Regularity and uniqueness of Kelvin-Voigt models for nonhomogeneous and incompressible fluids

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Abstract. Generalized Kelvin-Voigt equations governing nonhomogeneous and incompressible fluids are considered in this work. In this work, we establish the existence of weak solutions to the considered model. Sufficient conditions ensuring more regular solutions and its uniqueness are also considered.

1. Introduction

This paper is devoted to the study of the regularity and uniqueness of the solutions to generalized Kelvin-Voigt equations that describe flows of incompressible and nonhomogeneous fluids. Let us consider the cylinder $Q_T := \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with its boundary denoted by $\partial \Omega$, and let $\Gamma_T := \partial \Omega \times (0, T)$, where $T > 0$. The problem we shall study here is the following: given the initial velocity field $v_0$, the initial density $\rho_0$ and the forces field $f$, to find the velocity field $v$, the pressure $\pi$ and the density $\rho$ satisfying to

$$
\frac{\partial (\rho v)}{\partial t} + \text{div} (\rho v \otimes v) = \rho f - \nabla \pi + \text{div} \left( \mu |D(v)|^{p-2} D(v) + \kappa |D(v)|^{q-2} \frac{\partial D(v)}{\partial t} \right) + \gamma |v|^{m-2}v \quad \text{in } Q_T, \tag{1}
$$

$$
\frac{\partial \rho}{\partial t} + \text{div} (\rho v) = 0, \quad \text{div } v = 0 \quad \text{in } Q_T, \tag{2}
$$

$$
\rho v = \rho_0 v_0 \quad \text{and} \quad \rho = \rho_0 \quad \text{in } \Omega \times \{0\}, \quad v = 0 \quad \text{on } \Gamma_T. \tag{3}
$$

Here, $\kappa$ denotes the relaxation time and $\mu$ is the fluid viscosity, both of which are considered to be positive constants. The exponents $p$, $q$ and $m$ are positive constants satisfying to $p$, $q$, $m \in (1, \infty)$ and $\gamma$ is assumed to be a constant with no predefined sign. Physical justifications of this problem, in the case of $p = q = 2$ and $\gamma = 0$, can be found in the works by Oskolkov [7] and by Pavlovsky [13] (see also Zvyagin and Turbin [15]). This paper continues the study of the mathematical correctness of the problem (1)-(3), which has been studied by the authors in their previous works [1, 2, 3, 4, 5, 6]. The Kelvin-Voigt problem (1)-(3) in the case $\rho = \text{const}$ (homogeneous fluids), $p = q = 2$ and $\gamma = 0$ have been firstly considered by Oskolkov [7] and then studied by several authors (see Zvyagin and Turbin [15] and the references cited therein).
In our previous works\cite{1, 2, 3}, the problem describing a homogeneous incompressible fluid, which corresponds to the case when $\rho = \text{const}$ in (1)-(3), was studied with respect to the existence and uniqueness of weak solutions. The large time behavior of the solutions, under different conditions on the diffusion, relaxation and convection, have been also investigated by the authors [4, 5, 6]. However, the uniqueness of solutions to the problem (1)-(3) and their further regularity are still open problems. In the present work we analyze the regularity and uniqueness of the solutions to the problem (1)-(3) in the particular case of $p = q = 2$. We investigate the dependence of the smoothness of the solutions on the smoothness of the given data of the problem. Sufficient conditions for the uniqueness of the solution the problem (1)-(3), in the case $p = q = 2$, are also derived.

2. Preliminaries
We address the reader to the monographs\cite{8, 11} for the definitions and main notations used in this work. We just fix the following notations,

\[ V := \{ v \in C_0^\infty(\Omega) : \text{div} v = 0 \}, \]
\[ H := \{ \text{closure of} \ V \ \text{in the norm of} \ L^2(\Omega) \}, \]
\[ V^m_p := \{ \text{closure of} \ V \ \text{in the norm of} \ W^{m,p}(\Omega) \}. \]

If $p = 2$, we denote $V^m_p$ by $V^m$, and, if $m = 1$, we denote $V^m_p$ and $V^m$ simply by $V_p$ and $V$, respectively.

**Definition 1.** Let $d \geq 2$, $1 < q$, $p$, $m < \infty$ and assume that $f \in L^2(Q_T)$. A pair of functions $(v, \rho)$ is a weak solution to the problem (1)-(3), if:

(i) $v \in L^\infty(0, T; H \cap V_q) \cap L^p(0, T; V_p) \cap L^m(Q_T)$;
(ii) $\rho > 0$ a.e. in $Q_T$, $\rho \in C([0, T]; L^\lambda(\Omega))$ for all $\lambda \in [1, \infty)$ and $\rho|v|^2 \in L^\infty(0, T; L^1(\Omega))$;
(iii) $v(0) = v_0$ and $\rho(0) = \rho_0$, with $\rho_0 \geq 0$ a.e. in $\Omega$;
(iv) For every $\phi \in V$ there holds for a.a. $t \in [0, T]$

\[
\frac{d}{dt} \left( \int_\Omega \rho(t)v(t) \cdot \phi \, dx \right) + \int_\Omega \mu |D(v(t))|^{p-2}D(v(t)) : D(\phi) \, dx + \int_\Omega (\rho(t)v(t) \cdot \nabla)v(t) \cdot \phi \, dx \]

\[
= \int_\Omega \rho(t)f(t) \cdot \phi \, dx + \int_\Omega (\rho(t)v(t) \cdot \nabla)v(t) \cdot \phi \, dx. \]

(v) For every $\phi \in C_0^\infty(\Omega)$ there holds for a.a. $t \in [0, T]$

\[
\frac{d}{dt} \int_\Omega \rho(t) \phi \, dx + \int_\Omega (\rho(t)v(t) \cdot \nabla)\phi \, dx = 0. \]

3. Existence of weak solutions
In this section, we establish the existence of weak solutions to the problem (1)-(3). Henceforward we assume that

\[
0 < M_1 := \inf_{x \in \Omega} \rho_0(x) \leq \rho_0(x) \leq \sup_{x \in \Omega} \rho_0(x) =: M_2 < \infty \quad \forall \ x \in \overline{\Omega}, \]

for some positive constants $M_1$ and $M_2$, with $M_1 \leq M_2$. 


Theorem 1. Let $\Omega$ be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz-continuous boundary $\partial \Omega$, and assume that $p = q = 2$ and $\gamma = 0$ in (1)-(3). If (5) and

$$f \in L^2(Q_T), \quad v_0 \in H \cap V$$

are fulfilled, then there exists, at least, a weak solution $(v, \rho)$ to the problem (1)-(3). Moreover, such weak solutions satisfy to the following estimates,

$$0 < M_1 \leq \rho(x, t) \leq M_2 < \infty \quad \forall (x, t) \in Q_T,$$

(6)

$$\sup_{t \in [0, T]} \left( ||v(t)||^2_{L^2(\Omega)} + ||\nabla v(t)||^2_{L^2(\Omega)} \right) + ||\nabla v(t)||^2_{L^2(Q_T)} + ||v(t)||^2_{L^2(Q_T)} \leq M_3,$$

(7)

where $M_3$ is a positive constant that depends only on the problem data.

Proof. The proof follows from Theorems 2.2 and 2.3 of [4], and from Theorems 3.1 and 4.1 of [5], in the case of $p = q = 2$ and $\gamma = 0$. A solution to the problem (1)-(3) is constructed as the limit of the Galerkin approximations, defined in the form

$$v^n(x, t) = \sum_{j=1}^{n} c^n_j(t) \psi_j(x), \quad \psi_j \in V^n, \quad v^n_0(x) = \sum_{j=1}^{n} c^n_{j,0} \psi_j(x), \quad c^n_{j,0} \in \mathbb{R},$$

and of the Cauchy problem

$$\rho^n_t + v^n \cdot \nabla \rho^n = 0, \quad \rho^n(0) = \rho_0.$$

Here, $V^n$ is a $n$-dimensional space spanned by $\psi_1, \ldots, \psi_n \in \{\psi_k\}_{k \in \mathbb{N}}$, and $\{\psi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in $H$ of eigenfunctions of the Stokes operator $\Delta : H^2(\Omega) \cap V \rightarrow H$, defined by $\Delta \varphi := -\nabla \cdot (\nabla \varphi)$ for any $\varphi \in H^2(\Omega) \cap V$, and where $P : L^2(\Omega) \rightarrow H$ is the Leray projection (see, e.g., Chapter IV of Boyer and Fabrie [10]). The constant $M_3$ of (7) depends on $\|v_0\|_{L^2(\Omega)}$, $\|\nabla v_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2(Q_T)}$.

In the next result, we recover the pressure $\pi$ from the weak formulation (4) of the problem (1)-(3).

Theorem 2. Let $\Omega$ be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz-continuous boundary $\partial \Omega$ and let $(v, \rho)$ be a weak solution to the problem (1)-(3). Then there exists a unique $\pi \in C_w([0, T]; L^1(\Omega))$, with $\int_\Omega v(t) \, dx = 0$ for all $t \in [0, T]$, such that

$$\frac{d}{dt} \int_\Omega [\rho(t)v(t) : \nabla \varphi] \, dx + \int_\Omega [\mu \nabla v(t) - \rho(t)v(t) \otimes \nabla v(t)] : \nabla \varphi \, dx$$

$$- \int_\Omega \rho(t) f(t) \cdot \varphi \, dx = \int_\Omega \pi(t) \text{div} \varphi \, dx \quad \forall \varphi \in W_0^{1,r}(\Omega), \quad r \geq \max \left\{ 2, \frac{d}{2} \right\}.$$ (8)

in the distribution sense on $(0, T)$. Moreover, there exists a positive constant $M_4$ such that

$$\|\pi\|_{L^\infty([0, T]; L^1(\Omega))} \leq M_4.$$ (9)

Here, $C_w([0, T]; V)$ denotes the subspace of $L^\infty([0, T]; V)$ formed by weakly continuous functions from $[0, T]$ onto $V$.

Proof. The proof is based on a variant of de Rham’s lemma due to Bogovskiĭ [9] and Pileckas [14], and will be carried out elsewhere. The constant $M_4$ of (9) depends on $\|\nabla v_0\|_{L^2(\Omega)}$, $\|v\|_{L^\infty((0,T);V)}$, $\|\nabla v\|_{L^2(Q_T)}$ and $\|f\|_{L^2(Q_T)}$. 

\[\square\]
4. Regularity of the velocity

In this section, we shall prove that the velocity and density are sufficiently regular so that further on we can prove a uniqueness result.

**Lemma 1.** Assume that, in addition to the conditions of Theorem 1, \( \partial \Omega \) is of class \( C^2 \), \( 2 \leq d < 4 \) and \( \mathbf{v}_0 \in \mathbf{V}^2 \). Then there exists a positive constant \( M_5 \) such that

\[
\sup_{t \in [0, T]} \left( \| \Delta \mathbf{v}(t) \|_{L^2(\Omega)} + \| \mathbf{v}(t) \|_{C^2(\overline{\Omega})} \right) + \| \Delta \mathbf{v} \|_{L^2(0, T; L^2(\Omega))} \leq M_5, \tag{10}
\]

**Proof.** Using standard methods, we obtain the following relation for the Galerkin approximations,

\[
\frac{\kappa}{2} \frac{d}{dt} \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 + \mu \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 \leq \int_{\Omega} \rho(t) \left[ (\mathbf{v}_0^n(t) + (\mathbf{v}^n(t) \cdot \nabla) \mathbf{v}^n(t) - \mathbf{f}(t)) \Delta \mathbf{v}^n(t) \right] dx,
\]

which holds for all \( t \in [0, T] \). We evaluate the right-hand side of this identity by using the Hölder and Cauchy inequalities,

\[
\frac{\kappa}{2} \frac{d}{dt} \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 + \mu \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 \leq M_2 \left( \| \mathbf{v}_0^n(t) \|_{L^2(\Omega)}^2 + \| (\mathbf{v}^n(t) \cdot \nabla) \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 + \| \mathbf{f}(t) \|_{L^2(\Omega)}^2 \right) \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2.
\]

Next, using the estimates (6)-(7), we can prove that

\[
\kappa \frac{d}{dt} \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 + \mu \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 \leq C \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2.
\]

Applying the Gronwall inequality, we obtain from here that

\[
\sup_{t \in [0, T]} \| \Delta \mathbf{v}^n(t) \|_{L^2(\Omega)}^2 + \| \Delta \mathbf{v}^n(t) \|_{L^2(Q_T)}^2 \leq C \left( \| \mathbf{v}_0^n \|_{W^{2,2}(\Omega)}^2 + \| \mathbf{f} \|_{L^2(Q_T)}^2 \right) := M_6. \tag{11}
\]

Hence, from Ladyzhenskaya [11, Corollary 4.2.3], we can prove that (11) implies (10), and the constant \( M_5 \) depends on \( \| \mathbf{v}_0 \|_{W^{2,2}(\Omega)} \) and \( \| \mathbf{f} \|_{L^2(Q_T)}^2 \).

**Theorem 3.** Let the conditions of Theorem 1 be fulfilled, and assume that \( \partial \Omega \) is of class \( C^2 \), \( 2 \leq d < 4 \) and \( \mathbf{v}_0 \in \mathbf{V}^2 \). In addition to (5), assume also the following conditions,

\[
\mathbf{v}_0 \in W^{2, r}(\Omega) \cap \mathbf{V}, \quad \mathbf{f} \in L^2(0, T; L^r(\Omega)), \quad |\nabla \rho_0| \leq C_{\rho_0}, \tag{12}
\]

hold for \( d < r < \frac{2d}{d-2} \) and \( d < 4 \), and for some positive constant \( C_{\rho_0} \). Then

\[
\sup_{t \in [0, T]} \left( \| \Delta \mathbf{v}(t) \|_{L^2(\Omega)} + \| \nabla \mathbf{v}(t) \|_{C^0,0,0(\overline{\Omega})} \right) + \| \nabla \pi \|_{L^2(0, T; L^2(\Omega))} \leq M_7, \tag{13}
\]

\[
\| \nabla \rho \|_{C(\overline{Q_T})} + \rho_0 \|_{C(\overline{Q_T})} \leq M_8 \tag{14}
\]

for some positive constants \( M_7 \) and \( M_8 \) that depend only on the problem data.

**Proof.** Using the transformation \( \mathbf{u} := \mathbf{v} + \alpha \mathbf{v}_1 \), where \( \alpha := \frac{\mu}{\kappa} \), in the Kelvin-Voigt problem (1)-(3), where \( p = q = 2 \) and \( \gamma = 0 \), we obtain the following Stokes system

\[
\text{div} \mathbf{u}(t) = 0 \quad \text{in} \quad \Omega, \tag{15}
\]

\[
- \mu \Delta \mathbf{u}(t) + \nabla \pi(t) = \rho [\mathbf{f}(t) - \mathbf{v}_1(t) - (\mathbf{v}(t) \cdot \nabla) \mathbf{v}(t)] \quad \text{in} \quad \Omega, \tag{16}
\]

\[
\mathbf{u}(t) = 0 \quad \text{on} \quad \partial \Omega, \tag{17}
\]
which holds for all \( t \in (0, T) \). From standard results on the Stokes problem (15)-(17) (see, e.g., Ladyzhenskaya [Theorems 3.5.2-3][11]), we obtain

\[
\| \Delta u(t) \|_{L^r(\Omega)} + \| \nabla \pi(t) \|_{L^r(\Omega)} \leq C \left( \| \nabla v(t) \|_{L^2(\Omega)} + \| \Delta v(t) \|_{L^r(\Omega)} + \| f(t) \|_{L^r(\Omega)} \right)
\]  

(18)

for \( d < r < q \leq \frac{2d}{d-2} \). Introducing the new functions

\[
Y(t) := \left( \int_{\Omega} \left| e^{\frac{2t}{r}} \Delta v(t) \right|^r \, dx \right)^{\frac{1}{r}} \quad \text{and} \quad Z(t) := Y(t)^{\frac{1}{2}},
\]

and using (18), we can prove that \( Z'(t) \leq C \left( \| \nabla v(t) \|_{L^2(\Omega)} + e^{-\frac{2t}{r}} Z(t) + \| f(t) \|_{L^r(\Omega)} \right) \).

From Theorems 1 and 2, we already know that for each weak solution \( (\rho, v) \), we get

\[
\left\| \Delta v \right\|_{L^r(\Omega)} \leq M_7, \quad \left\| \nabla \pi \right\|_{L^r(\Omega)} \leq M_9, \quad \left\| v \right\|_{C^{0,\alpha}(\Omega)} \leq M_8, \quad \left\| \nabla \rho \right\|_{L^r(\Omega)} \leq M_9.
\]

5. Uniqueness of the density and velocity

From Theorems 1 and 2, we already know that for each weak solution \( (\rho, v) \) to the problem (1)-(3) there exists a unique \( \pi \in C_{\infty}(0, T]; L^r(\Omega)) \), with \( \int_0^T \pi(t) \, dx = 0 \) for all \( t \in [0, T] \) such that (8) holds true. Therefore, it only remains to prove the uniqueness of \( \rho \) and \( v \).

**Theorem 4.** Let \((\mathbf{v}_1, \rho_1)\) and \((\mathbf{v}_2, \rho_2)\) be two different solutions to the problem (1)-(3) in the conditions of Theorem 1. If, in addition, the conditions of Theorem 3 are verified, then \( \mathbf{v}_1 = \mathbf{v}_2 \) and \( \rho_1 = \rho_2 \).

**Proof.** Let \((\mathbf{v}_i, \pi, \rho_i)\), with \( i = 1, 2 \), be two solutions to the problem (1)-(3). Setting \( \mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2 \) and \( \rho = \rho_1 - \rho_2 \), we obtain by elementary calculus

\[
\rho_1 \partial_t \mathbf{v}_1 + \rho_1 (\mathbf{v}_1 \cdot \nabla) \mathbf{v} - \kappa \Delta \mathbf{v}_1 - \mu \Delta \mathbf{v} = \rho (\mathbf{f} - \mathbf{v}_2 \cdot \nabla) \mathbf{v}_2 - \rho_1 (\mathbf{v} \cdot \nabla) \mathbf{v}_2,
\]

(23)

\[
\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho_1 + (\mathbf{v}_1 \cdot \nabla) \rho = 0.
\]

(24)
Multiplying (23) by \( \mathbf{v} \), and (24) by \( \rho \), respectively, next integrating over \( \Omega \), and then adding up the resulting equations, we obtain for all \( t \in [0, T] \)

\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho_1(t)} \mathbf{v}(t) \|^2_{L^2(\Omega)} + \kappa \| \nabla \mathbf{v}(t) \|^2_{L^2(\Omega)} + \| \rho(t) \|^2_{L^2(\Omega)} \right) + \mu \| \nabla \mathbf{v}(t) \|^2_{L^2(\Omega)} = \\
\int_{\Omega} \rho(t) f(t) \cdot \mathbf{v}(t) \, d\mathbf{x} - \int_{\Omega} \rho(t) \nabla v_2(t) \cdot \mathbf{v}(t) \, d\mathbf{x} - \int_{\Omega} \rho(t) (\nabla) v_2(t) \cdot \mathbf{v}(t) \, d\mathbf{x} - \int_{\Omega} \rho(t) \rho_2(t) \cdot \mathbf{v}(t) \, d\mathbf{x} \tag{25}
\]

Estimating the terms on the right-hand side of (25), we can show that

\[
\frac{d}{dt} \left( \| \sqrt{\rho_1(t)} \mathbf{v}(t) \|^2_{L^2(\Omega)} + \kappa \| \nabla \mathbf{v}(t) \|^2_{L^2(\Omega)} + \| \rho(t) \|^2_{L^2(\Omega)} \right) \leq G(t) \left( \| \sqrt{\rho_1(t)} \mathbf{v}(t) \|^2 + \kappa \| \nabla \mathbf{v}(t) \|^2 + \| \rho(t) \|^2 \right) \tag{26}
\]

for some positive constant \( C \), and where \( G(t) := 1 + \| f(t) \|^2_{L^2(\Omega)} + \| \nabla v_2(t) \|^2_{L^2(\Omega)} + \| \Delta v_2(t) \|^2_{L^2(\Omega)} + \| \nabla v_2(t) \|^2_{L^2(\Omega)} + \| \nabla \rho_2(t) \|^2_{L^2(\Omega)} \). From the assumption (12) and estimates (7), (10) and (13)-(14), we easily can see that \( G \in L^1(0, T) \). Therefore, Gronwall’s inequality applied to the differential inequality (26) assures us that

\[
\| \sqrt{\rho_1(t)} \mathbf{v}(t) \|^2_{L^2(\Omega)} + \kappa \| \nabla \mathbf{v}(t) \|^2_{L^2(\Omega)} + \| \rho(t) \|^2_{L^2(\Omega)} = 0 \quad \forall \ t \in [0, T].
\]

Hence \( \mathbf{v}_1 = \mathbf{v}_2 \) and \( \rho_1 = \rho_2. \)

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References
[1] Antontsev S N and Khompysh Kh 2017 J. Math. Anal. Appl. 456 (1) 99–116
[2] Antontsev S N and Khompysh Kh 2017 J. Math. Anal. Appl. 446 (2) 1255–73
[3] Antontsev S N, de Oliveira H B and Khompysh Kh 2019 J. Math. Anal. Appl. 473 1122–54
[4] Antontsev S N, de Oliveira H B and Khompysh Kh 2019 J. Phys.: Conf. Ser. 1268 012008
[5] Antontsev S N, de Oliveira H B and Khompysh Kh 2019 Commun.Math.Sci. 17 (7) 1915–48
[6] Antontsev S N, de Oliveira H B and Khompysh Kh 2020 Asymptotic Analysis 2020 (Pre-press) 1–33
[7] Oskolkov A P 1973 Zap. Naucn. Sem. LOMI 38 98–136 (in Russian)
[8] Antontsev S N, Kazhikhov A V and Monakhov V N 1990 Boundary Value Problems in Mechanics of Nonhomogeneous Fluids (Amsterdam: North-Holland Publishing Co.)
[9] Bogovskii M E 1980 Solution of some problemsof vector analysis related to the operators div and grad Trudy Sem. S. L. Sobolev no. 1 (Novosibirsk: Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat.) pp 5–40
[10] Boyer F and Fabrie P 2013 Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models (New York: Springer)
[11] Ladyzhenskaya O A 1969 The Mathematical Theory of Viscous Incompressible Flow 2nd ed. (Moscow: Nauka)
[12] Ladyzhenskaya O A and Solonnikov V A 1975 Zap. Naucn. Sem. LOMI 52 (8) 52–109 (in Russian)
[13] Pavlovsky V A 1971 Dokl. Akad. Nauk SSSR 200 (4) 809–12
[14] Pileckas K 1984 Proc. Steklov Math Inst. 159 141–54
[15] Zvyagin V G and Turbin M V 2010 J. Math. Sci. 168 (2) 157–308