ABSTRACT

The recent failure of the scenario which uses only $10, \overline{126}$ Higgs multiplets to yield large enough neutrino masses in the $SO(10)$ MSGUT motivates an alternative scenario[1] where the $120$ -plet collaborates with the $10$ -plet to fit the dominant charged fermion masses. The small $\overline{126}$ -plet couplings give appreciable contributions only to light charged fermion masses and enhance the Type I seesaw masses to viable values. We analyze the 2-3 generation core of the complete hierarchical fermion mass system in the CP conserving approximation. Ansatz consistency requires $m_b - m_s = m_\tau - m_\mu$ at the GUT scale $M_X$ and predicts near maximal (PMNS) mixing in the leptonic sector for central values of charged fermion parameters and for wide ranges of the other relevant parameters: righthanded neutrino masses and relative strength of contributions of the two doublet pairs from the $120$-plet to the effective MSSM Higgs pair. These features are preserved in the CP preserving 3 generation system whose results are previewed: an additional consistency requirement $\theta_{13}^c = \theta_{12}^c \theta_{23}^c$ on the CKM angles at $M_X$ arises for 3 generations. Right handed neutrino masses in such scenarios are all less than about $10^{12} GeV$.

1 Introduction

Renormalizable supersymmetric $SO(10)$ GUTs[2, 3] have recently[4] been a focus of intense interest both as regards the GUT scale symmetry breaking[4], spectra[5, 6]...
RG evolution\[8\] and matter Higgs Clebsches\[5,8\]. Even more interest arose due to the demonstration of the feasibility\[9\] of the generic Babu-Mohapatra(BM) program\[10\] for a completely realistic fit of all the fermion mass and mixing data using only the 10 and 126-plet Fermion Mass (FM) Higgs. This scenario was initially also considered to be “predictive” in the neutrino sector. Detailed analysis\[11,12,13\] has shown, however, that successful charged fermion fits in fact require (arbitrary) choices of the (many) free phases present in the problem. Thus the successful fits can at most be considered indicative of likely \(MSSM \subset GUT\) embedding angles/phases\[14\] when the leptonic mixing and measured neutrino mass splittings are taken as data. These embedding angles are physical in the sense that they affect\[14\] the baryon violation and Lepton flavour violation rates in the Susy GUT and may therefore eventually be measurable. The renewed interest and excitement in the Babu-Mohapatra scenario began with the demonstration\[9\] that a natural linkage existed –at least regarding magnitude if not exact maximality - between Type II Seesaw\[10\] dominated neutrino masses and the manifest approximate \(b - \tau\) unification at scales \(\sim M_{GUT}\) observed in the MSSM and the near maximal neutrino (PMNS) mixing\[17\] in the 2-3 sector. The nearly successful generalizations of the generic fits to the 3 generation (real) and realistic 3 generation (complex) case\[11\] and the “completely” satisfactory generic fits when a small perturbation of the charged fermion masses by the 120-plet representation was allowed\[12\] seemed to elevate this generic fitting scenario to near canonical status : particularly due to the natural ease of the Bajc Senjanovic Vissani \(b - \tau\)-Type II seesaw- large mixing linkage; even though the near exact maximality of the 2-3 sector leptonic mixing does not have any obvious cause within such scenarios. The early observation\[3,19\] that the Type I and Type II neutrino masses tend to be too small (and therefore\[19\]) may be considered to motivate more complicated GUT Higgs sectors) did not dampen the general enthusiasm. However, nearly simultaneously with the demonstration\[12\] that the Type II weak \(10 \oplus T_{26} \oplus_{\text{weak}} 120\)-plet FM fit was completely satisfactory at least as far as neutrino mixing angles and mass squared splitting ratios are concerned, came the surprising finding\[13\] that Type I and Type I \(\oplus\) Type II fits are -in principle -equally viable. Furthermore we showed, nearly simultaneously\[14\], that Type II seesaw fits were most likely always far sub-dominant to Type I seesaw and Type I itself was too weak in the BM\[10\] scenario, at least in the fully specified and calculable context of the MSGUT\[4,8\], and under an assumption of the genericity of the magnitude of a seesaw matrix derived from a successful Type I seesaw fit available in the literature\[13,1\], even at the special points where the seesaws are weighted strongly\[4,8\]. Furthermore, with improved parametrization of the MSGUT symmetry breaking\[20\] we showed\[1\], by a detailed survey of the parameter space of the MSGUT -assuming the typical values of the \(10 - T_{26}\) parameters available in the literature\[12,13\] were generic - that, not only was Type II seesaw subdominant to Type I Seesaw everywhere in the parameter space, but also Type I seesaw itself could not yield neutrino masses in excess of about \(0.005\)eV. This failure
was traced\textsuperscript{[1]} to the double load borne by the $\bf{126}$ FM Higgs irrep and motivated us to pro-
pose the scenario that the $\bf{10}$-plet and $\bf{120}$-plet shoulder the main burden of the charged fermion fit, while the $\bf{126}$-plet coupling to the matter $\bf{16}$-plet is small so as to enhance its contribution to the Type I Seesaw (and lower that to Type II) without any strong (i.e involving 2-3 generation masses) lower bound on the magnitude of the $\bf{126}$-plet coupling due to the requirements of the charged fermion fit. Although the $\bf{120}$-plet has been considered—somewhat cursorily—previously in the literature \textsuperscript{[21]} and has also recently attracted attention not only as a perturbation to correct “small defects” in the generic Babu-Mohapatra(BM) fits\textsuperscript{[12]} but also as a dominant contributor to the charged fermion fits\textsuperscript{[22]} in non-supersymmetric theories with radiative neutrino masses arising\textsuperscript{[23]} from exchange of $\bf{16}_{\text{H}}$-plet Higgs. Yet the combination of $\bf{10}−\bf{120}−\bf{126}$-plets with the particular—and experimentally well motivated—assignment of roles, in the context of the renormalizable Supersymmetric $\text{SO}(10)$ GUT, that we proposed\textsuperscript{[1]} has not, to our knowledge, been previously considered in the literature.

This is the first paper of a series in which we examine we examine this scenario \textit{ab initio} and show that it enjoys a number of the same virtues manifest in the BM-Type II case \textsuperscript{[9]} and also shown to exist in the BM-Type I case\textsuperscript{[13]} while being free of their overall neutrino mass debility. Moreover it leads to a restrictive type of $b−τ = s−μ$ unification that radically reduces the latitude in choosing $m_b(M_X)$ to the same level as the uncertainty in $m_τ(M_X)$ yet proves not only compatible with $1σ$ limits but also leads to an extremely robust prediction of very near to maximal (sin$^2θ_{12} ≥ .95$) 2-3 generation lepton mixing completely compatible with experiment and unlikely to be modified when first generation masses and CP violation are also accounted for!

We first briefly summarize the essence of the reasons for the failure of the $\bf{10}−\bf{126}$ scenario\textsuperscript{[14]}\textsuperscript{[1]} and give the fermion mass formulae in the case where a $\bf{120}$-plet is also present. We then analyze these formulae in the case where $\bf{10}−\bf{120}$-plet completely dominate the charged fermion sector for the real two ( 2nd and 3d) generation case: where a completely explicit and analytic treatment is possible. This provides a insight into the the dominant core of the fermion Hierarchy and thus a clear paradigm for the analysis of realistic 3 generation case whose complications can be tackled by a perturbative treatment for which the 2−3 sector we analyze is a very stable,robust and non singular core and support which dominates the perturbative equations up to cubic order in the Fermion hierarchy parameter $ε ≃ θ_{12}^2 ≃ \sqrt{θ_{23}} ≃ 0.2$. We formulate the fitting equations in a way adapted to determining their solution by an expansion in this $ε$ parameter. For dominant 2-3 sector neglect of the $\bf{126}$ couplings in the charged fermion sector is shown to yield an acceptable scenario with $b−τ = s−μ$ unification and maximal neutrino mixing and masses for precisely the expected values of the 2-3 generation masses at $M_X$ and a wide and plausible range of the remaining parameters ($M^R_{ν}$ and GUT doublet/MSM doublet Higgs fractions) ! We conclude with an preview of the results of paper II where the 3 generation real case is analyzed.
using the perturbation method based on our results here.

2 Difficulties of Babu-Mohapatra Seesaw in the MSGUT

In [1] we wrote the Type I and Type II seesaw [16] Majorana masses of the light neutrinos as:

\[ M^I_\nu = (1.70 \times 10^{-3} \text{eV}) \ F_I \ \hat{n} \sin \beta \]
\[ M^{II}_\nu = (1.70 \times 10^{-3} \text{eV}) \ F^{II}_I \ \hat{f} \sin \beta \]
\[ \hat{n} = (\hat{h} - 3 \hat{f}) \hat{f}^{-1}(\hat{h} - 3 \hat{f}) \]

where \( v = 174 \text{GeV} \), \( \hat{h}, \hat{f} \) are the Yukawa coupling matrices of \( 10, \overline{126} \) to the 16plets containing fermion families and \( \beta \) is the MSSM Higgs doublet mixing angle. The functions \( F_I, F^{II}_I \) are defined (upto irrelevant phases) as

\[ F_I = 10^{-\Delta x} \frac{\gamma g}{\sqrt{2} \eta \lambda} \frac{|p_2 p_3 p_5|}{z_{16}} \sqrt{\frac{2g}{x(1 + x^2)}} \frac{(1 - 3x) q'_3}{x(1 - 3x)(4x - 1) q_3^2} \]
\[ F^{II}_I = 10^{-\Delta x} \frac{2 \sqrt{2} \gamma g (x - 1)}{\sqrt{\eta \lambda}} \frac{p_2 p_3 p_5}{z_{16}} \frac{x^2 + 1}{x(1 - 3x)} \frac{4x - 1}{q'_3 q_2 p_5} \]

(1)

Here \( \gamma, \lambda, \eta, g \) are couplings \( \sim 1 \) while the \( p_i, q_i, z_i \) are certain polynomials [4, 20, 1] in the variable \( x \) which is the only “fast” parameter controlling the GUT scale symmetry breaking in the MSGUT [3, 6]. This MSGUT is based on the \( 210 - 10 - 126 - 126 \) Higgs system and is tuned to keep one pair of Electroweak doublets light in the MSSM that emerges as the low energy effective theory. The modification \( \Delta_X = \text{Log}_{10}(M_X/\text{GeV}) - 16.25 \) of the one loop unification scale due to GUT scale threshold corrections is constrained by the proton lifetime to be greater than \(-1\).

The ‘proof’ of [14, 1] proceeds by first observing that in typical BM-Type II fits [11, 12, 13] the maximal value of \( \hat{f} \) eigenvalues is \( \sim 10^{-2} \) while the corresponding values for \( \hat{h} \) are about \( 10^2 \) times larger. As a result \( \hat{n} \sim 10^2 \hat{f} \). This implies that \( R = F_I / F^{II}_I \leq 10^{-3} \) for the pure BM-Type II not to be overwhelmed by the BM-Type I values it implies. Furthermore in the BM-Type I fit of [13] the requirement of large neutrino mixing yields, typically, \( \hat{n}_{\text{max}} \sim 5 \hat{f}_{\text{max}} \sim .3 \) for the maximal eigenvalues of \( \hat{n} \). Thus \( \hat{n}_{\text{max}} \) is much smaller than the naive estimate \( (h_{33} - f_{33})^2 / f_{33} \sim 10^1 \) so that magnitudes \( F_I > 10^2 \) are required for neutrino masses \( \sim .05 \text{eV} \). This deserves further
examination on the basis of either a catalog of Type I fits of the type found in [13] or an analytic perturbative treatment of the fitting problem [24, 33]. However, the reason for the lowering of the eigenvalues of $\hat{n}$ below the naive values appears to be the necessity to have roughly equal elements in the 2-3 sector to allow large mixing and so should be robust.

The complete control we have over the MSGUT [4, 5, 8, 6, 7, 32, 20] allows us to demonstrate [14, 1] that both these conditions are not achievable anywhere over the complex $x$ plane except where they also violate some aspect of successful unification (typically the requirement that $\Delta_X > -1$ when combined with the above requirements of the completely determined coefficient functions of the MSGUT ensures the failure to generate adequately large neutrino masses: see [1] for details).

3 2 Generation FM fits in NMSGUTs

The above recounted failure of the Babu-Mohapatra scenario, at least in generic cases of the MSGUT, could provoke one [19] to entertain more complicated scenarios which give more freedom to choose the parameters that enter the neutrino mass formulae. The likelihood of Type II failure, already noticed in [19], there motivated the introduction of additional 54 Higgs as a work-around of the obstruction posed. However, as we already noted in [14], a proper demonstration would then require a complete recalculation, of the various mass matrices and coefficient functions which is—and may remain—unavailable, not to speak of the new essentially ‘fast” parameters introduced by the modified -and considerably more complex GUT scale SSB scenario of such work-arounds. To us, allowing in the arbitrarily excluded 120-plet, particularly since it does not destroy the hard-won solution of the GUT scale SSB problem, is far more palatable and cogent.

If this is done the main effect is to introduce two additional doublet pairs, from the $(1, 2, 2)$ and $(15, 2, 2)$ Pati-Salam submultiplets of the 120-plet. The GUT scale SSB is undisturbed. The Dirac masses in such GUTs are then generically given by [5, 1]

$$
\begin{align*}
    m^u &= v(\hat{h} + \hat{f} + \hat{g}) \\
    m^\nu &= v(\hat{h} - 3\hat{f} + (r_5 - 3)\hat{g}) \equiv v(\hat{h} - 3\hat{f} + r'_5\hat{g}) \\
    m^d &= v(r_1\hat{h} + r_2\hat{f} + r_6\hat{g}) \\
    m^l &= v(r_1\hat{h} - 3r_2\hat{f} + (\bar{r}_5 - 3\bar{r}_6)\hat{g})
\end{align*}
$$

(2)

The form [5, 1] of the new generic coefficients $r_5, \bar{r}_5, \bar{r}_6$ in the particular case of the NMSGUT is of some interest for future work [18]:

$$
\begin{align*}
    r_5 &= \frac{4i\sqrt{3}\alpha_5}{\alpha_6 + i\sqrt{3}\alpha_5} ; \quad \bar{r}_5 = \frac{4i\sqrt{3}\bar{\alpha}_5}{\alpha_6 + i\sqrt{3}\alpha_5} \cot \beta
\end{align*}
$$
\[ \bar{r}_6 = \bar{\alpha}_6 + i\sqrt{3}\bar{\alpha}_5 \cot\beta; \quad \hat{g} = 2ig\sqrt{\frac{2}{3}}(\alpha_6 + i\sqrt{3}\bar{\alpha}_5)\sin\beta \] (3)

here \( g_{AB} \) is the \textbf{120} - \textbf{16} - \textbf{16} coupling, and the \( r_i \) are constants determined in terms of the \( \alpha_i(\bar{\alpha}_i) \) which are the fractions of the Electroweak doublet \( H[1, 2, 1] \) contributed by the two [1,2,1] \{[1.2,-1]\} doublets in the 120-plet. We have omitted similar details for the ‘old’ coefficients \( r_i, \bar{r}_i; i = 1, 2 \). Note that with free \( r_i \) our analysis is \textit{generic} and applies to any \textit{SO}(10) theory with the \textbf{10} - \textbf{120} - \textbf{126} FM Higgs system.

The right handed neutrino mass is \( M_{\nu} = \hat{f}\hat{\sigma} \) and the type I seesaw formula is

\[ M'_{\nu} = vr_4\hat{n}; \quad \hat{n} = (\hat{h} - 3\hat{f})\hat{f}^{-1}(\hat{h} - 3\hat{f}) \] (4)

where \( \hat{\sigma} = -\frac{i\bar{\sigma}\sqrt{3}}{\alpha_2\sin\beta} \) and \( \hat{\sigma} \) is the GUT scale vev of the \textbf{126}.

The essence of our proposal is to neglect the presence of \( \hat{f} \) in the Dirac masses of the 2-3 generations on the basis of an assumption that \( \hat{h}, \hat{g} >> \hat{f} \). Notice that this implies the strengthening of Type I at the expense of Type II Seesaw mechanism. When one analyzes the complete 3 generation case one finds that dominance of \( \hat{h}, \hat{g} \) can only be strong enough to justify the neglect of \( \hat{f} \) to \( O(e^3) \). At \( O(e^4) \) non zero values of \( \hat{f} \) are \textit{required} in order to preserve the consistency of the charged fermion mass fit at and beyond fourth order. Note that such small \( \hat{f} \) values imply that the typical right handed neutrino masses would lie in the range below \( 10^{12} GeV \). Since these masses are also thought to obey the Davidson-Ibarra \[25\] bound \( M_{\nu_r} \) based on reheating constraints in the popular leptogenesis scenario \[26\] for baryon asymmetry generation (which is so natural in theories supporting Seesaw neutrino masses). This may be a useful clue provided by renormalizable MSGUTs towards constraining the parameter space of the leptogenesis scenarios based on the Type I seesaw mechanism. In any case in the present 2-3 generation Real Core analysis we will set \( \hat{f} \) to zero and take all parameters and Unitary matrices to be real. The required generalizations to the 3 generation case are quite easy -although computationally tedious - and are given in the next paper of this series \[24\].

The mass terms above must be matched to the renormalized mass matrices of the MSSM evaluated at the GUT scale. While doing so one must allow \[14\] for the possibility that the fields of the GUT are only unitarily related to those of the MSSM at \( M_X \). This introduces several \( 3 \times 3 \) unitary matrices as well as various phase matrices into the fitting problem. Thus when matching the charged fermion dirac mass matrices (in which we neglect the contribution of the \textbf{126}) we get

\[
\begin{align*}
m^u &= v(h + \hat{g}) = V_u^T D_u Q \\
m^d &= (v_{1h} + v_{6\hat{g}}) V_d^T D_d R \\
m^l &= v(v_{1h} + v_{7\hat{g}}) = V_l^T D_l L \\
r_7 &= \bar{r}_5 - 3\bar{r}_6
\end{align*}
\] (5)
Where \( D_{u,d,l} \) are the charged fermion masses at \( M_X \) and \( V_u, Q, V_d, R = C^\dagger Q, L, V_l \) are arbitrary unitary matrices (\( C \) is the CKM matrix) which are to be fixed by convention – where allowed by conventional ambiguities – or determined by the fitting procedure in terms of the low energy data, or left as parameters to be determined by future experiments sensitive to degrees of freedom and couplings (e.g. baryon violating couplings) that the low energy data is not.

To put our equations in a form transparent enough to clearly separate the contributions of the \( 10, 120 \) (and -eventually- \( 126 \)) plethora\(^{[24]} \) we write the matrices \( V_{u,d,l} \) associated with the anti-fermion fields as new unitary matrices \( \Phi_{u,d,l} \) times the fermion field matrices \( Q, R, L \).

\[
V_d = \Phi_d R ; \quad V_u = \Phi_u Q ; \quad V_l = \Phi_l L \tag{6}
\]

We then separate symmetric and antisymmetric parts for each charged fermion equation

\[
Z = \Phi^T D + D\Phi \quad ; \quad A = \Phi^T D - D\Phi \tag{7}
\]

Then it is easy to solve for the matrices \( \hat{h}, \hat{g} \) using (say) just the d-type quark equations and substitute in the other two equations to obtain the equations in the form

\[
C^*Z_dC^\dagger = r_1 Z_u \quad ; \quad Z_d = DZ_lD^T \quad ; \quad D = R^*L^T \tag{8}
\]

\[
C^*A_dC^\dagger = r_6 A_u \quad ; \quad r_7 A_d = r_6 DA_lD^T \tag{9}
\]

In the 2 generation case the antisymmetric equations\(^{[9]} \) serve only to fix the parameters \( r_6, r_7 \) and thus play no further role. On the other hand since \( C, D \) are orthogonal matrices it follows that \( r_1 = Tr(Z_d)/Tr(Z_u) \) so that we can write these two equations in the (dimensionless) form

\[
\tilde{S}_1 = \frac{CZ_dC^T}{TrZ_d} - \frac{Z_u}{TrZ_u} = 0
\]

\[
\tilde{S}_2 = \frac{Z_d - DZ_lD^T}{TrZ_d} = 0 \tag{10}
\]

The convenience of this form is that the hierarchy within each generation will now structure the problem and render its solution almost trivial in the two generation case and tractable in the 3 generation case. It is amenable to an expansion in the hierarchy parameter \( \epsilon \) which reduces the problem of finding a sufficiently accurate fit to the \( \sim 10^4 \) intra-generational variation in mass to an exercise in going to sufficiently high order in the expansion parameter \( \epsilon \).\(^{[24]} \)
3.1 Analytic Solution of the Charged Fermion Mass Relations

We parametrize the matrices $\Phi_{u,d,l}, C, D$ as

\[
\Phi_{u,d,l} = \begin{pmatrix} \cos \chi_{u,d,l} & \sin \chi_{u,d,l} \\ -\sin \chi_{u,d,l} & \cos \chi_{u,d,l} \end{pmatrix},
\]

\[
C, D = \begin{pmatrix} \cos \chi_{c,D} & \sin \chi_{c,D} \\ -\sin \chi_{c,D} & \cos \chi_{c,D} \end{pmatrix}
\]

and define $\Delta u = u_3 - u_2$, $T_u = u_3 + u_2$, $\Delta d = d_3 - d_2$, $T_d = d_3 + d_2$, $\Delta l = l_3 - l_2$, $T_l = l_3 + l_2$.

Then the 2 independent equations in $\hat{S}_1 = 0$ reduce to

\[
tan \chi_u = \tan(\chi_d - 2\chi_c)
\]

\[
\cos \chi_d = \alpha \cos(\chi_d - 2\chi_c)
\]

\[
\alpha = \frac{(\Delta d)(T_u)}{(T_d)(\Delta u)} = \frac{(d_3 - d_2)(u_3 + u_2)}{(d_3 + d_2)(d_3 - u_2)}
\]

Where $\chi_c$ is the Cabbibo angle in the 2-3 sector i.e in the toy CKM matrix $C$ and $u_{2,3}, d_{2,3}, l_{2,3}$ are the charged fermion masses of the 2nd and 3d generations up to arbitrary signs. Since $\chi_u = \chi_d - 2\chi_c$, and $\chi_d \sim O(\epsilon^2)$, $\chi_{ud} \approx \chi_d$ to order in $\epsilon^2$. The second equation in (13) equation can be solved for $\tan \chi_d$:

\[
\tan \chi_d = \frac{\csc 2\chi_c}{\alpha} - \cot 2\chi_c
\]

Since $\alpha = 1 + (\epsilon^2)$, it is clear that the leading ($\sim \epsilon^{-2}$) contributions cancel leaving behind an $O(1)$ result. Similarly the three equations $\hat{S}_2 = 0$ yield

\[
tan(\chi_l - 2\chi_d) = \tan \chi_d
\]

\[
\cos \chi_l = \frac{T_d}{T_l} \cos \chi_d = \frac{d_3 + d_2}{l_3 + l_2} \cos \chi_d
\]

Thus

\[
\chi_D = (\chi_l - \chi_d)/2 ; \quad \chi_l = \pm \tilde{\chi}_l
\]

\[
\tilde{\chi}_l = \cos^{-1} \frac{T_d \cos \chi_d}{T_l}
\]

To leading order $\chi_l = \pm \chi_d$. However note at at next to leading order the ratio $T_d/T_l$ must not become so large ($\tan \chi_d \sim .9$) as to make $\chi_l$ the inverse cosine of a number bigger than 1.
The third equation yields the all important consistency condition

\[ d_3 - d_2 = \Delta d = \Delta l = l_3 - l_2 \quad (16) \]

i.e

\[ m_b(M_X) - m_\tau(M_X) = m_s(M_X) - m_\mu(M_X) \quad (17) \]

This is one of the main results of this paper. It should be understood that there is a sign/phase ambiguity–arising from the arbitrariness in the unknown Unitary matrices that relate the SGUT and MSSM fields (see above)–for each of the masses, while their magnitudes are determined by the RG flow of the MSSM into the UV.

\section{b - \tau = s - \mu Unification}

Several questions arise regarding the status of this version of \( b - \tau \) unification which is imposed by the consistency of the assumption that the 10, 120 multiplets dominate the heavy fermion mass matrices. Firstly it should be noted that the necessity to the zeroth order (in \( \eta = \epsilon^2 \)) version of this relation i.e \( m_b(M_X) - m_\tau(M_X) = 0 \) was already noticed by Bajc and Senjanovic\cite{22} while studying a technically related model but with a but quite different (non-supersymmetric) scenario. In that model the 10 - 120 are used to fit the 2-3 generation masses and neutrino masses are generated radiatively\cite{23} due to exchanges involving a a SO(10) spinorial Higgs 16\text{-plet}. Thus the motivation, origin and properties of that model are totally distinct from our proposal which is supersymmetric (and thus protected from radiative corrections of its Superpotential), employs no Higgs 16-plet and has a tree level Type I seesaw using the 126 (which also necessarily contributes to the fermion masses at order \( \epsilon^4 \)) in the three (\textit{but not two}) generation case. We have here analyzed completely the two generation case and to order \( \epsilon^6 \) for the 3 generation (CP conserving case) in \cite{24}.

The first question that arises is whether this relation will be preserved when the first generation and then CKM CP violation are also introduced. The explicit analytic solution trivially derivable from the above equations given above coincides, order by order in perturbation theory, \textit{including the} \( b - \tau = s - \mu \) \textit{constraint}, with the solution found \cite{24} by expanding all angles \( \chi \) in powers of \( \epsilon \) and solving the equations \( \dot{S}_1 = 0 = \dot{S}_2 \) order by order in \( \epsilon \) (and truncating all expansions by dropping undetermined coefficients left after solving to desired order). The \( b - \tau = s - \mu \) unification constraint arises at order \( \eta = \epsilon^2 \). In the three generation case the analytic solution is not available but the complete regularity observed in the perturbation theory in \( \epsilon \) and the extreme smallness of the first generation perturbations makes an expansion in \( \epsilon \) well motivated. Then we find that the \( b - \tau = s - \mu \) unification constraint is reproduced at \( O(\epsilon^2) \). One may also expect that this constraint will persist when the CP violation is introduced and a perturbation theory in \( \epsilon \) used to solve the the
fermion mass relations since it is well known that CP violating effects are Wolfenstein
suppressed. In that case, however, the relation will also contain phase uncertainties
generalizing the sign uncertainties present in the real case. Thus \( b - \tau = s - \mu \)
unification may be considered as one of the signature “tunes” of the fermion mass (FM)
fit in the NMSGUT scenario we proposed\[1\] and have begun elaborating on here.

Moreover the pursuit of the perturbation theory to \( O(\epsilon^3) \) gives\[24\] an additional
remarkable constraint:

\[
\theta_{13}^e = \theta_{12}^e \theta_{23}^e \tag{18}
\]

between the CKM rotation angles! No further parameter constraint was found up
to 5th order in \( \epsilon \). However at 4th order we found that the contribution of the \( \mathbf{126} \)
plet becomes necessary to avoid inconsistency in the \( \hat{S}_1, \hat{S}_2 \) fitting equations, which
remain combined in a single equation \( a\hat{S}_1 + b\hat{S}_2 = 0 \) rather than vanishing separately,
while the coupling of the \( \mathbf{126} \)-plet is determined to be \( \sim \hat{S}_2 \sim \epsilon^4 \). Given enough
computer power we see no reason why the perturbation cannot be carried to sixth or
even seventh order in \( \epsilon \); which should be more than sufficient to ensure convergence
to the accuracy to which the lepton masses at \( M_X \) are known: given our uncertainty
regarding the crucial quantities \( M_S, \tan \beta \).

Since \( d_2/l_3, l_2/l_3, \chi_c \sim \epsilon^2 \) and \( u_2/u_3 \sim \epsilon^4 \) it immediately follows that the solution
of eqn(13) to leading order in epsilon is

\[
\tan \chi^0_u = \tan \chi^0_d = \frac{d_2}{l_3 \chi_c} + O(\epsilon^2) \tag{19}
\]

Thus the magnitude of \( \tan \chi^0_d \) is determined to lie (see data values below) in the range

\[
|\tan \chi^0_d| = 0.4974(+.19)(-.1) \tag{20}
\]

Thus the accuracy of the leading order is \( \sim 10\% \), while the experimental uncertainties
are larger. For all practical purposes we may eliminate \( l_3 \) in favour of \( \tan \chi^0_d \) as the
basic parameter and let it vary in the range above with either positive or negative
sign. Note that these values will prove crucial in showing that the leptonic mixing is
near maximal over a wide range of parameters.

4.1 Numerical Values

Even the very precisely known low energy lepton masses become smudged by uncer-
tainties when RG evolved past the unknown thresholds of order \( M_S \) and \( M_R \) with
unknown \( \tan \beta \). Since the effects of CP violation on the \( O(\epsilon^3) \) relation \( \theta_{13}^e = \theta_{12}^e \theta_{23}^e \)
may well be non-negligible, the precise numerical implementation of the constraints
seems premature. Still a survey of the typical fermion mass-mixing data sets is quite
comforting since it exhibits at least approximate compatibility with the constraints
found by us and makes clear the huge parameter space available for satisfying them. Moreover implementation of the necessary \( b - \tau = s - \mu \) constraint implies a radical \((10 - 100 \text{ fold!})\) reduction in the uncertainty allowed in \( m_b(M_X) \) due to its strict correlation with \( m_\tau(M_X) \): which is much more precisely known (see below).

In [27] the following values are given for the two loop renormalized running fermion masses (in units of GeV) at \( \mu = M_X = 2 \times 10^{16}\text{GeV} \) for the representative high \( \tan \beta \) value \( \tan \beta(M_S = 1\text{TeV}) = 55 \)

\[
\begin{align*}
  m_u &= (0.7244^{+0.1219}_{-0.1466}) \times 10^{-3}; \quad m_c = 0.2105049^{+0.0151077}_{-0.0211538}; \quad m_t = 95.1486^{+69.2836}_{-20.659} \\
  m_d &= (1.4967^{+0.4157}_{-0.2278}) \times 10^{-3}; \quad m_s = (29.8135^{+4.1795}_{-4.4967}) \times 10^{-3}; \quad m_b = 1.4167^{+0.4803}_{-0.1944} \\
  m_e &= (0.3565^{+0.001}_{-0.0002}) \times 10^{-3}; \quad m_\mu = (75.2938^{+0.1912}_{-0.0615}) \times 10^{-3}; \quad m_\tau = 1.6292^{+0.0443}_{-0.0294} \\
  \tan \beta &= 52.0738^{-16.5475}_{+4.3757}; \quad v_u = 117.7947^{-46.7214}_{+19.2752}; \quad v_d = 2.2620^{-0.2615}_{+0.1661}
\end{align*}
\]

It is apparent that after including the sign ambiguity \(|d_2 - l_2|\) can be anywhere from .046GeV to .105GeV with a notional accuracy limited by the low energy error induced uncertainty in \( m_s(M_X) \) of about .004GeV. On the other hand \(|d_3 - l_3|\) is about .2 GeV ±.5 GeV (!) (evidently \(d_3, l_3\) must have the same sign !). There is a long way to go before the validity of the constraint can be verified, even granted that no additional ambiguities due to CP violating phases arise.

Our approach is thus to take as reference central values the relatively accurate values of \( m_{\mu, \tau, s} \), at large \( \tan \beta \) (obviously favoured by SO(10) GUTs) (say \( \tan \beta = 55 \)) for definiteness) and assume \( d_3 = m_\tau \pm m_s \pm m_\mu \approx 1.63 + \text{sign}[d_2], 0.03 - \text{sign}[l_2], 0.75 \) to satisfy the consistency constraint, secure in the knowledge that the data allows \( m_b \) to lie in the range \((1.23, 1.9)\) for \( \tan \beta = 55 \) and somewhat lower values for lower \( \tan \beta \). However we find that \( \text{sign}[d_3] = +, \text{sign}[l_2] = - \) is impermissible since it leads to an imaginary value for \( \chi_1 \) when one solves eqn(14).

The values of \( d_3 \) required in our scenario by the combination of \( b - \tau = s - \mu \) unification and near maximal leptonic mixing tend (at \( d_3 \approx 1.585 \)) to be somewhat larger than the central value quoted for \( m_b \) above but still comfortably within the \( 1\sigma \) error bars. This can be considered a prediction of this model which may become constraining as limits improve. Note particularly that the implication of this type of novel constrained unification leaves only the relatively tiny \( 1\sigma \) uncertainty \((\approx 0.001 - 0.05\text{GeV})\) in \( m_\tau(M_X) \) as a fudge factor while \( d_3 \) whose \((i.e. m_b(M_X)'s)\) uncertainty could have been 10-100 times larger is tied down to vary in tandem with \( m_\tau(M_X) \). Yet we will see that the prediction of the maximal mixing is still extremely robust.

The values of the mixing angles at \( M_X \) given by [27] (who assume \( \delta_{CKM} = \pi/2 \) for convenience) are \( \theta_{12} = 0.221, \theta_{23} = 0.037, \theta_{13} = 0.003 \), with no uncertainty quoted. The product \( \theta_{12}\theta_{23} = 0.008 \pm 0.002 \) is of the same order of magnitude as \( \theta_{13} \). Since this relation arises in the 3 generation case at \( O(\epsilon^3) \), by which stage the CP violation also enters it seems legitimate to hope for a considerable \((\approx 100\%)\) role for \( \delta_{CKM} \) in modifying the relation \( \theta_{13}^2 = \theta_{12}^2 \theta_{23}^2 \) between angles.
5 Neutrino Masses and Mixing

In this section we will first determine the angle in the PMNS$^{[15]}$ mixing matrix and then proceed to show that this mixing angle can can easily satisfy the near maximality constraints$^{[17]}$ imposed by experiment.

5.1 PMNS Matrix

Under the assumption that the couplings $\hat{h}, \hat{g}$ of the $10$ and $120$ completely dominate coupling $\hat{f}$ of the $126$, the Type I seesaw masses in this theory are given by

$$M'_\nu \approx \frac{-v^2}{2\sigma} (\hat{h} + r_5' \hat{g})^T \hat{f}^{-1}(\hat{h} + r_5' \hat{g})$$

$$= r_4 R^T \left( \frac{Z_d}{r_1} + \frac{r_5'}{r_6} A_d \right)^T R S^{-1} D_f^{-1} S^{-1T} R^T \left( \frac{Z_d}{r_1} + \frac{r_5'}{r_6} A_d \right) R$$

$$\equiv r_4 R^T Y_d^T R S^{-1} D_f^{-1} S^{-1T} R^T Y_d R$$

$$\equiv R^T F R$$

$$= L^T \mathcal{P} D_\nu \mathcal{P}^T L \quad (21)$$

Where $\mathcal{P}$ is the Lepton mixing (PMNS) matrix in the basis with diagonal leptonic charged current, $D_\nu$ the light neutrino masses extrapolated to $M_X$, and $Y_d$ a convenient dimensionless form of the linear combination of $Z_d, A_d$ that determines the PMNS mixing.

The explicit form of $Y_d$ is

$$Y_d = \begin{pmatrix}
\frac{2d}{\Delta d} \cos \chi_d & -\sin \chi_d - r_5' \frac{T_u}{\Delta u} \sin \chi_u \\
-\sin \chi_d + r_5' \frac{T_u}{\Delta u} \sin \chi_u & 2(1 + \frac{d_2}{\Delta d}) \cos \chi_d
\end{pmatrix} \quad (22)$$

So far we have not used the family basis ambiguity of the SO(10) GUT that allows us to perform an arbitrary unitary redefinition of the matter $16$-plets at will. Since $\hat{f}$ is symmetric it may be written as $\hat{f} = S^T D_f S$ where $S$ is unitary and $D_f$ is diagonal and real. The basis ambiguity can be fixed by a choice of $S$ to be any given unitary matrix : for example to be unity. Here we make the convenient choice $S = R$ since this removes the obscuring factors $RS^{-1}$ in the Type I mass formula eqn(21) above, leaving behind the two eigenvalues of $f$ as free parameters. One of these can be extracted into the overall scale say $\hat{f}_2$ leaving behind the parameter $\rho = \hat{f}_2 / \hat{f}_3 = M^{(2)} / M^{(3)}$ to affect the determination of the mixing angle $\chi_P$.

Imposing the family basis choice and diagonalizing the matrix $F$ we immediately obtain the neutrino mixing and masses to be

$$F \equiv Y_d^T D_f \equiv \mathcal{F} \text{Diag}(F_2, F_3) \mathcal{F}^T$$

$$\mathcal{P} = D^{\dagger}_\nu \mathcal{F} \quad ; \quad D_\nu = \frac{r_4}{\hat{f}_2} \text{Diag}(F_2, F_3) \quad (23)$$
The mixing angles $\chi_{F}^{\pm}$ of the 2-d rotation matrix $F$ which diagonalizes $F$ to $\text{Diag}(F_{+},F_{-})$ are given by

$$\chi_{F}^{\pm} = \tan^{-1}(\omega_{F} \mp \sqrt{1 + \omega_{F}^{2}})$$
$$\omega_{F} = \frac{F_{11} - F_{22}}{2F_{12}}$$

The eigenvalues $F_{\pm}$ obey

$$F_{\pm} = \frac{1}{2}(\text{Tr}(F) \pm \sqrt{\text{Tr}(F)^{2} - \text{Det}[F]})$$
$$\text{sign}[\text{Tr}(F)] = \text{sign}[F_{+}^{2} - F_{-}^{2}]$$

Note that since (to leading order in $\epsilon$)

$$\text{Tr}(F) = \sin^{2}\chi_{d}(4 + 16\rho + r_{5}^{2}(1 + \rho) - 4r_{5}(1 + 2\rho) + 4\rho \cot^{2}\chi_{d})$$

the hierarchy will invert for

$$\rho < - \frac{(r_{5} - 2)^{2}}{(r_{5} - 4)^{2} + 4\cot[\chi_{d}]^{2}} \equiv \rho_{\text{inv}}$$

### 5.2 Maximal 2-3 PMNS Mixing

We now have a formula for the leptonic mixing in terms of two quite arbitrary parameters $\rho,r_{5} = r'_{5} + 3$. The value $\rho = 1$ corresponds to completely degenerate right handed (heavy) neutrinos. While $|\rho| << 1$ and $|\rho| >> 1$ correspond to hierarchical righthanded neutrino masses, $r_{5} = 0$ corresponds to negligible contribution from the SU(4) singlet pair of MSSM doublets from the $\mathbf{120}$-plet (i.e Pati-Salam sub-representation representation $(1,2,2) \subset \mathbf{120}$) to the low energy symmetry breaking. Thus it may be realizing a Georgi-Jarlskog mechanism for the $\mathbf{10} \oplus \mathbf{120}$ based fit of the 2-3 generation charged fermion masses. We will show that

- For any value of $r_{5}$ there exists a range of values of $\rho$ for which the the leptonic mixing is near maximal ($1 \geq \sin^{2}2\chi_{P} \geq .9$).

- There exist ranges of $r_{5}$ where the width of the $\rho$-band where $1 \geq \sin^{2}2\chi_{P} \geq .9$ becomes very large.

Thus large mixing is always achievable and inevitable in certain broad regions of the $r_{5},\rho$ plane.

From the analytic solution of the charged fermion fit (eqns(13,14) and eqn(22)) we find that the two possible values of the total mixing are

$$\chi_{P}^{(0)} = \chi_{F}^{-} \quad ; \quad \chi_{P}^{(1)} = \chi_{F}^{-} + \chi_{d}$$
So that the two possible values of the mixing parameter \( \sin^2 2\chi_P \) (which is indifferent to inversion of the \( \nu_L \) hierarchy) are

\[
\sin^2 2\chi^{(0)}_P = \frac{1}{1 + \omega_F^2}, \\
\sin^2 2\chi^{(1)}_P = \frac{(-1 + 2\omega_F \tan \chi_d + \tan^2 \chi_d)^2}{(1 + \omega_F^2)(1 + \tan^2 \chi_d)^2}
\]

(29)

\( \sin^2 2\chi^{(0)} \) is maximal at \( \omega_F = \omega_F^{(0)} = 0 \), while \( \sin^2 2\chi^{(1)} \) is maximal at

\[
\omega_F = \omega_F^{(1)} = \frac{2 \tan \chi_d}{2 \tan^2 \chi_d - 1}
\]

(30)

Moreover the range of \( \omega_F \) values which leads to near maximal mixing (1 \( \geq \) \( \sin^2 2\chi_P \) \( \geq \) .9) is \( \omega_F \in (-1/3, 1/3) \). While for 1 \( \geq \) \( \sin^2 2\chi_P \) \( \geq \) .9 this range lies between

\[
\omega^{(1, \pm .9)} = \frac{\pm t^2 + 6 t + 1}{3 t^2 \mp 2 t - 3}
\]

(31)

There are in principle different combinations of signs : \( \text{sgn}\{d_2, l_2, u_3, u_2, \chi_c\} \) that can be chosen for the various parameters. They make no difference to the mixing at the leading order. However, the combination \( d_2 \) positive, \( l_2 \) negative, leads to imaginary \( \chi_l \) in the full theory (i.e beyond leading order where \( \chi_l = \pm \chi_d \)) and hence must be discarded. The remaining sign choices give results that are only marginally different from those where we take all masses and \( \chi_c \) as positive. So will not here discuss their minor differences and focus only on the “all positive” sign choice in the interests of clarity.

### 5.3 Depiction of the Large Mixing Parameter Regions

The regions of the \( r_5, \rho \) plane that support large PMNS mixing can be fairly completely and accurately delineated by working to leading order in \( \epsilon \). Then \( \tan \chi_d \approx m_s/\chi_c m_\tau \) is the only input parameter required. In leading (and determinative) order the formulae are not obscured by the other parameters whose role in determining the PMNS mixing is quite marginal. Note, however, that since we have an exact analytic solution numerical work can always use it to confirm and refine the features visible at leading order.

It is easy to show that to leading order in \( \epsilon \)

\[
\omega_F = \frac{\left[(-(-2 + r_5)^2 + (-4 + r_5)^2 \rho - 4 \rho \cot(\chi_d)^2)\right] \tan(\chi_d)}{4 (-4 + r_5) \rho}
\]

(32)

Note particularly the singularities at \( \rho = 0 \) and at \( r_5 = 4 \) and the fact that the result is dependent only on \( r_5, \rho, \tan \chi_d \). There is clearly never a large mixing \( \chi_P^{(0)} \)
solution for $\rho = 0$ or $r_5 = 4$, and it is easy to check that for these values the limit as $\omega \to \infty$, $\sin^2 2\chi_P^{(1)} \to \frac{4 \tan^2 \chi_d^{(1)}}{(1 + \tan^2 \chi_d^{(1)})^2}$ which never yields large mixing for $\tan \chi_d \in (.35, .6)$ which is the experimentally allowed range. Clearly eqn(33) implies that for arbitrary $(r_5, \tan \chi_d)$ one can always find a value of $\rho^{(0)}_{\text{max}}$ where the mixing $\sin^2 2\chi_P^{(0)}$ is maximal.

$$\rho^{(0)}_{\text{max}} = \frac{(-2 + r_5)^2}{(-4 + r_5)^2 - 4 \cot(\chi_d)^2}$$

(33)

For example, when $r_5 = 0$, $\tan \chi_d = .4974$, $\rho^{(0)}_{\text{max}} = -23.8511$ (while $\rho^{(0)}_{\text{inv}} = -0.124348$, so the hierarchy is inverted but not acutely so). Moreover even for $\rho = 1, r_5 = 0$ (degenerate $\nu_R$, Georgi-Jarlskog point) one finds

$$\omega_F = \frac{1}{4}(\cot \chi_d - 3 \tan \chi_d)$$

(34)

which gives $\sin^2 2\chi_P^{(0)} = .9835$ at $\tan \chi_d = d_2/(l_3 \chi_c) = .4974$. Giving already a hint of the robustness of large mixing.

For given $r_5, \tan \chi_d$ the edges $\rho^{(0,\pm 1/3)}$ of the large mixing band for the mixing $\chi_P^{(0)}$ are

$$\rho^{(0,+,.9)} = \frac{3 (-2 + r_5)^2 \tan(\chi_d)}{-4 (-4 + r_5 + 3 \cot(\chi_d)) + 3 (-4 + r_5)^2 \tan(\chi_d)}$$

$$\rho^{(0,-,.9)} = \frac{3 (-2 + r_5)^2 \tan(\chi_d)}{-12 \cot(\chi_d) + (-4 + r_5) (4 + 3 (-4 + r_5) \tan(\chi_d))}$$

(35)

while the width of the band is

$$\Delta^{(0,.9)} = \rho^{(0,+.9)} - \rho^{(0,-.9)}$$

$$= \frac{24 (-4 + r_5) (-2 + r_5)^2 \tan(\chi_d)}{144 \cot(\chi_d)^2 + (-4 + r_5)^2 (-88 + 9 (-4 + r_5)^2 \tan(\chi_d)^2)}$$

(36)

An exactly parallel discussion can be given for the other case, we omit the tedious details.

The (simple) poles of $\Delta^{(0,.9)}$ are at $r_5 \to -1.57871, 1.10189, 6.89811, 9.57871$. In Fig.1 we plot the width of the large mixing stripe versus $r_5$ to illustrate its large variation. It is apparent that it is precisely the poles of $\Delta^{(0,.9)}$ that cause the width to become large and that the whole region between $r_5 = -1.6$ and $r_5 = 10$ (particularly $r_5 \in (6.5, 10)$ supports very large regions where the mixing is maximal.

This is clearly seen in the corresponding contour plot Fig. 2 where we have truncated the extent in the $\rho$ direction to avoid obscuring the small mixing band around $\rho = 0$.

An exactly parallel discussion can be given for the mixing $\sin^2 2\chi_P^{(1)}$, we omit the details and give only the contour plot in Fig. 3, which has an analogous structure to
Figure 1: Plot of $\Delta^{(0,9)}$ vs $r_5$, at lowest order in $\epsilon$, at fixed $\tan \chi_d = 0.4974$. The poles of $\Delta^{(0,9)}$ cause regions of a very wide range of $\rho$ to support large PMNS mixing.

that for $\sin^2 2\chi_P^{(0)}$. Note that the very narrow width of the large mixing band for $r_5 \in (1.3, 5)$ has caused it to become invisible due to the large range on the $\rho$ axis. It is easy to see this relatively narrow band by magnifying the plot. In Figure 4 we show an example of the variation of the mixing parameter with $\tan \chi_d$ keeping $r_5, \rho$ fixed at values in the maximal mixing regions ($\sin^2 2\chi_P^{(0)} \geq 0.95$).

Although the exact formulae given above have a completely regular expansion in the small parameter $\eta = 0.04$ it is of interest to see how much the leading order results are modified when we include all orders in $\eta$ and also to examine the effects of varying the other charged fermion masses e.g $m_{top}$. The explicit formulae -although trivial to write down- are not very transparent. However plots (Figs 5-7) analogous to the ones for the leading order formulae tell almost the whole story:

- The contour plots for the exact results are essentially identical to those for the leading order case apart from minor changes in fine structure and a slight squeezing of the maximal mixing regions. The similarity is so great that we show only one example as Fig. 5: the contour plot $\sin^2 2\chi_P^{(1)}$. We show the plot for a different sign choice for $d_2, l_2$ to emphasize that all these (sign choice and higher order effects) cause very minor modifications as is apparent when one compares Fig. 3 and Fig. 5.
Figure 2: Contour Plot of the mixing parameter $\sin^2 2\theta_{23} = \sin^2 2\chi_P^{(0)}$, to lowest order in $\epsilon$, on the $(r_5, \rho)$ plane at fixed $\tan \chi_d$. The contours shown are at $\sin^2 2\chi_P^{(0)} = .85, .9, .95$ with the shading becoming darker towards smaller values.

- The variation with $m_{\text{top}}(M_X)$ is quite marginal. An example of how it affects the mixing near a maximal mixing point is shown in Fig. 6.
- Similarly the variation with $\chi_c$ near a maximal mixing point is shown in Fig. 7.
- The role of $m_e$ is essentially negligible since it is $O(\epsilon^4)$ relative to $m_t$ and the equations are sensitive only to their ratio.
Figure 3: Contour Plot of the mixing parameter $Sin^2\theta_{23} = Sin^2\chi_P^{(1)}$, to lowest order in $\epsilon$, on the $(r_5, \rho)$ plane at fixed $\tan \chi_d$. The contours shown are at $Sin^2\chi_P^{(0)} = .85, .9, .95$ with the shading becoming darker towards smaller values.

6 Discussion, Conclusions and Outlook

The work presented in this paper is based upon the very productive insight [9] that it is the mixing in the 2-3 sector of the Fermion mass matrices that is the true stable, organizing core of the entire complex hierarchical fermion mass matrix. Possibly due to the fact that the mixing angle $\theta_{23}^c$ in the CKM matrix is much smaller than the Cabbibo angle $\theta_{12}^c$ this realization has been slow to be digested by the many workers
fascinated by the problem of finding a non-trivial insight into the apparently finely crafted inner workings of this wondrous “primal artefact”.

Counter to the fact that $\theta_{23}^c << \theta_{12}^c$ one must consider the extreme tininess of the first generation masses relative to those of the 2-3 generations, which ensures that we must begin by understanding the smaller mixing angle. It was not until the discovery of the tiniest mass (differences), via the observation of neutrino oscillations, precisely in the range expected from a seesaw between the Electro-Weak and GUT scales that the deep inner connections between the physics of the Largest (GUT), Smallest ($m_\nu$), and “Everyday Electroweak” ($M_W$) mass scales emerged into plain view\[31, 14, 1\]. The initial surprise at the natural explanations lurking in the analysis of the simple 2-3 system and the apparently unique and natural simplicity \[9\] of the Type-II Seesaw, were followed by further surprises at the demonstration that similar large mixing could\[13\] also be obtained (“sporadically”?) in some of the possible solutions of the generic BM\[10\] fitting problem in Type I seesaw (or a combination of both seesaws). The price paid for this “numerical discovery” was that it was based on tuning the many unknown phases present in these models to achieve the desired large neutrino mixing.

Unfortunately the BM program appears to meet\[14, 1, 29, 20\] obstacles to its yield of large enough neutrino masses both in its Type I \[3, 14, 1\] and Type II
Figure 5: Contour plot of the exact solution for the mixing parameter $\sin^2 2\theta_{23} = \sin^2 2\chi_P^{(1)}$ on the $r_5, \rho$ plane, at central values of the charged fermion parameters (given on plot). The contours shown are at $\sin^2 2\chi_P = .85, .9, .95$ with the shading becoming darker towards smaller values. The white region corresponds to $\sin^2 2\chi_P < 0.95$. Note the $\{\text{sgn}[d_2], \text{sgn}[l_2]\} = \{-,-\}$ sign choice. This may be compared with Fig. 3.

versions at least in the most desirable context of the MSGUT. Still this experience with the fitting problem provided the valuable lesson that the phase complications of the full complex CKM matrix problem would respect in large part the pattern discovered in the real two and three generation fitting exercise[13]: which expectation is one of the operational assumptions of our work.
Figure 6: Plot of the exact solution for the mixing parameter $\sin^2 \theta_{23} = \sin^2 \chi_P^{(0)}$, vs $m_{\text{top}}(M_X)$, with sign choice $\{\text{sgn}[d_2], \text{sgn}[l_2]\} = \{++\}$

The likely smallness of the seesaw masses was noticed long before our work \cite{3,19} and was considered in \cite{19} to be a motivation for constructing more elaborate GUT scale Higgs structures to evade the tight constraints\cite{11,8,14,1} imposed on seesaw coefficients by the very simple yet fully functional (and fully calculable\cite{5,6,7}) structure of the MSGUT. Our contribution was a systematic survey of the entire parameter space of the MSGUT to show that the Type II contribution could not be saved by enhancement at the special points of the MSGUT parameter space and that the same also went for Type I. These conclusions were based on the genericity of the values extracted from Type I,II fits available in the literature and need to be confirmed on a wider sample generic fits or more satisfactorily by an analytic (perturbative) solution of the fitting problem in the MSGUT\cite{24}. We are personally reluctant to enthuse about the assumption of more complex AM Higgs (i.e GUT scale SSB) structures. A “shotgun” programmatic attitude towards decoding the enigmas of unification seems questionable, because if there is no tight discipline restraining “multiplication of hypotheses” (representations) while speculating on extensions of established theories then it is difficult, if not impossible, to know what one would be testing, if one ever could begin to do so: the wideness of the space of extensions or any specific unavoidable implication of theory. Since our knowledge about nature is sure to
arrive only incrementally and after hard struggle to define and test every additional observable, it should be kept in mind that extensions beyond those “minimally” necessary and forced upon us by the inner logic of Theory as we understand it in its essences - rather than a patchwork hypothetical “fix” of its symptoms - should guide our willingness to entertain new models: given that the “enveloping theory space” of established Theory is almost by definition multiply infinite. If we seek to avoid the (in our view very fortunately found) tight corners into which we are successful in driving the simplest - otherwise viable - model by changing the model itself so completely that the entire previous simplifying and constraining understanding must be jettisoned then we foreclose any opportunity of hearing the “tiny voice within” (the theory that is !) by which nature may seek to inform us of the path to the decoding of these ultimate enigmas.

The above pontifications may seem too sententious, pretentious or subjective, since one theorist’s minimality is another theorist’s monstrous pedantry! So we put it in more concrete terms: changing the MSGUT’s AM Higgs structure by adding additional 54’s or other Higgs multiplets will likely completely destroy the GUT scale simplicity and calculability achieved - after tedious labour - in the current formulation of the MSGUT. We think it is preferable to first allow the 120-plet arbitrarily excluded by the BM-scenario (justifiably so at that early stage of searching
for the simplest viable model but now not an assumption worth protecting by \textit{ad hoc} measures): which leaves the GUT scale SSB unchanged. In this way one may hope to arrive at definite statements and falsifiability about the simplest viable model so as to define the next necessary (and falsifiable!) one. On the other hand extending the GUT scale structure to avoid the conflict with the FM data merely extends the catalog of non-falsifiable models without adding insight into the inner logic of the minimal theoretical structures adequate to encompass the data.

Indeed, as shown in \cite{1} the constrained analysis in the framework of the MSGUT points to the root of the problem lying in the multiple contradictory constraints faced by the $\mathbf{T_{26}}$ Yukawa couplings. A very simple calculable alternative, namely $\mathbf{10} - \mathbf{120}$ domination of the charged fermion sector coupled with very weak $\mathbf{T_{26}}$ couplings that accentuate the Type I seesaw– due to its inverse dependence on these couplings through the large mass of the integrated out heavy right handed neutrinos – emerges as an obvious and well motivated resolution of the problem. Surprisingly just this possibility seems not to have been used to simplify the \textit{prima facie} intractable nature of the fitting problem involving both the two symmetric Yukawa matrices of the $\mathbf{10, 120}$ \textit{and} the antisymmetric couplings of the $\mathbf{120}$-plet. Indeed, guided by this insight, analysis of the scenario quickly yields rather novel insights into the structure of the fermion mass hierarchy. In particular, as shown by us in this paper, the mysterious maximality ($\sin^2 \theta_{23} \sim 1.02 \pm 0.04$ \cite{17}) of the atmospheric mixing angle– which has no naturalness even in the BM Type I and Type II seesaw fits \cite{9, 11, 12, 13} – here emerges as a direct consequence of the most accurately known fermion mass data at the GUT scale and the well motivated structural pattern chosen for the theory due to its previous debacle \cite{1}!

Usually such “coincidences” are explained by means of somewhat thinly motivated additional discrete symmetries or textures. Yet here Nature seems once again to hint that the solution is both simple and “commonsensical” and at the same time more deep and profound than anything arbitrary speculation might have rigged. The fermion spectra at $M_X$ seem seem to be fully compatible with the observed maximality of mixing without any assumption besides allowing all $\text{SO}(10)$ FM Higgs types and a pattern of dominance (and “division of labour”) that assigns each Higgs a consistently bearable “workload” : all serendipitously arranged so as to robustly yield maximal atmospheric mixing angle fixed at a nearly “geometric” value!

Moreover the analysis has proved to be fully compatible with and robust under a systematic perturbation theory in which the CKM angle $\theta_{23}$ is considered as a common small number for structuring magnitudes of masses and angles (i.e universal Wolfenstein parameter) in the whole hierarchy, leaving behind magnitudes $O(1)$ or smaller to be determined order by order in the expansion of the fitting problem. An expansion of of the three generation CP conserving case in this small parameter yields \cite{24} a solution of the fitting problem that is based upon and respects the 2-generation analytic solution found in this paper as its robust core. Besides the remarkable $b - \tau = s - \mu$
unification constraint—which strengthens the rough $b - \tau$ equality long canonical in SUSY GUTs into the viable yet imminently falsifiable (i.e pending refinements in the bottom quark mass measurement etc) $b - \tau$ pinning described above we find at $O(\epsilon^3)$ an additional constraint between angles: $\theta^c_{13} = \theta^c_{12}\theta^c_{23}$, which is compatible with observation at least as regards order of magnitude. Moreover since it arises at $O(\epsilon^3)$, by which order we expect CKM CP violation to rear its head, there is every reason to hope that this constraint will be modified to a more viable one involving the CKM phase when the full 3 generation analysis is completed. Once the formalism of generic fitting problem has been clarified it will be ready for application in the Nu (realistic) MSGUT (NMSGUT) whose mass spectrum, zero mode Higgs couplings, and Baryon violating operators have been calculated within the framework and conventions of our decomposition of SO(10) group theory to the maximal unitary sub group $SU(4) \times SU(2)_L \times SU(2)_R$ (i.e the Pati-Salam group).

Finally we note that the very nature of our ansatz dictates that the righthanded neutrino masses lie in the range below $10^{12} GeV$ due to the smallness of the $\frac{1}{126}$ couplings. Combined with the lower bound that arises in the Leptogenesis models naturally associated with SO(10) and the tight yet achievable pinning of $m_b(M_X)$ to the more accurately known $m_{\tau}(M_X)$, we may hope that our approach has added two significant stable elements to the slowly but surely emerging picture of the intricate interconnections and tightly interdependent structures concealed within the Fermion mass Hierarchy.

We earlier likened the circular linkages and mutual balance between the Large, Small and Geometric Mean mass scales to the fabled cosmic serpent that swallows its own tail: an ancient hermetic symbol expressing the same sentiment of wonder at the “boundary less” balance and intricate self sufficiency of the cosmic order that we seek to decipher. The wondrously patterned and enigmatic Fermion Mass pattern is then perhaps the supremely fascinating and valuable diadem that the oυρβ´ουρος 1 is fabled to wear. What was still only the ‘spoor of a grail’ now gleams -to our hopeful eyes - with the iridescence of the ouroborotic nagachudamani 2 which must -of course - also be our true philosopher’s stone !.

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1
**Gr**: Tail-devourer

2
**Skr**: (Cosmic) Snake crestjewel/diadem
8 Note Added

On receiving an advance version of this paper, B. Bajc informed me that in their latest work on the 10 – 120 system including complex couplings for the 2 generation case they have also found restrictions on \( m_b - m_\tau \) very similar to those that form one of the central results of this paper [31].

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