How quantum correlations enhance prediction of complementary measurements

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Immediately after the discovery of quantum mechanics, it was realized that quantum correlations between two particles exhibit interesting counterintuitive features. Assuming a pair of maximally entangled qubits and M, the results of complementary measurements on qubit S can be, in principle, perfectly predicted from two appropriate measurements on qubit M. Later, it was shown that quantum mechanics predicts different values of certain correlations of measurement results than local realistic theories. Inequalities, which have to be satisfied within the local realism, were derived by Bell. The predictions of quantum mechanics were already satisfactorily experimentally confirmed using pairs of photons entangled in polarizations. In this Letter, we analyze in detail how the predictions between the qubits prepared in a general mixed state enhance our ability to predict the results of complementary projective measurements on one qubit when we know the measurement results on the other one. This enhancement can be described by the quantity that we will call complementary knowledge excess. We derive a non-trivial constraint restricting them is derived. For any mixed state and for arbitrary measurements the knowledge excesses are bounded by a factor depending only on the maximal violation of Bell’s inequalities. This result is experimentally verified on two-photon Werner states prepared by means of spontaneous parametric down-conversion.

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states are determined by the corresponding vectors $\vec{m}_l$ and $\vec{n}_l$. Any mixed state $\rho_{SM}$ can be uniquely converted to a state $\rho_{SM}$ using appropriate local unitary operations $U$. Thus, just two orderings of the diagonal elements, $t_{11}^2 \geq t_{22}^2$ or $t_{11}^2 \leq t_{22}^2$, remain to be discussed.

Let us suppose the measurements $\Pi_S$ and $\Pi_M$ are constructed from projectors to the vectors of the bases in which $|\beta_3\rangle$ is maximal. Then $\Delta \tilde{D} = \max(0, |\beta_3\rangle - |n_3\rangle)$. For $t_{11}^2 \geq t_{22}^2$, let the measurements $\Pi'_S$ and $\Pi'_M$ be related to the bases in which $|\alpha_1\rangle$ is maximal so that $\Delta \tilde{D}' = \max(0, |\alpha_1\rangle - |n_1\rangle)$. Simultaneously, we express the violation of any Bell’s inequalities employing the criterion from Ref. $[8]$: A state $\tilde{\rho}_{SM}$ violates Bell’s inequalities if its maximal Bell factor $B_{\text{max}} = 2\sqrt{t_{11}^2 + t_{22}^2}$ lies in the interval $[2, 2\sqrt{2}]$ (notice also that $B_{\text{max}}$ is invariant under local unitary transformations). Analogous results can be derived for $t_{11}^2 \leq t_{22}^2$: $\Delta \tilde{D} = \max(0, |\beta_2\rangle - |n_2\rangle)$. Maximal Bell factor is then $B_{\text{max}} = 2\sqrt{t_{11}^2 + t_{22}^2}$. Finally, we obtain an inequality $\Delta \tilde{D} + \Delta \tilde{D}' \leq (B_{\text{max}}/2)^2$ valid for an arbitrary state $\rho_{SM}$. The equality occurs for states with zero a-priori knowledges. For such states a non-zero knowledge can be obtained only through the measurement on $M$.

Now we generalize these results to any state $\rho_{SM}$ as well as for arbitrary measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$, where $\Pi_S, \Pi'_S$ are complementary unitary transformations. As pointed out, any mixed two-qubit state can be uniquely prepared from some state $\tilde{\rho}_{SM}$ (of a special form discussed above) by appropriate local unitary transformations $U_S, U_M$ acting on qubits $S$ and $M$, respectively. Further, the transformation of the above chosen measurements $\Pi_S$ and $\Pi'_S$ to arbitrary (but still complementary) measurements $\Pi_S$ and $\Pi'_S$ corresponds exactly to the extra local unitary transformation $U_B$ acting on the qubit $S$. Since distinguishabilities $\Delta \tilde{D}(\Pi_S)$ and $\Delta \tilde{D}(\Pi'_S)$ are invariant under any local unitary transformation on the qubit $M$, it is sufficient to take into account only a joint unitary transformation $\hat{U}_S = U_B U_S$ acting on qubit $S$.

For any unitary transformations $U$ there is a unique rotation $O$ such that $U(\vec{n} \cdot \vec{\sigma})U^\dagger = (O\vec{n}) \cdot \vec{\sigma}$. If a state $\tilde{\rho}_{SM}$ with diagonal $T$ is subjected to the $U_S \otimes U_M$ transformation its correlation matrix transforms as follows $T = O_S T O_M'$. Thus a joint unitary transformation $\hat{U}_S$ can be represented as a transformation of the correlation tensor $T = O_S T$, where $O_S$ is a matrix of rotation in $R^3$ space.

First, we will explicitly calculate $\Delta \tilde{D}(\Pi_S)$ and $\Delta \tilde{D}(\Pi'_S)$ for any mixed state using the transformation $T = O_S T$. Assuming $t_{11}^2 \geq t_{22}^2$ we obtain $\Delta \tilde{D}(\Pi_S) = \max(0, \sqrt{t_{11}^2 + t_{22}^2} - |n_3\rangle)$ and $\Delta \tilde{D}(\Pi'_S) = \max(0, \sqrt{t_{11}^2 + t_{22}^2} + \sqrt{t_{11}^2} - |n_1\rangle)$. Then we straightforwardly get $\Delta \tilde{D}^2(\Pi_S) + \Delta \tilde{D}^2(\Pi'_S) \leq (B_{\text{max}}/2)^2$. By analogous calculations we obtain the same result for $t_{11}^2 \leq t_{22}^2$. Finally, since $\Delta \tilde{K}(\Pi_M \rightarrow \Pi_S) \leq \Delta \tilde{D}(\Pi_S)$ and $\Delta \tilde{K}(\Pi'_M \rightarrow \Pi'_S) \leq \Delta \tilde{D}(\Pi'_S)$ we can conclude that

$$\Delta \tilde{K}^2(\Pi_M \rightarrow \Pi_S) + \Delta \tilde{K}^2(\Pi'_M \rightarrow \Pi'_S) \leq \left(\frac{B_{\text{max}}}{2}\right)^2. \quad (2)$$

Thus the maximal Bell factor represents a non-trivial bound on the sum of the squares of knowledge excesses which can be extracted from a pair of measurements on the “meter” qubit. Assuming $\Pi_M = \Pi'_M$ we can also derive an inequality analogous to that given in Ref. $[8]$: $\Delta \tilde{K}^2(\Pi_M \rightarrow \Pi_S) + \Delta \tilde{K}^2(\Pi'_M \rightarrow \Pi'_S) \leq 1$. Our analysis shows that for $\Pi_M \neq \Pi'_M$ the unit value on the right-hand side may be overstepped. Note also that $(B_{\text{max}}/2)^2 > 1$ only if the state violates Bell inequalities. For details of the proofs see Ref. $[8]$.

A natural question is how inequality $[2]$ can be saturated. For the class of states with vanishing a-priori knowledges for any measurements $\Pi_S, \Pi'_S$ it can be saturated just by the appropriate choice of measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$. In fact, it corresponds to the transformation of the given state to the state with diagonal correlation tensor. It was recently shown that there are such local (stochastically reversible) filtering operations $F_S, F_M$ applicable on a single copy of a qubit pair $(F_S^t F_S \leq 1_S$ and $F_M^t F_M \leq 1_M)$ that transform (with a non-zero probability) any two-qubit mixed state into a state which is (i) diagonal in Bell basis and (ii) has the Bell factor $B_{\text{max}}^t \geq B_{\text{max}}$ $[8]$. Since these Bell-diagonal states have the both local states maximally disordered the a-priori knowledges vanish. Thus – because the inequality $[2]$ is satisfied also after the filtering – we can always saturate it with the upper bound given by $B_{\text{max}}^t$ just by an appropriate choice of the measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$ after the appropriate local filtering.

We have verified inequality $[2]$ experimentally for two Werner states of qubits, $\rho = \frac{1}{2}|\Psi\rangle\langle\Psi| + \frac{1}{2}|\Omega\rangle\langle\Omega|$ (each qubit was represented by a polarization of a photon $[10]$). The parameter of the first Werner state ($p_1 \approx 0.82$) has been chosen so that the state was entangled and violated Bell inequalities, the parameter of the second one ($p_2 \approx 0.45$) so that it was entangled but did not violate Bell inequalities. The scheme of our experimental setup is shown in Fig. $4$. A krypton-ion cw laser (413.1 nm, 90 mW) is used to pump a 10-mm-long LiIO$_3$ nonlinear crystal cut for degenerate type-I parametric downconversion. We exploit the fact that the pairs of photons generated by spontaneous parametric downconversion (SPDC) manifest tight time correlations. In our setup the photons produced by SPDC have horizontal linear polarizations. Different linear-polarization states are prepared by means of half-wave plates ($\lambda/2$). The two photons impinge on two input ports of a beamsplitter (BS) forming a Hong-Ou-Mandel (HOM) interferometer $[11]$. A scanning mirror is used in one interferometer arm in order to balance the length of both arms, as indicated by an arrow in Fig. $4$. A glass plate (GP), that introduces polarization dependent losses, serves to compensate a non-ideal splitting ratio of the beam-splitting cube (it is about 51:49 for vertical and 55:45 for horizontal polarization). HOM interferometer
enables us to prepare conditionally polarization singlet states (i.e., $|\Psi^\mp\rangle$ Bell states). The simplest theoretical model of the beamsplitter leads to the conclusion that if one fetches Bell states at the input the only one of them that results in a coincident detection at two different outputs of the beamsplitter is the singlet state $|\Psi^\mp\rangle$. However, in case of a “real” beam-splitting cube one must take into account that the two photons strike upon a beamsplitter in opposite directions. So, the mutual phase (at the interface plane) of the horizontal components of the electric-field vectors from the two opposite inputs is shifted by 180° just for geometrical reasons. Therefore it is the triplet state $|\Psi^+\rangle$ that leads to a coincident detection at different outputs. However, it is easy to change $|\Psi^+\rangle$ to $|\Psi^-\rangle$ by means of a half-wave plate placed in one output arm of the BS.

The measurement block in each output arm consists of a half-wave plate and polarizing beamsplitter (PBS). It enables measurement in any linear-polarization basis. Behind the PBS the beams are filtered by cut-off filters and fed into multi-mode optical fibers leading to detectors $D_1, D_2, D_3, D_4$ (Perkin-Elmer single-photon counting modules; quantum efficiency $\eta \approx 50\%$, dark counts about 100 s$^{-1}$).

The Werner states were prepared as a “mixture” of three kinds of inputs. First we measured coincidences with horizontal and vertical polarizations in the individual inputs of a HOM interferometer (measurement time for each point in the following graphs was 22 s), then we added the results of measurement with two horizontally polarized input photons (this measurement period took 10 s), and finally we measured with two vertically polarized input photons (13 s). The different times of measurement compensated the influence of a glass plate (GP) for the vertical-vertical and horizontal-horizontal input polarizations. The different values of parameter $p$ were obtained changing the position of the scanning mirror. Namely, we have measured at 0 $\mu$m and 30 $\mu$m from the dip center.

The measurement $\Pi_M$ on the “meter” qubit was represented by a measurement in different linear polarization bases parametrized by an angle $\vartheta$: $\Pi_M = \{\Pi_M^+, \Pi_M^-\} \equiv \{\psi\rangle, \rho\rangle\} \equiv \{\psi\rangle, \rho\rangle\}$, where $\psi\rangle = \cos \vartheta |H\rangle + \sin \vartheta |V\rangle$ and $\rho\rangle = \sin \vartheta |H\rangle - \cos \vartheta |V\rangle$. The angle $\vartheta$ was set by a properly rotated half-wave plate. Similarly, two measurements on the “signal” qubit, $\Pi_S$ and $\Pi'_S$, were represented by polarization measurements in two bases rotated by 45°: $\Pi_S = \{\Pi_S^+, \Pi_S^-\} \equiv \{|H\rangle, |V\rangle\}, \Pi'_S = \{\Pi'_S^+, \Pi'_S^-\} \equiv \{|X\rangle, |Y\rangle\}$, where $|X\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|Y\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$. In practice we measured coincidence rates between outputs $\Pi'_M$ and $\Pi'_S$ (it is denoted $C^{++}$), between $\Pi'_M$ and $\Pi_S$ (it is denoted $C^{-+}$), etc. (the first sign concerns the $M$-qubit, the second one the $S$-qubit). Then the knowledge $\mathbf{K}(\vartheta) = \frac{|C^{++} - C^{-+}| + |C^{+-} - C^{--}|}{C^{++} + C^{-+} + C^{+-} + C^{--}}$. (3)

Analogously the a-priori knowledge $\mathbf{P}[\Pi_M^+; \Pi_S^-] = \frac{|C^{++} - C^{-+}| - |C^{+-} - C^{--}|}{C^{++} + C^{-+} + C^{+-} + C^{--}}$ (4)

The knowledge excess is given as $\Delta \mathbf{K}(\vartheta) = \mathbf{K}(\vartheta) - \mathbf{P}$.

The quantities $\mathbf{K}'(\vartheta)$, $\mathbf{P}'$, and $\Delta \mathbf{K}'(\vartheta)$ are obtained in the same way just with $\Pi'_S$ instead of $\Pi_S$. The maximal violation of Bell inequalities, $B_{\text{max}}$, can be obtained by measuring correlation functions for two different polarization bases on each side. Namely, for the Werner states one can choose these bases rotated by 22.5° and 67.5° (with respect to the vertical axis) on the one side and 45° and 0° on the other side:

$B_{\text{max}} = |C(22.5°, 45°)| + C(67.5°, 45°) + C(22.5°, 0°) - C(67.5°, 0°)|$, (5)

where the correlation function $C(\vartheta_1, \vartheta_2)$ is estimated from the measured data as

$C^{++} + C^{+-} - C^{--} - C^{+-}$. (6)

Let us note that for Werner states the theoretical predictions of regarded quantities read: $\mathbf{K} = p |\cos(2\vartheta)|$, $\mathbf{K}' = p |\sin(2\vartheta)|$, $\mathbf{P} = \mathbf{P}' = 0$, $B_{\text{max}} = p 2\sqrt{2}$. Clearly, maximal value of $\Delta \mathbf{K}'(\vartheta)$ should appear for $\vartheta = 0°$ (and 90°), $\vartheta' = 45°$.

The following graphs display our experimental results. In Fig. 2 there are plotted the squares of the knowledge excesses $\Delta \mathbf{K}'(\vartheta)$, $\Delta \mathbf{K}'^2(\vartheta)$ and the sum measured for the Werner state with parameter $p \approx 0.82$ (this parameter was estimated from the best fit accordingly to the theoretical predictions for Werner states). The error bars show statistical errors. The accuracy of polarization-angle settings was about ±1°. Fig. 3 shows the sum $\Delta \mathbf{K}'(\vartheta) + \Delta \mathbf{K}'^2(\vartheta')$ as a function of two angle variables for the same Werner state. The maximal displayed
The squares of the knowledge excesses and their sum measured for the Werner state with $p \approx 0.82$. Symbols show experimental values, full lines theoretical predictions (for $p = 0.82$).

The measured values of the sum $\Delta K^2(\vartheta) + \Delta K'{}^2(\vartheta')$ as a function of two angle variables for the Werner state with $p \approx 0.82$. The maximal displayed value of the vertical axis shows the measured value of $\left(\frac{B_{\text{max}}}{2}\right)^2$.

The same kind of measurement is presented in Fig. 3 but now for the Werner state with $p \approx 0.45$. The corresponding measured maximal Bell factor is $B_{\text{max}} = 1.32 \pm 0.02$ (theoretical value for $p = 0.45$ is 1.273). As can be seen, for the both measured states the experiment has verified inequality [2].

The measurement on the one of two correlated particles give us a power of prediction of the measurement results on the other one. Of course, one can never predict exactly the results of two complementary measurements at once. However, knowing what kind of measurement we want to predict on “signal” particle, we can choose the optimal measurement on the “meter” particle. But there is still a fundamental limitation given by the sort and amount of correlations between the particles. Both these kinds of constraints are quantitatively expressed by our inequality.

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