Casimir forces between spheres

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Abstract

We discuss the calculation of Casimir forces between a collection of \( N \) dielectric spheres. This is done by evaluating directly the force on a sphere constructed from a stress tensor, rather than an interaction energy. Two- and three-body forces between the spheres are evaluated for setups of two- and three-sphere systems, respectively. An approximate large-\( N \) limit is also obtained for the functional dependence on the number of spheres.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

In a recent set of papers [1–4], Casimir interaction energies have been evaluated for collections of compact objects interacting with different force-carrying fields (electromagnetic and scalar). The approach taken has been to evaluate a suitable energy functional integral using a T-matrix, whereby an interaction energy can be deduced, normalized with respect to their energy when separated at infinity. This is a clean and appealing way of evaluating Casimir energies once the effective action and integration measures are known. See also [5, 6] for related earlier work on spherical geometry issues.

Looking more generally, one may try to evaluate Casimir interactions (forces or energies) with definitions that are not necessarily equivalent. For example, the Minkowski stress tensor is not compatible with the Lorentz force law; it is rather the standard vacuum expression that is compatible. The fact that the definitions lead to different forces means that they are amenable to experimental testing; three-body forces can be measured between dielectric spheres [7] and thus discernible outcomes can be tested for the best candidate theory. Typically the differences will show up in both the scale of the forces and the higher order curvature corrections.

In this paper, I summarize recent work we have done on calculating the Casimir force on a single sphere, in an \( N \)-sphere setup [8]. By using a multiple scattering approach to evaluate essentially the classical Green’s function of the configuration, we are able to evaluate the force directly on the sphere. It is given in terms of Mie scattering coefficients together with translation matrices that map TE and TM modes between different scattering centres. This is calculated explicitly for the case of two- and three-sphere setups (with overall evaluation being performed in Maple). For the case where we have a large number of spheres that interact weakly, we can form a new coupling constant for the perturbative expansions and look at the dependence on \( N \) rather than the details of the configurational setup.

2. The \( N \)-sphere configuration

The problem we are addressing is how to calculate the force on a particular sphere as a result of all the interactions with the remaining spheres in a particular static configuration. Pictorially, the setup we have for the configuration of \( N \) spheres is shown in figure 1. We shall take sphere 1 to be located at the coordinate origin.

The Casimir force on sphere 1 (in the \( j \)-direction) due to the effects of the \( N \)-sphere system of differing material properties is given by

\[
F_j(1|N-1) = \int_{B^2} d^3x \nabla_i T_{ij}(x),
\]

(1)

Here \( B^2 \) is the volume of sphere 1. The stress tensor is given by the standard vacuum expression (which is consistent with the Lorentz force law [9])

\[
T_{ij}(x) = E_i(x)E_j(x) + B_i(x)B_j(x) - \frac{1}{2} \delta_{ij}(|E(x)|^2 + |B(x)|^2),
\]

(2)

where \( x \in B^2 \) and it is understood that we are taking the limit for the initial and final points. We then need to evaluate...
the scattering correlation functions (while dropping the direct modes of propagation), namely,

$$\lim_{y \to x} E_i(x)E_j(y) = \int_0^\infty \int_0^\infty \frac{d\omega \, d\omega'}{(\omega'^2 - \omega^2)} \langle E_i^{\text{in}}(x; \omega)E_j^{\text{in}}(y; \omega') \rangle,$$

and similarly for magnetic fields. To construct the scattering two-point function, we write the fields in a mode decomposition [10] of spherical vector wave functions that are centred on each sphere centre. Then by applying the standard continuity equations at each of the spheres’ surfaces, one can calculate the out modes in terms of the in modes and scattering (Mie) coefficients. Care must be taken when evaluating effectively the classical scattering Green’s tensor for the N spheres. It is necessary to choose the centre of the sphere where the force is being evaluated (so as to be able to evaluate the eigenfunctions on the sphere). Assuming that the background in which we are evaluating this is filled with quantum noise such that the noise–current two-point function is non-zero, we find for the N-body force on sphere 1 (suppressing the SO(3) indices)

$$\mathbf{F}[1|N-1] = (-1)^N \frac{\hbar}{4\pi} R[1]\int_0^\infty \frac{d\omega}{\omega} \coth(\hbar\omega/2k_BT) \times \langle \mathbf{1}[\alpha^\dagger(\omega R[1]) \sum_{i=2}^N A^{1,i}(\mathbf{r}[i, 1], \cdot) \cdot \alpha^\dagger(\omega R[i]) \rangle$$

$$\times \cdots \sum_{j=2}^N A^{j,1}(\mathbf{r}[i, j], \cdot) \cdot \alpha^\dagger \sum_{j=2}^N \nabla_{\mathbf{r}[i, 1]} A^{i,1} \times (\mathbf{r}[j, 1], \cdot) j(kR[1])h^+(kR[1])W(\omega R[1]|I).$$

(4)

Since this expression is rather unwieldy, we now define all the terms that constitute it. The $\alpha^\dagger(\omega R[i])$ are the Mie scattering coefficients in the SO(3) basis for sphere i with radius $R[i]$; they depend on both the dielectric permittivity of the i\textsuperscript{th} sphere and the background, which in turn are functions of frequency. With due care, the limit of perfect conductors can also be obtained (see for instance [1]). $A^{i,j}(\mathbf{r}[i, j])$ are the translation matrices mapping the modes between spheres $i$ and $j$, which depend on the background dielectric permittivity alone via the wave vector $k = \sqrt{\epsilon_\text{B}/\omega/c}$ (see [10] for definitions). The dots indicate that we are forming a continuous chain of scattering events consisting of a scattering coefficient together with the appropriate translation matrix, for all the intermediate scattering events. This is constrained to give an order N polynomial in the scattering coefficients, since we are considering the total N-body force.

The functions $j$ and $h^+$ are the spherical Bessel functions evaluated on the surface of sphere 1 with the asymptotic conditions of regularity at the origin and falling off at infinity, respectively. $W$ is (see [8]) just a factor arising from the four different contributions that constitute the stress tensor in equation (2). The vectors $|I\rangle$ give the truncation in the L angular momentum quantum number. They are $(2L_{\text{cut}} + 1)$-dimensional vectors populated entirely with 1s that one must use to obtain a finite number of terms in the expansion. This enables the multipole-type expansion to be evaluated explicitly. For further details and clarifying remarks, see [8]. Note that the explicit form of the translation matrices involves exponentials of the intersphere separations [8] and thus it is the total path length that plays a key role in understanding the variables of the system.

3. Two- and three-sphere forces

For simple setups we can evaluate equation (4) perturbatively. In the case of two spheres aligned along the z-axis, one can calculate the force at $T = 0$ as a multipole expansion and at $T > 0$ by residues. An example plot is given in figure 2. In

Figure 1. The N-sphere system consists of N dielectric spheres of radii $R[1]$, ..., $R[N]$ each centred on N separate coordinate systems $\Sigma_1$, ..., $\Sigma_N$, all contained in a background dielectric.

Figure 2. A plot of the intersphere retarded force between two dielectric spheres in the empty vacuum at $T = 0\,\text{K}$. Here, the relative dielectric permittivity of the polystyrene spheres is $\epsilon_1 = \epsilon_2 = 2.6$. The angular momentum series representation has been truncated at $L_{\text{cut}} = 3$. 

For simple setups we can evaluate equation (4) perturbatively. In the case of two spheres aligned along the z-axis, one can calculate the force at $T = 0$ as a multipole expansion and at $T > 0$ by residues. An example plot is given in figure 2. In
a similar fashion, when the configuration consists of three spheres we can evaluate the three-body force (where the three 2-body forces have been subtracted). In figure 3, a plot of the potential (derived from the integral of force with respect to the appropriate separation vectors) on one of the spheres as a function of the position of one of the others (the other being held fixed) is given. In both cases the truncation has been done at a very low order of angular momentum as a proof of principle rather than a detailed numerical evaluation. It will be a good truncation provided we are at large separations compared to the sphere radii. As the spheres are moved closer together, it is necessary to consider higher and higher orders of angular momentum.

4. A large $N$ weak coupling limit

Consider the case of $N$ identical spheres arranged in some fashion as $N \to \infty$, $\alpha' = \alpha \to 0$, while $\lambda := N \alpha s$ is held fixed ($\alpha = \alpha_S \omega^3 R^3/c^3$). Also assume that there is some representative separation, $s$, between the spheres and a corresponding orientation. In this case there are $(N - 1)!$ irreducibly connected diagrams that contribute approximately equally to the $N$-body scattering. The expression for the associated potential reduces to (considering the case $T = 0$ to be specific)

$$V[1|N-1] \approx \pm (-1)^N \frac{\hbar c}{N s} (N-1)! \int_0^\infty dX e^{-X} [\alpha \hat{A}(X, s)]^X,$$

where $\hat{A}$ is a simple polynomial in $X$ obtained from the translation matrices $A$ after extracting the exponential prefactor. Taking the limit $N \to \infty$ we can extract the dominant $L = 1$ term, which reduces to (using Stirling’s approximation)

$$V[1|N-1] \sim \pm (-1)^N \hbar c \frac{e^{-N}}{N^3} \lambda N \frac{R^{3N}}{s^{1+3N}}. \quad (6)$$

The use of this formula would be in adding an additional sphere to the system and measuring the resulting oscillation of the force compared to the absolute force before the sphere is added. Alternatively, it could be used for quantifying the error of a truncated series representation for large numbers of weakly interacting particles.

5. Conclusions

In this paper, I have summarized the recent work we have done on calculating Casimir forces between spheres using a multiple scattering approach. This has been done at both zero temperature and finite temperature. The total closed path length being always greater than the respective length scales of the associated radii is what enables the perturbative evaluation to be an accurate approximation. For large $N$ and weakly scattering spheres, we have deduced a scaling-type formula for the force based on an average set of properties for the configuration. This could be useful for forces where we want to understand the behaviour as a function of $N$ rather than the detailed configuration.

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