Knizhnik–Zamolodchikov equations and spectral flow in $AdS_3$ string theory

Sylvain Ribault

King’s College London
Department of Mathematics
Strand, London WC2R 2LS
United Kingdom
ribault@mth.kcl.ac.uk

ABSTRACT: I generalize the Knizhnik–Zamolodchikov equations to correlators of spectral flowed fields in $AdS_3$ string theory. If spectral flow is preserved or violated by one unit, the resulting equations are equivalent to the KZ equations. If spectral flow is violated by two units or more, only some linear combinations of the KZ equations hold, but extra equations appear. Then I explicitly show how these correlators and the associated conformal blocks are related to Liouville theory correlators and conformal blocks with degenerate field insertions, where each unit of spectral flow violation removes one degenerate field. A similar relation to Liouville theory holds for noncompact parafermions.


### Contents

1. **Introduction and overview**

2. **KZ equations and spectral flow**
   - 2.1 Preliminaries: definition of the spectral flowed fields
   - 2.2 KZ-type equations for spectral flow-preserving correlators
   - 2.3 KZ-type equations for spectral flow-violating correlators
   - 2.4 Sklyanin’s separation of variables

3. **Correlation functions with spectral-flowed states**
   - 3.1 The $H^+_3$ model, Liouville theory, and their structure constants
     - 3.1.1 The three-point function of the $H^+_3$ model
     - 3.1.2 Comparison with Liouville theory
     - 3.1.3 The operator product expansion
   - 3.2 $H^+_3$ correlators from Liouville theory
     - 3.2.1 Results
     - 3.2.2 Proof
   - 3.3 $H^+_3$ conformal blocks from Liouville theory

4. **Outlook**

---

### 1. Introduction and overview

The Knizhnik–Zamolodchikov equations are an essential tool in the study of conformal field theories with affine Lie algebra symmetry [1]. All correlation functions of affine primary fields obey this system of linear differential equations, which determine their dependence on worldsheet coordinates.

However, in string theory in $AdS_3$, whose associated conformal field theory has an affine $SL(2, \mathbb{R})$ symmetry, Maldacena and Ooguri have shown that the physical spectrum cannot be built only from affine primaries [2]. Instead, one should also include spectral-flowed fields. Correlation functions involving such fields are not expected to obey the KZ equations.

Nevertheless, such spectral-flowed fields are obtained from affine primaries via the spectral flow automorphism of the affine Lie algebra. This will enable me to derive generalized KZ equations for their correlation functions. The other main purpose of this work is to explicitly relate these correlation functions to Liouville theory correlations functions. This amounts to solving the generalized KZ equations in terms of Virasoro conformal blocks.

Let me now sketch the results. If a correlation function respects spectral flow conservation, then it will satisfy a system of equations (2.18) which turns out to be equivalent to the KZ equations...
via a simple twist. (Actually, this conclusion also holds if spectral flow conservation is violated by one unit, due to the global group symmetry.) If a correlation function violates spectral flow conservation, then it will satisfy only some specific linear combinations (2.27) of the KZ equations. However, the missing equations will be replaced with simpler constraints (2.29) which do not involve derivatives wrt worldsheet coordinates.

Therefore, the equations obeyed in the case when spectral flow is not conserved are in some sense simpler than the original KZ equations. This will become clear after I show how to perform Sklyanin’s separation of variables for such equations. Each unit of spectral flow violation leads to the disappearance of one variable-separated equation, until there are none left in the case of maximal violation.

These variable-separated equations are actually identical to Belavin–Polyakov–Zamolodchikov equations (2.40). I will exploit this in order to derive a relation between correlation functions of \( n \) spectral-flowed fields in the \( H_3^+ \) model and correlation functions in Liouville theory. (The \( H_3^+ \) model, or string theory in the Euclidean \( AdS_3 \), is introduced here for technical reasons.) If spectral flow conservation is violated by \( r \) units, the relevant Liouville correlation functions will have \( n - 2 - r \) degenerate field insertions (3.26). This shows why the maximal spectral flow violation \( n - 2 \) is equal to the number of Liouville degenerate fields needed to reproduce an unflowed \( H_3^+ \) correlator. A similar relation with Liouville theory also holds for the \( SL(2, \mathbb{R})/U(1) \) coset model (3.29).

Deriving such relations between correlation functions involves not only the KZ equations, but also the structure constants of the \( H_3^+ \) model. The ordinary, spectral flow-preserving structure constant is known to be equal to the Liouville structure constant associated with a Liouville vertex dressed with one degenerate field [3]. Here I will show that the \( H_3^+ \) spectral flow-violating vertex corresponds to an ordinary Liouville vertex, at the levels of structure constants (3.15) (3.19), operator product expansions (3.24) (3.25), and conformal blocks (3.36). In particular, I will argue for the existence of a spectral flow-violating operator product expansion in the \( H_3^+ \) model, as an alternative to the ordinary operator product expansion. This is shown schematically in the diagrams below.

\[
\begin{array}{c|c}
\text{Euclidean } AdS_3 & \text{Liouville theory} \\
\hline
\text{Flow – preserving vertex} & \text{Vertex dressed with one degenerate field} \\
\text{Flow – violating vertex} & \text{Ordinary vertex} \\
\end{array}
\]

(1.1)

In an Outlook, I will mention possible applications of these results to string theory in \( AdS_3 \) and to the definition of a fusing matrix for the \( H_3^+ \) model.

2. KZ equations and spectral flow

In this section I derive which modifications of the KZ equations apply to correlation functions involving spectral flowed fields.
2.1 Preliminaries: definition of the spectral flowed fields

Let me consider a conformal field theory with an chiral affine Lie algebra symmetry \( \hat{s} \ell_2 \) at level \( k > 2 \), living on the Riemann sphere parametrized by complex coordinates \( z, \bar{z} \). In this section, I will be concerned only with the holomorphic sector. The symmetry of this sector is a “left-moving” copy of the algebra \( s \ell_2 \),

\[
\begin{align*}
[J^3_n, J^3_m] &= -\frac{k}{2} n \delta_{n+m,0}, \\
[J^3_n, \bar{J}^\pm_m] &= \pm J^\pm_{n+m}, \\
[J^+_n, \bar{J}^-_m] &= -2J^3_{n+m} + kn \delta_{n+m,0}.
\end{align*}
\]  

(2.1)

The generators \( J^a_n \) can be encoded in holomorphic currents \( J^a(z) \):

\[
J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n, \quad J^a_n = \frac{1}{2\pi i} \oint_0 dz \, z^n J^a(z),
\]  

(2.2)

where \( \oint_0 \) stands for the integral along a contour encircling the point \( z = 0 \). The conformal symmetry generators \( L_n \) are built from the \( s \ell_2 \) generators via the standard Virasoro construction, which yields the central charge \( c = \frac{3k}{k-2} \).

The spectral flowed field \( \Phi^j, w(z) \) of spin \( j \) and spectral flow number \( w \in \mathbb{Z} \) is defined as a primary with respect to the spectral flowed currents \( \bar{J}(z) \). These currents can be defined via their modes \( \bar{J}^a_n \), which are then used to build a spectral flowed copy \( \bar{L}_n \) of the Virasoro algebra [3]:

\[
\begin{align*}
\bar{J}^3_n &= J^3_n - \frac{k}{2} w \delta_{n,0}, \\
\bar{J}^\pm_n &= J^{\pm}_n, \\
\bar{L}_n &= L_n + w J^3_n - \frac{k}{4} w^2 \delta_{n,0}.
\end{align*}
\]  

(2.3)

Namely, the state \( |j, w \rangle \) corresponding to the field \( \Phi^j, w(z) \) is assumed to obey

\[
\begin{align*}
\bar{J}^a_{n>0} |j, w \rangle &= 0, \\
\bar{J}^0 |j, w \rangle &= -t^a |j, w \rangle.
\end{align*}
\]  

(2.4)

Equivalently, the field \( \Phi^j, w(z) \) has the following operator product expansion with the ordinary currents \( J^a(z) \):

\[
\begin{align*}
J^3(z) \Phi^j, w(y) &\sim -\frac{t^3 \Phi^j, w(y)}{z-y} + \frac{k w}{2} \Phi^j, w(y), \\
J^+ (z) \Phi^j, w(y) &\sim -t^a \Phi^j, w(y), \\
J^- (z) \Phi^j, w(y) &\sim -t^a \Phi^j, w(y).
\end{align*}
\]  

(2.5)

Here \( t^a \) are generators of the \( s \ell_2 \) algebra. The field \( \Phi^j, w(z) \) indeed carries a representation of \( s \ell_2 \) of spin \( j \) and Casimir \( \frac{1}{2}(t^+ t^- + t^- t^+ - 2 t^3 y^3) = -j(j+1) \), although the corresponding degrees of freedom are not spelt out explicitly so far. I will later assume that \( \Phi^j, w(z) \) belongs to a principal continuous series representation with spin \( j \in \mathbb{R} + i \mathbb{R} \), whose states can be labelled using a complex parameter \( \mu \) such that

\[
\begin{align*}
t^+ &= \mu, \\
t^3 &= \mu \frac{\partial}{\partial \mu}, \\
t^- &= \mu \frac{\partial}{\partial \mu} - \frac{j(j+1)}{\mu}.
\end{align*}
\]  

(2.6)
Another basis for the continuous representation is obtained by diagonalizing $t^3$ with eigenvalue $-m$ and considering $t^\pm$ as raising and lowering operators. Then the state $|j, w, m\rangle$ corresponding to the field $\Phi_{j,w}^m(z)$ satisfies

$$\tilde{j}_0^3 = -t^3 = J_0^3 - \frac{kw}{2} = m.$$  \hspace{1cm} (2.7)

An advantage of the $m$-basis fields is that they happen to be eigenvalues of the original dilatation operator $L_0$ and therefore scale as follows:

$$\left( z \frac{\partial}{\partial z} + \Delta_{m}^{j,w} \right) \Phi_{j,w}^m(z) = 0,$$

$$\Delta_{m}^{j,w} = \Delta_j - w m - \frac{k}{2} w^2,$$

$$\Delta_j = -\frac{j(j+1)}{k-2}.$$  \hspace{1cm} (2.8)

Moreover, the $m$-basis fields are simply related to parafermionic fields $\Psi_j^m$ of the coset model $SL(2, \mathbb{R})/U(1)$ [2],

$$\Phi_{j,w}^m(z) = e^{i(m+\frac{1}{2}w)\sqrt{2} \phi(z)} \Psi_j^m,$$  \hspace{1cm} (2.9)

where $\phi(z)$ is a free boson such that $J^3(z) = -i \sqrt{\frac{k}{2}} \frac{\partial}{\partial z} \phi(z)$.

From the relation with parafermions (2.9), it may seem easy to compute the correlation function of $n$ spectral flowed fields and to determine the differential equations it satisfies, by relating it to a correlation function with no spectral flow. The only dependence on the spectral flow $w$ is indeed in the free boson factor. However, the corresponding free boson correlation function does make sense only if the total charge vanishes, $\sum_{i=1}^{n} (m_i + \frac{k}{2} w_i) = 0$. In the case with no spectral flow $w_i = 0$, this implies $\sum_{i=1}^{n} m_i = 0$. In the case with spectral flow, the last two equalities imply spectral flow conservation $\sum_{i=1}^{n} w_i = 0$. Therefore, only spectral flow-preserving correlators are related to correlators without spectral flow. I will use this in order to check the KZ-type equations which I will derive for them.

### 2.2 KZ-type equations for spectral flow-preserving correlators

Each of the $n$ KZ equations determines the dependence of a correlation function with respect to the worldsheet position of one field $z_i$. This is done by inserting the worldsheet translation operator $L_{-1}$,

$$L_{-1} \Phi_{j_i,w_i}^m(z_i) = \frac{\partial}{\partial z_i} \Phi_{j_i,w_i}^m(z_i).$$  \hspace{1cm} (2.10)

The next step is to express $L_{-1}$ in terms of the currents $J^a$. Having better control on the action of the spectral flowed currents $\tilde{J}^a$ on the spectral flowed field $\Phi_{j,w}^m(z)$, it is actually more convenient to use

$$\left( (k-2)\tilde{L}_{-1} + \tilde{J}_0^a \right) \Phi_{j,w}^m(z) = 0,$$  \hspace{1cm} (2.11)

and to rely on equation (2.3) to relate $\tilde{L}_{-1}$ to the translation operator $L_{-1}$. The resulting equation is:

$$\left[ (k-2) \frac{\partial}{\partial z_i} + 2\tilde{J}_{-1}^3 \left( -t^3_i + \frac{k-2}{2} w_i \right) + \tilde{J}_{-1}^\pm t_i^\pm + \tilde{J}_{-1}^\mp t_i^\mp \right] \Phi_{j,w}^m(z) = 0.$$  \hspace{1cm} (2.12)
Now let me insert this null-vector equation into an \( n \)-point correlation function. The operators \( \tilde{J}^a_{\ell-1} \) have to be expressed in terms of Lie algebra generators \( t^a \) acting on the other fields \( \Phi^{i,\ell}(z_\ell) \) for \( \ell \neq i \). In the case of \( J^+ \), this is possible thanks to the equation:

\[
\left\langle \frac{1}{2\pi i} \oint_\infty \frac{dz}{z-z_\ell} \prod_{\ell=1}^n (z-z_\ell)^{w_\ell} J^+ (z) \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0.
\] (2.13)

This equation holds provided \( \sum_{\ell=1}^n w_\ell \leq 0 \). This indeed implies that the function \( \prod_{\ell=1}^n (z-z_\ell)^{w_\ell} \) is bounded near \( z = \infty \) and allows closing the contour there, knowing \( J^+(z) \sim \frac{1}{z} \).

Starting with equation (2.13), the contour of integration can be contracted into small loops around each point \( z_\ell \). With the help of the operator product expansion \( J^a(z)\Phi^{j,w}(y) \) (2.5), this yields:

\[
\left[ \rho_i J^+_{1,i} - \rho_i \sum_{\ell \neq i} \frac{w_\ell}{z_i - z_\ell} t^+_{1,i} - \sum_{\ell \neq i} \frac{\rho_{\ell}}{z_\ell - z_i} t^+_{\ell} \right] \left\langle \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0,
\] (2.14)

where the index \( i \) in \( J^+_{1,i} \) and \( t^+ \) indicates which field they act on, and

\[
\rho_i \equiv \prod_{\ell \neq i} (z_i - z_\ell)^{w_\ell}.
\] (2.15)

Similar manipulations are possible with \( J^- \) provided \( \sum_{\ell} w_\ell \geq 0 \), and yield:

\[
\left[ \rho_i^{-1} J^-_{1,i} + \rho_i^{-1} \sum_{\ell \neq i} \frac{w_\ell}{z_i - z_\ell} t^-_{1,i} - \sum_{\ell \neq i} \frac{\rho_{\ell}^{-1}}{z_\ell - z_i} t^-_{\ell} \right] \left\langle \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0.
\] (2.16)

In the case of \( J^3 \), the following equation does not require any constraint on \( w_\ell \):

\[
\left[ J^3_{1,i} - \sum_{\ell \neq i} \frac{1}{z_\ell - z_i} \left( i^3 - \frac{k}{2} w_\ell \right) \right] \left\langle \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0.
\] (2.17)

In the spectral flow-preserving case \( \sum_{\ell=1}^n w_\ell = 0 \), the three equations (2.14),(2.16),(2.17) hold. Plugging them into equation (2.12) yields the following generalization of the KZ equations:

\[
\tilde{E}_i \left\langle \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0 \text{ with } \tilde{E}_i \equiv \left( (k-2) \frac{\partial}{\partial z_i} + \sum_{j \neq i} \tilde{Q}_{ij} \right),
\]

\[
\tilde{Q}_{ij} \equiv -2t^3 i^3_j + t^+_i t^-_j \frac{\rho_j}{\rho_i} + t^-_i t^+_j \frac{\rho_i}{\rho_j} + (k-2)(w_j t^3_i + w_i t^3_j) - \frac{k(k-2)}{2} w_i w_j.
\] (2.18)

These equations are related to the ordinary KZ equations \( E_i = 0 \) by a twist of the correlation function:

\[
E_i \kappa^{-1} \left\langle \prod_{\ell=1}^n \Phi^{i,\ell}(z_\ell) \right. = 0,
\] (2.19)

with \( E_i \equiv \left( (k-2) \frac{\partial}{\partial z_i} + \sum_{j \neq i} Q_{ij} \right) \), \( Q_{ij} \equiv -2t^3 i^3_j + t^+_i t^-_j + t^-_i t^+_j \),

\[
\kappa \equiv \prod_{j<i} (z_j - z_i)^{w_j t^3_i + w_i t^3_j - \frac{k}{2} w_i w_j}.
\] (2.20)
This is shown by a direct computation which uses the spacetime $SL(2)$ invariance of the vacuum,

$$
\left\langle \frac{1}{2\pi i} \oint_{\infty} dz J^3(z) \prod_{\ell=1}^{n} \Phi_{j_{\ell},w_{\ell}}(z_{\ell}) \right\rangle = 0 \Rightarrow \sum_{i=1}^{n} \left( t_{i}^3 - \frac{k}{2} w_{i} \right) = 0 \Rightarrow \sum_{i=1}^{n} t_{i}^3 = 0. \tag{2.22}
$$

A check of the equation (2.20) can be performed using the relation of spectral flowed fields to parafermionic fields (2.9). This relation implies that the correlation function with spectral-flowed fields

$$
\left\langle \prod_{\ell=1}^{n} \Phi_{j_{\ell},w_{\ell}}(z_{\ell}) \right\rangle = \kappa \left\langle \prod_{\ell=1}^{n} \Phi_{j_{\ell}}(z_{\ell}) \right\rangle , \tag{2.23}
$$

where $m_{\ell}$ is by definition the eigenvalue of $-t_{\ell}^3$.

Therefore, the spectral flow-preserving correlators satisfy the ordinary KZ equations modulo a simple twist. I will now generalize the methods used to derive these equations to the spectral flow-violating case. The same twist will provide notable simplifications of the equations, without reducing them to the ordinary KZ equations.

### 2.3 KZ-type equations for spectral flow-violating correlators

Consider an $n$-point correlator which violates spectral flow by $r \geq 1$ units, say $\sum_{\ell=1}^{n} w_{\ell} = -r$. The reasoning of the previous subsection which led to KZ-type equations now fails because equation (2.16), which expressed the action of $J_{-1}^{\ell}$ on a field in terms of the action of $t_{-}$ on the other fields, no longer holds. To derive such an equation would require

$$
\frac{1}{2\pi i} \oint_{\infty} dz \prod_{\ell=1}^{n} \frac{dz}{z - z_{\ell}} \prod_{i=1}^{r} \frac{1}{(z - z_{i})^{w_{i}} J_{-}^{i}(z)} = 0, \tag{2.24}
$$

where the l.h.s. behaves near $z = \infty$ as $\frac{1}{2\pi i} \oint_{\infty} dz z^{-2}$. Actually, in the case $r = 1$, the spacetimes $SL(2)$ symmetry of the vacuum is able to save the day. This symmetry indeed reads

$$
\frac{1}{2\pi i} \oint_{\infty} dz J^{\rho}(z) = 0, \tag{2.25}
$$

which implies eq. (2.24). But, for $r \geq 2$, it is impossible to derive $n$ equations governing the $z_{i}$ dependence of the correlators. Instead of equation (2.24), it is however possible to use the weaker equations:

$$
\frac{1}{2\pi i} \oint_{\infty} dz \prod_{\alpha=1}^{r} \frac{dz}{\prod_{\ell=1}^{n} (z - z_{\ell})} \prod_{\ell=1}^{n} (z - z_{\ell})^{-w_{\ell}} J_{-}^{i}(z) = 0, \tag{2.26}
$$

for any choice of $r$ distinct indices $i_{1}, i_{2} \cdots i_{r}$. This leads to an expression for a linear combination of $J_{-1,i_{1}}, J_{-1,i_{2}} \cdots J_{-1,i_{r}}$ in terms of $t_{\ell}^{i}, \ell = 1 \cdots n$. Then it is possible to derive a differential equation for the spectral flow-violating $n$-point correlator, whose $z$-derivative part is a linear combination of $\frac{\partial}{\partial z_{i_{1}}} , \frac{\partial}{\partial z_{i_{2}}} , \frac{\partial}{\partial z_{i_{r}}}$. It is not necessary to go into much detail here: these manipulations
actually also hold in the spectral flow-preserving case, and they can therefore yield nothing but a linear combination of the KZ-type equations (2.18) which hold in that case. The actual combination can easily be read from eq. (2.26),

\[ \tilde{E}_{\{i_\alpha\}} \equiv \sum_{\alpha=1}^{r} (\rho_{i_\alpha} t_{\kappa_{i_\alpha}}^+)^{-1} \frac{1}{\prod_{\beta \neq \alpha} (z_{i_\alpha} - z_{i_\beta})} \tilde{E}_{i_\alpha} \quad \text{for all } \{i_1, i_2 \cdots i_r\} \subset \{1, 2 \cdots n\}, \quad (2.27) \]

where \(\rho_{i}\) was defined in eq. (2.15). That only such combinations of \(r\) equations hold, means that \(r - 1\) KZ equations have been lost because of spectral flow violation \(\sum_{\ell=1}^{n} w_\ell = -r\). This was because less equations could be obtained from the \(J^-\) current. Conversely, it is now possible to obtain new equations from the \(J^+\) current, using

\[ \oint_{\infty} dz \prod_{\ell=1}^{n} (z - z_\ell)^{w_\ell} z^j J^+(z) = 0 , \quad j = 0, 1 \cdots r. \quad (2.28) \]

This results in \(r + 1\) equations,

\[ \sum_{\ell=1}^{n} z_j^\ell \rho_\ell t^\ell_{\kappa_{i_\alpha}} \left\langle \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right\rangle = 0. \quad (2.29) \]

The \(j = 0\) equation already held in the spectral flow-preserving case as a consequence of the spacetime \(SL(2)\) symmetry. The other equations, however, are specific to the spectral flow-violating case.

To conclude this subsection, let me gather the equations satisfied by the spectral flow-violating correlators, while simplifying them by applying the twist by the function \(\kappa (2.21)\),

\[ E_{\{i_\alpha\}} \kappa^{-1} \left\langle \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right\rangle = 0 \quad \text{for } \{i_1, i_2 \cdots i_r\} \subset \{1, 2 \cdots n\}, \quad (2.30) \]

\[ \sum_{i=1}^{n} z_j^i \kappa^{-1} \left\langle \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right\rangle = 0 \quad \text{for } 0 \leq j \leq r, \quad (2.31) \]

\[ \left( \sum_{i=1}^{n} t_i^3 + \frac{k}{2} \right) \left\langle \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right\rangle = 0, \quad (2.32) \]

where \(E_{\{i_\alpha\}}\) is a combination of \(r\) ordinary KZ equations \(E_i (2.20)\),

\[ E_{\{i_\alpha\}} \equiv \sum_{\alpha=1}^{r} (t_{\kappa_{i_\alpha}}^+)^{-1} \frac{1}{\prod_{\beta \neq \alpha} (z_{i_\alpha} - z_{i_\beta})} E_{i_\alpha}, \quad (2.33) \]

and the last equation (2.32) is the \(J^3\) part of the spacetime \(SL(2)\) symmetry. (The other parts are implicitly included in the previous equations.)

### 2.4 Sklyanin’s separation of variables

The ordinary KZ equations, as well as the modified (combinations of) KZ equations for correlators of spectral-flowed fields, involve Lie algebra generators \(t_i^a\) acting on all the fields \(i = 1, 2 \cdots n\).
However, Sklyanin has shown how to separate them by a change of variables [4]. Since it maps the KZ equations to the Belavin–Polyakov–Zamolodchikov equations [5], this change of variables leads to a relation between correlators in the Euclidean AdS$_3$ and correlators in Liouville theory [3]. In preparation for the extension of such a relation to correlators of spectral-flowed fields, I will now show how to perform the separation of variables in the equations (2.30)-(2.31).

I now assume that the field $\Phi_{j,w}(z)$ belongs to the principal continuous series $j \in -\frac{1}{2} + i\mathbb{R}$, and choose the $\mu$-basis for this representation (see eq. (2.6)). This amounts to diagonalizing the operator $t^+$, with eigenvalue $\mu$. This could be made explicit by using the notation $\Phi_{j,w}(z) = \Phi_{j,w}(\mu|z)$. Then the equation (2.31) simply becomes

$$u_j \equiv \sum_{\ell=1}^n \mu_\ell z_\ell^j = 0 \quad \text{for} \quad 0 \leq j \leq r.$$  (2.34)

Let me define new variables as the zeroes of the rational function

$$R(t) = \sum_{\ell=1}^n \frac{\mu_\ell}{t - z_\ell}. \quad (2.35)$$

The number of zeroes of $R(t)$ is found by reducing it to the same denominator,

$$R(t) = \frac{\sum_{d=0}^{n-1} \left( \sum_{j=0}^{d} P_j u_d - j \right) t^{n-d}}{\prod_{\ell=1}^n (t - z_\ell)} \quad \text{where} \quad \prod_{\ell=1}^n (t - z_\ell) = \sum_{j=0}^n p_j t^{n-j} \quad \text{defines} \ p_j. \quad (2.36)$$

Since $u_j = 0$ for $0 \leq j \leq r$, the denominator of $R(t)$ has degree $n - 2 - r$. This defines the new variables $y_\alpha$ as

$$\sum_{\ell=1}^n \frac{\mu_\ell}{t - z_\ell} = u_{r+1} \prod_{\alpha=1}^{n-2-r} (t - y_\alpha) \prod_{\ell=1}^n (t - z_\ell). \quad (2.37)$$

Now I am in a position to perform the change of variables

$$\left(\mu_1, \mu_2, \cdots, \mu_n\right)|_{u_0 = u_1 = \cdots = u_r = 0} \quad \text{(n variables subject to r + 1 constraints)} \quad \rightarrow \quad \left(y_1, y_2, \cdots, y_{n-2-r}, u_{r+1}\right) \quad \text{(n - r - 1 variables)} \quad (2.38)$$

It is also convenient to perform a change of unknown function by explicitly solving the equations $u_0 = u_1 = \cdots = u_r = 0$ (2.31),

$$\left\langle \prod_{\ell=1}^n \Phi_{j,w}(\mu_\ell|z_\ell) \right\rangle = \kappa \prod_{j=0}^r \delta(u_j) \Omega_{n,r} \left( u_{r+1}, y_1, y_2, \cdots, y_{n-2-r} \middle| z_1, z_2, \cdots, z_n \right). \quad (2.39)$$
Claim 1. The system of linear combinations of KZ equations (2.30) satisfied by $\kappa^{-1} \left< \prod_{\ell=1}^{n} \Phi_{j\ell, w\ell}(\mu_{\ell} | z_{\ell}) \right>$, which amounts to $n + 1 - r$ differential equations, is equivalent to $\Omega_{n,r}$ satisfying the $n - 2 - r$ BPZ equations characteristic of the Liouville correlator $\left< \prod_{\ell=1}^{n} V_{\alpha_{\ell}}(z_{\ell}) \prod_{a=1}^{n-2-r} V_{\frac{1}{2\pi} y_{a}}(y_{a}) \right>$

\[ b^2 \frac{\partial^2}{\partial y_a^2} + \sum_{a' \neq a} \left( \frac{1}{y_{a'} - y_a} + \frac{\Delta - \frac{1}{2}}{y_{a'}^2} \right) + \sum_{i=1}^{n} \left( \frac{1}{y_a - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_{\alpha_i}}{(y_a - z_i)^2} \right) \] \[ b^{2\ell} \frac{\partial^2}{\partial y_a^2} + \sum_{a' \neq a} \left( \frac{1}{y_{a'} - y_a} + \frac{\Delta - \frac{1}{2}}{y_{a'}^2} \right) + \sum_{i=1}^{n} \left( \frac{1}{y_a - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_{\alpha_i}}{(y_a - z_i)^2} \right) \] \[ \Theta_{n,r}^{\Omega_{n,r}} = 0, \quad (2.40) \]

plus the three worldsheet SL(2) equations

\[ \sum_{i=1}^{n} z_i^{0,1,2} e_i \Omega_{n,r} = 0. \quad (2.41) \]

Notations: $b = (k - 2)^{-\frac{1}{2}}$, $\alpha_i = b(j_i + 1) + \frac{1}{2\pi}$, $\Delta_{\alpha_i} = \alpha(b + b^{-1} - \alpha)$, $\Delta = \frac{1}{2} - \frac{3}{4\pi}$,

\[ \Theta_{n,r} = \frac{\prod_{i<j} z_i z_{j'}}{\prod_{a} z_i - y_a} \prod_{a} y_{a'}. \quad (2.42) \]

The rest of the subsection is devoted to proving this claim.

First, notice that $\Omega_{n,r}$ satisfies $E_{\{i_0\}} \Omega_{n,r} = 0$. This follows from the equations (2.30), (2.32) and from

\[ [E_{\{i_0\}}, \prod_{j=0}^{r} \delta(u_{j})] = 0 \mod \sum_{\ell=1}^{n} \ell^{3} + \frac{k}{2}. \quad (2.43) \]

This can be proved by a direct if tedious computation.

Then, rewrite the equations $E_{\{i_0\}}$ (2.33) as

\[ \left( \prod_{\ell \neq i_1, i_2, \ldots, i_r} \frac{z_i - z_\ell}{z_i - y_a} \right) \Omega_{n,r} = 0 \quad \text{for} \quad \{i_1, i_2, \ldots, i_r\} \subset \{1, 2 \ldots n\}, \quad (2.44) \]

using $t_i^+ = u_{i+1} \prod_{\ell \neq i} (z_i - y_a)$. Taking linear combinations of these equations for different choices of $\{i_1, i_2, \ldots, i_r\}$ yields the equivalent system

\[ \left( \sum_{i=1}^{n} \frac{z_i}{z_i - y_a} \right) \Omega_{n,r} = 0, \quad \text{for} \quad j = 0, 1, \ldots, n - r. \quad (2.45) \]

Further linear combinations of these equations lead to the worldsheet SL(2) equations (2.41) and to the equations

\[ \sum_{i=1}^{n} \frac{1}{z_i - y_a} e_i = 0 \quad \text{for} \quad a = 1, 2, \ldots, n - 2 - r. \quad (2.46) \]

Now these equations are equivalent to the BPZ equations (2.40), by the same argument as in the spectral flow-preserving case, see [3].

As a check, I computed the $z$-scaling behaviour of $\Omega_{n,r}$ by using the equation $\sum_{i=1}^{n} z_i e_i \Omega_{n,r} = 0$.

The result agrees with the scaling

\[ \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \kappa^{-1} \left< \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right> = \left( -\sum_{i=1}^{n} \Delta_{j_i} - \frac{k}{2} \right)^2 \] \[ \kappa^{-1} \left< \prod_{\ell=1}^{n} \Phi_{j_\ell, w_\ell}(z_\ell) \right>, \quad (2.47) \]

which is expected from the conformal dimensions of the operators $\Phi_{j_\ell, w_\ell}$ (2.8).
3. Correlation functions with spectral-flowed states

Until now I have considered general properties of conformal field theories with a chiral $\widehat{sl}_2$ symmetry. In this section I plan to exploit these properties in the case of particular models. The most physically interesting model with $\widehat{sl}_2$ symmetry is string theory in $AdS_3$. However, this theory has a complicated spectrum including discrete states, and directly addressing it is difficult. Therefore, I will consider the Euclidean version of that model, also known as the $H^+_3$ model. Although non unitary [6], this model has the advantages of being Euclidean and of having a purely continuous spectrum.

My purpose is therefore to explicitly relate all correlation functions and conformal blocks of the $H^+_3$ model on a sphere to similar quantities in Liouville theory, a simpler non-rational conformal field theory. $H^+_3$ physical correlators were already related to Liouville theory in [3]; now I want to extend this relation to conformal blocks, and to correlators involving spectral-flowed fields. Such fields are unphysical in the $H^+_3$ model since they do not appear in the spectrum, but they play an important rôle in $AdS_3$ string theory.

In order to fully characterize $H^+_3$ correlators, the chiral results of the last section (namely the differential equations they satisfy) have to be supplemented with two types of information: how the left-moving and right-moving sectors are put together, and which structure constants appear in the operator product expansions. These data are already known, but in the next subsection I will recast them in a form which emphasizes the reflection symmetry and the relation to Liouville theory. Moreover, I will interpret them in terms of two alternative operator product expansions in the $H^+_3$ model. More details on these models can be found in [3] and references therein.

3.1 The $H^+_3$ model, Liouville theory, and their structure constants

3.1.1 The three-point function of the $H^+_3$ model

The $H^+_3$ model is a conformal field theory with symmetry algebra $\widehat{sl}_2 \times \widehat{sl}_2$. The spectrum is made of physical fields $\Phi^j(z, \bar{z})$, $j \in -\frac{1}{2} + i\mathbb{R}$ transforming as vectors in the principal continuous series representation of spin $j$ of both $\widehat{sl}_2$ algebras. Therefore, the physical fields transform as products $\Phi^j(z, \bar{z}) \sim \Phi^j(z)\Phi^j(\bar{z})$ of the chiral fields of the previous section; however this chiral factorization fails at the level of the zero modes [8]. The spectral flowed fields $\Phi^{j,w \neq 0}(z, \bar{z})$ do not belong to the spectrum.

Two different bases for the spin $j$ representation will appear: the $\mu$-basis (see eq. (2.6)) and the $m$-basis, whose elements diagonalize the operators $t^+, \bar{t}^+$ and $t^3, \bar{t}^3$ respectively. They are related by

$$
\Phi^{j,w}_{m,\bar{m}}(z, \bar{z}) = N^j_{m,\bar{m}} \int \frac{d^2 \mu}{\mu|\mu|^2} \mu^m \bar{\mu}^\bar{m} \Phi^{j,w}(\mu, \bar{\mu}|z, \bar{z}) , \quad N^j_{m,\bar{m}} = \frac{\Gamma(-j-m)}{\Gamma(j+1+m)}. \quad (3.1)
$$

(Note the change of convention $m \rightarrow -\bar{m}$ wrt [3].) In later formulas, the antiholomorphic dependence on $\bar{z}$ and $\bar{\mu}$ may be omitted. The physical values of $m, \bar{m}$ obey $m - \bar{m} \in \mathbb{Z}$ and $m + \bar{m} \in i\mathbb{R}$.
The $H_3^+$ two-point function has to preserve spectral flow. Therefore, the flowed two-point function can be deduced from the unflowed one by using formula (2.23):

$$\left\langle \Phi^{j_1, w} \left( \mu_1 | z_1 \right) \Phi^{j_2, -w} \left( \mu_2 | z_2 \right) \right\rangle = |z_{12}|^{-4\Delta_{j_1} + kw^2} |\mu_1|^2 \delta(2) \left( \mu_1 + (-1)^w z_{12} \mu_2 \right) \times \left[ \delta(j_2 + j_1 + 1) + R^H(j_1) \delta(j_2 - j_1) \right],$$

$$\left\langle \Phi^{j_{m_1}, w} \left( \mu_{m_1} | z_1 \right) \Phi^{j_{m_2}, -w} \left( \mu_{m_2} | z_2 \right) \right\rangle = |z_{12}|^{-4\Delta_{j_{m_1}} \left( -1 \right)^{m_1 - m_2}} \delta(2) (m_1 + m_2) \times \left[ \delta(j_2 + j_1 + 1) + R^H(j_1) \Gamma(-j_1 + m_1) \Gamma(-j_1 - m_1) \frac{\delta(j_2 - j_1)}{\Gamma(j_1 + 1 + m_1) \Gamma(j_1 + 1 - m_1)} \right],$$

where $\Delta_{j_{m_1}}$ is defined in eq. (2.8), and, using the notation $b^2 = \frac{1}{k+2}$,

$$R^H(j) = -\left( \frac{1}{\pi b^2} \gamma(b^2) \right)^{-2j+1} \frac{\Gamma(j + b^2(2j + 1)) \Gamma(-2j + 1)}{\Gamma(-j + b^2(2j + 1)) \Gamma(-b^2(2j + 1))}.$$ \hspace{1cm} (3.4)

This $H_3^+$ reflection coefficient is actually identical to the Liouville theory reflection coefficient $R^L(\alpha)$, provided $b^2 = \frac{1}{k+2}$ is interpreted as the usual parameter of Liouville theory (such that $c = 1 + 6Q^2$ with $Q = b + b^{-1}$), and the Liouville momentum $\alpha$ is given by

$$\alpha = b(j + 1) + \frac{1}{2b},$$ \hspace{1cm} (3.5)

The $H_3^+$ three-point function can either preserve spectral flow or violate it by one unit. Let me start by recalling the sectoral flow-preserving three-point function, while introducing new notations for the structure constants:

$$\left\langle \prod_{\ell=1}^{3} \Phi^{j_{\ell}, w} \left( \mu_{\ell} | z_{\ell} \right) \right\rangle = \sum_{w=0} |z_{12}|^{-2\Delta_{j_{2}} - kw_1 w_2} |z_{13}|^{-2\Delta_{j_{3}} - kw_1 w_3} |z_{23}|^{-2\Delta_{j_{3}} - kw_2 w_3} \times \delta(2) \left( \mu_1 \rho_1 + \mu_2 \rho_2 + \mu_3 \rho_3 \right) D^H \left[ \begin{array}{ccc} \frac{j_1}{\mu_1 \rho_1} & \frac{j_2}{\mu_2 \rho_2} & \frac{j_3}{\mu_3 \rho_3} \end{array} \right] C^H(j_1, j_2, j_3),$$ \hspace{1cm} (3.6)

with $\rho_i$ is defined by (2.13) and $\Delta_{j_2} = \Delta_{j_1} + \Delta_{j_2} - \Delta_{j_3}$. I define the structure constant $C^H$ as

$$C^H(j_3, j_2, j_1) = -\frac{1}{2\pi^3 b} \left[ \frac{\gamma(b^2)b^{2-2b}}{\pi} \right]^{-2\Sigma j_i} \frac{\Upsilon_b(0)}{\Upsilon_b(-b(j_{123} + 1)) \Gamma(-j_{123} - 1)} \times \frac{\Upsilon_b(-b^2(2j_1 + 1)) \Upsilon_b(-b^2(2j_2 + 1)) \Upsilon_b(-b^2(2j_3 + 1))}{\Upsilon_b(-b^2(j_{12}) \Gamma(-j_{12}) \Upsilon_b(-b^2(j_{13}) \Gamma(-j_{13}) \Upsilon_b(-b^2(j_{23}) \Gamma(-j_{23}))},$$ \hspace{1cm} (3.7)

where $j_{12} = j_1 + j_2 - j_3$ and $j_{123} = j_1 + j_2 + j_3$ and the definition of the special function $\Upsilon_b$ can be found in [3]. Notice the extra $\Gamma$ factors with respect to the standard definition [3, 8]. They are added so that $C^H$ is reflection-covariant like the three-point function itself,

$$C^H(j_1, j_2, j_3) = R^H(j_3)C^H(j_1, j_2, -j_3 - 1).$$ \hspace{1cm} (3.8)
Therefore, the factor $D^H$, and the three-point conformal block, are now reflection-invariant:

\[
D^H \left[ \begin{array}{cccc} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{array} \right] = \pi \left| \frac{\mu_1}{\mu_2} \right|^{2j_1+2} |\mu_2|^2 \times \\
\times \sum_{\gamma = \pm} \gamma^{j_1,j_2,j_3} \left| \frac{\mu_3}{\mu_2} \right|^{-2j_3} 2F_1(1 - j_2 - j_3, j_1 + j_2 - j_3 + 1, -2j_3, \frac{\mu_3}{\mu_2}),
\]

where $j^+ = j, j^- = -j - 1$, and $2F_1(a, b, c, z) = F(a, b, c)F(a, b, z)$. In this formula, the permutation symmetry of $D^H$ is not manifest, indeed $j_3$ plays a privileged rôle and the formula could be called a “$j_3$-decomposition” of $D^H$ into two terms $\eta = \pm$. The other possible decompositions associated with $j_1$ and $j_2$ naturally give the same result; this is a consequence of the monodromy properties of the hypergeometric function $2F_1$, which will shortly be interpreted as Liouville braiding.

Now let me consider the $H^+_3$ spectral flow-violating three-point function. This was determined in [9, 10]:

\[
\left\langle \prod_{\ell=1}^{3} \Phi^{j_\ell,w_\ell}_{\mu_\ell,\bar{\mu}_\ell}(z_\ell) \right\rangle = \sum_{u=-1}^{3} \left| \sum_{\ell=1}^{3} \frac{\Delta_{12}^{j_\ell} \Delta_{13}^{j_\ell} - \Delta_{23}^{j_\ell} - \Delta_{12}^{j_\ell} \Delta_{13}^{j_\ell}}{z_{12} z_{13} z_{23}} \right|^2 \\
\times \delta(2) \left( \sum_{\ell=1}^{3} m_\ell - k \right) \prod_{\ell=1}^{3} N^{j_\ell}_{m_\ell,\bar{m}_\ell} \times \tilde{C}^H(j_1, j_2, j_3),
\]

A useful notation. For conciseness, the modulus squared of $m$-dependent expressions means

\[
|Y(z, m)|^2 = Y(z, m) \times Y(z \to \bar{z}, m \to \bar{m}),
\]

that is a product of factors depending on $m$ and $\bar{m}$, although $m$ and $\bar{m}$ are not complex conjugates.

The spectral flow-violating structure constant has been determined up to a $k$-dependent normalization:

\[
\tilde{C}^H \simeq \left[ \frac{1}{\pi} \left( \frac{b^2}{2b^2} \right) \right]^{-j_{123}} \frac{\Gamma(b)(-b(2j_1 + 1))\Gamma(b(-b(2j_2 + 1))\Gamma(b(-b(2j_3 + 1))}{\Gamma(b(j_{123} + 1 + \frac{k}{2})\Gamma(b(j_{12} + \frac{k}{2})\Gamma(b(j_{13} + \frac{k}{2})\Gamma(b(j_{23} + \frac{k}{2}))}. \]

In the $\mu$ basis, the spectral flow-violating three-point function is:

\[
\left\langle \prod_{\ell=1}^{3} \Phi^{j_\ell,w_\ell}(\mu_\ell|z_\ell) \right\rangle = \sum_{u=-1}^{3} \left| \sum_{\ell=1}^{3} \frac{1}{4\pi^2} \delta(2) \left( \mu_\ell \rho_\ell \right) \delta(2) \left( \sum_{\ell=1}^{3} \mu_\ell \rho_\ell z_\ell \right) \right|^2 \\
\times \tilde{C}^H(j_1, j_2, j_3). \]

The $\mu$-dependence in this formula is derived with the help of the results in section 2 in particular the equation (2.29). But of course the $\mu$-basis result is equivalent to the previous $m$-basis formula.
3.1.2 Comparison with Liouville theory

The $H_3^+$ structure constants $C^H(j_1, j_2, j_3)$ and $\gamma_{j_3}^{j_1, j_2}$ are related to Liouville structure constants with momenta $\alpha = b(j + 1) + \frac{1}{2b}$ as follows [3]:

$$C^H(j_1, j_2, j_3) \gamma_{j_3}^{j_1, j_2} = -\frac{2\pi^3}{b} C^L(\alpha_1, \alpha_2, \alpha_3 + \frac{\eta}{2b}) C_{-\eta}(\alpha_3)$$

(3.15)

Here the diagrams illustrate relations between structure constants, which will later be promoted into relations between conformal blocks. These relations mean that the two terms $\eta = \pm$ of an $H_3^+$ vertex in the $j_3$-decomposition are equal to the two contributions to a Liouville vertex dressed with one degenerate field $\alpha = -\frac{1}{2b}$, associated with the two fusion channels $\alpha_3 \times -\frac{1}{2b} \rightarrow \alpha_3 + \frac{\eta}{2b}$. Fusing the degenerate field with $\alpha_1$ or $\alpha_2$ would yield the $j_1$- or $j_2$-decompositions of the $H_3^+$ vertex respectively. The different decompositions are therefore related by Liouville braiding as claimed above.

For completeness, let me recall the expressions [11, 12] for the Liouville structure constants which appear in eq. (3.15): (In the relation with the $H_3^+$ model, the Liouville interaction strength is fixed to $\mu_L = \frac{b^2}{\pi^2}$.)

$$C^L(\alpha_3, \alpha_2, \alpha_1) = \left[\pi \mu_L \hat{\gamma}(b^2) b^2 \right]^{\frac{1}{2}} \left[\frac{\hat{Y}_b(0) \hat{Y}_b(2\alpha_1) \hat{Y}_b(2\alpha_2) \hat{Y}_b(2\alpha_3)}{\Gamma(1 + b(2\alpha_1 - Q)) \Gamma(1 + b(2\alpha_2 - Q)) \Gamma(1 + b(2\alpha_3 - Q))} \right],$$

(3.16)

$$C^L_{-\eta}(\alpha) = R_L(\alpha) R_L(Q - \alpha - \frac{1}{2b}), \quad C^L_{+\eta}(\alpha) = 1,$$

(3.17)

$$R_L(\alpha) = (\pi \mu_L \hat{\gamma}(b^2) \frac{1}{2})^{-\frac{1}{2}} \frac{\Gamma(1 + b(2\alpha - Q)) \Gamma(1 + b^{-1}(2\alpha - Q))}{\Gamma(1 - b(2\alpha - Q)) \Gamma(1 - b^{-1}(2\alpha - Q))}.$$ 

(3.18)

The comparison of the spectral flow-violating structure constant $\tilde{C}^H$ with Liouville theory yields a very simple result: (The authors of [13] hint at such an equality but do not write it explicitly.)

$$\tilde{C}^H(j_1, j_2, j_3) = c_k C^L(\alpha_1, \alpha_2, \alpha_3)$$

(3.19)

Here $c_k$ is a $k$-dependent constant, whose determination would require a more precise calculation of $\tilde{C}^H$. 

– 13 –
3.1.3 The operator product expansion

Let me deduce the operator product expansion in the \( H_3^+ \) model from the two-point and three-point correlation functions. Using the two-point function (3.22), one can write:

\[
\Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2) \sim \int_{z_1\to 0} \int_{\frac{2}{3}+i\mathbb{R}} \frac{dz_1}{\mu s} |z_1\mu_s|^{4\Delta_{j_2}-k w_2^2} \\
\times \left\langle \Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2)\Phi^{-j_2-1,-w_2}(-(-1)^{w_2}z_2^{-2w_2}\mu_s|z_2) \right\rangle \Phi^{j_2,w_2}(\mu_2|z_2). \tag{3.20} \]

Here, \( z_s \) is an auxiliary worldsheet coordinate which disappears as \( z_{12} \to 0 \).

It is now necessary to discuss the domain of validity of the OPE eq. (3.20), and in particular the value of \( w_s \). When there is no spectral flow, \( w_1 = w_2 = 0 \), it is known \([8]\) that the OPE holds for \( w_s = 0 \): the OPE of unflowed fields yields only unflowed fields. The natural generalization is: the OPE preserves spectral flow, and eq. (3.20) holds for \( w_s = w_1 + w_2 \). However, the ordinary OPE \( \Phi^{j_1,0}\Phi^{j_2,0} \sim \int d\mathbf{s} \Phi^{j_2,0} \) may not hold when the fields are applied to states outside the physical spectrum, like states created by spectral flowed fields. For instance, using this OPE to compute a flow-violating three-point function \( \langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,-1} \rangle \) would yield an incorrect zero answer. The correct answer can however be obtained by using the OPE eq. (3.20) with \( w_s = w_1 + w_2 + 1 \).

In string theory in the Minkowskian \( AdS_3 \), all values of the spectral flow number \( w_s \) appear in the spectrum and are therefore expected to contribute to the OPE, \( \Phi^{j_1,w_1}\Phi^{j_2,w_2} \sim \int d\mathbf{s} \sum_{w_s \in \mathbb{Z}} \Phi^{j_3,w_s} \). This cannot happen in the \( H_3^+ \) model, because the spectrum is much smaller. Indeed, according to \([8]\), the \( H_3^+ \) four-point function \( \langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,0}\Phi^{j_4,0} \rangle \) can be decomposed by using the ordinary, flow-preserving OPE. No extra terms of the type \( \langle \Phi^{j_3,1}\Phi^{j_4,0}\Phi^{j_4,0} \rangle \) appear.

There are also cases, like the correlator \( \langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,0}\Phi^{j_4,-1} \rangle \) with one unit of spectral flow violation, where both OPEs \( \Phi^{j_1,0}\Phi^{j_2,0} \sim \int d\mathbf{s} \Phi^{j_2,0} \) and \( \Phi^{j_1,0}\Phi^{j_2,0} \sim \int d\mathbf{s} \Phi^{j_2,1} \) can be used and yield the same result for the correlator in question. This will be demonstrated in the next subsection, and is evidence for the following hypothesis:

**Hypothesis 1.** The \( H_3^+ \) operator product expansion eq. (3.20) can hold with either \( w_s = w_1 + w_2 - 1 \) or \( w_s = w_1 + w_2 \) or \( w_s = w_1 + w_2 + 1 \), depending on which correlator the expansion is inserted in. These possibilities are not exclusive, in particular both \( w_s = w_1 + w_2 \) and \( w_s = w_1 + w_2 + 1 \) expansions can be used in a correlator with spectral flow violation \( 0 < r < n - 2 \).

To prove this hypothesis would require a study of the fields \( \Phi^{j,w} \) as differential operators and of the domains they act on, which I shall not attempt. Further evidence will however come with the result eq. (3.20) for the \( H_3^+ \) correlators, derived using the hypothesis, and which is compatible with both choices of OPEs \( w_s = w_1 + w_2 \) and \( w_s = w_1 + w_2 + 1 \), when such a choice is available.

Moreover, the hypothesis is consistent with the definition of spectral flowed correlators reported in \([10, 13]\) and attributed to Fateev, Zamolodchikov and Zamolodchikov. This definition indeed involves the insertion of the \( H_3^+ \) operator of spin \( -\frac{1}{2} \):

\[
\langle \Phi^{j_1,1}(z_1)\Phi^{j_2,0}(z_2)\cdots \rangle \propto \lim_{u\to 0} \left\langle \Phi^{j_1}(z_1)\Phi^{j_2}(z_2)\Phi^{-\frac{1}{2}}(u)\cdots \right\rangle \propto \left\langle \Phi^{-j_1-\frac{1}{2}}(z_1)\Phi^{j_2}(z_2)\cdots \right\rangle . \tag{3.21} \]

In the limit \( z_1 \to z_2 \), this leads to the flow-violating OPE:

\[
\lim_{z_1 \to z_2} \left\langle \Phi^{j_1,1}(z_1)\Phi^{j_2,0}(z_2)\cdots \right\rangle \propto \lim_{z_1 \to z_2} \left\langle \Phi^{-j_1-\frac{1}{2}}(z_1)\Phi^{j_2}(z_2)\cdots \right\rangle \propto \int d\mathbf{s} \left\langle \Phi^{j_2,0}\cdots \right\rangle . \tag{3.22} \]
where the last operation was an ordinary $H_3^+$ OPE which yielded an unflowed field. However inverting the limits $z_1 \to z_2$ and $u \to z_1$ leads to

$$
\lim_{u \to z_1} \lim_{z_1 \to z_2} \left\langle \Phi^{j_1}(z_1) \Phi^{j_2}(z_2) \Phi^{-\frac{H}{2}}(u) \cdots \right\rangle \propto \lim_{u \to z_1} \int d_{j_s} \left\langle \Phi^{j_s}(z_1) \Phi^{-\frac{H}{2}}(u) \cdots \right\rangle
$$

$$
\propto \int d_{j_s} \left\langle \Phi^{j_s,1}(z_1) \cdots \right\rangle, \quad (3.23)
$$

i.e. a spectral flow-preserving OPE.

The hypothesis above suggests that the states $|j, w \neq 0\rangle$ created by spectral-flowed operators in the $H_3^+$ model are similar to the states $|j \not\in -\frac{1}{2} + i\mathbb{R}, 0\rangle$: they do not belong to the physical spectrum and do not appear in the physical OPE, but they have a non-vanishing three-point function with physical states, which can be accounted for by a non-physical OPE. However, in contrast to the flow-violating OPE, the non-physical OPE involving $j \not\in -\frac{1}{2} + i\mathbb{R}$ is obtained from the physical OPE by deforming the contour of integration $\int_{-\frac{1}{2} + i\mathbb{R}} d_{j_s}$.

Now here are explicit expressions for the OPEs derived from eq. (3.20), obtained by inserting the explicit expression for the three-point function. This yields the spectral flow-preserving OPE in the case $w_s = w_1 + w_2$,

$$
\Phi^{j_1,w_1}(\mu | z_1) \Phi^{j_2,w_2}(\mu | z_2) \sim \int_{z_1 \to z_2} \frac{1}{\pi |\mu_s|^{2H}} \left[ \frac{j_1}{\mu_1 z_{12}^{w_2}} \frac{j_2}{\mu_2 z_{21}^{w_1}} \right] \Phi^{j_s,w_1+w_2}(\mu = \mu_1 z_{12}^{w_2} + \mu_2 z_{21}^{w_1} | z_1), \quad (3.24)
$$

and the spectral flow-violating OPE in the case $w_s = w_1 + w_2 + 1$,

$$
\Phi^{j_1,w_1}(\mu | z_1) \Phi^{j_2,w_2}(\mu | z_2) \sim \int_{z_1 \to z_2} \frac{1}{4\pi^2} \delta^{(2)}(\mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1}) \left| \mu_1 z_{12}^{w_2+1} \right|^{2-k} \times
$$

$$
\int_{-\frac{1}{2} + i\mathbb{R}} d_{j_s} \tilde{C}^H(j_1, j_2, j_s - 1) |z_1-z_2|^{-2\Delta^+_{12} + \frac{H}{2} - kw_{12}w_2} \Phi^{j_s,w_1+w_2+1}(\mu_1 z_{12}^{w_2+1} | z_1). \quad (3.25)
$$

### 3.2 $H_3^+$ correlators from Liouville theory

#### 3.2.1 Results

The relations between the $H_3^+$ model and Liouville theory at the levels of structure constants eq. (3.15), (3.19), and differential equations reflecting chiral symmetry eq. (2.40), lead to the following expression for the $H_3^+$ correlation functions:

$$
\left\langle \prod_{\ell=1}^{n} \Phi^{j_\ell,w_\ell}(\mu|z_\ell) \right\rangle \sum_{w=-r \leq 0} \frac{\pi}{2} (-\pi)^{-n} b c^\ell_k \times
$$

$$
\prod_{j=0}^{r} \delta^{(2)}(\sum_{\ell} \mu_\ell z_\ell^j) \sum_{\sum_{\ell} \mu_\ell z_\ell^{j+1}} |2+2r-kr|^{2r-kr} |\Theta_{n,r}|^{k-2} \left\langle \prod_{\ell=1}^{n} V_{\alpha_{\ell}}(z_\ell) \prod_{a=1}^{n} V_{-\frac{1}{2}}(y_a) \right\rangle
$$

Let me recall the notations involved in this formula: $V_{\alpha}(z)$ is the Liouville vertex operator of conformal weight $\Delta_\alpha = \alpha(b + b^{-1} - \alpha)$, where the Liouville parameter is $b = (k - 2)^{-\frac{1}{4}}$ and
the interaction strength is $\mu_L = b^2/\pi^2$; the Liouville momenta $\alpha = b(j + 1) + \frac{1}{2R}$ are such that $\Delta_\alpha = \Delta_j + \frac{k}{2}$; and the positions of the Liouville degenerate fields $y_a$ are defined by
\[
\sum_{\ell=1}^{n} \frac{\mu_\ell \rho_\ell}{t - z_\ell} = \left( \sum_{\ell=1}^{n} \mu_\ell \rho_\ell z_\ell^{r+1} \right) \prod_{\ell=1}^{n-2-r}(t - y_a) \prod_{\ell \neq j}^{n}(t - z_\ell) \quad \text{with} \quad \rho_\ell = \prod_{j \neq \ell}^{n} z_\ell^{w_j}.
\] (3.27)

The factor $\Theta_{n,r}$ was defined in eq. (2.43), and the $k$-dependent factor $c_k$ is not known. ($c_k$ is related but not equal to the $c_k$ of eq. (3.19); other unknown $c_k$s will appear below.)

The $m$-basis $H_3^+$ correlators are related to Liouville correlators by applying the change of basis (3.1) and the change of variables (2.38), whose Jacobian is:
\[
\prod_{i=1}^{n} d^2\mu_i \delta^{(2)}(\sum_\mu_i z_\mu^r) \cdots \delta^{(2)}(\sum_\mu_i) \equiv \frac{d^2u}{|u|^{4+2}}, \prod_{a=1}^{n-2-r} d^2 y_a \prod_{a < a'} |y_{aa'}|^2 \prod_{i < i'} |z_{ii'}|^2.
\] (3.28)

The result is:

\[
\left\langle \prod_{\ell=1}^{n} \Phi_{j_\ell,m_\ell}^{i_\ell}(z_\ell) \right\rangle = \sum_{w=-r \leq 0}^{2} \! \! \frac{2\pi^{3-2n}b c_k}{(n-2-r)!} \prod_{\ell=1}^{n} N_{j_\ell,m_\ell} \times \delta^{(2)}(\sum m_\ell - k/2) \left( \prod_{\ell < \ell'} \bar{z}_{\ell\ell'}^{\beta_{\ell\ell'}} \right) \left( \prod_{a=1}^{n-2-r} d^2 y_a \frac{\prod_{a < a'} |y_{aa'}|^k}{\prod_{\ell,a}(z_\ell - y_a)^{-\frac{k}{2} - m_\ell}} \right)^2 \left( \prod_{\ell=1}^{n} V_{\alpha_\ell}(z_\ell) \prod_{a=1}^{n} V_{\frac{1}{2b}}(y_a) \right)^{\frac{n-2-r}{2}}
\] (3.29)

where the combinatorial factor $\frac{1}{(n-2-r)!}$ comes from the invariance of the $\mu_\ell$ wrt permutations of the $y_a$s, and the exponent $\beta_{\ell\ell'}$ is defined by
\[
\beta_{\ell\ell'} \equiv \frac{k}{2} \sum_{\ell < \ell'} w_\ell w_\ell' - w_i m_\ell - w_\ell m_i - m_\ell - m_i.
\] (3.30)

The formula (3.29) can be rephrased in the language of the parafermions (2.9) and it gives the $n$-point function $\left\langle \prod_{\ell=1}^{n} \Psi_j^{i_\ell,m_\ell} \right\rangle$ provided $\beta_{\ell\ell}$ is replaced with
\[
\beta'_{\ell\ell} = \frac{2}{k} (m_\ell - \frac{k}{2}) (m_\ell - \frac{k}{2}).
\] (3.31)

The resulting expression for the parafermionic correlators agrees with the unpublished results of Fateev [14], obtained by free field methods.

The integrals over $y_a$ in eq. (3.29) may have singularities at $y_a = z_i$, depending on the values of $m_\ell, m_\ell$. The physical values in the $H_3^+$ model are $m - \bar{m} \in \mathbb{Z}$, $m + \bar{m} \in i\mathbb{R}$ and would make the integrals converge, but they are forbidden by the constraints $\sum m = \sum \bar{m} = k/2$. However, assuming the internal spins are physical $j \in -\frac{1}{2} + i\mathbb{R}$, the integrals actually converge provided $\Re(m_\ell + \bar{m}_\ell) > -1$. This mild condition from the point of view of the $H_3^+$ model becomes a problem when Wick-rotating to string theory in $AdS_3$, whose physical spectrum satisfies $m_\ell + \bar{m}_\ell \in \mathbb{R}$.
3.2.2 Proof

Here I show how to prove the relation between $H_3^+$ correlators in the $\mu$ basis and Liouville theory correlators, eq. (3.26). In the case when there is no spectral flow $w_\ell = 0$, this has been done in [3]. Then, in the spectral flow-preserving case $\sum w_\ell = -r = 0$, this is a simple consequence of the formula (2.23), where the action of the differential operator $\kappa$ (2.21) on $\mu_\ell$ accounts for the $\rho_\ell$ factors. The general case will now be proved by induction on the spectral flow violation number $r$. A correlator with $r$ units of spectral flow violation reduces to a correlator with $r-1$ units by taking the limit $z_{12} \to 0$ and using the spectral flow-violating OPE (3.25). Assuming the relation with Liouville theory holds in the limit $z_{12} \to 0$, then it holds for all $z_1, z_2$ by using the results of Section 2 on the relation between KZ and BPZ equations (2.40). This is because these differential equations are first order in $z_1, z_2$.

Therefore, all I have to prove is that the proposed relation with Liouville theory eq. (3.26) is compatible with the spectral flow-violating OPE eq. (3.25). The comparison between the spectral flow-violating structure constant $\tilde{C}_{H}$ and Liouville theory (3.19) shows that the OPE coefficients agree. Now consider the quantities $u_j \equiv \sum_{\ell=0}^{\ell} \mu_\ell \rho_\ell z_{12}^j$ appearing in an $n$-point function, and

$$u'_j \equiv \mu_1 z_{12}^{w_1+1} \prod_{\ell \geq 3} z_{1\ell}^{w_\ell} + \sum_{i \geq 3} \mu_i z_{i1}^{w_1+w_2+1} \prod_{\ell \geq 3, \ell \neq i} z_{i\ell}^{w_\ell},$$

(3.32)

which appears in the $n-1$-point function obtained by a spectral flow-violating OPE. Direct computation leads to

$$u'_j \sim_{z_{12} \to 0} u_{j+1} - z_1 u_j.$$  

(3.33)

This equation is the key to showing that the positions $y_\alpha$ of the $n-2-r$ auxiliary fields are not affected by the OPE, and that the delta-function factors behave correctly:

$$\delta^{(2)} \left( \mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1} \right) \prod_{j=0}^{r-1} \delta^{(2)}(u'_j) \sim_{z_{12} \to 0} \delta^{(2)}(z_{12} u_0) \prod_{j=0}^{r-1} \delta^{(2)}(u_{j+1} - z_1 u_j) = |z_{12}|^{-2} \prod_{j=0}^{r} \delta^{(2)}(u_j).$$  

(3.34)

These manipulations with $\delta$-functions will not be very rigorous as long as I do not define the domain of these distributions. An alternative is to prove the equivalent $m$-basis result (3.29) instead of the $\mu$-basis result. This is actually quite straightforward, but I have given the argument in the $\mu$-basis in order to illustrate the $\mu$-basis OPE. One reason to insist on the use of the $\mu$-basis is that conformal blocks in this basis are much simpler than in any other basis, as will be demonstrated in the next subsection.

3.3 $H_3^+$ conformal blocks from Liouville theory

The relation (3.26) between $H_3^+$ correlators and Liouville correlators can be decomposed into relations between the structure constants of the theories, which I already wrote (3.15), (3.19), and relations between the conformal blocks.
Consider an $n$-point correlator in $H_3^+$ with $r$ units of spectral flow violation, $\sum w = -r \leq 0$. I will consider decomposition in conformal blocks which use vertices with winding violation 0 or $-1$. If vertices with winding violation $+1$ were included, there would probably be no simple relation to Liouville theory. Moreover, for ease of writing I will only consider a specific case $n = 6, r = 2$, which involves $n - 2 - r = 2$ Liouville degenerate field insertions.

A basis of $H_3^+$ non-chiral conformal blocks is defined as follows:

$$\mathcal{C}^{j_3, j_4, j_5, j_6} = \int \prod_{a=1}^{n} d\gamma_{j_a} \left( \sum_{m_{\ell}} \mathcal{C}_{m_{\ell}} \right)$$

The pictorial representation for the conformal block leaves its dependence on $\mu_{\ell}, z_{\ell}, j_1, w_{\ell}$ implicit. Note that with our definition of the structure constant $C^H$ (3.7), the conformal block is invariant wrt reflection of external spins $j_1 \cdots j_6$ and internal spins $j_{12}, j_{34}, j_{56}$.

This $H_3^+$ non-chiral conformal block can now be decomposed into Liouville chiral conformal blocks in the following way:

$$\mathcal{C}^{j_3, j_4, j_5, j_6} = \frac{2}{(2\pi)^{-n}} b c_k^r \prod_{j=0}^{r-1} \delta^{(2)}(\sum_{\mu_{\ell} p_{\ell}} z_{\ell}) \left| \sum_{\mu_{\ell} p_{\ell}} z_{\ell} + 1 \right|^{2r - 2k + 2}$$

where the indices $\eta_4, \eta_5$ indicate the fusion channels of the two degenerate fields, $\alpha_4 + \frac{8\ell}{2}$ and $\alpha_5 + \frac{\eta_5}{26}$, and the $z_{\ell}, y_{\alpha}$ dependence of the Liouville conformal block on the positions of the fields is omitted. Alternative positionings of the degenerate field insertions are possible, as long as they remain around the Liouville vertices which correspond to the spectral flow-preserving vertices in $H_3^+$. Naturally, $H_3^+$ conformal blocks in the $m$-basis (and thus parafermionic conformal blocks) can also be expressed in terms of Liouville theory conformal blocks in a similar manner:

$$\mathcal{C}^{j_3, j_4, j_5, j_6} = \frac{2\pi^{3-2m} b c_k^r}{(n - 2 - r)!} \prod_{\ell=1}^{n} N_{j_{\ell}, m_{\ell}} \times \delta^{(2)}(\sum m_{\ell} - \frac{k}{2}) \left| \sum_{\ell < \ell'} \delta_{\ell, \ell'} \right|^2$$

$$\times \sum_{\eta_4, \eta_5} \frac{\gamma_{j_3, j_4} \gamma_{j_5, j_6}}{26} \int d^2 y_{\alpha} \frac{\prod_{a=1}^{n} y_{a a'}}{\prod_{\ell, a} (z_{\ell} - y_{a})^{k - m_{\ell}} \eta_{j_a}}.$$
What is however unique to the \( \mu \)-basis is the possibility of explicitly writing the \( H_3^+ \) conformal blocks in some limits, obtained by performing multiple OPEs (3.24), (5.25), for instance

\[
\begin{align*}
\frac{1}{16\pi^2} \delta^{(2)}(\mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1}) \\
x \times \delta^{(2)}(\mu_{12} z_{13}^{w_3+w_4+1} - \mu_{34} z_{31}^{w_2+w_2+2}) \delta^{(2)}(\mu_{12} z_{13}^{w_3+w_4+1} z_{15}^{w_5} + \mu_5 z_{51}^{w_5-w_6} + \mu_6 z_{61}^{2w_6}) \\
x \times |\mu_1|^{2-2k} |z_{12}|^{-2\Delta_{12}^{12}} |z_{34}|^{-2\Delta_{34}^{34}} |z_{13}|^{-2\Delta_{13}^{13}} |z_{15}|^{-2\Delta_{15}^{15}} |z_{16}|^{-4\Delta_6} \delta^{(2)}(\sum \mu_\ell) |\mu_{12}|^{-2} |\mu_{34}|^{-2} |\mu_{56}|^{-2}
\end{align*}
\]

\( \times D^H \left[ \frac{j_3}{\mu_3 z_{13}^{w_3}} \frac{j_4}{\mu_4 z_{43}^{w_3}} \frac{j_{34}}{-\mu_{34}} \right] D^H \left[ \frac{j_{56}}{\mu_{12} z_{13}^{w_3+w_4+1} z_{15}^{w_5}} \frac{j_5}{\mu_5 z_{51}^{w_5-w_6}} \frac{j_6}{\mu_6 z_{61}^{2w_6}} \right], \quad (3.38)
\]

where \( \mu_{34} = \mu_3 z_{34}^{w_4} + \mu_4 z_{43}^{w_3} \) and \( \mu_{12} = \mu_1 z_{12}^{w_2+1} \). In the case where all spectral flow numbers \( w_\ell \) vanish, the conformal blocks reduce in such limits to \( SL(2, \mathbb{C}) \) coinvariants which have an explicit expression as products of the \( D^H \) coefficients:

\[
\begin{align*}
|z_{12}|^{-2\Delta_{12}^{12}} |z_{34}|^{-2\Delta_{34}^{34}} |z_{13}|^{-2\Delta_{13}^{13}} |z_{15}|^{-2\Delta_{15}^{15}} |z_{16}|^{-4\Delta_6} \delta^{(2)}(\sum \mu_\ell) |\mu_{12}|^{-2} |\mu_{34}|^{-2} |\mu_{56}|^{-2}
\end{align*}
\]

\( \times D^H \left[ \frac{j_1}{\mu_1} \frac{j_2}{\mu_2} \frac{j_{12}}{-\mu_{12}} \right] D^H \left[ \frac{j_3}{\mu_3} \frac{j_4}{\mu_4} \frac{j_{34}}{-\mu_{34}} \right] D^H \left[ \frac{j_5}{\mu_5} \frac{j_6}{\mu_6} \frac{j_{56}}{-\mu_{56}} \right] D^H \left[ \frac{j_{12}}{\mu_{12}} \frac{j_{34}}{\mu_{34}} \frac{j_{56}}{\mu_{56}} \right], \quad (3.39)
\]

where \( \mu_{\ell\ell'} = \mu_\ell + \mu_{\ell'} \). The \( H_3^+ \) conformal blocks are fully determined by their behaviour in such limits, plus the KZ equations.

4. Outlook

The present results may hopefully be useful in the study of string theory in \( AdS_3 \). Correlators in this theory should be obtained from \( H_3^+ \) correlators in the \( m \)-basis (5.29) by Wick-rotation of \( m_\ell + \bar{m}_\ell \), in the spirit of [10]. The integrals in eq. (5.29) then become divergent, and regularizing them should lead to the appearance of discrete states in the intermediate channels, even if all external states are continuous.

The expression (3.36) of \( H_3^+ \) conformal blocks in terms of Liouville theory conformal blocks suggests a way to define and compute the fusing matrix in the \( H_3^+ \) model. This fusing matrix is still not well understood, see [15]. However, knowing all \( H_3^+ \) correlators in terms of Liouville correlators makes this issue less crucial for the \( H_3^+ \) model proper. Nevertheless, an \( H_3^+ \) fusing matrix should be a very interesting object in itself, which may have an interpretation in terms of harmonic analysis on the quantum group \( U_q(s\ell_2) \).
Acknowledgments

I am grateful to Thomas Quella, Andreas Recknagel and Gérard Watts for interesting conversations. Some ideas in this work arose while collaborating with Joerg Teschner on related issues, and I also thank him for comments on this manuscript. Moreover, I wish to thank Vladimir Fateev for a copy of his unpublished note [14].

I am supported by the EUCLID European network, contract number HPRN-CT-2002-00325, and also in part by the PPARC rolling grant PPA/G/O/2002/00475.

References

[1] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and wess-zumino model in two dimensions, Nucl. Phys. B247 (1984) 83–103.

[2] J. M. Maldacena and H. Ooguri, Strings in AdS(3) and SL(2,R) WZW model. I, J. Math. Phys. 42 (2001) 2929–2960 [hep-th/0001053].

[3] S. Ribault and J. Teschner, $H_3^+$ correlators from Liouville theory, hep-th/0502049.

[4] E. K. Sklyanin, Separation of variables in the gaudin model, J. Sov. Math. 47 (1989) 2473–2488.

[5] A. V. Stoyanovsky, A relation between the knizhnik–zamolodchikov and belavin-polyakov–zamolodchikov systems of partial differential equations, math-ph/0012013.

[6] K. Gawedzki, Noncompact wzw conformal field theories, hep-th/9110076.

[7] J. Teschner, The mini-superspace limit of the SL(2,C)/SU(2) WZNW model, Nucl. Phys. B546 (1999) 369–389 [hep-th/9712258].

[8] J. Teschner, Operator product expansion and factorization in the $H_3^+$ WZNW model, Nucl. Phys. B571 (2000) 555–582 [hep-th/9906215].

[9] G. Giribet and C. Nunez, Correlators in ads(3) string theory, JHEP 06 (2001) 010 [hep-th/0105200].

[10] J. M. Maldacena and H. Ooguri, Strings in AdS(3) and the SL(2,R) WZW model. III: Correlation functions, Phys. Rev. D65 (2002) 106006 [hep-th/0111180].

[11] H. Dorn and H. J. Otto, Two and three point functions in liouville theory, Nucl. Phys. B429 (1994) 375–388 [hep-th/9403141].

[12] A. B. Zamolodchikov and A. B. Zamolodchikov, Structure constants and conformal bootstrap in liouville field theory, Nucl. Phys. B477 (1996) 577–605 [hep-th/9506136].

[13] G. Giribet and Y. Nakayama, The stoyanovsky-ribault-teschner map and string scattering amplitudes, hep-th/0505203.

[14] V. Fateev, Relation between Sine-Liouville and Liouville correlation functions, unpublished note.

[15] B. Ponsot, Monodromy of solutions of the Knizhnik-Zamolodchikov equation: SL(2,C)/SU(2) WZNW model, Nucl. Phys. B642 (2002) 114–138 [hep-th/0204085].