Modelling Gossip Interactions in Open Multi-Agent Systems
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Abstract—We consider open multi-agent systems, which are systems subject to frequent arrivals and departures of agents while the studied process takes place. We study the behavior of all-to-all pairwise gossip interactions in such open systems. Arrivals and departures of agents imply that the composition and size of the system evolve with time, and in particular prevent convergence. We describe the expected behavior of the system by showing that the evolution of scale-independent quantities can be characterized exactly by a fixed-size linear dynamical system. We apply this approach to characterize the evolution of the two first moments (and thus also of the variance) for open systems of fixed and variable size. Our approach is based on the continuous-time modelling of random asynchronous events impacting the systems (gossip steps, arrivals, departures, and replacements), and can be extended to other types of events.

Index Terms—Open multi-agent systems, Agents and autonomous systems, Cooperative control, Linear systems, Markov processes.

I. INTRODUCTION

Flexibility and scalability are among the most cited and desired features of multi-agent systems. Real-life examples include flock of birds [1], ad-hoc networks of mobile-devices [2], the Internet, or vehicle coordination [3], for which potential agent failures or new agent arrivals are expected to be handled by the system. However, the classical models used to study multi-agent systems assume that their composition, as complex as it can be, remains unchanged over time, allowing for asymptotic results such as convergence or synchronization.

If arrivals and departures of agents are sufficiently rare as compared to the time-scale of the process that is studied by the system, this apparent contradiction may be justified, as the system is expected to be able to incorporate the effect of such event before the next one occurs. Nevertheless, this may not be the case for certain large systems, as the arrival or departure probability of an agent and the characteristic length of a process both typically grow with the number of agents. Living systems with birth processes or human societies are examples where growth is proportional to the size. Extreme environments where communication within the system may be difficult or infrequent can also lead to slow convergence rates naturally comparable to agent failure rates, and thus also challenge this assumption. Moreover, some systems are inherently open, such as e.g., a stretch of road that vehicles keep joining and leaving forever.

We consider here open multi-agent systems, which are subject to permanent arrivals and departures of agents during the execution of the process that is considered, see e.g., Figure 1. These arrivals and departures result in new challenges for the design and analysis of such systems.

Firstly, every arrival and departure changes the system dimension, posing new challenges for its analysis. Moreover, the system states suffer from repeated potentially important changes, preventing asymptotic convergence to some specific state (see e.g., Figure 1). Instead, the general behavior of the system is expected to reach some form of steady state, which can be described by relevant quantities; as different quantities may behave very differently, their choice is not neutral.

Secondly, arrivals and departures significantly impact the design of decentralized algorithms. On the one hand, departures often imply losses of information that could be needed depending on the nature of the problem. On the other hand, the desired output of the algorithm can be defined by the values held by the agents presently in the system, and thus vary over time: it can then become necessary to eliminate outdated information. Algorithms in open systems must thus be robust to arrivals and departures, and able to handle potentially variable objectives. Such algorithms were already explored for instance for the MAX-consensus problem in [5] or the median consensus problem in [6], both subject to arrivals and departures. Moreover, algorithms designed for open systems cannot be expected to provide “exact” results. Fundamental limitations on the performance of averaging algorithms in open
systems were for instance exposed in [7], [8]. Hence, it may be preferable to maintain an approximate answer robust to perturbations rather than ensuring asymptotic convergence to an exact answer if the composition were to remain constant.

A. Contribution and preliminary results

We focus on the analysis of open multi-agent systems subject to random pairwise gossip interactions [4], with the goal of developing an approach applicable to general open systems. We consider all-to-all (possible) pairwise interactions, focusing on systems where departures and arrivals take place at random times, see Section II for a complete definition.

We show that the evolution of open systems can be analyzed in terms of “scale-independent” quantities, which we call descriptors. We provide a general methodology to analyze their evolution in Section IV-A, following from the observation that our system is a particular case of a general model involving Markov jump processes, whose properties can thus be used. We apply this general approach to our system for two particular cases. In Section IV-B-A we obtain an exact result for the evolution of the descriptors for fixed-size systems subject to only gossip interactions and replacements. In Section IV-B-B we focus on varying-size systems where arrivals and departures are decoupled; for simplicity and due to space limitation we directly work on the variance and obtain an upper bound on its evolution.

Preliminary versions of these results, presented in [9], [10], focused on specific discrete-time systems subject to deterministic or probabilistic events. By contrast, we now provide continuous-time tools that formalize our methodology and allow for more general analyses, relying on simpler and more concise proofs.

B. Other works on open multi-agent systems

Consensus has been considered in different works, e.g., in [11], [12]. Since usual notions of consensus and stability cannot be applied in open systems, those works define alternatives for these criteria, respectively in systems subject to activations and deactivations of nodes in the former, and in systems subject to arrivals and departures in the same sense as our work for the latter. Our approach differs however from those, as it aims at describing the “expected” behavior to which the system is expected to converge, by following relevant scale-independent quantities that capture its “averaged” behavior.

More generally, the possibility for agents to join and leave the system has been recognized in computer science, e.g., through specific architectures designed to deploy large-scale open multi-agent systems [13], or with algorithms designed to maintain connectivity into P2P networks subject to arrivals and departures [14]. Similar frameworks were also considered e.g. in [15] for dealing with autonomous cars joining cross-sections, or in the context of trust and reputation where the reliability of joining agents must be evaluated in [16].

Self-stabilizing protocols in [1], [17], which could also be modelled as open systems, considered the possibility for agents to undergo temporary or permanent failures: the objective is then to ensure asymptotic stability on the desired answer if the system “closes”. Empirical and simulation-based analyses have also been conducted in [18], [19] and in [20] respectively for opinion dynamics and social phenomena subject to arrivals and departures.

Finally, openness is getting attention in the context of decentralized optimization, e.g., in [21] where varying objective functions are considered, and in [22] where the stability of the decentralized gradient descent in open system is analyzed.

II. System description and preliminaries

We consider a multi-agent system whose composition evolves with time. We use integers to label agents, and denote by $N(t) \subset \mathbb{N}$ the set of agents present in the system at time $t$ and by $n(t) = |N(t)|$ the number of agents. Observe that the dimension of the system can change with time. Each agent $i$ holds a value $x_i(t) \in \mathbb{R}$; we assume that the initial value held by the agents at $t = 0$ is a bounded value randomly and independently drawn from some distribution $D$ with mean 0 and variance $\sigma^2$. Our results can immediately be adapted to distributions $D$ with arbitrary constant mean $\mu$. Finally, we define the vector $x(t) \in \mathbb{R}^{n(t)}$ containing the values $x_i(t)$ of all the agents $i \in N(t)$ present in the system at that time.

A. Evolution of the system

The evolution of the system is characterized by four types of instantaneous modifications of both $n(t)$ and $x(t)$: gossip interactions, arrivals, departures and replacements. Those modifications, whose effects are described below, are triggered by independent Poisson clocks with size-dependent rates $\lambda_{Gos_n}, \lambda_{Arr_n}, \lambda_{Dep_n}$ and $\lambda_{Rep_n}$ respectively. In the sequel, we use $y$ and $y^+$ to denote the value of the time-varying quantity $y$ before and after its modification at the activation of a clock.

(a) A gossip interaction among $n$ agents, denoted $Gos_n$, consists in selecting uniformly randomly and independently $i, j \in N(t)$ (with possibly $i = j$) to update their values $x_i$ and $x_j$ by performing a pairwise average $x_i^+ = x_j^+ = \frac{x_i + x_j}{2}$. Hence there holds $n^+ = n$.

(b) An arrival among $n$ agents, denoted $Arr_n$, consists in one “new” agent $i \not\in N(s), \forall s \leq t$ joining the system, so that $N^+ = N \cup \{i\}$, and thus $n^+ = n + 1$. The initial value of the arriving agent is then independently and randomly drawn from the distribution $D$ used to initialize the system.

(c) A departure among $n > 0$ agents, denoted $Dep_n$, consists in selecting an agent $i \in N(t)$ to leave the system, so that $N^+ = N \setminus \{i\}$ and $n^+ = n - 1$.

(d) A replacement among $n > 0$ agents, denoted $Rep_n$, consists in the simultaneous occurrence of both a departure and an arrival. There thus holds $n^+ = n$.

Observe that the way we model arrivals and departures implies that agents leaving the system never join it again (unlike systems subject to temporary disconnections of agents, see e.g., [23]). Moreover, we did not specify the way we select the leaving agent at a departure. In this work, we consider random departures, where the leaving agent is uniformly randomly chosen, but other definitions might be considered, e.g., adversarial departures where no assumption can be made on that choice. We will see later that other or additional clocks,
Lemma 1. In the setting described in Section II, and for any fixed time \( t \geq 0 \), there holds
\[
\mathbb{E}(\bar{x}^2|n(t) = j) \leq \frac{1}{j} \sigma^2; \quad \mathbb{E}(\bar{x}^2|n(t) = 1) \leq \sigma^2,
\]
where we remind \( n(t) \) is the system size at time \( t \).

Observe that at the beginning of the process, there holds \( \mathbb{E}(\bar{x}^2(0)|n(0) = j) = \frac{j}{n^2} \) and \( \mathbb{E}(\bar{x}^2(0)|n(0) = 1) = \sigma^2 \). Hence, the above lemma states that the expected values of the descriptors cannot exceed their initial expected value.

C. Evolution in expectation of the descriptors

Let us define the vector \( X = (\bar{x}^2 \quad \bar{x}^2)^\top \) containing the descriptors, so that \( \text{Var}(x) = \begin{pmatrix} -1 & 1 \end{pmatrix} X = \bar{x}^2 - \bar{x}^2 \). We now show that the evolution of the descriptors in expectation upon activation of the different clocks defining the evolution of the system is governed by a 2-dimensional affine system, from which we also derive the evolution of \( \mathbb{E}\text{Var}(x) \). We use the same notation as in Section II-A in the following lemmas, whose proofs are in Appendix B.

Lemma 2 (Gossip step). When \( \text{Gos}_n \) activates there holds
\[
\mathbb{E}(X^+|X, \text{Gos}_n) = A_{\text{Gos}_n} X = \begin{pmatrix} 1 & 0 \\ \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix} X.
\]

Hence, \( \mathbb{E}(\text{Var}(x^+)|\text{Var}(x), \text{Gos}_n) = (1 - \frac{1}{n}) \text{Var}(x) \).

Lemma 3 (Arrival). When \( \text{Arr}_n \) activates there holds
\[
\mathbb{E}(X^+|X, \text{Arr}_n) = A_{\text{Arr}_n} X + b_{\text{Arr}_n} = \frac{n}{n+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} X + \begin{pmatrix} \frac{1}{n+1} \sigma^2 \\ 1 \end{pmatrix} \sigma^2.
\]

Hence, \( \mathbb{E}(\text{Var}(x^+)|\text{Var}(x), \text{Arr}_n) \leq \frac{n+1}{n} \left( \text{Var}(x) + \frac{\sigma^2}{n} \right) \) holds under the conditions of Lemma 2.

Lemma 4 (Departure). When \( \text{Dep}_n \) activates there holds
\[
\mathbb{E}(X^+|X, \text{Dep}_n) = A_{\text{Dep}_n} X = \left( 1 - \frac{1}{(n-1)^2} \right) \begin{pmatrix} 0 \\ \frac{1}{n} \end{pmatrix} X.
\]

Hence, \( \mathbb{E}(\text{Var}(x^+)|\text{Var}(x), \text{Dep}_n) = \left( 1 - \frac{1}{(n-1)^2} \right) \text{Var}(x) \).

We remind that Lemma 4 holds for random departures, where the leaving agent is randomly uniformly chosen among those in the system before the departure. We now consider the random replacement, which consists of a random departure immediately followed by an arrival. The next result follows from a combination of Lemmas 3 and 5 the latter applied to a system of size \( n-1 \) joined by a \( n^{th} \) agent.

Lemma 5 (Replacement). When \( \text{Rep}_n \) activates there holds
\[
\mathbb{E}(X^+|X, \text{Rep}_n) = A_{\text{Rep}_n} X + b_{\text{Rep}_n} = \begin{pmatrix} n-2 \\ n^2 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sigma^2.
\]

Hence, under the conditions of Lemma 2 there holds
\[
\mathbb{E}(\text{Var}(x^+)|\text{Var}(x), \text{Rep}_n) \leq \frac{n^2 - n - 1}{n^3} \text{Var}(x) + \frac{n^2 - 1}{n^3} \sigma^2.
\]

III. EVOLUTION OF THE DESCRIPTORS AS AN ODE

A. Generalization of the system description

Our system is a particular case of the following more general model: Let \( S(t) = (n(t), x(t)) \) be the state of the system, with \( n(t) \in \mathbb{N} \) the system size at time \( t \), and \( x(t) \in \mathbb{R}^{n(t)} \) the time-varying vector containing the values of all the agents present in the system at that time. The state \( S(t) \) follows a Markov jump process, i.e., a continuous-time Markov process evolving with instantaneous changes at random times, called “jumps”. For each value of \( n(t) \) we associate a finite set of independent Poisson clocks. The activation of a clock \( \epsilon \), which we call “event of \( \epsilon \)”, happens with rate \( \lambda_\epsilon \) and triggers a jump in the process that modifies \( n(t) \) and \( x(t) \). The modification of \( n(t) \) is deterministic, and we can therefore define \( s(\epsilon) \in \mathbb{N} \) and \( a(\epsilon) \in \mathbb{N} \) such that an event of \( \epsilon \) changes \( n(t) \) from \( s(\epsilon) \) to \( a(\epsilon) \). On the other hand, \( x(t) \) is randomly modified according to a distribution that depends on the values of both \( n(t) \) and \( x(t) \). We use \( \Xi \) to denote the set of all the clocks in the system (i.e., for all values of \( n \)).

In our system described in Section II the clocks correspond to the gossip interactions, arrivals, departures and replacements, and we have \( \Xi = \bigcup_{n \in \mathbb{N}} \{ \text{Gos}_n, \text{Arr}_n, \text{Dep}_n, \text{Rep}_n \} \). An event of \( \text{Arr}_n \) for instance happens with rate \( \lambda_\text{Arr}_n \), if the system size is \( s(\text{Arr}_n) = n \), resulting in a jump of the process that changes the system size into \( a(\text{Arr}_n) = n + 1 \) and modifies \( x(t) \) by adding a new agent whose initial value is randomly drawn from the distribution \( D \).
B. General descriptors evolution

The notion of descriptors can also be more generally defined as functions of the system state \( f(S(t)) \), such as \( x^2 \) and \( x^2 t^2 \) that can be formulated as functions of both \( n(t) \) and \( x(t) \). Hence, our goal is to analyze the evolution in expectation of a given bounded function \( f : E \rightarrow \mathbb{R}^d \), where \( E \) is the state space of \( S(t) \) (the state space over which \( S(t) \) is defined). For that purpose, we provide variations of standard results on Markov processes in the following propositions, that are connected with Kolmogorov equations, and that are proved in Appendix C. These results will be the basis for establishing the expected evolution of our descriptors conditioned by the system size.

Proposition 1. Let \( (E, \mathcal{P}(E)) \) be a measurable space, and \( S(t) = (n(t), x(t)) \) a Markov jump process as defined in Section III-A with state-space \( E \). Let \( f : E \rightarrow \mathbb{R}^d \) for some \( d \in \mathbb{N} \) be a measurable bounded function. Moreover, for \( j \in \mathbb{N} \), let \( F_j(t) = \mathbb{E}[f(S(t)) \mid n(t) = j] \) and \( \pi_j(t) = \mathcal{P}[n(t) = j] \). If for all \( \epsilon \in \Xi \) there exist \( A_\epsilon \in \mathbb{R}^{d \times d} \) and \( b_\epsilon \in \mathbb{R}^d \) such that

\[
\mathbb{E}\left[f(S(t^+)) \mid f(S(t)), \epsilon\right] = A_\epsilon f(S(t)) + b_\epsilon,
\]

where \( S(t^+) \) denotes the state of \( S(t) \) after an event of \( \epsilon \) at time \( t \), then there holds

\[
\frac{d}{dt}(F_j(t)\pi_j(t)) = \sum_{\epsilon : \alpha(\epsilon) = j} \lambda_\epsilon \pi_{\alpha(\epsilon)}(t) (A_\epsilon F_{\alpha(\epsilon)}(t) + b_\epsilon) - \sum_{\epsilon : s(\epsilon) = j} \lambda_\epsilon \pi_j(t) F_j(t),
\]

(9)

Proposition 2. In the same setting as Proposition 1 if

\[
\mathbb{E}\left[f(S(t^+)) \mid f(S(t)), \epsilon\right] \leq A_\epsilon f(S(t)) + b_\epsilon
\]

holds instead of (8) for all \( \epsilon \in \Xi \), then there holds

\[
\frac{d}{dt}(F_j(t)\pi_j(t)) \leq \sum_{\epsilon : \alpha(\epsilon) = j} \lambda_\epsilon \pi_{\alpha(\epsilon)}(t) (A_\epsilon F_{\alpha(\epsilon)}(t) + b_\epsilon) - \sum_{\epsilon : s(\epsilon) = j} \lambda_\epsilon \pi_j(t) F_j(t).
\]

(11)

Remark 1. Proposition 2 describes, as a flow equation, how the evolution of a given state of \( n(t) \) contributes to that of \( \mathbb{E}[f(S(t))] = \sum_{j \in \mathbb{N}} F_j(t)\pi_j(t) \). Consider for instance the simple case where \( f(S(t)) = 1 \), then (9) reduces to simple Markov transitions between the states of \( n(t) \). Hence, \( f(S(t)) \) can be seen as some weight distribution on the different states of \( n(t) \), whose evolution is conditioned by that of \( n(t) \).

IV. APPLICATIONS

We now apply the general methodology of the previous section on our system described in Section II, i.e., with \( \Xi = \bigcup_{n \in \mathbb{N}} \{ \text{Gos}_n, \text{Arr}_n, \text{Dep}_n, \text{Rep}_n \} \). We restrict to two particular characterizations of that setting: a fixed-size system where \( n(t) = n \) and a more general setting where agents leave and join the system independently.

A. Fixed-size systems

For illustrative purpose, we first consider a simple setting with only gossip interactions and replacements, so that the number of agents remains constant, i.e., \( n(t) = n \). They respectively take place with rates \( \lambda_{\text{Gos}} = n_\lambda_\gamma \) and \( \lambda_{\text{Rep}} = n_\lambda_\gamma \) for some \( \lambda_\gamma, \lambda_r \geq 0 \), and \( \lambda_{\text{Arr}} = \lambda_{\text{Dep}} = 0 \) as no arrival nor departure ever happens. This means that on average \( n\lambda_\gamma \) gossip interactions and \( n\lambda_r \) replacements happen in the whole system per unit of time, and thus the rate of any event for a given individual agent is independent of \( n \). Moreover, the expected number of gossip interactions taking place between two replacements \( \rho = \lambda_\gamma / \lambda_r \) and the probability of a random event being a replacement \( p = \frac{1}{1+\rho} \) both remain constant as \( n \) grows.

We can now apply Proposition 1 combined with Lemmas 2 and 5 to derive the expected evolution of the descriptors for a system subject to gossip interactions and replacements (i.e., where the replaced agent is uniformly randomly chosen).

Theorem 1. In a system of fixed size subject to random replacements and gossip, there holds

\[
\frac{d}{dt} \mathbb{E}[X(t)] = \left( -2\frac{\lambda_\gamma}{\lambda_r} - \frac{2\lambda_\gamma}{\lambda_r + \lambda_\gamma} \right) \mathbb{E}[X(t)] + \frac{\lambda_\gamma}{\lambda_r + \lambda_\gamma} \mathbb{E}[X(t)]^2. 
\]

(12)

Proof. We consider the function \( f(S(t)) = X(t) \) that computes the descriptors. The result follows from the application of Proposition 1 combined with Lemmas 2 and 5, where we remind that \( \lambda_{\text{Gos}} = n_\lambda_\gamma \) and \( \lambda_{\text{Rep}} = n_\lambda_r \), and where we observe that \( \pi_n(t) = 1 \) and \( F_n(t) = \mathbb{E}[X(t)] \) for any \( t \), so that

\[
\frac{d}{dt} \mathbb{E}[X(t)] = n\lambda_r \left( A_{\text{Rep}} \mathbb{E}[X(t)] + b_{\text{Rep}} \right) + n\lambda_\gamma A_{\text{Gos}} \mathbb{E}[X(t)] - n \left( \lambda_r + \lambda_\gamma \right) \mathbb{E}[X(t)],
\]

where \( A_{\text{Rep}} \) and \( b_{\text{Rep}} \) come from Lemma 5 and \( A_{\text{Gos}} \) from Lemma 2. A few algebraic steps then conclude the proof.

Remark 2. A similar yet weaker result than that of Theorem 1 can be obtained by applying Proposition 2 directly on the variance (denoted \( V(t) \) to lighten notations), yielding

\[
\frac{d}{dt} \mathbb{E}[V(t)] \leq - \left( \lambda_\gamma + \frac{n+1}{n} \lambda_r \right) \mathbb{E}[V(t)] + \frac{n^2 - 1}{n^2} \lambda_r \sigma^2.
\]

(13)

By contrast, Theorem 1 yields an exact equality for \( \frac{d}{dt} \mathbb{E}[V(t)] \) since \( \mathbb{E}[V(t)] = \left( -1 - \frac{1-\rho}{1+\rho} \right) \mathbb{E}[X(t)] \), which thus highlights how following two descriptors allows for a more detailed analysis in this case, even if one were only interested in the variance.

We now analyze the fixed point of (12) given by

\[
\mathbb{E}x^2\big|_{eq} = \frac{2 + \rho}{2n(1 + \rho)} - \frac{\rho}{\sigma^2}, \quad \mathbb{E}x^2\big|_{eq} = \frac{2n + \rho}{2n(1 + \rho)} - \frac{\rho}{\sigma^2},
\]

leading to a variance

\[
\mathbb{E}\text{Var}(x)\big|_{eq} = \frac{1 - \frac{1}{n} + \frac{\rho}{2n}}{1 + \frac{\rho}{2}} \sim \frac{\sigma^2}{\rho + 1},
\]

(15)

where we remind \( \rho = \lambda_\gamma / \lambda_r \) is the ratio between gossip and replacement rates. As gossip becomes less frequent \( (\rho \rightarrow 0) \) the system eventually consists of agents with \( n \) i.i.d. values.
whereas the rates of gossip interactions and departures scale and departures are decoupled, so that the system size changes with 

B. Varying-size systems

We illustrate the results above in Figure 2 for a system of 50 agents subject to replacements and gossips in such a way that on average one event in twenty is a replacement ($\rho = 5$). In particular, increasing the replacement rate makes it increase. By contrast, as replacements become less frequent ($\rho \rightarrow 0$), the system starts behaving as a closed system with $\rho \rightarrow 0$. Moreover, as the system size grows the expected variance becomes $\mathbb{E}\text{Var}(x(t)) \rightarrow \frac{\gamma \sigma^2}{1+\rho} = p\sigma^2$ (where we remind $p$ is the probability for an event to be a replacement), consistently with results derived for DT systems in a preliminary version of this work [10]. In particular, increasing the gossip rate then makes the expected asymptotic variance decay, as the system gets closer to consensus; conversely, increasing the replacement rate makes it increase.

A similar analysis of the convergence rate can be conducted by analyzing the eigenvalues and eigenvectors of the matrix in (12). It is however omitted here due to space limitations.

We illustrate the results above in Figure 2 for a system of $n = 50$ agents subject to replacements and gossips in such a way that on average one event in twenty is a replacement (i.e., $\rho = 19$). The agent initial values are randomly drawn from a normal distribution with zero mean and constant variance $\sigma^2 = 1$. Figure 2(a) shows how convergence is prevented by replacements even if the agents occasionally get close to consensus with a single realization of the system. Figures 2(b)-(d) show that the theoretical results of Theorem 1 and the fixed point in (14) accurately match with the expected evolution of the descriptors from the simulation.

B. Varying-size systems

We now consider a more general case where arrivals and departures are decoupled, so that the system size changes with time. The rate of arrivals is independent of the system size, whereas the rates of gossip interactions and departures scale linearly with it, so that when the system size is $n$ one has $\lambda_{\text{Arr}} = \lambda_a$, $\lambda_{\text{Gos}} = n\lambda_g$ and $\lambda_{\text{Dep}} = n\lambda_d$ for some constants $\lambda_a, \lambda_d, \lambda_g \geq 0$ (and $\lambda_{\text{Rep}} = 0$ as we assume no replacement happens). It implies that the rates of gossip and departures per individual agent remain independent of $n$.

We could derive an exact expression for the evolution of the descriptors following the same approach as for the fixed-size system. However, for simplicity and due to space limitations, we directly work on the expected variance, which only allows obtaining the following upper bound on its evolution.

**Theorem 2.** In a system of variable size subject to arrivals, departures and gossips as described above, there holds

$$
\frac{d}{dt} \mathbb{E}\text{Var}(x(t)) \leq -\lambda_a \mathbb{E}\text{Var}(x(t)) + \mathbb{E}\left[ \frac{\lambda_a}{n(t) + 1} \right] \sigma^2.
$$

**Proof.** Let $f(S(t)) = \text{Var}(x(t))$. The result follows from applying Proposition 2 combined with Lemmas 2 to 4, yielding

$$
\frac{d}{dt} \mathbb{E}\text{Var}(x(t)) \leq -\lambda_a \mathbb{E}\text{Var}(x(t)) + \sum_{j=0}^{\infty} \lambda_a \frac{\sigma^2}{j+1} \mathbb{E}\text{Var}(x(t)) + \sum_{j=1}^{\infty} \frac{\lambda_a}{j+1} \mathbb{E}\text{Var}(x(t)) \pi_j,
$$

with $V = \mathbb{E}\text{Var}(x(t))$ and $\pi_j = P(n(t) = j)$, where we omit the dependence to the time to lighten the notations. Summing up over all values of $j$ then yields

$$
\frac{d}{dt} \mathbb{E}\text{Var}(x(t)) \leq -\lambda_a \mathbb{E}\text{Var}(x(t)) + \sum_{j=0}^{\infty} \frac{\lambda_a \sigma^2}{j+1} e^{-\lambda_a j} \pi_j.
$$

The result above provides an upper bound on the expected evolution of the variance and is parallel to that provided in (13) for the fixed-size system case. Interestingly, it overestimates the effect of arrivals on the variance, while neglecting the slight favourable impact of departures, thus introducing conservatism in the bound.

Observe that the system size evolves following a birth-death process as depicted in Figure 3 with birth and death rates respectively defined by the arrival and departure rates. Hence, we have the following lemma, where one can show $\bar{n} = \lambda_a/\lambda_d$ is the characteristic system size, such that $\lim_{t \rightarrow \infty} \mathbb{E}n(t) = \bar{n}$.

**Lemma 6.** Assume $\bar{n} := \lambda_a/\lambda_d < \infty$, then for all $i \in \mathbb{N}$ there holds $\pi_i^* = \lim_{t \rightarrow \infty} \pi_i(t) = \frac{n}{\bar{n}} e^{-\bar{n}}$.

**Proof.** Standard results on birth-death processes show that if $n(t)$ is ergodic, then $\pi_i^* = \lim_{t \rightarrow \infty} \pi_i(t)$ exists for all $i$ and satisfies $\pi_i^* = \pi_0^* \prod_{j=1}^{\infty} \frac{\lambda_a}{\lambda_d} = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{\lambda_a}{\lambda_d}} = e^{-\bar{n}}$, yielding $\pi_i^* = \frac{n}{\bar{n}} e^{-\bar{n}}$. The conclusion then follows from the fact that $n(t)$ is ergodic if $\lambda_a/\lambda_d < \infty$.

**Lemma 6** allows the derivation of the following corollary.

**Corollary 1.** Let $\bar{n} = \lambda_a/\lambda_d$ and $\gamma = \lambda_g/\lambda_d$, there holds

$$
\lim_{t \rightarrow \infty} \mathbb{E}\text{Var}(x(t)) \leq \frac{1 - e^{-\bar{n}}}{\gamma} \sigma^2.
$$
γ analysis, omitted here, would yield more accurate results.

that its derivation is rather straightforward, and a more detailed
lim λ becomes to zero when

Figure 3. Continuous-time Markov Chain representing the evolution of the
system size as a birth-death process where the birth and death rates are
respectively given by λa and λd when the system size is n(t) = j.

Figure 4. Evolution of the expected asymptotic variance of a system of
initially 5 agents subject to arrivals, departures and gossips as described in
Section V-B with λa = λd = 1 and for different values of λg with respect
to γ = λg/λa. The simulation (in dotted blue) is performed over 1000 events
and compared with the theoretical result [7] from Corollary 1.

Proof. Let us define h(t) := E \( \frac{\lambda_g^2}{n(t)+1} \) = λa \( \sum_{j=0}^{\infty} \frac{\pi_j(t)}{j+1} \sigma^2 \).

Then, using Lemma [6] for λd \neq 0 there holds

\[ \lim_{t \to \infty} h(t) = \lambda_a e^{-\lambda_a} \sum_{j=0}^{\infty} \frac{n}{j+1} \sigma^2 = \lambda_a 1 - \frac{e^{-n}}{n} \sigma^2. \]

Grönwall’s lemma then yields

\[ \mathbb{E} \text{Var}(x(t)) \leq e^{-\lambda_d} \mathbb{E} \text{Var}(x(0)) + \int_0^t h(s)e^{-\lambda_g(t-s)}ds. \]

Standard results from dynamical systems then yield

\[ \lim_{t \to \infty} \int_0^t h(s)e^{-\lambda_g(t-s)}ds = \lim_{t \to \infty} \frac{h(t)}{\lambda_g} = \frac{1 - \frac{e^{-n}}{n} \sigma^2}{\lambda_g}. \]

which concludes the proof.

As soon as \( \bar{n} \) is not very small, the bound on Corollary 1
is approximately \( \frac{\sigma^2}{\lambda_d} \), and hence the expected variance decays
to zero when \( \gamma \) becomes large (i.e., for gossip steps more frequent
than departures). Interestingly, this bound is inversely proportional to the expected number of gossip interactions
involving a given agent before it leaves the system \( \gamma \), similarly to [15] obtained in Section V-A for the fixed-size case (up
to a constant 1). More generally, when λd → 0, then [17]
becomes \( \lim_{t \to \infty} \mathbb{E} \text{Var}(x(t)) \to 0 \). One can actually show that when λd = 0 (i.e., for an only growing system), then
\( \lim_{t \to \infty} \mathbb{E} \text{Var}(x(t)) = 0 \), using a similar proof directly applied
with λd = 0.

Figure 4 compares the result of Corollary 1 with simulated results. It appears that our bound is larger than the actual performance by a factor around 4. This comes from the fact that its derivation is rather straightforward, and a more detailed analysis, omitted here, would yield more accurate results.

V. Conclusions

In this paper, we considered open systems where agents can join and leave, which raise several new challenges including dimension variations and the absence of usual convergence. We have shown that such systems subject to agent interactions can be characterized in terms of scale-independent descriptors whose evolution is governed by a fixed-size linear system. We applied this approach to analyze the behavior of all-to-all pairwise gossips in fixed-size systems subject to replacements, and have shown that the expected evolution of the two first moments (and thus of the variance) is characterized exactly. Our approach relies on continuous-time tools that can be applied to general settings, relying on more complex definitions of arrivals and departures and different types of interactions. We believe our methodology can also be applied to the analysis of interactions restricted to a graph: the challenge is then to find an appropriate set of descriptors that exploit the graph topology, and tight results might not be guaranteed with an inappropriate choice (e.g., for complete bipartite graph, an appropriate choice would be the two first moments for both parts as well as the covariance). Other challenges left untackled consist in establishing the probability distribution of our descriptors instead of only their expected value, and in the introduction of noise and delay in the analysis.

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APPENDIX

A. Proof of Lemma 1 (Upper bound on the descriptors)

Proof. Let $S$ denote the set of agents in the system at time $t$. We first prove that $\mathbb{E} \left( \sum_{i\in S} x_i(t) \mid |S| \right)^2 \leq |S| \sigma^2$ and $\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \mid |S| \right) \leq |S| \sigma^2$ conditional to a particular sequence of events $\omega$, which is equivalent to considering a fixed sequence. We then obtain the final result by taking its expectation over all possible sequences.

Let us fix some time $t \geq 0$ and some sequence of events $\omega$ starting at time $0$ such that the set of agents at time $t$ is $\mathcal{N}(t) = S$, and let $\mathcal{T}$ be the set of the labels of all the agents that have been in the system at some time between times $0$ and $t$ (hence $S \subseteq \mathcal{T}$). Those are determined by $\omega$.

Moreover, let $a_i$ and $d_i$ respectively denote the arrival and departure times of agent $i$ in the system, and we define the vector $\xi(s) \in \mathbb{R}^{|T|}$ for $0 \leq s \leq t$ such that for all $i \in \mathcal{T}$:

$$\xi_i(s) = \begin{cases} x_i(a_i) & \text{if } s \leq a_i \\ x_i(s) & \text{if } a_i \leq s \leq d_i \\ x_i(d_i) & \text{if } d_i \leq s \end{cases}$$

Arrivals and departures have no direct effect on $\xi(s)$, whereas gossip interactions result in the multiplicity of $\xi$ by a doubly-stochastic matrix. Let $A$ be the product of all those matrices since time $0$, then $A$ is doubly-stochastic, and there holds $\xi(t) = A\xi(0)$. One then has $\sum_{i\in S} x_i(t) = \sum_{i\in S} x_i(t) = \sum_{i\in S} \sum_{j\in T} A_{ij} x_j(0)$. Since $A$ is doubly-stochastic, and since $S \subseteq \mathcal{T}$, we have $w_j \leq w_i$. Moreover, we thus $w_i^2 \leq w_j^2$. Hence $\sum_{i\in S} A_{ij} x_j(0) \leq w_j$, and hence

$$\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \right) = \mathbb{E} \left( \sum_{i\in S} w_j \xi_i(0)^2 \right) \leq \sigma^2 \sum_{i\in S} w_j \mathbb{E} \left| \xi_i(0) \right|^2$$

$$\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \right) \leq \sigma^2 \sum_{i\in S} w_j$$.

Thus, we obtain the final result by taking its expectation over all possible sequences.

where the absence of correlation between the initial values $\xi(0)$ were used to nullify the crossed products. Finally, since $\sum_{j\in T} w_j = \sum_{i\in S} \sum_{j\in T} A_{ij} = |S|$, (18) and (19) become

$$\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \right) \leq |S| \sigma^2; \quad \mathbb{E} \left( \sum_{i\in S} \mathbb{E} \left| x_i(t)^2 \right| \right) \leq |S| \sigma^2$$.

Assume now that $\omega$ is a stochastic sequence of events such that $n(t) = |S|$, then $\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \right) \mid \omega, |S| \leq |S| \sigma^2$ and $\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \right) \mid \omega, |S| \leq |S| \sigma^2$ hold from (20).

Since $\omega$ is independent of the agents initial values, taking the expectation over all possible $\omega$ yields

$$\mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \mid |S| \right) \leq |S| \sigma^2; \quad \mathbb{E} \left( \sum_{i\in S} x_i(t)^2 \mid |S| \right) \leq |S| \sigma^2$$.

and it directly follows that

$$\mathbb{E} \left\{ (\hat{x})^2 | n(t) = j \right\} = \mathbb{E} \left\{ \left( \sum_{i\in S} x_i(t)^2 \right) \mid |S| = j \right\} \leq \frac{1}{j} j \sigma^2;$$

$$\mathbb{E} \left\{ (\hat{x})^2 | n(t) = j \right\} = \mathbb{E} \left\{ \left( \sum_{i\in S} x_i(t)^2 \right) \mid |S| = j \right\} \leq \frac{1}{j} j \sigma^2$$.

which concludes the proof.

B. Proofs of Lemmas 2, 3, 4 and 5 (Effect of events)

1) Proof of Lemma 2 (Gossip step): Let us first fix the nodes $i, j$ involved in the gossip step. Firstly, observe that $x_i^2 + x_j^2 = 2 x_i x_j + 2 x_i^2 = x_i + x_j$, and that $x_k^2 = x_k$ for all $k \neq i, j$. Hence $\hat{x} = \bar{x}$, which establishes the first line of (2).

Secondly, since $x_k = x_k^2$ for every $k \neq i, j$, there holds

$$\mathbb{E} \left\{ (\hat{x})^2 \right\} = \frac{1}{n} \sum_{k=1}^{n} (x_k^2)^2 = \bar{x}^2 + \frac{1}{n} \left( 2 \left( \frac{x_i + x_j}{2} \right)^2 - x_i^2 - x_j^2 \right)$$

$$= \bar{x}^2 + \frac{1}{n} \left( x_i x_j - \frac{1}{2} x_i^2 - \frac{1}{2} x_j^2 \right)$$.

(21)

Observe that $\mathbb{E} \left\{ (\hat{x})^2 | x \right\} = \mathbb{E} \left\{ (x_j^2) | x \right\} = \bar{x}^2$ and $\mathbb{E} \left\{ x_i x_j | x \right\} = \bar{x}^2$. Taking the expectation with respect to $i$ and $j$ in (21) yields

$$\mathbb{E} \left\{ (\bar{x})^2 \right\} = \mathbb{E} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2 \right) \mid |S| = j \right\} \leq \frac{1}{j} j \sigma^2$$.

from which the second line of (2) follows. The result on the variance follows from computing $\mathbb{E} \left\{ (\bar{x})^2 \right\}$.

2) Proof of Lemma 3 (Arrival): We label $n+1$ the arriving agent for simplicity, so that $x_k^+ = x_k$ for all $k \leq n$. We begin by computing the new average:

$$\bar{x}^+ = \frac{1}{n+1} \left( x_{n+1} + \sum_{k=1}^{n} x_k \right) = \frac{n}{n+1} \bar{x} + \frac{1}{n+1} x_{n+1}.$$

(22)

Since $x_{n+1} = 0$, we have $\mathbb{E} \left\{ (\bar{x})^2 \right\} = \frac{n}{n+1} \bar{x}^2$. By exactly the same reasoning but using $\mathbb{E} \left\{ (\bar{x})^2 \right\} = \sigma^2$ we also obtain

$$\mathbb{E} \left\{ (\bar{x})^2 \right\} = \frac{n}{n+1} \bar{x}^2 + \frac{1}{n+1} \sigma^2.$$
from which the second line of (3) follows. Turning to the first line, we obtain from (22)
\[
\mathbb{E}((\bar{x}^+)^2 | x) = \frac{1}{(n-1)^2} \left( n^2 \bar{x}^2 + n \bar{x} \mathbb{E} x_{n+1}^+ + \mathbb{E}(x_{n+1}^+)^2 \right)
\]
\[
= \frac{n^2}{(n-1)^2} \bar{x}^2 + 0 + \frac{1}{(n-1)^2} \sigma^2.
\]
Computing \((-1) \mathbb{E}(X^+ | X, \text{Gos}_n)\) then yields the variance, where we use Lemma 1 to obtain the inequality.

3) Proof of Lemma 2 (Departure): Let \(j\) be the randomly selected agent that leaves the system. It follows
\[
\bar{x}^+ = \frac{1}{n-1} \left( \sum_{k=1}^{n} x_k - x_j \right) = \frac{1}{n-1} (n \bar{x} - x_j).
\]
By exactly the same reasoning, there holds \((\bar{x}^+)^2 = \frac{1}{n-1} (n^2 \bar{x}^2 - 2 n \bar{x} x_j + x_j^2)\). Since \(j\) is randomly selected, \(\mathbb{E}(x_j^2) = \bar{x}^2\).

Hence, \(\mathbb{E}(\bar{x}^+)^2 | x) = \frac{1}{(n-1)^2} \left( n^2 \bar{x}^2 - 2 n \bar{x} \mathbb{E}(x_j^2) + (\mathbb{E}(x_j^2))^2 \right) = \frac{n^2}{(n-1)^2} \bar{x}^2 + \frac{1}{(n-1)^2} \sigma^2.
\]
Computing \((-1) \mathbb{E}(X^+ | X, \text{Gos}_n)\) yields the variance.

4) Proof of Lemma 3 (Replacement): The matrix equality (6) follows from a combination of Lemmas 2 and 3, the latter applied to a system of size \(n-1\) joined by an \(n^{th}\) agent.

The inequality on the variance follows from
\[
\mathbb{E}(\text{Var}(x^+) | x, \text{Rep}_n) = -\frac{n-2}{n} \bar{x}^2 - \frac{1}{n^2} \bar{x}^2 - \frac{\sigma^2}{n} - \frac{n-1}{n} \bar{x}^2 + \frac{\sigma^2}{n}
\]
\[
= \frac{n^2}{n^2} \bar{x}^2 - \frac{\sigma^2}{n} + \frac{n^2}{n^2} \mathbb{E}(x^+) + \frac{n^2}{n^2} \mathbb{E}(x^+) - \frac{\sigma^2}{n}
\]
\[
\leq \frac{n^2}{n^2} \bar{x}^2 + \frac{n^2}{n^2} \mathbb{E}(x^+) + \frac{n^2}{n^2} \mathbb{E}(x^+) - \frac{\sigma^2}{n},
\]
where we used Lemma 1 for the last inequality.

C. Proofs of Propositions 1 and 2

We first provide the following definitions.

Definition 1 (Transition kernel). Let \((E, \mathcal{E})\) be a measurable space (i.e., with \(E\) a collection of subsets of \(E\)) and \(S(t)\) an homogeneous Markov process defined over the state space \(E\), then the transition kernel or transition function of \(S(t)\) is the probability measure \(\kappa_x: E \times \mathcal{E} \to [0,1]\) which satisfies for all \(z \in E\), \(B \in \mathcal{E}\) and all time \(t\)
\[
\kappa_x(z, B) = P[S(t) \in B \mid S(t) = z].
\]

Definition 2 (Inifinitesimal generator operator). Let \((E, \mathcal{E})\) be a measurable space and \(S(t)\) an homogeneous continuous-time Markov process with state space \(E\), Let \(F_0(E)\) the set of measurable bounded functions \(f: E \to \mathbb{R}\), then the infinitesimal generator operator of \(S\) is the operator \(\mathcal{L}: F_0(E) \to F_0(E)\) such that for all \(f \in F_0(E)\) and \(x \in E\) there holds
\[
\mathcal{L} f(x) = \lim_{\tau \to 0} \frac{\mathbb{E}[f(S(t+\tau)) \mid S(t) = x] - f(x)}{\tau}.
\]

In the particular case of continuous-time homogeneous Markov jump processes, one can define the nonnegative measure \(q: E \times \mathcal{E} \to [0, +\infty)\) called instantaneous jump rate of \(S\), which satisfies \(\kappa_x(z, B) = q(z, B) \tau + o(\tau)\) for all \(z \in E, B \in \mathcal{E}\) and all \(\tau \geq 0\). Hence \(q\) measures the rate at which \(S(t)\) jumps from \(z\) to one of the states contained in \(B\). In that case, there holds
\[
\mathcal{L} f(x) = \int (f(y) - f(x)) q(x, dy)
\]