Exact quantum states of a general time-dependent quadratic system from classical action

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Abstract

A generalization of driven harmonic oscillator with time-dependent mass and frequency, by adding total time-derivative terms to the Lagrangian, is considered. The generalization which gives a general quadratic Hamiltonian system does not change the classical equation of motion. Based on the observation by Feynman and Hibbs, the propagators (kernels) of the systems are calculated from the classical action, in terms of solutions of the classical equation of motion: two homogeneous and one particular solutions. The kernels are then used to find wave functions which satisfy the Schrödinger equation. One of the wave functions is shown to be that of a Gaussian pure state. In every case considered, we prove that the kernel does not depend on the way of choosing the classical solutions, while the wave functions depend on the choice. The generalization which gives a rather complicated quadratic Hamiltonian is simply interpreted as acting an unitary transformation to the driven harmonic oscillator system in the Hamiltonian formulation.

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I. Introduction

Time-dependent quantum mechanical systems continue of great interest. In particular, the system described by an explicit time-dependent quadratic Hamiltonian has attracted considerable attention. One of the typical examples is the harmonic oscillator with time-dependent mass and/or frequencies. Those studies have many applications such as in quantum optics [1], Paul trap [2], and the analyses of quantum fields in curved spacetime [3], and they are closely related to the theory of quantum dissipation [4].

In the Hamiltonian formulation of a time-dependent harmonic oscillator, Lewis and Riesenfeld (LR) [5, 6] have shown that there exists quantum mechanically invariant operator whose exact form is determined by an auxiliary function. The invariant operator can then be used to find exact wave functions of quantum states. During the past several years, this LR method has been widely used for the study of a general quadratic Hamiltonian systems [7, 8, 9, 10, 11]. For the harmonic oscillator with time-dependent mass and frequency, the wave functions and the kernel (propagator) have been found [7, 8, 11]. Through Heisenberg-picture approach, Ji et al. [9] have refined the derivation, and the wave functions are given. The Heisenberg-picture approach have then been used [10], with successive unitary transformations [2], to find exact wave functions of the general quadratic Hamiltonian system [11]. The auxiliary functions in the LR method were defined through differential equations related to the equation of motion. We also note that the LR method can be applied for more general systems [12].

In this article, we will study the quadratic system in the Lagrangian formulation of
The Lagrangian we will consider is

$$L = \frac{1}{2} M(t) \dot{x}^2 - \frac{1}{2} M(t) w^2(t) x^2 + F(t)x + \frac{d}{dt} (M(t) a(t) x^2) + \frac{d}{dt} (b(t) x) + f(t), \quad (1)$$

where $w^2(t), F(t), a(t), b(t), f(t)$ are arbitrary real functions of $t$ and $M(t)$ is also arbitrary real but always positive. The last three terms in the right hand side (r.h.s.) of Eq.(1) have no effect in classical dynamics and classical trajectory $\bar{x}$ of the coordinate $x$ will satisfy the equation:

$$\frac{d}{dt} (M \dot{\bar{x}}) + M(t) w^2(t) \bar{x} = F(t). \quad (2)$$

The most general solution of this differential equation may be composed of a particular solution and two linearly independent homogeneous solutions. The corresponding quantum Hamiltonian may be written as

$$H = \frac{\hat{p}^2}{2M(t)} - a(t)[\hat{p} \hat{x} + \hat{x} \hat{p}] + \frac{1}{2} M(t) c(t) \hat{x}^2 - \frac{b(t)}{M(t)} \hat{p} + d(t) \hat{x} + (\frac{b^2(t)}{2M(t)} - f(t)), \quad (3)$$

where

$$c(t) = w^2 + 4a^2 - 2\dot{a} - 2\frac{\dot{M}}{M} a,$$

$$d(t) = 2ab - \dot{b} - F. \quad (4)$$

In the sense of differentiation of operator in Ref.[14], the quantum equation of motion for the operator $\hat{x}$ is again given as

$$\frac{d}{dt} (M \dot{\hat{x}}) + M(t) w^2(t) \hat{x} = F(t).$$

Feynman and Hibbs have shown that the coordinate dependent part of the kernel is determined from the classical action (Ch.3-5 of Ref.[13]). In this article, it will be shown
that the remained part of the kernel for the system will be completely determined from the Schrödinger equation and the initial condition which the kernel should satisfy. In this way the kernel for the system of Eq.(1) will evaluated. By the method of Ref.[15], the wave functions will then be evaluated from the kernel. This Lagrangian formulation has a clear advantage over the Hamiltonian formulation in showing how the last three terms in the r.h.s. of Eq.(1) which have no effect on classical dynamics affect the wave functions. The classical action will be evaluated in terms of the two linearly independent homogeneous solutions and one particular solution, and so are the kernel and the wave functions. We will prove that the kernel do not depend on the way of choosing the classical solutions, while the wave functions are not unique and depend on the choice of the classical solutions. By comparing with the results on the Gaussian pure states of Ref.[16], it is suggested that choosing different classical solutions might amount to acting unitary transformations to the annihilation operator.

In the next section, we will consider the harmonic oscillator with time-dependent mass and frequency, mainly to expose our method. It will also be shown that the kernel does not depend on the choice of two homogeneous solutions, whilst the wave functions depends on the choice. In Sec. III, the driven harmonic oscillator will be considered. In Sec. IV, the system of a general quadratic Lagrangian in Eq.(1) will be considered and some previous errors will be corrected. The general system will be shown equivalent to the driven harmonic oscillator through an unitary transformation. Sec. V will be devoted to a summary and discussions. We add an appendix to explain how to determine the time dependent part of kernel from the Schrödinger equation and the initial condition.
II. The harmonic oscillator with time-dependent mass and frequency

In this section, we will apply our method to the harmonic oscillator without driving force.

For this model the Lagrangian is written as;

\[ L^S = \frac{1}{2} M(t) \dot{x}^2 - \frac{1}{2} M(t) w^2(t) x^2. \]  

(5)

The action (integral) from time \( t_a \) to time \( t_b \) is written as;

\[ S = \int_{t_a}^{t_b} L dt \]

(6)

which gives the equation of motion for the classical trajectory of the model considered

\[ \frac{d}{dt} (M \ddot{x}) + M(t) w^2(t) \ddot{x} = 0. \]

(7)

To find a simple expression for the classical action that is the integral along classical trajectory, we can rewrite the action as

\[ S^S = \frac{1}{2} M \dot{x} x \bigg|_{t_a}^{t_b} - \frac{1}{2} \int_{t_a}^{t_b} x \left[ \frac{d}{dt} (M \ddot{x}) + M w^2 x \right]. \]

(8)

The classical action is then simply given as

\[ S^S_{cl}(a, b) = \frac{1}{2} M(t_b) x_b \dot{x}_b - \frac{1}{2} M(t_a) x_a \dot{x}_a, \]

(9)

where \( x_a (x_b) \) and \( \dot{x}_a (\dot{x}_b) \) denote fixed end point and \( \frac{dx}{dt} \) at \( t = t_a \) (\( t = t_b \)), respectively.

Suppose that \( u(t) \) and \( v(t) \) are two linearly independent real solutions of Eq.(7), so that \( \ddot{x} \) can be written by a linear combination of them. From the two linearly independent
solutions, one can always find \( v_s(t) \), a linear combination of the two solutions which satisfies \( v_s(t_a) = 0 \), so that \( u(t) \) and \( v_s(t) \) are again two linearly independent solutions. One can then easily find the quantities \( \Omega_s \) and \( \Omega \) defined as

\[
\Omega_s = M(t)[\ddot{v}_s(t)u(t) - \dot{u}(t)v_s(t)], \quad \Omega = M(t)[\ddot{v}(t)u(t) - \dot{u}(t)v(t)]
\]

(10)
do not depend on time.

The \( \bar{x}(t) \) with two fixed end points \( x_a, x_b \) can be written as

\[
\bar{x}(t) = x_a \frac{u(t)}{u(t_a)} + [x_b - x_a \frac{u(t_b)}{u(t_a)}] v_s(t_a)
\]

(11)

Making use of this expression of \( \bar{x} \), one can rewrite the classical action in Eq.(9) as

\[
S_{cl}^S(a,b) = \frac{x_a^2}{2}M(t_a)[\ddot{u}(t_a) - \dot{u}(t_a)\dot{v}_s(t_a)] + \frac{x_b^2}{2}M(t_b)\ddot{v}_s(t_b)
\]

\[
+ \frac{x_a x_b}{2}[M(t_b)(\frac{\ddot{u}(t_b)}{u(t_a)} - \frac{u(t_b)\dot{v}_s(t_b)}{u(t_a)v_s(t_b)}) - M(t_a)\ddot{v}_s(t_a)],
\]

(12)
The fact that classical dynamics are deterministic implies \( S_{cl}^S \) is unique, as can be explicitly proved; First, one find that \( S_{cl}^S \) does not depend on the scaling of \( u(t) \) or \( v_s(t) \) by multiplying constant factors. Second, the classical action in Eq.(12) is invariant under the substitution of \( u(t) \) by \( u(t) + C v_s(t) \) with an arbitrary constant \( C \). These two observations lead us to a conclusion that \( S_{cl}^S \) do not depend on the particular choice of \( u(t) \) or \( v_s(t) \) as long as \( v_s(t_a) = 0 \).

Then from the formula (3-51) of Ref.[13], the kernel can be written as;

\[
K^S(b,a) = \exp[i\hbar(S_{cl}^S(a,b) + D^S(t_a,t_b))],
\]

(13)
where, as shown in appendix, \( D(t_a,t_b) \) can be completely determined from the initial condition and Schrödinger equation. Making use of the formulas in appendix, one can
find the expression of $K^S(b,a)$ in terms of $u$ and $v_s$ as

$$K^S(b,a) = \sqrt{\frac{M(t_a)}{2\pi i \hbar}} \frac{\dot{v}_s(t_a)}{v_s(t_b)} \exp\left[\frac{i}{2\hbar}(M(t_a)(-\ddot{u}(t_a) + \frac{u(t_b)}{u(t_a)}\dot{v}_s(t_a))x_a^2 + M(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}x_b^2 - 2x_a x_b M(t_a)\frac{\dot{v}_s(t_a)}{v_s(t_b)})\right].$$  \(14\)

Since the kernel is uniquely determined from the classical action, our argument on the uniqueness of the classical action implies that $K^S(b,a)$ does not depend on the way of choosing $u(t)$ or $v_s(t)$.

To find the wave functions from the kernel along the method of Ref.\[15\], we define two functions $\rho(t)$ and $z(t)$ as;

$$\rho(t) = \sqrt{u^2(t) + v_s^2(t)}, \quad (15)$$

$$z(t) = \frac{u(t) - iv_s(t)}{\rho(t)}. \quad (16)$$

After a little algebra, one can find that the kernel can be written as

$$K^S(b,a) = \frac{1}{\sqrt{\pi \hbar}} \sqrt{\frac{\Omega_s}{\rho(t_a)\rho(t_b)}} \times \exp\left[\frac{x_a^2}{2\hbar}(-\frac{\Omega_s}{\rho^2(t_a)} - iM(t_a)\frac{\dot{\rho}(t_a)}{\rho(t_a)}) + \frac{x_b^2}{2\hbar}(-\frac{\Omega_s}{\rho^2(t_b)} + iM(t_b)\frac{\dot{\rho}(t_b)}{\rho(t_b)})\right] \times \sum_{n=0}^{\infty} \frac{z^{n+\frac{1}{2}}(t_b)}{2^n n!} H_n(\sqrt{\frac{\Omega_s}{\hbar} \rho(t_a)}) H_n(\sqrt{\frac{\Omega_s}{\hbar} \rho(t_b)}), \quad (17)$$

where $H_n$ is the $n$-th order Hermite polynomial. From now on the definition of $\rho(t)$ is modified as $\rho(t) = \sqrt{u^2(t) + v^2(t)}$. From the well-known fact that

$$K(x_b, t_b; x_a, t_a) = \sum_n \psi_n(x_b, t_b)\psi_n^*(x_a, t_a), \quad \text{for} \quad t_b > t_a, \quad (18)$$

one can find the $n$-th order wave function:

$$\psi_n^S(x, t) = \frac{1}{\sqrt{2^n n!}} (\frac{\Omega}{\pi \hbar})^{\frac{1}{4}} \frac{1}{\sqrt{\rho(t)}} \left[\frac{u(t) - iv(t)}{\rho(t)}\right]^{n+\frac{1}{2}} e^{\frac{x^2}{2\hbar}(-\frac{\rho(t)}{\rho(t)} + iM(t)\frac{\dot{\rho}(t)}{\rho(t)})} H_n(\sqrt{\frac{\Omega}{\hbar} \rho(t)}).$$  \(19\)
which satisfies the Schrödinger equation:

\[ i\hbar \frac{\partial \psi_n^S}{\partial t} = -\frac{\hbar^2}{2M(t)} \frac{\partial^2 \psi_n^S}{\partial x^2} + \frac{M(t)w^2(t)}{2} \psi_n^S. \]  

(20)

The \( \psi_n^S \) does depend on the choice of two homogeneous solutions, and any set of two linearly independent solutions can be used to construct the wave functions which satisfy the Schrödinger equation of Eq.(20).

To have a physical interpretation of the fact that different choice of \( \{u, v\} \) may give different set of wave functions \( \{\psi_n^S, \ n = 0, 1, 2, \cdots\} \), we consider the simplest case: the simple harmonic oscillator where \( M(t) = m_0 \) and \( w(t) = w_0 \). In this case, if we take \( \{u, v\} \) as \( \{C \cos w_0 t, C \sin w_0 t\} \) with arbitrary non-zero constant \( C \), then the \( \psi_n^S \) reduces to the usual stationary wave functions whose ground \( (n = 0) \) state is given as

\[ \tilde{\psi}_0 = \left( \frac{m_0 \hbar w_0}{\pi} \right)^{1/4} e^{-m_0 \hbar w_0 x^2 / 2\hbar}. \]

The choice of \( \{u, v\} \) as \( \{\cos w_0 t, C \sin w_0 t\} \) with \( C \neq 1 \), however, gives the wave functions of probability distribution pulsating as time passes.

In general case, by defining \( \gamma \) as \( \gamma_1 + i\gamma_2 \) where

\[ \gamma_1 = \frac{\Omega}{\hbar \rho^2} \quad \text{and} \quad \gamma_2 = -\frac{M \dot{\rho}}{\hbar \rho}, \]

we can rewrite the \( \psi_0^S \) as

\[ \psi_0^S = \left( \frac{\gamma_1}{\pi} \right)^{1/4} \exp(i\delta_0(t)) \exp\left[-\frac{1}{2} \gamma x^2\right], \]  

(21)

with a real function \( \delta_0 \) of \( t \). Therefore, \( \psi_0^S \) is one of the wave functions of the Gaussian pure states extensively studied in Ref.[16]. There, it has been shown that any Gaussian pure state is the eigenstate of a certain linear combination of creation and annihilation operator. If we choose different classical solutions, then we could have different \( \gamma \). The
studies of Ref. [16] suggest that choosing different classical solutions might amount to acting unitary transformations to the annihilation operator of the representation system.

III. Driven harmonic oscillator

In this section we will consider the system described by the Lagrangian:

$$L^F = \frac{1}{2} M(t) \dot{x}^2 - \frac{1}{2} M(t) w^2(t) x^2 + F(t)x. \quad (22)$$

Let’s denote the particular solution of Eq.(2) as $x_p(t)$, so that $x_p(t)$ satisfies the equation:

$$\frac{d}{dt}(M \dot{x}_p) + M(t)w^2(t)x_p = F(t).$$

Then one can rewrite the Lagrangian as

$$L^F = \frac{1}{2} \frac{d}{dt}[M(t)(x - x_p)(\dot{x} - \dot{x}_p)] + \frac{d}{dt}[M(t)\dot{x}_p(x - x_p)] - \frac{1}{2}(x - x_p)[\frac{d}{dt}(M(\dot{x} - \dot{x}_p)) + Mw^2(x - x_p)] + \frac{1}{2} \frac{d}{dt}[M\dot{x}_p x_p] + \frac{d}{dt}Y_x(t), \quad (23)$$

where $Y(t)$ is defined as

$$Y_{x_p}(t) = \int_{t_0}^{t} \frac{1}{2} x_p(t') F(t') dt'$$

with arbitrary constant $t_0$. The classical action $S_{cl}^F(a, b)$ from time $t_a$ to $t_b$ can be written as

$$\tilde{S}_{cl}^F(a, b; x_p) = S_{cl}^F(a, b) - \Delta S_1(x_p(t), Y_x(t)) |^{b}_{t_a}$$

$$= \left[\frac{1}{2} M(t)(\bar{x} - x_p)(\dot{\bar{x}} - \dot{x}_p) + M(t)\dot{x}_p(\bar{x} - x_p)\right] |^{b}_{t_a}. \quad (25)$$

By adding a homogeneous solution $Cu(t) + Dv(t)$ to the particular $x_p(t)$ with arbitrary constants $C$ and $D$, one can have a new particular solution $x'_p(t)$. By rewriting the classical
action as

\[ S_{cl}^F(a, b) = \int_{t_a}^{t_b} \left[ \frac{1}{2} \frac{d}{dt} (M \ddot{x}) - \frac{1}{2} \frac{d}{dt} (M \dot{x}_p) + \frac{1}{2} \frac{d}{dt} (M \dot{x}_p \ddot{x}) + \frac{1}{2} x_p F \right] dt, \]  

one can easily find that the classical action does not depend on the choice of particular solution. That is,

\[ S_{cl}^F(a, b) = \tilde{S}_{cl}^F(a, b; x_p) + \Delta S_1(x_p(t), Y_{x_p}(t)) \bigg|_{t_a}^{t_b} = \tilde{S}_{cl}^F(a, b; x'_p) + \Delta S_1(x'_p(t), Y_{x'_p}(t)) \bigg|_{t_a}^{t_b}. \]  

Through the same methods of previous section, one can find the end points dependent part of classical action:

\[ \tilde{S}_{cl}^F(a, b; x_p) = \frac{M(t_a)[x_a - x_p(t_a)]^2}{2} \left[ -\frac{\dot{u}(t_a)}{u(t_a)} + \frac{u(t_b)\dot{v}_s(t_b)}{u(t_a) v_s(t_b)} \right] + \frac{M(t_b)[x_b - x_p(t_b)]^2 \dot{v}_s(t_b)}{2 v_s(t_b)} \]
\[ - (x_a - x_p(t_a))(x_b - x_p(t_b)) \frac{M(t_a)\dot{v}_s(t_a)}{v_s(t_b)} \]
\[ + M(t_b) \dot{x}_p(t_b) x_b - M(t_a) \dot{x}_p(t_a) x_a. \]  

The kernel can be written as

\[ K^F(a, b) = \exp \left[ \frac{i}{\hbar} (S_{cl}^F(a, b) + D^F(t_a, t_b)) \right] = \exp \left[ \frac{i}{\hbar} (\tilde{S}_{cl}^F(a, b) + \tilde{D}^F(t_a, t_b)) \right]. \]  

Since the \( S_{cl}^F \) does not depend on the choice of the classical solutions within the given restriction and \( D^F \) is uniquely determined from the \( S_{cl}^F \), the kernel is again unique. For the explicit evaluation, we require the particular solution to satisfy \( x_p(t_a) = 0 \). In the notations of appendix, \( B \) and \( \beta \) is then given as,

\[ B = \frac{M(t_b) \dot{v}_s(t_b)}{2 v_s(t_b)}, \]
\[ \beta = -M(t_b) x_p(t_b) \frac{\dot{v}_s(t_b)}{v_s(t_b)} - M(t_b) \dot{x}_p(t_b), \]
and the kernel is written as

\[ K^F(b, a) = \frac{M(t_a) \dot{v}_s(t_a)}{2\pi i h \dot{v}_s(t_b)} \]

\[ \times \exp \left[ \frac{i}{2h} \left[ x_a^2 M(t_a) \left( -\frac{\ddot{u}(t_a)}{u(t_a)} + \frac{u(t_b)\dot{v}_s(t_a)}{u(t_a)v_s(t_b)} \right) + (x_b - x_p(t_b))^2 M(t_b) \frac{\dot{v}_s(t_b)}{v_s(t_b)} \right. \]

\[ \left. - 2x_a(x_b - x_p(t_b)) M(t_a) \frac{\dot{v}_s(t_a)}{v_s(t_b)} \right] \]

\[ + 2M(t_b)x_p(t_b)x_b - 2M(t_a)x_p(t_a)x_a - M(t_b) \frac{\dot{v}_s(t_b)}{v_s(t_b)} x^2_p(t_b) \]

\[ - \int_{t_a}^{t_b} M(t) \left[ x_p(t) \dot{v}_s(t) - \dot{x}_p(t)v_s(t) \right] dt \].

(32)

From the expression of the kernel in Eq.(32), as in the previous section, one can find the n-th order wave function as

\[ \psi_n^F(x, t) = \frac{1}{\sqrt{2^n n!}} \frac{\Omega}{\pi h} \frac{1}{\sqrt{\rho(t)}} \exp \left[ \left( \frac{u(t) - iv(t)}{\rho(t)} \right)^n \right] \exp \left[ \left( \frac{(x - x_p(t))^2}{2h} - \frac{\Omega}{\rho^2(t)} + iM(t) \dot{\rho}(t) \right) \right] \]

\[ \times H_n\left(\sqrt{\frac{\Omega}{h} \frac{x - x_p(t)}{\rho(t)}}\right) \exp \left[ i \left( M(t) \dot{x}_p(t)x - \frac{M(t)}{2v(t)} x^2_p(t) \right) \right. \]

\[ \left. - \frac{1}{2} \int_{t_0}^{t} M(z) \left( x_p(z) \frac{\dot{v}(z)}{v(z)} - \dot{x}_p(z) \right)^2 dz \right]. \]

(33)

In Eq.(33), \(\{u, v\}\) is the set of arbitrary linear independent homogeneous solutions, and \(x_p\) is an arbitrary particular solution. The wave functions, again, depend on the way of choosing classical solutions and one can explicitly find that these wave functions indeed satisfy the Schrödinger equation:

\[ i\hbar \frac{\partial \psi_n^F}{\partial t} = -\frac{\hbar^2}{2M(t)} \frac{\partial^2 \psi_n^F}{\partial x^2} + \frac{M(t)w^2(t)}{2} x^2 \psi_n^F - F(t)x \psi_n^F = H_F \psi_n^F. \]

(34)

**IV. The general quadratic system**

In this section we will consider the general quadratic system described by the Lagrangian of Eq.(1). As in the previous sections, one can find the end points dependent part of
classical action:

\[
\tilde{S}_c^G(a, b; x_p) = \frac{M(t_a)[x_a - x_p(t_a)]^2}{2} \left[ -\frac{\dot{u}(t_a)}{u(t_a)} + \frac{u(t_b)\dot{v}_s(t_a)}{u(t_a)v_s(t_b)} \right] + \frac{M(t_b)[x_b - x_p(t_b)]^2}{2} \frac{\dot{v}_s(t_b)}{v_s(t_b)}
\]

\[
- (x_a - x_p(t_a))(x_b - x_p(t_b))M(t_a)\dot{v}_s(t_a) \frac{v_s(t_b)}{v_a(t_b)}
\]

\[
+ M(t_b)\dot{x}_p(t_b)x_b - M(t_a)\dot{x}_p(t_a)x_a
\]

\[
+ M(t_b)a(t_b)x^2_b - M(t_a)a(t_a)x^2_a + b(t_b)x_b - b(t_a)x_a.
\] 

(35)

The only difference of \(\tilde{S}_c^G\) from \(\tilde{S}_c^F\) is the last four terms in the r.h.s. of Eq.(35). Again, by requiring \(x_p(t_a) = 0\), one can evaluate the kernel in terms of classical solutions;

\[
K^G(b, a) = \sqrt{\frac{M(t_a)\dot{v}_s(t_a)}{2\pi i\hbar v_s(t_b)}}
\]

\[
\times \exp\left[\frac{i}{2\hbar} \left\{ x^2_a M(t_a)(-u(t_a)\dot{u}(t_a) + u(t_b)\dot{v}_s(t_a)) + (x_b - x_p(t_b))^2 M(t_b)\frac{\dot{v}_s(t_b)}{v_s(t_b)}
\right.
\]

\[
- 2x_a(x_b - x_p(t_b)) M(t_a)\frac{\dot{v}_s(t_a)}{v_s(t_b)} + 2M(t_b)\dot{x}_p(t_b)x_b - 2M(t_a)\dot{x}_p(t_a)x_a
\]

\[
+ 2M(t_b)a(t_b)x^2_b - 2M(t_a)a(t_a)x^2_a + 2b(t_b)x_b - 2b(t_a)x_a
\]

\[
- M(t_b)\dot{v}_s(t_b)x^2_b
\]

\[
- \int_{t_a}^{t_b} \left[ -2f(t) + \frac{M(t)}{v^2_s(t)} [x_p(t)\dot{v}_s(t) - \dot{x}_p(t)v_s(t)]^2 dt \right].
\] (36)

whose difference from \(K^F(b, a)\) is just from the above-mentioned four terms and an integral of \(f\). As in the previous sections, one can prove that this kernel does not depend on the way of choosing classical solutions.

The n-th order wave function \(\psi_n\) can be found from the kernel as;

\[
\psi_n^G(x, t) = \frac{1}{\sqrt{2^nn!}} \left( \frac{\Omega}{\pi \hbar} \right)^{\frac{1}{2}} \frac{1}{\sqrt{\rho(t)}} \left[ u(t) - iv(t) \right]^{n+\frac{1}{2}}
\]

\[
\times \exp\left[\frac{i}{\hbar} \left\{ M(t)a(t)x^2 + (M(t)\dot{x}_p(t) + b(t))x \right\} \right]
\]

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\[ \times \exp \left[ \frac{(x - x_p(t))^2}{2\hbar} \left( -\frac{\Omega}{\rho^2(t)} + iM(t)\frac{\dot{\rho}(t)}{\rho(t)} \right) \right] H_n \left( \sqrt{\frac{\Omega}{\hbar}} \frac{x - x_p(t)}{\rho(t)} \right) \]

\[ \times \exp \left[ \frac{i}{\hbar} \left[ -\frac{M(t)}{2} \frac{\dot{v}(t)}{v(t)} x_p^2(t) \right. \right. \]

\[ + \left. \left. \int_{t_0}^t \left( f(z) - \frac{M(z)}{2} \left( \frac{\dot{v}(z)}{v(z)} x_p(z) - \dot{x}_p(z) \right)^2 \right) dz \right] \right]. \quad (37) \]

Again, \( \{u, v\} \) is the set of arbitrary linear independent homogeneous solutions, and \( x_p \) is an arbitrary particular solution. One can explicitly apply the Schrödinger equation to this wave functions, to find that they indeed satisfy the equation.

In the Lagrangian of Eq.(1), the conjugate momentum \( p \) of the coordinate \( x \) is written as \( p = M\dot{x} + 2Mx + b \). One may interpret \( x_p \) as the classical coordinate, and the conjugate momentum is then written as

\[ p_p = M\dot{x}_p + 2Mx_p + b. \quad (38) \]

As in the Sec. II, we define a \( \gamma' \) as \( \gamma'_1 + i\gamma'_2 \) where

\[ \gamma'_1 = \gamma_1 \quad \text{and} \quad \gamma'_2 = -\frac{M}{\hbar} (2a + \frac{\dot{\rho}}{\rho}). \]

Then, the wave function \( \psi^G_n \) can be simply written as

\[ \psi^G_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\gamma_1}{\pi} \right)^{\frac{n}{2}} e^{i\delta(t)} \exp\left[ -\frac{\gamma'_1}{2} (x - x_p)^2 + \frac{i}{\hbar} x_p \right] H_n \left( \sqrt{\gamma_1} (x - x_p) \right), \quad (39) \]

where \( \delta(t) \) is a real function of \( t \). The wave functions agree with those in Ref.[11] except the fact \( \delta(t) \) is real which is necessary for the conservation of total probability \( \int \psi^G_n^* \psi^G_n dx \).

The expression of \( \psi^G_n \) in Eq.(39) shows that \( \psi^G_0 \) is a wave function of a Gaussian pure state \([10] \), so the discussions of Sec. II are still valid in the general case.

With the wave functions, one can calculate the expectation values of operators, and
the uncertainty relations read;

\[ m < (\Delta x)^2 >_m \quad m < (\Delta p)^2 >_m = (m < x^2 >_m - m < x >_m^2)(m < p^2 >_m - m < p >_m^2) \]

\[ = (m + \frac{1}{2})^2 \hbar^2 [1 + \frac{1}{\Omega^2}(2M \rho^2 + M \dot{\rho})^2], \tag{40} \]

\[ m+1 < (\Delta x)^2 >_m \quad m+1 < (\Delta p)^2 >_m \]

\[ = \frac{1}{\sqrt{2}} \left( \frac{(m+1)\hbar}{\Omega} \right)^{\frac{3}{2}} (u + iv)^3 \left[ \frac{2\sqrt{2Mx_p}}{\sqrt{(m+1)\hbar(u + iv)}} - 1 \right] [2Ma + M \dot{\rho} + i\frac{\Omega}{\rho^2}] \]

\[ \times [p_p - \frac{1}{2} \sqrt{\frac{(m+1)\hbar}{2\Omega}}(u + iv)(2Ma + M \dot{\rho} + i\frac{\Omega}{\rho^2})], \tag{41} \]

\[ m+2 < (\Delta x)^2 >_m \quad m+2 < (\Delta p)^2 >_m \]

\[ = (m+2)(m+1)\left( \frac{\hbar}{2\Omega} \right)^2 (u + iv)^4 (2Ma + M \dot{\rho} + i\frac{\Omega}{\rho^2})^2, \tag{42} \]

with the notation \( n < O >_m = \int_{-\infty}^{\infty} \psi_n^G(x,t)O\psi_m^G(x,t). \) If we take \( u = \rho \cos \theta \) and \( v = \rho \sin \theta, \) then \( \Omega = M \rho^2 \dot{\theta} \) and the functions \( \rho(t), \theta(t) \) should satisfy:

\[ \ddot{\theta} + 2\frac{\dot{\rho}}{\rho} \dot{\theta} + \frac{\dot{M}}{M} \dot{\theta} = 0, \]

\[ \ddot{\rho} + \frac{\dot{M}}{M} \dot{\rho} - \rho \dot{\theta}^2 + w^2 \rho = 0. \]

With these notations, the uncertainty relations in Eq.(40-42) are written as;

\[ m < (\Delta x)^2 >_m \quad m < (\Delta p)^2 >_m = (m + \frac{1}{2})^2 \hbar^2 [1 + \frac{1}{\dot{\theta}^2}(2a + \frac{\dot{\theta}}{\rho})^2], \tag{43} \]

\[ m+1 < (\Delta x)^2 >_m \quad m+1 < (\Delta p)^2 >_m \]

\[ = \frac{(m+1)^2}{4} \hbar^2 e^{4i\theta} \frac{1}{\dot{\theta}^2} [1 - \frac{2\sqrt{2M\dot{\theta}}}{\sqrt{(m+1)\hbar}} x_p e^{-i\theta}][2a + \frac{\dot{\theta}}{\rho} + i\dot{\theta}] \]

\[ \times [2a + \frac{\dot{\theta}}{\rho} + i\dot{\theta} - \frac{2\sqrt{2\dot{\theta}}}{\sqrt{(m+1)M\hbar}} p_p e^{-i\theta}] \tag{44} \]

\[ m+2 < (\Delta x)^2 >_m \quad m+2 < (\Delta p)^2 >_m = \frac{(m+1)(m+2)}{4} \hbar^2 e^{4i\theta} \frac{1}{\dot{\theta}^2} [2a + \frac{\dot{\theta}}{\rho} + i\dot{\theta}]^2, \tag{45} \]
respectively. The uncertainty relations of Eq.(43,45) *exactly* agree with those of Ref.[11], but the uncertainty relation of Eq.(44) differs from the corresponding one there.

The terms which do not affect the classical dynamics of the model in Eq.(1) can be written as $L - LF$. The effects of those terms on the wave functions could simply be represented by writing $\psi_n^G$ as;

$$\psi_n^G(x,t) = \exp\left[\frac{i}{\hbar} \int_0^t (L - LF)(x, \frac{dx}{dz}, z) dz\right] \times \psi_n^F(x,t).$$

(46)

This relation suggests that $\psi_n^G$ can be obtained from $\psi_n^F$ by acting unitary operator $U$;

$$U = \exp\left[\frac{i}{\hbar} (M(t)a(t)x^2 + b(t)x + \int_0^t f(z) dz)\right].$$

(47)

By defining operator $O_F, O_G$ as

$$O_F = -i\hbar \frac{\partial}{\partial t} + H_F, \quad O_G = -i\hbar \frac{\partial}{\partial t} + H,$$

(48)

one may find the relation

$$UO_FU^\dagger = O_G,$$

(49)

which proves that the Schrödinger equation of general quadratic system is equivalent to that of the driven harmonic oscillator through the unitary transformation.

V. Summary

The Feynman and Hibbs formulation (or an observation) on the quadratic Lagrangian system gives a good explanation on the fact that the quantum wave function can be written in terms of solutions of classical equation of motion. By developing the observation, we
find the kernel and wave functions of the general quadratic system in terms of classical solutions. Furthermore, the kernel is shown to be independent from the choice of classical solutions. These results are then used to show that the general quadratic system is equivalent to the driven harmonic oscillator through a unitary transformation. This fact [17] shows that unitary transformation (or, canonical transformation, its classical correspondent) could make the problem simpler or more complicated, and could change the uncertainty relations as in Eq.(40-42).
Appendix

In this appendix, it will be shown that, if

\[ K(a, b) = \exp\left[ \frac{i}{\hbar} (A x_a^2 + B x_b^2 + h x_a x_b + \alpha x_a + \beta x_b + s) \right] \]  

(A.1)

where \( A, B, h, \alpha \) and \( \beta \) are already known functions of \( t_a, t_b \), then the function \( s(t_a, t_b) \) is uniquely determined from the initial condition and Schrödinger equation. For the system of Hamiltonian given in Eq.(3), the kernel should satisfy the Schrödinger equation:

\[
\frac{i \hbar}{\partial t_b} K = \left[ \frac{1}{2M(t_b)} \left( \frac{h}{i} \frac{\partial}{\partial x_b} \right)^2 - 2a(t_b) \frac{h}{i} x_b \frac{\partial}{\partial x_b} + \frac{M(t_b) c(t_b)}{2} x_b^2 \right. \\
- \frac{b(t_b)}{M(t_b)} \frac{h}{i} \frac{\partial}{\partial x_b} + d(t_b) x_b \left. + \left( \frac{b^2(t_b)}{2M(t_b)} - f(t_b) + i h a(t_b) \right) \right] K, \tag{A.2}
\]

which gives the following differential equations;

\[
\frac{\partial A}{\partial t_b} = - \frac{h^2}{2M(t_b)} \tag{A.3}
\]
\[
\frac{\partial B}{\partial t_b} = - \frac{2B^2}{M(t_b)} + 4a(t_b)B - \frac{M(t_b)c(t_b)}{2} \tag{A.4}
\]
\[
\frac{\partial h}{\partial t_b} = - \frac{2Bh}{M(t_b)} + 2a(t_b)h \tag{A.5}
\]
\[
\frac{\partial \alpha}{\partial t_b} = - \frac{h \beta}{M(t_b)} + \frac{b(t_b)}{M(t_b)} h \tag{A.6}
\]
\[
\frac{\partial \beta}{\partial t_b} = - \frac{2B \beta}{M(t_b)} + 2a(t_b) \beta + 2 \frac{b(t_b)}{M(t_b)} B - d(t_b) \tag{A.7}
\]
\[
\frac{\partial s}{\partial t_b} = - \frac{h}{i} \frac{B}{M(t_b)} - \frac{\beta^2}{2M(t_b)} + \frac{b(t_b)}{M(t_b)} \beta - \frac{b^2(t_b)}{2M(t_b)} + f(t_b) - i h a(t_b). \tag{A.8}
\]

With the explicit expressions of \( A, B, h, \alpha \) and \( \beta \), one may check that Eqs.(A.3-7) are satisfied. For example, in the general quadratic system considered in section IV, \( B \) is given as \( B(t_a, t_b) = \frac{M(t_b)}{2} \frac{\dot{v}_a(t_a)}{v_a(t_a)} + M(t_b)a(t_b) \) which satisfies Eq.(A.4).
\( s(t_a, t_b) \) can be determined from Eq. (A.8) up to a function \( g(t_a, t_0) \):

\[
s(t_a, t_b) = g(t_a, t_0) - \frac{\hbar}{i} \int_{t_0}^{t_b} \frac{B}{M(t)} dt - \int_{t_a}^{t_b} \left[ \frac{\beta^2}{2M(t)} - \frac{b(t)}{M(t)} \beta + \frac{b^2(t)}{2M(t)} - f(t) + i\hbar a(t) \right] dt.
\]  

(A.9)

A wave function \( \psi \) satisfies the integral equation

\[
\psi(x_b, t_b) = \int_{-\infty}^{\infty} K(x_b, t_b; x_a, t_a) \psi(x_a, t_a) dx_a.
\]  

(A.10)

In the limit of \( t_b \to t_a \), the classical actions approaches to

\[
\frac{M(t_a)}{2(t_b - t_a)} (x_b - x_a)^2.
\]

In order that the relation of Eq. (A.10) be satisfied in the limit, the kernel should satisfy the relation:

\[
K(b, a) \to \sqrt{\frac{M(t_a)}{2\pi i \hbar (t_b - t_a)}} \exp\left[ \frac{iM(t_a)}{2\hbar (t_b - t_a)} (x_a - x_b)^2 \right] \text{ as } t_b \to t_a.
\]  

(A.11)

This initial condition determines the \( g \) uniquely.
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The total probability of the wave function in Eq.(3.25), \( \int \psi_n^*(q,t)\psi_n(q,t) dq \), is given as \( \exp\left[-\delta_q q_0^2 + \frac{\delta_{p_0}}{\delta q_0^2}\right] \), and could depend on time unless \( p_0 = \pm |\delta| \hbar q_0 \).

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