ON TYCHONOFF-TYPE HYPERTOPOLOGIES

GEORGI DIMOV, FRANCO OBERSNEL, AND GINO TIRONI

Abstract. In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space, similar to the Tychonoff topology but weaker than it. In 1998, G. Dimov and D. Vakarelov [8] used a generalized version of this new topology, calling it Tychonoff-type topology. The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family \( \mathcal{M} \) of subsets of a set \( X \). When \( \mathcal{M} \) contains all singletons, a description of all Tychonoff-type topologies \( \mathcal{O} \) on \( \mathcal{M} \) is given. The continuous maps of a special form between spaces of the type \((\mathcal{M}, \mathcal{O})\) are described in an isomorphism theorem. The problem of commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology is investigated as well. Some topological properties of the hyperspaces \((\mathcal{M}, \mathcal{O})\) with Tychonoff-type topologies \( \mathcal{O} \) are briefly discussed.

1. Introduction

In 1975, M. M. Choban [5] introduced a new topology on the set of all closed subsets of a topological space for obtaining a generalization of the famous Kolmogoroff Theorem on operations on sets. This new topology is similar to the Tychonoff topology (known also as upper Vietoris topology, or upper semi-finite topology ([13]), or kappa-topology) but is weaker than it. In 1998, G. Dimov and D. Vakarelov [8] used a generalized version of this new topology for proving an isomorphism theorem for the category of all Tarski...
consequence systems. This generalized version was called Tychonoff-type topology. The present paper is devoted to a detailed study of Tychonoff-type topologies on an arbitrary family \( M \) of subsets of a set \( X \). When \( M \) is a natural family, i.e. it contains all singletons, a description of all Tychonoff-type topologies \( O \) on \( M \) is given (see Proposition 2.32). For doing this, the notion of \( T \)-space is introduced. The natural morphisms for \( T \)-spaces are not enough to describe all continuous maps between spaces of the type \((M, O)\), where \( M \) is a natural family and \( O \) is a Tychonoff-type topology on it; we obtain a characterization of those continuous maps which correspond to the morphisms between \( T \)-spaces. This is done by defining suitable categories and by proving that these categories are isomorphic (see Theorem 2.37). In such a way we extend to any natural family \( M \) on \( X \) the corresponding result obtained in [8] for the family \( F \) in \((X)\) of all finite subsets of \( X \). We investigate also the problem of commutability between hyperspaces and subspaces with respect to a Tychonoff-type topology, i.e. when the hyperspace of any subspace \( A \) of a topological space \( Y \) is canonically representable as a subspace of the hyperspace of \( Y \). Such investigations were done previously by H.-J. Schmidt [14] for the lower Vietoris topology, by G. Dimov [7, 6] for the Tychonoff topology and for the Vietoris topology, and by B. Karaivanov [12] for other hypertopologies. We study also such a problem for a fixed subspace \( A \) of \( Y \). Some results of [6, 7, 15] are generalized. Finally, we study briefly some topological properties (separation axioms, compactness, weight, density, isolated points, \( P_\infty \)) of the hyperspaces \((M, O)\) with Tychonoff-type topologies \( O \). Some results of [10, 8] are generalized.

Notations 1.1. We denote by \( \omega \) the set of all positive natural numbers, by \( \mathbb{R} \) — the real line, and by \( \mathbb{Z} \) — the set of all integers. We put \( \mathbb{N} = \omega \cup \{0\} \).

Let \( X \) be a set. We denote by \( \mathcal{P}(X) \) the set of all subsets of \( X \). Let \( M, \mathcal{A} \subseteq \mathcal{P}(X) \) and \( A \subseteq X \). We will use the following notations:

- \( A^+_M := \{ M \in M : M \subseteq A \} \);
- \( A^+_M := \{ A \in \mathcal{A} : A \subseteq \mathcal{A} \} \);
- \( \mathcal{F} \text{in}(X) := \{ M \subseteq X : |M| < \aleph_0 \} \);
- \( \mathcal{F} \text{in}_n(X) := \{ M \subseteq X : |M| \leq n \} \), where \( n \in \omega \).

We will denote by \( A^\cap \) (respectively by \( A^\cup \)) the closure under finite intersections (unions) of the family \( \mathcal{A} \). In other words,

- \( A^\cap := \{ \bigcap_{i=1}^k A_i : k \in \omega, A_i \in A \} \) and
- \( A^\cup := \{ \bigcup_{i=1}^k A_i : k \in \omega, A_i \in A \} \).

Let \((X, \mathcal{T})\) be a topological space. We put

- \( \mathcal{C}L(X) := \{ M \subseteq X : M \text{ is closed in } X, M \neq \emptyset \} \) and
- \( \text{Comp}(X) := \{ M \subseteq X : M \text{ is compact} \} \).

The closure of a subset \( A \) of \( X \) in \((X, \mathcal{T})\) will be denoted by \( c_X A \) or \( \overline{A}^X \); as usual, for \( U \subseteq A \subseteq X \), we put
Ex_{A, X} U := X \setminus \operatorname{cl}_X (A \setminus U).

By a base of \((X, \mathcal{T})\) we will always mean an open base. The weight (resp., the density) of \((X, \mathcal{T})\) will be denoted by \(w(X, \mathcal{T})\) (resp., \(d(X, \mathcal{T})\)).

If \(\mathcal{C}\) denotes a category, we write \(X \in |\mathcal{C}|\) if \(X\) is an object of the category \(\mathcal{C}\).

For all undefined here notions and notations, see [9] and [11].

2. Hypertopologies of Tychonoff-type

**Fact 2.1.** Let \(X\) be a set and \(\mathcal{M}, A \subseteq \mathcal{P}(X)\). Then:

(a) \(\bigcap A_{\mathcal{M}} = (\bigcap A)^{+}_{\mathcal{M}}\);
(b) \(A \subseteq B\) implies that \(A^{+}_{\mathcal{M}} \subseteq B^{+}_{\mathcal{M}}\) for all \(A, B \subseteq X\).

**Definition 2.2.** Let \((X, \mathcal{T})\) be a topological space and let \(\mathcal{M} \subseteq \mathcal{P}(X)\). The topology \(\mathcal{O}_{\mathcal{T}}\) on \(\mathcal{M}\), having as a base the family \(\mathcal{T}_{\mathcal{M}}^{+}\), will be called **Tychonoff topology on \(\mathcal{M}\) generated by \((X, \mathcal{T})\)**. When \(\mathcal{M} = \mathcal{C}_{\mathcal{L}}(X)\), then \(\mathcal{O}_{\mathcal{T}}\) is just the classical Tychonoff topology on \(\mathcal{C}_{\mathcal{L}}(X)\).

Let \(X\) be a set and \(\mathcal{M} \subseteq \mathcal{P}(X)\). A topology \(\mathcal{O}\) on \(\mathcal{M}\) is called a **Tychonoff topology on \(\mathcal{M}\)** if there exists a topology \(\mathcal{T}\) on \(X\) such that \(\mathcal{T}_{\mathcal{M}}^{+}\) is a base of \(\mathcal{O}\).

**Definition 2.3.** Let \(X\) be a set and \(\mathcal{M} \subseteq \mathcal{P}(X)\). A topology \(\mathcal{O}\) on the set \(\mathcal{M}\) is called a **topology of Tychonoff-type on \(\mathcal{M}\)** if the family \(\mathcal{O} \cap \mathcal{P}(X)^{+}_{\mathcal{M}}\) is a base for \(\mathcal{O}\).

Clearly, a Tychonoff topology on \(\mathcal{M}\) is always a topology of Tychonoff-type on \(\mathcal{M}\), but not vice versa (see Example 2.42).

**Fact 2.4.** Let \(X\) be a set, \(\mathcal{M} \subseteq \mathcal{P}(X)\) and \(\mathcal{O}\) be a topology of Tychonoff-type on \(\mathcal{M}\). Then the family \(\mathcal{B}_{\mathcal{O}} := \{A \subseteq X : A^{+}_{\mathcal{M}} \in \mathcal{O}\}\) is closed under finite intersections, \(X \in \mathcal{B}_{\mathcal{O}}\), and, hence, \(\mathcal{B}_{\mathcal{O}}\) is a base for a topology \(\mathcal{T}_{\mathcal{O}}\) on \(X\). The family \((\mathcal{B}_{\mathcal{O}})^{+}_{\mathcal{M}}\) is a base of \(\mathcal{O}\).

**Definition 2.5.** Let \(X\) be a set, \(\mathcal{M} \subseteq \mathcal{P}(X)\) and \(\mathcal{O}\) be a topology of Tychonoff-type on \(\mathcal{M}\). We will say that the topology \(\mathcal{T}_{\mathcal{O}}\) on \(X\), introduced in Fact 2.4, is induced by the topological space \((\mathcal{M}, \mathcal{O})\).

**Proposition 2.6.** Let \(X\) be a set and \(\mathcal{M} \subseteq \mathcal{P}(X)\). A topology \(\mathcal{O}\) on \(\mathcal{M}\) is a topology of Tychonoff-type if and only if there exists a topology \(\mathcal{T}\) on \(X\) and a base \(\mathcal{B}\) for \(\mathcal{T}\) (which contains \(X\) and is closed under finite intersections) such that \((\mathcal{B}_{\mathcal{T}})^{+}_{\mathcal{M}}\) is a base for \(\mathcal{O}\).

**Proof.** Suppose \(\mathcal{O}\) is a topology of Tychonoff-type on \(\mathcal{M}\). Then the topology \(\mathcal{T}_{\mathcal{O}}\) induced by the topological space \((\mathcal{M}, \mathcal{O})\) (see Fact 2.4 and Definition 2.3) and the base \(\mathcal{B}_{\mathcal{O}}\) have the required property.

Conversely, suppose \(\mathcal{T}\) and \(\mathcal{B}\) are given as in the statement. Then \((\mathcal{B}_{\mathcal{T}})^{+}_{\mathcal{M}}\) is a base for \(\mathcal{O}\), and therefore also \(\mathcal{O} \cap \mathcal{P}(X)^{+}_{\mathcal{M}}\) is a base for \(\mathcal{O}\). \(\square\)
Definition 2.7. Let $X$ be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. When $\mathcal{B}^+_\mathcal{M}$ is a base for a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$, we will say that $\mathcal{B}$ generates a topology on $\mathcal{M}$. (Obviously, the topology $\mathcal{O}_\mathcal{B}$ is of Tychonoff-type.

Proposition 2.8. Let $X$ be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. The family $\mathcal{B}$ generates a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$ if and only if the family $\mathcal{B}$ satisfies the following conditions:

(MB1) For any $M \in \mathcal{M}$ there exists a $U \in \mathcal{B}$ such that $M \subseteq U$;
(MB2) For any $U_1, U_2 \in \mathcal{B}$ and any $M \in \mathcal{M}$ with $M \subseteq U_1 \cap U_2$ there exists a $U_3 \in \mathcal{B}$ such that $M \subseteq U_3 \subseteq U_1 \cap U_2$.

Proof. It follows from Proposition 1.2.1 [9]. □

Corollary 2.9. Let $X$ be a set and $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{B} = \mathcal{B}^\cap$ and $X \in \mathcal{B}$, then $\mathcal{B}$ generates a Tychonoff-type topology on $\mathcal{M}$.

Definition 2.10. Let $X$ be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a natural family in $X$ if $\{x\} \in \mathcal{M}$ for all $x \in X$.

Corollary 2.11. Let $X$ be a set and $\mathcal{M}$ be a natural family in $X$. If $\mathcal{B} \subseteq \mathcal{P}(X)$ generates a topology on $\mathcal{M}$ (see Definition 2.7), then $\mathcal{B}$ is a base for a topology on $X$.

Proof. By Proposition 2.8, $\mathcal{B}$ satisfies the conditions (MB1) and (MB2). Since $\mathcal{M}$ is natural, this clearly implies that $\mathcal{B}$ satisfies the hypothesis of Proposition 1.2.1 [9]. So $\mathcal{B}$ is a base for a topology on $X$. □

Remark 2.12. Trivial examples show that there exist sets $X$ and (non-natural) families $\mathcal{M}, \mathcal{B} \subseteq \mathcal{P}(X)$ such that $\mathcal{B}^+_\mathcal{M}$ is a base for a topology on $\mathcal{M}$ but

(a) $\bigcup \mathcal{B} \neq X$, so that $\mathcal{B}$ cannot serve even as subbase of a topology on $X$ (take $X = \{0, 1\}, \mathcal{M} = \mathcal{B} = \{\{0\}\}$);
(b) $\mathcal{B}$ is not a base of a topology of $X$, although $\bigcup \mathcal{B} = X$ (take $X = \{0, 1, 2\}, \mathcal{M} = \mathcal{B} = \{\{0, 1\}, \{0, 2\}\}$).

The example of (b) shows also that if we substitute in 2.11 naturality of $\mathcal{M}$ with the condition “$\bigcup \mathcal{M} = X$” then we cannot prove that $\mathcal{B}$ is a base of a topology on $X$; however, it is easy to show that the condition “$\bigcup \mathcal{M} = X$” implies that $\bigcup \mathcal{B} = X$, i.e. $\mathcal{B}$ can serve as a subbase of a topology on $X$.

Of course, as it follows from Fact 2.7, if $\mathcal{B}^+_\mathcal{M}$ is a base of a topology $\mathcal{O}$ on $\mathcal{M}$, then $\tilde{\mathcal{B}} = \mathcal{B}^\cap \cup \{X\}$ is a base for a topology on $X$ and $\tilde{\mathcal{B}}^+_\mathcal{M}$ is a base of $\mathcal{O}$.

Corollary 2.13. Let $(X, T)$ be a topological space and let $\mathcal{B} \subseteq T$ be a base of $(X, T)$, closed under finite unions. Then $\mathcal{B}$ generates a topology of Tychonoff-type on $\text{Fin}(X)$ and $\text{Comp}(X)$.
Proof. It follows easily from Proposition 2.8. □

**Proposition 2.14.** Let $X$ be a set, $\mathcal{M}, \mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{P}(X)$, and suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ generate, respectively, some topologies (of Tychonoff-type) $\mathcal{O}_{\mathcal{B}_1}$ and $\mathcal{O}_{\mathcal{B}_2}$ on $\mathcal{M}$. Then $\mathcal{O}_{\mathcal{B}_1} = \mathcal{O}_{\mathcal{B}_2}$ if and only if the following conditions are satisfied:

- (CO1) For any $M \in \mathcal{M}$ and any $U_1 \in \mathcal{B}_1$ such that $M \subseteq U_1$ there exists $U_2 \in \mathcal{B}_2$ with $M \subseteq U_2 \subseteq U_1$;
- (CO2) For any $M \in \mathcal{M}$ and any $U_2 \in \mathcal{B}_2$ such that $M \subseteq U_2$ there exists $U_1 \in \mathcal{B}_1$ with $M \subseteq U_1 \subseteq U_2$.

**Proof.** It follows from 1.2.B. □

**Corollary 2.15.** Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}_1, \mathcal{B}_2$ be bases of $(X, \mathcal{T})$, closed under finite unions. Then they generate (see Corollary 2.13) equal topologies on $\text{Fin}(X)$ and $\text{Comp}(X)$. In particular, every topology of Tychonoff-type on $\text{Fin}(X)$ or on $\text{Comp}(X)$, generated by a base of $(X, \mathcal{T})$ which is closed under finite unions, coincides with the Tychonoff topology generated by $(X, \mathcal{T})$ on the corresponding set.

**Proof.** Check that conditions (CO1) and (CO2) of Proposition 2.14 are satisfied. □

**Corollary 2.16.** Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{B} \subseteq \mathcal{B}$, and suppose that $\mathcal{B}$ generates a topology of Tychonoff-type $\mathcal{O}$ on $\mathcal{M}$. Then $\mathcal{O}$ is the Tychonoff topology on $\mathcal{M}$ generated by $(X, \mathcal{T})$ if and only if for all $M \in \mathcal{M}$ and for all $V \in \mathcal{T}$ such that $M \subseteq V$, there exists $U \in \mathcal{B}$ with $M \subseteq U \subseteq V$. In this case we will say that $\mathcal{B}$ is an $\mathcal{M}$-base for $(X, \mathcal{T})$.

**Proof.** Put $\mathcal{B}_1 := \mathcal{T}$ and $\mathcal{B}_2 := \mathcal{B}$. Then condition (CO2) of Proposition 2.14 is trivially satisfied. The condition required in the statement is exactly condition (CO1). □

**Definition 2.17.** Let $X$ be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. A family $\mathcal{U} \subseteq \mathcal{P}(X)$ will be called an $\mathcal{M}$-cover of $A$ if $A = \bigcup \mathcal{U}$ and for all $M \in \mathcal{M}$ with $M \subseteq A$ there exists some $U \in \mathcal{U}$ such that $M \subseteq U$.

**Proposition 2.18.** Let $X$ be a set and $\mathcal{M}, A \subseteq \mathcal{P}(X)$. Then the following conditions are equivalent:

- (U1) For all $U \in A$ and for all $x \in U$, there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U$.
- (U2) For any $U \in A \cup \mathcal{M}$ and for any subfamily $\{U_\delta : \delta \in \Delta\}$ of $A \cup \mathcal{M}$, the equality $U_\Delta^+ = \bigcup_{\delta \in \Delta} (U_\delta)_\Delta^+$ holds if and only if the family $\{U_\delta\}_\delta \in \Delta$ is an $\mathcal{M}$-cover of $U$. 
Proof. Observe that, trivially, in condition (U1) we can replace the requirement ‘for all $U \in \mathcal{A}$’ with “for all $U \in \mathcal{A} \cup \mathcal{M}$”.

(U1)$\Rightarrow$(U2). Let $U^+_M = \bigcup_{\delta \in \Delta} (U^+_\delta)^+_M$, with $U^+_\delta \in \mathcal{A} \cup \mathcal{M}$ for all $\delta \in \Delta$. We will prove first that $\bigcup_{\delta \in \Delta} U^+_\delta = U$.

Let $x \in \bigcup_{\delta \in \Delta} U^+_\delta$. Then there exists a $\delta \in \Delta$ such that $x \in U^+_\delta$. By assumption, there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U^+_\delta$. Hence $M \in (U^+_\delta)^+_M$. Since $U^+_M = \bigcup_{\delta \in \Delta} (U^+_\delta)^+_M$, we obtain that $M \subseteq U$. Thus $x \in U$. Therefore, $\bigcup_{\delta \in \Delta} U^+_\delta \subseteq U$.

Conversely, let $x \in U$. By assumption, there exists an $M \in \mathcal{M}$ such that $x \in M \subseteq U$. Hence $M \in U^+_M = \bigcup_{\delta \in \Delta} (U^+_\delta)^+_M$. Therefore there exists a $\delta \in \Delta$ such that $M \in (U^+_\delta)^+_M$, i.e. $M \subseteq U^+_\delta$ and $x \in U^+_\delta \subseteq \bigcup_{\delta \in \Delta} U^+_\delta$.

We have verified that $\bigcup_{\delta \in \Delta} U^+_\delta = U$.

Suppose $M \in \mathcal{M}$ and $M \subseteq U$. Then $M \in U^+_M$ and therefore there exists some $\gamma \in \Delta$ with $M \in (U^+_\gamma)^+_M$. Hence $M \subseteq U^+_\gamma$.

This shows that the family $\{U^+_\delta\}_{\delta \in \Delta}$ is an $\mathcal{M}$-cover of $U$.

The other implication can be easily proved. (Let’s remark that condition (U1) is not used in the proof of this last implication.)

(U2)$\Rightarrow$(U1). Suppose $U \in \mathcal{A}$ and $x \in U$. Clearly, we have

$$U^+_M = \bigcup \{M^+_M : M \in \mathcal{M}, M \subseteq U\}.$$

Then, by assumption, the family $\{M \in \mathcal{M} : M \subseteq U\}$ is an $\mathcal{M}$-cover of $U$. Therefore $U = \bigcup \{M : M \in \mathcal{M}, M \subseteq U\}$. Hence there exists an $M \in \mathcal{M}$ with $x \in M \subseteq U$.

\[\mathbf{Proposition\ 2.19.}\ Let\ X\ be\ a\ set\ and\ \mathcal{M} \subseteq \mathcal{P}(X).\ Then\ the\ following\ conditions\ are\ equivalent: \]

(a) $\mathcal{M}$ is a natural family;

(b) For any $U \subseteq X$ and for any subfamily $\{U^+_\delta : \delta \in \Delta\}$ of $\mathcal{P}(X)$, the equality $U^+_M = \bigcup_{\delta \in \Delta} (U^+_\delta)^+_M$ holds if and only if the family $\{U^+_\delta\}_{\delta \in \Delta}$ is an $\mathcal{M}$-cover of $U$.

\[\mathbf{Proof.}\ Put\ A = \mathcal{P}(X)\ in\ Proposition\ 2.18.\]

\[\mathbf{Proposition\ 2.20.}\ Let\ X\ be\ a\ set,\ \mathcal{M} \subseteq \mathcal{P}(X),\ \mathcal{O}\ be\ a\ Tychonoff\ topology\ on\ \mathcal{M}\ generated\ by\ a\ topology\ \mathcal{T}\ on\ X\ and\ \mathcal{M}\ be\ a\ network\ in\ the\ sense\ of\ Arhangel’skiĭ\ for\ \mathcal{T}_\mathcal{O}.\ Then\ \mathcal{T} = \mathcal{B}_\mathcal{O}\ and\ \mathcal{O}\ is\ generated\ by\ a\ unique\ topology\ on\ X,\ namely\ by\ \mathcal{T}_\mathcal{O}\ (see\ Fact\ 2.4\ for\ the\ notation\ \mathcal{B}_\mathcal{O}\ and\ \mathcal{T}_\mathcal{O}).\]

\[\mathbf{Proof.}\ We\ only\ need\ to\ show\ that\ \mathcal{B}_\mathcal{O} \subseteq \mathcal{T}.\ Assume\ A \in \mathcal{B}_\mathcal{O}.\ Then\ A^+_M \in \mathcal{O}.\ Since\ \mathcal{T}\ generates\ \mathcal{O},\ we\ have\ A^+_M = \bigcup_{\delta \in \Delta} (U^+_\delta)^+_M,\ where\ U^+_\delta \in \mathcal{T}\ for\ all\ \delta \in \Delta.\ Clearly,\ U^+_\delta \in \mathcal{B}_\mathcal{O}\ for\ all\ \delta \in \Delta.\ By\ Proposition\ 2.18,\ we\ obtain\ that\ A = \bigcup_{\delta \in \Delta} U^+_\delta\ and\ therefore\ A \in \mathcal{T}.\]

\[\mathbf{Remark\ 2.21.}\ Trivial\ examples\ show\ that\ there\ exist\ sets\ X,\ families\ \mathcal{M} \subseteq \mathcal{P}(X)\ and\ Tychonoff\ topologies\ on\ \mathcal{M}\ which\ are\ generated\ by\ more\ than\ one\ topology\ on\ X.\]
Corollary 2.22. Let $X$ be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$ and $\mathcal{N}$ be a network in the sense of Arhangel’skiĭ for $\mathcal{T}_\mathcal{O}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_\mathcal{O} = \mathcal{T}_\mathcal{O}$ (see Fact 2.4 for the notations $\mathcal{B}_\mathcal{O}$ and $\mathcal{T}_\mathcal{O}$).

Proof. Suppose $\mathcal{B}_\mathcal{O} = \mathcal{T}_\mathcal{O}$. Then the topology $\mathcal{O}$ is generated by the topology $\mathcal{T}_\mathcal{O}$ on $X$ and hence, by definition, $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$.

Suppose $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$. Then $\mathcal{O}$ is generated by some topology $\mathcal{T}$ on $X$. By Proposition 2.20, we get $\mathcal{T} = \mathcal{B}_\mathcal{O}$. Hence $\mathcal{T}_\mathcal{O} = \mathcal{B}_\mathcal{O}$. □

Corollary 2.23. Let $X$ be a set, $\mathcal{M}$ be a natural family in $X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. Then $\mathcal{O}$ is a Tychonoff topology on $\mathcal{M}$ if and only if $\mathcal{B}_\mathcal{O} = \mathcal{T}_\mathcal{O}$.

Proof. A natural family $\mathcal{M}$ satisfies the hypothesis of Corollary 2.22. □

Proposition 2.24. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in $X$, $\mathcal{B} \subseteq \mathcal{T}$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$. Then
\[
\mathcal{B}_\mathcal{O}_\mathcal{B} = \{ A \subseteq X : A \text{ is } \mathcal{M}\text{-covered by some subfamily of } \mathcal{B} \}
\]
$\mathcal{B}_\mathcal{O}_\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{B}^\circ \subseteq \mathcal{B}_\mathcal{O}_\mathcal{B}$ (see Fact 2.4 for the notation $\mathcal{B}_\mathcal{O}_\mathcal{B}$).

Proof. It follows from Proposition 2.19 and Fact 2.3. □

Proposition 2.25. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M}$ be a natural family in $X$, $\mathcal{B}$ be a base for $(X, \mathcal{T})$ and suppose that $\mathcal{B}$ generates a topology $\mathcal{O}_\mathcal{B}$ on $\mathcal{M}$. Let $\mathcal{T}_\mathcal{O}_\mathcal{B}$ be the topology on $X$ induced by $(\mathcal{M}, \mathcal{O}_\mathcal{B})$. Then $\mathcal{T}_\mathcal{O}_\mathcal{B} = \mathcal{T}$.

Proof. We have, by Proposition 2.24, that $\mathcal{B} \subseteq \mathcal{B}_\mathcal{O}_\mathcal{B} \subseteq \mathcal{T}_\mathcal{O}_\mathcal{B}$. Thus $\mathcal{T} \subseteq \mathcal{T}_\mathcal{O}_\mathcal{B}$. As it is shown in Proposition 2.24, $\mathcal{B}_\mathcal{O}_\mathcal{B} \subseteq \mathcal{T}$ and hence $\mathcal{T}_\mathcal{O}_\mathcal{B} \subseteq \mathcal{T}$. So, $\mathcal{T} = \mathcal{T}_\mathcal{O}_\mathcal{B}$. □

Example 2.26. Let us show that in Proposition 2.25 the requirement “$\mathcal{M}$ is a natural family” is essential.

Let $X = (0, 1) \subset \mathbb{R}$ be the open unit interval with the usual topology, $\mathcal{M} = \{[a, b] : 0 < a < b < 1\}$, $\mathcal{B} = \{(a, b) : 0 < a < b < 1\}$. Then the family $\mathcal{B}$ satisfies conditions (MB1) and (MB2). Consider the set $A = (\frac{1}{2}, \frac{3}{4}) \subset \mathcal{M}$. We have $A^+_{\mathcal{M}} = (\frac{1}{2}, \frac{3}{4})^+_{\mathcal{M}} \subseteq \mathcal{B}^+_{\mathcal{M}}$ and therefore $A \in \mathcal{B}_\mathcal{O}_\mathcal{B}$ even though $A$ is not open in $X$.

Definition 2.27. Let $X$ be a set and $\mathcal{M}, \mathcal{U} \subseteq \mathcal{P}(X)$. We will say that $\mathcal{U}$ is an $\mathcal{M}$-closed family if for all $A \subseteq X$ such that $A$ is $\mathcal{M}$-covered by some subfamily of $\mathcal{U}$, we have that $A \in \mathcal{U}$.

Proposition 2.28. Let $X$ be a set and $\mathcal{M}, \mathcal{M}', \mathcal{U} \subseteq \mathcal{P}(X)$, $\mathcal{M} \subseteq \mathcal{M}'$. Suppose that $\mathcal{U}$ is an $\mathcal{M}$-closed family. Then $\mathcal{U}$ is an $\mathcal{M}'$-closed family too.
Proof. Suppose \( A \subseteq X \) is \( M' \)-covered by some subfamily \( U' \) of \( U \). Since \( M \subseteq M' \), the set \( A \) is also \( M \)-covered by \( U' \). By the hypothesis, \( U \) is an \( M \)-closed family. Hence \( A \in U \). \( \square \)

**Proposition 2.29.** Let \( X \) be a set, \( M \) be a natural family in \( X \) and \( O \) be a topology of Tychonoff-type on \( M \). Then \( B_O \) is an \( M \)-closed family (see Fact 2.4 for the notation \( B_O \)).

**Proof.** It follows from Proposition 2.19 \( \square \)

**Proposition 2.30.** Let \( (X,T) \) be a topological space, \( M \) be a natural family in \( X \) and \( B \subseteq T \) be an \( M \)-closed base of \( (X,T) \). Suppose that \( B \) generates a topology \( O_B \) on \( M \). Then \( B_O = B \), \( X \in B \) and \( B^\cap = B \) (see Fact 2.4 for the notation \( B_O \)).

**Proof.** Obviously, \( B_O \supseteq B \). Let us show that \( B_O \subseteq B \).

Let \( A \in B_O \). Then, by Proposition 2.24, \( A \) is \( M \)-covered by some subfamily of \( B \) and, since \( B \) is \( M \)-closed, we conclude that \( A \in B \). So, \( B_O = B \). Now Fact 2.4 implies that \( X \in B \) and \( B^\cap = B \). \( \square \)

**Definition 2.31.** Let \( X \) be a set and \( M, B \subseteq P(X) \). The ordered triple \((X,B,M)\) will be called a \( T \)-space if \( B \) is an \( M \)-closed family, \( X \in B \) and \( B^\cap = B \).

Note that if \( (X,B,M) \) is a \( T \)-space, then \( B \) is a base for a topology on \( X \).

**Proposition 2.32.** Let \( X \) be a set and \( M \) be a natural family in \( X \). Let \( TTT(T,X,M) \) be the set of all topologies of Tychonoff-type on \( M \). Denote by \( T-Sp(X,M) \) the set of all \( T \)-spaces of the form \((X,B,M)\). Then there is a bijective correspondence between the sets \( TTT(T,X,M) \) and \( T-Sp(X,M) \). Namely, consider the function \( \alpha : TTT(T,X,M) \to T-Sp(X,M) \), defined by \( \alpha(O) = (X,B,O,M) \) (see Fact 2.4 for the notation \( B_O \)), and the function \( \beta : T-Sp(X,M) \to TTT(T,X,M) \), defined by \( \beta((X,B,M)) = O_B \) (see Corollary 2.3 and Definition 2.4); then \( \alpha \) and \( \beta \) are bijections and each one is the inverse of the other one.

**Proof.** Let us show that the function \( \alpha \) is well-defined. Let \( O \in TTT(T,X,M) \). By Fact 2.4, the family \( B_O \) is closed under finite intersections and \( X \in B_O \). By Proposition 2.24, \( B_O \) is \( M \)-closed. Hence \((X,B,O,M) \in T-Sp(X,M) \).

We will prove now that the function \( \beta \) is well defined. Let \( B \subseteq P(X) \) be an \( M \)-closed family, closed under finite intersections and such that \( X \in B \). Then, by Corollary 2.3, \( O_B \) is a topology of Tychonoff-type on \( M \).

Proposition 2.30 gives that \( B_O = B \), i.e. \( \alpha \circ \beta = id_{T-Sp(X,M)} \).

To show that \( \beta \circ \alpha = id_{TTT(T,X,M)} \), let \( O \) be a topology of Tychonoff-type on \( M \). Then \( \beta(\alpha(O)) = O_B \). Since \( O \) is a topology of Tychonoff-type, \((B_O)^{\uparrow}_M \) is a base of \( O \). On the other hand, \((B_O)_M^{\uparrow} \) is, by definition, a base of \( O_B \). Hence \( O = O_B \). \( \square \)
Definition 2.33. We denote by $\mathcal{HT}$ (Hypertopologies of Tychonoff-type) the category defined as follows: its objects are all ordered triples $(X, \mathcal{M}, \mathcal{O})$ where $X$ is a set, $\mathcal{M}$ is a natural family in $X$ and $\mathcal{O}$ is a topology of Tychonoff-type on $\mathcal{M}$. To define the morphisms of $\mathcal{HT}$, let $(X, \mathcal{M}, \mathcal{O})$, $(X', \mathcal{M}', \mathcal{O}')$ be objects of $\mathcal{HT}$ and $f : X \to X'$ be a function between the sets $X$ and $X'$. We will say that $f$ generates a morphism $f_H$ of $\mathcal{HT}$ between $(X, \mathcal{M}, \mathcal{O})$ and $(X', \mathcal{M}', \mathcal{O}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{O}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) := f(M)$ (where the $M$ on the left-handside is regarded as an element of $\mathcal{M}$ and the $M$ on the right-handside is regarded as a subset of $X$) is continuous. The morphisms of $\mathcal{HT}$ are defined to be all $f_H$ generated in this way.

Remark 2.34. It is easy to see that not any continuous map between spaces of the type $(\mathcal{M}, \mathcal{O})$ appears as some $f_H$ (see Definition 2.33 for the notations). Indeed, let $(X, T)$ be a discrete space having more than one point; then the constant function $c : (CL(X), \mathcal{O}_T) \to (CL(X), \mathcal{O}_T)$, defined by $c(F) = X$ for all $F \in CL(X)$, is continuous but is not of the type $f_H$ (here $\mathcal{O}_T$ is the classical Tychonoff topology on $CL(X)$ (see Definition 2.3)).

Definition 2.35. We denote by $\mathcal{TH}$ the category defined as follows: its objects are all $T$-spaces $(X, \mathcal{B}, \mathcal{M})$ (see Definition 2.31). To define the morphisms of $\mathcal{TH}$, let $(X, \mathcal{B}, \mathcal{M})$, $(X', \mathcal{B}', \mathcal{M}')$ be objects of $\mathcal{TH}$ and $f : X \to X'$ be a function between the sets $X$ and $X'$. We will say that $f$ generates a morphism $f^T$ of $\mathcal{TH}$ between $(X, \mathcal{B}, \mathcal{M})$ and $(X', \mathcal{B}', \mathcal{M}')$ if $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. The morphisms of $\mathcal{TH}$ are defined to be all $f^T$ generated in this way.

Lemma 2.36. Let $X$, $X'$ be sets, $\mathcal{M} \subseteq \mathcal{P}(X)$ and $\mathcal{M}' \subseteq \mathcal{P}(X')$. Let $f : X \to X'$ be a function such that $f(\mathcal{M}) \subseteq \mathcal{M}'$ and let $f_m : \mathcal{M} \to \mathcal{M}'$ be defined by $f_m(M) := f(M)$. Then, for all $A' \subseteq X'$, we have

$$f_m^{-1}((A')^+_{\mathcal{M}'}) = (f^{-1}(A'))^+_{\mathcal{M}}.$$  

Proof. Let $M \in f_m^{-1}((A')^+_{\mathcal{M}'})$. Then $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $f(M) \subseteq A'$. Since $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(A')$, we obtain $M \in (f^{-1}(A'))^+_{\mathcal{M}}$.

Let $M \in (f^{-1}(A'))^+_{\mathcal{M}}$. Then $M \subseteq f^{-1}(A')$ and hence $f(M) \subseteq A'$. Therefore $f_m(M) \in (A')^+_{\mathcal{M}'}$, i.e. $M \in f_m^{-1}((A')^+_{\mathcal{M}'})$.

Theorem 2.37. The categories $\mathcal{HT}$ and $\mathcal{TH}$ are isomorphic.

Proof. We define a functor $F : \mathcal{HT} \to \mathcal{TH}$ as follows: for all $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{HT}|$, we put $F((X, \mathcal{M}, \mathcal{O})) := (X, B_\mathcal{O}, \mathcal{M})$ (see Fact 2.4 for the notation $B_\mathcal{O}$); for all morphisms $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$, we put $F(f_H) := f^T$ (see Definitions 2.33 and 2.35 for the notations $f_H$ and $f^T$).

We define a functor $G : \mathcal{TH} \to \mathcal{HT}$ as follows: for all $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{TH}|$, we put $G((X, \mathcal{B}, \mathcal{M})) := (X, \mathcal{M}, \mathcal{O}_\mathcal{B})$ (see Definition 2.7 for the notation $\mathcal{O}_\mathcal{B}$); for all morphisms $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$, we put $G(f^T) := f_H$. 


Let us check that $F$ and $G$ are well-defined.

Let $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{M}$. By Proposition 2.32, the triple $(X, \mathcal{B}_\mathcal{O}, \mathcal{M})$ is a $T$-space, so that $F(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{T}_\mathcal{H}|$.

Let $f_H : (X, \mathcal{M}, \mathcal{O}) \to (X', \mathcal{M}', \mathcal{O}')$ be a morphism in $\mathcal{H}_\mathcal{T}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and the induced function on $\mathcal{M}$, $f_m : (\mathcal{M}, \mathcal{O}) \to (\mathcal{M}', \mathcal{O}')$, defined by $f_m(M) = f(M)$, is continuous. To check that

$$f^T : F(X, \mathcal{M}, \mathcal{O}) \to F(X', \mathcal{M}', \mathcal{O}')$$

(i.e. $f^T : (X, \mathcal{B}_\mathcal{O}, \mathcal{M}) \to (X', \mathcal{B}_\mathcal{O}', \mathcal{M}')$) is a morphism in $\mathcal{T}_\mathcal{H}$, we need to show that $f^{-1}(\mathcal{B}_\mathcal{O}') \subseteq \mathcal{B}_\mathcal{O}$. Let $B' \in \mathcal{B}_\mathcal{O}$. Then $(B')_{\mathcal{M}'} \in \mathcal{O}'$. By the continuity of $f_m$, we have that $f_m^{-1}((B')_{\mathcal{M}'}) \in \mathcal{O}$. By Lemma 2.36,

$$f_m^{-1}((B')_{\mathcal{M}'}) = (f^{-1}(B'))_{\mathcal{M}'}.$$  

Hence $(f^{-1}(B'))_{\mathcal{M}'} \in \mathcal{O}$, i.e. $f^{-1}(B') \in \mathcal{B}_\mathcal{O}$. So, we have proved that $F$ is well-defined. Clearly, $F$ is a functor.

Let now $(X, \mathcal{B}, \mathcal{M}) \in |\mathcal{T}_\mathcal{H}|$. Then, by Proposition 2.32, the topology $\mathcal{O}_\mathcal{B}$ is a Tychonoff-type topology on $\mathcal{M}$. Hence, $G((X, \mathcal{B}, \mathcal{M})) \in |\mathcal{T}_\mathcal{H}|$.

Let $f^T : (X, \mathcal{B}, \mathcal{M}) \to (X', \mathcal{B}', \mathcal{M}')$ be a morphism in $\mathcal{T}_\mathcal{H}$. Then $f(\mathcal{M}) \subseteq \mathcal{M}'$ and $f^{-1}(B') \subseteq \mathcal{B}$. To check that

$$f_H : (X, \mathcal{M}, \mathcal{O}_\mathcal{B}) \to (X', \mathcal{M}', \mathcal{O}_\mathcal{B}')$$

is a morphism in $\mathcal{H}_\mathcal{T}$, we need to check that the function $f_m : (\mathcal{M}, \mathcal{O}_\mathcal{B}) \to (\mathcal{M}', \mathcal{O}_\mathcal{B}')$, defined by $f_m(M) = f(M)$, is continuous. By definition (see 2.7), $(B')_{\mathcal{M}'}$ is a base of the topology $\mathcal{O}_\mathcal{B}'$. Let $B' \in \mathcal{B}'$. Then $(B')_{\mathcal{M}'} \in (B')_{\mathcal{M}'}$. By assumption, $f^{-1}(B') \in \mathcal{B}$. Hence $(f^{-1}(B'))_{\mathcal{M}} \in \mathcal{O}_\mathcal{B}$. Since, by Lemma 2.36, $(f^{-1}(B'))_{\mathcal{M}} = f_m^{-1}((B')_{\mathcal{M}'})$, we obtain that $f_m^{-1}((B')_{\mathcal{M}'}) \in \mathcal{O}_\mathcal{B}$. Therefore, the function $f_m$ is continuous. So, $G$ is well-defined. Obviously, $G$ is a functor.

By Proposition 2.32, we have $F \circ G = \text{id}_{\mathcal{T}_\mathcal{H}}$ and $G \circ F = \text{id}_{\mathcal{T}_\mathcal{H}}$ on the objects. The equalities are clearly true for the morphisms. Hence $F$ and $G$ are isomorphisms. \hfill \Box

We recall that a topological space $(X, \mathcal{T})$ is called a $P_\infty$-space (see [1, 3]) if $\mathcal{T}$ is closed under arbitrary intersections.

**Lemma 2.38.** A space $(X, \mathcal{T})$ is a $P_\infty$-space if and only if it has a base $\mathcal{B}$ closed under arbitrary intersections.

**Proof.** Assume $\mathcal{T}$ has a base $\mathcal{B}$ closed under arbitrary intersections. Let $U \subseteq \mathcal{T}$. Since $\emptyset$ is an open set, we can assume that $\bigcap \mathcal{U} \neq \emptyset$. Let $x \in \bigcap \mathcal{U}$. For any $U \in \mathcal{U}$, let $B_U \in \mathcal{B}$ be such that $x \in B_U \subseteq U$. Then

$$x \in \bigcap \{B_U : U \in \mathcal{U}\} \subseteq \bigcap \mathcal{U}$$

and, by assumption, $\bigcap \{B_U : U \in \mathcal{U}\} \in \mathcal{B}$. Hence, $\bigcap \mathcal{U} \in \mathcal{T}$. \hfill \Box

**Proposition 2.39.** Let $(X, \mathcal{M}, \mathcal{O}) \in |\mathcal{H}_\mathcal{T}|$. The topological space $(\mathcal{M}, \mathcal{O})$ is a $P_\infty$-space if and only if the family $\mathcal{B}_\mathcal{O}$ is closed under arbitrary intersections. If $(\mathcal{M}, \mathcal{O})$ is a $P_\infty$-space then $(X, \mathcal{T}_\mathcal{O})$ is a $P_\infty$-space. If $\mathcal{O}$ is a
Tychonoff topology on $M$, then $(M, O)$ is a $P_\infty$-space if and only if $(X, T_O)$ is a $P_\infty$-space (see Fact 2.4 for the notations $B_O$ and $T_O$).

**Proof.** The first two assertions follow from Lemma 2.38, Fact 2.1(a) and the definitions of $B_O$ and $T_O$. The last assertion follows now from Corollary 2.22. □

**Corollary 2.40.** Let $\mathcal{H}_T^\infty$ be the full subcategory of $\mathcal{H}_T$ having as objects all triples $(X, M, O) \in |\mathcal{H}_T|$ for which the space $(M, O)$ is a $P_\infty$-space. Let $\mathcal{T}_H^\infty$ be the full subcategory of $\mathcal{T}_H$ whose objects are all $(X, B, M) \in |\mathcal{T}_H|$ such that the family $B$ is closed under arbitrary intersections. Then $\mathcal{H}_T^\infty$ and $\mathcal{T}_H^\infty$ are isomorphic.

**Proof.** It follows from (the proof of) Theorem 2.37 and Proposition 2.39. □

**Example 2.41.** We will show that there exists $(X, M, O) \in |\mathcal{H}_T|$ such that $(X, T_O)$ is a $P_\infty$-space but $(M, O)$ is not a $P_\infty$-space.

Let $X = \omega$, 

$$B = \{\{n\} : n \in \omega\} \cup \{A \subseteq \omega : |\omega \setminus A| < \aleph_0\} \cup \{\emptyset\},$$

and $\mathcal{P}(\omega) \supseteq M \supseteq Fin_2(\omega) \setminus \{\emptyset\}$. The family $B$ is a base of the discrete topology $\mathcal{T}$ on $\omega$, it is closed under finite intersections (but not under infinite intersections) and $X \in B$. Let us show that $B$ is $M$-closed.

Let $\{B_\delta\}_{\delta \in \Delta}$ be a subfamily of $B \setminus \{\emptyset\}$ which is an $M$-cover of a subset $B$ of $X$. Without loss of generality, we can assume that there exist at least two indices $\delta_1$ and $\delta_2$ such that $B_{\delta_1} \neq B_{\delta_2}$. Then there is at least one $\delta \in \Delta$ such that $|\omega \setminus B_\delta| < \aleph_0$; otherwise we would have $B_{\delta_1} = \{n_{\delta_1}\}$, $B_{\delta_2} = \{n_{\delta_2}\}$ and the set $F = \{n_{\delta_1}, n_{\delta_2}\}$, which belongs to $M$, would be contained in $B_\delta$ for any $\delta \in \Delta$. Therefore $|\omega \setminus B| < \aleph_0$ and, hence, $B \in M$.

Put $O := B_B$. Then, by Proposition 2.30, $B = B_O$ and hence $\mathcal{T} = T_O$. Since $B$ is not closed under arbitrary intersections, we obtain, by Proposition 2.39, that $(M, O)$ is not a $P_\infty$-space. Clearly, the space $(X, T)$ is a $P_\infty$-space because it is discrete. Observe that $O$ is not a Tychonoff topology since $B_O \neq T_O$.

**Example 2.42.** Two more examples of Tychonoff-type, non Tychonoff topologies on some families $M \subseteq \mathcal{P}(X)$.

Let $X$ be a set with more than two elements. Let $M = Fin(X) \setminus \{\emptyset\}$ and let $O = \{\{x\} : x \in A\} : A \subseteq X \cup \{M\} \cup \{\emptyset\}$. Then $O$ is a topology on $M$.

$O$ is a topology of Tychonoff-type since 

$$O \cap \mathcal{P}(X)_M^+ = \{\{x\} : x \in X\} \cup \{X_M^+\} \cup \{\emptyset\}$$

is a base for $O$. Clearly $(M, O)$ is a $P_\infty$-space.
We have \( B_\mathcal{O} = \{\{x\} : x \in X\} \cup \{X\} \cup \{\emptyset\} \), and therefore \( \mathcal{T}_\mathcal{O} \) is the discrete topology. By Proposition 2.22, since \( B_\mathcal{O} \neq \mathcal{T}_\mathcal{O} \), \( \mathcal{O} \) is not a Tychonoff topology on \( \mathcal{M} \).

Observe that \((\mathcal{M}, \mathcal{O})\) is not a \( T_0 \)-space. In fact, the only neighbourhood of any \( F \in \mathcal{M} \) such that \(|F| \geq 2\) is \( \mathcal{M} \), and we are assuming \(|X| > 2\).

We consider now the natural family \( \mathcal{M}' = \mathcal{P}(X) \setminus \{\emptyset\} \) and we define \( \mathcal{O} \) as above. \( \mathcal{O} \) is a Tychonoff-type topology on \( \mathcal{M}' \) but it is not a Tychonoff topology. Again \( \mathcal{T}_\mathcal{O} \) is the discrete topology on \( X \). Hence \( \mathcal{M}' = \mathcal{C}(\mathcal{X}, \mathcal{T}_\mathcal{O}) \).

We observe as before that \((\mathcal{M}', \mathcal{O})\) is not a \( T_0 \)-space. Note also that the family \( B_\mathcal{O} \) is \( \mathcal{M}'\)-closed for every natural family \( \mathcal{M}' \) in \( X \) which contains all two-points subsets of \( X \) and \( \emptyset \notin \mathcal{M}' \).

We will briefly discuss now some topological properties of the hyperspaces \((\mathcal{M}, \mathcal{O})\) with Tychonoff-type topologies \( \mathcal{O} \).

**Fact 2.43.** Let \( X \) be a set, \( \mathcal{M} \subseteq \mathcal{P}(X) \) and \( \mathcal{O} \) be a topology of Tychonoff-type on \( \mathcal{M} \) generated by a subfamily \( \mathcal{B} \) of \( \mathcal{P}(X) \). Then:

(a) the topological space \((\mathcal{M}, \mathcal{O})\) is a \( T_0 \)-space (resp., \( T_1 \)-space) if and only if for any \( F, G \in \mathcal{M} \) with \( F \neq G \), there exists a \( B \in \mathcal{B} \) such that either \( F \subseteq B \) and \( G \not\subseteq B \), or \( G \subseteq B \) and \( F \not\subseteq B \) (resp., \( F \subseteq B \) and \( G \not\subseteq B \)).

(b) if for any \( x \in X \) and for any \( F \in \mathcal{M} \) with \( x \notin F \), there exists a \( B \in \mathcal{B} \) such that \( F \subseteq B \) and \( x \notin B \), then \((\mathcal{M}, \mathcal{O})\) is a \( T_0 \)-space.

**Remark 2.44.** Let us note that Fact 2.43(b) implies the following assertion, which was mentioned in [5], section 2 (after Lemma 3) (the requirement that \( X \in \Omega \) has to be added there): if \((X, \mathcal{T})\) is a regular \( T_1 \)-space, \( \mathcal{M} \) is a family consisting of closed subsets of \((X, \mathcal{T})\) and \( \mathcal{B} \) is a base of \((X, \mathcal{T})\) such that \( \mathcal{B} = \mathcal{B}^\circ \) and \( U \in \mathcal{B} \) implies that \( X \setminus \overline{U} \in \mathcal{B} \), then \((\mathcal{M}, \mathcal{O}_\mathcal{B})\) is a \( T_0 \)-space.

**Fact 2.45.** Let \((X, \mathcal{M}, \mathcal{O}) \in |\mathcal{H}\mathcal{T}|\). Then the correspondence \((X, \mathcal{T}_\mathcal{O}) \to (\mathcal{M}, \mathcal{O}), x \mapsto \{x\}\), is a homeomorphic embedding. Hence, we have, in particular, that:

(a) \( w(X, \mathcal{T}_\mathcal{O}) \leq w(\mathcal{M}, \mathcal{O}) \);

(b) if \((\mathcal{M}, \mathcal{O})\) is a \( T_0 \)-space then \((X, \mathcal{T}_\mathcal{O})\) is a \( T_0 \)-space.

**Fact 2.46.**

(a) Let \( X \) be a set, \( \mathcal{M} \subseteq \mathcal{P}(X) \) be a family such that there exist \( F, G \in \mathcal{M} \) with \( F \subseteq G \) and \( F \neq G \), and let \( \mathcal{O} \) be a topology of Tychonoff-type on \( \mathcal{M} \). Then \((\mathcal{M}, \mathcal{O})\) is not a \( T_1 \)-space.

(b) If \((X, \mathcal{M}, \mathcal{O}) \in |\mathcal{H}\mathcal{T}|\) then \((\mathcal{M}, \mathcal{O})\) is a \( T_1 \)-space if and only if \((X, \mathcal{T}_\mathcal{O})\) is a \( T_1 \)-space and \( \mathcal{M} = \{\{x\} : x \in X\} \).

**Fact 2.47.** Let \((X, \mathcal{M}, \mathcal{O}) \in |\mathcal{H}\mathcal{T}|\). Then \((\mathcal{M}, \mathcal{O})\) is a compact space if and only if any \( \mathcal{M} \)-cover of \( X \), consisting of elements of \( \mathcal{B}_\mathcal{O} \), has a finite \( \mathcal{M} \)-subcover.
Proof. It follows from Proposition 2.19.

Examples 2.48. There are many examples of “very nice” spaces $X$ with non-$T_0$-hyperspaces $(\mathcal{M}, \mathcal{O}_B)$ (see Examples 2.42 and 3.19). As an example of a non-$T_0$-space $(X, \mathcal{T})$ with a $T_0$-hyperspace $(\mathcal{M}, \mathcal{O}_T)$, consider the two-points space $X = \{0, 1\}$, with $\mathcal{T} = \mathcal{M} = \{\emptyset, X\}$.

There exist non-compact spaces $X$ such that $(\mathcal{CL}(X), \mathcal{O}_B)$ is a compact non-$T_0$-space (e.g., the space $(\mathcal{CL}(\mathbb{R}), \mathcal{O}_B)$, described in Example 3.19).

To get an example of a non-compact space $X$ and a natural family $\mathcal{M}$ such that $(\mathcal{M}, \mathcal{O}_B)$ is a compact $T_0$-space, consider $X := \mathbb{R}$ with its natural topology, $\mathcal{M} := \mathcal{Fin}_2(\mathbb{R}) \cup \{\mathbb{R}\}$ and take $\mathcal{B}$ as in Example 3.19.

As an example of a compact space $(X, \mathcal{T})$ with a non-compact hyperspace $(\mathcal{M}, \mathcal{O}_T)$, consider the unit interval $X = [0, 1]$ with its natural topology and put $\mathcal{M} = \{\{x\} : x \in (0, 1]\}$.

The next three propositions are generalizations of, respectively, Propositions 1, 2 and 3 of [10], and have proofs similar to those given in [10]. (Let us note that in Proposition 2 of [10] the requirement “$\emptyset \notin C$” has to be added.)

Proposition 2.49. Let $(X, \mathcal{M}, \mathcal{O}) \in \mathcal{HT}$, $w(\mathcal{M}, \mathcal{O}) = \aleph_0$, $(X, \mathcal{T}_\mathcal{O})$ be a $T_1$-space, $\mathcal{B}_\mathcal{O}$ be closed under countable unions and $\mathcal{M}$ contain all infinite countable closed subsets of $(X, \mathcal{T}_\mathcal{O})$. Then $(X, \mathcal{T}_\mathcal{O})$ is a compact space.

Proposition 2.50. Let $(X, \mathcal{M}, \mathcal{O}) \in \mathcal{HT}$ and $\emptyset \notin \mathcal{M}$. Then $d(\mathcal{M}, \mathcal{O}) = d(X, \mathcal{T}_\mathcal{O})$.

Proposition 2.51. Let $(X, \mathcal{M}, \mathcal{O}) \in \mathcal{HT}$ and $\emptyset \notin \mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ has isolated points if and only if $(X, \mathcal{T}_\mathcal{O})$ has isolated points.

3. On $\mathcal{O}$-commutative spaces

3.1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Recall that $A$ is said to be $2$-combinatorially embedded in $X$ (see [4]) if the closures in $X$ of any two disjoint closed in $A$ subsets of $A$ are disjoint.

Definition 3.2. Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ and $\mathcal{B} \subseteq \mathcal{P}(X)$. We will say that $A$ is $2_\mathcal{B}$-combinatorially embedded in $X$ if for any $F \in \mathcal{CL}(A)$ and for any $U \in \mathcal{B}$ with $F \subseteq U$, there exists a $V \in \mathcal{B}$ such that $\mathcal{T}_X \subseteq V$ and $V \cap A \subseteq U$.

Proposition 3.3. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then $A$ is $2$-combinatorially embedded in $X$ if and only if $A$ is $2_T$-combinatorially embedded in $X$.

Proof. $(\Rightarrow)$ Let $H \in \mathcal{CL}(A)$, $V \in \mathcal{T}$ and $H \subseteq V$. We put $U = V \cap A$ and $F = A \setminus U$. Then $F$ and $H$ are two disjoint closed subsets of $A$. Hence,
by assumption, they have disjoint closures in $X$, i.e. $\overline{F}^X \cap \overline{G}^X = \emptyset$. Let $W = X \setminus \overline{F}^X$. Then $W$ is open in $X$, $W \cap A = U = V \cap A$ and $\overline{F}^X \subseteq W$.

$(\Leftarrow)$ Let $F$ and $G$ be two disjoint closed subsets of $A$. Put $V = X \setminus \overline{G}^X$. Then $V$ is open in $X$ and $F \subseteq V$. Hence, by assumption, there exists an open set $W$ such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Let $U = A \setminus G$. Then $Ex_{A,X} \subseteq V$. Hence $V \cap A = U$ and $W \subseteq V$. We conclude that $\overline{F}^X \subseteq W \subseteq V = X \setminus \overline{G}^X$, i.e. $\overline{F}^X \cap \overline{G}^X = \emptyset$. \hfill $\Box$

Remark 3.4. In Example 3.22 below we will show that there exist spaces $(X, \mathcal{T})$, subspaces $A$ of $X$ and bases $\mathcal{B}$ of $\mathcal{T}$ such that $A$ is $2_B$-combinatorially embedded in $X$ but $A$ is not 2-combinatorially embedded in $X$.

**Proposition 3.5.** Let $(X, \mathcal{T})$ be a $T_1$-space, $\mathcal{O}$ be a topology of Tychonoff-type on $CL(X)$ and $A \subseteq X$. Put $B_A = \{U \cap A : U \in \mathcal{B} \}$ (see Fact 2.3 for the notation $B_\mathcal{O}$). The family $B_A$ generates a topology of Tychonoff-type $\mathcal{O}_A$ on $CL(A)$. The function $i_{A,X} : (CL(A), \mathcal{O}_A) \to (CL(X), \mathcal{O})$, defined by $i_{A,X}(F) := \overline{F}^X$, is inversely continuous (i.e. it is injective and its inverse, defined on $i_{A,X}(CL(A))$, is a continuous function) if and only if the set $A$ is $2_{B_\mathcal{O}}$-combinatorially embedded in $X$.

**Proof.** The family $\mathcal{B} := B_\mathcal{O}$ is closed under finite intersections and $X \subseteq B$ (see Fact 2.4). Hence the family $B_A$ is closed under finite intersections and $A \subseteq B_A$. Therefore, by Corollary 2.9, $B_A$ generates a topology of Tychonoff-type $\mathcal{O}_A$ on $CL(A)$.

The function $i_{A,X} : (CL(A), \mathcal{O}_A) \to (CL(X), \mathcal{O})$ is clearly injective. Denote by $g$ its inverse defined on $i_{A,X}(CL(A))$, i.e. $g : i_{A,X}(CL(A)) \to CL(A)$.

$(\Rightarrow)$ Let $H \in CL(A)$, $U \in \mathcal{B}$ and $H \subseteq U$. Then $H \in (U \cap A)^+_{CL(A)} \in \mathcal{O}_A$. Since $g(\overline{F}^X) = H$, the continuity of $g$ implies that there exists a $V \in \mathcal{B}$ such that $\overline{F}^X \subseteq V$ and $g(V^+_{CL(X)} \cap i_{A,X}(CL(A))) \subseteq (U \cap A)^+_{CL(A)}$.

Then $V \cap A \subseteq U \cap A$. Indeed, let $x \in V \cap A$. Since $X$ is a $T_1$-space, we obtain that $\{x\} \in V^+_{CL(X)} \cap i_{A,X}(CL(A))$ and $g(\{x\}) = \{x\}$.

Hence $x \in U \cap A$. So, $A$ is $2_B$-combinatorially embedded in $X$.

$(\Leftarrow)$ Let $F \in i_{A,X}(CL(A))$ and $g(F) = H$. Then $F = \overline{F}^X$ and $H \in CL(A)$. Let $U \in B_A$ be such that $H \subseteq U$. Then there exists a $V \in \mathcal{B}$ with $V \cap A = U$. Hence $H \subseteq V$. Since $A$ is $2_B$-combinatorially embedded in $X$, there exists a $W \in \mathcal{B}$ such that $F = \overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A = U$. Then $F \in W^+_{CL(X)} \in \mathcal{O}$. We will show that $g(W^+_{CL(X)} \cap i_{A,X}(CL(A))) \subseteq U^+_{CL(A)}$. $\Box$
Indeed, let $K \in \mathcal{C}L(A)$, $G = \overline{K}^X$ and $G \subseteq W$. Then $g(G) = K$ and
$$K = G \cap A \subseteq W \cap A \subseteq V \cap A = U,$$
i.e. $K \in U^+_{\mathcal{C}L(A)}$, as we have to show. Hence, $g$ is a continuous function. □

**Corollary 3.6** ([6], Theorem 2.1). *If in Proposition 3.5 we take $\mathcal{O}$ to be the Tychonoff topology on $\mathcal{C}L(X)$ generated by $(X, \mathcal{T})$ then the function $i_{A,X}$ is inversely continuous if and only if $A$ is 2-combinatorially embedded in $X$.*

**Proof.** It follows from Propositions 3.5, 2.20 and 3.3. □

**Corollary 3.7.** Let $(X, \mathcal{T})$ be a $T_2$-space, $A \subseteq X$ and $\mathcal{O}$ be a topology of Tychonoff-type on $\mathcal{C}L(X)$ generated by a subfamily of $\mathcal{T}$. Let $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Assume that the following condition is satisfied:

(*) For any $U \in \mathcal{T}$ and for all countable $F \in \mathcal{C}L(A)$ such that $|A \setminus F| \geq \aleph_0$ and $F \subseteq U$, there exists a $V \in \mathcal{B}_\mathcal{O}$ with $F \subseteq V \subseteq U$.

Then the set $A$ is sequentially closed.

**Proof.** Put $\mathcal{B} := \mathcal{B}_\mathcal{O}$. Then, by Proposition 2.24, $\mathcal{B} \subseteq \mathcal{T}$. Assume that the set $A$ is not sequentially closed. Then there exists a sequence $(x_n)_{n \in \omega}$ in $A$ and a point $x \in X \setminus A$ such that $\lim_{n \to \infty} x_n = x$. Without loss of generality we can assume $x_n \neq x_m$ for all $n \neq m$.

Let us consider the sets $F = \{x_{2n} : n \in \omega\}$ and $G = \{x_{2n-1} : n \in \omega\}$. Put $U = X \setminus \overline{G}^X$. Then $F$ is a countable closed subset of $A$, $|A \setminus F| \geq \aleph_0$, $F \subseteq U$ and $U \in \mathcal{T}$. By (*), there exists a $V \in \mathcal{B}$ such that $F \subseteq V \subseteq U$. Since we are assuming that the function $i_{A,X}$ is inversely continuous, we obtain, by Proposition 3.3, that the set $A$ is $2_\mathcal{B}$-combinatorially embedded in $X$. Hence there exists a $W \in \mathcal{B}$ such that $\overline{F}^X \subseteq W$ and $W \cap A \subseteq V \cap A$. Then $x \in W$, because $x \in \overline{F}^X$. Since $W \in \mathcal{T}$ and $x$ is a limit point of $G$, we have $G \cap W \neq \emptyset$. However this is a contradiction because
$$W \cap A \subseteq V \cap A \subseteq U = X \setminus \overline{G}^X,$$
and hence $G \cap W = \emptyset$. Therefore, $A$ is sequentially closed. □

**Remark 3.8.** In Example 3.22 below we will show that condition (*) of Corollary 3.7 is essential, i.e., if we omit it, then the set $A$ could fail to be sequentially closed.

**Corollary 3.9** ([6], Corollary 2.3). *Let $(X, \mathcal{T})$ be a $T_2$-space, $A \subseteq X$, $\mathcal{O}$ be the Tychonoff topology on $\mathcal{C}L(X)$ generated by $(X, \mathcal{T})$ and $i_{A,X}$ be inversely continuous (see Proposition 3.5 for the notation $i_{A,X}$). Then the set $A$ is sequentially closed.*
Proof. We have, by Proposition 2.20, that $B_O = T$. Then condition (*) of Corollary 3.7 is trivially satisfied. Hence, by Corollary 3.7, $A$ is sequentially closed. □

**Corollary 3.10.** Let $(X,T)$ be a sequential $T_2$-space, $A \subseteq X$ and $O$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $T$. Assume that condition (*) of Corollary 3.7 is satisfied. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):

(a) $i_{A,X}$ is a homeomorphic embedding;
(b) $i_{A,X}$ is inversely continuous;
(c) $A$ is closed in $X$.

Proof. It is clear that (a) implies (b). The implication (c)$\Rightarrow$(a) is true for any $X$, because if $A$ is a closed subset of $X$ then $i_{A,X}$ is the inclusion map. Let us show that (b) implies (c). By Corollary 3.7, $A$ is sequentially closed. Since $X$ is a sequential space, we obtain that the set $A$ is closed. □

**Corollary 3.11** ([6], Corollary 2.4). Let $(X,T)$ be a sequential $T_2$-space, $A \subseteq X$ and $O$ be the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X,T)$. Then the following conditions are equivalent (see Proposition 3.5 for the notation $i_{A,X}$):

(a) $i_{A,X}$ is a homeomorphic embedding;
(b) $i_{A,X}$ is inversely continuous;
(c) $A$ is closed in $X$.

**Definition 3.12.** Let $(X,T)$ be a topological space and let $O$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$. The space $(X,T)$ is called $O$-commutative if for any $A \subseteq X$ the function $i_{A,X}$, defined in Proposition 3.5, is a homeomorphic embedding.

When $O$ is the Tychonoff topology on $\mathcal{CL}(X)$ generated by $(X,T)$, the notion of “$O$-commutative space” coincides with the notion of “commutative space”, introduced in [4, 5].

**Corollary 3.13.** Let $(X,T)$ be a sequential $T_2$-space, $O$ be a topology of Tychonoff-type on $\mathcal{CL}(X)$ generated by a subfamily of $T$ and let condition (*) of Corollary 3.7 be satisfied for every subspace $A$ of $X$. Then $X$ is $O$-commutative if and only if $X$ is discrete.

Proof. It follows from Corollary 3.10. □

**Corollary 3.14** ([6], Corollary 2.5). If $X$ is a sequential $T_2$-space then $X$ is commutative if and only if $X$ is discrete.

**Example 3.15.** Let us show that there exist spaces $X$ and topologies $O$ of Tychonoff-type on $\mathcal{CL}(X)$ that are not Tychonoff topologies and that satisfy all hypothesis of Corollary 3.13.
Let \( X = \mathcal{D}(\aleph_1) \) be the discrete space of cardinality \( \aleph_1 \). Let
\[
\mathcal{B} = \{ A \subseteq X : |A| \leq \aleph_0 \} \cup \{ X \}
\]
and \( \mathcal{M} = \text{CL}(X) \). Then \( \mathcal{M} \) is a natural family on \( X \), \( \mathcal{B} \) is \( \mathcal{M} \)-closed, \( \mathcal{B}^\cap = \mathcal{B} \), \( X \in \mathcal{B} \) and \( \mathcal{B} \) is a base for the discrete topology on \( X \). Let \( \mathcal{O}_\mathcal{B} \) be the topology on \( \mathcal{M} \) generated by \( \mathcal{B} \). Then \( \mathcal{O}_\mathcal{B} \) is a topology of Tychonoff-type on \( \mathcal{M} \), however it is not a Tychonoff topology. In fact, by Proposition 2.30, \( \mathcal{B} = \mathcal{B}_{\mathcal{O}_\mathcal{B}} \). Hence \( \mathcal{B}_{\mathcal{O}_\mathcal{B}} \neq \mathcal{T}_{\mathcal{O}_\mathcal{B}} \) and, by Corollary 2.23, \( \mathcal{O} \) cannot be a Tychonoff topology. Obviously, \( \mathcal{B} \) satisfies condition (*) of Corollary 3.7 for any subspace \( A \) of \( X \).

**Definition 3.16.** Let \((X, \mathcal{T})\) be a topological space and \( \mathcal{O} \) be a topology of Tychonoff-type on \( \text{CL}(X) \). The space \((X, \mathcal{T})\) is called \( \mathcal{O} \)-HS-space if, for any \( A \subseteq X \), the function \( i_{A,X} \), defined in Proposition 3.5, is continuous.

When \( \mathcal{O} \) is the Tychonoff topology on \( \text{CL}(X) \) generated by \((X, \mathcal{T})\), the notion of “\( \mathcal{O} \)-HS-space” coincides with the notion of “HS-space”, introduced in [2, 3].

**Corollary 3.17.** Let \((X, \mathcal{T})\) be a \( T_1 \)-space and \( \mathcal{O} \) be a topology of Tychonoff-type on \( \text{CL}(X) \). Then \( X \) is an \( \mathcal{O} \)-commutative space if and only if \( X \) is an \( \mathcal{O} \)-HS-space and every subset \( A \) of \( X \) is 2\( _{\mathcal{O}_\mathcal{B}} \)-combinatorially embedded in \( X \).

**Proof.** It follows from Proposition 3.3.
\( \square \)

**Corollary 3.18** ([3], Corollary 2.2). A \( T_1 \)-space \( X \) is commutative if and only if \( X \) is an HS-space and every subspace of \( X \) is 2-combinatorially embedded in \( X \).

**Proof.** It follows from Corollary 3.17 and Proposition 3.3.
\( \square \)

**Example 3.19.** We will describe two Tychonoff-type, non Tychonoff topologies on two different subfamilies of \( \mathcal{P}(\mathbb{R}) \) generated by the family \( \mathcal{B} \) of all open intervals of \( \mathbb{R} \). One of the resulting spaces will be \( T_0 \) and the other one will not.

Let \( \mathcal{T} \) be the natural topology on \( X := \mathbb{R} \). Then the family \( \mathcal{B} \) of all open intervals in \( X \) is a base for \( \mathcal{T} \), it is closed under finite intersections and \( X \in \mathcal{B} \). Put \( \mathcal{M} := \text{CL}(X, \mathcal{T}) \) and \( \mathcal{M}' := \text{Fin}_2(X) \). They are natural families. The family \( \mathcal{B} \) is both \( \mathcal{M}' \)-closed and \( \mathcal{M} \)-closed. Indeed, let \( U \subseteq X \) be \( \mathcal{M}' \)-covered by a subfamily \( \mathcal{B}_U \) of \( \mathcal{B} \). Then \( U \in \mathcal{T} \) and for every \( x, y \in U \) there exists an open interval \((\alpha, \beta) \in \mathcal{B}_U \) containing the points \( x \) and \( y \). Hence \( U \) is a connected open set in \( \mathbb{R} \), i.e. \( U \in \mathcal{B} \). Therefore, \( \mathcal{B} \) is an \( \mathcal{M}' \)-closed family. Since \( \mathcal{M}' \subseteq \mathcal{M} \), we obtain, by Proposition 2.28, that \( \mathcal{B} \) is an \( \mathcal{M} \)-closed family as well.

By Corollary 2.9, \( \mathcal{B} \) generates Tychonoff-type topologies \( \mathcal{O}_\mathcal{B} \) on \( \mathcal{M} \) and \( \mathcal{O}_\mathcal{B}' \) on \( \mathcal{M}' \). As it follows from Proposition 2.30, \( \mathcal{B}_{\mathcal{O}_\mathcal{B}} = \mathcal{B}_{\mathcal{O}_\mathcal{B}'} = \mathcal{B} \neq \mathcal{T} \). Hence, by Corollary 2.23, \( \mathcal{O}_\mathcal{B} \) and \( \mathcal{O}_\mathcal{B}' \) are not Tychonoff topologies on \( \mathcal{M} \), respectively \( \mathcal{M}' \).
It is easy to see that \((M', \mathcal{O}_B')\) is a \(T_0\)-space. Indeed, let \(\{x, y\}\) and \(\{u, v\}\) be two distinct elements in \(M'\). We can assume \(x < x + \varepsilon < u \leq v\) for some \(\varepsilon > 0\). Consider the interval \(B = (x + \varepsilon, +\infty)\). Then \(\{u, v\} \in B^+_M\) but \(\{x, y\} \notin B^+_M\).

Let’s prove that \((M, \mathcal{O}_B)\) is not a \(T_0\)-space. Put \(F = \{2k : k \in \mathbb{Z}\}\) and \(G = \{2k + 1 : k \in \mathbb{Z}\}\). Then \(F, G \in M\) and \(F \neq G\) but the only neighbourhood of both \(F\) and \(G\) in \(M\) is \(X^+_M = M\).

**Example 3.20.** In the notations of Example 3.19, we will show that \((\mathbb{R}, \mathcal{T})\) is an \(\mathcal{O}_B\)-HS-space.

We are working now with the space \((M, \mathcal{O}_B)\) from Example 3.19. We will write simply \(\mathcal{O}\) instead of \(\mathcal{O}_B\).

Let \(A \subseteq X\). We have to show that the function

\[
i_{A,X} : (\mathcal{CL}(A), \mathcal{O}_A) \to (\mathcal{CL}(X), \mathcal{O}),
\]

where the topology \(\mathcal{O}_A\) on \(\mathcal{CL}(A)\) is generated by the family

\[
\mathcal{B}_A = \{A \cap U : U \in \mathcal{B}\},
\]

is continuous (see Proposition 3.5 for the notation \(i_{A,X}\)). Let \(B \in \mathcal{B}\). We will show that \(i_{A,X}^{-1}(B^+_M)\) is an open set. Take an \(F \in i_{A,X}^{-1}(B^+_M)\). Then \(F \in \mathcal{CL}(A)\) and \(\overline{F}^X \subseteq B\). There exists an \(E \in \mathcal{B}\) such that

\[
\overline{F}^X \subseteq E \subseteq \overline{E}^X \subseteq B
\]

(this is clear if \(F\) is bounded, since in this case \(\overline{F}^X\) is compact; if \(F\) is unbounded below, but is bounded above, then \(B = (-\infty, \beta)\), for some \(\beta \in \mathbb{R}\), and we can pick \(E = (-\infty, \gamma)\) with \(\sup F < \gamma < \beta\); similarly if \(F\) is unbounded above but not below; if \(F\) is unbounded both above and below then we have \(B = \mathbb{R}\) and we put \(E := B\). Then

\[
F \in (E \cap A)^+_{\mathcal{CL}(A)} \subseteq i_{A,X}^{-1}(B^+_M).
\]

Indeed, let \(G \in (E \cap A)^+_{\mathcal{CL}(A)}\). Then

\[
\overline{G}^X \subseteq \overline{E}^X \subseteq B,
\]

i.e. \(i_{A,X}(G) \in B^+_M\).

**Remark 3.21.** Let us note that a similar proof shows that every subspace \(Y\) of \((\mathbb{R}, \mathcal{T})\) is an \(\mathcal{O}_B\)-HS-space (see Examples 3.19 and 3.20 for the notations).

More generally, let \(Y\) be a topological space and \(\mathcal{D}\) be a base of \(Y\). We will say that \(Y\) is \(\mathcal{D}\)-normal if for every \(F \in \mathcal{CL}(Y)\) and for every \(D \in \mathcal{D}\) such that \(F \subseteq D\) there exists an \(E \in \mathcal{D}\) with \(F \subseteq E \subseteq \overline{E}^Y \subseteq D\). Now, arguing as in Example 3.20, we can prove that if \(Y\) is a \(\mathcal{D}\)-normal space, \(\mathcal{D} = \mathcal{D}^\cap\) and \(Y \in \mathcal{D}\), then \(Y\) is an \(\mathcal{O}\)-HS-space, where \(\mathcal{O}\) is the Tychonoff-type topology on \(\mathcal{CL}(Y)\) generated by \(\mathcal{D}\). This generalizes the result of M. Sekanina [15] that any normal space is a HS-space.
Example 3.22. In the notations of Examples 3.19 and 3.20, we will show that the function $i_{A,Y}$ is a homeomorphic embedding for any open interval $A$.

We will argue for $A = (0, 1)$; the proof for any other open interval is similar. We know, by Example 3.20, that the function $i_{A,Y}$ is continuous. Therefore we only need to prove that $i_{A,Y}$ is inversely continuous. By Proposition 3.5, it is enough to show that the set $A$ is 2-combinatorially embedded in $X$. So, let $H$ be a closed subset of $(0, 1)$ and let $B = (\alpha, \beta) \in B$ be such that $H \subseteq B$. We have to find a $D \in B$ such that $\overline{D} \cap [0, 1] \subseteq B$. Clearly, $\overline{D} \subseteq [0, 1]$. If $\overline{D} \cap [0, 1]$, we can take $D = B$.

If $0 \in \overline{D}^X$ but $1 \notin \overline{D}^X$, then $\alpha \leq 0$ and we can put $D = (-1, \beta)$. If $1 \in \overline{D}^X$ but $0 \notin \overline{D}^X$, then $\beta \geq 1$ and we can put $D = (\alpha, 2)$. If $0, 1 \in \overline{D}^X$ then $\alpha \leq 0$ and $\beta \geq 1$ and we put $D = (-1, 2)$. Therefore, $A$ is 2-combinatorially embedded in $(\mathbb{R}, T)$.

Note that $A$ is not 2-combinatorially embedded in $(\mathbb{R}, T)$.

Observe that the triple $((\mathbb{R}, T), A, O)$ satisfies all hypothesis of Corollary 3.7 except for condition (*), but $A$ is not sequentially closed.

Example 3.23. Let $Y \subseteq \mathbb{R}$. We will say, as usual, that a point $x \in Y$ is isolated from the right (left) (in $Y$) if there exists an $\varepsilon > 0$ such that if we put $U = (x, x + \varepsilon)$ ($U = (x - \varepsilon, x)$) then $U \cap Y = \emptyset$. Now, in the notations of Examples 3.19 and 3.20, we have: a subspace $Y$ of $(\mathbb{R}, T)$ is $O_{B_Y}$-commutative if and only if every point of $Y$ is either isolated from the right or from the left.

We first show that a space $Y$ that has a point $y_0$ which is non-isolated both from the left and from the right cannot be $O_{B_Y}$-commutative. Indeed, put $A = Y \setminus \{y_0\}$. We will prove that $A$ is not 2-combinatorially embedded in $Y$. By Proposition 3.5, this will imply that the function $i_{A,Y}$ is not inversely continuous and hence the space $Y$ will be not $O_{B_Y}$-commutative. Let $H = \{y_0 : n \in \omega\}$ be a decreasing sequence in $Y$ converging to $y_0$. Then $H$ is a closed subset of $A$ and $H \subseteq (y_0, +\infty) \cap Y$. Suppose that there exists $B \in B$ such that $cl_Y H \subseteq B$ and $B \cap A \subseteq (y_0, +\infty)$. Since $y_0 \in cl_Y H \subseteq B$ and $y_0$ is not isolated from the left, we have that $B \cap A \setminus (y_0, +\infty) \neq \emptyset$, which is a contradiction. Hence, $A$ is not 2-combinatorially embedded in $Y$.

Now we will show that a space $Y$ having only points which are isolated either from the left or from the right is $O_{B_Y}$-commutative. Let $A \subseteq Y$. We know, by Remark 3.21, that the function $i_{A,Y}$ is continuous. Hence it is enough to show that it is inversely continuous, i.e., according to Proposition 3.5, that $A$ is 2-combinatorially embedded in $Y$. So, let $H \in CL(A)$ and let $H \subseteq B \cap Y$ for some $B = (\alpha, \beta)$. We have to find a $D \in B$ such that $\overline{D} \subseteq D \cap Y$ and $D \cap A \subseteq B$. We have $cl_Y H \subseteq \overline{B}^Y = [\alpha, \beta]$. If $cl_Y H \subseteq B$, we can take $D = B$ and we are done. If $\alpha \in cl_Y H$ and $\beta \notin cl_Y H$ then $\alpha \in Y$ and $\alpha$ is not isolated from the right, being a limit point of $H$. Hence, by
the assumption, $\alpha$ is isolated from the left. Thus there exists a $\gamma < \alpha$ such that $(\gamma, \alpha) \cap Y = \emptyset$. Then $D = (\gamma, \beta)$ is the required interval. The other two possible cases are treated analogously.

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Department of Mathematics, University of Sofia, Blvd. J. Bourchier 5, 1126 Sofia, Bulgaria
E-mail address: gdimov@fmi.uni-sofia.bg

Department of Mathematical Sciences and Computer Science, University of Trieste, Via A. Valerio 12/1, 34127 Trieste, Italy
E-mail address: obersnel@mathsun1.univ.trieste.it
E-mail address: tironi@univ.trieste.it