Scale dependence of halo and galaxy bias: effects in real space

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We examine the scale dependence of dark matter halo and galaxy clustering on very large scales (0.01 < \( k [h \text{Mpc}^{-1}] \) < 0.15), due to non-linear effects from dynamics and halo bias. We pursue a two line offensive: high resolution numerical simulations are used to establish some old and some new results, and an analytic model is developed to understand their origins. Our simulations show: (i) that the \( z = 0 \) dark matter power spectrum is suppressed relative to linear theory by \( \sim 5\% \) on scales \( 0.05 < k [h \text{Mpc}^{-1}] < 0.075 \); (ii) that, indeed, halo bias is non-linear over the scales we probe and that the scale dependence is a strong function of halo mass. High mass haloes show no suppression of power on scales \( k < 0.07 [h \text{Mpc}^{-1}] \), and only show amplification on smaller scales, whereas low mass haloes show strong, \( \sim 5 - 10\% \), suppression over the range \( 0.05 < k [h \text{Mpc}^{-1}] < 0.15 \). These results were primarily established through the use of the cross-power spectrum of dark matter and haloes, which circumvents the thorny issue of shot-noise correction. The halo-halo power spectrum, however, is highly sensitive to the shot-noise correction; we show that halo exclusion effects make this sub-Poissonian and a new correction is presented. Our results have special relevance for studies of the baryon acoustic oscillation features in the halo power spectra. Non-linear mode-mode coupling: (i) damp these features on progressively larger scales as halo mass increases; (ii) produces small shifts in the positions of the peaks and troughs which depend on halo mass. We show that these effects on halo clustering are important over the redshift range relevant to such studies (0 < \( z < 2 \)), and so will need to be accounted for when extracting information from precision measurements of galaxy clustering.

Our analytic model is described in the language of the ‘halo-model’. The halo-halo clustering term is propagated into the non-linear regime using ‘1-loop’ perturbation theory and a non-linear halo bias model. Galaxies are then inserted into haloes through the Halo Occupation Distribution. We show that, with non-linear bias parameters derived from simulations, this model produces predictions that are qualitatively in agreement with our numerical results. We then use it to show that the power spectra of red and blue galaxies depend differently on scale, thus underscoring the fact that proper modeling of nonlinear bias parameters will be crucial to derive reliable cosmological constraints. In addition to showing that the bias on very large scales is not simply linear, the model also shows that the halo-halo and halo-dark matter spectra do not measure precisely the same thing. This complicates interpretation of clustering in terms of the stochasticity of bias. However, because the shot noise correction is non-trivial, evidence for this in the simulations is marginal.

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I. INTRODUCTION

Statistical analysis of the large scale structures observed in galaxy surveys can provide a wealth of information about the cosmological parameters, the underlying mass distribution and the initial conditions of the Universe. The information is commonly extracted through measurement of the two-point correlation function \( \overline{\delta(\vec{r}) \delta(\vec{r}')} \) or its Fourier space analogue the Power spectrum \( \overline{\delta^2(k)} \). When further combined with high precision measurements of the temperature anisotropy spectrum from the Cosmic Microwave Background very strong constraints can be imposed on the initial conditions, the energy content, shape and evolution of the Universe [8].

For homogeneous and isotropic Gaussian Random fields, such as is supposed for the post inflationary density field of Cold Dark Matter (hereafter CDM) fluctuations, each Fourier mode is independent, and thus all of the statistical properties of the field are governed by

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the power spectrum. However, non-linear evolution of matter couples the Fourier modes together, and power is transferred from large to small scales \[9, 10\]. Consequently, it is non-trivial to relate the observed structures to the physics of the initial conditions. Further, since one typically measures not the mass, but the galaxy fluctuations, some understanding of the mapping from one to the other is required. This mapping, commonly referred to as galaxy bias, encodes the salient physics of galaxy formation.

One last complication must be added: since galaxy positions are inferred from recession velocities using Hubble’s law, and because each galaxy possesses its own peculiar velocity relative to the expansion velocity, a non-trivial distortion is introduced to the clustering on all scales from the velocity field. These velocity effects are commonly referred to as redshift space distortions. Thus one must accurately account for non-linear evolution of matter fluctuations, bias and redshift space distortions in order to extract precise information from large scale structure surveys and this remains one of the grand challenges for modern physical cosmology.

In this paper, we investigate the issue of bias in some detail, through both numerical and analytic means. We focus on real space effects and reserve our results from redshift space for a subsequent paper. Our numerical work focuses on the generation of multiple realizations of the same cosmological model, in two different box sizes. This allows us to construct halo catalogues spanning a large dynamic range in mass that are largely free from discreteness fluctuations and the multiple realizations allow us to derive errors that are ‘true errors from the ensemble’. We use this data to show that not only is halo clustering on very large scales scale dependent, but that the scale dependence is a strong function of halo mass. These results are completed expected given the standard theoretical understanding of dark matter haloes based on Zel’dovich’s law, and because each galaxy possesses its own peculiar velocity relative to the expansion velocity, a non-trivial distortion is introduced to the clustering on all scales from the velocity field. These velocity effects are commonly referred to as redshift space distortions. Thus one must accurately account for non-linear evolution of matter fluctuations, bias and redshift space distortions in order to extract precise information from large scale structure surveys and this remains one of the grand challenges for modern physical cosmology.

Our analytic approach to modelling these trends can be summarized as follows.

- Haloes are biased tracers of the mass distribution. To describe this bias, we assume that the bias relation between the halo density field and the dark matter field is non-linear, local and deterministic. This allows us to use the formalism of \[13, 16\]. In order for the local model to hold, one must integrate out small scales where locality is almost certainly violated. This can be done in real space by smoothing small-scale fluctuations, or in Fourier space by considering small wavenumbers \[94\].
- The underlying CDM density field is then propagated into the non-linear regime using standard Eulerian perturbation theory techniques \[16\].
- Galaxies are then assumed to form only in haloes above a given mass \[33\]. and these are inserted into each halo using the ‘halo model’ approach \[13, 20, 27, 34, 35\].

Because we write the PT evolved halo density field as a series expansion we refer to this method as ‘Halo-PT’ theory.

Our results are particularly relevant for studies which intend to use the baryon acoustic oscillation feature (hereafter BAO) in the low redshift clustering of galaxies to derive constraints on the dark energy equation of state \[2\]. The CDM transfer function that we have adopted throughout contains a significant amount of BAOs, and we give an accounting of the possible non-linear corrections from mass evolution and biasing that might influence the detection and interpretation of such features. Previous work in this direction has primarily focused on analysis of numerical simulations \[36, 37, 38, 39, 40\], although several analytic works have recently been presented: \[43\] derive the exact damping of BAOs in the Zel’dovich approximation and calculate it in the exact dynamics by re-summing perturbation theory; \[45\] consider real space corrections to the power spectrum from one-loop PT; \[44\] use the halo model, also in real space, to explore systematics; \[43, 46\] consider a model of Lagrangian displacements fit to simulations.

In Section \[II\] we discuss how the approach we have developed here is complementary to and expands on these studies. In Section \[III\] we discuss the numerical simulations and present our measurements of scale dependence in the dark matter, halo center and halo-dark matter cross power spectra. We also present the evidence for large scale non-linear bias. Then in Section \[IV\] we outline some key notions concerning the halo model of large scale structure, as this is the frame work within which we work. In Section \[IV.C\] we describe the non-linear bias model that we employ. In Section \[IV.D\] we use the 3rd Order Eulerian perturbation theory to describe the evolved Eulerian density field in terms of the initial Lagrangian fluctuations. In Section \[V\] we use the 3rd order halo density fields to produce an analytic model for the ‘1-Loop’ halo and halo-cross dark matter power spectra. In Section \[VI\] we explore the predictions of the analytic model for a range of different halo masses. In Section \[VII\] we compare our analytic model to the nonlinear bias seen in the numerical simulations. We use the analytic model to examine the galaxy power spectrum in Section \[VIII\] and present our conclusions in Section \[IX\].

Throughout, we assume a flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model with energy density at late times dominated by a cosmological constant (\( \Lambda \)) and a sea of collisionless cold dark matter particles as the dominant mass density. We take \( \Omega_m = 0.27 \) and \( \Omega_\Lambda = 0.73 \), where these are the ratios of the energy density in matter and a cosmological constant to the critical density, respectively. We use a linear theory power spectrum generated from \texttt{cmfast} \[47\], with baryon content of \( \Omega_b = 0.046 \) and \( h = 0.72 \). The normalization of fluctuations is set through \( \sigma_8 = 0.9 \), which is the initial value of the r.m.s. variance of fluctuations in spheres of comoving radius \( 8h^{-1}\) Mpc extrapolated to \( z = 0 \) using linear theory.
II. MOTIVATION

A number of recent papers have attempted to quantify the scale dependence of galaxy bias. A subset of these have forwarded simple analytic models to remove the scale-dependent biases in the power spectrum estimator. We discuss some of these below so as to set the stage for our work.

A. Cole et al. (2005)

Based on analysis of mock galaxy catalogues from the Hubble volume simulation, these authors proposed a simple analytic model to account for the non-linear scale dependence:

\[ P_{gg}(k) = b^2 P_{\text{Lin}}(k) \frac{1 + A_2 k^2}{1 + A_1 k} \]  

(1)

The parameter \( A_1 = 1.4 \), and \( A_2 \) was allowed to vary over a narrow range, which was then marginalized-over in the fitting procedure. When \( A_1 k \ll 1 \), this model has

\[ P_{gg}(k) \approx b^2 P_{\text{Lin}}(k) \left[ 1 - A_1 k + A_2 k^2 \right]. \]  

(2)

We show below that the bracketed terms are suggestive of the \( P_{13} + P_{22} \) terms from the dark matter perturbation theory, but with incorrect dependence on \( k \). In addition, ignoring the fact that \( A_1 \) may depend on galaxy type is a serious inconsistency. For instance, our results indicate that \( A_1 \) for LRG-like galaxies is smaller than 1.4.

A further concern regarding this model is that no accounting for non-Poisson shot noise has been made. If galaxy formation takes place only in haloes, then galaxies are not Poisson samples of the mass distribution. The analytic model we develop shows that it is important to account for this, and how. We are therefore skeptical about the blind use of equation (1), particularly with regard to its use in the analysis of LRGs [5, 50].

B. Seo & Eisenstein (2005)

These authors examined the scale dependence of halo bias in a large ensemble of low-resolution numerical simulations. They proposed

\[ P_{gg}(k) = b^2 P_{\text{Lin}}(k) + A_1 k + A_2 k^2 + A_0, \]  

(3)

which bares some similarity to that of Cole et al., as can be seen by re-writing \( A_1 \rightarrow A_1 / (b^2 P_{\text{Lin}}) \) and \( A_2 \rightarrow A_2 / (b^2 P_{\text{Lin}}) \). The effective spectral index of the linear power spectrum evolves from \( 0 > n_{\text{eff}} > -1 \) over the range of \( k \) of interest, so we can think of \( P_{\text{Lin}} \) as being approximately constant. Then, the inclusion of the \( b^2 \) term decreases the effective \( A_1 \) as required, but it also decreases the effective \( A_2 \); our results indicate that this is inappropriate.

There is an important difference between this model and the previous one—the inclusion of the constant power term \( A_0 \). This was introduced to account for ‘anomalous power’ – by which was meant effects envisaged by [20] – and or non-Poisson shot noise following [48]. Our analysis strongly supports the inclusion of this term.

C. Seljak (2001); Schulz & White (2006); Guzik, Bernstein & Smith (2006)

These authors explored the scale dependence of galaxy bias in the halo model focusing their attention on the 1-Halo term. They showed that if this was taken simply as a non-Poisson shot-noise correction, then it would be a significant source of scale dependence in the large scale galaxy power spectrum. We have confirmed this in our study. These studies lead one to suggest a base form of the kind:

\[ P_{gg}(k) = b^2 P_{\text{Lin}}(k) + A_0. \]  

(4)

On top of this base form we need to include modifications that capture the true non-linear evolution of the halo field.

D. Huff et al. (2006)

These authors used a set of three large cosmological simulations to investigate the scale dependence of the BAOs. They suggested

\[ P_{gg}(k) = b^2 P_{\text{Lin}}(k) \exp \left[ - (A_1 k)^2 \right] + A_0, \]  

(5)

where the exponential damping term was introduced to account for halo profiles. Again examining the large scale behaviour, \( A_1 k \ll 1 \), we see that this equation may be re-written

\[ P_{gg}(k) \approx b^2 P_{\text{Lin}} \left[ 1 - (A_1 k)^2 \right] + A_0. \]  

(6)

If \( 0 < A_1 < 1 \) (the range they considered), then this formula will suppress the power spectrum on scales \( k < 0.1 h/Mpc^{-1} \) by a percent at most. If this model is to account for the non-linear corrections that we see, then \( A_1 > 1 \). However, the strong exponential damping makes it unlikely that this model will properly characterize the transition from the 2- to 1-Halo term. This is because, as we show below, the nonlinear evolution of halo centers includes an additional boost at intermediate \( k \) which this model does not capture.

E. A necessary model

Our results suggest that a necessary model will have the following properties: the model should be able to pro-
duce a pre-virialization feature and a small scale nonlinear boost with $k$-dependencies motivated by physical arguments; nonlinear corrections should depend on galaxy type; a constant power term should be added to account for non-Poisson shot noise. We therefore expect a reasonable starting point for any empirical modelling of the large scale, scale dependence of the galaxy power spectrum to be,

$$P_{gg}(k, T) = b^2(T) \left[ P_{\text{lin}}(k) e^{-A_1(T) k^2} + A_2(T) k^{m(T)} \right]$$

$$\times \left| W(k/k_\alpha) \right| + A_0(T) \left| W(k/k_\beta) \right| + \frac{1}{\bar{n}_g}(T).$$

Our notation makes explicit that the coefficients have the following properties: $b > 0$, $A_0 > 0$, $A_1 > 0$, $A_2 > 0$ and $m \geq 0$, and that all depend on galaxy type $T$. The first term is composed of two pieces: in the first piece, we have modeled the damping of BAOs using a Gaussian as derived in [41, 42] for the dark matter case; for the second piece, we have added a $k$-dependent boost that models the power added by mode-mode coupling and nonlinear bias. Our simple power-law form (with two parameters $A_2, m$) is meant to describe this effect over a restricted range of scales. The fact that this term is additive as opposed to multiplicative, is meant to emulate the fact that in PT this corresponds to the $P_{\text{gg}}^{\text{lin}}$ term which arises from the convection of linear power on different scales and is therefore smooth possessing no information on BAOs. For weakly nonlinear scales it has a positive spectral index (note, however, that in the limit $k \to 0$, $m = 4$ is expected from momentum conservation arguments). The second term corresponds to Poisson shot noise from unequal weighting of haloes. The last term corresponds to the Poisson shot noise from the galaxy point distribution. We have included filter terms $W(k/k_\alpha)$ and $W(k/k_\beta)$ to indicate the damping due to density profiles, which will occur for $k > 1/r_{\text{vir}}$. This function may be greatly simplified by examining the case $k \ll 1/h\text{Mpc}^{-1}$, for which it reduces to

$$P_{gg}(k, T) = b^2(T) \left\{ P_{\text{lin}} \left[ 1 - A_1(T) k^2 \right] + A_2(T) k^{m(T)} \right\}$$

$$+ A_0(T), \quad (k \ll 1), \quad (8)$$

where the parameter $A_0(T)$ subsumes all sources of constant large scale power.

### III. SCALE DEPENDENT HALO BIAS FROM NUMERICAL SIMULATIONS

#### A. Simulation details and halo catalogues

We have performed a series of high-resolution, collisionless dark matter $N$-body simulations, where $N = 512^3$ equal mass particles. Each simulation was performed using Gadget2 [51]; the internal parameter settings can be found in Table 1 of [52], where more details about the runs themselves are available. The initial conditions were set-up using the 2nd-order Lagrangian Perturbation theory at redshift $z = 49$ [52], with linear theory power spectrum taken from cambfast [47], with the cosmological model being the same as that used throughout this paper. We will present results from two different box sizes: 20 smaller higher resolution box (hereafter HR) for which the volume is $V = L^3 = (512h^{-1}\text{Mpc})^3$, and 8 realizations of a larger, lower resolution box (hereafter LR) for which $L = 1024h^{-1}\text{Mpc}$ box.

Haloes were identified in the $z = 0$ outputs using the friends-of-friends algorithm with linking-length parameter $l = 0.2$. Halo masses were corrected for the error introduced by discretization of the halo density structure [53]. Since the error in the estimate of the halo mass diverges as the number of particles sampling the density field decreases, we only study haloes containing 50 particles or more. For the HR and LR simulations this corresponds to haloes with $M > 4.0 \times 10^{12}h^{-1}M_\odot$ and $M > 4.0 \times 10^{13}h^{-1}M_\odot$, respectively. We then constructed four non-overlapping sub-samples of haloes with roughly equal numbers per sub-sample. The two low mass bins were harvested from the HR simulations and the two high mass ones were taken from the LR runs. Further details may be found in Table I.

#### B. Mass, halo and halo-Mass power spectra

For each realization and each bin in halo mass we measured the following quantities: the power spectrum of the dark matter $P^{\delta\delta}(k)$; the power spectrum of dark matter haloes $P^{hh}(k)$; and the cross power spectrum of dark matter and dark matter haloes, $P^{ih}(k)$. The power spectra may be generally defined

$$\langle \delta^\alpha(k) \delta^\beta(k') \rangle \equiv (2\pi)^3 \delta^D(k + k') P^{\delta\delta}(k), \quad (9)$$

where

$$\delta^\alpha(k) \equiv \left[ \delta(k) \right]^{\alpha} ; \quad \delta^\beta(k) = \left[ P^{ij}(k) \right]^{\beta} ; \quad (10)$$

with $\delta(k)$ and $\delta(k)$ being the Fourier transforms of the mass and halo density perturbation fields,

$$\rho^i(x) = \rho^i \left[ 1 + \delta^i(x) \right], \quad (11)$$

| $N_{\text{real}}$ | $L$ [Mpc h$^{-1}$] | $\bar{N}_{\text{h}} \times 10^4$ | $\bar{n}_h$ [Mpc$^{-3}$ h$^3$] |
|-----------------|------------------|-----------------|-----------------|
| LR Bin 1 $^a$  | 8                | 1024            | 3.6863          | 3.43331 $\times 10^{-4}$ |
| LR Bin 2 $^b$  | 8                | 1024            | 7.3530          | 6.8480 $\times 10^{-5}$ |
| HR Bin 3 $^c$  | 20               | 512             | 6.9287          | 5.1623 $\times 10^{-4}$ |
| HR Bin 4 $^d$  | 20               | 512             | 5.5415          | 4.1287 $\times 10^{-4}$ |

$a$ Mass bin 1 = $M > 1.0 \times 10^{14}h^{-1}M_\odot$

$b$ Mass bin 2 = $1.0 \times 10^{14}h^{-1}M_\odot > M > 4.0 \times 10^{13}h^{-1}M_\odot$

$c$ Mass bin 3 = $4.0 \times 10^{13}h^{-1}M_\odot > M > 7.0 \times 10^{12}h^{-1}M_\odot$

$d$ Mass bin 4 = $7.0 \times 10^{12}h^{-1}M_\odot > M > 4.0 \times 10^{12}h^{-1}M_\odot$
FIG. 1: Halo power spectrum measurements and predictions, ratioed with a smooth ‘No Baryon’ dark matter power spectrum, for four bins in halo mass: results for massive haloes (the top two panels) are from the LR simulations, whereas lower masses (the two bottom panels) are from the HR simulations. Filled circles in the top, middle and bottom sections of each panel show the ensemble average non-linear $P_{\delta\delta}(k)$, $P_{hh}(k)$, and $P_{\delta h}$, respectively. The open circles in the middle sections show $P_{hh}(k)$ with the non-standard shot noise subtraction described in Appendix A. In all panels the linear theory dark matter, halo-halo and halo-cross power spectra are shown as dashed lines. The top panel also shows predictions from halofit (solid lines) [54] and 1-Loop perturbation theory (dot-dash lines). Solid lines in the middle and bottom panels show our new analytic model, Halo-PT.
where the index \( i \) again distinguishes between dark matter and haloes, e.g. \( \rho^1 \equiv \bar{\rho} \) and \( \rho^2 \equiv \rho^b \).

We estimate the spectra through the conventional Fast Fourier Transform (FFT) method \( \text{[93]} \) (for a detailed discussion see \( \text{[54]} \)). The mean power and 1-\( \sigma \) errors on the spectra were estimated from the ensemble neglecting the bin-to-bin covariances. Inspection of an estimate of the covariance matrix from the 20 HR simulations showed that this is reasonable. There is, however, a small degree of off diagonal covariance, but the number of simulations was insufficient to make a precise estimate.

Figure \( \text{[4]} \) shows the three types of power spectra measured from the ensemble of simulations for each of the four bins in halo mass described in Table \( \text{[1]} \). The filled circles and associated error bars in the top, middle and bottom sections of each panel show \( \hat{P}^{\delta \delta}(k) \), \( \hat{P}^{hh}(k|M) \), and \( \hat{P}^{\delta h}(k|M) \). The open circles show the result of applying a non-standard shot-noise correction to \( \hat{P}^{\delta h} \), which we describe in Appendix \( \text{[A]} \).

To emphasize the non-linear evolution of the spectra and the BAOs in particular, we have divided each spectrum by a smooth linear theory spectrum, which we shall refer to as our ‘No Baryon’ model. This was constructed by performing a chi-squared fit of the spectrum by a smooth linear theory spectrum, which we shall describe in Appendix \( \text{[A]} \).

Distinguish between the bias from the amplitude matrix \( \mathcal{A} \). The mean power and 1-\( \sigma \) estimates of the halo bias on very large scales, \( \hat{a} \), were insufficient to make a precise estimate.

On maximizing the likelihood function, we find the following estimator for the halo bias matrix

\[
\hat{A} = \frac{\sum_{i=1}^{N_{\text{dat}}} P_{\text{Lin}}(k_i) \hat{P}_{ij} / \sigma_i^2}{\sum_{i=1}^{N_{\text{dat}}} [P_{\text{Lin}}(k_i)]^2}.
\]

We construct error estimates through further differentiation of the Gaussian likelihood function:

\[
\sigma_A = \left( \frac{\partial^2 \log L}{\partial A^2} \right)^{-1/2} = \left( \sum_{i=1}^{N_{\text{dat}}} \left[ \frac{P_{\text{Lin}}(k_i)}{\sigma_i} \right]^2 \right)^{-1/2}.
\]

Lastly, since we observe scale-dependent non-linear effects in the matter power spectrum for \( k > 0.05h\text{Mpc}^{-1} \), we only use modes with \( k < 0.04h\text{Mpc}^{-1} \) in the fitting of the amplitude matrix \( \mathcal{A} \). Estimates of the large scale bias parameters \( b^h \) and \( b^h \) are presented in Table \( \text{[2]} \).

The dark matter power spectra in Fig. \( \text{[4]} \) show significant deviations away from linear theory prediction: at \( 0.05 < k [h\text{Mpc}^{-1}] < 0.075 \), there is a suppression of power relative to linear theory, whereas at \( k > 0.075 [h\text{Mpc}^{-1}] \) there is an amplification. Perturbation theory studies \( \text{[10]} \) refer to the suppression effect (caused by tidal terms) as ‘pre-virialization’. Recently, this has been understood in much more detail as a result of the damping of linear features by nonlinearities, leading to an exponentially decaying propagator that measures the loss of memory of the density field to the initial conditions \( \text{[41, 42]} \).

Although the effect in the power is rather well-known and has been observed in recent numerical simulations of CDM spectra \( \text{[4, 54, 56, 57]} \), our results constitute a rather precise measurement of this effect, with realistic errors drawn from the ensemble. A complete assessment of the damping of linear features such as BAOs is done by studying the propagator \( \text{[41, 42]} \) rather than the power spectrum. See \( \text{[58]} \) for further discussion on this.

The solid and dot-dashed lines show predictions based on halofit \( \text{[54]} \), and on 1-Loop perturbation theory (PT). The PT results do well compared to the simulations on very large scales, for \( k > 0.07 \) they increasingly over-predict the power. We note also that PT appears to predict the pre-virialization feature in the simulations, adding additional support to the claim that \( P^{\delta h}(k) \) requires non-linear corrections on the scales of interest. However, qualitatively, it under-predicts the magnitude of the effect. We note that halofit does reasonably well at capturing the behavior for \( k < 0.07 h\text{Mpc}^{-1} \), but it appears to under predict the measured data.

Before moving on to \( P^{hh} \) and \( P^{\delta h} \), we think it is worth noting that the BAOs in \( P^{\delta h} \) on \( k > 0.1h\text{Mpc}^{-1} \) have been erased. The large-box LR measurements show that the third peak is gone, and the height of the second peak has dropped so that it appears more as a plateau. However, the behavior at \( k < 0.05 h\text{Mpc}^{-1} \) appears unchanged.

Discreteness corrections for \( P^{\delta h}(k) \) have been studied in some depth \( \text{[54]} \). However, for \( P^{hh} \), the appropriate
correction is more complicated because haloes are rare, highly clustered, and spatially exclusive. In Appendix A we show that the standard Poisson shot noise correction for the cluster power spectrum results in \textit{negative power} at high \( k \). This lead us to propose a new method for making the ‘shot-noise’ correction that accounts for exclusion, which we discuss therein. The open circles in the middle sections of each panel in Figure 1 show the result of this new correction. Filled circles show the uncorrected power, and stars show the standard correction — clearly, the choice of correction is crucial. Note that owing to the arbitrary normalization things for the standard shot noise method look better than they actually are. Unfortunately, the residual uncertainties in our new procedure prevent us from making strong statements about the scale dependence of halo-halo clustering.

Whilst the discreteness correction is troublesome for \( P^\text{hh} \) it is almost negligible for \( P^\text{sh} \) (the halo model arguments which follow allow us to quantify this). Our estimates of \( P^\text{sh} \) are shown in the bottom sections of the panels in Figure 1. Notice that the scale dependence of halo bias is a strong function of halo mass. \( P^\text{sh} \) for the most massive haloes shows no deviations from linear theory until \( k > 0.7h\text{Mpc}^{-1} \). However, the pre-viralization feature appears and gets enhanced as one goes to lower masses. Indeed, for our lowest mass bin, \( P^\text{sh} \) is sub-linear until \( k > 0.15h\text{Mpc}^{-1} \).

This has important consequences for the BAOs. In the highest mass bin (top left panel), the oscillations in \( P^\text{sh} \) at \( k > 0.07h\text{Mpc}^{-1} \) have been erased. However, the first trough, at \( k \sim 0.04h\text{Mpc}^{-1} \), is unaffected. For the next mass bin (top right panel), the first peak and trough are unmodified, and that the second peak is becoming noticeable. This trend continues as we decrease mass; there is even a hint of the third peak in the bottom panels. These measurements indicate that non-linear dynamics can erase oscillations on progressively larger scales as halo mass increases and that small displacements to the positions of the peaks and troughs may occur and that these will also be dependent on halo mass. If the locations of these peaks and troughs are to play an important role in constraining cosmological parameters, our measurements suggest that understanding and quantifying these displacements will be very important.

Before continuing, we comment on the possible explanation of these shifts through scatter from cosmic variance. We remark that it is certainly possible to reconcile some of the shifts in the peak positions through this. However, we draw attention to the fact that all of the points \( k \geq 0.05 \) in the cross-power spectra of the low mass haloes are systematically lower than expected from the linear theory. We also re-iterate that the derived error bars are the errors on the means for 20 realizations. One caveat is that since the spectra are normalized by the very-large scale modes of the power spectrum, where cosmic variance errors are larger, we expect some small fluctuations in the relative amplitudes of the theory predictions as more data is acquired. Estimates of the error in the present LR simulations suggest changes of the order \( \sim 1\% \) to be acceptable; and this increases to \( \sim 3\% \) for the HR simulations. If the amplitudes for the theory curves are too low by \( \sim 3\% \), then some of these discrepancies may be alleviated. However, it is unquestionable that non-linear effects are present on these scales and we must therefore firmly accept that it is likely that these may cause some shifting of the harmonic series. Only a wider and expanded numerical study will be able to address and answer these questions more completely.

The solid lines in the middle and bottom sections of each panel show predictions from the analytic model described in the following sections. In all cases this model provides a better description than does linear theory.

### C. Scale dependence of the bias

Next, we examine the scale dependence of halo bias. We will consider

\[
\hat{b}^\text{NL}(k_i) = \sqrt{\frac{P^\text{sh}(k_i)}{\delta^3(k_i)}}; \quad \hat{b}^\text{NL}(k_i) = \hat{P}^\text{sh}(k_i) / \hat{P}^\delta(k_i), \tag{18}
\]

as well as

\[
\hat{b}^\text{sh}(k_i) = \sqrt{\frac{P^\text{sh}(k_i)}{P^\text{Lin}(k_i)}}; \quad \hat{b}^\text{sh}(k_i) = \frac{\hat{P}^\text{sh}(k_i)}{\hat{P}^\text{Lin}(k_i)}. \tag{19}
\]

For any particular realization the wave modes of the halo and dark matter density fields are almost perfectly correlated. Because the first set of estimators are derived from taking the ratio of measured power spectra, they are insensitive to this source of cosmic variance. In this sense, the second set of estimators are non-optimal. However, they are the ones which will be used with real data, since \( P^\delta \) is generally not observable.

In Figure 2 we show the results of measuring these quantities for the same halo mass bins as in the previous section. The top and bottom parts of each panel show (18) and (19), respectively. The error bars, which were derived from the ensemble to ensemble variations, are significantly larger for (19) than for (18), as expected. The

| Bin | \( \hat{b}^\text{sh} \) | \( \hat{b}^\text{sh} \) | \( \hat{b}^\text{sh} \) | \( \hat{b}^\text{sh} \) | \( \hat{b}^\text{sh} \) | \( \hat{b}^\text{sh} \) |
|-----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1   | 2.28 ± 0.03              | 2.19                      | -0.94                    | -0.98                    | 2.23                      | 1.68                     | -1.08                    |
| 2   | 1.49 ± 0.04              | 1.53                      | -0.30                    | 0.94                     | 1.41                      | 0.94                     | -1.29                    |
| 3   | 1.02 ± 0.03              | 1.13                      | -0.46                    | 1.47                     | 1.04                      | -0.85                    | 0.37                     |
| 4   | 0.87 ± 0.03              | 0.98                      | -0.44                    | 1.44                     | 0.91                      | -0.74                    | 0.55                     |

*See Table I for definition of bins.*
FIG. 2: Large scale halo bias derived directly from $N$-body simulations for four bins in halo mass. The top sections of each panel show the estimators $\hat{b}_{\delta h}^{NL}$ (solid blue points) and $\hat{b}_{hh}^{NL}$ with and without shot noise correction (large and small red stars, respectively). The bottom panels show the same, but for $\hat{b}_{\delta h}^{lin}$ and $\hat{b}_{hh}^{lin}$ (See equations 18 and 19 for definitions). The solid blue and red lines in each panel show the predictions for the bias from our Halo-PT model.
solid lines show the predictions from the new analytic model described in the next section.

For $b_{hh}$ we show both the shot-noise corrected (large stars) and un-corrected (small stars) results. As was the case for the halo-halo power spectra, we see that this correction is important, so it must be known rather accurately. The estimators for $b_{hh}$ are also shown (filled circles). Except for the highest mass bin, $b_{hh} \geq b_{th}$. Indeed, as we shall argue later, there are compelling theoretical reasons why the biases derived from $P_{hh}$ and $P_{bh}$ are not in fact the same, and that, one generally expects $b_{bh} \geq b_{th}$. However owing to the uncertainty regarding the shot noise correction, no firm statement can be drawn from the current data.

Two possible explanations why the highest mass halo bin appears to behave differently are: Firstly, if the shot noise correction to $P_{hh}$ is too aggressive, we may have underestimated the true $P_{bh}$ and therefore the bias. Alternatively, we have made no shot-noise correction to $P_{bh}$; if one should be applied (we think this is unlikely, owing to the large number of dark matter particles), then our current estimate of $b_{bh}$ may be biased high.

The estimators based on equation (19) show more scale dependence than those based on (18), especially for the two high mass bins. As noted by [44], this is because the BAOs are erased on larger and larger scales as higher and higher mass haloes are considered. Thus on dividing the halo spectra by a linear theory BAO spectrum, we are in fact introducing scale dependence from the linear model. For the highest mass haloes, $b_{bh}^{hh}$ is constant at $k < 0.04h$ Mpc$^{-1}$, but it increases monotonically as $k$ increases. This is a direct consequence of the absence of pre-virialization in $P_{bh}$ on intermediate scales and the rapid onset of non-linear power on smaller scales, compared to $P_{bh}$. The bias $b_{bh}^{hh}$ is flat for haloes with masses $(4.0 \times 10^{13} < M[h^{-1} M_{\odot}] < 1.0 \times 10^{14})$, suggesting that small clusters and group mass haloes are linearly biased tracers of the non-linear dark matter. The smallest two bins in halo mass show the reverse trend: the bias decreases at large $k$, by $\sim 10\%$ compared to the approximately constant value at smaller $k$.

IV. THEORETICAL MODEL

We now describe a model for interpreting the trends seen in the previous section. Our discussion is based on the ‘halo-model’, which we briefly summarize below. See [63] for a more detailed review.

A. The ‘Halo Model’ of large scale structure

The halo model may be described by the simple statement:

• All dark matter in the Universe is contained within a distribution of CDM haloes, with masses drawn from some mass function and with the density profile of each halo being drawn from some universal stochastic profile.

The model attains its full potential when the second assumption is stated:

• All galaxies exist only in isolated dark matter haloes, with more massive haloes hosting multiple galaxies.

In essence the model has been in existence for several decades [12, 33, 60, 61, 62, 63, 64]. However it was not until the advent of large numerical $N$-body simulations and accurate characterization of halo phenomenology that its true value was realized. Namely, given an appropriate Halo Occupation Distribution (HOD – i.e., a prescription for the number and spatial distribution of galaxies within a halo) the model successfully reproduces the real-space form of the two-point correlation function of galaxies over a wide range of scales. It predicts subtle deviations from a power-law which have recently been seen in observations [65] and provides a framework for describing the luminosity [66, 67] and environmental dependence of galaxy clustering [68, 69]. It also enables new tests of the CDM paradigm to be constructed [70, 71].

B. Power spectrum

In the model the density fields of haloes and dark matter may be written as a sums over haloes,

$$\rho^j(x) \equiv \rho^j \left[ 1 + \delta^j(x) \right] = \sum_{i=1}^{N_i(M_j)} M_j U_j(x - x_j | M_j) ,$$

where $i = \{1,2\}$ distinguishes between haloes and dark matter and $N_1 = N$ and $N_2 = N_i$ are the total number of haloes and the number of haloes in some restricted range in mass. $M_j$ and $x_j$ are the mass and center of mass of the $j$th halo and $U_j \equiv \rho_j(x_j | M_j)$ is the mass normalized density profile. Following [63], the power spectra $P_{bh}$, $P_{hh}$ and $P_{bh}$ can be written as the sum of two terms:

$$P_{ij}(k) = P_{ijh}(k) + P_{ijb}(k) .$$

The first term, $P_{ijh}$, referred to as the ‘1-Halo’ term, describes the intra-clustering of dark matter particles within single haloes; the second, $P_{ijb}$, referred to as the ‘2-Halo’ term, describes the clustering of particles in distinct haloes. They have the explicit forms:

$$P_{ijh}(k) = \frac{1}{\rho^j \rho^j} \int_0^\infty dM n(M) M^2 |U(k|M)|^2 \times \Theta_{ij}(M, M) ;

$$

$$P_{ijb}(k) = \frac{1}{\rho^j \rho^j} \int_0^\infty \prod_{l=1}^2 \{dM_l n(M_l) M_l U_l(k|M_l)\} \times P_{bh}^c(k|M_1, M_2) \Theta_{ij}(M_1, M_2) ,$$
where \( n(M) \) is the halo mass function, which gives the number density of haloes with masses in the range \( M \) to \( M + dM \), per unit mass. The \( \Theta_{ij} \) matrix carves out the halo density field to be considered, e.g. for haloes with mass \( M > M_{\text{cut}} \) the matrix is

\[
\Theta_{ij}(M_1, M_2) \equiv \begin{bmatrix}
\Theta(M_1)\Theta(M_2) & \Theta(M_1)\Theta(M_2 - M_{\text{cut}}) \\
\Theta(M_1 - M_{\text{cut}})\Theta(M_2) & \Theta(M_1 - M_{\text{cut}})\Theta(M_2 - M_{\text{cut}})
\end{bmatrix},
\]

where \( \Theta(x) \) is the Heaviside Step Function. More complicated halo selections can easily be described through the \( \Theta_{ij} \) notation. Lastly, \( P_{c}^{hh}(k|M_1, M_2) \) is the power spectrum of halo centers with masses \( M_1 \) and \( M_2 \). This function contains all of the information for the inter-clustering of haloes: precise knowledge of this term is required to make accurate predictions on large scales.

In principle, \( P_{c}^{hh}(k|M_1, M_2) \) is a complicated function of \( M_1 \), \( M_2 \) and \( k \). Initial formulations of the halo model [15, 24, 34, 72] assumed that it could be well-approximated by

\[
P_{c}^{hh}(k|M_1, M_2) = b_1(M_1)b_1(M_2)P_{\text{lin}}(k),
\]

(24)

where all the scale dependence is in \( P_{\text{lin}}(k) \), which is taken from linear theory, and all the mass dependence is in the scale-independent bias parameter \( b_1(M) \) [12, 13]. In this approximation, \( P_{c}^{hh}(k|M_1, M_2) \) is a separable function of \( M_1 \), \( M_2 \) and \( k \). As we show below, comparison with numerical simulations shows that this simple model over-predicts power on very large scales and provides insufficient power on intermediate scales. In both cases, this is a ten percent effect.

This discrepancy is not unexpected [33, 73, 74]. A simple correction results from setting

\[
P_{c}^{hh}(k|M_1, M_2) = b_1(M_1)b_1(M_2)P_{\text{NL}}(k),
\]

(25)

where \( P_{\text{NL}} \) is the non-linear rather than the linear matter power spectrum, and, in addition, imposing an exclusion constraint:

\[
\xi_{c}^{hh}(r|M_1, M_2) = -1 ; \quad (r < r_{\text{vir}1} + r_{\text{vir}2}),
\]

(26)

where \( r_{\text{vir}} \) is the virial radius of the halo.

The success of this approach is demonstrated by comparing \( P_{c}^{hh}(k) \) measured in the \( z = 0 \) output of the Hubble Volume simulation [73] with the halo model calculation. The open and filled symbols in the top panel of Figure 3 show the measurement before and after subtracting a Poisson shot noise term (which is shown by the triple dot-dashed line.) The dot-short dash line shows the linear theory prediction, and the other two dot-dashed curves show two estimates of the 2-halo term: the one which drops more sharply at large \( k \) is based on equation (25) and the other one is based on the original approximation of equation (24). Since equation (25) requires the use of a non-linear power spectrum, we used the one provided by [54].

The symbols in the bottom panel show the measurements divided by the halo model calculation which uses equation (24) for the 2-Halo term (i.e. the initial linear theory-based approximation). Notice how they drop below unity at \( k \sim 0.1h\text{Mpc}^{-1} \). The solid line shows the halo model calculation which is based on equation (25)—it reproduces this pre-virialization feature well.

As an interesting aside, we note that the fitting formula halofit does very well at matching the pre-virialization feature. Whilst it is not apparent from the figure we also point out that the transfer function of the Hubble volume simulation does contain BAOs; thus, our results demonstrate that the fitting formula of [54] appears to be accurate, for this data, for BAO models to roughly \( \sim 5\% \). In light of this, we note that the discrepancy between halofit and the mass power spectra from our smaller box HR simulations is somewhat puzzling. We highlight this issue for further study, one possible explanation is the difference in initial power spectra, on the other hand, also note that a calculation within the framework of Renormalized Perturbation Theory [41] suggests that even for these simulation volumes on expects small effects due to absence of coupling to large scales.

To fix the small discrepancies which remain, some authors have advocated making the halo bias factors scale dependent [76], but the implementation has been based on fitting formulae rather than fundamental theory. While equation (25) appears to fare better than the original approximation (24), as we will soon show, in going from (24) to (25), one is making the assumption that halo bias is linear even when the mass density field is not. If this is not the case then the method is incorrect. In addition, there is an unpleasant circularity in requiring prior knowledge of \( P_{\text{NL}}(k) \) in order to predict \( P_{\text{NL}}(k) \).

C. Halo bias: The non-linear local bias model

The discussion above makes clear that a rigorous treatment of the 2-Halo term is currently lacking. This term requires a description of how dark matter haloes cluster. Whereas current models seek to describe halo clustering as a biased version of dark matter clustering, the scale dependence of halo bias is still rather poorly understood [11, 12, 13, 14, 20, 76, 77, 78, 79, 80, 81, 82]. In the following sections we develop a model to understand its main properties. In particular, we will discuss a general non-linear, deterministic, local bias model for dark matter haloes. This model is exactly analogous to that derived for galaxy biasing by [16] and first applied to dark matter haloes by [13].

To begin, consider the density field of all haloes with masses in the range \( M \) to \( M + dM \), smoothed with some filter of scale \( R \). We now assume that this field can be related to the underlying dark matter field, smoothed with the same filter, through some deterministic mapping and that this mapping should apply independently of the
We now assume that there is a certain filter scale above which \( F_{(M,R)} \) is independent of both the scale considered and also the exact shape of the filter function. Hence,

\[
\delta^b(x|R, M) = \sum_{i=0}^{\infty} \frac{b_i(M)}{i!} \langle \delta(x|R) \rangle^i ,
\]

where the bias coefficients are

\[
b_i(M) = \frac{\partial^i F_{(M,R)}[\delta]}{\partial \chi^i} \Big|_{\chi=0} .
\]

The linear bias model has \( b_1 = 0 \) for all \( i > 1 \).

The bias coefficients from the Taylor series are not independent, but obey two constraints. The first arises from the fact that \( \langle \delta^b(x|R) \rangle = 0 \), which leads to

\[
b_0 = -\frac{b_2}{2} \langle \delta^2 \rangle - \frac{b_3}{3!} \langle \delta^3 \rangle - \cdots - \frac{b_i}{i!} \langle \delta^i \rangle .
\]

Thus, in general \( b_0 \) is non-vanishing and depends on the hierarchy of moments. This allows us to re-write equation (30) as

\[
\delta^b(x|R, M) = \sum_{i=1}^{\infty} \frac{b_i(M)}{i!} \left\{ \langle \delta(x|R) \rangle^i - \langle \delta^i(x|R) \rangle \right\} .
\]

Nevertheless, we may remove \( b_0 \) from further consideration by transforming to the Fourier domain, where it only contributes to \( \delta(k = 0) \).

The second constraint states that a sum over all halo density fields \( \delta^h(x|M) \) weighted by halo mass and abundance must recover the dark matter density field \( \delta^h \). This requires that

\[
\frac{1}{N} \int dM n(M) M b_i(M) = \begin{cases} 1 & (i = 1) \\ 0 & (i = 0, 2, 3, \ldots) \end{cases} .
\]

For CDM models whose initial density perturbations are Gaussian Random, the bias coefficients may either be derived directly through the ‘peak–background split’ argument or measured directly from N-body simulations. Figure 4 shows the halo bias parameters up to third order, derived in the context of the Sheth-Tormen mass function; see [13] for the analytic expressions. We compare these with measurements from our simulations in Appendix B.

A practical application of this method rests squarely upon our ability to truncate the Taylor series at some particular order. However, since the procedure that we have adopted for doing this requires some further knowledge, we shall reserve our discussion until Section V B.

**D. HaloPT: Evolution of halo fields**

We now evolve the halo density field as expressed by equation (30) into the non-linear regime via perturbation...
theory techniques. For a short discussion of these methods see Appendix C and for a full and detailed review see [10]. The main idea that we require from perturbation theory is that each Fourier mode of the density field may be expanded as a series,

$$\delta(k, a) = \sum_i D_i^f(a) \delta_i(k),$$

where $\delta_i(k)$ is the $i$th order Eulerian perturbation and $D_i(a)$ is the linear growth factor. Thus, on Fourier transforming the halo bias relation of equation (30), truncated at third order, and on inserting the PT expansion from above, we arrive at (keeping up to cubic terms)

$$\delta^h(k|M, R) = b_1(M) \left[ \delta_1(k|R) + \delta_2(k|R) + \delta_3(k|R) \right] + \frac{b_2(M)}{2!} \int \frac{d^3q_1}{(2\pi)^3} \left[ \delta_1(q_1|R) \delta_1(k-q_1|R) + 2 \delta_1(q_1|R) \delta_2(k-q_1|R) \right]$$

$$+ \frac{b_3(M)}{3!} \int \frac{d^3q_1 d^3q_2 d^3q_3}{(2\pi)^6} \delta_1(q_1|R) \delta_1(q_2|R) \delta_1(q_3|R) \times \delta_1(k-q_1-q_2-R),$$

where $\delta_i(q|R) = W(|q|R) \delta_i(q)$. We next insert the solutions for each order of perturbation, which are presented in equation (34) of the Appendix, into equation (36). On re-arranging terms and collecting powers of $\delta$, the mildly non-linear density field of dark matter haloes may be written as a PT series expansion of the dark matter density. This series is

$$\delta^h(k, a|M, R) = \sum_{n=1}^\infty D_i^n(a) \left[ \delta^h(k|M) \right]_n ;$$

$$\left[ \delta^h(k|M, R) \right]_n = \int \frac{d^3q_1 \delta_i(q_1)}{(2\pi)^3} \left[ \delta^D(k) \right]_n \times F^n_h(q_1, \ldots, q_n|M, R) \tag{38}$$

where $[\delta^h(k|M)]_n$ is the $n$th order perturbation to the halo density field, and where the short-hand notation $[\delta^D(k)]_n = \delta^D(k - q_1 - \cdots - q_n)$ has been used. The functions $F^n_h(q_1, \ldots, q_n|M, R)$ are the Halo-PT kernels, symmetrized in all of their arguments. The first three may be written in terms of the dark matter PT kernels:

$$F^h_1 = b_1(M) W(|k|R) F_1 ;$$

$$F^h_{1,2} = b_1(M) W(|k|R) F_{1,2} + \frac{b_2(M)}{2} W(|q_1|R) W(|q_2|R) F_1 F_2 \tag{40}$$

$$F^h_{1,2,3} = b_1(M) W(|k|R) F_{1,2,3} + \frac{b_3(M)}{3} \left[ W(|q_1 + q_2|R) W(|q_3|R) F_1 F_2 F_3 + 2 \text{ cyc} \right]$$

$$+ \frac{b_3(M)}{6} W(|q_1|R) W(|q_2|R) W(|q_3|R) F_1 F_2 F_3 \tag{41}$$

where $F^n_j(q_1, \ldots, q_j|M, R)$. Thus equations (37) and (38) can be used to describe the mildly non-linear evolution of dark matter halo density fields to arbitrary order in the dark matter perturbation, and equations (39)–(41) make explicit the halo evolution up to 3rd order. Together, these ideas define our meaning of the term ‘Halo-PT’.

It is now apparent that halo clustering studies which assume a linear bias model and take the power spectrum to be the fully non-linear one, are effectively assuming that $\delta^h(k|M) = b_1(M) \sum_i \delta_i(k)$, with $b_i(M) \ll b_1(M)$ for $i \neq 1$. However, for CDM, the peak-background split argument informs us that this never happens, unless the density field itself is linear [11,12,13,14,13]. We must therefore conclude that extrapolating the linear bias relation into the weakly non-linear regime, without full consideration of the non-linearity of halo bias is incorrect.

V. THE 1-LOOP HALO MODEL

A. Halo center power spectra

We now use the Halo-PT to calculate the power spectrum of halo centers in the mildly non-linear regime. We define the power spectrum of halo centers for haloes with masses $M_1$ and $M_2$ to be

$$\langle \delta^h_c(k_1|M_1) \delta^h_c(k_2|M_2) \rangle = (2\pi)^3 \delta^D(k_{12}) P^h_c(k_1|M_1, M_2),$$

where $P^h_c(k_1|M_1, M_2)$
On inserting the Halo-PT solutions for each order of the perturbation we find that \( P_{c}^{\text{hh}}(k) \) can be written as the sum of three terms

\[
P_{c}^{\text{hh}}(k|M_1, M_2) = P_{c,11}^{\text{hh}}(k|M_1, M_2) + P_{c,22}^{\text{hh}}(k|M_1, M_2) + 2P_{c,13}^{\text{hh}}(k|M_1, M_2),
\]

where

\[
P_{c,11}^{\text{hh}}(k|M_1, M_2) = F_{1}^{b}(k|M_1)F_{1}^{b}(k|M_2)P_{11}(k);
\]

\[
P_{c,22}^{\text{hh}}(k|M_1, M_2) = 2\int \frac{d^3q}{(2\pi)^3} P_{11}(q)P_{11}(|k - q|)
\]
\[\times F_{2}^{b}(q, k - q|M_1)F_{2}^{b}(q, k - q|M_2);
\]

\[
P_{c,13}^{\text{hh}}(k|M_1, M_2) = 6P_{11}(k)\int \frac{d^3q}{(2\pi)^3} P_{11}(q)
\]
\[\times F_{1}^{b}(k, q|M_1, M_2).
\]

Here \( P_{11}(k) \) is equivalent to the linear theory power spectrum and

\[
\tilde{F}_{13}^{\text{hh}}(k, q|M_1, M_2) = \frac{1}{2} [F_{1}^{b}(k|M_1)F_{3}^{b}(k, q_1, -q_1|M_2)
\]
\[+ F_{1}^{b}(k|M_2)F_{3}^{b}(k, q_1, -q_1|M_1)].
\]

When these expressions are averaged over all halo masses, weighted by the respective cosmic abundances \( M_{1}n(M_1) \) and \( M_{2}n(M_2) \), then the constraint equation \((54)\) guarantees that they reduce to the standard ‘1-loop’ expression for the PT power spectrum of dark matter \((10)\):

\[
P_{1\text{Loop}}(k) = P_{11}(k) + P_{22}(k) + P_{13}(k).
\]

Strictly speaking the 1-Loop power spectrum refers to \( P_{22} + P_{13} \), we shall break convention and use equation \((45)\) to define what we mean. Explicit details of the 1-Loop expressions may be found in Appendix \(C\).

The theory may be further developed by directly substituting the Halo-PT kernels, given by equations \((39)\)–\((41)\), into equations \((44)–(46)\). A little algebra shows that

\[
\frac{P_{c}^{\text{hh}}(k|M_1, M_2, R)}{|W(kR)|^2} = b_{1}(M_1)b_{1}(M_2)P_{1\text{Loop}}(k) + \tilde{b}_{1,3}(M_1, M_2)\sigma^2(R)P_{11}(k)
\]
\[+ 2\tilde{b}_{1,2}(M_1, M_2)\int \frac{d^3q}{(2\pi)^3} \frac{W(qR)W(|k - q|R)}{W(kR)} P_{11}(q)
\]
\[\times P_{11}(|k - q|F_{2}(q, k - q) + 2P_{11}(k)F_{2}(k, -q))\]
\[+ \frac{b_{2}(M_1)b_{2}(M_2)}{2}\int \frac{d^3q}{(2\pi)^3} \frac{|W(qR)|^2|W(|k - q|R)|^2}{|W(kR)|^2} P_{11}(q)P_{11}(|k - q|),
\]

where

\[
\tilde{b}_{i,j}(M_1, M_2) \equiv \frac{1}{2} [b_{i}(M_1)b_{j}(M_2) + b_{j}(M_1)b_{i}(M_2)].
\]

Before continuing, we point out and answer an important question that naturally arises at this junction: How does one compare the filtered theory with the unfiltered observations? We forward the proposition that the unfiltered nonlinear power spectrum can be recovered through the following simple operation:

\[
P(k|R)/|W(kR)|^2 = P(k).
\]

This is unquestionably true for an observed non-linear field. It will therefore also be true for the correct theoretical model.

This rather lengthy expression may be more readily digested through the examination of two limiting cases. But first, notice the important fact that it is still a separable product of mass dependent terms and scale dependent terms.

When the two halo masses are identical, then

\[
P_{1\text{Loop}}(k) = P_{11}(k) + P_{22}(k) + 2P_{13}(k).
\]
Both of these points were independently noted by \[24\] markable features:

Inspection of these two limiting cases reveals three remarkable features:

- Firstly, if the non-linear bias parameters \(b_2(M)\) and \(b_3(M)\) are non-vanishing then the bias on large scales is not \(b_1(M)\).
- Secondly, the halo-halo spectrum has a term that corresponds to constant power on very large scales, whereas the cross spectrum does not.

Both of these points were independently noted by \[24\] and \[83\], but for the case of non-linear galaxy biasing (also see \[29\]).

- Thirdly, the large scale bias derived from the halo-dark matter cross power spectrum is not \(b_1\), nor is it given by the bias derived from the halo-halo power spectrum.

To see these points more clearly we take the \(k \to 0\) limit of equations \(49\), \(52\) and \(53\):

\[
\frac{P_{c}^{hh}(k|M_1 = M_2, R)}{|W(kR)|^2} = b_1^2(M) P_{1\text{Loop}}(k) + b_1(M) b_3(M) \sigma^2(R) P_{11}(k) + b_1(M) b_2(M) \int \frac{d^3q}{(2\pi)^3} \frac{W(qR) W(|k - q|R)}{W(kR)} P_{11}(q) \left\{ 2P_{11}(|k - q|) F_2(q, k - q) + 4P_{11}(k) F_2(k, -q) \right\} + \frac{b_2^2(M)}{2} \int \frac{d^3q}{(2\pi)^3} \frac{|W(qR)|^2 |W(|k - q|R)|^2}{|W(kR)|^2} P_{11}(q) P_{11}(|k - q|) .
\]

(52)

\[
\frac{P_{c}^{hh}(k|M, R)}{|W(kR)|^2} = \left\{ b_1(M) P_{1\text{Loop}}(k) + \frac{1}{2} b_3(M) \sigma^2(R) P_{11}(k) \right\} + b_2(M) \int \frac{d^3q}{(2\pi)^3} \frac{W(qR) W(|k - q|R)}{W(kR)} P_{11}(q) \left[ P_{11}(|k - q|) F_2(q, k - q) + 2P_{11}(k) F_2(k, -q) \right] .
\]

(53)

\[
P_{c}^{hh}(k|M_1, M_2, R) = b_1(M_1) b_1(M_2) P_{11}(k) \left\{ 1 + \sigma^2(R) \frac{34}{21} \left( \frac{b_2(M_1)}{b_1(M_1)} + \frac{b_2(M_2)}{b_1(M_2)} \right) + \frac{1}{2} \left( \frac{b_3(M_1)}{b_1(M_1)} + \frac{b_3(M_2)}{b_1(M_2)} \right) \right\} + \frac{b_2(M_1) b_2(M_2)}{2} \int \frac{d^3q}{(2\pi)^3} \left[ P_{11}(q) |W(qR)|^2 \right]^2 ;
\]

(54)

\[
P_{c}^{hh}(k|M, R) = b_1^2(M) P_{11}(k) \left\{ 1 + \sigma^2(R) \frac{68}{21} \frac{b_2(M)}{b_1(M)} + \frac{b_3(M)}{b_1(M)} \right\} + \frac{b_2^2(M)}{2} \int \frac{d^3q}{(2\pi)^3} \left[ P_{11}(q) |W(qR)|^2 \right]^2 ;
\]

(55)

\[
P_{c}^{hh}(k|M, R) = b_1(M) P_{11}(k) \left\{ 1 + \sigma^2(R) \frac{34}{21} \frac{b_2(M)}{b_1(M)} + \frac{1}{2} \frac{b_3(M)}{b_1(M)} \right\} .
\]

(56)

These expressions make the first point noted above trivially obvious: The large scale bias is modulated by
the Halo-PT correction terms, and these depend on the non-linear bias parameters $b_2$ and $b_3$ and also on the filtered variance of fluctuations.

The second point noted above originates specifically from the quadratic non-linear bias terms found in equations (54) and (55), e.g. terms containing $b_2(M_1) b_2(M_2)$ and $b_2^2(M)$. For a linear power spectrum that obeys the limit $P_{11}(k \to 0) \to 0$, these expressions reduce to the constant

$$P^h_{11}(k \to 0|M) = \frac{b_2^2(M)}{2} \int \frac{d^3 q}{(2\pi)^3} [P_{11}(q)|W(qR)|^2]^2.$$  

(57)

This term was discussed in great detail for the case of galaxy biasing by [24].

The third point noted above can be understood by constructing the linear bias, e.g. dividing equations (56) and (55) by $P_{11}$. On squaring the bias recovered from (55) and subtracting it from the bias from (55), we find

$$\{[b^h_{\text{Lin}}]_{12} - [b^h_{\text{Lim}}]_{12}\}^2 = \frac{b_2^2(M)}{2P_{11}(k)} \int \frac{d^3 q}{(2\pi)^3} [P_{11}(q)|W(qR)|^2]^2 - \sigma^4(R) \left( \frac{34}{21} b_2(M) + \frac{1}{2} b_3(M) \right)^2,$$  

(58)

We now see that, because $P^h_{11}$ approaches a constant on very large scales, on dividing through by $P_{11}(k|M)$ the bias function $b^h_{\text{Lin}}$ diverges at the origin as $1/P_{11}^{1/2}$ diverges.

Figure 5 shows our expressions for $b^h_{\text{Lin}}$ and $b^h_{\text{Lim}}$ (from equations 54 and 55). In this particular case, we assume the nonlinear bias parameters derived from the Sheth-Tormen mass function by [13]. We use a Gaussian filter, for which $W(kR_G) = \exp[-(kR_G)^2/2]$. For $R_G = 20h^{-1}\text{Mpc}$ and for our fiducial cosmology, we find $\sigma^2(R_G = 20) \approx 0.046$. To inspect the differences more closely, we take the ratio of the predictions with respect to the tree-level theory, e.g. $b_1 = P^h_{11}/P_{11} = \sqrt{P^h_{11}/P_{11}}$. The figure demonstrates two of the points raised above. Firstly the large scale bias is not simply $b_1$: halo-halo bias (solid through to dotted curves) does not converge as one considers larger and larger scales; however bias from the cross power spectrum is very close to linear for all except the most massive haloes, where $b_2$ and $b_3$ are very strongly rising functions (see Fig. 3). Secondly, it is now obvious that $b^h_{\text{Lin}}$ and $b^h_{\text{Lim}}$ are not the same. Note also that the magnitude of the expected scale dependence: the bias varies by at most five percent when $k$ is changed by an order of magnitude. The mass dependence of $b^h_{\text{Lin}}$ shown in Fig. 5 is simply driven by that of $b_2^2(M)$ (again see Fig. 3). As halo mass increases the bias slowly increases until it reaches a maximum at $M \sim 10^{13} h^{-1} M_\odot$. It then decreases to $M \sim 10^{14} h^{-1} M_\odot$ after which it shoots up dramatically for larger masses.

B. Convergence of the power spectrum

We now return to the issue of truncation and applicability of the Taylor series expansion of the halo field. A first requirement for the Taylor series to converge after a finite number of terms is that the filter scale be large enough so that the rms dark matter fluctuations be much less than unity: $\sigma(R) \ll 1$. Consider the case where $R$ is very large and convergence occurs at first order, we then have: $\delta^h(x|R) = b_1(M) \delta_1(x|R)$. As the filter scale $R$ is slowly decreased the rms fluctuations in $\delta$ increase and a larger and larger number of terms are required to accurately map the underlying bias function. Finally, as $\sigma(R) \to 1$ all terms in the series are required. At this point the method has no merit.

Since a robust criterion for truncation is out of reach at present, we propose an ad-hoc criterion for convergence that must plausibly be obeyed, that is

$$b_1(M) \sigma(R) < 1.$$  

(59)

In Appendix B we shall also discuss an empirical method for testing convergence.

C. Returning to the halo model

We now translate these ideas back into the language of the halo model. To begin, we shall restrict our attention to the halo model in the large scale limit, more precisely we consider scales where $U(k|M) \sim 1$. Since the Fourier
transform of the mass normalized profile may be written

\[ U(k|M) = \int_0^{r_{\text{vir}}} d^3r U(r|M) \frac{\sin[kr]}{kr}, \quad (60) \]

our large scale condition simply becomes \( kr_{\text{vir}} \ll 1 \). In the above equation we have, for convenience, assumed spherical density profiles. If halo mass and virial radius are related through \( M = 4\pi 200 \rho_{\text{crit}}^\text{vir}/3 \), then for the largest collapsed objects in the Universe \( M \approx 10^{15} h^{-1} M_\odot \), and the above condition translates to the inequality \( k \lesssim 0.4 h \text{Mpc}^{-1} \). If we assume an NFW density profile \( \delta \) with concentration parameter \( c \sim 6 \), then for \( k \sim 0.15 h \text{Mpc}^{-1} \), we find that \( U(k) \approx 0.994 \). We therefore assume that this is an excellent approximation over the scales that we are interested in.

Hence for scales \( kr_{\text{vir}} \ll 1 \), our equations (22) and (23), at the 1-Loop level in Halo-PT, now take the forms:

\[
\frac{P_{ij}^{\text{H}}(k|R)}{|W(k|R)|^2} = \frac{1}{\bar{\rho}^2} \int_0^\infty dMn(M) M^2 \times \Theta_{ij}(M,M) ;
\]

\[
\frac{P_{ij}^{\text{hh}}(k|R)}{|W(k|R)|^2} = \frac{1}{\bar{\rho}^2} \int_0^\infty \prod_{i=1}^2 \{dM_i n(M_i) M_i\} \\
\times \frac{P^{\text{hh}}(k,M_1,M_2,R)}{|W(k|R)|^2} \Theta_{ij}(M_1,M_2), \tag{61}
\]

where we have explicitly included a filter on the 1-Halo term. (Recall that the halo center power spectrum in the 2-Halo term already includes such a filter.)

We now see that, because the \( b_i(M) \) are the only mass dependent functions, on insertion of equation (49) into equation (61) the integrals over mass may be immediately computed. Thus,

\[
P_{ij}^{\text{H}}(k|R) = P_{ij}^{\text{hh}}(k|R), \tag{63}
\]

where \( P_{ij}^{\text{hh}}(k|R) \) is equivalent to equation (49), except that we have replaced all of the mass dependent bias parameters by the average ones: i.e.

\[
\langle b_i \rangle = \frac{\int_0^\infty dM n(M) M b_i(M) \Theta(M-M_{\text{cut}})}{\int_0^\infty dM n(M) M \Theta(M-M_{\text{cut}})}. \tag{64}
\]

We note that if we wish to weight by halo number density rather than mass density then we simply replace the mass weighting in the numerator and denominator of \( \langle b_i \rangle \).

**VI. EVALUATION OF THE THEORY**

In this section we present the results from the direct computation of the 1-Loop halo center expressions for \( P^{\text{H}}_c \) and \( P^{\text{hh}}_c \), as given by equations (12) and (33), respectively. This section is almost entirely pedagogical; we urge those who are only interested in the direct comparison with the numerical work to press on to Section VII.

Recall that it is necessary to adopt some filter scale \( R \).

We have studied two choices: \( R_G = 20 h^{-1}\text{Mpc} \) and \( R_G = 10 h^{-1}\text{Mpc} \) for which the linear theory, Gaussian filtered, variances are \( \sigma^2(R) = 0.046 \) and 0.177, respectively. The larger smoothing scale is required for the more massive haloes.

**A. Halo–dark matter cross power spectra**

In Figure 6 we show the predictions for the scale dependence of \( P^{\text{hh}} \) at the 1-Loop level, as a function of wave number. The different panels show results for different halo masses and smoothing scales. In all four panels, we see that, as expected, there is a small (few percent) positive offset from the linear theory bias value \( b_1 \). The largest offset occurs for the cluster mass haloes, but here it may be the case that, owing to the bias being large, the filter scale that we have adopted for these objects may still be too small for adequate convergence. We also note that the offset for the \( M = 10^{14} h^{-1} M_\odot \) haloes is negligible. This can be attributed to the fact that \( b_2 \approx b_3 \approx 0 \) (see Fig. 3). For the lower mass haloes the offsets are roughly \( \sim 2\% \) in excess of linear.

Considering the predictions for the highest mass haloes, we see that the spectrum is scale independent up to \( k \sim 0.07 h \text{Mpc}^{-1} \), where the non-linear amplification from the \( P_{22} \) term becomes dominant. For this case, the absence of the pre-virialization feature may be understood as follows: Firstly, we note that \( b_1 \) and \( b_2 \) are positive, whereas \( b_3 \) is negative. However, since \( b_3 \) is an order of magnitude smaller than the others, it plays no significant part in determining the shape of the spectrum. The \( P_3 \) term in the 1-Loop power spectrum, which is the main cause of the pre-virialization feature, is thus overwhelmed by the action of the quadratic bias \( b_2 \) term. This suggests that \( P^{\text{hh}} \) should be scale independent up to \( k \sim 0.07 h \text{Mpc}^{-1} \).

Next, we collectively consider the predictions for the lower mass haloes, as they show many similar traits. Firstly, we find that when \( k < 0.01 h \text{Mpc}^{-1} \), then the ratios of the 1-Loop to tree level (or linear) spectra are flat. However, for \( 0.01 h \text{Mpc}^{-1} < k < 0.07 h \text{Mpc}^{-1} \), significant scale dependence is apparent: the pre-virialization feature is present and it appears to become stronger as halo mass decreases. On smaller scales still, the non-linear boost from the \( P_{22} \) term amplifies the power spectrum and breaks all scale independence. Interestingly, the onset of \( P_{22} \) is pushed to smaller scales as halo mass decreases. These effects can be understood as follows. For these objects \( b_1 \) and \( b_3 \) are positive, whereas \( b_2 \) is negative. On large scales, we see that \( b_2 \) and \( b_3 \) are nearly equivalent, but \( b_3 \) is slightly dominant, and this results in a small positive correction. Whereas on smaller scales this trend reverses and \( b_2 \) becomes dominant. The overall correction is then negative and this leads to the enhanced pre-virialization feature and delay of the onset of \( P_{22} \).

It is also interesting to note the imprint of the BAO fea-
FIG. 6: The real space cross power spectrum of haloes and cold dark matter at the 1-Loop level on large scales. The four panels show predictions for haloes with masses: $M = \{10^{15}, 10^{14}, 10^{13}, 10^{12}\} h^{-1} M_\odot$. In the upper plot of each panel, the thick (black) curve represents the total spectrum as given by equation (53). The dash curve gives the 1-Loop contribution from the linear bias parameter $b_1$ term; the dotted curve and dot-dash curves give the 1-Loop contributions from the non-linear bias parameters $b_2$ and $b_3$, and the thickness (color) of the lines indicates their sign, with thick (red) lines being positive and thin (blue) lines being negative contributions. The thin solid (green) line gives the smoothed, linearly biased, linear power spectrum. The lower plot of each panel shows the ratio of the total cross spectrum with the smoothed linear theory cross spectrum.
FIG. 7: The real space halo-halo power spectrum at the 1-Loop level on large scales. The four panels show predictions for haloes with masses: $M = \{10^{15}, 10^{14}, 10^{13}, 10^{12}\} \, h^{-1} M_\odot$. In the upper panels of each sub-figure, the thick (black) curve represents the total spectrum as given by equation (52), the dash curve represents the 1-Loop contribution from the pure linear bias ($b_2^2$) term, the dotted and dot-dash curves represent the 1-Loop contributions from the $b_2 b_1$ and $b_3 b_1$ terms, respectively. The triple dot-dash curve represents the quadratic nonlinear bias term $b_2^2$. Lastly, the thin solid (green) line denotes the smoothed, linearly biased linear theory spectrum. As before, line thickness (color) distinguishes positive and negative contributions. Bottom panels show the ratio with the linear spectrum.
tures in the ratios of the power spectra. The strength of the signal appears to depend on halo mass and increases as halo mass decreases. As we discussed in Section III C, this can be attributed to the fact that non-linear evolution suppresses BAOs on small scales, and on taking the ratio with a linear theory spectrum, we are artificially introducing oscillations.

B. Halo-Halo power spectra

In Figure 7 we show the predictions for the scale dependence of $P_{hh}$, at the 1-Loop level, as a function of wavenumber. The four panels show results for haloes with masses in the same range as Figure 6. Again, upper panels show the contributions from each of the Halo-PT terms in equation (52), and where the sign of each contribution is distinguished through line thickness/color. Sub-panels are as before.

Some obvious similarities exist between the auto-halo spectra and the cross spectra. In particular: the large scale bias is not given by $b_1$; the highest mass halo spectrum shows no sign of the pre-virialization power decrement; the non-linear boost occurs increasingly at smaller scales as halo mass decreases; with the exception of the highest mass haloes, the ratio of the 1-Loop spectra to the tree level spectra have BAOs imprinted. These effects may all be understood through the explanations from the previous sub-section.

We also notice some important differences between $P_{hh}$ and $P_{bh}$. Firstly, the addition of the quadratic non-linear bias term, $[b_2(M)]^2$, modifies the results on the largest scales. As was discussed in Section V, $P_{1Loop}^{hh}$ becomes a white-noise power spectrum as $P_{11} \to 0$, unless $b_2 = 0$. In the figure, the contributions from this term are denoted by the triple dot dash lines. On considering all four panels and paying special attention to the ratios, we see that, with the exception of the case $M \sim 10^{14} h^{-1} M_{\odot}$, there is an upturn in power on scales $k < 0.01 h Mpc^{-1}$. For the $M \sim 10^{14} h^{-1} M_{\odot}$ haloes this effect is not found, this owes to the fact that $b_2 \sim 0$ (see Fig 4).

Secondly, we note that on smaller scales the effect of the $b_2^2$ term is to boost the power across all scales. Thus the pre-virialization power decrement is no longer a decrement relative to the linear theory. However, relative to the power measured on say $k \sim 0.01 h Mpc^{-1}$, there is a very real decrement. Moreover, the $k$-dependence of the terms multiplying $b_2^2$ will make the decrement appear larger than we would expect from the non-linear matter spectrum.

C. Non-linear evolution of BAOs

We now briefly consider how mode-mode coupling and non-linear biasing affect the evolution of the BAOs. Figure 8 compares the 1-Loop predictions for $P_{bh}$ with the linear theory predictions. We again consider the set of halo masses: $M = \{10^{15}, 10^{14}, 10^{13}, 10^{12}\} h^{-1} M_{\odot}$, and, to emphasize the evolution of the BAO features, we have taken the ratio of each spectrum with the ‘No Baryon’ model of equation (12). In addition, because we are purely interested in the scale dependence, we re-normalize all ratios to unity at $k \sim 0.01 h Mpc^{-1}$.

The top left panel shows results for the highest mass haloes: all but the first of the BAO troughs has been removed through non-linear evolution. This is a much more aggressive non-linear evolution than one expects from considerations of the dark matter alone. Secondly, the trough appears to have been displaced towards lower frequencies. These effects arise because both the quadratic bias ($b_2^2$) and $b_2 b_1$ terms are positive, and so dominate over the negative $b_3$ and $P^{13}$ terms. This fact, coupled with the injection of power from $P_{22}$, means that the non-linear boost occurs at $k \sim 0.05 h Mpc^{-1}$. Thus, the overall effect on the halo spectrum is to shift the first trough to smaller $k$.

All the spectra of the lower mass haloes display a pre-virialization feature, the strength of which increases as halo mass decreases. In addition, the number of peaks and troughs which remain in the evolved spectra increases as halo mass decreases, because the strong non-linear amplification from the $P_{22}$ term is delayed by the negative $b_2 b_1$ corrections. However, in contrast to the high mass haloes, for the low mass haloes the first acoustic trough is shifted towards larger $k$ because, when $0.01 < k < 0.05$, then the negative correction from the $b_2 b_1$ term is dominant and so subtracts power from the higher frequency side of the oscillation. This acts to shift the overall pattern to higher frequencies.

VII. COMPARISON WITH SIMULATIONS

In this Section, we compare the predictions from the theoretical model with the results from the numerical simulations presented in Section III.

A. Halo center power spectra re-visited

Returning to our analysis of Figure 11. We are now in a position to comment on the success of the Halo-PT model in comparison to the numerical simulations. The analytic model was evaluated using the semi-empirical bias parameters that were determined as described in Appendix B. Since here we are concerned purely with the scale-dependence of the spectra and not their overall amplitude, we allowed the very large scale normalizations to be considered as free parameters. These were fit-for-in exactly the same way as was done for the linear theory models in equation (13). Whilst this re-scaling would not be necessary if the bias parameters that we were adopting were precise and accurate, since we have not been able to establish this in a robust manner we feel that this approach is acceptable, given that it has been equally ap-
FIG. 8: Non-Linear evolution of the BAOs as traced by the halo-halo power spectrum for four different halo masses. The solid (red) line denotes the the non-linear (1-Loop) spectrum as given by equation (52); the dash (blue) line denotes the linear theory (tree level) power spectrum. We have taken the ratio of each spectrum with a smooth ‘No Baryon’ model as described in the text, and have renormalized each spectrum so that they are unity on a scale $k \sim 0.01 h^{-1} \text{Mpc}$.

These predictions are shown as the solid lines in Figure 1. Comparison with the measured $P_{\delta h}^{c}$, shows that the model and the shot-noise corrected data show reasonable correspondence over scales $k < 0.07 h^{-1} \text{Mpc}$. On smaller scales the match is poor. However, the uncertainty in the shot-noise correction makes it very difficult to draw firm conclusions.

Considering $P_{\delta h}$, we find that the analytic model is in good agreement with the simulations over a much larger range of scales. The model correctly captures the mass dependence of the pre-virialization feature – low mass haloes show greater loss of memory to the initial density fluctuations. The shifting of the nonlinear boost to smaller scales as halo mass decreases is also matched rather well.

To assess whether $P_{\delta h}^{\text{c, Loop}}$ is a better fit to the simulation data than is linear theory, we have performed a likelihood ratio test, assuming that the likelihood functions are Gaussian. In this case, a necessary statistic for model selection is for

$$LR \equiv \frac{\mathcal{L}(\{x_i, y_i\} | [P_{\text{Loop}}^{ij}]_{\text{max}})}{\mathcal{L}(\{x_i, y_i\} | [P_{\text{Lin}}^{ij}]_{\text{max}})} > 1 .$$

where the subscript ‘max’ refers to the parameter choices in the models that maximize the likelihood. Restricting the information to be $k < 0.1 h^{-1} \text{Mpc}^{-1}$, we find, going from low to high mass haloes, that $LR = \{1.06, 1.03, 1.05, 1.07\}$, respectively.

B. Halo bias re-visited

We now examine how well the Halo-PT model does at matching the nonlinear scale dependent bias of the halo centers as measured in the numerical simulations (Figure 2). The top section of each panel shows that the analytic model (solid blue and red lines correspond to $b_{\text{c, NL}}^{\delta h}$ and $b_{\text{c, NL}}^{\delta h}$, respectively) captures, qualitatively, the scale dependence of the bias. The model shows a bias that increases with $k$ for the high mass haloes but decreases with $k$ for lower masses. However, there are some notable discrepancies: The model under predicts and then overshoots the measured relationship for the most massive haloes; for the next bin in halo mass (top-right panel), the measured bias is flat, whereas the model predicts a down turn after $k \sim 0.1 h^{-1} \text{Mpc}^{-1}$; the model fares better in the two lowest mass bins, but the down-turn at high $k$ is not seen in the data. However, we must stress that, with the exception of the $M \sim 10^{14} h^{-1} M_{\odot}$ haloes, the model out-performs linear theory, which would predict constant bias on all scales.

Having extolled the virtues of our model we now draw attention to its short-comings. Whilst the predictions provide a good match to the halo power spectra, they
do not simultaneously provide a good match to the scale dependence of the bias. If we re-consider our measurements of the non-linear matter power spectrum (upper sections in Fig 1) $P_{\delta \delta}$, we see that the 1-Loop model (dot-dash lines) over predicts the LR simulations on scales $k > 0.05h$Mpc$^{-1}$ and the HR simulations on scales $k > 0.07h$Mpc$^{-1}$. It is therefore unlikely that the model as presented here can be made to work precisely.

**VIII. GALAXY POWER SPECTRUM**

How does the scale dependence of the bias depend on galaxy type? The answer has important consequences for future galaxy surveys that will measure the clustering of specific sub-classes of objects. We may address this question using the halo model by changing the mass weighting in the integrals to:

$$
n(M)M \rightarrow n(M) \langle N|M \rangle ;
$$

$$
n(M)M^2 \rightarrow n(M) \langle N(N-1)|M \rangle , \quad (66)
$$

where $\langle N|M \rangle$ and $\langle N(N-1)|M \rangle$ are the first two factorial moments of the halo occupation probability function $P(N|M)$, which gives the probability for a halo of mass $M$ to host $N$ galaxies. Use of the factorial moments of $P(N|M)$ subtracts off a term corresponding to the self-correlation of galaxies e.g. $\xi(r \rightarrow 0) = \delta^D(r)/\bar{n}$. This corresponds to the Poisson shot noise term in Fourier space. Secondly, the mean density profile of dark matter is changed to the mean density profile of galaxies in the halo: $U(k|M) \rightarrow U^g(k|M)$. Following the discussion in Section [V.C](#V.C) it is a very good approximation to set $U^g(k|M) = 1$ when $k r_{\text{vir}} \ll 1$. Thirdly, the constant pre-factors transform as $1/\bar{n}^2 \rightarrow 1/\bar{n}_g^2$, where

$$
\bar{n}_g = \int dM n(M) \langle N|M \rangle \Theta(M - M_{\text{cut}}) . \quad (67)
$$

And finally, the galaxy bias parameters are

$$
b_i^g = \frac{1}{\bar{n}_g} \int dM n(M) \langle N(M) \rangle b_i(M) . \quad (68)
$$

These changes in equations (61) and (62) yield the 1-Loop halo-model prediction for the galaxy power spectrum. Explicitly:

$$
\frac{P_{2H}^g(k|R)}{|W(kR)|^2} = \frac{1}{\bar{n}^2} \int dM n(M) \langle N(N-1)|M \rangle \times \Theta(M - M_{\text{cut}}) ; \quad (69)
$$

$$
\frac{P_{2H}^g(k|R)}{|W(kR)|^2} = \frac{1}{\bar{n}^2} \int \prod_{i=1}^2 \left\{ dM_i n(M_i) \langle N|M_i \rangle \times \Theta(M_i - M_{\text{cut}}) \right\} \times \frac{P_{\text{c,1Loop}}(k|M_1, M_2, R)}{|W(kR)|^2} , \quad (70)
$$

where again we have explicitly included a filter function on the 1-Halo term. On inserting our expression for $P_{\text{c,1Loop}}$ from equation (62) into the 2-Halo term, and again noticing that in the large scale limit the mass integrals may be performed directly, we find:

$$
\frac{P_{2H}^g(k|R)}{|W(kR)|^2} = \big\{ b_0^g \big\}^2 P_{1\text{Loop}}(k) + b_1^g b_0^g \sigma^2(R) P_{11}(k) \big\} + \frac{\big\{ b_0^g \big\}^2}{2} \int \frac{d^3q}{(2\pi)^3} P_{11}(q) |W(qR)|^2 |W(|k-q|R)|^2 \{ P_{11}(|k-q|) F_2(q, k-q) + 2 P_{11}(k) F_2(k, -q) \} , \quad (71)
$$

The resulting expression for the galaxy power spectrum is identical to that given by [24 83], with one important difference—we include the large scale constant power originating from $P_{2H}^g(k|R)$. To study the expected differences between red and blue galaxies, we use the parametric forms for $\langle N(M) \rangle$ measured by [83] in the semi-analytic models of [86]:

$$
\langle N_B|M \rangle = 0.7 \left( \frac{M}{M_B} \right)^{\alpha_B} ; \quad \langle N_R|M \rangle = \left( \frac{M}{M_R} \right)^{\alpha_R} . \quad (72)
$$

The blue galaxy parameters are: $M_B = 4 \times 10^{12}h^{-1}M_\odot$ for haloes with masses in the range $10^{11} \leq M/h^{-1}M_\odot \lesssim 4.0 \times 10^{12}$ then $\alpha_B = 0.0$, for larger mass haloes $\alpha_B = 0.8$. The red galaxy parameters are: $M_R = 2.5 \times 10^{12}h^{-1}M_\odot$; for haloes with masses greater than the cut-off mass $\alpha_R = 0.9$. For the second moment of the HOD we follow the model of Kravtsov et al. [87], so that $\langle N(N-1)|M \rangle = \langle N|M \rangle^2 - 1$. This makes $P(N|M)$ sub-Poissonian as suggested from the observations [27, 67, 74] and the semi-analytic models [15, 26, 85, 88, 89]; and secondly, this choice allows the first moment alone to fully specify the hierarchy of moments.

In practice, we use the models above but impose a lower mass cutoff of $M_{\text{min}} = 10^{12}h^{-1}M_\odot$. This yields $\bar{n}_B = 4.10 \times 10^{-3}h^3$Mpc$^{-3}$, $\bar{n}_R = 7.93 \times 10^{-3}h^3$Mpc$^{-3}$, $\{ b_1^B = 1.20, b_2^B = -0.14, b_3^B = 1.06 \}$ and $\{ b_1^R = 1.39, b_2^R = 0.09, b_3^R = 0.89 \}$. This estimate of $b_2^B$ agrees with the observational determinations of simi-
FIG. 9: Scale dependence of the power spectrum of blue and red galaxies (thick solid blue and red lines) on very large scales. Top left panel shows the results obtained from the evaluation of equations (69) and (71). The 1-Halo term is represented by the thin solid lines. For the 2-Halo term we have the following contributions: $b_2^1$ terms denoted by dash lines; $b_2^2$ terms denoted by dotted lines; $b_3^1$ terms denoted by dot-dash lines; $b_2^2$ terms denoted by triple-dot-dash lines. Note again for the 2-Halo contributions, line thickness dictates sign. The middle panel shows the ratio of the red and blue galaxy 2-Halo terms (thick red and thin blue lines, respectively) with the no-Baryon linear model of equation (12). The dash line shows the linear theory model with BAOs. The bottom panel shows the same as the middle only this time we show the effect of including the 1-Halo term. The right hand panel is similar to the left, only we exchange $P_{\text{1Loop}}$ in the 2-Halo term for the halofit model [54].

lar galaxies from the PSCz survey [32].) These values are easily understood by noting how $b_i(M)$ depends on halo mass (e.g. Figure 4), the weightings given in equation (72), and recalling that the halo mass function declines exponentially at $M > 10^{13.9}h^{-1}M_\odot$. Notice that the red galaxy bias parameters are all positive, whereas $b_2^2$ for the blue galaxies is negative. Therefore, while we expect to see a pre-virialization feature in $P_B(k)$, we do not for the red galaxies.

The left panel of Figure 9 shows the power spectrum of blue and red galaxies evaluated using the 1-Loop Halo model. The top section shows the individual contributions from all the linear and nonlinear terms. Note that, for the blue galaxies the non-linear correction terms $b_2^1b_2^2$ (blue dotted lines) and $b_1^2b_3^2$ (blue dot-dash lines) are roughly the same order of magnitude as the 1-Halo term (thin blue line). However, for the red galaxies, the 1-Halo term (thin red line) dominates over the non-linear bias corrections by factors of a few. For both populations the quadratic bias terms $[b_2^2]^2$ (triple-dot-dash lines) appear to be negligible.

The middle section of the left hand panel shows the ratios of the 1-Loop $P_{\text{2H}}$ with the No-Baryon linear model (from equation 12). Again, since we are primarily interested in the scale dependence of the bias, we re-normalize all curves to be unity at $k \sim 0.01h\text{Mpc}^{-1}$. As expected, the red galaxy power spectrum appears to trace the linear theory matter fluctuations (green dash line) very well when $k < 0.07h\text{Mpc}^{-1}$. The non-linear boost breaks this accordance at larger $k$. However, for the blue galaxies, the scale dependence is more complicated, having an increased pre-virialization feature and a delayed nonlinear boost. We also note that, for the red galaxies, the second and third BAOs have been almost completely suppressed, whereas only the third peak has been removed for the blue galaxies. The peaks and troughs, however,
appear to be in the right places.

So far, we have neglected the contribution from the constant power 1-Halo term. In the bottom section of the panel we now take this into account and show $P_{1h}^{gg} + P_{2h}^{gg}$ ratioed to the No-Baryon model. For the red galaxies, the agreement between the predictions and the linear theory that was noticed before is now broken on much larger scales, $k \sim 0.04h\text{Mpc}^{-1}$. The trough of the first BAO has been shifted slightly to smaller $k$ and the non-linear boost occurs at a larger scale. Because the 1-Halo term is about 5 times smaller for the blue galaxies, the modifications are not as severe. The addition of this term offsets the suppression of power caused by the negative $b_1^B b_2^B$ term and the blue galaxies now appear to trace the linear theory on scales $k < 0.07h\text{Mpc}^{-1}$ quite well. At larger $k$ the linear spectrum is a poor match to the predictions. We also note that the BAOs are further suppressed and the second trough has been shifted to lower frequencies.

Because $P_{1\text{Loop}}$ does not provide a very accurate model for the true non-linear power spectrum, we have studied the effect of exchanging $P_{1\text{Loop}}$ for the halofit [5] power spectrum. The results are shown in the right hand panel of Figure 9. Although the predictions are qualitatively very similar, halofit predicts enhanced pre-virialization and smaller non-linear boosts. In the middle panel, where the 1-Halo term is not included, we see that the red and blue galaxies do not match the linear theory as well on large scales. In particular the blue galaxy power is suppressed on all scales except the largest. We also see that the BAOs have been better preserved. Although there are slight shifts in the positions of the second trough and third peak. However, the bottom panel shows that once the 1-Halo term has been included, the red galaxy predictions are almost as before. The blue galaxies still show a reasonably strong pre-virialization feature, but, because of the weak nonlinear boost, they match the linear theory rather well over nearly all the scales considered.

We note that this modification is not entirely self-consistent and as such is not meant to be blindly trusted, since the non-linear bias terms are still derived from the 1-Loop Halo-PT. However, we use this operation to highlight that a more advanced understanding of the nonlinear power spectrum does change the results quantitatively. This implies that a more advanced model of the scale dependence of the bias will also further modify and improve the predictions.

 IX. SCALE DEPENDENT BIAS AT $z > 0$

We have shown that the BAO harmonic series in the power spectrum at ($z = 0$) is affected by nonlinear effects from bias and gravitational evolution. Although naively one might expect that at higher redshift nonlinear effects are less important, this is not necessarily so. First, one must pick a criterion for how to compare things at different redshifts. A natural choice is to use objects of the same number density. In this case, as it is well-known, at higher redshifts objects of the same number density are more biased, leading to stronger nonlinear bias effects, even though the dark matter has less nonlinear evolution. Therefore, overall it is not clear a priori whether the situation improves or not. Figure 10 shows the halo power spectra at $z = 1$ and $z = 2$ measured from our 8 LR simulations, where the halo samples were harvested so that they would have the same fixed comoving number density as the Bin 1 sample at $z = 0$ (see from Table II). The power spectrum analysis was identical to that as described in Section III. This figure clearly demonstrates that the non-linear bias effects that are present at $z = 0$ (Fig. 1), remain present in the high redshift halo samples.

In light of this, we anticipate that low mass halo samples at higher redshift, constructed so that $M < M^* (z > 0)$, will likewise show enhanced pre-virialization ($M^* (z)$ is defined to be the halo mass at which $\sigma (M) = 1$). We reserve further details of this issue for future work.

 X. CONCLUSIONS

In this paper we have explored in detail, through both numerical and analytic means, the scale dependence of the nonlinear dark matter, halo and galaxy power spectra on very large scales $k < 0.15h\text{Mpc}^{-1}$. For our numerical work we used and ensemble of 20 simulations in boxes of side $512h^{-1}\text{Mpc}$ and 8 simulations in boxes of side $\sim 1h^{-1}\text{Gpc}$. Each simulation contained more than 134 million particles. We have found that:

- The non-linear matter power spectrum is suppressed relative to the linear theory by $5\sim 10\%$ on scales between $0.05 < k/(h\text{Mpc}^{-1}) < 0.075$ at the 2-$\sigma$ level.

- The halo-dark matter cross power spectrum shows clearly that the bias of halo centers is non-linear on very large scales. The form of the non-linearity depends strongly on halo mass: for high mass haloes no pre-virialization suppression is seen; whereas there is an apparent $\sim 10\%$ suppression of power relative to linear theory for lower mass haloes.

- To make robust statements concerning their large scale clustering of haloes, it is essential to characterize the shot noise correction to high precision. For high mass haloes this correction is sub-Poissonian, so the simple and widely used $1/\bar{n}$ model must be inappropriate. Halo exclusion effects lead to a plausible explanation for this phenomenon, which we used to motivate an alternate correction. However further work is required to establish this robustly. In addition the true answer may need to take into account the way in which haloes are identified in simulations.

- The large-scale bias of $P_{1h}$ is not expected to be the same as that of $P_{2h}$ due to nonlinear deter-
FIG. 10: Dark matter, halo–halo and halo–dark matter cross power spectra at $z = 1$ (left panel) and $z = 2$ (right panel) as a function of wavenumber. Symbol styles are as in Fig 1. At both epochs we computed the halo spectra using the same fixed number density of haloes as those in Bin 1 sample from $z = 0$: $\bar{n} = 3.43 \times 10^{-5}[h/{\text{Mpc}}]^3$. Non-linear mass and bias corrections remain present at the level of 5-10% in the halo spectra.

ministic bias; this complicates studies of stochastic bias. The difficulty of performing the halo-halo shot noise correction means we are unable to make a strong statement about the non-equivalence of $b_{\delta h}$ and $b_{hh}$.

- As wavenumber increases, low mass haloes are increasingly anti-biased and high mass haloes are increasingly positively biased. Therefore, the non-linear scale dependence of halo bias is not simply due to the nonlinear evolution of the matter fluctuations.

- Baryon acoustic oscillation features in the power spectrum are erased on progressively larger scales as halo mass is increased. In addition, small shifts in the positions of the higher order peaks and troughs occur which depend on halo mass.

In the second half of this paper we developed a ‘physical model’ to explain and reproduce these results. The model was constructed within the framework of the halo model and we focused our attention on the clustering of the halo centers. The halo-halo clustering term was carefully propagated into the non-linear regime using ‘1-loop’ perturbation theory and a non-linear halo bias model.

Our model can be summarized as follows: The density field of haloes was assumed to be a function of the local dark matter density field. Under the condition of small fluctuations, it was then expanded as a Taylor series in the dark matter density with non-linear bias coefficients $b_i(M)$ \cite{12}. The density field was then evolved under 3rd order Eulerian perturbation theory and this provided the 3rd order Eulerian perturbed halo density field. We then used the model to derive the halo-halo and halo-dark matter cross power spectra up to the 1-Loop level. This lead to the following conclusions:

- For $b_i$ non-vanishing, the effective bias on very, very large scales for the halo spectra is not simply $b_1$, but also depends on $b_2$, $b_3$ and the variance of fluctuations on scale $R$. The halo center power spectrum contains a term that corresponds to constant power on very large scales. This implies that as $k \rightarrow 0$, halo bias should diverge as $[P_{Lin}]^{-1/2}$. The halo-dark matter cross power spectrum does not exhibit this behavior. The predicted bias from this statistic approaches a constant value on large scales.

- When evaluated for a realistic cosmological model, with non-linear bias parameters taken from the Sheth-Tormen mass function, the theory is in broad
agreement with numerical simulations for a wide range of halo masses.

- The non-linear evolution of the BAOs was also examined. The model shows that non-linear bias and non-linear mode-mode coupling increasingly damp the BAOs as halo mass is increased. In addition, the positions of the peaks and troughs can be shifted by small amounts which depend on halo mass.

Using the ensemble of simulations we constructed scatter plots of the halo versus dark matter over densities, contained in top-hat spheres of size $R$ (see Appendix B). From these, it was shown that for filter scales $R < 60h^{-1}\text{Mpc}$ the bias was indeed non-linear, and that, whilst the scatter increased as $R$ decreased, the mean of the relationship did not change until $R < 20h^{-1}\text{Mpc}$. We also examined whether the nonlinear bias parameters derived from the Sheth & Tormen mass function \cite{14, 15} provided a reasonable match to the empirical halo bias.

Using semi-empirical bias parameters as inputs for the analytic model it was shown that the model well reproduced the scale dependence of the halo-dark matter cross power spectra in the simulations, and for all bins in halo mass. However, it was only qualitatively able to reproduce the scale dependence of the non-linear halo bias.

The 1-Loop halo center power spectrum was then inserted into the halo model framework and defined the '1-Loop Halo Model'. This was used to predict the scale dependence of blue and red galaxy power spectra. Plausible models for the Blue and Red Galaxy HODs were used and the results showed complicated scale dependence.

Significant work still remains to be performed for this analytic approach to be sharpened into a tool for precision cosmology. Some possible improvements are: exchanging the 1-Loop matter power spectrum for an accurate analytic fitting formula i.e. after the fashion of halofit, but designed purely for large scales; an application of the new re-normalized perturbation theory techniques \cite{11, 42, 90} coupled with the non-linear bias model should certainly produce better results.

The analytic and numerical results that we have developed here are concerned purely with the clustering in real space. In a subsequent paper we shall extend our analysis to explore the more realistic situation of the scale dependence of dark matter, halo and galaxy power spectra in redshift space.

When this paper was nearing completion, a preprint appeared \cite{91} with similar calculations of how nonlinear bias changes the power spectrum.

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1. Poisson model

It is necessary to correct the measured dark matter power spectra for shot-noise errors. These arise through approximating the continuous CDM fluid by a point process. If the discretization of the density field obeys a Poisson process, i.e., dark matter particles are placed with probability $\propto \bar{\rho}(1 + \delta(x))\delta V$, then the 'discreteness' correction is $\delta$:

$$P_{\text{true}}(k) = P_{\text{obs}}(k) - P_{\text{shot}}; \quad P_{\text{shot}} = 1/\bar{n}.$$  \hspace{1cm} (73)

Since the number of particles in our simulations is large, 512$^3$, this correction, on the scales of interest, is insignificant, e.g. $P_{\text{shot}}/P(0.1h\text{Mpc}^{-1}) \sim 10^{-3}$. However, one must also correct $P^{hh}$ for discreteness. If the dark matter haloes are also regarded as a Poisson smoothing of the smoothed halo density field, then the correction will be the same but using the appropriate number density $\bar{n}_{h}$. In Figure I we show the effect of this standard correction on our halo power spectra and we plot the ensemble of the shot-noise corrected halo power. The negative power values that result at high $k$ demonstrate that this model must not be exactly correct and therefore for accurate measurements it must be modified in some way. We now discuss a possible explanation for this and propose a new shot correction.
APPENDIX A: HALO DISCREteness CORRECTIONS

1. Halo exclusion effects

As we have seen Poisson sampling must not be exact for haloes, particularly those of large mass. A possible reason is that by using a friends of friends algorithm, one is automatically imposing that haloes are never separated by distances smaller than about the sum of their radii, else they would have been linked as a bigger halo. As a result of this exclusion effect, the two-point correlation function of haloes drops dramatically from a value $\xi_h \gg 1$ at $r \sim r_e$ to $\xi_h = -1$ for $r < r_e$, where $r_e$ is the exclusion radius. This sharp drop of $\xi_h$ has an impact on the large-scale power spectrum, as we now show.

Absent exclusion, the power spectrum would have been

$$P_h(k) = \int_0^\infty \frac{\sin(kr)}{kr} \xi_h^4 \pi r^2 dr.$$  \hspace{1cm} (A1)

However, exclusion effects mean that

$$P_h(k) = \int_{r_e}^\infty \frac{\sin(kr)}{kr} \xi_h^4 \pi r^2 dr - v_{TH}(k),$$  \hspace{1cm} (A2)
where \( v_{TH}(k) = (4\pi/3)r_e^3 W_{TH}(kr_e) \) with \( W_{TH} \) the Fourier transform of a top-hat window in real space. Hence, the difference in power due to exclusion is

\[
\Delta P^h(k) = \int_0^r \frac{\sin(kr)}{kr} (1 + \xi^h) 4\pi r^2 \, dr.
\]

To estimate this, we must model \( \xi^h \) at scales smaller than the exclusion imposed by the friends of friends definition of a halo. We do so simply by approximating \( \xi^h \) by a power-law \( \xi^h = \xi_e(r/r_e)^{-\gamma} \) obtained from fitting to the \( r \geq r_e \) measurements, where \( r_e \) is the exclusion scale. In this approximation

\[
\Delta P^h(k) = v_{TH}(k) + v(k) \xi_e,
\]

where, in the large-scale approximation \( kr_e \ll 1 \),

\[
v(k) \simeq \frac{4\pi r_e^3}{3 - \gamma} - \frac{2\pi r_e^3}{3(5 - \gamma)} (kr_e)^2.
\]

Setting \( \gamma = 0 \) gives \( v_{TH} \) in the same limit. Thus, we see that exclusion makes the halo power spectrum smaller by the amount given by Eq. (A3). In contrast, Poisson shot noise makes the power larger by \( 1/n_h \). If Poisson noise is subtracted from halo power spectra in simulations, then the result becomes negative at high \( k \). So it may be that adding back the power lost to exclusion will make the power positive again. This is our procedure. Note that the noise in Eq. (A3) is not white.

As a final note we reemphasize that all of these troubling issues may, to a certain extent, be neatly side stepped, if we measure the halo-dark matter cross power spectrum. The clear advantage of this approach is that the 'shot-noise' is dramatically reduced (there are many more particles than haloes) and exclusion no longer plays a role.

**APPENDIX B: MEASURING THE NON-LINEAR BIAS PARAMETERS**

The accuracy of 1-Loop Halo PT depends on the accuracy of the halo bias parameters \( b_1, b_2 \) and \( b_3 \). Figure 4 shows these parameters derived from the conditional Sheth & Tormen mass function [15]. There are a number of ways to test the accuracy of the bias parameters. The most direct is to smooth the halo and mass fields with a filter of scale \( R \), and to then plot \( \delta^h(x|R) \) versus \( \delta(x|R) \) in the smoothed fields (e.g. [78, 92]). From the resulting scatter plot, one then directly fits for the parameters \( b_i \) up to some order, subject to the constraint that \( \langle \delta^h \rangle = \langle \delta \rangle = 0 \).

Figure 12 shows such scatter plots for the four bins in halo mass described earlier (Table I). In all cases, unlike elsewhere in the paper, the fields were filtered with a ‘top-hat’ window function. Solid blue, red, green and magenta lines show results for four filter scales \( R_{TH} = \{20, 30, 40, 50\} h^{-1}\)Mpc. The upper and lower dotted lines show the \( 1 - \sigma \) errors about the mean relation (solid lines). (Smoothing with a Gaussian filter instead does not significantly change the mean, but does slightly change the scatter around the mean relation.) The figure shows that, above a certain scale, \( \langle \delta^h \delta \rangle \) defines a universal curve which is independent of \( R \); it is this curve which the bias coefficients are supposed to describe. The figure also shows that the scatter around this mean curve decreases as \( R \) increases. The assumption that bias is deterministic is equivalent to assuming that this scatter, this stochasticity, is negligible. Clearly, this is a reasonable approximation only for large smoothing scales \( R \), and \( R \) must be bigger for the rarer, more massive haloes before the bias can be called deterministic.

The main problem with this approach is that, when \( R \) is large, then the distribution of \( \delta \) becomes sharply peaked about its mean value (zero!), so estimating the higher order bias coefficients becomes difficult—only the linear bias term can be reliably measured. In effect, determination of the bias coefficients corresponds to fitting a polynomial to these mean curves, and the best-fitting coefficients will be correlated. The results of this exercise are tabulated in Table I which also shows the values of \( b_1 \) predicted from the Sheth-Tormen mass function. Some of the discrepancy is a consequence of the fact that the Sheth-Tormen mass function under-predicts the abundance of high mass haloes in the HR and LR simulations by up to 20% (see [22] for an explanation as to why this happens). But note that, because of the correlation between the fitted coefficients, the discrepancy with the theory values is difficult to assess. The solid thick (cyan) line in the figure shows the mean bias relation derived from the Sheth-Tormen mass function.

In light of these issues we use the following prescription when generating the theory curves: \( b_1 \) coefficients are to be measured directly from the data as described in Section III B and the parameters \( b_2 \) and \( b_3 \) are to be derived through computing the following integrals over the relevant bin width:

\[
\overline{b}_i(M) = \frac{1}{n(M_1, M_2)} \int_{M_1}^{M_2} dM n(M)b_i(M),
\]

(B1)
FIG. 12: Scatter plot of halo number-over-density vs the over density of dark matter for four bins in halo mass. The fluctuations were measured in top-hat spheres of radii $R_{TH} = \{20, 30, 40, 50\} h^{-1}$Mpc. The means and 1-sigma errors measured from the ensemble are denoted by the solid blue, dash red, dot-dash green and dotted magenta lines, respectively. The solid thick cyan line shows the predictions from the Sheth & Tormen model. The thick triple dot-dash lines show the results from the cubic fit in $\delta$ as described in the text.

where $n(M)$ and $b_i(M)$ are the Sheth & Tormen mass function and bias parameters. We shall refer to this approach as the semi-empirical method. It shall be left to our future work to provide a more self-consistent solution to this problem.

APPENDIX C: EULERIAN PT

1. Real space kernels

The growth of density inhomogeneities in an expanding universe may be explored in the single stream approximation using the Eulerian fluid equations. In Fourier space the equations governing the evolution of the fluctuations to the
density \((\delta(k, t))\) and divergence of the peculiar velocity field \((\theta(k, t) \equiv \nabla \cdot \mathbf{v})\) are written \[^{10}\]:

\[
\frac{\partial \delta(k, t)}{\partial t} + \theta(k, t) = - \int \frac{d^3q_1 d^3q_2}{(2\pi)^3} [\delta^D(k)]_2 \alpha(q_1, q_2) \theta(q_1, t) \delta(q_2, t) \tag{C1}
\]

\[
\frac{\partial \theta(k, t)}{\partial t} + H(a) \theta(k, t) + \frac{3}{2} \Omega_m H^2(t) \delta(k, t) = - \int \frac{d^3q_1 d^3q_2}{(2\pi)^3} [\delta^D(k)]_2 \beta(q_1, q_2) \theta(q_1, t) \theta(q_2, t) \tag{C2}
\]

where

\[
\alpha_{i,j} \equiv \alpha(q_i, q_j) = \frac{k \cdot q_i}{q_i^2} ; \quad \beta_{i,j} \equiv \beta(q_i, q_j) = \frac{k^2 (q_i \cdot q_j)}{2q_i^2 q_j^2} ; \quad k = q_i + q_j .
\]

As shown in \[^{10}\], for the case of the Einstein-de Sitter model the density and velocity divergence fields may be expanded as a perturbation series and the solutions at each order may then be written down in terms of the density and velocity perturbations from all lower orders: e.g.

\[
\delta(k) = \sum_{n=1}^{\infty} D^n_1(a) \delta_n(k) ; \quad \delta_n(k) = \int \frac{\prod_{i=1}^{n} \{d^3k_i \delta_1(k_i)\}}{(2\pi)^{3n-3}} [\delta^D(k)]_n F_n^{(s)}(k_1, \ldots, k_n) \tag{C4}
\]

\[
\theta(k) = H(a) f(a) \sum_{n=1}^{\infty} D^n_1(a) \theta_n(k) ; \quad \theta_n(k) = \int \frac{\prod_{i=1}^{n} \{d^3k_i \delta_1(k_i)\}}{(2\pi)^{3n-3}} [\delta^D(k)]_n G_n^{(s)}(k_1, \ldots, k_n) ,
\]

where \(H(a)\) is the dimensionless Hubble parameter, \(D_1(a) = a\) is the linear growth factor and \(f(a) = d \log D / d \log a \approx \Omega_m^{5/9}\) gives the velocity growth suppression factor. The functions \(F_n^{(s)}\) and \(G_n^{(s)}\) are the PT kernels for \(\delta\) and \(\theta\) symmetrized in all of their arguments, respectively. The notation \([\delta^D(k)]_n = \delta^D(k - q_1 - \cdots - q_n)\) was adopted. The first three symmetrized density kernels are:

\[
F_1^{(s)}(q) = 1 ; \tag{C6}
\]

\[
F_2^{(s)}(q_1, q_2) = \frac{5}{14} [\alpha_{1,2} + \alpha_{2,1}] + \frac{2}{7} \beta_{1,2} ; \tag{C7}
\]

\[
F_3^{(s)}(q_1, q_2, q_3) = \frac{7}{54} \left[ F_1^{(s)} \alpha_{3,12} + F_2^{(s)} \alpha_{1,23} + F_3^{(s)} \alpha_{2,31} + G_1^{(s)} \alpha_{12,3} + G_2^{(s)} \alpha_{23,1} + G_3^{(s)} \alpha_{31,2} \right]
\]

\[
+ \frac{2}{27} \left[ G_1^{(s)} \beta_{12,3} + G_2^{(s)} \beta_{23,1} + G_3^{(s)} \beta_{31,2} \right] .
\]

The first three symmetrized velocity divergence PT kernels are:

\[
G_1^{(s)}(q) = 1 ; \tag{C9}
\]

\[
G_2^{(s)}(q_1, q_2) = \frac{3}{14} [\alpha_{1,2} + \alpha_{2,1}] + \frac{4}{7} \beta_{1,2} ; \tag{C10}
\]

\[
G_3^{(s)}(q_1, q_2, q_3) = \frac{1}{18} \left[ F_1^{(s)} \alpha_{3,12} + F_2^{(s)} \alpha_{1,23} + F_3^{(s)} \alpha_{2,31} + G_1^{(s)} \alpha_{12,3} + G_2^{(s)} \alpha_{23,1} + G_3^{(s)} \alpha_{31,2} \right]
\]

\[
+ \frac{2}{9} \left[ G_1^{(s)} \beta_{12,3} + G_2^{(s)} \beta_{23,1} + G_3^{(s)} \beta_{31,2} \right] .
\]

where we have adopted the compact notation

\[
F_n^{(s)}(q_1, \ldots, q_n) ; \quad G_n^{(s)}(q_1, \ldots, q_n) \tag{C12}
\]

and

\[
\alpha_{i_1 \cdots i_n j_1 \cdots j_m} = \alpha(q_{i_1} + \cdots + q_{i_m}, q_{j_1} + \cdots + q_{j_m}) ; \quad \beta_{i_1 \cdots i_n j_1 \cdots j_m} = \beta(q_{i_1} + \cdots + q_{i_m}, q_{j_1} + \cdots + q_{j_m}) .
\]

Exact analytic solutions for arbitrary cosmological models have not yet been found. However, as was shown in \[^{10}\], under the assumption that \(D_n(a) \propto [D_1(a)]^n\) and \(f(\Omega) \sim \Omega_m^{0.5}\), the solutions are identical to the Einstein-de Sitter solutions, but the growth factors are changed to those for the particular cosmological model in question. All other changes are small corrections \[^{10}\].
2. Evaluation of 1-Loop expressions

Some care is required in the numerical evaluation of the Halo-PT expressions because fine cancellations can occur between negative and positive terms. The approach that we adopt throughout this paper can be demonstrated through the following example. Consider the 1-Loop power spectrum for CDM given by equation (48). The PT corrections $P_{13}$ and $P_{22}$ maybe analytically developed up to the following points:

$$P_{13}(k) = 6P_{11}(k) \int \frac{dq}{(2\pi)^3} q^2 P_{11}(q) \int d\bar{q} \tilde{F}_3(k, \bar{q}, -q)$$

$$= \frac{P_{11}(k)^3}{252(2\pi)^2} \int_0^{\infty} dx x^2 P_{11}(xk) \left\{ -42x^2 + 100 \frac{158}{x^2} + \frac{12}{x} (1 - x^2)^3(7x^2 + 2) \log \frac{x + 1}{|x - 1|} \right\} ; \quad (C14)$$

$$P_{22}(k) = 2 \int \frac{d\bar{q}}{(2\pi)^3} P_{11}(q)P_{11}(|k - \bar{q}|) [F_2(q, k - \bar{q})]^2$$

$$= 2 \int_0^{\infty} \frac{dq}{(2\pi)^2} dq P_{11}(q) \int_{-1}^{1} d\mu P_{11}(k\psi(x, \mu)) \left\{ \frac{5}{7} + \frac{1}{2} \frac{\mu - x}{\psi(x, \mu)} \left[ \frac{x}{\psi(x, \mu)} - \frac{\psi(x, \mu)}{x} \right] + \frac{2}{7} \left[ \frac{\mu - x}{\psi(x, \mu)} \right]^2 \right\}^2 , \quad (C15)$$

where $x = q/k$ and where $\psi^2(x, \mu) = 1 + x^2 - 2x\mu$. If the $P_{13}$ integral is truncated on large and small scales to inhibit infra-red and ultra-violet divergences, as may occur for some power spectra, then identical constraints must also be placed on the $P_{22}(k)$. Explicitly, if we adopt

$$P(q) = 0 \quad \text{for} \quad \left\{ \begin{array}{l} k < k_{\text{fun}} \\
 k > k_{\text{cut}} \end{array} \right.$$ 

then the angular integral for $P_{22}$ must necessarily have the new limits

$$\int_{-1}^{1} d\mu \to \int_{\mu_1}^{\mu_2} : \quad \mu_2 = \min \left[ 1, \sqrt{(k^2 + q^2 - k_{\text{cut}}^2)/2kq} \right] ; \quad \mu_1 = \max \left[ -1, \sqrt{(k^2 + q^2 - k_{\text{fun}}^2)/2kq} \right] . \quad (C16)$$

Lastly, the particular values for the limits $k_{\text{fun}}$ and $k_{\text{cut}}$ were selected so that variance integrals, $\sigma^2(R)$, would be convergent within the finite range.
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[94] Several broad classes of bias model may be defined: local and non-local; linear or non-linear; and deterministic or stochastic. With the exception of stochastic, local linear biasing, all these prescriptions result in some non-trivial degree of scale dependence. Note that when weakly nonlinear scales are discussed, as we do in this paper, it may not be entirely consistent to neglect non-locality, which could potentially alter the predictions we present. Nonlocal bias can be looked for in observations by using higher-order statistics. However, we may take comfort in the fact that there is always a strong correlation between haloes and dark matter, since the former are built from the latter!
[95] The dark matter particles/halo centers were assigned to a regular cubic grid using a fourth-order interpolation scheme, and each point on the grid was given equal weight. The FFT of the grided density field was then computed. Each resulting Fourier mode was corrected for convolution with the grid by dividing by the Fourier
transform of the mass assignment window function. The power spectra on scale $k_l$ are then estimated by performing the following sums,

$$\hat{P}^{ij}(k_l) = \frac{1}{M} \sum_{l=1}^{M} \left| \delta^i(k_l)(\delta^j(k_l))^\ast \right|^2,$$

where $M$ are the number of Fourier modes in a spherical shell in $k$-space of thickness $\Delta k$, and $\ast$ denotes complex conjugation.

[96] The procedure was also performed separately for the no baryon model spectrum, this was done in order for ratios to be taken.