Cross Ratios, Surface Groups, $SL(n, \mathbb{R})$ and $C^{1,h}(S^1) \rtimes Diff^h(S^1)$.

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Abstract

We present in this article relations between cross ratios and representations of surface groups. More specifically, we describe a connected component of representations into $SL(n, \mathbb{R})$ as the set of cross ratios on the boundary at infinity of the group which satisfy some functional relations depending on $n$. We also show that representations into $C^{1,h}(S^1) \rtimes Diff^h(S^1)$ can be described as cross ratios. We exhibit a "character variety" of such representations. We show that this character variety contains all character varieties for $SL(n, \mathbb{R})$ as well as the set of all negatively curved metrics on the surface.

1 Introduction

In this article, $S$ is a closed surface with genus at least 2, $\pi_1(S)$ is the fundamental group of $S$. We denote by $\partial_\infty \pi_1(S)$ the boundary at infinity of $\pi_1(S)$. We recall that $\partial_\infty \pi_1(S)$ is a one dimensional compact connected Hölder manifold, hence Hölder homeomorphic to the circle $S^1$. The group $\pi_1(S)$ acts by Hölder homeomorphisms on $\partial_\infty \pi_1(S)$.

We first give a definition, closely related to the beautiful one introduced by Otal in [7].

A (strict) cross ratio is a $\pi_1(S)$-invariant function $b$ defined on

$$\partial_\infty \pi_1(S)^4 \ast = \{(x, y, z, t) \in \partial_\infty \pi_1(S)^4, x \neq t \text{ and } y \neq z\}$$

which satisfies some algebraic rules: (cf. Equations (3.1)).

\begin{align*}
    b(x, y, z, t) &= b(z, t, x, y) \\
    b(x, y, z, t) &= 0 \iff x = y \text{ or } z = t \\
    b(x, y, z, t) &= b(x, y, z, w) b(x, w, z, t) \\
    b(x, y, z, t) &= b(x, y, w, t) b(w, y, z, t) \\
    b(x, y, z, t) &= 1 \iff x = z \text{ or } y = t
\end{align*}

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Whenever \( S \) has a hyperbolic metric, \( \partial_\infty \pi_1(S) \) is identified with \( \mathbb{RP}^1 \). Thus, the usual cross ratio of the projective line gives rise to a cross ratio on \( \partial_\infty \pi_1(S) \). It follows that every cocompact representation of \( \pi_1(S) \) in \( PSL(2, \mathbb{R}) \) give rises to a cross ratio. Moreover, these cross ratios can be characterised as those which satisfies the following extra relation

\[
1 - b(x, y, z, t) = b(t, y, z, x). \tag{1}
\]

In this paper, we will in particular generalise this construction when one replaces \( SL(2, \mathbb{R}) \) with \( SL(n, \mathbb{R}) \). We will also comment on an infinite dimensional version of this construction that appears when one replaces \( SL(n, \mathbb{R}) \) with \( C^{1,\mathrm{h}}(S^1) \rtimes \text{Diff}^h(S^1) \). We will also show that the corresponding “character variety” of representations in \( C^{1,\mathrm{h}}(S^1) \rtimes \text{Diff}^h(S^1) \) contains a connected component of the character variety of \( SL(n, \mathbb{R}) \) as well the moduli space of negatively curved metrics. We now explain some background and give more precise results in the next two paragraphs.

In [5], we define a \( n \)-Fuchsian representation of \( \pi_1(S) \) to be a representation \( \rho \) which may be written as \( \rho = \iota \circ \rho_0 \), where \( \rho_0 \) is a cocompact representation of \( \pi_1(S) \) with values in \( PSL(2, \mathbb{R}) \) and \( \iota \) is the irreducible representation of \( PSL(2, \mathbb{R}) \) in \( PSL(n, \mathbb{R}) \).

In [3], Hitchin proves the remarkable result that the connected components \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R})) \) of the space of reducible representations of \( \pi_1(S) \) containing \( n \)-Fuchsian representations are diffeomorphic to balls. Such a connected component is called a Hitchin component. It is denoted by \( \text{Rep}_H(\pi_1(S), PSL(n, \mathbb{R})) \). A representation which belongs to a Hitchin component is called a \( n \)-Hitchin representation. In other words it is a representation that can be deformed to a \( n \)-Fuchsian representation.

In [5], we give a geometric description of Hitchin representations, which was later completed by the work of O. Guichard [2] (cf. Section [3]). In particular, we show that if \( \rho \) is a Hitchin representation and \( \gamma \) a nontrivial element of \( \pi_1(S) \), then \( \rho(\gamma) \) is real split (Theorem 1.5 of [5]).

We finally state another construction about cross ratios: associated to a nontrivial element \( \gamma \) of \( \pi_1(S) \) and a cross ratio \( b \) is a real number \( l_b(\gamma) \) called the period of \( \gamma \) (cf. Equation [11]). For the case of cross ratios associated to hyperbolic metrics, these periods are the length of the associated closed geodesics.

**Cross ratios and Hitchin representations: a correspondence**

Our first result describes Hitchin representations in terms of cross ratios, generalising the situation about \( PSL(2, \mathbb{R}) \) which we briefly described in the previous paragraph.

We first introduce more complicated functions built out of cross ratio. For every \( p \), let \( \partial_\infty \pi_1(S)^p \) be the set of pairs of \( p+1 \)-uples \( (e_0, e_1, \ldots, e_p), (u_0, u_1, \ldots, u_p) \) in \( \partial_\infty \pi_1(S) \) such that

\[
j > i > 0 \implies e_j \neq e_i \neq u_0, u_j \neq u_i \neq e_0.
\]
Let \( b \) be a cross ratio. Let \( \chi^b_\rho \) be the map from \( \partial_\infty \pi_1(S)^P \) to \( \mathbb{R} \) defined by

\[
\chi^b_\rho(e, u) = \det_{i,j>0} ((b(e_i, u_j, e_0, u_0)).
\]

Our main result is the following.

**Theorem 1.1** There exists a bijection between the set of \( n \)-Hitchin representations and the set of cross ratios such that

1. \( \forall e, u, \chi^n(e, u) \neq 0 \)
2. \( \forall e, u, \chi^{n+1}(e, u) = 0 \).

Furthermore, if \( \rho \) is a \( n \)-Hitchin representation, \( b \) its associated cross ratio, and \( \gamma \) a nontrivial element of \( \pi_1(S) \) then the period of \( \gamma \) is given by

\[
l_b(\gamma) = \log(\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}),
\]

where \( \lambda_{\max}(\rho(\gamma)) \) and \( \lambda_{\min}(\rho(\gamma)) \) are respectively the eigenvalues of respectively maximum and minimum absolute values of the element \( \rho(\gamma) \).

For \( n = 2 \), it turns out that the functional relations described in this Theorem amounts to Relation \( \Simplify \).

The *limit curve*, drawn in \( \mathbb{P}(\mathbb{R}^n) \) and described in Paragraph 2.1.2, is the link between cross ratio and representations.

**A ”Character variety” containing all Hitchin representations**

Since Hitchin representations are irreducible, the natural embedding of \( PSL(n, \mathbb{R}) \) in \( PSL(n+1, \mathbb{R}) \) does not give rise to an embedding of the corresponding Hitchin components. Therefore there is no natural algebraic way - by an injective limit procedure say - to build a limit when \( n \) goes to infinity of Hitchin components. However, it follows from the previous paragraph that all Hitchin components sit in the same ”moduli space”: the space of all cross ratios.

We now explain the second construction of this article: all Hitchin components lie in a ”character variety” of \( \pi_1(S) \) into an infinite dimensional group \( G \). More precisely, let \( C^{1,h}(S^1) \) be the vector space of \( C^1 \)-functions with Hölder derivatives on the circle, and let \( Diff^h(S^1) \) be the group of \( C^1 \)-diffeomorphisms with Hölder derivatives of the circle. We observe that \( Diff^h(S^1) \) acts naturally on \( C^{1,h}(S^1) \). Let

\[
G = C^{1,h}(S^1) \times Diff^h(S^1).
\]

In Paragraph 6.1.2, we explain that this group \( G \) has a natural action by Hölder homeomorphisms on \( J^1(S^1) \), the space of 1-jets of functions on the circle.

Our first result is to build a ”character variety” for homomorphisms of \( \pi_1(S) \) in \( G \). Namely, we define in Paragraph 7.0.2 \( \infty \)-Hitchin homomorphisms of \( \pi_1(S) \) in \( G \). For such a \( \infty \)-Hitchin homomorphism \( \rho \), the quotient \( J^1(S^1)/\rho(\pi_1(S)) \) is compact. Moreover, in Paragraph 7.1.4 we associate a real number \( l_\rho(\gamma) \) called
the \( \rho \)-length of \( \gamma \) to every nontrivial element \( \gamma \) of \( \pi_1(S) \), and every \( \infty \)-Hitchin representation \( \rho \). The marked collection of \( \rho \)-lengths is the spectrum of \( \rho \). In Paragraph \[3\] we also associate to every \( \infty \)-Hitchin homomorphism a cross ratio.

We denote by \( \text{Hom}_H \) the set of all \( \infty \)-Hitchin representations. Let \( Z(G) \) be the center of \( G \). Our first result describe the action on \( G \) on \( \text{Hom}_H \).

**Theorem 1.2** The set \( \text{Hom}_H \) is an open set in the set \( \text{Hom}(\pi_1(S), G) \). Moreover, \( G/Z(G) \) acts properly on \( \text{Hom}_H \) and the quotient \( \text{Hom}(\pi_1(S), G)/G \) is Hausdorff. Two representations with the same spectrum and the same cross ratio are conjugated.

The two properties of \( \infty \)-Hitchin representations stated in the previous theorem show they are good candidates to be avatars of reducible representations. We denote by \( \text{Rep}_H \) the character variety of \( \infty \)-Hitchin representations \( \text{Hom}_H/G \). The following result relates Hitchin components to this character variety.

**Theorem 1.3** There exists a continuous injective map

\[ \psi : \text{Rep}_H(\pi_1(S), \text{SL}(n, \mathbb{R})) \rightarrow \text{Rep}_H, \]

such that, if \( \rho \in \text{Rep}_H(\pi_1(S), \text{SL}(n, \mathbb{R})) \) then

- for any \( \gamma \) in \( \pi_1(S) \), we have
  \[ l_{\psi(\rho)}(\gamma) = \log\left( \frac{\lambda_{\text{max}}(\rho(\gamma))}{\lambda_{\text{min}}(\rho(\gamma))} \right). \tag{2} \]

Here, \( l_{\psi(\rho)}(\gamma) \) is the \( \psi(\rho) \)-length of \( \gamma \), and \( \lambda_{\text{max}}(a) \) (resp. \( \lambda_{\text{min}}(a) \)) denote the maximum (resp. minimum) real eigenvalue of the endomorphism \( a \) in absolute value.

- The cross ratio associated to \( \rho \) and \( \psi(\rho) \) coincide.

In some sense, this result says that \( C^{1,h}(S^1) \times \text{Diff}^h(S^1) \) is a version of \( \text{SL}(\infty, \mathbb{R}) \).

We finally state in this introduction another result that explains that our character variety contains yet another interesting space.

**Theorem 1.4** Let \( \mathcal{M} \) be the space of negatively curved metrics on the surface \( S \). There exists a continuous injective map \( \psi \) from \( \mathcal{M} \) to \( \text{Rep}_H \). Furthermore, \( \psi \) preserves the length: for any \( \gamma \) in \( \pi_1(S) \)

\[ l_g(\gamma) = l_{\psi(g)}(\gamma). \]

Here \( l_g(\gamma) \) is the length of the closed geodesic for \( g \) freely homotopic to \( \gamma \), and \( l_{\psi(g)}(\gamma) \) is the \( \psi(g) \)-length of \( \gamma \). Finally, \( \psi(g_0) = \psi(g_1) \), if and only if there exists a diffeomorphism \( F \) of \( S \), homotopic to the identity, such that \( F^*(g_0) = g_1 \).
Both results are consequences of a general conjugation result: Theorem 8.1.

We finish this general introduction by stating a question about our construction: can one characterise
\[ F = \bigcup_{n} \text{Rep}_H(\pi_1(S), \text{PSL}(n, \mathbb{R})) \]
in \( \text{Rep}_H \)? For instance does \( F \) contains \( \mathcal{M} \)?

Structure of the article

We now describe shortly the content of this article.

2 Curves and hyperconvex representations. We recall results of [5] and explain how Hitchin representations are related to special curves in projectives spaces.

3 Cross ratio, definitions and first properties. We give the precise definition of a cross ratio and of the related quantities (periods and triple ratio).

4 Examples of cross ratio. We explain various constructions of cross ratio: the classical cross ratio on the projective line, cross ratio associated to curves in projective spaces, dynamical cross ratio and the original construction of J.-P. Otal for negatively curved metrics.

5 Hitchin representations and cross ratios. We prove Theorem 1.1.

6 The jet space \( J^1(S^1, \mathbb{R}) \). We describe the geometry of this jet space and the action of \( C^{1,\text{h}}(S^1) \ltimes \text{Diff}^b(S^1) \) on it.

7 Homomorphisms of \( \pi_1(S) \) in \( C^{1,\text{h}}(S^1) \ltimes \text{Diff}^b(S^1) \). We give the precise definitions of Anosov and \( \infty \)-Hitchin representations of \( \pi_1(S) \) and \( C^{1,\text{h}}(S^1) \ltimes \text{Diff}^b(S^1) \), and of the spectrum and cross ratio associated to these representations. We prove a refinement of Theorem 1.2.

8 A Conjugation Theorem. We explain Theorem 8.1 which shows how to build \( \infty \)-Hitchin representations of \( \pi_1(S) \) in \( C^{1,\text{h}}(S^1) \ltimes \text{Diff}^b(S^1) \).

9 Negatively curved metrics. We prove Theorem 1.3.

10 Hitchin component. We prove Theorem 1.3.

11 Appendix A: Filtrated Spaces, Holonomy. We prove results about foliations by affine spaces.

12 Appendix B: The symplectic nature of cross ratio. We explain the relation of cross ratios are related and symplectic geometry, and give constructions of a whole family of cross ratio related to Hitchin representations.
2 Curves and hyperconvex representations

We recall results and definitions from [5].

2.1 Hyperconvex representations

2.1.1 Fuchsian representations

A \( n \)-Fuchsian representation of \( \pi_1(S) \) is a representation \( \rho \) which may be written as \( \rho = \iota \circ \rho_0 \), where \( \rho_0 \) is a cocompact representation with values in \( \text{PSL}(2, \mathbb{R}) \) and \( \iota \) is the irreducible representation of \( \text{PSL}(2, \mathbb{R}) \) in \( \text{PSL}(n, \mathbb{R}) \).

2.1.2 Hyperconvex curves

A continuous curve \( \xi \) with values in \( \mathbb{P}(\mathbb{R}^n) \) is hyperconvex if, for any distinct points \( (x_1, \ldots, x_n) \), the following sum is direct

\[
\xi(x_1) + \ldots + \xi(x_n).
\]

We say a representation \( \rho \) of \( \pi_1(S) \) is \( n \)-hyperconvex, if there exists a \( \rho \)-equivariant hyperconvex curve from \( \partial_\infty \pi_1(S) \) in \( \mathbb{P}(\mathbb{R}^n) \). Actually, such a curve is unique and is called the limit curve of the representation. We say a representation is Hitchin if may be deformed in a \( n \)-Fuchsian representation.

In [5] we prove the following result.

**Theorem.** Let \( \rho \) be a Hitchin representation. Then \( \rho \) is hyperconvex. Each such representation is discrete, faithful. Finally, for each \( \gamma \) in \( \pi_1(S) \) different from the identity, \( \rho(\gamma) \) is real split with distinct eigenvalues

We explain later a refinement of this result (Theorem 2.2). We observe that the Veronese embedding is a hyperconvex curve equivariant under all Fuchsian representations. Therefore, a Fuchsian representation is indeed hyperconvex.

According to the previous result, many representations, at least those in Hitchin components, are hyperconvex.

Conversely, completing our work, O. Guichard [2] has shown the following result

**Theorem 2.1 [Guichard]** Every hyperconvex representation is Hitchin.

2.2 Frenet curves

We say a hyperconvex curve \( \xi \) is a Frenet curve, if there exists a family of maps \( (\xi^1, \xi^2, \ldots, \xi^{n-1}) \), called the osculating flag, such that

- \( \xi^p \) takes values in the Grassmannian of \( p \)-planes,
- \( \forall x, \quad \xi^p(x) \subset \xi^{p+1}(x) \)
\[ \xi = \xi^1, \]

- \( \xi = \xi^1 \)

- if \( (n_1, \ldots, n_l) \) are positive integers such that \( \sum_{i=1}^{l} n_i \leq n \), if \( (x_1, \ldots, x_l) \) are distinct points, then the following sum is direct
  \[ \xi^{n_1}(x_1) + \ldots + \xi^{n_l}(x_l); \quad (3) \]

- finally, for every \( x \), let \( p = n_1 + \ldots + n_l \), then
  \[ \lim_{(y_1, \ldots, y_l) \to x, y_i \text{ all distinct}} (\bigoplus_{i=1}^{l} \xi^{n_i}(y_i)) = \xi^p(x). \quad (4) \]

We call \( \xi_{n-1} \) the osculating hyperplane. We observe that for a Frenet hyperconvex curve, \( \xi^1 \) completely determines \( \xi^p \). Moreover, if \( \xi^1 \) is \( C^\infty \), then \( \xi^p(x) \) is completely generated by the derivatives at \( x \) of \( \xi^1 \) up to order \( p-1 \). However, in general, a Frenet hyperconvex curve has no reason to be \( C^\infty \) although its image is obviously a \( C^1 \)-submanifold.

### 2.3 Hyperconvex representations and Frenet curves

We list here several properties of hyperconvex representations proved in [5].

**Theorem 2.2** Let \( \rho \) be an hyperconvex representation of \( \pi_1(S) \) in \( SL(E) \), with limit curve \( \xi \).

1. Then for each \( \gamma \) in \( \pi_1(S) \) different from the identity, \( \rho(\gamma) \) is purely real split with distinct eigenvalues.
2. Furthermore, \( \xi \) is a hyperconvex Frenet curve.
3. Let \( \xi^* \) be its osculating hyperplane, then \( \xi^* \) is hyperconvex.
4. The osculating flag is Hölder.
5. Finally, if \( \gamma^+ \) is the attracting fixed point of \( \gamma \) in \( \partial_\infty \pi_1(S) \), then \( \xi(\gamma^+) \), \( (\text{resp. } \xi^*(\gamma^+)) \) is the unique attracting fixed point of \( \rho(\gamma) \) in \( \mathbb{P}(E) \) (resp. \( \mathbb{P}(E^*) \)).

### 3 Cross ratio, definitions and first properties

#### 3.1 Cross ratio

Let \( S \) be the circle. Let

\[ S^{4*} = \{(x, y, z, t) \in S^4 \mid x \neq t, \text{ and } y \neq z\}. \]
A cross ratio on $S$ is a H"older function $b$ on $S^4$ with values in $\mathbb{R}$ which satisfies the following rules
\begin{align}
  b(x, y, z, t) &= b(z, t, x, y) \\
  b(x, y, z, t) &= 0 \iff x = y \text{ or } z = t \\
  b(x, y, z, t) &= b(x, y, z, w)b(x, w, z, t) \\
  b(x, y, z, t) &= b(x, y, w, t)b(w, y, z, t)
\end{align}

(5) (6) (7) (8)

If $S = \partial_{\infty} \pi_1(S)$, we assume furthermore that $b$ is invariant under the diagonal action of $\pi_1(S)$:
\begin{align}
  \forall \gamma \in \pi_1(S), \quad b(\gamma x, \gamma y, \gamma z, \gamma t) &= b(x, y, z, t) \\
\end{align}

(9)

Furthermore, we say a cross ratio on $S$ is strict if
\begin{align}
  b(x, y, z, t) &= 1 \iff x = z \text{ or } y = t.
\end{align}

(10)

The classical cross ratio on $\mathbb{RP}^1$ is an example of a strict cross ratio. It is a well known fact that the classical cross ratio can be characterised as the unique cross ratio satisfying an extra functional rule (cf Proposition 4.1). We give more examples and constructions in Section 4.

Remark:
The definition given above does not coincide with the usual definition given for instance in [4], [6] (even after taking an exponential): indeed, first we require $b(z, t, x, y) = b(x, y, z, t)$, (which amounts to time reversibility), more importantly we do not require $b(x, y, z, t) = b(y, x, t, z)$. However we may observe that if $b(x, y, z, t)$ is a cross ratio with our definition, so is $b^*(x, y, z, t) = b(y, x, t, z)$, and finally $bb^*$ is a cross ratio according to the classical definitions quoted above. We explain Otal’s construction and the relation with negatively curved metrics in Section 4.3.

3.2 Periods

Let $b$ be a cross ratio $b$ and $\gamma$ be a nontrivial element in $\pi_1(S)$. The period $l_b(\gamma)$ is defined as follows. Let $\gamma^+$ (resp. $\gamma^-$) the attracting (resp. repelling) fixed point of $\gamma$ on $\partial_\infty \pi_1(S)$. Let $y$ be an element of $\partial_{\infty} \pi_1(S)$. Let’s define
\begin{align}
  l_b(\gamma, y) &= \log |b(\gamma^-, \gamma y, \gamma^+, y)|.
\end{align}

(11)

It is immediate to check that $l_b(\gamma) = l_b(\gamma, y)$ does not depend on $y$. Moreover, by Equation(4), $l_b(\gamma) = l_b(\gamma^{-1})$.

3.3 Triple ratios

We introduce a new feature associated to a cross ratio: for every quadruple of pair-wise distinct points $(x, y, z, t)$, one easily checks that the expression
\begin{align}
  b(x, y, z, t)b(z, x, y, t)b(y, z, x, t),
\end{align}
is independent of the choice of \( t \). We call such a function a \textit{triple ratio}. Indeed, in some cases, it is related to the triple ratios introduced by A. Goncharov in \cite{1}. It turns out, although we do not use this remark, that a triple ratio satisfies the (multiplicative) cocycle identity and hence defines a bounded cohomology class in \( H^2_b(\Gamma) \).

4 Examples of cross ratio

4.1 Cross ratio on the projective line

Let \( E \) be a vector space with \( \dim(E) = 2 \). We recall that the "classical" cross ratio is defined on \( \mathbb{P}(E) \), identified with \( \mathbb{R} \cup \{\infty\} \) using projective coordinates, by:

\[
b(x, y, z, t) = \frac{(x - y)(z - t)}{(x - t)(z - y)}.
\]

It is easy to check that this "classical" cross ratio is a strict cross ratio. The "classical" cross ratio on the projective line satisfies the rules 3.1 as well as the following extra rule:

\[
1 - b(f, v, e, u) = b(u, v, e, f).
\]

We will later on explain the well known fact that this extra relation completely characterises the classical cross ratio. Furthermore, it turns out that this (simple) relation is equivalent to the following more sophisticated one, which we may generalise to higher dimensions

**Proposition 4.1** For a cross ratio \( b \), Relation \( \text{(12)} \) is equivalent to,

\[
(b(f, v, e, u) - 1)(b(g, w, e, u) - 1) = (b(f, w, e, u) - 1)(b(g, v, e, u) - 1).
\]

Furthermore, if a cross ratio \( b \) satisfies Relation \( \text{(13)} \), there exists an embedding \( f \), unique up to left composition to projective transformations, of \( S \) in \( \mathbb{RP}^1 \), such that in projective coordinates,

\[
b(x, y, z, t) = \frac{(f(x) - f(y))(f(z) - f(t))}{(f(x) - f(t))(f(z) - f(y))}.
\]

**Proof:** First, if \( b \) satisfies Relation \( \text{(12)} \), then thanks to Relation \( \text{(7)} \), it satisfies \( \text{(13)} \). Conversely, assume it satisfies \( \text{(13)} \), then

\[
b(f, v, e, u) = \frac{(b(f, w, e, u) - 1)(b(g, v, e, u) - 1)}{b(g, w, e, u) - 1} + 1.
\]

Setting \( g = v \), we obtain

\[
b(f, v, e, u) = \frac{1 - b(f, w, e, u)}{b(v, w, e, u) - 1} + 1
\]

\[
= \frac{b(v, w, e, u) - b(f, w, e, u)}{b(v, w, e, u) - 1}.
\]

9
We have

\[ b(f, v, k, z) = \frac{b(f, v, e, z)}{b(k, v, e, z)} \text{ by (8)} \] (17)

\[ = \left( \frac{b(f, v, e, u)}{b(k, v, e, u)} \right) \left( \frac{b(k, z, e, u)}{b(k, v, e, u)} \right) \text{ by (7)} \] (18)

\[ = \frac{b(f, v, e, u)b(k, z, e, u)}{b(k, v, e, u)b(f, z, e, u)}. \] (19)

Finally, applying Relation (10) to the four left terms of Equation (19) we obtain

\[ b(f, v, k, z) = \frac{(b(v, w, e, u) - b(f, w, e, u))(b(z, w, e, u) - b(k, w, e, u))}{(b(v, w, e, u) - b(k, w, e, u))(b(z, w, e, u) - b(f, w, e, u))}. \]

The final statement follows when we set

\[ f = f(w, e, u) \left\{ \begin{array}{c} S \\ x \end{array} \right\} \rightarrow \mathbb{R} \cup \{\infty\} \]

Q.E.D.

### 4.2 Cross ratios and curves in projective spaces

We now extend the previous discussion. Let \( E \) be an \( n \)-dimensional vector space. Let \( \xi \) and \( \xi^* \) be two curves from \( S \) to \( \mathbb{P}(E) \) and \( \mathbb{P}(E^*) \) respectively. We assume furthermore

\[ \langle \xi^*(z), \xi(y) \rangle = 0 \Leftrightarrow z = y. \] (20)

This in particular true for an equivariant hyperconvex curve \( \xi \) and its osculating hyperplane \( \xi^* \).

For every \( x \), we choose an arbitrary nonzero vector \( \hat{\xi}(x) \) (resp. \( \hat{\xi}^*(x) \)) in the line \( \xi(x) \) (resp. \( \xi^*(x) \)).

We define the cross ratio associated to this pair of curves by

\[ b_{\xi, \xi^*}(x, y, z, t) = \frac{\langle \hat{\xi}(x), \hat{\xi}^*(y) \rangle \langle \hat{\xi}(z), \hat{\xi}^*(t) \rangle}{\langle \hat{\xi}(z), \hat{\xi}^*(y) \rangle \langle \hat{\xi}(x), \hat{\xi}^*(t) \rangle}. \]

It is easy to check that

- this definition does not depend on the choice of \( \hat{\xi} \) and \( \hat{\xi}^* \),
- \( b_{\xi, \xi^*} \) satisfies the axioms of a formal cross ratio.
- Let \( V = \xi(x) \oplus \xi(z) \). Let \( \eta(m) = \xi^*(m) \cap V \). Let \( b_V \) be the classical cross ratio on \( \mathbb{P}(V) \), then

\[ b_{\xi, \xi^*}(x, y, z, t) = b_V(\xi(x), \eta(y), \xi(z), \eta(t)). \]
It follows that $b_{\xi,\xi^*}$ is strict if furthermore, for all quadruple of pairwise distinct points $(x, y, z, t)$,

$$\text{Ker}(\xi^*(z)) \cap (\xi(x) \oplus \xi(y)) \neq \text{Ker}(\xi^*(t)) \cap (\xi(x) \oplus \xi(y)).$$

Finally, we have

**Lemma 4.2** Let $(\xi, \xi^*)$ and $(\eta, \eta^*)$ be two pairs of curves satisfying Condition (20). Assume that $\xi^*$ and $\eta^*$ are hyperconvex. Assume furthermore that $b_{\eta,\eta^*} = b_{\xi,\xi^*}$.

Then there exists a linear map $A$ such that $\xi = A \eta$.

**Proof**: We assume the hypothesis of the theorem. Let $(x_0, x_1, \ldots, x_n)$ be a tuple of $n + 1$ pair-wise distinct points of $S$. We choose a vector $z_0$ in $\xi^*(x_0)$. Let $U = (u_1, \ldots, u_n)$ be the basis of $E^*$ such that $u_i = \xi^*(x_i)$ and $\langle z_0, u_i \rangle = 1$.

The projective coordinates of $\xi(y)$ in the dual basis are

$$[\ldots : \langle \xi(y), u_i \rangle : \ldots] = [\ldots : \frac{\langle \xi(y), u_i \rangle}{\langle \xi(y), u_1 \rangle} : \ldots] = [\ldots : \langle \xi(y), u_i \rangle \langle z_0, u_1 \rangle : \ldots] = [\ldots : b_{\xi,\xi^*}(y, x_i, x_0, x_1) : \ldots].$$

Symmetrically, we choose a vector $y_0$ in $\eta^*(x_0)$, and let $V = (v_1, \ldots, v_n)$ be the basis of $E^*$ such that $v_i = \eta^*(x_i)$ and $\langle y_0, v_i \rangle = 1$.

It follows that if $A$ is the linear map that send the dual basis of $V$ to the dual basis of $U$, then $\xi = A \eta$. Q.E.D.

We state the following Proposition.

**Proposition 4.3** Let $\rho$ be a a hyperconvex representation of $\pi_1(S)$ in $SL(n, \mathbb{R})$, with limit curve $\xi$. Let $\xi^*$ be its osculating hyperplane (cf. Theorem 2.2). Then $b_{\rho} = b_{\xi,\xi^*}$ is a strict cross ratio defined on $\partial_{\infty} \pi_1(S)$ and its periods are

$$l_b(\gamma) = \log(|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}|),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are respectively the eigenvalues of respectively maximum and minimum absolute value of the purely loxodromic element $\rho(\gamma)$. 

11
PROOF: We already now that $b_\rho$ is a formal cross ratio. We will prove in Proposition 10.5 that $b_\rho$ is strict. We now compute the periods. By Theorem 22 if $\gamma^+$ is the attracting fixed point of $\gamma$ in $\partial_\infty \pi_1(S)$, then $\xi(\gamma^+)$ (resp. $\xi^*(\gamma^-)$) is the unique attracting (resp. repelling) fixed point of $\rho(\gamma)$ in $\mathbb{P}(E)$. In particular

$$
\rho(\gamma)\xi(\gamma^+) = \lambda_{\max}\xi(\gamma^+)
$$

$$
\rho(\gamma)\xi(\gamma^-) = \lambda_{\min}\xi(\gamma^-)
$$

Therefore,

$$
l_b(\gamma) = \log |b(\gamma^-, y, \gamma^+, \gamma^{-1} y)|
$$

$$
= \log \left| \frac{\langle \xi(\gamma^-), \xi^*(y) \rangle \langle \xi(\gamma^+), \xi^*(\gamma^{-1} y) \rangle}{\langle \xi(\gamma^-), \xi^*(\gamma^{-1} y) \rangle \langle \xi(\gamma^+), \xi(y) \rangle} \right|
$$

$$
= \log \left| \frac{\langle \xi(\gamma^-), \xi^*(y) \rangle \langle \xi(\gamma^+), \rho(\gamma)^*\xi^*(y) \rangle}{\langle \xi(\gamma^-), \rho(\gamma)^*\xi^*(y) \rangle \langle \xi(\gamma^+), \xi(y) \rangle} \right|
$$

$$
= \log \left| \frac{\langle \rho(\gamma)\xi(\gamma^-), \xi^*(y) \rangle \langle \rho(\gamma)\xi(\gamma^+), \xi^*(y) \rangle}{\langle \rho(\gamma)\xi(\gamma^-), \xi^*(y) \rangle \langle \xi(\gamma^+), \xi(y) \rangle} \right|
$$

$$
= \log \frac{\lambda_{\max}}{\lambda_{\min}}
$$

Q.E.D.

**Remark:** Let $\rho^*$ be the contragredient representation of $\rho$ defined by

$$
\rho^*(\gamma) = (\rho(\gamma^{-1}))^*.
$$

Let $b$ be a cross ratio. We define $b^*$ by $b^*(x, y, z, t) = b(y, x, t, z)$. Then $(b_\rho)^* = b_\rho^*$ and $b_\rho$ and $b_\rho^*$ have the same periods.

### 4.3 Dynamical cross ratio and other examples

We recall Otal’s construction of cross ratio in the case of negatively curved metrics on surfaces [7]. Let $S$ be equipped with a negatively curved metric. We lift this metric to the universal cover $\tilde{S}$ of $S$. Let $(a_1, a_2, a_3, a_4)$ be a quadruple of points on $\partial_\infty \pi_1(S) = \partial_\infty \tilde{S}$. Let $c_{ij}$ be the geodesic from $a_i$ to $a_j$. We choose nonintersecting horoballs $H_i$ ”centred” at each point $a_i$. Let $l_{ij}$ be the length of the following geodesic arc

$$
c_{ij} \cap (\tilde{S} \setminus (H_i \cup H_j)).
$$

Otal’s cross ratio is

$$
O(a_1, a_2, a_3, a_4) = l_{12} - l_{23} + l_{34} - l_{41}.
$$
Let’s make the link with our definition: the cross ratio of a *cyclically oriented $4$-uple* of distinct points is just the exponential of Otal’s cross ratio

$$b(a_1, a_2, a_3, a_4) = e^{l_{12} - l_{23} + l_{34} - l_{41}}.$$  

For noncyclically oriented 4-uple, we introduce a sign compatible with the sign of the usual cross ratio.

With this definition the cross ratio agrees with the cross ratio defined on $\mathbb{RP}^1$ in the case of hyperbolic surfaces.

**Remarks:**

1. Actually, this construction can be extended to Anosov flows on unit tangent bundle of surfaces. Conversely, it is very easy to show in the case of surfaces, that a cross ratio comes from a flow. Indeed, given a cross ratio $b$ on $\partial_{\infty} \pi_1(S)^{4*}$, and any real number $t$, we define a flow map $\phi_t$ from $\partial_{\infty} \pi_1(S)^{3*}$ to itself by

$$\phi_t(x_-, x_0, x_+),$$

where

$$b(x_+, x_0, x_-) = e^t.$$

Then, Equation (7) exactly says that $t \mapsto \phi_t$ is a one parameter group.

2. The formal rules 3.1 of a cross ratio are easily satisfied

3. The associated period of an element $\gamma$ is the length of the closed orbit in the free homotopy class.

4. the cross ratio of a negatively curved metric satisfies an extra symmetry:

$$b(x, y, z, t) = b(y, x, t, z).$$

This symmetry is required in Otal’s definition but not in ours. We pay this generalisation by the fact a cross ratio is not uniquely determined by its periods.

5. Of course it is well known that this construction is yet another instance of the "symplectic construction" explained in Section 12. Indeed the space of geodesics of the universal cover of a negatively curved manifold is equipped with a symplectic structure which comes from the symplectic reduction of the tangent space. Furthermore, this space is identified with the space pair of distinct points of the boundary at infinity. Hence, it inherits a real polarisation and one checks that the cross ratio defined as in Section 12 coincides with the dynamical one.

For a complete description of various aspects of dynamical cross ratios, one is advised to read François Ledrappier’s presentation [6].
5 Hitchin representations and cross ratios

We aim to generalise Proposition 4.1 in higher dimensions.

Let $b$ be a cross ratio on a set $S$. We now introduce more sophisticated functions. Let $\{e_0, e_1, \ldots, e_n\}$ be $n + 1$ points on $S$, and $\{u_0, u_1, \ldots, u_n\}$ be $n + 1$ other points. Assume that,

$$\forall i, e_i \neq u_0, \forall i, u_i \neq e_0.$$ 

Let $B(e; u)$ be the $n \times n$-matrix whose coefficients are $b_{ij} = b(e_i, u_j, e_0, u_0)$ with $1 \leq i, j \leq n$.

We set

$$\chi_n^b(e; u) = \det(B(e; u)).$$

When the context makes it obvious, we omit the subscript $b$. We now prove.

**Theorem 5.1** Let $\xi$ and $\xi^*$ be two hyperconvex curves from $S^1$ to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ such that

$$x = y \Leftrightarrow \langle \xi^*(x), \xi(y) \rangle = 0.$$ 

Let $b = b_{\xi, \xi^*}$ be the associated cross ratio. Then, for all pairs of $p$-uples $(e_0, e_1, \ldots, e_p), (u_0, u_1, \ldots, u_p)$ with

$$j > i > 0 \implies e_j \neq e_i \neq u_0, u_j \neq u_i \neq e_0,$$

we have,

1. if $p \leq n$, $\chi_n^b(e; u) \neq 0$,
2. if $p > n$, $\chi_n^b(e; u) = 0$.

Conversely, let $b$ be a continuous strict cross ratio on $S^1$. Assume that the cross ratio satisfies the above two conditions, then there exist two hyperconvex curves $\xi$ and $\xi^*$ with values in $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively, unique up to projective transformations, such that $b = b_{\xi, \xi^*}$ and $x = y \Leftrightarrow \langle \xi^*(x), \xi(y) \rangle = 0$.

The proof of the Theorem is given in Paragraph 5.1.

Remarks:

- We shall very soon prove that for all cross ratios, the nullity of $\chi^p(e; u)$ for a given $e$ and $u$ does not depend on $e_0$ and $u_0$.
- It is interesting to see what happens when $n = 2$. In this case, using the previous remark, let's take $u_0 = u_1 = u, e_0 = e_1 = e$. Then

$$\chi^2(e, e, f; u, u, v) = \begin{vmatrix} 1 & 1 \\ 1 & b(f, v, e, u) \end{vmatrix} = b(f, v, e, u) - 1 \neq 0.$$
Also,
\[
\chi^3(e, e, f, g; u, u, v, w) = \begin{vmatrix}
1 & 1 & 1 \\
1 & b(f, v, e, u) & b(g, v, e, u) \\
1 & b(f, w, e, u) & b(g, w, e, u)
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 \\
1 & b(f, v, e, u) - 1 & b(g, v, e, u) - 1 \\
1 & b(f, w, e, u) - 1 & b(g, w, e, u) - 1
\end{vmatrix}
= (b(f, v, e, u) - 1)(b(g, w, e, u) - 1) - (b(f, w, e, u) - 1)(b(g, v, e, u) - 1).
\]
Therefore, the condition that \(\xi^p = 0\) extends Equation (13).

5.1 The expression \(\chi\)

We first prove the following result of independent interest.

Proposition 5.2 The nullity of \(\chi^n(e, u)\) does not depend on \(e_0\) and \(u_0\).

Proof: From the formal rules of computation of cross ratios, we have
\[
b(e_i, u_j, e_0, u_0) b(e_i, u_0, e_0, v_0) = b(e_i, u_j, e_0, v_0) = b(e_i, u_j, f_0, v_0) b(f_0, u_j, e_0, v_0)
\]
Therefore
\[
\chi(e_0, e_1, \ldots, e_n; u_0, u_1, \ldots, u_n) = \left( \prod_{i,j} b(f_0, u_j, e_0, v_0) \right) \chi(f_0, e_1, \ldots, e_n; v_0, u_1, \ldots, u_n).
\]
Since we have assumed
\[
f_0 \neq v_0 \neq e_i, u_j \neq e_0 \neq f_0,
\]
The proposition immediately follows. Q.E.D.

5.2 Proof of Theorem 5.1

5.2.1 Cross ratio associated to curves

Let \(\xi\) and \(\xi^*\) be two hyperconvex curves with values in \(\mathbb{P}(E)\) and \(\mathbb{P}(E^*)\) such that
\[
x = y \Leftrightarrow \langle \xi(x), \xi(y) \rangle = 0.
\]
Let \(e_0\) and \(u_0\) be two distinct points of \(S^1\). Let \(E_0\) be a nonzero vector in \(\xi(e_0)\). Let \(U_0\) be a covector in \(\xi^*(u_0)\) such that \(\langle U_0, E_0 \rangle = 1\). We now lift the curves \(\xi\) and \(\xi^*\) with values in \(\mathbb{P}(E)\) and \(\mathbb{P}(E^*)\) to continuous curves \(\hat{\xi}\) and \(\hat{\xi}^*\) from \(S^1 \setminus \{e_0, u_0\}\) to \(E\) and \(E^*\), such that
\[
\langle \hat{\xi}^*(v), E_0 \rangle = 1 = \langle U_0, \hat{\xi}(e) \rangle.
\]
Then the associated cross ratio is
\[ b(f, v, e_0, u_0) = \langle \xi^*(v), \xi(f) \rangle \]
From this expression it follows that \( \chi^{n+1} = 0 \) and that by hyperconvexity \( \chi^n \neq 0 \).

### 5.2.2 Curves associated to cross ratios

We now suppose that we have a strict cross ratio \( b \) such that \( \chi^{n+1}_b = 0 \). Assume also that for every pair \( e = (e_0, e_1, \ldots, e_n) \), \( u = (u_0, u_1, \ldots, u_n) \) of \( n \)-uples satisfying
\[ j > i > 0 \implies e_j \neq e_i \neq u_0, \ u_j \neq u_i \neq e_0, \]
we have
\[ \chi^p_b(e, u) \neq 0. \]

We denote by \([\ldots : \ldots] \) the projective coordinates on \( \mathbb{P}(\mathbb{R}^n) \). Let \((e, u)\) be as above. We define two maps
\[
\xi \left\{ \begin{array}{ccc}
S^1 \setminus \{u_0, e_0\} & \to & \mathbb{P}(\mathbb{R}^n) \\
 f & \mapsto & (b(f, u_1, e_0, u_0) : \ldots : b(f, u_n, e_0, u_0))
\end{array} \right.
\]
\[
\xi^* \left\{ \begin{array}{ccc}
S^1 \setminus \{u_0, e_0\} & \to & \mathbb{P}(\mathbb{R}^n) \\
v & \mapsto & (b(e_1, v, e_0, u_0) : \ldots : b(e_n, v, e_0, u_0))
\end{array} \right.
\]

We now prove that \( b = b_{\xi, \xi^*} \). By Lemma 5.2 this will show that the curves \( \xi \) and \( \xi^* \) are unique up to projective transformations, and in particular do not depend on the choice of \( e \) and \( u \).

By our construction,
\[ \chi^p_b(e, u) = \chi^p_b(e, u). \]

Moreover, for every \( f \) and \( v \)
\[
b(f, u_i, e_0, u_0) = b_{\xi, \xi^*}(f, u_i, e_0, u_0),
\]
\[
b(e_i, v, e_0, u_0) = b_{\xi, \xi^*}(e_i, v, e_0, u_0). \tag{21}
\]

We now choose \( e_n = f, \ u_n = v \). Developing the determinant \( \chi^{n+1}(e, u) \) along the last line, the equation \( \chi^{n+1}(e, u) = 0 \) leads to
\[ b(f, v, e_0, u_0)\chi^p_b(e, u) = F(\ldots, b(f, u_i, e_0, u_0), \ldots, b(e_i, v, e_0, u_0), \ldots) \]
where the right hand term is a polynomial in \( b(f, u_i, e_0, u_0) \) and \( b(e_i, v, e_0, u_0) \).

For the same reason we have
\[ b_{\xi, \xi^*}(f, v, e_0, u_0)\chi^p_b(e, u) = F(\ldots b_{\xi, \xi^*}(f, u_i, e_0, u_0), \ldots, b_{\xi, \xi^*}(e_i, v, e_0, u_0), \ldots). \]
Therefore, using Equations \(21\), \(21\) and \(21\) we obtain
\[
b(f, v, e_0, u_0)\chi_n^0(e, u) = F(\ldots, b_{\xi \xi}, (f, u_0, e_0, u_0), \ldots, b_{\xi \xi}, (e, v, e_0, u_0), \ldots),
\]
\[
= b_{\xi \xi} (f, v, e_0, v_0) \chi_{b_{\xi \xi}}^u (e, u),
\]
\[
= b_{\xi \xi} (f, v, e_0, v_0) \chi_n^0 (e, u).
\]
It follows that \(b(f, v, e_0, u_0) = b_{\xi \xi} (f, v, e_0, u_0)\). Applying Relation \(7\) twice, we easily obtain that \(b = b_{\xi \xi} \). Finally since \(\xi^u (e, u) \neq 0\), \(\xi\) is hyperconvex as well as \(\xi^*\).

6 The jet space \(J^1(S^1, \mathbb{R})\)

In this preliminary section, we aim to explain the geometry of \(J = J^1(S^1, \mathbb{R})\), the space of one-jet of functions on \(S^1\), and how the group \(C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)\) acts on \(J\). We then explain that this geometric description characterises \(J\). Finally, we describe a homomorphism of \(\text{PSL}(2, \mathbb{R})\) in \(C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)\), whose image acts faithfully and transitively on \(J\).

6.1 Description of the jet space

We describe in this section the geometric features of \(J\) that we shall use later on, namely:

- projections on \(S^1\), \(T^*S^1\) and \(S^1 \times \mathbb{R}\),
- the action of \(C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)\) on \(J\),
- a foliation of \(J\) by affine leaves,
- a flow,
- a contact form.

6.1.1 Projections

Let \(J = J^1(S^1, \mathbb{R})\) be the space of one-jet of functions on \(S^1\). We denote by \(j^1_x(f)\) the one-jet of the function \(f\) at the point \(x\). We define the projections

\[
\begin{align*}
\pi & : \{ & J & \rightarrow S^1 \\
& & j^1_x(f) & \mapsto x \\
\pi_1 & : \{ & J & \rightarrow S^1 \times \mathbb{R} \\
& & j^1_x(f) & \mapsto (x, f(x)) \\
\pi_2 & : \{ & J & \rightarrow T^*S^1 \\
& & j^1_x(f) & \mapsto d_x f
\end{align*}
\]

We observe that each fibre of \(\pi\) carries an affine structure.
6.1.2 Action of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$

We denote by $\text{Diff}^h(S^1)$ the group of $C^1$-diffeomorphisms of $S^1$ with Hölder derivatives. Let $C^{1,h}(S^1)$ the space of $C^1$-functions on $S^1$ with Hölder derivatives. The group $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ acts naturally on $J$ by Hölder homeomorphisms in the following fashion: for every $C^1$-diffeomorphism $\phi$ of $S^1$, for every function $h$ in $C^{1,h}(S^1)$, we define the homeomorphism of $J$

$$F = (h, \phi) : j^1_h(f) \mapsto j^1_{h(\theta)}((h + f) \circ \phi^{-1}).$$

Alternatively, if we choose a coordinate $\theta$ on $S^1$ and consider the identification

$$\begin{cases}
J & \to S^1 \times \mathbb{R} \times \mathbb{R} \\
 j^1_h(f) & \mapsto (\theta, r, f) = (\theta, \frac{\partial f}{\partial \theta}, f(\theta)),
\end{cases}$$

then

$$F = (h, \phi) : (\eta, r, f) \mapsto (\phi(\eta), \frac{\partial \phi^{-1}}{\partial \theta}(\eta)(\frac{\partial h}{\partial \theta}(\eta) + r), f + h(\eta)). \quad (22)$$

The homeomorphism $F$ has the following properties

- $F$ preserves the fibres of $\pi: \pi \circ F = \phi \circ \pi$.
- For every $x$ in $S^1$, $F$ restricted to $\pi^{-1}(x)$ is affine, in particular $C^\infty$, and the derivatives of $F|_{\pi^{-1}(x)}$ vary continuously on $J$.

6.1.3 Foliation

Let $\mathcal{F}$ be the 1-dimensional foliation of $J$ given by the fibres of $\pi_1$. We observe that the fibre through $z$ is identified with $T_x^*S^1$ for $x = \pi(z)$. This fibre also carries an affine structure invariant by the action of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ described above.

6.1.4 Canonical flow

We define the canonical flow of $J$ to be the flow

$$\phi_t(j^1(f)) = j^1(f + t),$$

where we identify the real number $t$ with the constant function that takes value $t$. The canonical flow commutes with the action of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ on $J$. Notice also that

$$J/\phi_t = T^*S^1,$$

and this identification turns $\pi_2$ in a principal $\mathbb{R}$-bundle.
6.1.5 Contact form

We finally recall that \( J \) admits a contact form \( \beta \). If we choose a coordinate \( \theta \) on \( S^1 \) and consider the identification

\[
\begin{align*}
J & \to S^1 \times \mathbb{R} \times \mathbb{R} \\
\hat{j}_0(f) & \mapsto (\theta, r, f) = (\theta, \frac{\partial f}{\partial \theta}, f(\theta))
\end{align*}
\]

then

\[ \beta = df - rd\theta. \]

Note that a legendrian curve for \( \beta \) which is locally a graph above \( S^1 \) is the graph of one-jet of a function. Moreover, the canonical flow \( \phi_t \) preserves \( \beta \). Finally, we observe that \( \beta \) is a connection form for the \( \mathbb{R} \)-principal bundle defined by \( \pi_2 \), whose curvature form is the canonical symplectic form of \( T^*S^1 \). We prove.

**Proposition 6.1** Let \( \alpha \) be a connection one-form on \( J \). Assume that the curvature of \( \alpha \) is the canonical symplectic form on \( T^*S^1 \). Then there exists a diffeomorphism \( \xi \) of \( J \) which

- commutes with the action of the canonical flow,
- preserves the fibres of \( \pi \) and \( \pi_2 \),
- is above a symplectic diffeomorphism of \( T^*S^1 \),
- satisfies \( \xi^* \beta = \alpha \).

**Proof:** We choose a coordinate \( \theta \) of \( S^1 \). Thus, \( T^*S^1 \) is identified with \( S^1 \times \mathbb{R} \). Since \( \alpha \) is a connection form there exists functions \( \alpha_r \) and \( \alpha_\theta \) such that

\[ \alpha = df + \alpha_r(r, \theta)dr + \alpha_\theta(r, \theta)d\theta. \]

Let

\[ \gamma = \alpha - \beta = \alpha_r(r, \theta)dr + (\alpha_\theta(r, \theta) - r)d\theta. \]

Since the curvature of \( \alpha \) is \( d\theta \wedge dr \), \( \gamma \) is closed. Therefore, there exist a function \( h \) and a constant \( \lambda \) such that

\[ \gamma = dh + \lambda d\theta. \]

Let \( \mu_\lambda \) be the symplectic diffeomorphism of \( T^*S^1 \) such that

\[ \mu_\lambda(\theta, r, f) = (\theta, r - \lambda, f). \]

It is a straightforward check that

\[ \xi = \mu_\lambda \circ (h, id), \]

satisfies the condition of our proposition. Q.E.D.
6.2 A Geometric Characterisation of $C^\infty(S^1) \times \text{Diff}^\infty(S^1)$

The following proposition suggests that $\mathcal{F}$, $\beta$ and $\phi_t$ characterise the action of $C^{1,h}(S^1) \times \text{Diff}^h(S^1)$, or at least the smooth diffeomorphisms in $C^{1,h}(S^1) \times \text{Diff}^h(S^1)$.

**Proposition 6.2** Let $\psi$ be a $C^\infty$-diffeomorphism of $J$. Assume that

1. $\psi$ commutes with $\phi_t$,
2. $\psi$ preserves $\beta$,
3. $\psi$ preserves $\mathcal{F}$,

then $\psi = (f, \phi) \in C^{1,h}(S^1) \times \text{Diff}^h(S^1)$, where $f$ and $\phi$ are $C^\infty$.

In Corollary 8.7 we characterise $C^{1,h}(S^1) \times \text{Diff}^h(S^1)$.

**Proof:** We use the notations and assumptions of the Proposition. By (1) and (3), we obtain that $\psi$ preserves the fibres of $\pi$. Thus, there exists a $C^\infty$-diffeomorphism $\phi$ of $T^*S^1$ such that

$$\pi \circ \psi = \phi \circ \pi.$$  

Replacing $\psi$ by $\psi \circ (0, \phi^{-1})$, we may as well assume that $\phi = id$. By (1), $\psi$ also preserves the fibres of $\pi_2$.

We choose coordinate in $S_1$ and from now on, we use the identification $J = S^1 \times \mathbb{R} \times \mathbb{R}$ given in Paragraph 6.1.5. The previous discussion shows that

$$\psi(\theta, r, f) = (\theta, F(\theta, r), H(\theta, r, f)).$$

Since $\psi^* \beta = \beta$, we obtain that

$$dH - Fd\theta = df - rd\theta.$$  

Hence

$$\frac{\partial H}{\partial f} = 1, \quad F - \frac{\partial H}{\partial \theta} = r, \quad \frac{\partial H}{\partial r} = 0.$$  

Therefore, there exists a function $g$ such that $H = f + g(\theta)$. It follows that

$$\psi(\theta, r, f) = (\theta, r + \frac{\partial g}{\partial \theta}, f + g(\theta)).$$

This exactly means that

$$\psi(\theta, r, f) = (g, id) \cdot j^1(f).$$

Q.E.D.
6.3 \( PSL(2, \mathbb{R}) \) and \( C^\infty(S^1) \ltimes Diff^\infty(S^1) \)

We consider the 3-manifold \( \mathcal{J} = PSL(2, \mathbb{R}) \). It has a biinvariant Lorentz metric \( g \) coming from the Killing form. We introduce the following objects.

- Let \( \phi_t \) be the one-parameter group of diagonal matrices acting on the right on \( M \).
- Let \( X \) be the vector field generating \( \phi_t \).
- Let \( \mathcal{F} \) the foliation by the right action of the one-parameter group of upper triangular matrices \( B \),
- Let \( \beta = i_X g \).

Then

**Proposition 6.3** The one-form \( \beta \) is a contact form. Furthermore, there exists a \( C^\infty \)-diffeomorphism \( \Psi \) from \( \mathcal{J} \) to \( \mathcal{J} \) that sends \( (\phi_t, \mathcal{F}, \beta) \) to \( (\phi_t, \mathcal{F}, \beta) \) respectively. Moreover, the diffeomorphism \( \Psi \) is unique up to left composition by a \( C^\infty \)-diffeomorphism element of \( C^1, h(S^1) \ltimes Diff^h(S^1) \).

As an immediate application, we have

**Definition 6.4** By Propositions 6.2 and 6.3 and since \( PSL(2, \mathbb{R}) \) preserves \( \phi_t \), \( \mathcal{F} \) and \( \beta \), we have a group homomorphism

\[
\iota : \begin{cases}
PSL(2, \mathbb{R}) & \to C^\infty(S^1) \ltimes Diff^\infty(S^1) \\
g & \mapsto \Psi \circ g \circ \Psi^{-1},
\end{cases}
\]

well defined up to conjugation by a \( C^\infty \)-diffeomorphism element of \( C^{1, h}(S^1) \ltimes Diff^{h}(S^1) \). The corresponding representation is called standard.

**Proof :** It is immediate to check that

\[
PSL(2, \mathbb{R}) \to PSL(2, \mathbb{R})/\phi_t = \mathcal{W},
\]

is a \( \mathbb{R} \) principal bundle, whose connexion form is \( \beta \) and whose curvature is symplectic. It follows that \( \beta \) is a contact form.

Let \( \mathcal{F} \) be the projection

\[
\mathcal{W} \to \mathbb{R}P^1 = PSL(2, \mathbb{R})/B.
\]

We observe that \( \mathcal{W} \) is diffeomorphic to the annulus \( S^1 \times \mathbb{R} = T^*S^1 \) in such a way \( \mathcal{F} \) coincides with the projection \( \pi \) on the first factor. We may actually choose this diffeomorphism \( \psi_0 \) to be symplectic, still sending the fibres of \( \mathcal{F} \) to the fibres of \( \pi \).

Let now \( \xi \) be the symplectic diffeomorphism of \( T^*S^1 \) obtained by Proposition 6.1 applied to \( \alpha = (\psi_0^{-1})^*\beta \). It follows \( \Psi = \xi \circ \psi_0 \) has all the properties
required. Finally, by Proposition 6.2, \( \Psi \) is well defined up to multiplication with an element of \( C^\infty(S^1) \ltimes \text{Diff}^\infty(S^1) \).

Since the right action of \( \text{PSL}(2, \mathbb{R}) \) preserves \( \mathcal{F}, \, \bar{\beta}, \) and \( \bar{\phi}_t \), we obtain a representation of \( \text{PSL}(2, \mathbb{R}) \) well defined up to conjugation

\[
\begin{align*}
    \text{PSL}(2, \mathbb{R}) & \rightarrow C^\infty(S^1) \ltimes \text{Diff}^\infty(S^1) \\
    g & \mapsto \Psi \circ g \circ \Psi^{-1}
\end{align*}
\]

Q.E.D.

Remark: The action of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathbb{R}P^1 \) gives rise to an embedding of \( \text{PSL}(2, \mathbb{R}) \) in \( \text{Diff}^\infty(S^1) \). The above standard representation is a nontrivial extension of this representation. Indeed, the natural lift of the action of \( \text{Diff}^\infty(S^1) \) on \( T^*S^1 \) does not act transitively since it preserves the zero section. On the contrary, \( \text{PSL}(2, \mathbb{R}) \) does act transitively through the standard representation.

7 Homomorphisms of \( \pi_1(S) \) in \( C^{1,h}(S^1) \ltimes \text{Diff}^h(S^1) \)

Our aim in this section is to describe a "good" set of homomorphisms of \( \pi_1(S) \) in \( C^{1,h}(S^1) \ltimes \text{Diff}^h(S^1) \). For a semi-simple Lie group \( G \), a "good" set of homomorphisms of \( \pi_1(S) \) in \( G \) is the set of reductive homomorphisms (i.e. such that the Zariski closure of the image is reductive) it satisfies the following properties:

- it is open in the set of all homomorphisms,
- \( G \) acts properly on it.

As a consequence, the set of "good" representations, that is the set of "good" homomorphisms up to conjugacy, is a Hausdorff space.

We define "good" (actually \( \infty \)-Hitchin) homomorphisms in Paragraph 7.0.2 and show in Theorem 7.6 that they satisfy analogous properties to those of reductive homomorphisms in semi-simple Lie groups that we just described. We also show that to a \( \infty \)-Hitchin representation is associated a cross ratio. In Theorem 7.16, we finally explain how two representations with the same cross ratio are related.

7.0.1 Fuchsian and \( H \)-Fuchsian homomorphisms

Definition 7.1 [Fuchsian homomorphism] Let \( \rho \) be a Fuchsian homomorphism of \( \pi_1(S) \) in \( \text{PSL}(2, \mathbb{R}) \). We say that the composition of \( \rho \) by the standard representation \( \iota \) (cf Definition 6.4) is \( \infty \)-Fuchsian, or in short Fuchsian when there is no ambiguity.

Definition 7.2 [\( H \)-Fuchsian action on \( S^1 \)] We say an action of \( \pi_1(S) \) on \( S^1 \) is \( H \)-Fuchsian if it is an action by \( C^1 \)-diffeomorphisms with Hölder derivatives, Hölder conjugate to the action of a cocompact group of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathbb{R}P^1 \).
7.0.2 Anosov and $\infty$-Hitchin representations

We begin with a general definition.

**Definition 7.3** For a foliation $\mathcal{F}$ on a compact space, we denote by $d_{\mathcal{F}}$ a Riemannian distance along the leaves of $\mathcal{F}$ which comes from a leafwise continuous metric. In particular,

$$d_{\mathcal{F}}(x, y) < \infty \text{ iff } x \text{ and } y \text{ are in the same leaf.}$$

We say a flow $\phi_t$ contracts uniformly the leaves of a foliation $\mathcal{F}$, if $\phi_t$ preserve $\mathcal{F}$ and if moreover

$$\forall \epsilon > 0, \exists \alpha > 0, \exists t_0 : \forall t \geq t_0, d_{\mathcal{F}}(x, y) \leq \alpha \Rightarrow d_{\mathcal{F}}(\phi_t(x), \phi_t(y)) \leq \epsilon$$

This definition does not depend on the choice of $d_{\mathcal{F}}$.

We now use the notations of Section 6.1.

**Definition 7.4** [Anosov homomorphisms] A homomorphism $\rho$ of $\pi_1(S)$ in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ is Anosov if:

- $\rho(\pi_1(S))$ acts with a compact quotient on $J = J_1(S^1, \mathbb{R})$,
- The flow induced by the canonical flow $\phi_t$ on $J/\rho(\pi_1(S))$ contracts uniformly the leaves of $\mathcal{F}$ (cf. definition above)
- The induced action on $S^1$ is H-Fuchsian.

We denote the space of Anosov homomorphims by $\text{Hom}^*$.

**Remarks:**

1. If $\rho$ is Anosov, one should observe that in general $\rho(\pi_1(S))$ only acts by homeomorphisms on $J$. However, if we consider $J$ as a $C^\infty$-filtrated space (See the definitions in the Appendix), whose nested foliated structures are given by the fibres of the two projections $\pi_2$ and $\pi$, then $\rho(\pi_1(S))$ acts by $C^\infty$-laminated maps (i.e smoothly along the leaves with continuous derivatives). Therefore, $J/\rho(\pi_1(S))$ has the structure of a $C^\infty$-lamination such that the flow induced by $\phi_t$ is $C^\infty$-leafwise.

2. $\infty$-Fuchsian representations of $\pi_1(S)$ in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ provide our first examples of elements of Anosov homomorphisms.

**Definition 7.5** [$\infty$-Hitchin homomorphims] Let $\text{Hom}_H$ be the connected component of $\text{Hom}^*$ containing the Fuchsian homomorphisms. Elements of $\text{Hom}_H$ are called $\infty$-Hitchin homomorphisms.
The group $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ acts by conjugation on $\text{Hom}_H$. The canonical flow $\phi_t$ is in the center of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ and hence acts trivially on $\text{Hom}_H$. It follows that $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)/\phi_t$ acts on $\text{Hom}_H$.

The main result of this section is the following:

**Theorem 7.6** The set $\text{Hom}_H$ is open in the space of all homomorphisms. Furthermore, the action of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)/\phi_t$ by conjugation on $\text{Hom}_H$ is proper and the quotient is Haussdorff.

We prove later on a refinement (Theorem 7.16) of the second part of this result. We shall denote by $\text{Rep}_H$ the quotient $\text{Hom}_H/C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ where the right action is by conjugation.

The proof of the Theorem falls into two parts. We first show that $\text{Hom}_H$ is open. Then, we prove that $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ acts properly with a Haussdorff quotient. In this proofs, we use the definitions and propositions obtained in the Appendix A. Accordingly, from now on, we shall consider $J$ as a $C^\infty$-filtrated space, whose nested foliated structures are given by the fibres of the two projections $\pi_2$ and $\pi$.

### 7.1 Openness of $\text{Hom}^*$ and a Stability Lemma

Let $\rho$ be a representation in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$. We denote by $\overline{\rho}$ the associated representation in $\text{Diff}^h(S^1)$.

The following Stability Lemma implies immediately the openness of $\text{Hom}^*$.

Moreover, using corollaries of this Lemma, we associate to every $\infty$-Hitchin representation a cross ratio and a spectrum in Paragraph 7.1.3 and 7.1.4.

**Lemma 7.7** Let $\rho_0$ be an Anosov representation. Then for $\rho$ close enough to $\rho_0$, there exists a Hölder homeomorphism $\Phi$ of $J$ close to the identity which is a $C^\infty$-filtrated immersion as well as its inverse such that

$$\forall \gamma \in \pi_1(S), \rho_0(\gamma) = \Phi^{-1} \circ \rho(\gamma) \circ \Phi.$$ 

### 7.1.1 Minimal action on the circle

We prove the following preliminary Lemma which is independent of the rest of the article.

**Lemma 7.8** Let $\Gamma$ be a group acting on $S^1$ by homeomorphisms. Suppose that every orbit of the action on $S^1 \times S^1$ is dense. Suppose that every element of $\Gamma$ different from the identity has exactly two fixed points, one attractive the other repulsive. Let $\rho$ be a representation of $\Gamma$ in the group $\text{Homeo}(S^1)$ of homeomorphisms of $S^1$. Let $f$ be a continuous map of degree different from zero, from $S^1$ to $S^1$ such that

$$\forall \gamma \in \Gamma, f \circ \gamma = \rho(\gamma) \circ f.$$ 

Then $f$ is a homeomorphism.
Proof: Since \( f \) has nonzero degree, it is surjective.

We now prove \( f \) is injective. Let \( a \) and \( b \) two points of \( S^1 \) such that \( a \neq b \) and \( f(a) = f(b) = c \). Let \( I \) and \( J \) be the connected components of \( S^1 \setminus \{a, b\} \).

Since \( f \) has a nonzero degree, either \( S^1 \setminus \{c\} \subset f(I) \) or \( S^1 \setminus \{c\} \subset f(J) \). Assume \( S^1 \setminus \{c\} \subset f(I) \). By the density of orbits on \( S^1 \times S^1 \), we can find an element \( \gamma \) in \( \Gamma \) such that \( \gamma^+ \in I \) and \( \gamma^- \in J \).

We observe that for all \( n \), \( f(\gamma^n(I)) = S^1 \setminus \{\rho(\gamma^n(c))\} \). It follows we can find \( \epsilon > 0 \), and two sequences \( \{x_n\}_{n \in \mathbb{N}}, \{x_n\}_{n \in \mathbb{N}} \) such that

- \( x_n \in \gamma^n(I), y_n \in \gamma^n(I) \),
- \( d(f(x_n), f(y_n)) > \epsilon \).

The contradiction follows since

\[ \lim_{n \to \infty} d(x_n, y_n) = 0. \]

Q.E.D.

7.1.2 Proof of Lemma 7.7

Proof: Let \( \rho_0 \) be an element of \( \text{Hom}_H \).

\[ \rho_0 : \pi_1(S) \to C^{1,h}(S^1) \times \text{Diff}^h(S^1). \]

By assumption, \( P = \rho_0(\pi_1(S)) \setminus J \) is compact. The topological space \( P \) is a \( C^0 \)-manifold which is \( C^\infty \)-laminated by the fibres of the projection \( \pi : J \to S^1 \) and \( \pi_2 : J \to T^*S^1 \), since \( C^{1,h}(S^1) \times \text{Diff}^h(S^1) \) acts as \( C^\infty \)-laminated maps on \( J \). The covering \( J \to P \) is a Galois covering.

Let now \( \rho \) be a representation of \( \pi_1(S) \) close enough to \( \rho_0 \). Let \( \Gamma = \rho_0(\pi_1(S)) \). According to Theorem 11.10, we obtain a \( \rho \circ \rho_0^{-1} \)-equivariant \( C^\infty \)-laminated immersion \( \Phi \) from \( J \) to \( J \), arbitrarily close to the identity on compact sets. It remains to show that \( \Phi \) is a homeomorphism.

Since \( \Phi \) is a laminated map, we obtain a H"older map \( f \) close to the identity from \( S^1 \) to \( S^1 \) such that

\[ \pi \circ \Phi = f \circ \pi, \]

and

\[ \forall \gamma \in \Gamma, f \circ \rho_0(\gamma) = \rho(\gamma) \circ f \]

We recall that \( \rho_0 \) is Anosov, therefore \( \rho_0(\gamma) \) is a \( H \)-Fuchsian representation, and satisfies the hypothesis of Lemma 7.8. By this Lemma, and since \( f \) being close to the identity is not of degree 0, we obtain that \( f \) is a homeomorphism. Thus, \( \rho_1 \) is also \( H \)-Fuchsian.

This laminated equivariant immersion \( \Phi \) induces an affine structure on the leaves of \( P \), according to the definition of Paragraph 11.11. By compactness of \( P \), we deduce there exists an action of a one-parameter group \( \psi_t \) such that, on \( J \)

\[ \Phi \circ \psi_t = \phi_t \circ \Phi, \]
where \( \phi_t \) is the canonical flow. We observe that \( \psi_t \) preserves any given leaf and acts as a one-parameter group of translation on it.

Recall that the canonical flow contracts uniformly the leaves of \( \mathcal{F} \) for on \( J/\rho_0(\pi_1(S)) \). It follows the same holds for \( \psi_t \), for \( \Phi \) close enough to the identity.

By Lemma \( \text{(11.7)} \), we obtain that the affine structure induced by \( \phi_t \) is leaf-wise complete. In particular, for every \( x \) in \( S^1 \), \( \Phi \) is an affine bijection from \( \pi^{-1}\{x\} \) to \( \pi^{-1}\{f(x)\} \). Since \( f \) is a homeomorphism, we deduce that \( \Phi \) itself is a homeomorphism, its inverse being also a filtrated immersion.

This concludes the proof. Q.E.D.

We have the following immediate Corollary,

**Corollary 7.9** Let \( \rho_0 \) and \( \rho_1 \) be two elements of \( \text{Hom}_H \). Then there exists a Hölder homeomorphism \( \phi \) of \( J \) which is a filtrated immersion as well as its inverse such that

\[
\forall \gamma \in \pi_1(S), \rho_0(\gamma) = \phi^{-1} \circ \rho_1(\gamma) \circ \phi.
\]

Recall that \( C^{1,h}(S^1) \times \text{Diff}^h(S^1) \) acts on \( T^*S^1 = J/\phi_t \). We finally prove

**Proposition 7.10** Let \( \rho \) be an element in \( \text{Hom}_H \). Then there exists a unique \( \rho(\pi_1(S))-\text{equivariant} \) Hölder homeomorphism \( \Theta_\rho \) from \( T^*S^1 \) to \( \partial_\infty \pi_1(S)^{2*} = \partial_\infty \pi_1(S)^2 \setminus \{(x,x) / x \in \partial_\infty \pi_1(S)\} \)

such that

\[
\exists f : S^1 \to \partial_\infty \pi_1(S), \Theta_\rho(T^*_xS^1) \subset \{f(x)\} \times \partial_\infty \pi_1(S).
\]

**Proof:** By Corollary \( \text{(7.9)} \) it suffices to show this result for the action of a cocompact subgroup of \( \text{PSL}(2, \mathbb{R}) \). In this case the statement is obvious. Q.E.D.

### 7.1.3 Cross ratio associated to elements of \( \text{Hom}_H \)

We show that we can associate a cross ratio to every element \( \rho \) of \( \text{Hom}_H \).

Let \((a, b, c)\) be a triple of distinct elements of \( \partial_\infty \pi_1(S) \). We define \([b, c]_a\) to be the closure of the connected component of \( \partial_\infty \pi_1(S) \setminus \{b, c\} \) not containing \( a \). Let \( q = (x, y, z, t) \) be a quadruple of elements of \( \partial_\infty \pi_1(S) \), we define \( \widehat{q} \) to be following closed curve, embedded in \( \partial_\infty \pi_1(S)^{2*} \),

\[
\widehat{q} = (\{x\} \cup \{y\} \cup \{z\} \cup \{t\}) \cup (\{x\} \cup \{z\} \cup \{y\}) \cup (\{y\} \cup \{z\} \cup \{x\})
\]

We choose the orientation on \( \widehat{q} \) such that \((x, y), (x, t), (z, t), (z, y)\) are cyclically oriented. We define \( \epsilon_q \in \{-1, 1\} \) to be the sign of the classical cross ratio of \( q \) on \( \partial_\infty \pi_1(S) \).

Let \( \Theta_\rho \) be the map from \( T^*S^1 \) to \( \partial_\infty \pi_1(S)^{2*} \) as defined in Proposition \( \text{(7.10)} \).
Definition 7.11 [Associated cross ratio] We define the associated cross ratio to $\rho$ by

$$\hat{b}_\rho(x, y, z, t) = \epsilon_q \exp\left(\frac{1}{2} \int_{\Theta}^1 \rho_q^{-1}(\tilde{q}) r d\theta\right).$$

Remark: We observe that the cross ratio just depends on the action of $\pi_1(S)$ on $T^*S^1$. Here is another formulation of this observation. Let $\Omega^h(S^1)\ltimes \text{Diff}^h(S^1)$ be the space of H"older one-form on $S^1$. Note that $\Omega^h(S^1)\ltimes \text{Diff}^h(S^1)$ naturally acts on $T^*S^1$. We also have a natural homorphism

$$d: \left\{\begin{array}{c} C^1(S^1) \ltimes \text{Diff}^h(S^1) \to \Omega^h(S^1) \ltimes \text{Diff}^h(S^1) \\ (f, \phi) \mapsto (df, \phi) \end{array}\right.,$$

whose kernel is the canonical flow. Then two representations $\rho_1$ and $\rho_2$ such that $d\rho_1 = d\rho_2$ have the same associated cross ratio. In other words, the cross ratio only depends on the representation as with values in $\Omega^h(S^1)\ltimes \text{Diff}^h(S^1)$.

7.1.4 Spectrum

Definition 7.12 [$\rho$-length] Let $\rho$ be a representation in $\text{Hom}_1$. Let $\gamma$ be an element in $\pi_1(S)$, the $\rho$-length of $\gamma$, $l_\rho(\gamma)$ is the positive number $t$ such that

$$\exists u \in J, \phi_t(u) = \rho(\gamma)u.$$ 

We observe that the existence and uniqueness of such a number $t$ is follows by Corollary (7.9) and the description of standard representations.

In other words, $l_\rho(\gamma)$ is the length of the periodic orbit of $\phi_t$ in $J/\rho(\pi_1(S))$ freely homotopic to $\gamma$. It is clear that $l_\rho(\gamma)$ just depends on the conjugacy class of $\gamma$.

Definition 7.13 [Spectrum] The marked spectrum of $\rho$ is the map

$$l_\rho: \gamma \mapsto l_\rho(\gamma).$$

We say a representation is symmetric if $l_\rho(\gamma) = l_\rho(\gamma^{-1})$.

7.1.5 Compatible representations and Ghys deformations

Definition 7.14 [Compatible representation] We say an element $\rho$ is compatible if its marked spectrum coincides with the periods of the associated cross ratio.

We observe that every compatible representation is symmetric. Conversely, we prove in Proposition [1,4] that if $b_\rho$ is the period of the cross ratio associated to the representation $\rho$, then

$$l_{b_\rho}(\gamma) = \frac{1}{2}(l_\rho(\gamma) + l_\rho(\gamma^{-1})).$$
Consequently, every symmetric representation is compatible.

We shall later obtain lot’s of compatible representations.

**Definition 7.15 [Ghys deformation]** If $\rho$ is a compatible representation, if $\omega$ is a small enough nontrivial element in $H^1(\Gamma, \mathbb{R})$, then $\rho^\omega$ defined by

$$\rho^\omega(\gamma) = e^{\omega(\gamma)} \cdot \rho(\gamma).$$

is an element of $\text{Hom}_H$. We call such a representation $\rho^\omega$ a Ghys deformation of $\rho$.

We observe that $\rho^\omega$ and $\rho$ have the same associated cross ratio since they have the same action on $T^*S^1$. However

$$l_{\rho^\omega}(\gamma) = l_{\rho}(\gamma) + \omega(\gamma).$$

It follows that two representations can have the same cross ratio but different marked spectrum.

### 7.2 Action of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ on $\text{Hom}_H$

We prove in this section that $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)/\phi_1$ acts properly on $\text{Hom}_H$ and more precisely

**Theorem 7.16** The group $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)/\phi_1$ acts properly on $\text{Hom}_H$ with a Haussdorf quotient.

Moreover, if two representations $\rho_0$ and $\rho_1$ elements of $\text{Hom}_H$ have the same associated cross ratio, then there exists $\omega \in H^1(\pi_1(S))$ such that $\rho_0$ and $\rho_1^\omega$ are conjugated by some element in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$.

Consequently two representations with the same cross ratio and the same spectrum are conjugated in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$.

**Remark:** We shall see later that two representations with the same spectrum are not necessarily conjugated

#### 7.2.1 A preliminary lemma

Let $\Omega^h(S^1)$ be the space of Hölder one-form on $S^1$. We observe first that $\Omega^h(S^1) \rtimes \text{Diff}^h(S^1)$ acts naturally on $T^*S^1$ and preserves the area.

**Lemma 7.17** Let $G$ be an area preserving homeomorphism of $T^*S^1$. Assume there exists a homeomorphism $f$ such that

$$G(T_xS^1) \subset T_{f(x)}S^1.$$ 

Then $G$ belongs to $\Omega^h(S^1) \rtimes \text{Diff}^h(S^1)$.
Proof : We use the coordinates \((r, \theta)\) on \(T^*S^1\). By hypothesis,
\[
G(r, \theta) = (g(r, \theta), f(\theta)).
\]
Since \(G\) preserves the area, we obtain
\[
(r_0 - r_1)(\theta_0 - \theta_1) = \int_{f(\theta_0)}^{f(\theta_1)} \int_{g(r_0, \theta)}^{g(r_1, \theta)} dr d\theta 
= \int_{f(\theta_0)}^{f(\theta_1)} (g(r_0, \theta) - g(r_1, \theta)) d\theta
\]  
(23)
Hence \(g(r, \theta)\) is affine in \(r\):
\[
g(r, \theta) = \omega(\theta) + r\beta(\theta),
\]
where \(\omega(\theta)\) and \(\beta\) are Hölder. Since \(G\) is a homeomorphism, we observe that for all \(\theta\), \(\beta(\theta) \neq 0\). By Equation (22) we obtain that
\[
f^{-1}(\theta_0) - f^{-1}(\theta) = \int_{\theta_0}^{\theta_1} \beta(\theta) d\theta.
\]
It follows that \(f^{-1}\) is in \(C^{1, \text{h}}(S^1)\) with \(df^{-1} = \beta\). Since \(\beta\) never vanishes, \(f\) is actually a diffeomorphism, and \(G\) belongs to \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\). Q.E.D.

7.2.2 Conjugation in \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\)

Let \(\rho_0\) and \(\rho_1\) be two representations of \(\pi_1(S)\) in \(C^{1, \text{h}}(S^1) \times \text{Diff}^h(S^1)\) with same associated cross ratio.

Let \(\Omega^h(S^1)\) be the space of Hölder one-form on \(S^1\). We recall that \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\) acts naturally on \(T^*S^1\) and that \(C^{1, \text{h}}(S^1) \times \text{Diff}^h(S^1)\) naturally maps to \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\). We first prove

**Proposition 7.18** Let \(\rho_0\) and \(\rho_1\) be two representations of \(\pi_1(S)\) in \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\) with same associated cross ratio. Then there exists a unique element \(G = G^{\rho_0, \rho_1}\) in \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\) such that
\[
\forall \gamma \in \pi_1(S), \forall y \in T^*S^1, \quad G(\rho_0(\gamma)(y)) = \rho_1(\gamma)(G(y)).
\]
Moreover, for any representation \(\rho\), for any neighbourhood \(V\) of the identity in \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\), there exists a neighbourhood \(U\) of \(\rho\), such that if two representations \(\rho_1\) and \(\rho_0\) in \(U\) have the same cross ratio, then \(G^{\rho_0, \rho_1} \in V\).

Proof : Applying Proposition 7.10 twice, we obtain a unique Hölder homeomorphism \(G\) of \(T^*S^1\) and a homeomorphism \(f\) of \(S^1\) such that
\[
\forall \gamma \in \pi_1(S), \quad G \circ \rho_0(\gamma) = \rho_1(\gamma) \circ G,
\]
\[
G(T_{f(x)}S^1) \subset T_{f(x)}S^1.
\]
If \(\rho_0\) and \(\rho_1\) have the same associated cross ratio, then \(G\) preserves the area. By Lemma 7.14, \(G\) belongs to \(\Omega^h(S^1) \times \text{Diff}^h(S^1)\). This concludes the proof of the first part of the Proposition. The second part follows from Lemma 7.16.

29
7.2.3 Second step: from $\Omega^h(S^1) \times Diff^h(S^1)$ to $C^{1,h}(S^1) \times Diff^h(S^1)$

We prove the following Lemma

**Lemma 7.19** Let $\phi: \gamma \to \phi_{\gamma}$ be a representation of $\pi_1(S)$ in $Diff^h(S^1)$ with nonzero Euler class. Let $\alpha$ a a continuous one-form. Let

$$
\begin{align*}
\pi_1(S) & \to C^1(S^1) \\
\gamma & \mapsto f_\gamma
\end{align*}
$$

Assume that

$$
\begin{align*}
\phi^*_\gamma(f_\eta) + f_\gamma &= f_{\eta \gamma}, \\
\phi^*_\gamma(\alpha) - \alpha &= df_\gamma.
\end{align*}
$$

Then $\alpha$ is exact.

**Proof:** Let 

$$
\kappa = \int_{S^1} \alpha.
$$

We write $S^1 = \mathbb{R}/\mathbb{Z}$. We lift all our data to $\mathbb{R}$, and use $\tilde{u}$ to describe a lift of $u$. In particular, we have

$$
\tilde{\phi}_{\gamma \eta}(x) = \tilde{\phi}_{\gamma} \circ \tilde{\phi}_{\eta}(x) + c(\gamma, \eta) \in \mathbb{Z},
$$

where

$$
c \in H^2(\pi_1(S), \mathbb{Z}),
$$

is a representative of the Euler class of the representation $\phi$. We now write $\tilde{\alpha} = dh$. We observe that

$$
\forall m \in \mathbb{Z}, h(x + m) = h(x) + mk.
$$

Let

$$
\eta: \pi_1(S) \to \mathbb{R},
$$

be the function such that

$$
\tilde{\phi}^*_\eta(h) - h = \tilde{f}_\gamma + \eta(\gamma).
$$

We obtain

$$
c(\gamma, \eta)\kappa = h \circ \tilde{\phi}_{\gamma \eta} - h \circ \tilde{\phi}_{\gamma} \circ \phi_{\eta}
$$

$$
= \tilde{\phi}^*_\gamma(h) - \tilde{\phi}^*_\eta \tilde{\phi}^*_\gamma(h)
$$

$$
= (\tilde{\phi}_{\gamma \eta}(h) - h) - (\tilde{\phi}_{\eta \gamma}(h) - \tilde{\phi}^*_\gamma(h)) - (\tilde{\phi}^*_\eta(h) - h)
$$

$$
= \tilde{f}_{\gamma \eta} - \tilde{f}_\eta + \eta(\gamma) - \eta(\eta).
$$

But, since the Euler class is nonzero in cohomology by hypothesis, we obtain that $\kappa = 0$, hence $\alpha$ is exact. Q.E.D.
7.2.4 Proof of Theorem 7.16

We first prove that if two representations have the same cross ratio then, after a Ghys deformation, they are conjugate by an element of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$.

Let $\rho_0$ and $\rho_1$ be two elements of $\text{Hom}_H$ with the same cross ratio. Write $\rho_i(\gamma) = (f_i, \phi_i(\gamma))$.

By Proposition 7.18, we know that, as representations in $\Omega^h(S^1) \rtimes \text{Diff}^h(S^1)$ they are conjugated in $\Omega^h(S^1) \rtimes \text{Diff}^h(S^1)$, by some element $G = (\alpha, F)$, where $F \in \text{Diff}^h(S^1)$.

In particular, after conjugating by $(0, F) \in C^{1,h}(S^1)$, we may assume that $\phi_0^* \gamma = \phi_1^* \gamma := \phi_\gamma$.

Since $\rho_i$ is a representation

$\phi_i^*(f_i) + f_\gamma = f_i^\gamma$.

By Proposition 7.18 there exists $\alpha \in \Omega^h(S^1)$ such that

$\phi_i^* \alpha - \alpha = df_i^\gamma - df_0^\gamma$.

Applying Lemma 7.19 to $f_i^\gamma = f_1^\gamma - f_0^\gamma$, we obtain that $\alpha = dg$.

Conjugating by $(g, \text{Id})$, we may now as well assume that $\alpha = 0$. It follows that there exists $\omega : \pi_1(S) \to \mathbb{R}$, such that

$f_1^\gamma - f_0^\gamma = \omega(\gamma)$.

This exactly means that $\rho_0 = \rho_1^\gamma$.

This proves that two representations with the same cross ratio are conjugated after a Ghys deformation. Consequently two representations with the same cross ratio and spectrum are conjugated.

Finally, the proof of properness goes as follows. Suppose that we have a sequence of representation $\{\rho_n\}_{n \in \mathbb{N}}$ converging to $\rho_0$, a sequence of elements $\{\psi_n\}_{n \in \mathbb{N}}$ in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ such that $\{\psi_n^{-1} \circ \rho_n \circ \psi_n\}_{n \in \mathbb{N}}$ converges to $\rho_1$. Let $d$ be the homomorphism $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1) \to \Omega^h(S^1) \rtimes \text{Diff}^h(S^1)$. We aim to prove that $\{d\psi_n\}_{n \in \mathbb{N}}$ converges.

Since $\rho_n$ and $\psi_n^{-1} \circ \rho_n \circ \psi_n$ have the same spectrum and cross ratio, it follows that $\rho_1$ and $\rho_0$ have the same spectrum and cross ratio. From the first part of the proof it follows that there exists $F$ such that

$\rho_0 = F^{-1} \circ \rho_1 \circ F$.

We may therefore as well assume that $\rho_0 = \rho_1$.

From Proposition 7.18 $\{d\psi_n\}_{n \in \mathbb{N}}$ converges to the identity. Q.E.D.
8 A Conjugation Theorem

We will work in the following setting.

1. Let $\kappa$ be a Hölder homeomorphism of $S^1$. We write $D_\kappa = S^1 \times S^1 \setminus \{(s,t)/\kappa(s) = t\}$.

2. Let $p : M \to D_\kappa$ be a principal $\mathbb{R}$-bundle over $D_\kappa$ equipped with a connection $\nabla$.

We define

- Let $\mathcal{L}$ be the foliation of $M$ by parallel vector fields along the curves $c_s : t \mapsto (s,t)$.

We suppose

- Let $\omega$ be the curvature $\omega$ of $\nabla$. Let $f$ be such that $\omega = f(s,t)ds \wedge dt$. We suppose that $f$ is positive and Hölder.

In this section, we aim to prove the following Theorem.

**Theorem 8.1** Assume that $\pi_1(S)$ acts on $M$ by $C^1$-diffeomorphisms with Hölder derivatives. We suppose that this action preserves $\nabla$, and that

1. $M/\pi_1(S)$ is compact.
2. The action of $\mathbb{R}$ on $M/\pi_1(S)$ contracts the leaves of $\mathcal{L}$ (cf definition 7.3).
3. There exists two $H$-Fuchsian representations $\rho_1$ and $\rho_2$ of $\pi_1(S)$, with $\rho_2 = \kappa^{-1}\rho\kappa$ such that

$$\forall \gamma \in \pi_1(S), \quad p(\gamma u) = (\rho_1(\gamma)(p(u)), \rho_2(\gamma)(p(u))).$$

Then there exists homeomorphism $\hat{\psi}$ from $M$ to $J$, unique up to right composition by an element of $C^{1,h}(S^1) \times \text{Diff}^h(S^1)$ an area preserving homeomorphism $\psi$ from $D$ to $T^*S^1$ such that

- $\pi_2 \circ \hat{\psi} = \psi \circ p$,
- $\hat{\psi}$ commutes with the $\mathbb{R}$-action,
- $\hat{\psi}$ sends $\mathcal{L}$ to $\mathcal{F}$,

and a representation $\rho$ of $\pi_1(S)$ in $C^{1,h}(S^1) \times \text{Diff}^h(S^1)$ such that

$$\hat{\psi} \circ \gamma = \rho(\gamma) \circ \hat{\psi}.$$ 

In particular, $\rho$ is Anosov.

Finally, assume that our data $\kappa$, $\nabla$ depends continuously on a parameter. Then, we can choose $\hat{\psi}$ to depend continuously on this parameter.
In the first three sections we give a geometric description of elements of $C^{1,h}(S^1) \ltimes \text{Diff}^h(S^1)$ by their action on $J$. Then, we conclude the proof in the last two sections.

To simplify the notations, we fix a trivialisation of $M = \mathbb{R} \times D_\kappa$ so that the vector fields $U_{(s,u)} : t \mapsto (u,s,t)$ are parallel along $c_s$. Let

$$
\begin{align*}
\pi & : \left\{ \\
& \begin{array}{ccc}
M & \rightarrow & S^1 \\
(u,s,t) & \mapsto & t,
\end{array} \\
\pi_1 & : \left\{ \\
& \begin{array}{ccc}
M & \rightarrow & \mathbb{R} \times S^1 \\
(u,s,t) & \mapsto & (u,s),
\end{array} \\
p & : \left\{ \\
& \begin{array}{ccc}
M & \rightarrow & D_\kappa \\
(u,s,t) & \mapsto & (s,t).
\end{array}
\end{align*}
$$

In particular, the foliation $\mathcal{L}$ is the foliation by the fibres of $\pi_1$.

For later use, using the above notations, we state the following obvious result which allows us to restate Condition 2 of Theorem 8.1.

**Proposition 8.2** Suppose $M/\pi_1(S)$ is compact. Then the following conditions (2) and (2') are equivalent

1. The action of $\mathbb{R}$ on $M/\pi_1(S)$ contracts the leaves of $\mathcal{L}$.
2. Let $\{t\}_{m \in \mathbb{N}}$ be a sequence of real numbers going to $+\infty$. Let $\{\gamma\}_{m \in \mathbb{N}}$ be a sequence of elements of $\pi_1(S)$. Let $(u,s,t)$ be an element of $M$ such that $\{\gamma_m(u + t_m, s, t)\}_{m \in \mathbb{N}}$ converges to $(u_0, s_0, t_0)$ in $M$, then for all $w$ in $S^1$, with $w \neq \kappa(s)$, then $\{\gamma_m(s, w)\}_{m \in \mathbb{N}}$ converges to $(s_0, t_0)$.

8.1 $\pi$-exact symplectic Homeomorphisms of the Annulus

We aim to describe in a geometric way the group $C^{1,h}(S^1) \ltimes \text{Diff}^h(S^1)$ generalising Proposition 6.3. Roughly speaking, we will describe it as a subgroup of the central extension of exact symplectic "homeomorphisms" of the Annulus (i.e. $T^*S^1$). However, the notion of exact symplectic homeomorphism does not make sense in an obvious way, but some homeomorphisms may be coined as "exact symplectic" as we shall see.

8.1.1 Line bundle

We consider the real vector line bundle $p : L \mapsto \mathbb{T}^*S^1$ equipped with a connection $\nabla$ of curvature $\omega$. If $\beta$ is the Louville form, we identify $L$ with $T^*S^1 \times \mathbb{R}$, with the connection

$$
\nabla = D + \beta.
$$
It is well known, and we shall recall the construction in Section 8.2, that the action of every Hamiltonian diffeomorphism of $T^*S^1$ lifts to a connection preserving action on $L$. This lift is defined up to an homothety on $L$.

Let $J \to T^*S^1$ be the frame bundle of $L$. It is by construction a 3-dimensional contact manifold, with contact form $\beta$ and equipped with an $\mathbb{R}$ action whose flow is a Reeb flow of the contact form.

From the discussion of Paragraph 6.1, we conclude that $J$ is identified with $J^1(S^1, \mathbb{R})$ with its contact form and $\mathbb{R}$-action.

### 8.1.2 Towards exact symplectic homeomorphisms

The aim of the following sections is

- to recall basic facts about symplectic and Hamiltonian actions on the Annulus, and in particular that every exact symplectic action lift to a connexion preserving action on $L$,

- to extend this construction to a situation with less regularity, and in particular to give a "symplectic" interpretation of the action of $C^{1,h}(S^1) \ltimes Diff^h(S^1)$, on $T^*S^1$.

### 8.2 Exact symplectomorphisms

In this section, we recall basic facts about symplectic diffeomorphisms. Let $(M, \omega)$ be a symplectic manifold, such that $\omega$ is the curvature of an orientable real line bundle $L$ equipped with a connection $\nabla$. For any curve $\gamma$ joining $x$ and $y$ we denote by $Hol(\gamma) : L_x \to L_y$ the holonomy of the connection $\nabla$ along $\gamma$. If $\gamma$ is a closed curve (i.e: $x = y$), we identify $GL(L_x)$ with $\mathbb{R}$ and consider $Hol(\gamma)$ as a real number.

Let $\sigma$ be a nonzero section of $L$. Let $\beta$ be the primitive of $\omega$ defined by

$$\nabla_X \sigma = \beta(X) \sigma.$$ 

We observe that if $\gamma$ is a smooth closed curve, then

$$Hol(\gamma) = e^{\int_\gamma \beta}.$$ 

### 8.2.1 Exact symplectic diffeomorphisms

A symplectic diffeomorphism is of $M$ is exact if for any closed curve

$$Hol(\sigma) = Hol(\phi(\sigma)).$$

(25)

The following Proposition clarifies this last condition

**Proposition 8.3** Let $\phi$ be a symplectic diffeormorphism. The following conditions are equivalent
• for any curve \( \sigma \), \( \text{Hol}(\sigma) = \text{Hol}(\phi(\sigma)) \).
• \( \phi^* \beta - \beta \) is exact,
• The action of \( \phi \) lifts to a connection preserving action on \( L \).

Furthermore if \( M = T^*S^1 \) and \( \sigma_0 \) is the zero section of \( T^*S^1 \to S^1 \), then the map

\[
\gamma : \begin{cases} H &\to \mathbb{R} \\ \phi &\mapsto \text{Hol}(\phi(\sigma_0)) \end{cases}
\]

is a group homomorphism, whose kernel is the group of exact diffeomorphisms.

**Proof:** The only point which is not obvious is the fact that for an exact symplectic diffeomorphism \( \phi \), the action lift to an action on the line bundle \( L \).

We define a lift \( \hat{\phi} \) in the following way. We first choose a base point \( x_0 \) in \( T^*S^1 \) and \( \eta \) a curve joining \( x_0 \) to \( \phi(x_0) \). For any point \( x \) in \( T^*S^1 \), we choose a curve \( \gamma \) joining \( x_0 \) to \( x \), then we define

\[
\hat{\phi}_x = \text{Hol}(\phi(\gamma))\text{Hol}(\eta)\text{Hol}(\gamma)^{-1} : L_x \to L_{\phi(x)}.
\]

Since for every closed curve

\[
\text{Hol}(\sigma) = \text{Hol}(\phi(\sigma)),
\]

the linear map \( \hat{\phi}_x \) is independent of the choice of \( \gamma \). Finally we define

\[
\hat{\phi}(x,u) = (\phi(x), \hat{\phi}_x u).
\]

By construction, \( \hat{\phi} \) preserve the parallel transport along any curve \( \gamma \), that is

\[
\hat{\phi}(\text{Hol}(\gamma)u) = \text{Hol}(\phi(\gamma))\hat{\phi}(u).
\]

Q.E.D.

### 8.3 \( \pi \)-Symplectic Homeomorphism

The action of \( \text{Diff}(S^1) \) on \( S^1 \) lifts to an action by area preserving homeomorphisms on \( T^*S^1 = T^*S^1 \). On the other hand \( \Omega^1(S^1) \), the vector space of continuous one-forms on \( S^1 \) also acts on \( T^*S^1 \) by area preserving homeomorphism in the following way

\[
fd\theta(\theta,t) = (\theta, t + f(\theta)).
\]

We observe that \( \text{Diff}(S^1) \) normalizes this action.

We define the group of \( \pi \)-symplectic homeomorphisms to be the group \( \Omega^1(S^1) \rtimes \text{Diff}(S^1) \). Here is an immediate characterisation of \( \pi \)-symplectic homeomorphisms whose proof is identical as that of Lemma 7.17.

**Proposition 8.4** An area preserving homeomorphism \( \phi \) of \( T^*S^1 \) is \( \pi \)-symplectic, if there is \( \psi \) in \( \text{Diff}(S^1) \), such that

\[
\pi \circ \phi = \psi \circ \pi.
\]
8.3.1 \( \pi \)-curves, holonomy of \( \pi \)-curves

A \( \pi \)-curve is a continuous curve \( c = (c_\theta, c_r) \) with values in \( T^*S^1 \) such that \( c_\theta \) is \( C^1 \). We define

\[
\int_c \beta = \int c_r dc_\theta,
\]
and

\[
\text{Hol}(c) = e^{\int_c \beta}.
\]

We collect in the following Proposition a few elementary facts.

**Proposition 8.5**

1. The image of a \( \pi \)-curve by a \( \pi \)-symplectic homeomorphism is a \( \pi \)-curve,

2. Let \( \Phi = (\alpha, \phi) \) be a \( \pi \)-symplectic homeomorphism. Let \( c \) be a closed \( \pi \)-curve, then

\[
\text{Hol}(\Phi(c)) = \text{Hol}(c)e^{\int_{S^1} \alpha}.
\]

8.3.2 \( \pi \)-exact symplectomorphism

We finally define the group of \( \pi \)-exact symplectomorphism to be the group

\[
\text{Ex}^\pi = (C^1(S^1)/\mathbb{R}) \rtimes \text{Diff}(S^1),
\]
where \( C^1(S^1)/\mathbb{R} \) is identified with the space of continuous exact one-forms on \( S^1 \). Similarly, we finally the group of \( \pi \)-Hamiltonian to be the group

\[
\text{Ham}^\pi = (C^1(S^1)/\mathbb{R}) \rtimes \text{Diff}^+(S^1),
\]

By the above remarks, we deduce immediately, reproducing the proof of Proposition 8.5

**Proposition 8.6**

1. The group of \( \pi \)-exact symplectomorphisms is the group of \( \pi \)-symplectic homomorphisms \( \phi \) such that for any closed \( \pi \)-curve \( c \),

\[
\text{Hol}(c) = \text{Hol}(\phi(c)).
\]

2. The action of any \( \pi \)-exact symplectomorphism \( \Phi \) lift to an action of a homeomorphism \( \hat{\Phi} \) on \( L \) such that for any \( \pi \)-curve

\[
\hat{\Phi}(\text{Hol}(c)) = \text{Hol}(\phi(c))\hat{\Phi}.
\]

3. Furthermore, \( \hat{\Phi} \) is determined by Equation (26) up to a multiplicative constant. As a consequence there exists a element \((f, \phi) \) of \( C^1(S^1) \times \text{Diff}(S^1) \) such that the induced action of \( \hat{\Phi} \) on \( J \), identified with the frame bundle of \( L \) is \((f, \phi) \). We observe that \( f \) is defined uniquely up to an additive constant.

4. Finally, if \( \Phi \) is Hölder, so is \( \hat{\Phi} \).
From this we obtain the following characterisation of $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ which generalises Proposition 6.2.

**Corollary 8.7** A Hölder homeomorphism of $J$ belongs to $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ if and only if

1. it preserves $\mathcal{F}$,
2. it commutes with the canonical flow,
3. it is above a $\pi$-exact symplectomorphism of $T^*S^1$.

**8.3.3 Width**

We generalise the notion of width for a $\pi$-exact symplectomorphism $\phi$ in the following obvious way.

**Definition 8.8** [Width, Action difference] Let $x$ and $y$ two fixed points of $\phi$. Let $c$ be a $\pi$-curve joining $x$ to $y$. We define the action difference of $x$ and $y$ by

$$\delta(\phi; x, y) = \text{Hol}(c \cup \phi(c)) = e^{l_{\phi}(c)\beta - l_{\phi}(c)\beta}.$$ 

By Proposition 8.6 it follows that this quantity does not depend on $c$. Furthermore

$$\delta(\phi; x, y) = \hat{\phi}(x)/\hat{\phi}(y).$$ 

Finally, the width of $\phi$ is

$$w(\phi) = \sup_{x,y} \delta(\phi; x, y).$$

This quantity is invariant by conjugation under $\pi$-symplectic homeomorphisms. We identify this with the period of some cross ratio.

**Proposition 8.9** Let $\rho$ be a representation in $\text{Hom}_H$. Let $l_\rho$ be the spectrum of $\rho$ (cf Paragraph 7.1.3) and $b_\rho$ be its associated cross ratio (cf Paragraph 7.1.4) with periods $l_{b_\rho}$. Then

$$\log(w(\rho(\gamma))) = l_\rho(\gamma) + l_\rho(\gamma^{-1}) = 2l_{b_\rho}(\gamma).$$

**Proof:** Indeed, from the definition of the spectrum, for the attractive fixed point $\gamma^+$ of $\rho(\gamma)$ we have

$$\hat{\phi}(\gamma^+) = e^{l_{\rho}(\gamma)}.$$ 

Since the repulsive point of $\rho(\gamma)$ is the attractive fixed point of $\rho(\gamma^{-1})$, we obtain that

$$\log(w(\rho(\gamma))) = l_\rho(\gamma) + l_\rho(\gamma^{-1}).$$

We now recall that according to Proposition 7.10 there exists a $\pi_1(S)$-equivariant map $\Theta$ from $T^*S^1$ to $\partial_\infty \pi_1(S)^{2*}$ which sends the fibres of $T^*S^1 \to S^1$ to the first factor. We define the inverse images of the second factor as horizontal curves.
If two points $a$ and $b$ belong to the same fibre of $T^* S^1$, $[a, b]$ is the arc joining $a$ and $b$ long that fibre.

Let now $x \in T_\gamma^+ S^1$ and $y \in T_\gamma^- S^1$ be the two fixed points of $\rho(\gamma)$ in $T^* S^1$. Let $t$ be a point in $T_\gamma^+ S^1$. Let $u$ be a point in $T_\gamma^- S^1$ which is joined to $t$ by an horizontal arc $c$.

We note that by definition

$$l_{\rho}(\gamma) = \frac{1}{2} \log(\text{Hol}(c \cup [t, \rho(\gamma)(t)] \cup [\gamma(c) \cup [\rho(\gamma)(u), u]))).$$

Let now $\tilde{c} = [x, t] \cup c \cup [u, y]$. We then have

$$l_{\rho}(\gamma) = \log(\text{Hol}(\tilde{c} \cup \rho(\gamma)(\tilde{c}))) = \log(w(\rho(\gamma))).$$

The Assertion follows Q.E.D.

### 8.4 Conjugation Lemma

We state the hypothesis and notations of the result we prove in this paragraph.

- Let $\kappa$ be a a Hölder homeomorphism of $S^1$. Let
  $$D_\kappa = S^1 \times S^1 \setminus \{(s, t)/\kappa(s) = t\}.$$  

We define

- Let $\pi_D$ be the projection from $D_\kappa$ on the first $S^1$ factor.
- Let $ds$ (resp. $dt$) be Lebesgue probability measure on the first (resp. second) factor of $S^1 \times S^1$.
- Finally, let $d\theta \otimes dr$ be the measure associated to the canonical symplectic form of $T^* S^1$.

We also consider

- Let $f$ be a positive continuous Hölder function such that
  $$\forall s, t \text{ such that } \kappa(s) \neq t, \int_{\kappa(s)}^{\kappa(t)} f(s, u)du = \int_{\kappa(s)}^{t} f(s, u)du = \infty. \quad (27)$$

Finally,

- Let $\omega = f(s, t)ds \otimes dt$. Let
  $$\beta(s, t) = \left(\int_{g(s)}^{t} f(s, u)du\right)ds.$$  

Note that $\beta$ is a primitive of $\omega$. 

38
For any \(C^1\)-closed curve \(c\), let \(\text{Hol}_\omega(c) = e^{\int_c \beta}\).

**Lemma 8.10** Assume that \(f\) satisfies Hypothesis [27]. Then there exists a Hölder homeomorphism \(\psi\) from \(D_\kappa\) to \(T^*S^1\) such that

- \(\psi_*\omega = d\theta \otimes dr\),
- \(\pi \circ \psi = \pi_D\).
- if \(\gamma_1, \gamma_2\) are two \(C^1\) diffeomorphisms with Hölder derivatives of the circle such that \(\gamma = (\gamma_1, \gamma_2)\) preserves \(\omega\) then
  \[\psi \circ \gamma \circ \psi^{-1}\]
  is Hölder and is \(\pi\)-symplectic above \(\gamma_1\),
- if \(c\) is a \(C^1\)-curve in \(D\), then \(\psi(c)\) is a \(\pi\)-curve. Furthermore if \(c\) is a closed curve then \(\text{Hol}_\omega(c) = \text{Hol}(\psi(c))\).

The homeomorphism \(\psi\) is unique up to right composition with a \(\pi\)-exact symplectomorphism. Finally \(\psi\) depends continuously on \(\kappa\) and \(f\).

**Proof:** The uniqueness part of this statement follows by the characterisation of \(\pi\)-exact hamiltonian given in Proposition [24]. The proof is completely explicit. Let \(g\) be a \(C^1\)-diffeomorphism such that \(\forall s, g(s) \neq \kappa(s)\) We consider

\[\psi : \begin{cases} D &\rightarrow T^*S^1 = S^1 \times \mathbb{R} \\ (s, t) &\rightarrow (s, \int_{g(s)}^t f(s, w)dw) \end{cases}\]

It is immediate to check that

- \(\psi_*\omega = d\theta \otimes dr\),
- \(\pi \circ \psi = \pi_D\).

We also observe that \(\psi\) is a Hölder map and a homeomorphism. We prove now that \(\psi^{-1}\) is Hölder. We have

\[\psi^{-1}(s, u) = (s, \alpha(s, u))\]

Where

\[\int_{g(s)}^{\alpha(s, u)} f(s, w)dw = u\]

It is enough to prove that \(\alpha\) is Hölder. We work locally, so that \(f\) is bounded from below by \(k\). Firstly,

\[|u - v| = \left|\int_{\alpha(s, v)}^{\alpha(s, u)} f(s, w)dw\right| \geq k|\alpha(s, v) - \alpha(s, u)|.\]
This proves that $\alpha$ is Hölder with respect to the second variable. Let us take care of the first variable. We have

$$\int_{g(s)}^{\alpha(s,u)} f(s,w)dw = u = \int_{g(t)}^{\alpha(t,w)} f(t,w)dw.$$ 

Hence,

$$\int_{\alpha(s,u)}^{\alpha(t,u)} f(t,w)dw = \int_{g(s)}^{g(t)} f(t,w)dw + \int_{g(t)}^{\alpha(t,u)} (f(t,w) - f(s,w))dw.$$ 

Since we work locally, we may assume that $k \leq |f(t,w)| \leq K$ and

$$|f(t,w) - f(s,w)| \leq C|t - s|^\beta,$$

we have

$$k|\alpha(s,u) - \alpha(t,u)| \leq K|t - s| + C|t - s|^\beta.$$ 

Therefore $\alpha$ is a Hölder function.

By the construction of $\psi$, if $c$ is a $C^1$-curve in $D$, then $\psi(c)$ is a $\pi$-curve. Recall that

$$\beta(s,t) = \left( \int_{g(s)}^{t} f(s,u)du \right) ds$$

is a primitive of $\omega$. It follows that if $c: s \mapsto (c_1(s), c_2(s))$ is a $C^1$ curve in $D_\kappa$. Then

$$\log(Hol_\omega(c)) = \int_c \beta = \int_{g(s)}^{c_2(s)} f(s,u)c_1(s)duds = \log(Hol(\psi(c))).$$

It remains to check that $\psi \circ \gamma \circ \psi^{-1}$ is $\pi$-symplectic and Hölder. But this is immediate by construction. The continuity of $\psi$ on $\kappa$ and $f$ follows from the construction. Q.E.D.

### 8.5 Proof of Theorem 8.1

#### 8.5.1 A preliminary lemma

We need the following lemma

**Lemma 8.11** Let $\rho_1$ and $\rho_2$ be two $H$-Fuchsian representations of $\pi_1(S)$ in $\text{Diff}^h(S^1)$. Let $\kappa$ be a Hölder homeomorphism of $S^1$, such that $\kappa \circ \rho_1 = \rho_2 \circ \kappa$. Let $f(s,t)$ be a positive continuous function on

$$D_\kappa = S^1 \times S^1 \setminus \{(s,t)/\kappa(s) = t\}.$$ 

Assume that, for all $\gamma$ in $\pi_1(S)$, $\omega = f(s,t)ds \otimes dt$ is invariant under the action of $(\rho_1(\gamma), \rho_2(\gamma))$. Then

$$\forall(s,t) \in D_\kappa, \int_t^{\kappa(s)} f(s,u)du = \int_{\kappa(s)}^{t} f(s,u)du = \infty.$$ 

$40$
Proof: For the sake of simplicity, we write $\rho_i(\gamma) = \gamma^i$. We first observe that the invariance of $\omega$ yields

$$f(s, t) = \frac{d\gamma_1}{ds}(s) \frac{d\gamma_2}{dt}(t) f(\gamma_1(s), \gamma_2(t)).$$

For any $(s, t)$ in $D_\kappa$, we may find a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \gamma_1^n(s) = s_0, \quad \lim_{n \to \infty} \gamma_2^n(t) = t_0 \neq \kappa(s_0), \quad \lim_{n \to \infty} \frac{d\gamma_1}{ds}(s) = +\infty.$$

Now using the invariance of $\omega$ by $(\gamma^1, \gamma^2)$, we obtain that

$$\int_{\kappa(s)}^t f(s, u) du = \int_{\kappa(s)}^t \frac{d\gamma_1}{ds}(s) \frac{d\gamma_2}{du}(u) f(\gamma_1^n(s), \gamma_2^n(u)) du$$

$$= \frac{d\gamma_1}{ds}(s) \int_{\kappa(s)}^t \frac{d\gamma_2}{du}(u) f(\gamma_1^n(s), \gamma_2^n(u)) du$$

$$= \frac{d\gamma_1}{ds}(s) \int_{\gamma_1^n(s)}^{\gamma_2^n(t)} f(\gamma_1^n(s), u) du$$

$$= \frac{d\gamma_1}{ds}(s) \int_{\kappa(\gamma_1^n(s))}^{\gamma_2^n(t)} f(\gamma_1^n(s), u) du.$$ 

From Assertions (28) and (29), we deduce that

$$\lim_{n \to \infty} \int_{\kappa(\gamma_1^n(s))}^{\gamma_2^n(t)} f(\gamma_1^n(s), u) du = \int_{\kappa(s_0)}^{t_0} f(s_0, u) du > 0.$$

Hence Assertion (30) shows that

$$\int_{\kappa(s)}^t f(s, u) du = \infty.$$

A similar argument shows that

$$\int_{\kappa(s)}^\infty f(s, u) du = \infty.$$

Q.E.D.

### 8.5.2 Proof of Theorem 8.1

**Proof:** Recall our hypothesis and notations. Let $\kappa$ be a Hölder homeomorphism of $S^1$. Let

$$D_\kappa = S^1 \times S^1 \setminus \{(s, t)/\kappa(s) = t\}.$$
such that $\kappa \circ \rho_1 = \rho_2 \circ \kappa$. Let $p : M = \mathbb{R} \times D_\kappa \rightarrow D_\kappa$ be a principal $\mathbb{R}$-bundle over $D_\kappa$ equipped with a connection $\nabla$. Assume the curvature $\omega$ of $\nabla$ is such that $\omega = f(s,t)ds \wedge dt$ with $f$ positive and Hölder. Let

\[
\begin{align*}
\pi & : \left\{ \begin{array}{l}
M \rightarrow S^1 \\
(u, s, t) \mapsto t
\end{array} \right. \\
\pi_1 & : \left\{ \begin{array}{l}
M \rightarrow \mathbb{R} \times S^1 \\
(u, s, t) \mapsto (u, s)
\end{array} \right. \\
p & : \left\{ \begin{array}{l}
M \rightarrow D_\kappa \\
(u, s, t) \mapsto (s, t)
\end{array} \right.
\end{align*}
\]

Assume that $\pi_1(S)$ acts on $M$ by $C^1$-diffeomorphisms with Hölder derivatives. Assume this action preserves $\nabla$, and that

- $M/\pi_1(S)$ is compact
- The action of $\mathbb{R}$ on $M/\pi_1(S)$ contracts the fibres of $\pi_1$.
- There exists two $H$-Fuchsian representations $\rho_1$ and $\rho_2$ of $\pi_1(S)$, such that $p(\gamma u) = (\rho_1(\gamma)(p(u)), \rho_2(\gamma)(p(u)))$.

We want to prove there exists a $\mathbb{R}$-commuting Hölder homeomorphism $\hat{\psi}$ from $M$ to $J$ over a homeomorphism $\psi$ from $D$ to $T^*S^1$, a representation $\rho$ of $\pi_1(S)$ in $C^{1,h}(S^1) \ltimes \text{Diff}^h(S^1)$ element of $\text{Hom}_H$, such that

\[
\hat{\psi} \circ \gamma = \rho(\gamma) \circ \hat{\psi}.
\]

Notice first that by Lemma 8.11

\[
\forall(s, t) \in D_\kappa, \int_{\kappa(s)}^t f(s, u)du = \int_t^t f(s, u)du = \infty.
\]

It follows the hypotheses of Lemma 8.10 are satisfied. Thus, there exists a Hölder homeomorphism $\psi$ from $D$ to $T^*S^1$ unique up to right composition with a $\pi$-exact hamiltonian, such that

1. $\psi_* \omega = d\theta \otimes dr,$
2. $\pi \circ \psi = \pi_D.$
3. if $\gamma_1, \gamma_2$ are two $C^1$ diffeomorphisms with Hölder derivatives of the circle such that $\gamma = (\gamma_1, \gamma_2)$ preserves $\omega$ then

\[
\psi \circ \gamma \circ \psi^{-1},
\]

is Hölder and $\pi$-symplectic above $\gamma_1$.
4. if $c$ is a $C^1$-curve in $D$, then $\psi(c)$ is a $\pi$-curve and $\text{Hol}_\omega(c) = \text{Hol}(\psi(c)).$
Let \( \gamma \) be an element of \( \pi_1(S) \). Since \( \gamma \) acts on \( M \) preserving the connection \( \nabla \), it follows that \( f(\gamma) = (\rho_1(\gamma), \rho_2(\gamma)) \) acts on \( D_\kappa \) in such a way that for all \( C^1 \)-curves \( \text{Hol}_\nabla(c) = \text{Hol}_\nabla(f(\gamma)c) \).

We now show that
\[
\text{Hol}_\omega(c) = \text{Hol}_\omega(f(\gamma)c) \tag{31}
\]
We choose a trivialisation of \( L \). In this trivialisation \( \nabla = D + \beta + \alpha \), where \( D \) is the trivial connection and \( \alpha \) is a closed form. Then
\[
\text{Hol}_\nabla(c) = \text{Hol}_\omega(c)e^{\int_c \alpha}.
\]
To prove Equality (31), it suffices to show
\[
\int_{f(\gamma)(c)} \alpha = \int_c \alpha. \tag{32}
\]
But \( \rho_1(\gamma) \) preserves the orientation of \( S^1 \), hence is connected to the identity by a family of mapping \( f_t \). It follows that \( f(\gamma) \) is also homotopic to the identity through the family \( (f_t, \kappa f_t \kappa^{-1}) \). This implies that \( f(\gamma) \) acts trivially on the homology. Hence Equality (32).

It follows from (3) and (4) that \( g(\gamma) = \psi \circ f(\gamma) \circ \psi^{-1} \) is a \( \pi \)-exact symplectomorphism.

Finally, we define a map \( \hat{\psi} \) from \( M \) to \( J \) above \( \psi \), commuting with \( \mathbb{R} \) action, ”preserving the holonomy” well defined up right composition by an element of \( \mathbb{R} \). Let fix an element \( y_0 \) of the fibre of \( p \) above \( x_0 \) in \( D_\kappa \) and an element \( z_0 \) of the fibre of \( \pi_2 \) above \( \psi(x_0) \). Let \( y \) be an element of the fibre of \( p \) above some point \( x \) of \( D_\kappa \). Let \( c \) be path joining \( x_0 \) to \( x \). We observe that there exists \( \mu \) in \( \mathbb{R} \) such that
\[
y = \mu + \text{Hol}_\nabla(c)y_0.
\]
We define
\[
\hat{\psi}(y) = \mu + \text{Hol}(\psi(c))z_0.
\]
Since for any closed curve \( \text{Hol}_\nabla(c) = \text{Hol}(\psi(c)) \), it follows that \( \hat{\psi}(y) \) is independent of the choice of \( c \).

By construction, \( \hat{\psi} \) satisfies the required conditions. The uniqueness statement follows by the Corollary 8.7 The continuity statement follows by the corresponding continuity statement of Lemma 8.10 and the construction. Q.E.D.

9 Negatively curved metrics
We first use our conjugation Theorem 8.1 to prove the space \( \text{Rep}_H \) contains an interesting space.

**Theorem 9.1** Let \( \mathcal{M} \) be the space of negatively curved metrics on the surface \( S \). Then there exists a continuous injective map \( \psi \) from \( \mathcal{M} \) to \( \text{Rep}_H \). Furthermore, for any \( \gamma \) in \( \pi_1(S) \)
\[
l_\psi(\gamma) = l_{\psi(\gamma)}(\gamma).
\]
Here $l_g(\gamma)$ is the length of the closed geodesic for $g$ freely homotopic to $\gamma$, and $l_{\psi(g)}$ is the $\psi(g)$-length of $\gamma$. Finally, $\psi(g_0) = \psi(g_1)$, if and only if there exists a diffeomorphism $F$ of $S$, homotopic to the identity, such that $F^*(g_0) = g_1$.

We first recall some facts about the geodesic flow and the boundary at infinity of negatively curved manifolds.

### 9.1 The boundary at infinity and the geodesic flow

Let $S$ be a compact surface equipped with a negatively curved metric. Let $\tilde{S}$ be its universal cover. Let $U\tilde{S}$ (resp. $US$) be the unitary tangent bundle of $\tilde{S}$ (resp. $S$). Let $\phi_t$ be the geodesic flow on these bundles. Let $\partial_\infty \tilde{S}$ be the boundary at infinity of $\tilde{S}$.

We collect in the following Proposition well known facts:

**Proposition 9.2**

1. $\partial_\infty \tilde{S} = \partial_\infty \pi_1(S)$.

2. $\partial_\infty \tilde{S}$ has a $C^1$-structure depending on the choice of the metric such that the action of $\pi_1(S)$ on it is by $C^{1,\alpha}$-diffeomorphisms. The action of $\pi_1(S)$ is Hölder conjugate to a Fuchsian one.

3. Let $G$ be the space of geodesics of the universal cover of $S$. Then $G$ is $C^{1,\alpha}$-diffeomorphic to $(\partial_\infty \pi_1(S))^3$.

4. $U\tilde{S} \to G$, is a $\mathbb{R}$-principal bundle (with the action of the geodesic flow). Furthermore the Liouville form is a connection form for this action invariant under $\pi_1(S)$, and its curvature is symplectic.

The following well known Proposition explains the identification of the unitary tangent bundle with a suitable product of the boundary at infinity.

**Proposition 9.3** There exist a homeomorphism $f$ of $\partial_\infty \pi_1(S)^3 = \{ \text{oriented } (x,y,z) \in (\partial_\infty \pi_1(S))^3 / x \neq y \neq z \neq x \}$ with $U\tilde{S}$ such that if $\psi_t = f^{-1} \circ \phi_t \circ f$ then,

- $\psi_t(z,x,y) = (w,x,y)$,

- let $\{t\}_{m \in \mathbb{N}}$ be a sequence of real numbers going to infinity, let $(x,y,z) \in \partial_\infty \pi_1(S)^3$, let $\{\gamma\}_{m \in \mathbb{N}}$ be a sequence of elements of $\pi_1(S)$, such that

  \[ \{\gamma_m \circ \psi_{t_m}(z,x,y)\}_{n \in \mathbb{N}} \text{ converges to } (z_0,x_0,y_0). \]

Then for any $w, v$ such that $(w,x,v) \in \partial_\infty \pi_1(S)^3$, there exists $v_0$ such that

\[ \{\gamma_m \circ \psi_{t_m}(w,x,v)\}_{n \in \mathbb{N}} \text{ converges to } (v_0,x_0,y_0). \]
Proof : We explain first the construction of the map $f$. Let $(z, x, y) \in \partial_\infty \pi_1(S)^3$. Let $\gamma$ the geodesic in $\tilde{S}$ going from $x$ to $y$. Let $\gamma(t_0)$ be the projection of $y$ on $\gamma$, that is the unique minimum on $\gamma$ of the horospherical function associated to $z$. We set $f(z, x, y) = \dot{\gamma}(t_0)$. Then the first property of $f$ is obvious, and the second one is a classical consequence of negative curvature, namely that two geodesics with the same endpoints at infinity go exponentially closer and closer. Q.E.D.

9.2 Proof of Theorem 9.1

We first have to check the hypotheses of Theorem 8.1 are satisfied. This follows at once by Proposition 9.2 and 9.3, using Proposition 8.2. Therefore associated to a negatively curved metric we obtain a representation in $\text{Hom}^\ast$. For a hyperbolic metric, we obtain precisely a $\infty$-Fuchsian representation. Since the space of negatively curved metric is connected, all the representation are actually in $\text{Hom}_H$.

Finally if $\psi(g_0) = \psi(g_1)$, then the two metrics have the same length spectrum and therefore are isometric by Otal’s Theorem [7]. This proves injectivity. The continuity follows by the continuity statement of Theorem 8.1.

10 Hitchin component

Let $\text{Rep}_H(\pi_1(S), \text{SL}(n, \mathbb{R}))$ be a Hitchin component, i.e. a connected component of the space of representations of $\pi_1(S)$ in $\text{SL}(n, \mathbb{R})$ containing the Fuchsian representations as defined in [3] (see also [5]).

We now prove that $\text{Rep}_H$ contains all these Hitchin components.

Theorem 10.1 There exists a continuous injective map

$$\psi : \text{Rep}_H(\pi_1(S), \text{SL}(n, \mathbb{R})) \rightarrow \text{Rep}_H,$$

such that, if $\rho \in \text{Rep}_H(\pi_1(S), \text{SL}(n, \mathbb{R}))$ then

- for any $\gamma$ in $\pi_1(S)$, we have

$$l_{\psi(\rho)}(\gamma) = \log(\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}).$$

(33)

Here, $l_{\psi(\rho)}(\gamma)$ is the $\psi(\rho)$-length of $\gamma$, and $\lambda_{\max}(a)$ (resp. $\lambda_{\min}(a)$) denote the maximum (resp. minimum) real eigenvalue of the endomorphism $a$ in absolute value.

- The cross ratio associated to $\rho$ and $\psi(\rho)$ coincide.

- $\psi(\rho)$ is compatible (cf Definition 7.17).

In some sense, this result says that $C^{1,\lambda}(S^1) \times \text{Diff}^h(S^1)$ is a version of $\text{SL}(\infty, \mathbb{R})$.
10.1 Hyperconvex curves

For every Hitchin representation $\rho$ in $SL(n, \mathbb{R})$, there exists a $\rho$-equivariant hyperconvex map $\xi$ from $\partial_{\infty} \pi_1(S)$ to $\mathbb{P}(\mathbb{R}^n)$. More precisely, in [5] we proved there exists a $\rho$-equivariant Hölder map $(\xi^1, \xi^2, \ldots, \xi^{n-1})$ from $\partial_{\infty} \pi_1(S)$ to the flag manifold $F$, called the osculating flag of $\xi$ or the limit curve of $\rho$, such that

- $\xi = \xi^1$,
- $\xi^p$ is with values in the Grassmannian of $p$-planes,
- if $(n_1, \ldots, n_l)$ are positive integers such that
  $$\sum_{i=1}^l n_i \leq n,$$
  then the following sum is direct
  $$\xi^{n_1}(x_1) \oplus \ldots \oplus \xi^{n_l}(x_l);$$
- finally, for every $x$, let $p = n_1 + \ldots + n_l$, then
  $$\lim_{(y_1, \ldots, y_l) \to x, y_i \text{ all distinct}} \left( \bigoplus_{i=1}^l \xi^{n_i}(y_i) \right) = \xi^p(x).$$

It follows in particular that $\xi^1(\partial_{\infty} \pi_1(S))$ (resp. $\xi^{n-1}(\partial_{\infty} \pi_1(S))$) is a $C^1$-curve with Hölder derivatives in $\mathbb{P}(\mathbb{R}^n)$ (resp. $\mathbb{P}(\mathbb{R}^{n*})$). Furthermore, for $x \neq y$, the sum $\xi^1(x) \oplus \xi^{n-1}(y)$ is direct.

We may rephrase these properties in the following way

**Proposition 10.2** Let $\rho$ be a hyperconvex representation. Let $\xi = (\xi^1, \ldots, \xi^{n-1})$ be the limit curve of $\rho$. Then, there exist

- Two $C^1$ embeddings with Hölder derivatives, $\eta_1$ and $\eta_2$, of $S^1$ in respectively $\mathbb{P}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{R}^{n*})$,
- Two representations $\rho_1$ and $\rho_2$ of $\pi_1(S)$ in $Diff^h(S^1)$, the group of $C^1$-diffeomorphisms of $S^1$ with Hölder derivatives,
- a Hölder homeomorphism $\kappa$ of $S^1$,

such that

1. $\eta_1(S^1) = \xi^1(\partial_{\infty} \pi_1(S))$ and $\eta_2(S^1) = \xi^{n-1}(\partial_{\infty} \pi_1(S))$,
2. $\eta_i \circ \rho_i = \rho \circ \eta_i$,
3. if $\kappa(s) \neq t$, then the sum $\eta_1(s) \oplus \eta_2(t)$ is direct.
4. $\kappa \circ \rho_1 = \rho_2 \circ \kappa$.
Proof: Let \( \eta_1 \) (resp. \( \eta_2 \)) be the arc-length parameterisation of \( \xi^1(\partial_\infty \pi_1(S)) \) (resp. \( \xi^{n-1}(\partial_\infty \pi_1(S)) \)). Let

\[
\kappa = (\eta_2)^{-1} \circ \xi^1 \circ (\xi^{n-1})^{-1} \circ \eta_1.
\]

The result follows. Q.E.D.

For later use we introduce the continuous map

\[
\dot{\eta} = (\eta_1^{-1} \circ \xi^1, \eta_2^{-1} \circ \xi^{n-1}),
\]

from \( \partial_\infty \pi_1(S)^{2*} \) to \( D_\kappa \).

10.2 Proof of Theorem 10.1

We use in this section the independent results proved in the Appendix 12.

Let \( \rho \) be a hyperconvex representation. Let \( \eta_1, \eta_2 \) and \( \kappa \) as in Proposition 10.2. Let \( \eta = (\eta_1, \eta_2) \).

We observe that \( \pi_1(S) \) acts by \( \dot{\rho} = (\rho_1, \rho_2) \) on \( D_\kappa \). It follows by Proposition 10.2 that \( \eta \) is a \( \rho \)-equivariant (with respect to the action on \( D_\kappa \) given by \( \dot{\rho} \)) \( C^1 \) map to

\[
\mathbb{P}(n)^{2*} = \mathbb{P}(\mathbb{R}^n) \times \mathbb{P}(\mathbb{R}^{n*}) \setminus \{(D, P) \mid D \subset P\}.
\]

In Section 12.4 of the Appendix, we show there exists a \( SL(n, \mathbb{R}) \) invariant \( \mathbb{R} \)-principal bundle \( L \) on \( \mathbb{P}(n)^{2*} \) equipped with a connection whose curvature is a symplectic form \( \Omega \).

- Let \( M \) be the induced bundle by \( \eta \): \( M = \eta^*L \). It is a \( \mathbb{R} \)-principal bundle over \( D_\kappa \) equipped with an action of \( \pi_1(S) \) coming from the \( SL(n, \mathbb{R}) \) action on \( L \).
- Let \( \phi_t \) be the flow of the induced \( \mathbb{R} \)-action on \( M \).
- Let \( \mathcal{L} \) be the foliation of \( M \) by parallel vector fields along the curves \( c_s : t \mapsto (s, t) \) in \( D_\kappa \).

We observe the following

Proposition 10.3 Let \( \gamma \) be an element of \( \pi_1(S) \). Let \( x = \dot{\eta}(\gamma^+, \gamma^-) \) be a fixed point of \( \gamma \) in \( D_\kappa \). Let \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) be the largest and smallest eigenvalues (in absolute values) of \( \rho(\gamma) \). Then the action of \( \gamma \) on the fibre \( M_x \) of \( M \) above \( x \) is given by the translation by

\[
\log \left| \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \right|.
\]

Proof: This follows from the last point of Proposition 12.3 Q.E.D.

To complete the proof of Theorem 10.1 we prove
• $\omega = \eta^* \Omega$ is symplectic and Hölder (in Proposition 10.4),
• $M/\pi_1(S)$ is compact (in Proposition 10.6)
• The action of $\mathbb{R}$ on $M$ contracts the leaves of $\mathcal{L}$ (in Proposition 10.8).

We explain now how these properties imply our Theorem: by Theorem 8.1, there exists a homeomorphism $\hat{\psi}$ from $M$ to $J$, an area preserving homeomorphism $\psi$ from $D$ to $T^* S^1$ such that

• $\hat{\psi}$ commutes with the $\mathbb{R}$-action,
• $\pi_2 \circ \hat{\psi} = \psi \circ p$,
• $\hat{\psi}$ sends $\mathcal{L}$ to $\mathcal{F}$,

and a representation $\rho$ of $\pi_1(S)$ in $C^{1,h}(S^1) \rtimes \text{Diff}^h(S^1)$ such that

$$\hat{\psi} \circ \gamma = \rho(\gamma) \circ \hat{\psi}.$$ 

In particular, $\rho$ belongs to $\text{Hom}^\ast$.

Let us prove that $\rho$ is actually in $\text{Hom}_H$. From the definition of Hitchin component, and the fact that we can choose $\psi$ to depend continuously on our parameters, it suffices to check that for any $n$-Fuchsian representation, i.e. a representation that factors through a Fuchsian representation in $\text{PSL}(2, \mathbb{R})$ and the irreducible representation $\iota$ of this latter group in $\text{PSL}(n, \mathbb{R})$. But in this case, the action of $\pi_1(S)$ extends to a transitive action of $\text{PSL}(2, \mathbb{R})$ and is by definition in $\text{Hom}_H$.

The statement about the spectrum follows by Proposition 10.3. In particular the spectrum is symmetric. Therefore, by definition, $\psi(\rho)$ is symmetric. Finally, by construction and Proposition 12.4 the two cross ratios coincide.

By Theorem 7.19 a $\infty$-Hitchin representation which is symmetric is determined by its cross ratio. It follows that $\psi$ is injective.

Finally the continuity statement follows by the continuity statement of Theorem 8.1.

### 10.2.1 Symplectic form

We first prove the following

**Proposition 10.4** The two-form $\omega = \eta^* \Omega$ is symplectic and Hölder.

**Proof**: The regularity of $\eta$ implies $\eta^* \Omega$ is Hölder. We begin by an observation. Let $D$ be a line in $\mathbb{R}^n$, $P$ a line in $\mathbb{R}^n\ast$, such that $D \oplus P = \mathbb{R}^n$. Let $W$ be a two-plane containing $D$. Let

$$\hat{W} = T_D \mathbb{P}(W) \subset T_D \mathbb{P}(\mathbb{R}^n).$$

Let $V$ a $n-2$-plane contained in $P$. Let

$$\hat{V} = T_P \mathbb{P}(W^\perp) \subset T_P \mathbb{P}(\mathbb{R}^n\ast).$$
From the definition of $\Omega$, we check that if

$$V \oplus W = \mathbb{R}^n,$$

Then

$$\Omega|_{\hat{V} \oplus \hat{W}} \neq 0.$$

In the case of hyperconvex curves,

$$T_{\xi^1(x)}(S^\infty) = (\xi^2(x)), \quad T_{\xi^{n-1}(x)}(S^\infty) = (\xi^{n-2}(x)).$$

Since by hyperconvexity $\xi^2(x) \oplus \xi^{n-2}(y) = \mathbb{R}^n$ for $x \neq y$, we conclude that $\eta^* \omega$ is symplectic. Q.E.D.

As a corollary, we get

**Proposition 10.5** Let $\rho$ be a hyperconvex representation, then $b_\rho$ is strict.

**Proof:** Let $(x, y, z, t)$ be a quadruple of pairwise distinct points. Let $\hat{\eta}$ defined as in (36). Let $Q$ be the square in $\partial^\infty \pi_1(S)^2$ whose vertices are $(x, y), (z, y), (x, t), (z, t)$. We know that

$$|b_\rho(x, y, z, t)| = e^{\int_{\eta(Q)} \omega}.$$

Since $\omega$ is symplectic, and $Q$ has a nonempty interior, we have $\int_{\eta(Q)} \omega \neq 0$. Hence $b_\rho(x, y, z, t) \neq 1$. Q.E.D.

### 10.2.2 Compact quotient

We consider now the $\mathbb{R}$-principal bundle $M$ over $D_\kappa$, defined by $M = \eta^* L$. $M$ comes equipped with a connection, whose curvature form is $\omega$. Furthermore $\pi_1(S)$ acts on $M$, by the pull back of the action of $SL(n, \mathbb{R})$ on $L$. We now prove

**Proposition 10.6** The quotient $M/\pi_1(S)$ is compact.

Recall that $\pi_1(S)$ acts with a compact quotient on

$$\partial_\infty \pi_1(S)^3 = \{ \text{oriented triples } (x, y, z) \in (\partial_\infty \pi_1(S))^3, x \neq y, y \neq z, x \neq z \},$$

We first prove :

**Proposition 10.7** There exists a continuous onto $\pi_1(S)$-equivariant proper map $l$ from $\partial_\infty \pi_1(S)^3$ to $M$. Moreover, the map $l$ is above the map $\hat{\eta} = (\hat{\eta}_1, \hat{\eta}_2)$ from $\partial_\infty \pi_1(S)^2$ to $D_\kappa$, i.e $l(z, x, y) \in M_{\hat{\eta}(x, y)}$.

**Proof:** Let $(z, x, y)$ be an element of $S^3 = \partial_\infty \pi_1(S)^3$. The two transverse flags $\xi(x)$ and $\xi(y)$ defines a decomposition

$$\mathbb{R}^n = L_1(x, y) \oplus L_2(x, y) \oplus \ldots \oplus L_n(x, y),$$
such that $\xi^i(x) = L_1(x, y)$ and
\[
\xi^{n-1}(y) = L_2(x, y) \oplus \ldots \oplus L_n(x, y).
\] (37)

Let $u$ be a nonzero element of $\xi^1(z)$. Let $u_i$ be the projection of $u$ on $L_i(x, y)$. By hyperconvexity, $u_i \neq 0$. We choose $u_i$ up to sign, so that
\[
|u_1 \wedge \ldots \wedge u_n| = 1.
\]
Finally we choose $f$ in $\xi^{n-1}(y)^\perp$ so that $(f, u_1) = 1$. The pair $(u_1, f)$ is well defined up to sign, and hence defines a unique element $l(z, x, y)$ of the fibre of $M$ above $\eta(x, y)$.

We prove that $l :$
\[
\begin{align*}
S^3^* &\to M \\
(z, x, y) &\mapsto l(z, x, y),
\end{align*}
\]
is proper. Let $\{(z_m, x_m, y_m)\}_{m \in \mathbb{N}}$ be a sequence of elements of $S^3^*$ such that $\{l(z_m, x_m, y_m) = (u_m, f_m)\}_{m \in \mathbb{N}}$ converges to $(u_0, f_0)$ with $(f_0, u_0) = 1$. In particular, we have that $\{(x_m, y_m)\}_{m \in \mathbb{N}}$ converges to $(x_0, y_0)$, with $x_0 \neq y_0$. We may assume after extracting a subsequence that $\{z_m\}_{m \in \mathbb{N}}$ converges to $z_0$. To prove $l$ is proper, it suffices to show that $x_0 \neq z_0$ and $y_0 \neq z_0$.

Assume not, and let us suppose first that $z_0 = x_0$. Let $\pi_m$ be the projection of $\xi^1(z_m)$ on $\xi^1(x_m)$ along $\xi^{n-1}(y_m)$. We observe that $\pi_m$ converges to the identity from $\xi^1(z_0) = \xi^1(x_0)$ to $\xi^1(x_0)$. Let $v_m \in \xi^1(z_m)$ such that $\pi_m(v_m) = u_m$. Since $\{u_m\}_{m \in \mathbb{N}}$ converges to a nonzero element $u_0$, $\{v_m\}_{m \in \mathbb{N}}$ converges to $0$. As a consequence, all the projections $\{v_m^i\}_{m \in \mathbb{N}}$ of $v_m$ on $L_i(x_m, y_m)$ converge to zero for $i > 1$. Hence,
\[
1 = |v_1^1 \wedge \ldots \wedge v_n^m| \to 0,
\]
and the contradiction.

Suppose now that $z_0 = y_0$. Using the volume form of $\mathbb{R}^n$, we identify $\xi^{n-1}(y)^\perp$ with $L_2(x, y) \wedge \ldots \wedge L_n(x, y)$.

We use the same notations as in the previous paragraph. Then $v_1^2 \wedge \ldots \wedge v_n^m$ is identified with $f_m$. It follows that
\[
v_1^2 \wedge \ldots \wedge v_n^m \to f_0. \quad (38)
\]
Recall that by hyperconvexity
\[
\xi^1(z_m) \oplus \xi^p(y_m) \to \xi^{p+1}(y_0).
\]
Recall also that,
\[
\xi^p(y) = L_{n-p+1}(x, y) \oplus \ldots \oplus L_n(x, y).
\]
Since $\xi^1(z_m) \to \xi^1(y_0)$, it follows that for all $k$
\[
\frac{\|v_m^k\|}{\|v_m^k\|} \to 0.
\]

50
In particular
\[ \frac{\|v_1^1 \|^n - 1}{\|v_2^m \wedge \ldots \wedge v_n^m\|} \to 0. \]

Thanks to Assertion (38), we finally get that \( \|v_1^1\| \to 0. \) Hence
\[ 1 = \frac{\|v_1^1 \|^m \wedge \ldots \wedge v_n^m\|}{\|v_1^1\| \|v_2^2 \wedge \ldots v_n^n\|} \to 0, \]
and the contradiction. Therefore \((z_0, x_0, y_0) \in S^{3*}\) and \(l\) is proper.

It remains to prove that \(l\) is onto. First, for every \((x,y)\),
\[ L(x,y) = l(S^{3*}) \cap M_{\eta(x,y)}, \]
is a closed interval, being the image of an interval. If \((\gamma_+, \gamma_-)\) is a fixed point of \(\gamma\), then \(L(\gamma_+, \gamma_-)\) is invariant by \(\rho(\gamma)\) that acts as a translation on \(M(\gamma_+, \gamma_-)\).

It follows that \(L(\gamma_+, \gamma_-) = M(\gamma_+, \gamma_-).\)

Since the set of fixed points of element of \(\pi_1(S)\) is dense in \(\partial_{\infty} \pi_1(S)\), and \(l(S^{3*})\) is closed by properness, we conclude that \(l(S^{3*}) = M\) and that \(l\) is onto. Q.E.D.

**Proof of Proposition 10.7**

Since \(\pi_1(S)\) acts properly on \(S^{3*}\), and \(l\) is proper and onto, the action of \(\pi_1(S)\) on \(M\) in proper: indeed for any compact \(K\) in \(M\)
\[ \{\gamma \in \pi_1(S), \gamma(K) \cap K \neq \emptyset \} \subset \{\gamma \in \pi_1(S), \gamma(f^{-1}(K)) \cap f^{-1}(K) \neq \emptyset \}, \]
Hence,
\[ \sharp\{\gamma \in \pi_1(S), \gamma(K) \cap K \neq \emptyset \} \leq \sharp\{\gamma \in \pi_1(S), \gamma(f^{-1}(K)) \cap f^{-1}(K) \neq \emptyset \} < \infty. \]

Next since \(\pi_1(S)\) has no torsion elements and acts properly, it follows that the action of \(\pi_1(S)\) on \(M\) is free. The space \(M/\pi_1(S)\) is therefore a topological manifold. Finally, the quotient map of \(l\) from \(S^{3*}/\pi_1(S)\) (which is compact) being onto, it follows \(M/\pi_1(S)\) is compact.

**10.2.3 Contracting the leaves**

We notice that \(M\) is topologically a trivial bundle. Let \(\mathcal{L}\) be the foliation of \(M\) by parallel vector fields along the curves \(c_s: t \mapsto (s, t)\) in \(D_\kappa\). We choose a \(\pi_1(S)\)-invariant metric on \(M\). We prove now:

**Proposition 10.8** The action \(\phi_t\) of \(\mathbb{R}\) on \(M\) contracts the leaves of \(\mathcal{L}\).

**Proof:** In the proof of Proposition 10.7 we exhibited a proper onto continuous \(\pi_1(S)\)-equivariant map \(l\) from \(\partial_{\infty} \pi_1(S)^{3*}\) to \(M\). This map is such that
\[ l(z, x, y) \in M_{\eta(x,y)}. \quad (39) \]

From Proposition 38 by choosing a hyperbolic metric on \(S\) we have a flow \(\psi_t\) by proper homeomorphisms on \(\partial_{\infty} \pi_1(S)^{3*}\) such that

51
1. $\psi_t(z, x, y) = (w, x, y)$.

2. Let $\{t\}_{m \in \mathbb{N}}$ be a sequence of real numbers going to infinity, let $(z, x, y) \in \partial_\infty \pi_1(S)^3$, let $\{\gamma\}_{m \in \mathbb{N}}$ be a sequence of elements of $\pi_1(S)$, such that

$$\{\gamma_m \circ \psi_{t_m}(z, x, y)\}_{n \in \mathbb{N}} \text{ converges to } (z_0, x_0, y_0).$$

Then for any $w, v$ such that $(w, x, v) \in \partial_\infty \pi_1(S)^3$, there exists $v_0$ such that

$$\{\gamma_m \circ \psi_{t_m}(w, x, v)\}_{n \in \mathbb{N}} \text{ converges to } (w_0, x_0, y_0).$$

From the compactness of $S$, the properness of $l$ and the flows, and we obtain positive constants $a$ and $b$ such that

$$\forall u, \forall T, \exists t \in [T/a - b, Ta + b], l(\psi_t(u)) = \phi_T(l(u)).$$

Therefore, we deduce the following assertion holds: let $\{t\}_{m \in \mathbb{N}}$ be a sequence of real numbers going to infinity, let $u \in M$, let $\{\gamma\}_{m \in \mathbb{N}}$ be a sequence of elements of $\pi_1(S)$, such that

$$\{\gamma_m \circ \phi_{t_m}(u)\}_{n \in \mathbb{N}} \text{ converges to } u_0 \in M_{\phi(x_0, y_0)}.$$

Then for any $w, s$ such that $w \in M_{\phi(x, s)}$, there exists $v_0$ such that

$$\{\gamma_m \circ \phi_{t_m}(w)\}_{n \in \mathbb{N}} \text{ converges to } v_0 \in M_{\phi(x_0, y_0)}.$$

Hence the hypothesis of Proposition holds for the $\mathbb{R}$-action on $M$ and in particular this action contracts the leaves of $L$. Q.E.D.

### Appendix A : Filtrated Spaces, Holonomy.

We are going to describe a notion of a topological space with "nested" laminations. This require some definitions. First we introduce some notations, Let

$$Z = Z_1 \times \ldots \times Z_p$$

For $k < p$, we note $p_k$ the projection

$$Z \to Z_1 \times \ldots \times Z_k$$

If $U$ is a subset of $Z$ and $x$ a point of $U$, we define the $k^{th}$-leaf through $x$ in $U$ to be

$$U^k_x = U \cap p_k^{-1}\{p_k(x)\}.$$ 

We notice that $U^{k+1} \subset U^k_x$. The higher dimensional leaf is $U_1$. If $\phi$ is a map from $U$ to to a set $V$, we note

$$\phi^k_x = \phi|_{U^k_x}.$$ 

If $V \subset W = W_1 \times \ldots \times W_k$, we say $\phi$ is a filtrated map if

$$\forall k, \phi(U^k_x) \subset V^k_{\phi(x)}.$$
11.1 Filtrated Space, Lamination

In this section, we define a notion of filtrated space for which it does make sense to say some maps are "smooth along leaves with derivatives varying continuously".

**Definition 11.1 [C^∞-filtrated space]** We say a metric space $P$ is $C^\infty$-filtrated if there exist

- a covering of $P$ by open sets $U_i$, called charts,
- Hölder homeomorphisms, called coordinates, $\phi_i$ of $U_i$ with $V_1^i \times V_2^i \times \ldots \times V_p^i$ where for $k > 1$, $V_k^i$ is an open set in a finite dimensional affine space,

We furthermore make the following assumption about coordinates changes. Let $\phi_{ij} = \phi_i \circ (\phi_j)^{-1}$, defined from $W_{ij} = \phi_j(U_i \cap U_j)$, to $W_{ji}$. We suppose

- $\phi_{ij}$ is a filtrated map,
- $(\phi_{ij})_x$ is a $C^\infty$-map whose derivatives depends continuously on $x$.

We observe by the last assumption, for all $k$, $(\phi_{ij}^k)_x$ is a $C^\infty$-map whose derivatives depends continuously on $x$.

When $p = 2$, we speak of a laminated space. We may want to specify the number of nested leaves, in which case we talk of $p$-filtrated objects.

The Hölder hypothesis is somewhat irrelevant, but is needed in the applications we have in mind.

We now extend the definitions of the previous paragraph. Let $P$ be a $C^\infty$-$p$-filtrated space. Let $k < p$.

**Definition 11.2 [Leaves]** The $k^{\text{th}}$-leaves are the equivalence classes of the equivalence relation generated by

$$y R x \iff p_k(\phi_i(y)) = p_k(\phi_i(x)).$$

Let $P$ and $\overline{P}$ be two $C^\infty$-$p$-filtrated spaces. Let $\phi_i$ be coordinates on the $P$ and $\overline{\phi}_j$ be coordinates on $\overline{P}$.

**Definition 11.3 [Filtrated maps and immersions]** Let $\psi$ be a map from $\overline{P}$ to $P$. We say $\psi$ is a $C^\infty$-Filtrated maps if

- $\psi$ is Hölder,
- $\psi$ send $k^{\text{th}}$-leaves into $k^{\text{th}}$-leaves,
- $(\phi_i \circ \psi \circ \overline{\phi}_j)_x^k$ are $C^\infty$ and their derivatives vary continuously with $x$. 

53
A filtrated immersion is a filtrated map $\psi$ whose leafwise tangent map is injective: in other words, $(\phi_i \circ \psi \circ \phi_j)^k$ is an immersion.

**Definition 11.4 [Convergence of filtrated maps]** Finally let $\{\psi_n\}_{n \in \mathbb{N}}$ be a sequence of $C^\infty$-filtrated maps from $\mathbb{T}$ to $P$. We say it $C^\infty$ converges on every compact set if

- it converges uniformly on every compact set
- all the derivatives of $(\phi_i \circ \psi_n \circ \phi_j)^k$ converges uniformly on every compact set (as a function of $x$).

**Definition 11.5 [Affine Leaves]** We say that $P$ is $C^\infty$-laminated by affine leaves or carries a leafwise affine structure if furthermore $\phi_{ij}|_{U_x}$ is an affine map.

### 11.2 Holonomy Theorem

We now prove the following result which generalises Ehresmann-Thurston Holonomy Theorem.

**Theorem 11.6** Let $P$ be a $p$-filtrated space. Let $G$ be a group of $C^\infty$-filtrated leafwise immersions of $P$, equipped with the topology of $C^\infty$-convergence on compact sets. Let $V$ be a compact $p$-filtrated space and $\tilde{V}$ be a connected Galois covering with finitely generated Galois group $\Gamma$. Let $U$ the set of homomorphisms $\rho$ from $\Gamma$ to $G$ such that there exists a $\rho$-equivariant filtrated immersion from $\tilde{V}$ to $P$. Then $U$ is open.

Moreover, suppose $\rho_0$ belongs to $U$. Let $f_0$ be a $\rho_0$-equivariant filtrated immersion from $V$ to $P$. If $\rho$ is close enough to $\rho_0$, we may choose a $\rho$-equivariant filtrated immersion $f$ arbitrarily close to $f_0$ on compact sets.

We note again that we could have ripped the word “Hölder” from all the definitions and still have a valid theorem.

**Proof:** We choose a finite covering $U^i$ of $V$ such that

- $U^i$ are charts on $P$,
- $U^i$ are trivialising open sets for the covering $\pi : \tilde{V} \to V$.

We choose an open chart $U^1$ as well as a subset $\tilde{U}^1 \subset \tilde{V}$, such that $\pi$ is a homeomorphism from $\tilde{U}^1$ to $U^1$. We make the following temporary definitions.

A *loop* is a sequence of indices $i_1, \ldots, i_l$ such that $i_l = i_1 = 1$ and

$$U^{i_1} \cap U^{i_1+1} \neq \emptyset.$$  

A loop defines uniquely a sequence of open sets $\tilde{U}^j$ such that

- $\pi$ is a homeomorphism from $\tilde{U}^j$ to $U^{i_j}$,
- $\tilde{U}^j \cap \tilde{U}^{j+1} \neq \emptyset$. 


54
A loop \(i_1, \ldots, i_l\) is trivialising if \(\tilde{U}^1 = \tilde{U}^l\). In general, we associate to a loop the element \(\gamma\) of \(\Gamma\) such that \(\gamma(\tilde{U}^1) = \tilde{U}^p\). The group \(\Gamma\) is the group of loops (with the product structure given by concatenation) modulo trivialising loops. This is just a way to choose a presentation of \(\Gamma\) adapted to the charts \(U^i\).

A coycle is a finite sequence of \(g = \{g^{ij}\}\) of elements of \(G\) such that for every trivialising loop \(i_1, \ldots, i_p\) we have
\[
g^{i_1 i_2} \cdots g^{i_{p-1} i_p} = 1.
\]
It follows that every cocycle \(g\) defines uniquely a homomorphism \(\rho_g\) from \(\Gamma\) to \(G\), and furthermore the map \(g \mapsto \rho_g\) is open.

Let \(g\) be a cocycle, a \(g\)-equivariant map \(f\) is a finite collection \(\{f_i\}\) such that
- \(f_i\) is a filtrated map from \(U^i\) to \(P\),
- on \(U^i \cap U^j\), \(f_j = g^{ij} f_i\).

It is easy to check there is a one to one correspondence between \(\rho_g\)-equivariant laminated maps and \(g\)-equivariant maps.

We now observe the following fact which follows from the existence of partition of unity:

Let \(W_0, W_1, W_2\) be three open set in \(V\), such that
\[
W_0 \subset W_1 \subset W_2.
\]
Let \(h\) a filtrated map defined on \(W_2\), then there exists \(\epsilon\) such that if \(h_1\) is filtrated map defined on \(W_1\) and \(\epsilon\)-close to \(h\) on \(W_1\) then there exists \(h_0\), \(2\epsilon\)-close to \(h\) on \(W_2\), which coincides with \(h_1\) on \(W_0\).

Let us now begin the proof. Let \(g\) be a cocycle associated to a covering \(\mathcal{U} = (U_1, \ldots, U_m)\). Let \(f\) be a \(g\)-equivariant immersion. Let now \(\overline{g}\) be a cocycle arbitrarily close to \(g\).

We proceed by induction to build a \(\overline{g}\)-equivariant map \(\overline{f}\) defined on a smaller covering \(\mathcal{V} = (V_1, \ldots, V_m)\) and close to \(f\). Suppose
\[
\overline{f} = \{\overline{f}_1, \ldots, \overline{f}_{i-1}\}
\]
is a \(\overline{g}\)-equivariant map defined on \(\mathcal{V}^i = (V_1^i, \ldots, V_{i-1}^i)\), with \(V_k^i \subset U_k\).

Suppose also that \(\overline{f}\) is close to \(f\) on \(V_i^j\), with \(l < i\). Let
\[
W_i = V_1^i \cup \ldots \cup V_{i-1}^i.
\]
We now build \(\overline{f}_i\) on a smaller subset \(V_i^i\) of \(U_i\) in the following way: let \(h_i = \overline{g}_i \overline{f}_i\) on \(W_i \cap U_i\). The map \(h_i\) is well defined on \(W_i \cap U_i\) and close to \(f_i\). We use our preliminary observation to build \(\overline{f}_i\) close to \(f_i\) on \(U_i\) and coinciding with \(h_i\) on a slightly smaller open subset \(Z_i\) of \(W_i \cap U_i\). We finally define \(V_{i+1}^i = V_i^i \cap Z_i\).

This completes the induction.

In the end, we obtain a \(\overline{g}\)-equivariant map \(\overline{f}\) defined on slightly smaller open subsets of \(U_i\), and close to \(f\). Therefore, it follows \(\overline{f}\) is an immersion.

The construction above proves also the last part of the statement. Q.E.D.
11.3 Completeness of affine structure along leaves

Let $V$ be a compact space $C^\infty$-laminated by affine leaves. It follows that every leaf carries an affine structure. We shall say $V$ is leafwise complete if the universal cover of every leaf is isomorphic, in the affine category, to the affine space.

We want to prove the following

**Lemma 11.7** Let $V$ be a compact space $C^\infty$-laminated by affine leaves. Let $E$ be the vector bundle over $V$ whose fibre at $x$ is the tangent space at $x$ of the leaf $\mathcal{L}_x$. Assume there exists a one-parameter group $\phi_t$ of homeomorphisms of the leaves such that

- for every leaf $\mathcal{L}_x$, $\phi_t$ preserves $\mathcal{L}_x$ and acts as a one parameter group of translation on $\mathcal{L}_x$, generated by the vector field $X$.
- Let $L = \mathbb{R}.X$, then the action of the lift of $\phi_t$ on the vector bundle $F = E/L$ is uniformly contracting.

Then $V$ is leafwise complete.

**Proof:** For every $x$ in $V$,

$$O_x = \{u \in E_x/x + u \in \mathcal{L}_x\}.$$

Let

$$O = \bigcup_{x \in V} O_x.$$

We observe that $O$ is an open subset of $E$, which is invariant by $\phi_t$. By hypothesis, we have

$$L \subset O.$$

Since $\phi_t$ is contracting on $F = E/L$ and $V$ is compact, $L$ admits a $\phi = \phi_1$ invariant supplementary $F_0$. Let us recall the classical and well known proof of this fact. We choose a supplementary $F_1$ to $L$. Then $\phi^*F_1$ is the graph of an element $\omega$ in $K = F_1^* \otimes L$. We now identify $F_1$ with $F$ using the projection. Since the action of $\phi_t$ is uniformly contracting on $F$, the action of $\phi_{-t}$ is uniformly contracting on $K = F_1^* \otimes L$, hence exponentially contracting by compactness.

It follows the following element of $K = F_1^* \otimes L$ is well defined:

$$\alpha = \sum_{p=-\infty}^{p=-1} (\phi^p)^*\omega.$$

This section $\alpha$ satisfies the cohomological equation

$$\phi^*\alpha - \alpha = \omega.$$

This last equation exactly means that the graph $F_0$ of $\alpha$ is $\phi$ invariant.

Let $u \in E$. Write $u = v + \lambda X$, with $v \in F_0$. It follows that for $n$ large enough, we have $\phi^n(u) \in O$. Since $O$ is invariant by $\phi$, we deduce that $u \in O$. Hence $O = E$ and $V$ is leafwise complete. Q.E.D.
12 Appendix B: the symplectic nature of cross ratio

In this appendix, we explain how to construct different cross ratio from hyper-convex curves from a symplectic point of view. We concentrate on the case of projective spaces, although the construction can be extended to flag manifolds to produce a whole family of cross ratios. However in this case, for the moment, we do not know how to characterise these cross ratios using functional relations, as we did in the case of curves in the projective space.

We also give more precisions concerning cross ratios associated of hyperconvex curves and prove the result used in the proof.

12.1 A symplectic construction

All the examples of cross ratio we have defined may be interpreted from the following "symplectic" construction. Let $V$ and $W$ be two manifolds of the same dimension and $O$ be an open set of $V \times W$ equipped with an exact symplectic structure, or more generally an exact two-form $\omega$. We assume furthermore that the two foliations coming from the product structure

$$
\mathcal{F}_w^+ = O \cap (V \times \{w\}) \\
\mathcal{F}_v^- = O \cap (\{v\} \times W),
$$

satisfy the following properties

1. Leaves are connected.

2. The first cohomology groups of the leaves are reduced to zero.

3. $\omega$ restricted to the leaves is zero.

4. finally, let *squares* be closed curves of the form $c_1 \cup c_2 \cup c_3 \cup c_4$ where $c_1$ and $c_3$ are along $\mathcal{F}_w^+$ and $c_2$ and $c_4$ are along $\mathcal{F}_v^-$; we assume that squares are homotopic to zero.

Remarks:

- When $\omega$ is symplectic, by definition every leaf $\mathcal{F}_w^\pm$ is lagrangian. Then it is a standard fact that it carries a flat affine structure. If every leaf is simply connected and complete from the affine point of view, condition (4) above is satisfied: one may "straighten" the edges of the square, that is deforming them into geodesics of the affine structure, then use these straightening to define a homotopy.

- We shall give later on examples of this situation.
Associated to the above data is a function $B$, called the \textit{polarised cross ratio} defined on

$$U = \{(e, u, f, v) \in V \times W \times V \times W/(e, u), (f, u), (e, v), (f, v) \in O\},$$
in the following way. We consider a map $G$ from the square $[0, 1]^2$ to $O$ such that

- the image of the vertexes $(0, 0), (0, 1), (1, 1), (1, 0)$ are respectively $(e, u), (f, u), (e, v), (f, v)$,
- the image of every edge on the boundary of the square lies in a leaf of $\mathcal{F}^+$ or $\mathcal{F}^-$.

We define the \textit{polarised cross ratio} to be the function defined on $U$ by

$$B(e, u, f, v) = e \int_0^1 G \omega.$$ 

It is easy to check that the definition of $B$ does not depend on the choice of the specific map $G$.

Let now $\xi$ and $\xi^*$ be two maps from $S$ to $V$ and $W$ respectively such that for all distinct $x$ and $y$, $(\xi(x), \xi^*(y))$ lies in $O$. Then, we have the following immediate

\textbf{Proposition 12.1} \textit{The function $b$ defined by}

$$b(x, y, z, t) = B(\xi(x), \xi^*(y), \xi(z), \xi^*(t)).$$

satisfies

$$b(x, y, z, t) = b(z, t, x, y)$$

$$b(x, y, z, t) = b(x, y, z, w)b(x, w, z, t)$$

$$b(x, y, z, t) = b(x, y, w, t)b(w, y, z, t)$$

This function $b$ is not defined for $x = y$ and $z = t$. It follows from the above proposition that it extends to a crossratio provided that

$$\lim_{y \to x} b(x, y, z, t) = 0.$$

We explain quickly a similar construction for triple ratio. We consider a sextuple $(e, u, f, v, g, w)$ in $V \times W \times V \times W \times V \times W$. Let now $\phi$ be a map from the interior of the regular hexagon $H$ in $V \times W$ such that the image of the edges lies in $\mathcal{F}^+$ or $\mathcal{F}^-$, and the (ordered) image of the vertices are $(e, u), (f, u), (f, v), (g, v), (g, w), (e, w)$. We check that the following quantity does not depend on the choice of $\phi$:

$$T(e, u, f, v, g, w) = e \int_H \phi^* \omega.$$ 

Finally using the same notations as above, we check that

$$t(x, y, z) = T(\xi(x), \xi^*(z), \xi(y), \xi^*(x), \xi(z), \xi^*(y)),$$

is the triple ratio as defined in Paragraph 13.3.
12.1.1 Period and action difference

Let $\gamma$ be an exact Hamiltonian diffeomorphism of $O$. Let $\alpha$ and $\beta$ be two fixed points of $\gamma$ and $c$ a curve joining $\alpha$ and $\beta$. Since $\gamma$ is isotopic to the identity, it follows that $c \cup \gamma(c)$ bounds a disc $D$. We define the action difference to be

$$\delta_\gamma(\alpha, \beta, c) = \exp(\int_D \omega).$$

We first recall the

**Proposition 12.2** The quantity $\delta := \delta_\gamma(\alpha, \beta, c)$ just depends on the homotopy class of $c$.

**PROOF**: In our case, this follows from the fact $\omega$ is exact. In general the action difference depends also on a path joining $\gamma$ to the identity. Q.E.D.

In our case, we have a preferred class of curves joining two points as we now explain: let $\alpha = (a, b)$ and $\beta = (\bar{a}, \bar{b})$ be two points of $O$. We notice that since squares are homotopic to zero, we have a well defined homotopy class $c_{a,b,\bar{a},\bar{b}}$ for curves from $(a, b)$ to $(\bar{a}, \bar{b})$, namely curves homotopic to $c^+ \cup c^- \cup \bar{c}^+$, where $c^+$ is a curve along $F^+$ going from $(a, b)$ to $(y, b)$, $c^-$ a curve along $F^-$ going from $(y, b)$ to $(y, \bar{b})$, and $c^-$ a curve along $F^+$ going from $(y, \bar{b})$ to $(\bar{a}, \bar{b})$. By convention we set $\delta_\gamma(\alpha, \beta) = \delta_\gamma(\alpha, \beta, c_{a,b,\bar{a},\bar{b}})$.

Let $\phi = (\rho, \bar{\rho})$ be a representation of $\pi_1(S)$ in the group of Hamiltonian diffeomorphism of $O$ which are restriction of elements of $Diff(V) \times Diff(W)$. Let $(\xi, \xi^*)$ be a $\phi$-equivariant map of $\partial \pi_1(S)$ in $O$. We now prove using the notations of the previous paragraph

**Proposition 12.3** Let $\gamma$ be an element of $\pi_1(S)$ then,

$$b_{\xi, \xi^*}(\gamma^+, y, \gamma^-, \gamma y)^2 = \delta_\phi(\gamma)((\xi(\gamma^+), \xi^*(\gamma^-)), (\xi(\gamma^-), \xi^*(\gamma^+))).$$

In particular,

$$b_{c_{+,+,+}}(\gamma) = \frac{1}{2} \log |\delta_\phi(\gamma)((\xi(\gamma^+), \xi^*(\gamma^-)), (\xi(\gamma^-), \xi^*(\gamma^+)))|.$$

**PROOF**: Let $f = (g, \bar{g})$ be a Hamiltonian diffeomorphism of $O$, restriction of an element of $Diff(V) \times Diff(W)$. Let $(a, b)$ and $(\bar{a}, \bar{b})$ be two fixed points of $f$. Let as before $c = c^+ \cup c^- \cup \bar{c}^+$ composition of

- $c^+$ a curve along $F^+$ from $(a, b)$ to $(y, b)$,
- $c^-$ a curve along $F^-$ from $(y, b)$ to $(y, \bar{b})$,
- and $c^-$ a curve along $F^+$ from $(y, \bar{b})$ to $(\bar{a}, \bar{b})$.

Assume $(a, b)$ and $(\bar{a}, \bar{b})$ be fixed points of $f$. Then $c \cup \gamma(c)$ is a “square”, i.e the composition of...
• a curve along $\mathcal{F}^+$ from $f(y, b) = (g(y), b)$ to $(y, b)$,
• a curve along $\mathcal{F}^−$ from $(y, b)$ to $(y, \bar{b})$,
• a curve along $\mathcal{F}^+$ from $(y, \bar{b})$ to $f(y, \bar{b}) = (g(y), \bar{b})$,
• a curve along $\mathcal{F}^+$ from $(g(y), \bar{b})$ to $(g(y), b)$

Let $D$ be a disk whose boundary is this square. By definition

$$\delta_f ((a, b), (\bar{a}, \bar{b})) = \int_D \omega.$$ 

The proposition follows from the definition of the cross ratio associated to $(\xi, \xi^*)$ when we take $f = (\rho(\gamma), \rho^*(\gamma))$ and

$$(a, b, \bar{a}, \bar{b}) = (\xi(\gamma^+), \xi^*(\gamma^-), \xi(\gamma^-), \xi^*(\gamma^-)).$$

Q.E.D.

### 12.2 Projective spaces

As a specific example of the previous situation, we wish to discuss the following case which makes the link with Section 2.3. Let $E$ be a vector space. We identify $\mathbb{P}(E^*)$ with the set of hyperplanes of $E$. Let

$$\mathbb{P}^2 = \mathbb{P}(E) \times \mathbb{P}(E^*) \setminus \{(D, P)/D \subset P\}$$

Using the identification of $T_{(D, P)}\mathbb{P}^2$ with $\text{Hom}(D, P) \oplus \text{Hom}(P, D)$, let

$$\Omega((f, g), (h, j)) = tr(f \circ j) - tr(h \circ g).$$

Let $L$ be the $\mathbb{R}$-bundle over $\mathbb{P}(n)^2$, whose fibre at $(D, P)$ is

$$L_{(D, P)} = \{u \in D, f \in P^\perp/\langle f, u \rangle = 1\}/\{+1, -1\}.$$

Then

**Proposition 12.4** There exists a connection form $\beta$ on $L$ such that

- Its curvature is symplectic and equal to $\Omega$.

- Let $u \in D \subset E$, then the section

  $\xi_u : P \mapsto (u, f)$ such that $\langle u, f \rangle = 1$, and $f \in P^\perp$

  is parallel for $\beta$ above $\{D\} \times (\mathbb{P}(E^*) \setminus \{P/D \subset P\})$.

- If $h \in \mathbb{P}(F)$, we denote by $\hat{h}$ a nonzero element of $h$. The polarised cross ratio associated to $2\Omega$ is

  $$b(u, f, v, g) = \frac{\langle \hat{f}, \hat{v} \rangle \langle \hat{g}, \hat{u} \rangle}{\langle \hat{f}, \hat{u} \rangle \langle \hat{g}, \hat{u} \rangle}.$$
• Let \( f \) be an element of \( SL(n, \mathbb{R}) \). Let \( D \) (resp. \( \bar{D} \)) be an eigenspace of dimension one for the eigenvalue \( \lambda \) (resp. \( \mu \)). Then \((D, \bar{D})\) is a fixed point of \( f \) in \( \mathbb{P}^2 \). The action of \( f \) on \( L_{(D, \bar{D})} \) is the translation by \( \log |\lambda/\mu| \).

**Proof:** We consider the standard symplectic form \( \Omega^0 \) on \( E \times E^*/\{1, -1\} \). We observe that \( \Omega^0 = d\beta^0 \) where \( \beta^0_{(u,v)}(v,g) = \langle u, g \rangle \). We have a symplectic action of \( \mathbb{R} \) given by \( \lambda.((u,f)) = (\lambda^{-1}u, \lambda f) \), with moment map \( \mu((u,f)) = \langle f, u \rangle \).

We observe that \( L = \mu^{-1}(1) \). Therefore, we obtain that \( \beta = \beta_0|_L \) is a connection form for the \( \mathbb{R} \)-action, whose curvature \( \Omega \) is the symplectic form obtained by reduction of the Hamiltonian action of \( \mathbb{R} \).

We now compute explicitly \( \Omega \). Let \((D, P)\) be an element of \( \mathbb{P}^2 \). Let \( \pi \) be the projection onto \( P \) parallel to \( D \). We identify \( T_{(D, P)} \mathbb{P}^2 \) with \( \text{Hom}(D, P) \oplus \text{Hom}(P^\perp, D^\perp) \). Let \((f, \hat{g})\) be an element of \( T_{(D, P)} \mathbb{P}^2 \). Let \( u \in D \), \( \alpha \in P^\perp \) with \( \langle u, \alpha \rangle \in L \), then \((f(u), \hat{g}(\alpha))\) is an element of \( T_{(u, \alpha)} L \) which projects to \((f, \hat{g})\). By definition of the symplectic reduction if \((f, \hat{g})\) and \((h, \hat{l})\) are elements of \( T_{(D, P)} \mathbb{P}^2 \), then

\[
\Omega((f, \hat{g}), (h, \hat{l})) = \langle \hat{l}(\alpha), f(u) \rangle - \langle \hat{g}(\alpha), h(u) \rangle.
\]

Finally, let \( \pi \) be the projection onto \( P \) in the \( D \) direction. We define

\[
\begin{align*}
\text{Hom}(P, D) & \to \text{Hom}(P^\perp, D^\perp) \\
f & \mapsto \hat{f} = (f \circ \pi)^*.
\end{align*}
\]

In particular

\[
\langle \hat{l}(\alpha), f(u) \rangle = \text{tr}(l \circ f) \langle \alpha, u \rangle
\]

The second point follows immediately by the explicit formula for \( \beta^0 \). Using this, a computation of the holonomy of this connection shows the formula about the cross ratio.

The last point is obvious. Q.E.D.

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