Indications for Criticality at Zero Curvature in a 4d Regge Model of Euclidean Quantum Gravity

Wolfgang Beirl\textsuperscript{a} and Bernd A. Berg\textsuperscript{b,c}

\textsuperscript{a}Market Research Inc., Paradise Island, Bahamas
\textsuperscript{b}Department of Physics, The Florida State University, Tallahassee, FL 32306, USA
\textsuperscript{c}School of Computational Science and Information Technology, The Florida State University, Tallahassee, FL 32306, USA

We re-examine the approach to four-dimensional Euclidean quantum gravity based on the Regge calculus. A cut-off on the link lengths is introduced and consequently the gravitational coupling and the cosmological constant become independent parameters. We determine the zero curvature, $\langle R \rangle = 0$, line in the coupling constant plane by numerical simulations. When crossing this line we find a strong, probably first order, phase transition line with indications of a second order endpoint. Beyond the endpoint the transition through the $\langle R \rangle = 0$ line appears to be a crossover. Previous investigations, using the Regge or the Dynamical Triangulation approach, dealt with a limit in which the first order transition prevails.

1. Introduction

Euclidean quantum gravity in four dimensions is formally defined by the path integral

$$ Z = \int Dg e^{-S_{EH}[g]} . \quad (1) $$

The Einstein-Hilbert action $S_{EH}[g]$ is a function of the 4-geometry $g$, $S_{EH}[g] = -L_p^{-2}R[g] + \Lambda V[g]$. The Planck length $L_p$ determines the gravitational coupling constant for the total curvature $R[g]$ and the total 4-volume $V[g]$ enters the action if the cosmological constant $\Lambda$ is non-zero. In order to replace the path integral (1) by a finite integral on the simplicial lattice,

$$ Z = \int Dq e^{-S[q]} . \quad (2) $$

filled for every (sub-)simplex. In the following we will use the squared lengths $q_l$ associated with each link $l$ as the basic variables and a given set of such variables defines a 4-geometry $\{q\}$ for a given simplicial lattice. Variation of the variables $q_l$ allows us to approximate the path integral (1). One can calculate the Regge action $R[q] = \sum_t A_t \delta_t$ as a sum over all triangles $t$ with area $A_t$ multiplied by their associated deficit angles $\delta_t$. The 4-volume is calculated as a sum over the volume of all 4-simplices $s$, $V[q] = \sum_s V_s$. We then approximate the gravitational action by $S_{EH} \rightarrow S[q] = -L_p^{-2}R[q] + \Lambda V[q]$. It has been demonstrated how the geometrical variables of the Regge calculus converge towards the corresponding variables of General Relativity in the classical continuum limit [1,4]. It requires an upper bound on the link lengths, $q_l < Q$, and a limit $F$ on the 'fatness' $\phi_s = V_s / \max(q_i^2)$ of each 4-simplex, $\phi_s > F$. The continuum limit is then reached for $N_0 \rightarrow \infty$, $Q \rightarrow 0$ with $F > 0$ (Cheeger et al. [4]).

In order to replace the path integral (1) by a finite integral on the simplicial lattice,

$$ Z = \int Dq e^{-S[q]} . \quad (2) $$
we need to define the integration measure. A simple choice is the uniform measure

\[ Dq = \int_{0}^{Q} \prod_{l} dq_{l} \Theta_{F}[q] \]  

(3)

where the function \( \Theta_{F} \) equals 1 when all triangle inequalities are fulfilled and \( \phi_{s} > F \) for all 4-simplices; it is 0 otherwise. The integration limit \( Q \) is necessary for a well-defined integral, but it has no physical meaning in itself. Indeed, simple re-scaling \( q_{l} \rightarrow Q q_{l} \) formally removes it and leads to the integral

\[ Z(\beta, \lambda) = \int_{0}^{1} \prod_{l} dq_{l} \Theta_{F}[q] e^{\beta R[q]-\lambda V[q]} \]  

(4)

where \( \beta = Q L_{c}^{-2} \) and \( \lambda = Q^{2} \Lambda \).

In previous numerical investigations [5,6] the upper limit \( Q \) was infinite which corresponds to the limit \( \beta \rightarrow \infty \) and \( \lambda \rightarrow \infty \) in the integral (4).

In the following we will numerically evaluate the integral (4) in order to determine the phase structure by varying the coupling parameters \( \beta \) and \( \lambda \). Our results suggest a physically interesting continuum limit for finite values of the coupling parameters.

2. Results

The numerical computations have been performed for the integral

\[ Z(\beta, \lambda) = \int_{0}^{c} \prod_{l} dq_{l} \Theta_{F}[q] e^{\beta R[q]-\lambda V[q]} \]  

(5)

with \( c = 10 \) instead of (4) with \( c = 1 \). This means that the coupling constants and expectation values need to be adjusted accordingly, e.g. \( \beta \rightarrow c \beta \), \( \lambda \rightarrow c^{2} \lambda \) etc. We set \( F = 10^{-6} \) but our results seem to be independent of the specific value for a relatively wide range \( F = 10^{-8} \) to \( 10^{-5} \). We used a hypercubic regular triangulation of the 4-torus [2,3], with \( N_{0} = 4^{4}, 4^{3} \times 10, 6^{4} \) and \( 8^{4} \) vertices.

From simulations with \( \lambda = 1, 2, 3, 4 \) and varying \( \beta \) we find expectation values \( < R > \), which are (almost) zero for \( \beta_{0}(\lambda) \) values; \( \beta_{0} \) initially decreases for increasing \( \lambda \). Then, as depicted in figure 1, we find that \( \beta_{0} \) increases for \( \lambda = 4, 5, 6, 7 \).

Furthermore, the transition from \( < R > < 0 \) to \( < R > > 0 \) is no longer defined clearly, because we have to deal with a significant hysteresis for large \( \lambda \). The line \( \beta_{0}(\lambda) \) is depicted in figure 2 for different lattice size. The curve \( \beta_{0}(\lambda) \) is interesting, since a vanishing curvature is a desired ingredient for a physical continuum limit.

As we approach the turning point \( \lambda_{c} \sim 4.0 \) from smaller \( \lambda \), the susceptibility \( \chi_{q} = N_{1}^{-1} (< q^{2} > - < q >^{2}) \) with \( q = N_{1}^{-1} \sum q_{l} \), increases significantly; this increase is more pronounced for larger lattice size as depicted in figure 3.

Next, we performed calculations varying \( \lambda \) at \( \beta = 0.175 \) and 0.172 for the lattice size \( 4^{3} \times 10 \).

As expected, we find that the curvature \( < R > \) increases initially, exhibiting a transition from negative to positive values, but when \( \lambda \) increases further a second transition back to negative values is
Figure 3. Susceptibility $\chi_q$ as function of $\lambda$ at $\beta_0$ for increasing lattice size.

Figure 4. Susceptibility $\chi_R$ as function of $\lambda$ for two $\beta$ values.

We depict $\chi_R = N_j^{-1}(<p^2> - <R^2>)$, measuring curvature fluctuations, in figure 4; for $\beta = 0.175$ one can distinguish two peaks corresponding to the two transitions. These peaks merge into one at $\beta = 0.172$.

We also measured the 2-point correlation function $K(d) = <q_0q_d> - <q_0>^2$ along the elongated direction on the $4^3 \times 10$ lattice for $\beta_0(\lambda)$ and found that the function $K(d)$ is significantly different from zero for $d > 1$. Negative values alternate with positive values, which indicates a non-trivial interaction. The correlations increase for $\lambda \to \lambda_c$.

Although we are not able to provide a complete finite size scaling analysis, our data suggest the following: In the region $\lambda > \lambda_c$, with $\lambda \sim 4.0$, one deals with a strong phase transition, which is likely of first order. In the region $\lambda < \lambda_c$ we are currently unable to distinguish between a crossover transition or a higher-order phase transition. However the end-point $\lambda_c$ itself is a natural candidate for a second-order phase transition. The increase of susceptibility $\chi_q$ and the link correlations $K(d)$ approaching $\lambda_c$ from smaller values confirm this picture.

3. Summary and Conclusions

Our simulations differ from those of the previous literature by introducing the limit $Q$ in the functional integral (3). While a simple re-scaling removes $Q$ formally from the partition function (4), our model remains more general than those simulated in the literature, which are recovered in our limit of large $\lambda$. Remarkably, the new model appears to have a second order fixed point in the $\lambda - \beta$ plane, which deserves further study.

REFERENCES

1. T. Regge, Nuovo Cimento 9 (1961) 558.
2. B.A. Berg, Phys. Rev. Lett. 55 (1985) 904.
3. H.W. Hamber and R.M. Williams, Phys. Lett. B 157 (1985) 368.
4. R. Friedberg and T.D. Lee, Nucl. Phys. B 242 (1984) 145; G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, Nucl. Phys. B 245 (1984) 343; H. Cheeger, W. Müller and R. Schrader, Commun. Math. Phys. 92 (1984) 405.
5. M. Gross and H. Hamber, Nucl. Phys. B 364 (1991) 703; C. Holm and W. Janke, Phys. Lett. B 335 (1994) 143.
6. W. Beirl, P. Homolka, B. Krishnan, M. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 42, 710 (1995); J. Riedler, W. Beirl, E. Bittner, A. Hauke, P. Homolka and H. Markum Class. Quant. Grav. 16, 1163 (1999).