On the Biharmonic Curves in the Special Linear Group $\text{SL}(2, \mathbb{R})$

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Abstract. We characterize the biharmonic curves in the special linear group $\text{SL}(2, \mathbb{R})$. In particular, we show that all proper biharmonic curves in $\text{SL}(2, \mathbb{R})$ are helices and we give their explicit parametrizations as curves in the pseudo-Euclidean space $\mathbb{R}^4_2$.

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1. Introduction

Let $\phi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. The tension field of $\phi$ is, by definition, $\tau(\phi) = \text{trace} \nabla d\phi$. According to Eells and Lemaire, see [8], $\phi$ is biharmonic if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g.$$  

The first variation formula for $E_2$ was computed by Jiang in [9, 10] as

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \text{trace} R^N (d\phi, \tau(\phi)) d\phi = 0,$$

where $\Delta^\phi$ denotes the rough Laplacian acting on $C(\phi^{-1}TN)$, that with respect to a local orthonormal frame field $\{E_i\}_{i=1}^m$ on $M$ is defined by

$$\Delta^\phi = -\text{trace} \left( \nabla^2 \phi \right)^2 = -\sum_{i=1}^m \left\{ \nabla^\phi_{E_i} \nabla^\phi_{E_i} - \nabla^\phi_{\nabla^h_{E_i} E_i} \right\},$$

where $\nabla^\phi$ is the connection in $C(\phi^{-1} TN)$ induced by the Levi–Civita connection of $(N, h)$.

The field $\tau_2(\phi)$ is named bitension field of $\phi$.

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A curve \( \gamma : I \to (N, h) \), parametrized by arc-length, is biharmonic if \( \gamma \) is a biharmonic map, that is if
\[
\nabla^3_{\gamma'} \gamma' - R(\gamma', \nabla_{\gamma'} \gamma') \gamma' = 0.
\]

As a geodesic curve \((\tau(\phi) = \nabla_{\gamma'} \gamma' = 0)\) is biharmonic, we are interested in biharmonic curves that are not geodesics, i.e. proper biharmonic curves.

The study of the proper biharmonic curves on a curved surface starts with [5] where are described such curves for a surface, proving that biharmonic curves on a surface of non-positive Gaussian curvature are geodesics.

For 3-dimensional Riemannian manifolds with constant sectional curvature, the cases of null and negative curvature are considered in [3, 7] and it is shown that the only biharmonic curves are the geodesics. Moreover, in [2], it is considered the case of positive curvature showing that biharmonic curves have constant geodesic curvature and geodesic torsion.

Besides the spaces forms, the most interesting 3-dimensional homogeneous Riemannian spaces are those with 4-dimensional isometry group: the Berger spheres, the Heisenberg group, the special linear group \( SL(2, \mathbb{R}) \), and the Riemannian products \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \), where \( S^2 \) and \( \mathbb{H}^2 \) are the 2-dimensional sphere and the hyperbolic plane, respectively. A crucial feature in these spaces is that they admit a Riemannian submersion onto a surface of constant Gaussian curvature, called the Hopf fibration.

Balmuş [1] determined the parametric equations of all proper biharmonic curves on the Berger sphere \( S^3 \) as curves in \( \mathbb{R}^4 \) and gave a geometric interpretation for those curves in the unit Euclidean sphere \( S^3 \). In [6] the authors proved that any proper biharmonic curve in the Heisenberg group is a helix and gave its explicit parametrization.

Also, in [4] the authors considered the proper biharmonic curves in the Bianchi–Cartan–Vranceanu spaces \( \widetilde{SL}(2, \mathbb{R}) \), \( SU(2) \), \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \), proving that these curves are helices and giving their parametric equations.

In this paper we study the proper biharmonic curves in the special linear group \( SL(2, \mathbb{R}) \) endowed with a suitable 1-parameter family \( g_\tau \) of metrics that we shall describe in Sect. 2. Using the same technique given in [1] (for the case of the Berger spheres) and in [6] (for the Heisenberg group), we conclude that the biharmonic curves of \( SL(2, \mathbb{R}) \) make a constant angle \( \vartheta \) with the vector field tangent to the Hopf fibration. Moreover, in Theorem 3.4, we prove that the differential equation
\[
\gamma^{IV} + (b^2 - 2a) \gamma'' + a^2 \gamma = 0,
\]
where \( a \) and \( b \) are real constants depending on \( \vartheta \) and \( \tau \), must be satisfied by any proper biharmonic curve in \( SL(2, \mathbb{R}) \), as a curve in the pseudo-Euclidean space \( \mathbb{R}^4_2 \). We separate the study in three cases depending on the sign of the constant \((b^2 - 4a)\) obtaining, in each case, the expressions of these curves as curves in \( \mathbb{R}^4_2 \).
2. Preliminaries

Let $\mathbb{R}^4$ denote the 4-dimensional pseudo-Euclidean space endowed with the semi-definite inner product of signature $(2, 2)$ given by

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 - v_3 w_3 - v_4 w_4, \quad v, w \in \mathbb{R}^4.$$  

We identify the special linear group with

$$\text{SL}(2, \mathbb{R}) = \{(z, w) \in \mathbb{C}^2: |z|^2 - |w|^2 = 1\} = \{v \in \mathbb{R}^4: \langle v, v \rangle = 1\} \subset \mathbb{R}^4_2$$

and we shall use the Lorentz model of the hyperbolic plane with constant Gauss curvature $-4$, that is

$$H^2(-4) = \{(x, y, z) \in \mathbb{R}^3_1: x^2 + y^2 - z^2 = -1/4\},$$

where $\mathbb{R}^3_1$ is the Minkowski 3-space. Then the Hopf map $\psi: \text{SL}(2, \mathbb{R}) \to H^2(-4)$ given by

$$\psi(z, w) = \frac{1}{2} (2z\bar{w}, |z|^2 + |w|^2)$$

is a submersion, with circular fibers, and if we put

$$X_1(z, w) = (iz, iw), \quad X_2(z, w) = (i\bar{w}, i\bar{z}), \quad X_3(z, w) = (\bar{w}, \bar{z}),$$

we have that $X_1$ is a vertical vector field, while $X_2, X_3$ are horizontal. The vector $X_1$ is called the Hopf vector field.

We shall endow $\text{SL}(2, \mathbb{R})$ with the 1-parameter family of metrics $g_\tau, \tau > 0$, given by

$$g_\tau(X_i, X_j) = \delta_{ij}, \quad g_\tau(X_1, X_1) = \tau^2, \quad g_\tau(X_1, X_j) = 0, \quad i, j \in \{2, 3\},$$

which renders the Hopf map $\psi: (\text{SL}(2, \mathbb{R}), g_\tau) \to H^2(-4)$ a Riemannian submersion. With respect to the inner product in $\mathbb{R}^4_2$ the metric $g_\tau$ is given by

$$g_\tau(X, Y) = -\langle X, Y \rangle + (1 + \tau^2)\langle X, X_1 \rangle\langle Y, X_1 \rangle. \quad (4)$$

From now on, we denote $(\text{SL}(2, \mathbb{R}), g_\tau)$ with $\text{SL}(2, \mathbb{R})_{\tau}$. Obviously

$$E_1 = -\tau^{-1} X_1, \quad E_2 = X_2, \quad E_3 = X_3, \quad (5)$$

is an orthonormal basis on $\text{SL}(2, \mathbb{R})_{\tau}$. The Levi-Civita connection $\nabla^\tau$ of $\text{SL}(2, \mathbb{R})_{\tau}$ is given by:

$$\nabla^\tau_{E_1} E_1 = 0, \quad \nabla^\tau_{E_2} E_2 = 0, \quad \nabla^\tau_{E_3} E_3 = 0,$$

$$\nabla^\tau_{E_1} E_2 = -\tau^{-1} (2 + \tau^2) E_3, \quad \nabla^\tau_{E_1} E_3 = \tau^{-1} (2 + \tau^2) E_2,$$

$$\nabla^\tau_{E_2} E_1 = -\tau E_3, \quad \nabla^\tau_{E_2} E_3 = \tau E_2, \quad \nabla^\tau_{E_3} E_2 = -\tau E_1 = -\nabla^\tau_{E_2} E_3. \quad (6)$$

Using the conventions

$$R(X, Y)Z = \nabla^\tau_X \nabla^\tau_Y Z - \nabla^\tau_Y \nabla^\tau_X Z - \nabla^\tau_{[X,Y]} Z$$

and

$$R(X, Y, W, Z) = g_\tau(R(X, Y)Z, W),$$
the nonzero components of the Riemannian curvature are
\[ R_{1212} = \tau^2, \quad R_{1313} = \tau^2, \quad R_{2323} = -(4 + 3\tau^2), \] (7)
where \( R_{ijkl} = R(E_i, E_j, E_k, E_l) \).

Finally, we recall that the isometry group of \( SL(2, \mathbb{R})_{\tau} \) is the 4-dimensional indefinite unitary group \( U_1(2) \) that can be identified with:
\[
U_1(2) = \{ A \in O_2(4) : AJ_1 = \pm J_1 A \},
\]
where \( J_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \), \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and \( O_2(4) = \{ A \in GL(4, \mathbb{R}) : A^t = \epsilon A - \epsilon \} \), \( \epsilon = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \), \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), is the indefinite orthogonal group.

3. Biharmonic Curves in \( SL(2, \mathbb{R})_{\tau} \)

Let \( \gamma : I \to SL(2, \mathbb{R})_{\tau} \) be a differentiable curve parametrized by arc length and let \( \{ T, N, B \} \) be the orthonormal frame field tangent to \( SL(2, \mathbb{R})_{\tau} \) along \( \gamma \) defined as follows: we denote by \( T \) the unit vector field tangent to \( \gamma \), by \( N \) the unit vector field in the direction of \( \nabla_{\gamma} T \) normal to \( \gamma \), and we choose \( B \) so that \( \{ T, N, B \} \) is a positive oriented orthonormal basis. Then we have the following Frenet equations:
\[
\begin{align*}
\nabla_{\gamma} T &= k_1 N, \\
\nabla_{\gamma} T N &= -k_1 T + k_2 B, \\
\nabla_{\gamma} T B &= -k_2 N,
\end{align*}
\] (8)
where \( k_1 = |\nabla_{\gamma} T| \) is the geodesic curvature of \( \gamma \) and \( k_2 \) its geodesic torsion.

**Theorem 3.1.** Let \( \gamma : I \to SL(2, \mathbb{R})_{\tau} \) be a differentiable curve parametrized by arc length. Then \( \gamma \) is proper biharmonic if and only if
\[
\begin{align*}
k_1 &= \text{constant} \neq 0, \\
k_1^2 + k_2^2 &= \tau^2 - 4(1 + \tau^2) B_1^2, \\
k_2' &= -4(1 + \tau^2) N_1 B_1.
\end{align*}
\] (9)

**Proof.** Consider a curve \( \gamma : I \to SL(2, \mathbb{R})_{\tau} \) parametrized by arc length. In this case the Eq. (2) becomes
\[
(\nabla_{\gamma} T)^3 T - R(T, \nabla_{\gamma} T T) T = 0.
\] (10)
Using the Frenet equations into (10), we obtain the conditions
\[
\begin{align*}
k_1 &= \text{constant} \neq 0, \\
k_1^2 + k_2^2 &= R(T, N, T, N), \\
k_2' &= -R(T, N, T, B).
\end{align*}
\] (11)
Writing
\[
T = \sum_{i=1}^{3} T_i E_i, \quad N = \sum_{i=1}^{3} N_i E_i, \quad B = \sum_{i=1}^{3} B_i E_i, \tag{12}
\]
and using (7), we have that
\[
R(T, N, T, N) = \tau^2 - 4(1 + \tau^2)B_1^2,
\]
\[
R(T, N, T, B) = 4(1 + \tau^2)N_1 B_1.
\]

Proposition 3.2. If \( \gamma : I \to \text{SL}(2, \mathbb{R})_\tau \) is a proper biharmonic curve parameterized by arc length, then its geodesic curvature and torsion are constant.

Proof. From the Frenet equations it results that
\[
g_\tau(\nabla^r T, E_1) = -g_\tau(k_2 N, E_1) = -k_2 N_1.
\]
On the other hand, using (6), we get
\[
g_\tau(\nabla^r T, E_1) = g_\tau(B'_1 E_1 + T_2 B_3 \nabla^r E_2 E_3 + T_3 B_2 \nabla^r E_3 E_2, E_1) \\
= B'_1 + \tau(T_2 B_3 - T_3 B_2) \\
= B'_1 - \tau N_1.
\]
Combining these two equations, we have
\[
B'_1 = (\tau - k_2) N_1. \tag{13}
\]
Now, using (9) we obtain
\[
k_2 k'_2 = -4(1 + \tau^2) B_1 B'_1. \tag{14}
\]
From (13) and (14) it results that \((\tau - 2k_2)B_1 N_1 = 0\). Therefore, we have two possibilities: \(B_1 N_1 = 0\) that, together with (9), implies \(k'_2 = 0\); or \(k_2 = \frac{\tau}{2}\). In both cases \(k_2\) is constant. \qed

Proposition 3.3. If \( \gamma : I \to \text{SL}(2, \mathbb{R})_\tau \) is a proper biharmonic curve parameterized by arc length, then it makes a constant angle with the Hopf vector field \( E_1 \) and its tangent vector field can be written as:
\[
\gamma'(s) = T(s) = \cos \vartheta E_1 + \sin \vartheta \sin \beta(s) E_2 + \sin \vartheta \cos \beta(s) E_3, \tag{15}
\]
where \( \vartheta \in (0, \pi/2] \) and \( \beta : I \to \mathbb{R} \) is a smooth function.

Proof. First we note that \( B_1 \neq 0 \). Indeed if \( B_1 = 0 \) and \( N_1 = 0 \), then the curve is the integral curve of the vector field \( E_1 \) and it is a geodesic. Moreover, if \( B_1 = 0 \) and \( N_1 \neq 0 \), from (13) we get \( k_2 = \tau \) that, together with the second equation of (9), gives \( k_1 = 0 \).

Since \( B_1 \neq 0 \), the third equation of (9) and the Proposition 3.2 imply \( N_1 = 0 \). Now, using the Eqs. (6) and (8) we get
\[
k_1 N_1 = g_\tau(\nabla^r T, E_1) = T'_1.
\]
We conclude that \( T_1 = \text{constant} \) and we obtain the expression (15). \qed

Using the previous result we have the following
Theorem 3.4. Let $\gamma : I \to \text{SL}(2, \mathbb{R})_\tau \subset \mathbb{R}^4_2$ be a curve parametrized by arc length. Then $\gamma$ is proper biharmonic if and only if, as a curve in $\mathbb{R}^4_2$, it satisfies

$$\gamma^{IV} + (b^2 - 2a)\gamma'' + a^2 \gamma = 0,$$

(16)

where $a$ and $b$ are the constants given by:

$$\begin{cases}
a = \frac{1}{2}(-\tau^{-2} + 1 - (1 + \tau^{-2}) \cos 2\vartheta) - \tau^{-1} \cos \vartheta \beta', \\
b = \beta' = -\tau^{-1}(2 + \tau^2) \cos \vartheta \pm \sqrt{(4 + 5\tau^2) \cos^2 \vartheta - 4(1 + \tau^2)},
\end{cases}$$

(17)

with

$$\frac{4(1 + \tau^2)}{(4 + 5\tau^2)} \leq \cos^2 \vartheta < 1.$$

Proof. Writing

$$\gamma(s) = (x_1(s), x_2(s), x_3(s), x_4(s)),$$

from (15) we have that the coordinates functions of $\gamma$ in $\mathbb{R}^4_2$ satisfy

$$\begin{cases}
x_1' = \tau^{-1} \cos \vartheta x_2 + \sin \vartheta \cos \beta x_3 + \sin \vartheta \sin \beta x_4, \\
x_2' = -\tau^{-1} \cos \vartheta x_1 + \sin \vartheta \sin \beta x_3 - \sin \vartheta \cos \beta x_4, \\
x_3' = \sin \vartheta \cos \beta x_1 + \sin \vartheta \sin \beta x_2 + \tau^{-1} \cos \vartheta x_4, \\
x_4' = \sin \vartheta \sin \beta x_1 - \sin \vartheta \cos \beta x_2 - \tau^{-1} \cos \vartheta x_3,
\end{cases}$$

(18)

Deriving (18), it results that

$$\begin{cases}
x_1'' = ax_1 - bx_2', \\
x_2'' = ax_2 + bx_1', \\
x_3'' = ax_3 - bx_4', \\
x_4'' = ax_4 + bx_3',
\end{cases}$$

(19)

where

$$\begin{cases}
a = \frac{1}{2}(-\tau^{-2} + 1 - (1 + \tau^{-2}) \cos 2\vartheta) - \tau^{-1} \cos \vartheta \beta', \\
b = \beta' = -\tau^{-1}(2 + \tau^2) \cos \vartheta \pm \sqrt{(4 + 5\tau^2) \cos^2 \vartheta - 4(1 + \tau^2)}.
\end{cases}$$

Now, we shall prove that $b$ is constant and we determine its expression. Computing $\nabla^\tau T T$, using (15) and (6), the geodesic curvature and the normal vector field are given by

$$k_1 = \pm \sin \vartheta(\beta' + 2\tau^{-1}(1 + \tau^2) \cos \vartheta), \quad N = \pm(\cos \beta E_2 - \sin \beta E_3).$$

(20)

Then

$$B = T \wedge N = \pm(- \sin \vartheta E_1 + \cos \vartheta \sin \beta E_2 + \cos \vartheta \cos \beta E_3),$$

$$k_2 = g_\tau(\nabla^\tau N, B) = \tau - \cos \vartheta(\beta' + 2\tau^{-1}(1 + \tau^2) \cos \vartheta).$$

(21)

Substituting the expressions of $k_1$, $k_2$ and $B_1$ in the second equation of (9), it results that

$$\beta' = -\tau^{-1}(2 + \tau^2) \cos \vartheta \pm \sqrt{(4 + 5\tau^2) \cos^2 \vartheta - 4(1 + \tau^2)}.$$
Now deriving twice (19), and using (18), we obtain the Eq. (16). Also, as the curve $\gamma$ is not harmonic, from (20), $\cos \vartheta \neq 1$. □

**Remark 3.5.** Using (18) and (19), we find that:

$$
\langle \gamma, \gamma \rangle = 1, \quad \langle \gamma', \gamma' \rangle = \tilde{B}, \quad \langle \gamma, \gamma' \rangle = 0,
$$

$$
\langle \gamma', \gamma'' \rangle = 0, \quad \langle \gamma'', \gamma'' \rangle = D, \quad \langle \gamma, \gamma'' \rangle = -\tilde{B},
$$

$$
\langle \gamma', \gamma''' \rangle = -D, \quad \langle \gamma'', \gamma''' \rangle = 0, \quad \langle \gamma, \gamma''' \rangle = 0,
$$

$$
\langle \gamma''', \gamma''''' \rangle = E,
$$

where

$$
\tilde{B} = (1 + \tau^{-2}) \cos^2 \vartheta - 1, \quad D = a^2 + b^2 \tilde{B} + 2ab \tau^{-1} \cos \vartheta,
$$

$$
E = a(a - 2b^2)\tilde{B} + b^2D - 2a^2b \tau^{-1} \cos \vartheta.
$$

In addition, as

$$
J_1\gamma = X_1|_{\gamma} = -\tau E_1|_{\gamma},
$$

using (15) and (19), we obtain the following identities

$$
\langle J_1\gamma, \gamma' \rangle = -\tau^{-1} \cos \vartheta,
$$

$$
\langle J_1\gamma, \gamma'' \rangle = 0, \quad \langle J_1\gamma', \gamma'' \rangle = -a \tau^{-1} \cos \vartheta - b \tilde{B} := \tilde{I},
$$

$$
\langle J_1\gamma', \gamma''' \rangle = 0, \quad \langle J_1\gamma', \gamma''''' \rangle = 0,
$$

$$
\langle J_1\gamma'', \gamma''' \rangle + \langle J_1\gamma, \gamma''''' \rangle = 0.
$$

To determine the expression of the position vector of $\gamma$ in $\mathbb{R}^4$, we integrate (16), dividing the study in three cases, according to the three possibilities:

(i) $b^2 = 4a$;
(ii) $b^2 > 4a$;
(iii) $b^2 < 4a$.

### 4. The Case $b^2 = 4a$

**Theorem 4.1.** Let $\gamma : I \rightarrow \text{SL}(2, \mathbb{R})_\tau \subset \mathbb{R}^4_2$ be a proper biharmonic curve parametrized by arc length such that $b^2 = 4a$. Then

$$
b = -\tau^{-1}(2 + \tau^2) \cos \vartheta + \sqrt{(4 + 5\tau^2) \cos^2 \vartheta - 4(1 + \tau^2)},
$$

with

$$
\cos^2 \vartheta = \frac{(2 + \tau^2)^2}{4 + 5\tau^2 + \tau^4}.
$$

Also,

$$
\gamma(s) = A \left( \cos(\sqrt{a} s) + g_{14} s \sin(\sqrt{a} s), -\sin(\sqrt{a} s) + g_{14} s \cos(\sqrt{a} s),
$$

$$
-g_{14} s \cos(\sqrt{a} s), g_{14} s \sin(\sqrt{a} s) \right),
$$

(25)
where \( g_{14} \) is the constant given by

\[
g_{14} = \frac{\tau}{\sqrt{4 + 5\tau^2 + \tau^4}}
\]

and \( A \in O_2(4) \) is a \( 4 \times 4 \) indefinite orthogonal matrix which commutes with \( J_1 \).

**Proof.** As \( b^2 = 4a \), the differential equation (16) becomes

\[
\gamma''''(s) + 2a \gamma''(s) + a^2 \gamma(s) = 0.
\]

(26)

Integrating (26) we have

\[
\gamma(s) = \cos(\sqrt{a} s) g_1 + \sin(\sqrt{a} s) g_2 + s \cos(\sqrt{a} s) g_3 + s \sin(\sqrt{a} s) g_4,
\]

(27)

where \( g_1, g_2, g_3 \) and \( g_4 \) are constant vectors of \( \mathbb{R}^4 \).

A direct calculation shows that \( b^2 = 4a \) occurs in two cases: for \( \vartheta = 0 \) and for

\[
\cos^2 \vartheta = \frac{(2 + \tau^2)^2}{4 + 5\tau^2 + \tau^4},
\]

and in both cases \( b \) must have the expression given in (24). Since the first case produces harmonic curves, we study only the second one.

Using the relations (22) we get

\[
\langle g_1, g_1 \rangle = \langle g_2, g_2 \rangle = 1,
\]

\[
\langle g_3, g_3 \rangle = \langle g_4, g_4 \rangle = 0,
\]

\[
\langle g_1, g_4 \rangle = -\langle g_2, g_3 \rangle = \frac{\tau}{\sqrt{4 + 5\tau^2 + \tau^4}},
\]

\[
\langle g_1, g_2 \rangle = \langle g_1, g_3 \rangle = \langle g_2, g_4 \rangle = \langle g_3, g_4 \rangle = 0,
\]

(28)

while from (23) we obtain

\[
\langle J_1 g_1, g_2 \rangle = -1,
\]

\[
\langle J_1 g_2, g_4 \rangle = \langle J_1 g_1, g_3 \rangle = \frac{\tau}{\sqrt{4 + 5\tau^2 + \tau^4}},
\]

\[
\langle J_1 g_1, g_4 \rangle = \langle J_1 g_2, g_3 \rangle = \langle J_1 g_3, g_4 \rangle = 0.
\]

(29)

Now, putting

\[
\begin{align*}
e_1 &= g_1, \\
e_2 &= g_2, \\
e_3 &= g_3 - \frac{g_3}{\langle g_2, g_3 \rangle} - g_2, \\
e_4 &= g_4 - \frac{g_4}{\langle g_1, g_4 \rangle} - g_1,
\end{align*}
\]

we have that \( \{e_i\}_{i=1}^4 \) is an orthonormal basis of \( \mathbb{R}^4 \) that satisfies:

\[
\langle J_1 e_1, e_2 \rangle = \langle J_1 e_3, e_4 \rangle = -1,
\]

\[
\langle J_1 e_1, e_3 \rangle = \langle J_1 e_1, e_4 \rangle = \langle J_1 e_2, e_3 \rangle = \langle J_1 e_2, e_4 \rangle = 0.
\]

We conclude that \( e_2 = -J_1 e_1 \) and \( e_4 = J_1 e_3 \). So if we consider the orthonormal basis \( \tilde{E}_i \) of \( \mathbb{R}^4 \) given by

\[
\tilde{E}_1 = (1, 0, 0, 0), \quad \tilde{E}_2 = (0, -1, 0, 0), \quad \tilde{E}_3 = (0, 0, 1, 0), \quad \tilde{E}_4 = (0, 0, 0, 1),
\]
there must exist a matrix \( A \in O_2(4) \), with \( J_1 A = A J_1 \), such that \( e_i = A \tilde{E}_i, i \in \{1,2,3,4\} \). Finally, putting \( \langle g_1,g_4 \rangle = g_{14} \), we can rewrite (27) as (25). 

\[ \Box \]

5. The Case \( b^2 > 4a \)

**Theorem 5.1.** Let \( \gamma : I \rightarrow \text{SL}(2,\mathbb{R}) \subset \mathbb{R}^4_2 \) be a proper biharmonic curve parametrized by arc length, such that \( b^2 > 4a \). Then there are two possibilities:

(i) 
\[
b = -\tau^{-1}(2 + \tau^2) \cos \vartheta + \sqrt{(4 + 5\tau^2)} \cos^2 \vartheta - 4(1 + \tau^2)
\]
and
\[
\frac{4(1 + \tau^2)}{(4 + 5\tau^2)} \leq \cos^2 \vartheta < \frac{(2 + \tau^2)^2}{4 + 5\tau^2 + \tau^4}.
\]

(ii) 
\[
b = -\tau^{-1}(2 + \tau^2) \cos \vartheta - \sqrt{(4 + 5\tau^2)} \cos^2 \vartheta - 4(1 + \tau^2)
\]
and
\[
\frac{4(1 + \tau^2)}{(4 + 5\tau^2)} \leq \cos^2 \vartheta.
\]

In both cases, the expression of \( \gamma \) as a curve in \( \mathbb{R}^4_2 \) is
\[
\gamma(s) = A\left( \sqrt{C_{33}} \cos(\alpha_2 s), \sqrt{C_{33}} \sin(\alpha_2 s), \sqrt{-C_{11}} \cos(\alpha_1 s), \sqrt{-C_{11}} \sin(\alpha_1 s) \right),
\]
where
\[
\alpha_{1,2} = \sqrt{\frac{(b^2 - 2a) \pm \sqrt{b^2(b^2 - 4a)}}{2}}
\]
and
\[
C_{11} = \frac{\tilde{B} - \alpha_2^2}{\alpha_1^2 - \alpha_2^2}, \quad C_{33} = \frac{-\tilde{B} + \alpha_1^2}{\alpha_1^2 - \alpha_2^2}
\]
are real constants and \( A \in O_2(4) \) is a \( 4 \times 4 \) indefinite orthogonal matrix anticommuting with \( J_1 \).

**Proof.** First, observe that the condition \( b^2 > 4a \) gives the two possibilities (i) and (ii). Also, a direct integration of (16) gives the solution
\[
\gamma(s) = \cos(\alpha_1 s) C_1 + \sin(\alpha_1 s) C_2 + \cos(\alpha_2 s) C_3 + \sin(\alpha_2 s) C_4,
\]
where
\[
\alpha_{1,2} = \sqrt{\frac{(b^2 - 2a) \pm \sqrt{b^2(b^2 - 4a)}}{2}}
\]
are real constants, while the \( C_i, i \in \{1,2,3,4\} \), are constants vectors of \( \mathbb{R}_2^4 \).
Putting $C_{ij} = \langle C_i, C_j \rangle$, and evaluating the relations (22) in $s = 0$, we obtain:

$$C_{11} + C_{33} + 2C_{13} = 1, \quad (31)$$

$$\alpha_1^2 C_{22} + \alpha_2^2 C_{44} + 2\alpha_1\alpha_2 C_{24} = \tilde{B}, \quad (32)$$

$$\alpha_1 C_{12} + \alpha_2 C_{14} + \alpha_1 C_{23} + \alpha_2 C_{34} = 0, \quad (33)$$

$$\alpha_1^2 C_{12} + \alpha_1\alpha_2 C_{23} + \alpha_2^2 C_{14} + \alpha_3 C_{34} = 0, \quad (34)$$

$$\alpha_1^4 C_{11} + \alpha_2^4 C_{33} + 2\alpha_1^2\alpha_2^2 C_{13} = D, \quad (35)$$

$$\alpha_1^2 C_{11} + \alpha_2^2 C_{33} + (\alpha_1^2 + \alpha_2^2) C_{13} = \tilde{B}, \quad (36)$$

$$\alpha_1^4 C_{22} + (\alpha_1^3\alpha_2 + \alpha_1\alpha_2^3) C_{24} + \alpha_2^4 C_{44} = D, \quad (37)$$

$$\alpha_1^5 C_{12} + \alpha_1^3\alpha_2^2 C_{23} + \alpha_1^2\alpha_2^3 C_{14} + \alpha_2^5 C_{34} = 0, \quad (38)$$

$$\alpha_1^3 C_{12} + \alpha_1^3 C_{23} + \alpha_1^3 C_{14} + \alpha_2^3 C_{34} = 0, \quad (39)$$

$$\alpha_1^6 C_{22} + \alpha_2^6 C_{44} + 2\alpha_1^3\alpha_2^3 C_{24} = E. \quad (40)$$

From (33), (34), (38), (39), it follows that  
$$C_{12} = C_{14} = C_{23} = C_{34} = 0.$$  
Also, from (31), (35) and (36), we obtain  
$$C_{11} = \frac{\tilde{B} - \alpha_2^2}{\alpha_1^2 - \alpha_2^2}, \quad C_{13} = 0, \quad C_{33} = \frac{-\tilde{B} + \alpha_1^2}{\alpha_1^2 - \alpha_2^2}.$$  
Finally, using (32), (37) and (40), we get  
$$C_{22} = \frac{D - \tilde{B}\alpha_2^2}{\alpha_1^2(\alpha_1^2 - \alpha_2^2)}, \quad C_{24} = 0, \quad C_{44} = \frac{-D + \tilde{B}\alpha_1^2}{\alpha_2^2(\alpha_1^2 - \alpha_2^2)}.$$  
We observe that as  
$$\frac{4(1 + \tau^2)}{(4 + 5\tau^2)} \leq \cos^2 \vartheta,$$  
then  
$$C_{11} = C_{22} < 0, \quad C_{33} = C_{44} > 0.$$  
Since $\{C_i\}_{i=1}^4$ are mutually orthogonal and  
$$||C_1|| = ||C_2|| = \sqrt{-C_{11}}, \quad ||C_3|| = ||C_4|| = \sqrt{C_{33}},$$  
we obtain a pseudo-orthonormal basis of $\mathbb{R}_2^4$ putting $e_i = C_i/||C_i||$, $i \in \{1, 2, 3, 4\}$, and we can write:

$$\gamma(s) = \sqrt{-C_{11}} (\cos(\alpha_1 s) e_1 + \sin(\alpha_1 s) e_2) + \sqrt{C_{33}} (\cos(\alpha_2 s) e_3 + \sin(\alpha_2 s) e_4). \quad (41)$$
Now, evaluating in $s = 0$ the identities (23), we have:

\[
\begin{align*}
\alpha_2 C_{33} \langle J_1 e_3, e_4 \rangle - \alpha_1 C_{11} \langle J_1 e_1, e_2 \rangle \\
+ \sqrt{-C_{11} C_{33}} (\alpha_1 \langle J_1 e_3, e_2 \rangle + \alpha_2 \langle J_1 e_1, e_4 \rangle) &= -\tau^{-1} \cos \vartheta, \quad (42) \\
\langle J_1 e_1, e_3 \rangle &= 0, \\
\alpha_2^2 C_{33} \langle J_1 e_3, e_4 \rangle - \alpha_1^2 C_{11} \langle J_1 e_1, e_2 \rangle \\
+ \sqrt{-C_{11} C_{33}} (\alpha_1 \alpha_2 \langle J_1 e_3, e_2 \rangle + \alpha_1^2 \alpha_2 \langle J_1 e_1, e_4 \rangle) &= -I, \quad (43) \\
\langle J_1 e_2, e_4 \rangle &= 0, \\
\alpha_1 \langle J_1 e_2, e_3 \rangle + \alpha_2 \langle J_1 e_1, e_4 \rangle &= 0, \quad (44) \\
\alpha_2 \langle J_1 e_2, e_3 \rangle + \alpha_1 \langle J_1 e_1, e_4 \rangle &= 0. \quad (45)
\end{align*}
\]

We point out that to obtain the previous identities we have divided by

\[
\alpha_1^2 - \alpha_2^2 = \sqrt{b^2(b^2 - 4a)},
\]

which is always different from zero. From (44) and (45), taking into account

that $\alpha_1^2 - \alpha_2^2 \neq 0$, it results that

\[
\langle J_1 e_3, e_2 \rangle = 0, \quad \langle J_1 e_1, e_4 \rangle = 0. \quad (46)
\]

Then, $J_1 e_1 = \pm e_2$ and $J_1 e_3 = \pm e_4$. So, the position vector of $\gamma$ is given by:

\[
\gamma(s) = \sqrt{-C_{11}} (\cos(\alpha_1 s) e_1 \pm \sin(\alpha_1 s) J_1 e_1) \\
+ \sqrt{C_{33}} (\cos(\alpha_2 s) e_3 \pm \sin(\alpha_2 s) J_1 e_3). \quad (47)
\]

Evaluating (19) in $s = 0$, we get $J_1 e_1 = -e_2$ and $J_1 e_3 = -e_4$. Therefore, if

we fix the orthonormal basis of $\mathbb{R}^4$ given by:

\[
E_1 = (0, 0, 1, 0), \quad E_2 = (0, 0, 0, 1), \quad E_3 = (1, 0, 0, 0), \quad E_4 = (0, 1, 0, 0),
\]

there must exists a matrix $A \in O_2(4)$, with $J_1 A = -A J_1$, such that $e_i = A E_i$, $i \in \{1, 2, 3, 4\}$. Replacing $e_i = A E_i$ in (41) we obtain (30). $\square$

6. The Case $b^2 < 4a$

**Theorem 6.1.** Let $\gamma : I \to \text{SL}(2, \mathbb{R}) \subset \mathbb{R}^4$ be a proper biharmonic curve parametrized by arc length, such that $b^2 < 4a$. Then

\[
b = -\tau^{-1}(2 + \tau^2) \cos \vartheta + \sqrt{(4 + 5\tau^2) \cos^2 \vartheta - 4(1 + \tau^2)}, \quad (48)
\]

\[
\frac{(2 + \tau^2)^2}{4 + 5\tau^2 + \tau^4} < \cos^2 \vartheta < 1, \quad (49)
\]
and the expression of \( \gamma \) as a curve in \( \mathbb{R}^4_2 \) is

\[
\gamma(s) = A \left( \cos \left( \frac{b}{2} s \right) \cosh(\mu s) + w_{14} \sin \left( \frac{b}{2} s \right) \sinh(\mu s) \right),
\sin \left( \frac{b}{2} s \right) \cosh(\mu s) - w_{14} \cos \left( \frac{b}{2} s \right) \sinh(\mu s),
\cos \left( \frac{b}{2} s \right) \sinh(\mu s) \sqrt{1 + w_{14}^2},
\sin \left( \frac{b}{2} s \right) \sinh(\mu s) \sqrt{1 + w_{14}^2},
\right),
\]

where

\[
\mu = \frac{\sqrt{4a - b^2}}{2}, \quad w_{14} = \frac{b \tau + 2 \cos \vartheta}{2 \tau \mu}
\]

are real constants and \( A \in O_2(4) \) is a \( 4 \times 4 \) indefinite orthogonal matrix commuting with \( J_1 \).

**Proof.** From \( b^2 < 4a \), it results that \( b \) is given by \( (48) \) and \( \vartheta \) satisfies \( (49) \). Also, a direct integration of \( (16) \) gives

\[
\gamma(s) = \cos \left( \frac{b}{2} s \right) (\cosh(\mu s) w_1 + \sinh(\mu s) w_3) + \sin \left( \frac{b}{2} s \right) (\cosh(\mu s) w_2 + \sinh(\mu s) w_4),
\]

where

\[
\mu = \frac{\sqrt{4a - b^2}}{2},
\]

while the \( w_i, i \in \{1, 2, 3, 4\} \), are constant vectors in \( \mathbb{R}^4_2 \). If \( w_{ij} := \langle w_i, w_j \rangle \), evaluating the relations \( (22) \) in \( s = 0 \), we obtain

\[
w_{11} = 1, \quad \frac{b^2}{4} w_{22} + \mu^2 w_{33} + \mu b w_{23} = \tilde{B},
\]

\[
\frac{b}{2} w_{12} + \mu w_{13} = 0,
\]

\[
\frac{b}{2} \left( \mu^2 - \frac{b^2}{4} \right) w_{12} + \mu^2 b w_{34} + \mu \frac{b^2}{2} w_{24} + \mu \left( \mu^2 - \frac{b^2}{4} \right) w_{13} = 0,
\]

\[
\left( \mu^2 - \frac{b^2}{4} \right)^2 w_{11} + \mu^2 b^2 w_{44} + 2 \mu b \left( \mu^2 - \frac{b^2}{4} \right) w_{14} = D,
\]

\[
\left( \mu^2 - \frac{b^2}{4} \right) w_{11} + \mu b w_{14} = -\tilde{B},
\]

\[
\frac{b^2}{4} \left( 3\mu^2 - \frac{b^2}{4} \right) w_{22} + \mu^2 \left( \mu^2 - \frac{b^2}{4} \right) w_{33} + \mu \frac{b}{2} (4\mu^2 - b^2) w_{23} = -D,
\]
\[
\begin{align*}
\frac{b}{2} \left( 3\mu^2 - \frac{b^2}{4} \right) & \left( \mu^2 - \frac{b^2}{4} \right) w_{12} + b \mu^2 \left( \mu^2 - 3\frac{b^2}{4} \right) w_{34} \\
+ \mu \left( \mu^2 - 3\frac{b^2}{4} \right) & \left( \mu^2 - \frac{b^2}{4} \right) w_{13} + \mu \frac{b^2}{2} \left( 3\mu^2 - \frac{b^2}{4} \right) w_{24} = 0, \\
(59) \\
\frac{b}{2} \left( 3\mu^2 - \frac{b^2}{4} \right) & \left( \mu^2 - \frac{b^2}{4} \right) w_{12} + \mu \left( \mu^2 - 3\frac{b^2}{4} \right) w_{13} = 0, \\
(60) \\
\frac{b^2}{4} \left( 3\mu^2 - \frac{b^2}{4} \right)^2 & w_{22} + \mu^2 \left( \mu^2 - 3\frac{b^2}{4} \right)^2 w_{33} \\
+ \mu b \left( 3\mu^2 - \frac{b^2}{4} \right) & \left( \mu^2 - 3\frac{b^2}{4} \right) w_{23} = E. \\
(61)
\end{align*}
\]

From (52), (56) and (57), it follows that
\[
\begin{align*}
w_{11} &= -w_{44} = 1, \\
w_{14} &= \frac{b\tau + 2\cos \vartheta}{2\tau \mu}.
\end{align*}
\]
Also, from (54) and (60), we obtain
\[
w_{12} = w_{13} = 0
\]
and, therefore, from (55) and (59),
\[
w_{24} = w_{34} = 0.
\]
Moreover, using (53), (58) and (61), we get
\[
\begin{align*}
w_{22} &= -w_{33} = 1, \\
w_{23} &= -\frac{b\tau + 2\cos \vartheta}{2\tau \mu}.
\end{align*}
\]
Then, we can define the following pseudo-orthonormal basis of \(\mathbb{R}^4_2\):
\[
\begin{align*}
e_1 &= w_1, \\
e_2 &= w_2, \\
e_3 &= \frac{w_3 + w_{14} w_2}{\sqrt{1 + w_{14}^2}}, \\
e_4 &= \frac{w_4 - w_{14} w_1}{\sqrt{1 + w_{14}^2}},
\end{align*}
\]
with \(\langle e_1, e_1 \rangle = 1 = \langle e_2, e_2 \rangle \) and \(\langle e_3, e_3 \rangle = -1 = \langle e_4, e_4 \rangle\).

Evaluating the identities (23) in \(s = 0\), and taking into account that
\[
\begin{align*}
\gamma(0) &= w_1, \\
\gamma'(0) &= \frac{b}{2} w_2 + \mu w_3, \\
\gamma''(0) &= \left( \mu^2 - \frac{b^2}{4} \right) w_1 + \mu b w_4, \\
\gamma'''(0) &= \frac{b}{2} \left( 3\mu^2 - \frac{b^2}{4} \right) w_2 + \mu \left( \mu^2 - 3\frac{b^2}{4} \right) w_3, \\
\gamma^{IV}(0) &= \left( \mu^4 - \frac{3}{2} \mu^2 b^2 + \frac{b^4}{16} \right) w_1 + 2\mu b \left( \mu^2 - \frac{b^2}{4} \right) w_4,
\end{align*}
\]
we conclude that
\[ \langle J_1 w_1, w_2 \rangle = -\langle J_1 w_3, w_4 \rangle = 1, \]
\[ \langle J_1 w_3, w_2 \rangle = \langle J_1 w_1, w_4 \rangle = 0, \]
\[ \langle J_1 w_1, w_3 \rangle = \langle J_1 w_2, w_4 \rangle = -w_{14}. \]

Then,
\[ \langle J_1 e_1, e_2 \rangle = -\langle J_1 e_3, e_4 \rangle = 1, \]
\[ \langle J_1 e_1, e_4 \rangle = \langle J_1 e_1, e_3 \rangle = \langle J_1 e_2, e_3 \rangle = \langle J_1 e_2, e_4 \rangle = 0. \]

Therefore, we obtain that
\[ J_1 e_1 = e_2, \quad J_1 e_3 = e_4. \]

Consequently, if we consider the orthonormal basis \( \{ E_i \}_{i=1}^4 \) of \( \mathbb{R}^4 \) given by
\[ E_1 = (1, 0, 0, 0), \quad E_2 = (0, 1, 0, 0), \quad E_3 = (0, 0, 1, 0), \quad E_4 = (0, 0, 0, 1), \]
there must exist \( A \in O_2(4) \), with \( J_1 A = A J_1 \), such that \( e_i = A E_i, \ i \in \{1, 2, 3, 4\} \). Therefore, using (51) and (62) we obtain (50).

\[ \square \]

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