Strong quantum nonlocality in general multipartite quantum systems

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The concept of strong quantum nonlocality is introduced by Halder et. al. in [Phys. Rev. Lett. 122, 040403 (2019)]. An orthogonal set of states in multipartite systems is called to be strong quantum nonlocality if it is locally irreducible under every bipartition of the subsystems. In this work, we give two constructions of strongly nonlocal sets in multipartite quantum systems with arbitrary local dimensions. This provides a complete answer to an open question raised by Halder et. al and a recent paper [Phys. Rev. A 105, 022209 (2022)]. Compared with all previous relevant proofs, our proof here is quite concise. Our results show that this kind of strong quantum nonlocality is an universal property in quantum mechanics which does not depend on the number of participants and local dimensions.

I. INTRODUCTION

Quantum state discrimination is a fundamental task in quantum information processing. The task is to determine the identity of a randomly choosing state from a known set of quantum states. It is well known that the set of states can be perfectly identified via global measurement if and only if the states are mutually orthogonal [1]. However, if the quantum states are distributed in composite systems, due to the limitation of physical spacing, we may not perform a global measurement. All we can perform are local operations and classical communication (LOCC). Therefore, an orthogonal set of states can not be always distinguished perfectly under LOCC. If there is an LOCC protocol that can distinguish the set perfectly, we say that the set is locally distinguishable, otherwise, locally indistinguishable. If an orthogonal set is locally indistinguishable, we also call it has nonlocality as a global measurement reveals more information about the state than any LOCC protocols. Bennett et al. [2] presented the first example of locally indistinguishable set of product states. Since then, this kind of local-discrimination-based nonlocality has been studied extensively (here we list some representative references [3–24] for an incomplete list). The local indistinguishability of quantum states have been found applications in quantum data hiding [25, 26] and secret sharing [27, 28].

Recently, Halder et al. [29] introduced a stronger form of quantum nonlocality through the notion of local irreducibility. An orthogonal set of multipartite quantum states is said to be locally irreducible if it is not possible to eliminate one or more states from that set while performing orthogonality preserving local measurement. The local irreducibility of a set implies it is also local indistinguishable. In Ref. [29], they provided examples of orthogonal product bases in \(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3\) and \(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4\) that are locally irreducible for each bipartition of the corresponding tripartite systems and named such phenomenon as strong quantum nonlocality. This motivates several works on this kind of strong quantum nonlocality (see Refs. [30–36]). All these results consider systems with at most five subsystems. Therefore, it is interesting to consider whether this kind of strong nonlocality is a universal property in the sense that its existence does not depend on the number of parties and local dimensions. A recent work indicates that this property does not depend on the number of parties. In this work, we will show that it does not depend on the local dimensions.

II. PRELIMINARY AND TWO BASIC LEMMAS

For any positive integer \(d \geq 2\), we denote \(\mathbb{Z}_d\) as the set \(\{0, 1, \ldots, d-1\}\). Let \(\mathcal{H}_d\) be an \(d\) dimensional Hilbert space. We always assume that \(\{\left|0\right>, \left|1\right>, \ldots, \left|d-1\right>\}\) is the computational basis of \(\mathcal{H}_d\). A positive operator-valued measure (POVM) on \(\mathcal{H}_d\) is a set of positive semidefinite operators \(\{E_x\}_{x \in X}\) such that \(\sum_{x \in X} E_x = \mathbb{I}_{\mathcal{H}_d}\) where \(\mathbb{I}_{\mathcal{H}_d}\) is the identity operator on \(\mathcal{H}_d\). Throughout this paper, we do not normalize states for simplicity.

A measurement is trivial if all its POVM elements are proportional to the identity operator. Note that every LOCC protocol that distinguishes a set of orthogonal states is a sequence of orthogonality preserving local measurements (OPLM). There is a sufficient condition to prove that an orthogonal set is nonlocality (strongly nonlocality): each subsystem can only perform a trivial orthogonality preserving local measurement (the subsystems are corresponding to each bipartition of the original subsystems).

Generally, it is difficult to show that the orthogonality preserving local measurement for each subsystem is trivial. We list two useful lemmas (developed in Ref. [34]) for verifying the trivialization of such measurement.

**Lemma 1 (Block Zeros Lemma)** Let an \(n \times n\) matrix \(E = (a_{i,j})_{i,j \in \mathbb{Z}_n}\) be the matrix representation of an operator \(E\) under the basis \(\mathcal{B} := \{|0\>, |1\>, \ldots, |n-1\>\}\).
Given two nonempty disjoint subsets $S$ and $T$ of $B$, assume that $\{\{|\psi_i\rangle\}_{i=1}^{s}\}, \{\{|\phi_i\rangle\}_{i=1}^{t}\}$ are two orthogonal sets spanned by $S$ and $T$ respectively, where $s = |S|$, and $t = |T|$. If $\langle\psi_i|E|\phi_j\rangle = 0$ for any $i \in Z_s, j \in Z_t$, then $\langle x|E|y\rangle = \langle y|E|x\rangle = 0$ for $|x\rangle \in S$ and $|y\rangle \in T$.

**Lemma 2 (Block Trivial Lemma)** Let an $n \times n$ matrix $E = (a_{ij})_{i,j \in Z_n}$ be the matrix representation of an operator $E$ under the basis $B := \{0\}, \{1\}, \ldots, \{n-1\}$. Given a nonempty subset $S$ of $B$, let $\{\{|\psi_i\rangle\}_{i=1}^{s}\}$ be an orthogonal set spanned by $S$. Assume that $\langle\psi_i|E|\phi_j\rangle = 0$ for any $i \neq j \in Z_s$.

**Proof:** First, we show that $\hat{A}_1 := A_1 A_2 \cdots A_N$ can only perform a trivial orthogonality preserving measurement (OPM). Suppose $\{M_l|E|M_l\}_{l \in A}$ is an orthogonality preserving measurement with respect to the set $S$ which is performed by $\hat{A}_1$, i.e., $\langle M_l|E|M_m\rangle = 0$ for any two different $|\Psi\rangle, |\Phi\rangle \in S$. Set $E := \hat{A}_1 \otimes M_l|E|M_l$. Let $k, l \in Z_N$ and suppose $k \neq l$. As $C_k \cap C_l = \emptyset$, applying Block Zeros Lemma to the sets of base vectors corresponding to $C_k$ and $C_l$, we obtain that

$$\langle i_1|E|i_2\rangle = \langle j_1|E|j_2\rangle = 0$$

for any $i_k \in C_k$ and $j_1 \in C_l$. Now we claim that for any $k \in \{0, 1, \cdots, N-1\}$ if $i_k, i_k'$ are two different strings of $C_k$, then we also have

$$\langle i_k|E|i_k'\rangle = \langle i_k'|E|i_k\rangle = 0.$$ 

Moreover, $\langle i_k|E|i_k\rangle = \langle j_k|E|j_k\rangle = 0$. As $C_{<N} := \cup_{k<N} C_k$ contains $\{0\} \times Z_{d_1} \times \cdots \times Z_{d_N}$ from the above relations, one could conclude that $M_l|E|M_l \propto I_A$.

In the following, we will give a proof of the above claim by induction. First, the claim is true for $k = 0$. Now we assume that this claim is true for $0 \leq k < N - 1$. Let $l = k + 1$ and fix any $j_1 = (j_1, j_2, \cdots, j_N) \in C_l$ such that $j_1 \neq 0$. For any $i_k = (i_1, i_2, \cdots, i_N) \in C_l$ which is different from $j_1$. If $i_1 \neq j_1$,

$$\langle j_1|E|i_1\rangle = \langle j|\hat{A}_1 \otimes M_l|E|M_l|j_1\rangle = 0.$$ 

If $i_1 = j_1$, set $j_k := (0, j_2, \cdots, j_N)$ and $i_k := (0, i_2, \cdots, i_N)$, by definition, they are different strings of $C_k$. Moreover,

$$\langle j_k|E|i_k\rangle = \langle j_2 \cdots j_N|M_l|E|M_l|j_1\rangle = \langle j_k|E|i_k\rangle = 0$$

by induction. Applying Block Trivial Lemma to the set of base vectors corresponding to $C_l$, the set $\{|\Psi_{i_1, i_2}\rangle\}_{i_1 \in Z_{c_1}}$ and the vector $|j_1\rangle$, we obtain that for any different strings $i_1, i_1'$ in $C_l$,

$$\langle i_1|E|i_1'\rangle = \langle i_1'|E|i_1\rangle = 0, \quad \text{and} \quad \langle i_1|E|i_1\rangle = \langle j_1|E|i_1\rangle.$$ 

Note that $\langle j_1|E|i_1\rangle$ equals to

$$\langle j_2 \cdots j_N|M_l|E|M_l|j_1\rangle = \langle j_k|E|i_k\rangle = \langle 0|E|0\rangle.$$ 

This completes the proof of the claim. Therefore, the last $(N - 1)$-parties could only start with a trivial OPM. By the symmetric construction, one can also show that any $(N - 1)$ parties could only start with a trivial OPM. This statement also implies that any $k$ (where $1 \leq k \leq N - 1$) parties could only start with a trivial OPM.

Note that the elements in $S$ are not always with genuine entanglement. In fact, $|\Psi_0\rangle = |0\rangle$ is fully product states. Now we claim that except this state, all others are with genuine entanglement. We only need to show that $|\Psi_{k,i}\rangle$ ($1 \leq k \leq N - 1, i \in Z_{c_k}$) is entangled for any bipartition of the subsystems. We assume that the bipartition is $\{A_i|i \in I\}|\{A_j|j \in J\}$ where $I, J$ are nonempty subsets of $\{1, 2, \cdots, N\}$, disjoint and $I \cup J = \{1, 2, \cdots, N\}$. Let $A$ and $B$ denote the computational bases of the systems $\{A_i|i \in I\}$ and $\{A_j|j \in J\}$ respectively. Suppose that $|\Psi_{k,i}\rangle = \sum_{\alpha \in A} \sum_{\beta \in B} \psi_{\alpha,\beta}|\alpha\rangle|\beta\rangle$. It sufficient to prove that the rank of the matrix $\langle \psi_{\alpha,\beta}\rangle$ is greater than one. Clearly, $k$ can be expressed as two different forms $k = s + t$ such that $0 \leq s \leq |I|$ and $0 \leq t \leq |J|$. Suppose $k = s_1 + t_1 = s_2 + t_2$ such that $0 \leq s_1 < s_2 \leq |I|$ and $0 \leq t_2 < t_1 \leq |J|$. Choose any subsets $\mathcal{I} \subset I$ and $\mathcal{J} \subset J$ such that $\mathcal{I} = \{x\}$ for $x = 1, 2, 3$. For $y = 1, 2, 3$. We define

$$\mathcal{I}, \mathcal{J} := \left(\{x \in \mathcal{I}\}A_x\right) \otimes \left(\{x \in \mathcal{J}\}A_x\right) \in A, \quad \mathcal{I}, \mathcal{J} := \left(\{y \in \mathcal{J}\}A_y\right) \otimes \left(\{y \in \mathcal{I}\}A_y\right) \in B,$$ 

and then this would be a contradiction. Therefore, we get $\langle \psi_{\alpha,\beta}\rangle > 1$.
where $x, y \in \{1, 2\}$. The matrix $\langle \psi_{\alpha, \beta} \rangle$ has the $2 \times 2$ minor

$$
\begin{pmatrix}
|J, J_1\rangle & |J, J_2\rangle \\
|I, I_1\rangle & |I, I_2\rangle
\end{pmatrix}
$$

where $\alpha \beta \neq 0$. Therefore, the Schmidt rank of $|\Psi_{k,i}\rangle$ across this partition $\{A_i| i \in I\} \{A_j| j \in J\}$ is greater than 1. Hence it is entangled.

In Theorem 1, if we replace the set $S_0$ by two states $|\Psi_{\pm}\rangle := |0\rangle \pm |1\rangle$ where $|0\rangle = \otimes_{i=1}^{N}|0\rangle_{A_i}$, $|1\rangle = \otimes_{i=1}^{N}|1\rangle_{A_i}$ and denote the new total set as $S_G$, then the set $S_G$ contains only genuinely entangled states. Our result gives a complete answer to an open question raised in Ref. [34]. Our result results in Ref. [33], where only bipartite systems are considered.

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**IV. CONCLUSION AND DISCUSSION**

We studied the local-discrimination-based strong quantum nonlocality for general quantum systems. We present two constructions of sets that has the property of strong nonlocality. Its proof can be directly verified based on two basic lemmas developed in Ref. [34]. One of the set contains a fully product state. The other set contains only genuinely entangled states. Our result gives a complete answer to an open question raised in Ref. [29] and Ref. [37]. Moreover, our results on strong nonlocality of sets in general quantum systems greatly generalize the results in Ref. [33], where only bipartite systems are considered.

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