Three Point Functions for a Class of Chiral Operators
in Maximally Supersymmetric CFT at Large $N$

by

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Abstract

We present a calculation of three point functions for a class of chiral operators, including the primary ones, in $d = 3, \mathcal{N} = 8; d = 6, \mathcal{N} = (2, 0)$ and $d = 4, \mathcal{N} = 4$ superconformal field theories at large $N$. These theories are related to the infrared world-volume descriptions of $N$ coincident M2, M5 and D3 branes, respectively. The calculation is done in the framework of the AdS/CFT correspondence and can be given a unified treatment employing a gravitational action in arbitrary dimensions $D$, coupled to a $p + 1$ form and suitably compactified on $\text{AdS}_{D-2-p} \times S_{2+p}$. The interesting cases are obtained setting $(D, p)$ to the values $(11, 5), (11, 2)$ and $(10, 3)$.

Keywords: String Theory, Conformal Field Theory, Geometry.
PACS no.: 0240, 0460, 1110.
0. Introduction

The AdS/CFT correspondence [1–3] is a useful tool to study strongly coupled conformal field theories (CFT). It relates the generating functional for correlation functions of gauge invariant operators in certain large $N$ CFT to the on-shell value of a corresponding dual supergravity action suitably compactified on $\text{AdS} \times \text{M}$ spaces. When the dual pair is known, a tree-level computation on the supergravity side gives a prediction on the otherwise inaccessible strongly coupled CFT. The most studied case relates IIB supergravity on $\text{AdS}_5 \times \text{S}_5$ to $d = 4, \mathcal{N} = 4$ large $N$ $\text{SU}(N)$ super Yang–Mills (SYM) theory, where a variety of correlation functions have already been computed using the correspondence (see the review [4] also for a list of references). In particular, three point functions for a set of chiral primary operators (CPO), namely those corresponding to the so-called single trace operators, have been computed in [5]. Other three point functions involving the CPOs have been computed more recently in [6,7], in part with the aim of addressing eventually the more complicated case of their four point functions [8].

The $d = 4, \mathcal{N} = 4$ SYM theory, which can be thought of as describing the low energy dynamics of D3 branes in type IIB superstring theory, is not the only example with 16 supersymmetric charges. Other dual pairs involving maximally supersymmetric CFT have been identified in [1], namely “11D supergravity on $\text{AdS}_7 \times \text{S}_4$ / $d = 6, \mathcal{N} = (2,0)$ large $N$ SCFT” and “11D supergravity on $\text{AdS}_4 \times \text{S}_7$ / $d = 3, \mathcal{N} = 8$ large $N$ SCFT”. These pairs describe the low energy physics of $N$ coincident M5 and M2 branes of M-theory, respectively. In this respect, the AdS/CFT correspondence has been used in [9] to identify CPO three point functions in the $d = 6, \mathcal{N} = (2,0)$ SCFT case, while the present authors produced in [10] the set of CPO three point functions for both cases. In this last reference, we could treat simultaneously both cases by employing a gravitational action in arbitrary dimensions $D$, coupled to a $1+p$ form, suitably compactified on $\text{AdS}_{D-2-p} \times \text{S}_{2+p}$. Setting at the end $(D,p)$ to the values $(11,2)$ and $(11,5)$ allowed us to recover the M5 and M2 cases, respectively. We also noticed, with some surprise, that the general model with $(D,p) = (10,3)$ was able to reproduce the super Yang–Mills case worked out in [5].

As promised in [10], in this paper we present more details on the calculation leading to the CPO three point functions for all of these maximally supersymmetric CFTs. In addition, we extend it to include another class of scalar chiral operators sitting in the same short supermultiplets identified by the chiral primaries. Including them gives us the chance of producing quite large a class of three point functions with minor efforts. Note that obtaining the additional three point functions from the CPO ones by using the superconformal algebra seems possible in principle, but may be rather cumbersome. The
sources for the CPOs are given according to the AdS/CFT correspondence by the Kaluza–Klein tower of scalar fields with lowest mass, to be denoted by $s_I$, while the extra primary operators we wish to consider are scalar operators which couple to another Kaluza–Klein tower of scalars to be denoted by $t_I$. These $s_I - t_I$ fields are identified as mass eigenstates contained in the perturbations of the trace of the metric on $S_{2+p}$ and of the $1 + p$ form on $S_{2+p}$. Also, we show why our general model is expected to reproduce the super Yang–Mills case by setting $(D, p) = (10, 3)$. This gives us the opportunity of computing a new class of three point functions for the latter theory.

The paper is organized as follows. In sect. 1 we show why and how all these various cases of maximally supersymmetric CFT can be treated in a unified scheme which employs a general gravitational action. In sect. 2 we present the calculation identifying the bulk cubic couplings of those physical fluctuations which couple to the primary operators of our interest. In sect. 3 we use the couplings to compute the normalized three point functions for the given set of primary operators. Finally, in sect. 4 we present our conclusions and leave appendix A to set some notations for the scalar spherical harmonics on the sphere $S_n$.

1. Justification of the method

We consider first the case of branes in M-theory. According to the AdS/CFT correspondence principles [1–4], the low energy world volume conformal field theory of $N$ coincident M5 (M2) branes at large $N$ is described by $D = 11$ supergravity compactified on AdS$_7 \times S_4$ (AdS$_4 \times S_7$) [11–13]. The conformal operators of the CFT on the AdS boundary are related by duality in a precise way to the operators of the bulk AdS field theory representing the fluctuations of the supergravity fields around a maximally supersymmetric Freund–Rubin background [14]. As explained in the introduction, the aim of this paper is the computation of a class of bosonic three point functions of these CFTs. Thus, we need to consider only a certain subset of bosonic fluctuations described below.

The action of the bosonic sector in the usual 3 form formulation of $D = 11$ supergravity on $M_{11}$ is given by

$$I = \frac{1}{4\kappa^2} \int_{M_{11}} \left[ R(g) *_g 1 - F_4 \wedge *_g F_4 + \frac{2^4}{3} A_3 \wedge F_4 \wedge F_4 \right].$$

(1.1)

Here, $R(g)$ is the Ricci scalar of the metric $g$ and $F_4 = dA_3$ is the 4 form field strength of the 3 form field $A_3$. 
For the M5 theory, \( M_{11} = \text{AdS}_7 \times S_4 \). The Freund–Rubin background \( \bar{g}, \bar{A}_3 \) is such that \( \bar{g} \) is factorized and \( \bar{F}_4 = \bar{F}_{(0,4)} \). The fluctuations of \( g \) and \( A_3 \) around the background, relevant in the analysis that follows, are such that \( g \) remains factorized and \( F_4 = \bar{F}_{(0,4)} + da_{(0,3)} \). By dimensional reasons, for such fluctuations the integrand of the Chern–Simons term vanishes identically. Thus, we may use the truncated action

\[
I = \frac{1}{4\kappa^2} \int_{\text{AdS}_7 \times S_4} \left[ R(g) *_g 1 - F_4 \wedge *_g F_4 \right],
\]

\[
g = g' \oplus g'', \quad F_4 = \bar{F}_{(0,4)} + da_{(0,3)}.
\]

(1.2)

(1.3)

For the M2 theory, \( M_{11} = \text{AdS}_4 \times S_7 \). The Freund–Rubin background \( \bar{g}, \bar{A}_3 \) is such that \( \bar{g} \) is factorized and \( \bar{F}_4 = \bar{F}_{(4,0)} \). The relevant fluctuations of \( g \) and \( A_3 \) around the background are such that \( g \) remains factorized and \( F_4 = \bar{F}_{(4,0)} + da_{(3,0)} \). The integrand of the Chern–Simons term vanishes identically for such fluctuations in this case as well by dimensional reasons. Thus, we may use the action

\[
I = \frac{1}{4\kappa^2} \int_{\text{AdS}_4 \times S_7} \left[ R(g) *_g 1 - F_4 \wedge *_g F_4 \right],
\]

\[
g = g' \oplus g'', \quad F_4 = \bar{F}_{(4,0)} + da_{(3,0)}.
\]

(1.4)

(1.5)

Superficially, this seems to parallel the M5 case closely. However, a closer inspection reveals that the relevant scalar fluctuation contained in \( a_{(3,0)} \) comes about as the solution of the constraint

\[
d * \bar{g}' a_{(3,0)} = 0
\]

entailed by gauge fixing at quadratic level, which is difficult to implement in an off-shell fashion. This problem can be solved by means of a standard dualization trick. On notes that the action \( \int (-dA_3 \wedge *_g dA_3) \) is the reduction of the more general action \( \int (-F_4 \wedge *_g F_4 + 2F_7 \wedge (F_4 - dA_3)) \) upon substituting the \( F_7 \) field equation \( F_4 - dA_3 = 0 \). Substituting first instead the \( A_3 \) and \( F_4 \) field equations \( dF_7 = 0, F_7 = *_g F_4 \), one gets the equivalent dual action \( \int (-dA_6 \wedge *_g dA_6) \), where \( A_6 \) is the 6 form field solving the Bianchi identity

\[\text{(1.6)}\]

\[\text{Consider the } D \text{ dimensional space time } M_D = \text{AdS}_{D-2-p} \times S_{2+p}. \text{ We say that a metric } g \text{ on } M_D \text{ is factorized if } g \text{ has the block structure } g = g' \oplus g'', \text{ where } g', g'' \text{ are metrics on } \text{AdS}_{D-2-p}, S_{2+p}, \text{ respectively. We denote form degree on } M_D \text{ by a suffix, e.g. } \omega_r \text{ is a } r \text{ form on } M_D. \text{ Similarly, we denote form degree on the factors } \text{AdS}_{D-2-p}, S_{2+p} \text{ by a pair of suffixes, e.g. } \omega_{(r,s)} \text{ denotes a } r+s \text{ form on } M_D \text{ that is a } r \text{ form on } \text{AdS}_{D-2-p} \text{ and a } s \text{ form on } S_{2+p}.\]
\[ dF_7 = 0. \] In the dual formulation, the Freund–Rubin background \( \bar{g}, \bar{A}_6 \) is such that \( \bar{g} \) is again factorized and \( \bar{F}_7 = \bar{F}(0,7) \). The relevant fluctuations of \( g \) and \( A_6 \) around the background are such that \( g \) remains factorized and \( F_7 = \bar{F}(0,7) + da_{(0,6)} \). To summarize, the action is

\[
I = \frac{1}{4\kappa^2} \int_{\text{AdS}_4 \times S_5} \left[ R(g) \ast_g 1 - F_7 \wedge \ast_g F_7 \right],
\]

\[
g = g' \oplus g'', \quad F_7 = \bar{F}(0,7) + da_{(0,6)}. \tag{1.8}
\]

We now turn to the case of D3 branes in type IIB superstring theory. According to the AdS/CFT correspondence, the low energy world volume conformal field theory of a large number of coincident D3 branes is described by \( D = 10 \) type IIB supergravity compactified on \( \text{AdS}_5 \times S_5 \). Again, the conformal operators of the CFT on the AdS boundary are related by duality in a precise way to the operators of the bulk AdS field theory representing the fluctuations of the IIB supergravity fields around a maximally supersymmetric selfdual Freund–Rubin background [15]. To compute the set of bosonic three point functions we are interested in, we need to consider only a certain subset of bosonic fluctuations described below. We may try to proceed as we did above dealing with M theory and begin our discussion from the fully covariant action of the bosonic sector of type IIB supergravity worked out in ref. [16], but this does not seem to bring us that far. Thus, we follow a different path which we are now going to explain.

For the D3 brane, space time is \( M_{10} = \text{AdS}_5 \times S_5 \). The relevant fields are the metric \( g \) and the Ramond–Ramond 4 form field \( A_4 \) with selfdual field strength \( F_{5}^{sd} = dA_4 \)

\[
F_{5}^{sd} = \ast_g F_{5}^{sd}. \tag{1.9}
\]

Further, Einstein’s field equations hold

\[
R(g)_{ij} \ast_g 1 = F_{5}^{sd} i \wedge \ast_g F_{5}^{sd} j, \tag{1.10}
\]

where \( R(g)_{ij} \) is the Ricci tensor and \( F_{5}^{sd} i = \frac{1}{4!} F_{ijklm}^{sd} dx^j \wedge dx^k \wedge dx^l \wedge dx^m \).

The selfdual Freund–Rubin background \( \bar{g}, \bar{A}_4 \) is such that \( \bar{g} \) is factorized and \( \bar{F}_5^{sd} = 2^{-\frac{1}{2}}(\bar{F}(0,5) + \ast_{\bar{g}}\bar{F}(0,5)) \). The relevant fluctuations of \( g \) and \( A_4 \) around the background are such that \( g \) is factorized, as usual, and \( F_{5}^{sd} = \tilde{F}_{5}^{sd} + da_4 \), where \( a_4 = 2^{-\frac{1}{2}}(a(4,0) + a_{(0,4)}) \). The selfduality equations relate the fluctuations \( a_{(4,0)}, a_{(0,4)} \) as

\[
\ast_g (d_{(1,0)}a_{(0,4)}) - d_{(0,1)}a_{(4,0)} = 0, \tag{1.11a}
\]

\[
\ast_g (d_{(0,1)}a_{(0,4)}) - d_{(1,0)}a_{(4,0)} + (\ast_g - \ast_{\bar{g}})\bar{F}(0,5) = 0. \tag{1.11b}
\]
Therefore, \(a_{(4,0)}\) is not independent of \(a_{(0,4)}\) as eq. (1.11a) allows in principle to express \(a_{(4,0)}\) in terms of \(a_{(0,4)}\). Taking the selfduality equations into account, one can check that Einstein’s equation can be written in terms of \(a_{(0,4)}\) only

\[
R(g)_{ij} * g 1 \equiv (\bar{F}_{(0,5)} + da_{(0,4)})_i \wedge * g (\bar{F}_{(0,5)} + da_{(0,4)})_j - \frac{1}{2} g_{ij} (\bar{F}_{(0,5)} + da_{(0,4)}) \wedge * g (\bar{F}_{(0,5)} + da_{(0,4)}). \tag{1.12}
\]

Let us examine the above field equations from another point of view. We consider an action of the form

\[
I = \int_{\text{AdS}_5 \times S_5} \left[ R(g) * g 1 - F_5 \wedge * g F_5 \right], \tag{1.13}
\]

where \(g\) is a metric and \(F_5 = dA_4\) is a 5 form field strength, not necessarily selfdual. We pick a Freund–Rubin like background \(\tilde{g}, \tilde{A}_4\) where \(\tilde{g}\) is factorized as usual and \(F_5 = \tilde{F}_{(0,5)}\). We restrict to fluctuations for which \(g\) remains factorized and \(F_5 = \tilde{F}_{(0,5)} + da_{(0,4)}\). The field equations deduced from this action for the \(a_{(0,4)}\) fluctuations are

\[
d_{(0,1)} ( * g (d_{(1,0)} a_{(0,4)}) ) = 0, \tag{1.14a}
\]

\[
d_{(0,1)} ( * g (d_{(0,1)} a_{(0,4)}) + ( * g - * \tilde{g} ) \tilde{F}_{(0,5)} ) + d_{(1,0)} ( * g (d_{(1,0)} a_{(0,4)}) ) = 0. \tag{1.14b}
\]

From (1.14a), recalling that \(H^1_{\text{deRham}}(S_5) = 0\), one has

\[
* g (d_{(1,0)} a_{(0,4)}) = d_{(0,1)} a_{(4,0)} \tag{1.15}
\]

for some 4 form \(a_{(4,0)}\). Substituting this relation in (1.14b), one gets

\[
d_{(0,1)} ( * g (d_{(0,1)} a_{(0,4)}) ) - d_{(1,0)} a_{(4,0)} + ( * g - * \tilde{g} ) \tilde{F}_{(0,5)} = 0. \tag{1.16}
\]

Using that \(H^0_{\text{deRham}}(S_5) = \mathbb{R}\), this equation can be integrated once, yielding

\[
* g (d_{(0,1)} a_{(0,4)}) - d_{(1,0)} a_{(4,0)} + ( * g - * \tilde{g} ) \tilde{F}_{(0,5)} = \omega_{(5,0)}, \tag{1.17}
\]

where the 5 form \(\omega_{(5,0)}\) satisfies

\[
d_{(0,1)} \omega_{(5,0)} = 0, \tag{1.18}
\]

but is otherwise arbitrary. As \(H^5_{\text{deRham}}(\text{AdS}_5) = 0\), there is a 4 form \(\omega_{(4,0)}\) such that

\[
\omega_{(5,0)} = d_{(1,0)} \omega_{(4,0)}, \quad d_{(0,1)} \omega_{(4,0)} = 0. \tag{1.19}
\]

Now, from (1.15), it is evident that \(a_{(4,0)}\) is not uniquely defined and might be redefined into \(a_{(4,0)} - \omega_{(4,0)}\). By doing so, we can assume that

\[
\omega_{(5,0)} = 0 \tag{1.20}
\]
in eq. (1.17). We could have arrived to eq. (1.20) more directly by noting that we need to consider only small perturbations around the chosen background. Taking the limit of vanishing perturbations in eq. (1.17) and requiring that the background satisfies the resulting equation of motion fixes \( \omega(5,0) = 0 \). In conclusion, eqs. (1.14a), (1.14b) have the same content as eqs. (1.11a), (1.11b). The Einstein’s field equations following from (1.13) are the same as (1.12). Thus, by the above reasoning, the selfdual dynamics of the relevant bosonic type IIB fluctuations is reduced to that of the field \( g \) and \( a_{(0,4)} \) with action

\[
I = \frac{1}{4\kappa^2} \int_{\text{AdS}_5 \times S_5} \left[ R(g) *_{g} 1 - F_{5} \wedge *_{g} F_{5} \right],
\]

\[
g = g' \oplus g'', \quad F_{5} = \tilde{F}_{(0,5)} + da_{(0,4)}. \tag{1.21}
\]

The upshot of the above discussion is that all three models described above may be treated on the same footing as follows, in spite of their different physical content.

Space time \( M_D \) is \( \text{AdS}_{D-2-p} \times S_{2+p} \). The relevant fields are the metric \( g \) and the \( 1+p \) form field \( A_{1+p} \). The Freund–Rubin background \( \bar{g}, \bar{A}_{1+p} \) is such that \( \bar{g} \) is factorized and \( \bar{F}_{2+p} = \tilde{F}_{(0,2+p)} \). The relevant bosonic fluctuations of \( g \) and \( A_{1+p} \) around the background are such that \( g \) remains factorized and \( F_{2+p} = \tilde{F}_{(0,2+p)} + da_{(0,1+p)} \). The action effectively describing the dynamics of the fluctuations is

\[
I = \frac{1}{4\kappa^2} \int_{\text{AdS}_{D-2-p} \times S_{2+p}} \left[ R(g) *_{g} 1 - F_{2+p} \wedge *_{g} F_{2+p} \right],
\]

\[
g = g' \oplus g'', \quad F_{2+p} = \tilde{F}_{(0,2+p)} + da_{(0,1+p)}. \tag{1.23}
\]

The physically important cases are those for which \( (D,p) = (11,2), (11,5), (10,3) \).

### 2. Implementation of the method

We are now going to concretely implement the method described in the former section. Our starting point is the general action in eq. (1.23). One can easily verify that the field equations admit the standard \( \text{AdS}_{D-2-p} \times S_{2+p} \) solution \( \bar{g}_{ij}, \bar{A}_{i_{1+p}} \) generated by the Freund Rubin ansatz [14]. For this, the only non vanishing components of the Riemann tensor and field strength are given by

\[
R_{\kappa\lambda\mu\nu} = -a_{1}(\bar{g}_{\kappa\mu}\bar{g}_{\lambda\nu} - \bar{g}_{\kappa\nu}\bar{g}_{\lambda\mu}), \quad a_{1} = \frac{(1+p)}{(D-2)(D-3-p)}e^2, \tag{2.1a}
\]

\[
\bar{R}_{\alpha\beta\gamma\delta} = a_{2}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}), \quad a_{2} = \frac{(D-3-p)}{(D-2)(1+p)}e^2, \tag{2.1b}
\]

\[
\bar{F}_{\alpha_{1}...\alpha_{2+p}} = e\tilde{\varepsilon}_{\alpha_{1}...\alpha_{2+p}}, \quad \text{other } \bar{F} = 0. \tag{2.2}
\]
where $\epsilon_{\alpha_1 \cdots \alpha_{2+p}}$ denotes the standard volume form on the unit sphere and $e$ is an arbitrary mass scale parametrizing the compactification $^2$.

We expand the action in fluctuations around the background $g_{ij}$, $A_{i_1 \cdots i_1+p}$. We parametrize the fluctuations $\delta g_{ij}$, $\delta A_{i_1 \cdots i_1+p}$ of the fields $g_{ij}$, $A_{i_1 \cdots i_1+p}$ around the background as in [11]

$$\delta g_{\kappa\lambda} = h_{\kappa\lambda} - \frac{1}{D-4-p} \bar{g}_{\kappa\lambda} \pi,$$

$$\delta g_{a\kappa} = k_{a\kappa} + \bar{\nabla}_a l_\kappa,$$

$$\bar{\nabla}^\gamma k_{\gamma\kappa} = 0,$$

$$\delta g_{a\beta} = m_{a\beta} + \bar{\nabla}_a n_\beta + \bar{\nabla}_\beta n_a + (\bar{\nabla}_a \bar{\nabla}_\beta - \frac{1}{2+p} \bar{g}_{a\beta} \bar{\nabla}^\gamma \bar{\nabla}_\gamma) q + \frac{1}{2+p} \bar{g}_{a\beta} \pi,$$

$$m^\gamma_{\gamma} = 0, \quad \bar{\nabla}^\gamma m_{\gamma a} = 0, \quad \bar{\nabla}^\gamma n_\gamma = 0,$$

$$\delta A_{\alpha_1 \cdots \alpha_{1+p}} = (1+p) \bar{\nabla}_{[\alpha_1} a_{\alpha_2 \cdots \alpha_{1+p}]} + \epsilon_{\alpha_1 \cdots \alpha_{1+p}} \bar{\nabla}^\gamma b;$$

$$\bar{\nabla}^\gamma a_{\gamma \alpha_3 \cdots \alpha_{1+p}} = 0.$$  \hspace{1cm} (2.3a) - (2.4)

Fluctuations of the other components of $A_{i_1 \cdots i_1+p}$ can be disregarded as they are independent from the ones we are interested in, and we can set them directly to zero. Before describing the calculation, we note that one could partially fix the gauge by eliminating those gauge invariances that do not correspond to the usual reparametrization and form gauge invariances from the AdS$_{D-2-p}$ perspective. This could be done by imposing

$$\bar{\nabla}^\beta (\delta g_{\beta a} - \frac{1}{2+p} \bar{g}_{\beta a} \delta g_{\gamma \gamma}) = 0, \quad \bar{\nabla}^\beta \delta g_{\beta \kappa} = 0;$$

$$\bar{\nabla}^\beta \delta A_{i_1 \cdots i_1+p} = 0,$$  \hspace{1cm} (2.5a) - (2.5b)

as shown in [11]. These conditions imply in particular that $\bar{\nabla}_a l_\kappa = 0$, $\bar{\nabla}_a n_\beta + \bar{\nabla}_\beta n_a = 0$, $(\bar{\nabla}_a \bar{\nabla}_\beta - \frac{1}{2+p} \bar{g}_{a\beta} \bar{\nabla}^\gamma \bar{\nabla}_\gamma) q = 0$ and $a_{\alpha_1 \cdots a_p} = 0$. So, upon gauge fixing, the field equations of the fields $l_\kappa$, $n_\alpha$, $q$ and $a_{\alpha_1 \cdots a_p}$ must be enforced by hand as constraints. However, in the sequel we will not need to proceed this way. Rather, we will identify the action describing the dynamics of the physical scalar fields contained in the $\pi$-$b$ fluctuations by taking care of the constraints associated with the various gauge invariances using field redefinitions.

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$^2$ In this paper, we adopt the following conventions. Latin lower case letters $i, j, k, l, \ldots$ denote $M_D$ indices. Late Greek lower case letters $\kappa, \lambda, \mu, \nu, \ldots$ denote AdS$_{D-2-p}$ indices. Early Greek lower case letters $\alpha, \beta, \gamma, \delta, \ldots$ denote S$_{2+p}$ indices.
The quadratic and cubic actions of the fluctuation fields are given by

\[
I_{[2]} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{D-2-p}} d^{D-2-p}y (-\bar{g}_D-2-p)^{1/2} \int_{S^2_{2+p}} d^{2+p}x (\bar{g}_{2+p})^{1/2} \left\{ \frac{1}{2} \bar{\nabla}^\alpha h_{\kappa \mu} \bar{\nabla}_\alpha h_{\kappa \mu} - \frac{1}{2} \bar{\nabla}^\kappa h_{\kappa \mu} \bar{\nabla}_\kappa h_{\rho \mu} + \frac{1}{4} \bar{\nabla}^\kappa h^\lambda_\kappa \bar{\nabla}_\kappa h_{\mu \lambda} - \frac{1}{4} \bar{\nabla}^\kappa h_{\lambda \mu} \bar{\nabla}_\kappa h_{\kappa \lambda} \\
+ \frac{1}{4} \bar{\nabla}^\alpha h_{\kappa \lambda} \bar{\nabla}_\alpha h_{\kappa \lambda} + \frac{1}{2} \bar{\nabla}^\alpha h_{\kappa \lambda} \bar{\nabla}_\alpha h_{\kappa \lambda} \right\} + \frac{1 + p}{2(D - 2)} e^2 \left[ \frac{1}{2} h^\kappa_{\kappa \lambda} h_{\lambda \mu} - h^\kappa_{\lambda \mu} h_{\kappa \lambda} \right]
\]

and

\[
I_{[3]} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{D-2-p}} d^{D-2-p}y (-\bar{g}_D-2-p)^{1/2} \int_{S^2_{2+p}} d^{2+p}x (\bar{g}_{2+p})^{1/2} \left\{ \frac{1}{4} h^\nu_\nu \left[ \bar{\nabla}^\kappa h^\lambda_\kappa \bar{\nabla}_\lambda h_{\kappa \mu} - \bar{\nabla}^\kappa h_{\kappa \lambda} \bar{\nabla}^\lambda_{\mu} h_{\kappa \mu} + \frac{1}{2} \bar{\nabla}^\kappa h^\lambda_\kappa \bar{\nabla}_\kappa h_{\mu \lambda} - \frac{1}{4} \bar{\nabla}^\kappa h_{\lambda \mu} \bar{\nabla}_\kappa h_{\kappa \lambda} \right] \\
+ \frac{1}{4} h^\kappa_{\kappa \lambda} h_{\kappa \mu} \left[ \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu \nu} - \bar{\nabla}^\mu \bar{\nabla}_\mu h_{\nu \lambda} \right] - \frac{1}{2} h^\kappa_{\kappa \lambda} \left[ 2 \bar{\nabla}^\kappa h_{\mu \nu} \bar{\nabla}_\mu h_{\lambda \nu} \right] \\
+ \bar{\nabla}^\mu h_{\kappa \nu} \bar{\nabla}_\mu h_{\lambda \nu} - \bar{\nabla}^\nu h_{\kappa \mu} \bar{\nabla}_\mu h_{\lambda \nu} - \bar{\nabla}^\nu h_{\kappa \mu} \bar{\nabla}^\mu h_{\lambda \nu} - \bar{\nabla}^\mu h_{\kappa \nu} \bar{\nabla}_\mu h_{\lambda \nu} \\
+ \frac{1}{4} \bar{\nabla}^\kappa h_{\mu \nu} \bar{\nabla}^\lambda_\kappa h_{\mu \nu} - \frac{1}{4} \bar{\nabla}^\kappa h_{\mu \nu} \bar{\nabla}^\lambda_\mu h_{\kappa \nu} + \frac{1}{4} \left[ h^\kappa_{\kappa \lambda} h_{\kappa \lambda} - \frac{1}{4} h^\kappa_{\kappa \lambda} h_{\lambda \mu} \right] \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu \nu} \right] + \frac{1}{2} h^\kappa_{\kappa \lambda} \bar{\nabla}^\alpha h_{\mu \nu} - \frac{1}{8} h^\kappa_{\kappa \lambda} \bar{\nabla}^\alpha h_{\kappa \mu} \bar{\nabla}_\alpha h_{\lambda \mu} \\
+ \frac{1 + p}{D - 2} e^2 \left[ \frac{1}{12} h^\kappa_{\kappa \lambda} h^\lambda_\mu h_{\mu \kappa} - \frac{1}{4} h^\kappa_{\kappa \lambda} h_{\mu \lambda} h_{\mu \kappa} + \frac{2}{3} h^\kappa_{\kappa \lambda} h_{\mu \lambda} h_{\mu \kappa} \right] \\
+ \frac{1}{2(D - 4 - p)} \pi \left[ h^\kappa_{\kappa \lambda} \bar{\nabla}^\mu \bar{\nabla}_\mu h_{\kappa \lambda} - h^\kappa_{\kappa \lambda} \bar{\nabla}^\mu \bar{\nabla}_\kappa h_{\mu \lambda} + h^\kappa_{\kappa \lambda} \bar{\nabla}^\mu h_{\kappa \lambda} \bar{\nabla}_\mu h_{\lambda \kappa} \\
+ h^\kappa_{\kappa \lambda} \bar{\nabla}^\mu h_{\kappa \lambda} - 2 h^\kappa_{\kappa \lambda} \bar{\nabla}^\mu h_{\kappa \lambda} \right] + \frac{D - 6 - 2p}{4(2 + p)(D - 4 - p)} \pi h^\kappa_{\kappa \lambda} \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\lambda \kappa} \lambda \\
- \frac{1}{2(2 + p)} \pi h^\kappa_{\kappa \lambda} \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\kappa \lambda} - \frac{D - 2}{4(2 + p)(D - 4 - p)} \pi \bar{\nabla}^\alpha h^\kappa_{\kappa \lambda} \bar{\nabla}_\alpha h_{\kappa \lambda} \\
+ \frac{1 + p}{(D - 2)(D - 4 - p)} e^2 \left[ \frac{1}{2} h^\kappa_{\kappa \lambda} h_{\lambda \mu} - h^\kappa_{\kappa \lambda} h_{\lambda \mu} \right]
\]
By construction, the field $\phi$ is the field of interest.

The scalar fields $s$ and $t$ which couple to the chiral operator of the boundary conformal field theory of our interest.

Instead of fixing the gauge right away, we isolate the relevant scalar degrees of freedom by performing suitable field redefinitions, as done in [17,18]. We write the AdS metric fluctuations as

$$h_{\kappa\lambda} = \phi_{\kappa\lambda} + (\tilde{\nabla}_{\kappa} \tilde{\nabla}_{\lambda} - \frac{1}{D-4-p} \tilde{g}_{\kappa\lambda} \tilde{\nabla}^{\alpha} \tilde{\nabla}_{\alpha}) \varphi,$$  

where

$$\varphi = \left(-\tilde{\nabla}^{\alpha} \tilde{\nabla}_{\alpha} + \frac{(1+p)(D-4-p)}{(D-2)(D-3-p)} e^{2}\right)^{-1} \times \left(\frac{D-2}{2(p+1)(D-3-p)} \pi + (-1)^{p} \frac{2(D-4-p)}{D-3-p} e^{b}\right).$$

By construction, the field $\phi_{\kappa\lambda}$ decouples from the fields $\pi, b$ at the quadratic level.

The scalar fields $s, t$ are given by linear functionals of $\pi, b$ non local in $S_{2+p}$ defined as follows. One expands $s, t$ as well as the scalar fields $\pi, b$ with respect to an orthonormal basis $\{Y_I\}$ of scalar spherical harmonics of $S_{2+p}$ (cfr. appendix A1)

$$\pi = \sum_{I} \pi_I Y_I, \quad b = \sum_{I} b_I Y_I.$$
Then,

\[ s = \sum_i s_I Y_I, \quad t = \sum_I t_I Y_I, \quad (2.11a) - (2.11b) \]

where

\[ s_I = \frac{1}{2k + 1 + p} \left( \frac{1}{2(2 + p)(D - 3 - p)} \pi I + \frac{(-1)^p(k + 1 + p)}{(1 + p)(D - 2)} e b_I \right), \quad (2.12a) \]

\[ t_I = \frac{1}{2k + 1 + p} \left( \frac{1}{2(2 + p)(D - 3 - p)} \pi I - \frac{(-1)^p k}{(1 + p)(D - 2)} e b_I \right). \quad (2.12b) \]

The action obtained in this way is of the general form

\[
I = \int_{\text{AdS}_{d+1}} dy^{d+1} (-\tilde{g}_{d+1})^{\frac{1}{2}} \left\{ -\sum_i \frac{A_i}{2} \left[ \nabla_\kappa \psi_i \nabla_\kappa \psi_i + m_i^2 \psi_i \psi_i \right] + \sum_{ijk} \left[ \lambda_{ijk} \psi_i \nabla_\kappa \nabla_\kappa \psi_j \nabla_\lambda \psi_k + \mu_{ijk} \psi_i \psi_j \nabla_\kappa \nabla_\kappa \nabla_\lambda \psi_k \\
+ \rho_{ijk} \psi_i \psi_j \nabla_\kappa \nabla_\kappa \psi_k + \sigma_{ijk} \psi_i \psi_j \psi_k \right] + \cdots \right\},
\]

where the \( \psi_i \) are scalar fields and \( \lambda_{ijk} = \lambda_{ikj}, \mu_{ijk} = \mu_{jik}, \rho_{ijk} = \rho_{jik}, \sigma_{ijk} = \sigma_{jik} = \sigma_{ikj} = \cdots \) etc. and the ellipses denote terms in the \( \psi_i \) of order larger than 3. By performing successively the field redefinitions

\[
\psi_i = \psi_i' - \frac{1}{A_i} \sum_{jk} \left[ \lambda_{jki} \psi_j' \nabla_\kappa \nabla_\kappa \psi_k + \mu_{jki} \nabla_\kappa \nabla_\kappa \left( \psi_j' \psi_k' \right) \right],
\]

\[
\psi_i' = \psi_i'' - \sum_{jk} \left( \rho_{jki} + \lambda_{jki} m_k^2 + \mu_{jki} m_i^2 \right) \psi_j' \psi_k' \psi_k',
\]

one can bring the action in the form

\[
I = \int_{\text{AdS}_{d+1}} dy^{d+1} (-\tilde{g}_{d+1})^{\frac{1}{2}} \left\{ -\sum_i \frac{A_i}{2} \left[ \nabla_\kappa \psi_i'' \nabla_\kappa \psi_i'' + m_i^2 \psi_i'' \psi_i'' \right] + \sum_{ijk} g_{ijk} \psi_i'' \psi_j'' \psi_k'' + \cdots \right\},
\]

where the totally symmetric coupling constants \( g_{ijk} \) are given by

\[
g_{ijk} = \frac{1}{3} (\lambda_{ijk} m_j^2 m_k^2 + \lambda_{jki} m_k^2 m_i^2 + \lambda_{kij} m_i^2 m_j^2) + \frac{1}{6} (\mu_{ijk} m_j^4 + \mu_{jki} m_i^4 + \mu_{kij} m_j^4) + \frac{1}{3} (\rho_{ijk} m^2 + \rho_{jki} m^2 + \rho_{kij} m^2) + \sigma_{ijk}.
\]
Then, one finds that, after performing the indicated field redefinitions, the action of \(s\) and \(t\) to cubic order is given by (suppressing double primes for simplicity)

\[
I_{[s,t]}^{st} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{d+1}} d^{d+1}y (-\bar{g}_{d+1})^{\frac{1}{2}} \left\{ \sum_I \left[ A_I^s \left( -\frac{1}{2} \nabla^\kappa s_I \nabla^\kappa s_I - \frac{1}{2} m_{sI}^2 s_I s_I \right) + A_I^t \left( -\frac{1}{2} \nabla^\kappa t_I \nabla^\kappa t_I - \frac{1}{2} m_{tI}^2 t_I t_I \right) \right] + \sum_{I_1 I_2 I_3} \left[ g_{I_1 I_2 I_3}^{ss} s_{I_1} s_{I_2} s_{I_3} + g_{I_1 I_2 I_3}^{st} s_{I_1} t_{I_2} s_{I_3} + g_{I_1 I_2 I_3}^{ts} t_{I_1} t_{I_2} s_{I_3} + g_{I_1 I_2 I_3}^{tt} t_{I_1} t_{I_2} t_{I_3} \right] \right\}, \tag{2.13}
\]

with \(d = D - 3 - p\). The various constants appearing in the actions are given by the following expressions.

\[
A_I^s = \frac{2
u k(k + 1)(2k + 1 + p)}{k + \gamma_s} z_I e^{-2-p}, \tag{2.14a}
\]

\[
A_I^t = \frac{2\nu (k + 1 + p)(k + 2 + p)(2k + 1 + p)}{k + \gamma_t} z_I e^{-2-p}; \tag{2.14b}
\]

\[
m_{sI}^2 = k(k - 1)p \epsilon^2; \tag{2.15a}
\]

\[
m_{tI}^2 = (k + 1 + p)(k + 2 + 2p) \epsilon^2; \tag{2.15b}
\]

\[
g_{I_1 I_2 I_3}^{ss} = \zeta \frac{\alpha_1 \alpha_2 \alpha_3 (\alpha - \frac{1}{2} (1 + p)) (\alpha + \frac{1}{2} (1 + p))}{3(k_1 + \gamma_s)(k_2 + \gamma_s)(k_3 + \gamma_s)} \times \left\{ (\alpha - 1) \left( \alpha + \frac{1 + p}{D - 3 - p} \right) \left( \alpha + \frac{(1 + p)(-D + 4 + 2p)}{2(D - 2)} \right) \right. \\
+ \frac{\theta}{\nu} \left[ (1 + p)(D - 3 - p)(\alpha_1^2 \alpha_2 + \alpha_1 \alpha_2^2 \alpha_1 + \alpha_2^2 \alpha_1 + \alpha_3^2 \alpha_2 + \alpha_3^2 \alpha_1 + \alpha_1^2 \alpha_3) \right. \\
\left. + (3D - 8 + (2D - 8)p - 2p^2) \alpha_1 \alpha_2 \alpha_3 \right. \\
\left. + (1 + p)(-\frac{1}{2} D + 2 + p)(\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1) \right\} a_{I_1 I_2 I_3} \langle c_{I_1} c_{I_2} c_{I_3} \rangle e^{-p}, \tag{2.16a}
\]

\[
g_{I_1 I_2 I_3}^{st} = \zeta \frac{(\alpha_1 + \frac{1}{2} (1 + p))(\alpha_2 + \frac{1}{2} (1 + p))(\alpha_3 (\alpha_3 - 1 - p) (\alpha + \frac{1}{2} (1 + p))}{(k_1 + \gamma_s)(k_2 + \gamma_s)(k_3 + \gamma_t)} \times \left\{ (\alpha_3 - 1) \left( \alpha_3 + \frac{(1 + p)(-D + 3 + p)}{D - 2} \right) \left( \alpha_3 + \frac{(1 + p)(-D + 4 + p)}{D - 3 - p} \right) \right. \\
- \frac{\theta}{\nu} \left[ (1 + p)(D - 3 - p)(\alpha_1^2 \alpha_3 + \alpha_1 \alpha_3^2 \alpha_1 + \alpha_2^2 \alpha_3 + \alpha_3^2 \alpha_2) \right. \\
\left. + (-1 + (D - 4)p - p^2) \alpha_1^2 \alpha_2 + \alpha_2^2 \alpha_1 \right\} a_{I_1 I_2 I_3} \langle c_{I_1} c_{I_2} c_{I_3} \rangle e^{-p}, \tag{2.16b}
\]

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+ (D - 4 + (2D - 8)p - 2p^2)\alpha_1\alpha_2\alpha_3 \\
+ (1 + p)p(D - 3 - p)(\alpha_1\alpha_3 + \alpha_2\alpha_3) \\
+ (1 + p)(-D + 2 + (D - 3)p - p^2)\alpha_1\alpha_2 \\
+ (1 + p)(D - 3 - p)(-\alpha_1^2 - \alpha_2^2 + (1 + p)\alpha_3^2) \\
+ (1 + p)^2(-D + 3 + p)(\alpha_1 + \alpha_2 + \alpha_3) \right\} a_{t_1,t_2,t_3} \langle C_{t_1} C_{t_2} C_{t_3} \rangle \bar{e}^{-p},

(2.16b)

g_{t_1,t_2,t_3}^{tts} = \zeta \frac{\alpha_1\alpha_2(\alpha_3 + \frac{1}{5}(1 + p))(\alpha_3 + \frac{3}{5}(1 + p))((\alpha + 1 + p)(\alpha + 1 + \alpha_3) \\
\times \left\{ (\alpha + 2 + p)\left((\alpha + \frac{1+p}{2}(2D - 8 - 2p))\left(\alpha + \frac{(1+p)(D - 4 + p)}{2(D - 3 - p)}ight) \\
\right.

- \frac{\theta}{\nu} \left[(1 + p)(D - 3 - p)(\alpha_1^2\alpha_2 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_2 + \alpha_3^2\alpha_1 + \alpha_1^2\alpha_3) \\
+ (3D - 8 + (2D - 8)p - 2p^2)\alpha_1\alpha_2\alpha_3 \\
+ (1 + p)(4D - 13 + (3D - 14)p - 3p^2)(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) \\
+ (1 + p)^2(D - \frac{7}{2} - p)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \\
+ (1 + p)^2(3D - 11 + (2D - \frac{21}{2})p - 2p^2)(\alpha_1 + \alpha_2 + \alpha_3)

(2.16c)
\begin{equation}
\left. + (2 + p)(1 + p)^3(D - 4 - p) \right\} a_{I_1 I_2 I_3} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \bar{e}^{-p}, \tag{2.16d}
\end{equation}

where

\begin{align}
\nu &= (D - 2)(1 + p)(D - 3 - p), \tag{2.17a} \\
\gamma_s &= \frac{1 + p}{D - 3 - p}, \tag{2.17b} \\
\gamma_t &= \frac{(1 + p)(D - 4 - p)}{D - 3 - p}, \tag{2.17c} \\
\theta &= -D + 1 + (D - 4)p - p^2, \tag{2.17d} \\
\zeta &= 16(D - 2)^2(1 + p)(D - 3 - p), \tag{2.17e} \\
\bar{e}^2 &= \frac{(D - 3 - p)}{(D - 2)(1 + p)}e^2, \tag{2.18}
\end{align}

and

\begin{align}
\alpha_1 &= \frac{1}{2}(k_2 + k_3 - k_1), \quad \text{etc.}, \tag{2.19a} \\
\alpha &= \frac{1}{2}(k_1 + k_2 + k_3), \tag{2.19b}
\end{align}

and \( z_I, a_{I_1 I_2 I_3}, \langle C_{I_1} C_{I_2} C_{I_3} \rangle \) are defined in appendix A1.

As previously mentioned, the fields \( q \) and \( l_\kappa \) act as Lagrange multipliers enforcing certain constraints involving \( h_{\kappa \lambda}, \pi, b \). Such constraints, however, do not affect the quadratic and cubic terms of the fields \( s \) and \( t \) in the action, which are thus gauge invariant. This can be shown as follows.

To linear order, the constraints can be read off from the part \( I_{[2]}^{\text{constr}} \) of the quadratic action \( I_{[2]} \) which is linear in the fields directly involved by gauge fixing such as \( l_\kappa, n_\alpha, q, \) etc. By direct computation, one can check that only \( l_\kappa, q \) contribute to \( I_{[2]}^{\text{constr}} \)

\begin{equation}
I_{[2]}^{\text{constr}} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{D-2-p}} d^{D-2-p}y(-\bar{g}_{D-2-p})^{\frac{1}{2}} \int_{S_{2+p}} d^{2+p}x(\bar{g}_{2+p})^{\frac{1}{2}} \left\{ \frac{1}{2} \left[ 1 + p \nabla^\alpha \nabla_\alpha + \frac{D - 3 - p}{D - 2} e^2 \right] \nabla^\beta \nabla_\beta q \left[ h^\kappa_\kappa - \frac{2(D - 2)}{(2 + p)(D - 4 - p)} \pi \right] \\
+ \nabla^\alpha \nabla_\alpha t^\kappa \left[ \frac{(D - 2)}{(2 + p)(D - 4 - p)} \nabla_\kappa \pi + 2(-1)^p e \nabla_\kappa b + \nabla^\lambda h_{\lambda \kappa} - \nabla_\kappa h^\lambda_\lambda \right] \right\}. \tag{2.20}
\end{equation}

Perform the field redefinition (2.9) and expand \( \phi_{\kappa \lambda} l_\kappa, q \) in scalar harmonics of \( S_{2+p} \)

\begin{equation}
\phi_{\kappa \lambda} = \sum_I \phi_{I \kappa \lambda} Y_I, \tag{2.21}
\end{equation}

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\[ q = \sum_I q_I Y_I, \quad l_\kappa = \sum_I l_{I\kappa} Y_I. \] (2.22a) – (2.22b)

Then \( I_{[2]}^{\text{constr}} \) takes the form

\[
I_{[2]}^{\text{constr}} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{d+1}} d^{d+1}y (-\bar{g}_{d+1})^{\frac{1}{2}} \sum_I \left\{ u_I q_I \left[ \phi^I_{\kappa \kappa} + v^s_I \left( \bar{\nabla}^s \bar{\nabla} s_I - m_s s_I \right) \right] + v^t_I \left( \bar{\nabla}^t \bar{\nabla} t_I - m_t t_I \right) \right\} - w_I l^\kappa_I \left[ \bar{\nabla}^\lambda \phi^I_{\kappa \lambda \kappa} - \bar{\nabla}^\kappa \phi^I_{\kappa \lambda \lambda} \right],
\]

(2.23)

where

\[
u^s_I = \frac{2(D - 2)}{k + \gamma_s} \bar{\epsilon}^{-2}, \quad \nu^t_I = \frac{2(D - 2)}{k + \gamma_t} \bar{\epsilon}^{-2}, \quad w_I = k(k + 1 + p)z_I \bar{\epsilon}^{-p}, \]

(2.25a) – (2.25b)

\( \gamma_s, \gamma_t \) are given by (2.17b), (2.17c) and \( z_I \) is defined in appendix A1. The gauge fixing conditions (2.5) would amount to

\[
l_{I\kappa} = 0, \quad k \geq 1, \quad q_I = 0, \quad k \geq 2. \]

(2.27)

The constraint associated to \( q_I \) yields an apparent mixing of \( \phi^I_{\kappa \kappa}, s_I, t_I \). The mixing can be removed, before fixing the gauge, by performing the following field redefinition in the action \( I_{[\leq 3]}^{st} \) and \( I_{[2]}^{\text{constr}} \) (cfr eqs. (2.13) and (2.23)):

\[
s_I = s'_I - c^s_I q_I, \quad t_I = t'_I - c^t_I q_I,
\]

(2.28)

where

\[
c^s_I = \frac{(k + 1 + p)(k + 2 + p)}{2(D - 3 - p)(2 + p)(2k + 1 + p)} \bar{\epsilon}^2, \quad c^t_I = \frac{k(k - 1)}{2(D - 3 - p)(2 + p)(2k + 1 + p)} \bar{\epsilon}^2.
\]

(2.29a) – (2.29b)

Then, the \( st \) action \( I_{[\leq 3]}^{st} \) remains of the form (2.13) while \( I_{[2]}^{\text{constr}} \) becomes

\[
I_{[2]}^{\text{constr}} = \frac{1}{4\kappa^2} \int_{\text{AdS}_{d+1}} d^{d+1}y (-\bar{g}_{d+1})^{\frac{1}{2}} \sum_I \left\{ u_I q_I \phi^I_{\kappa \kappa} - w_I l^\kappa_I \left[ \bar{\nabla}^\lambda \phi^I_{\kappa \lambda \kappa} - \bar{\nabla}^\kappa \phi^I_{\kappa \lambda \lambda} \right] \right\}. \quad (2.30)
\]
In this way, the linear order constraints are simply conditions on $\phi_{k\lambda}$ and decouple from $s$ and $t$. At higher orders, the constraints yielded by gauge fixing are therefore of the form

\[ \phi^{k\kappa} = C(\phi, s, t) + \ldots, \]  
\[ \bar{\nabla}^\lambda \phi_{\lambda\kappa} - \bar{\nabla}_\kappa \phi^{\lambda} = D(\phi, s, t) + \ldots, \]  

where $C(\phi, s, t), D(\phi, s, t)$ are certain composites of the fields $\phi_{k\lambda}$, $s$, $t$ of polynomial degree at least 2 and the ellipses denote contributions containing other fields. Thus, it is evident that the $\phi\phi$ terms of the quadratic action yield no $sss$, $sst$, $ttt$ couplings at cubic level via the constraints. Since there are no $\phi\pi$, $\phi\eta$ terms in the quadratic action, the constraints cannot generate any $sss$, $sst$, $ttt$ couplings from these at cubic level.

Higher order terms involving $\phi$ produce obviously no contribution to the cubic coupling terms of $s$ and $t$. Thus, the $s$, $t$ action $I_{st}^{(\leq 3)}$ is not affected by the constraints associated with gauge fixing as announced.

From (2.13) and (2.17a), it appears that the kinetic term of the fields $s_I$ with $k = 0, 1$ vanishes. Therefore, such fields are non propagating gauge degrees of freedom. Consistency requires that they should decouple from all the propagating physical fields. Indeed it is easy to check that the coupling constants $g_{s_i}^{ss}, g_{s_j}^{ss}, g_{s_k}^{tt}, (cfr. eqs. (2.16a)-(2.16c))$ all vanish whenever the fields $s_I$ with $k = 0, 1$ are involved. To this end, one has to use the property that the triple contraction $\langle C_{I_1}C_{I_2}C_{I_3} \rangle$ vanishes unless the values of the non negative integers $k_1, k_2, k_3$ are such to allow complete contraction of the $SO(3+p)$ indices of the $C_{I}$'s (see appendix A1).

The expressions of the coupling constants $g_{I_1I_2I_3}$ simplify considerably when the constant $\theta$ (cfr. eq. (2.17d)) vanishes. This happens precisely for $(D, p) = (11, 2), (11, 5), (10, 3)$. These values correspond to the physically interesting cases of AdS$_7 \times$ S$_4$, AdS$_4 \times$ S$_7$, AdS$_5 \times$ S$_5$.

Writing in eq. (2.13)

\[ A_I = \bar{A}_I z_I \bar{e}^{-2-p}, \]  
\[ m_I^2 = \bar{m}_I^2 \bar{e}^2, \]  
\[ g_{I_1I_2I_3} = \bar{g}_{I_1I_2I_3} a_{I_1I_2I_3} \langle C_{I_1}C_{I_2}C_{I_3} \rangle \bar{e}^{-p}, \]  

one has:

\[ \text{AdS}_7 \times \text{S}_4 \]
\[ \tilde{A}_s^I = \frac{324k(k-1)(2k+3)}{k+\frac{1}{2}}, \]  
(2.35a)  
\[ \tilde{A}_t^I = \frac{324(k+3)(k+4)(2k+3)}{k+\frac{5}{2}}; \]  
(2.35b)  
\[ m_{sI}^2 = k(k-3), \]  
(2.36a)  
\[ \bar{m}_{tI}^2 = (k+3)(k+6); \]  
(2.36b)  
\[ \bar{g}_{I_1I_2I_3}^{sss} = \frac{7776\alpha_1\alpha_2\alpha_3(\alpha - 1)(\alpha^2 - \frac{1}{4})(\alpha^2 - \frac{9}{4})}{(k_1 + \frac{1}{2})(k_2 + \frac{1}{2})(k_3 + \frac{1}{2})}, \]  
(2.37a)  
\[ \bar{g}_{I_1I_2I_3}^{sst} = \frac{23328(\alpha_1 + \frac{3}{2})(\alpha_2 + \frac{3}{2})\alpha_3(\alpha_3 - 1)(\alpha_3 - 2)(\alpha_3 - \frac{3}{2})(\alpha_3 - 3)(\alpha + \frac{3}{2})}{(k_1 + \frac{1}{2})(k_2 + \frac{1}{2})(k_3 + \frac{5}{2})}, \]  
(2.37b)  
\[ \bar{g}_{I_1I_2I_3}^{tts} = \frac{23328\alpha_1\alpha_2(\alpha_3 + \frac{3}{2})(\alpha_3 + \frac{5}{2})(\alpha_3 + \frac{7}{2})(\alpha_3 + 4)(\alpha_3 + \frac{9}{2})(\alpha + 3)}{(k_1 + \frac{1}{2})(k_2 + \frac{1}{2})(k_3 + \frac{1}{2})}, \]  
(2.37c)  
\[ \bar{g}_{I_1I_2I_3}^{ttt} = \frac{7776(\alpha_1 + \frac{3}{2})(\alpha_2 + \frac{3}{2})(\alpha_3 + \frac{3}{2})(\alpha + 3)(\alpha + 4)(\alpha + 5)(\alpha + \frac{11}{2})(\alpha + 6)}{(k_1 + \frac{1}{2})(k_2 + \frac{1}{2})(k_3 + \frac{5}{2})}; \]  
(2.37d)  

**AdS\(_4\) \times S\(_7\)**

\[ \tilde{A}_s^I = \frac{324k(k-1)(2k+6)}{k+2}, \]  
(2.38a)  
\[ \tilde{A}_t^I = \frac{324(k+6)(k+7)(2k+6)}{k+4}; \]  
(2.38b)  
\[ m_{sI}^2 = k(k-6), \]  
(2.39a)  
\[ \bar{m}_{tI}^2 = (k+6)(k+12); \]  
(2.39b)  
\[ \bar{g}_{I_1I_2I_3}^{sss} = \frac{7776\alpha_1\alpha_2\alpha_3(\alpha + 2)(\alpha^2 - 1)(\alpha^2 - 9)}{(k_1 + 2)(k_2 + 2)(k_3 + 2)}, \]  
(2.40a)  
\[ \bar{g}_{I_1I_2I_3}^{sst} = \frac{23328(\alpha_1 + 3)(\alpha_2 + 3)\alpha_3(\alpha_3 - 1)(\alpha_3 - 2)(\alpha_3 - 4)(\alpha_3 - 6)(\alpha + 3)}{(k_1 + 2)(k_2 + 2)(k_3 + 4)}, \]  
(2.40b)  
\[ \bar{g}_{I_1I_2I_3}^{tts} = \frac{23328\alpha_1\alpha_2(\alpha_3 + 3)(\alpha_3 + 4)(\alpha_3 + 5)(\alpha_3 + 7)(\alpha_3 + 9)(\alpha + 6)}{(k_1 + 4)(k_2 + 4)(k_3 + 2)}, \]  
(2.40c)  
\[ \bar{g}_{I_1I_2I_3}^{ttt} = \frac{7776(\alpha_1 + 3)(\alpha_2 + 3)(\alpha_3 + 3)(\alpha + 6)(\alpha + 7)(\alpha + 8)(\alpha + 10)(\alpha + 12)}{(k_1 + 4)(k_2 + 4)(k_3 + 4)}; \]  
(2.40d)
AdS$_5 \times S_5$

\[
\bar{A}_I^s = \frac{256k(k-1)(2k+4)}{k+1}, \\
\bar{A}_I^t = \frac{256(k+4)(k+5)(2k+4)}{k+3};
\]

\[
\bar{m}_{xI}^2 = k(k-4), \\
\bar{m}_{yI}^2 = (k+4)(k+8);
\]

\[
\bar{g}_{I_1I_2I_3}^{sss} = \frac{16384\alpha_1\alpha_2\alpha_3\alpha(\alpha^2 - 1)(\alpha^2 - 4)}{3(k_1+1)(k_2+1)(k_3+1)}, \\
\bar{g}_{I_1I_2I_3}^{sst} = \frac{16384(\alpha_1+2)(\alpha_2+2)\alpha_3(\alpha_3 - 1)(\alpha_3 - 2)(\alpha_3 - 3)(\alpha_3 - 4)(\alpha + 2)}{(k_1+1)(k_2+1)(k_3+3)}, \\
\bar{g}_{I_1I_2I_3}^{tts} = \frac{16384\alpha_1\alpha_2(\alpha_3 + 2)(\alpha_3 + 3)(\alpha_3 + 4)(\alpha_3 + 5)(\alpha_3 + 6)(\alpha + 4)}{(k_1+3)(k_2+3)(k_3+1)}, \\
\bar{g}_{I_1I_2I_3}^{ttt} = \frac{16384(\alpha_1+2)(\alpha_2+2)(\alpha_3 + 2)(\alpha_3 + 4)(\alpha + 5)(\alpha + 6)(\alpha + 7)(\alpha + 8)}{3(k_1+3)(k_2+3)(k_3+3)}.
\]

3. Application of the method

We are now ready to compute two and three point functions in the SCFTs using the AdS$_{d+1}$/CFT$_d$ correspondence. The general formulas derived in [19] work with AdS radius set to 1. Assume that the AdS scalar fields $\phi_i$ correspond to the CFT local field $O_i$. The mass $m_i$ of $\phi_i$ and the conformal dimension $\Delta_i$ of $O_i$ are related as

\[
\Delta_i = \frac{1}{2} \left[ d + \left( d^2 + 4m_i^2 \right)^{\frac{1}{2}} \right].
\]

Then

\[
\langle O_i(x) O_j(y) \rangle = \frac{2}{\pi^{\frac{d}{2}}} \eta_i \frac{\Delta_i - \frac{d}{2}}{\Gamma(\Delta_i - \frac{d}{2})} \frac{\Gamma(\Delta_i + 1)}{\Gamma(\Delta_i - \frac{d}{2})} \frac{\Gamma(\Delta_i + 1)}{\Gamma(\Delta_i - \frac{d}{2})} |x-y|^{2\Delta_i},
\]

where $\eta_i$ is the coefficient of the canonically normalized kinetic term of the bulk field $\phi_i$, and

\[
\langle O_i(x) O_j(y) O_k(z) \rangle = \frac{R_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k} |y-z|^{\Delta_j+\Delta_k-\Delta_i} |z-x|^{\Delta_k+\Delta_i-\Delta_j}|},
\]

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with
\[ R_{ijk} = \frac{1}{2\pi^d} \lambda_{ijk} \frac{\Gamma\left(\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)\right) \Gamma\left(\frac{1}{2}(\Delta_j + \Delta_k - \Delta_i)\right) \Gamma\left(\frac{1}{2}(\Delta_k + \Delta_i - \Delta_j)\right)}{\Gamma\left(\Delta_i - \frac{d}{2}\right) \Gamma\left(\Delta_j - \frac{d}{2}\right) \Gamma\left(\Delta_k - \frac{d}{2}\right)} \]
\[ \Gamma\left(\frac{1}{2}(\Delta_i + \Delta_j + \Delta_k - d)\right) w_i w_j w_k, \]
(3.4)

where \( \lambda_{ijk} \) is the cubic coupling constant of \( \phi_i, \phi_j, \phi_k \) multiplied by the appropriate symmetry factor. The factors \( w_i \) parametrize unknown proportionality constants which relate the fields \( \phi_i \) to the sources of the operators \( O_i \), as in [5]. These factors can presumably be fixed by carefully studying absorption processes on the branes [20]. However, for the present purposes we follow ref. [5] and fix them to normalize the two point functions as
\[
\langle O_i(x)O_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta_i}}.
\]
(3.5)

With this canonical normalization the three point functions are readily computed.

In our case, \( d = D - 3 - p \). Imposing that the AdS radius is 1 fixes the value of \( \bar{e} \) to be
\[
\bar{e} = \frac{d}{1 + p}.
\]
(3.6)

We denote by \( O^s_I, O^t_I \) the CFT\(_d\) operators corresponding to the AdS\(_{d+1}\) scalars \( s_I, t_I \) in the AdS\(_{d+1}\)/CFT\(_d\) duality. Their dimensions are given by
\[
\Delta_s^k = \frac{dk}{1 + p},
\]
(3.7a)
\[
\Delta_t^k = \frac{d(k + 2 + 2p)}{1 + p}.
\]
(3.7b)

From (2.13), it appears that in the present case
\[
\eta^s_I = \frac{1}{4\kappa^2} A^s_I, \text{ etc.},
\]
(3.8)
\[
\lambda^{ss}_{I_1 I_2 I_3} = \frac{3!}{4\kappa^2} g^{ss}_{I_1 I_2 I_3}, \quad \lambda^{st}_{I_1 I_2 I_3} = \frac{2!}{4\kappa^2} g^{st}_{I_1 I_2 I_3}, \text{ etc.}
\]
(3.9)

We find the following expressions.

\underline{AdS\(_7\) \times S\(_4\)}

In this case one has \( \frac{1}{4\kappa^2} = \frac{2N^3}{\pi^3} \) and \( \bar{e} = 2 \). Set
\[
\sigma(k) = [(2k - 2)!]^{-\frac{1}{2}},
\]
(3.10a)
\[
\tau(k) = 16 \left[ \frac{(2k + 1)!(2k + 4)!(2k + 7)!}{(2k + 6)!(2k + 8)!(2k + 9)!(2k + 11)!} \right]^{\frac{1}{2}}.
\]
(3.10b)
Then,

\[ R_{I_1I_2I_3}^{ss} = \frac{1}{4(\pi N)^{\frac{4}{3}}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \sigma(k_1) \sigma(k_2) \sigma(k_3) 2^{2\alpha} \Gamma(\alpha) \]
\[ \times \Gamma(\alpha_1 + \frac{1}{2}) \Gamma(\alpha_2 + \frac{1}{2}) \Gamma(\alpha_3 + \frac{1}{2}), \quad (3.11a) \]

\[ R_{I_1I_2I_3}^{st} = \frac{1}{4(\pi N)^{\frac{4}{3}}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \sigma(k_1) \sigma(k_2) \tau(k_3) 2^{2\alpha} \Gamma(\alpha + 2) \]
\[ \times \frac{\Gamma(2\alpha_1 + 6)\Gamma(2\alpha_2 + 6)}{\Gamma(2\alpha_1 + 2)\Gamma(2\alpha_2 + 2)} \Gamma(\alpha_1 + \frac{5}{2}) \Gamma(\alpha_2 + \frac{5}{2}) \Gamma(\alpha_3 - \frac{3}{2}), \quad (3.11b) \]

\[ R_{I_1I_2I_3}^{ts} = \frac{1}{4(\pi N)^{\frac{4}{3}}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \tau(k_1) \tau(k_2) \sigma(k_3) 2^{2\alpha} \frac{\Gamma(2\alpha + 9)}{\Gamma(2\alpha + 5)} \Gamma(\alpha + 4) \]
\[ \times \frac{\Gamma(2\alpha_3 + 10)\Gamma(2\alpha_3 + 12)}{\Gamma(2\alpha_3 + 2)\Gamma(2\alpha_3 + 8)} \Gamma(\alpha_1 + \frac{1}{2}) \Gamma(\alpha_2 + \frac{1}{2}) \Gamma(\alpha_3 - \frac{9}{2}), \quad (3.11c) \]

\[ R_{I_1I_2I_3}^{tt} = \frac{1}{4(\pi N)^{\frac{4}{3}}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \tau(k_1) \tau(k_2) \tau(k_3) 2^{2\alpha} \frac{\Gamma(2\alpha + 13)\Gamma(2\alpha + 15)}{\Gamma(2\alpha + 5)\Gamma(2\alpha + 11)} \Gamma(\alpha + 6) \]
\[ \times \frac{\Gamma(2\alpha_1 + 6)\Gamma(2\alpha_2 + 6)\Gamma(2\alpha_3 + 6)}{\Gamma(2\alpha_1 + 2)\Gamma(2\alpha_2 + 2)\Gamma(2\alpha_3 + 2)} \Gamma(\alpha_1 + \frac{5}{2}) \Gamma(\alpha_2 + \frac{5}{2}) \Gamma(\alpha_3 + \frac{5}{2}). \quad (3.11d) \]

In this case one has \( \frac{1}{4\pi^2} = \frac{N^\frac{2}{3}}{2\pi^3} \) and \( \bar{\epsilon} = \frac{1}{2} \). Set

\[ \sigma(k) = [(k + 1)!]^\frac{1}{2}, \quad (3.12a) \]

\[ \tau(k) = \frac{1}{4} \left[ k!(k + 2)!(k + 3)!(k + 5)! \right]^\frac{1}{2}. \quad (3.12b) \]

Then,

\[ R_{I_1I_2I_3}^{ss} = \frac{\pi}{2} \left( \frac{2}{N} \right)^{\frac{4}{3}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \sigma(k_1) \sigma(k_2) \sigma(k_3) 2^{-\alpha} \frac{1}{\Gamma(\frac{1}{2}\alpha + 1)} \]
\[ \times \frac{1}{\Gamma(\frac{1}{2}(\alpha_1 + 1))\Gamma(\frac{1}{2}(\alpha_2 + 1))\Gamma(\frac{1}{2}(\alpha_3 + 1))}. \quad (3.13a) \]

\[ R_{I_1I_2I_3}^{st} = \frac{\pi}{2} \left( \frac{2}{N} \right)^{\frac{4}{3}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \sigma(k_1) \sigma(k_2) \tau(k_3) 2^{-\alpha} \frac{1}{\Gamma(\frac{1}{2}\alpha + 2)} \]
\[ \times \frac{1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\frac{1}{2}(\alpha_1 + 3))\Gamma(\frac{1}{2}(\alpha_2 + 3))\Gamma(\frac{1}{2}(\alpha_3 - 1))}. \quad (3.13b) \]

\[ R_{I_1I_2I_3}^{ts} = \frac{\pi}{2} \left( \frac{2}{N} \right)^{\frac{4}{3}} \langle C_{I_1} C_{I_2} C_{I_3} \rangle \tau(k_1) \tau(k_2) \tau(k_3) 2^{-\alpha} \frac{\Gamma(\alpha + 8)}{\Gamma(\alpha + 4)\Gamma(\frac{1}{2}\alpha + 3)} \]
\[ \times \frac{1}{\Gamma(\alpha + 1)\Gamma(\alpha_2 + 1)\Gamma(\frac{1}{2}(\alpha_1 + 3))\Gamma(\frac{1}{2}(\alpha_2 + 3))\Gamma(\frac{1}{2}(\alpha_3 - 1))}. \quad (3.13b) \]
\[ R^{ttt}_{1,1,2} = \frac{\pi}{2} \left( \frac{2}{N} \right)^{\frac{1}{4}} \langle C_1 C_2 C_3 \rangle (k_1) \tau(k_2) \tau(k_3) 2^{-\alpha} \frac{\Gamma(\alpha + 8) \Gamma(\alpha + 14)}{\Gamma(\alpha + 4) \Gamma(\alpha + 6) \Gamma(\frac{1}{2} \alpha + 4)} \cdot (3.13d) \]

In this case one has \( \frac{1}{4\alpha^2} = \frac{N^2}{8\pi^2} \) and \( \bar{e} = 1. \) Set

\[ \sigma(k) = k^{\frac{1}{2}}, \] (3.14a)
\[ \tau(k) = \left[ \frac{k!(k+1)!(k+2)!}{(k+5)!(k+6)!(k+7)!} \right]^{\frac{1}{2}}. \] (3.14b)

Then,

\[ R^{ss}_{1,1,2} = \frac{1}{N} \langle C_1 C_2 C_3 \rangle \sigma(k_1) \sigma(k_2) \sigma(k_3), \] (3.15a)
\[ R^{st}_{1,1,2} = \frac{1}{N} \langle C_1 C_2 C_3 \rangle \sigma(k_1) \sigma(k_2) \tau(k_3) \times \frac{\Gamma(\alpha_1 + 3) \Gamma(\alpha_1 + 4) \Gamma(\alpha_2 + 3) \Gamma(\alpha_2 + 4)}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_1 + 2) \Gamma(\alpha_2 + 1) \Gamma(\alpha_2 + 2)}, \] (3.15b)
\[ R^{tt}_{1,1,2} = \frac{1}{N} \langle C_1 C_2 C_3 \rangle \tau(k_1) \tau(k_2) \sigma(k_3) \frac{\Gamma(\alpha + 5) \Gamma(\alpha + 6)}{\Gamma(\alpha + 3) \Gamma(\alpha + 4)} \times \frac{\Gamma(\alpha + 7) \Gamma(\alpha + 8)}{\Gamma(\alpha + 3) \Gamma(\alpha + 2)}, \] (3.15c)
\[ R^{ttt}_{1,1,2} = \frac{1}{N} \langle C_1 C_2 C_3 \rangle \tau(k_1) \tau(k_2) \tau(k_3) \frac{\Gamma(\alpha + 9) \Gamma(\alpha + 10)}{\Gamma(\alpha + 3) \Gamma(\alpha + 4)} \times \frac{\Gamma(\alpha + 3) \Gamma(\alpha + 4) \Gamma(\alpha_2 + 3) \Gamma(\alpha_2 + 4) \Gamma(\alpha_3 + 3) \Gamma(\alpha_3 + 4)}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_1 + 2) \Gamma(\alpha_2 + 1) \Gamma(\alpha_2 + 2) \Gamma(\alpha_3 + 1) \Gamma(\alpha_3 + 2)}. \] (3.15d)

4. Conclusions

We have derived three point functions for a set of chiral operators, including the primary ones, for the large \( N \) limit of maximally supersymmetric CFT\( d \) in \( d = 3, 4, 6 \) using the AdS/CFT correspondence. In obtaining such results we have used a general gravitational action which could treat the different cases simultaneously. We have obtained the three point couplings for the bulk fields on AdS working at the level of the action. However, we
have also checked the correctness of our results by computing with a different method a subset of the couplings. This method consists in identifying the quadratic corrections to the linearized equations of motion and integrating them into an action, as done in [5].

Some of the final results for the three point correlation functions are not new, and can be used as a check on our lengthy calculation and procedure. In particular, in the super Yang-Mills case \((\text{AdS}_5 \times S_5)\) we have reproduced exactly the correlation functions for the CPO \(\langle O_I^s(x)O_J^s(y)O_K^s(z) \rangle\) worked out in [5], and the correlators \(\langle O_I^s(x)O_J^s(y)O_K^t(z) \rangle\) recently computed in [6] (the relevant bulk coupling are also given in [7]). Other correlators are new predictions from the AdS/CFT correspondence. They could in principle be obtained from the CPO correlators by a systematic use of the superconformal algebra [21], though we have not attempted to do so. For the \(d = 6, \mathcal{N} = (2, 0)\) SCFT (\(\text{AdS}_7 \times S_4\)) we can compare our results for the CPOs with ref. [9]. Actually, their results differ slightly from ours, though those authors seem to agree with our findings [22]. All other results are new as new is the case of the \(d = 3, \mathcal{N} = 8\) SCFT (\(\text{AdS}_4 \times S_7\)).

We should mention that in obtaining the values of the correlation functions for certain extremal cases we have implicitly used an analytic continuation in the conformal dimensions of the operators [23]. These extremal cases have the property that the bulk three point couplings of the supergravity source fields apparently vanish. However this zero is compensated by a divergent integral over AdS to produce a finite final result. A recent analysis on these issues has appeared also in [24], supporting the use of analytic continuation.

**A1. Scalar Spherical Harmonics**

We describe the \(n\)-sphere of radius \(\rho = \bar{e}^{-1}\) by \(S_n \equiv \{z^2 = \rho^2 | z \in \mathbb{R}^{n+1}\}\), and use scalar spherical harmonics defined by \(Y_I = \mathcal{C}_{Ii_1...i_k} x^{i_1}...x^{i_k}\), where the coordinates \(x^i = \bar{e}z^i\) live on the unit sphere and the tensors \(\mathcal{C}_{Ii_1...i_k}\) form an orthonormal basis of the completely symmetric traceless tensors so that \(\mathcal{C}_{Ii_1...i_k} \mathcal{C}_{Ji_1...i_k} = \delta_{IJ}\). The \(Y_I\) are eigenfunctions of the \(S_n\) Laplacian

\[
\nabla^a \nabla_a Y_I = -\bar{e}^2 k(k + n - 1) Y_I. \tag{A1.1}
\]

One has

\[
\int_{S_n} d^n z \sqrt{g} Y_{I_1} Y_{I_2} = z_{I_1} \delta_{I_1 I_2}, \tag{A1.2a}
\]

\[
\int_{S_n} d^n z \sqrt{g} Y_{I_1} Y_{I_2} Y_{I_3} = a_{I_1 I_2 I_3} \langle \mathcal{C}_{I_1} \mathcal{C}_{I_2} \mathcal{C}_{I_3} \rangle, \tag{A1.2b}
\]
where

\[ z_I = \omega_n \frac{(n-1)!!k!}{(2k+n-1)!!} \bar{e}^{-n}, \]  
(A1.3a)

\[ a_{I_1 I_2 I_3} = \omega_n \frac{(n-1)!!}{(2\alpha+n-1)!!} \frac{k_1!k_2!k_3!}{\alpha_1!\alpha_2!\alpha_3!} \bar{e}^{-n}. \]  
(A1.3b)

\[ \langle C_{I_1} C_{I_2} C_{I_3} \rangle \] denotes the unique $SO(n + 1)$ scalar contraction of three tensors $C_{i_1...i_k}$ and $\omega_n$ is the volume of the unit sphere

\[ \omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \]  
(A1.4)

In particular, the $Y_I$ form an orthonormal basis of the Hilbert space of scalar functions on $S_n$. 

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