The Starting and Stopping Problem under Knightian Uncertainty and Related Systems of Reflected BSDEs

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Abstract

This article deals with the starting and stopping problem under Knightian uncertainty, i.e., roughly speaking, when the probability under which the future evolves is not exactly known. We show that the lower price of a plant submitted to the decisions of starting and stopping is given by a solution of a system of two reflected backward stochastic differential equations (BSDEs for short). We solve this latter system and we give the expression of the optimal strategy. Further we consider a more general system of $m$ ($m \geq 2$) reflected BSDEs with interconnected obstacles. Once more we show existence and uniqueness of the solution of that system.

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0. Introduction: We first introduce through an example the standard starting and
stopping (or switching) problem which has attracted a lot of interests during the last
decades (see the long list of bibliography and the references therein).

Assume that a power plant produces electricity whose selling price, as we know,
fluctuates and depends on many factors such as consumer demand, oil prices, weather
and so on. It is also well known that electricity cannot be stored and when produced
it should be almost immediately consumed. Therefore for obvious economic reasons,
electricity is produced only when there is enough profitability in the market. Other-
wise the power station is closed up to time when the profitability is coming back, i.e.,
till the time when the market selling price of electricity reaches a level which makes
the production profitable again. Then for this power station there are two modes,
operating and closed. Accordingly, a management strategy of the station is an in-
creasing sequence of stopping times $\delta = (\tau_n)_{n \geq 0}$ ($\tau_0 = 0$ and for any $n \geq 0$, $\tau_n \leq \tau_{n+1}$).
At time $\tau_n$, the manager switches the mode of the station from its current one to the
other. However making a change of mode is not free and generates expenditures.

Suppose now that we have an adapted stochastic process $X = (X_t)_{t \leq T}$ which
stands for either the market electricity price or factors which determine the price.
When the power station is run under a strategy $\delta = (\tau_n)_{n \geq 0}$, its yield is given by
a quantity denoted $J(\delta)$ which depends also on $X$ and many other parameters such
as utility functions, expenditures, ... . Therefore the main problem is to find a
management strategy $\delta^* = (\tau^*_n)_{n \geq 1}$ such that for any $\delta$ we have $J(\delta^*) \geq J(\delta)$, i.e.
$J(\delta^*) = \sup_{\delta} J(\delta)$. Once determined, the strategy $\delta^*$ gives the optimal way of running
the power plant and, as a by-product, the real constant $J(\delta^*)$ is nothing else but the
fair price of the power plant in the energy market.

The two-mode starting and stopping problems attracted a lot of research activity
(see e.g. [1, 2, 3, 7, 8, 10, 11, 12, 13, 16, 18, 19, 22, 25, 26, 29, 28, 30], ... and the
references therein).

Recently, Hamadène and Jeanblanc [18] consider a finite horizon two-modes when
the price processes are only adapted to the filtration generated by a Brownian motion.
Porchet et al. in [25] have considered the same problem with exponential utilities and
allow for the manager the possibility to invest in a financial market. Djehiche and
Hamadène [8] studied also this problem but the model integrates the risk of default of

\[ \delta^n = \{ \tau^n \} \geq 0 \]
the economic unit. Let us also mention the work by Hamadène and Hdhiri [19] where the set up of those latter papers is extended to the case where the price processes of the underlying commodities are adapted to a filtration generated by a Brownian motion and an independent Poisson process.

Finally note that this two-mode switching problem models also industries, like copper or aluminium mines, ..., where parts of the production process are temporarily reduced or shut down when e.g. fuel, electricity or coal prices are too high to be profitable to run them. A further area of applications includes Tolling Agreements (see Carmona and Ludkovski [5] and Deng and Xia [7] for more details).

The natural extension of the two mode starting and stoping problem, is the case where there are more than two modes for the production. This problem has been recently considered by several authors amongst we can quote Carmona and Ludkovski [5], Djehiche et al. [9] and Porchet et al. [26].

The studies quoted above, however, assume that future uncertainty is characterized by a certain probability measure $P$ over the states of nature. This turn out to assume that the firm is in a way certain that future market conditions are governed by this particular probability measure $P$. The notion of Knightian uncertainty introduced by F.H. Knight [21] assumes that it is not granted that future uncertainty is characterized by a single probability measure $P$ but other probabilities $P^u$, $u \in U$, are also likely. Usually those probabilities $P^u$ are supposed not far from $P$. This notion will be defined later. Therefore one of the main issues is, e.g., related to the fair price of the power plant in the market. If this latter quantity does not exist what could be the lower price of the plant in accordance with the sur-replication concepts well-known in mathematical finance.

To make things more clear suppose that the process $X$ is the price of electricity in the energy market and assume that its dynamics is given by the following standard differential equation:

$$dX_t = X_t(r_t dt + \sigma_t dB_t), \ t \leq T \ and \ X_0 = x > 0$$

where $(B_t)_{t \leq T}$ is a Brownian motion, $r \overset{\Delta}{=} (r_t)_{t \leq T}$ is the spot interest rate and finally $(\sigma_t)_{t \leq T}$ the volatility of the electricity price. So if the parameters $r$ and $\sigma$ are known then the price of the power plant is just given by $\sup_\delta J(\delta)$. However usually it
happens that the process \( r \) is not precisely known. We just have on it some confidence \( i.e. \) we know that \( P - a.s., \) for any \( t \in [0, T], r_t \in [-\kappa, \kappa] \) where \( \kappa \) is a positive real constant which describes the degree of Knightian uncertainty (\( \kappa \)-ignorance in the terminology of Chen-Epstein (see [6])). Therefore possible dynamics of the electricity price are the following:

\[
dX_t = X_t(u_t dt + \sigma_t dB_t), \ t \leq T \text{ and } X_0 = x > 0
\]

where \( B \) is once more a Brownian motion and \( u \triangleq (u_t)_{t \leq T} \) is an adapted stochastic process which takes its values in the compact set \([-\kappa, \kappa]\). In this case, things go on like incompleteness in financial markets, we are just able to speak about the lower price of the power plant which is given by the quantity:

\[
J^* \triangleq \sup_{\delta} \inf_u J(\delta, u), \quad (0.1)
\]

where \( J(\delta, u) \) is the yield of the power plant when run under the strategy \( \delta \) and the future evolves according to the probability \( P^u \) for which \( B \) is a Brownian motion. Mainly in this work we aim at evaluating the quantity \( J^* \) and providing a pair \((\delta^*, u^*)\) such that \( J^* = J(\delta^*, u^*) \).

So in order to tackle our problem, using systems of reflected BSDEs with oblique reflection, we first provide a verification theorem which shapes the problem under consideration. We show that when the solution of the system exists it provides an optimal strategy \((\delta^*, u^*)\) of the switching problem under Knightian uncertainty. Then we deal with a general system of \( m \) \((m \geq 2)\) reflected BSDEs with oblique reflection for which we provide a solution. As a by-product, we obtain that the verification theorem is satisfied and therefore the switching problem solved. Further we address the difficult issue of uniqueness of the solution of the general system. Basically it turns out that the solution of that system can be characterized as an optimal value for an appropriate switching problem. Henceforth it is unique.

The idea of using reflected BSDEs in starting and stopping problems with two modes appeared already in a previous work by Hamadène & Jeanblanc [18]. Then there were several works on this subject using the same tool (see \textit{e.g.} [5, 26]). In [5], the authors consider the multi-mode starting and stopping problem. However they left open the question of the existence of the solution of the system of reflected
BSDEs with oblique reflection, associated with the multi-state switching problem. This question of existence/uniqueness is solved by Djehiche et al. in [9]. Independent of our work, very recently Hu & Tang [20] considered a quite more general, w.r.t. the one introduced in [5], multi-dimensional reflected BSDE with oblique reflection. They show existence and uniqueness of the solution. However their framework is still somehow narrow since, due to their techniques based on the use of local times and Tanaka’s formula, the assumptions they put on the data are rather stringent.

In this paper, using the notions of Snell envelope of processes [14, 17] and the notion of smallest $g$-supermartingales introduced by Mingyu & Peng [23] we provide new results, w.r.t. the ones of [20], on existence/uniqueness of the solution for the system of reflected BSDEs with oblique reflection.

This paper is organized as follows. In Section 1, we introduce the problem and give some properties of the model. The quantities $J(\delta, u)$ are expressed by means of solutions of standard BSDEs whose coefficients are not square integrable. Then we provide a verification theorem which shapes the problem via systems of reflected BSDEs with interconnected obstacles. The solution of the system provides the pair $(\delta^*, u^*)$ which achieves the $\sup \inf$ in (0.1). In Section 2, we consider a more general system of reflected BSDEs, and show the existence of its solution. Finally in Section 3 we characterize the solution as the optimal reward over some appropriate set of strategies. This implies uniqueness of the solution of the system.

1 The starting and stopping problem

1.1 The model

Throughout this paper $(\Omega, \mathcal{F}, P)$ will be a fixed complete probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$ whose natural filtration is $(\mathcal{F}_t^0) \equiv \sigma\{B_s, s \leq t\}_{0 \leq t \leq T}$. Let $\mathbf{F} \triangleq (\mathcal{F}_t)_{0 \leq t \leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0 \leq t \leq T}$ with the $P$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions, i.e., it is right continuous and complete. Furthermore, let:

- $\mathcal{P}$ be the $\sigma$-algebra on $[0, T] \times \Omega$ of $\mathbf{F}$-progressively measurable sets;

- $\mathcal{H}^{p,l}$ be the set of $\mathcal{P}$-measurable and $\mathbb{R}^l$-valued processes $\eta = (\eta_t)_{t \leq T}$ such that
\[ E[\int_0^T |\eta_s|^p ds] < \infty \ (p \geq 1) ; \]

- \( S^2 \) be the set of \( \mathcal{P} \)-measurable, continuous, \( R \)-valued processes \( \eta = (\eta_t)_{t \leq T} \) such that \( E[\sup_{t \leq T} |\eta_t|^2] < \infty \); we denote by \( \mathcal{A} \) the subset of \( S^2 \) which contains non-decreasing processes \( (K_t)_{t \leq T} \) such that \( K_0 = 0 \);

- for any stopping time \( \tau \in [0, T] \), \( T_\tau \) denotes the set of all stopping times \( \theta \) such that \( \tau \leq \theta \leq T \), \( P \)-a.s.

- the class \( \mathcal{D} \) be the set of \( \mathcal{P} \)-measurable \( rcll \) (right continuous with left limits) processes \( V = (V_t)_{t \leq T} \) such that the set of random variables \( \{V_\tau, \tau \in T_0\} \) is uniformly integrable.

- for any stopping time \( \lambda \), \( E_\lambda \) is the conditional expectation with respect to \( \mathcal{F}_\lambda \), i.e., \( E_\lambda[.] \triangleq E_\lambda[. | \mathcal{F}_\lambda] \).

Let us now fix the data of the problem.

(i) Let \( X \triangleq (X_t)_{0 \leq t \leq T} \) be an \( \mathcal{P} \)-measurable process with values in \( \mathbb{R}^k \) such that each component belongs to \( S^2 \) (then \( X \) is continuous). It stands for factors which determine the market electricity price.

(ii) For \( i = 1, 2 \), let \( \psi_i : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \psi_i(t, x) \in \mathbb{R} \), be Borelean functions for which there exists a constant \( C \) such that \( |\psi_i(t, x)| \leq C(1 + |x|) \), \( i = 1, 2 \). \( \psi_1 \) (resp. \( \psi_2 \)) represents the utility function for the power plant when it is in its operating (resp. close) mode. Actually in a small interval \( dt \), when the power plant is in its operating (resp. closed) mode it generates a profit equal to \( \psi_1(t, X_t)dt \) (resp. \( \psi_2(t, X_t)dt \)).

(iii) The switching of the power plant from one mode to another is not free. Actually if at a stopping time \( \tau \), the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one, the sunk cost is equal to \( \varphi_1(\tau, X_\tau) \) (resp. \( \varphi_2(\tau, X_\tau) \)) where the non-negative functions \( \varphi_1, \varphi_2 : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \varphi_1(t, x), \varphi_2(t, x) \in \mathbb{R}^+ \) are continuous and linearly growing, i.e., there exists a constant \( C \) such that \( |\varphi_i(t, x)| \leq C(1 + |x|) \), \( i = 1, 2 \). Additionally they verify \( \varphi_1(t, x) + \varphi_2(t, x) > 0 \) for any \( (t, x) \in [0, T] \times \mathbb{R}^k \). This latter requirement means that it is not free to make two instantaneous switching at any time \( t \leq T \).
(iv) Let $\delta = (\tau_n)_{n \geq 0}$ be an admissible management strategy of the plant, i.e., the $\tau_n$’s are $\mathcal{F}$-stopping times such that $\tau_n \leq \tau_{n+1}$ ($\tau_0 = 0$) for any $n \geq 0$ and $\lim_{n \to \infty} \tau_n = T$, $\mathbb{P}$-a.s.. The set of all admissible strategies will be denoted by $\mathcal{D}$. We assume that the power plant is in its operating mode at the initial time $t = 0$. Therefore $\tau_{2n+1}$ (resp. $\tau_{2n}$) are the times where the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one.

In the conventional model, i.e., if we know that the future will be governed by the probability measure $\mathbb{P}$ the mean yield of the power plant when run under the strategy $\delta = (\tau_n)_{n \geq 0}$ is given by:

$$J(\delta) \triangleq \mathbb{E}_\mathbb{P}\left\{ \int_0^T \psi^\delta(t, X_t)dt - A_T^\delta \right\},$$

where $\mathbb{E}$ is the expectation under the probability measure $\mathbb{P}$.

\begin{equation}
\begin{aligned}
\psi^\delta(t, x) &\triangleq \sum_{n \geq 0} \left[ \psi_1(t, x) \mathbb{1}_{[\tau_{2n}, \tau_{2n+1})}(t) + \psi_2(t, x) \mathbb{1}_{[\tau_{2n+1}, \tau_{2n+2})}(t) \right], \\
A_T^\delta &\triangleq \sum_{n \geq 0} \left[ \varphi_1(\tau_{2n+1}, X_{\tau_{2n+1}}) \mathbb{1}_{\{\tau_{2n+1}<t\}} + \varphi_2(\tau_{2n+2}, X_{\tau_{2n+2}}) \mathbb{1}_{\{\tau_{2n+2}<t\}} \right].
\end{aligned}
\end{equation}

Therefore the price of the power plant in the energy market is just $\sup_{\delta \in \mathcal{D}} J(\delta)$.

Knightian uncertainty amounts to suppose that we are not sure that the future will evolve under the probability $\mathbb{P}$ but other probabilities $\mathbb{P}^u$, $u \in \mathcal{U}$ (which we will precise later) are also likewise. However we will suppose that those possible probabilities $\mathbb{P}^u$ are not far from $\mathbb{P}$ in the sense that $\mathbb{P}$ and $\mathbb{P}^u$ are equivalent. Actually we will assume that:

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = L_T^u \triangleq \exp\left( \int_0^T b(s, X_s, u_s)dB_s - \frac{1}{2} \int_0^T |b(s, X_s, u_s)|^2 ds \right)$$

where:

(i) $u \triangleq (u_t)_{t \leq T}$ is an $\mathcal{P}$-measurable process with values in some compact set $\mathcal{U}$. Hereafter $u$ will be called an admissible control and the set of those controls will be denote by $\mathcal{U}$.

(ii) $b : (t, x, u) \in [0, T] \times C([0, T], \mathbb{R}^k) \times U \mapsto b(t, x, u) \in \mathbb{R}^d$ is a Borel measurable and bounded function. Moreover we assume that for any $(t, x)$, the mapping $u \in U \mapsto b(t, x, u) \in \mathbb{R}^k$ is continuous and for any $u \in \mathcal{U}$ the process $(b(t, X_s, u_t))_{t \leq T}$ is $\mathcal{P}$-measurable.
Note that since the function $b$ is bounded then the random variable $L^u_t$ has moment of any order, i.e., for any $p \geq 1$, $E[(L^u_t)^p] < \infty$ and if we set, for $t \leq T$, $L_t \triangleq E[L^u_t|\mathcal{F}_t]$ then the process $(L^u_t)_{t \leq T}$ satisfies the following standard stochastic differential equation:

$$dL^u_t = L^u_t b(t, X_t) dB_t, \quad t \leq T; \quad L^u_0 = 1.$$ 

As previously mentioned, if the future evolves according to the probability law $P^u$, $u \in U$, then the fair price of the power station in the energy market is given by:

$$J(u) = \sup_{\delta \in \mathcal{D}} J(\delta, u)$$

where

$$J(\delta, u) \triangleq E^u \left\{ \int_0^T \psi^\delta(t, X_t) dt - A^\delta_T \right\},$$

and $E^u$ is the expectation under $P^u$ and $\psi^\delta, A^\delta_T$ are defined by (1.2). However all the probability measures are likewise therefore the selling lower price of the power plant in the energy market is given by:

$$J^* \triangleq \sup_{\delta \in \mathcal{D}} J(\delta); \quad J(\delta) \triangleq \inf_{u \in U} J(\delta, u).$$

Actually the quantity $J^*$ stands for the optimal yield of the power plant in the worst case of evolution of the future. Therefore the problem we are interested in is to assess the value $J^*$ and to find a pair $(\delta^*, u^*)$ such that

$$J^* = J(\delta^*) = J(\delta^*, u^*) = \inf_{u \in U} J(\delta^*, u).$$

We note that, for any $u$, $J(\delta^*, u) \geq J^*$. However, for an arbitrary $\delta$, in general we do not have $J(\delta, u) \geq J(\delta, u^*)$. □

**Remark 1** In the particular case where the process $X$ is the solution of the following standard functional stochastic differential equation:

$$dX_t = a(t, X_t) dt + \sigma(t, X_t) dB_t, \quad t \leq T \text{ and } X_0 = x$$

with appropriate assumptions on the functions $a$ and $\sigma$ in order to guarantee existence and uniqueness of the solution of (1.5), then thanks to Girsanov’s Theorem we have:

$$dX_t = (a(t, X_t) + \sigma(t, X_t)b(t, X_t)) dt + \sigma(t, X_t) dB^u_t, \quad t \leq T \text{ and } X_0 = x$$

where $B^u_t = B_t - \int_0^t b(s, X_s, u_s) ds, t \leq T$, which is well known that it is a Brownian motion under the probability measure $P^u$. □
1.2 Properties of the model

We are going to simplify the problem and to show that we can focus only on a restricted set of strategies which satisfy appropriate integrability conditions. So for any admissible strategy \( \delta = (\tau_n)_{n \geq 0} \in \mathcal{D} \) let us recall (1.2), (1.3) and (1.4). Note that \( \psi^\delta, A^\delta \) do not depend on \( u \) and \( A^\delta \) is rcll.

Now for \( p \geq 1 \) let us set
\[
\mathcal{D}_p \triangleq \{ \delta \in \mathcal{D}, \text{ such that } \sup_{u \in U} E^\delta[(A^\delta_T)^p] < \infty \}; \quad \mathcal{D}' \triangleq \cup_{p \geq 1} \mathcal{D}_p.
\]

It follows that if \( \delta \in \mathcal{D} - \mathcal{D}_1 \), we have \( J(\delta) = -\infty \) since the process \( (\psi^\delta(t, X_t))_{t \leq T} \) belongs to \( L^1(dt \times dP) \) due to the facts that \( (\psi_i(t, X_t))_{t \leq T}, i = 1, 2 \), belongs to \( \mathcal{H}^{2,1} \) and that the random variable \( L^\delta_T \) has moments of any order with respect to the probability measure \( P \). As a consequence, in our objective to evaluate and characterize the quantity \( J^* = \sup_{\delta \in \mathcal{D}} \inf_{u \in U} J(\delta, u) \), we can discard the admissible strategies \( \delta \) which do not belong to \( \mathcal{D}_1 \).

Next we introduce the Hamiltonian of the problem which is defined by: for any \((t, x, u, z) \in [0, T] \times C([0, T], \mathbb{R}^k) \times U \times \mathbb{R}^d \),
\[
H(t, x, u, z) \triangleq z b(t, x, u) \quad \text{and} \quad H^*(t, x, z) \triangleq \inf_{u \in U} H(t, x, u, z).
\]

Since \( b(t, x, u) \) is bounded then the function \( H \) and \( H^* \) are uniformly Lipschitz w.r.t. \( z \). Additionally, thanks to Benes’s selection Theorem, there exists a measurable function \( u^* : (t, x, z) \in [0, T] \times C([0, T], \mathbb{R}^k) \times \mathbb{R}^d \mapsto u^*(t, x, z) \in U \) such that:
\[
H^*(t, x, z) = [z b(t, x, u)]_{u=u^*(t,x,z)}.
\]

We are now going to express the yields \( J(\delta, u) \) by the means of solutions of BSDEs whose coefficients are not square integrable. Actually we have:

**Proposition 1.1** (i) Let \( \delta \in \mathcal{D}_1 \) and \( u \in U \), then there exists a unique pair of processes \( (Y^\delta, u, Z^\delta, u) \) such that the process \( (Y^\delta, u - A^\delta) L^u \) is of class \( [D], \int_0^T |Z^\delta, u_s|^2 ds < \infty \) a.s., and finally for any \( t \leq T \) we have:
\[
Y^\delta_t = \int_t^T (\psi^\delta(s, X_s) + H(s, X_s, u_s, Z^\delta, u_s))ds - \int_t^T Z^\delta, u_s dB_s - (A^\delta_T - A^\delta_t)
\]
Moreover for any \( t \leq T \) we have:

\[
Y_t^{\delta,u} = E^u\big[ \int_t^T \psi^\delta(s, X_s)ds - (A_T^\delta - A_t^\delta) \big| \mathcal{F}_t \big].
\]

(ii) For any \( \delta \in \mathcal{D}' \), there exist \( q > 1 \) and a unique pair of processes \((Y^\delta, Z^\delta)\) such that:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
E\{ \sup_{t \leq T} |Y_t^\delta|^q + (\int_0^T |Z_s^\delta|^2 ds)^{q/2} \} < \infty; \\
Y_t^\delta = \int_t^T (\psi^\delta(s, X_s) + H^\star(s, X, Z_s^\delta))ds - \int_t^T Z_s^\delta dB_s - (A_T^\delta - A_t^\delta), t \leq T.
\end{array} \right.
\end{aligned}
\]

(1.6)

Moreover for any \( t \leq T \), \( Y_t^\delta = \text{essinf}_{u \in \mathcal{U}} Y_0^{\delta,u} \). In particular, \( J(\delta) = Y_0^\delta \) and the optimal argument is \((u^*(t, X, Z_t^\delta))_{t \leq T}\).

**Proof:** (i) Let \( \delta \) be a strategy which belongs to \( \mathcal{D}_1 \) and \( u \in \mathcal{U} \). Therefore we have \( E[L_T^u A_T^\delta] = E^u[A_T^\delta] < \infty \). Besides the process \((L_t^u \psi^\delta(t, X_t))_{t \leq T}\) belongs to \( L^1(dt \otimes dP) \).

Henceforth thanks to the result by Briand et al. ([4], Theorem 6.3, pp.18) related to solutions of BSDEs whose coefficients belong only to \( L^1 \), there exists a unique pair of processes \( \tilde{Y}^\delta,u \) of class \([D]\) and \( \tilde{Z}^\delta,u \) such that \( E[(\int_0^T |\tilde{Z}_s^\delta,u|^2 ds)\gamma] < \infty \), for any \( \gamma \in ]0,1[, \) which satisfy:

\[
\tilde{Y}_t^{\delta,u} = -L_t^u A_t^\delta + \int_t^T L_s^u \psi^\delta(s, X_s)ds - \int_t^T \tilde{Z}_s^{u,\delta} dB_s, t \leq T.
\]

Let us set now for \( t \leq T \),

\[
Y_t^{\delta,u} \triangleq \tilde{Y}_t^{\delta,u}(L_t^u)^{-1} + A_t^\delta; \quad Z_t^{\delta,u} \triangleq (L_t^u)^{-1} [\tilde{Z}_t^{u,\delta} - \tilde{Y}_t^{u,\delta} b(t, X_t, u_t)].
\]

First note that \( Y_t^{\delta,u} \) is finite since \( A_T^\delta < \infty \), P-a.s. due to the equivalence of the probability measures \( P \) and \( P^u \). Moreover \( \int_0^T |Z_s^{\delta,u}|^2 ds < \infty \), P-a.s. Finally the process \((Y^{\delta,u} - A^\delta)L^u \) is just \( \tilde{Y}^{\delta,u} \) which belongs to class \([D]\). Using now Itô’s formula for \( Y^{\delta,u} \) we get: \( \forall t \leq T \),

\[
Y_t^{\delta,u} = \int_t^T (\psi^\delta(s, X_s) + H(s, X_s, u_s, Z_s^{\delta,u}))ds - \int_t^T Z_s^{u,\delta} dB_s - (A_T^\delta - A_t^\delta).
\]

It remains to show that \( Y_t^{\delta,u} \) is just the conditional payoff after \( t \). Actually let \( \lambda_n \) be the following stopping time:

\[
\lambda_n \triangleq \inf\{ t \geq 0, \int_0^t |Z_s^{\delta,u}|^2 ds \geq n \} \wedge T.
\]
Therefore
\[
Y_{t \wedge \lambda_n}^{\delta,u} = E^u \{ Y_{\lambda_n}^{\delta,u} - A_{\lambda_n}^{\delta} + \int_{t \wedge \lambda_n}^{\lambda_n} \psi^{\delta}(s, X_s) ds + A_{t \wedge \lambda_n}^{\delta} \mid \mathcal{F}_{t \wedge \lambda_n} \}. 
\]

But the sequence of stopping times \((\lambda_n)_{n \geq 0}\) converges to \(T\) and \(L^u(Y_{\delta,u}^{\delta} - A_t^{\delta})\) belongs to class \([D]\), therefore \(Y_{\lambda_n}^{\delta,u} - A_{\lambda_n}^{\delta} \to -A_T^{\delta}\) in \(L^1(dP^u)\). Besides the second term in the conditional expectation converges also in \(L^1(dP^u)\) to \(\int_t^T \psi^{\delta}(s, X_s) ds + A_t^{\delta}\). It follows that:
\[
Y_t^{\delta,u} = E^u \{ \int_t^T \psi^{\delta}(s, X_s) ds - (A_T^{\delta} - A_t^{\delta}) \mid \mathcal{F}_t \}, \quad \forall t \leq T,
\]
which is the desired result.

Let us now focus on \((ii)\). Let \(\delta\) be a strategy of \(\mathcal{D}'\), therefore there exists \(p > 1\) such that \(\sup_{u \in \mathcal{U}} E^u[(A_T^{\delta})^p] < \infty\). As the moments of any order of \((L_T^u)^{-1}\), \(u \in \mathcal{U}\), exists then there exists \(q > 1\) such that \(E[(A_T^{\delta})^q] < \infty\). Now using once more the result by Briand et al. ([4], Theorem 4.2, pp.11) related to BSDEs in \(L^q\) \((q \in ]1, 2[)\) there exists a pair of processes \((\tilde{Y}^{\delta}, Z^{\delta})\) such that:
\[
\begin{align*}
E \{ \sup_{t \leq T} |\tilde{Y}^{\delta}_t|^q + (\int_0^T |Z^\delta_s|^2 ds)^q \} &< \infty; \\
\tilde{Y}^{\delta}_t &= -A_T^{\delta} + \int_t^T (\psi^{\delta}(s, X_s) + H^*(s, X_s, Z^\delta_s)) ds - \int_t^T Z^\delta_s dB_s, \quad t \leq T.
\end{align*}
\]
Now let us set \(Y^\delta = \tilde{Y}^{\delta} + A_T^{\delta}\), then the pair \((Y^\delta, Z^\delta)\) is solution of the BSDE (1.6).

Next for any \(t \leq T\), \(H^*(t, X, Z^\delta_t) = H(t, X, Z^\delta_t, u^*(t, X, Z^\delta_t))\) and since \((Y^\delta - A^\delta)L^{u^*}\) belongs to class \([D]\) (note that \(u^* = (u^*(t, X, Z^\delta_t))_{t \leq T}\) then thanks to \((i)\) we have:
\[
Y^\delta_t = E^{u^*} \{ \int_t^T \psi^{\delta}(s, X_s) ds - (A_T^{\delta} - A_t^{\delta}) \mid \mathcal{F}_t \} = Y^{\delta,u^*}_t, \quad \forall t \leq T.
\]
Next let \(u \in \mathcal{U}\). Then for any \(t \leq T\),
\[
Y_t^\delta - Y_t^{\delta,u} = \int_t^T (H^*(s, X, Z^\delta_s)) - H(s, X, u_s, Z^{\delta,u}_s)) ds - \int_t^T (Z^\delta_s - Z^{\delta,u}_s) dB_s
\]
\[
= \int_t^T (H^*(s, X, Z^\delta_s)) - H(s, X, u_s, Z^{\delta}_s)) ds - \int_t^T (Z^\delta_s - Z^{\delta,u}_s) dB_s.
\]
As \((Y^\delta - Y^{\delta,u})L^u\) is of class \([D]\) and since \(H^*(s, X, Z^\delta_s) - H(s, X, u_s, Z^{\delta}_s) \leq 0\) therefore, arguing as previously by using appropriate stopping times, we obtain \(Y^\delta_t - Y_t^{\delta,u} \leq 0\) for any \(t \leq T\). Henceforth it holds that:
\[
Y^\delta_t = \text{essinf}_{u \in \mathcal{U}} Y_t^{\delta,u}, \quad t \leq T,
\]
and the optimal argument is $u^* = (u^*(t, X, Z_t))_{t \leq T}$.

We are now going to prove that the suprema of $J(\delta)$ over $D_1$ and $D'$ are the same. Actually we have:

**Proposition 1.2** $\sup_{\delta \in D_1} J(\delta) = \sup_{\delta \in D'} J(\delta)$.

**Proof:** For any $\delta \in D_1$ and any $n$, let $\delta^n \triangleq \{\tau^n_i\}_{i \geq 0}$, where

$$
\lambda_n \triangleq \inf\{t \geq 0 : A_t^\delta \geq n\} \wedge T; \quad \tau^n_i \triangleq \begin{cases} 
\tau_i, & \text{if } \tau_i < \lambda_n; \\
T, & \text{if } \tau_i \geq \lambda_n.
\end{cases}
$$

It is obvious that the stopping times $\lambda_n \uparrow T$, and $A_T^\delta \leq n$ and then $\delta^n \in D'$.

For any $u \in U$,

$$
J(\delta, u) = E^u \left\{ \int_0^T \psi^\delta(t, X_t)dt - A_T^\delta \right\} \leq E^u \left\{ \int_0^T \psi^\delta(t, X_t)dt - A^n_T \right\} \triangleq J_n(\delta, u).
$$

Note that

$$
|J_n(\delta, u) - J(\delta^n, u)| \leq E^u \left\{ \int_{\lambda^n}^T |\psi(t, X_t) - \psi(t, X_t)| dt \right\} \\
\leq 2 \left\{ E^u \left[ \max_{i=1,2} |\psi_i(t, X_t)|^p dt \right] \right\}^{1/p} \left\{ E^u[(T - \lambda^n)] \right\}^{1/q}
$$

where $p \in [1, 2]$ and $q$ is its conjugate. But the right-hand side converges uniformly in $u \in U$ to 0 as $n \to \infty$ since the processes $(\psi_i(t, X_t))_{t \leq T}$ belong to $H_{2,1}^2$, $L_n^{u, T}$ have moments of any order and $(b(t, X, u_t))_{t \leq T}$ is a uniformly bounded process. Therefore we have:

$$
\lim_{n \to \infty} \sup_{u \in U} |J_n(\delta, u) - J(\delta^n, u)| = 0.
$$

It follows that:

$$
J(\delta, u) \leq J_n(\delta, u) = J_n(\delta, u) - J(\delta^n, u) + J(\delta^n, u) \leq \sup_{u \in U} |J_n(\delta, u) - J(\delta^n, u)| + J(\delta^n, u).
$$

Minimizing now both hand-sides over $u \in U$, we get:

$$
J(\delta) \leq \sup_{u \in U} |J_n(\delta, u) - J(\delta^n, u)| + J(\delta^n) \leq \sup_{u \in U} |J_n(\delta, u) - J(\delta^n, u)| + \sup_{\delta \in D'} J(\delta).
$$

Finally taking the limit as $n \to \infty$ to obtain the desired result.■
1.3 A verification theorem. Connection with reflected BS-DEs

In order to tackle the problem which is described in the previous part we are going to use the notion of systems of backward stochastic differential equations with reflecting barriers which we introduce now.

Let us consider the following two dimensional reflected BSDEs:

\[
\begin{aligned}
Y^1_t, Y^2_t &\in \mathcal{S}^2, \ Z^1_t, Z^2_t \in \mathcal{H}^{2,d} \text{ and } K^1, K^2 \in \mathcal{A}, \\
Y^1_t &= \int_t^T [\psi_1(s, X_s) + H^*(s, X_s, Z^1_s)] ds - \int_t^T Z^1_s dB_s + K^1_T - K^1_t; \\
Y^2_t &= \int_t^T [\psi_2(s, X_s) + H^*(s, X_s, Z^2_s)] ds - \int_t^T Z^2_s dB_s + K^2_T - K^2_t; \\
Y^1_t &\geq Y^2_t - \varphi_1(t, X_t); \quad [Y^1_t - Y^2_t + \varphi_1(t, X_t)]dK^1_t = 0; \\
Y^2_t &\geq Y^1_t - \varphi_2(t, X_t); \quad [Y^2_t - Y^1_t + \varphi_2(t, X_t)]dK^2_t = 0.
\end{aligned}
\]  

(1.7)

For the moment we suppose that the processes $Y^i, Z^i, K^i, i = 1, 2$ exist. We leave the well-posedness and computation of (1.7) to next section. Our main result of this section is the following theorem.

**Theorem 1.3** Assume $\varphi_1(t, x) + \varphi_2(t, x) > 0$. Then $Y^1_0 = \sup_{\delta \in \mathcal{D}_1} \inf_{u \in \mathcal{U}} J(\delta, u)$. Moreover, the optimal strategy $\delta^*$ which belongs to $\mathcal{D}_1$ is given by $\tau_0^* \triangleq 0$ and, for $n = 0, \cdots$,

\[
\begin{aligned}
\tau_{2n+1}^* &\triangleq \inf \{ t \geq \tau_{2n}^* : Y^1_t = Y^2_t - \varphi_1(t, X_t) \} \wedge T; \\
\tau_{2n+2}^* &\triangleq \inf \{ t \geq \tau_{2n+1}^* : Y^2_t = Y^1_t - \varphi_2(t, X_t) \} \wedge T.
\end{aligned}
\]

**Proof.** First let us point out that thanks to Proposition 1.2, it is enough to show that $Y^1_0 = \sup_{\delta \in \mathcal{D}'} \inf_{u \in \mathcal{U}} J(\delta, u)$. So let $\delta = (\tau_n)_{n \geq 0} \in \mathcal{D}'$ ($\tau_0 = 0$) and let us show that we have $Y^1_0 \geq Y^\delta_0$. To this end, we define for $t \leq T$:

\[
\begin{aligned}
\bar{Y}^\delta_t &\triangleq \sum_{n=0}^\infty \left[ Y^1_t 1_{[\tau_{2n+1}, \tau_{2n+2})}(t) + Y^2_t 1_{[\tau_{2n+1}, \tau_{2n+2}])(t) \right]; \\
Z^\delta_t &\triangleq \sum_{n=0}^\infty \left[ Z^1_t 1_{[\tau_{2n+1}, \tau_{2n+2})}(t) + Z^2_t 1_{[\tau_{2n+1}, \tau_{2n+2})}(t) \right].
\end{aligned}
\]

Note that there is no problem of definition of the processes $\bar{Y}^\delta$ and $Z^\delta$ since the series are convergent (at least pointwise). Besides $\bar{Y}^\delta$ is rcll and uniformly square integrable.
and $\bar{Z}^\delta$ belongs to $\mathcal{H}^{2,d}$ for any admissible strategy $\delta$. Moreover we have:

\[
\bar{Y}^\delta_0 = Y^1_0 = Y^1_{\tau_1} + \int_0^{T_1} \left[ \psi_1(s, X_s) + H^*(s, X_s, Z^1_s) \right] ds - \int_0^{T_1} Z^1_s dB_s + K^1_{\tau_1}
\]

\[
\geq Y^1_{\tau_1} - \varphi_1(\tau_1, X_{\tau_1}) \bf{1}_{\{\tau_1 < T\}} + \int_0^{T_1} \left[ \psi_1(s, X_s) + H^*(s, X_s, \bar{Z}^\delta_s) \right] ds + \int_0^{T_1} \bar{Z}^\delta_s dB_s
\]

\[
= Y^2_{\tau_1} + \int_{\tau_1}^{T_1} \left[ \psi_1(s, X_s) + H^*(s, X_s, Z^2_s) \right] ds - \int_{\tau_1}^{T_1} Z^2_s dB_s + K^2_{\tau_2} - K^2_{\tau_1}
\]

\[
- \varphi_1(\tau_1, X_{\tau_1}) \bf{1}_{\{\tau_1 < T\}} + \int_0^{T_1} \left[ \psi_1(s, X_s) + H^*(s, X_s, \bar{Z}^\delta_s) \right] ds - \int_0^{T_1} \bar{Z}^\delta_s dB_s
\]

\[
\geq Y^1_{\tau_2} - \varphi_2(\tau_2, X_{\tau_2}) \bf{1}_{\{\tau_2 < T\}} - \varphi_1(\tau_1, X_{\tau_1}) \bf{1}_{\{\tau_1 < T\}}
\]

\[
+ \int_0^{T_2} \left[ \psi_1(s, X_s) + H^*(s, X_s, \bar{Z}^\delta_s) \right] ds - \int_0^{T_2} \bar{Z}^\delta_s dB_s
\]

Repeat the procedure as many times as necessary we get: for any $n \geq 0$,

\[
\bar{Y}^\delta_0 \geq Y^1_{\tau_{2n+2}} - \sum_{k=0}^{n} \left[ \varphi_1(\tau_{2k+1}, X_{\tau_{2k+1}}) \bf{1}_{\{\tau_{2k+1} < T\}} + \varphi_2(\tau_{2k+2}, X_{\tau_{2k+2}}) \bf{1}_{\{\tau_{2k+2} < T\}} \right]
\]

\[
+ \int_0^{T_{2n+2}} \left[ \psi_1(s, X_s) + H^*(s, X_s, \bar{Z}^\delta_s) \right] ds - \int_0^{T_{2n+2}} \bar{Z}^\delta_s dB_s
\]

Taking now the limit as $n \to \infty$ and noting that $\tau_n \uparrow T$, we obtain:

\[
\bar{Y}^\delta_0 \geq \int_0^T \left[ \psi_1(s, X_s) + H^*(s, X, \bar{Z}^\delta_s) \right] ds - \int_0^T \bar{Z}^\delta_s dB_s - A^\delta_T.
\]

Following the same arguments we get, for any $t \leq T$,

\[
\bar{Y}^\delta_t \geq \int_t^T \left[ \psi_1(s, X_s) + H^*(s, X, \bar{Z}^\delta_s) \right] ds - \int_t^T \bar{Z}^\delta_s dB_s - [A^\delta_t - A^\delta_t]
\]

(1.10)

Here let us emphasize that up to now we did not use the fact that the strategy $\delta$ belongs to $\mathcal{D}'$ but only the fact that $\delta$ is admissible. This remark will be useful later.

At this level we need $\delta$ to be an element of $\mathcal{D}'$. Actually let us consider the process $Y^\delta$ defined in (1.6). Then for any $t \leq T$ we have,

\[
Y^\delta_t - Y^\delta_t \geq \int_t^T \left[ H^*(s, X, \bar{Z}^\delta_s) - H^*(s, X, Z^\delta_s) \right] ds - \int_t^T (\bar{Z}^\delta_s - Z^\delta_s) dB_s.
\]

(1.11)

where $\bar{B} \triangleq B_s - \int_0^s \gamma_s ds$ with

\[
\gamma_s \triangleq \frac{H^*(s, X, \bar{Z}^\delta_s) - H^*(s, X, Z^\delta_s)}{Z^\delta_s - \bar{Z}^\delta_s} \bf{1}_{[\bar{Z}^\delta_s - Z^\delta_s \neq 0]}
\]

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which is a bounded $\mathcal{P}$-measurable process since the mapping $z \mapsto H^*(t, X, z)$ is uniformly Lipschitz. Therefore, thanks to Girsanov’s Theorem, $\tilde{B}$ is a new Brownian motion under a new probability measure $\tilde{P}$ equivalent to $P$ whose density w.r.t. $P$ is given by $\tilde{L}$ which satisfies:

$$d\tilde{L}_t = \tilde{L}_t \gamma_t dB_t, \quad \tilde{L}_0 = 1.$$ 

Note that since the process $\gamma$ is bounded then the random variable $\tilde{L}_T$ has moment of any order w.r.t. $P$. Next we know that there exists a real constant $q > 1$ such that $E[(\int_0^T \{|Z_s^\delta|^2 + |Z_s^\delta|^2\} ds)^{q/2}] < \infty$, then there exists another real constant $q' > 1$ such that $\tilde{E}[(\int_0^T \{|Z_s^\delta|^2 + |Z_s^\delta|^2\} ds)^{q'/2}] < \infty$. Therefore the stochastic integral $\int_0^T (Z_s^\delta - Z_s^\delta) dB_s$ is actually a martingale. Going back now to (1.11), taking expectation w.r.t. $\tilde{P}$ we obtain that $\tilde{Y}_t^\delta - \bar{Y}_t^\delta \geq 0 \tilde{P}$-a.s. and then also $P$-a.s. since the probabilities are equivalent. As this inequality is valid for any $t \leq T$ and the processes $\bar{Y}^\delta$ and $\bar{Y}^\delta$ are rcll then $P$-a.s., for any $t \leq T$, $\bar{Y}_t^\delta \geq Y_0^\delta = \text{essinf}_{u \in U} Y_t^{\delta, u}$.

It remains to prove $\bar{\delta}^* = (\tau_n^*)_{n \geq 0}$ is optimal. First let us show that $\bar{\delta}^*$ is admissible, i.e., $P$-a.s. $\lim_{n \to \infty} \tau_n^* = T$. Actually let $\omega$ be such that $\lim_{n \to \infty} \tau_n^*(\omega) = \tau^*(\omega) < T$. As the processes $Y^1, Y^2, (\varphi_1(t, X_t))_{t \leq T}$ and $(\varphi_2(t, X_t))_{t \leq T}$ are continuous then for any $n \geq 0$ we have:

$$Y^1_{\tau_{2n+1}}(\omega) = Y^2_{\tau_{2n+1}}(\omega) - \varphi_1(\tau_{2n+1}(\omega), X_{\tau_{2n+1}}(\omega))$$

$$Y^2_{\tau_{2n+2}}(\omega) = Y^1_{\tau_{2n+2}}(\omega) - \varphi_2(\tau_{2n+2}(\omega), X_{\tau_{2n+2}}(\omega)).$$

We now let $n$ tends to $+\infty$ and we obtain

$$Y^1_\tau(\omega) = Y^2_\tau(\omega) - \varphi_1(\tau^*(\omega), X_{\tau^*}(\omega))$$

$$Y^2_\tau(\omega) = Y^1_\tau(\omega) - \varphi_2(\tau^*(\omega), X_{\tau^*}(\omega))$$

which obviously implies that $\varphi_1(\tau^*(\omega), X_{\tau^*}(\omega)) + \varphi_2(\tau^*(\omega), X_{\tau^*}(\omega)) = 0$ which is impossible. Therefore $P[\omega : \lim_{n \to \infty} \tau_n^*(\omega) < T] = 0$ and the strategy $\bar{\delta}^*$ is admissible.

On the other hand, note that by definition $(Y^1, Y^2)$ are continuous processes, then

$$Y^1_{\tau_1} = Y^2_{\tau_1} - \varphi_1(\tau_1^*, X_{\tau_1^*}) \mathbb{1}_{\{\tau_1^* < T\}}; \quad Y^2_{\tau_2} = Y^1_{\tau_2} - \varphi_1(\tau_2^*, X_{\tau_2^*}) \mathbb{1}_{\{\tau_2^* < T\}}.$$ 

Moreover,

$$K^1_{\tau_1} = 0; \quad K^2_{\tau_2} = K^2_{\tau^*}.$$
Therefore the inequalities (1.8) and (1.9) become equalities. Following similar arguments and since $\delta^*$ is admissible we have: for any $t \leq T$,

$$Y_t^{\delta^*} = \int_t^T \Big[ \psi^{\delta^*}(s, X_s) - H(s, X_s, \bar{Z}_s^{\delta^*}) \Big] ds - \int_t^T \bar{Z}_s^{\delta^*} dB_s - [A_T^{\delta^*} - A_t^{\delta^*}].$$

Writing the equation for $t = 0$ we deduce that $E[(A_T^{\delta^*})^2] < \infty$ since $Y_t^{\delta^*}$ is uniformly square integrable and $\bar{Z}_s^{\delta^*}$ belongs to $\mathcal{H}^{2,d}$. It follows that there exists a constant $p \in ]1,2[$ such that $\sup_{u \in \mathcal{U}} E^u[(A_T^{\delta^*})^p] < \infty$ and then $\delta^*$ belongs to $\mathcal{D}'$. By the well-posedness of (1.6) for elements of $\mathcal{D}'$, we get $Y_t^{\delta^*} = Y_t^{\delta^*}$ since $Y_t^{\delta^*}$ and $\bar{Z}_s^{\delta^*}$ are adapted processes. In particular, $Y_0^1 = Y_0^{\delta^*} = Y_0^{\delta^*} = \sup_{\delta \in \mathcal{D}'\prime} Y_0^{\delta} = \sup_{\delta \in \mathcal{D}'\prime} \inf_{u \in \mathcal{U}} J(\delta, u) = \sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} J(\delta, u) = J^*$. Additionally $\delta^*$ is optimal in $\mathcal{D}_1$ since $\mathcal{D}' \subset \mathcal{D}_1$. $\square$

**Remark 1.4**: Thanks to Proposition 1.1-(ii), the control $u^* = (u^*(t, X, \bar{Z}^\delta))_{t \leq T}$ combined with the strategy $\delta^*$ satisfy:

$$Y_0^1 = \bar{Y}_0^{\delta^*} = Y_0^{\delta^*} = J(\delta^*, u^*) = \inf_{u \in \mathcal{U}} J(\delta^*, u) = \sup_{\delta \in \mathcal{D}} \inf_{u \in \mathcal{U}} J(\delta, u).$$

### 2 High Dimensional Reflected BSDEs: Existence

As stated in Theorem 1.3, the solution of our original problem turns into solving the system of two reflected BSDEs (1.7) whose obstacles are inter-connected and depend on the solution. Therefore in what follows we are going to deal with general systems of reflected BSDEs such that (1.7) is just a particular case. Actually let us consider the following general system of RBSDEs: for $j = 1, \ldots, m$,

$$
\begin{align*}
Y^j_t &\in S^2, \quad Z^j_t \in \mathcal{H}^{2,d} \text{ and } K^j_t \in \mathcal{A}, \\
Y^j_t = \xi^j_j + \int_t^T f^j(s, Y^1_s, \ldots, Y^m_s, Z^1_s) ds - \int_t^T Z^j_s dB_s + K^j_T - K^j_t; \\
Y^j_t \geq \max_{i \in A^j_i} h_{j,i}(t, Y^i_t); \quad [Y^j_t - \max_{i \in A^j_i} h_{j,i}(t, Y^i_t)] dK^j_t = 0;
\end{align*}
$$

(2.1)

where $A^j \subset \{1, \ldots, m\} - \{j\}$, and the coefficients $f^j, h_{j,i}$ can depend upon $\omega$. For simplicity we denote $\bar{Y}_t \overset{\Delta}{=} (Y^1_t, \ldots, Y^m_t)$, and similarly for other vectors. We emphasize that here $A^j$ can be empty and if so we take the convention that the maximum over the empty set, denoted as $\emptyset$, is $-\infty$. Then in this case $Y^j$ has no lower barrier.
and then we take $K^j = 0$. Consequently, $Y^j$ satisfies the following BSDE without reflection:

$$Y^j_t = \xi^j + \int_t^T f^j(s, \overline{Y}_s, Z^j_s)ds - \int_t^T Z^j_s dB_s, \ t \leq T.$$ 

Also, for any $j$ we define

$$h^j_{j,j}(t, y) \triangleq y.$$  

(2.2)

We note that the $Y^j$ of the solution of (2.1) satisfies

$$Y^j_t \geq \max_{i \in A_j \cup \{j\}} h^j_{j,i}(t, Y^i_t).$$  

(2.3)

**Remark 2.1** The system we consider in (2.1) is appropriate for multi-dimensional switching problems when from one mode $j$ of the plant we are allowed to switch only to the modes which belong to $A_j$.

Throughout this section we shall adopt the following assumption.

**Assumption 2.2** For any $j = 1, \ldots, m$, it holds that:

(i) $E\left\{ \int_0^T \sup_{y^j = 0} |f^j(t, \overline{y}, 0)|^2 dt + |\xi^j|^2 \right\} < \infty$.

(ii) $f^j(t, \overline{y}, z)$ is uniformly Lipschitz continuous in $(y, z)$ and is continuous and increasing in $y_i$ for any $i \neq j$.

(iii) For $i \in A_j$, $h^j_{j,i}(t, y)$ is continuous in $(t, y)$ increasing in $y$, and $h^j_{j,i}(t, y) \leq y$.

Moreover, if $j_2 \in A_{j_1}, \ldots, j_k \in A_{j_{k-1}}, j_1 \in A_{j_k}$, for any $y$, denote

$$y_k \triangleq h^j_{j_k,j_{k-1}}(t, y), \quad y_{k-1} \triangleq h^j_{j_{k-1},j_{k}}(t, y_k), \ldots, y_1 \triangleq h^j_{j_1,j_2}(t, y_2).$$

Then we have

$$y_1 < y.$$  

(2.4)

(iv) For any $j = 1, \ldots, m$, $\xi_j \geq \max_{i \in A_j} h^j_{i,j}(T, \xi_i)$.

**Remark 2.3** The condition (2.4) means that it is not free to make a circle of instantaneous switchings. It is satisfied if for example for any $i, j$, $h^j_{i,j}(\omega, t, y) = y - c_{i,j}(\omega, t)$ with $c_{i,j}(\omega, t) > 0, \ \forall t \leq T$. □

Our main result is:
Theorem 2.4 Assume Assumption 2.2 holds true. Then RBSDE (2.1) has at least one solution.

Proof: We use Picard iteration. First let us denote:

\[ f_j(t, y, z) \triangleq \inf_{\overrightarrow{y} : y_j = y} f_j(t, \overrightarrow{y}, z) \quad \text{and} \quad \bar{f}_j(t, y, z) \triangleq \sup_{\overrightarrow{y} : y_j = y} f_j(t, \overrightarrow{y}, z). \]

By Assumption 2.2 (i) and (ii), \( f_j, \bar{f}_j \) are uniformly Lipschitz continuous in \((y, z)\) and

\[ E\left\{ \int_0^T \left[ |f_j(t, 0, 0)|^2 + |\bar{f}_j(t, 0, 0)|^2 \right] dt \right\} < \infty. \]

Next, let \((Y^{j,0}, Z^{j,0})\) be the solution to the following BSDE without reflection:

\[ Y^{j,0}_t = \xi_j + \int_t^T f_j(s, Y^{j,0}_s, Z^{j,0}_s) ds - \int_t^T Z^{j,0}_s dB_s, \quad j = 1, \ldots, m. \quad (2.5) \]

For \( j = 1, \ldots, m \) and \( n = 1, 2, \ldots \), recursively define \( Y^{j,n} \) via the following RBSDEs whose solution exists thanks to the result by El-Karoui et al. [15]:

\[
\begin{cases}
Y^{j,n}_t = \xi_j - \int_t^T Z^{j,n}_s dB_s + K^{j,n}_t - K^{j,n}_t \\
+ \int_t^T f_j(s, Y^{1,n-1}_s, \ldots, Y^{j-1,n-1}_s, Y^{j,n}_s, Y^{j+1,n-1}_s, \ldots, Y^{m,n-1}_s, Z^{j,n}_s) ds; \\
Y^{j,n}_t \geq \max_{i \in A_j} h_j(t, Y^{i,n-1}_t); \quad [Y^{j,n}_t - \max_{i \in A_j} h_j(t, Y^{i,n-1}_t)] dK^{j,n}_t = 0.
\end{cases}
\]

(2.6)

Note that, given \( Y^{i,n-1}, i = 1, \ldots, m \), for each \( j \) (2.6) is a one dimensional BSDE or reflected BSDE. Under Assumption 2.2, (2.6) has a unique solution. Moreover, by comparison theorem (see e.g. [15], Theorem 4.1.) it is obvious that \( Y^{j,1} \geq Y^{j,0} \). Then by induction one can easily show that \( Y^{j,n} \) is increasing as \( n \) increases.

In order to obtain uniform estimates of \( Y^{j,n} \), denote:

\[ \xi \triangleq \sum_{j=1}^m |\xi_j| \quad \text{and} \quad \tilde{f}(t, y, z) \triangleq \sum_{j=1}^m |\tilde{f}_j(t, y, z)|. \]

Let \((\tilde{Y}, \tilde{Z})\) be the solution to the following BSDE:

\[ \tilde{Y}_t = \xi + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dB_s. \]

Denote, for \( j = 1, \ldots, m \),

\[ \tilde{Y}^j_t \triangleq \tilde{Y}_t, \quad \tilde{Z}^j_t \triangleq \tilde{Z}_t, \quad K^j_t \triangleq 0. \]
Obviously \( Y_t^{i,0} \leq \bar{Y}_t^j \). Note that \((\bar{Y}^j, \bar{Z}^j, \bar{K}^j)\) satisfies
\[
\begin{align*}
\bar{Y}_t^j &= \xi + \int_t^T \bar{f}(s, \bar{Y}_s^j, \bar{Z}_s^j) - \int_t^T \bar{Z}_s^j dB_s + \bar{K}_T^j - \bar{K}_t^j; \\
\bar{Y}_t^j &\geq \max_{i \in A_j} h_{j,i}(t, \bar{Y}_t^i); \quad [\bar{Y}_t^j - \max_{i \in A_j} h_{j,i}(t, \bar{Y}_t^i)]d\bar{K}_t^j = 0.
\end{align*}
\]

Once more apply the comparison theorem repeatedly, we get
\[
Y_t^{j,n} \leq \bar{Y}_t, \quad \forall n.
\]

Recall that \( Y_t^{j,n} \geq Y_t^{j,0} \). Then
\[
\sum_{j=1}^m E \left\{ \sup_{0 \leq t \leq T} |Y_t^{j,n}|^2 \right\} \leq C < \infty, \quad \forall n. \tag{2.7}
\]

Moreover,
\[
E \left\{ \sup_{0 \leq t \leq T} |(\max_{i \in A_j} h_{j,i}(t, Y_t^{i,n-1}))|^2 \right\} \leq E \left\{ \sup_{0 \leq t \leq T} |(\max_{i \in A_j} Y_t^{i,n-1})|^2 \right\} \leq C.
\]

This further implies that
\[
E \left\{ \int_0^T |Z_t^{j,n}|^2 dt + |K_T^{j,n}|^2 \right\} \leq C, \quad \forall j, n. \tag{2.8}
\]

Now let \( Y^j \) denote the limit of \( Y_t^{j,n} \). By Peng’s monotonic limit theorem [24] or [23], we know \( Y^j \) is an rcll process, and following similar arguments there one can easily show that there exist \((\bar{Z}^j, K^j)\) such that
\[
\begin{align*}
Y_t^j &= \xi + \int_t^T f_j(s, \bar{Y}_s^j, \bar{Z}_s^j) ds - \int_t^T \bar{Z}_s^j dB_s + K_T^j - K_t^j; \\
Y_t^j &\geq \max_{i \in A_j} h_{j,i}(t, Y_t^i).
\end{align*}
\tag{2.9}
\]

Consider now the following RBSDEs whose solution exits thanks to the result by Hamadène [17] or Mingyu & Peng [23]:
\[
\begin{align*}
\tilde{Y}_t^j &= \xi - \int_t^T \tilde{Z}_s^j dB_s + \tilde{K}_T^j - \tilde{K}_t^j \\
&\quad + \int_t^T f_j(s, Y_s^1, \ldots, Y_s^j, \bar{Y}_s^j, Y_s^{j+1}, \ldots, Y_s^m, \bar{Z}_s^j) ds; \tag{2.10}
\end{align*}
\]

\[
\tilde{Y}_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i); \quad [\tilde{Y}_t^j - \max_{i \in A_j} h_{j,i}(t, Y_t^i)]d\tilde{K}_t^j = 0.
\]

We note that (2.9) and (2.10) have the same lower barrier. Since \( \tilde{Y}_t^j \) is the smallest \( f_j \)-supermartingale with lower barrier \( \max_{i \in A_j} h_{j,i}(t, Y_t^i) \), we have \( \tilde{Y}_t^j \leq Y_t^j \) (see [23],
Theorem 2.1. On the other hand, since \( Y_t^{i,n-1} \leq Y_t^i \) for any \((i, n - 1)\), by the monotonicity of \( h_{j,i} \) we get

\[
\max_{i \in A_j} h_{j,i}(t, Y_t^{i,n-1}) \leq \max_{i \in A_j} h_{j,i}(t, Y_t^i).
\]

Then once more by comparison theorem for RBSDEs we have \( Y_t^{j,n} \leq \tilde{Y}_t^j \), which implies that \( Y_t^j \leq \tilde{Y}_t^j \). Therefore, \( \tilde{Y}_t^j = Y_t^j \). This further implies that \( dt \otimes dP\)-\( \tilde{Z}_t^j = Z_t^j \) and P-a.s. for any \( t \leq T \), \( K_t^j = K_t^j \), and that

\[
\begin{cases}
Y_t^j = \xi_j + \int_t^T f_j(s, Y_s^j, Z_s^j)ds - \int_t^T Z_s^j dB_s + K_t^j - K_t^j, \\
Y_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i), \\
[Y_t^j - \max_{i \in A_j} h_{j,i}(t, Y_t^i)]dK_t^j = 0. 
\end{cases}
\tag{2.11}
\]

Finally we show that \( Y^j \) is continuous. We first note that, by (2.11), \( \Delta Y_t^j = -\Delta K_t^j \leq 0 \), and if \( \Delta K_t^j \neq 0 \), then \( Y_t^j = \max_{i \in A_j} h_{j,i}(t, Y_t^i) \). It is obvious that \( Y^j \) is continuous when \( A_j = \emptyset \). We now assume \( \Delta Y_t^{j_1} \neq 0 \) for some \( j_1 \) and \( t \). Then \( A_{j_1} \neq \emptyset \) and \( \Delta Y_t^{j_1} < 0 \). Note that in this case \( \Delta K_t^{j_1} > 0 \), which further implies that

\[
Y_t^{j_1} = \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_t^i).
\]

Let \( j_2 \in A_{j_1} \) be the optimal index, then

\[
h_{j_1,j_2}(t, Y_t^{j_2}) = Y_t^{j_1} > Y_t^{j_1} \geq \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_t^i) \geq h_{j_1,j_2}(t, Y_t^{j_2}).
\]

Thus \( \Delta Y_t^{j_2} < 0 \), and therefore \( A_{j_2} \neq \emptyset \). Repeat the arguments we obtain \( j_k \in A_{j_{k-1}} \) and \( \Delta Y_t^{j_k} < 0 \) for any \( k \). Since each \( j_k \) can take only values \( 1, \ldots, m \), we may assume, without loss of generality that \( j_1 = j_{k+1} \) for some \( k \geq 2 \) (note again that \( j_1 \notin A_{j_1} \) and thus \( j_2 \neq j_1 \)). Then we have

\[
Y_t^{j_1} = h_{j_1,j_2}(t, Y_t^{j_2}), \ldots, Y_t^{j_{k-1}} = h_{j_{k-1},j_k}(t, Y_t^{j_k}), \quad Y_t^{j_k} = h_{j_k,j_1}(t, Y_t^{j_1}).
\]

This contradicts with (2.4). Therefore, all processes \( Y^j \) are continuous.

By applying comparison theorem repeatedly, the following two results are direct consequence of Theorem 2.4, and their proofs are omitted.

**Corollary 2.5** The solution \( \bar{Y} \) constructed in Theorem 2.4 is the minimum solution to (2.1). That is, if \( \tilde{Y} \) is another solution to (2.1), then \( Y_t^j \leq \tilde{Y}_t^j, j = 1, \ldots, m. \)
Corollary 2.6 Assume \((\tilde{\xi}_j, \tilde{f}_j)\) also satisfy Assumption 2.2, and
\[ f_j \leq \tilde{f}_j, \quad \xi_j \leq \tilde{\xi}_j. \]

Let \(\overline{Y}\) and \(\overline{\tilde{Y}}\) denote the solution to (2.1) constructed in Theorem 2.4, with coefficients \((\xi_j, f_j, h_{j,i})\) and \((\tilde{\xi}_j, \tilde{f}_j, \tilde{h}_{j,i})\), respectively. Then \(Y^j_t \leq \tilde{Y}^j_t, j = 1, \cdots, m.\)

We now turn to the system (1.7) and we have:

Theorem 2.7 The system of reflected BSDEs (1.7) has a unique solution.

Proof: Existence is an immediate consequence of Theorem 2.4 through the properties satisfied by \(\psi_1, \psi_2, \varphi_1, \varphi_2\) and finally \(H^*\) which make Assumptions 2.2 fulfilled, especially the fact that \(\varphi_1(t, x) + \varphi_2(t, x) > 0\) for any \((t, x)\). Uniqueness of \(Y^1_0\) comes from Theorem 1.3. Similarly one can prove the uniqueness of \((Y^1_t, Y^2_t)\). Uniqueness of \(Z^1, Z^2\) is a consequence of Doob-Meyer Decomposition, therefore we have thoroughly uniqueness of \(K^1\) and \(K^2\).

Another by-product of Theorem 2.4 is that it provides also existence of a solution of the system (2.1) considered between two stopping times. This result is in particular useful to show uniqueness of (2.1).

Actually let \(\lambda_1\) and \(\lambda_2\) be two stopping times such that \(P\)-a.s., \(0 \leq \lambda_1 \leq \lambda_2 \leq T\) and let us consider the following RBSDE over \([\lambda_1, \lambda_2]\): for \(j = 1, \cdots, m,\) \(P\)-a.s.,

\[
\begin{cases}
(Y^j_t)_{t \in [\lambda_1, \lambda_2]} \text{ continuous, } (K^j_t)_{t \in [\lambda_1, \lambda_2]} \text{ continuous and nondecreasing,} \\
K^j_{\lambda_1} = 0, \text{ and } E \left\{ \sup_{t \in [\lambda_1, \lambda_2]} |Y^j_t|^2 + \int_{\lambda_1}^{\lambda_2} |Z^j_s|^2 ds + (K^j_{\lambda_2})^2 \right\} < \infty; \\
Y^j_t = \xi^j_{\lambda_2} + \int_{\lambda_1}^{\lambda_2} f_j(s, \overline{Y}_s, Z^j_s) ds - \int_{\lambda_1}^{\lambda_2} Z^j_s dB_s + K^j_{\lambda_2} - K^j_{\lambda_1}, \forall t \in [\lambda_1, \lambda_2]; \\
Y^j_t \geq \max_{i \in A_j} h_{j,i}(t, Y^i_t) \text{ and } [Y^j_t - \max_{i \in A_j} h_{j,i}(t, Y^i_t)] dK^j_t = 0, \forall t \in [\lambda_1, \lambda_2].
\end{cases}
\]

Then we have:

Theorem 2.8: Assume Assumption 2.2 holds true and that for \(j = 1, \cdots, m, \xi^j_{\lambda_2} \in \mathcal{F}_{\lambda_2}\) and satisfies:

\[ E\{|\xi^j_{\lambda_2}|^2\} < \infty \text{ and } \xi^j_{\lambda_2} \geq \max_{i \in A_j} h_{j,i}(\lambda_2, \xi^i_{\lambda_2}). \]

Then the RBSDE (2.12) has a solution.
3 Uniqueness

We now focus on uniqueness of the solution of RBSDE (2.12), hence that of RBSDE (2.1). To do that we need a stronger assumption.

Assumption 3.1 (i) \( f_j \) is uniformly Lipschitz continuous in all \( y_i \).

(ii) If \( i \in A_j, k \in A_i \), then \( k \in A_j \cup \{j\} \). Moreover,

\[
 h_{j,i}(t, h_{i,k}(t,y)) < h_{j,k}(t,y). \tag{3.1}
\]

(iii) For any \( i \in A_j \),

\[
 |h_{j,i}(t,y_1) - h_{j,i}(t,y_2)| \leq |y_1 - y_2|. \tag{3.2}
\]

Assume these assumptions are satisfied if \( A_j = \{1, \ldots, m\} - \{j\} \) for any \( j = 1, \ldots, m \) and \( h_{ij}(\omega, t, y) = y - c_{ij}(\omega, t) \) with \( c_{ij}(\omega, t) > 0 \) for any \( t \leq T \), \( \mathbb{P} \)-a.s.

Theorem 3.2 (Uniqueness)

(i) Assume Assumptions 2.2 and 3.1 are in force. Then the solution to (2.12) is unique.

(ii) Moreover, assume for \( j = 1, \ldots, m \), \( \tilde{f}_j \) satisfies Assumptions 2.2 and 3.1, and \( \tilde{\xi}_j \) satisfies (2.13). Let \((\tilde{Y}^j, \tilde{Z}^j)\) be the solution to RBSDE (2.12) corresponding to \((\tilde{f}_j, \tilde{\xi}_j)\). For \( j = 1, \ldots, m \), denote,

\[
 \Delta Y^j_t \triangleq Y^j_t - \tilde{Y}^j_t, \quad \Delta \xi^j \triangleq \xi^j - \tilde{\xi}^j, \quad ||\Delta f_t|| \triangleq \sum_{j=1}^{\mathcal{A}} \sup_{(y,z)} |[f_j - \tilde{f}_j](t, y, z)|. \tag{3.3}
\]

Then there exists a constant \( C \), which is independent of \( \lambda_1, \lambda_2 \), such that:

\[
 \max_{1 \leq j \leq m} |\Delta Y^j_{\lambda_1}|^2 \leq E_{\lambda_1} \left\{ e^{C(\lambda_2 - \lambda_1)} \max_{1 \leq j \leq m} |\Delta \xi^j|^2 + C \int_{\lambda_1}^{\lambda_2} ||\Delta f_t||^2 dt \right\}. \tag{3.4}
\]

The proof will be obtained after intermediary results. However basically it uses an induction argument and a characterization of \( Y^j \) as a supremum over strategies \( \delta \) of some processes \( Y^{j,\delta} \) which are uniquely defined.

So assume Assumptions 2.2 and 3.1 hold. Let \( \mu \) denote the number of nonempty sets \( A_j \) in (2.12), that is, the number of reflections in (2.12). We proceed by induction on \( \mu \). First, when \( \mu = 0 \), (2.12) becomes an \( m \)-dimensional BSDE without reflection. By standard arguments one can easily show that Theorem 3.2 holds true. Now assume it is true for \( \mu = m_1 - 1 \) for some \( 1 \leq m_1 \leq m \). For \( \mu = m_1 \), let \((Y^j, Z^j, K^j)\) be an arbitrary solution to (2.12).
3.1 Admissible strategies

We want to extend the arguments in Theorem 1.3 to this case. The idea is to express $Y_t^j$ as the supremum of $Y_t^{j,\delta}$, where $\delta$ is an admissible strategy which we are going to define soon, and $Y_t^{j,\delta}$ is the solution to a system of RBSDEs with $m_1 - 1$ reflections. Thus by induction $Y_t^{j,\delta}$ is unique for each $(j, \delta)$ and therefore $Y_t^j$ is unique.

To motivate the definition of admissible strategy, we heuristically discuss how to find the “optimal strategy”, an analogue of the $\tau_n^*$ in Theorem 1.3. A rigorous and more detailed argument will be given in §3.3.

Let $\tau_0^* \overset{\Delta}{=} \lambda_1$, and without loss of generality assume $A_1 \neq \emptyset$. Set

$$
\tau_1^* \overset{\Delta}{=} \inf\{t \geq \tau_0^* : Y_t^1 = \max_{i \in A_1} h_{1,i}(t, Y_t^i)\} \wedge \lambda_2.
$$

When $\tau_1^* < \lambda_2$, we have

$$
Y_t^1 = \max_{i \in A_1} (\tau_1^*, Y_{\tau_1^*}^i).
$$

That is, there exists an index, denoted as $\eta_1 \in A_1$, such that

$$
Y_t^1 = h_{1,\eta_1}(\tau_1^*, Y_{\tau_1^*}^\eta_1).
$$

So, besides the stopping time $\tau_1^*$, we need to keep track of the “optimal index” $\eta_1$. At this point, let us denote $\eta_0 \overset{\Delta}{=} 1$. Note that, over $[\tau_0^*, \tau_1^*]$, it holds that:

$$
\begin{cases}
Y_t^j = Y_{\tau_1^*}^j + \int_t^{\tau_1^*} f_j(s, \bar{Y}_s, Z_s^j)ds - \int_t^{\tau_1^*} Z_s^jdB_s + K_s^j - K_t^j, \ j \neq \eta_0; \\
Y_t^j \geq \max_{k \in A_j} h_{j,k}(t, Y_t^k); \ [Y_t^j - \max_{k \in A_j} h_{j,k}(t, Y_t^k)]dK_t^j = 0, \ j \neq \eta_0; \\
Y_t^{\eta_0} = Y_{\tau_1^*}^{\eta_0} + \int_t^{\tau_1^*} f_{\eta_0}(s, \bar{Y}_s, Z_s^{\eta_0})ds - \int_t^{\tau_1^*} Z_s^{\eta_0}dB_s.
\end{cases}
$$

This is a system with only $m_1 - 1$ reflections.

Now for $(\tau_1^*, \eta_1)$, we need to consider two different cases.

Case 1. Assume $A_{\eta_1} \neq \emptyset$. Then by considering $Y^{\eta_0}$ over $[\tau_1^*, \lambda_2]$ instead of $Y^{\eta_0}$ over $[\tau_0^*, \lambda_2]$, similarly one can define $\tau_2^*$ and $\eta_2 \in A_{\eta_1}$, and see that $\bar{Y}$ satisfies a system with $m_1 - 1$ reflections over $[\tau_1^*, \tau_2^*]$, where the $\eta_1$-th equation has no reflection.

Case 2. Assume $A_{\eta_1} = \emptyset$. In this case, the $\eta_1$-th equation has no reflection. Note that $Y_{\tau_1^*}^{\eta_0} = h_{\eta_0,\eta_1}(\tau_1^*, Y_{\tau_1^*}^{\eta_0})$. Choose $\tau_2^*$ “close” to $\tau_1^*$, then for any $t \in [\tau_1^*, \tau_2^*]$, we have $Y_t^{\eta_0} \approx h_{\eta_0,\eta_1}(\tau_1^*, Y_t^{\eta_0})$. On the other hand, by (3.1) one can see that $Y_t^j > h_{j,\eta_0}(\tau_1^*, Y_{\tau_1^*}^\eta_0)$.
for any \( j \) such that \( \eta_0 \in A_j \). Since \( \tau_2^* \) is close to \( \tau_1^* \), let us assume \( Y_t^j > h_{j, \eta_0}(\tau_1^*, Y_0) \) for \( t \in [\tau_1^*, \tau_2^*] \). So approximately, over \([\tau_1^*, \tau_2^*] \), \( Y^j, j \neq \eta_0 \) satisfy

\[
\begin{align*}
Y_t^j &\approx Y_{\tau_2^*}^j + \int_{t}^{\tau_2^*} f_j(s, h_{1, \eta_i}(\tau_1^*, Y_s), Y_s^2, \cdots, Y_s^m, Z_s^j)ds - \int_{t}^{\tau_2^*} Z_s^j dB_s + K_t^j - K_{\tau_2^*}^j; \\
Y_t^j &\geq \max_{k \in A_j - \{\eta_0\}} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k \in A_j - \{\eta_0\}} h_{j,k}(t, Y_t^k)]dK_t^k = 0.
\end{align*}
\]

This is a system of \( m - 1 \) equations with \( m_1 - 1 \) reflections, where we remove the equation for \( Y^{\eta_0} \) completely.

In order to move forward, we need to define \( \eta_2 \) so that \( A_{\eta_2} \neq \emptyset \). It turns out that the best way is to set \( \eta_2 \triangleq \eta_0 \). Then we can continue the procedure.

Based on the above argument, let us introduce the following:

**Definition 3.3** \( \delta = (\tau_0, \cdots, \tau_n; \eta_0, \cdots, \eta_m) \) is called an admissible strategy if

(i) \( \lambda_1 = \tau_0 \leq \cdots \leq \tau_n \leq \lambda_2 \) is a sequence of stopping times;

(ii) \( \eta_0, \cdots, \eta_m \) are random index taking value in \( \{1, \cdots, m\} \) such that \( \eta_i \in F_i \);

(iii) \( A_{\eta_i} \neq \emptyset \);

(iv) If \( A_{\eta_i} \neq \emptyset \), then \( \eta_{i+1} \in A_{\eta_i} \);

(v) \( A_{\eta_i} = \emptyset \), then \( \eta_{i+1} \triangleq \eta_{i-1} \).

**Remark 3.4** By Definition 3.3 (iii), \( A_{\eta_i} = \emptyset \) implies that \( i \geq 1 \). Then (v) makes sense. Moreover, in this case \( A_{\eta_{n+1}} = A_{\eta_{n-1}} \neq \emptyset \).

### 3.2 Construction of \( Y^\delta \)

For an admissible strategy \( \delta \), we construct \((Y^\delta_j, Z^\delta_j)\) as follows. First, for \( t \in [\tau_n, \lambda_2] \) and \( j = 1, \cdots, m \), set

\[
Y_t^{\delta_j} \triangleq Y_{\tau_n}^{\delta_j}, \quad Z_t^{\delta_j} \triangleq Z_{\tau_n}^{\delta_j},
\]

where \((Y^{\delta_0}, Z^{\delta_0})\) is the solution to (2.12) constructed in §2. Then in particular we have

\[
Y_{\tau_n}^{\delta_j} \geq \max_{i \in A_j} h_{j,i}(\tau_n, Y_{\tau_n}^{\delta_i}), \quad j = 1, \cdots, m.
\] (3.6)

For \( i = n - 1, \cdots, 0 \), assume we have constructed \( Y_{\tau_i}^{\delta_j} \) for \( j = 1, \cdots, m \), which we will do later. Note that \( Y^{\delta_j} \) may be discontinuous at \( \tau_{i+1} \). We define \((Y^{\delta_j}, Z^{\delta_j})\) over \([\tau_i, \tau_{i+1}]\) in two cases.
**Case 1.** If \( A_\eta \neq \emptyset \), assume,

\[
Y_{\tau_{i+1}^-}^{\delta,j} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,k}), \quad j \neq \eta_i. \tag{3.7}
\]

We consider the following RBSDE by removing the constraint of the \( \eta_i \)-th equation:

\[
\begin{aligned}
&\begin{cases}
Y_t^{\delta,j} = Y_{\tau_{i+1}^-}^{\delta,j} + \int_t^{\tau_{i+1}} f_j(s, Y_s^\delta, Z_s^{\delta,j}) ds - \int_t^{\tau_{i+1}} Z_s^{\delta,j} dB_s + K_t^{\delta,j} - K_{\tau_{i+1}}^{\delta,j}, \quad j \neq \eta_i; \\
Y_t^{\delta,j} \geq \max_{k \in A_j} h_{j,k}(t, Y_t^{\delta,k}), [Y_t^{\delta,j} - \max_{k \in A_j} h_{j,k}(t, Y_t^{\delta,k})] dK_t^{\delta,k} = 0, \quad j \neq \eta_i;
\end{cases} \\
Y_t^{\delta,\eta_i} = Y_{\tau_{i+1}^-}^{\delta,\eta_i} + \int_t^{\tau_{i+1}} f_{\eta_i}(s, Y_s^\delta, Z_s^{\delta,\eta_i}) ds - \int_t^{\tau_{i+1}} Z_s^{\delta,\eta_i} dB_s.
\end{aligned}
\tag{3.8}
\]

It is obvious that the \( f_j, h_{j,i}, A_j \) here satisfy Assumptions 2.2 and 3.1. Since (3.8) has only \( m_1 - 1 \) reflections, by induction (3.8) has a unique solution \((Y^{\delta,j}, Z^{\delta,j}), j = 1, \ldots, m \) over \([\tau_i, \tau_{i+1}]\). □

**Case 2.** If \( A_\eta = \emptyset \), by Remark 3.4 we have \( i \geq 1 \) and \( A_{\eta_{i-1}} \neq \emptyset \). Assume

\[
Y_{\tau_{i+1}^-}^{\delta,j} \geq \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,k}), \quad j \neq \eta_{i-1}. \tag{3.9}
\]

We now omit the \( \eta_{i-1} \)-th equation and consider the following \( m - 1 \) dimensional RBSDE with at most \( m_1 - 1 \) reflections: for \( j \neq \eta_{i-1} \),

\[
\begin{aligned}
&\begin{cases}
Y_t^{\delta,j} = Y_{\tau_{i+1}^-}^{\delta,j} - \int_t^{\tau_{i+1}} Z_s^{\delta,j} dB_s + K_t^{\delta,j} - K_{\tau_{i+1}}^{\delta,j} \\
+ \int_t^{\tau_{i+1}} \tilde{f}_j(s, Y_s^{\delta,1}, \ldots, Y_s^{\delta,\eta_{i-1}-1}, Y_s^{\delta,\eta_{i-1}+1}, \ldots, Y_s^{\delta,m}, Z_s^{\delta,j}) ds;
\end{cases} \\
Y_t^{\delta,j} \geq \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(t, Y_t^{\delta,k}), [Y_t^{\delta,j} - \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(t, Y_t^{\delta,k})] dK_t^{\delta,k} = 0.
\end{aligned}
\tag{3.10}
\]

Here:

\[
\tilde{f}_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \ldots, y_n, z) \triangleq f_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, h_{\eta_{i-1}+1}(\tau_i, y_{\eta_i}), y_{\eta_{i-1}+1}, \ldots, y_n, z).
\tag{3.11}
\]

One can easily check that \( \tilde{f}_j, h_{j,i}, A_j - \{\eta_{i-1}\} \) here satisfy Assumptions 2.2 and 3.1. Since (3.10) has at most \( m_1 - 1 \) reflections, by induction (3.10) has a unique solution \((Y^{\delta,j}, Z^{\delta,j}), j \neq \eta_{i-1}\) over \([\tau_i, \tau_{i+1}]\). □

It remains to construct \( Y_{\tau_{i+1}^-}^{\delta,j} \) satisfying (3.7) or (3.9). First, if \( i + 1 = n \), set \( Y_{\tau_{i+1}^-}^{\delta,j} \triangleq Y_{\tau_n}^{\delta,j} \); and if \( \tau_{i+1} = \lambda_2 \), set \( Y_{\tau_{i+1}^-}^{\delta,j} \triangleq \xi_{\lambda_2}^{\delta,j} \). By (3.5) and (2.13) we know both
(3.7) and (3.9) hold true. Now assume \( i < n - 1 \) and \( \tau_{i+1} < \lambda_2 \). Assume we have solved either (3.8) or (3.10) over \([\tau_{i+1}, \tau_{i+2}]\).

**Case 2.** Assume \( A_{\eta_i} = \emptyset \). By Remark 3.4 we know \( i \geq 1, \eta_{i+1} = \eta_{i-1}, \) and \( A_{\eta_i} \neq \emptyset \).

Then we obtain \( Y_{\tau_{i+1}}^{\delta, j} \) from (3.8) over \([\tau_{i+1}, \tau_{i+2}]\) satisfying:

\[
Y_{\tau_{i+1}}^{\delta, j} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, k}), \quad j \neq \eta_{i+1} = \eta_{i-1}.
\] (3.12)

Define

\[
Y_{\tau_{i+1}}^{\delta, j} - \triangleq Y_{\tau_{i+1}}^{\delta, j}, \quad j \neq \eta_i.
\] (3.13)

Then (3.9) follows immediately from (3.12). \( \square \)

**Case 1.** Assume \( A_{\eta_i} \neq \emptyset \). We further discuss two cases.

**Case 1.1.** Assume \( A_{\eta_{i+1}} = \emptyset \). Then we obtain \( Y_{\tau_{i+1}}^{\delta, j} \) from (3.10) over \([\tau_{i+1}, \tau_{i+2}]\) satisfying

\[
Y_{\tau_{i+1}}^{\delta, j} \geq \max_{k \in A_j - \{\eta_i\}} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, k}), \quad j \neq \eta_i.
\] (3.14)

Define

\[
Y_{\tau_{i+1}}^{\delta, j} - \triangleq Y_{\tau_{i+1}}^{\delta, j}, \quad j \neq \eta_i; \quad Y_{\tau_{i+1}}^{\delta, \eta_i} - \triangleq h_{\eta_i, \eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, \eta_{i+1}}).
\] (3.15)

By (3.14), to prove (3.7) it suffices to show that

\[
Y_{\tau_{i+1}}^{\delta, j} \geq h_{j, \eta_i}(\tau_{i+1}, h_{\eta_i, \eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, \eta_{i+1}})), \quad \text{if} \quad \eta_i \in A_j.
\] (3.16)

By (3.1), we have

\[
h_{j, \eta_i}(\tau_{i+1}, h_{\eta_i, \eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, \eta_{i+1}})) < h_{j, \eta_i}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, \eta_{i+1}}).
\]

When \( \eta_i \in A_j \), by Assumption 3.1 (ii), we have \( \eta_{i+1} \in [A_j - \{\eta_i\}] \cup \{j\} \). If \( \eta_{i+1} \in A_j - \{\eta_i\} \), then (3.16) follows (3.14). If \( \eta_{i+1} = j \), then (3.16) follows (2.2). So in both cases (3.16) holds true, then so does (3.7). \( \square \)

**Case 1.2.** Assume \( A_{\eta_{i+1}} \neq \emptyset \). Then we obtain \( Y_{\tau_{i+1}}^{\delta, j} \) from (3.8) over \([\tau_{i+1}, \tau_{i+2}]\) satisfying:

\[
Y_{\tau_{i+1}}^{\delta, j} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, k}), \quad j \neq \eta_{i+1}.
\] (3.17)

Define

\[
Y_{\tau_{i+1}}^{\delta, j} - \triangleq Y_{\tau_{i+1}}^{\delta, j}, \quad j \neq \eta_i, \eta_{i+1};
\]

\[
Y_{\tau_{i+1}}^{\delta, \eta_{i+1}} - \triangleq Y_{\tau_{i+1}}^{\delta, \eta_{i+1}} \lor \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1}, k}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, k});
\]

\[
Y_{\tau_{i+1}}^{\delta, \eta_i} - \triangleq h_{\eta_i, \eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\delta, \eta_{i+1}}).
\] (3.18)
We now check (3.7) for \( j \neq \eta_i \). First, for \( j = \eta_{i+1} \), by (3.18),

\[
Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}} \geq \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1},k}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,k}).
\]

Moreover, if \( \eta_i \in A_{\eta_{i+1}} \), by (3.1) and (2.2) we have

\[
h_{\eta_{i+1},\eta_i}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}) = h_{\eta_{i+1},\eta_i}(\tau_{i+1}, h_{\eta_{i+1},\eta_i}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}})) < Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}.
\]

So (3.7) holds true for \( j = \eta_{i+1} \).

Next, assume \( j \neq \eta_i, \eta_{i+1} \), by (3.17) and the first line in (3.18) we have

\[
Y_{\tau_{i+1}^-}^{\delta,j} \geq \max_{k \in A_j - \{\eta_i, \eta_{i+1}\}} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,k}).
\]

If \( \eta_{i+1} \in A_j \), recall the definition of \( Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}} \) in (3.18). First, by (3.17) we have

\[
h_{j,\eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}) \leq Y_{\tau_{i+1}^-}^{\delta,j} = Y_{\tau_{i+1}^-}^{\delta,j}.
\]

Second, for any \( k \in A_{\eta_{i+1}} - \{\eta_i\} \), similar to (3.16) one can easily prove

\[
h_{j,\eta_{i+1}}(\tau_{i+1}, h_{\eta_{i+1},k}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,k})) \leq Y_{\tau_{i+1}^-}^{\delta,j}.
\]

Thus

\[
h_{j,\eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}) \leq Y_{\tau_{i+1}^-}^{\delta,j}.
\]

Finally, if \( \eta_i \in A_j \), since \( \eta_{i+1} \in A_{\eta_i} \), by Assumption 3.1 (ii) we have \( \eta_{i+1} \in A_j \cup \{j\} \). Then by (3.20) and (2.4) we have

\[
h_{j,\eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}) = h_{j,\eta_{i+1}}(\tau_{i+1}, h_{\eta_{i+1},\eta_i}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}})) < h_{j,\eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}^-}^{\delta,\eta_{i+1}}) \leq Y_{\tau_{i+1}^-}^{\delta,j}.
\]

This, together with (3.19) and (3.20), proves (3.7) for \( j \neq \eta_i, \eta_{i+1} \). □

Now for each \( i \), either (3.8) or (3.10) is well defined. Therefore, over each \([\tau_i, \tau_{i+1})\), either (3.8) or (3.10) is wellposed. By applying Corollary 2.6 and comparison theorem repeatedly, one can easily show that:

**Lemma 3.5** For any admissible strategy \( \delta \) and any \( j \), we have \( Y_{t}^{\delta,j} \leq Y_{t}^{j} \) whenever \( Y_{t}^{\delta,j} \) is well defined.
3.3 Verification Theorem

Moreover, we have:

**Theorem 3.6** For \( j = 1, \ldots, m \), we have \( Y^j_{\lambda_i} = \text{esssup}_\delta Y^{\delta j}_{\lambda_i} \).

**Proof.** Fix \( \varepsilon > 0 \) and let \( D_\varepsilon \triangleq \{i \varepsilon : i = 0, 1, \ldots\} \). We construct an approximately optimal admissible strategy \( \delta \triangleq \delta^\varepsilon \) as follows. First, let \( \tau_0 \triangleq \lambda_1 \) and choose \( \eta_0 \) such that \( A_{\eta_0} \neq \emptyset \). For \( i = 0, 1, \ldots \), we define \((\tau_{i+1}, \eta_{i+1})\) in two cases.

**Case 1.** If \( A_{\eta_i} \neq \emptyset \), set

\[
\tau_{i+1} \triangleq \inf\{t \geq \tau_i : Y^{|\eta_i|}_{t} = \max_{k \in A_{\eta_i}} h_{\eta_i, k}(t, Y^k_t)\} \land \lambda_2.
\]

If \( \tau_{i+1} < \lambda_2 \), set \( \eta_{i+1} \in A_{\eta_i} \) be the smallest index such that

\[
Y^{|\eta_i|}_{\tau_{i+1}} = h_{\eta_i, \eta_{i+1}}(\tau_{i+1}, Y^{|\eta_i|}_{\tau_{i+1}}).
\]

(3.21)

Otherwise choose arbitrary \( \eta_{i+1} \in A_{\eta_i} \).

**Case 2.** If \( A_{\eta_i} = \emptyset \), since \( A_{\eta_0} \neq \emptyset \), we have \( i \geq 1 \). Set \( \eta_{i+1} \triangleq \eta_{i-1} \). If \( \tau_i = \lambda_2 \), define \( \tau_{i+1} \triangleq \lambda_2 \). Now assume \( \tau_i < \lambda_2 \). It is more involved to define \( \tau_{i+1} \) in this case. By the definition of \( \eta_i \), one can check that in this case we must have \( A_{\eta_i-1} \neq \emptyset \), and thus by Case 1, \( \eta_i \in A_{\eta_i-1} \) and

\[
Y^{|\eta_i|}_{\tau_i} = h_{\eta_i, \eta_i-1}(\tau_i, Y^{|\eta_i|}_{\tau_i}).
\]

We claim that, for any \( j \) such that \( \eta_{i-1} \in A_j \),

\[
Y^j_{\tau_i} > h_{j, \eta_{i-1}}(\tau_i, Y^{|\eta_i|}_{\tau_i}).
\]

(3.22)

In fact, if not, by Assumption 3.1 (ii), \( \eta_i \in A_j \cup \{j\} \) and

\[
Y^j_{\tau_i} = h_{j, \eta_{i-1}}(\tau_i, Y^{|\eta_i|}_{\tau_i}) = h_{j, \eta_{i-1}}(\tau_i, h_{\eta_{i-1}, \eta_i}(\tau_i, Y^{|\eta_i|}_{\tau_i})) < h_{j, \eta_i}(\tau_i, Y^{|\eta_i|}_{\tau_i}).
\]

This contradicts with (2.3). We now define

\[
\tau_{i+1} \triangleq \tau_{i+1} \land \lambda_2;
\]

where \( \tau_{i+1}^1 \) is the smallest number in \( D_\varepsilon \) such that \( \tau_{i+1}^1 > \tau_i \); and

\[
\tau_{i+1}^2 \triangleq \inf\{t > \tau_i : \exists j \text{ s.t. } \eta_{i-1} \in A_j, Y^j_t = h_{j, \eta_{i-1}}(t, Y^{|\eta_i|}_{t-1})\}.
\]
We claim that, for a.s. \( \omega \), \( \tau_n = \lambda_2 \) for \( n \) large enough. In fact, if \( \tau_n < \lambda_2 \) for all \( n \), let \( \tau_\infty \triangleq \lim_{n \to \infty} \tau_n \). In Case 1, (3.21) holds true. In Case 2, if \( \tau_{i+1} = \tau_{i+1}^1 \), then \( \tau_{i+1} \in D_\varepsilon \); and if \( \tau_{i+1} = \tau_{i+1}^2 \), then there exists \( \hat{\eta}_{i+1} \) such that \( \eta_{i+1} \in A_{\hat{\eta}_{i+1}} \) and

\[
Y_{\tau_{i+1}}^{\hat{\eta}_{i+1}} = h_{\hat{\eta}_{i+1},\eta_{i+1}}^{\tau_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\eta_{i+1}}).
\]

(3.23)

Since \( \tau_i < \infty \) for all \( i \), there can be only finitely many \( i \) such that \( \tau_{i+1} \in D_\varepsilon \). Therefore, there exists some \( n_0 \) such that for all \( i \geq n_0 \), either (3.21) or (3.23) holds true. The vector \((\hat{\eta}_{i+1}, \eta_{i+1}, \eta_i)\) can take only finitely many values, then there exist \((j_1, j_2, j_3)\) and an infinite sequence of \( \delta_k \) such that \( j_2 \in A_{j_1}, j_3 \in A_{j_2} \) and

\[
\hat{\eta}_{k+1} = j_1, \quad \eta_{k-1} = j_2, \quad \eta_k = j_3, \quad \forall k.
\]

By (3.23) and (3.21) we get

\[
Y_{\tau_{ik}}^{j_1} = h_{j_1,j_2}(\tau_{ik+1}, Y_{\tau_{ik+1}}^{j_2}), \quad Y_{\tau_{ik}}^{j_2} = h_{j_2,j_3}(\tau_{ik}, Y_{\tau_{ik}}^{j_3}), \quad \forall k.
\]

Send \( k \to \infty \), we have

\[
Y_{\tau_{\infty}}^{j_1} = h_{j_1,j_2}(\tau_{\infty}, Y_{\tau_{\infty}}^{j_2}), \quad Y_{\tau_{\infty}}^{j_2} = h_{j_2,j_3}(\tau_{\infty}, Y_{\tau_{\infty}}^{j_3}).
\]

Then, by Assumption 3.1 \((ii)\), \( j_3 \in A_{j_1} \cup \{j_1\} \) and

\[
Y_{\tau_{\infty}}^{j_1} = h_{j_1,j_2}(\tau_{\infty}, h_{j_2,j_3}(\tau_{\infty}, Y_{\tau_{\infty}}^{j_3})) < h_{j_1,j_3}(\tau_{\infty}, Y_{\tau_{\infty}}^{j_3}).
\]

This contradicts with (2.3). Therefore, \( \tau_n = \lambda_2 \) for \( n \) large enough.

We now set \( \delta^{n,\varepsilon} \triangleq (\tau_0, \ldots, \tau_n; \eta_0, \ldots, \eta_n) \). Recall Definition 3.3. One can easily check that \( \delta^{n,\varepsilon} \) is an admissible strategy. Denote

\[
\Delta Y_t^j \triangleq Y_t^j - Y_t^{\delta^{n,\varepsilon,j}}.
\]

If \( i + 1 = n \), it is obvious that

\[
|Y_{\tau_{i+1}}^{j} - Y_{\tau_{i+1}}^{\delta^{n,\varepsilon,j}}| = |\Delta Y_{\tau_{i+1}}^{j}|.
\]

(3.24)

We now assume \( i + 1 < n \).

\textbf{Case 1.} Note that \((Y^j, Z^j, K^j)\) satisfies

\[
\begin{align*}
Y_t^j &= Y_{\tau_{i+1}}^j + \int_{\tau_{i+1}}^{\tau_{i+1}} f_j(s, Y_s, Z_s^j) ds - \int_{\tau_{i+1}}^{\tau_{i+1}} Z_s^j dB_s + K_t^j, \quad j \neq \eta_i; \\
Y_t^j &\geq \max_{k \in A_{j_1}} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k \in A_{j_1}} h_{j,k}(t, Y_t^k)] dK_t^j = 0, \quad j \neq \eta_i; \\
Y_t^{\eta_i} &= Y_{\tau_{i+1}}^{\eta_i} + \int_{\tau_{i+1}}^{\tau_{i+1}} f_{\eta_i}(s, Y_s, Z_s^{\eta_i}) ds - \int_{\tau_{i+1}}^{\tau_{i+1}} Z_s^{\eta_i} dB_s.
\end{align*}
\]

(3.25)
Compare (3.25) and (3.8). By induction we have
\[
\max_{1 \leq j \leq m} |\Delta Y^j_{\tau_i}|^2 \leq E_{\tau_i} \left\{ e^{C(\tau_{i+1} - \tau_i)} \max_{1 \leq j \leq m} |Y^j_{\tau_{i+1}} - Y^j_{\tau_{i+1} - 1}|^2 \right\}. \tag{3.26}
\]
If \(\tau_{i+1} = \lambda_2\), then
\[
|Y^j_{\tau_{i+1}} - Y^j_{\tau_{i+1} - 1}| = |\xi^j_{\lambda_2} - \xi^j_{\lambda_2}| = 0, \quad \forall j.
\tag{3.27}
\]
Assume \(\tau_{i+1} < \lambda_2\). Note that \(Y^{d^{n,\epsilon},j}_{\tau_{i+1}}\) is defined by either (3.15) or (3.18). In the former case, by (3.2) we have
\[
\max_{j \neq \eta_i} |Y^j_{\tau_{i+1}} - Y^{d^{n,\epsilon},j}_{\tau_{i+1} - 1}| = \max_{j \neq \eta_i} |\Delta Y^j_{\tau_{i+1}}|;
\]
\[
|Y^{\eta_i}_{\tau_{i+1}} - Y^{d^{n,\epsilon},\eta_i}_{\tau_{i+1} - 1}| = |h_{\eta_i,\eta_i+1}(\tau_{i+1}, Y^{\eta_i}_{\tau_{i+1}}) - h_{\eta_i,\eta_i+1}(\tau_{i+1}, Y^{d^{n,\epsilon},\eta_i}_{\tau_{i+1} - 1})| \leq |\Delta Y^{\eta_i}_{\tau_{i+1}}|.
\]
Then
\[
\max_{1 \leq j \leq m} |Y^j_{\tau_{i+1}} - Y^{d^{n,\epsilon},j}_{\tau_{i+1} - 1}| \leq \max_{j \neq \eta_i} |\Delta Y^j_{\tau_{i+1}}|. \tag{3.28}
\]
In the latter case, recalling Lemma 3.5 and (3.1), we have
\[
\max_{j \neq \eta_i,\eta_i+1} |Y^j_{\tau_{i+1}} - Y^{d^{n,\epsilon},j}_{\tau_{i+1} - 1}| = \max_{j \neq \eta_i,\eta_i+1} |\Delta Y^j_{\tau_{i+1}}|;
\]
\[
|Y^{\eta_i}_{\tau_{i+1}} - Y^{d^{n,\epsilon},\eta_i}_{\tau_{i+1} - 1}| \leq |\Delta Y^{\eta_i}_{\tau_{i+1}}|;
\]
\[
|Y^{\eta_i}_{\tau_{i+1}} - Y^{d^{n,\epsilon},\eta_i}_{\tau_{i+1} - 1}| = |h_{\eta_i,\eta_i+1}(\tau_{i+1}, Y^{\eta_i}_{\tau_{i+1}}) - h_{\eta_i,\eta_i+1}(\tau_{i+1}, Y^{d^{n,\epsilon},\eta_i}_{\tau_{i+1} - 1})| \leq |\Delta Y^{\eta_i}_{\tau_{i+1}}|.
\]
Thus (3.28) also holds true. Therefore, in all the cases we get
\[
\max_{1 \leq j \leq m} |\Delta Y^j_{\tau_i}|^2 \leq E_{\tau_i} \left\{ e^{C(\tau_{i+1} - \tau_i)} \max_{j \neq \eta_i} |\Delta Y^j_{\tau_{i+1}}|^2 \right\}. \tag{3.29}
\]

**Case 2.** Note that \((Y^j, Z^j, K^j), j \neq \eta_{-1}\) satisfies
\[
\begin{cases}
Y^j_t = Y^j_{\tau_{i+1}} - \int_{\tau_{i+1}}^{\tau_{i+1}} Z^j_s dB_s + K^j_{\tau_{i+1} - 1} - K^j_t + \int_{\tau_{i+1}}^{\tau_{i+1}} \hat{f}_j(s, Y^1_s, \ldots, Y^{\eta_i-1}_s, Y^{\eta_i-1}_s, Y^{m-1}_s, \ldots, Y^m_s, Z^j_s) ds; \\
Y^j_t \geq \max_{k \in A_j - \{\eta_{-1}\}} h_{j,k}(t, Y^k_t); \quad [Y^j_t - \max_{k \in A_j - \{\eta_{-1}\}} h_{j,k}(t, Y^k_t)] dK^k_t = 0;
\end{cases} \tag{3.30}
\]
where
\[
\hat{f}_j(t, y_1, \ldots, y_{\eta_{-1}-1}, y_{\eta_{-1}-1}, \ldots, y_n, z) \triangleq \hat{f}_j(t, y_1, \ldots, y_{\eta_{-1}-1}, y_{\eta_{-1}}, \ldots, y_n, z) + I^j_t;
\]
\[
I^j_t \triangleq \hat{f}_j(t, Y^1_t, \ldots, Y^{\eta_i-1}_t, h_{\eta_{-1},\eta_i}(\tau_t, Y^\eta_t), Y^{\eta_i-1}_t, \ldots, Y^n_t, Z^j_t).
\]
We note that here $I_i^j$ is considered as a random coefficient. Compare (3.30) and (3.10). Recalling (3.13), by induction we get
\[ \text{max}_{j \neq n_{i-1}} |\Delta Y^{j}_{n_{i-1}}|^2 \leq E_{\tau_{i}} \left\{ e^{C(\tau_{i} + 1 - \tau_{i})} \text{max}_{j \neq n_{i-1}} |\Delta Y^{j}_{\tau_{i+1}}|^2 + C \sum_{j \neq n_{i-1}} \int_{\tau_{i}}^{\tau_{i+1}} |I_i^j|dt \right\}. \] (3.33)

Note that $Y^{n_{i-1}}_{\tau_{i}} = h_{n_{i-1}, n_{i}}(\tau_{i}, Y^{n_{i}}_{\tau_{i}})$. Then
\[ |I_i^j| \leq C \left| Y^{n_{i-1}}_{\tau_{i}} - h_{n_{i-1}, n_{i}}(\tau_{i}, Y^{n_{i}}_{\tau_{i}}) \right|^2 \]
\[ \leq C \left[ |Y^{n_{i-1}}_{\tau_{i}} - Y^{n_{i-1}}_{t_i}|^2 + |h_{n_{i-1}, n_{i}}(\tau_{i}, Y^{n_{i}}_{\tau_{i}}) - h_{n_{i-1}, n_{i}}(\tau_{i}, Y^{n_{i}}_{t_i})|^2 \right] \]
\[ \leq C \left[ |Y^{n_{i-1}}_{\tau_{i}} - Y^{n_{i-1}}_{t_i}|^2 + |Y^{n_{i}}_{\tau_{i}} - Y^{n_{i}}_{t_i}|^2 \right] \leq C \sum_{k=1}^{m} |Y^{k}_{t_i} - Y^{k}_{\tau_{i}}|^2. \]

Note that in this case $\tau_{i+1} - \tau_{i} \leq \varepsilon$. Then
\[ |I_i^j| \leq C \sum_{k=1}^{m} \sup_{\lambda_{1} \leq t_1 < t_2 \leq r_{i+1} - t_1 \leq \varepsilon} |Y^{k}_{t_1} - Y^{k}_{t_2}|^2 \triangleq I_{t_i}. \] (3.34)

Thus (3.33) implies
\[ \text{max}_{j \neq n_{i-1}} |\Delta Y^{j}_{\tau_{i}}|^2 \leq E_{\tau_{i}} \left\{ e^{C(\tau_{i} + 1 - \tau_{i})} \text{max}_{j=1, m} |\Delta Y^{j}_{\tau_{i+1}}|^2 + I_{t_i} \right\}. \] (3.35)

Now given $A_{r_{i}} \neq \emptyset$, if $A_{r_{i+1}} = \emptyset$, by (3.29) and (3.35) we have
\[ \text{max}_{1 \leq j \leq m} |\Delta Y^{j}_{\tau_{i}}|^2 \leq E_{\tau_{i}} \left\{ e^{C(\tau_{i} + 2 - \tau_{i})} \text{max}_{1 \leq j \leq m} |\Delta Y^{j}_{\tau_{i+2}}|^2 + I_{t_i} \right\}. \] (3.36)

By Definition 3.3 (v), we have $A_{r_{i+2}} \neq \emptyset$. Therefore, if $A_{r_{i}} \neq \emptyset$, then either $A_{r_{i+1}} \neq \emptyset$ and (3.29) holds true, or $A_{r_{i+2}} \neq \emptyset$ and (3.36) holds true. Since $A_{r_{0}} \neq \emptyset$, one gets immediately that
\[ \text{max}_{1 \leq j \leq m} |\Delta Y^{j}_{r_{0}}|^2 \leq CE_{\tau_{0}} \left\{ \text{max}_{1 \leq j \leq m} |\Delta Y^{j}_{r_{1}}|^2 + I_{\varepsilon} \right\} = CE_{\tau_{1}} \left\{ \text{max}_{1 \leq j \leq m} |Y^{0,j}_{r_{1}} - Y^{j}_{r_{1}}|^2 + I_{\varepsilon} \right\}.
\]

First send $n \to \infty$. Since $\tau_{n} \to \lambda_{2}$, we get
\[ Y^{0,j}_{\tau_{n}} \to \xi_{\lambda_{2}}^{j}, \quad Y^{j}_{\tau_{n}} \to \xi_{\lambda_{2}}^{j}. \]

By Dominating Convergence Theorem we have
\[ \text{max}_{1 \leq j \leq m} |\Delta Y^{j}_{\lambda_{2}}|^2 \leq CE_{\lambda_{2}} \{ I_{\varepsilon} \}. \]

Now send $\varepsilon \to 0$. Since $Y^{j}$ is continuous, by Dominating Convergence Theorem again we get
\[ \lim_{n \to \infty} E_{\lambda_{2}} \{ I_{\varepsilon} \} = 0. \]

This proves the theorem. \[ \blacksquare \]
3.4 Proof of Theorem 3.2

As mentioned before, we prove the theorem by induction. Assume Theorem 3.2 holds true for $\mu = m_1 - 1$. Now assume $\mu = m_1$.

(i) By Theorem 3.6, $Y^j_{\lambda_1}$ is unique. Similarly $Y^j_t$ is unique for any $t \in [\lambda_1, \lambda_2]$. By the uniqueness of the Doob-Meyer decomposition we get $Z^j$ is unique, which further implies the uniqueness of $K^j$ immediately.

(ii) For any admissible strategy $\delta$, define $\tilde{Y}^\delta j$ similarly and denote

$$\Delta Y^\delta j_t \triangleq Y^\delta j_t - \tilde{Y}^\delta j_t.$$ 

If $A_{\eta_1} \neq \emptyset$, recalling (3.8), (3.15), and (3.18), by induction we have:

$$\max_{1 \leq j \leq m} |\Delta Y^\delta j_{\tau_i}|^2 \leq E_{\tau_i}\left\{ e^{C(\tau_{i+1} - \tau_i)} \max_{j \neq \eta_i} |\Delta Y^\delta j_{\tau_{i+1}}|^2 + C \int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt \right\}.$$

If $A_{\eta_1} = \emptyset$, recalling (3.10) and (3.13), by induction we have:

$$\max_{j \neq \eta_{i-1}} |\Delta Y^\delta j_{\tau_i}|^2 \leq E_{\tau_i}\left\{ e^{C(\tau_{i+1} - \tau_i)} \max_{1 \leq j \leq m} |\Delta Y^\delta j_{\tau_{i+1}}|^2 + C \int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt \right\}.$$

Put together and note that $A_{\eta_0} \neq \emptyset$, we get:

$$\max_{1 \leq j \leq m} |\Delta Y^\delta j_{\lambda_1}|^2 \leq E_{\lambda_1}\left\{ e^{C(\lambda_2 - \lambda_1)} \max_{1 \leq j \leq m} |\Delta \xi^j_{\lambda_2}|^2 + C \int_{\lambda_1}^{\lambda_2} \|\Delta f_t\|^2 dt \right\}.$$

Then (ii) follows from Theorem 3.6 immediately.

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