SUPERTROPICAL $\text{SL}_n$

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Abstract. Extending earlier work on supertropical adjoints and applying symmetrization, we provide a symmetric supertropical version $\text{SLS}_n$ of the special linear group $\text{SL}_n$, which we partially decompose into submonoids, based on "quasi-identity" matrices, and we display maximal sub-semigroups of $\text{SLS}_n$. We also study the monoid generated by $\text{SLS}_n$ and its natural submonoids. Several illustrative examples are given of unexpected behavior. We describe the action of elementary matrices on $\text{SLS}_n$, which enables one to connect different matrices in $\text{SLS}_n$, but in a weaker sense than the classical situation.

Introduction

This paper rounds out [20, 21], its main objective being to lay out the foundations of the theory of $\text{SL}_n$ in tropical linear algebra. Given any semiring $R$, one can define the matrix semiring, comprised of matrices $A = (a_{i,j})$ with entries in $R$, where the addition and multiplication of matrices are induced from $R$ as in the familiar ring-theoretic matrix construction.

The classical definition of $\text{GL}_n$ is the set of invertible matrices, which coincides with the set of nonsingular matrices. Then, the set $\text{SL}_n \subseteq \text{GL}_n$ is the set of matrices with determinant 1, in which case $A^{-1} = \text{adj}(A)$. In particular, this is the group generated by the elementary matrices $E_{i,j}$, which differ from the identity matrix by one nondiagonal nonzero entry in the $(i, j)$ position. These elementary matrices play a fundamental role in linear algebra and K-theory. Our basic goal is to find the tropical analog, containing the elementary matrices and preferably all matrices of determinant 1, which raises various difficulties. Tropical algebra is based on the max-plus algebra, for which negation does not exist and its underlying semiring structure is idempotent. For purposes of motivation, we consider matrices over an ordered semifield $F$ (i.e., $F \setminus \{0\}$ is a multiplicative group), such as the max-plus algebra $\mathbb{Q}_{\text{max}}$ or $\mathbb{R}_{\text{max}}$ (noting that in this case the multiplicative identity 1 is 0, and the additive identity 0 is $-\infty$). Later on we switch to the supertropical language, which is more convenient.

Invertibility of matrices (in its classical sense) is quite restricted in the (super)tropical setting. In view of Remark 2.1 below, the matrices $E_{i,j}$ are not invertible, and therefore do not generate any permutation matrices. Nevertheless, the matrices $E_{i,j}$ are tropically nonsingular (to be defined presently) of determinant 1. Applying a permutation matrix to a set of vectors in $F^n$ merely rearranges the coordinates, whereas applying a diagonal matrix rotates the rays, or thought another way, rescales the coordinates. From this point of view, $E_{i,j}$ has considerable geometric significance, and should be in any serious tropical version of $\text{SLS}_n$.

The classical determinant of $A = (a_{i,j})$ is no longer available in the tropical setting, due to its lack of negations. One of the challenges of tropical matrix theory has been to introduce a viable analog of the determinant, given these limitations. In [26], the determinant was defined as usual, using tropical operations and permutation signs. In [19], the permanent (called tropical determinant) was used as a
substitute, given as
\[ \det(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}, \]  
(0.1)
and formulated in [5] as the optimal assignment problem. This approach has roots going back to [27, 31], and [26] also studied the optimal assignment problem by means of the permanent.

Using the permanent leads to a corresponding definition of the adjoint matrix and the matrix (cf. [13])
\[ A^\mathcal{V} := \det(A)^{-1} \text{adj}(A), \]
and was used in [20, 21] to build a theory parallel to the classical theory. In particular, a matrix \( A \) is nonsingular if \( \det(A) \) is “tangible,” and these matrices are exactly those of full row rank, by [20, Corollary 6.6]. So one is led to define \( \text{SLS}_n \) to be the set of nonsingular matrices with determinant \( \mathbb{1} \), in which case \( A^\mathcal{V} = \text{adj}(A) \).

Although the supertropical language is not strictly needed for our definition of \( \text{SLS}_n \), it makes the statements easier, and “supertropical matrix theory” has led to results in linear algebra unavailable in other tropical versions, such as equality of matrix ranks, a natural analog of the Cayley-Hamilton theorem, solutions of eigenvalues, etc., as indicated in [18, 20, 21].

**Definition 0.1.** A matrix \( A = (a_{i,j}) \) is definite if the identity permutation is the unique dominant permutation in \([0,1]\), with \( a_{i,i} = 1 \) for all \( i \). If \( A \) is definite and \( a_{i,j} \leq \mathbb{1} \) for all \( i \neq j \), then \( A \) is normal. \( A \) is strictly normal when all these inequalities are strict.

All matrices in \( E_{i,j} \) are definite for all \( i, j \), and every definite matrix is in \( \text{SLS}_n \). The set \( \text{SLS}_n \) also contains all permutation matrices and all diagonal matrices of determinant \( \mathbb{1} \). It has long been known (see [31], for instance), that when \( A_1A_2 \) is nonsingular, then it has a unique dominating permutation and \( \det(A_1A_2) = \det(A_1)\det(A_2) \). Thus, a nonsingular product of two matrices in \( \text{SLS}_n \) is in \( \text{SLS}_n \), and a nonsingular product of two definite matrices is definite.

Unfortunately, \( \text{SLS}_n \) is no longer a group (or even a monoid), since it need not be closed under tropical matrix multiplication; for example, for non-definite matrices,
\[ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 + ab & a \\ b & 1 \end{pmatrix}. \]
(0.2)
Nevertheless, tropical matrix multiplication is closed if the multiplicands are strictly normal.

Thus, one of our main objectives is to study \( \text{SLS}_n \) via related monoids. We need a suitable monoid to work with, cf. Definition 0.3 in order to have a proper algebraic structure to progress with K-theory.

One can expand \( \text{SLS}_n \) a bit by means of an approach of Akian, Gaubert, and Guterman [2], and the Max-plus group [26]. They had already refined the determinant by distinguishing between the even and odd permutations in defining the bideterminant; also see [4]. A related approach is given in [3]. In this context, a matrix is (symmetrically) singular if
\[ \sum_{\text{odd } \pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} = \sum_{\text{even } \pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}. \]

This yields a symmetrized version of \( \text{SLS}_n \) in Definition 0.4 permitting symmetrically nonsingular matrices.

This also leads to a subtle distinction, since a singular matrix in the supertropical sense (which is a tropicalization of a singular matrix over a Puiseux series) could be nonsingular in the symmetrized sense.

Consider for example the singular Puiseux matrix \( A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ (2-i) & 0 & (i-2) \end{pmatrix} \). Although its tropicalization \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is symmetrically nonsingular over the max-plus algebra, it is supertropical singular. So one could be misled to a wrong interpretation without taking the supertropical structure into account. By [20, Theorem 3.5], \( \text{SLS}_n \) yields a monoid under “ghost surpasses,” whose subset of nonsingular elements is precisely \( \text{SLS}_n \). Our first goal then is to find the smallest natural monoid \( \overline{\text{SLS}}_n \) which contains \( \text{SLS}_n \), defined in Definition 0.3. \( \text{SLS}_2 \) is generated by \( \text{SLS}_2 \), but for \( n \geq 3 \), there are matrices in \( \overline{\text{SLS}}_n \) that are not factorizable, and in particular are not products of matrices from \( \text{SLS}_n \) (Corollary 0.6).

We also investigate \( \text{SLS}_n \) from within, by approaching four natural questions:
(i) What are the submonoids of $\text{SLS}_n$ contained in $\text{SLS}_n$? For example, what are the maximal such submonoids?

(ii) If $A \in \text{SLS}_n$, is it (2-sided) invertible in a suitable submonoid of $\text{SLS}_n$?

(iii) We observe that the set $\text{SLS}_n$ is too broad (not closed under multiplication), and that the set of invertible tropical matrices, the generalized permutation matrices, also called “monomials,” is too narrow (does not include $E_{i,j}$). Can we find a maximal monoid “between” them?

(iv) Where precisely between the maximal monoid of (iii) and $\text{SLS}_n$ do we lose multiplicativity?

Concerning (i), $\text{SLS}_n$ contains the important submonoid generated by the $E_{i,j}$. In Theorem 3.21 we determine this submonoid in terms of upper and lower triangular elementary matrices. $\text{SLS}_n$ itself has several obvious submonoids, such as the subgroup of generalized permutation matrices, and the upper triangular matrices. More interesting is Example 3.11(iii), which yields a maximal nonsingular submonoid, cf. Theorems 3.16 and 3.19 below, built from “strictly normal” matrices (cf. Definition 0.1).

Question (ii) is perhaps more intriguing, leading to various intricacies tied to concepts from [19, 20]. One of the more intriguing aspects of tropical algebra, is that the classical theory does not always pass to the tropical. Nonsingular matrices other than generalized permutation matrices cannot be invertible, but we can get inversion by replacing the identity matrix by a more general version. A quasi-identity matrix $T_A^\ell := AA^\triangledown$ has many properties of the identity matrix, being nonsingular idempotent with $\det(T_A^\ell) = 1$, even though the product of quasi-identity matrices need not be idempotent (Example 2.18).

Thus, it is natural to try to write $\text{SLS}_n$ as the union of monoids having unit element $T_A^\ell$ for various nonsingular matrices $A$. But we have an immediate obstacle: $T_A^\ell := A^\triangledown A$ might not equal $T_A^\ell$, cf. Example 2.28. This situation is remedied when $A$ is reversible, by which we mean $T_A^\ell T_A^r T_A^r = T_A^r T_A^r T_A^r$. Although this condition may look technical, it is satisfied whenever $T_A^\ell$ and $T_A^r$ commute, which occurs rather frequently, and is the most general condition that we know which leads to workable submonoids, in Theorem 3.21. It holds for $2 \times 2$ matrices when $T_A^\ell \in \text{SLS}_2$, cf. Example 2.29 but not for $3 \times 3$ matrices, cf. Example 2.30.

In view of (iii), perhaps the most interesting monoids arise via Definition 3.14 and Lemma 3.26. Definition 3.14 introduce $\text{SLS}_n^0$ as the set of normal matrices, up to products by monomial matrices, which is shown in Theorem 3.16 and Theorem 3.19 to be a maximal submonoid of $\text{SLS}_n$. That is, $\text{SLS}_n^0$ aims to the nonsingularity property of matrices, rather than their invertability, which provides a clear and natural approach to future study of tropical $\text{GL}_n$.

Lemma 3.26 defines for any $A \in \text{SLS}_n$ a sub-semigroup of $\text{SLS}_n$, with left unit element $T_A^\ell := A \text{adj}(A)$, which contains $T_A^\ell A$. This reflects the important role of quasi-identities $T_A^\ell$ in [21], and “almost” partitions $\text{SLS}_n$ naturally into a union of submonoids. We also consider the natural conjugation $B \mapsto A^\triangledown BA$ in [21]. Although some basic properties expected for conjugation fail in this setting, they do hold when $A$ is “strictly normal”.

In the last section we bring in the role of elementary matrices, which is rather subtle. In Lemma 5.2 we see that a Gaussian transformation can turn a nonsingular matrix into a singular matrix, which addresses point (iv). Then we show in Theorem 5.3 that although not every matrix in $\text{SLS}_n$ is itself a product of elementary matrices, all matrices in $\text{SLS}_n$ are equivalent with respect to multiplication by elementary matrices.

## 1. Supertropical structures

### 1.1. Supertropical semirings and semifields

**Definition 1.1.** A **supertropical semiring** is a quadruple $R := (R, T, G, \nu)$ where $R$ is a semiring, $T \subset R$ is a multiplicative submonoid, and $G_0 := G \cup \{0\} \subset R$ is an ordered semiring ideal, together with a map $\nu : R \to G_0$, satisfying $\nu^2 = \nu$ as well as the conditions:

$$a + b = \begin{cases} a, & \nu(a) > \nu(b), \\ \nu(a), & \nu(a) = \nu(b). \end{cases}$$

Note that $R$ contains the “absorbing” element 0, satisfying $a + 0 = a$ and $a0 = 0a = 0$ for all $a \in R$. The tropical theory works for $R \setminus \{0\}$, but it is convenient to assume the existence of 0 when working with matrices. We denote the multiplicative unit of $R$ (and $T$) as 1.
Interpretation: The monoid $T$ is called the monoid of **tangible elements**, while the elements of $G$ are called **ghost elements**, and $\nu : R \to G \cup \{ \emptyset \}$ is called the **ghost map**. Intuitively, the tangible elements correspond to the original max-plus algebra, although now $a + a = \nu(a)$ instead of $a + a = a$. The ideal $G_0$ could be identified with the max-plus algebra together with $-\infty$, but our main tropical interest is in the tangible elements, which under the extra conditions of Definition 1.3 below “cover” the ghost elements by means of the ghost map $\nu$.

We write $a^\nu$ for $\nu(a)$; $a \equiv_\nu b$ stands for $a^\nu = b^\nu$. We define the $\nu$-order on $R$ by

$$a \geq_\nu b \iff a^\nu \geq b^\nu \text{ and } a >_\nu b \iff a^\nu > b^\nu;$$

The **ghost surpassing relation** on $R$ is given by defining

$$a \not\geq_\nu g$$

Remark 1.2. We recall some basic properties concerning the ghost map, for $a, b \in R$ and $c \in T$:

(i) $(a + b)^\nu \geq a^\nu + b^\nu$;

(ii) $(ab)^\nu = a^\nu b = ab^\nu = a^\nu b^\nu$;

(iii) $a \geq_\nu bc^{-1}$ implies $ac \geq_\nu b$;

(iv) $a \geq_\nu b$ and $b \geq_\nu a$ implies $a \equiv_\nu b$;

(v) $c \not\geq_\nu a$ collapses to the standard equality $c = a$.

Definition 1.3. A supertropical semiring $R$ is a **supertropical semifield** when $T$ is an Abelian group, $R = T \cup G_0$, and the restriction $\nu|_T : T \to G$ is onto.

Example 1.4. Our main supertropical example is the **extended tropical semiring** (cf. [13]), that is,

$$R = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu,$$

with $T = \mathbb{R}$, $G = \mathbb{R}^\nu$, where the restriction of the ghost map $\nu|_T : \mathbb{R} \to \mathbb{R}^\nu$ is a natural isomorphism. Addition and multiplication are induced respectively by the maximum and standard summation of the real numbers [13]. This supertropical semifield extends the familiar max-plus semifield [1], and serves in all of our numerical examples, in **logarithmic notation** (in particular $1 = 0$ and $0 = -\infty$).

Remark 1.5. Gaubert [8], F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat [4], and Akian, Gaubert, and Guterman [2] Definition 4.1 introduced the “symmetrized semiring” which serves as a common generalization of [13] and their earlier work. This is a useful semiring, which is the additive monoid $\hat{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \times \mathbb{R} \cup \{-\infty\}$ (two copies of the max-plus algebra, taking $\nu$ to be the identity map), with multiplication given by $(a_1, a_2)(b_1, b_2) = (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1)$. It follows [2] Remark 4.5] that $G' := \{(a, a) : a \in \hat{\mathbb{R}}\}$ is an ideal of $\hat{\mathbb{R}}$.

**Lemma 1.6.** The extended tropical semiring $R$ of [13] is a homomorphic image of the “symmetrized” semiring $\hat{\mathbb{R}}$, under the map $(a, b) \mapsto a + b$. In fact, taking $T' = \{\emptyset, a \in \hat{\mathbb{R}}\}$, one sees that $T' + G'$ is a sub-semiring of $\hat{\mathbb{R}}$ mapping onto $R$, with $T' \mapsto T$ and $G' \mapsto G$.

**Proof.** All the verifications are easy, since $(a, a) \mapsto a + a = a^\nu$ and

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1) \mapsto a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1 = (a_1 + b_1)(a_2 + b_2).$$

□

Here one would identify $T$ with the first component of $\hat{\mathbb{R}}$, and $G$ with $G'$. This map is not 1:1, and there is no isomorphism from $\hat{\mathbb{R}}$ to $R$ (since the multiplicative monoid of $\hat{\mathbb{R}}$ is generated by $\{\emptyset\} \times \mathbb{R}$, whereas the multiplicative monoid of $R$ is the group $\mathbb{R} \times \mathbb{R}$). $G'$ behaves very similarly in $\hat{\mathbb{R}}$ to $G$ in $R$, as indicated in [2 Corollaries 4.18 and 4.19]. There are some significant differences, which justify utilizing the supertropical structure:

- The supertropical semiring also includes other important cases from the tropical theory, such as (nonarchimedean) valuations of the Puiseux series field $\mathbb{K}$, where $T = \mathbb{K}$, $G$ is the value group, and $\nu$ is the valuation.
• As noted in the introduction, linear independence of vectors is determined in \[20\] in terms of the supertropical structure, not the symmetrized structure.

• Factorization of supertropical polynomials corresponds to decompositions of affine varieties, \[19\].

2. Matrices

In this paper we fix a supertropical semifield \(F\), and work exclusively in the set \(\text{Mat}_n(F)\) of all \(n \times n\) matrices over \(F\). We consider \(\text{Mat}_n(F)\) as a multiplicative monoid, with matrix multiplication induced from the operations on \(F\). Its unit element is the identity matrix \(I\) with \(\mathbb{1}\) on the main diagonal and whose off-diagonal entries are 0. We say that a matrix is tangible if its entries are all in \(\mathcal{T} \cup \{0\}\), and ghost if its entries are all in \(G_0\). We write \(\text{Mat}_n(G_0)\) for the monoid of all ghost matrices. Also we rely implicitly on Remark \[12\] throughout the proofs of this section.

2.1. Supertropical singularity.

The tropical determinant of a matrix \(A = (a_{i,j})\) is defined as the permanent:

\[
\det(A) = \sum_{\pi \in S_n} \prod_{i=1}^n a_{i,\pi(i)},
\]

where \(S_n\) is the set of permutations of \(\{1, \ldots, n\}\).

Invertibility of matrices (in its classical sense) is limited in the (super)tropical setting.

Remark 2.1. The only invertible tropical matrices are the generalized permutation matrices, defined as the product of an invertible diagonal matrix and a permutation matrix \(P_\pi\), such that \((P_\pi)_{i,j} = 1\) when \(j = \pi(i)\), and 0 otherwise. This venerable result going back to \[29\] and \[6\]. (Note that \(P_\pi^{-1} = P_{\pi^{-1}}\). See \[7\] for a rather general version of this result.)

Thus, limiting nonsingularity to invertible matrices is too restrictive for a viable matrix theory, and leads to following definition.

Definition 2.2. We define a matrix \(A \in \text{Mat}_n(F)\) to be (supertropical) nonsingular if \(\det(A) \in \mathcal{T}\); otherwise \(A\) is (supertropical) singular (in which case \(\det(A) \in G_0\)).

Consequently, a matrix \(A \in \text{Mat}_n(F)\) is singular if \(\det(A) \not\in G^s\). This definition does not match the semigroup notion of regularity.

Given matrices \(A = (a_{i,j})\) and \(B = (b_{i,j})\) in \(\text{Mat}_n(F)\), we write \(B \geq_\nu A\) if \(b_{i,j} \geq_\nu a_{i,j}\) for all \(i, j\), and \(B \equiv_\nu A\) if \(B \geq_\nu A\) and \(B \leq_\nu A\). The ghost surpassing relation extends naturally to matrices, defined as \(A \gg B\) if \(A = B + G\) for some ghost matrix \(G \in \text{Mat}_n(G_0)\). (When \(A\) is tangible, \(\not\gg\) collapses to the standard equality \(A = B\).)

Lemma 2.3. If \(A_1 \geq_\nu A_2\) and \(B_1 \geq_\nu B_2\), then \(A_1 + B_1 \geq_\nu A_2 + B_2\) and \(A_1 B_1 \geq_\nu A_2 B_2\). In particular, \(AB \geq_\nu A\) and \(BA \geq_\nu A\) if \(B \geq_\nu I\).

Moreover, if \(A_1 \gg A_2\) and \(B_1 \gg B_2\), then \(A_1 + B_1 \gg A_2 + B_2\) and \(A_1 B_1 \gg A_2 B_2\), and in particular, \(AB \gg A\) and \(BA \gg A\) if \(B \gg I\).

Proof. Check the components in the multiplication. \(\square\)

2.2. Dominant permutations.

Definition 2.4. A permutation \(\pi \in S_n\) is dominant for \(A\) if \(\det(A) \equiv_\nu a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}\). A dominant permutation \(\pi\) is strictly dominant if \(\prod_i a_{i,\pi(i)} >_\nu \prod_i a_{i,\sigma(i)}\) for any \(\sigma \neq \pi\) in \(S_n\). A strictly dominant permutation \(\pi \in S_n\) is uniformly dominant if \(a_{1,\pi(1)} = a_{i,\pi(i)} \forall i\) and \(a_{i,j} <_\nu a_{i,\pi(i)} \forall j \neq \pi(i)\).

Clearly the matrix \(A\) is nonsingular if and only if it has a strictly dominant permutation, all of whose corresponding entries are tangible.
Example 2.5. The permutation \( \pi \) is uniformly dominant for the permutation matrix \( P_\pi \).

We specify some useful classes of matrices, to be used in the present paper, following the terminology of [5 §3]. (It is only one of several usages of the terminology “definite” in the literature.)

A strictly normal matrix (Definition 0.1) is always nonsingular, while a normal matrix (and thus also a definite matrix) can be singular. However, for any of these matrices we have \( \det(A) \cong_\nu 1 \).

Lemma 2.8. If \( A_i \) are uniformly dominant for matrices \( A_i \) for \( 1 \leq t \leq \ell \), then \( \pi := \pi_\ell \circ \ldots \circ \pi_1 \) is uniformly dominant for \( A = A_1 \cdots A_\ell \), and \( \det(A) = \prod_{t=1}^\ell \det(A_t) \).

Proof. If \( \alpha \) is the entry of \( A \) for its uniformly dominant permutation \( \pi \) (for \( 1 \leq i \leq n \)), then \( \det(A) = \alpha^n \). On the other hand, the matrix entries contributing to \( \det(A) \) are all of the form

\[
\alpha_{i_1, \pi_1(i_1)} \alpha_{i_2, \pi_2(i_2)} \cdots \alpha_{i_\ell, \pi_\ell(i_\ell)} = \alpha_1 \cdots \alpha_\ell,
\]

since all other entries are clearly less. Hence \( \pi := \pi_\ell \circ \ldots \circ \pi_1 \) is uniformly dominant for \( A \), and \( \det(A) = \alpha_1^n \cdots \alpha_\ell^n \).

Trying to weaken the hypothesis would bring us into confrontation with Proposition 3.18 below.

We recall the basic fact [28 Theorem 3.5] that \( \det(AB) \cong_{gs} \det(A) \det(B) \). As pointed out in [8 Proposition 2.17], this result can be seen by means of transfer principles ([2 Theorems 3.3 and 3.4]), and likewise is sharpened in [3 Corollary 4.18]; the basic idea already appears in [27]. We shall return to this issue in Theorem 2.9. We denote by \( S^* \) the subset of invertible matrices (in classical sense) in a set \( S \).

Proposition 2.7. \( \det(AB) = \det(A) \det(B) = \det(BA) \) whenever \( B \in \text{Mat}_n(R)^* \).

Proof. \( \det(AB) \cong_{gs} \det(A) \det(B) \), and

\[
\det(A) = \det(ABB^{-1}) \cong_{gs} \det(AB) \det(B^{-1}) = \det(AB) \det(B)^{-1}.
\]

Hence \( \det(A) \det(B) \cong_{gs} \det(AB) \), and thus \( \det(AB) = \det(A) \det(B) \). The proof that \( \det(BA) = \det(A) \det(B) \) is analogous.

In particular, this holds when \( B \) is a generalized permutation matrix.

2.3. Symmetrization.

Following [1] [26] and [2 Example 4.11], we define the \textit{symmetrized semiring} \( \hat{R} \), defined to have the same module structure as \( R \times R \), but with multiplication

\[
(a_1, a_2)(a_1', a_2') = (a_1 a_1' + a_2 a_2', a_1 a_2' + a_2 a_1')
\]

(motivated by viewing the first component to be in the “positive” copy of \( R \) and the second component to be in the negative copy). Define \( R^o = \{(a_1, a_2) \in R : a_1 \cong_\nu a_2 \} \), easily seen to be an ideal of \( \hat{R} \). Then one defines \( (a_1, a_2) \cong_o (b_1, b_2) \) in \( \hat{R} \) if there are \( c_1 \in R \) with \( a_i \cong_\nu c_2 \), such that \( a_i = b_i + c_i \) for \( i = 1, 2 \).

In other words, \( (a_1, a_2) = (b_1, b_2) + (c_1, c_2) \) where \( (c_1, c_2) \in R^o \).

Lemma 2.8. If \( (a_1, a_2) \cong_o (b_1, b_2) \), then \( a_1 + a_2 \cong_{gs} b_1 + b_2 \).

Proof. If \( a_i = b_i + c_i \), then \( a_1 + a_2 = b_1 + b_2 + c'_1 \), and hence \( a_1 + a_2 \cong_{gs} b_1 + b_2 \). \( \Box \)

2.3.1. The bideterminant and symmetric singularity. One defines

\[
\det^+(A) = \sum_{\pi \in S_n: \text{sgn}(\pi) = 1} \prod_{i=1}^n a_{i, \pi(i)}, \quad \det^-(A) = \sum_{\pi \in S_n: \text{sgn}(\pi) = -1} \prod_{i=1}^n a_{i, \pi(i)},
\]

and the \textit{bideterminant} \( \text{bidet}(A) = (\det^+(A), \det^-(A)) \). Note that \( \det(A) = \det^+(A) + \det^-(A) \).

Gaubert proved [8 Proposition 2.1.7]:
Proposition 2.9. bidet(AB) ≥₀ bidet(A) bidet(B).

(This is seen most readily by means of what Akian, Gaubert, and Guterman [2] call the strong transfer principle.) This result leads us to a more refined definition of “nonsingular matrix.”

Definition 2.10. A matrix \( A \in \text{Mat}_n(R) \) is **symmetrically singular** if \( \text{bidet}(A) \in G^\circ_0 \).

Lemma 2.11. Every symmetrically singular matrix is singular.

**Proof.** \( \text{bidet}(A) \in G^\circ_0 \) implies \( \det(A) \in G_0 \), since both components are equal. □

But a singular matrix \( A \) with tangible entries can be symmetrically nonsingular, viz. \( A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \).

2.4. The adjoint matrix.

As in the classical theory of matrices over a field, the adjoint matrix is defined over any semiring, and has a major role in supertropical matrix algebra, as noted in [20, 21].

Definition 2.12. The \((i, j)\)-minor \( A_{i,j} \) of a matrix \( A = (a_{i,j}) \) is obtained by deleting the \( i \)-th row and the \( j \)-th column of \( A \). The adjoint matrix \( \text{adj}(A) \) of \( A \) is defined as \( \left( a'_{i,j} \right) \), where \( a'_{i,j} = \det(A_{j,i}) \).

Proposition 2.13 ([20, Proposition 4.8]). \( \text{adj}(AB) \mid_{gs} = \text{adj}(B) \text{adj}(A) \) for any \( A, B \in \text{Mat}_n(F) \).

One need not have equality, as indicated in [20, Example 4.7], but by [25, Lemma 5.7], \( \text{adj}(AB) = \text{adj}(B) \text{adj}(A) \text{adj}(B) = \text{adj}(A) \text{adj}(B) \) for any generalized permutation matrix \( B \).

This result leads us to the question as to whether \( AB \) nonsingular implies \( BA \) is nonsingular. But this fails (cf. [14, Remark 2.12.]), even when \( B = A^t \), the transpose matrix:

Example 2.14. (Inspired by an idea of Guy Blachar.)

Let \( A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 2 \end{pmatrix} \), given in logarithmic notation (cf. Example 1.4). Then \( A^t = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \) and \( AA^t = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \) which is nonsingular of determinant 10, whereas \( A^tA = \begin{pmatrix} 4 & 6 \\ 6 & 8 \end{pmatrix} \) is singular of determinant \( 12^\circ \) (even symmetrically singular of determinant \( 12^\circ \)).

2.5. Quasi-identity matrices and the \( \nabla \)-operation.

Since so few matrices are invertible, we need to replace the identity matrix by a more general notion.

Definition 2.15. A matrix \( E \) is (multiplicatively) idempotent if \( E^2 = E \). A **quasi-identity matrix** is a nonsingular idempotent matrix.

Remark 2.16. The fact that a quasi-identity matrix \( I \) is idempotent implies that it is definite, with its off-diagonal entries in \( G_0 \), so this matches [20, Definition 4.1].

We define the set of all quasi-identity matrices

\[ QI_n(F) := \{ \text{Quasi-identity matrices} \} \subset \text{Mat}_n(F) , \]

each simulating the role of the identity matrix.

Remark 2.17 ([20, Proposition 4.17]). \( \text{adj}(I) = I \) for every quasi-identity matrix \( I \).

On the other hand, \( QI_n(F) \) is not a monoid.

Example 2.18.

Let \( I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b^\nu \\ 0 & 0 & 1 \end{pmatrix} \), \( I_2 = \begin{pmatrix} 1 & a^\nu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) with \( a^\nu, b^\nu \neq 0 \). These matrices are quasi-identities, but

\[ I_1I_2 = \begin{pmatrix} 1 & a^\nu & 0 \\ 0 & 1 & b^\nu \\ 0 & 0 & 1 \end{pmatrix} \]

is not idempotent, even though it is nonsingular.
Our next task is to find a matrix to replace the inverse. As in classical theory, the following matrices have an important role in supertropical matrix algebra.

**Definition 2.19.** When $\det(A) \in T$, we define the matrix

$$A^\triangledown := \det(A)^{-1} \text{adj}(A).$$

Let us collect some information about $A^\triangledown$. We write $A_{\triangledown}^\triangledown$ for $(A^\triangledown)^\triangledown$.

**Lemma 2.20.**

(i) \cite[Theorem 4.9]{20}. $A^\triangledown$ is nonsingular if $A$ is nonsingular.

(ii) $A^\triangledown = A^{-1}$ if $A$ is invertible (in view of Remark 2.27).

(iii) \cite[Proposition 4.17]{21}. $I^\triangledown = I$ for every quasi-identity matrix $I$.

(iv) \cite[Lemma 2.17]{21}. $\det(A) \text{adj}(A) \geq \nu \text{adj}(A)A \text{adj}(A)$ for any matrix $A \in \text{Mat}_n(F)$, and thus $A^{\triangledown} \geq \nu A^V A A^V$ for $A$ nonsingular.

(v) \cite[Remark 4.2]{21}, \cite[Theorem 3.5]{24}, \cite[Example 4.16]{20}. $A^{\triangledown} \vartriangleright A$, but $A^{\triangledown} \vartriangleright A$ in general.

(vi) \cite[Remark 2.18]{21}. $A^\triangledown$ is definite (resp. strictly normal) whenever the matrix $A$ is definite (resp. strictly normal).

**Definition 2.21.** For any nonsingular $A \in \text{Mat}_n(F)$, we define

$$I_A^\triangledown := AA^\triangledown, \quad I_A^{\triangledown} := A^\triangledown A.$$

The following facts are crucial.

**Theorem 2.22.**

(i) \cite[Theorem 4.12]{20}. $I_A^\triangledown$ and $I_A^{\triangledown}$ are idempotent (although not necessarily equal).

(ii) \cite[Remark 2.21]{21}. $\det(I_A^\triangledown) \geq \nu 1$ and $\det(I_A^{\triangledown}) \geq \nu 1$.

(iii) \cite[Theorem 4.3]{20}. $I_A^\triangledown$ and $I_A^{\triangledown}$ are quasi-identities.

(iv) \cite[Corollary 4.7]{21}. $I_A^{\triangledown} = I_A^\triangledown = I_A^{\triangledown} = I_A^\triangledown$.

In this way, $A^\triangledown$ could be called a “right quasi-inverse” with respect to $I_A^\triangledown$, and a “left quasi-inverse” with respect to $I_A^{\triangledown}$. This raises the major question, “What is the relation between $I_A^\triangledown$ and $I_A^{\triangledown}$?”

**Lemma 2.23.** $I_A^\triangledown = I_A^{\triangledown}$ for any definite matrix $A$.

**Proof.** By Lemma 2.20, we have

$$I_A^\triangledown = (AA^\triangledown)^2 = AA^\triangledown A A^\triangledown \geq \nu AA^\triangledown A \geq \nu A^\triangledown A = I_A^{\triangledown},$$

and, by symmetry, $I_A^{\triangledown} \geq \nu I_A^\triangledown$, implying that $I_A^\triangledown \cong \nu I_A^{\triangledown}$. The off-diagonal entries of $I_A^\triangledown$ and $I_A^{\triangledown}$ are the same (since they are all ghosts), whereas the diagonal entries are $1$; thus $I_A^\triangledown = I_A^{\triangledown}$.

Even when $I_A^\triangledown \neq I_A^{\triangledown}$, there is one nice situation worth mentioning.

**Definition 2.24.** For any nonsingular matrix $A$, we define

$$I_A = I_A^\triangledown I_A^{\triangledown} I_A, \quad \widetilde{I}_A = I_A^\triangledown I_A^{\triangledown} I_A.$$  

We say that $A$ is **reversible** if $I_A = \widetilde{I}_A$.

**Lemma 2.25.** $I_A = \widetilde{I}_A^{\triangledown}.$

**Proof.** $I_A^{\triangledown} I_A^{\triangledown} I_A = I_A^{\triangledown} I_A^{\triangledown} I_A^{\triangledown} \widetilde{I}_A^{\triangledown} = I_A^{\triangledown}.$

Reversibility gains interest from the following result.

**Proposition 2.26.** If $A$ is reversible and $I_A$ is nonsingular, then $I_A$ is a quasi-identity.

**Proof.** $I_A$ is idempotent since

$$I_A^2 = I_A^{\triangledown} I_A^{\triangledown} I_A^{\triangledown} I_A = (I_A^{\triangledown} I_A^{\triangledown} I_A^{\triangledown}) I_A = (I_A^{\triangledown} I_A^{\triangledown} I_A^{\triangledown}) I_A = I_A^{\triangledown} I_A^{\triangledown} I_A^{\triangledown} = I_A.$$  

and $\det(I_A) = \det(I_A^{\triangledown}) \det(I_A^{\triangledown}) \geq \nu \frac{1}{3} \cdot \frac{1}{1} = 1$ by Theorem 2.22.

\qed
**Proposition 2.27.** If $\mathcal{I}_A^t \mathcal{I}_A = \mathcal{I}_A^t \mathcal{I}_A$, then $A$ is reversible, and $\mathcal{I}_A = \mathcal{I}_A^t \mathcal{I}_A$.

**Proof.** $\mathcal{I}_A^t \mathcal{I}_A \mathcal{I}_A^t = \mathcal{I}_A^t \mathcal{I}_A \mathcal{I}_A^t = \mathcal{I}_A^t \mathcal{I}_A = \mathcal{I}_A^t \mathcal{I}_A = \mathcal{I}_A^t \mathcal{I}_A^t \mathcal{I}_A$. \hfill $\square$

But here is an example, obtained by modifying an example from [13] showing the complexity of the situation in general.

**Example 2.28.** (logarithmic notation)

Take $A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ whose determinant is $0 (= 1)$, whereas $A^2 = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ is singular. We have $A^\nu = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, and the quasi-identity matrices

$$\mathcal{I}_A^t = A A^\nu = \begin{pmatrix} 0 & (-2)^\nu \\ 1 & 0 \end{pmatrix}, \quad \mathcal{I}_A = A^\nu A = \begin{pmatrix} 0 & 0^\nu \\ -1 & 0 \end{pmatrix}.$$  

We see that

$$\mathcal{I}_A^t \mathcal{I}_A = \begin{pmatrix} 0 & 0^\nu \\ 1 & 1^\nu \end{pmatrix} \neq \begin{pmatrix} 1^\nu & 0^\nu \\ 1^\nu & 0 \end{pmatrix} = \mathcal{I}_A^t \mathcal{I}_A.$$  

Here is the general situation for $2 \times 2$ matrices, in algebraic notation.

**Example 2.29.** Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose determinant $ad + bc$ is $1$. Then $A^\nu = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$, and we get the quasi-identity matrices

$$\mathcal{I}_A = A A^\nu = \begin{pmatrix} 1 & (ab)^\nu \\ (cd)^\nu & 1 \end{pmatrix} \neq \mathcal{I}_A^t = A^\nu A = \begin{pmatrix} 1 & (bd)^\nu \\ (ac)^\nu & 1 \end{pmatrix}.$$  

We see that

$$\mathcal{I}_A^t \mathcal{I}_A = \begin{pmatrix} 1 + (a^2 bc)^\nu & b^\nu(a + d) \\ c^\nu(a + d) & 1 + (bcd^2)^\nu \end{pmatrix} \quad \text{whereas} \quad \mathcal{I}_A^t \mathcal{I}_A = \begin{pmatrix} 1 + (bcd)^\nu & b^\nu(a + d) \\ c^\nu(a + d) & 1 + (a^2 bc)^\nu \end{pmatrix} \quad \text{(in algebraic notation)},$$  

but when either is nonsingular, then they are both equal to $\begin{pmatrix} 1 & b^\nu(a + d) \\ c^\nu(a + d) & 1 \end{pmatrix}$, implying $A$ is reversible in this case.

This raises hope that the theory works when we only encounter tangible matrices, but a troublesome example exists for $3 \times 3$ matrices.

**Example 2.30.** (logarithmic notation, where $-$ denotes $-\infty$)

Take $A = \begin{pmatrix} -5 & 0 \\ 0 & -5 \\ -5 & 0 \end{pmatrix}$ whose determinant is $0 = 1$. Then $A^\nu = \begin{pmatrix} -6 & 0 \\ -0 & 5 \end{pmatrix}$, so

$$\mathcal{I}_A^t = A A^\nu = \begin{pmatrix} 0 & -5^\nu \\ -0 & 0 \\ -0 & 0 \end{pmatrix}, \quad \mathcal{I}_A = A^\nu A = \begin{pmatrix} 0 & -5 \\ -0 & 0 \\ -0 & -5^\nu \end{pmatrix},$$  

which are both definite (and would be strictly normal if we took $-5$ instead of $5$). But

$$\mathcal{I}_A^t \mathcal{I}_A = \begin{pmatrix} 0 & 10^\nu & 5^\nu \\ -0 & 0 & -5^\nu \\ -5^\nu & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -5^\nu \\ -0 & 0 \\ -0 & -5^\nu \end{pmatrix} = \mathcal{I}_A^t \mathcal{I}_A.$$  

Furthermore, $\mathcal{I}_A^t \mathcal{I}_A$ is idempotent and nonsingular, and thus a quasi-identity, so $\mathcal{I}_A^t \mathcal{I}_A = (\mathcal{I}_A^t \mathcal{I}_A)^\nu$ which does not equal $\mathcal{I}_A^t \mathcal{I}_A = (\mathcal{I}_A^t \mathcal{I}_A)^\nu$. The quasi-identities $\mathcal{I}_A^t$ and $\mathcal{I}_A$ always satisfy a nice relation in the $2 \times 2$ case. We say that $2 \times 2$ matrices $\mathcal{I} = \begin{pmatrix} 1 & u^\nu \\ u^\nu & 1 \end{pmatrix}$ and $\mathcal{T} = \begin{pmatrix} 1 & u^\nu \\ u^\nu & 1 \end{pmatrix}$ (in algebraic notation) are paired if $u^\nu v^\nu = u^\nu v^\nu$. 
Lemma 2.31. For any $2 \times 2$ matrix $A$ of determinant $1$, the quasi-identities $I_A$ and $I_A'$ are paired. Conversely, if $F$ is closed under square roots then, given paired quasi-identity matrices $I$ and $I'$, there is a $2 \times 2$ matrix $A$ of determinant $1$, such that $I = I_A$ and $I' = I_A'$.

Proof. After a permutation, we may write $A = \begin{pmatrix} a & b \\ c & a^{-1} \end{pmatrix}$ with $bc <_\nu 1$. Then

$$I_A' = \begin{pmatrix} 1 \\ (a^{-1}c)''(ab)^{''''} \\ 1 \end{pmatrix} \quad \text{and} \quad I_A'' = \begin{pmatrix} 1 \\ (ac)''(a^{-1}b)^{''''} \\ 1 \end{pmatrix}$$

are paired since $(ab)(a^{-1}c) = bc = (a^{-1}b)(ac)$. Conversely, given $uv' = u'v$ we take $b = \sqrt{uv'}$, $c = \sqrt{vv'}$, and $a = \sqrt{u}$ to get $ab = u$, $a^{-1}b = u'$, $ac = \sqrt{uvv'} = \sqrt{v^2} = v$, and $a^{-1}c = v'$.

Lemma 2.32 ([11, Lemma 2.17]). $A^\nabla \leq A^\nabla A A^\nabla$ for any matrix $A \in \operatorname{Mat}_n(F)$.

Definition 2.33. A matrix $A$ is $\nabla$-regular if $A = A A^\nabla A$.

Example 2.34. $A A^\nabla A = I_A A = (I_A') A'$ is $\nabla$-regular (but not necessarily reversible, nor nonsingular). Every quasi-identity matrix is $\nabla$-regular as well as reversible.

Since $A A^\nabla A$ shares many properties with $A$ (for example, yielding the same quasi-identities $I_A'$ and $I_A''$ and other properties concerning solutions of equations in [21]), the passage to $A A^\nabla A$ is a closure operation which is of particular interest to us.

3. Special linear supertropical matrices

As stated earlier, our main objective is to pinpoint the most viable tropical version of $\operatorname{SL}_n$. The obvious attempt is the set

$$\operatorname{SLS}_n(F) := \{ A \in \operatorname{Mat}_n(F) : \det(A) = 1 \}$$

of matrices with supertropical determinant $1$, which we call special linear matrices.

3.1. The monoid generated by $\operatorname{SLS}_n(F)$.

$\operatorname{SLS}_n(F)$ is not a monoid, as noted in Equation (11.2). Thus, we would like to determine the monoid generated by $\operatorname{SLS}_n(F)$, as well as the submonoids of $\operatorname{SLS}_n(F)$.

Remark 3.1. For the matrices $A \in \operatorname{SLS}_n$ and $B \in \operatorname{SLS}_n^\times$, we have $AB, BA \in \operatorname{SLS}_n$, by Proposition 2.7.

Thus, any difficulty involves noninvertible matrices of $\operatorname{SLS}_n(F)$. The following observation ties this discussion to definite matrices.

Lemma 3.2.

(i) Any nonsingular matrix $A$ is the product $PA_1$ of a generalized permutation matrix $P$ with a definite matrix $A_1$.

(ii) Any matrix $A$ of $\operatorname{SLS}_n(F)$ is the product $PA_1$ of a generalized permutation matrix $P \in \operatorname{SLS}_n(F)$ with a definite matrix $A_1 \in \operatorname{SLS}_n(F)$. Likewise we can write $A = A_2 Q$ for a generalized permutation matrix $Q$ in $\operatorname{SLS}_n$ and $A_2$ a definite matrix.

Proof. Multiplying by a permutation matrix puts the dominant permutation of $A$ on the diagonal, which we can make definite by multiplying by a diagonal matrix. If $A \in \operatorname{SLS}_n(F)$ then $A_1 \in \operatorname{SLS}_n(F)$, in view of Proposition 2.7.

The point of this lemma is that the process of passing a matrix of $\operatorname{SLS}_n$ to definite form takes place entirely in $\operatorname{SLS}_n$, so the results of [21] are applicable in this paper, as we shall see.

Definition 3.3.

$$\overline{\operatorname{SLS}}_n(F) := \{ A \in \operatorname{Mat}_n(F) : \det(A) \vdash 1 \}.$$

We write $\operatorname{SLS}_n$ and $\overline{\operatorname{SLS}}_n$ for $\operatorname{SLS}_n(F)$ and $\overline{\operatorname{SLS}}_n(F)$ respectively, when $F$ is clear from the context.
In the spirit of [20 Proposition 3.9], but using symmetrization, we can turn to \( \geq_c \), and define:

**Definition 3.4.**

\[ \text{SLS}_n(F) := \text{SLS}_n(F) \cup \{ A \in \text{SLS}_n(F) : \text{bidet}(A) = (\alpha, \beta) \text{ where } \alpha = 1^\nu > \nu \beta \text{ or } \beta = 1^\nu > \nu \alpha \}. \]

Thus \( \text{SLS}_n(F) \subset \text{SLS}_n(F)_o \subset \text{SLS}_n(F) \), so we could increase the scope of the theory by considering \( \text{SLS}_n(F)_o \) instead of \( \text{SLS}_n(F) \).

Here is a generic sort of example.

**Example 3.5.** Consider two rank 1 matrices \( \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \) and \( \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \). Their product is \( \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \) whose bideterminant is \( (abcd) \).

Although the first two matrices are singular, they “explain” the following modification: The product of the matrices \( \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \) and \( \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \), both from \( \text{SLS}_n(F) \), is \( \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \) when \( abcd > 1 \). Put another way, given any \( u, v, u', v' \in F \) satisfying \( uv' = u'v \), we can find two matrices whose product is \( \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \), namely take \( a = 1, c = u, d = u' \), and \( b = \frac{1}{u} \). Thus, every \( 2 \times 2 \) matrix in \( \text{SLS}_2(F) \) is a product of two matrices in \( \text{SLS}_2(F) \).

This yields:

**Proposition 3.6.** \( \text{SLS}_2(F) \) is the submonoid of matrices generated by \( \text{SLS}_2(F) \).

**Proof.** The key computation is \( \begin{pmatrix} 1 \\ ab^{-1} & b \end{pmatrix} = \begin{pmatrix} 1 \\ ab^{-1} & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) when \( a > \nu 1 \), a special case of the previous example.

On the other hand, for larger \( n \), we have room for obstructions.

**Example 3.7.** For \( n \geq 3 \), suppose \( A \) has the form
\[
\begin{pmatrix}
 a_{1,1} & a_{1,2} & 0 & 0 & \ldots & 0 \\
 0 & a_{2,2} & a_{2,3} & 0 & \ldots & 0 \\
 0 & 0 & a_{3,3} & a_{3,4} & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 0 & \ldots & 0 & 0 & a_{n-1,n-1} & a_{n-1,n} \\
 a_{n,1} & 0 & \ldots & 0 & 0 & a_{n,n}
\end{pmatrix}.
\]

Then \( A \) cannot be factored into \( A_1 A_2 \) unless one of the \( A_i \) is invertible.

More generally, we have:

**Proposition 3.8.** [25 Proposition 3.2] Suppose \( \pi, \sigma \in S_n \) such that there exists an integer \( 0 < t < \frac{n}{2} \) for which, for all \( i \), \( \pi(i) \equiv \sigma(i) + t \pmod{n} \). Then any \( n \times n \) matrix \( A = \sum_{i=1}^{n} (a_{i,\pi(i)} e_{i,\pi(i)} + a_{i,\sigma(i)} e_{i,\sigma(i)}) \) (with invertible coefficients \( a_{i,\pi(i)}, a_{i,\sigma(i)} \)) is not factorizable.

For \( n = 3 \) this example is not so bad, since \( A \) has no odd permutations contributing to the determinant. But for \( n \) even, \( A \) has one odd permutation and one even permutation which contribute.

**Corollary 3.9.** For even \( n \geq 4 \), \( \text{SLS}_n(F) \) is not a product of elements of \( \text{SLS}_n(F) \).

**Proof.** The permutation \( (1 2 \cdots n) \) is odd, and so we get an element of \( \text{SLS}_n(F) \setminus \text{SLS}_n(F) \) which is not factorizable, and in particular is not a product of elements of \( \text{SLS}_n(F) \). \( \square \)

### 3.2. Nonsingular submonoids.

Although \( \text{SLS}_n \) is not a monoid, it does have interesting submonoids.

**Definition 3.10.** A matrix monoid is **nonsingular** if it consists of nonsingular matrices.

A subset \( S \subseteq \text{Mat}_n(F) \) is **\( \nabla \)-closed** if \( A^\nabla \in S \) for all \( A \in S \).
Geometric and combinatorial characterizations of nonsingular tropical matrix monoids are provided in [11, 16], where these monoids admit nontrivial (universal) semigroup identities [14]. Most of the sets we consider are $\nabla$-closed.

Example 3.11.  
(i) The set of generalized permutation matrices in $\text{SLS}_n$ is a nonsingular subgroup of $\text{Mat}_n(F)^\times$ (with unit element $I$).  
(ii) The upper triangular matrices of $\text{SLS}_n$ are a submonoid (with unit element $I$).  
(iii) If $A$ is strictly normal, then the monoid generated by $A$ is nonsingular. (Indeed, $A^k$ is nonsingular for every $k < n$, which means that there is only one way of getting a maximal diagonal entry in any power of $A$, which is by taking a power of $a_{i,i} = 1$, and the non-diagonal entries will be smaller.)

We continue with (iii), and appeal to a more restricted version of $\text{SLS}_n$.

Definition 3.12. $\text{SN}_n$ denotes the set of all strictly normal $n \times n$ matrices in $\text{Mat}_n(F)$.

Lemma 3.13. $\text{SN}_n$ is a nonsingular $\nabla$-closed monoid, also closed under transpose.

Proof. A straightforward verification, using Lemma 2.20(vi). \hfill $\Box$

3.2.1. The $\mathbb{1}$-special linear monoid.

We enlarge the monoid $\text{SN}_n$ via the left and right action of permutation matrices.

Definition 3.14. Given a set $S$, we define its permutation closure to be 
\[
\{PJQ : J \in S \text{ and } P,Q \text{ are permutation matrices}\}.
\]

The $\mathbb{1}$-special linear monoid $\text{SLS}^\mathbb{1}_n$ is the permutation closure of the monoid $\text{SN}_n$ of strictly normal matrices.

Remark 3.15. A matrix $A$ of $\text{Mat}_n(F)$ is in $\text{SLS}^\mathbb{1}_n$ if and only if $A = (a_{i,j})$ has a uniformly dominant permutation $\pi$ with $a_{i,\pi(i)} = 1$ for all $i$.

Theorem 3.16. $\text{SLS}^\mathbb{1}_n$ is a $\nabla$-closed submonoid of $\text{SLS}_n$.

Proof. Write $B = A_1A_2$ where $A_1 = P_{\pi_1}J_{\tau_1}Q_{\tau_2}$, $A_2 = P_{\pi_3}J_{\tau_2}Q_{\pi_4} \in \text{SLS}^\mathbb{1}_n$, with $J_1, J_2$ strictly normal. Thus $B$ is a product of matrices with respective uniformly dominant permutations $\pi_1, \text{id}, \pi_2, \pi_3, \text{id}, \pi_4$, and we see from Lemma 2.4 that $\tau = \pi_1 \text{id} \pi_2 \pi_3 \text{id} \pi_4$ is uniformly dominant for $B$, in which $b_{i,\tau(i)} = 1$, $\forall i$. Hence, $B \in \text{SLS}^\mathbb{1}_n$, and we have proved that $\text{SLS}^\mathbb{1}_n$ is a monoid.

By Lemma 3.13 and Proposition 2.7 it follows that 
\[
A^\nabla = (P_{\pi_1}JQ_{\pi_2})^\nabla = Q_{\pi_2}J^\nabla P_{\pi_1}^\nabla = Q_{\pi_2}^{-1}J^\nabla P_{\pi_1}^{-1},
\]
for every $A \in \text{SLS}^\mathbb{1}_n$, and thus $\text{SLS}^\mathbb{1}_n$ is $\nabla$-closed. \hfill $\Box$

Assume that the semifield $F$ is dense in the sense that if $a > b$ in $F$ then there is $u \leq 1$ such that $ua > b$. The next lemma and proposition show why matrices not in $\text{SLS}^\mathbb{1}_n$ and permutation matrices do not mix well.

Lemma 3.17. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or $A = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ where $a < 1 < d$ and $bc < ad$. There exists $U \in \text{SLS}^\mathbb{1}_n$ such that $U^tAU$ is symmetrically singular.

Proof. There is $u$ with $a < u < 1$ for which $du^2 > a$, and thus, taking $U = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, and $U^t = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, in the first case we have 
\[
U^tAU = U^t(AU) = U^t\begin{pmatrix} a + bu & b \\ c + du & d \end{pmatrix} = \begin{pmatrix} (b + c)u + du^2 & b + du \\ c + du & d \end{pmatrix}
\]
whose bideterminant is $((b + c)du + d^2u^2, (b + c)du + d^2u^2)$. The second case works in the same way. $\Box$
Theorem 3.21. \( \text{AB} \) can rearrange any product of elementary matrices to a product \( \text{E} \).

The relations in Lemma 3.20 enable us to move all \( \text{MUM} \) monoids of Example 3.11.

Proposition 3.18. Suppose \( M \in \text{SLS}_n \) but not in \( \text{SLS}^2_n \). Then there exists a matrix \( U \in \text{SLS}_n \) such that either \( \text{MUM} \) is symmetrically singular with \( U \) a permutation matrix, or \( U^t \text{MU} \) is symmetrically singular with \( U \) a strictly normal matrix.

Proof. We write the dominant track of \( M = (a_{i,j}) \) as \( a_{i,\pi(i)} \leq a_{i,\pi(i_2)} \leq \cdots \leq a_{i,\pi(i_n)} \). Reordering the indices we may assume that \( a_{i,\pi(1)} \leq a_{2,\pi(2)} \leq \cdots \leq a_{n,\pi(n)} \).

First assume that all the \( a_{i,\pi(i)} = 1 \). By hypothesis \( a_{i,\pi(j)} \geq 1 \) for some \( j \neq i \). Consider the \( 2 \times 2 \) matrix

\[
B := \begin{pmatrix} a_{i,\pi(i)} & a_{i,\pi(j)} \\ a_{j,\pi(i)} & a_{j,\pi(j)} \end{pmatrix} = \begin{pmatrix} 1 & a_{i,\pi(j)} \\ a_{j,\pi(i)} & 1 \end{pmatrix}.
\]

Since \( M \) is nonsingular, we must have \( a_{i,\pi(j)} a_{j,\pi(i)} < 1 \), so \( a_{j,\pi(i)} < 1 \). Let \( P \) be the \( 2 \times 2 \) permutation matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), so \( PB = \begin{pmatrix} a_{j,\pi(i)} & 1 \\ 1 & a_{i,\pi(j)} \end{pmatrix} \), and

\[
BPB = \begin{pmatrix} a_{i,\pi(j)} & a_{i,\pi(j)}^2 \\ a_{i,\pi(j)} & a_{i,\pi(j)} \end{pmatrix},
\]

which is symmetrically singular. Extending \( P \) to the \( n \times n \) permutation matrix \( U \) corresponding to the transposition \((i,j)\), yields \( \text{MUM} \) symmetrically singular.

Thus we may assume that some \( a_{i,\pi(i)} < 1 \), and some \( a_{i,\pi(j)} > 1 \). Then Lemma 3.17 is applicable taking \( A = B \). Therefore, \( U^t \text{MU} \) is symmetrically singular, taking \( U \) to be the elementary matrix \( E_{i,j}(u) \), where \( u \) as in Lemma 3.17.

\( \square \)

Theorem 3.19. The monoid \( \text{SLS}^2_n \) is a maximal nonsingular submonoid of \( \text{SLS}_n \).

Proof. \( \text{SLS}^2_n \) is a nonsingular submonoid of \( \text{SLS}_n \) by Theorem 3.16. \( \text{SLS}^2_n \) is maximal nonsingular by Proposition 3.18, since for \( M \in \text{SLS}_n \) but not in \( \text{SLS}^2_n \), there exists \( U \in \text{SLS}^2_n \), such that either \( U^t \text{MU} \notin \text{SLS}_n \) or \( \text{MUM} \notin \text{SLS}_n \).

But \( \text{SLS}^2_n \) is not the only maximal nonsingular submonoid of \( \text{SLS}_n \), since it does not contain the other monoids of Example 3.11.

3.3. Submonoids of \( \text{SLS}_n(F) \).

Define \( T^a \) to be the set of products of \( E_{i,j} \) with \( i < j \), and \( T^b \) to be the set of products of \( E_{i,j} \) with \( i > j \), the respective sets of upper and lower triangular matrices. These are both monoids, and we want to consider \( T^a T^b \). Toward this objective, we commute elements of \( T^a \) and \( T^b \).

The following argument, based on the Steinberg relations of the \( E_{i,j} \). We write \( E_{i,j}(a) \) for \( I + ae_{i,j} \).

Lemma 3.20.

(i) \( E_{i,j}(a)E_{k,\ell}(b) = E_{k,\ell}(b)E_{i,j}(a) \) for \( i \neq \ell, j \neq k \).

(ii) \( E_{i,j}(a)E_{j,i}(b) = E_{j,i}(b)E_{i,j}(a) \) for \( ab \neq 1 \).

(iii) \( E_{i,j}(a)E_{j,k}(b) = \begin{cases} E_{i,k}(ab)E_{j,k}(b)E_{i,j}(a) & \text{for } k < i < j, \\ E_{j,k}(b)E_{i,j}(a)E_{i,k}(ab) & \text{for } i < k < j. \end{cases} \)

Proof. Direct computation. \( \square \)

Theorem 3.21. Any product of \( E_{i,j} \) matrices contained in \( \text{SLS}_n \) is in \( T^a T^b \).

Proof. The relations in Lemma 3.20 enable us to move all \( E_{i,j} \) for \( i < j \) to the right, so by induction we can rearrange any product of elementary matrices to a product \( AB \) where \( A \in T^a \) and \( B \in T^b \). \( \square \)

Of course, in our situation, this is a submonoid of \( \text{SLS}_n(F) \).

Example 3.22. Let \( E \) denote the set of \( 2 \times 2 \) definite matrices, and \( E_{\text{sing}} \) denote the set of matrices of the form \( c \begin{pmatrix} ab & a \\ b & 1 \end{pmatrix} \) or \( c \begin{pmatrix} 1 & a \\ b & ab \end{pmatrix} \) such that \( ab, c \geq 1 \). The monoid generated by \( E_{1,2} \) and \( E_{2,1} \) is \( E \cup E_{\text{sing}} \):

\[
\begin{pmatrix} 1 + a_1a_2 & a_1 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_1 \\ b_2 & 1 + b_1b_2 \end{pmatrix} = \begin{pmatrix} 1 + a_1(a_2 + b_2) & a_1 + b_1 \end{pmatrix} \begin{pmatrix} 1 + a_1(a_2 + b_2) & a_1 + b_1 \end{pmatrix} \begin{pmatrix} a_2 + b_2 & 1 + b_1(a_2 + b_2) \end{pmatrix} \].
The LU-factorization, attributed to Turing [32], is one of the pillars of classical matrix algebra, cf. [30] Theorems 1E and 1F. Theorem 3.21 gives us the LU-factorization for nonsingular products of elementary matrices. We also recall that all nonsingular definite $2 \times 2$ matrices have an LU-factorization (see [25, Example 2.9]), but this fails already in the $3 \times 3$ case, as seen by Proposition 3.8. Nevertheless, by [25, Corollary 6.6], we have \( \{ A^\nabla : A \in \text{SLS}_n \} \subseteq T^\nabla T^\nabla u_n \). We do not know if this inclusion is strict.

**Conjecture 3.23.** If \( B \) is a non-triangular definite matrix in \( T^\nabla T^\nabla u_n \), then \( B \in \{ A^\nabla : A \in \text{SLS}_n \} \).

It has been recently proved in [9] that the monoid generated by Jacobi matrices \( E_{i,j} \pm 1 \), is the set of tropical totally nonnegative matrices (defined by means of sign-singularity and dominant permutation parity) with non-0 determinant. However, the question of what is generated by all the \( E_{i,j} \) remains open.

We further study the action of these tropical nonsingular noninvertible elementary matrices in [9].

### 3.4. Semigroup unions in \( \text{SLS}_n \)

Our objective here is to carve \( \text{SLS}_n \) into monoids, each of which has a multiplicative unit \( I \), where \( I \) is a quasi-identity. Although we cannot quite do this, the process works for \( \nabla \)-regular matrices.

**Definition 3.24.** For any \( A \in \text{Mat}_n(F) \) with \( \det(A) \neq 0 \):

(i) \( \text{SLS}^\ell_{A,n} = \{ B \in \text{SLS}_n : I^\ell_A B = B \} \);

(ii) \( \text{SLS}^r_{A,n} = \{ B \in \text{SLS}_n : BI^r_A = B \} \).

(iii) \( \text{SLS}_{A,n} = \{ B \in \text{SLS}_n : I_A B = BI_A = B \} \).

In particular, for a quasi-identity \( I \),

\[
\text{SLS}^\ell_{I,n} = \{ B \in \text{SLS}_n : I B = B \} \quad \text{and} \quad \text{SLS}^r_{I,n} = \{ B \in \text{SLS}_n : B I = B \}.
\]

**Lemma 3.25.** If \( A \) is \( \nabla \)-regular, then \( A \in \text{SLS}^\ell_{A,n} \cap \text{SLS}^r_{A,n} \).

**Proof.** \( A = A A^\nabla A = I^\ell_A A = A I^r_A \). \( \square \)

**Lemma 3.26.** \( \text{SLS}^\ell_{A,n} \) is a sub-semigroup of \( \text{SLS}_n \) with left unit element \( I^\ell_A \) and right unit element \( I^r_A \).

**Proof.** First note that if \( I^\ell_A B_1 = B_1 \) then for any \( B_2 \) we have \( I^\ell_A (B_1 B_2) = B_1 B_2 \). Thus \( \text{SLS}^\ell_{A,n} \) is closed under multiplication on the right by any matrix. In particular, \( \text{SLS}^\ell_{A,n} \) is a sub-semigroup of \( \text{SLS}_n \). The other assertion holds since \( I^\ell_A \) and \( I^r_A \) are idempotent. \( \square \)

This provides the intriguing situation in which we have a natural semigroup with left and right identities which could be unequal. The situation is better when \( A \) is reversible.

**Theorem 3.27.** Every reversible element \( A \) of \( \text{SLS}_n \) defines a submonoid \( \text{SLS}^\ell_{A,n} \) with unique unit element \( I_A \), and which contains \( I_A A \). The union of these submonoids contains every reversible \( \nabla \)-regular element of \( \text{SLS}_n \), and in particular, every quasi-identity matrix.

**Proof.** If \( A \) is reversible, then \( I_A \in \text{SLS}^\ell_{A,n} \) is the (unique) unit element, in view of Proposition 2.26 so \( \text{SLS}^\ell_{A,n} \) is a monoid. Furthermore, \( I_A A = A A^\nabla A \in \text{SLS}^\ell_{A,n} \). The last assertion follows at once. \( \square \)

### 4. The conjugate action

For any nonsingular matrix \( A \) and any matrix \( B \), we define

\[
A B = A^\nabla B A.
\]

This is the closest we have to conjugation by supertropical matrices. (Note that \( A I = A^\nabla I A = I^\ell_A \).

We continue with an example of a nonsingular matrix having a singular conjugate.
Example 4.1. Take $A = \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix}$, $B = \begin{pmatrix} x & z \\ w & y \end{pmatrix}$, where $x >_\nu y$, $xy >_\nu zw \geq_\nu 0$, and $\alpha, \beta <_\nu 1$ such that $\alpha \beta >_\nu \frac{1}{2}$. Then

$$A^\triangledown BA = \begin{pmatrix} x\alpha + z\beta + w\alpha + y & x\beta + z\beta^2 + y\beta + w \\ x\alpha + w\alpha^2 + y\alpha + z & z\beta + y\alpha \beta + w\alpha + x \end{pmatrix} = \begin{pmatrix} x\alpha \beta + z\beta + w\alpha & x\beta + z\beta^2 + w \\ x\alpha + w\alpha^2 + z & z\beta + w\alpha + x \end{pmatrix},$$

for which

$$\det(A^\triangledown BA) = (wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha \beta + wx\alpha^2 \beta + z^2\beta^2 + xz\alpha \beta^2),$$

since $x^2\alpha \beta >_\nu xy \geq_\nu wz >_\nu w\alpha \beta >_\nu w\alpha^2 \beta^2$. Thus $A^\triangledown BA$ is singular. Obviously, this holds for any nonsingular matrix $B$ with $y = x^{-1}$, namely when $\det(B) = 1$.

Given a nonempty set $S \subset \text{Mat}_n(F)$ of matrices and a matrix $A$ with $I_A \in S$, we write

$$A^S = \{ A^\triangledown BA : B \in S \}.$$

If $S$ is a monoid, then $A^S$ also is a monoid. But when $A \in \text{SLS}_n$ is not invertible, we get into difficulties, even in the $2 \times 2$ case.

Example 4.2. (logarithmic notation)

$$B = \begin{pmatrix} 0 & -\infty \\ 1 & 0 \end{pmatrix}$$

is definite, and $A = \begin{pmatrix} 0 & 5\nu \\ -\infty & 0 \end{pmatrix}$ is a quasi-identity matrix, but $BAB = \begin{pmatrix} 6\nu & 5\nu \\ 7\nu & 6\nu \end{pmatrix}$ is singular.

Note that if $S$ is a nonsingular matrix submonoid of $\text{Mat}_n(F)$, then $P^S$ also is a nonsingular submonoid of $\text{Mat}_n(F)$, for any permutation matrix $P$. On the other hand, these often do not mix well, as seen in Lemma 5.2 below.

The following example also shows that nonsingularity need not be preserved under multiplication in $\text{SLS}^\ell_{A,n}$, even when we conjugate by diagonal matrices.

Example 4.3. (logarithmic notation)

If $B = \begin{pmatrix} 0 & -\infty \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & -\infty \\ -\infty & -1 \end{pmatrix}$, then $BDB^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is singular. In view of Proposition 2.7, $B(DB^tD^{-1})$ is singular.

Here is one consolation.

Lemma 4.4. If $I_A \in \text{SN}_n$, then $\{ A^\triangledown JA : J \in \text{SN}_n \}$ is a monoid.

Proof. By Theorem 2.10.

The situation improves significantly when we restrict our attention to the submonoid $\text{SLS}^\ell_{A,n}$ and the space on which it acts. We define

$$V_A = \{ v \in F^{(n)} : I_A^t v = v \}.$$ 

Lemma 4.5. $\text{SLS}^\ell_{A,n} F^{(n)} = V_A = \text{SLS}^\ell_{A,n} V_A.$

Proof. If $B \in \text{SLS}^\ell_{A,n}$ and $v \in F^{(n)}$, then $I_A^t (Bv) = (I_A^t B)v = Bv$. On the other hand, if $v \in V_A$ then $v = I_A^t v \in \text{SLS}^\ell_{A,n} V_A$. Thus, equality holds since $\text{SLS}^\ell_{A,n} F^{(n)} \subseteq V_A \subseteq \text{SLS}^\ell_{A,n} V_A \subseteq \text{SLS}^\ell_{A,n} F^{(n)}.$

Proposition 4.6. For any nonsingular $A$, left multiplication by $A^\triangledown$ yields a module map from $V_A$ to $V_A^\triangledown$, which commutes with conjugation by $A$.

Proof. If $B \in \text{SLS}^\ell_{A,n}$, then letting $v' = A^\triangledown v$ we have $(A^\triangledown BA)v' = (A^\triangledown BA)A^\triangledown v = A^\triangledown Bv.$
5. Tropical elementary matrices

Unlike the situation over a field, the tropical concepts of singularity, invertibility, and factorability into elementary matrices do not coincide, cf. [28]. Over a field, the fact that a nonsingular matrix can be written as the product of elementary matrices means that one can pass between any two nonsingular matrices using elementary operations. In the tropical case, even though factorability fails, we show in Theorem 5.4 below that one still can pass between nonsingular matrices, in a certain sense.

In analogy with the classical definition, we define three types of tropical elementary matrices of SLS$_n$:

- **Transposition matrices**, which switch two rows (resp. columns);
- **Diagonal multipliers**, which multiply a row (resp. a column) by some element of $T$;
- **Gaussian matrices**, which add one row (resp. column), multiplied by a scalar, to another row (resp. column).

**Definition 5.1.** A nonsingular matrix is defined to be **(tropically) factorizable** if it can be written as a product of tropical elementary matrices.

As noted earlier, the product of nonsingular matrices could be singular. Since the transposition and diagonal multipliers are invertible, the difficulty must lie in the Gaussian matrices, which are identified with the $E_{i, j}(a)$ defined earlier. In the next lemma we pinpoint the elementary operation that causes a nonsingular matrix which is non-invertible to become singular.

**Lemma 5.2.** For every non-invertible matrix $A$ in SLS$_n$, there exists an elementary Gaussian matrix $E$ such that $EA$ is singular.

**Proof.** First we recall that if $A$ is a factorizable matrix, then we can find a factorization in which the Gaussian matrices are at the right of its factorization (see [25]). Therefore, in view of Lemma 5.2 it suffices to prove the lemma for a definite matrix $A$. Hence, det$(A) = 1$ is attained solely by the diagonal.

Since $A$ is non-invertible, there exists at least one off-diagonal entry $a_{i, j} \neq 0$. We let $E = E_{j, i}(a_{i, j}^{-1})$. Then

$$
\det(EA) = \sum_{\sigma \in S_n} a_{1, \sigma(1)} \cdots a_{i, \sigma(i)} \cdots a_{n, \sigma(n)} + \sum_{\sigma \in S_n} a_{1, \sigma(1)} \cdots (a_{i, j}^{-1}) a_{i, \sigma(j)} \cdots a_{n, \sigma(n)}
$$

$$
= \det(A) + \sum_{\sigma \in S_n} a_{1, \sigma(1)} \cdots (a_{i, j}^{-1}) a_{i, \sigma(j)} \cdots a_{n, \sigma(n)}.
$$

The summand in the right side given by $\sigma = (i, j)$ is 1, which together with det$(A)$ yields $1''$. Moreover, by [20, Theorem 3.5], any larger dominant term on the right sum must be ghost. Since det$(A) = 1$, the assertion follows. \hfill \Box

We recall the well-known connection between tropical matrices and digraphs. Any $n \times n$ matrix $A$ is associated with a weighted digraph $G_A$ over $n$ vertices having edge $(i, j)$ of weight $a_{i, j}$ whenever $a_{i, j} \neq 0$ cf. [20, §3.2]. From this viewpoint the $(i, j)$-entry of the matrix adj$(A)$ equals the maximal weight of all paths from $i$ to $j$ in the graph $G_{\text{adj}(A)}$. We utilize this identification and work with nonsingular definite matrices, in which case $A^\vee \cong_{\nu} A^\nabla$ by [24, Corollary 6.2]. A path is called **simple** if each vertex appears only once.

**Proposition 5.3.** For any matrix $A \in \text{SLS}_n$ there exists a product $E$ of elementary Gaussian matrices such that $A^\nabla = EA$.

**Proof.** In view of Lemma 5.2 we may assume that $A$ is definite; indeed, writing $A = PA_1$, for $A_1$ definite, we would have $(PA_1)^\nabla = (A_1^\nabla P^{-1})^\nabla = PA_1^\nabla = PEA_1 = (PEP^{-1})PA_1$.

Now let $A = (A_{i, j})$, $A^\vee = (A_{i, j}^\vee)$ and let $A^\nabla = (A_{i, j}^\nabla)$. Then $A^\nabla \models A$ by Remark 5.2. Since $A$ is nonsingular definite, we have det$(A) = 1 = \prod_i A_{i, i}$, where its dominant permutation is the identity. It follows that any nontrivial cycle has weight $< 1$. Any product including such a cycle is strictly dominated by the product where this cycle is replaced by the (weight 1) identity permutation on the corresponding set of indices.
Thus, we may assume that each \((i, j)\)-entry of \(\text{adj}(A)\) is \(\nu\)-equivalent to the sum of weights of simple paths from \(i\) to \(j\):
\[
A_{i,j}^{\nabla} \cong_\nu \sum_{\pi \in S_n: \pi(j)=i} A_{i,\pi(i)} A_{\pi(i),\pi^2(i)} \cdots A_{\pi^{-1}(j),j} = A_{i,j} + \sum_{(i,j) \neq \pi \in S_n: \pi(j)=i} A_{i,\pi(i)} A_{\pi(i),\pi^2(i)} \cdots A_{\pi^{-1}(j),j}.
\]
Applying this identity to \([25, \text{Corollary 6.2}]\): \(A^{\nabla} \cong_\nu A^{\nabla\nabla}\), yields for every entry such that \(A_{i,j}^{\nabla} \neq A_{i,j}\):
\[
\begin{bmatrix}
E_{i,j}(A_{i,j}^{\nabla\nabla}) A
\end{bmatrix}
= \begin{cases}
A_k, \ell, & \forall k \neq i \\
A_{i,\ell} + A_{i,j}^{\nabla\nabla} A_{j,\ell}, & k = i.
\end{cases}
\]
When \(j = \ell\), we get \(A_{i,j} + A_{i,j}^{\nabla\nabla} \cdot 1 = A_{i,j}^{\nabla\nabla}\). If \(j \neq \ell\), then
\[
A_{i,j} A_{j,\ell} = \sum_{(i,j) \neq \pi \in S_n: \pi(j)=i} A_{i,\pi(i)} A_{\pi(i),\pi^2(i)} \cdots A_{\pi^{-1}(j),j} A_{j,\ell}
\]
is either a closed path dominated by \(A_{i,i} = 1\) when \(\ell = i\), or is the product of cycles with an elementary path from \(i\) to \(\ell\) dominated by \(A_{i,j}^{\nabla\nabla}\). Therefore, applying \(E_{i,j}(A_{i,j}^{\nabla\nabla})\) for every \(j \neq i\) such that \(A_{i,j}^{\nabla\nabla} \neq A_{i,j}\) transforms all entries of \(A\) to entries of \(A^{\nabla\nabla}\) and we get
\[
A^{\nabla\nabla} = \left( \prod_{A_{i,j}^{\nabla\nabla} \neq A_{i,j}} E_{i,j}(A_{i,j}^{\nabla\nabla}) \right) \left( \prod_{A_{i,j}^{\nabla\nabla} = A_{i,j}} E_{i,j}(A_{i,j}^{\nabla\nabla}) \right) A,
\]
that is, where the elementary operations \(E_{i,j}(A_{i,j}^{\nabla\nabla})\) are applied to upper entries first, lower entries later, and \(\{i, j\}\) is lexicographically ordered.

\qed

Let \(A\) and \(B\) be nonsingular matrices. Over a field, in classical linear algebra, \(A\) and \(B\) can be written as products of elementary matrices. Thus, one can pass from \(A\) to \(B\) by applying elementary operations. In the tropical case, whereas we do not have factorizability into elementary matrices, cf. \([25, \text{Example 4.5}]\), we do have the second implication, described in the following theorem.

**Theorem 5.4.** For any two nonsingular matrices \(A, B\), there exist matrices \(E_1, E_2, E_3, E_4\) which are products of elementary matrices of \(\text{SLS}_n\), such that \(E_1 A E_2 = E_3 B E_4\).

**Proof.** Using Lemma \([22, \text{Lemma 3.2}]\), we write \(A^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P\) and \(B^{\nabla\nabla} = \overline{B}^{\nabla\nabla} Q\) where \(\overline{A}, \overline{B}\) are definite, and \(P, Q\) are invertible matrices chosen so that \(A = \overline{A}P\) and \(B = \overline{B}Q\).

By Remark \([22, \text{Remark 2.21(v)}]\), the matrices \(\overline{A}^{\nabla\nabla}\) and \(\overline{B}^{\nabla\nabla}\) respectively ghost-surpass \(\overline{A}\) and \(\overline{B}\), and noting that \(P^{\nabla\nabla} = P\) and \(Q^{\nabla\nabla} = Q\), we get that
\[
A^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P \quad \text{and} \quad B^{\nabla\nabla} = \overline{B}^{\nabla\nabla} Q.
\]
Recalling \([25, \text{Lemma 6.5}]\), the matrices \(\overline{A}^{\nabla\nabla}\) and \(\overline{B}^{\nabla\nabla}\) are factorizable, and therefore, \(A^{\nabla\nabla}\) and \(B^{\nabla\nabla}\) are factorizable. Clearly \(I A^{\nabla\nabla} B^{\nabla\nabla} = A^{\nabla\nabla} B^{\nabla\nabla} I\), which provides the assertion for \(A^{\nabla\nabla}\) and \(B^{\nabla\nabla}\).

We denote by \(E, E'\) the elementary products such that \(\overline{A}^{\nabla\nabla} = E\overline{A}\) and \(\overline{B}^{\nabla\nabla} = E'\overline{B}\), whose existence are guaranteed by Proposition \([5, \text{Proposition 5.3}]\), and have
\[
E A B^{\nabla\nabla} = E \overline{A} P B^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P B^{\nabla\nabla} = A^{\nabla\nabla} B^{\nabla\nabla} = A^{\nabla\nabla} B^{\nabla\nabla} Q = A^{\nabla\nabla} E' \overline{B} Q = A^{\nabla\nabla} E' B.
\]
But \(E, B^{\nabla\nabla}, A^{\nabla\nabla}\), and \(E'\) are products of elementary matrices. \(\square\)

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