Polygonal approximation and energy of smooth knots

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Abstract

We establish a fundamental connection between smooth and polygonal knot energies, showing that the Minimum Distance Energy for polygons inscribed in a smooth knot converges to the Möbius Energy of the smooth knot as the polygons converge to the smooth knot. However, the polygons must converge in a “nice” way, and the energies must be correctly regularized. We determine an explicit error bound between the energies in terms of the number of the edges of the polygon and the Ropelength of the smooth curve.

Key words: Polygonal Knots, Möbius Energy, Ropelength, Knot Energy, Physical Knot Theory

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1 Introduction

Given a knot $K$ in 3-space, there are several ways to define an “energy function” that measures how complicated the knot is in its spatial conformation. In this paper, we establish a fundamental approximation theorem, showing that when both are appropriately normalized, the Minimum Distance Energy for polygons inscribed in a smooth curve converge to the Möbius Energy of the curve as the polygons converge to the smooth knot. We do a careful analysis, and determine an explicit error bound (Theorem 1), from which the approximation (Theorem 2) follows immediately.

In Section 2, we state the main theorems and agree on notation for the whole paper. In Section 3 we present a number of lemmas. These establish useful properties of curves and chords, so they may be of independent interest. In Section 4, we outline the proof of the error bound (Theorem 1), especially how to divide the problem into several cases (more precisely, divide the domains into different “zones”) for which different analyses are needed. In Section 5, we give the detailed analyses for the various cases, and in Section 6, we combine the results from Section 5 to obtain the overall bound.

Of course the error depends on how well the polygon approximates the smooth curve. However, there are more subtle issues to confront in controlling the error: One must reckon with the amount of curvature the knot has, and how close it is to being self-intersecting. These are captured by the thickness radius $r(K)$ (see later in this section for definition). Our error bound is developed in terms of the total arc-length $\ell(K)$, the number of edges of the inscribed polygon $n$, the mesh size $\delta = \frac{\ell(K)}{n}$, the thickness radius $r(K)$, and the ratio $E_L(K) = \frac{\ell(K)}{r(K)}$. Since these quantities are interrelated, there are various ways to write the bound: the one we give in Theorem 1 is stated in terms of $n$ and $E_L(K)$ to emphasize that it is invariant under change of scale.

Let $t \to x(t)$ be a unit-speed parameterization of $K$ with domain a circle $C$. The Möbius Energy or O’Hara Energy is

$$E_0(K) = \int_{C \times C} \frac{1}{|x(t) - y(s)|^2} - \frac{1}{|s - t|^2} \, ds \, dt.$$  

The energy $E_0$ was defined and studied in [1, 2, 3, 4]. The subscript 0 in $E_0$ reminds us that this version of the energy is exactly zero if $K$ is a circle.

By visualizing a smooth knot as being made of some “rope”, with a positive thickness, we obtain a fundamental measure of knot complexity. Hold the core knot $K$ fixed and thicken the rope until the moment of self-contact. Call that sup radius the thickness radius or injectivity radius, $r(K)$. Here is
a more precise definition: For small enough $r$, the knot $K$ has a solid torus neighborhood consisting of pairwise disjoint disks of radius $r$ centered at the points of $K$ and orthogonal to $K$ at those centers. Gradually increase $r$ until some meridional disks touch; we call that supremum of good radii $r(K)$. The ratio

$$E_L(K) = \frac{\text{arc-length of } K}{r(K)},$$

called the Rope-Length of $K$, is a scale-invariant numerical measure of knot compaction.

The basic theorems on thickness appear in (5), although the energy $E_L$ first appeared in (6). We recall the properties of $E_L$ in Section 3.3.

Let $P$ be a polygon with $n$ edges. The Minimum Distance Energy of $P$ is defined (7) as follows: For each pair $X, Y$ of nonconsecutive edges of $K$, compute the minimum distance between the segments $MD(X, Y)$, define $U_{md}(X, Y) = \frac{\text{length}(X) \cdot \text{length}(Y)}{[MD(X,Y)]^2}$, and sum:

$$U_{md}(P) = \sum_{\text{all edges } X} \sum_{Y \neq X \text{ or adjacent}} U_{md}(X, Y).$$

This version of $U_{md}$ counts each edge-pair twice, analogous to a double integral over (most of) $K \times K$. We write $U_{md}'$ just to distinguish from the original version (7) that counted each pair once. To consider knots with varying numbers of segments, we regularize by subtracting the energy associated to a standard regular $n$-gon (8) (or see (9)). Note that $U_{md}'$ is scale invariant, so we can use any regular $n$-gon and get the same number. We define

$$E_{md}(P) = U_{md}'(P) - U_{md}'(\text{regular } n\text{-gon}).$$

The energy $U_{md}$ has been implemented in several software systems (10; 11; 12), and studied in (13; 14; 15).

We shall show that for suitable polygonal approximations $P$ of a smooth curve $K$, $E_{md}(P) \approx E_0(K)$. While the Möbius Energy is defined for $C^{1,1}$ curves, our proof requires that the knot $K$ be $C^2$ smooth.

In order for the approximation to work, we need to be careful about what it means to say, “the polygon $P$ is a close approximation of $K$”. First, we need to prevent extreme changes in the edge lengths of $P$ (see Figure 1). Suppose $P$ is a polygon closely inscribed in $K$. We can slide vertex $v_3$ along $K$ towards vertex $v_4$, making edge $e_3$ arbitrarily short, making edge $e_2$ longer, and keeping the other edges of $P$ fixed. This will make the contribution of the edge-pair $(e_2, e_4)$ to $E_{md}$ arbitrarily large. Thus, we can make polygons $P'$ that also seem like close approximations of $K$, yet $E_{md}(P') >> E_0(K)$. 
extreme differences in edge lengths causes $E_{md}$ to become large despite having the polygon close to the smooth curve. In such a case, $E_{md}$ will be much larger than $E_0$.

Fig. 1. Portions of the smooth curve can be arbitrarily close, causing $E_0$ to be very large while $E_{md}$ of the inscribed polygon remains fixed. In such a case, $E_0$ is much larger than $E_{md}$.

To prevent this problem, we have to limit the variation in edge lengths of polygons inscribed in $K$; we do this by having the vertices equally spaced in arc-length along $K$. One can modify our arguments to handle other tractable approximating polygons, e.g. equal edge lengths or “equal time” subdivisions of a regularly parameterized curve.

Conversely, we can find situations where $E_{md}(P) << E_0(K)$. Let $P$ be the polygon (not drawn) $\langle v_1, v_2, v_3, v_4, v_1 \rangle$ in Figure 2. We construct the quadrilateral so that the arcs between consecutive vertices are of equal length. Keeping the vertices fixed, deform the arc $\tilde{v}_1\tilde{v}_2$ and the arc $\tilde{v}_3\tilde{v}_4$ slightly so they get arbitrarily close to intersecting (where one crosses over the other in the figure). This makes $E_0(K) - E_{md}(P)$ arbitrarily large. This problem is detected by the fact that $r(K)$ decreases to 0, since normal disks of smaller and smaller radii will intersect. This is why the error bound in Theorem 1 must take into account the geometric quantity $r(K)$.

To avoid situations as in Figure 3, we assume the phrase “inscribed polygon” means the vertices of $P$ occur in the same order as they occur along $K$.

Finally, note that the regularizations play an essential role, making the proof more delicate than may be at first evident. See the discussion in Section 4.
2 Statement of main results and notation

We shall use the following notation throughout the paper.

- \( K \) is a \( C^2 \) smooth simple closed curve in \( \mathbb{R}^3 \).
- \( \ell(K) \) is the total arc-length of \( K \).
- \( r(K) \) is the thickness radius of \( K \).
- \( E_L(K) = \frac{\ell(K)}{r(K)} \) is the Rope-Length.
- \( \delta = \frac{\ell(K)}{n} \) is the mesh size of the inscribed polygon.
- \( K \) is subdivided into \( n \) arcs of equal length. \( \delta = \frac{\ell(K)}{n} \).
- \( v_1, \ldots, v_n \) are the subdivision points along \( K \).
- \( P_n \) is the polygon formed by connecting the points \( v_1, \ldots, v_n, v_1 \) in order.
- \( \text{arc}(x, y) \) is the length of the shorter of the two arcs of \( K \) connecting \( x \) and \( y \).
- \( |e| \) is the length of the line segment \( e \).

Additional notation used in the proofs is listed at the beginning of Section 4.

**Theorem 1 (Error Bound)** For any smooth knot \( K \), if \( P_n \) are inscribed polygons as above and \( n \) is large enough that \( n > E_L(K) \), i.e. \( \delta < r(K) \), then

\[
|E_0(K) - E_{md}(P_n)| \leq \Phi(n, E_L(K)),
\]

where \( \Phi \) is a linear combination (see final page of the paper) of six fractions of the form \( \frac{E_L(K)^a}{n^b} \) for various \( a > 0 \) and \( b > 0 \). By combining some terms, we can take \( \Phi = 550\frac{E_L(K)^{5/4}}{n^{1/4}} + 10\frac{E_L(K)^4}{n} \). For very large \( n \) (\( n > E_L(K)^{11/3} \)), we can use \( \Phi = 560\frac{E_L(K)^{5/4}}{n^{1/4}} \).

**Remark.** There are other ways to write this scale-invariant error bound, using the identity

\[
\frac{E_L(K)}{n} = \frac{\delta}{r(K)}.
\]

From Theorem 1, we have immediately:

**Theorem 2 (Approximation Theorem)** For any smooth knot \( K \), if \( P_n \)
are inscribed polygons as above, then as \( n \to \infty \), \( E_{md}(P_n) \to E_0(K) \).

**PROOF.** The supporting lemmas and the proof of the theorem occupy the rest of the paper. The lemmas are in Section 3. In Section 4, we outline the proof and explain how the domains will be divided into zones for which different analyses are needed. We give the analysis for each zone in Section 5 and put them all together in Section 6.

There are numerous coefficients in the calculations; we constantly round up and pick the worst-case values, to keep the claims accurate and the numbers simple.

## 3 The lemmas

In this section, we prove the lemmas needed for the proof of the main theorem.

### 3.1 Lemmas about the cosine function, also chords and arcs of circles

**Lemma 3** If \( 0 < \phi \leq \pi \), then the following hold:

(a) \( 1 - \frac{1}{2} \phi^2 \leq \cos \phi \leq 1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4 \),

(b) \( \phi^2 - \frac{1}{12} \phi^4 \leq 2 - 2 \cos \phi \leq \phi^2 \),

(c) \( 1 - \frac{1}{12} \phi^2 \leq \frac{2 - 2 \cos \phi}{\phi^2} \leq 1 \),

(d) \( \frac{\phi^2}{2 - 2 \cos \phi} \leq 1 + \frac{1}{2} \phi^2 \).

**PROOF.** For (a), consider the Taylor series for \( \cos(\phi) \). Parts (b), (c), (d) follow immediately.

**Lemma 4**

(a) On the unit circle \( C \), for any points \( x, y \),

\[
\frac{1}{12} < \frac{1}{|x - y|^2} - \frac{1}{\arccos(x, y)^2} \leq \frac{1}{4} - \frac{1}{\pi^2}.
\]

(b) On a circle of radius \( R \),

\[
\frac{1}{12} \frac{1}{R^2} < \frac{1}{|x - y|^2} - \frac{1}{\arccos(x, y)^2} \leq \left(\frac{1}{4} - \frac{1}{\pi^2}\right) \frac{1}{R^2}.
\]
PROOF. Let $\phi$ be the angle ($\leq \pi$) between points $x$ and $y$ on the circle. Since $C$ is the unit circle, $\text{arc}(x, y) = \phi$ and $|x - y|^2 = 2 - 2 \cos \phi$. The function \[
\frac{1}{2 - 2 \cos \phi} - \frac{1}{\phi^2}
\]
is monotone, has a maximum at $\phi = \pi$, and is bounded below by the limiting value as $\phi \to 0$. Part (b) is similar.

Next we want to compare the quantities $\frac{1}{|x - y|^2}$ and $\frac{1}{|X - Y|^2}$, where the points lie on circles of different sizes.

**Lemma 5** Suppose $r < R$ are radii of circles and $0 < a < \pi r$. Construct any arcs of (the same) length $a$ on the two circles and let $x, y$ and $X, Y$ be the endpoints of the two arcs. Then

$$0 < \frac{1}{|x - y|^2} - \frac{1}{|X - Y|^2} < \left(\frac{1}{4} - \frac{1}{\pi^2}\right) \frac{1}{r^2}.$$ 

**PROOF.** Chord length is always less than arc-length. For a fixed arc-length, as the radius gets larger, the chord length gets closer to the arc-length. Thus $|X - Y| > |x - y|$. On the other hand, applying Lemma 4(b) to each circle, we have

$$\frac{1}{|x - y|^2} - \frac{1}{|X - Y|^2} < \left(\frac{1}{4} - \frac{1}{\pi^2}\right) \frac{1}{r^2} - \frac{1}{12} \frac{1}{R^2}.$$ 

3.2 Lemmas about chords and arcs of general curves

We rely a lot on Schur’s Theorem. Here is the version we need:

**Lemma 6** Let $K$ be a $C^2$ smooth curve in $\mathbb{R}^3$ whose curvature everywhere is $\leq$ some number $\kappa$. Let $C$ be a circle of curvature $\kappa$, i.e. of radius $r = \frac{1}{\kappa}$. Let $x, y \in K, s, t \in C$ such that $\text{arc}(x, y) = \text{arc}(s, t) \leq \pi r$. Then the chord distances satisfy

$$|x - y| \geq |s - t|.$$ 

When we write the chord length on $C$ in terms of the central angle, this becomes

$$|x - y| \geq r \left(2 - 2 \cos \left(\frac{\text{arc}(s, t)}{r}\right)\right)^{1/2}.$$
PROOF. See Schur’s Theorem in [16].

**Lemma 7** Let $K$ be a $C^2$ smooth curve in $\mathbb{R}^3$, with minimum radius of curvature $r$. Suppose $x : [0, \pi r] \to \mathbb{R}^3$ is a unit speed parameterization of an arc of $K$ of length $\pi r$. Then the function $|x(t) - x(0)|$ is monotone increasing. That is to say: As points move farther apart along the curve, they also move farther apart in space, so long as the arc-distance is no greater than $\pi r$.

**PROOF.** Let $f(t) = |x(t) - x(0)|^2 = (x(t) - x(0)) \cdot (x(t) - x(0))$. We claim $\frac{df}{dt} > 0$ for $t \in (0, \pi r)$. The derivative $\frac{df}{dt} = 2 (x(t) - x(0)) \cdot x'(t)$. Thus we need to show that this dot product is positive, for all points $x(t)$ in the interior of the arc. The proof uses the same central idea as the proof of Schur’s theorem.

We have

$$x(t) - x(0) = \int_0^t x'(s) \, ds ,$$

so

$$(x(t) - x(0)) \cdot x'(t) = \int_0^t x'(s) \cdot x'(t) \, ds .$$

The dot product $x'(s) \cdot x'(t)$ is just the cosine of the angle $\leq \pi$ between the two velocity vectors. This angle is measured by the length of the geodesic arc on the unit sphere between the unit vectors $x'(s)$ and $x'(t)$. The trace of $x'(u)$, as $u$ runs from $s$ to $t$, is another path on the unit sphere between the same vectors. The length of that path gives an upper bound for the length of the geodesic path. Thus, since $|x''(u)| \leq 1/r$ (recall $r =$ minimum radius of curvature),

$$\angle(x'(s), x'(t)) \leq \int_s^t |x''(u)| \, du \leq \frac{(t - s)}{r} .$$

Since $0 \leq s \leq t \leq \pi r$, and the cosine function is decreasing on $[0, \pi]$, we have

$$\cos(\angle(x'(s), x'(t))) \geq \cos \frac{(t - s)}{r} .$$

Thus

$$(x(t) - x(0)) \cdot x'(t) \geq \int_0^t \cos \frac{(t - s)}{r} \, ds = r \sin(t/r) .$$

For $0 < t < r \pi$, $\sin(t/r) > 0$.

**Lemma 8** Let $K$ be a $C^2$ smooth curve in $\mathbb{R}^3$, with minimum radius of curvature $r$. Let $x : [0, \ell(K)] \to \mathbb{R}^3$ be a unit speed parameterization of $K$. Suppose $0 \leq a < b < c < d \leq \pi r$, so $x(a), x(b), x(c), x(d)$ are four points in order along $K$, contained in an arc of total length $\leq \pi r$.  

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Then the minimum spatial distance between line segments \( \langle x(a)x(b) \rangle \) and \( \langle x(c)x(d) \rangle \) is realized at the closest endpoints. Taking into account Lemma 7, this says,

\[
MD(\langle x(a)x(b) \rangle, \langle x(c)x(d) \rangle) = |x(c) - x(b)|.
\]

**Proof.** Without loss of generality, rescale the curve to have \( r = 1 \). Then the four points lie in an arc of total length \( \leq \pi \).

Let \( A \) denote the segment \( \langle x(a)x(b) \rangle \) and \( C \) the segment \( \langle x(c)x(d) \rangle \). We shall show that for each point \( x \in A \), the point \( x(c) \) is the closest point of \( C \) to \( x \); so \( x(c) \) is the closest point of \( C \) to \( A \). By a symmetric argument, the point \( x(b) \) is the closest point of \( A \) to \( C \).

Fix a point \( y \in C, y \neq x(c), x(d) \). For any \( x \in A \), construct the directed line segment from \( x \) to \( y \). We claim that the vectors satisfy

\[
(y - x) \cdot (x(d) - x(c)) > 0.
\]

If this dot product is positive, then moving \( y \) along \( C \) closer to \( x(d) \) will increase the distance to \( x \), and moving \( y \) closer to \( x(c) \) will decrease the distance to \( x \). Thus \( x(c) \) must be the closest point of \( C \) to \( x \).

We now show that the above dot product is positive for each \( x, y \). It is convenient to think for a moment of fixing \( y \) and varying \( x \). Let \( P_y \) be the plane through \( y \) perpendicular to \( C \). Rotate the entire ensemble so that the vector \( x(d) - x(c) \) points “up”. Then the dot product inequality is equivalent to the assertion that the entire line segment \( A \) lies below \( P_y \). It suffices to show that each vertex \( x(a), x(b) \) lies below \( P_y \).

But in fact, if \( x(a) \) and \( x(b) \) lie below \( P_{x(c)} \), then they lie below \( P_y \). We have now reduced the lemma to the following claim, an inequality that involves only the given points on \( K \). The inequality is stated for parameter value \( a \), and is identical for \( b \). If \( 0 \leq a < c < d \leq \pi \), then

\[
(x(c) - x(a)) \cdot (x(d) - x(c)) > 0.
\]

The rest of the proof is similar to the proof of Lemma 7 with some trigonometry at the end. We first express the difference vectors as integrals of derivatives,

\[
(x(c) - x(a)) \cdot (x(d) - x(c)) = \int_a^c \int_c^d x'(s) \cdot x'(t) \, dt \, ds.
\]
Since the cosine function is decreasing on \([0, \pi]\), we have
\[
\cos(\angle(x'(s), x'(t))) \geq \cos(t - s) .
\]

As in the proof of Lemma 7, \(x'(s) \cdot x'(t) \geq \cos(t - s)\), so
\[
(x(c) - x(a)) \cdot (x(d) - x(c)) \geq \int_a^c \int_c^d \cos(t - s) \ dt \ ds .
\]

The integral evaluates to
\[
\cos(d - c) - \cos(d - a) - 1 + \cos(c - a) ,
\]
which is positive.

The previous two lemmas tell us that for arcs that are near each other in arc-length along a curve, the minimum spatial distance between the arcs is the same as the minimum distance between their inscribed chords. For more general pairs of arcs, the minimum distances usually will not be equal, but they still are related.

**Lemma 9** Suppose \(\alpha, \beta\) are smooth arcs in \(\mathbb{R}^3\), each of length \(\delta\), and each having radius of curvature everywhere \(\geq r \geq \delta\). Let \(e\) be the chord joining the endpoints of \(\alpha\) and \(f\) the corresponding chord for \(\beta\).

(a) The maximum distance between \(\alpha\) and \(e\) (likewise between \(\beta\) and \(f\)) is
\[
\leq \frac{1}{\sqrt{48}} \frac{\delta^2}{r} .
\]

(b) If \(\text{MD}(\alpha, \beta)\) is the minimum spatial distance between \(\alpha\) and \(\beta\), and \(\text{MD}(e, f)\) is the minimum distance between the chords, then
\[
|\text{MD}(e, f) - \text{MD}(\alpha, \beta)| \leq \frac{\sqrt{3}\delta^2}{6r} \leq \frac{\sqrt{3}}{6} r .
\]

**Proof.** For part (a), imagine the chord \(e\) as a rod with “string” of length \(\delta\) attached at either end, and ask, “What configuration allows the string to reach as far as possible from the rod?” The answer is when the string is pulled out to form two equal sides of an isosceles triangle, with the rod as the base. The maximum distance that any point of \(\alpha\) can be from \(e\) is the altitude \(h\) of this isosceles triangle, so \(h^2 = (\frac{\delta}{2})^2 - (|e|\frac{3}{2})^2\). Since \(\delta \leq r\), in particular \(\delta \leq \pi r\), we can apply Schur’s theorem: By Lemma 6 and Lemma 3(b),
\[
\frac{1}{4} |e|^2 \geq \frac{1}{4} \delta^2 - \frac{1}{48} \delta^4 ,
\]
so

\[ h^2 \leq \frac{1}{48} \frac{\delta^4}{r^2}. \]

Part (b) follows from part (a), the triangle inequality, and the fact that \( \delta \leq r \).

### 3.3 Lemmas about the thickness of a curve

The first lemma is a characterization of the thickness radius \( r(K) \) in terms of curvature and the critical self-distance.

Fix a point \( x_0 \in K \) and consider points \( y \) that start at \( x_0 \) and gradually move along \( K \). A point \( y \) is a critical point for the function \( |y - x_0| \) when \( y = x_0 \) or when \( \langle xy \rangle \perp y' \). We define the critical self-distance of \( K \) (an idea attributed by J. O’Hara to N. Kuiper) to be

\[ \text{sd}(K) = \min \{ |y - x| : x \neq y \in K \text{ and } \langle xy \rangle \perp y' \} . \]

**Lemma 10** The thickness of a smooth knot is bounded by the minimum radius of curvature and half the critical self-distance. In fact,

\[ r(K) = \min \left\{ \text{MinRad}(K), \frac{1}{2} \text{sd}(K) \right\}. \]

**PROOF.** See [5].

The next lemma is a consequence of Lemmas 6 and 10, and is proven in [17].

**Lemma 11** Suppose \( K \) is a smooth knot of thickness radius \( r(K) = r \). For any \( x, y \in K \) with \( \text{arc}(x, y) \geq \pi r \), we must have \( |y - x| \geq 2r \).

**Lemma 12** Let \( K \) be a \( C^2 \) smooth closed curve in \( \mathbb{R}^3 \), with minimum radius of curvature \( r \). Let \( C \) be a circle whose total arc-length is the same as \( K \), and \( R \) be the radius of \( C \). Then \( r \leq R \) and (from Lemma 10) the thickness radius \( r(K) \leq R \).

**PROOF.** Since \( r \) is the minimum radius of curvature of \( K \), the maximum curvature of \( K \) is \( \frac{1}{r} \), so the total curvature of \( K \) is at most \( \frac{\ell(K)}{r} \). On the other hand, by Fenchel’s theorem ([18]), the total curvature of \( K \) is at least \( 2\pi \). Thus \( 2\pi r \leq \ell(K) = 2\pi R \).
Fig. 4. The objects of study: smooth knot $K$ with arc $\alpha_i$ and vertex $v_i$, inscribed polygon $P$ with vertex $v_i$, circle $C$ with arc $\beta_i$ and vertex $b_i$ corresponding to $\alpha_i$ and $v_i$ respectively, and inscribed regular $n$-gon $Q$ with vertex $b_i$.

**Lemma 13** For any $C^2$ smooth closed curve $K$, $E_L(K) \geq 2\pi$.

**PROOF.** By Lemma 10, the curvature of $K$ is everywhere $\leq 1/r(K)$. Thus the total curvature of $K$ is $\leq \ell(K)/r(K) = E_L(K)$. But the total curvature of a closed curve is $\geq 2\pi$.

4 Notation and outline of proof of Theorem 1

We have four objects of interest: the knot $K$, the circle $C$, the inscribed $n$-gon $P$, and the regular $n$-gon $Q$. In the following list, refer to Figure 4

- $K$ is a $C^2$ smooth simple closed curve in $\mathbb{R}^3$.
- $C$ is a circle with total arc-length $\ell(C) = \ell(K)$.
- $r(K)$ is the thickness radius of $K$.
- $K$ is subdivided into $n$ arcs of equal length $\delta = \frac{\ell(K)}{n}$, and we are assuming $\delta < r(K)$ (so $n > E_L(K)$).
- $v_1, \ldots, v_n$ are the subdivision points along $K$.
- $\alpha_i$ is the arc of $K$ with endpoints $v_i$ and $v_{i+1}$. We number the vertices modulo $n$, so $\alpha_n$ is the arc from $v_n$ to $v_1$.
- $R$ is the radius of $C$, so $R = \frac{\ell(K)}{2\pi}$.
- $t \to x(t)$ is a unit speed parameterization of $K$ from $C$.
- $b_1, \ldots, b_n$ are evenly spaced points along $C$ such that $x(b_i) = v_i$. 
• $\beta_i$ is the arc of $C$ corresponding to $\alpha_i$.
• $P$ is the polygon formed by connecting the points $v_i$ in order.
• $e_i$ is the edge of $P$ from $v_i$ to $v_{i+1}$, with length denoted $|e_i|$.
• $Q$ is the regular polygon inscribed in $C$, with vertices $b_1, \ldots, b_n$.
• $f_i$ is the edge of $Q$ with vertices $b_i, b_{i+1}$, with length $|f_i|$.

Just to have all the important parameters specified in one place, we also define two integers, $m$ and $p$, whose role will be evident later in this section.

• $m = \lfloor \pi r(K) \delta \rfloor$. For a vertex $v_i$, the vertices $v_i, v_{i+1}, \ldots, v_{i+m}$ are a maximal list that lie in an arc of $K$ of length $\leq \pi r(K)$.
• $p = \lfloor m^{3/4} \rfloor$. For a list of $m$ vertices as specified in the previous item, we will need to distinguish an initial bunch from the rest. It turns out that the number we need to separate off should be some fractional power of $m$ strictly greater than $1/2$, and we take $3/4$ for simplicity.

We shall analyze the energies in terms of individual pairs of arcs and/or edges.

The energies are

$$E_0(K) = \int_{x \in K} \int_{y \in K} \frac{1}{|x-y|^2} - \frac{1}{|s-t|^2},$$

where

$$E_0(\alpha_i, \alpha_j) = \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{|x-y|^2} - \frac{1}{|s-t|^2},$$

and

$$E_{md}(P) = U_{md}'(P) - U_{md}'(Q)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} U_{md}(e_i, e_j) - U_{md}(f_i, f_j) \quad (j \neq i - 1, i, i + 1).$$

Sometimes we need to treat $E_0$ as the difference between two integrals, so we also define

$$E(\alpha_i, \alpha_j) = \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{|x-y|^2},$$

and likewise for $E(\beta_i, \beta_j)$ for arcs on $C$. As one might expect, our overall plan is to show that the various terms in the sum (1) are close to the corresponding terms in (2). However, some terms in (1) have no corresponding term in (2); and even when they do, there are different cases requiring different analyses.

We shall, in fact, consider four kinds of pairs $(i, j)$, bound each contribution to the error, and add them to get a full error bound.

Here is a “schematic diagram” of our situation: We want to show that something of the form $f(W - X)$ is close to something of the form $(Y - Z)$. For the
edge pairs where $E_0$ has a contribution and $E_{md}$ is not defined, we show the $E_0$ contribution is small. For other edge pairs, we sometimes show that $f(W - X)$ and $(Y - Z)$ each is small, and sometimes show that $|Y - f W|$ and $|Z - f X|$ both are small. The analysis has to involve this kind of complication because the unregularized polygon energy $U'_{md}(P)$ is not a good approximation of the divergent integral $\int_{K \times K} \frac{1}{|y - x|^2}$, that is $|Y - f W|$ does not get negligibly small for arc pairs (and their corresponding segment pairs) that are extremely close together along $K$. Here is a simple example to illustrate the difficulty: Consider two segments $A = [0, \epsilon]$ and $B = [2\epsilon, 3\epsilon] \subset \mathbb{R}$. Then $U_{md}(A, B) = 1$. On the other hand, $\int_{x \in A} \int_{y \in B} \frac{1}{|y - x|^2} \, dy \, dx = \ln \frac{4}{3}$. For segments close together along the curves, we need to understand the regularizing terms rather than show the two energies are close to each other.

Following are the four types of pairs (of indices $(i, j)$, edges or arcs) that determine our four “zones” for separate analysis. The definitions are symmetric, so $(i, j)$ and $(j, i)$ are of the same type.

1. **Adjacent Pairs**: $j = i - 1, i, i + 1$
   
   For these arc pairs, we bound $\sum_{i,j} E_0(\alpha_i, \alpha_j)$. Since $U_{md}$ is only defined for non-adjacent edges, there are no corresponding edge pairs for these arc pairs.

2. **Near Pairs**: non-diagonal pairs $(i, j)$ for which the arcs $\alpha_i$ and $\alpha_j$ are contained in an arc of $K$ of length $\leq \pi r(K)$.
   
   Within the Near Zone, we make an additional distinction between “Very Near” and “Moderately Near”: For each vertex $v_i$, let $A$ be either of the arcs of $K$ starting at $v_i$ and having length $\ell(A) = \pi r(K)$. The vertices contained in the arc $A$ are a sequence $v_i, v_{i+1}, \ldots, v_{i+m}$ (for the other arc, we count in the other direction). The arcs contained in $A$ are $\alpha_i, \ldots, \alpha_{i+m-1}$. The vertex $v_{i+m}$ may or may not be an endpoint of $A$.
   
   A. For $j = i + 2, \ldots, i + p$, we call $(i, j)$ a very near pair.
      
      For such $(i, j)$, we bound $\sum_{i,j} (U_{md}(e_i, e_j) - U_{md}(f_i, f_j))$ and $\sum_{i,j} E_0(\alpha_i, \alpha_j)$.
   
   B. For $j = i + p + 1, \ldots, i + m - 1$, we call $(i, j)$ a moderately near pair.
      
      For such $(i, j)$, we shall bound $\sum_{i,j} (E(\alpha_i, \alpha_j) - U_{md}(e_i, e_j))$ and $\sum_{i,j} (E(\beta_i, \beta_j) - U_{md}(f_i, f_j))$.

3. **Far Pairs**: The pairs $(i, j)$ that are neither adjacent nor near are called far.
   
   For such pairs, we shall also bound $\sum_{i,j} E(\alpha_i, \alpha_j) - U_{md}(e_i, e_j)$ and $\sum_{i,j} E(\beta_i, \beta_j) - U_{md}(f_i, f_j)$, but we need an argument different from the moderately near pairs.

See Figure 5 for an example of the zone pairings where $m = 17$. We use the same terminology for corresponding pairs of arcs in $C$; that is, if $(i, j)$ are far [resp. adjacent, very near, moderately near] on $K$, then we call them far [resp. adjacent, very near, moderately near] on $C$.

In the next section, we establish the explicit error bounds in each of the different zones. In Section 6, we collect all of the errors to determine the total
For this arc
Adjacent zone
Very Near Zone
Moderately Near Zone
Far Zone starts here

Fig. 5. The four types of zones on which we do our analysis. Note that this is just a schematic to show the arrangement of the zones with respect to a fixed arc.

error bound.

5 Proofs for the different zones

5.1 Bounds for $E_0$ in Adjacent and Very Near Zones

We establish the error bound for the combined contributions of the Adjacent and Very Near Zones to the Möbius Energy.

**Proposition 14** In the Adjacent and Very Near Zone,

$$\left| \sum_{i,j} E_0(\alpha_i, \alpha_j) \right| < 1.06 \frac{E_L(K)^{5/4}}{n^{1/4}}$$

**PROOF.** If $x, y$ are contained in diagonal or very near arcs, then $\text{arc}(x, y) \leq (p+1)\delta$. Thus it suffices to bound

$$\left| \int_{x \in K} \int_{y \in K, \text{arc}(x, y) \leq (p+1)\delta} \frac{1}{|x-y|^2} - \frac{1}{|s-t|^2} \right|. \quad (3)$$

The calculation is independent of the choice of $x$, so we analyze

$$\left| 2 \ell(K) \int_{y = x}^{x+(p+1)\delta} \frac{1}{|x-y|^2} - \frac{1}{|s-t|^2} dy \right|,$$

where the limits of integration are meant to indicate that we are integrating along an arc of $K$ of length $(p+1)\delta$ starting from $x$. 

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We are going to find upper and lower bounds for the integrand, observe that the upper bound is positive and the lower bound is negative, and conclude that the magnitude of the integrand is bounded by the difference between the upper and lower bounds. To simplify subsequent expressions, let \( r \) denote \( r(K) \) and \( a \) denote arc\((x, y)\).

Since \( s \) and \( t \) lie on a circle of radius \( R \),

\[
\frac{1}{|s-t|^2} = \frac{1}{R^2(2-2\cos(a/R))}.
\]

First we get the upper bound. Since \( \delta \leq r \), in particular \( m \geq 2 \), we have \( p < m \) and \((p+1)\delta \leq \pi r\). Thus we can apply Lemma 6 to conclude

\[
|x-y|^2 \geq r^2(2 - 2\cos(a/r)).
\]

So we have

\[
\frac{1}{|x-y|^2} - \frac{1}{|s-t|^2} \leq \frac{1}{r^2(2-2\cos(a/r))} - \frac{1}{R^2(2-2\cos(a/R))}.
\]

By Lemma 12, \( r \leq R \). By Lemma 6 applied to circles of different radii, or the argument in Lemma 5, this upper bound is nonnegative.

Now we get the lower bound. Since arc-length on any curve must be at least as large as chord length,

\[
|x-y|^2 \leq a^2.
\]

Thus

\[
\frac{1}{|x-y|^2} - \frac{1}{|s-t|^2} \geq \frac{1}{a^2} - \frac{1}{|s-t|^2} = \frac{1}{a^2} - \frac{1}{R^2(2-2\cos(a/R))},
\]

which is negative since chord length < arc-length on a circle.

Taking the difference between the nonnegative upper bound and the negative lower bound, we have

\[
\left| \frac{1}{|x-y|^2} - \frac{1}{|s-t|^2} \right| \leq \frac{1}{r^2(2-2\cos(a/r))} - \frac{1}{a^2}.
\]

So

\[
(3) \leq 2 \ell(K) \int_0^{(p+1)\delta} \frac{1}{r^2(2-2\cos(a/r))} - \frac{1}{a^2} \, da,
\]
where now we are just integrating a function of a real variable. Applying Lemma 4(b), we have

\[(3) \leq 2 \ell(K) (p + 1) \delta \left( \frac{1}{4} - \frac{1}{\pi^2} \right) \frac{1}{r^2}.\]

Since \(m \geq 3, p \geq 2\), so \((p + 1) < 1.5p\). Combining the constants, we have

\[(3) < 0.45 \frac{\ell(K)p\delta}{r^2} \leq 0.45 \frac{\ell(K)(\frac{\pi r}{2})^{3/4}\delta}{r^2} \leq 1.06 \frac{E_L(K)^{5/4}}{n^{1/4}}\]

as desired.

5.2 Bound for \(E_{md}\) in the Very Near Zone

**Proposition 15** In the Very Near Zone,

\[|E_{md}(P)| = |U'_md(P) - U'_md(Q)| < 2.76 \frac{E_L(K)^{5/4}}{n^{1/4}}.\]

**PROOF.**

\[E_{md}(\text{very near}) = 2 \sum_{i=1}^{n} \sum_{j=i+2}^{i+p} \frac{|e_i| |e_j|}{MD(e_i, e_j)^2} - \frac{|f_i| |f_j|}{MD(f_i, f_j)^2}.\]

We shall bound the inner sums uniformly in \(i\), that is bound

\[\left| \sum_{k=1}^{p-1} \frac{|e_i| |e_{i+k+1}|}{MD(e_i, e_{i+k+1})^2} - \frac{|f_i| |f_{i+k+1}|}{MD(f_i, f_{i+k+1})^2} \right| \quad (4)\]

for arbitrary \(i\), and then multiply that bound by \(2n\). Here \(k = j - i - 1\) is the number of edges separating the two edges. As in Proposition 14, we find a positive upper bound for each difference term, and a negative lower bound; so the difference between the upper and lower bounds is a bound for the absolute value.

On the circle \(C\) of radius \(R\), the edge lengths are \(|f_i| = |f_j| = \sqrt{R^2(2 - 2 \cos(\delta/R))}\), and \(MD(f_i, f_{i+k+1}) = \sqrt{R^2(2 - 2 \cos(k\delta/R))}\). So

\[(4) = \left| \sum_{k=1}^{p-1} \frac{|e_i| |e_{i+k+1}|}{MD(e_i, e_{i+k+1})^2} - \frac{R^2(2 - 2 \cos(\delta/R))}{R^2(2 - 2 \cos(k\delta/R))} \right| .\]
To simplify subsequent expressions, let \( r \) denote \( r(K) \). If we compare \( K \) locally with a circle of radius \( r \), Lemma 8 and Lemma 6 say \( MD(e_i, e_{i+k+1})^2 \geq r^2(2 - 2 \cos(k\delta/r)) \). The longest an edge can be is the arc-length, so \( (|e_i|, |e_j|) \leq \delta^2 \). Thus, an upper bound for each summand is

\[
\text{summand} \leq \frac{\delta^2}{r^2(2 - 2 \cos(k\delta/r))} - \frac{R^2(2 - 2 \cos(\delta/R))}{R^2(2 - 2 \cos(k\delta/R))}.
\]

We claim this upper bound is positive. First, \( \delta^2 > R^2(2 - 2 \cos(\delta/R)) \) since arc-length (now on the big circle \( C \)) is always \( > \) chord length. Furthermore, \( r^2(2 - 2 \cos(k\delta/r)) \leq R^2(2 - 2 \cos(k\delta/R)) \) by Lemma 5.

We next obtain a lower bound. By Lemma 10, \( r \leq \) minimum radius of curvature of \( K \). So we can apply Lemma 6 and Lemma 8 to any points that lie in an arc of \( K \) of length \( \leq \pi r \). By Lemma 6, we have \( (|e_i|, |e_{i+k+1}|) \geq r^2(2 - 2 \cos(\delta/r)) \). For the denominator, Lemma 8 gives us that \( MD(e_i, e_{i+k+1}) = |v_{i+k+1} - v_{i+1}| \), the distance between points of \( K \) whose arc-distance is \( k\delta \).

Since chord length \( \leq \) arc-length, we thus have \( MD(e_i, e_{i+k+1})^2 \leq (k\delta)^2 \). So a lower bound for the summand is

\[
\text{summand} \geq \frac{r^2(2 - 2 \cos(\delta/r))}{k^2\delta^2} - \frac{R^2(2 - 2 \cos(\delta/R))}{R^2(2 - 2 \cos(k\delta/R))}.
\]

Comparing numerators and denominators as we did for the upper bound, we see that this lower bound is always negative.

Thus, we can bound the absolute value of the summand by the difference between the upper and lower bounds:

\[
|\text{summand}| \leq \frac{\delta^2}{r^2(2 - 2 \cos(k\delta/r))} - \frac{r^2(2 - 2 \cos(\delta/r))}{k^2\delta^2} = \frac{1}{k^2} \left( \frac{k^2\delta^2}{r^2(2 - 2 \cos(k\delta/r))} - \frac{r^2(2 - 2 \cos(\delta/r))}{\delta^2} \right) \quad (5)
\]

We now appeal to our lemmas on cosines and chords. To clarify how lemmas will be used, introduce angles \( \theta = \delta/r \) and \( \phi = k\delta/r \). Thus, the bound (5) can be written

\[
(5) = \frac{1}{k^2} \left( \frac{\phi^2}{2 - 2 \cos \phi} - \frac{2 - 2 \cos \theta}{\theta^2} \right).
\]

By Lemma 3(d), \( \frac{\phi^2}{2 - 2 \cos \phi} \leq 1 + \frac{1}{2} \phi^2 \). By Lemma 3(c), \( \frac{2 - 2 \cos \theta}{\theta^2} \geq 1 - \frac{1}{12} \theta^2 \). Thus,

\[
|\text{summand}| \leq \frac{1}{k^2} \left( \frac{\phi^2}{2} + \frac{1}{12} \theta^2 \right).
\]
We return to the original double sum and see that

\[
2n \sum_{k=1}^{p-1} \frac{|e_i||e_{i+k+1}|}{MD(e_i, e_{i+k+1})^2} - \frac{|f_i||f_{i+k+1}|}{MD(f_i, f_{i+k+1})^2} \leq 2n \sum_{k=1}^{p-1} \frac{1}{2} \phi^2 + \frac{1}{12} \theta^2 \\
= 2n \sum_{k=1}^{p-1} \frac{1}{2} \frac{k^2 \phi^2 + \frac{1}{12} \theta^2}{k^2} \\
= 2n \frac{\delta^2}{r^2} \sum_{k=1}^{p-1} \left( \frac{1}{2} + \frac{1}{12 k^2} \right) \\
\leq 2n \frac{\delta^2}{r^2} (p-1) \left( \frac{7}{12} \right) \\
< \frac{7}{6} n \frac{\delta^2}{r^2} p \\
< 2.76 \frac{E_L(K)^{5/4}}{n^{1/4}}.
\]

**Remark.** For the Very Near Zone, we could use \( p \leq \) any fractional power \( m^q \). It is in the Moderately Near Zone that we need \( p > 1/2 \).

### 5.3 Bound for \(|E_0(K) - E_{md}(P)| in the Moderately Near Zone

In this section, we determine the error bounds in the Moderately Near Zone for \(|E(K) - U'_{md}(K)|\) and \(|E(C) - U'_{md}(C)|\). Recall that the Moderately Near Zone consists of pairs \((i, j)\) where \(\alpha_i, \alpha_j\) [resp. \(\beta_i, \beta_j\)] are contained in an arc of \(K\) [resp. \(C\)] of length \(\pi r(K)\) but are separated by at least \(p\) other arcs; that is \(k = j - i - 1\) runs from \(p\) to \((m - 2)\). The keys to the analysis in this zone are:

- The minimum distance between a given pair of arcs, or a given pair of chords, is realized at the closest endpoints along the curve.
- That vertex-to-vertex distance is bounded away from zero by Schur’s theorem.

**Proposition 16** *In the Moderately Near Zone,*

\[
|\text{total error}| < 3.00 \frac{E_L(K)^{11/4}}{n^{7/4}} + 542.84 \frac{E_L(K)^{3/2}}{n^{1/2}}.
\]

**PROOF.** As before, we use \(r\) to abbreviate \(r(K)\). We first analyze the error on \(K\),

\[
2 \sum_{i=1}^{n} \sum_{k=p}^{m-2} \left( \frac{|e_i||e_j|}{MD(e_i, e_j)^2} - \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{|x - y|^2} dy dx \right).
\]

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Note: The expressions seem more clear if we use both \( k \) and \( j \), where \( j = i + k + 1 \).

As in the previous case, the analysis is independent of \( i \), so we work with a general \( i \) and multiply that bound by \( n \). To bound the above sum of differences, we introduce a third term (larger than each of the two we are studying) and use the triangle inequality.

**Claim 1.**

\[
2n \sum_{k=p}^{m-2} \left( \frac{\delta^2}{MD(e_i, e_j)^2} - \frac{|e_i| |e_j|}{MD(e_i, e_j)^2} \right) \leq 1.50 \frac{E_L(K)^{11/4}}{n^{7/4}}.
\]

**Claim 2.**

\[
2n \sum_{k=p}^{m-2} \left( \frac{\delta^2}{MD(e_i, e_j)^2} - \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{|x-y|^2} dy \, dx \right) \leq 271.42 \frac{E_L(K)^{3/2}}{n^{1/2}}.
\]

**Proof of Claim 1.** Since chord length \( \leq \) arc-length, \( |e_i||e_j| \leq \delta^2 \). So the summand without absolute value is non-negative, and any upper bound will bound the absolute value.

Since we are still within the Near Zone, Lemma 6 and Lemma 3(b) give

\[
|e_i||e_j| \geq r^2 (2 - 2 \cos(\delta/r)) \geq \delta^2 - \frac{1}{12} \frac{\delta^4}{r^2}.
\]

Now consider the denominator. By Lemmas 8, 6, and 3(b)

\[
MD(e_i, e_j)^2 = |v_{j} - v_{i+1}|^2 \geq r^2 (2 - 2 \cos(k\delta/r))
\]

\[
\geq k^2 \delta^2 - \frac{1}{12} \frac{k^4 \delta^4}{r^2}
\]

Thus

\[
2n \sum_{k=p}^{m-2} \left( \frac{\delta^2}{MD(e_i, e_j)^2} - \frac{|e_i| |e_j|}{MD(e_i, e_j)^2} \right) \leq \frac{1}{6} n \delta^2 \sum_{k=p}^{m-2} \frac{1}{k^2} \left( r^2 - \frac{1}{12} k^2 \delta^2 \right) .
\]

We next bound this denominator away from 0. In the Near Zone, \( k\delta < \pi r \), so \( r^2 - \frac{1}{12} k^2 \delta^2 > r^2 (1 - \frac{1}{12} \pi^2) \), which gives

\[
\frac{1}{6} \frac{1}{r^2(1 - \frac{1}{12} \pi^2)} < 0.94 \frac{1}{r^2}.
\]
We thus have

\[
2n \sum_{k=p}^{m-2} \left( \frac{\delta^2}{MD(e_i, e_j)^2} - \frac{|e_i||e_j|}{MD(e_i, e_j)^2} \right) \leq 0.94 n \frac{\delta^2}{r^2} \sum_{k=p}^{m-2} \frac{1}{k^2} < 0.94 n \frac{\delta^2}{r^2} \frac{\infty}{p-1}.
\]

We want to bound \( \frac{1}{p-1} \) in terms of \( \delta \) and \( r \). Recall that \( p = \lfloor m^{3/4} \rfloor \) and \( m = \lfloor \frac{\pi r}{\delta} \rfloor \). Since \( \delta < r \), we have \( m \geq 3 \) and \( p \geq 2 \). So \( \frac{1}{p-1} \leq \frac{3}{p+1} \) and \( \frac{1}{m} \leq \frac{1}{3m+1} \). Thus,

\[
\frac{1}{p-1} \leq 3 \leq m^{3/4} \leq 3 \left( \frac{4}{3} \right)^{3/4} \frac{1}{(m+1)^{3/4}} < 3.73 \left( \frac{\delta}{\pi r} \right)^{3/4} < 1.59 \left( \frac{\delta}{r} \right)^{3/4}.
\]

Then

\[
2n \sum_{k=p}^{m-2} \left( \frac{\delta^2}{MD(e_i, e_j)^2} - \frac{|e_i||e_j|}{MD(e_i, e_j)^2} \right) < (0.94)(1.59) n \frac{\delta^2}{r^2} \left( \frac{\delta}{r} \right)^{3/4} < 1.50 \frac{E_L(K)^{11/4}}{n^{7/4}}.
\]

This completes the proof of Claim 1.

**Proof of Claim 2.**

We need to bound

\[
2n \sum_{k=p}^{m-2} \left| \frac{\delta^2}{MD(e_i, e_j)^2} - \int_{\alpha_i} \int_{\alpha_j} \frac{1}{|x-y|^2} \right|.
\]

By Lemma 8, \( MD(e_i, e_j) = MD(\alpha_i, \alpha_j) = |v_j - v_{i+1}| \). Since the arcs have length \( \delta \), we know that the summands without absolute value are nonnegative; so, as in Claim 1, we bound the absolute value by finding an upper bound. We are dealing with something that looks like a Riemann Sum upper estimate of a finite integral. But as \( n \) increases, we are changing the domain, not just subdividing the same set and we want to control the size of the error, not just say it goes to zero as \( n \to \infty \). This is where we use the choice of \( p \) as a fractional power \( m^q \) where \( q \) is strictly larger than 1/2.
For brevity, let \( md \) denote \( MD(\alpha_i, \alpha_j) = |v_j - v_{i+1}| \), where \( \text{arc}(v_{i+1}, v_j) = k\delta \).

Since
\[
|x - y| \leq md + 2\delta
\]
we have
\[
\frac{\delta^2}{md^2} - \int_{\alpha_i}^{\alpha_j} \frac{1}{|x - y|^2} \leq \frac{\delta^2}{md^2} - \frac{\delta^2}{(md + 2\delta)^2} < 4\delta^3 \frac{1}{md^3}.
\]

Thus
\[
(6) \leq 8n\delta^3 \sum_{k=p}^{m-2} \frac{1}{md^3}.
\]

As before, by Lemmas 8, 6, and 3(b), since \( \text{arc}(v_{i+1}, v_j) = k\delta \),
\[
md^2 = |v_j - v_{i+1}|^2 \geq r^2(2 - 2\cos(k\delta/r))
\]
\[
\geq k^2\delta^2 - \frac{1}{12} \frac{k^4\delta^4}{r^2}
\]
\[
= k^2\delta^2 \left( 1 - \frac{1}{12} \frac{k^2\delta^2}{r^2} \right)
\]
\[
\geq k^2\delta^2 \left( 1 - \frac{1}{12}\pi^2 \right) \text{ since } k\delta \leq \pi r.
\]

Thus, \( md \geq 0.42k\delta \), so
\[
\frac{1}{md^3} < 13.42 \frac{1}{k^3\delta^3}.
\]

With the above observation, the \( \delta^3 \)'s cancel and we have
\[
(6) < 107.36n \sum_{k=p}^{m-2} \frac{1}{k^3}
\]
\[
< 107.36n \sum_{k=p}^{\infty} \frac{1}{k^3}
\]
\[
< 107.36n \frac{1}{(p-1)^2}.
\]

We showed in the proof of the prior claim that \( \frac{1}{p-1} < 1.59 \left( \frac{2}{\pi} \right)^{3/4} \). Thus,
\[
(6) \leq 107.36n \frac{1}{(p-1)^2} \leq 271.42 \frac{E_{\ell}(K)^{3/2}}{n^{1/2}}
\]

Note that in the above analysis the exponent 3/4 needs to be strictly greater than 1/2, so that when we double it, the power of \( n \) in the denominator will more than cancel the leading factor \( n \).
We now need to bound the contribution from \( C \), that is \(|E(C) - U'_md(P)|\). The radius of \( C \), \( R \), is the thickness radius \( r(C) \). Also, we know from Lemma 5 that \( R \geq r \). So if arcs \( \alpha_i, \alpha_j \) of \( K \) are near, then the corresponding arcs \( \beta_i, \beta_j \) lie within an arc of \( C \) of length \( \leq \pi R \). Thus, the various steps in our analysis of \( K \) can be carried out on \( C \). We could obtain sharper bounds for \( C \), but we will settle for the same bound since they dominate anyway.

For Claim 1, we have

\[
|f_i| |f_j| = R^2 (2 - 2 \cos(\delta/R)) \geq \delta^2 - \frac{1}{12} \delta^4 R^4 \geq \delta^2 - \frac{1}{12} \delta^4 r^2,
\]

and

\[
MD(f_i, f_j)^2 \geq k^2 \delta^2 - \frac{1}{12} \frac{k^4 \delta^4}{r^2} \geq k^2 \delta^2 - \frac{1}{12} \frac{k^4 \delta^4}{r^2},
\]

exactly as for \( K \). Now continue the proof of Claim 1 verbatim.

For Claim 2,

\[
md(f_i, f_j)^2 = R^2 (2 - 2 \cos(k\delta/r)) \geq k^2 \delta^2 \left(1 - \frac{1}{12} \frac{k^2 \delta^2}{R^2}\right) \geq k^2 \delta^2 \left(1 - \frac{1}{12} \frac{k^2 \delta^2}{r^2}\right),
\]

and the rest follows verbatim.

Thus, our final bound for the total error in this zone is just double the values obtained in Claims 1 and 2.

5.4 Bounds for \(|E_0(K) - E_{md}(P)|\) in the Far Zone

As before, we use \( r \) to abbreviate \( r(K) \). In the Near Zones, we just needed a value for \( r \leq \) minimum radius of curvature of \( K \). But in the Far Zone, we need both aspects of the thickness radius.

The argument here is somewhat similar to the Moderately Near Zone, but we control the denominators in a different way. In each situation, we need to know that spatial distances between points are bounded away from zero in some way depending on their arc-length distances along \( K \). For the Far Zone, we use the fact that thickness controls critical self-distance, in particular Lemma 11, together with local analysis (Lemma 9), to relate chord-chord distances to arc-arc distances. Also, we continue to use the hypothesis \( \delta \leq r \).
Remark on notation heuristics. In the following paragraphs and Lemma 17, think of \((\alpha, \beta)\) as \((\alpha_i, \alpha_j)\) and \((e, f)\) as \((e_i, e_j)\).

**Lemma 17** Suppose \((\alpha, \beta)\) is a pair of far arcs (on \(K\) or on \(C\)), with \((e, f)\) the inscribed chords joining their endpoints. Then

\[
\text{md}(\alpha, \beta) > 1.08r,
\]

and

\[
\text{md}(e, f) > 0.79r.
\]

**Proof.** We analyze \(K\), and note that the same bound will work for \(C\) since \(r \leq R\). We establish the lower bound for arcs, then use that to bound the distance for chords. If the minimum distance between a pair of arcs is realized at points that are interior to one or both arcs, then we are dealing with singly- or doubly-critical pairs of points, so, by Lemma 10, \(\text{md}(\alpha, \beta) \geq 2r\). Thus we just need to bound the end-point distances. Let \(\alpha_0\) and \(\alpha_1\) be the endpoints of the arc \(\alpha\) and \(\beta_0\) and \(\beta_1\) the endpoints of the arc \(\beta\). Choose the labels so that \(\alpha_1\) and \(\beta_0\) are the points which are closest with respect to arc-length. In the worst case, the arc-length from \(\alpha_0\) to \(\beta_1\) is \(\geq \pi r\), but the arc-lengths of the arcs \(\overline{\alpha_0\beta_0}, \overline{\alpha_1\beta_0}, \text{ and } \overline{\alpha_1\beta_1}\) are less than \(\pi r\). In such a case, we have the following situation:

- \(|\alpha_0 - \beta_0|\)  
  \[\pi r \geq \text{arc}(\alpha_0, \beta_0) \geq \pi r - \delta \implies |\alpha_0 - \beta_0|^2 \geq r^2(2 - 2 \cos(\pi - 1))\]  
  by Lemma 6, and the fact that \(\delta \leq r\). So \(|\alpha_0 - \beta_0| > 1.75r\).

- \(|\alpha_0 - \beta_1|\)  
  \[\text{arc}(\alpha_0, \beta_1) \geq \pi r \implies |\alpha_0 - \beta_1| \geq 2r\] by Lemma 11.

- \(|\alpha_1 - \beta_1|\) same bound as \(|\alpha_0 - \beta_0|\).

- \(|\alpha_1 - \beta_0|\)  
  \[\text{arc}(\alpha_1, \beta_0) \geq \pi r - 2\delta \implies \text{arc}(\alpha_1, \beta_0) > (\pi - 2)r, \text{ since } \delta \leq r\]. Thus, by Lemma 6,  
  \[|\alpha_1 - \beta_0|^2 \geq r^2(2 - 2 \cos(\pi - 2)) \implies |\alpha_1 - \beta_0| > 1.08r\].

In other scenarios, the arc pair \((\alpha, \beta)\) yields three of the above four cases, but we lose the smallest. For “most” arc pairs \((\alpha, \beta)\), we have all point-to-point distances at least \(2r\).

We now obtain the lower bound on chord-to-chord distances using Lemma 9:

\[
\text{md}(e, f) \geq \text{md}(\alpha, \beta) - \frac{\sqrt{3}}{6}r > 1.08r - \left(\frac{\sqrt{3}}{6}\right)r > 0.79r.
\]
Proposition 18  The total error in the Far Zone is bounded by

\[
0.56 \frac{E_L(K)^4}{n^2} + 1.60 \frac{E_L(K)^5}{n^2} + 7.76 \frac{E_L(K)^4}{n}.
\]

Proof. We first analyze the error on \( K \),

\[
2 \sum_{i=1}^{n} \sum_{j=i+m}^{n} \left| \frac{|e_i| |e_j|}{md(e_i, e_j)^2} - \frac{1}{|x-y|} \int_{x \in \alpha_i} \int_{y \in \alpha_j} \right| \right| dy \ dx
\]

We do this in three steps: Compare \( \frac{|e_i| |e_j|}{md(e_i, e_j)^2} \) to \( \frac{\delta^2}{md(e_i, e_j)^2} \), that to \( \frac{\delta^2}{md(\alpha_i, \alpha_j)^2} \), and that to \( \int \frac{1}{|x-y|} \). After we do each step for \( K \), we double that to include the contribution from \( C \). Note \( \frac{\delta^2}{md(e_i, e_j)^2} = \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{md(e_i, e_j)^2} \) dy dx and similarly for \( \frac{\delta^2}{md(\alpha_i, \alpha_j)^2} \).

Claim 1:

\[
2 \sum_{i=1}^{n} \sum_{j=i+m}^{n} \left| \frac{\delta^2}{md(e_i, e_j)^2} - \frac{|e_i| |e_j|}{md(e_i, e_j)^2} \right| \leq 0.28 \frac{E_L(K)^4}{n^2}.
\]

Claim 2.

\[
2 \sum_{i=1}^{n} \sum_{j=i+m}^{n} \left| \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{md(e_i, e_j)^2} - \frac{1}{md(\alpha_i, \alpha_j)^2} \right| dy \ dx \leq 0.80 \frac{E_L(K)^5}{m^2}. \tag{7}
\]

Claim 3.

\[
2 \sum_{i=1}^{n} \sum_{j=i+m}^{n} \left| \int_{x \in \alpha_i} \int_{y \in \alpha_j} \frac{1}{md(\alpha_i, \alpha_j)^2} - \frac{1}{|x-y|} \right| dy \ dx \leq 3.88 \frac{E_L(K)^4}{n}. \tag{8}
\]

Proof of Claim 1.

Since arc-length \( \geq \) chord length, each summand is nonnegative without taking the absolute value, so we just need to bound the terms from above. By Lemma 6 and Lemma 3(b), \( \delta^2 - \frac{1}{12} \frac{\delta^4}{r^2} \leq |e_i| |e_j| \). Thus,

\[
\delta^2 \frac{1}{md(e_i, e_j)^2} - \frac{|e_i| |e_j|}{md(e_i, e_j)^2} \leq \frac{1}{12} \frac{\delta^4}{r^2 md(e_i, e_j)^2}.
\]

But Lemma 17 gives us that \( md(e_i, e_j)^2 > (0.79)^2 r^2 \), so

\[
\delta^2 \frac{1}{md(e_i, e_j)^2} - \frac{|e_i| |e_j|}{md(e_i, e_j)^2} < 0.14 \frac{\delta^4}{r^2}.
\]
Multiplying by $2n^2$ gives

$$2 \sum_{i=1}^{n} \sum_{j=m}^{n} \left( \frac{\delta^2}{md(e_i, e_j)^2} - \frac{|e_i||e_j|}{md(e_i, e_j)^2} \right) < 0.28 \frac{n^2\delta^4}{r^4} = 0.28 \frac{E_L(K)^4}{n^2}$$

Proof of Claim 2.

The sum (7) is bounded by $(2n^2\delta^2)$ (worst error in integrands). We will use Lemma 9(b) to bound that. To make the algebra more evident, let $\epsilon = md(e_i, e_j)$ and $\gamma = md(\alpha_i, \alpha_j)$. The term we wish to bound is

$$\left| \frac{1}{\epsilon^2} - \frac{1}{\gamma^2} \right| = \left| \frac{\gamma^2 - \epsilon^2}{\epsilon^2\gamma^2} \right| < 1.38 \frac{\gamma - \epsilon}{\epsilon^2} \left( \frac{\gamma + \epsilon}{r^4} \right) \leq 0.40 \frac{(\gamma + \epsilon)\delta^2}{r^5},$$

since $\epsilon > 0.79r$ and $\gamma > 1.08r$ by Lemma 17, and $|\gamma - \epsilon| \leq \frac{\sqrt{3} \delta^2}{6r}$ by Lemma 9(b).

Now $\epsilon, \gamma$ are minimum distances between sets that include points of $K$, so $\epsilon, \gamma \leq \ell(K)/2$ and $\gamma + \epsilon \leq \ell(K)$. Thus,

$$\left| \frac{1}{\epsilon^2} - \frac{1}{\gamma^2} \right| \leq \frac{0.40 \ell(K)\delta^2}{r^5}.$$

Multiplying by $2n^2\delta^2$, we get

$$2 \sum_{i=1}^{n} \sum_{j=m}^{n} \int_{x \in \alpha_i} \int_{y \in \alpha_j} \left| \frac{1}{md(e_i, e_j)^2} - \frac{1}{md(\alpha_i, \alpha_j)^2} \right| dy \, dx \leq 0.80 \frac{n^2\delta^4\ell(K)}{r^5} = 0.80 \frac{E_L(K)^5}{n^2}.$$

Proof of Claim 3.

The sum (8) is bounded by $(2n^2\delta^2)$ (worst error in integrand).

Let $\gamma$ denote $md(\alpha_i, \alpha_j)$. So for particular $x, y$ on $\alpha_i$ and $\alpha_j$, we have $|x - y| = \gamma + t$ for some $0 \leq t \leq 2\delta$. The largest error is then

$$\frac{1}{\gamma^2} - \frac{1}{(\gamma + t)^2} = \frac{t(2\gamma + t)}{\gamma^2(\gamma + t)^2} \leq \frac{t(2\gamma + t)}{\gamma^4} < \frac{t(2\gamma + t)}{(1.08)^4r^4},$$

since $\gamma \geq 1.08r$ by Lemma 17.
Now $t \leq 2\delta$ and $\gamma \leq \ell(K)/2$. Thus,

$$\frac{t(2\gamma + t)}{1.08^4 r^4} \leq \frac{2\delta(\ell(K) + 2\delta)}{1.08^4 r^4}$$

$$= \frac{2}{1.08^4} \frac{\delta(n\delta + 2\delta)}{r^4}$$

$$= \frac{2}{1.08^4} \frac{\delta^2(n + 2)}{r^4}$$

$$\leq \frac{2}{1.08^4} \frac{\delta^2 (2\pi + 2)n}{2\pi}$$

since $n > E_L(K) \geq 2\pi$

$$< 1.94 \frac{\delta^2 n}{r^4}$$.

Thus,

$$2 \sum_{i=1}^{n} \sum_{j=i+m}^{n} \int_{x \in \alpha_i} \int_{y \in \alpha_j} \left| \frac{1}{md(\alpha_i, \alpha_j)^2} - \frac{1}{|x - y|^2} \right| dy dx \leq 3.88 n^2 \delta^2 \frac{\delta^2 n}{r^4}$$

$$= 3.88 \frac{E_L(K)^4}{n}$$

6 Putting it all together

Here we combine the bounds from the various zones.

From Propositions 14, 15, 16, and 18, we have

$$|E_0(K) - E_{ma}(P)| \leq 3.82 E_L(K) \left( \frac{E_L(K)}{n} \right)^{1/4}$$

$$+ 3.00 E_L(K) \left( \frac{E_L(K)}{n} \right)^{7/4}$$

$$+ 542.84 E_L(K) \left( \frac{E_L(K)}{n} \right)^{1/2}$$

$$+ 0.56 E_L(K)^2 \left( \frac{E_L(K)}{n} \right)^2$$

$$+ 1.60 E_L(K)^3 \left( \frac{E_L(K)}{n} \right)^2$$

$$+ 7.76 E_L(K)^3 \left( \frac{E_L(K)}{n} \right)^3$$

Since $E_L(K) \geq 2\pi > 1$, and $n > E_L(K)$, we see that certain terms dominate
others. So,

$$|E_0(K) - E_{md}(P)| < 550 \frac{E_L(K)^{5/4}}{n^{1/4}} + 10 \frac{E_L(K)^4}{n}.$$  

If $n > E_L(K)^{11/3}$, then the total error is less than $560 \frac{E_L(K)^{5/4}}{n^{1/4}}$.

This completes the proof of Theorem 1.

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