Type-II quantum spin Hall effect in two-dimensional metals

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Abstract

The quantum spin Hall (QSH) effect has been observed in topological insulators and long quantum wells using spin–orbit coupling as the probe, but it has not yet been observed in a metal. An experiment is proposed to measure the different Type-II QSH effect of an electron or hole in a two-dimensional (2D) metal by using the previously unexplored but relativistically gauge-invariant form of the generated 2D QSH Hamiltonian. Instead of using the electric field in the surface of the spin-polarized bands of a topological insulator or across the quantum well width as the probe, ones uses an applied azimuthal vector potential and an applied radial electric field as the tools to generate a spontaneously quantized spin current in an otherwise spin unpolarized 2D metal. A long cylindrical solenoid lies normally through the inner radius of a 2D metallic Corbino disk. The current \(I_S\) surrounding the solenoid produces an azimuthal magnetic vector potential but no magnetic field in the disk. In addition, a radial electric field is generated across the disk by imposing either a potential difference \(\Delta v\) or a radial charge current \(I\) across its inner and outer radii. Combined changes in \(I_S\) and in either \(\Delta v\) or \(I\) generate spontaneously quantized azimuthal charge and spin currents. The experiment is designed to measure these quantized azimuthal charge and spin currents in the disk consistently. The quantum Hamiltonians for both experiments are solved exactly. A method to control the Joule heating is presented, which could potentially allow the Type-II QSH measurements to be made at room temperature.

Keywords: two-dimensional metals, quantum spin Hall effect, magnetic vector potential, quantized spin and charge currents, Joule heating control

(Some figures may appear in colour only in the online journal)

1. Introduction

In the hydrogen atom, one of the leading relativistic corrections to the non-relativistic limit is proportional to \((p \times E) \cdot \sigma\) \([1–4]\), where \(p\) is the quantum-mechanical momentum of the electron, the electric field \(E = -\nabla \Phi\), where \(\Phi\) is the radial electrostatic potential in spherical coordinates, and the components of \(\sigma\) are the Pauli matrices representing the electron spin. Since \(E| r\) and the angular momentum \(L = r \times p\), such a term in the Hamiltonian represents spin–orbit coupling, but differs from the classical Hall effect in three-dimensional metals that results from both an applied \(E\) and an applied magnetic induction \(B\), but does not include the particle’s spin.

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Since those early studies of the atomic effects of spin–orbit scattering, there was a very large interest in the quantum Hall effect in a two-dimensional (2D) degenerate electron gas in the inversion layer of a metal-oxide-semiconductor field-effect transistor (MOSFET) in a strong perpendicular $B$ and variations thereof [5–11], and in the fractional quantum Hall effect in a 2D GaAsGa$_x$Al$_{1-x}$. As MOSFET in a yet stronger perpendicular $B$ [12–14], both of which have been discussed extensively and presented in textbooks [15–17]. In those experiments, the electrical conductance was found to be quantized in integral or fractional multiples of $e^2/h$, where $e$ and $h$ are the electric charge and Planck’s constant, respectively.

More recently, there has been a very large interest in the first type of quantum spin Hall (QSH) effect, which we denote as the Type-I QSH effect, in thin topological insulators (the Type-IA QSH effect) and insulating quantum wells (the Type-IB QSH effect) [15]. In both of those systems, the model Hamiltonian was also proportional to $(p \times E) \cdot \sigma$, and the electrons in those topological insulators only travel on or near to the sample surface, with crossing Dirac surface states that are spin polarized for $E \neq 0$. In both Type-I QSH experiments, the important component of $E$ is perpendicular to $p$. Moreover in topological insulators, including that 1$'$ form of monolayer WTe$_2$ studied with regard to the QSH [22], the Dirac-cone electronic dispersion locks the electron spins onto their momenta, and the protected edge currents are insensitive to backscattering from defects or from travelling around the corners of the top surface. In HgTe quantum wells, which are essentially long channels of HgTe thicker than the critical thickness $d_c$, sandwiched between Hg$_{1-x}$Cd$_x$Te barriers, the inverted conductance and valence band electronic structure of the other II–VI semiconductors causes the conducting spin–orbit-coupled $p$-orbitals to align along the well edges, and the electric field generated by the potential gradients at the well edges causes one-dimensional spin-polarized motion along opposite well edges with opposite spin polarizations [23, 24]. In both of those experiments, the conductance was also observed to be quantized in integral units of $e^2/h$, as for the integral quantum Hall effect [15–17]. In addition, a proposal to study the Rashba spin–orbit coupling in 2D metallic quantum dots with both 0 $\neq |E|/|B| \perp p$ was made [25].

Simultaneously, at the opposite end of the conductivity spectrum, there has also been a large interest in 2D and layered superconductors [26–35]. Generally a 2D metal or superconductor is either atomically thin or has the smallest thickness appropriate for its chemical composition. Particular interest has been in monolayer FeSe, in monolayer and few-layer samples of the transition metal dichalcogenide 2H-NbSe$_2$, in gated bulk samples of the transition metal dichalcogenide 2H-MoS$_2$, which also resulted in effective monolayer superconductors [26–29], and in twisted-bilayer, twisted-trilayer, and twisted multilayer graphene, which was surprisingly also shown to be superconducting for the appropriate magic twist angles $\sim 1.1^\circ$–$1.4^\circ$ and for certain induced carrier densities [30–32].

But is there anything more that should be studied for materials in the center of the conductivity spectrum? More recently, some very interesting 2D examples are the metallic phases of magic-angle, twisted-bilayer and twisted-trilayer graphene obtained either with different induced carrier densities than those used to study the superconductivity [30–32], or for temperatures $T$ exceeding the superconducting transition temperature $T_c \lesssim 2.1 \text{ K}$ [32]. More generally, by reducing the thickness of a large variety of semiconducting or insulating transition metal dichalcogenides (MX$_2$ with $M = Mo$, V, W, Ta and $X = S$, Se, Te) with octahedral 1$T$ or distorted octahedral 1$T'$ structures [33], to single MX$_2$ layer thicknesses, the resulting materials, possibly except for monolayer 1$T'$-WTe$_2$, turned out to be very surprisingly metallic [36–44]. Presently, there appear to be a variety of reasons for this, among which is the removal of the charge-density waves present in bulk materials [35]. In addition, monolayer MoS$_2$ has been shown to be metallic [45, 46], and first-principles predictions of monolayer metallicity in Au$_2$B and MoSi$_2$ have been presented [47, 48]. The more recent growth of such crystals on Au substrates appears to yield more uniform monolayer metals than their earlier growth on NaCl microcrystals [43, 44], which suggests that the number of high-quality monolayer metallic systems could increase significantly in the near future. In addition, ultrathin single crystals of ultrapure Al could be effectively 2D metals suitable for experiments to measure quantum effects. Those studies suggested that such non-magnetic, metallic monolayers might have many practical uses [36–48].

However, one could also ask if there might be some interesting new physics that could be extracted from such 2D metals. Here we show that the answer is yes! The general QSH interaction is obtained from the gauge-invariant Dirac equation for an electron or hole by expanding near the non-relativistic limit [1–4], and may be written in gauge-invariant form as

\[ H^{QSH} = -\frac{\mu_B}{2m_1c^2}[E \times (p - qA)] \cdot \sigma, \]

where $\mu_B = qh/(2m)$ is the Bohr magneton for electrons or holes, $q \equiv \pm e$ for electrons or holes, respectively, $c$ is the speed of light in vacuum, $h = h/(2\pi)$, $m_1$ is the effective mass of the carrier for motion parallel to the 2D metal or topological insulator, $E$ is the electric field, $A$ is the magnetic vector potential, and the components of $\sigma$ are the Pauli matrices representing the spin of the electron or hole [1–4]. We note that the early expansions only kept the gauge invariance in the leading term of the kinetic energy, correctly obtaining the Zeeman interaction, but omitting it in all higher order terms [2, 3]. More recently, the gauge invariance was included in all terms up to fourth order in the inverse of the Einstein rest energy [4].

The conventional (Type-I) QSH interactions present in either a topological insulator or a quantum well channel have $A = 0$ and a non-vanishing spin–orbit interaction proportional to $(p \times E) \cdot \sigma$ leading to spin-polarized edge states with charges moving in opposite directions under the electric Lorentz force, either by the applied $E|\mathbf{r}$, where in the Type-IA topological insulators, $E$ is generated by the gradient of the potential difference across the well, which is also parallel to $\mathbf{r}$.\[ J. Phys.: Condens. Matter 34 (2022) 485302 A Zhao et al]
The similar (Type-IB) QSH interaction in quantum well channels has the charges in spin-polarized well edge states moving in opposite directions by the well potential. In both Type-I cases studied, $A = 0$ and $E = |p \perp p$, as in spin–orbit coupling. In the proposed Type-II QSH experiments, $A \neq 0$ and $E = |p \perp A$. Although both Type-I and the new proposed Type-II QSH experiments arise from the same Hamiltonian, equation (1), the Type-II QSH effect does not involve spin–orbit coupling.

The new unconventional Type-II QSH effect does not arise from a special band structure such as the linearly crossing Dirac-like bands of a topological insulator or the potential difference in an inverted semiconductor structure of a quantum well, but can exist in any 2D material. For example, in a non-interacting 2D electron gas on a square lattice of lattice constant $a$, for low band filling, the electronic Fermi ‘surface’ is a circle about the $\Gamma$ point at the center of the square first Brillouin zone of area $(2\pi/a)^2$, or for high band fillings, it is an effectiveldiamond-shaped hole Fermi ‘surface’ curve about one of the corner $K$ points [16]. With tight-binding intersite hopping, such Fermi surfaces would be modified. Similar arguments apply for different 2D lattices. For appropriate dopings, even metallic phases of magic-angle twisted multilayer graphene [32, 49, 50], with no crystal lattice symmetry, can have an effective Fermi surface that has a very large density of states. But in such cases, the interlayer electron or hole transport must be much weaker than the intralayer transport, as is easiest to visualize with near-neighbor single-particle hopping interactions [15–17, 33, 49–51]. In addition, we require that there are no magnetic particle–particle interactions for this model Hamiltonian, equation (1), to be valid, so that the number of electrons or holes with each of the two spin $\frac{1}{2}$ states is identical when $E = A = 0$. However, with nominally non-spin-polarized electrons or holes for $A = E = 0$, when the proper geometry for application of $A$ and $E$ is chosen, spontaneously quantized spin currents can be generated and observed.

In section 2, we describe the details of two proposed experiments to test the Type-II QSH effect. In both experiments, a central solenoid inside a 2D metallic Corbino disk generates an azimuthal magnetic vector potential and a radial electric field is generated either by a potential difference or by an applied charge current across the disk. In addition, a radial vector potential is generated in both experiments by the driving radial charge sheet current, but it does not contribute significantly to the Type-II QSH effect. In section 3, we present the quantum Hamiltonian for both of those experiments, show that they are both exactly soluble, and that the radial vector potential can be removed by a gauge transformation. The continuity equation in the presence of the Type-II QSH Hamiltonian is also shown to be satisfied. In section 4, we discuss finite temperature effects and discuss additional experimental techniques needed to reduce Joule heating. In section 5, we summarize our results.

2. The proposed experiments

We propose two different Type-II QSH experiments that employ the previously untested but relativistically-generated Hamiltonian, equation (1), that does not make use of its spin–orbit coupling component. In both experiments, a cylindrical solenoid of radius $\rho_S$ is normally placed in the center of a 2D metallic Corbino disk of radius $\rho_i$ and inner radii $\rho_o$, as sketched in figure 1(a). The experimenter applies a charge current $I_2$ in a wire tightly wrapped around the solenoid, which generates an azimuthal magnetic vector potential

$$A_S(\rho) = \hat{\Phi}/(2\pi\rho)$$

in the disk, where $B_S = \nabla \times A_S$, $\hat{\Phi} = \pi |B_S|^2/\rho^2$ is the magnetic flux in the solenoid, and due to the Biot–Savart law, $\Phi_S = \mu_0 B_S^2/2$, where $\mu_0$ is the magnetic permeability of vacuum. The solenoid must be long enough that $B_S \approx 0$ everywhere in the disk, as in the Aharonov–Bohm experiment [52, 53].

2.1. The first experiment

In the first experiment, a uniform potential difference $\Delta V = V_o - V_i$ is applied across the outer and inner disk radii $\rho_o$ and $\rho_i$, respectively, as pictured in figure 1(b). The circular electrodes must be much better conductors than the 2D metal in the Corbino disk. This has the result of imposing the electric potentials $V_o$ and $V_i$ on $\rho_o$ and $\rho_i$, respectively.

Since the electric potential $\Phi(\rho)$ in the disk satisfies the Laplace equation, $\nabla^2 \Phi = 0$, its solution in polar coordinates is easily found to be

$$\Phi_1(\rho) = \frac{V_o - V_i}{\ln(\rho_o/\rho_i)},$$

and

$$\Delta V_i = (V_o - V_i)/\ln(\rho_o/\rho_i),$$

so that

$$E_1 = -\nabla \Phi(\rho) = -\hat{\rho}/(\Delta V_i/\rho).$$

The combined generations of the azimuthal $A_S$ and radial $E_1$ given respectively by equations (2) and (5) are employed to perform the first Type-II QSH measurements. For this experiment, $E_1 = j_1/\sigma$ generates a radial sheet current $K_1 = K_1 \hat{\rho} = j_1/(2\pi\rho)$, which also generates a radial magnetic vector potential $A_1$ surrounding the disk. The exact wave functions and energies for independent electrons or holes in the disk are presented in section 3.

2.2. The second experiment

In the second Type-II QSH experiment pictured in figure 1(c), the potential difference $\Delta V$ pictured in figure 1(b) is replaced by an applied radial current $I$, resulting in a uniform radial sheet current $K_2$ across the disk, that also generates a radial magnetic vector potential $A_2$ surrounding the disk. In this case, after thermal equilibrium is attained, the applied radial sheet current $\hat{K}_2 = \hat{K}_2 = j_2/(2\pi\rho)$, and due to Ohm’s law, a time-independent radial electric field $E_2 = j_2/\sigma = K_2/(2\pi\rho\sigma)$, where $\sigma$ is the electrical conductivity of the metallic disk. As for the first
both cases, the combined radial and the different equilibrium potentials $E_{radial}$ sheet charge current the electrodes at between both the radially central FM and NM electrodes and either connected radially central NM electrode. The radial voltages between that radially central NM electrode and its electrically detectably different from the charge current detected by the voltage radially central normal metallic (NM) electrode, which should be electron or hole spins, leading to a spin imbalance that generates an applied radial current inducing the radial disk. (b) (top view) In the first experiment, the electric potentials $B$ applied to the tightly wrapped wire coil surrounding a long cylindrical solenoid of radius $\rho_1$ that is normally placed inside a 2D metallic Corbino disk of inner radius $\rho_i > \rho_o$, generating a constant $B_s$ inside the solenoid and only an azimuthal vector potential $A_k$ in the disk. (b) (top view) In the first experiment, the electric potentials are fixed respectively at $v_i$ and $v_o$ on $\rho_i$ and $\rho_o$, respectively, inducing the radial $E$. (c) (top view) In the second experiment, an applied radial current $I$ source across $\rho_i$ and $\rho_o$ induces a uniform radial sheet charge current $K$, which induces the radial electric field $E$ and the different equilibrium potentials $v_i$ and $v_o$ on $\rho_i$ and $\rho_o$. In both cases, the combined radial $E$ and azimuthal $A$ couple to the electron or hole spins, leading to a spin imbalance that generates an azimuthal spin current detected by the voltage between the radially central ferromagnetic (FM) electrode and the electrically connected radially central normal metallic (NM) electrode, which should be detectably different from the charge current detected by the voltage between that radially central NM electrode and its electrically connected radially central NM electrode. The radial voltages between both the radially central FM and NM electrodes and either the electrodes at $\rho_i$ or $\rho_o$ should detect the same radial charge current, but no radial spin current. See text.-experiment, equations (3)–(5) apply, but in this case $\Delta v_2 = -K_2/(2\pi \sigma)$, so that equation (3) is replaced by

$$\Phi_2(\rho) = v_i + \Delta v_2 \ln(\rho/\rho_i) = v_i - K_2/2\pi \sigma \ln(\rho/\rho_i),$$

where we have chosen the arbitrary integration constant to be $v_i$ in order that the two potentials may be written as

$$\Phi_2(\rho) = v_i + \Delta v_i \ln(\rho/\rho_i)$$

for $\ell = 1, 2$. For both $\ell = 1, 2$, $\nabla \cdot E = (\nabla \times E_\ell) \cdot \sigma = 0$.

In both Type-II QSH experiments, the radial sheet current $K_\ell$ also generates a radial vector potential $A_\ell$ for $\ell = 1, 2$, due to Ampère’s law, as discussed in section 2.3.

2.3. The induced radial vector potential in the two experiments

For the cases $\ell = 1, 2$ pictured in figures 1(b) and (c), respectively, in which either a uniform potential difference or a radial current is applied across $\rho_i$ and $\rho_o$, there is an additional complication due to the induced radial vector potential given in general coordinates by [54]

$$\vec{A}_\ell(x) = \frac{\mu_0}{4\pi} \int \frac{j_\ell(x')d^3x'}{|x-x'|},$$

where $\mu_0$ is the magnetic permeability of vacuum. Since in cylindrical coordinates, we have

$$j_\ell(x') = \frac{K_\ell}{2\pi \rho'} \delta_{\ell,0} \Theta(\rho'-\rho_i) \Theta(\rho_o-\rho'),$$

where $\Theta(x)$ is the Heaviside step function, we obtain the general expression for a long solenoid

$$\vec{A}_\ell(\rho,z) = \frac{\mu_0 K_\ell}{2\pi} \int_{\rho_i}^{\rho_o} \frac{K(k)d\rho'}{\sqrt{\rho^2 + (\rho')^2 + z^2}},$$

where $K(k)$ is the elliptic integral of the first kind, which should not be confused with the sheet currents $K_\ell = K_\ell \rho$ for $\ell = 1, 2$. It is most likely easier experimentally to apply the current across those radii, $\ell = 2$, but an experimental measure of this correction could be made by doing the potential difference experiment, $\ell = 1$, at least once. In figure 2, we plot $\Delta A(\rho_1/\rho_o, z/\rho_i) = 2\pi^2 A_\ell(\rho_1/\rho_o, z/\rho_i)/(\mu_0 K_\ell)$ for $\rho_1/\rho_o = 10$ as a function of $z/\rho_i$ from 1 to 20 for the indicated values of $z/\rho_i$.

In non-magnetic metals, it is now well established that a spin current can be generated into the metal by injecting a charge current from a FM electrode, and measuring the voltage between two different FM electrodes [55–58]. In the present proposed experiments, the quantized spin current in

![Figure 1](image1.png)

**Figure 1.** Sketch of a side view of the non-thermally-managed version of the proposed experimental setup. (a) A current $I_k$ is applied to the tightly wrapped wire coil surrounding a long cylindrical solenoid of radius $\rho_1$ that is normally placed inside a 2D metallic Corbino disk of inner radius $\rho_i > \rho_o$, generating a constant $B_s$ inside the solenoid and only an azimuthal vector potential $A_k$ in the disk. (b) (top view) In the first experiment, the electric potentials are fixed respectively at $v_i$ and $v_o$ on $\rho_i$ and $\rho_o$, respectively, inducing the radial $E$. (c) (top view) In the second experiment, an applied radial current $I$ source across $\rho_i$ and $\rho_o$ induces a uniform radial sheet charge current $K$, which induces the radial electric field $E$ and the different equilibrium potentials $v_i$ and $v_o$ on $\rho_i$ and $\rho_o$. In both cases, the combined radial $E$ and azimuthal $A$ couple to the electron or hole spins, leading to a spin imbalance that generates an azimuthal spin current detected by the voltage between the radially central ferromagnetic (FM) electrode and the electrically connected radially central normal metallic (NM) electrode, which should be detectably different from the charge current detected by the voltage between that radially central NM electrode and its electrically connected radially central NM electrode. The radial voltages between both the radially central FM and NM electrodes and either the electrodes at $\rho_i$ or $\rho_o$ should detect the same radial charge current, but no radial spin current. See text.

![Figure 2](image2.png)

**Figure 2.** Magnetic vector potential generated by the radial applied current or potential. Plots of the dimensionless $\Delta A(\rho/\rho_o, z/\rho_i) = 2\pi^2 A_\ell(\rho/\rho_o, z/\rho_i)/(\mu_0 K_\ell)$ for $\rho_1/\rho_o = 10$ in each curve, and for $z/\rho_i = 10^{-4}$ (solid black), 0.5 (dashed red), 1.0 (dashed blue), 5.0 (dashed orange), and 10.0 (brown dot-dashed).
the non-magnetic Corbino disk is spontaneously generated by the radial electric field \( E_r \) generated either by the applied \( \Delta V \) or by the applied radial \( I \) and by the azimuthal \( A_{\theta} \) generated by the applied current \( I_s \) in the solenoid, and their combined Type-II QSH coupling of \( E_r \) and \( p - qA_s \) to the metallic electron or hole spins.

As shown in section 3, the nonvanishing Type-II QSH Hamiltonian in 2D spontaneously generates azimuthal charge and spin current densities, but does not generate any radial currents. To measure the generated currents, two NM electrodes and one FM electrode are as azimuthally equally spaced as experimentally possible at \( \rho_{\text{exp}} \), also chosen to be as close to the midpoints between \( \rho_i \) and \( \rho_o \) as experimentally possible, as sketched in figures 1(b) and (c). To measure the spontaneously generated azimuthal charge and spin current densities \( j_\phi^\pm (\rho, \phi) \), the experimenter measures the voltage \( V \) between the two neighboring radially central NM electrodes and between the FM electrode and its neighboring radially central NM electrode, respectively, all at \( \rho_{\text{exp}} \). To measure the total radial charge and vanishingly small spin current densities \( j_r^\pm (\rho, \phi) \), the experimenter respectively measures the voltage across either the NM or FM electrodes and the circular NM electrode on either \( \rho_i \) or \( \rho_o \), for both signs of \( I \) or \( \Delta V \), as pictured in figures 1(b) and (c). Performing both of these measurements allows the experimenter to correct for slight variations in the values of \( \rho_{\text{exp}} \) for those two electrodes.

3. Theory of the experiments

3.1. The Hamiltonian and particle continuity

For the experiment pictured in figures 1(a) and either (b) or (c), the Hamiltonian for a single electron or hole in a metallic 2D Corbino disk with an isotropic planar effective mass \( m_{||} \) is

\[
H_{2D}^{\ell} = H_{2D}^{T(\ell)} + q \Phi_{\ell}(\rho) + H_{2D}^{QSH(\ell)},
\]

where \( \ell = 1, 2 \),

\[
H_{2D}^{T(\ell)} = \frac{\beta(p - q(A_{\perp} + A_{\ell}))^2}{2m_{||}}
\]

is the gauge-invariant kinetic energy, \( p = -i\hbar \nabla \) is the quantum-mechanical momentum in polar coordinates, \( p - q(A_{\perp} + A_{\ell}) \) is the total mechanical momentum \([52, 59–61], \beta = \pm 1 \) for electrons or holes, respectively, \( q\Phi_{\ell}(\rho) \) is the potential energy for both experiments, where \( A_{\perp} \) is given by equation (2), the \( \Phi_{\ell}(\rho) \) are given by equations (3) and (6), and the \( H_{2D}^{QSH(\ell)} \) are given by

\[
H_{2D}^{QSH(\ell)} = -\frac{\mu_B}{2m_{||}} [E_\ell \times (p - qA_{\perp})] \cdot \sigma,
\]

which is the quantum-mechanical gauge-invariant QSH Hamiltonian in 2D \([1–4]\). With \( A_s \) and \( E_{\ell} \) respectively in the azimuthal and radial directions of a 2D Corbino disk, the only relevant Pauli matrix is \( \sigma_z \), and the generated \( A_{\ell} \) do not enter this quantum spin Hall Hamiltonian. For the particular 2D metal under study, \( m_{||} \) can be measured by cyclotron resonance with an applied \( B \) normal to the film. Particle-particle interactions are neglected.

The density operators \( \rho_{\ell} = \sum_{s} |\Psi_{\ell}^s(\rho, \phi)|^2 \) for the \( \ell = 1, 2 \) experiments satisfy \([61]\)

\[
\frac{\partial \rho_{\ell}}{\partial t} = \frac{1}{i\hbar} \sum_{s} [\Phi_{\ell}^{(s)} H_{2D}^{T(\ell)} \Psi_{\ell}^s - \left( H_{2D}^{T(\ell)} \Psi_{\ell}^s \right)^\dagger \Psi_{\ell}^s],
\]

leading to the 2D continuity equation

\[
\frac{\partial \rho_{\ell}}{\partial t} + \nabla \cdot \mathbf{j}_{\ell} = 0,
\]

where the conserved particle currents \( \mathbf{j}_{\ell} \) are given by

\[
\mathbf{j}_{\ell} = \frac{1}{m_{||}} \mathcal{R} \left( \sum_{s} \Phi_{\ell}^{(s)}(\rho, \phi) \left[ \rho \left( \frac{\hbar \partial}{\partial \rho} - qA_{\ell} \right) \right. \right.
\]

\[
+ \left. \left. \Phi \left( 1 - \frac{\mu_B E_{\ell}s}{c^2} \right) \left( \frac{1}{\rho} \frac{\hbar \partial}{\partial \phi} - qA_{\perp} \right) \right] \Psi_{\ell}^s(\rho, \phi) \right).
\]

We note that the conserved particle currents contain both radial and azimuthal particle currents, and an azimuthal spin current driven solely by \( H_{2D}^{QSH(\ell)} \).

At this point, a strong analogy with the Aharonov–Bohm experiment is evident. The radial \( A_x = \rho A_{\perp}(\rho) \) can be removed from \( H_{2D}^{T} \) in equation (13) and in equation (17) for the conserved current by a gauge transformation,

\[
\Psi_{\ell}^s(\rho, \phi) = \tilde{\Psi}_{\ell}^s(\rho, \phi) \exp \left[ \frac{iq}{\hbar} \int_{\rho_i}^{\rho} A_x(\rho') d\rho' \right],
\]

where \( \tilde{\Psi}_{\ell}^s \) is now the wave function without \( A_{\perp} \) and the full wave function with \( \Psi_{\ell}^s \). On the other hand, the azimuthal vector potential \( A_{\theta} \) is quantized, and cannot be removed by a gauge transformation, as detailed in the following.

With the gauge transformation, the conserved particle currents satisfy

\[
\mathbf{j}_{\ell} = \frac{1}{m_{||}} \mathcal{R} \left( \sum_{s} \tilde{\Psi}_{\ell}^{s*}(\rho, \phi) \left[ \rho \left( \frac{\hbar \partial}{\partial \rho} - qA_{\ell} \right) \right. \right.
\]

\[
+ \left. \left. \Phi \left( 1 - \frac{\mu_B E_{\ell}s}{c^2} \right) \left( \frac{1}{\rho} \frac{\hbar \partial}{\partial \phi} - qA_{\perp} \right) \right] \tilde{\Psi}_{\ell}^s(\rho, \phi) \right).
\]

Before writing the Hamiltonian for the wave functions \( \tilde{\Psi}_{\ell}^s \) in polar \((\rho, \phi)\) coordinates, we first break up the Hamiltonian into two parts, one exactly solvable, the other solvable in first order perturbation theory, and all higher order perturbations vanish. We write

\[
H_{2D}^{\ell} = H_{2D,0}^{\ell} + \epsilon H_{2D,1}^{\ell},
\]

where

\[
H_{2D,0}^{\ell} = q \nu_i \left[ \frac{\beta^2 h^2}{2m_{||}^2} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial \phi^2} \right) \right. \right.
\]

\[
- \delta \left( \delta + 2i \frac{\partial}{\partial \phi} \right) + \frac{\gamma}{2 \beta} \left( \delta + i \frac{\partial}{\partial \phi} \right),
\]

and

\[
H_{2D,1}^{\ell} = q \nu_i \left[ \frac{\beta^2 h^2}{2m_{||}^2} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial \phi^2} \right) \right. \right.
\]

\[
- \delta \left( \delta + 2i \frac{\partial}{\partial \phi} \right) + \frac{\gamma}{2 \beta} \left( \delta + i \frac{\partial}{\partial \phi} \right) \right],
\]

\[
H_{2D}^{T(\ell)} = H_{2D,0}^{T(\ell)} + \epsilon H_{2D,1}^{T(\ell)},
\]

\[
H_{2D}^{QSH(\ell)} = H_{2D,0}^{QSH(\ell)} + \epsilon H_{2D,1}^{QSH(\ell)}.
\]
where
\[ \delta = \frac{\Phi_s}{\Phi_0}, \]
\[ \gamma_\ell = \frac{q\Delta V_\ell}{mc^2}, \]
\[ \Phi_0 = \hbar/q \]
is the flux quantum for a hole or minus the flux quantum for an electron, and
\[ H_{2D}^{(f)} = q\Phi_\ell(\rho) - qv_i = q\Delta V_\ell \ln(\rho/\rho_i) = \gamma_\ell mc^2 \ln(\rho/\rho_i), \]

and since \(|\gamma_\ell| \ll 1\), this is a small perturbation for the entire range of experimental values of \(q\Delta V_\ell\), which is described in detail in appendix A.

We then write the time-independent Schrödinger equation for the bare (solvable) Hamiltonian as
\[ H_{2D}^{(0)} \Psi_{0,\ell}(\rho, \varphi) = E_{0,\ell} \Psi_{0,\ell}(\rho, \varphi). \]

We first multiply \( H_{2D}^{(0)} \Psi_{0,\ell}(\rho, \varphi) = E_{0,\ell} \Psi_{0,\ell}(\rho, \varphi) \) by \( \rho^2 \), and then use the standard ‘separation of variables’ technique to write \( \Psi_{0,\ell}(\rho, \varphi) = R_{\ell,0}(\rho)\chi(\varphi) \) [52, 60], and divide both sides of the equation by \( R_{\ell,0}(\rho)\chi(\varphi) \). Of course, we also require the boundary condition
\[ \chi(\varphi + 2\pi) = \chi(\varphi). \]

As shown in appendix B,
\[ \chi(\varphi) = e^{iv_f}, \]
and the radial wave functions \( R_{\ell,0}(\rho) \) satisfy
\[ R_{\ell,0}(\rho) = E_{0,\ell} R_{\ell,0}(\rho), \]

where the effective radial potential \( V_{eff,\ell}(\rho) \) is given by
\[ V_{eff,\ell}(\rho) = \frac{\beta E_{\ell,0}}{2m_0} + qv_i, \]
\[ \varepsilon_{n,\ell} = (n - \delta)(n - \delta + s\beta_\gamma_\ell/2). \]

If \( \delta = n \), the two spin states are degenerate. The experimenter has a significant amount of flexibility in choosing the spin \( s \) value of the dominant probed carriers. Since
\[ \beta_\gamma_\ell = -\frac{|e|\Delta V_\ell}{mc^2} \]
for both electrons and holes, the dominant spin state just depends upon the sign of the potential difference (or the direction of \( E \)) and the sign of \( n - \delta \). If \( n - \delta > 0 \), the lower energy state is that for \( s\beta_\gamma_\ell < 0 \). In figure 1(a), the \( s = + \) (up) spin state for either electrons or holes has lower energy than does the down spin state for \( \Delta V > 0 \). For \( \Delta V < 0 \), the \( s = - \) (down) spin state is lower in energy for both electrons and holes. On the other hand, if \( n - \delta < 0 \), those interpretations hold for the opposite signs of \( \Delta V \). A sketch of the use of the \( \delta \) to spontaneously generate azimuthal spin currents \( j_{z,\ell}^{(\text{eff})}(\rho, \varphi) \) is shown in figure 3.

Although this is qualitatively similar to the Zeeman interaction, here \( B = 0 \) and the spin states for both electrons and holes are distinguished by changing the signs of \( \Delta V \) and of \( n - \delta \). There is a very interesting interchange between the roles of the flux in the solenoid and the electric potential difference upon the spin states of the electrons or holes.

It addition, it is shown in appendix A that the only corrections to the energies \( E_{n,\ell}^{(f)} \) are first order in \( \gamma_\ell \), and there are no perturbative corrections to the wave functions. Hence, the Hamiltonian given by equation (20) is exactly soluble for any value of \( \gamma_\ell \).

3.2. The exact wave functions
When \( \Delta V_\ell = 0 \), \( \gamma_\ell = 0 \), and \( \varepsilon_{n,\ell} = (n - \delta)^2 \), so the energies of the two spin states are degenerate, both for electrons and holes. However, for \( \Delta V_\ell \neq 0 \), the combined signs of \( \delta - n \) and \( \gamma_\ell \) flip the spins of the ground state. During the experiments, both signs of \( \varepsilon_{n,\ell} \) are obtained by properly varying the sign and magnitude of \( I_0 \) and the sign of either \( \Delta V \) or \( I \).

Although \( \varepsilon_{n,\ell} \) can be of either sign, it is convenient to set
\[ \varepsilon_{n,\ell}^{(f)} = \left(\nu_f^{(\text{eff})}\right)^2, \]
where \( \varepsilon_{n,\ell}^{(f)} \) is given by equation (30) and
\[ \nu_f^{(\text{eff})} = \nu_{s,1}^{(\text{eff})} + i\nu_{s,2}^{(\text{eff})}. \]
That is, if $\nu_{x,n}^s > 0$, $\nu_s^x = \nu_{x,1}^s$ is real, and if $\nu_{x,n}^s < 0$, $\nu_s^x = i\nu_{x,2}^s$ is pure imaginary.

Since there are two distinct spin states characterized by $\nu_{x,n}^s$, we write $H_{0,\nu}^\ell \cdot R_{\nu,ho}(\rho) = E_{0,\nu}^\ell \cdot R_{\nu,ho}(\rho)$, and after multiplying by $-2m_i|\beta|^2/\hbar^2$, we have

$$\rho^2 R_{\nu,ho}^\ell (\rho) + \rho R_{\nu,ho}''(\rho) - (\nu_s^x)^2 R_{\nu,ho}(\rho) = -\frac{2m_i|\beta|^2}{\hbar^2} R_{\nu,ho}(\rho), \tag{34}$$

where $R_{\nu,ho}^\ell$ and $R_{\nu,ho}''$ are the first and second derivatives of $R_{\nu,ho}$ with respect to $\rho$, and $E_{0,\nu}^\ell = E_{0,\nu}^\ell(\rho) - q\nu_s$. Then, by setting

$$E_{0,\nu}^\ell = \beta \hbar^2 (k^0_x)^2/(2m_i), \tag{35}$$

and letting the dimensionless variables be $x_s^x = k^0_x \rho$, we have

$$(x_s^x)^2 R_{\nu,ho}^\ell (x_s) + x_s^x R_{\nu,ho}''(x_s) + [(x_s^x)^2 - (\nu_s^x)^2] R_{\nu,ho}(x_s) = 0, \tag{36}$$

the solutions of which are the Bessel functions.

The Bessel functions relevant to this problem are the Hankel functions $H_n^{(1)}(z) = J_n(z) + iN_n(z)$ and $H_n^{(2)}(z) = [H_n^{(1)}(z^*)]^*$ [62, 63], where the $J_n(z)$ and the $N_n(z)$ are the Bessel functions of the first and second kind, respectively, and $z = k^0_x \rho$ is real. Hence, the general wave functions are

$$\tilde{\Psi}_{n,s}^{(1)}(\rho,\phi) = [B_n^x H_n^{(1)}(k^0_x \rho) + C_n^x H_n^{(2)}(k^0_x \rho)] e^{i\nu_s^x \phi}, \tag{37}$$

where $B_n^x$ and $C_n^x$ are constants that depend upon the experimental conditions, and $H_n^{(1)}(k^0_x \rho)$ and $H_n^{(2)}(k^0_x \rho)$ are respectively the outward and inward radial waves. We note that $x_s^x$ implicitly depends upon $n$ and $\delta$ from equations (32) and (30).

Since $\rho_s$ and $\rho_s$ are macroscopic quantities, and the $k^0_x$ are the wave vectors of the metallic 2D annulus, we expect $k^0_x \sim \frac{2\pi}{na}$, where $N \geq 1$ and $a$ is the order of a lattice constant, which is much less than $\rho_s$. Then, for all $\rho_s \leq \rho \leq \rho_s', \rho_s' \gg 1$, and the $k^0_x \gg 1$ asymptotic forms of the Hankel functions are valid,

$$H_n^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(\pi/4 - i\pi x/2)}, \tag{38}$$

Then, the asymptotic forms of the Hankel functions that can describe either sign of $\nu_{x,n}^s$ are [62, 63]

$$H_n^{(1)}(x_s^x) \approx \sqrt{\frac{2}{\pi x_s^x}} \exp[i(x_s^x - \eta_{s}^x) + \pi \nu_s^x/2], \tag{39}$$

$$H_n^{(2)}(x_s^x) = [H_n^{(1)}(x_s^x)]^* \tag{40}$$

$$\approx \sqrt{\frac{2}{\pi x_s^x}} \exp[-i(x_s^x - \eta_{s}^x) - \pi \nu_s^x/2], \tag{41}$$

$$\eta_{s}^x = \pi \left( \nu_{s,1}^x + \frac{1}{2} \right).$$

The experimenters first need to provide an independent measurement of $m_i$ and the Fermi wave vector $k_F$ when both $\Delta_{\nu} = 0$ for $\ell = 1, 2$, which can be respectively measured by cyclotron resonance and angle-resolved photoemission experiments on an identical sample of the same 2D metal. They are then ready to perform the main experiments, which without proper thermal management of a conventional, three-dimensional device, normally would be done at low T, to minimize heating effects. There are two probes to force the spontaneously generated quantized currents: $\delta$, or the flux $\Phi_s$ in the solenoid, controlled by the current in the wire wrapped around it, and either the applied $\Delta \nu$ or the applied radial current $I$.

### 3.3. The charge and spin currents

For both experiments, the gauge-invariant radial and azimuthal charge and spin current densities are generalizations to poloidal coordinates of the gauge-invariant one-dimensional particle current density [52, 59–61],

$$j_{\rho}^{(1)}(\rho,\phi) = \frac{\hbar q}{m_i} \text{Im} \sum_{n=\pm} \tilde{\Psi}_{n,s}^{(1)}(\rho,\phi) \frac{\partial}{\partial \rho} \tilde{\Psi}_{n,s}^{(1)}(\rho,\phi), \tag{42}$$

$$j_{\rho}^{(2)}(\rho,\phi) = \frac{\hbar q}{m_i} \text{Im} \sum_{n=\pm} s(\tilde{\Psi}_{n,s}^{(2)}(\rho,\phi) \frac{\partial}{\partial \rho} \tilde{\Psi}_{n,s}^{(2)}(\rho,\phi) \tag{43}$$

$$j_{\rho}^{(s)}(\rho,\phi) = \frac{\hbar q}{m_i} \text{Im} \sum_{n=\pm} s(\tilde{\Psi}_{n,s}^{(s)}(\rho,\phi) \frac{\partial}{\partial \phi} \tilde{\Psi}_{n,s}^{(s)}(\rho,\phi) \tag{44}$$

and

$$j_{\rho}^{(s)}(\rho,\phi) = -\frac{\mu q E_F}{c^2 \rho} \text{Im} \sum_{n=\pm} s(\tilde{\Psi}_{n,s}^{(s)}(\rho,\phi) \frac{\partial}{\partial \phi} \tilde{\Psi}_{n,s}^{(s)}(\rho,\phi) \tag{45}$$

where $\tilde{\Psi}_{n,s}^{(s)}(\rho,\phi)$ is given by equation (37).

After setting $\Delta \nu = 0$ for $\ell = 1, 2$, so that $\gamma = 0$, the experimenters should first measure the quantum Hall effect by varying $\delta$ until it is an integer $n$, for which the spontaneous jump in the azimuthal current will occur, as sketched by the solid black lines in figure 3. Then, depending upon the sign of the carrier charge $q$, they should either slightly decrease or increase $\delta$ from $n$, and increase or decrease $I$ or $\Delta \nu$ and hence $\gamma$. Then, due to the Type-II QSH effect, the spontaneous jump in the azimuthal current will occur at $\delta = n + \beta \gamma/2$. Depending upon which $s$ value corresponds to $E_F$, this will either be at $\delta = n + \beta \gamma/2$ or at $\delta = n - \beta \gamma/2$. By changing the sign of $\gamma$ and repeating the experiment, the jump will switch to the other possibility. These points correspond to the dark vertical lines in figure 3. Details of the azimuthal and radial and spin currents using the asymptotic forms of the Hankel functions are given in appendix C.
4. Experimental techniques

4.1. Finite temperature effects

The single particle states of the non-interacting electron or hole gas in the 2D Corbino disk are then given by
\[ \epsilon_i(k_F) = \sum_x [E_0^x(k_F) + E_1^x(k_F)], \] (46)
where the \( E_0^x(k_F) \) and \( E_1^x(k_F) \) are given respectively by and just before equation (35) and by equation (51) in appendix A, and their states are occupied according to the Fermi–Dirac distribution function
\[ f[\epsilon_i(k_F)] = \frac{1}{e^{\frac{\epsilon_i(k_F) - \mu_i(T)}{k_B T}} + 1}, \] (47)
where \( k_B \) is Boltzmann’s constant, and for a free-particle \( \epsilon_i(k_F) \) in 2D, an excellent approximation to the present model, the chemical potential \( \mu_i(T) \) is given by [16]
\[ \mu_i(T) = \mu_i(0) + k_B T \ln[1 - e^{-\mu_i(0)/(k_B T)}] \approx \mu_i(0) - k_B T e^{-\mu_i(0)/(k_B T)}, \] (48)
which is nearly independent of \( T \), so that
\[ \mu_i(0) \approx E_F, \] (49)
where
\[ E_F = \max_x [E_0^x(k_F,x) + E_1^x(k_F,x)] \] (50)
is the Fermi energy, the ground state energy of the 2D metallic Corbino disk. Equation (50) applies for both electrons and holes. However, equation (49) also implies that the \( k_F,x \) have additional dependencies upon \( v_l \) and \( v_o \), as well as upon \( \rho_l, \rho_o, \) and the \( v_0 - v_i \) dependence of \( \mu_i \) in \( E_0^x(k_F,x) \). Since for most 2D metals, \( k_B T < E_F/3 \) would be in the low-\( T \) regime, one could in principle perform the experiment at room temperature with an appropriate metal.

Of course, one could also perform the measurements at temperatures well below room temperature, provided that a phase transition from the assumed non-spin-polarized metallic to some non-metallic state (such as an insulating, a semimetallic, a FM, an antiferromagnetic, a spin-polarized half metallic, a spin- and valley-polarized quarter-metallic, or a superconducting) state did not occur. If one or more of these or any other phase transitions were found to occur in the nominal 2D metal under study, the temperature range for the Type-II QSH measurements would have to be in the material’s ordinary metallic temperature range.

In addition, we note that since the 2D material under study is assumed to have an ordinary, non-spin-polarized metallic state in the absence of the external probes pictured in figure 1, we predict that the external probes will spontaneously generate a quantized azimuthal spin current that is present in the entire Corbino disk, with a radial distribution given by equation (68) in appendix C, which is primarily inversely proportional to \( \rho^2 \), so it is generally strongest near to \( \rho_i \). This is qualitatively different from the topological surface currents used in the Type-1A QSH measurements and from the spin-polarized currents generated along the edges of the quantum well channels in the Type-IB QSH measurements. In the Type-II QSH measurements, topological features are not predicted to occur.

4.2. Thermal management

However, with large applied currents and the resulting radial voltage difference across the disk, Joule heating could be a problem, unless the experimenters found a way to significantly reduce it. Such heat removal is now standard with the high-frequency temperature \( T_c \) superconducting Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (Bi2212) terahertz emitters by coating the top and bottom of each device with Au [64, 65], and sandwiching the emitter between sapphire plates with properly placed gold electrodes [66], allowing it to operate in liquid \( N_2 \) [67]. Although the metallic disks cannot be coated with Au except in the specific NM electrode positions, a similar design could work to control the Joule heating in this experiment. The thermally-managed solenoid can be constructed by tightly wrapping a very fine electrically insulated wire coil around a cylindrical sapphire rod, which is tightly covered with a thin cylindrical sapphire sheath. Then, the 2D metallic Corbino disk should be sandwiched between two identical sapphire Corbino disks. The Au voltage and current electrodes are deposited on the top of the bottom sapphire disk, which will make electrical contact with the three small electrodes on the bottom of the 2D metallic disk when the sapphire disks are tightly glued together [66]. Such a design of the thermally-managed Corbino disk is sketched in figure 4. Such or similar heat control procedures could allow the experiment to be performed at easily accessible \( T \) values [66, 67], such as in liquid \( N_2 \) or conceivably even up to room temperature.

Very recently, a very different QSH experiment employing spin–orbit coupling in both of the neighboring cuprate layers in the non-superconducting \( d_{x^2−y^2} \) density-wave phase of Bi2212 was proposed [68].

We note that changes in the applied radial current \( I \) makes changes in \( v_l \) and \( v_o \) and linear changes in \( E_F \) that are easy to evaluate and should be measurable. In addition, equation (59) shows that the quadratic dispersion of the particles only differs for each spin by a constant, so that equation (48) is highly accurate, implying that the experiment should be possible at room temperature, even when including the perturbations to the ground state energies.

4.3. Proposed measurements

The experimenters should first make independent measurements of the Fermi wave vector \( k_F,0 \) in the absence of \( v_l \) and \( v_o \) and of \( m_{||} \) (such as by angle-resolved photoemission measurements and cyclotron resonance experiments) for
the 2D metal under study. Then they should perform four current measurements. They should measure \(J_x^s(\rho_{\text{expr}})\) and \(J_z^s(\rho_{\text{expr}})\) from the two appropriate voltage leads pictured in figures 1(b) and (c), and they should also measure \(j_{\rho}^s\) and check to see that \(J_x^s\) is sufficiently small, from the NM and FM electrodes at \(\rho_{\text{expr}}\) (roughly midway between \(\rho_i\) and \(\rho_o\)) and the NM electrodes at \(\rho_i\) and at \(\rho_o\). As mentioned previously, they should do each of these measurements both for positive and negative \(\beta\) and \(\gamma\), which respectively result in non-vanishing \(\nu_{s,1}^o\) and \(\nu_{s,2}^o\). From these measurements, experimental values for all of the wave function parameters can be determined.

By changing \(\delta\) and \(\gamma\) in \(\nu_{s}\), the experimenters can distinguish the special cases \(\nu_{s} = 0\), which can occur in three ways: either \(\delta = n, \delta + \beta \gamma/2 = n, \) or \(\delta - \beta \gamma/2 = n\). Since the latter two cases apply simultaneously for opposite spins of the electrons or holes, it is actually rather easy to do the experiment. Since \(\gamma \ll 1\), the experimenters should choose \(\delta\) very near to an integer, and then the experiment will be very sensitive to \(\gamma\) variations. For the special region for which one fraction of the electrons or holes satisfies \(z_{n,1}^o < 0, \nu_{s,1}^o = 0, \) and \(\nu_{s,2}^o \neq 0\), and the other fraction satisfies \(z_{n,1}^o > 0, \nu_{s,1}^o \neq 0, \) and \(\nu_{s,2}^o = 0\), and the spontaneous azimuthal currents from the particles with different spins will have different phases. These different phases will be present simultaneously, and can be probed by varying \(\Delta \nu\). It is possible to measure the overall phase shift \(\zeta\) between the outgoing and incoming waves, which should be the same at each quantum jump in \(J_z^s(\rho_{\text{expr}})\).

5. Summary and conclusions

In conclusion, an experiment is proposed to measure the Type-II quantum spin Hall effect in 2D metallic films that is qualitatively different from the effect in topological insulators and quantum wells, as it makes use of a radial \(E\) and an azimuthal \(A\) but not of spin–orbit coupling. The apparatus is a thermally-managed 2D metallic Corbino disk surrounding a thermally-managed cylindrical solenoid, the applied current around which generates an azimuthal \(A\) in the disk, and by applying either a uniform radial current \(I\) or a potential difference \(\Delta \nu\) between the inner and outer radii \(\rho_i\) and \(\rho_o\), both a radial electric field \(E\) and fixed potentials \(v_i\) and \(v_o\) on those respective radii are generated. Then, by varying the flux in the solenoid and the potentials on the inner and outer radii, quantized azimuthal charge and spin currents are spontaneously generated. This quantized azimuthal current can be studied by adjusting the solenoid flux to be either an integral number of flux quanta or slightly different from an integral number of flux quanta combined with either positive or negative values of the applied radial current \(I\) that couples to the carrier spins by the Type-II QSH interaction. The relevant quantum Hamiltonian for this system is exactly soluble. Provided that the thermal management design sketched in figure 4, or a modified version of it, functions as desired, the experiment could in principle be performed at room temperature. An individual Corbino disk surrounding a solenoid could function as a qubit in a quantum computer potentially operating at room temperature.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Author contributions

A Z and R A K did the theoretical analysis and wrote the paper. T J B designed the experimental setup. Q G and T J H...
supervised the overall operation, suggested points to demonstrate, and contributed to the final editing of the manuscript.

Appendix A

We remark that $H_{2D}^{(1)}$ given by equation (24) can be treated in first-order perturbation theory, and that higher order corrections to the wave function contain the expectation values $\langle \Psi_{n,s}^{(1)}|H_{2D}^{(1)}|\Psi_{n',s'}^{(1)} \rangle$ for $n' \neq n$ and/or $s' \neq s$. However, we note that $\delta$ and $\beta\gamma$ in $\gamma_{n,s}$ are experimental parameters, so the only quantized objects are $n$, $n'$, $s$, and $s'$. But since $H_{2D}^{(1)}$ is independent of $\varphi$ and of the spin $s$, we must have $n' = n$ and $s' = s$, so all higher-order corrections to the wave function vanish, as do the second and all higher-order corrections to the energy. Hence, the exact energies $E_k^s = E_k^{(s)} + E_k^{(s')}$, where the $E_k^{(s')} = (\Psi_{n,s}^{(s')}|H_{2D}^{(s')}|\Psi_{n,s}^{(s')})$ are given by equation (35) and

$$E_k^{(s')} = \left( \Psi_{n,s}^{(s')}|H_{2D}^{(s')}|\Psi_{n,s}^{(s')} \right)$$

$$= \frac{1}{n_f} \int_0^{\pi} d\varphi \int_0^{\rho_0} \rho d\rho \Psi_{n,s}^{(s')} (\rho, \varphi) H_{2D}^{(s')} (\rho) \Psi_{n,s}^{(s')} (\rho, \varphi)$$

$$= q\Delta \ln (\rho_0/\rho) g_s^{(s')} (k^s)$$

$$g_s^{(s')} (k^s) = \frac{4\rho}{k_s^2 s^2} \int_0^{\rho_0} dx \left[ \ln (\rho_0/\rho) \right]^2$$

$$+ e^{-\pi s^2} \left[ C_s^2 + 2 \Re \left( B_s^2 C_s^2 e^{2\pi s^2} \right) \right]$$

$$E_{k_s'}^{(s')} = \left( \Psi_{n,s}^{(s')}|H_{2D}^{(s')}|\Psi_{n,s}^{(s')} \right)$$

where we have used the standard Dirac notation for the expectation value [52, 60], the wave functions $\Psi_{n,s}^{(s')} (\rho, \varphi)$ are given by equations (18) and (37) with the asymptotic forms for the Hankel functions, equation (39), $H_{2D}^{(s')} (\rho)$ is given by equation (24), $n_f^s$ is the number of particles of spin $s$ given by equation (70), $\eta_{s'} = \eta_{s,s'}$ is given by equation (41), and we have renormalized the integration variable to the dimensionless $x = \rho/\rho_0$. We note that $g_s^{(s')}$ could easily be integrated by parts to obtain exact solutions in terms of the sines and cosines integral functions. But, equation (52) is already in a closed form for the exact perturbation energy $E_{k_s'}^{(s')}$ for $k_s' \rho_0 \gg 1$.

We now consider a special case of this perturbation correction. For the general case, we have

$$E_{k_s'}^{(s')} = \max \left\{ \frac{\beta h^2 (k_s')^2}{2m} + q v + q \Delta \ln (\rho_0/\rho) g_s^{(s')} (k_s') \right\}$$

$$g_s^{(s')} (k_s') = \left( \Psi_{n,s}^{(s')}|H_{2D}^{(s')}|\Psi_{n,s}^{(s')} \right)$$

where $g_s^{(s')} (k_s')$ is given by equation (52) with $k_s' \rightarrow k_{\rho_0}$. Since the radial spin current density at fixed $\rho_\text{exp}$ given by equation (66) vanishes,

$$C_s^s = B_s^s e^{ip_s^s}$$

Note that this does not mean that the quantum-induced azimuthal charge and spin currents vanish, as they are given by equations (67) and (68). In this particular case, at low $T$ where one spin state dominates the other near to

![Figure 5: Plots of $g_{0,s}(k_{\rho_0}^s, \rho_0/\rho, \eta_{s'})$ with vanishingly small radial current. The solid and dashed curves are plots of $g_{0,s}(k_{\rho_0}^s, \rho_0/\rho, \eta_{s'})$ given by equation (55) for $k_{\rho_0}^s \rho_0$ from 1000 to 1000.05 and $\eta_{s'} = \pi/2$ (solid) and $3\pi/2$ (dashed), respectively.](image-url)

Using the identity $\cos^2 z = \frac{1}{2} [1 + \cos (2z)]$ in equation (55), integrating the denominator and the non-oscillatory integral in the numerator exactly, integrating the oscillatory term in the numerator by parts, and expanding the results in powers of $(2k_{\rho_0}^s \rho_0)^{-1}$, it is easy to show that

$$g_{0,s}(k_{\rho_0}^s, \rho_0/\rho) = \frac{C_1}{2k_{\rho_0}^s \rho_0} + O \left( \frac{1}{(2k_{\rho_0}^s \rho_0)^2} \right)$$

where

$$\lim_{s' \rightarrow \infty} g_{0,s}(k_{\rho_0}^s, \rho_0/\rho) = g_{\infty}(\rho_0/\rho)$$

$$= \frac{1}{1 - \rho_0/\rho_0 - \ln (\rho_0/\rho_0)}$$

and $C_1$ contains only terms that are oscillatory in $2k_{\rho_0}^s \rho_0$ and $2k_{\rho_0}^s \rho_0$.

In figure 5, plots of $g_{0,s}(k_{\rho_0}^s, \rho_0/\rho, \eta_{s'})$ for $\rho_0/\rho_0 = 10$ are shown for $\eta_{s'} = \pi/2$ (solid) and $3\pi/2$ (dashed) in the range of $k_{\rho_0}^s \rho_0$ from 1000 to 10000.05, respectively, in order to display the weak oscillations clearly. We note that for $\rho_0/\rho_0 = 10$, the values of $g_{0,s}$ in these plots are in good agreement with $g_{\infty}(10) = \frac{1}{10} - 1/\ln (10) \approx 0.686 816 6292$, as given by equation (58). Thus, to a good approximation, when the radial induced charge and spin currents vanish, we have
Appendix B

Starting with equation (25) in the text

\[ H_{2D_{s}}^{(t)}(\tilde{\Psi}_{0,s}) = H_{2D_{s}}^{(t)}(\tilde{\Psi}_{0,s}) = E_{0,s}^{(t)}(\rho,\phi). \]  

(60)

We first multiply \( H_{2D_{s}}^{(t)}(\tilde{\Psi}_{0,s}) = E_{0,s}^{(t)} \) by \( \rho \), and then use the standard ‘separation of variables’ technique to write

\[ \tilde{\Psi}_{0,s}(\rho,\phi) = R_{0,s}(\rho) \chi(\phi) \]  

[52, 59–61], and divide both sides of the equation by \( R_{0,s}(\rho) \). It is easily seen that

\[ F_{t,s}(\rho) = G_{t,s}(\rho) + C_{t,s}, \]

(61)

where

\[ F_{t,s}(\rho) = -\frac{\beta \hbar^2}{2m} \left( \frac{\rho^{2} E_{0,s}(\rho)}{\rho} \right) + \rho^{2} \left( qv_{i} - E_{0,s}(\rho) \right), \]

\[ G_{t,s}(\rho) = \frac{\beta \hbar^2}{2m} \left[ \chi'(\phi) - \left( \frac{2\delta - \gamma_1}{2\beta} \right) s \right], \]

\[ C_{t,s} = \frac{\beta \hbar^2}{2m} \left( -\delta^2 + \frac{\gamma_1 \delta}{2\beta} \right), \]

(62)

and where the primes and the double prime refer to the first and second derivatives with respect to the relevant spatial variable. We note that since \( s = \pm 1 \) arises from the diagonal Pauli matrix \( \sigma_z \), these equations are implicitly the diagonal elements of rank-2 matrices, so that terms not containing \( s \) are implicitly proportional to \( 1 \), the rank-2 identity matrix, so \( F_{t,s}(\rho) \), \( G_{t,s}(\rho) \), and \( C_{t,s} \), and the overall Hamiltonian are diagonal rank-2 matrices, or they could be rank two column vectors in the Nambu representation.

The fundamental assumption of the separation of variables technique is that \( G_{t,s}(\phi) = C_{t,s} \) and \( F_{t,s}(\rho) = C_{t,s} + C_{t,s} \) must both be constants, independent of either variable \( \rho \) or \( \phi \) [52, 59–61]. Since we must require \( \chi(\phi + 2\pi) = \chi(\phi) \) to be invariant under rotations by \( 2\pi \), we then may set

\[ \chi(\phi) = e^{i\alpha \phi}, \]

(63)

Then, using equation (63) in the expression for \( G_{t,s}(\phi) \), and combining that result with the expression for \( C_{t,s} \), we obtain

\[ C_{t,s} = -\frac{\beta \hbar^2}{2m} \left( n - \delta \right) \left( n - \delta + \frac{\gamma_1}{2\beta} \right). \]

(64)

For simplicity, we may set \( \beta = 1/\beta \), which is satisfied for both electrons and holes. Then, by multiplying the resulting expression for \( F_{t}(\rho) \) in equation (62) by \( \rho^2 \), it is then elementary to obtain the time-independent radial Schrödinger wave equation for \( R_{t,s}(\rho) \), given by equation (28).

Appendix C

From the asymptotic forms of the radial wave functions, equations (39) and (41), the radial and azimuthal charge and spin currents are independent of \( \phi \) and have the forms

\[ j_{r}(\rho) = 2g_{\rho} \left( \frac{2\hbar}{\pi m_{||}^{1/2} \rho} \sum_{s = \pm} \left( e^{\pi i s_{2}} |B_{s}|^2 - e^{-\pi i s_{2}} \right) \right), \]

(65)

\[ j_{\phi}(\rho) = 2g_{\phi} \left( \frac{2\hbar}{\pi m_{||}^{1/2} \rho} \sum_{s = \pm} \left( e^{\pi i s_{2}} |B_{s}|^2 - e^{-\pi i s_{2}} \right) \right), \]

(66)

where \( n \) is an integer, which can be 0 or of either sign, and let its normalization constant be included in \( R(\rho) \). We note that more complicated forms for \( \chi(\phi) \) satisfying the required rotational invariance by \( 2\pi \) such as \( \chi(\phi) = a_{1} e^{i\alpha \phi} + a_{2} e^{-i\alpha \phi} \) are not allowed. Although such a solution would satisfy \( \chi'^{2}(\phi)/\chi(\phi) = -n^2 \), a constant, \( \chi'(\phi)/\chi(\phi) \) for that form with both nonvanishing \( a_{1} \) and \( a_{2} \) would depend strongly upon \( \phi \), violating the fundamental assumption of the separation of variables technique that \( G_{t,s}(\phi) \) be a constant. Hence, there is only one quantum number \( n \) in the exponential expression for \( \chi(\phi) \), equation (63). We note that equation (66) implies that the \( s = + \) and \( s = - \) terms in equation (65) are identical.

When both spin states are occupied, the wave functions can be normalized to the total particle number \( \overline{N} \) for each spin state in the Corbino disk

\[ \overline{n}^2 = \sum_{s = \pm} \overline{n}_{s}^2, \]

(69)
\[
\bar{n}_c^T = \frac{4}{k_F \nu_s} \int_{\rho_0}^{\rho_i} d\rho \left( e^{i\nu_s \ell} |B|^2 \left( e^{i\nu_s \ell} |C|^2 + e^{-i\nu_s \ell} |C^*|^2 \right) + 2 \text{Re} \left[ B^* C^* e^{2i(k_F \rho - n_c \ell)} \right] \right).
\]

(70)

Note that the total charge in the disk is \( q \). The experimenter must be able to work in both regions of positive and negative \( z_{n,s} \). For \( z_{n,s} > 0 \), \( \nu_{s,1} = 0 \) and \( \nu_{s,1} = i \nu_{s,2} \), which is imaginary. The experimenter introduces a potential difference \( \Delta V_{\ell} \) in the Corbino disk, as sketched in figures 1(b) and (c). Then by measuring the voltages from the NM and from the FM electrodes to the electrode on the outer perimeter at \( \rho_0 \), separately for \( \nu_{s,2} = 0 \) and \( \nu_{s,2} \neq 0 \), the experimenter can determine values for \( \ell_{\rho} \) between \( \rho_0 \) and \( \rho_{\text{expt}} \) and between \( \rho_{\text{expt}} \) and \( \rho_0 \), where \( \rho_{\text{expt}} \) is the radial position(s) of the centrally located NM and FM electrodes. Since the sheet current is uniform, the overall current density is inversely proportional to \( \rho \). Since there should be no radial spin current, the values obtained from the radial NM and FM electrodes to \( \rho_0 \) electrode should be the same, as should the values obtained from the radial NM and FM electrodes to \( \rho_0 \) electrode.

These measurements should provide information of the radial function \( D_{s,1}^T(\rho) \) given by

\[
D_{s,1}^T(\rho) = \frac{1}{\rho} \left( e^{i\nu_{s,2}|B|^2} - e^{-i\nu_{s,2}|C|^2} \right)
- \frac{\alpha_A}{k_{F,s} \nu_s} \left( e^{i\nu_{s,2}/2 B_s^* e^{i(k_{F,s} \rho - n_c \ell)}} \right)
+ e^{-i\nu_{s,2}/2 C_s e^{-i(k_{F,s} \rho - n_c \ell)}}\right|^{2},
\]

(71)

which should be the same for \( s = + \) and \( s = - \).

Then, by measuring the voltage difference between the two neighboring NM electrodes and between the FM and its neighboring NM electrode, also separately for \( \nu_{s,2} = 0 \) and for \( \nu_{s,2} \neq 0 \) near different \( n \) values, the experimenter can infer values for \( \ell_{\rho} \) and \( \rho_{\text{expt}} \), and obtain measurements of the parameters

\[
D_{s,2}^T = \left( \frac{e^{i\nu_{s,2}|B|^2} - e^{-i\nu_{s,2}|C|^2}}{k_{F,s} \nu_s \rho_{\text{expt}}^2} \right)
+ e^{-i\nu_{s,2}/2 C_s e^{-i(k_{F,s} \rho_{\text{expt}} - n_c \ell)}}\right|^{2}.
\]

(72)

The combination of these measured \( D_{s,1} \) values would give rise to measured values of \( |B|^2, |C|^2, k_{F,s}, n_{c,s}, \nu_{s,2} \), and to the phase difference \( \nu_{s} \) given by

\[
e^{i\nu_{s}} = \frac{B^* C^* e^{2i(k_{F,s} \rho_{\text{expt}} - n_c \ell)}}{|B|^2 |C|^2}.
\]

(73)

measured at \( \rho_{\text{expt}} \). This second set of experiments at the fixed \( \rho_{\text{expt}} \) is the most important, and by changing the values of \( E_{\ell} \), one can quantify the quantized azimuthal spin current generated by the Type-II QSH Hamiltonian. This is sketched in figure 3 in the text.
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