MULTIFRACTAL FORMALISM FOR INVERSE MEASURES OF RANDOM WEAK GIBBS MEASURES

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Abstract. Any Borel probability measure supported on a Cantor set of zero Lebesgue measure on the real line possesses a discrete inverse measure. We study the validity of the multifractal formalism for the inverse measures of random weak Gibbs measures supported on the attractor associated with some $C^1$ random dynamics encoded by a random subshift of finite type, and expanding in the mean. The study requires, in particular, to develop in this context of random dynamics a suitable extension of the results known for heterogeneous ubiquity associated with deterministic Gibbs measures.

1. Introduction

This paper investigates the validity of the multifractal formalism for discrete measures naturally arising in some random dynamical systems, namely the inverse measures of random weak Gibbs measures supported on random dynamical attractors in the line, to be defined below.

Let us recall the basic framework of multifractal formalism. Given a positive, finite, and compactly supported Borel measure $\mu$ on $\mathbb{R}$, whose topological support is denote by $\text{supp}(\mu)$, one defines its $L^q$-spectrum $\tau_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$\tau_\mu(q) = \liminf_{r \to 0} \frac{\log \sup \{ \sum \mu(B_i) \}^{q}}{\log r},$$

where the supremum is taken over all families of disjoint closed balls $B_i$ of radius $r$ with centers in $\text{supp}(\mu)$. One also defines for any $x \in \text{supp}(\mu)$ the lower local dimension and the upper local dimension of $\mu$ at $x$ by

$$\dim_{\text{loc}}(\mu, x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{\dim}_{\text{loc}}(\mu, x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

and the local dimension of $\mu$ at $x$ by $\dim_{\text{loc}}(\mu, x) = \underline{\dim}_{\text{loc}}(\mu, x) = \overline{\dim}_{\text{loc}}(\mu, x)$ if the last equality holds. Then, one considers the level sets associated with these quantities, i.e., for $d \in \mathbb{R}$, the sets

$$E(\mu, d) = \{ x \in \text{supp}(\mu) : \dim_{\text{loc}}(\mu, x) = d \},$$

$$\mathcal{E}(\mu, d) = \{ x \in \text{supp}(\mu) : \underline{\dim}_{\text{loc}}(\mu, x) = d \},$$

$$E(\mu, d) = \mathcal{E}(\mu, d) \cap \mathcal{E}(\mu, d).$$

One also defines the lower Hausdorff dimension of $\mu$:

$$\dim_H(\mu) = \sup \{ s : \dim_{\text{loc}}(\mu, x) \geq s \text{ for } \mu\text{-almost every } x \in \text{supp}(\mu) \}.$$

2010 Mathematics Subject Classification. Primary:37C45; Secondary:37Hxx,28A78.

Key words and phrases. Multifractal, Hausdorff dimension, local dimension, random dynamical attractor, random weak Gibbs measure, inverse measure, conditioned ubiquity.
An equivalent definition is (see [8, chapter 10]) \( \dim_H(\mu) = \inf \{ \dim_E E : E \text{ Borel set}, \mu(E) > 0 \} \). One always has, for all \( d \in \mathbb{R} \) (see [19, 15])

\[
\dim_H E(\mu, d) \leq \min(\dim_H E(\mu, d), \dim_H \overline{E}(\mu, d))
\leq \max(\dim_H E(\mu, d), \dim_H \overline{E}(\mu, d)) \leq \tau^*_\mu(d),
\]

where \( \dim_H \) stands for the Hausdorff dimension, a negative dimension means that the set is empty, and we recall that the Legendre transform \( f^* \) of any function \( f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) with non-empty domain is defined on \( \mathbb{R} \) by

\[
f^*(d) = \inf_{q \in \mathbb{R}} \{ dq - f(q) \} \in \mathbb{R} \cup \{-\infty\}.
\]

The Hausdorff spectrum of \( \mu \) is defined by

\[
d \in \mathbb{R} \mapsto \dim_H E(\mu, d),
\]
while the lower and upper Hausdorff spectra are defined similarly with the sets \( E(\mu, d) \) and \( \overline{E}(\mu, d) \) respectively, in place of \( E(\mu, d) \).

**Definition 1.1.** One says that the multifractal formalism holds on a subset \( I \) of \( \mathbb{R} \) if \( \dim_H E(\mu, d) = \tau^*_\mu(d) \) for all \( d \in I \) and that it holds strongly on \( I \) if \( \dim_H E(\mu, d) = \tau^*_\mu(d) \) for all \( d \in I \). If \( I = \mathbb{R} \), one simply says that the formalism holds.

The study of the validity of the multifractal formalism for discrete measures started in [16], and was developed further in [21]. In these papers, the authors study the relations between the multifractal behavior of a Borel probability measure supported on \([0, 1]\) and its inverse measure defined as follows:

**Definition 1.2.** Let \( \mu \) be a Borel probability measure supported on \([0, 1]\), and let \( F_\mu \) be its distribution function, i.e. \( F_\mu(t) = \mu([0, t]) \). The inverse measure \( \nu \) of \( \mu \) is the unique Borel probability measure on \([0, 1]\) such that for all \( x \in [0, 1] \), \( F_\nu(x) = \sup \{ t \in [0, 1] ; F_\mu(t) \leq x \} \).

The authors of [21] use \( \lim_{I \to \{x\}} \frac{\log(\mu(I))}{\log(|I|)} \) as a definition of the local dimension, where \( I \) is a non trivial interval containing \( x \). With this definition, they observe that for a Gibbs measure on a cookie-cutter set, while it is well known that the strong multifractal formalism holds, they can establish its failure on a non trivial interval for the discrete inverse measure of such a measure (they obtained the same type of failure for discrete in-homogeneous self-similar measures, see also [20]). Later, the validity of the multifractal formalism as defined above was obtained in [7], where the authors used the so-called heterogeneous, or conditioned, ubiquity theory, which combines ergodic theory and metric approximation theory, and was developed in [4]. This tool makes it possible to study a broad class of multifractal discrete measures [2, 6].

It is worth mentioning that the multifractal analysis of discrete measures via the Hausdorff dimensions of the level sets \( E(\mu, d) \), and with no consideration of multifractal formalism, started with homogeneous sums of Dirac masses [1, 11, 12, 9], in particular the derivative of Lévy subordinators [12], and that originally heterogeneous ubiquity was elaborate with the multifractal analysis of Lévy processes in multifractal time as a target [5].

In [22], we consider, on a base probability space \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\), random weak Gibbs measures \( \{\mu_\omega\}_{\omega \in \Omega} \) on some class of attractors \( \{X_\omega\}_{\omega \in \Omega} \) included in \([0, 1]\) and associated with \( C^1 \) random dynamics conjugate (up to countably many points), or semi-conjugate to a random subshift of finite type. We provide a study of the multifractal nature of these measures, including the validity of the strong multifractal formalism, the calculation of...
Hausdorff and packing dimensions of the so-called level sets of divergent points, and a $0$-$\infty$ law for the Hausdorff and packing measures of the level sets of the local dimension.

In the present work, we study the multifractal nature of the discrete measures obtained as the inverse measures of the random weak Gibbs measures $\{\mu_\omega\}_{\omega \in \Omega}$, when the attractors have zero Lebesgue measure. The precise definitions of these objects and our main result, theorem 2.1, are exposed in the next section. Let us just mention that the randomness and the fact that we work on a subshift rather than a fullshift are two sources of serious complications with respect to the study achieved for inverse measures of deterministic Gibbs measures in [7]. In particular, the fundamental geometric tool provided by [4] must be revisited.

2. Setting and main result

We first need to expose basic facts from random dynamical systems and thermodynamic formalism.

2.1. Random subshift and random weak Gibbs measures. Random subshift. Denote by $\Sigma$ the symbolic space $(\mathbb{Z}_+)^\mathbb{Z}$, and endow it with the standard ultrametric distance: for any $\underline{y} = u_0u_1 \cdots$ and $\underline{y} = v_0v_1 \cdots$ in $\Sigma$, $d(u,v) = e^{-\inf\{n \in \mathbb{N}: u_n \neq v_n\}}$, with the convention $\inf(\emptyset) = +\infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\sigma$ a $\mathbb{P}$-preserving ergodic map. The product space $\Omega \times \Sigma$ is endowed with the $\sigma$-field $\mathcal{F} \otimes \mathcal{B}(\Sigma)$, where $\mathcal{B}(\Sigma)$ stands for the Borel $\sigma$-field of $\Sigma$.

Let $l$ be a $\mathbb{Z}_+^\times$ valued random variable such that $\int \log(l) \, d\mathbb{P} < \infty$ and $\mathbb{P}([\omega \in \Omega : l(\omega) \geq 2]) > 0$. Let $A = \{A(\omega) = (A_{r,s}(\omega)) : \omega \in \Omega\}$ be a random transition matrix such that $A(\omega)$ is a $l(\omega) \times l(\sigma \omega)$-matrix with entries 0 or 1. We suppose that the map $\omega \mapsto A_{r,s}(\omega)$ is measurable for all $(r,s) \in \mathbb{Z}_+^\times \times \mathbb{Z}_+$ and each $A(\omega)$ has at least one non-zero entry in each row and each column. Let $\Sigma_\omega = \{\underline{y} = v_0v_1 \cdots : 1 \leq v_k \leq l(\sigma^k(\omega)) \text{ and } A_{v_k,v_{k+1}}(\sigma^k(\omega)) = 1 \text{ for all } k \in \mathbb{N}\}$, and $F_\omega : \Sigma_\omega \to \Sigma_{\sigma \omega}$ be the left shift $(F_\omega \underline{y})_i = v_{i+1}$ for any $\underline{y} = v_0v_1 \cdots \in \Sigma_\omega$. Define $\Sigma_0 = \{(\omega, \underline{y}) : \omega \in \Omega, \underline{y} \in \Sigma_\omega\}$ which is endowed with the $\sigma$-field obtained as the trace of $\mathcal{F} \otimes \mathcal{B}(\Sigma)$. Define the map $F : \Sigma_0 \to \Sigma_0$ as $F((\omega, \underline{y})) = (\sigma \omega, F_\omega \underline{y})$. The corresponding family $F = \{F_\omega : \omega \in \Omega\}$ is called a random subshift. We assume that this random subshift is topologically mixing, i.e. there exists a $\mathbb{Z}_+^\times$-valued r.v. $M$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-almost every (a.e.) $\omega$, $A(\omega)A(\sigma(\omega)) \cdots A(\sigma^{M(\omega)-1}(\omega))$ is positive.

For each $n \geq 1$, define $\Sigma_{\omega,n}$ as the set of words $v = v_0v_1 \cdots v_{n-1}$ of length $n$, i.e. such that $1 \leq v_k \leq l(\sigma^k(\omega))$ for all $0 \leq k \leq n - 1$ and $A_{v_k,v_{k+1}}(\sigma^k(\omega)) = 1$ for all $0 \leq k \leq n - 2$. Define $\Sigma_{\omega,s} = \bigcup_{n \in \mathbb{N}} \Sigma_{\omega,n}$. For $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, we write $|v|$ for the length of $n$, and we define the cylinder $[v]_\omega$ as $[v]_\omega = \{w \in \Sigma_\omega : w_i = v_i \text{ for } i = 0, \ldots, n-1\}$.

For any $s \in \Sigma_{\omega,1}$, $p \geq M(\omega)$ and $s' \in \Sigma_{\sigma^p+1,\omega,1}$, there is at least one word $v(s,s') \in \Sigma_{\sigma^p,s',\omega,p+1}$ such that $sv(s,s')s' \in \Sigma_{\sigma^p,s',\omega,p+1}$. We fix such a $v(s,s')$ and denote the word $sv(s,s')s'$ by $s \ast s'$. Similarly, for any $w = w_0w_1 \cdots w_{n-1} \in \Sigma_{\omega,n}$ and $w' = w_0'w_1' \cdots w'_m-1 \in \Sigma_{\sigma^{n-1}(\omega),\omega,m}$ with $n, p, m \in \mathbb{N}$ and $p \geq M(\sigma^{n-1}(\omega))$, we fix $v(w_{n-1},w'_0) \in \Sigma_{\sigma^n\omega,p}$ (a word depending on $w_{n-1}$ and $w'_0$ only) so that $w \ast w' := w_0w_1 \cdots w_{n-1}v(w_{n-1},w'_0)w'_1w'_2 \cdots w'_{m-1} \in \Sigma_{\omega,n+m+p-1}$.

Random weak Gibbs measures. We say that a measurable function $\Phi$ on $\Sigma_0$ is in $L^1_{\Sigma_0}(\Omega, C(\omega))$ if
Also define \( \omega \in [13, 18] \).

**Proposition 1.**

Removing from \( \Omega \) a set of \( \mathbb{P} \)-probability 0 if necessary, for all \( \omega \in \Omega \) there exists \( \lambda(\omega) = \lambda^\Phi(\omega) > 0 \) and a probability measure \( \hat{\nu}_\omega = \hat{\nu}_\omega^\Phi \) on \( \Sigma_\omega \) such that \((\mathcal{L}_\Phi^\sigma)^* \hat{\nu}_\sigma = \lambda(\omega) \hat{\nu}_\omega \).

**Proposition 2.**

For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \),

\[
\lim_{n \to \infty} \frac{\log(\prod_{i=0}^{n-1} \lambda^\Phi(\sigma^i \omega))}{n} = P(\Phi).
\]

We call the family \( \{\hat{\nu}_\omega^\Phi : \omega \in \Omega\} \) a random weak Gibbs measure on \( \{\Sigma_\omega : \omega \in \Omega\} \) associated with \( \Phi \).

### 2.2. A model of random dynamical attractor

We present the model of random dynamical attractor in the real line defined and illustrated in [22]. It is more general than examples considered until now in the literature dedicated to multifractal analysis of random Gibbs measures on \( \mathbb{R} \) [14, 18, 10].

For any \( \omega \in \Omega \), let \( U_1^\omega = [a_{\omega,1}, b_{\omega,1}], U_2^\omega = [a_{\omega,2}, b_{\omega,2}], \ldots \) be closed non trivial intervals with disjoint interiors and \( b_{\omega,s} \leq a_{\omega,s+1} \). We assume that for each \( s \geq 1 \), \( \omega \mapsto (a_{\omega,s}, b_{\omega,s}) \) is measurable, as well as \( a_{\omega,1} \geq 0 \) and \( b_{\omega,1}(\omega) \leq 1 \). Let \( f_\omega^s(x) = \frac{x-a_{\omega,s}}{b_{\omega,s}-a_{\omega,s}} \) and consider a measurable mapping \( \omega \mapsto T^s_\omega \) from \( (\Omega, \mathcal{F}) \) to the space of \( C^1 \) diffeomorphisms of \([0,1]\) endowed with its Borel \( \sigma \)-field. We consider the measurable \( C^1 \) diffeomorphism \( T^s_\omega : U^s_\omega \to [0,1] \) by \( T^s_\omega = T^s_\omega \circ f_\omega^s \). We denote the inverse of \( T^s_\omega \) by \( g_\omega^s \). We also define

\[
U^s_\omega = g_\omega^0 \circ g_\omega^{n-1} \circ \cdots \circ g_\omega^1([0,1]), \forall v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n},
\]

\[
X_\omega = \bigcup_{n \geq 1} \bigcup_{v \in \Sigma_{\omega,n}} U^v_\omega,
\]

\[
X_\Omega = \{(\omega, x) : \omega \in \Omega, x \in X_\omega\},
\]

and for all \( \omega \in \Omega \), \( s \geq 1 \) and \( x \in U^s_\omega \),

\[
\psi(\omega, s, x) = -\log |(T^n_\omega)'(x)|.
\]
We say that a measurable function $\tilde{\psi}$ defined on $\tilde{U}_\Omega = \{(\omega,s,x) : \omega \in \Omega, 1 \leq s \leq l(\omega), x \in U_\omega^s\}$ is in $L_{X_\Omega}^1((\Omega, \tilde{C}([0,1]))$ if

1. $\int_\Omega \|\tilde{\psi}(\omega)\|_\infty d\mathbb{P}(\omega) < \infty$, where $\|\tilde{\psi}(\omega)\|_\infty := \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_\omega^s} |\tilde{\psi}(\omega, s, x)|$,
2. for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\var(\tilde{\psi}, \omega, \varepsilon) \to 0$ as $\varepsilon \to 0$, where

$$\var(\tilde{\psi}, \omega, \varepsilon) = \sup_{1 \leq s \leq l(\omega)} \sup_{x,y \in U_\omega^s} |\tilde{\psi}(\omega, s, x) - \tilde{\psi}(\omega, s, y)|.$$

We will make the following assumption:

$\psi \in L_{X_\Omega}^1((\Omega, \tilde{C}([0,1]))$ and $\psi$ satisfies the contraction property in the mean

$$c_\psi := -\int_\Omega \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_\omega^s} \psi(\omega, s, x) d\mathbb{P}(\omega) > 0.$$ 

Under this assumption, there is $\mathbb{P}$-almost surely a natural projection $\pi_\psi : \Omega \to \pi_\psi(\Omega)$ defined as

$$\pi_\psi(\omega) = \lim_{n \to \infty} g^{n_\omega}_0 \circ g^{n_\omega}_1 \circ \cdots \circ g^{n_\omega}_{n-1}(0).$$

This mapping may not be injective, but any $x \in X_\omega$ has at most two preimages in $\pi_\psi(\Omega)$. Let

$$\Psi(\omega, v) = \psi(\omega, v_0, \pi(\omega))$$

for $v = v_0 v_1 \cdots \in \pi_\psi(\Omega)$.

By construction we know $\Psi \in L_{X_\Omega}^1((\Omega, \tilde{C}(\Sigma)))$. Using a standard approach, it can be easily proven that for $\mathbb{P}$-a.e. $\omega \in \Omega$, the Bowen-Ruelle formula holds, i.e. $\dim_H X_\omega = t_0$ where $t_0$ is the unique root of the equation $P(t\Psi) = 0$.

### 2.3. Main result.

Let $\phi \in L_{X_\Omega}^1((\Omega, \tilde{C}([0,1]))$ and consider the function

$$\Phi(\omega, v) = \phi(\omega, v_0, \pi(\omega)) \quad (v = v_0 v_1 \cdots \in \pi_\psi(\Omega)).$$

We have $\Phi \in L_{X_\Omega}^1((\Omega, \tilde{C}(\Sigma))$. Let $\mu$ be the random weak Gibbs measure on $\{X_\omega : \omega \in \Omega\}$ obtained as $\mu_\omega = \pi_\omega \cdot \tilde{\mu}_\omega := \pi^{-1}_\omega \cdot \tilde{\mu}_\omega$, where $\tilde{\mu}$ is obtained from proposition 1 with respect to $\Phi$. Without changing the random measures $\tilde{\mu}_\omega$ and $\mu_\omega$, we can assume $P(\Phi) = 0$.

Then, due to equation (4), for any $q \in \mathbb{R}$, there exists a unique $T(q) \in \mathbb{R}$ such that $P(q\Phi - T(q)\Psi) = 0$, and the mapping $T$ is concave and increasing. In [22], we showed that with $\mathbb{P}$-probability 1, the strong multifractal formalism holds for $\mu_\omega$ with $\tau_{\mu_\omega} = T$.

Now we assume

$$c_\phi := -\int_\Omega \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_\omega^s} (\phi(\omega, s, x)) d\mathbb{P}(\omega) > 0,$$

and

$$\mathbb{P}(\{\omega \in \Omega : \text{The Lebesgue measure of } X_\omega \text{ is equal to } 0\}) = 1.$$ 

The first property ensures that for any $q \in \mathbb{R}$, there exists a unique $T(q) \in \mathbb{R}$ such that $P(q\Phi - T(q)\Psi) = 0$; moreover, the mapping $T$ is concave and increasing. The second property is equivalent to requiring that the inverse measure of $\mu_\omega$ is discrete. We notice that while it is not hard to construct examples with $\dim_H X_\omega < 1$ (which implies (6)), no example with $\dim_H X_\omega = 1$ and $\text{Leb}(X_\omega) = 0$ is known.

**Theorem 2.1.** For $\mathbb{P}$-a.e. $\omega \in \Omega$, let $\nu_\omega$ be the inverse measure of $\mu_\omega$. We have the following properties:

1. The multifractal formalism holds for $\nu_\omega$, with $\tau_{\nu_\omega} = \min(T, 0)$.  

Remark 1. (1) The flavor of theorem 2.1(i) is similar to that of the result obtained in [7] for the inverse of Gibbs measures on cookie-cutter sets: for the level sets of the lower local dimension, the Hausdorff spectrum is composed of two parts: a linear part with slope \( \dim_H X_\omega \), which is established thanks to conditioned ubiquity theory, and a concave part which mainly reflects the multifractal structure of weak Gibbs measures or, equivalently, ratios of Birkhoff averages. The properties stated in Theorem 2.1 are not considered in [7]. Also, in [7] the level set \( E(\nu_\omega, T'(\infty)) \), corresponding to the maximal lower local dimension, is not treated when \( T'(\infty) = 0 \).

(2) Although the main lines of the proof of theorem 2.1(i) are similar to those used to treat the case of deterministic Gibbs measures [7], the study of \( \nu_\omega \) requires the tools developed in [22] to study the multifractal nature of random weak Gibbs measures. Also, it is made structurally more complex because the weak Gibbs measures are constructed on a random subshift; this is reflected in the expression of \( \nu_\omega \) as a weighted sum of Dirac masses (see propositions 4 and 5). Moreover, we need to establish a version of the heterogeneous ubiquity theorem of [4] adapted to our more general context. Indeed, until now the sufficient conditions required to directly apply the main result of [4] have been checked to hold in connection with a random Gibbs measure only in the random fullshift case and when \( \mathbb{P} \) is a product measure [3], and our investigations lead us to conclude that such conditions can be verified in our more general setting only if the dynamical system \( (\Omega, \sigma, \mathbb{P}) \) possesses rapid decay of correlations.

(3) It is easy to see that there is a strong relationship between \( T \) and \( T' \): a short calculation yields \( T^*(d) = dT^*(1/d) \) for all \( d \) in the support of \( T^* \).

The rest of the paper is organized as follows. Section 3 provides an explicit writing of the measure \( \nu_\omega \) and some useful estimate of the mass of its atoms. In section 4, we start the multifractal analysis of \( \nu_\omega \) by examining the possible scenarios which lead to a given lower local dimension. This yields a first, not everywhere sharp, but very useful for the sequel, upper bound for the lower Hausdorff spectrum. Indeed, it is already related to conditioned ubiquity properties associated with the sets of atoms, and thus it provides a beginning of concrete explanation of the origin of the linear part in the lower Hausdorff spectrum. In section 5, we derive the sharp upper bound for the \( L^q \)-spectrum of \( \nu_\omega \), in which ubiquity properties remain hidden. Section 6 introduces basic properties related to the approximation of \( (\Phi, \Psi) \) by Hölder continuous random potentials, and used in next sections. Sections 7 to 9 obtain the sharp lower bound for the lower Hausdorff spectrum. This, combined with the result of section 5 gives the equality \( \dim_H E(\mu, d) = \tau_{\nu_\omega}(d) \) for all \( d \in \mathbb{R} \), hence the validity of the multifractal formalism, as well as the equality \( \tau_{\nu_\omega} = \min(\mathcal{T}, 0) \) by the duality property of Legendre transforms of concave functions. Specifically, section 7 derives the sharp lower bound for the lower Hausdorff spectrum in its non linear part, section 8 provides the conditioned ubiquity theorem used in section 9 to get the sharp lower bound for the lower Hausdorff spectrum in the linear part. Finally, section 10 deals with the Hausdorff dimension of the level sets \( E(\nu_\omega, d) \) and \( \overline{E}(\nu_\omega, d) \).
3. Writing of \( \nu_\omega \) as a sum of Dirac masses and estimates for the point masses

We begin with a useful proposition established in [22], which provides estimates of the \( \mu_\omega \) mass and the diameter of any set of the form \( X^v_\omega \):

**Proposition 3** ([22], Proposition 3). For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), there are non increasing sequences \( (\epsilon(\psi, \omega, n))_{n \geq 0} \) and \( (\epsilon(\phi, \omega, n))_{n \geq 0} \), that we also denote as \( (\epsilon(\Psi, \omega, n))_{n \geq 0} \) and \( (\epsilon(\Phi, \omega, n))_{n \geq 0} \), converging to 0 as \( n \to +\infty \), such that for all \( n \in \mathbb{N} \), for all \( v = v_0 v_1 \ldots v_n \in \Sigma_{\omega,n} \), we have (the diameter of a set \( E \) is denoted by \( |E| \)):

(1) For all \( z \in \bar{U}^v_\omega \),

\[
\exp(S_n \psi(\omega, z) - ne(\psi, \omega, n)) \leq |U^v_\omega| \leq \exp(S_n \psi(\omega, z) + ne(\psi, \omega, n)),
\]

hence for all \( \varphi \in [v]_\omega \),

\[
\exp(S_n \Psi(\omega, \varphi) - n\epsilon(\Psi, \omega, n)) \leq |U^v_\omega| \leq \exp(S_n \Psi(\omega, \varphi) + n\epsilon(\Psi, \omega, n)).
\]

Consequently, for all \( \varphi \in X^v_\omega \),

\[
|X^v_\omega| \leq |U^v_\omega| \leq \exp(S_n \Psi(\omega, \varphi) + n\epsilon(\Psi, \omega, n)).
\]

(2) For any \( \Upsilon \in \mathcal{L}^1_\mathbb{P}(\Omega, C(\Sigma)) \), for any \( v \in \Sigma_{\omega,n} \) with \( n \in \mathbb{N} \), we have

\[
\exp(-n\epsilon(\Upsilon, \omega, n)) \leq \frac{\tilde{\mu}_\Upsilon([v]_\omega)}{\exp(S_n \Phi(\omega, \varphi) - nP(\Upsilon))} \leq \exp(n\epsilon(\Upsilon, \omega, n)),
\]

for all \( \varphi \in [v]_\omega \), where \( P(\Upsilon) \) denote the topological pressure for \( \Upsilon \).

Noticing \( P(\Phi) = 0 \), we get

\[
\exp(S_n \Phi(\omega, \varphi) - n\epsilon(\Phi, \omega, n)) \leq \tilde{\mu}_\omega([v]_\omega) \leq \exp(S_n \Phi(\omega, \varphi) + n\epsilon(\Phi, \omega, n)),
\]

hence for all \( z \in \bar{U}^v_\omega \),

\[
\exp(S_n \phi(\omega, z) - n\epsilon(\phi, \omega, n)) \leq \mu_\omega(X^v_\omega) \leq \mu_\omega(U^v_\omega),
\]

as well as \( \mu_\omega(U^v_\omega) \leq \exp(S_n \phi(\omega, z) + n\epsilon(\phi, \omega, n)) \) if \( \tilde{\mu}_\omega \) is atomless.

Proposition 3 (ii) and assumption (5) imply that \( \tilde{\mu}_\omega \) is atomless \( \mathbb{P} \)-almost surely. Without loss of generality, we assume that this is the case for all \( \omega \in \Omega \) and the sequences \( (n\epsilon(\Psi, \omega, n))_{n \geq 0} \) and \( (n\epsilon(\Phi, \omega, n))_{n \geq 0} \) are increasing as \( n \) increasing to \( \infty \). Furthermore, we can also ask \( \sum_{n=1}^\infty \text{var}_{n-1} \Upsilon(\sigma^n \omega) \leq \epsilon(\Phi, \omega, n) \) for any \( \Upsilon \in \mathcal{L}_\mathbb{P}(\Omega, C(\Sigma)) \).

Next we introduce a few notations required to explicitly write \( \nu_\omega \) as a sum of weighted Dirac measures.

For \( \omega \in \Omega, n \geq 1, v \in \Sigma_{\omega,n} \) and \( k \geq 1 \) we define

\[
S(\omega, v, k) = \{ w \in \Sigma_{\sigma^n \omega,k} : vw \in \Sigma_{n+k}(\omega) \},
\]

the set of words in \( \Sigma_{\sigma^n \omega,k} \) which can be a suffix of \( v \). Next we consider the set of words \( w \) in \( S(\omega, v, k) \) such that \( U^{vw} \) has a right neighboring interval \( U^{\bar{w}v} \), with \( \bar{w} \in S(\omega, v, k) \):

\[
S'(\omega, v, k) = \left\{ w \in S(\omega, v, k) \mid \text{there exists } \bar{w} \in S(\omega, v, k) \text{(necessarily unique)} \text{ such that } U^{\bar{w}v}_{\omega} \text{ is the nearest right neighboring interval of } U^{\bar{w}v}_x \right\}.
\]

**Remark 2.** Notice that \( S'(\omega, v, k) \) may be empty, while this is not possible in the deterministic case, i.e. when the attractor is a cookie cutter set.
Lemma 3.1. \(\{\omega \in \Omega, \nu_\omega \vert_{S} \}\) is 1 and \(\nu_\omega(S) \geq 1\) with probability 1.

For every \(v \in \Sigma_{\omega,s}, k \geq 1\) and \(w \in S(\omega, v, k)\), define
\[
m_{\omega}^{vw} = \min X_{\omega}^{vw} \quad \text{and} \quad M_{\omega}^{vw} = \max X_{\omega}^{vw},
\]
For any \(v \in \Sigma_{\omega,s}\), we define
\[
I_{\omega}^{v} := [F_{\mu_\omega}(m_{\omega}^{v})], F_{\mu_\omega}(M_{\omega}^{v})] = F_{\mu_\omega}(X_{\omega}^{v}) \setminus \{F_{\mu_\omega}(M_{\omega}^{v})\}.
\]
Since the support of \(\mu_\omega\) restricted to the interval \([m_{\omega}^{v}, M_{\omega}^{v}]\) (or \(U_{\omega}^{v}\)) is \(X_{\omega}^{v}\), by construction \(I_{\omega}^{v}\) is a non-empty interval of length \(|I_{\omega}^{v}| = \mu_\omega(X_{\omega}^{v}) = \mu_\omega([v])\).

Also, since \(\supp(\mu_\omega) = X_{\omega}\) and \(\bigcup_{v \in \Sigma_{\omega,s}} X_{\omega}^{v} = X_{\omega}\), the families of intervals \(F_{\omega}^{v} = \{I_{\omega}^{v}\}_{v \in \Sigma_{\omega,s}}, \ n \geq 1\), form a sequence of refined partitions of \([0, 1]\) into intervals.

For any \(v \in \Sigma_{\omega,s}\) and \(s \in S'(\omega, v, 1)\), we define
\[
x_{\omega}^{vs} = F_{\mu_\omega}(M_{\omega}^{vs}).
\]
We also define \(m_{\omega}^{\min} = \min X_{\omega}\) and \(M_{\omega}^{\max} = X_{\omega}\).

Notice that by construction we have
\[
\nu_\omega(I_{\omega}^{v}) \leq |X_{\omega}^{v}| \leq |U_{\omega}^{v}|
\]
for any \(v \in [v]\) and \(v \in \Sigma_{\omega,s}\).

We can now give the following explicit form for the inverse of the random weak Gibbs measures \(\{\mu_\omega : \omega \in \Omega\}\).

**Proposition 4** (The inverse measure \(\nu_\omega\) of \(\mu_\omega\)). The inverse measure \(\nu_\omega\) of the random weak Gibbs measure \(\mu_\omega\) is the discrete probability measure on \([0, 1]\) given by the following weighted sum of Dirac measures:

\[
\nu_\omega = m_{\omega}^{\min} \cdot \delta_0 + \sum_{v \in \Sigma_{\omega,s}} \sum_{s \in S'(\omega, v, 1)} (m_{\omega}^{vs} - M_{\omega}^{vs}) \cdot \delta_{v,s} + (1 - M_{\omega}^{\max}) \delta_1.
\]

This proposition can be easily proved if we notice the following two facts. On the one hand, from the definition we can get that at each point \(x_{\omega}^{vs}\), the point mass is at least \(m_{\omega}^{vs} - M_{\omega}^{vs}\). On the other hand, we know the total mass of \(\nu_\omega\) is 1 and \(m_{\omega}^{\min} + \sum_{v \in \Sigma_{\omega,s}} \sum_{s \in S'(\omega, v, 1)} (m_{\omega}^{vs} - M_{\omega}^{vs}) + (1 - M_{\omega}^{\max}) = 1\) from equation (6).

**Remark 3.** Notice that even if \(S'(\omega, v, 1) \neq \emptyset\), the weight \(m_{\omega}^{vs} - M_{\omega}^{vs}\) may vanish if there is no gap between \(X_{\omega}^{vs}\) and \(X_{\omega}^{vs}\). For instance, it is not difficult to see that in the full shift case this situation occurs with probability 1, infinitely many times, if and only if with probability 1 we have \(\ell(\omega) \geq 3\), \(\min(U_{\omega}^{i}) = 0\), \(\max(U_{\omega}^{l(\omega)}) = 1\), and \(\{1 \leq i \leq \ell(\omega) - 1 : U_{\omega}^{i} \text{ and } U_{\omega}^{i+1} \text{ are contiguous}\} \neq \emptyset\).

We end this section with a non trivial lower bound estimate for some point masses associated with \(\nu_\omega\) (Proposition 5). For every \(k \geq 1\) define
\[
\text{gap}(\omega, k) = \inf_{v \in \Sigma_{\omega,1}} \sup_{1 \leq m \leq k} \sup_{w \in S'(\omega, v, m)} \{m_{\omega}^{vw} - M_{\omega}^{vw}\}.
\]

**Lemma 3.1.** We have
\[
\mathbb{P}(\{\omega \in \Omega : \text{gap}(\omega, k) > 0\}) > 0.
\]
Consequently, setting \(\text{Gap}(k, \gamma) = \{\omega \in \Omega : \text{gap}(\omega, k) > \gamma\}\), there exist some \(k_\gamma > 0\) and \(\gamma_\psi > 0\) such that \(\mathbb{P}(\text{Gap}(k_\gamma, \gamma_\psi)) > 0\).
Remark 4. We notice that property (6) is not necessary to get lemma 3.1. We only need that $X_\omega$ differs from $[0,1]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. Suppose, by contradiction, that the result does not hold. Then for $\mathbb{P}$-a.e. $\omega$, there exists $v \in \Sigma_{\omega,1}$ such that

$$\sup_{m \in \mathbb{N}} \sup_{w \in S'(\omega,v,m)} m^-_\omega - M^-_\omega = 0.$$ 

This implies that $X_\omega$ has no gap. Then $X_\omega$ is either a point or an interval. Since $X_\omega$ has a Lebesgue measure 0, we get that it is a point.

Now, defining

$$B = \{ \omega \in \Omega : M(\omega) \leq M', \ l(\omega) \geq 2 \},$$

we have $\mathbb{P}(B) > 0$ for $M'$ large enough. For any $\omega \in \Omega$, define $b_k(\omega)$ the $k$-th return time of $\omega$ to the set $B$ by the map $\sigma$. From ergodic theorem we have $\lim_{k \to \infty} b_k(\omega) = \frac{1}{\mathbb{P}(B)}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Define $\Omega' = \{ \omega \in \Omega : \lim_{k \to \infty} b_k(\omega) = \frac{1}{\mathbb{P}(B)} \}$.

For any $\omega \in \Omega'$, we know that there are at least four words in $\Sigma_{\omega,b_{M'+2}}$ with the prefix $v \in \Sigma_{\omega,1}$, and we denote them by $w_1, w_2, w_3$ and $w_4$. We can assume that these intervals appear from the left to the right as $U_{\omega}^{w_1}, U_{\omega}^{w_2}, U_{\omega}^{w_3}$ and $U_{\omega}^{w_4}$. The sets $X_{\omega}^{w_i} \subset U_{\omega}^{w_i}$, $i = 1, 2, 3, 4$, are not empty since by definition the random transition matrix $A$ has at least one non-zero entry in each row and each column. Choose $x_i \in X_{\omega}^{w_i} \subset U_{\omega}^{w_i}$, $i = 1, 2, 3, 4$. Since $U_{\omega}^{w_i}$, $i = 1, 2, 3, 4$ are intervals, we have that $x_4 - x_1 > 0$, which contradicts the fact that $X_\omega$ is a singleton. 

Proposition 5. For $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $n \in \mathbb{N}$, for all $v \in \Sigma_{\omega,n}$, there exists $k_v$ and $w \in S'(\omega,v,k_v)$ such that

$$m^-_\omega - M^-_\omega \geq \exp(S_n(\Psi(\omega, v)) - o(n))$$

for any $v \in [v]_{\omega}$. Here $o(n)$ is independent of $v$, and we have $k_v = o(n)$ independently on $v$ as well.

Proof. For any $N \in \mathbb{N}$, let

$$\Omega_N = \left\{ \omega : M(\omega) < N, \ \frac{1}{n} \sum_{k=0}^{n-1} \sup_{1 \leq s \leq l(\sigma^k \omega)} |\psi(\omega, s, x)| \leq 2C_\psi, \forall n \geq N \right\}.$$ 

Choose $N$ large enough so that $\mathbb{P}(\Omega_N) > 0$.

For $\mathbb{P}$-a.e. $\omega \in \Omega$, for $n$ large enough, denote by $H_1(n)$ the smallest integer such that $\sigma^{n+H_1(n)} \omega \in \Omega_N$, and $H_2(n)$ the smallest integer such that $\sigma^{n+H_1(n)+H_2(n)} \omega \in \text{Gap}(k_\psi, \gamma_\psi)$ with $H_2(n) \geq N$. Since $\mathbb{P}(\Omega_N) > 0$ and $\mathbb{P}(\text{Gap}(k_\psi, \gamma_\psi)) > 0$, from ergodic theorem we can get that $\lim_{n \to \infty} \frac{H_1(n)+H_2(n)}{n} = 0$. Moreover, since $\sigma^{n+H_1(n)+H_2(n)} \omega \in \text{Gap}(k_\psi, \gamma_\psi)$, there exists $1 \leq s \leq l(\sigma^{n+H_1(n)+H_2(n)} \omega)$ and $v' \in S'(\sigma^{n+H_1(n)+H_2(n)} \omega, s, k)$ with $k \leq k_\psi$ such that $m^-_{\sigma^{n+H_1(n)+H_2(n)} \omega} - M^-_{\sigma^{n+H_1(n)+H_2(n)} \omega} > \gamma_\psi$.

For any $v \in \Sigma_{\omega,n}$, since $M(\sigma^n \omega) \leq M(\sigma^{n+H_1(n)} \omega) + H_1(n) < H_1(n) + H_2(n)$, recalling the operation $\ast$ defined in section 2.1, there exists a word $s$ such that $v \ast s \in \Sigma_{\omega,n+H_1(n)+H_2(n)+1}$, and by construction $U_{\omega}^{v \ast s v'}$ and $U_{\omega}^{v \ast s v'}$ are contiguous intervals. We simply denote the left one by $U_{\omega}^{v w}$ and the right by $U_{\omega}^{v'}$. We have $T_{\omega}([M^-_\omega, m^-_\omega]) = \cdots$
\[ M^u v \in \left[ M_{\sigma^n + H_1(n) + H_2(n), \omega}^\tilde{\nu}, M_{\sigma^n + H_1(n) + H_2(n), \omega}^\tilde{\nu} \right]. \]

Now using Lagrange’s finite-increment theorem we can get
\[ m_w^v - M^v \geq \gamma \exp(S_n \Psi(\omega, v) - o(n)). \]

for any \( v \in [v], \) since \( H_1(n) + H_2(n) \) is a \( o(n) \). Then the result holds since \( \gamma \) is a constant. Moreover \( k_w = |w| = H_1(n) + H_2(n) + k \leq H_1(n) + H_2(n) + k \) is also \( o(n) \). □

Remark 5. Proposition 5 implies that for \( \mathbb{P}-a.e. \) \( \omega \in \Omega, \) for all \( n \in \mathbb{N}, \) for any \( v \in \Sigma_{\omega, n}, \) there exist some point \( x \) of the form \( x^u v \in \tilde{I}^v, \) with \( |w| = o(n), \) such that
\[ \nu(\omega) \{ x \} \geq \exp(S_n \Psi(\omega, v) - o(n)) \]
for any \( v \in [v]. \) For each \( v \in \Sigma_{\omega, s}, \) we fix such a point and denote it by \( z^v. \) These points will play a crucial role in proving the sharp lower bound for the lower Hausdorff spectrum of \( \nu. \)

Arguments similar to those leading to proposition 5 lead to the following remark.

Remark 6. For \( \mathbb{P}-a.e. \) \( \omega \in \Omega, \) for all \( n \in \mathbb{N} \) and \( v \in \Sigma_{\omega, n}, \) for any \( v \in [v], \)
\[ |X_v^u| \geq \exp(S_n \Psi(\omega, v) - o(n)), \]
where the \( o(n) \) does not depend on the choice of \( v. \) Consequently, we have
\[ \exp(S_n \Psi(\omega, v) - o(n)) \leq |X_v^u| \leq \exp(S_n \Psi(\omega, v) + o(n)). \]

Next section provides first information on the lower local dimension of \( \nu_\omega \) and the lower Hausdorff spectrum.

4. Pointwise behavior of \( \nu_\omega \) and an upper bound for the lower Hausdorff spectrum without using the multifractal formalism

The following definitions will be essential in making explicit the connection between the lower local dimension of \( \nu_\omega \) and the conditioned ubiquity which partly governs the multifractal structure of \( \nu_\omega. \)

Definition 4.1. For \( v \in \Sigma_{\omega, s}, \) we set
\[ \ell^v_\omega = 2|I^v_\omega| = 2\tilde{\mu}_\omega([v]), \quad \alpha^v_\omega = \frac{\tilde{\Psi}(\omega, v)}{\log |I^v_\omega|}, \]
where
\[ \tilde{\Psi}(\omega, v) = \sup_{\xi \in [v]} \{ S_{[\xi]} \Psi(\omega, v) \} \]
(we define \( \tilde{\Phi} \) similarly). We notice that due to proposition 3(2) we have
\[ \alpha^v_\omega = \frac{\tilde{\Psi}(\omega, v)}{\tilde{\Phi}(\omega, v)} + o(1), \]
where \( o(1) \) tends uniformly in \( v \) to 0 as \( |v| \) tends to \( \infty. \)

For \( x \in [0, 1] \) and \( n \geq 1, \) let \( v(\omega, n) \) be the unique element \( v \) in \( \Sigma_{\omega, n} \) such that \( x \in I^v_\omega. \)
If \( x = 1, \) \( v(\omega, n, 1) \) is the unique \( v \in \Sigma_{\omega, n} \) such that \( 1 \in I^v_\omega. \) If there is no confusion we will denote \( v(\omega, n, x) \) by \( v(n, x) \) or \( x|n \) for short.

Define \( \alpha^v_\omega(x) = \alpha^v_\omega(x|n) \) and \( \alpha_\omega(x) = \liminf_{n \to \infty} \alpha^v_\omega(x). \)
For $x \in [0, 1] \setminus \{x^v_s : v \in \Sigma_{x,s}, s \in S'(\omega, v, 1)\}$, the approximation degree $\xi^x_\omega$ by the system $\{(x^v_s, f^v_s) : v \in \Sigma_{x,s}, s \in S'(\omega, v, 1)\}$ is defined as

$$
\xi^x_\omega = \limsup_{n \to \infty} \sup_{s \in S'(\omega, x|_n, 1)} \frac{\log |x - x^v_s|}{\log \ell^v_s}.
$$

Set

$$
\Xi_\omega = \{x^v_s : v \in \Sigma_{x,s}, s \in S'((\omega, v, 1)) \}
$$

and

$$
\Xi'_\omega = \{\text{atoms of } \nu_\omega\} = \{x^v_s : v \in \Sigma_{x,s}, s \in S'(\omega, v, 1), \text{ and } m^v_s - M^v_s > 0\} \subset \Xi_\omega.
$$

**Proposition 6.**

1. If $x \in \Xi'_\omega$, then $\nu_\omega\{\{x\}\} > 0$, thus $\dim_{\text{loc}}(\nu_\omega, x) = 0$.
2. For any $x \in [0, 1]$, if $x \notin \Xi_\omega$, then

$$
\frac{\alpha_\omega(x)}{\xi^x_\omega} \leq \dim_{\text{loc}}(\nu_\omega, x) \leq \alpha_\omega(x),
$$

with the convention that if $\xi^x_\omega = +\infty$ then $\frac{\alpha_\omega(x)}{\xi^x_\omega} = 0$.

**Proof.** (i) is obvious. Let us prove (ii).

Let $x \in [0, 1] \setminus \Xi_\omega$ and $r > 0$. If $r$ is small enough, the integer

$$
n^k_\omega = \max\{n : \exists \nu \in \Sigma_{x,n} \text{ such that } B(x, r) \subset I^v_n\}
$$

is well defined, and by definition of $c_\psi$ (see (4)) we have $n^k_\omega \leq -2\log_\omega r$. Moreover, $n^k_\omega \to \infty$ as $r \to 0$. By definition of $n^k_\omega$, if we denote by $v(x, r)$ the word $v(\omega, n^k_\omega x, r)$, there exists an unique element $s$ of $S'(\omega, v(x, r), 1)$ such that

$$
x^v(x,r)_s \in B(x, r) \subset I^v(x,r).
$$

The inclusion $B(x, r) \subset I^v(x,r)$ and proposition 3(i) imply

$$
\nu_\omega(B(x, r/2)) \leq \nu_\omega(I^v(x,r) \setminus \{\max I^v_\omega, \min I^v_\omega\}) \leq |X^v(x,r)| \leq |I^v_\omega|
$$

$$
\leq \exp(S[v(x,r)]\psi(\omega, v) + |v(x,r)|\epsilon(\Phi, \omega, |v(x,r)|)) \text{ where } v \in [v(x,r)]_{\omega}.
$$

Now for any $\epsilon > 0$, by definition of $\xi^x_\omega$, for $r$ small enough we have

$$
r \geq |x - x^v(x,r)_s| \geq 2|I^v_\omega|^{\xi^x_\omega + \epsilon}.
$$

Moreover, again for $r$ small enough, we have

$$
\exp(\tilde{\psi}(\omega, v(x,r))) \leq |I^v_\omega|^{\alpha_\omega(x) - \epsilon}
$$

by definition of $\alpha_\omega(x)$. These estimates yield

$$
\nu_\omega(B(x, r/2)) \leq \exp(\tilde{\psi}(\omega, v(x,r))) + o(|v(x,r)|) \leq r^{\frac{\alpha_\omega(x) - \epsilon}{\xi^x_\omega + \epsilon}} \exp(o(n^k_\omega r)),
$$

and by letting $r$ tend to zero, since $n^k_\omega \leq -2\log_\omega r$, it follows that $\dim_{\text{loc}}(\nu_\omega, x) \geq \frac{\alpha_\omega(x) - \epsilon}{\xi^x_\omega + \epsilon}$.

From the arbitrariness of $\epsilon$ we get that $\dim_{\text{loc}}(\nu_\omega, x) \geq \frac{\alpha_\omega(x)}{\xi^x_\omega}$.

For the second inequality, let $\{p_i\}_{i \geq 1}$ be an increasing sequence of integers such that $\exp(\tilde{\psi}(\omega, x|_{p_i})) \geq |I^v_\omega|^{\alpha_\omega(x) + \epsilon}$ for all $i \geq 1$. Now recall remark 5. Since $x^v_{p_i} \in B(x, 2|I^v_\omega|)$, we have

$$
\nu_\omega(B(x, 2|I^v_\omega|)) \geq \nu_\omega\{x^v_{p_i}\} \geq \exp(S_{p_i} \psi(\omega, v)) \exp(-o(p_i)) \geq |I^v_{p_i}|^{\alpha_\omega(x) + \epsilon} \exp(-o(p_i)).
$$
Also, $|I^d_{\omega}| \leq \exp(-\frac{c_\omega p_i}{2})$ for $p_i$ large enough and $\epsilon$ is arbitrarily small. Consequently, 
$$\dimloc(\nu_{\omega}, x) \leq \alpha_{\omega}(x).$$

**Remark 7.** Arguments similar to those used to get proposition 6 show that
$$\overline{\dimloc}(\nu_{\omega}, x) \leq \limsup_{n \to \infty} \alpha^n_{\omega}(x).$$

**Definition 4.2.** Let $\alpha > 0, \xi \geq 1$ and $\varepsilon > 0$. A real number $x \in [0, 1]$ is said to satisfy the property $P(\omega, \alpha, \xi, \varepsilon)$ if there exists an increasing sequence of positive integers $(n_k)_{k \geq 1}$ such that for every $k \geq 1$, there exists $v \in \Sigma_{\omega, n_k}$ and $s \in S'(\omega, v, 1)$, such that 
$$x \in B(x^{\nu_s}_\omega, (\nu_s)^{\xi-\varepsilon})$$
and $\alpha^n_{\omega} \in [\alpha - \varepsilon, \alpha + \varepsilon]$.

We now introduce new sets.

**Definition 4.3.** For $d \geq 0$, let
$$F(\omega, d) = \left\{ x \in (0, 1) \mid \forall \varepsilon > 0, \exists \alpha \in \mathbb{Q}^+, \exists \xi \in \mathbb{Q}, \xi \geq 1 \text{ such that } \alpha/\xi \leq d + 2\varepsilon \text{ and } x \text{ satisfies the property } P(\omega, \alpha, \xi, \varepsilon) \right\}.$$

Now, the following proposition explores the relationship between the level sets $E(\nu_{\omega}, d)$ and the sets $F(\omega, d)$.

**Proposition 7.** For $\mathbb{P}$-a.e. $\omega$, for any $d \geq 0$, we have $(E(\nu_{\omega}, d) \setminus \Xi_d) \subset F(\omega, d)$.

**Proof.** Fix $d \geq 0$, $x \in E(\nu_{\omega}, d)$ and $\varepsilon > 0$. By definition of $\dimloc(\nu_{\omega}, x)$, there exists a sequence $(r_k)_{k \geq 1}$ of positive numbers decreasing to zero such that for all $k \geq 1$ we have 
$$\nu_{\omega}(B(x, r_k/2)) \geq (r_k/2)^{d+\varepsilon}.$$ Now recall the definition of $n^{x,r}_\omega$ given in (9). If $x \notin \{x^{\nu_s}_\omega : v \in \Sigma_{\omega,s}, s \in S'(\omega, v, 1)\}$, then $n^{x,r}_\omega \to \infty$ as $r \to 0$. Since $n^{x,r}_\omega$ is maximal, there exist $v = v(x, r)$ and $s \in S'(\omega, v(x, r), 1)$ such that 
$$x^{(x,r)s}_\omega \in B(x, r) \subset I^{(x,r)}_\omega.$$ Then 
$$\nu_{\omega}(B(x, r/2)) \leq \nu_{\omega}(I^{(x,r)}_\omega) \leq \max I^{(x,r)}_\omega, \min I^{(x,r)}_\omega) \leq |U^{(x,r)}_\omega|,$$ so
$$|r_k/2|^{d+\varepsilon} \leq \nu_{\omega}(B(x, r_k/2)) \leq \exp(\tilde{\Psi}(\omega, v(x, r_k)) + o(|v(x, r_k)|)),$$ where in the last inequality we used proposition 3. Consequently, due to the definition of $\alpha^n_{\omega}$ and proposition 3 again, we have
$$|I^{(x,r)}_\omega| \alpha^{x^{(x,r)s}_\omega} \omega(1) = \exp(\tilde{\Psi}(\omega, v(x, r_k))) \geq (r_k/2)^{d+\varepsilon}.$$ Since 
$$x^{(x,r)s}_\omega \in B(x, r), |x - x^{(x,r)s}_\omega| \leq r_k.$$ Writing 
$$r_k \geq |x - x^{(x,r)s}_\omega| = (2(|I^{(x,r)}_\omega|))^{\xi_k} \geq 2(|I^{(x,r)}_\omega|)^{\xi_k},$$
we get
$$|I^{(x,r)}_\omega| \alpha^{x^{(x,r)s}_\omega} \omega(1) \geq (|I^{(x,r)}_\omega|)^{\xi_k(d+\varepsilon)},$$ and $\xi_k \geq 1$.

If $\limsup_{k \to \infty} \xi_k < \infty$, there exists $(\alpha, \xi) \in \mathbb{Q}^+ \times (\mathbb{Q} \cap [1, +\infty))$ and an increasing sequence of integers $(k_s)_{s \geq 1}$ such that 
$$|\alpha^{x^{(x,r_k)s}_\omega} \omega - \alpha| \leq \varepsilon, |\xi_{k_s} - \xi| \leq \varepsilon, \text{ and } \alpha/\xi \leq d + 2\varepsilon.$$ This means $x \in G(\omega, \alpha, \xi, \varepsilon)$, $(\alpha, \xi) \in \mathbb{Q}^+ \times (\mathbb{Q} \cap [1, +\infty))$ and $\alpha/\xi \leq h + 2\varepsilon$.

If $\limsup_{k \to \infty} \xi_k = \infty$, there exists $\alpha \in \mathbb{Q}^+$ and an increasing sequence of integer number $(k_s)_{s \geq 1}$ such that 
$$|\alpha^{x^{(x,r_k)s}_\omega} \omega - \alpha| \leq \varepsilon \text{ and } \xi_{k_s} \to \infty.$$
Since \( \omega^{(0,\infty)} \) is bounded (for \( \mathbb{P} \)-a.e. \( \omega \)), there exists some \( \xi \in \mathbb{Q} \cap [1, +\infty) \) with \( \alpha/\xi \leq d+2\varepsilon \) such that \( x \) satisfies \( P(\alpha, \varepsilon, \xi) \) (because if \( \xi_1 \leq \xi_2 \) then \( P(\alpha, \varepsilon, \xi_2) \) implies \( P(\alpha, \varepsilon, \xi_1) \)).

Finally,
\[
(10) \quad (E(\nu, d) \setminus \Xi_\omega) \subset F(\omega, d).
\]

**Definition 4.4.** For every \( \alpha, \varepsilon > 0 \) and \( \xi \geq 1 \), let
\[
G(\omega, \alpha, \varepsilon, \xi) = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{v \in \Sigma_{\omega,n}, s \in S'(\omega, v, 1) : \alpha \varepsilon \in [\alpha-\varepsilon, \alpha+\varepsilon]} B(x_n^v, (\ell_n^v)^\xi).
\]

It is easily seen that
\[
F(\omega, d) \subset \bigcup_{\alpha \in \mathbb{Q}^+} \bigcup_{\xi \in \mathbb{Q} \cap [1, +\infty), \alpha/\xi \leq d+2\varepsilon} G(\omega, \alpha, \varepsilon, \xi).
\]

**Lemma 4.5.** There exists \( C > 0 \) such that for \( \mathbb{P} \)-a.e. \( \omega \), for \( \varepsilon > 0 \) small enough, for all rationals \( \alpha > 0 \) and \( \xi \geq 1 \),
\[
\dim_H G(\omega, \alpha, \varepsilon, \xi) \leq C\varepsilon + \frac{\max(T^*(\alpha-\varepsilon), T^*(\alpha), T^*(\alpha+\varepsilon))}{\xi}.
\]

**Proof.** It is enough to prove the result for fixed \( \varepsilon > 0 \) and rational numbers \( \alpha > 0 \) and \( \xi \geq 1 \). For any \( N \geq 1 \), let \( \delta_N = \sup_{v \in \Sigma_{\omega,N}} \ell_n^v \). By construction, if \( G(\omega, \alpha, \varepsilon, \xi) \neq \emptyset \), given \( s \in \mathbb{R} \) we have
\[
H^s_{\delta_N}(G(\omega, \alpha, \varepsilon, \xi)) \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n} : \alpha-\varepsilon \leq \alpha \varepsilon \leq \alpha+\varepsilon} l(\sigma^n\omega)2^s(\ell_n^v)^\xi
\]
where we naturally extend the definition of \( H^s_{\delta} \) to negative \( s \).

Here, to avoid confusions, we recall that \( l(\omega) \) is the number of types in the subshift, and \( \ell_n^v \) is the length of the interval \( I_n^v \).

**Case 1:** \( \alpha \leq T'(0-) - \varepsilon \). Since \( \alpha_n^v = \tilde{\Psi}(\omega, v)/\log |I_n^v| \), then for any \( q \geq 0 \) one has:
\[
H^s_{\delta_N}(G(\omega, \alpha, \varepsilon, \xi)) \leq 4^s \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n} : \alpha-\varepsilon \leq \alpha \varepsilon \leq \alpha+\varepsilon} l(\sigma^n\omega)(\ell_n^v)^\xi
\]
\[
\leq 4^s \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} l(\sigma^n\omega) \exp(q \Psi(\omega, v) - q(\alpha+\varepsilon) - q(\alpha-\varepsilon) - q(\alpha+\varepsilon) - q(\alpha-\varepsilon)) \cdot \exp(o(n))
\]
Now take \( s = (\eta + T^*(\alpha+\varepsilon))/\xi \) with \( \eta > 0 \), we can get
\[
H^s_{\delta_N}(G(\omega, \alpha, \varepsilon, \xi)) \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} \exp(q \Psi(\omega, v) - (q(\alpha+\varepsilon) - q(\alpha-\varepsilon) - q(\alpha+\varepsilon) - q(\alpha-\varepsilon)) \cdot \exp(o(n))
\]
Since \( \alpha \leq T'(0-) - \varepsilon \), that is \( \alpha+\varepsilon \leq T'(0-) \), there exists \( q \geq 0 \) such that
\[
T^*(\alpha+\varepsilon) = (\alpha+\varepsilon)q - T(q) - \gamma_q,
\]
with \( 0 \leq \gamma_q \leq \frac{q}{2} \).

Then,
\[
H^s_{\delta_N}(G(\omega, \alpha, \varepsilon, \xi)) \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} \exp(q \Psi(\omega, v) - (T(q) - \eta/2) \Phi(\omega, v)) \cdot \exp(o(n)),
\]
where $g$ is any element of $[v]_\omega$. Then
\[ \mathcal{H}^s_{\delta_N}(G(\omega, \alpha, \varepsilon, \xi)) \leq \sum_{n \geq N} \sum_{v \in \Sigma_n} \mu^\Phi_{\omega} T(\Phi([v]_\omega)) \exp(-\eta c_\Phi n/2 + o(n)) \leq \sum_{n \geq N} \exp(-\eta c_\Phi n/4) \]
for $n$ large enough (recall $c_\Phi = c_\Phi > 0$). Consequently, $\lim_{N \to \infty} \mathcal{H}^s_{\delta_N} G(\omega, \alpha, \varepsilon, \xi) = 0$. However, if $T^*(\alpha + \varepsilon) < 0$, we can choose $\eta$ and $q$ such that $s < 0$, in which case it is necessary that $\lim_{N \to \infty} \mathcal{H}^s_{\delta_N} G(\omega, \alpha, \varepsilon, \xi)$ is not empty. Consequently, if $T^*(\alpha + \varepsilon) < 0$, then $G(\omega, \alpha, \varepsilon, \xi) = \emptyset$. Otherwise, $\dim_H G(\omega, \alpha, \varepsilon, \xi) \leq (\eta + T^*(\alpha + \varepsilon))/\xi$. This holds for all $\eta > 0$, so $\dim_H G(\omega, \alpha, \varepsilon, \xi) \leq T^*(\alpha + \varepsilon)$.

Case 2: $\alpha \geq T'(0+) + \varepsilon$. It is almost the same as before except that one needs to use $q \leq 0$ and $q \tilde{\Psi}(\omega, v) \geq q(\alpha - \varepsilon) \log |v|_\omega$.

Case 3: $\alpha \in (T'(0-) - \varepsilon, T'(0+) + \varepsilon)$. Two situations must be considered. If $T'(0-) - T'(0+) > 0$, we can assume $\varepsilon < T'(0-) - T'(0+)$, then $T'(0-) - \varepsilon > T'(0+) + \varepsilon$, so that Case 3 is empty. If $T'(0-) = T'(0+)$, then $T$ is differentiable at 0. Take $s = \frac{T'(0)}{\xi}$ with $\eta > 0$. Then
\[ \mathcal{H}^s_{\delta_N} G(\omega, \alpha, \varepsilon, \xi) \leq \sum_{n \geq N} \exp(-\frac{mc_\Phi}{2} - nP(T(0)\Phi)) = \sum_{n \geq N} \exp(-\frac{mc_\Phi}{2}) < \infty. \]

Here we used the fact that by definition we have $P(-T(0)\Phi) = 0$.

This yields $\dim_H G(\omega, \alpha, \varepsilon, \xi) \leq \frac{T'(0)}{\xi}$ since we can choose $\eta$ arbitrarily close to 0. Since $T^*$ is concave, for $\varepsilon$ small enough there exists some $C > 0$ such that $-\frac{T'(0)}{\xi} = \frac{T^*(T'(0))}{\xi} \leq C\varepsilon + \frac{T^*(\alpha)}{\xi}$. □

As a consequence of proposition 7 and lemma 4.5, the following corollary will provide us with a first upper bound for $\dim_H E(\nu_\omega, d)$ which will turn out to be sharp on $[0, T'(t_0-)]$, recalling that $t_0$ is the unique root of the equation $P(t\Psi) = 0$ and $\dim_H X_\omega = t_0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

**Corollary 1.** For $\mathbb{P}$-a.e. $\omega$, for all $d \geq 0$,
\[ \dim_H E(\nu_\omega, d) \leq \dim_H F(\omega, d) \leq d \cdot \sup_{\alpha > 0} \frac{T^*(\alpha)}{\alpha} = d \cdot t_0. \]

**Proof.** For any $\varepsilon > 0$, we saw that
\[ F(\omega, d) \subset \cup_{\alpha \in \mathbb{Q}^+} \cup_{\xi \in \mathbb{Q} \cap [1, +\infty), \alpha \xi \leq d + 2\varepsilon} G(\omega, \alpha, \varepsilon, \xi). \]
Thus, from lemma 4.5 one has
\[ \dim_H F(\omega, d) \leq \sup_{\alpha \in \mathbb{Q}^+, \xi \in \mathbb{Q} \cap [1, +\infty), \alpha \xi \leq d + 2\varepsilon} C\varepsilon + \max(\frac{T^*(\alpha - \varepsilon), T^*(\alpha), T^*(\alpha + \varepsilon)}{\xi}). \]
Letting $\varepsilon$ tends to 0 yields
\[ \dim_H F(\omega, d) \leq \sup_{0 \leq \xi \leq \frac{d}{\alpha}} \frac{T^*(\alpha)}{\xi} \leq d \cdot \sup_{\alpha > 0} \frac{T^*(\alpha)}{\alpha} = d \cdot t_0. \]
To getting the last equality, at first we notice that since $T(t_0) = 0$, we have $\sup_{\alpha > 0} \frac{T^*(\alpha)}{\alpha} \geq \frac{T(t_0)}{\alpha} = \frac{T(t_0)}{\alpha} = t_0$. Next, we know that for any $\alpha > 0$, $\inf_q \{q\alpha - t_0\alpha - T(q)\} \leq 0$, so $\inf_q (q\alpha - T(q)) \leq t_0$, which is $\frac{T^*(\alpha)}{\alpha} \leq t_0$. Finally $\sup_{\alpha > 0} \frac{T^*(\alpha)}{\alpha} \leq t_0$. Now, recall (10). Since the set of atoms of $\nu_\omega$ is countable, we get the desired conclusion for $\dim_H E(\nu_\omega, d)$. □
5. LOWER BOUND FOR THE $L^q$-SPECTRUM AND UPPER BOUND FOR THE LOWER
HAUSDORFF SPECTRUM

Proposition 8. For $\mathbb{P}$-a.e. $\omega \in \Omega$, for every $q \in \mathbb{R}$, we have $\tau_{\nu_\omega}(q) \geq \min(\mathcal{T}(q), 0) := \overline{T}(q)$.

Due to (2), Proposition 8 gives the sharp upper bound for the lower Hausdorff spectrum.

Since the functions $\tau_{\nu_\omega}$ and $\overline{T}$ are both continuous, we just need to prove that the inequality of proposition 8 holds on a dense and countable subset of $\mathbb{R}$, which amounts to prove it for any fixed $q \in \mathbb{R}$, almost surely.

Proof. Let $r > 0$ and consider $\mathcal{B} = \{B_i\}$, a packing of $[0, 1]$. that is a family of disjoint intervals $B_i$ with radius $r$ and centers in $[0, 1]$.

Case 1: $q \leq 0$. Set $B_i := B(x_i, r)$. There exists a unique $v(x_i, r) \in \Sigma_{\omega,n}$ such that $x_i \in I_{\omega}^{(v(x_i, r))} \subset B_i$ and $I_{\omega}^{(v(x_i, r))} \not\subset B_i$. Here the notation $\ast$ means that we delete the last character of the word. Then

$$2r \geq |I_{\omega}^{(v(x_i, r))}| = \tilde{\mu}_\omega([v(x_i, r)])_\omega \geq \exp(S_n(\Phi(\omega, v) - n\epsilon(\Phi, \omega, n))) \geq \exp(-2nC_F),$$

for $r$ small enough. On the other hand, since $I_{\omega}^{(v(x_i, r))} \not\subset B_i$, we have

$$r \leq |I_{\omega}^{(v(x_i, r))}| \leq \exp(S_{n-1}(\Phi(\omega, v) + (n-1)\epsilon(\Phi, \omega, n-1))) \leq \exp(-\frac{(n-1)c_F}{2})$$

(recall that $c_F = c_\Phi$ was defined in (5)). Consequently, $\frac{\log 2r}{-c_\Phi} \leq n \leq \frac{2\log r}{c_\Phi} + 1$.

Also, $\nu_\omega(B_i) \geq \nu_\omega(I_{\omega}^{(v(x_i, r))}) \geq |X_{\omega}^{(v(x_i, r))}|$. Since $q < 0$, we get (using proposition 3 applied to $q\Psi - \mathcal{T}(q)\Phi$ and its associated random weak Gibbs measure $\tilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}$). Noticing $\mathcal{T}(q)$ for $q \leq 0$ and $\lim_{n \to \infty} \|\Phi(\sigma^{n-1}\omega)\|_\infty / n = 0$, i.e. $\|\Phi(\sigma^{n-1}\omega)\|_\infty = o(n)$, we get

$$\nu_\omega(B_i)^q \leq |X_{\omega}^{(v(x_i, r))}|^q \leq \exp(q(S_n(\Phi(\omega, v) - n\epsilon(\Phi, \omega, n))) \quad (\forall v \in [v(x_i, r)])_\omega$$

$$\leq \exp((S_n(q(\Psi - \mathcal{T}(q)\Phi)(\omega, v) + \mathcal{T}(q)S_n(\Phi(\omega, v) - qn\epsilon(\Psi, \omega, n))) \leq \tilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v(x_i, r)])_\omega)_{\mathcal{T}(q)} \exp(-qn\epsilon(\Psi, \omega, n) + o(n))$$

$$\leq \tilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v(x_i, r)])_{\mathcal{T}(q)} \exp(o(-\log r)).$$

Thus $\sum_{B_i \in \mathcal{B}} \nu_\omega(B_i)^q \leq r^{\mathcal{T}(q)} \exp(o(-\log r))$, and letting $r \to 0$ yields $\tau_{\nu_\omega}(q) \geq \mathcal{T}(q)$.

Case 2: $q \in (0, t_0) \subset (0, 1)$. Recall that $t_0 = \dim_H X_\omega$ is the unique real number such that $P(t\Psi) > 0$. Define

$$\nu_\omega(V(\omega, n, r)) = \{v \in \Sigma_{\omega,n} : |I_{\omega}^{v}| \geq 2r, \exists s \text{ such that } vs \in \Sigma_{\omega,n+1}, |I_{\omega}^{vs}| < 2r\},$$

$$\nu_\omega(V'(\omega, n, r)) = \{v \in V(\omega, n, r) : \text{there is no } k \in \mathbb{N} \text{ such that } v|_k \in V(\omega, k, r) \text{ with } k < n\},$$

as well as $V(\omega, r) = \bigcup_{n \geq 1} V'(\omega, n, r)$, $n_r = \max\{|v| : v \in V(\omega, r)\}$, and $n'_r = \min\{|v| : v \in V(\omega, r)\}$.

We have $n_r = O(-\log r) = O(n'_r)$ and for any $v \in V(\omega, r)$ we have

$$|I_{\omega}^{v}| \leq |I_{\omega}^{v}| \exp(|\Phi(\sigma^{|v|})|_\infty + |v|\epsilon(\Phi, \omega, |v|) + (|v| + 1)\epsilon(\Phi, \omega, |v| + 1))).$$
so that
\begin{equation}
|I_v^w| \leq 2r \exp(o(- \log r))
\end{equation}
For $v \in V(\omega, r)$, $I_v^w$ meets at most $\exp(o(- \log r))$ intervals $B_i$ of the packing $B$, and for every $B_i$ there are at most two intervals $I_v^w$ and $I_v^{w'}$ such that $B_i \subset I_v^w \cup I_v^{w'}$ and $v, v' \in V(\omega, r)$. Using the sub-additivity of the function $s \geq 0 \mapsto s^q$, we get
\[
\nu_\omega(B_i)^q \leq \begin{cases} 
\nu_\omega(1)^q + \nu_\omega(I_v^w)^q + \nu_\omega(I_v^{w'})^q & \text{if } 1 \in B_i, \\
\nu_\omega(I_v^w)^q + \nu_\omega(I_v^{w'})^q & \text{otherwise}
\end{cases}
\]
Recalling the definition of the inverse measure $\nu_\omega$ and proposition 4 we know that $\nu_\omega(I_v^w) \leq |X_v^w|$. Since $0 \leq q \leq t_0$, $T(q) \leq 0$, then we get:
\[
\nu_\omega(I_v^w)^q \leq |X_v^w|^q \leq \exp(qS_{\nu}[\Psi(\omega, v) + q|v|e(\Psi, \omega, |v|)]) \quad (\forall \nu \in [v]_\omega)
\]
\[
\leq \exp((S_{\nu}[q(\Psi - T(q)\Phi(\omega, v)] + T(q)S_{\nu}\Phi(\omega, v) + q|v|e(\Psi, \omega, |v|)))
\]
\[
\leq \bar{\mu}^q_{\omega} - T(q)\Phi([v]_\omega)|I_v^w|^q
\]
\[
\times \exp(q|v|e(\Psi, \omega, |v|) + q|v|e(q(\Psi - T(q)\Phi, \omega, |v|) - T(q)|v|e(\Phi, \omega, |v|))
\]
\[
\leq \bar{\mu}^q_{\omega} - T(q)\Phi([v]_\omega)r^{T(q)} \exp(o(- \log r))( \text{see (11) and } |v| = O(- \log r)).
\]
It follows that
\[
\sum_{B_i \in B} \nu_\omega(B_i)^q \leq \nu_\omega(1)^q + \exp(o(- \log r)) \left( \sum_{n=n'}^{n_r} \sum_{v \in V(\omega, r) \cap \Sigma_{\omega, n}} \nu_\omega(I_v^w)^q 
\right.
\]
\[
\left. + \sum_{n=0}^{n_r} \sum_{v \in \Sigma_{\omega, n}} \sum_{s \in S(\omega, v, 1)} \nu_\omega(\{x_s^v\}^q) \right).
\]
Now, on the one hand,
\[
\sum_{n=n'}^{n_r} \sum_{v \in V(\omega, r) \cap \Sigma_{\omega, n}} \nu_\omega(I_v^w)^q \leq r^{T(q)} \exp(o(- \log r)),
\]
and on the other hand, for any $n \leq n_r$, with a similar way, we have that
\[
\sum_{v \in \Sigma_{\omega, n}} \exp(qS_{\nu}[\Psi(\omega, v) + o(n)]) (\forall \nu \in [v]_\omega)
\]
\[
\leq \sum_{v \in \Sigma_{\omega, n}} \bar{\mu}^q_{\omega} - T(q)\Phi([v]_\omega)r^{T(q)} \exp(o(- \log r)) \leq r^{T(q)} \exp(o(- \log r)).
\]
Consequently,
\[
\sum_{n=0}^{n_r} \sum_{v \in \Sigma_{\omega, n}} \sum_{s \in S(\omega, v, 1)} \nu_\omega(\{x_s^v\}^q)
\]
\[
\leq \sum_{n=0}^{n_r} \sum_{v \in \Sigma_{\omega, n}} \sum_{s \in S(\omega, v, 1)} (m_{\omega}^{v_s} - M_{\omega}^{v_s})^q
\]
\[
\leq \sum_{n=0}^{n_r} \sum_{v \in \Sigma_{\omega, n}} l(\sigma^n \omega) \exp(qS_{\nu}[\Psi(\omega, v) + o(n)]) (\forall \nu \in [v]_\omega)
\]
\[
\leq r^{T(q)} \exp(o(- \log r)) (\text{noticing the fact that } \log n + \log l(\sigma^n \omega) = o(- \log r)).
\]
Now we obtain
\[ \sum_{B_i \in \mathcal{B}} \nu_\omega(B_i)^q \leq r^{T(q)} \exp(o(-\log r)), \]
and letting \( r \to 0 \), we get \( \tau_\nu(q) \geq T(q) \).

**Case 3:** \( q \geq t_0 = \dim_H X_\omega \). Since \( \nu_\omega \) is discrete, we can easily get \( \tau_\nu(q) = 0 \) for every \( q \geq 1 \). For \( q = t_0 \), one has \( \tau_\nu(q) \geq T(q) = 0 \). Since the function \( \tau_\nu \) is concave, we get \( \tau_\nu(q) = 0 \) for every \( q \geq t_0 \).

Next we collect information associated with the approximation of \((\Phi, \Psi)\) by pairs of Hölder continuous potentials.

6. Basic properties related to the approximation of \((\Phi, \Psi)\) by Hölder continuous random potentials

The material of this section, which can be skipped in a first reading, is borrowed from [22] (here we just permute the roles of \( \Phi \) and \( \Psi \); this is natural since we work with the inverse measures of random weak Gibbs measures). Some facts will be collected in order to construct suitable measures supported on the level sets of the lower local dimension of \( \nu_\omega \).

We fix two sequences \( \{D_i\}_{i \geq 1} \) and \( \{\Phi_i\}_{i \geq 1} \) of random Hölder potentials as in [22, section 3], which converge to \( \Phi \) and \( \Psi \) respectively. Since we assumed that \( c_\Phi = c_\Psi > 0 \), for each \( i \in \mathbb{N} \) there exists a function \( \mathcal{T}_i \) such that for any \( q \in \mathbb{R} \) one has \( P(q_\Psi_i - \mathcal{T}_i(q)\Phi_i) = 0 \), and we have:

**Lemma 6.1.**

1. \( \mathcal{T}_i \) converges pointwise to \( \mathcal{T} \) as \( i \to \infty \).
2. \( \mathcal{T}_i^* \) converges pointwise to \( \mathcal{T}^* \) over the interior of the domain of \( \mathcal{T}^* \) as \( i \to \infty \).

Let \( \mathcal{D} \) be a dense and countable subset of \((\mathcal{T}'(+\infty), \mathcal{T}'(-\infty))\), so that for any \( d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)] \), there exists \( \{d_k\}_{k \in \mathbb{N}} \subset \mathcal{D}^\mathbb{N} \) such that \( \lim_{k \to \infty} d_k = d \) and \( \lim_{k \to \infty} \mathcal{T}^*(d_k) = \mathcal{T}^*(d) \).

Let \( \{\mathcal{D}_i\}_{i \in \mathbb{N}} \) be a sequence of sets such that
- \( \mathcal{D}_i \) is a finite set for each \( i \in \mathbb{N} \),
- \( \mathcal{D}_i \subset \mathcal{D}_{i+1} \), for each \( i \in \mathbb{N} \),
- \( \cup_{i \in \mathbb{N}} \mathcal{D}_i = \mathcal{D} \).

Let us fix a positive sequence \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) decreasing to 0. For each \( i \), there exists \( j_i \) large enough such that for any \( d_i \in \mathcal{D}_i \), there exists \( q_i \in \mathbb{R} \) such that

1. \( \mathcal{T}_{j_i}(q_i) = d_i \),
2. \( |\mathcal{T}_{j_i}^*(d_i) - \mathcal{T}^*(d_i)| \leq (\varepsilon_i)^4 \),
3. \( \int \varphi_j \Psi d\mathbb{P} \leq (\varepsilon_i)^4 \) and \( \int \varphi_j \Phi d\mathbb{P} \leq (\varepsilon_i)^4 \).

Define \( Q_i = \{q_i, d_i \in \mathcal{D}_i\} \) and assume that \( j_{i+1} > j_i \) for each \( i \in \mathbb{N} \). For any \( q \in Q_i \), we define
\[ \Lambda_{i,q} := \tilde{\Lambda}_{j_i,q} = q \Phi_{j_i} - \mathcal{T}_{j_i}(q)\Psi_{j_i}. \]

With \( \Lambda_{i,q} \) is associated a random Gibbs measure \( \{\tilde{\mu}_{\omega}^{\Lambda_{i,q}}\}_{\omega \in \Omega} \) and \( \{\mu_{\omega}^{\Lambda_{i,q}}\}_{\omega \in \Omega} \) (see for instance [22] for the definition). For any \( n \in \mathbb{N} \), for any \( v \in \Sigma_{\omega,n} \), define
\[ \zeta_{\omega}^{\Lambda_{i,q}}(v) = \mu_{\omega}^{\Lambda_{i,q}}(U^v_n) = \tilde{\mu}_{\omega}^{\Lambda_{i,q}}([v]_{\omega}). \]
**Fact 1.** For each $i \geq 1$, for any $\epsilon > 0$, there exist $C > 0$ (large enough), $M_i \in \mathbb{N}$ and a measurable set $\Omega(i) \subset \Omega$ such that the following basic properties (denoted by (BP)) hold for $\omega$, i.e.

1. $\mathbb{P}(\Omega(i)) > 3/4$;
2. $M(\omega) + 1 \leq M_i$;
3. for all $\omega \in \Omega(i)$, there is an increasing sequence $\{n^i_k = n^i_k(\omega)\}_{n \in \mathbb{N}}$ in $\mathbb{N}^M$ such that $\sigma_{n^i_k + M_i}^n \omega \in \Omega(i)$
4. for any $\omega \in \Omega(i)$, for $\Upsilon \in \{\Phi, \Psi\}$, $\forall n \in \mathbb{N}$, $\forall \nu \in \Sigma_\omega$, we have
5. for any $\epsilon_i > 0$, for all $\omega \in \Omega(i)$, there exists an integer $\mathcal{K}_i = \mathcal{K}_i(\omega)$ such that for any $d_i \in D_i$, there exists $q_i \in Q_i$ with $\mathcal{T}_i(q_i) = d_i$ and $E_{i,q_i} = E_{i,q_i}(\sigma_{M_i}^n \omega) \subset [0, 1]$ such that the following hold:
   - the measure $\mu_{\sigma_{M_i}^n}^{\mathcal{K}_i}$ is well defined, and thus so is $\zeta_{\sigma_{M_i}^n}^{\mathcal{K}_i}$ with $\zeta_{\sigma_{M_i}^n}^{\mathcal{K}_i}(E_{i,q_i}) > 1/2$;

   $M_i \leq n^i_{\mathcal{K}_i}(\epsilon_i)^4$;

   for $\Upsilon \in \{\Phi, \Psi\}$, for $p \geq n^i_{\mathcal{K}_i}$, we have

   $\left| \frac{S_p \text{var}_j \Upsilon(\sigma_{M_i}^n \omega)}{p} - \int_{\Omega} \text{var}_j \Upsilon d\mathbb{P} \right| \leq (\epsilon_i)^4$,

   and $\forall \nu \in \Sigma_\omega$, we have

   $S_p \Upsilon(E_{M_i}^n(\omega, \nu)) \leq \frac{p C}{2}$;

   for any $k \geq \mathcal{K}_i$, we have

   $\frac{n^i_k - n^i_{k-1}}{n^i_{k-1}} \leq (\epsilon_i)^4$;

   for any $x \in E_{i,q_i}(\sigma_{M_i}^n \omega)$, for any $k \geq \mathcal{K}_i$, for any $v \in \Sigma_{\sigma_{M_i}^n \omega,n^i_k}$ such that $x \in I^v_{\omega}$, one has $|v \vee v^+| \geq n^i_{k-1}$ and $|v \vee v^-| \geq n^i_{k-1}$. Furthermore, for any $x \in [v]_{\sigma_{M_i}^n \omega} \cup [v^+]_{\sigma_{M_i}^n \omega} \cup [v^-]_{\sigma_{M_i}^n \omega}$, one has

   $\left| \frac{S_{n^i_k} \Psi_j(\sigma_{M_i}^n, v)}{S_{n^i_k} \Phi_j(\sigma_{M_i}^n, v)} - d_i \right| \leq (\epsilon_i)^2$,

   and for $v' \in \{v, v+, v^-, v^-\}$, one has

   $\left| \log \zeta_{\sigma_{M_i}^n}^{n^i_k} \left( \frac{I_{\sigma_{M_i}^n}^{v'}}{\Phi_j(\sigma_{M_i}^n, v')} \right) - \mathcal{T}^*(d_i) \right| \leq (\epsilon_i)^2$.

**Fact 2.** We can change $\Omega(i)$ to $\Omega_i \subset \Omega(i)$ a bit smaller such that $\mathbb{P}(\Omega_i) \geq 1/2$ and there exist two integers $\kappa_i$ and $W(i)$ such that for any $\omega \in \Omega_i$, $\mathcal{K}_i(\omega) \leq \kappa_i$, $n^i_{\kappa_i}(\omega) \leq W(i)$, and the properties listed in fact 1 still hold.

Then, we denote by $\theta(i, \omega, s)$ the $s$-th return time of the point $\omega$ to the set $\Omega_i$, that is

$\theta(i, \omega, 1) =: \min\{n \in \mathbb{N} : \sigma^n \omega \in \Omega_i\}$,

$\forall s \in \mathbb{N}, s > 1, \theta(i, \omega, s) =: \min\{n \in \mathbb{N} : n > \theta(i, \omega, s - 1) \text{ and } \sigma^n \omega \in \Omega_i\}$
Proof. The approach used in [22] also yields the validity of the multifractal formalism for this is rather long and tedious, so we omit it.

Remark 8. It is possible to give a self-contained proof of the proposition based only on the basic properties of random weak Gibbs measures listed in the previous section. However, this is rather long and tedious, so we omit it.

7. A FIRST LOWER BOUND FOR THE LOWER HAUSDORFF SPECTRUM

In this section we establish the following lower bound for the lower Hausdorff spectrum.

Proposition 9. For any $d \in [T'(\infty), T'(-\infty)]$,
\[
\dim_H(E(\nu_\omega, d)) \geq \dim_H(E(\nu_\omega, d)) \geq T^*(d),
\]
This bound will prove to be sharp for $d \in [T'(t_0^-), T'(-\infty)]$.

Proof. The approach used in [22] also yields the validity of the multifractal formalism for $\mu_\omega$ if one considers the sets of the form \{ $x \in X_\omega : \lim_{x \in I_\omega | I \to 0} \frac{\log(\mu_\omega(I))}{\log(|I|)} = 1/d$ \}, like in [21]. Then using the general theory of [21, theorem 21], one gets
\[
\dim_H \{ x \in [0, 1] : \lim_{x \in I_\omega | I \to 0} \frac{\log(\nu_\omega(I))}{\log(|I|)} = d \} = T^*(d).
\]
Since $E(\nu_\omega, d) \supset E(\nu_\omega, d) \supset \{ x \in [0, 1] : \lim_{x \in I_\omega | I \to 0} \frac{\log(\nu_\omega(I))}{\log(|I|)} = d \}$,
\[
\dim_H(E(\nu_\omega, d)) \geq \dim_H(E(\nu_\omega, d)) \geq T^*(d).
\]
\[
\square
\]
8. Conditioned ubiquity

In this section we extend to our context the conditioned ubiquity result obtained in [4].
The statement takes the same form, but as explained in remark 1(2), the proof must be
revisited in order to cover the more general situation we face. The result will be applied to
get the sharp lower bound for the lower Hausdorff spectrum on $[0, T'(t_0-)]$ (proposition 10
in next section).

Recall remark 5. We are interested in the ubiquity of the family of points $\{z^v_\omega\}_{v \in \Sigma_\omega}$
relatively to the radii $\{\ell_\omega\}_{v \in \Sigma_\omega}$, and conditionally on the behavior of $\frac{S_{\nu,v}\Psi(\omega,v)}{S_{\nu,v}\Psi(\omega,d)}$ in $[v]\omega$.

**Definition 8.1.** If $d \geq 0$, $\xi \geq 1$ and $\tilde{c} = \{c_i\}_{i \in \mathbb{N}}$ is a positive sequence decreasing to 0 as
$i \to \infty$, we set
\[
S(\omega, d, \xi, \tilde{c}) := \bigcap_{N \geq 1} \bigcup_{i \geq N} \bigcup_{v \xi, \tilde{c}} B(z^v_\omega, (\ell^v_\omega)\xi) .
\]

**Theorem 8.2.** For $\mathbb{P}$-a.e. $\omega \in \Omega$, for any $\xi \geq 1$ and any exponent $d \in [T'(\infty), T'(-\infty)]$,
there exists a sequence $\tilde{c}(\omega) \{c_i(\omega)\}_{i \in \mathbb{N}}$ decreasing to 0 as $i \to \infty$, as well as a set $K^d(\xi) \subset S(\omega, d, \xi, \tilde{c})$ and a Borel probability measure $m^d_\xi$ supported on $K^d(\xi)$ such that
\[
\dim_H(m^d_\xi) \geq T^*(d).
\]

**Remark 9.** In fact we can choose $\tilde{c}$ independent of $\omega$.

**Remark 10.** For any $x \in S(\omega, d, \xi, \tilde{c})$ there are infinitely many $n_i$ and $v(i) \in \Sigma_\omega, n_i$ such that $x \in B(z^{v(i)}_\omega, (\ell^{v(i)}_\omega)\xi)$ and $\alpha_{n_i}^v(x) \leq d + \epsilon_i$, so
\[
\dim_{\text{loc}}(\nu_\omega, x) \leq \liminf_{i \to \infty} \frac{\log \nu_\omega(B(x, (\ell^{v(i)}_\omega)\xi))}{\xi \log \ell^{v(i)}_\omega} \leq \liminf_{i \to \infty} \frac{\log \nu_\omega(\{z^{v(i)}_\omega\})}{\xi \log \ell^{v(i)}_\omega} \leq \liminf_{i \to \infty} \frac{\alpha_{n_i}^v(x)}{\xi} \leq d + \epsilon_i
\]
by remark 5 $\frac{\log \nu_\omega(\{z^{v(i)}_\omega\})}{\log \ell^{v(i)}_\omega}$ is asymptotically not bigger than the Birkhoff average $\frac{\sum_{i=1}^\infty \Psi(\omega,v)}{\sum_{i=1}^\infty \varphi}\tilde{c}$. Consequently, $S(\omega, d, \xi, \tilde{c}) \subset \bigcup_{h \leq \tilde{c}} E(\nu_\omega, h)$.

**Proof of theorem 8.2.** Again, we will use the properties collected in section 6. We can
assume without loss of generality that for all $i \geq 1$, for all $d_i \in D_i$, we have
\[
\epsilon_i \leq \min \left\{ \frac{c_i}{10 \xi (C+2)(C+1) + 2c_i}, \frac{1}{4C}, \frac{1}{3+\xi}, \frac{14c_i}{20(d_i+1)} \right. \text{ with } d_i \in D_i \right\}
\]
(21) take $(\epsilon_i)_{i \geq 1}$ which goes quickly to 0, while $\min\{T^*(d_i) : d_i \in D_i\}$ goes slowly to $\inf\{T^*(T'(\cdot))\}$. Recall that $\tilde{\theta}(\theta(i, \omega, k)$ is the $k$-th return time of $\omega$ to the set $\Omega_k$ under the mapping $\sigma$.

We start by constructing a generalized Cantor set $K(\xi, \tilde{d})$ for each $\xi > 1$ and each
sequence $\tilde{d} = (d_i)_{i=1}^\infty \in \prod_{i=1}^\infty D_i$.

**Step 1:** Let $\omega \in \Omega'$, choose $n = \theta(1, \omega, s)$ large enough such that:
- $\log 4 \Gamma_1 \leq 1$, where $\Gamma_1$ is the constant in Besicovich's covering theorem on $\mathbb{R}$.
- for any $p \geq n$, for any $v \in \Sigma_\omega, p$, one has $\epsilon(\Phi, \omega, p) \leq (\epsilon_1)^4$ and (20).
- $W(1) \leq n(\epsilon_1)^4$.

Fix $w \in \Sigma_\omega, n$. Recall that for each $d_i \in D_i$ there is $q_i \in Q_i$ with $T_{q_i}^*(q_i) = d_i$.
From fact 1 and fact 2, we know that properties (BP) hold for $\sigma^n\omega \in \Omega_1$. In this
step, $k_1, n_k^1$ are defined with respect to $\sigma^n\omega$, and also $\xi_{\omega, w, q_1}$ and $E(\omega,1,w,q_1)$
are well defined.
For \( k \geq \kappa_1 \) and \( x \in E(\omega, 1, w, q_1) \), let \( v(\omega, 1, q_1, n_k^1, x) \) be the unique word such that \( x \in I_{\omega}^{w,v(\omega,1,q_1,n_k^1,x)} \) and \( v(\omega, 1, q_1, n_k^1, x) \in \Sigma_{\sigma^1 + M_1, \omega, n_k^1} \). For any \( k \geq \kappa_1 \), define

\[
\mathcal{F}_1(q_1, n + M_1 + n_k^1) = \left\{ B(y, 2\ell_{\omega}^{w,v(\omega,1,q_1,n_k^1,y)} : y \in E(\omega, 1, w, q_1) \right\}.
\]

Then \( \mathcal{F}_1(q_1, n + n_k^1) \) is a covering of \( E(\omega, 1, w, q_1) \). By Besicovitch’s covering theorem [17, theorem 2.7], there are \( \Gamma_1 \) families of balls \( \mathcal{F}_1'(q_1, n + n_k^1), \cdots, \mathcal{F}_1^{\Gamma_1}(q_1, n + n_k^1) \subset \mathcal{F}_1(q_1, n + n_k^1) \), such that \( E(\omega, 1, w, q_1) \subset \bigcup_{\gamma=1}^{\Gamma_1} \bigcup_{B \in \mathcal{F}_1'(q_1, n + n_k^1)} B \) and for any \( B, B' \in \mathcal{F}_1'(q_1, n + n_k^1) \), if \( B \neq B' \) one has \( B \cap B' = \emptyset \) (where \( \Gamma_1 \) is a constant depending on the dimension 1 of the Euclidean space \( \mathbb{R} \)).

Since \( \zeta_{\omega,w,q_1}(E(\omega,1,w,q_1)) > 1/2 \), there exists \( s \) such that

\[
\zeta_{\omega,w,q_1} \left( \bigcup_{B \in \mathcal{F}_1'(q_1,k)} B \right) \geq \frac{1}{2\Gamma_1}.
\]

Among the intervals of \( \mathcal{F}_1'(k) \) we can choose a finite subset

\[
D^w(1, d_1, k) = \{ B_1, \cdots, B_{d_1} \}
\]

such that

\[
\zeta_{\omega,w,q_1} \left( \bigcup_{B_i \in D^w(1,d_1,k)} B_i \right) \geq \frac{1}{4\Gamma_1}.
\]

For any \( B_i \in D^w(1,d_1,k) \), there exists \( y_i \in E(\omega, 1, w, q_1) \) such that \( B_i = B(y_i, 2\ell_{\omega}^{v(\omega,1,q_1,n_k^1)}) \), where \( v(k,l) := v(\omega,1,q_1,n_k^1,y_i) \). Since

\[
B \left( z_{\omega}^{w,v(k,l)}, (\ell_{\omega}^{w,v(k,l)})^{\xi} \right) \subset B \left( z_{\omega}^{w,v(k,l)}, \ell_{\omega}^{w,v(k,l)} \right) \subset B(y_i, 2\ell_{\omega}^{w,v(k,l)}) = B_i,
\]

Choose \( \kappa' > \kappa_1 \) large enough such that:

- for any \( v \in \Sigma_{\sigma^1 + M_1, \omega, n_k^1} \), one has \( 2\ell_{\omega}^{w,v} \leq |I_{\omega}^w| \exp(-\varepsilon_1)^2 \);
- for any \( j \geq n + M_1 + n_k^1 \), one has \( \epsilon(\Phi, \omega, j) \leq (\varepsilon_2)^4 \);
- \( W(2) \leq (\varepsilon_2)^4(n + n_k^1) \);
- for any \( s \) such that the return time \( \theta(2, \omega, s) \) satisfies \( \theta(2, \omega, s) \geq n + M_1 + n_k^1 \), one also has
  \[
  \frac{\theta(2, \omega, s) - \theta(2, \omega, s - 1)}{\theta(2, \omega, s - 1)} \leq (\varepsilon_1)^4.
  \]

For any \( k \geq \kappa' \), we can get (this will be justified in the next step, see (25)):

\[
(22) \quad \zeta_{\omega,w,q_1} \left( B(y_i, 2\ell_{\omega}^{w,v(k,l)}) \right) \leq \frac{4\ell_{\omega}^{w,v(k,l)}}{|I_{\omega}^w|} \left( \frac{T^*(d_1) - \frac{\xi_1}{2}}{|I_{\omega}^w|} \right).
\]

Let \( s_2 = s_2(\omega, w) \) be the smallest \( s \) such that there exists \( v \in \Sigma_{\omega,\theta(2,\omega,s_2)} \) such that:

- \( z_{\omega}^{w,v(k,l)} \) belongs to the closure of the interval \( I_{\omega}^v \);
- \( I_{\omega}^v \subset B \left( z_{\omega}^{w,v(k,l)}, (\ell_{\omega}^{w,v(k,l)})^{\xi} \right) \).
By definition of $s_2$, setting $v' = |v|^2(2, \omega, s_2 - 1) \in \Sigma_0, \theta(2, \omega, s_2 - 1)$, we have $|I^v_{w}| \geq \xi$. Now let $K_1$ be the largest $k$ such that $n + M_1 + n_k^1 \leq \theta(2, \omega, s_2 - 1)$ (by construction we have $K_1 \geq \kappa'_1$). Due to (17), we have
\[
\theta(2, \omega, s_2 - 1) - n - M_1 - n_k^1 \leq n_k^1 \leq (\xi)^4n_k^1.
\]
Then (this will be justified in the next step in (30))
\[
2(\xi)\xi \leq |I^v_{w}| \leq \xi|I^v_{w}|^{1-\xi} \leq |I^v_{w}|^{1-\xi}.
\]
Define $J_t$ to be $T_v$, the closure of the interval $I^v$, and denote $B_1 = J_t$ and $B_2 = \tilde{J}_t$.

We get:
\[
|J_t| \leq |\tilde{J}_t|^{\xi} \leq |J_t|^{1-\xi}.
\]

Now using lemma 8.4 of the next step, we can get that for $k$ large enough so that $n_k^1 \geq \frac{n}{(\xi)^2}$, then for any $v \in [w * v(k, l)]_{\omega}$ we have:
\[
\frac{|S_{n+n_k^1}(\omega, v)|}{|S_{n+n_k^1}(\omega, v)| - d} \leq \epsilon_1.
\]

For $k > \kappa'_1$ large enough with $n_k^1 \geq \frac{n}{(\xi)^2}$, define $G^w(1, d_1, k) = \{B_1, B_2 \in D^w(1, d_1, k)\}$.

If $J_1$ and $J_2$ are two distinct elements of $G^w(1, d_1, k)$ then their distance is at least $\max_{j \in \{1, 2\}}(|J_j|/2 - |\tilde{J}_j|/2)^{\xi}$, which is larger than $\max_{j \in \{1, 2\}}|\tilde{J}_j|/3$ for $k$ large enough (since $\xi > 1$).

Now we can define $m_{\xi}^{d_1}$ with $d_1 \in D_1$ as follows:
\[
m_{\xi}^{d_1}(J) = \frac{\zeta_{w, w, q_1}(\tilde{J})}{\sum_{J \in G^w(1, d_1, k)} \zeta_{w, w, q_1}(J)}.
\]

For any $J \in G^w(1, d_1, k)$, by inequality (22) and (23), we can get
\[
\zeta_{w, w, q_1}(\tilde{J}) \leq |J|^{\frac{\tau^*(d_1) - 2\mu}{\xi}} |I^v_{w}|^{-\tau^*(d_1)}.
\]

Then, the inequality $\sum_{J \in G^w(1, d_1, k)} \zeta_{w, w, q_1}(\tilde{J}) \geq \frac{1}{\pi_1}$ yields, $\forall J \in G^w(1, d_1, k)$:
\[
m_{\xi}^{d_1}(J) \leq 4\Gamma_1 |J|^{-\frac{\tau^*(d_1) - 2\mu}{\xi}} |I^v_{w}|^{-\tau^*(d_1)}.
\]

Choose $k_1 > \mathcal{N}_1^0$ large enough with $n_k^1 \geq \frac{n}{(\xi)^2}$. Then, for any $d_1 \in D_1$ and any $J \in G^w(1, d_1, k_1)$, one has $4\Gamma_1 |I^v_{w}|^{-\tau^*(d_1)} \leq |J|^{-\frac{2\mu}{\xi}}$ (this will follow from a general estimate in the next step).

Finally,
\[
\forall J \in G^w(1, d_1, k_1), m_{\xi}^{d_1}(J) \leq |J|^{-\frac{\tau^*(d_1) - 2\mu}{\xi}}.
\]

For any $d_1 \in D_1$ define:
\[
G(d_1) = G^w(1, d_1, k_1),
\]
and
\[
G_1 = \bigcup_{d_1 \in D_1} G^w(1, d_1, k_1).
\]
Step 2: Suppose that $G_i = \bigcup_{d_1,\ldots,d_i} \prod_{j=1}^{d_i} G(d_1,\ldots,d_i)$ is well defined and for any \(\{d_j\}_{1 \leq j \leq i} \in \prod_{1 \leq j \leq i} D_j\), the set function \(m_\xi^{(d_j)}\) is well defined on the set \(G(d_1,\ldots,d_i)\).

For any \(w\) such that \(J\), the closure of \(I^w\), belongs to \(G(d_1 \cdots d_i) \subset G_i\), we set \(n = |w|\). In this step \(n_{k+1}\) stands for \(n_{k+1}(\sigma^n\omega)\).

By construction:
1. \(\sigma^n\omega \in \Omega_{k+1}\),
2. for \(p \geq n\), we have

\[
(24) \quad e(\Phi,\omega,p) \leq (\varepsilon_{i+1})^4,
\]

(3) \(W(i+1) \leq n(\varepsilon_{i+1})^4\),

For any \(d_{i+1} \in D_{i+1}\), take \(q_{i+1} \in Q_{i+1}\) such that \(T_{j+1}(q_{i+1}) = d_{i+1}\). From fact 1 and fact 2, we know that (BP) hold for \(\sigma^n\omega \in \Omega_{i+1}\). Also \(\zeta_{\omega,w,q_{i+1}}\) and \(E(\omega,i+1,w,q_{i+1})\) are well defined.

For any \(k \geq k_{i+1}\), let

\[
F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1}) = \{B(y,2^{w+1}(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y)) : y \in E(\omega,i+1,w,q_{i+1})\},
\]

where \(v(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y)\) is the unique word such that \(y \in L^{w+1}(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y)\) and \(v(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y) \in \Sigma^{w+1}M_{i+1}\omega,n_{k+1}^{i+1}\). Then \(F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1})\) is a covering of \(E(\omega,i+1,w,q_{i+1})\).

From the Besicovitch’s covering theorem, \(\Gamma_1\) families of disjoint balls, namely

\[
F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1}), \cdots, F_{i+1}(n + M_{i+1} + n_{k+1}^{i+1})
\]

can be extracted from \(F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1})\) so that

\[
E(\omega,i+1,w,q_{i+1}) \subset \bigcup_{s=1}^{\Gamma_1} B_{s \in F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1})} B.
\]

Since \(\zeta_{\omega,w,q_{i+1}}(E(\omega,i+1,w,q_{i+1})) \geq 1/2\), there exists \(s\) such that

\[
\zeta_{\omega,w,q_{i+1}}\left(\bigcup_{B \in F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1})} B\right) \geq \frac{1}{2}\Gamma_1.
\]

Again, we extract from \(F_{i+1}(q_{i+1}, n + M_{i+1} + n_{k+1}^{i+1})\) a finite family of pairwise disjoint intervals \(D^w(i+1,d_{i+1},k) = \{B_1,\cdots,B_y\}\) such that

\[
\zeta_{\omega,w,q_{i+1}}\left(\bigcup_{B \in D^w(i+1,d_{i+1},k)} B\right) \geq \frac{1}{4\Gamma_1}.
\]

For each \(B_l \in D^w(i+1,q_{i+1},k)\), there exists \(y_l \in E(\omega,i+1,w,q_{i+1})\) such that \(B_l = B(y_l,2^{w+1}(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y_l))\). Set \(v(k,l) = v(\omega,i+1,q_{i+1},n_{k+1}^{i+1}y_l)\). Moreover,

\[
B(\zeta_{\omega,w+v(k,l)}(\xi_{\omega,w+v(k,l)}),\zeta_{\omega,w+v(k,l)}(\xi_{\omega,w+v(k,l)})) \subset B(\zeta_{\omega,w+v(k,l)}(\xi_{\omega,w+v(k,l)}),\zeta_{\omega,w+v(k,l)}(\xi_{\omega,w+v(k,l)})) \subset B(y_l,2^{w+1}(\omega,i+1,w,q_{i+1})) = B_l.
\]

We can get the following lemma.
Lemma 8.3. For any \( y \in E(\omega, i + 1, w, q_{i+1}) \), for \( r \leq |I^w_\omega| \exp(-n(\varepsilon_{i+1})^2) \), we have

\[
\zeta_{\omega, w, q_{i+1}}(B(y, r)) \leq \left( \frac{r}{|I^w_\omega|} \right)^{T^*(d_{i+1})-\frac{n_{i+1}^k}{2}}.
\]

Proof of the lemma. The idea of the proof is the following. Noticing \( y \in E(\omega, i + 1, w, q_{i+1}) \). If \( r \) small enough, we will know that \( v(n + M_{i+1} + n_{i+1}^{k+1}, y) \) is a comment prefix \( v(n + M_{i+1} + n_{i+1}^{k+1}, y) \) and its left and right. This will implies \( B(x, r) \subset v(n + M_{i+1} + n_{i+1}^{k+1}, y) \) and \( r \) is comparable with \( |v(n + M_{i+1} + n_{i+1}^{k+1}, y)| \). The result follows since we have a good control of the measure \( v(n + M_{i+1} + n_{i+1}^{k+1}, y) \).

First, for any \( y \in E(\omega, i + 1, w, q_{i+1}) \), for any \( k \geq \kappa_{i+1} \) such that \( y \in I^w_\omega \) with \( v \in \Sigma_{\omega}^{\alpha + M_{i+1} + \alpha} \), for any \( v' \in \Sigma_{\alpha + M_{i+1} + \alpha} \), with \( v'|_{\alpha} = v \), we have

\[
\frac{|I^w_\omega v'|}{|I^w_\omega v|} = \frac{\tilde{\mu}_\omega([w * v']_\omega)}{\tilde{\mu}_\omega([w * v]_\omega)} \geq \exp\left( S_{n_{k+1}^{i+1}}(\Phi(F^{n+M_{i+1}+n_{i+1}^{k+1}}(\omega, v')) - 2(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right) (\text{see (24)})
\]

\[
\geq \exp\left( - (n_{i+1}^{k+1} + n_{i+1}^{k+1})C - 2(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right)
\]

\[
(\text{noting } \sigma^{n+M_{i+1}+n_{i+1}^{k+1}}(\omega) \in \Omega_{i+1} \text{ and (13)})
\]

\[
(26) \geq \exp\left( - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right) (\text{see (17)}).
\]

Next, choose the largest \( k \) and \( v \in \Sigma_{\alpha + M_{i+1} + \alpha} \) such that \( y \in I^w_\omega \) and \( r \leq |I^w_\omega| \exp(-n(\varepsilon_{i+1})^2) \). We have, applying the shorthand \( F^\alpha = F^{n+M_{i+1}} \).

\[
\frac{r}{|I^w_\omega|} \leq \frac{|I^w_\omega| \exp\left( - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right)}{|I^w_\omega|}
\]

\[
= \frac{\tilde{\mu}_\omega([w * v]_\omega) \exp\left( - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right)}{\tilde{\mu}_\omega([w]_\omega)}
\]

\[
\leq \frac{\tilde{\mu}_\omega([w * v]_\omega) \exp\left( - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right)}{\tilde{\mu}_\omega([w]_\omega)}
\]

\[
\leq \exp\left( S_{n_{k+1}^{i+1}}(\Phi(F^{n+M_{i+1}}(\omega, v')) + (n + M_{i+1})c(\Phi, \omega, n + M_{i+1})) \right)
\]

\[
\cdot \exp\left( (n + n_{i+1}^{k+1})c(\Phi, \omega, n + M_{i+1} + n_{i+1}^{k+1}) - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right)
\]

\[
\leq \exp\left( S_{n_{k+1}^{i+1}}(\Phi(F^{n+M_{i+1}}(\omega, v')) \right)
\]

\[
\cdot \exp\left( 2(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 - (C + 2)(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right) \text{ (due to (24))}
\]

\[
\leq \exp\left( - \frac{(n_{i+1}^{k+1})c_\Phi}{2} - C(n + M_{i+1} + n_{i+1}^{k+1})(\varepsilon_{i+1})^4 \right) \text{ (due to (16))}
\]

\[
\leq \exp\left( - \frac{n_{i+1}^{k+1}c_\Phi}{2} \right) \text{ (due to (14))},
\]
so

\[
n_{k}^{i+1} \leq -\frac{2\log(r)}{c_{\Phi}}.
\]

Also, since \(y \in E(\omega, i + 1, w, q_{i+1})\), for \(v' \in \Sigma_{q_{n}+\mathbf{M}_{i+1},(p_{n+1})_{\omega}}\), we know that \(v\) is a common prefix of \(v', v^{'} - 1\), and \(v'\). Thus, due to (26) and our choice for \(k\), \(B(y, r) \subset (I_{\omega}^{w,v'} - \mathbf{M}_{k}^{i} I_{\omega}^{w,v'} + I_{\omega}^{w,v'}) \subset I_{\omega}^{w,v}\) and \(r \geq |I_{\omega}^{w,v'}| \exp(-(C + 2)(n + M_{i} + n_{k}^{i+1})(\varepsilon_{i+1})^{4})\). From the last inequality we can get

\[
\frac{r}{|I_{\omega}^{w}|} \geq \frac{|I_{\omega}^{w,v'}| \exp(-(C + 2)(n + M_{i} + n_{k}^{i+1})(\varepsilon_{i+1})^{4})}{|I_{\omega}^{w}|}
= \frac{\tilde{\mu}_{\omega}([w * v']_{\omega}) \exp(-(C + 2)(n + M_{i} + n_{k}^{i+1})(\varepsilon_{i+1})^{4})}{\mu_{\omega}([w]_{\omega})}
\geq \exp \left(- (C + 2)(n + M_{i} + n_{k}^{i+1})(\varepsilon_{i+1})^{4}\right) \cdot \exp \left(S_{n_{k}^{i+1} \Phi(\tilde{F}^{n}(\omega, v'))} - 2(n + M_{i} + n_{k}^{i+1})(\varepsilon_{i+1})^{4}\right)
\geq \exp \left(S_{n_{k}^{i+1} \Phi(\tilde{F}^{n}(\omega, v'))} - 2n_{k}^{i+1}(\varepsilon_{i+1})^{4}\right)
\geq \exp \left( - (C + 4)(n + n_{k}^{i+1} + n_{k}^{i+1})(\varepsilon_{i+1})^{4}\right) \cdot \exp \left( - (C + 4)(n + n_{k}^{i+1}(\varepsilon_{i+1})^{4} + n_{k}^{i+1}(1 + (\varepsilon_{i+1})^{4}))\varepsilon_{i+1})^{4}\right)
\geq \exp \left( - (C + 4)(n + n_{k}^{i+1}(\varepsilon_{i+1})^{4} + n_{k}^{i+1}(1 + (\varepsilon_{i+1})^{4}))\varepsilon_{i+1})^{4}\right)
\]

(see (14) to control \(M_{i} + 1\) and (17) for \(n_{k}^{i+1} - n_{k}^{i+1}\) since \(k \geq k_{i+1}\)).

Thus, noticing that \(\varepsilon_{i+1} < 1/2\), we get

\[
\frac{r}{|I_{\omega}^{w}|} \geq \exp \left(S_{n_{k}^{i+1} - M_{i+1} \Phi(\tilde{F}^{n}(\omega, v'))} - (4C + 8)(n + n_{k}^{i+1})(\varepsilon_{i+1})^{4}\right).
\]

Since \(r \leq |I_{\omega}^{w}| \exp(-n(\varepsilon_{i+1})^{2})\), we deduce

\[
n(\varepsilon_{i+1})^{2} \leq -S_{n_{k}^{i+1} \Phi(\tilde{F}^{n}(\omega, v'))} + (4C + 8)(n + n_{k}^{i+1})(\varepsilon_{i+1})^{4}
\leq Cn_{k}^{i+1} + (4C + 8)(n + n_{k}^{i+1})(\varepsilon_{i+1})^{4}.
\]

This yields \(n_{k}^{i+1} \geq \frac{n(\varepsilon_{i+1})^{2} - (4C + 8)n(\varepsilon_{i+1})^{4}}{C + (4C + 8)(\varepsilon_{i+1})^{4}} \geq \frac{n(\varepsilon_{i+1})^{2}}{2C}, \) so \(n \leq 2Cn_{k}^{i+1}/((\varepsilon_{i+1})^{2})\), then,

\[
\frac{r}{|I_{\omega}^{w}|} \geq \exp \left(S_{n_{k}^{i+1} \Phi(\tilde{F}^{n}(\omega, v'))} - (4C + 8)(2C + 1)n_{k}^{i+1}(\varepsilon_{i+1})^{2}\right).
\]
Consequently,
\[
\zeta_{\omega,w,q_i+1}(B(y,r)) 
\leq \zeta_{\omega,w,q_i+1}(I^w_{n+M_i+1}) 
\leq \exp((T^*)^{d_i+1} - (\varepsilon_i+1)^2)S_{n_i+1}(i^+\Phi(i^+\Phi(i^+\Phi(i^+\Phi(I^w_{\omega,w})))) \text{ (see (19))}
\leq \exp((T^*)^{d_i+1} - (\varepsilon_i+1)^2)S_{n_i+1}(i^+\Phi(i^+\Phi(I^w_{\omega,w})))) + 2n_i^{+1}(\varepsilon_i+1)^4\}
\leq \left(\frac{r}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2}\cdot \exp((T^*)^{d_i+1} - (\varepsilon_i+1)^2)(4C + 8)(2C + 1)n_i^{+1}(\varepsilon_i+1)^2 + 2n_i^{+1}(\varepsilon_i+1)^4,
\]
where we have used (28). Then,
\[
\zeta_{\omega,w,q_i+1}(B(y,r)) \leq \left(\frac{r}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2}\cdot \exp((8(C + 2)(C + 1)n_i^{+1}(\varepsilon_i+1)^2)
\leq \left(\frac{r}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2} (\frac{r}{|I^w_{\omega,w}|}) \cdot \exp((8(C + 2)(C + 1)n_i^{+1}(\varepsilon_i+1)^2)
\leq \left(\frac{r}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2} (\frac{r}{|I^w_{\omega,w}|}) \cdot \exp((4C + 8)(2C + 1)n_i^{+1}(\varepsilon_i+1)^2)
\leq \left(\frac{r}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2}.
\]
(from the choice \(\varepsilon_i+1 \leq \frac{C}{16(2C + 1 + 2e_i)}\).

Choose \(\kappa_i^{+1} > \kappa_i^{+1}\) large enough so that:
• for any \(v \in \Sigma_{\omega,n+M_i+1,w,n_i^{+1}}\) one has \(2\lambda_{\omega,w} \leq |I^w_{\omega,w}| \exp(-n(\varepsilon_i+1)^2)\);
• for any \(j \geq n + n_i^{+1}\), one has \(e(\Phi_i, \omega, j) \leq (\varepsilon_i+2)^4\);
• \(W(i + 2) \leq (\varepsilon_i+2)^4(n + M_i + 2 + n_i^{+1})\);
• for any \(s\) such that the return time \(\theta(i + 2, \omega, s)\) satisfies \(\theta(i + 2, \omega, s) \geq n + M_i + 2 + n_i^{+1}\), one also has
\[
\frac{\theta(i + 2, \omega, s) - \theta(i + 2, \omega, s - 1)}{\theta(i + 2, \omega, s - 1)} \leq (\varepsilon_i+1)^4.
\]

For any \(k \geq \kappa_i^{+1}\), from (25) we can get:
\[
\zeta_{\omega,w,q_i+1}(B(y_i, 2\lambda_{\omega,w}^{w+1}(k,l))) \leq \left(\frac{4\lambda_{\omega,w}^{w+1}(k,l)}{|I^w_{\omega,w}|}\right)^{(T^*)^{d_i+1} - (\varepsilon_i+1)^2}.
\]

Let \(s_{i+2} = s_i^2(\omega, w)\) be the smallest \(s\) such that there exists \(v \in \Sigma_{\omega,\theta(i + 2, \omega, s_{i+2})}\) such that:
• \(z_{\omega,w}^{w+1}(k,l)\) belongs to the closure of the interval \(I^w_{\omega}\),
• \(I^w_{\omega} \subseteq B(z_{\omega,w}^{w+1}(k,l), (2\lambda_{\omega,w}^{w+1}(k,l))/\delta^2)\)

Define \(J_l\) to be \(I^w_{\omega}\), the closure of the interval \(I^w_{\omega}\), and denote \(B_{\xi} = J_{\xi}\) and \(B_{\xi} = \hat{J}_{\xi}\). From the construction, we can claim:
\[
|J_l| \leq |\hat{J}_{\xi}| \leq |J_l|^{1-\varepsilon_i+1)^2}.
\]
Since $s_{i+2}$ is the smallest one, so for $v' = v|_{y(i+2,\omega,s_{i+2}-1)} \in \Sigma_{\omega_n}(i+2,\omega,s_{i+2}-1)$, we have $|I_{\omega}^{v'}| \geq (\omega_{\theta(i+2,\omega,s_{i+2}-1)})^{\xi}$. Now let $K_{i+1}$ be the largest $k$ such that $n + M_{i+1} + n_k^{i+1} \leq \theta(i + 2, \omega, s_{i+2} - 1)$ (by construction we have $k \geq \kappa_{i+1}$). Due to (17), we have 
\[\theta(i + 2, \omega, s_{i+2} - 1) - n - M_{i+1} - n_k^{i+1} \leq n_k^{i+1} - n_{K_{i+1}}^{i+1} \leq (\varepsilon_{i+1})^4 n_{K_{i+1}}^{i+1}.\]

Using a similar method as in the proof of (17), we can get 
\[\frac{|I_{\omega}^{v'}|}{|I_{\omega}^n|} \leq \exp \left(\frac{2(C + 1)\theta(i + 2, \omega, s_{i+2}) (\varepsilon_{i+1})^4}{c_\eta} - 4(C + 1)(\varepsilon_{i+1})^4\right) \leq \frac{|I_{\omega}^n|}{|I_{\omega}^n|^{-(\varepsilon_{i+1})^3}},\]
so 
\[(2\theta(i+2,\omega,s_{i+2}))^\xi \leq 2^\xi |I_{\omega}^{v'}| \leq 2^\xi |I_{\omega}^n|^{1-(\varepsilon_{i+1})^3} \leq |I_{\omega}^n|^{1-(\varepsilon_{i+1})^2}.\]
So that (30) follows.

For $k$ large enough so that $n_k^{i+1} \geq \frac{n}{(\varepsilon_{i+1})^7}$, define
\[G_n(i + 1, d_{i+1}, k) = \{B_t, B_t \in D_n(i + 1, d_{i+1}, k)\}.\]

If $J_1$ and $J_2$ are two distinct elements of $G_n(i + 1, d_{i+1}, k)$ then their distance is at least $\max_{i \in [1, 2]} (|\bar{J}_i|/2 - (|\bar{J}_i|/2)^\xi)$, which is larger than $\max_{i \in [1, 2]} |\bar{J}_i|/3$ for $k$ large enough (since $\xi > 1$).

We can define $m_{\xi}^{(d_j)}(j)_{i \leq j \leq i+1}$ with $d_{i+1} \in D_{i+1}$ as follows,
\[m_{\xi}^{(d_j)}(j)_{i \leq j \leq i+1}(J) = \frac{\zeta_{\omega, w, q_{i+1}}(\bar{J})}{\sum_{J_i \in G_n(i+1, d_{i+1}, k)} \zeta_{\omega, w, q_{i+1}}(\bar{J})} \left(m_{\xi}^{(d_j)}(j)_{i \leq j \leq i+1}(T_{\omega}^n)\right).\]

For any $J \in G_n(i + 1, d_{i+1}, k)$, from the inequality (29) we get obtain
\[\zeta_{\omega, w, q_{i+1}}(\bar{J}) \leq \frac{|\bar{J}|}{|I_{\omega}^n|^{\xi - \frac{\varepsilon_{i+1}}{\xi}}} \leq |J|^{\xi - \frac{\varepsilon_{i+1}}{\xi}} |I_{\omega}^n|^{-\xi - \frac{\varepsilon_{i+1}}{\xi}} \leq |J|^{\frac{\varepsilon_{i+1}}{\xi}} |I_{\omega}^n|^{-\xi - \frac{\varepsilon_{i+1}}{\xi}}.\]
Then, the inequality
\[\sum_{J_i \in G_n(i+1, d_{i+1}, k)} \zeta_{\omega, w, q_{i+1}}(\bar{J}) \geq \frac{1}{4\Gamma_1},\]
yields, $\forall J \in G_n(i + 1, d_{i+1}, k)$:
\[m_{\xi}^{(d_j)}(j)_{i \leq j \leq i+1}(J) \leq 4\Gamma_1 |J|^{\frac{\varepsilon_{i+1}}{\xi}} |I_{\omega}^n|^{-\xi - \frac{\varepsilon_{i+1}}{\xi}}.\]
Let $k_{i+1} > k'_{i+1}$ large enough so that $n_{k_{i+1}} > \frac{n}{(\varepsilon_{i+1})^3}$. For any $d_{i+1} \in D_{i+1}$ and for any $J \in G^w(i+1, d_{i+1}, k_{i+1})$, one has 

$$4\Gamma_1 |{w}| - T^*(d_{i+1})$$

$$\leq 4\Gamma_1 \exp(-Cn) - T^*(d_{i+1})$$

$$\leq \exp(CT^*(d_{i+1}) + \log(4\Gamma_1))$$

$$\leq \exp \left( \frac{c_\Phi}{6 \xi} \cdot \frac{n}{\varepsilon_{i+1}} \right)$$

(noticing that $T^* \leq -T(0) = 1, \log(4\Gamma_1) \leq n$ and $\varepsilon_{i+1} \leq \frac{c_\Phi}{6\xi(C+1)}$)

$$\leq \exp \left( \frac{c_\Phi}{6 \xi} \cdot \frac{n + M_{i+1} + n_{k_{i+1}}}{\varepsilon_{i+1}} \right) = \left( \exp \left( -\frac{c_\Phi}{2} (n + M_{i+1} + n_{k_{i+1}}) \right) \right)$$

$$\leq \left| J \right|^{-\frac{\varepsilon_{i+1}}{6\xi}} \leq \left| J \right|^{-\frac{\varepsilon_{i+1}}{6\xi}} \right.,$$

where the second inequality in the last line comes from (20) and we have chosen $n$ large enough in the first step. Consequently,

$$\forall J \in G^w(i+1, d_{i+1}, k_{i+1}), \ n_{k_{i+1}} \leq \left| J \right|^{-\frac{T^*(d_{i+1}) - \varepsilon_{i+1}}{\xi}}.$$ 

For $(d_j)_{1 \leq j \leq i+1} \in \prod_{1 \leq j \leq i+1} D_j$ define:

$$G(d_1, d_2, \cdots, d_{i+1}) = \bigcup_{w \in G(d_1, d_2, \cdots, d_i)} G^w(i+1, d_{i+1}, k_{i+1}),$$

and

$$G_{i+1} = \bigcup_{w \in G(i)} \bigcup_{w \in G(i)} G^w(i+1, d_{i+1}, k_{i+1}).$$

The definition of $m^w_{\xi} \{ d_j \}_{1 \leq j \leq i+1}$ can be extended to the algebra generated by

$$\bigcup_{s \leq i+1} G(d_1, d_2, \cdots, d_s),$$

and for any $J = I^w_{\xi} \in G(d_1, d_2, \cdots, d_s)$,

$$m^w_{\xi} \{ d_j \}_{1 \leq j \leq i+1}(J) \leq \left| J \right|^{-\frac{T^*(d_{i+1}) - \varepsilon_{i+1}}{\xi}}.$$ 

**Step 3:** For any $\tilde{d} = \{ d_j \}_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} D_i$, for any $J \in G(d_1, \cdots, d_i)$, define $m^w_{\xi}(J) = m^w_{\xi} \{ d_j \}_{1 \leq j \leq i}(J)$. This yields a probability measure $m^w_{\xi}$ on the algebra generated by $\bigcup_{i \in \mathbb{N}} G(d_1, \cdots, d_i)$.

For any $i \in \mathbb{N}$, the elements in $G(d_1, \cdots, d_i)$ are closed and disjoint intervals. Also, for any $J \in G(d_1, \cdots, d_i)$, let $\tilde{J}$ be the ball associated with $J$. We have the following properties:

1. $J \subset \tilde{J}$, for any $J \in G(d_1, \cdots, d_i)$;
2. for any $J \in G(d_1, \cdots, d_i)$

$$|J| \leq |\tilde{J}|^\xi \leq |J|^{1-(\xi)^3};$$

- if $J_1 \neq J_2$ belong to $G(d_1, \cdots, d_i)$, their distance is at least $\max_{i \in \{1, 2\}} |\tilde{J}_i|/\xi$;
- The intervals $\tilde{J}_1, \tilde{J}_2 \in G(d_1, \cdots, d_i)$, are disjoint.

2. For any $J$ in $G(d_1, d_2, \cdots, d_i)$, $\tilde{J} \cap E(\omega, i, w, q_i) \neq \emptyset$, where $q_i \in Q_i$ is such that $T_{\tilde{J}}(q_i) = d_i$ and $E(\omega, i, w, q_i)$ is the set used in step 2.
(3) For any \( J \in G(d_1, d_2, \cdots, d_i) \),

\[
m^i_\xi(J) \leq |J|^{\frac{T^*(d_i) - \epsilon_i}{\zeta}}.
\]

(4) Any \( J \in G(d_1, d_2, \cdots, d_i) \) is contained in some element \( L = T_w^i \in G(d_1, d_2, \cdots, d_{i-1}) \) such that

\[
m^i_\xi(J) \leq 4\Gamma_1 m^i_\xi(L) \zeta_{\omega,w,q_i}(\tilde{J}),
\]

where \( q_i \in Q_i \) is such that \( T^i_{j_i}(q_i) = d_i \).

Because of the separation property 1, we get a probability measure \( m^\tilde{\xi} \) on \( \sigma(J : J \in \bigcup_{i \geq 1} G(d_1, d_2, \cdots, d_i)) \) such that properties 1 to 4 hold for every \( i \geq 1 \). We now define

\[
K(\xi, \tilde{d}) = \bigcap_{i \geq 1} \bigcup_{J \in G(d_1, \cdots, d_i)} J,
\]

then, \( m^\tilde{\xi}(K(\xi, \tilde{d})) = 1 \). The measure \( m^\tilde{\xi} \) can be extended to \([0, 1]\) by setting, for any \( B \in \mathcal{B}([0, 1]) \), \( m^\tilde{\xi}(B) := m^\tilde{\xi}(B \cap K(\xi, \tilde{d})). \)

**Step 4:** Fix a sequence \( \tilde{d} = \{d_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} D_i \) such that

\[
\lim_{i \to \infty} d_i = d, \quad \lim_{i \to \infty} T^*(d_i) = T^*(d).
\]

Define \( K^d(\xi) = K(\xi, \tilde{d}) \), and \( m^\xi_\tilde{\xi} = m^\tilde{\xi} \).

From the construction, we claim that: \( K^d(\xi) \subset S(\omega, d, \xi, \tilde{c}) \). In fact we just need to prove the following lemma:

**Lemma 8.4.** For any \( w \in G(d_1, d_2, \cdots, d_i) \) with \( n = |w| \), for any \( v \in G^w(i + 1, d_{i+1}, k) \) with \( n^i_k + \geq \frac{n}{(i + 1)^2} \), for any \( \tilde{y} \in [w \ast v]_\omega \), we have:

\[
\left| \frac{S_{n+M_i+1+n^i_k+1}^i \Psi(\omega, \tilde{y})}{S_{n+M_i+1+n^i_k+1}^i \Phi(\omega, \tilde{y})} - d \right| \leq \epsilon_{i+1}.
\]

where \( \epsilon_{i+1} = |d - d_{i+1}| + 2\epsilon_{i+1} \).

**Proof.** First, for any \( \Upsilon \in \{\Phi, \Psi\} \),

\[
\left| S_{n+M_i+1+n^i_k+1}^i \Upsilon(\omega, \tilde{y}) - S_{n^i_k+1}^i \Upsilon_{j_i+1}(F^{n+M_i+1}(\omega, \tilde{y})) \right|
\]

\[
\leq 2(n + M_i+1)C + \left| S_{n^i_k+1}(\Upsilon - \Upsilon_{j_i+1})(F^{n+M_i+1}(\omega, \tilde{y})) \right|
\]

\[
\leq 2(n + M_i+1)C + S_{n^i_k+1}^i \text{var}_{j_i+1} \Upsilon(\sigma^{n+M_i+1} \omega)
\]

\[
\leq 4n^i_k (\epsilon_{i+1})^2 + 2n^i_k (\epsilon_{j_i+1})^4 \quad (\text{see (15)} \quad \text{and} \quad \int_{\Omega} \text{var}_{j_i+1} \Upsilon \ d\mathbb{P} \leq (\epsilon_{i+1})^4)
\]

\[
\leq 5n^i_k (\epsilon_{j_i+1})^2.
\]
Next, applying again the shorthand $\tilde F^n = F^{n+M_i+1}$,

$$\frac{S_{n+M_i+1+n_k+1}(\omega, \nu)}{S_{n+M_i+1+n_k+1}(\omega, \nu)} - d \leq \left| \frac{S_{n+M_i+1+n_k+1}(\omega, \nu) - d_{i+1}}{S_{n+M_i+1+n_k+1}(\omega, \nu)} \right| + |d - d_{i+1}|$$

$$\leq \left| \frac{S_{n+M_i+1+n_k+1}(\omega, \nu) - S_{n_k+1}(\tilde F^n(\omega, \nu))}{S_{n+M_i+1+n_k+1}(\omega, \nu)} \right| + d_{i+1}$$

$$\left. \leq \left| \frac{S_{n_k+1}(\tilde F^n(\omega, \nu)) - d_{i+1}S_{n_k+1}(\tilde F^n(\omega, \nu))}{S_{n+M_i+1+n_k+1}(\omega, \nu)} \right| + |d - d_{i+1}| \right.$$  

where we have used (18) and (15). Thus,

$$\frac{S_{n+M_i+1+n_k+1}(\omega, \nu)}{S_{n+M_i+1+n_k+1}(\omega, \nu)} - d \leq (\varepsilon_{j+1})^2 + 5(d_{i+1} + 1)\varepsilon_{j+1}^2 \left| \frac{\varepsilon_{j+1}^4 + 5(d_{i+1} + 1)\varepsilon_{j+1}^2}{(n+M_i+1+n_k+1)c_\Phi} \right| + |d - d_{i+1}|$$

$$\leq (\varepsilon_{j+1})^2 + 10(\varepsilon_{j+1})^4 + 10\varepsilon_{j+1}^2 |d - d_{i+1}|$$

$$\leq (\varepsilon_{j+1})^2 + \varepsilon_{j+1} + |d - d_{i+1}| \leq \varepsilon_{i+1}' .$$

Now we turn to estimate the lower Hausdorff dimension of $m_B^\xi$. If $T^*(d) = 0$, there is nothing need to prove. So we assume that $T^*(d) > 0$.

For any $J \in G(d_1, d_2, \cdots, d_\ell)$, define $g(J) = i$. Let us fix $B$ a subinterval of $[0, 1]$ of length smaller than that of every element in $G(d_1)$, and assume that $B \cap K_i^\xi \neq \emptyset$. Let $L = \overline{T^n}$ be the element of largest diameter in $\bigcup_{j \geq 1} G(d_1 \cdots d_i)$ such that $B$ intersects at least two elements of $G(d_1 \cdots d_g(L) + 1)$ and is included in $L \in G(d_1 \cdots d_g(L))$. We remark that this implies that $B$ does not intersect any other element of $G(d_1, d_2, \cdots, d_i)$, where $i = g(L)$, and as a consequence $m_B^\xi(B) \leq m_B^\xi(L)$.

Let us distinguish three cases:

- $|B| \geq |L|$: then

$$m_B^\xi(B) \leq m_B^\xi(L) \leq |L| \frac{(T^*(d_i)-\varepsilon_2)}{\xi} \leq |B| \frac{(T^*(d_i)-\varepsilon_2)}{\xi} .$$

(32)
We have finished the proof of theorem 8.2. Assume $L_1, \ldots, L_p$ are the elements of $G_{i+1}$ which have non-empty intersection with $B$. From property 4, we can choose $q_{i+1} \in Q_{i+1}$ so that $T_{j+1}(q_{i+1}) = d_{i+1}$, and get

$$m^d(B) = \sum_{l=1}^p m^d(B \cap L_l) \leq 4\Gamma_1 m^d(L) \sum_{i=1}^p \zeta_{\omega,w,q_{i+1}}(\hat{L}_l).$$

From property 1 we can also deduce that $\max\{|\hat{L}_l|: 1 \leq l \leq p\} \leq 3|B|$. From property 2 we can get $E(\omega, i+1, w, q_{i+1}) \cap \hat{L}_l \neq \emptyset$. If $y$ is taken in the intersection, we have $B(y, 4|B|) \supset \left(\bigcup_{l=1}^p \hat{L}_l\right)$.

Now we notice that $L = \bar{I}_w^w$ is the closure of $I_w^w$ for some $w \in \Sigma_{\omega,n}$ with $n \in \mathbb{N}$, and we have $\sigma^n \omega \in \Omega_{i+1}$. Using (25) in lemma 8.3, we can get:

$$\zeta_{\omega,w,q_{i+1}}(B(y, 4|B|)) \leq \left(\frac{4|B|}{|I_w^w|}\right)^{\tau^*(d_{i+1})-\frac{\varepsilon_{i+1}}{\xi}}.$$

Now, since $L = \bar{I}_w^w$ is the closure of $I_w^w$, we have:

$$m^d(B) \leq 4\Gamma_1 m^d(L) \sum_{i=1}^p \zeta_{\omega,w,q_{i+1}}(\hat{L}_l) \leq 4\Gamma_1 m^d(L) \zeta_{\omega,w,q_{i+1}}(B(y, 4|B|)) \leq 4\Gamma_1 L \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})} \leq 4\Gamma_1 (4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})} \leq 4\Gamma_1 (4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})},$$

where $\alpha_i = \tau^*(d_{i+1})-\frac{\varepsilon_{i+1}}{\xi}$ is positive for $i$ large enough since $\lim_{i \to \infty} \tau^*(d_i) = \tau^*(d) > 0$. Moreover, $4|B|/|L| \leq 1$, so

$$m^d(B) \leq 4\Gamma_1 (4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})}.$$  (33)

$$\frac{1}{2}|L| \exp(-|w|(\varepsilon_{i+1})^2) \leq |B| \leq |L|.$$  (32)

We need at most $M(B) = [4 \exp(|w|(\varepsilon_{i+1})^2)]+1$ contiguous intervals $(B(k))_{1 \leq k \leq M(B)}$ with diameter $\frac{1}{2}|L| \exp(-|w|(\varepsilon_{i+1})^2)$ to cover $B$. For these intervals we have the estimate above. Consequently,

$$m^d(\hat{B}) \leq \sum_{k=1}^{M(B)} 4\Gamma_1 (4|B(k)|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})} \leq 4\Gamma_1 M(B)(4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})} \leq 20\Gamma_1 \exp(|w|(\varepsilon_{i+1})^2)(4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})}.$$

Since $|B| \leq |L| \leq \exp\left(-\frac{\kappa e_0}{2}\right)$, we get $\exp(|w|(\varepsilon_{i+1})^2) \leq |L|^{-\frac{2(\varepsilon_{i+1})^2}{e_0}} \leq |B|^{-\frac{2(\varepsilon_{i+1})^2}{e_0}}$. Finally,

$$m^d(\hat{B}) \leq 20\Gamma_1 (4|B|) \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})} \leq \frac{\tau^*(d_{i+1})-\varepsilon_{i+1}}{\tau^*(d_{i+1})}.$$  (34)

It follows from the estimations (32),(33) and (34) that $\dim_H(m^d) \geq \tau^*(d) - \varepsilon_{i+1} / \tau^*(d)$. We have finished the proof of theorem 8.2.
9. Conclusion on the lower bound for the lower Hausdorff spectrum

Next proposition is both a complement to proposition 9, and an improvement over the interval $[\mathcal{T}'(+\infty), \mathcal{T}'(t_0-)]$.

**Proposition 10.** For $\mathbb{P}$-a.e. $\omega$, for any $d \in [0, \mathcal{T}'(t_0-)]$, one has

$$\dim_H(E(\nu_\omega, d)) \geq t_0d = (\dim_H X_\omega) \cdot d.\]

**Proof.** If $d \in (0, \mathcal{T}'(t_0-)]$, we write $d = \mathcal{T}'(t_0-)/\xi$ with $\xi \geq 1$. We can find a suitable sequence $\tilde{\xi}$ such that theorem 8.2 and remark 10 hold. This provides us with a positive Borel measure $m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}$ on $\mathcal{K}^{\mathcal{T}'(t_0-)}(\xi)$, with the following properties:

- $m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}(\mathcal{K}^{\mathcal{T}'(t_0-)}(\xi)) = 1$ and $\dim_H(m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}) \geq \frac{\mathcal{T}'(t_0-)}{\xi} = d t_0$.

- $m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}(E) = 0$ as soon as $\dim_H E < d t_0$.

- For any $x \in \mathcal{K}^{\mathcal{T}'(t_0-)}(\xi)$, we have that $\dim_{\text{loc}}(\nu_\omega, x) \leq d$.

It follows from lemma 7 that

$$(\mathcal{K}^{\mathcal{T}'(t_0-)}(\xi) \setminus \bigcup_{0 \leq h < d} F(h)) \subset (E(\nu_\omega, d) \cup \Xi_\omega).$$

Also, corollary 1 tells dim$_H F(h) \leq h t_0 < d t_0$ for all $0 \leq h < d$, so $m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}(F(h)) = 0$ for all $0 \leq h < d$. Moreover, the family of sets $(F(h))_{0 < h < d}$ is nondecreasing. Thus, we have

$$m_{\tilde{\xi}}^{\mathcal{T}'(t_0-)}(E(\nu_\omega, d) \cup \Xi_\omega) > 0,$$

hence

$$\dim_H(E(\nu_\omega, d) \cup \Xi_\omega) \geq d t_0.$$

Finally, dim$_H E(\nu_\omega, d) \geq d t_0$ since $\Xi_\omega$ is a countable set.

If $d = 0$ or $t_0 = 0$, we have

$$\emptyset \neq \Xi'_\omega \subset E(\nu_\omega, 0),$$

thus dim$_H E(\nu_\omega, d) \geq d t_0$ for $d = 0$. \hfill \Box

Next proposition collects all the information required to conclude regarding the lower bound for the lower Hausdorff spectrum. Its claim (iii) is the desired sharp lower bound.

**Proposition 11.** For $\mathbb{P}$-a.e. $\omega$:

1. if $d \in [0, \mathcal{T}'(t_0-)]$, then dim$_H(E(\nu_\omega, d)) \geq t_0d$,
2. if $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, then dim$_H(E(\nu_\omega, d)) \geq \mathcal{T}^*(d)$,
3. for any $d \in [0, \mathcal{T}'(-\infty)]$, dim$_H(E(\nu_\omega, d)) \geq \mathcal{T}^*(d)$.

**Proof.** (i) and (ii) come from proposition 10 and proposition 9.

To prove (iii), since $\overline{T}(q) = \min\{T(q), 0\}, T(t_0) = 0$ and $T$ is increasing,

$$\mathcal{T}^*(d) = \inf_{q \in \mathbb{R}} \{qd - \overline{T}(q)\} = \begin{cases} t_0d, & d \in [0, \mathcal{T}'(t_0-)], \\ \mathcal{T}^*(d), & d \in [\mathcal{T}'(t_0-), \mathcal{T}'(-\infty)]. \end{cases}$$

\hfill \Box
10. Hausdorff Dimensions of the Level Sets $E(\nu_\omega, d)$ and $E(\nu_\omega, d)$

Recall that $v(\omega, n, x)$ has been defined in definition 4.1. We need to introduce another approximation rate.

For any $x \in [0, 1] \setminus \{x^v_\omega : v \in \Sigma_\omega, s \in S'(\omega, v, 1)\}$, define

$$\hat{\xi}(\omega, n, x) = \frac{\log(\inf\{|x - x^v_\omega| : |v| \leq n, s \in S'(\omega, v, 1)\})}{\log |I^v_{\omega}(\omega, n, x)|}$$

and then

$$\hat{\xi}(\omega, x) = \liminf_{n \to \infty} \hat{\xi}(\omega, n, x).$$

The desired conclusion on the sets $E(\nu_\omega, d)$ and $E(\nu_\omega, d)$ will follow from two lemmas and one proposition.

**Lemma 10.1.** For $P$-a.e. $\omega \in \Omega$, we have

$$\{x \in [0, 1] \setminus \Xi_\omega : \hat{\xi}(\omega, x) > 1\} = \emptyset.$$

In other words, for any $x \in [0, 1]$, if $x \notin \Xi_\omega$, then $\hat{\xi}(\omega, x) = 1$.

**Proof.** We just need to prove that for any $k \in \mathbb{Z}^+$,

$$\{x \in [0, 1] \setminus \Xi_\omega : \hat{\xi}(\omega, x) > 1 + 1/k\} = \emptyset.$$

For any $x \in [0, 1] \setminus \Xi_\omega$ such that $\hat{\xi}(\omega, x) > 1 + 1/k$, there exists $N(x) \in \mathbb{Z}^+$ such that for any $n \geq N(x)$ one has

$$\inf\{|x - x^v_\omega| : |v| \leq n, s \in S'(\omega, v, 1)\} \leq |I^v_{\omega}(\omega, n, x)|^{1+1/k}.$$

Furthermore, the infimum must be attained at a point $x^v_\omega$ which is in the closure of $I^v_{\omega}(\omega, n, x)$.

We denote $v$s by $w(\omega, n, x)$. We just need to prove that $x$ is the point $x^{w(\omega, n, x)}_\omega$ for $n$ large enough. And $x^{w(\omega, n, x)}_\omega \in \Xi_\omega$ yields the lemma.

The choice of $x^{w(\omega, n + 1, x)}_\omega$ must be made in $\{x^{v(\omega, n + 1, x)}_\omega : s \in S'(\omega, v(\omega, n + 1, x), 1)\} \cup \{x^{w(\omega, n, x)}_\omega\}$. Otherwise it is easily seen that it is in contradiction with the choice of $w(\omega, n, x)$ and $x \in I^v_{\omega}(\omega, n, x)$.

![Figure 1. The choice for $w(\omega, n + 1, x)$](image)

We have

$$\inf\{|x - x^{v}_\omega| : |v| \leq n + 1, s \in S'(\omega, v, 1)\} = |x^{w(\omega, n + 1, x)}_\omega - x| \leq |I^v_{\omega}(\omega, n, x)|^{1 + 1/k} \leq |I^v_{\omega}(\omega, n, x)|^{1 + 1/k}.$$

Now suppose that $x^{w(\omega, n + 1, x)}_\omega \neq x^{w(\omega, n, x)}_\omega$. On the one hand, since $x^{w(\omega, n, x)}_\omega$ must be an endpoint of $I^v_{\omega}(\omega, n, x)$, there exists $s \in S(\omega, v(\omega, n + 1, x), 1)$ such that

$$|I^v_{\omega}(\omega, n, x)| \leq |x^{w(\omega, n, x)}_\omega - x^{w(\omega, n, x)}_\omega| \leq |x^{w(\omega, n, x)}_\omega - x| + |x^{w(\omega, n, x)}_\omega| \leq 2|I^v_{\omega}(\omega, n, x)|^{1 + 1/k},$$
Lemma 10.2. For \( P \text{-a.e. } \omega \in \Omega \), for any \( x \in [0,1] \setminus \Xi_{\omega} \), we have \( \dim_{\text{loc}}(\nu_{\omega}, x) \geq \liminf_{n \to \infty} \frac{\log \nu_{\omega}(I^{v(\omega, n, x)}_{\omega})}{\log |I^{v(\omega, n, x)}_{\omega}|} \), where \( v(\omega, n, x) \) is defined after definition 4.1 in section 4.

Proof. Suppose that the conclusion of lemma 10.1 holds for \( \omega \). If \( x \in [0,1] \setminus \Xi_{\omega} \), there exists a subsequence \( \{ n_k \}_{k \in \mathbb{Z}^+} \) such that \( \tilde{\xi}(\omega, n_k, x) \to 1 \) as \( k \to \infty \). Now,

\[
\limsup_{r \to 0} \frac{\log \nu_{\omega}(B(x, r))}{\log r} \geq \limsup_{k \to \infty} \frac{\log \nu_{\omega}(B(x, I^{v(\omega, n_k, x)}_{\omega} \tilde{\xi}(\omega, n_k, x)+1/n_k))}{\log |I^{v(\omega, n_k, x)}_{\omega} \tilde{\xi}(\omega, n_k, x)+1/n_k|} \\
\geq \limsup_{k \to \infty} \frac{\log \nu_{\omega}(|I^{v(\omega, n_k, x)}_{\omega}|)}{\log |I^{v(\omega, n_k, x)}_{\omega}|} \geq \liminf_{n \to \infty} \frac{\log \nu_{\omega}(|I^{v(\omega, n, x)}_{\omega}|)}{\log |I^{v(\omega, n, x)}_{\omega}|}.
\]

The second inequality follows from the fact that \( \tilde{\xi}(\omega, n_k, x) \to 1 \) as \( k \to \infty \) and

\[
B(x, |I^{v(\omega, n_k, x)}_{\omega} \tilde{\xi}(\omega, n_k, x)+1/n_k|) \subset I^{v(\omega, n_k, x)}_{\omega}
\]

by definition of \( \tilde{\xi}(\omega, n_k, x) \). \( \square \)

Proposition 12. For \( P \text{-a.e. } \omega \in \Omega \),

1. if \( d \in [T'(+\infty), T'(t_0-)] \), then \( \dim_{H}(\{ x \in [0,1] : \dim_{\text{loc}}(\nu_{\omega}, x) \leq d \}) \leq T^*(d) \);
2. if \( x \in [0,1] \setminus \Xi_{\omega}' \) then \( \dim_{\text{loc}}(\nu_{\omega}, x) \geq T'(+\infty) \).
3. \( E(\nu_{\omega}, 0) = \Xi(\nu_{\omega}, 0) = \Xi_{\omega}' \), so \( \dim_{H}(E(\nu_{\omega}, 0)) = \dim_{H}(E(\nu_{\omega}, 0)) = 0 \).

Proof. (i) For any \( d \in [T'(+\infty), T'(t_0-)] \), for any \( \varepsilon > 0 \), there exists \( q \geq 0 \) such that \( T^*(d) \geq qd - T(q) - \varepsilon/2 \). Choose \( \varepsilon > 0 \) such that \( q\varepsilon \leq \varepsilon/4 \). Due to the previous lemma, for any \( N \in \mathbb{N} \) we have

\[
\{ x \in [0,1] : \dim_{\text{loc}}(\nu_{\omega}, x) \leq d \} \subset \left\{ x \in [0,1] : \liminf_{n \to \infty} \frac{\log \nu_{\omega}(I^{v(\omega, n, x)}_{\omega})}{\log |I^{v(\omega, n, x)}_{\omega}|} \leq d \right\} \\
\subset \bigcup_{n \geq N} \bigcup_{v \in I_{\omega}} \bigcup_{i \in [0,1]} I^{v}_{\omega}.
\]
For any $\delta > 0$, for $N$ large enough, and $v \in \Sigma_{\omega,n}$, one has $|I^v_{\omega}| < \delta$. Choosing $h = T^*(d) + \epsilon$ we get for $N$ large enough,

$$
\mathcal{H}^h_{\delta}(\{x \in [0,1] \setminus \Xi_{\omega} : \dim_{\text{top}}(\nu_\omega, x) \leq d\}) \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} |I^v_{\omega}|^{T^*(d) + \epsilon} \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} |I^v_{\omega}|^{-T(q)} + \epsilon/2 \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} |I^v_{\omega}|^{-T(q)} + \epsilon/2 - qe^{\nu_\omega(I^v_{\omega})}q^q \leq \sum_{n \geq N} \sum_{v \in \Sigma_{\omega,n}} \exp(nP(q\Psi - T(q)\Phi - \frac{ncp\epsilon}{8}) \leq \sum_{n \geq N} \exp(-\frac{ncp\epsilon}{8}).
$$

Here we used the fact that $\nu_\omega(I^v_{\omega}) \leq |X^v_{\omega}| \leq |U^v_{\omega}| \leq \exp(S_n\Psi(\omega, x) + o(n))$ for any $\omega \in [v]_\omega$ and $v \in \Sigma_{\omega,n}$.

Letting $N$ go to $\infty$ we get $\mathcal{H}^h_{\delta}(\{x \in [0,1] \setminus \Xi_{\omega} : \dim_{\text{top}}(\nu_\omega, x) \leq d\}) = 0$ for any $\delta > 0$, so $\mathcal{H}^h(\{x \in [0,1] \setminus \Xi_{\omega} : \dim_{\text{top}}(\nu_\omega, x) \leq d\}) = 0$. This holds for any $h > -T^*(d)$, so $\dim_H(\{x \in [0,1] \setminus \Xi_{\omega} : \dim_{\text{top}}(\nu_\omega, x) \leq d\}) = T^*(d)$. Since $\Xi_{\omega}$ is countable we get the desired conclusion.

(ii) For any $v \in \Sigma_{\omega,n}$ define

$$
I^v_{\omega} = (F_{\mu_\omega}(m^v_{\omega}), F_{\mu_\omega}(M^v_{\omega})) = F_{\mu_\omega}(X^v_{\omega}) \setminus \{F_{\mu_\omega}(m^v_{\omega})\}.
$$

Then for $x \in [0,1]$ and $n \geq 1$, let $v'(\omega, n, x)$ be the unique element $v$ in $\Sigma_{\omega,n}$ such that $x \in I^v_{\omega}$. If $x = 0$, $v'(\omega, n, 0)$ is the unique $v \in \Sigma_{\omega,n}$ such that $1 \in \overline{T^v_{\omega}}$. In (i) we proved that $\{x \in [0,1] : \liminf_{n \to \infty} \frac{\log \nu_\omega(I^v_{\omega}(\omega, n, x))}{\log |I^v_{\omega}(\omega, n, x)|} \leq d\} = \emptyset$ for $d < T'(+\infty)$. Using the same arguments we can also prove that $\{x \in [0,1] : \liminf_{n \to \infty} \frac{\log \nu_\omega(I^{v'}_{\omega}(\omega, n, x))}{\log |I^{v'}_{\omega}(\omega, n, x)|} \leq d\} = \emptyset$ for $d < T'(+\infty)$. Thus, for any $x \in [0,1], (35)$

$$
\min \left( \liminf_{n \to \infty} \frac{\log \nu_\omega(I^{v}_{\omega}(\omega, n, x))}{\log |I^{v}_{\omega}(\omega, n, x)|}, \liminf_{n \to \infty} \frac{\log \nu_\omega(I^{v'}_{\omega}(\omega, n, x))}{\log |I^{v'}_{\omega}(\omega, n, x)|} \right) \geq T'(+\infty).
$$

Now we come to the proof of the assertion. From the proof of item 1, we just need to deal with the set $\Xi_{\omega} \setminus \Xi_{\omega}$. For $x \in \Xi_{\omega} \setminus \Xi_{\omega}$ and $r > 0$ small enough, there exist $n, n' \in \mathbb{N}$ such that $|I^v_{\omega}(\omega, n, x)| > r$, $|I^{v'}_{\omega}(\omega, n', x)| > r$ and $|I^v_{\omega}(\omega, n+1, x)| \leq r$, $|I^{v'}_{\omega}(\omega, n'+1, x)| \leq r$. Now we have $B(x, r) \subset I^v_{\omega}(\omega, n, x) \cup I^{v'}_{\omega}(\omega, n', x) \cup \{x\}$. Since $\nu_\omega(\{x\}) = 0$, we have

$$
\nu_\omega(B(x, r)) \leq \nu_\omega(I^v_{\omega}(\omega, n, x)) + \nu_\omega(I^{v'}_{\omega}(\omega, n', x)),$$

and it follows from (35) and the choices of $n$ and $n'$ that $\dim_{\text{top}}(\nu_\omega, x) \geq T'(+\infty).

(iii) It clearly follows from (ii). \qed

**Theorem 10.3.** For $\mathbb{P}$-a.e. $\omega \in \Omega$, for any $d \in [T'(+\infty), T'(-\infty)]$, we have

$$\dim_H E(\nu_\omega, d) = \dim_H \overline{E}(\nu_\omega, d) = T^*(d).$$

**Proof.** The expected lower bound for the Hausdorff dimensions of $E(\nu_\omega, d) \subset \overline{E}(\nu_\omega, d)$ was already obtained in the proof of proposition 9, while the upper bound was obtained in
the previous proposition for $d \in \left[ T', (\infty), T'(t_0-) \right]$, and it follows from the multifractal formalism for $d \in \left[ T'(t_0-), T'(-\infty) \right]$, since $\tau_{\nu_0}^* \leq T^*$.

□

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