Evolutionary Integro-Differential Equation Solution of Self-Consistent Beam Dynamics in Dielectric-Filled Wakefield Accelerating Structure

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Abstract. Self-coordinated transverse dynamics of the high current relativistic electronic bunches used for generation of wake fields in accelerating structures with dielectric filling is investigated. The Laplace transform method is used to find the analytical solution of the problem of self-consistent beam dynamics.

1. Introduction
Wakefield acceleration in dielectric wakefield waveguide structures is one of the most intensively developing direction among new methods of particle acceleration. This principle is of major concern for linear colliders (ILC [1], CLIC [2]), and FEL (LCLS-II, X-FEL) [3, 4] and other accelerator projects.
Dielectric wakefield accelerating structures are single or multilayer dielectric cylindrical waveguides with outer metallic coating and vacuum channel along the axis. A driving bunch moving through the waveguide creates a Cherenkov field with longitudinal component which is used for acceleration of the less intensive witness bunch. One of the main problems in realization of the wakefield method is to keep an intensive electronic bunch in the channel of the waveguide and to prevent particles from hitting its wall. Along with longitudinal fields there are transverse fields, leading to the bunch deflection from the axis of the waveguide and particles from hitting its wall. It makes impossible to continue the acceleration. The cumulative beam breakup instability in linear accelerators occurs if the beam is injected into an accelerator with a lateral offset or an angular divergence. This instability is caused by the dipole modes of the accelerating structures [5].
In this regard, a key task of the wakefield acceleration method is modelling of the self-coordinated movement of the relativistic electronic bunch passing through dielectric structure in the Cherenkov field created by it.
In [6-11] analytical solutions for describing the development of the cumulative instability of the free beam in one mode regime of waveguide excitation are obtained. In [12, 13] two mode and multimode regimes of waveguide excitation are described. The self-consistent dynamics of the beam under the influence of external force varying according to the harmonic law [14] and quasi-elastic force [6-8, 15] was investigated. This paper is devoted to obtain an analytical solution for the beam dynamics in the waveguide excited in multimode regime under the influence of external forces depending on time.
2. Beam Dynamics Equation
The description of movement of an electronic bunch was carried out on the basis of the equations of relativistic dynamics in the assumption of lack of the azimuthal movement of particles [3]:

\[ \frac{d(m,v,\gamma)}{dt} = F_j - e\delta \sum_{i,j} [\psi_{F_{i,j}}(k_{r,i,j},r(\zeta,t)) \frac{\hat{z}}{0} f(\zeta_0) \sin(k_{z,i,j}(\zeta - \zeta_0)) I_i(k_{r,i,j},r(\zeta_0,t))d\zeta_0], \]

where \( r(\zeta,t) \) is a bunch deflection from waveguide axes, \( \zeta = z - vt \) is a distance behind the bunch head, \( F_j \) is an external (for example focusing) force, \( e \) and \( m_\zeta \) are charge and mass of electron, \( q \) and \( \gamma \) are charge and relativistic factor of the bunch, \( k_{z,i,j} \) and \( k_{r,i,j} \) are longitudinal and radial components of wave vector, \( \psi_{F_{i,j}} \) are coefficients of series, depending on inner \( R_e \) and outer \( R_w \) radii of dielectric waveguide, wave guide filling permittivity \( \varepsilon \) and initial charge place, \( f(\zeta_0) \) is a function describing longitudinal charge distribution, \( I_i(x) \) are modified Bessel function of \( i \)-th order.

The task of the description of macroparticles movement is self-coordinated: the mutual provision of particles in ensemble influences a field created by particles which, in turn, leads to change of their position. We consider an analytical method of the solution of the integro-differential equation of self-coordinated dynamics at the following simplifying assumptions:

1. Let's consider that the charge in the bunch is distributed evenly in the longitudinal direction, thus \( f(\zeta_0) = 1/l \), where \( l \) is the length of the bunch.
2. Let's neglect change of a relativistic factor over time \( \gamma(t) = \gamma_0, \ \gamma = const. \)
3. Let's neglect change of a relativistic factor over time \( \gamma(t) = \gamma_0, \ \gamma = const. \)

In considered cases \( k_{r,i,j}r(\zeta_0,t) << 1 \), in rejecting field at small deviations of a bunch from an axis the overwhelming contribution is brought by the 1st azimuthal mode \( i = 1 \). Nonlinear component of the force is negligible. Thus, it is possible to consider that the force operating on charges in the radial direction, depends on \( r \) linearly \( I_1(kr) \approx kr/2, \ I'_1(kr) \approx 1/2 \).

3. Let the external force \( F_j \) depends only on time. This corresponds to the case when the period of the focusing force is much less than the flight range at which there is an appreciable displacement of the beam particles under the action of their own forces.

According to these assumptions the equation of radial beam dynamics has a form:

\[ \frac{\partial^2 r(\zeta,t)}{\partial t^2} - \sum_{j=1} A_j \int_0^\infty k_{z,j} \sin(k_{z,j}(\zeta - \zeta_0)) r(\zeta_0,t)d\zeta_0 = \frac{F_j}{\gamma_0 m_\zeta} = g(t), \]

where \( A_j > 0 \), noting that electron beam charge is negative, \( k_{z,j} = k_{z,1,j} \).

Let initial conditions are

\[ r(\zeta,0) = r_0, \quad \frac{dr(\zeta,t)}{dt} \bigg|_{t=0} = v_{r,0}. \]

Let the function \( g(t) \) be infinitely differentiable in some neighborhood near zero. We expand it in a Taylor series:

\[ g_f(t) = \sum_{k=0}^\infty \frac{g_f^{(k)}(0)}{k!} t^k = \sum_{k=0}^\infty \frac{g_k}{k!} t^k. \]

3. Solution of the equation of self-consistent dynamics
To reduce of the received integro-differential equation to the integrated equation we will apply Laplace's transformation on time.

\[ r^*(\zeta,p) - \sum_{j=1}^N A_j \int_0^\zeta \frac{\sin(k_{z,j}(\zeta - \zeta_0))}{p^j} r^*(\zeta_0, p)d\zeta_0 = \tilde{g}(p). \]
where $\tilde{g}(p) = \frac{r_0}{p} + \frac{v_0}{p^2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{g_k}{p^{k+1}}$.

The received integrated equation has the solution received on the basis of transformation of Laplace on longitudinal coordinate $[16]$.

$$r^*(\zeta, p) = \tilde{g}(p) \left[ 1 + \sum_{j=1}^{n} \frac{C_j}{x_j} \left( \cosh (\lambda_j \zeta) - 1 \right) + \sum_{j=n+1}^{\infty} \frac{C_j}{x_j} \left( \cos (\lambda_j \zeta) - 1 \right) \right],$$  \hspace{1cm} (1)

where $\lambda_j = \sqrt{|x_j|}$, $x_j$ are roots of the algebraic equation

$$\sum_{j=1}^{n} \frac{k_j A_j}{x + k_j^2} = p^2,$$  \hspace{1cm} (2)

which, after reduction to the common denominator, reduces to the problem of determining the roots of the characteristic polynomial of degree $n$. We assume that all the roots $x_j$ of the equation are real, distinct and not equal to zero. In this case, all the roots, depending on their sign, are divided into two groups: $x_1, x_2, \ldots, x_n > 0$ (positive roots); $x_{n+1}, x_{n+2}, \ldots, x_{s} < 0$ (negative roots).

Coefficients $C_j$ are found from the system of linear algebraic equations

$$\sum_{j=1}^{n} \frac{C_j}{k_j^2 m + x} = 1, \hspace{1cm} m = 1, 2, \ldots, n.$$  \hspace{1cm} (3)

To find the time dependence of the radial coordinate, we apply the inverse Laplace transform

$$r(\zeta, t) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} r^*(\zeta, p) e^{pt} dp.$$  \hspace{1cm} (4)

The image $r^*(\zeta, p)$ is an analytic function for all $p > c$, and is of order less than $-1$, so the inverse transformation for it exists and is continuous for all values of the argument. To find the integral in the inverse Laplace transform, we complete the closure of the integration line at infinity to the contour enclosing the left half-plane. Points $x_j = 0$ are removable singular points of the image. With negative roots $x_{n+1}, x_{n+2}, \ldots, x_{s} < 0$ the image $r^*(\zeta, p)$ is bounded and decomposes into a Taylor series without contributing to the integral. With positive roots $x_1, x_2, \ldots, x_n > 0$, the image $r^*(\zeta, p)$ is described by a relationship containing a hyperbolic cosine. Since the hyperbolic cosine function is analytic in the whole region except for an infinitely distant point that is essentially singular, the image $r^*(\zeta, p)$ has only an essentially singular point at some root $x_j \rightarrow \infty$.

We will look for the root $x_1$ of equation (2) in the form of an expansion in powers $p^2$:

$$x_1 = \frac{B_1}{p^2} + B_0 + B_1 p^2 + B_2 p^4 + B_3 p^6 + \ldots = \sum_{j=0}^{\infty} B_{j+1} p^{2j-2}$$  \hspace{1cm} (5)
Let us find the derivative in the denominator.

Substituting \( x_i \) in (2), expanding the left side of the equation in a Taylor series and equating the expansion coefficients in the left and right sides of the equation, we obtain

\[
B_{j+1} = \frac{k_{j+1}^{i}}{A_{i}^{j+1}}b_{j}, \quad j = 0, 1, 2, \ldots
\]

where with the notation \( \kappa_{j} = k_{j}/k_{1}, \quad a_{j} = A_{i}/A \), we have

\[
b_{0} = \sum_{j=1}^{n}(\kappa_{j}a_{j}), \quad b_{1} = -\frac{1}{b_{0}}\sum_{j=1}^{n}a_{j}\kappa_{j}^{3}, \quad b_{2} = \frac{1}{b_{0}}\sum_{j=1}^{n}(\kappa_{j}a_{j}(b_{1} + \kappa_{j}^{2})^{3}), \quad b_{3} = -\frac{1}{b_{0}}\sum_{j=1}^{n}(\kappa_{j}a_{j}(b_{1} + \kappa_{j}^{2})^{3}),
\]

\[
b_{4} = \frac{1}{b_{0}}\sum_{j=1}^{n}(\kappa_{j}a_{j}(b_{1} + \kappa_{j}^{2})^{4}) - 2\frac{b_{2}^{2}}{b_{0}}, \quad b_{5} = -\frac{1}{b_{0}}\sum_{j=1}^{n}(\kappa_{j}a_{j}((b_{1} + \kappa_{j}^{2})^{5})) - \frac{5b_{2}b_{3}}{b_{0}}, \ldots
\]

We now determine the coefficient \( C_{i} \) corresponding to the root \( x_{i} \) of system (3). Using Cramer's formula for solving system (3), we obtain \( C_{i} = \Delta_{i}/\Delta \), where \( \Delta \) is the determinant of the system of equations (3) and \( \Delta_{i} \) is the determinant obtained from it when replacing the first column by ones:

\[
\Delta = \begin{vmatrix}
\frac{k_{1}^{1} + x_{1}}{1} & \frac{k_{1}^{2} + x_{2}}{1} & \cdots & \frac{k_{1}^{n} + x_{n}}{1} \\
\frac{k_{2}^{1} + x_{1}}{1} & \frac{k_{2}^{2} + x_{2}}{1} & \cdots & \frac{k_{2}^{n} + x_{n}}{1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{k_{n}^{1} + x_{1}}{1} & \frac{k_{n}^{2} + x_{2}}{1} & \cdots & \frac{k_{n}^{n} + x_{n}}{1}
\end{vmatrix}, \quad \Delta_{i} = \begin{vmatrix}
\frac{k_{i}^{1} + x_{1}}{1} & \frac{k_{i}^{2} + x_{2}}{1} & \cdots & \frac{k_{i}^{n} + x_{n}}{1} \\
\frac{k_{i+1}^{1} + x_{1}}{1} & \frac{k_{i+1}^{2} + x_{2}}{1} & \cdots & \frac{k_{i+1}^{n} + x_{n}}{1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{k_{n}^{1} + x_{1}}{1} & \frac{k_{n}^{2} + x_{2}}{1} & \cdots & \frac{k_{n}^{n} + x_{n}}{1}
\end{vmatrix}
\]

Expanding the determinants \( \Delta \) and \( \Delta_{i} \), we obtain

\[
C_{i} = \frac{(k_{i}^{1} + x_{1})(k_{i}^{2} + x_{2}) \cdots (k_{i}^{n} + x_{n})}{(x_{1} - x_{2}) \cdots (x_{i} - x_{n})}.
\] (6)

To find the product of root differences in the denominator (6), we represent equation (2) in the form

\[
L(x) = \left(1 - \sum_{j=1}^{n} \frac{k_{j}A}{p^{2}(x + k_{j}^{2})}\right) \prod_{j=1}^{n}(x + k_{j}^{2}) = \prod_{j=1}^{n}(x + k_{j}^{2}) - \sum_{j=1}^{n} \left(\frac{k_{j}A}{p^{2}} \cdot \prod_{j=1}^{n}(x + k_{j}^{2})\right) = 0.
\]

The left-hand side of this equation is a polynomial of degree \( n \) relative to \( x \) and can be rewritten in the form

\[
L(x) = (x - x_{1})(x - x_{2}) \cdots (x - x_{n}).
\]

Taking the derivative and substituting the root \( x = x_{i} \), we obtain:

\[
\frac{d}{dx}L(x) \bigg|_{x=x_{i}} = L'(x_{i}) = (x_{i} - x_{2}) \cdots (x_{i} - x_{n}).
\] (7)

Substituting (7) to (6), we have:

\[
C_{i} = \frac{(k_{i}^{1} + x_{1})(k_{i}^{2} + x_{2}) \cdots (k_{i}^{n} + x_{n})}{L'(x_{i})}.
\] (8)

Let us find the derivative in the denominator:

\[
L'(x) = \left(\sum_{j=1}^{n} \frac{k_{j}A}{p^{2}(x + k_{j}^{2})}\right) \prod_{j=1}^{n}(x + k_{j}^{2}) + \left(1 - \sum_{j=1}^{n} \frac{k_{j}A}{p^{2}(x + k_{j}^{2})}\right) \left(\prod_{j=1}^{n}(x + k_{j}^{2})\right)\).
\]

Taking into account that \( x_{i} \) is the root (2), the second term vanishes:
\[ L'(x_i) = \left( \sum_{j=1}^{n} \frac{k_{x_j} A_j}{p^2 (x_i + k_{x_j}^2)^2} \right) \prod_{j=1}^{n} (x_i + k_{x_j}^2). \]

Then, substituting the expression found in (8), we have
\[ C_1 = \left( \sum_{j=1}^{n} \frac{k_{x_j} A_j}{p^2 (x_i + k_{x_j}^2)^2} \right)^{-1}. \]

Substituting in the resulting expression the expansion for \( x_i \) (5) and expanding \( C_1 \) in a Taylor series, we obtain:
\[ C_1 = \frac{B_1}{p^3} + o(p^{-2}) + 2B_2 p^2 - 3B_3 p^4 - 4B_4 p^6 - 5B_5 p^8 - \ldots = \sum_{j=0}^{\infty} (1-j) B_{j+1} p^{2j-2}. \] (9)

We now find the time dependence of the radial coordinate of the beam particles. Substituting (1) into (4), and taking the integral according to the deduction theorem, we obtain:
\[ r(\zeta,t) = \frac{r_0 + v_{r0} t + \int_{0}^{t} g_j(t) dt}{\left[ 1 + \frac{\text{Res}_{j=0} \left[ \hat{g}(p)C_1 \left( \cosh(\sqrt{x_j \zeta}) - 1 \right) \right]}{z_1} e^{pt} \right]}. \]

We expand the integrand in the Laurent series.
\[ \frac{\cosh(\sqrt{x_j \zeta}) - 1}{x_j} = \sum_{j=0}^{\infty} \left( \frac{x_j^{\zeta^{2j}}}{\zeta^2} \right) \left( \frac{x_j^{\zeta^{2j+2}}}{(2s+2)!} \right). \]

We preserve the terms of orders up to \( p^{2k-2} \) inclusive in the expansion for \( x_i \). Then
\[ x_i = \sum_{j=0}^{k} B_{j+1} p^{2j-1} + o(p^{2k-1}). \]

We use the Newtonian polynomial expansion:
\[ x_i = \sum_{m_0, \ldots, m_k} \left( \frac{s!}{m_0! \cdots m_k!} \right) B_{m_0}^{m_0} \cdots B_{m_k}^{m_k} \left( B_{k+1} p^{2k+1} \right)^{m_k}, \]
where \( \left( \frac{s!}{m_0! \cdots m_k!} \right) \) are multinomial coefficients. The sum is over all n-tuples of nonnegative integers \( (m_0, \ldots, m_k) \), \( m_j \geq 0 \) satisfying the constraint \( s = m_0 + \ldots + m_k = \sum_{j=0}^{k} m_j \). Then
\[ \frac{\cosh(\sqrt{x_j \zeta}) - 1}{x_j} = \sum_{j=0}^{s} \frac{x_j^{\zeta^{2j+2}}}{(2s+2)!} \sum_{m_0, \ldots, m_k} \left( \frac{s!}{m_0! \cdots m_k!} \right) \left( \prod_{j=0}^{k} B_{m_j}^{m_j} \right) \left( \sum_{j=0}^{\infty} \left( \frac{x_j^{\zeta^{2j+2}}}{(2s+2)!} \right) \left( \frac{s!}{m_0! \cdots m_k!} \right) \left( \prod_{j=0}^{k} (B_{m_j}^{m_j} p^{2j+1}) \right)^{m_k} \right). \]

Expanding the exponent in a Taylor series and substituting it in the expression for the original, we get:
\[ r(\zeta,t) = \frac{r_0 + v_{r0} t + \int_{0}^{t} g_j(t) dt}{\left[ 1 + \frac{\text{Res}_{j=0} \left[ \hat{g}(p)C_1 \left( \cosh(\sqrt{x_j \zeta}) - 1 \right) \right]}{z_1} e^{pt} \right]}. \]

The required residue is the coefficient of the Laurent series obtained at \( p^{-1} \). We denote the resulting expression using dimensionless complexes \( k_{1z} \zeta \) and \( A_{t^2/k_{1z}} \):
The solution obtained allows us to predict the position of the beam at an arbitrary time moment.

\[ r(\zeta, t) = \mathbf{r}_0 + v_{o}t + \int_{0}^{t} g_{i}(t) dt + \sum_{j=0}^{k} \left[ (1-j) \mathbf{b}_{j} \sum_{i=0}^{\infty} \left[ \frac{(k \zeta)^{2s+2i}}{(2s+2i)!} \cdot \sum_{l=0}^{m} \left( \mathbf{m}_{1}, \ldots, \mathbf{m}_{l} \right) \prod_{i=0}^{l} \left( h_{m_{i}} \right) \right] \right] \cdot \left( \frac{A_{i}^{2} \beta^{2}}{k_{1}} \right)^{1-j} \sum_{m=0}^{s} \frac{M_{j}^{s-j}}{s!} \left[ 2 \left( 1-j - \sum_{i=0}^{m} (i-1) \mathbf{m}_{i} \right) \right] \cdot \left[ 1+2 \left[ 1-j - \sum_{i=0}^{m} (i-1) \mathbf{m}_{i} \right] \right] + \sum_{l=0}^{\infty} \left[ 2 \left[ 1-j - \sum_{i=0}^{l} (i-1) \mathbf{m}_{i} \right] + l \right] \right] \]

In the expression obtained, it is necessary to carry out the summation when the conditions \( m_{i} \geq 0, \)

\[ \sum_{i=0}^{k} m_{i} = s, \quad 1-j - \sum_{i=0}^{k} (i-1)m_{i} \geq 0, \quad m_{i} \geq j-1 \text{ and } s \geq j-1. \]

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