AN ELEMENTARY PROOF OF THE $A_2$ BOUND

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ABSTRACT. A martingale transform $T$, applied to an integrable locally supported function $f$, is pointwise dominated by a positive sparse operator applied to $|f|$, the choice of sparse operator being a function of $T$ and $f$. As a corollary, one derives the sharp $A_p$ bounds for martingale transforms, recently proved by Thiele-Treil-Volberg, as well as a number of new sharp weighted inequalities for martingale transforms. The (very easy) method of proof (a) only depends upon the weak-$L^1$ norm of maximal truncations of martingale transforms, (b) applies in the vector valued setting, and (c) has an extension to the continuous case, giving a new elementary proof of the $A_2$ bounds in that setting.

1. INTRODUCTION

Our subject is weighted inequalities in Harmonic Analysis, a subject which started in the 1970’s, and has been quite active in recent years. A succinct history begins in 1973, when Hunt-Muckenhoupt-Wheeden [HMW] proved that the Hilbert transform $H$ was bounded on $L^2(w)$, for non-negative weight $w$, if and only if the weight satisfied the Muckenhoupt $A_2$ condition [M]. One could have asked, even then, what the sharp dependence on the norm of $H$ is, in terms of the $A_2$ characteristic of the weight. This was not resolved until 2007, by Petermichl [P], showing that norm of the Hilbert transform is linear in the $A_2$ characteristic. Five years later, among competing approaches, Hytönen [H1] established the $A_2$ Theorem: For any Calderón-Zygmund operator $T$, and any $A_2$ weight, the norm estimate is linear in $A_2$ characteristic. Hytönen’s original method depended upon the Hytönen Representation Theorem, [H1, Thm. 4.2], a novel expansion of $T$ into a rapidly convergent series of discrete dyadic operators. Subsequently, Lerner [L4] gave an alternate proof, cleverly exploiting his local mean oscillation inequality [L2] to prove a remarkable statement: The norm of Calderón-Zygmund operators on a Banach lattice is dominated by the operator norm of a class of very simple positive sparse operator $S|f|$. See [L4, (1.2)]. The $A_2$ bound is very easy to establish for the sparse operators. For more details on the history, see [H2, L4].

The purpose of this note is to establish the pointwise control of $Tf$ by a sparse operator in a direct elementary fashion. For each $T$ and suitable $f$, there is a choice of sparse operator $S$ so that $|Tf| \lesssim S|f|$. This improves on the main result of Lerner in [L3] by removing the notion of complexity for the sparse operators, and on [L4] which does not need complexity, but only bounds the operator norm. Moreover, the recursive proof only

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requires elementary facts about dyadic grids, and the weak integrability of the maximal truncations of $T$. (That is, we do not need Lerner’s mean oscillation inequality.)

The argument is especially transparent in the setting of martingale transforms §2. The main result, Theorem 2.4, proved in §3, is new, and cannot be proved by Lerner’s mean oscillation inequality. As corollaries, we deduce the $A_2$ bound for martingale transforms recently established by Thiele-Treil-Volberg [TTV1], who employ more sophisticated techniques. A number of new, sharp weighted inequalities for martingales also follow as corollaries. The argument can be formulated in the Euclidean setting, giving a new self-contained proof of the $A_2$ Theorem in that setting, which we do in §5. Remarks and complements conclude the paper.

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2. Martingale Transforms

We introduce standard notation required for a discrete time martingale. Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, and let $\{Q_n : n \in \mathbb{Z}\}$ be collections of measurable sets $Q_n \subset \mathcal{A}$ such that

1. Each $Q_n$ is a partition of $\Omega$, and $0 < \mu(Q) < \infty$ for all $Q \in Q_n$.
2. For each $n$, and $Q \in Q_n$, the set $Q$ is the union of sets $Q' \in Q_{n+1}$ (And $Q'$ is allowed to equal $Q$.)
3. The collection $Q := \bigcup_{n \in \mathbb{Z}} Q_n$ generates $\mathcal{A}$.

The set $Q$ is a tree with respect to inclusion. (And the branching of the tree is unbounded in general.) Set $Q^a$ to be the minimal element $P \in Q$ which strictly contains $Q$. In words, $Q^a$ is the parent of $Q$. Denote by $\text{ch}(P)$ those $Q \in Q$ such that $Q^a = P$, that is the children of $P$.

The conditional expectation associated with $Q_n$ is

$$E(f | Q_n) := \sum_{Q \in Q_n} \mu(Q)^{-1} \int_Q f \, d\mu \cdot 1_Q.$$ 

Below, we will frequently use the abbreviation $\langle f \rangle_Q := \mu(Q)^{-1} \int_Q f \, d\mu$. The martingale difference associated with $P \in Q$ is

$$\Delta_P f := \sum_{Q \in \text{ch}(P)} 1_Q (\langle f \rangle_Q - \langle f \rangle_P).$$

One should note that $P \in Q_n$ is allowed to be in $Q_{n+1}$, so that the difference above is equal to $(E(f 1_P | Q_{n+1}) - \langle f \rangle_P)1_P$ only if $P \in Q_n$, but not $Q_{n+1}$.

There are two kinds of operators we consider. The first are martingale transforms, namely operators of the form

$$Tf = \sum_{P \in P} \epsilon_P \Delta_P f, \quad |\epsilon_P| \leq 1.$$ 

By the joint orthogonality of the martingale differences, $T$ is an $L^2$-bounded operator.
And the second category are sparse operators. An operator $S$ is *sparse* if

$$\tag{2.2} Sf = \sum_{P \in S} \langle f \rangle_P 1_P$$

where the collection $S$ satisfies this *sparseness condition*: For each $P \in S$,

$$\tag{2.3} \sum_{Q \in \text{ch}_S(P)} \mu(Q) \leq \frac{1}{2} \mu(P).$$

The sum on the left is restricted to the $S$-children of $P$: The maximal elements of $S$ that are strictly contained in $P$. It is well-known that both classes of operators are bounded in $L^p$, with constants uniform over the class of operators.

One of the main results is this extension of the Lerner inequality [L4] to the martingale setting, with a very simple proof, see §3.

**Theorem 2.4.** There is a constant $C > 0$ so that for all functions $f \in L^1(\mu)$ supported on $Q_0 \in Q$, and martingale transforms $T$, there is a sparse operator $S$ so that

$$\tag{2.5} 1_{Q_0} |Tf| \leq C \cdot S |f|.$$  

The same inequality holds for the maximal truncations, $T^\sharp$, as defined in (3.2).

From this Theorem, one can deduce a range of sharp inequalities for martingale transforms by simply repeating the proofs for sparse operators on Euclidean space, using for instance the arguments in [L3, §2].

Consider two weights $w, \sigma$. For pair, we have the joint $A_p$ condition, $1 < p < \infty$, given by

$$[\sigma, w]_{A_p} := \sup_{Q \in Q} \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q = \sup_n \left[ \|E(\sigma | Q_n)\|^{p-1} E(\sigma | Q_n) \right].$$

If $w$ is non-negative $\mu$-almost everywhere, and $w^{1/(1-p)}$ is also a weight (that is, $w^{1/(1-p)}$ is locally integrable), then $[w^{1/(1-p)}, w]_{A_p} = [w]_{A_p}$ is the $A_p$ characteristic of $w$.

This is the $A_2$ Theorem in the martingale setting, as proved by [TTV1]. We will give a self-contained, short and elementary proof in §4.

**Theorem 2.6.** For any martingale transform or sparse operators $T$, for any $1 < p < \infty$, and $A_p$ weights $w$,

$$\|T : L^p(w) \to L^p(w)\| \leq [w]_{A_p}^{\min(1,1/(p-1))}.$$  

One draws the corollary that the martingale differences $\Delta Q$ are unconditionally convergent in $L^p(w)$, for all $1 < p < \infty$, and all $A_p$ weights $w$, which fact has been known since the 1970’s. We return to this topic in §6.

Many finer results are also corollaries, and the reader can use [L3, §2], and [HL, L1, NRTV, L4, TV1] as guides to these results.

One further application is to certain martingale paraproduct operators. The Theorem below follows from an easy modification of the proof that follows, but we suppress the
details. Define an operator \( \Pi f := \sum_{Q \in \mathcal{Q}} \langle f \rangle_Q : b_Q, \) where \( \{b_Q : Q \in \mathcal{Q}\} \) are a sequence of functions which satisfy \( \Delta_Q b_Q = b_Q, \) and this normalization condition.

\[
\sup_P \mu(P)^{-1} \sum_{Q : Q \subset P} \|b_Q\|_2 \mu(Q) \leq 1.
\]

It is an easy consequence of the Carleson embedding theorem that \( \Pi \) is an \( L^2 \)-bounded operator. They are considered in [TV1, Thm 2.5], and they too can be dominated by a sparse operator.

**Theorem 2.7.** Let \( f \in L^1(\Omega) \) be supported on \( Q_0 \in \mathcal{Q} \). Then, there is a sparse operator \( S \) such that \( 1_{Q_0} |\Pi f| \lesssim S f \).

A number of weighted inequalities for these operators is an immediate consequence, including both of the main results of [TV1].

### 3. Domination by Sparse Operators: Martingales

We give the proof of Theorem 2.4, with the basic fact being the weak-\( L^1 \) inequality for maximal truncations of martingale transforms.

**Theorem 3.1.** [Burkholder [B, Thm 6]] For any martingale transform \( T \) we have

\[
\sup_{\lambda > 0} \lambda \mu(T^\sharp f > \lambda) \lesssim \lambda^{-1} \|f\|_{L^1(\mu)},
\]

where \( T^\sharp f \) is the maximal truncation, given by

\[
T^\sharp f := \sup_{Q'} \left| \sum_{Q \in P, Q' \supset Q} \epsilon_Q \Delta_Q f \right|.
\]

**Proof of Theorem 2.4.** We concentrate on the case of \( T \) being a martingale transform as in (2.1), with the changes to accommodate the maximal truncations being easy to provide.

Apply the weak \( L^1 \) inequality for the maximal function \( Mf \), and the maximal truncations of the martingale transform. Thus, there is a constant \( C_0 \) so large that the set

\[
E := \{ \max \{ Mf + T^\sharp f \} > \frac{1}{2} C_0 \| f \rangle_{Q_0} \}
\]

satisfies \( \mu(E) \leq \frac{1}{2} \mu(Q_0) \). Let \( \mathcal{E} \) be the collection of maximal elements of \( Q \) contained in \( E \). We claim that

\[
|Tf(x)| 1_{Q_0} \leq C_0 \| f \rangle_{Q_0} + \sum_{P \in \mathcal{E}} |T_P f(x)|,
\]

where \( T_P f := \epsilon_{P_0} \langle f \rangle_P 1_P + \sum_{Q : Q \subset P} \epsilon_Q \Delta_Q f. \)
Here, \( P^a \) is the parent of \( P \). It is clear that we will add \( Q_0 \) to \( S \), the sparse collection defining \( S \). And, one should recurse on the \( T_PF \), at which stage the collection \( \mathcal{E} \) becomes the \( S \)-children of \( Q_0 \). Continuing the recursion will complete the proof.

If \( x \in Q_0 \setminus E \), certainly (3.3) is true. For \( x \in E \) then there is a unique \( P \in \mathcal{E} \) with \( x \in P \), for which we can write

\[
(3.4) \quad Tf(x) = \sum_{Q : P^a \subseteq Q} e_Q \Delta_P f(x) - e_{P^a} (f)_{P^a} + T_P f(x).
\]

Note that the martingale difference \( \Delta_P f(x) \) is split between the second and third terms above. The first two terms on the right in (3.4) are, by construction, bounded by \( \frac{1}{2} C_0 \langle |f| \rangle_{Q_0} \). Thus (3.3) follows.

\[ \square \]

4. Proof of the \( A_p \) estimates

We give a proof of Theorem 2.6 which is an extension of the \( L^2 \) proof in [L3, §2.1]. The advantage of the current proof is that it applies to all \( 1 < p < \infty \). (No auxiliary extrapolation argument is needed.) By Theorem 2.4 it suffices to prove the estimate for sparse operators \( S \). There is a general remark, that for \( w \in A_p \), the 'dual' weight is \( \sigma := w^{1/(1-p)} \), and it equivalent to show that

\[
(4.1) \quad \|S(\sigma f)\|_{L^p(w)} \lesssim [\sigma, w]_{A_p}^{\max(1,1/(p-1))} \|f\|_{L^p(\sigma)}.
\]

Consider first the case of \( 1 < p \leq 2 \). Note that the bound above is the same as

\[
(4.2) \quad \|S(\sigma f)^{p-1}\|_{L_p'(w)} \lesssim [\sigma, w]_{A_p} \|f\|_{L_p(\sigma)}^{p-1}.
\]

Here \( p' = p/(p-1) \) is the conjugate index to \( p \). The sparse operator is defined by the sparse collection \( S \). For \( P \in S \), we set \( E_P \) to be those \( x \in P \) which are not contained in any \( S \)-child of \( P \). Then, \( \mu(E_P) \geq \frac{1}{4} \mu(P) \), and these sets are pairwise disjoint in \( P \). But also note that an \( A_p \) weight is necessarily non-zero \( \mu \)-a.e., hence by Hölder’s inequality,

\[
(4.3) \quad \mu(E_P) = \int_{E_P} w^{-1/p} \cdot w^{1/p} \, d\mu \leq \sigma(E_P)^{1/p'} w(E_P)^{1/p}.
\]

Using the subadditivity of \( t \mapsto t^{p-1} \), we have for non-negative \( f \),

\[
[S(\sigma f)]^{p-1} \leq \sum_{P \in S} (f|_P^{p-1} 1_P.
\]
Below, we will write $\langle f\sigma \rangle_p = \langle f\rangle_p^\sigma \langle \sigma \rangle_p$, where $\langle f\rangle_p^\sigma = \sigma(P)^{-1} \int_P f \, d\sigma$ is the $\sigma$-average of $f$ on $P$. To prove (4.2), we proceed by duality. Thus, for non-negative $g \in L^p(w)$,

$$
\langle S(f\sigma)^{p-1}, gw \rangle \leq \sum_{P \in S} \left( \langle f\rangle_p^\sigma \langle \sigma \rangle_p \right)^{p-1} \int_P g \, dw
\leq 2 \sum_{P \in S} \left( \langle f\rangle_p^\sigma \langle \sigma \rangle_p \right)^{p-1} \langle g \rangle_p^w \langle w \rangle_Q \mu(Q_E)
\leq 2 \langle \sigma, w \rangle_{A_p} \sum_{P \in S} \left( \langle f\rangle_p^\sigma \langle \sigma \rangle_p \right)^{p-1} \langle g \rangle_p^w \mu(Q_E).
$$

Notice that we have obtained the $A_p$ characteristic of $w$. The remaining sum is, by (4.3) and Hölder’s inequality again, not more than

$$
\left[ \sum_{P \in S} \left( \langle f\rangle_p^\sigma \langle \sigma \rangle_p \right)^p \sigma(E_P) \right]^{1/p'} \left[ \sum_{P \in S} \left( \langle g \rangle_p^w \langle w \rangle \right)^p w(E_P) \right]^{1/p}.
$$

Appealing to the disjointness of the sets $E_P$, both terms are bounded by a martingale maximal function, with a change in measure, namely

$$
\sum_{P \in S} \langle g \rangle_p^w \langle w \rangle(E_P) \leq \int \sup_{P \in Q} \langle g \rangle_p^w \langle w \rangle(E_P) \, dw \lesssim \|g\|_{L^p(w)}^p.
$$

This with the similar estimate for $f$ completes the proof of (4.2).

It remains to consider the case of $2 < p < \infty$, in which case the bound is linear in the $A_p$ characteristic. For $w$ an $A_p$ weight, the dual weight $\sigma \in A_{p'}$, and moreover, $[\sigma]_{A_{p'}} = [w]_{A_p}^{-1}$, as is easy to check. Thus by duality, the linear bound follows from (4.1), in the case of $1 < p < 2$.

5. Domination by Sparse Operators: Euclidean Case

We extend the martingale proof of (2.5) to the Euclidean setting, yielding an inequality that applies to (a) the most general class of Calderón-Zygmund operators, and (b) in the vector valued setting.

Let $T$ be an $L^2(\mathbb{R}^d)$ bounded operator, of norm 1, and with kernel $K(x, y)$. Namely for all $f, g \in C_0(\mathbb{R}^d)$, whose closed supports do not intersect, we have

$$
\langle Tf, g \rangle = \iint K(x, y)f(y)g(x) \, dx \, dy.
$$

Let $\omega : [0, \infty) \rightarrow [0, 1)$ be a modulus of continuity, that is a monotone increasing and subadditive function. The operator $T$ is said to be a Calderón-Zygmund operator if the kernel $K(x, y)$ satisfies

$$
|K(x, y) - K(x', y)| \leq \omega\left(\frac{|x - x'|}{|x - y|}\right) \frac{1}{|x - y|^d},
$$

where $|x| \leq |x'|$.
and the dual condition, with the roles of $x$ and $y$ reversed, holds. The classical condition to place on $\omega$ is the Dini condition

$$\int_0^1 \omega(t) \frac{dt}{t} < \infty.$$ \hfill (5.1)

The Theorem below is a stronger version of Lerner’s inequality [L4], and its extension in [CAR, Corollary A1]. Moreover, it only assumes the Dini condition, while prior approaches [HLP, L4, CAR] require $1/t$ in the Dini integral be replaced by $(\log 2/t)/t$.

**Theorem 5.2.** Let $T$ be a Calderón-Zygmund operator for which the modulus of continuity satisfies the Dini condition (5.1). Then, for any compactly supported function $f \in L^1(\mathbb{R}^d)$, we have $|Tf| \leq S|f|$, where $S = S(f, T)$ is a sum of at most $3^d$ operators, each of which are sparse relative to a choice of dyadic grid on $\mathbb{R}^d$. The same conclusion holds for the maximal truncations $T^\#_x$, as defined in (5.4).

To explain the conclusion of the Theorem, we say that $D$ is a dyadic grid of $\mathbb{R}^d$, if $D$ is a collection of cubes $Q \subset \mathbb{R}^d$ so that for each integer $k \in \mathbb{Z}$, the cubes $D_k := \{Q \in D : |Q| = 2^{-kd}\}$ partition $\mathbb{R}^d$, and these collections form an increasing filtration on $\mathbb{R}^d$. Then, a sparse operator is one that is sparse in the sense of (2.3), relative to a dyadic grid and Lebesgue measure.

To prove the Theorem, we use the following standard fact, that the maximal truncations of $T$ satisfy a weak-L$^1$ inequality, see [S, §4.2].

**Lemma 5.3.** For a Calderón-Zygmund operator $T$ with Dini modulus of continuity, we have $\lambda |\{T_x f > \lambda\}| \leq \|f\|_1$, for all $\lambda > 0$, where $T_x$ are the maximal truncations

$$T_x f(x) := \sup_{\delta > 0} \left| \int_{|x-y| > \delta} K(x, y) f(y) \, dy \right|.$$ \hfill (5.4)

There are a bounded number of choices of grids, which well-approximate any cube in $\mathbb{R}^d$. This old observation has a proof in [HLP, Lemma 2.5].

**Lemma 5.5.** There are choices of dyadic grids $D_u$, for $1 \leq u \leq 3^d$, so that for any cube $P \subset \mathbb{R}^d$, there is a choice of $1 \leq u \leq 3^d$, and dyadic $Q \in D_u$ such that $P \subset Q$, and $\ell Q \leq 6\ell P$, where $\ell P = |P|^{1/d}$ is the side length of $P$.

Abusing common terminology, we say that $Q$ is a dyadic cube if it is in any of the collections $D_u$, $1 \leq u \leq 3^d$. And, we will write $Q \in D_{u(Q)}$. The grids of the Lemma are defined by appropriate shifts of a dyadic grid.

Since the proof is recursive, we need to adapt the definitions of maximal functions and maximal truncations to a cube $P$. Set, for $x \in P$,

$$M_t f(x) := \sup_{0 < t < \text{dist}(x, \partial P)} A_t f(x), \quad A_t f(x) := \int |f(x-y)|\psi(y/t) \, dy/t^d.$$
Here and below, $\psi$ is a smooth function with $1_{(-1/2,1/2)^d} \leq \psi \leq 1_{(-1,1)^d}$. Likewise, the maximal truncations to be adapted to a particular cube $P$: For $x \in P$, define

$$T_{s,t}f(x) := \sup_{0 < s < t < \frac{1}{12} \text{dist}(x, \partial P)} |T_{s,t}f(x)|, \quad x \in P,$$

where $T_{s,t}f(x) := \int K(x,y)f(y)\left[\psi(x - y/t) - \psi((x - y)/s)\right]dy$.

The truncation levels are taken to stay inside $P$, as we measure the distance with respect to the $\ell^\infty$ norm on $\mathbb{R}^d$. For $x \notin P$, set $M Pf(x) = T_{s,t}f(x) = 0$. It follows that this definition is monotone, an important fact for us: For $P \subset Q$ (5.6) $T_{s,t}f = T_{s,t}f1_P \leq T_{s,t}f$.

Both $M P$ and $T_{s,t}$ still satisfy the weak $L^1$ inequality.

The main step in the recursion is in this next Lemma.

**Lemma 5.7.** There is a finite constant $C$ so that this holds. For all integrable functions $f$ supported on cube $P$, there there is a collection $P(P)$ of dyadic cubes $Q \subset P$, so that these conditions hold.

1. (The cubes have small measure.) There holds

$$\sum_{Q \in P(P)} |Q| < 3^{-3d-3}|P|.$$

2. (The cubes are approximately disjoint.) If $Q' \subset Q$, and $Q', Q \in P(P)$, then $Q' = Q$.

3. (Tf is controlled.) There holds

$$T_{s,t}f \leq C\langle|f|\rangle_P + \max_{Q \in P(P)} T_{s,t}Qf.$$

The last point is an extension of (3.3).

**Proof.** For a large constant $C_0$, set

$$E(C_0) := \{x \in P : \max\{M Pf(x), T_{s,t}f(x)\} > C_0\langle|f|\rangle_P\}.$$

It follows from Lemma 5.3, that $|E(C_0)| \lesssim C_0^{-1}|P|$. For $x \in E$, letting $0 < \sigma_x \leq \text{dist}(x, \partial Q_0)$ be the largest choice of $s > 0$ such that there is a $s < \tau_x \leq \frac{1}{12} \text{dist}(x, \partial Q_0)$ with

$$\max\{|T_{s,t}x}f(x)|, A_x f(x)\} = C_0\langle|f|\rangle_P.$$

We have this (standard) lower semi-continuity result: There is an absolute $0 < \rho < 1$ so that

$$\max\{|T_{s,t}x}f(x')|, A_x f(x')\} > \frac{1}{2}C_0\langle|f|\rangle_P, \quad |x - x'| < \rho \sigma_x.$$
For the averaging operator $A_{\chi}f(x)$, this is very easy. But, for the truncated singular integral, we have to rely upon the Dini condition: Abbreviating a familiar calculation, and assuming that $t \leq \omega(t)$, for $0 < t < 1$, as we can do,

$$
|T_{\sigma,\tau}f(x') - T_{\sigma,\tau}f(x)| \leq \int_{|x - y| > \sigma/2} \omega\left(\frac{\rho\sigma}{x - y}\right) \frac{|f(y)|}{|x - y|^q} \, dy
$$

$$
\leq \langle |f| \rangle \int_{\sigma/2}^{\infty} \omega\left(\frac{\rho\sigma}{t}\right) \frac{dt}{t}
$$

$$
\leq \langle |f| \rangle \int_{0}^{\rho/2} \omega(s) \frac{ds}{s}.
$$

Thus, under the Dini condition (5.1), we see that for $\rho = \rho(\psi, \omega)$ sufficiently small, (5.9) holds.

There is a second upper semi-continuity result. Set $B_\chi := x + \sigma_{[-1/2, 1/2]}^d \subset P$. For any $y \in B_\chi$, and any $\frac{1}{2}\sigma_{x} < s < t < \frac{1}{4}\text{dist}(x, \partial P)$,

$$
(5.10) \quad \left| \int K(y, z)f(z)[\psi(y - z/t) - \psi((y - z)/s)] \, dz \right| \leq C_0 \langle |f| \rangle.
$$

That is, on $B_\chi$, very large values of $T_{\tau,t}f(y)$ cannot arise from $f1_{(2B_\chi)\psi}$.

We have just seen that $\rho B_\chi \subset E(C_0/2)$. Thus

$$
|E_0| \leq \rho^{-d}C_0^{-d}|P|,
$$

$$
E_0 := \bigcup_{x \in E(C_0)} B_\chi.
$$

Now, if $y \in P \setminus E_0$, then $|T_{\tau,t}f(y)| \leq C_0 \langle |f| \rangle$.

The path from the cubes $B_\chi$ to dyadic cubes uses Lemma 5.5. To each cube $B_\chi$, associate a choice of dyadic cube $Q_\chi \supset B_\chi$ with $\ell Q_\chi \leq 6\sigma_{x}$. Since $12\sigma_{x} < \text{dist}(x, \partial P)$, it follows that $Q_\chi \subset P$. Then, we require

$$
|E_1| \leq 3^{-3d-3}|P|,
$$

$$
E_1 := \bigcup_{x \in E_0} Q_\chi,
$$

which is true for large enough $C_0$, thus specifying this constant.

Now, let $E'_1$ be the set of $x$ for which $Q_\chi$ is maximal with respect to inclusion among all such cubes, $\{Q_\gamma : y \in E_0\}$. The set $E'_1$ is clearly countable, and these associated cubes are those of the conclusion of the Lemma. Properties (1) and (2) are immediate, and it remains to check (3), namely to bound $T_{\tau,t}f(y)$. If $y \in P \setminus E_0$, then $T_{\tau,t}f(y) \leq C_0 \langle |f| \rangle$. And, otherwise, $y \in B_\chi \subset Q_\chi$, for some $x \in E'_1$. But it follows from (5.10) and monotonicity again that we have

$$
T_{\tau,t}f(y) \leq C_0 \langle |f| \rangle + T_{\tau,\rho_{B_\chi}}f(y) \leq C_0 \langle |f| \rangle + T_{\tau,\rho_{Q_\chi}}f(y).
$$

And, this completes our proof. \qed
Proof of Theorem 5.2. Take a compactly supported function \( f \in L^1(\mathbb{R}^d) \), and a dyadic cube \( P \in \mathcal{D}_{u(P)} \) large enough that \((T_f^s)1_P \leq T_{u,P}f\), and off of \( P \), we have

\[
T_f^s(x)1_{P^c}(x) \leq \frac{\|f\|_1}{[\ell P + \text{dist}(x,P)]^d}.
\]

Apply Lemma 5.5 to the cubes \( 2^n P, n \geq 1 \), to select dyadic \( P_n \) meeting the conclusion of the Lemma. Add \( P \) to \( \mathcal{S}_{u(P)} \), and \( P_n \) to \( \mathcal{S}_{u(P_n)} \), where \( \mathcal{S}_u \subset \mathcal{D}_u \) is the sparse collection of dyadic cubes that defines \( \mathcal{S}_u \). Then, it follows that

\[
T_f^s(x) \lesssim 3^d \sum_{u=1}^{3^d} S_u |f|(x), \quad x \not\in P.
\]

It remains to dominate \( T_{u,P}f \). Apply Lemma 5.7, so that in particular (5.8) holds. We will rewrite (5.8) in the form below, to set up the recursion.

\[
T_{u,P}f \leq C \sum_{u=1}^{3^d} S_u^0 |f| + \max_{Q \in P_0} T_{\sigma,Q}(f)
\]

Above, all of the sparse operators are zero, except for \( S_u^0 \), and the associated sparse collection is \( S_u^0 = \{P\} \). We will refer to the cubes in \( \mathcal{P}(P) = P_0 \) as \( z \)-descendants of \( P \).

The inductive step is to apply Lemma 5.7 to those \( Q \in P_0 \) with maximal side lengths. The details are as follows. Suppose that we have this estimate below for some integer \( t \geq 0 \).

(5.11) \( T_{u,P}f(x) \leq C \sum_{u=1}^{3^d} S_u^t |f| + \max_{Q \in P_t} T_{\sigma,Q}(f) \)

where these properties hold:

**T0:** \( S_u^t \) is an operator defined by collection \( S_u^t \subset \mathcal{D}_u \), for \( 1 \leq u \leq 3^d \), as in the equation (2.2). (The operator will be sparse, but we do not assert that at this point.)

**T1:** All cubes \( Q \in P_t \) are dyadic. If \( Q' \subset Q \) for any two \( Q, Q' \in P_t \), then \( Q' = Q \).

**T2:** If \( Q \cap S \neq \emptyset \), where \( Q \in P_t \) and \( S \in S_t := \bigcup_{u=1}^{3^d} S_u^t \), then \( \ell Q \leq \ell S \).

Then, the recursive step is as follows. Let \( P_t^* \) be those \( Q \in P_t \) such that \( \ell Q \) is maximal among all side lengths of the cubes \( \{Q : Q \in P_t\} \). Apply Lemma 5.7 to each \( Q \in P_t^* \). We have

\[
T_{\sigma,Q}(f) \leq C \langle |f| \rangle_Q + \max_{Q' \in \mathcal{P}(Q)} T_{\sigma,Q'}(f),
\]

where the cubes \( \mathcal{P}(Q) \) satisfy the conclusions of Lemma 5.7.
AN ELEMENTARY PROOF OF THE $A_2$ BOUND

It follows from (5.11) that

$$T_{\sharp} f(x) \leq C \sum_{u=1}^{3^d} S^t_u |f| + \max \left\{ \max_{Q \in \mathcal{P}_{t+1}} T_{\sharp} Q(f), \max_{Q \in \mathcal{P}_t} \left\{ C(\|f\|_Q 1_Q + \max_{Q' \in \mathcal{P}(Q)} T_{\sharp} Q'(f)) \right\} \right\}$$

$$\leq C \sum_{u=1}^{3^d} S^{t+1}_u |f| + \max_{Q \in \mathcal{P}_{t+1}} T_{\sharp} Q(f).$$

In the last line, the new operator $S^{t+1}_u$ is obtained from $S^t_u$ by adding to $S^t_u$ those cubes $\{Q \in \mathcal{P}_{t+1} : u(Q) = u\}$. By monotonicity (5.6), we take the collection $\mathcal{P}_{t+1}$ to be those that are maximal, with respect to inclusion, from the collection

$$(\mathcal{P}_t \setminus \mathcal{P}_t^*) \cup \bigcup_{P_{t+1}} \mathcal{P}(Q).$$

After the recursion finishes, it only remains to show that the limiting operators $S_u$ are sparse. Take $S \in S_u$, and fix the smallest integer $t$ such that $S \in S^t_u$. There are three sources of $S_u$-children for $S$, either directly, or through further $\sharp$-descendants:

**Type A:** Cubes in $\mathcal{P}(S')$, for some $S' \in S^t_u$, with $\ell S' > \ell S$.

**Type B:** Cubes in $\mathcal{P}(S')$, for some $S' \in S^t_u$ with $\ell S' = \ell S$.

**Type C:** Cubes $Q_k \in \mathcal{P}_t$, that are not of Type A or Type B.

Cubes of Type A cannot occur. Since $\ell S' > \ell S$, the smallest integer $t'$ such that $S' \in S^t_{u'}$ is strictly smaller than $t$. Let $Q \in \mathcal{P}(S')$ be contained in $S$. If at any time in the recursion from $t' \leq s < t$, both cubes $Q$ and $S$ were potential members of $\mathcal{P}_s$, then $Q$ would have been eliminated by monotonicity, namely using (5.6).

Concerning cubes of Type B, there are at most $2^d \cdot 3^d$ cubes $S' \in \bigcup_{v=1}^{3^d} S^s_{u'}$ of the same side length as $S$, that also intersect $S$. The total measure of all $\sharp$-descendants of each $S'$ is at most $3^{-3d-3}|S|$, so the total measure of potential children from this source is at most $3^{-d-3}|S|$.

In the case of Type C, for a cube $Q \in \mathcal{P}_t$ not of Type B, we must have $Q \cap S \neq \emptyset$, $\ell Q < \ell S$, but $Q \not\subset S$. Therefore, any $S_u$-child of $S$ must be a $\sharp$-descendent of $Q$. But, then note that the total volume of all such possible cubes is at most

$$\sum_{Q \text{ is dyadic} : \ell Q < \ell S, Q \cap S \neq \emptyset, Q \not\subset S} 3^{-3d-3}|Q| < \frac{1}{5}|S|.$$ 

Therefore, all the operators $S_u$ we have constructed are sparse, so the proof is complete. □
6. Complements

In late 1970’s, several authors considered martingale analogs of the $A_p$ theory. For instance, Izumisawa-Kazamaki [IK], proved a variant of the Muckenhoupt maximal function result [M] in this setting. When it came to martingale transforms, the distinction between the homogeneous and non-homogeneous cases was already recognized by these authors. Nevertheless, norm inequalities for martingale transforms were proved by Bonami-Lépingle [BL, Th. 1] in 1978.¹ The proof is just a good-$\lambda$ approach. For instance, it is easy to check that with our current notations, for martingale transform $T$, and

$$\mu( |Tf| > 2\lambda, Mf < \epsilon \lambda ) \leq \epsilon \mu( |Tf| > \lambda ).$$

From this, and the weighted inequality for $Mf$, and the $A_\infty$ property of $A_p$ weights, it is easy to demonstrate that $Tf$ is bounded on $L^p(w)$, for all $w \in A_p$. The familiar details are omitted, as these techniques do not yield sharp estimates.

The Lerner mean oscillation inequality [L2] gives a pointwise bound on an arbitrary measurable function $\phi$ by a sum over a sparse dyadic collection of cubes, $Q \in S$. Applying this inequality Calderón-Zygmund operators $T$ applied to $L^1$ function $f$, one needs to bound sums over $Q \in S$ of

$$1_Q \inf \{ |c| : |\{ x \in Q : |Tf(x) - c| \} | < \epsilon_d |Q| \},$$

where $\epsilon_d$ is a dimensional constant. Clearly, one can use the weak $L^1$ bound to control the term associated with $T(f1_{2Q})$, leading to a sparse operator. But, the complementary term is more subtle, involving a number of ‘tail issues’, because one knows very little about how the cubes $Q$ are selected. (See [HLP, L4] for how to address them.)

The proof just described recursively selects advantageous sparse cubes, requiring only the weak-$L^1$ inequality for maximal truncations, and basic facts about dyadic grids. Thus, the technique works with only modest changes for UMD valued functions. (In contrast, the mean oscillation inequality requires a concept of ‘median’, see [HH].) We anticipate that the proof strategy herein can be extended to the setting of homogeneous spaces [NRV, HK].

By an observation of Treil-Vol’berg [TV2], the argument in § 5 can be adapted to the non-homogeneous Calderón-Zygmund theory. In particular, a variant of Theorem 5.2 holds in this setting, and it implies this result, stated in the language of [OP, NTV].

**Theorem 6.1.** [Thiele, Treil, Volberg [TTV2]] Let $(X,d)$ be a geometrically doubling space, Let $\mu$ be a measure on $X$ which is of order $m$, that is $\mu(B(x,r)) \leq r^m$, where $B(x,r) := \{ y : d(x,y) \leq r \}$. Let $T$ be an order $m$ $L^2$-bounded Calderón-Zygmund operator on $(X, \mu)$. For any locally finite weight $w$ on $X$, positive almost everywhere,

¹ These references do not seem to be as well known as they should be. Beyond the setting of discrete martingales of this paper, some of these references consider continuous time martingales, with cadlag sample paths.
such that $\sigma := w^{-1}$ is also a weight, there holds

$$\|T : L^2(w) \mapsto L^2(w)\| \leq \sup_{x,r} \frac{w(B(x,r)) \sigma(B(x,r))}{\mu(B(x,3r)) \mu(B(x,r))}.$$ 

Further applications to the multi-linear Calderón-Zygmund theory [LMS, DLP] should also be possible. In particular, [CAR, Corollary A.1] dominates a multi-linear Calderón-Zygmund operator by a sparse operator, and from there deduces weighted inequalities. The modulus of continuity of the operator is assumed to satisfy a logarithmic Dini condition, which can can presumably be relaxed to just a Dini condition.

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