RANK ONE CONNECTIONS ON ABELIAN VARIETIES, II

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ABSTRACT. Given a holomorphic line bundle $L$ on a compact complex torus $A$, there are two naturally associated holomorphic $\Omega_A$–torsors over $A$: one is constructed from the Atiyah exact sequence for $L$, and the other is constructed using the line bundle $(p_1^*L^*) \otimes (\alpha^*L)$, where $\alpha$ is the addition map on $A \times A$, and $p_1$ is the projection of $A \times A$ to the first factor. In [BHR], it was shown that these two torsors are isomorphic. The aim here is to produce a canonical isomorphism between them through an explicit construction.

1. Introduction

Let $A$ be a complex abelian variety and $L$ a holomorphic line bundle over $A$. The sheaf of holomorphic connections on $L$ defines a torsor $C_L$ on $A$ for the holomorphic cotangent bundle $\Omega_A$.

Let $\alpha, p_1, p_2 : A \times A \to A$ be the addition map and the projections respectively. The holomorphic line bundle

$$L := (p_1^*L^*) \otimes (\alpha^*L) \to A \times A \xrightarrow{p_2} A$$

will be considered as a holomorphic family, parametrized by $A$, of topologically trivial holomorphic line bundles on $A$. We have an $\Omega_A$–torsor $Z_L \to A$ whose holomorphic sections over any open subset $U \subset A$ are the holomorphic families of relative holomorphic connections on $L|_{A \times U}$.

In [BHR] it was crucially used that the two torsors $C_L$ and $Z_L$ are holomorphically, or equivalently, algebraically, isomorphic (see [BHR] Proposition 2.1]). The proof of Proposition 2.1 of [BHR] was carried out by comparing the cohomological invariants associated to the torsors.

Our aim here is to give an explicit construction of a holomorphic isomorphism between the two torsors. The isomorphism is canonical in the sense that its construction does not require making any choices.

We work with a compact complex torus; we do not need $A$ to be algebraic.

2. A criterion for isomorphism of torsors

Let $M$ be a connected complex manifold. Let $V$ be a holomorphic vector bundle over $M$.

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A $\mathcal{V}$–torsor on $M$ is a holomorphic fiber bundle $p : Z \to X$ and a holomorphic map from the fiber product
$$\varphi : Z \times_M \mathcal{V} \to Z$$
such that

1. $p \circ \varphi = p \circ p_Z$, where $p_Z$ is the natural projection of $Z \times_M \mathcal{V}$ to $Z$, 
2. the map $Z \times_M \mathcal{V} \to Z \times_M \mathcal{V}$ defined by $p_Z \times \varphi$ is an isomorphism, 
3. $\varphi(\varphi(z,v),w) = \varphi(z,v+w)$.

A $\mathcal{V}$–torsor $(Z,p,\varphi)$ is called trivializable if there is a holomorphic isomorphism
$$\beta : \mathcal{V} \to Z$$
such that $p \circ \beta$ is the natural projection of $\mathcal{V}$ to $M$, and
$$\beta^{-1} \circ \varphi \circ (\beta \times \text{Id}_\mathcal{V}) : \mathcal{V} \times_M \mathcal{V} \to \mathcal{V}$$
is the fiberwise addition homomorphism on $\mathcal{V} \times_M \mathcal{V}$. A holomorphic isomorphism $\beta$ satisfying the above conditions is called a trivialization of $Z$.

Any $\mathcal{V}$–torsor $Z$ has a $C^\infty$ section $M \to Z$ because the fibers of $Z$ are contractible. It has a holomorphic section if and only if it is trivializable.

Take a $\mathcal{V}$–torsor $(Z,p,\varphi)$. Let
$$\sigma : M \to Z$$
be a $C^\infty$ section, so $p \circ \sigma = \text{Id}_M$. Let
$$d\sigma : T^R M \to \sigma^* T^R Z$$
be the differential of $\sigma$, where $T^R$ is the real tangent bundle. The almost complex structures on $M$ and $Z$ will be denoted by $J_M$ and $J_Z$ respectively. Let
$$\tilde{\sigma} : T^R M \to \sigma^* T^R Z, \quad v \mapsto d\sigma(J_M(v)) - J_Z(d\sigma(v))$$
be the obstruction for $\sigma$ to be holomorphic. Since the projection $p$ is holomorphic, we have
$$dp(\tilde{\sigma}(v)) = dp(d\sigma(J_M(v))) - dp(J_Z(d\sigma(v))) = J_M(v) - J_M(dp(d\sigma(v))) = 0,$$
where $dp$ is the differential of $p$. Hence $\tilde{\sigma}(v)$ is an element of $\mathcal{V}_x$ if $v \in T^R_x M$ (the vertical tangent subbundle of $T^R Z$ for $p$ is identified with $p^* \mathcal{V}$). Note that $\tilde{\sigma}(J_M(v)) = -J_Z(\tilde{\sigma}(v))$.

Define
$$\hat{\sigma} : T^{0,1} M \to \mathcal{V}, \quad v + \sqrt{-1} \cdot J_M(v) \leftrightarrow \tilde{\sigma}(v).$$

(2.1)

So $\hat{\sigma}$ is a smooth $(0,1)$–form with values in $\mathcal{V}$. Clearly, $\hat{\sigma} = 0$ if and only if $\sigma$ is holomorphic. Equivalently, $\hat{\sigma} = 0$ if and only if $\sigma$ is a trivialization of $Z$.

Let $(Z_1,p_1,\varphi_1)$ and $(Z_2,p_2,\varphi_2)$ be two $\mathcal{V}$–torsors. Let $\sigma$ and $\tau$ be $C^\infty$ sections of $Z_1$ and $Z_2$ respectively. We have a unique $C^\infty$ isomorphism of $\mathcal{V}$–torsors
$$\gamma : Z_1 \to Z_2, \quad \gamma \circ \varphi_1(\sigma(x),v) = \varphi_2(\tau(x),v), \quad x \in M, v \in \mathcal{V}_x.$$

(2.2)

So, $\gamma \circ \sigma = \tau$.

**Proposition 2.1.** If $\hat{\sigma} = \hat{\tau}$ (constructed as in (2.1)), then $\gamma$ in (2.2) is holomorphic.
Proof. Let \( q : H \to M \) be the holomorphic fiber bundle whose fiber over any \( x \in M \) is the space of all isomorphisms \( \phi : (Z_1)_x \to (Z_2)_x \) such that \( \phi \circ \varphi_1(z, v) = \varphi_2(\phi(z), v) \) for all \( v \in V_x \). Note that we have a map

\[ \varphi_x : H_x \times V_x \to H_x, \quad \varphi_x(\phi, v)(z) = \varphi_2(\phi(z), v) = \phi \circ \varphi_1(z, v). \]

There is a complex structure on \( H \) uniquely determined by the condition that a section of \( H \) defined over any open subset \( U \subset M \) is holomorphic if the corresponding map \( Z_1|_U \to Z_2|_U \) is holomorphic. The triple \((H, q, \varphi)\) is a \( V \)-torsor, where \( \varphi|_{H_x \times V_x} := \varphi_x \).

The map \( \gamma \) in (2.2) defines a \( C^\infty \) section \( M \to H \), which will also be denoted by \( \gamma \). It is straightforward to check that \( \hat{\gamma} = \hat{\tau} - \hat{\sigma} \). Therefore, if \( \hat{\tau} = \hat{\sigma} \), then \( \hat{\gamma} = 0 \). As noted before, \( \hat{\gamma} = 0 \) if and only if \( \gamma \) is holomorphic. \( \square \)

3. Line bundles on a complex torus

Let \( A \) be a compact complex torus. Let

\[ \alpha : A \times A \to A \]

be the addition map. Let \( p_i : A \times A \to A, i = 1, 2 \), be the projection to the \( i \)-th factor. Take a holomorphic line bundle \( L \) over \( A \). The holomorphic line bundle

\[ \mathcal{L} := (p_1^* L^*) \otimes (\alpha^* L) \to A \times A \xrightarrow{p_2} A \]

will be considered as a holomorphic family of holomorphic line bundles on \( A \) parametrized by \( A \). For any \( x \in A \), consider the holomorphic line bundle

\[ \mathcal{L}^x := \mathcal{L}|_{A \times \{x\}} \to A. \]

It is topologically trivial, so \( \mathcal{L}^x \) admits holomorphic connections.

The holomorphic cotangent bundle of \( A \) will be denoted by \( \Omega_A \). Define \( V := H^0(A, \Omega_A) \), and let

\[ V := A \times V \to A \]

be the trivial holomorphic vector bundle. The space of all holomorphic connections on \( \mathcal{L}^x \) is an affine space for \( V \). We have a \( V \)-torsor

\[ p_Z : Z_L \to A \]

whose fiber over any point \( x \in X \) is the space of all holomorphic connections on \( \mathcal{L}^x \). A holomorphic section of \( Z_L \) defined over an open subset \( U \subset A \) is a holomorphic family of relative holomorphic connections on \( \mathcal{L}|_{A \times U} \). This condition determines uniquely the complex structure of \( Z_L \).

Let

\[ 0 \to \mathcal{O}_A \to \text{At}(L) \to TA \to 0 \]
be the Atiyah exact sequence for \(L\) (see [At]); here \(TA\) is the holomorphic tangent bundle. Let
\[
0 \rightarrow \Omega_A \rightarrow \text{At}(L)^* \xrightarrow{\lambda} \mathcal{O}_A \rightarrow 0
\]
be the dual of the Atiyah exact sequence. Let \(1_A\) be the section of \(\mathcal{O}_A\) given by the constant function 1. From (3.6) it follows that
\[
(3.7) \quad p_C : C_L := \lambda^{-1}(1_A) \rightarrow A
\]
is an \(\Omega_A\)-torsor. Note that the vector bundle \(\Omega_A\) is canonically identified with \(V\) (defined in (3.4)) using the evaluation map on sections. Therefore, \(C_L\) is a \(V\)–torsor on \(A\).

**Theorem 3.1.** The two \(V\)–torsors \(Z_L\) and \(C_L\), constructed in (3.5) and (3.7) respectively, are canonically holomorphically isomorphic.

**Proof.** We will show that both \(Z_L\) and \(C_L\) have tautological \(C^\infty\) sections.

There is a unique translation invariant, with respect to \(\alpha\) in (3.1), \((1, 1)\)–form \(\omega\) on \(A\) representing \(c_1(L) \in H^2(A, \mathbb{Q})\) (any translation invariant form on \(A\) is closed). There is a unique unitary complex connection \(\nabla_L\) on \(L\) such that the curvature of \(\nabla_L\) is \(\omega\). The hermitian structure on \(L\) for \(\nabla_L\) is determined uniquely up to multiplication by a constant positive real number. Since complex connections on \(L\) are \(C^\infty\) splittings of (3.6), the connection \(\nabla_L\) defines a \(C^\infty\) section of the \(V\)–torsor \(C_L\) in (3.7). Let
\[
(3.8) \quad \sigma : A \rightarrow C_L
\]
be this \(C^\infty\) section.

Consider the holomorphic line bundle \(L\) in (3.2). The connection \(\nabla_L\) on \(L\) pulls back to connections on both \(p_1^*L\) and \(\alpha^*L\), and the connection on \(p_1^*L\) produces a connection on \(p_1^*L^*\). Therefore, we get a unitary complex connection on \(L\) from \(\nabla_L\); this connection on \(L\) will be denoted by \(\nabla_L\).

For any point \(x \in A\), let \(\nabla^x_L\) be the restriction of \(\nabla_L\) to the line bundle \(L^x\) defined in (3.3). Since the curvature of the connection \(\nabla_L\) is translation invariant, the curvature of \(\nabla^x_L\) vanishes identically. Hence \(\nabla^x_L\) is a holomorphic connection on \(L^x\). Consequently, we get a \(C^\infty\) section of the \(V\)–torsor \(Z_L\) in (3.5).

\[
(3.9) \quad \tau : A \rightarrow Z_L, \quad x \mapsto \nabla^x_L.
\]

In view of Proposition (2.1), to prove the theorem it suffices to show that \(\hat{\sigma} = \hat{\tau}\), where \(\tau\) and \(\sigma\) are constructed in (3.9) and (3.8) respectively. Note that a smooth \((0, 1)\)–form on \(M\) with values in \(V := H^0(A, \Omega_A) = \Omega_A\) is a \((1, 1)\)–form on \(A\).

As before, \(\omega\) denotes the curvature of \(\nabla_L\). It is standard that
\[
(3.10) \quad \hat{\sigma} = \omega
\]
(it is a general expression of the curvature of a connection in terms of the splitting of the Atiyah exact sequence for the connection).

Consider the connection \(\nabla_L\) on \(L\) constructed above. Since the curvature of \(\nabla_L\) is \(\omega\), it follows immediately that the curvature of the connections on \(p_1^*L^*\) and \(\alpha^*L\) are \(-p_1^*\omega\)
and $\alpha^\ast \omega$ respectively. Hence the curvature of $\nabla_L$ is
\[ K(\nabla_L) = \alpha^\ast \omega - p_1^\ast \omega \]
(see (3.2)). For any $y \in A$, let
\[ f_y : A \rightarrow A \times A \]
be the section of $p_2$ defined by $x \rightarrow (y, x)$. We have
\[ f_y^\ast K(\nabla_L) = f_y^\ast \alpha^\ast \omega - f_y^\ast p_1^\ast \omega = \omega \]
(3.11)
because $\omega$ is translation invariant. From (3.11) it follows that
\[ \hat{\tau} = \omega, \]
(3.12)
where $\tau$ is constructed in (3.9) (see (2.1)); the curvature $K(\nabla_L)$ is the obstruction for the $C^\infty$ family of holomorphic connections $\nabla^\xi_z$ to be holomorphic. From (3.10) and (3.12) we conclude that $\hat{\tau} = \hat{\sigma}$. Therefore, the proof of the theorem is complete by Proposition 2.1.

Theorem 3.1 immediately implies Proposition 2.1 of [BHR] by giving the isomorphism $\eta$ in the proposition explicitly. The factor $-1$ in Proposition 2.1 of [BHR] arises because $C_L$ and $Z_L^\ast$ is being compared (instead of $C_L$ and $Z_L$). Note that there is a natural holomorphic isomorphism $\delta : Z_L \rightarrow Z_L^\ast$ between the total spaces such that $\delta(z + v) = \delta(z) - v$ for all $v \in \mathcal{V}$; this is because there is a natural bijection between the connections on a line bundle $\zeta$ and its dual $\zeta^\ast$. For the same reason, there is a holomorphic isomorphism $\delta' : C_L \rightarrow C_L^\ast$ between the total spaces such that $\delta'(z + v) = \delta'(z) - v$ for all $v \in \mathcal{V}$.

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