Maximal $q$-Subharmonicity in $\mathbb{C}^n$

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Abstract In this paper, we study maximal $q$-subharmonic functions in $\mathbb{C}^n$. We prove that maximality of $q$-subharmonic functions is a local notion and give a condition to check the maximality of $C^2$ $q$-subharmonic functions.

Keywords $q$-Subharmonic functions · Maximal $q$-subharmonic functions

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1 Introduction

Let $\Omega$ be an open set in $\mathbb{C}^n$. An upper semi-continuous function $u : \Omega \to [-\infty, +\infty)$ is called plurisubharmonic on $\Omega$ if for every complex line $\ell$ of $\mathbb{C}^n$, $u|_{\ell \cap \Omega}$ is a subharmonic function on $\ell \cap \Omega$.

The set of plurisubharmonic functions on $\Omega$ is denoted by $PSH(\Omega)$.

Now as in [2] and [7], there is a class of plurisubharmonic functions playing an important role in pluripotential theory. This is a class of maximal plurisubharmonic functions. We recall the following definition given in [2].

Definition 1 A plurisubharmonic function $u$ on $\Omega$ is called maximal plurisubharmonic (briefly, $u \in MPSH(\Omega)$) if for every $v \in PSH(\Omega)$, $v \leq u$ outside a compact subset of $\Omega$ implies $v \leq u$ on $\Omega$.

As in [2] and [7], a locally bounded plurisubharmonic function $u$ is in $MPSH(\Omega)$ if and only if it satisfies the homogeneous complex Monge–Ampère equation $(dd^c u)^n = 0$. 

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Moreover, a recent result in [3] claimed that maximality is a local notion for locally bounded plurisubharmonic functions (see Corollary 1 in [3]). Also in [3], Blocki extended the above result for the class $E(\Omega)$ introduced and investigated by Cegrell in [4] recently.

The aim of this paper is to extend the class of maximal plurisubharmonic functions to the class of maximal $q$-subharmonic functions, where $q$ is an integer with $1 \leq q \leq n$ and to give some results about maximal $q$-subharmonic functions. Notice that for studying maximality of plurisubharmonic functions one often approaches by using the complex Monge–Ampère operator. But defining the complex Monge–Ampère operator for $q$-subharmonic functions is impossible. Hence one needs to find another approach for studying maximal $q$-subharmonic functions. In this paper, by using a new method, we prove that the maximality of $q$-subharmonic functions is equivalent to a local notion. We also provide a condition to check when a $C^2 q$-subharmonic function $u$ is maximal $q$-subharmonic.

The note is organized as follows. Besides the introduction, the note has two sections. Section 2 is devoted to study $q$-subharmonic functions and to establish some results concerning this class. Section 3 deals with maximal $q$-subharmonic functions and proves the local property of this class.

2 $q$-Subharmonic Functions in $\mathbb{C}^n$

First, we recall the following definition of $q$-subharmonic functions which has been introduced by H. Ahn and N.Q. Dieu in [1] (also see [5]).

Definition. Let $\Omega$ be an open set in $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$, $u \neq -\infty$ is called $q$-subharmonic if for every $q$-dimensional complex plane $L$ in $\mathbb{C}^n$, $u|_L$ is a subharmonic function on $L \cap \Omega$. This means that for every compact subset $K \subseteq L \cap \Omega$ and every continuous harmonic function $h$ on $K$ such that $u \leq h$ on $\partial K$ it follows that $u \leq h$ on $K$.

The set of $q$-subharmonic functions on $\Omega$ is denoted by $SH_q(\Omega)$.

Compared with subharmonic and plurisubharmonic functions in potential theory and pluripotential theory, it is easy to see that 1-subharmonic functions are plurisubharmonic and $n$-subharmonic functions are subharmonic.

The following basic properties of $q$-subharmonic functions can be proved in the same way as for subharmonic functions.

Proposition 1. Let $\Omega$ be an open set in $\mathbb{C}^n$ and $1 \leq q \leq n$. Then the following hold:

1. $SH_q(\Omega)$ is a convex cone.
2. If $\{u_\alpha\}$, $\alpha \in A$ is a family of $q$-subharmonic functions and $u = \sup_{\alpha \in A} u_\alpha < +\infty$, $u$ is upper semi-continuous then $u$ is a $q$-subharmonic function.
3. If $\{u_j\}_{j=1}^\infty$ is a decreasing sequence of $q$-subharmonic functions then so is $u = \lim_{j \rightarrow +\infty} u_j$.
4. If $u$ is a $q$-subharmonic function in $\Omega$ then $u_\varepsilon := u \ast \varrho_\varepsilon$ is smooth $q$-subharmonic in $\Omega_\varepsilon$, where $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial \Omega) > \varepsilon\}$ and $\varrho_\varepsilon = \varrho(z/\varepsilon)/|\varepsilon|^{2n}$, $\varrho$ is a nonnegative smooth function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \varrho \, dV_n = 1$. Moreover, $u \ast \varrho_\varepsilon$ decreasingly tends to $u$ when $\varepsilon \downarrow 0$.
5. If $\chi$ is a convex increasing function in $\mathbb{R}$ and $u$ is $q$-subharmonic in $\Omega$ then so is $\chi \circ u$.
6. If $u$ is a $q$-subharmonic function then for any unitary change of coordinates $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$, the function $u \circ \varphi \in SH_q(\Omega)$. 

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Let \( u \) be a \( q \)-subharmonic function in \( \Omega \) and \( v \in SH_q(G) \) satisfying \( \lim_{G \ni \xi \to z \in \partial G} v(\xi) \leq u(z) \) for all \( z \in \partial G \). Then the function

\[
w(z) = \begin{cases} \max(u(z), v(z)), & z \in G, \\ u(z), & z \in \Omega \setminus G \end{cases}
\]
is \( q \)-subharmonic on \( \Omega \).

Now we give the following.

**Proposition 2** Let \( u \) be an upper-semicontinuous function on \( \Omega \subset \mathbb{C}^n \) and \( u \in L^1(\Omega, \text{loc}) \), where \( L^1(\Omega, \text{loc}) \) denotes the set of locally integrable functions on \( \Omega \). Then the following statements are equivalent:

(a) \( u \) is a \( q \)-subharmonic function in \( \Omega \).

(b) \( i \partial u \wedge \omega^{q-1} \geq 0 \) in the sense of currents, where \( \omega := i \partial \bar{\partial} \mid z \mid^2 \).

In particular, if \( u \in C^2(\Omega) \) then \( u \in SH_q(\Omega) \) if and only if its complex Hessian has the sum of \( q \) smallest eigenvalues nonnegative at each point.

**Proof** First, we assume that \( u \in C^2(\Omega) \). Let \( L \subset \mathbb{C}^n \) be a \( q \)-dimensional complex plane of \( \mathbb{C}^n \) with \( z_0 \in L \). Then by the hypothesis \( u \vert_{\Omega \cap L} \) is subharmonic on \( \Omega \cap L \) and so \( \sum_{k \in K} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0) \geq 0 \) for all \( K = \{k_1, k_2, \ldots, k_q\} \subset \{1, 2, \ldots, n\} \). It follows that \( i \partial \bar{\partial} u(z_0) \wedge \omega^{q-1} \geq 0 \). Hence, \( i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0 \) for all \( z \in \Omega \).

Conversely, assume that \( i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0 \) for all \( z \in \Omega \). Let \( L \) be a \( q \)-dimensional subspace of \( \mathbb{C}^n \). Since \( i \partial \bar{\partial} u(z) \wedge \omega^{q-1} \geq 0 \), \( z \in \Omega \) it follows that \( u \in SH(\Omega \cap L) \). Hence, \( u \in SH_q(\Omega) \). Thus the conclusion is true in the case \( u \in C^2(\Omega) \).

Assume that \( u \) is as in the statement of the proposition. By putting \( u_\varepsilon = u \ast \varrho_\varepsilon \) and applying the above results to \( u_\varepsilon \), we obtain the desired conclusion.

**Example 1** We give an example of a \( q \)-subharmonic function which is not plurisubharmonic. Let \( d > 1 \) and \( 1 < q \leq d \). Consider the function

\[
\varphi(z) = |z|^2 - q |z_1|^2 = \sum_{j=1}^d |z_j|^2 - q |z_1|^2, \quad z \in \mathbb{C}^d.
\]

It is easy to see that \( \sum_{j=1}^q \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j}(z) = 0 \) and by (b) of Proposition 2 it follows that \( \varphi \) is \( q \)-subharmonic. However, \( \varphi \) is not plurisubharmonic. Indeed, let \( \ell = \{(z_1, 0, \ldots, 0)\} \subset \mathbb{C}^d \) be a complex line. Then \( \varphi \vert_\ell = (1 - q) |z_1|^2 \) is not subharmonic, and the desired conclusion follows.

### 3 Maximal \( q \)-Subharmonic Functions

The following definition is similar as in the situation of maximal plurisubharmonic functions presented in [2] and [7].
Definition 3 A function $u \in SH_q(\Omega)$ is said to be maximal $q$-subharmonic if for every $v \in SH_q(\Omega)$, $v \leq u$ outside a compact subset of $\Omega$ implies that $v \leq u$ in $\Omega$.

The set of all maximal $q$-subharmonic functions in $\Omega$ is denoted by $MSH_q(\Omega)$.

We give the following.

Proposition 3 Let $\Omega$ be an open subset in $\mathbb{C}^n$ and let $u \in SH_q(\Omega) \cap L_{loc}^\infty(\Omega)$. Then the following conditions are equivalent:

(a) $u \in MSH_q(\Omega)$.
(b) $u + g \in MSH_q(\Omega)$ for all pluriharmonic functions $g$ in $\Omega$.
(c) For every open subset $G \Subset \Omega$ and every $v \in SH_q(G)$ such that $\limsup_{G \ni w \to z \in \partial G}(v - u)(w) \leq 0$ it follows that $v \leq u$ in $G$.
(d) For every open subset $G \Subset \Omega$ and every $v \in SH_q(\Omega)$ we have

$$\sup_{G}(v - u) \leq \sup_{\Omega \setminus G}(v - u).$$

Proof From Definition 3, we infer that (a) $\iff$ (b) $\iff$ (c) is obvious. We prove that (a) $\iff$ (d).

Assume that $u \in MSH_q(\Omega)$. Let $G \Subset \Omega$ and $v \in SH_q(\Omega)$. Put $M = \sup_{\Omega \setminus G}(v - u)$. If $M = +\infty$ then (1) holds. Now assume that $M < +\infty$. Then $v - M \in SH_q(\Omega)$ and $v - M \leq u$ on $\Omega \setminus G$. Hence, by (a), we get $v - M \leq u$ on $\Omega$ and (1) follows.

Conversely, let $v \in SH_q(\Omega)$ and $G \Subset \Omega$ such that $v \leq u$ on $\Omega \setminus G$. Take an open subset $G_1$ such that $G \Subset G_1 \Subset \Omega$. We have $v \in SH_q(G_1)$. By (d), we have

$$\sup_{G}(v - u) \leq \sup_{G_1 \setminus G}(v - u) \leq 0,$$

and the desired conclusion follows. \hfill \square

Now we are in a position to prove the local property of maximal $q$-subharmonic functions. Namely, we have the following.

Theorem 1 Let $\Omega \subset \mathbb{C}^n$ be an open set, $q$ be an integer with $1 \leq q \leq n$ and $u \in SH_q(\Omega) \cap L_{loc}^\infty(\Omega)$. Then $u$ is maximal $q$-subharmonic if and only if $u$ is local maximal $q$-subharmonic in $\Omega$ (i.e., for every $z \in \Omega$ there is an open neighborhood $V_z \subset \Omega$ of $z$ such that $u|_{V_z}$ is maximal $q$-subharmonic on $V_z$).

Proof The proof of the necessity is obvious. Now we give the proof of the sufficiency. Assume that $G \Subset \Omega$ and $v$ is a $q$-subharmonic function on $\Omega$ such that $v \leq u$ on $\Omega \setminus G$.

We have to prove that $v \leq u$ on $G$. Choose $z^j \in \Omega$, $j = 1, \ldots, m$ and open subsets $K_j \Subset V_{z^j} \Subset \Omega$, $j = 1, \ldots, m$ such that $z^j \in V_{z^j}$, $G \subset \bigcup_{j=1}^m K_{z^j}$ and $u$ is maximal $q$-subharmonic on $V_{z^j}$ for all $j = 1, \ldots, m$.

We split the proof into two steps.

Step 1. We prove that if $K_j$, $j = 1, \ldots, m$ are open subsets in $\Omega$ and $\overline{K}_j$, $j = 1, \ldots, m$ are relatively compact open subsets such that $K_j \Subset \overline{K}_j \Subset V_j \Subset \Omega$, $j = 1, \ldots, m$ then the following inequality holds

$$\sup_{\bigcup_{j=1}^m K_j}(v - u) \leq \sup_{\bigcup_{j=1}^m \overline{K}_j}(v - u) \leq \sup_{\bigcup_{j=1}^m V_j \setminus \bigcup_{j=1}^m K_j}(v - u).$$

\hfill \square
It suffices to prove that (2) holds for $m = 2$. Let $V = V_1 \cup V_2$. To get a contradiction, without loss of generality we may assume that

$$\sup_{V \setminus K_1 \cup K_2} (v - u) < 0 < \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u) < +\infty.$$  

For $\varepsilon > 0$, put $u_{(\varepsilon)}(z) = u(z) - \varepsilon |z|^2$, $z \in V$ and choose $\varepsilon$ sufficiently small such that

$$\sup_{V \setminus K_1 \cup K_2} (v - u_{(\varepsilon)}) < 0 < \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) < +\infty.$$  

Then there exists a sequence $\{p_j\} \subset \tilde{K}_1 \cup \tilde{K}_2$, $p_j \rightarrow p \in \tilde{K}_1 \cup \tilde{K}_2$ such that

$$\sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) = \lim_{j \rightarrow \infty} (v - u_{(\varepsilon)})(p_j).$$  

Now we consider the following two cases.

Case 1. $p \notin \tilde{K}_1 \cup \tilde{K}_2$. Then $p \in (V \setminus \tilde{K}_1) \cap (V \setminus \tilde{K}_2)$. Hence there exists $j_0$ such that $p_j \in (V \setminus \tilde{K}_1) \cap (V \setminus \tilde{K}_2)$ for every $j \geq j_0$. It follows that

$$0 < \sup_{\tilde{K}_1 \cup \tilde{K}_2} (v - u_{(\varepsilon)}) = \lim_{j \rightarrow \infty} (v - u_{(\varepsilon)})(p_j) \leq \sup_{(V \setminus \tilde{K}_1) \cap (V \setminus \tilde{K}_2)} (v - u_{(\varepsilon)})$$

$$\leq \sup_{(V \setminus \tilde{K}_1) \cap (V \setminus \tilde{K}_2)} (v - u_{(\varepsilon)}) = \sup_{V \setminus (K_1 \cup K_2)} (v - u_{(\varepsilon)}) < 0,$$

and we get a contradiction.

Case 2. $p \in \tilde{K}_1 \cup \tilde{K}_2$. We may assume that $p \in \tilde{K}_1 \subseteq \tilde{K}_2$. We choose balls $\tilde{B} = B(p, \frac{\varepsilon}{4}) \subset B_0 = B(p, r) \subseteq B_1 = B(p, r_1) \subseteq B_2 = B(p, r_2) \subseteq \tilde{K}_1$. Moreover, we may assume that $p_j \in \tilde{B}$ for all $j \geq 1$. For all $j$ we can write

$$u_{(\varepsilon)}(z) = u(z) - \varepsilon |z|^2 = u(z) + \varepsilon |p_j|^2 - 2\varepsilon \Re \langle z, p_j \rangle - \varepsilon |z - p_j|^2.$$  

Note that since $\Re \langle z, p_j \rangle$ is plurisubharmonic and $u$ is a maximal $q$-subharmonic function in $V_1$, hence, by (b) of Proposition 3, $u_{(\varepsilon)}(z) + \varepsilon |z - p_j|^2 = u(z) + \varepsilon |p_j|^2 - 2\varepsilon \Re \langle z, p_j \rangle$ is maximal $q$-subharmonic in $V_1$. From (d) of Proposition 3 we get

$$(v - u_{(\varepsilon)})(p_j) \leq \sup_{z \in B_2 \setminus \overline{B}_1} (v(z) - u_{(\varepsilon)}(z) - \varepsilon |z - p_j|^2).$$

On the other hand, because $z \in B_2 \setminus \overline{B}_1$, it follows that $|z - p_j|^2 \geq \frac{r^2}{4}$ for all $j \geq 1$. Hence

$$-u_{(\varepsilon)}(z) - \varepsilon |z - p_j|^2 \leq -u_{(\varepsilon)}(z) - \varepsilon \frac{r^2}{4}$$

for all $z \in B_2 \setminus \overline{B}_1$ and for all $j \geq 1$. It follows that

$$(v - u_{(\varepsilon)})(p_j) \leq \sup_{z \in B_2 \setminus \overline{B}_1} (v(z) - u_{(\varepsilon)}(z) - \varepsilon |z - p_j|^2) \leq \sup_{z \in B_2 \setminus \overline{B}_1} (v - u_{(\varepsilon)})(z) - \varepsilon \frac{r^2}{4}$$

$$\leq \sup_{K_1} (v - u_{(\varepsilon)}) - \varepsilon \frac{r^2}{4} \leq \sup_{K_1 \cup K_2} (v - u_{(\varepsilon)}) - \varepsilon \frac{r^2}{4}.$$  

Letting $j \rightarrow \infty$, we infer that

$$\sup_{K_1 \cup K_2} (v - u_{(\varepsilon)}) \leq \sup_{K_1 \cup K_2} (v - u_{(\varepsilon)}) - \varepsilon \frac{r^2}{4},$$
and we get a contradiction. Hence, (2) is proved.

\textit{Step 2.} Let $x_0 \in G$. We need to prove

\begin{equation}
 v(x_0) \leq u(x_0).
\end{equation}

By Step 1, we get

\begin{align*}
 (v - u)(x_0) &\leq \sup_{G}(v - u) \\
 &\leq \sup_{\bigcup_{j=1}^{m} K_j}(v - u) \\
 &\leq \sup_{\Omega \setminus G}(v - u) \\
 &\leq 0.
\end{align*}

This shows that (3) is true and the proof is complete. \hfill \Box

From the above theorem we get the following useful corollary.

\textbf{Corollary 1} Assume that $u \in C(\Omega)$. Then $u \in MSH_q(\Omega)$ if and only if for every open subset $G \subseteq \Omega$ and every $v \in SH_q(G) \cap C^2(G)$ the following holds

\begin{equation}
 \sup_{K}(v - u) \leq \sup_{G \setminus K}(v - u),
\end{equation}

where $K \subseteq G$ is an arbitrary relatively compact open subset of $G$.

\textbf{Proof} From Theorem 1 and (d) of Proposition 3, it follows that the necessity is clear. Thus it suffices to prove that if (4) holds then $u \in MSH_q(\Omega)$. Let $v \in SH_q(\Omega)$, $G \subseteq \Omega$. By (d) of Proposition 3, we have to prove

\begin{equation}
 \sup_{G}(v - u) \leq \sup_{\Omega \setminus G}(v - u).
\end{equation}

To get a contradiction, we assume that

\begin{equation}
 \sup_{\Omega \setminus G}(v - u) < \delta < \sup_{G}(v - u).
\end{equation}

Choose open subsets $G_1, G_2$ of $\Omega$ such that $G \subseteq G_1 \subseteq G_2 \subseteq \Omega$. Since $u \in C(\Omega)$ so we have $\sup_{G_2}|u_{\varepsilon} - u| < \delta$ for all $\varepsilon$ sufficiently small, where $u_{\varepsilon} = u \ast \varrho_{\varepsilon}$. Moreover, because $v - u < 0$ on $\Omega \setminus G$, so we have

$$v_{\varepsilon} < u_{\varepsilon} < u + \delta \quad \text{on} \quad G_2 \setminus G_1$$

for all $\varepsilon$ sufficiently small. Hence by (4) we have

\begin{align*}
 \sup_{G}(v - u) &\leq \sup_{G_1}(v_{\varepsilon} - u) \\
 &\leq \sup_{G_2 \setminus G_1}(v_{\varepsilon} - u) \\
 &\leq \sup_{G_2}(v_{\varepsilon} - u) < \delta < \sup_{G}(v - u),
\end{align*}

and we get a contradiction. Thus,

\begin{equation}
 \sup_{G}(v - u) \leq \sup_{\Omega \setminus G}(v - u),
\end{equation}
and Proposition 3 implies that \( u \in MSH_q(\Omega) \). The proof is complete.

Compared to Proposition 1.4.9 in [2] we have the following.

**Corollary 2** Assume that \( \{u_j\}_{j=1}^\infty \) is a decreasing sequence of maximal \( q \)-subharmonic functions in \( \Omega \). Then \( u = \lim_{j \to \infty} u_j \) either is a maximal \( q \)-subharmonic function or \( \equiv -\infty \) on \( \Omega \).

**Proof** Assume that \( u \not\equiv -\infty \). By (d) of Proposition 3, it is enough to prove that for every open subset \( G \subseteq \Omega \) and every \( v \in SH_q(\Omega) \) we have

\[
\sup_G (v - u) \leq \sup_{\Omega \setminus G} (v - u) \tag{5}
\]

Let \( \{b_k\}_{k=1}^\infty \subset G \) be such that \( \sup_G (v - u) = \lim_{k \to \infty} (v - u)(b_k) \). Fix a \( k \). Since \( u_j \in MSH_q(\Omega) \) so by (d) of Proposition 3 we have

\[
(v - u_j)(b_k) \leq \sup_G (v - u_j) \leq \sup_{\Omega \setminus G} (v - u_j) \leq \sup_{\Omega \setminus G} (v - u)
\]

for every \( j = 1, 2, \ldots \). Letting \( j \to \infty \), we get

\[
(v - u)(b_k) \leq \sup_{\Omega \setminus G} (v - u).
\]

Hence we have

\[
\sup_G (v - u) = \lim_{k \to \infty} (v - u)(b_k) \leq \sup_{\Omega \setminus G} (v - u).
\]

Thus (5) is proved and the desired conclusion follows.

The following fact is well known.

**Lemma 1** Assume that \( A = (a_{jk})_{j,k=1}^n \) is a complex \( n \times n \)-matrix such that \( A = \overline{A}^t \). Put \( B = (a_{jk})_{j,k=1}^{n-1} \). Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( A \) and \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \) the eigenvalues of \( B \). Then

\[
\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.
\]

In particular, if the matrix \( B \) has one nonnegative eigenvalue then so does the matrix \( A \).

**Proof** Without loss of generality, we may assume that \( B = \text{diag}(\mu_1, \ldots, \mu_{n-1}) \) and

\[
A = \begin{bmatrix}
\mu_1 & 0 & \cdots & 0 & \overline{x_1} \\
0 & \mu_2 & \cdots & 0 & \overline{x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \mu_{n-1} & \overline{x_{n-1}} \\
x_1 & x_2 & \cdots & x_{n-1} & b
\end{bmatrix}
\]

First, we prove that

\[
\det A = \mu_1 \mu_2 \cdots \mu_{n-1} b - \sum_{j=1}^{n-1} \mu_1 \cdots \widehat{\mu_j} \cdots \mu_{n-1} |x_j|^2. \tag{6}
\]
Indeed, we have

$$\det A = \mu_1 \pmatrix{\mu_2 & \ldots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \mu_{n-1} & \bar{x}_{n-1} \\ x_2 & \ldots & x_{n-1} & b} + (-1)^n x_1 \pmatrix{\mu_2 & \ldots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \mu_{n-1} & \bar{x}_{n-1}}\pmatrix{0 & \ldots & 0 & \bar{x}_1}$$

$$= \mu_1 \pmatrix{\mu_2 & \ldots & 0 & \bar{x}_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \mu_{n-1} & \bar{x}_{n-1} \\ x_2 & \ldots & x_{n-1} & b} - |x_1|^2 \mu_2 \cdots \mu_{n-1}.$$

Hence by induction we get

$$\det A = \mu_1 \mu_2 \cdots \mu_{n-1} b - \sum_{j=1}^{n-1} \mu_1 \cdots \hat{\mu}_j \cdots \mu_{n-1} |x_j|^2.$$

Thus, (6) follows.

Now consider the equation $f(\lambda) = \det(A - \lambda I) = 0$. By (6), we get

$$f(\lambda) = \prod_{j=1}^{n-1} (\mu_j - \lambda)(b - \lambda) - \sum_{j=1}^{n-1} (\mu_1 - \lambda) \cdots \hat{\mu}_j \cdots (\mu_{n-1} - \lambda) |x_j|^2.$$

Since $f(\mu_j) \cdot f(\mu_{j+1}) \leq 0$ for all $j = 1, 2, \ldots, n - 2$, and moreover $f(-\infty) \cdot f(\mu_1) \leq 0$, $f(\mu_{n-1}) \cdot f(+\infty) \leq 0$, hence there exist $n$ solutions $\lambda_j$, $j = 1, 2, \ldots, n$ of the equation $f(\lambda) = 0$ such that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

The proof is complete.

We need the following fact.

**Lemma 2** Let $u \in C^2(\Omega)$ and assume that its complex Hessian has least one nonnegative eigenvalue at each point. Then for every open subset $G \Subset G_1 \Subset \Omega$ we have

$$\sup_G u \leq \sup_{G_1 \setminus G} u.$$

**Proof** From the hypothesis and Lemma 2.6 in [6], it follows that $u$ is an $(n-1)$-plurisubharmonic function in $\Omega$. The desired conclusion of the lemma follows from Lemma 2.7 in [6].

Next we give a condition under which a $C^2$ $q$-subharmonic function is maximal $q$-subharmonic.

**Theorem 2** Let $\Omega$ be an open set in $\mathbb{C}^n$ and let $q$ be an integer with $1 \leq q \leq n - 1$. Assume that $u \in C^2(\Omega)$. Then $u \in \text{MSH}_q(\Omega)$ if and only if its complex Hessian has the sum of $q$ smallest eigenvalues equal to 0 at each point.
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Proof Suppose that we arrange the eigenvalues of $\left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} (z) \right)_{j,k=1,...,n}$ as follows: $\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_n(z)$. We have to prove that $u \in MSH_q(\Omega)$ if and only if $\lambda_1(z) + \lambda_2(z) + \cdots + \lambda_q(z) = 0$ for all $z \in \Omega$.

Necessity. Assume that $u \in MSH_q(\Omega)$. Then by Proposition 2 it follows that $\lambda_1(z) + \lambda_2(z) + \cdots + \lambda_q(z) \geq 0$ for all $z \in \Omega$. To get a contradiction, we assume that there exists $a \in \Omega$ such that $\lambda_1(a) + \lambda_2(a) + \cdots + \lambda_q(a) > 0$. Choose $\varepsilon > 0$ such that

$$
\lambda_1(a) + \lambda_2(a) + \cdots + \lambda_q(a) > \varepsilon q > 0.
$$

Hence there exists $r_\varepsilon > 0$ such that $u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \in SH_q(\mathbb{B}(a, r_\varepsilon))$. It is clear that

$$
u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \leq u \quad \text{in } \Omega \setminus \mathbb{B}(a, r_\varepsilon).
$$

Since $u \in MSH_q(\Omega)$ so it follows that

$$
u - \varepsilon(|z - a|^2 - r_\varepsilon^2) \leq u \quad \text{in } \mathbb{B}(a, r_\varepsilon),
$$

and we get a contradiction. Hence $\lambda_1(z) + \lambda_2(z) + \cdots + \lambda_q(z) = 0$ for all $z \in \Omega$.

Sufficiency. Assume that $\lambda_1(z) + \lambda_2(z) + \cdots + \lambda_q(z) = 0$ for all $z \in \Omega$. By Corollary 1, it is enough to prove that for every open subset $G \subset \Omega$ and every $v \in SH_q(G) \cap C^2(G)$ we have

$$
sup_{K \subset G} (v - u) \leq sup_{G \setminus K} (v - u)
$$

for all open subsets $K \subset G$. Moreover, by Lemma 2, it suffices to check that the matrix

$$
\left( \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} (z) - \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} (z) \right)_{j,k=1}^{n}
$$

has at least one nonnegative eigenvalue at each $z \in G$.

Let $a \in G$ and fix $a$. By a unitary change of coordinates, we can assume that $\lambda_j(a) = \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} (a)$ and $\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} (a) = 0$ for all $j \neq k$. Let $L = a + \mathbb{C}^q \times (0, \ldots, 0) \subset \mathbb{C}^n$. Then $L$ is a $q$-dimensional complex subspace in $\mathbb{C}^n$ containing $a$. By the hypothesis, we have $v|_{L \cap G} \in SH(L \cap G)$. Since $\Delta u|_{L \cap G} (a) = \sum_{j=1}^q \lambda_j(a) = 0$ it follows that

$$
\Delta (v - u)|_{L \cap G} (a) = \Delta v|_{L \cap G} (a) \geq 0.
$$

Hence the matrix $\left( \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} (a) - \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} (a) \right)_{j,k=1}^{n}$ has at least one nonnegative eigenvalue. Thus, by Lemma 1, the matrix $\left( \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} (a) - \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} (a) \right)_{j,k=1}^{n}$ has at least one nonnegative eigenvalue, and the desired conclusion follows. \hfill \Box

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