Degeneracies when T=0 Two Body Interaction Matrix Elements are Set Equal to Zero:

Talmi’s method of calculating coefficients of fractional parentage to states forbidden by the Pauli principle.

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Abstract

In a previous work we studied the effects of setting all two body T=0 matrix elements to zero in shell model calculations for $^{43}$Ti ($^{43}$Sc) and $^{44}$Ti. The results for $^{44}$Ti were surprisingly good despite the severity of this approximation. In this approximation degeneracies arose in the $T=\frac{1}{2} I= (\frac{1}{2})_{-1}^{-1}$ and $(\frac{13}{2})_{-1}^{-1}$ states in $^{43}$Sc and the $T=\frac{1}{2} I = (\frac{15}{2})_{2}^{+}, (\frac{17}{2})_{1}^{-},$ and $(\frac{19}{2})_{1}^{-}$ in $^{43}$Sc. The $T=0 3_{2}^{+}, 7_{2}^{+}, 9_{1}^{+},$ and $10_{1}^{+}$ states in $^{44}$Ti were degenerate as well. The degeneracies can be explained by certain 6j symbols and 9j symbols either vanishing or being equal as indeed they are. Previously we used Regge symmetries of 6j symbols to explain these degeneracies. In this work a simpler more physical method is used. This is Talmi’s method of calculating coefficients of fractional parentage for identical particles to states which are forbidden by the Pauli principle. This is done for both one particle cfp to handle 6j symbols and two particle cfp to handle 9j symbols. The states can be classified by the dual quantum numbers ($J_{\pi}, J_{\nu}$).
I. INTRODUCTION

In a previous work [1] we considered both states in a single j shell and a full FP calculation for $T=\frac{1}{2}$ states in $^{43}$Ti ($^{43}$Sc) and $T=0$ states in $^{44}$Ti. We compared the results in which the full interaction was used and one in which all the $T=0$ two body interaction matrix elements were set equal to zero. Despite the apparent severity of the latter, the results in $^{44}$Ti were surprisingly good, especially for states of even angular momentum. The odd I states were somewhat too low and got shifted up (almost uniformly) when the $T=0$ matrix elements were put back in.

When the $T=0$ matrix elements were set equal to zero certain degeneracies appeared. eg $I=(1\frac{1}{2})^{-1}$, $(1\frac{3}{2})^{-1}$ in $^{43}$Ti ($^{43}$Sc) as well as $T=\frac{1}{2}$ $I=(1\frac{3}{2})^{-1}$, $(1\frac{7}{2})^{-1}$, and $(1\frac{9}{2})^{-1}$. In $^{44}$Ti the $3^{+}_{2}, 7^{+}_{2}, 9^{+}_{1},$ and $10^{+}_{1}$ states were degenerate. These degeneracies required certain 6j and 9j symbols to vanish and others to be equal. This was verified by simply looking them up and some insight as to why they vanished was obtained using Regge’s 6j symmetries relations.

In that method a given 6j symbol was shown to be equal to one which had at least one small number up to two. For such 6j expressions there are analytical expressions which can be used.

The wavefunctions of the $I=\frac{13}{2}^{-}$ state in $^{43}$Ti ($^{43}$Sc) can be written as

$$
\psi_1 = a[J_\pi = 4, j_\nu = \frac{7}{2} I = \frac{13}{2}^{-}] + b[J_\pi = 6, j_\nu = \frac{7}{2} I = \frac{13}{2}^{-}]
$$

(1)

$$(a^2 + b^2 = 1)$$

In general $a$ and $b$ are finite. However when the $T=0$ two body interaction matrix elements are set equal to zero we find $a=1$ $b=0$ i.e. the eigenfunctions are $[J_\pi = 4, j_\nu = \frac{7}{2} I = \frac{13}{2}^{-}]$ and $[J_\pi = 6, j_\nu = \frac{7}{2} I = \frac{13}{2}^{-}]$. Furthermore the first $\frac{13}{2}^{-}$ state is degenerate with the $I=\frac{1}{2}^{-}$ state $[J_\pi = 4, j_\nu = \frac{7}{2} I = \frac{1}{2}^{-}]$. The fact that both the $\frac{1}{2}^{-}$ and $\frac{13}{2}^{-}$ states have the same structure $[4, \frac{7}{2}]$ gives us some insight into why they are degenerate.

There are two conditions to be met:

a) no mixing condition

$$
\left\{ j \ j \ 4 \right\} = 0 \\
\left\{ j \ \frac{13}{2} \ 6 \right\} = 0
$$

b) degeneracy

$$
\left\{ j \ j \ 4 \right\} = \left\{ j \ j \ \frac{13}{2} \right\} \\
\left\{ j \ \frac{13}{2} \ 4 \right\} = \left\{ j \ \frac{13}{2} \ \frac{13}{2} \right\} \\
\left\{ j \ j \ \frac{13}{2} \right\} = \left\{ j \ j \ 6 \right\} \\
\left\{ j \ \frac{13}{2} \ 6 \right\} = \left\{ j \ \frac{17}{2} \ 6 \right\} = \left\{ j \ \frac{19}{2} \ 6 \right\}
$$

These conditions are true. They can be generalized to

a) $\left\{ j \ j \ (2j-3) \right\} = 0$

b) $\left\{ j \ (3j-4) \ (2j-1) \right\} = 0$
b) \[ \binom{j}{j} \binom{(2j-1)}{(2j-1)} \binom{j}{(2j-1)} = \frac{(-1)^{2j}}{(2j-2)} \] for I = (3j-1), (3j-2), and (3j-4). These hold for both half integer and integer J.

From degeneracy of the 9\textsuperscript{+} and 10\textsuperscript{+} states in 44\textsuperscript{Ti} we get conditions on the 9j symbols

\[
\begin{align*}
\binom{j}{j} & \binom{6}{6} \\
\binom{j}{j} & \binom{6}{6} \\
4 & 6 I
\end{align*}
\]

This decouples states with \((J_\pi, J_\nu) = (4,6)+(6,4)\) from states (6,6).

This can be generalized to

\[
\begin{align*}
\binom{j}{j} & \binom{(2j-1)}{(2j-3)(4j-4)} \binom{j}{(2j-1)} \binom{(2j-1)}{(2j-3)(4j-4)}
\end{align*}
\]

We also get the diagonal condition

\[
\begin{align*}
\binom{j}{j} & \binom{(2j-3)}{(2j-1)} \binom{j}{(2j-1)} \binom{(2j-3)}{(2j-1)} I
\end{align*}
\]

\[
\frac{1}{16(2j-2)(2j-1)-12} = \frac{1}{4(2j-5)(4j-1)}
\]

for I = (4j-4),(4j-5), and (4j-7). (10, 9 and 7 in the above example).

Note that we are dealing only with \(T=\frac{1}{2}\) states in 43\textsuperscript{Ti} and \(T=0\) states in 44\textsuperscript{Ti}. In the next section we will show how these conditions arise. One clue is that the degenerate states in 43\textsuperscript{Ti} occur for angular momentum for which there are no (single j) \(T=\frac{3}{2}\) states and in 44\textsuperscript{Ti} for angular momentum for which there are no \(T=2\) states. Note that when the \(T=0\) two body matrix elements are set equal to zero we can classify the states by the dual quantum number \((J_\pi, J_\nu)\).

II. TALMI’S METHOD OF CFP TO PAULI FORBIDDEN STATES

In order to explain the properties of 6j and 9j symbols in the previous section we shall take what may appear a strange detour and consider systems of identical particles. Although we are concerned with degeneracies of \(T=\frac{1}{2}\) states and \(T=0\) states we shall now examine \(T=\frac{3}{2}\) states in 43\textsuperscript{Ca} and \(T=2\) states in 44\textsuperscript{Ca}.

The Pauli principle imposes severe constraints on the \(j^3\) configuration of 3 neutrons. Although 3 non-identical particles could couple to a maximum spin of \(\frac{21}{2}\), for identical particles the maximum is \(\frac{15}{2}\). This is easy to see by trying to construct the maximum of \(j_3\). To satisfy the Pauli principle \(M_{max} = \frac{7}{2} + \frac{5}{2} + \frac{3}{2} = \frac{15}{2}\). Thus we can form a state \(I=\frac{15}{2}\) \(M=\frac{15}{2}\).

One then notes there is only one way to form \(M=\frac{13}{2}\) namely \(\frac{7}{2} + \frac{5}{2} + \frac{1}{2}\).

This must correspond to \(I=\frac{15}{2}\) \(M=\frac{13}{2}\) state. So there is no \(M=\frac{13}{2}\) left for the \(I=\frac{13}{2}\) state. Hence the \(I=\frac{13}{2}\) state is forbidden by the Pauli principle.
This is all in Talmi’s book [2]. Of particular interest is the fact stated therein by Talmi that one can extract useful information by trying to construct coefficients of fractional parentage for this forbidden \( I = \frac{13}{2} \) state and then setting the cfp equal to zero.

We can form a \( I = \frac{13}{2} \) state as follows:

\[
\psi^0 = [[j(1)j(2)]I_0j(3)]I
\]

(3)

here \( I_0 \) must be even so that particles 1 and 2 are antisymmetrized. For \( I = \frac{13}{2} \), \( I_0 \) can be either 4 or 6. Note that this wavefunction is not overall antisymmetric. The antisymmetric wave function is \( N(1 - P_{13} - P_{23})\psi^0 \). The act of antisymmetrization means that \( I_0 \) is merely a starting index and is usually not unique. The cfp expansion is \( \psi_{antisymmetric} = \Sigma I_1 (j^2I_1j)j^3I)[[j(1)j(2)]I_1j(3)]I \). Each term in the expansion is not antisymmetric but the total expression is. Clearly the act of putting particle 3 to the extreme right is going to involve some Racah coefficients.

In fact the explicit expression is

\[
(j^2I_1j)j^3I] = N[\delta_{I_1I_0} \pm 2\sqrt{2I_0 + 1}(2I_1 + 1)] \frac{1}{18}
\]

(4)

\[
N = [3 + 6(2I_0 + 1) \frac{1}{18}]
\]

(5)

However as Talmi points our since the \( I = \frac{13}{2} \) state does not exist all the above cfp must vanish [2].

Let us pick the starting \( I_0 \) to be 4, then the \( I_1 \) can be either 4 or 6. If \( I_1 \) equals 4 we get

\[
\left\{ \begin{array}{c}
j & j & 4 \\
I & j & 4 
\end{array} \right\} = \frac{-1}{18}
\]

for \( I = \frac{13}{2} \). Here we note that this 6j is independent of I for all I that are forbidden by the Pauli principle i.e. \( \frac{1}{2} \) and \( \frac{13}{2} \). If \( I_1 \) equals 6 we get

\[
\left\{ \begin{array}{c}
j & j & 4 \\
I & j & 6 
\end{array} \right\} = 0
\]

But these are exactly the conditions we derived for the degeneracies of \( I = \frac{1}{2} \) and \( \frac{13}{2} \) \((T = \frac{1}{2})\) states of \(^{43}\text{Sc}\). The vanishing of the second 6j gives the decoupling of \( [J_\pi = 4, j_\nu = \frac{7}{2}\{I = \frac{13}{2}\}] \) from \( [J_\pi = 6, j_\nu = \frac{7}{2}\{I = \frac{13}{2}\}] \) and the first 6j gives the condition that \( [J_\pi = 4, j_\nu = \frac{7}{2}\{I = \frac{13}{2}\}] \) state and \( [J_\pi = 4, j_\nu = \frac{7}{2}\{I = \frac{13}{2}\}] \) are degenerate.

These results can be easily generalized to the following relations:

\[
\left\{ \begin{array}{c}
j & j & (2j-3) \\
(3j-4) & j & (2j-1) 
\end{array} \right\} = 0
\]

this holds for both half integer and integer j. By a Regge symmetry this 6j is also equal to

\[
\left\{ \begin{array}{c}
(2j-2) & (2j-3) & 2 \\
(2j-2) & (2j-1) & (2j-2) 
\end{array} \right\}.
\]
There are simple analytic expressions for 6j symbols where one of the entries is two or less so it is easy to show that this 6j vanishes. The other condition is
\[
\begin{pmatrix}
  j & j & I_0 \\
  I & j & I_0
\end{pmatrix} = \frac{(-1)^{2j}}{2(2I_0+1)}
\]
for states \(I\) which are forbidden by the Pauli principle for three identical particles. The important thing to point out is that for such states the 6j is independent of the total angular momentum \(I\). This is necessary for states of different \(I\) to be degenerate.

For states of angular momentum \(I\) which are not allowed by the Pauli principle there are alternative ways of writing some of the relations i.e.
\[
\begin{pmatrix}
  j & j & (2j - 1) \\
  j & I & (2j - 1)
\end{pmatrix} = \frac{(-1)^{2j}}{(8j-2)}
\]
for \(I=(3j-1), (3j-2), \text{and} (3j-4)\).

The argument extends to integer \(j\). Consider as an example \(j=4\). Then the vanishing 6j \(\begin{pmatrix} j & j & (2j - 3) \\
(3j - 4) & j & (2j - 1) \end{pmatrix} = \begin{pmatrix} 4 & 4 & 5 \\
8 & 4 & 7 \end{pmatrix}\) where we can regard \(j\) now as the orbital angular momentum. We try to form an antisymmetric state for \(I=8\) using \((1-P_{13} - P_{23})\)\([\begin{pmatrix} j & j \\
I_A & I_B \end{pmatrix}]^{(6)}\). But we can show that there is no \(L=8\) antisymmetric state of 3 \(L=4\) particles. The highest \(M\) one can have is \(M=4+3+2=9\). We can thus have an \(L=9\) \(M=9\) state. There is only one way to form \(M=8\) i.e \(4+3+1\). This must correspond to the \(L=9\) \(M=8\) states. This leaves no \(M=8\) to form an \(L=8\) state. Hence the cfp for such a state much vanish. Thus for the starting angular momentum \(L_A = 5\) we must have the cfp for \(L_1=7\) to vanish i.e. \(\begin{pmatrix} 4 & 4 & 5 \\
8 & 4 & 7 \end{pmatrix} = 0\).

### III. Extension of the Arguments to Two Particle Fractional Parentage Coefficients

To explain the degeneracies of certain \(T=0\) states in \(^{44}\text{Ti}\), we consider a system of 4 identical particles i.e. \(^{44}\text{Ca}\) and as an obvious extension of the previous section we consider the two particle cfp’s.

They are defined by
\[
\Psi_{\text{antisymmetrized}} = \Sigma_{I_1, I_2} (j_{1}^{n-2} I_{1} I_{2} | j_{2}^{n} I) [(j_{1}^{n-2} I_{1} (j_{2}^{2} I_{2}) I_{1} I_{2})^{I}]
\]

We can form an antisymmetric state for 4 neutrons in the single \(j\) shell as follows
\[
(1 - P_{13} - P_{23} - P_{14} - P_{24})[[j(1)j(2)] I_{A} [j(3)j(4)] I_{B}]^{I}
\]
with the starting angular momentum \(I_A\) and \(I_B\) even.

Consider the \(P_{13}\) term - call it C
\[
C = -[[j(3)j(2)] I_{A} [j(1)j(4)] I_{B}]^{I}
\]
We must bring particle 3 to the right side and particle 1 to the left side. Thus

\[ C = -(-1)^{I_A} [j(2) j(3)] I_A [j(1) j(4)] I_B ]^I \]

\[ C = \sum_{I_1, I_2} (-1)^{I_A + I_1 + 1} < (jj) I_A (jj) I_B | (jj) I_1 (jj) I_2 >^I [j(1) j(2)] I_1 [j(3) j(4)] I_2 ]^I \]

At this point \( I_1 \) and \( I_2 \) can be even or odd. However when we perform the complete antisymmetrization \( I_1 \) and \( I_2 \) will be even. We find

\[ (j^2 I_1 j^2 I_2) j^4 I) = N [\delta_{I_1 I_1} \delta_{I_2 I_2} - 4 \sqrt{(2 I_A + 1)(2 I_B + 1)((2 I_1 + 1)(2 I_2 + 1)} \]

\[ \left\{ \begin{array}{c}
  j \\
  j \\
  I_A \\
  j \\
  j \\
  I_B \\
  I_1 \\
  I_2 \\
  I
\end{array} \right\} \]

\[ (11) \]

Let us consider the states with angular momentum \( I=10 \). We recall that when all the \( T=0 \) two body matrix elements were set equal to zero there was a decoupling of the states for which \((J_\pi, J_\nu)\) were \((6,4)\) and \((4,6)\) from the state \((6,6)\). This demanded that \[ \left\{ \begin{array}{c}
  j \\
  j \\
  6 \\
  j \\
  j \\
  6 \\
  4 \\
  6 \\
  10
\end{array} \right\} = 0 \]

as indeed it is.

The only allowed states for 4 neutrons in the \( f_{\frac{7}{2}} \) shell are

- \( v=0 \) \( I=0 \)
- \( v=2 \) \( I=2,4,6 \)
- \( v=4 \) \( I=2,4,5,8 \)

In the above \( v \) is the seniority quantum number. Since \( I=10 \) is not allowed all the 2 particle cfp above must vanish.

Let us choose the starting angular momentum to be \( I_A = 4 \) and \( I_B = 6 \). We get two conditions from the fact that the two particle cfp vanish

- a) \( I_1 = 4 \) \( I_2 = 6 \)
  \[ \left\{ \begin{array}{c}
  j \\
  j \\
  4 \\
  j \\
  j \\
  6 \\
  4 \\
  6 \\
  10
\end{array} \right\} = \frac{1}{368} \]

- b) \( I_1 = 6 \) \( I_2 = 6 \)
  \[ \left\{ \begin{array}{c}
  j \\
  j \\
  4 \\
  j \\
  j \\
  6 \\
  6 \\
  6 \\
  I
\end{array} \right\} = 0 \]

The second condition is the one required for the decoupling of \((J_\pi, J_\nu)\)=\((4,6)\)+\((6,4)\) from \((6,6)\). The first condition is better stated as \[ \left\{ \begin{array}{c}
  j \\
  j \\
  4 \\
  j \\
  j \\
  6 \\
  4 \\
  6 \\
  I
\end{array} \right\} \] is independent of \( I \) for states which
are not allowed by the Pauli principle for 4 neutrons and which can be formed by the vector sum \( \vec{4} + \vec{6} \). These are 3, 7, 9, and 10, precisely the states for which we get degeneracies.

The above results can be generalized to

\[
\begin{pmatrix}
  j & j & (2j - 1) \\
  j & j & (2j - 1) \\
(2j - 1) & (2j - 3) & (4j - 4)
\end{pmatrix}
\]

\[
= 0
\]

\[
\begin{pmatrix}
  j & j & (2j - 3) \\
  j & j & (2j - 1) \\
(2j - 3) & (2j - 1) & I
\end{pmatrix}
\]

\[
= \frac{1}{16(2j-2)(2j-1)-12} = \frac{1}{4(4j-5)(4j-1)}.
\]

Both of these relations hold for integer \( j \) as well as half integer \( j \). We can now make a different choice for the starting angular momentum i.e. \( I_A = I_B = 6 \) and consider the case where \( I_1 = 6 \) and \( I_2 = 6 \). This “diagonal” term is independent of \( I \) for the states not allowed by the Pauli principle, a condition necessary for degeneracy. Without going into the detail, this will lead to the result that the second 10\(^+\) state in \( ^{44}\text{Ti} \) is degenerate with the unique 12\(^+\) state.

**IV. ELECTROMAGNETIC TRANSITIONS BETWEEN EVEN SPIN AND ODD SPIN STATES IN \( ^{44}\text{Ti} \)**

We now consider the electromagnetic transitions from the even spin - even parity states to the odd spin-even parity states. We take the example of the 10\(^+\) \( \rightarrow \) 9\(^+\). The transition can occur either via the M1 or E2 modes. In the single \( j \) shell approximation \( B(\text{M1}) \) will vanish. This is because the M1 operator in the single \( j \) shell becomes

\[
\sqrt{\frac{3}{4\pi}} \left( \frac{g_{jj} + g_{j\nu}}{2} \right) \vec{I}
\]

and the total angular momentum \( \vec{I} \) cannot induce M1 transitions. Even when configuration mixing is included the value of \( B(\text{M1}) \) will be very small because the isoscalar M1 coupling is very weak. This is due to the fact that the proton and neutron have magnetic moments of opposite sign.

The E2 transitions are not strongly collective but they should still dominate over the M1’s. The values of \( B(\text{E2})_{10\rightarrow9} \) are 7.167 e\(^2\)fm\(^4\) in the single shell and 6.589 e\(^2\)fm\(^4\) in a full f-p calculation. The corresponding values of \( B(\text{M1}) \) are zero and 1.07x10\(^{-4}\) \( \mu_n^2 \).

**V. SUMMARY**

In summary we find that we can explain the many degeneracies in single \( j \) shell calculations for \( T=\frac{1}{2} \) states in \( ^{43}\text{Ti}(^{43}\text{Sc}) \) and \( T=0 \) states in \( ^{44}\text{Ti} \) when all the two body interaction
matrix elements are set equal to zero. These degeneracies involve states of angular momentum $I$, which because of the Pauli principle cannot exist for $T=\frac{3}{2}$ in $^{43}\text{Ti}^{(43}\text{Sc})$ or $T=2$ states in $^{44}\text{Ti}$ (or alternatively for systems of identical particles e.g. $^{43}\text{Ca}$, $^{44}\text{Ca}$). These degeneracies require certain $6j$ and $9j$ symbols to vanish and others to be equal. These conditions can be derived using Talmi’s method of calculating coefficients of fractional parentage to states that are forbidden by the Pauli principle and setting them equal to zero. One can apply this method to other single $j$ shells e.g. $g_{\frac{9}{2}}$. Thus for $^{97}\text{Cd}$ there would be degeneracy of $I=\frac{19}{2}^+, \frac{23}{2}^+$, and $\frac{25}{2}^+$ and in $^{96}\text{Cd}$ $I=13^+$ and $14^+$.

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