

Constructing analytical solutions of linear perturbations of inflation with modified dispersion relations

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(Dated: May 11, 2014)

We develop a technique to construct analytical solutions of the linear perturbations of inflation with a nonlinear dispersion relation, due to quantum effects of the early universe. Error bounds are given and studied in detail. The analytical solutions describe the exact evolution of the perturbations extremely well even when only the first-order approximations is used.

I. INTRODUCTION

The inflationary cosmology \cite{1} provides a framework for solving several fundamental and conceptual problems of the standard big bang cosmology \cite{2}. Most importantly, it provides a causal mechanism for generating structures in the universe and the spectrum of cosmic microwave background (CMB) anisotropies. These are matched to observations with unprecedented precision \cite{3}, especially after the recent release of more precise results from the Planck satellite \cite{4}.

However, such successes are contingent on the understanding of physics in much earlier epochs when temperatures and energies were much higher than what we are able to access elsewhere \cite{5}. In particular, if the inflationary period is sufficiently long, the physical wavelength of fluctuations observed at the present time may well originate with a wavelength smaller than the Planck length at the beginning of the inflation - the trans-Planckian issues \cite{6}. Then, questions arise as to whether usual predictions of the scenario still remain robust, due to the ignorance of physics in such a small scale, and more interestingly, whether they have left imprints for future observations.

Such considerations have attracted lots of attention, and various approaches have been proposed \cite{7,8,9}. One of them is to replace the linear dispersion relation by a nonlinear one in the equations of the perturbations. This approach was initially applied to inflation as a toy model \cite{7}, motivated from the studies of the dependence of black hole radiation on Planck scale physics \cite{10}. Later, it was naturally realized \cite{11} in the framework of the Horava-Lifshitz gravity, a candidate of the ultraviolet complete theory of quantum gravity \cite{12,13}.

Then, obtaining approximate analytical solutions of the perturbations becomes one of the crucial steps in understanding the quantum effects on inflation, including the power spectra of the perturbations, non-Gaussianities, primordial gravitational waves, temperature and polarization of CMB, and has been intensively investigated in the past decade \cite{7,8,9,11}. However, these studies were carried out mainly by using the Brandenberger-Martín method, in which the evolution of the perturbations is divided into several epochs, and in each of them the approximate analytical solution can be obtained either by the WKB approximations when the adiabatic condition is satisfied, or by the linear combination of the exponentially decaying and growing modes, otherwise. Then, the individual solutions were matched together at their boundaries. While this often yields reasonable analytical approximations, its validity in more general cases has been questioned recently, and shown that it is valid only when $k \gg aH$ \cite{14}, where $k$ is the comoving wavenumber, $a$ the expansion factor of the universe, and $H \equiv \dot{a}/a$ with $\dot{a} \equiv da/dt$. In addition, the errors are not known in these approximates. However, with the arrival of the precision era of cosmological measurements, accurate calculations of cosmological variables are highly demanded \cite{15}.

In this Letter, we propose another method, the uniform asymptotic approximation, to construct analytical solutions of the linear (scalar, vector or tensor) perturbations of inflation with modified dispersion relations. We construct explicitly the error bounds and study them in detail. Because of the understanding and control of the errors, such constructed solutions describe the exact evolutions of the perturbations extremely well, even when only the first-order approximation is used [cf. Fig. 1].

It should be noted that the uniform asymptotic approximation has been used to study the mode function by Habib et al \cite{16}, and later applied to some particular models \cite{17}. However, their treatments are applicable only to the case where the dispersion relation is linear $\beta_i = 0$, where $\beta_i$ are defined in Eq. (3), so that $g(\eta) = 0$ has only one single root [cf. Eq. (13)]. It cannot be applied to the more interesting cases with several roots, and in particular, to those where some roots may be double, triple or even high-multipole roots. The method to be developed below shall treat all these cases in a unified way, which is mathematically quite different from that of \cite{10}, and reduces to it as $\beta_i = 0$. 


II. UNIFORM ASYMPTOTIC APPROXIMATION

In the slow-roll inflation, we have \(a(\eta) \simeq -(1 - \varepsilon)/(\eta H)\), with \(\eta\) and \(\varepsilon \equiv -\dot{H}/H^2\) being, respectively, the conformal time and slow-roll parameter. Then, the perturbations (of scalar, vector or tensor) are given by \(g\),

\[
\mu_k''(y) = [g(y) + q(y)]\mu_k(y),
\]

where \(y \equiv -k\eta\), \(\mu_k(y)\) denotes the mode function, a prime the derivative with respect to \(y\), and

\[
g(y) + q(y) = \frac{\nu^2(y) - 1/4}{y^2} - \omega_k^2(y). \tag{2}
\]

Here \(\nu(y)\) depends on the background and types of perturbations. The modified dispersion relation \(\tilde{\omega}_k^2(y)\) takes the form,

\[
\tilde{\omega}_k^2(y) = 1 - b_1\epsilon_1^2y^2 + b_2\epsilon_2^4y^4, \tag{3}
\]

where \(\epsilon_1 \equiv H/M_*\), with \(M_*\) being the energy scale, above which the quantum effects become important. To the first-order approximations of the slow-roll inflation, one can treat \(\nu(y)\), \(H\) and \(b_1\) as constants for all types of perturbations. For details, see for example [9].

Equation (2) shows that \(g(y)\) and \(q(y)\) in general have two poles, at \(y = 0^+\) and \(y = +\infty\), respectively. In addition, \(g(y)\) has multiple turning points (or roots of the equation \(g(y) = 0\)). From the theory of the second-order linear differential equations, one finds that the asymptotic solutions of Eq. (1) depend on the behavior of \(g(y)\) and \(q(y)\) around the poles and turning points. To develop a unified way to treat all the cases together, let us first introduce the Liouville transformations with two new variables \(U\) and \(\xi\)

\[
U(\xi) = \chi^{1/4}\mu_k(y), \quad \chi = \frac{|g(y)|^{1/4}}{f^{(1)}(\xi)2} = \left(\frac{d\xi}{dy}\right)^2, \tag{4}
\]

where

\[
f(\xi) = \int^y \sqrt{|g(y)|} dy, \quad f^{(1)}(\xi) = \frac{df(\xi)}{d\xi}. \tag{5}
\]

Note that \(\chi\) must be regular and not vanish in the intervals of interest. Consequently, \(f(\xi)\) should be chosen so that \(f^{(1)}(\xi)\) has zeros and singularities of the same type as \(g(y)\). As shown below, such requirements play an essential role in determining the approximate analytical solutions. In terms of \(U\) and \(\xi\), Eq. (1) takes the form,

\[
\frac{d^2U}{d\xi^2} = \left[ \pm f^{(1)}(\xi)^2 + \psi(\xi) \right]U, \tag{6}
\]

where

\[
\psi(\xi) = \frac{g(y)}{\chi} - \chi^{-3/4} \frac{d^2(\chi^{-1/4})}{dy^2}, \tag{7}
\]

and the signs \(\pm\) correspond to the cases of \(g(y) > 0\) and \(g(y) < 0\), respectively. Considering \(\psi(\xi) = 0\) as the first-order approximation, one can choose \(f^{(1)}(\xi)\) so that the first-order approximation can be as close to the exact solutions as possible with the guidelines of the error functions constructed below, and solved in terms of known functions.

Such a choice crucially depends on the behavior of the function \(g(y)\) near the poles and turning points. In particular, in the neighborhoods of the two poles, we can choose \(f^{(1)}(\xi)^2 = \text{const.}\). Without loss of generality, we take this constant to be unity. Then, we find that

\[
\xi = \int^y \sqrt{\pm g(y)} dy, \tag{8}
\]

here \(\pm\) correspond to the poles \(y = 0^+, +\infty\), respectively. Then, as the first-order approximation, neglecting the \(\psi(\xi)\) term in Eq. (6) we find

\[
\mu_k^\pm(y) = \frac{c^\pm}{|\pm g(y)|^{1/4}} e^{\int^y \sqrt{\pm g(y)} dy} \left( 1 + \epsilon_1^\pm \right) \tag{9}
\]

\[
+ \frac{d^\pm_1}{|\pm g(y)|^{1/4}} e^{-\int^y \sqrt{\pm g(y)} dy} \left( 1 + \epsilon_2^\pm \right),
\]

where \(c_1^\pm\) and \(c_2^\pm\) represent the errors of the asymptotic solutions, \(c_1\) and \(d_1\) are the integration constants, and \(s = 0\) \((s = 1)\) at the pole \(y = 0^+\) \((y = +\infty)\). The corresponding error bounds are given by [20],

\[
|\epsilon_1^+|, \quad \frac{1}{2} |g(y)|^{-1/2} \left| \frac{d^+}{dy} \right| \leq e^{\frac{1}{2}Y_{0^+,y}(F)} - 1,
\]

\[
|\epsilon_1^-|, \quad |g(y)|^{-1/2} \left| \frac{d^-}{dy} \right| \leq e^{\frac{1}{2}Y_{0^+,y}(F)} - 1,
\]

where \(Y_{x_1, x_2}(F) \equiv \int_{x_1}^{x_2} |dF(y)/dy| dy\), and the error control function \(F(y)\) is defined as,

\[
F(y) = \int^y \left( |g|^{-1/4} \frac{d^2}{dy^2} |g|^{-1/4} - q |g|^{-1/2} \right) dy. \tag{10}
\]

From the above expressions one can see that the errors sensitively depend on the choice of \(g(y)\) and \(q(y)\). To fix them uniquely, let us consider the above error bounds. Let us first expand

\[
g(y) = y^{-\infty} \sum_{s=0}^{\infty} g_s y^s, \quad q(y) = y^{-\infty} \sum_{s=0}^{\infty} q_s y^s. \tag{11}
\]

about \(y = 0^+\). Then, the LG approximations are valid only when \(g(y)\) has a pole of order \(m \geq 2\) [33]. However, when \(m > 0\), Eq. (2) shows that \(|q(y)| < |g(y)|\) does not hold. Therefore, in the present case we must choose \(m = 2\), for which the condition \(|q(y)| < |g(y)|\) requires \(n \leq 2\). Then, the convergence of \(F(y)\) requires \(n = 2\), \(q_0 = -1/4\), while the condition \(|q(y)| < |g(y)|\) leads to \(q_1 = q_2 = 0\). On the other hand, at the pole \(y = \infty\), we can make similar expansions, i.e.,

\[
g(y) = y^{-\infty} \sum_{s=0}^{\infty} g_s y^{-s}, \quad q(y) = y^{-\infty} \sum_{s=0}^{\infty} q_s y^{-s}. \tag{12}
\]
Following similar arguments given at \( y = 0^+ \), we find that \( \tilde{m} = 4, \tilde{n} < 1 \). Thus, \( q(y) \) must take the form,
\[
q(y) = -1/(4y^2) + q_2, \quad \text{where } q_2 = -(1 + g_2) \text{ and } |q_2| < |g_2|.
\]
Then, without loss of generality, we can always set \( q_2 = 0 \) and finally obtain \[20\],
\[
q(y) = -\frac{1}{4y^2}, \\
q(y) = -\frac{\nu^2}{y^2} - 1 + b_1\epsilon^2 y^2 - b_2\epsilon^4 y^4. \tag{13}
\]

The LG approximate solutions are valid only in the region where \( q(y) \neq 0 \). Once \( q(y) \) are zero, both \( \psi(\xi) \) and \( \mu_k(y) \) diverge, and the LG approximations become invalid. In order to get the asymptotic solutions around these turning points, we need to choose a different \( f(1)(\xi)^2 \) in Eq. \[1\]. However, such choice depends on the nature of the turning points. But nevertheless, since \( g(0) = 0 \) is in general a cubic equation, it can be always cast in the form
\[
b_2x^6 - b_1x^4 + x^2 - \nu^2\epsilon^2 = 0, \tag{14}
\]
where \( x = \epsilon y \). Let
\[
\Delta \equiv (Y - 1)^3 + \frac{1}{4} (2 - 3\nu + 3b_1)^2\nu^2\epsilon^2, \tag{15}
\]
and \( Y \equiv 3b_2/b_1^2 \). Then, when \( \Delta < 0 \), there exist three distinct real single roots, denoted by \( y_i \) \((i = 0, 1, 2)\), respectively. Without loss of generality, we further assume \( y_0 < y_1 < y_2 \). When \( \Delta = 0 \), we have one real single root \( y_0 \), and one real double root \( y_1 = y_2 \), with \( y_0 < y_1 \). However, in this case it is impossible to have all three roots equal. When \( \Delta > 0 \), there exists only one real single root \( y_0 \), while \( y_1 \) and \( y_2 \) become single complex roots with \( y_1 = y_2 \). In all the three cases, we have \( y_0 \sim \mathcal{O}(1) \), while the magnitudes of the roots \( y_0 \) and \( y_1 \) depends on \( \epsilon \). Physically, we expect \( \epsilon \ll 1 \). Then, we have \( y_1, y_2 \gg 1 \) for the cases \( \Delta \leq 0 \). However, the difference between \( y_1 \) and \( y_2 \) generally depends on the choice of \( b \) and \( \epsilon \).

To process further, let us consider the three conditions \[18\] \[19\] \[21\] \[22\]:
(a) \(|q(y)| < |g(y)|\) in regions except for the neighborhoods of \( y_i(i = 0, 1, 2)\); (b) \(|q(y)| < |g(y)/(y-y_1)|\) in the neighborhoods of \( y_i \); and (c) \(|q(y)| < |g(y)/(y-y_1)(y-y_2)|\) in the neighborhoods of \( y_1 \) and \( y_2 \).

In the neighborhood of \( y_0 \), conditions (a) and (b) are satisfied, and \( y_0 \) can be treated as a simple turning point. Then, we can introduce a monotone increasing or decreasing function \( \xi \) as \( f(1)(\xi)^2 = \pm \xi \), where \( \xi(y_0) = 0 \). Without loss of generality, we can always choose \( \xi \) to have the same sign as \( g(y) \), and thus \( \xi \) is a monotone decreasing function around \( y_0 \) and is given by
\[
\xi = \begin{cases} 
-\left(\frac{1}{2} \int_{y_0}^{y} \sqrt{-g(y)} \, dy \right)^{\frac{3}{4}}, & y \geq y_0, \\
\left(\frac{1}{2} \int_{y_0}^{y} \sqrt{g(y)} \, dy \right)^{\frac{3}{4}}, & y \leq y_0.
\end{cases}
\]

Neglecting the \( \psi(\xi) \) term in the first-order approximation of Eq. \[6\], we obtain
\[
\mu_k(y) = \alpha_0 \left(\frac{\xi}{g(y)}\right)^{1/4} \left(\text{Ai}(\xi) + \epsilon_3(y)\right) + \beta_0 \left(\frac{\xi}{g(y)}\right)^{1/4} \left(\text{Bi}(\xi) + \epsilon_4(y)\right), \tag{16}
\]
where \( \text{Ai}(\xi) \) and \( \text{Bi}(\xi) \) are the Airy functions, and \( \epsilon_{3,4}(y) \) denote the errors of the approximate solutions. In particular, the error bounds now read \[20\],
\[
\left|\frac{\epsilon_3}{M(\xi)}\right|, \left|\frac{\partial \epsilon_3/\partial \xi}{N(\xi)}\right| \leq E^{-1}(\xi) \frac{\lambda}{\lambda} \left[\exp\left\{\lambda\mathcal{V}_{\alpha}(H)\right\} - 1\right],
\]
\[
\left|\frac{\epsilon_4}{M(\xi)}\right|, \left|\frac{\partial \epsilon_4/\partial \xi}{N(\xi)}\right| \leq E(\xi) \frac{\lambda}{\lambda} \left[\exp\left\{\lambda\mathcal{V}_{\alpha}(H)\right\} - 1\right],
\]
where \( \xi \in [a_4, a_3] \), \( H(\xi) \) denotes the corresponding error control function, given by
\[
H(\xi) = \int_{\xi}^{\xi} |v|^{-1/2} \rho(v) \, dv, \tag{17}
\]
and \( M(\xi), N(\xi), \lambda \) are given explicitly in \[20\] \[23\].

Near the turning points \( y_1 \) and \( y_2 \), conditions (a) and (c) are always satisfied. When condition (b) is also satisfied, we can treat \( y_1 \) and \( y_2 \) as single turning points, and similar to \( y_0 \), we can get the asymptotic solutions near them. However, when \( y_2 - y_1 \ll 1 \), condition (b) is not satisfied, and the method used for \( y_0 \) is no longer valid. Following Olver \[21\], we adopt a method to treat all these cases together. The crucial step is to choose \( f(1)(\xi)^2 = \xi^2 - \xi^2_0 \), where \( \xi \) is an increasing variable and \( \xi(y_2) = -\xi(y_1) \equiv \xi_0 \). The case \( y_1 = y_2 \) corresponds to \( \xi_0 = 0 \), and the one with a pair of complex conjugate roots corresponds to \( \xi_0 < 0 \). Then, \( \xi(\xi) \) is given by
\[
\int_{y_1}^{y} \sqrt{|g(y)|} \, dy = \int_{\xi}^{\xi} \left|\xi^2 - \xi^2_0\right| \, d\xi', \tag{18}
\]
where \( \xi_0 = \pm/(2/\pi) \int_{y_1}^{y} \sqrt{|g(y)|} \, dy \), and “+” (”-“) corresponds to real (complex) \( y_1/2 \). Thus, we obtain
\[
\mu_k(y) = \alpha_1 \left(\frac{\xi^2 - \xi^2_0}{g(y)}\right)^{1/4} \left[W \left(\frac{1}{2} \xi^2_0, \sqrt{2}\xi\right) + \epsilon_5(\xi)\right]
+ \beta_1 \left(\frac{\xi^2 - \xi^2_0}{g(y)}\right)^{1/4} \left[W \left(\frac{1}{2} \xi^2_0, -\sqrt{2}\xi\right) + \epsilon_6(\xi)\right], \tag{19}
\]
where \( W \left(\frac{1}{2} \xi^2_0, \pm/\sqrt{2}\xi\right) \) are the parabolic cylinder functions, with \( \xi \) now being given by Eq. \[18\]. Then, the error bounds read \[20\],
\[
\left|\frac{\epsilon_5}{M(\xi_0^2 + 0^2)}\right|, \left|\frac{\partial \epsilon_5/\partial \xi}{\sqrt{2}N(\xi_0^2 + 0^2)}\right| \leq \frac{\kappa}{\lambda E(\xi_0^2 + 0^2)} \left[\exp\left\{\lambda\mathcal{V}_{\alpha}(I)\right\} - 1\right],
\]
\[
\frac{|\epsilon_0|}{M} \frac{|\partial \epsilon_0/\partial \xi|}{\sqrt{2} \mathcal{N} \left( \frac{1}{2} \xi_0^2 \sqrt{2} \xi \right)} \leq \frac{\kappa E}{\sqrt{2} \mathcal{N} \left( \frac{1}{2} \xi_0^2 \sqrt{2} \xi \right)} \left[ \exp \left\{ \lambda \mathcal{V}_{0, \xi}(I) \right\} - 1 \right],
\]

for \( \xi > 0 \), where \( a_5 \) is the upper bound of \( \xi \), \( I(\xi) \) is the error control function, now defined as

\[
I(\xi) = \int_0^{\xi} |v|^{-1} \psi(v) dv,
\]

(20)

and \( \lambda, M \left( \frac{1}{2} \xi_0^2 \sqrt{2} \xi \right), \kappa, N \left( \frac{1}{2} \xi_0^2 \sqrt{2} \xi \right), \) and \( E \left( \frac{1}{2} \xi_0^2 \sqrt{2} \xi \right) \) are given explicitly in [20] [21]. It is easy to get the error bounds for \( \xi < 0 \) by replacing the above \( \xi \) by \( -\xi \).

So far, using the Liouville transformations, we have found the analytical approximate solutions near the poles \( y = 0^+, -\infty \), given by Eq. \([9]\), and in the neighborhoods of the turning points \( y_k \), given, respectively, by Eqs. \([16]\) and \([19]\). We now move onto determining the integration constants from the initial conditions by matching them on their boundaries. In this Letter, we assume that the universe was initially at the adiabatic vacuum [2, 9].

\[
\lim_{y \to +\infty} \mu_k(y) = \frac{1}{\sqrt{2} \omega_k(y)} e^{-i \int_0^\eta \omega_k(\xi) d\xi},
\]

where \( \mu_k(y) \) also satisfies the Wronskian,

\[
\mu_k(y) \mu_k'(y) - \mu_k'(y) \mu_k(y) = i.
\]

(21)

Applying them to the solution \( \mu_k(y) \), we find

\[
c_+ = 0, \quad d_+ = \frac{1}{\sqrt{2} k}.
\]

(22)

On the other hand, to consider the matching between the solution \( \mu_k(y) \) and the one given by Eq. \([19]\), we note that for positive and large \( \xi \) the cylindrical functions take the asymptotic forms [24],

\[
W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \approx \left( \frac{2 \xi^2}{\xi_0^2 - \xi^2} \right)^{1/4} \cos \mathcal{D},
\]

\[
W \left( \frac{1}{2} \xi_0^2, -\sqrt{2} \xi \right) \approx \left( \frac{2 \xi^2}{\xi_0^2 - \xi^2} \right)^{1/4} \sin \mathcal{D},
\]

(23)

where \( j(\xi_0) \equiv \sqrt{1 + e^{2 \pi} \xi_0^2} - e^{2 \pi} \xi_0^2 \), and

\[
\mathcal{D} \equiv \frac{1}{2} \xi_0^2 \sqrt{\xi_0^2 - \xi^2} - \frac{1}{2} \xi_0^2 \ln \left( \xi + \sqrt{\xi_0^2 - \xi^2} \right) + \frac{1}{2} \xi_0^2 \ln |\xi| + \frac{\pi}{4} + \phi \left( \frac{1}{2} \xi_0^2 \right),
\]

(24)

with

\[
\phi(x) \equiv \frac{x}{2} - \frac{x}{4} \ln x^2 + \frac{1}{2} \text{ph} \Gamma \left( \frac{1}{2} + ix \right),
\]

(25)

where the phase \( \text{ph} \Gamma \left( \frac{1}{2} + ix \right) \) is zero when \( x = 0 \), and determined by continuity, otherwise. Inserting the above into Eq. \([19]\), and then comparing the resulting solution with \( \mu_k(y) \), we find that continuity of the mode function and its first derivative with respect to \( \eta \) leads to

\[
\alpha_1 = k^{-1/2} \frac{j(\xi_0)}{2^{1/4}} \sin \frac{1}{2} j(\xi_0)^{1/2}, \quad \beta_1 = -k^{-1/2} \frac{j(\xi_0)^{1/2}}{2^{1/4}}.
\]

(26)

To determine the coefficients \( \alpha_0 \) and \( \beta_0 \), we match the solutions \([16]\) and \([19]\) within the region \( y \in (y_0, y_1) \). In this region, \( |y - y_0| \) is very large, as mentioned above, and \( \xi \) is negative. Thus, from the asymptotic formula \([23]\) of \( W \left( \frac{1}{2} \xi_0^2, \sqrt{2} \xi \right) \), and the asymptotic form of the Airy functions,

\[
\text{Ai}(-x) = \frac{1}{\pi^{1/2} x^{1/4}} \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right),
\]

\[
\text{Bi}(-x) = \left( \frac{1}{\pi^{1/2} x^{1/4}} \right) \sin \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right),
\]

(27)

for \( x \gg 1 \), we find

\[
\alpha_0 = \sqrt{\frac{\pi}{2k}} \left[ j^{-1}(\xi_0) \sin \mathcal{B} - i \cdot j(\xi_0) \cos \mathcal{B} \right],
\]

\[
\beta_0 = \sqrt{\frac{\pi}{2k}} \left[ j^{-1}(\xi_0) \cos \mathcal{B} + i \cdot j(\xi_0) \sin \mathcal{B} \right],
\]

(28)

where

\[
\mathcal{B} \equiv \int_{y_0}^{y_1} \sqrt{-g} dy + \phi(\xi_0^2/2).
\]

(29)
Finally, we consider the matching between $\mu_k^+(y)$ given by Eq. (9) and the one given by Eq. (16) in the region $y \in (0, y_0)$. It can be shown that the continuity condition yields

$$d_+ = \left( \frac{\alpha_0 \beta_0}{2\pi} \right) c_+^{-1} = \frac{\alpha_0}{2\sqrt{\pi}} \exp \left( - \int_{y_0}^{y_0} \sqrt{g} dy \right). \quad (30)$$

Once we have uniquely determined the integration constants from the initial conditions, let us turn to consider some representative cases. In particular, in the case with three different single turning points, the numerical (exact) and our analytical approximate solutions are plotted in Fig. 1(a). The cases with two and one turning point(s) are plotted, respectively, in Figs. 1(b) and 1(c). From these figures, one can see how well the exact solutions are approximated by our analytical ones. In fact, we have considered many other cases, and found that in all those cases the exact solutions are extremely well approximated by the analytical ones.

### III. Conclusions

In this Letter, we have extended the uniform approximation method of [16] to construct analytical solutions of linear perturbations of inflation, in which the dispersion relations are generically nonlinear and include high-order momentum terms, due to the quantum effects of the early universe. The explicit error bounds are constructed for error terms associated with the approximations. Consequently, the errors are well understood and controlled, and the analytical solutions describe the exact evolution of the perturbations extremely well even when the first-order approximation is used, as shown in Fig. 1. Thus, with this method it is expected that the accuracy of the calculations of cosmological variables, such as the power spectra, non-Gaussianity, primordial gravitational waves, temperature and polarization of CMB, shall be significantly improved.

It must be noted that, although in this Letter we have considered only the case where the maximal number of roots of the equation $g(y) = 0$ is three, our method can be easily generalized to the case with any number, and each of which can be a single, double, triple or even higher multiple root. An interesting case is when the parity is violated in the early universe [25], and terms like the Chern-Simons and fifth-order derivatives do appear [11].

In addition, high-order approximations can also be constructed [16, 18, 19]. All these can be done not only in the case within the slow-roll inflation considered above, but also in more general inflationary backgrounds.

Moreover, the case $b_i = 0$ was studied extensively by using various methods, including the Green-function one [29]. It would be interesting to compare the results obtained by our method and those obtained elsewhere. Such considerations are clearly out of the scope of this Letter, and we hope to address them in forthcoming discussions.

### Acknowledgements

We thank Yongqing Huang, Jiro Soda, Qiang Wu, and Wen Zhao for valuable discussions/comments. This work is supported in part by DOE, DE-FG02-10ER41692 (AW), Ciência Sem Fronteiras, No. 004/2013 - DRI/CAPES (AW), NSFC Nos: 11375153 (AW), 11173021 (AW), 11047008 (TZ), 1105120 (TZ), and 11205133 (TZ).

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