A system of equations involving the fractional p-Laplacian and doubly critical nonlinearities

Abstract: This article deals with existence of solutions to the following fractional p-Laplacian system of equations:

\[
\begin{align*}
(-\Delta_p)^s u &= |u|^{s^* - 2}u + \frac{\gamma_1}{p_s^*} |u|^{\alpha - 2}u |v|^\beta \quad \text{in} \ \Omega, \\
(-\Delta_p)^s v &= |v|^{s^* - 2}v + \frac{\gamma_2}{p_s^*} |v|^{\beta - 2}v |u|^\alpha \quad \text{in} \ \Omega,
\end{align*}
\]

where \( s \in (0, 1) \), \( p \in (1, \infty) \) with \( N > sp \), \( \alpha, \beta > 1 \) such that \( \alpha + \beta = p_s^* = \frac{Np}{N - sp} \) and \( \Omega \subset \mathbb{R}^N \) or smooth bounded domains in \( \mathbb{R}^N \). When \( \Omega = \mathbb{R}^N \) and \( \gamma = 1 \), we show that any ground state solution of the aforementioned system has the form \((U, \tau_1 V)\) for certain \( \tau > 0 \) and \( U \) and \( V \) are two positive ground state solutions of \((-\Delta_p)^s u = |u|^{s^* - 2}u \) in \( \mathbb{R}^N \). For all \( \gamma > 0 \), we establish existence of a positive radial solution to the aforementioned system in balls. When \( \Omega = \mathbb{R}^N \), we also establish existence of positive radial solutions to the aforementioned system in various ranges of \( \gamma \).

Keywords: fractional p-Laplacian, doubly critical, ground state, existence, system, least energy solution, Nehari manifold

MSC 2020: 35B09, 35B33, 35E20, 35D30, 35J50, 45K05

1 Introduction

We consider the following fractional p-Laplacian system of equations in \( \mathbb{R}^N \):

\[
\begin{align*}
(-\Delta_p)^s u &= |u|^{s^* - 2}u + \frac{\alpha}{p_s^*} |u|^{\alpha - 2}u |v|^\beta \quad \text{in} \ \mathbb{R}^N, \\
(-\Delta_p)^s v &= |v|^{s^* - 2}v + \frac{\beta}{p_s^*} |v|^{\beta - 2}v |u|^\alpha \quad \text{in} \ \mathbb{R}^N, \\
u, v \in W^{s,p}(\mathbb{R}^N),
\end{align*}
\]

where \( 0 < s < 1 \), \( p \in (1, \infty) \), \( N > sp \), and \( \alpha, \beta > 1 \) such that \( \alpha + \beta = p_s^* = \frac{Np}{N - sp} \). Here, \((-\Delta_p)^s\) denotes the fractional p-Laplacian operator, which can be defined for the Schwartz class functions \( S(\mathbb{R}^N) \) as follows:

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\((-\Delta_p)^s u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,\)

where P.V. denotes the principle value sense. Consider the following homogeneous fractional Sobolev space

\[ W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\}. \]

The space \( W^{s,p}(\mathbb{R}^N) \) is a Banach space with the corresponding Gagliardo norm

\[ \|u\|_{W^{s,p}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

For simplicity of the notation, we write \( \|u\|_{W^{s,p}(\mathbb{R}^N)} \) instead of \( \|u\|_{W^{s,p}(\mathbb{R}^N)}^p \).

**Definition 1.1.** We say a pair \((u, v) \in X\) is a positive weak solution of the system \((S)\) if \(u, v > 0\), and for every \((\phi, \psi) \in X\), it holds

\[
\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy \\
+ \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(\psi(x) - \psi(y))}{|x - y|^{N+sp}} \, dx \, dy \\
= \int_{\mathbb{R}^N} |u|^{p-2} u \phi \, dx + \int_{\mathbb{R}^N} |v|^{p-2} v \psi \, dx + \frac{a}{p_s} \int_{\mathbb{R}^N} |u|^{p-2} u |\phi|^\beta \, dx + \frac{b}{p_s} \int_{\mathbb{R}^N} |v|^{p-2} v |\psi|^\alpha \, dx.
\]

Define

\[ S = S_{u,v,p} = \inf_{u \in W^{s,p}(\mathbb{R}^N), u \neq 0} \left\{ \frac{\|u\|^p_{W^{s,p}(\mathbb{R}^N)}}{\left( \int_{\mathbb{R}^N} |u|^\alpha \, dx \right)^\frac{\alpha}{\beta}} \right\}. \quad (1.1) \]

In the limit case \( p = 1 \), the sharp constant \( S \) has been determined in [18, Theorem 4.1] (see also [8, Theorem 4.10]). The relevant extremals are given by the characteristic functions of balls, exactly as in the local case. For \( p > 1 \), (1.1) is related to the study of the following nonlocal integro-differential equation

\[
\begin{cases}
(-\Delta_p)^s u = S u^{\gamma - 1} & \text{in } \mathbb{R}^N, \\
u > 0, & u \in W^{s,p}(\mathbb{R}^N).
\end{cases}
\]

In the Hilbertian case \( p = 2 \), it is known by [14, Theorem 1.1], the best Sobolev constant \( S \) is attained by the family of functions

\[ U_t(x) = t^{\frac{N}{p-2}} \left( 1 + \left( \frac{|x - x_0|}{t} \right)^{\frac{2p-N}{2}} \right)^{\frac{N}{p-2}}, \quad x_0 \in \mathbb{R}^N, \quad t > 0. \]

Moreover, the family \( U_t \) is the only set of minimizers for the best Sobolev constant \([11]\). However, for \( p \neq 2 \), the minimizers of \( S \) are not yet known, and it is not known whether (1.1) has any unique minimizer. In [9], Brasco et al. have conjectured that the optimizers of \( S \) in (1.1) are given by
U_t(x) = C t^{-\beta/\gamma} \left[ 1 + \left( \frac{|x - x_0|}{t} \right)^{\frac{p}{p-1}} \right]^{\frac{\gamma}{\gamma - p}}, \quad x_0 \in \mathbb{R}^N, \quad t > 0,

but it remains as an open question till date. However, in [9, Theorem 1.1], it has been proved that if $U$ is any minimizer of $S$, then $U$ is of constant sign, radially symmetric and monotone function with

$$\lim_{|x| \to \infty} |x|^{\gamma/p} U(x) = U_\infty,$$

for some constant $U_\infty \in \mathbb{R}\setminus\{0\}$.

For $p = 2, a = \beta$ and $u = v$, the system $(S)$ reduces to the fractional Laplacian equation with purely critical exponent. In [21], authors have studied existence and convergence properties of least-energy symmetric solutions $u_\sigma$ ($s$ is a varying parameter) in symmetric bounded domains. For scalar equation, we also refer [7,23] where existence/multiplicity of solutions for a class nonlinear elliptic equation with mixed fractional Laplacians have been studied.

Peng et al. in [25] studied system $(S)$ for $p = 2$ and $s = 1$, and among the other results, they proved uniqueness of least energy solution. In the local case $s = 1$, a variant of system $(S)$ (with $p = 2$) appears in various contexts of mathematical physics e.g. in Bose-Einstein condensates theory, nonlinear wave-wave interaction in plasma physics, nonlinear optics, and for more details, see [1,3,26] and the references therein.

With the system of elliptic $p$-Laplacian type equations with weakly coupled nonlinearities, we also cite [19] and the references therein. In the nonlocal case, there are not so many articles, in which weakly coupled systems of equations have been studied. We refer to [12,13,16,20,22], where Dirichlet systems of equations in bounded domains have been treated. In [20], existence and multiplicity of solutions to system of equations with critical and concave nonlinearities have been studied (see [6,10,24] for similar problem in the case of scalar equations).

For the nonlocal systems of equations in the entire space $\mathbb{R}^N$, we cite [5,17,27] and the references therein.

For $p = 2$ and $s \in (0, 1)$ Bhakta et al. in [4] studied the following system:

$$\begin{align*}
(-\Delta)^s u &= \frac{a}{2s} |u|^{p_a-2} u |v|^\beta + f(x) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{\beta}{2s} |v|^{p_v-2} v |u|^\alpha + g(x) \quad \text{in } \mathbb{R}^N,
\end{align*}

(1.3)$$

where $f, g$ belongs to the dual space of $W^{s,\beta}(\mathbb{R}^N)$. Among other results, the authors proved that when $f = 0 = g$, any ground state solution of (1.3) has the form $(Bw, Cw)$, where $C/B = \sqrt{\beta/\alpha}$ and $w$ is the unique solution of (1.2) (corresponding to $p = 2$).

Being inspired by the aforementioned works, in this article, we generalize some of the aforementioned results in the fractional-$p$-Laplacian case.

**Definition 1.2.**

(i) We say a weak solution $(u, v)$ of $(S)$ is of the synchronized form if $u = \lambda w, v = \mu w$ for some constants $\lambda, \mu$ and a common function $w \in \dot{W}^{s,p}(\mathbb{R}^N)$.

(ii) We say a weak solution $(u, v)$ of $(S)$ is a ground state solution if $(u, v)$ is a minimizer of $S_{a,b}$ (see (1.4)).

Define

$$S_{a,b} = \inf_{\substack{(u,v)\in X, \\ (u,v)\neq(0,0)}} \frac{||u||_{W^{s,p}}^a + ||v||_{W^{s,p}}^b}{\int_{\mathbb{R}^N} \left( |u|^p + |v|^p + |u|^\alpha |v|^\beta \right) dx}.$$

(1.4)

Suppose that $(S)$ has a positive solution of the synchronized form $(\lambda U, \mu U)$ for some $\lambda > 0, \mu > 0$ and $U \in \dot{W}^{s,p}(\mathbb{R}^N)$ is a ground state solution of (1.2). Then it holds

$$\lambda \beta^{\alpha-\beta} + \frac{a}{p_a} \mu^\alpha \lambda^{\alpha-\beta} = 1 = \mu \alpha^{\beta-\mu} + \frac{\beta}{p_v} \mu^\beta \lambda^{\beta-\mu}.$$
Now setting \( \mu = \tau \lambda \), we obtain \( \lambda^{p^* - p} = -\frac{\nu^*}{P} \) and \( \tau \) satisfies
\[
p_s^* + \alpha \tau^\beta - \beta \tau^{p - p} - P_s^* \tau^{p - p} = 0. \tag{1.5}
\]
On the other hand, we find that if \( \tau \) satisfies (1.5), then \( (\lambda U, \tau \lambda U) \) solves (S).

Therefore, the natural question arises: Are all the ground state solutions of (S) is of the synchronized form \( (\lambda U, \tau \lambda U) \)?

If the answer of the aforementioned question is affirmative, then it will hold
\[
S_{0, \beta} = \frac{1 + \tau^p}{(1 + \tau^p + \tau \mu^p)^{p/\mu}} S.
\]

This inspires us to define the following function:
\[
h(\tau) = \frac{1 + \tau^p}{(1 + \tau^p + \tau \mu^p)^{p/\mu}}. \tag{1.6}
\]

Note that \( h(\tau_{\min}) = \min_{\tau \geq 0} h(\tau) \leq 1 \).

Below we state the main results of this article, we present the following:

**Theorem 1.3.** Let \((u_0, v_0)\) be any positive ground state solution of (S). If one of the following conditions hold
(i) \( 1 < \beta < p \),
(ii) \( \beta = p \) and \( \alpha < p \),
(iii) \( \beta > p \) and \( \alpha < p \),
then, there exists unique \( \tau_{\min} > 0 \) satisfying
\[
h(\tau_{\min}) = \min_{\tau \geq 0} h(\tau) < 1,
\]
where \( h \) is defined by (1.6). Moreover,
\[
(u_0, v_0) = (\lambda U, \tau_{\min} \lambda V),
\]
where \( U \) and \( V \) are two positive ground state solutions of (1.2). Further, \( \lambda^{p^* - p} = \frac{\nu^*}{P^* \tau_{\min}^\mu} \).

**Remark 1.4.** Since for \( p \neq 2 \), uniqueness of ground state solutions of (1.2) is not yet known, we are not able to conclude whether any ground state solution of (S) is of the synchronized form, i.e., of the form of \( (\lambda U, \tau_{\min} \lambda V) \) or not.

Next, we consider (S) with a small perturbation \( \gamma > 0 \), namely, we consider the system
\[
\begin{aligned}
(-\Delta_p)u &= |u|^{p^* - 2}u + \frac{\alpha y}{P_s} |u|^{p - 2}u |v|^\beta \quad \text{in } \mathbb{R}^N, \\
(-\Delta_p)v &= |v|^{p^* - 2}v + \frac{\beta y}{P_s} |v|^{p - 2}v |u|^\alpha \quad \text{in } \mathbb{R}^N, \\
u, v &\in W^{s,p}(\mathbb{R}^N).
\end{aligned} \tag{\tilde{S}_\gamma}
\]

and prove existence of positive solutions to (\tilde{S}_\gamma) in various range of \( \gamma \). The corresponding energy functional of the problem (\tilde{S}_\gamma), given by for \((u, v) \in X \)
\[
\mathcal{J}(u, v) = \frac{1}{p} (||u||_{W_{s,p}}^p + ||v||_{W_{s,p}}^p) - \frac{1}{p_s^*} \int_{\mathbb{R}^N} (|u|^{p^*} + |v|^{p^*} + \gamma |u|^{\alpha} |v|^\beta) dx. \tag{1.7}
\]

We define
\[
\mathcal{N} = \left\{ (u, v) \in X : u \neq 0, v \neq 0, ||u||_{W_{s,p}}^p = \int_{\mathbb{R}^N} (|u|^{p^*} + \frac{\alpha y}{P_s} |u|^{\alpha} |v|^\beta) dx, ||v||_{W_{s,p}}^p = \int_{\mathbb{R}^N} (|v|^{p^*} + \frac{\beta y}{P_s} |u|^{\alpha} |v|^\beta) dx \right\}. \tag{1.8}
\]
It is easy to see that $\mathcal{N} \neq \emptyset$ and that any nontrivial solution of $(\tilde{S}_\gamma)$ belongs to $\mathcal{N}$. Set

$$A = \inf_{(u,v) \in \mathcal{N}} \mathcal{J}(u,v).$$

Consider the nonlinear system of algebraic equations

$$\begin{align*}
\frac{k^{\gamma,s}}{k} + \frac{\alpha y}{p^s_k} k^\gamma \ell^s \gamma = 1, \\
\frac{\gamma^{\gamma,s}}{\ell} + \frac{\gamma{\beta y}}{p^s_k} \ell^\gamma \gamma^s \ell = 1, \\
k, \ell > 0.
\end{align*}$$

(1.9)

**Theorem 1.5.** Assume that one of the following conditions hold:

(i) If $\frac{N}{2} < p < \frac{N}{\gamma}$, $\alpha > p$ and

$$0 < \gamma \leq \frac{p^s_k(p^s_k - p)}{p} \min \left\{ \frac{1}{\alpha} \left( \frac{a - p}{\beta - p} \right)^{\frac{p}{\gamma}}, \frac{1}{\beta} \left( \frac{\beta - p}{a - p} \right)^{\frac{p}{\gamma}} \right\}. \quad (1.10)$$

(ii) If $\frac{2N}{N + 2} < p < \frac{N}{\gamma}$, $\alpha < p$ and

$$\gamma \geq \frac{p^s_k(p^s_k - p)}{p} \max \left\{ \frac{1}{\alpha} \left( \frac{p - \beta}{p - a} \right)^{\frac{p}{\gamma}}, \frac{1}{\beta} \left( \frac{p - a}{\beta - \beta} \right)^{\frac{p}{\gamma}} \right\}. \quad (1.11)$$

Then the least energy $A = \frac{2}{N}(k_0 + \ell_0)S^{N/p}$ and $A$ is attained by $(k_0^1 U, \ell_0^1 U)$, where $U$ is a minimizer of (1.1), $k_0, \ell_0$ satisfies (1.9) and

$$k_0 = \min\{k : (k, \ell) \text{ satisfies (1.9)}\}. \quad (1.12)$$

**Theorem 1.6.** Assume that $\frac{2N}{N + 2} < p < \frac{N}{\gamma}$ and $\alpha < p$. There exists $y_1 > 0$ such that for any $\gamma \in (0, y_1)$, there exists a solution $(k(\gamma), \ell(\gamma))$ of (1.9) such that $(k(\gamma)^{1/p} U, \ell(\gamma)^{1/p} U)$ is a positive solution of system $(\tilde{S}_\gamma)$ with $\mathcal{J}(k(\gamma)^{1/p} U, \ell(\gamma)^{1/p} U) > \tilde{A}$, where $U$ is a minimizer of (1.1),

$$\tilde{A} = \inf_{(u,v) \in \mathcal{N}} \mathcal{J}(u,v)$$

and

$$\mathcal{N} = \{ (u, v) \in X : ||u||^p_{W^{s,p}_0} + ||v||^p_{W^{s,p}_0} = \int_{\mathbb{R}^N} (|u|^p + |v|^p + \gamma |u|^p |v|^p) dx \}.$$

**Theorem 1.7.** Assume that $\frac{2N}{N + 2} < p < \frac{N}{\gamma}$ and $\alpha < p$. Then the following system of equations

$$\begin{align*}
(-\Delta_p)^s u &= |u|^{s-2} u + \frac{\alpha y}{p^s_k} |u|^{s-2} u |v|^{s} \text{ in } B_\delta(0), \\
(-\Delta_p)^s v &= |v|^{s-2} v + \frac{\alpha y}{p^s_k} |v|^{s-2} v |u|^{s} \text{ in } B_\delta(0), \quad (1.13)
\end{align*}$$

admit a radial positive solution $(u_0, v_0)$.

The organization of the rest of the article is as follows: In Section 2, we prove Theorem 1.3. Section 3 deals with the proof of Theorems 1.5, 1.7, and 1.6.
2 Proof of Theorem 1.3

Lemma 2.1. Suppose $\alpha, \beta > 1$ such that $\alpha + \beta = p^*_\alpha$. Then

(i) $S_{a,b} = h(\tau_{\min}).$

(ii) $S_{a,b}$ has minimizers $(U, \tau_{\min}U)$, where $U$ is a ground state solution of (1.2) and $\tau_{\min}$ satisfies

$$\tau^{p_* - 1}(p^*_\alpha + \alpha \tau^\beta - \beta \tau^{p^*_\beta - p} - p^*_\alpha \tau^{p^* - p}) = 0.$$ 

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence in $X$ for $S_{a,b}$. Choose $\tau_n > 0$ such that $|v_n|_{L^{\alpha}(\mathbb{R}^n)} = \tau_n |u_n|_{L^{\alpha}(\mathbb{R}^n)}$. Now set, $u_n = \frac{u_n}{\tau_n}$. Therefore, $|u_n|_{L^{\alpha}(\mathbb{R}^n)} = ||w_n||_{L^{\alpha}(\mathbb{R}^n)}$ and applying Young’s inequality,

$$\int_{\mathbb{R}^n} |u_n|^\alpha |w_n|^\beta \, dx \leq \frac{\alpha}{p^*_\alpha} \int_{\mathbb{R}^n} |u_n|^p \, dx + \frac{\beta}{p^* \beta} \int_{\mathbb{R}^n} |w_n|^p \, dx = \int_{\mathbb{R}^n} |u_n|^\alpha |w_n|^\beta \, dx.$$ 

Therefore,

$$S_{a,b} + o(1) = \frac{||u_n||_{L^{p^*\alpha}_\delta} + ||v_n||_{L^{p^*\beta}_\delta}}{\left( \int_{\mathbb{R}^n} (|u_n|^\alpha + |v_n|^\beta + |u_n|^a |v_n|^b) \, dx \right)^{\frac{1}{p^*}}}$$

$$= \frac{||u_n||_{L^{p^*\alpha}_\delta}^\alpha + \tau^{p^*_\alpha} |u_n|^\alpha + \tau^{p^*_\beta} |u_n|^a |w_n|^b \, dx \right)^{\frac{1}{p^*}}}$$

$$+ \frac{\tau^{p^*_\alpha} |w_n||_{L^{p^*_\beta}_\delta}^\beta}{\left( \int_{\mathbb{R}^n} (|u_n|^\alpha + |v_n|^\beta + |u_n|^a |w_n|^b) \, dx \right)^{\frac{1}{p^*}}}$$

$$\geq \left(1 + \tau^{p^*_\alpha} + \tau^{p^* \beta - p} \right)^{\frac{1}{p^* - 1}} \frac{\left( ||u_n||_{L^{p^*\alpha}_\delta}^a + \tau^{p^*_\beta} |w_n||_{L^{p^*_\beta}_\delta}^\beta \right)}{\left( \int_{\mathbb{R}^n} (|u_n|^\alpha + |v_n|^\beta + |u_n|^a |w_n|^b) \, dx \right)^{\frac{1}{p^*}}}$$

$$\geq \frac{1}{(1 + \tau^{p^*_\alpha} + \tau^{p^* \beta - p} \right)^{\frac{1}{p^* - 1}}} S \geq \min h(\tau)S.$$ 

Thus, as $n \to \infty$, we have $h(\tau_{\min})S \leq S_{a,b}$. For the reverse inequality, we choose $u = U$, $v = \tau_{\min}U$ to obtain $h(\tau_{\min})S \geq S_{a,b}$. In Lemma 2.2, we will show that point $\tau_{\min}$ exists. This proves (i).

(ii) Taking $(u, v) = (U, \tau_{\min}U)$, a simple computation yields that

$$\frac{||u||_{L^{p^*\alpha}_\delta}^\alpha + ||v||_{L^{p^*\beta}_\delta}^\beta}{\left( \int_{\mathbb{R}^n} (|u|^\alpha + |v|^\beta + |u|^a |v|^b) \, dx \right)^{\frac{1}{p^*}}} = h(\tau_{\min})S.$$ 

By using (i), we infer that $(U, \tau_{\min}U)$ is a minimizer of $S_{a,b}$. Further, since $\tau_{\min}$ is a critical point of $h$, computing $h'(\tau_{\min}) = 0$ yields that $\tau_{\min}$ satisfies

$$\tau^{p^*_\alpha - 1}(p^*_\alpha + \alpha \tau^\beta - \beta \tau^{p^*_\beta - p} - p^*_\alpha \tau^{p^* - p}) = 0.$$ 

We observe from (1.6) that $h(0) = 1$ and $\lim_{\tau \to -\infty} h(\tau) = 1$. Therefore, to ensure the existence of $\tau_{\min}$ (i.e., minimum point of $h$ does not escape at infinity), $\tau_{\min}$ is uniquely defined and $\tau_{\min} > 0$, we need to investigate the solvability of the following equation:

$$g(\tau) = p^*_\alpha + \alpha \tau^\beta - \beta \tau^{p^*_\beta - p} - p^*_\alpha \tau^{p^* - p} = 0.$$ 

(2.1)
Lemma 2.2. Let $a, \beta > 1$, and $a + \beta = p_s^*$. Then (2.1) always has at least one root $\tau > 0$, and for any root $\tau > 0$, the problem (S) has positive solutions $(\lambda U, \mu U)$, where

$$\mu = \tau \lambda, \quad \lambda^{p_s^*-p} = \frac{p_s^*}{\tau p_s + \alpha \tau^{p_s}}.$$ 

Moreover, if one of the following conditions hold

(i) $1 < \beta < p$.
(ii) $\beta = p$ and $a < p$.
(iii) $\beta > p$ and $a < p$,

then, $\tau_{\text{min}} > 0$ and $h(\tau_{\text{min}}) < 1$. In all other cases, $\tau_{\text{min}} = 0$.

Proof. Clearly, if $\tau > 0$ solves

$$\begin{align*}
(p_s^* + \alpha \tau^p)\lambda^{p_s^*-p} &= p_s^*, \\
(p_s^* + \beta \tau^{p_s^*-p})\lambda^{p_s^*-p} &= p_s^*,
\end{align*}$$

then $(\lambda U, \mu U)$ with $\mu = \tau \lambda$ solves (S). Thus to prove the required result, it is enough to show that (2.1) has positive roots $\tau$, which we discuss in the following cases.

**Case 1:** If $1 < \beta < p$.

Therefore, $\lim_{\tau \to 0} g(\tau) = -\infty$.

Now, if $a \geq p$, then $g(1) = a - \beta > 0$. Thus, there exists $\tau \in (0, 1)$ such that $g(\tau) = 0$.

If $1 < a < p$, then we have $p_s^* - p < p_s^* - a = \beta$, and consequently, $\lim_{\tau \to a} g(\tau) = \infty$. Thus, there exists $\tau > 0$ such that $g(\tau) = 0$.

Also observe that, by direct computation, we obtain

$$h'(\tau) = f(\tau) g(\tau), \quad \text{where } f(\tau) = \frac{\tau^{1-p} \beta}{p_s^*(1 + \tau^p + \tau^{1+p})}. $$

If $\tau > 0$, $f(\tau) > 0$ for all $\tau > 0$ and $f(0) = 0$. This together with the fact that $\lim_{\tau \to 0} g(\tau) = -\infty$ implies $h'(\tau) < 0$ in $\tau \in (0, \varepsilon)$ for some $\varepsilon > 0$. This means $h$ is a decreasing function near 0. Combining this with the fact that $h(0) = 1$ and $h(\tau) = 1$, we conclude that there exists a point $\tau_{\text{min}} \in (0, \infty)$ such that $\min_{\tau > 0} h(\tau) = h(\tau_{\text{min}}) < 1$, and this holds for all $a > 1$.

**Case 2:** If $\beta = p$.

In this case, $g$ becomes $g(\tau) = a(1 + \tau^p) - p_s^* \tau^p$. Hence, $g(0) = a > 0$ and $g(1) = a - p$.

(i) If $a = p$, then $N = 2sp$ and $g(\tau) = p - \tau^p p$. Thus, there exists a unique root $\tau_1 = 1$ of $g$. Also note that $h$ is increasing near 0. Hence, $\tau_1$ is the maximum point of $h$ with $h(\tau_1) > 1$. In this case, $\min_{\tau > 0} h(\tau) = h(\tau_{\text{min}}) = h(0)$.

(ii) If $1 < a < p$, then we have $2sp < N < sp(p + 1)$. Observe that $g(0) = a > 0$, $g(1) = a - p < 0$, and $\lim_{\tau \to 0} g(\tau) = +\infty$. Also note that, $g$ is decreasing in $\left(0, \left(p_s^*/p)^1/p\right)\right]$ and increasing in $\left((p_s^*/p)^1/p, \infty\right]$. Therefore, $g$ has exactly one critical point $(p_s^*/p)^1/p$ and two roots $\tau_i (i = 1, 2)$ with $\tau_i \in \left(0, (p_s^*/p)^1/p\right)$, $\tau_i \in \left((p_s^*/p)^1/p, \infty\right)$. 

\[\text{(a) The case 2(i)}\]
Further, note that in this case \( h \) is increasing near 0, which leads that first positive critical point of \( h \), i.e., \( \tau_1 \) is the local maximum for \( h \) and \( h(\tau_1) > 1 \). Further, as \( \lim_{\tau \to \infty} h(\tau) = 1 \), the second root of \( g \), i.e., \( \tau_2 \), becomes the second and last critical point of \( h \) and \( h(\tau_2) < 1 \). Therefore, in this case, \( \tau_{\min} = \tau_2 > 0 \) is the minimum point of \( h \) with \( h(\tau_{\min}) < 1 \).

(iii) If \( a > p \), then \( sp < N < 2sp \) and \( g(0) > 0 \). We see that \( g \) is increasing in \( [0, (p/p^*_1)^{1/p^*_2}) \) and decreasing in \( \left( (p/p^*_1)^{1/p^*_2}, \infty \right) \). This together with the fact \( \lim_{\tau \to \infty} g(\tau) = -\infty \) leads that there exists a unique \( \tau > 0 \) such that \( g(\tau) = 0 \).

Since in this case, \( h \) is increasing near 0, so at \( \tau \), \( h \) attains the maximum with \( h(\tau) > 1 \). Hence, \( h \) has no other critical point, and therefore, \( h(\tau_{\min}) = h(0) \).

**Case 3:** If \( \beta > p \).

If \( 1 < a \leq p \), then \( g(1) = a - \beta \leq 0 \). Since \( g(0) > 0 \), there is a \( \tau \in (0, 1] \) such that \( g(\tau) = 0 \). If \( a > p \) and \( a > \beta \), then \( g(1) > 0 \) and \( \lim_{\tau \to \infty} g(\tau) = -\infty \). Thus, there exists \( \tau \in (1, \infty) \) such that \( g(\tau) = 0 \). If \( a > p \) and \( a \leq \beta \), then \( g(1) \leq 0 \). As \( g(0) > 0 \), thus there exists \( \tau \in (0, 1] \) such that \( g(\tau) = 0 \). Next we analyze \( \tau_{\min} \) in case 3 in the following three subcases.

(i) \( \beta > p \) and \( a > p \).

Observe that in this case we have

\[
\beta < p^*_1 - p \quad \text{and} \quad a < p^*_1 - p. \tag{2.2}
\]

Hence, without loss of generality, we can assume \( a \geq \beta \).

Claim 1: \( g(\tau) > 0 \) for \( \tau \in (0, 1] \). Indeed, by using (2.2), \( \tau \in (0, 1] \) implies \( \tau^a, \tau^\beta > \tau^{p^*_1-p} \). Therefore,

\[
g(\tau) > p^*_1 + \alpha \tau^\beta - \beta \tau^{p^*_1-p} - p^*_1 \tau^\beta = p^*_1 + (a - p^*_1) \tau^\beta - \beta \tau^{p^*_1-p} > p^*_1 + a - p^*_1 - \beta \tau^{p^*_1-p} \quad \text{(as} \ a < p^*_1 \text{ and} \ \tau^\beta < 1)\]

\[
= a - \beta \tau^{p^*_1-p} > 0,
\]

where in the last inequality, we have used the fact that \( \tau^{p^*_1-p} < 1 \Rightarrow \tau^{p^*_1-p} < \beta \leq a \). This proves claim 1.

Claim 2: \( g \) is monotonically decreasing for \( \tau \geq 1 \). Indeed, \( \tau \geq 1 \) implies \( \tau^a \geq \tau^\beta \). Therefore, by using (2.2), we have
This proves claim 2. Also observe that \( g(1) \geq 0 \) and \( g(\tau) \rightarrow -\infty \) as \( \tau \rightarrow \infty \). Combining these facts along with claim 1 and 2 proves that \( g \) has only one root say \( \tau \) in \((0, \infty)\), which in turn implies \( h \) has only one critical point \( \tau \) in \((0, \infty)\). Since \( \beta > p \) implies \( h \) is increasing near 0, so at \( \tau \), \( h \) attains the maximum with \( h(\tau) > 1 \). Combining this with \( \lim_{\tau \to \infty} h(\tau) = 1 \) proves that \( h(\tau_{\text{min}}) = h(0) = 1 \), i.e., \( \tau_{\text{min}} = 0 \).

(ii) \( \beta > p \) and \( a < p \).

In this case, \( g(0) > 0, g(1) < 0 \), and we claim \( g \) is strictly decreasing in \((0, 1)\). Indeed, \( a < p \Rightarrow \tau^p < \tau^a \) for \( \tau \in (0, 1) \). Also \( \beta > p \Rightarrow a < p^*_x - p \). Therefore,

\[
g'(\tau) = \tau^{\beta-p-1}[a\beta \tau^p - p^*_x(p^*_x - p)\tau^a - \beta(\beta - p)]
\leq \tau^{\beta-p-1}(p^*_x(p^*_x - p)\tau^a - \beta(\beta - p))
< 0.
\]

Claim: \( g \) has only one critical point in \((1, \infty)\). Indeed,

\[
g'(\tau) = \tau^{\beta-p-1}[a\beta \tau^p - p^*_x(p^*_x - p)\tau^a - \beta(\beta - p)]
\leq \tau^{\beta-p-1}(p^*_x(p^*_x - p)\tau^a - \beta(\beta - p))
< 0.
\]

So to prove that \( g \) has only critical point in \((1, \infty)\), it is enough to show that \( g_1 \) has only one root in \((1, \infty)\). Observe that, \( g_1(0) < 0, \lim_{\tau \to \infty} g_1(\tau) = \infty \) and a straightforward computation yields that \( g_1 \) is a decreasing function in \( \left[0, \left(\frac{p^*_x(p^*_x - p)}{p^*_x}\right)^{\frac{1}{p^*_x - 1}}\right] \) and \( g_1 \) is an increasing function in \( \left(\frac{p^*_x(p^*_x - p)}{p^*_x}\right)^{\frac{1}{p^*_x - 1}}, \infty \). Thus, \( g_1 \) has only one root.

Hence, the claim follows. Next, we observe that \( a < p \Rightarrow \beta > p^*_x - p \), and therefore, \( \lim_{\tau \to \infty} g(\tau) = \infty \).

Combining all the aforementioned observations and claim, it follows that \( g \) has only one critical point in \((0, \infty)\) and two roots \( \tau_1, \tau_2 \) with \( \tau_1 \in (0, 1) \) and \( \tau_2 \in (1, \infty) \). Hence, \( h \) has exactly two critical points \( \tau_1, \tau_2 \). Since \( h \) is increasing near 0 leads to the conclusion that first positive critical point of \( h \), i.e., \( \tau_1 \) is the local maximum for \( h \) and \( h(\tau_1) > 1 \) and since \( \lim_{\tau \to \infty} h(\tau) = 1 \) at the second critical point of \( h \), i.e., at \( \tau_2 \), we have \( h(\tau_2) < 1 \). Therefore, in this case, \( \tau_{\text{min}} = \tau_2 > 0 \) is the minimum point of \( h \) with \( h(\tau_{\text{min}}) < 1 \).

(iii) \( \beta > p, a = p \).

In this case, \( g(0) > 0 \) and \( a = p \Rightarrow \beta = p^*_x - p \). Therefore,
\[
g'(\tau) = \tau^{\beta-p} [a\beta \tau^p - p^s(p^s - p)\tau^a - \beta(\beta - p)]
\]
\[
= \tau^{\beta-p} [a - p^s] \beta \tau^a - \beta(\beta - p)]
\]
\[
< 0,
\]
i.e., \( g \) is a strictly decreasing function. Also, observe that \( \lim_{\tau \to \infty} g(\tau) = -\infty \). Hence, \( g \) has only one root in \((0, \infty)\), i.e., \( h \) has only critical point \( \tau \) in \((0, \infty)\). Since \( \beta > p \) implies \( h \) is increasing near 0, at \( \tau \), \( h \) attains the maximum with \( h(\tau) > 1 \). Combining this with \( \lim_{\tau \to 0} h(\tau) = 1 \) proves that \( h(\tau_{\min}) = h(0) = 1 \), i.e., \( \tau_{\min} = 0 \). \( \square \)

To prove Theorem 1.3, next we introduce an auxiliary system of equations with a positive parameter \( \eta \),

\[
\begin{cases}
(-\Delta)^{\alpha} u = \eta |u|^{\alpha-2} u + \frac{\alpha}{P^s} |u|^{\alpha-2} u |v|^{^\beta} & \text{in } \mathbb{R}^N, \\
(-\Delta)^{\beta} v = |v|^{\beta-2} v + \frac{\beta}{P^s} |v|^{\beta-2} v |u|^\alpha & \text{in } \mathbb{R}^N, \\
u, v \in W^{s,\beta}(\mathbb{R}^N).
\end{cases}
\]

We define the following minimization problem associated to \((S)_\eta\):

\[
S_{0,\alpha,\beta} = \inf_{(u,v) \in X, (u,v) \neq 0} \left\{ \frac{\|u\|_{W^{s,\beta}}^\alpha + \|v\|_{W^{s,\beta}}^\beta}{\int_{\mathbb{R}^N} (\eta |u|^{\alpha} + |v|^{\beta}) + |u|^\alpha |v|^{\beta} dx} \right\}.
\]

Similarly for \( \tau > 0 \), we define

\[
f_\tau(\tau) = \frac{1 + \tau^p}{(\eta + \tau^\beta + \tau^\beta - p^s p^s - p^s \tau^\beta - p^s \tau^\beta - p^s)} , \quad f_\eta(\tau_{\min}) = \min_{\tau \geq 0} f_\eta(\tau).
\]

Proceeding as in the proof of Lemma 2.2, we find \( \varepsilon \in (0, 1) \) small such that \( \tau^s_{\min}(\eta), \lambda^s(\eta), \mu^s(\eta) \) are unique for \( \eta \in (1 - \varepsilon, 1 + \varepsilon) \) and \( \tau^s_{\min}(\eta) \) satisfies

\[
\tau^\beta - 1(\eta p^s + \alpha \tau^\beta - \beta \tau^\beta - p^s p^s - p^s \tau^\beta - p^s \tau^\beta - p^s) = 0.
\]

Moreover, \( \tau^s_{\min}(\eta), \lambda^s(\eta), \mu^s(\eta) \) are \( C^1 \) for \( \eta \in (1 - \varepsilon, 1 + \varepsilon) \) and \( \varepsilon > 0 \) small. Indeed, if we denote

\[
F(\eta, \tau) = \eta p^s + \alpha \tau^\beta - \beta \tau^\beta - p^s p^s - p^s \tau^\beta - p^s \tau^\beta - p^s.
\]

Then,

\[
\frac{\partial F}{\partial \eta} = \tau^{\beta-p} [a \beta \tau^p - p^s(p^s - p)\tau^a - \beta(\beta - p)].
\]

Since \( \eta_{\min} \) is the minimum of \( h \), direct computation yields \( g(\eta_{\min}) = 0, \quad g'(\eta_{\min}) > 0 \). Therefore, \( F(1, \eta_{\min}) = 0 \), \( \frac{\partial F}{\partial \eta}(1, \eta_{\min}) > 0 \). Consequently, by implicit function theorem, we obtain that \( \lambda^s(\eta), \mu^s(\eta) \) are \( C^1 \) for \( \eta \in (1 - \varepsilon, 1 + \varepsilon) \).

**Proof of Theorem 1.3.** Let \((u_0, v_0)\) is a ground state solution of \((S)\). First, we claim that
To prove this, we define the following min–max problem associated to \((S_{\eta})\)

\[
B(\eta) = \inf_{(u,v)\in X_\eta (0)} \max_{t > 0} E_t'(tu, tv),
\]

where

\[
E_t(u, v) = \frac{1}{p} \left( \|u\|_{W^{p\ast}} + \|v\|_{W^{p\ast}} \right) - \frac{1}{p^\ast} \int_{\mathbb{R}^N} \left( (\eta u^\prime)^{p^\ast} + (\eta v^\prime)^{p^\ast} + (u^\prime)(v^\prime)^{p^\ast} \right) dx.
\]

Observe that there exists \(t(\eta) > 0\) such that \(\max_{t>0} E_t'(tu_0, tv_0) = E_0(t(\eta)u_0, t(\eta)v_0)\), and moreover, \(t(\eta)\) satisfies \(H(\eta, t(\eta)) = 0\), where \(H(\eta, t) = t^{p\ast - p}(\eta G + D) - C\) with

\[
C = \|u_0\|_{W^{p\ast}} + \|v_0\|_{W^{p\ast}}, \quad D = \int_{\mathbb{R}^N} (|v_0|^\beta + |u_0|^\beta) dx \quad \text{and} \quad G = \int_{\mathbb{R}^N} |u_0|^{\beta} dx.
\]

As \((u_0, v_0)\) is a least energy solution of \((S)\), then

\[
H(1, 1) = 0, \quad \frac{\partial H}{\partial t}(1, 1) > 0 \quad \text{and} \quad H(\eta, t(\eta)) = 0.
\]

Thus, by the implicit function theorem, there exists \(\varepsilon > 0\) such that \(t(\eta) : (1 - \varepsilon, 1 + \varepsilon) \to \mathbb{R}\) is \(C^1\) and

\[
t'(\eta) = \frac{\partial H}{\partial t} \bigg|_{t = 1 + t} = -\frac{G}{(p^\ast - p)(G + D)}.
\]

By Taylor expansion, we also have \(t(\eta) = 1 + t'(1)(\eta - 1) + O(|\eta - 1|^2)\) and thus

\[
t(\eta) = 1 + pt'(1)(\eta - 1) + O(|\eta - 1|^2).
\]

\(H(1, 1) = 0\) implies \(C = G + D\), and \(H(\eta, t(\eta)) = 0\) implies \(C = t(\eta)^{p\ast - p}(\eta G + D)\). Therefore, by definition of \(B(\eta)\) and the aforementioned observation, we obtain

\[
B(\eta) \leq E_0(t(\eta)u_0, t(\eta)v_0)
\]

\[
= \frac{t(\eta)^p}{p} \left( C - \frac{t(\eta)^{p^\ast}}{p^\ast} (\eta G + D) \right)
\]

\[
= \frac{t(\eta)^p}{p^\ast} C = t(\eta)^p B(1)
\]

\[
= B(1) - \frac{pGB(1)}{(p^\ast - p)(G + D)} (\eta - 1) + O(|\eta - 1|^2).
\]

Now, let us compute \(B(1)\) from the definition
$$B(1) = \inf_{(u,v) \in \mathcal{X}} E(t_{\max} u, t_{\max} v), \quad \text{where} \quad t_{\max}^{p^* - p} = \frac{\|u\|_{W^{1,p}}^p + \|v\|_{W^{1,p}}^p}{\int_{\mathbb{R}^n} (|u|^{p^*} + |v|^{p^*} + |u|^p |v|^p) \, dx}$$

$$= \frac{s}{N} \inf_{(u,v) \in \mathcal{X}} \left( \frac{\|u\|_{W^{1,p}}^p + \|v\|_{W^{1,p}}^p}{\int_{\mathbb{R}^n} (|u|^{p^*} + |v|^{p^*} + |u|^p |v|^p) \, dx} \right)^{\frac{p^*}{p}}$$

$$= \frac{s}{N} \frac{p^*}{N^{1/p^*}}$$

$$= \frac{s}{N} \frac{p^*}{N^{1/p^*}} \frac{\|u_0\|_{W^{1,p}}^p + \|v_0\|_{W^{1,p}}^p}{\int_{\mathbb{R}^n} (|u_0|^{p^*} + |v_0|^{p^*} + |u_0|^p |v_0|^p) \, dx}$$

$$= \frac{s}{N} (G + D).$$

By using this in (2.4), we obtain

$$B(\eta) \leq B(1) - \frac{G}{p^*} (\eta - 1) + O(\eta - 1^2).$$

Therefore, we have

$$\frac{B(\eta) - B(1)}{\eta - 1} \begin{cases} \leq - \frac{G}{p^*} + O(\eta - 1) & \text{if } \eta > 1, \\ \geq - \frac{G}{p^*} + O(\eta - 1) & \text{if } \eta < 1. \end{cases}$$

This implies that

$$B'(1) = - \frac{G}{p^*} = - \frac{1}{p^*} \int_{\mathbb{R}^n} |u_0|^\beta \, dx.$$ (2.5)

Arguing similarly as in the proof of Lemma 2.1, it follows that $S_{\eta, a, \beta}$ is attained by $(tU, \tau(\eta)tu)$. Therefore,

$$B(\eta) = \frac{s}{N} \left( \frac{1 + \tau(\eta)^p}{(\eta + \tau(\eta)^\beta + \tau(\eta)^\beta)^{\frac{p}{p^*}}} \right)^{\frac{p^*}{p}} \int_{\mathbb{R}^n} |U|^\beta \, dx$$

$$= \frac{s}{N} \left( \frac{1 + \tau(\eta)^p}{(\eta + \tau(\eta)^\beta + \tau(\eta)^\beta)^{\frac{p}{p^*}}} \right)^{\frac{p^*}{p}} \int_{\mathbb{R}^n} |U|^\beta \, dx.$$ (2.6)

Then, from a simple computation, it follows

$$B'(\eta) = - \frac{(1 + \tau(\eta)^p)^{\frac{p^*}{p}}}{p^* (\eta + \tau(\eta)^\beta + \tau(\eta)^\beta)^{\frac{p}{p^*}}} \left( \tau(\eta)^{p^* - 1} (\eta \tau^p + \alpha \tau(\eta)^\beta - \beta \tau(\eta)^{p^* - p} - p^* \tau(\eta)^{\beta - 1}) - 1 - \tau(\eta)^p \right) \int_{\mathbb{R}^n} |U|^\beta \, dx.$$ (2.7)

Note that for $\eta = 1$, $\tau(1)$ satisfies the equation $g(\tau) = 0$, where $g(\tau)$ is given by (2.1); thus, we obtain $\tau(1) = \tau_{\min}$. Consequently,
\[ B'(1) = -\frac{1}{p_*} \left( 1 + \tau_{\min}^{\beta} \right)^{\frac{-\beta}{p_*}} \int_{\mathbb{R}^N} |u|^{p_*} dx = -\frac{\lambda^{\beta}}{p_*} \int_{\mathbb{R}^N} |u|^{p_*} dx. \] (2.6)

By combining (2.5) and (2.6), we conclude (2.3). By a similar argument as in the proof of (2.3), we show that
\[
\int_{\mathbb{R}^N} |v_0|^p dx = \lambda^{\beta} \int_{\mathbb{R}^N} |v_0|^p dx, \quad \int_{\mathbb{R}^N} |u_0|^p |v_0|^\beta dx = \lambda^{\beta} \int_{\mathbb{R}^N} |u_0|^p dx.
\] (2.7)

Therefore, by (2.3) and (2.7), we obtain
\[
\int_{\mathbb{R}^N} |u_0|^p |v_0|^\beta dx = \lambda^{\beta} \int_{\mathbb{R}^N} |u_0|^p dx, \quad \int_{\mathbb{R}^N} |u_0|^p |v_0|^\beta dx = \lambda^{\beta} \int_{\mathbb{R}^N} |v_0|^p dx.
\]

Again, since \((\lambda U, \mu U)\) solves the problem \(S\), we obtain
\[
\lambda^{\beta} = \frac{\alpha}{p_*} \mu^{\beta} \lambda^{\alpha}. \quad \text{(2.8)}
\]

Now define \((u_1, v_1) = \left( \frac{u_0}{\Lambda}, \frac{v_0}{\Lambda} \right)\). By using (2.3), (2.7), and (2.8), we have
\[
||u_1||_{W^{s,p}} = \lambda^{\beta} ||u_0||_{W^{s,p}},
\]
\[
= \lambda^{\beta} \int_{\mathbb{R}^N} \left( |u_0|^p + \frac{\alpha}{p_*} |u_0|^a |v_0|^\beta \right) dx
\]
\[
= \lambda^{\beta} \left( \lambda^{\beta} + \frac{\alpha}{p_*} \mu^{\beta} \lambda^{\alpha} \right) \int_{\mathbb{R}^N} |u_0|^p dx
\]
\[
= ||u_0||_{W^{s,p}}.
\]

Similarly, we obtain \(||v_1||_{W^{s,p}} = ||u_1||_{W^{s,p}} ||v_1||_{W^{s,p}}. Therefore, we have
\[
||u_1||_{W^{s,p}} = ||u_0||_{W^{s,p}}, \quad ||v_1||_{W^{s,p}} = ||v_0||_{W^{s,p}}.
\] (2.9)

Also, by (2.3),
\[
\int_{\mathbb{R}^N} |u_1|^p dx = \int_{\mathbb{R}^N} |u_0|^p dx, \quad \text{(2.10)}
\]
and by (2.7),
\[
\int_{\mathbb{R}^N} |v_1|^p dx = \int_{\mathbb{R}^N} |v_0|^p dx.
\] (2.11)

Thus, from (2.9) and (2.10), we conclude that \(u_1\) achieves \(S\). Further, from (1.1), (2.9) and (2.11) imply that \(v_1\) also achieves \(S\) in (1.1). This completes the proof. \(\square\)

3 Proof of Theorems 1.5, 1.6, and 1.7

In this section, we study the system \((\tilde{S}_\gamma)\) that we introduced in the introduction. For the reader's convenience, we recall \((\tilde{S}_\gamma)\):
\[
(-\Delta)\gamma u = |u|^{p_*-2} u + \frac{\alpha y}{p_*} |u|^{a-2} u |v|^\beta \quad \text{in} \quad \mathbb{R}^N,
\]
\[
(-\Delta)\gamma v = |v|^{p_*-2} v + \frac{\beta y}{p_*} |v|^{p-2} v |u|^a \quad \text{in} \quad \mathbb{R}^N,
\]
\[\text{in} \quad \mathbb{R}^N,\quad u, v \in W^{s,p}(\mathbb{R}^N).\] \((\tilde{S}_\gamma)\)
We also recall that (see (1.7)) the energy functional associated to the aforementioned system is
\[ J(u, v) = \frac{1}{p} \left( ||u||_{W^{1,p}}^p + ||v||_{W^{1,p}}^p \right) - \frac{1}{p^*} \int_{\mathbb{R}^N} (||u||^{p^*} + ||v||^{p^*} + \gamma |u|^a|v|^b)dx, \quad (u, v) \in X. \]

The definition of Nehari manifold (1.8) is
\[ N = \left\{(u, v) \in X : u \neq 0, v \neq 0, ||u||_{W^{1,p}}^p + ||v||_{W^{1,p}}^p \right\} = \int_{\mathbb{R}^N} (||u||^{p^*} + ||v||^{p^*} + \gamma |u|^a|v|^b)dx. \]

Therefore, it follows
\[ A = \inf_{(u, v) \in N} J(u, v) \]
\[ = \inf_{(u, v) \in N} \frac{S}{N} (||u||_{W^{1,p}}^p + ||v||_{W^{1,p}}^p) \]
\[ = \inf_{(u, v) \in N} \frac{S}{N} \int_{\mathbb{R}^N} (||u||^{p^*} + ||v||^{p^*} + \gamma |u|^a|v|^b)dx. \]

Proposition 3.1. Assume that \( c, d \in \mathbb{R} \) satisfy
\[
\begin{align*}
\frac{p^* - p}{c - p} + \frac{ay}{p^* d^*} & \geq 1, \\
\frac{p^* - p}{d - p} + \frac{by}{p^* d^*} & \geq 1, \\
c, d & > 0.
\end{align*}
\]

If \( \frac{N}{25} < p < \frac{N}{s}, a, b > p \) and (1.10) hold, then \( c + d \geq k + \ell \), where \( k, \ell \in \mathbb{R} \) satisfy (1.9).

Proof. We use the change of variables \( y = c + d, x = c/d, y_0 = k + \ell \), and \( x_0 = k/\ell \) into (3.1) and (1.9), we obtain
\[ y^\frac{p^* - p}{c - p} \geq \frac{(x + 1)^{p^* - p}}{x^{p^* - p}} + \frac{ay}{p^* d^*} = f_1(x), \quad y_0^{p^* - p} = f_1(x_0), \]
\[ y^\frac{p^* - p}{d - p} \geq \frac{(x + 1)^{p^* - p}}{1 + \frac{by}{p^* d^*}} = f_2(x), \quad y_0^{p^* - p} = f_2(x_0). \]

Then, one has
\[ f'_1(x) = \frac{ay(x + 1)^{p^* - p}}{p^* \left( x^{p^* - p} + \frac{ay}{p^* d^*} \right)^{\frac{1}{p^*}}} \left\{ p^* (p^* - p) x^{\frac{1}{p^*}} + bx - a + p \right\}, \]
\[ = \frac{ay(x + 1)^{p^* - p}}{p^* \left( x^{p^* - p} + \frac{ay}{p^* d^*} \right)^{\frac{1}{p^*}}} g_1(x), \]
\[ f'_2(x) = \frac{by(x + 1)^{p^* - p}}{p^* \left( 1 + \frac{by}{p^* d^*} \right)^{\frac{1}{p^*}}} \left\{ p^* (p^* - p) x^{\frac{1}{p^*}} + (b - p)x^2 - ax^p \right\}, \]
\[ = \frac{by(x + 1)^{p^* - p}}{p^* \left( 1 + \frac{by}{p^* d^*} \right)^{\frac{1}{p^*}}} g_2(x). \]
Hence, we obtain \( x_1 = \left( \frac{\text{pay}}{p^*(p^* - p)} \right)^\frac{\beta}{p} \) from \( g_1'(x) = 0 \) and similarly, for \( g_2 \), we have \( x_2 = \frac{\alpha - p}{\beta - p} \). Now by using (1.10), we conclude that

\[
\max_{x>0} g_1(x) = g_1(x_1) = \left( \frac{\text{pay}}{p^*(p^* - p)} \right)^\frac{\beta}{p} (\beta - p) - (\alpha - p) \leq 0,
\]

\[
\min_{x>0} g_2(x) = g_2(x_2) = \left( \frac{\text{pay}}{p^*(p^* - p)} \right)^\frac{\alpha}{p} - \left( \frac{\alpha - p}{\beta - p} \right)^\frac{\alpha}{p} \geq 0.
\]

Therefore, we conclude that the function \( f_1 \) is decreasing in \((0, \infty)\), and on the other hand, the function \( f_2 \) is increasing in \((0, \infty)\). Thus, we have

\[
y^{\frac{\beta}{p}} \geq \max\{f_1(x), f_2(x)\} \geq \min(\max\{f_1(x), f_2(x)\}) = \min(\max\{f_1(x), f_2(x)\}) = y_0^{\frac{p^* - p}{p}}.
\]

Hence, the result follows. \(\square\)

We define the functions

\[
F_1(k, \ell) = k^{\frac{\beta}{p}} + \frac{\text{pay}k^{\frac{\alpha}{p}}}{p^*} - 1, \quad k > 0, \ell \geq 0,
\]

\[
F_2(k, \ell) = \ell^{\frac{\beta}{p}} + \frac{\text{pay}k^{\frac{\alpha}{p}}}{p^*} - 1, \quad k \geq 0, \ell > 0,
\]

\[
\dot{\ell} = \frac{\left( k^{\frac{\alpha}{p}} \right)^\frac{\beta}{p}}{\text{pay}} k^{\frac{\alpha}{p}} \left( 1 - k^{\frac{\alpha}{p}} \right)^\frac{\beta}{p}, \quad 0 < k \leq 1,
\]

\[
k = \frac{\left( k^{\frac{\alpha}{p}} \right)^\frac{\beta}{p}}{\text{pay}} \ell^{\frac{\alpha}{p}} \left( 1 - \ell^{\frac{\alpha}{p}} \right)^\frac{\beta}{p}, \quad 0 < \ell \leq 1.
\]

Then \( F_1(k, \ell(k)) = 0 \) and \( F_2(k(\ell), \ell) = 0 \).

**Lemma 3.2.** Assume that \( \frac{2N}{N+2s} < p < \frac{N}{2s} \) and \( \alpha, \beta < p \). Then

\[
F_1(k, \ell) = 0, F_2(k, \ell) = 0, \quad k, \ell > 0
\]

has a solution \((k_0, \ell_0)\) such that \( F_2(k, \ell(k)) < 0 \) for all \( k \in (0, k_0) \), that is, \((k_0, \ell_0)\) satisfies (1.12). Similarly, (3.3) has a solution \((k_0, \ell_1)\) such that \( F_1(k(\ell), \ell) < 0 \) for all \( \ell \in (0, \ell_1) \), that is, \((k_0, \ell_1)\) satisfies (1.9) and \( \ell_1 = \min(\ell : (k, \ell) \text{ satisfies (1.9)}) \).

**Proof.** The proof is exactly similar to [19, Lemma 3.2]. \(\square\)

**Lemma 3.3.** Assume that \( \frac{N}{N+2s} < p < \frac{N}{2s} \), \( \alpha, \beta < p \) and (1.11) holds. Then \( k_0 + \ell_0 < 1 \), where \((k_0, \ell_0)\) is same as in Lemma 3.2 and

\[
F_1(k(\ell), \ell) < 0 \quad \forall \ell \in (0, \ell_0), \quad F_2(k, \ell(k)) < 0 \quad \forall k \in (0, k_0).
\]

**Proof.** By using (3.2), we obtain
\[ e'(k) = \left( \frac{p^*}{a^*} \right) \frac{p^*}{k^*} \left( 1 - \frac{p^*}{k^*} \right) - \left( \frac{p - a}{\beta} - k^* \right), \]

and then we have

\[ e''(k) = \frac{(p - \beta)(p^* - p)}{p^*} \left( \frac{p^*}{k^*} \right) \frac{p^*}{k^*} \left( 1 - \frac{p^*}{k^*} \right) \times \left( k^* - \frac{p - a}{\beta} + k^* \frac{p - a}{\beta} - k^* \right). \]

Note that \( e'(1) = 0 = e''(k) > 0 \) for \( 0 < k < \left( \frac{p - a}{\beta} \right)^{\frac{1}{p}} \), whereas \( e'(k) < 0 \) for \( \left( \frac{p - a}{\beta} \right)^{\frac{1}{p}} < k < 1 \).

From \( e''(k) = 0 \), we obtain \( k = \left( \frac{p^* - p}{p^* - p} \right)^{\frac{1}{p}} \). Then by (1.11), we obtain

\[ \min_{k \in (0, 1)} e'(k) = \min_{k \in \left\{ \frac{a^*}{p}, \frac{p^*}{p} \right\}} e'(k) = e'(\bar{k}) = \left( \frac{p^* - p}{p^* - p} \right)^{\frac{1}{p}} \left( \frac{p - a}{\beta} \right)^{\frac{1}{p}} \geq -1. \]

The remaining proof follows from [19, Lemma 3.3] by considering \( \mu_1 = 1 = \mu_2 \) in their proof. \( \square \)

**Lemma 3.4.** Assume that \( N_{\alpha + \beta} < p < N_{\beta} \), \( \alpha, \beta < p \), and (1.11) holds. Then

\[ \begin{aligned}
k + \ell &\leq k_0 + \ell_0 \\
F_1(k, \ell) &\geq 0, \quad F_2(k, \ell) \geq 0 \\
k, \ell &\geq 0, \quad (k, \ell) \neq (0, 0)
\end{aligned} \]

has a unique solution \( (k, \ell) = (k_0, \ell_0) \), where \( F_1 \) and \( F_2 \) are given by (3.2).

**Proof.** The proof follows from [19, Proposition 3.4]. \( \square \)

**Proof of Theorem 1.5.** By using (1.9), we have \( (k_0^{1/p} U, \ell_0^{1/p} U) \in \mathcal{N} \) is a nontrivial solution of \( (S) \) and

\[ A \leq \mathcal{F}(k_0^{1/p} U, \ell_0^{1/p} U) = \frac{S}{N}(k_0 + \ell_0)S. \]  \( \text{(3.4)} \)

Now, suppose \( \{u_n, v_n\} \in \mathcal{N} \) be a minimizing sequence for \( A \) such that \( \mathcal{F}(u_n, v_n) \to A \) as \( n \to \infty \). Let \( c_n = ||u_n||_{L^p(\mathbb{R}^n)} \) and \( d_n = ||v_n||_{L^p(\mathbb{R}^n)} \). Then by Hölder’s inequality, we have

\[ S c_n \leq ||u_n||_{W^{s,p}} = \int_{\mathbb{R}^n} |u_n|^{p^*} + \frac{ay}{p^*} |u_n|^{p^*} |v_n|^{\beta} \right) \, dx \leq c_n^{p^*} + \frac{ay}{p^*} c_n^{p^*} d_n^{\beta}. \]  \( \text{(3.5)} \)

This implies that

\[ \tilde{c}_n^{p^*} + \frac{ay}{p^*} \tilde{c}_n^{p^*} d_n^{\beta} \geq 1 \quad \text{i.e.,} \quad F_1(\tilde{c}_n, \tilde{d}_n) \geq 0, \]

where \( \tilde{c}_n = \frac{c_n}{S^{p^*}}, \tilde{d}_n = \frac{d_n}{S^{p^*}}. \) Similarly, we obtain

\[ S d_n \leq ||v_n||_{W^{s,p}} = \int_{\mathbb{R}^n} |v_n|^{p^*} + \frac{\beta y}{p^*} |u_n|^{p^*} |v_n|^{\beta} \right) \, dx \leq d_n^{p^*} + \frac{\beta y}{p^*} c_n^{p^*} d_n^{\beta}, \]  \( \text{(3.6)} \)

and thus, \( F_2(\tilde{c}_n, \tilde{d}_n) \geq 0. \) Then for \( \alpha, \beta > p \), by Proposition 3.1, we have \( \tilde{c}_n + \tilde{d}_n \geq k + \ell = k_0 + \ell_0 \), and on the other hand for \( \alpha, \beta < p \), by Lemma 3.4, we have \( \tilde{c}_n + \tilde{d}_n = k_0 + \ell_0 \). Hence,
\[ c_n + d_n \geq (k_0 + \epsilon_0)S_{\frac{Np}{2}}^{\frac{Np}{2}}. \]  

(3.7)

Since \( J(u_n, v_n) = \frac{s}{N} \left( || u_n ||_{W^{s, p}}^p + || v ||_{W^{s, p}}^p \right) \), by using (3.4)–(3.6), we have

\[ S(c_n + d_n) \leq \frac{N}{s} J(u_n, v_n) = \frac{N}{s} A(1) \leq (k_0 + \epsilon_0)S_{\frac{Np}{2}}^{\frac{Np}{2}} + o(1). \]

This implies that

\[ c_n + d_n \leq (k_0 + \epsilon_0)S_{\frac{Np}{2}}^{\frac{Np}{2}} + o(1). \]

(3.8)

By combining (3.7) and (3.8), we obtain

\[ A = \lim_{n \to \infty} J(u_n, v_n) = \frac{s}{N} \lim_{n \to \infty} (c_n + d_n) = \frac{s}{N} (k_0 + \epsilon_0)S_{\frac{Np}{2}}^{\frac{Np}{2}}. \]

Therefore,

\[ A = \frac{s}{N} (k_0 + \epsilon_0)S_{\frac{Np}{2}}^{\frac{Np}{2}} = J(k_{0}^{\frac{1}{p}}U, \epsilon_{0}^{\frac{1}{p}}U). \]

This completes the proof of Theorem 1.5. \( \square \)

Next, we prove existence of solutions of (1.13), namely, Theorem 1.7. For this, define

\[ W^{s, p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\}, \]

\[ X(B_0(0)) = W^{s, p}(B_0(0)) \times W^{s, p}(B_0(0)), \]

where \( W^{s, p}(B_0(0)) = \{ u \in W^{s, p}(\mathbb{R}^N) : u = 0 \ \text{in} \ \mathbb{R}^N \setminus B_0(0) \} \) with the norm \( ||u||_{W^{s, p}} \), and

\[ \tilde{N}(R) = \left\{ (u, v) \in X(B_0(0)) \setminus \{(0, 0)\} : ||u||_{W^{s, p}}^p + ||v||_{W^{s, p}}^p = \int_{B_0(0)} (|u|^s + |v|^s + y |u|^q |v|^p) \, dx \right\}, \]

and set \( \tilde{A}(R) = \inf_{(u, v) \in \tilde{N}(R)} J(u, v) \). We also define

\[ \hat{N} = \left\{ (u, v) \in X \setminus \{(0, 0)\} : ||u||_{W^{s, p}}^p + ||v||_{W^{s, p}}^p = \int_{\mathbb{R}^N} (|u|^s + |v|^s + y |u|^q |v|^p) \, dx \right\}. \]

Set \( \hat{A} = \inf_{(u, v) \in \hat{N}} J(u, v) \). Since \( N \subset \hat{N} \), it follows \( \hat{A} \leq A \) and by the fractional Sobolev embedding \( \hat{A} > 0 \).

For \( \varepsilon \in (0, \min(a, \beta) - 1) \), consider

\[\begin{cases}
(-\Delta)^s u = |u|^{a-2-\varepsilon} u + \frac{(a - \varepsilon)\varepsilon}{p^s - 2\varepsilon} |u|^{a-2-\varepsilon} u |v|^\beta \varepsilon & \text{in } B_0(0), \\
(-\Delta)^s v = |v|^{\beta-2-\varepsilon} v + \frac{(\beta - \varepsilon)\varepsilon}{p^s - 2\varepsilon} |v|^{\beta-2-\varepsilon} v |u|^a \varepsilon & \text{in } B_0(0),
\end{cases}\]

(3.9)

The corresponding energy functional of the system (3.9) is given by

\[ J_{\varepsilon}(u, v) = \frac{1}{p} (|| u ||_{W^{s, p}}^p + || v ||_{W^{s, p}}^p) - \frac{1}{p^s - 2\varepsilon} \int_{B_0(0)} (|u|^{s-2} + |v|^{s-2} + y |u|^{a-\varepsilon} |v|^{\beta-\varepsilon}) \, dx. \]

Define
\[ \tilde{\mathcal{N}}_\varepsilon(R) = \left\{ (u, v) \in X(B_0(0)) \setminus \{(0, 0)\} : G_\varepsilon(u, v) = ||u||^p_{W^{s,p}} + ||v||^p_{W^{s,p}} \right\} \]

\[ - \int_{B_0(0)} (|u|^{p^*-2}\varepsilon + |v|^{p^*-2}\varepsilon + y|u|^{a-\varepsilon}|v|^{\beta-\varepsilon})\,dx = 0, \]

and set \( \tilde{A}_\varepsilon(R) = \inf_{(u,v)\in\tilde{\mathcal{N}}_\varepsilon(R)} \mathcal{F}_\varepsilon(u, v). \)

**Lemma 3.5.** For any \( \varepsilon_0 \in (0, \min(\alpha - 1, \beta - 1, (p^*-p)/2)) \), there exists a constant \( C_{\varepsilon_0} > 0 \) such that

\[ \tilde{A}_\varepsilon(R) \geq C_{\varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0]. \]

**Proof.** Let \((u, v) \in \tilde{\mathcal{N}}_\varepsilon(R)\). Then

\[ \mathcal{F}_\varepsilon(u, v) = \left( \frac{1}{p} - \frac{1}{p^*} \right) \left( ||u||^p_{W^{s,p}} + ||v||^p_{W^{s,p}} \right), \]

so it suffices to show that \( ||u||^p_{W^{s,p}} + ||v||^p_{W^{s,p}} \) is bounded away from zero. We have

\[
\begin{align*}
||u||^p_{W^{s,p}} + ||v||^p_{W^{s,p}} &= \int_{B_0(0)} (|u|^{p^*-2}\varepsilon + |v|^{p^*-2}\varepsilon + y|u|^{a-\varepsilon}|v|^{\beta-\varepsilon})\,dx \\
&\leq |B_0(0)|^{2/e} ||u||^{p^*} ||v||^p \left( \left\{ \int_{B_0(0)} |u|^{p^*} \,dx \right\}^{(p^*-2\varepsilon)/p^*} + \left\{ \int_{B_0(0)} |v|^{p} \,dx \right\}^{\beta-\varepsilon}/p^* \right) \\
&\quad + y \left( \left\{ \int_{B_0(0)} |u|^{p^*} \,dx \right\}^{(a-\varepsilon)/p^*} + \left\{ \int_{B_0(0)} |v|^{p} \,dx \right\}^{\beta-\varepsilon}/p^* \right) \\
&\leq |B_0(0)|^{2/e} ||u||^{p^*} S^{-((p^*-2\varepsilon)/p)} ||u||^{p^*} ||v||^p + y||u||^{p^*} ||v||^p,
\end{align*}
\]

by the Hölder and Sobolev inequalities. By Young's inequality,

\[ ||u||^{p^*} ||v||^p \leq \frac{a - \varepsilon}{p^*-2\varepsilon} ||u||^{p^*} + \frac{\beta - \varepsilon}{p^*-2\varepsilon} ||v||^p \leq ||u||^{p^*} ||v||^p. \]

Therefore, (3.10) gives

\[ ||u||^{p^*} ||v||^p \leq (1 + y) |B_0(0)|^{2/e} ||u||^{p^*} S^{-((p^*-2\varepsilon)/p)} ||u||^{p^*} + ||v||^p. \]

Since \((p^*-2\varepsilon)/p > 1\),

\[ ||u||^{p^*} + ||v||^p \leq (||u||^{p^*} + ||v||^p) S^{((p^*-2\varepsilon)/p)}, \]

thus (3.11) gives

\[ ||u||^{p^*} + ||v||^p \geq \frac{S^{((p^*-2\varepsilon)/p)}}{(1 + y) |B_0(0)|^{2/e} ||u||^{p^*}} ||u||^{p^*} + ||v||^p. \]

The desired conclusion follows from this since \( p^*-p-2\varepsilon \geq p^*-p - 2\varepsilon_0 > 0 \) and the function

\[ h(t) = \left( \frac{S^{((p^*-2\varepsilon)/p)}}{(1 + y) |B_0(0)|^{2/e} ||u||^{p^*}} \right)^{p/(p^*-p-2\varepsilon)} \]

is continuous and positive in \([0, \varepsilon_0]\). \( \square \)

**Lemma 3.6.** Assume that \( \frac{2N}{N+2} \leq p < \frac{N}{2} \) and \( a, \beta < p \). For \( \varepsilon \in (0, \min(\alpha, \beta) - 1) \), it holds
\[ \hat{A}_e(R) < \min\{ \inf_{(u,0) \in \tilde{N}_e(R)} \mathcal{J}_e(u,0), \inf_{(0,v) \in \tilde{N}_e(R)} \mathcal{J}_e(0,v) \}. \]

**Proof.** Clearly, \( 2 < p_s^* - 2\varepsilon < p_s^* \) from \( \min\{\alpha, \beta\} \leq \frac{2}{p_s} \). Then we may assume that \( u_1 \) is a least energy solution of

\[
\begin{cases}
(\Delta_{p})^y u = |u|^{p_s^*-2\varepsilon} u \\
u \in W_0^{p,\varepsilon}(B(0)).
\end{cases}
\]

Set

\[ \mathcal{J}_e(u_1,0) = a_{01} = \inf_{(u,0) \in \tilde{N}_e(R)} \mathcal{J}_e(u,0), \mathcal{J}_e(0,u_1) = a_{01} = \inf_{(0,v) \in \tilde{N}_e(R)} \mathcal{J}_e(0,v). \]

We claim that for any \( \sigma \in \mathbb{R} \), there exists a unique \( t(\sigma) > 0 \) such that \( (t(\sigma))^{1/p} u_1, (t(\sigma))^{1/p} \sigma u_0) \in \tilde{N}_e(R) \).

\[
(\sigma)^{p_s^*-2\varepsilon} = \frac{||u_1||_{W_{p_s^*}}^p + |\sigma| p_{s^*}}{\int_{B(0)} (|u_1|^{p_s^*-2\varepsilon} + |\sigma u_0|^{p_s^*-2\varepsilon} + \beta |u_1|^{\beta-\varepsilon} |\sigma u_0|^{\beta-\varepsilon}) dx}
\]

where \( q = \frac{p(p_s^*-2\varepsilon)}{p_s^*-p-2\varepsilon} \), i.e., \( \frac{1}{q} = \frac{1}{p_s} - \frac{1}{p_s^*-2\varepsilon} \). Note that \( t(0) = 1 \), we have

\[
\lim_{\sigma \to 0} t'(\sigma) = -\left( \frac{\beta-\varepsilon}{a_{01}(p_s^*-2\varepsilon)} \right) |\sigma|^{\beta-\varepsilon} \sigma (1 + o(1)).
\]

This implies that as \( \sigma \to 0 \)

\[
t'(\sigma) = -\left( \frac{\beta-\varepsilon}{a_{01}(p_s^*-2\varepsilon)} \right) |\sigma|^{\beta-\varepsilon} \sigma (1 + o(1)).
\]

Then

\[
t(\sigma) = 1 - \left( \frac{\beta-\varepsilon}{a_{01}(p_s^*-2\varepsilon)} \right) |\sigma|^{\beta-\varepsilon} (1 + o(1)) \text{ as } \sigma \to 0,
\]

and therefore,

\[
t(\sigma)^{p_s^*-2\varepsilon} = 1 - \left( \frac{\beta-\varepsilon}{a_{01}(p_s^*-2\varepsilon)} \right) |\sigma|^{\beta-\varepsilon} (1 + o(1)) \text{ as } \sigma \to 0.
\]

We obtain for \( \sigma \) small enough

\[
\hat{A}_e(\sigma) \leq \mathcal{J}_e(t(\sigma))^{1/p} u_1, (t(\sigma))^{1/p} \sigma u_0)
\]

\[
= \frac{1}{q} \left( qa_{01} + qa_{01} |\sigma|^{p_s^*-2\varepsilon} + |\sigma|^{\beta-\varepsilon} \int_{B(0)} |u_1|^{p_s^*-2\varepsilon} dx \right) (\sigma)^{p_s^*-2\varepsilon}
\]

\[
= a_{01} - \frac{1}{p_s^*-2\varepsilon} |\sigma|^{\beta-\varepsilon} \int_{B(0)} |u_1|^{p_s^*-2\varepsilon} dx + o(|\sigma|^{\beta-\varepsilon}) < a_{01}.
\]

Similarly, we see that \( \hat{A}_e(\sigma) < a_{01} \). This completes the proof. \( \square \)

Note that, similarly to Lemma 3.6, we obtain

\[
\tilde{A} < \min\{ \inf_{(u,0) \in N} \mathcal{J}(u,0), \inf_{(0,v) \in N} \mathcal{J}(0,v) \} = \min\{ \mathcal{J}(U,0), \mathcal{J}(0,U) \} = \frac{S}{N} S^\frac{m}{p_s}.
\] (3.12)
Proposition 3.7. For $0 < \epsilon < \min\{\min(a, \beta) - 1, \frac{s^*-p}{2}\}$, system (3.9) has a positive least energy solution $(u_\epsilon, v_\epsilon)$, where both $u_\epsilon, v_\epsilon$ are radially symmetric nonincreasing functions.

Proof. By Lemma 3.5, $\tilde{A}_\epsilon(R) > 0$. Let $(u, v) \in \tilde{\mathcal{N}}(R)$ with $u, v \geq 0$. Let $u^*, v^*$ be Schwartz symmetrization of $u, v$, respectively. Then by nonlocal Pólya-Szegö inequality [2] and properties of the Schwartz symmetrization, we obtain

$$\|u^*\|_{W^{s,p}}^p + \|v^*\|_{W^{s,p}}^p \leq \int_{B_0(0)} (|u^*|^{p^*-2} + |v^*|^{p^*-2})dx.$$

Also, note that $\mathcal{J}(t^s_1/pu^*, t^s_1/pv^*) \leq \mathcal{J}(u, v)$ for some $t_\epsilon \in (0, 1]$ such that $((u_n, v_n)) \subset \tilde{\mathcal{N}}(R)$ of $\tilde{A}_\epsilon$ such that $(u_n, v_n) = (u^*_n, v^*_n)$ for any $n$ and $\mathcal{J}(u_n, v_n) \to \tilde{A}_\epsilon$ as $n \to \infty$. Thus, we obtain both the sequences $[u_n]$ and $[v_n]$ that are bounded in $W_0^{s,p}(B_0(0))$. $W_0^{s,p}(B_0(0))$ is a reflexive Banach space, up to a subsequence, $u_n \rightharpoonup u_\epsilon, v_n \rightharpoonup v_\epsilon$ weakly in $W_0^{s,p}(B_0(0))$. Moreover, as $W_0^{s,p}(B_0(0)) \hookrightarrow L^{s^*-2p}(B_0(0))$ is a compact embedding, it follows $u_n \to u_\epsilon, v_n \to v_\epsilon$ strongly in $L^{s^*-2p}(B_0(0))$. Therefore,

$$\int_{B_0(0)} (|u_n|^{p^*-2} + |v_n|^{p^*-2})dx \leq \lim_{n \to \infty} \mathcal{J}(u_n, v_n) = \frac{p(p^*_s - 2\epsilon)}{p^*_s - 2\epsilon - n} \tilde{A}_\epsilon(R) > 0,$$

and this yields that $(u_\epsilon, v_\epsilon) \neq (0, 0)$ and also $u_\epsilon, v_\epsilon$ are nonnegative radially symmetric decreasing. By using the weak lower semicontinuity property of the norm, we also have

$$\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \leq \lim_{n \to \infty} (\|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p),$$

and therefore,

$$\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p \leq \int_{B_0(0)} (|u_\epsilon|^{p^*-2} + |v_\epsilon|^{p^*-2})dx.$$

Therefore, there exists $t_\epsilon \in (0, 1]$ such that $(t^s_1/pu_\epsilon, t^s_1/pv_\epsilon) \in \tilde{\mathcal{N}}_\epsilon$, and hence,

$$\tilde{A}_\epsilon(R) \leq \mathcal{J}(t^s_1/pu_\epsilon, t^s_1/pv_\epsilon) = \frac{t_\epsilon(p^*_s - 2\epsilon - p)}{p^*_s - 2\epsilon} \lim_{n \to \infty} (\|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p) \leq \frac{t_\epsilon(p^*_s - 2\epsilon - p)}{p^*_s - 2\epsilon} \tilde{A}_\epsilon(R),$$

which yields that $t_\epsilon = 1, (u_\epsilon, v_\epsilon) \in \tilde{\mathcal{N}}_\epsilon(R), \tilde{A}_\epsilon(R) = \mathcal{J}(u_\epsilon, v_\epsilon)$, and

$$\|u_\epsilon\|_{W^{s,p}}^p + \|v_\epsilon\|_{W^{s,p}}^p = \lim_{n \to \infty} (\|u_n\|_{W^{s,p}}^p + \|v_n\|_{W^{s,p}}^p).$$

This proved that $u_n \to u_\epsilon, v_n \to v_\epsilon$ strongly in $W_0^{s,p}(B_0(0))$. Now by Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that
\[ \mathcal{F}'(u_\varepsilon, v_\varepsilon) + \lambda \mathcal{G}'(u_\varepsilon, v_\varepsilon) = 0. \]

Since \( \mathcal{F}'(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = \mathcal{G}(u_\varepsilon, v_\varepsilon) = 0 \) and
\[ G'(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = -(p^* - 2\varepsilon - p) \int_{\mathbb{R}^d} \left( |u_\varepsilon|^{p^* - 2\varepsilon} + |v_\varepsilon|^{p^* - 2\varepsilon} + \gamma |u_\varepsilon|^{a-\varepsilon} |v_\varepsilon|^{p^{*\varepsilon}} \right) dx < 0, \]
we obtain \( \lambda = 0 \) and hence, \( \mathcal{F}'(u_\varepsilon, v_\varepsilon) = 0 \). Since \( \widehat{A}(R) = \mathcal{F}(u_\varepsilon, v_\varepsilon) \) and by Lemma 3.6, we have \( u_\varepsilon, v_\varepsilon \neq 0 \). By maximum principle [15, Lemma 3.3] we conclude the desired result.

**Lemma 3.8.** For any \((u, v) \in \tilde{N}\), there is a sequence \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) such that \((u_n, v_n) \to (u, v)\) in \(X\) as \(n \to \infty\).

**Proof.** By density, there is a sequence \((\tilde{u}_n, \tilde{v}_n) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)\) such that \((\tilde{u}_n, \tilde{v}_n) \to (u, v)\) in \(X\) as \(n \to \infty\). Let
\[ t_n = \left( \frac{|\tilde{u}_n|_p^p + |\tilde{v}_n|_p^p}{\int_{\mathbb{R}^d} (|\tilde{u}_n|_p^p + |\tilde{v}_n|_p^p + \gamma |\tilde{u}_n|^{a} |\tilde{v}_n|^{p^{\varepsilon}}) dx} \right)^{1/(p^* - p)} \]
and note that \( t_n \to 1 \) since \((u, v) \in \tilde{N}\). Then \((u_n, v_n) = (t_n \tilde{u}_n, t_n \tilde{v}_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) and \((u_n, v_n) \to (u, v)\) in \(X\).

**Lemma 3.9.** There is a minimizing sequence \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) for \( \widehat{A} \).

**Proof.** Let \((\tilde{u}_n, \tilde{v}_n) \in \tilde{N}\) be a minimizing sequence for \( \widehat{A} \), i.e., \( \mathcal{F}(\tilde{u}_n, \tilde{v}_n) \to \widehat{A} \). By the continuity of \( \mathcal{F} \) and Lemma 3.8, there is a \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) such that
\[ |\mathcal{F}(u, v_n) - \mathcal{F}(\tilde{u}, \tilde{v})| < \frac{1}{n}. \]
Then \( \mathcal{F}(u_n, v_n) \to \widehat{A} \), so \((u_n, v_n) \in \tilde{N} \cap (C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N))\) is a minimizing sequence for \( \widehat{A} \).

**Proof of Theorem 1.7.** First, we prove that
\[ \widehat{A}(R) = \widehat{A} \quad \text{for every } R > 0. \quad (3.13) \]
Let \( R_0 < R_2 \), then \( \tilde{N}(R_0) \subset \tilde{N}(R_2) \), and hence, by definition, we have \( \widehat{A}(R_2) \leq \widehat{A}(R_0) \). To prove reverse inequality, let \((u, v) \in \tilde{N}(R_2)\) and define
\[ (u_\varepsilon(x), v_\varepsilon(x)) = \left( \frac{R_2}{R_0} \int_{\frac{R_2}{R_0} x}^{\infty} \right) \begin{bmatrix} R_2 \\ R_0 \end{bmatrix} \left( \begin{bmatrix} u \\ v \end{bmatrix} \right), \]
Clearly, \((u_\varepsilon, v_\varepsilon) \in \tilde{N}(R_0)\). Therefore, we obtain
\[ \widehat{A}(R_2) = \mathcal{F}(u, v), \quad \text{for any } (u, v) \in \tilde{N}(R_2), \]
and this implies that \( \widehat{A}(R_2) \leq \widehat{A}(R_0) \). So, we obtain \( \widehat{A}(R_2) = \widehat{A}(R_0) \). Let \((u_n, v_n) \in \tilde{N}\) be a minimizing sequence of \( \widehat{A} \). In view of Lemma 3.9, we may assume that \( u_n, v_n \in W_0^{p, p}(B_{R_0}(0)) \) for some \( R_0 > 0 \). Then, \((u_n, v_n) \in \tilde{N}(R_0)\) and
\[ \widehat{A} = \lim_{n \to \infty} \mathcal{F}(u_n, v_n) \geq \lim_{n \to \infty} \widehat{A}(R_0) = \widehat{A}(R), \]
and hence, (3.13) holds.

Let \((u, v) \in \tilde{N}(R)\) be arbitrary, then there exists \( t_0 > 0 \) with \( t_0 \to 1 \) as \( \varepsilon \to 0 \) such that \((t_\varepsilon^{1/p} u, t_\varepsilon^{1/p} v) \in \tilde{N}_\varepsilon(R)\). Therefore, we have
\[ \limsup_{\varepsilon \to 0} \widehat{A}_\varepsilon(R) \leq \limsup_{\varepsilon \to 0} \mathcal{F}(t_\varepsilon^{1/p} u, t_\varepsilon^{1/p} v) = \mathcal{F}(u, v). \]
Thus, by using (3.13), we obtain
\[
\limsup_{\varepsilon \to 0} \tilde{A}_\varepsilon(R) \leq \tilde{A}(R) = \tilde{A}.
\] (3.14)

By Proposition 3.7, let \((u_\varepsilon, v_\varepsilon)\) be a positive least energy solution of (3.9), which is radially symmetric non-increasing. Then by Lemma 3.5, for any \(\varepsilon_0 \in (0, \min\{\alpha - 1, \beta - 1, (p^*_p - p)/2\})\), there exists a constant \(C_{\varepsilon_0} > 0\) such that

\[
\tilde{A}_\varepsilon(R) = \frac{p^*_p - p - 2\varepsilon}{p(p^*_p - 2\varepsilon)} \left( ||u_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} + ||v_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} \right) \geq C_{\varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0].
\] (3.15)

Therefore, from (3.14), we obtain \(u_\varepsilon, v_\varepsilon \in W^{p^*_p,p}_0(B_0(0))\) are uniformly bounded. Thus, by reflexivity upto a subsequence, \(u_\varepsilon \rightharpoonup u_0\) and \(v_\varepsilon \rightharpoonup v_0\) weakly in \(W^{p^*_p,p}_0(B_0(0))\) as \(\varepsilon \to 0\). Since (3.9) is a subcritical system in bounded domain, passing the limit \(\varepsilon \to 0\), it follows that \((u_0, v_0)\) is a solution of the following system:

\[
\begin{align*}
(-\Delta)^s_u &= |u|^{p^*_p-2}u + \frac{\alpha y}{p^*_p} |u|^{p-2}u |v|^\beta \quad \text{in } B_0(0), \\
(-\Delta)^s_v &= |v|^{p^*_p-2}v + \frac{\beta y}{p^*_p} |v|^{p-2}v |u|^\alpha \quad \text{in } B_0(0), \\
u, v &\in W^{p^*_p,p}_0(B_0(0)).
\end{align*}
\]

Also note that \(u_0\) and \(v_0\) are nonnegative and from (3.15), we see that \((u_0, v_0) \neq (0, 0)\). We may now assume that \(u_0 \neq 0\). Therefore, by strong maximum principle [15], we obtain \(u_0 > 0 \text{ in } B_0(0)\). Further, we claim, \(v_0 \neq 0\). If \(v_0 \equiv 0\), then substituting \((u_0, v_0)\) in the aforementioned system of equation shows that \(u_0\) is a positive solution to \((-\Delta)^s u = |u|^{p^*_p-2}u \text{ in } B_0(0)\). Since \(u_0 \in W^{p^*_p,p}_0(B_0(0))\), it follows

\[
\mathcal{J}(u_0, 0) = \frac{1}{p} ||u_0||^{p^*_p}_{W^{p^*_p,p}} - \frac{1}{p^*_p} \int_{B_0(0)} u_0^{p^*_p} dx = \frac{1}{p} ||u_0||^{p^*_p}_{W^{p^*_p,p}} - \frac{1}{p^*_p} \int_{B_0(0)} u_0^{p^*_p} dx = \frac{S}{N} ||u_0||^{p^*_p}_{W^{p^*_p,p}}.
\] (3.16)

We also observe that \((u_0, 0), (0, u_0) \in \hat{N}\). Therefore, by using (3.12), we have

\[
\tilde{A} < \min \left\{ \inf_{(u_0, 0) \in \hat{N}} \mathcal{J}(u_0, 0), \inf_{(0, v) \in \hat{N}} \mathcal{J}(0, v) \right\} \leq \min\{\mathcal{J}(u_0, 0), \mathcal{J}(0, u_0)\} = \mathcal{J}(u_0, 0).
\] (3.17)

Combining (3.16) and (3.17) together yields

\[
\tilde{A} < \frac{S}{N} ||u_0||^{p^*_p}_{W^{p^*_p,p}}.
\] (3.18)

Further, by (3.14) and the fact that \((u_\varepsilon, v_\varepsilon)\) is a positive least energy solution of (3.9), it follows

\[
\tilde{A} \geq \limsup_{\varepsilon \to 0} \tilde{A}_\varepsilon(R)
\]

\[
= \limsup_{\varepsilon \to 0} \mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon)
\]

\[
= \limsup_{\varepsilon \to 0} \left[ \frac{1}{p} \left( ||u_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} + ||v_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} \right) - \frac{1}{p^*_p} \int_{B_0(0)} (u_\varepsilon^{p^*_p-2} + v_\varepsilon^{p^*_p-2} + yu_\varepsilon^{\alpha-2}v_\varepsilon^{\beta-2}) dx \right]
\]

\[
= \limsup_{\varepsilon \to 0} \left[ \frac{1}{p} - \frac{1}{p^*_p} \right] \left( ||u_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} + ||v_\varepsilon||^{p^*_p}_{W^{p^*_p,p}} \right)
\]

\[
\geq \frac{S}{N} \left( ||u_0||^{p^*_p}_{W^{p^*_p,p}} + ||v_0||^{p^*_p}_{W^{p^*_p,p}} \right)
\]

\[
= \frac{S}{N} ||u_0||^{p^*_p}_{W^{p^*_p,p}} > \tilde{A} \quad \text{(by (3.18))},
\]

which is a contradiction. Hence, \(v_0 \neq 0\), and again by strong maximum principle, we obtain \(v_0 > 0 \text{ in } B_0(0)\). Moreover, as \((u_\varepsilon, v_\varepsilon)\) is radial and \(u_\varepsilon \rightharpoonup u_0 \text{ a.e. and } v_\varepsilon \rightharpoonup v_0 \text{ a.e. (up to a subsequence)}\), we also have \(u_0, v_0\) are radial functions. Hence, \((u_0, v_0)\) is a positive radial solution to (1.13). \(\square\)
Proof of Theorem 1.6. To prove the existence of \((k(y), \ell(y))\) for small \(y > 0\), recalling (3.2), we denote \(F_i(k, \ell, y)\) instead of \(I_i(k, \ell)\), \(i = 1, 2\) in this case. Let \(k(0) = 1 = \ell(0)\), then \(F_i(k(0), \ell(0), 0) = 0\), \(i = 1, 2\). Clearly, we have
\[
\frac{\partial F_1}{\partial k}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial \ell}(k(0), \ell(0), 0) = \frac{p^*_s - p}{p} > 0
\]
and
\[
\frac{\partial F_1}{\partial \ell}(k(0), \ell(0), 0) = \frac{\partial F_2}{\partial k}(k(0), \ell(0), 0) = 0.
\]
Therefore, the Jacobian determinant is \(J_F(k(0), \ell(0), 0) = \frac{(p^*_s - p)^2}{p^2} > 0\), where \(F = (F_1, F_2)\). Therefore, by the implicit function theorem, \(k(y)\) and \(\ell(y)\) are well-defined functions and of class \(C^1\) in \((-y_2, y_2)\) for some \(y_2 > 0\) and \(F(k, \ell, y) = 0\) for \(y \in (-y_2, y_2)\). Then \((k(y)^{1/p}U, \ell(y)^{1/p}U)\) is a positive solution of \((S_\gamma)\). Since \(\lim_{\gamma \to 0} (k(y) + \ell(y)) = 2\). Thus, there exists \(y_1 \in (0, y_2)\) such that \(k(y) + \ell(y) > 1\) for all \(y \in (0, y_1)\). Therefore, by (3.12), we obtain
\[
J(k(y)^{1/p}U, \ell(y)^{1/p}U) = \frac{S}{N}(k(y) + \ell(y))S_\gamma^{\frac{1}{p}} > \frac{S}{N}S_\gamma^{\frac{1}{p}} = \tilde{A}.
\]
This completes the proof. □

Funding information: The research of M. Bhakta is partially supported by the SERB WEA grant (WEA/2020/000005) and DST Swarnajaynti fellowship (SB/SJF/2021-22/09). K. Perera was partially supported by the Simons Foundation grant 962241. F. Sk was partially supported by the SERB grant WEA/2020/000005.

Conflict of interest: The authors declare that there is no conflict of interest in this article.

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