Some Notes on Constructions of Binary Sequences with Optimal Autocorrelation

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Abstract

Constructions of binary sequences with low autocorrelation are considered in the paper. Based on recent progresses about this topic, several more general constructions of binary sequences with optimal autocorrelations and other low autocorrelations are presented.

Index Terms. sequences, interleaved method, optimal autocorrelation, almost difference set.

1 Introduction

Pseudo random sequences with low cross correlation have important applications in code-division multiple-access (CDMA) communications and cryptology. The pseudo random sequences employed in CDMA communications with low cross correlation may be separated from the others in the family and can successfully combat interference from the other users who share a common channel. On the other hand, the sequences with low cross correlation employed in either stream cipher cryptosystems as key stream generators or in digital signature algorithms as pseudo random number generators can resist correlation attacks.

Given two binary sequences $a = (a(t))$ and $b = (b(t))$ of period $N$, the periodic correlation between $a$ and $b$ is defined by

$$R_{ab}(\tau) = \sum_{t=0}^{N-1} (-1)^{a(t)+b(t+\tau)}, 0 \leq \tau < N$$ (1)
where the addition \( t + \tau \) is performed modulo \( N \). Define the symbol \( L^m(a) = \{a_i+m\} \). Then we have

**Lemma 1** Let \( m \) be an integer. Correlation of sequences satisfies the following properties:

1. \( R_{L^m(ab)}(\tau) = R_{ab}(\tau-m), \)
2. \( R_{aL^m(b)}(\tau) = R_{ab}(\tau+m), \)
3. \( R_{ab}(\tau) = R_{ab}(\tau+N) = R_{ba}(N-\tau), \)
4. \( R_{ab}(\tau) + R_{\bar{a}b}(\tau) = R_{ab}(\tau) + R_{\bar{a}b}(\tau) = 0. \)

If \( a = b \), \( R_{ab}(\tau) \) is called the (periodic) autocorrelation function of \( a \), denoted by \( R_a(\tau) \), or simply \( R(\tau) \) if the context is clear, otherwise, \( R_{ab}(\tau) \) is called the (periodic) cross-correlation function of \( a \) and \( b \). For the autocorrelation of the sequence \( a \), we have

**Lemma 2** Let \( m \) be an integer. \( R_{L^m(a)}(\tau) = R_{\bar{a}}(\tau) = R_a(\tau) \).

Let \( s = \{s_i\} \) denote a binary sequence of period \( N \). The set \( C_s = \{0 \leq t \leq N-1 : s(t) = 1\} \) is called the support of \( s \). If \( k = |C_s| \), then the periodic autocorrelation of \( s(t) \) can be given by

\[
R_s(\tau) = N - 4(k - |(\tau + C_s) \cap C_s|),
\]
where \( k = |C_s| \). The smallest possible values for the out-of-phase autocorrelation function of a binary sequence are listed below in Table 1 depending on value of \( N \) modulo 4 \[16,19\]. An autocorrelation function with one of those values is called the perfect autocorrelation.

The next smallest values for the out-of-phase autocorrelation of a binary sequence of period \( N \) is listed below in Table 2 \[16,19\], a sequence with one of those autocorrelation is called a sequence with optimal autocorrelation.

For more details about ideal sequences and optimal autocorrelation, the reader is referred to \[16,19,3,4,17\].

Let \( t = (t(0), t(1), t(2), t(3)) \) be a binary sequence of period 4, and the sequence

\[
u = I(a_0 + t(0), a_1 + t(1), a_2 + t(2), a_3 + t(3))\]

be an interleaved with \( a_i + t(i) \) as its column sequences. Thus \( u \) has period \( 4N \). Some known binary sequences with optimal autocorrelation are listed in the following Table 3 where \( L \) denotes the left shift operator and \( H(t) \) the hamming weight of the sequence \( t \).

This paper contributes to give a general construction different from the one in \[2\], which is a generalization of the construction in \[18\]. By inputting some perfect sequences and three pairs of sequences in \[2\] into our new construction respectively, several kinds of sequences with optimal autocorrelation or other low autocorrelation can be produced.
Table 1: Perfect Autocorrelation Values for $\tau \neq 0 \mod N$

| $N = i \mod 4$ | $R(\tau)$ | Comments |
|---------------|-----------|----------|
| 0             | $\{0\}$  | perfect sequence, only exists for $N = 4$, 0111, searched for $N < 108900$ |
| 1             | $\{1\}$  | corresponding to $(2u^2 + 2u + 1; u^2; u(u-1)/2)$ cyclic difference sets, exist for $u = 1$ and 2, not exist for $3 \leq u < 100$. |
| 2             | $\{2\}$ or $\{-2\}$ | $R(\tau) = 2$ does not exist for $N$ between 7 and 12545; $R(\tau) = -2$ only exists as the sequence 01 or 10. |
| 3             | $-1$      | idea 2-level autocorrelation sequences, corresponding to cyclic Hadamard difference sets. |

Table 2: Optimal Autocorrelation Values for $\tau \neq 0 \mod N$

| $N = i \mod 4$ | $R(\tau)$ | Comments |
|---------------|-----------|----------|
| 0             | $\{0, -4\}$ | Sidelnikov sequences of period $q - 1$, $q \equiv 1 \mod 4$, Arasu-Ding-Helleseth-Kumar-Martinsen sequences, and some interleaved sequences in Table 3 |
| 1             | $\{1, -3\}$ | Legendre sequences of period $p = 1 \mod 4$, Ding-Helleseth-Lam sequences of period $p$, $p = x^2 + 4$ and $p \equiv 1 \mod 4$, generalized cyclotomic sequences of period $p(p + 4)$ |
| 2             | $\{2, -2\}$ | Sidelnikov sequences of period $q - 1$, $q \equiv 3 \mod 4$, Ding-Helleseth-Martinsen sequences |
| 3             | $\{-1, 3\}$ | Cai-Ding sequences |
Table 3: Progressive process for finding \((a_0, a_1, a_2, a_3)\) for being equal to 
\((b_0, L^{1/4+s}(b_1), L^{1/2+s}(b_2), L^{3/4+s}(b_3))\)

| \((b_0, b_1, b_2, b_3)\) | \(R_{\alpha}(\tau)\) | Comments |
|-------------------------|-----------------|----------|
| \((a, a, a, a)\)        | \{0, -4\}       | **a** 2-level auto Arash *et al.* 2001 \(t = 0111\) [3]. This form obtained by Yul and Gong 2008, and \(s = 0\), product sequence [7]. |
| \((a', a', a, a)\)      | \{0, ±4\}       | **a** and \(a'\) are paired m-sequences, \(s = 0\) and \(t_0 + t_1 = 1\) by Yul and Gong, 2008, [7]. |
|                         |                 | The following cases are due to Tang and Gong, 2010, [2]. |
| \((a', a', a, a)\)      | \{0, ±4\}       | **a** and \(a'\) are paired GGMW or twin prime paired sequences. |
|                         |                 | **a** and \(a'\) are paired Legendre sequences [2], |
| \((a, a', a, a')\)      | \{0, -4\}       | \(t_0 = 0\), \((t_1, t_2, t_3) \in \{001, 111\}\), \(s = 0\) known before |
| \((a, a, a', a')\)      | \{0, ±4\}       | \(t_0 = 0\), \(H(t_1, t_2, t_3) = 1\) or 3 |
| \((a, a, a, a')\)       | \{0, ±4\}       | \(t_0 = 0\), \(H(t(1), t(2), t(3)) = 1\) or 3 |
| \((a, b, a, b)\)        | \{0, -4\}       | **a** and \(b\) are 2-level, \(s = 0\), and \(t = 0001\), Tang and Ding, 2010 [2]. Equivalently, it is also true for \(t = 0010, 1101\) or 1110. |
2 Three interleaved sequences and their modifications

Let \( s = (s(0), s(1), \cdots, s(N-1)) \) be a binary sequence of period \( N \) and can be denoted by the following interleaved constructions:

Construction A: \( s = I(0_K, a_1, a_2, \cdots, a_{T-1}) \) is a generalized GMW construction defined in [2], where

- (1) \( 0_K \) is an all zero sequence of period \( K \).
- (2) \( a_i, 1 \leq i \leq T-1 \), are shift equivalent and possess ideal autocorrelation.

By replacing the first column sequence \( 0_K \) in the sequence \( s \) with \( 1_K \), we get the following modified construction of Construction A:

Construction B: \( s' = I(1_K, a_1, a_2, \cdots, a_{T-1}) \).

Define \( d(a_i) = | \{ k | a_i(k) = 1, k = 0, 1, 2, \cdots, K-1 \} | - | \{ k | a_i(k) = 0, k = 0, 1, 2, \cdots, K-1 \} | . \)

Then we have obtained the following results:

Lemma 3 [6] For the sequences \( s \) and \( s' \),

\[
R_{s's'}(\tau) = R_{ss'}(\tau) \iff d(a_{T-\tau_2}) = d(a_{\tau_2}).
\]

Lemma 4 [2] Let \( s \) and \( s' \) be the sequences defined above, \( T = 2^n + 1, K = 2^n - 1 \).

\[
R_s(\tau) = \begin{cases} 
2^{2n} - 1 & \text{if } \tau = 0, \\
-1 & \text{otherwise}.
\end{cases}
\]

\[
R_{s'}(\tau) = \begin{cases} 
2^{2n} - 1 & \text{if } \tau = 0, \\
-1 & \text{if } \tau \equiv 0 \mod 2^n + 1, \tau \neq 0, \\
3 & \text{otherwise}.
\end{cases}
\]

\[
R_{s's'}(\tau) = R_{ss'}(\tau) = \begin{cases} 
2^{2n} - 2^n + 1 & \text{if } \tau = 0, \\
-2^{n+1} + 1 & \text{if } \tau \equiv 0 \mod 2^n + 1, \tau \neq 0, \\
1 & \text{otherwise}.
\end{cases}
\]

Let \( p \) be an odd prime and Legendre function

\[
\left( \frac{t}{p} \right) = \begin{cases} 
1 & \text{if } t \in QR_p, \\
-1 & \text{if } t \in NQR_p.
\end{cases}
\]

where \( QR_p \) and \( NQR_p \) denote the quadratic residue and nonquadratic residue of \( p \).

A Legendre sequence \( l(t) \) is defined as

\[
l(t) = \begin{cases} 
0 \text{ or } 1 & \text{if } t = 0, \\
\frac{1}{2}(1 - \left( \frac{t}{p} \right)) & \text{otherwise}.
\end{cases}
\]
l(t) is called the first type Legendre sequence if \( l(0) = 1 \) otherwise the second type Legendre sequence (denoted by \( l'(t) \)).

**Lemma 5** Legendre sequences \( l(t) \) and \( l'(t) \) possess the following autocorrelation.

If \( p \equiv 3 \mod 4 \), \( l(t) \) and \( l'(t) \) possess ideal autocorrelation.

If \( p \equiv 1 \mod 4 \),

\[
R_l(\tau) = \begin{cases} 
  p & \text{if } \tau = 0, \\
  1 & \text{if } \tau \in QR_p, \\
  -3 & \text{if } \tau \in NQR_p.
\end{cases}
\]

\[
R_{l'}(\tau) = \begin{cases} 
  p & \text{if } \tau = 0, \\
  -3 & \text{if } \tau \in QR_p, \\
  1 & \text{if } \tau \in NQR_p.
\end{cases}
\]

**Lemma 6** Legendre sequences \( l(t) \) and \( l'(t) \) possess the following crosscorrelation.

If \( N \equiv 1 \mod 4 \),

\[
R_{ll'}(\tau) = R_{l'l}(\tau) = \begin{cases} 
  N - 2 & \text{if } \tau = 0, \\
  -1 & \text{otherwise}.
\end{cases}
\]

If \( N \equiv 3 \mod 4 \),

\[
R_{ll'}(\tau) = \begin{cases} 
  N - 2 & \text{if } \tau = 0, \\
  1 & \text{if } \tau \in QR_p, \\
  -3 & \text{if } \tau \in NQR_p.
\end{cases}
\]

\[
R_{l'l} = \begin{cases} 
  N - 2 & \text{if } \tau = 0, \\
  -3 & \text{if } \tau \in QR_p, \\
  1 & \text{if } \tau \in NQR_p.
\end{cases}
\]

For the twin-prime sequence

\[
\mathbf{t} = I(0_p, L^{e_1}(a_1) + b(1), \cdots L^{e_{p+1}}(a_{p+1}) + b(p + 1))
\]

where \( e_i = i(p + 2)^{-1} \mod p \), \( p \) and \( p + 2 \) are two primes, \( b(i) = 1 \) if \( i \in QR_{p+2} \) otherwise \( b(i) = 0 \), and \( a_i = l' \) if \( i \in QR_{p+2} \) otherwise \( a_i = l, i = 1, 2, \cdots, p + 1 \). The modified type of the twin-prime sequence \( t \)

\[
\mathbf{t'} = I(1_p, L^{e_1}(a_1) + b(1), \cdots L^{e_{p+1}}(a_{p+1}) + b(p + 1))
\]
Lemma 7 The twin-prime sequence and its modified possess the following correlation properties.

\[
R_t(\tau) = \begin{cases} 
p(p + 2) & \text{if } \tau = 0, \\
-1 & \text{otherwise}.
\end{cases}
\]

\[
R_t'(\tau) = \begin{cases} 
p(p + 2) & \text{if } \tau = 0, \\
-1 & \text{if } \tau \equiv 0 \mod (p + 2), \tau \neq 0, \\
3 & \text{otherwise}.
\end{cases}
\]

\[
R_{tt'}(\tau) = \begin{cases} 
p^2 & \text{if } \tau = 0, \\
-2p - 1 & \text{if } \tau \equiv 0 \mod (p + 2), \tau \neq 0, \\
1 & \text{otherwise}.
\end{cases}
\]

3 Correlation Properties of Sequences in Two Difference Constructions

For two binary sequences \(a = \{a_i\}\) and \(b = \{b_i\}\) of period \(N\), a sequence \(u = \{u_j\} \triangleq \{a_i \parallel b_i\} = a \parallel b\) is defined as

\[
u_j = \begin{cases} 
a_i & \text{if } j = 2i, \\
b_i & \text{if } j = 2i + 1.
\end{cases}
\]

Define another two binary sequences \(c = \{c_i\}\) and \(d = \{d_i\}\) of period \(N\).

Lemma 8 The correlation

\[
R_{(a \parallel b)(c \parallel d)}(\tau) = \begin{cases} 
R_{ac}(\frac{\tau}{2}) + R_{bd}(\frac{\tau}{2}) & \text{if } \tau \text{ is even}, \\
R_{ad}(\frac{\tau - 1}{2}) + R_{bc}(\frac{\tau + 1}{2}) & \text{if } \tau \text{ is odd}.
\end{cases}
\] (4)

Assuming \(a = c\) and \(b = d\) in the above Lemma 8, then we get

Lemma 9 The autocorrelation of \(u = a \parallel b\)

\[
R_u(\tau) = \begin{cases} 
R_a(\frac{\tau}{2}) + R_b(\frac{\tau}{2}) & \text{if } \tau \text{ is even}, \\
R_{ab}(\frac{\tau - 1}{2}) + R_{ba}(\frac{\tau + 1}{2}) & \text{if } \tau \text{ is odd}.
\end{cases}
\] (5)

Specially, let \(b = L^m(a)\) in the above Lemma 9

Lemma 10 The autocorrelation of \(u = a \parallel L^m(a)\)

\[
R_u(\tau) = \begin{cases} 
2R_a(\frac{\tau}{2}) & \text{if } \tau \text{ is even}, \\
R_a(\frac{\tau - 1}{2} + m) + R_a(\frac{\tau + 1}{2} - m) & \text{if } \tau \text{ is odd}.
\end{cases}
\] (6)
If $N$ is odd and $m = \frac{N+1}{2}$, then the autocorrelation of $\mathbf{u} = \mathbf{a} \parallel L^m(\mathbf{a})$

$$R_u(\tau) = \begin{cases} 2R_a(\frac{\tau}{2}) & \text{if } \tau \text{ is even,} \\ 2R_a(\frac{\tau+N}{2}) & \text{if } \tau \text{ is odd.} \end{cases}$$ \quad (7)

Specially, let $\mathbf{b} = L^m(\bar{a})$ in the above Lemma 9.

**Lemma 11** The autocorrelation of $\mathbf{u} = \mathbf{a} \parallel L^m(\bar{a})$

$$R_u(\tau) = \begin{cases} 2R_a(\frac{\tau}{2}) & \text{if } \tau \text{ is even,} \\ -R_a\left(\frac{\tau-1}{2} + m\right) - R_a\left(\frac{\tau+1}{2} - m\right) & \text{if } \tau \text{ is odd.} \end{cases}$$ \quad (8)

If $N$ is odd and $m = \frac{N+1}{2}$, then the autocorrelation of $\mathbf{u} = \mathbf{a} \parallel L^m(\bar{a})$

$$R_u(\tau) = \begin{cases} 2R_a(\frac{\tau}{2}) & \text{if } \tau \text{ is even,} \\ -2R_a(\frac{\tau}{2}) & \text{if } \tau \text{ is odd.} \end{cases}$$ \quad (9)

**Lemma 12** The correlation

$$R_{(\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a}))(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))}(\tau) = R_{\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a})}(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))(\tau) = 0.$$ \quad (10)

**Proof.** If $\tau$ is even, then, from Lemmas 8 and 11

$$R_{(\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a}))(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))}(\tau) = R_{\mathbf{ab}}(\frac{\tau}{2}) + R_{L \frac{N+1}{2}(\bar{a}) L \frac{N+1}{2}(\mathbf{b})}(\tau)$$ \quad (11)

$$= R_{\mathbf{ab}}(\frac{\tau}{2}) + R_{\mathbf{ab}}(\frac{\tau}{2}) = 0.$$

If $\tau$ is odd, then, from Lemmas 8 and 11

$$R_{(\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a}))(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))}(\tau) = R_{\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a})}(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))(\tau)$$ \quad (12)

$$= R_{\mathbf{ab}}\left(\frac{\tau-1}{2} + \frac{N+1}{2}\right) + R_{\mathbf{ab}}\left(\frac{\tau+1}{2} - \frac{N+1}{2}\right)$$

$$= R_{\mathbf{ab}}\left(\frac{\tau+N}{2}\right) + R_{\mathbf{ab}}\left(\frac{\tau-N}{2}\right)$$

$$= 0.$$

Thus, from Equations 11 and 12,

$$R_{(\mathbf{a} \parallel L \frac{N+1}{2}(\bar{a}))(\mathbf{b} \parallel L \frac{N+1}{2}(\mathbf{b}))}(\tau) = 0.$$ \quad (13)
And, from Lemma 1 and Equation 13,

\[ R_{\| L^{N+1/2} (a) \| L^{N+1/2} (b) } (\tau) = R_{\| L^{N+1/2} (b) \| L^{N+1/2} (b) } (2N - \tau) = 0. \]

**Definition 1** Define a new binary interleaved sequence \( v \) as the following

\[ v(t) = (a, b, L^{N+1/2} (\bar{a}), L^{N+1/2} (b)). \]

where \( a \) and \( b \) are two binary sequences with length \( N, N \equiv 1, 3 \text{ mod } 4.\)

**Theorem 1** Let \( \tau = 4\tau_1 + \tau_2, 0 \leq \tau_2 \leq 3. \) Autocorrelation of the sequence \( v \) is

\[ R_v(\tau) = \begin{cases} 
2R_a(\tau_1) + 2R_b(\tau_1) & \text{if } \tau_2 = 0, \\
0 & \text{if } \tau_2 = 1, 3, \\
-2R_a(\tau_1 + \frac{N+1}{2}) + 2R_b(\tau_1 + \frac{N+1}{2}) & \text{if } \tau_2 = 2. 
\end{cases} \]

**Proof.** Actually, the sequence \( v \) can be seen as \( (a \parallel L^{N+1/2} (\bar{a}) \parallel (b \parallel L^{N+1/2} (b)). \) Thus, from Lemma 11,

\[ R_v(\tau) = \begin{cases} 
R_{\| L^{N+1/2} (a) \| L^{N+1/2} (b) } (\tau_2) & \text{if } \tau \text{ is even,} \\
R_{\| L^{N+1/2} (a) \| L^{N+1/2} (b) } (\tau_2) + R_{\| a \| L^{N+1/2} (b) \| L^{N+1/2} (a) } (\tau_2) & \text{if } \tau \text{ is odd.} 
\end{cases} \]

For the case \( \tau \) is even, if \( \tau_2 = 0, \) then \( \tau \) is even, from Lemmas 9, 10, 11, and 2,

\[ R_v(\tau) = R_a(\tau_1) + R_b(\tau_1) = 2R_a(\tau_1) + 2R_b(\tau_1). \]

if \( \tau_2 = 2, \) then \( \tau \) is odd, from Lemmas 8, 10, and 11,

\[ R_v(\tau) = -2R_a(\frac{4\tau_1 + 2 + 2N}{4}) + 2R_b(\frac{4\tau_1 + 2 + 2N}{4}) = -2R_a(\tau_1 + \frac{N+1}{2}) + 2R_b(\tau_1 + \frac{N+1}{2}). \]

For the case \( \tau \) is odd, from Lemma 12, \( R_v(\tau) = 0. \)

Based on Theorem 1, many binary sequences with low autocorrelation can be obtained as the following Theorems 2–6.

**Theorem 2** If \( N \equiv 3 \text{ mod } 4, \) then \( u(t) \) possess optimal autocorrelation \( R_u(\tau) \in \{4N, -4, 0\} \) if and only if \( a \) and \( b \) in Definition 2 are two binary sequences with ideal autocorrelation.
At this time, the autocorrelation function of $u(t)$

$$R_u(\tau) = \begin{cases} 
4N & \text{if } \tau = 0, \\
-4 & \text{if } \tau_2 = 0 \\
& \text{and } \tau \neq 0, \\
0 & \text{otherwise.}
\end{cases}$$

**Proof.** Sufficiency can be verified directly by Theorem 1. For the necessity, from Theorem 1 the statement $s$ possesses optimal autocorrelation $R_s(\tau) \in \{4N, -4, 0\}$ requires that $R_a, R_b \in \{0, \pm 2, \pm 1\}$. But, by Equation (2), neither $R_a$ nor $R_b$ of period $N \equiv 3 \mod 4$ can take values $0, 1, \pm 2$. Thus there exists only one choice that $R_a = R_b = -1$.

**Remark 1** It is easy to prove the above Theorem 2 is equivalent to the Construction B in [18] and is also true in the cases $t = 0001, 1101$ or $1110$. The corresponding almost different set has been given in Theorem 9 [18].

Similarly to the proof of the above Theorem 2 we can prove the following Theorem 3

**Theorem 3** If $N \equiv 1 \mod 4$, $u(t)$ possesses optimal autocorrelation $R_s \in \{4N, 4, 0\}$ if and only if $a$ and $b$ possess optimal autocorrelation $R_a, R_b \in \{N, 1\}$. At this time, the autocorrelation function of $u(t)$

$$R_u(\tau) = \begin{cases} 
4N & \text{if } \tau = 0, \\
4 & \text{if } \tau_2 = 0 \\
& \text{and } \tau \neq 0, \\
0 & \text{otherwise.}
\end{cases}$$

Let $a = 01000$, $b = 10000$, then $u(t) = 01101010001100000010$ possesses optimal autocorrelation $R_a \in \{4N, 4, 0\}$. If $b = 10000$ is replaced by its shift $b' = 00010$, then the corresponding $u'(t) = 0011010001001000010$ also possesses optimal autocorrelation in $\{4N, 4, 0\}$. Obviously, $u(t) \neq u'(t)$. But their supports are the same almost difference set $(20, 7, 3, 2)$. Since there only exist two known binary sequences possess optimal autocorrelation in $\{N, 1\}$, from which we can only get finite sequences with optimal autocorrelation in $\{4N, 4, 0\}$. But, as we know, these are only binary sequences with this type of optimal autocorrelation.

By Lemma 4 and Theorem 1, we have

**Theorem 4** If $s$ and $s'$ are generalized GMW sequence Construction A and its modified type in Construction B, and $(a, b) = (L^{n_1}(s), L^{n_2}(s'))$ (or $(L^{n_1}(s'), L^{n_2}(s))$, where $0 \leq n_1, n_2 \leq 2^{2n} - 2$, then $u(t)$
possesses autocorrelation as the following

\[
R_u(\tau) = \begin{cases}  
2^{2n+2} - 4 & \text{if } \tau = 0, \\
-4 & \text{if } \tau_1 \equiv 0 \text{ mod } 2^n + 1, \tau_2 \equiv 0 \text{ mod } 4, \tau \neq 0, \\
4 & \text{if } \tau_1 \neq 0 \text{ mod } 2^n + 1, \tau_2 \equiv 0 \text{ mod } 4, \\
0 & \text{if } \tau_2 \equiv 1, 3 \text{ mod } 4, \\
0 & \text{if } \tau_1 + 2^{2n-1} \equiv 0 \text{ mod } 2^n + 1, \tau_2 = 2, \\
8 (or -8) & \text{if } \tau_1 + 2^{2n-1} \not\equiv 0 \text{ mod } 2^n + 1, \tau_2 = 2,
\end{cases}
\]

Note that \( \frac{1}{2} = 2^{2n-1} \) in the ring \( \mathbb{Z}_{2^{2n}-1} \).

By Lemma 5 and Theorem 1, we have

**Theorem 5** Let \( l', l \) be two types of Legendre sequence respectively of period \( p, \) \( p \equiv 1 \mod 4, \) and \( (a, b) = (L^n(l'), L^n(l)) \) (or \( ((L^n(l), L^n(l'))) \)), where \( 0 \leq \eta_1, \eta_2 \leq p-1, \) then \( u(t) \) have autocorrelation as the following

\[
R_u(\tau) = \begin{cases}  
4p & \text{if } \tau = 0, \\
-4 & \text{if } \tau \neq 0, \tau_2 = 0, \\
0 & \text{if } \tau_2 = 1, 3, \\
0 & \text{if } \tau_1 + \frac{1}{2} = 0, \tau_2 = 2, \\
8 \text{ (or -8) } & \text{if } 0 \neq \tau_1 + \frac{1}{2} \in QR_p, \\
\text{ and } \tau_2 = 2, \\
-8 \text{ (or 8) } & \text{if } 0 \neq \tau_1 + \frac{1}{2} \in NQR_p, \\
\text{ and } \tau_2 = 2.
\end{cases}
\]

By Lemma 7 and Theorem 1, we have

**Theorem 6** Let \( t \) and \( t' \) be the twin-prime sequence and its modification, and \( (a, b) = (L^n(t), L^n(t')) \) (or \( ((L^n(t'), L^n(t))) \)), where \( 0 \leq \eta_1, \eta_2 \leq p(p+2) - 1, \) then the interleaved sequence \( u(t) \) have autocorrelation as the following

\[
R_u(\tau) = \begin{cases}  
4p(p+2) & \text{if } \tau = 0, \\
-4 & \text{if } \tau \neq 0, \tau_2 = 0, \tau_1 \equiv 0 \text{ mod } (p+2), \\
4 & \text{if } \tau_2 = 0, \tau_1 \not\equiv 0 \text{ mod } (p+2), \\
0 & \text{if } \tau_2 = 1, 3, \\
0 & \text{if } \tau_1 + \frac{1}{2} \equiv 0 \text{ mod } (p+2), \tau_2 = 2, \\
8 \text{ (or -8) } & \text{if } \tau_1 + \frac{1}{2} \not\equiv 0 \text{ mod } (p+2), \tau_2 = 2.
\end{cases}
\]
Definition 2 Define a new binary interleaved sequence \( w \) as the following

\[
w(t) = (a, L^n(a), b, L^n(b)).
\]

where \( a \) and \( b \) are two binary sequences with length \( N, N \equiv 1, 3 \mod 4 \).

Theorem 7 Let \( \tau = 4\tau_1 + \tau_2, 0 \leq \tau_2 \leq 3 \). Autocorrelation of the sequence \( w \) is

\[
R_w(\tau) = \begin{cases} 2R_a(\tau_1) + 2R_b(\tau_1) & \text{if } \tau_2 = 0, \\
-R_a(\tau_1 + \eta) + R_b(\tau_1 + \eta) & \text{if } \tau_2 = 1, \\
-R_{ab}(\tau_1 - \eta) + R_{ba}(\tau_1 - \eta) & \text{if } \tau_2 = 2, \\
0 & \text{if } \tau_2 = 3, \\
+R_{ab}(\tau_1 + \eta) - R_{ba}(\tau_1 + \eta) & \text{if } \tau_2 = 3. 
\end{cases}
\]

Proof. The sequence \( w \) can be seen as

\[
(a \parallel b) \parallel (L^n(a) \parallel L^n(b)) = (a \parallel b) \parallel (L^n(a) \parallel b)).
\]

Thus, from Lemma 10

\[
R_w(\tau) = \begin{cases} R_{a\parallel b}(\frac{\tau}{4}) + R_{L^n(\bar{a}\parallel\bar{b})}(\frac{\tau}{4}) & \text{if } \tau \text{ is even}, \\
R_{(a\parallel b)(L^n(\bar{a}\parallel\bar{b}))(a\parallel b)}(\frac{\tau}{4}) & \text{if } \tau \text{ is odd}. 
\end{cases}
\]

For the case \( \tau \) is even, if \( \tau_2 = 0 \), then \( \frac{\tau}{4} \) is even, from Lemmas 9 and 2 and Equation (14),

\[
R_w(\tau) = R_a(\frac{\tau}{4}) + R_b(\frac{\tau}{4}) + R_a(\frac{\tau}{4}) + R_b(\frac{\tau}{4})
\]

\[
= 2R_a(\tau_1) + 2R_b(\tau_1).
\]

if \( \tau_2 = 2 \), then \( \frac{\tau}{2} \) is odd, from Lemmas 2 and 1 and Equation (14),

\[
R_w(\tau) = R_{ab}(\frac{\tau - 2}{4}) + R_{ba}(\frac{\tau + 2}{4}) + R_{ab}(\frac{\tau - 2}{4}) + R_{ba}(\frac{\tau + 2}{4})
\]

\[
= 0.
\]

For the case \( \tau \) is odd, if \( \tau_2 = 1 \), then \( \frac{\tau - 1}{2} \) is even, from Lemmas 8 and 2 and Equation (14), we have

\[
R_w(\tau) = R_{aL^n(\bar{a})}(\frac{\tau - 1}{4}) + R_{bL^n(a)}(\frac{\tau - 1}{4}) + R_{L^n(\bar{a})b}(\frac{\tau - 1}{4}) + R_{L^n(\bar{b})a}(\frac{\tau + 3}{4})
\]

\[
= -R_a(\tau_1 + \eta) + R_b(\tau_1 + \eta) - R_{ab}(\tau_1 - \eta) + R_{ba}(\tau_1 + 1 - \eta).
\]

\[
= -R_a(\tau_1 + \eta) + R_b(\tau_1 + \eta) - R_{L^n(\bar{a})b}(\tau_1 + \frac{N + 1}{2} - \eta) + R_{ba}(\tau_1 + \frac{N + 1}{2} - \eta)
\]

\[
= -R_a(\tau_1 + \eta) + R_b(\tau_1 + \eta) - R_{ab}(\tau_1 - \eta) + R_{ba}(\tau_1 + \frac{N + 1}{2} - \eta).
\]
if $\tau_2 = 3$, then $\frac{\tau_2 - 1}{2}$ is odd, $\frac{\tau_2 + 1}{2}$ is even, from Lemmas 8 and Equation (13),

$$R_w(\tau) = R_{aL\eta(b)}(\frac{\tau - 3}{4}) + R_{bL\eta(ab)}(\frac{\tau + 1}{4}) + R_{L\eta(ab)}(\frac{\tau + 1}{4}) + R_{L\eta(b)}(\frac{\tau + 1}{4})$$

$$= R_{ab}(\frac{\tau - 3}{4} + \eta) - R_{ba}(\frac{\tau + 1}{4} + \eta) - R_{a}(\frac{\tau + 1}{4} - \eta) + R_{b}(\frac{\tau + 1}{4} - \eta)$$

$$= R_{L^{\frac{N+1}{2}}(a)b}(\tau_1 + \frac{N + 1}{2} + \eta) - R_{L^{\frac{N+1}{2}}(a)}(\tau_1 + \frac{N + 1}{2} + \eta) - R_{a}(\tau_1 + 1 - \eta) + R_{b}(\tau_1 + 1 - \eta)$$

$$= R_{ab}(\tau_1 + \eta) - R_{ba}(\tau_1 + \eta) - R_{a}(\tau_1 + 1 - \eta) + R_{b}(\tau_1 + 1 - \eta).$$

From Theorem 7, we can induce three ideals to decrease the autocorrelation of the sequence $w$:

Ia: If $R_a(\tau_1 + 1 - \eta) = R_{b}(\tau_1 + 1 - \eta)$, we decrease the value $|R_{ab}(\tau_1 + \eta) - R_{ba}(\tau_1 + \eta)|$.

Ib: If $R_{ab}(\tau_1 + \eta) = R_{ba}(\tau_1 + \eta)$, we decrease the value $|R_a(\tau_1 + 1 - \eta) - R_b(\tau_1 + 1 - \eta)|$.

Ic: Otherwise, we have to decrease the values of $|R_{ab}(\tau_1 + \eta) - R_{ba}(\tau_1 + \eta)|$ and $|R_{ab}(\tau_1 + \eta) - R_{ba}(\tau_1 + \eta)|$ analogously.

It is obvious that all these ideals are based on the fact that $2R_a(\tau_1) + 2R_b(\tau_1)$ for $\tau_2 = 0$ possess low values appropriately.

For the generalized GMW construction $s$ and its modifications $s'$, if $a = s, b = s'$, then, by the ideal Ib, we have the out-of-phase autocorrelation

$$R_w(\tau) = \begin{cases} -4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \equiv 0 \text{ mod } 2^n + 1, \\
4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \not\equiv 0 \text{ mod } 2^n + 1, \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 + \eta \equiv 0 \text{ mod } 2^n + 1, \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 + \eta \not\equiv 0 \text{ mod } 2^n + 1, \\
0 & \text{if } \tau_2 = 2, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \equiv 0 \text{ mod } 2^n + 1, \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \not\equiv 0 \text{ mod } 2^n + 1; \end{cases}$$

if $a = s', b = s$, then, by the ideal Ib, we have the out-of-phase autocorrelation

$$R_w(\tau) = \begin{cases} -4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \equiv 0 \text{ mod } 2^n + 1, \\
4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \not\equiv 0 \text{ mod } 2^n + 1, \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \equiv 0 \text{ mod } 2^n + 1, \\
-4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \not\equiv 0 \text{ mod } 2^n + 1, \\
0 & \text{if } \tau_2 = 2, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \equiv 0 \text{ mod } 2^n + 1, \\
-4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \not\equiv 0 \text{ mod } 2^n + 1. \end{cases}$$

For Legendre sequences $l(t)$ and $l_0(t)$ for $p \equiv 3 \text{ mod } 4$, if $a = l(t), b = l_0(t)$, then, by the ideal Ia, we have the out-of-phase autocorrelation of $w$.
\[
R_w(\tau) = \begin{cases} 
-4 & \text{if } \tau_2 = 0, \\
-4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \in QR_p, \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta = 0, \\
0 & \text{if } \tau_2 = 2, \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \in QR_p, \\
-4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta = 0. 
\end{cases}
\]

if \( a = l_0(t), b = l(t) \), then, by the ideal \( I_a \), we have the out-of-phase autocorrelation of \( w \)

\[
R_w(\tau) = \begin{cases} 
-4 & \text{if } \tau_2 = 0, \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \in QR_p, \\
-4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 2, \\
-4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \in QR_p, \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta = 0. 
\end{cases}
\]

For Legendre sequences \( l(t) \) and \( l_0(t) \) for \( p \equiv 1 \text{ mod } 4 \), if \( a = l(t), b = l_0(t) \), then, by the ideal \( I_b \), we have the out-of-phase autocorrelation of \( w \)

\[
R_w(\tau) = \begin{cases} 
-4 & \text{if } \tau_2 = 0, \\
-4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 + \eta \in QR_p, \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 + \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta = 0, \\
0 & \text{if } \tau_2 = 2, \\
-4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \in QR_p, \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta = 0. 
\end{cases}
\]

if \( a = l_0(t), b = l(t) \), then, by the ideal \( I_b \), we have the out-of-phase autocorrelation of \( w \)
For the twin-prime sequence $t$ and its modifications $t'$, if $a = t$, $b = t'$, then, by the ideal $I_b$, we have the out-of-phase autocorrelation

\[
R_w(\tau) = \begin{cases} 
-4 & \text{if } \tau_2 = 0, \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 \in QR_p, \\
-4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 + \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 - \eta = 0, \\
0 & \text{if } \tau_2 = 2, \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \in QR_p, \\
-4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \in NQR_p, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta = 0.
\end{cases}
\]

if $a = t'$, $b = t$, then, by the ideal $I_b$, we have the out-of-phase autocorrelation

\[
R_w(\tau) = \begin{cases} 
-4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \equiv 0 \mod (p + 2), \\
4 & \text{if } \tau_2 = 0 \text{ and } \tau_1 \not\equiv 0 \mod (p + 2), \\
0 & \text{if } \tau_2 = 1 \text{ and } \tau_1 \equiv 0 \mod (p + 2), \\
4 & \text{if } \tau_2 = 1 \text{ and } \tau_1 \not\equiv 0 \mod (p + 2), \\
0 & \text{if } \tau_2 = 2, \\
0 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \equiv 0 \equiv 0 \mod (p + 2), \\
4 & \text{if } \tau_2 = 3 \text{ and } \tau_1 + 1 - \eta \not\equiv 0 \mod (p + 2).
\end{cases}
\]

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