Abstract. I give a short and completely elementary proof of Takagi’s 1921 theorem on the zeros of a composite polynomial \( f(d/dz)g(z) \).

Many theorems in the analytic theory of polynomials \([2, 8, 10, 11]\) are concerned with locating the zeros of composite polynomials. More specifically, let \( f \) and \( g \) be polynomials (with complex coefficients) and let \( h \) be a polynomial formed in some way from \( f \) and \( g \); under the assumption that the zeros of \( f \) (respectively, \( g \)) lie in a subset \( S \) (respectively, \( T \)) of the complex plane, we wish to deduce that the zeros of \( h \) lie in some subset \( U \). The theorems are distinguished by the nature of the operation defining \( h \), and the nature of the subsets \( S, T, U \) under consideration.

Here we shall be concerned with differential composition: \( h(z) = f(d/dz)g(z) \), or \( h = f(D)g \) for short. In detail, if \( f(z) = \sum_{i=1}^{m} a_i z^i \) and \( g(z) = \sum_{j=1}^{n} b_j z^j \), then \( h(z) = \sum_{i=1}^{m} a_i g^{(i)}(z) \); and \( D \) denotes the differentiation operator, i.e., \( Dg = g' \). The following important result was found by Takagi \([13]\) in 1921, subsuming many earlier results:\footnote{See Honda \([4]\), Iyanaga \([5, 6]\), Kaplan \([7]\), and Miyake \([9]\) for biographies of Teiji Takagi (高木貞治, Takagi Teiji, 1875–1960). Takagi’s papers published in languages other than Japanese (namely, English, German, and French) have been collected in \([14]\).}

**Theorem 1 (Takagi).** Let \( f \) and \( g \) be polynomials with complex coefficients, with \( \deg f = m \) and \( \deg g = n \). Let \( f \) have an \( r \)-fold zero at the origin \( (0 \leq r \leq m) \), and let the remaining zeros (with multiplicity) be \( \alpha_1, \ldots, \alpha_{m-r} \neq 0 \). Let \( K \) be the convex hull of the zeros of \( g \). Then either \( f(D)g \) is identically zero, or its zeros lie in the set \( K + \sum_{i=1}^{m-r} [0, n-r]z \alpha_i^{-1} \).

Here we have used the notations \( A + B = \{a + b: a \in A \text{ and } b \in B\} \) and \( AB = \{ab: a \in A \text{ and } b \in B\} \).

Takagi’s proof was based on Grace’s apolarity theorem \([3]\), a fundamental but somewhat enigmatic result in the analytic theory of polynomials.\footnote{For discussion of Grace’s apolarity theorem and its equivalents—namely Walsh’s coincidence theorem and the Schur–Szegő composition theorem—see Marden \([8, \text{ Chapter 18}]\), Oubrechhoff \([10, \text{ pp. 135–136}]\), and Rahman and Schmeisser \([11, \text{ Chapter 3}]\).} This proof is also given in the books of Marden \([8, \text{ Section 18}]\), Oobrechhoff \([10, \text{ Chapter VII}]\), and Rahman
and Schmeisser [11, Sections 5.3 and 5.4]. Here I give a short and completely elementary proof of Takagi’s theorem.

The key step—as Takagi [13] observed—is to understand the case of a degree-1 polynomial \( f(z) = z - \alpha \):

**Proposition 2 (Takagi).** Let \( g \) be a polynomial of degree \( n \), and let \( K \) be the convex hull of the zeros of \( g \). Let \( \alpha \in \mathbb{C} \), and define \( h = g' - \alpha g \). Then either \( h \) is identically zero, or all the zeros of \( h \) are contained in \( K \) if \( \alpha = 0 \), and in \( K + [0, n] \alpha^{-1} \) if \( \alpha \neq 0 \).

The case \( \alpha = 0 \) is the celebrated theorem of Gauss and Lucas [8, Section V], [10, Chapter V], and [11, Section 2.1], which is the starting point of the modern analytic theory of polynomials. My proof for general \( \alpha \) will be modeled on Cesàro’s [1] 1885 proof of the Gauss–Lucas theorem [11, pp. 72–73], with a slight twist to handle the case \( \alpha \neq 0 \).

**Proof of Proposition 2.** Clearly, \( h \) is identically zero if and only if either (a) \( g \equiv 0 \) or (b) \( g \) is a nonzero constant and \( \alpha = 0 \). Moreover, if \( g \) is a nonzero constant and \( \alpha \neq 0 \), then the zero set of \( h \) is empty. So we can assume that \( n \geq 1 \).

Let \( \beta_1, \ldots, \beta_n \) be the zeros of \( g \) (with multiplicity), so that \( g(z) = b_n \prod_{i=1}^{n} (z - \beta_i) \) with \( b_n \neq 0 \). If \( z \notin K \), then \( g(z) \neq 0 \), and we can consider

\[
\frac{h(z)}{g(z)} = \frac{g'(z) - \alpha g(z)}{g(z)} = \sum_{i=1}^{n} \frac{1}{z - \beta_i} - \alpha.
\]

If this equals zero, then by taking complex conjugates we obtain

\[
0 = \sum_{i=1}^{n} \frac{1}{\bar{z} - \bar{\beta_i}} - \bar{\alpha} = \sum_{i=1}^{n} \frac{|z - \beta_i|^2}{|z - \beta_i|} - \bar{\alpha},
\]

which can be rewritten as

\[
z = \sum_{i=1}^{n} \lambda_i \beta_i + \kappa \bar{\alpha} \text{ where } \lambda_i = \frac{|z - \beta_i|^2}{\sum_{j=1}^{n} |z - \beta_j|^2}, \quad \kappa = \frac{1}{\sum_{j=1}^{n} |z - \beta_j|^2}.
\]

Then \( \lambda_i > 0 \) and \( \sum_{i=1}^{n} \lambda_i = 1 \), so \( \sum_{i=1}^{n} \lambda_i \beta_i \in K \); and of course \( \kappa > 0 \). Moreover, by the Schwarz inequality we have

\[
|\alpha|^2 = \left| \sum_{i=1}^{n} \frac{1}{z - \beta_i} \right|^2 \leq \sum_{i=1}^{n} |z - \beta_i|^{-2} = \frac{n}{\kappa},
\]

so \( \kappa \leq n|\alpha|^{-2} \). This implies that \( \kappa \bar{\alpha} \in [0, n]|\alpha|^{-1} \) and hence that \( z \in K + [0, n]|\alpha|^{-1} \).

We can now handle polynomials \( f \) of arbitrary degree by iterating **Proposition 2**:

**Proof of Theorem 1.** From \( f(z) = a_m \left( \prod_{i=1}^{m-r} (z - \alpha_i) \right) z^r \) it is easy to see that \( f(D) = a_m \left( \prod_{i=1}^{m-r} (D - \alpha_i) \right) D^r \). We first apply \( D^r \) to \( g \), yielding a polynomial of degree \( n - r \) whose zeros also lie in \( K \) (by the Gauss–Lucas theorem); then we repeatedly apply (in any order) the factors \( D - \alpha_i \), using **Proposition 2**.

**Remark.** When \( \alpha = 0 \), the zeros of \( h = g' \) lie in \( K \); so one might expect that when \( \alpha \) is small, the zeros of \( h = g' - \alpha g \) should lie near \( K \). But when \( \alpha \) is small and nonzero, the set \( K + [0, n]|\alpha|^{-1} \) arising in **Proposition 2** is in fact very large. What is going on here?
Here is the answer: Suppose that \( \deg g = n \). When \( \alpha = 0 \), the polynomial \( h = g' \) has degree \( n - 1 \); but when \( \alpha \neq 0 \), the polynomial \( h = g' - \alpha g \) has degree \( n \). So, in order to make a proper comparison of their zeros, we should consider the polynomial \( g' \) corresponding to the case \( \alpha = 0 \) as also having a zero “at infinity.” This zero then moves to a value of order \( \alpha^{-1} \) when \( \alpha \) is small and nonzero.

This behavior is easily seen by considering the example of a quadratic polynomial \( g(z) = z^2 - \beta^2 \). Then the zeros of \( g' - \alpha g \) are

\[
z = \frac{1 \pm \sqrt{1 + \alpha^2 \beta^2}}{\alpha} = -\frac{\beta^2}{2} + O(\alpha^3), \quad 2\alpha^{-1} + O(\alpha).
\]

So there really is a zero of order \( \alpha^{-1} \), as Takagi’s theorem recognizes.

In the context of Proposition 2, one expects that \( g' - \alpha g \) has one zero of order \( \alpha^{-1} \) and \( n - 1 \) zeros near \( K \) (within a distance of order \( \alpha \)). More generally, in the context of Theorem 1, one would expect that \( h \) has \( m - r \) zeros of order \( \alpha^{-1} \), with the remaining zeros near \( K \). It is a very interesting problem — and one that is open, as far as I know — to find strengthenings of Takagi’s theorem that exhibit these properties. There is an old result that goes in this direction [8, Corollary 18.1], [11, Corollary 5.4.1(ii)], but it is based on a disc \( D \) containing the zeros of \( g \), which might in general be much larger than the convex hull \( K \) of the zeros.

**Postscript.** A few days after finding this proof of Proposition 2, I discovered that an essentially identical argument is buried in a 1961 paper of Shisha and Walsh [12, pp. 127–128 and 147–148] on the zeros of infrapolynomials. I was led to the Shisha–Walsh paper by a brief citation in Marden’s book [8, pp. 87–88, Exercise 11]. So the proof given here is not new; but it deserves to be better known.

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A Generalization of Euler’s Limit

Euler’s limit is defined as \( \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n = e \). We establish a generalization of this limit in the following proposition.

**Proposition.** Let \( A_n \) be a strictly increasing sequence of positive numbers satisfying the asymptotic formula \( A_{n+1} \sim A_n \), and let \( d_n = A_{n+1} - A_n \). Then

\[
\lim_{n \to \infty} \left( \frac{A_{n+1}}{A_n} \right)^{\frac{A_n}{d_n}} = e. \tag{1}
\]

**Proof.** Let us consider the function \( \ln x \) on the interval \([A_n, A_{n+1}]\) for all \( n \in \mathbb{N} \). By the mean value theorem, we have \( \ln A_{n+1} - \ln A_n = \frac{1}{c} (A_{n+1} - A_n) \) for some \( c \) with \( A_n < c < A_{n+1} \). Hence (since \( \frac{1}{A_{n+1}} < \frac{1}{c} < \frac{1}{A_n} \))

\[
\frac{A_{n+1} - A_n}{A_{n+1}} < \ln A_{n+1} - \ln A_n < \frac{A_{n+1} - A_n}{A_n}.
\]

Since \( A_{n+1} \sim A_n \), we have

\[
1 \leftarrow \frac{A_n}{A_{n+1}} < \frac{\ln A_{n+1} - \ln A_n}{A_{n+1} - A_n} < 1;
\]

that is,

\[
\lim_{n \to \infty} \ln \left( \frac{A_{n+1}}{A_n} \right)^{\frac{A_n}{A_{n+1} - A_n}} = 1.
\]

This completes the proof.

It can be seen that generalization (1) gives Euler’s limit when \( A_n = n \).

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