ON THE SEMIGROUP $B^x_n$ WHICH IS GENERATED BY THE FAMILY $F_n$ OF FINITE BOUNDED INTERVALS OF $\omega$

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ABSTRACT. We study the semigroup $B^x_n$, which is introduced in the paper [O. Gutik and M. Mykhalenyh, On some generalization of the bicyclic monoid, Visnyk Lviv. Univ. Ser. Mech.-Mat. 90 (2020), 5–19 (in Ukrainian)], in the case when the family $F_n$ generated by the set $\{0, 1, \ldots, n\}$. We show that the Green relations $\mathcal{D}$ and $\mathcal{J}$ coincide in $B^x_n$, the semigroup $B^x_n$ is isomorphic to the semigroup $\mathcal{F}^{n+1}(\omega)$ of partial convex order isomorphisms of $(\omega, \leq)$ of the rank $\omega$ of $n+1$, and $B^x_n$ admits only Rees congruences. Also, we study shift-continuous topologies on the semigroup $B^x_n$. In particular we prove that for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B^x_n$ every non-zero element of $B^x_n$ is an isolated point of $(B^x_n, \tau)$, $B^x_n$ admits the unique compact shift-continuous $T_1$-topology, and every $\omega$-compact shift-continuous $T_1$-topology is compact. We describe the closure of the semigroup $B^x_n$ in a Hausdorff semitopological semigroup and prove the criterium when a topological inverse semigroup $B^x_n$ is $H$-closed in the class of Hausdorff topological semigroups.

1. INTRODUCTION, MOTIVATION AND MAIN DEFINITIONS

We shall follow the terminology of [11, 14, 15, 17, 36]. By $\omega$ we denote the set of all non-negative integers.

Let $\mathcal{P}(\omega)$ be the family of all subsets of $\omega$. For any $F \in \mathcal{P}(\omega)$ we put $n - m + F = \{n - m + k: k \in F\}$ if $F \neq \emptyset$ and $n - m + \emptyset = \emptyset$. A subfamily $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called $\omega$-closed if $F_1 \cap (-n + F_2) \in \mathcal{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathcal{F}$.

We denote $[0; 0] = \{0\}$ and $[0; k] = \{0, \ldots, k\}$ for any positive integer $k$. The set $[0; k]$, $k \in \omega$, is called an initial interval of $\omega$.

A partially ordered set (or shortly a poset) $(X, \leq)$ is the set $X$ with the reflexive, antisymmetric and transitive relation $\leq$. In this case relation $\leq$ is called a partial order on $X$. A partially ordered set $(X, \leq)$ is linearly ordered or is a chain if $x \leq y$ or $y \leq x$ for any $x, y \in X$. A map $f$ from a poset $(X, \leq)$ onto a poset $(Y, \leq)$ is said to be an order isomorphism if $f$ is bijective and $x \leq y$ if and only if $f(x) \leq f(y)$. A partial order isomorphism $f$ from a poset $(X, \leq)$ into a poset $(Y, \leq)$ is an order isomorphism from a subset $A$ of a poset $(X, \leq)$ into a subset $B$ of a poset $(Y, \leq)$. For any elements $x$ of a poset $(X, \leq)$ we denote

$$\uparrow_{\leq} x = \{y \in X: x \leq y\} \quad \text{and} \quad \downarrow_{\leq} x = \{y \in X: y \leq x\}.$$

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the mapping $\text{inv}: S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a band (or the band of $S$). Then the semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. By $(\omega, \min)$ or $\omega_{\min}$ we denote the set $\omega$ with the semilattice operation $x \cdot y = \min\{x, y\}$.

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If \( S \) is an inverse semigroup then the semigroup operation on \( S \) determines the following partial order \( \preceq \) on \( S \): \( s \preceq t \) if and only if there exists \( e \in E(S) \) such that \( s = te \). This order is called the natural partial order on \( S \) [40].

For semigroups \( S \) and \( T \) a map \( h: S \to T \) is called a homomorphism if \( h(s_1 \cdot s_2) = h(s_1) \cdot h(s_2) \) for all \( s_1, s_2 \in S \).

A congruence on a semigroup \( S \) is an equivalence relation \( \mathcal{C} \) on \( S \) such that \((s,t) \in \mathcal{C} \) implies that \((as, at), (sb, tb) \in \mathcal{C} \) for all \( a, b \in S \). Every congruence \( \mathcal{C} \) on a semigroup \( S \) generates the associated natural homomorphism \( \mathcal{C}: S \to S/\mathcal{C} \) which assigns to each element \( s \) of \( S \) its congruence class \([s]_\mathcal{C} \) in the quotient semigroup \( S/\mathcal{C} \). Also every homomorphism \( h: S \to T \) of semigroups \( S \) and \( T \) generates the congruence \( \mathcal{C}_h \) on \( S \): \((s_1, s_2) \in \mathcal{C}_h \) if and only if \( h(s_1) = h(s_2) \).

A nonempty subset \( I \) of a semigroup \( S \) is called an ideal of \( S \) if \( SIS = \{as: s \in I, a, b \in S\} \subseteq I \). Every ideal \( I \) of a semigroup \( S \) generates the congruence \( \mathcal{C}_I = (I \times I) \cup \Delta_S \) on \( S \), which is called the Rees congruence on \( S \).

Let \( \mathcal{I}_\lambda \) denote the set of all partial one-to-one transformations of \( \lambda \) together with the following semigroup operation:

\[
x(\alpha \beta) = (x\alpha)\beta \quad \text{if} \quad x \in \text{dom}(\alpha \beta) = \{y \in \text{dom} \alpha: y\alpha \in \text{dom} \beta\}, \quad \text{for} \quad \alpha, \beta \in \mathcal{I}_\lambda.
\]

The semigroup \( \mathcal{I}_\lambda \) is called the symmetric inverse semigroup over the cardinal \( \lambda \) (see [14]). For any \( \alpha \in \mathcal{I}_\lambda \) the cardinality of \( \text{dom} \alpha \) is called the rank of \( \alpha \) and it is denoted by \( \text{rank} \alpha \). The symmetric inverse semigroup was introduced by V. V. Wagner [40] and it plays a major role in the theory of semigroups.

Put \( \mathcal{I}_\lambda^\alpha = \{\alpha \in \mathcal{I}_\lambda: \text{rank} \alpha \leq n\} \), for \( n = 1, 2, 3, \ldots \). Obviously, \( \mathcal{I}_\lambda^\alpha \) (\( n = 1, 2, 3, \ldots \)) are inverse semigroups, \( \mathcal{I}_\lambda^n \) is an ideal of \( \mathcal{I}_\lambda \), for each \( n = 1, 2, 3, \ldots \). The semigroup \( \mathcal{I}_\lambda^n \) is called the symmetric inverse semigroup of finite transformations of the rank \( \leq n \) [26]. Formally,

\[
\begin{pmatrix}
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_n
\end{pmatrix}
\]

we denote a partial one-to-one transformation that maps \( x_1 \) onto \( y_1 \), \( x_2 \) onto \( y_2 \), \ldots , and \( x_n \) onto \( y_n \). Obviously, in such case we have \( x_i \neq x_j \) and \( y_i \neq y_j \) for \( i \neq j \) (\( i, j = 1, 2, 3, \ldots , n \)). The empty partial map \( \emptyset: \lambda \to \lambda \) is denoted by \( \emptyset \). It is obvious that \( \emptyset \) is zero of the semigroup \( \mathcal{I}_\lambda^\alpha \).

For a partially ordered set \( (P, \preceq) \), a subset \( X \) of \( P \) is called order-convex, if \( x \preceq z \preceq y \) and \( \{x, y\} \subseteq X \) implies that \( z \in X \), for all \( x, y, z \in P \) [31]. It is obvious that the set of all partial order isomorphisms between convex subsets of \((\omega, \preceq)\) under the composition of partial self-maps forms an inverse subsemigroup of the symmetric inverse semigroup \( \mathcal{I}_\lambda \) over the set \( \omega \). We denote this semigroup by \( \mathcal{I}_\omega(\mathsf{conv}) \).

Put \( \mathcal{I}_\omega(\mathsf{conv}) = \mathcal{I}_\omega(\mathsf{conv}) \cap \mathcal{I}_\omega^n \) and it is obvious that \( \mathcal{I}_\omega(\mathsf{conv}) \) is closed under the semigroup operation of \( \mathcal{I}_\lambda^n \) and the semigroup \( \mathcal{I}_\omega(\mathsf{conv}) \) is called the inverse semigroup of convex order isomorphisms of \((\omega, \preceq)\) of the rank \( \leq n \).

The bicyclic monoid \( \mathcal{C}(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The semigroup operation on \( \mathcal{C}(p, q) \) is determined as follows:

\[
q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.
\]

It is well known that the bicyclic monoid \( \mathcal{C}(p, q) \) is a bisimple (and hence simple) combinatorial \( E \)-unitary inverse semigroup and every non-trivial congruence on \( \mathcal{C}(p, q) \) is a group congruence [14].

On the set \( B_\omega = \omega \times \omega \) we define the semigroup operation “\( \cdot \)” in the following way

\[
(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leq i_2; \\
(i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geq i_2. 
\end{cases}
\]

It is well known that the semigroup \( B_\omega \) is isomorphic to the bicyclic monoid by the mapping \( h: \mathcal{C}(p, q) \to B_\omega, q^k p^l \mapsto (k, l) \) (see: [14, Section 1.12] or [35, Exercise IV.1.11(ii)]).

By \( \mathbb{R} \) and \( \omega_0 \) we denote the set of real numbers with the usual topology and the infinite countable discrete space, respectively.

Let \( Y \) be a topological space. A topological space \( X \) is called:

- compact if any open cover of \( X \) contains a finite subcover;
• countably compact if each closed discrete subspace of \( X \) is finite;
• \( Y \)-compact if every continuous image of \( X \) in \( Y \) is compact.

A topological (semitopological) semigroup is a topological space together with a continuous (separately continuous) semigroup operation. If \( S \) is a semigroup and \( \tau \) is a topology on \( S \) such that \((S, \tau)\) is a topological semigroup, then we shall call \( \tau \) a semigroup topology on \( S \), and if \( \tau \) is a topology on \( S \) such that \((S, \tau)\) is a semitopological semigroup, then we shall call \( \tau \) a shift-continuous topology on \( S \). An inverse topological semigroup with the continuous inversion is called a topological inverse semigroup.

Next we shall describe the construction which is introduced in [23].

Let \( B_\omega \) be the bicyclic monoid and \( \mathcal{F} \) be an \( \omega \)-closed subfamily of \( \mathcal{P}(\omega) \). On the set \( B_\omega \times \mathcal{F} \) we define the semigroup operation "\( \cdot \)" in the following way
\[
(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} 
(i_1 - j_1 + i_2, j_2, (i_1 - j_1 + F_1) \cap F_2), & \text{if } j_1 \leq i_2; \\
(i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geq i_2.
\end{cases}
\]

In [23] it is proved that if the family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) is \( \omega \)-closed then \((B_\omega \times \mathcal{F}, \cdot)\) is a semigroup. Moreover, if an \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) contains the empty set \( \emptyset \) then the set \( I = \{(i, j, \emptyset) : i, j \in \omega \} \) is an ideal of the semigroup \((B_\omega \times \mathcal{F}, \cdot)\). For any \( \omega \)-closed family \( \mathcal{F} \subseteq \mathcal{P}(\omega) \) the following semigroup
\[
B_\omega^{\mathcal{F}} = \begin{cases} 
(B_\omega \times \mathcal{F}, \cdot)/I, & \text{if } \emptyset \in \mathcal{F}; \\
(B_\omega \times \mathcal{F}, \cdot), & \text{if } \emptyset \notin \mathcal{F}
\end{cases}
\]
is defined in [23]. The semigroup \( B_\omega^{\mathcal{F}} \) generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proved in [23] that \( B_\omega^{\mathcal{F}} \) is a combinatorial inverse semigroup and Green’s relations, the natural partial order on \( B_\omega^{\mathcal{F}} \) and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup \( B_\omega^{\mathcal{F}} \) and when \( B_\omega^{\mathcal{F}} \) has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular in [23] is proved that the semigroup \( B_\omega^{\mathcal{F}} \) is isomorphic to the semigroup of \( \omega \times \omega \)-matrix units if and only if \( \mathcal{F} \) consists of a singleton set and the empty set.

The semigroup \( B_\omega^{\mathcal{F}} \) in the case when the family \( \mathcal{F} \) consists of the empty set and some singleton subsets of \( \omega \) is studied in [21]. It is proved that the semigroup \( B_\omega^{\mathcal{F}} \) is isomorphic to the subsemigroup \( \mathcal{B}_\omega^{\mathcal{F}}(F_{\min}) \) of the Brandt \( \omega \)-extension of the subsemilattice \((F, \min)\) of \((\omega, \min)\), where \( F = \bigcup \mathcal{F} \). Also topologizations of the semigroup \( B_\omega^{\mathcal{F}} \) and its closure in semitopological semigroups are studied.

For any \( n \in \omega \) we put \( \mathcal{F}_n = \{[0; k] : k = 0, \ldots, n\} \). It is obvious that \( \mathcal{F}_n \) is an \( \omega \)-closed family of \( \omega \).

In this paper we study the semigroup \( B_\omega^{\mathcal{F}_n} \). We show that the Green relations \( \mathcal{D} \) and \( \mathcal{J} \) coincide in \( B_\omega^{\mathcal{F}_n} \), the semigroup \( B_\omega^{\mathcal{F}_n} \) is isomorphic to the semigroup \( \mathcal{J}_\omega^{n+1}(\text{conv}) \), and \( B_\omega^{\mathcal{F}_n} \) admits only Rees congruences. Also, we study shift-continuous topologizations of the semigroup \( B_\omega^{\mathcal{F}_n} \). In particular we prove that for any shift-continuous \( T_1 \)-topology \( \tau \) on the semigroup \( B_\omega^{\mathcal{F}_n} \) every non-zero element of \( B_\omega^{\mathcal{F}_n} \) is an isolated point of \((B_\omega^{\mathcal{F}_n}, \tau)\), \( B_\omega^{\mathcal{F}_n} \) admits the unique compact shift-continuous \( T_1 \)-topology, and every \( \omega_0 \)-compact shift-continuous \( T_1 \)-topology is compact. We describe the closure of the semigroup \( B_\omega^{\mathcal{F}_n} \) in a Hausdorff semitopological semigroup and prove the criterion when a topological inverse semigroup \( B_\omega^{\mathcal{F}_n} \) is \( H \)-closed in the class of Hausdorff topological semigroups.

2. Algebraic properties of the semigroup \( B_\omega^{\mathcal{F}_n} \)

An inverse semigroup \( S \) with zero is said to be 0-E-unitary if \( 0 \neq e \leq s \), where \( e \) is an idempotent in \( S \), implies that \( s \) is an idempotent [32]. The class of 0-E-unitary semigroups was first defined by Maria Szendrei [37], although she called them \( E^* \)-unitary. The term 0-E-unitary appears to be due to Meakin and Sapir [33].

In the following proposition we summarise properties which follow from properties of the semigroup \( B_\omega^{\mathcal{F}_n} \) in the general case. These properties are corollaries of the results of the paper [23].

Proposition 1. For any \( n \in \omega \) the following statements hold:

1. \( B_\omega^{\mathcal{F}_n} \) is an inverse semigroup, namely \( 0^{-1} = 0 \) and \((i, j, [0; k])^{-1} = (j, i, [0; k])\), for any \( i, j, k \in \omega \);
2. \((i, j, [0; k]) \in B_\omega^{\mathcal{F}_n} \) is an idempotent if and only if \( i = j \);
(3) \((i_1, i_1, [0; k_1]) \preceq (i_2, i_2, [0; k_2])\) in \(E(B_\omega^{\mathcal{F}_n})\) if and only if \(i_1 \geq i_2\) and \(i_1 + k_1 \leq i_2 + k_2\) and this natural partial order on \(E(B_\omega^{\mathcal{F}_n})\) is presented on Fig. 1;

\[
\begin{array}{cccccccc}
(0, 0, [0; n]) & (1, 1, [0; n]) & (2, 2, [0; n]) & (3, 3, [0; n]) & (4, 4, [0; n]) & \cdots & (i, i, [0; n]) & (i+1, i+1, [0; n]) \\
(0, 0, [n-1]) & (1, 1, [n-1]) & (2, 2, [n-1]) & (3, 3, [n-1]) & (4, 4, [n-1]) & \cdots & (i, i, [n-1]) & (i+1, i+1, [n-1]) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
(0, 0, [3]) & (1, 1, [3]) & (2, 2, [3]) & (3, 3, [3]) & (4, 4, [3]) & \cdots & (i, i, [3]) & (i+1, i+1, [3]) \\
(0, 0, [2]) & (1, 1, [2]) & (2, 2, [2]) & (3, 3, [2]) & (4, 4, [2]) & \cdots & (i, i, [2]) & (i+1, i+1, [2]) \\
(0, 0, [1]) & (1, 1, [1]) & (2, 2, [1]) & (3, 3, [1]) & (4, 4, [1]) & \cdots & (i, i, [1]) & (i+1, i+1, [1]) \\
(0, 0, [0]) & (1, 1, [0]) & (2, 2, [0]) & (3, 3, [0]) & (4, 4, [0]) & \cdots & (i, i, [0]) & (i+1, i+1, [0]) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & & & & & & & \\
\end{array}
\]

**Figure 1.** The natural partial order on the band \(E(B_\omega^{\mathcal{F}_n})\)

(4) \((i, i, [0; n])\) is a maximal idempotent of \(E(B_\omega^{\mathcal{F}_n})\) for any \(i \in \omega\);

(5) \((i, i, [0; 0])\) is a primitive idempotent of \(E(B_\omega^{\mathcal{F}_n})\) for any \(i \in \omega\);

(6) \((i_1, j_1, [0; k_1])\mathcal{R}(i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\) if and only if \(i_1 = i_2\) and \(k_1 = k_2\);

(7) \((i_1, j_1, [0; k_1])\mathcal{L}(i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\) if and only if \(j_1 = j_2\) and \(k_1 = k_2\);

(8) \((i_1, j_1, [0; k_1])\mathcal{H}(i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\) if and only if \(i_1 = i_2\), \(j_1 = j_2\), \(i\) \(k_1 = i\) \(k_2\);

(9) \((i_1, j_1, [0; k_1])\mathcal{D}(i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\) if and only if \(k_1 = k_2\);

(10) \(\mathcal{D} = \mathcal{J}\) in \(B_\omega^{\mathcal{F}_n}\);

(11) \((i_1, j_1, [0; k_1]) \prec (i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\) if and only if \(i_1 \geq i_2\), \(i_1 - j_1 = i_2 - j_2\) and \(i_1 + k_1 \leq i_2 + k_2\);

(12) \(B_\omega^{\mathcal{F}_n}\) is a 0-E-unitary inverse semigroup.

**Proof.** Statements (1)–(5) are trivial. Statements (6)–(8) follow from Proposition 3.2.11 of [32] and corresponding statements of Theorem 2 in [23].

(9) \((\Rightarrow)\) Suppose that \((i_1, j_1, [0; k_1])\mathcal{D}(i_2, j_2, [0; k_2])\) in \(B_\omega^{\mathcal{F}_n}\). Then there exists \((i_0, j_0, [0; k_0]) \in B_\omega^{\mathcal{F}_n}\) such that \((i_1, j_1, [0; k_1])\mathcal{L}(i_0, j_0, [0; k_0])\) and \((i_0, j_0, [0; k_0])\mathcal{R}(i_2, j_2, [0; k_2])\). By statement (6) we have that \(i_0 = i_2\) and \(k_0 = k_2\), and by (7) we get that \(j_0 = j_1\) and \(k_1 = k_0\). This implies that \(k_1 = k_2\).

\((\Leftarrow)\) Let \((i_1, j_1, [0; k])\) and \((i_2, j_2, [0; k])\) be elements of the semigroup \(B_\omega^{\mathcal{F}_n}\). By statements (6) and (7) we have that \((i_1, j_1, [0; k])\mathcal{L}(i_2, j_2, [0; k])\mathcal{R}(i_1, j_1, [0; k])\) and hence \((i_1, j_1, [0; k])\mathcal{D}(i_2, j_2, [0; k])\) in \(B_\omega^{\mathcal{F}_n}\).

(10) It is obvious that the \(\mathcal{D}\)-class of the zero \(0\) coincides with \(\{0\}\). Also the \(\mathcal{J}\)-class of the zero \(0\) coincides with \(\{0\}\).

Fix an arbitrary non-zero element \((i_0, j_0, [0; k_0])\) of \(B_\omega^{\mathcal{F}_n}\). By (9) the \(\mathcal{D}\)-class of \((i_0, j_0, [0; k_0])\) is the following set \(\mathcal{D} = \{(i, j, [0; k_0]) \colon i, j \in \omega\}\). By (3) every two distinct idempotents of the set \(\mathcal{D}\) are
incomparable, and hence every idempotent of the \(D\)-class of \((i_0, j_0, [0; k_0])\) is minimal with the respect to the natural partial order on \(B^\mathcal{F}_\omega\). By Proposition 3.2.17 from [32] if the \(D\)-class \(D_y\) has a minimal element then \(D_y = J_y\) and hence the \(D\)-class of \((i_0, j_0, [0; k_0])\) coincides with its \(\mathcal{F}\)-class. Therefore we obtain that \(D = \mathcal{F}\) in \(B^\mathcal{F}_\omega\).

(11) By Proposition 2 of [23] the inequality \((i_1, j_1, [0; k_1]) \preceq (i_2, j_2, [0; k_2])\) is equivalent to the conditions

\[ [0; k_1] \subseteq i_2 - i_1 + [0; k_2] = j_2 - j_1 + [0; k_2], \]

which are equivalent to

\[ i_2 - i_1 = j_2 - j_1 \leq 0 \quad \text{and} \quad k_1 \leq i_2 - i_1 + k_2. \]

It is obvious that the last conditions are equivalent to

\[ i_1 \geq i_2, \quad i_1 - j_1 = i_2 - j_2 \quad \text{and} \quad i_1 + k_1 \leq i_2 + k_2, \]

which completes the proof of the statement.

Statement (12) follows from (11). \(\square\)

**Lemma 1.** Let \(n \in \omega\). Then \(\uparrow_{\omega}(i_0, j_0, [0; k_0])\) and \(\downarrow_{\omega}(i_0, j_0, [0; k_0])\) are finite subsets of the semigroup \(B^\mathcal{F}_\omega\) for any its non-zero element \((i_0, j_0, [0; k_0]), i_0, j_0 \in \omega, k_0 \in \{0, \ldots, n\}\).

**Proof.** By Proposition 1(11) there exist finitely many \(i, j \in \omega\) and \(k \in \{0, \ldots, n\}\) such that \((i, j, [0; k]) \preceq (i_0, j_0, [0; k_0])\) for some \(i, j \in \omega\) and hence the set \(\downarrow_{\omega}(i_0, j_0, [0; k_0])\) is finite.

The inequality \(k \leq n\) and Proposition 1(11) imply that there exist finitely many \(i, j \in \omega\) and \(k \in \{0, \ldots, n\}\) such that \((i_0, j_0, [0; k_0]) \preceq (i, j, [0; k])\), and hence the set \(\uparrow_{\omega}(i_0, j_0, [0; k_0])\) is finite, too. \(\square\)

**Lemma 2.** If \(n \in \omega\) then for any \(\alpha, \beta \in B^\mathcal{F}_\omega\) the set \(\alpha \cdot B^\mathcal{F}_\omega \cdot \beta\) is finite.

**Proof.** The statement of the lemma is trivial when \(\alpha = 0\) or \(\beta = 0\).

Fix arbitrary non-zero-elements \(\alpha = (i_\alpha, j_\alpha, [0; k_\alpha])\) and \(\beta = (i_\beta, j_\beta, [0; k_\beta])\) of \(B^\mathcal{F}_\omega\). If \(i \geq j_\alpha + n + 1\) or \(j \geq i_\beta + n + 1\) then for any \(k \in \{0, \ldots, n\}\) we have that

\[ (i_\alpha, j_\alpha, [0; k_\alpha]) \cdot (i, j, [0; k]) = (i_\alpha - j_\alpha + i, j, (j_\alpha - i + [0; k_\alpha]) \cap [0; k]) = 0 \]

and

\[ (i, j, [0; k]) \cdot (i_\beta, j_\beta, [0; k_\beta]) = (i, j - i_\beta + j_\beta, [0; k] \cap (i_\beta - j + [0; k_\beta]) = 0. \]

Hence there exist only finitely many \((i, j, [0; k]) \in B^\mathcal{F}_\omega\) such that \(\alpha \cdot (i, j, [0; k]) \cdot \beta \neq 0\). This implies the statement of the lemma. \(\square\)

**Lemma 3.** Let \(n \in \omega\). Then for any non-zero elements \((i_1, j_1, [0; k_1])\) and \((i_2, j_2, [0; k_2])\) of \(B^\mathcal{F}_\omega\) the sets of solutions of the following equations

\[ (i_1, j_1, [0; k_1]) \cdot \chi = (i_2, j_2, [0; k_2]) \quad \text{and} \quad \chi \cdot (i_1, j_1, [0; k_1]) = (i_2, j_2, [0; k_2]) \]

in the semigroup \(B^\mathcal{F}_\omega\) are finite.

**Proof.** Suppose that \(\chi\) is a solution of the equation \((i_1, j_1, [0; k_1]) \cdot \chi = (i_2, j_2, [0; k_2])\). The definition of the semigroup operation on the semigroup \(B^\mathcal{F}_\omega\) implies that \(\chi \neq 0\) and \(k_1 \geq k_2\). Assume that \(\chi = (i, j, [0; k])\) for some \(i, j \in \omega, k = 0, 1, \ldots, n\). Then we have that

\[ (i_2, j_2, [0; k_2]) = (i_1, j_1, [0; k_1]) \cdot (i_1, j_1, [0; k]) = \begin{cases} 
(i_1 - j_1 + i, j, (j_1 - i + [0; k_1]) \cap [0; k]), & \text{if } j_1 < i; \\
(i_1, j, [0; k_1] \cap [0; k]), & \text{if } j_1 = i; \\
(i_1, j_1 - i + j, [0; k_1] \cap (i - j_1 + [0; k])), & \text{if } j_1 > i.
\end{cases} \]

We consider the following cases.

(1) If \(j_1 < i\) then \(i = i_2 - i_1 + j, j = j_2, k \geq k_2\) and

\[ j_1 - i + k_1 = j_1 - i_2 + i_1 - j_1 + k_1 = i_1 - i_2 + k_1 \geq k. \]

(2) If \(j_1 = i\) then \(j = j_2\) and \(k \geq k_2\).
(3) If \( j_1 > i \) then \( i = i_2, j = j_2 - j_1 + i = j_2 - j_1 + i_2 \) and \( i - j_1 + k = i_2 - j_1 + k \geq k_2 \).

Since \( k \leq n \) the above considered cases imply that the equation \( (i_1, j_1, [0; k_1]) \cdot \chi = (i_2, j_2, [0; k_2]) \) has finitely many solutions.

The proof of the statement that the equation \( \chi \cdot (i_1, j_1, [0; k_1]) = (i_2, j_2, [0; k_2]) \) has finitely many solutions is similar. \( \square \)

**Theorem 1.** For an arbitrary \( n \in \omega \) the semigroup \( B_{\omega^n}^{\mathcal{F}_n} \) is isomorphic to an inverse subsemigroup of \( \mathcal{J}_\omega^{n+1} \), namely \( B_{\omega^n}^{\mathcal{F}_n} \) is isomorphic to the semigroup \( \mathcal{J}_\omega^{n+1}(\mathcal{Conv}) \).

**Proof.** We define a map \( \mathcal{J} : B_{\omega^n}^{F_n} \to \mathcal{J}_\omega^{n+1} \) by the formulae \( \mathcal{J}(0) = 0 \) and

\[
\mathcal{J}(i, j, [0; k]) = \left( \begin{array}{c} i + 1 \cdot i + k \\ j + 1 \cdot j + k \end{array} \right), \quad \text{for all } i, j \in \omega \text{ and } k = 0, 1, \ldots, n.
\]

It is obvious that so defined map \( \mathcal{J} \) is injective.

Next we shall show that \( \mathcal{J} : B_{\omega^n}^{\mathcal{F}_n} \to \mathcal{J}_\omega^{n+1} \) is a homomorphism.

It is obvious that

\[
\begin{align*}
\mathcal{J}(0 \cdot 0) &= \mathcal{J}(0) = 0 = 0 \cdot 0 = \mathcal{J}(0) \cdot \mathcal{J}(0), \\
\mathcal{J}(0 \cdot (i, j, [0; k])) &= \mathcal{J}(0) = 0 = 0 \cdot \left( \begin{array}{c} i + 1 \cdot i + k \\ j + 1 \cdot j + k \end{array} \right) = \mathcal{J}(0) \cdot \mathcal{J}(i, j, [0; k]),
\end{align*}
\]

and

\[
\mathcal{J}((i, j, [0; k]) \cdot 0) = \mathcal{J}(0) = 0 = \left( \begin{array}{c} i + 1 \cdot i + k \\ j + 1 \cdot j + k \end{array} \right) \cdot 0 = \mathcal{J}(i, j, [0; k]) \cdot \mathcal{J}(0),
\]

for any non-zero element \((i, j, [0; k])\) of the semigroup \( B_{\omega^n}^{\mathcal{F}_n} \).

Fix arbitrary \( i_1, i_2, j_1, j_2 \in \omega \) and \( k_1, k_2 \in \{0, \ldots, n\} \). In the case when \( k_1 \leq k_2 \) we have that

\[
\mathcal{J}((i_1, j_1, [0; k_1]) \cdot (i_2, j_2, [0; k_2])) = \begin{cases}
\mathcal{J}(0), \\
\mathcal{J}(i_1 - j_1 + i_2, j_2, [0; 0]), \\
\mathcal{J}(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1]), \\
\mathcal{J}(i_1, j_2, [0; k_1]), \\
\mathcal{J}(i_1, j_1 - i_2 + j_2, [0; k_1] \cap (i_2 - j_1 + [0; k_2])), \\
\mathcal{J}(i_1, j_1, [0; k_1]), \\
\mathcal{J}(i_1 - j_1 + i_2 + j_2, [0; i_1 - j_1 + k_2]), \\
\mathcal{J}(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2]), \\
\mathcal{J}(i_1, j_2, [0; 0]), \\
\mathcal{J}(0),
\end{cases}
\]

\[
= \begin{cases}
0, \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + j_2 \\ i_1 - j_1 + i_2 + j_2 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + j_2 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + j_2 + k_1 \\ i_1 - j_1 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + j_2 + k_1 \\ i_1 - j_1 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
0,
\end{cases}
\]

\[
= \begin{cases}
0, \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + k_1 + k_1 \\ i_1 - j_1 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 - j_1 + i_2 + k_1 + k_1 \\ i_1 - j_1 + k_1 \\ i_1 - j_1 + i_2 + k_1 + k_1 \end{array} \right), \\
0,
\end{cases}
\]

\[
= \begin{cases}
0, \\
\left( \begin{array}{c} i_1 + k_1 \\ i_1 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 + k_1 \\ i_1 + k_1 + k_1 \end{array} \right), \\
\left( \begin{array}{c} i_1 + k_1 \\ i_1 + k_1 + k_1 \end{array} \right), \\
0,
\end{cases}
\]
and

$$
\mathcal{I}(i_1, j_1, [0; k_1]) \cdot \mathcal{I}(i_2, j_2, [0; k_2]) = \begin{cases}
O, & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 < i_2;
(i_{j_1+k_1}, i_{j_2+k_1}), & \text{if } j_1 = i_2; \\
\mathcal{I}(i_1, j_1, [0; k_1]) & \text{if } j_1 + k_1 = i_2; \end{cases}
$$

In the case when $k_1 > k_2$ we have that

$$
\mathcal{I}((i_1, j_1, [0; k_1]) \cdot (i_2, j_2, [0; k_2])) = \begin{cases}
\mathcal{I}(0), & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 < 0; \\
\mathcal{I}(i_1 - j_1 + i_2, j_2, [0; 0]), & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 = 0; \\
\mathcal{I}(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1]), & \text{if } j_1 < i_2 \text{ and } 1 \leq j_1 + k_1 \leq i_2 + k_2; \\
\mathcal{I}(i_1 - j_1 + i_2, j_2, [0; k_2]), & \text{if } j_1 < i_2 \text{ and } j_1 + k_1 > i_2 + k_2; \\
\mathcal{I}(i_1, j_1 - i_2 + j_2, [0; j_2 - j_1 + k_2]), & \text{if } j_1 > i_2 \text{ and } j_2 - j_1 + k_2 > 0; \\
\mathcal{I}(i_1, j_1 - i_2 + j_2, [0; 0]), & \text{if } j_1 > i_2 \text{ and } j_2 - j_1 + k_2 = 0; \\
\mathcal{I}(0), & \text{if } j_1 > i_2 \text{ and } j_2 - j_1 + k_2 < 0.
\end{cases}
$$

and

$$
\mathcal{I}(i_1, j_1, [0; k_1]) \cdot \mathcal{I}(i_2, j_2, [0; k_2]) = \begin{cases}
O, & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 < 0; \\
(i_{j_1+k_1}, i_{j_2+k_1}), & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 = 0; \\
\mathcal{I}(0), & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 > 0; \\
\mathcal{I}(i_{j_1+k_1}, i_{j_2+k_1}), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 < i_2 + k_2; \\
\mathcal{I}(0), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 = i_2 + k_2; \\
\mathcal{I}(i_{j_1+k_1}, i_{j_2+k_1}), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 > i_2 + k_2; \\
\mathcal{I}(0), & \text{if } j_1 > i_2 \text{ and } j_1 + k_1 > i_2 + k_2.
\end{cases}
$$
By Lemma II.1.10 of [35] the homomorphic image $\mathcal{I}(B_\omega^{\mathbb{Rn}})$ is an inverse subsemigroup of $\mathcal{I}^{n+1}$. It is obvious that $\mathcal{I}(0)$ is the empty partial self-map of $\omega$ and it is by the assumption is an order convex partial isomorphism of $(\omega, \leq)$. Also the image $\mathcal{I}(i, j, [0; k]) = \{ \frac{i + k}{j + k} \}$ is an order convex partial isomorphism of $(\omega, \leq)$ for all $i, j \in \omega$ and $k = 0, 1, \ldots, n$. The definition of $\mathcal{I} : B_\omega^{\mathbb{Rn}} \to \mathcal{I}^{n+1}$ implies that its co-restriction on the image $\mathcal{I}_{n+1}(\mathbb{C}^n)$ is surjective, and hence $\mathcal{I} : B_\omega^{\mathbb{Rn}} \to \mathcal{I}^{n+1}(\mathbb{C}^n)$ is an isomorphism. □

Remark 1. Observe that the image $\mathcal{I}(B_\omega^{\mathbb{Rn}})$ does not contains all idempotents of the semigroup $\mathcal{I}^{n+1}$, especially $\left( \begin{array}{c} 0 \\ 0 \end{array} \right) \notin \mathcal{I}(B_\omega^{\mathbb{Rn}})$ for any $n \geq 1$. But by Proposition 4 of [23] the semigroup $B_\omega^{\mathbb{R0}}$ is isomorphic to the semigroup $\omega \times \omega$-matrix units, and hence $B_\omega^{\mathbb{R0}}$ is isomorphic to the semigroup $\mathcal{I}_\omega^1$.

A subset $D$ of a semigroup $S$ is said to be $\omega$-unstable if $D$ is infinite and for any $a \in D$ and an infinite subset $B \subseteq D$, we have $aB \cup Ba \not\subseteq D$ [20]. A basic example of $\omega$-unstable sets is given in [20]: for an infinite cardinal $\lambda$ the set $D = \mathcal{I}^{n} \setminus \mathcal{I}^{n-1}$ is an $\omega$-unstable subset of $\mathcal{I}^n$.

For any $n \in \omega$ the definition of the semigroup operation on $B_\omega^{\mathbb{Rn}}$ implies that its subsemigroup $B_\omega^{\mathbb{Rk}}$ is an ideal of $B_\omega^{\mathbb{Rn}}$ for any $k \in \{0, \ldots, n\}$. Also, since $\mathcal{I}^{k+1}(\mathbb{C}^n) \setminus \mathcal{I}^{k}(\mathbb{C}^n)$ is an infinite subset of $\mathcal{I}^{n+1}(\mathbb{C}^n)$ for any $k \in \{0, \ldots, n\}$, the above arguments and Theorem 1 imply the following lemma:

Lemma 4. For an arbitrary $n \in \omega$ the subsets $B_\omega^{\mathbb{R0}} \setminus \{0\}$ and $B_\omega^{\mathbb{Rk}} \setminus B_\omega^{\mathbb{Rk-1}}$ are $\omega$-unstable of $B_\omega^{\mathbb{Rn}}$ for any $k \in \{1, \ldots, n\}$.

Proof. We shall show that the set $B_\omega^{\mathbb{Rk}} \setminus B_\omega^{\mathbb{Rk-1}}$ is $\omega$-unstable, and the proof that the set $B_\omega^{\mathbb{R0}} \setminus \{0\}$ is $\omega$-unstable is similar.

Fix an arbitrary distinct $(i_1, j_1, [0; k]), (i_2, j_2, [0; k]) \in B_\omega^{\mathbb{Rk}} \setminus B_\omega^{\mathbb{Rk-1}}$. The definition of the semigroup operation of $B_\omega^{\mathbb{Rn}}$ implies that for any $(i, j, [0; k]) \in B_\omega^{\mathbb{Rk}} \setminus B_\omega^{\mathbb{Rk-1}}$ we have that

\[(i, j, [0; k]) \cdot (i_p, j_p, [0; k]) = \begin{cases} (i - j + i_p, j_p, (j - i_p + [0; k]) \cap [0; k]), & \text{if } j < i_p; \\ (i, j_p, [0; k] \cap [0; k]), & \text{if } j = i_p; \\ (i, j_p + j_p, [0; k] \cap (i_p - j + [0; k])), & \text{if } j > i_p \\ \end{cases} \]

for $p = 1, 2$. In the case when $i_1 \neq i_2$ we obtain that $(i, j, [0; k]) \cdot \{(i_1, j_1, [0; k]), (i_2, j_2, [0; k])\} \not\subseteq B_\omega^{\mathbb{Rk}} \setminus B_\omega^{\mathbb{Rk-1}}$. In the case when $j_1 \neq j_2$ the proof is similar. □

Definition 1 ([20]). An ideal series for a semigroup $S$ is a chain of ideals

\[I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S. \]

This ideal series is called tight if $I_0$ is a finite set and $D_k = I_k \setminus I_{k-1}$ is an $\omega$-unstable subset for each $k = 1, \ldots, m$.

Lemma 4 implies

Proposition 2. For an arbitrary $n \in \omega$ the following ideal series

\[\{0\} \subseteq B_\omega^{\mathbb{R0}} \subseteq B_\omega^{\mathbb{R1}} \subseteq \cdots \subseteq B_\omega^{\mathbb{Rn-1}} \subseteq B_\omega^{\mathbb{Rn}}\]

is tight.
Proposition 3. For any non-negative integer \( n \) and arbitrary \( p = 0, 1, \ldots, n - 1 \) the map \( h_p : B_\omega^{x_n} \to B_\omega^{x_n} \) defined by the formulae \( h_p(0) = 0 \) and

\[
h_p(i, j, [0; k]) = \begin{cases} 
0, & \text{if } k = 0, 1, \ldots, p; \\
(i, j, [0; k - p - 1]), & \text{if } k = p + 1, \ldots, n,
\end{cases}
\]

is a homomorphism which maps the semigroup \( B_\omega^{x_n} \) onto its subsemigroup \( B_\omega^{x_n-p} \).

Proof. First we shall show that the map \( h_0 : B_\omega^{x_n} \to B_\omega^{x_n} \) defined by the formulae \( h_0(0) = 0 \) and

\[
h_0(i, j, [0; k]) = \begin{cases} 
0, & \text{if } k = 0; \\
(i, j, [0; k - 1]), & \text{if } k = 1, \ldots, n,
\end{cases}
\]

is a homomorphism.

It is obvious that

\[
h_0(0) \cdot h_0(i, j, [0]) = 0 \cdot 0 = h_0(0) = h_0(0 \cdot (i, j, [0]))
\]

and

\[
h_0(i, j, [0]) \cdot h_0(0) = 0 \cdot 0 = h_0(0) = h_0((i, j, [0]) \cdot 0)
\]

for any \( i, j \in \omega \).

Fix arbitrary \( i_1, i_2, j_1, j_2 \in \omega \) and positive integers \( k_1 \) and \( k_2 \). In the case when \( k_1 \leq k_2 \) we have that

\[
h_0(i_1, j_1, [0; k_1]) \cdot h_0(i_2, j_2, [0; k_2]) = (i_1, j_1, [0; k_1 - 1]) \cdot (i_2, j_2, [0; k_2 - 1]) =
\]

\[
\begin{cases} 
0, & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 - 1 < 0; \\
(i_1 - j_1 + i_2, j_2, [j_1 - i_2 + k_1 - 1]), & \text{if } j_1 < i_2 \text{ and } 0 \leq j_1 - i_2 + k_1 - 1 \leq k_2 - 1; \\
(i_1, j_2, [0; k_1 - 1]), & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2, [0; k_1 - 1]), & \text{if } j_1 > i_2 \text{ and } k_1 - 1 \leq i_1 - j_1 + k_2 - 1; \\
(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]), & \text{if } j_1 > i_2 \text{ and } k_1 - 1 > i_1 - j_1 + k_2 - 1 \geq 0; \\
0, & \text{if } j_1 > i_2 \text{ and } k_1 - 1 > i_1 - j_1 + k_2 - 1 < 0
\end{cases}
\]

and

\[
h_0((i_1, j_1, [0; k_1]) \cdot (i_2, j_2, [0; k_2])) = \begin{cases} 
h_0(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + [0; k_1]) \cap [0; k_2]), & \text{if } j_1 < i_2; \\
h_0(i_1, j_2, [0; k_1]) \cap [0; k_2]), & \text{if } j_1 = i_2; \\
h_0(i_1, j_1 - i_2 + j_2, [0; k_1]) \cap (i_2 - j_1 + [0; k_2]), & \text{if } j_1 > i_2.
\end{cases}
\]
In the case when $k_1 \geq k_2$ we have that

$$b_0(i_1, j_1, [0; k_1]) \cdot b_0(i_2, j_2, [0; k_2]) = (i_1, j_1, [0; k_1 - 1]) \cdot (i_2, j_2, [0; k_2 - 1]) =$$

$$= \begin{cases} 
(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1) \cap [0; k_2 - 1)) & \text{if } j_1 < i_2; \\
(i_1, j_2, [0; k_1 - 1] \cap [0; k_2 - 1)) & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]) & \text{if } j_1 > i_2
\end{cases}$$

$$= \begin{cases} 
0, & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 - 1 < 0; \\
(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1) \cap [0; k_2 - 1)) & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 - 1 = 0; \\
(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1) \cap [0; k_2 - 1)) & \text{if } j_1 < i_2 \text{ and } 1 \leq j_1 - i_2 + k_1 - 1 \leq k_2 - 1; \\
(i_1 - j_1 + i_2, j_2, [0; k_2 - 1)) & \text{if } j_1 < i_2 \text{ and } k_2 - 1 < j_1 - i_2 + k_1 - 1; \\
(i_1, j_2, [0; k_1 - 1] \cap [0; k_2 - 1)) & \text{if } j_1 = i_2; \\
(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]) & \text{if } j_1 > i_2 \text{ and } i_2 - j_1 + k_2 - 1 > 0; \\
0, & \text{if } j_1 > i_2 \text{ and } i_2 - j_1 + k_2 - 1 < 0
\end{cases}$$

and

$$b_0((i_1, j_1, [0; k_1])) \cdot (i_2, j_2, [0; k_2]) = \begin{cases} 
b_0(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + [0; k_1]) \cap [0; k_2]), & \text{if } j_1 < i_2; \\
b_0(i_1, j_2, [0; k_1] \cap [0; k_2]), & \text{if } j_1 = i_2; \\
b_0(i_1, j_1 - i_2 + j_2, [0; k_1] \cap (i_2 - j_1 + [0; k_2])), & \text{if } j_1 > i_2
\end{cases}$$
statement that all elements of the subsemigroup for any \( p \in \mathbb{R} \).

**Proof.** Next we shall show the step of induction: if
\[
\begin{align*}
B_{\omega}^p &- \begin{cases}
0, & \text{if } j_1 < i_2 \text{ and } j_1 - i_2 + k_1 < 0; \\
(i_1 - j_1 + i_2, j_2, [0; k_2 - 1]), & \text{if } j_1 < i_2 \text{ and } k_2 < j_1 - i_2 + k_1 < 1; \\
(i_1, j_2, [0; k_2 - 1]), & \text{if } j_1 = i_2;
\end{cases}
\end{align*}
\]

Next observe that by induction we obtain that
\[
h_p = h_0 \circ \cdots \circ h_0 = h_0^{p+1}.
\]
for any \( p = 1, \ldots, n - 1 \).

Simple verifications show that the homomorphism \( h_p : B_{\omega}^n \rightarrow B_{\omega}^n \) maps the semigroup \( B_{\omega}^n \) onto its subsemigroup \( B_{\omega}^{n-p-1} \).

**Proposition 4.** For any positive integer \( n \) every congruence on the semigroup \( \mathcal{I}_{\omega}^n(\text{con}) \) is Rees.

**Proof.** First we observe that since the semigroup \( \mathcal{I}_{\omega}^n(\text{con}) \) has the zero \( 0 \) the identity congruence on \( \mathcal{I}_{\omega}^n(\text{con}) \) is Rees, and it is obvious that the universal congruence on \( \mathcal{I}_{\omega}^n(\text{con}) \) is Rees, too.

By induction we shall show the following: if \( \mathcal{C} \) is a congruence \( \mathcal{I}_{\omega}^n(\text{con}) \) such that for some \( k \leq n \) there exist two distinct \( \mathcal{C} \)-equivalent elements \( \alpha, \beta \in \mathcal{I}_{\omega}^k(\text{con}) \) with \( \max\{\text{rank } \alpha, \text{rank } \beta\} = k \), then all elements of subsemigroup \( \mathcal{I}_{\omega}^k(\text{con}) \) are equivalent.

In the case when \( k = 1 \) then it is obvious that the semigroup \( \mathcal{I}_{\omega}^1(\text{con}) \) is isomorphic to the semigroup \( \mathcal{I}_{\omega}^1 \) which is isomorphic to the semigroup of \( \omega \times \omega \)-matrix units \( B_{\omega} \). Since the semigroup \( B_{\omega} \) of \( \omega \times \omega \)-matrix units is congruence-free (see [24, Corollary 3]), the statement that any two distinct elements of the semigroup \( \mathcal{I}_{\omega}^1(\text{con}) \) are \( \mathcal{C} \)-equivalent implies that all elements of \( \mathcal{I}_{\omega}^1(\text{con}) \) are \( \mathcal{C} \)-equivalent. Hence the initial step of induction holds.

Next we shall show the step of induction: if \( \mathcal{C} \) is a congruence \( \mathcal{I}_{\omega}^n(\text{con}) \) such that there exist two distinct \( \mathcal{C} \)-equivalent elements \( \alpha, \beta \in \mathcal{I}_{\omega}^{k+1}(\text{con}) \) with \( \max\{\text{rank } \alpha, \text{rank } \beta\} = k + 1 \), then the statement that all elements of the subsemigroup \( \mathcal{I}_{\omega}^{k+1}(\text{con}) \) are \( \mathcal{C} \)-equivalent implies that all elements of the subsemigroup \( \mathcal{I}_{\omega}^{k+1}(\text{con}) \) are \( \mathcal{C} \)-equivalent, as well.

Next we consider all possible cases.
(I) Suppose that \( \alpha = (\frac{a}{b}, \frac{a+1}{a+1}, \ldots, \frac{a+k}{a+k}) \), \( \beta = 0 \) and \( \alpha \, \mathcal{E} \beta \). Since \( C \) is a congruence on \( \mathcal{J}_\omega^n(\con) \), for any element \( \gamma = (\frac{c}{d}, \frac{c+1}{d+1}, \ldots, \frac{c+k_1}{d+k_1}) \) of the subsemigroup \( \mathcal{J}_\omega^{k+1}(\con) \), where \( k_1 \leq k + 1 \), we have that

\[
\gamma = \left( \frac{c}{a}, \frac{c+1}{a+k_1} \right) \cdot \alpha \cdot \left( \frac{b}{d}, \frac{b+1}{d+k_1} \right)
\]

is \( \mathcal{E} \)-equivalent to

\[
\left( \frac{c}{a}, \frac{c+1}{a+k_1} \right) \cdot 0 \cdot \left( \frac{b}{d}, \frac{b+1}{d+k_1} \right) = 0,
\]

and hence \( \gamma \mathcal{E} 0 \).

(II) Suppose that \( \alpha = (\frac{a}{b}, \frac{a+1}{a+1}, \ldots, \frac{a+k}{a+k}) \) and \( \beta = (\frac{b}{b}, \frac{b+1}{b+1}, \ldots, \frac{b+k}{b+k}) \) are non-zero \( \mathcal{E} \)-equivalent idempotents of the subsemigroup \( \mathcal{J}_\omega^k(\con) \) such that \( k_1 \leq k \) and \( \beta \leq \alpha \). In this case we have that \( [b; b+k] \subseteq [a; a+k] \).

We put

\[
\varepsilon = \left\{ \begin{array}{ll}
\left( \frac{a+1}{a+1}, \frac{a+2}{a+2}, \ldots, \frac{a+k}{a+k} \right), & \text{if } a = b; \\
\left( \frac{a+1}{a+1}, \frac{a+2}{a+2}, \ldots, \frac{a+k}{a+k} \right), & \text{if } a+k = b+k_1
\end{array} \right.
\]

and \( \gamma = (\frac{a}{a+1}, \frac{a+1}{a+1}, \ldots, \frac{a+k}{a+k}) \) if \( a < b \) and \( b + k_1 < a + k \).

In the case when either \( a = b \) or \( a + k = b + k_1 \) we obtain that \( \varepsilon \alpha \) and \( \varepsilon \beta \) are distinct \( \mathcal{E} \)-equivalent idempotents of the subsemigroup \( \mathcal{J}_\omega^k(\con) \) and hence by the assumption of induction all elements of \( \mathcal{J}_\omega^k(\con) \) are \( \mathcal{E} \)-equivalent.

In the case when \( a < b \) and \( b + k_1 < a + k \) we obtain that \( \gamma \alpha \gamma^{-1} \) and \( \gamma \beta \gamma^{-1} \) are distinct \( \mathcal{E} \)-equivalent idempotents of the subsemigroup \( \mathcal{J}_\omega^k(\con) \), because they have distinct rank \( \leq k \). Hence by the assumption of induction all elements of \( \mathcal{J}_\omega^k(\con) \) are \( \mathcal{E} \)-equivalent.

In both above cases we get that \( \alpha \, \mathcal{E} 0 \), which implies that case (I) holds.

(III) Suppose that \( \alpha \) and \( \beta \) are distinct incomparable non-zero \( \mathcal{E} \)-equivalent idempotents of the subsemigroup \( \mathcal{J}_\omega^k(\con) \) of \( \mathcal{J}_\omega^n(\con) \) such that \( \rank \alpha = k + 1 \). Then \( \alpha = \alpha \, \mathcal{E} \alpha \beta \) and \( \alpha \beta \leq \alpha \) which implies that either case (II) or case (I) holds.

(IV) Suppose that \( \alpha \) and \( \beta \) are distinct non-zero \( \mathcal{E} \)-equivalent elements of the subsemigroup \( \mathcal{J}_\omega^k(\con) \) of \( \mathcal{J}_\omega^n(\con) \) such that \( \rank \alpha = k + 1 \). Then at least one of the following conditions holds \( \alpha \alpha^{-1} \neq \beta \beta^{-1} \) or \( \alpha^{-1} \alpha \neq \beta^{-1} \beta \), because by Proposition 1(8) and Theorem 1 all \( \mathcal{H} \)-classes in \( \mathcal{J}_\omega^n(\con) \) are singletons. By Proposition 2.3.4(1) of [32], \( \alpha \alpha^{-1} \beta \beta^{-1} \) and \( \alpha^{-1} \alpha \beta \beta^{-1} \), and hence at least one of cases (II) or (III) holds.

Theorem 1 and Proposition 4 imply the description of all congruences on the semigroup \( \mathcal{B}_\omega^n \):

**Theorem 2.** For an arbitrary \( n \in \omega \) the semigroup \( \mathcal{B}_\omega^n \) admits only Rees congruences.

**Theorem 3.** Let \( n \) be a non-negative integer and \( S \) be a semigroup. For any homomorphism \( h: \mathcal{B}_\omega^n \to S \) the image \( h(\mathcal{B}_\omega^n) \) is either isomorphic to \( \mathcal{B}_\omega^k \) for some \( k = 0, 1, \ldots, n \), or is a singleton.

**Proof.** By Theorem 2 the homomorphism \( h \) generates the Rees congruence \( \mathcal{C}_h \) on the semigroup \( \mathcal{B}_\omega^n \). By Proposition 1(9) the following ideal series

\[
\{0\} \subseteq \mathcal{B}_\omega^{m} \subseteq \mathcal{B}_\omega^{m-1} \subseteq \cdots \subseteq \mathcal{B}_\omega \subseteq \mathcal{B}_\omega^n
\]

is maximal in \( \mathcal{B}_\omega^n \), i.e., if \( \mathcal{J} \) is an ideal of \( \mathcal{B}_\omega^n \) then either \( \mathcal{J} = \{0\} \) or \( \mathcal{J} = \mathcal{B}_\omega^n \) for some \( m = 0, 1, \ldots, n \).

It is obvious that if \( \mathcal{J} = \{0\} \) then the Rees congruence \( \mathcal{C}_\mathcal{J} \) generates the injective homomorphism \( h_{\mathcal{J}} \), and hence the image \( h_{\mathcal{J}}(\mathcal{B}_\omega^n) \) is isomorphic to the semigroup \( \mathcal{B}_\omega^n \). Similarly in the case when \( \mathcal{J} = \mathcal{B}_\omega^n \) we have that the image \( h_{\mathcal{J}}(\mathcal{B}_\omega^n) \) is a singleton.

Suppose that \( \mathcal{J} = \mathcal{B}_\omega^{m-1} \) for some \( m = 0, 1, \ldots, n - 1 \). Then the Rees congruence \( \mathcal{C}_\mathcal{J} \) generates the natural homomorphism \( h: \mathcal{B}_\omega^n \to \mathcal{B}_\omega^n / \mathcal{J} \). It is obvious that \( \alpha \mathcal{C}_\mathcal{J} \beta \) if and only if \( h_m(\alpha) = h_m(\beta) \) for \( \alpha, \beta \in \mathcal{B}_\omega^n \) where \( h_m: \mathcal{B}_\omega^n \to \mathcal{B}_\omega^n \) is the homomorphism defined in Proposition 3. Then by Proposition 3 the image \( h(\mathcal{B}_\omega^n) \) is isomorphic to the semigroup \( \mathcal{B}_\omega^{n-m} \).
3. ON TOPOLOGIZATIONS AND CLOSURE OF THE SEMIGROUP $B_{\omega}^{\mathcal{F}_n}$

In this section we establish topologizations of the semigroup $B_{\omega}^{\mathcal{F}_n}$ and its compact-like shift-continuous topologies.

**Theorem 4.** Let $n$ be a non-negative integer. Then for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B_{\omega}^{\mathcal{F}_n}$ every non-zero element of $B_{\omega}^{\mathcal{F}_n}$ is an isolated point of $(B_{\omega}^{\mathcal{F}_n}, \tau)$ and hence every subset in $(B_{\omega}^{\mathcal{F}_n}, \tau)$ which contains zero is closed. Moreover, for any non-zero element $(i, j, [0; k])$ of $B_{\omega}^{\mathcal{F}_n}$ the set $\{\pi(i, j, [0; k])\}$ is open-and-closed in $(B_{\omega}^{\mathcal{F}_n}, \tau)$.

**Proof.** Fix an arbitrary non-zero element $(i, j, [0; k])$ of the semigroup $B_{\omega}^{\mathcal{F}_n}$, $i, j \in \omega$, $k \in \{0, \ldots, n\}$. Proposition 7 of [20] and Proposition 3 imply there exists an open neighbourhood $U_{(i,j,[0;k])}$ of the point $(i, j, [0; k])$ in $(B_{\omega}^{\mathcal{F}_n}, \tau)$ such that

- $U_{(i,j,[0;k])} \subseteq B_{\omega}^{\mathcal{F}_n} \setminus B_{\omega}^{\mathcal{F}_{i-1}}$ and $(i, j, [0; k])$ is an isolated point in $B_{\omega}^{\mathcal{F}_n}$ if $k \in \{1, \ldots, n\}$; and
- $U_{(i,j,[0;k])} \subseteq B_{\omega}^{\mathcal{F}_n} \setminus \{0\}$ and $(i, j, [0; k])$ is an isolated point in $B_{\omega}^{\mathcal{F}_0}$ if $k = 0$.

By separate continuity of the semigroup operation in $(B_{\omega}^{\mathcal{F}_n}, \tau)$ there exists an open neighbourhood $V_{(i,j,[0;k])}$ of $(i, j, [0; k])$ such that $V_{(i,j,[0;k])} \subseteq U_{(i,j,[0;k])}$ and

$$(i, i, [0; k]) \cdot V_{(i,j,[0;k])} \cdot (j, j, [0; k]) \subseteq U_{(i,j,[0;k])}.$$ 

We claim that $V_{(i,j,[0;k])} \subseteq \pi(i, j, [0; k])$. Suppose to the contrary that there exists $(i_1, j_1, [0; k_1]) \in V_{(i,j,[0;k])} \setminus \pi(i, j, [0; k])$. Then by Lemma 1.4.6(4) of [32] we have that

$$(i, i, [0; k]) \cdot (i_1, j_1, [0; k_1]) \cdot (j, j, [0; k]) \neq (i, j, [0; k]).$$

Since $B_{\omega}^{\mathcal{F}_k}$ is an ideal of $B_{\omega}^{\mathcal{F}_n}$ the above inequality implies that

$$(i, i, [0; k]) \cdot V_{(i,j,[0;k])} \cdot (j, j, [0; k]) \notin U_{(i,j,[0;k])},$$

a contradiction. Hence $V_{(i,j,[0;k])} \subseteq \pi(i, j, [0; k])$. By Lemma 1 the set $\pi(i, j, [0; k])$ is finite which implies that $(i, j, [0; k])$ is an isolated point of $(B_{\omega}^{\mathcal{F}_n}, \tau)$, because $(B_{\omega}^{\mathcal{F}_n}, \tau)$ is a $T_1$-space.

The last statement follows from the equality

$$\pi(i, j, [0; k]) = \{((a, b, [0; p]) \in B_{\omega}^{\mathcal{F}_n} : (i, i, [0; k]) \cdot (a, b, [0; p]) = (i, j, [0; k])\}$$

and the assumption that $\tau$ is a shift-continuous $T_1$-topology on the semigroup $B_{\omega}^{\mathcal{F}_n}$. \hfill \Box

**Corollary 1.** Let $n$ be a non-negative integer. Then for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B_{\omega}^{\mathcal{F}_n}$ the space $(B_{\omega}^{\mathcal{F}_n}, \tau)$ is scattered, 0-dimensional and collectionwise normal.

**Proof.** Theorem 4 implies that $(B_{\omega}^{\mathcal{F}_n}, \tau)$ is a scattered, 0-dimensional space.

Let $\{F_s\}_{s \in \mathcal{F}}$ be a discrete family of closed subsets of $(B_{\omega}^{\mathcal{F}_n}, \tau)$. By Theorem 4 every non-zero element of $B_{\omega}^{\mathcal{F}_n}$ is an isolated point of $(B_{\omega}^{\mathcal{F}_n}, \tau)$. In the case when every element of the family $\{F_s\}_{s \in \mathcal{F}}$ does not contain the zero $0$ of $B_{\omega}^{\mathcal{F}_n}$ by Theorem 5.1.17 from [17] the space $(B_{\omega}^{\mathcal{F}_n}, \tau)$ is collectionwise normal.

Suppose that $0 \notin F_{s_0}$ for some $s_0 \in \mathcal{F}$. Let $U(0)$ be an open neighbourhood of the zero $0$ of $B_{\omega}^{\mathcal{F}_n}$ which intersects at more one element of the family $\{F_s\}_{s \in \mathcal{F}}$. Put $U_{s_0} = U(0) \cup F_{s_0}$ and $U_s = F_s$ for all $s \in \mathcal{F} \setminus \{s_0\}$. Then $U_{s_0} \cap U_t = \emptyset$ for all distinct $s, t \in \mathcal{F}$ and hence by Theorem 5.1.17 from [17] the space $(B_{\omega}^{\mathcal{F}_n}, \tau)$ is collectionwise normal. \hfill \Box

**Example 1.** Let $n$ be a non-negative integer. We define a topology $\tau_{Ac}$ on the semigroup $B_{\omega}^{\mathcal{F}_n}$ in the following way. All non-zero elements of the semigroup $B_{\omega}^{\mathcal{F}_n}$ are isolated points of $(B_{\omega}^{\mathcal{F}_n}, \tau_{Ac})$ and the
family $\mathcal{B}_{\mathcal{A}}(0) = \{ A \subseteq B_{\omega}^n : 0 \in A \text{ and } B_{\omega}^n \setminus A \text{ is finite} \}$ determines the base of the topology $\tau_{\mathcal{A}}$ at the point $0$.

It is obvious that the topological space $(B_{\omega}^n, \tau_{\mathcal{A}})$ is homeomorphic to the Alexandroff one-point compactification of the discrete infinite countable space, and hence $(B_{\omega}^n, \tau_{\mathcal{A}})$ is a Hausdorff compact space. Then the space $(B_{\omega}^n, \tau_{\mathcal{A}})$ is normal and since it has a countable base, by the Urysohn Metrization Theorem (see [17, Theorem 4.2.9]) the space $(B_{\omega}^n, \tau_{\mathcal{A}})$ is metrizable.

Next we shall show that $(B_{\omega}^n, \tau_{\mathcal{A}})$ is a semitopological semigroup. Let $\alpha$ and $\beta$ be non-zero elements of the semigroup $B_{\omega}^n$. Since $\alpha$ and $\beta$ are isolated points in $(B_{\omega}^n, \tau_{\mathcal{A}})$, it is sufficient to show how to find for a fixed open neighbourhood $U_0$ open neighbourhoods $V_0$ and $W_0$ of the zero $0$ in $(B_{\omega}^n, \tau_{\mathcal{A}})$ such that

$$V_0 \cdot \alpha \subseteq U_0 \quad \text{and} \quad \beta \cdot W_0 \subseteq U_0.$$ 

Since the space $(B_{\omega}^n, \tau_{\mathcal{A}})$ is compact, any open neighbourhood $U_0$ of the zero $0$ is cofinite subset in $B_{\omega}^n$. By Lemma 2,

$$V_0 = \{ \gamma \in U_0 : \gamma \cdot \alpha \in U_0 \} \quad \text{and} \quad W_0 = \{ \gamma \in U_0 : \beta \cdot \gamma \in U_0 \}$$

are cofinite subsets of $U_0$ and hence by the definition of the topology $\tau_{\mathcal{A}}$ the sets $V_0$ and $W_0$ are required open neighbourhoods of the zero $0$ in $(B_{\omega}^n, \tau_{\mathcal{A}})$.

Since all non-zero elements of the semigroup $B_{\omega}^n$ are isolated points in $(B_{\omega}^n, \tau_{\mathcal{A}})$ and every open neighbourhood $U_0$ of the zero in $(B_{\omega}^n, \tau_{\mathcal{A}})$ has the finite complement in $B_{\omega}^n$, the inversion is continuous in $(B_{\omega}^n, \tau_{\mathcal{A}})$.

The following theorem describes all compact-like shift-continuous $T_1$-topologies on the semigroup $B_{\omega}^n$.

**Theorem 5.** Let $n$ be a non-negative integer. Then for any shift-continuous $T_1$-topology $\tau$ on the semigroup $B_{\omega}^n$ the following conditions are equivalent:

1. $(B_{\omega}^n, \tau)$ is a compact semitopological semigroup;
2. $(B_{\omega}^n, \tau)$ is topologically isomorphic to $(B_{\omega}^n, \tau_{\mathcal{A}})$;
3. $(B_{\omega}^n, \tau)$ is a compact semitopological semigroup with continuous inversion;
4. $(B_{\omega}^n, \tau)$ is an $\omega_0$-compact space.

**Proof.** Implications $(1) \Rightarrow (4)$, $(2) \Rightarrow (1)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are obvious. Since by Theorem 4 every non-zero element of the semigroup $B_{\omega}^n$ is an isolated point in $(B_{\omega}^n, \tau)$, statement (1) implies (2).

$(4) \Rightarrow (1)$ Suppose there exists a shift-continuous $T_1$-topology $\tau$ on the semigroup $B_{\omega}^n$ such that $(B_{\omega}^n, \tau)$ is an $\omega_0$-compact non-compact space. Then there exists an open cover $\mathcal{U} = \{ U_s \}$ of $(B_{\omega}^n, \tau)$ which has no a finite subcover. Let $U_{s_0} \in \mathcal{U}$ be such that $U_{s_0} \ni 0$. Then $B_{\omega}^n \setminus U_{s_0}$ is an infinite countable subset of isolated points of $(B_{\omega}^n, \tau)$. We enumerate the set $B_{\omega}^n \setminus U_{s_0}$ by positive integers, i.e., $B_{\omega}^n \setminus U_{s_0} = \{ \alpha_i : i \in \mathbb{N} \}$. Next we define a map $f : (B_{\omega}^n, \tau) \rightarrow \omega_0$ by the formula

$$f(\alpha) = \begin{cases} 0, & \text{if } \alpha \in U_{s_0}; \\ i, & \text{if } \alpha = \alpha_i \text{ for some } i \in \mathbb{N}. \end{cases}$$

By Theorem 4 the set $U_{s_0}$ is open-and-closed in $(B_{\omega}^n, \tau)$, and hence so defined map $f$ is continuous. But the image $f(B_{\omega}^n)$ is not a compact subset of $\omega_0$, a contradiction. The obtained contradiction implies the implication $(4) \Rightarrow (1)$.

The following proposition states that the semigroup $B_{\omega}^n$ has a similar closure in a $T_1$-semitopological semigroup as the bicyclic monoid (see [10] and [16]), the $\lambda$-polycyclic monoid [9], graph inverse semigroups [7,34], McAlister semigroups [8], locally compact semitopological 0-bisimple inverse $\omega$-semigroups with a compact maximal subgroup [18], and other discrete semigroups of bijective partial transformations [12,13,19,22,25,27–30].

**Proposition 5.** Let $n$ be a non-negative integer. If $S$ is a $T_1$-semitopological semigroup which contains $B_{\omega}^n$ as a dense proper subsemigroup then $I = (S \setminus B_{\omega}^n) \cup \{ 0 \}$ is an ideal of $S$. 

Proof. Fix an arbitrary element \( \nu \in I \). If \( \chi \cdot \nu = \zeta \notin I \) for some \( \chi \in B_{\omega_{\nu}}^\mathbb{F}_n \) then there exists an open
neighbourhood \( U(\nu) \) of the point \( \nu \) in the space \( S \) such that \( \{ \chi \} \cdot U(\nu) = \{ \zeta \} \subset B_{\omega_{\nu}}^\mathbb{F}_n \setminus \{ 0 \} \). By
Lemma 3 the open neighbourhood \( U(\nu) \) should contain finitely many elements of the semigroup \( B_{\omega_{\nu}}^\mathbb{F}_n \)
which contradicts our assumption. Hence \( \chi \cdot \nu \in I \) for all \( \chi \in B_{\omega_{\nu}}^\mathbb{F}_n \) and \( \nu \in I \). The proof of
the statement that \( \nu \cdot \chi \in I \) for all \( \chi \in B_{\omega_{\nu}}^\mathbb{F}_n \) and \( \nu \in I \) is similar.

Suppose to the contrary that \( \chi \cdot \nu = \omega \notin I \) for some \( \chi, \nu \in I \). Then \( \omega \in B_{\omega_{\nu}}^\mathbb{F}_n \) and the separate
continuity of the semigroup operation in \( S \) yields open neighbourhoods \( U(\chi) \) and \( U(\nu) \) of the points \( \chi \) and \( \nu \) in the space \( S \), respectively, such that \( \{ \chi \} \cdot U(\nu) = \{ \omega \} \) and \( U(\chi) \cdot \{ \nu \} = \{ \omega \} \). Since
both neighbourhoods \( U(\chi) \) and \( U(\nu) \) contain infinitely many elements of the semigroup \( B_{\omega_{\nu}}^\mathbb{F}_n \), equalities
\( \{ \chi \} \cdot U(\nu) = \{ \omega \} \) and \( U(\chi) \cdot \{ \nu \} = \{ \omega \} \) do not hold, because \( \{ \chi \} \cdot (U(\nu) \cap B_{\omega_{\nu}}^\mathbb{F}_n) \subseteq I \). The obtained
contradiction implies that \( \chi \cdot \nu \in I \).

For any \( k = 0, 1, \ldots, n + 1 \) we denote \( D_k = \{ \alpha \in \mathcal{S}_{\omega}^{n+1}(\text{conv}) : \text{rank } \alpha = k \} \). We observe that by Proposition 1(9) and Theorem 1, \( D = \{ D_k : k = 0, 1, \ldots, n + 1 \} \) is the family of all
\( \mathcal{D} \)-classed of the semigroup \( \mathcal{S}_{\omega}^{n+1}(\text{conv}) \).

The following proposition describes the remainder of the semigroup \( B_{\omega_{\nu}}^\mathbb{F}_n \) in a semitopological
group.

Proposition 6. Let \( n \) be a non-negative integer. If \( S \) is a \( T_1 \)-semitopological semigroup which contains
\( B_{\omega_{\nu}}^\mathbb{F}_n \) as a dense proper subsemigroup then \( \chi \cdot \chi = 0 \) for all \( \chi \in S \setminus B_{\omega_{\nu}}^\mathbb{F}_n \).

Proof. We observe that \( 0 \) is zero of the semigroup \( S \) by Lemma 4.4 of [18].

We shall prove the statement of the proposition for the semigroup \( \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) which by Theorem 1
is isomorphic to the semigroup \( B_{\omega_{\nu}}^\mathbb{F}_n \).

Fix an arbitrary \( \chi \in S \setminus \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) and any open neighbourhood \( U(\chi) \) of the point \( \chi \) in \( S \). Since \( B_{\omega_{\nu}}^\mathbb{F}_n \) is a dense proper subsemigroup of \( S \) the set \( U(\chi) \cap (\mathcal{S}_{\omega}^{n+1}(\text{conv}) \setminus \{ 0 \}) \) is infinite. Since the family
\( D \) is finite there exists \( i = 1, \ldots, n + 1 \) such that the set \( U(\chi) \cap D_i \) is infinite. This and the definition of
the semigroup \( \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) imply that at least one of the families

\[
\text{dom}(D_iU(\chi)) = \{ \text{dom } \alpha : \alpha \in U(\chi) \cap D_i \} \quad \text{or} \quad \text{ran}(D_iU(\chi)) = \{ \text{ran } \alpha : \alpha \in U(\chi) \cap D_i \}
\]

has infinitely many members. Assume that the family \( \text{dom}(D_iU(\chi)) \) is infinite. Then the definition of
the semigroup operation on \( \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) implies that there exist infinitely many \( \beta \in U(\chi) \cap \mathcal{S}_{\omega}^{n+1}(\text{conv}) \)
such that \( 0 \in \beta \cdot U(\chi) \), and since \( S \) is a \( T_1 \)-space we have that \( \beta \cdot \chi = 0 \) for some elements \( \beta \). Also, the
infiniteness of \( \text{dom}(D_iU(\chi)) \) and the semigroup operation of \( \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) imply the existence infinitely
many \( \gamma \in U(\chi) \cap \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) such that \( 0 \in U(\chi) \cdot \gamma \), and since \( S \) is a \( T_1 \)-space we have that \( \chi \cdot \gamma = 0 \)
for so elements \( \gamma \). In the case when the family \( \text{ran}(D_iU(\chi)) \) is infinite similarly we obtain that there exist
infinitely many \( \beta, \gamma \in U(\chi) \cap \mathcal{S}_{\omega}^{n+1}(\text{conv}) \) such that \( \beta \cdot \chi = 0 \) and \( \chi \cdot \gamma = 0 \).

Thus we show that \( 0 \in V(\chi) \cdot \chi \) and \( 0 \in \chi \cdot V(\chi) \) for any open neighbourhood \( V(\chi) \) of the point \( \chi \)
in \( S \). Since \( S \) is a \( T_1 \)-space this implies the required equality \( \chi \cdot \chi = 0 \) for all \( \chi \in S \setminus B_{\omega_{\nu}}^\mathbb{F}_n \).
Absolutely $H$-closed topological semigroups and algebraically $h$-complete semigroups were introduced by Stepp in [39], and there they were called *absolutely maximal* and *algebraic maximal*, respectively. Other distinct types of completeness of (semi)topological semigroups were studied by Banakh and Bardyla (see [1–6]).

Proposition 10 of [20] and Proposition 3 imply the following theorem.

**Theorem 6.** For any $n \in \omega$ the semigroup $B_{\omega}^{\mathcal{F}_{n}}$ is algebraically complete in the class of Hausdorff semi-topological inverse semigroups with continuous inversion, and hence in the class of Hausdorff topological inverse semigroups.

**Theorem 7.** Let $n$ be a non-negative integer. If $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ is a Hausdorff topological semigroup with the compact band then $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ is $H$-closed in the class of Hausdorff topological semigroups.

**Proof.** Suppose to the contrary that there exists a Hausdorff topological semigroup $T$ which contains $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ as a non-closed subsemigroup. Since the closure of a subsemigroup of a topological semigroup $S$ is a subsemigroup of $S$ (see [11, p. 9]), without loss of generality we can assume that $B_{\omega}^{\mathcal{F}_{n}}$ is a dense subsemigroup of $T$ and $T \setminus B_{\omega}^{\mathcal{F}_{n}} \neq \emptyset$. Let $\chi \in T \setminus B_{\omega}^{\mathcal{F}_{n}}$. Then 0 is the zero of the semigroup $T$ by Lemma 4.4 of [18], and $\chi \cdot \chi = 0$ by Proposition 6.

Since $0 \cdot \chi = \chi \cdot 0 = 0$ and $T$ is a Hausdorff topological semigroup, for any disjoint open neighbourhoods $U(\chi)$ and $U(0)$ of $\chi$ and 0 in $T$, respectively, there exist open neighbourhoods $V(\chi) \subseteq U(\chi)$ and $V(0) \subseteq U(0)$ of $\chi$ and 0 in $T$, respectively, such that $V(0) \cdot V(\chi) \subseteq U(0)$ and $V(\chi) \cdot V(0) \subseteq U(0)$.

By Theorem 4 every non-zero element of $B_{\omega}^{\mathcal{F}_{n}}$ is an isolated point in $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ and by Corollary 3.3.11 of [17] it is an isolated point of $T$, and hence the set $E(B_{\omega}^{\mathcal{F}_{n}}) \setminus V(0)$ is finite. Also Hausdorffness and compactness of $E(B_{\omega}^{\mathcal{F}_{n}})$ imply that without loss of generality we may assume that $V(\chi) \cap E(B_{\omega}^{\mathcal{F}_{n}}) = \emptyset$. Since the neighbourhood $V(\chi)$ contains infinitely many elements of the semigroup $B_{\omega}^{\mathcal{F}_{n}}$ and the set $E(B_{\omega}^{\mathcal{F}_{n}}) \setminus V(0)$ is finite, there exists $(i, j, [0; k]) \in V(\chi)$ such that either $(i, i, [0; k]) \in V(0)$ or $(j, j, [0; k]) \in V(0)$. Therefore, we have that at least one of the following conditions holds:

$$(V(0) \cdot V(\chi)) \cap V(\chi) \neq \emptyset \quad \text{and} \quad (V(\chi) \cdot V(0)) \cap V(\chi) \neq \emptyset.$$ 

Every of the above conditions contradicts the assumption that $U(\chi)$ and $U(0)$ are disjoint open neighbourhoods of $\chi$ and 0 in $T$. The obtained contradiction implies the statement of the theorem. \hfill $\square$

Since compactness preserves by continuous maps Theorems 3 and 7 imply

**Corollary 2.** Let $n$ be a non-negative integer. If $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ is a Hausdorff topological semigroup with the compact band then $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ is absolutely $H$-closed in the class of Hausdorff topological semigroups.

**Theorem 8.** Let $n$ be a non-negative integer and $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ be a Hausdorff topological inverse semigroup. If $(B_{\omega}^{\mathcal{F}_{n}}, \tau)$ is $H$-closed in the class of Hausdorff topological semigroups then its band $E(B_{\omega}^{\mathcal{F}_{n}})$ is compact.

**Proof.** We shall prove the statement of the proposition for the semigroup $\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V})$ which by Theorem 1 is isomorphic to the semigroup $B_{\omega}^{\mathcal{F}_{n+1}}$.

Suppose to the contrary that there exists a Hausdorff topological inverse semigroup $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$ with the non-compact band such that $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$ is $H$-closed in the class of Hausdorff topological semigroups. By Theorem 4 every non-zero element of $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$ is an isolated point in $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$ and hence there exists an open neighbourhood $U(0)$ of the zero 0 in $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$ such that the set $A = E(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V})) \setminus U(0)$ is infinite and closed in $(\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V}), \tau)$. Let $k$ be the smallest positive integer $\leq n + 1$ such that the set $A_k = A \cap \mathcal{F}_{\omega}^{k}(\text{con}\mathcal{V})$ is infinite for the subsemigroup $\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V})$ of $\mathcal{F}_{\omega}^{n+1}(\text{con}\mathcal{V})$. Without loss of generality we may assume that there exists an increasing sequence of non-negative integers $\{a_j\}_{j \in \omega}$ such that $a_0 \geq n + 1$ and

$$\tilde{A}_k = \{(a_j \ldots a_{j+k-1}) : j \in \omega\} \subseteq A_k.$$
The continuity of the semigroup operation in \( \mathcal{S}^{n+1}(\text{conv}) \) implies that there exists an open neighbourhood \( V(0) \subseteq U(0) \) of the zero \( 0 \) in \( \mathcal{S}^{n+1}(\text{conv}) \), \( \tau \) such that \( V(0) \cdot V(0) \subseteq U(0) \). By the definition of the semigroup operation on \( \mathcal{S}^{n+1}(\text{conv}) \) we have that the neighbourhood \( V(0) \) does not contain at least one of the points
\[
\left( a_j \cdots a_j + n \right) \quad \text{or} \quad \left( a_j - n + k - 2 \cdots a_j + k - 1 \right).
\]
Since the both above points belong to \( \mathcal{S}^{n+1}(\text{conv}) \) \( \setminus \mathcal{S}^n(\text{conv}) \), without loss of generality we may assume that there exists an increasing sequence of non-negative integers \( \{ b_j \}_{j \in \omega} \) such that \( b_j + n - 1 < b_{j+1} \) for all \( j \in \omega \) and
\[
\widetilde{B}_{n+1} = \left\{ \left( \left( b_j \cdots b_{j+n} \right) : j \in \omega \right) : j \in \omega \right\} \not\subseteq V(0).
\]

Since \( \mathcal{S}^{n+1}(\text{conv}) \) \( \tau \) is a Hausdorff topological inverse semigroup, the maps \( f_1 : \mathcal{S}^{n+1}(\text{conv}) \to E(\mathcal{S}^{n+1}(\text{conv})) \), \( \alpha \mapsto \alpha \cdot \alpha^{-1} \) and \( f_2 : \mathcal{S}^{n+1}(\text{conv}) \to E(\mathcal{S}^{n+1}(\text{conv})) \), \( \alpha \mapsto \alpha^{-1} \) are continuous, and hence the set \( \widetilde{B}_{n+1} = f_1^{-1}(\widetilde{B}_{n+1}) \cup f_2^{-1}(\widetilde{B}_{n+1}) \) is infinite and open in \( \mathcal{S}^{n+1}(\text{conv}) \), \( \tau \).

Let \( \chi \not\in \mathcal{S}^{n+1}(\text{conv}) \). Put \( S = \mathcal{S}^{n+1}(\\text{conv}) \cup \{ \chi \} \). We extend the semigroup operation from \( \mathcal{S}^{n+1}(\text{conv}) \) onto \( S \) in the following way:
\[
\chi \cdot \chi = \chi, \quad \alpha \cdot \chi = \chi, \quad \chi \cdot \alpha = \chi, \quad \text{for all } \alpha \in \mathcal{S}^{n+1}(\text{conv}).
\]
Simple verifications show that such defined binary operation is associative.

For any \( p \in \omega \) we denote
\[
\Gamma_p = \left\{ \left( \left( b_{j+1} \cdots b_{j+n} \right) : j \geq p \right) \right\}.
\]
We determine a topology \( \tau_S \) on the semigroup \( S \) in the following way:
\begin{enumerate}
\item for every \( \gamma \in \mathcal{S}^{n+1}(\text{conv}) \) the bases of topologies \( \tau \) and \( \tau_S \) at \( \gamma \) coincide;
\item \( \mathcal{B}(\chi) = \{ U_p(\chi) = \{ \chi \} \cup \Gamma_p : p \in \omega \} \) is the base of the topology \( \tau_S \) at the point \( \chi \).
\end{enumerate}
Simple verifications show that \( \tau_S \) is a Hausdorff topology on the semigroup \( \mathcal{S}^{n+1}(\text{conv}) \).

For any \( p \in \omega \) and any open neighbourhood \( V(0) \subseteq U(0) \) of the zero \( 0 \) in \( \mathcal{S}^{n+1}(\text{conv}) \), \( \tau \) we have that
\[
V(0) \cdot U_p(\chi) = U_p(\chi) \cdot V(0) = U_p(\chi) \cdot U_p(\chi) = \{ 0 \} \subseteq V(0).
\]
We observe that the definition of the set \( \Gamma_p \) implies that for any non-zero element \( \gamma = \left( \frac{c}{d} \cdots \frac{c+l}{d+l} \right) \) of the semigroup \( \mathcal{S}^{n+1}(\text{conv}) \) there exists the smallest positive integer \( j_\gamma \) such that \( c+l < b_{j_\gamma} \) and \( d+l < b_{j_\gamma+1} \).
Then we have that
\[
\gamma \cdot U_{j_\gamma}(\chi) = U_{j_\gamma}(\chi) \cdot \gamma = \{ 0 \} \subseteq V(0).
\]
Therefore \( (S, \tau_S) \) is a topological semigroup which contains \( (\mathcal{S}^{n+1}(\text{conv}), \tau) \) as a dense proper subsemigroup. The obtained contradiction implies that \( E(B_n^\omega) \) is a compact subset of \( (\mathcal{S}^{n+1}(\text{conv}), \tau) \).

\textbf{Theorem 9.} Let \( n \) be a non-negative integer and \( (B_n^\omega, \tau) \) be a Hausdorff topological inverse semigroup. Then the following conditions are equivalent:
\begin{enumerate}
\item \( (B_n^\omega, \tau) \) is \( H \)-closed in the class of Hausdorff topological semigroups;
\item \( (B_n^\omega, \tau) \) is absolutely \( H \)-closed in the class of Hausdorff topological semigroups;
\item the band \( E(B_n^\omega) \) is compact.
\end{enumerate}

\textit{Proof.} Implication \( (2) \Rightarrow (1) \) is obvious. Implications \( (1) \Rightarrow (3) \) and \( (3) \Rightarrow (1) \) follow from Theorem 8 and Theorem 7, respectively.

Since a continuous image of a compact set is compact, Theorem 3 implies that \( (3) \Rightarrow (2) \).

The following example shows that a counterpart of the statement of Theorem 8 does not hold when \( (B_n^\omega, \tau) \) is a Hausdorff topological semigroup.

\textbf{Example 2.} On the semigroup \( \mathcal{S}^1(\text{conv}) \) we define a topology \( \tau_1 \) in the following way. All non-zero elements of the semigroup \( \mathcal{S}^1(\text{conv}) \) are isolated points of \( \mathcal{S}^1(\text{conv}), \tau_1 \) and the family \( \mathcal{B}_1(0) = \{ U_k(0) : k \in \omega \} \), where \( U_k(0) = \{ 0 \} \cup \{ (\frac{2i}{2i+1}) : i \geq k \} \), determines the base of the topology \( \tau_1 \) at the
point $0$. It is obvious that $\tau_1$ is a Hausdorff topology on $\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V)$. Since $U_k(0) \cdot U_k(0) = \{0\}$ for any $k \in \omega$ and $U_q(0) \cdot \{(p)\} = \{(p)\} \cdot U_p(0) = \{0\}$ for any $p, q \in \omega$, $(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V), \tau_1)$ is a topological semigroup.

**Proposition 7.** $(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V), \tau_1)$ is $H$-closed in the class of Hausdorff topological semigroups.

*Proof.* Suppose to the contrary that there exists a Hausdorff topological semigroup $T$ which contains $(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V), \tau_1)$ as a non-closed subsemigroup. Since the closure of a subsemigroup of a topological semigroup $S$ is a subsemigroup of $S$ (see [11, p. 9]), without loss of generality we can assume that $\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V)$ is a dense proper subsemigroup of $T$. Let $\chi \in T \setminus \mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V)$. Then $0$ is the zero of the semigroup $T$ by Lemma 4.4 of [18], and $\chi \cdot \chi = 0$ by Proposition 6.

Fix disjoint open neighbourhoods $U(\chi)$ and $U_q(0)$ of $\chi$ and $0$ in $T$. By Proposition 6, $E(T) = E(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V))$. By Theorem 1.5 of [11], $E(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V))$ is a closed subset of $T$ and hence without loss of generality we can assume that $U(\chi) \cap E(\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V)) = \emptyset$. Then for any open neighbourhoods $V(\chi) \subseteq U(\chi)$ and $U_q(0) \subseteq U_p(0)$ the infiniteness of $V(\chi)$ and the definition of the semigroup operation on $\mathcal{S}_\omega(\overline{\operatorname{conv}}_\tau V)$ that imply that

$$V(\chi) \cdot U_q(0) \not\subseteq U_p(0) \quad \text{or} \quad U_q(0) \cdot V(\chi) \not\subseteq U_p(0),$$

which contradicts the continuity of the semigroup operation on $T$.  \hfill $\Box$

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