BOUNDDEDNESS OF THE IMAGES OF PERIOD MAPS

KEFENG LIU AND YANG SHEN

Abstract. We prove a conjecture of Griffiths on simultaneous normalization of all periods which asserts that the image of the lifted period map on the universal cover lies in a bounded domain in complex Euclidean space.

Introduction

First we introduce the notations in this paper. Following the terminology of Griffiths, all algebraic varieties are assumed to be smooth over \( \mathbb{C} \) and morphisms between algebraic varieties to be rational and holomorphic. In this paper, we will consider variation of Hodge structures arising from geometry. This means that we shall consider an algebraic family of polarized algebraic varieties, which is a proper morphism between algebraic varieties \( f: \mathcal{X} \to S \) with the following properties

(1) the varieties \( \mathcal{X} \) and \( S \) are smooth and connected, and the morphism \( f \) is non-degenerate, i.e. the tangent map \( df \) is of maximal rank at each point of \( \mathcal{X} \);
(2) \( \mathcal{X} \subseteq \mathbb{P}^N \) is quasi-projective with restricted polarization \( \mathcal{L} \) over \( \mathcal{X} \);
(3) each fiber \( X_s = f^{-1}(s), \ s \in S, \) is smooth, connected, and projective with the polarization \( L_s = \mathcal{L}|_{X_s} \).

In general, \( S \) is not compact, but it admits a smooth compactification, i.e. we have a proper morphism \( \overline{f}: \overline{\mathcal{X}} \to \overline{S} \) between smooth and complete varieties such that the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{f} & \overline{\mathcal{X}} \\
\downarrow & & \downarrow \overline{f} \\
S & \xleftarrow{} & \overline{S}
\end{array}
\]

commutes and \( \overline{\mathcal{X}} - \mathcal{X} \) and \( \overline{S} - S \) are divisors with simple normal crossings. That is to say, for example, that \( \overline{S} - S \) is locally defined by the equation \( z_1 \cdots z_k = 0 \) in local coordinates \( \{z_1, \cdots, z_n\} \) with \( k \leq \dim_{\mathbb{C}} S = n \). We will call the smooth divisors locally defined by \( z_1 = 0, \cdots, z_k = 0 \) respectively the irreducible branches at infinity.

Since each fiber \( X = X_s \) is projective with polarization \( L = \mathcal{L}|_X \in H^2(X, \mathbb{Z}) \), we can define the primitive cohomology group by

\[
H^n_{pr}(X, \mathbb{F}) = \ker(L: H^n(X, \mathbb{F}) \to H^{n+2}(X, \mathbb{F})),
\]

where \( n = \dim_{\mathbb{C}} X \) and \( \mathbb{F} \) can be \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{C} \). From Hodge theory we have the Hodge decomposition of the primitive cohomology group

\[
H^n_{pr}(X, \mathbb{C}) = H^{n,0}_{pr}(X) \oplus H^{n-1,1}_{pr}(X) \oplus \cdots \oplus H^{0,n}_{pr}(X),
\]
where $H^{k,n-k}_{pr}(X) = H^{k,n-k}(X) \cap H^n_{pr}(X, \mathbb{C})$. The Poincaré bilinear form $Q$ on $H^n_{pr}(X, \mathbb{C})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any $d$-closed primitive $n$-forms $u, v$ on $X$. Then $Q$ is nondegenerate and satisfies the Hodge-Riemann relations

(1) $Q(H^{k,n-k}_{pr}(X), H^{l,n-l}_{pr}(X)) = 0$ unless $k + l = n$, and

(2) $(\sqrt{-1})^{2k-n} Q(v, v) > 0$ for $v \in H^{k,n-k}_{pr}(X) \setminus \{0\}$.

Let $h^{k,n-k} = \text{dim}_\mathbb{C} H^{k,n-k}_{pr}(X)$ and $f^k = \sum_{i=0}^n h^{i,n-i}$. Define the decreasing filtration $H^n_{pr}(X, \mathbb{C}) = F^0 \supset \cdots \supset F^n = 0$ by taking $F^k = F^k(X) = H^{n,0}_{pr}(X) \oplus \cdots \oplus H^{k,n-k}_{pr}(X)$. We know that

(3) $\dim_{\mathbb{C}} F^k = f^k$,

(4) $H^n_{pr}(X, \mathbb{C}) = F^k \oplus F^{n-k+1}$, and $H^{k,n-k}_{pr}(X) = F^k \cap F^n$.

In terms of the Hodge filtration, the Hodge-Riemann relations (1) and (2) are

(5) $Q(F^k, F^{n-k+1}) = 0$, and

(6) $Q(Cv, v) > 0$ if $v \neq 0$,

where $C$ is the Weil operator given by $Cv = (\sqrt{-1})^{2k-n} v$ for $v \in H^{k,n-k}_{pr}(X)$. The period domain $D$ for polarized Hodge structures with data (3) is the space of all such Hodge filtrations

$$D = \{ F^n \subseteq \cdots \subseteq F^0 = H^n_{pr}(X, \mathbb{C}) \mid (3), (5) \text{ and } (6) \text{ hold} \}.$$ 

The compact dual $\hat{D}$ of $D$ is

$$\hat{D} = \{ F^n \subseteq \cdots \subseteq F^0 = H^n_{pr}(X, \mathbb{C}) \mid (3) \text{ and } (5) \text{ hold} \}.$$ 

One can prove that $\hat{D}$ is a complex manifold and the period domain $D \subseteq \hat{D}$ is an open submanifold. See Theorem 4.3 in [9] for a complete proof and Proposition 8.2 in [11] for an alternative proof.

From the definition of period domain we naturally get the Hodge bundles on $\hat{D}$ by associating to each point in $\hat{D}$ the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point.

For the family $f : \mathcal{X} \rightarrow S$ the period map is defined as a morphism $\Phi : S \rightarrow D/\Gamma$ by

$$s \mapsto (F^n_s \subseteq \cdots \subseteq F^0_s) \in D,$$

where $F^k_s = F^k(X_s)$ and $\rho : \pi_1(S) \rightarrow \Gamma \subseteq \text{Aut}(H_X, Q)$ is the monodromy representation. It is well-known that the period map has the following properties:

(i) locally liftable;
(ii) holomorphic: $\partial F^i_s/\partial \sigma \subseteq F^i_s$, $0 \leq i \leq n$;
(iii) Griffiths transversality: $\partial F^i_s/\partial s \subseteq F^{i-1}_s$, $1 \leq i \leq n$. 

Thanks to (i) we can lift the period map to \( \tilde{\Phi} : T \to D \) by taking the universal cover \( T \) of \( S \) such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\tilde{\Phi}} & D \\
\downarrow \pi & & \downarrow \pi \\
S & \xrightarrow{\Phi} & D/\Gamma
\end{array}
\]

is commutative. Without loss of generality we will assume \( \Gamma \) is torsion free, therefore \( D/\Gamma \) is smooth. Otherwise it is standard to take a torsion free subgroup and proceed on a cover of \( S \). For example, see the proof of Lemma IV-A, page 705-706 in [29].

In his paper [12], Griffiths raised the following conjecture as Conjecture 10.1 in Section 10, which is now the main theorem in our paper.

**Theorem 0.1.** (Griffiths Conjecture) Given \( f : X \to S \), there exists a simultaneous normalization of all the periods \( \Phi(X_s) \) (\( s \in S \)). More precisely, the image \( \tilde{\Phi}(T) \) lies in a bounded domain in a complex Euclidean space.

The main idea of our proof is Riemann extension theorem which asserts that

- Suppose that \( M \) is a complex manifold and \( V \subseteq M \) is an analytic subvariety with \( \text{codim}_CM \geq 1 \). Then any holomorphic function \( f \) defined on \( M \setminus V \), which is locally bounded, can be extended uniquely to a global holomorphic function \( \tilde{f} \) on \( M \) such that \( \tilde{f}|_{M \setminus V} = f \).

Based on this main idea, our proof can be divided into the following steps:

**Step 1:** Find an analytic subvariety of \( \text{codim}_CM \geq 1 \).

To explain the detail we need to review some results of period domain from Lie theory. We fix a point \( p \) in \( T \) and its image \( o = \tilde{\Phi}(p) \) as the reference points or base points. If we define the complex Lie group as

\[
G_C = \{ g \in GL(H_C) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_C \},
\]

then \( G_C \) acts transitively on \( \tilde{D} \) with stabilizer \( B = \{ g \in G_C | gF^k_p = F^k_p, \ 0 \leq k \leq n \} \). Let \( G_R \subseteq G_C \) be the real subgroup which maps \( H_R \) to \( H_R \), then we can realize \( D \) as \( D = G_R/V \) with \( V = B \cap G_R \). On the Lie algebra \( g \) of \( G_C \) we can define a Hodge structure of weight zero by

\[
g = \bigoplus_{k \in \mathbb{Z}} g^{k,-k} \quad \text{with} \quad g^{k,-k} = \{ X \in g | XH_1^{r,n-r} \subseteq H_1^{r+k,n-r-k}, \ 0 \leq r \leq n - k \}.
\]

By definition the Lie algebra of \( B \) is \( b = \bigoplus_{k \geq 0} g^{k,-k} \) and the holomorphic tangent space of \( \tilde{D} \) at the base point \( o \) is naturally isomorphic to \( g/b \cong \oplus_{k \geq 1} g^{-k,k} \cong \mathfrak{n}_+ \). We denote the unipotent group to be \( N_+ = \exp(\mathfrak{n}_+) \). Since \( N_+ \cap B = \{ \text{Id} \} \), we can identify the unipotent group \( N_+ \subseteq G_C \) with its orbit \( N_+(o) \subseteq \tilde{D} \) so that \( N_+ \subseteq \tilde{D} \) is meaningful. With this we define \( \tilde{T} = (\tilde{\Phi})^{-1}(N_+ \cap D) \) and show that \( T \setminus \tilde{T} \) is an analytic subvariety of \( T \) with \( \text{codim}_C(T \setminus \tilde{T}) \geq 1 \).

**Step 2:** Show that \( \tilde{\Phi}|_T : T \to N_+ \cap D \) is bounded.
Now we study in detail the structure of the Lie algebras involved. Suppose that the Weil operator \( \theta : \mathfrak{g} \rightarrow \mathfrak{g} \) is defined by
\[ \theta(X) = (-1)^k X, \quad \text{for } X \in \mathfrak{g}^{k,-k}. \]
Then we can decompose Lie algebra \( \mathfrak{g} \) into the union of eigenspaces of the Weil operator as
\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \]
where \( \mathfrak{k} \) and \( \mathfrak{p} \) correspond to the eigenvalues \( +1 \) and \( -1 \) respectively.

Let \( \mathfrak{g}_0 \subseteq \mathfrak{g} \) be the Lie algebra of \( G_\mathbb{R} \) and \( \mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0 \), then we get the decomposition of \( \mathfrak{g}_0 \) as
\[ \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0. \]
In fact, Schmid showed that this decomposition is Cartan decomposition, i.e. \( \mathfrak{g}_c \triangleq \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0 \) is a compact real semi-simple Lie algebra.

Next we choose a Cartan subalgebra \( \mathfrak{h}_0 \) of \( \mathfrak{g}_0 \) such that \( \mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0 \) and \( \mathfrak{h}_0 \) is also a Cartan subalgebra of \( \mathfrak{k}_0 \), where \( \mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0} \) is the Lie algebra of \( V \). In Lie theory, root system plays a central role in the structures of Lie algebras. With the root system, one has the decomposition of the Lie algebra as
\[ \mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^\varphi. \]

In our case, we decompose the root system \( \Delta \) into the union of \( \Delta_\mathfrak{k} \) and \( \Delta_\mathfrak{p} \) due to the corresponding space \( \mathfrak{g}^\varphi \subseteq \mathfrak{k} \) or \( \mathfrak{p} \). Then we introduce the notion of strongly orthogonal which says that two different roots \( \varphi, \psi \in \Delta \) are strongly orthogonal if and only if \( \varphi \pm \psi \notin \Delta \cup \{0\} \). The following two properties are key to our proof.

1. There exists a set of strongly orthogonal noncompact positive roots
\[ \Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq \Delta^+_\mathfrak{p} \]
such that
\[ \mathfrak{A}_0 = \sum_{i=1}^r \mathbb{R}(e_{\varphi_i} + e_{-\varphi_i}) \]
is a maximal abelian subspace in \( \mathfrak{A}_0 \).

2. If \( \mathfrak{A}_0 \) is any maximal abelian subspace of \( \mathfrak{p}_0 \), then there exists an element \( k \in K \), a maximal compact subgroup of \( G \), such that \( \text{Ad}(k) \cdot \mathfrak{A}_0 = \mathfrak{A}_0 \). See Section 2 for the definition of \( K \). Moreover, we have
\[ \mathfrak{p}_0 = \bigcup_{k \in K} \text{Ad}(k) \cdot \mathfrak{A}_0. \]

Let \( \mathfrak{a} \subseteq \mathfrak{n}_+ \) be the abelian subalgebra of \( \mathfrak{n}_+ \) determined by the period map \( \Phi \). Let \( A \triangleq \exp(\mathfrak{a}) \subseteq N_+ \) which is isomorphic to a complex Euclidean subspace, and \( P : N_+ \cap D \rightarrow A \cap D \) be the induced projection map. The restricted period map \( \tilde{\Phi} : \tilde{T} \rightarrow N_+ \cap D \) is composed with the projection map \( \tilde{P} \) gives a holomorphic map \( \Psi : \tilde{T} \rightarrow A \cap D \) by \( \Psi = \tilde{P} \circ \tilde{\Phi} \).

In Lemma 3.1 we prove that \( \Psi : \tilde{T} \rightarrow A \cap D \) is bounded with respect to the Euclidean metric on \( A \subseteq N_+ \). In Theorem 3.2 we prove the boundedness of \( \tilde{\Phi}(\tilde{T}) \) in \( N_+ \) by the
finiteness of the map $P|_{\mathcal{T}}$, where we essentially use the Griffiths transversality of the extended period map, which is introduced in Section II.

Step 3: By Step 1 and Step 2 and the Riemann Extension Theorem, we can finish the proof of Theorem 0.1.

We remark that without further assumptions, we can only prove that the image $\tilde{\Phi}(\mathcal{T})$ lies in $\mathbb{C}^N$ as a bounded subvariety. In many cases, however, we can embed $\tilde{\Phi}(\mathcal{T})$ in $\mathbb{C}^N$ as a complex sub-manifold, even as bounded open domain. We will come back to this in our future work.

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1. Period domains from the viewpoint of Lie theory

In this section we review the definitions and basic properties of period domains from Lie theory point of views. We consider the nilpotent Lie subalgebra $n_+$ and define the corresponding unipotent group to be $N_+ = \exp(n_+)$. Since $N_+ \cap B = \{\text{Id}\}$, we can identify the unipotent group $N_+ \subseteq G_{\mathbb{C}}$ with its orbit $N_+(o) \subseteq \hat{D}$. Then we define $\mathcal{T} = \tilde{\Phi}^{-1}(N_+ \cap D)$ and show that $\mathcal{T} \setminus \mathcal{F}$ is an analytic subvariety of $\mathcal{T}$ with $\text{codim}_{\mathbb{C}}(\mathcal{T} \setminus \mathcal{F}) \geq 1$.

First we fix a point $p$ in $\mathcal{T}$ and its image $o = \tilde{\Phi}(p)$ as the reference points or base points. Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. For the fixed point $p \in \mathcal{T}$ and $f^k = \dim F^k_p$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \{\xi_0, \cdots, \xi_{f^n-1}, \xi_{f^n}, \cdots, \xi_{f^{n-1}-1} \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_1, \cdots, \xi_{f^0-1}\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an adapted basis for the given Hodge decomposition

$$H^n_{pr}(X_p, \mathbb{C}) = H_{pr}^{n,0} \oplus H_{pr}^{n-1,1} \oplus \cdots \oplus H_{pr}^{1,n-1} \oplus H_{pr}^{0,n},$$

if it satisfies $H^{k,n-k}_{pr} = \text{Span}_{\mathbb{C}} \{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\}$ with $h^{k,n-k} = f^k - f^{k+1}$. We call a basis

$$\zeta = \{\zeta_0, \cdots, \zeta_{f^n-1}, \zeta_{f^n}, \cdots, \zeta_{f^{n-1}-1} \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_1, \cdots, \zeta_{f^0-1}\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an adapted basis for the given filtration

$$F^n_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p$$

if it satisfies $F^k_p = \text{Span}_{\mathbb{C}} \{\zeta_0, \cdots, \zeta_{f^{k+1}}\}$ with $\dim_{\mathbb{C}} F^k = f^k$. For the convenience of notations, we set $f^{n+1} = 0$ and $m = f^0$.

The blocks of an $m \times m$ matrix $T$ are set as follows: for each $0 \leq \alpha, \beta \leq n$, the $(\alpha, \beta)$-th block $T^{\alpha,\beta}$ is

$$T^{\alpha,\beta} = \{T_{ij} \}_{f^{-\alpha+\beta+1} \leq i \leq f^{\alpha-\alpha+1}, f^{-\beta+\beta+1} \leq j \leq f^{\beta-\beta+1}},$$

where $T_{ij}$ is the entries of the matrix $T$. In particular, $T = (T^{\alpha,\beta})$ is called a block lower triangular matrix if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

Let $H_{\mathbb{F}} = H^n_{pr}(X, \mathbb{F})$, where $\mathbb{F}$ can be chosen as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$
and the real one
\[ G_\mathbb{R} = \{ g \in GL(H_\mathbb{R}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_\mathbb{R} \}. \]

Griffiths in [9] showed that $G_\mathbb{C}$ acts on $\tilde{D}$ transitively and so does $G_\mathbb{R}$ on $D$. The stabilizer of $G_\mathbb{C}$ on $\tilde{D}$ at the fixed point $o$ is
\[ B = \{ g \in G_\mathbb{C} \mid gF_p^k = F_p^k, \ 0 \leq k \leq n \}, \]
and the one of $G_\mathbb{R}$ on $D$ is $V = B \cap G_\mathbb{R}$. Thus we can realize $\tilde{D}$ as
\[ \tilde{D} = G_\mathbb{C}/B, \text{ and } D = G_\mathbb{R}/V \]
so that $\tilde{D}$ is an algebraic manifold and $D \subseteq \tilde{D}$ is an open complex submanifold. One can find a complete proof of this result in Theorem 4.3 of [9], or Proposition 8.2 in [11] for an alternative proof.

The Lie algebra $g$ of the complex Lie group $G_\mathbb{C}$ is
\[ g = \{ X \in \text{End}(H_\mathbb{C}) \mid Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_\mathbb{C} \}, \]
and the real subalgebra
\[ g_0 = \{ X \in g \mid XH_\mathbb{R} \subseteq H_\mathbb{R} \} \]
is the Lie algebra of $G_\mathbb{R}$. Note that $g$ is a simple complex Lie algebra and contains $g_0$ as a real form, i.e. $g = g_0 \oplus \sqrt{-1}g_0$.

On the linear space $\text{Hom}(H_\mathbb{C}, H_\mathbb{C})$ we can give a Hodge structure of weight zero by
\[ g = \bigoplus_{k \in \mathbb{Z}} g^{k,-k} \text{ with } g^{k,-k} = \{ X \in g \mid XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}, \ 0 \leq r \leq n - k \}. \]

By definition of $B$ the Lie algebra $b$ of $B$ has the form $b = \bigoplus_{k \geq 0} g^{k,-k}$. Then the Lie algebra $v_0$ of $V$ is
\[ v_0 = g_0 \cap b = g_0 \cap b \cap \overline{b} = g_0 \cap g^{0,0}. \]
With the above isomorphisms, the holomorphic tangent space of $\tilde{D}$ at the base point is naturally isomorphic to $g/b$.

Let us consider the nilpotent Lie subalgebra $n_+ := \oplus_{k \geq 1} g^{-k,k}$. Then one gets the holomorphic isomorphism $g/b \cong n_+$. We denote the unipotent group to be
\[ N_+ = \exp(n_+). \]

As $\text{Ad}(g)(g^{k,-k})$ is in $\bigoplus_{i \geq k} g^{i,-i}$ for each $g \in B$, the subspace $b \oplus g^{-1,1}/b \subseteq g/b$ defines an $\text{Ad}(B)$-invariant subspace. By left translation via $G_\mathbb{C}$, $b \oplus g^{-1,1}/b$ gives rise to a $G_\mathbb{C}$-invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by $T_{o,h}^{1,0} \tilde{D}$, and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point $o$, restricted to $D$, determines a subbundle $T_{o,h}^{1,0}D$ of the holomorphic tangent bundle $T_{o,h}^{1,0}D$ of $D$ at the base point. The $G_\mathbb{C}$-invariance of $T_{o,h}^{1,0} \tilde{D}$ implies the $G_\mathbb{R}$-invariance of $T_{o,h}^{1,0}D$. As another interpretation of this holomorphic horizontal bundle at the base point, one has

\[ T_{o,h}^{1,0} \tilde{D} \cong T_{o,h}^{1,0} \tilde{D} \cap \bigoplus_{k=1}^n \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k). \]
In [26], Schmid call a holomorphic mapping \( \Psi : M \to \check{D} \) of a complex manifold \( M \) into \( \check{D} \) horizontal if at each point of \( M \), the induced map between the holomorphic tangent spaces takes values in the appropriate fibre \( T^{1,0} \check{D} \). It is easy to see that the period map \( \check{\Phi} : T \to \check{D} \) is horizontal since \( \check{\Phi}_* (T^{1,0}_p T) \subseteq T^{1,0}_{\check{a},h} \check{D} \) for any \( p \in T \). Since \( \check{D} \) is an open set in \( \check{D} \), we have the following relation:

\[
T^{1,0}_{\check{a},h} \check{D} = T^{1,0}_{\check{a},h} \check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1} / \mathfrak{b} \cong \mathfrak{n}^+.
\]

**Remark 1.1.** We remark that elements in \( \mathfrak{n}^+ \) can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in \( \mathfrak{b} \) can be realized as nonsingular block upper triangular matrices. If \( c, c' \in \mathfrak{n}^+ \) such that \( cB = c'B \) in \( \check{D} \), then \( c'c^{-1} \in \mathfrak{n}^+ \cap B = \{ I \} \), i.e. \( c = c' \). This means that the matrix representation in \( \mathfrak{n}^+ \) of the unipotent orbit \( \mathfrak{n}^+ (o) \) is unique. Therefore with a fixed base point \( o \in \check{D} \), we can identify \( \mathfrak{n}^+ \) with its unipotent orbit \( \mathfrak{n}^+ (o) \) in \( \check{D} \) by identifying an element \( c \in \mathfrak{n}^+ \) with \( [c] = cB \) in \( \check{D} \). Then \( \mathfrak{n}^+ \subseteq \check{D} \) is meaningful. In particular, when the base point \( o \) is in \( D \), we have \( \mathfrak{n}^+ \cap D \subseteq D \).

Now we define

\[
\tilde{T} = \check{\Phi}^{-1}(\mathfrak{n}^+ \cap D).
\]

Then we show that \( T \setminus \tilde{T} \) is an analytic subvariety of \( T \) with \( \text{codim}_C (T \setminus \tilde{T}) \geq 1 \).

**Lemma 1.2.** Let \( p \in T \) be the reference point with \( \Phi(p) = \{ F^m_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p \} \). Let \( q \in T \) be any point with \( \Phi(q) = \{ F^m_q \subseteq F^{n-1}_q \subseteq \cdots \subseteq F^0_q \} \), then \( \Phi(q) \in \mathfrak{n}^+ \) if and only if \( F^k_q \) is isomorphic to \( F^k_p \) for all \( 0 \leq k \leq n \).

**Proof.** For any \( q \in T \), we choose an arbitrary adapted basis \( \{ \zeta_0, \cdots, \zeta_{m-1} \} \) for the given Hodge filtration \( \{ F^m_q \subseteq F^{n-1}_q \subseteq \cdots \subseteq F^0_q \} \). Recall that \( \{ \eta_0, \cdots, \eta_{m-1} \} \) is the adapted basis for the Hodge filtration \( \{ F^m_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p \} \) for the base point \( p \). Let \( [A^{i,j}(q)]_{i \leq i,j \leq n} \) be the transition matrix between the basis \( \{ \eta_0, \cdots, \eta_{m-1} \} \) and \( \{ \zeta_0, \cdots, \zeta_{m-1} \} \) for the same vector space \( H_c \), where \( A^{i,j}(q) \) are the corresponding blocks. Then \( \Phi(q) \in \mathfrak{n}^+ = \mathfrak{n}^+ B / B \subseteq \check{D} \) if and only if its matrix representation \( [A^{i,j}(q)]_{i \leq i,j \leq n} \) can be decomposed as \( L(q) \cdot U(q) \), where \( L(q) \) is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and \( U(q) \) is a nonsingular block upper triangular matrix. By basic linear algebra, we know that \( [A^{i,j}(q)] \) has such decomposition if and only if \( \det([A^{i,j}(q)]_{i \leq i,j \leq n}) \neq 0 \) for any \( 0 \leq k \leq n \). In particular, we know that \( [A(q)^{i,j}]_{i \leq i,j \leq k} \) is the transition map between the bases of \( F^k_p \) and \( F^k_q \). Therefore, \( \det([A(q)^{i,j}]_{i \leq i,j \leq k}) \neq 0 \) if and only if \( F^k_q \) is isomorphic to \( F^k_p \).

**Proposition 1.3.** The subset \( \tilde{T} \) is an open dense submanifold in \( T \), and \( T \setminus \tilde{T} \) is an analytic subvariety of \( T \) with \( \text{codim}_C (T \setminus \tilde{T}) \geq 1 \).

**Proof.** From Lemma 1.2 one can see that \( \check{D} \setminus \mathfrak{n}^+ \subseteq \check{D} \) is defined as an analytic subvariety by equations

\[
\{ q \in \check{D} : \det([A^{i,j}(q)]_{i \leq i,j \leq k}) = 0 \text{ for some } 0 \leq k \leq n \}.
\]

Therefore \( \mathfrak{n}^+ \) is dense in \( \check{D} \), and that \( \check{D} \setminus \mathfrak{n}^+ \) is an analytic subvariety, which is closed in \( \check{D} \) and with \( \text{codim}_C (\check{D} \setminus \mathfrak{n}^+) \geq 1 \). We consider the period map \( \Phi : T \to \check{D} \) as a holomorphic
map to $\tilde{T}$, then $\mathcal{T} \setminus \tilde{T} = \Phi^{-1}(\tilde{D} \setminus N_+)$ is the preimage of $\tilde{D} \setminus N_+$ of the holomorphic map $\Phi$. Therefore $\mathcal{T} \setminus \tilde{T}$ is also an analytic subvariety and a closed set in $\mathcal{T}$. Because $\mathcal{T}$ is smooth and connected, $\mathcal{T}$ is irreducible. If $\dim(\mathcal{T} \setminus \tilde{T}) = \dim \mathcal{T}$, then $\mathcal{T} \setminus \tilde{T} = \mathcal{T}$ and $\tilde{T} = \emptyset$, but this contradicts to the fact that the reference point $p$ is in $\tilde{T}$. Thus we conclude that $\dim(\mathcal{T} \setminus \tilde{T}) < \dim \mathcal{T}$, and consequently $\text{codim}_C(\mathcal{T} \setminus \tilde{T}) \geq 1$.

\[\square\]

In the introduction, we have assumed that $S$ admits a compactification $\overline{S}$ such that $\overline{S}$ is projective and $\overline{S} \setminus S$ is a divisor with simple normal crossings. Let $S' \supseteq S$ be the subset of $\overline{S}$ to which the period map $\Phi : S \to D/\Gamma$ extends continuously and $\Phi' : S' \to D/\Gamma$ be the extended map. Then one has the commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{i} & S' \\
\downarrow \Phi & & \downarrow \Phi' \\
D/\Gamma & & D/\Gamma
\end{array}
$$

with $i : S \to S'$ the inclusion map.

**Lemma 1.4.** $S'$ is dense and open smooth submanifold in $\overline{S}$ and the complex codimension of $\overline{S} \setminus S'$ is at least one.

**Proof.** Here by abusing notation, in the following discussion we still denote by $\Gamma$ the discrete subgroup in $\text{Aut}(H_{\mathbb{Z}}, Q)$ containing the monodromy group, the image of $\pi_1(S)$. As a standard procedure, after going to a finite cover we may assume $\Gamma$ is torsion free, or even neat. See Theorem 3.6 in [22]. We will give two proofs of the lemma.

First we can use Theorem 9.6 of Griffiths in [11], see also Corollary 13.4.6 in [2], to get the Zariski open submanifold $S''$ of $\overline{S}$, where $S''$ contains all points of finite monodromy in $\overline{S}$. The extension of the period map $\Phi'' : S'' \to D/\Gamma$ is a proper holomorphic map. Since $\Gamma$ is torsion free, the isotropy groups of $\Gamma$ acting on $D$ are all finite, therefore trivial. From this we see that $S''$ is precisely $S'$ defined above. For related discussion, see page 705-706 in [29].

For the second proof, note that $\overline{S}$ is a smooth and $S \subseteq S$ is densely open in $\overline{S}$. To prove this we use the compactification space $\overline{D}/\Gamma$. There are several natural notions of the compactification space $\overline{D}/\Gamma$, see [32], [8, Page 2], [3] Page 29, 30], in which it is proved that the period map has continuous, even holomorphic extension. We can choose any one of them together with the continuous extension of the period map $\Phi : S \to D/\Gamma$ as

$$
\overline{\Phi} : \overline{S} \to \overline{D}/\Gamma.
$$

By the definition of $S'$, $S' = \overline{\Phi}^{-1}(D/\Gamma)$. Since $D/\Gamma$ is open and dense in the compactification in $\overline{D}/\Gamma$, $S'$ is therefore an open submanifold of $\overline{S}$.

\[\square\]
Let \( \mathcal{T}' \) be the universal cover of \( S' \) with the universal covering map \( \pi' : \mathcal{T}' \to S' \). We then have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_T} & \mathcal{T}' \\
\pi & & \pi' \\
S & \xrightarrow{i} & S' \xrightarrow{\Phi'} D/\Gamma,
\end{array}
\]

where \( i_T \) is the lifting of \( i \circ \pi \) with respect to the covering map \( \pi' : \mathcal{T}' \to S' \) and \( \tilde{\Phi}' \) is the lifting of \( \Phi' \circ \pi' \) with respect to the covering map \( \pi_D : D \to D/\Gamma \). Then \( \tilde{\Phi}' \) is continuous.

There are different choices of \( i_T \) and \( \tilde{\Phi}' \), but Lemma A.1 in the Appendix shows that we can choose \( i_T \) and \( \tilde{\Phi}' \) such that \( \tilde{\Phi} = \tilde{\Phi}' \circ i_T \). Let \( \mathcal{T}_0 \subseteq \mathcal{T}' \) be defined by \( \mathcal{T}_0 = i_T(\mathcal{T}) \).

**Lemma 1.5.** \( \mathcal{T}_0 = \pi'^{-1}(S) \).

*Proof.* From the diagram (10), we see that \( \pi'(\mathcal{T}_0) = \pi'(i_T(\mathcal{T})) = i(\pi(\mathcal{T})) = S \), hence \( \mathcal{T}_0 \subseteq \pi'^{-1}(S) \).

Conversely, for any \( q \in \pi'^{-1}(S) \), we need to prove that \( q \in \mathcal{T}_0 \). Let \( p = \pi'(q) \). If there exists a \( r \in \pi^{-1}(q) \) such that \( i_T(r) = q \), then we are done. Otherwise, we can draw a curve \( \gamma \) from \( i_T(r) \) to \( q \) for some \( r \in \pi^{-1}(q) \), as \( \mathcal{T}' \) is connected and thus path connected. Then we get a circle \( \Gamma = \pi'(\gamma) \) in \( S' \). But Lemma A.2 in the Appendix implies that we can choose \( \Gamma \) contained in \( S' \). Note that \( p \in \Gamma \). Since \( \pi : \mathcal{T} \to S \) is covering map, we can lift \( \Gamma \) to a unique curve \( \tilde{\gamma} \) from \( r \) to some \( r' \in \pi^{-1}(p) \). Notice that both \( \gamma \) and \( \tilde{\gamma} \) map to \( \Gamma \), that is \( \gamma \) is the lift of \( \tilde{\gamma} \) for the covering map \( \pi' : \mathcal{T}' \to S' \). By the uniqueness of homotopy lifting, \( i_T(r') = q \), i.e. \( q \in i_T(\mathcal{T}) = \mathcal{T}_0 \).

\( \square \)

Lemma 1.5 implies that \( \mathcal{T}_0 \) is an open submanifold of \( \mathcal{T}' \) and \( \text{codim}_S(\mathcal{T}' \setminus \mathcal{T}_0) \geq 1 \). Since \( \tilde{\Phi} = \tilde{\Phi}' \circ i_T \) is holomorphic, \( \tilde{\Phi}'|_{\mathcal{T}_0} : \mathcal{T}_0 \to D \) is also holomorphic. Since \( \tilde{\Phi}' : \mathcal{T}' \to D \) is continuous, \( \tilde{\Phi}' \) is locally bounded around \( \mathcal{T}' \setminus \mathcal{T}_0 \), then by applying Riemann extension theorem we have that \( \tilde{\Phi}' : \mathcal{T}' \to D \) is holomorphic.

**Lemma 1.6.** The extended holomorphic map \( \tilde{\Phi}' : \mathcal{T}' \to D \) satisfies the Griffiths transversality.

*Proof.* Let \( T'^{1,0}_h \) be the horizontal subbundle. Since \( \tilde{\Phi}' : \mathcal{T}' \to D \) is a holomorphic map, the tangent map

\[
\tilde{\Phi}'_* : T'^{1,0} \to T'^{1,0}_h
\]

is at least continuous. We only need to show that the image of \( \tilde{\Phi}'_* \) is contained in the horizontal tangent bundle \( T'^{1,0}_h \).

Since \( T'^{1,0}_h \) is closed in \( T'^{1,0} \), \( (\tilde{\Phi}'_*)^{-1}(T'^{1,0}_h) \) is closed in \( T'^{1,0} \). But \( \tilde{\Phi}'|_{\mathcal{T}_0} \) satisfies the Griffiths transversality, i.e. \( (\tilde{\Phi}'_*)^{-1}(T'^{1,0}_h) \) contains \( T'^{1,0}_h \cap \mathcal{T}_0 \), which is open in \( T'^{1,0} \). Hence \( (\tilde{\Phi}'_*)^{-1}(T'^{1,0}_h) \) contains the closure of \( T'^{1,0}_h \), which is \( T'^{1,0} \).

\( \square \)
2. Further results of period domains from Lie theory

In this section, we study the structure of the Lie algebra \( \mathfrak{g} \) by considering the root system, which is fundamental to our proof of Theorem 3.2 about the boundedness of the image of the restricted period map. Many results are from [14] and [26] to which we refer the reader for detailed proofs.

Now we define the Weil operator \( \theta : \mathfrak{g} \to \mathfrak{g} \) by
\[
\theta(X) = (-1)^k X, \quad \text{for } X \in \mathfrak{g}^{k,-k}.
\]
Then \( \theta \) is an involutive automorphism of \( \mathfrak{g} \), and defined over \( \mathbb{R} \). Let \( \mathfrak{k} \) and \( \mathfrak{p} \) be the \((+1)\) and \((-1)\) eigenspaces of \( \theta \) respectively. Considering the types, we have
\[
\mathfrak{k} = \bigoplus_k \mathfrak{g}^{k,-k}, \quad \mathfrak{p} = \bigoplus_k \mathfrak{g}^{k,-k}.
\]
Set
\[
\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.
\]
Then we have the decompositions
\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0
\]
with the property that
\[
[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.
\]

Let \( \mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0 \). Then \( \mathfrak{g}_c \) is also a real form of \( \mathfrak{g} \). Let us denote the complex conjugation of \( \mathfrak{g} \) with respect to the real form \( \mathfrak{g}_c \) by \( \tau_c \), and the complex conjugation of \( \mathfrak{g} \) with respect to the real form \( \mathfrak{g}_0 \) by \( \tau_0 \).

Recall that on the complex Linear space \( H_C \) we can define an Hermitian inner product \((\cdot, \cdot)\) induced by the Poincaré bilinear form \( Q \) as
\[
(u, v) = Q(Cu, \overline{v}) \quad u, v \in H_C,
\]
where \( C \) is the Weil operator on \( H_C \) and defined over \( \mathbb{R} \). Thus \( C \) can be considered as an element in \( G_{\mathbb{R}} \), whose adjoint action on \( \mathfrak{g} \) is just \( \theta \). For any \( Z = X + \sqrt{-1}Y \in \mathfrak{g}_c \), where \( X \in \mathfrak{k}_0 \) and \( Y \in \mathfrak{p}_0 \), we have that \( \forall u, v \in H_C \)
\[
(Z \cdot u, v) = Q(C((X + \sqrt{-1}Y) \cdot u), \overline{v})
\]
\[
= Q((X - \sqrt{-1}Y) \cdot Cu, \overline{v})
\]
\[
= -Q(Cu, (X - \sqrt{-1}Y) \cdot \overline{v})
\]
\[
= -Q(Cu, (X + \sqrt{-1}Y) \cdot \overline{v})
\]
\[
= -(u, Z \cdot v).
\]
Thus \( \mathfrak{g}_c \) is the intersection of \( \mathfrak{g} \) with the algebra of all skew Hermitian transforms with respect to the Hermitian inner product \((\cdot, \cdot)\). Using this result, Schmid in [26] proved that:

- \( \mathfrak{g}_c \) is a compact real form of \( \mathfrak{g} \), and the Killing form \( B \)
\[
B(X, Y) = \text{Trace}(\text{ad}X \circ \text{ad}Y), \quad X, Y \in \mathfrak{g}
\]
restricts to a negative definite bilinear form $B|_{\mathfrak{g}_c}$ on $\mathfrak{g}_c$. Moreover, one has an Hermitian inner product $-B(\cdot, \cdot)$ on $\mathfrak{g}$, making $\mathfrak{g}$ an Hermitian complex linear space.

Following this result, we have that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is Cartan decomposition.

Now we define the Lie subgroup $G_c \subseteq G_C$ corresponding to $\mathfrak{g}_c \subseteq \mathfrak{g}$. Then (12) implies that $G_c$ contains the elements in $G_C$ which preserve the Hermitian inner product $(\cdot, \cdot)$, i.e. $G_c$ is the unitary subgroup of $G_C$. Thus $G_c$ is compact. As noted by Schmid, a compact real form in a connected complex semisimple Lie group is always connected and is its own normalizer, which implies that $G_c$ is also connected.

The intersection $K = G_c \cap G_R$ is a compact subgroup of $G_R$ with Lie algebra $\mathfrak{g}_c \cap \mathfrak{g}_0 = \mathfrak{k}_0$. In pages 278-279 of [26], Schmid showed that:

- $K$ is a maximal compact subgroup of $G_R$ and it meets every connected component of $G_R$.
- $G_c \cap B = V$, which implies $V \subseteq K$ and their Lie algebras $\mathfrak{v}_0 \subseteq \mathfrak{k}_0$.

In [14], Griffiths and Schmid observed that:

- There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$.

Denote $\mathfrak{h}$ to be the complexification of $\mathfrak{h}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Now we review the root systems we need. Write $\mathfrak{h}_0^* = \text{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_R^* = \sqrt{-1} \mathfrak{h}_0^*$. Then $\mathfrak{h}_R^*$ can be identified with $\mathfrak{h}_R := \sqrt{-1} \mathfrak{h}_0$ by the restricted Killing form $B|_{\mathfrak{h}_R}$ on $\mathfrak{h}_R$. Let $\rho \in \mathfrak{h}_R^* \simeq \mathfrak{h}_R$, one can define the following subspace of $\mathfrak{g}$,

$$\mathfrak{g}^{\rho} = \{ x \in \mathfrak{g} | [h, x] = \rho(h)x \ \text{for all } h \in \mathfrak{h} \}.$$  

An element $\varphi \in \mathfrak{h}_R^* \simeq \mathfrak{h}_R$ is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}^{\varphi} \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_R^* \simeq \mathfrak{h}_R$ denote the space of nonzero $\mathfrak{h}$-roots. Then each root space

$$\mathfrak{g}^{\varphi} = \{ x \in \mathfrak{g} | [h, x] = \varphi(h)x \ \text{for all } h \in \mathfrak{h} \}$$

with respect to some $\varphi \in \Delta$ is one-dimensional over $\mathbb{C}$, generated by a root vector $e_{\varphi}$.

Since the involution $\theta$ is a Lie-algebra automorphism fixing $\mathfrak{t}$, we have

$$\theta([h, \theta(e_{\varphi})]) = [h, e_{\varphi}]$$

and hence

$$[h, \theta(e_{\varphi})] = \theta([h, e_{\varphi}]) = \varphi(h)\theta(e_{\varphi}),$$

for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_{\varphi})$ is also a root vector belonging to the root $\varphi$, so $e_{\varphi}$ must be an eigenvector of $\theta$. It follows that there is a decomposition of the roots $\Delta$ into the union $\Delta_k \cup \Delta_p$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_{\varphi} \subseteq \mathfrak{t}$ and $\mathfrak{p}$ respectively.

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^{\varphi}.$$
Let \( \{ \varphi_i : 1 \leq i \leq l \} \), \( l = \text{rank}(\mathfrak{g}) \) be any simple root system of \( \Delta \), then we can introduce the notion of positivity in \( \Delta \), i.e. we can define \( \Delta^+ \) to be the subset of \( \Delta \) consisting of the roots in \( \Delta \) which are positive integral linear combination of the simple roots. Let \( \{ h_i : 1 \leq i \leq l \} \) be the basis of \( \mathfrak{h}_\mathbb{R} \) corresponding to the simple roots. That is to say that \( \varphi_i(h) = B(h_i, h) \) for any \( h \in \mathfrak{h} \), where \( 1 \leq i \leq l \) and the Cartan matrix \( (B(h_i, h_j)) \) is positive definite.

Following Serre [27], we can choose a Weyl basis \( \{ e'_\alpha : \alpha \in \Delta \} \) such that

\[
[h, h] = 0; \\
[h, e'_\alpha] = \alpha(h)e'_\alpha, \ \forall h \in \mathfrak{h}; \\
[e'_\alpha, e'_{-\alpha}] = B(e'_\alpha, e'_{-\alpha})h_\alpha, \ \forall \varphi \in \Delta; \\
[e'_\alpha, e'_\beta] = N_{\alpha, \beta}e'_{\alpha + \beta}, \ \alpha + \beta \neq 0,
\]

where \( N_{\alpha, \beta} = 0 \) if \( \alpha + \beta \neq 0 \), \( \alpha + \beta \notin \Delta \); \( N_{\alpha, \beta} \neq 0 \) if \( \alpha + \beta \in \Delta \) with relation that

\[
N_{-\alpha, -\beta} = -N_{\alpha, \beta} \in \mathbb{R}.
\]

Now we define the real subspace of \( \mathfrak{g} \) as

\[
\mathfrak{g}'_c = \mathfrak{h}_0 + \sum_{\alpha \in \Delta} \mathbb{R}(e'_\alpha - e'_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}\sqrt{-1}(e'_\alpha + e'_{-\alpha}).
\]

In fact, Theorem 6.3 in Chapter III of [16] shows that \( \mathfrak{g}'_c \) is a compact real form of \( \mathfrak{g} \) with complex conjugation \( \tau' \). Again by Theorem 7.1 in Chapter III of [16] and its proof, there exists one-parameter subgroup \( \vartheta \) of automorphisms of \( \mathfrak{g} \) such that \( \vartheta_{1/2}(\mathfrak{g}'_c) \) is invariant under \( \tau_c \) and \( \vartheta_1 = (\tau_c \tau'_c)^2 \). Then the proof of Theorem 7.2 and Corollary 7.3 in Chapter III of [16] show that \( \mathfrak{g}_c = \vartheta_{1/2}(\mathfrak{g}'_c) \) and there exist some \( X \in \mathfrak{g} \) such that \( \vartheta_1 = \exp(t \cdot \text{ad} X) \). Since \( \exp(\text{ad} X) = (\tau_c \tau'_c)^2 \) and \( \tau_c|_{\mathfrak{h}_0} = \tau'_c|_{\mathfrak{h}_0} = \text{id} \), \( X \) must be in \( \mathfrak{h} \), hence \( \vartheta_1|_{\mathfrak{h}_0} = \exp(t \cdot \text{ad} X)|_{\mathfrak{h}_0} = \text{id} \). Therefore we have proved that there exists an automorphism \( \vartheta = \vartheta_{1/2} \) of \( \mathfrak{g} \) such that \( \mathfrak{g}_c = \vartheta(\mathfrak{g}'_c) \) and \( \vartheta|_{\mathfrak{h}_0} = \text{id} \).

For any \( \alpha \in \Delta \), let \( e_\alpha \triangleq \vartheta_{1/2}(e'_\alpha) \). Since \( \vartheta_{1/2} \) is an automorphism, \( \{ e_\alpha : \alpha \in \Delta \} \) is also a Weyl basis. Then under the Weyl basis \( \{ e_\alpha : \alpha \in \Delta \} \),

\[
\mathfrak{g}_c = \vartheta_{1/2}(\mathfrak{g}'_c) \\
= \mathfrak{h}_0 + \sum_{\alpha \in \Delta} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}\sqrt{-1}(e_\alpha + e_{-\alpha})
\]

\[
= \mathfrak{h}_0 + \sqrt{-1}\mathfrak{p}_0.
\]

Therefore, by considering types we have proved the following theorem.

**Theorem 2.1.** There exist a basis \( \{ h_i : 1 \leq i \leq l \} \) of \( \mathfrak{h}_\mathbb{R} \) and a Weyl basis \( \{ e_\varphi : \varphi \in \Delta \} \) such that

\[
\tau_c(h_i) = \tau_0(h_i) = -h_i, \quad \text{for any } 1 \leq i \leq l;
\]

\[
\tau_c(e_\varphi) = \tau_0(e_\varphi) = -e_{-\varphi}, \quad \text{for any } \varphi \in \Delta_i;
\]

\[
\tau_0(e_\varphi) = -\tau_c(e_\varphi) = e_{-\varphi}, \quad \text{for any } \varphi \in \Delta_p,
\]

\[
(13)
\]
and
\begin{align}
\mathfrak{t}_0 &= h_0 + \sum_{\varphi \in \Delta_t} \mathbb{R}(e_\varphi - e_{-\varphi}) + \sum_{\varphi \in \Delta_t} \mathbb{R}\sqrt{-1}(e_\varphi + e_{-\varphi}); \\
p_0 &= \sum_{\varphi \in \Delta_p} \mathbb{R}(e_\varphi + e_{-\varphi}) + \sum_{\varphi \in \Delta_p} \mathbb{R}\sqrt{-1}(e_\varphi - e_{-\varphi}).
\end{align}

**Lemma 2.2.** Let \( \Delta \) be the set of \( h \)-roots as above. Then for each root \( \varphi \in \Delta \), there is an integer \(-n \leq k \leq n\) such that \( e_\varphi \in g^{k,-k} \). In particular, if \( e_\varphi \in g^{k,-k} \), then \( e_{-\varphi} = -\tau_0(e_\varphi) \in g^{-k,k} \) for any \(-n \leq k \leq n\).

**Proof.** Let \( \varphi \) be a root, and \( e_\varphi \) be the generator of the root space \( g^\varphi \), then \( e_\varphi = \sum_{k=-n}^{n} e^{-k,k} \), where \( e^{-k,k} \in g^{-k,k} \). Because \( h \subseteq v \subseteq g^{0,0} \), the Lie bracket \([h,e^{-k,k}] \in g^{-k,k}\) for each \( k \).

Then the condition \([h,e_\varphi] = \varphi(h)e_\varphi\) implies that
\[\sum_{k=-n}^{n} [h,e^{-k,k}] = \sum_{k=-n}^{n} \varphi(h)e^{-k,k} \quad \text{for each} \ h \in h.\]

By comparing the type, we get
\[ [h,e^{-k,k}] = \varphi(h)e^{-k,k} \quad \text{for each} \ h \in h.\]

Therefore \( e^{-k,k} \in g^\varphi \) for each \( k \). As \( \{e^{-k,k}\}_{k=-n}^{n} \) forms a linear independent set, but \( g^\varphi \) is one dimensional, thus there is only one \(-n \leq k \leq n\) with \( e^{-k,k} \neq 0 \).

\( \square \)

Let us now introduce a lexicographic order (cf. pp.41 in [34] or pp.416 in [30]) in the real vector space \( h_\mathbb{R} \) as follows: we fix an ordered basis \( h_1, \cdots, h_l \) for \( h_\mathbb{R} \). Then for any \( h = \sum_{i=1}^{l} \lambda_i h_i \in h_\mathbb{R} \), we call \( h > 0 \) if the first nonzero coefficient is positive, that is, if \( \lambda_1 = \cdots = \lambda_k = 0, \lambda_{k+1} > 0 \) for some \( 1 \leq k < l \). For any \( h, h' \in h_\mathbb{R} \), we say \( h > h' \) if \( h - h' > 0 \) if \( h - h' < 0 \) and \( h = h' \) if \( h - h' = 0 \).

In particular, let us identify the dual spaces \( h_\mathbb{R}^* \) and \( h_\mathbb{R} \), thus \( \Delta \subseteq h_\mathbb{R} \). Let us choose a maximal linearly independent subset \( \{\varphi_1, \cdots, \varphi_s\} \) of \( \Delta_p \), then a maximal linearly independent subset \( \{\varphi_{s+1}, \cdots, \varphi_t\} \) of \( \Delta_t \). Then \( \{\varphi_1, \cdots, \varphi_s, \varphi_{s+1}, \cdots, \varphi_t\} \) forms a basis for \( h_\mathbb{R}^* \) since Span\(_{\mathbb{R}}\Delta = h_\mathbb{R}^* \). Then define the above lexicographic order in \( h_\mathbb{R}^* \simeq h_\mathbb{R} \) using the ordered basis \( \{\varphi_1, \cdots, \varphi_t\} \). In this way, we can also define
\[\Delta^+ = \{\varphi > 0 : \varphi \in \Delta\}; \quad \Delta^+_t = \Delta^+ \cap \Delta_t.\]

Similarly we can define \( \Delta^- \), \( \Delta^-_t \), \( \Delta^+_t \), and \( \Delta^-_t \).

**Lemma 2.3.** Using the above notation, we have
\begin{align}
\Delta_t \cap \Delta^+_t \cap \Delta \subseteq \Delta^+_t; \\
\Delta_t^+ \cap \Delta \subseteq \Delta^+_t; \quad (\Delta^+_t + \Delta^-_t) \cap \Delta = \emptyset; \quad (\Delta^-_t + \Delta^-_t) \cap \Delta = \emptyset.
\end{align}

If one defines
\[p^+ = \sum_{\varphi \in \Delta^+_p} g^\varphi \subseteq p,\]
then \( p = p^+ \oplus p^- \), \([p^+, p^-] \subseteq \mathfrak{t}\) and
\begin{align}
\mathfrak{t}, p^+ \subseteq p^+;
\end{align}
which is a contradiction. Now we assume that $\sum_i a_i \varphi_i + \sum_i b_i \varphi_i = 0$ \(\varphi \in \Delta_\mathfrak{p}^+\), then $\varphi + \psi = \sum_i (a_i + b_i) \varphi_i \in \Delta_\mathfrak{p}^+$ and $e_\varphi$, $e_\psi$, $e_{\varphi + \psi} \in \mathfrak{p}$. Suppose that $\varphi + \psi \in \Delta$, then

$$0 \neq [e_\varphi, e_\psi] = N_{\varphi + \psi} e_{\varphi + \psi} \in [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}.$$ 

Then $0 \neq e_{\varphi + \psi} \in \mathfrak{t} \cap \mathfrak{p} = \{0\}$, which is a contradiction. Therefore (17) holds, from which (19) follows.

**Definition 2.4.** Two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$, which is denoted by $\varphi \perp \psi$.

**Lemma 2.5.** There exists a set of strongly orthogonal noncompact positive roots $\Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq \Delta_\mathfrak{p}^+$ such that $0 < \varphi_1 < \cdots < \varphi_r$, and

$$\mathfrak{A}_0 = \sum_{i=1}^r \mathbb{R}(e_{\varphi_i} + e_{-\varphi_i})$$

is a maximal abelian subspace in $\mathfrak{p}_0$.

**Proof.** Let $\varphi_1$ be the minimum in $\Delta_\mathfrak{p}^+$, and $\varphi_2$ be the minimal element in $\{\varphi \in \Delta_\mathfrak{p}^+ : \varphi \perp \varphi_1\}$, then we obtain inductively an maximal ordered set of roots $\Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq \Delta_\mathfrak{p}^+$, such that for each $1 \leq k \leq r$

$$\varphi_k = \min\{\varphi \in \Delta_\mathfrak{p}^+ : \varphi \perp \varphi_j \text{ for } 1 \leq j \leq k - 1\}.$$ 

Because $\varphi_i \perp \varphi_j$ for any $1 \leq i < j \leq r$, we have $[e_{\pm \varphi_i}, e_{\pm \varphi_j}] = 0$. Therefore

$$\mathfrak{A}_0 = \sum_{i=1}^r \mathbb{R}(e_{\varphi_i} + e_{-\varphi_i})$$

is an abelian subspace of $\mathfrak{p}_0$.

We claim that $\varphi_1, \ldots, \varphi_r$ are $\mathbb{R}$-linearly independent. In fact, since $\varphi_i \pm \varphi_j \notin \Delta \cup \{0\}$ for $i \neq j$, we conclude that $B(\varphi_i, \varphi_j) = 0$, otherwise either $\frac{2B(\varphi_i, \varphi_j)}{B(\varphi_i, \varphi_i)}$ or $\frac{2B(\varphi_i, \varphi_j)}{B(\varphi_i, \varphi_i)}$ equals to $\pm 1$ and we assume that $\frac{2B(\varphi_i, \varphi_j)}{B(\varphi_i, \varphi_i)} = 1$, then the property of root system implies that

$$\varphi_i \mp \varphi_j = \varphi_i - \frac{2B(\varphi_i, \varphi_j)}{B(\varphi_j, \varphi_j)} \varphi_j \in \Delta,$$

which is a contradiction. Now we assume that $\sum_i k_i \varphi_i = 0$ with each $k_i \in \mathbb{R}$. Then

$$0 = B(\sum_i k_i \varphi_i, \sum_i k_i \varphi_i) = \sum_i k_i^2 B(\varphi_i, \varphi_i).$$

As $B|_{\mathfrak{p}_0}$ is negative definite, we conclude that each $k_i = 0$.

It remains to prove that $\alpha_0$ is maximal. If it is not so, we can find $X \in \mathfrak{p}_0$ with

$$X = \sum_{\alpha \in \Delta_\mathfrak{p}^+ \setminus \Lambda} \lambda_\alpha (e_\alpha + e_{-\alpha}) + \sum_{\alpha \in \Delta_\mathfrak{p}^- \setminus \Lambda} \mu_\alpha \sqrt{-1} (e_\alpha - e_{-\alpha}) \text{, \quad } \lambda_\alpha, \mu_\alpha \in \mathbb{R}$$
such that \([X, e_{\varphi_i} + e_{-\varphi_i}] = 0\) for \(1 \leq i \leq r\). Let \(c_\alpha = \lambda_\alpha + \sqrt{-1}\mu_\alpha\), then
\[
X = \sum_{\alpha \in \Delta^+_p \setminus \Lambda} (c_\alpha e_\alpha + \overline{c_\alpha} e_{-\alpha}) \text{ and for } 1 \leq i \leq r
\]
\[
0 = [X, e_{\varphi_i} + e_{-\varphi_i}]
\]
(20) \[
= \sum_{\alpha \in \Delta^+_p \setminus \Lambda} \left( c_\alpha (N_{\alpha+\varphi_i} e_{\alpha+\varphi_i} + N_{\alpha-\varphi_i} e_{\alpha-\varphi_i}) + \overline{c_\alpha} (N_{\alpha+\varphi_i} e_{\alpha+\varphi_i} + N_{\alpha-\varphi_i} e_{\alpha-\varphi_i}) \right).
\]

By Lemma 2.3 for any \(1 \leq i \leq r\), \(\alpha + \varphi_i \notin \Delta\), hence \(N_{\alpha+\varphi_i} = 0\). Similarly, \(N_{\alpha-\varphi_i} = 0\). Then (20) implies
\[
\sum_{\alpha \in \Delta^+_p \setminus \Lambda} \left( c_\alpha N_{\alpha-\varphi_i} e_{\alpha-\varphi_i} + \overline{c_\alpha} N_{\alpha+\varphi_i} e_{\alpha+\varphi_i} \right) = 0.
\]

For any \(\alpha \in \Delta^+_p \setminus \Lambda\), by the construction of \(\Lambda\) there exists an \(i\) such that \(\alpha - \varphi_i\) lies in \(\Delta\), and then \(N_{\alpha-\varphi_i} \neq 0\). Equation (21) then implies that \(c_\alpha = 0\). Hence \(X = 0\).

For further use, we also state a proposition about the maximal abelian subspace of \(p_0\) as Lemma 6.3 in Chapter V of [16].

**Proposition 2.6.** Let \(\mathfrak{A}_0\) be an arbitrary maximal abelian subspaces of \(p_0\), then there exists an element \(k \in K\) such that \(\text{Ad}(k) \cdot \mathfrak{A}_0 = \mathfrak{A}_0\). Moreover, we have

\[
\mathfrak{p}_0 = \bigcup_{k \in K} \text{Ad}(k) \cdot \mathfrak{A}_0,
\]
where \(\text{Ad}\) denotes the adjoint action of \(K\) on \(\mathfrak{A}_0\).

3. Boundedness of the period map

In the previous two sections, we have discussed the basic properties of the period domains. In particular, we have found the analytic subvariety \(\mathcal{T} \setminus \mathcal{T}\) with \(\text{codim}_\mathbb{C}(\mathcal{T} \setminus \mathcal{T}) \geq 1\). In this section, we will prove the boundedness of the restricted period map \(\Phi|_\mathcal{T} : \mathcal{T} \to N_+ \cap D\) by using the structure theory of the complex semi-simple Lie algebra \(\mathfrak{g}\).

Recall that we have fixed the reference points \(p \in \mathcal{T}\) and \(o = \Phi(p) \in D\). Then \(N_+\) can be viewed as a subset in \(D\) by identifying it with its orbit \(N_+(o)\) in \(D\). At the base point \(\Phi(p) = o \in N_+ \cap D\), the tangent space \(T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+\) and the exponential map \(\exp : \mathfrak{n}_+ \to N_+\) is an isomorphism. Then the Hodge metric on \(T_o^{1,0}D\) induces an Euclidean metric on \(N_+\) so that \(\exp : \mathfrak{n}_+ \to N_+\) is an isometry.

Also recall that we have defined \(\mathcal{T} = \Phi^{-1}(N_+ \cap D)\) and have shown that \(\mathcal{T} \setminus \mathcal{T}\) is an analytic subvariety of \(\mathcal{T}\) with \(\text{codim}_\mathbb{C}(\mathcal{T} \setminus \mathcal{T}) \geq 1\). In this section, we prove that \(\tilde{\Phi} : \mathcal{T} \to N_+ \cap D\) is bounded in \(N_+\) with respect to the Euclidean metric on \(N_+\).

Let \(\mathfrak{a} \subseteq \mathfrak{n}_+\) be the abelian subalgebra of \(\mathfrak{n}_+\) determined by the tangent map of period map

\[
\tilde{\Phi}_* : T^{1,0}\mathcal{T} \to T^{1,0}D.
\]

By Griffiths transversality, \(\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}\) is an abelian subspace. Let

\[
A \triangleq \exp(\mathfrak{a}) \subseteq N_+
\]
and \( P : N_+ \cap D \rightarrow A \cap D \) be the projection map induced by the projection from \( N_+ \) to its subspace \( A \). Then \( A \), as a complex Euclidean space, has the induced Euclidean metric from \( N_+ \). We can consider \( A \) as an integral submanifold of the abelian algebra \( \mathfrak{a} \) passing through the base point \( o = \Phi(p) \), see page 248 in [4]. For the basic properties of integral manifolds of horizontal distribution, see Chapter 4 of [4], or [1] and [2 1].

The restricted period map \( \tilde{\Phi} : \tilde{T} \rightarrow N_+ \cap D \) composed with the projection map \( P \) gives a holomorphic map

\[
\Psi : \tilde{T} \rightarrow A \cap D,
\]

that is \( \Psi = P \circ \tilde{\Phi}|_{\tilde{T}}. \)

**Lemma 3.1.** The image of the holomorphic map \( \Psi : \tilde{T} \rightarrow A \cap D \) is bounded in \( A \) with respect to the Euclidean metric on \( A \subseteq N_+ \).

**Proof.** Let \( a_0 \triangleq a + \tau_0(a) \subseteq p_0 \) be the abelian subspace of \( p_0 \). Then \( a_0 \) is contained in some maximal abelian subspace \( \mathfrak{A}'_0 \). By Proposition 2.6 maximal abelian subspaces are conjugate to each other. Thus without lose of generality, we may assume that \( a_0 \) is contained in \( \mathfrak{A}_0 \), which is defined in lemma 2.5.

The following proof is an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see page 94 in [23].

Let \( \Lambda = \{ \varphi_1, \ldots, \varphi_r \} \subseteq \Delta^+ \) be a set of strongly orthogonal roots given in Lemma 2.5.

We denote \( x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i} \) and \( y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i}) \) for any \( \varphi_i \in \Lambda \). Then

\[
\mathfrak{A}_0 = \mathbb{R}x_{\varphi_1} \oplus \cdots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad \mathfrak{A}_c = \mathbb{R}y_{\varphi_1} \oplus \cdots \oplus \mathbb{R}y_{\varphi_r},
\]

are maximal abelian spaces in \( p_0 \) and \( \sqrt{-1}\mathfrak{p}_0 \) respectively. For any \( X \in \mathfrak{a}_0 \subseteq \mathfrak{A}_0 \) there exists \( \lambda_i \in \mathbb{R} \) for \( 1 \leq i \leq r \) such that

\[
X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \cdots + \lambda_r x_{\varphi_r}.
\]

Since \( \mathfrak{A}_0 \) is commutative, we have

\[
\exp(tX) = \prod_{i=1}^{r} \exp(t\lambda_i x_{\varphi_i}).
\]

Now for each \( \varphi_i \in \Lambda \), we have \( \text{Span}_\mathbb{C}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C}) \) with

\[
h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};
\]

and \( \text{Span}_\mathbb{R}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R}) \) with

\[
\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y_{\varphi_i} \mapsto \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}.
\]

Since \( \Lambda = \{ \varphi_1, \ldots, \varphi_r \} \) is a set of strongly orthogonal roots, we have that

\[
\mathfrak{g}_\mathbb{C}(\Lambda) = \text{Span}_\mathbb{C}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^{r} \simeq (\mathfrak{sl}_2(\mathbb{C}))^r,
\]

and \( \mathfrak{g}_\mathbb{R}(\Lambda) = \text{Span}_\mathbb{R}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^{r} \simeq (\mathfrak{sl}_2(\mathbb{R}))^r. \)

In fact, we know that for any \( \varphi, \psi \in \Lambda \) with \( \varphi \neq \psi \), \( [e_{\pm\varphi}, e_{\pm\psi}] = 0 \) since \( \varphi \) is strongly orthogonal to \( \psi \); \( [h_{\varphi}, h_{\psi}] = 0 \), since \( \mathfrak{h} \) is abelian; and

\[
[h_{\varphi}, e_{\pm\psi}] = [[e_{\varphi}, e_{-\varphi}], e_{\pm\psi}] = -[[e_{-\varphi}, e_{\pm\psi}], e_{\varphi}] - [e_{\pm\psi}, e_{\varphi}] - [e_{\pm\psi}, e_{\varphi}] = 0.
\]
Then we denote $G_C(\Lambda) = \exp(g_C(\Lambda)) \cong (SL_2(\mathbb{C}))^r$ and $G_R(\Lambda) = \exp(g_R(\Lambda)) = (SL_2(\mathbb{R}))^r$, which are subgroups of $G_C$ and $G_R$ respectively. With the fixed reference point $o = \Phi(p)$, we denote $D(\Lambda) = G_R(\Lambda)(o)$ and $S(\Lambda) = G_C(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

\begin{align*}
(22) \quad D(\Lambda) &= G_R(\Lambda) \cdot B/B \cong G_R(\Lambda)/G_R(\Lambda) \cap V, \\
(23) \quad S(\Lambda) \cap (N_+ B/B) &= (G_C(\Lambda) \cap N_+) \cdot B/B \cong G_C(\Lambda) \cap N_+.
\end{align*}

With the above notations, we will show that

(i) $D(\Lambda) \subseteq S(\Lambda) \cap (N_+ B/B) \subseteq D$;

(ii) $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+ B/B)$.

By Lemma 2.2, we know that for each pair of roots \{\(e_{\varphi_i}, e_{-\varphi_i}\}\), there exists an integer $k$ such that either $e_{\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ and $e_{-\varphi_i} \in \mathfrak{g}^{k,-k}$, or $e_{\varphi_i} \in \mathfrak{g}^{k,-k}$ and $e_{-\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$. For notation simplicity, for each pair of root vectors \{\(e_{\varphi_i}, e_{-\varphi_i}\}\), we may assume the one in $\mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ to be $e_{\varphi_i}$ and denote the one in $\mathfrak{g}^{k,-k}$ by $e_{-\varphi_i}$. In this way, one can check that \{\(\varphi_1, \cdots, \varphi_r\)\} may not be a set in $\Delta^+_p$, but it is a set of strongly orthogonal roots in $\Delta_p$.

Therefore, we have the following description of the above groups,

\begin{align*}
G_R(\Lambda) &= \exp(g_R(\Lambda)) = \exp(\text{Span}_R\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1} h_{\varphi_1}, \cdots, x_{\varphi_r}, y_{\varphi_r}, \sqrt{-1} h_{\varphi_r}\}) \\
G_R(\Lambda) \cap V &= \exp(g_R(\Lambda) \cap v_0) = \exp(\text{Span}_R\{\sqrt{-1} h_{\varphi_1}, \cdots, \sqrt{-1} h_{\varphi_r}\}) \\
G_C(\Lambda) \cap N_+ &= \exp(g_C(\Lambda) \cap n_+) = \exp(\text{Span}_C\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}).
\end{align*}

Thus by the isomorphisms in (22) and (23), we have

\begin{align*}
D(\Lambda) &\cong \prod_{i=1}^{r} \exp(\text{Span}_R\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1} h_{\varphi_i}\})/\exp(\text{Span}_R\{\sqrt{-1} h_{\varphi_i}\}), \\
S(\Lambda) \cap (N_+ B/B) &\cong \prod_{i=1}^{r} \exp(\text{Span}_C\{e_{\varphi_i}\}).
\end{align*}

Let us denote $G_C(\varphi_i) = \exp(\text{Span}_C\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}) \cong SL_2(\mathbb{C})$, $S(\varphi_i) = G_C(\varphi_i)(o)$, and $G_R(\varphi_i) = \exp(\text{Span}_R\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1} h_{\varphi_i}\}) \cong SL_2(\mathbb{R})$, $D(\varphi_i) = G_R(\varphi_i)(o)$.

Now each point in $S(\varphi_i) \cap (N_+ B/B)$ can be represented by

$$
\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.
$$
Thus $S(\varphi_i) \cap (N_+ B/B) \simeq \mathbb{C}$. In order to see $D(\varphi_i)$ in $G_C/B$, we decompose each point in $D(\varphi_i)$ as follows. Let $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$\exp(ax_{\varphi_i} + by_{\varphi_i}) = \left[ \frac{\cosh |z|}{|z|} \begin{array}{c} \frac{\sinh |z|}{|z|} \\ \frac{\sinh |z|}{|z|} \end{array} \right]$$

$$= \left[ \frac{1}{|z|} \tanh |z| \begin{array}{c} 0 \\ 1 \end{array} \right] \left[ \begin{array}{c} \cosh |z| \\ 0 \end{array} \right] \left( \cosh |z| \right)^{-1} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$$

$$= \exp \left( \frac{z}{|z|} \tanh |z| e_{-\varphi_i} \right) \exp \left[ \log(\cosh |z|) \right] h_{\varphi_i} \exp \left( \frac{z}{|z|} \tanh |z| e_{\varphi_i} \right)$$

$$\equiv \exp \left( \frac{z}{|z|} \tanh |z| e_{\varphi_i} \right) \pmod{B}.$$ 

So elements of $D(\varphi_i)$ in $G_C/B$ can be represented by $\exp((z/|z|)(\tanh |z|)e_{\varphi_i})$, i.e.

$$\left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \frac{z}{|z|} \tanh |z|,$$

in which $\frac{z}{|z|} \tanh |z|$ is a point in the unit disc $\mathcal{D}$ of the complex plane. Therefore the $D(\varphi_i)$ is a unit disc $\mathcal{D}$ in the complex plane $S(\varphi_i) \cap (N_+ B/B)$. Therefore

$$D(\Lambda) \simeq \mathcal{D}^r \quad \text{and} \quad S(\Lambda) \cap N_+ \simeq \mathbb{C}^r.$$ 

So we have obtained both (i) and (ii). As a consequence, we get that for any $q \in \mathcal{T}$, $\Psi(q) \in D(\Lambda)$. This implies

$$d_E(\Psi(p), \Psi(q)) \leq \sqrt{r}$$

where $d_E$ is the Euclidean distance on $S(\Lambda) \cap (N_+ B/B)$.

To complete the proof, we only need to show that $S(\Lambda) \cap (N_+ B/B)$ is totally geodesic in $N_+ B/B$. In fact, the tangent space of $N_+$ at the base point is $n_+$ and the tangent space of $S(\Lambda) \cap N_+ B/B$ at the base point is $\text{Span}_\mathbb{C}\{e_{\varphi_1}, e_{\varphi_2}, \ldots, e_{\varphi_r}\}$. Since $\text{Span}_\mathbb{C}\{e_{\varphi_1}, e_{\varphi_2}, \ldots, e_{\varphi_r}\}$ is a sub-Lie algebra of $n_+$, and the corresponding orbit $S(\Lambda) \cap N_+ B/B$ is a sub-Lie algebra of $n_+$, and the corresponding orbit $S(\Lambda) \cap N_+ B/B$ is totally geodesic in $N_+ B/B$. Here the basis $\{e_{\varphi_1}, e_{\varphi_2}, \ldots, e_{\varphi_r}\}$ is an orthonormal basis with respect to the pull-back Euclidean metric.

Now let $p_h = p/p \cap b \subseteq n_+$ denote the horizontal tangent space of $D$ at the base point $o$, which is viewed as an Euclidean subspace of $n_+$, and consider the natural projection $P_h : D \cap N_+ \to D \cap \exp(p_h)$ by viewing $\exp(p_h)$ as an Euclidean subspace of $N_+$. It is easy to see that the natural projection $\pi : D \to G_R/K$, restricted to $D \cap \exp(p_h)$ is surjective onto $G_R/K$.

We consider $D \cap A$ as submanifolds of $D \cap \exp(p_h)$. Then there exists a neighborhood $U$ of the base point $o$ in $D$ such that the image of $U \cap \tilde{\Phi}(\mathcal{T})$ under this projection $P_h$ lies inside $D \cap A$, since they both are integral submanifolds of the horizontal distribution given by the abelian algebra $a$. From this we see that $P_h(\tilde{\Phi}(\mathcal{T}))$ must lie inside $D \cap A$ completely, since $\tilde{\Phi}(\mathcal{T})$ is a connected irreducible complex analytic variety, which is induced from the irreducibility of $\mathcal{T}$. This shows that any fiber of the projection $\pi : D \to G_R/K$, which intersects $\tilde{\Phi}(\mathcal{T})$, must intersect $D \cap A$.
**Theorem 3.2.** The image of the restriction of the period map \(\tilde{\Phi} : \tilde{T} \to N_+ \cap D\) is bounded in \(N_+\) with respect to the Euclidean metric on \(N_+\).

**Proof.** In Lemma 3.1, we already proved that the image of \(\Psi = P \circ \tilde{\Phi}\) is bounded with respect to the Euclidean metric on \(A \subseteq N_+\). Now together with the Griffiths transversality, we will deduce the boundedness of the image of \(\tilde{\Phi} : \tilde{T} \to N_+ \cap D\) from the boundedness of the image of \(\Psi\).

In fact, \(A \cap D\) is an integral submanifold in \(D\) of the abelian algebra \(a \subseteq \mathfrak{g}^{-1,1}\) passing through the base point \(o = \Phi(p)\), so is \(\Phi(T)\) which is an analytic variety. See Theorem 9.6 of \([11]\). Restricted to \(\tilde{\Phi}(\tilde{T})\) and given any point \(z \in \Psi(\tilde{T})\), \(P|_{\tilde{\Phi}(\tilde{T})}\) is just the projection map from the intersection \(\tilde{\Phi}(\tilde{T}) \cap \pi^{-1}(z')\) to \(z\), where \(z' = \pi(z) \in G_{\mathbb{R}}/K\) and \(\pi : D \to G_{\mathbb{R}}/K\) is the natural projection map. Therefore from the discussion above Theorem 3.2 we get that, given any point \(z = \Psi(q) \in A \cap D\), \((P|_{\tilde{\Phi}(\tilde{T})})^{-1}(z)\) is the set of points in \(\tilde{\Phi}(\tilde{T}) \cap \pi^{-1}(z')\). Note that the Griffiths transversality implies that the projection map \(\pi\) is a locally one-to-one map when restricted to a horizontal slice, which is a small open neighborhood of the integral submanifold of the horizontal distribution. For discussion of the local structure of the integral manifolds of distributions, see for example, Proposition 19.16, page 500 of \([18]\) and Corollary 2.1 in \([19]\).

With the above in mind, our proof can be divided into two steps. It is an elementary argument to apply the Griffiths transversality on \(T'\).

(i) We claim that there are only finite points in \((P|_{\tilde{\Phi}(\tilde{T})})^{-1}(z)\), for any \(z \in \Psi(\tilde{T})\).

Otherwise, we have \(\{q_i\}_{i=1}^{\infty} \subseteq \tilde{T}\) and \(y_i = \tilde{\Phi}(q_i) \subseteq (P|_{\tilde{\Phi}(\tilde{T})})^{-1}(z)\) with limiting point \(y_{\infty} \in \pi^{-1}(z') \simeq K/V\), since \(K/V\) is compact. We project the points \(q_i\) to \(q'_i \in S\) via the universal covering map \(p : \mathcal{T} \to S\). There must be infinite many \(q'_i\)'s. Otherwise, we have a subsequence \(\{q_{j_k}\}\) of \(\{q_j\}\) such that \(p(q_{j_k}) = q'_{i_0}\) for some \(i_0\) and

\[
y_{j_k} = \tilde{\Phi}(q_{j_k}) = \gamma_k \tilde{\Phi}(q_{j_0}) = \gamma_k y_{j_0},
\]

where \(\gamma_k \in \Gamma\) is the monodromy action. Since \(\Gamma\) is discrete, the subsequence \(\{y_{j_k}\}\) is not convergent, which is a contradiction.

Now we project the points \(q_i\) on \(S\) via the universal covering map \(p : \mathcal{T} \to S\) and still denote them by \(q_i\) without confusion. Then the sequence \(\{q_i\}_{i=1}^{\infty} \subseteq S\) has a limiting point \(q_{\infty}\) in \(\overline{S}\), where \(\overline{S}\) is the compactification of \(S\) as defined in the introduction section. By continuity the period map \(\Phi : S \to D/\Gamma\) can be extended over \(q_{\infty}\) with \(\Phi(q_{\infty}) = \pi_D(y_{\infty}) \in D/\Gamma\), where \(\pi_D : D \to D/\Gamma\) is the projection map. Thus \(q_{\infty}\) lies \(S'\). Now we can regard the sequence \(\{q_i\}_{i=1}^{\infty}\) as a convergent sequence in \(S'\) with limiting point \(q_{\infty} \in S'\). We can also choose a sequence \(\{q_i\}_{i=1}^{\infty} \subseteq T'\) with limiting point \(\tilde{q}_{\infty} \in T'\) such that \(\tilde{q}_i\) maps to \(q_{i}\) via the universal covering map \(\pi' : T' \to S'\) and \(\tilde{\Phi}(\tilde{q}_i) = y_i \in D\), for \(i \geq 1\) and \(i = \infty\). Since the extended period map \(\tilde{\Phi} : T' \to D\) still satisfies the Griffiths transversality by Lemma 1.6, we can choose a small neighborhood \(U\) of \(\tilde{q}_{\infty}\) such that \(U\), and thus the points \(\tilde{q}_i\) for \(i\) sufficiently large are mapped into to a horizontal slice, which is a contradiction.

(ii) Let \(r(z)\) be the cardinality of the fiber \((P|_{\tilde{\Phi}(\tilde{T})})^{-1}(z)\), for any \(z \in \Psi(\tilde{T})\). We claim that \(r(z)\) is locally constant.
To be precise, for any $z \in \Psi(\mathcal{T})$, let $r = r(z)$ and choose points $x_1, \cdots, x_r$ in $N_+ \cap D$ such that $P|_{\tilde{\Phi}(\mathcal{T})}(x_i) = z$. Let $U_i$ be the horizontal slice around $x_i$ such that

$$P|_{\tilde{\Phi}(\mathcal{T})} : U_i \cap \tilde{\Phi}(\mathcal{T}) \to P|_{\tilde{\Phi}(\mathcal{T})}(U_i)$$

is injective, $i = 1, \cdots, r$. We choose the balls $B_i$, $i = 1, \cdots, r$ small enough in $N_+ \cap D$ such that $B_i$’s are mutually disjoint and $B_i \supseteq U_i$. We claim that there exists a small neighborhood $V \ni z$ in $A \cap D$ such that the restricted map

$$(24) \quad P|_{\tilde{\Phi}(\mathcal{T})} : (P|_{\tilde{\Phi}(\mathcal{T})})^{-1}(V) \cap B_i \to V$$

is injective, for $i = 1, \cdots, r$.

We define the pair of sequences $(z, y)$ for some $i$ as follows

$(\ast)_i$ \{ \{z_k\}_{k=1}^{\infty} \text{ is a convergent sequence in } A \cap D \text{ with limiting point } z. \} \{y_k\}_{k=1}^{\infty} \text{ is a convergent sequence in } N_+ \cap D \text{ with limiting point } x_i \text{ such that } P(y_k) = z_k \text{ and } y_k \in B_i \setminus U_i, \text{ for any } k \geq 1.$

We will prove that the pair of sequences $(z, y)$ as $(\ast)_i$ does not exist for any $i = 1, \cdots, r$, which implies that we can find a small neighborhood $V \ni z$ in $A \cap D$ such that

$$(P|_{\tilde{\Phi}(\mathcal{T})})^{-1}(V) \cap B_i \subseteq U_i.$$ 

Hence (24) holds and $r(z)$ is locally constant.

In fact, if for some $i$ there exists a pair of sequences $(z, y)$ as $(\ast)_i$, then we can find a sequence $\{q_k\}_{k=1}^{\infty}$ in $\mathcal{T}'$ with limiting point $q_\infty \in \mathcal{T}'$ by a similar argument as (i) such that

$\tilde{\Phi}'(q_k) = y_k$ for any $k \geq 1$ and $\tilde{\Phi}'(q_\infty) = x_i$. Since $\tilde{\Phi'} : \mathcal{T}' \to D$ still satisfies the Griffiths transversality due to Lemma 1.6, we can choose a small neighborhood $W \ni q_\infty$ in $\mathcal{T}'$ such that $\tilde{\Phi}'(W) \subseteq U_i$. Then for $k$ sufficiently large, $y_k = \tilde{\Phi}'(q_k) \in U_i$, which is a contradiction to $(\ast)_i$.

Therefore from (i) and (ii) we deduce that the image $\tilde{\Phi}(\mathcal{T}) \subseteq N_+ \cap D$ is also bounded. 

\[ \square \]

4. Proof of the Griffiths Conjecture

In this section, we prove the Griffiths conjecture by using the boundedness of the restricted period map $\tilde{\Phi}|_\mathcal{T} : \mathcal{T} \to N_+ \cap D$ and the Riemann extension theorem. This is our main result.

**Theorem 4.1.** (Main Theorem) The image of $\tilde{\Phi} : \mathcal{T} \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on $N_+$.

**Proof.** According to Proposition 1.3, $\mathcal{T} \setminus \mathcal{T}$ is an analytic subvariety of $\mathcal{T}$ and the complex codimension of $\mathcal{T} \setminus \mathcal{T}$ is at least one; by Theorem 3.2, the holomorphic map $\tilde{\Phi} : \mathcal{T} \to N_+ \cap D$ is bounded in $N_+$ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\tilde{\Phi}_\mathcal{T} : \mathcal{T} \to N_+$, such that $\tilde{\Phi}_\mathcal{T}|_\mathcal{T} = \tilde{\Phi}|_\mathcal{T}$. Since as holomorphic maps, $\tilde{\Phi}_\mathcal{T}$ and $\tilde{\Phi}$ agree on the open subset $\mathcal{T}$, they must be the same on the entire $\mathcal{T}$. Therefore, the image of $\tilde{\Phi}$ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on $N_+$. As a consequence, we also get $\mathcal{T} = \mathcal{T} = \tilde{\Phi}^{-1}(N_+ \cap D)$. 

\[ \square \]
From the proof of Theorem 3.3 and Theorem 4.1, we also have the following corollary, which improves the boundedness on $\mathcal{T}$ to its completion space $\mathcal{T}'$.

**Corollary 4.2.** The image of the extended period map $\tilde{\Phi}' : \mathcal{T}' \to D$ also lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on $N_+$.

**Appendix A. Two topological lemmas**

The two lemmas are elementary and may be well-known. We include their proofs here for the sake of completeness.

**Lemma A.1.** There exists a suitable choice of $i_T$ and $\tilde{\Phi}'$ such that $\tilde{\Phi}' \circ i_T = \tilde{\Phi}$.

*Proof.* Recall the following commutative diagram as in (10)

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_T} & \mathcal{T}' \\
\downarrow & & \downarrow \\
S & \xrightarrow{i} & S' \\
\end{array}
\begin{array}{ccc}
\tilde{\Phi} & \xrightarrow{\pi} & \tilde{\Phi}' \\
\downarrow & & \downarrow \\
D & \xrightarrow{\pi_D} & D/\Gamma,
\end{array}
$$

Fix a reference point $p \in \mathcal{T}$. The relations $i \circ \pi = \pi' \circ i_T$ and $\Phi' \circ \pi' = \pi_D \circ \tilde{\Phi}'$ imply that $\pi_D \circ \tilde{\Phi}' \circ i_T = \Phi' \circ i \circ \pi = \Phi \circ \pi$. Therefore $\tilde{\Phi}' \circ i_T$ is a lifting map of $\Phi$. On the other hand $\tilde{\Phi} : \mathcal{T} \to D$ is also a lifting of $\Phi$. In order to make $\tilde{\Phi}' \circ i_T = \tilde{\Phi}$, one only needs to choose the suitable $i_T$ and $\tilde{\Phi}'$ such that these two maps agree on the reference point, i.e. $\tilde{\Phi}' \circ i_T(p) = \tilde{\Phi}(p)$.

For an arbitrary choice of $i_T$, we have $i_T(p) \in \mathcal{T}'$ and $\pi'(i_T(p)) = i(\pi(p))$. Considering the point $i_T(p)$ as a reference point in $\mathcal{T}'$, we can choose $\tilde{\Phi}'(i_T(p))$ to be any point from $\pi_D^{-1}(\Phi'(i(\pi(p)))) = \pi_D^{-1}(\Phi(\pi(p)))$. Moreover the relation $\pi_D(\tilde{\Phi}(p)) = \Phi(\pi(p))$ implies that $\tilde{\Phi}(p) \in \pi_D^{-1}(\Phi(\pi(p)))$. Therefore we can choose $\tilde{\Phi}'$ such that $\tilde{\Phi}'(i_T(p)) = \tilde{\Phi}(p)$. With this choice, we have $\tilde{\Phi}' \circ i_T = \tilde{\Phi}$. $\square$

**Lemma A.2.** Let $\pi_1(S)$ and $\pi_1(S')$ be the fundamental groups of $S$ and $S'$ respectively, and suppose the group morphism

$$
\gamma : \pi_1(S) \to \pi_1(S')
$$

is induced by the inclusion $i : S \to S'$. Then $\gamma$ is surjective.

*Proof.* First notice that $S$ and $S'$ are both smooth manifolds, and $S \subseteq S'$ is open. Thus for each point $q \in S' \setminus S$ there is a disc $D_q \subseteq S'$ with $q \in D_q$. Then the union of these discs

$$
\bigcup_{q \in S' \setminus S} D_q
$$

forms a manifold with open cover \{$D_q : q \in \bigcup_i D_q$\}. Because both $S$ and $S'$ are second countable spaces, there is a countable subcover \{$D_i$\}$_{i=1}^{\infty}$ such that $S' = S \cup \bigcup_{i=1}^{\infty} D_i$, where the $D_i$ are open discs in $S'$ for each $i$. Therefore, we have $\pi_1(D_i) = 0$ for all $i \geq 1$. Letting $S_k = S \cup \bigcup_{i=1}^{k} D_i$, we get

$$
\pi_1(S_k) \ast \pi_1(D_{k+1}) = \pi_1(S_k) = \pi_1(S_{k-1} \cup D_k), \text{ for any } k.
$$
We know that \( \text{codim}_C(S' \setminus S) \geq 1 \). Therefore since \( D_{k+1} \setminus S_k \subseteq D_{k+1} \setminus S \), we have
\[
\text{codim}_C(D_{k+1} \setminus (D_{k+1} \setminus S_k)) \geq 1
\]
for any \( k \). As a consequence we can conclude that \( D_{k+1} \cap S_k \) is path-connected. Hence we can apply the Van Kampen Theorem on \( S_k = D_{k+1} \cup S_k \) to conclude that for every \( k \), the following group homomorphism is surjective:
\[
\pi_1(S_k) = \pi_1(S_k) * \pi_1(D_{k+1}) \rightarrow \pi_1(S_k \cup D_{k+1}) = \pi_1(S_{k+1}).
\]
Thus we get the directed system:
\[
\pi_1(S) \rightarrow \pi_1(S_1) \rightarrow \ldots \rightarrow \pi_1(S_k) \rightarrow \ldots
\]
By taking the direct limit of this directed system, we get the surjectivity of the group homomorphism \( \pi_1(S) \rightarrow \pi_1(S') \).

\[ \square \]

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