HOMOLOGICAL DIMENSIONS OF RIGID MODULES

OLGUR CELIKBAS, MOHSEN GHEIBI,
MAJID RAHRO ZARGAR AND ARASH SADEGHI

Abstract. We obtain various characterizations of commutative Noetherian local rings \((R, m)\) in terms of homological dimensions of certain finitely generated modules. For example, we establish that \(R\) is Gorenstein if the Gorenstein injective dimension of the maximal ideal \(m\) of \(R\) is finite. Furthermore we prove that \(R\) must be regular if a single \(\text{Ext}^n_R(I, J)\) vanishes for some integrally closed \(m\)-primary ideals \(I\) and \(J\) of \(R\) and for some integer \(n \geq \dim(R)\). Along the way we observe that local rings that admit maximal Cohen-Macaulay Tor-rigid modules are Cohen-Macaulay.

Contents

1. Introduction 4
2. Definitions 6
3. Projective and injective dimensions via rigid modules 10
4. Auslander’s transpose and remarks on Tor-rigidity 13
5. Main theorem 16
6. Corollaries of the main theorem 18
7. Gorenstein injective dimension of strongly-rigid modules 21
Appendix A. Some examples of test and rigid-test modules 22
Appendix B. Remarks on two-dimensional rational singularities 23
Acknowledgments 23
References 23

1. INTRODUCTION

Throughout \(R\) is a commutative Noetherian local ring with unique maximal ideal \(m\) and residue field \(k\), and all modules over \(R\) are assumed to be finitely generated.

It is well-known that the projective dimension of an \(R\)-module \(M\) is determined by the vanishing of \(\text{Ext}^n_R(M, k)\), i.e., if \(\text{Ext}^n_R(M, k) = 0\) for some positive integer \(n\), then \(\text{pd}(M) \leq n - 1\). In fact \(\text{pd}(M) = \sup \{i \in \mathbb{Z} : \text{Ext}^i_R(M, k) \neq 0\}\). Furthermore it follows from classical theorems of Auslander, Buchsbaum and Serre that finiteness of the projective or the injective dimension of the residue field \(k\) characterizes the ring itself: \(R\) is regular if \(\text{pd}(k) < \infty\) or \(\text{id}(k) < \infty\); see \([10]\) 2.2.7 and 3.1.26].
The main task in this paper is to introduce a class of modules, called rigid-test modules, that replace the residue field \( k \) in the aforementioned classical results; see [2, 3] for the definition. A special case of our main result, Theorem 5.8, can be summarized as follows; see also Corollaries 6.1 and 6.11.

**Theorem 1.1.** Let \((R, \mathfrak{m})\) be a local ring and let \(M\) and \(N\) be nonzero \( R \)-modules. Assume \(N\) is a rigid-test module (e.g., \( N = k \)).

(i) If \( \operatorname{Ext}^n_R(M, N) = 0 \) for some \( n \geq \operatorname{depth}(N) \), then \( \operatorname{pd}(M) \leq n - 1 \).

(ii) \( \operatorname{pd}(M) = \sup \{ i \in \mathbb{Z} : \operatorname{Ext}^i_R(M, N) \neq 0 \} \).

(iii) \( \operatorname{pd}(N) < \infty \) or \( \operatorname{id}(N) < \infty \), then \( R \) is regular.

To motivate our approach, let us remark Corso, Huneke, Katz and Vasconcelos [18, 3.3] established that integrally closed \( \mathfrak{m} \)-primary ideals are rigid-test modules; see (A.2). Thus, for such ideals \( I \) and \( J \), if \( \operatorname{Ext}^n_R(I, J) = 0 \) for some \( n \geq \dim(R) \), then it follows from Theorem 1.1 that \( \operatorname{pd}(I) < \infty \), and hence \( R \) is regular; see (2.3). Furthermore, if \( R \) is a two-dimensional complete normal local rational singularity with an algebraically closed residue field, by a result of Lipman [44, 7.1], \( I^i \) is integrally closed, and hence rigid-test, for all \( i \geq 1 \). Consequently Theorem 1.1 yields the following result; see also (B.3).

**Corollary 1.2.** Let \((R, \mathfrak{m})\) be a two-dimensional complete normal local domain with an algebraically closed residue field. Assume that \( R \) has a rational singularity. If \( \operatorname{Ext}^n_R(I^i, J^s) = 0 \) for some integrally closed \( \mathfrak{m} \)-primary ideals \( I \) and \( J \) of \( R \), and for some positive integers \( n, r, s \), then \( R \) is regular.

Examples of two-dimensional rational singularities include hypersurface rings (rational double points), such as \( R = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2) \).

Our argument has applications in several directions. We use Theorem 1.1 and deduce the following characterization of regularity from a beautiful result of Avramov, Hochster, Iyengar and Yao [9, 1.1]; see (6.7) and Corollary 6.11.

**Corollary 1.3.** Let \( R \) be a complete local ring of prime characteristic \( p \) with a perfect residue field, and let \( M \) and \( N \) be nonzero \( R \)-modules with \( \operatorname{Ext}^i_R(\varphi^e M, N) = 0 \) for some \( i \geq \operatorname{depth}(N) \) and \( n \geq 1 \). If \( N \) is a rigid-test module, then \( R \) is regular.

Here \( \varphi^e M \) is the \( R \)-module \( M \) with the \( R \)-action given by the \( n \)th iterate of the Frobenius endomorphism \( \varphi \); see (A.3). As an example, we note that nonzero modules of infinite projective dimension are rigid-test modules over \( R = \mathbb{F}_p[x, y, z]/(xy - z^2) \), with \( p \) an odd prime; thus Corollary 1.3 implies that \( \operatorname{Ext}^{p+1}_R(\varphi^e M, \varphi^n N) \neq 0 \) for all positive integers \( e, n, r \), and for all nonzero \( R \)-modules \( M, N \); see (A.4) and (A.6).

Theorem 1.1 determines the Gorensteinness of \( R \) via the Gorenstein injective dimension, a refinement of the usual injective dimension introduced by Enos and Jenda [29]. We prove in Corollary 7.10 that \( R \) is Gorenstein if the Gorenstein injective dimension \( \operatorname{Gid}(\mathfrak{m}) \) of the maximal ideal \( \mathfrak{m} \) of \( R \) is finite. This, combined with the results in the literature, seems to give a fairly complete picture: \( R \) is Gorenstein if and only if at least one of the dimensions \( \operatorname{Gid}(\mathfrak{m}), \operatorname{Gid}(R) \) or \( \operatorname{Gid}(k) \) is finite; see also (7.4) and Avramov’s remark following Question 1.1.

A rigid-test module is, by definition, Tor-rigid [1] and a test module (for projectivity) in the sense of [14]; see (2.3). Among those already discussed, there are quite a few motivations to study test and rigid-test modules: it was established in [13, 3.7] that if the dualizing module of a Cohen-Macaulay ring is a test module, then...
there are no nonfree totally reflexive modules. Proposition 1.12 extends 14 3.7 and establishes that there are no nonfree totally reflexive modules if there exists a nonzero test module over \(R\) – not necessarily maximal Cohen-Macaulay – of finite injective dimension. Another motivation for us to introduce rigid-test modules comes from the fact that the hypothesis – \(N\) is a test module – in Theorem 1.1 cannot be dropped in general. Tor-rigidity has remarkable consequences 11 21, but if \(N\) is a Tor-rigid module, which is not a test module, i.e., not a rigid-test module, then the vanishing of \(\text{Ext}^n_R(M,N)\), even for all \(n \gg 0\), does not necessarily force \(M\) to have finite projective dimension in general; see Theorem 1.1 and Example 6.3.

Cohen-Macaulay local rings admit maximal Cohen-Macaulay Tor-rigid modules, e.g., a high syzygy of the residue field is such an example. On the other hand, examples of non Cohen-Macaulay rings that admit maximal Cohen-Macaulay modules are easy to find; see 4.9. In proving our main result, we discover the following, which came as a surprise to us; see 4.7.

Observation 1.4. If a local ring \(R\) admits a (finitely generated) maximal Cohen-Macaulay Tor-rigid module, then \(R\) is Cohen-Macaulay.

We make various observations in sections 3 and 4, and prove our main result, Theorem 5.8 in section 5. Sections 6 and 7 are devoted to applications of our argument. We also collect examples of test and rigid-test modules from the literature in Appendix A; Appendix B collaborates a result of Iyama and Wemyss 33.

2. Definitions

2.1. ([1]) An \(R\)-module \(M\) is said to be Tor-rigid provided that the following holds for all \(R\)-modules \(N\):

If \(\text{Tor}^R_n(M,N) = 0\) for some \(n \geq 1\), then \(\text{Tor}^R_{n+1}(M,N) = 0\).

The notion of Tor-rigidity were initially used in the study of the Koszul complex; it was later formulated and analyzed for modules by Auslander; see [1]. An interesting result of Lichtenbaum 43 Theorem 3] shows that modules over regular local rings, and those of finite projective dimension over hypersurfaces – quotient of power series rings over fields – are Tor-rigid.

2.2. ([14 1.1]) An \(R\)-module \(M\) is said to be a test module for projectivity provided that the following holds for all \(R\)-modules \(N\):

If \(\text{pd}(N) = \infty\), then \(\text{Tor}^R_n(M,N) \neq 0\) for infinitely many integers \(n\).

We will call a test module for projectivity simply a test module.

Motivated by a question of Lichtenbaum 43 page 226, question 4], we define:

2.3. A Tor-rigid test module is called a rigid-test module. More precisely, \(M\) is called a rigid-test module provided that the following holds for all \(R\)-modules \(N\):

If \(\text{Tor}^R_n(M,N) = 0\) for some \(n \geq 1\), then \(\text{Tor}^R_{n+1}(M,N) = 0\) and \(\text{pd}(N) < \infty\).

Dao, Li and Miller 22 defined strong-rigidity to study Tor-rigidity of the Frobenius endomorphism over Gorenstein rings:

2.4. ([22 2.1]) An \(R\)-module \(M\) is said to be strongly-rigid provided that the following holds for all \(R\)-modules \(N\):

If \(\text{Tor}^R_n(M,N) = 0\) for some \(n \geq 1\), then \(\text{pd}(N) < \infty\).
It follows from the definition that rigid-test modules are strongly-rigid, but we do not know whether the converse is true in general. Most of our results work for strongly-rigid modules. However, to obtain the conclusion of Theorem 1.1(i) when \( n \geq \text{depth}(N) \), we need Tor-rigidity; see Theorem 5.8 and Corollary 6.1. This leads us to pose the following question for further study.

**Question 2.5.** Let \( R \) be a local ring and let \( M \) be an \( R \)-module. If \( M \) is strongly-rigid, then must \( M \) be a rigid-test module, or equivalently, must \( M \) be Tor-rigid?

We give some relations between the above definitions in a diagram form:

\[
\begin{array}{ccccccc}
\text{Tor-rigid} & \xrightarrow{(1)} & \text{test} & \xleftarrow{(3)} & \text{rigid-test} & \xleftarrow{(5)} & \text{strongly-rigid} \\
\xrightarrow{(4)} & \xleftarrow{(2)} & \xrightarrow{(6)} & \xleftarrow{(7)} &
\end{array}
\]

The implications in the diagram can be justified as follows:

1. and 3.: see Example 6.3
2. and 6.: see Example 6.4
4), 5) and 7): these follow from the definitions; see (2.1), (2.3) and (2.4).

### 3. Projective and injective dimensions via rigid modules

Let \( R \) be a local ring. If \( N \) is a nonzero rigid-test module over \( R \), then the vanishing of \( \text{Tor}_R^i(N, N) \) is not mysterious at all: it follows from the definition – unless \( R \) is regular – that \( \text{Tor}_R^i(N, N) \neq 0 \) for all \( i \geq 0 \); see (2.3). Hence it seems interesting to consider the vanishing of \( \text{Tor}_R^i(M, N) \) when \( N \) is a rigid-test module and \( M \) is an arbitrary \( R \)-module. In particular we seek to find whether the vanishing of \( \text{Tor}_R^i(M, N) \) for all \( i \gg 0 \) yields the exact value of the projective dimension of \( M \). Auslander remarked that, if \( \text{depth}(N) = 0 \) and \( \text{pd}(M) = s < \infty \), then \( \text{Tor}_R^s(M, N) \neq 0 \); see [1, 1.1]. Therefore an immediate observation is:

**3.1.** If \( R \) is a local ring and \( N \) is a test module such that \( \text{depth}(N) = 0 \), then it follows that \( \text{pd}(M) = \sup \{ i \in \mathbb{Z} : \text{Tor}_R^i(M, N) \neq 0 \} \); see (2.2).

A rigid-test module of positive depth does not necessarily detect the exact value of the projective dimension via the vanishing of Tor in general; cf. Theorem 1.1.

**Example 3.2.** Let \( R = \mathbb{k}[x, y, z]/(xy) \), \( T = R/(x) \) and let \( N = T \oplus \Omega T = R/(x) \oplus R/(y) \). Then \( \text{depth}(N) = 2 \) and \( N \) is Tor-rigid; see [19, 1.9]. Moreover, since \( \text{pd}(N) = \infty \), it follows that \( N \) is a rigid-test module; see (2.2). Setting \( M = R/(z) \), we see that \( 1 = \text{pd}(M) = \sup \{ i \in \mathbb{Z} : \text{Tor}_R^i(M, N) \neq 0 \} = 0 \).

If \( N \) is a rigid-test module that is not necessarily of depth zero, Proposition 3.3 can be useful to detect the projective dimension of \( M \); see also Remark 3.4. In the following \( \text{syz}(N) \) denotes the largest integer \( n \) for which \( N \) can be an \( n \)th syzygy module in a minimal free resolution of an \( R \)-module. The assumption that \( \text{depth}(N) = \text{syz}(N) \) in Proposition 3.3 holds, for example, when \( N \) is reflexive and \( N_p \) is free for all prime ideals \( p \) of \( R \) with \( p \neq m \); see [21, 3.9].
Proposition 3.3. Let $R$ be a local ring and let $M$ and $N$ be nonzero $R$-modules. Assume $N$ is a rigid-test module and that $\text{syz}(N) = \text{depth}(N) \leq \text{pd}(M)$. Then

$$\sup\{i \in \mathbb{Z} : \text{Tor}_i^R(M, N) \neq 0\} = \text{pd}(M) - \text{depth}(N).$$

Proof. We may assume $\text{pd}(M) < \infty$; see (2.3). Set $\text{pd}(M) = n$, $\text{depth}(N) = t$, and $q = \sup\{i \in \mathbb{Z} : \text{Tor}_i^R(M, N) \neq 0\}$. We proceed by induction on $t$ and prove that $\text{Tor}_i^R(M, N) \neq 0$ for all $i = 0, \ldots, n - t$. Notice, since $N$ is Tor-rigid, it is enough to show that $\text{Tor}_{n-t}^R(M, N) \neq 0$. If $t = 0$, then it follows from Auslander’s remark that $\text{Tor}_n^R(M, N) \neq 0$; see (3.3). Hence assume $t \geq 1$, and pick a non zero-divisor $x$ on $N$. Then one can see that there is a long exact sequence of the form:

$$\cdots \to \text{Tor}_n^R(M, N) \to \text{Tor}_n^R(M, N/xN) \to \text{Tor}_{n-1}^R(M, N) \to \cdots$$

It is easy to see that $N/xN$ is a rigid-test over $R$; see [11, 2.2]. Thus the induction hypothesis yields $\text{Tor}_{n-t+1}^R(M, N/xN) \neq 0$. Therefore $\text{Tor}_{n-t}^R(M, N) \neq 0$. In particular we have that $q \geq n - t$.

Let $p \in \text{Ass}(\text{Tor}_n^R(M, N))$. Then the depth formula [11, 1.2] implies that:

(3.3.1) $\text{pd}(M_p) - \text{depth}_{R_p}(N_p) = \text{depth}(R_p) - \text{depth}_{R_p}(M_p) - \text{depth}_{R_p}(N_p) = q$.

If $\text{depth}_{R_p}(N_p) \geq \text{depth}(N)$, then it follows that:

$$q = \text{pd}(M_p) - \text{depth}_{R_p}(N_p) \leq \text{pd}(M) - \text{depth}(N) = n - t.$$

This shows $q = n - t$, and hence completes the proof.

Next suppose $\text{depth}_{R_p}(N_p) < \text{depth}(N) = t$. Since $\text{depth}(N) = \text{syz}(N)$, we know that $N$ is a $t$-th syzygy module. This implies that $\text{depth}_{R_p}(N_p) \geq \min\{t, \text{depth}(R_p)\}$; see [10, 1.3.7]. Hence $\text{depth}_{R_p}(N_p) \geq \text{depth}(R_p)$. Therefore, by (3.3.1), we deduce that $q = 0$. Now the fact $\text{Tor}_{n-t}^R(M, N) \neq 0$ yields that $n - t = 0$, i.e., $q = n - t$. □

Remark 3.4. Jorgensen [35, 2.2] proved, if $M$ is a module over a local ring $R$ with $\text{pd}(M) < \infty$, then $q^R(M, N) = \sup\{i \in \mathbb{Z} : \text{Tor}_i^R(M, N) \neq 0\}$ is equal to $\sup\{\text{pd}(M_p) - \text{depth}_{R_p}(N_p) : p \in \text{Supp}(M \otimes_R N)\}$. Therefore, if $\text{pd}(M) < \infty$, one can deduce from Jorgensen’s result that $q^R(M, N) \geq \text{pd}(M) - \text{depth}(N)$. In case $\text{syz}(N) = \text{depth}(N) \leq \text{pd}(M) < \infty$, Proposition 3.3 establishes the equality $q^R(M, N) = \text{pd}(M) - \text{depth}(N)$ without appealing to [35, 2.2]. Notice, by Example 3.2, the hypothesis $\text{depth}(N) \leq \text{pd}(M)$ is required, but we do not know whether the condition $\text{syz}(N) = \text{depth}(N)$ is essential.

If $(R, m, k)$ is a local ring and $N$ is a nonzero $R$-module, then the $k$-vector spaces $\text{Ext}_R^a(k, N)$ are nonzero for all $n$, where $\text{depth}(N) \leq n \leq \text{id}(N)$; see [51, Theorem 2]. In other words, if $\text{Ext}_R^a(k, N) = 0$ for some $n \geq \text{depth}(N)$, then $\text{id}(N) < \infty$. Since $k$ is strongly-rigid, this leads us to pose the following question; see also (2.4).

Question 3.5. Let $R$ be a local ring and let $M$ and $N$ be nonzero $R$-modules. Assume $M$ is strongly-rigid and $\text{Ext}_R^a(M, N) = 0$ for some $n \geq \text{depth}(N)$. Then must $N$ have finite injective dimension?

In case $M$ is a test module (not necessarily strongly-rigid) and $R$ has a dualizing complex (i.e., $R$ is a homomorphic image of a Gorenstein ring), it follows from [14, 3.2] that $\text{id}(N) < \infty$ if and only if $\text{Ext}_R^i(M, N)$ vanishes for all $i \gg 0$. Here our aim is to examine the case where $M$ is strongly-rigid, and a single $\text{Ext}_R^a(M, N)$ vanishes
for some \( n \geq \text{depth}(N) \). In Proposition 3.6 we obtain a partial affirmative answer to Question 3.5 over Cohen-Macaulay rings. This, in particular, gives an affirmative answer when \( R \) is Artinian; see Corollary 3.8.

**Proposition 3.6.** Let \( (R, m) \) be a Cohen-Macaulay local ring with a dualizing module and let \( M \) and \( N \) be nonzero \( R \)-modules. Assume the following holds:

(i) \( \text{pd}_{R_p}(M_p) < \infty \) for all \( p \in \text{Spec}(R) - \{m\} \) (e.g., \( R \) has an isolated singularity.)

(ii) \( \text{Ext}^j_R(M, N) = 0 \) for some \( j \geq \text{dim}(R) + 1 \).

(iii) \( M \) is strongly-rigid.

Then \( \text{id}(N) < \infty \).

**Proof.** As \( R \) has a dualizing module, we can consider a maximal Cohen-Macaulay approximation of \( N \), i.e., a short exact sequence of \( R \)-modules

\[
0 \rightarrow Y \rightarrow C \rightarrow N \rightarrow 0, \tag{3.6.1}
\]

where \( C \) is maximal Cohen-Macaulay and \( \text{id}(Y) < \infty \). Set \( n = j - d + \text{depth}(M) \) where \( d = \text{dim}(R) \). Applying \( \text{Hom}_R(M, -) \) to (3.6.1), we get the following long exact sequence:

\[
\cdots \rightarrow \text{Ext}^j_R(M, Y) \rightarrow \text{Ext}^j_R(M, C) \rightarrow \text{Ext}^j_R(M, N) \rightarrow \cdots \tag{3.6.2}
\]

Note that \( \text{Ext}^j_R(M, Y) = 0 \). So it follows from (3.6.2) and (ii) that \( \text{Ext}^j_R(M, C) = 0 \). Observe, by (3.6.1), that \( \text{id}(C) < \infty \) if and only if \( \text{id}(N) < \infty \). Therefore we may assume \( N \) is maximal Cohen-Macaulay. Consider the following standard spectral sequence:

\[
E_2^{p,q} = \text{Ext}^p_R(\text{Tor}^R_q(N^\dagger, M), \omega) \implies H^{p+q} = \text{Ext}^{p+q}_R(M, N)
\]

Here \( N^\dagger = \text{Hom}(N, \omega) \) and \( \omega \) is the dualizing module of \( R \). Observe \( \text{Tor}^R_q(M, N^\dagger) \) has finite length for all \( q \geq 1 \): this follows from (i) and the fact that \( N^\dagger \) is maximal Cohen-Macaulay; see [22, 2.2]. Therefore \( E_2^{p,q} = 0 \) if \( q \geq 1 \) and \( p \neq d \). Furthermore:

\[
\text{Ext}^j_R(M, N) = H^j \cong E_2^{d,n} = \text{Ext}^d_R(\text{Tor}^R_n(M, N^\dagger), \omega).
\]

Now, by (ii), the local duality theorem [10, 3.5.11(b)] yields that \( \text{Tor}^R_n(M, N^\dagger) = 0 \). Thus (iii) gives the required conclusion: see also [25]. \( \square \)

We will use the following observation several times; see Corollaries 6.1 and 6.13.

**3.7.** Let \( R \) be a local ring and let \( M \) and \( N \) be nonzero \( R \)-modules. Assume \( \text{Ext}^n_R(M, N) = 0 \) for some \( n \geq \text{depth}(N) \). If \( n = 0 \), then \( \text{depth}(N) = 0 \) and hence \( \text{Hom}(M, N) \neq 0 \); see, for example, [10, 1.2.3]. Therefore \( n \) is positive.

**Corollary 3.8.** Let \( R \) be an Artinian ring and let \( M \) and \( N \) be nonzero \( R \)-modules. If \( M \) is strongly-rigid and \( \text{Ext}^n_R(M, N) = 0 \) for some \( n \geq 0 \), then \( N \) is injective.

**Proof.** In view of (3.7), the required result follows from Proposition 3.6. \( \square \)

If \( R \) is an Artinian hypersurface, that is quotient of a power series ring over a field, \( M \) is an \( R \)-module of infinite projective dimension and \( \text{Ext}^n_R(M, N) = 0 \) for some \( n \geq 0 \), then \([5, 5.12]\) and \([11, 4.7]\) show that \( N \) is injective. One can recover this result from Corollary 3.8 since each module of infinite projective dimension is strongly-rigid over such an Artinian hypersurface; see [A.4] and (A.6).
Corollary 3.9. Let $R$ be a $d$-dimensional excellent Cohen-Macaulay local ring, $N$ a nonzero $R$-module, and let $I$ be an integrally closed $\mathfrak{m}$-primary ideal of $R$. Assume $\text{Ext}^n_R(I, N) = 0$ for some $n \geq d$. Then $\text{id}(N) < \infty$.

Proof. It follows that $I \otimes_R \hat{R}$ is an integrally closed $\mathfrak{m}\hat{R}$-primary ideal of $\hat{R}$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of $R$; see, for example [32, 19.2.5]. So we may assume $R$ is complete with a dualizing module. Notice, by (3.7), we have that $\text{Ext}^n_R(I, N) = 0$ for some $n \geq d$. Moreover $I$ is strongly-rigid, and $\text{id}(N) < \infty$. An affirmative answer has been recently obtained in [12]:

3.11. (I2) Let $R$ be a local ring and let $M$ be a nonzero $R$-module. Then $M$ is a test module over $R$ if and only if $M \otimes_R \hat{R}$ is a test module over the $\mathfrak{m}$-adic completion $\hat{R}$ of $R$.

In light of (3.11), the excellent hypothesis on $R$ in Corollary 3.9 can be removed provided there is an affirmative answer to the following longstanding open problem:

Question 3.12. Let $R$ be a local ring. If $M$ is a Tor-rigid module over $R$, then must $M \otimes_R \hat{R}$ be Tor-rigid over $\hat{R}$?

Remark 3.13. It is established in Corollary 3.9 that, if $\text{Ext}^{n+1}_R(R/I, N) = 0$, then $\text{id}(N) < \infty$. Since $R/I$ has finite length, it is worth noting that the vanishing of $\text{Ext}^n_R(M, N)$ for an arbitrary $R$-module $M$ of finite length does not force $N$ to have finite injective dimension in general. For example, if $R = k[[x, y]]/(xy)$, $M = R/(x + y)$ and $N = k$, then $\text{Ext}^i_R(M, N) = 0$ for all $i \geq 2$, but $\text{id}(N) = \infty$.

4. Auslander’s transpose and remarks on Tor-rigidity

4.1. (2) Let $M$ be an $R$-module with a projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$. Then the transpose $\text{Tr} M$ of $M$ is the cokernel of $f^* = \text{Hom}_R(f, R)$ and hence is given by the exact sequence: $0 \to M^* \to P_0^* \to P_1^* \to \text{Tr} M \to 0$.

If $n$ is a positive integer, $\tau_n M$ denotes the transpose of the $(n-1)^{st}$ syzygy of $M$, i.e., $\tau_n M = \text{Tr} \Omega^{n-1} M$.

There are exact sequence of functors [2, 2.8]:

(i) $\text{Tor}^R_2(\tau_{n+1} M, -) \to (\text{Ext}^n_R(M, R) \otimes_R -) \to \text{Ext}^n_R(M, -) \to \text{Tor}^R_1(\tau_n M, -),$ 

(ii) $\text{Ext}^1_R(\tau_{n+1} M, -) \to \text{Tor}^R_1(M, -) \to \text{Hom}(\text{Ext}^n_R(M, R), -) \to \text{Ext}^n_R(\tau_n M, -).$

The next useful fact has been initially addressed by Auslander in his 1962 ICM talk [3]. Auslander’s original remark is for regular local rings but, in case Tor-rigidity holds, it also holds over arbitrary local rings; see (2.1).

4.2. (Auslander [3, Corollary 6]; see also Jothilingam [38]) Let $M$ and $N$ be nonzero $R$-modules. Assume $N$ is Tor-rigid. Assume further that $\text{Ext}^n_R(M, N) = 0$ for some nonnegative integer $n$. 

RIGID MODULES
It follows from (4.1)(i) that $\text{Tor}_i^R(T_{n+1}M, N) = 0$. This implies, since $N$ is Tor-rigid, that $\text{Tor}_i^R(T_{n+1}M, N) = 0$ for all $i \geq 1$. We can now use (4.1)(i) once more and conclude that $\text{Ext}_R(M, R) \otimes_R N = 0$. Therefore $\text{Ext}_R(M, R) = 0$.

In particular, if $M = N$, then $\text{Tor}_i^R(\text{Tr} \Omega^n M, \Omega^n M) = 0$ so that [60] 3.9 implies $\Omega^n M$ is free, i.e., $\text{pd}(M) \leq n - 1$.

As $\varphi_R$ is Tor-rigid over complete intersection rings, we deduce:

4.3. Assume $R$ is an F-finite local complete intersection ring with prime characteristic $p$. If $\text{Ext}_R^i(\varphi^s \cdot R, \varphi^s \cdot R) = 0$ for some positive integers $i$ and $n$, then it follows from (4.3)(iii) and (4.2) that $R$ is regular; cf. Corollary 1.3.

One can find remarkable applications of (4.2) in the literature. For example, Jorgensen [60, 2.1] proved that, if $R$ is a complete intersection ring and $M$ is an $R$-module such that $\text{Ext}_R^2(M, M) = 0$, then $\text{pd}(M) \leq 1$; (4.2) plays an important role in Jorgensen’s proof. On the other hand, Dao exploited (4.2) and obtained new results on the non-commutative crepant resolutions; see [20] for details.

We give two applications of Auslander’s rigidity result recorded in (4.2). The first one, (4.3), is an immediate observation, albeit it will be quite useful later; see the proof of Corollary 4.12. Our second application is given in Corollary 4.6: it yields a characterization of Cohen-Macaulay rings in terms of Tor-rigidity. We proceed by recalling a remarkable result of Foxby:

4.4. (Foxby [10, 3.1.25]) $R$ is Gorenstein if and only if there exists a nonzero finitely generated $R$-module $M$ such that $\text{pd}(M) < \infty$ and $\text{id}(M) < \infty$.

4.5. Let $M$ be a nonzero Tor-rigid $R$-module. If $\text{id}(M) < \infty$, then it follows from (4.2) that $\text{pd}(M) < \infty$ and hence, by (4.4), $R$ is Gorenstein.

Tor-rigidity hypothesis in (4.5) cannot be replaced with test property: a local ring admitting a nonzero test module of finite injective dimension is not necessarily Gorenstein, however such a ring $R$ is G-regular [55], i.e., $\text{G-dim}(M) = \text{pd}(M)$ for all $R$-modules $M$; see (2.2), Example 6.4 and Corollary 4.13.

The grade of a pair of nonzero modules $(M, N)$, denoted by $\text{grade}(M, N)$, is defined as $\inf \{i \in \mathbb{N} \cup \{0\} : \text{Ext}_R^i(M, N) \neq 0\}$. Setting $\text{grade}(M) = \text{grade}(M, R)$, we see that $\text{grade}(M) < \infty$.

Proposition 4.6. Let $R$ be a local ring and let $M$ and $N$ be nonzero $R$-modules. Assume $N$ is Tor-rigid. Set $\text{grade}(M) = n$ and $\text{grade}(M, N) = s$. Then $s \leq n$, and $\text{Ext}_R^i(M, N) \neq 0$ for all $i = s, \ldots, n$.

Proof. We have, by definition, that $\text{Ext}_R^s(M, R) \neq 0$. If $\text{Ext}_R^s(M, N) = 0$, then it follows from (4.2) that $\text{Ext}_R^s(M, R) = 0$, which is a contradiction. Therefore $\text{Ext}_R^s(M, N) \neq 0$, and hence $s \leq n$. Now suppose $\text{Ext}_R^s(M, N) = 0$ for some $s < i < n$. Set $r = \min \{j \in \mathbb{Z} : \text{Ext}_R^j(M, N) = 0 \text{ with } s < j < n\}$. We know, since $r < n$, that $\text{Ext}_R^r(M, R) = 0$. Therefore $T_r M$ is stably isomorphic to $\Omega T_{r+1} M$. Moreover it follows from (4.2) that $\text{Tor}_i^R(T_{r+1} M, N) = 0$ for all $i \geq 1$. This implies that $\text{Tor}_i^R(T_r M, N) = 0$ for all $i \geq 1$. Now we use (4.1)(i) and deduce that $\text{Ext}_R^{n-1}(M, N) = 0$. This contradicts the choice of $r$, and finishes the proof. \[\square\]

An immediate consequence of Proposition 4.6 is a characterization of local rings:

Corollary 4.7. Let $R$ be a local ring and let $N$ be a nonzero Tor-rigid $R$-module.
4.9. Assume $R$ is a local ring that is not Cohen-Macaulay.

(i) If $R$ is one-dimensional and $p$ is a minimal prime ideal of $R$, then $R/p$ is a maximal Cohen-Macaulay $R$-module that is not Tor-rigid. For example if we put $R = k[x, y]/(x^2, xy)$, then $R/(x)$ is not a Tor-rigid $R$-module. In fact $\text{Tor}_d^R(R/(x), R/(y)) = 0 \neq \text{Tor}_d^R(R/(x), R/(y))$; see [43 Question 3].

(ii) If $R$ is a two-dimensional complete domain, then the integral closure $\overline{R}$ of $R$ in its field of fractions is a (finitely generated) maximal Cohen-Macaulay $R$-module that is not Tor-rigid. For example, if $R = k[x^4, x^3y, xy^3, y^4]$, then $\overline{R} = R[[x^2, y^2]]$ is not a Tor-rigid $R$-module.

One can also find examples of three dimensional non Cohen-Macaulay local rings that admit maximal Cohen-Macaulay modules; see, for example, Hochster [25, 5.4, 5.6 and 5.9]. These modules are not Tor-rigid, for example, by [43].
Recall that an $R$-module module $C$ is called *semidualizing* if the natural map $R \to \text{Hom}(C, C)$ is bijective and $\text{Ext}^i_R(C, C) = 0$ for all $i \geq 1$. If $C$ is a semidualizing module such that $\text{id}(C) < \infty$, then $R$ is Cohen-Macaulay and $C$ is dualizing.

**4.10.** Let $C$ be a semidualizing module over $R$. Then the *C-projective dimension* $\text{Cpd}(M)$ of a nonzero $R$-module $M$ is defined as the infimum of the integers $n$ such that there exists an exact sequence

$$0 \to C^{b_0} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0$$

where each $b_i$ is a positive integer. It follows $\text{Cpd}(M) = \text{pd}(\text{Hom}_R(C, M))$, and the $\text{C-injective dimension}$ $\text{Cid}(M)$ of $M$ is defined similarly: $\text{Cid}(M) = \text{id}(C \otimes_R M)$; see [54, 2.9]. Notice, if $\text{pd}(C) < \infty$, then $C \cong R$ and hence $\text{Cpd}(N) = \text{pd}(N)$ and $\text{Cid}(N) = \text{id}(N)$.

**4.11.** (Takahashi and White [54, 2.9]) Let $M$ be a nonzero $R$-module and let $C$ be a semidualizing $R$-module. If $\text{Cpd}(M) < \infty$ (respectively, $\text{Cid}(M) < \infty$), then $\text{Ext}^i_R(C, M) = 0$ for all $i \geq 1$ (respectively, $\text{Tor}^i_R(C, M) = 0$ for all $i \geq 1$). In particular, if $M$ is a nonzero test module and $\text{Cid}(M) < \infty$ for some semidualizing $R$-module $C$, then $C \cong R$ and hence $\text{id}(M) < \infty$; see [24] and [4.10].

**Proposition 4.12.** Let $R$ be a local ring and let $C$ be a semidualizing $R$-module. Assume $M$ is a nonzero test module over $R$. Assume further that $\text{Cid}(M) < \infty$.

If $X$ is an $R$-module such that $\text{Ext}^i_R(X, R) = 0$ for all $i \gg 0$, then $\text{pd}(X) < \infty$.

**Proof.** Assume $X$ is an $R$-module with $\text{Ext}^i_R(X, R) = 0$ for all $i \gg 0$. Note, by [4.11], we have that $\text{id}(M) < \infty$. This yields $\text{RHom}(\text{RHom}(X, R), M) \cong X \otimes_R M$; see [15, A.4.24]. Therefore $\text{Tor}^i_R(M, X) = 0$ for all $i \gg 0$, so that $\text{pd}(X) < \infty$. □

It was proved in [14, 3.7] that, if the dualizing module of a Cohen-Macaulay local ring $R$ is a test module, then $R$ is $G$-regular, i.e., $\text{pd}(M) = G\text{-dim}(M)$ for all $R$-modules $M$ [55]. A straightforward application of Proposition 4.12 extends this:

**Corollary 4.13.** A local ring admitting a nonzero test module of finite injective dimension is $G$-regular.

Before we proceed to prove our main result, we give an overview of what has been established so far in terms of the injective dimension of test and rigid modules:

**4.14.** Let $R$ be a local ring and let $N$ be a nonzero $R$-module such that $\text{id}(N) < \infty$.

(i) If $N$ is Tor-rigid, then $R$ is Gorenstein; see [4.15].

(ii) If $N$ is a test module over $R$, then $R$ is $G$-regular; see Corollary 4.13.

(iii) If $N$ is a rigid-test module over $R$, then it follows from (i) and (ii) that $R$ is regular; see also Question 2.30 and Corollary 6.11.

5. Main theorem

This section is dedicated to a proof of our main result, Theorem 5.8. In the following $H\text{-dim}$ denotes a homological dimension of finitely generated modules; see [5.2] and, for example, Avramov’s expository article [8] 8.6 - 8.8 for details. The special case – where $H$ is the projective dimension $\text{pd}$ – is what we really need for the proof of Theorem 1.1 stated in the introduction. However one can follow our argument word for word by replacing $H\text{-dim}$ with projective dimension $\text{pd}$ so there is no extra penalty for this generality. Furthermore such a generality is useful to examine the *Gorenstein dimension* $G\text{-dim}$ of Tor-rigid modules; see Corollary 6.13.
5.1. ([2]) A finitely generated module $M$ over a commutative Noetherian ring $R$ is said to be \textit{totally reflexive} if the canonical map $M \to \text{Hom}(\text{Hom}(M, R), R)$ is bijective, and $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(M, \text{Hom}(M, R))$ for all $i \geq 1$.

The infimum of nonnegative integers $n$ for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$, such that each $X_i$ is totally reflexive, is called the Gorenstein dimension of $M$. If $M$ has Gorenstein dimension $n$, we write $G\text{-dim}(M) = n$. Therefore $M$ is totally reflexive if and only if $G\text{-dim}(M) \leq 0$, where it follows by convention that $G\text{-dim}(0) = -\infty$.

5.2. Throughout we assume $H\text{-dim}$ satisfies the following conditions:

(i) $G\text{-dim}(M) \leq H\text{-dim}(M) \leq \text{pd}(M)$ for all $R$-modules $M$.

(ii) If $H\text{-dim}(M) = 0$, then $\text{Tr}M = 0$ or $H\text{-dim}(\text{Tr}M) = 0$ for all $R$-modules $M$.

Although we will not use it, we note that the \textit{complete intersection dimension} [6] is an example of a homological dimension – in general distinct than the Gorenstein and projective dimension – that satisfies the conditions in (5.2).

5.3. For our purpose we recall a few properties of $H\text{-dim}$; see [8] 3.1.2, 8.7 and 8.8].

(i) If one of the dimensions in (5.2) is finite, then it equals the one on its left.

(ii) If $H\text{-dim}(M) < \infty$, then $H\text{-dim}(M) = \sup\{i \in \mathbb{Z} : \text{Ext}_R^i(M, R) \neq 0\}$.

(iii) If $H\text{-dim}(M) < \infty$, then $H\text{-dim}(M) \leq \text{depth}(R)$.

If $X$ and $Y$ are nonzero $R$-modules and $H\text{-dim}$ is a homological dimension of modules, we consider the following condition for $(X, Y, H)$:

5.4. If $\text{Tor}_1^R(X, Y) = 0$, then $H\text{-dim}(X) = \text{depth}(Y) - \text{depth}(X \otimes_R Y)$.

The condition in (5.4) is not restrictive for rigid modules. For example, Auslander [I 1.2] proved that, if $R$ is a local ring, and $X$ and $Y$ are nonzero $R$-modules where $\text{pd}(X) < \infty$ and $Y$ is Tor-rigid, then (5.4) holds for $(X, Y, \text{pd})$; see (2.1).

Recently Christensen and Jorgensen [13 5.3] established a similar result over AB rings: if $R$ is AB, and $X$ and $Y$ are nonzero $R$-modules either of which is Tor-rigid, then $(X, Y, G\text{-dim})$ satisfies the condition in (5.4). Recall that a Gorenstein local ring $R$ is said to be AB [30] if, for all $R$-modules $M$ and $N$, $\text{Ext}_R^i(M, N) = 0$ for all $i > 0$ implies that $\text{Ext}_R^i(M, N) = 0$ for all $i > \dim(R)$. The class of AB rings strictly contain that of complete intersections; see [30] 3.6 and [67] 4.5] for details.

Next we summarize the aformentioned two results:

5.5. Let $R$ be a local ring and let $X$ and $Y$ be nonzero $R$-modules. Then $(X, Y, H)$ satisfies the condition in (5.4) if at least one of the following conditions holds:

(i) $X$ is a rigid-test module, and $H\text{-dim} = \text{pd}$; see (2.3) and [I 1.2];

(ii) $R$ is AB, $X$ or $Y$ is Tor-rigid, and $H\text{-dim} = G\text{-dim}$; see (2.1) and [10 5.3].

Next is the key result we use for our proof of Theorem 5.8; see also (4.1).

5.6. (Auslander and Bridger; see [2 2.12, 2.15 and 2.17]) Let $R$ be a local ring, $M$ a nonzero $R$-module and let $n$ be a positive integer. If $M$ is $n$-torsion-free, then it is $n$-reflexive, i.e., if $\text{Ext}_R^i(\text{Tr}M, R) = 0$ for all $i = 1, \ldots, n$, then $M \approx \Omega_R^n(\text{Tr}_{R+1}(\text{Tr}M))$, i.e., $M$ is isomorphic to $\Omega_R^n(\text{Tr}_{R+1}(\text{Tr}M))$ up to a projective summand.

Recall that $\text{depth}(0) = \infty$. 

5.7. (2) Let $R$ be a local ring and let $M$ be an $R$-module. Then $M$ is said to satisfy $(\tilde{S}_n)$ if $\text{depth}_{R_p}(M_p) \geq \min\{n, \text{depth}(R_p)\}$ for all $p \in \text{Supp}(M)$.

If the ring is Cohen-Macaulay, $(\tilde{S}_n)$ coincides with Serre’s condition $(\tilde{S}_n)$ [24], but in general $(\tilde{S}_n)$ is a weaker condition. For example, if $R = k[x, y]/(x^2, xy)$, then, by definition, $R$ satisfies $(\tilde{S}_n)$ for all nonnegative integers $n$, but fails to satisfy $(S_1)$ since $\text{depth}(R) = 0$; see also the discussion following [45] Definition 10.

Theorem 5.8 is a generalization of a result of Jothilingham [39, Corollary 1]. We give the argument in two steps as the conclusion of part (1) may be of independent interest. It is already known that the vanishing of $\text{Ext}^i_R(\text{Tr}M, R)$ for all $i = 1, \ldots, r$ forces $M$ to be an $r$th syzygy module, and that forces $M$ to satisfy $(\tilde{S}_r)$; see (5.7) and [45] Propositions 11 and 40.

**Theorem 5.8.** Let $R$ be a local ring and let $M$ and $N$ be a nonzero $R$-modules. Assume $M$ satisfies $(\tilde{S}_r)$ for some nonnegative integer $r$.

1. If $\text{G-dim}(\text{Tr}M) < \infty$, then $\text{Ext}^i_R(\text{Tr}M, R) = 0$ for all $i = 1, \ldots, r$.
2. Assume $\text{Ext}^n_R(M, N) = 0$ for some positive integer $n$.
   
   (i) If $N$ is strongly-rigid and $\text{depth}(R) \leq n + r$, then $\text{pd}(M) \leq n - 1$.
   
   (ii) If $N$ is $\text{Tor}$-rigid, $\text{depth}(N) \leq n + r$ and $(\text{Tr}_{n+1}M, N, H)$ satisfies the condition in (7.3), then $\text{H-dim}(M) \leq n - 1$.

**Proof.** We proceed to prove (1). Assume $\text{G-dim}(\text{Tr}M) < \infty$ and $r \geq 1$. If $\text{G-dim}(\text{Tr}M) = 0$, then $\text{Ext}^i_R(\text{Tr}M, R) = 0$ for all $i \geq 1$ so that there is nothing to prove. So we assume $\text{G-dim}(\text{Tr}M) \geq 1$ and set $s = \inf\{i \geq 1 : \text{Ext}^i_R(\text{Tr}M, R) \neq 0\}$.

Let $p \in \text{Ass}_R(\text{Ext}^s_R(\text{Tr}M, R))$. Then we have $pR_p \in \text{Ass}_{R_p}(\text{Ext}^s_{R_p}(\text{Tr}_{R_p}M_p, R_p))$ and that $s = \inf\{i \geq 1 : \text{Ext}^i_{R_p}(\text{Tr}_{R_p}M_p, R_p) \neq 0\}$. It follows from the definition of the transpose, see (4.1), that there is an injection

$$0 \to \text{Ext}^s_{R_p}(\text{Tr}_{R_p}M_p, R_p) \hookrightarrow \text{Tr}_s(\text{Tr}_{R_p}M_p),$$

which shows that $pR_p \in \text{Ass}_{R_p}(\text{Tr}_s(\text{Tr}_{R_p}M_p))$. Hence $\text{depth}_{R_p}(\text{Tr}_s(\text{Tr}_{R_p}M_p)) = 0$. Since $\text{Ext}^i_{R_p}(\text{Tr}_{R_p}M_p, R_p) = 0$ for all $i = 1, \ldots, s - 1$, we conclude from (5.6) that

$$M_p \cong \Omega^{s-1}_{R_p}(\text{Tr}_s(\text{Tr}_{R_p}M_p)).$$

Note that $s \leq \text{G-dim}_{R_p}(\text{Tr}_{R_p}M_p) \leq \text{depth} R_p$. Therefore $\text{depth}_{R_p}(M_p) = s - 1$. Furthermore, since $M$ satisfies $(\tilde{S}_r)$, it follows that $\text{depth}_{R_p}(M_p) \geq \min\{r, \text{depth} R_p\}$. Hence $\text{depth} R_p \geq r + 1$ and $r \leq s - 1$. Consequently $\text{Ext}^i_R(\text{Tr}M, R) = 0$ for all $i = 1, \ldots, r$.

We now proceed to prove (2). If $\text{Tr}_{n+1}M = 0$, then $\Omega^n M$ is free and hence $\text{pd}(M) \leq n - 1$; see (4.4). In particular, this implies that $\text{H-dim}(M) \leq n - 1$; see (5.3)(i). Therefore we may assume $\text{Tr}_{n+1}M \neq 0$.

Since $M$ satisfies $(\tilde{S}_r)$, it follows that $\Omega^n M$ satisfies $(\tilde{S}_{n+r})$; see (5.7). Therefore, by the first part of the theorem, we conclude that:

(5.8.1) $\text{Ext}^i_{R}(\text{Tr}_{n+1}M, R) = 0$ for all $i = 1, \ldots, n + r$.

Notice, since $\text{Ext}^n_R(M, N) = 0$, we have by (4.1)(i) that:

(5.8.2) $\text{Tor}^i_{R}(\text{Tr}_{n+1}M, N) = 0$. 
If (i) holds, then it follows from (2.3) and (6.8.2) that \( \text{pd}(M) < \infty \). Therefore, since \( \text{depth}(R) \leq n + r \), we use (5.8) and deduce:

\[(5.8.3) \quad H\text{-dim}(M) = \text{pd}(M) \leq \text{depth}(R) \leq n + r.\]

On the other hand, if (ii) holds, then it follows from our assumption that

\[(5.8.4) \quad H\text{-dim}(M) = \text{depth}(N) - \text{depth}(M) \leq n + r.\]

This yields that

\[(6.8.1) \quad H\text{-dim}(M) = n + r.\]

Consequently, if either (i) or (ii) holds, then \( H\text{-dim}(M) = n + r \), where for part (i), \( H\text{-dim}(M) = \text{pd}(M) \); see (5.8.3) and (5.8.4).

Recall that \( H\text{-dim}(M) = \sup \{ i : \text{Ext}^i_R(M, N) \neq 0 \} < \infty \); see (5.3)(ii). Since \( H\text{-dim}(M) \leq n + r \), it follows from (5.8.1) that \( H\text{-dim}(M) = 0 \), i.e., \( H\text{-dim}(M) = n \); see (4.1). Thus, by (4.2)(i), we have that \( H\text{-dim}(M) = 0 \). This yields that \( H\text{-dim}(M) \leq n \).

If (i) holds, then, since \( \text{Ext}^n(M, N) = 0 \), we conclude that \( H\text{-dim}(M) = n - 1 \); see, for example, [46], Chapter 19, Lemma 1(iii)]. On the other hand, if (ii) holds, then, since \( N \) is Tor-rigid, it follows from (5.8.2) and (4.1)(i) that \( \text{Ext}^n(M, R) = 0 \). Therefore we see that \( H\text{-dim}(M) \leq n - 1 \). □

6. Corollaries of the main theorem

In this section we give various applications of Theorem 5.8 and examine homological dimensions of test and rigid-test modules. Corollary 6.1, a reformulation of Theorem 5.8, is fundamental to our work: it shows that one can use an arbitrary nonzero strongly-rigid, or a rigid-test module \( N \), just like the residue field \( k \), to determine the exact value of the projective dimension of \( M \) via the vanishing of \( \text{Ext}^n_R(M, N) \). Besides this, Corollary 6.1 yields a series of related results. Among those is Corollary 7.3 which proves, in particular, that \( R \) is Gorenstein if the Gorenstein injective dimension \( \text{Gpd}(R) \) of the maximal ideal \( m \) is finite.

Corollary 6.1 is well-known for the special case where \( N = k \). Recall that a rigid-test module is, by definition, strongly-rigid, but we do not know whether or not all strongly-rigid modules are rigid-test; see Question 2.5.

**Corollary 6.1.** Let \( R \) be a local ring, and let \( M \) and \( N \) be nonzero \( R \)-modules.

(i) Assume \( \text{Ext}^n_R(M, N) = 0 \) for some \( n \geq \text{depth}(R) \). Assume further \( N \) is strongly-rigid. Then \( \text{pd}(M) = \sup \{ i \in \mathbb{Z} : \text{Ext}^i_R(M, N) \neq 0 \} \leq n - 1 \).

(ii) Assume \( \text{Ext}^n_R(M, N) = 0 \) for some integer \( n \geq \text{depth}(N) \). Assume further \( N \) is a rigid-test module. Then \( \text{pd}(M) = \sup \{ i \in \mathbb{Z} : \text{Ext}^i_R(M, N) \neq 0 \} \leq n - 1 \).

**Proof.** Note that, for part (i) and part (ii), it suffices to prove that \( \text{pd}(M) \) cannot exceed \( n - 1 \); see, for example, [46], Chapter 19, Lemma 1(iii)].

Assume (i). If \( n = 0 \) then \( \text{depth}(R) = 0 \). Since \( \text{Hom}_R(M, N) = 0 \) and \( N \) is strongly-rigid, it follows from (1.7) that \( \text{depth}(R) \geq \text{depth}(N) \geq 1 \), which is a contradiction. Hence \( n \geq 1 \). Setting \( r = 0 \) in Theorem 5.8(2)(i), we conclude that \( \text{pd}(M) \leq n - 1 \). Next assume (ii). Notice, by (5.7), \( n \) is a positive integer. Moreover, since \( N \) is a rigid-test module, (5.4) holds for \( (T_{n+1}, N, \text{pd}) \); see (5.5)(i). Therefore we obtain the required conclusion by setting \( r = 0 \) in Theorem 5.8(2)(ii). □

The conclusion of Corollary 6.1 is sharp: Example 6.2 shows that the condition on \( n \) cannot be removed. Examples 6.3 and 6.4 respectively, highlight the fact that
the assumption “$N$ is Tor-rigid” or “$N$ is a test module” is not merely enough to deduce that $M$ has finite projective dimension; see (2.4), (2.2) and (2.3).

**Example 6.2.** Let $k$ be a field, $R = k[x, y, z]/(xy - z^2)$, $M = k$ and $N = \Omega^2 k$. Then $N$ is a rigid-test module, $\text{Ext}^1_R(M, N) = 0$, $\text{depth}(N) = 2$ and $\text{pd}(M) = \infty$.

**Example 6.3.** Let $k$ be a field, $R = k[x, y]/(xy)$, $M = k$ and $N = R/(x + y)$. Then, since $\text{pd}(N) = 1$, $N$ is Tor-rigid. Furthermore, since $R$ is not regular, $N$ is not a test module. Consequently $N$ is not a rigid-test, or a strongly-rigid module. Note that $\text{pd}(M) = \infty$ and $\text{Ext}^i_R(M, N) = 0$ for all $i \geq 2$.

**Example 6.4.** Let $k$ be a field and put $R = k[[x, y, z]]/(y^2 - xz, x^2y - z^2, x^3 - yz)$. Let $M = m$ and $N = \omega$, the canonical module of $R$. As $R$ is a one-dimensional domain with minimal multiplicity, it is Golod [4, 5.2.8]. Hence, since $\text{pd}(N) = \infty$, it follows from (A.3) that $N$ is a test-module. Huneke and Wiegand [31, 4.8] proved that there exists an $R$-module $M$ such that $M \otimes_R N$ is torsion-free and $M$ has torsion. We now follow the proof of [31 1.1]:

Let $\overline{M}$ be the torsion-free part of $M$. Then, since $R$ is a domain, there is an exact sequence $0 \to \overline{M} \to F \to C \to 0$, where $F$ is a free $R$-module. Tensoring this short exact sequence with $N$, we obtain an injection $\text{Tor}^R_i(C, N) \to \overline{M} \otimes_R N$. Since $\overline{M} \otimes_R N \cong M \otimes_R N$ and $\text{Tor}^R_i(C, N)$ is torsion, we see that $\text{Tor}^R_i(C, N) = 0$. Note that $\text{pd}(M) = \infty$: otherwise $\overline{M}$ is free, and this would force $M$ to be free. Therefore $N$ is not a strongly-rigid module. For completeness, we also remark that $N$ is not Tor-rigid; see [15]. Notice $\text{Ext}^i_R(M, N) = 0$ for all $i \geq 1$ and $\text{pd}(M) = \infty$.

Corollary [6.1] can be useful to determine the depth of $\text{Hom}(M, N)$:

**Corollary 6.5.** Let $R$ be a Cohen-Macaulay local ring and let $M$ and $N$ be nonzero $R$-modules. Assume that the following conditions hold:

(i) $R$ has an isolated singularity, i.e., $R_p$ is regular for all $p \in \text{Spec}(R) - \{m\}$.

(ii) $M$ is nonfree and maximal Cohen-Macaulay.

(iii) $N$ is strongly-rigid with $\text{depth}(N) \geq 2$.

Then $\text{depth}(\text{Hom}_R(M, N)) = 2$.

**Proof.** Note that $\text{depth}(\text{Hom}_R(M, N)) \geq \min\{2, \text{depth}(N)\} = 2$; see [10] 1.4.19. Thus it suffices to prove $\text{depth}(\text{Hom}_R(M, N)) \leq 2$. Assume not, i.e., assume $\text{depth}(\text{Hom}_R(M, N)) \geq 3$. Then, by [31 1.1], we see either $\text{Ext}^1_R(M, N) = 0$, or $1 \leq \text{depth}(\text{Ext}^1_R(M, N)) < \infty$. Since $\text{Ext}^1_R(M, N)$ has finite length, it follows that $\text{Ext}^1_R(M, N) = 0$. Set $d = \text{dim}(R)$. Then $M = \Omega^d(X)$ for some finitely generated $R$-module $X$; see [11 A.15]. This yields $\text{Ext}^{d+1}_R(X, N) = 0$. Now Corollary [6.1] shows that $\text{pd}(X) < \infty$, i.e., $M$ is free. So we conclude that $\text{depth}(\text{Hom}_R(M, N)) = 2$. $\square$

**Corollary 6.6.** Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring over a perfect field $k$, and $f$ is a nonconstant polynomial in $Q$. Set $R = A_m$, where $m = (x_1, \ldots, x_{2s+1})A$ and assume $A_p$ is a regular for all $p \in \text{Spec}(A) - \{m\}$. If $M$ and $N$ are nonfree maximal Cohen-Macaulay $R$-modules, then $\text{depth}(\text{Hom}_R(M, N)) = 2$.

**Proof.** In light of (A.3) and (A.4), the required conclusion follows immediately from Corollary [6.3] cf. [19 3.4]. $\square$

Corollary [A.3] stated in the introduction, is now a direct consequence of our argument and the following special case of [9 1.1]; see also (A.3)(i).
6.7. (Avramov, Hochster, Iyengar and Yao; see [9, 1.1]) Let $R$ be a local ring of prime characteristic $p$ and let $M$ be a nonzero finitely generated $R$-module. If $\tilde{\varphi}M$ has finite flat dimension for some positive integer $e$, then $R$ is regular.

**Corollary 6.8.** Let $R$ be an $F$-finite local ring of prime characteristic $p$ and let $N$ be a nonzero rigid-test module over $R$. If $\text{Ext}_R^n(\tilde{\varphi}M, N) = 0$ for some nonzero $R$-module $M$, and for some integers $n \geq 1$ and $j \geq \text{depth}(N)$, then $R$ is regular.

**Proof.** Notice, as $R$ is $F$-finite, $\tilde{\varphi}M$ is a finitely generated $R$-module. Thus it follows from Corollary 6.1(ii) that $\text{pd}(\tilde{\varphi}M) < \infty$. Now, by (6.7), $R$ is regular. \qed

6.9. (Proof of Corollary 6.3) As $R$ is complete and $k$ is perfect, it follows that $R$ is $F$-finite; see, for example, [10, page 398]. So the result follows from Corollary 6.8.

A special case of Corollary 6.1 and 6.8 has been established in [50] Theorem B: if $R$ is a complete intersection ring of prime characteristic $p$, and $\text{Ext}_R^n(M, \tilde{\varphi}R) = 0$ for some $n \geq \text{depth}(R)$, then $\text{pd}(M) < \infty$; see [A.3](iii). We should note that [50] Theorem B does not require an $F$-finite ring and relies on methods different from ours. As discussed in the introduction, our argument is not specific to rings of characteristic $p$, and gives useful information regarding the Frobenius endomorphism even if the ring considered is not a complete intersection. For example the next result, in view of [A.3](ii), is immediate from Corollary 6.1 cf. [50] Theorem A.

**Corollary 6.10.** Let $R$ be a one-dimensional $F$-finite Cohen-Macaulay local ring of prime characteristic $p$, and let $M$ be an $R$-module. Then $\text{Ext}_R^n(M, \tilde{\varphi}R) = 0$ for some $n \geq 1$ and some $i \gg 0$ if and only if $\text{pd}(M) < \infty$.

Corollary 6.1 yields a characterization of regularity in terms of $C$-$\text{pd}$ and $C$-$\text{id}$ dimensions of strongly-rigid modules; see also [24] and [410].

**Corollary 6.11.** Let $R$ be a local ring, $C$ a semidualizing $R$-module and let $M$ be a nonzero strongly-rigid $R$-module. Assume either $C$-$\text{pd}(M) < \infty$ or $C$-$\text{id}(M) < \infty$. Then $R$ is regular.

**Proof.** We start by noting that $M$ is a test module; see (2.2). Assume first $C$-$\text{id}(M) < \infty$. Then it follows from (4.11) that $\text{id}(M) < \infty$, i.e., $\text{Ext}_R^i(k, M) = 0$ for all $i \gg 0$. Now Corollary 6.1(i) implies that $\text{pd}(k) < \infty$ so that $R$ is regular.

Next assume $C$-$\text{pd}(M) < \infty$. Then it follows from (4.11) that $\text{Ext}_R^i(C, M) = 0$ for all $i \geq 1$. Hence we can use Corollary 6.1(i) once more and deduce that $\text{pd}(M) < \infty$. This implies that $R$ is regular. \qed

A special case of Corollary 6.11 is a characterization of regularity in terms of integrally closed $m$-primary ideals; see [A.2].

**Corollary 6.12.** Let $(R, m)$ be a local ring and let $I$ be an integrally closed $m$-primary ideal of $R$. Then $R$ is regular if and only if there exists a semidualizing $R$-module $C$ such that $C$-$\text{id}(\Omega^n I) < \infty$ or $C$-$\text{pd}(\Omega^n I) < \infty$ for some nonnegative integer $n$. In particular, $R$ is regular if and only if $\text{id}(I) < \infty$.

The conclusions of next corollaries, 6.13 and 6.14, are known over complete intersection rings; see [52, 3.6]. Here we are able to show that these results carry over to AB rings. Recall that every complete intersection ring is AB, but not vice versa; see the paragraph preceding (6.5). Furthermore, in Corollary 6.15, we obtain a nonvanishing result for Ext over hypersurfaces that are in the form of (A.8).
Corollary 6.13. Let $R$ be a local AB ring, and let $M$ and $N$ be nonzero $R$-modules. Assume $N$ is Tor-rigid and that $\text{Ext}^n_R(M, N) = 0$ for some $n \geq \text{depth}(N)$. Then $\text{sup}\{i \in \mathbb{Z} : \text{Ext}^i_R(M, N) \neq 0\} = \text{G-dim}(M) = \text{depth}(R) - \text{depth}(M) \leq n - 1$.

Proof. The bound on $\text{G-dim}(M)$ follows from Theorem 5.8(ii): the conditions in (5.4) hold for $(\mathbb{T}_{n+1}M, N, \text{G-dim})$; see (5.5)(ii). Hence, since $R$ is an AB ring, it suffices to prove by [16, 3.2 and 6.1] that $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$.

As $\text{Ext}^n_R(M, N) = 0$ and $N$ is a Tor-rigid module, it follows from the exact sequence (1.1)(i) that $\text{Tor}^R_i(\mathbb{T}_{n+1}M, N) = 0$ for all $i \geq 1$. This shows, by [16, 3.2], that Tate Tor groups $\text{Tor}^R_i(\mathbb{T}_{n+1}M, N)$ vanish for all $i \in \mathbb{Z}$. Note that $\text{G-dim}(M) \leq n - 1$, and $\text{G-dim}(\Omega^n M) = 0$; see [2, 3.13]. We now use [5, 4.4.7] and conclude, for all $i \geq n + 1$, that:

$$\text{Ext}^i_R(M, N) \cong \text{Ext}^i_R(\Omega^n M, N) \cong \widehat{\text{Ext}}^{i-n}_R(\Omega^n M, N)$$

$$\cong \text{Tor}_{-i+n+1}^R(\text{Hom}(\Omega^n M, R), N)$$

$$\cong \text{Tor}_{-i+n+1}^R(\mathbb{T}_{n+1}M, N).$$

Therefore we have that $\text{Ext}^i_R(M, N) = 0$ for all $i \geq n$, and this proves our claim. \qed

Corollary 6.14. Let $R$ be a local AB ring, and let $M$ and $N$ be nonzero $R$-modules such that $N$ is Tor-rigid (e.g., $N = k$). Assume $\text{depth}(N) \leq \text{G-dim}(M)$. Then $\text{Ext}^i_R(M, N) \neq 0$ for all $i$, where $\text{depth}(N) \leq i \leq \text{G-dim}(M)$. In particular, if $\text{depth}(M) = 0$, then $\text{Ext}^i_R(M, N) \neq 0$ for all $i$, where $\text{depth}(N) \leq i \leq \text{dim}(R)$.

Proof. The first part is clear from Corollary 6.13. If $\text{depth}(M) = 0$, then $\text{G-dim}(M) = \text{dim}(R)$ so that the second part follows. \qed

If $R$ is a hypersurface (quotient of an equi-characteristic regular local ring) and $N$ is an $R$-module such that $\text{id}(N) < \infty$, then $\text{pd}(N) < \infty$ and hence it follows from a result of Lichtenabum [43, Theorem 3] that $N$ is Tor-rigid. Consequently, when $R$ is such a hypersurface, and $M$ and $N$ are nonzero $R$-modules such that $\text{pd}(M) < \infty$, $\text{depth}(M) = 0$ and $\text{id}(N) < \infty$, Corollary 6.14 implies that $\text{Ext}^i_R(M, N) \neq 0$ for all $i$, where $\text{depth}(N) \leq i \leq \text{dim}(R)$. Over certain hypersurfaces, we know that all modules are Tor-rigid so that the vanishing of $\text{Ext}^i_R(M, N)$ occurs without any restriction on $N$. For example, Corollary 6.14 and (A.8) yield:

Corollary 6.15. Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring over a perfect field $k$, and $f$ is a nonconstant polynomial in $Q$. Set $R = A_m$, where $m = (x_1, \ldots, x_{2s+1})A$ and assume $A_p$ is a regular for all $p \in \text{Spec}(A) - \{m\}$. If $M$ and $N$ are nonzero $R$-modules such that $\text{depth}(M) = 0$, then $\text{Ext}^i_R(M, N) \neq 0$ for all $i$, where $\text{depth}(N) \leq i \leq \text{dim}(R)$.

7. Gorenstein injective dimension of strongly-rigid modules

Let $(R, m, k)$ be a local ring. If $R$ is Gorenstein and the injective dimension $\text{id}(m)$ of the maximal ideal $m$ is finite, so is the injective dimension of $k$, and hence the Auslander-Buchsbaum formula implies that $R$ is regular; see [10, 3.1.26]. Our results in this section originated in an attempt to answer the following question:

Question 7.1. Let $(R, m)$ be a local ring. Assume $\text{id}(m) < \infty$. Then must $R$ be Gorenstein, or equivalently, must $R$ be regular?
Levin and Vasconcelos [12, Theorem 1.1] proved that $R$ is regular if $\text{pd}(mM) < \infty$ for an $R$-module $M$ with $mM \neq 0$. They also remarked that an argument analogous to that of [12, Theorem 1.1] would work just as well for finite injective dimension.

Avramov [17] pointed out that an affirmative answer to Question 7.1 came out in a discussion with himself and H.-B. Foxby in the summer of 1983. He also referred us to Lescot’s explicit computation of the Bass series of $m$ for an example of a published treatment of this fact; see [40, 1.8]. Avramov [7, Theorem 4] proved that any submodule $L$ of a finitely generated $R$-module $M$ satisfying $L \supseteq mM \supseteq mL$ has the same injective complexity and curvature as the residue field $k$. It follows, for example, if $m^nM \neq 0$ and $\text{id}(m^nM) < \infty$, then $R$ is regular; see also [7, Corollary 5] and the remark following it. A very special case of this result – the case where $n = 1$ and $M = R$, i.e., the case where $\text{id}(m) < \infty$ – also follows from (4.5).

The Gorenstein injective dimension, introduced by Enochs and Jenda [23], is a refinement of the classical injective dimension. We use Corollary 6.1 and prove that $R$ is Gorenstein if the Gorenstein injective dimension of an integrally closed $m$-primary ideal of $R$ is finite; see Corollary 7.6. This, in particular, refines Question 7.1 and establishes that $R$ is Gorenstein if and only if the Gorenstein injective dimension of the maximal ideal $m$ is finite. We proceed by recalling some definitions:

7.2. ([23]; see [15, 6.2.2]) An $R$-module $M$ is said to be Gorenstein injective if there is an exact sequence $I_\bullet = (\cdots \to I_1 \xrightarrow{\partial_1} I_0 \xrightarrow{\partial_0} I_{-1} \to \cdots)$ of injective $R$-modules such that $M \cong \ker(\partial_0)$ and $\text{Hom}_R(E, I_\bullet)$ is exact for any injective $R$-module $E$. The Gorenstein injective dimension of $M$, $\text{Gid}(M)$, is defined as the infimum of $n$ for which there exists an exact sequence $0 \to M \to I_0 \to \cdots \to I_{-n} \to 0$, where each $I_i$ is Gorenstein injective.

The Gorenstein injective dimension is a refinement of the classical injective dimension: $\text{Gid}(M) \leq \text{id}(M)$, with equality if $\text{id}(M) < \infty$; see [15, 6.2.6]. It follows that every module over a Gorenstein ring has finite Gorenstein injective dimension. Hence, if $(R, m)$ is Gorenstein, but not regular, then $\text{Gid}(k) < \infty = \text{id}(k)$.

Cohen-Macaulay local rings that admit a nonzero strongly-rigid module of finite Gorenstein injective dimension are Gorenstein.

**Proposition 7.3.** Let $R$ be a Cohen-Macaulay local ring. If $\text{Gid}(M) < \infty$ for some nonzero strongly-rigid $R$-module $M$, then $R$ is Gorenstein.

**Proof.** Assume $\text{Gid}(M) < \infty$ for some nonzero strongly-rigid $R$-module $M$. Then, since $R$ is Cohen-Macaulay, there exists a nonzero finitely generated $R$-module $N$ such that $\text{id}(N) < \infty$ and $\text{Ext}^i_R(N, M) = 0$ for all $i > 0$; see [26, 2.22]. Therefore Corollary 6.1(i) shows that $\text{pd}(N) < \infty$, i.e., $R$ is Gorenstein; see (4.4). \[\square\]

In general it is not known whether or not a local ring admitting a nonzero module of finite Gorenstein injective dimension must be Cohen-Macaulay; see [15]. Hence, in view of the foregoing result, it seems worth raising the following question:

**Question 7.4.** Let $R$ be a local ring and let $M$ be a nonzero strongly-rigid module over $R$. If $\text{Gid}(M) < \infty$, then must $R$ be Gorenstein?

We are able to give an affirmative answer to Question 7.4 when $M$ is an integrally closed $m$-primary ideal. For that we need the following result of Yassemi:
7.5. (Yassemi [57, 1.3]) Let $R$ be a local ring and let $M$ be a nonzero $R$-module. Assume $\text{Gid}(M) < \infty$. Assume further that $\dim(M) = \dim(R)$. Then $R$ is Cohen-Macaulay.

An integrally closed $m$-primary ideal $I$ of $(R, m)$ is a strongly-rigid module with $\dim(I) = \dim(R)$; see (7.2). So we deduce from Proposition 7.3 and (7.5) that:

**Corollary 7.6.** A local ring $(R, m)$ is Gorenstein if and only if $\text{Gid}(I) < \infty$ for some integrally closed $m$-primary ideal $I$ of $R$. In particular, $(R, m)$ is Gorenstein if and only if $\text{Gid}(m) < \infty$.

**Remark 7.7.** It is known that if $\text{Gid}(k) < \infty$ or $\text{Gid}(R) < \infty$, then $R$ is Gorenstein; see [15, 6.2.7] and [27, 2.1]. However we do not know whether the finiteness of $\text{Gid}(m)$ directly implies the finiteness of $\text{Gid}(k)$ or $\text{Gid}(R)$ via the short exact sequence $0 \to m \to R \to k \to 0$. Thus, as far as we know, Corollary 7.6 even for the special case where $I = m$, is new.

Our final aim is to show in Proposition 7.13 that the finiteness of $\text{C-Gid}(m)$ for a semidualizing module $C$ detects the dualizing module, i.e., forces $C$ to be dualizing. We first record a few preliminary results.

7.8. Let $C$ be a semidualizing $R$-module; see (4.10). Then the $C$-Gorenstein injective dimension $\text{C-Gid}(M)$ of a nonzero $R$-module $M$ can be defined as $\text{Gid}_{R \times C}(M)$, where $R \times C$ is the trivial extension of $R$ by $C$; see Henrik and Jørgensen [28, 2.16]. In particular, if $C = R$, then $\text{C-Gid}(M) = \text{Gid}(M)$; see (7.2).

Proposition 7.9 is used for our proof of Proposition 7.13 for the case where $R$ is Artinian; see also (2.4).

**Proposition 7.9.** Let $R$ be a Cohen-Macaulay local ring with a dualizing module and let $C$ be a semidualizing $R$-module. Assume $M$ is a nonzero strongly-rigid module over $R$. Assume further $\text{C-Gid}(M) < \infty$. Then $C$ is dualizing.

**Proof.** Note that $\text{Ext}^i_R(C, \omega) = 0$ for all $i \geq 1$, where $\omega$ is the dualizing module. Hence $X = R\text{Hom}_R(C, \omega) \simeq \text{Hom}_R(C, \omega)$ is a maximal Cohen-Macaulay $R$-module. It follows from [28, 4.6] that $M$ is in the Bass class of $R$ with respect to $X$. In particular, $\text{Ext}^i_R(X, M) = 0$ for all $i > 0$; see [28, 4.1]. Therefore Corollary 6.1(i) implies that $C \cong \omega$. \hfill $\square$

7.10. Let $M$ be a nonzero $R$-module and let $C$ be a semidualizing $R$-module. Assume $\text{C-Gid}(M) < \infty$. If $\dim(M) = \dim(R)$, then $\dim_{R \times C}(M) = \dim(R \times C)$ so that, by (7.6), $R$ is Cohen-Macaulay. Therefore, if $M$ is a strongly-rigid module, $\dim(M) = \dim(R)$ and $\text{depth}(R) = 0$, then $R$ is Artinian and it follows from Proposition 7.9 that $C$ is dualizing.

For the rest of our arguments, $\bar{X}$ denotes $X/xX$, where $X$ is an $R$-module and $x$ is a non-zerodivisor on $R$.

7.11. Let $(R, m)$ be a local ring and let $x \in m - m^2$ be a non-zerodivisor on $R$. Then the surjective $R$-linear map $f: m/xm \to m/xR$, given by $f(y + zm) = y + xR$ for all $y \in m$, splits; see, for example, the proof of [46, 19.2]. Therefore there exists an $R$-module $N$ such that $\bar{m} \cong N \oplus m/xR$.
Let $R$ be a local ring and let $x \in R$. Assume $x$ is a non-zerodivisor on $R$. Then $x$ is also a non-zerodivisor on $R \times C$, and hence the following holds:

\[(7.12.1) \quad R \times C \cong \overline{R} \times C\]

Let $M$ be an $R$-module. Assume $x$ is also a non-zerodivisor on $M$. Assume further $\text{C-Gid}(M) < \infty$ for some semidualizing $R$-module $C$. Then, in view of (7.12.1), we deduce from [53, Lemma 2] that:

\[(7.12.2) \quad \text{Gid}_{R \times C}(M) = \text{Gid}_{\overline{R} \times \overline{C}}(M) = \text{C-Gid}_{\overline{R}}(M) < \infty.\]

**Proposition 7.13.** Let $R$ be a local ring. Assume at least one of the following conditions holds:

(i) $\text{C-Gid}(m) < \infty$ for some semidualizing $R$-module $C$.

(ii) $\text{C-Gid}(M) < \infty$ for some maximal Cohen-Macaulay strongly-rigid module $M$.

Then $C$ is dualizing.

**Remark 7.14.** We already know from (7.10) that $R$ must be Cohen-Macaulay in case (i) or (ii) holds in Proposition 7.13.

**Proof of Proposition 7.13** We proceed by induction on $d = \text{depth } R$. If $d = 0$, then (7.10) gives the required conclusion for both case (i) and (ii). Hence we assume $d \geq 1$ and pick a non-zerodivisor $x$ on $R$ such that $x \in m - m^2$.

First suppose (i) holds, i.e., $\text{C-Gid}(m) < \infty$. It follows from (7.11) that there exists an $\overline{R}$-module $N$ such that $m \cong N \oplus m/xR$. Therefore we conclude from (7.12.2) that $\text{Gid}_{R \times C}(N \oplus m/xR) = \text{Gid}_{R \times C}(m) < \infty$. Then [26, 2.6] implies that $\text{Gid}_{R \times C}(m/xR) < \infty$. Hence we obtain the following; see also (7.10) and (7.12.1).

\[\text{Gid}_{R \times C}(m/xR) = \text{Gid}_{R \times C}(m/xR) = \text{C-Gid}_{\overline{R}}(m/xR) < \infty.\]

Now the induction hypothesis forces $\overline{C}$ to be dualizing over $\overline{R}$, i.e., $\text{id}_{\overline{R}}(\overline{C}) < \infty$. Consequently $\text{id}_{R}(C) < \infty$ and hence $C$ is dualizing over $R$.

Next assume (ii). Then it follows from (7.12.2) that $\text{C-Gid}_{\overline{R}}(M) < \infty$. Moreover, since $M$ is maximal Cohen-Macaulay, $x$ is a non-zerodivisor on $M$. We can easily observe, similar to [13, 2.2], that $M$ is strongly-rigid over $\overline{R}$; here we include an argument since [13] deals with only test modules; see also (2.2) and (2.4).

Suppose $\text{Tor}_n^\overline{R}(M, L) = 0$ for some $\overline{R}$-module $L$, and for some positive integer $n$. Then, since $M$ is a strongly-rigid module over $R$, and $\text{Tor}_n^R(M, L) \cong \text{Tor}_n^\overline{R}(M, L)$ for all $i \geq 0$, we see that $\text{pd}_R(L) < \infty$. Now the fact $x \notin m^2$ implies that $\text{pd}_R(L) < \infty$; see [4, 3.3.5(1)]. This proves that $M$ is strongly-rigid over $\overline{R}$. Thus the induction hypothesis implies that $\overline{C}$ is dualizing over $\overline{R}$, and so $C$ is dualizing over $R$. \qed

**Question 7.15.** Let $(R, m)$ be a local ring and let $C$ be a semidualizing $R$-module. If $\text{C-Gid}(M) < \infty$ for some nonzero strongly-rigid $R$-module $M$, then must $R$ be Cohen-Macaulay? What if $M$ is an integrally closed $m$-primary ideal?

The *Cohen-Macaulay injective dimension* $\text{CMid}(M)$ [29, 2.3] of an $R$-module $M$ is defined as $\inf \{ \text{C-Gid}(M) : C$ is a semidualizing $R$-module $\}$; see also (7.8). In view of this notation, Proposition 7.13 characterizes Cohen-Macaulay rings in terms of the finiteness of the Cohen-Macaulay injective dimension of the maximal ideal $m$, i.e., if $\text{CMid}(m) < \infty$, then $R$ is Cohen-Macaulay with a dualizing module $C$. 
Appendix A. Some Examples of Test and Rigid-Test Modules

There are quite a few examples of test and rigid-test modules in the literature. In this section we catalogue a few of them; see also [13, 1.4] for a characterization of test modules over complete intersection rings. We start by pointing out that test and Tor-rigid modules are distinct in general; see (2.1) and (2.2).

A.1. Let $S = k[[x, y, z]]$ be the formal power series over a field $k$, and let $R$ be the subring of $S$ generated by monomials of degree 2, that is, the 2nd Veronese subring. Then $S = R \oplus_R L$, where $L$ generates the class group of $R$; see, for example [13, 3.16]. Since $S$ is a finite extension of $R$, [14, 2.4] shows that $L$ is a test module over $R$. On the other hand, by Dao’s remark [20, 2.6], $L$ is not Tor-rigid.

Recall that all rigid-test modules are strongly-rigid; see (2.3) and (2.4).

A.2. If $I$ is an integrally closed $m$-primary ideal of $R$ and $\text{Tor}^R_i(R/I, N) = 0$, then $\text{pd}(N) \leq n - 1$, i.e., $\Omega^i(R/I)$ is a rigid-test module for all $i \geq 1$; see [18, 3.3].

A.3. Let $R$ be an $F$-finite local ring of prime characteristic $p$, and let $\varphi^n : R \to R$ be the $n$th iterate of the Frobenius endomorphism defined by $r \mapsto r^p$ for $r \in R$. If $M$ is a finitely generated $R$-module, $\varphi^n M$ denotes the (finitely generated) $R$-module $M$ with the $R$-action given by $r \cdot m = \varphi^n(r)m$.

(i) $\varphi^n R$ is a test module over $R$ for all $n \gg 0$; see (2.2) and [47, 2.2.8].
(ii) If $R$ is a one-dimensional Cohen-Macaulay local ring, then $\varphi^n R$ is a rigid-test module over $R$ for all $n \gg 0$; see (2.3) and [47, 2.1.3 and 2.1.4].
(iii) If $\mathfrak{m}^3 = 0$, or $R$ is a complete intersection ring, then $\varphi^n R$ is a rigid-test module over $R$ for all $n \geq 1$; see (2.3) and [47, 2.2.9, 2.2.10 and 5.1.1].

A.4. Let $R$ be a Golod ring (e.g., $R$ is a hypersurface). If $M$ is an $R$-module with $\text{pd}(M) = \infty$, then $M$ is a test-module over $R$; see [4, section 5] and [55, 3.1].

A.5. Let $(R, \mathfrak{m}, k)$ be a two-dimensional complete normal local domain with an algebraically closed residue field $k$. Assume $R$ has a rational singularity, i.e., there exists a resolution of singularities $X \to \text{Spec}(R)$, a proper birational morphism where $X$ is a regular scheme, such that $H^1(X, \mathcal{O}_X) = 0$; see [41, 6.32] or [44]. It follows that $R$ is a Cohen-Macaulay ring with minimal multiplicity; see, for example, [41, 6.36]. Thus $R$ is Golod [4, 5.2.8]. Hence each $R$-module $M$ with $\text{pd}(M) = \infty$ is a test module over $R$; see (A.4). In particular, if $R$ is not Gorenstein, then the dualizing module of $R$ is a test module.

A.6. Let $R = k[[x_1, \ldots, x_n]]/(f)$, where $k$ is a field and $n \geq 3$. Assume $R$ has an isolated singularity, i.e., $R_p$ is regular for all $p \in \text{Spec}(R) - \{\mathfrak{m}\}$. Let $M$ be an $R$-module. If $n = 3$ (i.e., $\dim(R) = 2$), or $\dim(M) \leq 1$ (e.g., $M$ has finite length), then $M$ is Tor-rigid; see [21, 2.10, 3.4 and 3.6].

A technical but rather important point for us is that a rigid-test module, unlike the residue field $k$, may have arbitrary depth and, even if its depth is zero, it does not have to have finite length in general. Here is such an example:

A.7. Let $k$ be a field, $R = k[[x, y, z]]/(yx - z^2)$ and $M = \mathfrak{m}/x\mathfrak{m}$. Then $M$ is a rigid-test module with $\text{depth}(M) = 0$ and $\dim(M) = 1$; see (A.4) and (A.0).

The next result, in view of Dao [21, 2.10], was already known when $f$ is homogeneous. Recently Walker [56] removed the homogeneity assumption; see [48] and also the paragraph preceding [56, 1.2].
A.8. Let $A = Q/(f)$, where $Q = k[x_1, \ldots, x_{2s+1}]$, with $s \geq 1$, is a polynomial ring (with standard grading) over a perfect field $k$, and $f$ is a nonconstant polynomial in $Q$. Set $R = A_m$, where $m = (x_1, \ldots, x_{2s+1})A$ and assume $A_p$ is a regular for all $p \in \text{Spec}(A) - \{m\}$. Then all $R$-modules are Tor-rigid; see [21, 2.10] and [50, 1.3].

**Appendix B. Remarks on Two-Dimensional Rational Singularities**

In this section we prove an extension of Corollary [13, 4.9] and compare our conclusion to a vanishing result obtained in [13]. Furthermore we improve a result of Iyama and Wemyss [33] over two-dimensional rational singularities; see (B.8).

B.1. Let $R$ be a rational singularity as in Corollary [1, 2] and let $M$ and $N$ be nonzero $R$-modules. Assume $N$ is Tor-rigid and $pd(N) = \infty$. Then it follows that $N$ is a rigid-test module; see (A.4) and (A.5). Therefore, if $\text{Ext}_R^n(M, N) = 0$ for some $n \geq \text{depth}(N)$, then Corollary [B.1] implies that $pd(M) \leq n - 1$.

If $R$ is as in Corollary [1, 2] any power of an integrally closed $m$-primary ideal is Tor-rigid, and has infinite projective dimension unless $R$ is regular; see the paragraph following Theorem [1.1] and (A.2). Therefore (B.1) subsumes Corollary [1, 2].

The following vanishing result is from [13]:

B.2. ([13, 4.9]) Let $R$ be a two-dimensional normal local domain such that the class group of $R$ is torsion. Let $M$ and $N$ be nonzero finitely generated $R$-modules. Assume $\text{Cl-dim}(M) < \infty$. Set $c = c_\text{cl}(M)$, the complexity of $M$ (e.g., $c$ is the codimension of $R$); see [4, 4.2]. If $\text{Ext}_R^n(M, N) = \cdots = \text{Ext}_R^{n+c-1}(M, N) = 0$ for some $n \geq 3 - \text{depth}(M)$, then $\text{Ext}_R^n(M, N) = 0$ for all $i \geq 3 - \text{depth}(M)$.

The result of Iyama and Wemyss, stated in (B.4), leads to the important result that the category of maximal Cohen-Macaulay $R$-modules does not contain any

B.3. Assume $R$ is complete and contains an algebraically closed field $k$ of characteristic zero. Assume further that $R$ is as in (B.2), or equivalently, $R$ is a rational singularity as in Corollary [1, 2]; see [13, 2.6].

Then $R$ is Gorenstein; see (A.5). Since $\text{Cl-dim}(M) < \infty$, it follows that either $pd(M) < \infty$, or $pd(M) = \infty$ and $R$ is a hypersurface that is quotient of a power series ring over $k$ (rational double point); see [4, 5.3.3(2)] and [11, 6.18 and 6.37]. If $pd(M) < \infty$, then the vanishing result stated in (B.2) is vacuous. So assume $pd(M) = \infty$ and that $R$ is a rational double point.

Suppose $pd(N) = \infty$. Then $N$ is a rigid-test module; see (A.4) and (A.6). Hence, if $\text{Ext}_R^n(M, N) = 0$ for some $n \geq 2$, then Corollary [6, 1] implies that $pd(M) < \infty$. Thus $pd(N) < \infty$ and $\text{Ext}_R^i(M, N) = 0$ for all $i \geq 3 - \text{depth}(M)$; see also [5, 4.7].

Next our aim is to improve the following result of Iyama and Wemyss:

B.4. ([Iyama and Wemyss [33, 3.1(1)]]) Let $R$ be a two-dimensional complete normal local domain such that $k$ is algebraically closed and has characteristic zero. Assume $R$ has a rational singularity. Assume further $R$ is not Gorenstein. If $X$ is a maximal Cohen-Macaulay $R$-module such that $\text{Ext}_R^n(X, R) = \text{Ext}_R^n(X, R) = 0$, then $X$ is free.
n-cluster tilting objects for \( n \geq 3 \); see [33, 4.15(3)] for details. In [B.8] we improve (B.4) and show that the vanishing of \( \text{Ext}^1_R(X, R) \) is not necessary, i.e., we prove that \( X \) is free provided that \( \text{Ext}^2_R(X, R) = 0 \).

The proof of (B.4), given in [33, 3.1(1)], relies upon a deep result of Christensen, Piepmeyer, Strüli and Takahashi [18, 4.3]. As we have already noted that two-dimensional rational singularities are Golod (A.3), we can reprove (B.4) by giving a short and straightforward argument without appealing to [18].

**B.5.** Assume \( R \) is as in (B.4) and let \( X \) be a maximal Cohen-Macaulay \( R \)-module such that \( \text{Ext}^1_R(X, R) = \text{Ext}^2_R(X, R) = 0 \). Iyama and Wemyss establishes in [33, 3.1(1)] that \( \text{Ext}^1_R(X, R) = 0 \) for all \( i \geq 1 \). We briefly explain their argument for completeness:

As \( \text{Ext}^1_R(X, R) = 0 \), one has \( \Omega X \cong X^* \) where \( X^* = \text{Hom}(M, R) \); see [33, 2.2]. Similarly, since \( \text{Ext}^2_R(X, R) = \text{Ext}^1_R(\Omega X, R) = \text{Ext}^1_R(X^*, R) = 0 \), it follows that \( \Omega X^* \cong X^{**} \). Notice \( X \) is reflexive; see, for example, [10, 1.4.1(b)]. Thus there are exact sequences \( 0 \rightarrow X^* \rightarrow F \rightarrow X \rightarrow 0 \) and \( 0 \rightarrow X \rightarrow G \rightarrow X^* \rightarrow 0 \), where \( F \) and \( G \) are free \( R \)-modules. Consequently, using dimension shifting along those exact sequences, one concludes that \( \text{Ext}^i_R(X, R) = 0 \) for all \( i \geq 1 \).

**B.6.** Let \( R \) be a local ring, \( M \) a nonzero test module and let \( N \) be a nonzero Cohen-Macaulay \( R \)-module. Assume \( \text{Ext}^i_R(M, N) = 0 \) for all \( i \gg 0 \). Then \( \text{id}(N) < \infty \).

**Proof.** We proceed by induction on \( \text{dim}(N) \). It follows from our assumption that \( \text{Tor}^R_i(M, \text{Hom}(N, E)) = 0 \) for all \( i \gg 0 \), where \( E \) is the injective hull of \( k \). Hence the case where \( \text{dim}(N) = 0 \) follows by Matlis duality. Next assume \( \text{dim}(N) \geq 1 \), and choose \( x \in R \) such that \( x \) is a non zero-divisor on \( N \). The short exact sequence \( 0 \rightarrow N \xrightarrow{x} N \rightarrow N/xN \rightarrow 0 \) induces the following long exact sequence:

\[
\cdots \rightarrow \text{Ext}^i_R(M, N) \xrightarrow{x} \text{Ext}^i_R(M, N) \rightarrow \text{Ext}^i_R(M, N/xN) \rightarrow \cdots
\]

Therefore \( \text{Ext}^i_R(M, N/xN) = 0 \) for all \( i \gg 0 \). It now follows from the induction hypothesis that \( \text{id}(N/xN) < \infty \). Hence, applying \( \text{Hom}_R(k, -) \) to the short exact sequence above, we see that \( \text{Ext}^i_R(k, N) = 0 \) for all \( i \gg 0 \), i.e., \( \text{id}(N) < \infty \). \( \square \)

**Alternate proofs of (B.4).** As \( R \) is Golod and not Gorenstein, the vanishing of \( \text{Ext}^i_R(X, R) \) for all \( i \geq 1 \) implies that \( \text{pd}(X) < \infty \); see [15, 5.5] and [37, 1.4]. This forces \( X \) to be free as it is a maximal Cohen-Macaulay \( R \)-module.

As an alternate argument, one can use (B.6): suppose \( X \) is not free. Then, as \( R \) is Golod, \( X \) is a test module; see (A.4). Moreover \( \text{Ext}^i_R(X, R) = 0 \) for all \( i \geq 1 \); see (B.4). It now follows from (B.6) that \( R \) is Gorenstein. Thus \( X \) must be free. \( \square \)

**B.7.** (Auslander and Bridger; see [2, 2.17 and 2.21]) Let \( R \) be a Noetherian ring and let \( X \) be a nonzero finitely generated \( R \)-module. Then there exists an exact sequence \( 0 \rightarrow F \rightarrow Y \rightarrow X \rightarrow 0 \) of finitely generated \( R \)-modules, where \( F \) is free, \( Y = \text{Tr} \Omega \text{Tr} X \) and \( \text{Ext}^i_R(Y, R) = 0 \); see (B.5).

**B.8.** Let \( R \) be a two-dimensional rational singularity as in (B.3). Assume \( X \) is a maximal Cohen-Macaulay \( R \)-module such that \( \text{Ext}^2_R(X, R) = 0 \). Then \( X \) is free.

**Proof.** There exists an exact sequence \( 0 \rightarrow F \rightarrow Y \rightarrow X \rightarrow 0 \), where \( F \) is free and \( Y = \text{Tr} \Omega \text{Tr} X \); see (B.7) and also (1.11). This shows that \( Y \) is maximal Cohen-Macaulay (since \( X \) is maximal Cohen-Macaulay) and induces the long exact sequence: \( \cdots \rightarrow \text{Ext}^2_R(X, R) \rightarrow \text{Ext}^2_R(Y, R) \rightarrow \text{Ext}^2_R(F, R) \rightarrow \cdots \). Hence
\( \text{Ext}_R^2(Y, R) = 0 \). Furthermore, by [2.17], we have that \( \text{Ext}_R^1(Y, R) = 0 \). Therefore \( \text{Ext}_R^2(Y, R) = \text{Ext}_R^1(Y, R) = 0 \). Applying (B.4), we conclude that \( Y \) is free. Consequently, since \( Y \) and \( F \) is free, \( X \) is free. \( \square \)

An immediate consequence of (B.8) is:

**B.9.** Let \( R \) be a rational singularity as in (B.4). If \( X \) is an \( R \)-module such that \( \text{Ext}_R^n(X, R) = 0 \) for some \( n \geq 4 \), then \( \text{pd}(X) < \infty \).

**Acknowledgments**

We are grateful to Luchezar Avramov, David Buchsbaum, Kamran Divaani-Aazar, Henrik Holm, Li Liang, Saeed Nasseh, Greg Piepmeyer, William Sanders, Alex Tchernev, Sean Sather-Wagstaff, Yang Zheng, Ryo Takahas hi and Mark E. Walker for valuable information and discussions related to this work during its preparation.

**References**

[1] M. Auslander. Modules over unramified regular local rings. *Illinois J. Math.*, 5:631–647, 1961.

[2] M. Auslander and M. Bridger. *Stable module theory*. Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969.

[3] Maurice Auslander. Modules over unramified regular local rings. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 230–233. Inst. Mittag-Leffler, Djursholm, 1963.

[4] L. L. Avramov. Infinite free resolutions. In *six lectures on commutative algebra (Bellaterra, 1996)*, volume 166 of *Progr. Math.*, pages 1–118. Birkhäuser, Basel, 1998.

[5] L. L. Avramov and R.-O. Buchweitz. Support varieties and cohomology over complete intersections. *Invent. Math.*, 142(2):285–318, 2000.

[6] L. L. Avramov, V. Gasharov, and I. Peeva. Complete intersection dimension. *Inst. Hautes Études Sci. Publ. Math.*, (86):67–114 (1998), 1997.

[7] Luchezar L. Avramov. Modules with extremal resolutions. *Math. Res. Lett.*, 3(3):319–328, 1996.

[8] Luchezar L. Avramov. Homological dimensions and related invariants of modules over local rings. In *Representations of algebra. Vol. I, II*, pages 1–39. Beijing Norm. Univ. Press, Beijing, 2002.

[9] Luchezar L. Avramov, Melvin Hochster, Srikant B. Iyengar, and Yongwei Yao. Homological invariants of modules over contracting endomorphisms. *Math. Ann.*, 353(2):275–291, 2012.

[10] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[11] O. Celikbas and H. Dao. Asymptotic behavior of Ext functors for modules of finite complete intersection dimension. *Math. Z.*, 269, 1005–1020, 2011.

[12] O. Celikbas and S. Sather-Wagstaff. Test complexes of finite Gorenstein dimension. preprint, 2014.

[13] Olgur Celikbas and Hailong Dao. Necessary conditions for the depth formula over Cohen-Macaulay local rings. *J. Pure Appl. Algebra*, 218(3):522–530, 2014.

[14] Olgur Celikbas, Hailong Dao, and Ryo Takahashi. Modules that detect finite homological dimensions. *Kyoto J. Math.*, 54(2):295–310, 2014.

[15] Lars Winther Christensen. *Gorenstein dimensions*, volume 1747 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[16] Lars Winther Christensen and David A. Jorgensen. Vanishing of Tate homology and depth formulas over local rings. *J. Pure Appl. Algebra*, 219(3):464–481, 2015.

[17] Private communication.

[18] Alberto Corso, Craig Huneke, Daniel Katz, and Wolmer V. Vasconcelos. Integral closure of ideals and annihilators of homology. In *Commutative algebra*, volume 244 of *Lect. Notes Pure Appl. Math.*, pages 33–48. Chapman & Hall/CRC, Boca Raton, FL, 2006.
[19] H. Dao. Some observations on local and projective hypersurfaces. Math. Res. Lett., 15(2):207–219, 2008. [11]

[20] Hailong Dao. Remarks on non-commutative crepant resolutions of complete intersections. Adv. Math., 224(3):1021–1030, 2010. [5, 20]

[21] Hailong Dao. Decent intersection and Tor-rigidity for modules over local hypersurfaces. Trans. Amer. Math. Soc., 365(6):2803–2821, 2013. [3, 20, 21]

[22] Hailong Dao, Jinjia Li, and Claudia Miller. Gorenstein injective and projective modules. Math. Z., 220(4):611–633, 1995. [6, 19]

[23] Edgar E. Enochs and Overtoun M. G. Jenda. Gorenstein injective and projective modules. Math. Z., 214(3):387–400, 1993. [2, 13, 17]

[24] E. G. Evans and P. Griffith. Syzygies, volume 106 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1985. [4, 12]

[25] Henrik Holm. Gorenstein homological dimensions. J. Pure Appl. Algebra, 189(1-3):167–193, 2004. [17, 19]

[26] Henrik Holm. Rings with finite Gorenstein injective dimension. Proc. Amer. Math. Soc., 132(5):1279–1283 (electronic), 2004. [15]

[27] Henrik Holm and Peter Jørgensen. Semi-dualizing modules and related Gorenstein homological dimensions. J. Pure Appl. Algebra, no. 2:423–445, 2006. [15]

[28] Henrik Holm and Peter Jørgensen. Cohen-Macaulay homological dimensions. Rend. Semin. Mat. Univ. Padova, 117:87–112, 2007. [19]

[29] C. Huneke and D. A. Jorgensen. Symmetry in the vanishing of Ext over Gorenstein rings. Math. Scand., 93(2):161–184, 2003. [11]

[30] David A. Jorgensen. A generalization of the Auslander-Buchsbaum formula. J. Pure Appl. Algebra, 144(2):145–155, 1999. [5, 20]

[31] David A. Jorgensen. Finite projective dimension and the vanishing of Ext. Comm. Algebra, 36(12):4461–4471, 2008. [5]

[32] David A. Jorgensen and Liana M. Sega. Nonvanishing cohomology and classes of Gorenstein rings. Adv. Math., 188(2):479–490, 2004. [11, 22]

[33] Osamu Iyama and Michael Wemyss. A new triangulated category for rational surface singularities. Illinois J. Math., 55(1):325–341 (2012), 2011. [3, 22, 22]

[34] M. I. Jinnah. Reflexive modules over regular local rings. Arch. Math. (Basel), 26(4):367–371, 1975. [11]

[35] David A. Jorgensen. A generalization of the Auslander-Buchsbaum formula. J. Pure Appl. Algebra, 144(2):145–155, 1999. [5, 20]

[36] David A. Jorgensen. Finite projective dimension and the vanishing of Ext. Math. Scand., 93(2):161–184, 2003. [11]

[37] C. Huneke and R. Wiegand. Tensor products of modules and the rigidity of Tor. Math. Ann., 299(3):449–476, 1994. [17]

[38] Craig Huneke and Irena Swanson. Integral closure of ideals, rings, and modules, volume 336 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006. [7]

[39] Osamu Iyama and Michael Wemyss. A new triangulated category for rational surface singularities. Illinois J. Math., 55(1):325–341 (2012), 2011. [3, 22, 22]

[40] M. I. Jinnah. Reflexive modules over regular local rings. Arch. Math. (Basel), 26(4):367–371, 1975. [11]

[41] David A. Jorgensen. A generalization of the Auslander-Buchsbaum formula. J. Pure Appl. Algebra, 144(2):145–155, 1999. [5, 20]

[42] David A. Jorgensen. Finite projective dimension and the vanishing of Ext. Comm. Algebra, 36(12):4461–4471, 2008. [5]

[43] David A. Jorgensen and Liana M. Sega. Nonvanishing cohomology and classes of Gorenstein rings. Adv. Math., 188(2):470–490, 2004. [11, 22]

[44] P. Jothilingam. A note on grade. Nagoya Math. J., 59:149–152, 1975. [7]

[45] P. Jothilingam. Syzygies and Ext. Math. Z., 188:278–282 (1985). 1985. [12]

[46] Jack Lescoat. La série de Bass d’un produit fibré d’anneaux locaux. In Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982), volume 1029 of Lecture Notes in Math., pages 120–152. Lecture Notes in Math., Vol. 311. Springer, Berlin, 1973. [3]
[46] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[47] Claudia Miller. The Frobenius endomorphism and homological dimensions. In *Commutative algebra (Grenoble/Lyon, 2001)*, volume 331 of *Contemp. Math.*, pages 207–234. Amer. Math. Soc., Providence, RI, 2003.

[48] W. F. Moore, G. Piepmeyer, S. Spiroff, and M.E. Walker. Hochster’s theta invariant and the Hodge-Riemann bilinear relations. *Adv. Math.*, no. 2:1692–1714, 2011.

[49] M. P. Murthy. Modules over regular local rings. *Illinois J. Math.*, 7:558–565, 1963.

[50] Saeed Nasseh, Masoud Tousi, and Siamak Yassemi. Characterization of modules of finite projective dimension via Frobenius functors. *Manuscripta Math.*, 130(4):425–431, 2009.

[51] P. Roberts. Two applications of dualizing complexes over local rings. *Ann. Sci. École Norm. Sup. (4)*, 9(1):103–106, 1976.

[52] A. Sadeghi. Vanishing of cohomology over complete intersection rings. *to appear in Glasgow Mathematical Journal; posted at arxiv:1210.5882*.

[53] Shokrollah Salarian, Sean Sather-Wagstaff, and Siamak Yassemi. Characterizing local rings via homological dimensions and regular sequences. *J. Pure Appl. Algebra*, 207(1):99–108, 2006.

[54] R. Takahashi and D. White. Homological aspects of semidualizing modules. *Math. Scand.*, 10.

[55] Ryo Takahashi. On G-regular local rings. *Comm. Algebra*, 36(12):4472–4491, 2008.

[56] M. E. Walker. Chern characters for twisted matrix factorizations and the vanishing of the higher Herbrand difference. *preprint, posted at arXiv:1404.0352*.

[57] Siamak Yassemi. A generalization of a theorem of Bass. *Comm. Algebra*, 35(1):249–251, 2007.

[58] Siamak Yassemi, Leila Khatami, and Tirdad Sharif. Grade and Gorenstein dimension. *Comm. Algebra*, 29(11):5085–5094, 2001.

[59] Ken-ichi Yoshida. Tensor products of perfect modules and maximal surjective Buchsbaum modules. *J. Pure Appl. Algebra*, 123(1-3):313–326, 1998.

[60] Y. Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings*, volume 146 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

Olgur Celikbas, Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: olgur.celikbas@uconn.edu

Mohsen Gheibi, Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA
E-mail address: mohsen.gheibi@gmail.com

Majid Rahro Zargar, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.
E-mail address: zargar9077@gmail.com

Arash Sadeghi, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.
E-mail address: sadeghiarash61@gmail.com