Distribution function

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Abstract

From a known result of diophantine equations of the first degree with 2 unknowns we simply find the results of the distribution function of the sequences of positive integers generated by the functions at the origin of the problems $3x + 1$ and $5x + 1$.

1 Introduction

We analyze and demonstrate the properties of the distribution (or density) function $F(k)$ introduced by Riho Terras [1] in 1976 and taken up by several authors including Lagarias [2]. The detailed analysis of the function does not solve the conjecture linked to the $3x+1$ problem, but it remains probably one of the biggest advances in this quest, without ever using notions of probability. In our opinion, the path used by Terras is the most complete. However, this requires 5 definitions, 11 theorems, 8 corollaries and 4 propositions. In our paper we only use three theorems to reach the same results and, in addition, we raise the conjecture advanced by Terras at the end of this paper concerning the stopping times.

Terras proves that this function is well defined and has very interesting properties. We use his reasoning until the remarkable result which appears with the theorem of periodicity. Thereafter we use a completely different path. Our reasoning is much simpler and leads to results which coincide with those expected. We believe that this exercise justifies by itself the use of diophantine equations for the solution of problems like the two treated in this paper, namely the problems $3x + 1$ and $5x + 1$.

2 Functions $T_3$ and $T_5$

Mappings can be define on integers represented by functions such that each element of the set $\mathbb{Z}$ is connected to a single element of this set. The iterative application of these functions produces a sequence of integers called trajectories.

Let

$$(n, f(n), f^{(2)}(n), f^{(3)}(n), \ldots, f^{(i)}(n), \ldots),$$

with $f^{(i+1)}(n) = f\{f^{(i)}(n)\}$, $i = 0, 1, 2, 3, \ldots$ and $f^{(0)}(n) = n$, a trajectory generates by a function $f$ on an integer $n$.

A sequence of integers forms a loop when there exists a number of iterations $k \geq 1$ such that
\[ f^{(k)}(n) = n. \]  

If all integers in the sequence are different two by two, we have by definition a cycle of length \( p = k \), so the trajectory \((n, f(n), f^{(2)}(n), f^{(3)}(n), \ldots, f^{(k-1)}(n))\). Generally, we note the trajectory characterizing a cycle starting with the smallest integer.

There are a multitude of functions that have these properties. The function \( g(n) \) giving rise to the original Collatz problem and the 3x + 1 function \( T_3(n) \) [2], the 5x + 1 function \( T_5(n) \) and the accelerated 3x + 1 function [2], are some examples. Except for the last function, the others come from a group called Generalized 3x + 1 Mappings [3].

The two functions dealt with in this paper are defined by

\[ T_{m_i}(n) \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod{2}, \\ \frac{m_i \cdot n + 1}{2}, & \text{if } n \equiv 1 \pmod{2} \end{cases} \]  

with \( m_3 = 3 \) for the 3x + 1 problem and \( m_5 = 5 \) for the 5x + 1 problem.

The general expression giving the result of \( k \) iterations of the function \( T_{m_i} \), which we will simply call \( T \), on an integer \( n \) is

\[ T^{(k)}(n) = \lambda_{k_1,k_2} n + \rho_{k}(n), \]  

where

\[ \lambda_{k_1,k_2} = \left( \frac{1}{2} \right)^{k_1} \left( \frac{m_i}{2} \right)^{k_2} \]  

and

\[ k = k_1 + k_2, \]  

with \( k_1 \) the number of transformations of the form \( n/2 \) and \( k_2 \), transformations of the form \((m_i n + 1)/2\).

Unlike parameter \( \lambda_{k_1,k_2} \), \( \rho_{k}(n) \) depend on the order of application of the transformations.

Let \( n \) and \( T^{(k)} \) be replaced by the variables \( x \) and \( y \),

\[ 2^k \rho_k(n) = 2^k y - 1^{k_1} m_i^{k_2} x. \]

In this form we have a diophantine equation of first degree at two unknowns,

\[ c = by - ax, \]  

where

\[ a = 1^{k_1} m_i^{k_2}, \quad b = 2^k \quad \text{and} \quad c = 2^k \rho_k(n). \]

From a well-known result of diophantine equations theory we have the theorem
Theorem 2.1 Let the diophantine equation \( c = by - ax \) of first degree at two unknowns. If the coefficients \( a \) and \( b \) of \( x \) and \( y \) are prime to one another (if they have no divisor other than 1 and \(-1\) in common), this equation admits a infinity of solutions to integer values. If \((x_0, y_0)\) is a specific solution, the general solution will be \((x = x_0 + bq, y = y_0 + aq)\), where \( q \) is any integer, positive, negative or zero.

Proof

References: on the web and [4]. ■

We may to assign to every integer of a trajectory generates by the function \( T(n) \) a number \( t_j = 0 \) if \( T^{(j)}(n) \) is even, and \( t_j = 1 \) if it is odd. Then, the iterative application of the function \( T \) to an integer \( n \) give a diadic sequence \( w_l = (t_0, t_1, t_2, t_3, \ldots, t_j, \ldots, t_{l-1}) \), with \( l \geq 1 \).

For a given length \( l \) there are \( 2^l \) different diadic sequences \( w_l \) of 0 and 1.

The representation of the trajectories in terms of \( t_j \) leads to an important theorem which makes it possible to bring out an intrinsic property, namely the periodicity. This property has already been observed by Terras [1] and Everett [6] concerning the process of iterations of the function \( T_3(n) \) generating the problem \( 3x + 1 \), and appears in a theorem which they have demonstrated by induction. We will prove it differently, using the previous theorem.

Theorem 2.2 All diadic sequences \( w_l \) of length \( l = k \geq 1 \) generated by any \( 2^k \) consecutive integers are different and are repeated periodically.

Proof

Let \( k = l \geq 1 \) the number of iterations applied to a given integer \( n \). The trajectories

\[
\begin{align*}
(T^{(0)}(n), T^{(1)}(n)) \\
(T^{(0)}(n), T^{(1)}(n), T^{(2)}(n)) \\
& \quad \cdots \\
(T^{(0)}(n), T^{(1)}(n)), \ldots, T^{(k)}(n))
\end{align*}
\]

correspond respectively to the diadic sequences

\[
\begin{align*}
w_1 &= (t_0) \\
w_2 &= (t_0, t_1) \\
& \quad \cdots \\
w_{l=k} &= (t_0, t_1, \ldots, t_{k-1}).
\end{align*}
\]

For a given number \( l \) we have \( 2^k \) different diadic sequences \( w_l \) possible.

According to theorem 2.1 each of the \( 2^k \) diadic sequences will be performed for \( k = l \). Indeed, the 0 and the 1 of these sequences correspond to the operations on the even and odd integers. We build \( 2^k \) different diophantine equations characterized by \( 2^k \) different combinations of the parameters \( a, b \) and \( c \), whose solutions will be given by \((x = x_0 + 2^k q, y = y_0 + m_{i^2} q)\). Therefore, all the integers \( x_0 + 2^k q \) starting a trajectory of length \( k + 1 \) correspond to the same sequence \( w_k \). In a sequence of \( 2^k \) consecutive integers, each integer must start a different sequence \( w_k \), otherwise the \( 2^k \) different diadic sequences will not be performed. ■

We will use another property of the diophantine equations generated by functions like \( T_3 \) and \( T_5 \).
Theorem 2.3 Let the trajectories of the integers (of length \( L \)) that are connected to each other by the operations \( n/2 \) or \((m,n+1)/2\). The diophantine equation connecting the first integer \( x \) and the last integer \( y \) of a sequence can be expressed in the general form \( c = by - ax \) where the parameters \( a, b, c \), always positive, depend on the operations themselves and in which orders they are applied.

If \( b > a \), \( x \geq y \) and, if \( b < a \), \( x < y \).

Proof
Let \( k_1, k_2 = 0, 1, 2, \ldots \) and \( k = k_1 + k_2 = L - 1 \), with \( L \geq 2 \).
Then, \( a = m_1^{k_2}, b = 2^k \) and \( c \geq 0 \).

As the factors \( a \) and \( b \) of \( x \) and \( y \) are prime to one another, the diophantine equation admits an infinity of solutions to integer values. If \((x_0, y_0)\) is a specific solution, the general solution will be \((x_0 + bq, y_0 + aq)\), where \( q \) is any integer, positive, negative or zero.

Two cases are possible, \( b > a \) or \( b < a \).

First case : \( b > a \)
Suppose that a particular solution \((x_0, y_0)\) is such that \( x_0 < y_0 \). We have the general solution
\[
y = y_0 + aq \quad \text{and} \quad x = x_0 + bq.
\]

As \( b > a \) and \( x_0 < y_0 \), beyond a certain value of \( q \), we will have \( x > y \). The equation
\[
c = by - ax,
\]
eventually lead to a negative \( c \) value. But, the parameter \( c \) must always be positive. Therefore
\[
x > y \quad \text{when} \quad b > a.
\]
If \( x = y \) then \( c = x(b - a) \). Since \( c \) must always be positive, then \( b > a \).

Second case : \( b < a \)
Let the equation
\[
c = by - ax.
\]

As \( c \) is always positive and \( b < a \), \( x \) must necessarily always be smaller than \( y \) (\( x < y \)).

For example, for \( L = 2 \), we have \( k = k_1 + k_2 = L - 1 = 1 \). Two cases are possible, \( k_1 = 1 \) and \( k_2 = 0 \) or, \( k_1 = 0 \) and \( k_2 = 1 \). Then, if \( m_1 = m_3 = 3 \), we have \( a = 3^1 = 3 \) or \( a = 3^0 = 1 \). Since \( b = 2^k = 2 \), we write the diophantine equations
\[
0 = 2y - x \quad \text{or} \quad 1 = 2y - 3x. \tag{8}
\]

where \((b = 2, a = 1) \ (b > a) \) in the first case and \((b = 2, a = 3) \ (b < a) \) in the other case.
The general solutions \((x, y)\) are respectively \((2 + 2q, 1 + 1q)\) with \((x > y)\), and \((1 + 2q, 2 + 3q)\) with \((x < y)\).

3 Distribution function \( F(k) \)

Let us define the distribution function \( F(k) \) as
\[
F(k) = \lim_{m \to \infty} (1/m) \mu\{n \leq m \mid \chi(n) \geq k\}, \tag{9}
\]
where \( \mu \) is the number of positive integers \( n \leq m \) with \( m \) that tends towards infinity. \( \chi(n) \) is called the "stopping time", and corresponds to the smallest positive integer such that the iterative application \((k \text{ times})\) of function \( T_3 \) (equation 2) on a integer \( n \) gives the result \( T_3^{(k)}n < n \).

Terras [1] proves that the distribution function \( F(k) \) is well defined for any value of \( k \) and that it tends towards 0 for \( k \) tending towards infinity.

Lagarias [2] redoes the demonstration using the function we will call \( G(k) \),

\[
G(k) = \lim_{x \to \infty} \frac{1}{x} \frac{1}{\sum n \leq x \text{ and } \sigma(n) \leq k},
\]

(10)

where \( \sigma(n) \) is the "stopping time". This function \( G(k) \) is in away almost the reciprocal of the function \( F(k) \), and tends towards 1 when \( k \) tends towards infinity. The properties inherent in these functions will be clarified in the following examples.

The application of theorem 2.2 on periodicity can be interpreted as follows.

Let \( k \) be a number of iterations applied to any \( 2^k \) consecutive integers. We will have all possible combinations \( 2^k \) of operations \( n/2 \) on the even integers and \( (3n + 1)/2 \) on the odd integers of the diadic sequences generated by the function \( T_3(n) \) and each combination appears only once. For a given \( k \), all the integers \( m \) of the form \( m = n + 2^k q \) will have the same combination of operations. The distribution of different combinations is then binomial versus the operations.

For example, let \( k = 1 \) and the \( 2^k = 2^1 = 2 \) consecutive positive integers 1 and 2. The trajectories of length \( k + 1 = 2 \) generated by the function \( T_3(n) \) will be

\[
(1, 2) (2, 1) (3, 5) (4, 2) (5, 8) (6, 3) \cdots,
\]

where we have added the numbers 3, 4, 5 and 6 after the two consecutive numbers 1 and 2 starting the trajectories, so as to bring out the periodicity.

If we use the diadic sequences of the 0 and 1 representing respectively the even and odd operations, we will have

\[
(1) (0) (1) (0) (1) (0) \cdots,
\]

all repeating periodically for every two consecutive trajectories. This result follows from the fact that the all integers alternate between the even and odd integers.

We have already writed the diophantine equations for \( k = 1 \) (equations 3) which give the first integer \( x \) of the trajectory versus the last integer (here the second).

In the first case we have all the trajectories starting with an even positive integer \( x \) and ending with a smaller integer \( y \) after 1 iteration. The stopping time is equal to the number of iterations \( k = 1, \) so \( \chi(n = \text{even}) = k = 1. \) In the second case we have all trajectories starting with an odd positive integer \( x \) and ending with a greater integer \( y \) after 1 iteration and, \( \chi(n = \text{odd}) > k. \) The stopping time meets the condition \( \chi(n) \geq k = 1 \) in two cases and all integers contribute to the distribution function \( F(k), \) so \( F(k = 1) = 1. \) Unlike Terras, we will not count the integers with \( \chi = k \) because in these cases, we have reached the condition \( T_3^{(k)}n < n. \) It will create a slight gap with the results of Terras. Then, the distribution function \( F(k) \) with \( \chi > k \) instead \( \chi \geq k \) really becomes the reciprocal of the function \( G(k) \) defined by Lagarias, and the new function \( F_{\text{new}}(k = 1) = 1/2. \) We write

\[
F_{\text{new}}(k) = \lim_{m \to \infty} \frac{1}{m} \mu \{ n \leq m \mid \chi(n) > k \}.
\]

(11)
Let another example. Take $k = 2$ and the $2^k = 2^2 = 4$ consecutive positive integers 3, 4, 5 and 6. The trajectories of length $k + 1 = 3$ generated by the function $T(n)$ will be

$$(3, 5, 8), (4, 2, 1), (5, 8, 4), (6, 3, 5), (7, 11, 17), (8, 4, 2), \ldots,$$

where we have added the numbers 7 and 8 after the four consecutive numbers 3, 4, 5 and 6 starting the trajectories, so as to bring out the periodicity.

The diadic sequences are

$$(1, 1), (0, 0), (1, 0), (0, 1), (1, 1), (0, 0), \ldots,$$

all repeating periodically for every four consecutive integers starting a trajectory.

We can write the $2^k = 2^2 = 4$ diophantine equations in the same way as before. But, we will do it differently here. In fact the diadic sequences and the theorem 2.3 we will help to deduce whether or not the stopping time is equal, greater or less than the number of iterations $k = 2$.

In the general case, the parameter $b = 2^k$ and the parameter $a = 3^{k_2} \cdot 1^{k_1} = 3^{k_2}$ with $k$ the total number of iterations, $k_1$ the number of operations on the even integers, and $k_2 = k - k_1$ the number of operations on the odd integers.

As the first two diadic sequences correspond to the trajectories starting with an even integer, we do not count them in $F(k)$. The third diadic sequence, so $(1, 0)$ which is generated by the integers $5 + 4q$, is such that $\chi(5 + 4q) = k = 2$. The fourth diadic sequence, so $(1, 1)$ which is generated by the integers $3 + 4q$, is such that $\chi(3 + 4q) > k = 2$. These data appear in the table 1. Then, the distribution function $F_{\text{new}}(k)$ with $\chi > k$ instead $\chi \geq k$ really becomes $F_{\text{new}}(k = 2) = 1/4$. The original function $F(k = 2)$ would correspond to $1/2$.

And so on for different values of the number of iterations $k$.

| diadic sequences | $k_1$ | $k_2$ | $b = 2^k$ | $a = 3^{k_2}$ | $b \ vs \ a$ | $x \ vs \ y$ | stopping time $\chi(n)$ |
|-----------------|-------|-------|------------|---------------|-------------|----------------|-----------------|
| $(0, 0)$        | 2     | 0     | 4          | 1             | $b > a$     | $x > y$        | $-$             |
| $(0, 1)$        | 1     | 1     | 4          | 3             | $b > a$     | $x > y$        | $-$             |
| $(1, 0)$        | 1     | 1     | 4          | 3             | $b > a$     | $x > y$        | $\chi = k$     |
| $(1, 1)$        | 0     | 2     | 4          | 9             | $b < a$     | $x < y$        | $\chi > k$     |

Table 1: Stopping time for $k = 2$

The number of different sequences is given by $b = 2^k$, and the number of different parameters $a$ is calculated by the binomial coefficients $\binom{k}{k_2}$. Binomial coefficients can be represented in a Pascal triangle (table 2).
| $k_2 \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | \ldots |
|------------------|----|--|--|--|--|--|--|--|--|--------|
| 0                | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | \ldots |
| 1                | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 8  | \ldots |
| 2                | 1  | 3  | 6  | 10 | 15 | 21 | 28 | 28 | 28 | \ldots |
| 3                | 1  | 4  | 10 | 20 | 35 | 56 | 56 | 56 | 56 | \ldots |
| 4                | 1  | 5  | 15 | 35 | 70 | 70 | 70 | 70 | 70 | \ldots |
| 5                | 1  | 6  | 21 | 56 | 56 | 56 | 56 | 56 | 56 | \ldots |
| 6                | 1  | 7  | 28 | 28 | 28 | 28 | 28 | 28 | 28 | \ldots |
| 7                | 1  | 8  | \ldots |
| 8                | 1  | \ldots |
| 9                | \ldots |

Table 2: Pascal triangle - Binomial coefficients

We use a similar table (table 3) which will contain the number of integers $n(i, j)$ by $2^k$ consecutive integers which satisfy the condition that the stopping time $\chi$ is greater than the number of iterations $k$. We have

| $k_2 \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | \ldots |
|------------------|----|--|--|--|--|--|--|--|--|--|--|--------|
| 0                | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 1                | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 2                | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 3                | 1  | 2  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 4                | 1  | 3  | 3  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 5                | 1  | 4  | 7  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | \ldots |
| 6                | 1  | 5  | 12 | 12 | 0  | \ldots |
| 7                | 1  | 6  | 18 | 30 | \ldots |
| 8                | 1  | 7  | 25 | \ldots |
| 9                | 1  | 8  | \ldots |
| 10               | \ldots |

Table 3: Pascal triangle - Number of integers $n(i, j)$ by $2^k$ consecutive integers with $\chi > k$

The index $j$ for the columns of the table is the exponent $k$ (the number of iterations) of 2 in the parameter $b = 2^k$. The index $i$ for the rows is the exponent $k_2$ of 3 in the parameter $a = 3^{k_2}$. As $k_2$ correspond to the number of operations on the odd integers, this value is in fact the number of 1 in the diadic sequences and varies of 0 to $k$. The various data in this table are calculated recursively.

The first data is trivial and indicates that all the integers satisfy the condition $\chi > k$ and this, because the number of iterations is $k = 0$. The case $k = 1$ has already been analyzed and we perform the following initialization, so $n(0, 1) = 0$ and $n(1, 1) = 1$. After 1 iteration, all positive even integer go to a smaller integer ($n = 0$) and, all positive odd integer go to a greater integer ($n = 1$).

From $k = 2$ we proceed recursively in the calculation of $n(i, k)$.

We use the principle that each sequence is generated so that the new parameter $b$ (for $k$) is the precedent (for $k - 1$) time 2, and the new parameter $a$ is the precedent time 1 or 3.
For example, for $k = 2$, we have two $n$ which precede (for $k = 1$), so $n(0, 1) = 0$ and $n(1, 1) = 1$. As $n(0, 1) = 0$, the sequences starting with $a$ even positive integer for $k = 2$ will not contribute to $F(k)$ and $n(0, 2) = 0$. On the other hand, the sequences generated by the integers with $n(1, 1) = 1$ can contribute to $n(1, 2)$ and $n(2, 2)$. The new parameter $b$ will be $b = 2 \cdot 2$ and the new parameter $a$ will be $a = 3 \cdot 1$ or $a = 3 \cdot 3$ (table 1). In the first case, $b > a$, $x \geq y$ and $\chi = k$. Then $n(1, 2) = 0$. In the second case, $b < a$, $x < y$ and $\chi > k$. Then $n(2, 2) = 1$. And so on for different values of $k$.

The sum on the index $i$ of $n(i, k)/2^k$ for a given $k$ gives the value of the distribution function $F_{\text{new}}(k)$ for this number of iterations $k$. Knowing that non-zero values must satisfy inequality $b < a$ ($x < y$), with $b = 2^k$ and $a = 3^{k_2}$, the sum begins with $i = k_2 > k\theta$,

$$F_{\text{new}}(k) = \sum_{i=0}^{k} \frac{n(i, k)}{2^k} = \sum_{i>k\theta}^{k} \frac{n(i, k)}{2^k}, \text{ with } \theta = \frac{\ln 2}{\ln 3} \approx 0.63093.$$  \hspace{1cm} (12)

It is then easy to build the computer programs starting from the recursive function worked out by Terras and by the previous process which makes it possible to fill the table 3. The results of these two programs are compiled in the table 4. For a given $k$, if we add $n(i, k - 1)$ to the sum in the equation 12 when $n(i, k) = 0$, we obtain exactly the same results as Terras.

We have also extended the programs to the distribution function $F_5(k)$ generated by the $5x + 1$ function $T_5$ (table 4).

It should not be forgotten that these exercises have never made it possible to exclude other cycles than the trivial cycle. On this last subject, we refer readers to the two papers [7, 8] preprint on arXiv.

| $k$ | $n$ | $k$ | $n$ |
|-----|-----|-----|-----|
| 10  | 7.4219 $\times 10^{-2}$ | 100 | 2.6396 $\times 10^{-4}$ |
| 20  | 2.8591 $\times 10^{-2}$ | 200 | 3.3187 $\times 10^{-6}$ |
| 30  | 1.1894 $\times 10^{-2}$ | 300 | 5.7714 $\times 10^{-8}$ |
| 40  | 6.5693 $\times 10^{-3}$ | 400 | 1.2191 $\times 10^{-9}$ |
| 50  | 3.3573 $\times 10^{-3}$ | 500 | 2.7866 $\times 10^{-11}$ |
| 60  | 1.9222 $\times 10^{-3}$ | 600 | 6.7168 $\times 10^{-13}$ |
| 70  | 1.1644 $\times 10^{-3}$ | 700 | 1.5719 $\times 10^{-14}$ |
| 80  | 7.0744 $\times 10^{-4}$ | 800 | 4.0963 $\times 10^{-16}$ |
| 90  | 4.1078 $\times 10^{-4}$ | 900 | 1.0837 $\times 10^{-17}$ |

Table 4: Distribution function $F_3(k)$

4 Property of the distribution function function $F(k)$

The inherent properties of the distribution function flow directly from the properties generated by the construction of the Pascal triangles. We recall the fact that the column number $j$ corresponds to the exponent $k$ of the parameter $b = 2^k$, so the number of iterations, and the row number $i$ to the exponent $k_2$ of the parameter $a = 3^{k_2}$ (problem $3x + 1$) or $a = 5^{k_2}$ (problem $5x + 1$), where $k_2$ is the number of transformations on odd integers. We have $k_2 = 0, 1, \ldots, k$. 

8
In the first Pascal triangle (table 2) we find the number of different possible transformations, that we will call BC(i, j) = BC(k2, k), on the even and odd integers while taking into account the order of application. These are the binomial coefficients. The concept of diadic sequences allows us to be easily determine all the combinations of transformations. The first data is trivial. We have two values for one iteration k = 1; BC(0, 1) = 1, corresponding to the diadic sequence (0) (transformation on the even integers) and, BC(1, 1) = 1 for the diadic sequence (1) (transformation on the odd integers). For two iterations (k = 2), the diadic sequences are (0, 0), (0, 1), (1, 0) and (1, 1), giving BC(0, 2) = 1, BC(1, 2) = 2 and BC(2, 2) = 1. As expected in a Pascal triangle, each value of BC(0, k) = 1 and, BC(i, k) = BC(i − 1, k − 1) + BC(i, k − 1) for i ≠ 0, so the sum of two previous terms.

In the second Pascal triangle (table 3) we used a similar table, with the same row and column numbering, which contain the number of integers n(i, j) by 2^k consecutive integers and satisfying the condition that the the stopping time χ is greater than the number of iterations k. The trivial first data for k = 0 indicates that all the integers satisfy to the condition χ > k. For k = 1, n(0, 1) = 0 and n(1, 1) = 1 meaning that after one iteration all positive even integers go to a smaller integer (χ = k = 1) and, all positive odd integers go to a greater integer (χ > k).

The other values in this table have the following properties.

If n(i, j) = 0 for a given combination i = k2 and j = k, then n(i, j) = 0 for i = k2 fixed and j > k. For example, n(0, 1) = 0 (b > a) implies that n(0, 2), n(0, 3), · · ·, equal to 0, because for each new value of k the parameter b is the previous one multiplied by 2. The parameter b = 2^k increases while the parameter a = 3^k2 = 3^0 remains constant, implying that b is always greater than a and x ≥ y by the theorem 2.3. Then χ < k and the new n(i, j) = 0.

If n(i, j) ≠ 0 for a given combination i = k2 and j = k, then n(i, j) ≠ 0 for i > k2 and j = k fixed. For example, n(6, 9) = 12 ≠ 0 (b < a) implies that n(7, 9), n(8, 9) and n(9, 9) are different from 0, because for each new value of k2 the parameter a is the previous one multiplied by 3. The parameter a = 3^k2 increases while the parameter b = 2^k = 2^0 remains constant, implying that b is always smaller than a and x < y by the theorem 2.3. Then χ > k and the new n(i, j) ≠ 0.

Now let’s look at the cases where n(i, j) = 0 and n(i + 1, j) ≠ 0. Then, n(i + 1, j + 1) = 0 or n(i + 1, j + 1) ≠ 0. The first condition implies that b/a = 2^k/3^i > 1 and the second condition implies that b/a = 2^k/3^{i+1} < 1. Then, we write

| k | Terras | new | k | Terras | new |
|---|--------|-----|---|--------|-----|
| 10 | 0.2734375 | 0.25976563 | 100 | 0.1808772 | 0.18060217 |
| 20 | 0.22122192 | 0.22122192 | 200 | 0.17688689 | 0.17685114 |
| 30 | 0.20572651 | 0.20572651 | 300 | 0.17622449 | 0.17621811 |
| 40 | 0.19784735 | 0.19625785 | 400 | 0.17572622 | 0.17562622 |
| 50 | 0.19116563 | 0.19116563 | 500 | 0.17602715 | 0.17602715 |
| 60 | 0.18811449 | 0.18811449 | 600 | 0.17603033 | 0.17603024 |
| 70 | 0.18573498 | 0.18513014 | 700 | 0.17604079 | 0.17604048 |
| 80 | 0.18317774 | 0.18317774 | 800 | 0.17607927 | 0.17607775 |
| 90 | 0.18192180 | 0.18192180 | 900 | 0.17622449 | 0.17621811 |

Table 5: Distribution function $F_5(k)$
The quotient \( b/a = 2^{k+1}/3^{i+1} \) for \( n(i+1,j+1) \) must meet the condition

\[
\frac{2}{3} \cdot \frac{1}{3} < \frac{2^k}{3^i} \cdot \frac{2}{3} \cdot \frac{1}{3} < 2.
\]

So, \( b > a \) or \( b < a \) and \( n(i+1,j+1) = 0 \) or \( n(i+1,j+1) \neq 0 \).

The quotient \( b/a = 2^{k+1}/3^{i+2} \) for \( n(i+2,j+1) \) must meet the condition

\[
\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} < 2.
\]

So, \( b < a \) and \( n(i+2,j+1) \neq 0 \).

We have \( F(k) \leq F(k-1) \). If the first non-zero value of \( n(i,k-1) \) for a given \( (k-1) \) is \( a \), the second \( b \), the third \( c \), \( \cdots \), and using the fact that \( n(i,k) = n(i-1,k-1) + n(i,k-1) \), we have

\[
F(k-1) = \frac{(a + b + c + \cdots)}{2^{k-1}}
\]

and,

\[
F(k) = \frac{((0 + a) + (a + b) + (b + c) + \cdots)}{2^k} = 2(a + b + c + \cdots) = \frac{2(a + b + c + \cdots)}{2 \cdot 2^{k-1}} = F(k-1)
\]

or

\[
F(k) = \frac{(0 + 0) + (a + b) + (b + c) + \cdots}{2^k} = \frac{2(a + b + c + \cdots) - a}{2 \cdot 2^{k-1}} = F(k-1) - \frac{a}{2^k} < F(k-1).
\]

The distribution of positive integers \( F(k) \), which can simply be called density, decreases constantly without, however, reaching the zero value.

Another interesting property in Pascal’s triangle (table 3) is the one related to \( n(k,k) \), and containing only the transformations on the odd integers. The first integer (the smallest) of a trajectory is \((2^k - 1, 3 \cdot 2^{k-1} - 1, 3^2 \cdot 2^{k-2} - 1, 3^3 \cdot 2^{k-3} - 1, \cdots, 3^k - 1)\).

According to the theorem 2.2 (periodicity), all integers \( 2^k - 1 + q \cdot 2^k \) start identical trajectories (containing only the transformations on the odd integers) of length \( k+1 \).

Finally, Terras defined a second stopping time \( \tau(n) = k \) if \( k > 0 \) is the smallest integer such \( \lambda_k(n) < 1 \), and states the following conjecture: “The stopping time relation \( \tau(n) = \chi(n) \) holds for all integers \( n \geq 2 \).” From the equations \( 4 \) and \( 7 \)

\[
\lambda_{k_1,k_2}(n) = \frac{a}{b}.
\]

By the theorem 2.3 if \( b > a, x \geq b \) and, if \( b < a, x < y \). The two stopping times are therefore interrelated. The conditions leading to a given conclusion will be the same. For example, \( \chi > k \) corresponds to \( b < a \) and \( \lambda_{k_1,k_2}(n) < 1 \).
5 Conclusion

The theorem on the periodicity bring out the regularity in the trajectories generated by the $3x + 1$ function $T_3(n)$, the $5x + 1$ function $T_5(n)$ and many other functions [7]. The application of this theorem to the various problems like those generated by the functions derived from generalized $3x + 1$ mappings is surely an essential key to the implementation of their solutions together with the basic properties of the diophantine equations.

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