TANNAKIAN CLASSIFICATION OF EQUIVARIANT PRINCIPAL BUNDLES ON TORIC VARIETIES

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Abstract. Let $X$ be a complete toric variety equipped with the action of a torus $T$, and $G$ be a reductive algebraic group, defined over $\mathbb{C}$. We introduce the notion of a compatible $\Sigma$–filtered algebra associated to $X$, generalizing the notion of a compatible $\Sigma$–filtered vector space due to Klyachko. We combine Klyachko’s classification of $T$–equivariant vector bundles on $X$ with Nori’s Tannakian approach to principal $G$–bundles, to give an equivalence of categories between $T$–equivariant principal $G$–bundles on $X$ and certain compatible $\Sigma$–filtered algebras associated to $X$.

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1. Introduction

Let $X$ be a toric variety under the action of a torus $T$, and $G$ be a reductive algebraic group; all defined over an algebraically closed field $K$. A $T$–equivariant vector bundle $E$ is a vector bundle on $X$ endowed with a lift of the $T$–action which is linear on fibers. The $T$–equivariant vector bundles over a nonsingular toric variety were first classified by Kaneyama [12]. This classification result for toric vector bundles is up to isomorphism and involves both combinatorial and linear algebraic data modulo an equivalence relation. Recently this work has been generalized for $T$–equivariant principal $G$–bundles [2, 3], also see [4] [5], when $K$ is the field $\mathbb{C}$ of complex numbers.

Later Klyachko gave an alternative description of equivariant vector bundles on arbitrary toric varieties (possibly non-smooth) over any algebraically closed field [8]. His correspondence gives an equivalence between the category $\mathcal{V}ec^T(X)$ of equivariant vector bundles on $X$ and the category $\mathcal{V}ec(\Sigma)$ of finite dimensional vector spaces with collection of decreasing $\mathbb{Z}$–graded filtrations, indexed by rays of $\Sigma$, satisfying a certain compatibility condition, where $\Sigma$ is the associated fan of the toric variety $X$. Klyachko used this
classification theorem to compute the Chern characters and sheaf cohomology of equivariant vector bundles. As a major application, later he used this classification theorem for equivariant vector bundles over \( \mathbb{P}^2 \) to prove Horn’s conjecture on eigenvalues of sums of Hermitian matrices \([9]\). Another interesting and more recent application is the theorem of Payne \([17]\) that the moduli of rank 3 toric vector bundles satisfy Murphy’s law. Klyachko’s classification theorem has also been generalized for equivariant torsion-free and equivariant pure sheaves by Perling \([18]\) and Kool \([10]\) respectively.

In this paper, our aim is to prove an analogue of Klyachko’s result for \( T \)-equivariant principal \( G \)-bundles over a complex toric variety \( X \). The first step in our formulation is an equivariant Nori theorem (Theorem 2.2), where we identify \( T \)-equivariant principal \( G \)-bundles with functors, from the category of finite dimensional \( G \)-modules to the category of \( T \)-equivariant vector bundles over \( X \), satisfying Nori’s four conditions; see §2. In fact this theorem holds not only for \( T \) but for any algebraic group \( \Gamma \) acting on an algebraic variety \( X \) defined over an algebraically closed field \( K \). Then we use a crucial fact, that a \( T \)-equivariant principal \( G \)-bundle over any complex affine toric variety is equivariantly trivializable \([3]\).

Next we introduce the notion of a compatible \( \Sigma \)-filtered \( K \)-algebra which is a \( K \)-algebra endowed with a collection of decreasing \( \mathbb{Z} \)-graded filtrations indexed by the rays of \( \Sigma \), that satisfy certain additive and multiplicative compatibility conditions, see Definitions 3.5 and 3.6. Let \( \text{Calg}_G(\Sigma) \) be the category such filtered \( K \)-algebras, \( G \)-equivariantly isomorphic to \( K[G] \) (under the standard action), that satisfy: For every top dimensional cone \( \sigma \in \Sigma \) the \( K \)-algebra admits an action of \( T \) which is compatible with the filtrations and commutes with the \( G \)-action.

Now assume that every maximal cone in the fan of \( X \) is of top dimension. Using the two results mentioned above, we prove a categorical equivalence between the category \( \mathcal{P}\text{bun}^T_G(X) \) of \( T \)-equivariant principal \( G \) bundles over \( X \), and the category \( \text{Calg}_G(\Sigma) \) (Theorem 5.6). The most intriguing step in our proof is the commutativity of the \( T \) and \( G \) actions on the \( K \)-algebras in the definition of \( \text{Calg}_G(\Sigma) \), see Lemma 4.6. As a corollary to Theorem 5.6, we obtain a necessary and sufficient condition for equivariant reduction of structure group (Theorem 6.1).

When \( G = \text{Gl}(n, \mathbb{C}) \), Klyachko’s filtration data for equivariant vector bundle may be recovered from our filtered algebra description (see proof of Lemma 5.4). Moreover, granted an equivariant trivialization, the arguments presented here would yield a similar classification result for toric principal bundles over any algebraically closed field.

Recently, Ilten and Süss \([7]\) have obtained a Klyachko–type classification of torus equivariant vector bundles over \( T \)-varieties, and related it to Hartshorne’s conjecture on splitting of rank two bundles over projective spaces. It seems natural that our classification of equivariant principal \( G \)-bundles should generalize for \( T \)-varieties.

2. Nori’s Correspondence

Let \( K \) be an algebraically closed field. For any \( K \)-variety \( X \), we denote the ring of \( K \)-valued regular functions on \( X \) by \( K[X] \).

The category of finite dimensional vector spaces over \( K \) will be denoted by \( \mathcal{V}\text{ec} \). Let \( G \) be an affine algebraic group defined over \( K \). The category of algebraic left representations of \( G \) that are finite dimensional \( K \)-vector spaces will be denoted by \( G \)-mod.

Let \( \mathcal{T} \) be a tensor category over \( K \). A functor

\[
H : G\text{-mod} \rightarrow \mathcal{T}
\]
is said to satisfy properties F1–F4 if the following hold (see [14] for more detailed description of these properties):

(1) F1: $\mathcal{H}$ is a $K$–additive exact functor,
(2) F2: $\mathcal{H} \circ \otimes = \otimes \circ (\mathcal{H} \times \mathcal{H})$,
(3) F3: furthermore,
   (a) $\mathcal{H}$ respects associativity of tensor products,
   (b) $\mathcal{H}$ respects commutativity of tensor products,
   (c) $\mathcal{H}$ takes the identity object of $(G\text{–mod}, \otimes)$, meaning the trivial $G$–module $K$ goes to the identity object of $(\mathcal{F}, \otimes)$, and
(4) F4: the functor $\mathcal{H}$ is faithful.

Note that, unlike [14], we do not require a rank condition in F4 as we do not employ the full force of the Tannakian categories.

Let $X$ be a $K$–scheme. Let $\text{Vec}(X)$ be the category of vector bundles over $X$. Whenever convenient, we will identify a vector bundle on $X$ with the locally free coherent sheaf on $X$ given by its local sections. Consider the category $\text{Nor}(X)$ of “Nori functors” whose

• objects are functors $E : G\text{–mod} \rightarrow \text{Vec}(X)$ that satisfy F1-F4, and
• morphisms are natural isomorphisms of functors.

Let $\text{Pbun}_G(X)$ denote the category of principal $G$–bundles over $X$. Let

$$N_0 : \text{Pbun}_G(X) \rightarrow \text{Nor}(X) \quad (2.1)$$

be the functor that sends any principal $G$–bundle $E_G$ to the object given by the functor that sends any $G$–module $V$ to the associated vector bundle $E_G \times_G V$.

Let $\text{Qco}(X)$ be the category of quasi-coherent sheaves of $\mathcal{O}_X$–modules. In [14, 15], Nori showed that any functor $E \in \text{Nor}(X)$ admits a unique and natural extension to a functor $E$ from affine $G$–schemes to $\text{Qco}(X)$. He showed that $E(G)$ is a principal $G$–bundle over $X$. This defines a functor

$$N_1 : \text{Nor}(X) \rightarrow \text{Pbun}_G(X) \quad (2.2)$$

that sends any $E$ to the principal $G$–bundle $E(G)$. He went on to show that $N_0$ and $N_1$ are quasi-inverses, proving that the categories $\text{Nor}(X)$ and $\text{Pbun}_G(X)$ are equivalent.

In this section, we will establish an equivariant analogue of the above equivalence. Let $\Gamma$ be an affine algebraic group defined over $K$, and let

$$\eta : \Gamma \times X \rightarrow X$$

be an algebraic left action of $\Gamma$ on $X$. A $\Gamma$–equivariant vector bundle on $X$ is a pair $(W, \tilde{\eta})$, where $W$ is an algebraic vector bundle on $X$ and

$$\tilde{\eta} : \Gamma \times W \rightarrow W$$

is an algebraic left action of $\Gamma$ on the total space of $W$ such that

• $\tilde{\eta}$ is a lift of $\eta$, and
• $\tilde{\eta}$ preserves the linear structure on $W$, in particular, it is fiberwise linear.

Similarly, a $\Gamma$–equivariant principal $G$–bundle on $X$ is a pair $(E_G, \tilde{\eta})$, where $E_G$ is an algebraic principal bundle on $X$ and

$$\tilde{\eta} : \Gamma \times E_G \rightarrow E_G$$

is an algebraic left action of $\Gamma$ on the total space of $E_G$ such that

• $\tilde{\eta}$ is a lift of $\eta$, and
\[ \textbullet \ \tilde{\eta} \text{ commutes with the right action of } G \text{ on } E_G. \]

Let \( \mathfrak{Vec}^\Gamma(X) \) (respectively, \( \mathfrak{Bun}_G^\Gamma(X) \)) be the category of \( \Gamma \)-equivariant vector bundles (respectively, \( \Gamma \)-equivariant principal \( G \)-bundles) over \( X \). Let

\[ \mathfrak{Norr}^\Gamma(X) \]

be the category whose

\[ \textbullet \ \text{objects are functors } E : G-\text{mod} \longrightarrow \mathfrak{Vec}^\Gamma(X) \text{ satisfying F1–F4, and} \]

\[ \textbullet \ \text{morphisms are natural isomorphisms of functors.} \]

Take any \( E \in \mathfrak{Norr}^\Gamma(X) \). For any \( V \in G-\text{mod} \), let \( E(V) \) denote the underlying vector bundle of \( E(V) \), and let \( \tilde{\eta}(V) \) denote the action of \( \Gamma \) on \( E(V) \). For any homomorphism of \( G \)-modules \( \phi : V \longrightarrow W \), the following diagram is commutative:

\[ \begin{array}{ccc}
\Gamma \times E(V) & \xrightarrow{\tilde{\eta}(V)} & E(V) \\
\downarrow \text{id} \times E(\phi) & & \downarrow E(\phi) \\
\Gamma \times E(W) & \xrightarrow{\tilde{\eta}(W)} & E(W)
\end{array} \] (2.4)

Also, we have \( E(V \otimes W) = E(V) \otimes E(W) \) and \( \tilde{\eta}(V \otimes W) = \tilde{\eta}(V) \otimes \tilde{\eta}(W) \).

We say that a (possibly infinite dimensional) \( G \)-module \( V \) is \emph{locally finite} if given any vector \( v \in V \), there exists a finite dimensional \( G \)-submodule \( V \subset V \) with \( v \in V \). Let \( G-\text{mod} \) denote the category whose objects are locally finite \( G \)-modules with morphisms being \( G \)-module homomorphisms.

It is well-known that any affine algebraic group \( G \) and any affine \( G \)-scheme \( X \), the \( G \)-module \( K[X] \) is locally finite (cf. [11, Proposition 8.6], [13, Lemma 3.1]).

**Lemma 2.1.** Let \( \mathfrak{Alg}^\Gamma(X) \) denote the category of \( \Gamma \)-equivariant sheaves of commutative associative \( \mathcal{O}_X \)-algebras, and let \( G-\text{sch} \) be the category of affine \( G \)-schemes. Let \( E \) be an object of \( \mathfrak{Norr}^\Gamma(X) \). Then there exists a unique extension of \( E \) to a functor

\[ \overline{E} : G-\text{sch} \longrightarrow \mathfrak{Alg}^\Gamma(X). \]

**Proof.** Let \( \Omega\text{co}^\Gamma(X) \) be the category of all \( \Gamma \)-equivariant quasi coherent sheaves of \( \mathcal{O}_X \)-modules. First, observe that there is a unique extension \( \overline{E} : G-\text{mod} \longrightarrow \Omega\text{co}^\Gamma(X) \) that satisfies properties F1-F4 (cf. [15, Lemma(2.1)]). For \( V \in G-\text{mod} \), denote the underlying sheaf of \( \mathcal{O}_X(V) \) by \( \overline{E}(V) \). Define \( \overline{E}(V) \) to be the direct limit of \( E(V) \), where \( V \) varies over all finite dimensional \( G \)-submodules of \( V \).

Use (2.4) to take direct limit of the morphisms \( \tilde{\eta}(V) : \Gamma \times E(V) \longrightarrow E(V) \) as \( V \) varies over all finite dimensional \( G \)-submodules of \( V \). In this way we obtain an action

\[ \overline{\eta}(V) : \Gamma \times \overline{E}(V) \longrightarrow \overline{E}(V). \] (2.5)

Suppose \( \overline{\phi} : \overline{U} \longrightarrow \overline{V} \) is a morphism of locally finite \( G \)-modules. To define \( \overline{E}(\overline{\phi}) \), consider any \( u \in \overline{U} \). There exists a finite dimensional \( G \)-module \( U \subset \overline{U} \) such that we have \( u \in U \). Let \( V \) denote the image \( \overline{\phi}(U) \) with \( V \longrightarrow \overline{V} \) being the inclusion map. Note that \( V \) is a finite dimensional \( G \)-module. Let \( \psi : U \longrightarrow V \) be the unique homomorphism such that \( \overline{\phi}|_U = i_V \circ \psi \). Define

\[ \overline{E}(\overline{\phi})(u) = [\overline{E}(\psi)(u)], \]

to be the equivalence class of \( \overline{E}(\psi)(u) \in V \) in the direct limit \( \overline{V} \). It is straightforward to check that this is indeed well-defined. Since the operation of direct limit commutes with tensor product, the extension preserves tensor product.
Following Nori, consider a commutative $G$–algebra $A$ as a locally finite $G$–module together with a homomorphism $m : A \otimes A \to A$. Then $E(m)$ defines the structure of a $\Gamma$–equivariant commutative, associative $O_X$–algebra on $E(A)$.

A similar argument shows that if $\phi : A \to B$ is a homomorphism of $G$–algebras, then $E(\phi)$ is a homomorphism of $\Gamma$–equivariant sheaves of $O_X$–algebras. □

It was shown by Nori that $E(K[G])$ is a sheaf of $O_X$–algebras that corresponds to a principal $G$–bundle over $X$. We denote by $E(K[G])$ the principal $G$–bundle on $X$ corresponding to $E(K[G])$.

The right $G$–action on $E(K[G])$ is constructed as follows. Consider $G'$ to be a copy of $G$ with trivial $G$–action. Note that $E(K[G'])$ is the trivial principal $G'$–bundle $X \times G' \to X$ with trivial $\Gamma$–action on fibers. Let

$$a : G \times G' \to G$$

be the multiplication map of $G$. This $a$ produces an action of $G'$ on $G$. Then $E(a)$ induces a morphism

$$E(a) : E(K[G]) \times_X E(K[G']) = E(K[G]) \times G' \to E(K[G]).$$

This induces the required fiber-wise action of $G'$ on $E(K[G])$. Note that $E(a)$ is a morphism of $\Gamma$–equivariant sheaves. Therefore, the actions of $\Gamma$ and $G'$ on $E(K[G])$ commute. Consequently, we have $E(K[G]) \in \mathcal{Pbun}_G^\Gamma(X)$.

It follows that $N_1$ in (2.2) produces a functor

$$N_1^\Gamma : \mathfrak{N}or^\Gamma(X) \to \mathcal{Pbun}_G^\Gamma(X), \ E \mapsto (E(K[G]), \tilde{\eta}(K[G])), $$

where $\tilde{\eta}$ is constructed in (2.5).

The functor $N_0$ in (2.4) produces a functor $N_0^\Gamma : \mathcal{Pbun}_G^\Gamma(X) \to \mathfrak{N}or^\Gamma(X)$.

An analogue of the following result when $\Gamma$ is a finite group has appeared before in [1].

**Theorem 2.2.** The above two functors $N_0^\Gamma$ and $N_1^\Gamma$ are mutually quasi-inverses that induce an equivalence of categories between $\mathcal{Pbun}_G^\Gamma(X)$ and $\mathfrak{N}or^\Gamma(X)$.

**Proof.** Consider any $E \in \mathfrak{N}or^\Gamma(X)$. Let $\Pi : \mathfrak{N}or^\Gamma(X) \to \mathfrak{N}(X)$ be the forgetful functor that forgets the action of $\Gamma$. For every $V \in G$–mod, the object $E(V)$ consists of the vector bundle $\Pi(E)(V)$ and an action $\tilde{\eta}(V) : \Gamma \times \Pi(E)(V) \to \Pi(E)(V)$. Let $\tilde{\eta}_h(V)$ be the restriction of this map $\tilde{\eta}(V)$ to $\{h\} \times \Pi(E)(V)$, where $h \in \Gamma$. Setting $E(V) = \Pi(E)(V)$ in (2.3), we conclude that the maps $\tilde{\eta}_h(V)$ induce an isomorphism of functors $\tilde{\eta}_h : \Pi(E) \to \Pi(E)$. It is straightforward to check that

$$\tilde{\eta}_h = \tilde{\eta}_g \circ \tilde{\eta}_h.$$

(2.8)

Observe that having an element $E \in \mathfrak{N}or^\Gamma(X)$ is same as having the element $\Pi(E) \in \mathfrak{N}(X)$ and natural isomorphisms $\tilde{\eta}_h : \Pi(E) \to \Pi(E)$ for every $h \in \Gamma$ that satisfy (2.8).

By Nori's work that there exists a natural isomorphism of functors

$$\Phi : 1_{\mathfrak{N}or(X)} \to N_0 \circ N_1.$$

(2.9)

Given $E \in \mathfrak{N}or^\Gamma(X)$, this $\Phi$ induces

- an isomorphism $\Phi(E) : \Pi(E) \to N_0 \circ N_1(\Pi(E))$, and
• commutative diagrams

$$\begin{align*}
\Pi(E) & \xrightarrow{\Phi(E)} N_0 \circ N_1(\Pi(E)) \\
\eta_h & \downarrow \quad N_0 \circ N_1(\eta_h) \\
\Pi(E) & \xrightarrow{\Phi(E)} N_0 \circ N_1(\Pi(E))
\end{align*}$$

(2.10)

Note that $N_0 \circ N_1(\Pi(E)) = \Pi(N_0^\Gamma \circ N_1^\Gamma(E))$ by construction. Similarly,

$$N_0 \circ N_1(\eta_h(V)) = N_0^\Gamma \circ N_1^\Gamma(\eta_h(V))$$

for all $V \in G$–mod. So $N_0 \circ N_1(\eta_h) = N_0^\Gamma \circ N_1^\Gamma(\eta_h)$. Then (2.10) implies that $\Phi$ induces a natural isomorphism between the functors $E$ and $N_0 \circ N_1(\Pi(E))$.

The other direction is proved similarly. \qed

3. Filtration functor for vector bundles

Let $X$ be a complex toric variety defined over, corresponding to a fan $\Sigma$ in a lattice $N$ (see \cite{6,16} for details). Let $T$ denote the algebraic torus whose one-parameter subgroups are indexed by $N$. Then $X$ admits an action of $T$ with an open dense $T$–orbit $O$. Denote the set of all $d$–dimensional cones of $\Sigma$ by $\Sigma(d)$. Let $|\Sigma(1)|$ be the set of primitive integral generators of elements of $\Sigma(1)$.

Define $M = \text{Hom}_{N}(N, \mathbb{Z})$. Then $M$ is isomorphic to the group of characters of $T$. For any $\sigma \in \Sigma$, denote the corresponding affine toric subvariety of $X$ by $X_\sigma$; also define

$$\sigma^\perp = \{ u \in M \mid u(n) = 0 \ \forall \ n \in \sigma \}$$

and $M_\sigma := M/\sigma^\perp$. Then $M_\sigma$ is the character group of the maximal sub-torus $T_\sigma \subset T$ that has a fixed point in $X_\sigma$. The set of $k$ dimensional cones of $\Sigma$ will be denoted by $\Sigma(k)$.

Let $\underline{Vec}$ be the category of $K$–vector spaces of countable dimension; the morphisms are $K$–linear homomorphisms.

**Definition 3.1.** A *decreasing filtration* $V$ on a $K$–vector space $V \in \underline{Vec}$ is a collection $\{V(i) \mid i \in \mathbb{Z}\}$ of subspaces of $V$ such that $V(i) \supseteq V(i+1)$ for each $i$. We say $V$ is full if given any $v \in V$ there exists an integer $i$ depending on $v$ such that $v \in V(i)$.

**Definition 3.2.** A *$\Sigma$–filtration* on a vector space $V \in \underline{Vec}$ is a collection of full decreasing filtrations

$$V^\rho : \cdots \supseteq V^\rho(i-1) \supseteq V^\rho(i) \supseteq V^\rho(i+1) \supseteq \cdots$$

on $V$, where $\rho \in |\Sigma(1)|$. We denote the data $(V, \{V^\rho(i)\})$ by $V^\bullet$ and say that $V^\bullet$ is a $\Sigma$–filtered vector space on $X$. If the vector space $V$ is finite dimensional then $V^\bullet$ is said to be finite dimensional.

A *morphism* of $\Sigma$–filtered vector spaces $\phi : V^\bullet \rightarrow W^\bullet$ is a homomorphism of vector spaces $\phi : V \rightarrow W$, such that $\phi(V^\rho(i)) \subseteq W^\rho(i)$ for each $i$ and $\rho$. Such a morphism is injective (respectively, surjective) if the underlying homomorphism of vector spaces is injective (respectively, surjective).

The category of $\Sigma$–filtered vector spaces is a tensor category with the following tensor product:

$$V^\bullet \otimes W^\bullet = \{V \otimes W, (V \otimes W)^\rho(j)\}$$

(3.3)
where
\[(V \otimes W)_{\chi}^\rho(j) = \sum_{p+q=j} V^\rho(p) \otimes W^\rho(q). \quad (3.4)\]

**Definition 3.5.** Let \((A, m)\) be a commutative, associative, \(K\)-algebra, where \(A \in \mathfrak{Vec}\) is the underlying \(K\)-vector space of the algebra, and \(m : A \otimes A \to A\) is the multiplication operation. A \(\Sigma\)-filtration on \((A, m)\) is a \(\Sigma\)-filtration \(A^\bullet = (A, \{A^\rho(i)\})\) on the vector space \(A\) such that
\[m(A^\rho(i) \otimes A^\rho(j)) \subseteq A^\rho(i+j)\]
for every \(\rho \in |\Sigma(1)|\) and \(i, j \in \mathbb{Z}\).

The above data \((A, \{A^\rho(i)\}, m)\) is denoted by \((A^\bullet, m)\), and is called a \(\Sigma\)-filtered algebra on \(X\).

A morphism of \(\Sigma\)-filtered algebras \((A^\bullet_1, m_1) \to (A^\bullet_2, m_2)\) is a homomorphism of underlying \(K\)-algebras that respects the filtrations. Equivalently, it is a morphism of \(\Sigma\)-filtered vector spaces
\[\phi : A^\bullet_1 \to A^\bullet_2\]
such that \(\phi \circ m_1 = m_2 \circ (\phi \otimes \phi)\).

**Definition 3.6.** A compatible \(\Sigma\)-filtered vector space on \(X\) is a \(\Sigma\)-filtered vector space \(F^\bullet = (F, \{F^\rho(i)\})\) such that for every \(\sigma \in \Sigma\), there exists a decomposition of the vector space
\[F = \bigoplus_{[u] \in M_\sigma} F^\sigma_{[u]}, \quad (3.7)\]
with the following property: For each \(\sigma\) and for each \(\rho \in \sigma \cap |\Sigma(1)|\)
\[F^\rho(i) = \bigoplus_{u(\rho) \geq i} F^\sigma_{[u]}. \quad (3.8)\]

Similarly a compatible \(\Sigma\)-filtered algebra on \(X\) is a \(\Sigma\)-filtered algebra \((F^\bullet, m)\) whose underlying \(\Sigma\)-filtered vector space \(F^\bullet\) is compatible, and the subspaces \(F^\sigma_{[u]}\) in \((3.7)\) satisfy
\[\sum_{[u]+[v]=[w]} m(F^\sigma_{[u]} \otimes F^\sigma_{[v]}) \subseteq F^\sigma_{[w]} \quad (3.9)\]

A morphism between compatible \(\Sigma\)-filtered vector spaces (respectively, algebras) is simply a morphism between the underlying \(\Sigma\)-filtered vector spaces (respectively, algebras).

**Remark 3.10.** In the above definition it is enough to require that a decomposition \((3.7)\) satisfying \((3.8)\) exists for every maximal cone \(\sigma\) in the fan \(\Sigma\). A decomposition corresponding to a maximal cone induces decompositions corresponding to its subcones.

**Remark 3.11.** A decomposition as in \((3.7)\) corresponds to an action of \(T_\sigma\) on \(F\).

Given a \(\Sigma\)-filtered vector space \((F, \{F^\rho(i)\})\), a decomposition \((3.7)\) that satisfies \((3.8)\), will be called a compatible decomposition.

Let \(\mathfrak{Vec}(\Sigma)\) and \(\mathcal{Vec}(\Sigma)\) denote the categories of \(\Sigma\)-filtered vector spaces and compatible \(\Sigma\)-filtered vector spaces on \(X\) respectively. Their finite dimensional counterparts are denoted by
\[\mathfrak{Vec}(\Sigma)\quad \text{and}\quad \mathcal{Vec}(\Sigma) \quad (3.12)\]
respectively. These are pre-additive categories.
The category \( \overline{\text{Vec}}(\Sigma) \) is a tensor category with product as in [3.4]: Suppose \( V^\bullet \) and \( W^\bullet \) are compatible \( \Sigma \)-filtered vector spaces. Let \( V = \bigoplus_{[u] \in M_\sigma} V_{[u]}^\sigma \) and \( W = \bigoplus_{[u] \in M_\sigma} W_{[u]}^\sigma \) be compatible decompositions for \( V^\bullet \) and \( W^\bullet \) respectively. Define

\[
(V \otimes W)_{[u]}^\sigma = \bigoplus_{[u_1] + [u_2] = [u]} V_{[u_1]}^\sigma \otimes W_{[u_2]}^\sigma.
\]

Then

\[
V \otimes W = \bigoplus_{[u] \in M_\sigma} (V \otimes W)_{[u]}^\sigma
\]

is a compatible decomposition for \( V^\bullet \otimes W^\bullet \).

Let \( \mathcal{V}ec^T(X) \) (respectively, \( \mathfrak{gb}^T(X) \)) denote the category of \( T \)-equivariant vector bundles (respectively, \( T \)-equivariant principal \( G \)-bundles) on \( X \). There exists a fully faithful, surjective functor

\[
F : \mathcal{V}ec^T(X) \to \mathcal{V}ec(\Sigma)
\]

(see [8] Theorem 2.2.1). We will sketch the construction of \( F \). Let \( \xi \in \mathcal{V}ec^T(X) \) be a bundle of rank \( r \). Fix a closed point \( x_0 \) in the principal \( T \)-orbit \( O \subset X \). Denote by \( F \) the fiber \( \xi(x_0) \). Let \( \sigma \) be a cone of \( \Sigma \) and \( X_\sigma \) the corresponding affine toric variety. Denote by \( \xi_\sigma \) the restriction of \( \xi \) to \( X_\sigma \). Consider the action of \( T \) on the space of sections of \( \xi_\sigma \) defined by

\[
(t \cdot s)(x) = ts(t^{-1}x)
\]

for any point \( x \in X_\sigma \), any element \( t \in T \), and any section \( s \) of \( \xi_\sigma \). A section \( s \) is said to be semi-equivariant if \( t \cdot s = u(t)s \) for some character \( u \) of \( T \).

It was shown by Klyachko [8] Proposition 2.1.1] that there exists a framing (which is not unique) of \( \xi_\sigma \) by semi-equivariant sections. Fix such a framing \( (s_1, \ldots, s_r) \). Let \( S_\sigma \) be the \( T \)-submodule of \( H^0(X_\sigma, \xi_\sigma) \) generated by the semi-equivariant sections \( s_1, \ldots, s_r \). Evaluation at \( x_0 \) gives an isomorphism of vector spaces \( ev_0 : S_\sigma \to F \). This isomorphism induces a \( T \)-module structure on \( F \), or equivalently, a decomposition

\[
F = \bigoplus_{u \in M} F_u^\sigma.
\]

Restricting to the action of \( T_\sigma \) on \( \xi_\sigma \), we similarly get a decomposition

\[
F = \bigoplus_{[u] \in M_\sigma} F_{[u]}^\sigma.
\]

The decompositions (3.13) and (3.14) may depend on the choice of the semi-equivariant framing of \( \xi_\sigma \). However, for each \( \rho \in |\Sigma(1)| \), the subspaces

\[
F^\rho(i) := \bigoplus_{[u] \in M_\sigma, \mu(\rho) \geq i} F_{[u]}^\sigma,
\]

where \( \sigma \) is such that \( \rho \in |\Sigma(1)| \cap \sigma \),

are independent of the choice of \( \sigma \) containing \( \rho \) as well as the framing (see [8]).

Then \( F(\xi) \) is defined to be the compatible \( \Sigma \)-filtered vector space \( F^\bullet = (F, \{F^\rho(i)\}) \) on \( X \).

**Lemma 3.1.** The functor \( F : \mathcal{V}ec^T(X) \to \mathcal{V}ec(\Sigma) \) satisfies

\[
F(\xi_1) \otimes F(\xi_2) = F(\xi_1 \otimes \xi_2)
\]

for all \( \xi_1, \xi_2 \in \mathcal{V}ec^T(X) \). 

Proof. Let $V = \xi_1(x_0)$ and $W = \xi_2(x_0)$ with $r_i = \dim(\xi_i(x_0))$. Clearly $V \otimes W = (\xi_1 \otimes \xi_2)(x_0)$. Denote, 
$$F(\xi_1) = (V, \{V^\rho(j)\}), \quad F(\xi_2) = (W, \{W^\rho(j)\}), \quad \text{and} \quad F(\xi_1 \otimes \xi_2) = (V \otimes W, \{(V \otimes W)^\rho(j)\}).$$
By (3.3) and (3.4), we need to show that 
$$\left(V \otimes W\right)^\rho(j) = \sum_{p+q=j} V^\rho(p) \otimes W^\rho(q). \quad (3.15)$$

Consider any $\rho \in |\Sigma(1)|$. Let $\sigma$ be any cone that contains $\rho$. Fix semi-equivariant frames $s^1_1, \ldots, s^1_{r_1}$ of $(\xi_1)_\sigma$. Let $[u^1_k] \in M_\sigma$ be the character corresponding to the action of $T_\sigma$ on $s^1_k$. We have compatible decompositions 
$$V = \bigoplus_{1 \leq k \leq r_1} V^\sigma_{[u^1_k]} \quad \text{and} \quad W = \bigoplus_{1 \leq l \leq r_2} W^\sigma_{[u^2_l]}$$
induced by these frames. Note that $\{s^1_k \otimes s^2_l\}$ is a semi-equivariant frame of $(\xi_1)_\sigma \otimes (\xi_2)_\sigma$, which induces a compatible decomposition 
$$V \otimes W = \bigoplus_{[u] \in M_\sigma} (V \otimes W)^\sigma_{[u]},$$
where 
$$(V \otimes W)^\sigma_{[u]} = \bigoplus_{[u^1_k]+[u^2_l]=[u]} V^\sigma_{[u^1_k]} \otimes W^\sigma_{[u^2_l]}.\quad \text{Note that}$$
$$V^\rho(p) = \bigoplus_{u^1_k(\rho) \geq p} V^\sigma_{[u^1_k]}, \quad W^\rho(q) = \bigoplus_{u^2_l(\rho) \geq q} W^\sigma_{[u^2_l]}, \quad (V \otimes W)^\rho(j) = \bigoplus_{u(\rho) \geq j} (V \otimes W)^\sigma_{[u]}.\quad \text{It is straightforward to check that}$$
$$V^\rho(p) \otimes W^\rho(j-p) \subset (V \otimes W)^\rho(j) \quad \text{for any} \ p \in \mathbb{Z}.\quad \text{To satisfy (3.15), need to verify that}$$
$$(V \otimes W)^\rho(j) \subset \sum_{p+q=j} V^\rho(p) \otimes W^\rho(q).$$

Take any $w \in (V \otimes W)^\rho(j)$. Then $w = \sum w_t$, where each $w_t \in (V \otimes W)^\rho_{[u_t]}$ for some $[u_t] \in M_\sigma$ such that $u_t(\rho) \geq j$.

It suffices to show that each $w_t \in \sum_{p+q=j} V^\rho(p) \otimes W^\rho(q)$. Since $\{s^1_k(x_0) \otimes s^2_l(x_0)\}$ is a basis for $V \otimes W$, we have 
$$w_t = \sum_{[u^1_k]+[u^2_l]=[u_t]} a^t_{kl} s^1_k(x_0) \otimes s^2_l(x_0) \quad \text{where} \quad a^t_{kl} \in K.\quad \text{Note that} \ [u^1_k]+[u^2_l]= [u_t] \implies u^1_k(\rho) + u^2_l(\rho) = u_t(\rho) \geq j.\quad \text{Since} \ u^1_k(\rho) \text{and} \ u^2_l(\rho) \text{are integers, there exist integers} \ p \text{and} \ q \text{such that} \ u^1_k(\rho) \geq p, \ u^2_l(\rho) \geq q, \text{and} \ p+q = j.\quad \text{It follows that}$$
$$s^1_k(x_0) \otimes s^2_l(x_0) \in V^\rho(p) \otimes W^\rho(q), \quad \text{and} \quad w_t \in \sum_{p+q=j} V^\rho(p) \otimes W^\rho(q).\quad \text{This completes the proof.} \quad \square$$
Since $F : \mathcal{Vec}^T(X) \longrightarrow \mathcal{Vec}(\Sigma)$ is an equivalence of categories, it is an exact functor. The quasi-inverse $K : \mathcal{Vec}(\Sigma) \longrightarrow \mathcal{Vec}^T(X)$ of $F$, constructed by Klyachko in [8], respects direct sums and tensor products. Being an equivalence of categories, it is also exact and faithful.

Consider the category $\mathcal{C}_{nor}(\Sigma)$ whose objects are functors

$$M : G\text{-mod} \longrightarrow \mathcal{Vec}(\Sigma)$$

(see (3.12)) that satisfy properties F1-F4, and whose morphisms are natural isomorphisms of functors.

**Theorem 3.2.** There exists an equivalence of categories between $\mathcal{C}_{nor}(\Sigma)$ (defined above) and $\mathcal{P}\text{bun}^T_G(X)$ (the category of $T$–equivariant principal $G$–bundles on the toric variety $X$).

**Proof.** Let

$$\mathcal{M}or^T(X) \quad (3.16)$$

be the category in (2.3) obtained by the substituting $T$ in place of $\Gamma$. Consider the functor $F : \mathcal{M}or^T(X) \longrightarrow \mathcal{C}_{nor}(\Sigma)$ defined by composition with $F$,

$$F_*(E) = F \circ E \quad \text{for any } E \in \mathcal{M}or^T(X).$$

Similarly, composition with $K$ gives a functor $K_* : \mathcal{C}_{nor}(\Sigma) \longrightarrow \mathcal{M}or^T(X)$,

$$K_*(M) = K \circ M \quad \text{for any } M \in \mathcal{C}_{nor}(\Sigma).$$

It is easily observed from the construction of $K$ that $F \circ K = 1_{\mathcal{C}_{vec}(\Sigma)}$. Therefore,

$$F_* \circ K_* = 1_{\mathcal{C}_{nor}(\Sigma)}.$$ 

Since $F$ and $K$ are fully faithful, so are $F_*$ and $K_*$. Hence, they induce an equivalence of categories between $\mathcal{C}_{nor}(\Sigma)$ and $\mathcal{M}or^T(X)$. Then, by Theorem 2.2 $\mathcal{C}_{nor}(\Sigma)$ and $\mathcal{P}\text{bun}^T_G(X)$ are equivalent categories. □

4. **Filtered algebra associated to an equivariant principal bundle**

Henceforth, we assume that $K = \mathbb{C}$, the field of complex numbers, and $X$ is a complex toric variety. The proof of Lemma 4.5 below depends heavily on equivariant triviality over affine toric variety, which is at the moment known to be true only over complex numbers. It will be interesting if this condition on the base field $K$ can be removed.

Let $G$ be a reductive complex algebraic group. Let $E_G$ be a $T$–equivariant principal $G$–bundle over $X$.

Given any $E \in \mathcal{M}or^T(X)$, define $E_x \in \mathcal{C}_{nor}(\Sigma)$ by

$$E_x = F \circ E.$$ 

It is easily checked that $E_x$ is faithful. Moreover, it preserves tensor products as a consequence of Lemma 3.1.

Let $O : \mathfrak{Vec}(\Sigma) \longrightarrow \mathcal{Vec}$ be the forgetful functor that maps a $\Sigma$–filtered vector space to its underlying vector space. Define $E_x := O \circ E_x$. Note that $E_x(V) = E(V)(x_0)$. It is evident that $E_x$ preserves tensor products.

Let $\phi_1, \phi_2 : V \longrightarrow W$ be two morphisms of $G$–modules such that $E_x(\phi_1) = E_x(\phi_2)$. Then $E_x(\phi_1) = E_x(\phi_2)$. By faithfulness of $E_x$, $\phi_1 = \phi_2$. Therefore, $E_x$ is faithful.
Lemma 4.1. There exists a unique extension of $E_\sharp$ (respectively, $E_\ast$) to a functor $\overline{E}_\sharp : G\text{-mod} \rightarrow \overline{\text{Cor}}(\Sigma)$ (respectively, $\overline{E}_\ast : G\text{-mod} \rightarrow \overline{\mathfrak{C}}(\Sigma)$) that preserves direct limits and tensor products.

Proof. It is easily observed that the category of $\Sigma$–filtered vector spaces over $X$ admits direct limit. For any $V$ in $G\text{-mod}$, define $\overline{E}_\sharp(V)$ (respectively, $\overline{E}_\ast(V)$) to be the direct limit of $E_\sharp(V)$ (respectively, $E_\ast(V)$) as $V$ varies over all finite dimensional $G$–submodules of $V$. Note that direct limit commutes with tensor product. So it follows from Lemma 3.1 that $\overline{E}_\sharp$ preserves tensor products.

To understand the compatibility condition, consider the isotypical decomposition

$$V = \bigoplus_{i \in I} V_i \otimes \text{Hom}_G(V_i, V)$$

of $V$. Here $I$ denotes the set of isomorphism classes of irreducible $G$–submodules of $V$. Since $G$ is reductive, each $V_i$ is finite dimensional. The module $\text{Hom}_G(V_i, V)$ has trivial action of $G$. Therefore, $\overline{E}_\sharp(\text{Hom}_G(V_i, V))$ has trivial $\Sigma$–filtration. Therefore it may be assigned the trivial compatible decomposition comprising the subspaces

$$(\overline{E}_\sharp(\text{Hom}_G(V_i, V)))^\sigma_u = \begin{cases} \overline{E}_\sharp(\text{Hom}_G(V_i, V)) & \text{if } |u| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now a choice of compatible decomposition for each $E_\sharp(V_i)$ determines a compatible decomposition for $\overline{E}_\sharp(V)$. The construction of $\overline{E}_\sharp(\overline{\phi})$ for a morphism $\overline{\phi} : \overline{U} \rightarrow \overline{V}$ is similar to Lemma 2.1.

Lemma 4.2. The functors $\overline{E}_\sharp$ (respectively, $\overline{E}_\ast$) satisfy properties F2, F3 and F4.

Proof. Since direct limit commutes with tensor product, $\overline{E}_\sharp$ satisfies F2 and F3.

Suppose there exist morphisms $\overline{\phi}_j : \overline{U} \rightarrow \overline{V}$, $j = 1, 2$, such that $\overline{E}_\sharp(\overline{\phi}_1) = \overline{E}_\sharp(\overline{\phi}_2)$. To prove $\overline{E}_\sharp$ is faithful, it is enough to show that $\overline{\phi}_1 = \overline{\phi}_2$. Consider any element $u \in \overline{U}$. Then there exists a finite dimensional $G$–submodule $U$ of $\overline{U}$ such that $u \in U$. Let $i_U : U \rightarrow \overline{U}$ be the inclusion map. Let $\phi_j = \overline{\phi}_j \circ i_U$. Then

$$\overline{E}_\sharp(\phi_1) = \overline{E}_\sharp(\phi_2). \quad (4.1)$$

There exists a finite dimensional $G$–module $V \subset \overline{V}$ such that $\phi_j(U) \subset V$ for each $j$. Let $i_V : V \rightarrow \overline{V}$ be the inclusion map. Let $\psi_j : U \rightarrow V$ be the unique map such that $\phi_j = i_V \circ \psi_j$. So, by (4.1),

$$\overline{E}_\sharp(i_V) \circ \overline{E}_\sharp(\psi_1) = \overline{E}_\sharp(i_V) \circ \overline{E}_\sharp(\psi_2). \quad (4.2)$$

The map $i_V$ may be regarded as a direct limit of inclusion maps. Since $\overline{E}_\sharp$ is faithful, $\overline{E}_\sharp(i_V)$ is also a direct limit of inclusion maps. Therefore, $\overline{E}_\sharp(i_V)$ is injective. Hence it follows from (4.2) that

$$\overline{E}_\sharp(\psi_1) = \overline{E}_\sharp(\psi_2). \quad (4.3)$$

Hence, by faithfulness of $\overline{E}_\sharp$ and (4.3), we have $\overline{\psi}_1 = \overline{\psi}_2$. It follows that $\overline{\phi}_1 = \overline{\phi}_2$, and hence $\overline{\phi}_1(u) = \overline{\phi}_2(u)$. Since $u$ is arbitrary, $\overline{\phi}_1 = \overline{\phi}_2$.

The proof for $\overline{E}_\ast$ is similar.

Lemma 4.3. $\overline{E}_\sharp$ (respectively, $\overline{E}_\ast$) defines a functor from affine $G$–schemes to $\Sigma$–filtered algebras (respectively, $K$–algebras). We denote this functor by $\overline{E}_\sharp$ (respectively, $\overline{E}_\ast$) as well.
Proof. Suppose $A$ is a finitely generated $K$–algebra on which $G$ acts. Following Nori, we view $A$ as a locally finite $G$–module, and its multiplication as a morphism $m_A : A \otimes A \to A$ in $G$–mod.

Note that $\overline{E}_g(A \otimes A) = \overline{E}_g(A) \otimes \overline{E}_g(A)$. Therefore we have a morphism $\overline{E}_g(m_A) : \overline{E}_g(A) \otimes \overline{E}_g(A) \to \overline{E}_g(A)$.

Since $\overline{E}_g$ is faithful and satisfies property F3, $\overline{E}_g(m_A)$ defines a nontrivial multiplication on $\overline{E}_g(A)$ which is commutative and associative.

For any two finite dimensional $G$–submodules $V$ and $W$ of $A$, by (3.4) we have

$$E_g(V)^p(i) \otimes E_g(W)^p(j) \subseteq (E_g(V) \otimes E_g(W))^p(i + j).$$

Since $E_g(V \otimes W) = E_g(V) \otimes E_g(W)$,

$$(E_g(V) \otimes E_g(W))^p(i + j) = (E_g(V \otimes W))^p(i + j).$$

Then applying Lemma 4.1, we have

$$\overline{E}_g(A)^p(i) \otimes \overline{E}_g(A)^p(j) \subseteq (\overline{E}_g(A \otimes A))^p(i + j).$$

As $\overline{E}_g(m_A)$ is a morphism of $\Sigma$–filtered vector spaces, we have

$$\overline{E}_g(m_A)(\overline{E}_g(A \otimes A))^p(i + j) \subseteq \overline{E}_g(A)^p(i + j).$$

By (4.4) and (4.5),

$$\overline{E}_g(m_A)(\overline{E}_g(A))^p(i) \otimes (\overline{E}_g(A))^p(j) \subseteq (\overline{E}_g(A))^p(i + j).$$

Thus $\overline{E}_g(A)$ is a $\Sigma$–filtered algebra.

Now suppose that $\phi : A \to B$ is a morphism of $G$–algebras, so that $\phi \circ m_A = m_B \circ (\phi \otimes \phi)$.

By functoriality,

$$\overline{E}_g(\phi) \circ \overline{E}_g(m_A) = \overline{E}_g(m_B) \circ (\overline{E}_g(\phi) \otimes \overline{E}_g(\phi)).$$

Thus $\overline{E}_g(\phi)$ is a morphism of $\Sigma$–filtered algebras. \qed

Lemma 4.4. The algebra $\overline{E}_g(K[G])$ is the algebra of functions of the fiber $\overline{E}(K[G])(x_0)$ of the principal bundle $\overline{E}(K[G])$. Moreover, $\overline{E}_g(K[G])$ admits a $G$–action which makes it equivariantly isomorphic to $K[G]$.

Proof. Let $\mathcal{O}$ be the forgetful functor that takes a $\Sigma$–filtered algebra to its underlying $K$–algebra. Then

$$\overline{E}_g = \mathcal{O} \circ \overline{E}_g.$$ (4.6)

Indeed, this follows from $E_g = \mathcal{O} \circ E_g$ combined with Lemma 4.2 and the uniqueness of the extensions of $E_g$ and $E_g$ to $G$–mod.

Let $q : \mathcal{O}_X \to K$ be the evaluation map corresponding to the closed point $x_0 \in X$.

Using $q$, to any $\mathcal{O}_X$–module $M$ we may associate a $K$–vector space $M \otimes_{\mathcal{O}_X} K$. Now recall that, for any $V \in G$–mod,

$$E_g(V) = E(V) \otimes_{\mathcal{O}_X} K.$$

Then by the uniqueness of extensions we have

$$\overline{E}_g(K[G]) = \overline{E}(K[G]) \otimes_{\mathcal{O}_X} K.$$
It is then clear that \( \overline{E}_2(K[G]) \) is the algebra of functions of the fiber at \( x_0 \) of the principal bundle \( \overline{E}(K[G]) \). Note that, by (4.6), this \( \overline{E}_2(K[G]) \) is the underlying algebra of \( \overline{E}_2(K[G]) \). This completes the proof of the first part of the lemma.

Next note that there is a natural \( G \)-action on the principal bundle \( \overline{E}(K[G]) \) which is free and transitive on each fiber. This yields the required \( G \)-action on \( \overline{E}_2(K[G]) \) by the first part of the lemma.

\[ \square \]

**Lemma 4.5.** Let \( X \) be a complex toric variety. Then the \( \Sigma \)-filtered algebra \( \overline{E}_2(K[G]) \) is compatible.

**Proof.** By Remark 3.10, it is enough to concentrate on a maximal cone \( \sigma \). It has been shown in [3] that \( E_G \) admits a section \( s \) over \( X_\sigma \), such that

\[ t s(x) = s(tx) \rho_s(t) \text{ for every } x \in X_\sigma \text{ and } t \in T, \text{ where } \rho_s : T \rightarrow G \text{ is a group homomorphism.} \]

(For nonsingular \( X \), the existence of such a section was proved in [2].)

Then, for any locally finite \( G \)-module \( V \), the homomorphism \( \rho_s \) and the given action of \( G \) defines a \( T \)-action on \( V \) which we denote by \( \rho_s \) again without confusion. An eigenvector \( v \) of this action with weight \( \chi(t) \) gives rise to a semi-equivariant section \( [(s(x), v)] \) of \( \overline{E}(V) \) with the same weight. Such sections, corresponding to a choice of an eigen-basis of \( V \), induce a compatible \( T \)-decomposition of \( \overline{E}_2(V) \). Moreover, this decomposition does not depend on the choice of the eigen-basis.

Now consider the \( G \)-module \( K[G] \). The action of \( T \) on \( K[G] \) induced by \( \rho_s \) satisfies the condition that

\[ \rho_s(t)f(\cdot) = f(\cdot \rho_s(t)). \]

It follows that if \( f_1, f_2 \in K[G] \) are \( T \)-eigenvectors with weights \( \chi_1(t) \) and \( \chi_2(t) \) respectively, then the product \( f_1f_2 \) is a \( T \)-eigenvector with weight \( \chi_1(t)\chi_2(t) \). This implies that the compatible \( T \)-decomposition on \( \overline{E}_2(K[G]) \) respects the multiplication of the algebra \( K[G] \) (cf. [3]).  

\[ \square \]

**Lemma 4.6.** For every \( n \)-dimensional cone \( \sigma \), an action of \( T \) on \( \overline{E}_2(K[G]) \) which is compatible with the \( \Sigma \)-filtration, commutes with the action of \( G \).

**Proof.** We revisit the \( G \)-action on \( \overline{E}_2(K[G]) \). Recall the multiplication map \( a : G \times G' \rightarrow G \) of \( G \) defined in (2.6). Let \( a^* : K[G] \rightarrow K[G] \otimes K[G'] \) be the algebra morphism corresponding to \( a \). Then we have a map

\[ \overline{E}_2(a^*) : \overline{E}_2(K[G]) \rightarrow \overline{E}_2(K[G]) \otimes \overline{E}_2(K[G']). \tag{4.7} \]

of \( \Sigma \)-filtered algebras. Note that the underlying algebra \( \overline{E}_2(K[G']) \) of \( \overline{E}_2(K[G]) \) is \( K[G'] \). This follows from the fact observed in Section 2 that \( \overline{E}(K[G']) = X \times G' \), and the first part of Lemma 4.6. Thus the map \( \overline{E}_2(a^*) \) induces an action of \( G \) on \( \text{Spec}(\overline{E}_2(K[G])) = \overline{E}(K[G])(x_0) \), which agrees with the action of \( G \) given by \( \overline{E}(a) \) (cf. (2.7)).

It follows from property F3(c) that the bundle \( \overline{E}(K[G']) \) has trivial \( T \)-action for any \( E \in \mathfrak{g} \text{tor}_T(X) \). Therefore, the \( \Sigma \)-filtration on \( \overline{E}_2(K[G']) \) satisfies

\[ \overline{E}_2(K[G'])^{\rho}(i) = \begin{cases} K[G'] & \text{if } i \leq 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{4.8} \]

for every \( \rho \in |\Sigma(1)| \). Since the filtrations are decreasing, it follows (cf. 3.1) that for every \( \rho \),

\[ (\overline{E}_2(K[G]) \otimes \overline{E}_2(K[G']))^{\rho}(i) = \overline{E}_2(K[G])^{\rho}(i) \otimes K[G']. \tag{4.9} \]
As \( E_\sharp(a^*) \) respects the \( \Sigma \)-filtration, we have
\[
E_\sharp(a^*)(E_\sharp(K[G])^{\rho}(i)) \subset E_\sharp(K[G])^{\rho}(i) \otimes K[G'].
\]

Let \( \sigma \) be an \( n \)-dimensional cone. Note that \( T_\sigma = T \). Fix a decomposition (equivalently, a compatible \( T \)-action)
\[
E_\sharp(K[G]) = \bigoplus_{\chi \in \mathcal{M}} E_\sharp(K[G])^\chi, \tag{4.10}
\]
such that for any \( \rho \in \sigma \cap \Sigma(1) \),
\[
E_\sharp(K[G])^{\rho}(i) = \sum_{\chi(\rho) \geq i} E_\sharp(K[G])^\chi.
\]

We will show that
\[
E_\sharp(a^*)(E_\sharp(K[G])^\chi) \subseteq E_\sharp(K[G])^\chi \otimes K[G'] \tag{4.11}
\]
for every \( \chi \).

Suppose \( f \in E_\sharp(K[G])^\chi \). Since \( G \) is reductive, \( K[G] \otimes K[G'] \) is locally finite. So we may write \( E_\sharp(a^*)(f) \) as a finite sum,
\[
E_\sharp(a^*)(f) = \sum f_j \otimes b_j \tag{4.12}
\]
where \( f_j \in E_\sharp(K[G])^\chi_j \) and \( b_j \in K[G'] \). Note that
\[
f \in E_\sharp(K[G])^{\rho}(\chi(\rho)) \text{ for every } \rho \in \sigma \cap \Sigma(1).
\]

Since \( E_\sharp(a^*) \) preserves the \( \Sigma \)-filtration, we must have \( \chi_j(\rho) \geq \chi(\rho) \) for every \( \rho \in \sigma \cap \Sigma(1) \).

Suppose, if possible, \( \chi_j \neq \chi \) for some value \( j_0 \) of \( j \). Then since \( \sigma \cap \Sigma(1) \) spans \( N \otimes \mathbb{R} \), there exists \( \rho_0 \in \sigma \cap \Sigma(1) \) such that \( \chi_j(\rho_0) > \chi(\rho_0) \).

Given any \( h \in G' \), consider the \( G \)-map
\[
\phi_h : G \longrightarrow G \times G'
\]
defines by \( g \mapsto (g, h) \). The induced map
\[
\phi_h^* : K[G] \otimes K[G'] \longrightarrow K[G]
\]
satisfies \( \phi_h^*(x \otimes y) = y(h) x \). Identifying \( K[G] \) with \( K[G] \otimes_K K \), we can write
\[
\phi_h^* \otimes id = ev_h
\]
where
\[
ev_h : K[G'] \longrightarrow K
\]
is the evaluation map at \( h \). Therefore,
\[
E_\sharp(\phi_h^*) = \text{id} \otimes E_\sharp(ev_h).
\]

Note that \( E(K) \) is the trivial line bundle with trivial \( T \)-action by property F3(c). Therefore, \( E_\sharp(K) = K \). Moreover, for any two \( G \)-algebras \( A, B \) with trivial \( G \)-action, and any homomorphism \( \theta : A \longrightarrow B, E_\sharp(\theta) = \theta \). Hence, \( E_\sharp(ev_h) = ev_h \). Thus we have,
\[
E_\sharp(\phi_h^*) = \text{id} \otimes ev_h : E_\sharp(K[G]) \otimes K[G'] \longrightarrow E_\sharp(K[G]).
\]

Hence,
\[
E_\sharp(\phi_h^*) \left( \sum f_j \otimes b_j \right) = \sum b_j(h)f_j.
\]
Then using (4.12) we have
\[ \mathcal{E}_t(\phi_h^* \circ a^*)(f) = \sum b_j(h) f_j. \]
Since \( b_{j_0} \neq 0 \), there exists \( h_0 \) such that \( b_{j_0}(h_0) \neq 0 \). Let \( i_0 = \chi(\rho_0) \). Then
\[ b_{j_0}(h_0) f_{j_0} \in \mathcal{E}_t(K[G])^{\rho_0}(i_1) \]
where \( i_1 = \chi_{j_0}(\rho_0) > i_0 \).

Note that the composition of maps
\[ (\phi_{h_0}^* \circ a^*) \circ (\phi_{h_0}^* \circ a^*) = \text{id}. \]
Since \( \mathcal{E}_t(\phi_{h_0}^* \circ a^*) \) is also filtration preserving,
\[ \mathcal{E}_t(\phi_{h_0}^* \circ a^*)(b_{j_0}(h_0) f_{j_0}) \in \mathcal{E}_t(K[G])^{\rho_0}(i_1). \]
Therefore,
\[ \mathcal{E}_t(\phi_{h_0}^* \circ a^*)(b_{j_0}(h_0) f_{j_0}) = \mathcal{E}_t(K[G])^{\rho_0}(i_1) \]
unless \( \mathcal{E}_t(\phi_{h_0}^* \circ a^*)(b_{j_0}(h_0) f_{j_0}) = 0 \). But
\[ \mathcal{E}_t(\phi_{h_0}^* \circ a^*) \circ \mathcal{E}_t(\phi_{h_0}^* \circ a^*)(f) = f \in \mathcal{E}_t(K[G])^{\rho_0}. \]
Therefore,
\[ \mathcal{E}_t(\phi_{h_0}^* \circ a^*)(b_{j_0}(h_0) f_{j_0}) = 0. \]
This is a contradiction since \( \mathcal{E}_t(\phi_{h_0}^* \circ a^*) \) is an isomorphism. Thus no such \( j_0 \) exits.

Therefore, using (4.12), we obtain (4.11). This implies that the actions of \( G \) and \( T \) on \( \mathcal{E}_t(K[G]) \), induced by (4.7) and (4.10) respectively, commute. The lemma follows.

We may associate to any equivariant principal bundle \( E_G \), the compatible \( \Sigma \)–filtered algebra \( \mathcal{E}_t(K[G]) \), where \( E = \mathcal{N}_G^T(E_G) \).

5. Correspondence between equivariant \( G \)–bundles and \( \Sigma \)–filtered algebras

Let \( X \) be a complex toric variety. Assume that every maximal cone in the fan \( \Sigma \) of \( X \) is of top dimension. We note that this assumption is always satisfied when \( X \) is a complete toric variety.

**Definition 5.1.** Let \( \text{Calg}_G(\Sigma) \) be the category whose objects are compatible \( \Sigma \)–filtered \( K \)–algebras \( B \) such that

- \( B \) admits a \( G \)–action with respect to which it is \( G \)–equivariantly isomorphic to the algebra \( K[G] \),
- For every top dimensional cone in the fan, \( B \) admits a compatible action of \( T \) that commutes with the action of \( G \) on \( B \).

The morphisms of \( \text{Calg}_G(\Sigma) \) are \( G \)–equivariant isomorphisms of compatible \( \Sigma \)–filtered \( K \)–algebras.

**Lemma 5.1.** The association \( E \mapsto \mathcal{E}_t(K[G]) \) induces a functor
\[ \mathcal{A} : \mathcal{N}_{G}^T(X) \longrightarrow \text{Calg}_G(\Sigma), \]
where \( \mathcal{N}_{G}^T(X) \) and \( \text{Calg}_G(\Sigma) \) are as in (4.10) and Definition 5.1 respectively.
Proof. It follows from Section 4 that \( E_{\Sigma}^1(K[G]) \) is an object in \( \mathcal{C}alg_{G}(\Sigma) \).

Let \( \Psi : E^1 \rightarrow E^2 \) be a morphism in \( \mathfrak{Hor}^T(X) \). For any morphism \( f : V \rightarrow W \) in \( G \)-mod, we have a commuting diagram.

\[
\begin{array}{ccc}
E^1_\Sigma(V) & \xrightarrow{F \circ \Psi(V)} & E^2_\Sigma(V) \\
E^2_\Sigma(f) & \downarrow & E^2_\Sigma(f) \\
E^1_\Sigma(W) & \xrightarrow{F \circ \Psi(W)} & E^2_\Sigma(W)
\end{array}
\]

So the direct limit of the morphisms \( F \circ \Psi(V) \) exists as \( V \) runs over all finite dimensional \( G \)-submodules of \( K[G] \). We denote this limit by \( A(\Psi) : E_{\Sigma}^1(K[G]) \rightarrow E_{\Sigma}^2(K[G]) \).

Let \( G' \) be a copy of \( G \) with trivial \( G \)-action as in \( (2.6) \). We will denote the limit of \( F \circ \Psi(V) \) as \( V \) varies over all finite dimensional \( G \)-submodules of \( K[G'] \) by \( A'(\Psi) \). Note that \( E_{\Sigma}^j(K[G']) \) is the algebra \( K[G'] \) with the trivial filtration \( (4.8) \) and \( A'(\Psi) = id \) using property F3(c).

Since the \( F \circ \Psi(V) \)'s are morphisms of filtered vector spaces and the filtration on \( E_{\Sigma}^j(K[G]) \) is the direct limit of the filtrations on \( E_{\Sigma}^j(V) \), it follows that \( A(\Psi) \) is a morphism of \( \Sigma \)-filtered vector spaces.

Since \( \Psi \), by definition, respects F1-F4, it follows that \( A(\Psi) \) is a morphism of algebras. Regard the action of \( G \) on \( K[G] \) as a morphism of algebras \( a^* : K[G] \rightarrow K[G] \otimes K[G'] \).

By \( (4.9) \), the \( G \)-action on \( E_{\Sigma}^j(K[G]) \) is given by

\[
E_{\Sigma}^j(a^*) : E_{\Sigma}^j(K[G]) \rightarrow E_{\Sigma}^j(K[G]) \otimes K[G'].
\]

Again, using functoriality, we have a commutative diagram

\[
\begin{array}{ccc}
E_{\Sigma}^1(K[G]) & \xrightarrow{E_{\Sigma}^1(a^*)} & E_{\Sigma}^1(K[G]) \otimes K[G'] \\
A(\Psi) & \downarrow & A(\Psi) \otimes id \\
E_{\Sigma}^2(K[G]) & \xrightarrow{E_{\Sigma}^2(a^*)} & E_{\Sigma}^2(K[G]) \otimes K[G']
\end{array}
\]

This shows that \( A(\Psi) \) is \( G \)-equivariant. \qed

Lemma 5.2. The functor \( A : \mathfrak{Hor}^T(X) \rightarrow \mathcal{C}alg_{G}(\Sigma) \) is faithful.

Proof. Let \( \Psi_j : E^1 \rightarrow E^2 \), \( j = 1, 2 \), be two morphisms. By Lemma \( (4.4) \), the underlying algebra of \( A(E^j) \) is the coordinate algebra

\[
K[E^j(K[G])(x_0)] = K[N_T^j(E^j)(x_0)].
\]

If \( A(\Psi_1) = A(\Psi_2) \), then

\[
N_T^1(\Psi_1)|_{x_0} = N_T^1(\Psi_1)|_{x_0}.
\]

Now, by \( T \)-equivariance \( N_T^1(\Psi_1) \) and \( N_T^1(\Psi_2) \) must agree over the principal \( T \)-orbit and hence by continuity they must agree over \( X \). Since \( N_T^1 \) is faithful, we conclude that \( \Psi_1 = \Psi_2 \). \qed

Lemma 5.3. Consider an object \( B \) in \( \mathcal{C}alg_{G}(\Sigma) \). Fix an embedding \( \theta \) of \( G \) in \( GL(V) \). Then the \( \Sigma \)-filtration on \( B \) induces a compatible \( \Sigma \)-filtration on \( (B \otimes K[V])^G \).
Proof. Suppose $\sigma$ is a maximal cone. Consider an action of $T_\sigma = T$ on $B$ which is compatible with the $\Sigma|_\sigma$–filtration. Let

$$B = \bigoplus_{u \in M_\sigma} B^\sigma_u.$$ 

be the corresponding isotypical decomposition.

We first claim that

$$(B \otimes K[V])^G = \bigoplus_{u \in M_\sigma} (B^\sigma_u \otimes K[V])^G.$$ \hspace{1cm} (5.2)

Let $x \in (B \otimes K[V])^G$. We may write $x$ uniquely as a finite sum

$$x = \sum x_u,$$ \hspace{1cm} (5.3)

where $x_u \in B^\sigma_u \otimes K[V]$. Since the actions of $T$ and $G$ on $B$ commute, $B^\sigma_u$ is $G$–invariant. This implies that $gx_u \in B^\sigma_u \otimes K[V]$ for any $g \in G$. Since $gx = x$, we have

$$x = \sum gx_u$$

which is another decomposition of $x$ with components in $B^\sigma_u \otimes K[V]$. By the uniqueness of (5.3), we must have $gx_u = x_u$ for all $u$ and $g$. This means that $x_u \in (B^\sigma_u \otimes K[V])^G$. Hence,

$$(B \otimes K[V])^G \subseteq \bigoplus_{u \in M_\sigma} (B^\sigma_u \otimes K[V])^G.$$ Clearly

$$(B \otimes K[V])^G \supseteq \bigoplus_{u \in M_\sigma} (B^\sigma_u \otimes K[V])^G.$$ Hence (5.2) follows. Using it we conclude that

$$((B \otimes K[V])^G)^\sigma = (B^\sigma_u \otimes K[V])^G.$$ \hspace{1cm} (5.4)

By the compatibility of the $\Sigma$–filtration on $B$, the decomposition

$$B^\rho(i) = \bigoplus_{u(\rho) \geq i} B^\sigma_u$$

is independent of the choice of $\sigma$ such that $\rho \in \sigma$. By (5.4),

$$((B \otimes K[V])^G)^\rho(i) = \bigoplus_{u(\rho) \geq i} (B^\sigma_u \otimes K[V])^G.$$ \hspace{1cm} (5.5)

To show the independence of (5.5) of the choice of maximal cone $\sigma$, we want to show that

$$((B \otimes K[V])^G)^\rho(i) = (B^\rho(i) \otimes K[V])^G.$$ \hspace{1cm} (5.6)

If $v(\rho) \geq i$, then

$$(B^\sigma_v \otimes K[V])^G \subseteq ((\bigoplus_{u(\rho) \geq i} B^\sigma_u) \otimes K[V])^G = (B^\rho(i) \otimes K[V])^G.$$ Therefore, by (5.5),

$$((B \otimes K[V])^G)^\rho(i) \subseteq (B^\rho(i) \otimes K[V])^G.$$ On the other hand, suppose $x \in (B^\rho(i) \otimes K[V])^G$. Then $x$ admits a unique decomposition

$$x = \sum_{u(\rho) \geq i} x_u$$
where \( x_u \in B_u^\sigma \otimes K[V] \). Then by using the \( G \)-invariance of \( x \) and the uniqueness of the decomposition as before, we conclude that \( gx_u = x_u \) for all \( g \in G \). Hence \( x_u \in (B_u^\sigma \otimes K[V])^G \), and consequently,

\[
x \in \bigoplus_{u(\rho) \geq i} (B_u^\sigma \otimes K[V])^G = ((B \otimes K[V])^G)^{\rho(i)}.
\]

Hence,

\[
((B \otimes K[V])^G)^{\rho(i)} \supseteq (B^{\rho(i)} \otimes K[V])^G.
\]  

By (5.7) and (5.8), equation (5.6) holds, concluding the proof.

**Lemma 5.4.** Assume all maximal cones in \( \Sigma \) are top dimensional. Then the functor 
\( \mathcal{A} : \mathcal{Hor}^T(X) \to \mathcal{Calg}_G(\Sigma) \) is essentially surjective.

**Proof.** Consider an object \( B \) in \( \mathcal{Calg}_G(\Sigma) \). Consider a top-dimensional cone \( \sigma \) in \( \Sigma \). Note that \( T_\sigma = T \).

Fix a \( T \)-action on \( B \) which is compatible with the \( \Sigma \)-filtration and commutes with the \( G \)-action on \( B \). Define \( E^G_\sigma = X_\sigma \times \text{Spec}(B) \). With the induced actions of \( T \) and \( G \), this \( E^G_\sigma \) is a \( T \)-equivariant principal \( G \)-bundle over \( X_\sigma \).

Fix a closed point \( y_0 \) in \( \text{Spec}(B) \). Recall the closed point \( x_0 \in O \) used in the construction of \( F \). Let \( e = (x_0, y_0) \in E^K_\sigma \). The \( T \)-action on \( \text{Spec}(B) \) may be represented by a homomorphism

\[
\rho_\sigma : T \to G, \quad \text{defined by} \quad ty_0 = y_0 \cdot \rho_\sigma(t) \text{ for any } t \in T.
\]

We claim that for any two top-dimensional cones \( \sigma \) and \( \tau \), the functions

\[
\rho_\sigma \rho_\tau^{-1} : T \to G
\]

extend to regular \( G \)-valued functions over \( X_\sigma \cap X_\tau \) under the standard identification of \( T \) with the principal orbit \( O \) in \( X \).

Fix an embedding \( \theta \) of \( G \) in \( GL(V) \). Let \( E^\sigma = E^G_\sigma \times_G V \) be the associated \( T \)-equivariant vector bundle. The actions of \( T \) and \( G \) on \( \text{Spec}(B) = E^K_\sigma(x_0) \) commute and hence induce a \( T \)-action on \( E^\sigma(x_0) = \text{Spec}(B) \times_G V \). Using a specific isomorphism

\[
\text{Spec}(B) \times_G V \cong V \quad \text{induced by the rule} \quad [(y_0, v)] \mapsto v,
\]

we obtain an induced \( T \)-action and therefore a \( \Sigma|_\sigma \)-filtration on \( V \). We claim that as \( \sigma \) varies, this induces a compatible \( \Sigma \)-filtration on \( V \). We will derive this from the compatibility of the \( \Sigma \)-filtration on the algebra \( B \).

The \( T \)-action on \( B \), for any fixed \( \sigma \), induces a \( T \)-action on \((B \otimes K[V])^G \). Here the action of \( G \) on \( K[V] \) is induced by \( \theta \), and the action of \( T \) on \( K[V] \) is assumed to be trivial. As \( \sigma \) varies, from Lemma 5.3 it follows that these actions yield a compatible \( \Sigma \)-filtration on \((B \otimes K[V])^G \).

Since \( G \) is reductive and \( \text{Spec}(B) \times V \) is affine, \( \text{Spec}(B) \times_G V = \text{Spec}((B \otimes K[V])^G) \). The isomorphism (5.9) induces a specific isomorphism \((B \otimes K[V])^G \cong K[V] \). This gives a \( \Sigma \)-filtration on \( K[V] \). By linearity of the \( G \)-action on \( V \), the cone-wise \( T \)-actions on \( K[V] \) are determined by cone-wise \( T \)-actions on the dual \( V^* \) of \( V \). These determine a \( \Sigma \)-filtration on \( V^* \), which is compatible as it is the restriction of the compatible \( \Sigma \)-filtration on \( K[V] \). We have an induced compatible dual \( \Sigma \)-filtration on \( V \) (see [7], section 6.3). This \( \Sigma \)-filtration on \( V \) agrees with the \( \Sigma \)-filtration on \( V \) derived immediately after (5.9). This proves the claim regarding the compatibility of that \( \Sigma \)-filtration.
Note that the $T$–action on $V$, associated to the cone $\sigma$, is given by $\theta(\rho_\sigma)$. Since the $\Sigma$–filtration on $V$ is compatible, by Klyachko [8], it gives rise to a $T$–equivariant vector bundle over $X$ and the $\text{GL}(V)$–valued functions $\theta(\rho_\sigma\rho_\tau^{-1})$ extend regularly over $X_\sigma \cap X_\tau$. It follows that the functions $\rho_\sigma\rho_\tau^{-1}$ extend regularly over $X_\sigma \cap X_\tau$. Since $G$ is closed in $\text{GL}(V)$, the extensions are $G$–valued. This allows us to construct a $T$–equivariant principal $G$–bundle $E_G$ over $X$ by gluing the bundles $\{E_G^0\}$ using $\{\rho_\sigma\rho_\tau^{-1}\}$ as transition functions.

It is straightforward to show that $A(N_0^T(E_G)) \cong B$: As an algebra, $A(N_0^T(E_G)) = \text{Spec}(B) \times_G K[G]$. We have an isomorphism

$$\alpha_* : \text{Spec}(B) \times_G K[G] \longrightarrow K[G],$$

induced by the $G$–equivariant isomorphism

$$\alpha : \text{Spec}(B) \longrightarrow G, \quad \text{where } \alpha(y_0) = 1_G.$$ Moreover, the map $\alpha$ also induces an isomorphism

$$\alpha^* : K[G] \longrightarrow B.$$ Thus we have an isomorphism

$$\alpha^* \circ \alpha_* : \text{Spec}(B) \times_G K[G] \longrightarrow B.$$

Given a right $T$–action $\rho_\sigma$ on $\text{Spec}(B)$, let $\rho_\sigma^*$ be the induced $T$–action on $B$. These actions, as $\sigma$ varies, produce the $\Sigma$–filtration on $B$. The $\Sigma|_{\sigma}$–filtration or $T_\sigma$–action on $\text{Spec}(B) \times_G K[G]$ is induced from $\rho_\sigma$ by $\alpha$. Let us call this action $\alpha_* (\rho_\sigma)$. Note that the action on $B$ induced from $\alpha_* (\rho_\sigma)$ by the isomorphism $\alpha^*$ is same as $\rho_\sigma^*$. Thus $\alpha^* \circ \alpha_*$ induces the required isomorphism of $\Sigma$–filtrations.

**Lemma 5.5.** Let $X$ be a complex toric variety such that every maximal cone in its fan is top-dimensional. Then the functor

$$A : \mathcal{N}_T^\Sigma(X) \longrightarrow \mathcal{CAlg}_G(\Sigma)$$

(see [3.16], Definition [5.1] and Lemma [5.1]) is full.

**Proof.** Let $E^1, E^2 \in \mathcal{N}_T^\Sigma(X)$ and $\phi : A(E^2) \longrightarrow A(E^1)$ be a morphism in $\mathcal{CAlg}_G(\Sigma)$. We need to show that there exists a morphism $\Psi : E^1 \longrightarrow E^2$ in $\mathcal{N}_T^\Sigma(X)$ such that $A(\Psi) = \phi$.

Note that the underlying algebra of $A(E^j)$ is the algebra of functions on the fiber of the principal $G$–bundle $\mathcal{T}^j(K[G])$ at $x_0$. Then $\phi$ induces a $G$–equivariant morphism of varieties

$$\phi' : \mathcal{T}^1(K[G])(x_0) \longrightarrow \mathcal{T}^2(K[G])(x_0).$$

The latter, by $T$–equivariance induces a morphism of principal bundles

$$\phi'_* : \mathcal{T}^1(K[G]) \longrightarrow \mathcal{T}^2(K[G])$$

over the principal orbit $O$ of $X$. This induces a continuous family

$$\phi_* = \{\phi^j_* : K[\mathcal{T}^j(K[G])](x) \longrightarrow K[\mathcal{T}^j(K[G])](x)\}$$

of isomorphisms of the coordinate algebras of the fibers of the bundles for $x \in O$.

For any finite dimensional $\Sigma$–filtered subspace $V$ of $A(E^2)$, the restriction

$$\phi|_V : V \longrightarrow \phi(V)$$

is an isomorphism of $\Sigma$–filtered vector spaces. The corresponding restriction of $\phi_*$ induces a $T$–equivariant isomorphism of vector bundles over $O$. But since $\phi|_V$ respects the filtrations,
by the arguments of Klyachko [8], this extends to a $T$–equivariant isomorphism of vector bundles over $X$. Then taking direct limit over all finite dimensional $\Sigma$–filtered subspaces of $A(E^2)$, we observe that $\phi_*$ extends to a $T$–equivariant isomorphism between two families of (infinite dimensional) vector spaces over $X$.

Since $\phi_*$ respects the fiberwise algebra structures over $O$, and $\phi_*$ as well as the family of algebras are continuous, $\phi_*$ must respect the fiberwise algebra structures over the boundary divisors as well. More precisely, for $j = 1, 2$, let

$$m_j^x : K[\overline{E}^j(K[G])(x)] \otimes K[\overline{E}^j(K[G])(x)] \rightarrow K[\overline{E}^j(K[G])(x)]$$

denote the multiplication in the fiber over $x \in X$ of the bundle $\overline{E}^j(K[G])$. Then by assumption $\phi_*^{x_0} = \phi$ commutes with $m_j^{x_0}$,

$$m_2^{x_0} \circ (\phi_*^{x_0} \otimes \phi_*^{x_0}) = \phi_*^{x_0} \circ m_1^{x_0}.$$

By $T$–equivariance of $m_j$ and $\phi_*$, we then have

$$m_2^x \circ (\phi_*^x \otimes \phi_*^x) = \phi_*^x \circ m_1^x,$$

for all $x \in O$. Finally we extend this relation to all $x \in X$ by continuity of $m_j^x$ and $\phi_*^x$.

We may similarly argue that $\phi_*$ is $G$–equivariant. Thus $\phi_*$ induces an isomorphism $\hat{\phi}_* : \overline{E}^1(K[G]) \rightarrow \overline{E}^2(K[G])$ of $T$–equivariant principal $G$–bundles. This induces an isomorphism

$$N_0^T(\hat{\phi}_*) : N_0^T(\overline{E}^1(K[G])) \rightarrow N_0^T(\overline{E}^2(K[G])).$$

By functoriality of the isomorphism $\Phi^{-1} : N_0^T \circ N_1^T \rightarrow 1_{\text{det}^r_2}(x)$ (cf. (2.9)), we have the required morphism

$$\Psi = \Phi^{-1}(N_0^T(\hat{\phi}_*)) : \overline{E}^1 \rightarrow \overline{E}^2.$$

The following theorem is a consequence of Theorem 2.2 and Lemmas 5.2, 5.4 and 5.5.

**Theorem 5.6.** Let $X$ be a complex toric variety such that every maximal cone in its fan is top-dimensional. Then there is an equivalence of categories between $\mathcal{PBun}_G^T(X)$ and $\mathcal{Cal}_G(\Sigma)$, where $G$ is a reductive group.

### 6. Reduction of structure group

**Theorem 6.1.** Suppose $H$ is a reductive subgroup of $G$, and let $E_G$ be a $T$–equivariant principal $G$–bundle over $X$. Let $S = \overline{E}_2(K[G])$, where $E = N_0^T(E_G)$. Then $E_G$ admits a $T$–equivariant reduction of structure group to $H$ if and only if there exists a filtered algebra $R \in \mathcal{Cal}_H(\Sigma)$ such that $(R \otimes K[G])^H \cong S$ in $\mathcal{Cal}_G(\Sigma)$.

**Proof.** If $E_G$ admits a $T$–equivariant reduction of structure group to $E_H \in \mathcal{PBun}_G^T(X)$, there exists an isomorphism

$$E_H \times_H G \cong E_G. \quad (6.1)$$

Let $R$ be the $\Sigma$–filtered algebra in $\mathcal{Cal}_H(\Sigma)$ associated to $E_H$. Then (6.1) yields an isomorphism

$$(R \otimes K[G])^H \cong S$$

in $\mathcal{Cal}_G(\Sigma)$.

On the other hand, a bundle $E_G$ with filtered algebra $(R \otimes K[G])^H$ is isomorphic to $E_H \times_H G$ where $E_H$ is the bundle associated to $R$. This follows from the isomorphism...
at the level of fibers at the closed point $x_0$ in the principal $T$–orbit $O$, together with the isomorphism of filtrations, as in the proof of Lemma 5.5.

\[ \square \]

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