Magnetization bound for classical spin models on graphs

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Abstract

In this paper we prove the existence of phase transitions at finite temperature for $O(n)$ classical ferromagnetic spin models on infrared finite graphs. Infrared finite graphs are infinite graphs with $\lim_{m \to 0^+} \text{Tr}(L + m)^{-1} < \infty$, where $L$ is the Laplacian operator of the graph. The ferromagnetic couplings are only requested to be uniformly bounded by two positive constants. The proof, inspired by the classical result of Fröhlich, Simon and Spencer on lattices, is given through a rigorous bound on the average magnetization. The result holds for $n \geq 1$ and it includes as a particular case the Ising model.

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1 Introduction

The study of statistical models on graphs requires the introduction of new techniques and concepts with respect to the well known case of lattices. The lack of translational invariance in general implies inhomogeneity in local variables and makes useless the introduction of such a powerful tool as Fourier transforms in space. In addition, due to the absence of a natural definition of dimension, the statement itself of general results and theorems is rather problematic. The key point is to find a connection between geometry and physical properties. This problem can be studied in the framework of algebraic graph theory. However, phase transitions only occur on infinite graphs and the algebraic theory of infinite graphs is a very recent field of research in mathematics [9, 10]. Up to now only few classical results on lattices have been extended to graphs.

In this paper we deal with the generalization of the proof of existence of spontaneous magnetization at finite temperature for ferromagnetic models on lattices in $d \geq 3$. This result was proven in 1976 by Fröhlich, Simon and Spencer [5] by a general approach based on Gaussian domination of correlation functions in the infrared regime. Such approach, as well as the formulation of the result itself, is deeply related to the translation invariance of lattices. Therefore here we modify the mathematical techniques and we use an alternative definition of the order parameter. In particular the concept of dimension is replaced by the asymptotic behavior of the Laplacian spectral density at low eigenvalues, a choice which is meaningful also in many other contexts. The structure of the paper is as follows: in section 2 we recall the basic definitions and theorems of graph theory to be used in the proof; in section 3 we give the fundamental known results about statistical models on graphs and state our theorem on the magnetization bound; finally in section 4 we present the proof.

2 Some mathematical properties of graphs

Definition 1 A graph $G$ consists of a countable set of vertices (points, sites) $V(G) = \{i, j, k, \ldots\}$ and of a set $E(G)$ of unordered pairs of vertices. The generic element $(i, j)$ of $E(G)$ is called a link and its vertices are said to be nearest neighbors (adjacent). If $V(G)$ is finite, $G$ is called a finite graph and we will denote by $N$ the number of vertices of $G$.

Definition 2 A path connecting two vertices $i$ and $j$ is an alternating sequence of vertices and links $[i, (i, k), k, (k, h), h \ldots, w, (w, j), j]$.

Definition 3 A graph $G$ is connected if, given any two vertices $i, j \in V(G)$, it exists a path between $i$ and $j$.

Here we will deal only with connected graphs. The topology of $G$ is algebraically described by its adjacency matrix.
Definition 4 The adjacency matrix of $G$, $A_{ij}$, is given by:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

(1)

Most physical models on $G$ can be defined in terms of the Laplacian matrix $L_{ij}$:

Definition 5 The Laplacian matrix of $G$, $L_{ij}$ is given by:

$$L_{ij} = z_i \delta_{ij} - A_{ij}$$

(2)

where $z_i \equiv \sum_{j \in V(G)} A_{ij}$ is called the coordination number of $i$.

In the following we will consider graphs with coordination numbers bounded from above: $z_i \leq z_{\text{max}} \in \mathbb{N} \ \forall i \in V(G)$. If $G$ is infinite, i.e. $V(G)$ is infinite, $L_{ij}$ can be considered as the representation of the Laplacian operator $L$ acting on $\ell_2(V(G))$. Under the previous conditions for $G$, it can be shown that $L$ is symmetric, non negative and bounded \cite{1}. A relevant property of $L$ we will exploit in the following is the Schwinger-Dyson identity \cite{7}. Let us consider a diagonal bounded operator $\eta_{ij} \equiv \eta_i \delta_{ij}$ with $\eta_i \in \mathbb{C}$ such that $(L + \eta)_{ij}^{-1}$ exists. Then:

$$\sum_{i \in V(G)} \eta_i (L + \eta)_{ij}^{-1} = 1.$$  

(3)

This identity follows from the property $\sum_i L_{ij} = 0$.

The graph $G$ is naturally provided with an intrinsic metric defined by the chemical distance on $G$.

Definition 6 The chemical distance $r_{i,j} \in \mathbb{N}$ between two vertices $i$ and $j$ is the number of links in the shortest path connecting $i$ and $j$.

The intrinsic metric is the fundamental tool to define the thermodynamical limit on an infinite graph $G$, allowing us to introduce the generalized Van Hove spheres.

Definition 7 The Van Hove sphere $S_{i,r}$ of center $i$ and radius $r$ is the subset of $V(G)$, $S_{i,r} = \{j \in V(G) \mid r_{ij} \leq r\}$. We will define $N_{i,r}$ as the number of vertices in $S_{i,r}$.

The behaviour of $N_{i,r}$ as a function of $r$ characterizes the growth of $G$. In particular if

$$A_i r^c \leq N_{i,r} \leq B_i r^c$$

where $c$, $A_i$ and $B_i$ are positive constants $\forall i$.

(4)

$G$ is said to have polynomial growth and we define the connectivity exponent $c$ to be the inf of the set of all $c$ satisfying \cite{3}. Here we will consider graphs with polynomial growth. Indeed this condition is sufficient to guarantee the convergence of the restriction of a bounded operator $O$ defined on $G$ to $O$ itself for $r \to \infty$. 

2
**Definition 8** Let $G$ be an infinite graph satisfying all previous conditions. Given a complex function of the vertices $\phi : V(G) \to \mathbb{C}$ we define the thermodynamic limit average $\overline{\phi}$ of $\phi$ as:

$$\overline{\phi} = \lim_{r \to \infty} \frac{1}{N_{i,r}} \sum_{j \in S_{i,r}} \phi_j.$$  

$\overline{\phi}$ can be easily shown to be independent of $i$.

In this work we will use also on finite graphs the symbol $\overline{\phi}$ to indicate the average over all vertices. We will define the average on a subset $V_\lambda \subseteq V(G)$:

$$\chi_{V_\lambda} \phi = \lim_{r \to \infty} \frac{1}{N_{i,r}} \sum_{i \in S_{i,r}} \chi_{V_\lambda}(i) \phi_i.$$  

where $\chi_{V_\lambda}(i)$ is the characteristic function of $V_\lambda$. We define the measure $\mu(V_\lambda)$ of $V_\lambda$:

$$\mu(V_\lambda) = \chi_{V_\lambda}.$$  

More generally, the thermodynamic limit average of a $k$ variable function $\Phi : V((G))^k \to \mathbb{C}$ is defined by:

$$\overline{\Phi} = \lim_{r \to \infty} \frac{1}{N_{i,r}} \sum_{i_1, \ldots, i_k \in S_{i,r}} \Phi_{i_1, \ldots, i_k}.$$  

We define the thermodynamic limit average trace of an infinite matrix $B_{ij}$ as:

$$\text{Tr} B = \overline{\Phi}$$  

where $b_i = B_{ii}$, and its restriction to a subset $V_\lambda \subseteq V(G)$:

$$\text{Tr}_{V_\lambda} B = \chi_{V_\lambda} \overline{\Phi}$$

The large scale topology affecting the critical behavior of statistical models on graphs can be characterized by the properties of the trace of $(L + M)^{-1}$, where $M_{ij} = m \delta_{ij}$ is a real and positive diagonal matrix.

**Definition 9** An infinite graph is infrared finite if

$$\lim_{m \to 0^+} \text{Tr}(L + M)^{-1} < \infty.$$  

An infinite graph is infrared infinite if:

$$\lim_{m \to 0^+} \text{Tr}(L + M)^{-1} = \infty.$$  

It can be shown that if we substitute the constant matrix $M$ with $M'_{ij} = m b_i \delta_{ij}$ where $0 < \epsilon < b_i < K$ the infrared behavior for $m \to 0^+$ does not change.

In dealing with statistical models the adjacency matrix $A_{ij}$ is often generalized to the ferromagnetic interaction matrix $J_{ij}$:
Definition 10

\[ J_{ij} = J_{ji} = \begin{cases} J_{ij} & \text{with } 0 < \epsilon \leq J_{ij} \leq J \text{ if } A_{ij} = 1 \\ 0 & \text{if } A_{ij} = 0 \end{cases} \]  

In this case the Laplacian generalizes to:

\[ L_{ij} = z_i \delta_{ij} - J_{ij} \]

where \( z_i = \sum_j J_{ij} \).

The generalized Laplacian has the same properties of the Laplacian associated to \( A \): it is symmetric, positive and bounded, it satisfies the Schwinger-Dyson identity and finally it has the same infrared behaviour [1].

3 O(n) classical spin models on graphs

An important class of classical statistical spin models on graphs is defined by the Hamiltonian:

\[ H' = -\frac{1}{2} \sum_{i,j \in V(G)} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - \vec{h} \cdot \sum_{i \in V(G)} \vec{\sigma}_i \]

where \( J_{ij} \) is a ferromagnetic interaction matrix defined on the graph \( G \) satisfying conditions (13) and \( \vec{\sigma}_j \) are \( n \)-dimensional real unit vectors \( \vec{\sigma}_i \equiv (\sigma_1^i, \ldots, \sigma_n^i) \) defined on each vertex \( i \) and satisfying the constraints:

\[ \vec{\sigma}_i^2 = 1 \quad \forall i. \]  

For \( n = 1 \) \( H' \) defines the Ising model which is invariant under the discrete symmetry group \( \mathbb{Z}_2 \), while for \( n \geq 2 \) \( H' \) represents an \( O(n) \) model with continuous symmetry. Finally \( \vec{h} \equiv (h,0,\ldots,0) \) is an external magnetic field coupled to \( \vec{\sigma}_i \). In the following we will set \( h > 0 \).

Due to (10) \( H' \) is equivalent up to an additive constant to:

\[ H = \frac{1}{4} \sum_{ij} J_{ij} (\vec{\sigma}_i - \vec{\sigma}_j)^2 - \vec{h} \cdot \sum_i \vec{\sigma}_i = \frac{1}{2} \sum_{ij} L_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - \vec{h} \cdot \sum_i \vec{\sigma}_i \]

where the Laplacian operator defined in (14) has been introduced.

The Boltzmann measure \( \mu_{\beta,h}(\vec{\sigma}) \) is given by:

\[ d\mu_{\beta,h}(\vec{\sigma}) = \prod_i d\vec{\sigma}_i e^{-\beta H} \]

where \( \beta = 1/k_B T \), with \( \vec{\sigma}_i \) satisfying the constraints (16). The order parameter of the model is the average magnetization in the \( \vec{h} \) direction:

\[ M(\beta,h) = \frac{\int d\mu_{\beta,h}(\vec{\sigma}) \vec{\sigma}^T}{\int d\mu_{\beta,h}(\vec{\sigma})}. \]
On lattices, in the thermodynamical limit for \( n \geq 2 \) it is known that the limit for \( h \to 0 \) of the order parameter vanishes for \( T > 0 \) in the infrared infinite case (i.e. if the Euclidean dimension \( d \) is 2 or 1) [8]. In the infrared finite case \((d \geq 3)\) for low enough temperatures the limit is positive [5], implying the existence of phase transitions. On graphs we have \( \lim_{h \to 0} M(\beta, h) = 0 \) for every \( \beta \), if for each positive measure subset \( V_\lambda \subseteq V(G) \) we have [4]:

\[
\lim_{b \to 0^+} \mathrm{Tr}_{V_\lambda}(L + b)^{-1} = \infty .
\]  

(20)

For Ising models on lattices the limit of the order parameter vanishes for all temperatures only in one dimension. There are no analogous results for generic graphs.

In this work we will show that, for the classical spin models (15) on infrared finite graphs, below a certain temperature \( M(\beta, h) \) satisfies the bound:

\[
\lim_{h \to 0} M(\beta, h) > k > 0.
\]  

(21)

This result for \( O(n) \) models is not the exact inversion of [4]. Indeed one should prove that if there is a positive measure subset \( V_\lambda \), for which the limit in (20) is finite, then the bound (21) is satisfied. Here the proof is given only for the case where the limit in (20) is finite for \( V_\lambda \equiv G \). A complete inversion of [4] will be given in a following paper [3]. We remark that this is the first result for the Ising model on a generic graph.

4 The magnetization bound

In this section we will prove the magnetization bound (21) for the classical spin model (15) according to the following strategy:

(A) first of all we will prove some useful inequalities for the Laplacian on finite graphs;

(B) then we will introduce for the constraint (16) an integral representation allowing us to perform a Gaussian integral with respect to the spin variables \( \vec{\sigma}_i \);

(C) we will take the thermodynamic limit of the inequalities (A) and in the expressions for \( M \) and for the global constraint;

(D) exploiting a saddle point technique for large \( \beta \), we will apply the inequalities in (C) to \( M \) getting the bound (21), for infrared finite graphs.

Lemma 1 The following inequalities hold for finite graphs, with \( h_{ij} \equiv h \delta_{ij} \) and \( \alpha \equiv \alpha_i \delta_{ij} \) with \( \alpha_i \in \mathbb{R} \ \forall i \):

\[
1 \geq \mathrm{Re} \left( \frac{h}{N} \sum_{ij} (L + h + i\alpha)^{-1}_{ij} \right) \geq \mathrm{Re} \left( \frac{h^2}{N} \sum_{ij} (L + h + i\alpha)^{-2}_{ij} \right) .
\]  

(22)

\[
0 \leq \frac{1}{Nh} \sum_{ikj} \alpha_i (L + h - i\alpha)_{ik}^{-1}(L + h + i\alpha)_{kj}^{-1} \alpha_j \leq \mathrm{Re} \left( \frac{1}{Nh} \sum_{ij} \alpha_i (L + h + i\alpha)^{-1}_{ij} \alpha_j \right)
\]  

(23)
\[ 0 \leq \frac{1}{N} \text{ReTr}_{V_{\lambda}} \left( (L + h + i\alpha)^{-1} \right) \leq \frac{1}{N} \text{ReTr}_{V_{\lambda}} \left( (L + h)^{-1} \right) \] \[ \leq \frac{1}{N} \text{ReTr} \left( (L + h)^{-1} \right) \] (24)

\[ \frac{1}{N} |\text{ImTr}_{V_{\lambda}} (L + h + i\alpha)^{-1}| \leq \frac{1}{2N} \text{Tr}_{V_{\lambda}} (L + h)^{-1} \leq \frac{1}{2N} \text{Tr} (L + h)^{-1} \] (25)

where \( V_{\lambda} \) is a generic subset \( V(G) \) and

\[ \text{Tr}_{V_{\lambda}} B \equiv \sum_i B_{ii} \chi_{V_{\lambda}}(i) , \] (26)

\[ \frac{1}{N} \sum_i \alpha_i^2 \leq f(\alpha) \left( \frac{1}{N} \sum_{ikj} \alpha_i (L + h - i\alpha)_{ik}^{-1} (L + h + i\alpha)_{kj}^{-1} \alpha_j \right)^{1/2} \] (27)

where \( f(\alpha) \leq 1/N \sum_i (\alpha_i^2 (l_{\text{max}} + h)^2 + \alpha_i^4) \) and \( l_{\text{max}} \) is the maximum eigenvalue of \( L \).

The proof of these inequalities will be given at the end of this section. \( \square \)

As for (B), we prove the following:

**Lemma 2** On a finite graph the expression of the average magnetization \( [14] \) and of the constraint:

\[ 1 = \frac{\int d\vec{\sigma}^2 \text{d}\mu_{\beta,h}(\vec{\sigma})}{\int \text{d}\mu_{\beta,h}(\vec{\sigma})} \] (28)

can be written in the following way:

\[ M(\beta,h) = \frac{1}{Z} \int \text{d}\mu_{\beta,h}(\alpha) \frac{h}{N} \sum_{kj} (L + h + i\alpha)^{-1}_{kj} \] (29)

\[ 1 = \frac{1}{Z} \int \text{d}\mu_{\beta,h}(\alpha) \left( \frac{n}{\beta N} \text{Tr} (L + h + i\alpha)^{-1} + \frac{h^2}{N} \sum_{ij} (L + h + i\alpha)_{ij}^{-2} \right) \] (30)

where:

\[ \text{d}\mu_{\beta,h}(\alpha) \equiv \prod_{i \in G} d\alpha_i e^{iS_{\beta,h}} \] (31)

\[ iS_{\beta,h} \equiv \frac{n}{2} \text{Tr} (\ln(L + h + i\alpha)) + \frac{\beta}{2} \left( i \sum_i \alpha_i + h^2 \sum_{ij} (L + h + i\alpha)_{ij}^{-1} \right) \]

and

\[ Z \equiv \int \text{d}\mu_{\beta,h}(\alpha_i) \]

Let us write the constraints \( [16] \) using the complex integral representation of the delta function:

\[ \delta(x^2) = \int d\alpha \exp(-i\alpha x^2 / 2 - \epsilon x^2 / 2) \]
where $\epsilon$ is a real arbitrary constant. We will put $\epsilon = h\beta$. Substituting the expression for $\delta$ in (19) and (28) we obtain:

$$M(\beta, h) = \frac{\int d\vec{\alpha} \, d\sigma \, \exp \left( -\beta \left( \frac{1}{2} \sum_{ij} L_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - h \sum_i \sigma_i^1 \right) - \sum_i \left( \frac{i\alpha_i}{2} (\sigma_i^2 - 1) - \frac{h\beta}{2} \sigma_i^2 \right) \right)}{\int d\vec{\sigma} \, d\alpha \, \exp \left( -\beta \left( \frac{1}{2} \sum_{ij} L_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - h \sum_i \sigma_i^1 \right) - \sum_i \left( \frac{i\alpha_i}{2} (\sigma_i^2 - 1) - \frac{h\beta}{2} \sigma_i^2 \right) \right)}$$

and

$$1 = \frac{\int d\vec{\sigma} \, d\alpha \, \exp \left( -\beta \left( \frac{1}{2} \sum_{ij} L_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - h \sum_i \sigma_i^1 \right) - \sum_i \left( \frac{i\alpha_i}{2} (\sigma_i^2 - 1) - \frac{h\beta}{2} \sigma_i^2 \right) \right)}{\int d\vec{\sigma} \, d\alpha \, \exp \left( -\beta \left( \frac{1}{2} \sum_{ij} L_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j - h \sum_i \sigma_i^1 \right) - \sum_i \left( \frac{i\alpha_i}{2} (\sigma_i^2 - 1) - \frac{h\beta}{2} \sigma_i^2 \right) \right)}$$

where $d\alpha \equiv \prod_i d\alpha_i$ and $d\vec{\sigma} \equiv \prod_i d\vec{\sigma}_i$. Substituting in both integral $\alpha_i$ with $\beta\alpha_i$ and $\vec{\sigma}_i$ with $\vec{\sigma}_i/\sqrt{\beta}$, we can perform the Gaussian integral on the variables $\vec{\sigma}_i$, obtaining in this way (29) and (30). $\square$

Now we consider an infinite graph $G$ and using the thermodynamic limit, we will extend to $G$ the Lemmas (1) and (2). This is point (C) in our plan.

**Lemma 3** For an infinite graph the inequalities (22), (23), (24), (25), (26) become:

$$1 \geq \text{Re} \left( \frac{1}{(\alpha(L + h + i\alpha)^{-1})} \right) \geq \text{Re} \left( \frac{1}{(h^2(L + h + i\alpha)^{-2})} \right).$$

(32)

$$0 \leq \alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha \leq \text{Re} \left( \alpha(L + h + i\alpha)^{-1} \alpha \right)$$

(33)

$$0 \leq \text{Re} \left( \text{Tr}_{V_\lambda}(L + h + i\alpha)^{-1} \right) \leq \text{Re} \left( \text{Tr}_{V_\lambda}(L + h)^{-1} \right) \leq \text{Re} \left( \text{Tr}(L + h)^{-1} \right)$$

(34)

$$|\text{Im} \text{Tr}_{V_\lambda}(L + h + i\alpha)^{-1}| \leq \frac{1}{2} \text{Tr}_{V_\lambda}(L + h)^{-1} \leq \frac{1}{2} \text{Tr}(L + h)^{-1}$$

(35)

here $V_\lambda$ is a generic positive measure subset of $G$.

$$\alpha^2 \leq f(\alpha) \left( \alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha \right)^{1/2}$$

(36)

where $f(\alpha) \leq \left( \alpha^2(l_{\text{max}} + h)^2 + \alpha^4 \right)$.

These inequalities are easily obtained by applying the inequalities (32-36) to the restriction to the Van Hove spheres $S_{i,r}$ of $L$, $\alpha$ and $h$ and taking the thermodynamical limit by letting $r \to \infty$. $\square$

Now we will prove the main theorem showing the existence of the bound for the magnetization.
Theorem 1 In an infinite and infrared finite graph $G$, $\forall \epsilon > 0 \exists \beta > 0$ such that $\forall \beta \geq \beta$:

$$\lim_{h \to 0} M(\beta, h) \geq 1 - \epsilon$$

We consider, in the thermodynamic limit, the expressions (29) and (30) for $M(\beta, h)$ and for the global constraint, with the measure given by (31). At the end of this section we will prove the following lemma:

**Lemma 4** Consider a positive measure subset $V_\lambda$ of $V(G)$. If $G$ is infrared finite, the quantity

$$\chi_{V_\lambda} S'_{\beta,h}$$

where $S'_{\beta,h}(i) = \frac{\partial}{\partial \alpha_i} S_{\beta,h}$ is bounded for all $V_\lambda$ when $\beta \to \infty$ only if the $\alpha_i$ satisfy the condition:

$$\alpha^2 = 0.$$ (39)

This condition is true also in the limit $h \to 0$. □

We will call the set of values $\alpha_i \equiv \{\alpha_1, \alpha_2, \ldots\}$ satisfying (39) the stationary point of the statistical weight $d \mu_{\beta,h}(\alpha)$. Now we separate the space of integration variables $\alpha_i$ into two subsets $\Gamma_{\beta,h}(\alpha)$ and its complement $\Gamma_{\beta,h}(\alpha)$ such that $\Gamma_{\beta,h}(\alpha)$ is the region around the stationary point where the real part of the measure $d \mu_{\beta,h}(\alpha)$ is positive. We define:

$$\epsilon(\beta, h) \equiv \frac{1}{Z} \int_{\Gamma_{\beta,h}(\alpha)} d\text{Re}(\mu_{\beta,h}(\alpha))(L + h + i\alpha)^{-1} + \frac{1}{Z} \int d\text{Im}(\mu_{\beta,h}(\alpha))(L + h + i\alpha)^{-1}$$

$$\epsilon'(\beta, h) \equiv \frac{1}{Z} \int_{\Gamma_{\beta,h}(\alpha)} d\text{Re}(\mu_{\beta,h}(\alpha)) \left( \frac{n}{\beta} \text{Tr}(L + h + i\alpha)^{-1} + h^2(L + h + i\alpha)^{-2} \right)$$

$$+ \frac{1}{Z} \int d\text{Im}(\mu_{\beta,h}(\alpha)) \left( \frac{n}{\beta} \text{Tr}(L + h + i\alpha)^{-1} + h^2(L + h + i\alpha)^{-2} \right)$$

Lemma 4 and the saddle-point theorem imply that:

$$\forall \epsilon > 0 \exists \beta' \in \mathbf{R}$$ such that $\forall \beta \geq \beta'$: $\lim_{h \to 0} |\epsilon(\beta, h)| \leq \epsilon/3$ and $\lim_{h \to 0} |\epsilon'(\beta, h)| \leq \epsilon/3$. (41)

Now for $M(\beta, h)$ we have:

$$M(\beta, h) = \epsilon(\beta, h) + \frac{1}{Z} \int_{\Gamma_{\beta,h}(\alpha)} d\text{Re}(\mu_{\beta,h}(\alpha))(L + h + i\alpha)^{-1}$$

$$= \epsilon(\beta, h) + \frac{1}{Z} \int_{\Gamma_{\beta,h}(\alpha)} d\text{Re}(\mu_{\beta,h}(\alpha)) \text{Re} \left( h(L + h + i\alpha)^{-1} \right)$$

The imaginary part of the second term in (42) does not provide any contribution to the integral because $M(\beta, h)$ is a real quantity. In analogous way for the constraint (31) we get:

$$1 = \epsilon'(\beta, h) + \frac{1}{Z} \int_{\Gamma_{\beta,h}(\alpha)} d\text{Re}(\mu_{\beta,h}(\alpha)) \text{Re} \left( \frac{n}{\beta} \text{Tr}(L + h + i\alpha)^{-1} + h^2(L + h + i\alpha)^{-2} \right)$$
Now in (43) we deal with a real quantity averaged with respect to a positive measure and we can use the inequalities of Lemma 3. In particular applying (32) in (43) we get:

\[ M(\beta, h) \geq \epsilon(\beta, h) + \frac{1}{Z} \int_{\Gamma_{\beta, h}(\alpha)} dRe(\mu_{\beta, h}(\alpha)) \ Re \left( h^2 (L + h + i\alpha)^{-2} \right) \]

With this inequality and the expression for the global constraint (44) we get:

\[ M(\beta, h) \geq 1 - \epsilon'(\beta, h) + \epsilon(\beta, h) - \int_{\Gamma_{\beta, h}(\alpha)} d\alpha \ Re(\mu_{\beta, h}(\alpha)) \frac{n}{\beta} \ Re \left( \text{Tr}(L + h + i\alpha)^{-1} \right). \]

(45)

Now we take the limit \( h \to 0 \) and we study the terms of this inequalities for large \( \beta \). For \( \epsilon(\beta, h) \) and \( \epsilon'(\beta, h) \) the behaviour is given by (41). For the integral term in (45) we can exploit the inequality (34) getting:

\[ 0 \leq \int_{\Gamma_{\beta, h}(\alpha)} d\alpha \ Re(\mu_{\beta, h}(\alpha)) \frac{n}{\beta} \ Re \left( \text{Tr}(L + h + i\alpha)^{-1} \right) \leq \frac{n}{\beta} \text{Tr}(L + h)^{-1} \int_{\Gamma_{\beta, h}(\alpha)} d\alpha \ Re(\mu_{\beta, h}(\alpha)). \]

The saddle-point theorem implies that:

\[ \exists \beta'' \in \mathbb{R} \text{ such that } \forall \beta \geq \beta'' : \lim_{h \to 0} \frac{1}{Z} \int_{\Gamma_{\beta, h}(\alpha)} dRe(\mu_{\beta, h}(\alpha)) \geq 1/2. \]

(46)

If for each \( \epsilon \) we choose

\[ \beta \geq \beta'' \text{ and } \beta \geq \frac{3}{\epsilon} \lim_{h \to 0} \text{Tr}(L + h)^{-1} = \beta''' \]

(47)

we get:

\[ \lim_{h \to 0} \frac{n}{\beta Z} \int_{\Gamma_{\beta, h}(\alpha)} dRe(\mu_{\beta, h}(\alpha)) \text{Re} \left( \text{Tr}(L + h + i\alpha)^{-1} \right) \leq \epsilon/3. \]

(48)

Finally fixed \( 0 \leq \epsilon \leq 1 \), if we choose \( \tilde{\beta} = \max(\beta', \beta'', \beta''') \), we have that \( \forall \beta > \tilde{\beta} \) \( M(\beta, h) \) satisfies condition (37). Note that from (17) we have that \( \beta''' \) and then \( \tilde{\beta} \) are bounded only for infrared finite graphs. ∎

### 4.1 Inequalities

Now we will prove the inequalities (22), (23), (24), (25) and (27) for finite graphs. On a finite graph the set of all real functions of the vertices \( \phi_i \) has a vector space structure. Let us define \( \langle \phi \rangle = (\phi_1, \ldots, \phi_N) \) and \( |\phi\rangle = (\phi_1, \ldots, \phi_N)^t \). The scalar product \( \langle \phi | \psi \rangle \) is defined by:

\[ \langle \phi | \psi \rangle = \sum_i \phi_i \psi_i. \]

(49)

The matrices \( L, A, h \) and \( \alpha \) are operators on this space. The matrix \( L \) is diagonalizable by a real transformation, and its eigenvalues \( l \) satisfy \( 0 \leq l \leq l_{\text{max}} \). Let us prove the following:

**Lemma 5** On a finite graph, it is possible to introduce a real operator \( B \) defined by:

\[ B^l (L + h) B = I \quad B^l \alpha B = c \]

(50)
where $c$ is a real diagonal operator and $I$ is the identity operator. Furthermore $B$ satisfies the properties:

$$(L + h + i\alpha)^{-1} = B(1 + ic)^{-1}B^t \quad (L + h)^{-1} = BB^t$$ (51)

where

$$\|B^tB\| = \frac{1}{h}$$ (52)

The existence of $B$ is proved in [6] where it is also shown that is real. Properties (51) can be immediately obtained by (50). Therefore it is easy to get the exact expression for $B$:

$$B = TAT^\prime$$ (53)

where $T$ is the orthogonal transformation that diagonalize $L$; $A$ is the matrix $A_{km} = (1/\sqrt{l_k + h})\delta_{km}$, where $l_k$ is the eigenvalue of $L$ relative to the eigenvector $k$; finally $T^\prime$ is the orthogonal operator that diagonalize the symmetric matrix $AT^t\alpha TA$. $B$ is not an orthogonal transformation but we compute its norm:

$$\|B^tB\| = \sup_{\phi} \frac{\langle \phi | B^tB | \phi \rangle}{\langle \phi | \phi \rangle} = \sup_{\phi} \frac{\langle \phi | A^2 | \phi \rangle}{\langle \phi | \phi \rangle} = \frac{1}{h}$$

This proves property (52). $\Box$

Proof of Lemma 1 Let $\langle \phi \rangle$ be a generic vector of the space of the functions of vertices. We have:

$$\text{Re} \langle \phi | h^2(L + h + i\alpha)^{-2} | \phi \rangle = \text{Re} h^2 \langle \phi | B(1 + ic)^{-1}B^tB(1 + ic)^{-1}B^t | \phi \rangle$$

$$= h^2 \langle \phi | B(\text{Re}(1 + ic)^{-1})B^tB(\text{Re}(1 + ic)^{-1})B^t | \phi \rangle$$

$$- h^2 \langle \phi | B(\text{Im}(1 + ic)^{-1})B^tB(\text{Im}(1 + ic)^{-1})B^t | \phi \rangle$$

$$\leq h\langle \phi | B(\text{Re}(1 + ic)^{-1})B^tB(\text{Re}(1 + ic)^{-1})B^t | \phi \rangle$$

$$\leq h\langle \phi | B(1 + c^2)^{-2}B^t | \phi \rangle$$

$$\leq h\langle \phi | B(1 + c^2)^{-1}B^t | \phi \rangle$$

where we used properties (51) and (52). Then we have:

$$\text{Re} \langle \phi | h(L + h + i\alpha)^{-1} | \phi \rangle = h\langle \phi | B(\text{Re}(1 + ic)^{-1})B^t | \phi \rangle$$

$$= h\langle \phi | B(1 + c^2)^{-1}B^t | \phi \rangle$$

$$\leq h\langle \phi | BB^t | \phi \rangle \leq \langle \phi | \phi \rangle$$

so we get:

$$\text{Re} \langle \phi | h^2(L + h + i\alpha)^{-2} | \phi \rangle \leq \text{Re} \langle \phi | h(L + h + i\alpha)^{-1} | \phi \rangle \leq \langle \phi | \phi \rangle.$$ (54)
Now choosing in (54) \( \langle \phi \rangle \) to be the constant vector \( \alpha_i = 1 \), we prove (22).

To prove the inequality (23) we consider the following expression:

\[
0 \leq \langle \phi \rangle (L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} |\phi\rangle = \langle \phi \rangle B (1 - i\alpha)^{-1} B^t (1 + i\alpha)^{-1} B^t |\phi\rangle \\
\leq \frac{1}{h} \langle \phi \rangle B (1 - i\alpha)^{-1} (1 + i\alpha)^{-1} B^t |\phi\rangle \\
\leq \frac{1}{h} \langle \phi \rangle B (1 + c^2)^{-1} B^t |\phi\rangle \\
\leq \frac{1}{h} \text{Re} \langle \phi \rangle B (1 + i\alpha)^{-1} B^t |\phi\rangle \\
\leq \frac{1}{h} \text{Re} \langle \phi \rangle (L + h + i\alpha)^{-1} |\phi\rangle.
\]

(55)

where we used (51) and (52). Putting \( \phi_i = \alpha_i \) in (53), we get (23).

To get inequalities (24) and (25) we introduce the base \( \langle k \rangle \), \( k = 1, \ldots, N \) in the vector space of the functions of the sites given by:

\[
\langle k \rangle = \left\{ \begin{array}{ll}
    k_i = 1 & \text{if } i = k \\
    0 & \text{in all others vertices.}
\end{array} \right.
\]

We have:

\[
(L + h + \alpha)^{-1}_{kk} = \langle k \rangle (L + h + \alpha)^{-1} |k\rangle = \langle k \rangle B (1 + i\alpha)^{-1} B^t |k\rangle
\]

(56)

For the real part of (56), with properties (51) we get:

\[
0 \leq \text{Re} (L + h + \alpha)^{-1}_{kk} = \langle k \rangle |B (1 + c^2)^{-1} B^t |k\rangle \leq \langle k \rangle B B^t |k\rangle = (L + h)^{-1}_{kk}
\]

If we sum over all \( k \in V_\lambda \) we get:

\[
0 \leq \text{Re} \text{Tr}_{V_\lambda} (L + h + \alpha)^{-1} \leq \text{Tr}_{V_\lambda} (L + h)^{-1} \leq \text{Tr} (L + h)^{-1}
\]

This proves (24). Now for the imaginary part of (56) we have:

\[
|\text{Im} (L + h + \alpha)^{-1}_{kk}| = |\langle k \rangle |B c (1 + c^2)^{-1} B^t |k\rangle| \leq |\langle k \rangle |B c (1 + c^2)^{-1} B^t |k\rangle| \leq \frac{1}{2} \langle k \rangle |B B^t |k\rangle = \frac{1}{2} (L + h)^{-1}_{kk}
\]

If we sum over all \( k \in V_\lambda \) we get:

\[
|\text{Im} \text{Tr}_{V_\lambda} (L + h + \alpha)^{-1}| \leq \sum_{i \in V_\lambda} |\text{Im} (L + h + \alpha)^{-1}_{kk}| \frac{1}{2} \text{Tr}_{V_\lambda} (L + h)^{-1} \leq \frac{1}{2} \text{Tr} (L + h)^{-1}
\]

In this way we proved inequality (25).

For (27) we have:

\[
\sum_i \alpha_i^2 = \langle \alpha | \alpha \rangle = \langle \alpha | (L + h + i\alpha)(L + h + i\alpha)^{-1} |\alpha\rangle \leq
\]

\[
\leq \langle \alpha | (L + h + i\alpha)(L + h - i\alpha)|\alpha\rangle^{1/2} \langle \alpha | (L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} |\alpha\rangle^{1/2}
\]

\[
\leq f(\alpha) \left( \sum_{i \neq j} \alpha_i (L + h - i\alpha)^{-1}_{ik} (L + h + i\alpha)^{-1}_{kj} \alpha_j \right)^{1/2} = 0
\]

where \( f(\alpha) = \langle \alpha | (L + h)^2 + \alpha^2 |\alpha\rangle^{1/2} \leq \sum_i (\alpha_i^2 (l_{\max} + h^2 + \alpha_i^2))^{1/2} \) and this completes the proof of Lemma 1. □
4.2 The saddle-point condition

Proof of Lemma 4. Let us compute the quantity (38). We have:

\[
\chi V \lambda S'_{\beta,h} = -\frac{n}{2} \text{Tr} V \lambda (L + h + i\alpha)^{-1} + \frac{\beta}{2} \left( \mu(V) - h^2(L + h + i\alpha)^{-1} \chi(V)(L + h + i\alpha)^{-1} \right)
\]

where \( \chi(V)_{ij} = \chi \delta_{ij} \). The first term in this expression is always bounded, also when \( h \to 0 \) for infrared finite graphs; inequalities (24) and (25) can be used for the proof. Then for \( \beta \to \infty \) (38) is bounded only when the condition:

\[
\mu(V) = h^2(L + h + i\alpha)^{-1} \chi(V)(L + h + i\alpha)^{-1}
\]

is satisfied. Let us consider the particular case where we put in (57) \( V = G \). We get

\[
1 = h^2(L + h + i\alpha)^{-2}
\]

Let us show that (38) is satisfied only when (39) holds. First of all we have that (58) implies:

\[
\text{Re} \left( h(L + h + i\alpha)^{-1} \right) = 1.
\]

Indeed with the inequality (32) we get:

\[
1 = \text{Re} \left( h^2(L + h + i\alpha)^{-2} \right) \leq \text{Re} \left( h(L + h + i\alpha)^{-1} \right) \leq 1.
\]

Now using the Schwinger-Dyson identity (3) in (59) we have:

\[
1 = \text{Re} \left( h(L + h + i\alpha)^{-1} \right) = 1 + \text{Re} \left( (L + h + i\alpha)^{-1}(-i\alpha) \right)
\]

and then:

\[
0 = \text{Re} \left( (L + h + i\alpha)^{-1}(-i\alpha) \right) = \text{Re} \left( \frac{\pi}{h} + \frac{1}{h}(-i\alpha)(L + h + i\alpha)^{-1}(-i\alpha) \right)
\]

\[
= \text{Re} \left( \frac{1}{h} \alpha(L + h + i\alpha)^{-1} \alpha \right).
\]

where in the first step we used again the Schwinger-Dyson identity. Now applying the inequality (33) to (60) we get:

\[
0 \leq \alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha \leq \text{Re} \left( \frac{1}{h} \alpha_i(L + h + i\alpha)^{-1} \alpha \right)
\]

\[
\frac{\alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha}{\alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha} = 0
\]

Let us consider the inequality (34). We have:

\[
\alpha^2 \leq f(\alpha) \left( \frac{\alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha}{\alpha(L + h - i\alpha)^{-1}(L + h + i\alpha)^{-1} \alpha} \right)^{1/2} = 0.
\]
Notice that $f(\alpha)$ is bounded if $\alpha^4$ is bounded. So we proved that the condition (39) for the variables $\alpha_i$ must be satisfied, if the quantity (38) is bounded when $\beta \to \infty$. This proof also holds when $h \to 0$. Indeed in this case the stationary condition (58) can be expressed by (61) and the inequality (36) holds.

Now in order to complete our proof we must verify that also for a generic $V_{\lambda}$ (38) is bounded at the stationary point when $\beta \to \infty$. In this way we will show that (39) is not only a necessary condition but also a sufficient one. If we evaluate (38) with the constraint (39) we get:

$$i\left(-\frac{n}{2} \text{Tr}_{V_{\lambda}}(L + h)^{-1} + \frac{\beta}{2} \left(\mu(V_{\lambda}) - h^2(L + h)^{-1}\chi(V_{\lambda})(L + h)^{-1}\right)\right)$$

$$= i\left(-\frac{n}{2} \text{Tr}_{V_{\lambda}}(L + h)^{-1} + \frac{\beta}{2} \left(\mu(V_{\lambda}) - \chi(V_{\lambda})\right)\right)$$

$$= -i \frac{n}{2} \text{Tr}_{V_{\lambda}}(L + h)^{-1}$$

where we used again the Schwinger-Dyson identity. For infrared finite graphs the last expression is always bounded (also when $h \to 0$), see (11). This completes our proof. \(\blacksquare\).

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13