Propagation of small fluctuations in electromagnetically induced transparency. Influence of Doppler width.

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The propagation of a pair of quantized fields inside a medium of three-level atoms in Λ configuration is analyzed. We calculate the stationary quadrature noise spectrum of the field after propagating through the medium in the case where the field has a general (but small) noise spectrum and the atoms are in a coherent population trapping state and show electromagnetically induced transparency (EIT). Although the mean values of the field remain unaltered as the field propagates, there is an oscillatory interchange of noise properties between the probe and pump fields. Also, as the field propagates, there is an oscillatory creation and annihilation of correlations between the probe and pump quadratures. We further study the field propagation of squeezed states when there is two-photon resonance, but the field has a detuning δ from atomic resonance. We show that the field propagation is very sensitive to δ. The propagation in this case can be explained as a combination of a frequency dependent rotation of maximum squeezed quadrature with an interchange of noise properties between pump and probe fields. It is also shown that the effect of the Doppler width in a squeezed state propagation is considerable.

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I. INTRODUCTION

Electromagnetically induced transparency (EIT) [1] emerges when coherence between electronic states of an atom suppresses the absorption of incident light. An usually opaque medium consisting of such atoms becomes transparent. Three electronic levels in a Λ-shaped configuration are a paradigm for showing EIT when the two lower states are coupled in two-photon resonance via the common excited state: In this case destructive interference between the two excitation paths suppresses absorption of photons. EIT has found many applications in coherent transfer of atoms [2], laser cooling [3, 4], and recently it was proposed to serve as a quantum memory device for applications in quantum information technology [5, 6].

The Λ-configuration is exemplified in Fig. 1, where two (meta-)stable states, |1⟩ and |2⟩, are both coupled to an excited state |e⟩ by dipole interaction with the electromagnetic fields of illuminating laser light. Spontaneous emission rates from the excited state with linewidth γ into |i⟩ are denoted by γ_i and the detunings of the lasers’ carrier frequency from the atomic transitions are labeled by δ_i. Detuned two-photon resonance is present if δ_1 = δ_2 = δ ≠ 0.

In the usual setup, the so-called pump laser drives one transition, e.g. |1⟩ ↔ |e⟩, while the probe laser, interacting with |2⟩ ↔ |e⟩, is tested for transparency [7]. The Rabi frequencies associated with pump and probe are denoted by Ω_1 and Ω_2. The linear response of the absorption of the probe by the medium is described by the imaginary part of the electric susceptibility χ which is proportional to the mean value of the imaginary part of the electric dipole. In the case of two-photon resonance, δ_1 = δ_2, the imaginary part of the electric susceptibility χ vanishes and the medium becomes transparent for the classical field. In Fig. 2 we plot Im χ as a function of the probe detuning δ_2 for the cases where the pump is in resonance (δ_1 = 0, solid line) and detuned (δ_1 = γ, dashed line). The maximum absorption frequency of the probe field increases monotonically with the Rabi frequencies.

There has been recent interest in understanding the behavior of the quantum and noise properties of a field which propagates in EIT media. The studies concentrate...
on two cases: (i) when the strength of the intensity of the pump field if much bigger than the probe field and (ii) when the intensity of the pump field is of the same order as the probe field.

For case (i), assuming a classical pump field and neglecting the small absorption inside the EIT window, calculation shows that an incoming quantum state is the same after propagation in the medium [5]. For the case where both fields are quantized, with the resonant pump field being a coherent state and the incoming probe state a squeezed vacuum, the absorption of squeezing from the different frequencies of a broadband squeezed vacuum follows the classical EIT window [8]. Different experiments that measure propagation of a probe squeezing vacuum use this fact in order to explain the propagated field [9,10].

When the absorption of the squeezed vacuum follows the classical EIT window, we can write the effect of the medium on the field in the following way [9]. Let \( a(\omega) \) be the annihilation operator for frequency \( \omega \). Then, after propagation:

\[
a''(\omega) = T(\omega)a(\omega) + \sqrt{1 - |T(\omega)|^2}v(\omega),
\]

where \( v(\omega) \) is the vacuum contribution and \( T(\omega) \) is the transmission function from the classical measurement.

In some cases it is necessary to include the atomic noise generated by the atoms due to decoherence effects in the base levels in order to explain the experimental results [11,12].

For case (ii) a theoretical study of general phase noise propagating in EIT was carried out in references [13,14], in the stationary regime. In these studies, the spectrum of the difference between the phase noise of both fields is treated. They found that as the field propagates the phase noise of both fields correlate and tend to be the same. Inside the validity domain of the approximation, the length of the formation of this correlation follows the classical EIT transparency curve. The more transparent the medium for the mean values, the longer the distance the field has to propagate in order to get correlated. Note that this result, although it does not give the noise spectrum for each field, tells us that the field indeed changes as it propagates and the scale of this change is given by the scale of the build-up of the correlations. Nevertheless, changing the field strength can make this scale very long. This result is similar to that in Ref. [8], in the sense that the distance scale where the field changes follows the classical EIT transparency curve.

There are also several experimental studies of the propagation of probe and pump field noise and its correlations [15,16]. They explain the results using the known effect of transformation of the incoming laser phase-noise to intensity noise. This transformation happens due to small absorption by the atoms because of decoherence effects. In the case of perfect EIT, this process would not exist, and the correlations would not emerge.

Recent calculations, where both fields, probe and pump, are treated quantum mechanically, show what happens in the case where the probe field is initially a squeezed state. It is shown, in the stationary regime and where there is no decoherence between the base levels, that although the medium is transparent for the mean values of the field, initial quantum fluctuations are not necessarily preserved after interaction. In the case of a cavity filled with atoms in \( \Lambda \) configuration driven by a squeezed pump field and a coherent pump field, the quantum properties of the output field can be very different from the quantum properties of the input field [17,18].

Similar calculations were done in the case of a probe squeezed state and a pump coherent state propagating in a medium showing EIT. When the probe and pump carrier frequencies drive the atoms in resonance with their respective transition, there is an oscillatory interchange of noise properties between the initially squeezed probe field and the initially coherent pump field as the state propagates inside the medium. The length of this interchange of squeezing can be much smaller than the length where the phase noise of both fields correlate [17,19]. This means that the state measured after interacting with the EIT medium can be totally different to the incoming state and that the kind of analysis expressed by Eq. (1) is not always valid. Note that this interchange of squeezing between probe and pump fields is not related to absorption or noise generated by the atoms.

In this paper, treating both fields quantum mechanically, we study (i) the spectrum of the probe and pump quadrature and the correlations between them when the pump and probe field have some initial general noise, (ii) the case where there is two-photon resonance, but the carrier frequencies of the fields are not in resonance with the corresponding atomic transitions \( (\delta_1 = \delta_2 = \delta \neq 0) \) [31], and (iii) the effect of the atoms’ Doppler width in the field propagation. We find that the interchange of noise properties between squeezed states and coherent states described in [19] extends to the case of general noise. Thus, for some spectrum frequency and distance, the probe noise becomes equal to the initial pump noise, and vice versa. Also, as the field propagates, there is an oscillatory creation and annihilation of correlations be-
between the pump and probe fields. Our results allow, for example, to study the propagation of a probe squeezed state with a pump laser with some phase noise. Although this is usually the case, it has not been treated theoretically before, except for the case where only difference in the phase noise was studied [14]. Our results allows us to predict the quadrature of each field and the correlation between them and not only the spectrum of the phase difference. Furthermore, we do not make an adiabatic approximation: Our solution is valid for all frequencies.

We find also new qualitative behavior that appears neither in the semiclassical nor in the resonance case. If the carrier frequencies of the pump and probe fields drive the atoms with the same detuning from their respective dipolar transition then, the mean value of the field stays unaltered. Basically this means that as long as there is two-photon resonance, the photon detuning from the atomic transition does not have a strong effect in the propagation of classical pulses in EIT. Nevertheless, our analysis shows that a detuned (from the atom transition) two-photon resonance of the carrier frequencies have a large impact on the propagation of the field state.

Our results show that the propagation of general states in EIT media is richer than was usually believed. In particular we show that the maximum squeezed quadrature of an initial broad squeezed vacuum rotates as it propagates in the medium, the velocity of rotation depends on the detuning from the atom transition and the spectrum frequency of interest. This means that after some propagation, the maximum squeezed quadrature of the propagated field is different for each frequency. This spectrum-frequency velocity dependent rotation of the quadratures implies that the output field is different to the incoming field for length scales where the mean values of the field are almost unchanged.

Note that the reduced velocity of pulse propagation in an EIT medium implies a rotation of the maximum squeezed quadrature. This quadrature rotation is due to the phase difference between the pulse, after propagating inside the medium, with respect to the local oscillator used to measure the quadratures. This phase difference depends on the pulse velocity. Nevertheless this phase difference is global and because of that the quadratures for each spectrum-frequency are rotated the same amount. The fact that we obtain that the velocity of rotation depends on the spectrum frequency makes this result qualitatively different from the quadrature rotation due to slow light propagation.

We also show that in the case where the probe and pump field have the same Rabi frequency, propagation is a combination of the rotation of the squeezed quadrature plus interchange of noise properties between the probe and pump field.

We also present the influence of detuned two-photon resonances on EIT media with some Doppler width. The movement of the atoms in a medium causes a shift of the frequency that the atoms “see” in their reference frame. Due to this Doppler effect, the atom experiences a velocity-dependent frequency shift relative to the carrier frequency of the fields measured in the laboratory frame. For a thermal vapor cell at room temperature, the Doppler width can be several times the decay rate. If the pump and probe fields propagate in the same direction, then the Doppler effect consists of detuning the frequency of both fields by the same amount. This implies that the propagation of the mean value of the field would be negligibly disturbed, because, independently of the Doppler width, all the atoms would be in two-photon resonance and the media shows EIT [7]. This observation is one reason why conventional (mean value measurement of the field) EIT experiments can be performed at room temperature. In contrast, a detuned two-photon resonance has a strong effect on the propagation of quantum states. This implies a significant impact of the atoms’ Doppler width on the quadrature spectra measured after propagation. There are many recent experiments where the propagation of vacuum squeezed states is studied [9, 20, 21, 22, 23]. Our results impose a restriction on the possibility of conducting EIT experiments with squeezed states in denser thermal clouds, even when we can neglect decoherence between the base levels.

The paper is organized as follows: in section III we state the equations for the field and for the atoms, in section IV we study the case where the carrier frequencies of the probe and pump field are in resonance and the incoming field has some general noise, in section V we study the case where the carrier frequencies of the probe and pump field are in a detuned two-photon resonance ($\delta_1 = \delta_2 = \delta \neq 0$), in section VI we study the effect of the Doppler width on the propagation of the field.

II. THEORETICAL DESCRIPTION OF THE PROPAGATING FIELDS

The dynamics of the composite system of atoms and pump and probe fields are described by Heisenberg’s equation of motion [14].

\[ \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial z} \right) a_j = -ig_j N \sigma_{je}, \quad \text{(2a)} \]

and

\[ \frac{\partial}{\partial t} \varpi_1 = \frac{1}{3} (-\gamma_1 - \gamma) (1 + \varpi_1 + \varpi_2) - 2ig_1 (\sigma_{e1} a_1 - a_1^\dagger \sigma_{e1}) \]
\[ \quad - ig_2 (\sigma_{e2} a_2 - a_2^\dagger \sigma_{e2}) + f_{\varpi_1}, \]
\[ \frac{\partial}{\partial t} \varpi_2 = \frac{1}{3} (-\gamma_2 - \gamma) (1 + \varpi_1 + \varpi_2) - ig_1 (\sigma_{e1} a_1 - a_1^\dagger \sigma_{e1}) \]
\[ \quad - 2ig_2 (\sigma_{e2} a_2 - a_2^\dagger \sigma_{e2}) + f_{\varpi_2}, \]
\[ \frac{\partial}{\partial t} \sigma_{e1} = \left( -\frac{\gamma}{2} + i\delta_1 \right) \sigma_{e1} + ig_1 \varpi_1 a_1 - ig_2 \sigma_{e2} a_2 + f_{\sigma_{e1}}, \]
\[ \frac{\partial}{\partial t} \sigma_{e2} = \left( -\frac{\gamma}{2} + i\delta_2 \right) \sigma_{e2} + ig_2 \varpi_2 a_2 - ig_1 \sigma_{e1} a_1 + f_{\sigma_{e2}}, \]
\[ \frac{\partial}{\partial t} \sigma_{21} = \left( -\gamma_{12} - i(\delta_1 - \delta_2) \right) \sigma_{21} - ig_1 a_1^\dagger \sigma_{e2} + ig_2 \sigma_{e1} a_2 + f_{21}. \quad \text{(2b)} \]
where \( j = 1, 2 \). This description relies on a multimode decomposition of the electromagnetic fields \( \tilde{E}_j = \tilde{E}_j \alpha_j(z,t) \exp[\text{i} k_j z - \text{i} \omega_j t] + \text{h.c.} \) around the carrier frequencies \( \omega_j = \epsilon_k j \), where \( |\tilde{E}_j| \) is the corresponding vacuum electric field. The detuning with respect to the atomic transition \( \omega_{je} \) is given by \( \delta_j = \omega_j - \omega_{je} \). In this notation, the field envelope operators \( \alpha_j(z,t) \) are slowly varying in space and time, allowing us to write Maxwell’s equations in the form \( [24] \), where the atomic polarization \( \sigma \) is given by

\[
\sigma_{\mu\nu}(z) = \lim_{\Delta z \to 0} \frac{L}{N \Delta z} \sum_{j=1}^{N} \sigma_{\mu\nu}^{(j)}(z) \quad \text{and} \quad \sigma_{\nu
u} = \sigma_{\mu\epsilon} - \sigma_{\epsilon
u} \quad \text{are written in the continuum limit, where} \quad \sigma_{\mu
u}^{(j)} = |\mu\rangle \langle \nu| \quad \text{is the atomic operator of atom} \quad j \quad \text{at position} \quad z_j. \quad \text{Here} \quad N \quad \text{is the number of atoms,} \quad L \quad \text{the length of the medium and} \quad \Delta z \quad \text{a space region around} \quad z. \quad \text{This approximation is justified if the inter-atomic distance is smaller than the length scale introduced by the wavelength of the carrier fields.}
\]

The coupling between atoms and field relies on a dipole interaction with coupling constant \( g_j = \vec{\gamma} \cdot \vec{E}_j / \hbar \), where \( \vec{\gamma} \) is the atomic dipole moment. In order to arrive at the form of equations \( [24] \) and \( [25] \), the rotating wave approximation was performed and all the operators are in a reference frame rotating with the corresponding carrier frequencies. The laboratory frame notation can be obtained by the transformations \( \sigma'_{\mu\nu} = \sigma_{\mu\nu} \exp[-\text{i} (\delta_j + \omega_{je}) t] \) and \( \sigma'_1 = \sigma_{12} \exp[-\text{i} (\delta_1 - \delta_2)] \).

To account for the noise introduced by the coupling of the atomic system to the free radiation field, delta-correlated, collective Langevin operators, \( f_j \), were introduced. They have vanishing mean values and correlation functions of the form \( \langle f_z(z,t) f_y(z',t') \rangle = \frac{\hbar}{2} D_{xy} \delta(t-t') \delta(z-z') \). The diffusion coefficients \( D_{xy} \) can be obtained from the generalized Einstein equations \( [24] \). Their explicit form can be found, for example, in \( [25] \).

In order to obtain the spectrum of the quadrature fluctuations, the system of equations is solved in the small-noise approximation using the standard technique of transforming them into a system of c-number stochastic differential equations \( [20] \). The initial conditions for our analysis are: The probe and pump field has \( \langle \alpha_j \rangle = \alpha_j \), thus driving the atomic transitions with Rabi frequency \( \Omega_j = |g_j \alpha_j| \).

Denoting the fluctuation of the \( \theta \) quadrature of field \( j = 1, 2 \) as

\[
\delta Y_j^\theta(z,t) = \delta a_j(z,t) \exp(-\text{i} \theta) + \delta a_j^\dagger(z,t) \exp(\text{i} \theta), \quad (3)
\]

where \( \delta \theta = \theta - \langle \theta \rangle \), the \( \theta \)-quadrature noise spectrum, in the stationary regime, is given by

\[
\mathcal{S}_\theta(z,\omega) = \int_{-\infty}^{\infty} e^{-\text{i} \omega t} \langle \delta Y_j^\theta(z,t) \delta Y_j^\theta(z,0) \rangle dt, \quad (4)
\]

and the correlations noise spectrum between, the probe \( \theta_1 \)-quadrature and the pump \( \theta_2 \)-quadrature, is given by

\[
\mathcal{S}_\omega(z,\omega) = \int_{-\infty}^{\infty} e^{-\text{i} \omega t} \langle \delta Y_1^\theta(z,t) \delta Y_2^\theta(z,0) \rangle dt, \quad (5)
\]

where \( \omega = 0 \) corresponds to the carrier frequency of the field in accordance with the co-rotating reference frame we use. The initial conditions for the fluctuations can be written as

\[
\mathcal{S}_\theta(z,0,\omega) = 1 + 2 g_1(\omega) \cos(2 \theta) + 2 f_1(\omega), \quad (6)
\]

where \( f_j(\omega) \) and \( g_j(\omega), \quad j = 1, 2 \) are real.

### III. CARRIER FREQUENCIES IN RESONANCE WITH ATOMIC TRANSITIONS (\( \delta_1 = \delta_2 = 0 \))

We solve equations \( [19] \) with \( \delta_1 = \delta_2 = 0 \) and noise spectrum given by \( [6] \), following the treatment used in \( [19] \). We list in appendix A the analytical expressions for the quadrature noise spectra of the pump \( (j = 1) \) and probe \( (j = 2) \) field, and the correlations between them.

The behavior of the spectrum is characterized by two quantities: as the field propagates \( Q^{(i)} \) characterizes an exponential decay of noise properties and \( Q^{(r)} \) characterizes the coherent propagation of the field which corresponds to an oscillatory evolution of noise properties.

The behavior of \( Q^{(i)} \) as a function of the parameters resembles the behavior of the classical transmission spectrum \( (T(\omega) \text{ in Eq.} \ (1)) \). The absorption of the initial properties of the field is then characterized by this quantity and allow us to define a length scale, \( z_{abs} \), where these absorptions are important,

\[
z_{abs} \approx 1/(|Q^{(i)}|). \quad (7)
\]

#### A. Propagation of the field quadratures

When \( Q^{(i)} z \ll 1 \), that is, when we consider positions \( z \) where the exponential absorption of the fluctuations can be neglected, we can easily understand the effect of the oscillatory behavior characterized by \( Q^{(r)} \). Let us assume that \( \alpha_1 = \alpha_2 \), then, as the field propagates, there is a complete oscillatory interchange of noise properties:

\[
S_2(Q^{(r)}) = S_1(Q^{(r)}), \quad (8)
\]

\[
S_j(Q^{(r)}) = S_2(Q^{(r)}), \quad (9)
\]

where \( k \) is an integer and \( j = 1, 2 \).

In Fig. \( 3a \) we have plotted the fluctuation spectrum for an initially squeezed probe (solid) and an initially
FIG. 3: Quantum properties of probe and pump fields propagating through an EIT medium. (a) The fluctuation spectrum of the probe (solid), initially at \( z = 0 \) in a squeezed state with squeezing parameter \( \xi = 1 \) and of the pump (dashed), initially in a coherent state, as a function of position for \( \theta = 0 \) and no absorption \( (\gamma_i = 0) \). (b) Same as (a) but with absorption. The amplitude of the oscillations decays \( 1/e \) after \( z_{\text{abs}}/|z_{\text{osc}}| = 2.02 \) periods.

They studied the particular case of phase difference fluctuations. We first obtain a generalization (for the no-decoherence case) of the known result of the spectrum of phase difference fluctuations given in [14]. We present here the result for no decoherence in the base level, but valid for all frequencies (no adiabatic approximation). Let \( \phi_j \) represent the phase of field \( j \). We use the fact that for small noise the phase quadrature noise is proportional to the phase noise [26]. From Eqs. (A1), (A2) and (A3), we obtain \( (\alpha_1 = \alpha_2 = \alpha) \):

\[
S_\phi(\omega, z) = (\delta\phi_1 - \delta\phi_2)^2 \\
\approx \frac{1}{\alpha^2} \left( S_1^{\theta = \pi/2}(\omega, z) + S_2^{\theta = \pi/2}(\omega, z) - 2S_1^{\theta_1 = \theta_2 = \pi/2}(\omega, z) \right) \\
= \frac{1}{\alpha^2} e^{-Q^{(i)}(\omega)\zeta} S_\phi(\omega, z = 0). \tag{10}
\]

Observe that as the field propagates, the difference between the phase noise of the fields disappears, which implies that correlations between the phase noise of both fields builds up. For the case of no decoherence, Eq. (10) is an extension of Eq.(12) in Ref. [13], in the sense that it is valid for all \( \omega \). It is easy to see that the length of the fading of the phase difference between the fields goes like \( z_{\text{abs}} \), Eq. (7). For the phase noise difference, we do not observe any oscillatory behavior, as in the case of the propagation of the field quadratures.

Nevertheless, the oscillatory transfer of initial noise properties between the pump and probe field generates also an oscillatory creation and annihilation of correlations (defined in Eq. (5)). When \( z \ll z_{\text{abs}} \), from Eq. (A3)
we obtain

\[ S_c(z, \omega) = \sin \left( Q^{(i)} z \right) \left[ \sin (\theta_1 - \theta_2) (f_2(\omega) - f_1(\omega)) + \sin (\theta_1 + \theta_2) (g_1(\omega) - g_2(\omega)) \right]. \tag{11} \]

Equation (11) shows that if the noise properties of the pump and probe field are different, then correlations between the fields oscillates for length scales small with respect to the absorption scale. Due to \( Q^{(i)} \) frequency-dependence, these correlations are different for each spectrum frequency, showing a rich coherent interaction between the probe and pump fields. Note that for the spectra of phase noise difference, nothing happens for \( z < z_{abs} \), implying that the correlation oscillations shown in Eq. (11) are qualitatively different from the correlations implied in Ref. [19] and Eq. (10).

For \( z > z_{abs} \), the correlations reach a constant value given by:

\[ S_c(z, \omega) = \frac{1}{2} \left( \cos (\theta_1 - \theta_2) (f_1(\omega) + f_2(\omega)) + \cos (\theta_1 + \theta_2) (g_1(\omega) + g_2(\omega)) \right). \tag{12} \]

Note that, except for initial coherent states (\( f_i = g_i = 0 \)), there is always a formation of correlations.

We can interpret the generation of the \( z > z_{abs} \) constant correlations as a signature of initial noise in the field, and the oscillatory correlations between the fields as a signature that the initial noise spectrum is different for each field.

The interchange of noise properties and the oscillatory creation and annihilation of correlations can be explained by the fact that the two fields are coupled via the atomic medium. Due to EIT this coupling is such that the field mean values are unaltered and only the fluctuations get coupled. The atoms base level coherence plays a key role in this behavior, the oscillatory interchange of noise is complete and the generated correlations reach a maximum when the base level coherence of the atoms, \( \sigma_{12} \), is maximal. When \( \langle \sigma_{12} \rangle \) goes to zero there is neither interchange of noise nor generation of correlations between the fields.

When \( \alpha_1 = \alpha_2 \) the oscillatory interchange of noise properties seems to resemble the noise interchanges that occurs when two oscillators, with the same frequency, are coupled (see Ref. [27] for an example with squeezed states). Differently to the case we are studying, in the two coupled oscillators the transfer of noise does not depend on the mean value of the fields: it is always complete. Also in this case the field mean value is altered during the interaction.

IV. CARRIER FREQUENCIES IN A DETUNED TWO-PHOTON RESONANCE (\( \delta_1 = \delta_2 \neq 0 \))

In this section we study the case where the carrier frequencies of the pump and probe field are in a detuned two-photon resonance \( \delta = \delta_1 = \delta_2 \neq 0 \). The pump field is initially in a coherent state and the probe field has initially some general noise.

We solve equations (2) with \( \delta_1 = \delta_2 = \delta \) and \( f_1(\omega) = g_1(\omega) = 0 \) in the initial conditions Eq. (6) (i.e., a coherent pump field) following the treatment used in [19]. In appendix A2 we list the analytical expressions for the quadrature noise spectra of the pump \((j = 1)\) and probe \((j = 2)\) field, and the correlations between them.

A. Asymptotic behavior

As \( Q^{(i)} \) is always negative, for large propagation lengths, \( z \to \infty \), we find

\[ S_1(z, \omega) \approx 1 + (S_2(z = 0, \omega) - 1) \frac{\alpha_1^2 \alpha_2^2}{(\alpha_1^2 + \alpha_2^2)^2}, \tag{13} \]
\[ S_2(z, \omega) \approx 1 + (S_2(z = 0, \omega) - 1) \frac{\alpha_1^2}{(\alpha_1^2 + \alpha_2^2)^2}, \tag{14} \]

which are reminiscent to similar correlations known from cavity EIT [28] and the effect of pulse-matching [1, 13]. The asymptotic behavior does not depend on \( \delta \). The distance where this asymptotic behavior is dominant is governed by the exponentials in Eqs. (A5) and (A6), and is of the order of

\[ z_{abs} \approx 1/(\text{Max}(|Q^{(i)}_+|, |Q^{(i)}_-|)). \tag{15} \]

When \( \alpha_1 \neq 0 \), the asymptotic quadrature fluctuations of the pump field shows a fraction of the noise properties of the probe field. Also the noise of the probe field diminishes if \( S_2(z = 0, \omega) > 1 \) and increases otherwise.

For equal field intensities \( \alpha_1 = \alpha_2 \), both fields have asymptotically the same fluctuations.

B. The probe field is a vacuum squeezed state

For a broadband vacuum squeezed state \((\alpha_2 = 0)\) the initial conditions (6) reduce to

\[ S_1(z = 0, \omega) = 1, \tag{16} \]
\[ S_2(z = 0, \omega) = e^{-2\xi} \cos^2 \theta + e^{2\xi} \sin^2 \theta, \tag{17} \]

where \( \xi \) is a real number characterizing the amount of squeezing.

From Eq. (A5) can be concluded that the pump field quadratures remains as in the initial condition, \( S_1(\theta, \omega) = 1 \), as the field propagates. We also know that in the case where both fields are in resonance the medium is transparent for the squeezed vacuum, except for the expected absorption for frequencies detuned from resonance (see section III and Ref. [29]). Nevertheless, when the fields are detuned but in two-photon resonance,
although there is no change for the mean values, the vacuum squeezed state is altered. We can see this by substituting $\alpha_2 = 0$ in Eq. (A6):

$$S_2(z, \omega) = 1 + \left(e^{2Q_+^{(i)}z} + e^{2Q_-^{(i)}z}\right) \sinh^2(\xi) - e^{Q_+^{(i)}z + Q_-^{(i)}z} \cos(Q_+^{(r)}z - Q_-^{(r)}z + 2\theta) \sinh(2\xi).$$

(18)

The propagation of a squeezed vacuum in an EIT medium for the case of detuned two-photon resonance is characterized by a frequency dependent quadrature rotation and absorption, characterized by $Q_+^{(r)}$ and $Q_-^{(i)}$, respectively. The spectrum absorption of the squeezing is thus different from the spectrum absorption of the mean value.

1. Quadrature rotation

To better understand the behavior of the coherent part (i.e., neglecting the absorption part) of the propagation, let us suppose that $z \ll z_{\text{abs}}$. In that case we can replace the exponentials in Eq. (A6) with 1. Comparing the result with the incoming field, Eq. (17), we can write:

$$S_2(z, \omega, \theta = 0) = S_2(z = 0, \omega, \theta = (Q_-^{(r)} - Q_+^{(r)})z/2).$$

(19)

This last equation shows that the propagation of a squeezed probe vacuum in EIT, is equivalent to the rotation of the angle of maximum squeezing. The velocity of this rotation is given by $Q_+^{(r)}$ and is a function of the detuned two-photon resonance parameter $\delta$ and spectrum frequency $\omega$. Due to the dependence of the velocity of rotation on $\omega$, after propagation the maximum squeezed probe quadrature would be different for each spectrum frequency. Note that this behavior can not be modeled using Eq. (1). Since the $\theta = 0$ quadrature fluctuation of the propagated field is different for each frequency, we can say that the media would not be transparent for the incoming broadband $\theta = 0$ vacuum squeezed state.

Note that this result is qualitatively different from the known case [30], where as a result of the pump detuning all the quadratures are rotated the same amount, independently of $\omega$. The difference between our results and the results in [30] are due to the approximations used. In particular we do not make any adiabatic approximation. Note also that the frequency-dependence on the velocity of quadrature rotation cannot be explained only by a global phase rotation due to slow light propagation.

2. Spectrum of squeezing absorption

The effect of the atom decay rate is, as the field propagates, to dampen the squeezing of the quadrature with maximum squeezing of the probe field. For the resonance case, the absorption of the maximum squeezed quadrature follows the classical absorption spectrum. This is not the case when there is two photon resonance but each field is detuned from the atomic transition. In Eq. (18) it can be observed that the absorption is proportional to $|Q_+^{(i)}|$ and the sum $|Q_+^{(r)} + Q_-^{(i)}|$. The two maxima $\omega_\pm$ of each of the $Q_\pm$ are located at

$$\omega_\pm = \delta + \frac{1}{2}(\pm\delta \pm \sqrt{\delta^2 + 4\Omega^2}).$$

(20)

Figure 4 shows the absorption spectrum. The absorption spectrum of the noise differs qualitatively from the absorption spectrum of the mean values Fig. 2 only for $\omega = \delta$ there is no absorption in either case. We denote the region around $\omega = \delta$ where absorption is negligible as $\Delta\omega_\delta$. The length of this spectral region, $\Delta\omega_\delta$, decreases as $\delta$ increases. When $\delta$ increases, the difference between $\omega_+^+ - \omega_-^- = \omega_+^+ - \omega_-^-$ increases. When this difference is much bigger than $\gamma$, a spectral region appears between $\omega_+^+$ and $\omega_+^-$, and between $\omega_-^+$ and $\omega_-^-$ where absorption is negligible. This is shown as the dotted line in Fig. 4(b).

Then, when $\delta \gg \gamma, \Omega$, instead of the three regions of transparency that appears in the spectrum absorption for the mean values (see Fig. 2), there are five regions of transparency: (i) around $\omega = \delta$; (ii) between $\omega_+^+$ and $\omega_-^-$; (iii) between $\omega_+^+$ and $\omega_-^-$; (iv) when $\omega \gg \omega_+^+$ and (v) $\omega \ll \omega_-^-$. The width of the transparency window for case (i) is of order $\Delta\omega_\delta \approx 2\Omega^2/\delta$, for case (ii) and (iii) is of order $\delta$.

Note that in Ref. [8 section II-C], it is concluded that maximum absorption takes place at $\omega = \delta$, and that there is an absorption of squeezed states around $\omega = \delta$, and transparency elsewhere. This contradiction between the results presented here and those from the reference is due to the fact that they make the $\delta \gg \gamma$ approximation before solving the equation. As a result of that approximation, they do not obtain the transparency region around $\omega = \delta$.

An example of the interplay between the rotation of the angle of maximum squeezing and the absorption of the initial squeezing due to the linewidth of the excited level, given by Eq. (18), is plotted in Fig. 5.

C. The probe field is a squeezed state with $\alpha_1 = \alpha_2$

We discuss now the propagation of a squeezed probe field in the case where $\alpha_1 = \alpha_2$. We separate this discussion into three parts. From the form of the $Q_\pm^{(r,i)}$, Eq. (A5), we can identify three regions of qualitatively different behavior, denoted by the Roman numbers I, II, and III in Fig. 6. We will therefore divide the study of the propagation of squeezed fields in EIT media into these three parts for the case $\delta > 0$, the other case being symmetric.
We analyze here the propagation of the field in the near-resonant case, $|\omega \delta| \ll \Omega^2$. A qualitatively different behavior from the case $\delta = 0$ appears.

### 1. Near-resonant case (Region I)

![Absorption spectrum for squeezed states](image)

**FIG. 4:** Absorption spectrum for squeezed states for the case where there is a detuned two-photon resonance. a) $|Q^{(i)}_+|$ (solid line), $|Q^{(r)}_+|$ (dotted line). b) $|Q^{(i)}_+ + Q^{(r)}_-|$ for $\delta = 1\gamma$ (solid line), $\delta = 3\gamma$ (dotted line). Note how five regions of transparency start to appear in figure b), see text for details. Parameter: $\Omega = 0.6\gamma$.

Figs. [4] and [8(a)] show a typical example of how the $\theta = 0$ quadrature of the field propagates, according to Eqs. (A5), (A6), for the case of negligible absorption. In Fig. [8] we show the effect of the decay rate $\gamma$. The black area in the plots represents the fast oscillations. There are three scales that can be seen in Figs. [7] and [8] (i) a quick scale $z_{osc}$, where we have oscillating transfer of squeezing between pump and probe fields for some regions, (ii) an intermediate oscillating scale, $z_{int}$, given by the oscillation of the envelope of the fast oscillations; and (iii) the absorption scale $z_{abs}$ which damps the oscillation behavior, given by Eq. (13). On the intermediate scale, the probe $\theta = 0$ quadrature goes from a squeezed value to $Q^{(r)}_+$ and back to the squeezed value. It is not difficult to show that for $e^{2\xi} \gg 1$ the maximum of the $\theta = 0$ probe quadrature is $S_2(\theta = 0, z_{max}) \approx e^{2\xi}/4$ and the minimum is $e^{-2\xi}$.

We proceed now to obtain analytical expressions for the scales. When $\Omega^2 > |\omega(\omega \pm \delta)|$ and $Q^{(r)}_-, Q^{(r)}_+ > 0$, we can extract fast oscillating terms (given by $Q^{(r)}_-, Q^{(r)}_+$)
and slow oscillating terms (given by \(Q^{(r)} - Q^{(r)}\)) from Eqs. (A5) and (A6). \(z_{\text{osc}}\) is given by the fast oscillating terms when \(\omega_0 \ll \Omega^2\),

\[
z_{\text{osc}} \approx \pi/Q^{(r)}_\pi \approx \pi/Q^{(r)}_+.
\]

The intermediate oscillating scale, \(z_{\text{int}}\), is given by the slow oscillating terms:

\[
z_{\text{int}} = 2\pi/(Q^{(r)}_\pi - Q^{(r)}_+).
\]

To gain more insight into the propagation of an initially \(\theta = 0\) quadrature squeezed state in EIT media, it is necessary to study all the quadratures. Excess noise in the \(\theta = 0\) quadrature after some propagation does not necessarily imply that the state is no longer a squeezed state. The squeezed quadrature could have been rotated (see section IV B). It is worth then, instead of studying the propagation of a specific quadrature, to study the propagation of the quadrature where the minimal and maximal fluctuations are achieved. To calculate these quadratures we minimize Eqs. (A5) and (A6) for \(\theta\). We will assume that \(z \ll z_{\text{abs}}\). Simple expressions can be obtained for the angle \(\theta_{\text{min}}\) of the probe quadrature where the minimum value is reached and the angle \(\theta_{\text{max}}\) where the maximum value is reached:

\[
\theta_{\text{min},n} = n\pi + \frac{(Q^{(r)}_+ - Q^{(r)}_\pi)z}{4},
\]

\[
\theta_{\text{max},n} = \theta_{\text{min},n} + \pi/2,
\]

where \(n\) is an integer. Similarly to the case where the probe field is a vacuum squeezed state (see section IV B), the quadrature of minimal fluctuations rotates with propagation.

In Fig. 9 we compare the \(\theta_{\text{min}}\) quadrature with the \(\theta_{\text{max}}\) quadrature. We observe that for some values of \(Q^{(r)}_+ z\), the minimum quadrature and the maximum quadrature of the probe field reach the same value. This means that for these particular values of \(Q^{(r)}_+ z\), all the quadratures have the same value. These values of \(Q^{(r)}_+ z\) can be calculated analytically by minimizing Eqs. (A5) and (A6) with respect to \(z\) such that they are independent of \(\theta\). We obtain

\[
Q^{(r)}_+ z_{2,+}(m) = 4\pi m + \pi,
\]

\[
Q^{(r)}_- z_{2,-}(m) = 4\pi m + \pi,
\]

where \(m\) is an integer.

The previous results tell us that for an initially squeezed probe field, the propagation has the effect of distributing the noise between the quadratures until, for distances defined by Eq. (25), the noise is equally distributed between all the quadratures. Then, as the field continues to propagate, the inverse process starts until a particular mode quadrature is squeezed. This process repeats until the initial condition of the mode (a \(\theta = 0\) quadrature squeezed state for the probe field, a coherent state for the pump field) is reached. We understand this dynamics as a combination of the rotation of the quadrature of minimal fluctuations due to detuning (see Eq. (23)) and the interchange of noise between the probe and pump fields characterized by \(z_{\text{osc}}\).

A similar process occurs for the pump field but with the distances \(z_{1,\pm}(m)\) where all the quadratures have the same value given by

\[
Q^{(r)}_+ z_{1,+}(m) = 4\pi m,
\]

\[
Q^{(r)}_- z_{1,-}(m) = 4\pi m.
\]

The value of all the quadratures for these distances oscillates and is given by:

\[
S_1(z_{1,\pm}(m)) = S_2(z_{2,\pm}(m)) = 1 + \frac{1}{2}(\cos(Q^{(r)}_+ z_{2,\pm}(m)) + 1) \sinh^2(\xi).
\]

2. Intermediate detuned two-photon resonance (region II)

In region II (see Fig. 6(a)), it is not difficult to see that exponential absorption, in the vicinity of the maximum of \(Q^{(i)}_+(\delta)\), makes all the properties of propagation described in the previous section fade away for distances \(z < z_{\text{osc}}, z_{\text{int}}\).

3. Large detuned two-photon resonance (region III)

We now study the case of a large detuned two-photon resonance \(|\delta\omega| \gg \Omega^2, \omega^2\) (region III). For that case,

\[
Q^{(r)}_+ \approx -Q^{(r)}_- \approx -\frac{C\delta\omega^2}{4} + \frac{\delta^2\omega^2}{2},
\]

\[
Q^{(i)}_+ \approx Q^{(i)}_- \approx -\frac{C\gamma\omega^2}{2(\frac{3\omega^2}{4} + \delta^2\omega^2)}.
\]

If \(\delta \gg \gamma\) then \(|Q^{(r)}_+| \gg |Q^{(i)}_+|\) and a simple expression can be found from Eqs. (A6) and (A5) for \(z < z_{\text{abs}}\).

\[
S_2(z, \omega) = S_1(z + \pi/Q^{(r)}_+, \omega) \approx e^{-2\xi} \cos\left(\frac{Q^{(r)}_+ z}{2}\right)
\]

\[
+ \sin^2\left(\frac{Q^{(r)}_+ z}{2}\right) + \frac{1}{4} e^{2\xi} \sin^2(Q^{(r)}_+ z).
\]

From the former equation, it can be concluded that interchange of quantum fluctuation between pump and probe happens when \(Q^{(r)}_+ z = n\pi, n\) being an integer. Differently to the near-resonant case, described in section IV C 1, the distances where both fields have the same excess noise is reached before the interchange of noise between the fields occurs. In Fig. 10 an example of propagation for large detuned two-photon resonance is shown.
FIG. 7: Quantum properties of probe and pump fields propagating through an EIT medium with a detuned two-photon resonance. It is shown how the fluctuation spectrum of the probe field, initially (at $z = 0$) in a squeezed state with squeezing parameter $\xi = 1$ propagates along the $z$-direction for different decay rates. The black area of the plots represents the fast oscillations. a) No decay rate ($Q^{(i)}_+ = 0$), b)-c) $|Q^{(i)}_+ / Q^{(i)}_+| = 449\pi, 224\pi, 45\pi$ ($Q^{(i)}_+$ is proportional to the decay rate). The $z_{\text{min}}$ scale from minimum $e^{-2\xi}$ to $e^{2\xi}/4$ and back can be clearly seen in a). From b) to c) the effect of the absorption scale can be clearly seen. Parameters: $Q^{(i)}_+/Q^{(i)}_- = 1.014$, $Q^{(i)}_+/Q^{(i)}_- = 1.029$, $\alpha_1 = \alpha_2$.

V. MOVING ATOMS AND TWO-PHOTON RESONANCE: DOPPLER EFFECT

In this section we study numerically the effect of the atoms' Doppler width in the spectrum of quadrature fluctuation.

As we studied in the previous section, if the pump field is not in resonance with the dipole transition, then the propagation of the state is hugely altered and the final propagated state can be very different from the initial state. This means that for moving atoms, due to the Doppler width, groups of atoms with different directions and velocities would see different detuned two photon resonance, and each group would have a different influence on the propagation of the initial state. Using the superposition principle of the field and the fact that we are working in a linear approximation, the propagated final field would be a sum of the propagated field for each...
group of atoms \( a(t) = \int_0^\infty \rho(\delta) a_\delta(t) d\delta \), where \( a_\delta(t) \) is the solution of equation (24) when the atoms' velocity is such that the detuned two-photon resonance is \( \delta \), and \( \rho(\delta) \) is the density of atoms with velocities and directions such that the detuned two-photon resonance is \( \delta \). The final quadrature spectrum is then given by

\[
S_2(\theta, \omega)_{\Delta \delta} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\delta_1) \rho(\delta_2) S_2(\theta, \omega, \delta_1, \delta_2) d\delta_1 d\delta_2 ,
\]

where

\[
S(\theta, \omega, \delta_1, \delta_2) = \int_{-\infty}^{\infty} e^{-i\omega t} \langle \delta Y_{\delta_1}(t) \delta Y_{\delta_2}^\dagger(0) \rangle dt ,
\]

and \( \delta Y_{\delta}(t) = \delta a_\delta(t) \exp(-i\theta) + \delta a_\delta^\dagger(t) \exp(i\theta) \). We will assume that \( \rho(\delta) \) is a Gaussian with variance \( \Delta \delta \).

To obtain the effect of the Doppler width in the quadrature spectrum we calculate, from Eqs. (2), \( S(\theta, \omega, \delta_1, \delta_2) \) following a method similar to the one used in [19]. Then we numerically integrate Eq. (28).

The frequency \( \omega \) obtained from Fourier transforming Eqs. (2) is the spectrum measured from the field carrier frequencies. For fixed \( \omega \), changing \( \delta \) also changes the spectrum frequency with respect to the dipole transition of the atom. In order to correctly obtain the Doppler effect in the quadrature, before integrating Eq. (28) we substituted \( \omega \) by \( \omega - \delta \). After this substitution, and in this section, \( \omega \) represents the spectrum frequency with respect to the atomic transition.

A. The probe field is squeezed vacuum

We now suppose that the probe field is in a broadband squeezed vacuum. This means that \( \alpha_2 = 0 \). In this case, for atoms at rest, and a coherent pump field in resonance with its corresponding transition, the medium is transparent for the squeezed vacuum (see section [II] and Ref. [29]). In Fig. 11 we show the numerically calculated probe \( \theta = 0 \) quadrature spectrum for an initially squeezed vacuum propagating in a medium composed of atoms with Doppler distribution with width \( \Delta \delta \). For \( \Delta \delta = 0.01 \gamma \), the detuned two-photon resonance due to the Doppler effect is small and does not affect the propagation. As \( \Delta \delta \) increase, the effect of the different quadrature rotation on the incoming field, due to atoms with different detuned two-photon resonance, begins to be important. When \( \Delta \delta \gtrsim 25 \gamma \) we even have excess noise in the \( \theta = 0 \) quadratures for some distances. The excess noise is larger the larger \( \Delta \delta \). In the limit \( z \to \infty \) it can be calculated from Eq. (14) that the \( \theta = 0 \) quadrature will be 1.

B. The probe field is squeezed state

We study now how the Doppler effect affects the propagation of a squeezed state as a probe initial condition. We will suppose that the mean value of the probe field is equal to the mean value of the pump field, \( \alpha_1 = \alpha_2 \). We will also suppose that the carrier frequencies of both fields are in resonance with their respective dipole transitions of the atoms at rest.

Numerical results are presented in Fig. 12 for \( g_1 = g_2 = \gamma/10 \), \( \Omega_1 = \Omega_2 = 1\gamma \), \( \omega = \gamma/10, N = 10^{12}, \xi = 2 \). In Fig. 12(a) for a Doppler width of \( \Delta \delta = 0.01 \gamma \), the behavior of the state propagation is very similar to the studied in section [III]. From Fig. 12(b) to Fig. 12(c) it can be seen how the increase of the Doppler width starts to destroy the interchange of squeezing properties between the probe and pump field and the differences between both fields vanish. In Fig. 12(d) both fields have practically the same \( \theta = 0 \) quadrature spectra for \( zC/\gamma > 50 \).

These results impose restrictions in the allowed atoms’ Doppler width for studying quantum state propagation in EIT media.

VI. CONCLUSIONS

We have shown that in the propagation of pump and probe fields through an EIT medium, the initial noise properties of the field are not conserved, except for the carrier frequencies which drive the atoms on two-photon resonance. We found a series of novel behavior in the propagations.

The results reported in Ref. [19] extend to any initial spectrum of noise. The effect of coherent propagation (i.e., neglecting the exponential decay terms) is to inter-
change the noise properties between the probe and pump field as the field propagates. The interchange is maximal when both fields have comparable intensities. The frequency of the interchange of noise properties depends on the spectrum frequency. This implies that the noise spectrum of the outcoming field can be completely different to the noise spectrum of the incoming field.

When the initial noise spectrum is different between the fields, as the field propagates there is an oscillatory creation and annihilation of correlations between the pump and probe fields.

The oscillatory interchange of noise properties between the two fields is reminiscent to the case of two coupled harmonic oscillators [27]. However, in the case presented here, the coupling is realized by a quantum system, namely the atomic media, which shows some special properties due to EIT: In terms of mean values, there is no excited state population, and thus the fields and the atomic media are uncoupled. Consequently, the mean values are stationary, in contrast to the case of the two oscillators. Coupling is enabled by the quantum fluctuations of the atomic dipole moments, and the ground state coherence conveys the interaction between the two fields. Indeed, the generated correlations are most strongly present when $\langle \sigma_{12} \rangle$ is maximal. Apart from the coherent dynamics, the atomic media introduces noise due to the finite lifetime of the excited state level. This is reflected in the decay of the oscillatory interchange of noise properties between the fields at a certain time scale.

The effect of a detuned two-photon resonance in the propagation of an initially broad band vacuum squeezed state as the probe field and a coherent state as the pump field, is to rotate the quadrature of maximum squeezing as the field propagates. The velocity of the rotation is a function of the detuned two-photon resonance and spectrum frequency. With a frequency-dependent velocity of rotation, the outcoming field can be completely different to the incoming field. This frequency-dependent phase is different to the expected global phase (the same phase for all frequencies) due to slow velocity propagation inside the medium.

When both fields have comparable Rabi frequencies, with the probe field initially in a squeezed state and the pump field in a coherent state, the effect of a detuned two photon resonance in the propagation can be described as a combination of a rotation of the maximum squeezed quadrature with an interchange of squeezing properties between the pump and probe fields. The effect of this combination is to redistribute the noise in such a way that gives rise to mode-dependent propagation distances where all the quadratures have the same noise. As the field continues to propagate we recover the mode squeezed state.

Besides the fields' coherent propagation, the squeezing properties of the quadratures are absorbed as the field propagates. The absorption spectrum for the squeezing properties is different to the absorption spectrum for the mean values. It is interesting to note that the transparency window around spectrum frequency $\omega = \delta$ is inversely proportional to the detuned two-photon resonance and can be very narrow.

An important result, is the influence of the atoms’ Doppler width on the propagation of a squeezed probe field. Differently to the propagation of the mean values, where the Doppler effect has a small influence, and due to the considerable effect that detuned two-photon resonance has for states propagating in an EIT medium, the atoms’ Doppler width can destroy the initial squeezing properties of the field as it propagates.

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APPENDIX A: ANALYTICAL EXPRESSIONS FOR THE NOISE AND CORRELATION SPECTRA.

We solve equations (2) with $\delta_1 = \delta_2 = 0$ and noise spectrum given by (6), following the treatment used in [19]. We obtain the following analytical results.
1. The resonance case

For the resonance case, $\delta_1 = \delta_2 = 0$, and noise spectrum given by (6), we obtain the following expressions for

$$
S_1(z, \omega) = \frac{1}{(\alpha_1^2 + \alpha_2^2)^2} \left\{ 4e^{-Q^{(i)}z} \cos \left( Q^{(r)}z \right) \alpha_2^2 (f_1(\omega) - f_2(\omega) + \cos(2\theta) (g_1(\omega) - g_2(\omega))) \alpha_1^2 \\
\quad + 2e^{-2Q^{(i)}z} \alpha_2^2 \left( f_2(\omega)\alpha_1^2 + \alpha_2^2 f_1(\omega) + \cos(2\theta) (g_2(\omega)\alpha_1^2 + \alpha_2^2 g_1(\omega)) \right) \\
\quad + 2 \left( f_1(\omega)\alpha_1^2 + \alpha_2^2 f_2(\omega) + \cos(2\theta) (g_1(\omega)\alpha_1^2 + \alpha_2^2 g_2(\omega)) \right) \alpha_1^2 \right\} + 1,
$$

(A1)

$$
S_2(z, \omega) = \frac{1}{(\alpha_1^2 + \alpha_2^2)^2} \left\{ 4e^{-Q^{(i)}z} \cos \left( Q^{(r)}z \right) \alpha_2^2 \alpha_1^2 \left( -f_1(\omega) - f_2(\omega) + \cos(2\theta) (g_2(\omega) - g_1(\omega)) \right) \\
\quad + 2e^{-2Q^{(i)}z} \alpha_2^2 \left( f_2(\omega)\alpha_1^2 + \alpha_2^2 f_1(\omega) + \cos(2\theta) (g_2(\omega)\alpha_1^2 + \alpha_2^2 g_1(\omega)) \right) \\
\quad + 2\alpha_2^2 \left( f_1(\omega)\alpha_1^2 + \alpha_2^2 f_2(\omega) + \cos(2\theta) (g_1(\omega)\alpha_1^2 + \alpha_2^2 g_2(\omega)) \right) \right\} + 1.
$$

(A2)

For the case, $g_1 = g_2$ and $\alpha_1 = \alpha_2$ we have for the correlation spectrum

$$
S_c(z, \omega) = \frac{1}{2} \left( \cos(\theta_1 - \theta_2) (f_1(\omega) + f_2(\omega)) + \cos(\theta_1 + \theta_2) (g_1(\omega) + g_2(\omega)) \\
\quad + 2e^{-Q^{(i)}z} \sin \left( Q^{(r)}z \right) \left( \sin(\theta_1 - \theta_2) (f_2(\omega) - f_1(\omega)) + \sin(\theta_1 + \theta_2) (g_1(\omega) - g_2(\omega)) \right) \\
\quad -e^{-Q^{(i)}z} \cos(\theta_1 - \theta_2) (f_1(\omega) + f_2(\omega)) + \cos(\theta_1 + \theta_2) (g_1(\omega) + g_2(\omega))) \right),
$$

(A3)

with

$$
Q = \frac{\omega C}{\Omega^2 - \omega^2 + i\omega \gamma / 2}, \quad (A4a)
$$

$$
Q^{(r)} \equiv \text{Re} Q = \frac{\omega C}{(\Omega^2 - \omega^2)^2 + \omega^2 \gamma^2 / 4}, \quad (A4b)
$$

$$
Q^{(i)} \equiv \text{Im} Q = \frac{-\omega^2 \gamma / 2}{(\Omega^2 - \omega^2)^2 + \omega^2 \gamma^2 / 4}, \quad (A4c)
$$

$$
C = \frac{N(\Omega_1^2 + \Omega_2^2)}{\Omega^2 c}, \quad \Omega = \sqrt{\Omega_1^2 + \Omega_2^2}.
$$

2. Detuned two-photon resonance case

For the detuned two-photon resonance case, $\delta_1 = \delta_2 = \delta$, and assuming that $f_1 = g_1 = 0$ in Eqs. (6), we obtain

$$
S_1(z, \omega) = \frac{\alpha_1^2 \alpha_2^2}{(\alpha_1^2 + \alpha_2^2)^2} \left\{ f_2(\omega) \left[ 2 + e^{2Q^{(i)}z} + e^{2Q^{(i)}z} - 2e^{Q^{(i)}z} \cos Q^{(r)}z - 2e^{Q^{(i)}z} \cos Q^{(r)}z \right] \\
\quad + 2q_2(\omega) \left[ \cos 2\theta + e^{Q^{(i)}z} + \cos(Q^{(r)}z - Q^{(r)}z + 2\theta) \\
\quad - e^{Q^{(i)}z} \cos(Q^{(r)}z - 2\theta) - e^{Q^{(i)}z} \cos(Q^{(r)}z + 2\theta) \right] \right\},
$$

(A5)
\[
S_2(z, \omega) = \frac{1}{(\alpha_1^2 + \alpha_2^2)^2} \left\{ \alpha_1^4 \left\{ f_2(\omega) \left( e^{2Q_-^{(r)}z} + e^{2Q_+^{(r)}z} \right) + 2g_2(\omega) e^{Q_+^{(r)}z+Q_-^{(r)}z} \cos(Q_+^{(r)}z - Q_-^{(r)}z + 2\theta) \right\} + \alpha_2^2 \alpha_1^2 \left\{ 2f_2(\omega) \left( e^{Q_-^{(r)}z} \cos(Q_+^{(r)}z + e^{Q_+^{(r)}z} \cos(Q_+^{(r)}z) \right) + 2g_2(\omega) \left( e^{Q_+^{(r)}z} \cos(Q_+^{(r)}z - 2\theta) + e^{Q_-^{(r)}z} \cos(Q_+^{(r)}z + 2\theta) \right) \right\} + 2\alpha_2^4 \left\{ f_2(\omega) + g_2(\omega) \cos(2\theta) \right\} \right\}.
\]

For the correlations we have, assuming \( \alpha_1 = \alpha_2 \) and \( g_1 = g_2 \)

\[
S_c(z, \omega, \theta_1, \theta_2) = -\frac{1}{2} e^{Q_-^{(r)}z} \left\{ \cos(\theta_1 - \theta_2) \left( f_1(\omega) + f_2(\omega) \right) + 2\sin(Q_-^{(r)}z) \sin(\theta_1 - \theta_2) \left( f_1(\omega) - f_2(\omega) \right) + \cos \left( Q_-^{(r)}z + \theta_1 + \theta_2 \right) \left( g_1(\omega) - g_2(\omega) \right) \right\} + \frac{1}{2} e^{Q_+^{(r)}z} \cos \left( Q_+^{(r)}z - \theta_1 - \theta_2 \right) \left( g_1(\omega) - g_2(\omega) \right) - \frac{1}{2} e^{Q_-^{(r)}z+Q_+^{(r)}z} \cos \left( Q_-^{(r)}z - Q_+^{(r)}z + \theta_1 + \theta_2 \right) \left( g_1(\omega) + g_2(\omega) \right) + \frac{1}{2} \left( \cos(\theta_1 - \theta_2) \left( f_1(\omega) + f_2(\omega) \right) + \cos(\theta_1 + \theta_2) \left( g_1(\omega) + g_2(\omega) \right) \right)
\]

(A7)

with

\[
Q_\pm = \frac{1}{\omega C} \frac{\Omega^2 - \omega(\omega \pm \delta) + i\omega \gamma/2}{\Omega^2 - \omega(\omega \pm \delta)^2 + \omega^2 \gamma^2/4}
\]

(A8a)

\[
Q_\pm^{(r)} = \text{Re} Q_\pm = \frac{\Omega^2 - \omega(\omega \pm \delta)}{[\Omega^2 - \omega(\omega \pm \delta)]^2 + \omega^2 \gamma^2/4}
\]

(A8b)

\[
Q_\pm^{(i)} = \text{Im} Q_\pm = \frac{-\omega^2 \gamma/2}{[\Omega^2 - \omega(\omega \pm \delta)]^2 + \omega^2 \gamma^2/4}
\]

(A8c)

\[
\mathcal{C} = \frac{N(g_1^2 \Omega_2^2 + g_2^2 \Omega_1^2)}{\Omega^2_c}
\]

\[
\Omega = \sqrt{\Omega_1^2 + \Omega_2^2}
\]

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