A Global Maximum Principle for the Stochastic Optimal Control Problem with Delay *

Weijun Meng† Jingtao Shi‡

November 7, 2019

Abstract: In this paper, we solve an open problem for the stochastic optimal control problem with delay where the control domain is nonconvex and the diffusion term contains both control and its delayed term. Inspired by previous results by Øksendal and Sulem [A maximum principle for optimal control of stochastic systems with delay, with applications to finance. In J. M. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal control and partial differential equations, ISO Press, Amsterdam, 64-79, 2000] and Chen and Wu [Maximum principle for the stochastic optimal control problem with delay and application, Automatica, 46, 1074-1080, 2010], we generalize the Peng’s general stochastic maximum principle [A general stochastic maximum principle for optimal control problems, SIAM J. Control Optim., 28, 966-979, 1990] to the time delayed case, which we called the global maximum principle. A new adjoint equation is introduced to deal with the cross terms, when applying the duality technique, though assumption imposed on it is a little strong. Comparing with the classical result, the maximum condition contains an indicator function, in fact it is the characteristic of the stochastic optimal control problem with delay. A solvable linear-quadratic example is also discussed.

Keywords: Stochastic differential equations with delay; anticipated backward stochastic differential equation; optimal control; maximum principle

Mathematics Subject Classification: 93E20, 60H10, 34K50

1 Introduction

The study of stochastic optimal control problems has been an important topic in recent years, and Pontryagin’s type maximum principle has been recognized and drawn special attention to

---

*This work is financially supported by the National Key R&D Program of China (2018YFB1305400), and the National Natural Science Foundations of China (11971266, 11571205, 11831010).

†School of Mathematics, Shandong University, Jinan 250100, P.R. China, E-mail: 201611337@mail.sdu.edu.cn

‡Corresponding author, School of Mathematics, Shandong University, Jinan 250100, P.R. China, E-mail: shijingtao@sdu.edu.cn
the researchers as one of the main tools. Since Fleming [10], a lot of significant results have been obtained. For a forward stochastic control system with Brownian motion, Kushner [15] and Bismut [4] proposed the definition of the adjoint processes and its characterization by Itō-type equations. Based on which, Bensoussan [3] gave the maximum principle under the case where either the control domain is convex or the diffusion term contains control variable with nonconvex control domain. Subsequently, Hu [11] derived the maximum principle for Markov processes, and Hu and Peng [12] investigated the maximum principle for semilinear stochastic evolution control systems. But these literatures still didn’t solve the maximum principle when the control domain is not convex and the control variable appears in the diffusion term. Until 1990, Peng [19] completely solved the problem applying the second-order variation method in calculating the variation of the cost functional caused by the spike variation of the given optimal control. Systematically, Yong and Zhou [25] summarized these context.

In the above systems, system states only depend on the current time value. However, recently people have come to realize that the development of some problems in the real world depends not only on the current state, but also on its past history. Such problems should be characterized by system equations whose states depend on the past, which we call stochastic differential delay equations (abbreviated as SDDE). More detailed research about SDDE can be referred to Mohammed [16] and [17]. Due to its wide application in engineering, life sciences and finance (see [2], [14], etc.), the SDDE has become a hot issue in modern research.

However, delay response brings many difficulties to studying the stochastic control problems with delay, not only the corresponding problems become the infinite problems, but also so far we lack the Itō formula to deal with the delay part of the state. Øksendal and Sulem [18] studied a class of stochastic control problem with delay, where in their model, not only the current value but also the average value of the past duration will affect the growth of wealth at time $t$. Due to the particularity of the selected models, they can deal with infinite dimensional problems into finite dimensional ones and get the sufficient maximum principle of the problem. But they required that the third adjoint variable is always equal to 0. Subsequently, Chen and Wu [7] discussed the stochastic control system involving both delays in the state variable and the control variable with convex control domain, and they derived the maximum principle by an anticipated backward stochastic differential equation (in short, ABSDE) as the adjoint equation, where the ABSDE was first introduced by Peng and Yang [20] in 2009. Next, Chen and Wu [6] obtained the maximum principle for stochastic optimal control problem of forward-backward system with delay, although the control domain is non-convex, the diffusion term of the forward equation does not contain control and control delay term. Recent progress for stochastic optimal control problems with delay, please refer to Yu [24], Agram et al. [1], Du et al. [9], Chen and Huang [5], Zhu and Zhang [26], Chen and Yu [8], Wu and Wang [23], Wang and Wu [22] and the references therein.

Inspired by [18] and [7], in this paper we derived the global maximum principle for a con-
trolled stochastic system whose diffusion term can contain the control and the control domain is nonconvex. In fact, the key difficulty is how to deal with the cross term of state and state delay term appearing in the variational inequality, in this paper we solve the problem by introducing a new adjoint equation. Comparing the classical result, the maximum condition contains an indicator function, which is the characteristic of the stochastic optimal control problem with delay. Besides, we also give the estimate of SDDE for the completeness of the content.

The rest of this paper is organized as follows. In Section 2, some results concerning the SDDE and ABSDE are displayed. The problem is formulated and the spike variational method is used to introduce the variational equations for SDDE in Section 3. Section 4 mainly focuses on the adjoint equations for SDDE and derives the maximum principle. In part 5, an example is solved using the theoretic results. Finally, some concluding remarks are given in Section 6.

2 Preliminaries

In this section, we first display some preliminary results concerning the SDDE and ABSDE.

Suppose that \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) is a complete filtered probability space and \(\{\mathcal{F}_t\}_{t \geq 0}\) is generated by the \(d\)-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\). Let \(E[\cdot]\) denote the mathematical expectation with respect to the probability \(P\) and \(T > 0\) be the finite time duration.

We first define some spaces which will be used later:

\[
C([0, T]; \mathbb{R}^n) := \left\{ \text{\(\mathbb{R}^n\)-valued continuous function } \phi(t); \sup_{0 \leq t \leq T} |\phi(t)| < \infty \right\},
\]

\[
L^2_{\mathcal{F}}([0, T]; \mathbb{R}^n) := \left\{ \text{\(\mathbb{R}^n\)-valued } \mathcal{F}_t\text{-adapted process } \phi(t); E \int_0^T |\phi(t)|^2 dt < \infty \right\},
\]

\[
S^2_{\mathcal{F}}([0, T]; \mathbb{R}^n) := \left\{ \text{\(\mathbb{R}^n\)-valued } \mathcal{F}_t\text{-adapted process } \phi(t); E\left[\sup_{0 \leq t \leq T} |\phi(t)|^2\right] < \infty \right\}.
\]

Consider the following SDDE:

\[
\begin{aligned}
&dX(t) = b(t, X(t), X(t - \delta))dt + \sigma(t, X(t), X(t - \delta))dB(t), \ t \geq 0, \\
&X(t) = \varphi(t), \ t \in [-\delta, 0].
\end{aligned}
\]  

(2.1)

where \(\delta > 0\) is a given finite time delay, \(\varphi \in C([-\delta, 0]; \mathbb{R})\) is the given initial path of the state \(X(\cdot)\). \(b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d}\) are measurable functions satisfying:

(H1) \(|b(t, x, x') - b(t, y, y')| + |\sigma(t, x, x') - \sigma(t, y, y')| \leq D(|x - y| + |x' - y'|), \forall x, x', y, y' \in \mathbb{R}^n, t \in [0, T],\) for some constant \(D > 0\);

(H2) \(\sup_{0 \leq t \leq T}(|b(t, 0, 0)| + |\sigma(t, 0, 0)|) < +\infty.\)

Then by the Theorem 2.2 of [7], we have the following result.
Theorem 2.1. Suppose (H1) and (H2) hold, let \( \varphi : \Omega \to C([-\delta, 0]; \mathbb{R}^n) \) be \( \mathcal{F}_0 \)-measurable and \( \mathbb{E} \left[ \sup_{-\delta \leq t \leq 0} |\varphi(t)|^2 \right] < \infty \). Then SDDE (2.1) admits a unique continuous \( \mathcal{F}_t \)-adapted solution \( X(\cdot) \in \mathcal{S}_T^2([0, T]; \mathbb{R}^n) \).

In the following result, an estimate of the solution to SDDE is given.

Theorem 2.2. Suppose (H1) and (H2) hold, let \( \varphi : \Omega \to C([-\delta, 0]; \mathbb{R}^n) \) be \( \mathcal{F}_0 \)-measurable and \( \mathbb{E} \left[ \sup_{-\delta \leq t \leq 0} |\varphi(t)|^2 \right] < \infty \). Then for \( p \geq 2 \), the solution to SDDE (2.1) satisfies the following estimate:

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X(t)|^p \right] \leq C(\delta, T, D, p) \mathbb{E} \left\{ \left( \int_0^T |b(r, 0, 0)|^2dr \right)^{\frac{p}{2}} + \sup_{-\delta \leq r \leq t} \mathbb{E} \left| \varphi(r) \right|^p \right\}.
\]

where \( C(\delta, T, D, p) \) is a constant depends on \( \delta, T, D, p \).

Proof. By (H1) and the Burkholder-Davis-Gundy inequality, we have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X(t)|^p \right] \leq C(p) \mathbb{E} \left( \int_0^T |b(r, X(r), X(r - \delta))|^pdr \right) + C(p) \mathbb{E} \left( \int_0^T |\sigma(r, X(r), X(r - \delta))|^2dr \right)^{\frac{p}{2}} + C(p) \mathbb{E} |X(0)|^p
\]

\[
\leq C(p) \mathbb{E} |X(0)|^p + C(p) \mathbb{E} \left[ \left( \int_0^T \left( |b(r, 0, 0)| + D|X(r)| + D|X(r - \delta)| \right) dr \right]^p \right] + C(p) \mathbb{E} \left[ \left( \int_0^T |\sigma(r, 0, 0)|^2dr \right)^{\frac{p}{2}} \right] \]

\[
\leq C(p) \mathbb{E} |X(0)|^p + C(p) \mathbb{E} \left[ \left( \int_0^T |b(r, 0, 0)|^pdr \right) + C(p) \mathbb{E} \left[ \int_0^T |\sigma(r, 0, 0)|^2dr \right] \right] + C(p, D, T) \mathbb{E} \left[ \int_0^T |X(r)|^pdr \right] + C(p, D, T) \mathbb{E} \left[ \int_0^T |X(r - \delta)|^pdr \right].
\]

Noting

\[
\mathbb{E} \left[ \int_0^T |X(r - \delta)|^pdr \right] = \mathbb{E} \left[ \int_{-\delta}^{T-\delta} |X(r)|^pdr \right] \leq \delta \mathbb{E} \left[ \sup_{-\delta \leq r \leq t} |\varphi(r)|^p \right] + \mathbb{E} \left[ \int_0^T |X(r)|^pdr \right],
\]

and applying the Gronwall’s inequality, thus we complete the proof of (2.2). \( \square \)

Now we consider the following ABSDE:

\[
\begin{aligned}
- dY(t) &= f(t, Y(t), Z(t), Y(t + \delta(t)), Z(t + \zeta(t)))dt - Z(t)dB(t), \quad t \in [0, T], \\
Y(t) &= \xi(t), \quad Z(t) = \eta(t), \quad t \in [T, T + K],
\end{aligned}
\]

(2.3)
where \( \delta(\cdot) \) and \( \zeta(\cdot) \) are \( \mathbb{R}^+ \)-valued functions defined on \([0, T]\) satisfying:

(i) There exists a constant \( K \geq 0 \) such that for all \( s \in [0, T] \), \( s + \delta(s) \leq T + K \), \( s + \zeta(s) \leq T + K \);

(ii) There exists a constant \( L \geq 0 \) such that for all \( t \in [0, T] \) and for all nonnegative and integrable function \( g(\cdot) \), the following holds:

\[
\int_t^T g(s + \delta(s))ds \leq L \int_t^{T + K} g(s)ds, \quad \int_t^T g(s + \zeta(s))ds \leq L \int_t^{T + K} g(s)ds.
\]

We impose the following assumptions to the generator of ABSDE:

- **(H3)** \( f(s, \omega, y, z, \xi, \zeta) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_F^2(s, T + K; \mathbb{R}^m) \times L_F^2(s, T + K; \mathbb{R}^{m \times d}) \to L^2(F_s; \mathbb{R}^m) \) for all \( s \in [0, T] \) and \( \mathbb{E}[\int_0^T |f(s, 0, 0, 0)|^2 ds] < +\infty \).

- **(H4)** There exists a constant \( C > 0 \) such that for all \( s \in [0, T] \), \( y, y' \in \mathbb{R}^m \), \( z, z' \in \mathbb{R}^{m \times d} \), \( \xi, \xi' \in L_F^2(s, T + K; \mathbb{R}^m) \), \( \eta, \eta' \in L_F^2(s, T + K; \mathbb{R}^{m \times d}) \), \( r, r' \in [s, T + K] \), we have

\[
|f(s, y, z, \xi, \eta) - f(s, y', z', \xi', \eta')| \leq C \left( |y - y'| + |z - z'| + \mathbb{E}^{F_s} [||\xi - \xi'|| + ||\eta - \eta'||] \right),
\]

where \( \mathbb{E}^{F_s} [\cdot] \equiv \mathbb{E} [\cdot | \mathcal{F}_s] \) denotes the conditional expectation, for \( s \geq 0 \).

The following theorem can be found in [20].

**Theorem 2.3.** Suppose that \( f \) satisfies (H3) and (H4), and \( \delta, \zeta \) satisfy (i) and (ii). Then for any given \( \xi \in \mathcal{S}^2_F(T, T + K; \mathbb{R}^m) \) and \( \eta \in L_F^2(T, T + K; \mathbb{R}^{m \times d}) \), the ABSDE (2.3) admits a unique \( \mathcal{F}_t \)-adapted solution \((Y(\cdot), Z(\cdot)) \in \mathcal{S}^2_F(0, T + K; \mathbb{R}^m) \times L^2_F(0, T + K; \mathbb{R}^{m \times d}) \).

## 3 Problem formulation and variational equations

In this section, we state the problem which will be studied and apply the spike variation technique to give the variational equations.

Suppose \( U \subseteq \mathbb{R}^k \) is nonempty and \( \delta > 0 \) is a given time delay parameter, we consider the following stochastic control systems with delay with:

\[
\begin{aligned}
\begin{cases}
    dX(t) = b(t, X(t), X(t - \delta), v(t), v(t - \delta))dt + \sigma(t, X(t), X(t - \delta), v(t), v(t - \delta))dB(t), & t \geq 0, \\
    X(t) = \varphi(t), & t \in [-\delta, 0],
\end{cases}
\end{aligned}
\]

along with the cost functional

\[
J(v(\cdot)) = \mathbb{E} \left[ \int_0^T l(t, X(t), X(t - \delta), v(t), v(t - \delta))dt + h(x(T)) \right],
\]

where \( b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^n \), \( \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^{n \times d} \), \( l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^n \times U \times U \rightarrow \mathbb{R} \), \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) are given functions.

The admissible control set is defined as follows:

\[
U_{ad} := \{ v(\cdot) | v(\cdot) \text{ is a } U\text{-valued, square-integrable, } \mathcal{F}_t\text{-predictable process} \}.
\]
Our object is to find a control \( u(\cdot) \) over \( U_{ad} \) such that \( (3.1) \) is satisfied and \( (3.2) \) is minimized. Any \( u(\cdot) \in U_{ad} \) that achieves the above infimum is called an optimal control and the corresponding solution \( x(\cdot) \) is called the optimal trajectory.

Throughout the paper, we impose the following assumptions.

(A1) (i) The functions \( b = b(t, x, x_\delta, v, v_\delta) \), \( \sigma = \sigma(t, x, x_\delta, v, v_\delta) \) are twice continuously differentiable with respect to \((x, x_\delta)\) and their derivatives are bounded.

(ii) There exists a constant \( C \) such that for \( \phi = b, \sigma \),

\[
|\phi(t, x, x_\delta, v, v_\delta)| \leq C(1 + |x| + |x_\delta|), \quad \forall x, x_\delta \in \mathbb{R}^n, \ v, v_\delta \in U, \ t \geq 0.
\]

(iii) The function \( \varphi : \Omega \to C([-\delta, 0]; \mathbb{R}^n) \) is \( \mathcal{F}_0 \)-measurable and \( \mathbb{E}\left[ \sup_{-\delta \leq t \leq 0} |\varphi(t)|^2 \right] < \infty \).

(iv) \( b, b_t, b_{xx}, T_b, x_{x_\delta}, \sigma, \sigma_t, \sigma_{xx}, \sigma_{x_\delta}, \sigma_{xx_\delta} \) are continuous in \((x, x_\delta, v, v_\delta)\).

Under (A1), for any admissible control \( v(\cdot) \), \( (3.1) \) admits a unique adapted solution \( x(\cdot) \in S_x^2([0, T]; \mathbb{R}^n) \) by Theorem 2.1.

Next we introduce the variational equation. Let \( u(\cdot) \) be the optimal control and \( x(\cdot) \) is the optimal trajectory. Since the control domain is nonconvex, we solve the optimality problem by applying the spike variation method. Suppose \( u^\varepsilon(\cdot) \) is an admissible control of form

\[
u^\varepsilon(t) = \begin{cases} u(t), & t \notin [\tau, \tau + \varepsilon], \\ v(t), & t \in [\tau, \tau + \varepsilon], \end{cases}
\]

for all \( t \in [0, T] \), \( \varepsilon > 0 \).

and \( x^\varepsilon(\cdot) \) is the corresponding trajectory, where \( v(\cdot) \) is any admissible control.

For simplicity of presentation, in the following of this paper we only discuss the one-dimensional case, namely, \( n = k = d = 1 \). However, the multi-dimensional case can be obtained without difficulty.

We introduce the first-order and second-order variational equations for \( x(\cdot) \) as follows:

\[
\begin{aligned}
dx_1(t) &= \left[ b_x(t)x_1(t) + b_{x_\delta}(t)x_1(t - \delta) + b_b(t) \right] dt \\
&\quad + \left[ \sigma_x(t)x_1(t) + \sigma_{x_\delta}(t)x_1(t - \delta) + \delta \sigma(t) \right] dB(t), \quad t \geq 0,
\end{aligned}
\]

\[
x_1(t) = 0, \quad -\delta \leq t \leq 0,
\]

\[
\begin{aligned}
dx_2(t) &= \left[ b_x(t)x_2(t) + b_{x_\delta}(t)x_2(t - \delta) + \frac{1}{2} b_{xx}(t)x_1(t)x_1(t) \right] dt \\
&\quad + \left[ \sigma_x(t)x_2(t) + \sigma_{x_\delta}(t)x_2(t - \delta) + \frac{1}{2} \sigma_{xx}(t)x_1(t)x_1(t) \right] dB(t), \quad t \geq 0,
\end{aligned}
\]

\[
x_2(t) = 0, \quad -\delta \leq t \leq 0,
\]
Lemma 3.1. Let assumption (A1) hold. Suppose \( x(\cdot) \) is the optimal trajectory, \( x^e(\cdot) \) is the trajectory corresponding to the admissible control \( u^e(\cdot) \), then for any \( p \geq 1 \),

\[
E \left[ \sup_{0 \leq t \leq T} |x^e(t) - x(t)|^2p \right] = O(\varepsilon^p),
\]

\[
E \left[ \sup_{0 \leq t \leq T} |x_1(t)|^2p \right] = O(\varepsilon^p),
\]

\[
E \left[ \sup_{0 \leq t \leq T} |x_2(t)|^p \right] = O(\varepsilon^p),
\]

\[
E \left[ \sup_{0 \leq t \leq T} |x^e(t) - x(t) - x_1(t)|^2p \right] = o(\varepsilon^p),
\]

\[
E \left[ \sup_{0 \leq t \leq T} |x^e(t) - x(t) - x_1(t) - x_2(t)|^p \right] = o(\varepsilon^p).
\]

Proof. For simplicity of presentation, let \( E_\varepsilon = [\tau, \tau + \varepsilon] \cup [\tau + \delta, \tau + \delta + \varepsilon] \). Noting when \( t \in E_\varepsilon \), we have \( \delta b(t) \neq 0 \), etc. If we choose \( \varepsilon \) enough small, then \( |E_\varepsilon| = 2\varepsilon \). In the whole proof, \( C > 0 \) is a generic constant, which change from line to line.
Recall Theorem 2.2, applying the assumption (A1)(ii), we obtain
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^p \right] \leq C \mathbb{E} \left[ \left( \int_0^T |\delta b(t)| dt \right)^{2p} \right] + C \mathbb{E} \left[ \left( \int_0^T |\delta \sigma(t)|^2 dt \right)^p \right]
\]
\[
\leq C \mathbb{E} \left[ \left( \int_{E_\varepsilon} \left( 1 + |x(t)| + |x(t - \delta)| + |x^{\varepsilon}(t)| + |x^{\varepsilon}(t - \delta)| \right) dt \right)^{2p} \right] + C \mathbb{E} \left[ \left( \int_{E_\varepsilon} \left( 1 + |x(t)| + |x(t - \delta)| + |x^{\varepsilon}(t)| + |x^{\varepsilon}(t - \delta)| \right) dt \right)^p \right]
\]
\[
\leq C(\varepsilon^p + \varepsilon^{2p}) \mathbb{E} \left[ 1 + \sup_{-\delta \leq t \leq T} |x(t)|^{2p} + \sup_{-\delta \leq t \leq T} |x^{\varepsilon}(t)|^{2p} \right] + C(\varepsilon^p + \varepsilon^{2p}) \mathbb{E} \left[ 1 + \sup_{-\delta \leq t \leq T} \varphi(t)^{2p} \right] = O(\varepsilon^p).
\]

By (2.2), the proof of (3.8) is completed.

Similarly, with assumption (A1)(i), we deduce
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_2(t)|^p \right]
\]
\[
\leq C \mathbb{E} \left[ \left( \int_0^T \frac{1}{2} b_{xx}(t)x_1(t)x_1(t) + \frac{1}{2} b_{x x}(t)x_1(t - \delta)x_1(t - \delta) + b_{x x}(t)x_1(t)x_1(t - \delta) \right) dt \right)^p \right]
\]
\[
+ C \mathbb{E} \left[ \left( \int_0^T \frac{1}{2} \sigma_{xx}(t)x_1(t)x_1(t) + \frac{1}{2} \sigma_{xx}(t)x_1(t - \delta)x_1(t - \delta) + \sigma_{xx}(t)x_1(t)x_1(t - \delta)
\]
\[
+ \delta \sigma_x(t)x_1(t) + \delta \sigma_x(t)x_1(t - \delta) \right)^2 dt \right)^p \right]
\]
\[
\leq C \mathbb{E} \left[ \sup_{-\delta \leq t \leq T} |x_1(t)|^{2p} \right] + C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^p \left( \int_0^T |\delta \sigma_x(t)|^2 dt \right)^{\frac{p}{2}} \right]
\]
\[
+ C \mathbb{E} \left[ \sup_{-\delta \leq t \leq T} |x_1(t)|^p \left( \int_0^T |\delta \sigma_x(t)\right) |^2 dt \right)^{\frac{p}{2}} \right] \leq C\varepsilon^p + C\varepsilon^p = C\varepsilon^p.
\]

Next we try to prove (3.10) and (3.11). Write
\[
\xi(t) := x^{\varepsilon}(t) - x(t) - x_1(t), \quad \eta(t) := \xi(t) - x_2(t),
\]
then $\xi(\cdot)$ and $\eta(\cdot)$ satisfy the following SDEs:
\[
\begin{cases}
  d\xi(t) = [b_x(t)\xi(t) + b_{x x}(t)\xi(t - \delta) + (b^\theta_x(t) - b_x(t))(x^{\varepsilon}(t) - x(t))
  
  + (b^\theta_x(t) - b_{x x}(t))(x^{\varepsilon}(t - \delta) - x(t - \delta))] dt
  
  + [\sigma_x(t)\xi(t) + \sigma_{x x}(t)\xi(t - \delta) + (\sigma^\theta_x(t) - \sigma_x(t))(x^{\varepsilon}(t) - x(t))
  
  + (\sigma^\theta_x(t) - \sigma_{x x}(t))(x^{\varepsilon}(t - \delta) - x(t - \delta))] dB(t),
\end{cases}
\]
\[
\xi(t) = 0, \quad t \in [-\delta, 0],
\]
\[
(3.12)
\]
\[
\begin{aligned}
\text{dn}(t) = & \left\{ b_x(t)\eta(t) + b_{xx}(t)\eta(t-\delta) + \delta b_x(t)(x^\varepsilon(t) - x(t)) \\
& + \delta b_{xx}(t)(x^\varepsilon(t-\delta) - x(t-\delta)) + \bar{b}_x(t)\left[(x^\varepsilon(t)-x(t))^2 - x^2(t)\right] \\
& + \left[\bar{b}_{xx}(t) - \frac{1}{2}b_{xx}(t)\right]x^2(t) + (2\bar{b}_{xx}(t) - b_{xx}(t))x_1(t)x_1(t-\delta) \\
& + \left[\bar{b}_{xx}(t) - \frac{1}{2}b_{xx}(t)\right]x^2(t-\delta) \\
& + \bar{b}_{xx}(t)\left[(x^\varepsilon(t-\delta) - x(t-\delta))^2 - x^2(t-\delta)\right] \\
& + 2\bar{b}_{xx}(t)[(x^\varepsilon(t) - x(t))(x^\varepsilon(t-\delta) - x(t-\delta)) - x_1(t)x_1(t-\delta)] \\
& + (2\bar{\sigma}_{xx}(t) - \sigma_{xx}(t))x_1(t)x_1(t-\delta) \right\} dB(t), \quad t \geq 0,
\end{aligned}
\]

where

\[
b^\beta_x(t) := \int_0^1 b_x(t, x(t) + \theta(x^\varepsilon(t) - x(t)), x(t-\delta) + \theta(x^\varepsilon(t-\delta) - x(t-\delta)), u^\varepsilon(t), u^\varepsilon(t-\delta))d\theta,
\]

\[
\bar{b}_{xx}(t) := \int_0^1 \int_0^1 \frac{1}{2}\theta b_{xx}(t, x(t) + \lambda\theta(x^\varepsilon(t) - x(t)), x(t-\delta) + \lambda\theta(x^\varepsilon(t-\delta) - x(t-\delta)), u^\varepsilon(t), u^\varepsilon(t-\delta))d\lambda d\theta,
\]

and \(b^\beta_{xx}, \sigma^\beta_{xx}, \bar{b}_{xx}, \bar{\sigma}_{xx}, \sigma_{xx}, \bar{\sigma}_{xx}, \bar{\sigma}_{xx}\) are similarly defined.

Then applying theorem 2.2, we get the following estimate with the assumption (A1) (iv):

\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |\xi(t)|^{2p}\right] \leq C\mathbb{E}\left[\left(\int_0^T \left[(b^\beta_x(t) - b_x(t))(x^\varepsilon(t) - x(t)) \\
+ (b^\beta_{xx}(t) - b_{xx}(t))(x^\varepsilon(t-\delta) - x(t-\delta))\right] dt\right)^{2p}\right]
+ C\mathbb{E}\left[\left(\int_0^T \left[|\sigma^\beta_x(t) - \sigma_x(t)|^2 |x^\varepsilon(t) - x(t)|^2 \\
+ |\sigma^\beta_{xx}(t) - \sigma_{xx}(t)|^2 |x^\varepsilon(t-\delta) - x(t-\delta)|^2\right] dt\right)^p\right]
\]

9
For the same reason, we obtain
\[
\leq C \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^p \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \left( \int_0^T |b_x^\varepsilon(t) - b_x(t)| dt \right)^4 \right] \right\}^{\frac{1}{2}} + C \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^p \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \left( \int_0^T |\sigma_x^\varepsilon(t) - \sigma_x(t)|^2 dt \right)^2 \right] \right\}^{\frac{1}{2}} = o(\varepsilon^p).
\]

Finally, noting the boundedness of all the derivatives, we have
\[
\mathbb{E} \left[ \left( \int_0^T |\delta b_x(t)(x^\varepsilon(t) - x(t))| dt \right)^p \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^p \left( \int_0^T |\delta b_x(t)| dt \right)^p \right] = o(\varepsilon^p).
\]

For the same reason, we obtain
\[
\mathbb{E} \left[ \left( \int_0^T |\delta b_{xx}(t)(x^\varepsilon(t) - x(t)) - x(t) - \delta(t)| dt \right)^p \right] = o(\varepsilon^p),
\]
\[
\mathbb{E} \left[ \left( \int_0^T |\delta \sigma_x(t)\xi(t)|^2 dt \right)^{\frac{p}{2}} \right] = o(\varepsilon^p),
\]
\[
\mathbb{E} \left[ \left( \int_0^T |\delta \sigma_{xx}(t)\xi(t) - \delta(t)|^2 dt \right)^{\frac{p}{2}} \right] = o(\varepsilon^p).
\]

Using (3.7) and (3.8), we get
\[
\mathbb{E} \left[ \left( \int_0^T \tilde{b}_{xx}(t) \left[ (x^\varepsilon(t) - x(t))^2 - x^2_{1}(t) \right] dt \right)^p \right] \leq C \mathbb{E} \left[ \left( \int_0^T \xi(t)(x^\varepsilon(t) - x(t) + x_1(t)) dt \right)^p \right] \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\xi(t)|^p \left[ \sup_{0 \leq t \leq T} |x^\varepsilon(t) - x(t)|^p + \sup_{0 \leq t \leq T} |x_1(t)|^p \right] \right] = o(\varepsilon^p).
\]

In the same way, we have
\[
\mathbb{E} \left[ \left( \int_0^T \tilde{b}_{xx}(t) \left[ (x^\varepsilon(t) - x(t) - \delta(t))^2 - x^2_1(t) - \delta(t) \right] dt \right)^p \right] = o(\varepsilon^p),
\]
\[
\mathbb{E} \left[ \left( \int_0^T |\tilde{\sigma}_{xx}(t)|^2 \left[ (x^\varepsilon(t) - x(t))^2 - x_1(t)^2 \right]^2 dt \right)^{\frac{p}{2}} \right] = o(\varepsilon^p),
\]
\[
\mathbb{E} \left[ \left( \int_0^T |\tilde{\sigma}_{xx}(t)|^2 \left[ (x^\varepsilon(t) - x(t) - \delta(t))^2 - x_1(t - \delta)^2 \right]^2 dt \right)^{\frac{p}{2}} \right] = o(\varepsilon^p).
\]
Using (3.7), (3.8) and (3.10), we have

\[ \mathbb{E} \left[ ( \int_0^T \tilde{b}_{xx}^2(t) (x^\varepsilon(t) - x(t))(x^\varepsilon(t - \delta) - x(t - \delta)) - x_1(t)x_1(t - \delta) \, dt)^p \right] \]

\[ \leq C \mathbb{E} \left[ ( \int_0^T |x^\varepsilon(t)(x^\varepsilon(t - \delta) - x(t - \delta)) + x_1(t)(x^\varepsilon(t - \delta) - x(t - \delta)) \, dt)^p \right] \]

\[ \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^\varepsilon(t)|^p \right] \mathbb{E} \left[ \sup_{-\delta \leq t \leq T} |x(t)|^p \right] \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^p \right] \mathbb{E} \left[ \sup_{-\delta \leq t \leq T} |x(t)|^p \right] = o(\varepsilon^p), \]

and similarly

\[ \mathbb{E} \left[ ( \int_0^T |\tilde{\sigma}_{xx}(t)|^2 (x^\varepsilon(t) - x(t))(x^\varepsilon(t - \delta) - x(t - \delta)) - x_1(t)x_1(t - \delta) \, dt)^\frac{p}{2} \right] = o(\varepsilon^p). \]

By the continuity of \( b_{xx} \), we deduce

\[ \mathbb{E} \left[ ( \int_0^T \left( \tilde{b}_{xx}(t) - \frac{1}{2} b_{xx}(t) \right) x_1^2(t) \, dt)^p \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^2 \left( \int_0^T \left( \tilde{b}_{xx}(t) - \frac{1}{2} b_{xx}(t) \right) \, dt \right)^p \right] = o(\varepsilon^p). \]

Using the same method, we get

\[ \mathbb{E} \left[ ( \int_0^T (2\tilde{b}_{xx}(t) - b_{xx}(t)) x_1(t)x_1(t - \delta) \, dt)^p \right] = o(\varepsilon^p), \]

\[ \mathbb{E} \left[ ( \int_0^T \left( \tilde{b}_{xx}(t) - \frac{1}{2} b_{xx}(t) \right) x_1^2(t) \, dt)^p \right] = o(\varepsilon^p), \]

\[ \mathbb{E} \left[ ( \int_0^T |\tilde{\sigma}_{xx}(t)| - \frac{1}{2} \sigma_{xx}(t)|^2 |x_1(t)|^4 \, dt)^\frac{p}{2} \right] = o(\varepsilon^p), \]

\[ \mathbb{E} \left[ ( \int_0^T |\tilde{\sigma}_{xx}(t)| - \frac{1}{2} \sigma_{xx}(t)|^2 |x_1(t - \delta)|^4 \, dt)^\frac{p}{2} \right] = o(\varepsilon^p), \]

\[ \mathbb{E} \left[ ( \int_0^T (2\tilde{\sigma}_{xx}(t) - \sigma_{xx}(t))^2 |x_1(t)|^2 |x_1(t - \delta)|^2 \, dt)^\frac{p}{2} \right] = o(\varepsilon^p). \]

According to all above estimates, by Theorem 2.2 we complete the proof of (3.11). \( \square \)

Based on Lemma 3.2, we give the variational inequality.

**Lemma 3.2.** Let assumption (A1) hold. Suppose \((u(\cdot), x(\cdot))\) is the optimal pair, \(x^\varepsilon(\cdot)\) is the trajectory corresponding to the admissible control \(u^\varepsilon(\cdot)\), then the following variational inequality holds:

\[ \mathbb{E} \left[ \int_0^T \left( \delta l(t) + l_x(t)x_1(t) + x_2(t) + l_{xx}(t)x_1(t)(t - \delta) + x_2(t - \delta) + \frac{1}{2} l_{xx}(t)x_1(t)x_1(t) ight) \right. \]

\[ + \frac{1}{2} l_{xx}(t)x_2(t)(t - \delta)x_1(t)(t - \delta) + l_{xx}(t)x_1(t)x_1(t - \delta) \left. \right] \, dt + h_x(x(T))(x_1(T) + x_2(T)) \]  \( (3.14) \)

\[ + \frac{1}{2} h_{xx}(x(T))x_1(T)x_1(T) \geq o(\varepsilon). \]

where \( \delta l, l_x, l_{xx}, l_{xx}(t), l_{xx}, h_x, h_{xx} \) are defined as (3.6).
4 Global maximum principle

Now we consider the global maximum principle. Suppose \( u(\cdot) \) is the optimal control and \( u^\varepsilon(\cdot) \) is the perturbed admissible control.

We introduce the following three adjoint equations to deal with \( x_1(\cdot) + x_2(\cdot) \), \( x_3(\cdot) \) and \( x_1(\cdot)x_1(\cdot - \delta) \), as follows:

\[
\begin{align*}
-dp(t) &= \left\{ b_{x_3}(t)p(t) + \sigma_{x_3}(t)q(t) + l_{x_3}(t) + \mathbb{E}^{\mathcal{F}_t} \left[ b_{x_3|t+\delta}p(t + \delta) + \sigma_{x_3|t+\delta}q(t + \delta) \right] \right. \\
&\quad + l_{x_3}(t + \delta) \right\} dt - q(t)dB(t), \ t \in [0, T], \\
p(T) &= h_x(x(T)), \ p(t) = 0, \ t \in (T, T + \delta], \ q(t) = 0, \ t \in [T, T + \delta].
\end{align*}
\]

(4.1)

\[
\begin{align*}
-dP(t) &= \left\{ 2b_{x_3}(t)P(t) + \sigma_{x_3}(t)P(t) + 2\sigma_x(t)Q(t) + b_{xx}(t)p(t) + \sigma_{xx}(t)q(t) + l_{xx}(t) \\
&\quad + \mathbb{E}^{\mathcal{F}_t} \left[ \sigma_{x_3|t+\delta}P(t + \delta) + b_{x_3x_3|t+\delta}p(t + \delta) + \sigma_{x_3x_3|t+\delta}q(t + \delta) \right] + l_{x_3x_3|t+\delta} \right\} dt - Q(t)dB(t), \ t \in [0, T], \\
P(T) &= h_{xx}(x(T)), \ P(t) = 0, \ t \in (T, T + \delta], \ Q(t) = 0, \ t \in [T, T + \delta],
\end{align*}
\]

(4.2)

\[
\begin{align*}
-dK(t) &= \left\{ b_{x_3}(t)P(t) + \sigma_x(t)\sigma_{x_3}(t)P(t) + \sigma_{x_3}(t)Q(t) + b_{xx}(t)p(t) \\
&\quad + \sigma_{xx}(t)q(t) + l_{xx}(t) \right\} dt, \ t \in [0, T], \\
K(t) &= 0, \ t \in [T, T + \delta],
\end{align*}
\]

(4.3)

respectively.

Remark. The adjoint variable \( K(\cdot) \) is introduced mainly in order to deal with the cross terms of \( x_1(\cdot) \) and \( x_1(\cdot - \delta) \), which does not appear in the literatures as we know.

Apparently, the above three adjoint equations are linear ABSDEs. Since all derivatives are bounded, by Theorem \( 2.3 \) (4.1) admits a unique solution \( (p(\cdot), q(\cdot)) \in \mathcal{S}_2^2(0, T+\delta; \mathbb{R}) \times L^2_F(0, T+\delta; \mathbb{R}) \), (4.2) admits a unique solution \( (P(\cdot), Q(\cdot)) \in \mathcal{S}_2^2(0, T+\delta; \mathbb{R}) \times L^2_F(0, T+\delta; \mathbb{R}) \).

The following task is to give the global maximum principle. Since \( x_1(t) = 0, \ t \in [-\delta, 0] \) and \( p(t) = q(t) = 0, \ t \in (T, T + \delta] \), we have

\[
\begin{align*}
\mathbb{E} \int_0^T x_1(t - \delta) \left[ b_{x_3}(t)p(t) + \sigma_{x_3}(t)q(t) \right] dt \\
= \mathbb{E} \int_{-\delta}^{T-\delta} x_1(t) \left[ b_{x_3}(t + \delta)p(t + \delta) + \sigma_{x_3}(t + \delta)q(t + \delta) \right] dt \\
= \mathbb{E} \int_0^T x_1(t) \left[ b_{x_3}(t + \delta)p(t + \delta) + \sigma_{x_3}(t + \delta)q(t + \delta) \right] dt \\
- \mathbb{E} \int_{T-\delta}^T x_1(t) \left[ b_{x_3}(t + \delta)p(t + \delta) + \sigma_{x_3}(t + \delta)q(t + \delta) \right] dt
\end{align*}
\]
Similarly, suppose \( l_{x_3}(t) = 0 \) for \( t \in (T, T + \delta) \), then we have
\[
\mathbb{E} \int_0^T l_{x_3}(t)[x_1(t - \delta) + x_2(t - \delta)]dt = \mathbb{E} \int_0^T \mathbb{E}^{F_t} [l_{x_3}(t + \delta)] [x_1(t) + x_2(t)]dt.
\]
Applying Itô’s formula to \( p(t)(x_1(t) + x_2(t)) + \frac{1}{2} P(t)x_1^2(t) \) and substituting it into (3.14), we obtain
\[
\mathbb{E} \left\{ \int_0^T [\delta l(t) + p(t)\delta b(t) + q(t)\delta \sigma(t) + \frac{1}{2}P(t)|\delta \sigma(t)|^2 \right. \\
+ x_1(t - \delta) [q(t)\delta \sigma_x(t) + P(t)\sigma_x(t)\delta \sigma(t)] \\
+ x_1(t) [q(t)\delta \sigma_x(t) + P(t)\sigma_x(t)\delta \sigma(t) + Q(t)\delta \sigma(t)] \\
+ x_1(t)x_1(t - \delta) [l_{x_3}(t) + b_{x_3}(t)P(t) + \sigma_x(t)\sigma_{x_3}(t)P(t) + \sigma_{x_3}(t)Q(t) \\
+ b_{xx_3}(t)p(t) + \sigma_{xx_3}(t)q(t)] dt \} \geq o(\varepsilon).
\]
Applying again Itô’s formula to \( K(t)x_1(t) \), we get
\[
d[K(t)x_1(t)] = \left[ -x_1(t) [b_{x_3}(t)P(t) + x(t)\sigma_{x_3}(t)P(t) + \sigma_{x_3}(t)Q(t) + b_{x_3}(t)p(t) \\
+ \sigma_{xx_3}(t)q(t) + l_{x_3}(t)] + K(t) [b_x(t)x_1(t) + b_{x_3}(t)x_1(t - \delta) + \delta b(t)] \right] dt \\
+ K(t) [\sigma_x(t)x_1(t) + \sigma_{x_3}(t)x_1(t - \delta) + \delta \sigma(t)] dB(t).
\]
If \( K(t) = 0 \) for all \( t \in [0, T + \delta] \), we have
\[
0 = -\int_0^\delta x_1(t) [b_{x_3}(t)P(t) + x(t)\sigma_{x_3}(t)P(t) + \sigma_{x_3}(t)Q(t) + b_{x_3}(t)p(t) \\
+ b_{xx_3}(t)p(t) + \sigma_{xx_3}(t)q(t) + l_{x_3}(t)] dt \\
= -\int_0^\delta x_1(t + \delta) [b_{x_3}(t + \delta)P(t + \delta) + x(t + \delta)\sigma_{x_3}(t + \delta)P(t + \delta) + \sigma_{x_3}(t + \delta)Q(t + \delta) + b_{xx_3}(t + \delta)p(t + \delta) + \sigma_{xx_3}(t + \delta)q(t + \delta) + l_{x_3}(t + \delta)] dt, \ \delta \leq r \leq T.
\]
Noting (3.4), we have
\[
x_1(r - \delta) = \int_0^{r-\delta} [b_x(t)x_1(t) + b_{x_3}(t)x_1(t - \delta) + \delta b(t)] dt \\
+ \int_0^{r-\delta} [\sigma_x(t)x_1(t) + \sigma_{x_3}(t)x_1(t - \delta) + \delta \sigma(t)] dB(t).
\]
Under the filtration $F_{r-\delta}$, applying Itô’s formula to $K(r)x_1(r)x_1(r - \delta)$, by the above two equalities we get

$$0 = \int_0^{r-\delta} -x_1(t)x_1(t + \delta) [b_{x_3}(t + \delta)P(t + \delta) + \sigma_x(t + \delta)\sigma_{x_3}(t + \delta)P(t + \delta)$$

$$+ \sigma_{x_3}(t + \delta)Q(t + \delta) + b_{xx_3}(t + \delta)p(t + \delta) + \sigma_{xx_3}(t + \delta)q(t + \delta) + l_{xx_3}(t + \delta)]dt.$$ 

Now choose $r = T$ and recall $x_1(t - \delta) = 0$ for $t \in [0, \delta]$, we can derive

$$0 = -\int_0^{T} x_1(t)x_1(t - \delta) [b_{x_3}(t)P(t) + \sigma_x(t)\sigma_{x_3}(t)P(t) + \sigma_{x_3}(t)Q(t)$$

$$+ b_{xx_3}(t)p(t) + \sigma_{xx_3}(t)q(t) + l_{xx_3}(t)]dt. \quad (4.5)$$

Substituting (4.5) into (4.4), we obtain

$$\mathbb{E} \int_0^{T} \left[ \delta l(t) + p(t)\delta b(t) + q(t)\delta \sigma(t) + \frac{1}{2} P(t)\delta \sigma(t)^2 \right]$$

$$+ x_1(t - \delta) \left[ q(t)\delta \sigma_{x_3}(t) + P(t)\sigma_{x_3}(t)\delta \sigma(t) \right]$$

$$+ x_1(t) \left[ q(t)\delta \sigma_x(t) + P(t)\delta b(t) + P(t)\sigma_x(t)\delta \sigma(t) + Q(t)\delta \sigma(t) \right] dt \geq o(\varepsilon). \quad (4.6)$$

Next we try to prove that

$$\mathbb{E} \int_0^{T} \left[ x_1(t - \delta) \left[ q(t)\delta \sigma_{x_3}(t) + P(t)\sigma_{x_3}(t)\delta \sigma(t) \right] + x_1(t) \left[ q(t)\delta \sigma_x(t) + P(t)\delta b(t) \right.$$

$$+ P(t)\sigma_x(t)\delta \sigma(t) + Q(t)\delta \sigma(t) \left. \right] \right] dt = o(\varepsilon). \quad (4.7)$$

In fact, by the boundedness of $\sigma_x$, we have

$$\mathbb{E} \int_0^{T} |x_1(t)q(t)\delta \sigma_x(t)| dt$$

$$\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)| \left( \int_{E_\varepsilon} |q(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{E_\varepsilon} |\delta \sigma_x(t)|^2 dt \right)^{\frac{1}{2}} \right]$$

$$\leq C \varepsilon^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \int_{E_\varepsilon} |q(t)|^2 dt \right\}^{\frac{1}{2}} = o(\varepsilon). \quad (4.8)$$

Recall the assumption (A1)(ii) and the estimate of the solution to SDDE (3.1), we obtain

$$\mathbb{E} \int_0^{T} |x_1(t)P(t)\delta b(t)| dt$$

$$\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)| \left( \int_{E_\varepsilon} |P(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{E_\varepsilon} |\delta b(t)|^2 dt \right)^{\frac{1}{2}} \right]$$

$$\leq \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_1(t)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \int_{E_\varepsilon} |P(t)|^2 dt \int_{E_\varepsilon} C (1 + |x(t)|^2 + |x(t - \delta)|^2$$

$$+ |x_\varepsilon(t)|^2 + |x_\varepsilon(t - \delta)|^2) dt \right\}^{\frac{1}{2}} = o(\varepsilon). \quad (4.9)$$
Then, we obtain the following main result in this paper.

The other terms of (4.7) can be verified in a similar method. Hence we can simplify (4.6) to

$$\mathbb{E} \int_0^T \left[ \delta l(t) + p(t)\delta b(t) + q(t)\delta \sigma(t) + \frac{1}{2}P(t) |\delta \sigma(t)|^2 \right] dt \geq o(\varepsilon).$$

(4.10)

Now, define the Hamilton function as

$$H(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta), p(\tau), q(\tau), P)$$

$$= l(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta)) + pb(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta)) + \frac{1}{2}P\sigma^2(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta))$$

$$- P\sigma(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta)) \sigma(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta)).$$

(4.11)

Then, we obtain the following main result in this paper.

**Theorem 4.1.** Let assumption (A1) hold. Suppose \((x(\cdot), u(\cdot))\) is the optimal pair. Let \(l_{x, \delta}(t) = l_{x, 2\delta}(t) = 0\) hold for \(t \in (T, T + \delta)\). Suppose \((p(\cdot), q(\cdot)) \in S^2_{\mathbb{F}}([0, T]; \mathbb{R}) \times L^2_{\mathbb{F}}([0, T]; \mathbb{R})\) and \((P(\cdot), Q(\cdot)) \in S^2_{\mathbb{F}}([0, T]; \mathbb{R}) \times L^2_{\mathbb{F}}([0, T]; \mathbb{R})\) satisfy (4.1) and (4.2), respectively. Besides, suppose \(K(t) = 0\) for all \(t \in [0, T + \delta]\). Then

$$H(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta), p(\tau), q(\tau), P)$$

$$+ \mathbb{E}^{F_{\tau}} \left[ H(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v, p(\tau + \delta), q(\tau + \delta), P(\tau + \delta)) \right] I_{[0, T - \delta)}(\tau)$$

$$\geq H(\tau, x(\tau), x(\tau - \delta), v, u(\tau - \delta), p(\tau), q(\tau), P(\tau))$$

$$+ \mathbb{E}^{F_{\tau}} \left[ H(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v, p(\tau + \delta), q(\tau + \delta), P(\tau + \delta)) \right] I_{[0, T - \delta)}(\tau),$$

$$\forall v \in U, \text{ a.e. } \tau \in [0, T], \mathbb{P}\text{-a.s.},$$

where \(H\) is the Hamiltonian function defined by (4.11).

**Remark.** When \(b, \sigma, l\) do not contain the delay term, the maximum condition (4.12) becomes the classical maximum condition in Peng [19] or Yong and Zhou [25].

**Proof.** Let \(x^\varepsilon(\cdot)\) be the state trajectory corresponding to the perturbed control \(u^\varepsilon(\cdot)\) defined in (4.3). Under the assumptions of Theorem 4.1, we can get (4.10). Noting \(u^\varepsilon(t) \neq u(t)\) for \(t \in [\tau, \tau + \varepsilon]\) and \(u^\varepsilon(t - \delta) \neq u(t - \delta)\) for \(t \in [\tau + \delta, \tau + \delta + \varepsilon]\), thus we have

$$\mathbb{E} \int_{\tau}^{\tau + \varepsilon} \left\{ l(t, x(t), x(t - \delta), v(t), u(t - \delta)) - l(t, x(t), x(t - \delta), u(t), u(t - \delta)) $$

$$+ p(t) \left[ b(t, x(t), x(t - \delta), v(t), u(t - \delta)) - b(t, x(t), x(t - \delta), u(t), u(t - \delta)) \right]$$

$$+ q(t) \left[ \sigma(t, x(t), x(t - \delta), v(t), u(t - \delta)) - \sigma(t, x(t), x(t - \delta), u(t), u(t - \delta)) \right]$$

$$+ \frac{1}{2}P(t) \left[ \sigma^2(t, x(t), x(t - \delta), v(t), u(t - \delta)) - \sigma^2(t, x(t), x(t - \delta), u(t), u(t - \delta)) \right]$$

$$- P(t) \sigma(t, x(t), x(t - \delta), u(t), u(t - \delta)) \left[ \sigma(t, x(t), x(t - \delta), v(t), u(t - \delta))$$

$$- \sigma(t, x(t), x(t - \delta), u(t), u(t - \delta)) \right] \right\} dt$$
\[ + \mathbb{E} \int_{t+\delta}^{t+\delta+\varepsilon} \left\{ l(t, x(t), x(t-\delta), u(t), v(t-\delta)) - l(t, x(t), x(t-\delta), u(t), u(t-\delta)) \\
+ p(t) \left[ b(t, x(t), x(t-\delta), v(t)) - b(t, x(t), x(t-\delta), u(t), u(t-\delta)) \right] \\
+ q(t) \left[ \sigma(t, x(t), x(t-\delta), v(t)) - \sigma(t, x(t), x(t-\delta), u(t), u(t-\delta)) \right] \\
+ \frac{1}{2} P(t) \left[ \sigma^2(t, x(t), x(t-\delta), v(t)) - \sigma^2(t, x(t), x(t-\delta), u(t), u(t-\delta)) \right] \\
- P(t) \sigma(t, x(t), x(t-\delta), u(t), u(t-\delta)) \right\} dt I_{[0, T-\delta]}(t) \geq o(\varepsilon). \]

Dividing both sides simultaneously by \( \varepsilon \), we obtain

\[ \mathbb{E} \left\{ l(\tau, x(\tau), x(\tau-\delta), v(\tau), u(\tau-\delta)) - l(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \\
+ p(\tau) \left[ b(\tau, x(\tau), x(\tau-\delta), v(\tau), u(\tau-\delta)) - b(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right] \\
+ q(\tau) \left[ \sigma(\tau, x(\tau), x(\tau-\delta), v(\tau), u(\tau-\delta)) - \sigma(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right] \\
+ \frac{1}{2} P(\tau) \left[ \sigma^2(\tau, x(\tau), x(\tau-\delta), v(\tau)) - \sigma^2(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right] \\
- P(\tau) \sigma(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right\} I_{[0, T-\delta]}(\tau) \geq 0, \text{ a.e. } \tau \in [0, T]. \]

Choose \( v(\tau) = vI_A + u(\tau)I_{A^c}, A \in \mathcal{F}_\tau, v \in \mathbf{U} \), then we get

\[ \mathbb{E} \left\{ l(\tau, x(\tau), x(\tau-\delta), v, u(\tau-\delta)) - l(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \\
+ p(\tau) \left[ b(\tau, x(\tau), x(\tau-\delta), v, u(\tau-\delta)) - b(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right] \\
+ q(\tau) \left[ \sigma(\tau, x(\tau), x(\tau-\delta), v, u(\tau-\delta)) - \sigma(\tau, x(\tau), x(\tau-\delta), u(\tau), u(\tau-\delta)) \right] \\
+ \frac{1}{2} P(\tau) \left[ \sigma^2(\tau, x(\tau), x(\tau-\delta), v) - \sigma^2(\tau, x(\tau), x(\tau-\delta), u) \right] \\
- P(\tau) \sigma(\tau, x(\tau), x(\tau-\delta), u, u(\tau-\delta)) \right\} I_{[0, T-\delta]}(\tau) \geq 0, \text{ a.e. } \tau \in [0, T]. \]
+ \mathbb{E}^{\mathcal{F}_\tau}\left\{ l(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v) - l(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau)) \\
+ p(\tau + \delta) [b(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v) - b(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau))] \\
+ q(\tau + \delta) [\sigma(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v) - \sigma(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau))] \\
+ \frac{1}{2} P(\tau + \delta) \left[ \sigma^2(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v) \\
- \sigma^2(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau)) \right] \\
- P(\tau + \delta) \sigma(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau)) \left[ \sigma(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), v) \\
- \sigma(\tau + \delta, x(\tau + \delta), x(\tau), u(\tau + \delta), u(\tau)) \right] \right\} \mathbf{1}_{[0, T]}(\tau) \geq 0, \text{ a.e. } \tau \in [0, T], \mathbb{P}\text{-a.s.}

Hence we complete the proof of the theorem.

\begin{remark}
Comparing with the classical stochastic maximum principle, (1.12) has an indicator function, which is the characteristic of the stochastic optimal control problem with delay.
\end{remark}

\begin{remark}
Noting in the above theorem, it is rigorous that \( l_{x,\delta}(t) = l_{x,x,\delta}(t) = 0 \) holds for \( t \in (T, T + \delta) \). However, in [7], they assume that \( l \) doesn’t contain the state delay term \( x_\delta \) (See equation (4) on page 1075 of [7]).
\end{remark}

\begin{remark}
It is interesting to find that the Hamilton function \( H \) does not contain the adjoint variable \( K(\cdot) \) in a explicit form. But in fact, the value of \( P(\cdot) \) is relative to the value of \( K(\cdot) \), thus the Hamilton function \( H \) actually implicitly contains the adjoint variable \( K(\cdot) \). This is one of the main contribution of this paper.
\end{remark}

5 A solvable LQ example

In this section, we solve a simple example using the global maximum principle obtained in the previous section.

Consider the following linear controlled stochastic system with delay:

\[
\begin{cases}
\quad dx(t) = [Bv(t) + \bar{B}v(t - \delta)] dt + [Cx(t - \delta) + Dv(t) + \bar{D}v(t - \delta)] dB(t), \ t \in [0, T], \\
\quad x(\theta) = \varphi(\theta), \ \theta \in [-\delta, 0],
\end{cases}
\]

(5.1)

with the quadratic cost functional

\[
J(v(\cdot)) = \mathbb{E} \left[ \frac{1}{2} \int_0^T \left[ Nv(t)^2 + \bar{N}v^2(t - \delta) \right] dt + x(T) \right],
\]

(5.2)

where \( B, D, C, \bar{D}, N, \bar{N} \in \mathbb{R} \) and \( N, \bar{N} > 0 \).

The admissible control set is defined as \( \mathcal{U}_{ad} := \{ v(\cdot) : [0, T] \times \Omega \rightarrow \mathbf{U} | v(\cdot) \text{ is a } \mathcal{F}_t\text{-predictable, square-integrable process} \} \), where \( \mathbf{U} \subseteq \mathbb{R} \) is a nonconvex set.

Our object is to find the control \( u(\cdot) \in \mathcal{U}_{ad} \) such that (5.1) is satisfied and (5.2) is minimized.
First we introduce the following adjoint equations:

\[
\begin{cases}
-dp(t) = CE^F_t[q(t + \delta)]dt - q(t)dB(t), \quad t \in [0, T], \\
p(T) = 1, q(T) = 0, \quad p(t) = q(t) = 0, \quad t \in (T, T + \delta],
\end{cases}
\]

\( (5.3) \)

\[
\begin{cases}
-dP(t) = C^2E^F_t[P(t + \delta)]dt - Q(t)dB(t), \quad t \in [0, T], \\
P(T) = 0, Q(T) = 0, \quad P(t) = Q(t) = 0, \quad t \in (T, T + \delta],
\end{cases}
\]

\( (5.4) \)

\[
\begin{cases}
-dK(t) = CQ(t)dt, \quad t \in [0, T], \\
K(t) = 0, \quad t \in (T, T + \delta].
\end{cases}
\]

\( (5.5) \)

By Theorem 2.3, the above three ABSDEs admit the unique solutions, respectively. In general, the solution to ABSDE can be given part by part. In fact, in our case we have for \( t \in [0, T + \delta] \),

\[
p(t) \equiv 1, \quad q(t) \equiv 0, \quad P(t) \equiv 0, \quad Q(t) \equiv 0, \quad K(t) \equiv 0.
\]

\( (5.6) \)

Define the Hamilton function as follows:

\[
H(\tau, x, x_\delta, v, v_\delta, p, q, P) = Nv^2 + \bar{N}v_\delta^2 + p(Bv + \bar{B}v_\delta) + q(Cx_\delta + Dv + \bar{D}v_\delta) + \frac{1}{2}P(Cx_\delta + Dv + \bar{D}v_\delta)^2 - P[Cx(\tau - \delta) + Du(\tau) + \bar{D}u(\tau - \delta)](Cx_\delta + Dv + \bar{D}v_\delta).
\]

\( (5.7) \)

Then according to Theorem 4.1, we obtain

\[
[N + \bar{N}I_{[0,T-\delta]}(\tau)]u^2 + [B + \bar{B}I_{[0,T-\delta]}(\tau)]u \geq [N + \bar{N}I_{[0,T-\delta]}(\tau)]u^2(\tau) + [B + \bar{B}I_{[0,T-\delta]}(\tau)]u(\tau), \quad \forall v \in U, \ a.e. \ \tau \in [0, T], \ \mathbb{P}\text{-a.s.}
\]

\( (5.8) \)

Hence the optimal control is of the form as

\[
u(\tau) = -\frac{B + \bar{B}I_{[0,T-\delta]}(\tau)}{2[N + \bar{N}I_{[0,T-\delta]}(\tau)]}, \quad a.e. \ \tau \in [0, T].\]

\( (5.9) \)

6 Concluding remarks

In this paper, we have discussed the stochastic optimal control problem with delay where the control domain is nonconvex and the diffusion term contains both control and its delayed term. Inspired by [18] and [7], we could derive the global maximum principle. Comparing with the classical maximum principle without delay, the maximum condition (4.12) contains an indicator function, in fact it is the characteristic of the stochastic optimal control problem with delay. Furthermore, it is interesting to find that the Hamilton function \( H \) does not explicitly contain the new adjoint variable \( K(\cdot) \), and it is relative to the adjoint variables \( p(\cdot), q(\cdot) \) and \( P(\cdot) \). However, to derive the maximum principle, we have to assume that \( K(\cdot) \equiv 0 \), which is apparently a little
strong assumption. Finally, we give an LQ example to explain the application of the maximum principle. In the future, we hope to find a method to weaken the condition $K(\cdot) \equiv 0$. Possible extension to the controlled forward-backward stochastic systems with delay (Chen and Wu [6], Huang, Li and Shi [13], Shi, Xu and Zhang [21]), to find the global maximum principle, is a natural research topic. We wish to deal with this problem in our near future research.

References

[1] N. Agram, S. Haadem, B. Øksendal and F. Proske, Maximum principle for infinite horizon delay equations, *SIAM J. Math. Anal.*, 45, 2499-2522, 2013.

[2] M. Arriojas, Y. Z. Hu, S. E. A. Monhammed and G. Pap, A delayed Black and Scholes formula, *Stoch. Anal. Appl.*, 25, 471-492, 2007.

[3] A. Bensoussan, Nonlinear Filtering and Stochastic Control, Lectures on Stochastic Control, *Lecture Notes in Mathematics*, 972, Proceedings, Contona, 1982.

[4] J. M. Bismut, An introductory approach to duality in optimal stochastic control, *SIAM Rev.*, 20, 62-78, 1978.

[5] L. Chen and J. H. Huang, Stochastic maximum principle for controlled backward delayed system via advanced stochastic differential equation, *J. Optim. Theory Appl.*, 167, 1112-1135, 2015.

[6] L. Chen and Z. Wu, Maximum principle for stochastic optimal control problem of forward-backward system with delay, *Proc. Joint 48th IEEE CDC and 28th CCC*, 2899-2904, Shanghai, December 16-18, 2009.

[7] L. Chen and Z. Wu, Maximum principle for the stochastic optimal control problem with delay and application, *Automatica*, 46, 1074-1080, 2010.

[8] L. Chen, Z. Y. Yu, Maximum principle for nonzero-sum stochastic differential game with delays, *IEEE Trans. Autom. Control*, 60, 1422-1426, 2015.

[9] H. Du, J. H. Huang and Y. L. Qin, A stochastic maximum principle for delayed mean-field stochastic differential equations and its applications, *IEEE Trans. Autom. Control*, 58, 3212-317, 2013.

[10] W. H. Fleming, Optimal continuous-parameter stochastic control, *SIAM Rev.*, 11, 470-509, 1969.

[11] Y. Hu, Maximum principle of optimal control for Markov processes, *Acta Mathematica Sinica*, 33, 43-56, 1990.
[12] Y. Hu and S. G. Peng, Maximum principle for semilinear stochastic evolution control systems, *Stoch. & Stoch. Rep.*, 33, 159-180, 1990.

[13] J. H. Huang, X. Li and J. T. Shi, Forward-backward linear quadratic stochastic optimal control problem with delay, *Syst. & Control Lett.*, 61, 623-630, 2012.

[14] Y. I. Kazmerchuk, A. V. Swishchuk and J. H. Wu, The pricing of options for securities markets with delayed response, *Math. Compu. Simul.*, 75, 69-79, 2007.

[15] H. J. Kushner, Necessary conditions for continuous parameter stochastic optimization problems, *SIAM J. Control*, 10, 550-565, 1972.

[16] S. E. A. Mohammed, *Stochastic Functional Differential Equations*, Pitman, 1984.

[17] S. E. A. Mohammed, *Stochastic differential equations with memory: theory, examples and applications*, Progress in Probability, Stochastic Analysis and Related Topics 6, The Geido Workshop, Birkhauser, 1996.

[18] B. Øksendal and A. Sulem, A maximum principle for optimal control of stochastic systems with delay, with applications to finance. In J. M. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal control and partial differential equations, ISO Press, Amsterdam, 64-79, 2000.

[19] S. G. Peng, A general stochastic maximum principle for optimal control problems, *SIAM J. Control Optim.*, 28, 966-979, 1990.

[20] S. G. Peng and Z. Yang, Anticipated backward stochastic differential equation, *Anna. Proba.*, 37, 877-902, 2009.

[21] J. T. Shi, J. J. Xu and H. S. Zhang, Stochastic recursive optimal control problems with time delay and applications, *Math. Control Rela. Fields*, 5, 859-888, 2015.

[22] Y. Wang and Z. Wu, Necessary and sufficient conditions for near-optimality of stochastic delay systems, *Inter. J. Control*, 91, 1730-1744, 2018.

[23] S. Wu and G. C. Wang, Optimal control problem of backward stochastic differential delay equation under partial information, *Syst. & Control Lett.*, 82, 71-78, 2015.

[24] Z. Y. Yu, The stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls, *Automatica*, 48, 2420-2432, 2012.

[25] J. M. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, USA, 1999.

[26] W. L. Zhu and Z. S. Zhang, Verification theorem of stochastic optimal control with mixed delay and applications to finance, *Asian J. Control*, 17, 1285-1295, 2015.