Cayley 4-Frames and a Quaternion Kähler Reduction Related to Spin(7)

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Abstract. The object of this note is CAYLEY, the Grassmannian of the oriented 4-planes in $\mathbb{R}^8$ that are closed under the three-fold cross product. We describe an action of $U(1) \times Sp(1)$ on the quaternionic projective space $H\mathbb{P}^7$, that allows to obtain a $\mathbb{Z}_2$-quotient of CAYLEY by quaternion Kähler reduction.

1. Introduction

The existence of only two exceptional cross products - in $\mathbb{R}^7$ and in $\mathbb{R}^8$, with two and three factors respectively - attracted the interest of Alfred Gray in the sixties, and this was one of his approaches to the study of holonomies $G_2$ and Spin(7) on Riemannian manifolds $[5], [10]$. About one decade later, the symmetric space structure of the Grassmannians of those planes in $\mathbb{R}^7$ or $\mathbb{R}^8$ that are closed under such cross products was recognized in the positive quaternion Kähler manifolds $\frac{G_2}{SO(4)}$ and $\frac{Spin(7)}{(Sp(1) \times Sp(1) \times Sp(1))/\mathbb{Z}_2}$ $[14], [11]$. The latter of these manifolds is in fact isometric to the Grassmannian $Gr_4(\mathbb{R}^7) = \frac{SO(7)}{SO(4) \times SO(4) \times SO(3)}$ of oriented 4-planes in $\mathbb{R}^7$, but its rôle as ”exceptional Grassmannian” of some distinguished 4-planes in $\mathbb{R}^8$ is so interesting to make it deserving of notations like CAY or CAYLEY in papers on calibrations $[6], [9]$.

In this note we describe how CAYLEY can be obtained – up to a $\mathbb{Z}_2$-quotient – through a quaternion Kähler reduction of the projective space $H\mathbb{P}^7$, acted on by a group isomorphic to $U(1) \times Sp(1)$. This action is similar to that used by P. Kobak and A. Swann in $H\mathbb{P}^6$, obtaining a $\mathbb{Z}_3$-quotient of $\frac{G_2}{SO(4)}$ by quaternion Kähler reduction $[12]$. In the later note $[13]$ a different action of the same group in $H\mathbb{P}^7$ is described, obtaining this time as a reduction a $\mathbb{Z}_2$-quotient of $Gr_4(\mathbb{R}^7)$. Our reduction is in fact equivalent to this latter, via the isometry CAYLEY $\cong Gr_4(\mathbb{R}^7)$, although our definition of the action is much closer to the point of view of the former article $[12]$. We observe also that, through the same isometry with this

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real Grassmannian, one can obtain \textit{CAYLEY} as a reduction of \( \mathbb{H}P^6 \) acted on by \( \text{Sp}(1) \), following one of the classical procedures described in [1] (and in this way no finite quotient is involved). However, the reduction we are going to describe here – without making use of the isometry \( \text{CAYLEY} \cong \text{Gr}_4(\mathbb{R}^7) \) – has the advantage of admitting two interesting generalizations.

One of them is a consequence of the possibility of introducing weights in the action of the \( U(1) \)-factor. This allows to obtain a family of 12-dimensional quaternion Kähler orbifolds, some of them admitting smooth 3-Sasakian manifolds over them. These smooth 15-dimensional manifolds are, together with similar 11-dimensional manifolds related to \( G_2 \), the first examples of 3-Sasakian manifolds which are neither homogeneous nor toric [3].

The other possibility of extending the present reduction is obtained by looking at two other quaternion Kähler Wolf spaces. Since \( \text{CAYLEY} \) can be regarded as the manifold of the hypercomplex 4-planes in \( \mathbb{R}^8 \) identified with the real vector space of Cayley numbers \( \mathbb{C}a \) (Proposition 3.1 below), higher dimensional analogues of it are the manifolds \( \frac{\text{Spin}(9)}{\text{Spin}(2) \times \text{Sp}(1) \times \text{Sp}(1) / \mathbb{Z}_2} \cong \text{Gr}_4(\mathbb{R}^9) \) and the exceptional Wolf space \( \frac{\mathbb{H}P^1}{\text{Spin}(4) \times \text{Sp}(1)} \), geometrically the manifolds of the \( \mathbb{H}P^1 \subset \mathbb{C}a \) and of the \( \mathbb{H}P^2 \subset \mathbb{C}a \mathbb{P}^2 \), respectively. Some issues related to this second generalization will be studied in a future work [4].

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\section{Preliminaries on Cayley numbers}

Let \( \mathbb{C}a \) be the algebra of Cayley numbers, and let \( \{1, i, j, k, e, f = ie, g = je, h = ke\} \) be its canonical basis over \( \mathbb{R} \). The multiplication is given by

\[ xy = (ac - bd) + (b \sigma + da)e, \]

where \( x = a + be, y = c + de \in \mathbb{C}a \) are written through the identification \( \mathbb{C}a \cong \mathbb{H}^2 \) with pairs of quaternions. The quaternionic conjugation (already used in the previous formula) induces a conjugation in \( \mathbb{C}a \): \( \overline{x} = \overline{a} - \overline{b}e \), allowing to write the non-commutativity rule: \( \overline{xy} = y \overline{x} \).

The non-associativity of \( \mathbb{C}a \) gives rise to the associator \( [x, y, z] = (xy)z - x(yz) \), alternating form that vanishes whenever two of its arguments are either equal or conjugate. Geometrically, the associator defines the class of \textit{associative 3-planes} in \( \mathbb{R}^7 \cong \text{Im}\mathbb{C}a \), defined in orthonormal bases by \( [x, y, z] = 0 \). They are characterized as the 3-planes of \( \mathbb{R}^7 \) closed with respect to the \textit{two-fold cross product}:

\[ x \times y = \text{Im}(\overline{y}x), \]

that is so more generally defined for any \( x, y \in \mathbb{R}^8 \cong \mathbb{C}a \). Note that if \( x, y \) are orthogonal and in \( \mathbb{R}^7 \cong \text{Im}\mathbb{C}a \), the cross product is simply \( xy \).

The \textit{three-fold cross product} in \( \mathbb{C}a \cong \mathbb{R}^8 \) is defined by the formula

\[ x \times y \times z = \frac{1}{2}(x(\overline{y}z) - z(\overline{y}x)), \]

reducing to \( x(\overline{y}z) \) for \( x, y, z \) orthogonal.
The following properties hold whenever $x, y$ are orthogonal and for any $w \in \mathbb{Ca}$:
\begin{equation}
(2.1) \quad x(\overline{w}) = -y(\overline{w}), \quad (w\overline{w})x = -(w\overline{w})y.
\end{equation}

Moreover, for any $x, y, z \in \mathbb{Ca}$:
\begin{equation}
(2.2) \quad (xy)(zx) = x(yz)x.
\end{equation}

(A reference for these preliminaries is [11], Appendix IV).

3. The Stiefel manifold of Cayley 4-frames

A 4-plane $\zeta$ of $\mathbb{R}^8$ that is closed under the three-fold cross product is called a Cayley 4-plane and it is oriented by choices of bases $\{w = x \times y \times z, x, y, z\}$. The manifold of the Cayley 4-planes in $\mathbb{R}^8$ is CAYLEY = $\mathbb{Sp}(1) \times \mathbb{Sp}(1) / \mathbb{Z}_2$, 12-dimensional quaternionic submanifold of the Grassmannian $Gr_4(\mathbb{R}^8)$ of oriented 4-planes in $\mathbb{R}^8$ ([14], p. 262 or [11], p. 123).

Proposition IV,1.27 in [11] states that a 4-plane $\zeta$ in $\mathbb{R}^8$ is Cayley if and only if $-\zeta$ is closed under the complex structures defined by the 2-planes $\alpha \subset \zeta$. This fact can be reformulated as follows.

**Proposition 3.1.** A 4-plane $\zeta$ in $\mathbb{R}^8$ is Cayley if and only if any triple of mutually orthogonal 2-planes $\alpha, \beta, \gamma \subset \zeta$, all intersecting in a line, defines a hypercomplex structure on $\zeta$.

**Proof.** For any $\zeta$, $\dim(\zeta \cap \text{Im}\mathbb{Ca})$ is either 3 or 4. Thus, if $\zeta \in \text{CAYLEY}$ we may select orthonormal imaginary octonions $x, y, z \in \zeta$ such that $\{x \times y \times z, x, y, z\}$ is an oriented basis of $\zeta$. If $u = x \times (x \times y \times z) = y \times z, v = y \times (x \times y \times z) = z \times x, w = z \times (x \times y \times z) = x \times y$ we have $u, v, w \in S^6$, and their corresponding complex structures $J_u, J_v, J_w$ are associated to the 2-planes $\alpha = \text{span}\{x \times y \times z, x\}, \beta = \text{span}\{x \times y \times z, y\}, \gamma = \text{span}\{x \times y \times z, z\}$. Since $J_u v = -z, J_v x = z, J_w x = -y$, then $(J_u, J_v, J_w)$ satisfy $J_v \circ J_u = -J_u \circ J_v = J_w$, i.e. it is a hypercomplex structure on $\zeta$. The converse follows from the aforementioned characterization in [11], p. 119. 

Our construction of $(u, v, w)$ out of $\zeta = \text{span}\{x \times y \times z, x, y, z\}$ corresponds to the isometry $\sim : \text{CAYLEY} \rightarrow Gr_2(\mathbb{R}^7)$ of [3], p. 11. The image under $\sim$ of $\zeta \in \text{CAYLEY}$ can be interpreted as a tricomplex section of $S^6$, oriented orthonormal bases of the $\zeta^\circ$ being triples $u, v, w$ of unit octonions non necessarily satisfying the hypercomplex relations. The non-associativity of $\mathbb{Ca}$ allows and ensures that such triples define a hypercomplex structure on $\zeta$. An example is the Cayley 4-plane $\zeta = \text{span}\{1 - h, i + g, j - f, k + e\}$: our procedure gives the tricomplex triple $(u, v, w) = (i, j, e)$, whose associated $(J_u, J_v, J_w)$ is hypercomplex on $\zeta$.

This discussion permits to describe the inverse of the isometry $\sim$, as follows.

**Corollary 3.1.** Given a tricomplex section of $S^6$ with oriented orthonormal basis $(u, v, w)$, there is a unique Cayley 4-plane $\zeta$ in $\mathbb{R}^8$ on which $(J_u, J_v, J_w)$ is hypercomplex.

**Definition 3.1.** A Cayley 4-frame in $\mathbb{R}^8$ is an oriented orthonormal 4-frame in a Cayley 4-plane $\zeta$, hence a frame $\{x, I_1 x, I_2 x, I_3 x\}$, where $(I_1, I_2, I_3)$ is the hypercomplex structure of $\zeta$.

By the action of $\text{Spin}(7) \supset G_2 \supset \text{SU}(3)$ on the spheres $S^7 \supset S^6 \supset S^5$, the latter with isotropy $\text{SU}(2) \cong \text{Sp}(1)$, we have:
PROPOSITION 3.2. The Stiefel manifold of Cayley 4-frames in $\mathbb{R}^8$ is the homogeneous space $V = \frac{\text{Spin}(7)}{\text{Sp}(1)}$.

Observe finally that:

PROPOSITION 3.3. An orthonormal frame $\{f_1, f_2, f_3, f_4\}$ in $\mathbb{R}^8$ is a Cayley 4-frame if and only if $\bar{F}_2f_1 = \bar{F}_3f_4$.

4. CAYLEY and a reduction of $\mathbb{H}P^7$.

We now show how a $\mathbb{Z}_2$-quotient of CAYLEY can be obtained as quaternion Kähler reduction of $\mathbb{H}P^7$ by the action of $U(1) \times \text{Sp}(1)$. According to [2], we first reduce (by the same group) the 3-Sasakian manifold which stands over $H\text{Kähler}$ reduction of $S^7$. Then we interpret the quotient of $S^{31}$ by $U(1) \times \text{Sp}(1)$ as the total space of an $\text{SO}(3)$-bundle over a quaternion Kähler orbifold which is the quotient of $\mathbb{H}P^7$ by the same group.

The 3-Sasakian sphere $S^{31}$ is acted on by $U(1) \times \text{Sp}(1)$ as follows. The factor $\text{Sp}(1)$ acts by right multiplication on $\bar{h} = (h_\alpha) \in S^{31} \subset \mathbb{H}^8$, and the moment map $\mu : S^{31} \rightarrow \mathbb{R}^9$ of the action reads:

$$\mu(\bar{h}) = \left( \sum_{\alpha=1}^{8} h_\alpha h_\alpha, \sum_{\alpha=1}^{8} \bar{h}_\alpha h_\alpha, \sum_{\alpha=1}^{8} \bar{h}_\alpha k h_\alpha \right).$$

By writing $\bar{h} = \bar{a} + \bar{b}i + \bar{c}j + \bar{d}k$, it is easy to see that $\mu^{-1}(0)$ coincides with the Stiefel manifold of oriented (renormalized) orthonormal 4-frames in $\mathbb{R}^8$ [2].

We then act by the factor $U(1)$, rotating pairs of coordinates. This is explicitly described by $\bar{h} \mapsto \text{diag}(\cos \theta, A(\theta), A(\theta), A(\theta)) \cdot \bar{h}$, where $A(\theta) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right)$, $\theta \in \mathbb{R}$. The associated moment map $\nu : S^{31} \rightarrow \mathbb{R}^3$ is now:

$$\nu(\bar{h}) = \sum_{\beta=1}^{4} (\bar{h}_{2\beta-1} h_{2\beta} - \bar{h}_{2\beta} h_{2\beta-1}).$$

We are interested in the common zero set $N = \mu^{-1}(0) \cap \nu^{-1}(0)$.

PROPOSITION 4.1. $N = U(1) \cdot V$, where $\cdot$ is the action of $U(1)$ on Cayley 4-frames.

PROOF. The inclusion $V \subset N$ can be checked either by direct computation, using Proposition 3.3, or by a standard choice of the frame, like $(1,1,j,k)$, and the observation that $\nu(\bar{h}) = \nu(\bar{a} + \bar{b}i + \bar{c}j + \bar{d}k) = 0$ is invariant under right multiplication of $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ by any $u \in S^6$, and hence by $\text{Spin}(7)$ (cf. [11], p. 121). It follows also $U(1) \cdot V \subset N$, by the $(U(1))$-equivariance of $\nu$.

Conversely, to see that $N \subset U(1) \cdot V$, refer to a standard choice of three vectors to be substituted in the moment map equation $\nu(\bar{h}) = \nu(\bar{a} + \bar{b}i + \bar{c}j + \bar{d}k) = 0$, assuming $f_2 = \bar{b} = j, f_3 = \bar{c} = e, f_4 = \bar{d} = g$, (cf. the similar proof of the $G_2$-case in [12]). Then the equation $\nu(\bar{h}) = 0$ and the orthonormality of the frame give $\bar{f}_1 = \bar{a} = \cos \theta + \sin \theta i$. Then it is easy to check that the element $e^{-i\frac{\pi}{2}}$ of $U(1)$ transforms $(\cos \theta + \sin \theta i, j, e, g)$ into a Cayley 4-frame. □

Observe now that $U(1) \cap \text{Spin}(7) = U(1) \cap \text{SU}(4) = \mathbb{Z}_4$ with generator $\tau = e^{i\frac{\pi}{2}}$, and that under the action of $\tau$ on $V$, a Cayley 4-frame $(f_1, f_2, f_3, f_4)$ is transformed into another frame of the same Cayley 4-plane if and only if $(f_1, f_2, f_3, f_4)$ is complex
unitary, i.e. an element of $\frac{\SU(4)}{\Sp(1)}$. Also, of course $\tau^2 = -1$, so that any Cayley 4-plane is fixed under it. This explains the following description of the orbits of the $U(1) \times \Sp(1)$-action on $\mathcal{N}$: points in $\frac{\SU(4)}{\Sp(1)} \subset V$ generate orbits that are the fixed points of an induced action of $\mathbb{Z}_2$ on all the orbits of $\mathcal{N}$, and a 3-Sasakian orbifold $\mathbb{Z}_2 \setminus \Spin(7) / \Spin(4)$ is obtained as quotient. We state the corresponding quaternion Kähler reduction.

**Theorem 4.1.** The quaternion Kähler quotient of $\mathbb{H}P^7$ by the described action of $U(1) \times \Sp(1)$ is an orbifold $\mathbb{Z}_2 \setminus \text{CAYLEY}$, with a singular stratum isometric to the complex Grassmannian $\text{SU}(4) / \text{SU}(2) \times \text{U}(2)$.

**Remark 4.2.** By identifying any $\zeta$ with its orthogonal complement $\zeta^\perp$, one obtains a smooth $\mathbb{Z}_2$-quotient of CAYLEY. Since $\perp$ corresponds to the change of orientation on 4-planes in $\mathbb{R}^7$ ([6], p. 11), this smooth $\mathbb{Z}_2$-quotient of CAYLEY is the locally quaternion Kähler Grassmannian of unoriented 4-planes in $\mathbb{R}^7$.

The $\mathbb{Z}_2 \setminus \text{CAYLEY}$ given by Theorem 4.1 is not smooth, its construction yielding the stratified space $\mathcal{M}_{\text{reg}} \cup \text{Gr}_2(\mathbb{C}^4)$. The singular stratum $\text{Gr}_2(\mathbb{C}^4)$ corresponds, under the isometry CAYLEY $\cong \text{Gr}_4(\mathbb{R}^7)$, to the standard $\text{Gr}_4(\mathbb{R}^6) \subset \text{Gr}_4(\mathbb{R}^7)$. Thus the orbifold $\mathbb{Z}_2 \setminus \text{CAYLEY}$ in Theorem 4.1 is isometric to the singular quotient $\text{Gr}_4(\mathbb{R}^7) / \sigma_{\mathbb{R}^6}$ by the symmetry $\sigma_{\mathbb{R}^6}$ with respect to $\mathbb{R}^6 \subset \mathbb{R}^7$.

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