu}{N-\mu}}. \tag{42}
$$

Assume that $\beta \geq \frac{N-\mu+2s}{4N-2\mu} \max \{\alpha_1, \alpha_2\}$. Let $\{(u_n,v_n)\} \in N$ be a minimizing sequence for $A$, that is, $I(u_n,v_n) \to A$. Define

$$
c_n = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x,0)|^{2s} |u_n(z,0)|^{2s}}{|x-z|^\mu} \, dx \, dz \right)^{\frac{1}{2s}},
$$

$$
d_n = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x,0)|^{2s} |v_n(z,0)|^{2s}}{|x-z|^\mu} \, dx \, dz \right)^{\frac{1}{2s}}.
$$

By (11) and (39), we have

$$
S_C c_n \leq \int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla u_n|^2 \, dx \, dy \leq \alpha_1 c_n^{\frac{2s}{\mu}} + \beta c_n^{\frac{2s}{\mu}} d_n^{\frac{2s}{\mu}}, \tag{43}
$$

$$
S_C d_n \leq \int_{\mathbb{R}^N_{+}} y^{1-2s} |\nabla v_n|^2 \, dx \, dy \leq \alpha_2 d_n^{\frac{2s}{\mu}} + \beta c_n^{\frac{2s}{\mu}} d_n^{\frac{2s}{\mu}}. \tag{44}
$$

This means

$$
S_C (c_n + d_n) \frac{4N - 2\mu}{N - \mu + 2s} I(u_n, v_n) \leq (k_0 + l_0) S_C^{-\frac{2N-\mu}{N-\mu}} + o(1), \tag{45}
$$

$$
\alpha_1 c_n^{\frac{2s}{\mu}} + \beta c_n^{\frac{2s}{\mu}} d_n^{\frac{2s}{\mu}} \geq S_C, \tag{46}
$$

$$
\alpha_2 d_n^{\frac{2s}{\mu}} + \beta c_n^{\frac{2s}{\mu}} d_n^{\frac{2s}{\mu}} \geq S_C. \tag{47}
$$

Notice that $c_n, d_n$ are uniformly bounded. Passing to a subsequence, we may assume that $c_n \to c$ and $d_n \to d$ as $n \to \infty$. Then by (43)-(44) we have $\alpha_1 c_n^{\frac{2s}{\mu}} + 2\beta c_n^{\frac{2s}{\mu}} d_n^{\frac{2s}{\mu}} + \alpha_2 d_n^{\frac{2s}{\mu}} \geq \frac{4N-2\mu}{N-\mu+2s} A$. Hence, without loss of generality, we assume that $c \neq 0$. If $d = 0$, then (45) implies $c \leq (k_0 + l_0) S_C^{-\frac{N-2s}{N-\mu+2s}}$. By (46) and Lemma 2.3 we get

$$
S_C \leq \alpha_1 c^{\frac{2s}{\mu}} \leq \alpha_1 (k_0 + l_0)^{\frac{2s}{\mu} - 1} S_C < S_C.
$$
a contradiction. Therefore, \( c \neq 0 \) and \( d \neq 0 \). Let \( k = \frac{c}{s^2 c^{\frac{N-2s}{2N-2s}}} \) and \( l = \frac{d}{s^2 c^{\frac{N-2s}{2N-2s}}} \), then by (45)-(47) we see that \((k,l)\) satisfies (38). By Lemma 2.4, we see that \((k,l) = (k_0, l_0)\). We see that \( c_n \rightarrow k_0 S_c^{\frac{N-2s}{2N-2s}} \) and \( d_n \rightarrow l_0 S_c^{\frac{N-2s}{2N-2s}} \) as \( n \rightarrow +\infty \) and
\[
\frac{4N-2\mu}{N-\mu+2s} A = \lim_{n \rightarrow +\infty} \frac{4N-2\mu}{N-\mu+2s} I(u_n, v_n) \geq \lim_{n \rightarrow +\infty} S_c(c_n + d_n) = (k_0 + l_0) S_c^{\frac{N-\mu+2s}{2N-2s}}.
\]
This implies that \( A = \frac{N-\mu+2s}{N-2\mu} (k_0 + l_0) S_c^{\frac{N-\mu+2s}{2N-2s}} = I(\sqrt{k_0} U_z, \sqrt{l_0} U_z) \) and so \((\sqrt{k_0} U_z, \sqrt{l_0} U_z)\) is a least energy solution of (16).

For any \( 0 < \beta < \frac{N-2s}{N-\mu+2s} \) \( \max\{\alpha_1, \alpha_2\} \), next we can prove (16) has a least energy solution. The following proof works for all \( \beta > 0 \). Therefore, we assume that \( \beta > 0 \) and define \( A' := \inf_{(u,v) \in N'} I(u, v) \), where \( N' := \{(u, v) \in D\setminus\{0, 0\}, I'(u, v)(u, v) = 0\} \). Note that \( N' \subset N' \), we have \( A' \leq A \). Define \( B_{R}^{N+1} := \{x \in \mathbb{R} : |x| < R, y > 0\} \), \( B_{R}^{N} := \{x \in \mathbb{R} : |x| < R, y = 0\} \) and \( H(0, R) = X_0^2(B_{R}^{N+1}) \times X_0^2(B_{R}^{N+1}) \). Consider
\[
\begin{align*}
\begin{cases}
-\text{div}(y^{1-2s} \nabla u) = 0, & \text{in } B_{R}^{N+1}, \\
-\text{div}(y^{1-2s} \nabla v) = 0, & \text{in } B_{R}^{N+1}, \\
\frac{\partial u}{\partial N} = \alpha_1 \int_{B_{R}^{N}} \frac{|u(z, 0)|^{2_s^*}}{|x-z|^\mu} dz |u|^{2_s^*-2} u + \beta \int_{B_{R}^{N}} \frac{|v(z, 0)|^{2_s^*}}{|x-z|^\mu} dz |u|^{2_s^*-2} u, & \text{in } B_{R}^{N}, \\
\frac{\partial v}{\partial N} = \alpha_2 \int_{B_{R}^{N}} \frac{|u(z, 0)|^{2_s^*}}{|x-z|^\mu} dz |v|^{2_s^*-2} v + \beta \int_{B_{R}^{N}} \frac{|v(z, 0)|^{2_s^*}}{|x-z|^\mu} dz |v|^{2_s^*-2} v, & \text{in } B_{R}^{N}, \\
u = v = 0, & \text{on } \partial B_{R}^{N+1}.
\end{cases}
\end{align*}
\]
Define \( A'(R) := \inf_{(u,v) \in N'(R)} I(u, v) \), where
\[
N'(R) := \{(u, v) \in H(0, R) \setminus \{0, 0\}, \int_{B_{R}^{N+1}} y^{1-2s} (|\nabla u|^2 + |\nabla v|^2) dx dy \\
- \int_{B_{R}^{N}} \int_{B_{R}^{N}} \alpha_1 |u(x, 0)|^{2_s^*} |u(z, 0)|^{2_s^*} |u(x, 0)|^{2_s^*} |v(z, 0)|^{2_s^*} dx dz \\
- \int_{B_{R}^{N}} \int_{B_{R}^{N}} 2\beta |u(x, 0)|^{2_s^*} |v(z, 0)|^{2_s^*} dx dz = 0\}\}
\]

**Lemma 2.6.** \( A'(R) = A' \) for all \( R > 0 \).

**Proof.** Take any \( R_1 > R_2 \). By \( N'(R_2) \subset N'(R_1) \), we have \( A'(R_1) \leq A'(R_2) \). On the other hand, for any \((u, v) \in N'(R_1)\), we define
\[
(u_1(x, y), v_1(x, y)) := \left( \frac{R_1}{R_2} \right)^{\frac{N-2s}{2N-2s}} u \left( \frac{R_1}{R_2} (x, y) \right), \left( \frac{R_1}{R_2} \right)^{\frac{N-2s}{2N-2s}} v \left( \frac{R_1}{R_2} (x, y) \right),
\]
and obviously \((u_1, v_1) \in N'(R_2)\). Then we get \( A'(R_2) \leq I(u_1, v_1) = I(u, v), \forall (u, v) \in N'(R_1) \). That is, \( A'(R_2) \leq A'(R_1) \) and so \( A'(R_2) = A'(R_1) \). Clearly \( A' \leq A'(R) \). Let \((u_n, v_n) \in N' \) be a minimizing sequence of \( A' \). We may assume that \( u_n, v_n \in X_0^2(B_{R_n}^{N+1}) \) for \( R_n > 0 \). Then \((u_n, v_n) \in N'(R_n)\) and \( A' = \lim_{n \rightarrow \infty} I(u_n, v_n) \geq \lim_{n \rightarrow \infty} A'(R_n) = A'(R) \). Therefore, \( A' = A'(R) \) for all \( R > 0 \). \( \square \)
Let $0 \leq \varepsilon < \frac{N-\mu+2s}{N-2s}$, $\theta = 2\mu - \varepsilon$. Consider

\[
\begin{aligned}
& -\text{div}(y^{1-2s}\nabla u) = 0, \\
& -\text{div}(y^{1-2s}\nabla v) = 0, \\
& \frac{\partial u}{\partial \nu^s} = \alpha_1 \int_{B^N_1} \frac{|u(z,0)|^\theta}{|x-z|^\mu} dz |u|^{\theta-2}u + \beta \int_{B^N_1} \frac{|v(z,0)|^\theta}{|x-z|^\mu} dz |u|^{\theta-2}u, \quad \text{in } B^{N+1}_1, \\
& \frac{\partial v}{\partial \nu^s} = \alpha_2 \int_{B^N_1} \frac{|v(z,0)|^\theta}{|x-z|^\mu} dz |v|^{\theta-2}v + \beta \int_{B^N_1} \frac{|u(z,0)|^\theta}{|x-z|^\mu} dz |v|^{\theta-2}v, \quad \text{in } B^{N+1}_1, \\
& u = v = 0, \quad \text{on } \partial B^{N+1}_1.
\end{aligned}
\]

and define $A_\varepsilon := \inf_{(u,v)\in \mathcal{N}_\varepsilon} I_\varepsilon(u,v)$, where

\[
I_\varepsilon(u,v) = \frac{1}{2} \int_{B^{N+1}_1} y^{1-2s}(|\nabla u|^2 + |\nabla v|^2) dx dy - \frac{1}{2\theta} \left( \int_{B^N_1} \int_{B^{N}_1} \frac{\alpha_1 |u(x,0)|^\theta |u(z,0)|^\theta}{|x-z|^\mu} dx dz \right.
\]

\[
+ \int_{B^N_1} \int_{B^{N}_1} \frac{2\beta |u(x,0)|^\theta |v(z,0)|^\theta}{|x-z|^\mu} dx dz + \int_{B^N_1} \int_{B^{N}_1} \frac{\alpha_2 |v(x,0)|^\theta |v(z,0)|^\theta}{|x-z|^\mu} dx dz) \bigg),
\]

\[
\mathcal{N}_\varepsilon = \{(u,v) \in H(0,1) \setminus (0,0), \quad H_\varepsilon(u,v) := I'_\varepsilon(u,v)(u,v) = 0\}.
\]

**Lemma 2.7.** For any $0 < \varepsilon < \frac{N-\mu+2s}{N-2s}$, there holds

\[
A_\varepsilon < \min \left\{ \inf_{(u,0)\in \mathcal{N}_\varepsilon} I_\varepsilon(u,0), \inf_{(0,v)\in \mathcal{N}_\varepsilon} I_\varepsilon(0,v) \right\}.
\]

**Proof.** Fix any $0 < \varepsilon < \frac{N-\mu+2s}{N-2s}$. Recall that $2 < \frac{2(2N-\mu)}{N-2s} - 2\varepsilon < \frac{2(2N-\mu)}{N-2s}$, we may let $u_i$ be a least energy solution of

\[
\begin{aligned}
& -\text{div}(y^{1-2s}\nabla u) = 0, \\
& \frac{\partial u}{\partial \nu^s} = \alpha_1 \int_{B^N_1} \frac{|u(z,0)|^\theta}{|x-z|^\mu} dz |u|^{\theta-2}u, \quad \text{in } B^N_1, \\
& u = 0, \quad \text{on } \partial B^{N+1}_1.
\end{aligned}
\]

Then $I_\varepsilon(u_1,0) = c_1 := \inf_{(u,0)\in \mathcal{N}_\varepsilon} I_\varepsilon(u,0)$, $I_\varepsilon(0,u_2) = c_2 := \inf_{(0,v)\in \mathcal{N}_\varepsilon} I_\varepsilon(0,v)$. The follow proof is inspired by [1]. For any $m \in \mathbb{R}$, there exists a unique $t(m) > 0$ such that $(t(m)u_1,t(m)mu_2) \in \mathcal{N}_\varepsilon$. In fact

\[
t(m)^{2\theta-2} = \frac{p'c_1 + m^2p'c_2}{p'c_1 + m^2p'c_2 + |m|^{\theta}2\beta \int_{B^N_1} \int_{B^{N}_1} \frac{|u_1(x,0)|^\theta |u_2(z,0)|^\theta}{|x-z|^\mu} dx dz},
\]

where $p' = \frac{2\left(2N-\mu-\varepsilon(N-2s)\right)}{N-\mu-2s-\varepsilon(N-2s)}$. Note $t(0) = 1$. Recall that $1 < \frac{2N-\mu-\varepsilon(N-2s)}{N-2s} < 2$, by direct computations we have

\[
\lim_{m \to 0} \frac{t'(m)}{|m|^{\theta-2}m} = -\frac{2\beta \int_{B^N_1} \int_{B^{N}_1} \frac{|u_1(x,0)|^\theta |u_2(z,0)|^\theta}{|x-z|^\mu} dx dz}{(2\theta - 2)p'c_1},
\]
that is, as \( m \to 0 \),
\[
t'(m) = -\frac{2^{2\beta} \int_{B_1^N} \int_{B_1^N} \frac{|u(x,0)|^\theta |w_2(x,0)|^\theta}{|x-z|^\mu} \, dx 
\]
have \( t(m) = 1 - \frac{2^{2\beta} \int_{B_1^N} \int_{B_1^N} \frac{|u(x,0)|^\theta |w_2(x,0)|^\theta}{|x-z|^\mu} \, dx}{2c_1} \). This implies that
\[
\frac{\alpha_1}{N-\mu + 2s} \omega < \frac{N-\mu + 2s}{4N-2\mu} \omega
\]
Recalling \( w_{\alpha_1} \) in the proof of Theorem 1.1-(1), similarly as Lemma 2.7, we have
\[
A_\varepsilon \leq \inf_{(u,v) \in \mathcal{N}_\varepsilon^*} \mathcal{I}_\varepsilon(u,v) = \inf_{(0,v) \in \mathcal{N}_\varepsilon^*} \mathcal{I}_\varepsilon(0,v).
\]
By a similar argument, we have \( A_\varepsilon < c_2 = \inf_{(0,v) \in \mathcal{N}_\varepsilon^*} \mathcal{I}_\varepsilon(0,v) \).

Recalling \( w_{\alpha_1} \) in the proof of Theorem 1.1-(1), similarly as Lemma 2.7, we have
\[
A' < \min \left\{ \inf_{(v_0) \in \mathcal{N}_\varepsilon^*} \mathcal{I}_\varepsilon(u,v), \quad \inf_{(0,v) \in \mathcal{N}_\varepsilon^*} I_\varepsilon(0,v) \right\}
\]
\[
= \min \left\{ \mathcal{I}(w_{\alpha_1}, 0), \mathcal{I}(0, w_{\alpha_2}) \right\}
\]
\[
= \min \left\{ \frac{N-\mu + 2s}{4N-2\mu} \omega, \frac{N-\mu + 2s}{4N-2\mu} \omega \right\}
\]
(51)

**Theorem 2.8.** For any \( 0 < \varepsilon < \frac{N+2s}{N-2s} \), (50) has a classical least energy solution \((u_\varepsilon, v_\varepsilon)\), and \(u_\varepsilon, v_\varepsilon\) are both radially symmetric and decreasing.

**Proof.** Fix any \( 0 < \varepsilon < \frac{N+2s}{N-2s} \), it is easy to see that \( A_\varepsilon > 0 \). For \((u, v) \in \mathcal{N}_\varepsilon^*\), we denote by \((u^*, v^*)\) as its Schwartz symmetrization. Then by the properties of Schwartz symmetrization and \( \beta > 0 \), we have
\[
\int_{B_1^{N+1}} y^{1-2s} (|\nabla u|^2 + |\nabla v|^2) \, dxdy 
\]
\[
\alpha_1 \int_{B_1^N} \int_{B_1^N} \frac{|u(x,0)|^\theta |u(z,0)|^\theta}{|x-z|^\mu} \, dx 
\]
\[
+ \alpha_2 \int_{B_1^N} \int_{B_1^N} \frac{|v(x,0)|^\theta |v(z,0)|^\theta}{|x-z|^\mu} \, dx 
\]
\[
\leq \alpha_1 \int_{B_1^N} \int_{B_1^N} \frac{|u^*(x,0)|^\theta |u^*(z,0)|^\theta}{|x-z|^\mu} \, dx 
\]
\[
+ \alpha_2 \int_{B_1^N} \int_{B_1^N} \frac{|v^*(x,0)|^\theta |v^*(z,0)|^\theta}{|x-z|^\mu} \, dx, 
\]
so we have

\[ \int_{B_1^{N+1}} y^{1-2s}(|\nabla u|^2 + |\nabla v|^2) dx \, dy \leq \alpha_1 \int_{B_1^N} \int_{B_1^N} \frac{|u^*(x,0)|^\theta |u^*(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz + 2\beta \int_{B_1^N} \int_{B_1^N} \frac{|u^*(x,0)|^\theta |v^*(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz + \alpha_2 \int_{B_1^N} \int_{B_1^N} \frac{|v^*(x,0)|^\theta |v^*(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz. \]

Therefore, there exists 0 < t* ≤ 1 such that (t*u*, t*v*) ∈ N, and then

\[ I_\varepsilon(t*u*, t*v*) \leq \left( \frac{1}{2} - \frac{1}{2\theta} \right) \int_{B_1^{N+1}} y^{1-2s}(|\nabla u|^2 + |\nabla v|^2) dx \, dy = I_\varepsilon(u, v). \] (52)

We may take a minimizing sequence (u_n, v_n) ∈ N of A_\varepsilon such that (u_n, v_n) = (u_n^*, v_n^*) and I_\varepsilon(u_n, v_n) → A_\varepsilon. We see from (52) that u_n, v_n are uniformly bounded in X_0^\mu(B_1^{N+1}). Passing to a subsequence, still denoted by u_n, there exists u_\varepsilon ∈ X_0^\mu(B_1^{N+1}) such that u_n → u_\varepsilon in X_0^\mu(B_1^{N+1}) and u_n → u_\varepsilon in L_2^2(B_1^{N+1}) as n → +∞. Then \( |u_n|^\theta \rightarrow |u_\varepsilon|^\theta \) in \( L_2^{2N - 2(N-2)\mu}(B_1^N) \) as \( n \rightarrow +\infty \). By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \( L_2^{2N - 2(N-2)\mu}(B_1^N) \) to \( L_2^{2N - 2(N-2)\mu}(B_1^N) \), we know that

\[ |x|^{-\mu} |u_n(z, 0)|^\theta \rightarrow |x|^{-\mu} |u_\varepsilon(z, 0)|^\theta \text{ in } L_2^{2N - 2(N-2)\mu}(B_1^N) \text{ as } n \rightarrow +\infty. \]

Combining with the fact that \( |u_n|^\theta - 2u_n \rightarrow |u_\varepsilon|^\theta - 2u_\varepsilon \) in \( L_2^{2N - 2(N-2)\mu}(B_1^N) \) as \( n \rightarrow +\infty \), we have \( (|x|^{-\mu} |u_n(z, 0)|^\theta) |u_n|^\theta - 2u_n \rightarrow (|x|^{-\mu} |u_\varepsilon(z, 0)|^\theta) |u_\varepsilon|^\theta - 2u_\varepsilon \) in \( L_2^{2N - 2(N-2)\mu}(B_1^N) \) as \( n \rightarrow +\infty \). Similarly, we get \( (|x|^{-\mu} |u_n(z, 0)|^\theta) |v_n|^\theta - 2v_n \rightarrow (|x|^{-\mu} |u_\varepsilon(z, 0)|^\theta) |v_\varepsilon|^\theta - 2v_\varepsilon \) in \( L_2^{2N - 2(N-2)\mu}(B_1^N) \) as \( n \rightarrow +\infty \). Then, we can obtain

\[ \int_{B_1^N} \int_{B_1^N} \frac{\alpha_1 |u_n(x,0)|^\theta |u_n(z,0)|^\theta + \alpha_2 |v_n(x,0)|^\theta |v_n(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz + \lim_{n \rightarrow \infty} \left( \int_{B_1^N} \int_{B_1^N} \frac{\alpha_1 |u_n(x,0)|^\theta |u_n(z,0)|^\theta + \alpha_2 |v_n(x,0)|^\theta |v_n(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz \right. \]

\[ \left. + \int_{B_1^N} \int_{B_1^N} \frac{2\beta |u_n(x,0)|^\theta |v_n(z,0)|^\theta}{|x-z|^{\mu}} \, dx \, dz \right) = 4N - 2\mu - 2\varepsilon(N - 2s) \lim_{n \rightarrow \infty} I_\varepsilon(u_n, v_n) = \frac{4N - 2\mu - 2\varepsilon(N - 2s)}{N - \mu + 2s - \varepsilon(N - 2s)} A_\varepsilon > 0, \]

which implies \( (u_\varepsilon, v_\varepsilon) \neq (0, 0) \). Moreover, \( u_\varepsilon, v_\varepsilon \) are radially symmetric. By Futu Lemma we have \( \int_{B_1^{N+1}} y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \, dy \leq \lim_{n \rightarrow \infty} \int_{B_1^{N+1}} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2) dx \, dy \leq \int_{B_1^{N+1}} y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx \, dy \].

(53)
Therefore, there exists $0 < t_\varepsilon \leq 1$ such that $(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) \in \mathcal{N}_\varepsilon'$, and then

$$A_\varepsilon \leq \lim_{n \to \infty} \left( \frac{\theta - 1}{2\theta} \right) \int_{B_{1}^{N+1}} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2) dxdy = \lim_{n \to \infty} I_\varepsilon(u_n, v_n) = A_\varepsilon.$$ 

Therefore, $t_\varepsilon = 1$ and $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon'$ with $I(u_\varepsilon, v_\varepsilon) = A_\varepsilon$. Moreover,

$$\int_{B_{1}^{N+1}} y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dxdy = \lim_{n \to \infty} \int_{B_{1}^{N+1}} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2) dxdy,$$

that is, $u_n \to u_\varepsilon$ and $v_n \to v_\varepsilon$ in $X_0^{2s}(B_{1}^{N+1})$. There exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $I'_\varepsilon(u_\varepsilon, v_\varepsilon) - \gamma H'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$. Since $I'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = H'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ and $H'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) \neq 0$, we get $\gamma = 0$, then $I'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$, by lemma 2.7, we see that $u_\varepsilon \neq 0$ and $v_\varepsilon \neq 0$. This means that $(u_\varepsilon, v_\varepsilon)$ is a least energy solution of (48) and $u_\varepsilon, v_\varepsilon$ are radially symmetric and decreasing.

Completion of the proof (2) in Theorem 1.1, we need to establish a Pohožaev type formula of (48). We adapt the method of Brezis and Kato [11] as in [40] and obtain the regularity of the weak solutions by using the Morse iteration technique, we can obtain $(u, v) \in H(0, 1) \cap L^\infty(0, 1)$, where $H(0, 1) = X_0^{2s}(B_{1}^{N+1}) \times X_0^{2s}(B_{1}^{N+1})$ and $L^\infty(0, 1) = L^\infty(B_{1}^{N+1}) \times L^\infty(B_{1}^{N+1})$. We skip the details.

**Lemma 2.9.** If $(u, v)$ is a weak solution of (48) in $H(0, 1) \cap L^\infty(0, 1)$, then $(u, v)$ satisfies the Pohožaev type identity:

$$\frac{N - 2s}{2} \int_{B_{1}^{N+1}} y^{1-2s} |\nabla u|^2 dxdy + \frac{N - 2s}{2} \int_{B_{1}^{N+1}} y^{1-2s} |\nabla v|^2 dxdy$$

$$+ \frac{1}{2} \int_{\partial B_{1}^{N+1}} y^{1-2s} |\nabla u|^2(x, \nu) d\sigma + \frac{1}{2} \int_{\partial B_{1}^{N+1}} y^{1-2s} |\nabla v|^2(x, \nu) d\sigma$$

$$= \frac{2N - \mu}{2 \cdot 2^*_s} \left( \alpha_1 \int_{B_{1}^{N} \times \{0\}} \int_{B_{1}^{N} \times \{0\}} |u(x, 0)|^{2^*_s} |u(z, 0)|^{2^*_s} \frac{|x - z|^\mu}{|x - z|^{\mu}} dxdz \right.$$  

$$+ \frac{2}{2} \int_{B_{1}^{N} \times \{0\}} \int_{B_{1}^{N} \times \{0\}} |u(x, 0)|^{2^*_s} |v(z, 0)|^{2^*_s} \frac{|x - z|^\mu}{|x - z|^{\mu}} dxdz$$

$$+ \alpha_2 \int_{B_{1}^{N} \times \{0\}} \int_{B_{1}^{N} \times \{0\}} |v(x, 0)|^{2^*_s} |v(z, 0)|^{2^*_s} \frac{|x - z|^\mu}{|x - z|^{\mu}} dxdz \right).$$
Thus, by (48), we know that in $H(0, 1) \cap L^\infty(0, 1)$ and $X = (x, y)$.
The following identity is known:
\[
\div\{y^{-2s}(X, \nabla u)\nabla u + (X, \nabla v)\nabla v\} - \frac{1}{2}y^{1-2s}X(|\nabla u|^2 + |\nabla v|^2)
\]
\[
+ \left(\frac{N - 2s}{2}\right)y^{1-2s}(|\nabla u|^2 + |\nabla v|^2)
\]
= $(X, \nabla u)\div(y^{1-2s}\nabla u) + (X, \nabla v)\div(y^{1-2s}\nabla v)$.

Thus, by (48), we know that in $B_1^{N+1}$,
\[
\div\{y^{1-2s}(X, \nabla u)\nabla u - \frac{1}{2}y^{1-2s}X|\nabla u|^2\} + \left(\frac{N - 2s}{2}\right)y^{1-2s}|\nabla u|^2
\]
\[
+ \div\{y^{1-2s}(X, \nabla v)\nabla v - \frac{1}{2}y^{1-2s}X|\nabla v|^2\} + \left(\frac{N - 2s}{2}\right)y^{1-2s}|\nabla v|^2
= 0.
\]

Integrating the above equation over $B_1^{N+1}$, we see
\[
\frac{1}{2}\int_{\partial B_1^{N+1}} y^{1-2s}|\nabla u|^2(x, \nu)\,d\sigma + \frac{1}{2}\int_{\partial B_1^{N+1}} y^{1-2s}|\nabla v|^2(x, \nu)\,d\sigma
\]
\[
+ \int_{B_1^{N}\times\{R\}} \left\{ y^{1-2s}((x, \nabla_x u) + R\partial_y u) - Rg^{1-2s}\frac{|\nabla u|^2}{2}\right\}\,dx
\]
\[
+ \int_{B_1^{N}\times\{R\}} \left\{ y^{1-2s}((x, \nabla_x v) + R\partial_y v) - Rg^{1-2s}\frac{|\nabla v|^2}{2}\right\}\,dx
\]
\[
+ \int_{B_1^{N}\times\{0\}} y^{1-2s}(x, \nabla_x u)(\nabla u, \nu)\,dx + \int_{B_1^{N}\times\{0\}} y^{1-2s}(x, \nabla_x v)(\nabla v, \nu)\,dx
\]
\[
+ \frac{N - 2s}{2}\int_{B_1^{N+1}} y^{1-2s}|\nabla u|^2\,dxdy + \frac{N - 2s}{2}\int_{B_1^{N+1}} y^{1-2s}|\nabla v|^2\,dxdy = 0.
\]

Form Proposition 6.2 of [21], we get
\[
\int_{B_1^{N}\times\{0\}} \left( x \cdot \nabla_x u(x, 0) \right) \left( \int_{B_1^{N}\times\{0\}} \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu} \,dz \right)|u(x, 0)|^{2^*_\mu - 1}\,dx
\]
\[
= -N \int_{B_1^{N}\times\{0\}} \frac{|u(x, 0)|^{2^*_\mu}}{|x - z|^\mu} \left| \int_{B_1^{N}\times\{0\}} \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu} \,dz \right| |u(x, 0)|^{2^*_\mu - 1}\,dx
\]
\[
- (2^*_\mu - 1) \int_{B_1^{N}\times\{0\}} x \cdot \nabla_x u(x, 0) \int_{B_1^{N}\times\{0\}} \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu} \,dz |u(x, 0)|^{2^*_\mu - 1}\,dx
\]
\[
+ \mu \int_{B_1^{N}\times\{0\}} \int_{B_1^{N}\times\{0\}} x \cdot (x - z) \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu+2} |u(x, 0)|^{2^*_\mu} \,dz\,dx.
\]

This implies that
\[
2^*_\mu \int_{B_1^{N}\times\{0\}} \left( x \cdot \nabla_x u(x, 0) \right) \left( \int_{B_1^{N}\times\{0\}} \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu} \,dz \right)|u(x, 0)|^{2^*_\mu - 1}\,dx
\]
\[
= -N \int_{B_1^{N}\times\{0\}} \int_{B_1^{N}\times\{0\}} \frac{|u(x, 0)|^{2^*_\mu}}{|x - z|^\mu} \,dz |u(x, 0)|^{2^*_\mu - 1}\,dx
\]
\[
+ \mu \int_{B_1^{N}\times\{0\}} \int_{B_1^{N}\times\{0\}} x \cdot (x - z) \frac{|u(z, 0)|^{2^*_\mu}}{|x - z|^\mu+2} |u(x, 0)|^{2^*_\mu} \,dz\,dx,
\]
\[2^\mu \int_{B_N^N \times \{0\}} (z \cdot \nabla_z u(z, 0)) \left( \int_{B_N^N \times \{0\}} \frac{|u(x, 0)|^{2^\mu}}{|x-z|^\mu} \right) |u(z, 0)|^{2^\mu-1} dz = -N \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{|u(z, 0)|^{2^\mu} |u(x, 0)|^{2^\mu}}{|x-z|^\mu} dz dx + \mu \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} z \cdot (z-x) \frac{|u(x, 0)|^{2^\mu}}{|x-z|^\mu+2} |u(z, 0)|^{2^\mu} dx dz.\]

Consequently, we get
\[\int_{B_N^N \times \{0\}} (x \cdot \nabla_x u(x, 0)) \left( \int_{B_N^N \times \{0\}} \frac{|u(x, 0)|^{2^\mu}}{|x-z|^\mu} dz \right) |u(x, 0)|^{2^\mu-1} dx = \frac{\mu - 2N}{2 \cdot 2^\mu} \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{|u(x, 0)|^{2^\mu} |u(z, 0)|^{2^\mu}}{|x-z|^\mu} dx dz.\]

Similarly,
\[\int_{B_N^N \times \{0\}} (x \cdot \nabla_x v(x, 0)) \left( \int_{B_N^N \times \{0\}} \frac{|v(x, 0)|^{2^\mu}}{|x-z|^\mu} dz \right) |v(x, 0)|^{2^\mu-1} dx = \frac{\mu - 2N}{2 \cdot 2^\mu} \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{|v(x, 0)|^{2^\mu} |v(z, 0)|^{2^\mu}}{|x-z|^\mu} dx dz,\]

Then using (48), we have
\[\int_{B_N^N \times \{0\}} (x, \nabla_x u) y^{1-2s} (\nabla u, v) dx + \int_{B_N^N \times \{0\}} (x, \nabla_x v) y^{1-2s} (\nabla v, v) dx + \int_{B_N^N \times \{0\}} (x, \nabla_z v) \left( \frac{|u(x, 0)|^{2^\mu}}{|x-z|^\mu} dz \right) |v(z, 0)|^{2^\mu-2} v dx + \beta \int_{B_N^N \times \{0\}} \frac{|u(z, 0)|^{2^\mu}}{|x-z|^\mu} dz |v(z, 0)|^{2^\mu-2} u dx = \left( \frac{\mu - 2N}{2 \cdot 2^\mu} \right) \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{\alpha_1 |u(x, 0)|^{2^\mu} |u(z, 0)|^{2^\mu}}{|x-z|^\mu} dx dz \]

\[+ \left( \frac{\mu - 2N}{2 \cdot 2^\mu} \right) \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{2\beta |u(x, 0)|^{2^\mu} |v(z, 0)|^{2^\mu}}{|x-z|^\mu} dx dz + \frac{\mu - 2N}{2 \cdot 2^\mu} \int_{B_N^N \times \{0\}} \int_{B_N^N \times \{0\}} \frac{\alpha_2 |v(x, 0)|^{2^\mu} |v(z, 0)|^{2^\mu}}{|x-z|^\mu} dx dz.\]

Form Lemma 3.1 of [49], there exists a sequence \( R_n \to \infty \) such that
\[\lim_{n \to \infty} \left( \int_{B_N^N \times \{R_n\}} \left\{ y^{1-2s} (x, \nabla_x u) + R_n \partial_y u - R_n y^{1-2s} |\nabla u|^2 \right\} dx \right. \]
\[\left. + \int_{B_N^N \times \{R_n\}} \left\{ y^{1-2s} (x, \nabla_x v) + R_n \partial_y v - R_n y^{1-2s} |\nabla v|^2 \right\} dx \right) = 0.\]
Thus, by (54), (55) and (56), we can obtain the Pohožaev type identity.

Recalling (49), for any \((u, v) \in N'(1)\), there exists \(t_\varepsilon > 0\) such that \((t_\varepsilon u, t_\varepsilon v) \in N'_\varepsilon\) with \(t_\varepsilon \to 1\) as \(\varepsilon \to 0\). Then \(\lim_{\varepsilon \to 0} A_\varepsilon \leq \limsup_{\varepsilon \to 0} I_\varepsilon(t_\varepsilon u, t_\varepsilon v) = I(u, v), \ \forall (u, v) \in N'(1)\). By Lemma 2.6 we have

\[
\limsup_{\varepsilon \to 0} A_\varepsilon \leq A'(1) = A'.
\]

By Theorem 2.8, let \((u_\varepsilon, v_\varepsilon)\) be a least energy solution of (50), which is radially symmetric and decreasing. By \(I'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = 0\), it is easily seen that, for \(\forall \ 0 < \varepsilon < \frac{N-2\mu}{N-2s}\), we have

\[
\frac{4N-2\mu-2\varepsilon(N-2s)}{N-\mu+2s-\varepsilon(N-2s)} A_\varepsilon = \int_{B^N_{\varepsilon+1}} y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) \, dx dy \geq C,
\]

where \(C\) is a positive constant independent of \(\varepsilon\). Then \(u_\varepsilon, v_\varepsilon\) are uniformly bounded in \(X^s_0(B^N_{1+1})\). Passing to a subsequence, we may assume that \(u_\varepsilon \rightharpoonup u_0\) and \(v_\varepsilon \rightharpoonup v_0\) in \(X^s_0(B^N_{1+1})\). Then \((u_0, v_0)\) is a solution of (48). Assume by contradiction that \(\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty\) is uniformly bounded, then by the Dominated Convergent Theorem, we get that

\[
\lim_{\varepsilon \to 0} \int_{B^N_{\varepsilon+1}} \int_{B^N_1} \frac{|u_\varepsilon(x, 0)|^\theta |u_\varepsilon(z, 0)|^\theta}{|x-z|^\mu} \, dx dz = \int_{B^N_1} \int_{B^N} \frac{|u_\varepsilon(x, 0)|^2 |u_\varepsilon(z, 0)|^{2\theta}}{|x-z|^\mu} \, dx dz,
\]

\[
\lim_{\varepsilon \to 0} \int_{B^N_{\varepsilon+1}} \int_{B^N_1} \frac{|v_\varepsilon(x, 0)|^\theta |v_\varepsilon(z, 0)|^\theta}{|x-z|^\mu} \, dx dz = \int_{B^N_1} \int_{B^N} \frac{|v_\varepsilon(x, 0)|^2 |v_\varepsilon(z, 0)|^{2\theta}}{|x-z|^\mu} \, dx dz,
\]

\[
\lim_{\varepsilon \to 0} \int_{B^N_{\varepsilon+1}} \int_{B^N_1} \frac{|v_\varepsilon(x, 0)|^\theta |v_\varepsilon(z, 0)|^\theta}{|x-z|^\mu} \, dx dz = \int_{B^N_1} \int_{B^N} \frac{|v_\varepsilon(x, 0)|^2 |v_\varepsilon(z, 0)|^{2\theta}}{|x-z|^\mu} \, dx dz.
\]

Combining these with \(I'_\varepsilon(u_\varepsilon, v_\varepsilon) = I'(u_0, v_0) = 0\), it is standard to show that \(u_\varepsilon \rightharpoonup u_0\) and \(v_\varepsilon \rightharpoonup v_0\) in \(X^s_0(B^N_{1+1})\). Then by (58), we see that \((u_0, v_0) \neq (0, 0)\). We may assume that \(u_0 \neq 0\) in \(X^s_0(B^N_{1+1})\). Combining these with Pohožaev identity in Lemma 2.9 and \((u_0, v_0)\) be a nontrivial of (48), we get a contradiction

\[
0 < \frac{1}{2} \int_{\partial B^N_{1+1}} y^{1-2s} |\nabla u_0|^2 (x, \nu) \, d\sigma + \frac{1}{2} \int_{\partial B^N_{1+1}} y^{1-2s} |\nabla v_0|^2 (x, \nu) \, d\sigma = 0,
\]

where \(\nu\) denotes the outward unit normal vector on \(B^N_{1+1}\). Therefore, \(\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \to \infty\) as \(\varepsilon \to 0\). We define

\[
u = \max_{B^N_{1+1}} u(x, y), v(x, 0) = \max_{B^N_{1+1}} v(x, y)\]

and

\[K_\varepsilon := \max\{\|u_\varepsilon(0, 0)\|, \|v_\varepsilon(0, 0)\|\},\]

we see that \(K_\varepsilon \to +\infty\). Define

\[
U_\varepsilon(x, y) = K^{-1}_\varepsilon u_\varepsilon \left( K^{-\theta-2}_\varepsilon x, K^{-\theta-2}_\varepsilon y \right),
\]

\[
V_\varepsilon(x, y) = K^{-1}_\varepsilon v_\varepsilon \left( K^{-\theta-2}_\varepsilon x, K^{-\theta-2}_\varepsilon y \right).
\]

Then we have

\[
1 = \max\{U_\varepsilon(0, 0), V_\varepsilon(0, 0)\} = \max_{(x, y) \in B^N_{1+1}} U_\varepsilon(x, y), \max_{(x, y) \in B^N_{1+1}} V_\varepsilon(x, y)\}
\]
and $U_\varepsilon, V_\varepsilon$ satisfy
\[
\begin{aligned}
-\text{div}(y^{1-2s}\nabla U_\varepsilon) &= 0, & \text{in } B^{N+1}_{K^{d-2}_\varepsilon}, \\
-\text{div}(y^{1-2s}\nabla V_\varepsilon) &= 0, & \text{in } B^N_{K^d_\varepsilon}, \\
\frac{\partial U_\varepsilon}{\partial \nu^s} &= \alpha_1 \int_{B^N_{K^d_\varepsilon}} \frac{|U_\varepsilon(z,0)|^\theta}{|x-z|^s} d\varepsilon U_\varepsilon^{\theta-2}U_\varepsilon, & \text{in } B^N_{K^d_\varepsilon}, \\
+\beta \int_{B^N_{K^d_\varepsilon}} \frac{|V_\varepsilon(z,0)|^\theta}{|x-z|^s} d\varepsilon U_\varepsilon^{\theta-2}V_\varepsilon, & \text{in } B^N_{K^d_\varepsilon}, \\
\frac{\partial V_\varepsilon}{\partial \nu^s} &= \alpha_2 \int_{B^N_{K^d_\varepsilon}} \frac{|V_\varepsilon(z,0)|^\theta}{|x-z|^s} d\varepsilon V_\varepsilon^{\theta-2}V_\varepsilon, & \text{in } B^N_{K^d_\varepsilon}, \\
+\beta \int_{B^N_{K^d_\varepsilon}} \frac{|U_\varepsilon(z,0)|^\theta}{|x-z|^s} d\varepsilon V_\varepsilon^{\theta-2}V_\varepsilon, & \text{in } B^N_{K^d_\varepsilon}, \\
U_\varepsilon &= V_\varepsilon = 0, & \text{on } \partial B^{N+1}_{K^{d-2}_\varepsilon}.
\end{aligned}
\]

From $\int_{\mathbb{R}^d_+} y^{1-2s}|\nabla U_\varepsilon|^2 dx dy \leq \int_{\mathbb{R}^d_+} y^{1-2s}|\nabla V_\varepsilon|^2 dx dy \leq \int_{\mathbb{R}^d_+} y^{1-2s}|\nabla v_\varepsilon|^2 dx dy$, we get that $(U_\varepsilon, V_\varepsilon)$ is bounded in $D = X^{2s}(\mathbb{R}^N_+ \times \mathbb{R}^{N+1})$. By elliptic estimates, for a subsequence we have $(U_\varepsilon, V_\varepsilon) \to (U, V) \in D$ uniformly in every compact subset of $\mathbb{R}^N_+$ as $\varepsilon \to 0$ and $(U, V)$ satisfies (16), we have $I'(U, V) = 0$. By (59) we have $(U, V) \neq (0, 0)$, and so $(U, V) \in \mathcal{N}$. Then we deduce from (57) that
\[
A' \leq I(U, V) = \left(\frac{1}{2} - \frac{1}{2\theta}\right) \int_{\mathbb{R}^d_+} y^{1-2s}(|\nabla U|^2 + |\nabla V|^2) dx dy
\]
\[
\leq \liminf_{\varepsilon \to 0} \left(\frac{1}{2} - \frac{1}{2\theta}\right) \int_{B^N_{K^d_\varepsilon}} y^{1-2s}(|\nabla U_\varepsilon|^2 + |\nabla V_\varepsilon|^2) dx dy
\]
\[
\leq \liminf_{\varepsilon \to 0} \left(\frac{1}{2} - \frac{1}{2\theta}\right) \int_{B^{N+1}_{K^{d-2}_\varepsilon}} y^{1-2s}(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2) dx dy = \lim_{\varepsilon \to 0} A_\varepsilon \leq A'.
\]

This implies that $I(U, V) = A'$. By (51) we have that $U \neq 0, V \neq 0$ are radially symmetric and decreasing, we also have $(U, V) \in \mathcal{N}$, so we get $I(U, V) \geq A \geq A'$, that is,
\[
I(U, V) = A = A',
\]
and $(U, V)$ is a least energy solution of (16), which is radially symmetric and decreasing.

Finally, we show the existence of $(k(\beta), l(\beta))$ for $\beta > 0$ small. Recall (32)-(33), we denote $\varphi_i(k, l)$ by $\varphi_i(k, l, \beta)$ here. Define $k(0) = \alpha_1^{-\frac{1}{2\mu-1}}$, $l(0) = \alpha_2^{-\frac{1}{2\mu-1}}$, then $\varphi_i(k(0), l(0), 0) = 0, i = 1, 2$. Note that
\[
\partial_k \varphi_1(k(0), l(0), 0) = (2\mu - 1)\alpha_1 k(0)^{2\mu - 2}, \quad \partial_l \varphi_1(k(0), l(0), 0) = (2\mu - 1)\alpha_2 l(0)^{2\mu - 2},
\]
\[
\partial_k \varphi_2(k(0), l(0), 0) = \partial_l \varphi_2(k(0), l(0), 0) = 0,
\]
which implies that
\[
\det \begin{pmatrix} \partial_k \varphi_1(k(0), l(0), 0) & \partial_l \varphi_1(k(0), l(0), 0) \\ \partial_k \varphi_2(k(0), l(0), 0) & \partial_l \varphi_2(k(0), l(0), 0) \end{pmatrix} > 0.
\]

Therefore, by the implicit function theorem, $k(\beta), l(\beta)$ are well defined and class $C^1$ on $(-\beta_2, \beta_2)$ for some $\beta_2 > 0$, and $\varphi_i(k(\beta), l(\beta), \beta) = 0, i = 1, 2$. This implies that $(\sqrt{k(\beta)} U_\varepsilon, \sqrt{l(\beta)} U_\varepsilon)$ is a solution of (16). Note $\lim_{\beta \to 0} (k(\beta) + l(\beta)) =$
\[ k(0) + l(0) = \alpha_1 \frac{\lambda}{2^{\frac{\mu+1}{2}}} + \alpha_2 \frac{1}{2^{\frac{\mu+1}{2}}}, \] 
that is, there exists \( 0 < \beta_1 \leq \beta_2 \), such that \( k(\beta) + l(\beta) > \min\{\alpha_1 \frac{\lambda}{2^{\frac{\mu+1}{2}}}, \alpha_2 \frac{1}{2^{\frac{\mu+1}{2}}}\}, \forall \beta \in (0, \beta_1). \) Combining this with (42) and (51), we have \( I(U, V) = A' = A < I(\sqrt{k(\beta)}U, \sqrt{l(\beta)}V), \forall \beta \in (0, \beta_1) \), that is, \((\sqrt{k(\beta)}U, \sqrt{l(\beta)}V)\) is different solution of (16) with respect to \((U, V)\). This completes the proof.

3. Proof of Theorem 1.2. The idea comes from [47]. We assume that \(-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0\). By (11) it is standard to see that \( B > 0 \). Since \( \beta > 0 \), by Lemma 2.1 equation (18) has a solution \((k_0, l_0)\). Recall (30), we see that \((\sqrt{k_0}w, \sqrt{l_0}w)\) is a nontrivial solution of (25) and

\[ 0 < B \leq E(\sqrt{k_0}w, \sqrt{l_0}w) = (k_0 + l_0)B_1. \]  
(61)

Now we assume that \( B > (\frac{N-\mu+2s}{2N-\mu})\max\{\alpha_1, \alpha_2\} \), then we will prove that \( B = E(\sqrt{k_0}w, \sqrt{l_0}w) \). Let \((u_n, v_n) \subset M \) be a minimizing sequence for \( B \), that is, \( E(u_n, v_n) \to B \). Recall \( c_n \) and \( d_n \) in Section 2, by (31) and (39), we have

\[ \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} c_n \leq \alpha_1 c_n^\ast + \beta c_n^\ast d_n^\ast. \]  
(62)

Similarly, we have

\[ \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} d_n \leq \alpha_2 d_n^\ast + \beta c_n^\ast d_n^\ast. \]  
(63)

Since

\[ E(u_n, v_n) = \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y^{1-2s}|\nabla u_n|^2 dx dy + \lambda \int_\Omega u_n(x, 0)^2 dx, + \int_C |\nabla v_n|^2 dx dy + \lambda \int_\Omega v_n(x, 0)^2 dx \right), \]

by (61) we have

\[ \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} (c_n + d_n) \leq \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} (k_0 + l_0)B_1 + o(1), \]
(64)

\[ \alpha_1 c_n^{\ast-1} + \beta c_n^{\ast-1} \frac{2\ast}{d_n} \geq \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}}, \]  
(65)

\[ \alpha_2 d_n^{\ast-1} + \beta d_n^{\ast-1} \frac{2\ast}{c_n} \geq \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} \]  
(66)

Similarly as the proof of Theorem 1.1. It follows that \( c_n \to k_0 \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}}, \)

\[ d_n \to l_0 \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} \]
as \( n \to +\infty \), and

\[ \frac{4N - 2\mu}{N - \mu + 2s}B \geq \lim_{n \to +\infty} \left( \frac{4N - 2\mu}{N - \mu + 2s}B_1 \right)^{\frac{N-\mu+2s}{2N-\mu}} (c_n + d_n) \]

\[ \quad = \frac{4N - 2\mu}{N - \mu + 2s} (k_0 + l_0)B_1. \]

Combining this with (61), one has that \( B = (k_0 + l_0)B_1 = E(\sqrt{k_0}w, \sqrt{l_0}w) \) and so \((\sqrt{k_0}w, \sqrt{l_0}w)\) is a least energy solution of (25).
4. **Proof of Theorem 1.3.** In this section, we also assume that $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0$. Let $\beta > 0$ and define

$$\mathcal{B} := \inf_{k \in \Gamma} \max_{t \in [0,1]} E(h(t)),$$

where $\Gamma \in \{ h \in C([0,1], H) : h(0) = (0,0), E(h(1)) < 0 \}$. By (27), we see that for any $(u, v) \in H, (u, v) \neq (0, 0)$ and

$$\max_{t > 0} E(tu, tv) = E(\hat{t}u, \hat{t}v)$$

$$= \frac{N - \mu + 2s}{4N - 2\mu} \frac{\bar{t}^2}{2} \left( \int_C y^{1 - 2s} |\nabla u|^2 dx dy + \lambda_1 \int_\Omega u(x, 0)^2 dx \right)$$

$$+ \frac{\bar{t}^2}{2} \frac{\lambda_2}{2} \int_\Omega v(x, 0)^2 dx$$

$$= \frac{N - \mu + 2s}{4N - 2\mu} \frac{\bar{t}^2}{2} \left( \int_\Omega \int_\Omega \frac{2\beta |u(x, 0)|^{2\mu} |v(z, 0)|^{2\mu}}{|x - z|^\mu} dxdz \right)$$

$$+ \int_\Omega \int_\Omega \frac{\alpha_1 |u(x, 0)|^{2\mu} |v(z, 0)|^{2\mu} + \alpha_2 |v(x, 0)|^{2\mu} |v(z, 0)|^{2\mu}}{|x - z|^\mu} dxdz$$

where $\hat{t} > 0$ satisfies

$$\hat{t}^2 \frac{\bar{t}^2}{2} = \frac{\int_C y^{1 - 2s} |\nabla u|^2 dx dy + \lambda_1 \int_\Omega u(x, 0)^2 dx \int_C y^{1 - 2s} |\nabla v|^2 dx dy + \lambda_2 \int_\Omega v(x, 0)^2 dx}{Q}.$$

where

$$Q = \int_\Omega \int_\Omega \frac{\alpha_1 |u(x, 0)|^{2\mu} |v(z, 0)|^{2\mu} + \alpha_2 |v(x, 0)|^{2\mu} |v(z, 0)|^{2\mu}}{|x - z|^\mu} dxdz$$

$$+ \int_\Omega \int_\Omega \frac{2\beta |u(x, 0)|^{2\mu} |v(z, 0)|^{2\mu}}{|x - z|^\mu} dxdz$$

Note that $(\hat{t}u, \hat{t}v) \in \mathcal{M}'$, where

$$\mathcal{M}' = \left\{ (u, v) \in H \setminus \{(0, 0)\} \right\}, \quad G(u, v) := \int_C y^{1 - 2s} |\nabla u|^2 dx dy + \lambda_1 \int_\Omega u(x, 0)^2 dx + \int_C y^{1 - 2s} |\nabla v|^2 dx dy + \lambda_2 \int_\Omega v(x, 0)^2 dx$$

$$\int_\Omega \int_\Omega \frac{\alpha_1 |u(x, 0)|^{2\mu} |v(z, 0)|^{2\mu} + \alpha_2 |v(x, 0)|^{2\mu} |v(z, 0)|^{2\mu}}{|x - z|^\mu} dxdz = 0 \right\},$$

it is easy to check that

$$\mathcal{B} = \inf_{H \ni (u, v) \neq (0, 0)} \max_{t > 0} E(tu, tv) = \inf_{(u, v) \in \mathcal{M}'} E(u, v).$$

Note that $\mathcal{M} \subset \mathcal{M}'$, and so $\mathcal{B} \leq \mathcal{B}$.

For $\beta \geq (2s_\mu - 1) \max\{\alpha_1, \alpha_2\}$, we define $g : [(2s_\mu - 1) \max\{\alpha_1, \alpha_2\}, \infty)$ by

$$g(\beta) := (2s_\mu - 1) \alpha_1 \alpha_2 \beta^{\frac{2}{\mu} - 2} + \beta^{\frac{2}{\mu}}.$$
Then we have

$$g'(\beta) = 2^\mu \beta^{\frac{1}{2^\mu} - 3} (\beta^2 - (2^\mu - 1)^2 \alpha_1 \alpha_2) > 0.$$  

By direct computations, we get

$$g((2^\mu - 1) \max\{\alpha_1, \alpha_2\}) \leq 2^\mu (2^\mu - 1)^{\frac{1}{2^\mu} - 1} \max\left\{\frac{\beta}{\alpha_1}, \frac{\beta}{\alpha_2}\right\}.$$  

Therefore, there exists a unique $\beta_0 \geq (2^\mu - 1) \max\{\alpha_1, \alpha_2\}$ such that

$$g(\beta_0) = 2^\mu (2^\mu - 1)^{\frac{1}{2^\mu} - 1} \max\left\{\frac{\beta}{\alpha_1}, \frac{\beta}{\alpha_2}\right\}.$$  

(73)

For $\forall \beta > \beta_0$, there hold

$$g(\beta) > 2^\mu (2^\mu - 1)^{\frac{1}{2^\mu} - 1} \max\left\{\frac{\beta}{\alpha_1}, \frac{\beta}{\alpha_2}\right\}.$$  

(74)

Moreover, if $\alpha_1 = \alpha_2$, we have

$$\beta_0 = (2^\mu - 1) \max\{\alpha_1, \alpha_2\}. \quad (75)$$

**Lemma 4.1.** Fix any $\alpha_1, \alpha_2 > 0$ and $\beta > \beta_0$. Let $(u_0, v_0)$ be a least energy solution of (25) proved in Theorem 1.4 and $(\sqrt{k_0}w, \sqrt{k_0}w)$ be the least energy solution obtained in Theorem 1.2, then we have

$$\int_{\Omega} \int_{\Omega} \frac{|u_0(x,0)|^2^\mu |u_0(z,0)|^2^\mu}{|x-z|^\mu} dxdz = k_0^2 \int_{\Omega} \int_{\Omega} \frac{|v(x,0)|^2^\mu |v(z,0)|^2^\mu}{|x-z|^\mu} dxdz. \quad (76)$$

**Proof.** Fix any $\alpha_1, \alpha_2 > 0$ and $\beta > \beta_0$ where $\beta_0 := \beta_0(\alpha_1, \alpha_2)$ is completely determined by $\alpha_1, \alpha_2$. There exists $0 < \varepsilon < \alpha$ such that for any $\alpha \in (\alpha_1 - \varepsilon, \alpha_1 + \varepsilon)$, one has $\beta > \beta_0(\alpha, \alpha)$. Then by Lemma 2.1, Lemma 4.2 in [17] and the implicit function theorem, when $\alpha_2$ is replaced by $\alpha$, functions $k_0(\alpha)$ and $l_0(\alpha)$ are well defined and class $C^1$ for $\alpha \in (\alpha_1 - \varepsilon, \alpha_1 + \varepsilon)$ for some $0 < \varepsilon_1 \leq \varepsilon$. Notice that $E, M$ and $B$ depend on $\alpha$, then we use notations $E_\alpha, M_\alpha$ and $B(\alpha)$. Then $B(\beta) = (k_0(\alpha) + l_0(\alpha)) B_1 \in C^1((\alpha_1 - \varepsilon, \alpha_1 + \varepsilon), \mathbb{R})$. In particular, $B'_{(\alpha_1)} := \frac{d}{d\alpha} B(\alpha)$ exists. Theorem 1.4 says that $B = B_\beta, \forall \beta > 0$. Then by (71) we have

$$B(\alpha) = \inf_{H \ni (u, v) \neq (0, 0)} \max_{t > 0} E_{\alpha}(tu, tv).$$

We denote

$$Q = \int_C y^{1-2^\mu}(|\nabla u_0|^2 + |\nabla v_0|^2) dx dy + \int_\Omega \left(\lambda_1 u_0(x,0)^2 + \lambda_2 v_0(x,0)^2\right) dx,$$

$$D = 2\beta \int_\Omega \int_\Omega \frac{|u_0(x,0)|^2^\mu |u_0(z,0)|^2^\mu}{|x-z|^\mu} dxdz + \alpha_2 \int_\Omega \int_\Omega \frac{|v_0(x,0)|^2^\mu |v_0(z,0)|^2^\mu}{|x-z|^\mu} dxdz,$$

$$G = \int_\Omega \int_\Omega \frac{|u_0(x,0)|^2^\mu |u_0(z,0)|^2^\mu}{|x-z|^\mu} dxdz$$

and

$$h(\alpha, t) := t^{2^\mu - 2}(\alpha G + D) - Q.$$ 

Observe that there exists $t(\alpha) > 0$ such that $\max_{t > 0} E_{\alpha}(tu_0, tv_0) = E_{\alpha}(t(\alpha)u_0, t(\alpha)v_0)$, we know $t(\alpha) > 0$ satisfies $h(\alpha, t(\alpha)) = 0$. Note that $h(\alpha_1, 1) = 0, \frac{d}{d\alpha} h(\alpha_1) = (2 - 2^\mu)(\alpha_1 G + D) > 0$, and $h(\alpha, t(\alpha)) \equiv 0$. By the implicit function theorem, there exists $0 < \varepsilon_2 \leq \varepsilon_1$, such that $t(\alpha) \in C^\infty((\alpha_1 - \varepsilon_2, \alpha_1 + \varepsilon_2), \mathbb{R})$. By $h(\alpha_1, t(\alpha)) \equiv 0$ we see that $t'(\alpha_1) = -\frac{G}{(2^\mu - 2)(\alpha_1 G + D)}$. By Taylor expansion one has $t(\alpha) = 1 +
\( t'(\alpha_1)(\alpha - \alpha_1) + O((\alpha - \alpha_1)^2) \), and so \( t^2(\alpha) = 1 + 2t'(\alpha_1)(\alpha - \alpha_1) + O((\alpha - \alpha_1)^2) \).

Since \( Q = \alpha_1 G + D = \frac{4N-2\mu}{N-\mu+2s} B(\alpha_1) \), by (68) we know that

\[
B(\alpha) \leq E_\alpha(t_0 u_0, t_0 u_0) = \frac{N-\mu + 2s}{4N-2\mu} t^2(\alpha) Q = t^2(\alpha) B(\alpha_1)
\]

\[
= B(\alpha_1) - \frac{2GB(\alpha_1)}{(2 \cdot 2\mu - 2)(\alpha_1 G + D)} (\alpha - \alpha_1) + O((\alpha - \alpha_1)^2)
\]

\[
= B(\alpha_1) - \frac{G}{2 \cdot 2\mu} (\alpha - \alpha_1) + O((\alpha - \alpha_1)^2).
\]

If \( \alpha \geq \alpha_1 \), we know \( \frac{B(\alpha) - B(\alpha_1)}{\alpha - \alpha_1} \geq -\frac{G}{2 \cdot 2\mu} + O(\alpha - \alpha_1) \), so \( B'(\alpha_1) \geq -\frac{G}{2 \cdot 2\mu} \). Similarly, if \( \alpha \leq \alpha_1 \), we have \( \frac{B(\alpha) - B(\alpha_1)}{\alpha - \alpha_1} \leq -\frac{G}{2 \cdot 2\mu} + O(\alpha - \alpha_1) \), that is, \( B'(\alpha_1) \leq -\frac{G}{2 \cdot 2\mu} \).

Consequently, \( B'(\alpha_1) = -\frac{G}{2 \cdot 2\mu} = -\frac{1}{2 \cdot 2\mu} \int \int \frac{|u_0(x,0)|^{2\mu}_2|u_0(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz \). By Theorem 1.2, \( (\sqrt{\nu_0} u, \sqrt{\nu_0} v) \) is also a least energy solution of (3). Therefore, \( B'(\alpha_1) = -\frac{k_0^{\mu}}{2 \cdot 2\mu} \int \int \frac{|w(x,0)|^{2\mu}_2|w(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz \), that is, (76) holds.

**Proof of Theorem 1.3.** Let \( (u, v) \) be any least energy solution of (25). By Lemma 4.1, we have

\[
\int \int_{\Omega} \frac{|u(x,0)|^{2\mu}_2|u(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz = k_0^{\mu} \int \int_{\Omega} \frac{|w(x,0)|^{2\mu}_2|w(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz,
\]

we also have

\[
\int \int_{\Omega} \frac{|v(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz = \rho^{\mu}_0 \int \int_{\Omega} \frac{|w(x,0)|^{2\mu}_2|w(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz,
\]

\[
\int \int_{\Omega} \frac{|u(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz = \frac{1}{k_0^{\mu}} \int \int_{\Omega} \frac{|v(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz.
\]

Therefore,

\[
\int \int_{\Omega} \frac{|u(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz = \rho^{\mu}_0 \frac{1}{k_0^{\mu}} \int \int_{\Omega} \frac{|u(x,0)|^{2\mu}_2|u(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz,
\]

\[
\int \int_{\Omega} \frac{|u(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz = \rho^{\mu}_0 \frac{1}{k_0^{\mu}} \int \int_{\Omega} \frac{|v(x,0)|^{2\mu}_2|v(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz.
\]

Define \( (\bar{u}, \bar{v}) := (\frac{1}{\sqrt{k_0^{\mu}}} u, \frac{1}{\sqrt{\rho^{\mu}_0}} v) \). By \( \varphi_1(k_0, l_0) = \varphi_2(k_0, l_0) = 0 \) and (77) we get

\[
\int_C y^{-2\mu} |\nabla \bar{u}|^2 dxdy + \lambda_1 \int \bar{u}(x,0)^2 dx = \int \int \frac{|\bar{u}(x,0)|^{2\mu}_2|\bar{u}(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz,
\]

\[
\int_C y^{-2\mu} |\nabla \bar{v}|^2 dxdy + \lambda_2 \int \bar{v}(x,0)^2 dx = \int \int \frac{|\bar{v}(x,0)|^{2\mu}_2|\bar{v}(z,0)|^{2\mu}_2}{|x-z|^{\mu}} dxdz.
\]

Then by (31) we have

\[
\frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y^{-2\mu} |\nabla \bar{u}|^2 dxdy + \lambda_1 \int \bar{u}(x,0)^2 dx \right) \geq B_1,
\]

\[
\frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y^{-2\mu} |\nabla \bar{v}|^2 dxdy + \lambda_2 \int \bar{v}(x,0)^2 dx \right) \geq B_1.
\]
and so
\[
B = (k_0 + l_0)B_1
\]
\[
= \frac{N - \mu + 2s}{4N - 2\mu} \left( \int_C y_1^{1-2s} |\nabla u|^2 dxdy + \lambda_1 \int_{\Omega} u(x,0)^2 dx \right.
\]
\[
+ \int_C y_1^{1-2s} |\nabla \bar{u}|^2 dxdy + \lambda_2 \int_{\Omega} \bar{u}(x,0)^2 dx \Big) \geq (k_0 + l_0)B_1.
\]

This implies that
\[
\frac{N - \mu + 2s}{4N - 2\mu} \int_C y_1^{1-2s} |\nabla \bar{u}|^2 dxdy + \lambda_1 \int_{\Omega} \bar{u}(x,0)^2 dx = B_1,
\]
\[
\frac{N - \mu + 2s}{4N - 2\mu} \int_C y_1^{1-2s} |\nabla \bar{v}|^2 dxdy + \lambda_2 \int_{\Omega} \bar{v}(x,0)^2 dx = B_1.
\]

Combining this with (78), we see \( \bar{u} \) and \( \bar{v} \) are both least energy solutions of (29).

Then we see from \((u,v)\) satisfies (25) that
\[
\begin{cases}
\text{div}(y_1^{1-2s} \nabla \bar{u}) = 0, & \text{in } C, \\
\bar{u} = 0, & \text{on } \partial C, \\
\frac{\partial \bar{u}}{\partial \nu^*} = \left( \int_{\Omega} |\bar{u}(z,0)|^{2^*_\mu} d\mu \right) |\bar{u}(x)|^{2^*_\mu - 2} \bar{u}(x) - \lambda \bar{u}, & \text{in } \Omega,
\end{cases}
\]
we can get \( \bar{u} = \bar{v} \). Denote \( U = \bar{u} \), then \((u,v) = (\sqrt{k_0} U, \sqrt{l_0} U)\), where \( U \) is a least energy solution of (29).

5. Proof of Theorem 1.4. In this section, without loss of generality, we assume that \(-\lambda_1(\Omega) < \lambda_1 \leq \lambda_2 < 0\). Since
\[
\int_C y_1^{1-2s} |\nabla u|^2 dxdy + \lambda_i \int_{\Omega} u(x,0)^2 dx \geq \left(1 + \frac{\lambda_i}{\lambda_1(\Omega)} \right) \int_C y_1^{1-2s} |\nabla u|^2 dxdy, \quad i = 1, 2,
\]
it is standard to see that \( B > 0 \). The Brezis–Nirenberg problem
\[
\begin{cases}
\text{div}(y_1^{1-2s} \nabla u) = 0, & \text{in } C, \\
u = 0, & \text{on } \partial C, \\
\frac{\partial u}{\partial \nu^*} = \alpha_i \left( \int_{\Omega} |u(z,0)|^{2^*_\mu} d\mu \right) |u|^{2^*_\mu - 2} u - \lambda_i u, & \text{in } \Omega,
\end{cases}
\]
Thus define function \( \rho \)

\[
\frac{N - \mu + 2s}{4N - 2\mu} \left( 1 + \frac{\lambda_i}{\lambda_1(\Omega)} \right)^{\frac{2N - \mu}{N + 2s}} \alpha_i^{\frac{2s - N}{2(\mu - 2s)}} C \leq \frac{1}{2} \left( \int_C y^{1-2s} \left| \nabla u_{\alpha_i} \right|^2 \, dx \, dy + \lambda_i \int_{\Omega} u_{\alpha_i}(x,0)^2 \, dx \right) + \frac{N - 2s}{4N - 2\mu} \alpha_i \int_{\Omega} \left| u_{\alpha_i}(x,0) \right|^2 \left| u_{\alpha_i}(z,0) \right|^2 \frac{dx \, dz}{|x - z|^\mu} \tag{80}
\]

\[\leq \frac{N - \mu + 2s}{4N - 2\mu} \alpha_i^{\frac{2s - N}{2(\mu - 2s)}} C \leq \beta \rho, \quad i = 1, 2.
\]

The next lemma is very important, where we need the assumption \( \lambda_1, \lambda_2 < 0 \).

**Lemma 5.1.** Let \( \beta < 0 \), then

\[B < \min \left\{ B_{\alpha_1} + \frac{N - \mu + 2s}{4N - 2\mu} \alpha_2^{\frac{2s - N}{2(\mu - 2s)}} C \leq \beta \rho, \quad B_{\alpha_2} + \frac{N - \mu + 2s}{4N - 2\mu} \alpha_1^{\frac{2s - N}{2(\mu - 2s)}} C \leq \beta \rho, \quad A \right\}. \]

**Proof.** Let \( \beta < 0, t > t_0 > 0 \) such that

\[
\frac{N - \mu}{N - \mu + 2s} B_{\alpha_1} t^2 - \frac{N - 2s}{2(N - \mu + 2s)} B_{\alpha_1} t^{\frac{4N - 2s}{N - 4s}} + \frac{N - \mu + 2s}{4N - 2\mu} \left\{ \alpha_2^{\frac{2s - N}{2(\mu - 2s)}} C \right\} \leq \beta \rho < 0. \tag{81}
\]

Since \( u_{\alpha_1} \in C(\bar{C}) \) and \( u_{\alpha_1} \equiv 0 \) on \( \partial \Omega \), without loss of generality, we can denote that

\[B^+ = \{(x,y) | (x,y) < \rho \text{ and } y > 0 \}
\]

such that

\[
\delta := \max_{B^+} u_{\alpha_1} \leq \min \left\{ \left( \frac{\alpha_2}{2|\beta|} \right)^{\frac{N - 2s}{N + 2s}}, \left( \frac{\lambda_1 + \lambda_1(\Omega)}{2|\beta|} \right)^{\frac{N - 2s}{N + 2s}} \right\}. \tag{82}
\]

Given any \( \rho \), let \( B_{\rho} \) be the ball of radius \( \rho \) centered at the origin in \( \mathbb{R}^N \) and \( B^+_{\rho} \) be the half ball of radius \( \rho \) in \( \mathbb{R}^{N+1}_+ \) satisfying \( B^+_{\rho} \subset C \cup \Omega \). Choose a smooth cutoff function \( \psi \in C^\infty(C) \), and for small fixed \( \rho \),

\[
\begin{cases}
\psi(x, y) = \begin{cases}
1 & (x,y) \in B^+_{\rho/2}, \\
0 & (x,y) \not\in B^+_{\rho},
\end{cases} \\
0 \leq \psi(x, y) \leq 1, & \forall (x,y) \in C, \\
|D\psi(x, y)| \leq C = \text{const}. & \forall (x,y) \in C.
\end{cases}
\]

Thus define \( \nu \epsilon = \psi U_{\epsilon} \in X_0^2(C, \mathbb{C}) \), where \( U_{\epsilon} \) is defined in (13) and (14). Next it is known that \( S_C \) is achieved by the extremal functions \( U_{\epsilon} \). We see that

\[
\int_{\mathbb{R}^N \times \{0\}} \int_{\mathbb{R}^N \times \{0\}} \frac{|U_{\epsilon}(x,0)|^{2s} |U_{\epsilon}(z,0)|^{2s}}{|x - z|^\mu} \, dx \, dz = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \epsilon^{2N - \mu} \frac{|x|^{2 + 1}}{2} \frac{|x|^{2 + 1}}{2} \frac{dx \, dz}{\rho^{2N - \mu}} \tag{83}
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{(|x|^2 + 1)^{2N - \mu} |x - z|^\mu (|z|^2 + 1)^{2N - \mu}} \, dx \, dz =: K_1.
\]
We have
\[
\int_{\Omega \times (0)} \int_{\Omega \times (0)} \frac{|\psi U_z(x,0)|^2}{|x-z|^\mu} |\psi U_z(z,0)|^2 dx dz
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^2 - 1)(|\psi(z,0)|^2 - 1)}{|x-z|^\mu(|z|^2 + \varepsilon^2)^{2N-\mu}} dx dz
+ 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^2 \varepsilon^{2\mu - \mu}}{|x-z|^\mu(|z|^2 + \varepsilon^2)\varepsilon^{2N-\mu}} dx dz - K_1
= A + 2B - K_1,
\]
where
\[
A = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\psi(x,0)|^2 - 1)(|\psi(z,0)|^2 - 1)}{|x-z|^\mu(|z|^2 + \varepsilon^2)^{2N-\mu}} dx dz,
B = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x,0)|^2 \varepsilon^{2\mu - \mu}}{|x-z|^\mu(|z|^2 + \varepsilon^2)\varepsilon^{2N-\mu}} dx dz.
\]
By direct computation, we know
\[
A = O(\varepsilon^{2N-\mu}) \int_{\mathbb{R}^N/B_\varepsilon^+} \int_{\mathbb{R}^N/B_\varepsilon^+} \frac{1}{|x-z|^\mu |z|^2N-\mu} dx dz = O(\varepsilon^{2N-\mu}),
\]
\[
B = -O(\varepsilon^{2N-\mu}) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{(|x|^2 + \varepsilon^2)^{2N-\mu} |x-z|^\mu(|z|^2 + \varepsilon^2)\varepsilon^{2N-\mu}} dx dz + K_1
= -O(\varepsilon^{2N-\mu}) + K_1.
\]
It follows from (83) to (85) that
\[
\int_{\Omega} \int_{\Omega} \frac{|\psi U_z(x,0)|^2 |\psi U_z(z,0)|^2}{|x-z|^\mu} dx dz = K_1 - O(\varepsilon^{2N-\mu}).
\]
Hence we see
\[
\left( \int_{\Omega} \int_{\Omega} \frac{|v_z(x,0)|^2 |v_z(z,0)|^2}{|x-z|^\mu} dx dz \right)^{\frac{N-2s}{N-s}} = K_1^{\frac{N-2s}{N-s}} - O(\varepsilon^{N-2s}).
\]
Let \(K_2 := \int_{\mathbb{R}^N \setminus B_{\varepsilon}^+} y^{1-2s} |\nabla U_z|^2 dx dy\). Since \(U_z\) are minimizers for \(S_C\), we have that \(S_C = \frac{K_3}{K_1^{N-2s}}\). We see \(\int_{\mathbb{R}^N} y^{1-2s} |\nabla U_z|^2 dx dy = \psi^2 \int_{\mathbb{R}^N} y^{1-2s} |\nabla U_z|^2 dx dy + O(\varepsilon^{N-2s})\), since in the second term all integrals are computed in \(B_{\rho}^+ \setminus B_{\varepsilon}^+\). Thus, by uniform integrability of \(\varepsilon^{2s-N} |\nabla U_z|^2\) in \(\mathbb{R}^{N+1}_{+} \setminus B_{\varepsilon}^+\), we deduce
\[
\int_{\mathbb{R}^N} y^{1-2s} |\nabla U_z|^2 dx dy = \int_{\mathbb{R}^N} y^{1-2s} |\nabla (\psi U_z)|^2 dx dy = K_2 + O(\varepsilon^{N-2s}).
\]
On the other hand, we have for all \(\varepsilon << \frac{\rho}{2}\),
\[
\int_{\Omega} v_z(x,0)^2 dx = C\varepsilon^{2s} + O(\varepsilon^{N-2s}).
\]
Since \(\text{supp}(v_\varepsilon) \subset B_\rho^+\), by (82), Hardy-Littlewood-Sobolev inequality and sobolev embedding theorem, for \(t, s > 0\), we have

\[
2|\beta|^2 t^{2^*_\nu} m^{2^*_\nu} \int_\Omega \int_\Omega \frac{|u_{\alpha_1}(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \\
\leq C|\beta|^2 t^{2^*_\nu-1} t^{2^*_\nu} \int_\Omega |u_{\alpha_1}(x,0)|^2 \, dx \\
+ |\beta|^2 t^{2^*_\nu-1} m^{2^*_\nu} \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \\
\leq \frac{1}{2} t^{2^*_\nu} \alpha_1 \int_\Omega \int_\Omega \frac{|u_{\alpha_1}(x,0)|^{2^*_\nu} |u_{\alpha_1}(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \\
+ \frac{1}{2} t^{2^*_\nu} \alpha_2 \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz
\]

and so

\[
E(t u_{\alpha_1}, m v_\varepsilon) \leq \frac{1}{2} t^{2^*_\nu} \left( \int_C y^{1-2s}(|\nabla u_{\alpha_1}|^2 dx + \lambda_1 \int_\Omega u_{\alpha_1}(x,0)^2 dx) \\
- \frac{N - 2s}{4(2N - \mu)} t^{2^*_\nu} \alpha_1 \int_\Omega \int_\Omega \frac{|u_{\alpha_1}(x,0)|^{2^*_\nu} |u_{\alpha_1}(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \\
+ \frac{1}{2} m^{2^*_\nu} \alpha_2 \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \right) \\
= f(t) + g(m).
\]

By (87), (88), (89) and \(\lambda_2 < 0\), it is easy to check that

\[
\left( \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \right)^\frac{N-2s}{N-\mu} \leq \frac{S_C + C \varepsilon^{2s} K_{1}^{\frac{2s}{N-\mu}} + O(\varepsilon^{N-2s})}{1 - O(\varepsilon^{N-2s})} < S_C.
\]

Then we can obtain

\[
\max_{m \geq 0} g(m) < \frac{N - \mu + 2s}{4N - 2\mu} \left( \frac{\alpha_2}{2} \right)^{-\frac{N-2s}{N-\mu+2s}} S_C^{\frac{2N-\mu}{N-\mu+2s}}.
\]

By (80) we see that \(f(t) = \frac{2N-\mu}{N-\mu+2s} B_{\alpha_1} t^2 - \frac{N-2s}{2(2N-\mu+2s)} B_{\alpha_1} t^{2^*_\nu}\). Combining these with (81), we get that \(f(t) + g(m) < 0\), \(\forall t > t_0\), \(m > 0\), and so it follows from (91) that \(E(t u_{\alpha_1}, m v_\varepsilon) = \max_{0 < t \leq t_0, m > 0} E(t u_{\alpha_1}, m v_\varepsilon)\). Define

\[
g_t(m) := \frac{1}{2} m^{2^*_\nu} \left( \int_C y^{1-2s} |\nabla v_\varepsilon|^2 \, dx + \lambda_2 \int_\Omega v_\varepsilon(x,0)^2 \, dx \\
- \frac{N - 2s}{2(2N - \mu)} m^{2^*_\nu} \alpha_2 \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \right),
\]

\[
\lambda_1 \int_\Omega u_{\alpha_1}(x,0)^2 \, dx \\
- \frac{N - 2s}{4(2N - \mu)} m^{2^*_\nu} \alpha_1 \int_\Omega \int_\Omega \frac{|u_{\alpha_1}(x,0)|^{2^*_\nu} |u_{\alpha_1}(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz \\
+ \frac{1}{2} m^{2^*_\nu} \alpha_2 \int_\Omega \int_\Omega \frac{|v_\varepsilon(x,0)|^{2^*_\nu} |v_\varepsilon(z,0)|^{2^*_\nu}}{|x-z|^\mu} \, dx \, dz.
\]
where \( m > 0 \). Then there exists a unique \( m(\varepsilon) > 0 \), for \( \varepsilon \) small enough, such that \( g'_\varepsilon(m(\varepsilon)) = 0 \) with

\[
m(\varepsilon)^{2 - 2\mu, -2} \geq \left(1 + \frac{\lambda_2}{\lambda_1(\Omega)}\right) \frac{\int_{\mathbb{R}^N} y^{1 - 2s} |\nabla U_\varepsilon|^2 dx dy}{\alpha_2 \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x,0)|^{2\mu} |U_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dz dx} \geq \left(1 + \frac{\lambda_2}{\lambda_1(\Omega)}\right) \frac{S_c}{2\alpha_2} := m_0^{2 - 2\mu, -2}.
\]

Since \( g_\varepsilon \) is increasing for \( 0 < m < m(\varepsilon) \), then for any \( 0 < m \leq m_0 \), we have \( g_\varepsilon(m) < g_\varepsilon(m_0) \) and so \( E(tu_{\alpha_1, m\varepsilon}) < E(tu_{\alpha_1, m_0\varepsilon}) \). That is

\[
\max_{0 \leq t \leq m_0, 0 \leq \varepsilon \leq \varepsilon_0} E(tu_{\alpha_1, m\varepsilon}) = \max_{0 < t \leq m_0, \varepsilon \leq \varepsilon_0} E(tu_{\alpha_1, m\varepsilon})). \tag{95}
\]

For \( 0 < t \leq t_0 \) and \( m \geq m_0 \), we see from (82),

\[
|\beta|^{2\mu} m_0^{2\mu} \int_{\Omega} \int_{\Omega} \frac{|u_{\alpha_1}(x,0)|^{2\mu}|v_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \\
\leq |\beta|^{2\mu} m_0^{2\mu} \left( \int_{\Omega} \int_{\Omega} \frac{|u_{\alpha_1}(x,0)|^{2\mu}|u_{\alpha_1}(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \right)^{\frac{1}{2}} \\
\left( \int_{\Omega} \int_{\Omega} \frac{|v_\varepsilon(x,0)|^{2\mu}|v_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \right)^{\frac{1}{2}} \\
\leq C |\beta|^{2\mu} m_0^{2\mu} \delta^{2\mu} \left( \int_{B_\mu} \int_{B_\mu} \frac{|U_\varepsilon(x,0)|^{2\mu}|U_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \right)^{\frac{1}{2}} \\
\leq C
\]

and so

\[
E(tu_{\alpha_1, m\varepsilon}) \leq \frac{1}{2} t^2 \left( \int_{\mathbb{R}^N} y^{1 - 2s}(|\nabla u_{\alpha_1}|^2 dx dy + \lambda_1 \int_{\Omega} u_{\alpha_1}(x,0)^2 dx) \\
- \frac{N - 2s}{4N - 2\mu} m_0^{2\mu, \alpha_1} \int_{\Omega} \int_{\Omega} \frac{|u_{\alpha_1}(x,0)|^{2\mu}|u_{\alpha_1}(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \\
+ \frac{1}{2} m^2 \left( \int_{\mathbb{R}^N} y^{1 - 2s}|\nabla v_\varepsilon|^2 dx dy + \lambda_2 \int_{\Omega} v_\varepsilon(x,0)^2 dx \right) \\
- \frac{N - 2s}{4N - 2\mu} m_0^{2\mu, \alpha_2} \int_{\Omega} \int_{\Omega} \frac{|v_\varepsilon(x,0)|^{2\mu}|v_\varepsilon(z,0)|^{2\mu}}{|x-z|^\mu} dx dz \right) \\
=: f_1(t) + g_1(m).
\]

Note that \( \max_{t > 0} f_1(t) = f_1(1) = B_{\alpha_1} \). Due to \( \varepsilon \) small enough, we can get

\[
\max_{m > 0} g_1(m) < \frac{N - \mu + 2s}{4N - 2\mu} \theta^{\frac{N - 2s}{N - \mu + 2s}} S_{\varepsilon}^{\frac{2N - \mu}{N - \mu + 2s}}.
\]

Combining these with (95) and (96), it is easy to show that

\[
\max_{0 < t \leq t_0, m \geq m_0} E(tu_{\alpha_1, m\varepsilon}) = \max_{0 < t \leq t_0, m \geq m_0} E(tu_{\alpha_1, m\varepsilon}) \\
\leq \max_{t > 0} f_1(t) + \max_{m > 0} g_1(m) \\
< B_{\alpha_1} + \frac{N - \mu + 2s}{4N - 2\mu} \theta^{\frac{N - 2s}{N - \mu + 2s}} S_{\varepsilon}^{\frac{2N - \mu}{N - \mu + 2s}}.
\]
Now, we claim that there exists $t_\varepsilon, s_\varepsilon > 0$ such that $(t_\varepsilon u_{\alpha_1}, m_\varepsilon v_\varepsilon) \in \mathcal{M}$. Similarly as (90) we have
\[
\left( \int \int_{\Omega} \frac{\beta |u_{\alpha_1}(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right)^2 \\
\leq |\beta|^2 \delta^2 2^{2\nu-2} \left( \int \int \frac{|u_{\alpha_1}(x,0)||u_{\alpha_1}(z,0)|}{|x-z|^{\mu}} dx \, dz \right) \\
\left( \int \int \frac{|v_\varepsilon(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right) \\
\leq \frac{C |\beta|^2 \delta^2 2^{2\nu-2}}{(\lambda_1 + \lambda_1(\Omega))\alpha_2} \left( \alpha_1 \int \int \frac{|u_{\alpha_1}(x,0)|^{2\nu} |u_{\alpha_1}(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right) \\
\left( \alpha_2 \int \int \frac{|v_\varepsilon(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right) \\
< \left( \alpha_1 \int \int \frac{|u_{\alpha_1}(x,0)|^{2\nu} |u_{\alpha_1}(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right) \\
\left( \alpha_2 \int \int \frac{|v_\varepsilon(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz \right).
\]

For convenience we denote
\[
D_1 = \alpha_1 \int \int_{\Omega} \frac{|u_{\alpha_1}(x,0)|^{2\nu} |u_{\alpha_1}(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz, \\
D_2 = \beta \int \int_{\Omega} \frac{|u_{\alpha_1}(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz, \\
D_3 = \alpha_2 \int \int_{\Omega} \frac{|v_\varepsilon(x,0)|^{2\nu} |v_\varepsilon(z,0)|^{2\nu}}{|x-z|^{\mu}} dx \, dz, \\
D_4 = \int \int_{\Omega} y^{1-2s} |\nabla v_\varepsilon|^2 dx \, dy + \lambda_2 \int \int_{\Omega} v_\varepsilon(x,0)^2 dx.
\]

Then $D_2 < 0$ and $D_1 D_3 - D_2^2 > 0$. Furthermore, $(t u_{\alpha_1}, m v_\varepsilon) \in \mathcal{M}$ for some $t, m > 0$ is equivalent to
\[
t^{2-2s} D_1 = t^{2s} D_1 + m^{2s} D_2, \quad m^{2-2s} D_4 = m^{2s} D_3 + t^{2s} D_2. \tag{98}
\]

Note that $1 < 2^{2s} = \frac{2N-\mu}{N-2s} < 2$, by $m^{2s} = (t^{2-2s} - t^{2s}) \frac{D_1}{D_2} > 0$ we have $t > 1$. Therefore, (98) is equivalent to
\[
f_3(t) := D_4 \left( \frac{D_1}{|D_2|} \right)^{\frac{2s}{2s-2}} - \frac{D_1 D_3 - D_2^2}{|D_2|} t^{2s-2} + \frac{D_1 D_3}{|D_2|}, \quad t > 1. \tag{99}
\]

Since $f_3(1) > 0$ and $\lim_{t \to +\infty} f_3(t) < 0$, (99) has a solution $t > 1$. Hence (98) has a solution $t_\varepsilon > 0, m_\varepsilon > 0$. That is, $(t_\varepsilon u_{\alpha_1}, m_\varepsilon v_\varepsilon) \in \mathcal{M}$ and from (97) we get
\[
B \leq E(t_\varepsilon u_{\alpha_1}, m_\varepsilon v_\varepsilon) \leq \max_{t, m > 0} E(t u_{\alpha_1}, m v_\varepsilon) < B_{\alpha_1} + \frac{N - \mu + 2s}{4N - 2\mu} \alpha_2 \frac{N - 2s}{N - 2\mu} N_{\varepsilon, m}. \]
By a similar argument, we can also prove that $B < B_{\alpha_2} + \frac{N-\mu+2s}{4N-2\mu} \alpha_1 \frac{\alpha_2^2}{4N-2\mu} S_C^{2N-\mu}$.

By (41) and (80), we have

$$A > \max \left\{ B_{\alpha_1} + \frac{N-\mu+2s}{4N-2\mu} \alpha_2 \frac{3s-N}{4N-2\mu} S_C^\frac{2N-\mu}{N-\mu}, B_{\alpha_1} + \frac{N-\mu+2s}{4N-2\mu} \alpha_1 \frac{\alpha_2^2}{4N-2\mu} S_C^{2N-\mu} \right\},$$

This completes the proof. \( \square \)

**Lemma 5.2.** Assume that $\beta < 0$, then there exists $C_2 > C_1 > 0$, such that for any $(u, v) \in \mathcal{M}$ with $E(u, v) \leq A$, there holds

$$C_1 \leq \int_{\Omega} \int_{\Omega} \frac{|u(x, 0)|^2}{|x - z|^\mu} |u(z, 0)|^2 dxdz, \quad \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^2}{|x - z|^\mu} |v(z, 0)|^2 dxdz \leq C_2,$$

where $C_1, C_2$ are constant.

**Proof.** Due to (11), (28), $E(u, v) \leq A$ and $\beta < 0$, we have

$$\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)} \frac{1}{\lambda_1(\Omega)} S_C \left( \int_{\Omega} \int_{\Omega} \frac{|u(x, 0)|^2}{|x - z|^\mu} |u(z, 0)|^2 dxdz \right)^{\frac{1}{p}} \leq \alpha_1 \int_{\Omega} \int_{\Omega} \frac{|u(x, 0)|^2}{|x - z|^\mu} |u(z, 0)|^2 dxdz,$$

$$\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)} \frac{1}{\lambda_1(\Omega)} S_C \left( \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^2}{|x - z|^\mu} |v(z, 0)|^2 dxdz \right)^{\frac{1}{p}} \leq \alpha_2 \int_{\Omega} \int_{\Omega} \frac{|v(x, 0)|^2}{|x - z|^\mu} |v(z, 0)|^2 dxdz,$$

this completes the proof. \( \square \)

**Proof of Theorems 1.4 for the case $\beta < 0$.** Assume that $\beta < 0$. Note that $E$ is coercive and bounded from below on $\mathcal{M}$. Then by the Ekeland variational principle, there exists a minimizing sequence $\{ (u_n, v_n) \} \subset \mathcal{M}$ satisfying

$$E(u_n, v_n) \leq \min \left\{ B + \frac{1}{n}, A \right\}, \quad (100)$$

and

$$E(u, v) \geq E(u_n, v_n) - \frac{1}{n} \| (u_n, v_n) - (u, v) \|, \quad \forall (u, v) \subset \mathcal{M}, \quad (101)$$

here $\| (u, v) \| := \left( \int_{\mathcal{C}} y^{1-2s} |(\nabla u|^2 + |\nabla v|^2) dxdy \right)^{\frac{1}{2}}$ is the norm of $H$. Then $\{ (u_n, v_n) \}$ is bounded in $H$. For any $(\eta, \phi) \in H$ with $\| \eta \| \leq 1$, $\| \phi \| \leq 1$ and each $n \in \mathbb{N}$, we define the functions $h_n$ and $g_n$ by

$$h_n(t, m, l) = \int_{\mathcal{C}} y^{1-2s} |(\nabla (u_n + t\eta + mu_n))^2 dxdy + \lambda_1 \int_{\Omega} |(u_n + t\eta + mu_n)(x, 0)|^2 dx$$

$$- \alpha_1 \int_{\Omega} \int_{\Omega} \frac{|(u_n + t\eta + mu_n)(x, 0)|^2}{|x - z|^\mu} |(u_n + t\eta + mu_n)(z, 0)|^2 dxdz$$

$$- \beta \int_{\Omega} \int_{\Omega} \frac{|(u_n + t\eta + mu_n)(x, 0)|^2}{|x - z|^\mu} |(v_n + t\phi + lv_n)(z, 0)|^2 dxdz, \quad (102)$$
Define the matrix $\beta < l - C$ where $m = (0, t, n)$ is independent of $t$. Let $0 = (0, 0, 0)$, then $h_n, g_n \in C^1(\mathbb{R}^3, \mathbb{R})$, $h_n(0) = g_n(0) = 0$ and

$$
\frac{\partial h_n}{\partial m}(0) = -(2 \cdot 2^* - 2) \left( \int_{\mathcal{C}} y^{1-2s} |\nabla u_n|^2 dx dy + \lambda_1 \int_{\Omega} u_n(x, 0)^2 dx \right) + 2^* \beta \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^{2^*_p} |v_n(z, 0)|^{2^*_p}}{|x - z|^\mu} dx dz,
$$

$$
\frac{\partial h_n}{\partial l}(0) = \frac{\partial g_n}{\partial m}(0) = -2^* \beta \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^{2^*_p} |v_n(z, 0)|^{2^*_p}}{|x - z|^\mu} dx dz,
$$

$$
\frac{\partial g_n}{\partial l}(0) = -(2 \cdot 2^* - 2) \left( \int_{\mathcal{C}} y^{1-2s} |\nabla v_n|^2 dx dy + \lambda_2 \int_{\Omega} v_n(x, 0)^2 dx \right) + 2^* \beta \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^{2^*_p} |v_n(z, 0)|^{2^*_p}}{|x - z|^\mu} dx dz.
$$

Define the matrix

$$
F_n := \left( \begin{array}{cc} \frac{\partial h_n}{\partial m}(0) & \frac{\partial h_n}{\partial l}(0) \\ \frac{\partial g_n}{\partial m}(0) & \frac{\partial g_n}{\partial l}(0) \end{array} \right),
$$

since $\beta < 0$, it follows from Lemma 5.2 that

$$
\text{det}(F_n) \geq (2 \cdot 2^* - 2)^2 \left( \int_{\mathcal{C}} y^{1-2s} |\nabla u_n|^2 + |\nabla v_n|^2 \right) dx dy
$$

$$
+ \int_{\Omega} \left( \lambda_1 u_n(x, 0)^2 + \lambda_2 v_n(x, 0)^2 \right) dx
$$

$$
\geq CS_2^3 \left( \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^{2^*_p} |u_n(z, 0)|^{2^*_p}}{|x - z|^\mu} dx dz \right)^{\frac{1}{2^*_p}}
$$

$$
\left( \int_{\Omega} \int_{\Omega} \frac{|v_n(x, 0)|^{2^*_p} |v_n(z, 0)|^{2^*_p}}{|x - z|^\mu} dx dz \right)^{\frac{1}{2^*_p}} \geq C > 0,
$$

where $C$ is independent of $n$. By the implicit function theorem, functions $m_n(t)$ and $l_n(t)$ are well defined and of class $C^1$ on some interval $(-\delta_n, +\delta_n)$ for $\delta_n > 0$. Moreover, $m_n(0) = l_n(0) = 0$ and

$$
h_n(t, m_n(t), l_n(t)) = 0, \quad g_n(t, m_n(t), l_n(t)) = 0, \quad t \in (-\delta, +\delta).
$$

This imply

$$
\begin{align*}
\frac{\partial h_n}{\partial l}(0) + \frac{\partial h_n}{\partial m}(0) m_n'(0) + \frac{\partial h_n}{\partial l}(0) l_n'(0) &= 0, \\
\frac{\partial g_n}{\partial l}(0) + \frac{\partial g_n}{\partial m}(0) m_n'(0) + \frac{\partial g_n}{\partial l}(0) l_n'(0) &= 0
\end{align*}
$$
and so
\[
\begin{aligned}
&\left\{\begin{array}{l}
m'_n(0) = \frac{1}{\det(F_n)} \left( \frac{\partial g_n}{\partial t}(0) \frac{\partial h_n}{\partial t}(0) - \frac{\partial g_n}{\partial t}(0) \frac{\partial h_n}{\partial m}(0) \right), \\
v'_n(0) = \frac{1}{\det(F_n)} \left( \frac{\partial g_n}{\partial m}(0) \frac{\partial h_n}{\partial t}(0) - \frac{\partial g_n}{\partial t}(0) \frac{\partial h_n}{\partial m}(0) \right).
\end{array}\right.
\end{aligned}
\]

Since \( \{u_n, v_n\} \) is bounded in \( H \), we have
\[
\left| \frac{\partial h_n}{\partial t}(0) \right| = \left| 2 \int y^{1-2k} \sum_{\Omega} u_n \nabla x \sum_{\Omega} \eta dxdy + 2 \lambda_1 \int_{\Omega} u_n(x, 0) \eta(x, 0) dx \right|
\]
\[
= 2 \cdot 2^* \alpha_1 \left( \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^2 |u_n(z, 0)|^{2-1} \eta(z, 0)}{|x-z|^\mu} dxdz \right)
\]
\[
- 2^* \beta \left( \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^2 |v_n(z, 0)|^{2-1} \eta(z, 0)}{|x-z|^\mu} dxdz \right)
\]
\[
- 2^* \beta \left( \int_{\Omega} \int_{\Omega} \frac{|u_n(x, 0)|^{2-1} \eta(x, 0)|v_n(z, 0)|^{2-1}}{|x-z|^\mu} dxdz \right) \leq C,
\]
where \( C \) is independent of \( n \), similarly, \( \left| \frac{\partial g_n}{\partial t}(0) \right| \leq C \). From Lemma 5.2, we also have
\[
\left| \frac{\partial h_n}{\partial m}(0) \right|, \left| \frac{\partial h_n}{\partial l}(0) \right|, \left| \frac{\partial g_n}{\partial m}(0) \right|, \left| \frac{\partial g_n}{\partial l}(0) \right| \leq C.
\]
Hence, combining these with (104), we have
\[
\left| m'_n(0) \right|, \left| v'_n(0) \right| \leq C,
\]
where \( C \) is independent of \( n \).

Define \( \eta_{n,t} := u_n + t \eta + \frac{m_n(t)}{2} u_n \), \( \phi_{n,t} := v_n + t \phi + \frac{l_n(t)}{2} v_n \), then \( (\eta_{n,t}, \phi_{n,t}) \in M \) for \( t \in (-\delta_n, \delta_n) \). It follows that (106) we have
\[
E(\eta_{n,t}, \phi_{n,t}) - E(u_n, v_n) \geq - \frac{1}{n} \| (t \eta + \frac{m_n(t)}{2} u_n, t \phi + \frac{l_n(t)}{2} v_n) \|.
\]
Note that
\[
E'(u_n, v_n)(u_n, 0) = E'(u_n, v_n)(0, v_n) = 0.
\]
By Taylor Expansion, we have
\[
E(\eta_{n,t}, \phi_{n,t}) - E(u_n, v_n) = E'(u_n, v_n)(t \eta + \frac{m_n(t)}{2} u_n, t \phi + \frac{l_n(t)}{2} v_n) + r(n, t)
\]
\[
= t E'(u_n, v_n)(\eta, \phi) + r(n, t),
\]
where \( r(n, t) = o(||t \eta + \frac{m_n(t)}{2} u_n, t \phi + \frac{l_n(t)}{2} v_n||) \) as \( t \to 0 \). By (105) we see that
\[
\lim_{t \to 0} \sup \| (t \eta + \frac{m_n(t)}{2} u_n, \phi + \frac{l_n(t)}{2} v_n) \| \leq C,
\]
where \( C \) is independent of \( n \). Hence, \( r(n, t) = o(t) \). By (106), (107), (108) and letting \( t \to 0 \), we get that \( |E'(u_n, v_n)(\eta, \phi)| \leq \frac{C}{n} \), where \( C \) is independent of \( n \). Hence \( \lim_{n \to +\infty} E'(u_n, v_n) = 0. \)
Thus, we deduce

\[ u_n \to u, \quad v_n \to v, \quad \text{weakly in } X_0^2(C), \]
\[ u_n(\cdot,0) \to u(\cdot,0), \quad v_n(\cdot,0) \to v(\cdot,0), \quad \text{strongly in } L^q(\Omega), \]
\[ 2 \leq q < 2^*_s, \quad u_n(x,0) \to u(x,0), \quad v_n(x,0) \to v(x,0), \quad \text{a.e. in } \Omega, \]

on the other hand,

\[ (|x|^{-\mu} * |u_n(z,0)|^{2^*_s})|u_n|^{2^*_s-2}u_n \to (|x|^{-\mu} * |u(z,0)|^{2^*_s})|u|^{2^*_s-2}u, \quad \text{in } L^{\frac{2N}{2N+2\mu}}(\Omega), \]
\[ (|x|^{-\mu} * |v_n(z,0)|^{2^*_s})|v_n|^{2^*_s-2}v_n \to (|x|^{-\mu} * |v(z,0)|^{2^*_s})|v|^{2^*_s-2}v, \quad \text{in } L^{\frac{2N}{2N+2\mu}}(\Omega), \]
\[ (|x|^{-\mu} * |u_n(z,0)|^{2^*_s})|v_n|^{2^*_s-2}v_n \to (|x|^{-\mu} * |u(z,0)|^{2^*_s})|v|^{2^*_s-2}v, \quad \text{in } L^{\frac{2N}{2N+2\mu}}(\Omega). \]

Thus, we deduce \( E'(u,v) = 0. \)

Set \( w_n = u_n - u \) and \( \sigma_n = v_n - v. \) Using the Brezis-Lieb Lemma and weak convergence, we have

\[ \int_{\Omega} y^{1-2s}|\nabla w_n|^2 \, dx \, dy = \int_{\Omega} y^{1-2s}|\nabla u_n|^2 \, dx \, dy - \int_{\Omega} y^{1-2s}|\nabla u|^2 \, dx \, dy + o(1), \quad (109) \]
\[ \int_{\Omega} y^{1-2s}|\nabla \sigma_n|^2 \, dx \, dy = \int_{\Omega} y^{1-2s}|\nabla v_n|^2 \, dx \, dy - \int_{\Omega} y^{1-2s}|\nabla v|^2 \, dx \, dy + o(1). \]

On the other hand, from Lemma 2.2 in [21], it follows that

\[ \int_{\Omega} y^{1-2s}|\nabla w_n|^2 \, dx \, dy - \alpha_1 \int_{\Omega} \int_{\Omega} \frac{|w_n(x,0)|^{2^*_s}|w_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz \]
\[ \quad - \beta \int_{\Omega} \int_{\Omega} \frac{|w_n(x,0)|^{2^*_s}|\sigma_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz = o(1), \quad (110) \]
\[ \int_{\Omega} y^{1-2s}|\nabla \sigma_n|^2 \, dx \, dy - \alpha_2 \int_{\Omega} \int_{\Omega} \frac{|\sigma_n(x,0)|^{2^*_s}|\sigma_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz \]
\[ \quad - \beta \int_{\Omega} \int_{\Omega} \frac{|w_n(x,0)|^{2^*_s}|\sigma_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz = o(1), \quad (111) \]
\[ E(u_n,v_n) = E(u,v) + I(w_n,\sigma_n) = o(1). \]

Passing to a subsequence, we may assume that

\[ \lim_{n \to +\infty} \int_{\Omega} y^{1-2s}|\nabla w_n|^2 \, dx \, dy = b_1, \quad \lim_{n \to +\infty} \int_{\Omega} y^{1-2s}|\nabla \sigma_n|^2 \, dx \, dy = b_2. \]

Then by (110) and (111) we have \( I(w_n,\sigma_n) = \frac{N-\mu+2s}{4N-2\mu}(b_1 + b_2) + o(1). \) As \( n \to \infty, \) thus we get

\[ 0 \leq E(u,v) \leq E(u,v) + \frac{N-\mu+2s}{4N-2\mu}(b_1 + b_2) = \lim_{n \to +\infty} E(u_n,v_n) = B. \quad (113) \]

**Case 1.** \( u = 0, \quad v = 0. \)

By Lemma 5.2 and (113), we have \( b_1 \neq 0 \) and \( b_2 \neq 0, \) then we can assume that \( w_n \neq 0 \) and \( \sigma_n \neq 0 \) for \( n \) large. Then by (110) and (111), as \( n \) large, we get

\[ \left( \alpha_1 \int_{\Omega} \int_{\Omega} \frac{|w_n(x,0)|^{2^*_s}|w_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz \right) \left( \alpha_2 \int_{\Omega} \int_{\Omega} \frac{|\sigma_n(x,0)|^{2^*_s}|\sigma_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz \right) \]
\[ - \left( \beta \int_{\Omega} \int_{\Omega} \frac{|w_n(x,0)|^{2^*_s}|\sigma_n(z,0)|^{2^*_s}}{|x-z|\mu} \, dx \, dz \right)^2 > 0. \]
Then by a similar argument as Lemma 5.1, as \( n \) large, there exists \( (t_n w_n, m_n \sigma_n) \in \mathcal{N} \). Up to a subsequence, we claim that
\[
\lim_{n \to +\infty} (|t_n - 1| + |m_n - 1|) = 0. \tag{114}
\]
Denote
\[
B_{n,1} = \int_C y^1 - 2^s |\nabla w_n|^2 dx dy \to b_1, \quad B_{n,2} = \int_C y^1 - 2^s |\nabla \sigma_n|^2 dx dy \to b_2;
\]
\[
C_{n,1} = \alpha_1 \int_\Omega \int_\Omega \left| w_n(x,0) \right|^{2^*} \left| w_n(z,0) \right|^{2^*} \frac{1}{|x-z|^\mu} dz dx,
\]
\[
C_{n,2} = \alpha_2 \int_\Omega \int_\Omega \left| \sigma_n(x,0) \right|^{2^*} \left| \sigma_n(z,0) \right|^{2^*} \frac{1}{|x-z|^\mu} dz dx,
\]
\[
D_n = |\beta| \int_\Omega \int_\Omega \left| w_n(x,0) \right|^{2^*} \left| \sigma_n(z,0) \right|^{2^*} \frac{1}{|x-z|^\mu} dz dx.
\]
Passing to a subsequence, we may assume that \( C_{n,1} \to c_1 < +\infty \), \( C_{n,2} \to c_2 < +\infty \) and \( D_n \to d < +\infty \). By (110) and (111) we have
\[
c_1 = b_1 + d > b_1 > 0, \quad c_2 = b_2 + d > b_2 > 0,
\]
\[
t^2_n B_{n,1} = t_n^{2 - 2^*_\mu} C_{n,1} - t_n \sigma_n^{2^*} D_n, \quad m_n^2 B_{n,2} = m_n^{2 - 2^*_\mu} C_{n,2} - t_n \sigma_n^{2^*} D_n. \tag{115}
\]
This implies that
\[
t_n^{2 - 2^*_\mu} \geq \frac{B_{n,1}}{C_{n,1}} \to \frac{b_1}{c_1} > 0, \quad m_n^{2 - 2^*_\mu} \geq \frac{B_{n,2}}{C_{n,2}} \to \frac{b_2}{c_2} > 0. \tag{117}
\]
Assume that, up to a subsequence, \( t_n \to +\infty \) as \( n \to \infty \), then by (116)
\[
t_n^{2 - 2^*_\mu} C_{n,1} - t_n^{2 - 2^*_\mu} B_{n,1} = m_n^{2 - 2^*_\mu} C_{n,2} - m_n^{2 - 2^*_\mu} B_{n,2},
\]
we also have \( m_n \to +\infty \). Then
\[
d^2 = \lim_{n \to \infty} D_n^2 = \lim_{n \to \infty} (C_{n,1} - t_n^{2 - 2^*_\mu} B_{n,1})(C_{n,2} - m_n^{2 - 2^*_\mu} B_{n,2})
\]
\[
= \frac{b_1}{c_1} \cdot \frac{b_2}{c_2} (b_1 + d)(b_2 + d) > d^2,
\]
a contradiction. Therefore, \( t_n, m_n \) are uniformly bounded. Passing to a subsequence, by (117) we may assume that \( t_n \to t_0 \geq \left( \frac{b_1}{c_1} \right)^\frac{1}{2 - 2^*_\mu} \) and \( m_n \to m_0 \geq \left( \frac{b_2}{c_2} \right)^\frac{1}{2 - 2^*_\mu} \). Then we can get \( m_0^{2^*} d = t_0^{2^*} c_1 - t_0^{2 - 2^*_\mu} b_1, \ t_0^{2^*} d = m_0^{2^*} c_1 - m_0^{2 - 2^*_\mu} b_2. \)
If \( d = 0 \), then \( c_1 = b_1 \) and \( t_0 = m_0 = 1 \). That is (114) hold. Now consider \( d > 0 \).
Define \( h(t) = t^{2^*} c_1 - t^{2 - 2^*_\mu} b_1 \), then for \( t \geq \left( \frac{b_1}{c_1} \right)^\frac{1}{2 - 2^*_\mu} \), we get
\[
h'(t) = 2^*_\mu c_1 t^{2^*_\mu - 1} - (2 - 2^*_\mu) b_1 t^{1 - 2^*_\mu} > (2 - 2^*_\mu) t^{1 - 2^*_\mu} (c_1 t^{2 - 2^*_\mu} - b_1) \geq 0,
\]
that is, \( h \) is increasing. If \( t_0 < 1 \), then \( m_0^{2^*} d = h(t_0) < h(1) = c_1 - b_1 = d \), that is, \( m_0 < 1 \) and we see from (115) that
\[
d^2 = \frac{2^*_\mu c_1 - t_0^{2 - 2^*_\mu} b_1}{m_0^{2^*}} \cdot \frac{2^*_\mu c_2 - m_0^{2 - 2^*_\mu} b_2}{t_0^{2^*}} = (c_1 - t_0^{2 - 2^*_\mu} b_1)(c_2 - m_0^{2 - 2^*_\mu} b_2)
\]
\[
= (d + b_1 - t_0^{2 - 2^*_\mu} b_1)(d + b_2 - m_0^{2 - 2^*_\mu} b_2) < d^2,
\]
a contradiction. If $t_0 > 1$, since $1 \geq \left( \frac{9}{c_9} \right)^{\frac{m}{2}}$, we have $m_0^2 d = h(t) > h(1) = c_1 - b_1 = d$, that is, $m_0 > 1$, and so $d^2 = (d + b_1 - t_0^{2-2s} b_1)(d + b_2 - m_0^{2-2s} b_2) > d^2$, a contradiction. Therefore, $t_0 = m_0 = 1$. That is (114) hold. This implies that

$$\frac{N - \mu + 2s}{4N - 2\mu} (b_1 + b_2) = \lim_{n \to +\infty} I(\sigma_n) = I(t_n \sigma_n, m_n \sigma_n) \geq A.$$  

Combining this with (113) we get that $B \geq A$, a contradiction with Lemma 5.1. Therefore, Case 1 is impossible.

**Case 2.** $u \neq 0, v = 0$ or $u = 0, v \neq 0$.

Without loss of generality, we assume that $u \neq 0, v = 0$. Then $b_2 > 0$. By Case 1 we may assume that $b_1 = 0$. Then $\lim_{n \to +\infty} \int \int \frac{|u_n(x,0)|^2}{|x|} dxdy = 0$ and so

$$\int \int y^{1-2s} |\nabla \sigma_n|^2 dxdy \leq \alpha_2 (S_C)^{-2s} \left( \int \int y^{1-2s} |\nabla \sigma_n|^2 dxdy \right)^{2s} + o(1).$$

This implies that $b_2 \geq \frac{N - \mu + 2s}{4N - 2\mu} b_2 \geq B_{\alpha_1} + \frac{N - \mu + 2s}{4N - 2\mu} \frac{N - \mu + 2s}{S_C^{2N - 2s}}$, a contradiction with Lemma 5.1. Therefore, Case 2 is impossible.

**Case 3.** $u \neq 0, v \neq 0$.

Since $(u, v) \in \mathcal{M}$, by (113) we have $E(u, 0) \geq B_{\alpha_1}$, by Lemma 2.5, $(u, v)$ is a solution of (25). Therefore, $(u, v)$ is a least energy solution of (25). This completes the proof.

It remains to prove Theorem 1.4 for the case $\beta > 0$. We should introduce the following Lemma.

**Lemma 5.3.** Let $\beta > 0$, then $B < \min\{B_{\alpha_1}, B_{\alpha_1}, A\}$.

**Proof.** Without loss of generality, we can denote that $B^+_{\rho} = \{(x, y) | (x, y) < \rho \text{ and } y > 0 \} \subset \Omega$. Let $\psi \in C^0_0(B(0, \rho))$ be a nonnegative function with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ for $(x, y) \in B^+_{\rho/2}$. We recall that $(U, V)$ in Theorem 1.1, for $\varepsilon > 0$, we define

$$U_{\varepsilon}(x, y) := \varepsilon^{2-N} U \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right), \quad u_{\varepsilon}(x, y) := \psi(x, y) U_{\varepsilon}(x, y),$$

$$V_{\varepsilon}(x, y) := \varepsilon^{2-N} V \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right), \quad v_{\varepsilon}(x, y) := \psi(x, y) V_{\varepsilon}(x, y).$$

Let $0 < \varepsilon \ll \rho$, similarly as Lemma 5.1, we can obtain

$$\int \int y^{1-2s} |\nabla u_{\varepsilon}|^2 dxdy = \int \int y^{1-2s} |\nabla U|^2 dxdy + O(\varepsilon^{N-2s}), \quad (118)$$

$$\int \int \frac{|u_{\varepsilon}(x, 0)|^{2s}}{|x - z|^{\mu}} dxdz = \int \int \frac{|U(x, 0)|^{2s}}{|x - z|^{\mu}} dxdz - O(\varepsilon^{N-2s}), \quad (119)$$

$$\int \int \frac{|u_{\varepsilon}(x, 0)|^{2s}|v_{\varepsilon}(z, 0)|^{2s}}{|x - z|^{\mu}} dxdz = \int \int \frac{|U(x, 0)|^{2s}|V(z, 0)|^{2s}}{|x - z|^{\mu}} dxdz - O(\varepsilon^{N-2s}), \quad (120)$$

$$\int \int u_{\varepsilon}(x, 0)^2 dxdy \geq C \varepsilon^{2s} + O(\varepsilon^{N-2s}), \quad (121)$$
where $C$ is a positive constant. Similarly, we also can get
\[
\int_C y^{1-2s} |\nabla v_\varepsilon|^2 \, dxdy = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V|^2 \, dxdy + O(\varepsilon^{N-2s}), \tag{122}
\]
\[
\int_{\Omega} \int_{\Omega} \frac{|v_\varepsilon(x,0)|^{2^*_\varepsilon} |v_\varepsilon(z,0)|^{2^*_\varepsilon}}{|x-z|^{\mu}} \, dx \, dz = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|V(x,0)|^{2^*_\varepsilon} |V(z,0)|^{2^*_\varepsilon}}{|x-z|^{\mu}} \, dx \, dz - O(\varepsilon^{\frac{2N-\mu}{2}}), \tag{123}
\]
\[
\int_{\Omega} v_\varepsilon(x,0)^2 \, dx \geq C\varepsilon^{2s} + O(\varepsilon^{N-2s}). \tag{124}
\]
Recall that $I(U, V) = A$, we have
\[
\frac{4N - 2\mu}{N - \mu + 2s} A = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla U|^2 + \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla V|^2 \, dxdy
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \alpha_1 |U(x,0)|^{2^*_\mu} |U(z,0)|^{2^*_\mu} + \alpha_2 |U(x,0)|^{2^*_\mu} |V(z,0)|^{2^*_\mu} \, dx \, dz
\]
\[
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2\beta |U(x,0)|^{2^*_\mu} |V(z,0)|^{2^*_\mu}}{|x-z|^{\mu}} \, dx \, dz.
\]
Combining this with (118)–(124), $\lambda_1, \lambda_2 < 0$, $t > 0$ and $\varepsilon > 0$ small enough, we have
\[
E(tu_\varepsilon, tv_\varepsilon) \leq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \left( \frac{4N - 2\mu}{N - \mu + 2s} A - C\varepsilon^{2s} + O(\varepsilon^{N-2s}) \right) t^2
\]
\[
\leq \frac{N - 2s}{4N - 2\mu} \left( \frac{4N - 2\mu}{N - \mu + 2s} A - C\varepsilon^{2s} + O(\varepsilon^{N-2s}) \right)
\]
\[
\leq \frac{4N - 2\mu}{N - \mu + 2s} \left( \frac{4N - 2\mu}{N - \mu + 2s} A - C\varepsilon^{2s} + O(\varepsilon^{N-2s}) \right)^{\frac{N-2s}{N-\mu+2s}} < A.
\]
Hence, by (68) and (71) there holds $B \leq \max_{t>0} E(tu_\varepsilon, tv_\varepsilon) < A$. Next we will prove $B < B_{\alpha_1}$. The similar to proof of (1) in Theorem 1.1. Recall (70) and (80), we define $t(m) := t_{u_{\alpha_1}, m u_{\alpha_1}}$, then
\[
t(m)^{2^*_\mu-2}
\]
\[
= \int_{\mathbb{R}^{N+1}_+} y^{1-2s} (|\nabla u_{\alpha_1}|^2 + m^2 |\nabla u_{\alpha_1}|^2) \, dxdy + \int_{\Omega} (\lambda_1 u_{\alpha_1}(x,0)^2 + m^2 \lambda_2 u_{\alpha_1}(x,0)^2) \, dx.
\]
\[
(\alpha_1 + 2\beta m^{2^*_\mu} + \alpha_2 m^{2^*_\mu}) \int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_{\alpha_1}(x,0)|^{2^*_\mu} |u_{\alpha_1}(z,0)|^{2^*_\mu}}{|x-z|^{\mu}} \, dx \, dz.
\]
Note that $t(0) = 1$ and $1 < 2^*_\mu < 2$, by direct computations we have $\lim_{m \to 0} \frac{t'(m)}{|m|^{2^*_\mu-2}m} = -\frac{2^*_\mu}{(2^*_\mu-2)\alpha_1}$, that is, $t'(m) = -\frac{2^*_\mu}{(2^*_\mu-2)\alpha_1} |m|^{2^*_\mu-2}m \left(1 + o(1)\right)$ as $m \to 0$, we have
\[
t(m) = 1 - \frac{2^*_\mu}{(2^*_\mu-2)\alpha_1} |m|^{2^*_\mu} \left(1 + o(1)\right)\]
as $m \to 0$. This implies that $t(m)^{2^*_\mu-2} = 1 - \frac{2^*_\mu}{(2^*_\mu-1)\alpha_1} |m|^{2^*_\mu} \left(1 + o(1)\right)$. Therefore, as $|m| > 0$ small enough, by (68) and
\[
B_{\alpha_1} = \left( \frac{N - \mu + 2s}{4N - 2\mu} \right) \alpha_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\alpha_1}(x,0)|^{2^*_\mu} |u_{\alpha_1}(z,0)|^{2^*_\mu}}{|x-z|^{\mu}} \, dx \, dz,
\]
we have
\[ B \leq E(t(m)u_{\alpha_1}, t(m)mu_{\alpha_1}) \]
\[ B_{\alpha_1} - 2\beta \left( \frac{N - 2s}{4N - 2\mu} \right) |m|^2 \int_\Omega \int_\Omega \frac{|u_{\alpha_1}(x,0)|^2 |u_{\alpha_1}(z,0)|^2}{|x-z|^\mu} \, dx \, dz + o(|m|^2) \]
\[ < B_{\alpha_1}. \]
By a similar argument, we have \( B < B_{\alpha_2} \). This completes the proof. \( \square \)

**Proof of Theorems 1.4 for the case \( \beta > 0 \).** Assume that \( \beta > 0 \). Since the functional \( E \) has a mountain pass structure, by the mountain pass theorem there exists \( \{(u_n, v_n)\} \subset H \) such that \( \lim_{n \to \infty} E(u_n, v_n) = B \), \( \lim_{n \to \infty} E'(u_n, v_n) = 0 \). It is easy to see that \( \{(u_n, v_n)\} \) is bounded in \( H \), and so we may assume that \( (u_n, v_n) \rightharpoonup (u, v) \) weakly in \( H \). Set \( w_n = u_n - u \) and \( \sigma_n = v_n - v \), similarly as in the proof of Theorem 1.4 for the case \( \beta < 0 \), we know that \( E'(u, v) = 0 \) and (110)-(112) are also hold. Moreover,
\[ 0 \leq E(u, v) \leq E(u, v) + \frac{N - \mu + 2s}{4N - 2\mu} (b_1 + b_2) = \lim_{n \to \infty} E(u_n, v_n) = B. \quad (125) \]

**Case 1.** \( u = 0, v = 0 \). By (125), we have \( b_1 + b_2 > 0 \). Then we may assume that \( (w_n, \sigma_n) \neq (0, 0) \) for \( n \) large. By (111), there exists \( t_n > 0 \) such that \( (t_n u_n, t_n v_n) \subset N' \) and \( t_n \to 1 \) as \( n \to \infty \). Then by (60) and (125), we have
\[ B = \frac{N - \mu + 2s}{4N - 2\mu} (b_1 + b_2) = \lim_{n \to \infty} I(w_n, \sigma_n) = \lim_{n \to \infty} I(t_n u_n, t_n v_n) \geq A' = A, \]
a contradiction with Lemma 5.3. Therefore, Case 1 is impossible.

**Case 2.** \( u \neq 0, v = 0 \) or \( u = 0, v \neq 0 \). Without loss of generality, we assume that \( u \neq 0, v = 0 \). Note that \( u \) is a nontrivial solution of (79), we have \( B \geq E(u, 0) \geq B_{\alpha_1} \), a contradiction with Lemma 5.3. Therefore, Case 2 is impossible.

**Case 3.** \( u \neq 0, v \neq 0 \). Since \( (u, v) \subset M \), by \( B \leq B \) and (125) we have \( E(u, v) = B = B \). By (70) and (71), there exists a Lagrange multiplier \( \gamma \in \mathbb{R} \) such that \( E'(u, v) - \gamma G'(u, v) = 0 \). Since \( E'(u, v)(\gamma, v) = G(u, v) = 0 \) and \( G'(u, v)(u, v) \neq 0 \), we get \( \gamma = 0 \), then \( E'(u, v) = 0 \). This means that \( (u, v) \) is a least energy solution of (25).

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