Adaptive fuzzy fixed-time control for a class of strict-feedback stochastic nonlinear systems

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ABSTRACT

This paper studies fixed-time tracking problems for stochastic nonlinear systems in strict-feedback form. Different from previous results, the practical fixed-time bounded theorem for stochastic nonlinear systems is given. The unknown functions of the stochastic nonlinear systems are approximated by the Fuzzy logic system (FLS) which has a universal approximation. Then by using a back-stepping method, a novel adaptive fuzzy fixed-time controller is designed for stochastic nonlinear systems based on the fixed-time bounded theorem. The states of the stochastic nonlinear systems are guaranteed to converge into an equilibrium point contained compact set semi-globally in fixed-time by the designed controller. Finally, a numerical example and a vehicle tracking model example are provided to illustrate the proposed strategy.

1. Introduction

Adaptive fuzzy control strategy via back-stepping design strategy is a powerful tool in handling the tracking problem or stabilization task for nonlinear systems. Fruitful results have been obtained in Sun, Yi et al. (2020), Na et al. (2020), Wang et al. (2015), Wang et al. (2020), Wang et al. (2021, 2022) and Ma et al. (2021) after fuzzy basis function (FBF) proposed firstly in Wang and Mendel (1992) which proves universal approximation of FLS by using Stone–Weierstrass theorem. This method shows great potential in improving control performance for nonlinear systems not only in theoretical analysis but also in industrial applications, such as hydraulic control, servo mechanism control, and aerospace control. To mention a few, the scholars in Wang et al. (2015) design an adaptive fuzzy controller by using FLS for pure-feedback nonlinear systems with state constraints and input saturation. In order to alleviate the number of updated control inputs, an event-triggered strategy is applied to reduce the communication burden. In applications, the scholars in Sun, Yi et al. (2020) develop an adaptive fuzzy finite-time control strategy for air-breathing hypersonic vehicles by utilizing an interval type-2 FLS, and the examples shown in representative scenarios verify the robustness of developed method. For active suspension systems with unknown dynamics, the authors in Na et al. (2020) use FLS to approximate unknown functions of the systems. Then, an estimation error-based adaptive algorithm is proposed to improve the transient suspension response performance and guarantee the suspension error achieves convergence.

Most results aforementioned are for deterministic nonlinear systems, a mount of latest researches are also obtained for more complicated stochastic nonlinear systems in recent years. Combining Itô stochastic differential equation with stochastic Lyapunov stability theory, the authors in Deng and Krstić (1997) introduce the quadratic Lyapunov function and infinitesimal generator to propose a global asymptotic stability in probability control strategy for the first time via back-stepping method. Based on the result, the authors in (2022) and Khoo et al. (2013) propose different adaptive fuzzy control strategies for high-order stochastic nonlinear systems and stochastic nonlinear systems in strict-feedback form, respectively. All the stochastic variables of these closed-loop nonlinear systems are guaranteed to be bounded in probability. Nevertheless, stability is not the only criteria to evaluate the control strategies designed for the stochastic nonlinear systems, the convergence rate and the transient performance are also important. Combine with the finite-time control method which is proposed in Bhat and Bernstein (2000) and widely concerned in Min et al. (2021), Sui et al. (2019), Jiang et al. (2019) and 2022 (2022), the authors in Sui et al. (2019) solve
the stochastically adaptive fuzzy control problem for the first-order stochastic nonlinear systems in non-triangular form. As for high-order stochastic nonlinear systems of which orders are given in odd positive integers, the authors in Jiang et al. (2019) and (2022) design finite-time controllers with full-state constraints and inverse dynamics, respectively.

However, in the finite-time control strategy, the setting-time function which refers to the convergence time to the equilibrium point not only depends on the chosen parameters but also closely associates with the initial values of nonlinear systems. In order to remove this restriction, the author in Polyakov (2012) proposes a fixed-time nonlinear feedback design method for linear control systems by utilizing block control principles. Then, this method extends to the nonlinear systems in Polyakov et al. (2015) by using theorems on implicit Lyapunov functions. After that, the fixed-time control method has been widely used in nonlinear control theories and applications such as the synchronized consensus control for multi-agent systems in Li et al. (2021) and air-breathing hypersonic vehicles via sliding mode observers in Sun, Pu et al. (2020). Even though some results have been received on fixed-time control for deterministic nonlinear systems in recent years in Chen et al. (2021), Ba et al. (2019) and Hua et al. (2017), a few studies focus on the tracking problems for stochastic nonlinear systems except (2021; Yu et al., 2019), which motivate this study. The main contributions of this article are emphasized as following:

1. A novel adaptive fuzzy fixed-time control strategy is proposed for the stochastic nonlinear system in this paper. Compare with classical adaptive fuzzy control and finite-time control, the fixed-time control method proposed in this paper has a stronger converge ability. Different from the sliding mode control used in previous works, the adaptive fuzzy method, which extends the potential using for nonlinear systems with unknown uncertainties, is utilized as the universal approximation performance. Thus, the practical fixed-time bounded theorem for stochastic nonlinear systems is given for the existence of approximate errors.

2. Different from deterministic nonlinear systems, stochastic nonlinear systems are studied in this paper. Ito formula and infinitesimal generator are utilized in this paper as the Riemann calculus is not suitable for stochastic systems. Then quartic Lyapunov function is constructed to handle the Hessian item which is caused by the use of stochastic calculus and brings the second partial derivative of the state x directly.

The rest of this paper is organized as follows. Section 2 gives the preliminaries of this paper. The adaptive fuzzy fixed-time control strategy is proposed in Section 2. Section 4 presents the simulation studies via a numerical and a vehicle tracking model example. The conclusions of the paper are presented in Section 5.

2. Preliminaries

2.1. Problem formulation

The stochastic nonlinear systems in this paper are described by the stochastic differential equation (SDE) as following

\[
\begin{align*}
\dot{x}_i &= (x_{i+1} + f(\tilde{x}_i)) \, dt + g_i(\tilde{x}_i) \, d\omega_i, \quad 1 \leq i \leq n - 1 \\
\dot{x}_n &= (u + f(\tilde{x}_n)) \, dt + g_n(x) \, d\omega \\
y &= x_1
\end{align*}
\]

where \( x_i \in \mathbb{R}(i = 1, \ldots, n) \) are state variables of the stochastic nonlinear systems which can be measured directly. \( u \in \mathbb{R} \) and \( y \in \mathbb{R} \) are input and measured output signals of system (1), respectively. \( \omega \) represents for an independent \( r \)-dimensional standard Brownian motion defined on a complete probability space \((\mathcal{S}, \mathcal{F}, \mathbb{P})\). \( f(\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) denote for smooth nonlinear functions which are unknown and satisfy Lipschitz conditions.

The objective of this paper is to design a novel adaptive state-feedback controller for stochastic nonlinear (1) such that

1. All signals of the stochastic systems (1) are semi-globally ultimately uniformly bounded in fixed-time;
2. The tracking error of output signal and reference signal can converge into an equilibrium point contained compact set within setting time \( T \) which is not dependent on the initial values of the states.

To this end, the following assumptions are introduced.

**Assumption 2.1:** The given reference signal \( y_r(t) \) and its \( n \)th order derivatives are assumed to be bounded, i.e. \(|y_r(t)| \leq y_0 \) and \(|y_r^{(i)}(t)| \leq y_1\) hold.

**Assumption 2.2:** The Brownian motion in this paper is mutually independent for \( \omega_i \) and \( \omega_j \) and is assumed to has a bounded covariance \( \mathbb{E}([d\omega_i \cdot d\omega_j]^T) = \vartheta^T(t)\vartheta(t) \).

Hence, the smooth functions \( g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) satisfy \( G^T(x)\vartheta \hat{\vartheta}^T G(x) = \bar{\vartheta} \bar{\vartheta}^T \), where \( \hat{\vartheta} \) is the upper bound of \( \vartheta(t) \) which is assumed to be known and \( G(x) \) is defined as \( G(x) = [g_1(x), \ldots, g_n(x)]^T \).
**Definition 2.1 (Itô formula, Deng & Krstić, 1997):** Consider the following nonlinear SDE

$$dx = f(x, u) \, dt + g(x) \, d\omega$$  \hspace{1cm} (2)

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the system state and input vector, respectively. By introducing a positive definite $C^2$ Lyapunov function $V(x)$, the infinitesimal generator $LV(x)$ of $V(x)$ is defined as

$$LV(x) = \frac{\partial V(x)}{\partial x} \cdot f(x) + \frac{1}{2} \text{Tr} \left\{ g^T(x) \frac{\partial^2 V(x)}{\partial x^2} g(x) \right\}. \hspace{1cm} (3)$$

**Definition 2.2 (Sui et al., 2019):** Consider SDE $dx = f(x, u) \, dt + g(x) \, d\omega$, if for initial state $x(t_0, \omega) = x_0$ in compact set $U$, there exist a small positive constant $\epsilon$ and setting-time $T_s = T(\epsilon, x_0) \leq \frac{V(x(t_0))}{\epsilon}$ such that $\mathbb{E}[|x(t)|] \leq \epsilon$ when $t \geq t_0 + T_s$, then the equilibrium point $x = 0$ is semi-global finite-time stable in probability (SGFSP).

**Definition 2.3 (Sui et al., 2019):** If there exists positive defined $C^2$ Lyapunov function $V(x)$, positive numbers $C$, $D$ and $0 < \eta < 1$, infinitesimal generator $LV(x)$ defined in Definition 2.2, and functions $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ in $K_\infty$, such that

$$\kappa_1(|x|) \leq V(x) \leq \kappa_2(|x|)$$

$$LV(x) \leq -CVp(x) + D \hspace{1cm} (4)$$

for $t > t_0$, the SDE (2) in Definition 2.1 is guaranteed to be SGFSP. Then there is a unique strong solution that satisfies the following inequality

$$\mathbb{E}[V(x(t))] \leq V(x_0) e^{-Ct} + \frac{D}{C} \hspace{1cm} (5)$$

**Lemma 2.1 (2021):** For SDE given in (2), if there exists positive defined $C^2$ Lyapunov function $V(x)$ and the infinitesimal generator $LV(x)$ defined in Definition 2.1 satisfies following

$$LV(x) \leq -\alpha Vp(x) - \beta Vq(x) \hspace{1cm} (6)$$

where $0 < p < 1$, $q > 1$, $\alpha$ and $\beta$ are positive numbers, then the equilibrium of the system is practically fixed-time stable in probability. Furthermore, the setting-time $T_s$ is bounded by $T_s < T_{\max} := \frac{1}{\alpha - ap} + \frac{1}{\beta - bq}$.

**Remark 2.1:** Fixed-time control is a sort of finite-time control strategy with strong convergence performance. The setting time in Definition 2.2 is the function of initial condition $x(t_0)$ and designed parameters, while the setting time function in Lemma 2.1 is the function of parameters that are not related to the initial condition. Lemma 2.1 shows that the states of stochastic nonlinear systems are asymptotic converges to the equilibrium point within a setting-time function $T_s$. However, it is not necessary to (sometimes can not) converge to the equilibrium point but a compact set of the neighbourhood of equilibrium point in theoretical analyses and practical applications.

**Theorem 2.1:** If the following inequality holds for stochastic nonlinear system (2)

$$LV(x) \leq -\alpha Vp(x) - \beta Vq(x) + D \hspace{1cm} (7)$$

where $p \in (0, 1)$, $q > 1$, $\alpha$, $\beta$ and $D$ are positive numbers, the state of the stochastic nonlinear system can converges to an equilibrium point contained compact set semi-globally in fixed-time. The setting-time $T_s$ satisfies $T_s < T_{\max} := \frac{1}{\alpha - ap} + \frac{1}{\beta - bq}, 0 < \eta < 1$. The residual set is given by $x \in \{ V(x) \leq \min\{\frac{D}{\alpha(1-\eta)}\} \}$.

**Proof:** As $0 < \eta < 1$, we have following from (7)

$$LV(x) \leq -\eta \alpha Vp(x) - (1 - \eta)\alpha Vp(x) - \beta Vq(x) + D \hspace{1cm} (8)$$

We partition the domain of definition of $x$ into two subsets: $\Omega_1 = \{ x | V(x) > (\frac{D}{\alpha(1-\eta)})^\frac{1}{p} \}$ and $\Omega_2 = \{ x | V(x) \leq (\frac{D}{\alpha(1-\eta)})^\frac{1}{p} \}$. Then we have following

Case 1: For $x(t, \omega) \in \Omega_1$, it yields

$$LV(x) \leq -\eta \alpha Vp(x) - \beta Vq(x) \hspace{1cm} (9)$$

Combine with Lemma 2.1, the solution is practically fixed-time stable in probability, and the setting-time function $T_1$ is bounded by $T_1 < T_{\max} := \frac{1}{\alpha - ap} + \frac{1}{\beta - bq}$.

Case 2: For $x(t, \omega) \in \Omega_2$, the trajectories of state varies $x(t, \omega)$ do not exceed of set $\Omega_2$, which also means state varies are bounded.

The following inequality can obtain from (7)

$$LV(x) \leq -\alpha Vp(x) - \eta \beta Vq(x) - (1 - \eta)\beta Vq(x) + D \hspace{1cm} (10)$$

Similarly, we partition the domain of definition of $x$ into another two subsets: $\Omega_2 = \{ x | V(x) > (\frac{D}{\beta(1-\eta)})^\frac{1}{q} \}$ and $\Omega_2 = \{ x | V(x) \leq (\frac{D}{\beta(1-\eta)})^\frac{1}{q} \}$. Then we have following two cases

Case 3: For $x(t, \omega) \in \Omega_2$, we have

$$LV(x) \leq -\alpha Vp(x) - \eta \beta Vq(x) \hspace{1cm} (11)$$

Then, the solution is practically fixed-time stable in probability, and setting-time function $T_2$ is bounded by $T_2 < T_{\max} := \frac{1}{\alpha - ap} + \frac{1}{\beta - bq}$.

Case 4: For $x(t, \omega) \in \Omega_2$, the trajectories of state varies $x(t, \omega)$ is bounded by the set $\Omega_2$.

As $\eta \in (0, 1)$, we have $\frac{1}{\alpha(1-\eta)} < \frac{1}{\alpha(1-\eta)\eta}$ and $\frac{1}{\beta(1-\eta)} < \frac{1}{\beta(1-\eta)\eta}$. Combined with case 1–4, Theorem 2.1 is proved. □
Remark 2.2: Similarly, inequality (7) can also be written as follows by introducing positive constants $\eta_1$ and $\eta_2$

$$\mathcal{L}V(x) \leq -\eta_1 \alpha \psi^p(x) - \eta_2 \beta \psi^q(x) - (1 - \eta_1) \alpha \psi^p(x) - (1 - \eta_2) \beta \psi^q(x) \leq \delta_0$$ (12)

where $0 < \eta_1 < 1$ and $0 < \eta_2 < 1$. Similar with Theorem 2.1, the states of stochastic nonlinear systems can converge to a compact set $\Omega_1 = \{x(1 - \eta_1) \alpha \psi^p(x) + (1 - \eta_2) \beta \psi^q(x) < D\}$ in setting time $T_s < T_{\max} := \frac{1}{\alpha \eta_1 (1 - p)} + \frac{1}{\beta \eta_2 (1 - q)}$.

Lemma 2.2 (Cauchy–Schwarz inequality, Chen et al., 2021): For any positive number $m$, the following holds

$$\left(\sum_{i=1}^{n} m_i\right)^2 \leq n \times \sum_{i=1}^{n} m_i^2$$ (13)

Lemma 2.3 (Chen et al., 2021): For positive number $m \in [0,1]$, we have the following inequality for real number $m \in \mathbb{R}$

$$(|m_1| + \cdots + |m_n|)^p \leq |m_1|^p + \cdots + |m_n|^p$$ (14)

where $i = 1, \ldots, n$.

Lemma 2.4 (Chen et al., 2021): For any given constant $m \in \mathbb{R}$ and a positive number $\kappa$, we have the following

$$0 \leq |m| \leq \frac{m^2}{(m^2 + \kappa^2)^{1/2}} + \kappa$$ (15)

Lemma 2.5 (Young’s inequality, 2022): For any variables $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, we have the following

$$|a||b| \leq \frac{\epsilon^p}{p} |a|^p + \frac{1}{q \epsilon^q} |b|^q, \quad \text{where} \ \epsilon > 0 \quad \text{is a positive number.}$$ (16)

Lemma 2.6 (Fuzzy Logic Systems, 2022): $f(x)$ in (1) is assumed to be an unknown continuous function which can be approximate by FLS as $f(x) = \Theta^T \psi(x)$, then the approximate error can be expressed as follows

$$\sup_{x \in U} |f(x) - \hat{\psi}^T \hat{x}^T \varphi(x)| \leq \delta_0$$ (17)

where $\delta_0$ is a positive number, $\theta^*$ is the ideal parameter vector, $U$ is the domain of $\theta$ of definition of $f(x)$. The approximation error $\epsilon_i$ which satisfies $|i| \leq \epsilon_i^*$ is

$$\epsilon_i = f_i(\hat{x}_i) - f_i(\hat{x}_i)(\theta^*)_i, \quad 1 \leq i \leq n$$ (18)

Furthermore, by defining estimation error as $\tilde{\psi}_i$, we have $\tilde{\psi}_i = \theta_i^* - \theta_i$.

3. Adaptive fuzzy fixed-time control strategy

In this section, fixed-time controller design strategy via backstepping technique will be present in the following $n$ steps. Firstly, by introducing output tracking error $\hat{\xi}_1$ as $\hat{\xi}_1(t) = y(t) - y_r(t)$, changes of coordinates are defined as follows

$$\hat{\xi}_1(t) = x_1 - y_r,$$

$$\hat{\xi}_2(t) = x_2 - \alpha_1,$$ (19)

where $\alpha_1(i = 2, \ldots, n)$ is the virtual controller.

Step 1: According to Itô formula and the stochastic nonlinear system (1), we have

$$d\hat{\xi}_1(t) = (x_2 + f_1(x_1) - y_r) dt + g_1(x_1) d\omega$$ (20)

The Lyapunov function of stochastic nonlinear system is selected as $V_1 = \frac{1}{2} \hat{\xi}_1^2(t) + \frac{1}{2} \hat{\xi}_2^2(t)$. Then, combine the Lyapunov function with (19) and Itô formula, we have

$$\mathcal{L}V_1 = \frac{3}{2} \hat{\xi}_1^2(t)(\hat{\xi}_2^2 + \alpha_1 + f_1(x_1) - y_r) - \frac{1}{\delta_1} \hat{\psi}_1^T \hat{\psi}_1 \hat{\xi}_1 + \frac{3}{4} \frac{\hat{\xi}_1^2(t)g_1^T(x_1)\hat{\psi}_1^T \hat{\psi}_1 g_1(x_1)}{\delta_1} \quad (21)$$

By using Lemma 2.5, we have

$$\frac{3}{4} \hat{\xi}_1^2(t)\hat{\xi}_2^2(t) - \frac{3}{4} \hat{\xi}_1^2(t) \hat{\xi}_2^2(t) \leq \frac{3}{4} \hat{\xi}_1^2(t) \hat{\xi}_2^2(t) + \frac{3}{4} (\hat{\psi}^T \hat{\psi})^2$$ (22)

By substitute the inequalities above into (21), it yields

$$\mathcal{L}V_1 \leq \frac{3}{4} \hat{\xi}_1^2(t) \left(\alpha_1 + \frac{3}{2} \hat{\xi}_1(t) + f_1(x_1) - y_r\right) + \frac{1}{4} \hat{\xi}_1^2(t) \hat{\xi}_1$$ (23)

By defining $\Pi_1(X)$ as $\Pi_1(X_1) = \frac{3}{2} \frac{\hat{\xi}_1^2(t) + f_1(x_1) - y_r}{\delta_1}$, where $X_1 = (x_1, y_r, \hat{\xi}_1)$, which can be approximated by the FLS: $\Pi_1(X_1) = \Theta^T \psi(X_1) + \epsilon_1$, we have

$$\frac{3}{4} \hat{\xi}_1^2 \Pi_1(X) = \frac{3}{4} \hat{\xi}_1^2 \Theta^T \psi(X_1) + \frac{3}{4} \hat{\xi}_1^2 \epsilon_1 \leq \frac{1}{2} \hat{\xi}_1^2(\Theta^T \psi(X_1))^2 + \frac{1}{2} \gamma_1^2 + \frac{3}{4} \hat{\xi}_1^2 + \frac{1}{4} \epsilon_1^2$$ (24)

where $\theta^*$ is defined as $\theta^* = ||\Theta^T||^2 / \gamma_1$, $\gamma_1$ is a positive number to be designed; $\psi(X_1)$ is the fuzzy basis function abbreviated to $\hat{\psi}_1$. Substituting (25) into (24), we have:

$$\mathcal{L}V_1 \leq \frac{3}{4} \hat{\xi}_1^2(t) \left(\alpha_1 + \frac{1}{2} \frac{3}{4} \hat{\xi}_1^2(\Theta^T \psi(X_1) + \frac{3}{4} \hat{\xi}_1(t)) + \frac{1}{4} \hat{\xi}_1^2(t) \hat{\xi}_1 \right) - \frac{1}{\delta_1} \hat{\psi}_1^T \hat{\psi}_1 + \hat{D}_1$$ (26)

where $\hat{D}_1 = \frac{1}{2} \gamma_1^2 + \frac{1}{4} \epsilon_1^2 + \frac{3}{4} (\hat{\psi}^T \hat{\psi})^2$. 


In order to guarantee error variable converges into an equilibrium point contained compact set in fixed-time, \( \alpha_1 \) and \( \theta_1 \) are designed as follows by introducing the intermediate variable \( \alpha_{1c} \):

\[
\alpha_1 = -\frac{\xi_1^3(t)\alpha_{1c}^2}{\sqrt{\xi_1^3(t)\alpha_{1c}^2 + k_1^2}}
\]

\[
\alpha_{1c} = c_{11} \left( \frac{1}{4} \right)^\frac{3}{4} \text{sgn}(\xi_1) + c_{12} \left( \frac{1}{4} \right)^2 \xi_1^5(t) + \frac{3}{4} \xi_1^3(t)
\]

\[
+ \frac{1}{2\gamma_1^2} \xi_1^3(t)\theta_1^T\varphi_1
\]

\[
\dot{\theta}_1 = -\frac{\delta_1}{2\gamma_1} \xi_1^6 \varphi_1 - \tau_{11} \theta_1 - \frac{\tau_{12}}{\delta_1} \theta_1^3 \tag{27}
\]

where \( \kappa_1, c_{11}, c_{12}, \tau_{11}, \tau_{12} \) are positive numbers to be designed.

Combine with Lemma 2.4, we have

\[
\xi_1^3(t)\alpha_1 = -\frac{\xi_1^5(t)\alpha_{1c}^2}{\sqrt{\xi_1^3(t)\alpha_{1c}^2 + k_1^2}} \leq \kappa_1 - \xi_1^3(t)\alpha_{1c} \tag{28}
\]

Substitute the virtual control signal \( \alpha_1 \), adaptive law \( \theta_1 \) and (28) into (26), we have

\[
\mathcal{L}V_1 \leq -c_{11} \left( \frac{1}{4} \xi_1^4(t) \right)^\frac{3}{4} - c_{12} \left( \frac{1}{4} \xi_1^4(t) \right)^2 + \frac{\tau_{11}}{\delta_1} \theta_1^T \theta_1 + \frac{\tau_{12}}{\delta_1^2} \theta_1^3 + D_1 \tag{29}
\]

where \( D_1 \equiv \bar{D}_1 + \kappa_1 \).

**Remark 3.1:** According to Theorem 2.1, the parameter \( p \) and \( q \) of the inequality (7) we choose in this paper are \( p = \frac{3}{4} \) and \( q = 2 \). Then the state variable of the nonlinear stochastic system is semi-globally bounded in fixed-time and the setting-time \( T_1 \) in Theorem 2.1 is chosen as \( T_1 \leq T_{\text{max}} := \frac{1}{a(1-p)\eta} + \frac{1}{b(q-1)\eta} = \frac{4}{a\eta} + \frac{1}{b\eta} \).

**Step i:** The following inequality is supposed to be obtained from step \( i-1 \)

\[
\mathcal{L}V_{i-1} \leq -\sum_{j=1}^{i-1} c_{j1} \left( \frac{1}{4} \xi_j^4(t) \right)^\frac{3}{4} - \sum_{k=1}^{i-1} c_{j2} \left( \frac{1}{4} \xi_j^6(t) \right)^2 + \frac{\tau_{11}}{\delta_1} \sum_{j=1}^{i-1} \theta_j^T \theta_j + \frac{\tau_{12}}{\delta_1^2} \theta_j^3 + D_{i-1} \tag{30}
\]

According to Itô formula and dynamic nonlinear system (1), we have

\[
d\xi_j(t) = dx_j - d\alpha_{i-1} = (x_{j+1} + f_j(\bar{x}_j)) \, dt + g_j(\bar{x}_j) \, dw - d\alpha_{i-1}
\]

\[
= \left( x_{j+1} + f_j(\bar{x}_j) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \theta_j} \bar{\theta}_j - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (x_{j+1} + f_j(\bar{x}_j)) \right) \, dt
\]

\[
+ \frac{1}{2} \bar{\theta}_j \theta_j^T \theta_j^T \theta_j \, dt + \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j \partial x_k} g_j^T(\bar{x}_j) \theta_j \theta_j^T g_j(\bar{x}_j) \, dt + \Lambda_i \, dw.
\]

where \( \Lambda_i = g_j(\bar{x}_j) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j(\bar{x}_j) \).

By choosing Lyapunov function as \( V_i = V_{i-1} + \frac{1}{4} \xi_i^4(t) + \frac{1}{2\gamma_1} \theta_j^T \theta_j \), we have the following equality the same with (21)

\[
\mathcal{L}V_i = \mathcal{L}V_{i-1} + \xi_i^4(t) \left( x_{i+1} + \alpha_i + f_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \theta_j} \bar{\theta}_j - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (x_{j+1} + f_j(\bar{x}_j)) \right)
\]

\[
- \frac{1}{2} \sum_{j=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j^2} g_j^T(\bar{x}_j) \theta_j \theta_j^T g_j(\bar{x}_j) \theta_j^T \theta_j \theta_j^T \theta_j \, dt + \frac{3}{2} \xi_i^2(t) \Lambda_i \theta_j \theta_j^T \theta_j - \frac{1}{\delta_1^2} \theta_j^T \theta_j. \tag{32}
\]

By using Lemma 2.5, it yields

\[
\frac{3}{2} \xi_i^2(t) \Lambda_i \theta_j \theta_j^T \theta_j \leq \frac{3}{4} \xi_i^4(t) + \frac{3}{4} \xi_{i+1}^4(t) \tag{33}
\]

\[
- \frac{1}{2} \xi_i^3(t) \sum_{j=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j^2} g_j^T(\bar{x}_j) \theta_j \theta_j^T g_j(\bar{x}_j) \theta_j^T \theta_j \theta_j^T \theta_j \leq \frac{3}{4} \xi_i^4(t) \mathcal{H}_i + \frac{3}{4} \theta_j \theta_j^T \theta_j^T \theta_j \tag{34}
\]

\[
\frac{3}{2} \xi_i^2(t) \Lambda_i \theta_j \theta_j^T \theta_j \theta_j^T \theta_j + \frac{1}{4} \xi_i^4(t) \mathcal{H}_i \leq \frac{3}{4} \xi_i^4(t) \mathcal{H}_i + \frac{3}{4} \theta_j \theta_j^T \theta_j \theta_j^T \theta_j, \tag{35}
\]

where \( \mathcal{H}_i = \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_j \partial x_k} g_j^T(\bar{x}_j) \theta_j \theta_j^T g_j(\bar{x}_j) \).
Then substitute the inequalities above and $\mathcal{L} V_{i−1}$ into (32), it yields

$$
\mathcal{L} V_i \leq - \sum_{j=1}^{i-1} c_{j1} \left( \frac{1}{4} \xi_j^4 (t) \right)^{\frac{3}{2}} - \sum_{k=1}^{i-1} c_{k2} \left( \frac{1}{4} \xi_k^4 (t) \right)^2 \\
+ \frac{1}{4} \xi_{i+1}^4 (t) + \sum_{j=1}^{i-1} \frac{\tau_j}{\delta_j} \tilde{\theta}_j ^T \theta_j \\
+ \sum_{k=1}^{i-1} \frac{\tau_k}{\delta_k} \tilde{\theta}_k ^T \theta_k + \xi_i^3 (t) \left( \alpha_i + M_i \xi_i (t) + f_i (x_i) \right)
$$

$$
- \sum_{j=1}^{i-1} \frac{\partial \alpha_{i−1}}{\partial \theta_j} \tilde{\theta}_j + \frac{7}{4} \xi_i (t) \\
- \sum_{j=1}^{i-1} \frac{\partial \alpha_{i−1}}{\partial x_j} \left( x_{j+1} + f_j (x_j) \right) + \sum_{j=1}^{i-1} \frac{3}{4} H_i \xi_i (t) \\
- \frac{1}{\delta_i} \tilde{\theta}_i + \tilde{D}_i, 
$$

where $\tilde{D}_i = D_{i−1} + \frac{i−1}{8} (\tilde{\theta}^T \tilde{\theta})^4 + \frac{3 \xi_i^2}{4} (\tilde{\theta}^T \tilde{\theta})^2$.

Similar with (25), we define $\Pi_i (x_i)$ as $\Pi_i (x_i) = \xi_i (t) ( M_i + \frac{1}{2} \xi_i + \frac{3}{4} H_i ) = f_i (x_i) - \sum_{j=1}^{i} \frac{\partial \alpha_{i−1}}{\partial x_j} \theta_j - \sum_{k=1}^{i} \frac{\partial \alpha_{i−1}}{\partial x_k} (x_{j+1} + f_j (x_j))$. By utilizing FLS, we have $\Pi_i (x_i) = \Theta_i ^T \varphi_i (x_i) + \xi_i$. Combine FLS with Young's inequality, it yields:

$$
\xi_i^3 \Pi_i (x_i) = \xi_i^3 \Theta_i ^T \varphi_i (x_i) + \xi_i^3 \xi_i \\
\leq \frac{1}{2 \gamma_i^2} \xi_i^6 (\Theta_i ^T \varphi_i (x_i))^2 + \frac{1}{2 \gamma_i^2} \frac{3}{4} \xi_i^4 + \frac{1}{4} \xi_i^4 \\
\leq \frac{1}{2 \gamma_i^2} \xi_i^6 \Theta_i ^T \varphi_i \xi_i + \frac{1}{2 \gamma_i^2} \frac{3}{4} \xi_i^4 + \frac{1}{4} \xi_i^4 
$$

Then we have

$$
\mathcal{L} V_i \leq - \sum_{j=1}^{i-1} c_{j1} \left( \frac{1}{4} \xi_j^4 (t) \right)^{\frac{3}{2}} - \sum_{k=1}^{i-1} c_{k2} \left( \frac{1}{4} \xi_k^4 (t) \right)^2 \\
+ \frac{1}{4} \xi_{i+1}^4 (t) + \sum_{j=1}^{i-1} \frac{\tau_j}{\delta_j} \tilde{\theta}_j ^T \theta_j + \sum_{k=1}^{i-1} \frac{\tau_k}{\delta_k} \tilde{\theta}_k ^T \theta_k \\
+ \xi_i^3 (t) \left( \alpha_i + \frac{3}{4} \xi_i (t) + \frac{1}{2 \gamma_i^2} \xi_i^3 \theta_i ^T \varphi_i \theta_i \right) \\
- \frac{1}{\delta_i} \tilde{\theta}_i + \frac{7}{4} \xi_i (t) + \frac{3}{4} H_i \xi_i (t) \\
- \frac{1}{\delta_i} \tilde{\theta}_i + \frac{1}{2 \gamma_i} \xi_i^4 + \tilde{D}_i. 
$$

The virtual controller $\alpha_i$ and adaptive law $\theta_i$ are selected as following

$$
\alpha_i = - \frac{\xi_i^3 (t) \alpha_i^2}{\sqrt{\xi_i^6 (t) \alpha_i^2 + \kappa_i^2}} \\
\alpha_{ic} = c_{i1} \left( \frac{1}{4} \right) \text{sgn} (\xi_i) + c_{i2} \left( \frac{1}{4} \right) \xi_i^5 (t) + \frac{1}{2 \gamma_i^2} \xi_i^3 (t) \theta_i ^T \varphi_i \theta_i \\
\dot{\theta}_i = - \frac{\tau_j}{\delta_j} \xi_i^6 (t) \theta_i ^T \varphi_i - \tau_{i1} \theta_i - \tau_{i2} \theta_i^3, 
$$

where $\kappa_i$, $c_{i1}$, $c_{i2}$, $\tau_{i1}$, $\tau_{i2}$ are positive numbers to be designed.

Combine with Lemma 2.4, we have

$$
\xi_i^3 (t) \alpha_i = - \frac{\xi_i^6 (t) \alpha_i^2}{\sqrt{\xi_i^6 (t) \alpha_i^2 + \kappa_i^2}} \leq \kappa_i - \xi_i^3 (t) \alpha_{ic}. \tag{40} 
$$

Substitute virtual control signal $\alpha_i$, adaptive law $\theta_i$ and (40) into (38), we have

$$
\mathcal{L} V_i \leq - \sum_{j=1}^{i} c_{j1} \left( \frac{1}{4} \xi_j^4 (t) \right)^{\frac{3}{2}} - \sum_{k=1}^{i} c_{k2} \left( \frac{1}{4} \xi_k^4 (t) \right)^2 \\
+ \frac{1}{4} \xi_{i+1}^4 (t) + \sum_{j=1}^{i} \frac{\tau_j}{\delta_j} \tilde{\theta}_j ^T \theta_j + \sum_{k=1}^{i} \frac{\tau_k}{\delta_k} \tilde{\theta}_k ^T \theta_k + \xi_i^3 (t) \left( \alpha_i + \frac{3}{4} \xi_i (t) + \frac{1}{2 \gamma_i^2} \xi_i^3 \theta_i ^T \varphi_i \theta_i \right) \\
- \frac{1}{\delta_i} \tilde{\theta}_i + \frac{7}{4} \xi_i (t) + \frac{3}{4} H_i \xi_i (t) \\
- \frac{1}{\delta_i} \tilde{\theta}_i + \frac{1}{2 \gamma_i} \xi_i^4 + \tilde{D}_i, \tag{41} 
$$

where $\tilde{D}_i = \tilde{D}_i + \frac{1}{2 \gamma_i} \xi_i ^4 + \frac{1}{4} \xi_i^4 + \kappa_i$. The inequality has the same format with (29) in step 1 which means (30) holds for the $i−1$th step.

**Step n:** We consider the same Lyapunov function used in step $i$, i.e. $V_n = V_{n−1} + \frac{1}{4} \xi_i^4 (t) + \frac{1}{2 \gamma_i} \tilde{\theta}_i ^T \tilde{\theta}_i$. Then we have

$$
\mathcal{L} V_n = \mathcal{L} V_{n−1} + \xi_i^3 (t) \left( u + f_i (\tilde{x}_n) - \sum_{j=1}^{n−1} \frac{\partial \alpha_{n−1}}{\partial \theta_j} \theta_j \\
- \sum_{j=1}^{n−1} \frac{\partial \alpha_{n−1}}{\partial x_j} \left( \tilde{\theta}_j ^T g_j (x_j) \right) ^T \\
x \theta g_j (x_j) + x_{j+1} + f_j (x_j) \right) \right) + \frac{3}{2} \xi_n^2 (t) \Lambda_n ^T \theta ^T \Lambda_n \\
- \frac{1}{\delta_n} \tilde{\theta}_n \theta_n, \tag{42} 
$$

where $\Lambda_n = g_n (\tilde{x}_n) - \sum_{j=1}^{n−1} \frac{\partial \alpha_{n−1}}{\partial x_j} g_j (x_j)$.

$$
\mathcal{L} V_n \leq \xi_n^4 (t) M_n + \frac{n−1}{8} (\tilde{\theta} ^T \tilde{\theta})^4 \tag{43} 
$$
\[
\frac{3}{2} \xi_n^2(t) \Lambda_n \theta \cdot \Lambda_n \leq \frac{3}{4} \xi_n^4(t) \lambda_n + \frac{3n}{4} (\bar{\theta}^T \bar{\theta})^2
\]  
(44)

where \( \lambda_n = \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \left( \frac{\partial \alpha_{n-1}}{\partial \theta_j} \right) \left( \frac{\partial \alpha_{n-1}}{\partial \theta_k} \right) + 2 \sum_{j=1}^{n-1} \left( \frac{\partial^2 \alpha_{n-1}}{\partial \theta_j^2} \right)^2 + 1, \lambda_n = \frac{3}{4} \sum_{j=1}^{n-1} \left( \frac{\partial \alpha_{n-1}}{\partial \theta_j} \right)^2 \).

Similarly with step i, we substitute the inequalities into (42) and denote \( \Pi_n(X_n) \) as \( \xi_n(t)(M_n + \frac{\xi_n}{4} + \frac{3}{4} \lambda_n) + f_n(X_n) - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \theta_j} \theta_j - \sum_{j=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial \theta_j^2} (\bar{\theta}_j + 1)(\bar{\theta}_j) \). By using the same technique in (37), the following inequality is obtained

\[
LV_n \leq -\sum_{j=1}^{n-1} c_{j1} \left( \frac{1}{4} \xi_j^4(t) \right)^{\frac{3}{4}} - \sum_{k=1}^{n-1} c_{k2} \left( \frac{1}{4} \xi_k^4(t) \right)^{\frac{3}{4}}
- \frac{1}{\delta_n} \xi_n^3(t) \theta_n \cdot \theta_n + \sum_{j=1}^{n-1} \frac{\tau_j}{\delta_j} \xi_j^4 \xi_n^4 + \xi_n^3 \xi_j^4 \left( u + \frac{1}{2} \frac{\xi_n^4}{\lambda_n} \xi_n^4 \right) \frac{\xi_n^4}{\lambda_n} \xi_n^4 + \bar{D}_n
\]  
(45)

where \( \bar{D}_n = D_{n-1} + n \left( \frac{\xi_n^4}{\lambda_n} \xi_n^4 + \frac{3n}{4} (\bar{\theta}^T \bar{\theta})^2 \right) + \frac{1}{4} \xi_n^4 \).

In this step \( u \) and \( \theta_n \) are selected as following

\[
u = -\frac{\xi_n^3(t) \alpha_n}{\sqrt{\xi_n^4(t) \alpha_n^2 + \kappa_n^2}}
\]

\[
\alpha_n = c_n \left( \frac{1}{4} \right) \xi_n^3(t) \xi_n^4(t) + c_n \left( \frac{1}{2} \right) \xi_n^3(t) + \frac{1}{4} \xi_n^3(t)
\]

\[
\theta_n = -\frac{\delta_n}{\xi_n^3(t) \xi_n^4(t) \xi_n^4(t) - \tau_n \theta_n - \tau_n \theta_n - \tau_n \theta_n}
\]  
(46)

where \( \kappa_n, \alpha_n, \beta_n, \gamma_n, \tau_n, \tau_n \) are positive constants to be designed.

Combine with Lemma 2.4, we have

\[
\xi_n^3(t) \alpha_n = -\frac{\xi_n^3(t) \alpha_n}{\sqrt{\xi_n^4(t) \alpha_n^2 + \kappa_n^2}} \leq \kappa_n - \xi_n^3(t) \alpha_n.
\]  
(47)

Substitute actual control signal \( u \), adaptive law \( \theta_n \) and (47) into (45), we have

\[
LV_n \leq -\sum_{j=1}^{n} c_{j1} \left( \frac{1}{4} \xi_j^4(t) \right)^{\frac{3}{4}} - \sum_{k=1}^{n} c_{k2} \left( \frac{1}{4} \xi_k^4(t) \right)^{\frac{3}{4}}
+ \sum_{j=1}^{n} \frac{\tau_j}{\delta_j} \xi_j^4 \xi_n^4 + \sum_{j=1}^{n} \frac{\tau_k}{\delta_k} \xi_k^4 \xi_n^4 + D_n
\]  
(48)

where \( D_n = \bar{D}_n + \kappa_n \).

Then we begin to handle with each item of (48) in the following step by using inequality techniques.

By utilizing Lemma 2, we have

\[
-\sum_{j=1}^{n} c_{j1} \left( \frac{1}{4} \xi_j^4(t) \right)^{\frac{3}{4}} \leq -\tilde{c}_1 \sum_{j=1}^{n} \left( \frac{1}{4} \xi_j^4(t) \right)^{\frac{3}{4}}
\]  
(49)

where \( \tilde{c}_1 = \min(c_{11}, \ldots, c_{1n}), c_1 = c_1 \).

By using Lemma 2.3, we have

\[
-\sum_{k=1}^{n} c_{k2} \left( \frac{1}{4} \xi_k^4(t) \right)^{\frac{3}{4}} \leq -\tilde{c}_2 \sum_{k=1}^{n} \left( \frac{1}{4} \xi_k^4(t) \right)^{\frac{3}{4}}
\]  
(50)

where \( \tilde{c}_2 = \min(c_{21}, \ldots, c_{2n}), c_2 = \frac{1}{4} \xi_2 \).

By using Lemma 2.5, it yields

\[
\sum_{j=1}^{n} \frac{\tau_j}{\delta_j} \xi_j^4 \xi_n^4 \leq \sum_{j=1}^{n} \frac{\tau_j}{2\delta_j} \xi_j^4 \xi_n^4 + \sum_{j=1}^{n} \frac{\tau_j}{2\delta_j} \theta_j^2 + \tilde{c}
\]  
(51)

\[
\sum_{j=1}^{n} \frac{\tau_j}{\delta_j} \xi_j^4 \xi_n^4 \leq \sum_{j=1}^{n} \xi_j^4 \xi_n^4 \left( \frac{n}{4} \right)^{\frac{3}{4}} \left( \frac{n}{4} \right)^{\frac{3}{4}} + \frac{\rho_1}{\tilde{c}}
\]  
(52)

where \( \rho_1 \) is choosing as \( \rho_1 = \frac{4}{3} \) in the inequality, \( \tilde{c} = \frac{1}{4} \left( \frac{3}{2} \right)^{\frac{3}{4}} \). Then we have

\[
\sum_{j=1}^{n} \frac{\tau_j}{\delta_j} \xi_j^4 \xi_n^4 \leq \left( \sum_{j=1}^{n} \frac{\tau_j}{2\delta_j} \xi_j^4 \xi_n^4 \right)^{\frac{3}{4}} + \sum_{j=1}^{n} \frac{\tau_j}{2\delta_j} \theta_j^2 + \tilde{c}
\]  
(53)

where \( \tilde{c} = \min(\frac{3}{4}, \ldots, \frac{3}{4}), j = 1, \ldots, n \).
Since \( \ddot{\theta}^T \theta^3 = \ddot{\theta} (\theta^* - \bar{\theta})^3 = \ddot{\theta} (\theta^* - 3 \theta^* \ddot{\theta} + 3 \theta^* \ddot{\theta}^2 - \ddot{\theta}^3) \), we have the following by using Lemma 2.5

\[
\sum_{k=1}^{n} \frac{3 \tau_k^2 \ddot{\theta}_k^3}{\delta_k^4} \leq \sum_{k=1}^{n} \frac{\tau_k^2}{\delta_k^4} \left( \frac{9 \rho_2^2 \theta_k^4}{4} + \frac{3 \theta_k^4}{4 \rho_2^2} \right) \tag{54}
\]

\[
\sum_{k=1}^{n} \frac{\tau_k^2 \tilde{\theta}_k \theta_k^2}{\delta_k^2} \leq \sum_{k=1}^{n} \frac{\tau_k^2}{\delta_k^2} \left( \frac{\rho_3 \theta_k^2 \theta_k^2}{2 \rho_2^2} + \frac{1}{2 \rho_2^2} \theta_k^4 \right) \leq \sum_{k=1}^{n} \frac{\tau_k^2}{\delta_k^2} \left( \frac{\theta_k^2 \theta_k^2}{2 \delta_k^2} + \frac{1}{12} \theta_k^4 \right) \tag{55}
\]

where \( \rho_3 \) is choosing as \( \rho_3 = 6 \) in the inequality. Then we have

\[
\sum_{k=1}^{n} \frac{\tau_k^2 \tilde{\theta}_k \theta_k^3}{\delta_k^4} \leq \sum_{k=1}^{n} \frac{\tau_k^2}{\delta_k^4} \left( \frac{9 \rho_2^2 - 4 \theta_k^4}{4} + \frac{9 \rho_2^{-1} + 1}{12} \theta_k^4 \right) \leq \tilde{\tau}_2 \left( \frac{n}{\delta_k^2} \right)^2 + \sum_{k=1}^{n} \frac{9 \rho_2^{-1} + 1}{12} \theta_k^4 \tag{56}
\]

where \( \tilde{\tau}_2 = \min \{ (4 - 9 \rho_2) \tau_k^2 \} \), \( 0 < \rho_2 < \frac{4}{3} \), \( k = 1, \ldots, n \).

Substitute (49), (50), (53) and (56) into (48), we have

\[
\begin{align*}
\mathcal{L}V_n &\leq -c_1 \left( \sum_{j=1}^{n} \frac{1}{4} \tilde{\dot{e}}_j^4(t) \right)^{\frac{1}{4}} - c_2 \left( \sum_{j=1}^{n} \frac{1}{4} \tilde{e}_j^4(t) \right)^{\frac{1}{2}} \\
&\quad - \tilde{\tau}_1 \left( \frac{n}{\delta_j^2} \right)^{\frac{1}{4}} - \tilde{\tau}_2 \left( \frac{n}{\delta_k^2} \right)^{\frac{1}{2}} + D \tag{57}
\end{align*}
\]

where \( D = D_0 + \sum_{j=1}^{n} \frac{n}{12} \theta_j^2 \gamma_j^2 + \sum_{j=1}^{n} \frac{7}{12} \theta_j^{-1} \theta_j^3 + \tilde{\gamma} \).

Noting that the Lyapunov function chosen in this paper is \( V_n = \sum_{j=1}^{n} \frac{1}{4} \tilde{\dot{e}}_j^4(t) + \sum_{j=1}^{n} \frac{1}{20} \tilde{\dot{\gamma}}_j \tilde{\theta}_j \). Then, by using Lemmas 2.2 and 2.3, we have the following:

\[
\begin{align*}
\mathcal{L}V_n &\leq -\mu_1 \left( \left( \sum_{j=1}^{n} \frac{1}{4} \tilde{\dot{e}}_j^4(t) \right)^{\frac{1}{4}} + \left( \sum_{j=1}^{n} \frac{\tilde{\dot{\gamma}}_j^2}{\delta_j^2} \right)^{\frac{1}{2}} \right) \\
&\quad - \mu_2 \left( \sum_{k=1}^{n} \frac{1}{4} \tilde{e}_k^4(t) \right)^{\frac{1}{2}} + \sum_{k=1}^{n} \frac{\tilde{\dot{\gamma}}_k^2}{\delta_k^2} + \frac{1}{2} + D \tag{58}
\end{align*}
\]

where \( \mu_1 = \min \{ c_1, \tilde{\tau}_1 \} \) \( \mu_2 = \min \{ c_2, \tilde{\tau}_2, \tilde{\gamma} \} \).

According to Theorem 2.1, the states of the stochastic nonlinear systems are guaranteed to converge to a compact set semi-globally in fixed-time.

4. Simulation example

Two examples (a numerical example and a vehicle tracking model example) will be given to verify the effectiveness of proposed practical fixed-time control strategy for stochastic nonlinear systems.

Example 4.1: The vehicle tracking system model which is also called ACC (adaptive cruise control) is partly borrowed from ([2022] is shown in Figure 1(a). ACC is one of the most significant advanced driving assistant systems. ACC can aim to maintain a host vehicle behind a preceding vehicle at a certain distance and in this way to improve the safety and also the comfort. ACC technique substantially reduces human perception errors and in this way improve the driving safety and also the driving comfort. The model can be expressed as follows:

\[
\begin{align*}
\dot{s} &= V \quad (54) \\
V &= \frac{1}{m} F - \frac{1}{2 m} \rho_0 A V^2 - g \sin \theta - \mu g \cos \theta
\end{align*}
\]

where \( m \) and \( A \) are the vehicle mass and frontal area, which are chosen as 1000 kg and 1.75 m², respectively. \( \rho_0 \) \( \mu \) are the aerodynamic drag coefficient and rolling friction coefficient, which are chosen as 0.3 and 0.6. \( F \) is the error torque between driving torque \( F_d \) and braking torque \( F_b \), \( g = 9.8 \text{ m/s}^2 \) is gravitational acceleration and \( \rho = 1.22 \text{ kg/m}^3 \) is density of air of 15°C at standard atmospheric pressure. \( \theta \) is the road inclination, we assume the vehicle running at a level and smooth road while \( \theta = 0 \). \( S \) and \( V \) are the safe distance and speed of the vehicle. The initial condition of the distance is given as \( S(0) = 45 \text{ m} \) and the desired distance is \( S_0 = 40 \text{ m} \). We consider the stochastic disturbance of \( S \) and \( V \). The equilibrium point is chosen as \( S_0 = 40 \text{ m} \) and \( V_0 = 15 \text{ m/s} \), then the vehicle tracking problem becomes a stabilization task for the following nonlinear stochastic system.

A simply adaptive fuzzy controller is designed as follows in the simulation to compare with the method proposed in this paper

\[
\begin{align*}
\dot{u} &= 1000 \left( -c_1 \dot{x}_1(t) - c_2 \dot{x}_2(t) - y_2 \tilde{\dot{\theta}}_2(t) \psi_2(x_2) \right) \\
\dot{\theta}_2 &= -y_2 \tilde{\dot{\theta}}_2 - \frac{\tau_2}{\delta_2} \dot{\theta}_2(t) \psi_2(x_2) 
\end{align*}
\]

The initial conditions for the state vectors are given as \( x_1(0) = 5, x_2(0) = 0.001 \). The stochastic item functions are given as \( g_1(x) = g_2(x) = 0.5 \sin x_1 + 2 \). Other initial conditions are given as zero. The design parameters for the two methods are selected as \( c_{11} = c_{12} = c_{21} = c_{22} = 0.2, k_1 = k_2 = 0.01, y_1 = 0.04, \delta_2 = 1, \frac{\tau_{21}}{\delta_2} = 1, \frac{\tau_{22}}{\delta_2} = 1 \).

The results of different methods are presented in Figure 2. In order to emphasize the fixed-time stability of this article, trajectories of stochastic variables \( x_1 \) and \( x_2 \)
Figure 1. The vehicle tracking model and tracking performance. (a) The vehicle tracking system model and (b) the system states with our method and adaptive fuzzy method.

Figure 2. Different control input with our method and adaptive fuzzy method. (a) The control input signal $u(t)$ with our proposed strategy and (b) the control input signal $u(t)$ with adaptive fuzzy method.

Figure 3. Trajectories of output signal $y$, reference signal $r$, system states $x_1(t)$ and $x_2(t)$ with $c_{11} = c_{12} = c_{21} = c_{22} = 10$. (a) Trajectories of output signal $y$ and the reference signal $r$ and (b) trajectories of system states $x_1(t)$ and $x_2(t)$. 
with our proposed strategy and adaptive fuzzy method are shown in Figure 1(b). The curve graphs of the control input signal $u$ in different methods are present in Figures 2(a,b), respectively. The trajectory of the adaptive estimation error is shown in Figure 5(a).

**Example 4.2:** The numerical example in this simulation is the following which is partly borrowed from Chen et al. (2021) and take the stochastic item into consideration

$$
\begin{align*}
    dx_1 &= (0.1x_1^2 + x_2) \, dt + g_1(x) \, d\omega \\
    dx_2 &= (u + 0.1x_1x_2 - 0.2x_1) \, dt + g_2(x) \, d\omega \\
    y &= x_1
\end{align*}
$$

In this simulation, we choose different parameters to show the influence on the performance of the proposed strategy. $r(t)$ is selected as $r(t) = 0.5 \sin(t)$. We choose the initial conditions for the state vectors as $x_1(0) = 0.5, x_2(0) = -0.5$. The stochastic item functions are given as $g_1(x) = g_2(x) = 0.5x_1 \sin(t) + 0.2$. The design parameters are selected as $x_1 = x_2 = 0.01, \gamma_1 = \gamma_2 = 0.5, \delta_1 = \delta_2 = 1, \tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 1$. We choose three groups of parameters as $c_{11} = c_{12} = c_{21} = c_{22} = 2, c_{11} = c_{12} = c_{21} = c_{22} = 10$ and $c_{11} = c_{12} = c_{21} = c_{22} = 15$, respectively.

The simulation results are shown in Figures 3–4. Figure 3(a) shows the trajectories of output signal $y$ and the reference signal $r$ and Figure 3(b) shows the trajectories stochastic variables $x_1$ and $x_2$ with $c_{11} = c_{12} = c_{21} = c_{22} = 10$. With the different parameters chosen above, the trajectories of input signal $u$ and trajectories of output error $\xi_1$ are shown in Figure 4(a,b), respectively. The trajectories of the adaptive estimation error for the unknown functions are shown in Figure 5(b). The fixed-time controller can get a better performance with high gain parameters. We can obtain from two examples that the high convergence rate of the controller is always along with a higher consumption of the input, which is the disadvantage of the proposed method.
5. Conclusions
This paper studies the fixed-time tracking problem for stochastic nonlinear systems. The practical fixed-time bounded theorem for stochastic nonlinear systems is proofed before the controller design process. FLS are used to approximate unknown functions in the considered nonlinear systems. Adaptive fuzzy control via the back-stepping method is used in the process of the controller design. All signals of the systems are guaranteed to be bounded in fixed-time. Two examples given in the simulation show the effectiveness of the proposed strategy.

However, from the results of the simulation, we can obtain that the tracking performance and convergence time of the designed controller are closely related to the designed parameters and the values of input signal, which may be the limitation for the applications. The fixed-time controller can reach a higher convergence rate with a large consumption of control input. This will lead to the actuator saturation and the instability for the practical systems. Then, how to achieve better performance in the limited input signal is the further study we will focus on.

Disclosure statement
No potential competing interest was reported by the authors.

Data availability statement
Data sharing is not applicable to this article as no new data were created or analysed in this study.

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