Abstract. In this paper, we establish Liouville type theorems for stable solutions on the whole space $\mathbb{R}^N$ to the fractional elliptic equation

$$(-\Delta)^su = f(u)$$

where the nonlinearity is nondecreasing and convex. We also obtain a classification of stable solutions to the fractional Lane-Emden system

$$\begin{cases}
(-\Delta)^su = v^p \text{ in } \mathbb{R}^N \\
(-\Delta)^sv = u^q \text{ in } \mathbb{R}^N
\end{cases}$$

with $p > 1$ and $q > 1$. In our knowledge, this is the first classification result for stable solutions of the fractional Lane-Emden system in literature.

1. Introduction

Let $s$ be a positive real number satisfying $0 < s < 1$. The fractional Laplacian is defined on the space of rapidly decreasing functions by

$$(-\Delta)^su(x) = c_{N,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $c_{N,s}$ is the normalization constant and $B_\epsilon(x)$ is the ball centered at $x$ with radius $\epsilon$. Notice further that, see e.g., [32], $(-\Delta)^su(x)$ is well-defined at any $x \in \mathbb{R}^N$ when $u \in C^{2\sigma}(\mathbb{R}^N) \cap \mathcal{L}_s(\mathbb{R}^N)$ for some $\sigma > s$ with

$$\mathcal{L}_s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} ; \int_{\mathbb{R}^N} \frac{|u(y)|}{(1 + |y|)^{N+2s}} dy < \infty \right\}.$$

We are, in this paper, interested in the nonexistence of nontrivial nonnegative stable solutions to the fractional elliptic equation

$$(-\Delta)^su = f(u) \text{ in } \mathbb{R}^N$$

(1.1)

with general nonlinearity and the fractional Lane-Emden system

$$\begin{cases}
(-\Delta)^su = v^p \text{ in } \mathbb{R}^N \\
(-\Delta)^sv = u^q \text{ in } \mathbb{R}^N
\end{cases}$$

(1.2)

with $p > 1$ and $q > 1$. In what follows, all solutions of (1.1) and (1.2) are considered in the space $C^{2\sigma}(\mathbb{R}^N) \cap \mathcal{L}_s(\mathbb{R}^N)$ for some $\sigma > s$.

1991 Mathematics Subject Classification. Primary: 35B53, 35J60; Secondary: 35B35.

Key words and phrases. Liouville type theorems, Stable solutions, Fractional Lane-Emden system, Fractional elliptic equation.
1.1. **Fractional elliptic equation.** The first topic in this paper is concerned with the classification of nontrivial nonnegative stable solutions to the problem (1.1). Here, a solution to (1.1) is called stable if

\[
\int_{\mathbb{R}^N} f'(u)\phi^2 \, dx \leq \frac{CN_s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} \, dy \, dx \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^N). \tag{1.3}
\]

In the local case \( s = 1 \), the nonexistence of nontrivial stable solutions of (1.1) has been much studied in the last two decades. For instance, in the typical case \( f(u) = |u|^{p-1}u \) or \( f(u) = e^u \), the classification of stable solutions to (1.1) was completely established in the pioneering articles [18, 19], see also [5, 6, 13]. In [18], among other things, it was shown that the equation

\[-\Delta u = |u|^{p-1}u \text{ in } \mathbb{R}^N \]

admits nontrivial stable solutions \( u \in C^2(\mathbb{R}^N) \) if and only if \( N \geq 11 \) and

\[ p \geq \frac{(N-2)^2 - 4N + 8\sqrt{(N-1)}}{(N-10)(N-2)}. \]

It was also proved in [19] that the equation

\[-\Delta u = e^u \text{ in } \mathbb{R}^N \]

has no nontrivial stable solution when \( N < 10 \). This result is sharp. In [42], the author obtained an optimal classification of stable weak solutions to the Hénon type elliptic equations.

In the case of general nonlinearities, the nonexistence of nontrivial bounded radial stable solutions to (1.1) was obtained when \( N \leq 10 \) and \( f \in C^1(\mathbb{R}) \), see [1, 41]. In non-radial case, the classification of bounded stable solutions to (1.1) in low dimension was established in [8, 15, 20]. In particular, Dupaigne and Farina proved that, under the assumption \( f \in C^1(\mathbb{R}) \), \( f \geq 0 \) and \( N \leq 4 \), any bounded stable solution \( u \in C^2(\mathbb{R}) \) of (1.1) with \( s = 1 \) must be constant. In higher dimensions, the Liouville type theorem for bounded stable solutions of (1.1) with \( s = 1 \) has been established in [15], see also [14]. Let us recall the assumptions on the nonlinearity used in [15]. Consider \( f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_+) \). For \( t > 0 \), define

\[ q(t) = \begin{cases} \frac{(f')^2}{ff''}(t) & \text{if } ff''(t) \neq 0, \\ +\infty & \text{if } ff''(t) = 0. \end{cases} \]

Assume that there exists the limit

\[ q_0 = \lim_{t \to 0^+} q(t) \in \mathbb{R}. \tag{1.4} \]

As shown in [15] that when \( f \) is nondecreasing, convex and \( f(0) = 0 \), then \( q_0 \geq 1 \).

The following classification in arbitrary dimension is proved in [15].

**Theorem A.** Let \( s = 1 \), \( f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_+) \) is nondecreasing, convex, \( f > 0 \) in \( \mathbb{R}^+_+ \) and (1.4) holds. Assume that \( u \in C^2(\mathbb{R}^N) \) is a bounded, nonnegative stable solution to (1.1). Then \( u \equiv 0 \) if one of the following conditions is satisfied:

i) \( N < 10 \).

ii) \( N = 10 \) and \( p_0 < +\infty \), where \( p_0 \) is the conjugate exponent of \( q_0 \), i.e.,

\[ \frac{1}{p_0} + \frac{1}{q_0} = 1 \tag{1.5} \]
iii) $N > 10$ and $p_0 < p_c(N,1)$, where $p_0$ is given in (1.5) and

$$p_c(N,1) = \frac{(N - 2)^2 - 4N + 8\sqrt{(N - 1)}}{(N - 10)(N - 2)}.$$ 

It is worth to mentioning that, the classification of stable solutions to quasilinear elliptic equation has been also investigated recently, see e.g., [3,7,29].

Let us now consider (1.1) in the nonlocal case $0 < s < 1$. A natural question in studying the equations with fractional Laplacian is that whether one can obtain similar classifications to the case of Laplace operator. The pioneering work in the classification of stable solutions to the fractional Lane-Emden equation, i.e., (1.1) with $f(u) = |u|^{p-1}u$, is due to Dávila, Dupaigne and Wei [9] where the authors exploited the monotonicity formula and some nonlinear integral estimates. This technique has been used and generalized in [22,23,36] to some fractional elliptic equations with polynomial nonlinearities and weights.

In the case of general nonlinearity, the authors of [16] obtained a fractional version of a result in [15] in low dimensional case. More precisely, under the same assumptions of $f$, i.e., $f \in C^1(\mathbb{R})$, $f \geq 0$, it was shown that (1.1) has no nontrivial bounded stable solution in dimension $N \leq 2$ if $0 < s < \frac{1}{2}$ and in dimension $N \leq 3$ if $\frac{1}{2} \leq s < 1$. However, in order to obtain a fractional version of Theorem A, the techniques in [9,16] seem not applicable. In this paper, we develop a new technique which allows one to use a non-compactly supported function as test function. From this technique and delicate nonlinear integral estimates on half space $\mathbb{R}^{N+1}_+$, we obtain a nonexistence result of nontrivial stable solutions of (1.1) given in the following theorem.

**Theorem 1.1.** Let $0 < s < 1$. Assume that $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^+$ and (1.4) holds. Then the problem (1.1) has no nontrivial bounded nonnegative stable solution if one of the following conditions is satisfied:

i) $N < 10s$.

ii) $N = 10s$ and $p_0 < +\infty$, where $p_0$ is given in (1.5).

iii) $N > 10s$ and $p_0 < p_c(N,s)$, where $p_0$ is given in (1.5) and

$$p_c(N,s) = \frac{(N - 2s)^2 - 4sN + 8s\sqrt{s(N - s)}}{(N - 10s)(N - 2s)}.$$ 

Remark that, recently, the authors of the present paper have obtained a nonexistence of stable solutions of the fractional Gelfand equation, i.e., $f(u) = e^u$ under the assumption that $N < 10s$, see [11]. Very recently, the complete classification of stable weak solutions to the fractional Gelfand equation has been proved in [28].

1.2. **Fractional Lane-Emden system.** The second topic in this paper is to study the nonexistence of positive stable solutions to the fractional Lane-Emden system (1.2). Motivated by [4,21,33], a positive solution $(u,v) \in (C^{2\sigma}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)) \times (C^{2\sigma}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N))$ of (1.2) is called stable if there are two positive functions $\zeta_1$ and $\zeta_2$ satisfying

$$\begin{cases}
(-\Delta)^s\zeta_1 = pv^{p-1}\zeta_2 \text{ in } \mathbb{R}^N \\
(-\Delta)^s\zeta_2 = qu^{q-1}\zeta_1 \text{ in } \mathbb{R}^N.
\end{cases}$$

(1.6)

In this topic, let us begin with the local case. When $s = 1$, (1.2) is known as the Lane-Emden system which has received considerably attention in recent years, see the pioneering works [30,31,38,39] and recent results [4,24–27,34,40]. Concerning
the class of classical positive solutions, the well-known conjecture states that the system (1.2) with \( s = 1 \) admits solutions if and only if
\[
p > 0, q > 0 \quad \text{and} \quad \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.
\]
This conjecture has been solved only for the radial solutions in any dimension, see \([30, 31, 38, 39]\) and for the nonradial solutions in low dimensions \( N \leq 4 \), see \([31, 39, 40]\). Some partial results in higher dimensions were also obtained in \([40]\).

Concerning the class of stable positive solutions of (1.2) with \( s = 1 \), a nonexistence result was shown in the pioneering work of Cowan \([4]\).

**Theorem B.** Let \( s = 1 \),
\[
t_+ = \sqrt{\frac{pq(q+1)}{p+1}} + \sqrt{\frac{pq(q+1)}{p+1} - \frac{pq(q+1)}{p+1}},
\]
and
\[
t_- = \sqrt{\frac{pq(q+1)}{p+1}} - \sqrt{\frac{pq(q+1)}{p+1} - \frac{pq(q+1)}{p+1}}.
\]

i) Suppose that \( 2 < q \leq p \) and
\[
N - 2 - \frac{4p+4}{pq-1} t_+ < 0.
\] Then there has no positive stable solution of (1.2). In particular, there has no positive stable solution of (1.2) for any \( 2 \leq q \leq p \) if \( N \leq 10 \).

ii) Suppose that \( 1 < q \leq 2 \), \( t_- < \frac{q}{p} \) and (1.7). Then there has no positive stable solution of (1.2).

The main idea used in \([4]\) is a combination of stability inequality, comparison principle and bootstrap argument. After that, this idea was exploited by many authors in studying various elliptic systems \([10, 12, 24–27]\). In \([25]\), the author has obtained a classification of positive stable solutions to the weighted Lane-Emden system
\[
\begin{aligned}
-\Delta u &= (1 + |x|^2)\frac{\alpha}{2} v^p \\
-\Delta v &= (1 + |x|^2)\frac{\alpha}{2} u^q
\end{aligned}
\quad \text{in } \mathbb{R}^N,
\]
where \( \alpha > 0 \) and \( \frac{4}{\alpha} < q \leq p \) or \( 1 < q \leq \min(p, \frac{4}{\alpha}) \) with additional assumption.

In \([24]\), the authors have established a Liouville type theorem for the Lane-Emden system with general weights
\[
\begin{aligned}
-\Delta u &= \rho(x) v^p \\
-\Delta v &= \rho(x) u^q
\end{aligned}
\quad \text{in } \mathbb{R}^N,
\]
where \( \rho \) is a radial function satisfying \( \rho(x) \geq C(1 + |x|^2)\frac{\alpha}{2} \) at infinity. In \([24]\), a new inverse comparison principle is introduced for bounded positive stable solutions in order to deal with the case \( 1 < p \leq \frac{4}{\alpha} \). In particular, the range of nonexistence result in \([24]\) is larger than that in \([4, 25]\). We should also mention some nonexistence results of positive stable solutions to elliptic systems involving advection terms have been also studied in \([10, 26]\) by developing the approach of Cowan.

To the best of our knowledge, there has no works in literature classifying stable solutions to fractional elliptic systems. In fact, some serious difficulties arise when one wants to classify positive stable solutions to systems involving the fractional
Laplacian. The first one is that, in order to obtain an a priori estimate, the standard test-function method does not work well with the fractional Laplacian since \((-\Delta)^s \phi\) is not, in general, compactly supported for \(\phi \in C_0^\infty(\mathbb{R}^N)\). In addition, the bootstrap argument—a key step to get better exponent in the study of Lane-Emden system—becomes a challenging problem since one has no estimates on compact sets in the nonlocal case and one needs to transform nonlinear integral estimates on half space \(\mathbb{R}^{N+1}_+\) to that on \(\mathbb{R}^N\). Besides that, one also needs to establish a comparison principle for the system (1.2).

In this paper, we obtain the following theorem.

**Theorem 1.2.** Let \(0 < s < 1\).

i) If \(p \geq q > \frac{4}{3}\) and

\[
N < 2s + \frac{4s(p + 1)}{pq - 1} t_+, \tag{1.8}
\]

then the system (1.2) has no stable positive solution.

ii) If \(1 < q \leq \min(p, \frac{4}{3})\), \(t_- < \frac{2}{q}\) and (1.8) holds, then the system (1.2) has no stable positive solution.

Notice that when \(p = q\), the condition (1.8) is equivalent to \(p < p_c(N, s)\) where \(p_c(N, s)\) is given in Theorem 1.1.

Let us now sketch the outline of the proof of Theorems 1.1 and 1.2. Concerning Theorem 1.1, we divide the proof into two cases according to the range of \(q_0\). First, when \(q_0 > \frac{N}{2}\) then the conjugate exponent \(p_0 < \frac{N}{N-2}\). In this case, we make use of some comparison and the strong maximum principle to get the desired result. When \(q_0 \leq \frac{N}{2}\), the proof becomes more involved. The difficulty arises when we need to compare integrals on half space \(\mathbb{R}^{N+1}_+\) to \(\mathbb{R}^N\). Let \(u\) be a bounded nonnegative stable solution of (1.1) and \(U\) be the extension of \(u\) in the sense of [2]. We first establish an integral estimate

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^{N+1}_+} \eta^2 dx dt \leq C \int_{\mathbb{R}^{N+1}_+} f^{2\alpha}(U)(|\nabla \eta|^2 + |L_s \eta|^2) t^{1-2s} dx dt,
\]

where \(\eta \in C_0^\infty(\mathbb{R}^{N+1}_+), \eta(x) = \bar{\eta}(x,0), q_1 < q_0, \alpha \in [1, 1 + 1/\sqrt{q_0}]\) and \(L_s\) is the second order differential operator given in (2.6). We next control the right hand side of the inequality above and choose suitably test function \(\bar{\eta}\) to get

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^{N+1}_+} \eta^{2s} (u) \rho_{N+2s} (x/R) dx \leq CR^{-2s} \int_{\mathbb{R}^N} f^{2\alpha}(u(y)) \rho_{N+2s}(y/R) dy,
\]

where \(R > 0\) and \(\rho_{N+2s}(x) = (1 + |x|^2)^{-\frac{N+2s}{2}}\). This is, in fact, the most difficult step in the proof. Finally, some computations and the H"older inequality give the desired result.

The proof of Theorem 1.2 consists of the following main steps:

- Establish a stability inequality and give an a priori estimate of solutions.
- Prove a comparison principle between \(u\) and \(v\).
- Use bootstrap argument to get better result.

One obtains the stability inequality in Lemma 3.1 by similar argument as in the local case. Nevertheless, the a priori estimate of solutions is more delicate since we must prove nonlinear integral estimate on the whole space \(\mathbb{R}^N\), see Proposition 3.2. In Proposition 3.3, we establish a comparison principle in nonlocal setting which is more or less new. As mentioned above, a serious difficulty in dealing with
fractional system is that the bootstrap argument in the local case does not work in the nonlocal case. To overcome this difficulty, we make use of the bootstrap argument for the extensions $U,V$ of $u,v$ on $\mathbb{R}_+^{N+1}$ in the sense [2] and develop the technique in [11] which allows us to reduce the estimates on half space $\mathbb{R}_+^{N+1}$ to that on $\mathbb{R}^N$.

The rest of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we prove the stability inequality and the comparison principle for the system (1.2). The proof of Theorem 1.2 is given in Section 4.

2. PROOF OF THEOREM 1.1

This section is devoted to prove Theorem 1.1. We first notice that if $u$ is a nontrivial nonnegative solution of (1.1) then $u$ is a positive solution of (1.1) by the strong maximum principle for fractional Laplacian. So, in order to prove Theorem 1.1, it is enough to show that the equation (1.1) does not possess any positive stable solution under the conditions in this theorem. In what follows, $C$ denotes a generic positive constant which may change from line to line or even in the same line.

Suppose, in contrary, that $u$ is a positive stable solution of (1.1). We shall point out a contradiction by considering two different cases of $q_0$.

**Case 1.** $q_0 > \frac{N}{2s}$.

We have $p_0 < \frac{N}{N-2s}$. By using (1.4), there exist $p < \frac{N}{N-2s}$ and $c_1 > 0$ such that, see [15, Formula 2.5],

$$f(t) \geq c_1 t^p$$

for $t$ near 0. (2.1)

Let $\varphi$ be a nonnegative function satisfying

$$\begin{cases}
(-\Delta)^s \varphi &= c_1 \varphi^p \text{ in } B_1 \\
\varphi &= 0 \text{ in } \mathbb{R}^N \setminus B_1
\end{cases},$$

where $B_R$ denotes the ball centered at the origin of radius $R$. It follows from Proposition 1.4 in [37] that $\varphi \in C^\beta(\mathbb{R}^N)$ for some $\beta \in (0,1)$. So, $\varphi$ is bounded. For $R > 0$, put $\varphi_R = R^{\frac{N-2s}{2}} \varphi(x/R)$. Hence, $\varphi_R$ satisfies

$$\begin{cases}
(-\Delta)^s \varphi_R &= c_1 \varphi_R^p \text{ in } B_R \\
\varphi_R &= 0 \text{ in } \mathbb{R}^N \setminus B_R
\end{cases}$$

and

$$R^{N-2s} \| \varphi_R \|_{L^\infty(B_R)} \leq R^{N-2s - \frac{2s}{N-2s}} \| \varphi \|_{L^\infty(B_1)} \to 0 \text{ as } R \to \infty. \quad (2.2)$$

We next prove that

$$u(x) \geq c_2 |x|^{-N+2s} \text{ for } x \in \mathbb{R}^N \setminus B_1. \quad (2.3)$$

Indeed, let $U$ be the extension of $u$ in the sense of [2], i.e., for $(x,t) \in \mathbb{R}_+^{N+1}$

$$U(x,t) = \int_{\mathbb{R}^N} P_s(x-z,t) u(z) dz, \quad (2.4)$$

where $P_s(x,t)$ is the Poisson kernel

$$P_s(x,t) = C(N,s) \frac{t^{2s}}{(|x|^2 + t^2)^{\frac{N+2s}{2}}}.$$
and \( C(N, s) \) is the normalization constant. Then, \( U \in C^2(\mathbb{R}_+^{N+1}) \cap C(...)

Consequently \( \partial B \). Here we have used the fact that \( \min \). Letting \( R \). We choose the constant \( c \). Using \[ \text{Formula (4.2)} \] with \( \alpha = \frac{2s-N}{2} \) and \( (2.5) \), we have for \( (x, t) \in \mathbb{R}_+^{N+1} \setminus B_1^+ \):

\[ L_s(x, t, \partial_{u} \partial_{v}) = \Delta_x + \frac{2s}{t} \partial_t. \]  

Using \[ \text{Formula (4.2)} \] with \( \alpha = \frac{2s-N}{2} \) and \( (2.5) \), we have for \( (x, t) \in \mathbb{R}_+^{N+1} \setminus B_1^+ \):

\[ L_s(U - c_2 \Upsilon) = L_sU = 0. \]

Put \( W(x, t) := U(x, t) - c_2 \Upsilon(x, t) \). Let \( R > 1 \), the strong maximum principle implies that, for \( (x, t) \in B_R^+ \setminus B_1^+ \):

\[ W(x, t) \geq \min_{\partial(B_R^+ \setminus B_1^+)} W = \min_{\partial B_1^+} \min_{B_1^+} W \in \{ (x, t) : 1 < |x| < R \}. \]

If there is \( x_0 \in \mathbb{R}^N \), \( 1 < |x_0| < R \) such that

\[ W(x_0, 0) = \min_{\partial(B_R^+ \setminus B_1^+)} W = \min_{B_1^+} W, \]

then we use the L’Hospital rule and then \[ \text{Formula (4.2)} \] with \( \alpha = \frac{2s-N}{2} \) to arrive at

\[ 0 \leq 2s \lim_{t \to 0^+} \frac{W(x_0, t) - W(x_0, 0)}{t^2s} = \lim_{t \to 0^+} t^{1-2s} \partial_t(U - c_2 \Upsilon)(x_0, t) = -(-\Delta)^s u(x_0) = -f(u(x_0)) < 0 \]

which is impossible. Thus, \( W \geq \min_{\partial B_1^+} W, \min_{\partial B_1^+} W \) on \( B_R^+ \setminus B_1^+ \). Fix \( (x, t) \in \mathbb{R}_+^{N+1} \setminus B_1^+ \), for any \( R > ||(x, t)|| \) we have

\[ W(x, t) \geq \min_{\partial B_1^+} W, \min_{\partial B_1^+} W. \]

Letting \( R \to +\infty \) in this inequality, we obtain

\[ W(x, t) \geq \lim_{R \to +\infty} \min_{\partial B_1^+} W, \min_{\partial B_R^+} W = \lim_{R \to +\infty} \min_{\partial B_1^+} W, \lim_{R \to +\infty} \min_{\partial B_R^+} W \geq 0. \]

Here we have used the fact that \( \min_{\partial B_1^+} W \geq 0, U \geq 0 \) and \( \lim_{R \to +\infty} \sup_{\partial B_R^+} \Upsilon = 0. \)

Consequently \( U(x, t) \geq c_2 \Upsilon(x, t) \) for \( (x, t) \in \mathbb{R}_+^{N+1} \setminus B_1^+ \) which gives \( (2.3) \).

It results from \((2.2)\) and \((2.3)\) that there exists \( R > 0 \) so that

\[ u(x) \geq \varphi_R(x). \]

Furthermore, from \((2.1)\) we get \[ f(\varphi_R) \geq c_1 \varphi_R^p. \]
Hence,
\[ (-\Delta)^s(u - \varphi_R) \geq f(u) - f(\varphi_R) \geq 0. \]
This combined with the strong maximum principle gives
\[ u(x) > \varphi_R(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{2.7} \]
Given any unit directional vector \( e \), define \( \varphi_{R,t}(x) = \varphi_R(x + te) \). We claim that
\[ u \geq \varphi_{R,t} \quad \text{for all } t \geq 0. \]
Put
\[ T = \sup\{t \in [0, +\infty); u > \varphi_{R,t} \text{ for all } l \in [0, t]\}. \]
Since \( u > \varphi_R = \varphi_{R,0} \), we deduce that \( T > 0 \). Assume that \( T < \infty \) then \( u \geq \varphi_{R,T} \) and there is \( x_0 > 0 \) such that \( u(x_0) = \varphi_{R,T}(x_0) \). However, we still have
\[ (-\Delta)^s(u - \varphi_{R,T}) \geq f(u) - f(\varphi_{R,T}) \geq 0. \]
Thus, the strong maximum principle implies that \( u > \varphi_{R,T} \) which is a contradiction.

**Case 2.** \( q_0 \leq \frac{N}{2}. \)

Let \( U \) be the extension of \( u \) in the sense of (2.5). Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a convex function and
\[ \psi(u) = \int_0^u (\phi')^2 dt, \quad K(u) = \int_0^u \psi(t) dt. \]
Take a radial test function \( \bar{\eta} \in C^\infty_c(\mathbb{R}^{N+1}) \), supp\( \bar{\eta} \subset \{(x, t) \in \mathbb{R}^{N+1}; |x|^2 + |t|^2 \leq 4\} \) and put \( \eta(x) = \bar{\eta}(x, 0) \). Using the weak form of the first equation in (2.5) with the test function \( \psi(U)\bar{\eta}^2 \), we get
\[
\kappa_s \int_{\mathbb{R}^N} f(u)\psi(u)\eta^2 dx = \int_{\mathbb{R}^{N+1}} \nabla U \cdot \nabla \left( \psi(U)\eta^2 \right) t^{1-2s} dx dt
\]
\[
= \int_{\mathbb{R}^{N+1}} \phi'(U)|\nabla U|^2 \eta^2 t^{1-2s} dx dt + \int_{\mathbb{R}^{N+1}} \psi(U)|\nabla U| \cdot \nabla \eta^2 t^{1-2s} dx dt
\]
\[
= \int_{\mathbb{R}^{N+1}} (\phi'(U))^2 |\nabla U|^2 \eta^2 t^{1-2s} dx dt - \int_{\mathbb{R}^{N+1}} K(U)L_s \eta^2 t^{1-2s} dx dt,
\]
where \( L_s \) is given in (2.6) and in the last equality, we have used an integration by parts and the fact that
\[ \lim_{\ell \to 0} t^{1-2s} \partial_t \bar{\eta} = 0. \]
As a consequence,
\[ \int_{\mathbb{R}^{N+1}} (\phi'(U))^2 |\nabla U|^2 \eta^2 t^{1-2s} dx dt \leq \int_{\mathbb{R}^{N+1}} K(U)|L_s \eta^2 t^{1-2s} dx dt \]
\[ + \kappa_s \int_{\mathbb{R}^N} f(u)\psi(u)\eta^2 dx. \tag{2.8} \]
We now exploit the stability inequality (1.3) with the test function \( \phi(u)\eta; \)
\[ \kappa_s \int_{\mathbb{R}^N} f(u)\phi^2(u) \eta^2 dx \leq \kappa_s \|\phi(u)\eta\|^2_{H^s(\mathbb{R}^N)} \leq \int_{\mathbb{R}^{N+1}} |(\nabla(\phi(U))\eta)|^2 t^{1-2s} dx dt. \tag{2.9} \]
In addition,
\[
\int_{\mathbb{R}^{N+1}} |(\nabla (\phi(U)\bar{\eta}))|^2 t^{1-2s} dx dt = \int_{\mathbb{R}^{N+1}} (\phi'(U))^2 |\nabla U|^2 \eta^2 t^{1-2s} dx dt
\]
\[
+ \int_{\mathbb{R}^{N+1}} (\phi(U))^2 |\nabla \eta|^2 t^{1-2s} dx dt - \frac{1}{2} \int_{\mathbb{R}^{N+1}} (\phi(U))^2 L_s \eta^2 t^{1-2s} dx dt.
\] (2.10)

Combining (2.8), (2.9) and (2.10), we obtain
\[
\kappa_s \int_{\mathbb{R}^N} (f'(u)\phi^2(u) - f(u)\psi(u))\eta^2 dx \leq \int_{\mathbb{R}^{N+1}} K(U)|L_s \eta^2| |t|^{1-2s} dx dt
\]
\[
+ \int_{\mathbb{R}^{N+1}} (\phi(U))^2 |\nabla \eta|^2 t^{1-2s} dx dt - \frac{1}{2} \int_{\mathbb{R}^{N+1}} (\phi(U))^2 L_s \eta^2 t^{1-2s} dx dt.
\] (2.11)

It follows from the convexity of \(\phi\) that \(\psi(u) \leq \phi'(u)\phi(u)\) and then \(K(u) \leq \frac{1}{2} \phi^2(u)\). Thus, (2.11) gives
\[
\kappa_s \int_{\mathbb{R}^N} (f'(u)\phi^2(u) - f(u)\psi(u))\eta^2 dx \leq C \int_{\mathbb{R}^{N+1}} (\phi(U))^2 (|\nabla \eta|^2 + |L_s \eta^2|) t^{1-2s} dx dt.
\] (2.12)

For any \(\alpha \in [1, 1 + \frac{1}{\sqrt{q}}]\), choose \(\phi = f^\alpha\). Then, using [15, Formulas (36), (38)], one has
\[
f'(u)\phi^2(u) - f(u)\psi(u) \geq Cf'(u)\phi^2(u) \geq Cf^\frac{1}{q_1+2\alpha}(u)\]
for some \(q_1 < q_0\) fixed.

This estimate and (2.12) follow that
\[
\int_{\mathbb{R}^N} f_{\frac{1}{q_1+2\alpha}}(u)\eta^2 dx \leq C \int_{\mathbb{R}^{N+1}} f^{2\alpha}(U)(|\nabla \eta|^2 + |L_s \eta^2|) t^{1-2s} dx dt.
\] (2.13)

Let \(\tilde{\phi} \in C^\infty_c(\mathbb{R})\) such that \(\tilde{\phi}(t) = 1\) when \(|t| \leq 1\) and \(\tilde{\phi}(t) = 0\) when \(|t| \geq 2\). Let \(R\) and \(\tilde{R}\) be positive numbers. Put
\[
\Phi_{\tilde{R}}(x,t) = \tilde{\phi} \left( \frac{|x|+t}{\tilde{R}} \right), \quad \zeta(x,t) = (1 + |x|^2 + t^2)^{-\frac{N+2s}{2}}
\]
and
\[
\bar{\eta}_{\tilde{R}}(x,t) = \Phi_{\tilde{R}}(x,t)\zeta(x,t)
\]
for \((x,t) \in \mathbb{R}^{N+1}_+\). Then
\[
\eta_{\tilde{R}}(x) := \bar{\eta}_{\tilde{R}}(x,0) = \tilde{\phi} \left( \frac{|x|}{\tilde{R}} \right) \rho_{N+2s}^\frac{4}{q_1+2\alpha} (x),
\]
where \(\rho_{N+2s}(x) = (1 + |x|^2)^{-\frac{N+2s}{2}}\).

Replacing \(\bar{\eta}(x)\) and \(\eta(x)\) by \(\bar{\eta}_{\tilde{R}} \left( \frac{x}{\tilde{R}} \right)\) and \(\eta_{\tilde{R}} \left( \frac{x}{\tilde{R}} \right)\) in (2.13), one obtains
\[
\int_{\mathbb{R}^N} f_{\frac{1}{q_1+2\alpha}}(u)\eta^2_{\tilde{R}} \left( \frac{x}{\tilde{R}} \right) dx
\]
\[
\leq CR^{-2} \int_{\mathbb{R}^{N+1}} f^{2\alpha}(U) \left( \left| \nabla \bar{\eta}_{\tilde{R}} \left( \frac{x}{\tilde{R}} \right) \right|^2 + \left| L_s \bar{\eta}_{\tilde{R}}^2 \left( \frac{x}{\tilde{R}} \right) \right| \right) t^{1-2s} dx dt.
\] (2.14)
On the other hand, it follows from the Jensen inequality that
\[
f^{2\alpha}(U(x, t)) \leq \int_{\mathbb{R}^N} P_s(x - y, t)f^{2\alpha}(u(y)) dy.
\]
Here \(P_s\) is the Poisson kernel given above. Therefore,
\[
I := R^{-2}\int_{\mathbb{R}^{N+1}} f^{2\alpha}(U) \left( \left| \nabla \eta R \right| \left( \frac{x}{R} - \frac{t}{R} \right)^2 + \left| L_s \eta R^2 \right| \left( \frac{x}{R} - \frac{t}{R} \right) \right) t^{1-2s} dx dt
\]
\[
= R^{N-2s}\int_{\mathbb{R}^{N+1}} f^{2\alpha}(U(Rx, Rt)) \left( \left| \nabla \eta R \right|^2 + \left| L_s \eta R^2 \right| \right) t^{1-2s} dx dt
\]
\[
\leq R^{N-2s}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} P_s(Rx - y, Rt)f^{2\alpha}(u(y)) dy \right) \left( \left| \nabla \eta R \right|^2 + \left| L_s \eta R^2 \right| \right) t^{1-2s} dx dt
\]
\[
= R^{-2s}\int_{\mathbb{R}^N} f^{2\alpha}(u(y)) \left( \int_{\mathbb{R}^N} P_s(x - y, t) \left( \left| \nabla \eta R \right|^2 + \left| L_s \eta R^2 \right| \right) t^{1-2s} dx dt \right) dy,
\]
where in the last equality we have used the Fubini theorem and the homogeneity of \(P_s\). Denote by
\[
h_{\tilde{R}}(y) := \int_0^\infty \int_{\mathbb{R}^N} P_s(x - y, t) \left( \left| \nabla \eta R \right|^2 + \left| L_s \eta R^2 \right| \right) t^{1-2s} dx dt
\]
\[
= C(N, s) \int_0^\infty \int_{\mathbb{R}^N} \frac{t\left| \nabla \eta R \right|^2 \left| L_s \eta R^2 \right|}{\left( |x-y|^2 + t^2 \right)^{N+2s}} dx dt.
\]
We estimate the first integral
\[
I_1 := \int_0^\infty \int_{\mathbb{R}^N} \frac{t\left| \nabla \eta R \right|^2}{\left( |x-y|^2 + t^2 \right)^{N+2s}} dx dt
\]
\[
\leq \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{t\left| \nabla \Phi \right|^2 \left| \zeta \right|^2}{\left( |x-y|^2 + t^2 \right)^{N+2s}} dx dt + \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \frac{t\left| \nabla \zeta \right|^2 \left| \Phi \right|^2}{\left( |x-y|^2 + t^2 \right)^{N+2s}} dx dt
\]
\[
=: \frac{1}{2} J_1(y) + \frac{1}{2} J_2(y) \leq C(1 + \tilde{R}^{-2s})\rho_{N+2s}(y),
\]
where \(C\) is independent of \(\tilde{R}\). Here, in the last inequality, we have used the following estimates from \cite{11}, Proof of Step 2 of Lemma 2.4
\[
J_1(y) \leq C\rho_{N+2s}(y) \text{ and } J_2(y) \leq C\tilde{R}^{-2s}\rho_{N+2s}(y).
\]
We now consider
\[
I_2 = \int_0^\infty \int_{\mathbb{R}^N} \frac{t\left| L_s \eta R^2 \right|}{\left( |x-y|^2 + t^2 \right)^{N+2s}} dx dt.
\]
A straightforward computation gives
\[
L_s \eta R^2 = 2\left| \nabla \eta R \right|^2 + 2\eta R L_s \eta R
\]
\[
= 2\left| \nabla \eta R \right|^2 + 2\left| \nabla \eta R \right| \left( \Phi \left| \Phi \right| \zeta + \Phi \zeta \right) + 2\nabla \Phi \cdot \nabla \zeta
\]
\[
= 2\left| \nabla \eta R \right|^2 + 2\left| \nabla \Phi \right| \left| L_s \Phi \right| \left| \zeta \right|^2 + 2\left| \nabla \Phi \right| \left| L_s \zeta \right| + 4\left| \Phi \right| \nabla \Phi \cdot \nabla \zeta
\]
Then, applying the Young inequality, we obtain
\[
\left| L_s \eta R^2 \right| \leq 6\left( \left| \nabla \Phi \right|^2 \left| \nabla \zeta \right|^2 + \left| \nabla \Phi \right| \left| \zeta \right|^2 \right) + 2\left| \Phi \right| \left| L_s \Phi \right| \left| \zeta \right|^2 + \Phi \left| \zeta \right|^2 \left| \left| L_s \zeta \right| \right|.
\]
It is now enough to decompose \( I_2 \) into four terms and then observe that the first and the last term are bounded by \( CJ_2 \), the second and the third term are bounded by \( CJ_1 \). Hence,

\[
I_2 \leq C(1 + \tilde{R}^{-2s})\rho_N(y), \tag{2.18}
\]

where \( C \) is independent of \( \tilde{R} \). Consequently, substituting (2.17) and (2.18) into (2.16) we arrive at

\[
h\tilde{R}(y) \leq C(1 + \tilde{R}^{-2s})\rho_N \tag{2.19}
\]

where \( C \) does not depend on \( \tilde{R} \). This together with (2.15) and (2.14) implies that

\[
\int_{\mathbb{R}^N} f^{\frac{1}{q_1}}(u)\eta^2_R(x/R) \, dx \leq CR^{-2s} \int_{\mathbb{R}^N} f^{2\alpha}(u(y))h\tilde{R}(y/R) \, dy \\
\leq CR^{-2s}(1 + \tilde{R}^{-2s}) \int_{\mathbb{R}^N} f^{2\alpha}(u(y))\rho_{N,2s}(y/R) \, dy.
\]

Letting \( \tilde{R} \to \infty \) in this estimate, we deduce that

\[
\int_{\mathbb{R}^N} f^{\frac{1}{q_1}}(u)\rho_{N,2s}(x/R) \, dx \leq CR^{-2s} \int_{\mathbb{R}^N} f^{2\alpha}(u(y))\rho_{N,2s}(y/R) \, dy. \tag{2.19}
\]

Recall that that \( q_1 < q_0 \) and \( 1 \leq \alpha < 1 + \frac{1}{\sqrt{q_0}} \). Applying the Hölder inequality in (2.19), we get

\[
\int_{\mathbb{R}^N} f^{\frac{1}{q_1}}(u)\rho_{N,2s}(x/R) \, dx \\
\leq CR^{-2s} \left( \int_{\mathbb{R}^N} f^{\frac{1}{q_1}+2\alpha}(u)\rho_{N,2s}(x/R) \, dx \right)^{\frac{2\alpha}{\frac{1}{q_1} + 2\alpha}} R^N \frac{1}{\frac{1}{q_1} + 2\alpha},
\]

or

\[
\int_{\mathbb{R}^N} f^{\frac{1}{q_1}+2\alpha}(u)\rho_{N,2s}(x/R) \, dx \leq CR^{N-2s-4\alpha q_1}. \tag{2.20}
\]

Under one of the conditions in Theorem 1.1, we shall show that the exponent in the right hand side of (2.20) is negative by choosing \( q_1 \) close to \( q_0 \) and \( \alpha \) close to \( 1 + \frac{1}{q_0} \). Indeed, it is enough to claim that

\[
N - 2s - 4s(q_0 + \sqrt{q_0}) < 0. \tag{2.21}
\]

We now consider \( N < 10s \). Then (2.21) is true since \( q_0 \geq 1 \). In the case \( N = 10s \) and \( p_0 < \infty \), (2.21) still holds since \( q_0 > 1 \). Finally, when \( N > 10s \), the condition \( p_0 < p_c(N,s) \) ensures that \( q_0 + \sqrt{q_0} - \frac{N-2s}{4s} > 0 \), i.e., (2.21) is also true.

Let \( R \to +\infty \) in (2.20), we obtain a contradiction. \( \square \)

3. Some technical lemmas for fractional Lane-Emden system

In this section, we prove the stability inequality for stable positive solutions and the comparison principle for positive solutions. The former is given in the following.

**Lemma 3.1.** Let \((u, v)\) be a stable positive solution of (1.2). Then for all \( \phi \in C_c^\infty(\mathbb{R}^N) \), we have

\[
\sqrt{pq} \int_{\mathbb{R}^N} \frac{u^{q-1}v^{p-1}}{p+q} \phi^2 \, dx \leq \frac{CN,s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{N+2s}} \, dy \, dx.
\]
Proof. The proof is based on the idea of Cowan [4]. Let \( \phi \in C_c^\infty(\mathbb{R}^N) \) be a test function. Multiplying the first equation in (1.2) by \( \frac{\phi^2}{\xi_1} \), one has

\[
\int_{\mathbb{R}^N} p v \phi^{p-1} \frac{\phi^2}{\xi_1} dx = \int_{\mathbb{R}^N} (-\Delta)^s \frac{\phi^2}{\xi_1} dx.
\]

Using an integration by parts to the right hand side of this equality and a simple inequality \(-a^2 - b^2 \leq -2ab\), one gets

\[
\int_{\mathbb{R}^N} p v \phi^{p-1} \frac{\phi^2}{\xi_1} dx = \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^s \frac{\phi^2}{\xi_1} + \frac{\phi^2}{\xi_1} (-\Delta)^s \left( \frac{\phi^2}{\xi_1} \right) dx
\]

\[
= \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\phi^2(x) - \phi^2(y) - \phi(y) \phi(x) + \phi(y) \phi(x)}{|x-y|^{N+2s}} dydx.
\]

\[
\leq c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\phi^2(x) - \phi(y) \phi(x)}{|x-y|^{N+2s}} dydx.
\]

By exchanging the role of \( x \) and \( y \), we also have

\[
\int_{\mathbb{R}^N} p v \phi^{p-1} \frac{\phi^2}{\xi_1} dx \leq c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\phi^2(y) - \phi(y) \phi(x)}{|x-y|^{N+2s}} dydx.
\]

As a consequence of the two inequalities above, there holds

\[
\int_{\mathbb{R}^N} p v \phi^{p-1} \frac{\phi^2}{\xi_1} dx \leq \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} dydx. \tag{3.1}
\]

Similarly, we also deduce from the second equation of (1.2) with the test function \( \frac{\phi^2}{\xi_2} \) that

\[
\int_{\mathbb{R}^N} q u^{q-1} \frac{\phi^2}{\xi_2} dx \leq \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} dydx. \tag{3.2}
\]

The inequalities (3.1) and (3.2) yield

\[
\int_{\mathbb{R}^N} \left( p v \phi^{p-1} \frac{\phi^2}{\xi_1} + q u^{q-1} \frac{\phi^2}{\xi_2} \right) dx \leq c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+2s}} dydx. \tag{3.3}
\]

Applying again the simple inequality \( a^2 + b^2 \geq 2ab \) to the left hand side of (3.3), we end the proof of Lemma. \( \square \)

From now on, we use the notation \( \rho_m(x) = (1 + |x|^2)^{-\frac{m}{2}} \) for some \( m > N \). We next establish an \textit{a priori} estimate for positive solutions of (1.2).

\textbf{Proposition 3.2.} Suppose that \( p, q > 1 \) and \( (u, v) \) is a positive solution of (1.2). Let \( R \) be a positive constant, then it holds

\[
\int_{\mathbb{R}^N} v^p \rho_{N+2s}(x/R) dx \leq CR^{N-\frac{2p(p+1)}{p-1}} \tag{3.4}
\]

\[
\int_{\mathbb{R}^N} u^q \rho_{N+2s}(x/R) dx \leq CR^{N-\frac{2q(q+1)}{q-1}}
\]

for some constant \( C \) depending only on \( N, s, p \) and \( q \).

Remark that on \( B_R \) we have \( \rho_{N+2s}(x/R) \sim C \) with some constant \( C \) depending only on \( N \) and \( s \). Hence, this proposition provides not only the same estimates on \( B_R \) which were obtained by Yang and Zou [43] but also the estimates outside the ball \( B_R \).
Proof. Let \( \varphi \in C_c^\infty(\mathbb{R}^N) \) be a cut-off function satisfying \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) on \( B_1 \) and \( \varphi = 0 \) outside \( B_2 \). For \( k > 0 \), define \( \varphi_k(x) = \varphi(x/k) \). Testing the system (1.2) by \( \varphi_k \rho_{N+2s} \in C_c^\infty(\mathbb{R}^N) \) we get

\[
\int_{\mathbb{R}^N} v^p \rho_{N+2s} \varphi_k \, dx = \int_{\mathbb{R}^N} u(-\Delta)^s(\varphi_k \rho_{N+2s}) \, dx
\]

and

\[
\int_{\mathbb{R}^N} u^q \rho_{N+2s} \varphi_k \, dx = \int_{\mathbb{R}^N} v(-\Delta)^s(\varphi_k \rho_{N+2s}) \, dx.
\]

Letting \( k \to \infty \) and using [11, Lemma 2.2] and the Lebesgue dominated convergence theorem, we obtain

\[
\int_{\mathbb{R}^N} v^p \rho_{N+2s} \, dx = \int_{\mathbb{R}^N} u(-\Delta)^s(\rho_{N+2s}) \, dx
\]

and

\[
\int_{\mathbb{R}^N} u^q \rho_{N+2s} \, dx = \int_{\mathbb{R}^N} v(-\Delta)^s(\rho_{N+2s}) \, dx.
\]

It follows from this and [11, Lemma 2.1] that

\[
\int_{\mathbb{R}^N} v^p \rho_{N+2s} \, dx \leq C \int_{\mathbb{R}^N} u \rho_{N+2s} \, dx \tag{3.5}
\]

and

\[
\int_{\mathbb{R}^N} u^q \rho_{N+2s} \, dx \leq C \int_{\mathbb{R}^N} v \rho_{N+2s} \, dx, \tag{3.6}
\]

with \( C \) depending only on \( N \) and \( s \). Notice that \( \rho_{N+2s} = \rho_s \rho_{N+s} \) and \( \rho_{N+s} \in L^1(\mathbb{R}^N) \). Hence by the Hölder inequality, we have

\[
\int_{\mathbb{R}^N} u \rho_{N+2s} \, dx \leq \left( \int_{\mathbb{R}^N} u^p \rho_{N+2s} \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} \rho_{N+2s} \, dx \right)^{\frac{q-1}{q}}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} u^q \rho_{N+(q+1)s} \, dx \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} u^q \rho_{N+2s} \, dx \right)^{\frac{1}{q}}, \tag{3.7}
\]

with \( C \) depending only on \( N, s \) and \( q \). Similarly, we also get

\[
\int_{\mathbb{R}^N} u \rho_{N+2s} \, dx \leq C \left( \int_{\mathbb{R}^N} v^p \rho_{N+2s} \, dx \right)^{\frac{1}{p}} \tag{3.8}
\]

with \( C \) depending only on \( N, s \) and \( p \). From (3.5), (3.6), (3.7) and (3.8) and \( pq > 1 \), we obtain

\[
\int_{\mathbb{R}^N} u^q \rho_{N+2s} \, dx \leq C \text{ and } \int_{\mathbb{R}^N} v^p \rho_{N+2s} \, dx \leq C, \tag{3.9}
\]

for some constant \( C \) depending only on \( N, s, p \) and \( q \). Remark that the estimate (3.9) holds for any positive solution \((u, v)\) of (1.2).

We now use a scaling argument. For any \( R > 0 \), then the functions

\[
u_R(x) = R^{2(\frac{q}{s} - 1)} u(Rx) \text{ and } v_R(x) = R^{\frac{pq-1}{s}} v(Rx),
\]

also form a positive solution of (1.2). By (3.9), we get

\[
\int_{\mathbb{R}^N} u^q_R \rho_{N+2s} \, dx \leq C \text{ and } \int_{\mathbb{R}^N} v^p_R \rho_{N+2s} \, dx \leq C.
\]
Making the change of variables, we obtain (3.4).

The comparison principle is given in the following.

**Proposition 3.3.** Let \( p \geq q > 1 \) and \((u,v)\) be a positive solution to (1.2). Then, there holds, point-wise in \( \mathbb{R}^N \),

\[
\frac{v^{p+1}}{p+1} \leq \frac{u^{q+1}}{q+1}
\]

**Proof.** For the simplicity of notations, put \( \tilde{\sigma} = \frac{q+1}{p+1} \leq 1 \) and \( l = \tilde{\sigma}^{-\frac{1}{p-1}} > 1 \). Then, we need to prove that

\[
w := v - lw \tilde{\sigma} \leq 0.
\]  

We first point out that

\[
(-\Delta)^s u \tilde{\sigma} \geq \tilde{\sigma} u^{\tilde{\sigma}-1} (-\Delta)^s u.
\]  

Indeed, we have

\[
(-\Delta)^s u \tilde{\sigma}(x) = \int_{\mathbb{R}^N} \frac{u \tilde{\sigma}(x) - u \tilde{\sigma}(y)}{|x - y|^{N+2s}} dy. \tag{3.12}
\]

In addition, \( f(t) = t^{\tilde{\sigma}}, t > 0 \), is concave. Then, 

\[
f(u(y)) \leq f(u(x)) + f'(u(x))(u(y) - u(x)),
\]

or

\[
u \tilde{\sigma}(y) - u \tilde{\sigma}(x) \leq \tilde{\sigma} u^{\tilde{\sigma}-1}(x)(u(y) - u(x)).
\]

Substituting this into (3.12), we obtain (3.11).

As a consequence of (3.11), there holds

\[
(-\Delta)^s w = (-\Delta)^s v - (-\Delta)^s u \tilde{\sigma} \leq u^q - lw \tilde{\sigma} u^{\tilde{\sigma}-1} v^p. \tag{3.13}
\]

Moreover \( u^q - lw \tilde{\sigma} u^{\tilde{\sigma}-1} v^p = lp \tilde{\sigma}^{-1}((lu \tilde{\sigma})^p - v^p) \leq 0 \) on the set \( \{x; w(x) \geq 0\} \). It then results from (3.13) that

\[
(-\Delta)^s w \leq 0 \text{ on the set } \{x; w(x) \geq 0\}.
\]

Let \( W \) be the extension of \( w \) in the sense of (2.4)-(2.5). By following the argument in [43, Lemma 3.1] which is inspired by the idea in [35], we also get \( W \leq 0 \). In particular, the restriction \( w \) of \( W \) is nonpositive which implies (3.10). This completes the proof of Proposition. \( \square \)

## 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by way of contradiction. Assume that \((u,v)\) is a positive stable solution of (1.2). Denote by \( U \) and \( V \) the extensions of \( u \) and \( v \) in the sense of (2.4)-(2.5), i.e., \( U, V \in C^2(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+}), t^{1-2s} \partial_t U, t^{1-2s} \partial_t V \in C(\overline{\mathbb{R}^{N+1}_+}) \) and

\[
\begin{cases}
-\text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
U = u & \text{on } \partial \mathbb{R}^{N+1}_+ \\
-\lim_{t \to 0} t^{1-2s} \partial_t U = \kappa_s (-\Delta)^s u & \text{on } \partial \mathbb{R}^{N+1}_+.
\end{cases} \tag{4.1}
\]
and
\[
\begin{cases}
-\text{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
V = v & \text{on } \partial\mathbb{R}^{N+1}_+ \\
-\lim_{t \to 0} t^{1-2s}\partial_t V = \kappa_s (-\Delta)^s v & \text{on } \partial\mathbb{R}^{N+1}_+.
\end{cases}
\]

In order to use the bootstrap argument, we need the following lemma.

**Lemma 4.1.** For any \( t_- < \gamma < t_+ \) and \( \Phi \in C^\infty_c(\mathbb{R}^{N+1}_+) \), there holds
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} dx dt \leq C \int_{\mathbb{R}^{N+1}_+} U^{2\gamma} |\nabla \Phi|^2 t^{1-2s} dx dt.
\]

(4.3)

Here \( t_+ \) and \( t_- \) are given in Theorem B.

**Proof.** Let \( \Phi \in C^\infty_c(\mathbb{R}^{N+1}_+) \) be a test function and \( t_- < \gamma < t_+ \). Define \( \phi(x) = \Phi(x, 0) \in C^\infty_c(\mathbb{R}^N) \). Multiplying the first equation in (4.1) by \( U^{2\gamma - 1} \Phi^2 \) and then integrating by parts, we have
\[
\kappa_s \int_{\mathbb{R}^N} v^p u^{2\gamma - 1} \phi^2 dx = \int_{\mathbb{R}^{N+1}_+} \nabla U \cdot \nabla (U^{2\gamma - 1} \Phi^2) t^{1-2s} dx dt
\]
\[
= (2\gamma - 1) \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 U^{2\gamma - 2} \Phi^2 t^{1-2s} dx dt + \frac{2}{\gamma} \int_{\mathbb{R}^{N+1}_+} \Phi \nabla (U^\gamma) \cdot \nabla \Phi U^{t-2s} dx dt
\]
\[
= \frac{2\gamma - 1}{\gamma^2} \int_{\mathbb{R}^{N+1}_+} |\nabla U^\gamma|^2 \Phi^2 t^{1-2s} dx dt + \frac{2}{\gamma} \int_{\mathbb{R}^{N+1}_+} \Phi \nabla (U^\gamma) \cdot \nabla \Phi U^{t-2s} dx dt.
\]

(4.4)

A straightforward computation leads to
\[
\int_{\mathbb{R}^{N+1}_+} |\nabla U^\gamma|^2 \Phi^2 t^{1-2s} dx dt = \int_{\mathbb{R}^{N+1}_+} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} dx dt
\]
\[
- 2 \int_{\mathbb{R}^{N+1}_+} \Phi \nabla (U^\gamma) \cdot \nabla \Phi U^{t-2s} dx dt - \int_{\mathbb{R}^{N+1}_+} U^{2\gamma} |\nabla \Phi|^2 t^{1-2s} dx dt.
\]

Plugging this into (4.4), one has
\[
\kappa_s \int_{\mathbb{R}^N} v^p u^{2\gamma - 1} \phi^2 dx = \frac{2\gamma - 1}{\gamma^2} \int_{\mathbb{R}^{N+1}_+} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} dx dt
\]
\[
- \frac{2(2\gamma - 1)}{\gamma^2} \int_{\mathbb{R}^{N+1}_+} \Phi \nabla (U^\gamma) \cdot \nabla \Phi U^{t-2s} dx dt - \frac{1}{\gamma^2} \int_{\mathbb{R}^{N+1}_+} U^{2\gamma} |\nabla \Phi|^2 t^{1-2s} dx dt
\]
\[
\geq \left( \frac{2\gamma - 1}{\gamma^2} - \frac{2(2\gamma - 1)\epsilon}{\gamma^2} \right) \int_{\mathbb{R}^{N+1}_+} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} dx dt
\]
\[
- \left( \frac{1}{\gamma^2} + \frac{2\gamma - 1}{2\gamma^2 \epsilon} \right) \int_{\mathbb{R}^{N+1}_+} U^{2\gamma} |\nabla \Phi|^2 t^{1-2s} dx dt.
\]

(4.5)

where in the last estimate, we have used \( ab \leq a^2 \epsilon + b^2 \) for any \( \epsilon > 0 \).
Choosing the test function $u^\gamma \phi$ in the stability inequality and then using the comparison principle, one gets

$$
\kappa_s \sqrt{pq} \left( \frac{p+1}{p} \right) \int_{\mathbb{R}^N} u^p u^{2\gamma-1} \phi^2 \, dx 
\leq \kappa_s \sqrt{pq} \int_{\mathbb{R}^N} v^{\frac{q+1}{q}} u^{\frac{q+1}{2}} u^{2\gamma} \phi^2 \, dx 
\leq \kappa_s \|u^\gamma \phi\|_{H^s(\mathbb{R}^N)} 
\leq \int_{\mathbb{R}^N} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} \, dx \, dt,
$$

where in the last inequality, we have used the fact that $U^\gamma \Phi$ has the trace $u^\gamma \phi$ on $\partial \mathbb{R}^{N+1}_+$. It results from (4.5) and (4.6) that

$$
\left( \frac{2\gamma - 1}{\gamma^2} - \frac{2(2\gamma - 1)\epsilon}{\gamma^2} - \sqrt{pq} \left( \frac{p+1}{p} \right) \right) \int_{\mathbb{R}^{N+1}_+} |\nabla (U^\gamma \Phi)|^2 t^{1-2s} \, dx \, dt 
\leq C \int_{\mathbb{R}^{N+1}_+} U^{2\gamma} |\nabla \Phi|^2 t^{1-2s} \, dx \, dt.
$$

Since $t_- < \gamma < t_+$ and $\epsilon$ is chosen small enough, we have

$$
\frac{2\gamma - 1}{\gamma^2} - \frac{2(2\gamma - 1)\epsilon}{\gamma^2} - \sqrt{pq} \left( \frac{p+1}{p} \right) > 0.
$$

Consequently, we get (4.3).

□

End of the proof of Theorem 1.2. Denote by

$$
k_s = \frac{N+2-2s}{N-2s}.
$$

Under the assumption on the exponent $q$, we fix a real positive number $\tau$ satisfying

$$
2t_- \leq 2\tau < q
$$

and let $m$ be a non-negative integer satisfying

$$
\tau k^m < t_+ \leq \tau k^m.
$$

We construct an increasing geometric sequence

$$
t_- < t_1 < t_2 < \ldots < t_m < t_+
$$

as follows

$$
2t_1 = 2\tau k, 2t_2 = 2\tau kk_s, \ldots, 2t_m = 2\tau kk_s^m-1,
$$

where $k \in [1, k_s]$ is chosen such that $t_m$ is arbitrarily close to $t_+$.

Take $\Phi \in C_c^\infty(\mathbb{R}^{N+1}_+)$ satisfying $\Phi = 1$ on $B^+_1$ and $\Phi = 0$ outside $B^+_2$. Recall that

$$
B^+_R = \{ (x, t) \in \mathbb{R}^{N+1}_+; |x|^2 + t^2 < R^2 \}.\]
From (4.3) and the Sobolev inequality, one has
\[
\left( \int_{B_1^+} U^{2t_m k_s} t^{1-2s} dx dt \right)^{\frac{1}{t_m k_s}} \leq \left( \int_{B_2^+} U^{2t_m k_s} \tilde{\Phi}^{2k_s} t^{1-2s} dx dt \right)^{\frac{1}{t_m k_s}}
\leq \left( \int_{B_2^+} |\nabla (U^{t_m} \Phi)|^2 t^{1-2s} dx dt \right)^{\frac{1}{t_m}}
\leq C \left( \int_{B_2^+} U^{2t_m} |\nabla \tilde{\Phi}|^2 t^{1-2s} dx dt \right)^{\frac{1}{t_m}}
\leq C \left( \int_{B_2^+} U^{2t_m - 1} t^{1-2s} dx dt \right)^{\frac{1}{t_m-1+s}}.
\]

By an induction argument, we arrive at
\[
\left( \int_{B_1^+} U^{2t_m k_s} t^{1-2s} dx dt \right)^{\frac{1}{t_m k_s}} \leq C \left( \int_{B_{2m-1}^+} U^{2t_m k_s} t^{1-2s} dx dt \right)^{\frac{1}{t_m k_s}}
\leq C \left( \int_{B_{2m-1}^+} U^{2t_m} |\nabla \tilde{\Phi}|^2 t^{1-2s} dx dt \right)^{\frac{1}{t_m}}
\leq C \left( \int_{B_{2m}^+} U^{2t_m} |\nabla \Phi|^2 t^{1-2s} dx dt \right)^{\frac{1}{t_m+1}}.
\]

(4.7)

where \( \Phi \in C^\infty_c (\mathbb{R}^{N+1}_+), \Phi = 1 \) on \( B_{2m-1}^+ \) and \( \Phi = 0 \) outside \( B_{2m}^+ \).

Our task is next to transform the estimate the right hand side of (4.7) on half space \( \mathbb{R}^{N+1}_+ \) to that on \( \mathbb{R}^N \). Applying the Jensen inequality, we get
\[
U^{2\tau}(x, t) \leq \int_{\mathbb{R}^N} P_s(x - y, t)(u(y))^{2\tau} dy,
\]
where \( P_s \) is the Poisson kernel. Consequently,
\[
\int_{\mathbb{R}^{N+1}_+} U^{2\tau} |\nabla \Phi|^2 t^{1-2s} dx dt
\leq \int_0^\infty \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} P_s(x - y, t)(u(y))^{2\tau} dy \right) |\nabla \Phi|^2 t^{1-2s} dx dt
= \int_{\mathbb{R}^N} (u(y))^{2\tau} \left( \int_0^\infty \int_{\mathbb{R}^N} P_s(x - y, t)|\nabla \Phi|^2 t^{1-2s} dx dt \right) dy.
\]
(4.8)

Let us put
\[
\varphi(y) = \int_0^\infty \int_{\mathbb{R}^N} P_s(x - y, t)|\nabla \Phi|^2 t^{1-2s} dx dt
= p(N, s) \int_0^\infty \int_{\mathbb{R}^N} \frac{t|\nabla \Phi|^2}{(|x - y|^2 + t^2)^{\frac{N+1}{2}}} dx dt.
\]

Recall that \( \Phi \in C^\infty_c (\mathbb{R}^{N+1}_+) \) and \( \Phi = 0 \) outside \( B_{2m}^+ \). In particular, \( \text{supp} |\nabla \Phi| = 0 \) outside \( B_{2m}^+ \). In order to estimate \( \varphi \), we need only consider the integral in the set \( B_{2m}^+ \).
Firstly, it is easy to see that \( \varphi \) is continuous on \( \mathbb{R}^N \) and \( \varphi(y) > 0 \) for all \( y \in \mathbb{R}^N \). Consequently, for \( |y| \leq 2^{m+1} \), we have \( \varphi \sim C \rho_{N+2s} \). In addition, when \( |y| > 2^{m+1} \) and \( |x| \leq 2^m \), there holds
\[
\frac{y}{2} \leq |x - y| \leq 2y.
\]

Thus, we obtain for \( |y| \geq 2^{m+1} \) that
\[
C_1 \rho_{N+2s}(y) \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 dx dt \leq \varphi(y) \leq C_2 \rho_{N+2s}(y) \int_0^\infty \int_{\mathbb{R}^N} |\nabla \Phi|^2 dx dt
\]
These above estimates imply that, for all \( y \in \mathbb{R}^N \),
\[
C_1 \rho_{N+2s}(y) \leq \varphi(y) \leq C_2 \rho_{N+2s}(y),
\]
for some \( C_1, C_2 > 0 \). We deduce from (4.7), (4.8) and (4.9) that
\[
\left( \int_{B^+_1} U^{2t_m k_s} t^{1-2s} dx dt \right)^\frac{1}{t_m k_s} \leq C \left( \int_{\mathbb{R}^N} (u(y))^{2\tau} \rho_{N+2s}(y) dy \right)^\frac{1}{\tau}.
\]

We next exploit a scaling argument. Let \( R \) be a large positive parameter. Then, as above, \((u_R, v_R)\) is also a stable positive solution of (1.2) with \( u_R(x) = R^{2s(p+1)} u(Rx) \) and \( v_R(x) = R^{2s(p+1)} v(Rx) \). Replacing \( U \) and \( u \) in (4.10) by \( U_R := R^{\frac{2s(p+1)}{p+1}} U(Rx, Rt) \) and \( u_R \), we arrive at
\[
\left( \int_{B^+_R} U^{2t_m k_s} (Rx, Rt) t^{1-2s} dx dt \right)^\frac{1}{t_m k_s} \leq C \left( \int_{\mathbb{R}^N} (u(Ry))^{2\tau} \rho_{N+2s}(y) dy \right)^\frac{1}{\tau}.
\]

This estimate implies that
\[
\left( R^{-N-2+2s} \int_{B^+_R} U^{2t_m k_s} (x, t) t^{1-2s} dx dt \right)^\frac{1}{t_m k_s} \leq C \left( \int_{\mathbb{R}^N} (u(Ry))^{2\tau} \rho_{N+2s}(y) dy \right)^\frac{1}{\tau}
\leq C \left( \int_{\mathbb{R}^N} (u(Ry))^{\gamma} \rho_{N+2s}(y) dy \right)^\frac{1}{\gamma}
= C \left( R^{-N} \int_{\mathbb{R}^N} (u(y))^{\gamma} \rho_{N+2s}(y/R) dy \right)^\frac{1}{\gamma}
\leq CR^{-\frac{4\gamma(p+1)}{p+1}}.
\]

Here we have used the Hölder inequality in the penultimate inequality and Proposition 3.2 in the last one. Hence,
\[
\left( \int_{B^+_R} U^{2t_m k_s} (x, t) t^{1-2s} dx dt \right)^\frac{1}{t_m k_s} \leq CR^{N+2-2s - \frac{4\gamma(p+1)}{p+1}} t_m k_s.
\]

Since \( k \) is chosen so that \( t_m \) is sufficiently close to \( t_+ \), the exponent in the right hand side of (4.11) is negative thanks to the assumptions of Theorem 1.2. Let \( R \) tend to infinity, we have a contradiction.

\[ \square \]

**Acknowledgments**

The first author was supported by Vietnam Ministry of Education and Training under grant number B2019-SPH-02.
References

[1] Cabré, X., and Caffarelli, A. On the stability of radial solutions of semilinear elliptic equations in all of $\mathbb{R}^n$. C. R. Math. Acad. Sci. Paris 338, 10 (2004), 769–774.

[2] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32, 7–9 (2007), 1245–1260.

[3] Castorina, D., Esposito, P., and Sciunzi, B. Low dimensional instability for semilinear and quasilinear problems in $\mathbb{R}^N$. Commun. Pure Appl. Anal. 8, 6 (2009), 1779–1793.

[4] Cowan, C. Liouville theorems for stable Lane-Emden systems with biharmonic problems. Nonlinearity 26, 8 (2013), 2357–2371.

[5] Cowan, C. Stability of entire solutions to supercritical elliptic problems involving advection. Nonlinear Anal. 104 (2014), 1–11.

[6] Cowan, C., and Fazly, M. On stable entire solutions of semi-linear elliptic equations with weights. Proc. Amer. Math. Soc. 140, 6 (2012), 2003–2012.

[7] Damascelli, L., Farina, A., Sciunzi, B., and Valdinoci, E. Liouville results for $m$-Laplace equations of Lane-Emden-Fowler type. Ann. Inst. H. Poincaré Anal. Non Linéaire 26, 4 (2009), 1099–1119.

[8] Dancer, E. N. Stable and finite Morse index solutions on $\mathbb{R}^n$ or on bounded domains with small diffusion. Trans. Amer. Math. Soc. 357, 3 (2005), 1225–1243.

[9] Dávalo, J., Dupaigne, L., and Wei, J. On the fractional Lane-Emden equation. Trans. Amer. Math. Soc. 369, 9 (2017), 6087–6104.

[10] Duong, A. T. A Liouville type theorem for non-linear elliptic systems involving advection terms. Complex Var. Elliptic Equ. 63, 12 (2018), 1704–1720.

[11] Duong, A. T., and Nguyen, V. H. A liouville type theorem for fractional elliptic equation with exponential nonlinearity. arXiv:1911.05966 (2019), 3, 6, 10, 13.

[12] Duong, A. T., and Phan, Q. H. Liouville type theorem for nonlinear elliptic system involving Grushin operator. J. Math. Anal. Appl. 454, 2 (2017), 785–801.

[13] Dupaigne, L. Stable solutions of elliptic partial differential equations, vol. 143 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2011.

[14] Dupaigne, L., and Farina, A. Liouville theorems for stable solutions of semilinear elliptic equations with convex nonlinearities. Nonlinear Anal. 70, 8 (2009), 2882–2888.

[15] Dupaigne, L., and Farina, A. Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$. J. Eur. Math. Soc. (JEMS) 12, 4 (2010), 855–882.

[16] Dupaigne, L., and Sire, Y. A Liouville theorem for non local elliptic equations. In Symmetry for elliptic PDEs, vol. 528 of Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 105–114.

[17] Fall, M. M. Semilinear elliptic equations for the fractional laplacian with hardy potential. Nonlinear Anal. In press (2008).

[18] Farina, A. On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^n$. J. Math. Pures Appl. (9) 87, 5 (2007), 537–561.

[19] Farina, A. Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$. C. R. Math. Acad. Sci. Paris 345, 2 (2007), 63–66.

[20] Farina, A., Sciunzi, B., and Valdinoci, E. Bernstein and De Giorgi type problems: new results via a geometric approach. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7, 4 (2008), 741–791.

[21] Fazly, M., and Ghoussoub, N. On the Hénon-Lane-Emden conjecture. Discrete Contin. Dyn. Syst. 34, 6 (2014), 2513–2533.

[22] Fazly, M., and Wei, J. On stable solutions of the fractional Hénon-Lane-Emden equation. Commun. Contemp. Math. 18, 5 (2016), 1650005, 24.

[23] Fazly, M., and Wei, J. On finite Morse index solutions of higher order fractional Lane-Emden equations. Amer. J. Math. 139, 2 (2017), 433–460.

[24] Hajlaoui, H., Harrabi, A., and Mtiri, F. Liouville theorems for stable solutions of the weighted Lane-Emden system. Discrete Contin. Dyn. Syst. 37, 1 (2017), 265–279.

[25] Hu, L.-G. Liouville type results for semi-stable solutions of the weighted Lane-Emden system. J. Math. Anal. Appl. 432, 1 (2015), 429–440.
Hu, L.-G. Liouville type theorems for stable solutions of the weighted elliptic system with
the advection term: $p \geq \vartheta > 1$. NoDEA Nonlinear Differential Equations Appl. 25, 1 (2018),
Art. 7, 30, 3, 4

[27] Hu, L.-G., and Zeng, J. Liouville type theorems for stable solutions of the weighted elliptic
system. J. Math. Anal. Appl. 437, 2 (2016), 882–901, 3, 4

[28] Hyder, A., and Yang, W. Classification of stable solutions to a non-local gelfand-liouville
equation. arXiv:1911.05966, (2020).

[29] Le, P. Liouville theorems for stable solutions of p-Laplace equations with convex nonlinearities.
J. Math. Anal. Appl. 443, 1 (2016), 431–444.

[30] Mitidieri, E. A Rellich type identity and applications. Comm. Partial Differential Equations
18, 1-2 (1993), 125–151.

[31] Mitidieri, E. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$.
Differential Integral Equations 9, 3 (1996), 465–479.

[32] Molica Bisci, G., Radulescu, V. D., and Servadei, R. Variational methods for nonlocal
fractional problems, vol. 162 of Encyclopedia of Mathematics and its Applications. Cambridge
University Press, Cambridge, 2016. With a foreword by Jean Mawhin.

[33] Montenegro, M. Minimal solutions for a class of elliptic systems. Bull. London Math. Soc.
37, 3 (2005), 405–416.

[34] Mtiri, F., and Ye, D. Liouville theorems for stable at infinity solutions of Lane-Emden
system. Nonlinearity 32, 3 (2019), 910–926.

[35] Quittner, P., and Souplet, P. Symmetry of components for semilinear elliptic systems.
SIAM J. Math. Anal. 44, 4 (2012), 2545–2559.

[36] Rahal, B., and Zaidi, C. On the classification of stable solutions of the fractional equation.
Potential Anal. 50, 4 (2019), 565–579.

[37] Ros-Oton, X., and Serra, J. The extremal solution for the fractional Laplacian. Calc. Var.
Partial Differential Equations 50, 3-4 (2014), 723–750.

[38] Serrin, J., and Zou, H. Non-existence of positive solutions of Lane-Emden systems.
Differential Integral Equations 9, 4 (1996), 635–653.

[39] Serrin, J., and Zou, H. Existence of positive solutions of the Lane-Emden system. Atti Sem.
Mat. Fis. Univ. Modena 46, suppl. (1998), 369–380. Dedicated to Prof. C. Vinti (Italian)
(Perguia, 1996).

[40] Souplet, P. The proof of the Lane-Emden conjecture in four space dimensions. Adv. Math.
221, 5 (2009), 1409–1427.

[41] Villegas, S. Asymptotic behavior of stable radial solutions of semilinear elliptic equations
in $\mathbb{R}^N$. J. Math. Pures Appl. (9) 88, 3 (2007), 241–250.

[42] Wang, C., and Ye, D. Some Liouville theorems for Hénon type elliptic equations. J. Funct.
Anal. 262, 4 (2012), 1705–1727.

[43] Yang, H., and Zou, W. Symmetry of components and Liouville-type theorems for semilinear
elliptic systems involving the fractional Laplacian. Nonlinear Anal. 180 (2019), 208–224.