Bounds for some entropies and special functions

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Abstract
We consider a family of probability distributions depending on a real parameter and including the binomial, Poisson and negative binomial distributions. The corresponding index of coincidence satisfies a Heun differential equation and is a logarithmically convex function. Combining these facts we get bounds for the index of coincidence, and consequently for Rényi and Tsallis entropies of order 2.

1 Introduction
For \( c \in \mathbb{R} \), let \( I_c := [0, -\frac{1}{c}] \) if \( c < 0 \), and \( I_c := [0, +\infty) \) if \( c \geq 0 \).

Let \( a \in \mathbb{R} \) and \( k \in \mathbb{N}_0 \); the binomial coefficients are defined as usual by
\[
\binom{a}{k} := \frac{a(a-1)\ldots(a-k+1)}{k!}
\]
if \( k \in \mathbb{N} \), and \( \binom{a}{0} := 1 \).

Consider also a real number \( n > 0 \) such that \( n > c \) if \( c \geq 0 \), and \( n = -cl \) for some \( l \in \mathbb{N} \) if \( c < 0 \).

For \( k \in \mathbb{N}_0 \) and \( x \in I_c \) define
\[
p_{n,k}^c(x) := \left( -\frac{c}{k} \right)^k (-cx)^k (1+cx)^{-\frac{n}{c}-k}, \text{ if } c \neq 0,
\]
\[
p_{n,k}^0(x) := \lim_{c \to 0} p_{n,k}^c(x) = \frac{(nx)^k}{k!} e^{-nx}.
\]

These functions were intensively used in Approximation Theory: see [3], [8], and the references therein.

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In particular,
\[ \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1, \]
so that \( \left(p_{n,k}^{[c]}(x)\right)_{k \geq 0} \) is a parameterized probability distribution.

Its index of coincidence (see [7]) is
\[ S_{n,c}(x) := \sum_{k=0}^{\infty} \left(p_{n,k}^{[c]}(x)\right)^2, \quad x \in I_c. \]

The Rényi entropy of order 2 and the Tsallis entropy of order 2 are given, respectively, by (see [18], [20])
\[ R_{n,c}(x) := -\log S_{n,c}(x); \quad T_{n,c}(x) := 1 - S_{n,c}(x), \]
while the associated Shannon entropy is
\[ H_{n,c}(x) := -\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x), \quad x \in I_c. \]

The cases \( c = -1, \ c = 0, \ c = 1 \) correspond, respectively, to the binomial, Poisson, and negative binomial distributions; see also [13], [14].

It was proved in [12], [15] that the index of coincidence \( S_{n,c} \) satisfies the Heun differential equation
\[ x(1 + cx)(1 + 2cx)S''_{n,c}(x) + (4(n + c)x(1 + cx) + 1) S'_n(x) + +2n(1 + 2cx)S_{n,c}(x) = 0, \quad x \in I_c. \tag{1} \]

It was conjectured in [11] and proved in several papers (for details see [1], [4], [10], [12], [15], [16] and the references given there) that \( S_{n,c} \) is a convex function, i.e.,
\[ S''_{n,c}(x) \geq 0, \quad x \in I_c. \tag{2} \]

It is easy to combine (1) and (2) in order to get
\[ S_{n,c}(x) \leq (4(n + c)x(1 + cx) + 1)^{-\frac{n}{2(n+1)}}, \quad x \in I_c. \tag{3} \]

Particular cases and related results can be found in [12], [15]. Let us remark also that \( S_{n,c}(0) = 1 \).

The upper bound for \( S_{n,c} \), given by (3), leads obviously to lower bounds for the Rényi entropy \( R_{n,c} \) and the Tsallis entropy \( T_{n,c} \).

The following conjecture was formulated in [12] and [15]:

**Conjecture 1.1** For \( c \in \mathbb{R} \), \( S_{n,c} \) is a logarithmically convex function, i.e.,
\( \log S_{n,c} \) is convex.
For $c \geq 0$, U. Abel, W. Gawronski and Th. Neuschel obtained a stronger result:

**Theorem 1.2** ([1]) For $c \geq 0$ the function $S_{n,c}$ is completely monotonic, i.e.,

$$(-1)^j S_{n,c}^{(j)}(x) > 0, \quad x \geq 0, \quad j \geq 0.$$  

Consequently, for $c \geq 0$, $S_{n,c}$ is logarithmically convex.

The following corollary can be found in [16]:

**Corollary 1.3** ([16])

i) Let $c \geq 0$. Then $R_{n,c}$ is increasing and concave, while $T_{n,c}'$ is completely monotonic on $[0, +\infty)$.

ii) $T_{n,c}$ is concave for all $c \in \mathbb{R}$.

Let us remark that the complete monotonicity for the Shannon entropy $H_{n,c}$ was investigated in [16], and for other entropies in [23].

In Sections 2 and 3 we shall use (1) in connection with the log-convexity of $S_{n,c}$, $c \geq 0$, in order to obtain upper-bounds for $S_{n,c}$, sharper than (3); they can be immediately converted into sharp lower-bounds for the Rényi entropy and the Tsallis entropy. Theorem 3.1 provides an upper bound for the modified Bessel function of first kind of order 0.

Section 4 is devoted to the case $c < 0$. In this case Conjecture 1.1 was proved in [17], so that it is again possible to obtain upper-bounds for $S_{n,c}$, sharper than (3).

On the other hand (see [10]), $S_{n,-1}$ is related to the Legendre polynomials $P_n$; using results from [10] we obtain bounds for $S_{n,-1}$ and $P_n$.

Sharp bounds on other entropies can be found in [2], [7], [19], [21] and the references therein.

2 The case $c > 0$

According to Theorem 1.2 log $S_{n,c}(x)$, $x \in [0, +\infty)$, is a convex function, i.e.,

$$S''_{n,c}(x) \geq \left( S_{n,c}'(x) \right)^2 / S_{n,c}(x), \quad x \in [0, +\infty).$$  

(4)

Denote $X := x(1 + cx)$, and therefore $X' = 1 + 2cx$. Then (11) becomes

$$XX' S''_{n,c}(x) + (4(n + c)X + 1) S_{n,c}'(x) + 2nX' S_{n,c}(x) = 0.$$  

(5)

From (4) and (5) we infer that

$$XX' \left( \frac{S_{n,c}'}{S_{n,c}} \right)^2 + (4(n + c)X + 1) \frac{S_{n,c}'}{S_{n,c}} + 2nX' \leq 0.$$  

(6)
This implies
\[
\frac{S'_{n,c}(x)}{S_{n,c}(x)} \leq \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2XX'},
\]  
(7)
and, since \(S_{n,c}(0) = 1\),
\[
\log S_{n,c}(t) \leq \int_0^t \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2XX'} \, dx.
\]  
(8)
Note that \(X^2 = 1 + 4cX\). Now (8) becomes
\[
\log S_{n,c}(t) \leq \int_0^T \frac{\sqrt{1 + 8cX + 16(n^2 + c^2)X^2} - 1 - 4(n + c)X}{2X(1 + 4cX)} \, dX.
\]  
(9)
where \(T := t + ct^2, t \geq 0\).
Moreover, denoting \(\rho := \sqrt{n^2 + c^2}\) and \(R := \sqrt{16\rho^2T^2 + 8cT + 1}\), we have

**Theorem 2.1** The following inequalities hold in the case \(c > 0\):
\[
S_{n,c}^2(t) \leq \frac{2}{1 + 4cT + R} \left( \frac{1}{R + 4nT} \right)^{n/c} \left( \frac{\rho R + 4\rho^2T + c}{\rho + c} \right)^{\rho/c} \leq \frac{1}{1 + 4cT} \left( 1 + 4(n + c)T \right)^{-n/c} \left( 1 + 8\sqrt{n^2 + c^2}T \right)^{\sqrt{n^2 + c^2}/c}.
\]  
(10)
Consequently,
\[
S_{n,c}(t) = \mathcal{O}\left( t^{\frac{\sqrt{n^2 + c^2} - n - c}{c}} \right), \quad t \to \infty.
\]  
(11)

**Proof**
The first inequality in (10) follows from (9) by a straightforward calculation. In order to get the second one it suffices to use the inequalities \(1 + 4cT \leq R \leq 1 + 4\rho T\).

**Remark 2.2** The inequality (10) is stronger than (5); therefore, the bound for \(S_{n,c}\) given in (10) is sharper than the bound given in (3). In particular, (3) yields
\[
S_{n,c}(t) = \mathcal{O}\left( t^{\frac{\sqrt{n^2 + c^2} - n - c}{c}} \right), \quad t \to \infty,
\]  
and comparing with (11) we see that
\[
\frac{\sqrt{n^2 + c^2} - n - c}{c} < -\frac{n}{n + c}.
\]
3 The case \( c = 0 \).

The relations (4) - (9) are still valid with obvious simplifications induced by \( c = 0 \). In particular, (9) reduces to

\[
\log S_{n,0}(t) \leq \int_0^t \frac{\sqrt{1 + 16n^2x^2} - 1 - 4nx}{2x} \, dx,
\]

and this yields

\[
S_{n,0}^2(t) \leq \frac{2 \exp \left( \sqrt{1 + 16n^2t^2} - 1 - 4nt \right)}{1 + \sqrt{1 + 16n^2t^2}}, \quad t \geq 0. \tag{12}
\]

This bound for \( S_{n,0} \) is sharper than the bound furnished by (3) with \( c = 0 \). By using (12) we get also

**Theorem 3.1** Let \( I_0(t) \), \( t \geq 0 \), be the modified Bessel function of first kind of order 0. Then

\[
I_0^2(t) \leq \frac{2 \exp \left( \sqrt{1 + 4t^2} - 1 \right)}{\sqrt{1 + 4t^2} + 1}, \quad t \geq 0. \tag{13}
\]

**Proof**

According to [15, (12)],

\[
I_0(t) = e^t S_{n,0} \left( \frac{t}{2n} \right). \tag{14}
\]

Now (13) is a consequence of (12) and (14).

4 The case \( c < 0 \).

As mentioned in the Introduction, in this case Conjecture 1.1 was proved in [17]. Consequently, with the same notation and the same proof as in Theorem 2.1, we get

**Theorem 4.1** The following inequality holds for all \( c < 0 \) and \( t \in \left[ 0, \frac{1}{\rho} \right] \):

\[
S_{n,c}^2(t) \leq \frac{2}{1 + 4cT + R} \left( \frac{1}{R + 4nT} \right)^{n/c} \left( \frac{\rho R + 4\rho^2T + c}{\rho + c} \right)^{\rho/c}.
\]

Since the log-convexity of \( S_{n,c} \) implies the convexity, the above inequality is sharper than (3). Remember that if \( c < 0 \), then \( n = -cl \) for some \( l \in \mathbb{N} \). It follows that

\[
S_{n,c}(t) = S_{l,-1}(-ct), \quad t \in \left[ 0, \frac{1}{c} \right].
\]

Consequently, in what follows we shall investigate only the function \( S_{n,-1}(x) \) with \( n \in \mathbb{N} \) and \( x \in [0, 1] \).
G. Nikolov proved in [10, Theorem 3] that the Legendre polynomials \( P_n(t) \) satisfy the inequalities
\[
\frac{n(n+1)}{2t + (n-1)\sqrt{t^2-1}} \leq \frac{P_n'(t)}{P_n(t)} \leq \frac{n^2(2n+1)}{(n+1)t + (2n^2-1)\sqrt{t^2-1}}, \quad t \geq 1. \tag{15}
\]

Let
\[ X := x(1-x), \quad t = \frac{2x^2-2x+1}{1-2x} = \frac{1-2X}{X'}, \quad x \in \left[0, \frac{1}{2}\right). \]

Then \( t \geq 1 \) and (see [13, (2.9)], [15, Section 4])
\[
\frac{P_n'(t)}{P_n(t)} = \frac{nX'}{2X} + \frac{1-4X S_{n-1}'(x)}{4X S_{n-1}(x)}. \tag{16}
\]

From (15) and (16) we obtain
\[
-\frac{2nX'}{1+(n-3)X} \leq \frac{S_{n-1}'(x)}{S_{n-1}(x)} \leq \frac{2n(n+1)X'}{n+1 + (4n^2-2n-4)X'}, \quad x \in \left[0, \frac{1}{2}\right]. \tag{17}
\]

Let \( t \in \left[0, \frac{1}{2}\right] \). By integrating in (17) with respect to \( x \in [0,t] \) it follows that
\[
(1 + (n-3)T)^{-\frac{2n}{n+1}} \leq S_{n-1}(t) \leq \left(1 + \frac{4n^2-2n-4}{n+1}T\right)^{-\frac{n(n+1)}{2n+3}}, \tag{18}
\]

where \( T = t(1-t) \) and for \( n = 3 \) the left-hand side is \( e^{-6T} \). Since \( S_{n-1}(1-t) = S_{n-1}(t) \), (18) is valid for \( t \in [0,1] \).

**Remark 4.2** For \( c = -1 \), (3) is a consequence of the inequality
\[
\frac{S_{n-1}'(x)}{S_{n-1}(x)} \leq -\frac{2nX'}{1+4(n-1)X'}, \quad x \in \left[0, \frac{1}{2}\right]. \tag{19}
\]

Comparing (19) with (17), we get
\[
-\frac{2n(n+1)X'}{n+1 + (4n^2-2n-4)X} \leq -\frac{2nX'}{1+4(n-1)X}, \quad x \in \left[0, \frac{1}{2}\right],
\]

and so the second inequality (18) is sharper than (3) with \( c = -1 \).

**Remark 4.3** According to [15, (29)],
\[
S_{n-1}(t) = \frac{1}{\pi} \int_0^1 \frac{(x + (1-x)(1-2t)^2)^n}{\sqrt{x(1-x)}}, \quad t \in [0,1].
\]
It follows that
\[ S_{n,-1}(t) \geq \frac{2}{\pi} \int_0^1 (x + (1 - x)(1 - 2t)^2)^n \, dx, \]
which leads to
\[ 1 - (1 - 4T)^{n+1} \leq S_{n,-1}(t), \quad t \in [0, 1]. \]

This inequality is comparable with the first inequality (18).

The following results can be found also in [13].

Consider the inequality
\[ P'_n(t) P_n(t) \leq \frac{2n^2}{t + (2n - 1)\sqrt{t^2 - 1}}, \quad t \geq 1, \quad n \geq 2, \quad (20) \]
which was established in [10, Theorem 2]. As remarked in [10], (15) is stronger than (20).

From (20) we get by integration
\[ P_n(t) \leq (t + \sqrt{t^2 - 1})^{\frac{n(2n - 1)}{2(2n - 1)}} \left( t + (2n - 1)\sqrt{t^2 - 1} \right)^{-\frac{n(n + 1)}{2n^2 - n - 2}}, \quad t \geq 1, n \geq 2. \]

The stronger inequality
\[ P_n(t) \leq (t + \sqrt{t^2 - 1})^{\frac{n(2n - 2)}{2n^2 - n - 2}} \left( t + \frac{2n^2 - 1}{n + 1} \sqrt{t^2 - 1} \right)^{-\frac{n(n + 1)}{2n^2 - n - 2}}, \quad t \geq 1 \]
is a consequence of (15).

5 Concluding remarks and further work

The index of coincidence \( S_{n,c} \) is intimately related with the Renyi entropy \( R_{n,c} \), Tsallis entropy \( T_{n,c} \) and Legendre polynomial \( P_n \). We established new bounds for \( S_{n,c} \) and, consequently, for \( R_{n,c}, T_{n,c} \) and \( P_n \). Certain convexity properties of \( S_{n,c} \) were instrumental in our proofs. In fact, \( S_{n,c} \) has also other useful convexity properties. For example, for each integer \( j \) in \([1, n]\), \( S_{n,-1} \) is \((2j - 1)-2\)-strongly convex with modulus

\[ 4j^n \binom{2j}{j} \binom{2n-2j}{n-j} \]
(see the pertinent definition in [5]), and for each \( j \geq 1 \), \( S_{n,0} \) is approximately \((2j - 1)-\)concave with modulus

\[ n^{2j} \binom{4j}{2j} \frac{1}{(2j)!} \]
(see the definition in [9]).
On the other hand, according to (1.1), $S_{n,c}$ is a Heun function. By comparing two different expressions of this Heun function it is possible to derive combinatorial identities generalizing some classical ones from [6]. Sample results are

$$
\sum_{j=k}^{n} \binom{j}{k} \binom{2j}{j} \binom{2n-2j}{n-j} = 4^{n-k} \binom{n}{k} \binom{2k}{k}, \quad 0 \leq k \leq n,
$$

$$
\sum_{i=0}^{n-j} \left( -\frac{1}{4} \right)^{i} \binom{n-j}{i} \binom{2i+2j}{i+j} = 4^{j-n} \binom{2j}{j} \binom{2n-2j}{n-j} \binom{n}{j}^{-1}, \quad 0 \leq j \leq n.
$$

All these investigations will be presented in forthcoming papers.

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