LOSING MOMENTUM IN CONTINUOUS-TIME STOCHASTIC OPTIMISATION

KEXIN JIN, JONAS LATZ, CHENGUANG LIU, AND ALESSANDRO SCAGLIOTTI

Abstract. The training of deep neural networks and other modern machine learning models usually consists in solving non-convex optimisation problems that are high-dimensional and subject to large-scale data. Here, momentum-based stochastic optimisation algorithms have become especially popular in recent years. The stochasticity arises from data subsampling which reduces computational cost. Moreover, both, momentum and stochasticity are supposed to help the algorithm to overcome local minimisers and, hopefully, converge globally. Theoretically, this combination of stochasticity and momentum is badly understood.

In this work, we propose and analyse a continuous-time model for stochastic gradient descent with momentum. This model is a piecewise-deterministic Markov process that represents the particle movement by an underdamped dynamical system and the data subsampling through a stochastic switching of the dynamical system. In our analysis, we investigate long-time limits, the subsampling-to-no-subsampling limit, and the momentum-to-no-momentum limit. We are particularly interested in the case of reducing the momentum over time: intuitively, the momentum helps to overcome local minimisers in the initial phase of the algorithm, but prohibits fast convergence to a global minimiser later. Under convexity assumptions, we show convergence of our dynamical system to the global minimiser when reducing momentum over time and let the subsampling rate go to infinity.

We then propose a stable, symplectic discretisation scheme to construct an algorithm from our continuous-time dynamical system. In numerical experiments, we study our discretisation scheme in convex and non-convex test problems. Additionally, we train a convolutional neural network to solve the CIFAR-10 image classification problem. Here, our algorithm reaches competitive results compared to stochastic gradient descent with momentum.

Keywords: stochastic optimisation, momentum-based optimisation, piecewise-deterministic Markov processes, stability of stochastic processes, deep learning.

1. Introduction

Machine learning and artificial intelligence play a fundamental role in modern scientific research and modern life. In many instances, the underlying learning process consists of solving a high-dimensional non-convex optimisation problem with respect to large-scale data. These problems have been approached by applicants and researchers using a large range of different algorithms. Methods are based on, e.g., stochastic approximation, statistical mechanics, and ideas from biological evolution. Many of these methods deviate from simply solving the optimisation problem, but are actually also used for regularisation or approximate uncertainty quantification. Unfortunately, many successfully employed methods are theoretically only badly understood.

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In this article, we investigate momentum-based stochastic optimisation methods in a continuous-time framework. We now introduce the optimisation problem, motivate stochastic and momentum-based optimisation, review past work, and summarise our contributions.

1.1. Optimisation and continuous dynamics. We study optimisation problems of the form

$$\min_{\theta \in X} \Phi(\theta),$$

(1.1)

on the space $X := \mathbb{R}^K$, where

$$\Phi(\theta) := \frac{1}{N} \sum_{i=1}^{N} \Phi_i(\theta).$$

We denote the number of terms $N \in \mathbb{N} := \{1, 2, \ldots\}$ and assume that the functions $\Phi_i$, with $i \in I := \{1, \ldots, N\}$, are continuously differentiable with Lipschitz gradients, see also Assumption 1. We assume that (1.1) is well-defined and denote $\theta^* \in \text{argmin} \Phi$. In this work, we study continuous dynamical systems that can be used to optimise functions of type $\Phi$ and to represent the dynamics of optimisation algorithms. The most basic of these dynamical systems is the gradient flow

$$\frac{d\zeta(t)}{dt} = -\nabla \Phi(\zeta(t)), \quad \zeta(0) = \theta_0 \in X,$$

(1.2)

The ODE solution $(\zeta(t))_{t \geq 0}$ converges to a stationary point of $\Phi$ under some convexity conditions. In practice, especially in modern machine learning, we often encounter optimisation problems that are non-convex. Here, $(\zeta(t))_{t \geq 0}$ may converge to a saddle point or a local minimiser. When discretising (1.2), the iterates actually only converge to (local) minimisers (see [23, 28]). However, as observed in [5], the gradient descent could take a lot of iterations to escape a saddle point.

The stochastic gradient descent (SGD) method going back to [31] has been a popular alternative to the standard gradient flow/descent. In SGD, we randomly select one of the $\Phi_i$ ($i \in I$) and optimise with respect to that function before switching to another $\Phi_j$ ($j \in I \setminus \{i\}$).

A continuous-time model for stochastic gradient descent has recently proposed by [20] and extended by [12] and [21]. There, stochastic gradient descent is represented through the dynamical system

$$\frac{d\theta(t)}{dt} = -\nabla \Phi_i(\theta(t)), \quad \theta(0) = \theta_0 \in X,$$

(1.3)

where the index process $(i(t))_{t \geq 0}$ is a homogeneous continuous-time Markov process on $I$; see Definition 1. The processes $(i(t))_{t \geq 0}$, $(\theta(t))_{t \geq 0}$ and any other stochastic processes and random variables throughout this work are defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $(\theta(t))_{t \geq 0}$ represents a randomised gradient flow whose potential is randomly replaced by another one when $(i(t))_{t \geq 0}$ jumps from one state to another. Both $(\theta(t))_{t \geq 0}$ and $(i(t), \theta(t))_{t \geq 0}$ are called stochastic gradient process. Since $\Phi$ often represents the averaging over data subsets, we refer to the process of replacing $\Phi$ by a $\Phi_i$ as subsampling.

Since we consider only one of the $\nabla \Phi_i$ at a time, we significantly reduce the computational cost when discretising the dynamical system. Moreover, the perturbation through the randomised sampling in stochastic gradient descent can help to overcome saddle points, see
However, as the method is purely gradient-based, we would again expect that the escaping saddle points is slow.

A different way to escape saddle points are momentum-based optimisation methods. Intuitively, momentum is used to vault the optimiser out of saddle points and local minimisers. The idea of considering momentum in optimisation dates back to the 1960s, when Polyak formulated the heavy ball method ([29, 30]) in the framework of convex optimisation. Here, the aim was to accelerate the convergence of the classical gradient descent. Few years later, in his seminal paper [26], Nesterov proposed a momentum-based class of accelerated methods for convex problems. Finally, the interplay between accelerated methods and dynamical systems is currently an active research field. In this regard, we report the important contribution of [36], and the more recent works [1 33 34 32].

In the next subsection, we give a motivating example where the usefulness of momentum in non-convex optimisation can be observed.

1.2. Momentum: a motivating example. In this passage we discuss a simple one-dimensional non-smooth non-convex optimisation problem inspired by the training of neural networks. As we shall see, the introduction of momentum can prevent the convergence of the gradient flow to a stationary point that is not the global minimiser.

In this introductory example we deal with deterministic methods only. Let us consider the function \( \Phi : \mathbb{R} \to \mathbb{R} \) defined as

\[
\Phi(x) := (\text{ReLU}(x) - 1)^2 + x^2 = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ 2x^2 - 2x + 1 & \text{if } x > 0 \end{cases} \quad (x \in \mathbb{R})
\]  

where \( \text{ReLU}(\cdot) := \max\{0, \cdot\} \). The function \( \Phi \) attains its global minimum at the point \( \hat{x} = \frac{1}{2} \). However, \( \bar{x} = 0 \) is the minimiser of \( \Phi \) restricted to the non-positive half-line \((-\infty, 0]\). Therefore, any solution of the gradient flow equation

\[
\begin{cases}
\frac{dx(t)}{dt} = -\nabla \Phi(x(t)) \\
x(0) = x_0
\end{cases}
\]  

with Cauchy datum \( x_0 < 0 \) converges to the “false” local minimiser \( \bar{x} \). In order to avoid the convergence to \( \bar{x} \), we can replace the gradient flow (1.5) by an underdamped dynamical system, i.e., introduce momentum. Namely, we can consider the differential equation

\[
m \frac{d^2 x(t)}{dt^2} + \nabla \Phi(x) = -\alpha \frac{dx(t)}{dt}, \quad x(0) = x_0, \quad \frac{dx(0)}{dt} = v_0.
\]  

Throughout this work, we refer to this differential equation as underdamped gradient flow. We recall that (1.6) models the motion of a particle of mass \( m > 0 \) in \( \mathbb{R} \), which is subjected to the force field generated by the potential energy \( \Phi \) and to the linear viscosity friction tuned by the parameter \( \alpha > 0 \). If the starting point \( x_0 \) of (1.6) is negative, then the equation of the motion reduces to

\[
m \frac{d^2 x(t)}{dt^2} + 2x(t) = -\alpha \frac{dx(t)}{dt},
\]  

whenever the massive particle stays in the negative half-line. At this point the natural question is whether there exist combinations of \( m \) and \( \alpha \) such that, assuming that \( x_0 < 0 \), the particle is not confined in the negative half-line for every time. Using elementary theory of linear
second-order differential equations, it turns out that the particle manages to overcome the “false” minimiser $\tilde{x} = 0$ if the following relation is satisfied:

$$\alpha^2 - 8m < 0,$$

(1.8)
i.e., the friction $\alpha$ is sufficiently small or the mass $m$ sufficiently large. In summary, momentum can be used to overcome stationary points of the target function. Next, we see how momentum is used within stochastic optimisation practice.

1.3. Momentum-based stochastic optimisation and the Adam algorithm. In practice, we combine a forward Euler discretisation of the underdamped gradient flow (1.6) with subsampling. We obtain, what we call (stochastic gradient descent with) classical momentum:

$$\begin{align*}
\theta_n &= \theta_{n-1} + v_{n-1}, \\
v_n &= \alpha v_{n-1} - \eta \nabla \Phi_i(\theta_{n-1}),
\end{align*}$$

(1.9)

where $\alpha, \eta > 0$ are hyperparameters and $i_1, i_2, \ldots \sim \text{Unif}(I)$ are independent and identically distributed (i.i.d.) random variables. We refer to $\eta$ as learning rate. This is probably the most basic momentum-based stochastic optimisation method. Especially, in deep learning, more advanced momentum-based descent methods have been widely used, including Adagrad [6], Adadelta [39], RMSprop [11], Adam [16], etc. Adam is one of the most commonly used optimizers when training neural networks. The updating rule is the following,

$$\begin{align*}
u_n &= \beta_2 v_{n-1} + (1 - \beta_2) [\nabla \Phi_i(\theta_{n-1})]^2, \\
\theta_n &= \theta_{n-1} - \alpha \frac{u_n/(1 - \beta_1^2)}{\sqrt{v_n/(1 - \beta_2^2) + \beta_3}}
\end{align*}$$

(1.10)

where $\beta_1$, $\beta_2$, and $\alpha$ are hyper-parameters, $\beta_3$ is a small constant, and, again, $i_1, i_2, \ldots \sim \text{Unif}(I)$ (i.i.d.). Comparing to the classical momentum method, Adam uses not only the first moment of the gradient but also an estimate of the second moment as the part of the adaptive learning rate. The variable $u_n$ is the biased first moment, similar to the one used in the classical momentum method. The variable $v_n$ is the biased second moment estimated using the gradient squared. When updating $\theta_n$,Adam normalizes $u_n$ and $v_n$ so that they become unbiased estimators. Moreover, it was shown in [4] that Adam converges in the sense of [41] with speed $O(\log(T)/\sqrt{T})$.

While momentum-based stochastic optimisation methods are popular in machine learning practice, they are overall rather badly understood; see the discussion in [25]. In the present work, we analyse a continuous-time stochastic gradient dynamic with momentum and propose an efficient discretisation technique for this dynamical system. Thus, we improve the understanding of momentum-based stochastic optimisation in a theoretical framework and machine learning practice. Before we summarise our contributions more precisely, we introduce our continuous-time stochastic gradient-momentum process.

1.4. The stochastic gradient-momentum process. Throughout this work, we discuss a continuous-time dynamical system that represents stochastic gradient descent with momentum – the stochastic gradient-momentum process. Due to the continuous-time nature of the system, it can equally represent stochastic gradient descent with momentum (1.9) and Adam (1.10).
We obtain this system by combining the underdamped dynamical system (1.6) with the stochastic gradient flow (1.3). We commence with the index process, the process that controls the subsampling.

**Definition 1** (index process). First, let \((i(t))_{t \geq 0}\) be a continuous-time Markov process on \(I\) with the transition rate matrix
\[
A_N = \Gamma_N - N\gamma I_N,
\]
where \(\gamma > 0\), \(\Gamma_N\) is an \(N \times N\) matrix whose entries are all equal to \(\gamma\), and \(I_N\) is the identity matrix. We assume that the initial distribution \(P(i(0) \in \cdot)\) is the uniform distribution on \(I\).

Moreover, when given a value \(\nu > 0\) or an appropriate, strictly increasing function \(\beta: [0, \infty) \to [0, \infty)\), we define the rescaled index processes \((i^\nu(t))_{t \geq 0}\) and \((i^\beta(t))_{t \geq 0}\) as index process.

The function \(\beta^{-1}\) and the scalar \(1/\nu\) play the role of a time-dependent and constant learning rate, respectively; see [12]. Thus, we sometimes refer to them as learning rate. To define the stochastic gradient process with momentum, we consider the following regularity assumption to be true.

**Assumption 1.** \(\Phi_i \in C^1(X: \mathbb{R})\), i.e., it is continuously differentiable, and \(\nabla_x \Phi_i\) is Lipschitz with Lipschitz constant \(L\), for \(i \in I\).

Throughout this work, we study multiple versions of the stochastic gradient-momentum process and introduce them in Definitions 2 – 4. We now introduce the prototype of the dynamical systems studied in this paper. Note here that we often choose the more compact form \(dz_t = f(z_t)dt\) to represent \(dz_t = f(z(t))\). The general stochastic gradient-momentum process (SGMP) is given by \((p^\dagger_t, q^\dagger_t, i^\beta(t))_{t \geq 0}\), where

\[
\begin{align*}
\frac{dq^\dagger_t}{dt} &= p^\dagger_t dt, \\
\int m(t)dp^\dagger_t &= -\nabla \Phi_i^\beta(q^\dagger_t)dt - \alpha p^\dagger_t dt, \\
p^\dagger(t = 0) &= p_0, \\
q^\dagger(t = 0) &= q_0.
\end{align*}
\]

Here, \((p^\dagger_t)_{t \geq 0}\) describes the velocity of the particle, \((q^\dagger_t)_{t \geq 0}\) its position, \((m(t))_{t \geq 0}\) its mass (that may depend on time), and \(\alpha > 0\) the viscosity friction. The function \(\beta\) controls the switching of the index process. In the following, we explain the importance behind the functions \(m\) and \(\beta\).

**Remark 1** (Mass \(m\)). The introduction of momentum allows the particle to be vaulted out of stationary points. An effect that is particularly pronounced if \(m\) is large or \(\alpha\) is small. While this behaviour is convenient when escaping local minimisers and saddle points, it can also occur when approaching global minimisers. We give a very simple example in Figure 4, where we see that the method shows oscillatory behaviour and very slow convergence when the mass \(m\) is large and constant. As a tuning of the mass \(m\) is difficult in practice, we suggest an alternative strategy: reducing the mass over time; rationale: a large mass in the beginning leads to a fast escaping of local minimisers, while a small mass later leads to fast convergence once the global minimiser is reached. This intuition is confirmed in Figure 4.
Figure I. We consider the minimisation of $\Phi(\theta) := \theta^2/2$. Let $\alpha = 1, m = 1, p_0 = 1, q_0 = 0$. Then, $p_t = \exp(-t/2) \left( \sin(\sqrt{3}t/2)/\sqrt{3} + \cos(\sqrt{3}t/2) \right) (t \geq 0)$ (solid line) oscillates around the solution and converges ultimately to $\theta^* = 0$. If we choose instead $m(t) = (1 + t)^{-1} (t \geq 0)$ (dashed lines), we have $p_t = \exp(-t^2/2)$, which does not oscillate, but converges very quickly to $\theta^*$.

Remark 2 (Learning rate and $\beta$). The switching between the potentials leads to a similar effect as the momentum. As the flow approaches the minimisers of different target functions in each step, we are unlikely to converge to a single point. While this, again, helps to escape local minimisers, it prohibits the convergence to a global minimiser. In several stochastic optimisation methods, we need to reduce the learning rate or step size throughout the algorithm to reduce the time in-between switches of data sets. We can reach this reduction of learning rate by rescaling the time in the index process $(i(t))_{t \geq 0}$ through an appropriate function $\beta$. The rescaling will lead to the waiting times between two switches becoming small over time in a certain sense.

A reduction of mass and learning rate is often necessary to reach fast convergence or even contraction of the optimisation algorithm to a point. In practice, however, this is sometimes purposefully disregarded – convergence to, e.g., a probability distribution is preferred. In stochastic optimisation this leads to an implicit regularisation of the optimisation problem. Especially in machine learning, this implicit regularisation can be necessary to get good generalisation results, see, for example, [14, 15, 27, 40, 42].

In cases, where the mass $m$ does not depend on $t$ or the function $\beta$ is linear, i.e. $\beta(u) = \nu u$, we refer to mass or learning rate as homogeneous, respectively. Since, we are otherwise always interested in reducing them over time, we speak of decreasing mass or learning rate, respectively.

1.5. Contributions and outline. We have introduced the stochastic gradient-momentum process above. In this work, we analyse this dynamical system, discuss its discretisation, and employ it in numerical experiments.
We are interested in both, convergence to global minimisers and implicit regularisation. Thus, we study three different stochastic gradient-momentum processes: (1) homogeneous mass and homogeneous learning rate, (2) decreasing mass and homogeneous learning rate, and (3) decreasing mass and decreasing learning rate, respectively. We now summarise our contributions:

- We study the connection between SGMP and the gradient flow, the underdamped gradient flow, as well as the stochastic gradient process. These connections allow us to investigate the longtime behaviour of the stochastic gradient-momentum process.

We obtain those connections by considering momentum-to-no-momentum asymptotics (letting a homogeneous mass $m$ approach 0), and random-to-deterministic asymptotics (for a homogeneous $\beta(u) = \nu u$, letting $\nu$ approach 0). Understanding the longtime behaviour, of course, allows us to study the longtime behaviour of the dynamical system and, thus, eventually the convergence of the algorithm. We show the following three convergence results under convexity assumptions:

- In the fully homogeneous case, we show that the stochastic gradient-momentum process is close to the stochastic gradient process when the mass is small.
- If the mass decreases over time, the stochastic gradient-momentum process converges to the stochastic gradient process in the longtime limit.
- If mass and learning rate are decreased over time, the stochastic gradient-momentum process converges to the minimiser of the full target function.

The connection to the stochastic gradient process is especially interesting, as its longtime behaviour in convex settings is well-understood, see [12, 20]. Thus, we can argue here that SGMP leads to a similar implicit regularisation when the mass is small or converging to zero.

In a wider sense, the connections to the other methods allow us to interpret the stochastic gradient-momentum process as a method that can freely interpolate between gradient flow, underdamped gradient flow, and stochastic gradient process – through adjusting learning rate $\nu^{-1}$ and mass $m$. We depict this relationship in Figure III.

Surprisingly, the continuous relationship shown in Figure III is lost when, e.g., comparing the algorithms SGD and SGD with classical momentum. This is mainly due to the forward Euler discretisation that is employed in classical momentum. Especially problematic is the instability of classical momentum when the mass is small. Here, the step size needs to be chosen as $O(m; m \downarrow 0)$. From a practical perspective, our contributions are the following:

- We propose a discretisation strategy for the stochastic gradient-momentum process. The strategy is a semi-implicit method that is explicit in $(q_t)_{t \geq 0}$, but implicit in $(p_t)_{t \geq 0}$. This discretisation technique allows us to choose the step size independently of the mass $m$ and, thus, is also stable for very small masses and such that converge to zero. Moreover, as opposed to the conventional forward Euler method, our strategy is symplectic, leading to a physically correct treatment of the system’s energy.
- We test our discretised algorithm in numerical experiments. We start with academic convex and non-convex optimisation problems in low and high dimensions. Then we study the training of a convolutional neural network (CNN), which we use to classify images from the CIFAR-10 dataset. The achieved train and test accuracy with
our stable discretisation is comparable to stochastic gradient descent with classical momentum.

This work is organised as follows. We introduce and investigate the homogeneous-in-time and heterogeneous-in-time stochastic gradient-momentum processes in Sections 2 and 3 respectively. In Section 4 we propose discretisation techniques which we then employ in the mentioned academic and deep learning problems in Section 5. We conclude the work in Section 6. Some auxiliary results are presented in Appendix A.

2. HOMOGENEOUS-IN-TIME

In this section, we study the stochastic gradient-momentum process with homogeneous momentum and learning rate. In the first part, Subsection 2.1 we investigate the interplay in between the underdamped gradient flow and the stochastic gradient-momentum process. In the second part, Subsection 2.2 we compare the stochastic gradient-momentum process and the stochastic gradient process.

Indeed we study the dynamical system described below.
Definition 2. For $\nu > 0$, the homogeneous stochastic gradient-momentum process (hSGMP) is a solution of the following stochastic differential equation,

$$
\begin{align*}
\frac{dq_{t}^{\nu,m}}{dt} & = p_{t}^{\nu,m}, \\
\frac{dp_{t}^{\nu,m}}{dt} & = -\nabla \Phi_{i}(q_{t}^{\nu,m}) - \alpha p_{t}^{\nu,m}, \\
q_{t}^{\nu,m}(t = 0) & = q_{0},
\end{align*}
$$

where $\alpha, m > 0$ are respectively the viscosity friction parameter and the mass, respectively, both kept constant. Finally, $\Phi_{j}$ satisfies Assumption 1 for every $j = 1, \ldots, N$, and we introduce the stochastic process $i^{\nu}(t) = i(t/\nu)$, where $\{i(t)\}_{t \geq 0}$ is defined in Definition 1. In the following, we denote $(i) (p_{t}^{\nu}, q_{t}^{\nu})_{t \geq 0} := (p_{t}^{1,1}, q_{t}^{1,1})_{t \geq 0}$, i.e., choosing a constant unit mass $m := 1$ and varying the learning rate $\nu$.

(ii) $(p_{t}^{m}, q_{t}^{m})_{t \geq 0} := (p_{t}^{m,\delta}, q_{t}^{m,\delta})_{t \geq 0}$, for some $\delta \in [0, 1)$, i.e., varying the mass, and choosing a constant or a mass-depending learning rate $\nu := m^{\delta}$; in this case, we additionally choose a unit viscosity friction $\alpha := 1$.

The Lipschitz condition on $\nabla \Phi_{i}$ guarantees the well-posedness of this system, which can be showed similarly as \cite{12, 20}. In Subsections 2.1 and 2.2 we consider the hSGMPs $(p_{t}^{\nu}, q_{t}^{\nu})_{t \geq 0}$ and $(p_{t}^{m}, q_{t}^{m})_{t \geq 0}$, respectively. In (i), (ii) can set $m, \alpha = 1$ without loss of generality.

2.1. Momentum with and without subsampling. First, we study the interplay between subsampling and not subsampling in the momentum dynamic. Indeed, we show that the stochastic gradient-momentum process can approximate the underdamped gradient flow at any accuracy. More precisely, we are interested in the limiting behavior of hSGMP $(p_{t}^{\nu}, q_{t}^{\nu})_{t \geq 0}$ as the learning rate approaches zero uniformly, i.e., we take $\nu \to 0$ in (2.1). We prove that the hSGMP converges to the solution to the underdamped gradient flow, which is given by

$$
\begin{align*}
\frac{dq_{t}}{dt} & = p_{t}, \\
\frac{dp_{t}}{dt} & = -\nabla \bar{\Phi}(q_{t}) - \alpha p_{t}, \\
p(t = 0) & = p_{0}, \\
q(t = 0) & = q_{0},
\end{align*}
$$

where we implicitly set the mass $m := 1$. In this case, we show that hSGMP is a stochastic approximation to the underdamped gradient flow.

After showing this approximation result, we discuss the longtime behaviour of underdamped gradient flow. This is for the sake of completeness to explain the behaviour of the object we approximate, but also since we need those results throughout this work.

Weak convergence. We now formulate the statement about the convergence of hSGMP to the underdamped gradient flow more particularly. In principal, we show convergence of a random path to a deterministic path – both are contained in the space of continuous functions from $[0, \infty)$ to $X^{2}$ which we denote by $C([0, \infty) : X^{2})$ and equip with the metric

$$
\rho \left( (\zeta_{t})_{t \geq 0}, (\xi_{t})_{t \geq 0} \right) := \int_{0}^{\infty} e^{-t} (1 \wedge \sup_{0 \leq s \leq t} \| \zeta_{s} - \xi_{s} \|) \, dt,
$$
where as usual \( a \wedge b := \min\{a, b\} \) for \( a, b \in \mathbb{R} \). Probabilistically, we show convergence in the weak sense. We state and prove the convergence result below.

**Theorem 1.** Let \((q^t, p^t)_{t \geq 0}\) and \((q_t, p_t)_{t \geq 0}\) solve \((2.1)\) and \((2.2)\). Then \((q^t, p^t)_{t \geq 0}\) converges weakly to \((q_t, p_t)_{t \geq 0}\) in \(C([0, \infty) : X^2)\) as \(\nu \to 0\), i.e. for any bounded continuous function \(F : C([0, \infty) : X^2) \to \mathbb{R}\),

\[
\mathbb{E}[F((p^t, q^t)_{t \geq 0})] \to \mathbb{E}[F((p_t, q_t)_{t \geq 0})] = F((p_t, q_t)_{t \geq 0}).
\]

**Proof.** We apply Theorem 4.3 from [19, Chapter 7, Theorem 4.3], which studies tightness and weak convergence property of solutions of a certain type of differential equations. The form of the differential equation is given in (4.1) from [19, Chapter 7], which corresponds to equation \((2.2)\) with \(\xi_t = i(t)\), \(\tilde{G}(x, y) = (y, -\nabla_x \Phi(x) - \alpha y)\), \(\tilde{G}(x, y, i) = (0, -\nabla_x \Phi_i(x) + \nabla_x \Phi(x))\), and \(F = 0\). In [19, Chapter 7, Theorem 4.3], the differential operator \(A\) in our case is defined in \((2.1)\), i.e. \(A h(x, y) = -\nabla h(x, y) \cdot (y, -\nabla_x \Phi(x) - \alpha y)\) for any \(h\) twice differentiable. If Assumptions (A4.2) to (A4.6) from [19] were checked, applying [19, Chapter 7, Theorem 4.3] and we obtain that \((q^t, p^t)_{t \geq 0}\) is tight and the limit of any weakly convergent subsequence solves \((2.1)\). Since \(q^0 = q_0\) and \(p^0 = p_0\), we have \((q^t, p^t)_{t \geq 0}\) converges weakly to \((q_t, p_t)_{t \geq 0}\). Therefore, we just need to verify Assumptions (A4.2) to (A4.6) from [19]. Assumption (A4.2) follows directly from Assumption 1. Assumption (A4.3) holds since \((i(t))\) is càdlàg and bounded. Assumptions (A4.4) and (A4.6) trivially hold since \(F = 0\). The only non-trivial part is to verify Assumption (A4.5), that is \(t, \tau \to \infty\),

\[
\frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}[-\nabla_x \Phi_{i(s)}(x) + \nabla_x \Phi(x) \{i(s')\}_{s' \leq t}] ds \to 0, \text{ a.s.}
\]

By the Markov property of \(\{i(t)\}_{t \geq 0}\) and since \(i(0)\) is stationary,

\[
\left| \frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}[-\nabla_x \Phi_{i(s)}(x) + \nabla_x \Phi(x) \{i(s')\}_{s' \leq t}] ds \right|
\]

\[
= \frac{1}{\tau} \int_0^{\tau} \mathbb{E}_{i(t)}[-\nabla_x \Phi_{i(s)}(x) + \nabla_x \Phi(x)] ds
\]

\[
= \frac{1}{\tau} \int_0^{\tau} \mathbb{E}_{i(t)}[\nabla_x \Phi_{i(s)}(x) - \nabla_x \Phi(x)] ds
\]

\[
\leq \sum_{i=1}^N \frac{1}{\tau} \int_0^{\tau} \mathbb{E}[\nabla_x \Phi_{i(s)}(x) - \nabla_x \Phi(x)] ds
\]

\[
= \frac{1}{\tau} \sum_{i=1}^N \left| \int_0^{\tau} \left( \sum_{j=1}^N \frac{1 - \exp(-Ns)}{N} \nabla_x \Phi_j(x) \right) + \exp(-Ns) \Phi_i(x) - \nabla_x \Phi(x) ds \right|
\]

\[
= \frac{1}{\tau} \sum_{i=1}^N \left| \int_0^{\tau} \exp(-Ns) \left( \nabla_x \Phi_i(x) - \nabla_x \Phi(x) \right) ds \right|
\]

\[
\leq \frac{1}{N\tau} \sum_{i=1}^N \left| \nabla_x \Phi_i(x) - \nabla_x \Phi(x) \right| \to 0 \text{ as } \tau \to \infty.
\]

\[\square\]
Hence, $\text{hSGMP} \left( q^t_r, p^t_r \right) t \geq 0$ is a stochastic approximation of the underdamped gradient flow $(q_t, p_t) t \geq 0$. This is a good moment to study some properties of the dynamical system that we approximate, especially its longtime behaviour.

**Longtime behavior of the underdamped gradient flow.** We show that the position in the dynamical system (2.2) converges to the global minimizer of the objective function $\Phi$ and the velocity converges to 0. Here, we need a technical assumption that has also been utilised in [7]. We discuss the assumption below.

**Assumption 2.** Assume that $\theta_*$ is a global minimizer of $\Phi(x)$, $\alpha > 0$ is the viscosity friction from (2.2), and that there exists some constant $\lambda \in (0, 1/4]$ such that

$$
(x - \theta_*) \cdot \nabla \Phi(x)/2 \geq \lambda(\Phi(x) - \Phi(\theta_*) + \alpha^2 \|x - \theta_*\|^2/4). \tag{2.3}
$$

We note that since $\Phi$ achieves a global minimum at $\theta_*$, $\theta_*$ is a critical point of $\Phi$. Inequality (2.3) implies that $\theta_*$ is the unique critical point of $\Phi$.

Assumption 2 holds, for instance, if $\Phi$ is strongly convex, i.e., if

$$
\langle x - y, \nabla \Phi(x) - \nabla \Phi(y) \rangle \geq \kappa \|x - y\|^2 \quad (x, y \in X),
$$

for some $\kappa > 0$ which is the convexity parameter. We prove this assertion in Lemma 7 in the appendix. Note however that while strong convexity implies Assumption 2, it is not equivalent. We give a counterexample below.

**Example 1.** Let $\Phi$ be an odd function on $\mathbb{R}$ satisfying

$$
\Phi(x) = \begin{cases} 
x^2, & x \in [0, 1], \\
2x - 1, & x \in (1, 2], \\
\frac{1}{2}x^2 + 1, & x \in (2, +\infty).
\end{cases}
$$

$\Phi$ satisfies Assumption 2 with $\lambda \in (0, 2/(8 + \alpha^2)]$. Moreover, it is convex but not strongly convex since it is linear on $(1, 2].$

We can now move on to studying the longtime behaviour of the damped dynamical system. Here, we use a technique similar to that of [7]. Indeed, we introduce the following Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$
V(x, y) := -\Phi(x) - \Phi(\theta_*) + \frac{\alpha^2}{4} \left( \|x - \theta_* + \alpha^{-1}y\|^2 + ||\alpha^{-1}y||^2 - \lambda \|x - \theta_*\|^2 \right) \tag{2.4}
$$

which we employ below. Using $V$, we can show Lyapunov exponential global stability for the underdamped gradient flow (2.2).

**Proposition 1.** Assume that $\Phi$ satisfies Assumption 1 and 2. Let $(p_t, q_t) t \geq 0$ be the solution to (2.2) and $V$ be the Lyapunov function defined in (2.4). Then we have the following inequality

$$
V(q_t, p_t) \leq V(q_0, p_0) e^{-\alpha \lambda t}.
$$

**Proof.** Notice that

$$
\nabla_x V = \nabla \Phi(x) + \frac{\alpha^2}{2} \left( (1 - \lambda)(x - \theta_*) + \alpha^{-1}y \right),
$$

$$
\nabla_y V = \frac{\alpha^2}{2} \left( 2\alpha^{-2}y + \alpha^{-1}(x - \theta_*) \right) = y + \frac{\alpha}{2}(x - \theta_*). \tag{2.5}
$$
Hence,

\[ dV(q_t, p_t) = \nabla_x V \cdot dq_t + \nabla_y V \cdot dp_t = \nabla_x V \cdot p_t dt - \nabla_y V \cdot (\nabla \Phi(q_t) + \alpha p_t) dt \]

By Grönwall’s inequality, we obtain \( V(q_t, p_t) \leq V(q_0, p_0)e^{-\alpha \lambda t} \).

In the previous theorem, we show that the Lyapunov function \( V(q_t, p_t) \) is bounded above by a number that exponentially decreases in \( t > 0 \). \( V(q_t, p_t) \) is zero, if \( q_t = \theta_* \) and \( p_t = 0 \), i.e., the particle is positioned at the unique minimiser of \( \Phi \) and the particle velocity is 0: the dynamical system is converged. However, \( V \) can have multiple zeros. Thus, we need some more work to show exponential convergence of the dynamical system.

**Corollary 2.** Under the condition of Proposition 1, we have

\[ \|p_t\|^2 + \|q_t - \theta_*\|^2 \leq C_{\alpha, \lambda}e^{-\alpha \lambda t}V(q_0, p_0). \]

**Proof.** Since \( \lambda \leq \frac{1}{4} \), we have, by the \( \varepsilon \)-Young inequality:

\[
V(x, y) \geq \frac{\alpha^2}{4} \left( (1 - \lambda) \|x - \theta_*\|^2 + 2 \|\alpha^{-1}y\|^2 + 2(x - \theta_*) \cdot (\alpha^{-1}y) \right) \\
\geq \frac{\alpha^2}{4} \left( (1 - \lambda) \|x - \theta_*\|^2 + 2 \|\alpha^{-1}y\|^2 - 2 \|\alpha^{-1}y\|^2 - \frac{1}{2} \|x - \theta_*\|^2 \right) \\
\geq \frac{\alpha^2}{4} \left( \frac{1}{2} - \lambda \right) \|q_t - \theta_*\|^2,
\]

which implies

\[ \|q_t - \theta_*\|^2 \leq \frac{8}{\alpha^2(1 - 2\lambda)} V(q_t, p_t) \leq \frac{8}{\alpha^2(1 - 2\lambda)} V(q_0, p_0)e^{-\alpha \lambda t}. \]
Similarly, for \( \|p_t\| \), by \( \varepsilon \)-Young inequality,

\[
V(x, y) \geq \frac{\alpha^2}{4} \left( (1 - \lambda) \|x - \theta_*\|^2 + 2 \|\alpha^{-1} y\|^2 \right) + \frac{1}{1 - \lambda} \|\alpha^{-1} y\|^2 - (1 - \lambda) \|x - \theta_*\|^2 \]

which implies

\[
\|p_t\|^2 \leq \frac{4(1 - \lambda)}{1 - 2\lambda} e^{-\alpha \lambda t} V(q_0, p_0) \leq e^{-\alpha \lambda t} V(q_0, p_0).
\]

Therefore, we conclude that

\[
\|q_t - \theta_*\|^2 + \|p_t\|^2 \leq C_{\alpha, \lambda} e^{-\alpha \lambda t} V(q_0, p_0),
\]

where \( C_{\alpha, \lambda} = 4 + \frac{8}{\alpha^2(1 - 2\lambda)} \).

In summary, we have shown in this section that hSGMP is a stochastic approximation of the underdamped gradient flow that – under Assumption 2 – converges to the minimiser of \( \bar{\Phi} \).

2.2. Subsampling with and without momentum. In a similar way to the last section, we now study the relation between the homogeneous stochastic gradient-momentum process and the stochastic gradient process. Indeed, we let the momentum in hSGMP go to zero by letting the mass \( m \) go to zero. We show that in the limit, we converge to the stochastic gradient process. We do this while either keeping a constant learning rate or while allowing the learning rate to depend on the mass and thus go to zero as well. Specifically, we consider the process \((p^m_t, q^m_t)\) introduced in Definition 2(ii). Namely, we set \( \delta \in [0, 1), \) learning rate \( \nu := m^\delta, \) and \( \alpha := 1. \) We then aim to show that \((q^m_t, p^m_t)\) in (2.2) converges uniformly to some stochastic gradient process as \( m \to 0. \) The associated stochastic gradient process is the following:

\[
\begin{align*}
d\theta^m_t &= -\nabla \Phi_i(t/m^\delta)(\theta^m_t) dt, \\
\theta^m_0 &= \theta_0.
\end{align*}
\]

We now first collect preliminary results about the momentum-to-no-momentum limit in the deterministic case by fixing the sample throughout the dynamical system. Then, we show the limiting result about the stochastic processes.

The fixed-sample case. We now consider the SGMP dynamic subject to a fixed sample \( i \in I \) that does not change throughout the algorithm. More specifically, for \( i \in I \) and \( m > 0, \) we define

\[
\begin{align*}
dq^i_{t, m} &= p^i_{t, m} dt, \\
mdp^i_{t, m} &= -\nabla \Phi_i(q^i_{t, m}) dt - p^i_{t, m} dt, \\
p^i(t = 0) &= p^i_0, \\
q^i(t = 0) &= q^i_0.
\end{align*}
\]

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When \( m \to 0 \), we have the following formal limiting equation
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d\theta^i_t}{dt} = -\nabla\Phi_i(\theta^i_t)dt, \\
\theta^i(0) = \theta_0.
\end{array} \right.
\end{aligned}
\]
(2.8)

This is the gradient flow with respect to the potential \( \Phi_i \). To prove this described limiting behaviour, we require three auxiliary results. In the first one we study, similarly to Corollary [2] the longtime behaviour of the deterministic damped dynamical system (2.7). This time with emphasis on the influence of the mass \( m \). Interestingly, we can see that the convergence rate is independent of the mass \( m \), if it is sufficiently small.

**Lemma 1.** Let \( \Phi_i \) satisfy Assumption [1] and [2] with \( 0 < \lambda_i \leq \frac{1}{4}, \alpha = 1 \) and critical point \( \theta^*_i \).

Let \( q^{i,m}_t \) be the solution to (2.7). We set \( \lambda := \min\{\lambda_1, \ldots, \lambda_N\} \). Then for \( 0 < m \leq 1 \), we have the following inequality
\[
\| q^{i,m}_t - \theta^*_i \|^2 \leq 16C_L e^{-\lambda t}(\| q^{i}_0 - \theta^*_i \|^2 + \| p^i_0 \|^2),
\]
(2.9)

where \( C_L := L + 2 \) and \( L \) is, again, the Lipschitz constant of the \( (\nabla\Phi_i)_{i \in I} \).

**Proof.** Recall the Assumption [2] for \( \Phi_i \) with \( \alpha = 1 \),
\[
(x - \theta^*_i) \cdot \nabla\Phi_i(x) / 2 \geq \lambda(\Phi_i(x) - \Phi_i(\theta^*_i)) + \| x - \theta^*_i \|^2 / 4,
\]
which implies
\[
(x - \theta^*_i) \cdot \nabla\Phi_i(x) / (2m) \geq \lambda(m^{-1}\Phi_i(x) - m^{-1}\Phi_i(\theta^*_i) + m^{-1} \| x - \theta^*_i \|^2 / 4).
\]
And since \( 0 < m \leq 1 \), we have
\[
\lambda(m^{-1}\Phi_i(x) - m^{-1}\Phi_i(\theta^*_i) + m^{-1} \| x - \theta^*_i \|^2 / 4) = \lambda m(m^{-2}\Phi_i(x) - m^{-2}\Phi_i(\theta^*_i) + m^{-2} \| x - \theta^*_i \|^2 / 4)
\]
\[
\geq \lambda m(m^{-1}\Phi_i(x) - m^{-1}\Phi_i(\theta^*_i) + m^{-2} \| x - \theta^*_i \|^2 / 4).
\]

So Assumption [2] implies
\[
(x - \theta^*_i) \cdot \nabla\Phi_i(x) / (2m) \geq \lambda m(m^{-1}\Phi_i(x) - m^{-1}\Phi_i(\theta^*_i) + m^{-2} \| x - \theta^*_i \|^2 / 4)
\]
which exactly means \( \Phi_i(x)/m \) satisfies Assumption [2] with \( \alpha = m^{-1} \) and \( \lambda m \). Define
\[
V^{i,m}(x, y) = \frac{\Phi_i(x) - \Phi_i(\theta^*_i)}{m} + \frac{1}{4m^2} \left( \| x - \theta^*_i + my \|^2 + \| my \|^2 - m\lambda \| x - \theta^*_i \|^2 \right).
\]
From Proposition [1] we have
\[
V^{i,m}(q^{i,m}_t, q^{i,m}_0) \leq V^{i,m}(q^i_0, p^i_0) e^{-\lambda t}.
\]

Next, we are going to show that
\[
c m^{-2} \| x - \theta^*_i \|^2 \leq V^{i,m}(x, y) \leq C_L \left( m^{-2} \| x - \theta^*_i \|^2 + \| y \|^2 \right),
\]
where \( c = \frac{1}{16} \) and \( C_L = L + 2 \).
Since \( \Phi_i(x) - \Phi_i(\theta_i^*) \geq 0 \) and \( \|x - \theta_i^* + my\|^2 + \|my\|^2 \geq \frac{\|x - \theta_i^*\|^2}{2} \), we have
\[
V^{i,m}(x, y) \geq \frac{1}{4m^2} \left( \|x - \theta_i^* + my\|^2 + \|my\|^2 - m\lambda \|x - \theta_i^*\|^2 \right)
\[
\geq \frac{1}{4m^2} \left( \frac{1}{2} - m\lambda \right) \|x - \theta_i^*\|^2 \geq \frac{\|x - \theta_i^*\|^2}{16m^2}.
\]
Notice that
\[
\Phi_i(x) - \Phi_i(\theta_i^*) = \int_0^1 (x - \theta_i^*) \left[ \nabla \Phi_i((x - \theta_i^*)s + \theta_i^*) - \nabla \Phi_i(\theta_i^*) \right] ds \leq L \frac{\|x - \theta_i^*\|^2}{2}.
\]
This implies that
\[
V^{i,m}(x, y) \leq \frac{L \|x - \theta_i^*\|^2}{2m} + \frac{1}{2m^2} \left( \|x - \theta_i^*\|^2 + \|my\|^2 \right) \leq C_L \left( m^{-2} \|x - \theta_i^*\|^2 + \|y\|^2 \right),
\]
where \( C_L \) is \( L + 2 \). Hence, we immediately get
\[
m^{-2} \left\| \dot{q}^{i,m}_t - \theta_i^* \right\|^2 \leq 16C_L e^{-\lambda t} \left( m^{-2} \left\| \dot{q}^0_i - \theta_i^* \right\|^2 + \left\| \dot{p}^0_i \right\|^2 \right),
\]
which implies
\[
\left\| \dot{q}^{i,m}_t - \theta_i^* \right\|^2 \leq 16C_L e^{-\lambda t} \left( \left\| \dot{q}^0_i - \theta_i^* \right\|^2 + m^2 \left\| \dot{p}^0_i \right\|^2 \right).
\]
We obtain the asserted result since \( m \leq 1 \) and all the terms are non-negative. \( \square \)

In the next auxiliary result, we show boundedness of the velocity, i.e., \( (\dot{p}^{i,m}_t)_{t \geq 0} \) that depends on the mass \( m \). Moreover, we show Lipschitz continuity of the particle position with respect to time. Here, the Lipschitz constant can be chosen independently of \( m \).

**Lemma 2.** Under the same assumptions as Lemma 1, we have for any \( 0 \leq s \leq t \),
\[
\left\| \dot{p}^{i,m}_t \right\| \leq \left( e^{\frac{\lambda s}{m}} + m \right) \left\| \dot{p}^0_i \right\| + C_L \left\| \dot{q}^0_i - \theta_i^* \right\|,
\]
\[
\left\| \dot{q}^{i,m}_t - \dot{q}^{i,m}_s \right\| \leq C_L (t - s) \left[ \left\| \dot{q}^0_i - \theta_i^* \right\| + \left\| \dot{p}^0_i \right\| \right],
\]
where, again, \( C_L := L + 2 \).

**Proof.** From (2.7), we have
\[
\frac{d(e^{\frac{s}{m}} \dot{p}^{i,m}_t)}{dt} = -m^{-1} e^{\frac{s}{m}} \nabla \Phi_i(q^{i,m}_t),
\]
which implies
\[
\dot{p}^{i,m}_t = e^{\frac{s}{m}} \dot{p}^0_i - m^{-1} e^{\frac{s}{m}} \int_0^t e^{\frac{s}{m}} \nabla \Phi_i(q^{i,m}_s) ds.
\]  \[ (2.10) \]
Hence, we have
\[\| p_t^{i,m} \| \leq \left\| e^{\frac{m}{2} t} p_0^i \right\| + m^{-1} e^{\frac{m}{2} t} \left\| \int_0^t e^{\frac{m}{2} s} \nabla \Phi_i(q_s^{i,m}) ds \right\| \]
\[\leq \left\| e^{\frac{m}{2} t} p_0^i \right\| + m^{-1} e^{\frac{m}{2} t} \int_0^t e^{\frac{m}{2} s} \| \nabla \Phi_i(q_s^{i,m}) - \nabla \Phi_i(\theta_s^i) \| ds \]
\[\leq e^{\frac{m}{2} t} \| p_0^i \| + m^{-1} e^{\frac{m}{2} t} L \int_0^t e^{\frac{m}{2} s} \| q_s^{i,m} - \theta_s^i \| ds \]
\[\leq \left( e^{\frac{m}{2} t} + m \right) \| p_0^i \| + L \| q_0^i - \theta_0^i \|. \]

This also implies \( \| p_t^{i,m} \| \leq C_L(\| \dot{p}_0^i \| + \| q_0^i - \theta_0^i \|) \). Then from (2.7), we immediately get
\[\| q_t^{i,m} - q_s^{i,m} \| = \left\| \int_s^t p_m^{i,m} dm \right\| \leq \int_s^t \| p_m^{i,m} \| \, dm \leq C_L(t-s) \left( \| q_0^i - \theta_0^i \| + \| p_0^i \| \right). \]

\[\square\]

The third auxiliary result is a bound on the time derivative of the velocity \((p_t^{i,m})_{t \geq 0}\), i.e. a bound on the particle’s acceleration.

**Lemma 3.** Under the same assumptions as Lemma 1 for any \( t \geq 0 \),
\[ \left\| \frac{dp_t^{i,m}}{dt} \right\| \leq C_L^{(0)} (1 + m^{-1} e^{-t/m}) \left[ \| q_0^i - \theta_0^i \| + \| p_0^i \| \right]. \tag{2.11} \]

where \( C_L^{(0)} = 2 + 4LC_L = 2 + 8L + 4L^2 \)

**Proof.** Recall (2.10), we have
\[ \frac{dp_t^{i,m}}{dt} = -e^{\frac{m}{2} t} p_0^i - \frac{\nabla \Phi_i(q_t^{i,m})}{m} + \frac{e^{\frac{m}{2} t}}{m^2} \int_0^t e^{\frac{m}{2} s} \nabla \Phi_i(q_s^{i,m}) ds. \tag{2.12} \]

We notice that
\[ \frac{\nabla \Phi_i(q_t^{i,m})}{m} = \frac{e^{-\frac{m}{2} t}}{m^2} \int_0^t e^{\frac{m}{2} s} \nabla \Phi_i(q_s^{i,m}) ds + \frac{e^{-\frac{m}{2} t}}{m} \nabla \Phi_i(q_t^{i,m}). \tag{2.13} \]
Then for Proposition 2. Let the same assumptions hold as in Lemma 1 and let also
In addition to the previous assumptions, we now also need to assume \( \Phi \)
\( \text{case, we show that the solution to (2.7) converges to the solution of the limiting equation (2.8).} \)

Hence combine with (2.12) and (2.13), we get

\[
\left\| \frac{dp_{i,m}^t}{dt} \right\| = e^{\frac{t}{m}} \left\| \frac{-p_0^i}{m} - \nabla \Phi_i(q_{i,m}^t) + \frac{1}{m^2} \int_0^t e^{\frac{s}{m}} (\nabla \Phi_i(q_s^i) - \nabla \Phi_i(q_{i,m}^t)) ds \right\|
\]

\[
\quad = e^{\frac{t}{m}} \left\| \frac{-p_0^i}{m} - \nabla \Phi_i(q_{i,m}^t) - \nabla \Phi_i(\theta_s^i) + \frac{1}{m^2} \int_0^t e^{\frac{s}{m}} (\nabla \Phi_i(q_s^i) - \nabla \Phi_i(q_{i,m}^t)) ds \right\|
\]

\[
\quad \leq e^{\frac{t}{m}} \left\| \frac{\|p_0^i\| + L \|q_{i,m}^t - \theta_s^i\|}{m} + \frac{L}{m^2} \int_0^t e^{\frac{s}{m}} \|q_s^i - q_{i,m}^t\| ds \right\|
\]

\[
\quad = m^{-1} e^{\frac{t}{m}} \left[ \|p_0^i\| + L \|q_{i,m}^t - \theta_s^i\| \right] + \frac{L}{m^2} \int_0^t e^{\frac{(t-s)}{m}} \|q_s^i - q_{i,m}^t\| ds
\]

\[
\quad \leq m^{-1} e^{\frac{t}{m}} \left[ \|p_0^i\| + L \|q_{i,m}^t - \theta_s^i\| \right] + \frac{LC_L}{m} \int_0^t e^{\frac{(t-s)}{m}} (t-s) ds \left[ \|q_0^i - \theta_s^i\| + \|p_0^i\| \right]
\]

\[
\quad = m^{-1} e^{\frac{t}{m}} \left[ \|p_0^i\| + L \|q_{i,m}^t - \theta_s^i\| \right] + LC_L \int_0^t e^{-\frac{s}{m}} \|q_0^i - \theta_s^i\| + \|p_0^i\| ds
\]

\[
\quad \leq \text{(a1)} \frac{m^{-1} e^{\frac{t}{m}}}{m} \left[ \|p_0^i\| + 4LC_L (\|q_0^i - \theta_s^i\| + \|p_0^i\|) \right] + LC_L \left[ \|q_0^i - \theta_s^i\| + \|p_0^i\| \right]
\]

\[
\quad \leq C_L^{(0)} (1 + m^{-1} e^{\frac{t}{m}}) \left[ \|q_0^i - \theta_s^i\| + \|p_0^i\| \right],
\]

where the inequality (a1) come form the inequality (2.9) and the fact \( \int_0^t e^{\frac{s}{m}} e^{-s} ds < 1 \).

The previous lemmas provide the boundedness results needed for proving the following proposition. Here, we show that the difference between the fixed sample SGMP (2.7) and the fixed sample stochastic gradient process (2.8) can be bounded by the sum of the distance between their initial values and a term that is linear in \( m \). Hence, in the fixed subsample case, we show that the solution to (2.7) converges to the solution of the limiting equation (2.8). In addition to the previous assumptions, we now also need to assume \( \Phi_i \) to be convex.

**Proposition 2.** Let the same assumptions hold as in Lemma 1 and let also \( \Phi_i \) be convex. Then for \( 0 < m \leq 1 \), we have

\[
\|q_{i,m}^t - \theta_i^t\| \leq \|q_0^i - \theta_0^i\| + C_L^{(0)} m (1 + t) \left[ \|p_0^i\| + \|q_0^i - \theta_0^i\| + \|\theta_0^i - \theta^i_0\| \right],
\]
Proof. We take the deviation of \( \| q_{t_i}^{i,m} - \theta_i \|^2 \),
\[
\frac{1}{2} \frac{d}{dt} \| q_{t_i}^{i,m} - \theta_i \|^2 = \left\langle q_{t_i}^{i,m} - \theta_i, \frac{dq_{t_i}^{i,m}}{dt} - \frac{d\theta_i}{dt} \right\rangle
\]
\[
= -\left\langle q_{t_i}^{i,m} - \theta_i, \nabla \Phi_i(q_{t_i}^{i,m}) - \nabla \Phi_i(\theta_i) \right\rangle - m \left\langle q_{t_i}^{i,m} - \theta_i, \frac{dp_{t_i}^{i,m}}{dt} \right\rangle
\]
\[
\leq -m \left\langle q_{t_i}^{i,m} - \theta_i, \frac{dp_{t_i}^{i,m}}{dt} \right\rangle \leq m \| q_{t_i}^{i,m} - \theta_i \| \| \frac{dp_{t_i}^{i,m}}{dt} \| \]
\[
\overset{2.1}{\leq} C_L^{(0)} m (1 + m^{-1} e^{-\frac{t_i}{m}}) \left[ \| q_0^i - \theta_0^i \| + \| p_0^i \| \right] \| q_{t_i}^{i,m} - \theta_i \|
\]
which implies
\[
\frac{d}{dt} \| q_{t_i}^{i,m} - \theta_i \|^2 = \frac{d}{dt} \left\langle q_{t_i}^{i,m} - \theta_i, \frac{dp_{t_i}^{i,m}}{dt} \right\rangle \leq C_L^{(0)} m (1 + m^{-1} e^{-\frac{t_i}{m}}) \left[ \| q_0^i - \theta_0^i \| + \| p_0^i \| \right].
\]
Integrating both sides, we get
\[
\| q_{t_i}^{i,m} - \theta_i \| \leq \| q_0^i - \theta_0^i \| + C_L^{(0)} \int_0^t (m + e^{-\frac{s}{m}}) ds \left[ \| p_0^i \| + \| q_0^i - \theta_0^i \| \right]
\]
\[
= \| q_0^i - \theta_0^i \| + C_L^{(0)} \left[ mt + m(1 - e^{-\frac{t}{m}}) \right] \left[ \| p_0^i \| + \| q_0^i - \theta_0^i \| \right]
\]
\[
\leq \| q_0^i - \theta_0^i \| + C_L^{(0)} m(1 + t) \left[ \| p_0^i \| + \| q_0^i - \theta_0^i \| \right]
\]
\[
\leq \| q_0^i - \theta_0^i \| + C_L^{(0)} m(1 + t) \left[ \| p_0^i \| + \| q_0^i - \theta_0^i \| + \| \theta_0^i - \theta_i \| \right].
\]
\[
\Box
\]

In the next subsection, we use this results to also understand the stochastic case.

Random subsampling. We establish the connection between \((q_{t_i}^{m})_{t \geq 0}\) and \((\theta_{t_i}^{m})_{t \geq 0}\) in the following theorem by iterating over jump times of the index process. In each of the iteration steps, we employ Proposition 2.

**Theorem 3.** Under the same assumptions as Proposition 2, let \((q_{t_i}^{m})_{t \geq 0}\) and \((\theta_{t_i}^{m})_{t \geq 0}\) solve (2.2) and (2.6), respectively. For \(0 \leq \delta < 1\) and \(T > 0\),
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| q_{t}^{m} - \theta_{t}^{m} \| \right] \leq \left( 1 + C_L^{'} m(1 + T) \right) e^{m^{1-\delta}T(1+T)\gamma N} \left( \| q_0 - \theta_0 \| + m \| p_0 \| \right)
\]
\[
+ \left[ e^{m^{1-\delta}T(1+T)\gamma N} - 1 \right] + C_L^{'} m(T + 1) K_{\Phi,T,\theta_0},
\]
where \(C_L^{'} , K_{\Phi,T,\theta_0} > 0\) are constants.

**Proof.** We now denote by \(\{\tau_{n}\}_{n \geq 0}\) and \(\{\tau_{n}^{m,\delta}\}_{n \geq 0}\) the sequences of the jump times of processes \(i(t)\) and \(i^{m,\delta}(t)\). It is obvious that \(\tau_0 = 0\). From the definition, we know the last jump time
before \( t \) is \( \tau_{N_t} \), where \( N_t \) is a Possion process with rate \( \gamma N \). And since \( i^{m,\delta}(t) = i(t/m^\delta) \), we have \( \tau_{n+1}^{m,\delta} = m^\delta \tau_n \). By Proposition \( \ref{prop} \) for \( \tau_n^{m,\delta} < t \leq \tau_{n+1}^{m,\delta} \),

\[
\| q_t^m - \theta_t^m \| \leq \| q_{\tau_n^{m,\delta}}^m - \theta_{\tau_n^{m,\delta}}^m \| + C_{L}(0) m(1 + t - \tau_n^{m,\delta})(\| q_{\tau_n^{m,\delta}}^m - \theta_{\tau_n^{m,\delta}}^m \| + \| \theta_{\tau_n^{m,\delta}}^m - \theta_{\tau_{n+1}^{m,\delta}}^m \|)
\]

and from the first inequality in Lemma \( \ref{lem} \)

\[
\| p_t^m \| \leq (1 + m) \| p_{\tau_n^{m,\delta}}^m \| + C_L \| q_{\tau_n^{m,\delta}}^m - \theta_{\tau_n^{m,\delta}}^m \| + \| \theta_{\tau_n^{m,\delta}}^m - \theta_{\tau_{n+1}^{m,\delta}}^m \|.
\]

From Lemma \( \ref{lem} \) in the Appendix,

\[
\| \theta_{\tau_n^{m,\delta}}^m - \theta_{\tau_{n+1}^{m,\delta}}^m \| \leq \| \theta_{\tau_n^{m,\delta}}^m \| + \| \theta_{\tau_{n+1}^{m,\delta}}^m \| \leq \| \theta_0 \| + C_{\Phi} T + K_{\Theta^*} := K_{\Phi,T,\theta_0}.
\]

So we have

\[
\sup_{0 \leq t \leq \tau_{n+1}^{m,\delta}} \| q_t^m - \theta_t^m \| \leq \max \left\{ \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| q_t^m - \theta_t^m \|, \sup_{\tau_n^{m,\delta} < t \leq \tau_{n+1}^{m,\delta}} \| q_t^m - \theta_t^m \| \right\}
\]

\[
\leq \sup_{0 \leq t \leq \tau_{n+1}^{m,\delta}} \| q_t^m - \theta_t^m \| + C_{L}(0) m(1 + \tau_{n+1}^{m,\delta} - \tau_n^{m,\delta})(\| q_{\tau_n^{m,\delta}}^m - \theta_{\tau_n^{m,\delta}}^m \| + \| p_{\tau_n^{m,\delta}}^m \| + K_{\Phi,T,\theta_0}).
\]

For \( T > 0 \), if we assume \( \tau_{n+1}^{m,\delta} \leq T \), we have

\[
\sup_{0 \leq t \leq \tau_{n+1}^{m,\delta}} \| q_t^m - \theta_t^m \| \leq \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| q_t^m - \theta_t^m \|
\]

\[
+ C_{L}(0) m(1 + T) \left[ \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| q_t^m - \theta_t^m \| + \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| p_t^m \| + K_{\Phi,T,\theta_0} \right].
\]

We denote

\[
A_{n}^{m,\delta} := \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| q_t^m - \theta_t^m \|, \quad B_{n}^{m,\delta} := \sup_{0 \leq t \leq \tau_n^{m,\delta}} \| p_t^m \|, \quad D_{n}^{m,\delta} = A_{n}^{m,\delta} + m B_{n}^{m,\delta}.
\]

We rewrite the inequalities

\[
A_{n+1}^{m,\delta} \leq A_{n}^{m,\delta} + m C_{L}(0) (1 + T) \left( A_{n}^{m,\delta} + B_{n}^{m,\delta} + K_{\Phi,T,\theta_0} \right),
\]

\[
B_{n+1}^{m,\delta} \leq (1 + m) B_{n}^{m,\delta} + C_{L} \left( A_{n}^{m,\delta} + K_{\Phi,T,\theta_0} \right).
\]

Since \( m < 1 \), there exists constant \( C_{L}' \) such that

\[
D_{n+1}^{m,\delta} \leq \left( 1 + C_{L}' m(1 + T) \right) D_{n}^{m,\delta} + C_{L}' m(1 + T) K_{\Phi,T,\theta_0}.
\]

Hence, for \( n \geq 0 \), we have

\[
D_{n}^{m,\delta} \leq \left( 1 + C_{L}' m(1 + T) \right)^n \left[ \| q_0 - \theta_0 \| + \| p_0 \| \right] + \left[ \left( 1 + C_{L}' m(1 + T) \right)^n - 1 \right] K_{\Phi,T,\theta_0}.
\]
Let \( n = N_{T/m} \). Form (2.15) and by using the moment-generating function of Possion random variable, we conclude:

\[
\mathbb{E}[D^m_{N_{T/m}}] \leq \left[ \|q_0 - \theta_0\| + \|p_0\| \right] \mathbb{E}\left[ \left( 1 + C'_L m(1 + T) \right)^{N_{T/m}} \right]
\]

\[
+ K_{\Phi,T,\theta_0} \mathbb{E}\left[ \left( 1 + C'_L m(1 + T) \right)^{N_{T/m}} - 1 \right]
\]

\[
= e^{m^{1-\delta}(1+T)\gamma N} \left[ \|q_0 - \theta_0\| + m \|p_0\| \right] + \left[ e^{m^{1-\delta}(1+T)\gamma N} - 1 \right] K_{\Phi,T,\theta_0}.
\]

For any \( T \geq 0 \), there exists an integer \( n \geq 0 \), such that \( \tau^{m,\delta}_n \leq T \leq \tau^{m,\delta}_{n+1} \) (i.e. \( n = N_{T/m} \)). Hence

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|q^m_t - \theta^m_t\| \right] \leq \mathbb{E}\left[ \sup_{0 \leq t \leq \tau^{m,\delta}_{n+1}} \|q^m_t - \theta^m_t\| \right] \leq \mathbb{E}[D^m_{N_{T/m}}+1] \]

\[
\leq \left( 1 + C'_L m(1 + T) \right) e^{m^{1-\delta}(1+T)\gamma N} \left( \|q_0 - \theta_0\| + m \|p_0\| \right)
\]

\[
+ \left[ e^{m^{1-\delta}(1+T)\gamma N} - 1 + C'_L m(T + 1) \right] K_{\Phi,T,\theta_0}.
\]

\( \Box \)

Thus, if we let the mass \( m \) go down to zero and the learning rate parameter \( \nu \) is independent of \( \mu \) or \( \nu \) goes to zero at a certain rate that is slower than \( m \), the stochastic gradient-momentum process with momentum converges to the stochastic gradient process. The speed of convergence is linear in \( m \), which had also been the case in the deterministic setting. Note that we show convergence in a possibly weaker sense as compared with the learning rate result in Theorem 1.

We finish our discussion of the homogeneous-in-time setting with a final remark regarding reducing mass and learning rate at the same time.

**Remark 3.** From [20, Theorem 1], we already know \((\theta^m_t)_{t \geq 0} \rightarrow (\zeta_t)_{t \geq 0} \) if \( m \rightarrow 0 \), i.e. the stochastic gradient process converges to the gradient flow as the learning rate approaches zero. Convergence is here in the weak sense probabilistically and the usual \( \infty \)-norm

\[
\rho'(\langle \zeta_t \rangle_{t \geq 0}, (\xi_t)_{t \geq 0}) := \sup_{s \geq 0} \| \zeta_s - \xi_s \|,
\]

in space. Combined with Theorem 3, we can easily see that also \((q^m_t)_{t \geq 0} \rightarrow (\zeta_t)_{t \geq 0} \) in the case \( \delta \in (0,1) \). Here, convergence is in the sense of Theorem 3.

### 3. Heterogeneous-in-time

In the last section, we have studied the effect of losing momentum and reducing the learning rate in a uniform fashion. Having a homogeneous mass and learning rate is unquestionably very popular in practice. However, while we could show convergence to the minimiser of the deterministic methods, this is typically not true for the stochastic methods. To obtain convergence in SGMP, we need to reduce both, learning rate and momentum, over time.
Hence, we need to discuss a heterogeneous version of the SGMP. Indeed, we now allow the mass \( m \) to be a non-constant function of time, especially to be reduced to zero over time, and the index process \((i^j(t))_{t \geq 0}\) to have waiting times that get smaller as time progresses. Thus we study the case where the stochastic gradient-momentum process converges to the stochastic gradient process and the gradient flow, respectively, at runtime. This may sometimes lead to faster convergence overall, as we have seen in Figure 1.

We again split this section into two parts. In Subsection 3.1 we let only the momentum depend on and decrease over time. We introduce the dynamical system, show well-posedness, and discuss the longtime behaviour of the new dynamical system. Here, we see that asymptotically, the dynamical system behaves like the stochastic gradient process. Then, in Subsection 3.2 we decrease mass and learning rate over time. In this case, we discuss a setting, in which we see convergence of the stochastic dynamical system to the minimiser \( \theta^* \) of the target function \( \Phi \).

### 3.1. Losing momentum over time

We consider the stochastic gradient-momentum process with a time-dependent, decreasing mass.

**Definition 3.** The decreasing-mass stochastic gradient-momentum process (dmSGMP) is a solution of the following stochastic differential equation,

\[
\begin{align*}
    dq_t &= p_t dt \\
    m(t)dp_t &= -\nabla \Phi_{i(t)}(q_t)dt - p_t dt, \\
    p(t = 0) &= p_0, \\
    q(t = 0) &= q_0,
\end{align*}
\]

where \( \Phi_j \) satisfies Assumption 1, \( j = 1, \cdots, N \). The stochastic process \( \{i(t)\}_{t \geq 0} \) is defined in Definition 1. The mass \( m(t) > 0 \) is strictly decreasing and differentiable with \( \lim_{t \to \infty} m(t) = 0 \).

The formal limit of (3.1) is the stochastic gradient process without momentum, as given in (1.3). As in the previous section, we commence our discussion considering the decreasing mass case with a fixed index, i.e. the following dynamical system:

\[
\begin{align*}
    dq^i_t &= p^i_t dt \\
    m(t)dp^i_t &= -\nabla \Phi_i(q^i_t)dt - p^i_t dt, \\
    p^i(t = 0) &= p^i_0, \\
    q^i(t = 0) &= q^i_0,
\end{align*}
\]

where the mass \( (m(t))_{t \geq 0} \) is chosen as before. In addition, we denote \( \mathcal{E}(t) := \int_0^t 1/m(s)ds \). The associated limiting equation is, again, given by (2.8). Next, we discuss the well-posedness of the system (3.2). Here, the mass must not decrease too quickly. We quantify this statement in the following assumption.

**Assumption 3.** There exists a constant \( \lambda \in (0, 1] \), such that \( |m'(t)| \leq \lambda m(t) \), \( \forall t > 0 \). For the initial value, we assume \( m(0) =: m_0 \leq 1 \).

It is easy to verify that \( m(t) = \frac{m_0}{\lambda^{-1} + t} \) and \( m(t) = m_0 e^{-\lambda t} \) satisfy Assumption 3. Setting \( m_0 = \lambda = 1 \), we retrieve the example in Figure 1. More generally, Assumption 3 implies \( m(t) \geq m_0 e^{-\lambda t} \).
Well-posedness. We study the well-posedness of the deterministic dynamical system (3.2). Well-posedness of the stochastic dynamical system (3.1) then follows from, e.g., Proposition 1 in [20].

**Proposition 3.** For any fixed \( i \in I \), let \( \nabla \Phi_i \) satisfy Assumption 1 with constant \( L \) and \( m(t) \) satisfy Assumption 3. Then the equation (3.2) admits a unique solution.

**Proof.** We first rewrite equation (3.2) as

\[
\begin{cases}
\frac{dq^i_t}{dt} = p^i_t dt \\
\frac{dp^i_t}{dt} = -\nabla \Phi_i(q^i_t)/m(t)dt - p^i_t/m(t)dt,
\end{cases}
\]

where \( 0 < t \leq t_1 \). Next, we are going to show \( \mathcal{T}_0 \) is a contraction for \( t_1 \) small enough. Since \( m(t) \) is strictly decreasing and positive, combining Assumption 3, we can conclude that

\[
(e^{\lambda t}m(t))' \geq 0.
\]

This implies

\[
m(t) \geq m_0 e^{-\lambda t}.
\]

We then conclude

\[
\|\mathcal{T}_0(q,p) - \mathcal{T}_0(q',p')\|_\infty \leq t_1 \|p - p'\|_\infty + L \|q - q'\|_\infty \int_0^{t_1} \frac{1}{m(s)} ds + \|p - p'\|_\infty \int_0^{t_1} \frac{1}{m(s)} ds
\]

\[
\leq (\|q - q'\|_\infty + \|p - p'\|_\infty) \left( \frac{L + 1}{\lambda m_0} (e^{\lambda t_1} - 1 + t_1) \right).
\]

Set \( t_1 = \frac{\lambda m_0}{2(L+1)(2\lambda+1)} =: C(\lambda, m_0, L) \), and since \( e^{\lambda x} - 1 \leq 2\lambda x \) when \( \lambda x \leq 1 \), we have

\[
\frac{L + 1}{\lambda m_0} (e^{\lambda t_1} - 1 + t_1) \leq \frac{L + 1}{\lambda m_0} (2\lambda + 1) t_1 \leq \frac{1}{2}.
\]

Then \( \mathcal{T}_0 \) is contracting with constant smaller 1/2. By Banach Fixed Point Theorem, we conclude equation (3.2) admits a unique solution on the interval \([0, t_1]\) with initial value \((q_0, p_0)\).

Let \( \mathcal{T}_n : C^1([t_n, t_{n+1}], \mathbb{R}) \times C^1([t_n, t_{n+1}], \mathbb{R}) \), with

\[
\mathcal{T}_n(x, y) = \left( q_{n+1} + \int_{t_n}^{t_{n+1}} y_s/m(s) ds, \ p_{n+1} - \int_{t_n}^{t_{n+1}} \nabla \Phi_i(x_s)/m(s) ds - \int_{t_n}^{t_{n+1}} y_s/m(s) ds \right).
\]

Then we have

\[
\|\mathcal{T}_n(q,p) - \mathcal{T}_n(q',p')\|_\infty \leq (t_{n+1} - t_n) \|p - p'\|_\infty + L \|q - q'\|_\infty \int_{t_n}^{t_{n+1}} \frac{1}{m(s)} ds + \|p - p'\|_\infty \int_{t_n}^{t_{n+1}} \frac{1}{m(s)} ds
\]

\[
\leq (\|q - q'\|_\infty + \|p - p'\|_\infty) \frac{L + 1}{\lambda m_0} e^{\lambda h_n} (e^{\lambda h_n} - 1 + h_n),
\]

where \( h_n \) is the step-size.
where \( h_n = t_{n+1} - t_n \). Set \( h_n = C(\lambda, m_0, L)e^{-\lambda t_n} = \frac{\lambda m_0 e^{-\lambda t_n}}{2(L+1)(2\lambda+1)} \), then

\[
\frac{L + 1}{\lambda m_0} e^{\lambda t_n} (e^{\lambda t_n} - 1 + h_n) \leq \frac{L + 1}{\lambda m_0} e^{\lambda t_n} (2\lambda + 1) h_n \leq \frac{1}{2}.
\]

This implies \( T_n \) is a contraction, and by Banach fixed-point theorem, we conclude equation (3.2) admits a unique solution on the interval \([t_n, t_{n+1}]\) with initial value \((q_{t_n}, p_{t_n})\). Therefore, by Lemma 10, \( t_n \to \infty \), equation (3.2) admits a unique solution globally. \( \square \)

As the unique solution to the dynamical system (3.2) is a strong solution, the solution is continuously differentiable. Thus, the map \((p_t^0, p_t^0) \mapsto (p_t^i, p_t^i)\) is continuous for any \( t \geq 0 \). Hence, the differential equations (3.2) and (3.1) are well-posed.

**Long time behavior.** Having understood the well-posedness of dmSGMP, we are now interested in the longtime behaviour of the dynamical system. By reducing the mass over time, we aim to reduce the momentum and converge (in an appropriate sense) to the stochastic gradient process (1.3).

To show this convergence result, we now collect a number of auxiliary results that will, again, mainly concern the deterministic system (3.2) that considers a single, fixed sample. Some of these results may remind the reader of similar results shown in Subsection 2.2 regarding the fixed sample, homogeneous momentum dynamic; we make these connections clear in the titles of the following results. The intermediate goal is to show a bound between the system (3.2) and its limiting equation (2.8). From there we will then go back to the randomised setting and discuss the longtime behaviour and convergence of dmSGMP.

We first show that the deterministic system converges at exponential speed to its unique stationary point.

**Lemma 4 (cf. Lemma 1).** Let \( q_t^i \) solve (3.2). Let \( \theta^*_t \) be the critical point of \( \Phi_t \). Under the Assumption 1 with \( \alpha = 2 \), and 3, we have

\[
\|q_t^i - \theta^*_t\|^2 \leq 8e^{-\lambda t} V^i(0, q_0^i, p_0^i),
\]

where \( V^i(t, x, y) = m(t)[\Phi_t(x) - \Phi_t(\theta^*_t)] + \frac{1}{4}\left( \|x - \theta^*_t + m(t)y\|^2 + \|m(t)y\|^2 \right) \).

**Proof.** By Lemma 11, we have \( \|q_t^i - \theta^*_t\|^2 \leq 8V^i(t, q_t^i, p_t^i) \leq 8e^{-\lambda t} V^i(0, q_0^i, p_0^i). \) \( \square \)

In the next auxiliary result, we show Lipschitz continuity of the position \((q_t^i)_{t \geq 0}\) and boundedness of the velocity \((p_t^i)_{t \geq 0}\).

**Lemma 5 (cf. Lemma 2).** Under the same assumptions as Lemma 4, for any \( 0 \leq s \leq t \), we have

\[
\|q_t^i - q_s^i\| \leq C_L^{(1)}(t-s) \left[ \|q_0^i - \theta^*_s\| + (m_0 + e^{-\varepsilon(s)}) \|p_0^i\| \right],
\]

\[
\|p_t^i\| \leq e^{-\varepsilon(t)} \|p_0^i\| + C_L^{(1)}(\|q_0^i - \theta^*_s\| + m_0 \|p_0^i\|),
\]

where \( C_L^{(1)} = 1 + 8L^2 + 8L \).
Proof. From (3.2), we have
\[
\frac{d(e^{\epsilon(t)} p^i)}{dt} = -e^{\epsilon(t)} \nabla \Phi_i(q^i_t)/m(t),
\]
which implies
\[
p^i_t = e^{-\epsilon(t)} p^i_0 - e^{-\epsilon(t)} \int_0^t e^{\epsilon(s)} \nabla \Phi_i(q^i_s)/m(s) ds.
\] (3.4)
Hence
\[
\|p^i_t\| \leq e^{-\epsilon(t)} \|p^i_0\| + e^{-\epsilon(t)} \int_0^t \frac{e^{\epsilon(s)}}{m(s)} \|\nabla \Phi_i(q^i_s)\| ds
\]
\[
= e^{-\epsilon(t)} \|p^i_0\| + e^{-\epsilon(t)} \int_0^t \frac{e^{\epsilon(s)}}{m(s)} \|\nabla \Phi_i(q^i_s) - \nabla \Phi_i(\theta^i_s)\| ds
\]
\[
\leq e^{-\epsilon(t)} \|p^i_0\| + L e^{-\epsilon(t)} \int_0^t \frac{e^{\epsilon(s)}}{m(s)} \|q^i_s - \theta^i_s\| ds
\]
\[
\leq e^{-\epsilon(t)} \|p^i_0\| + 8L e^{-\epsilon(t)} (V^i(0, q^i_0, p^i_0))^{1/2} \int_0^t \frac{e^{\epsilon(s)}}{m(s)} e^{-\epsilon(t)} ds
\]
\[
\leq e^{-\epsilon(t)} \|p^i_0\| + 8L (V^i(0, q^i_0, p^i_0))^{1/2}.
\]
The last inequality holds since \(\int_0^t \frac{e^{\epsilon(s)}}{m(s)} e^{-\epsilon(t)} ds \leq e^{\epsilon(t)}\), which can be seen by taking the derivative on both sides and comparing the initial value. And since \(V^i(0, q^i_0, p^i_0) \leq (1 + L)(\|q^i_0 - \theta^i_s\|^2 + m_0^2 \|p^i_0\|^2)\), we have
\[
\|p^i_t\| \leq e^{-\epsilon(t)} \|p^i_0\| + (1 + 8L^2 + 8L)(\|q^i_0 - \theta^i_s\| + m_0 \|p^i_0\|)
\]
Set \(C^{(1)}_L = (1 + 8L^2 + 8L)\), we have
\[
\|q^i_t - \theta^i_s\| = \left| \int_s^t p^i_m dm \right| \leq \int_s^t \|p^i_m\| dm \leq C^{(1)}_L (t-s)(\|q^i_0 - \theta^i_s\| + m_0 \|p^i_0\|) + \|p^i_0\|^2 \int_s^t e^{-\epsilon(s)} dx
\]
\[
\leq C^{(1)}_L (t-s) \left[ \|q^i_0 - \theta^i_s\| + (m_0 + e^{-\epsilon(s)}) \|p^i_0\| \right].
\]
\[
\Box
\]
In the third lemma, we give again a bound on the system’s acceleration \((dp^i_t/dt)_{t \geq 0}\).

Lemma 6 (cf. Lemma 3). Under the same assumptions as Lemma 4, we have
\[
\left| \frac{dp^i_t}{dt} \right| \leq C^{(2)}_L \left[ \frac{e^{-\epsilon(t)}}{m(t)} + 2 \right] \left( \|p^i_0\| + \|q^i_0 - \theta^i_s\| \right),
\] (3.5)
where \(C^{(2)}_L = 8(C^{(1)}_L)^2 = 8(1 + 8L^2 + 8L)^2\).

Proof. From (3.4), we have
\[
\frac{dp^i_t}{dt} = \frac{-e^{-\epsilon(t)} p^i_0}{m(t)} - \frac{\nabla \Phi_i(q^i_t)}{m(t)} + \frac{e^{-\epsilon(t)}}{m(t)} \int_0^t e^{\epsilon(s)} \nabla \Phi_i(q^i_s)/m(s) ds.
\]
Notice that
\[
\frac{\nabla \Phi_i(q^i_t)}{m(t)} = e^{-\xi(t)} \int_0^t e^{\xi(s)} \nabla \Phi_i(q^i_s)/m(s)ds + \frac{e^{-\xi(t)} \nabla \Phi_i(q^i_t)}{m(t)}.
\]
Then we could rewrite \(\frac{dp^i_t}{dt}\) as
\[
\frac{dp^i_t}{dt} = -e^{-\xi(t)}(p^i_0 + \nabla \Phi_i(q^i_t)) + \frac{e^{-\xi(t)}}{m(t)} \int_0^t e^{\xi(s)} [\nabla \Phi_i(q^i_s) - \nabla \Phi_i(q^i_t)]/m(s)ds.
\]
Hence
\[
\left\| \frac{dp^i_t}{dt} \right\| \leq \frac{e^{-\xi(t)}}{m(t)} \left[ \left\| p^i_0 + \nabla \Phi_i(q^i_t) \right\| + \int_0^t e^{\xi(s)} \left\| \nabla \Phi_i(q^i_s) - \nabla \Phi_i(q^i_t) \right\|/m(s)ds \right]
\]
\[
= \frac{e^{-\xi(t)}}{m(t)} \left[ \left\| p^i_0 + \nabla \Phi_i(q^i_t) - \nabla \Phi_i(\theta^*_t) \right\| + \int_0^t e^{\xi(s)} \left\| \nabla \Phi_i(q^i_s) - \nabla \Phi_i(q^i_t) \right\|/m(s)ds \right]
\]
\[
\leq (1 + L)e^{-\xi(t)} \frac{1}{m(t)} \left[ \left\| p^i_0 \right\| + \left\| q^i_t - \theta^*_t \right\| + \int_0^t e^{\xi(s)} \left\| q^i_s - q^i_t \right\|/m(s)ds \right]
\]
\[
\leq \frac{4C^{(1)}_L e^{-\xi(t)}}{m(t)} \left[ \left\| q^i_t - \theta^*_t \right\| + \left\| p^i_0 \right\| 
\right.
\]
\[
+ C^{(1)}_L \int_0^t (t - s)e^{\xi(s)}/m(s)ds \left[ \left\| q^i_0 - \theta^*_t \right\| + \left( m_0 + e^{-\xi(s)} \right) \left\| p^i_0 \right\| \right]
\]
\[
= \frac{4C^{(1)}_L e^{-\xi(t)}}{m(t)} \left[ \left\| q^i_t - \theta^*_t \right\| \left[ 1 + C^{(1)}_L \int_0^t (t - s)e^{\xi(s)}/m(s)ds \right]
\right.
\]
\[
+ \frac{4C^{(1)}_L e^{-\xi(t)}}{m(t)} \left[ \left\| p^i_0 \right\| \left[ 1 + C^{(1)}_L \int_0^t (t - s)e^{\xi(s)}(m_0 + e^{-\xi(s)})/m(s)ds \right]
\right.
\]
\[
\leq 4C^{(1)}_L \left[ \frac{e^{-\xi(t)}}{m(t)} + 2C^{(1)}_L \right] \left\| q^i_0 - \theta^*_0 \right\| + 4C^{(1)}_L \left[ \frac{e^{-\xi(t)}}{m(t)} + 4C^{(1)}_L \right] \left\| p^i_0 \right\|,
\]
where the last step follows since
\[
\int_0^t (t - s)e^{\xi(s)}/m(s)ds = \int_0^t (t - s)e^{\xi(s)}ds
\]
\[
= -te^{\xi(0)} + \int_0^t e^{\xi(s)}ds
\]
\[
\leq -te^{\xi(0)} + 2 \int_0^t (1 + m'(s))e^{\xi(s)}ds
\]
\[
= -t + 2[m(t)e^{\xi(t)} - m(0)]
\]
\[
\leq 2m(t)e^{\xi(t)}
\]

and the inequality \((a_2)\) follows from Lemmas 4 and 5 and the fact that \(V^i(0, q_0^i, p_0^i) \leq (1 + L)(\|q_0^i - \theta_*\|^2 + m_0^2 \|p_0^i\|^2)\). Hence

\[
\left\| \frac{dp_i^t}{dt} \right\| \leq 4C^{(1)}_L \left[ \frac{e^{-\varepsilon(t)}}{m(t)} + 4C^{(1)}_L \right] (\|p_0^i\| + \|q_0^i - \theta_*\|) \leq 8(C^{(1)}_L)^2 \left[ \frac{e^{-\varepsilon(t)}}{m(t)} + 2 \right] (\|p_0^i\| + \|q_0^i - \theta_*\|).
\]

Set \(C^{(2)}_L = 8(C^{(1)}_L)^2 = 8(1 + 8L^2 + 8L)^2\), we complete the proof.

We now reach the aforementioned intermediate goal where we show a bound between the system \((3.2)\) and its limiting equation \((2.8)\). Again, we need to additionally ask for some underlying convexity; strong convexity in this case.

**Proposition 4** (cf. Proposition 2). Let \(q_i^t\) solve \((3.2)\). Let \(\theta_*^i\) be the critical point of \(\Phi_i\). We assume \(\Phi_i\) is strongly convex with constant \(\kappa\) and let \(\Phi_i\) satisfy Assumption 1 and \(m(t)\) satisfy Assumption 2 with \(\lambda = (\kappa/4) \land (1/4)\). Then we have

\[
\|q_i^t - \theta_*^i\| \leq e^{-\kappa t/2} \|q_0^i - \theta_0^i\| + C^{(1)}_{L, \kappa}m_0 \left[ \|p_0^i\| + \|q_0^i - \theta_*^i\| \right],
\]

where \(C^{(1)}_{L, \kappa} = 2C^{(2)}_L \kappa^{-1/2} (1 + \kappa^{-1/2} + \kappa^{1/2}) + \kappa^{1/2}\), which is larger than \(\kappa^{1/2}\).

**Proof.** By Lemma 6 and Lemma 7 we have,

\[
\frac{1}{2} \frac{d}{dt} \|q_i^t - \theta_*^i\|^2 = \left< q_i^t - \theta_*^i, \frac{dq_i^t}{dt} - d\theta_*^i \right> = -\left< q_i^t - \theta_*^i, \nabla \Phi_i(q_i^t) - \nabla \Phi_i(\theta_*^i) \right> - m(t) \left< q_i^t - \theta_*^i, \frac{dp_i^t}{dt} \right>
\]

\[
\leq -\kappa \|q_i^t - \theta_*^i\|^2 + \frac{\kappa}{2} \left( \left\| q_i^t - \theta_*^i \right\| + \frac{m^2(t)}{2\kappa} \left\| \frac{dp_i^t}{dt} \right\|^2 \right)
\]

\[
\leq -\frac{\kappa}{2} \left( \left\| q_i^t - \theta_*^i \right\|^2 + (C^{(2)}_L)^2(\kappa^{-1}m^2(t)) \left[ \frac{e^{-2\varepsilon(t)}}{m^2(t)} + 4 \right] \left[ \|p_0^i\|^2 + \|q_0^i - \theta_*^i\|^2 \right] \right)
\]

which implies

\[
\|q_i^t - \theta_*^i\|^2 \leq e^{-\kappa t} \|q_0^i - \theta_0^i\|^2 + (C^{(2)}_L)^2(\kappa^{-1})e^{-\kappa t} \int_0^t \left( e^{-2\varepsilon(s)} + m^2(s) \right) e^{\kappa s} ds \left[ \|p_0^i\|^2 + \|q_0^i - \theta_*^i\|^2 \right].
\]

From Lemma 12 we know that \(e^{-\kappa t} \int_0^t m^2(s)e^{\kappa s} ds \leq 2\kappa^{-1}m^2(t) \leq 2\kappa^{-1}m_0^2 \leq 2\kappa^{-1}m_0\), and \(e^{-\kappa t} \int_0^t e^{-2\varepsilon(s)}e^{\kappa s} ds \leq (1 + \kappa)m_0\), which implies

\[
\|q_i^t - \theta_*^i\|^2 \leq e^{-\kappa t} \|q_0^i - \theta_0^i\|^2 + (C^{(1)}_{L, \kappa})^2m_0 \left[ \|p_0^i\|^2 + \|q_0^i - \theta_*^i\|^2 \right].
\]

From Lemma 5 and Proposition 4 we immediately have the following corollary.
Corollary 4. Under the same condition as Proposition 4, for $0 < m_0 \leq 1$ and any fixed $i \in I$, we have

$$
\|q^i_t - \theta_t\| \leq e^{-\kappa t/2} \|q^i_0 - \theta_0\| + C_{L, \kappa}^{(1)} m_0 \left[ \|p^i_0\| + \|q^i_0 - \theta_0\| + \|\theta^i_0 - \theta^*_i\| \right].
$$

We can now show the main statement of this section. The convergence of $d_m \text{SGMP}$ to SGP in the longtime limit. Importantly, we here assume a coupling between the processes through the index process $(i(t))_{t \geq 0}$ that is identical in both dynamical systems. Throughout this section, we collected evidence for a contractive behaviour in between the deterministic processes (2.8) and (3.2). We now use these results to study the randomised version piece-by-piece. This, of course, is again a very similar strategy to that used in Theorem 3.

Theorem 5. Let $\{\tau_n\}_{n \geq 1}$ be the sequences of the jump times of process $(i(t))_{t \geq 0}$ and let the same assumptions as Proposition 4 hold. If we assume $m_0 \leq \frac{C_{L, \kappa}^{(2)} - \kappa^2}{4(2\gamma N + \kappa)^2}$ where $C_{L, \kappa}^{(2)} = C_{L, \kappa}^{(1)} + C_L^{(1)}$. Then, we have $\mathbb{E}\left[ \|q_{\tau_n} - \theta_{\tau_n}\| \right] \to 0$, as $n \to \infty$. Furthermore, if we additionally assume that $m$ decays at least exponentially i.e. there exist $C, c > 0$ such that $m(t) \leq Ce^{-ct}$, we have

$$
\|q_t - \theta_t\| \to 0.
$$

almost surely and in expectation.

Remark 4. After Assumption 3, we had concluded that $m(t) \geq m_0 e^{-\lambda t}$. Thus, the constant $c$ mentioned above needs to be smaller than $\lambda$.

Proof. For any $n \geq 0$, by Corollary 4 and notice that the initial value of $m(t)$ in the interval $(\tau_n, \tau_{n+1}]$ is exactly $m(\tau_n)$,

$$
\|q_{\tau_{n+1}} - \theta_{\tau_{n+1}}\| \leq e^{-\kappa(\tau_{n+1} - \tau_n)/2} \|q_{\tau_n} - \theta_{\tau_n}\| + C_{L, \kappa}^{(1)} m(\tau_n) \left( \|q_{\tau_n} - \theta_{\tau_n}\| + \|p_{\tau_n}\| + \|\theta_{\tau_n} - \theta^i(\tau_n)\| \right)
$$

and

$$
\|p_{\tau_{n+1}}\| \leq \left( e^{-\mathcal{E}(\tau_{n+1}) + \mathcal{E}(\tau_n)} + C_L^{(1)} m(\tau_n) \right) \|p_{\tau_n}\| + C_L^{(1)} \left( \|q_{\tau_n} - \theta_{\tau_n}\| + \|\theta_{\tau_n} - \theta^i(\tau_n)\| \right).
$$

We take a look at Lemma 9 then

$$
\|\theta_{\tau_n} - \theta^i(\tau_n)\| \leq \|\theta_0\| + K_\Phi = K_{\Phi, \theta_0}.
$$

We denote $a_n := e^{-\kappa(\tau_{n+1} - \tau_n)/2}$. Hence

$$
\|q_{\tau_{n+1}} - \theta_{\tau_{n+1}}\| \leq a_n \|q_{\tau_n} - \theta_{\tau_n}\| + C_{L, \kappa}^{(1)} m(\tau_n) \left( \|q_{\tau_n} - \theta_{\tau_n}\| + \|p_{\tau_n}\| + K_{\Phi, \theta_0} \right)
$$

and also

$$
e^{-\mathcal{E}(\tau_{n+1}) + \mathcal{E}(\tau_n)} \leq e^{-t_{n+1} - t_n}/m(\tau_n) \leq e^{-(\tau_{n+1} - \tau_n)/m_0} \leq e^{-(\tau_{n+1} - \tau_n)(C_{L, \kappa}^{(2)})^2} \leq e^{-\kappa(\tau_{n+1} - \tau_n)/2} := a_n.
We then have the following iteration inequality,
\[ \| p_{\tau_{n+1}} \| \leq (a_n + C_L^{(1)} m(\tau_n)) \| p_{\tau_n} \| + C_L^{(1)} \left( \| q_{\tau_n} - \theta_{\tau_n} \| + K_{\Phi, \theta_0} \right). \]

Denote
\[ A_n := \| q_{\tau_n} - \theta_{\tau_n} \|, \quad B_n := \| p_{\tau_n} \|. \]

We then have the following iteration inequality,
\[ A_{n+1} \leq a_n A_n + C_L^{(1)} m(\tau_n) \left( A_n + B_n + K_{\Phi, \theta_0} \right), \]
\[ B_{n+1} \leq (a_n + C_L^{(1)} m(\tau_n)) B_n + C_L^{(1)} \left( A_n + K_{\Phi, \theta_0} \right). \]

We denote \( D_n = A_n + m_{1/2}^{1/2} (\tau_n) B_n, \) \( C_L^{(2)} = C_L^{(1)} + C_L^{(1)}. \) Hence
\[ D_{n+1} \leq (a_n + C_L^{(1)} m(\tau_n)) A_n + m_{1/2}^{1/2} (\tau_{n+1}) (a_n + C_L^{(1)} m(\tau_n)) B_n + C_L^{(1)} m(\tau_n) B_n + m_{1/2}^{1/2} (\tau_{n+1}) C_L^{(1)} A_n \]
\[ + C_L^{(1)} m(\tau_n) K_{\Phi, \theta_0} + C_L^{(1)} m_{1/2}^{1/2} (\tau_{n+1}) K_{\Phi, \theta_0} \leq \left( a_n + C_L^{(2)} m_{1/2}^{1/2} (\tau_n) \right) D_n + C_L^{(2)} m_{1/2}^{1/2} (\tau_n) K_{\Phi, \theta_0} \]
\[ \leq \left( a_n + C_L^{(2)} m_0 \right) D_n + C_L^{(2)} m_{1/2}^{1/2} (\tau_n) K_{\Phi, \theta_0} \]
(3.6)

And since \( \tau_{n+1} - \tau_n \) satisfy exponential distribution with parameter \( \gamma N \) and it is independent of \( \mathcal{F}_{\tau_n} \),
\[ \mathbb{E}[D_{n+1}] \leq \mathbb{E}[e^{-\kappa(\tau_{n+1} - \tau_n)/2}] + C_L^{(2)} m_{1/2}^{1/2} \mathbb{E}[D_n] + C_L^{(2)} K_{\Phi, \theta_0} \mathbb{E}[m_{1/2}^{1/2} (\tau_n)], \]  
(3.7)
\[ \mathbb{E}[D_{n+1}^2] \leq \mathbb{E}[(e^{-\kappa(\tau_{n+1} - \tau_n)/2} + C_L^{(2)} m_{1/2}^{1/2})^2 \mathbb{E}[D_n^2] + (C_L^{(2)} K_{\Phi, \theta_0})^2 \mathbb{E}[m(\tau_n)]] \]
\[ + C_L^{(2)} m_0 K_{\Phi, \theta_0} \mathbb{E}[D_n], \]  
(3.8)

where \( C_L^{(2)} = 2C_L^{(1)} (1 + C_L^{(2)}). \) Since \( m_0 \leq \frac{(C_L^{(1)})^2 (\gamma N)^2}{(2\gamma N + 2)^2}, \) we have \( \mathbb{E}[e^{-\kappa(\tau_{n+1} - \tau_n)/2}] + C_L^{(2)} m_{1/2}^{1/2} \leq c_1 \) where \( c_1 = \frac{C_L^{(2)} (2\gamma N + 2)}{2\gamma N + 2} < 1. \) From the iteration inequality and Lemma \[13\] we know that \( \lim_{n \to \infty} \mathbb{E}[D_n] = 0, \) which finish the proof of the first part.

If we assume \( m(t) \leq Ce^{-ct}, \) we then have
\[ \mathbb{E}[m_{1/2} (\tau_n)] \leq C \mathbb{E}[e^{-ct/2}] = C \mathbb{E} \left[ \prod_{i=1}^{n} e^{-c(\tau_n - \tau_{i-1})/2} \right] = C \prod_{i=1}^{n} \mathbb{E} \left[ e^{-c(\tau_n - \tau_{i-1})/2} \right] = C(c_2)^n \]
where \( c_2 := \mathbb{E} \left[ e^{-c(\tau_n - \tau_{i-1})/2} \right] = \frac{2\gamma N}{c + 2\gamma N} \) which is a constant does not depend on \( i \) and that is smaller than 1. We then rewrite \([3.7]\) as
\[ \mathbb{E}[D_{n+1}] \leq c_1 \mathbb{E}[D_n] + CC_L^{(2)} m_0 K_{\Phi, \theta_0} (c_2)^n, \]
which follows from the second part of Lemma \[13\] We then have \( \mathbb{E}[D_n] \leq B e^{-bn}. \) This also implies
\[ \mathbb{E}[D_{n+1}^2] \leq c_3 \mathbb{E}[D_n^2] + CC_L^{(2)} m_0 K_{\Phi, \theta_0} (c_4)^n. \]
Furthermore, by using Lemma 13 again, we have $\mathbb{E}[D_n^2] \leq \tilde{B}e^{-\tilde{b}n}$ for some $\tilde{B}, \tilde{b} > 0$. The Markov inequality implies that we have
\begin{equation}
\sum_{n=1}^{+\infty} \mathbb{P}(\|D_n\| \geq \delta) \leq \delta^{-1} \sum_{n=1}^{+\infty} \mathbb{E}[D_n] \leq \delta^{-1} \sum_{n=1}^{+\infty} \mathbb{E}[D_n] \leq \delta^{-1} \tilde{B} \sum_{n=1}^{+\infty} e^{-\tilde{b}n} < +\infty,
\end{equation}
for any $\delta > 0$. This implies $D_n \to 0$ almost surely as $n \to \infty$; since $N_t \to \infty$ almost surely as $t \to \infty$. We then have $D_{N_t} \to 0$ almost surely as $t \to \infty$.

By Corollary 4 for any $t \geq 0$, we have
\begin{equation}
\|q_t - \theta_t\| \leq e^{-\kappa(t-\tau_{N_t})/2} \|q_{\tau_{N_t}} - \theta_{\tau_{N_t}}\| + C^{(1)}_{L_\kappa}m(\tau_{N_t}) \left( \|q_{\tau_{N_t}} - \theta_{\tau_{N_t}}\| + \|p_{\tau_{N_t}}\| + \|\theta_{\tau_{N_t}} - \theta^i(\tau_{N_t})\| \right) \\
\leq (1 + C^{(3)}_{L_\kappa}) D_{N_t} + C^{(1)}_{L_\kappa}m(\tau_{N_t})K_{\Phi, \theta_0}.
\end{equation}
This implies $\|q_t - \theta_t\| \to 0$ almost surely as $t \to \infty$ since $D_{N_t} \to 0$ and $m(\tau_{N_t}) \to 0$ almost surely.

Also, in order to prove $\mathbb{E}[\|q_t - \theta_t\|] \to 0$, it is sufficient to show $\mathbb{E}[D_{N_t}] + \mathbb{E}[m(\tau_{N_t})] \to 0$ as $t \to +\infty$. Since
\begin{equation}
\mathbb{E}[D_{N_t}^2] = \sum_{k=0}^{\infty} \mathbb{E}[D_k^2 \mathbb{1}_{N_t=k}]^2 \leq \sum_{k=0}^{\infty} \mathbb{E}[D_k^2] < +\infty,
\end{equation}
which implies that $\{D_{N_t}\}_{t \geq 0}$ is uniformly integrable. This implies that $\mathbb{E}[D_{N_t}] \to 0$ from $D_{N_t} \to 0$ almost surely.

Note that $\lim_{t \to +\infty} m(\tau_{N_t}) = 0$. By the Dominated Convergence Theorem, we get
\begin{equation}
\lim_{t \to +\infty} \mathbb{E}[m(\tau_{N_t})] = 0.
\end{equation}
This completes the proof. \( \square \)

### 3.2. Losing, both, momentum and randomness over time.

In the previous subsection, we have seen that the stochastic gradient process with decreasing momentum approaches the stochastic gradient process. Asymptotically, the stochastic gradient process converges to a stationary distribution, not necessarily to a single point. As discussed before, to reach convergence to a single point, we usually need to decrease the learning rate over time, see [20]. We now study the following model.

**Definition 4.** The *decreasing-mass, decreasing-learning-rate stochastic gradient-momentum process (ddSGMP)* is a solution of the following stochastic differential equation,
\begin{equation}
\begin{cases}
 dq_t = p_t dt \\
 m(t) dp_t = -\nabla \Phi_{\beta(t)}(q_t) dt - p_t dt, \\
p(t=0) = p_0, \\
 q(t=0) = q_0,
\end{cases}
\end{equation}
where $\Phi_j$ satisfies Assumption 7, $j = 1, \cdots, N$. The mass $m(t) > 0$ is strictly decreasing and differentiable with $\lim_{t \to \infty} m(t) = 0$. The stochastic process \{i(t)\}_{t \geq 0} is defined in Definition 7. The re-scaled \(i^\beta(t)\) \(t \geq 0 = \{i(\beta(t))\}_{t \geq 0}\), where $\beta(t) = \int_0^t \mu(s)ds$, $t \geq 0$, and $\mu : [0, \infty) \to (0, \infty)$ be a non-decreasing continuously differentiable function.
As opposed to the previous section, we do not consider SGPC as the limiting system, but actually the associated stochastic gradient process with decreasing learning rate, denoted by
\[
\begin{cases}
    d\xi_t = -\nabla\Phi_i(t)(\xi_t)dt, \\
    \xi(t = 0) = \xi_0.
\end{cases}
\] (3.10)

Next, we define the sequence \(\{\tau_n\}\) to be the jump times of process \((i(t))_{t \geq 0}\) and \(\tau^\beta_n = \beta^{-1}(\tau_n)\), which are the jump times of \((i^\beta(t))_{t \geq 0}\). We denote by
\[
\Omega^\alpha_n := \left\{ \frac{\kappa}{2\mu(\tau^\beta_{n+1})} \geq \frac{\alpha_1}{\sqrt{n}} \text{ and } m(\tau^\beta_n) \leq \alpha_2e^{-\alpha_3\sqrt{n}} \right\},
\]
for \(n \in \mathbb{N}\) an event that is used to impose a growth condition on \(\beta\) and \(m\). Regarding this event, we now denote the following assumption.

**Assumption 4.** For \(n \geq k\), let \(W^\alpha_n = \cap_{i=k}^{n} \Omega^\alpha_i\). There exist \(\alpha_1, \alpha_2, \alpha_3 > 0\) such that \(\lim_{k \to +\infty} \mathbb{P}(W^\alpha,\infty) = 1\).

The event \(W^\alpha_n\) is increasing in \(k\) and decreasing in \(n\). Assumption 4 implies that the complement of \(\Omega^\alpha_n\) is eventually small. We describe a setting for which this assumption holds in Example 2 below, after stating and proving the main results of this section: the convergence of ddSGMP to the stochastic gradient process with decreasing learning rate and the convergence of ddSGMP to the minimiser of the target function. In neither of the cases, we obtain a convergence rate; we later study the speed of convergence when looking at numerical experiments in Section 5. But now we start with the first of the two aforementioned results.

**Theorem 6.** For any \(i = 1, \ldots, N\), we assume that \(\Phi_i\) is strongly convex with constant \(\kappa\) and let \(\Phi_i\) satisfy Assumption 1. Let \((m(t))_{t \geq 0}\) satisfy Assumption 3 with \(\lambda = (\kappa/4) \land (1/4)\). In addition, we assume that \((m(t))_{t \geq 0}\) and \((\beta(t))_{t \geq 0}\) satisfy Assumption 4. Then, we have
\[
\|q_t - \xi_t\| \to 0
\]
amongt surely, as \(t \to \infty\).

**Proof.** For any \(n \geq 0\), by Corollary 4, we have the following iteration inequality,
\[
\|q_{r_n} - \xi_{r_n}\| \leq e^{-\kappa(r_n - r_{n+1})/2} \|q_{r_n} - \xi_{r_n}\| + C^{(1)}_{L,\kappa} m(\tau_{r_n}) \left( \|q_{r_n} - \xi_{r_n}\| + \|q_{r_n} - \theta^i(\tau_{r_n})\| \right)
\]
and
\[
\|p_{r_n} - \xi_{r_n}\| \leq e^{-\kappa(r_n - r_{n+1})/2} + C^{(1)}_L m(\tau_{r_n}) \|p_{r_n}\| + C^{(1)}_{L,\kappa} \left( \|q_{r_n} - \xi_{r_n}\| + \|\xi_{r_n} - \theta^i(\tau_{r_n})\| \right).
\]

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Since $\frac{dβ}{dt}(t) = \frac{1}{μ(β^{-1}(t))}$, we have $τ^{β}_{n+1} - τ^{β}_{n} = β^{-1}(τ_{n+1}) - β^{-1}(τ_{n}) ≥ (τ_{n+1} - τ_{n})/μ(β^{-1}(τ_{n+1}))$.
Hence, under the event $W^α_{k,n}$, for $n ≥ k$, we have

$$
\left\| q^{β}_{τ^{β}_{n+1}} - ξ^{β}_{τ^{β}_{n+1}} \right\| ≤ e^{-κ(τ^{β}_{n+1} - τ^{β}_{n})/2} \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| + C^{(1)}_{L,κ} m(τ^{β}_{n}) \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| + \left\| p^{β}_{τ^{β}_{n}} \right\| + \left\| ξ^{β}_{τ^{β}_{n}} - θ^{(τ^{β}_{n})}_* \right\|)
$$

$$
≤ \left( e^{-α_1(τ^{β}_{n+1} - τ^{β}_{n})/\sqrt{n}} + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \right) \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \left\| p^{β}_{τ^{β}_{n}} \right\| + K_{Φ,ξ_0} α_2 e^{-α_3 \sqrt{n}}
$$

and

$$
\left\| p^{β}_{τ^{β}_{n}} \right\| ≤ \left( e^{-α_1(τ^{β}_{n+1} - τ^{β}_{n})/\sqrt{n}} + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \right) \left\| p^{β}_{τ^{β}_{n}} \right\| + C^{(1)}_{L,κ} \left( \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| + \left\| ξ^{β}_{τ^{β}_{n}} - θ^{(τ^{β}_{n})}_* \right\| \right).
$$

Since $τ^{β}_{n+1} - τ^{β}_{n}$ is independent of $\left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\|$ and $W^α_{k,n}$, for $n ≥ k$, we have

$$
\mathbb{E}\left[ \left\| q^{β}_{τ^{β}_{n+1}} - ξ^{β}_{τ^{β}_{n+1}} \right\| \mathbb{1}_{W^α_{k,n}} \right] ≤ \left( \frac{γN}{γN + α_1/\sqrt{n}} + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \right) \mathbb{E}\left[ \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right] + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \mathbb{E}\left[ \left\| p^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right] + K_{Φ,ξ_0} α_2 e^{-α_3 \sqrt{n}}
$$

and

$$
\mathbb{E}\left[ \left\| p^{β}_{τ^{β}_{n+1}} \right\| \mathbb{1}_{W^α_{k,n}} \right] ≤ \left( \frac{γN}{γN + α_1/\sqrt{n}} + C^{(1)}_{L,κ} α_2 e^{-α_3 \sqrt{n}} \right) \mathbb{E}\left[ \left\| p^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right] + C^{(1)}_{L,κ} \mathbb{E}\left[ \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right] + K_{Φ,ξ_0}.
$$

We denote

$$
A^β_n := \mathbb{E}\left[ \left\| q^{β}_{τ^{β}_{n}} - ξ^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right], \quad B^β_n := \mathbb{E}\left[ \left\| p^{β}_{τ^{β}_{n}} \right\| \mathbb{1}_{W^α_{k,n}} \right].
$$

Let $D^β_n = A^β_n + α_2 e^{-α_3 \sqrt{n}} B^β_n$, we could find some constant $0 < C_{γ, N, L, κ} < 1$ such that for $n ≥ k ≥ 1$ and $k$ large enough

$$
D^β_{n+1} ≤ \left( 1 - \frac{C_{γ, N, L, κ}}{\sqrt{n}} \right) D^β_n + C_{Φ, ξ_0, L} e^{-α_3 \sqrt{n}}. \tag{3.11}
$$

Let $C^n_k = \Pi^n_{i=k} \left( 1 - \frac{C_{γ, N, L, κ}}{\sqrt{i}} \right)$ with $C^n_1 = 1 - \frac{C_{γ, N, L, κ}}{\sqrt{n}} ≥ \frac{1}{2}$. From (3.11), we have

$$
D^β_{n+1} ≤ G^n_k D^β_k + 2C_{Φ, ξ_0, L} \sum^n_{i=k} e^{-α_3 \sqrt{i}} C^n_i.
$$
From Lemma 14, we have that there exist some constant $c > 0$ such that \( G_i^n \leq e^{-c(\sqrt{n} - \sqrt{i})} \). This implies
\[
D_{n+1}^\beta \leq C e^{-c\sqrt{n}} D_k^\beta + 2C\Phi,\xi_0,L \sum_{i=k}^{n} e^{-\alpha_3 \sqrt{i} e^{-c(\sqrt{n} - \sqrt{i})}},
\]
\[
\leq C e^{-c\sqrt{n}} D_k^\beta + 2C\Phi,\xi_0,L n e^{-c\alpha_3\sqrt{n}}.
\]

Hence
\[
\sum_{n=k}^{\infty} \mathbb{P} \left( \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| \mathbb{1}_{W_k^{n,n-1}} \geq \varepsilon \right) \leq \sum_{n=k}^{\infty} \frac{D_n^\beta}{\varepsilon} \leq \frac{C}{\varepsilon} \sum_{n=k}^{\infty} n e^{-c\alpha_3\sqrt{n}} < +\infty
\]

Then from Borel–Cantelli lemma, we have
\[
\lim_{n \to \infty} \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| \mathbb{1}_{W_k^{n,n-1}} = 0 \quad \text{almost surely.}
\]

Under the Assumption 4, we have
\[
\lim_{k \to \infty} \mathbb{1}_{(W_k^{n,\infty})^c} = 0 \quad \text{almost surely.}
\]

Since
\[
\left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| = \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| \mathbb{1}_{W_k^{n,n-1}} + \left( q_{r_n}^\beta - \xi_{r_n}^\beta \right) \mathbb{1}_{(W_k^{n,n-1})^c}
\]
\[
\leq \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| \mathbb{1}_{W_k^{n,n-1}} + \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| \mathbb{1}_{(W_k^{n,\infty})^c},
\]
for $k$ large enough, let $n \to \infty$, we have
\[
\lim_{n \to \infty} \left\| q_{r_n}^\beta - \xi_{r_n}^\beta \right\| = 0 \quad \text{almost surely.}
\]

Similarly, we have
\[
\lim_{n \to \infty} e^{-\alpha_3\sqrt{n}} \left\| p_{r_n}^\beta \right\| = 0 \quad \text{almost surely.}
\]

On event $W_k^{n,n}, m(r_n^\beta)$ is smaller that $\alpha_2 e^{-\alpha_3\sqrt{n}}$. Hence, we have
\[
\lim_{n \to \infty} m(r_n^\beta) \left\| p_{r_n}^\beta \right\| = 0 \quad \text{almost surely.}
\]

Since $\lim_{n \to \infty} r_n^\beta = +\infty$ and
\[
\left\| q_t - \xi_t \right\| \leq e^{-n(t-r_n^\beta)/2} \left\| q_{r_n^\beta}^\beta - \xi_{r_n^\beta}^\beta \right\| + C_{L,n}^{(1)} \left( \left\| q_{r_N^\beta}^\beta - \xi_{r_N^\beta}^\beta \right\| + \left\| p_{r_N^\beta}^\beta \right\| + \left\| \xi_{r_N^\beta}^\beta - \theta^\beta_{r_N^\beta} \right\| \right),
\]
32
where $\tau^\beta_{N_t} \leq t \leq \tau^\beta_{N_{t+1}}$, we have
\[\|q_t - \xi_t\| \leq (1 + C_{L,\kappa}^{(1)} m_0) \left\| q^\beta_{N_t} - \xi^{\beta}_{N_t} \right\| + m(\tau^\beta_{N_t}) \left( \left\| p^\beta_{N_t} \right\| + \left\| \xi^{\beta}_{N_t} - \theta^*_{\tau^\beta_{N_t}} \right\| \right).\]
This implies $\|q_t - \xi_t\| \to 0$ almost surely when $t \to \infty$. \qed

Using this result and prior knowledge about the stochastic gradient process, we can now finally show convergence to the minimiser of $\Phi$.

**Corollary 7.** Under the same conditions as Theorem 6, we have
\[\|q_t - \theta^*\| \to 0\]
in probability, as $t \to \infty$, where $\theta^*$ is the unique minimiser of the function $\Phi$.

**Proof.** From [20, Theorem 4], we have $\xi_t$ converges to $\theta^*$ weakly, which is equivalent to $\|\xi_t - \theta^*\| \to 0$ in probability since $\theta^*$ is deterministic. Then the result follows by applying Theorem 6. \qed

Hence, we have shown that when reducing – indeed, losing – momentum and decreasing the learning rate over time, we can show convergence of the stochastic gradient-momentum process to the minimiser $\theta^*$.

We finish this section, by discussing the non-trivial Assumption 4. Indeed, we give an example below for how $(\beta(t))_{t \geq 0}$ and $(m(t))_{t \geq 0}$ can be chosen to satisfy the assumption.

**Example 2.** Let $\beta(t) = t^2$ and $m(t) = m_0 e^{-m_1 t}$, where $0 < m_0 < 1$ is a constant. In this case $\mu(t) = 2t$ and $\beta^{-1}(t) = \sqrt{t}$. Hence,
\[\Omega_n^\alpha = \left\{ \frac{k}{2(\beta^{-1}(\tau_{n+1}))} \geq \frac{\alpha_1}{\sqrt{n}} \text{ and } m(\tau^\beta_n) \leq \alpha_2 e^{-\alpha_3 \sqrt{n}} \right\} = \left\{ \frac{k}{4(\tau_{n+1})} \geq \frac{\alpha_1}{\sqrt{n}} \text{ and } m_0 e^{-\lambda \sqrt{n}} \leq \alpha_2 e^{-\alpha_3 \sqrt{n}} \right\}.
By concentration inequality, we have
\[\mathbb{P}\left( \tau_n \leq \frac{n}{2\gamma N} \right) = \mathbb{P}\left( e^{-2\gamma N \tau_n} \geq e^{-n} \right) \leq e^n \mathbb{E}[e^{-2\gamma N \tau_n}] = \frac{e^n}{3^n}\]
and
\[\mathbb{P}\left( \tau_n \geq \frac{2n}{\gamma N} \right) = \mathbb{P}\left( e^{\gamma N \tau_n/2} \geq e^n \right) \leq e^{-n} \mathbb{E}[e^{\gamma N \tau_n/2}] = \frac{2^n}{e^n}.\] (3.12) (3.13)
When $n$ is big enough, $\frac{n}{2\gamma N} \leq \tau_n \leq \frac{2n}{\gamma N}$ implies the event $\Omega_n^\alpha$ holds for $\alpha_1 = \frac{2}{\kappa \sqrt{2\gamma N}}$, $\alpha_2 = m_0$, $\alpha_3 = \frac{\lambda}{\sqrt{2\gamma N}}$. Hence, when $n$ is big enough, $(\Omega_n^\alpha)^\complement \subset \{ \tau_n \leq \frac{n}{2\gamma N} \} \cup \{ \tau_n \geq \frac{2n}{\gamma N} \}$. Therefore, from (3.12) and (3.13) we have
\[\sum_{n=1}^\infty \mathbb{P}(\Omega_n^\complement) \leq \sum_{n=1}^\infty \left( \frac{e^n}{3^n} + \frac{2^n}{e^n} \right) < +\infty.
From Borel-Cantelli’s Lemma, this implies $\lim_{k \to +\infty} \mathbb{P}(\cup_{i=k}^\infty (\Omega_i^\alpha)^\complement) = 0$ which is equivalent to the Assumption 4.\]
4. Discretisation of the continuous-time system

After the previous theoretical study of stochastic gradient-momentum processes, we now propose a numerical scheme to discretise those dynamical systems. For practical optimisation, this passage from continuous-time dynamic to discrete time algorithm is crucial, see the discussion in [32, 33]: the continuous-time analysis given in Sections 2 and 3 is only worthwhile, when the discrete algorithm behaves similarly to the continuous-time dynamic. The most usual discretisation of SGMP with the forward Euler method would yield an algorithm very similar to stochastic gradient descent with classical momentum. However, in general it is not clear that classical momentum would behave similar to SGMP as the forward Euler method tends to be unstable and is not a symplectic integrator. In the following, we aim at obtaining a stable, symplectic, and efficient discretisation strategy that retains the correct longtime behaviour for the deterministic parts of SGMP.

Indeed, we start with the discretisation of the deterministic part of the SGMP. We discuss the discretisation of the underdamped gradient flow with constant or decreasing mass. For convenience, we briefly recall the ODE of interest:

\[
\begin{align*}
\frac{dq}{dt} &= p_t, \\
\frac{dp}{dt} &= -\nabla \Phi(q_t) - \alpha p_t.
\end{align*}
\]

(4.1)

Since our theoretical analysis contemplates the case \(m(t) \to 0\) as \(t \to \infty\), we need the discretisation scheme to especially be stable for small values of \(m(t)\). In this framework, a fully explicit method, such as the forward Euler method, would not be suitable for the purpose. In particular, since the evolution of \(t \mapsto p_t\) is directly affected by the value of the mass \(m(t)\), it is natural to consider an implicit discretisation for this variable:

\[
\begin{align*}
m_n \frac{p_{n+1} - p_n}{h} &= -\left(\nabla \Phi(q_n) + \alpha p_{n+1}\right),
\end{align*}
\]

for every \(n \geq 0\), yielding

\[
p_{n+1} = \frac{1}{m_n/h + \alpha} \left(\frac{m_n}{h} p_n - \nabla \Phi(q_n)\right),
\]

(4.2)

where \(h > 0\) denotes the discretisation step-size and \(m_n := m(nh)\). For the \(q\) variable in (4.1) we use the scheme

\[
q_{n+1} = q_n + hp_{n+1}
\]

(4.3)

for every \(n \geq 0\). Finally, combining (4.2) and (4.3), we obtain the following update rule for the discretised dynamical model:

\[
\begin{align*}
p_{n+1} &= \frac{1}{m_n/h + \alpha} \left(\frac{m_n}{h} p_n - \nabla \Phi(q_n)\right), & p_0 = 0, \\
q_{n+1} &= q_n + hp_{n+1}, & q_0 = x_0,
\end{align*}
\]

(4.4)

for every \(n \geq 0\) and for every initial point \(x_0\). As we consider \(q\) explicitly and \(p\) implicitly, this discretisation scheme falls in the category of semi-implicit methods. Moreover, the discretisation scheme is symplectic and preserves the energy in the system, see [9].

Remark 5. As suggested in the previous sections and depicted in Figure [7], the parameter \(m\) can be interpreted as an interpolation variable between the classical (stochastic) gradient flow
and the dynamical equations modeling the damped motion. We observe that it is still the case for the discretised dynamical system (4.4). Indeed, if we set \( \alpha = 1 \) and we let \( m \to 0 \), (4.4) becomes
\[
\begin{align*}
  p_{n+1} &= -\nabla \Phi(q_n), \\
  q_{n+1} &= q_n + h p_{n+1},
\end{align*}
\]
which is exactly the gradient method.

4.1. Step-size choice. The first natural question about the discrete-time optimization method (4.4) concern the choice of the discretisation step-size \( h > 0 \). Here we derive an heuristic rule, assuming for simplicity that \( m_n = m \) for every \( n \geq 1 \). From (4.4), we obtain that
\[
q_{n+1} = q_n + \frac{h^2}{m + \alpha h} \left( \frac{m}{h} p_n - \nabla \Phi(q_n) \right)
\]
for every \( n \geq 0 \). With a backward induction argument, assuming that \( p_0 = 0 \), we deduce that
\[
q_{n+1} = q_n - \frac{h^2}{m + \alpha h} \sum_{j=0}^{n} \left( \frac{m}{m + \alpha h} \right)^j \nabla \Phi(q_{n-j}).
\]
The previous identity suggests that the position at the step \( k + 1 \) is obtained through \( k + 1 \) evaluations of the gradient at the previous points of the discrete trajectory. Moreover, each evaluation is weighted by a power of the forgetting coefficient \( \frac{m}{m + \alpha h} < 1 \). Therefore, it is reasonable to ask that the sum of the weights is of the order \( \frac{1}{L} \), where \( L > 0 \) is an upper bound of the Lipschitz constant of \( \nabla \Phi \). More precisely we require
\[
\frac{h^2}{m + \alpha h} \sum_{j=0}^{\infty} \left( \frac{m}{m + \alpha h} \right)^j \approx \frac{1}{L}.
\]
Taking the limit as \( n \to \infty \) in the previous sum, we obtain
\[
\frac{1}{L} \approx \frac{h^2}{m + \alpha h} \sum_{j=0}^{\infty} \left( \frac{m}{m + \alpha h} \right)^j = \frac{h}{\alpha},
\]
or equivalently
\[
h \approx \frac{\alpha}{L}. \tag{4.5}
\]
We first note that the expression at the right-hand side of (4.5) does not depend on the parameter \( m \). Thus, according to the empirical argument presented above, there is no need to adjust the magnitude of the step-size according to \( m \), also not if \( m \) changes over time. This positive fact descends from the implicit discretisation of the \( t \mapsto p_t \) variable in (4.1). The second observation is that, when we set \( m = 0 \) and \( \alpha = 1 \), the discrete method (4.4) reduces to the classical gradient scheme, and (4.5) actually prescribes the correct step-size.

Remark 6. Let us assume that the objective function is multiplied by a positive constant \( \rho > 0 \), i.e., \( \Phi^\prime = \rho \Phi \). A natural question is how we should rescale the parameters of the method (4.4) such that the sequence of positions \( (q'_n)_{n \geq 0} \) coincides with the sequence \( (q_n)_{n \geq 0} \).
corresponding to the original objective. It turns out that it is sufficient to set \( h' = h/\rho \) and \( m'_n = m_n/\rho \). Indeed, if we consider the sequences \((q'_n)_{n\geq 0}\) \(\) and \((p'_n)_{n\geq 0}\) obtained with
\[
\begin{align*}
p'_{n+1} &= \frac{1}{\frac{n}{m}+\alpha} \left( \frac{m_n}{h'} p_n' - \nabla \Phi'(q'_n) \right), \quad p'_0 = 0 \\
q'_{n+1} &= q_n' + h' p'_{n+1}, \quad q'_0 = q_0,
\end{align*}
\]
then a direct computation yields
\[
q'_n = q_n, \quad p'_n = \rho p_n
\]
for every \( k \geq 1 \), where the sequences \((q_n)_{n\geq 0}\) \(\) and \((p_n)_{n\geq 0}\) are obtained using (4.4) with the original objective \( \Phi \) and the parameters \( h \) and \( m_n \).

4.2. Randomised version of the method. The stochastic dynamical systems discussed throughout this work consist of piecewise deterministic ODEs where the pieces are determined by random waiting times and a subsampling process. After having discussed the ODE discretisation in the previous subsections, we now move on to the stochasticity. Here, especially the random waiting times do not feel very natural when applying the algorithm in practical situations. The use of random waiting times in practice within this framework has been discussed by \[12, 20\]. In the present work, we replace those random waiting times by deterministic waiting times that coincide with the time steps of the algorithm. This is exactly the paradigm of the classical stochastic gradient descent method and also the classical momentum method.

Let again \( \bar{\Phi} : \mathbb{R}^n \to \mathbb{R} \) be a function given as \( \bar{\Phi} = \frac{1}{N} \sum_{i=1}^N \Phi_i \), where \( \Phi_1, \ldots, \Phi_N : \mathbb{R}^n \to \mathbb{R} \) satisfy Assumption 2, e.g., they are \( C^1 \)-regular with Lipschitz-continuous gradients. Therefore, we can use the following stochastic momentum method:
\[
\begin{align*}
p_{n+1} &= \frac{1}{\frac{n}{m}+\alpha} \left( \frac{m_n}{h'} p_n - \nabla \Phi_n(q_n) \right), \\
q_{n+1} &= q_n + h' p_{n+1}, \quad q_0 = q_0,
\end{align*}
\]
where we set
\[
\begin{align*}
v_n := \frac{1}{N} \frac{\ell}{\ell} \sum_{r=1}^\ell \nabla \Phi_{i_r}(q_n),
\end{align*}
\]
by picking a subset \( \{i_1, \ldots, i_\ell\} \) of \( I \) that is sampled uniformly without replacement. This so-called batch subsampling that does not choose a single but rather a set of \( \ell \) potentials at once is very useful in practice and, of course, completely contained in the our theory. To see this, one can just define a new set of potentials \( (\Phi'_K)_{K \in \ell} \), where \( \ell' := \{ K \subseteq I : \|K\| = \ell \} \) and \( \Phi'_K = \frac{1}{\ell} \sum_{i \in K} \Phi_i \) for \( K \in \ell' \). Then, the mean of the \( (\Phi'_K)_{K \in \ell} \) is again \( \Phi \).

5. Numerical Experiments

We now present numerical experiments in which we test the discretisation strategy proposed in Section 4. We start with academic convex and non-convex examples. Then, we employ SGMP for the training of a convolutional neural network regarding the classification of the CIFAR-10 data set. We aim to show how the method compares with the classical momentum method and standard stochastic gradient descent.
5.1. One-dimensional non-smooth stationary point. The first numerical experiment involves the one-dimensional example discussed in Subsection \[1.2\] We recall that we want to minimise the function

\[
\Phi(x) := (\text{ReLU}(x) - 1)^2 + x^2 = \begin{cases} 
  x^2 + 1 & \text{if } x \leq 0 \\
  2x^2 - 2x + 1 & \text{if } x > 0 
\end{cases} \quad (x \in \mathbb{R}), 
\]

which attains the global minimum at the point \(x^* = \frac{1}{2}\). The point \(\tilde{x} = 0\) is a non-smooth stationary point, since \(\Phi(x) \geq \Phi(\tilde{x})\) for every \(x \leq \tilde{x}\). In Subsection \[1.2\] we are able to show that the underdamped gradient flow can overcome the local minimiser \(\tilde{x}\) if \(\alpha^2 - 8m < 0\).

We now test this property using our discrete-time method \[4.4\]. The discretisation scheme asks us to choose the stepsize \(h\) according to the Lipschitz constant of \(\Phi'\). Actually, the derivative of \(\Phi\) is not continuous. However, in this framework, by “Lipschitz constant” of \(\Phi'\) we mean \(L_0 = \max\{\text{Lip}(\Phi'|_{x \leq 0}), \text{Lip}(\Phi'|_{x \geq 0})\}\). We test the scheme in this example using the precise Lipschitz constant and overestimated Lipschitz constants. The results are presented in Figure \[II\] Here, we see that the condition \(\alpha^2 - 8m < 0\) is also relevant for the discrete dynamical system. The inequality appears to be sharp whenever we overestimate the Lipschitz constant – unsurprisingly as a smaller stepsize leads to a more accurate discretisation of the underdamped gradient flow. Using the correct Lipschitz constant lets \(\alpha^2 - 8m < 0\) appear quite conservative. Indeed, a much larger range of \(\alpha, m\) allow for convergence to the global minimiser. A larger stepsize makes the method more robust.

5.2. Strongly convex example: quadratic objective. We now consider a target function of the form \(\Phi = \frac{1}{N} \sum_{i=1}^{N} \Phi_i\), where

\[
\Phi_i(x) = \frac{N}{2} x^T A_i x + N b_i^T x, 
\]

and where \(A_i \in \mathbb{R}^{K \times K}\) is a symmetric and positive definite matrix and \(b_i \in \mathbb{R}^K\), for \(i = 1, \ldots, N\). Importantly, all of the \((\Phi_i)_{i \in I}\) are strongly convex. We consider \(K = 500\) (dimension of the domain), and \(N = 100\). For every \(i = 1, \ldots, N\), we sample the eigenvalues of the matrix \(A_i\) using a uniform distribution in \([0.05, 15]\), and we obtained \(b_i\) using a normal distribution centered at the origin and with standard deviation \(\sigma = 2\). We look at a total of 100 different randomly generated problems and later average over the results. We compare SGD and the our method \[4.4\]. At each iteration we use a mini-batch of \(\ell = 10\) elements of \(\{\Phi_1, \ldots, \Phi_N\}\) to compute the stochastic approximation of the gradient of \(\Phi\). Finally, the learning rate/discretisation stepsize is polynomially decreasing, namely at the \(n\)-th iteration we set \(h_n = h_0/n\). The results are reported in Figure \[IV\] As we can see, the stochastic momentum method shows a faster convergence than the classical SGD scheme. In this case, the decrease of the mass parameter \(m\) leads to a deterioration of the performances. Indeed, consistently with the theoretical predictions, the behaviour of SGMP gets closer to SGP as \(m\) diminishes. We also study the case where the mass is non-constant and decreases over time as \(m_k = m_0(0.995)^k\). In that situation, the decrement of the mass leads to a slower convergence as well.
Figure III. The plots depict for which combinations of $m, \alpha$ the discrete-time version of (1.6) derived in (4.4) manages to overcome the “false minimiser”. The blue crosses represent convergence to the global minimiser of (1.4), the red ones to the origin. The black curve divides the $m, \alpha$ that do and do not satisfy (1.8). As the step-size $h = \frac{1}{L}$ gets smaller, the theoretical prediction (1.8) becomes more accurate. Finally, we observe that the gradient method (that corresponds to $m = 0$) never converges to the global minimiser.

5.3. Non-convex example: polynomial function. We studied the behaviour of our stochastic momentum method in the case of a polynomial non-convex function, again, given as the sum $\Phi = \frac{1}{N} \sum_{i=1}^{N} \Phi_i$, where for every $i = 1, \ldots, N$

$$\Phi_i(x) = -\frac{N}{2} x^T A_i x + N b_i x + \frac{1}{4} \sum_{j=1}^{n} x_j^4$$

and where $A_i \in \mathbb{R}^{K \times K}$ is a symmetric and positive definite matrix and $b_i \in \mathbb{R}^K$. The problem is non-convex, since the Hessian of $\Phi$ (as well as the one of each $\Phi_1, \ldots, \Phi_N$) is negative definite at the origin $x = 0$. On the other hand, outside a large enough compact, the objective function
\textbf{Figure IV.} Convergence rate comparison. The plot represents the decrease of the distance from the true minimiser achieved by the SGD and the stochastic momentum method introduced in (4.6). We consider the case where the mass is constant, but the learning rate/stepsizes \((h_n)_{n=0}^{\infty}\) is decreasing (left) and the case where the mass is decreased as \(m_k = m_0(0.995)^k\) and the learning rate/stepsizes decreases as before (right, see Subsection 3.1). The experiments are repeated 100 times (always resampling the potentials), and we reported the mean decrease achieved by each method, and the corresponding standard deviation.

\(\Phi\) is locally convex. We considered \(K = 500\) (dimension of the domain), and \(N = 100\). For every \(i = 1, \ldots, N\), we sampled the eigenvalues of the matrix \(A_i\) using a uniform distribution in \([0.05, 15]\), and we obtained \(b_i\) using a normal distribution centered at the origin and with standard deviation \(\sigma = 2\). At each iteration we use a mini-batch of 5 elements of \(\{\Phi_1, \ldots, \Phi_N\}\) to compute the stochastic approximation of the gradient of \(\Phi\). We compared the stochastic momentum method with the classical SGD. In this case, the learning rate is kept constant during the iterations. The results are reported in Figure V. In this case it seems that the stochastic momentum method tends to stabilise in correspondence of lower values of the objective function. Interestingly, we observe that the implementations with smaller \(m\) have a faster decay in the initial iterations – as opposed to the results obtained in the previous subsection.

5.4. Convolutional Neural Network (CNN). We now employ the discrete SGMP method (4.4) to solve the CIFAR-10 [17] image classification task with a convolutional neural network (CNN). The CIFAR-10 data set consists of \(6 \times 10^4\) colour images (\(32 \times 32\) pixels) which are split into \(5 \times 10^4\) training images and \(10^4\) test images. CIFAR-10 has 10 classes (e.g., airplane, dog, frog,...) with 6000 images per class. In the classification task, images with known class are used to train the CNN to automatically recognise the class of any image. We use a VGG-like CNN architecture [35]. More precisely, we use \(3 \times 3\) kernels with depth 32, 64, and 128. The network contains 6 convolutional layers, each of them followed by the ReLU activation, batch normalization, max-pooling, and drop-out layers. The train data is augmented as [24], that is a random horizontal flip and a \(32 \times 32\) random crop after a \(4 \times 4\) padding. The training is done with Google Colab using GPUs (often Tesla V100, sometime Tesla A100). We compare the classical SGD, the classical momentum, and SGMP (4.4) with different parameters, constant
Figure V. Decay of objective comparison. The plot represents the decrease of the objective function achieved by the SGD and the stochastic momentum method introduced in [4.6]. We consider the constant-mass regime (left) and the exponentially decreasing case $m_k = m_0(0.995)^k$ (right). We repeated the experiments 100 times, and we reported the mean objective decrease achieved by the methods, and the respective standard deviations.

| Method          | Parameters | Train Acc. | Test Acc. |
|-----------------|------------|------------|-----------|
| SGD             | $\eta = 0.01$ | 96.902     | 90.98     |
| Classical momentum | $\eta = 0.01$, $\rho = 0.9$ | 97.236     | 91.33     |
| hSGMP           | $\alpha = 1$, $m = 0.1$, $h = 1$ | 97.096     | 91.09     |
|                 | $\alpha = 10$, $m = 0.1$, $h = 1$ | 96.986     | 91.25     |
|                 | $\alpha = 15$, $m = 0.1$, $h = 1$ | 96.822     | 91.09     |
|                 | $\alpha = 10$, $m = 0.15$, $h = 1$ | 96.556     | 90.75     |
|                 | $\alpha = 10$, $m = 0.1$, $h = 1$ | 97.35      | 91.35     |
|                 | $\alpha = 10$, $m = 0.1$, $h = 2$ | 96.984     | 90.87     |
|                 | $\alpha = 10$, $m = 0.1$, $h = 0.1$ | 97.108     | 91.26     |
| dmSGMP          | $\alpha = 10$, $m_0 = 0.1$, $h = 1$ | 97.288     | 91.32     |

Table 1. Comparison of train and test accuracy(%) with different hyper-parameters over the CIFAR-10 dataset (best results presented in bold font).

mass, and decreasing mass. The experiments are set in the following way. We train for 800 epochs with batch size 100 and no weight decay. We use constant learning rate $\eta = 0.01$ for SGD and the classical momentum. In classical momentum, we set the momentum hyper-parameter $\rho = 0.9$. See Figure VII for the plots of train loss for constant mass. See Figure VII for the plots of train loss for decreasing mass. The decreasing rate is set to be $m = m_00.995^k$, where $k$ is the number of iterations and $m_0 = 0.1$ is the initial mass. See Table 1 for the train and test accuracy. The train loss for each epoch is calculated by averaging over batches. The accuracy for each method is computed using the model from the last epoch. We observe that the hSGMP achieves competitive test accuracy to the classical momentum. With decreasing mass, the dmSGMP achieves competitive train loss to the classical momentum.
Figure VI. Train loss comparison for SGD ($\eta = 0.01$), classical momentum ($\eta = 0.01, \rho = 0.9$), and hSGMP for CNN on CIFAR-10. We vary $\alpha$, $m$ and $h$ in each experiment, which are specified in the title of each plot.

6. Conclusions

In this work, we have proposed and analysed the stochastic gradient-momentum process, a continuous-time dynamic representing momentum-based stochastic optimisation. We have especially analysed limiting behaviour when reducing learning rate and/or particle mass. Here, learning rate and particle mass can either be reduced homogeneously or decrease over time. In those cases, we have shown pathwise or long time convergence to the underlying gradient flow or the stochastic gradient process, respectively. We have then proposed a stable discretisation strategy for the stochastic gradient-momentum process and tested the strategy in several numerical examples. In those, we especially saw that the stable discretisation of the stochastic gradient-momentum process can achieve a similar accuracy in a CNN training as (the possibly unstable) stochastic gradient descent with classical momentum algorithm.
Figure VII. Train loss comparison for SGD ($\eta = 0.01$), classical momentum ($\eta = 0.01, \rho = 0.9$), hSGMP ($\alpha = 10, m = 0.1, h = 1$), and dmSGMP ($\alpha = 10, m_0 = 0.1, m = m_00.995^k, h = 1$) for CNN on CIFAR-10.

Most of the theoretical results we have obtained throughout this work refer to the setting of convex optimisation. Convex optimisation is vital in, e.g., image reconstruction. The training of neural network usually requires non-convex optimisation and momentum-based methods are especially popular in non-convex settings. Hence, a natural future research direction are non-convex optimisation problems.

The stochastic gradient-momentum process does not represent the adaptivity in the Adam algorithm. To represent the adaptivity, we would need to study a two-sided dependence between $(i(t))_{t \geq 0}$ and $(p(t), q(t))_{t \geq 0}$ and a non-linear weighting in front of the gradient. Both these additions to the stochastic gradient-momentum process are a very interesting and challenging direction for future research.

Appendix A. Auxiliary results

Lemma 7. Let $\Phi \in C^1(X : \mathbb{R})$ be strongly convex with constant $\kappa$. Then $\Phi$ satisfy Assumption 2 with $\lambda = (\kappa/\alpha^2) \wedge (1/4)$.

Proof. Since $\Phi$ is strongly convex, we have

$$(x - \theta_s) \cdot (\nabla \Phi(x) - \nabla \Phi(\theta_s)) \geq \kappa \|x - \theta_s\|^2.$$ 

By Lagrange’s mean value theorem, there exist some $\xi$ lying on the line between $x$ and $\theta_s$, such that

$$\Phi(x) - \Phi(\theta_s) = (x - \theta_s) \cdot \nabla \Phi(\xi).$$

Let $\xi = \theta_s + t(x - \theta_s)$ for some $0 \leq t \leq 1$. By strong convexity, we have

$$(x - \xi) \cdot (\nabla \Phi(x) - \nabla \Phi(\xi)) \geq \kappa \|x - \xi\|^2.$$ 

Replace $\xi$ by $\theta_s + t(x - \theta_s)$, we get

$$(x - \theta_s) \cdot (\nabla \Phi(x) - \nabla \Phi(\xi)) \geq (1 - t)(x - \theta_s) \cdot (\nabla \Phi(x) - \nabla \Phi(\xi)) \geq \kappa \|x - \xi\|^2.$$ 

This implies

$$(x - \theta_s) \cdot \nabla \Phi(x) \geq (x - \theta_s) \cdot \nabla \Phi(\xi) + \kappa \|x - \xi\|^2 \geq \Phi(x) - \Phi(\theta_s).$$
Hence,
\[(x - \theta_*) \cdot \nabla \Phi(x)/2 \geq (\Phi(x) - \Phi(\theta_*))/4 + \kappa \|x - \theta_*\|^2/4).\]
By choosing \(\lambda = (\kappa/\alpha^2) \wedge (1/4)\), the proof is completed. \(\square\)

**Lemma 8.** Assume \(\Phi_i\) is convex and satisfies Assumption \([J]\) for all \(i \in I\). Let \(\theta^m_t\) be the solution to system \([2.6]\). Then we have
\[
\|\theta^m_t\| \leq \|\theta_0\| + C\Phi t.
\]

**Proof.** We differentiate \(\|\theta^m_t\|^2\),
\[
d\|\theta^m_t\|^2 = 2 \left\langle \theta^m_t, d\theta^m_t/dt \right\rangle = -2 \left\langle \theta^m_t, \nabla \Phi_{i,m,\delta}(\theta^m_t) \right\rangle
\]
\[
= -2 \left\langle \theta^m_t, \nabla \Phi_{i,m,\delta}(\theta^m_t) - \nabla \Phi_{i,m,\delta}(0) \right\rangle - 2 \left\langle \theta^m_t, \nabla \Phi(0) \right\rangle
\]
\[
\leq 2 \|\theta^m_t\| \|\nabla \Phi_{i,m,\delta}(0)\| \leq 2C\Phi \|\theta^m_t\|,
\]
which implies
\[
d\|\theta^m_t\| \leq C\Phi.
\]
Hence
\[
\|\theta^m_t\| \leq \|\theta_0\| + C\Phi t.
\]
\(\square\)

**Lemma 9.** Assume \(\Phi_i\) is strong convex with \(\kappa > 0\) for all \(i \in I\). Let \(\theta_t\) be the solution to system \([1.3]\). Then we have
\[
\|\theta_t\| \leq \|\theta_0\| e^{-\kappa t} + C\Phi.
\]

**Proof.** We differentiate \(\|\theta_t\|^2\),
\[
d\|\theta_t\|^2 = 2 \left\langle \theta_t, d\theta_t/dt \right\rangle = -2 \left\langle \theta_t, \nabla \Phi_{i,t}(\theta_t) \right\rangle
\]
\[
= -2 \left\langle \theta_t, \nabla \Phi_{i,t}(\theta_t) - \nabla \Phi_{i,t}(0) \right\rangle - 2 \left\langle \theta_t, \nabla \Phi(0) \right\rangle
\]
\[
\leq -2\kappa \|\theta_t\|^2 + 2 \|\theta_t\| \|\nabla \Phi_{i,t}(0)\| \leq -\kappa \|\theta_t\|^2 + \frac{1}{\kappa} \|\nabla \Phi_{i,t}(0)\|^2,
\]
By Grönwall’s inequality, we have
\[
\|\theta_t\|^2 \leq e^{-\kappa t} \|\theta_0\|^2 + \frac{1}{\kappa^2} \|\nabla \Phi_{i,t}(0)\|^2.
\]
Let \(C_\Phi = \frac{1}{\kappa^2} \sup_{i=1,\ldots,N} \{\|\nabla \Phi_i\|^2\}\) \(\square\)

**Lemma 10.** Let \(C\) and \(\lambda\) be two positive constants. \(\{t_n\}_{n \geq 0}\) is a sequence with \(t_0 \geq 0\) and satisfies the following recurrence relation
\[
t_{n+1} = t_n + Ce^{-\lambda t_n}.
\]
Then \(\lim_{n \to +\infty} t_n = +\infty\).
Proof. It is obvious that \( \{t_n\}_{n \geq 0} \) is a strictly increasing sequence. If \( \lim_{n \to +\infty} t_n < +\infty \), let \( A \) be the smallest upper bound of \( \{t_n\}_{n \geq 0} \). Then for any \( m > 0 \), there exist \( n_0 \) such that, for any \( n \geq n_0 \), \( t_n \geq A - m \). We have
\[
t_{n+1} = t_n + C e^{-\lambda t_n} \geq A - m + C e^{-A}.
\]
Set \( m \leq C e^{-A}/2 \), then \( t_{n+1} > A \). Contradiction. Hence \( \lim_{n \to +\infty} t_n = +\infty \). \( \square \)

**Lemma 11.** Let \( q_i^t \) and \( p_i^t \) solve (3.2). Under the Assumption 1-2 with \( \alpha = 2 \) and 3, we have the following convergence of the Lyapunov function,
\[
V^i(t, q_i^t, p_i^t) \leq e^{-\lambda t} V^i(0, q_0^t, p_0^t),
\]
where \( V^i(t, x, y) = m(t)[\Phi_i(x) - \Phi_i(\theta_*)] + \frac{1}{4} \left\| x - \theta_* + m(t) y \right\|^2 + \left\| m(t) y \right\|^2 \).

Proof. First, we differentiate \( V^i(t, q_i^t, p_i^t) \),
\[
dV^i(t, q_i^t, p_i^t)/dt = \frac{1}{4} d\left( \left\| q_i^t - \theta_* \right\|^2 + \left\| m(t) p_i^t \right\|^2 \right) / dt
\]
\[
= m'(t)[\Phi_i(q_i^t) - \Phi_i(\theta_*)] + m(t) \left\langle \nabla \Phi_i(q_i^t), dq_i^t \right\rangle + \frac{1}{2} \left( \left\langle q_i^t - \theta_* + m(t) p_i^t, dq_i^t \right\rangle + \left\langle m(t) p_i^t, dp_i^t \right\rangle \right)
\]
\[
= m'(t)[\Phi_i(q_i^t) - \Phi_i(\theta_*)] + m(t) \left\langle \nabla \Phi_i(q_i^t), p_i^t \right\rangle
\]
\[
+ \frac{1}{2} \left( \left\langle q_i^t - \theta_* + m(t) p_i^t, (p_i^t + m(t) dp_i^t) \right\rangle \right)
\]
\[
= m'(t)[\Phi_i(q_i^t) - \Phi_i(\theta_*)] + m(t) \left\langle \nabla \Phi_i(q_i^t), p_i^t \right\rangle
\]
\[
+ \frac{1}{2} \left( \left\langle q_i^t - \theta_* + m(t) p_i^t, (m'(t) p_i^t + m(t) dp_i^t) \right\rangle \right)
\]
\[
\leq m'(t)[\Phi_i(q_i^t) - \Phi_i(\theta_*)] - \frac{1}{2} \left\langle q_i^t - \theta_* + \nabla \Phi_i(q_i^t), (m'(t) p_i^t) \right\rangle + m(t) \left( m'(t) - \frac{1}{2} \right) \left\| p_i^t \right\|^2
\]
\[
\leq - \frac{1}{2} \left\langle q_i^t - \theta_* + \nabla \Phi_i(q_i^t), (m'(t) p_i^t) \right\rangle - \frac{m(t)}{2} \left\| p_i^t \right\|^2 + \frac{\lambda}{4} \left\| q_i^t - \theta_* \right\|^2 + \frac{(m'(t))^2}{4\lambda} \left\| p_i^t \right\|^2
\]
\[
\leq - \lambda \left( \Phi_i(q_i^t) - \Phi_i(\theta_*) + \left\| q_i^t - \theta_* \right\|^2 \right) - \frac{m(t)}{4} \left\| p_i^t \right\|^2 + \frac{\lambda}{4} \left\| q_i^t - \theta_* \right\|^2
\]
\[
\leq - \lambda \left( \Phi_i(q_i^t) - \Phi_i(\theta_*) + \frac{\left\| q_i^t - \theta_* + m(t) p_i^t \right\|^2}{4} + \frac{\left\| m(t) p_i^t \right\|^2}{4} + \frac{3\lambda m^2(t) - m(t)}{4} \right) \left\| p_i^t \right\|^2
\]
\[
\leq - \lambda V^i(t, q_i^t, p_i^t).
\]
The inequality \((\beta_1)\) is since Assumption 2 and 3 which the second assumption implies \(\frac{m'(t)^2}{4\lambda} - \frac{m(t)}{2} \|P_i\|^2 \leq \frac{\lambda^2 m(t)^2}{4\lambda} - \frac{m(t)}{4} \|P_i\|^2 \leq - \frac{m(t)}{4} \leq - \frac{m(t)}{4}.\) Finally, by Grönwall’s inequality we have

\[ V(t, q^i, p^i) \leq e^{-\lambda t} V(0, q^i_0, p^i_0). \]

\[ \square \]

**Lemma 12.** Let \(m(t) > 0\) be a strictly decreasing differentiable function and satisfy Assumption 3 with \(\lambda = k/4,\) where \(k > 0.\) Then we have the following inequalities,

\[
\int_0^t m^2(s)e^{ks}ds \leq 2\kappa^{-1} e^{\kappa t} m^2(t),
\]

\[
\int_0^t e^{-2\xi(s)}e^{ks}ds \leq (1 + \kappa)m_0 e^{\kappa t}.
\]

**Proof.** For the first inequality, by integration by parts, we have

\[
\int_0^t m^2(s)e^{ks}ds = \kappa^{-1} \int_0^t m^2(s)de^{ks} = \kappa^{-1} \left[ m^2(t)e^{\kappa t} - m_0 - 2 \int_0^t m(s)m'(s)e^{ks}ds \right].
\]

Under the Assumption 3

\[-2 \int_0^t m(s)m'(s)e^{ks}ds \leq \frac{k}{2} \int_0^t m^2(s)e^{ks}ds.
\]

Hence

\[
\int_0^t m^2(s)e^{ks}ds \leq \kappa^{-1} e^{\kappa t} m^2(t) + \frac{1}{2} \int_0^t m^2(s)e^{ks}ds.
\]

This implies

\[
\int_0^t m^2(s)e^{ks}ds \leq 2\kappa^{-1} e^{\kappa t} m^2(t).
\]

For the second inequality, we use integration by parts again and notice that

\[
\int_0^t e^{-2\xi(s)}e^{ks}ds = -\frac{1}{2} \int_0^t m(s)e^{ks}de^{-2\xi(s)}
\]

\[
= \frac{1}{2} \left( m_0 - m(t)e^{kt}e^{-2\xi(t)} \right) + \frac{1}{2} \int_0^t e^{-2\xi(s)}e^{ks}(km(s) + m'(s))ds
\]

\[
\leq \frac{m_0}{2} + \frac{3km_0e^{kt}}{8} \int_0^t e^{-2\xi(s)ds} \leq \frac{m_0}{2} + \frac{3km_0e^{kt}}{8} \int_0^t e^{-\frac{2s}{m_0}}ds \leq (1 + \kappa)m_0 e^{\kappa t}.
\]

This completes the proof. \[ \square \]

**Lemma 13.** Let \(\{m_n\}_{n \geq 0}\) be a non-negative decreasing sequence with \(\lim_{n \to \infty} m_n = 0.\) Let \(\{d_n\}_{n \geq 0}\) be a positive sequence satisfying the following induction

\[ d_{n+1} \leq cd_n + m_n, \]

where \(0 < c < 1.\) Then we have \(\lim_{n \to \infty} d_n = 0.\) Furthermore, if \(m_n \leq Be^{-bn}\) for some constant \(\bar{B}, \bar{b} > 0\) we have \(d_n \leq \bar{B}e^{-bn}.\)
Proof. It is obvious \( \{m_{n}\}_{n \geq 0} \) is bounded and we denote the upper bound by \( C \). Hence \( \{d_{n}\}_{n \geq 0} \) could be bounded by \( D := d_{0} + C/(1 - c) \). Since \( \lim_{n \to +\infty} m_{n} = 0 \), for any \( m > 0 \), there exist a \( N_{0} \), for any \( N \geq N_{0} \), \( m_{n} \leq m \). Hence, for any \( n \geq N_{0} \), we have

\[
d_{n+1} \leq cd_{n} + m_{n} \leq cd_{n} + m.
\]

which implies

\[
d_{n} \leq c^{n-N_{0}}d_{N_{0}} + \frac{m}{1-c}.
\]

Since \( m \) is arbitrary, we get \( \lim_{n \to \infty} d_{n} = 0 \), which finish the proof of the first part.

Under the assumption \( m_{n} \leq Be^{-bn} \), we note that \( d_{n+1} \leq cd_{n} + m_{n} \) implies

\[
d_{n+1} \leq \sum_{i=0}^{n} c^{n-i}m_{i} \leq B \sum_{i=0}^{n} c^{n-i}e^{-bi} = Be^{n} \sum_{i=0}^{n} e^{-b}\log\frac{1}{c}i.
\]

If \( b + \log c \geq 0 \), \( d_{n+1} \leq Bnc^{n} \) the result is obvious. For \( b + \log c < 0 \), we have

\[
d_{n+1} \leq Be^{n}e^{-b(n+1)} - 1 \leq Be^{n}e^{-b(n+1)} - 1 \leq \tilde{B}e^{-bn+1}.
\]

Lemma 14. Let \( G_{k}^{n} = \prod_{i=k}^{n} \left(1 - \frac{b}{\sqrt{i}}\right) \), where \( 0 < b < 1 \). Then there exist \( c > 0 \) such that for any \( 0 \leq k < n \), we have \( G_{k}^{n} \leq e^{-c(\sqrt{n} - \sqrt{k})} \).

Proof. It is obvious that for \( 0 \leq x < 1 \), \( \log(1 - x) \leq -x \), which implies

\[
\log G_{k}^{n} = \log \left(\prod_{i=k}^{n} \left(1 - \frac{b}{\sqrt{i}}\right)\right) = \sum_{i=k}^{n} \log \left(1 - \frac{b}{\sqrt{i}}\right) \leq -b \sum_{i=k}^{n} \frac{1}{\sqrt{i}}
\]

\[
= -b \sum_{i=k}^{n} \frac{1}{\sqrt{i}} \int_{i}^{i+1} dx \leq -b \sum_{i=k}^{n} \int_{i}^{i+1} \frac{1}{\sqrt{x}} dx = -b \int_{k}^{n+1} \frac{1}{\sqrt{x}} dx
\]

\[
= -b(\sqrt{n+1} - \sqrt{k}) \leq -b(\sqrt{n} - \sqrt{k})/2.
\]

Finally, by taking exponential both side and let \( c = -b/2 \), we finish the proof. \( \square \)

REFERENCES

[1] H. Attouch, Z. Chbani, J. Peypouquet, P. Redont. Fast convergence of inertial dynamics and algorithms with asymptotic vanishing viscosity. Math. Program., 168:123–175, 2018.
[2] B. Cloez, M. Hairer. Exponential ergodicity for markov processes with random switching. Bernoulli, 21(1):505-536, 2015.
[3] H. Daneshmand, J. Kohler, A. Lucchi, T. Hofmann. Escaping saddles with stochastic gradients. Proceedings of the 35th International Conference on Machine Learning, PMLR, pp. 1155-1164 , 2018.
[4] A. Défossez, L. Bottou, F. Bach, N. Usunier. A Simple Convergence Proof of Adam and Adagrad. arXiv:2003.02395 2020.
[5] S.S. Du, C. Jin, J.D. Lee, M.I. Jordan, A. Singh, B. Poczos. Gradient descent can take exponential time to escape saddle points. Conference on Neural Information Processing Systems (NeurIPS), 2017.
[6] J. Duchi, E. Hazan, Y. Singer. Adaptive Subgradient Methods for Online Learning and Stochastic Optimization. Journal of Machine Learning Research (JMLR), 2011.
[7] A. Eberle, A. Guillin, R. Zimmer. Couplings and quantitative contraction rates for Langevin dynamics. *The Annals of Probability* 2019, Vol. 47, No. 4, 1982–2010.

[8] K. Fukushima. Neocognitron: A Self-organizing Neural Network Model for a Mechanism of Pattern Recognition Unaffected by Shift in Position. *Biol. Cybernetics* 36, 193-202, 1980.

[9] E. Hairer, C. Lubich, G. Wanner. Geometric numerical integration illustrated by the Störmer-Verlet method. *Acta Numerica* 2003, 1-51.

[10] K. He, X. Zhang, S. Ren and J. Sun. Deep Residual Learning for Image Recognition. *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2016.

[11] G. Hinton. Lecture 6, *Online course in Neural Networks for Machine Learning*.

[12] K. Jin, J. Latz, C. Liu, C.-B. Schönlieb. A Continuous-time Stochastic Gradient Descent Method for Continuous Data [arXiv:2112.03754], 2021.

[13] C. Jin, P. Netrapalli, R. Ge, S. M. Kakade, M. I. Jordan. On Nonconvex Optimization for Machine Learning: Gradients, Stochasticity, and Saddle Points. *In International Conference on Machine Learning (ICML)*, 2017.

[14] S. Kale, A. Sekhari, K. Sridharan. SGD: The Role of Implicit Regularization, Batch-size and Multiple-epochs. *Conference on Neural Information Processing Systems (NeurIPS)*, 2021.

[15] N. S. Keskar, D. Mudigere, J. Nocedal, M. Smelyanskiy, P. T. P. Tang. On Large-Batch Training for Deep Learning: Generalization Gap and Sharp Minima. *International Conference on Learning Representations (ICLR)*, 2017.

[16] D. P. Kingma and J. Ba. Adam: A Method for Stochastic Optimization. *International Conference on Learning Representations (ICLR)*, 2015.

[17] A. Krizhevsky. Learning Multiple Layers of Features from Tiny Images, 2009.

[18] A. Krizhevsky, I. Sutskever, G. E. Hinton. ImageNet Classification with Deep Convolutional Neural Networks. *International Conference on Neural Information Processing Systems (NeurIPS)*, 2012.

[19] H. Kushner. Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems. *Systems & Control: Foundations & Applications*, Volume 3 Birkhäuser Boston, 1990.

[20] J. Latz. Analysis of stochastic gradient descent in continuous time *Statistics and Computing*, 31:39, 2021.

[21] J. Latz. Gradient flows and randomised thresholding: sparse inversion and classification [arXiv:2112.03754], 2022.

[22] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-Based Learning Applied to Document Recognition. *Proceedings of the IEEE*, Volume: 86, Issue: 11, Nov. 1998.

[23] J.D. Lee, M. Simchowitz, M.I. Jordan, B. Recht. Gradient descent only converges to minimizers. *JMLR: Workshop and Conference Proceedings*, 49:1-12, 2016.

[24] C. Lee, S. Xie, P. Gallagher, Z. Zhang and Z. Tu. Deeply-Supervised Nets. *Proceedings of the Eighteenth International Conference on Artificial Intelligence and Statistics*, PMLR 38:562-570, 2015.

[25] Y. Liu, Y. Gao, W. Yin. An Improved Analysis of Stochastic Gradient Descent with Momentum. *International Conference on Neural Information Processing Systems (NeurIPS)*, 2020.

[26] Y. Nesterov. A method of solving a convex programming problem with convergence rate $O(\frac{1}{k^2})$. *Sov. Math. Dokl.*, 27:372–376, 1983.

[27] B. Neyshabur, R. Tomioka, N. Srebro. In Search of the Real Inductive Bias: On the Role of Implicit Regularization in Deep Learning. *International Conference on Learning Representations (ICLR)*, 2015.

[28] I. Panageas, G. Piliouras. Gradient Descent Only Converges to Minimizers: Non-Isolated Critical Points and Invariant Regions. *8th Innovations in Theoretical Computer Science Conference (ITCS)*, 2017.

[29] B.T. Polyak. Gradient method for the minimization of functionals. *USSR Comput. Math. Math. Phys.*, 3(4):864–878, 1963.

[30] B.T. Polyak. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.*, 4(5):1-17, 1964.

[31] H. Robbins, S. Monro A Stochastic Approximation Method. *The Annals of Mathematical Statistics*, 22(3):400-407, 1951.

[32] A. Scagliotti, P. Colli Franzone. A piecewise conservative method for unconstrained convex optimization. *Comput. Optim. Appl.*, 81:251-288, 2022.
[33] B. Shi, S.S. Du, M.I. Jordan, W.J. Su. Acceleration via symplectic discretization of high-resolution differential equations. International Conference on Neural Information Processing Systems (NeurIPS), 2019.

[34] B. Shi, S.S. Du, M.I. Jordan, W.J. Su. Understanding the acceleration phenomenon via high-resolution differential equations. Math. Program., 2021.

[35] K. Simonyan, A. Zisserman. Very Deep Convolutional Networks for Large-Scale Image Recognition. International Conference on Learning Representations (ICLR), 2015.

[36] W.J. Su, S. Boyd, E. Candès. A differential equation for modeling Nesterov’s accelerated gradient method: theory and insights. Journal of Machine Learning Research (JMLR), 17(153):1-43, 2016.

[37] C. Szegedy, W. Liu, Y. Jia, P. Sermanet, S. Reed, D. Anguelov, D. Erhan, V. Vanhoucke and A. Rabinovich. Going Deeper with Convolutions. IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 2015.

[38] M. Tan and Q. V. Le. EfficientNet: Rethinking Model Scaling for Convolutional Neural Networks. International Conference on Machine Learning, PMLR, 2019.

[39] M.D. Zeiler. ADADELTA: An Adaptive Learning Rate Method. arXiv:1212.5701, 2012.

[40] C. Zhang, S. Bengio, M. Hardt, B. Recht, O. Vinyals. Understanding deep learning requires rethinking generalization. International Conference on Learning Representations (ICLR), 2017.

[41] M. Zinkevich. Online Convex Programming and Generalized Infinitesimal Gradient Ascent. Proceedings of the Twentieth International Conference on Machine Learning, 2003.

[42] D. Zou, J. Wu, V. Braverman, Q. Gu, D. P. Foster, S. M. Kakade. The Benefits of Implicit Regularization from SGD in Least Squares Problems. International Conference on Neural Information Processing Systems (NeurIPS), 2021.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544-1000, USA
Email address: kexinj@math.princeton.edu

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES AND SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES, HERIOT-WATT UNIVERSITY, EDINBURGH, EH14 4AS, UK
Email address: j.latz@hw.ac.uk

DELTIF INSTITUTE OF APPLIED MATHEMATICS, TECHNISCHE UNIVERSITET DELFT, 2628 DELFT, THE NETHERLANDS
Email address: C.Liu-13@tudelft.nl

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI, TRIESTE, ITALY
Email address: ascaglio@sissa.it

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