The Effect of Weak Magnetic Twist on Resonant Absorption of Slow Sausage Waves in Magnetic Flux Tubes

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Abstract

Observations show that twisted magnetic flux tubes are present throughout the Sun’s atmosphere. The main aim of this work is to obtain the damping rate of sausage modes in the presence of weak magnetic twist. Using the connection formulae obtained by Sakurai et al., we investigate resonant absorption of the sausage modes in the slow continuum under photospheric conditions. We derive the dispersion relation and solve it numerically, and consequently obtain the frequencies and damping rates of the slow surface sausage modes. We conclude that the magnetic twist can result in strong damping in comparison with the untwisted case.

Key words: magnetohydrodynamics (MHD) – Sun: activity – Sun: oscillations – Sun: photosphere

1. Introduction

One of the important questions regarding the Sun is how the solar corona reaches temperatures exceeding 1 MK. There are different theories trying to answer this problem. One of these is the propagation of magnetohydrodynamic (MHD) waves and their damping by the resonant absorption, proposed for the first time by Ionson (1978). At the observational point of view, there are some evidences for the propagation and damping of MHD waves (see, e.g., Yu et al. 2017a). Based on the propagating speed of MHD waves, they are classified into fast, slow, and Alfvén waves. Here, we focus on the slow sausage MHD waves, which have recently been observed by Dunn Solar Telescope (Grant et al. 2015).

Dorotovič et al. (2008) observed linear slow sausage waves in a magnetic pore of period 20 to 70 minutes with photospheric pore size using the Swedish Solar Telescope. Also, Morton et al. (2011) observed sausage modes in magnetic pores with the period ranging from as short as 30 s up to 450 s. Grant et al. (2015) reported the observational period of the sausage mode as 181–412 s. Moreover, Keys et al. (2018) observationally found that in solar magnetic pores, the number of surface modes is more than that of body modes. They also pointed out that surface modes appear to carry more energy compared to body modes. The slow waves are usually generated by the motions of sunspots, magnetic pores, and granules, and they play a significant role in heating the lower part of the Sun’s atmosphere (Yu et al. 2017a). Yu et al. (2017b) showed that resonant absorption plays an important role in the wave damping for slow surface sausage modes in the slow continuum.

Furthermore, there is ample evidence for the existence of a magnetic twist at and below the solar atmosphere. For instance, it has been suggested that magnetic flux tubes are twisted while rising through the convection zone (e.g., Murray & Hood 2008; Hood et al. 2009; Luoni et al. 2011). Also, observations show that magnetic twist is one of the important mechanisms in energy transport from the photosphere to the corona; see Wedemeyer-Böhm et al. (2012).

So far, many studies have been done about the effect of twisted magnetic fields on the kink and sausage MHD waves in magnetic flux tubes. For instance, connection formulae (or conservation laws) were first used for studying surface waves in cylindrical plasmas in the presence of magnetic twist by Goossens et al. (1992). Bennett et al. (1999) studied the sausage modes in a magnetic flux tube containing a uniform magnetic twist. They pointed out that in the presence of magnetic twist, an infinite set of body modes is generated.

Erdélyi & Carter (2006) considered magnetic twist just in the annulus for surface and hybrid modes. They found that when the magnetic twist increases, the hybrid modes include a wide range of phase speeds for the sausage modes.

Erdélyi & Fedun (2006) investigated the propagation of MHD waves in an incompressible twisted magnetic flux tube. They showed that an increase in the twist of the magnetic field from 0 to 0.3 could lead to an increase of 1%–2% in the period of the sausage waves. Erdélyi & Fedun (2007) extended their previous work, Erdélyi & Fedun (2006), to include the compressibility condition and concluded that the period of the sausage waves increases by 3%–5%.

Karami & Bahari (2010) considered the effect of a twisted magnetic field on the resonant absorption of MHD waves in coronal loops. They showed that the period ratio \( P_1/P_2 \) of the fundamental and its first-overtone surface waves for the kink (\( m = 1 \)) and flute (\( m = 2, 3 \)) modes is lower than 2 in the presence of a twisted magnetic field. Ebrahim & Karami (2016) investigated the resonant absorption of kink MHD waves by a magnetic twist in coronal loops. They concluded that the resonant absorption by the magnetic twist can justify the rapid damping of kink MHD waves observed in coronal loops.

Giagkosios et al. (2016) studied the resonant absorption of axisymmetric modes (\( m = 0 \)) in a twisted magnetic flux tube. They showed that in the presence of a twisted magnetic field, both the longitudinal magnetic field and the density have crucial roles in the wave damping. Giagkosios et al. (2015) also elaborated that the magnetic twist can remove the cutoff of fast body sausage modes. Aside from the works mentioned above, there are some further studies in the literature on the effect of magnetic twist on MHD waves; see, e.g., Erdélyi & Fedun (2010), Carter & Erdélyi (2007, 2008), Ruderman (2007, 2015), Karami & Barin (2009), Terradas & Goossens (2012), and Karami & Bahari (2012).
In the present work, our main goal is to study the effect of magnetic twist on the resonant absorption of the slow surface sausage modes in the Sun’s atmosphere. To do so, we apply the magnetic twist to the model of Yu et al. (2017b). To achieve this aim, in Section 2 we introduce the model and solve the equations of motion governing the slow surface sausage modes. In Section 3, we obtain the dispersion relation under magnetic pole conditions. In Section 4, using the connection formulae, we derive the damping rate for the slow surface waves. The numerical results are shown in Section 5. Finally, we conclude the paper in Section 6.

2. Equations of Motion and Model

The linearized ideal MHD equations are as follows (Kadomtsev 1966):

\[
\frac{\partial^2 \xi}{\partial t^2} = -\nabla p - \frac{1}{\mu_0} (\delta B \times (\nabla \times B) + B \times (\nabla \times \delta B)),
\]

(1a)

\[\delta p = -\xi \cdot \nabla p - \gamma p \nabla \cdot \xi,
\]

(1b)

\[\delta B = -\nabla \times (B \times \xi),
\]

(1c)

where \(\rho, p,\) and \(B\) are the background density, kinetic pressure, and magnetic field, respectively. Also, \(\xi\) is the Lagrangian displacement vector, and \(\delta p\) and \(\delta B\) are the Eulerian perturbations of the pressure and magnetic field, respectively. Here, \(\gamma\) is the ratio of specific heats (taken to be 5/3 in this work), and \(\mu_0\) is the permeability of free space.

In our model, we consider a magnetic field as follows:

\[B = (0, B_\phi(r), B_z(r)).\]

(2)

Now, the plasma pressure and magnetic field must satisfy the following magneto-hydrostatic equation in the \(r\)-direction as

\[
dr \left( \frac{p + B_\phi^2 + B_z^2}{2\mu_0} \right) + \frac{B_z^2}{\mu_0 r} = 0.
\]

(3)

Here, following Yu et al. (2017b), we consider the following profiles for the density and \(z\) component of the background magnetic field as

\[
\rho(r) = \begin{cases} p_i, & r \leq r_i, \\ p_i + (\rho_e - p_i) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\ \rho_e, & r \geq r_e, \end{cases}
\]

(4)

\[
B_z^2(r) = \begin{cases} B_{z1}^2, & r \leq r_i, \\ B_{z1}^2 + (B_{z\text{e}r}^2 - B_{z\text{e}i}^2) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\ B_{z\text{e}r}^2, & r \geq r_e, \end{cases}
\]

(5)

where \(p_i\) and \(\rho_e\) are the constant densities of the interior and exterior regions of the flux tube, respectively. Also, \(B_{z1}\) and \(B_{z\text{e}r}\) are the interior and exterior constant longitudinal magnetic fields, respectively. We further assume that the \(\phi\) component of the background magnetic field takes the form

\[
B_\phi^2(r) = \begin{cases} \frac{B_{\phi1}^2 r_i^2}{r_i^2}, & r \leq r_i, \\ \frac{B_{\phi1}^2 r_i^2 + (B_{\phi\text{e}r}^2 - B_{\phi\text{e}i}^2) (r - r_i)}{r_e - r_i}, & r_i < r < r_e, \\ B_{\phi\text{e}r}^2, & r \geq r_e, \end{cases}
\]

(6)

where \(B_{\phi1}\) and \(B_{\phi\text{e}r}\) are constant. Putting Equations (5) and (6) into the magnetohydrostatic Equation (3), we obtain the background gas pressure as follows:

\[
p(r) = \begin{cases} p_i - \frac{B_{\phi1}^2 (r/r_i)^2}{\mu_0}, & r \leq r_i, \\ A_1 + A_2 r + A_3 \ln(r/r_i), & r_i < r < r_e, \\ p_e - \frac{B_{sz}^2}{\mu_0} \ln(r/r_e), & r \geq r_e, \end{cases}
\]

(7)

where

\[
A_1 = \left( p_i - \frac{B_{\phi1}^2}{\mu_0} \right) - A_2 r_i, \\
A_2 = \frac{3(B_{\phi1}^2 - B_{\phi\text{e}i}^2) + (B_{\phi1}^2 - B_{\phi\text{e}r}^2)}{2\mu_0 (r_e - r_i)}, \\
A_3 = \frac{r_i B_{\phi1}^2 - r_e B_{\phi\text{e}i}^2}{\mu_0 (r_e - r_i)}, \\
p_e = \left( p_i - \frac{B_{\phi1}^2}{\mu_0} \right) + A_2 (r_e - r_i) + A_3 \ln(r_e/r_i),
\]

(8)

and \(p_i\) is an arbitrary constant. Also, the constants \(A_1\) and \(p_e\) have been obtained from the continuity of the gas pressure across the boundaries \(r = r_i\) and \(r = r_e\). Note that Equation (7), in the absence of twist, i.e., \(B_{\phi1} = B_{\phi\text{e}i} = 0\), recovers the gas pressure profile given by Yu et al. (2017b).

In addition, we define the following quantities:

\[
v_{z1}^2 = \frac{B_{z1}^2 r_i}{\mu_0 \rho_i}, \\
\gamma_{z1} = \frac{B_{z1}^2 (r_e)}{\mu_0 \rho_e},
\]

(9)

\[
\gamma_{\phi1} = \frac{p(r_i)}{\rho_i} = \frac{p_i - \frac{B_{\phi1}^2}{\mu_0}}{\rho_i},
\]

(10)

\[
v_{\phi1}^2 = \gamma_{\phi1} \rho_i = \gamma_{\phi1} p_i / \rho_i,
\]

(11)

\[
v_{z\text{e}r}^2 = \gamma_{\phi1} \rho_e = \gamma_{\phi1} p_e / \rho_e,
\]

(12)

\[
v_{z\text{e}i}^2 = \gamma_{\phi1} \rho_e = \gamma_{\phi1} p_e / \rho_e,
\]

(13)

where \(B^2 = B_{\phi1}^2 + B_{z1}^2\). Also, \(v_{\phi1}(r_i)\), \(v_{\phi1}(r_e)\), and \(v_{z\text{e}i}(r_e)\) are the interior/exterior Alfvén, sound, and cusp velocities, respectively. Furthermore, we define the parameter \(\beta\) as the ratio of the plasma pressure to the magnetic field pressure inside the
flux tube as
\[ \beta \equiv \frac{\rho(r_i)}{B^2(r_i)/(2\mu_0)} = 2\gamma \frac{\bar{B}}{\bar{v}_e} \frac{\rho_i}{\rho_i}. \]
With the help of parameter \( \beta \), Equation (14), the arbitrary constant \( p_i \) can be determined as
\[ \frac{p_i}{B_i^2/(2\mu_0)} = \beta + 2 \left( \frac{B_{0i}}{B_i} \right)^2, \]
where \( B_i = B(r_i) \). Using Equations (11), (12), and (14), one can find the density ratio \( \rho_e/\rho_i \) to be
\[ \frac{\rho_e}{\rho_i} = \left( \frac{\bar{v}_i}{\bar{v}_e} \right)^2 \frac{B_i}{B_i}, \]
\[ = \frac{2}{\beta} \frac{(v_i/v_e)}{\left( (p_i - B_i^2)/\mu_0) \right)} \frac{B_i}{B_i} \]
\[ + \left( 1 - \frac{r_e \ln(r_e/r_i)}{r_e - r_i} \right) \frac{B_i^2}{B_i} \left( 1 - \frac{r_e \ln(r_e/r_i)}{r_e - r_i} \right) \frac{B_i^2}{B_i}, \]
where \( \bar{B} = B/B_i \) and \( \bar{B}_i = 1 \). Figure 1 shows the background quantities containing the density, Equation (4); the magnetic field components, Equations (5) and (6); and the plasma pressure, Equation (7), for the twist parameter \( B_{0e}/B_0 = 0.3 \) under magnetic pore conditions.

The set of Equations 1(a)–(c) can be solved by Fourier decomposition of the perturbed quantities as follows
\[ (\xi, \delta p) \propto \exp(\imath(m\phi + k_z z - \omega t)), \]
where \( \omega \) is the angular frequency, \( m \) is the azimuthal wavenumber for which only integer values are allowed, and \( k_z \) is the longitudinal wavenumber in the \( z \) direction. Also, \( \delta p = \delta p + B \delta B/\mu_0 \) is the Eulerian perturbation of the total (gas and magnetic) pressure. Putting Equation (17) into Equations (1a)–(1c), we obtain the two coupled first-order differential equations,
\[ \frac{D}{dr} d\xi = C_1(r \xi) - r C_2 \delta p_r, \]
\[ \frac{D}{dr} d\delta p_r = \frac{1}{r} C_3(r \xi) - C_1 \delta p_r. \]

The above equations were derived earlier by Appert et al. (1974), and later on by Hain & Lust (1958), Goedbloed (1971), and Sakurai et al. (1991). Here, the multiplicative factors are defined as
\[ D \equiv \rho(\omega^2 - \omega^2_\lambda) C_4, \]
\[ C_1 \equiv \frac{2B_\phi}{\mu_0 r^2} \left( \omega^2 B_\phi + \frac{m^2}{r^2} F_\phi C_4 \right), \]
\[ C_2 \equiv \omega^2 - \left( \bar{k}_z^2 + \frac{m^2}{r^2} \right) C_4, \]
\[ C_3 \equiv \frac{\mu_0 r^2}{\rho_c} \left( \omega^2 - \omega^2_\lambda \right) + \frac{4B_\phi^2 \mu_0 r^2}{\rho_c} - \omega^2 B_\phi C_4, \]
\[ C_4 \equiv \left( \frac{v_i^2 + v_e^2}{\omega^2 - \omega^2_\lambda} \right) C_4, \]

where
\[ f_B = \frac{m}{r} B_\phi + k_z B_z, \]
\[ \omega^2_\lambda = \frac{f_B^2}{\mu_0 \rho_c}, \]
\[ \omega^2 = \frac{v_i^2 + v_e^2}{\omega^2_\lambda} \omega^2_\lambda, \]
and
\[ n^2 = \frac{\omega^4}{\left( v_i^2 + v_e^2 \right) \left( \omega^2 - \omega^2_\lambda \right)}. \]

Here, \( \omega_\lambda = k_z v_\lambda \) is the Alfvén angular frequency and \( \omega_i = k_z v_i \) is the cusp angular frequency. \( \omega_i \) and \( \omega_\lambda \) are defined as \( \omega_i = |B|/\sqrt{\mu_0 \rho} \) is the Alfvén speed, \( v_i = \sqrt{\rho^2/\mu_0} \) is the sound speed, and \( v_e = \frac{v_i}{v_e} \) is the flow velocity.

Combining Equations (18a) and (18b), one can obtain a second-order ordinary differential equation for the radial component of the Lagrangian displacement \( \xi \) as in Erdélyi & Fedun (2007) and Giagkiozis et al. (2016):
\[ \frac{d}{dr} \left[ \frac{D}{dr} \right] (r \xi) + r \left( C_3 - \frac{C_1^2}{C_2} \right) - \frac{C_1}{r} \frac{d}{dC_2} \left[ \frac{C_1}{C_2} \right] \xi = 0. \]

For the sausage modes \( m = 0 \), the solutions of Equation (21) in the interior \( (r \ll r_i) \) and exterior \( (r \gg r_e) \) regions are given...
by Giagkiozis et al. (2015):
\[
\xi_n(s) = A_i \frac{s^{1/2}}{E^{1/2}} e^{-s/2} M(a, b; s), \quad (22a)
\]
\[
\xi_{re}(r) = A_e K_r(k_re), \quad (22b)
\]
where \(A_i\) and \(A_e\) are constant. Also, \(M(\cdot)\) is the Kummer function, and \(K(\cdot)\) is the modified Bessel function of the second kind (Abramowitz & Stegun 2012). Using solutions (22a) and (22b) in Equation 18(b) gives
\[
\delta p_n(s) = A_i e^{-s/2} \left( \frac{k_o D_i}{n_i - k_z^2} \right) \left( \frac{n_i + k_z}{k_z} \right) s M(a, b; s) - 2M(a, b - 1; s), \quad (23a)
\]
\[
\delta p_{re}(r) = A_e \left( \frac{\mu_0 (1 - \nu) D_e - 2B_{re}^2 n_z^2}{\mu_0 r^2 (k_z^2 - n_z^2)} \right) K_r(k_re r) - \frac{D_e}{k_re} K_{r-1}(k_re r). \quad (23b)
\]
The parameters appearing in Equations 22(a)–23(b) are defined as
\[
s \equiv k_z^2 E^{1/2} r^2, \quad E \equiv \frac{4B_{re}^4 n_z^2}{\mu_0^2 k_z^2 D_e^2 r^2 (1 - \alpha^2)^2}, \quad (24)
\]
\[
k_a \equiv k_z (1 - \alpha^2)^{1/2}, \quad \alpha^2 \equiv \frac{4B_{re}^2 \omega_{Ai}}{\mu_0 r^2 \rho_i (\omega^2 - \omega_{Ai}^2)^2}. \quad (25)
\]
\[
a \equiv 1 + \frac{k_a^2}{4k_z^2 E^{1/2}}, \quad b = 2, \quad (26)
\]
\[
k_e \equiv \frac{\omega_A^2 - \omega^2}{(\nu_A^2 + \nu_e^2) (\omega^2 - \omega_A^2)}, \quad (27)
\]
\[
D_i \equiv \rho_i (\omega^2 - \omega_{Ai}^2), \quad D_e \equiv \rho_e (\omega^2 - \omega_{Ae}^2), \quad (28)
\]
\[
\nu^2(0, r) \equiv 1 + 2B_{re}^2 \left( \frac{2B_{re}^2 n_z^2 k_z^2 + \mu_0 \rho_e \omega_{Ae}^3 (3n_z^2 - k_z^2) - \omega^2 (n_z^2 + k_z^2))}{\mu_0 D_e^2} \right). \quad (29)
\]
Note that in the annulus region \((r_i < r < r_e)\), we do not solve the MHD equations. Instead, we relate the interior solutions to exterior ones by using the connection formula introduced in Section 4.

3. Dispersion Relation for the Case with No Inhomogeneous Layer

Here, we are interested in obtaining the dispersion relation for the sausage mode in the case with no annulus region. Solutions 22(a)–23(b) for inside and outside the flux tube must satisfy the following boundary conditions:
\[
\xi_n \mid_{r=R} = \xi_{re} \mid_{r=R}. \quad (30a)
\]
\[
\left( \frac{\delta p_n - B_{re}^2}{\mu_0 r} \xi_n \right) \mid_{r=R} = \left( \frac{\delta p_{re} - B_{re}^2}{\mu_0 r} \xi_{re} \right) \mid_{r=R}. \quad (30b)
\]
where \(R = r_i = r_e\) is the tube radius. The above relations show the continuity conditions for the Lagrangian displacement and Lagrangian changes of the total pressure across the tube boundary, respectively. Inserting solutions (22a)–(23b) into the boundary conditions, Equations (30a) and (30b), after some algebra one can find the following dispersion relation:
\[
- \frac{\mu_0 D_i}{k_i^2} \left( n_i + k_z \right) s - 2M(a, b - 1, s) \quad (31)
\]
\[
= \frac{\mu_0 D_e}{k_{re}^2} \left( 1 - \nu - k_{re} R K_{r-1}(k_{re} R) \right) - \left( 1 + 2k_{re}^2 \right) B_{re}^2 + B_{re}^2, \quad (31)
\]
where \(s \equiv k_z^2 E^{1/2} R^2\).

Now, we are interested in investigating the dispersion relation (31) in the limit of no twist inside and outside the tube, i.e., \(B_{ri} = B_{re} = 0\). For the small twist, from Equation (24) we have \(E \ll 1\) and then from the first relation of Equation (26), we get \(E^{1/2} \approx k_z^4 / (4\alpha^2 k_z^2)\). Also from Equation (25), we obtain \(\alpha^2 \ll 1\) and then \(k_a \approx k_e\). Consequently, from the first relation of Equation (24), we find \(s \approx k_z^2 R^2 / (4a)\). Using these approximations for the small twist, the Kummer functions appearing in the dispersion relation (31) behave as
\[
\lim_{a \to \infty} M(a, b - 1, s) = \lim_{a \to \infty} M \left( a, 1, \frac{k_z^2 R^2}{4a} \right) = \Gamma(1) I_0 \left( \frac{k_z^2 R^2}{4} \right) = I_0(k_{re} R), \quad (32)
\]
\[
\lim_{a \to \infty} M(a, b, s) = \lim_{a \to \infty} \left( a, 2, \frac{k_z^2 R^2}{4a} \right) = \Gamma(2) \left( \frac{k_z^2 R^2}{4 \alpha^2} \right)^{-1/2} I_0 \left( \frac{k_z^2 R^2}{4} \right) = \frac{2}{k_{re} R} I_0(k_{re} R), \quad (33)
\]
where we have used the following relation (Abramowitz & Stegun 2012):
\[
\lim_{a \to \infty} M(a, b, z/a) = \Gamma(b) z^{(1-b)/2} I_{b-1}(2\sqrt{z}), \quad (34)
\]
in which \(\Gamma(b)\) is the Gamma function and \(I(.)\) is the modified Bessel function of the first kind.

Now, putting Equations (32) and (33) into the dispersion relation (31) and using \(B_{ri} \to 0, B_{re} \to 0, s \to 0\), and \(\nu \to 1\) (see Equation (29)), one can find
\[
\frac{D_i}{k_i^2} I_0(k_{re} R) = - \frac{D_e}{k_{re}^2} K_0(k_{re} R) \quad (35)
\]
Using \(D_i\) and \(D_e\) from Equation (28) into the above relation, we get
\[
\rho_i (\omega^2 - \omega_{Ai}^2) + \frac{k_{ai}}{k_{re}} \rho_e (\omega^2 - \omega_{Ae}^2) Q_0 = 0, \quad (36)
\]
singularity occurs, where the phase speed of the slow surface sausage (sss) mode lies in the range of \( v_{ce} < v_{sas} < v_{ci} \).

Following Sakurai et al. (1991), in the resonant layer where the singularity occurs because of the magnetic twist, one does not need to solve Equation (21). Sakurai et al. (1991) showed that in the thin boundary approximation, the solutions inside and outside of the flux tube are related to each other via the following connection formula:

\[
[\xi_r] \equiv \xi_{rc}(r_c) - \xi_{ri}(r_i) = -i \pi \frac{\mu_0 \omega_A^2}{|D_r| B^2 \omega_A^2} \left[ \begin{array}{c} \delta p_{ri} - 2 B_0^2 \xi_{ri} \\ \mu_0 \rho \end{array} \right] \bigg|_{r=r_i} \text{,} \tag{38a}
\]

\[
[\delta p_r] \equiv \delta p_{rc}(r_c) - \delta p_{ri}(r_i) = -i 2 \pi \frac{\omega_A^2}{|D_r| B^2 \omega_A^2} \left[ \begin{array}{c} \delta p_{ri} - 2 B_0^2 \xi_{ri} \\ \mu_0 \rho \end{array} \right] \bigg|_{r=r_i} \text{,} \tag{38b}
\]

where \([\xi_r]\) and \([\delta p_r]\) are the jump conditions in the Lagrangian radial displacement and total pressure perturbation, respectively, across the inhomogeneous (resonant) layer. The subscript \(c\) denotes the position of the slow resonance \((r = r_c)\) and \(|D_r| \equiv \left| \frac{d \omega - \omega_A^2}{dr} \right| \bigg|_{r=r_c} \). Note that we will determine the cusp resonance point \(r_c\) later; see Equation (52).

Substituting solutions (22a)–(23b) into the connection formula, Equations (38a) and (38b), one can find the dispersion relation governing the sss modes in the presence of magnetic twist to be

\[
D_r \left[ n_i + k_z \mathcal{S} - 2 \frac{M(a, b - 1; s)}{M(a, b; s)} \right] + \frac{\mu_0 (1 - \nu) D_e - 2 B_0^2 \nu e_n^2}{\mu_0 \nu e_k^2} - \frac{D_e}{K_r} \left( K_r^{-1}(k_r e_r) \right) \text{,}
\]

\[
+ i \pi \mu_0 \omega_A^2 \left| \frac{1}{|D_r| B^2 \omega_A^2} \right|_{r=r_c} \bigg[ \frac{D_r}{k_r} \left[ n_i + k_z \mathcal{S} - 2 \frac{M(a, b - 1; s)}{M(a, b; s)} \right] + \frac{B_0^2}{\mu_0} \left( \frac{2 B_0^2}{\mu_0 \nu e_k^2} \right) - \frac{D_e}{K_r} \left( K_r^{-1}(k_r e_r) \right) \bigg] = 0. \tag{39}
\]

For the case where there is no twist, i.e., \(B_0 = B_{\phi e} = 0\), the dispersion relation (39) following the same approach that was used in the previous section takes the form

\[
\rho_k (\omega^2 - \omega_A^2) - \rho_k (\omega^2 - \omega_A^2) k_e^2 Q_0 
+ i \pi \frac{k_e^2}{\rho |D_r|} \left( \frac{\nu e}{\nu e + \nu_A} \right)^2 \rho_k (\omega^2 - \omega_A^2) (\omega^2 - \omega_A^2) G_0 k_e^2 = 0, \tag{40}
\]

where \(G_0 = \frac{K_i(k \omega_A)}{K_i(k \omega_A)} \). The above relation is the same as that obtained by Yu et al. (2017b) in the absence of magnetic twist.

Using Equations (4)–(12), one can obtain the quantities \(v_i = \sqrt{\gamma_p / \rho}, \; v_A = |B_i| / \sqrt{\mu_0 \rho}\), and the cusp velocity
Figure 3. Variations of the dimensionless velocity $v/v_{ci}$, Equations (41)–(43), as a function of $\delta \equiv (r_c - r)/(r_c - r_i)$ in the nonuniform (transitional) layer for the twist parameter $B_{si}/B_{ci} = 0.3$ under magnetic pore conditions. The auxiliary parameters are $v_{Ai} = 12 \text{ km s}^{-1}$, $v_{ci} = 0 \text{ km s}^{-1}$ (i.e., $B_{si} = B_{ci} = 0$), $v_{si} = 7 \text{ km s}^{-1}$, $v_{ci} = 11.5 \text{ km s}^{-1}$, $v_{ce} = 6.0464 \text{ km s}^{-1}$ ($\approx 0.8638 v_{so}$), and $v_{ce} = 0 \text{ km s}^{-1}$ (Grant et al. 2015).

$v_c \equiv \frac{v_{so}}{(v_c^2 + v_{so}^2)^{1/2}}$ in the inhomogeneous layer ($r_i < r < r_c$) as

$$v_c^2 = v_{Ai}^2 \left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right] / \left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right]$$

(41)

$$v_A^2 = v_{Ai}^2 \left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right] / \left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right]$$

(42)

$$v_s^2 = \frac{v_{si}^2 v_{Ai}^2}{\left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right]}$$

$$\times \frac{\left[ 1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta \right]}{v_{si}^2(1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta)} v_{Ai}^2(1 + \delta \left( \chi v_{Ai}^2 - 1 \right) + \zeta) \right]$$

(43)

where $\delta \equiv \frac{r_c - r}{r_c - r_i}$, $\chi \equiv \rho_c / \rho_i$, $v_{si} \equiv v_{si}/v_{ci}$, $v_{Ai} \equiv v_{Ai}/v_{ci}$, and

$$\zeta \equiv \gamma v_{Ai}^2 \left[ \frac{r_c B_{so}^2 - r_i B_{si}^2}{(r_c - r_i) B_{ci}^2} \right] (\ln(r_c/r_i) - \delta \ln(r_c/r_i))$$

(44)

Notice that in the absence of twist, i.e., $B_{si} = B_{so} = 0$, Equations (41)–(43) transform to the corresponding relations in Yu et al. (2017b).

In Figure 3, using Equations (41)–(43) we plot the sound, Alfvén, and cusp velocities for the twist parameters $B_{si}/B_{ci} = 0.3$ under magnetic pore conditions. Figure 3 shows that for $v_c < v_{ci}$ and $v_{ci} < v_c < v_{max}$ respectively, the surface and body sausage modes can resonantly damp in the slow continuum. Here, $v_{max}$ is the maximum value of the cusp velocity.

Note that according to Yu et al. (2017b), the position of the cusp resonance point $r_c$ is obtained by setting $\omega^2 = \omega_c^2 \left| r = r_c \right.= \chi^2 v_{so}^2 \left| r = r_c \right.$ in Equation (43). Consequently, the resulting equation in terms of the variable $\delta_c \equiv \delta_{r = r_c} = (r_c - r)/(r_c - r_i)$ yields the following second-order equation:

$$A \delta_c^2 + B \delta_c + C = 0$$

(45)

with

$$A \equiv 1 - \frac{v_c^2}{v_{ci}^2} - \frac{\gamma v_{Ai}^2}{v_{ci}^2} \left( \frac{r_c B_{so}^2 - r_i B_{si}^2}{(r_c - r_i) B_{ci}^2} \right) \left( 1 - \frac{v_c^2}{v_{ci}^2} \right)$$

$$\times \left[ \frac{1}{2} \left( \frac{r_c - r_i}{r_i} \right)^2 + \frac{r_c - r_i}{r_i} + \ln(r_c/r_i) \right]$$

$$+ \chi \left[ 2 v_{si}^2 v_{Ai}^2 \left( v_{si}^2 + v_{Ai}^2 \right) - \gamma v_{Ai}^2 \left( r_c B_{so}^2 - r_i B_{si}^2 \right) \right]$$

$$\times \left( \frac{v_{Ai}^2}{v_{ci}^2} \left( \frac{r_c - r_i}{r_i} - \ln(r_c/r_i) \right) \right)$$

$$- \chi^2 \frac{v_{si}^2 v_{Ai}^2}{v_{ci}^2} \left( v_{si}^2 + v_{Ai}^2 \right) \right)$$

(46)

$$B \equiv 2 \left( \frac{v_c^2}{v_{ci}^2} - 1 \right) + \frac{\gamma v_{Ai}^2}{v_{ci}^2} \left( r_c B_{so}^2 - r_i B_{si}^2 \right) \left( r_c - r_i \right) B_{ci}^2 \right)$$

$$\times \left[ 1 - \frac{v_{si}^2}{v_{ci}^2} \left( \frac{r_c - r_i}{r_i} \right) + \ln(r_c/r_i) \right]$$

$$- \chi \left[ \frac{v_{si}^2}{v_{ci}^2} \left( 1 + \frac{v_{so}^2 + v_{Ai}^2}{v_{so}^2 + v_{Ai}^2} \right) - (v_{si}^2 + v_{Ai}^2) \right]$$

(47)

$$C \equiv 1 - \frac{v_{so}^2}{v_{ci}^2}$$

(48)

where we have used the following approximation:

$$\ln(r_c/r_i) = \frac{1 + \left( \frac{r_c - r_i}{r_i} \right)^2 \delta_c}{\left( \frac{r_c - r_i}{r_i} \right)^2 \delta_c}$$

(49)

for $\left( \frac{r_c - r_i}{r_i} \right) \delta_c < 1$. We checked that keeping the higher order terms $O(\delta_c^2)$ does not affect the results. Equation (45) has two roots for $\delta_c$:

$$\delta_{c1} = - \frac{B}{2A} + \sqrt{\frac{B^2 - 4AC}{2A}}, \quad 0 < \delta_{c1} \leq \delta_{m},$$

(50)

$$\delta_{c2} = - \frac{B}{2A} - \sqrt{\frac{B^2 - 4AC}{2A}}, \quad \delta_{m} \leq \delta_{c2} < 1,$$

(51)

where $\delta_{m}$ is the value of $\delta$ when $v_c = v_{max}$. For instance, in Figure 3 for $B_{si}/B_{ci} = 0.3$, we have $v_{max} = 0.93 v_{so}$ and $\delta_{m} = 0.27$.

Under magnetic pore conditions, for the slow surface mode, we have only one root denoted by $\delta_{c1}$. Consequently, the cusp resonance position $r_c$ reads

$$r_c = r \left[ \left( \frac{r_c - r_i}{r_i} \right)^2 \delta_{c1} + 1 \right]$$

(52)

Next, we calculate the parameter $\Delta_1$ appearing in the dispersion relation (39). To this aim, using Equation (43) and
\( \omega_c^2(r_c) = k_z^2 v_A^2 \), we obtain

\[
\Delta_c \equiv \left[ \frac{d}{dr} (\omega^2 - \omega_c^2) \right]_{r=r_c} = -2 \left( \omega_c \frac{d\omega_c}{dr} \right)_{r=r_c},
\]

\[
= - \left( \omega_c^2(r_c) \right) \left\{ \frac{(\chi v_A^2 - 1) + \zeta'}{1 + \delta(\chi v_A^2 - 1) + \zeta} - 1 + \delta(\chi v_A^2 - 1) \right\} r=r_c,
\]

where

\[
\zeta' \equiv \frac{d\zeta}{dr} = \frac{v_A^2}{v_A^2} \left( \frac{r_B \omega_{ce} - r_c \omega_{ce}^2}{l^2 B_t^2} \right) \left( 1/r - \ln(r/c/r_c) \right).
\]

Note that in both Equations (39) and (53), due to having the cusp resonance, \( \delta(r = r_c) \) should be replaced by \( \delta_c \), Equation (51).

### 4.1. Weak Damping Limit—Slow Continuum

Here, we are interested in investigating the dispersion relation (39) in the limit of weak damping. To this aim, we first rewrite Equation (39) as follows:

\[
D_t + k_z^2 \frac{n_i + k_z S - \gamma M(a, b = 1; s)}{k_B \omega_A M(a, b, s)} + \frac{i \pi \mu_0 \omega_A}{|\Delta_c| B^2 \omega_A^2} \left\{ D_t \left[ \frac{n_i + k_z S - 2 M(a, b = 1; s)}{M(a, b, s)} \right] + \frac{2B_z \omega_A^2}{\mu_0} \right\}
+ \left( \frac{2B_z \omega_A^2}{\mu_0} \right)_{r=r_c} \left( \frac{\mu_0 (1 - v) D_r - 2B_z \omega_{ce}^2}{\mu_0 r_k^2} \right) - D_r \frac{K_{r-1}(k_{re} r_c)}{K_e(k_{re} r_c)} = 0,
\]

(55)

This can also be recast in the following compact form,

\[
D_{AR} + i D_M = 0,
\]

(56)

where \( D_{AR} \) and \( D_M \), respectively, are the real and imaginary parts of Equation (55), given by

\[
D_{AR} = \rho_0 (\omega^2 - \omega_c^2) - \rho_\omega (\omega^2 - \omega_{\Lambda c}^2) \frac{k_z}{k_{re}} Q,
\]

\[
D_M = \frac{\pi \rho_0 \rho_\omega k_z^2}{k_{re} \rho_\omega \Delta_c} \left( \frac{v_A^2}{v_A^2 + v_{ce}^2} \right)^2 \times (\omega^2 - \omega_{\Lambda c}^2) + Z (\omega^2 - \omega_{\Lambda c}^2) G,
\]

(57)

where

\[
Q \equiv \left( \frac{\mu_0 \rho_0 (1 - v) - 2B_z \omega_{ce}^2}{\mu_0 k_B r_k} - \frac{K_{r-1}(k_{re} r_c)}{K_e(k_{re} r_c)} \right),
\]

\[
G \equiv \left( \frac{2k_{re} B_z^2}{\mu_0 \rho_\omega r_k} \right)_{r=r_c} + \frac{\mu_0 (1 - v) D_r - 2B_z \omega_{ce}^2}{\mu_0 r_k k_{re} D_r} - \frac{K_{r-1}(k_{re} r_c)}{K_e(k_{re} r_c)}.
\]

(59)

(60)

Note that in Equations (57) and (58) we have the complex frequency \( \omega = \omega_c + \gamma \), in which \( \omega_c \) and \( \gamma \) are the cusp (slow) frequency and the damping rate, respectively. In the limit of weak damping, i.e., \( \gamma \ll \omega_c \), the damping rate \( \gamma \) is given as (Goossens et al. 1992)

\[
\gamma = -D_{AL}(\omega_c) \left( \frac{\partial D_{AR}}{\partial \omega} \right)_{\omega_c}^{-1}.
\]

(62)

With the help of Equation (62), one can obtain an analytical expression for \( \gamma \) (see Appendix A) in the slow (cusp) continuum as follows

\[
\gamma = -\frac{\pi \rho_0 k_z^2}{k_{re} \rho_\omega \Delta_c} \left( \frac{v_A^2 + v_{ce}^2}{\omega_{\Lambda c}^2 + v_{ce}^2} \right)^2 \times \left( (\omega^2 - \omega_{\Lambda c}^2) + Z (\omega^2 - \omega_{\Lambda c}^2) G \right) \times \left( 2\omega_c \left( 1 - \frac{k_z}{k_{re}} Q - \omega_c T \right) \right),
\]

(63)

where the quantity \( T \) is given by Equation (93). Note that in the limit of no twist, i.e., \( B_\phi = B_{\phi\epsilon} = 0 \), Equations (59)–(61), (86), and (87) reduce to

\[
Q = Q_0 \equiv \frac{\gamma_0^\epsilon(x) K_0(y)}{I_0(x) K_0(y)}
\]

\[
G = G_0 \equiv \frac{K_0(y)}{K_0(y)}
\]

\[
Z = 0,
\]

\[
P = P_0 \equiv \frac{I_0^\epsilon(x)}{I_0(x)} \frac{I_0^\epsilon(x)}{I_0(x)} K_0(y)
\]

(64)

\[
S = S_0 \equiv \frac{1 - K_0^\epsilon(y) K_0^\epsilon(y)}{K_0^\epsilon(y) K_0^\epsilon(y)} I_0(x)
\]

\[
\lim_{x \to \infty} \frac{M(a, b = 1; s)}{M(a, b, s)} = x \frac{I_0(x)}{2 I_0^2(x)}.
\]

(65)
Inserting relations (64) into Equation (63), the damping rate $\gamma$ in the absence of twist takes the form

$$
\gamma = -\frac{\pi \mu k_{r}^{2}}{k_{r} r_{i} |\Delta_{r}|} \left( \frac{\nu^{2}_{s}}{\nu^{2}_{A} + \nu^{2}_{s}} \right)^{2} \times \frac{(\omega^{2}_{r} - \omega^{2}_{A})(\omega^{2}_{s} - \omega^{2}_{A})G_{0}}{2\omega_{r}(1 - \chi_{I}^{2} z_{s} Q_{0}) - \omega_{r} \chi_{I} T_{0}},
$$

(66)

where

$$
T_{0} = \omega^{2}_{s} (\nu^{2}_{r} - \omega^{2}_{A}) \frac{k_{n}}{k_{r}^{2}} \left( \frac{(Q_{0} + x P_{0})(\omega^{2}_{r} - 2\omega_{c}^{2})}{(\omega^{2}_{r} - \omega^{2}_{A})(\omega^{2}_{s} - \omega^{2}_{A})} \right)
$$

$$
- \frac{(Q_{0} - y S_{0})(\omega^{2}_{r} - 2\omega_{c}^{2})}{(\omega^{2}_{r} - \omega^{2}_{A})(\omega^{2}_{s} - \omega^{2}_{A})(\omega^{2}_{r} - \omega^{2}_{c})}
$$

(67)

Notice that Equation (66) is the same as the dispersion relation (28) in Yu et al. (2017b) for the sss modes when the twist is absent.

4.2. Weak Damping Rate in the Long-wavelength Limit—Slow Continuum

Here, we try to examine the damping rate (63) in the long-wavelength limit, i.e., $k_{r}R \ll 1$ ($\nu_{i} \approx \nu_{c} = k_{r} \nu_{c}$). In this limit, one can show that Equation (63) takes the form (see Equation (96) in Appendix B).

$$
\gamma = -\frac{\pi \mu k_{r}^{2}}{|\Delta_{r}|} \omega^{2}_{A i} \frac{\nu^{2}_{s}}{2\omega_{c i}^{2} r_{i}^{2} Q_{0}} - \omega_{c i} r_{i} T_{0},
$$

(68)

where the quantities $T$, $Q$, $G$, and $Z$ are given by Equations (97)–(100), respectively. Under photospheric (magnetic) pore conditions, i.e., $v_{A i} = v_{A c e} = v_{c e} \approx 0$, one can show that Equation (68) reduces to (see Equation (115) in Appendix B).

$$
\gamma = -\pi \mu k_{r}^{2} \omega^{2}_{A i} \frac{\nu^{2}_{s}}{|\Delta_{r}|} \frac{1}{2} \frac{\nu^{2}_{s}}{\omega^{2}_{c i}} \left( \frac{\omega^{2}_{r}}{\omega^{2}_{A i}} \right) G_{0} - \nu_{c e} r_{i} T_{0},
$$

(69)

where now the quantities $T$, $Q$, $G$, and $Z$ are given by Equations (116)–(119), respectively.

Note that in the limit of long wavelength (i.e., $k_{r}R \ll 1$), from Equations (116)–(119), one can show that

$$
Q \simeq (O(k_{r}^{2}) + O(k_{r}^{2} \ln k_{r}))(1 + O(B_{0}^{2}))
$$

$$
\approx O(k_{r}^{2} \ln k_{r}),(1 + O(B_{0}^{2})),
$$

(70)

$$
k_{r}G \simeq O(k_{r}^{2}) + O(k_{r}^{2} \ln k_{r}) + O(B_{0}^{2}),
$$

(71)

$$
Z \simeq O(B_{0}^{2} k_{r}^{2}) \simeq O(B_{0}^{2} k_{r}^{2}),
$$

(72)

$$
P \simeq (O(k_{r} + O(k_{r} \ln k_{r}))(1 + O(B_{0}^{2})),
$$

(73)

$$
S \simeq (O(k_{r} + O(k_{r} \ln k_{r}))(1 + O(B_{0}^{2})),
$$

(74)

$$
T \simeq O(Q + xP) + O(Q - yS)
$$

$$
\approx (O(k_{r}^{2}) + O(k_{r}^{2} \ln k_{r}))(1 + O(B_{0}^{2})),
$$

(75)

where $k_{c}$ and $B_{0}$ are in terms of $R^{-1}$ and $B_{c i}$. Also, from Equation (53) we have

$$
|\Delta_{r}| \approx O(k_{r}^{2}).
$$

(76)

Substituting Equations (70) to (76) into Equation (69), we obtain

$$
\lim_{k_{r} \rightarrow 0} \frac{\gamma}{\omega_{r}} \approx \lim_{k_{r} \rightarrow 0} \frac{\gamma}{\omega_{i}}
$$

$$
\approx \lim_{k_{r} \rightarrow 0} - \frac{\pi \mu k_{r}^{2} \nu^{2}_{s}}{2|\Delta_{r}^{2}|} \left( \frac{\nu^{2}_{i} - \nu^{2}_{A i}}{2}(1 + O(B_{0}^{2})) + \frac{Z}{k_{r}^{2}} \right) k_{r} G_{0},
$$

(77)

which shows that in the presence of twist, the damping rate to frequency ratio in the long-wavelength limit and for small twist corresponds to $O(B_{0}^{2})$ up to the lowest order in twist. Notice that the result $\gamma/\omega_{r} \approx O(B_{0}^{2})$ for $k_{r}R \rightarrow 0$ is in agreement with our numerical results presented in Section 5 (see Figures 4–8). Also, for the case where there is no twist, from Equation (77) we have $\gamma/\omega_{r} \rightarrow 0$, which recovers the result obtained by Yu et al. (2017b).

Note that in the absence of twist (i.e., $B_{0 i} = B_{0 c e} = 0$), the weak damping rate $\gamma$ in the long-wavelength limit, Equation (68), reduces to (see Equation (137) in Appendix C).

$$
\gamma = \frac{2 \pi \chi^{3}}{|\Delta_{r}|} \left[ \frac{\omega^{2}_{A i} \omega^{2}_{s} (\omega^{2}_{r} - \omega^{2}_{A i})^{3}}{3 \omega^{10}_{A i} \omega^{2}_{s} + 8 \chi^{2} \omega^{8}_{A i} \omega^{2}_{s} (\omega^{2}_{r} - \omega^{2}_{A i}) \ln(k_{r}R)} \right] \times (k_{r}R)^{4} \ln^{3}(k_{r}R).
$$

(78)

For photospheric conditions (i.e., $v_{A i} = v_{A c e} = 0$), Equation (78) reads

$$
\gamma = \frac{2 \pi \chi^{3}}{|\Delta_{r}|} \left[ \frac{\omega^{11}_{A i} \omega^{2}_{s}}{3 \omega^{10}_{A i} + 8 \chi^{2} \omega^{8}_{A i} \omega^{2}_{s} \ln(k_{r}R)} \right] \times (k_{r}R)^{4} \ln^{3}(k_{r}R).
$$

(79)

It should be noted that Equations (78) and (79) without the terms $8 \chi^{2} \omega^{8}_{A i} \omega^{2}_{s} (\omega^{2}_{r} - \omega^{2}_{A i}) \ln(k_{r}R)$ and $8 \chi^{2} \omega^{2}_{A i} (\omega^{2}_{r} - \omega^{2}_{A i}) \ln(k_{r}R)$ appearing in their denominator are the same as Equations (36) and (37) in Yu et al. (2017b). This difference is because of an incorrect minus sign appearing in Equation (A.7) in Yu et al. (2017a).

5. Numerical Results

Here, we solve numerically the dispersion relation (39) to obtain the frequencies and damping rates of the sss modes. To this aim, it is convenient to recast Equation (39) in dimensionless form (see Equation (138) in Appendix D).

Under magnetic pore conditions, following Grant et al. (2015), we again set the model parameters as $v_{A i} = 12 \text{ km s}^{-1}$, $v_{A c e} = 0 \text{ km s}^{-1}$ (i.e., $B_{0 c e} = B_{0 c i} = 0$), $v_{s c e} = 7 \text{ km s}^{-1}$, $v_{c e} = 11.5 \text{ km s}^{-1}$, $v_{s c i} = 6.0464 \text{ km s}^{-1}$ ($\approx 0.8638 v_{s i}$), and $v_{c i} = 0 \text{ km s}^{-1}$. Our numerical results are shown in Figures 4–14.
The phase speed $v/v_{\|} \equiv \omega_r/\omega_{\|}$, (b) the damping rate to frequency ratio $|\gamma|/\omega_r$, and (c) the damping time to period ratio $\tau_D/T = \omega_r/(2\pi|\gamma|)$ of the slow surface sausage modes vs. $k_zR$ for $l/R = 0.1$ and different twist parameters $B_{\phi l}/B_{\phi z} = (0, 10^{-3}, 0.1, 0.2, 0.3)$. For comparison, the analytical results obtained using Equation (63) are shown by the dashed line curves. Auxiliary parameters are as in Figure 2. The results for $B_{\phi l}/B_{\phi z} = 0$ and $10^{-3}$ overlap with each other.

Figure 5. Same as Figure 4, but for $l/R = 0.2$. 
Figures 4–7 present variations of the phase speed (or normalized frequency) $v/v_{si} \equiv \omega_{r}/\omega_{si}$, the damping rate to frequency ratio $|\gamma|/\omega_{r}$, and the damping time to period ratio $\tau_{D}/T = \omega_{r}/(2\pi|\gamma|)$ of the sss modes versus $kzR$ for different twist parameters $B_{\phi}/B_{zi} = (0, 10^{-3}, 0.1, 0.2, 0.3)$ and different thicknesses of the inhomogeneous layer $l/R = (0.1, 0.2, 0.3, 0.4)$. The figures show clearly that (i) the minimum value of the phase speed $v/v_{si}$ decreases and shifts to smaller $kzR$ with increasing twist parameter $B_{\phi}/B_{zi}$. (ii) For a given $l/R$, in the short-wavelength limit ($kzR \gg 1$), we have asymptotically $v/v_{si} \to v_{ci}/v_{si} = 0.8638$ and $|\gamma|/\omega_{r} \to 0$. This shows that the effect of magnetic twist for a larger $kzR$ is negligible. (iii) The maximum value of $|\gamma|/\omega_{r}$ increases, and its position moves to Figure 6. Same as Figure 4, but for $l/R = 0.3$.

Figure 7. Same as Figure 4, but for $l/R = 0.4$. 

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smaller $k_z R$ when $B_0/B_{ci}$ increases. (iv) In the absence/presence of twist, the maximum value of $|\gamma|/\omega_r$ decreases and its position moves to smaller $k_z R$ when $l/R$ increases (see also Figure 8). (v) For a given $l/R$, the minimum value of $\tau_D/T$ decreases with increasing $B_0/B_{ci}$. For instance, for the case where $l/R = 0.1$, the minimum value of $\tau_D/T$ for $B_0/B_{ci} = 0.3$ changes by $\sim 38\%$ less than the case where there is no twist. (vi) The dashed line curves in Figures 4–7 present the analytical results of the damping rate to frequency ratio $|\gamma|/\omega_r$ evaluated using Equation (63). These curves show that for weak damping (i.e., $\gamma \ll \omega_r$) and in the long-wavelength limit (i.e., $k_z R \ll 1$), our numerical results are in good agreement with analytical ones. (vii) For $k_z R \rightarrow 0$, we see that the damping time to period ratio does not go to zero for finite values of the twist parameter, and it is approximately of order $|\gamma|/\omega_r \approx O(B_{ci}^2/B_z^2)$, which confirms the analytical relation obtained in Equation (77).

Here, it is useful to compare our results obtained for the photospheric conditions with those reported for the coronal ones. In the long-wavelength limit ($k_z R \ll 1$), our results show that under photospheric conditions, both the frequencies and the damping time to period ratio $\tau_D/T$ of the sausage modes decrease when the twist increases. This result is in agreement with that obtained by Giagkiozis et al. (2016) for the sausage modes but under coronal conditions. However, for the kink modes ($m = 1$) under coronal conditions, the story is a little bit different. For instance, according to Ebrahimi & Karami (2016), we conclude that with increasing twist parameter, the damping time to period ratio $\tau_D/T$ of kink modes (like the sausage ones) decreases but the frequencies of the kink modes (unlike the sausage ones) increase.

In Figures 9–13, we plot variations of $v/v_{si}, |\gamma|/\omega_r$, and $\tau_D/T$ versus the thickness of the inhomogeneous layer $l/R$ for different $B_0/B_{ci} = (0, 10^{-3}, 0.1, 0.2, 0.3)$ and $k_z R = (0.5, 1, 2, 4, 8)$. Figures show that (i) the maximum value of $|\gamma|/\omega_r$ increases and moves to smaller $l/R$ when $B_0/B_{ci}$ increases.

(ii) In the absence/presence of twist, the maximum value of $|\gamma|/\omega_r$ increases, and its position moves to smaller $l/R$ when $k_z R$ increases (see also Figure 14). (iii) For a given $k_z R$, the phase speed and the damping rate to frequency ratio, respectively, approach $v/v_{si} \rightarrow v_{ci}/v_{si} = 0.8638$ and $|\gamma|/\omega_r \rightarrow 0$ in the limit of larger values of $l/R$. (iv) For a given $k_z R$, the minimum value of $\tau_D/T$ decreases with increasing $B_0/B_{ci}$. For instance, for the case where $k_z R = 1$, the minimum value of $\tau_D/T$ for $B_0/B_{ci} = 0.3$ decreases $\sim 63.5\%$ in comparison to the case where there is no twist.

It is worth mentioning that for the case where there is no twist, $B_0/B_{ci} = 0$, the results of Figures 4–14 recover those obtained in Yu et al. (2017b). Note that the results for $B_0/B_{ci} = 0$ and $10^{-3}$ overlap with each other. Note that there is a limitation in extending our numerical results to the case of strong magnetic field. Because, following Priest (2014), to avoid sausage instability in a cylindrical magnetic flux tube in the solar atmosphere, one must have $B_{ci}/B_0 < 1.4$. The observation of sausage instability in the solar atmosphere has not been reported so far (see Aschwanden 2005). This means that the azimuthal magnetic field of a twisted tube never seems to reach the critical limit of $B_{ci}/B_0 > 1.4$. Also, Parker (1974) showed that if a flux tube were not unstable to sausage or kink
modes, for $B_{\phi i}/B_{zi} > 1$, it is no longer under tension and becomes unstable. As far as we know, there are no observational reports for magnetic twist values under photospheric (magnetic pore) conditions. However, for coronal
conditions, the observational magnetic twist values are on the order of $B_0/B_z \approx O(0.1)$ (see Kwon & Chae 2008; Wang et al. 2015). This is why we restrict ourselves to weak magnetic twist.

6. Conclusions

Here, we investigated the effect of weak magnetic twist on the resonant absorption of slow sausage waves in magnetic flux tubes under solar photospheric (or magnetic pore) conditions. We
considered a straight cylindrical flux tube with different magnetic twist profiles in the interior, annulus, and exterior regions. We also assumed the density and longitudinal magnetic field to be constant inside and outside the flux tube, but to be inhomogeneous in the interior region.

Figure 13. Same as Figure 9, but for $k_z R = 8$.

Figure 14. (a) The phase speed $v/v_{ci} \equiv \omega_{ri}/\omega_{ci}$, (b) the damping rate to frequency ratio $|\gamma|/\omega_{ri}$, and (c) the damping time to period ratio $\tau_d/\tau$ of the slow surface sausage modes vs. $l/R$ for different $k_z R = (0.5, 1, 2, 4, 8)$. Here, the dashed and solid line curves, respectively, are related to $B_{\phi i}/B_{zi} = 0$ and $B_{\phi i}/B_{zi} = 0.2$. Auxiliary parameters are as in Figure 2.
annulus layer. We presented the solutions of ideal MHD equations for the interior and exterior regions of the flux tube. In the case where there is no inhomogeneous (annulus) layer, we derived the dispersion relation, which recovers the result obtained by Edwin & Roberts (1983) and Yu et al. (2017b) in the absence of magnetic twist. In the presence of an inhomogeneous layer, with the help of the appropriate connection formula of the resonant absorption introduced by Sakurai et al. (1991), we obtained the jump conditions governing the solutions inside and outside the flux tube. Consequently, we derived the dispersion relation of the slow surface sausage modes in the presence of magnetic twist. Using this, we first obtained an analytical relation for the damping rate of the sss modes in the limit of weak damping and in the long-wavelength limit. Then, we showed that our analytical expression for the damping rate in the absence of twist recovers the result obtained by Edwin & Roberts (1983) and Yu et al. (2017b). In addition, we numerically solved the dispersion relation and obtained the phase speed (or normalized frequency) \( v/v_{ss} \approx \omega_2/\omega_{R2} \), the damping rate to frequency ratio \( |\gamma|/\omega_r \), and the damping time to period ratio \( \tau_D/T = \omega_r/(2\pi|\gamma|) \) of the slow surface sausage modes under photospheric (magnetic pore) conditions. Our results show the following:

1. For a given thickness of the inhomogeneous layer \( L/R \), with increasing twist parameter \( B_{\theta}/B_{\xi} \), (i) the minimum values of both the phase speed \( v/v_{ss} \) and the damping time to period ratio \( \tau_D/T = \omega_r/(2\pi|\gamma|) \) decrease and shift to smaller \( k,R \); (ii) the maximum value of \( |\gamma|/\omega_r \) increases and moves to smaller \( k,R \).

2. For a given \( L/R \), the phase speed and the damping rate to frequency ratio, approach \( v/v_{ss} \to v_{ss}/v_\| = 0.8638 \) and \( |\gamma|/\omega_r \to 0 \), respectively, in the short-wavelength limit \( (k,R \gg 1) \). This asymptotic behaviors also hold for a given \( k,R \) in the limit of larger values of \( L/R \).

3. For a given \( k,R \), the maximum value of \( |\gamma|/\omega_r \) (or minimum value of \( \tau_D/T \)) increases (or decreases) and moves to smaller \( L/R \) when the twist parameter increases.

4. For the case where \( L/R = 0.1 \), the minimum value of \( \tau_D/T \) for \( B_{\theta}/B_{\xi} = 0.3 \), for instance, changes \( \sim 38\% \) less than the case where there is no twist. Also, for \( k,R = 1 \), the minimum value of \( \tau_D/T \) for \( B_{\theta}/B_{\xi} = 0.3 \), for example, decreases by \( \sim 63.5\% \) in comparison to the case where there is no twist. These results show that the magnetic twist can considerably affect the resonant absorption of the slow surface sausage modes in magnetic flux tubes under photospheric conditions.

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Appendix A

Weak Damping Rate for the Surface Sausage Mode

Here, with the help of Equation (62), we obtain the damping rate of surface sausage modes in the weak damping limit, i.e., \( \gamma \ll \omega_r \). To this aim, we first calculate \( \partial D_{AR}/\partial \omega \) from Equation (57) as follows:

\[
\frac{\partial D_{AR}}{\partial \omega} = 2\rho \omega - 2\rho \omega^2 \frac{k_{ri}}{k_{re}} Q - \rho_\xi (\omega - \omega_\xi) \left( \frac{1}{k_{re}} \frac{dk_{ri}}{d\omega} - \frac{k_{ri}}{k_{re}} \frac{dk_{re}}{d\omega} \right) Q - \rho_\xi (\omega - \omega_\xi^2) \frac{k_{ri}}{k_{re}} \frac{dQ}{d\omega}.
\]

(80)

Now, from Equation (27), one can obtain

\[
\frac{dk_{ri}}{d\omega} = -\omega^3(\omega^2 - 2\omega_\xi^2)/(v_{\xi,1}^2 + v_{\xi,2}^2(\omega^2 - \omega_\xi^2))^2k_{ri},
\]

(81)

\[
\frac{dk_{re}}{d\omega} = -\omega^3(\omega^2 - 2\omega_\xi^2)/(v_{\xi,1}^2 + v_{\xi,2}^2(\omega^2 - \omega_\xi^2))^2k_{re}.
\]

(82)

Also, from Equation (59) for \( dQ/d\omega \), we get

\[
\frac{dQ}{d\omega} = \frac{Q}{x} \frac{dx}{d\omega} - \frac{d}{d\omega} \left[ \left( \frac{\rho_\xi D_\xi(1 - y) - 2R_{\xi}^2 n_\xi^2}{\rho_\xi D_\xi} - \frac{K_{\perp}(y)}{K_\perp(y)} \right) \frac{dn_{\perp}}{d\omega} + \left( \frac{\rho_\xi D_\xi(1 - y) - 2R_{\xi}^2 n_\xi^2}{\rho_\xi D_\xi} - \frac{K_{\perp}(y)}{K_\perp(y)} \right) \frac{dn_{\parallel}}{d\omega} \right] \left[ \frac{n_{\perp} + k_{\xi}}{k_{\xi}} - 2M(a,b-1,s)/M(a,b,s) \right]^2.
\]

(83)

After some algebra, we obtain

\[
\frac{dQ}{d\omega} = \frac{Q}{x} \frac{dx}{d\omega} + \frac{1 - y^2}{y^2 + \frac{2\rho_\xi D_\xi}{(\omega^2 - \omega_\xi^2)^2}} \left( \frac{k_{\perp,1}}{k_\perp} - \frac{k_{\perp,2}}{k_\perp} \right) \frac{dy}{d\omega} + \frac{1 - y^2}{y^2 + \frac{4\rho_\xi D_\xi}{(\omega^2 - \omega_\xi^2)^2}} \left( n_e \frac{dn_e}{d\omega} - \frac{\omega_\xi}{\omega_\xi} \right) \left[ \frac{n_{\perp} + k_{\xi}}{k_{\xi}} - 2M(a,b-1,s)/M(a,b,s) \right]^2
\]

\[
+ \left[ \frac{x d\xi}{k_\perp} + \frac{n_{\perp} + k_{\xi}}{k_{\xi}} ds - 2 \frac{d}{d\omega} \left( M(a,b-1,s)/M(a,b,s) \right) \left( \frac{\rho_\xi D_\xi(1 - y) - 2R_{\xi}^2 n_\xi^2}{\rho_\xi D_\xi} - \frac{K_{\perp}(y)}{K_\perp(y)} \right) \left[ \frac{n_{\perp} + k_{\xi}}{k_{\xi}} - 2M(a,b-1,s)/M(a,b,s) \right]^2 \right]
\]

(84)
where \( x = k_{r}r_{i} \) and \( y = k_{r}r_{e} \). This can be rewritten as

\[
\frac{dQ}{d\omega} = P \frac{dx}{dw} + S \frac{dy}{d\omega},
\]

where

\[
P = \frac{Q}{x} + x \left[ \frac{s}{k_{i}} \frac{dn_{i}}{d\omega} + n_{i} + k_{i} \frac{ds}{k_{i}} - 2 \frac{d}{dw} \left( M(a, b - 1; s) \right) \left[ \frac{\mu_{i} D_{i}(1 - 1/\nu) - 2R_{i}^{2} n_{i}^{2}}{\mu_{i} D_{i} y - K_{i}(y)} \right] \right],
\]

\[
S = x \left[ \frac{1 - \nu}{\nu^{2}} \left( \frac{2\nu^{2} n_{i}^{2}}{(\omega^{2} - \omega_{r}^{2})^{2}} \right) + \left( K_{i}^{2} - K_{e}^{2} \right) \right] \frac{dy}{d\omega} - \frac{1}{\nu} \frac{d}{dw} \left( \frac{4\nu^{2} \mu_{i} D_{i}}{(\omega^{2} - \omega_{r}^{2})} \right) \left( n_{i} + k_{i} \frac{dn_{i}}{d\omega} - \frac{\omega n_{i}^{2}}{(\omega^{2} - \omega_{r}^{2})} \right).
\]

In addition, from Equation (20), we have

\[
\frac{dn_{i}}{dw} = \frac{\omega^{3}(\omega^{2} - 2 \omega_{r}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})},
\]

\[
\frac{dn_{e}}{dw} = \frac{\omega^{3}(\omega^{2} - 2 \omega_{e}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})}.
\]

With the help of Equations (81) and (82), Equation (85) takes the form

\[
\frac{dQ}{d\omega} = xP \frac{\omega^{3}(\omega^{2} - 2 \omega_{r}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})} + yS \frac{\omega^{3}(\omega^{2} - 2 \omega_{e}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})}.
\]

Inserting this into Equation (80) yields

\[
\frac{\partial D_{AR}}{\partial \omega} = 2\rho_{i} \omega - 2\rho_{e} \omega \frac{k_{i}}{k_{re}} Q - \rho_{e} \omega^{3}(\omega^{2} - \omega_{r}^{2}) \frac{k_{r}}{k_{re}} \left( \frac{(Q + xP)(\omega^{2} - 2 \omega_{r}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})} \right)
\]

\[
- \frac{(Q - yS)(\omega^{2} - 2 \omega_{e}^{2})}{(\omega_{s}^{2} - \omega^{2})(\omega_{e}^{2} - \omega^{2})(\omega^{2} - \omega_{c}^{2})} \right).
\]

Finally, substituting Equations (58) and (91) into Equation (62), one can get the damping rate \( \gamma \) in the limit of weak damping for the surface sausage modes in the slow continuum as

\[
\gamma \vert_{\omega = \omega_{r}} = - \frac{\pi_{i} k_{i}^{2}}{k_{i} r_{i} \Delta_{l}} \left( \frac{\nu^{2}}{\nu^{2} + \nu^{2}} \right) \left( (\omega^{2} - \omega_{Ai}^{2}) + Z(\omega^{2} - \omega_{Ar}^{2})G \right) \frac{2\omega}{1 - \chi e^{-\frac{\omega}{\omega_{c}}}} - \omega T.
\]

where

\[
T = \omega_{r}^{2}(\omega_{e}^{2} - \omega_{Ar}^{2}) \frac{k_{e}}{k_{re}} \left( \frac{(Q + xP)(\omega_{r}^{2} - 2 \omega_{r}^{2})}{(\omega_{s}^{2} - \omega_{r}^{2})(\omega_{e}^{2} - \omega_{r}^{2})(\omega_{r}^{2} - \omega_{c}^{2})} \right) - \frac{(Q - yS)(\omega_{r}^{2} - 2 \omega_{r}^{2})}{(\omega_{s}^{2} - \omega_{r}^{2})(\omega_{e}^{2} - \omega_{r}^{2})(\omega_{r}^{2} - \omega_{c}^{2})} \right).
\]

**Appendix B**

**Weak Damping Rate in the Long-wavelength Limit**

Here, we turn to examine the dispersion relation (92) in the long-wavelength limit, i.e., \( k_{r}R \ll 1 \). In this limit, Equation (29) yields

\[
\nu^{2} = 1 + O(k_{r}^{2} R^{2}) \approx 1.
\]

Also, following Abramowitz & Stegun (2012), we have

\[
\lim_{k_{r} \to 0} M(a, b - 1; s) = 1 + \frac{a}{b} s + O(s^{2}),
\]

\[
\lim_{k_{r} \to 0} K_{0}(y) = -y \ln(y).
\]
In the limit \( k_z R \ll 1 \) (\( \omega_r \approx \omega_{ci} \)), one should note that the damping rate (92) at \( \omega = \omega_r \approx \omega_{ci} \) becomes singular. To avoid this singularity, we follow the approach of Yu et al. (2017b) in which one can assume \( \omega_r^2 = \omega_{ci}^2 - \alpha \), where \( \alpha \ll \omega_{ci}^2 \). Substituting \( \omega_r^2 = \omega_{ci}^2 - \alpha \) into Equation (92) and using Equations (94) and (95), one can obtain

\[
\gamma = \frac{\pi \rho_c k_z^4 \omega_{ci}^4 ((\omega_{ci}^2 - \omega_{Ae}^2) + Z) (\omega_{ci}^2 - \omega_{ci}^2) G}{[\Delta_c] \omega_{ci}^4 2 \omega_{ci} (\rho_i - \rho_e \frac{k_z^2}{k_{re}} Q) - \omega_{ci} \rho_c T},
\]

where we have used \( r_i \approx R \) and \( k_{re} = k_z \). Also,

\[
T = \omega_{ci}^2 (\omega_{ci}^2 - \omega_{Ae}^2) \frac{k_h}{k_{re}} \left( \frac{(Q + xP)n_r^2}{\omega_{ci}^2 (\omega_{ci}^2 -\omega_{Ae}^2) - \omega_{ci}^2} - \frac{(Q - yS)n_r^2}{\omega_{ci}^2 (\omega_{ci}^2 -\omega_{ci}^2) - \omega_{ci}^2} \right),
\]

\[
Q = -x \left[ \frac{-2 \frac{\omega_{ci}^2}{\mu_0} \omega_r^2}{n_r \omega_{ci}} - 2 \left( 1 + \frac{a}{2} \right) \right],
\]

\[
G = \left( -\frac{2k_{re}B_0^2}{D_r \mu_0 r} \right) - \frac{2B_0^2 \omega_r^2}{\mu_0 y D_r} + y \ln(y),
\]

\[
Z = \frac{2B_0^2 k_h^2}{\mu_0 r_{ci}^2} \left[ \frac{n_r \omega_{ci}}{2(1 + \frac{a}{2})} \right],
\]

\[
P = \frac{Q}{x} \left[ s \frac{dn_r \omega_{ci}}{k_{re} dw} + \frac{n_r \omega_r^2}{k_{re} \omega_{ci}} - 2 \left( 1 + \frac{a}{2} \right) \right],
\]

\[
S = x \left[ \frac{2 \frac{\omega_{ci}^2}{\mu_0} \omega_r^2}{(\omega_{ci}^2 -\omega_{Ae}^2) y^2} - \frac{k_z}{2} (1 + \ln(y)) \right] \frac{dy}{dw} - \frac{4 \frac{\omega_{ci}^2}{\mu_0} \omega_r^2}{(\omega_{ci}^2 -\omega_{Ae}^2) y} \left( \frac{n_r \omega_r^2}{\mu_0} - \frac{\omega_{ci}^2}{\omega_{ci}^2 -\omega_{Ae}^2} \right).
\]

Now, from Equations (24), (26), and (25), one can obtain

\[
s = \frac{v_{Ae}^2 n_k k_z}{(\omega_{ci}^2 - \omega_{Ae}^2)},
\]

\[
a = 1 + \frac{x^2 \left( \frac{\omega_{ci}^2}{\omega_{ci}^2 - \omega_{Ae}^2} \right)^2 - \frac{4 \frac{\omega_{ci}^2}{\mu_0} \omega_r^2}{8 \frac{\omega_{ci}^2}{\mu_0} n_k k_z \left( \omega_{ci}^2 - \omega_{Ae}^2 \right)}}{8 \omega_{ci}^2 \omega_{Ae} n_k k_z \omega_{ci}^2 \omega_{ci}^2}.
\]

where \( v_{Ae}^2 \equiv \frac{Bl_e}{\mu_0 P} \). With the help of Equations (103) and (104), one can get

\[
\frac{ds}{d\omega} = \frac{x^2}{4} + v_{Ae}^2 \left( \frac{2 n_k k_z (\omega_{ci}^2 - \omega_{Ae}^2) - x^2 r_i^2 \omega_{ci}^2}{(\omega_{ci}^2 - \omega_{ci}^2)^2} \right),
\]

\[
\frac{da}{d\omega} = \frac{1}{8 \frac{\omega_{ci}^2 n_k k_z}{v_{Ae}^2}} \left[ 2 x \left( \frac{\omega_{ci}^2}{\omega_{ci}^2 - \omega_{Ae}^2} \right) \frac{dx}{d\omega} + x^2 \left( \frac{(\omega_{ci}^2 - \omega_{Ae}^2)}{n_i} \frac{dn_i}{d\omega} \right) \frac{dx}{d\omega} \right] + \frac{2x^2 r_i^2}{(\omega_{ci}^2 - \omega_{ci}^2)^2} \left( \frac{2 \omega_{ci}^2}{(\omega_{ci}^2 - \omega_{ci}^2)^2} + \frac{1}{n_i (\omega_{ci}^2 - \omega_{Ae}^2)} \right),
\]

\[
\frac{ds}{d\omega} = 2 v_{Ae}^2 n_k k_z \left( \frac{1}{n_i (\omega_{ci}^2 - \omega_{ci}^2)} \frac{dn_i}{d\omega} - \frac{2 \omega_{ci}^2}{(\omega_{ci}^2 - \omega_{ci}^2)^2} \right).
\]
Under photospheric conditions, i.e., \( \nu_{As} = \nu_{A\omega} = \nu_c \approx 0 \), the weak damping rate \( \gamma \), Equation (96), in the long-wavelength limit reduces to

\[
\gamma = -\frac{\pi\rho_s k_R \omega_{ci}^5 (\omega_{ci}^2 - \omega_{Ai}^2) + Z G}{|\Delta|} \frac{\omega_{Ai}^2}{2}\left(\rho_s - \rho_e \frac{\omega_{ci}}{\omega_{Ai}} Q\right) - \rho_e T.
\]

Further, Equations (97)–(102), (113), and (114) take the forms

\[
T = \omega_{ci}^2 \frac{x}{y} \left( \frac{Q + xP}{\alpha} - \frac{(Q - yS)}{(\omega_{ci}^2 - \omega_{se}^2)} \right).
\]

Combining Equations (108) and (109) yields

\[
s \frac{da}{d\omega} + a \frac{ds}{d\omega} = \frac{x}{2} \frac{dx}{d\omega} + 4v_{A\omega}^2 \omega_{A\omega}^2 x^2 r_i^{-2} - \frac{2}{(\omega_{ci}^2 - \omega_{Ai}^2)^2} \frac{dx}{d\omega} + \frac{9\omega_{ci}}{4(\omega_{ci}^2 - \omega_{Ai}^2)^3} + \frac{3}{4n_i(\omega_{ci}^2 - \omega_{Ai}^2)^2} \frac{dn_i}{d\omega}
\]

\[
+ 2v_{A\omega}^2 n_i k_c \left( \frac{1}{n_i(\omega_{ci}^2 - \omega_{Ai}^2)} \right) \frac{dn_i}{d\omega} - \frac{2\omega_{ci}}{(\omega_{ci}^2 - \omega_{Ai}^2)^2}.
\]

In addition, we need to evaluate the quantity \( \alpha \). To this aim, following Yu et al. (2017b), we first insert \( \omega = \omega_{ci} - \alpha \) into Equation (27) and get

\[
k_{ri}^2 \approx \frac{k_{ri}^2 (\omega_{ci}^2 - \omega_{Ai}^2)(\omega_{ci}^2 - \omega_{Ae}^2)}{(\omega_{Ai}^2 + \omega_{Ae}^2)} = \frac{k_{ri}^2}{\alpha} \frac{\omega_{ci}}{\omega_{Ai}^2},
\]

where we have used the definition \( \omega_{ci}^2 \equiv (\omega_{ci}^2 - \omega_{Ae}^2)/\omega_{Ai}^2 \) in obtaining the second equality of the above relation. In the next, the dispersion relation (31) in the long-wavelength limit \( (k, R \ll 1) \) reads

\[
\frac{(\omega_{ci}^2 - \omega_{Ai}^2)}{k_{ri}^2} \left[ \frac{n_i}{k_c} \right] \left[ \left( \frac{n_i}{k_c} \right) s - 2 \left( 1 + \frac{a}{b} s \right) \right] = -\chi \left( \omega_{ci}^2 - \omega_{Ae}^2 \right) R^2 \ln(y) + \left( 1 + \frac{2n_i^2}{k_{re}^2} \right) \frac{B_{ce}^2}{\mu_0 \rho_i} - \frac{B_{ci}^2}{\mu_0 \rho_i}.
\]

Now, using \( k_{ri}^2 \) from Equation (111) in Equation (112), the quantity \( \alpha \) can be obtained as follows

\[
\alpha = -k_{ci}^2 \omega_{ci}^2 \omega_{Ai}^2 \left( \frac{n_i}{k_c} \right) s - 2 \left( 1 + \frac{a}{b} s \right) \chi \left( \omega_{ci}^2 - \omega_{Ae}^2 \right) R^2 \ln(y) - \left( 1 + \frac{2n_i^2}{k_{re}^2} \right) v_{A\omega}^2 + v_{A\omega i}^2.
\]

Substituting this into Equation (111) yields

\[
k_{ri}^2 = \frac{\omega_{ci}^2 \omega_{Ai}^2}{\omega_{Ai}^2} \left( \frac{n_i}{k_c} \right) s - 2 \left( 1 + \frac{a}{b} s \right) \chi \left( \omega_{ci}^2 - \omega_{Ae}^2 \right) R^2 \ln(y) - \left( 1 + \frac{2n_i^2}{k_{re}^2} \right) v_{A\omega}^2 + v_{A\omega i}^2.
\]
Finally, substituting Equations (116) to (123) into Equation (115) gives a long analytical expression for the weak damping $\gamma$ in the long-wavelength limit for photospheric conditions.

**Appendix C**

**Weak Damping Rate in the Long-wavelength Limit with No Twist**

In the limit of no twist, i.e., $B_{\phi 0} = B_{\phi e} = 0$, Equations (105) and (110) read

$$\alpha = -\frac{k_{nt}^2 \omega_{nt}^4}{\omega_{Ai}^4} \left( \frac{n_{ni}}{e_k} s - 2 \left(1 + \frac{a}{b} s \right) \right) \frac{k_{nt}^2 \omega_{ei}^4}{\omega_{Ai}^4} \left( \chi_{12}^2 R^2 \ln(y) + \chi_{10}^2 \right) \left( \frac{1}{2} - \frac{a}{4} s + \frac{v_{A0}^2 n_{10}^2}{2(\omega_{ei}^2 - \omega_{Ai}^2)} \right)$$

$$= \frac{k_{nt}^2 \omega_{ei}^4}{\omega_{Ai}^4} \left( \chi_{12}^2 R^2 \ln(y) + \chi_{10}^2 \right) \left( 1 - v_{A0}^2 \right) - \frac{2 v_{A0}^2 n_{10}^2}{(\omega_{ei}^2 - \omega_{Ai}^2)^2}$$

Finally, substituting Equations (116) to (123) into Equation (115) gives a long analytical expression for the weak damping $\gamma$ in the long-wavelength limit for photospheric conditions.
Substituting the above relations into Equations (98) to (102), (113), and (114), one can get

\[ Q = \frac{xy}{2} \ln(y), \]

\[ G = y \ln(y), \]

\[ Z = 0, \]

\[ P = \left( 1 - \frac{3y^2}{16} \right) y \ln(y), \]

\[ S = \frac{x}{2}(1 + \ln(y)). \]

\[ \alpha = \frac{x\omega^2}{2\omega_A^2} (\omega_{ci}^2 - \omega_{ce}^2) k^2 R^2 \ln(y), \]

\[ k^2_{ni} = -2 \frac{\omega_{ci}^2 \omega_A^2}{\chi \omega_{ci}^2 (\omega_{ci}^2 - \omega_{ce}^2) R^2 \ln(y)}. \]

Inserting Equations (126)–(128) into Equation (96) gives

\[ \gamma = -\frac{\pi \alpha k^2 R^2}{|\Delta_e|} \frac{\omega_{ci}^4 (\omega_{ci}^2 - \omega_{ce}^2) (\omega_{ci}^2 - \omega_{ci}) R \ln(y)}{2 \left( \rho_i - \rho_c \frac{x^2}{2} \ln(y) \right) - \rho_c T}. \]

Putting Equations (120) and (121) into Equation (97), one can get

\[ T = \omega_{ci}^2 (\omega_{ci}^2 - \omega_{ce}^2) \left( \frac{x^2 \ln(y) \omega_{ci}^2}{(\omega_{ci}^2 - \omega_{ce}^2) (\omega_A^2 - \omega_{ci}) \alpha} - \frac{3x^4 \ln(y) \omega_{ci}^2}{16 (\omega_{ci}^2 - \omega_{ce}^2) (\omega_A^2 - \omega_{ci}) \alpha} + \frac{x^2 (\omega_{ci}^2 - 2\omega_{ce}^2)}{2 (\omega_{ce}^2 - \omega_{ci}) (\omega_A^2 - \omega_{ci}) (\omega_{ci}^2 - \omega_{ce}^2)} \right). \]

Inserting Equations (131) and (132) into Equation (134) yields

\[ T = -\frac{4\omega^6_A}{\chi^2 \omega_{ci}^2 \omega_A^2 (\omega_{ci}^2 - \omega_{ce}^2) k^2 R^2 \ln(k_i R)} - \frac{3\omega^8_A}{2 \chi^3 \omega^4_A (\omega_{ci}^2 - \omega_{ce}^2)^2 k^2 R^2 \ln(k_i R)} \]

\[ - \frac{\omega^4_A (\omega_{ci}^2 - 2\omega_{ce}^2)}{\chi \omega_{ci}^2 (\omega_{ci}^2 - \omega_{ce}^2) (\omega_A^2 - \omega_{ci}) (\omega_{ci}^2 - \omega_{ce}^2) \ln(k_i R)}. \]

Note that in the long-wavelength limit \( k_i R \ll 1 \), the third term appearing in Equation (135) is small in comparison to the first two. Hence, Equation (135) reduces to

\[ T = -\frac{4\omega^6_A}{\chi^2 \omega_{ci}^2 \omega_A^2 (\omega_{ci}^2 - \omega_{ce}^2) k^2 R^2 \ln(k_i R)} - \frac{3\omega^8_A}{2 \chi^3 \omega^4_A (\omega_{ci}^2 - \omega_{ce}^2)^2 k^2 R^2 \ln(k_i R)}. \]

Finally, substituting Equation (136) into Equation (133) gives the weak damping rate in the long-wavelength limit with no twist as

\[ \gamma = \frac{2\pi x^3}{|\Delta_e| R} \left[ \frac{\omega_{ci}^7 \omega_A^2 (\omega_{ci}^2 - \omega_{ce}^2)^3}{3 \omega_A^2 \omega_{ci}^2 + 8 \chi \omega_A^2 \omega_{ci}^2 (\omega_{ci}^2 - \omega_{ce}^2) \ln(k_i R)} \right] (k_i R)^4 \ln^3(k_i R). \]
Appendix D

Dimensionless Dispersion Relation

In order to numerically solve the dispersion relation (39), we recast it in the following dimensionless form:

\[
v_F^2 - v_{Al}^2 \left[ (N_f + 1)s - 2\frac{M(a, b - 1; s)}{M(a, b; s)} \right] + \left( 1 - \nu \right) \chi \frac{(v_F^2 - v_{Al}^2)}{v_F^2} - 2\frac{\chi v_{Al,AE}^2 N_E^2}{r_E k_{rE} k_{rE}} - \frac{\chi k_z (v_F^2 - v_{Al}^2) K_{rE-1}(k_{rE} k_{rE})}{k_{rE} K_{rE}(k_{rE} k_{rE})} = 0,
\]

where

\[
a = 1 + \frac{k_{rE}^2 (v_F^2 - v_{Al}^2)^2 - 4v_{Al,AE}^2 v_{Al}^2)}{8v_{Al,AE} N_f (v_F^2 - v_{Al}^2)}, s = 2 - \frac{v_{Al,AE} N_f}{(v_F^2 - v_{Al}^2)}, v_F = \frac{\omega_r}{\omega_{si}}, v_{Al} = v_{AE} = v_{Ai} = v_{Al}, b = 2,
\]

\[
\chi = \frac{\rho_{ps}}{\rho_f}, v_{SE} = \frac{v_{Al}}{v_{si}}, v_{AE} = \frac{v_{Al}}{v_{si}}, \quad v_{Ai} = \frac{v_{Al}}{v_{si}}, \quad v_{sl} = 1, \quad v_{t}^2 = \frac{B_{0,si}^2}{\mu_{0,si}},
\]

\[
k_{rE} = k_2 k_{rE}, \quad k_1 = k_2 k_{rE}, \quad n_z^2 = k_2^2 N_E^2, \quad n_l^2 = k_2^2 N_f^2.
\]

and

\[
N_f^2 = \frac{v_F^4}{v_F^2 v_{Al} + (v_F^2 - 1)}, \quad N_E^2 = \frac{v_F^2}{v_F^2 v_{AE} + v_{SE}(v_F^2 - 1)}, \quad k_2^2 = \frac{(1 - v_F^2)(v_{AE}^2 - v_{t}^2)}{(1 + v_{AE}^2)(v_{AE}^2 - v_{t}^2)}, \quad k_{rE}^2 = \frac{(v_{SE}^2 - v_{Al}^2)(v_{AE}^2 - v_{t}^2)}{(v_{SE}^2 + v_{AE}^2)(v_{SE}^2 - v_{t}^2)}.
\]

\[
v_{SE} = \frac{v_{Al}}{v_{si}}, \quad v_{sl} = \frac{v_{Ai}}{v_{si}}, \quad \nu^2 = 1 + 2\frac{v_{Al,AE}^2}{(v_F^2 - v_{Al}^2)} \left[ 2v_{Al,AE}^2 N_E^2 + (v_{Al,AE}^2 (3N_E^2 - 1) - v_{t}^2(N_E^2 + 1)) \right].
\]

Note that in the absence of an inhomogeneous layer, Equation (138) reduces to

\[
v_F^2 - v_{Al}^2 \left[ (N_f + 1)s - 2\frac{M(a, b - 1; s)}{M(a, b; s)} \right] + \left( 1 - \nu \right) \chi \frac{(v_F^2 - v_{Al}^2)}{v_F^2} - 2\frac{\chi v_{Al,AE}^2 N_E^2}{r_E k_{rE} k_{rE}} = 0.
\]

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