Casimir effect for nucleon parity doublets

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Finite volume effects for the nucleon chiral partners are studied within the framework of the parity doublet model. Our model includes the vacuum energy shift for nucleons, that is the Casimir effect. We find that, for the antiperiodic boundary, the finite volume effect leads to chiral symmetry restoration, and the masses of the nucleon parity doublets degenerate. For the periodic boundary, the chiral symmetry breaking is enhanced, and the masses of the nucleons also increase. We also discuss the finite-temperature effect and the dependence on the number of compactified spatial dimensions.

I. INTRODUCTION

Chiral symmetry is a fundamental property of quarks in quantum chromodynamics (QCD). At low temperature and density, chiral symmetry is spontaneously broken by the chiral condensate, which affects the various properties of hadrons, such as masses and decay constants. On the other hand, at high temperature and/or density, chiral symmetry is restored by medium effects, and the hadronic observables are drastically modified. In particular, a useful concept to elucidate the relation between chiral symmetry and hadronic observables is the chiral-partner structure between hadrons. This structure means that the masses (or other observables) of the partners split in the chiral-broken phase and become degenerate in the chiral-restored phase.

The parity doublet model for nucleons was first proposed in Ref. [1] to understand a nucleon doublet [e.g., the positive-parity \( N \) and negative-parity \( N^*(1535) \)] as a chiral partner. This model has been applied not only to investigate the role of chiral-partner structures for baryons in vacuum [1–22] but also to elucidate various physics in nuclear environments such as \( \eta \) mesic nuclei [23–27], hadron modification in matter [28–31], the phase diagrams of the isospin symmetric nuclear matter [32–47], isospin asymmetric nuclear matter [34, 35, 43, 44, 48], thermal nuclear matter [38, 49–51], and magnetized nuclear matter [52], and neutron stars [32, 34, 35, 48, 53–56]. Recently, the degeneracy for the correlators (and masses) of the positive- and negative-parity nucleons was found from lattice QCD simulations at high-temperature phase above the chiral phase transition [57–59]. These results might indicate not only the validity of the parity-doublet picture but also the survival of the chiral invariance at high temperature.

The purpose of this work is to focus on finite volume effects for the nucleon parity-doublet structure. Within the parity doublet model, we consider nucleons inside a finite “box” with a boundary condition. Such a setup will enable us to compare results from the models with observables from lattice QCD simulations. Here, lattice QCD setup has two advantages: (i) We can compactify the arbitrary space-time dimensions, so that we can study not only finite volume effects in the usual 3 + 1 dimensional box but also physics in an “anisotropic box”, such as the (usual) Casimir effect [60] induced by one dimensional compactification, as shown in Fig. 1. (ii) We can choose arbitrary boundary conditions such as periodic and antiperiodic ones, which might modify the infrared part of the momentum of particles. Thus, our studies will be useful for giving us an intuitive interpretation of the role of chiral symmetry in a finite volume.

It should be noted that finite volume effects for the nucleon masses in a box could be estimated within the framework of the chiral perturbation theory (ChPT) with...
baryons [61–68], which have been devoted to compare results with artificial volume effects from lattice QCD simulations. We emphasize that our purpose in this work is to investigate the properties of the nucleon parity doublet in a finite box, where the finite volume effects for \( \sigma \) mean fields will be essential. This is a different situation from the ChPT, where the momentum discretization effects for pion loops would be dominant.

In fact, finite volume effects for the chiral symmetry breaking/restoration have been investigated by effective models such as the Nambu–Jona-Lasinio (NJL) model in 3+1 dimensions [73–80], 2+1 dimensions [81–83], and 1+1 dimensions (or the so-called chiral Gross-Neveu model) [84–91] and linear sigma (or quark-meson) model [92–100]. Also, in order to study the thermodynamics taking into account the deconfinement transition, we can utilize the models implemented the properties of the Polyakov loop, such as the Polyakov-Nambu–Jona-Lasinio (PNJL) Model [101–104] and Polyakov-linear-sigma model [105]. The thermodynamics of hadronic matter without quarks could be investigated by the sigma Model [105], The thermodynamics of hadronic matter without quarks could be investigated by the hadron resonance gas model in a finite volume [106–108]. For nucleon sectors, the Walecka model [109] (or sometimes called the \( \sigma \)-\( \omega \) model) is well-known as a conventional tool to study properties of chiral symmetry of the nuclear matter. Finite volume effects at zero and finite temperature from this model were studied in Ref. [110].

This paper is organized as follows. In Sec. II, we introduce the parity doublet model in a finite box. To compare different models, we also review the case of the Walecka model. After introducing each model, we also introduce the Lambda parameters of the Walecka model [111].

### Table I. Parameters of the Walecka model [111].

| Parameters | Values |
|------------|--------|
| \( m_N \) [MeV] | 939 |
| \( m_\sigma \) [MeV] | 550 |
| \( m_\omega \) [MeV] | 783 |
| \( g_\sigma \) | 10.3 |
| \( g_\omega \) | 12.7 |

interacts with meson fields by the coupling constants, \( g_\sigma \), and \( g_\omega \). For the mesonic part, we include the isoscalar-scalar \( \sigma \) and isoscalar-vector \( \omega \) fields:

\[
L^{\text{mes}}_{\text{Walecka}} = \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - m_\sigma \sigma^2) - \frac{1}{4} \omega_\mu \omega^{\mu \nu} + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu.
\]

By the mean-field approximation, we introduce the classical fields \( \sigma \rightarrow \bar{\sigma} \) and \( \omega^\mu \rightarrow \tilde{\omega}^\mu \). The effective nucleon mass \( M \) and the effective nucleon chemical potential \( \mu^* \) are given by

\[
M = m_N - g_\sigma \bar{\sigma},
\]

\[
\mu^* = m_N - g_\omega \tilde{\omega}^0.
\]

The numerical parameters are shown in Table I.

### B. Parity doublet model

The Lagrangian of the parity doublet model with the mirror assignment [1] is written as follows:

\[
L^{\text{Mirror}} = \bar{\psi}_1 i \gamma_5 \psi_1 + \bar{\psi}_2 i \gamma_5 \psi_2 + m_0 (\bar{\psi}_2 \gamma_5 \psi_1 - \bar{\psi}_1 \gamma_5 \psi_2) + g_1 \bar{\psi}_1 (\sigma + i \gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi_1 + g_2 \bar{\psi}_2 (\sigma - i \gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi_2 + L^{\text{mes}}_{\text{Mirror}},
\]

where \( \psi_1 (\psi_2) \) is a “bare” baryon field with positive (negative) parity, and \( m_0 \) is called the chiral invariant mass mixing \( \psi_1 \) and \( \psi_2 \). The baryon fields interact with the meson fields by the coupling constants, \( g_1 \) and \( g_2 \). For the mesonic part, we include the isoscalar-scalar \( \sigma \), isovector-pseudoscalar \( \vec{\pi} \), and isoscalar-vector \( \omega \) fields:

\[
L^{\text{mes}}_{\text{Mirror}} = \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma) + \frac{1}{2} (\partial_\mu \pi \partial^\mu \pi)
\]

\[
+ \frac{\mu^2}{2} (\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2)^2
\]

\[
+ \epsilon \sigma - \frac{1}{4} \omega_\mu \omega^{\mu \nu} + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu,
\]

where the term with \( \epsilon \sigma \) corresponds to the explicitly chiral symmetry breaking.

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1 In the Lagrangian, the chemical potentials for the bare baryon fields, \( \psi_1 \) and \( \psi_2 \), do not be defined, but we can introduce the baryon chemical potential \( \mu_N \) for the physical nucleon fields after diagonalizing the mass matrix.

For the early works of the finite-volume ChPT (without baryons) by Gasser and Leutwyler, see Refs. [69–72].
TABLE II. Parameters of the parity doublet model [33], where \( f_\pi = 93 \text{ MeV}, \) \( m_\pi = 138 \text{ MeV}, \) and \( \epsilon = m_2^2 f_\pi. \)

| Parameters | Values |
|-----------|--------|
| \( m_0 \) [MeV] | 790 |
| \( g_1 \) | 13.0 |
| \( g_2 \) | 6.97 |
| \( g_\omega \) | 6.79 |
| \( \bar{\mu} \) [MeV] | 199.26 |
| \( \lambda \) | 6.82 |

After applying the mean-field approximation for the scalar and vector fields, \( \sigma \to \bar{\sigma} \) and \( \omega^0 \to \bar{\omega}^0 \), and diagonalizing the mass formula of the nucleon parity doublet, we obtain the mass formulae for the nucleon parity doublet:

\[
M_{\pm} = \frac{1}{2} \left( \sqrt{(g_1 + g_2)^2\bar{\sigma}^2 + 4m_0^2} \mp (g_1 - g_2)\bar{\sigma} \right),
\]

where \( M_+ \) and \( M_- \) are the “physical” nucleon masses with the positive and negative parity, respectively. We note that the \( \bar{\sigma}^2 \) term of Eq. (7) lifts up both the masses \( M_+ \) and \( M_- \), while the linear \( (g_1 - g_2)\bar{\sigma} \) term splits the masses. As an interesting situation, when \( \bar{\sigma} \) is small enough \( (\bar{\sigma} \ll m_0(g_1 - g_2)/(g_1 + g_2)^2) \), the linear term contribution dominates the mass formula, so that the nucleon masses still split from \( m_0 \) and the mass of the positive-parity nucleon becomes smaller than \( m_0 \). Such a situation will be realized in our numerical results. The effective baryon chemical potential \( \mu^* \) are given by

\[
\mu^* = \mu_N - g_\omega \bar{\omega}^0.
\]

The numerical parameters based on Ref. [33] are shown in Table II.3

C. Thermodynamic potentials

The nucleonic part of the thermal potential (per volume \( V \)) at temperature \( T \) is

\[
\frac{\Omega_N(T,\mu_N)}{V} = -\sum_i \gamma_i \int \frac{d^3p}{(2\pi)^3} \left[ E_i(p) \right]
+ T \left\{ \ln \left[ 1 + e^{-\beta(E_i(p) - \mu_i^*)} \right] + \ln \left[ 1 + e^{-\beta(E_i(p) + \mu_i^*)} \right] \right\},
\]

where the index \( i \) of the nucleon degrees of freedom included in the model, labels only \( N \) for the Walecka model and \( N_+ \) and \( N_- \) for the parity doublet. \( \gamma_i = 2 \times 2 \) is the spin-isospin degeneracy factor, and \( E_i(p) = \sqrt{p^2 + M_i^2} \) is the energy of nucleons. The first term of Eq. (9) with the ultraviolet divergence corresponds to the free energy of the vacuum, and the second (third) term is the thermal and density effects for nucleons (antinucleons).

The mesonic parts of the thermodynamic potentials are

\[
\frac{\Omega_{\text{res}}^{\text{Walecka}}}{V} = -\frac{1}{2} m_\pi^2 \bar{\omega}_0^2 + \frac{1}{2} m_\pi^2 \bar{\sigma}^2,
\]

\[
\frac{\Omega_{\text{res}}^{\text{Mirror}}}{V} = -\frac{1}{2} m_\pi^2 \bar{\omega}_0^2 - \frac{1}{2} m_\pi^2 \bar{\sigma}^2 + \frac{1}{4} \lambda \bar{\sigma}^4 - \epsilon \bar{\sigma}.
\]

The potential for the whole system is defined by \( \Omega(T,\mu_N) \equiv \Omega_N(T,\mu_N) + \Omega_{\text{res}}^{\text{Walecka}}(T,\mu_N) \). We solve the gap equations for the mean fields \( \bar{\sigma} \) and \( \bar{\omega}_0 \), which are represented by \( \frac{\partial \Omega(T,\mu_N)}{\partial \bar{\sigma}} = 0 \) and \( \frac{\partial \Omega(T,\mu_N)}{\partial \bar{\omega}_0} = 0 \).

D. Finite volume effect

In the following, we introduce the finite volume effects for the compactified dimension \( \delta \) in the 3+1 dimensional spacetime. Here we focus on the compactification of the one spatial dimension \( \delta = 2 \), also see Fig. 1, which has the spatial \( R^2 \times S^1 \) topology. This setup is the so-called “two parallel plates” geometry and the same situation as the original Casimir effect. For the generalization to arbitrary compactified dimensions, see Appendix A.

For \( \delta = 2 \), we discretize the \( z \) component of the three momentum for nucleon fields:

\[
p_z \to p_z^{2n} = \frac{(2l + 1)\pi}{L},
\]

\[
p_z \to p_z^L = \frac{2\pi}{L},
\]

where \( l = 0, \pm 1, \cdots, \) for the antiperiodic boundary condition \([\psi(\tau, x, y, z) = 0] = -\psi(\tau, x, y, z = L)\) and for the periodic boundary condition \([\psi(\tau, x, y, z = 0) = \psi(\tau, x, y, z = L)\], respectively. The resulting energy is represented by \( E_i(p) = \sqrt{p_z^2 + p_\perp^2 + M_i^2} \), where \( p_\perp^2 = p_x^2 + p_y^2 \). The thermodynamic potential is rewritten as

\[
\frac{\Omega_N(T,\mu_N,L)}{V} = -\sum_i \gamma_i \int_{L=\infty}^{L=0} \int \frac{d^2p_\perp}{(2\pi)^2} \left[ E_i(p) \right]
+ T \left\{ \ln \left[ 1 + e^{-\beta(E_i(p) - \mu_i^*)} \right] + \ln \left[ 1 + e^{-\beta(E_i(p) + \mu_i^*)} \right] \right\},
\]

Here, as finite volume effects, we separate it into two parts: (i) An thermal energy shift for nucleon free energy, which corresponds to the second and third terms of Eq. (14), and (ii) An energy shift for the zero point energy that is the Casimir effect, which corresponds to the first term of Eq. (14).

3 Note that, in Ref. [33], the parameters were determined to reproduce the properties of the nuclear matter. Even though these parameters are applied, we could reproduce the physical quantities in vacuum, such as the decay widths of \( N^* \to \pi N \) by including additional higher-order derivative coupling terms.
The first term of Eq. (14) still includes the vacuum energy with the ultraviolet divergence, but by using a regularization scheme, we can estimate a finite energy shift by finite volume effect, that is the Casimir energy. For the antiperiodic boundary condition, the Casimir energy for massive fermions at zero temperature is given by [110, 112–114]

\[
\Omega_{\text{Cas}}^\text{lp}(L) = \sum_i \gamma_i \sum_{n=1}^\infty (-1)^n \left( \frac{M_i}{n\pi L} \right)^2 K_2(nM_iL), \quad (15)
\]

where \( K_2 \) is the modified Bessel function. For the periodic boundary condition,

\[
\Omega_{\text{Cas}}^\text{p}(L) = \sum_i \gamma_i \sum_{n=1}^\infty \left( \frac{M_i}{n\pi L} \right)^2 K_2(nM_iL). \quad (16)
\]

The convergence of this expansion by the modified Bessel function may practically important. The function is exponentially damping as \( K_2(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \) when \( x \) is large enough. As a result, at a large volume \( L \), the Casimir energy is suppressed, as intuitively expected. Also, the contribution from larger \( n \) terms in the summation can be neglect. In this work, we set the summation up to \( n = 5 \) for numerical calculations. Then the error from this truncation is estimated to be at worst \( \mathcal{O}(1\%) \) due to the factor \( 1/n^2 \).

III. NUMERICAL RESULTS

A. Finite \( L \) transition with antiperiodic boundary

The finite volume effects from the antiperiodic boundary condition are similar with effects from finite temperature. The leading \( (n=1) \) term of the Casimir energy in Eq. (15) has the minus sign for the thermodynamic potential. For a small \( L \), the term is dominated by the second term of \( K_2(x) = 2/x^2 - 1/2 + \mathcal{O}(x^2) \) and it is proportional to \( M_i^2 \). For this reason, smaller nucleon masses by modification of the \( \bar{\sigma} \) mean field are favored, which corresponds to the restoration of chiral symmetry in both the models.

In Figs. 2 and 3, we show the \( L \)-dependence of nucleon masses in the two models with the antiperiodic boundary condition.

FIG. 2. Finite-volume transition of nucleon masses with \( \delta = 2 \) and antiperiodic boundary condition. Upper: Walecka model. Lower: Parity doublet model.

FIG. 3. Finite-volume transition of nucleon masses with \( \delta = 2, 3, 4 \) and antiperiodic boundary condition at \( T = 0 \). Upper: Walecka model. Lower: Parity doublet model.
condition. From these figures, our findings are as follows:

1. In the Walecka model, as $L$ gets smaller, the nucleon mass decreases as shown in the upper panel of Fig. 2. This behavior is induced by the chiral symmetry restoration (or the increase of $\bar{\sigma}$) by finite-volume effect. At the small $L$ limit, the nucleon mass goes to zero.

2. In the parity doublet model, in the large $L$ region, the masses ($M_+$ and $M_-$) of the nucleon doublet split as shown in the lower panel of Fig. 2, which is consistent with those in the infinite-volume limit. As $L$ gets smaller, the masses degenerate, which is also induced by the chiral symmetry restoration (or the reduction of $\bar{\sigma}$) by finite volume effect. In the small $L$ region, the nucleon masses agree with the chiral-invariant mass $m_0$. Around the transition length, the mass splitting in $L \sim 0.8 \text{ fm}$ is dominated in linear $\bar{\sigma}$ term since $\bar{\sigma}$ is finite but small. In the larger $L$ than the transition length, both the masses are lifted up by $\bar{\sigma}^2$ term with large $\bar{\sigma}$ value.

3. In any case, at $T=0$, the transition length is about $L \sim 1 \text{ fm} \sim 0.005 \text{ MeV}^{-1}$. This energy scale is comparable to that of the chiral condensate ($\sim 200 \text{ MeV}$).

4. With increasing temperature, the transition length is shifted to the larger $L$. This is because chiral symmetry is partially restored by thermal effects, and the nucleon masses also decrease.

5. In Fig. 3, we compare the nucleon masses with different compactified dimensions ($\delta = 2, 3, 4$). In both the models, as $\delta$ increases, the transition length becomes larger. This is because the finite volume effect gets stronger by increasing the number of compactified dimensions.

B. Finite $L$ transition with periodic boundary

In contrast to the antiperiodic boundary, the finite volume effects from the periodic boundary condition lead to a characteristic behavior. The Casimir energy in Eq. (16) has the plus sign for the thermodynamics po-
tential, and eventually it is proportional to $-M_i^2$, using $K_0(x) = 2/x^2 - 1/2 + O(x^2)$ for a small $x$. For this reason, larger nucleon masses by modification of the $\bar{\sigma}$ mean field are energetically favored. This corresponds to the increase of the chiral condensate in QCD, which is originally induced by the domination of an infrared quark momentum (or the momentum “zero mode” as Eq. (13) for $l = 0$). Such a catalysis of chiral symmetry breaking by the periodic boundary condition for fermions has been observed also from other chiral effective models (e.g. see [82, 85, 93–95, 99]).

In Figs. 4 and 5, we show the $L$-dependence of nucleon masses with the periodic boundary condition at a fixed $T$. From these figures, our findings are as follows:

1. In the Walecka model, as $L$ gets smaller, the nucleon mass increases as shown in the upper panel of Fig. 4. This is corresponding to the enhancement of chiral symmetry breaking (or the decrease of $\bar{\sigma}$) induced by finite volume effects with the periodic boundary condition.

2. In the parity doublet model, as $L$ gets smaller, the masses of both the nucleons increase as shown in the lower panel of Fig. 4. This is also corresponding to the enhancement of chiral symmetry breaking (or the increase of $\bar{\sigma}$).

3. For both the models, in the large $L$ region, with increasing temperature, the nucleon masses decrease by the chiral symmetry restoration. On the other hand, in the small $L$ region below $L \sim 0.6$ fm $\sim 0.003$ MeV$^{-1}$, the nucleon masses are independent of temperature. This is because the nucleon mass shifts are dominated by the finite volume effects with a large scale ($\sim 330$ MeV) and thermal effects relatively do not contribute to the nucleons.

4. In Fig. 5, we compare the different compactified dimensions ($\delta = 2, 3, 4$). In both the models, as $\delta$
increases, the nucleon masses also increase. This is because the chiral symmetry breaking is enhanced by increasing the number of compactified dimensions.

\section*{C. Finite $T$ transition with antiperiodic boundary}

In this and next sections, we investigate finite volume effects for thermal phase transitions. Notice that, for our parameters, the thermal transition for $\bar{\sigma}$ of the Walecka model at infinite volume is a crossover.\textsuperscript{5} Furthermore, the order for the parity doublet model is also a crossover.\textsuperscript{6}

\textsuperscript{5} If the coupling constant $g_{\sigma}$ is larger, the thermal phase transition can be first order. In this case, finite volume effects have already been studied in Ref. \cite{110}. Therefore, in this work, we focus on the crossover transition.

\textsuperscript{6} If we use another setup for the parity doublet model, the thermal phase transition could be first order. For such a situation, see Appendix B, where a six-point scalar vertex is introduced.

In Figs. 6 and 7, we show the temperature dependence of the nucleon masses at a fixed $L$ with antiperiodic boundary condition. As we mentioned, the finite volume effect from antiperiodic boundary condition is similar to the finite temperature effect. From these figures, our findings are as follows:

1. In the Walecka model, as the $L$ decreases, the nucleon mass at low temperature (in the chiral broken phase) also decreases. The order of the phase transition is still a crossover.

2. In the parity doublet model, as $L$ gets smaller, the nucleon masses at low temperature decrease, and the transition temperature also decreases.

3. In Fig. 7, we compare the nucleon masses with different compactified dimensions ($\delta = 2, 3, 4$). In both the models, as $\delta$ increases, the nucleon masses decrease, and the transition temperature also decreases.
D. Finite $T$ transition with periodic boundary

In Figs. 8 and 9, we show the temperature dependence of nucleon masses at a fixed $L$ with periodic boundary condition. From these figures, our findings are as follows:

1. For both the models, as the $L$ decreases, the nucleon mass in the low-temperature phase increases as the result of chiral symmetry breaking, and the transition temperature also increases. The order of the phase transition becomes first order in small volume $L$.

2. In Fig. 9, we compare the nucleon masses with different compactified dimensions ($\delta = 2, 3, 4$). In both the models, as $\delta$ increases, the nucleon masses increase, and the transition temperature also increases. The order of the phase transition becomes first order in the more compactified case. Thus, a larger $\delta$ provides more substantial finite volume effects.

IV. CONCLUSION AND OUTLOOK

In this work, we have shown finite volume effects for the parity doublet model with the mirror assignment. We introduced the finite volume effects as the Casimir effects. For the antiperiodic boundary condition, the finite volume effects are similar to the effects from finite temperature. Chiral symmetry is restored in smaller volume $L$. We also consider the effect of the periodic boundary. This contribution lifts the masses of nucleons for small volume $L$. In addition, the transition order could change to the first order in small $L$.

In particular, the Walecka and parity doublet models are useful for studying finite-dimension systems, namely nuclear matter. Investigation of the finite volume (and Casimir) effects for the nuclear matter is left for future works [121]. In this situation, we can consider not only $\omega$ mean field but also other mean fields: the competition between the “usual” nuclear matter with the homogeneous $\sigma$ and $\omega$ mean fields and the “anomalous” phase with the inhomogeneous chiral condensate (or the so-called chiral density waves) are also interesting, as discussed in Refs. [40, 46].

In the framework of the parity doublet model, other additional degrees of freedom can be included. For example, the parity doublet model taking into account $\Delta$ isobar (the so-called chiral quartet scheme) was first suggested in Ref. [4], and the properties of symmetric and asymmetric nuclear matter including $\Delta$ isobars were investigated in Ref. [44]. Its thermal behaviors were investigated in Ref. [51]. Moreover, the extension of the parity doublet model to flavor SU(3) would also be interesting [2, 6, 10, 12, 13, 16–19, 38, 51]. To extend the parity doublet structure to the other symmetries [20] would also be useful for understanding the relation between baryon properties and chiral symmetry.

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Appendix A: Derivation of Casimir effects

In this appendix, we introduce the regularization scheme which is essential for the definition of the Casimir effect. Here, we summarize the regularization by the zeta-function.

In order to define the Casimir energy from the thermodynamic potential, we use the Epstein-Hurwitz inhomogeneous zeta function $Y(s)$ (see Ref. [122] for a textbook). For generality, we consider $D$ dimensional spacetime with compactified $\delta$-dimension spacetime. For example, the theory on the $3 + 1$ dimensional spacetime at finite temperature corresponds $D = 4$ and $\delta = 1$.

We consider the potential from the partition function for fields with a mass $M$ and chemical potential $\mu^*$

$$
\frac{\Omega_{V}}{V} = - \frac{\gamma}{\beta L^{\delta} \cdots L_{\delta-1}} \sum_{l_0, \cdots, l_{\delta-1} = -\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\gamma^{D-\delta}}{(2\pi)^{D-\delta}} f(q, p),
$$

where $V$, $\beta = 1/T$, $\gamma$, and $L_i$ are the volume, inverse temperature, degeneracy factor, and size of compactified dimensions, respectively. $q$ and $p$ are the continuous and discretized momenta. $l_0, \cdots, l_{\delta-1}$ is the mode indices for the discretized momenta, and

$$
f(q, p) = \ln \left[ \sum_{i=1}^{\delta-1} (p_i)^2 + \sum_{j=1}^{D-\delta} (q_j)^2 + (p_0 - i\mu^*)^2 + M^2 \right].
$$

Here, $p_i = \frac{2\pi}{L_i} \left( l_i + \frac{\alpha_i}{2} \right)$, $q_j$, and $p_0 = \frac{2\pi}{\beta} \left( l_0 + \frac{\alpha_0}{2} \right)$ are the discretized momentum in the $i$ th dimension, continuous momentum in the $j$ th dimension, and time component, respectively. We also introduce the parameter $\alpha_i$ for denoting a boundary condition. For the antiperiodic boundary condition, this symbol takes $\alpha_i = 1$, and the periodic boundary condition corresponds to $\alpha_i = 0$. This potential diverges by the sum of the contribution from high-momentum modes. We take analytical continuation in $D$ for using a regularization with the zeta function. After the regularization, we perform the $D-\delta$ dimensional integration in the polar coordinates using the relation $\int_0^\infty t^\nu (1 + t)^\delta dt = \frac{\Gamma(1+\nu)\Gamma(-\nu+\delta)}{\Gamma(-\nu)}$. 

$\Gamma$ denotes the related function.
Then the potential $\Omega$ is represented by

$$\Omega(\beta, \{L_i\}, \mu^*) = \frac{\gamma V}{\beta_1 \times \cdots \times L_{\delta-1}} \left[ \frac{\partial Y}{\partial s} \right]_{s=0}, \quad (A3)$$

where the function $Y(s)$ is introduced as

$$Y(s) = \frac{1}{(4\pi)^{(D-2)/2}} \frac{\Gamma(\nu)}{\Gamma(s)} \left[ \Gamma(s - \frac{D-\delta}{2})M^{D-2s} \right]^{\nu}$$

$$\times \sum_{i_0, \ldots, i_{\delta-1} = -\infty}^{\infty} \left[ \left( \frac{2\pi}{\beta} \left( l_0 + \frac{\alpha_0}{2} \right) - i\mu^* \right)^2 + \sum_{i=1}^{\delta-1} \left( \frac{2\pi}{\beta} \left( l_i + \frac{\alpha_i}{2} \right) \right)^2 + M^2 \right]^{\nu}, \quad (A4)$$

with the parameter $\nu = s - D - \delta$.

We expand the function $Y(s)$ by the modified Bessel function $K_n(x)$ for the regularization. The expansion is represented with the parameters $a_i$ and $b_i$ as follows [123]:

$$\sum_{i_0, \ldots, i_{\delta-1} = -\infty}^{\infty} \left[ \sum_{i=1}^{\delta-1} a_i (l_i - b_i)^2 + M^2 \right]^{\nu}$$

$$= \frac{2}{\Gamma(\nu)} \frac{\pi^{\delta/2}}{\prod_{i}^{\infty} 2\pi a_i} \left[ \Gamma(s - \frac{D-\delta}{2})M^{D-2s} \right]^{\nu}$$

$$+ 2 \sum_{a_i}^{\infty} \sum_{n_i=1} \cos(2\pi n_i b_i) \left( \frac{\pi n_i}{\sqrt{a_i} M} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} \left( \frac{2\pi n_i M}{\sqrt{a_i}} \right)$$

$$\times \left( \frac{\pi}{M} \sqrt{\frac{n_i^2}{a_i} + \frac{n_j^2}{a_j}} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} \left( \frac{2\pi M}{\sqrt{n_i^2 a_i + n_j^2 a_j}} \right)$$

$$+ \cdots$$

$$+ 2^{\delta} \sum_{n_0, \ldots, n_{\delta-1}=1}^{\infty} \prod_{i=0}^{\delta-1} \cos(2\pi n_i b_i) \left( \frac{\pi}{M} \sqrt{\sum_{i=0}^{\delta-1} n_i^2 a_i} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} \left( \frac{2\pi M}{\sqrt{\sum_{i=0}^{\delta-1} n_i^2 a_i}} \right). \quad (A5)$$

By using this expansion, we can get the $Y(s)$ explicitly,

$$\frac{\gamma V}{\beta L_1 \times \cdots \times L_{\delta-1}} Y(s)$$

$$= \frac{\gamma V}{(4\pi)^{D/2}} \left[ \Gamma(s - \frac{D}{2})M^{D-2s} \right]^{\nu}$$

$$+ 2 \sum_{i=0}^{\delta-1} \sum_{i=1}^{\infty} (1 - \eta_{i\alpha}) \cosh(n_i L_i \mu_i) \left( \frac{n_i L_i}{2M} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} (n_i mL_i)$$

$$+ 2^{\delta} \sum_{i<j=0}^{\infty} \sum_{n_i, n_j = 1} \cos(2\pi n_i b_i \cos(2\pi n_j b_j)) \left( \frac{\pi}{M} \sqrt{\frac{n_i^2 L_i^2 + n_j^2 L_j^2}{a_i}} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} \left( \frac{M \sqrt{n_i^2 L_i^2 + n_j^2 L_j^2}}{\sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}} \right)$$

$$+ \cdots$$

$$+ 2^{\delta} \sum_{n_0, \ldots, n_{\delta-1}=1}^{\infty} \prod_{i=0}^{\delta-1} \cos(2\pi n_i b_i) \left( \frac{\pi}{M} \sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2} \right)^{s - \frac{D}{2}} K_{s - \frac{D}{2}} \left( \frac{M \sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}}{\sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}} \right). \quad (A6)$$

In order to obtain $\left[ \frac{\partial Y}{\partial s} \right]_{s=0}$ of Eq. (A3), we use the relation for any regular function $G(s)$,

$$\lim_{s \to 0} \frac{d}{ds} \frac{G(s)}{\Gamma(s)} = G(0). \quad (A7)$$

We also use the property of the Bessel function, $K_{-\nu}(x) = K_{\nu}(x)$, and then the thermodynamic potential (A3) can be represented by

$$\frac{\Omega_E}{V} = \Omega_0$$

$$+ \frac{2\gamma}{(2\pi)^{\delta/2}} \left[ 2 \sum_{i=0}^{\delta-1} \sum_{n_i=1}^{\infty} (1 - \eta_{i\alpha}) \cosh(n_i L_i \mu_i) \right]$$

$$\times \left( \frac{M}{n_i L_i} \right)^{\delta/2} K_{\delta/2} (n_i mL_i)$$

$$+ \cdots$$

$$+ 2^{\delta} \sum_{n_0, \ldots, n_{\delta-1}=1}^{\infty} \prod_{i=0}^{\delta-1} (1 - \eta_{i\alpha}) \left[ \cosh(n_i L_i \mu_i) \right]$$

$$\times \left( \frac{M}{\sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}} \right)^{\delta/2} K_{\delta/2} \left( \frac{M \sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}}{\sqrt{\sum_{i=0}^{\delta-1} n_i^2 L_i^2}} \right). \quad (A8)$$

We omit the first term $\Omega_0$ which contains the ultraviolet divergence in infinite volume. This representation reproduces the Casimir energy for $D = 4$ and $\delta = 2$ which is shown in Eqs. (15) and (16).

When we analytically derive the gap equation, the following relation of the Bessel function is useful:

$$\frac{d}{dx} (x^{\alpha} K_\nu(ax)) = -ax^{\nu} K_{\nu-1}(ax). \quad (A9)$$
TABLE III. Parameters of the parity doublet model with the six-point scalar vertex [43], where \( f_\pi = 93\text{MeV}, m_\pi = 140\text{MeV}, \) and \( \epsilon = m_\pi^2 f_\pi \).

| Parameters | Values |
|------------|--------|
| \( m_0 \) [MeV] | 500 |
| \( g_1 \) | 15.4 |
| \( g_2 \) | 8.96 |
| \( g_\omega \) | 11.4 |
| \( \bar{\mu} \) [MeV] | 435 |
| \( \lambda \) | 40.5 |
| \( \lambda_0 \) [MeV\(^{-2}\)] | \( 1.88 \times 10^{-3} \) |

Finally, we comment on anisotropic finite volume as a more special situation. For example, we can consider finite volume for \( L_1 \ll L_2 \neq \infty \). Then, from the equation (A8), the finite volume effects are dominated by contribution from the smaller \( L_1 \). This situation is the same as competitions between finite volume and temperature, such as small volume at low temperature \( (L \ll \beta) \) and large volume at high temperature \( (\beta \ll L) \).

Appendix B: Parity doublet model with the six-point scalar vertex

In this appendix, we check the effect of the six-point scalar vertex in the parity doublet model. This interaction was first introduced in Ref. [43] to reproduce the incompressibility of nuclear matter. The nuclear matter without this interaction was investigated in the early works [32, 33, 49]. Instead of Eq. (6), we set the following mesonic part of the Lagrangian.

\[
\mathcal{L}_{\text{Mirror}}^{\text{mes}} = \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma) + \frac{1}{2} (\partial_\mu \bar{\pi} \partial^\mu \bar{\pi}) + \bar{\mu}^2 (\sigma^2 + \bar{\pi}^2 - \lambda^2 \frac{3}{4} (\sigma^2 + \bar{\pi}^2)^2) + \frac{\lambda_0}{6} (\sigma^2 + \bar{\pi}^2)^3 + \bar{\epsilon} \sigma - \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu, \tag{B1}
\]

where \( \lambda_0 \) is the six-point coupling constant. The numerical parameters based on Ref. [43] are shown in Table III.\(^7\)

In Fig. 10, we show the phase transitions for the antiperiodic boundary condition. In the setup with the six-point vertex, the orders of the finite-volume phase transition at \( T = 0 \) and the thermal phase transition at \( L = \infty \) are first order. We find that, as \( T \) increases, the finite-volume transition becomes a crossover. Similarly, as \( L \) decreases, thermal phase transition becomes a crossover.

In Fig. 11, we show the results for the periodic boundary condition. Notice that, in this figure, not only the minimum of the potential but also the maximum are

\(^7\) Notice that the parameters shown in Erratum of Ref. [43] include errors. The correct parameters are those in the original article.
shown as a solution to the gap equation. Therefore, among the multiple solutions, the lower lines are favored. For example, the lines starting from $M_+ \sim 940$ MeV and $M_- \sim 1500$ MeV at $T = 0$ are favored. In this case, we find a different behavior from the results without the six-point vertex as shown in Fig. 4. In the small $L$ region, we find the disappearance of the solution for the nucleon masses (or the $\bar{\sigma}$ mean field). This is because the six-point vertex term has a minus sign in the thermodynamic potential, so that the potential becomes unstable for a large value of $\bar{\sigma}$. When $L$ is large enough, there is a (local) minimum of the thermodynamic potential, which is stabilized by the four-point scalar vertex term with a positive sign. As $L$ decreases, the minimum is shifted to the larger $\bar{\sigma}$, and eventually, it becomes unstable by the six-point vertex. Thus, in the small $L$ (or large $\bar{\sigma}$) region for the periodic boundary condition, this setup leads to the instability. At least, we emphasize that, from Figs. 4 and 11, the results in the large $L$ region are consistent within the parity doublet model. Such an instability by the six-point vertex could be improved by introducing higher-order terms. In other words, the finite volume with a small $L$ is outside the scope of the parity doublet model with the six-point scalar vertex because of its implicit UV cutoff.

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