Mapping the Two-Component Atomic Fermi Gas to the Nuclear Shell-Model

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The physics of two-component fermi systems is now frequently addressed in laboratories. Usually this is done for large samples of tens to hundreds of thousands of particles. However, there are now groups working towards few-body systems (1-100 particles) in very tight traps where the shell structure of the external potential becomes important. This is analogous to the situation encountered in nuclear structure physics. Here the state-of-the-art method is large-scale diagonalization of the interaction hamiltonian in an appropriate model space. We map the atomic physics problem onto the nuclear shell-model and show that the two-component atomic fermi gas with zero-range interaction does not suffer from the famous fermionic sign problem, making Quantum Monte Carlo methods feasible.

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I. INTRODUCTION

The physics of ultracold gases has seen rapid developments over the past two decades [1, 2, 3]. One of the next goals is to trap few fermions to approach the regime where the trapping shell structure becomes prominent [4]. For small number of particles one expects large quantum fluctuations that invalidate the use of mean-field theories such as the BCS approach. Theoretical investigations beyond mean-field are therefore needed and on-going [5, 6]. In this paper we demonstrate how methods to solve the many-body problem in nuclei can be applied by mapping the atomic two-component fermion system in a tight trap into that of a single species of nucleon in an external field.

One particular successful approach to the nuclear shell-model is the so-called Shell-model Monte Carlo method [7]. This method overcomes the large-scale matrix diagonalization by transforming two-body terms into one-body terms and subsequently solves a multi-dimensional path integral using Monte Carlo integration. It has had many successes in nuclear application and it is hoped that one might adapt this method to also address small atomic fermi systems. However, there is an infamous sign problem that plagues Monte Carlo methods when dealing with fermions [8], and it is not a priori obvious that the atomic two-component Fermi system can be handled at all.

In this article we discuss this mapping in some detail and give the essentials of the shell-model Monte Carlo method. We then proceed to prove the absence of a sign problem for the case of a zero-range interaction. We also briefly discuss of some similarities and differences of pairing studies in atomic and nuclear physics. For the benefit of both nuclear and atomic physics communities, the explicit evaluation of the zero-range interaction relevant for the atomic physics system in terms of the matrix elements used in the nuclear shell model is given in considerable detail in an appendix.

II. MAPPING THE FERMI GAS TO THE NUCLEAR SHELL MODEL

The two-component ultracold fermionic atomic gas consists of neutral atoms, usually alkali ones, that occupy two different internal states. The actual internal states are hyperfine states of different projection that can be split by a magnetic field [3]. The hyperfine splitting is larger than any other energy scale in the problem so that no internal process in the gas can transfer atoms between the hyperfine levels, thus one may think of these as frozen degrees of freedom. Also, since the systems are usually dilute, the range of the atom-atom interactions is very short compared to the typical interparticle distance. Therefore the simple zero-range potential is a popular and highly successful model.

The $N$-body fermi system in an isotropic harmonic trap with a zero-range interaction of strength $V_0$ has the
hamiltonian

\[ H = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{1}{2} m \omega^2 r_i^2 + \sum_{[ij]} V_0 \delta(\vec{r}_i - \vec{r}_j), \]  

(1)

where \([ij]\) denotes the sum over all pairs of particles. In terms of two-body matrix elements we have

\[ \langle \psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2) | V_{\text{int}} | \psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2) \rangle, \]

(2)

where \(V_{\text{int}}\) is any two-body interaction. The two-body wavefunction \(\psi_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_2)\) and \(\psi_\alpha(\vec{r}_1)\bar{\psi}_\beta(\vec{r}_2)\) must be antisymmetric under exchange of the coordinates.

In the tight harmonic trap, the quantum numbers are \(n, l, m_l\). The two internal hyperfine states can now be mapped onto the spin of a single species of nucleon \(m_s = \pm 1/2\). Thus, a single-particle state is uniquely given by \(a = (n, l, m_s, m_l)\). Any two-body state constructed from these states will have external and internal parts that combine to determine the overall symmetry. Since we use a zero-range interaction, the spatial part must be symmetric under particle exchange to give a non-zero contribution. Therefore the internal (pseudospin) state must be antisymmetric, which in the spin 1/2 language is the spin-singlet state. This completes our mapping of the two-component fermi system in a trap onto the nuclear shell model with one species of nucleon (proton or neutron). The nuclear mean-field is simply replaced by the harmonic oscillator and the internal spin states of the nucleon are now hyperfine states for the atoms.

III. AUXILIARY-FIELD MONTE CARLO METHOD

The use of Quantum Monte Carlo for fermions is plagued by the well-known sign problem (see f.x. the review [8]). This basically arises because there is no way to choose a positive definite measure for the Monte Carlo sampling when taking into account the exchange symmetry of fermions. In fact it was shown in [9] that the problem is computationally NP hard. Various ways have been invented to deal with the problem by appropriately restricting the Hilbert space or interacting hamiltonian of the system under study. One possibility is to integrate out the fermions and get an effective bosonic theory [10, 11], another is to use variational wavefunctions with a fixed node structure [12, 13], a third way is the so-called determinant diagrammatic Monte Carlo which is also principally sign problem free [14]. Common to all the methods above is that they are mostly successful for homogeneous fermi systems. Here we are concerned with a very different limit of small systems in tight traps, thus the sensible quantum numbers are quite different. This corresponds to the situation in nuclear physics.

Here we review the Auxiliary-field Monte Carlo method for the fermionic many-body problem. In the context of the nuclear shell-model this approach is known as the Shell-Model Monte Carlo (SMMC) method [7]. This method is based on a linearization of the two-body part of the Hamiltonian using the Hubbard-Stratonovich transformation. Here we adopt a formulation of this transformation starting from a general Hamiltonian which is explicitly time-reversal invariant:

\[ H = \sum_\alpha (\epsilon_\alpha O_\alpha + \epsilon_\alpha^* \bar{O}_\alpha) + \frac{1}{2} \sum_\alpha V_\alpha \{ O_\alpha, \bar{O}_\alpha \}, \]

(3)

where \(O_\alpha\) are one-body operators in a convenient basis and the \(V_\alpha\) are real numbers. The bars denote time-reversed operators. The SMMC approach uses the Hubbard-Stratonovich (HS) transformation to linearize the many-body evolution operator \(e^{-\beta H}\), where \(\beta^{-1}\) may be interpreted as the temperature in the (grand)canonical ensemble. We first divide \(\beta\) into \(N_t\) time slices so that we can express individual terms at different time slices in \(e^{-\beta H} = [e^{-\Delta \beta H}]^{N_t}\) as

\[ e^{-\Delta \beta H} \approx e^{-\Delta \beta \sum_\alpha (\epsilon_\alpha O_\alpha + \epsilon_\alpha^* \bar{O}_\alpha)} \prod_\alpha e^{-\Delta \beta \frac{1}{2} [(O_\alpha + \bar{O}_\alpha)^2 - (O_\alpha - \bar{O}_\alpha)^2]} + \mathcal{O}(\Delta \beta)^2, \]

where we used \(2 \{ O_\alpha, \bar{O}_\alpha \} = (O_\alpha + \bar{O}_\alpha)^2 - (O_\alpha - \bar{O}_\alpha)^2\). Quadratic interaction terms can be effectively linearized through the Gaussian integral identity

\[ e^{-\Delta \beta \frac{1}{2} [(O_\alpha + \bar{O}_\alpha)^2 - (O_\alpha - \bar{O}_\alpha)^2]} = \frac{\Delta \beta |V_\alpha|}{4\pi} \int d\sigma_{\alpha} R d\sigma_{\alpha}^I e^{-\Delta \beta \frac{1}{2} [\sigma_{\alpha}^2 + (\sigma_{\alpha}^I)^2]} e^{-\Delta \beta \frac{1}{2} [s_{\alpha}^2 (O_\alpha + \bar{O}_\alpha) + i s_{\alpha}^2 (O_\alpha - \bar{O}_\alpha)]}, \]

(4)
where the integration variables $s_\alpha^R$ and $s_\alpha^I$ are the real auxiliary fields that give the method its name. The sign factors are $s_\alpha = \pm 1$ for $V_\alpha < 0$ and $s_\alpha = \pm i$ for $V_\alpha > 0$. Introducing complex fields for each time slice $\sigma_\alpha(\tau_n) = s_\alpha^R(\tau_n) + i s_\alpha^I(\tau_n)$, we arrive at the Hubbard-Stratonovich representation of the many-body evolution operator

$$e^{-\beta H} = \int \mathcal{D}[\sigma] G(\sigma) U_\sigma(\beta, 0).$$

Above,

$$\mathcal{D}[\sigma] = \prod_{\alpha,n} \frac{d\sigma_\alpha(\tau_n)d\sigma_\alpha^*(\tau_n)}{2i} \frac{\Delta \beta |V_\alpha|}{4\pi}$$

is the measure of the integral. $G(\sigma)$ is a Gaussian weight

$$G(\sigma) = e^{-\frac{\Delta \beta}{4} \sum_n |V_\alpha||\sigma_\alpha(\tau_n)|^2}.$$ 

Many-body propagator is now effectively reduced to a superposition of one-body propagators

$$U_\sigma(\beta, 0) = e^{-\Delta \beta h_\sigma(\tau_N)} \ldots e^{-\Delta \beta h_\sigma(\tau_1)},$$

where the linearized Hamiltonian as a function of the time-dependent auxiliary fields is given by

$$h_\sigma(\tau) = \sum_\alpha \left( \epsilon_\alpha + \frac{1}{2} s_\alpha V_\alpha \sigma_\alpha(\tau) \right) \sigma_\alpha + \left( \epsilon_\alpha^* + \frac{1}{2} s_\alpha^* V_\alpha \sigma_\alpha^*(\tau) \right) \sigma_\alpha.$$ 

In the SMMC, expectation value of an observable $\Omega$ is calculated by expressing both the numerator and the denominator of $\langle \Omega \rangle = \text{Tr}_N[\Omega e^{-\beta H}]/\text{Tr}_N e^{-\beta H}$ (where $\text{Tr}_N$ denote canonical traces for $N$-particle system) in the HS representation. In order to perform a Monte Carlo integration, a positive definite weight function is defined as $W(\sigma) = G(\sigma)|\text{Tr}_N U_\sigma(\beta, 0)|$. Thus, one can express the thermal expectation values by

$$\langle \Omega \rangle = \frac{\int \mathcal{D}[\sigma] W(\sigma) \Phi(\sigma) (\Omega)_{\sigma}}{\int \mathcal{D}[\sigma] W(\sigma) \Phi(\sigma)}$$

(10)

where $\Phi(\sigma) = \text{Tr}_N U_\sigma(\beta, 0)/|\text{Tr}_N U_\sigma(\beta, 0)|$ is the “sign” and $\langle \Omega \rangle_{\sigma} = \text{Tr}_N[\Omega U_\sigma(\beta, 0)]/\text{Tr}_N U_\sigma(\beta, 0)$. The observable $\langle \Omega \rangle$ is then computed in a Monte Carlo integration by selecting an ensemble of auxiliary fields $(\sigma_1, \ldots, \sigma_N)$ sampled according to the distribution function $W(\sigma)$, i.e.,

$$\langle \Omega \rangle \approx \frac{1}{N} \sum_n \Phi(\sigma_n) \langle \Omega \rangle_{\sigma_n}.$$ 

(11)

Success of the outlined method hinges on the sign $\Phi(\sigma)$ of the weight function $W(\sigma)$. Unfortunately, in the most general case, $\text{Tr}_N U_\sigma(\beta, 0)$ is not always positive hence $\Phi(\sigma)$ can be $\pm 1$. Such fluctuations causes significant cancellations in the denominator of Eq. 11 and renders the method ineffective due to large statistical uncertainties in $\langle \Omega \rangle$. In the literature, this problem is referred to as the Monte Carlo sign problem. For any Hamiltonian (Eq. 3) with all $V_\alpha < 0$, $h_\sigma$ are always time-reversal invariant, since all $s_\alpha$ are real (Eq. 1). As was shown by Lang et al. [12], time-reversal invariance of $h_\sigma$ implies that the eigenvalues of the matrix $U_\sigma$ come in complex-conjugate pairs which, in turn, ensures that the grand-canonical partition function $\text{Tr} U_\sigma$ is positive definite. In the canonical ensemble, projections on even number of particles always preserve the good sign as long as the grand canonical partition function is positive definite. However for systems with odd-number of particles, projections on to an odd number of particles usually reintroduces the sign problem at large values of $\beta$ even when the grand canonical partition function is positive definite.

### IV. SIGN PROPERTIES OF THE ZERO-RANGE INTERACTION

We now consider the zero-range interaction in the $J$-coupled representation which is discussed in full detail in appendix A. We write the two-body hamiltonian in the so-called pairing (or particle-particle) decomposition as $H_2$, which for the zero-range interaction gives

$$H_2 = \frac{1}{2} \sum_{abcd} \sum_{JM} V_{J}(ab,cd) A_{JM}^\dagger(ab) A_{JM}(cd),$$

(12)
Since this matrix is real symmetric, there is a basis of orthonormal eigenvectors. Let us denote this basis \( V \). We will have no sign problems when all eigenvalues of \( V \) is to diagonalize this latter matrix and look at the signs of the eigenvalues. As is demonstrated in [15], the interaction will have no sign problems when all eigenvalues of \( V \) are negative. Obviously the problem splits into blocks of given \( J \) so we work in a fixed \( J \) subspace.

The crucial observation is that \( V_{ij} ab, cd \) can be factorized in the following way. First define the quantity

\[
\langle f_i | T | f_j \rangle = \frac{1}{\sqrt{2}} (-1)^{l_a + j_a + 1/2} | | j_a | | j_b | | l_b | | l_a | | J \rangle (J M) e^{i \theta} \sqrt{|V_0|} R_{n_a l_a} (r) R_{n_b l_b} (r),
\]

where \( e^{2i \theta} = \text{sgn}(V_0) \). Notice that \( f_i (ab) e^{-i \theta} \) is a purely real number. In terms of the combined indices we now have

\[
V_{ij} = \int_0^\infty dr f_i (j) f_j (i).
\]

Since this matrix is real symmetric, there is a basis of orthonormal eigenvectors. Let us denote this basis \( u^k \) and the corresponding eigenvalues \( \lambda^k \). The dimension is given by the number of pairs in the given model space that can couple to total angular momentum \( J \). Consider now for a given \( k \) the product \( (u^k)^T V J u^k \), where \( T \) denotes the transpose. Inserting the explicit form of \( V \) we have

\[
u^T V J u = \sum_{ij} u^k (i) V_{ij} u^k (j) = \int_0^\infty dr \left[ \sum_{ij} f_i (i) f_j (i) \right]^2 = \lambda^k,
\]

where the last equality follows from the eigenvalue equation and the fact that \( u^k \) is normalized. We thus see that the eigenvalues are equal to some real number squared times a phase \( e^{2i \theta} = \text{sgn}(V_0) \). Therefore, the sign of \( V_0 \) is also the sign of the eigenvalues. We thus have the result that any attractive zero-range interaction \( (V_0 < 0) \) will have no sign problem, whereas the repulsive \( (V_0 > 0) \) case can never give a positive-definite path integral.

The simple form of \( V_{ij} \) always us to prove some further properties of its spectrum. Define (for fixed \( J \) not shown) the row vector \( f = [f_1 f_2 \ldots f_n] \), where \( n \) counts the pairs, such as to fulfill \( V J = \int dr f^T f \). Now pick a row vector orthogonal to \( f \) so that \( f g^T = 0 \). Then we see that \( V J g^T = \int dr f^T g^T = \int dr f^T (f g)^T = 0 \), thus all vectors orthogonal to \( f \) are in the null-space of \( V \). We therefore have only one non-zero eigenvalue for each \( J \) and \( n - 1 \) eigenvectors with zero eigenvalue. The sole non-zero eigenvalue has the value \( \int dr f^T f \) and the eigenvector \( f^T \). We thus see that the zero-range pairing interaction has a very simple structure after diagonalization.

As mentioned, the above proof was carried out in the so-called pairing decomposition with the operators \( A^\dagger_{JM} (ab) \) and \( A_{JM} (cd) \). In many nuclear applications of the method, the calculations are carried out in the density decomposition [7]. However, the exact path integral is independent of the particular representation and the above result will still hold. In particular, the change from pairing to density decomposition is in practice a re-coupling of the angular momenta involved (and a change of the one-body terms that we are not concerned with). Since re-couplings corresponds to changes of basis the result for the eigenvalues still holds. We have also done explicit numerical checks of this fact and confirmed the general statement.

The good sign properties of the zero-range pairing rested on the fact that it could be factorized, which is more commonly referred to as separability of the zero-range interaction. A non-zero range interaction would not have this property and positive eigenvalues with associated sign problems can be expected.

V. CONNECTION TO NUCLEAR PAIRING

Many nuclear pairing studies consider only pairs of particles in time-reversed states with an attractive zero-range interaction of constant magnitude \( g < 0 \). This effectively means that the radial part of the matrix element in eq. A10 is replaced by a constant. For the time-reversed states one has \( l_a = l_b \) and \( l_c = l_d \). A special case of the proof above thus shows that such a restriction preserves the good sign.

The restriction to this interaction is inspired by the BCS approach to the pairing of electrons in conventional low temperature superconductors. The BCS theory was applied to nuclei only one year after its invention [16]. In the
general case, the gap will be state-dependent, $\Delta_i$, where $i$ denotes single-particle levels in the nuclear mean-field (as arising for instance in a Hartree-Fock calculation). However, the matrix element in the gap equations is the one in eq. A.8 with $l_a = l_b$ and $l_c = l_d$ (see f.x. [17]). This approach is sometimes generalized into the Hartree-Fock-Bogoliubov (HFB) theory (also known as Bogoliubov-de Gennes in the atomic community), where pairing enters explicitly into the self-consistent equations for the optimal Slater-determinant state [18]. However, since the exact form of the pairing interaction in nuclei is not known, these studies are mostly phenomenological, and residual interaction derived from realistic nucleon-nucleon interactions cannot reproduce the magnitude of the experimentally observed gap [17].

Notice that the SMMC described above would reduce to a HFB approach if the auxiliary field was determined self-consistently. However, as it is integrated out, the SMMC is in principle exact in the model space chosen and thus goes beyond HFB theory. It would be very interesting to see the differences between the full interaction and that self-consistently. However, as it is integrated out, the SMMC is in principle exact in the model space chosen and thus goes beyond HFB theory. It would be very interesting to see the differences between the full interaction and that self-consistently. However, as it is integrated out, the SMMC is in principle exact in the model space chosen and thus goes beyond HFB theory. It would be very interesting to see the differences between the full interaction and that self-consistently. However, as it is integrated out, the SMMC is in principle exact in the model space chosen and thus goes beyond HFB theory. It would be very interesting to see the differences between the full interaction and that self-consistently.

VI. OUTLOOK

Studies of small two-component fermi systems in tight traps are currently being pursued experimentally [4] and theoretically [5, 6]. Here we demonstrates how the mapping of the atomic system to a nuclear physics problem can be done, and prove the very important feature that there is no sign problem with the typical choice of a zero-range interaction. As we have discussed, the atomic interaction between two internal hyperfine states is more general than the typical pairing force used in many nuclear investigations, and it was therefore not a priori clear that the transfer to nuclear shell-model Monte Carlo would be sign problem free. Of course one could also use large-scale shell-model diagonalization codes on the atomic problem. However, here the number of configurations grows exponentially with model space size, whereas in the SMMC it only grows quadratically [7]. Of course it might be possible to do reasonable truncations and use effective interactions to reduce the size of the problem. Since low-dimensional systems are currently under intense study in atomic physics, this might also help alleviate to reduce the matrix problem in size. However, at the moment the shell-model Monte Carlo method seems to be the only tractable approach in the full three-dimensional case.

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APPENDIX A: THE ZERO-RANGE FORCE IN $J$-COUPLED REPRESENTATION

The matrix element used in the $J$-coupled representation for nuclear problems is given by

$$V_J(ab,cd) = \langle \psi_{Ja}(\vec{r}_1) \times \psi_{Jb}(\vec{r}_2) \rangle^{JM} \left| V(\vec{r}_1,\vec{r}_2) \right| \langle \psi_{Ja}(\vec{r}_1) \times \psi_{Jd}(\vec{r}_2) \rangle^{JM},$$

(A1)

where $a, b, c, d$ have angular momentum $j_a, j_b, j_c$, and $j_d$, all of which are coupled from the external and internal quantum numbers. Notice that $V_J$ is independent of the total projection $M$ (this can be seen by applying the Wigner-Eckart theorem). In a typical nuclear model there is some external mean-field (here assumed spherical) with quantum numbers $(nml)$ and the internal spin half quantum numbers $(\frac{1}{2}m_s)$. Coupling these through $j = \ell + \tilde{s}$ gives $j = \ell \pm 1/2$ for the single-particle quantum numbers.

As discussed in the text, in the atomic case with zero-range interactions the internal state must be a spin singlet; $|S = 0, M_a = 0 \rangle$. In order to ensure that the interaction in the $J$-scheme respects this, it is necessary to change the coupling scheme above from $j_a$ and $j_b$ coupled to $J$ into $l_a$ and $l_b$ coupled to $L$ and $s_a$ and $s_b$ coupled to $S$. This is done through the standard techniques of angular momentum [19]:

$$|(l_als_a j_a, l_bl_b j_b, JM) \rangle = \sum_{L,S} |L| |S| |j_a| |j_b| \left\{ \begin{array}{ccc} l_a & s_a & j_a \\ l_b & s_b & j_b \\ L & S & J \end{array} \right\} |(l_als_a L, s_bl_b S, JM) \rangle.$$  

(A2)
Here we are interested in the $s_a = s_b = 1/2$ case, and, since the interaction contains a projection onto spin singlet states, only need the $S = 0$ component of this transformation. Using a reduction on the $9j$ symbol [19], the projection can be written

$$P_{S=0}(l_a \frac{1}{2}, j_a, (l_b \frac{1}{2})j_b, JM) = \sum_{L}[L][j_a][j_b] \left\{ \begin{array}{ccc} l_a & j_a & j_b \\ l_b & \frac{1}{2} & \frac{1}{2} \\ L & 0 & J \end{array} \right\} |(l_a l_b) L, (\frac{1}{2} \frac{1}{2})0, JM\rangle =$$

$$(-1)^{L+l_a+j_b+2j_a-1} \frac{1}{\sqrt{2}} \left\{ \begin{array}{ccc} L & j_a & j_b \\ \frac{1}{2} & l_a & l_b \end{array} \right\} \delta_{L,J} |(l_a l_b) L, (\frac{1}{2} \frac{1}{2})0, JM\rangle,$$

(A3)

where $P_{S=0} = (1 - \sigma_1 \cdot \sigma_2)/4$ is the projection onto the spin singlet state. The remaining zero-range interaction of course only acts on the external quantum states, thus we have to evaluate matrix elements between coupled states with operators acting on only one of the degree of freedom. Since the spin part is trivial for singlets we simply have the result (keeping both $L$ and $J$ for clarity even though $L = J$)

$$\langle (l_a l_b) L, (\frac{1}{2} \frac{1}{2})0, JM | V(\vec{r}_1 - \vec{r}_2) | (l_a l_b) L, (\frac{1}{2} \frac{1}{2})0, JM \rangle =$$

$$\langle (l_a l_b) LM | V(\vec{r}_1 - \vec{r}_2) | (l_a l_b) LM \rangle,$$

(A4)

where we have explicitly indicated the orbital angular momenta of all states involved. For the zero-range interaction $V(\vec{r}_1 - \vec{r}_2) = V_0 \delta(\vec{r}_1 - \vec{r}_2)$, the latter matrix element can be found in many textbooks (see for instance [19]) and is given by

$$\langle l_1 l_2 J M | V_0 \delta(\vec{r}_1 - \vec{r}_2) | l'_1 l'_2 J' M' \rangle =$$

$$\delta_{L,J'} \delta_{M,M'} (l_1) (l_2) | (l'_1) (l'_2) \rangle \left\{ \begin{array}{ccc} l_1 & l_2 & J \\ 0 & 0 & 0 \end{array} \right\} \left\{ \begin{array}{ccc} l'_1 & l'_2 & J' \\ 0 & 0 & 0 \end{array} \right\} \frac{V_0}{\sqrt{2}} \int_0^\infty \! dr r^2 R_{n_1l_1}(r) R_{n_2l_2}(r) R_{n_1'l_1'}(r) R_{n_2'l_2'}(r).$$

(A5)

We can now insert all these formulae into eq. (A1) to get an expression for the $J$-scheme interaction:

$$V_J(ab, cd) = \delta_{J,L} (-1)^{\frac{l_a + l_b + 2j_a + j_b + j_c + j_d - 1}{2}} \left\{ \begin{array}{ccc} L & j_a & j_b \\ \frac{1}{2} & l_a & l_b \end{array} \right\}$$

$$\left\{ \begin{array}{ccc} L & j_c & j_d \\ \frac{1}{2} & l_c & l_d \end{array} \right\} \langle (l_a l_b) L0 | V(\vec{r}_1 - \vec{r}_2) | (l_a l_b) L0 \rangle = (-1)^{j_a + j_b + j_c + j_d + l_a + l_b + l_c + l_d} \left\{ \begin{array}{ccc} J & j_a & j_b \\ \frac{1}{2} & l_a & l_b \end{array} \right\} \left\{ \begin{array}{ccc} J & j_c & j_d \\ \frac{1}{2} & l_c & l_d \end{array} \right\}$$

$$\langle l_a | l_b | l_c | l_d \rangle \left\{ \begin{array}{ccc} l_a & l_b & J \\ 0 & 0 & 0 \end{array} \right\} \left\{ \begin{array}{ccc} l_c & l_d & J \\ 0 & 0 & 0 \end{array} \right\} \frac{V_0}{\sqrt{2}} \int_0^\infty \! dr r^2 R_{n_1l_1}(r) R_{n_2l_2}(r) R_{n_1'l_1'}(r) R_{n_2'l_2'}(r),$$

(A6)

where the second equality comes from using the formula in eq. (A5). Notice that the phase can be written with $l_b + l_d$ instead of $l_a + l_c$ since $l_a + l_b + l_c + l_d$ is even due to the restrictions from the Clebsch-Gordon coefficients. This is the general interaction in the spin singlet state and $l_a$, $l_b$, $l_c$, and $l_d$ can in general be different as long as they couple pairwise to $L = J$. For the pairing interaction in the time-reversed states discussed in the text, we have $l_a = l_b$ and $l_c = l_d$.

The formula above explicitly shows that $l_a + l_b + J$ and $l_c + l_d + J$ must be even. However, the multipole expansion used to arrive at this expression implicitly requires that also $l_a + l_c + J$ and $l_b + l_d + J$ be even. Notice also that the factor $l_a + l_c$ means that pairing across two opposite parity major shells can be repulsive for $V_0 < 0$. This is well-known in nuclear pairing studies [7].

As discussed in [7], the physical matrix elements used in the nuclear shell model must be antisymmetrized. This can be achieved by using the definition

$$V_J^A(ab, cd) = \frac{1}{\sqrt{(1 + \delta_{ab})(1 + \delta_{cd})}} [V_I(ab, cd) - (-1)^{j_a + j_d - J} V_J(ab, dc)].$$

(A7)

However, as one can easily show, $V_J(ab, dc) = -(-1)^{j_a + j_d - J} V_J(ab, cd)$. Thus we have the simple result

$$V_J^A(ab, cd) = \frac{2}{\sqrt{(1 + \delta_{ab})(1 + \delta_{cd})}} V_J(ab, cd).$$

(A8)
This is not surprising since we argued that only the antisymmetric $S = 0$ spin-singlet component should have non-zero matrix elements. We have effectively enforced the Pauli principle in this manner.

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885(2008).
[2] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).
[3] W. Ketterle and M.W. Zwierlein: Ultracold Fermi Gases, Proceedings of the International School of Physics "Enrico Fermi", Course CLXIV, Varenna, 20 - 30 June 2006, edited by M. Inguscio, W. Ketterle, and C. Salomon (IOS Press, Amsterdam) 2008.
[4] Selim Jochim, private communication.
[5] N.T. Zinner, K. Mølmer, C. Özen, K. Langanke, and D.J Dean, [arXiv:0803.2861v2].
[6] N.T. Zinner, C. Özen et al., in preparation.
[7] S. E. Koonin, D. J. Dean, and K. Langanke, Phys. Rep. 278, 1 (1997).
[8] W. von der Linden, Phys. Rep. 220, 53 (1992).
[9] M. Troyer and U.-J. Wiese, Phys. Rev. Lett. 94, 170201 (2005).
[10] D. J. Scalapino and R. L. Sugar, Phys. Rev. Lett. 46, 519 (1981).
[11] J.-W. Chen and D.B. Kaplan, Phys. Rev. Lett. 92, 257002 (2004).
[12] J. Carlson, S.Y. Chang, V.R. Pandharipande, and K.E. Schmidt, Phys. Rev. Lett. 91, 050401 (2003).
[13] G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. 93, 200404 (2004).
[14] E. Burovski, N. Prokofev, B. Svistunov, and M. Troyer, New J. Phys. 8, 153 (2006).
[15] G.E. Lang, C.W. Johnson, S.E. Koonin, and W.E. Ormand, Phys. Rev. C 48, 1518 (1993).
[16] A. Bohr, B. Mottelson, and D. Pines, Phys. Rev. 110, 936 (1958).
[17] A.L. Fetter and J.D. Walecka: Quantum Theory of Many-Particle Systems, McGraw-Hill San Francisco, 1971.
[18] P.J. Siemens and A.S. Jensen: Elements of Nuclei: Many-Body Physics with the Strong Interaction, Redwood City, California, Addison-Wesley Pub. Co., 1987.
[19] D.M. Brink and G.R. Satchler: Angular Momentum, Oxford University Press Inc., New York, 1993.