Rigged Hilbert Space Resonances and Time
Asymmetric Quantum Mechanics*

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Abstract

The Rigged Hilbert Space (RHS) theory of resonance scattering and decay is reviewed and contrasted with the standard Hilbert space (HS) theory of quantum mechanics. The main difference is in the choice of boundary conditions. Whereas the conventional theory allows for the in-states $\phi^+$ and the out-states (observables) $\psi^-$ of the $S$-matrix elements $(\psi^-, \phi^+) = (\psi^{\text{out}}, S \phi^{\text{in}})$ any elements of the HS $\mathcal{H}$, $\{\psi^-\} = \{\phi^+\} (= \mathcal{H})$, the RHS theory chooses the boundary conditions: $\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_\times^-$, $\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_\times^+$, where $\Phi_-$ ($\Phi_+$) are Hardy class spaces associated to the lower (upper) half-plane of the second sheet of the analytically continued $S$-matrix. This can be phenomenologically justified by causality. The two RHS’s for states $\phi^+$ and observables $\psi^-$ provide new vectors which are not in $\mathcal{H}$, e.g. the Dirac-Lippmann-Schwinger kets $|E^\pm\rangle \in \Phi_\times^\pm$ (solutions of the Lippmann-Schwinger equation with $\pm i\epsilon$ respectively) and the Gamow vectors $|E_R - i\Gamma/2^\pm\rangle \in \Phi_\times^\pm$. The Gamow vectors $|E_R - i\Gamma/2^-\rangle$ have all the properties that one heuristically needs for quasistable states. In addition, they give rise to asymmetric time evolution expressing irreversibility on the microphyscial level.

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1 Introduction

Resonances and decaying states can really not be understood as autonomous elementary particles in Hilbert space quantum mechanics because the Hilbert space mathematics does not allow state vectors characterized by both an energy \( E_R \), and a lifetime \( \tau \) (or a width \( \Gamma = \hbar / \tau \)). This is in contrast to the way experimentalists analyze their data and list their results as Breit-Wigner peak value and width \((E_R, \Gamma)\) for resonances (large values of \( \Gamma / E_R \)) and as \((E_d, \tau)\) for decaying states if the (Breit-Wigner or Lorentzian) line shape cannot be resolved but the decay rate can be fitted to an exponential (small values of \( \hbar \tau / E_d \)). Since experimental data are finite in number and there are always experimental tolerances and interference with background, one will never be able to say that the experiment has precisely established an ideal Breit-Wigner or an exact exponential. However, Breit-Wigner energy distribution and exponential time evolution have been observed so many times for such an enormous number of different systems in all areas of physics and chemistry that one can safely take them as the defining signature of a quasistable state and attribute observed experimental deviations to the ever present background and calculated mathematical deviations to an unjustified mathematical idealization. For the mathematical physicists this means that they have to provide the suitable mathematical idealization. How and why to do that is the content of our discussion here.

2 The fundamental calculational tools of quantum mechanics

In quantum theory one has states denoted by

\[ \rho \text{ or } W, \text{ or } \phi \text{ for a pure state } \rho = |\phi\rangle\langle\phi|, \]

and one has observables denoted by

\[ A(= A^\dagger), \text{ or } \Lambda, \text{ or } P (P^2 = P), \text{ or } \psi, \text{ if } P = |\psi\rangle\langle\psi|, \text{ for properties}. \]

The vectors \( \phi, \psi \) form a linear scalar product space \( \Phi \). The operators \( \rho, W \) and \( A, \Lambda, P \) are linear operators in \( \Phi \). The space \( \Phi \) is usually called Hilbert space, but it is mostly treated like a pre-Hilbert space. For each “kind” of quantum physical system one takes one particular space \( \Phi \).
In an laboratory experiment the state of the quantum physical system is prepared by a preparation apparatus, e.g. an accelerator. The state $W$ (or $\phi$) is thus experimentally defined (“determined”) by the preparation apparatus. The quantum physical observables are observed or registered by a registration apparatus, e.g. a detector. The observable $\Lambda$ (or $\psi$) or $A$ are thus experimentally defined by the registration apparatus.

In experiments with quantum systems one measures ratios of large integers $N_i/N$ or $N(t)/N$, e.g. as ratios of detector counts of the $i$-th detector $N_i$ and counts of all detectors $N$ or as ratios of detector counts $N(t)$ in the time interval between $t = 0$ and $t = t$ and in “all” time $N = N(\infty)$. This ratio of large numbers is interpreted as probability, e.g. as probability $\mathcal{P}(P_i)$ for a property $P_i$

$$\frac{N_i}{N} \approx \mathcal{P}(P_i) \tag{3a}$$

or as probability for the observable $\Lambda(t)$ at a time $t$

$$\frac{N(t)}{N} \approx \mathcal{P}(\Lambda(t)) \tag{3b}$$

where the observables $P_i$ or $\Lambda$ are experimentally given (“defined”) by the detector. For a more general observable

$$A = \sum_i a_i P_i \tag{4}$$

one obtains the average value (of the eigenvalues $a_i$)

$$\sum_{i=1}^{\text{finite}} a_i \frac{N_i}{N} \approx \sum_{i=1}^{\infty} a_i \mathcal{P}(P_i) \tag{3c}$$

$$\sum_{i=1}^{\text{finite}} \frac{N_i}{N} \approx \sum_{i=1}^{\infty} \mathcal{P}(P_i) = 1.$$ 

The symbol $\approx$ denotes the association of the experimentally measured quantity on the left hand side with the theoretically calculated quantity on the right hand side.

In quantum theory the probability of an observable $\Lambda$ in the state $W$ at time $t$ is calculated as

$$\mathcal{P}(t) = \mathcal{P}(\Lambda(t)) = \text{Tr}(\Lambda(t)W_0) = \text{Tr}(\Lambda_0 W(t)) \tag{5a}.$$
If the state is pure, given by the state vector $\phi$, and if the observable is a property given by the one-dimensional projector $|\psi\rangle\langle\psi|$, or given by the “observable” vector $\psi$, then

$$P(t) = |\langle\psi|\phi(t)\rangle|^2 = |\langle\psi(t)|\phi\rangle|^2.$$  \hfill (5b)

The trace in (5) is calculated using any basis system of the space $\Phi$; either any discrete basis

$$\Phi \ni \phi = \sum_i |i\rangle\langle i| \phi ;$$  \hfill (6a)

or any continuous basis (Dirac basis vector expansion)

$$\Phi \ni \phi = \int d\lambda |\lambda\rangle\langle \lambda| \phi$$  \hfill (6b)

or any basis system consisting of discrete and continuous (generalized) eigenvectors of a complete system of commuting observables. Thus the trace is given by e.g.

$$\text{Tr}(\Lambda W) = \sum_i \langle i|\Lambda W|i\rangle \quad \text{or} \quad \text{Tr}(\Lambda W) = \int d\lambda \langle \lambda|\Lambda W|\lambda \rangle.$$  \hfill (7a)

For the special case (5b) the probability for the observably $\psi$ in the state $\phi$ is given by

$$\text{Tr}(|\psi\rangle\langle\psi|\phi\rangle\langle\phi|) = |\langle\psi|\phi\rangle|^2 = \left| \sum_{i=0}^{\infty} \langle \psi|i\rangle\langle i|\phi \rangle \right|^2,$$  \hfill (7b)

or by

$$= \left| \int d\lambda \langle \psi|\lambda\rangle\langle \lambda|\phi \rangle \right|^2$$  \hfill (7c)

if one uses a continuous basis of Dirac kets $|\lambda\rangle$.

The time evolution in (5a) and (5b) (dynamics of the quantum system)
is given by the Hamiltonian operator $H$ of the system; either as
\[
\frac{\partial W}{\partial t} = \frac{i}{\hbar} [H, W(t)] \quad (8a);
\]
\[
i\hbar \frac{\partial \phi(t)}{\partial t} = H \phi(t) \quad (8b);
\]
\[
\phi(t = 0) = \phi_0 \quad (\text{Schroedinger picture}),
\]
or by:
\[
\frac{\partial \Lambda}{\partial t} = -\frac{i}{\hbar} [H, \Lambda(t)] \quad (8c);
\]
\[
i\hbar \frac{\partial \psi}{\partial t} = -H \psi(t). \quad (8d)
\]
(Heisenberg picture)

None of the above equations is mathematically precise until we define the space $\Phi$, the kets $|\lambda\rangle$ or the integration $d\lambda$ in (6b) and (7c), and specify the initial-boundary conditions $\phi_0$ etc. for the equations (8). Before we choose these mathematical definitions we want to discuss some phenomenological properties of the above quantities which may influence these choices.

## 3 Empirical reasons for time asymmetric quantum mechanics

The time $t$ in (5) and in the operators $\Lambda(t)$ and $W(t)$ is usually allowed to take positive and negative values, i.e. one chooses time symmetric boundary conditions for the Schroedinger and Heisenberg equations of motion (8). This choice is not compelling from an empirical point of view for the following reason.

An obvious expression of causality is that a state $W_0$ (or $\phi_0$) has to be prepared first before an observable $\Lambda(t)$ or $\psi(t)$ can be measured in it. If one calls $t = 0$ the time at which the preparation of the state $W$ (or $\phi$) is completed then the time translation in $\Lambda(t)$ (or $\psi(t)$) in (5) makes physically sense only for $t \geq 0$. The registration counts $N(t)$ in (3b), by which $P(t)$ is measured, can be taken in the future $t > 0$, but not in the past. Thus, $\text{Tr}(\Lambda(t)W)$ will have an experimental counterpart for $t \geq 0$ but not for $t < 0$.

The experimental data for $P(t)$ would give that
\[
P(t) = \text{Tr}(\Lambda(t)W) \approx \frac{N(t)}{N} \quad \text{for } t > 0 \quad (9a)
\]

and
\[
P(t) = \text{Tr}(\Lambda(t)W) \approx 0 \quad \text{for } t < 0 \quad (9b)
\]
because if the detector would click before the state is prepared we would
discount this click as noise.

Often, such as for stationary states and/or time independent observables,
it does not matter at which time $P(t) = \text{Tr}(\Lambda(t)W) = \text{Tr}(\Lambda W(t))$ is calcu-
lated. But one cannot make it a general principle that time evolution of the
observable $\Lambda(t)$ (or of the state $W(t)$) must go into both directions. In fact
the most natural description of the experimental situation (9) would be to
require a time ordering in the probability formula

$$P(t) = \text{Tr}(\Lambda(t)W(0)) \quad t > t_0 = 0$$

(10)

and admit in the mathematical description only time translations of observ-
ables $\Lambda(t)$ relative to the state $W(t_0)$ by an amount $t-t_0 > 0$. This would not
only reflect the experimental situation, which forbids time translation of the
registration apparatus relative to the preparation apparatus to a time $t < t_0$.
It would also incorporate the notion of causality into the mathematical theory
on the operator level.

The time translation is quite different from other transformations of the
space-time symmetry group, e.g. rotations and space-translations. These
symmetry transformations of space-time are experimentally realized as trans-
formations of the registration apparatus (detector) relative to the preparation
apparatus (accelerator). Whereas space translations and rotations of the ap-
paratuses can be performed back and forth, the time at which the detector
is activated can only be shifted into the future not backward past the time
of preparation. Thus time translations of the apparatuses form only a semi-
group whereas rotations and translations of the apparatuses form a group.
Therefore it is natural to represent space translations and rotations by uni-
tary groups of operators in the space of physical states. But it is unjustified
to assume that time translations are also always represented by group oper-
ators in the space of states, i.e. to assume that in quantum mechanics only
states with reversible time evolution exist.

When we make the choices that give a precise mathematical meaning
to the vectors $\phi$ and $\psi$ and the operators $W$, $\Lambda$, $A$, $A^\dagger$ in (1) and (2) we
should therefore not restrict ourselves to those mathematical assumptions
that dictate a unitary group evolution.
4 The Hilbert space idealization has reversible time evolution

Of the vectors \( \phi \), \( \psi \) we have so far assumed that they fulfill the mathematical axioms of a linear space with scalar product \( \langle \phi | \psi \rangle \) and that \( W, \Lambda, \ldots \) are linear operators in this space. In addition, we needed some rules like (6) that permit the calculations in (7); we have not yet said anything about the convergence of infinite sequences and the meaning of integration in (7). In order to prove existence theorems mathematicians need their spaces to be complete (topologically)—they use spaces that also fulfill topological axioms. This gives infinite series, like the ones that occur in (3), (7) and (11) and on the right hand side of (3c), a well defined meaning of convergence. It also defines the meaning of such continuous operator functions of a parameter \( t \) as occur on the right hand side of (3c) and (3b). These topological properties (the definition of open sets) cannot be directly inferred from experimental data as one can see by comparing the right hand side (calculated theoretical quantities) of (3c) and (3b) with the left hand side of (3c) and (3c) (measured experimental quantities). The left hand side of (3c) changes as a function of \( t \) only in integer steps of \( \frac{1}{N} \), whereas the right hand side is presumed to be a continuous function of time. Similarly, the experimental expression on the left hand side of (3c) has a finite sum, in the theoretical expression on the right hand side the sum is infinite. Experiments necessarily involve a finite amount of data. Measured probabilities therefore are only approximate probabilities and cannot supply continuous functions. They cannot determine the meaning of continuous operator functions (like \( \Lambda(t) \)) or continuous vector functions (like \( \psi(t) \)) or the meaning of convergence of infinite sequences as in (3c) or in (7a) or other topological notions. The equality \( \approx \) between experimental and mathematical quantities has a meaning only within certain experimental errors and for large numbers \( N \). Therefore, one has to make some—more or less—arbitrary mathematical idealizations in order to obtain complete mathematical structures, like linear topological spaces and topological algebras of operators.

One such idealization is the Hilbert space (HS). This is obtained by adjoining to the linear scalar product space \( \Phi \) the limit element of infinite converging (Cauchy) sequences, with the Hilbert space convergence (or Hilbert
space topology $\tau_H$) defined by

$$\phi_\nu \xrightarrow{\tau_H} \phi \quad \text{iff} \quad \|\phi_\nu - \phi\| \to 0 \quad \text{for} \quad \nu \to \infty \quad (11)$$

where the norm $\|\phi\|$ is defined by the scalar product $\|\phi\| \equiv \sqrt{\langle \phi, \phi \rangle}$. Thus the infinite sequences of the discrete components $\{\langle i|\psi \rangle\}_{i=1}^\infty$, $\{\langle i|\phi \rangle\}_{i=1}^\infty$ of the vectors $\psi, \phi \in \mathcal{H}$ in (6b) are square summable. The continuous components (wave-functions) $\{\langle \lambda|\psi \rangle = \psi(\lambda) \mid \lambda \in \text{Spectrum of self-adjoint operator}\}$ and $\{\langle \lambda|\phi \rangle = \phi(\lambda) \mid \lambda \in \text{Spectrum}\}$ are square integrable functions. However, the integrals

$$\langle \psi, \phi \rangle = \int d\lambda \overline{\langle \lambda|\psi \rangle} \langle \lambda|\phi \rangle \quad (12)$$

are not Riemann but are Lebesgue integrals. This means the values of the functions $\psi(\lambda) = \langle \lambda|\psi \rangle$ at a particular point (or at all rational numbers) are not defined, which in turn means that the Dirac kets $|\lambda\rangle$ cannot be defined. The Lebesgue integration also makes the interpretation of the probability density $|\langle E|\psi \rangle|^2$ as the energy resolution of the detector for the observable $|\psi\rangle\langle \psi|$ rather unintuitive, at least for those $\psi \in \mathcal{H}$ that do not have a smooth function in the class of Lebesgue integrable functions $\{\langle E|\psi \rangle\}$ which represent $\psi$.

In the (complete) Hilbert space one can define self-adjointness and give the precise meaning of “spectral resolution” to equations like (4). One can make the hypothesis that observables are (not just hermitian or symmetric but) self-adjoint operators and one can prove existence theorems.

One of these existence theorems is the Gleason theorem which states that if $\mathcal{P}(P_i)$ is the function on the set of projectors $\{P_i\}$ which fulfills the axioms of probabilities then there exists a positive trace class operator (density operator) $\rho$ in $\mathcal{H}$ such that $\mathcal{P}(P_i) = \text{Tr}(P_i \rho) = \mathcal{P}_\rho(P_i)$. This operator $\rho$ thus defines the state. Since the converse (for any positive trace class operator $\rho$, $\mathcal{P}_\rho(P_i) \equiv \text{Tr}(P_i \rho)$ fulfills the axioms of probability theory) is

1For $\mathcal{P}(P_i)$ or $\mathcal{P}(\Lambda)$ to be a probability it has to fulfill the axioms of probability theory:

$$\mathcal{P}(P) \geq 0, \quad \mathcal{P}(1) = 1, \quad \mathcal{P}(P_i + P_j) = \mathcal{P}(P_i) + \mathcal{P}(P_j), \quad \{[P_i, P_j] = 0, P_i P_j = 0\).$$
simple to see, one was led to the conclusion \( \text{[8]} \) that there is one to one correspondence between

\[
\text{quantum physical states } \Leftrightarrow \text{ density operators } \rho \quad (13a)
\]

and in particular between

\[
\text{pure quantum physical states } \Leftrightarrow \text{ vectors } \phi \text{ (up to a phase) in } H \quad (13b)
\]

Another existence theorem (Stone-von Neumann operator calculus \( \text{[9]} \)) asserts that the Cauchy problem in quantum mechanics, e.g. in the form of the Schrödinger equation (8b) with the initial condition \( \phi(t = 0) = \phi_0 \in H \) and a selfadjoint Hamiltonian \( H \) with domain \( D(H) \) dense in \( H \), has a solution given by the one parameter group of (strongly continuous) unitary operators \( U^\dagger(t) \) in \( H \):

\[
\phi(t) = U^\dagger(t)\phi_0 \equiv e^{-iHt}\phi_0 \quad -\infty < t < \infty \quad (14a)
\]

where \( \phi(t) \) depends continuously on the initial condition. This means

\[
U^\dagger(t + \tau) = U^\dagger(t)U^\dagger(\tau) , \quad U^\dagger(-t) = (U^\dagger)^{-1}(t) = U(t) \quad (15)
\]

\[
\frac{d^p U(t)}{dt^p} \phi = (-iH)^p U^\dagger(t) \phi = U^\dagger(t)(-iH)^p \phi \quad \text{for } \phi \in D(H) . \quad (16)
\]

The same result holds for the time evolution of the observable vector \( \psi \) in \( H \)

\[
\psi(t) = U(t)\psi_0 = e^{iHt}\psi_0 \quad -\infty < t < \infty . \quad (14b)
\]

In terms of the operators \( W \) and \( \Lambda \) this is given by

\[
W(t) = U^\dagger(t)W_0U(t) = e^{-iHt}W_0 e^{iHt} \quad -\infty < t < +\infty \quad (17a)
\]

\footnote{The operator \( U^\dagger(t) \) is defined in all of \( H \) by the Stone-von Neumann calculus as \( U^\dagger(t) = \int_{E_0} e^{-iEt}dP(E) \) not by the exponential series

\[
e^{-iHt} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(-iH)^n
\]

which is defined (converges) only on a dense subspace \( A \subset D \subset H \) called analytic vectors.}
\( \Lambda(t) = e^{iHt} \Lambda_0 e^{-iHt} \quad -\infty < t < +\infty. \)  \hspace{1cm} (17b)

The conclusion that one draws from these two existence theorems is the following:

The choice to describe states in the HS theory is very restricted and can only be given by a density operator (positive definite trace-class) and (for a pure state) by a vector \( \phi \in \mathcal{H} \). The time evolution of these states must be the time symmetric reversible group evolution (14). There are no states in the HS quantum mechanics of closed systems (which fulfill (8a) or (8b)) which have asymmetric (irreversible) time evolution. The evolution of the quantum mechanical observables is also time symmetric, i.e. the observables \( \Lambda'(t - t_0) \) can evolve relative to the state \( W(t_0) \) by an arbitrary amount \( t - t_0 \geq 0 \) as well as \( t - t_0 \leq 0 \). For every \( U(t_2 - t_1) \) that time translates the observable \( \psi(t_1) \) to the observable \( \psi(t_2) \) there exists also a time translation \( U(t_1 - t_2) = U^{-1}(t_2 - t_1) \) that reverses the transformation.

The time symmetric dynamics (17a) and (17b) that is a mathematical consequence of the Hilbert space idealization (topology \( \tau_\mathcal{H} \) defined by (11)) contradicts the phenomenological time ordering for the observed probabilities (10). This time ordering is expressed by the preparation \( \rightarrow \) registration arrow of time: a state must be prepared first before an observable can be measured in it. It expresses causality in quantum mechanics. Quantum theory in HS is time symmetric and cannot express causality. A consequence of this result is in particular that in the HS theory there cannot exist a state vector \( \phi^D \) (or more general state \( W^D \)) which has been created at any finite time \( t_0 \neq -\infty \) (which we call \( t_0 = 0 \)) and then decays into decay products.

This mathematical consequence is not surprising if one keeps in mind that in the HS theory the \( t \)-evolution–where \( t \) is the relative time between state and observable–is given by a reversible unitary group. One cannot just impose on the group evolution an arbitrary condition like causality, chopping off one half of the theory, and expect that what remains is still a consistent theory.

The reversibility of quantum mechanics in HS and the violation of the causality principle is a consequence of the mathematical idealization given by the topology of the HS, i.e. by (11). It is not a consequence of the

\footnote{Irreversibility in conventional quantum theory is considered to be “non-quantum mechanical” and always thought of as being due to external influences upon “open” systems. It is described by an additional term on the right hand side of (8a) which is not given by the commutator with \( H \) of the quantum system.}
fundamental hypothesis of quantum physics as given by the interpretation (3) and by the dynamical equation (8). We shall therefore return to the quantum mechanical Cauchy problem (8) and modify the boundary conditions \( \phi_0 \in \mathcal{H} \) such that also semigroup time evolution is possible. Mathematically, the simplest modification is to go to strongly continuous semigroups in a Banach space. But since we have already a scalar product \((\ ,\ )\) in the linear space \( \Phi \) (and we need the scalar product to calculate such physical quantities like the probabilities \( |(\phi, \psi)|^2 \)), Banach space completion is no more an option.

5 The Rigged Hilbert Space idealization has irreversible semigroup evolution and Gamow vectors with exponential decay

We shall complete the linear scalar product space \( \Phi \) into a locally convex nuclear space with a topology stronger than the Hilbert space topology given by the scalar product. Specifically we shall choose a countable Hilbert space where the meaning of convergence, i.e. the topology \( \tau_\Phi \) is defined by a countable number of scalar products \((\ ,\ )_p, p = 0, 1, 2, \ldots \) where \((\ ,\ )_{p=0} = (\ ,\ )\) is the scalar product of the HS. Convergence with respect to \( \tau_\Phi \), \( \phi_\nu \xrightarrow{\tau_\Phi} \phi \), means:

\[
\phi_\nu \xrightarrow{\tau_\Phi} \phi \text{ iff } \left\| \phi_\nu - \phi \right\|_p^2 = (\phi_\nu - \phi, \phi_\nu - \phi)_p \to 0 \text{ for } \nu \to \infty
\]

for every \( p = 0, 1, 2, \ldots \).

This topology is stronger than \( \tau_\mathcal{H} \). The countable number of scalar products, i.e. the topology \( \tau_\Phi \), is usually chosen such that the algebra of observables for the physical system under consideration becomes an algebra of \( \tau_\Phi \)-continuous operators. If we denote the \( \tau_\Phi \)-completion of the linear space \( \Phi \) again by \( \Phi \) then we have the two complete topological spaces \( \Phi \subset \mathcal{H} \) with \( \Phi \) dense in \( \mathcal{H} \). Taking in addition the space \( \Phi^\times \) of \( \tau_\Phi \)-continuous antilinear functionals \( F(\phi) \) on \( \Phi \), and the space \( \mathcal{H}^\times \) of \( \tau_\mathcal{H} \)-continuous functionals \( f(h) = (h, f) \) which are given by the scalar product, we obtain the Gelfand triplet or Rigged Hilbert Space (RHS) \[10\]

\[
\Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times.
\]
We shall use the Dirac notation for the $\tau \Phi$-continuous functionals $F(\phi) \equiv \langle \phi|F \rangle$, because $F(\phi)$ is an extension of the scalar product $(h,f)$ to those $F \in \Phi^\times$ which are not in $\mathcal{H}$.

In these RHS’s (one for each kind of quantum physical system) Dirac’s formalism of kets (with a continuous set of eigenvalues) and the continuous basis vector expansion (23) attain a mathematical meaning and the integrals in (24) are Riemann integrals.

These RHS’s also allow for time-asymmetric solutions of the quantum mechanical Cauchy problem (25).

In the RHS formulation one can choose different subspaces of $\mathcal{H}$ to distinguish between states and observables. We call $\Phi^-$ the space that describes the states (called in-states in the scattering experiment) prepared by preparation apparatuses (e.g. accelerator). We call $\Phi^+$ the space that describes the observables (called out-states in scattering theory) registered by the registration apparatuses (e.g. detector). The two subspaces $\Phi^-$ and $\Phi^+$ are not disjoint. The HS formulation, in contrast, does not allow for this mathematical distinction into separate subspaces of states and observables. Thus there is one Hilbert space $\mathcal{H}$ and for each quantum mechanical (scattering) system two dense subspaces $\Phi^\pm$ and therewith two RHS’s

\begin{align*}
\Phi^- \subset \mathcal{H} \subset \Phi^-^\times \text{ with the physical interpretation as in-states and} & \quad (20a) \\
\Phi^+ \subset \mathcal{H} \subset \Phi^+^\times \text{ with the physical interpretation as out-observables}. & \quad (20b)
\end{align*}

Mathematically $\Phi^\pm$ are defined by their realization as function spaces for their energy wavefunctions:

\begin{align*}
\phi^+ & \in \Phi^- \iff \langle E|\phi^+ \rangle \in \mathcal{S} \cap \mathcal{H}^{-2}_{R^+} & \quad (21a) \\
\psi^- & \in \Phi^+ \iff \langle E|\psi^- \rangle \in \mathcal{S} \cap \mathcal{H}^{2+}_{R^+} & \quad (21b)
\end{align*}

where $\mathcal{S}$ denotes the Schwartz space of functions and $\mathcal{H}^2_{R^+}, \mathcal{H}^{2+}_{R^+}$ denotes Hardy class functions in the lower and upper complex plane, respectively. (Here complex half-planes refer to the second Riemann sheet of the $S$-matrix). The $\pm$ in $\langle E \rangle$ refers to the $\pm i\epsilon$ in the Lippmann-Schwinger equation for the eigenkets of $H = H_0 + V$.

The interpretation (20) of the mathematical spaces (21) can be inferred from the preparation $\rightarrow$ registration arrow of time [11].

In addition to the vectors $\phi^+ \in \Phi^- \subset \mathcal{H}$ and the $\psi^- \in \Phi^+ \subset \mathcal{H}$ defined by the experimental apparatus, the RHS formulation also provides elements
outside of $\mathcal{H}$, e.g. Dirac’s scattering states $|p\rangle = |E, \theta_p, \phi_p^\pm\rangle \in \Phi_{\pm}$ and the Gamow (decaying) states (or Gamow kets) $\psi^G = |E_R - i\Gamma/2^-\rangle \in \Phi_+^\times$. A Gamow ket is a generalized eigenvector of a self-adjoint Hamiltonian extended to $\Phi^\times$, with complex eigenvalue $z_R = E_R - i\Gamma/2$, precisely,

$$\langle H\psi^- | E_R - i\Gamma/2^- \rangle \equiv \langle \psi^- | H^x | E_R - i\Gamma/2^- \rangle = (E_R - i\Gamma/2) \langle \psi^- | E_R - i\Gamma/2^- \rangle$$

for all $\psi^- \in \Phi_+$ (the space of observed decay products). Here $E_R$ represents the resonance energy and $\Gamma$ the width of the Breit-Wigner energy distribution. The superscript $-$ in $|E_R - i\Gamma/2^-\rangle$ is inherited from the kets of the Lippmann-Schwinger equation, the subscripts on the corresponding space $\Phi_+^\times$ from the mathematicians’ notation for Hardy class functions (21). Scattering theory in physics and Hardy class functions in mathematics were developed independently of each other. Except for this discrepancy in the notation for the labels $\mp$, the Hardy class spaces $\Phi_+$ provide an excellent mathematical image for prepared states $\{\phi^+\}$ and the spaces $\Phi_+$ for the registered observables $\{\psi^-\}$ in the quantum theory of scattering and decay. This match is a wonderful example of what Wigner called “the miracle of the appropriateness of the language of mathematics for the formulation ... of physics” [12].

The Gamow kets have asymmetric time evolution that obeys an exact exponential law [3]

$$\psi^G(t) = e^\mp iH^x t |E_R - i\Gamma/2^-\rangle = e^{-iE_R t} e^{-\Gamma t/2} |E_R - i\Gamma/2^-\rangle ,$$

for $t \geq 0$ only. (23a)

The Gamow kets are solutions of the Schroedinger equation (8b) but do not fulfill the Hilbert space boundary condition. Instead they fulfill the time asymmetric boundary condition $\psi^G(t = 0) \in \Phi_+^\times$. Whereas the first part of (23a) can be formally verified from (22), the derivation of the time asymmetry $t \geq 0$ is highly non-trivial and requires specific properties of Hardy class functions. The semigroup $e^\mp iH^x t$ is only defined for positive values of the time, $t \geq 0$, ($H^x$ is the operator $H^\dagger$ extended into $\Phi^\times$ defined by the first equality of (22)). There is another semigroup $e^{-iH^x t}$, $t \leq 0$ and another Gamow ket $\tilde{\psi}^G = |E_R + i\Gamma/2^+\rangle \in \Phi_-^\times$ with the asymmetric time evolution

$$\tilde{\psi}^G(t) = e^{-iH^x t} |E_R + i\Gamma/2^+\rangle = e^{-iE_R t} e^{+\Gamma t/2} |E_R + i\Gamma/2^+\rangle ,$$

for $t \leq 0$ only. (23b)
The Gamow vectors are defined from the pole term of the analytically continued $S$-matrix at the resonance position at $z_R = E_R - i\Gamma/2$ (and at $z_R = E_R + i\Gamma/2$ for (23b)) in the second Riemann sheet. From this one obtains, using the Hardy class property, the Breit-Wigner energy distribution for their wave function [3, 13]

$$\langle -E|\psi^G \rangle = i\sqrt{\Gamma/2\pi} \frac{1}{E - (E_R - i\Gamma/2)}, \quad -\infty_{II} < E < +\infty.$$  \hspace{1cm} (24)

The variable $E$ extends over the physical values (upper rim of positive real axis first sheet = lower rim of positive real axis second sheet) and from $-\infty_{II}$ to 0 in the second sheet. This is an idealized Breit-Wigner in contrast to the standard Breit-Wigner for which $0 \leq E < \infty$.

The results (22) and (23) are derived from the pole term definition using the properties of the Hardy spaces (21) [3, 13].

If there are $N$ resonances in the system, each occurring as a pole of the $j$-th partial $S$-matrix at the positions $z_{R_i} = E_{R_i} - i\Gamma_i/2$, then one obtains $N$ Gamow vectors $\psi^G_i$.

The Gamow vectors $\psi^G_i$ are members of a “complex” basis vector expansion [13]. In place of the well known Dirac basis system expansion (6b) given for the Hamiltonian $H$ by

$$\phi^+ = \int_{0}^{+\infty} dE|E^+\rangle\langle +E|\phi^+ \rangle$$  \hspace{1cm} (25)

(where a discrete sum over bound states has been ignored), every state vector $\phi^+ \in \Phi_-$ can be expanded as

$$\phi^+ = -\sum_{i=1}^{N} |\psi^G_i \rangle \langle \psi^G_i |\phi^+ \rangle + \int_{-\infty_{II}}^{0} dE|E^+\rangle\langle +E|\phi^+ \rangle$$  \hspace{1cm} (26)

(where $-\infty_{II}$ indicates that the integration along the negative real axis (or other contours) is in the second Riemann sheet of the $S$-matrix). The “complex” basis vector expansion (24) is rigorous. This allows us to mathematically isolate the exponentially decaying states $\psi^G_i$. It also allows us an easy approximation by omitting the background integral in (26), and just using

$$\phi^+ = \sum_{i=1}^{N} |\psi^G_i \rangle c_i, \quad c_i = -\langle \psi^G_i |\phi^+ \rangle.$$  \hspace{1cm} (27)
Then one obtains the “effective” theories with finite complex Hamiltonian. For instance, for the $K_L^0 - K_S^0$ meson system with $N = 2$,

$$\phi^+ = \psi^G_S b_S + \psi^G_L b_L$$

(28)

and one obtains the Lee-Oehme-Yang theory [14]. The finite dimensional approximations (27) have been successfully applied to many areas of physics, in particular to nuclear physics [15, 2], which shows that to isolate the Gamow states can be a good approximation.

Since (23a) implies the exponential law for the decay rate $\dot{P}_\eta(t) = \Gamma_\eta e^{-\Gamma t}$ the width $\Gamma$ of the Breit-Wigner distribution (24) and the lifetime fulfill the exact relation $\tau = \frac{\hbar}{\Gamma}$. This has not been obtained before as an exact, precisely derived relation, though it has always been assumed on the basis of some “approximate derivations” [16].

Gamow vectors are ideally suited to describe resonances (the pair (23a) and (23b)) in a scattering process or quasistable particles (23a) that decay. Like the Dirac-Lippmann-Schwinger kets $|E^\pm\rangle$, from which they are constructed, they do not describe interaction-free in- or out- asymptotic states. In a theory that allows only asymptotic particles, they are therefore not admitted. Gamow states have all the properties that heuristically the unstable states need to possess. In addition and unintended they give rise to an asymmetric time evolution, which may not have been wanted but is in agreement with the empirical principle of causality.

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