Finite Canonical Measure for Nonsingular Cosmologies

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Abstract

The total canonical (Liouville-Henneaux-Gibbons-Hawking-Stewart) measure is finite for completely nonsingular Friedmann-Lemaître-Robertson-Walker classical universes with a minimally coupled massive scalar field and a positive cosmological constant. For a cosmological constant very small in units of the square of the scalar field mass, most of the measure is for nearly de Sitter solutions with no inflation at a much more rapid rate. However, if one restricts to solutions in which the scalar field energy density is ever more than twice the equivalent energy density of the cosmological constant, then the number of e-folds of rapid inflation must be large, and the fraction of the measure is low in which the spatial curvature is comparable to the cosmological constant at the time when it is comparable to the energy density of the scalar field.

The measure for such classical FLRWA-φ models with both a big bang and a big crunch is also finite. Only the solutions with a big bang that expand forever, or the time-reversed ones that contract from infinity to a big crunch, have infinite measure.
1 Introduction

Starting with the Louiville measure, the procedure of Henneaux [1] and, in more detail, of Gibbons, Hawking, and Stewart [2], provides a natural canonical measure on the set of classical universes. For a minisuperspace model of a $k = +1$ Friedmann-Lemaître-Robertson-Walker (FLRW) geometry (homogeneous and isotropic three-sphere spatial sections) with a time-dependent scale size $a(t)$ and a single minimally coupled homogeneous massive scalar field $\phi(t)$, the total measure is infinite, and Hawking and I showed [3] that all but a finite measure have arbitrarily small spatial curvature $1/a^2$ (or arbitrarily large size $a$) at any fixed positive value of the energy density of the scalar field. However, we also showed [3] that both the set of inflationary solutions and the set of noninflationary solutions have infinite measure, with an ambiguous ratio: the Louiville-Henneaux-Gibbons-Hawking-Stewart canonical classical measure gives an ambiguous prediction for the probability of inflation.

Gibbons and Turok [4] sought to remove this ambiguity by identifying universes in which the spatial curvature $1/a^2$ is too small to be distinguished, though in another way of looking at it, it is not clear why one should be justified in identifying universes that have scale sizes $a$ (and hence also spatial volumes that are proportional to $a^3$) that are so large and different. Indeed, Turok has abandoned this approach and is working on another [5]. In both of these procedures, one gets a classical probability of inflation that goes approximately inversely with the volume expansion factor during inflation, i.e., roughly as $\exp(-3N)$, where $N$ is the number of e-folds of inflation [4, 5].

Here a very simple alternative restriction of the set of classical solutions is considered that does not depend on any identifications of what might not be observationally distinguished and does not depend on any arbitrary choice of finite ranges. In particular, I consider the set of classical FLRW solutions that are completely nonsingular, with neither a big bang nor a big crunch. For FLRW solutions with a homogeneous minimally coupled massive scalar field and with a cosmological constant, nonsingular solutions with positive canonical measure occur only for the $k = +1$ FLRW geometries (allowing $a(t)$ to have a minimum value) and for a positive cosmological constant $\Lambda$ (allowing the universe to expand forever in both directions of time). (There is an uncountable set of perpetually bouncing solutions for a homogeneous massive scalar field minimally coupled to a $k = +1$ FLRW geometry with $\Lambda = 0$ [6], but this apparently fractal set is discrete and so has zero canonical measure.) This restriction of positive measure for nonsingular solutions to $k = +1$ and $\Lambda > 0$ is also true for any homogeneous minimally coupled scalar field with a canonical kinetic term and a potential term with one single extremum that is a minimum of zero value, but here we shall focus on the homogeneous free massive inflaton scalar field $\phi(t)$ of mass $m$, taking the cosmological constant to be $\Lambda \equiv 3/b^2 \equiv 3m^2\lambda > 0$ with a characteristic length scale $b$ and a dimensionless rescaled cosmological constant $\lambda$.

This paper shows that the set of such nonsingular classical FLRW universes has finite canonical measure. Therefore, if one restricts to such cosmologies with neither a big bang nor a big crunch, one can get unambiguous finite fractions for any subsets one chooses.
2 Friedmann-Lemaître-Robertson-Walker closed model with massive scalar field and cosmological constant

The Friedmann-Lemaître-Robertson-Walker (FLRW) closed model with massive scalar field and possibly also with a cosmological constant has been analyzed many times previously \([7, 8, 9, 6, 10, 11, 12, 13, 14, 15, 16]\), so many of the dynamical equations I shall give below have been previously given, along with much of the qualitative behavior.

The \(k = +1\) FLRW spacetime metric is

\[
ds^2 = -N^2(t)dt^2 + A^2(t)d\Omega^2_3 = \frac{1}{m^2}(-n^2(t)dt^2 + a^2(t)d\Omega^2_3),
\]

where \(N(t)\) is the lapse function, \(A(t)\) is the physical scale size (which is what is usually called \(a(t)\), but I am reserving that for the rescaled scale size), \(d\Omega^2_3\) is the metric on a unit 3-sphere that has volume \(2\pi\), \(n(t) \equiv mN(t)\) is a rescaled lapse function that is dimensionless if \(t\) is taken to be dimensionless, and \(a(t) \equiv mA(t)\) is a rescaled scale size that is also dimensionless.

Using units in which \(\hbar = c = 1\), but writing Newton’s constant \(G \equiv m_{\text{Pl}}^2\) or the Planck mass \(m_{\text{Pl}} \equiv G^{-1/2} \equiv \sqrt{\hbar c/G}\) explicitly, the Lorentzian action is (cf. \([6, 16]\), but note that here I am using \(A(t)\) for the physical scale factor and \(a(t)\) for the dimensionless rescaled scale factor, unlike the \(a(t)\) and \(r(t)\) used in \([16]\) for those two respective quantities)

\[
S = \int N dt 2\pi^2 A^3 \left \{ \frac{3}{8\pi G} \left[ -\left( \frac{1}{NA} \frac{dA}{dt} \right)^2 + \frac{1}{A^2} - \frac{A}{3} \right] + \frac{1}{2} \left( \frac{1}{N} \frac{d\phi}{dt} \right)^2 - \frac{1}{2} m^2 \varphi^2 \right \}
\]

\[
= \frac{3\pi}{4G} \int N dt A^3 \left\{ -\left( \frac{1}{NA} \frac{dA}{dt} \right)^2 + \left( \frac{1}{N} \frac{d\phi}{dt} \right)^2 + \frac{1}{A^2} - \frac{1}{b^2} - m^2 \varphi^2 \right \}
\]

\[
= \frac{1}{2} \left( \frac{3\pi}{2Gm^2} \right) \int ndt a^3 \left\{ -\left( \frac{1}{na} \frac{da}{dt} \right)^2 + \left( \frac{1}{n} \frac{d\varphi}{dt} \right)^2 + \frac{1}{a^2} - \lambda - \varphi^2 \right \}
\]

\[
= \frac{1}{2} S_0 \int ndte^{3\alpha} \left[ -n^{-2}(\alpha^2 - \varphi^2) + e^{-2\alpha} - \lambda - \varphi^2 \right]
\]

\[
= \frac{1}{4} S_0 \int dt \left\{ -\ddot{n}^{-1} \dot{\varphi} + \ddot{n} \left[ 1 - \left( \lambda + \frac{1}{16} \ln^2 \frac{v}{u} \right) \varphi^2 \right] \right \}
\]

\[
= \frac{1}{2} S_0 \int dt \left\{ -\left( \frac{4}{9} n^{-1} \dot{U} \dot{V} + n(UV)^{1/3} \right) \left[ 1 - \left( \lambda + \frac{1}{9} \ln^2 \frac{V}{U} \right) (UV)^{2/3} \right] \right \}
\]

\[
= \frac{1}{2} \int ndt \left[ \left( \frac{1}{n} \frac{ds}{dt} \right)^2 - \dot{V} \right] = \frac{1}{2} \int dt \left[ \frac{1}{\ddot{n}} \left( \frac{d\dot{s}}{dt} \right)^2 - \ddot{n} \right],
\]

where \(b \equiv \sqrt{3/\Lambda}\) is the radius of the throat of pure de Sitter with the same value
of the cosmological constant, \( \lambda \equiv \Lambda/(3m^2) \equiv 1/(mb)^2 \) is a dimensionless measure of the cosmological constant in units given by the mass of the inflaton, \( a \equiv e^{\alpha} \equiv mA \equiv (uv)^{1/4} \equiv (UV)^{1/3} \) and \( \varphi \equiv \sqrt{4\pi G/3\dot{\phi}} \equiv (1/4) \ln (v/u) \equiv (1/3) \ln (V/U) \) are dimensionless forms of the scale factor and inflaton scalar field, \( u = e^{2\alpha - 2\varphi} = a^2e^{-2\varphi} \) and \( v = e^{2\alpha + 2\varphi} = a^2e^{2\varphi} \) are a convenient choice of null coordinates on the minisuperspace (see, e.g., \[19\]), \( U = u^{3/4} = e^{(3/2)\alpha - (3/2)\varphi} = a^{3/2}e^{-2\varphi} \) and \( V = v^{3/4} = e^{(3/2)\alpha + (3/2)\varphi} = a^{3/2}e^{2\varphi} \) are an alternative choice of null coordinates, an overdot represents a derivative with respect to \( t \), \( S_0 \equiv (3\pi)/(2Gm^2) = (3\pi/2)(m_{Pl}/m)^2 \), the DeWitt metric \[20\] on the minisuperspace is

\[
\frac{V}{a^2} = \frac{3}{4} S_1 \equiv \frac{1}{(3 \pi)^2} \ln \left( \frac{3}{2} \varphi \right), \]

the ‘potential’ on the minisuperspace is

\[
\hat{V} = S_0 e^{3\alpha} (\varphi^2 + \lambda - e^{-2\alpha}) = -S_0(uv)^{1/4} \left[ 1 - \left( \lambda + \frac{1}{16} \ln^2 \frac{u}{v} \right) \sqrt{uv} \right],
\]

alternative rescaled lapse functions are \( \bar{n} = 2(uv)^{1/4} = 2an = 2m^2AN \) and \( \hat{n} \equiv n\hat{V} = mn\hat{V} \), and the conformal minisuperspace metric is

\[
d s^2 = S_0 e^{3\alpha} (-d\alpha^2 + d\varphi^2) = -\frac{1}{4} S_0(uv)^{-1/4} dudv = -\frac{4}{9} S_0 dU dV,
\]

The null coordinates \( u \) and \( v \) are chosen so that as one approaches the null boundaries of the minisuperspace, at \( u \geq 0 \), \( v = 0 \) where \( a = 0 \) with \( \varphi = -\infty \) (except at \( u = v = 0 \), where \( \varphi \) can have any value), and at \( u = 0 \), \( v \geq 0 \) where \( a = 0 \) with \( \varphi = +\infty \) (again except at \( u = v = 0 \)), the conformal minisuperspace metric Eq. (5) approaches \( (1/4)S_0^2 dudv \), so that \( u \) and \( v \) are \( 2/S_0 \) times null coordinates that are the local analogues of orthonormal Minkowski coordinates near the boundaries.

The Hamiltonian constraint equation and independent equation of motion can now be written as

\[
\left( \frac{1}{Na} \frac{dA}{dt} \right)^2 = \left( \frac{1}{N} \frac{d\varphi}{dt} \right)^2 + m^2 \varphi^2 + \frac{1}{b^2} - \frac{1}{A^2},
\]

\[
\frac{1}{N} \frac{d}{dt} \left( \frac{1}{N} \frac{d\varphi}{dt} \right) + \left( \frac{3}{NA} \frac{da}{dt} \right) \left( \frac{1}{N} \frac{d\varphi}{dt} \right) + m^2 \varphi^2 = 0,
\]

for general lapse function from the second form of the action above,

\[
\dot{a}^2 = a^2(\dot{\varphi}^2 + \varphi^2 + \lambda) - 1,
\]

\[
\dot{\varphi} + 3\frac{a^2}{\dot{a}} \dot{\varphi} + \varphi = 0,
\]
from the third form of the action with \( n = 1 \), which will henceforth be assumed unless otherwise indicated (e.g., by including the lapse \( N \) or \( n \) explicitly in a formula),

\[
\dot{\alpha}^2 - \dot{\varphi}^2 = \varphi^2 + \lambda - e^{-2\alpha}, \\
\ddot{\varphi} + 3\dot{\alpha}\dot{\varphi} + \varphi = 0,
\]

for the fourth form of the action, and

\[
\dot{u}\dot{v} = -4\sqrt{uv}\left[1 - \left(\lambda + \frac{1}{16}\ln^2\frac{v}{u}\right)\sqrt{uv}\right], \\
\frac{\ddot{U}}{U} - \frac{\ddot{V}}{V} = \ln \frac{V}{U},
\]

for the minisuperspace null coordinates \( u = e^{2\alpha - 2\varphi} \) and \( v = e^{2\alpha + 2\varphi} \), and the alternative null coordinates \( U = u^{3/4} = e^{(3/2)\alpha -(3/2)\varphi} \) and \( V = v^{3/4} = e^{(3/2)\alpha +(3/2)\varphi} \). This is with unit value for the rescaled lapse function, \( n = 1 \), but the last two equations appear simpler directly in terms of \( u \) and \( v \) if for just these equations we use the rescaled lapse \( \bar{n} = 1\) or \( n = 1/(2a) = (1/2)e^{-\alpha} = (1/2)(uv)^{-1/4} = (1/2)(UV)^{-1/3} \), which gives

\[
\dot{u}\dot{v} = -1 + \left(\lambda + \frac{1}{16}\ln^2\frac{v}{u}\right)\sqrt{uv}, \\
\frac{\ddot{u}}{u} - \frac{\ddot{v}}{v} = \frac{1}{4\sqrt{uv}}\ln \frac{v}{u},
\]

Although they are redundant equations, one may readily derive from Eqs. (7) and (8) that

\[
\ddot{a} = a(\varphi^2 - 2\dot{\varphi}^2 + \lambda) = a\left(\frac{\dot{\alpha}^2 + 1}{a^2} - 2\dot{\varphi}^2\right)
\]

and

\[
\ddot{\alpha} = e^{-2\alpha} - 3\dot{\varphi}^2
\]

when \( n = 1 \). Then when neither side of the constraint (first) equation part of Eqs. (8) vanishes (e.g., when \( \dot{V} \neq 0 \)), and when \( \dot{\varphi} \neq 0 \), one may define \( f' \equiv df/d\varphi = \dot{f}/\dot{\varphi} \) and reduce Eqs. (8) to the single second-order differential equation (cf. [6])

\[
\alpha'' = \frac{(\alpha'^2 - 1)(\varphi\alpha' + 3\varphi^2 + 3\lambda - 2e^{-2\alpha})}{\varphi^2 + \lambda - e^{-2\alpha}}.
\]

Alternatively, when \( \dot{V} \neq 0 \) (or equivalently \( \dot{\alpha}^2 \neq \dot{\varphi}^2 \)), but when \( \ddot{\alpha} \neq 0 \) instead of \( \dot{\varphi} \neq 0 \), one can write

\[
\frac{d^2\varphi}{d\alpha^2} = \frac{(d\varphi/d\alpha)^2 - 1}{\varphi^2 + \lambda - e^{-2\alpha}} \left[\left(3\varphi^2 + 3\lambda - 2e^{-2\alpha}\right)\frac{d\varphi}{d\alpha} + \varphi\right].
\]

5
Yet another way to get the equations of motion is to note that the seventh (penultimate) form of the action from Eq. (2) gives the trajectories of a particle of mass-squared \( \hat{V} \) in the DeWitt minisuperspace metric \( ds^2 = \hat{V} ds^2 \). When one goes to the gauge \( \hat{n} = 1 \), then \((ds/dt)^2 = -1\), so that along the classical timelike geodesics of \( ds^2 \), the Lorentzian action is \( S = -\int dt = -\int \sqrt{-ds^2} \), minus the proper time along the timelike geodesic of \( ds^2 \). However, one must note that the conformal metric \( ds^2 \) or the spacetime metric along this hypersurface (curve) in the two-dimensional minisuperspace \((\alpha, \varphi)\) under consideration. The second-order differential equations (13) and (14) also break down at \( \hat{V} = 0 \) and must be supplemented by the continuity of \( \dot{\alpha} \) and of \( \dot{\varphi} \) (in a gauge in which \( n \neq 0 \) is continuous there) across the \( \hat{V} = 0 \) hypersurface (curve).

To get reasonable numbers for the dimensionless constants in these equations, I shall follow [16] and set \( m \approx 1.5 \times 10^{-6}G^{-1/2} \approx 7.5 \times 10^{-6}(8\pi G)^{-1/2} \) [17] [18], so the prefactor of the action becomes \( S_0 \equiv (3\pi)/(2Gm^2) \equiv (3\pi/2)(m_{Pl}/m)^2 \approx 2 \times 10^{12} \), and the dimensionless measure of the cosmological constant is \( \lambda \equiv \Lambda/(3m^2) \equiv 1/(mb)^2 \approx 5 \times 10^{-111} \). Thus \( \lambda \) may be taken to be extremely tiny.

The constrained Hamiltonian system for this \( k = +1 \) FLRW-\( \Lambda \)-\( \phi \) model universe has an unconstrained 2d-dimensional phase space \( \Gamma_\alpha \) with \( d = 2 \) that may be labeled by the coordinates \( Q^i \) and conjugate momenta \( P_i \), which may be chosen to be any of the following sets:

\[
\{ A, \phi, p_A \} = -\frac{3\pi}{2Gm} \frac{A dA}{N dt}, \quad p_\phi = +\frac{2\pi^2 A^3 d\phi}{N dt}, \tag{15}
\]

\[
\{ a, \varphi, p_\alpha \} = -S_0 \frac{a da}{n dt}, \quad p_\varphi = +S_0 \frac{a^3 d\varphi}{n dt}, \tag{16}
\]

\[
\{ \alpha, \varphi, p_\alpha \} = -S_0 \frac{e^{3\alpha}}{n} \dot{\alpha}, \quad p_\varphi = +S_0 \frac{e^{3\alpha}}{n} \dot{\varphi}, \tag{17}
\]

\[
\{ u, v, p_u \} = -\frac{S_0}{4n} \dot{v}, \quad p_v = -\frac{S_0}{4n} \dot{u}, \tag{18}
\]

\[
\{ U, V, p_U \} = -\frac{2S_0}{9n} \dot{V}, \quad p_V = -\frac{2S_0}{9n} \dot{U}. \tag{19}
\]

The Hamiltonian on this phase space is then

\[
H = -\frac{G N}{3\pi A} p_A^2 + \frac{N}{4\pi^2 A^3} p_\phi^2 - \frac{3\pi N}{4G} A + \frac{\pi \Lambda N}{4G} A^3 + \pi^2 m^2 N A^3 \phi^2
\]

\[
= \frac{G m^2 n}{3\pi} \left( -\frac{p_\alpha^2}{a} + \frac{p_\varphi^2}{a^3} \right) + \frac{3\pi n}{4Gm^2} \left( -a + \lambda a^3 + a^3 \varphi^2 \right)
\]

\[
= \frac{1}{2S_0} n e^{-3\alpha} (-p_\alpha^2 + p_\varphi^2) + \frac{S_0}{2} n e^{3\alpha} \left(-e^{-2\alpha} + \lambda + \varphi^2 \right)
\]
\[ H = \frac{4}{S_0} \bar{n} p_u p_v - S_0 \bar{n} \left[ 1 - \left( \lambda + \frac{1}{16} \ln^2 \frac{u}{v} \right) \sqrt{uv} \right] \]
\[ = -\frac{9}{2 S_0} n p_u p_v - \frac{1}{2} S_0 n (UV)^{1/3} \left[ 1 - \left( \lambda + \frac{1}{9} \ln^2 \frac{V}{U} \right) (UV)^{2/3} \right] \]
\[ = \frac{1}{2} S_0 \left[ -a \dot{\alpha}^2 + a^3 \dot{\varphi}^2 \right] + n \left( -a + \lambda a^3 + a^3 \varphi^2 \right) \]
\[ = \frac{1}{2} S_0 n e^{3\alpha} \left[ -\left( \frac{\dot{\alpha}}{n} \right)^2 + \left( \frac{\dot{\varphi}}{n} \right)^2 - e^{-2\alpha} + \lambda + \varphi^2 \right] \]
\[ = \frac{1}{4} S_0 \left\{ -\bar{n}^{-1} \dddot{\bar{n}} - \bar{n} \left[ 1 - \left( \lambda + \frac{1}{16} \ln^2 \frac{V}{u} \right) \sqrt{uv} \right] \right\} \]
\[ = \frac{1}{2} S_0 \left\{ -\frac{4}{9} n^{-1} \dddot{U} - n (UV)^{1/3} \left[ 1 - \left( \lambda + \frac{1}{9} \ln^2 \frac{V}{U} \right) (UV)^{2/3} \right] \right\} . \quad (20) \]

The last four expressions are not in the canonical form as functions of the coordinates and momenta but are given to express the value of the Hamiltonian in terms of the coordinates and their time derivatives. Because the Hamiltonian constraint equation, obtained by varying the action \( S \) of Eq. (2) with respect to the lapse function \( n \) or \( \bar{n} \), is \( H = 0 \), these last four expressions for \( H \) can be easily seen to lead to the first equations in each of Eqs. (7), (8), (9), and (10) when one chooses the rescaled lapse function \( n \) to be 1 for Eqs. (7), (8), and (9) and chooses \( \bar{n} = 1 \) in Eq. (10). One can also write the Hamiltonian constraint equation \( H = 0 \) directly in terms of the canonical coordinates and momenta as

\[ a^2 p_a^2 - p_\varphi^2 = S_0^2 a^6 \left( \lambda + \varphi^2 - a^{-2} \right), \quad \text{(21)} \]
\[ p_\alpha^2 - p_\varphi^2 = S_0^2 e^{3\alpha} \left( \lambda + \varphi^2 - e^{-2\alpha} \right), \quad \text{(22)} \]
\[ p_a p_v = -\frac{1}{16} S_0^2 \left[ 1 - \left( \lambda + \frac{1}{16} \ln^2 \frac{V}{u} \right) \sqrt{uv} \right], \quad \text{(23)} \]
\[ p_v p_V = -\frac{1}{9} S_0^2 (UV)^{1/3} \left[ 1 - \left( \lambda + \frac{1}{9} \ln^2 \frac{V}{U} \right) (UV)^{2/3} \right] . \quad \text{(24)} \]

### 3 The canonical measure for \( k = +1 \) FLRWΛ-Φ

The natural symplectic structure for the \( k = +1 \) FLRWΛ-Φ constrained Hamiltonian system is the closed nondegenerate differential 2-form

\[ \omega_n = \omega_2 = dP_t \wedge dQ^i \]
\[ = dP_A \wedge dA + dP_\varphi \wedge d\varphi = -S_0 m^2 A d\dot{A} \wedge dA + 2 \pi^2 m d(A^3 \dot{\varphi}) \wedge d\varphi \]
\[ = dP_\alpha \wedge da + dP_\varphi \wedge d\varphi = S_0 \left( -a d\dot{\alpha} \wedge da + d(a^3 \dot{\varphi}) \wedge d\varphi \right) \]
\[ = dP_\alpha \wedge d\alpha + dP_\varphi \wedge d\varphi = S_0 e^{3\alpha} \left( -a d\dot{\alpha} \wedge d\alpha + d\dot{\varphi} \wedge d\varphi + 3 \dot{\varphi} d\alpha \wedge d\varphi \right) \]
\[ = dP_\alpha \wedge du + dP_\varphi \wedge dv = -\frac{S_0}{8} (uv)^{1/4} \left[ d\dot{v} \wedge du + d\dot{u} \wedge dv + \frac{1}{4} \left( \frac{\dot{v}}{v} - \frac{\dot{u}}{u} \right) du \wedge dv \right] \]
\begin{align}
\omega &= dp_U \wedge dU + dp_V \wedge dV = \frac{2}{9} S_0 \left( d\dot{V} \wedge dU + d\dot{U} \wedge dV \right),
\end{align}

where for the expressions in terms of the time derivatives, I have used the default option \( n = 1 \).

When this is pulled back to the \( H = 0 \) constraint hypersurface of dimension \( 2d - 1 = 3 \) in the unconstrained phase space \( \Gamma_d = \Gamma_2 \) of dimension \( 2d = 4 \) and further pulled back to an initial-data surface \( \Gamma_{d-1} = \Gamma_1 \) of dimension \( 2d - 2 = 2 \) that is transverse to the Hamiltonian flow in the 3-dimensional constraint hypersurface, it gives the symplectic structure differential form \( \omega \equiv \omega_{d-1} = \omega_1 \) on that 2-dimensional initial data surface. Since \( d - 1 = 1 \), it is the first power of this symplectic structure form that gives the canonical Liouville-Henneaux-Gibbons-Hawking-Stewart measure or volume (area) element \( \Omega \equiv \Omega_{d-1} = \omega_1 \) on the initial data surface \( \Sigma \).

Here we are restricting to nonsingular cosmologies, Friedmann-Lemaître-Robertson-Walker universes that have neither a big bang nor a big crunch, so the scale factor \( A \) or \( a \) never goes to zero. Except for a discrete set of zero canonical measure \[6\], all of these solutions will contract from infinite \( a \) at infinite past time and re-expand to infinite \( a \) at infinite future time. Therefore, they will each have a global minimum for \( a = e^{\alpha} \), which I shall label \( a_m \equiv \exp(\alpha_m) \), where \( da/dt = e^\alpha \dot{\alpha} = 0 \) and hence \( p_A = p_a = p_\alpha = 0 \). Let \( \varphi_m, \dot{\varphi}_m, \ddot{\alpha}_m, \) and \( p_{\varphi_m} \) be the values of \( \varphi, \dot{\varphi}, \ddot{\alpha}, \) and \( p_\varphi \) at this global minimum for \( a \) and hence also for \( \alpha \).

The Hamiltonian constraint \( H = 0 \), given by the first Eq. (7) with both sides equal to zero, implies that (using the default setting of the rescaled lapse function as \( n = 1 \))

\begin{align}
a_m &= (\varphi_m^2 + \dot{\varphi}_m^2 + \lambda)^{-1/2}, \\
\alpha_m &= -\frac{1}{2} \ln(\varphi_m^2 + \dot{\varphi}_m^2 + \lambda),
\end{align}

Therefore, initial data are given by values of \( \varphi_m \) and \( \dot{\varphi}_m \), with \( a_m = a_m(\varphi_m, \dot{\varphi}_m) \) given by Eq. (26), and then one has \( \dot{a} = 0 \) automatically at this point in the constrained phase space.

One can further readily see (cf. [12, 13]) that at an extremum for \( a \), where \( \dot{a} = 0 \), that one has

\begin{align}
\frac{\ddot{a}}{a} = \ddot{\alpha} = 3\varphi^2 + 3\lambda - 2e^{-2\alpha} = \varphi^2 - 2\dot{\varphi}^2 + \lambda.
\end{align}

Therefore, for an extremum to be at least a local minimum, one needs \( a_m \leq (2/3)^{1/2}(\varphi_m + \lambda)^{-1/2} \) or \( \dot{\varphi}_m^2 \leq (\varphi_m^2 + \lambda)/2 \), though there are additional conditions for such a local minimum to be a global minimum.
Since Eq. (26) gives a unique value for $a_m$ for each set of real values for $\varphi_m$ and $\dot{\varphi}_m$, it naively appears that there are no constraints on $\varphi_m$ and $\dot{\varphi}_m$. However, different sets of $\varphi_m$ and $\dot{\varphi}_m$ can lead to the same solution, since a solution may have more than one point along its trajectory (more than one time) where $\dot{a} = 0$, only one of which may be a global minimum in the generic case in which there are not more than one time with the same global minimum value of $a(t)$. Therefore, counting all possibilities for $\varphi_m$ and $\dot{\varphi}_m$ overcounts the trajectories that have $\dot{a} = 0$ somewhere along them. Furthermore, singular trajectories, with $a$ going to zero in the past or future, may also have points where $\dot{a} = 0$ and thus be counted if one counts all possible pairs of $\varphi_m$ and $\dot{\varphi}_m$. In the next Section we shall look at the restrictions on $\varphi_m$ and $\dot{\varphi}_m$ in order that this pair correspond to a global minimum of $a$ rather than some other extremum like a local minimum that is not a global minimum, or either a local or a global maximum. However, first we shall show that the total canonical measure of all solutions with any local nonzero extremum for $a$ (at $a = a_m$ or $\alpha = \alpha_m$ where $\dot{a} = 0$ and $\dot{\alpha} = 0$) has finite measure.

It is convenient to define two new variables $\beta$ and $\theta$ so that (with rescaled lapse function $n = 1$ as usual)

$$\varphi = e^{-\beta} \cos \theta,$$

$$\dot{\varphi} = e^{-\beta} \sin \theta.$$  \hspace{1cm} (29)

The Hamiltonian constraint equation Eq. (8) then becomes

$$\dot{\alpha}^2 = \lambda + e^{-2\beta} - e^{-2\alpha}. \hspace{1cm} (30)$$

One can also easily calculate that the time derivatives of $\beta$ and $\theta$ are

$$\dot{\beta} = 3\dot{\alpha} \sin^2 \theta = \frac{3}{2} \dot{\alpha}(1 - \cos 2\theta),$$

$$\dot{\theta} = -1 - 3\dot{\alpha} \sin \theta \cos \theta = -1 - \frac{3}{2} \dot{\alpha} \sin 2\theta. \hspace{1cm} (32)$$

When averaged over one period of $\theta$ in a regime in which the scalar field oscillates rapidly relative to the expansion (so that the time-average of the scalar field stress-energy tensor is approximately that of pressureless dust), $\beta$ changes by approximately $3/2$ as much as $\alpha$, so it is convenient to define a total rationalized dimensionless 'mass' (twice the energy density multiplied by the volume and divided by $S_0$ and by the scalar field mass $m$) that is nearly constant in the dustlike regime \cite{16}:

$$M \equiv e^{3\alpha - 2\beta} \equiv a^3(\dot{\varphi}^2 + \ddot{\varphi}^2), \hspace{1cm} (33)$$

obeying

$$\dot{M} = 3M\dot{\alpha} \cos 2\theta \hspace{1cm} (34)$$
or
\[
\frac{d \ln M}{d \alpha} = 3 \cos 2\theta.
\] (35)

The symplectic structure 2-form \( \omega_m = \omega_2 \) given by Eq. (25) is written in terms of the four independent 1-forms of the unconstrained phase space \( \Gamma_n = \Gamma_2 \) of dimension \( 2n = 4 \). When one imposes the Hamiltonian constraint \( H = 0 \), one of the four 1-forms appearing in \( \omega_2 \) can be written in terms of the other three. Thus one can write the symplectic structure 2-form in terms of three 1-forms that are independent on the constraint hypersurface \( H = 0 \). Choosing these three to be various combinations of the differentials of \( \alpha, \dot{\beta}, \varphi, \dot{\varphi}, \beta, \theta, \) and \( M \), one can write the 2-form on the constraint hypersurface as
\[
\omega = S_0 e^{3\alpha} \left( -\frac{d\dot{\varphi}}{d\alpha} d\alpha \wedge d\varphi + \frac{d\varphi}{d\alpha} d\alpha \wedge d\dot{\varphi} - d\varphi \wedge d\dot{\varphi} \right)
= S_0 e^{5\alpha} (\dot{\varphi} d\alpha \wedge d\varphi + \varphi d\alpha \wedge d\dot{\varphi} - \dot{\alpha} d\varphi \wedge d\dot{\varphi})
= S_0 e^{3\alpha-2\beta} \left( \frac{d\theta}{d\alpha} d\alpha \wedge d\beta - \frac{d\beta}{d\alpha} d\alpha \wedge d\theta + d\beta \wedge d\theta \right)
= S_0 e^{5\alpha-2\beta} \left( \dot{\beta} d\alpha \wedge d\beta - \dot{\beta} d\alpha \wedge d\theta + \dot{\alpha} d\beta \wedge d\theta \right)
= \frac{1}{2} S_0 \left( -\frac{d\theta}{d\alpha} d\alpha \wedge dM + \frac{dM}{d\alpha} d\alpha \wedge d\theta - dM \wedge d\theta \right). \tag{36}
\]

Here the derivatives with respect to time (with \( n = 1 \)) or to \( \alpha \) that are the coefficients of the basis 2-forms inside the parentheses are derivatives along the trajectories forming the cosmological spacetime solutions, unlike the basis 1-forms that make up the basis 2-forms that are differentials transverse to the trajectories.

On an initial data surface that is an extremum of the scale size or of the logarithm of the rescaled scale size, \( \alpha = \alpha_m \), where \( \dot{\alpha} = 0 \), one has \( \beta = \beta_m \), \( \theta = \theta_m \), and \( M = M_m = e^{3\alpha_m-2\beta_m} \), and from the two coordinates \( \beta_m \) and \( \theta_m \) on this initial data surface that I shall call \( S_e \), one gets \( \alpha_m = -(1/2) \ln (\lambda - e^{-2\beta_m}) \) or \( \beta_m = -(1/2) \ln (e^{-2\beta_m} - \lambda) \), then giving \( M_m = e^{\beta_m} (1 + \lambda e^{2\beta_m})^{-3/2} = e^{a_m} (1 - \lambda e^{2a_m}) \). Note that for an extremum we must have \( a_m \leq -(1/2) \ln \lambda \) or \( a_m \leq 1/\sqrt{\lambda} \) or \( \lambda a_m^2 \leq 1 \), but \( \beta_m \) can be an arbitrary real number (though one no longer has the full range of all real numbers for \( \beta \) if the extremum is required to be a global minimum for \( a \) or \( \alpha \)). One can alternatively label the initial data surface \( S_e \) of extrema (all points in the constrained hypersurface \( H = 0 \) where also \( \dot{\alpha} = 0 \)) by the two coordinates \( a_m \) and \( \theta_m \), both of which have finite ranges, \( 0 < a_m \leq 1/\sqrt{\lambda} \) and \( 0 \leq \theta_m < 2\pi \). Then on that initial data surface \( S_e \) one has \( a = a_m, \alpha = \alpha_m = \ln a_m, \beta = \beta_m = -(1/2) \ln (1/\alpha^2_m - \lambda), \varphi = \varphi_m = e^{-\beta_m} \cos \theta_m, \dot{\varphi} = \dot{\varphi}_m = e^{-\beta_m} \sin \theta_m, \dot{\alpha}_m = [1 - 3(1 - \lambda a^2_m) \sin^2 \theta_m]/a^2_m, \) and \( M_m = a_m (1 - \lambda a^2_m) \).

The pullback of \( \omega_2 \) to an initial data surface \( S_e \) that is at an extremum of \( a \) and of \( \alpha \), where \( \dot{a} = 0 \) and hence where \( a_m = a_m (\varphi_m, \dot{\varphi}_m) = (\lambda + \dot{\varphi}_m^2 + \varphi_m^2)^{-1/2} = 0 \)
\((\lambda + e^{-2\beta_m})^{-1/2} \leq 1/\sqrt{\lambda}\), is

\[
\omega = -S_0 e^{5\alpha_m} \dot{\alpha}_m d\dot{\varphi}_m \wedge d\varphi_m \\
= S_0 e^{5\alpha_m-2\beta_m} \dot{\alpha}_m d\beta_m \wedge d\theta_m \\
= S_0 \left[1 - 3 \left(1 - \lambda a_m^2\right) \sin^2 \theta_m \right] da_m \wedge d\theta_m \\
= \mu_0 \left[1 - 3 \left(1 - x^2\right) \sin^2 \theta_m \right] dx \wedge d\theta_m, \tag{37}
\]

where \(\mu_0 \equiv S_0/\sqrt{\lambda} \approx 3 \times 10^{67}\) and \(x \equiv \sqrt{\lambda} a_m\), which has the range \(0 < x \leq 1\).

Because both \(a_m\) (or \(x\)) and \(\theta_m\) have finite ranges, and because the integrand is bounded above within this range, the measure for the set of solutions with a nonzero extremum for \(a\) is finite. For the case with zero cosmological constant, Hawking and I showed \[3\] that the solutions with an extremum within a finite range of \(a\) have finite measure, but for that model there is no upper bound on \(a\) at an extremum, and almost all solutions have a maximum for \(a\), so the total measure for solutions with maxima is infinite. But in the present case, the positive cosmological constant imposes an upper limit on the value of \(a\) at an extremum.

Therefore, we see that the set of solutions with a nonzero extremum for \(a\) has a finite canonical measure, out of the infinite measure for all solutions for a \(k = +1\) FLRW cosmology with a minimally coupled massive scalar field and a positive cosmological constant. The finite measure of solutions includes both totally nonsingular solutions, which have a nonzero global minimum for \(a\), and also solutions with both a big bang and a big crunch, which have a finite global maximum for \(a\). It also includes solutions that start at a big bang and eventually expand forever, and their time reverses that contract from \(a = \infty\) to a big crunch, so long as they have a local extremum for \(a\), where \(\dot{a}\) or the Hubble variable \(\ddot{a}\) is zero. The only set of solutions that have infinite measure are those that expand monotonically from a big bang at \(a = 0\) to infinite size at \(a = \infty\), or the time reverses that contract monotonically from \(a = \infty\) to a big crunch at \(a = 0\).

4 Canonical measure for nonsingular cosmologies

We have seen that the total Liouville-Henneaux-Gibbons-Hawking-Stewart canonical measure for nonsingular Friedmann-Lemaître-Robertson-Walker cosmologies with a minimally coupled massive scalar field is finite. (For there to be a nonzero measure for such nonsingular cosmological solutions of the Einstein-scalar field equations, we need a closed cosmology with \(k = +1\) to allow \(a\) to have a minimum value, and we need a positive cosmological constant to allow \(a\) to go to infinity asymptotically in both directions of time.) Now let us calculate the measure for the nonsingular solutions.

The canonical Liouville-Henneaux-Gibbons-Hawking-Stewart measure or volume (area) element \(\omega\) on the initial data surface \(\Gamma_1\) with \(\dot{a} = 1\), given by Eq. \(37\), has
a sign of the coefficient of \( dx \wedge d\theta_m \) that is proportional to \( \ddot{a} = \ddot{a}_m = [1 - 3(1 - \lambda a_m^2) \sin^2 \theta_m] / a_m = \sqrt{\lambda} [1 - 3(1 - x^2) \sin^2 \theta_m] / x \), the acceleration of the scale factor \( a \) at its extremum. When this is positive, the extremum is a local minimum for the scale factor; when \( \ddot{a}_m < 0 \), the extremum is a local maximum for \( a \). If we integrate \( \omega \) over the range giving positive \( \ddot{a}_m \), we get the finite measure \( \mu_1 = (4\pi / \sqrt{27}) \mu_0 \approx 2.4184 \mu_0 \). If we reverse the sign of \( \omega \) and integrate it over the range giving negative \( \ddot{a}_m \), we get the same finite measure, \( \mu_2 = (4\pi / \sqrt{27}) \mu_0 \). Therefore, the total measure for solutions with extrema for \( a \) is not greater than \( \mu_3 = \mu_1 + \mu_2 = (8\pi / \sqrt{27}) \mu_0 \), finite.

However, solutions may have more than one extremum for \( a \), and \( \mu_3 \) counts all such solutions with a multiplicity given by the number of extrema that they have. Therefore, let us calculate what the measure is for nonsingular solutions by just taking the measure at the nonzero global minimum for \( a \), avoiding the overcounting of a multiplicity of minima. For this calculation we shall assume that \( \lambda \ll 1 \), as it indeed is in our part of the universe where \( \lambda \equiv \Lambda / (3m^2) \approx 5 \times 10^{-111} \), and hence drop correction terms proportional to positive powers of \( \lambda \).

Most of the measure for the nonsingular solutions will come from values of \( a \) not too much less than the maximum value for an extremum, which is at \( a = 1/\sqrt{\lambda} \) or \( x = \sqrt{\lambda} a_m = 1 \). Therefore, we can assume that \( x \) is not enormously smaller than unity for almost all of the measure. For a nonsingular solution that has a global minimum at \( \alpha \equiv \ln a = a_m = \ln x - (1/2) \ln \lambda \gg 1 \) (with \( -(1/2) \ln \lambda \approx 127 \gg 1 \)), the fact that Eq. (31) implies that \( \beta \) cannot decrease as \( \alpha \) increases implies that the Hamiltonian constraint equation Eq. (30) gives

\[
\dot{\alpha}^2 = \lambda + e^{-2\beta} - e^{-2\alpha} \leq \lambda + e^{-2\beta_m} - e^{-2\alpha} = e^{-2\alpha} - e^{-2\alpha} \leq \frac{\lambda}{x^2}.
\]  

(38)

For \( x \gg \sqrt{\lambda} \approx 7 \times 10^{-56} \), we thus get \( \dot{\alpha}^2 \ll 1 \).

As a result, Eq. (32) implies that \( \dot{\theta} \approx -1 \) to high accuracy, and Eq. (34) implies that \( M \equiv e^{3\alpha - 2\beta} \equiv a^2 (\varphi^2 + \dot{\varphi}^2) \) stays very nearly constant along most of the nonsingular trajectories. Eq. (31) implies that \( \beta \) is also very small, so over a number of oscillations of the scalar field that is not too large (a change in the phase \( \theta \) that is not too many times \( 2\pi \)), neither \( \alpha \) nor \( \beta \) change much, though after a very long time and a huge number of oscillations of the scalar field (enormous change in \( \theta \)), both \( \alpha \) and \( \beta \) grow indefinitely, while \( M \) and \( \psi = \theta + t \) stay nearly constant and indeed both approach precise constants in the infinite future, \( M_\infty \) and \( \psi_\infty \).

(To define \( \psi_\infty \) unambiguously, set \( t = 0 \) at the global minimum for \( a \) and require \( 0 \leq \theta_m < 2\pi \) there. One can make this definition not only for nonsingular solutions but also for big bang solutions that start at global minimum for \( a \) that is \( a = 1 \), where one can set \( t = 0 \), and then evolve to infinite \( a \) where \( \psi_\infty \) can be evaluated. To circumvent the jumps in \( \theta_m \) at the minimum that would occur with a sequence of solutions with \( \theta_m \) approaching \( 2\pi \) and then jumping back to 0, instead of defining the two real constants \( M_\infty \) and \( \psi_\infty \), it would be better to define the one complex
constant $Z = \sqrt{M_\infty e^{i\psi_\infty}}$, which is invariant under shifting $\theta_m$ and hence $\psi$ and $\psi_\infty$ by an integer multiple of $2\pi$. Any solution that evolves to $a = \infty$ will have a definite value for $Z$ that may be obtained by analytic integration of the equations of motion from any initial point in the constrained phase space, except for the nonsingular solutions that have two equal global minima for $a$ and therefore the ambiguity of which one to use for setting the zero of $t$, and the solutions that are the limits of sequences of solutions with bounces of arbitrarily large values of the scalar field [6]. Both of these types of particular solutions will have zero measure, so all but a set of measure zero of the solutions that evolve to $a = \infty$ will be integrable, having two real constants of motion that may be given by one complex constant $Z$, that are analytic functions over all but a set of hypersurfaces of measure zero in the constrained phase space. The same will be true for $k = 0$ and $k = -1$ FLRW-scalar models with a nonnegative cosmological constant to allow solutions to expand to $a = \infty$, though since they cannot have extrema of $a$ that in the $k = +1$ case can lead to hypersurfaces of the constrained phase space where the constants of motion are not analytic, it appears that the $k = 0$ and $k = -1$ FLRW-scalar models will be totally integrable over the entire constrained phase space. In fact, since these models give $a$ expanding monotonically from $a = 0$ to $a = \infty$, or the time reverse, one can not only define constants of motion by the asymptotic behavior of $M$ and $\psi$ at $a = \infty$ that gives rise to the complex constant $Z$, but also by the asymptotic behavior of $a$, such as the value of $v - u$ where either one of these null coordinates $u$ and $v$ goes to zero, and the value of the slope $dv/du$ there.)

During the oscillations of the scalar field while $\alpha$ and $\beta$ stay near their values at the extremum, one can write

$$\frac{d^2\alpha}{d\theta^2} \approx \frac{d^2\alpha}{dt^2} \approx \frac{\lambda}{2x^2} \left[ 3x^2 - 1 + 3(1 - x^2)(1 - \cos 2\theta) \right].$$

Integrating this gives

$$\frac{d\alpha}{d\theta} \approx \frac{\lambda(3x^2 - 1)}{2x^2} \left[ \theta - \theta_0 + B \sin 2\theta \right],$$

where

$$B \equiv \frac{3(1 - x^2)}{2(3x^2 - 1)}, \quad \theta_0 \equiv \theta_m + B \sin 2\theta_m,$$

and then finally

$$\alpha \approx \alpha_m + \frac{\lambda(3x^2 - 1)}{4x^2} \left[ (\theta - \theta_0)^2 - B \cos 2\theta - (\theta_0 - \theta_m)^2 + B \cos 2\theta_m \right],$$

One can then see that for $\theta = \theta_m$ to be not only a local minimum (which requires merely $3x^2 - 1 + 3(1 - x^2)(1 - \cos 2\theta_m) \geq 0$) but also a global minimum, one needs that $3x^2 - 1 \geq 0$ for a nonnegative coefficient of the quadratic term in $\theta$. Furthermore, by sketching the behavior of $\alpha(\theta)$, one can see that for $-\pi < 2\theta_0 < \pi$,
one needs $-\pi/2 < 2\theta_m < \pi/2$; for $\pi < 2\theta_0 < 3\pi$, one needs $3\pi/2 < 2\theta_m < 5\pi/2$; for $3\pi < 2\theta_0 < 5\pi$, one needs $7\pi/2 < 2\theta_m < 9\pi/2$; etc. Thus we cannot have a global minimum with $\pi/2 < 2\theta_m < 3\pi/2$, $5\pi/2 < 2\theta_m < 7\pi/2$, etc.

If we choose $\theta_m$ to lie in the range $0 \leq \theta_m < 2\pi$, then there are four allowed ranges of $\theta_m$ of width $\pi/4$ (covering half the full circle for $\theta_m$; the other half does not give extrema that are global minima) that give equal contributions to the measure. Let us focus on the first, which is that part of $0$ that one needs $-\pi/2 < 2\theta_m < \pi/2$.

Let us denote $\theta_m$ by $\gamma$ and $\theta_m + B \sin 2\theta_m < \pi/2$. Using Eq. (41) to express $B$ in terms of $x\sqrt{\Lambda}a_m$ allows one to convert this to a restriction on $x$ for $0 \leq \theta_m < \pi/4$:

$$x_m(\theta_m) \equiv \frac{\sqrt{\pi - 2\theta_m + 3\sin 2\theta_m}}{3\pi - 6\theta_m + 3\sin 2\theta_m} \leq x \leq 1.$$  \hfill (43)

If we now integrate the canonical Liouville-Henneaux-Gibbons-Hawking-Stewart measure or volume (area) element $\omega$ in Eq. (37) over the initial data surface, say $S_m$, that has $\gamma$ not only an extremum but also a global minimum (which is 4 times the integral of $\omega$ over the one region above, in order to include all possibilities for $0 \leq \theta_m < 2\pi$ which give a global minimum), we get (using $y = 2\theta_m$)

$$\mu_m \equiv \gamma \mu_0 = \int_{S_m} \omega = \mu_0 \left\{ \frac{2 - \int_0^{\pi/2} dy \left[ (3\cos y - 1)x_m + (1 - \cos y)x_m^3 \right]}{2 - \int_0^{\pi/2} dy \left[ (3\cos y - 1)x_m + (1 - \cos y)x_m^3 \right]} \right\}.$$  \hfill (44)

Doing the integral numerically with Maple 12 gave $\gamma \approx 0.86334$. Putting in the numbers given above for $S_0 \equiv (3\pi)/(2Gm^2) = (3\pi/2)(m_{Pl}/m)^2 \approx 2 \times 10^{12}$, $\lambda \equiv \Lambda/(3m^2) \equiv 1/(mb)^2 \approx 5 \times 10^{-11}$, and $\mu_0 = S_0/\sqrt{\Lambda} \approx 3 \times 10^{67}$ gives the measure for the nonsingular $k = +1$ FLRW cosmologies with the observed value of the cosmological constant and a scalar field mass of $m \approx 1.5 \times 10^{-6}m_{Pl}$ as

$$\mu_m \equiv \gamma \mu_0 \equiv \gamma \frac{S_0}{\sqrt{\lambda}} \equiv \gamma \frac{3\sqrt{3}\pi}{2Gm\sqrt{\Lambda}} \approx 0.86334\mu_0 \approx 3 \times 10^{67}.$$  \hfill (45)

To convert this to a number of quanta, say $N_m$, one divides the phase space measure by $h = 2\pi\hbar = 2\pi$ in our units with $\hbar = c = 1$ to get

$$N_m \equiv \frac{\mu_m}{2\pi} \equiv \frac{\gamma \mu_0}{2\pi} \equiv \frac{S_0}{2\pi\sqrt{\lambda}} \equiv \gamma \frac{3\sqrt{3}}{4Gm\sqrt{\Lambda}} \approx 0.13740\mu_0 \approx 4 \times 10^{66}.$$  \hfill (46)

One can compare this with the maximum number of scalar field quanta, say $N_M$, that one can have for a nonsingular $k = +1$ FLRW cosmology with a positive cosmological constant if the scalar field acted as pressureless dust, which is how it does act at late times when one averages over an integer number of oscillations of the scalar field. Then $M$ would stay constant. For a universe with a minimum value of $a$ that is $a_m$, one gets $M = a_m(1-\lambda a_m^2)$, which has a maximum value (when $1-3\lambda a_m^2 \equiv 1-3x^2 = 0$) of $M_M = 2/\sqrt{27\lambda}$. The physical energy density $(1/2)[(m\phi)^2 + (\dot{\phi}/N)^2] =$
\[
\frac{3}{(8\pi G)}m^2(\varphi^2 + \dot{\varphi}^2) \text{ (with } N = n/m = 1/m) \text{ multiplied by the physical 3-volume } 2\pi^2 A^3 = 2\pi^2 a^3/m^3 \text{ then gives a physical ‘mass’ } M = (S_0/2)mM, \text{ so the maximum number of scalar dust quanta of mass } m \text{ in a nonsingular } k = +1 \text{ FLRW cosmology with a positive cosmological constant is}
\]

\[
N_M = \frac{1}{2} S_0 M_M = \frac{S_0}{\sqrt{27\lambda}} \approx 0.19245 \mu_0 \approx 1.400607 N_m \approx 6 \times 10^{66}. \tag{47}
\]

Therefore, the number of quanta corresponding to the actual phase space measure over the nonsingular \( k = +1 \) FLRW cosmologies is \( \sqrt{27\gamma}/(2\pi) \approx 0.713976 \) times the maximum of that for a dust model with the same particle mass. In the actual scalar field model, the scalar field undergoes coherent oscillations in which the phase has gravitational consequences, so it cannot be accurately modeled by assuming that the scalar field is in a precise number eigenstate with a totally undetermined phase, which would give zero pressure for a homogeneous field such as is being assumed here.

## 5 Canonical measure for inflationary cosmologies

Nearly all of the finite total Liouville-Henneaux-Gibbons-Hawking-Stewart canonical measure for nonsingular Friedmann-Lemaître-Robertson-Walker cosmologies with a minimally coupled massive scalar field and a positive cosmological constant occurs for solutions that are not large deviations from empty de Sitter spacetime. For example, if one defines the rationalized dimensionless energy density to be

\[
\rho \equiv e^{-2\beta} \equiv \varphi^2 + \dot{\varphi}^2 \equiv \frac{M}{a^3}, \tag{48}
\]

which is \( (8\pi G)/(3m^2) \) times the physical energy density \( \dot{\rho} = (1/2)[(m\dot{\phi})^2 + (\dot{\phi}/N)^2] = \frac{3}{(8\pi G)}m^2(\varphi^2 + \dot{\varphi}^2) = \frac{3}{(8\pi G)}m^2\rho \) of the scalar field, and sets \( \rho = \rho_m = a_m^{-2} - \lambda \) at the global minimum for the scale factor \( a = a_m \), then under the approximation that \( M \) is constant, one has \( 1/\sqrt{3\lambda} \leq a_m \leq 1/\sqrt{\lambda} \) \( \rho \leq \rho_m \leq 2\lambda \) everywhere in the spacetime, so the physical energy density of the scalar field is never more than twice the physical energy density \( \rho_\Lambda = \Lambda/(8\pi G) = [3/(8\pi G)]m^2\lambda \) corresponding to the cosmological constant, that is, \( \dot{\rho} \leq 2\rho_\Lambda \).

If \( \rho_m = a_m^{-2} - \lambda > 2\lambda \ll 1 \) at an extremum of \( a \), then for \( \rho_m \ll 1, \dot{\alpha}^2 \ll 1 \) for the entire solution, so \( M \) stays nearly constant at a value less than its maximum value for nonsingular dust solutions (\( M = M_M = 2/\sqrt{27\lambda} \), and the resulting solutions collapse to \( a = 0 \) rather than expanding to infinity. To obtain solutions with \( \dot{\rho}_m > 2\rho_\Lambda \) that expand to infinity in both directions of time rather than collapsing to zero size, one need to have a period of inflation in which \( M \) grows larger to become larger than \( M_M = 2/\sqrt{27\lambda} \), as one can see from the following argument:
The Hamiltonian constraint equation in Eq. (7) can be written in terms of $a$ and $M$ as
\[ \dot{a}^2 = f(a) \equiv \lambda a^2 - 1 + \frac{M}{a}. \] (49)

After the end of a possible period of inflation (which requires $\rho \approx 1 \gg \lambda$), during which $M$ can grow exponentially, $M$ will become nearly constant as the scalar field starts oscillating with a period much less than the inverse of the Hubble expansion rate. Then the universe will expand forever if $f(a)$ stays positive for all larger $a$. For constant $M$, the extremum of $f(a)$ is at $a = [M/\langle 2\lambda \rangle]^{1/3} = (M/M_M)^{1/3}/\sqrt{3\lambda}$. The value at the extremum is $f(a) = (6.75M^2\lambda)^{1/3} - 1 = (M/M_M)^{2/3} - 1$, so if inflation ends before $a$ reaches the extremum, one needs $M > M_M$ in order for $\dot{a}$ to stay positive as $a$ passes through the extremum of $f(a)$. If inflation gives $M \leq M_M$, it will end far before this value of $a$ is reached, so one needs inflation to give $M > M_M = 2/\sqrt{27\lambda} \gg 1$ if one starts at an extremum with $\rho_m > 2\lambda$ and hence with $a_m < 1/\sqrt{3\lambda}$, where there are no noninflationary solutions ($M$ nearly constant) that expand to infinity in both directions of time and hence are nonsingular.

For a symmetric bounce ($\dot{\varphi}_m = 0$ or $\theta_m = 0$) at $\varphi_m = \varphi_b \gg 1$, I have calculated numerically [16] that the number of e-folds of inflation $N$ (not be be confused with the previous use of $N$ for the lapse function) is
\[ N(\varphi_b) \approx \frac{3}{2}\varphi_b^2 + \frac{1}{3}\ln \varphi_b - 1.0653 - \frac{3\pi^2 - 14}{36\varphi_b^2} - \frac{0.4}{\varphi_b^4}, \] (50)
and that the asymptotic value of $M$ is
\[ M_\infty(\varphi_b) \approx 0.1815\varphi_b^{-3}e^{3N(\varphi_b)} \approx \frac{0.08914e^{4.5\varphi_b^2}}{12\varphi_b^2 + 3\pi^2 - 14 + 24/\varphi_b^2}. \] (51)

To give $M > M_M$, this requires $N > N_M \approx 44.27$.

For $\ddot{\varphi} = \lambda + \rho_m(1 - 3\sin^2 \theta_m) \approx \rho_m(1 - 3\sin^2 \theta_m) > 0$, we need $|\sin \theta_m| < 1/\sqrt{3}$. If this is is true for $\rho_m \gg 1$ and $\theta_m$ is not too close to the boundary, then the equations of motion in this inflationary regime will generally cause $\varphi$ and hence $\theta$ to decay to near zero, and then one will get roughly $N \sim (3/2)\rho_m^{1/2}$ e-folds of inflation, though the coefficient $(3/2)$ will become some $\theta_m$-dependent number that is $(3/2)$ only at $\sin \theta_m = 0$. However, for a crude estimate of the measure for different amounts of inflation, let us ignore this effect. Then Eq. (57) with $a_m = (\lambda + \rho_m)^{-1/2} \approx \rho_m^{-1/2}$ gives the measure of varying numbers $N$ of e-folds of inflation as
\[
\mu = \int \omega \\
= -\frac{1}{2}S_0 \int (\lambda + \rho_m)^{-5/2} [\lambda + \rho_m(1 - 3\sin^2 \theta_m)] d\rho_m \wedge d\varphi_m \\
\approx S_0(1 - 3\sin^2 \theta_m)d(\rho_m^{-1/2}) \wedge d\varphi_m \\
= (\sqrt{6} - \cos^{-1} 3^{-3/2})S_0 \int d(\rho_m^{-1/2})
\]
\[ \sim (3 - \sqrt{3}/2 \cos^{-1} 3^{-3/2})S_0 \int d(N^{-1/2}). \]
\[ \sim S_0 \int d(N^{-1/2}). \] (52)

The second expression on the right hand side (after \( \int \omega \)) uses the very good approximation that \( \lambda \ll \rho_m \) for the inflationary values of \( \rho_m \) that are at least of the order of unity in our units that set effectively \( m = 1 \). The third evaluates the integral over \( \theta_m \). The fourth uses the approximation \( N \sim (3/2)\rho_m \) that is actually only true for \( \sin \theta_m = 0 \), so the fifth drops the uncertain numerical coefficient.

Thus we see that the measure for at least \( N \) e-folds of inflation is proportional to \( 1/\sqrt{N} \) for large \( N \). If we take the fraction of the total measure for nonsingular solutions, which was \( \gamma \mu_0 = \gamma S_0/\sqrt{\lambda} \approx 3 \times 10^{67} \), the fraction of the total measure for at least \( N \) e-folds of inflation is

\[ F \sim \frac{\sqrt{\lambda}}{\sqrt{N}} = \frac{\sqrt{\Lambda/3}}{m\sqrt{N}} \sim \frac{10^{-55}}{\sqrt{N}} \sim \frac{10^{-56}}{\sqrt{N/N_M}}. \] (53)

That is, the fraction of the measure for all nonsingular \( k = +1 \) FLRW cosmologies that have inflation (requiring \( N > N_M \approx 44 \) for the present toy model with just the observed value of the cosmological constant and a massive scalar field with \( m \approx 1.5 \times 10^{-6}m_{\text{Pl}} \); it would be higher if the scalar field could decay into radiation at the end of inflation) is about \( 10^{-56} \). However, the fraction goes down with the minimum number of e-folds required, \( N \), only by an inverse square root of \( N \), and not as \( e^{-3N} \), so there is no conflict with not observing the universe to have such a minimal amount of inflation that spatial curvature is observable.

6 Conclusions

Although the total canonical Liouville-Henneaux-Gibbons-Hawking-Stewart measure is infinite for Friedmann-Lemaître-Robertson-Walker classical universes with a minimally coupled massive scalar field and a positive cosmological constant, it is finite for completely nonsingular solutions (which have positive scale factor everywhere). Nearly all of the solutions have the energy density never more than twice the effective energy density of the cosmological constant, but the tiny fraction, \( \sim 10^{-56} \), of the measure in which the energy density ever exceeds this tiny amount has at least \( \sim 44 \) e-folds of inflation and gives a measure that decreases only very slowly (as an inverse square root) with the minimal number of e-folds required.
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References

[1] M. Henneaux, “The Gibbs Entropy Production in General Relativity,” Nuovo Cim. Lett. 38, 609-614 (1983).

[2] G. W. Gibbons, S. W. Hawking, and J. M. Stewart, “A Natural Measure on the Set of All Universes,” Nucl. Phys. B 281, 736-751 (1987).

[3] D. N. Page and S. W. Hawking, “How Probable Is Inflation?” Nucl. Phys. B 298, 789-809 (1988).

[4] G. W. Gibbons and N. Turok, “Measure Problem in Cosmology,” Phys. Rev. D 77, 063516 (2008).

[5] N. Turok, in preparation (2011).

[6] D. N. Page, “A Fractal Set of Perpetually Bouncing Universes?” Class. Quant. Grav. 1, 417-427 (1984).

[7] L. Parker and S. A. Fulling, “Quantized Matter Fields and the Avoidance of Singularities in General Relativity,” Phys. Rev. D 7, 2357-2374 (1973).

[8] A. A. Starobinsky, “On a Nonsingular Isotropic Cosmological Model,” Sov. Astron. Lett. 4, 82 (1978).

[9] S. W. Hawking, “Quantum Cosmology,” in Relativity, Groups and Topology II, Les Houches, 1983, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984), pp. 333-379.

[10] V. A. Belinsky, L. P. Grishchuk, I. M. Khalatnikov, and Ya. B. Zeldovich, “Inflationary Stages In Cosmological Models With a Scalar Field,” Phys. Lett. B 155, 232-236 (1985).

[11] V. A. Belinsky and I. M. Khalatnikov, “On the Degree of Generality of Inflationary Solutions in Cosmological Models with a Scalar Field,” Sov. Phys. JETP 66, 441 (1987).

[12] A. Yu. Kamenshchik, I. M. Khalatnikov, and A. V. Toporensky, “Simplest Cosmological Model with the Scalar Field,” Int. J. Mod. Phys. D6, 673-692 (1997).
[13] A. Yu. Kamenshchik, I. M. Khalatnikov, and A. V. Toporensky, “Simplest Cosmological Model with the Scalar Field II. Influence of Cosmological Constant” Int. J. Mod. Phys. D7, 129-138 (1998).

[14] N. J. Cornish and E. P. S. Shellard, “Chaos in Quantum Cosmology,” Phys. Rev. Lett. 81, 3571-3574 (1998). gr-qc/9708046.

[15] A. Yu. Kamenshchik, I. M. Khalatnikov, S. V. Savchenko, and A. V. Toporensky, “Topological Entropy for Some Isotropic Cosmological Models,” Phys. Rev. D 59, 123516 (1999) gr-qc/9809048.

[16] D. N. Page, “Symmetric-Bounce Quantum State of the Universe,” JCAP 0909, 026 (2009) arXiv:0907.1893.

[17] A. Linde, Particle Physics and Inflationary Cosmology (Harwood Academic Publishers, Chur, Switzerland, 1990).

[18] A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large-Scale Structure (Cambridge University Press, Cambridge, 2000).

[19] D. N. Page, “Minisuperspaces with Conformally and Minimally Coupled Scalar Fields,” J. Math. Phys. 32, 3427-3438 (1991).

[20] B. S. DeWitt, “Quantum Theory of Gravity. 1. The Canonical Theory,” Phys. Rev. 160, 1113-1148 (1967).