Quantum Extended Supersymmetries

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Abstract
We analyse some quantum multiplets associated with extended supersymmetries. We study in detail the general form of the causal (anti)commutation relations. The condition of positivity of the scalar product imposes severe restrictions on the (quantum) model. It is problematic if one can find out quantum extensions of the standard model with extended supersymmetries.

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1 Introduction

The construction of a model for the interaction of elementary particles should have the ultimate goal of providing a quantum model. Indeed, from the phenomenological point of view a classical Yang-Mills field is without relevance. The standard model of the elementary particles \cite{25} is basically a quantum model: to check it experimentally one needs only the Feynman rules i.e. the expressions for the propagators and the vertices. But to give the propagators of some model is equivalent to specify the set of quantum fields relevant for the model and the expression of the vertices is nothing but the interaction Lagrangian. So the phenomenological point of view fits very nicely with the Bogoliubov version of perturbation theory; in this approach the basic input is a given Hilbert space $\mathcal{H}$ (which is taken of Fock type generated from the vacuum state by some set of free quantum fields) and the interaction Lagrangian (which is some Wick polynomial in these fields). So, all considerations of the standard model as classical field theory followed by some quantization procedure should be viewed only as some auxiliary steps leading to the quantum model. It is not at all clear if a quantization procedure really provides a Hilbert space description of the classical model. This has to be checked in detail after the quantum model has been constructed. In particular the positivity of the scalar product is essential: otherwise we would have negative probabilities for some transition processes.

In two preceding papers \cite{12}, \cite{13} we have examined critically the possible supersymmetric extensions of the standard model for the case un-extended supersymmetry i.e. $N = 1$. The main point was that the construction of a quantum supersymmetric multiplet is more restrictive that the classical construction. The main difference lies in the requirement that the multiplet has a \emph{bona fide} representation in a Hilbert space. The formal construction of a quantum supersymmetric multiplet is done by applying some (free) quantum fields on the vacuum state: in the language of axiomatic field theory this means that we construct the Borchers algebra (see for instance \cite{6}). However, one has to provide a positively defined scalar product: only in this case one gets from the Borchers algebra a Hilbert space by a standard procedure. For free fields the scalar product can be obtained from the form of the causal (anti)commutation relations using Källan-Lehman representation theorem. But the supersymmetric algebra puts severe restrictions on the most general form of these causal (anti)commutators and it is not guaranteed that the positivity can be always enforced. In \cite{13} we have found out that this imposes severe restrictions on the free parameters of the model: in particular the vector model has a Hilbert space representation only for positive mass.

In this paper we investigate critically other supersymmetric models based on extended supersymmetries. We are interested only in models which can be used for a supersymmetric extension of the standard model (SM) and without particles of spin higher that 1. From the analysis of the irreducible representations of the supersymmetric algebra it is known that there are only five multiplets describing irreducible representations with particles of spin $s \leq 1$ namely (see for instance \cite{3}): for $N = 1$ the chiral multiplet and the vector multiplet; for $N = 2$ the...
hyper-multiplet and a vector multiplet; for $N = 4$ a vector multiplet. At the level of classical field theory these models have been studied in detail: see for instance [7], [10], [14], [23], [24].

In this paper we analyze in the spirit of [13] the last three cases. We find out that there are considerable difficulties to construct supersymmetric extensions of the standard model for $N = 2$ and $N = 4$ contrary to what it is asserted in the literature.

In Section 2 we provide a general discussion of extended supersymmetry for quantum models. In Section 3 we consider a $N = 2$ extended model without central charges containing spin 1 particles. In Section 4 we consider the so-called hyper-multiplet which can be used to describe matter. In Section 5 we study a $N = 4$ extended model without central charges containing spin 1 particles. In principle the results from Sections 3 and 5 could be used for construction supersymmetric extensions of the standard model of elementary particles [25] but we point out some problems in achieving this goal.

2 Extended Supersymmetries

We remind here the definition of a extended supersymmetric theory in a pure quantum context. We use the notations from [12].

The conventions are the following: (a) we use summation over dummy indices; (b) we raise and lower Minkowski indices with the Minkowski pseudo-metric $g_{\mu\nu} = g^{\mu\nu}$ with diagonal $1, -1, -1, -1$; (c) we raise and lower Weyl indices with the anti-symmetric $SL(2, \mathbb{C})$-invariant tensor $\epsilon_{ab} = -\epsilon^{ab}$; $\epsilon_{12} = 1$; (d) we denote by $\sigma^\mu$ the usual Pauli matrices with elements denoted by $\sigma_{ab}^\mu$ and the convention $\sigma^0 = 1$.

Suppose that we have a quantum theory of free fields; this means that we have the following construction:

- $\mathcal{H}$ is a Hilbert space of Fock type (associated to some one-particle Hilbert space describing some choice of elementary particles) with the scalar product $(\cdot, \cdot)$;
- $\Omega \in \mathcal{H}$ is a special vector called the vacuum;
- $U_{a,A}$ is a unitary irreducible representation of $inSL(2, \mathbb{C})$ the universal covering group of the proper orthochronous Poincaré group such that $a \in \mathbb{R}^4$ is translation in the Minkowski space and $A \in SL(2, \mathbb{C})$;
- $U \mapsto V_U$ is a unitary representation of the compact group $G$ (usually $SU(N)$ or $SO(N)$); $V_U$ commutes with $U_{a,A}$
- $b_J, \ J = 1, \ldots, N_B$ (resp. $f_A, \ A = 1, \ldots, N_F$) are the quantum free fields of integer (resp. half-integer) spin. We assume that the fields are linearly independent up to equations of motion;
• The equations of motion do not connect distinct fields.

In this paper, we consider only particles of spin \( s \leq 1 \). All fields verify Klein-Gordon equation; if the Fermionic fields are Majorana they verify Dirac equation.

Now we define the notion of extended supersymmetry invariance of the system of Bosonic and Fermionic fields considered above. Suppose that in the Hilbert space \( \mathcal{H} \) we also have the operators \( Q_{ja} \), \( j = 1, \ldots, N \), \( a = 1, 2 \) such that:

(i) the following relations are verified:

\[
Q_{ja} \Omega = 0, \quad \bar{Q}_{ja} \Omega = 0 \quad (2.1)
\]

\[
[Q_{ja}, P_\mu] = 0, \quad U_A^{-1}Q_{ja}U_A = A^a_b Q_{jb}, \quad \forall A \in SL(2, \mathbb{C})
\]

\[
V_U^{-1}Q_{ja}V_U = \rho(U)_{jk} Q_{ka}, \quad \forall U \in G \quad (2.2)
\]

and

\[
\{Q_{ja}, Q_{kb}\} = \epsilon_{ab} Z_{jk}, \quad \{Q_{ja}, \bar{Q}_{kb}\} = 2\delta_{jk} \sigma^\mu_{ab} P_\mu, \quad (2.3)
\]

Here \( P_\mu \) are the infinitesimal generators of the translation group given by

\[
[P_\mu, b] = -i \partial_\mu b, \quad [P_\mu, f] = -i \partial_\mu f, \quad (2.4)
\]

\[
Q_{jb} \equiv (Q_{jb})^\dagger, \quad (2.5)
\]

\( Z_{jk} \) are the so-called central charges and, by definition, they commute with all other SUSY generators and \( U \mapsto \rho(U) \) is \( N \)-dimensional a representation of the group \( G \). We will consider only the case \( \rho = Id \).

(ii) The following commutation relations are true:

\[
i [Q_{ja}, b] = p_j (\partial) f \quad \{Q_{ja}, f\} = q_j (\partial) b
\]

\[
i [Z_{jk}, b] = p_{jk} (\partial) b \quad i[Z_{jk}, f] = q_{jk} (\partial) f. \quad (2.6)
\]

Above \( b \) (resp. \( f \)) is the collection of all integer (resp. half-integer) spin fields and \( p, q \) are matrix-valued polynomials in the partial derivatives \( \partial_\mu \) (with constant coefficients). These relations express the tensor properties of the fields with respect to (infinitesimal) supersymmetry transformations.

If this conditions are true we say that \( Q_{ja} \) are super-charges and \( b, f \) are forming a supersymmetric multiplet. The notion of irreducibility can be defined for any supersymmetric multiplet if we consider the quantum fields as a modulus over the ring of partial differential operators. As emphasized in [12], the matrix-valued operators \( p \) and \( q \) are subject to some constraints which generalize the case \( N = 1 \) (see [12] and [13]).
• From the compatibility of (2.6) with Lorentz transformations it follows that these polynomials are Lorentz covariant.

• The equation of motion are supersymmetric invariant, i.e. if we take the commutator of the supercharges $Q_{ja}$ and $\bar{Q}_{k\bar{a}}$ with the equations for the Bosonic fields we obtain zero modulo the equation of motion for the Fermionic fields and vice-versa.

• To verify the validity of (2.3) it is necessary and sufficient to prove they commute with all the fields $b$ and $f$ of the model:

$$[[Q_{ja}, Q_{kb}] - \epsilon_{ab}Z_{jk}, b] = 0, \quad [[Q_{a}, \bar{Q}_{b}] - 2\sigma_{ab}^{\mu}P_{\mu}, b] = 0$$

$$[[Q_{a}, Q_{b}] - \epsilon_{ab}Z_{jk}, f] = 0, \quad [[Q_{a}, \bar{Q}_{b}] - 2\sigma_{ab}^{\mu}P_{\mu}, f] = 0;$$

(2.7)

this follows from (2.1) and the fact that the Hilbert space is generated by vectors of the type

$$\Psi = b_{j_{1}}(x_{1}) \cdots b_{j_{p}}(x_{p})f_{A_{1}}(y_{1}) \cdots f_{A_{q}}(y_{q})\Omega \in \mathcal{H}.$$  

(2.8)

Using the (graded) Jacobi identities it follows that we must check:

$$\{Q_{ja}, [\bar{Q}_{kb}, b]\} - \{\bar{Q}_{kb}, [b, Q_{ja}]\} + [b, \epsilon_{ab}Z_{jk}] = 0$$

$$\{Q_{ja}, \{Q_{kb}, f\}\} + [\bar{Q}_{kb}, \{f, Q_{ja}\}] + [f, \epsilon_{ab}Z_{jk}] = 0$$

$$\{Q_{ja}, [\bar{Q}_{kb}, b]\} - \{\bar{Q}_{kb}, [b, Q_{ja}]\} + [b, 2\delta_{jk}\sigma_{ab}^{\mu}P_{\mu}] = 0$$

$$[Q_{ja}, \{\bar{Q}_{kb}, f\}] + [\bar{Q}_{kb}, \{f, Q_{ja}\}] + [f, 2\delta_{jk}\sigma_{ab}^{\mu}P_{\mu}] = 0.$$  

(2.9)

• Causal (anti)commutation relations are verified by the free fields $b$ and $f$:

$$[b(x), b(y)] = -ip(\partial)D(x - y) \quad \{f(x), f(y)\} = -iq(\partial)D(x - y)$$

(2.10)

where $D$ is Pauli-Jordan causal distribution. Reasoning as above it follows that new consistency relations are valid following again from (graded) Jacobi identities; the non-trivial ones are:

$$[b(x), \{f(y), Q_{ja}\}] + \{f(y), [Q_{ja}, b(x)]\} = 0$$

(2.11)

• If one considers higher-spin fields (more precisely $s \geq 1$), is necessary to extend somewhat this framework: one considers in $\mathcal{H}$ besides the usual positive definite scalar product a non-degenerate sesqui-linear form $<\cdot, \cdot>$ which becomes positively defined when restricted to a factor Hilbert space $Ker(Q)/Im(Q)$ where $Q$ is some gauge charge. We denote with $A^\dagger$ the adjoint of the operator $A$ with respect to $<\cdot, \cdot>$.  

4
The gauge supercharge \( Q \) is usually determined by relations of the type (2.6) involving ghost fields also, so it means that we must impose consistency relations of the same type as above. Moreover, it is desirable to have

\[
\{ Q, Q_{ja} \} = 0, \quad \{ Q, \bar{Q}_{ja} \} = 0; \tag{2.12}
\]

this implies that the supersymmetric charges \( Q_{ja} \) and \( \bar{Q}_{ja} \) factorizes to the physical Hilbert space \( \mathcal{H}_{\text{phys}} = \text{Ker}(Q)/\text{Im}(Q) \). This implies new consistency relations of the type (2.9) with one of the supercharges replaced by the gauge charge:

\[
\{ Q_{ja}, [Q, b] \} + \{ Q, [Q_{ja}, b] \} = 0 \quad [Q_{ja}, \{ Q, f \}] + [Q, \{ Q_{ja}, f \}] = 0. \tag{2.13}
\]

- A relation of the type (2.13) must be also valid for the gauge charge:

\[
[b(x), \{ f(y), Q \}] + \{ f(y), [Q, b(x)] \} = 0. \tag{2.14}
\]

- To have \( Q^2 = 0 \) we must also impose

\[
\{ Q, [Q, b] \} = 0 \quad [Q, \{ Q, f \}] = 0. \tag{2.15}
\]

In the presence of a gauge charge one can relax (2.9): Indeed one can factorize the supercharges and the representation of the Poincaré group to the physical Hilbert space \( \mathcal{H}_{\text{phys}} = \text{Ker}(Q)/\text{Im}(Q) \) and require the (2.9) are valid only for these factorized operators [13].
3 \textit{N = 2 with the internal symmetry group SU(2) and without central charges}

\textbf{Theorem 3.1} \textit{Let us consider the multiplet \((z, D, F_{ab}, C_{jk}, \lambda_{ja}, \chi_{ja})\), \(j = 1, 2\) where:}

- \(z, D\) are two complex Bosonic scalar field which are SU(2) scalars;

- \(F_{ab}\) are complex Bosonic fields which are SU(2) scalars and such that
  \[ F_{ab} = F_{ba}; \]  
  \[(3.1)\]

- \(C_{jk}\) are complex Bosonic fields which verify
  \[ V^{-1}_U C_{jk} V_U = U_{jl} U_{km} C_{lm}, \quad \forall U \in SU(2) \]  
  \[ \text{and such that} \]  
  \[ C_{jk} = C_{kj}; \]  
  \[(3.2)\]

- \(\lambda_{ja}, \chi_{ja}\) are Fermionic Dirac spinor fields and verifying
  \[ V^{-1}_U \lambda_{ja} V_U = U_{jk} \lambda_k, \quad V^{-1}_U \chi_{ja} V_U = U_{jk} \chi_{ka}, \quad \forall U \in SU(2) \]  
  \[(3.3)\]

\textit{The action of the supercharges on these fields is well defined through:}

\[ i[Q_{ja}, z] = \chi_{ja} \quad [\bar{Q}_{j\bar{a}}, z] = 0. \]  
\[ (3.5) \]

\[ [Q_{ja}, F_{bc}] = \epsilon_{ab} \lambda_{jc} + \epsilon_{ac} \lambda_{jb} \]  
\[ (3.6) \]

\[ [\bar{Q}_{j\bar{a}}, F_{bc}] = -i \epsilon_{jk} (\sigma_{\mu b} \partial_\mu \chi_{kb} + \sigma_{\mu a} \partial_\mu \chi_{kc}) \]  
\[ (3.7) \]

\[ [Q_{ja}, C_{kl}] = - (\epsilon_{jk} \lambda_{la} + \epsilon_{jl} \lambda_{ka}) \]  
\[ (3.8) \]

\[ [\bar{Q}_{j\bar{a}}, C_{kl}] = i \sigma_{\mu b} \partial_\mu (\delta_{jk} \chi_{bl} + \delta_{jl} \chi_{bk}) \]  
\[ (3.9) \]

\[ [Q_{ja}, D] = 0 \]  
\[ (3.10) \]

\[ [\bar{Q}_{j\bar{a}}, D] = 2i \epsilon_{jk} \sigma_{\mu a} \partial_\mu \lambda_{kb} \]  
\[ (3.11) \]

\[ \{Q_{ja}, \chi_{kb}\} = \epsilon_{jk} F_{ab} + \epsilon_{ab} C_{jk} \]  
\[ (3.12) \]

\[ \{Q_{ja}, \chi_{kb}\} = 2 \delta_{jk} \sigma_{\mu a} \partial_\mu z^a \]  
\[ (3.13) \]

\[ \{Q_{ja}, \lambda_{kb}\} = \epsilon_{jk} \epsilon_{ab} D \]  
\[ (3.14) \]

\[ \{\bar{Q}_{j\bar{a}}, \lambda_{ka}\} = -i (\delta_{jk} \sigma_{\mu c} \epsilon^{cd} \partial_\mu F_{ad} + \epsilon_{jl} \sigma_{\mu a} \partial_\mu C_{lk}). \]  
\[ (3.15) \]

\textit{If the field \(z\) verifies the Klein-Gordon equation for mass \(m\) then all fields of the multiplet verify the same equation.}
**Proof:** One can start from the existence of the fields $z$ and $\chi_{kb}$ derive the other fields of the multiplet from the Jacobi identities (2.9).

1. From the relation involving $Q_{ja}, Q_{kb}$ and $z$ we get that the expression $\{Q_{ja}, \chi_{kb}\}$ is antisymmetric in the couples $ja$ and $kb$ so if we define

$$F_{ab} \equiv \frac{1}{2} \epsilon_{jk} \{Q_{ja}, \chi_{kb}\}, \quad C_{jk} \equiv \frac{1}{2} \epsilon^{ba} \{Q_{ja}, \chi_{kb}\}$$

we obtain the action of the supercharge $Q_{ja}$ on $\chi_{kb}$.

2. The relation involving $Q_{ja}, Q_{kb}$ and $z^*$ is an identity.

3. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $z$ gives the action of the supercharge $\bar{Q}_{k\bar{b}}$ on $\chi_{ja}$.

4. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $\chi_{lc}$ gives the action of the supercharge $\bar{Q}_{k\bar{b}}$ on $F_{ac}$ and $C_{jk}$.

5. The relation involving $Q_{ja}, Q_{kb}$ and $\bar{\chi}_{lc}$ is an identity.

6. The relation involving $Q_{ja}, Q_{kb}$ and $\chi_{lc}$ leads us to the introduction of a new field

$$\lambda_{ja} \equiv -\frac{1}{3} \epsilon_{kl} [Q_{ka}, C_{jl}]$$

and the action of the supercharge $Q_{ka}$ on $F_{bc}$ and $C_{jl}$ follows.

7. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $C_{lm}$ give the action of $\bar{Q}_{k\bar{b}}$ on $\lambda_{ma}$.

8. The relation involving $Q_{ja}, Q_{kb}$ and $C_{lm}$ leads us to the introduction of a new field

$$d_{ab} \equiv \epsilon_{kl} \{Q_{kb}, \lambda_{la}\};$$

one finds out the this expression must be antisymmetric in $a$ and $b$ so in fact the new field is the (complex) scalar field $D$ such that:

$$d_{ab} = \frac{1}{2} \epsilon_{ab} D;$$

we now have the action of $Q_{ja}$ on $\lambda_{kb}$.

9. The relation involving $\bar{Q}_{j\bar{a}}, \bar{Q}_{k\bar{b}}$ and $C_{lm}$ is an identity.

10. The relation involving $Q_{ja}$ (or $\bar{Q}_{j\bar{a}}$), $Q_{kb}$ (or $\bar{Q}_{k\bar{b}}$) and $F_{cd}$ are identities.

11. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $\lambda_{lc}$ gives the action of $\bar{Q}_{k\bar{b}}$ on $D$. 

7
12. The relation involving $Q_{ja}, Q_{kb}$ and $\lambda_{lc}$ gives the action of $Q_{kb}$ on $D$.

13. The relation involving $\bar{Q}_{j\bar{a}}, \bar{Q}_{k\bar{b}}$ and $\lambda_{lc}$ is an identity.

14. The relation involving $Q_{ja}$ (or $\bar{Q}_{j\bar{a}}$), $Q_{kb}$ (or $\bar{Q}_{k\bar{b}}$) and $D$ are identities.

Finally one notices the compatibility with the $SU(2)$ transformations of all the relations from the statement. ■

It is convenient to replace $F_{ab}$ by new fields; we proceed in two stages with two elementary proposition.

**Proposition 3.2** We can write uniquely

\[
F_{ab} = \sigma_{ab}^{\mu\nu} F_{\mu\nu} \tag{3.20}
\]

if we impose antisymmetry and the duality condition

\[
F_{\mu\nu} = F_{\nu\mu}, \quad \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = F^{\mu\nu}. \tag{3.21}
\]

Then the action of the supercharges on the new fields is:

\[
[Q_{ja}, F_{\mu\nu}] = \sigma_{ab}^{\mu\nu} \lambda_j^b \tag{3.22}
\]

\[
[\bar{Q}_{j\bar{a}}, F_{\mu\nu}] = \frac{1}{2} \epsilon_{jk}(\sigma_{ba}^{\mu} \partial^\nu \chi_k^b - \sigma_{ba}^{\nu} \partial^\mu \chi_k^b + i \epsilon^{\mu\nu\rho\sigma} \sigma_{ab} \partial_\rho \chi_k^b). \tag{3.23}
\]

Indeed, then we have:

\[
F_{\mu\nu} = -\frac{1}{2} \sigma_{ab}^{\mu\nu} F^{ab}. \tag{3.24}
\]

**Proposition 3.3** Let us define the real fields:

\[
F_{\mu\nu} \equiv \frac{i}{2} (F_{\mu\nu} - F^{\ast}_{\mu\nu}); \tag{3.25}
\]

then the correspondence $F_{ab} \leftrightarrow F_{\mu\nu}$ is one-one and the action of the supercharges on the new fields is:

\[
[Q_{ja}, F^{\mu\nu}] = i \sigma_{ab}^{\mu\nu} \lambda_j^b + \frac{i}{4} \epsilon_{jk}(\sigma_{ba}^{\mu} \partial^\nu \chi_k^b - \sigma_{ba}^{\nu} \partial^\mu \chi_k^b - i \epsilon^{\mu\nu\rho\sigma} \sigma_{ab} \partial_\rho \chi_k^b). \tag{3.26}
\]

Indeed the correspondence is one-one because we have

\[
F_{\mu\nu} = F_{\mu\nu} - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \tag{3.27}
\]

The preceding multiplet is reducible.
Theorem 3.4 The condition
\[ C^*_j k = \epsilon_{jl} \epsilon_{km} C_{lm} \] (3.28)
is compatible with the SU(2) transformations. It is consistent with the action of the supercharges from the preceding theorem iff we also have:
\[ \lambda_{ja} = i \epsilon_{jk} \sigma^\mu_{ab} \partial_\mu \bar{\chi}^b_k \] (3.29)
\[ D = -2i \partial^2 z^* \] (3.30)
\[ \epsilon^{\mu \nu \alpha \beta} \partial_\nu F_{\alpha \beta} = 0. \] (3.31)

Proof: One starts from the first constrain and by commuting with the supercharges gets new constraints. Then we one iterates the procedure and, hopefully, it will close after a finite number of steps.

1. We commute the constrain (3.28) with the supercharge \( Q_{ja} \) and obtain (3.29).
2. The commutator of (3.28) with the supercharge \( \bar{Q}_{kb} \) is an identity.
3. We anticommute the constraint (3.29) with the supercharge \( Q_{ja} \) and obtain (3.30).
4. We anticommutator the constraint (3.29) with the supercharge \( \bar{Q}_{kb} \) and we obtain (3.31).
5. The commutator of (3.30) and (3.31) with the supercharges \( Q_{ja}, \bar{Q}_{kb} \) are identities if we use the preceding constraints.

■

It follows that the reduced multiplet can be described as follows:

Theorem 3.5 The reduced multiplet is composed of the fields \( (z, F_{\mu \nu}, C_{jk}, \chi_{ja}) \) verifying the restrictions
\[ C_{jk} = C_{kj}, \quad C^*_j k = \epsilon_{jl} \epsilon_{km} C_{lm} \] (3.32)
\[ F_{\mu \nu} = -F_{\nu \mu}, \quad \epsilon^{\mu \nu \alpha \beta} \partial_\nu F_{\alpha \beta} = 0. \] (3.33)

The action of the supercharges on these fields is well defined by:
\[ i[Q_{ja}, z] = \chi_{ja}, \quad [\bar{Q}_{j\bar{a}}, z] = 0. \] (3.34)
\[ [Q_{ja}, F_{\mu \nu}] = \frac{i}{2} \epsilon_{jk} (\sigma^\mu_{ab} \partial_\nu \bar{\chi}^b_k - \sigma^\nu_{ab} \partial_\mu \bar{\chi}^b_k) \] (3.35)
\[ [Q_{ja}, C_{kl}] = -i \sigma^\mu_{ab} \partial_\mu (\epsilon_{jk} \epsilon_{lm} \bar{\chi}^b_m + \epsilon_{jl} \epsilon_{km} \bar{\chi}^b_m) \] (3.36)
\[ [\bar{Q}_{j\bar{a}}, C_{kl}] = i \sigma^\mu_{ba} \partial_\mu (\delta_{jk} \chi^b_k + \delta_{jl} \chi^b_k) \] (3.37)
\[ \{Q_{ja}, \chi_{kb}\} = -2i \epsilon_{jk} \sigma^\mu_{ab} F_{\mu \nu} + \epsilon_{ab} C_{jk} \] (3.38)
\[ \{Q_{ja}, \bar{\chi}_{kb}\} = 2 \delta_{jk} \sigma^\mu_{ab} \partial_\mu z^*. \] (3.39)
The preceding multiplet can be reduced even further. We have:

**Theorem 3.6** In the preceding conditions the relations

\begin{align*}
C_{jk} &= 0 \quad (3.40) \\
\sigma^\mu_{ab} \partial_\mu \bar{\chi}_j^b &= 0 \quad (3.41) \\
\partial^2 z &= 0 \quad (3.42) \\
\partial^\mu F_{\mu \nu} &= 0 \quad (3.43)
\end{align*}

are supersymmetric invariant. In particular we can consider the multiplet \((z, \chi_{ja}, F_{\mu \nu})\) of zero mass fields such that \(\chi_{ja}\) verifies Dirac equation of zero mass and \(F_{\mu \nu}\) verifies the restrictions

\[ F_{\mu \nu} = -F_{\nu \mu}, \quad \epsilon^{\mu \nu \alpha \beta} \partial_\nu F_{\alpha \beta} = 0 \quad \partial^\mu F_{\mu \nu} = 0. \quad (3.44) \]

The action of the supercharges is well defined by:

\begin{align*}
[i [Q_{ja}, z] = \chi_{ja} & \quad [\bar{Q}_{j\bar{a}}, z] = 0. \quad (3.45) \\
[Q_{ja}, F^{\mu \nu}] &= \frac{i}{2} \epsilon_{jk} (\sigma^\mu_{ab} \partial^\nu \bar{\chi}_k^b - \sigma^\nu_{ab} \partial^\mu \bar{\chi}_k^b) \quad (3.46) \\
\{Q_{ja}, \chi_{kb}\} &= -2i \epsilon_{jk} \sigma^\mu_{ab} F_{\mu \nu} \quad (3.47) \\
\{Q_{ja}, \bar{\chi}_{k\bar{b}}\} &= 2 \delta_{jk} \sigma^\mu_{ab} \partial_\mu z^\star. \quad (3.48)
\end{align*}

We call this multiplet the **new reduced multiplet**.

We can make now the connection with the notations of [7]. We have to show that the fields \(C_{jk}\) are transforming according to the vector representation of \(SU(2)\). Indeed we have

**Lemma 3.7** (i) In the preceding conditions, let us define the operator-valued \(2 \times 2\) matrix \(H\) with elements

\[ H_{jk} \equiv C_{jl} \epsilon_{lk}. \quad (3.49) \]

Then we have

\[ H^T = \epsilon \ H \epsilon \quad (3.50) \]

\[ V_U^{-1} H \ V_U = U \ H \ U^\star, \quad \forall U \in SU(2). \quad (3.51) \]

(ii) Any \(2 \times 2\) matrix \(H\) verifying the relation \(\text{(3.51)}\) can be uniquely written in the form

\[ H = \sum_{j=1}^3 C_j \sigma_j \quad (3.52) \]
where $\sigma_j$ are the Pauli matrices) so (3.51) is equivalent to

$$V_U^{-1} C_j V_U = \delta(U)_{jk} C_k.$$  \hfill(3.53)

Here $\delta : SU(2) \to SO(3)$ is the covering map.

**Proof:** The relations (i) are elementary and the writing (3.52) follows from

$$\sigma_j^T = \epsilon \sigma_j \epsilon, \quad \forall j = 1, 2, 3.$$  \hfill(3.54)

The relation (3.53) tells us that the multiplet of fields $C_j$ transforms according to the vector representation of $SU(2)$.

Now the multiplet described in [7] by the rules (12)+(16) coincides with the multiplet from our theorem 3.5: indeed, up to some factors we have the correspondence:

$$z = A + iB, \quad \chi_1 a = p_a, \quad \chi_2 a = p_a, \quad F^{\mu\nu} = V^{\mu\nu} \quad Q^a_1 = Q^a_1, \quad Q^a_2 = Q^a_1.$$  \hfill(3.55)

The new reduced multiplet corresponds to (18) from [7]. Let us remember that the action of the supercharges on the fields described in theorem 3.1 implies that we do have (2.3) without central charges; in fact, we have derived the action of the supercharges imposing this conditions, the relations (2.3) being generic for a supersymmetry algebra. A posteriori we have (2.3) for theorems 3.5 and 3.6. In [7] it is asserted that the anticommutator relations between supercharges corresponding to $j = 1$ and $j = 2$ are unconstrained; according to our analysis this is not true.

Now we determine the possible form of the causal (anti)commutator relations. We have the following result:

**Theorem 3.8** Let us suppose that for the multiplet of Theorem 3.1 the field $z$ is a complex scalar field of mass $m \geq 0$; in particular we have

$$[z(x), z(y)] = 0$$  \hfill(3.56)

$$[z(x), z^*(y)] = -i \ D_m(x - y).$$  \hfill(3.57)

Then we also have:

$$[F_{ab}(x), F_{cd}(y)] = i \alpha \ (\epsilon_{ac} \epsilon_{bd} + \epsilon_{ad} \epsilon_{bc}) D_m(x - y)$$  \hfill(3.58)

$$[F_{ab}(x), F^*_{cd}(y)] = 2i \ (\sigma^a_{ac} \sigma^b_{bd} + \sigma^a_{ad} \sigma^b_{bc}) \ \partial_\mu \partial_\nu D_m(x - y)$$  \hfill(3.59)

$$[C_{jk}(x), C_{lm}(y)] = -i \alpha (\epsilon_{jl} \epsilon_{km} + \epsilon_{jm} \epsilon_{kl}) D_m(x - y)$$  \hfill(3.60)

$$[C_{jk}(x), C^*_{lm}(y)] = -2i \ m^2 (\delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl}) D_m(x - y)$$  \hfill(3.61)

$$[D(x), D^*(y)] = -4i m^2 \ D_m(x - y).$$  \hfill(3.62)
\[ [z(x), D(y)] = \alpha \, D_m(x - y) \]  
(3.61)

\[ \{\chi_{ja}(x), \bar{\chi}_{kb}(y)\} = 2 \, \delta_{jk} \sigma_{ab}^\mu \partial_\mu D_m(x - y) \]  
(3.62)

\[ \{\lambda_{ja}(x), \bar{\lambda}_{kb}(y)\} = 2 \, \delta_{jk} m^2 \sigma_{ab}^\mu \partial_\mu D_m(x - y) \]  
(3.63)

\[ \{\chi_{ja}(x), \lambda_{bk}(y)\} = -i \alpha \, \epsilon_{ab} \, \epsilon_{jk} \, D_m(x - y), \]  
(3.64)

and all other (anti)commutators are zero. Here \( \alpha \in \mathbb{C} \) is a free parameter.

**Proof:** From Lorentz covariance considerations we must have:

\[ [z(x), F_{ab}(y)] = 0, \quad [z^*(x), F_{ab}(y)] = 0 \]  
(3.65)

because there is no partial differential operator \( P_{ab}(\partial) \) symmetric in \( a \) and \( b \) and Lorentz covariant. We also must have

\[ [z(x), D(y)] = \alpha \, D_m(x - y), \quad [z(x), D^*(y)] = \alpha' \, D_m(x - y) \]  
(3.66)

with \( \alpha, \alpha' \in \mathbb{C} \).

From consideration of Lorentz and \( SU(2) \) covariance we must also have:

\[ [z(x), C_{jk}(y)] = 0 \]  
(3.67)

because the only available combination \( \epsilon_{jk} D_m(x - y) \) is in conflict with the symmetry property.

Starting from the hypothesis and the preceding relations one determines by some long but straightforward computation all the causal (anti)commutators from the Jacobi identity (2.11). In particular we get \( \alpha' = 0 \). \( \blacksquare \)

Now we determine if the multiplets considered above do admit a representation in a Fock space. We proceed in the spirit of the reconstruction theorem from the axiomatic field theory. The key property is positivity. We suppose that all the fields are free of mass \( m \). In this case it is known that it is sufficient to verify the positivity of the 2-point function. We have immediately

**Corollary 3.9** The field \( D \) has a representation in a Fock space iff \( m > 0 \)

**Proof:** Let us consider the real scalar fields \( D_r, r = 1, 2 \) given by \( D = D_1 + i \, D_2 \). From the commutation relation for \( D \) we get:

\[ [D_r(x), D_s(y)] = -2i \, m^4 \, \delta_{rs} \, D_m(x - y) \]  
(3.68)

so the 2-points functions is obtained from the causal commutator with the substitution

\[ D_m \rightarrow D_m^{(+)} \]  
(3.69)
i.e. we keep only the positive frequency part of the Pauli-Villars commutator:

\[ \langle \Omega, D_r(x) D_s(y) \Omega \rangle = -2i m^2 \delta_{rs} D_m^{(+)}(x - y); \]  

(3.70)

this follows from Källan-Lehman representation for the 2-point distribution (see for instance [6], Introduction to QFT, Section 1.5). If we consider the norm of an arbitrary one-particle state we get something proportional to \( m^4 \) so the positivity condition follows.

To analyse the possibility of Hilbert space representations for the other field more easily we rewrite some of the commutation relations given above using new fields. The first rewriting is elementary in terms of the fields \( F_{\mu\nu} \).

Corollary 3.10  The causal commutation relations for the field \( F_{\mu\nu} \) are

\[
[F_{\mu\nu}(x), F_{\rho\sigma}(y)] = -\frac{i}{2} \left( g_{\mu\rho} \partial_\nu \partial_\sigma - g_{\nu\rho} \partial_\mu \partial_\sigma + g_{\nu\sigma} \partial_\mu \partial_\rho - g_{\nu\rho} \partial_\mu \partial_\sigma \right) D_m(x - y) \\
+ \frac{i}{8} \left[ \text{Re}(\alpha) - 2m^2 \right] (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) D_m(x - y) + \frac{i}{8} \text{Im}(\alpha) \epsilon_{\mu\rho\sigma} D_m(x - y). 
\]  

(3.71)

In particular the field has a representation in a Fock space if and only if \( \alpha = 2m^2 \).

Proof:  The commutation relation is obtained by a straightforward computation. The 2-point function is again obtained from the causal commutator with the substitution (3.69). The computation of the norm of an arbitrary one-particle state

\[
\Psi \equiv \int dx \: f^{\mu\nu}(x) F_{\mu\nu}(x) \Omega 
\]  

(3.72)

where \( f^{\mu\nu} \) are test function (antisymmetric in \( \mu \) and \( \nu \)) gives

\[
|\Psi|^2 = -2\pi \int d\alpha_m^+(p) \bar{f}^{\mu}(p) \bar{f}_\mu(p) \\
+ \frac{\pi}{2} \left[ \text{Re}(\alpha) - 2m^2 \right] \int d\alpha_m^+(p) \bar{f}^{\mu\nu}(p) \bar{f}_{\mu\nu}(p) - \frac{\pi}{4} \text{Im}(\alpha) \int d\alpha_m^+(p) \epsilon^{\mu\nu\rho\sigma} \bar{f}_{\mu\nu}(p) \bar{f}_{\rho\sigma}(p). 
\]  

(3.73)

Here \( \alpha_m^+ \) is the Lorentz-invariant measure on the upper hyperboloid of mass \( m \), \( f^\mu \equiv \partial_\nu f^{\mu\nu} \) and by \( \bar{f} \) we denote the Fourier transform of \( f \). Now it is clear that the last two contributions can have arbitrary signs so they must be identically zero. On the contrary, because of the transversality condition \( \partial_\mu f^\mu = 0 \iff p_\mu \bar{f}(p) = 0 \), the first contribution is positive.

To analyze the positivity condition for the fields \( C_{jk} \) we proceed in the same spirit as for the complex field \( D \); we make a decomposition into real and imaginary parts but in a \( SU(2) \) covariant way.
Corollary 3.11 Let us define the new fields
\[ C_{jk} \equiv \frac{1}{2}(C_{jk} + \epsilon_{jl}\epsilon_{km}C_{km}^*), \quad D_{jk} \equiv \frac{i}{2}(C_{jk} - \epsilon_{jl}\epsilon_{km}C_{km}^*). \tag{3.74} \]

The correspondence \( C_{jk} \leftrightarrow (C_{jk}, D_{jk}) \) is one-one. These new fields verify the reality conditions
\[ C_{jk}^* = \epsilon_{jl}\epsilon_{km}C_{lm}, \quad D_{jk}^* = \epsilon_{jl}\epsilon_{km}D_{lm} \tag{3.75} \]
and the causal commutators are:
\[ [C_{jk}(x), C_{km}^*(y)] = -\frac{i}{2}(2m^2 + \alpha)(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})D_m(x-y) \tag{3.76} \]
\[ [D_{jk}(x), D_{lm}^*(y)] = -\frac{i}{2}(2m^2 - \alpha)(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})D_m(x-y) \tag{3.77} \]
\[ [C_{jk}(x), D_{lm}(y)] = 0. \tag{3.78} \]

In particular the positivity condition is fulfilled iff \(-2m^2 < \alpha < 2m^2\).

Proof: The commutators are obtained by elementary computation. If we consider now the one-particle states
\[ \Psi_1 \equiv \int dx \ f_{jk}(x)C_{jk}^*(x)\Omega, \quad \Psi_2 \equiv \int dx \ g_{jk}(x)D_{jk}^*(x)\Omega \tag{3.79} \]
where \( f_{jk}, g_{jk} \) are test functions (antisymmetric in \( j \) and \( k \)) then the norms are:
\[ |\Psi_1|^2 = 2\pi(2m^2 + \alpha) \int d\alpha_m(p)|\tilde{f}_{jk}(p)|^2 \]
\[ |\Psi_2|^2 = 2\pi(2m^2 - \alpha) \int d\alpha_m(p)|\tilde{f}_{jk}(p)|^2 \tag{3.80} \]
and we get the inequalities from the statement. ■

Comparing the last two corollaries we arrive at the conclusion that the multiplet described in Theorem 3.1 does not have a representation in a Hilbert space (of Fock type). This conclusion makes the reduced multiplet more interesting because in this case we have positivity.

Theorem 3.12 The causal (anti)commutators for the reduced multiplet of Theorem 3.1 are
\[ [z(x), z^*(y)] = -i \ D_m(x-y). \tag{3.81} \]
\[ [F_{\mu\nu}(x), F_{\rho\sigma}(y)] = -\frac{i}{2}(g_{\mu\sigma}\partial_{\nu}\partial_{\rho} - g_{\nu\rho}\partial_{\mu}\partial_{\sigma} + g_{\nu\sigma}\partial_{\mu}\partial_{\rho} - g_{\mu\rho}\partial_{\nu}\partial_{\sigma}) \ D_m(x-y) \tag{3.82} \]
\[ [C_{jk}(x), C_{lm}(y)] = -2im^2(\epsilon_{jl}\epsilon_{km} + \epsilon_{jm}\epsilon_{kl})D_m(x-y) \tag{3.83} \]
\[ \{\chi_{ja}(x), \bar{\chi}_{kb}(y)\} = 2 \delta_{jk}\sigma_{ab}^{\mu} \partial_{\mu}D_m(x-y). \tag{3.84} \]

This corresponds to \( \alpha = 2m^2 \); in particular we have \( D_{jk} = 0 \) so \( C_{jk} = C_{jk} \). The positivity condition is verified in this case for \( m > 0 \).
Proof: The computations are elementary: from the constraints we have \( D = 2im^2z^* \); if we substitute this into the commutation relation of \( z \) with \( D \) we get the value of \( \alpha \). This in turn gives \( D_{jk} = 0 \) and the commutators from the statement follows. If \( m = 0 \) we would get that the fields \( C_{jk} \) are commuting among themselves so they cannot be represented as operators in a Hilbert space. (They cannot be c-number fields because this would contradict the relations from theorem 3.1).

Let us remark that in [7] the reduced multiplet is constructed for \( m = 0 \). Apparently one can make \( m = 0 \) in theorem 3.5 at the purely algebraic level and still obtain a good multiplet. However, according to our analysis the multiplet obtained after this limiting procedure will not have a representation in a Hilbert space.

For the new reduced multiplet we have similarly:

**Theorem 3.13** The causal (anti)commutators for the reduced multiplet of Theorem 3.6 are

\[
[z(x), z^*(y)] = -i \, D_m(x - y),
\]

\[
[F_{\mu\nu}(x), F_{\rho\sigma}(y)] = -i/4(g_{\mu\rho}\partial_\nu\partial_\sigma - g_{\mu\sigma}\partial_\nu\partial_\rho + g_{\nu\sigma}\partial_\mu\partial_\rho - g_{\nu\rho}\partial_\mu\partial_\sigma) \, D_m(x - y)
\]

\[
\{\chi_ja(x), \bar{\chi}_{kb}(y)\} = 2 \, \delta_{jk} \sigma^\mu_{ab} \, \partial_\mu D_m(x - y).
\]

The positivity condition is verified in this case.

The reduced (and the new reduced) multiplet describe particles of spin 1 through the fields \( F_{\mu\nu} \). Indeed it is easy to see that the one-particle Fock subspace \( \mathcal{H}^{(1)} \) generated from the vacuum by \( F_{\mu\nu} \) describes the irreducible representation \([m, 1]\) of the Poincaré group. For the reduced multiplet we must have \( m > 0 \) and for the new reduced multiplet we have \( m = 0 \).

One can associate with every field of the multiplet from Theorem 3.1 a superfield using a sandwich formula as in [12] and [13]: if \( f \) is any field of the multiplet we define:

\[
s(f) \equiv e^{iS} fe^{-iS}
\]

where

\[
S \equiv \epsilon_{jk}(\theta^a_j Q_{ka} - \bar{\theta}^a_j \bar{Q}_{ka});
\]

here \( \theta_{ja} \) are some arbitrary Grassmann parameters. The presence of the tensor \( \epsilon_{jk} \) makes the construction \( SU(2) \)-covariant. For instance, the superfield \( Z \equiv s(z) \) verifies the following covariance property \( \forall U \in SU(2) \):

\[
V^{-1}_U Z(x, \theta)V_U = Z(x, U \cdot \theta)
\]

where we have defined the following action of \( SU(2) \) on the Grassmann variables:

\[
(U \cdot \theta)_{ja} \equiv U_{jk} \theta_{ka}.
\]
The superfield $Z$ is chiral i.e. we have
\[ \mathcal{D}_{ja}Z = 0 \] (3.92)
where the covariant derivatives are defined as usual [12] for both values of the index $j$.

We also mention that the constraints (3.28)-(3.31) can be express compactly using the superfield $Z$
\[ \mathcal{D}_{ja} \bar{D}^a_k Z + \epsilon_{jl} \epsilon_{km} D^l a D^m a Z = 0. \] (3.93)

The explicit expression for the superfield $Z$ is rather complicated and so is the corresponding causal commutator.

Finally we investigate if it is possible to introduce in the game the the electromagnetic potential $A_\mu$.

Apparently this is possible for the reduced multiplet because the constraint (3.31) is the homogeneous Maxwell equation which tells us that there exists $A_\mu$ such that
\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \] (3.94)

One needs the action of the supercharges on $A_\mu$. There are two distinct possibilities. The simplest one is to observe that the “lift” the action of the supercharges on given by
\[ [Q_{ja}, A^\mu] = i \frac{1}{2} \epsilon_{jk} \sigma^\mu a b \bar{\chi}^b_k \] (3.95)
is compatible with (3.94) and the action on the supercharges from the Theorem 3.5. The closeness of the supersymmetric algebra is more subtle and can be understood as in [13], Sect. 8. However, if one computes the causal commutators one finds out that the fields $C_{jk}$ causally commutes with every other fields (including themselves) so we do not have a Hilbert space representation. The situation is better for the new reduced multiplet from Theorem 3.6 for which the non-trivial causal (anti)commutators are:
\[ [z(x), z^*(y)] = -i \ D_m (x - y). \] (3.96)
\[ [F_{\mu \nu}(x), F_{\rho \sigma}(y)] = \frac{-i}{4} (g_{\mu \rho} \partial_\nu \partial_\sigma - g_{\nu \rho} \partial_\mu \partial_\sigma + g_{\nu \sigma} \partial_\mu \partial_\rho - g_{\nu \rho} \partial_\mu \partial_\sigma) \ D_m (x - y) \] (3.97)
\[ \{\chi_{ja}(x), \bar{\chi}^b_k(y)\} = 2 \delta_{j k} \sigma^\mu_{ab} \partial_\mu D_m (x - y) \] (3.98)
so we do have a Hilbert space representations.

The second possibility of introducing the field $A_\mu$ is more logical. First one observes that for $m > 0$ one can always transform an index $a$ into an index $\bar{a}$ and vice-versa using the Dirac operator. In particular, if we start from the basic field $F_{ab}$ of the multiplet appearing in Theorem 3.1 we can define
\[ F_{ab} \equiv \sigma^\mu_{cb} \epsilon_{cd} \partial_\mu F_{ad}. \] (3.99)
One can check two facts: (a) The association $F_{ab} \to F_{\bar{a}b}$ one-one. Indeed, one easily get from the preceding definition

$$F_{ab} = \frac{1}{m^2} \sigma^\mu_{be} \epsilon^{\bar{c} \bar{d}} \partial_\mu F_{\bar{a}d}$$

(3.100)

which is the inverse of the map $F_{ab} \to F_{\bar{a}b}$. (b) The new field $F_{\bar{a}b}$ verifies the property

$$\sigma^\mu_{be} \epsilon^{\bar{c} \bar{d}} \partial_\mu F_{\bar{a}d} = a \leftrightarrow b.$$  

(3.101)

So the first step is to replace in the multiplet $F_{ab}$ by $F_{\bar{a}b}$. One can give the action of the supercharges for the transformed multiplet:

$$[Q_j a, F_{\bar{b}c}] = \epsilon_{ab} \sigma_i^\mu \partial_\mu \lambda_i^d - \sigma_i^\mu \partial_\mu \lambda_{jb}$$

(3.102)

$$[\bar{Q}_j \bar{a}, F_{\bar{b}c}] = -i \epsilon_{jk} (\sigma_i^\mu \sigma_i^\rho \partial_\mu \partial_\rho \chi_{kb} - m^2 \epsilon_{ib} \chi_{kb}).$$

(3.103)

We easily prove that for the reduced multiplet from Theorem 3.5 one can impose the reality condition:

$$(F_{ab})^* = F_{\bar{b}a}.$$  

(3.104)

Next, one defines

$$A_\mu = \frac{1}{4m^2} \sigma_i^\mu \epsilon^{ac} \epsilon^{\bar{d} \bar{e}} F_{\bar{a}d}.$$  

(3.105)

One can easily prove the following facts: (a) The association $F_{\bar{a}b} \to A_\mu$ is one-one. (b) The field $A_\mu$ verifies the transversality condition

$$\partial^\mu A_\mu = 0;$$

(3.106)

(this follows from (3.101)). (c) If the field $F_{a\bar{b}}$ is real then the field $A_\mu$ is also real and vice-versa.

This means that we can replace the (real) field $F_{a\bar{b}}$ by the (real) field $A_\mu$. The action of the supercharges in this new representation is:

$$i [Q_j a, A_\mu] = \epsilon_{ab} \sigma_i^\rho \left( g_{\mu \rho} + \frac{1}{m^2} \partial_\mu \partial_\rho \right) \overline{\chi}_k.$$  

(3.107)

One can also compute the causal commutation relations for the new field $A_\mu$ using the two successive transformations; one gets:

$$[A_\mu(x), A_\rho(y)] = \frac{i}{8} \left( g_{\mu \rho} + \frac{1}{m^2} \partial_\mu \partial_\rho \right) D_m(x - y).$$  

(3.108)

The preceding two relations are compatible with the transversality property. However one knows that in the usual formulation of the standard model one does not impose this transversality property \[20\] so this multiplet cannot be used for a supersymmetric extension of the standard model.
4 The $N = 2$ Hyper-multiplet

We consider now a multiplet with $SU(2)$ invariance and a non-trivial central charge. It can be argued easily that we must have in \( (2.3) \)

\[
Z_{jk} = \epsilon_{jk} Z
\]  

(4.1)

where $Z$ is a central charge and is a $SU(2)$ scalar. Moreover we can suppose that it is self-adjoint

\[
Z^* = Z.
\]  

(4.2)

We have now

**Theorem 4.1** Let us consider the multiplet \((\phi_j, f_j, \psi_a, \chi_a), \ j = 1, 2\) where:

- $\phi_j, f_j$ are complex Bosonic scalar field verify

\[
V_U^{-1} \phi_j V_U = U_{jk} \phi_k, \quad V_U^{-1} f_j V_U = U_{jk} f_k, \quad \forall U \in SU(2).
\]  

(4.3)

- $\psi_a, \chi_a$ are Fermionic Dirac spinor fields which are $SU(2)$ scalars.

The action of the supercharges and of the central charge on these fields is well defined through:

\[
i[Q_{ja}, \phi_k] = \epsilon_{jk} \psi_a
\]  

(4.4)

\[
i[\bar{Q}_{j\bar{a}}, \phi_k] = \delta_{jk} \bar{\chi}_{\bar{a}}
\]  

(4.5)

\[
[Q_{ja}, f_k] = i\epsilon_{jk} \sigma_{ab}^{\mu} \partial_{\mu} \bar{\chi}_{\bar{b}}
\]  

(4.6)

\[
[\bar{Q}_{j\bar{a}}, f_k] = -i\delta_{jk} \sigma_{ab}^{\mu} \partial_{\mu} \psi_b
\]  

(4.7)

\[
\{Q_{ja}, \psi_b\} = -2\epsilon_{ab} f_j
\]  

(4.8)

\[
\{Q_{ja}, \bar{\psi}_{\bar{b}}\} = 2\epsilon_{jk} \sigma_{ab}^{\mu} \partial_{\mu} \phi_k^*
\]  

(4.9)

\[
\{Q_{ja}, \chi_{\bar{b}}\} = 2\epsilon_{jk} \sigma_{ab}^{\mu} \partial_{\mu} \phi_k^*
\]  

(4.10)

\[
\{Q_{ja}, \bar{\chi}_{\bar{b}}\} = 2\sigma_{ab}^{\mu} \partial_{\mu} \phi_j
\]  

(4.11)

\[
i[Z, \phi_j] = f_j
\]  

(4.12)

\[
i[Z, f_j] = i\partial^2 \phi_j
\]  

(4.13)

\[
[Z, \psi_a] = i \sigma_{ba}^{\mu} \partial_{\mu} \bar{\psi}_{\bar{b}}^*
\]  

(4.14)

\[
[Z, \chi_a] = -i \sigma_{ba}^{\mu} \partial_{\mu} \bar{\psi}_{\bar{b}}^*
\]  

(4.15)

If the fields $\phi_j$ verifies the Klein-Gordon equation for mass $m$ then all fields of the multiplet verify the same equation.
Proof: One assume that the action of the supercharges and of the central charge on $\phi_j$ is given by the formulæ from the statement and derive the others using (2.9).

1. From the relation involving $Q_{ja}, Q_{kb}$ and $\phi_l$ we obtain the action of the supercharge $Q_{ja}$ on $\psi_b$.
2. The relation involving $Q_{ja}, Q_{kb}$ and $\phi_l^*$ gives the action of the supercharge $Q_{ja}$ on $\chi_b$.
3. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $\phi_l$ gives the action of the supercharge $\bar{Q}_{k\bar{b}}$ on $\psi_a$ and $\chi_b$.
4. The relation involving $Q_{ja}, Q_{kb}$ and $\bar{\psi}_c$ gives the action of the central charge $Z$ on $\psi_a$.
5. The relation involving $Q_{ja}, Q_{kb}$ and $\psi_c$ gives the action of the supercharge $Q_{ja}$ on $f_k$.
6. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $\psi_c$ gives the action of the supercharge $\bar{Q}_{j\bar{a}}$ on $f_k$.
7. The relation involving $Q_{ja}, Q_{kb}$ and $\bar{\chi}_c$ gives the action of the central charge $Z$ on $\chi_a$.
8. The relation involving $Q_{ja}, Q_{kb}$ and $\chi_c$ is an identity.
9. The relation involving $Q_{ja}, \bar{Q}_{k\bar{b}}$ and $\chi_c$ is an identity.
10. The relation involving $Q_{ja}, Q_{kb}$ and $f_l$ (or $f^*_l$) gives the action of the central charge $Z$ on $f_l$.
11. The relation involving $Q_{ja}, \bar{Q}_{j\bar{b}}$ and $f_c$ is an identity.
12. The relations involving a supercharge, the central charge and a fields are identities.

Finally one notices the compatibility with the $SU(2)$ transformations of all the relations from the statement. ■

Next, we analyze the possible causal (anti)commutation relation. We have

**Theorem 4.2** Let us suppose that for the multiplet of the preceding Theorem the fields $\phi_j$ are complex scalar fields of mass $m \geq 0$; in particular we have

\[
[\phi_j(x), \phi_k(y)] = 0 \quad (4.16)
\]

\[
[\phi_j(x), \phi_k^*(y)] = -i \delta_{jk} D_m(x - y). \quad (4.17)
\]

Then we also have:

\[
[f_j(x), f_k^*(y)] = -i m^2 \delta_{jk} D_m(x - y) \quad (4.18)
\]

\[
[\phi_j(x), f_k(y)] = \alpha \epsilon_{jk} D_m(x - y) \quad (4.19)
\]
\[ [\phi_j(x), f_k^*(y)] = \beta \delta_{jk} \, D_m(x - y) \] (4.20)

\[ \{\psi_a(x), \psi_b(y)\} = -2 \, i \, \epsilon_{ab} \, D_m(x - y) \] (4.21)

\[ \{\psi_a(x), \bar{\psi}_b(y)\} = 2 \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \] (4.22)

\[ \{\chi_a(x), \chi_b(y)\} = 2i \, \bar{\alpha} \, \epsilon_{ab} \, D_m(x - y), \] (4.23)

\[ \{\chi_a(x), \bar{\chi}_b(y)\} = 2 \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \] (4.24)

\[ \{\psi_a(x), \chi_b(y)\} = -2i \, \beta \, \epsilon_{ab} \, D_m(x - y) \] (4.25)

and all other (anti)commutators are zero. Here \( \alpha \in \mathbb{C} \) and \( \beta \in \mathbb{R} \) are free parameters.

**Proof:** From Lorentz and \( SU(2) \) covariance considerations we must have:

\[ [\phi_j(x), f_k(y)] = \alpha \epsilon_{jk} \, D_m(x - y), \quad [\phi_j(x), f_k^*(y)] = \beta \delta_{jk} \, D_m(x - y) \] (4.26)

Starting from the hypothesis and the preceding relations one determines by some computation all the causal (anti)commutators from the Jacobi identity (2.11). ■

Concerning the representability in a Hilbert space we have the following result:

**Theorem 4.3** The following multiplet has a representation in a Hilbert space iff

\[ |\alpha|^2 + \beta^2 \leq 2. \] (4.27)

**Proof:** For the Fermi sector we proceed as in [13] i.e we suppose that the Hilbert space is generated by the Majorana spinors \( f^{(A)} \) verifying the causal anticommutation relations:

\[ \{ f^{(A)}_a(x), f^{(B)}_b(y) \} = i \, \delta_{AB} \, \epsilon_{ab} \, m \, D_m(x - y), \] (4.28)

So, we must have

\[ \psi = \sum_A c_A \, f^{(A)} \quad \chi = \sum_A d_A \, f^{(A)} \] (4.29)

for some (complex) numbers \( \bar{c} = \{c_A\}, \bar{d} = \{d_A\} \). Like in [13] we find out

\[ \{\psi_a(x), \psi_b(y)\} = i \, m \, \bar{c}^2 \, \epsilon_{ab} \, D_m(x - y), \]  
\[ \{\psi_a(x), \bar{\psi}_b(y)\} = \bar{c} \cdot \bar{c}^* \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \]  
\[ \{\chi_a(x), \chi_b(y)\} = i \, m \, \bar{d}^2 \, \epsilon_{ab} \, D_m(x - y), \]  
\[ \{\chi_a(x), \bar{\chi}_b(y)\} = \bar{d} \cdot \bar{d}^* \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \]  
\[ \{\psi_a(x), \chi_b(y)\} = i \, m \, \bar{c} \cdot \bar{d} \, \epsilon_{ab} \, D_m(x - y), \]  
\[ \{\psi_a(x), \bar{\chi}_b(y)\} = \bar{c} \cdot \bar{d}^* \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y). \] (4.30)
By comparison we get
\[ m \vec{c}^2 = -2\alpha, \quad \vec{c} \cdot \vec{c}^* = 2, \quad m \vec{d}^2 = 2\alpha, \quad \vec{d} \cdot \vec{d}^* = 2, \quad m \vec{c} \cdot \vec{d} = -2\beta, \quad \vec{c} \cdot \vec{d}^* = 0. \] (4.31)

We separate the real and the imaginary part of the vectors \( \vec{c}, \vec{d} \) and consider all possible Cauchy-Schwartz inequalities. As a result we get
\[ |\alpha|^2 \leq m^2, \quad \beta^2 + \alpha_1^2 \leq m^2. \] (4.32)

For the Bosonic sector we simplify the reasoning by some field redefinitions. If we consider instead of the fields \( f_j \) the new fields
\[ F_j \equiv f_j + c_1 \phi_j + c_2 \epsilon_{jk} f_k^* + c_3 \epsilon_{jk} \phi_k^* \] (4.33)
then one can decouple the fields \( \phi_j \) from \( F_j \) if one chooses
\[ c_3 = i (\beta + \bar{\alpha} c_2), \quad c_3 = i (\alpha - \beta c_2) \] (4.34)
i.e. with this choice \( \phi_j \) causally commutes with \( F_j \) and \( F_j^* \). Now we still have to check the positivity for the new fields \( F_j \). One finds out that
\[ [F_j(x), F_k(y)] = -2i \alpha \beta \epsilon_{jk} D_m(x - y) \] (4.35)
\[ [F_j(x), F_k^*(y)] = -i (m^2 - \beta^2 - |\alpha|^2) \delta_{jk} D_m(x - y). \] (4.36)

We now make a new field transformation
\[ g_j \equiv F_j + c \epsilon_{jk} F_k^* \] (4.37)
and by a convenient choice of the constant \( c \) we arrive at the standard form:
\[ [g_j(x), g_k(y)] = 0 \] (4.38)
\[ [g_j(x), g_k^*(y)] = -i d \delta_{jk} D_m(x - y) \] (4.39)
for some constant \( d \) which can be computed explicitly. The positivity condition is \( d > 0 \) and give the relation from the statement which is stronger than the relation obtained in the Fermi sector. 

21
5  The $N = 4$ multiplet

We consider a model with $SU(4)$ invariance and no central charges.

**Theorem 5.1** Let us consider the multiplet $(\phi_{jk}, F_{ab}, \lambda_{ja})$, $j, k = 1, \ldots, 4$ where:

- $\phi_{jk}$ are complex Bosonic scalar fields antisymmetric in $j$ and $k$ and verifying
  \[ V_U^{-1} \phi_{jk} V_U = U_{jm} \phi_{lm}, \quad \forall U \in SU(4); \quad (5.1) \]
- $F_{ab}$ are complex Bosonic fields which are $SU(4)$ scalars;
- $\lambda_{ja}$ are spinor Fermionic fields and verifying
  \[ V_U^{-1} \lambda_{ja} V_U = \overline{U}_{jk} \lambda_k, \quad \forall U \in SO(4). \quad (5.2) \]

The action of the supercharges is well defined by:

\[
\begin{align*}
    i [Q_{ja}, \phi_{kl}] &= \epsilon_{jklm} \lambda_{ma} \\
    i [\bar{Q}_{j\bar{a}}, \phi_{kl}] &= \delta_{jk} \bar{\lambda}_{\bar{a}a} - \delta_{j\bar{a}} \bar{\lambda}_k \\
    \{Q_{ja}, \lambda_{kb}\} &= \delta_{jk} F_{ab} \\
    \{Q_{ja}, \bar{\lambda}_{\bar{k}b}\} &= 2 \sigma^{\mu}_{a\bar{b}} \partial_\mu \phi_{jk} \\
    [Q_{ja}, F_{bc}] &= 0 \\
    [\bar{Q}_{j\bar{a}}, F_{bc}] &= -2i \sigma^{\mu}_{c\bar{b}} \partial_\mu \lambda_{kb} \\
\end{align*}
\]

iff the following constraints are valid:

\[
\begin{align*}
    \partial^2 \phi_{jk} &= 0 \quad (5.9) \\
    \partial^2 F_{ab} &= 0 \quad (5.10) \\
    \sigma^{\mu}_{ab} \partial_\mu \lambda^b_j &= 0, \quad (5.11) \\
    F_{ab} &= F_{ba} \quad (5.12) \\
    \sigma^{\mu}_{ab} \epsilon^{ac}_{\mu} \partial_\mu F_{cd} &= 0. \quad (5.13) \\
    \epsilon_{jklm} \phi_{lm} &= 2 \phi^*_{jk} \quad (5.14)
\end{align*}
\]

**Proof:** One starts in a well known way from the first two relations and uses (2.9).

1. Consider the relation involving; $Q_{ja}, Q_{kb}$ and $\phi^*_{lm}$. If we define

\[ F_{ab} \equiv \{ Q_{jb}, \lambda_{ja} \} \quad (5.15) \]

we obtain the action of the supercharge $Q_{ja}$ on $\lambda_{kb}$ and the symmetry property (5.12).
2. The relation involving $Q_{ja}, Q_{kb}$ and $\phi_{lm}$ is an identity.

3. The relation involving $Q_{ja}, \bar{Q}_{kb}$ and $\phi^{*}_{lm}$ gives the action of the supercharge $\bar{Q}_{kb}$ on $\lambda_{ja}$ and the constraint (5.14).

4. The relation involving $Q_{ja}, Q_{kb}$ and $\lambda_{lc}$ gives the constraint (5.11).

5. The relation involving $Q_{ja}, Q_{kb}$ and $\phi^{*}_{lm}$ gives the action of $\bar{Q}_{kb}$ on $\lambda_{ja}$ and the constraint (5.13).

6. The relation involving $Q_{ja}, Q_{kb}$ and $F_{cd}$ is an identity.

7. The relation involving $Q_{ja}, Q_{kb}$ and $F_{*cd}$ gives the constraint (5.9).

8. The relation involving $Q_{ja}, Q_{kb}$ and $F_{cd}$ gives the constraint (5.13).

9. From (5.11) and the definition of $F_{ab}$ given above we also get (5.10). Now we take the (anti)commutators of the supercharges with the constraints and obtain no new identities. The $SU(4)$ consistency of the relation from the statement is easy to obtain. ■

Let us remark that the ansatz regarding the action of the supercharges on $\phi_{jk}$ is quite general. If we can consider only $SO(4)$ covariance a more general situation of the type

$$i[Q_{ja}, \phi_{kl}] = \alpha_{1}(\delta_{jk}\lambda_{la} - \delta_{jl}\lambda_{ka}) + \beta_{1}\epsilon_{jklm}\lambda_{ma}$$

(5.16)

and

$$i[\bar{Q}_{ja}, \phi_{kl}] = \alpha_{2}(\delta_{jk}\bar{\lambda}_{la} - \delta_{jl}\bar{\lambda}_{ka}) + \beta_{2}\epsilon_{jklm}\bar{\lambda}_{m\bar{a}}$$

(5.17)

is possible but one can prove that by clever redefinitions of the fields we we can make $\alpha_{1} = 0 = \beta_{2}$.

To verify the positivity condition it is convenient to replace $F_{ab}$ by $F_{\mu\nu}$ as in Section 3; the action of the supercharges on $F_{\mu\nu}$ is

$$[Q_{ja}, F_{\mu\nu}] = \frac{i}{4}(\sigma_{ab}^{\mu} \partial^{\nu}\bar{\lambda}_{k} - \sigma_{ab}^{\nu} \partial^{\mu}\bar{\lambda}_{k} - i\epsilon_{\mu\nu\rho\alpha}\sigma_{a\bar{b}}\partial^{\rho}\bar{\lambda}_{j})$$

(5.18)

and the field $F_{\mu\nu}$ must verify the consistency conditions

$$F_{\mu\nu} = -F_{\nu\mu}, \quad \partial^{\mu}F_{\mu\nu} = 0, \quad \epsilon^{\mu\nu\alpha\beta}\partial_{\beta}F_{\mu\nu} = 0.$$ 

(5.19)

Next, we consider the causal (anti)commutator relations. We have:

**Theorem 5.2** Let us suppose that for the preceding multiplet the field $\phi_{jk}$ are complex scalar fields of zero mass; in particular we have

$$[\phi_{jk}(x), \phi_{lm}(y)] = -i(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})D_{0}(x - y)$$

(5.20)
which is compatible with the \( SO(4) \) covariance properties. Then we also have:

\[
[F_{ab}(x), F_{cd}^*(y)] = 2i \left( \sigma^\mu_{ae} \sigma^\nu_{bd} + \sigma^\mu_{ad} \sigma^\nu_{be} \right) \partial_\mu \partial_\nu D_0(x - y)
\]  

(5.21)

\[
\{ \lambda_{ja}(x), \bar{\lambda}_{k\bar{b}}(y) \} = 2 \delta_{jk} \sigma^\mu_{ab} \partial_\mu D_0(x - y)
\]  

(5.22)

and all other (anti)commutators are zero. Alternatively, if we work with the field \( F_{\mu\nu} \) the relation (5.21) can be replaced by:

\[
[F_{\mu\nu}(x), F_{\rho\sigma}(y)] = -\frac{i}{2} \left( g_{\mu\rho} \partial_\nu \partial_\sigma - g_{\nu\rho} \partial_\mu \partial_\sigma + g_{\nu\sigma} \partial_\mu \partial_\rho - g_{\nu\rho} \partial_\mu \partial_\sigma \right) D_0(x - y)
\]  

(5.23)

which is compatible with the constraints (5.19). The positivity condition is verified for this model.

**Proof:** The compatibility of (5.20) with the \( SO(4) \) covariance properties is elementary. From Lorentz covariance considerations we must have:

\[
[\phi_{jk}(x), F_{ab}(y)] = 0, \quad [\phi_{jk}^*(x), F_{ab}(y)] = 0
\]  

(5.24)

because there is no partial differential operator \( P_{ab}(\partial) \) symmetric in \( a \) and \( b \) and Lorentz covariant.

Starting from the hypothesis and the preceding relations one determines by some computation all other causal (anti)commutators from the using Jacobi identity (2.11). One can easily check that these (anti)commutation relations are compatible with the constraints (5.9) - (5.14). Only the positivity of the field \( F_{\mu\nu} \) requires some work and it is done as in Corollary 3.10.

We investigate if it is possible to introduce in a consistent way the electromagnetic potential \( A_\mu \). The standard construction from the literature - see for instance [22] Sect. 13, formulæ (13.11) - consists of replacing \( F_{ab} \) by \( A_\mu \), and postulation the following action of the supercharges

\[
i[Q_{ja}, \phi_{kl}] = \delta_{jk} \lambda_{la} - \delta_{jl} \lambda_{ka}
\]  

(5.25)

\[
i[\bar{Q}_{ja}, \phi_{kl}] = \epsilon_{jklm} \bar{\lambda}_{m\bar{a}}
\]  

(5.26)

\[
\{Q_{ja}, \lambda_{kb}\} = -i \delta_{jk} \sigma_{ab}^{\mu\nu} F_{\mu\nu}
\]  

(5.27)

\[
\{Q_{ja}, \bar{\lambda}_{k\bar{b}}\} = 2 \sigma_{ab}^{\mu} \partial_\mu \phi^*_{jk}
\]  

(5.28)

\[
i[Q_{ja}, A^{\mu}] = \sigma_{ab}^{\mu} \bar{\lambda}_{j\bar{b}}
\]  

(5.29)

where

\[
F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.
\]  

(5.30)
One can prove as in [13], Section 8 that SUSY algebra is full-filed on the physical space \( \mathcal{H}_{\text{phys}} = \text{Ker}(Q)/\text{Im}(Q) \). However the model does not have a representation in a Hilbert space! The reason is that the Jacobi identity

\[
[A^\mu(x_1), \{\lambda_{ja}(x_2), Q_{kb}\}] + \{\lambda_{ja}(x_2), [Q_{kb}, A^\mu(x_1)]\} = 0
\]  
(5.31)
cannot be satisfied.

Alternatively if we want to solve the constraints (5.19) in terms of some electromagnetic potential \( A^\mu \) we first have from the third constraint that \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) and the second constraint gives \( \partial_\mu A_\mu = 0 \); but such a fields does not have a Hilbert space representation. So it seems that the \( N = 4 \) model considered above cannot be used for a supersymmetric extension of the standard model.
6 Conclusions

The analysis from this paper shows that a quantum supersymmetric extension of the standard model using extended supersymmetry is problematic. The main reason comes from the severe restrictions on the causal (anti)commutator relations imposed by the positivity condition (which give a well-defined scalar product of the model). This is a serious problem for string theory which seems to predict that the standard model should be necessarily be an extended supersymmetric one. For other models the situation is even more dramatic. Consider for instance the super-gravity multiplet [26] Ch. 9. There one tries to extend the linearized Einstein gravity i.e. one describes gravitation using the field $h_{\mu\nu}$ symmetric in the indices and which is the first order approximation of the metric tensor $g_{\mu\nu}$. From the analysis of the irreducible representations of the supersymmetric algebra one knows that the multiplet should also contain a Rarita-Schwinger field $\psi^a_\mu$ of spin $3/2$. The theory is considered at the classical level. It particular this means that in the relations (2.6) one should replace the (anti)commutators by an action of the supercharges on the supersymmetric manifold with coordinates $h_{\mu\nu}, \psi^a_\mu$ and their derivatives. At the level of a classical field theory the supersymmetric algebra closes (up to gauge transformations) only if one uses the (linearized) Einstein equations. However, if one tries to construct the corresponding quantum multiplet one finds out that it is not possible to quantize the field $h_{\mu\nu}$ such that the Einstein field equations are verified by the quantum operators: the most general form of the causal commutation relations for $h_{\mu\nu}$ is not compatible with the (linearized) Einstein equations. The argument remains the same even if one introduces the auxiliary fields $M, N, b_{\mu}$. (A good quantization procedure for the field $h_{\mu\nu}$ can be found in [9], [20]). This spoils completely the verification of the supersymmetric algebra!

So, we point out that a quantum construction of supersymmetric multiplet is more restrictive than the corresponding construction for the classical model. However, only the quantum model is relevant for any attempt of generalizing the standard model of elementary particles.

One can see that the restrictions leading to the negative results obtained in this paper and in the preceding one [13] come from the identity (2.11). One way to circumvent this restriction is to accept that the supersymmetry is spontaneously broken. This is in agreement with phenomenology which failed to find out supersymmetric partners of the known elementary particles of equal mass. It is obvious that in a broken supersymmetric theory the supercharges $Q_a, \bar{Q}_{\dot{a}}$ cannot exist. Indeed, the existence of the supercharges and the postulated relations (2.6) lead in all known cases to the equality of the masses of the Bosons and Fermions. This cannot be saved even if one modifies (2.6) by adding some constants in the right hand side as it is suggested in the literature. The standard literature on spontaneous broken symmetries suggests indeed that in such a case the charges do not exists but one hopes to to have the currents as well defined objects and the symmetry group of the model is replaced by a current algebra. If such a framework could be constructed in a supersymmetric model we would expect that (2.6) are replaced by other relations expressing the (anti)commutation relations of the
supercurrents and the fields. If in (2.9) and (2.11) one replaces the supercharges by the supercurrents then less severe restrictions would appear; in particular one would not be forced to have equal masses. This approach seems worthwhile investigating.

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