REMARKS ON PHOTONS AND THE AETHER

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Abstract. We expand upon some topics reviewed and sketched in [15] with more details, embellishments, and some new material of a speculative nature.

1. INTRODUCTION

In the book [15] we discussed many aspects of e.g. electromagnetism (EM), the aether, and the Schrödinger equation (SE) partly in connection with our study of the quantum potential (QP). We now want to examine further the nature of photons and radiation in connection with a putative aether. It is essential that we review some of the background from [15] in order to motivate the aether treatment in Section 6.

2. PHOTONS

We begin with [92] (which is also sketched in [15] in a somewhat different manner) and in Section 5 give a related description following [28]. We will also examine further various points of view concerning the massless Klein-Gordon (KG) equation, the SE, the Maxwell equations (ME), and the quantum vacuum. For background we mention here [5, 13, 15, 27, 32, 36, 37, 38, 43, 44, 46, 51, 47, 48, 49, 50, 56, 57, 62, 63, 64, 65, 70, 74, 78, 79, 80, 85, 86, 87, 91]. One takes massless photons as objects with energy $E$, momentum $P$, and internal angular momentum (or spin) $S$ with $E = c|P|$ and $S \times P = 0$. It is presumed to have velocity $c$ in the direction $k$ and to spin in a plane perpendicular to $k$, which is spanned by two vectors $e$ and $b$ where

\begin{align}
\mathbf{k} \cdot \mathbf{e} &= \mathbf{k} \cdot \mathbf{b} = 0; \quad \mathbf{k} \times \mathbf{e} = \mathbf{b}; \quad \mathbf{k} \times \mathbf{b} = -\mathbf{e}; \quad |\mathbf{e}| = |\mathbf{b}|; \quad \mathbf{e} \cdot \mathbf{b} = 0
\end{align}

One sets $\omega = e = b$ (frequency) and $E = \hbar \omega$ historically (with $|S| = \pm \hbar$) while $\lambda = 2\pi c/\omega$ (which will eventually be identified with a wave length). The photon is considered as following a right of left handed helix generated.

Date: July, 2005.
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by the tip of $e$ where the plane of $e$, $b$ moves along the direction $k$ with velocity $c$. These objects are exhibited via a photon tensor

\[
 f^{\mu\nu} = \begin{pmatrix}
 0 & e_1 & e_2 & e_3 \\
 -e_1 & 0 & b_3 & -b_2 \\
 -e_2 & -b_3 & 0 & b_1 \\
 -e_3 & b_2 & -b_1 & 0 \\
 \end{pmatrix}
\]

which is not a field like the EM tensor $F^{\mu\nu}$ (see Section 5 for more comments on the tensor nature of $f^{\mu\nu}$). The dual tensor is

\[
 f^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} f_{\sigma\rho} = \begin{pmatrix}
 0 & -b_1 & -b_2 & -b_3 \\
 b_1 & 0 & e_3 & -e_2 \\
 b_2 & -e_3 & 0 & e_1 \\
 b_3 & e_2 & -e_1 & 0 \\
 \end{pmatrix}
\]

with $f^{\mu\nu} f_{\mu\nu} = 2(e^2 - b^2) = 0$ and $f^{\mu\nu} f^{*\mu\nu} = -4e \cdot b = 0$. One works here in a Hilbert space $H = H^S \otimes H^K$ with $S \sim S \otimes 1$, $P \sim 1 \otimes P$, and $R \sim 1 \otimes R$. Now spin is colinear with momentum (recall $S \times P = 0$) and the spin eigenstates $\chi_{\pm}$ correspond to helicities $\pm 1$ satisfying

\[
 (k \cdot S) \chi_{\pm} = \pm \hbar \chi_{\pm}
\]

where $k$ is a unit vector in the direction of $P$. The spin operators will be expressed via

\[
 S_x = \hbar \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & -i \\
 0 & i & 0 \\
 \end{pmatrix};
 S_y = \hbar \begin{pmatrix}
 0 & 0 & i \\
 0 & 0 & 0 \\
 -i & 0 & 0 \\
 \end{pmatrix};
 S_z = \hbar \begin{pmatrix}
 0 & -i & 0 \\
 i & 0 & 0 \\
 0 & 0 & 0 \\
 \end{pmatrix}
\]

with $(S_j)_{k\ell} = -i\hbar \epsilon_{j\ell k}$. One must distinguish here $k \in H^K$ and $S \in H^S$; the 2-dimensional spin space is orthogonal to $k$ with $k \cdot S \sim \pm \hbar$ as indicated in (2.4). Now write $\psi_j \in H^S \otimes H^K$ with $j = 1, 2, 3$ denoting components in $H^S$ and set $k = p/|p|$. An operator leaving invariant a photon state $\chi_{\pm}^k \otimes \phi_p$ is $S \cdot P = k \cdot S \otimes |P|$ where $|P| \phi_p = |p| \phi_p$ (with $E = c|p| = c|p|$) and one has

\[
 S \cdot P \chi_{\pm}^k \otimes \phi_p = \pm \frac{\hbar E}{c} \chi_{\pm}^k \otimes \phi_p
\]

(the Hamiltonian is $H = (c/\hbar)S \cdot P$ and a minus sign should be interpreted as positive energy but negative helicity). Then the time evolution of a general photon state is

\[
 i\hbar \partial_t \psi_j = (H)_{jk} \psi_k = \frac{c}{\hbar} (S \cdot P)_{jk} \psi_k
\]
(H is a $3 \times 3$ matrix in $H^S$ whose components are operators in $H^K$). Putting $P = -i\hbar \nabla$ one has a SE for the photon, namely

$$i\frac{c}{\hbar} \partial_t \psi_j(t, \mathbf{r}) = -\epsilon_{jkl} \partial_k \psi_k(t, \mathbf{r})$$

(since $(c/\hbar)(-i\hbar \epsilon_{jkl})(-i\hbar \partial_l) = -c \hbar \epsilon_{jkl} \partial_l$). Note $\hbar$ has disappeared and although this is a QM equation it does not have a classical limit.

**REMARK 2.1.** It is pointed out in [92] that there are conceptual errors in writing $\psi_j = E_j + iB_j$ and deriving the Maxwell equations via $(i/c)\partial_t (E_j + iB_j) = -\epsilon_{jkl} \partial_k (E_k + iB_k)$ in the form

$$\frac{1}{c} \partial_t E_j = -\epsilon_{jkl} \partial_l B_k; \quad \frac{1}{c} \partial_t B_j = \epsilon_{jkl} \partial_l E_k$$

(e.g. $(1/c)(-i\hbar \epsilon_{jkl}(i\hbar \partial_l) = -c \hbar \epsilon_{jkl} \partial_l$). The equations are correct but the derivation is faulty since it identifies the 3-D space of states with the 3-D physical space! ■

**REMARK 2.2.** Using the momentum representation one can write as in [92]

$$\frac{\hbar}{c} \partial_t \psi_j(t, \mathbf{p}) = \epsilon_{jkl} \partial_k \psi_k(t, \mathbf{p})$$

but this is not $\vec{\psi} \times \vec{p}$ because the two vectors belong to different spaces. One can look also at stationary state solutions $\psi_j = \exp[-(i/\hbar)Et] \Phi_{j,E}$ where $(k = \mathbf{p}/|\mathbf{p}|)$

$$\epsilon_{\pm} = \frac{1}{\sqrt{2}} (\mathbf{e} + i\mathbf{b}); \quad \epsilon_{-} = \frac{1}{\sqrt{2}} (i\mathbf{e} + \mathbf{b})$$

where $\mathbf{e} = \omega \mathbf{e}$ and $\mathbf{b} = \omega \mathbf{b}$. Then write

$$\mathbf{e}_+(t) = \left( \frac{\omega}{\sqrt{2}} e^{i\omega t} + c.c. \right); \quad \mathbf{e}_-(t) = \left( \frac{\omega}{\sqrt{2}} e^{-i\omega t} + c.c. \right)$$
One defines then Hermitian operators
\[ H = \sum_s \int d^3 p \sqrt{\omega} \left( i a_s^\dagger (p) \epsilon_s e^{i (p \cdot r - Et)} + h.c. \right); \]

\[ B(r, t) = \frac{1}{2\pi h} \sum_s \int d^3 p \sqrt{\omega} \left( i a_s (p) (k \times \epsilon_s) e^{i (p \cdot r - Et)} + h.c. \right); \]

\[ A(r, t) = \frac{c}{2\pi h} \sum_s \int d^3 p \frac{1}{\sqrt{\omega}} \left( a_s (p) \epsilon_s e^{i (p \cdot r - Et)} + h.c. \right). \]

Then
\[ E = -\frac{1}{c} \partial_t A; \quad B = \nabla \times A; \quad H = \frac{1}{8\pi} \int d^3 r (E^2 + B^2); \]

\[ P = \frac{1}{8\pi c} \int d^3 r (E \times B - B \times E); \quad S = \frac{1}{8\pi c} \int d^3 r (E \times A - A \times E) \]

and one checks the Maxwell equations
\[ \nabla \times E = \frac{1}{c} \partial_t B; \quad \nabla \times B = \frac{1}{c} \partial_t E; \quad \nabla \cdot E = \nabla \cdot B = 0 \]

Thus photons are posited as the fundamental objects and they generate EM fields as a collective manifestation.
Next one defines the “singular” function (cf. [92] for details)

\[
D(\vec{\rho}, \tau) = \frac{-1}{(2\pi\hbar)^3} \int d^3p e^{i\vec{p}\cdot\vec{\rho}} \frac{Sin(\omega\tau)}{\omega} = \\
= \frac{-1}{8\pi^2 c} \left[ \delta(\rho - c\tau) - \delta(\rho + c\tau) \right]
\]

Here \( \rho = |\vec{\rho}| \) where \( \vec{\rho} \sim r_1 - r_2 \) and one can say that \( D(\vec{\rho}, \tau) \) has support on the light cone (cf. also [70]). This leads to

\[
[E_i(r_1, t_1), E_j(r_2, t_2)] = -4\pi i\hbar c^2 \left( \frac{\delta_{ij}}{c^2} \partial_{t_1} \partial_{t_2} + \partial_{r_1,i} \partial_{r_2,j} \right) D(r_1 - r_2, t_1 - t_2)
\]

\[
[B_i(r_1, t_1), B_j(bfr_2, t_2)] = -4\pi i\hbar c^2 \left( \frac{\delta_{ij}}{c^2} \partial_{t_1} \partial_{t_2} + \partial_{r_1,i} \partial_{r_2,j} \right) D(r_1 - r_2, t_1 - t_2)
\]

\[
[E_i(r_1, t_1), B_j(r_2, t_2)] = 4\pi i\hbar c \epsilon_{ijk} \partial_{t_1} \partial_{r_1,k} D(r_1 - r_2, t_1 - t_2)
\]

Note that the singular nature of \( D \) is really unacceptable in QM (e.g. because of the uncertainty principle) and one could conclude that the field strengths are not measurable quantities (cf. the first paper in [92]). On the other hand field averages can be accepted in QM. This is one feature leading to the approach in [92] based on the photon as fundamental. The EM fields are considered essentially as a classical macroscopic ideas and are not “basic”. Such an argument might be extendable quite generally to cast suspicion on many results involving singular behavior or generalized solutions of partial differential equations (distributions). The “classical” theory might require e.g. averaging of dependent variables or some new physics (not necessarily QM) in order to retain any meaning.

One looks next at the expectation values of fields in the quantum state describing a system of photons. For the vacuum described via \( \phi_0 \) one has

\[
(3.11) \quad <\phi_0, E(r, t)\phi_0 > = <\phi_0, B(r, t)\phi_0 > = 0
\]

as expected. However one can show that e.g.

\[
(3.12) \quad <\phi_0, E^2(r, t)\phi_0 > = \frac{2}{(2\pi\hbar)^2} \int d^3p \omega
\]

indicating that there are fluctuations of the electric field in vacuum. For a quantum state of \( n \) photons in the same state with fixed helicity and momentum one has (cf. [92])

\[
(3.13) \quad \phi = \phi_{n(s_1p_1)} = \frac{1}{\sqrt{n!}} (a^\dagger_{s_1}(p_1))^n \phi_0; \quad <\phi, E(r, t)\phi > = <\phi, B(r, t)\phi > = 0
\]
which is somewhat strange. However for an indefinite number of photons in a superposition of states $\psi = \sum_n C_n \phi_{n(s_1 p_1)}$ one has

$$<\psi, E(r, t)\psi> = \frac{\sqrt{\omega_1}}{2\pi \hbar} \left( i \sum_n C^*_n C_{n+1} \epsilon_{s_1} e^{(i/h)(p_1 \cdot r - E_1 t)} + c.c. \right)$$

One concludes here that the EM field of an indefinite number of photons all with the same helicity and momentum is a plane wave with circular polarization. The quantum state where all photons are in the same one photon state of fixed helicity and momentum is a Bose-Einstein condensate (?).

**REMARK 2.1.** We extract here from [74] for a few philosophical observations. The photon, as an elementary “particle” is unique; it is the only elementary particle of energy. A relativistic energy equation should be $E^2 = p^2 c^2 + m_0 c^2 = p^2 c^2$ since the rest mass $m_0 = 0$. In the frame of the moving photon the photon’s energy is stored as rotational (spin) energy where $E = h \nu = h c / \lambda$ with $\nu$ the frequency and $\lambda$ the wave length. Hence the greater the energy the smaller the wave length and one expects to find a lower bound for the wavelength. For a “particle” the angular momentum is $L = mrw$ limited by $L = mrc$ and replacing $L$ by the spin $S$ one has $\hbar = mrc$ where $m$ is a putative mass presumably “generated” by the spin (see here also [92] for toy models with extended energy distributions). Assume the concept of Schwartzschild radius $R$ is valid for the photon where for a black hole $R = 2Gm/c^2$ or $R/m = (2G/c^2)$. The right side of (♣) is a constant but for the photon the radius decreases as the “mass” increases; hence there is a unique value of radius and mass for which a photon can behave as a black hole. Combining (♣) with (♠) one finds $m = \sqrt{hc/2G}$ for the Planck mass, which here is the maximum “pseudomass” permitted for the photon. This corresponds to a maximum energy of $mc^2 = (\sqrt{hc/2G})c^2 = 8.61 \times 10^{22}$ MeV and the highest energy so far observed for a photon is apparently less than this. It is suggested that pair production or photon “splitting” will ensue at the energy limit.

4. THE ZERO POINT FIELD - ZPF

This is a murky subject and essentially involves understanding the quantum vacuum, which of course still retains some mysteries. We gave some hesitant and heuristic comments on ZPF in [15], based on [13, 32, 47, 48, 49, 50, 51, 57, 70, 78, 79, 80, 85, 86], which upon hindsight seem woefully inadequate. Some of this is also summarized and enhanced in a recent paper [85] (first paper). However we go here to the lovely collection of papers by J. Field (see e.g. [39, 40]) for an aperçu of basic physical connections between
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QM, thermodynamics, and special relativity. This will serve as a complement to Sections 2-3. We begin with [39] (first paper) which in a sense follows the spirit of Feynman’s QED where the fundamental concepts of QM are explained in terms of the interactions of photons and electrons. One recalls first the energy momentum vector \( P = m(dX/d\tau) \sim ((E/c), p_x, p_y, p_z) \) with \( X = (ct, x, y, z) \) and \( \tau \) the proper time (time observed in the rest frame). If the inertial frame \( S' \) is moving with uniform velocity \( \beta c \) relative to the frame \( S \) along the common \( x, x' \) axis with \( 0y \) parallel to \( 0y' \) then the 4-vectors as observed in \( S, S' \) are related by Lorentz transform (LT) equations

\[
\begin{align*}
    p'_x &= \gamma(p_x - \beta p_t); \\
    p'_y &= p_y; \\
    p'_z &= p_z; \\
    p'_t &= \gamma(p_t - \beta p_x);
\end{align*}
\]

\( \gamma = \frac{1}{\sqrt{1 - \beta^2}}; \quad p_t = \frac{E}{c} \)

As \( m \to 0 \) \( P \) is still well defined so one has an energy momentum vector say \( P_\gamma = [(E_\gamma/c), (E_\gamma/c)\cos(\phi), (E_\gamma/c)\sin(\phi), 0] \) for a photon of energy \( E_\gamma \) moving in the \( (x, y) \) plane in a direction making an angle \( \phi \) with the \( x \) axis. A plane EM wave will be associated with a large number of photons in general and for such a collection, all with the same 4-vector \( P_\gamma \), one finds from the LT equations an EM wave with

\[
\begin{align*}
    \nu' &= \nu \gamma (1 - \beta \cos(\phi)); \\
    E'_T &= E_T \gamma (1 - \beta \cos(\phi))
\end{align*}
\]

(4.2) (here \( \nu \sim \) frequency and \( E_T \sim \) total energy). Using (4.1) the energies of the photons in the EM wave transform via \( E'_\gamma = E_\gamma \gamma (1 - \beta \cos(\phi)) \). If \( n_\gamma \) is the total number of photons then \( E_T = n_\gamma E_\gamma \) which yields (4.2). Further one sees immediately that \( E_\gamma/\nu = E'_\gamma/\nu' = \text{constant} \) and calling this constant \( h \) one finds that \( E_\gamma = h\nu \) which identifies \( h \) with Planck’s constant. Consequently Planck’s constant arises from consistency between the relativistic kinematics of photons, considered to be massless particles, and the relativistic Doppler effect for classical EM waves. Note also that using \( \lambda = c/\nu \) and \( E_\gamma = h\nu = p_\gamma c \) one arrives at the deBroglie relation \( p_\gamma = h/\lambda = h\nu/c \).

Now from the energy density of a plane EM wave, namely

\[
\rho_W = \frac{E^2 + B^2}{8\pi}
\]

(4.3)

the photon interpretation gives immediately Poynting’s formula for the energy flow \( F \) per unit area per unit time, namely \( F = c\rho_W \) as well as the formula for the radiation pressure \( P_{rad} \) of a plane wave at normal incidence on a perfect reflector, namely \( P_{rad} = 2\rho_W \). Note the number of photons incident is \( F/E_\gamma \) per unit area per unit time so the total momentum transferred is then \( p_\gamma (F/E_\gamma) = F/c = \rho_W \); but a perfect reflector will not absorb energy so an equal number of photons are re-emitted, yielding the factor
of 2. Now consider a plane EM wave of wavelength \( \lambda \) moving in free space parallel to the positive \( x \) direction in the frame \( S \), written as

\[
\text{(4.4)} \quad E_y = E_0 e^{\Phi}; \quad H_z = H_0 e^{\Phi}; \quad \Phi = 2\pi i \frac{(x - ct)}{\lambda}; \quad E_0 = H_0 = A
\]

The time averaged energy density per unit volume \( \bar{\rho}_W \) is \( \bar{\rho}_W = (E_0^2/8\pi) = (H_0^2/8\pi) = (A^2/8\pi) \). Assuming that the wave consists of a beam of photons of energy \( h\nu \) the average number density of photons \( \bar{\rho}_\gamma \) in the wave is \( \bar{\rho}_W / h\nu \) so one gets \( \bar{\rho}_\gamma = A^2/8\pi h\nu \). This is the point where one now leaps across the chasm separating the classical and quantum worlds. First the use of a complex exponential to represent a classical EM wave is convenient but it is really the real or imaginary parts that come into play (Cosines and Sines); for QM the complex exponential is mandatory. Second one uses the definition of wavelength together with \( E_\gamma = h\nu \) and \( p_\gamma = h/\lambda \) to replace in the complex exponential the wave parameter \( \lambda \) by the particle parameters \( E_\gamma \) and \( p_\gamma \). The parameter \( c \) is part of both descriptions (photons and EM waves) and this leads to the complex exponential describing photons in the form

\[
\text{(4.5)} \quad u_p = u_0 e^{\sqrt{2\pi i \frac{h}{\hbar} (p_\gamma x - E_\gamma t)}} = \exp \left[ 2\pi i \frac{h}{\hbar} (P \cdot X) \right]; \quad u_0 = \frac{A}{\sqrt{8\pi E_\gamma}}
\]

In this situation \( \bar{\rho}_\gamma = |u_p|^2 = u_0^2 \) and for the case of very weak EM fields such that \( \bar{\rho}_\gamma << 1 \) it follows that \( |u_p|^2 dV \) can be thought of as the probability that a photon is in the volume \( dV \); for large numbers of photons or strong EM fields this probabilistic interpretation is not appropriate. Note also that one can write

\[
\text{(4.6)} \quad \mathcal{P}_x = -i \hbar \frac{\partial}{2\pi \partial x}; \quad \mathcal{E} = i \hbar \frac{\partial}{2\pi \partial t}; \quad \mathcal{P}_x u_p = p_\gamma u_p; \quad \mathcal{E} u_p = E_\gamma u_p
\]

Note also for \( f \) an arbitrary function on space-time

\[
\text{(4.7)} \quad \mathcal{P}_x (xf) = -i \hbar \frac{\partial (xf)}{2\pi \partial x} = -i \hbar \frac{\partial f}{2\pi} + x \mathcal{P}_x f
\]

Repeated use of (4.6), (4.7), etc. and the relation \( E_\gamma = p_\gamma c \) gives

\[
\text{(4.8)} \quad (c^2 \mathcal{P}_x^2 - \mathcal{E}^2) u_p = \left( \frac{\hbar c}{2\pi} \right) \Box u_p = 0; \quad \Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}
\]

so \( u_p \) will satisfy the Maxwell-Lorentz equation \( \Box u_p = 0 \). If one uses instead of \( E_\gamma = p_\gamma c \) the general energy momentum relation for massive particles \( E^2 = p^2 c^2 + m^2 c^4 \) one can arrive at the Klein-Gordon (KG) equation and expanding \( E \simeq mc^2 + (p^2/2m) + \cdots \) the Schrödinger equation (SE) will result.
Consider next \( p_\gamma \rightarrow h/\lambda \) and \( E_\gamma \rightarrow hc/\lambda \)

\[
\chi = \frac{1}{2}(u_p^0 + u_p^0) = \mathcal{R}(u_p) = u_0 \cos \left[ \frac{2\pi}{\hbar} (p_\gamma x - E_\gamma t) \right] \rightarrow u_0 \cos \left( \frac{2\pi(x - ct)}{\lambda} \right)
\]

This equation is then a bridge back across the chasm from QM to the classical world (cf. (1.5)). Just as the quantum wave function is only meaningful in the limit of very low photon density so the function \( \chi \) is meaningful only in the limit of high photon density. \( \chi \) is not an eigenfunction of either \( E_\gamma \) or \( p_\gamma \) and is a real function. The time average of \( \chi^2 \) is \( 1/2 \) the mean photon density \( \bar{\rho}_\gamma \) and \( \bar{\rho}_W = h\nu \bar{\rho}_\gamma \). In a typical situation \( \bar{\rho}_\gamma \Delta V \) is much larger than 1 and no probabilistic meaning can be attached to it.

We show next following [39] how to derive the Maxwell equations using only Coulomb’s inverse square law, special relativity, and Hamilton’s principle. Thus take two objects \( O_1 \) of masses \( m_i \) and electric charges \( q_i \) with no external forces. The spatial distance separating them in the common center of mass frame is \( x_{12} = x_1 - x_2 \). One constructs a most general Lorentz invariant Lagrangian in a nonrelativistic reference frame via \( (x_i \sim x_i) \)

\[
L(x_1, u_1, x_2, u_2) = -\frac{m_1 u_1^2}{2} - \frac{m_2 u_2^2}{2} - \frac{j_1 \cdot j_2}{c^2 \sqrt{(x_1 - x_2)^2}}
\]

where the \( j_i = q_i u_i \) are current 4-vectors; this is then put into the machinery of Hamilton’s principle so that

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial u_i^\mu} \right) - \frac{\partial L}{\partial x_i^\mu} = 0; \ (i = 1, 2; \ \mu = 1, 2, 3, 4)
\]

Since the Lagrangian is a Lorentz scalar this provides a description of the motion of the \( O_1 \) in any inertial reference frame. Note that if one introduces a 4-vector potential \( A_2 = j_2/c r_{12}, \ r_{12} = |r_{12}| \) the standard Lorentz invariant Lagrangian, describing the motion of \( O_1 \) in the EM field created by \( O_2 \), namely \( L(x_1, u_1) = -(m_1 u_1^2/2) - (1/c)q_1 u_1 \cdot A_2 \), is recovered (and similarly for motion of \( O_2 \) in the field of \( O_1 \)). Now write \( \partial_i = -\partial^1 \equiv (\partial/\partial x^i) \equiv \nabla_i \) and set \( p = m u \) along with

\[
E^i = \partial^i A^0 - \frac{1}{c} \frac{\partial A^i}{\partial t} = \partial^i A^0 - \partial^0 A^i; \ B^k = -\epsilon_{ijk}(\partial^i A^j - \partial^j A^i) = (\nabla \times A)^k
\]

Some calculation (cf. [39]) gives then the 3-D Lorentz force equation and a relativistic Biot-Savart Law in the form

\[
\frac{dp}{dt} = q \left[ \frac{v}{c} \times \mathbf{B} \right]; \ \mathbf{B} = \frac{q_2 \gamma_2 (v_2 \times r)}{c r^3} = \frac{j \times r}{c r^3};
\]

\[
E = \frac{j^0}{c r^3} - \frac{1}{c^2 r^3} \frac{dj_2}{dt} - \frac{j_2 (r \cdot v_2)}{c^2 r^3}
\]
where $r = r_{12}$. The Maxwell equations can be derived immediately from (4.12) along with the Faraday-Lenz law, Ampere’s law, etc. (cf. [39] for details).

Now concerning the ZPF we collect some background information as follows.

1. It seems well established that there is a unique Lorentz invariant spectral energy density in the EM vacuum of the form $\rho(\omega) = \rho_0(\omega) = \frac{h\omega^2}{2\pi^2c^3}$ (cf. [13, 70]). An observer moving with constant velocity in the EM vacuum perceives no force.

2. Following [30, 94] an object undergoing uniform constant acceleration $a$ in the vacuum perceives himself to be immersed in a thermal bath at temperature $T = \frac{h}{k} \sim$ Boltzman constant).

3. One recalls also that there is a zero point energy $(1/2)\hbar\omega$ attached to a quantum harmonic oscillator. Also since there are $(\omega^2/2\pi^2c^3)d\omega$ field nodes per unit volume in the frequency interval $[\omega, \omega + d\omega]$ one obtains the spectral density $\rho_0(\omega) = h\omega^2/2\pi^2c^3$ of Item 1 (cf. [70]).

4. In [13] one derives the Planck radiation law for the blackbody spectrum without the formalism of quantum theory. It is assumed only that (i) There is classical, homogeneous, and fluctuating EM radiation at absolute zero with Lorentz invariant spectrum. (ii) Classical EM theory holds for a dipole oscillator. (iii) A free particle in equilibrium with blackbody radiation has classical kinetic energy $(1/2)kT$ per degree of freedom. This leads then to the zero point energy density shown above and to Planck’s formula

$$\rho(\omega, T) = \frac{\omega^2}{\pi^2c^3} \left[ \frac{h\omega}{e^{(h\omega/kT)}-1} + \frac{1}{2}h\omega \right]$$

If the zero point energy is ignored one obtains the Rayleigh-Jeans formula

$$\rho(\omega, T) = \left( \frac{\omega^2}{\pi^2c^3} \right) kT$$

Here (the quantum number) $\hbar$ arises in (4.14) as a linear factor in calculating the Lorentz invariant spectral density and can later be identified with Planck’s constant (so the derivation is classical).

5. Going again to [13] one finds a lovely discussion involving entropy and energy fluctuations following and modifying Einstein’s arguments. Thus one considers a cavity containing thermal radiation separated into large and small volumes $V$ and $V'$. The energy $U$ of EM radiation in $V'$ between frequencies $\omega$ and $\omega + d\omega$ undergoes spontaneous fluctuations creating a change in the corresponding entropy. Let $\Sigma$ (resp. $S$) be the entropy contributed between $\omega$ and
\( \omega + d\omega \) for \( V \) (resp. \( \mathcal{V} \)). Then for \( \epsilon \) the entropy fluctuation in \( \mathcal{V} \)

\[
S(\epsilon) = \Sigma + \mathcal{S} = \Sigma_0 + \mathcal{S}_0 + (\partial_\epsilon \Sigma + \partial_\epsilon \mathcal{S})\epsilon + \frac{1}{2} \left( \frac{\partial^2 \Sigma}{\partial \epsilon^2} + \frac{\partial^2 \mathcal{S}}{\partial \epsilon^2} \right) \epsilon^2 + \cdots
\]

where \( \Sigma_0, \mathcal{S}_0 \) signify equilibrium entropies where the fluctuation is zero. The first derivatives vanish at \( \epsilon = 0 \) and if \( V >> \mathcal{V} \) one finds \( S(\epsilon) \approx \Sigma_0 + \mathcal{S}_0 + (1/2)(\partial^2 \mathcal{S}/\partial \mathcal{U}^2)\epsilon^2 \). Now there is probabilistic entropy \( (\clubsuit) S_{prob} = (S_{prob})_0 + k \log(W) \) (or \( W = \exp(S_{prob}/k) \)) where \( W \) is the number of microstates giving the same macrostate. There is also caloric entropy \( S_{cal} \) where

\[
dS_{cal} = \frac{dQ}{T}
\]

for reversible processes. Then write

\[
dW = c \exp \left[ \frac{S_{prob}}{k} \right] d\epsilon = \hat{c} \exp \left[ \frac{1}{2k} \partial^2 S_{prob}/\partial U^2 \right] d\epsilon
\]

Some classical argument (cf. [13], paper 2) involving \( \langle \epsilon^2 \rangle = \int \epsilon^2 dW \sim (\pi^2 c^3/\omega^2)\rho^2 \omega, \mathcal{U} = \rho \omega, \) and \( \partial^2 \mathcal{S}_{prob}/\partial \mathcal{U}^2 = -k/\langle \epsilon^2 \rangle \) leads then to \( (\spadesuit) \partial^2 S_{prob}/\partial E^2 = -(k/E^2) \) for average oscillator energy \( E \). Note in fact directly from the definition \( (\clubsuit) \partial S_{prob}/\partial E = k/E \) leading to \( (\spadesuit) \). Now Einstein assumed that \( S_{prob} = S_{cal} \) in \( (\spadesuit) \) and produced \( E = kT \) along with the Planck formula \( (4.14) E = \hbar \omega/\exp(\hbar \omega/kT) - 1 \) (with the zero point term missing). Note here (using \( 4.14 \)) that the average energy of an oscillator is

\[
\langle \epsilon \rangle = \frac{\pi^2 c^3}{\omega^2} \rho(\omega, T) = \frac{\hbar \omega}{\exp(\hbar \omega/kT) - 1} + \frac{1}{2} \hbar \omega = E
\]

Now Boyer modifies Einstein’s argument in a way which recovers the zero point term (and \( 4.18 \)). Indeed he writes \( \langle \epsilon^2 \rangle = \langle \epsilon^2 \rangle_{ZPF} + \langle \epsilon^2 \rangle_{cal} \) and finds that

\[
\frac{\partial^2 S_{cal}}{\partial E^2} = -\frac{k}{E^2 - (\hbar \omega/2)^2}
\]

leading to \( 4.18 \).

5. MORE ON PHOTONS

We go here to \[9, 10, 11, 28, 42, 54, 55, 67, 89\] for some interesting developments concerning the localization of photons and their structure. One knows of course that the methods of quantum field theory (QFT) work for a description of photon activity but we want to examine more direct connections to EM fields, Maxwell’s equations, and wave-particle duality. First from \[89\] one argues that a photon wave function can be introduced if one is willing to redefine in a physically meaningful manner what one wishes to mean by such a wave function. First one introduces a naive single photon.
wave function. Then one produces a second quantized many photon theory approached via many particle physics (which will correspond to the quantization of the free radiation field) and then recovers the naive single photon wave function by looking at the manifold of one photon states. There are apparently connections to the work of [11] to which we don’t have access at the moment.

Now photons can be of positive or negative helicity and being massless one has \( E = cp \) where \( p = |p| \). If one introduces probability amplitudes for photons of momentum \( p \) and helicity \( \pm \), namely \( \gamma_{\pm}(p, t) \) which would be expected to satisfy a Schrödinger type equation (\( \Phi \)) \( i\hbar \partial_t \gamma_{\pm}(p, t) = c p \gamma_{\pm}(p, t) \). Next for each \( p \) introduce two unit vectors \( \hat{e}_i(p) \) where \( \hat{p} = p/|p| \) such that \( \hat{e}_1, \hat{e}_2, \hat{p} \) form a right handed triad (cf. here Section 1). Then define helicity vectors \( \hat{e}_{\pm}(\hat{p}) = \mp (1/\sqrt{2})[\hat{e}_1 \pm i\hat{e}_2] \) and write

\[
\vec{\gamma}_{\pm} = \hat{e}_{\pm} \gamma_{\pm} \quad \text{with} \quad (5.1)
\]

for the probability of detecting a photon of positive helicity and momentum \( p \) between \( p \) and \( p + d p \) (similarly for negative helicity). Note also that

\[
\vec{\gamma}_{\pm} \cdot \vec{\gamma}_{\pm} = 0.
\]

Then define Fourier transforms

\[
(5.2) \quad \Phi_{\pm}(r, t) = \int \frac{dp}{(2\pi\hbar)^{3/2}} \gamma_{\pm}(p, t) e^{i p \cdot r/\hbar}
\]

One then checks that (\( \Phi \)) is satisfied if

\[
(5.3) \quad i\hbar \partial_t \Phi_{\pm}(r, t) = \pm c \hbar \nabla \times \Phi_{\pm}(r, t)
\]

From the assumption that one is dealing with a single photon there results

\[
(5.4) \quad \int [\vec{\gamma}_{\pm} \cdot \vec{\gamma}_{\pm} + \vec{\gamma}_{\pm} \cdot \vec{\gamma}_{\pm}]dp = 1 \equiv \int [\Phi_{\pm} \cdot \Phi_{\pm} + \Phi_{\pm} \cdot \Phi_{\pm}]dr = 1
\]

The dynamical equations (\( \Phi \)) or (5.3) guarantee that if these equations (5.4) are true at one time then they are satisfied at all later times. In fact there results

\[
(5.5) \quad \Phi_{\pm}(r, t) = \int \frac{dp}{(2\pi\hbar)^{3/2}} \gamma_{\pm}(p, 0) e^{-icpt/\hbar} e^{i p \cdot r/\hbar}
\]

There is then a temptation to try and identify the \( \Phi_{\pm} \) as position representation probability amplitudes for photons of positive or negative helicity or perhaps their sum \( \Phi_{\pm} + \Phi_{\pm} \) as a position representation of a photon. However photons are not localizable so this doesn’t work. One way around (cf. [61]) is to show that an operator representing the number of photons in an arbitrary volume \( V \) can be defined but not as the integral over \( V \) of a photon density operator. Another approach (cf. [68]) is to determine an operator representing the number of photons in a volume \( V \) as the integral over \( V \) of a so called detection operator which (when the linear
dimensions of V are large compared to the photon wavelengths) leads to a simple formula for the probability that n photons are present in V (cf. also [28] and Section 5.1 for coarse grained photon density and current density operators). Here one proceeds differently following [89] and looks for a probability amplitude for the photon energy to be detected about \(dr\) of \(r\) in the form \(\Psi^* \cdot \Psi dr\) with normalizations in the sense that

\[
\int \Psi^* \cdot \Psi d\mathbf{r} = \int cp[\gamma_+^* \cdot \gamma_+ + \gamma_-^* \cdot \gamma_-]d\mathbf{p}
\]

To do this one sets \(\Psi = \Psi_+ + \Psi_-\) with

\[
\Psi_\pm(r, t) = \int \sqrt{cp}d\mathbf{p}(2\pi\hbar)^{3/2} \gamma_\pm(p, t)e^{ip \cdot \mathbf{r}/\hbar}
\]

One notes then that \(i\hbar\partial_t\Psi_\pm = \pm c\hbar\nabla \times \Psi_\pm\) and one must satisfy (cf. (5.5)) an initial condition given by the \(t = 0\) case of

\[
\Psi_\pm(r, t) = \int \sqrt{cp}d\mathbf{p}(2\pi\hbar)^{3/2} \gamma_\pm(p, 0)e^{-icpt/\hbar}e^{ip \cdot \mathbf{r}/\hbar}
\]

Note here that \(p\) (the usual photon momentum) and \(r\), the position associated with the photon energy, are not conjugate variables. One then builds up a QFT of the free radiation field via many particle physics (not from a canonical formulation of the EM fields) and this is equivalent to standard canonical quantization. Moreover upon specializing to one photon the energy functions \(\Psi_\pm\) above are recovered. Thus it is reasonable to describe the single photon energy distribution in a region \(dr\) about \(r\) via \(\Psi^*(r, t) \cdot \Psi(r, t) dr\). Further it is shown that in a spontaneous emission process the wave function \(\Psi(r, t)\) generated is a causal field, propagating out from the emitting atom at the speed of light.

Now one goes to [54, 55] where the Bohr photon having a specific size and shape is discussed. This involves a circularly polarized photon being a monochromatic EM traveling wave confined within a circular ellipsoid of length equal to the wavelength (\(\lambda\)) and diameter \(\lambda/\pi\) propagating along the long axis of the ellipsoid. In this model the quantization of the photon’s angular momentum (corresponding to spin \(\hbar\)) arises from an appropriately chosen of Maxwell’s equations and the energy is quantized to be \(\hbar \nu\). In a sense, not entirely clear (cf. [22]), one can think here of an ellipsoidal soliton arising from the imposition of causality upon the solution of the linear Maxwell equations where EM energy \(E^2 + H^2\) integrated over the volume of the ellipsoid equals \(\hbar \nu\) leading to an average intensity within the photon-soliton of \(I_p = 4\pi \hbar c^2/\lambda^4\). The word wavicle is also used in [55]. For a wave traveling with the speed of light parallel to the z-axis the solution of Maxwell’s equations can be any function of \(z - ct\) and if monochromatic
one has a term \( S(z - ct) = \exp[2\pi i(z - ct)/\lambda] \). Setting \( x = r\cos(\phi) \) and \( y = r\sin(\phi) \) in the already separated d'Alembert equation then leads to

\[
\Phi(\phi) \frac{d^2\Phi}{d\phi^2} = \frac{m^2}{r^2} - \frac{1}{r} \frac{dR}{dr} + \frac{1}{r^2} \frac{d^2R}{dr^2} = m^2 \frac{R}{r^2}
\]

where \( m^2 \) is the real separation constant. The simple plane wave solutions with \( m^2 = 0 \) are rejected here since light is observed to travel along very narrow beams and for \( m^2 = 1 \) one has factors of \( r \) or \( 1/r \) with angular factors \( \exp(\pm i\phi) \). This corresponds to angular momentum \( L_z = (\hbar/i) \partial_\phi \) leading to solutions

\[
\psi(r, \phi, z - ct) = (\alpha r + \beta/r) (A e^{i\phi} + Be^{-i\phi}) e^{2\pi i(z - ct)/\lambda}
\]

This yields then

\[
E_z = H_z = 0; \quad E_x = (\alpha r + \beta/r) \left[ A e^{i\phi} + Be^{-i\phi} \right] e^{2\pi i(z - ct)/\lambda} = \mu_0 c H_y;
\]

\[
E_y = i(\alpha r - \beta/r) \left[ A e^{i\phi} - Be^{-i\phi} \right] e^{2\pi i(z - ct)/\lambda} = -\mu_0 c H_x
\]

Imposing causality leads to the result that if \( A \) or \( B \) is zero then the field must be contained within a circular ellipsoid of length \( \lambda \) and cross sectional diameter \( \lambda/\pi \) (cf. [55]). The amplitude is determined by integration of the energy \( E^2 + H^2 \) and the \( 1/r \) term is then discarded to preserve the ellipsoidal shape; there results \( A^2 + B^2 = 1 \) and \( \alpha^2 = 120\hbar h c \pi^2 \lambda^4/\epsilon_0 \lambda^6 \) (in suitable units). In addition one expects an evanescent wave decaying like \( 1/r \) (with \( \alpha = 0 \)) described via

\[
E_r = \frac{\beta}{r} [A + B] = \mu_0 c H_\phi; \quad E_\phi = -i \frac{\beta}{r} [A - B] = -\mu_0 c H_r
\]

where \( \alpha r = \beta/r \) for \( r = \lambda/2\pi \) and \( \beta^2 = (\lambda/2\pi)^4 \times 120n hc \pi^4/(\epsilon_0 \lambda^6) \). The evanescent wave is believed to be responsible for diffraction and interference and some experimental material is sketched.

5.1. PHOTON DYNAMICS. We sketch here from [29] where it is shown that one can define the notions of photon density and photon current density with certain limits. As a trivial example think of geometrical optics where light is treated as an ensemble of point photons moving along definite trajectories with speed \( c \). In the geometric limit the photon number density and current density are perfectly well defined as are the density and current density for any collection of point particles. As a second example one refers to [65] where Mandel defines an operator \( n_V \) representing the number of photons in \( V \) as the integral over \( V \) of the photon density \( D_M(x) = A^\dagger(x) \cdot A(x) \) where

\[
A(x) = L^{-3/2} \sum_{k, \lambda} \bar{\epsilon}_{k, \lambda} a_{k, \lambda} e^{i(k \cdot x - \omega t)}
\]
is the so-called detection operator (here \( \vec{e}, a, \) and \( \omega = ck \) are respectively the polarization unit vector, the annihilation operator and the frequency of a transverse photon of wave vector \( \mathbf{k} \) and polarization \( \lambda (= 1, 2) \) with \( L^3 \) the quantization volume). It is shown that when the linear dimensions of \( V \) are large compared to the photon wavelengths this definition of \( n_V \) yields a simple for the probability \( p_V(n) \) that \( n \) photons are present in \( V \).

It was later shown by Amrein [4] that \( n_V \) agrees with that derived from the theory of [61] and this all has motivated the study in [29] that a coarse grained photon density operator can exist even though a fine grained or microscopic one may not.

Thus one derives a photon density \( D(\mathbf{x}) \) and a photon current density \( C(\mathbf{x}) \) to satisfy

\[
\partial_t D + \nabla \cdot C = 0
\]

(conservation of photons ignoring absorption and emission). These will be defined in terms of vector field operators \( \vec{\psi}(\mathbf{x}) \) and \( \vec{\phi}(\mathbf{x}) \) which will be referred to as the photon field (cf. here also Section 2 again). For a volume \( V \) large compared to the photon wavelengths \( D(\mathbf{x}) \) will correctly predict the number statistics of photons in that volume while for a time interval \([t, t+T]\) long compared to \( \lambda/c \) \( C(\mathbf{x}) \) correctly predicts the statistics of the number of photons that cross the surface \( S \) in time \( T \). This was worked out in the first paper of [28] for a discrete situation and is redone in the second paper in a continuum context; we sketch this here for the free field case and, following [28], show that photon dynamics is a relativistically covariant theory. Thus write

\[
\vec{\psi}(\mathbf{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_{\lambda=1}^{2} \int d^3k \vec{e}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)};
\]

\[
\vec{\phi}(\mathbf{x}, t) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_{\lambda} \int d^3k \left( \frac{\mathbf{k}}{k} \times \vec{e}_\lambda(\mathbf{k}) \right) a_\lambda(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}
\]

where \( a_\lambda(\mathbf{k}) \) is the annihilation operator and \( \vec{e}_\lambda(\mathbf{k}) \) the polarization vector of a transverse photon of wave vector \( \mathbf{k} \) and polarization \( \lambda (= 1, 2) \). Evidently we have the free field equations

\[
\nabla \cdot \vec{\psi} = 0; \quad \nabla \cdot \vec{\phi} = 0; \quad \nabla \times \vec{\psi} + \frac{1}{c} \partial_t \vec{\phi} = 0; \quad \nabla \times \vec{\phi} - \frac{1}{c} \partial_t \vec{\psi} = 0
\]

Then one defines

\[
D = \vec{\psi}^\dagger \cdot \vec{\psi} + \vec{\phi}^\dagger \cdot \vec{\phi}; \quad C = c(\vec{\psi}^\dagger \times \vec{\phi}^\dagger - \vec{\phi}^\dagger \times \vec{\psi})
\]

Evidently (\( \bullet \)) \( \partial_t D + \nabla \cdot C = 0 \) as required. Note that \( D \) is a positive definite operator whose integral over all space is the usual photon number operator

\[
\int d^3x D(\mathbf{x}) = \sum_{\lambda} \int d^3k a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k})
\]
The interpretation of $\mathbf{C}$ as the photon current density is justified by (1) and by a calculation showing that the integral of the inward normal component of $\mathbf{C}$ over the surface of an ideal photon detector equals the counting rate of the detector (cf. [28] first paper). The operators for the number of photons in $V$ and the number of photons crossing a given surface in the time interval $[t, t+T]$ are

$$n_V = \int_V d^3x D(x); \quad n_T = \int_t^{t+T} dt' \int_S d\mathbf{a} \cdot \mathbf{C}(x,t)$$

($\mathbf{n}$ is the unit normal to $S$ in the direction of interest). For a volume $V$ large as described the probability that $V$ contains $m$ photons is 

$$p_V(m) = \text{Tr} [\rho : n^m V \exp(-n_V) : ] / m!$$

where $\rho$ is the density operator of the radiation field and $: :$ means normal ordering. Similarly for sufficiently large $T$ the probability that $m$ photons cross the surface $S$ in time $T$ is 

$$p_T(m) = \text{Tr} [\rho : n^m T \exp(-n_T) : ] / m!$$

where $S$ is the sensitive surface of an ideal photon detector (with one unit quantum efficiency) and $p_T(m)$ is the photon count distribution measured by the detector. For these calculations see the first paper of [28] and [68] ($n_V$ and $n_T$ are treated as number operators).

Now the transverse EM field operators $\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^-$ and $\mathbf{B} = \mathbf{B}^+ + \mathbf{B}^-$ can be expressed via

$$\mathbf{E} = \frac{i}{2\pi} \sum_\lambda \int d^3k (\hbar \omega)^{1/2} \mathbf{e}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)};$$

$$\mathbf{B} = \frac{i}{2\pi} \sum_\lambda \int d^3k (\hbar \omega)^{1/2} \left( \frac{\mathbf{k}}{k} \times \mathbf{e}(\mathbf{k}) \right) a_\lambda(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

and $\mathbf{E}^- = (\mathbf{E}^+)^\dagger$ with $\mathbf{B}^- = (\mathbf{B}^+)^\dagger$. One sees that the photon field vectors in (5.14) are obtained from $\mathbf{E}^+$ and $\mathbf{B}^+$ by multiplying the momentum components of $\mathbf{E}^+$ and $\mathbf{B}^+$ by $-i[4\pi \hbar \omega(\mathbf{k})]^{-1/2}$ which corresponds to a convolution in position space

$$\tilde{\mathbf{\psi}}(\mathbf{x},t) = \int d^3y g(\mathbf{x} - \mathbf{y}) \mathbf{E}^+(\mathbf{y},t); \quad \tilde{\mathbf{\phi}}(\mathbf{x},t) = \int d^3y g(\mathbf{x} - \mathbf{y}) \mathbf{B}^+(\mathbf{y},t)$$

where

$$g(\mathbf{x}) = -\frac{i}{(2\pi)^3} \int d^3k (4\pi \hbar \omega)^{-1/2} e^{-i\mathbf{k} \cdot \mathbf{x}}; \quad \int d^3y g^{-1}(\mathbf{x} - \mathbf{y}) g(\mathbf{y} - \mathbf{z}) = \delta^3(\mathbf{x} - \mathbf{z});$$

$$g^{-1}(\mathbf{x}) = -\frac{i}{(2\pi)^3} \int d^3k (4\pi \hbar \omega)^{1/2} e^{-i\mathbf{k} \cdot \mathbf{x}}$$

One has also

$$\mathbf{E}^+ = \int d^3y g^{-1}(\mathbf{x} - \mathbf{y}) \tilde{\mathbf{\psi}}(\mathbf{y},t); \quad \mathbf{B}^+ = \int d^3y g^{-1}(\mathbf{x} - \mathbf{y}) \tilde{\mathbf{\phi}}(\mathbf{y},t)$$
It is convenient now to express the photon field as a matrix

\[
\psi_{\mu\nu} = \begin{pmatrix}
0 & -\psi_1 & -\psi_2 & -\psi_3 \\
\psi_1 & 0 & \phi_3 & -\phi_2 \\
-\phi_3 & 0 & \phi_1 \\
\phi_2 & -\phi_1 & 0 \\
\end{pmatrix}
\]

Note here the similarity to (2.2) in Section 2 (apparently A. de la Torre was unaware of Cook’s work). An immediate relation to the EM field strength tensor is exhibited via

\[
F^{+\mu\nu} = \begin{pmatrix}
0 & -E_1^+ & -E_2^+ & -E_3^+ \\
E_1^+ & 0 & B_3^+ & -B_2^+ \\
E_2^+ & B_3^+ & 0 & B_1^+ \\
E_3^+ & B_2^+ & -B_1^+ & 0 \\
\end{pmatrix}
\]

One is using coordinates \(x^\mu = (ct, x, y, z)\) and metric \(g^{00} = 1, g^{ii} = -1\) and will raise and lower indices with \(g_{\alpha\beta}\) or \(g^{\alpha\beta}\) as in \(\psi_{\mu\nu}\) were a tensor (it turns out to transform as a tensor under displacements and spatial rotations but not for boosts - cf. [28]). Now define

\[
G(x) = \frac{-i}{(2\pi)^4} \int d^4k [4\pi\hbar\omega(k)]^{-1/2} e^{ikx}
\]

where \(x \sim x^\mu, k \sim k^\mu = (k^0, k)\), \(kx \sim k^\mu x_\mu\), and \(\omega(k) = c|k|\). Clearly \(G\) has an inverse with \(\int d^4y G^{-1}(x - y) G(y - z) = \delta^4(x - z)\) etc. Then one can write

\[
\psi_{\mu\nu}(x) = \int d^4y G(x - y) F^{+\mu\nu}(y); \quad F^{+\mu\nu}(x) = \int d^4y G^{-1}(x - y) \psi_{\mu\nu}(y)
\]

To see that these are equivalent to (5.20) and (5.22) note that since \(\omega(k) = c|k|\) does not depend on \(k^0\) the \(k^0\) integrals can be evaluated immediately to give \(G(x) = \delta(x^0) g(x)\) etc. Although (5.20) and (5.22) were derived from (5.14) and (5.19) for transverse photon and EM fields one assumes that they and hence (5.26) remain valid when the EM fields have a longitudinal component (for the free field this is of no concern). Now one shows that the photon field equations (5.22) are a direct consequence of the free field Maxwell equations

\[
\partial^\nu F^{+\mu\nu} = 0; \quad \partial_\alpha F^{+\beta\gamma} + \partial_\beta F^{+\gamma\alpha} + \partial_\gamma F^{+\alpha\beta} = 0
\]

The second of these equations is a general operator relation following from \(F^{+\mu\nu}_\rho = \partial_\nu A^+_{\rho\mu} - \partial_\mu A^+_{\rho\nu}\) while the first equation is valid in the sense that \(\partial^\nu F^{+\mu\nu} \geq 0\) for all physically admissable states \(|>\) (which are defined as those satisfying the Gupta-Bleuler condition (\(\bullet\)) \(\partial^\mu A^+_{\mu\nu} \geq 0\) - note the free field vector potential satisfies the wave equation \(\partial^\mu \partial_\nu A^+_{\mu\nu} = 0\). Now
\( \frac{\partial \psi_{\mu \nu}(x)}{\partial x^\alpha} = \int d^4y \frac{\partial G(x - y)}{\partial x^\alpha} F_{\mu \nu}^+(y) = - \int d^4y \frac{\partial G(x - y)}{\partial y^\alpha} F_{\mu \nu}^+(y) = \int d^4y G(x - y) \frac{\partial F_{\mu \nu}(y)}{\partial y^\alpha} \)

Neglect of the integrated part is justified via \( G(x) \to 0 \) for \( x \to \infty \) and \( F_{\mu \nu}(x) \to 0 \) at spatial infinity. From (5.27) one has then

\( \partial^\nu \psi_{\mu \nu} = 0; \partial_\alpha \psi_{\beta \gamma} + \partial_\beta \psi_{\gamma \alpha} + \partial_\gamma \psi_{\alpha \beta} = 0 \)

and these are equivalent to the original photon field equations (5.15); again one restricts to the subspace of physical states. Although these have the appearance of tensor equations they are not manifestly covariant since \( \psi_{\mu \nu} \) is not a tensor (cf. Section 2 for comments in this direction). Nevertheless the equations are shown to be invariant under Lorentz transformations because the photon field \( \psi_{\mu \nu} \) is defined in terms of the tensor \( F_{\mu \nu}^+ \) in the same way in each Lorentz frame (see here [28] for details).

Finally one considers the matrix of Hermitian operators

\( N^{\alpha \beta} = \psi_\lambda^\dagger \psi_\lambda^\beta + \psi_\lambda^\dagger \psi_\lambda^\alpha + \frac{1}{2} g^{\alpha \beta} \psi_\lambda^\dagger \psi_\lambda^\mu \psi_{\mu \nu} \)

with is analogous to the EM energy momentum tensor. One checks easily that (♣♣) \( \partial_\beta N^{\alpha \beta} = 0 \Rightarrow M^{\alpha} = \int d^3x N^{\alpha 0} \) is conserved as the photon field develops in time. In fact one can write

\[
N^{\alpha \beta} = \begin{pmatrix}
D & C_1/c & C_2/c & C_3/c \\
C_1/c & S_{11}/c^2 & S_{12}/c^2 & S_{13}/c^2 \\
C_2/c & S_{21}/c^2 & S_{22}/c^2 & S_{23}/c^2 \\
C_3/c & S_{31}/c^2 & S_{32}/c^2 & S_{33}/c^2 \\
\end{pmatrix}
\]

Here (♠♠) \( S_{ij} = c^2[D \delta_{ij} - (\psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i) - (\phi_i^\dagger \phi_j + \phi_j^\dagger \phi_i)] \) is a \( 3 \times 3 \) matrix analogous to the Maxwell stress tensor and (♣♣) now takes the form

\( \partial_t D + \nabla \cdot C = 0; \partial_t C_i + \frac{\partial S_{ij}}{\partial x^j} = 0; \)

\( M^0 = \int d^3x D(x) = \text{const.}; \ C^i = \int d^3xC_i(x = \text{const.}) \)

These equations express the local and global conservation of photons. One shows that \( N^{\alpha \beta} \) does not transform as a tensor (except for coordinate displacements and spatial rotations) and in fact there is no general transformation law relating the components of \( N^{\alpha \beta} \) in different Lorentz frames. Nevertheless (5.32) are covariant since the photon field equations are covariant and \( N^{\alpha \beta} \) is constructed from the photon field in the same way in each frame. In particular the number of photons \( M^0 \) is independent of time.
in each Lorentz frame and considerable calculation also shows that $M^0$ is a scalar (using the condition ($\star$)).

6. SOME SPECULATIONS ON THE AETHER

The aether has been reviewed in [15, 21] to a certain extent and in [15] some speculations were advanced concerning a possible geometry for the aether. These were based on work of [11, 5, 7, 15, 16, 18, 19, 20, 21, 34, 35, 59, 95] and we sketch here some variations and embellishments. First we note from (5.15) that the components $\psi_i$ and $\phi_i$ satisfy the massless KG equation so for analysis of photons one needs 6 components $(\psi_i, \phi_i)$ each satisfying a massless KG equation. However the equations (5.15) are exactly the same as the Maxwell equations (3.7) so one could also imagine introducing a vector $\Psi = (-A, \phi)$ with $A_{\mu\nu} = \Psi_{\mu,\nu} - \Psi_{\nu,\mu}$ to generate the photon equations for a free field with $\Box \Psi = 0$ (see e.g. [82]). In this spirit then one would have a 4-vector $\Psi$ satisfying the massless KG equation to serve as a generator of photon activity. In any event we will think of fields labeled $\psi_i$ for $i = 0, 1, 2, 3$ as characterizing photon dynamics with each component satisfying the massless KG equation. Then we will apply the machinery of $(x, \psi)$ duality of Faraggi-Matone and Vancea (see especially [34, 95]) to express the coordinates $x^\mu$ in terms of the fields $\psi_i$ arising from $\Psi$ (which will be called aether fields); they are seen to be “potential” fields for the photon fields $\psi_i, \phi_i$ of Section 5.1.

As background here we refer to a lovely paper [59] of P. Isaev where he makes conjectures, with supporting arguments, which arrive at a definition of the aether as a Bose-Einstein condensate of neutrino-antineutrino pairs of Cooper type (Bose-Einstein condensates of various types have been considered by others in this context - cf. [25, 33, 60]). The equation for the $\psi$-aether is then a solution of the massless Klein-Gordon (KG) equation (photon equation) $(\hbar^2 \Delta - (\hbar^2/c^2) \partial_t^2) \psi = 0$. This $\psi$ field heuristically acts as a carrier of waves (playground for waves) and one might say that special relativity (SR) is a way of including the influence of the aether on physical processes and consequently SR does not see the aether (cf. here also the idea of a Dirac aether in [23, 24, 31, 76] and Einstein-aether theories as in [33, 60] - this is discussed further in [15]). In the electromagnetic (EM) theory in [59] one looks at $\bar{\psi} = (\phi, \vec{A})$ with $\Box \psi_i = 0$ as the defining equation for a real $\psi$-aether, in terms of the potentials $\phi$ and $\vec{A}$ which therefore define the $\psi$-aether. EM waves are then considered as oscillations of the $\psi$ aether and wave processes in the aether accompanying a moving particle determine wave properties of the particle. Interesting examples involving standing EM waves in a spherical resonator are attributed to oscillations of the $\psi$ aether and references to superconductivity à la Volovik [96, 97] are
indicated.

In [34] Faraggi and Matone develop a theory of $x - \psi$ duality, related to Seiberg-Witten theory in the string arena, which was expanded in various ways in [11, 55, 7, 16, 19, 69, 95]. Here one works from a stationary SE $[-(\hbar^2/2m)\Delta + V(x)]\psi = E\psi$, and, assuming for convenience one space dimension, the space variable $x$ is determined by the wave function $\psi$ from a prepotential $\mathfrak{F}$ via Legendre transformations. The theory suggests that $x$ plays the role of a macroscopic variable for a statistical system with a scaling term involving $\hbar$. Thus define a prepotential $F_E(\psi) = \mathfrak{F}(\psi)$ such that the dual variable $\psi^D = \partial F/\partial \psi$ is a (linearly independent) solution of the same SE. Take $V$ and $E$ real so that $\bar{\psi} = \psi_E(x)$ qualifies and write

$$\partial_x F = \psi_D \partial_x \psi = \left(\frac{1}{2}\right) \left[ \partial_x (\psi D^2 \mathfrak{F}) + W \right]$$

where $W$ is the Wronskian. This leads to $\mathfrak{F} = (1/2)\psi\bar{\psi} + (W/2)x$ (setting the integration constant to zero). Consequently, scaling $W$ to $-2i\sqrt{2m}/\hbar$ one obtains

$$i\sqrt{2m}/\hbar x = \frac{1}{2} \psi D \partial_x \psi - \mathfrak{F} \equiv \frac{i\sqrt{2m}}{\hbar} x = \psi^2 \frac{\partial \mathfrak{F}}{\partial \psi^2} - \mathfrak{F}$$

which exhibits $x$ as a Legendre transform of $\mathfrak{F}$ with respect to $\psi^2$. Duality of the Legendre transform then gives also

$$\mathfrak{F} = \phi \partial_\phi \left( i\sqrt{2m} \frac{x}{\hbar} \right) - \left( i\sqrt{2m} \frac{x}{\hbar} \right) ; \quad \phi = \partial_{\psi^2} \mathfrak{F} = \frac{\psi}{2\psi}$$

so that $\mathfrak{F}$ and $(i\sqrt{2m}x/\hbar)$ form a Legendre pair. In particular one has $\rho = |\psi|^2 = \frac{2i\sqrt{2m}}{\hbar} x + 2\mathfrak{F}$ which also relates $\mathfrak{F}$ and the probability density. In any event one sees that the wave function $\psi$ specifically determines the location of the “particle” whose quantum evolution is described by $\psi$. We mention here also that the (stationary) SE can be replaced by a third order equation

$$4\mathfrak{F}'' + (V(x) - E)(\mathfrak{F}' - \psi \mathfrak{F}'')^3 = 0; \quad \mathfrak{F}' \sim \frac{\partial \mathfrak{F}}{\partial \psi}$$

and a dual stationary SE has the form

$$\frac{\hbar^2}{2m} \frac{\partial^2 x}{\partial \psi^2} = \psi[E - V] \left( \frac{\partial x}{\partial \psi} \right)^3$$

A noncommutative version of this is developed in the second paper of [95].

We mention [70, 78, 79] for some material on the aether and the vacuum and refer to the bibliography for other references. We sketch first some material from [11, 55, 7, 95] which extends the SE theory to the Klein-Gordon (KG) equation. Following [95] take a spacetime manifold $M$ with a metric field $g$ and a scalar field $\psi$ satisfying the KG equation. Locally
one has cartesian coordinates \( x^\alpha (\alpha = 0, 1, \ldots, n - 1) \) in which the metric is diagonal with \( g_{\alpha\beta}(x) = \eta_{\alpha\beta}(x) \) and the KG equation has the form 
\[ (\Box x + m^2)\psi(x) = 0 \] (\( \Box x \sim (\hbar^2/c^2)[(\partial_t^2/c^2) - \nabla^2] \)). Defining prepotentials such that \( \tilde{\psi}(\alpha) = \partial \tilde{\mathcal{S}}(\alpha)[\psi(\alpha)]/\partial \psi(\alpha) \) where \( \psi(\alpha) \) and \( \tilde{\psi}(\alpha) \) are two linearly independent solutions of the KG equation depending on parameters \( x^\alpha \) one has as above (with a different scaling factor)

\[
(6.5) \quad \frac{\sqrt{2m}}{\hbar} x^\alpha \equiv \frac{1}{2} \psi(\alpha) \frac{\partial \tilde{\mathcal{S}}(\alpha)[\psi(\alpha)]}{\partial \psi(\alpha)} - \tilde{\mathcal{S}}(\alpha); \quad [\partial^\alpha \partial_\alpha - V^\alpha] \psi^\alpha = 0
\]

This is suggested in \([34]\) and used in \([95]\); the factor \( \sqrt{2m}/\hbar \) is simply a scaling factor (possibly too stringent here) and it would be more productive to scale \( x^0 \sim ct \) differently or in fact to scale all variables as indicated in \([15]\) (cf. below for a general scaling). Locally \( \tilde{\mathcal{S}}(\alpha) \) satisfies the third order equation

\[
(6.6) \quad 4\tilde{\mathcal{S}}(\alpha)''' + [V(\alpha)(x^\alpha) + m^2](\psi(\alpha)\tilde{\mathcal{S}}(\alpha)''' - \tilde{\mathcal{S}}(\alpha)) = 0
\]

where \( ' \sim \partial/\partial \phi(\alpha) \) and a (quantum) potential \( V^\alpha \) has the form

\[
(6.7) \quad V(\alpha)(x^\alpha) = \left[ \frac{1}{\psi(x)} \sum_{\beta=0, \beta \neq \alpha}^{n-1} \partial^\beta \partial_\beta \psi(x) \right]_{x^\beta \neq \alpha \text{ fixed}}
\]

We go back to \([34]\) now and derive equations for the KG equation with \( m = 0 \) from the beginning (rather than rescaling and then taking \( m \to 0 \)). Further we proceed with more detail and show how a general scaling will involve insertion of some variable factors (cf. also \([16, 20]\) for various scaling factors). Thus consider \((1/c^2)\psi_{\text{tt}} - \Delta \psi = 0 \) with \( x^0 = ct \) and write out explicitly \((i = 1, 2, 3)\)

\[
(6.8) \quad \frac{1}{c^2} \partial^2_t \psi^0 - V^0 \psi^0 = 0; \quad V^0 = \frac{\Delta \psi}{\psi}
\]

\[
\partial^2_t \psi^i - V^i \psi^i = 0; \quad V^i = \left( \frac{1}{c^2} \partial^2_t \psi - \sum_{j \neq i} \partial^2_j \psi \right) / \psi
\]

Here \( V^i \) is thought of as \( V^i(x^i) \) (where in fact \( V^i = V^i(x^i, x^j, x^0) \) with \( j \neq i \) and \( x^0, x^j \) are considered as parameters). Similarly \( V^0 = V^0(x^0) \) (\( \equiv V^0(x^0, x^i) \)). Now e.g. for \( \psi^0 \) and \( \tilde{\psi}^0 \) linearly independent solutions of the first equation in \((6.8)\) one has \( \psi^0_t \tilde{\psi}^0 = \psi^0 \tilde{\psi}^0_t \) which implies

\[
(6.9) \quad W^0(t) = (\psi^0_t \tilde{\psi}^0(t) - \tilde{\psi}^0 \psi^0_t)(t) = 2c\gamma(x^i)
\]

Here, as specified above, \( \tilde{\psi}^0 = \partial \tilde{\mathcal{S}}^0 / \partial \psi^0 \), and

\[
(6.10) \quad \partial_t \tilde{\mathcal{S}}^0 = \tilde{\mathcal{S}}^0 \partial_t \psi^0 = \tilde{\psi}^0 \psi^0_t \Rightarrow
\]
\[
\Rightarrow \frac{1}{2} \partial_t (\psi^0 \tilde{\psi}^0) - \tilde{\psi}^0 \psi_t^0 = \frac{1}{2} (\psi^0 \tilde{\psi}^0_t - \psi_t^0 \tilde{\psi}^0) = \frac{1}{2} W^0 = c_\gamma(x^i)
\]
and consequently one can write

\[
(6.11)
\]

\[
c_\gamma(x^i) t = \frac{1}{2} \psi^0 \partial \tilde{\psi}^0 - \tilde{\psi}^0 \psi_t^0 = \frac{1}{2} W^0 = c_\gamma(x^i)
\]

This leads to (for \(\psi^0 \sim \phi\))

\[
(6.12)
\]

\[
c_\gamma(x^i) = \partial \tilde{c}_\gamma \frac{d\phi}{dt} = \left[ \frac{1}{2} \left( \tilde{\delta}_\phi^0 + \phi \partial^2 \tilde{\delta}_\phi^0 \right) - \tilde{\delta}_\phi^0 \right] \frac{d\phi}{dt} = \frac{1}{2} \left( \phi \partial^2 \tilde{\delta}_\phi^0 - \tilde{\delta}_\phi^0 \right) \frac{d\phi}{dt} = \frac{1}{2} E^0 d\phi \frac{d\phi}{dt}
\]

Similarly we write, using \(6.8\),

\[
(6.13)
\]

\[
\tilde{\psi}^i = \frac{\partial \tilde{\delta}_\phi^i}{\partial \psi^0}; \quad W^i = \psi^i \partial_t \tilde{\psi}^i - \tilde{\psi}^i \partial_t \psi^i; \quad \beta^i(x^0, x^j)x^i = \frac{1}{2} \psi^i \partial \tilde{\delta}_\phi^i - \tilde{\delta}_\phi^i = \mathbf{c}^i
\]

Consequently \((\psi_i \equiv \psi^i)\)

\[
(6.14)
\]

\[
\gamma dx^0 = c_\gamma dt = \frac{1}{2} E^0 d\psi^0; \quad \beta^i dx^i = \frac{1}{2} E^i d\psi^i = \frac{1}{2} \left( \psi^i \partial^2 \tilde{\delta}_\phi^i - \partial \tilde{\delta}_\phi^i \right) d\psi^i
\]

Since \(\partial_t = \partial_\phi (d\phi/dt)\), etc. one can write then

\[
(6.15)
\]

\[
\partial_t = \left( \frac{2c_\gamma}{E^0} \right) \partial_\phi; \quad \partial_i = \left( \frac{2\beta^i}{E^i} \right) \frac{\partial}{\partial \psi^i}
\]

The extraneous variables are considered as parameters when concentrating on one \(x^i\) or \(x^0\) and we note from \(6.11\) or \(6.13\) that \(x^0\) or \(x^i\) can be considered as a function of \(\phi = \psi^0\) or \(\psi^i\) and \(\tilde{\delta}_\phi^i\) is a function of \(\psi^i\) (satisfying ordinary differential equations as in \(6.6\) - with \(m = 0\)). Here \(6.14\)–\(6.15\) represents an induced parametrization on the spaces \(T_P(U)\) and \(T_P^*(U)\) \((P \in U\) - local tangent and cotangent spaces). Now using the linearity of the metric tensor field one sees that the components of the metric in the \(\{ (\psi^\alpha, \tilde{\delta}_\phi^\alpha) \}\) parametrization are \((\beta^0 = c_\gamma)\)

\[
(6.16)
\]

\[
G_{\alpha\sigma}(\psi) = \frac{E^\alpha E^\sigma}{4\beta^\alpha \beta^\sigma} \eta_{\alpha\sigma}(x)
\]

(cf. \[95\]). Now following \[95\] let \(z^\mu (\mu = 0, 1, \ldots, n - 1)\) be a general coordinate system in \(U\) and write the coordinate transformation matrices via

\[
(6.17)
\]

\[
A^\alpha_\mu = \frac{\partial x^\alpha}{\partial z^\mu}; \quad (A^{-1})^\mu_\alpha = \frac{\partial z^\mu}{\partial x^\alpha}
\]

The metric then takes the form

\[
(6.18)
\]

\[
g_{\mu\nu}(z) = \frac{4\beta^\alpha \beta^\sigma}{E^\alpha E^\sigma} A^\alpha_\mu A^\sigma_\nu G_{\alpha\sigma}(\psi)
\]
The components of the metric connection can be computed via

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(z) \sum_\mathcal{P} \epsilon_\mathcal{P} \mathcal{P} \left[ \frac{\partial g_{\sigma\nu}(z)}{\partial z^\mu} \right] \]

where \( \mathcal{P} \) is a cyclic permutation of the ordered set of indices \( \{\sigma\nu\mu\} \) and \( \epsilon_\mathcal{P} \) is the signature of \( \mathcal{P} \). Via the coordinate transformation (6.17) the function \( \psi^\alpha \) depends on all the \( z^\mu \). The metric connection (6.19) can be expressed in the \( \{\psi^\alpha, \bar{\psi}^\alpha\} \) parametrization and in [95] one computes also the components of the curvature tensor, the Ricci tensor, and the scalar curvature and gives an expression for the Einstein equations (we omit the details here). The same procedure apply to our formulas above which leads us to state heuristically

**THEOREM 6.1.** The formulas (6.14), (6.15), (6.16), (6.17), (6.18), and (6.19), and their continuations determine a geometry for a putative aether, expressed in terms of our so-called aether fields \( \psi_i \).

**REMARK 1.1.** These matters are taken up again in [5] for a general curved spacetime and some sufficient constraints are isolated which make the theory work. Also in both papers a quantized version of the KG equation is also treated and the relevant \( x - \psi \) duality is spelled out in operator form. We omit this also in remarking that the main feature here for our purposes is the fact that one can describe spacetime geometry (at least locally) in terms of (field) solutions of a KG equation and prepotentials (which are themselves functions of the fields). In other words the coordinates are programmed by fields and if the motion of some particle of mass \( m \) is involved then its coordinates are choreographed by the fields with a quantum potential eventually entering the picture via (6.7). In [1] a similar duality is worked out for the Dirac field and cartesian coordinates and to connect this with the aether idea one should examine such formulas for \( m \to 0 \).

**EXAMPLE 6.1.** One knows that general solutions of the massless KG equation will have the form \( \psi = \psi(a \cdot x - ct) \) with \( |a| = 1 \). For example take \( \psi = exp(\sum a_i x_i - ct) \) with \( (1/c^2)\psi_{tt} = \psi \) and \( \psi_{ii} = a_i^2 \psi \). This leads to

\[ V^0 = 1; \quad V^i = 1 - \sum_{j \neq i} a_j^2 \]

Hence

\[ \frac{1}{c^2} \partial_t^2 \psi^0 - \psi^0 = 0; \quad \partial_t^2 \psi^i - (1 - \sum_{j \neq i} a_j^2) \psi^i = 0 \]
On the other hand if $\psi = f(a \cdot x - ct)$ one gets

$$V^0 = \left(\frac{f''}{f}\right)(a \cdot x - ct); \quad V^i = \left(1 - \sum_{j \neq i} a_j^2\right)\left(\frac{f''}{f}\right)(a \cdot x - ct)$$

Setting $f''/f = g(x^i, x^0)$ one has

$$\partial_0^2 \psi^0 - g(x^i, x^0)\psi^0 = 0; \quad \partial_0^2 \psi^i - \left(1 - \sum_{j \neq i} a_j^2\right)g(x^i, x^0, x^0)\psi^i$$

Here the $x^i$ or $(x^i, x^0)$ are considered as parameters.

**EXAMPLE 6.2.** Consider a simple situation with two $x^i$ variables and $x^0 = ct$ and take $a_1 = a_2 = 1/\sqrt{2}$. Then $V^0 = 1$ and $V^i = 1 - (1/2) = 1/2$ with

$$\frac{1}{c^2} \partial_t^2 \psi^0 = \frac{1}{2} \psi^0; \quad \frac{\partial^2 \psi^i}{\partial (x^i)^2} = \frac{1}{2} \psi^i$$

Hence we can take

$$\psi^0 = A_0 e^{ct}; \quad \psi^i = A_i e^{(1/\sqrt{2})x^i}; \quad \tilde{\psi}^i = \tilde{A}_i e^{-(1/\sqrt{2})x^i}; \quad \tilde{\psi}^0 = \tilde{A}_0 e^{-ct}$$

Now $\psi^i \tilde{\psi}^i = \kappa_i$ for $i = 0, 1, 2$ so (recall $\beta^0 = \gamma$ and $x^0 = ct$)

$$\beta^i x^i = \frac{1}{2} \kappa_i - \tilde{\beta}^i = \epsilon^i (i = 0, 1, 2)$$

and

$$\frac{1}{2} \epsilon^i = \frac{\partial}{\partial \psi^i} \left(\frac{1}{2} \kappa_i - \tilde{\beta}^i\right) = -\frac{\partial \tilde{\beta}^i}{\partial \psi^i} = -\tilde{\psi}^i (i = 0, 1, 2)$$

(the $\beta^i$ here need not depend on other variables). Consequently one has

$$G_{\alpha\sigma}(\psi) = \frac{E^\alpha E^\sigma}{4\beta^\alpha \beta^\sigma} \eta_{\alpha\sigma}(x) = \frac{\tilde{\psi}^\alpha \tilde{\psi}^\sigma}{\beta^\alpha \beta^\sigma} \eta_{\alpha\sigma}(x)$$

and this exhibits in a simple example the manner in which the metric can depend on the fields.

**EXAMPLE 6.3.** We look now at the more complicated situation for $\psi = f(a \cdot x - ct)$ as in (6.22)-(6.23). Here $f''/f = g$ could be a fairly general function with argument $a \cdot x - ct$ and in the equations $\partial_0^2 \psi^i = \alpha_i g \psi^i$ the function $g_i$ is considered as a function of $x^i$ with the other $x^j$ as parameters. Let $\psi^i$ and $\tilde{\psi}^i$ be two solutions $(i = 0, 1, 2, 3$ say) and look at $(\psi_i \equiv \psi^i)$

$$\epsilon^i = \beta^i x^i = \frac{1}{2} \psi^i \frac{\partial \tilde{\beta}^i}{\partial \psi^i} - \tilde{\beta}^i; \quad E^i = \psi^i \frac{\partial^2 \tilde{\beta}^i}{\partial \psi^i} - \frac{\partial \tilde{\beta}^i}{\partial \psi^i}$$
Recall $\tilde{\psi}^i = \partial F / \partial \psi^i$ and we can write, from Item 3 in Section 2, $\psi^i = \partial F / \partial (\psi^j)^2 = \tilde{\psi}^i / 2\psi^i$ (although this will not be used here). In terms of the two fields $\psi$ and $\tilde{\psi}^i$ one has

$$E^i = \frac{1}{2} \psi^i \tilde{\psi}^i - 3^i = \beta^i x^i; \quad 3^i = 3^i(\psi^i, \tilde{\psi}^i, x^i, \beta^i);$$

(6.30)

In particular $E^i$ is expressed directly in terms of the fields $\psi^i$ and $\tilde{\psi}^i$; no extraneous variables are explicit. Now $\psi^i$ and $\tilde{\psi}^i$ are linearly independent solutions of $\partial^2 \psi^i = \alpha_i g \psi$ but they are linked by a Wronskian

$$W_i = (\partial x \psi^i) \tilde{\psi}^i - \psi^i (\partial x \tilde{\psi}^i) = -2\beta^i$$

where $\beta^i$ does not depend on $x^i$ (only perhaps on the other $x^j$). One can write now

$$\partial_x \left( \frac{\tilde{\psi}^i}{\psi^i} \right) = \frac{W_i}{\psi_i^2} \Rightarrow \tilde{\psi}^i = \psi^i \int^x dx \frac{W_i dx}{\psi_i^2} + c \psi^i$$

(6.31)

Formally this suggests

$$\partial_{\psi^i} \tilde{\psi}^i = -2\beta^i \int^x dx \frac{dx^i}{\psi_i^2} + 4\beta^i \psi^i \int^x dx \frac{dx^i}{\psi_i^3} + c$$

from which follows

$$E^i = \psi^i \left[ -2\beta^i \int^x dx \frac{dx^i}{\psi_i^2} + c + 4\beta^i \psi^i \int^x dx \frac{dx^i}{\psi_i^3} - \frac{2\beta^i dx^i}{\psi^3} \frac{d\psi^i}{d\psi^3} \right] +$$

$$+ 2\beta^i \psi^i \int^x dx \frac{dx^i}{d\psi^i} - c \psi^i = 4\beta^i \psi_i^2 \int^x dx \frac{dx^i}{\psi_i^3} - 2\beta^i \frac{E_i}{2\beta^i} \Rightarrow E^i = 2\beta^i \psi_i^2 \int^x dx \frac{dx^i}{\psi_i^3}$$

(6.33)

Thus $E^i$ can be expressed entirely in terms of the field $\psi^i$. 

One notes here that these arguments and results hold for any $\psi^\alpha$, $V^\alpha$ as in (6.5)-(6.7) so we state heuristically

**THEOREM** 6.2. The objects $E^\alpha$ used in constructing the geometry can be expressed in terms of fields $\psi^\alpha$ as in (6.34).
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