Epiregularity in generalized topological space

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Abstract
The notion of generalized epiregularity in generalized topological space is introduced and investigate some of its properties in this paper.

Keywords
Generalized Epiregular, Generalized Hausdorff, Generalized Completely Hausdorff, Generalized Paracompact.

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1. Introduction
One of the important development of general topology in recent years is the theory of generalized topological spaces defined by A. Csaszar \(^2\). In particular, he introduced the basic operators in generalized topology. Noiri and B. Roy \(^1\) in 2011 introduced a new kind of sets called generalized \(\mu\)-closed set in a topological space by using the concept of generalized \(\mu\)-open set introduced by Csaszar. In 2011, M. S. Sarsak \(^3\) studied separation axioms and B. Roy \(^1\) introduced regularity and normality on generalized topological spaces using generalized \(\mu\)-closed and \(\mu\)-open sets. Samirab A. Alzahrani\(^4\) introduced the concept of epiregularity in topological spaces and studied its topological properties. We introduced the concept of epiregular space in generalized topology and studied some of its basic properties using the concept of generalized \(\mu\)-open and generalized \(\mu\)-closed sets.

2. Preliminaries
In this section, we recall some definitions and basic results used throughout the paper.

Definition 2.1. \(^{14}\) Let topology \(\tau\) on a set \(Y\) contains another topology \(\tau'\) on \(Y\) (that is, every member of \(\tau'\) is a member of \(\tau\)), we say that \(\tau\) is a stronger or finer topology than \(\tau'\), or that \(\tau'\) is a weaker or coarser topology than \(\tau\).

Definition 2.2. \([3]\) A generalized topological space \((Y, \mu)\) is said to be generalized paracompact if every \(\mu\)-open covering of \(Y\) has \(\mu\)-locally finite \(\mu\)-open refinement that covers \(Y\).

Definition 2.3. \([4]\) A generalized topological space \((Y, \mu)\) is said to be generalized Hausdorff if for any two distinct points \(m\) and \(n\), there exists disjoint \(\mu\)-open sets \(A\) and \(B\) such that \(m \in A\) and \(n \in B\).

Definition 2.4. \([1]\) A generalized topological space \((Y, \mu)\) is said to be generalized regular (generalized \(T_3\)) if for each \(\mu\)-closed set \(F\) of \(Y\) not containing \(m\), there exist disjoint \(\mu\)-open sets \(A\) and \(B\) such that \(m \in A\) and \(F \subseteq B\).

Definition 2.5. \([1]\) A generalized topological space \((Y, \mu)\) is generalized normal (generalized \(T_4\)) if for any pair of disjoint \(\mu\)-closed subsets \(G\) and \(H\) of \(Y\), there exist disjoint \(\mu\)-open sets \(A\) and \(B\) such that \(G \subseteq A\) and \(H \subseteq B\).

3. Generalized Epiregularity

Definition 3.1. A generalized topological space \((Y, \mu)\) is called generalized epiregular if there is a generalized coarser topology \((Y, \mu')\) on \(Y\) such that \((Y, \mu')\) is generalized \(T_3\).

Example 3.2. Let \(Y = \{p, q, r\}\), \(\mu = \{\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, \{p, r\}, \{p, q, r\}\}\) and \(\mu' = \{\phi, \{p\}, \{q\}, \{p, q\}, \{q, r\}, \{p, q, r\}\}\). Hence \((Y, \mu')\) is generalized \(T_3\) (generalized regular Hausdorff) space. Here \((Y, \mu)\) is generalized epiregular.
Theorem 3.3. Every generalized epiregular space is generalized Hausdorff.

Proof. Let (Y, µ) be generalized epiregular space, and let m, n be any two distinct points in Y, then there exists generalized coarser topology µ on Y such that (Y, µ) is generalized T3. By generalized regularity of µ, there exists A, B ∈ (Y, µ) such that m ∈ A ⊆ cl (A), n ∈ B and cl (B) ∩ A = ∅. Thus (Y, µ) is generalized T3.

Remark 3.4. Converse of the above theorem need not be true which is given in the following example.

Example 3.5. Let Y = {p, q} and µ = {Φ, Y, {p}, {q}, {r}, {p, q}}. Here (Y, µ) is generalized Hausdorff but it is not generalized epiregular. Here there does not exist coarser topology (Y, µ′) such that (Y, µ′) is generalized T3.

Definition 3.6. A generalized topological space (Y, µ) is said to be generalized completely Hausdorff if for any x ≠ y, there exists disjoint µ-open sets A and B such that x ∈ cl (A), y ∈ cl (B).

Theorem 3.7. Every generalized epiregular space is generalized completely Hausdorff.

Proof. Let (Y, µ) be any generalized epiregular space, and let m, n be any two distinct points in Y, then there exists generalized coarser topology µ on Y such that (Y, µ) is generalized T3. By generalized regularity of µ, there exists A, B ∈ (Y, µ) such that m ∈ A ⊆ cl (A), n ∈ B and cl (B) ∩ A = ∅. Thus (Y, µ) is generalized completely Hausdorff.

Remark 3.8. Converse of the above theorem need not be true which is given in the following example.

Example 3.9. Let Y = {p, q} and µ = {Φ, {p}, {q}, {r}, {p, q}}. Here (Y, µ) is generalized completely Hausdorff but it is not generalized epiregular. Here there does not exist coarser topology (Y, µ′) such that (Y, µ′) is generalized T3.

Theorem 3.10. If the generalized coarser topology (Y, µ′) of the generalized epiregular space (Y, µ) is generalized paracompact, then (Y, µ) is generalized T₄.

Proof. Proof of the theorem follows from 2.3 and 2.4.

Theorem 3.11. Any generalized epiregular compact space is generalized T₄.

Proof. Let the generalized epiregular compact space be (Y, µ) then generalized coarser topology (Y, µ′) is generalized T₃ which implies that it is generalized T₂. Also every generalized epiregular compact space is generalized T₄.

Theorem 3.12. In an generalized epiregular space, for every generalized compact set G and every m ∉ G, there exist disjoint µ-open sets A, B such that G ⊆ A and m ∈ B.

Proof. Let (Y, µ) be generalized epiregular space, then there exists a generalized coarser topology µ on Y such that (Y, µ) is generalized T₃. Let G be any generalized compact set in (Y, µ) and let m ∉ G, hence G is µ′-closed in (Y, µ′) and m ∉ G, by generalized regularity of (Y, µ′), there exists A, B ∈ µ′ such that G ⊆ A, m ∈ B and A ∩ B = ∅.

Theorem 3.13. If G and H are disjoint generalized compact subsets of an generalized epiregular space (Y, µ), then there exists disjoint µ-open sets A and B such that G ⊆ A, H ⊆ B.

Proof. Let (Y, µ) be generalized epiregular space, then there exists a generalized coarser topology µ on Y such that (Y, µ) is generalized T₃. Let G, H be any disjoint generalized compact subsets of (Y, µ), hence they are disjoint generalized compact subsets of (Y, µ) and by theorem 3.10 for each p ∈ G and generalized compact set H, there exist µ-open sets Aₚ, Bₚ such that p ∈ Aₚ, H ⊆ Bₚ and Aₚ ∩ Bₚ = ∅. Consider G as an arbitrary generalized compact set disjoint from H. (By theorem 3.10), for each p ∈ G, disjoint µ-open set Aₚ, containing p and Bₚ containing H, such that A = ∪ₚ Aₚ, B = ∪ₚ Bₚ which a µ-open set containing H.

Theorem 3.14. Generalized epiregularity is a generalized topological property.

Proof. Let (Y, µ) be generalized epiregular space. Assume that (X, µ₁) ∼ (X, µ₂). Let µ′₁ be a generalized coarser topology on Y such that (Y, µ′₁) is generalized T₃. Let f : (Y, µ₁) → (X, µ₂) be a generalized homeomorphism. Define a generalized topology µ′₂ on by (X, µ₂) = {f(A) : A ∈ µ′₁}. Then µ′₂ is a coarser than µ₁ and (X, µ′₂) is generalized T₃. Hence (X, µ₁) is generalized epiregular.

Theorem 3.15. Generalized epiregularity is a generalized hereditary property.

Proof. Let (Y, µ) be generalized epiregular space, and let (X, µₓ) be a subspace of (Y, µ). Let µ′ be a generalized coarser topology on Y such that (Y, µ′) is generalized T₃. Since generalized T₃ is generalized hereditary, (X, µₓ) is generalized T₃ and µₓ′ ≤ µₓ. Therefore (X, µₓ) is generalized epiregular.

Theorem 3.16. Generalized epiregular is an generalized additive property.

Proof. Let (Yₐ, µₐ) be an generalized epiregular space for each a ∈ A, let µₐ′ be a generalized coarser topology on Yₐ, such that (Yₐ, µₐ′) is generalized T₃. Since generalized T₃ is generalized epiregularity are both generalized additive, ∪ₐ∈A (Yₐ, µₐ′) is generalized T₃, and its generalized topology is coarser than the generalized topology on ∪ₐ∈A (Yₐ, µₐ).

Theorem 3.17. Let {Yₐ, µₐ : a ∈ A} be a family of generalized topological spaces, and let Y = ∪ₐ∈A Yₐ. Then (Y, µ) is generalized epiregular, where µ is the Tychonoff product.
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generalized topology, if and only if \((Y_{\alpha}, \mu_{\alpha})\) is generalized epiregular for each \(\alpha \in \Lambda\).

Proof. Assume \((Y, \mu)\) is generalized epiregular, and let \(\beta \in \Lambda\), by theorem 3.13, every subspace of \((Y, \mu)\) is generalized epiregular. Then there is a subspace of \((Y, \mu)\) that is generalized homeomorphic to \((Y_{\beta}, \mu_{\beta})\). Since generalized epiregularity is a generalized topological property \((Y_{\beta}, \mu_{\beta})\) is generalized epiregular. Assume \((Y_{\alpha}, \mu_{\alpha})\) is generalized epiregular space for each \(\alpha \in \Lambda\), let \(\mu'_{\alpha}\) be a generalized topology on \(Y_{\alpha}\), coarser than \(\mu_{\alpha}\) such that \((Y_{\alpha}, \mu'_{\alpha})\) is generalized \(T_3\).

Since generalized \(T_3\) is multiplicative, \(\bigwedge_{\alpha \in \Lambda} Y_{\alpha}\) is generalized \(T_3\) with respect of the generalized product topology of \(\mu'_{\alpha}\)’s, and its generalized topology is coarser than the generalized topology on \(\bigwedge_{\alpha \in \Lambda} Y_{\alpha}\) is generalized epiregular.

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