Degenerate central factorial numbers of the second kind

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In this paper, we introduce the degenerate central factorial polynomials and numbers of the second kind which are degenerate versions of the central factorial polynomials and numbers of the second kind. We derive some properties and identities for those polynomials and numbers. We obtain, among other things, recursive formulas for the degenerate central factorial polynomials and numbers of the second kind.

1. Introduction

Various degenerate versions of special polynomials and numbers have drawn the attention of many mathematicians in recent years. The origin of these are the papers by Carlitz [2, 3] on degenerate Bernoulli and degenerate Euler polynomials and numbers. The degenerate Bernoulli polynomials were later rediscovered by Ustinov [13] under the name of Korobov polynomials of the second. Also, Korobov [11] introduced Korobov polynomials of the first kind which are in fact a degenerate version of Bernoulli polynomials of the second kind. All of them studied some arithmetic and combinatorial aspects of those degenerate special polynomials and numbers.

More recently, along the same line, studying various degenerate versions of many special polynomials and numbers regained attention of the present authors, their colleagues and some other people in connection with their interest not only in arithmetic and combinatorial properties but also in certain symmetric identities and differential equations [9, 10]. This idea of introducing some degenerate version of certain polynomials and numbers has been extended even to transcendental functions so that degenerate gamma functions were introduced in [7].

Here in this paper we study the degenerate central factorial polynomials and numbers of the second kind which are degenerate versions of the central factorial polynomials and numbers of the second kind. We derive some properties and identities for those polynomials and numbers. In particular, we will be able to find recursive formulas for the degenerate central factorial polynomials and numbers of the second kind. As to degenerate central factorial numbers of the second kind.

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first kind, we will be content with defining them.

In conclusion, we may say that studying some degenerate versions of certain special polynomials and numbers are promising area of research and that there are still many things yet to be uncovered.

For \( n \in \mathbb{N} \cup \{0\} \), it is known that the Stirling numbers of the first kind are defined as

\[
(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 1), \quad (x)_0 = 1. \tag{0.1}
\]

As shown in [12], the Stirling numbers of the first kind satisfy the relation

\[
S_1(n+1,k) = S_1(n,k-1) - nS_1(n,k), \quad (1 \leq k \leq n). \tag{0.2}
\]

Carlitz [2, 3] studied the degenerate Euler polynomials given by

\[
(1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) t^n, \quad (\lambda \in \mathbb{R}). \tag{0.3}
\]

Now, as a generalization of the falling factorial sequence in (0.1), the \( \lambda \)-analogue of the falling factorial sequence are defined as follows:

\[
(x)_0,\lambda = 1, \quad (x)_n,\lambda = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1). \tag{0.4}
\]

As defined in [7], the \( \lambda \)-Stirling numbers of the second kind are given by

\[
\frac{1}{k!}((1 + \lambda t)^{\frac{x}{\lambda}} - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \in \mathbb{N}). \tag{0.5}
\]

Note that, by taking the limit \( \lambda \to 0 \) in (0.5), we have \( \lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k), (n,k \geq 0) \). Here, as we can see in [5, 8, 12], \( S_2(n,k) \) are the Stirling numbers of the second kind given by

\[
x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0).
\]

As shown in [8], the \( \lambda \)-analogue of binomial expansion is given by

\[
(1 + \lambda t)^\frac{x}{\lambda} = \sum_{l=0}^{\infty} \binom{x}{l}_\lambda t^l, \tag{0.6}
\]

where \( \binom{x}{l}_\lambda = \frac{(x)_{l,\lambda}}{l!} = \frac{x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)}{l!}, \quad (l \geq 1), \quad \binom{x}{0}_\lambda = 1
\]

The central factorial \( x^{[n]} \) is defined by the generating function

\[
\sum_{n=0}^{\infty} x^{[n]} \frac{t^n}{n!} = \left( \frac{t}{2} + \sqrt{1 + \frac{t^2}{4}} \right)^{2x}. \tag{0.7}
\]
From (0.7), we note that
\[ x^{[n]} = x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2) \cdots (x - \frac{n}{2} + 1), \quad (n \geq 1), \quad x^{[0]} = 1. \]

As defined in [1, 4, 6, 14], for any nonnegative integer \( n \), the central factorial numbers of the first kind are given by
\[ x^{[n]} = \sum_{k=0}^{n} t(n, k)x^k. \] (0.8)

By (0.8), we easily get
\[ \frac{1}{k!} \left( 2 \log \left( \frac{t}{2} + \sqrt{1 + \frac{t^2}{4}} \right) \right)^k = \sum_{n=k}^{\infty} t(n, k) \frac{t^n}{n!}. \] (0.9)

Let \( f(t) = 2 \log \left( \frac{t}{2} + \sqrt{1 + \frac{t^2}{4}} \right) \). Then we have
\[ f^{-1}(t) = e^{\frac{t}{2}} - e^{-\frac{t}{2}}. \] (0.10)

In view of (0.9) and (0.10), we define the central factorial numbers of the second kind by
\[ \frac{1}{k!} \left( e^{\frac{t}{2}} - e^{-\frac{t}{2}} \right)^k = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!}. \] (0.11)

Thus, as shown in [1, 4, 6, 14], and from (0.11) we easily get
\[ x^n = \sum_{k=0}^{n} T(n, k)x^{[k]}. \] (0.12)

From (0.12), we note that
\[ T(n, k) = T(n - 2, k - 2) + \frac{k^2}{4} T(n - 2, k), \quad (n, k \geq 2). \]

2. A note on Central factorial numbers and polynomials of the second kind

The central difference operator \( \delta \) is defined by
\[ \delta f(x) = f(x + \frac{1}{2}) - f(x - \frac{1}{2}). \] (0.13)

By proceeding induction with (0.13), we can easily show that
\[ \delta^k f(x) = \sum_{l=0}^{k} \binom{k}{l} f(x + l - \frac{k}{2})(-1)^{k-l}, \quad (k \in \mathbb{N}). \] (0.14)

From (0.14), we note that
\[ \delta^k x^{m+1} = \sum_{l=0}^{k} \binom{k}{l} \left( x + l - \frac{k}{2} \right)^{m+1} (-1)^{k-l} = \sum_{l=0}^{k} \binom{k}{l} \left( x + l - \frac{k}{2} \right)^{m} (-1)^{k-l} \left( x + l - \frac{k}{2} \right) \]

\[ = (x - \frac{k}{2}) \delta^k x^m + k \sum_{l=1}^{k} \left( \binom{k-1}{l-1} \right) \left( x + l - \frac{k}{2} \right)^m (-1)^{k-l} \]

\[ = (x - \frac{k}{2}) \delta^k x^m + k \sum_{l=0}^{k} \left( \binom{k}{l} \right) \left( x + l - \frac{k}{2} \right)^m (-1)^{k-l} \]

\[ = (x - \frac{k}{2}) \delta^k x^m + k \delta^k x^m = (x + \frac{k}{2}) \delta^k x^m + k \delta^{k-1} \left( x - \frac{1}{2} \right)^m. \]

We define the degenerate central factorial polynomials of the second kind by

\[ \frac{1}{k!} (1 + \lambda t)^{\frac{k}{2}} \left( \frac{1}{1 + \lambda t} - \frac{1}{\lambda} \right)^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n, k | x) \frac{t^n}{n!}. \] (0.16)

When \( x = 0 \), \( T_{2,\lambda}(n, k) = T_{2,\lambda}(n, k | 0) \) are called the degenerate central factorial numbers of the second kind so that

\[ \frac{1}{k!} (1 + \lambda t)^{\frac{k}{2}} \left( \frac{1}{1 + \lambda t} - \frac{1}{\lambda} \right)^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n, k) \frac{t^n}{n!}. \] (0.17)

From (0.17) and with the notation in (0.4), we have

\[ \frac{1}{k!} (1 + \lambda t)^{\frac{k}{2}} \left( \frac{1}{1 + \lambda t} - \frac{1}{\lambda} \right)^k = \sum_{n=k}^{\infty} T_{2,\lambda}(n, k | x) \frac{t^n}{n!}. \] (0.18)

Therefore, by (0.16) and (0.18), we obtain the following theorem.

**Theorem 0.1.** For any nonnegative integers \( n, k \), with \( n \geq k \), we have

\[ T_{2,\lambda}(n, k | x) = \sum_{l=k}^{n} \binom{n}{l} T_{2,\lambda}(l, k) (x)_{n-l, \lambda}. \] (0.19)
Now, we observe that
\[
\frac{1}{k!} (1 + \lambda t)^k ((1 + \lambda t)^\frac{1}{k} - (1 + \lambda t)^\frac{1}{k})^k 
\]
\[
= \frac{1}{k!} (1 + \lambda t)^\frac{1}{k} (x - \frac{x}{k}) \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (1 + \lambda t)^\frac{1}{k}
\]
\[
= \frac{1}{k!} \sum_{l=0}^{k} \left( \binom{k}{l} (-1)^{k-l} e^{(x - \frac{x}{k}) l \log(1 + \lambda t)} \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) (x + l - \frac{k}{2})^m \left( \frac{1}{m!} \log(1 + \lambda t) \right)^m 
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \left( \frac{1}{k!} \delta_k x^m \right) \lambda^{n-m} S_1(n, m) \right).
\]

Therefore, by (0.16) and (0.20), we obtain the following theorem.

**Theorem 0.2.** For any nonnegative integers \( n, k \), we have
\[
\sum_{m=0}^{n} \left( \frac{1}{k!} \delta_k x^m \right) \lambda^{n-m} S_1(n, m) = \begin{cases} T_2,\lambda(n, k | x), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}
\] (0.21)

Letting \( x = 0 \) in (0.21) gives the next result.

**Theorem 0.3.** For any nonnegative integers \( n, k \), we have
\[
\sum_{m=0}^{n} \left( \frac{1}{k!} \delta_k x^m \right) \lambda^{n-m} S_1(n, m) = \begin{cases} T_2,\lambda(n, k), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}
\] (0.22)

By making use of (0.21) and (0.2), we note that
\[
T_2,\lambda(n + 1, k | x) = \sum_{m=0}^{n+1} \left( \frac{1}{k!} \delta_k x^m \right) \lambda^{n+1-m} S_1(n + 1, m) 
\]
\[
= \sum_{m=1}^{n+1} \left( \frac{1}{k!} \delta_k x^m \right) \lambda^{n+1-m} \left( S_1(n, m - 1) - nS_1(n, m) \right) 
\]
\[
= \sum_{m=0}^{n} \left( \frac{1}{k!} \delta_k x^{m+1} \right) \lambda^{n-m} S_1(n, m) - n\lambda T_2,\lambda(n, k | x).
\] (0.23)
On the other hand, by (0.15), we get
\[
\sum_{m=0}^{n} \left( \frac{1}{k!} \delta^{k} x^{m+1} \right) \lambda^{n-m} S_{1}(n, m)
\]
\[
= \sum_{m=0}^{n} \frac{1}{k!} \left( (x + \frac{k}{2}) \delta^{k} x^{m} + k \delta^{k-1} (x - \frac{1}{2})^{m} \right) \lambda^{n-m} S_{1}(n, m)
\]
\[
= (x + \frac{k}{2}) \sum_{m=0}^{n} \left( \frac{1}{k!} \delta^{k} x^{m} \right) \lambda^{n-m} S_{1}(n, m)
\]
\[
+ \sum_{m=0}^{n} \left( \frac{1}{(k-1)!} \delta^{k-1} (x - \frac{1}{2})^{m} \right) \lambda^{n-m} S_{1}(n, m)
\]
\[
= \left( x + \frac{k}{2} \right) T_{2, \lambda}(n, k \mid x) + T_{2, \lambda}(n, k - 1 \mid x - \frac{1}{2}).
\] (0.24)

Therefore, by (0.23) and (0.24), we obtain the following theorem.

**Theorem 0.4.** For any integers \( n, k \), with \( 1 \leq k \leq n \), we have
\[
T_{2, \lambda}(n + 1, k \mid x) = \left( x + \frac{k}{2} - n \lambda \right) T_{2, \lambda}(n, k \mid x) + T_{2, \lambda}(n, k - 1 \mid x - \frac{1}{2}).
\] (0.25)

Setting \( x = 0 \) in (0.25) yields the following result.

**Theorem 0.5.** For any integers \( n, k \), with \( 1 \leq k \leq n \), we have
\[
T_{2, \lambda}(n + 1, k) = \left( \frac{k}{2} - n \lambda \right) T_{2, \lambda}(n, k) + T_{2, \lambda}(n, k - 1 \mid - \frac{1}{2}).
\] (0.26)

Note that taking \( \lambda \to 0 \) in (0.26) gives us
\[
T(n + 1, k) = \frac{k}{2} T(n, k) + T(n, k - 1 \mid - \frac{1}{2}),
\]
where \( 1 \leq k \leq n \).

By (0.17), we get
\[
\frac{1}{k!} \left( (1 + \lambda t)^{\frac{1}{k}} - (1 + \lambda t)^{- \frac{1}{k}} \right) = \frac{1}{k!} (1 + \lambda t)^{- \frac{1}{k}} \left( (1 + \lambda t)^{\frac{1}{k}} - 1 \right)^{k}
\]
\[
= \frac{1}{k!} \sum_{l=0}^{k} \frac{k}{l} (-1)^{k-l} (1 + \lambda t)^{\frac{k}{l} - \frac{l}{2}}
\]
\[
= \sum_{n=0}^{\infty} \left( \frac{n!}{k!} \sum_{l=0}^{k} \frac{k}{l} \left( \frac{l - \frac{k}{2}}{n} \right) \lambda^{(l-k)^{n}/n!} \right)
\] (0.27)

Therefore, by (0.17) and (0.27), we obtain the following theorem.
Theorem 0.6. For any nonnegative integers \( n, k \), we have

\[
\frac{n!}{k!} \sum_{l=0}^{k} \binom{k}{l} \left( \frac{l - \frac{k}{2}}{n} \right) (-1)^{k-l} = \begin{cases} 
T_{2,\lambda}(n, k), & \text{if } n \geq k, \\
0, & \text{if } n < k.
\end{cases}
\]

The Carlitz degenerate Euler polynomials of higher-order (see (0.3)) are defined as

\[
\left( \frac{2}{(1 + \lambda t)^{\frac{1}{2}} + 1} \right)^r (1 + \lambda t)^{\frac{r}{2}} = \sum_{n=0}^{\infty} \mathcal{E}_{n, \lambda}^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{R}). \tag{0.28}
\]

Then (0.28) is also given by

\[
\left( \frac{2}{(1 + \lambda t)^{\frac{1}{2}} + 1} \right)^r (1 + \lambda t)^{\frac{r}{2}} = 2^r \left( (1 + \lambda t)^{\frac{r}{2}} + 1 \right)^{-r} (1 + \lambda t)^{\frac{r}{2}}
\]

\[
= \left( \frac{(1 + \lambda t)^{\frac{r}{2}} - 1}{2} + 1 \right)^{-r} (1 + \lambda t)^{\frac{r}{2}}
\]

\[
= \sum_{l=0}^{\infty} \binom{\frac{r}{2} + l - 1}{l} \left( -\frac{1}{2} \right)^l ((1 + \lambda t)^{\frac{r}{2}} - 1)^l (1 + \lambda t)^{\frac{r}{2}}
\]

\[
= \sum_{l=0}^{\infty} \binom{\frac{r}{2} + l - 1}{l} \left( -\frac{1}{2} \right)^l ((1 + \lambda t)^{\frac{r}{2}} - (1 + \lambda t)^{\frac{r}{2}} - \frac{1}{2})^l (x + \frac{1}{2})
\]

\[
= \sum_{l=0}^{\infty} \binom{\frac{r}{2} + l - 1}{l} \left( -\frac{1}{2} \right)^l (1 + \lambda t)^{\frac{r}{2}} (x + \frac{1}{2})^l
\]

Therefore, by (0.28) and (0.29), we obtain the following theorem.

Theorem 0.7. For any nonnegative integer \( n \), we have

\[
\mathcal{E}_{n, \lambda}^{(r)}(x) = \sum_{l=0}^{n} \binom{\frac{r}{2} + l - 1}{l} \left( -\frac{1}{2} \right)^l l! T_{2,\lambda}(n, l \mid x + \frac{1}{2}).
\]
By (0.17) and (0.5), we get
\[
\sum_{n=2k}^{\infty} T_{2,\lambda}(n, 2k) \frac{t^n}{n!} = \frac{1}{(2k)!} \left( (1 + \lambda t)^\frac{1}{\lambda} - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)^{2k}
\]
\[
= \frac{1}{(2k)!} \left( (1 + \lambda t)^\frac{k}{\lambda} + (1 + \lambda t)^{-\frac{k}{\lambda}} - 2 \right)^k
\]
\[
= \frac{1}{(2k)!} \sum_{l=0}^{k} \binom{k}{l} \left( (1 + \lambda t)^\frac{k}{\lambda} - 1 \right)^l \left( (1 + \lambda t)^{-\frac{k}{\lambda}} - 1 \right)^{k-l}
\]
\[
= \frac{k!}{(2k)!} \sum_{l=0}^{k} \sum_{n=0}^{\infty} \left( \sum_{i=l}^{n} \binom{n}{i} S_{2,\lambda}(i, l) S_{2,\lambda}(n - i, k - l)(-1)^{n-i} \right) \frac{t^n}{n!}
\]
Comparing the coefficients on both sides of (0.30), we have the following theorem.

**Theorem 0.8.** For any nonnegative integers \( n, k \), we have
\[
\sum_{l=0}^{k} \sum_{i=0}^{n} \binom{n}{i} S_{2,\lambda}(i, l) S_{2,\lambda}(n - i, k - l)(-1)^{n-i} = \begin{cases} 
\frac{k!}{(2k)!} T_{2,\lambda}(n, 2k), & \text{if } n \geq 2k, \\
0, & \text{if } n < 2k.
\end{cases}
\]

Finally, we would like to define the degenerate central factorial numbers of the first kind. We first recall that the \( \lambda \)-logarithmic function is defined by
\[
\log_\lambda t = \frac{t^\lambda - 1}{\lambda},
\]
where we note that \( \lim_{\lambda \to 0} \log_\lambda t = \log t \). Let \( g(t) = (1 + \lambda t)^\frac{1}{\lambda} - (1 + \lambda t)^{-\frac{1}{\lambda}} \).

Then, using (0.31) we see that the inverse \( g^{-1}(t) \) of \( g(t) \) is given by
\[
g^{-1}(t) = \log_\lambda \left( \frac{t}{2} + \sqrt{1 + \frac{t^2}{4}} \right)^2.
\]
In view of (0.17) and (0.32), we are led to define the degenerate central factorial numbers of the first kind by
\[
\frac{1}{k!} \left( \log_\lambda \left( \frac{t}{2} + \sqrt{1 + \frac{t^2}{4}} \right)^2 \right)^k = \sum_{n=k}^{\infty} t_{1,\lambda}(n, k) \frac{t^n}{n!}.
\]
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