On the Stability of Jordan-Brans-Dicke Static Universe

Sergio del Campo and Ramón Herrera

Instituto de Física, Pontificia Universidad Católica
de Valparaíso, Casilla 4059, Valparaíso, Chile.

Pedro Labraña

Departamento de Física, Universidad del Bío-Bío,
Avenida Collao 1202, Casilla 5-C, Concepción, Chile.

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Abstract

In this work we study the stability of the Jordan-Brans-Dicke (JBD) static universe. This is motivated by the possibility that the universe might have started out in an asymptotically JBD static state, in the context of the so called emergent universe scenario. We extent our previous results on stability of JBD static universe by considering spatially homogeneous Bianchi type IX anisotropic perturbation modes and by including more general perfect fluids. Contrary to general relativity, we have found that the JBD static universe, dominated by a standard perfect fluid, could be stable against isotropic and anisotropic perturbations. The implications of these results for the initial state of the universe and its pre-inflationary evolution are discussed.

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I. INTRODUCTION

Measures coming from three different independent sources (CMB, Type Ia supernova, and cluster abundances) strongly suggest that the expansion of the universe has been accelerating during recent epoch [1]. This discovery has become one of the main challenges for modern theoretical physics.

In general, dark energy or quintessence, responsible for the cosmic acceleration, determines the features of the evolution of the universe. Certainly, the nature of this sort of energy may lead to the improvement of our picture about particle physics and/or gravitation. Most of these studies have been done in the standard theory of gravity, i.e. general relativity theory (GR). However, motivated mainly from string theories, a less standard theory have been carried out, namely the so called scalar-tensor theory of gravity [2, 3, 4, 5]. An important advantage of these models is that they naturally allow [3, 5] a super-accelerating expansion of the universe where the effective dark energy equation of state $w = p/\rho$ crosses the phantom divide line $w = -1$. Such a crossing is consistent with current cosmological data [6] and if it is confirmed, it would become an enigmatic problem which can not been explained easily with standard quintessence models [7, 8].

The archetypical theory associated with scalar tensor models is the JBD gravity. The JBD theory [2] is a class of models in which the effective gravitational coupling evolves with time. The strength of this coupling is determined by a scalar field, the so-called JBD field, which tends to the value $G^{-1}$, the inverse of the Newton’s constant. The origin of JBD theory is in Mach’s principle according to which the property of inertia of material bodies arises from their interactions with the matter distributed in the universe. In modern context, JBD theory appears naturally in supergravity models, Kaluza-Klein theories and in all known effective string actions [9, 10, 11, 12, 13, 14, 15].

The study of static universe and its stability has always been of great interest since the pioneer work of Eddington [16]. For example, in the context of GR the stability of the Einstein static (ES) universe in the presence of conventional matter field has been studied in Refs. [17, 18, 19]. In the presence of ghost scalar field it was studied in Ref. [20]. Also, the ES universe has been studied in different gravitational theories. In GR a generalization which include a variable pressure have been analyzed in Ref. [21]. In the context of brane world models it was considered in Refs. [22, 23, 24, 25]. The study in the Einstein-Cartan
theory it is found in Ref. [26]. In loop quantum cosmology, this subject has been studied in Refs. [27, 28, 29]. The stability of the ES universe in a $f(R)$ gravity and in modified Gauss-Bonnet gravity theories have been studied in Refs. [30] and [31], respectively.

Recently, the stability of ES models has become relevant for the study of cosmological scenarios in which the ES universe corresponds to an initial state for a past-eternal inflationary cosmology, the so-called emergent universe scenario [32]. The original idea of an emergent universe [32, 33] is that in which the universe emerges from an ES universe state with radius $a_0 \gg l_p$ (where $a_0$ is the scale factor at some instant and $l_p$ is the Planck length), inflates and then is subsumed into a hot Big Bang era. Such models are appealing since they provide specific examples of nonsingular (geodesically complete) inflationary universes. Also, these models could avoid an initial quantum-gravity stage if the static radius is larger than the Planck length.

However, the emergent universe models based on GR, with ordinary matter, suffer from a number of important shortcomings. In particular, the instability of the ES state [23, 24, 28, 29] makes it extremely difficult to maintain such a state for an infinitely long time. The instability of the ES solution ensures that any perturbations, no matter how small, rapidly force the universe away from the static state, thereby aborting the scenario.

Some models have been proposed to solve the stability problem of the asymptotic static solution. They consider non-perturbative quantum corrections of the Einstein field equations, either coming from a semiclassical state in the framework of loop quantum gravity (LQG) [28, 29] or braneworld cosmology with a timelike extra dimension [23, 24]. Other possibilities to consider are the Starobinsky model or exotic matter [34, 35].

On the other hand, it has been shown that a scalar tensor theory could solve the problem of the instability of the emergent universe models. In particular, in Ref. [36], it was found that a self-interacting JBD theory presents a stable past eternal static solution, which eventually enters a phase where the stability of this solution is broken leading to an inflationary period, providing in this way, an explicit construction of an emergent universe scenario.

In this work, we study the stability of the JBD static universe. This is motivated by the possibility that the universe might have started out in an asymptotically JBD static state [30]. We extent our previous results on the stability of JBD static universe by consider spatially homogeneous Bianchi type IX anisotropic perturbation modes and by including more general perfect fluids. General anisotropic perturbations are important to be consider
because they could be the crucial destabilizing modes of a static universe, see Refs. \[18, 19\]. Contrary to the GR case we have found that the JBD static universe dominated by a perfect fluid could be stable against isotropic and anisotropic perturbations for some sort of matter components.

The paper is organized as follows. In Sect. II we review briefly the cosmological equations of the JBD model. In Sect. III the existence and nature of static solutions are discussed for universes dominated by a standard perfect fluid and by a scalar field. In Sect. IV we study the stability of the JBD static universe against small anisotropic perturbations. In Sect. V we focus in a particular example of a JBD potential which allow us to use a dynamical system approach to study the problem of stability of the JBD static universe. In Sect. VI we summarize our results.

II. THE MODEL

We consider the following JBD action for a self-interacting potential and matter, given by \[2\]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Phi R - \frac{1}{2} \frac{w}{\Phi} \nabla_\mu \Phi \nabla^\mu \Phi + V(\Phi) + \mathcal{L}_m \right],
\]

where \(\mathcal{L}_m\) denote the Lagrangian density of the matter, \(R\) is the Ricci scalar curvature, \(\Phi\) is the JBD scalar field, \(w\) is the JBD parameter and \(V(\Phi) = V\) is the potential associated to the field \(\Phi\). In this theory \(1/\Phi\) plays the role of the gravitational constant, which changes with time. This action also matches the low energy string action for \(w = -1\) \[15\].

From the Lagrangian density, Eq. (1), we obtain the field equations:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{w}{\Phi^2} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{\Phi} \nabla_\mu \nabla_\nu \Phi + g_{\mu\nu} \left( \frac{\Box \Phi}{\Phi} + \frac{w}{2\Phi^2} (\nabla \Phi)^2 - \frac{V(\Phi)}{\Phi} \right) = \frac{1}{\Phi} T_{\mu\nu},
\]

(2)

and

\[
\Box \Phi = \frac{1}{3 + 2w} T^\mu_\mu + \frac{2}{3 + 2w} \left[ 2V - \Phi \frac{dV}{d\Phi} \right],
\]

(3)
where we have consider that \( \Phi \) is a function of the cosmological time, \( t \), only. Units are such that \( c = \hbar = 1 \).

### III. THE STATIC UNIVERSE SOLUTION IN JBD THEORY

Let us start by considering the closed Friedmann-Robertson-Walker metric:

\[
\begin{align*}
    ds^2 &= dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right], \quad (4)
\end{align*}
\]

where \( a(t) \) is the scale factor, \( t \) represents the cosmic time. The matter content of the universe is modelled by a perfect fluid with effective equation of state given by \( P = (\gamma - 1) \rho \). In general, when the perfect fluid is described by a scalar field, it is found that the parameter \( \gamma \) becomes variable. Thus, by using the metric, Eq. (4), the set of field equations (2) and (3) become

\[
\begin{align*}
    H^2 + \frac{1}{a^2} + \frac{\dot{\Phi}}{\Phi} \frac{\dot{\Phi}}{\Phi} &= \frac{\rho}{3 \Phi} + \frac{w}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + \frac{V}{3 \Phi}, \quad (5)
\end{align*}
\]

\[
\begin{align*}
    2 \ddot{a} \frac{a}{\dot{a}} + H^2 + \frac{1}{a^2} + \frac{\ddot{\Phi}}{\Phi} + 2 H \frac{\dot{\Phi}}{\Phi} + \frac{w}{2} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - \frac{V}{\Phi} &= - \frac{P}{\Phi}, \quad (6)
\end{align*}
\]

and

\[
\begin{align*}
    \ddot{\Phi} + 3 H \dot{\Phi} &= \frac{(\rho - 3P)}{(2w + 3)} + \frac{2}{2w + 3} [2 V - \Phi V' \Phi'] . \quad (7)
\end{align*}
\]

The energy-momentum conservation implies that

\[
\dot{\rho} + 3H(\rho + P) = 0 , \quad (8)
\]

where \( V' = dV(\Phi)/d\Phi \). Dots mean derivatives with respect to the cosmological time.

In the context of JBD theory the static solutions are closed universes characterized by the conditions \( a = a_0 = Const., \dot{a}_0 = 0 = \ddot{a}_0 \) and \( \Phi = \Phi_0 = Cte., \dot{\Phi}_0 = 0 = \ddot{\Phi}_0 \), see Ref. [36].

Then the static solution for a universe dominated by a general perfect fluid is obtained if the following conditions are fulfilled

\[
\begin{align*}
    a_0^2 &= \frac{3}{V_0'}, \quad (9)
\end{align*}
\]

\[
\begin{align*}
    \rho_0 &= V_0' \Phi_0 - V_0 , \quad (10)
\end{align*}
\]
and

\[ \gamma_0 = \frac{2}{3} \left( 1 + \frac{V_0}{\rho_0} \right) \frac{2 \Phi_0}{a_0^2 \rho_0}, \]

(11)

where \( V_0 = V(\Phi_0) \) and \( V'_0 = \left( dV(\Phi)/d\Phi \right)_{\Phi=\Phi_0} \). These equations connect the equilibrium values of the scale factor and the JBD field with the energy density and the JBD potential at the equilibrium point.

Note that in order to obtain a static solution we need to have a non-zero JBD potential with a non-vanishing derivative at the static point \( \Phi = \Phi_0 \). The original Brans-Dicke model corresponds to \( V(\Phi) = 0 \). However, non-zero \( V(\Phi) \) is better motivated and appears in many particle physics models. In particular, \( V(\Phi) \) can be chosen in such a way that \( \Phi \) is forced to settle down to a non-zero expectation value, \( \Phi \to m_p^2/8\pi \), where \( m_p = 10^{19}\text{GeV} \) is the present value of the Planck mass. On the other hand, if \( V(\Phi) \) fixes the field \( \Phi \) to a non-zero value, then time-delay experiments place no constraints on the Brans-Dicke parameter \( w \) \[37\]. In particular, if we choose the JBD potential in such a way that \( \Phi \) will be stabilized at a constant value, let say \( \Phi_f \), at the end of the inflationary period (see Ref. \[36\] as an example), we can recover GR by setting \( \Phi_f = m_p^2/8\pi \), together with an appropriated value for the parameter \( w \) which will be in agreement with the solar system bound \[37, 38\].

A. JBD static universe dominated by a standard perfect fluid

The static solution is characterized by the Eqs. (9-11), from which we obtain \( \gamma \geq \frac{2}{3} \) as a condition for a static solution, if the JBD potential, \( V(\Phi) \), is positive \[36\]. Notice that this means that it is not possible to have a static solution if the universe is dominated by the cosmological constant (corresponding to \( \gamma = 0 \) in the equation of state), but it is possible to have a static universe when it is dominated by dust or radiation, among others possibilities.

Now, we study the stability of this solution against small homogeneous and isotropic perturbations. In order to do this, we consider small perturbations around the static solution for the scale factor and the JBD field. We set

\[ a(t) = a_0 \left[ 1 + \varepsilon(t) \right], \]

(12)

and

\[ \Phi(t) = \Phi_0 \left[ 1 + \beta(t) \right]. \]

(13)
Then, we have
\[ \rho = \rho_0 + \delta \rho(\varepsilon) \approx \rho_0 - 3\gamma \rho_0 \varepsilon, \]  
(14)
where \( \varepsilon \ll 1 \) and \( \beta \ll 1 \) are small perturbations. By introducing expressions (12), (13) and (14) into Eq. (6) and Eq. (7), and retaining terms at the linear order on \( \epsilon \) and \( \beta \), we obtain the following coupled equations
\[ \ddot{\varepsilon} - \left[ \frac{1}{a_0^2} + 3 \left( \frac{\gamma - 1}{a_0^2} \right) \right] \varepsilon - \frac{\dot{\beta}}{a_0^2} = 0, \]  
(15)
and
\[ (3 + 2w) \ddot{\beta} - \left( \frac{6}{a_0^2} - 2 \Phi_0 V'' \right) \beta + (4 - 3\gamma) \frac{6}{a_0^2} \varepsilon = 0, \]  
(16)
where \( V'' = \frac{d^2 V(\Phi)}{d\Phi^2} \). From the system of Eqs. (15) and (16) we can obtain the frequencies for small oscillations
\[ \omega_{\pm}^2 = \frac{1}{a_0^2 (3 + 2w)} \left[ a_0^2 \Phi_0 V'' - 6 + w (2 - 3\gamma) \right] \]  
\[ \pm \sqrt{\left[ -6 + a_0^2 \Phi_0 V'' + 2w - 3w \gamma \right]^2 + 2(3 + 2w) \left( -6 + a_0^2 \Phi_0 V'' [3\gamma - 2] \right)}. \]  
(17)
Note that the static solution is stable if the inequality, \( \omega_{\pm}^2 > 0 \), is fulfilled. Assuming that the parameter \( w \) satisfies the constraint, \( (3 + 2w) > 0 \), it is found that the following inequalities must be achieved in order to have a stable static solution
\[ \frac{2}{3} < \gamma < \frac{4}{3}, \]  
or \[ \frac{4}{3} < \gamma, \]  
(18)
\[ -\frac{3}{2} \leq w < -18 \frac{(\gamma - 1)}{(2 - 3\gamma)^2}, \]  
(19)
and
\[ 2(6 + w) - 3(3 + w)\gamma + \sqrt{3} |4 - 3\gamma| \sqrt{3 + 2w} < a_0^2 \Phi_0 V'' < \frac{6}{3\gamma - 2}. \]  
(20)
From these inequalities we can conclude that for a universe dominated by a standard perfect fluid (with \( \gamma > 2/3 \)), it is possible to find a solution where the universe is static and stable. Here, the only exception is radiation, where \( \gamma = 4/3 \), which becomes explicitly excluded by the latter inequalities. This peculiar behavior for radiation could be understood due to the particular way in which the perfect fluid appears in the equation for the JBD field,
Eq. (7), where it becomes independent of the energy density and pressure of the fluid. Then, we can note that in the radiation case $w_+^2$ are both real numbers. On the other hand, $w_-^2$ could be a positive number, but $w_-^2$ is always negative. Therefore, in this case, we have a saddle instability.

### B. Scalar fields

In the context of emergent universe models, the static JBD universe dominated by a scalar field (inflaton) was studied in Ref. [36]. Here, we reproduce the main results concerning this static solution.

The energy density, $\rho$, and the pressure, $P$, are expressed by the following equations

$$\rho = \frac{\dot{\Psi}^2}{2} + U(\Psi),$$

and

$$P = \frac{\dot{\Psi}^2}{2} - U(\Psi).$$

Here, $U(\Psi)$ represents the scalar potential associated to the scalar field $\Psi$.

We could write an effective equation of state for the scalar field, $\Psi$, expressed by the equation $P = (\gamma - 1) \rho$, where the equation of state ”parameter”, $\gamma$, could be written as

$$\gamma = 2 \left(1 - \frac{U(\Psi)}{\rho}\right).$$

During the static regimen, in the context of an emergent universe models, the matter potential $U(\Psi)$ is consider as a flat potential, that is $U(\Psi) = U_0 = Const.$ and the scalar field rolls along this potential with a constant velocity $\dot{\Psi}_0$. The conditions for static universes, Eqs. (9-11), imply that the following condition for the state parameter

$$\gamma_0 = 2 \frac{\Phi_0}{a_0\rho_0} \left(1 - \frac{U_0}{\rho_0}\right),$$

must be satisfied.

The velocity when the scalar field $\Psi$ is rolling along a constant potential, $U_0$, it becomes expressed in terms of the static values of the scale factor, $a_0$, and the JBD field, $\Phi_0$. It results to be

$$\dot{\Psi}_0^2 = 2 \frac{\Phi_0}{a_0^2}.$$
Assuming $(3 + 2w) > 0$, the following stability conditions were obtained

$$0 < a_0^2 \Phi_0 V_0'' < \frac{3}{2},$$  \hspace{1cm} (26)

and

$$-\frac{3}{2} < w < -\frac{1}{4} \left[ \sqrt{9 - 6a_0^2 \Phi_0 V_0''} + (3 + a_0^2 \Phi_0 V_0'') \right].$$  \hspace{1cm} (27)

In relation to the conditions (26) and (27) let us mention that the first inequality imposes a condition on the JBD potential, specifically for its first and second derivatives: $0 < V_0'' < V_0'/(2\Phi_0)$. The second inequality restricts the values of the JBD parameter. Notice that this inequality imposes that $w < 0$. JBD models with negative values of $w$ have been considered in the context of late acceleration expansion of the universe \[39, 40\], but also appear in low energy limits of string theory \[15\]. On the other hand, as was mentioned above, we choose the JBD potential, $V(\Phi)$, in such a way that $\Phi$ will be stabilized at a constant value, namely $\Phi_f = m_p^2/8\pi$.

Thus, from Eqs. (26) and (27) we can conclude that for a universe dominated by a scalar field it is possible to obtain a static solution, stable under homogenous and isotropic perturbation.

IV. ANISOTROPIC PERTURBATIONS

If we are interested in studying the stability of the static universe an important point, showed in Ref. \[19\], is that the crucial destabilizing modes are not only the conformal perturbations considered in the previous section. Anisotropic perturbations could be even more important. For example, in the case of the ES universe it is known that the static solution is neutrally stable to inhomogeneous scalar perturbations with high enough sound speed and to vector and tensor isotropic perturbations \[18, 19\]. However, this analysis does not cover spatially homogeneous, but anisotropic modes. It turns out that there are various unstable spatially homogeneous anisotropic modes \[19\]. This suggest that anisotropic perturbations could be the crucial destabilizing modes.

In this section we proceed to study the stability of the static solution found in the previous section against theses anisotropic perturbations modes. In particular, we consider the general case of spatially homogeneous Bianchi type IX perturbations modes.
In this context, the JBD static universe is a particular exact solution of the Bianchi type IX, or Mixmaster universe, containing a perfect fluid. The Mixmaster is a spatially homogeneous closed (compact space sections) universe of the most general type. It contains the closed isotropic Friedmann universes as particular cases when a fluid is present. Physically, the Mixmaster universe arises from the addition of expansion anisotropy and 3-curvature anisotropy to the Friedmann universe.

The diagonal type IX universe has three expansion scale factors, i.e., \(a(t), b(t)\) and \(c(t)\), and the diagonal Bianchi IX metric is expressed by

\[
ds^2 = dt^2 - \eta_{\alpha\beta}(t) w^\alpha w^\beta,
\]

where

\[
\eta_{\alpha\beta}(t) = \begin{pmatrix}
a^2(t) & 0 & 0 \\
0 & b^2(t) & 0 \\
0 & 0 & c^2(t)
\end{pmatrix},
\]

and the \(w^\alpha\) are differential 1-forms invariant under \(SO(3)\) transformation.

In the following we will consider a universe dominated by a general perfect fluid whose equation of state is \(P = (\gamma - 1)\rho\). We will assume that \(\Phi\) and \(\rho\) are function of the time \(t\), only. Then, by using the metric, Eq. (28), in the action (1), we obtain the following set of equations for the non-null components. The (0,0) component becomes

\[
\frac{1}{2a^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{a^2}{4b^2c^2} - \frac{b^2}{4a^2c^2} - \frac{c^2}{4a^2b^2} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{\dot{a}}{a} \frac{\dot{c}}{c} + \frac{\dot{b}}{b} \frac{\dot{c}}{c} = \rho - \left(\frac{\dot{\Phi}}{\Phi} + \frac{\ddot{\Phi}}{\Phi} + \frac{w}{2} \left(\frac{\ddot{\Phi}}{\Phi}\right)^2 + \frac{V(\Phi)}{\Phi}\right).
\]

The (1,1) component is given by

\[
- \frac{1}{2a^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{3a^2}{4b^2c^2} + \frac{1}{4a^2c^2} + \frac{1}{4a^2b^2} + \frac{\dot{b}}{b} \frac{\dot{c}}{c} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = \frac{P}{\Phi} - \left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) \frac{\Phi}{\Phi} - \left(\frac{\Phi}{\Phi}\right) - \frac{w}{2} \left(\frac{\ddot{\Phi}}{\Phi}\right)^2 + \frac{V(\Phi)}{\Phi}.
\]
The other two nonzero equations, components (2,2) and (3,3), are just cyclic changes in the scale factors \((a, b, c)\) in Eq. (31).

The equation for the JBD field, Eq. (3), becomes given by

\[
\ddot{\Phi} + \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) \dot{\Phi} = \frac{(\rho - 3P)}{3 + 2w} + \frac{2}{3 + 2w} \left[ 2V - \Phi \frac{dV}{d\Phi} \right].
\] (32)

On the other hand, the conservation of energy-momentum implies that

\[
\dot{\rho} + \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) (\rho + P) = 0.
\] (33)

The static solution discussed in the previous section correspond to the case where \(a(t) = b(t) = c(t) = \frac{1}{2} a_0\) and \(\Phi = \Phi_0\). Here, the constant values, \(a_0\) and \(\Phi_0\), satisfy the conditions for a static solution. This was discussed in Sect. [II]

In order to study the stability of this solution against anisotropic Bianchi type IX perturbations, we take small perturbations around the static solutions of the scale factors and the JBD field. We set

\[
a(t) = \frac{a_0}{2} \left[ 1 + \varepsilon_1(t) \right],
\] (34)

\[
b(t) = \frac{a_0}{2} \left[ 1 + \varepsilon_2(t) \right],
\] (35)

and

\[
c(t) = \frac{a_0}{2} \left[ 1 + \varepsilon_3(t) \right],
\] (36)

together with the perturbation associated to the JBD field, expressed by Eq.(13). Here, the parameters \(\varepsilon_i (i = 1, 2, 3)\), just like the parameter \(\beta\), are small perturbations. Therefore, they satisfy \(\varepsilon_i \ll 1\).

For the energy density, pressure and state parameter we take

\[
\rho = \rho_0 + \delta \rho(\varepsilon_1, \varepsilon_2, \varepsilon_3),
\] (37)

\[
P = P_0 + \delta P(\varepsilon_1, \varepsilon_2, \varepsilon_3),
\] (38)

and

\[
\gamma = \gamma_0 + \delta \gamma(\varepsilon_1, \varepsilon_2, \varepsilon_3),
\] (39)
respectively. The specific form of \( \delta \rho, \delta P \) and \( \delta \gamma \) depend on the kind of perfect fluid under consideration.

Now, introducing these latter expressions into Eqs. (30-33) and retaining only the linear terms in the perturbation parameters, we obtain that

\[
-6 \frac{\varepsilon_1}{a_0^2} + 2 \frac{\varepsilon_2}{a_0^2} + 2 \frac{\varepsilon_3}{a_0^2} + \ddot{\varepsilon}_2 + \ddot{\varepsilon}_3 = -\frac{\delta P}{\phi_0} + 2 \frac{\beta}{a_0^2} - \ddot{\beta},
\]  

(40)

\[
-6 \frac{\varepsilon_2}{a_0^2} + 2 \frac{\varepsilon_3}{a_0^2} + 2 \frac{\varepsilon_1}{a_0^2} + \ddot{\varepsilon}_3 + \ddot{\varepsilon}_1 = -\frac{\delta P}{\phi_0} + 2 \frac{\beta}{a_0^2} - \ddot{\beta},
\]  

(41)

\[
-6 \frac{\varepsilon_3}{a_0^2} + 2 \frac{\varepsilon_1}{a_0^2} + 2 \frac{\varepsilon_2}{a_0^2} + \ddot{\varepsilon}_1 + \ddot{\varepsilon}_2 - \frac{\delta P}{\phi_0} + 2 \frac{\beta}{a_0^2} - \ddot{\beta},
\]  

(42)

and

\[
(3 + 2w) \phi_0 \ddot{\beta} = (4 - 3\gamma_0) \delta \rho - 3\delta \gamma \rho_0 + 2 V'_0 \phi_0 \beta - 2 \phi_0^2 V''_0 \beta.
\]  

(43)

In the next subsections we study universes dominated by different type of perfect fluids.

**A. Standard perfect fluid**

Here, we consider the case of a universe dominated by a standard perfect fluid, where \( \gamma \) is a constant. Then, Eqs. (37) and (38) become

\[
\rho = \rho_0 + \delta \rho(\varepsilon_1, \varepsilon_2, \varepsilon_3) \approx \rho_0 - \gamma \rho_0 [\varepsilon_1 + \varepsilon_2 + \varepsilon_3],
\]  

(44)

and

\[
P = P_0 + \delta P(\varepsilon_1, \varepsilon_2, \varepsilon_3) \approx P_0 + \gamma (1 - \gamma) \rho_0 [\varepsilon_1 + \varepsilon_2 + \varepsilon_3],
\]  

(45)

respectively. By introducing these expressions into Eqs. (40-43), and retaining the linear order in the parameters \( \varepsilon_i (i = 1, 2, 3) \) and \( \beta \), we obtain a set of four coupled equations.

The general solution of this set of equations may be written as

\[
\begin{pmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t) \\
\varepsilon_3(t) \\
\beta(t)
\end{pmatrix} =
\begin{pmatrix}
\bar{\varepsilon}_1 \\
\bar{\varepsilon}_2 \\
\bar{\varepsilon}_3 \\
\bar{\beta}
\end{pmatrix} e^{iwt},
\]  

(46)
where \( \varepsilon_i \) and \( \beta \) are constants. The frequencies corresponding to small oscillations are given by

\[
\omega^2_1 = \frac{8}{a_0^2}, \quad (47)
\]
\[
\omega^2_2 = \frac{8}{a_0^2}, \quad (48)
\]

and

\[
\omega^2_\pm = \frac{1}{a_0^2(3 + 2w)} \left[ a_0^2 \Phi_0 V''_0 - 6 + w(2 - 3\gamma) \right] \pm \sqrt{\left[ -6 + a_0^2 \Phi_0 V''_0 + 2w - 3w\gamma \right]^2 + 2(3 + 2w) \left( -6 + a_0^2 \Phi_0 V''_0 \left[ 3\gamma - 2 \right] \right)}. \quad (49)
\]

The static solution is stable if \( \omega^2_\pm > 0 \). Assuming that \( 3 + 2w > 0 \) we find that this solution is stable against anisotropic perturbations, providing that Eqs. \((18-20)\) are fulfilled.

The oscillation mode which belongs to the frequency \( \omega_1 \) is given by

\[
\begin{pmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t) \\
\varepsilon_3(t) \\
\beta(t)
\end{pmatrix} = C_1 \begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix} e^{i\omega_1 t}. \quad (50)
\]

Similarly, the oscillation mode corresponding to \( \omega_2 \) is:

\[
\begin{pmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t) \\
\varepsilon_3(t) \\
\beta(t)
\end{pmatrix} = C_2 \begin{pmatrix}
-1 \\
1 \\
0 \\
0
\end{pmatrix} e^{i\omega_2 t}. \quad (51)
\]

Finally, the oscillation modes corresponding to \( \omega_\pm \) are:

\[
\begin{pmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t) \\
\varepsilon_3(t) \\
\beta(t)
\end{pmatrix} = C_\pm \begin{pmatrix}
A_\pm \\
A_\pm \\
A_\pm \\
1
\end{pmatrix} e^{i\omega_\pm t}, \quad (52)
\]
where \( C_i (i=1,2) \) and \( C_\pm \) are arbitrary constants. On the other hand, the constants \( A_\pm \) are given by

\[
A_\pm = \frac{1}{6(3\gamma - 4)} \left[ a_0^2 \Phi_0 V_0'' - w(2 - 3\gamma) \right.
\]

\[
\pm \sqrt{\left[ -6 + a_0^2 \Phi_0 V_0'' + 2w - 3w\gamma \right]^2 + 2(3+2w) \left( -6 + a_0^2 \Phi_0 V_0'' [3\gamma - 2] \right)}.
\]

Notice that the oscillation modes corresponding to the frequencies \( w_1 \) and \( w_2 \) are anisotropic oscillations around the equilibrium point. In these oscillations the JBD field remains static at its equilibrium point, \( \Phi_0 \). On the other hand, the oscillation modes, related to the frequencies \( w_\pm \) are isotropic oscillations around the same point, but where now the JBD field oscillate.

We note that this stability behavior is completely different wherewith it happens with the ES solution, where it was found that spatially homogeneous Bianchi type IX modes destabilize the static solution \[19\].

### B. Scalar field

In this case, we consider a universe dominated by a scalar field. Following a similar scheme to that of Sec. III, we take a flat matter potential, \( U(\Psi) \), with a scalar field \( \Psi \) rolling along its potential with a constant velocity satisfying the conditions for a static universe. We study the stability of this solution against anisotropic Bianchi type IX perturbation modes.

In this case the set of Eqs. (37-39) becomes

\[
\rho = \rho_0 + \delta \rho(\varepsilon_1,\varepsilon_2,\varepsilon_3) \approx \rho_0 - \gamma_0 \rho_0 \left[ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \right],
\]

\[
P = P_0 + \delta P(\varepsilon_1,\varepsilon_2,\varepsilon_3) \approx P_0 + \left( -\frac{2U_0}{\rho} + \gamma_0 (1 - \gamma_0) \rho_0 \right) \left[ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \right].
\]

and

\[
\gamma = \gamma_0 + \delta \gamma(\varepsilon_1,\varepsilon_2,\varepsilon_3) \approx \gamma_0 - 2\frac{\gamma_0 U_0}{\rho_0} \left[ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \right],
\]
We introduce these latter expressions into Eqs. (40)-(43) and, just like above, we retain the linear terms in the parameters $\epsilon_i$ and $\beta$. In this way, we obtain a set of four coupled equations. A general solution to this set of equations could be written as Eq. (46).

In this case, the frequencies for small oscillation are given by

$$w_1^2 = \frac{8}{a_0^2},$$  \hspace{1cm} (57)  

$$w_2^2 = \frac{8}{a_0^2},$$  \hspace{1cm} (58)

and

$$\omega^2_\pm = \frac{1}{a_0^2(3+2w)} \left[ a_0^2 \Phi_0 V''_0 - 2(3+2w) \right]$$  \hspace{1cm} (59)  

$$\pm \sqrt{\left[ a_0^2 \Phi_0 V''_0 + 4a_0^2 \Phi_0 V''_0(3+2w) + 8w(3+2w) \right]}.$$

If $\omega^2_\pm > 0$ the static solution is stable. Assuming that $(3+2w) > 0$, we find that this solution is stable against anisotropic perturbations, if the ranges expressed by expressions (26, 27) are fulfilled. Notice that these constrains are the same constrains which were found previously in Ref. [36], where the stability of this static solution against homogeneous and isotropic perturbations was studied.

The oscillation modes corresponding to these perturbations share similar properties than that the ones discussed in the previous section. In particular, they could be expressed by the same expressions, Eqs. (50, 51, 52), but where now $w_1$, $w_2$ and $w_\pm$ are given by Eqs. (57, 58, 59) respectively, and $A_\pm$ is given by

$$A_\pm = \frac{1}{12} \left[ a_0^2 \Phi_0 V''_0 + 4w \pm \sqrt{\left[ a_0^2 \Phi_0 V''_0 \right]^2 + 4a_0^2 \Phi_0 V''_0(3+2w) + 8w(3+2w)} \right].$$  \hspace{1cm} (60)

This modification of the stability behavior has important consequences for the emergent universe scenario, since it ameliorates the fine-tuning that arises from the fact that the ES model is an unstable saddle in GR.

V. POLYNOMIAL JBD POTENTIAL

As a particular but interesting example, we consider the case where the JBD potential is a polynomial in the scalar field $\Phi$. 

15
\[ V(\Phi) = C_0 + C_1 \Phi + C_2 \Phi^2, \]  
(61)

where \( C_0, C_1 \) and \( C_2 \) are constants. Also, we consider a homogenous and isotropic closed universe described by a Friedmann-Robertson-Walker metric Eq. (4). As a matter content we take a standard perfect fluid.

It is interesting to notice that under these consideration and following the scheme of Refs. [41, 42] we can rewrite the field equations of this model, Eqs. (5-7), as an autonomous system. In order to do so, we first rewrite Eqs. (5, 7) together with the conservation of energy equation, by means of the conformal time

\[ \eta = \int \frac{dt}{a(t)}. \]

Thus, we obtain that

\[
\left( \frac{a'}{a} + \frac{\Phi'}{2\Phi} \right)^2 + 1 = \rho \frac{a^2}{3\Phi} + \frac{2w + 3}{12} \left( \frac{\Phi'}{\Phi} \right)^2 + \frac{C_0 a^2}{3\Phi} + \frac{V_1 a^2}{3\Phi} + \frac{V_2 a^2}{3\Phi},
\]

(62)

\[
\frac{\Phi''}{a^2} + 2\Phi' \frac{a'}{a^3} = \frac{(4 - 3\gamma)}{(2w + 3)} \rho + \frac{2}{2w + 3} \left[ 2V_0 + V_1 \right],
\]

(63)

and the conservation of energy-momentum becomes

\[
\rho' + 3 \frac{a'}{a} \gamma \rho = 0,
\]

(64)

where, we have used the following definitions

\[ V_1 = C_1 \Phi, \]

(65)

and

\[ V_2 = C_2 \Phi^2. \]

(66)

Following Refs. [41, 42] we introduce the set of variables
\[ X = \sqrt{\frac{2w + 3}{12} \Phi'} = A \frac{\Phi'}{\Phi}, \quad (67) \]
\[ Y = \frac{\alpha'}{a} + \frac{\Phi'}{2\Phi}, \quad (68) \]
\[ Z_0 = \frac{C_0 a^2}{\Phi}, \quad (69) \]
\[ Z_1 = \frac{V_1 a^2}{\Phi}, \quad (70) \]

and
\[ Z_2 = \frac{V_2 a^2}{\Phi}. \quad (71) \]

Now, we rewrite Eqs. (62) and (63), together with the energy-momentum conservation in these variables as follows

\[ Y^2 + 1 = \frac{\rho a^2}{3\Phi} + X^2 + \frac{Z_0}{3} + \frac{Z_1}{3} + \frac{Z_2}{3}, \quad (72) \]

and

\[ X' = -2XY + \left(1 - \frac{3}{4}\right) \frac{\rho a^2}{3A\Phi} + \frac{Z_0}{6A} + \frac{Z_1}{6A}. \quad (73) \]

Differentiating Eq. (72) and from the equation for the \( X \) variable, together with the energy-momentum conservation, we obtain that

\[ X' = -2XY + \left(1 - \frac{3}{4}\right) \left[ \frac{Y^2 + 1 - X^2}{A} \right] + \frac{\gamma Z_0}{4A} \]
\[ + \left( \frac{\gamma}{4} - \frac{1}{6} \right) \frac{Z_1}{A} + \left( \frac{3}{4}\gamma - 1 \right) \frac{Z_2}{3A}, \quad (74) \]
\[ Y' = -2X^2 + \left(1 - \frac{3}{2}\right) \left[ Y^2 + 1 - X^2 \right] + \frac{\gamma}{2} (Z_0 + Z_1 + Z_2), \quad (75) \]
\[ Z_0' = 2Z_0 \left[ -\frac{X}{A} + Y \right], \quad (76) \]
\[ Z_1' = 2Z_1 \left[ -\frac{X}{2A} + Y \right], \quad (77) \]
and

\[ Z'_2 = 2Z_2 Y . \]  \hspace{1cm} (78)

Requiring that \( \rho > 0 \), we get from Eq. (72) that

\[ Y^2 - X^2 - \frac{Z_0}{3} - \frac{Z_1}{3} - \frac{Z_2}{3} + 1 \geq 0 . \]  \hspace{1cm} (79)

In the set of equations (74)-(78) we look for critical points. In particular, we are interested in critical points related to static universes which were discussed in Sect. III. Thus, from Eqs. (74)-(78) together with expression (67) and (68), the critical points correspond to

\[ X = Y = 0, \ Z_0 = \bar{Z}_0, \ Z_1 = \bar{Z}_1 \] \quad and \quad \[ Z_2 = \bar{Z}_2, \]

where

\[ \bar{Z}_0 = \frac{3}{2} - \frac{2}{\gamma} - \frac{\bar{Z}_1}{2}, \] \hspace{1cm} (80)

and

\[ \bar{Z}_2 = \frac{3}{2} - \frac{Z_1}{2}. \] \hspace{1cm} (81)

Then, we have a set of critical points which represents different static universes. They depend on the arbitrary value of \( \bar{Z}_1 \). Actually, the possibility of obtaining stable or instable critical points depends on the value of \( \bar{Z}_1 \). In the following, we will give a range for the parameter \( \bar{Z}_1 \) where the corresponding solutions are stable (see Eqs. (84) and (87)).

In order to study the nature of these critical points we linearize the set of equations (74)-(78) near the critical points. From the study of the eigenvalues of the system we found that the critical points could be centers or saddles points, depending on the values of the parameters of the model (\( \gamma \) and \( A \)) and on the value of \( \bar{Z}_1 \). Stable static solutions correspond to a center, and this imposes the following conditions for the parameters \( A \) and \( \gamma \), and for the value of \( \bar{Z}_1 \).

\[ \frac{2}{3} < \gamma < \frac{4}{3}, \] \hspace{1cm} (82)

\[ 0 < A < \frac{4 - 3\gamma}{6\gamma - 4}, \] \hspace{1cm} (83)

and

\[ \frac{9\gamma - 12}{3\gamma - 2} < \bar{Z}_1 < -6(1 + 2A(2 + A)) + \frac{9}{2}(1 + 2A)^2 \gamma, \] \hspace{1cm} (84)

or

18
FIG. 1: Plot showing the evolution of two numerical solutions for a universe dominated by dust.

\[
\frac{4}{3} < \gamma < \infty, \quad (85)
\]

\[
0 < A < \frac{3\gamma - 4}{6\gamma - 4}, \quad (86)
\]

and

\[
\frac{9\gamma - 12}{3\gamma - 2} < \bar{Z}_1 < -6(1 - 2A(2 - A)) + \frac{9}{2}(1 - 2A)^2 \gamma. \quad (87)
\]

These conditions are in agreement with the general stability conditions that were found previously in Sect. III (see Eqs. (18-20)).

In Fig. 1 it is shown a projection of the axis \(X\) and \(Z_1\). This represents the evolution of two numerical solutions for a universe dominated by dust. In order to satisfy the requirements of stability we have taken the values \(A = 0.008\) and \(\bar{Z}_1 = -2\). Here, the critical point, which in this graph corresponds to the point \(X = 0\) and \(Z_1 = \bar{Z}_1 = -2\), represents a center.

In Fig. 2 it is shown a projection of the axis \(X\) and \(Z_1\) of two numerical solutions for the case where the universe is dominated by a scalar field moving in a null scalar potential. In order to satisfy the requirements of stability we take \(A = 0.008\) and \(\bar{Z}_1 = 2\). As we expect,
FIG. 2: Plot showing the evolution of two numerical solutions for a universe dominated by a scalar field.

the critical point, which in this graph correspond to the point $X = 0$ and $Z_1 = \bar{Z}_1 = 2$, it is a center.

**VI. CONCLUSIONS**

In this paper, we have studied the stability of the JBD static universe model. This is motivated by the possibility that the universe might have started out in an asymptotically JBD static state, in the context of the so called emergent universe models.

We extent our previous results on stability of JBD static universe by considering spatially homogeneous Bianchi type IX anisotropic perturbation modes and by including more general perfect fluid. Contrary to GR we have found that the JBD static universe dominated by a standard perfect fluid could be stable against isotropic and anisotropic perturbations for some sort of perfect fluids, for example for dust or scalar field (inflaton). This modification of the stability behavior has important consequences for the emergent universe scenario, since it ameliorates the fine-tuning that arises from the fact that the ES model is an unstable saddle
in GR and prevent that small fluctuations, such as quantum fluctuations, will inevitably arise, forcing the universe away from its static state, thereby aborting the emergent universe scenario.

In particular we found that for a standard perfect fluid with a polytropic state equation satisfying $\gamma > 2/3$ it is possible to find a static solution which is stable against isotropic and anisotropic perturbations, with the only exception of radiation ($\gamma = 4/3$). This implies that for a universe dominated by dust ($\gamma = 1$), for example, we could find a solution where the universe is static and stable. The instability of the static universe dominated by radiation, although disturbing, seems not to be a problem, since in a pre-inflationary cosmological model, it might be possible that radiation be an element which is not dominant at all.

Also, we found that the static JBD universe described in Ref. [36], which correspond to a universe dominated by a scalar field moving in a flat potential, is stable against isotropic and anisotropic perturbations when the JBD potential and the JBD parameter satisfy a set of general conditions discussed in Sect. IV.

Finally, we focus on a particular example of a JBD potential, which is a polynomial in the JBD field. This kind of JBD potential allow us to use a dynamical system approach for studying the stability of the JBD static universe. In this respect, we have found that the JBD static universe solutions are center equilibrium points. We obtained numerical solutions for a universe dominated by standard perfect fluids and dominated by a scalar field. We have considered the cases where the universe starts from an initial state close to the equilibrium point. The numerical solutions showed a behavior just like the expected if the equilibrium points are centers.

We should stress that in this work we have studied the stability of the Jordan-Brans-Dicke static universe against spatially homogeneous isotropic and anisotropic perturbations, see Refs. [18, 19]. Of course, also it is possible to study the stability of the JBD static universe against spatially inhomogeneous perturbations (scalar, vector and tensor perturbations). The situation for the ES solution, becomes neutrally stable against inhomogeneous scalar perturbations (with high enough sound speed), vector and tensor isotropic perturbations [18, 19]. We expect that in the JBD case these inhomogeneous perturbations do not lead to additional instabilities in the same way that happens with the ES case. We intend to return to this point in the near future by working an approach analogous to that followed in Refs. [43, 44, 45, 46].
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