(m, n)-Hyperideals in ordered semihypergroups

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Abstract. In this paper, first we introduce the notions of an (m, n)-hyperideal and a generalized (m, n)-hyperideal in an ordered semihypergroup, and then, some properties of these hyperideals are studied. Thereafter, we characterize (m, n)-regularity, (m, 0)-regularity, and (0, n)-regularity of an ordered semihypergroup in terms of its (m, n)-hyperideals, (m, 0)-hyperideals and (0, n)-hyperideals, respectively. The relations \( I_m, I_n, H_m^n, \) and \( B_m^n \) on an ordered semihypergroup are, then, introduced. We prove that \( B_m^n \subseteq H_m^n \) on an ordered semihypergroup and provide a condition under which equality holds in the above inclusion. We also show that the (m, 0)-regularity [(0, n)-regularity] of an element induce the (m, 0)-regularity [(0, n)-regularity] of the whole \( H_m^n \)-class containing that element as well as the fact that (m, n)-regularity and (m, n)-right weakly regularity of an element induce the (m, n)-regularity and (m, n)-right weakly regularity of the whole \( B_m^n \)-class and \( H_m^n \)-class containing that element, respectively.

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1 Introduction and preliminaries

By an ordered semigroup, we mean an algebraic structure \((S, \cdot \leq)\), which satisfies the following conditions: (1) \(S\) is a semigroup with respect to the multiplication \(\cdot\); (2) \(S\) is a partially ordered set by \(\leq\); (3) if \(a\) and \(b\) are elements of \(S\) such that \(a \leq b\), then \(ac \leq bc\) and \(ca \leq cb\) for all \(c \in S\). Many authors, especially Alimov [1], Clifford [2–4], Hion [13], Conrad [5], and Kehayopulu [15] studied such semigroups with some restrictions.

In 1934, Marty [21] introduced the concept of a hyperstructure and defined hypergroup. Later on several authors studied hyperstructure in various algebraic structures such as rings, semirings, semigroups, ordered semigroups, \(\Gamma\)-semigroups and Ternary semigroups, etc. The concept of a semihypergroup is a generalization of the concept of a semigroup and many classical notions such as of ideals, quasi-ideals and bi-ideals defined in semigroups and regular semigroups have been generalized to semihypergroups (see [8, 9] for other related notions and results on semihypergroups). In [14], Heidari and Davvaz introduced the notion of an ordered semihypergroup as a generalization of the notion of an ordered semigroup. Davvaz et al. in [6, 7, 14, 22, 23, 25, 26] studied some properties of hyperideals and bi-hyperideals in ordered semihypergroups. Lajos [16] introduced the concept of \((m, n)\)-ideals in semigroups (see also [17–19]). In [12], the authors defined the notion of an \((m, n)\)-quasi-hyperideal in a semihypergroup and investigated several properties of these \((m, n)\)-quasi-hyperideals.

A hyperoperation on a non-empty set \(H\) is a map \(\circ : H \times H \to \mathcal{P}^*(H)\) where \(\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}\) (the set of all non-empty subsets of \(H\)). In such a case, \(H\) is called a hypergroupoid. Let \(H\) be a hypergroupoid and \(A, B\) be any non-empty subsets of \(H\). Then

\[ A \circ B = \bigcup_{a \in A, b \in B} a \circ b. \]

We shall write, in whatever follows, \(A \circ x\) instead of \(A \circ \{x\}\) and \(x \circ A\) instead of \(\{x\} \circ A\), for any \(x \in H\). Also, for simplicity, throughout the paper, we shall write \(A^n\) for \(A \circ A \circ \cdots \circ A\) \(n\) – copies of \(A\) for any \(n \in \mathbb{Z}^+\). Also the integers \(m, n\) will stand for positive integers throughout the paper until and unless otherwise specified. Moreover, the hypergroupoid \(H\) is called a semihypergroup if, for all \(x, y, z \in H\),

\[ (x \circ y) \circ z = x \circ (y \circ z) \]
that is, 

\[ \bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v. \]

A non-empty subset \( T \) of a semihypergroup \( H \) is called a subsemihypergroup of \( H \) if \( T \circ T \subseteq T \).

Let \( H \) be a non-empty set, the triplet \( (H, \circ, \leq) \) is called an ordered semihypergroup if \( (H, \circ) \) is a semihypergroup and \( (H, \leq) \) is a partially ordered set such that

\[ x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y \]

for all \( x, y, z \in H \). Here, if \( A \) and \( B \) are non-empty subsets of \( H \), then we say that \( A \leq B \) if for every \( a \in A \) there exists \( b \in B \) such that \( a \leq b \).

Let \( H \) be an ordered semihypergroup. For a non-empty subset \( A \) of \( H \), we denote \( (A] = \{x \in H \mid x \leq a \text{ for some } a \in A\} \). A non-empty subset \( A \) of \( H \) is called idempotent if \( A = (A \circ A) \). A non-empty subset \( A \) of \( H \) is called left (right)-hyperideal \[7\] of \( H \) if \( H \circ A \subseteq A \) \( (A \circ H \subseteq A) \) and \( (A] \subseteq A \). A non-empty subset \( J \) of \( H \) is called a hyperideal of \( H \) if \( J \) is both a left hyperideal and a right hyperideal of \( H \). A subsemihypergroup (non-empty subset) \( B \) of an ordered semihypergroup \( H \) is called a bi-hyperideal (generalized bi-hyperideal) of \( H \) if \( B \circ H \circ B \subseteq B \) \( (B] \subseteq B \). An ordered semihypergroup \( H \) is called regular (left-regular, right-regular) \[7\] if for each \( x \in H \), \( x \in (x \circ H \circ x)(x \in (H \circ x \circ x), x \in (x \circ x \circ H)) \).

**Lemma 1.1.** \[7\] Let \( H \) be an ordered semihypergroup and \( A, B \) be any non-empty subsets of \( H \). Then the following conditions hold:

(i) \( A \subseteq (A] \);
(ii) \( A \subseteq B \Rightarrow (A] \subseteq (B] \);
(iii) \((A] \circ (B] \subseteq (A \circ B] \);
(iv) \((A] \circ (B]) = (A \circ B] \);
(v) \((A] \cup (B] = (A \cup B] \).

\[2\] \( (m, 0) \)-hyperideals, \( (0, n) \)-hyperideals and \( (m, n) \)-hyperideals in ordered semihypergroups

In this section, the notions of \( (m, n) \)-hyperideals and generalized \( (m, n) \)-hyperideals in ordered semihypergroups are introduced. Moreover, important some properties of these hyperideals are studied.
Definition 2.1. Let $H$ be an ordered semihypergroup and $m, n$ be the positive integers. Then a subsemihypergroup (respectively, non-empty subset) $A$ of $H$ is called an (respectively, generalized) $(m, n)$-hyperideal of $H$ if

(i) $A^m \circ H \circ A^n \subseteq A$; and

(ii) $(A] \subseteq A$.

Note that in Definition 2.1, if $m = 1 = n$, then $A$ is called a (generalized) bi-hyperideal of $H$. Moreover, a (generalized) bi-hyperideal of an ordered semihypergroup $H$ is an (generalized) $(m, n)$-hyperideal of $H$ for all positive integers $m$ and $n$. It is clear that, for positive integers $m$ and $n$, the notion of (generalized) $(m, n)$-hyperideal of $H$ is a generalization of the notion of (generalized) bi-hyperideal of $H$. The following example shows that a generalized $(m, n)$-hyperideal of $H$ need not be an $(m, n)$-hyperideal and generalized bi-hyperideal of $H$.

Example 2.2. Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

| $\circ$ | $a$ | $b$ | $c$ | $d$ |
|---------|-----|-----|-----|-----|
| $a$     | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$     | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $c$     | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a, b\}$ |
| $d$     | $\{a\}$ | $\{a\}$ | $\{a, b\}$ | $\{a, b, c\}$ |

$\leq: = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$.

The covering relation $\prec$ and the figure of $H$ are as follows:

$\prec: = \{(a, b)\}$

Then $H$ is an ordered semihypergroup. The subset $\{a, d\}$ of $H$ is a generalized $(m, n)$-hyperideal of $H$ for all integers $m, n \geq 2$ which is neither an $(m, n)$-hyperideal nor a generalized bi-hyperideal of $H$. 
Definition 2.3. [20] Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. Then a subsemihypergroup $A$ of $H$ is called an $(m,0)$-hyperideal (respectively, $(0,n)$-hyperideal) of $H$ if

(i) $A^m \circ H \subseteq A$ (respectively, $H \circ A^n \subseteq A$); and

(ii) $(A] \subseteq A$.

In Definition 2.3, if $m = 1 = n$, then $A$ is called a right hyperideal (left hyperideal) of $H$. Clearly, each right hyperideal (respectively, left hyperideal) of $H$ is an $(m,0)$-hyperideal for each positive integer $m$ (respectively, $(0,n)$-hyperideal for each positive integer $n$), that is, the notion of an $(m,0)$-hyperideal ($(0,n)$-hyperideal) of $H$ is a generalization of the notion of a right hyperideal (respectively, left hyperideal) of $H$. Conversely, an $(m,0)$-hyperideal (respectively, $(0,n)$-hyperideal) of $H$ need not be a right hyperideal (respectively, left hyperideal) of $H$. We illustrate it by the following example.

Example 2.4. Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ and order $\leq$ on $H$ as follows:

\[
\begin{array}{c|cccc}
\circ & a & b & c & d \\
\hline
a & \{a\} & \{a\} & \{a\} & \{a\} \\
b & \{a\} & \{a\} & \{a\} & \{a\} \\
c & \{a\} & \{a\} & \{a, b\} & \{a, b\} \\
d & \{a\} & \{a\} & \{a, b\} & \{a\} \\
\end{array}
\]

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}$.

The covering relation $\prec$ and the figure of $H$ are as follows:

$\prec := \{(a, b), (a, c)\}$

[Diagram showing the covering relation with vertices labeled $a$, $b$, $c$, $d$ and edges connecting $b$ to $a$, $c$, $d$, and $c$ to $a$]
Then $H$ is an ordered semihypergroup. It is easy to verify that the subset $A = \{a, d\}$ of $H$ is an $(m, 0)$-hyperideal and a $(0, n)$-hyperideal of $H$ for all integers $m, n \geq 2$, but it is neither a right hyperideal nor a left hyperideal of $H$.

**Remark 2.5.** Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be any non-empty subset of $H$. Then $(B^m \cup B \circ H \circ B^m]$ is a (generalized) bi-hyperideal of $H$. Indeed, $(B^m \cup B \circ H \circ B^m] \circ (B^m \cup B \circ H \circ B^m] \subseteq ((B^m \cup B \circ H \circ B^m) \circ (B^m \cup B \circ H \circ B^m)] = (B^m \circ B^m \cup B \circ H \circ B^m \cup B \circ B^m \circ B \circ H \circ B^m \circ B \circ H \circ B^m \subseteq (B \circ H \circ B^m] \subseteq (B^m \cup B \circ H \circ B^m]$ and $(B^m \cup B \circ H \circ B^m] \circ H \circ (B^m \cup B \circ H \circ B^m] \subseteq (B^m \cup H \cup B \circ B^m \circ H \circ (B^m \cup B \circ H \circ B^m] \subseteq (B \circ H \circ B^m] \subseteq (B^m \cup B \circ H \circ B^m]$. 

Note that in Remark 2.5, if $m = 1$, then $(B \cup B \circ H \circ B)$ is a generalized bi-hyperideal of $H$ which is not a bi-hyperideal of $H$. Thus $(B^m \cup B \circ H \circ B^m)$ is a generalized bi-hyperideal of $H$ for each positive integer $m$.

**Theorem 2.6.** Let $B$ be a non-empty subset of an ordered semihypergroup $H$ and let $m \geq 2$ be any positive integer. Then the following are equivalent:

(i) $B$ is a $(1, m)$-hyperideal of $H$;

(ii) $B$ is a left hyperideal of some bi-hyperideals of $H$;

(iii) $B$ is a bi-hyperideal of some left hyperideals of $H$;

(iv) $B$ is a $(0, m)$-hyperideal of some right hyperideals of $H$;

(v) $B$ is a right hyperideal of some $(0, m)$-hyperideals of $H$.

**Proof.** (i) $\Rightarrow$ (ii) Let $B$ be a $(1, m)$-hyperideal of $H$. So $B \circ B \subseteq B, (B] \subseteq B$ and $B \circ H \circ B^m \subseteq B$. Therefore, $(B^m \cup B \circ H \circ B^m] \circ B = (B^m \cup B \circ H \circ B^m] \circ (B] \subseteq (B^m+1 \cup B \circ H \circ B^m+1] \subseteq (B^m+1 \cup B \circ H \circ B^m] \subseteq (B] = B$. If $b \in B$, then $h \in (B^m \cup B \circ H \circ B^m]$ such that $h \leq b$. As $h \in H$ and $B$ is a $(1, m)$-hyperideal of $H$, $h \in B$. Hence, $B$ is a left hyperideal of the bi-hyperideal $(B^m \cup B \circ S \circ B^m]$ of $H$.

(ii) $\Rightarrow$ (iii) Let $B$ be a left hyperideal of a bi-hyperideal $A$ of $H$. So $B \subseteq A, A \circ B \subseteq B$ and $A \circ H \circ A \subseteq A$. Therefore, $B \circ B \subseteq A \circ B \subseteq B$ and $B \circ (B \cup H \circ B] \circ B = (B] \circ (B \cup H \circ B] \circ (B] \subseteq (B \circ (B \cup H \circ B] \circ B] \subseteq (B^3 \cup H \circ B^2] \subseteq (A^2 \circ B \cup A \circ H \circ A \circ B] \subseteq (A \circ B \cup A \circ H \circ A \circ B] \subseteq (B \cup A \circ B] \subseteq (B] = B$. Let $b \in B, h \in (B \cup H \circ B]$ such that $h \leq b$. As $b \in B \subseteq A, b \in A$. So $h \in A$. Thus $h \in B$. Hence, $B$ is a bi-hyperideal of the left hyperideal $(B \cup H \circ B]$ of $H$. 

(iii) $\Rightarrow$ (iv) Let $B$ be a bi-hyperideal of a left hyperideal $L$ of $H$. Then $B \subseteq L$, $B \circ L \circ B \subseteq B$ and $H \circ L \subseteq L$. Therefore, $(B \cup B \circ H) \circ B^m \subseteq (B \cup B \circ H) \circ (B^m) \subseteq (B \cup B \circ H \circ B^m) \subseteq (B \cup B \circ (H \circ B^m) \circ B) \subseteq (B \cup B \circ B \circ H \circ B \circ B^m) = (B \cup B \circ (H \circ L) \circ B) \subseteq (B \cup B \circ B \circ L \circ B) = (B) = B$. Let $b \in B$, $h \in (B \cup B \circ H)$ such that $h \leq b$. As $B \subseteq L$, $b \in L$. So $h \in L$ and, thus, $h \in B$. Hence, $B$ is a $(0,m)$-hyperideal of the right hyperideal $(B \cup B \circ H]$ of $H$.

(iv) $\Rightarrow$ (v) Let $B$ be a $(0,m)$-hyperideal of a right hyperideal $R$ of $H$. So $B \subseteq R$, $R \circ B^m \subseteq B$ and $R \circ H \subseteq R$. Therefore, $B \circ (B \cup H \circ B^m) \subseteq (B \cup H \circ B^m) \subseteq (B \cup B \circ H \circ B^m) \subseteq (B \cup B \circ B \circ H \circ B^m) = (B) = B$. Let $b \in B$, $h \in (B \cup H \circ B^m)$ be such that $h \leq b$. As $B \subseteq R$, $b \in R$. So $h \in R$ which implies that $h \in B$. Hence, $B$ is a right hyperideal of the $(0,m)$-hyperideal $(B \cup H \circ B^m]$ of $H$.

(v) $\Rightarrow$ (i) Let $B$ be a right hyperideal of a $(0,m)$-hyperideal $A$ of $H$. Thus $B \subseteq A$, $B \circ A \subseteq B$ and $H \circ A^m \subseteq A$. Therefore, $B \circ H \circ B^m \subseteq B \circ H \circ A^m \subseteq B \circ A \subseteq B$. Let $b \in B$, $h \in H$ be such that $h \leq b$. As $B \subseteq R$, we have $b \in R$. Therefore, $h \in R$ and, thus, $h \in B$. Hence, $B$ is a $(1,m)$-hyperideal of $H$.

**Definition 2.7.** Let $H$ be an ordered semihypergroup, $m,n$ be positive integers and $A$ be any (generalized) $(m,n)$-hyperideal of $H$. Then $A$ is said to be a minimal (generalized) $(m,n)$-hyperideal of $H$ if for every (generalized) $(m,n)$-hyperideal $B$ of $H$, $B \subseteq A$ implies $B = A$.

Similarly, a minimal $(m,0)$-hyperideal and a minimal $(0,n)$-hyperideal of $H$ may be defined.

**Lemma 2.8.** Let $H$ be an ordered semihypergroup, $m \geq 2$ be any positive integer and $B$ be a non-empty subset of $H$. Then $B$ is a minimal (generalized) $(m,m-1)$-hyperideal of $H$ if and only if $B$ is a minimal (generalized) bi-hyperideal of $H$.

**Proof.** Let $H$ be an ordered semihypergroup and $B$ be a minimal $(m,m-1)$-hyperideal of $H$. Since $(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$, $((B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$ and $(B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$. Therefore, $(B^m \circ H \circ B^{m-1}]$ is a $(m,m-1)$-hyperideal of $H$ such that $(B^m \circ H \circ B^{m-1}] \subseteq B$. So by minimality of $(m,m-1)$-hyperideal $B$ of $H$, $(B^m \circ H \circ B^{m-1}] = B$. Now $B \circ B =
\((B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1}) \subseteq ((B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1})) \subseteq (B^m \circ H \circ B^{m-1}) = B\) and \(B \circ H \circ B = (B^m \circ H \circ B^{m-1}) \circ H \circ (B^m \circ H \circ B^{m-1}) \subseteq (B^m \circ H \circ B^{m-1}) = B\). Therefore, \(B\) is bi-hyperideal of \(H\). It remains to show that \(B\) is a minimal bi-hyperideal of \(H\), so assume that \(A\) is any bi-hyperideal of \(H\) contained in \(B\). Therefore, \(A\) is \((m, m - 1)\)-hyperideal of \(H\). Since \(B\) is a minimal \((m, m - 1)\)-hyperideal of \(H\), \(B = A\). Hence, \(B\) is a minimal bi-hyperideal of \(H\). For the converse, assume that \(B\) is a minimal bi-hyperideal of \(H\). As \(B^m \circ H \circ B^{m-1} = B \circ (B^m \circ H \circ B^{m-2}) \circ B \subseteq B \circ H \circ B \subseteq B\), \(B\) is an \((m, m - 1)\)-hyperideal of \(H\). To show that \(B\) is a minimal \((m, m - 1)\)-hyperideal of \(H\), let \(A\) be any \((m, m - 1)\)-hyperideal of \(H\) such that \(A \subseteq B\). As \((A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1}) \subseteq ((A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1})) \subseteq (A^m \circ H \circ A^{m-1})\) and \((A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1}) \subseteq ((A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1})) \subseteq (A^m \circ H \circ A^{m-1})\), \((A^m \circ H \circ A^{m-1})\) is a bi-hyperideal of \(H\). Since \(B\) is a minimal bi-hyperideal of \(H\) and \((A^m \circ H \circ A^{m-1}) \subseteq B\), \((A^m \circ H \circ A^{m-1}) = B\). As \((A^m \circ H \circ A^{m-1}) \subseteq A\), \(B \subseteq A\). Now, as \(A \subseteq B\), we have \(A = B\). Hence, \(B\) is a minimal \((m, m - 1)\)-hyperideal of \(H\).

**Theorem 2.9.** Let \(H\) be an ordered semihypergroup and \(\{A_i \mid i \in I\}\) be a set of \((m, n)\)-hyperideals of \(H\). If \(\bigcap_{i \in I} A_i \neq \emptyset\), then \(\bigcap_{i \in I} A_i\) is an \((m, n)\)-hyperideal of \(H\).

**Proof.** Assume that \(\bigcap_{i \in I} A_i \neq \emptyset\). Let \(x, y \in \bigcap_{i \in I} A_i\). Then, \(x, y \in A_i\) for each \(i \in I\). As for each \(i \in I\), \(A_i\) is an \((m, n)\)-hyperideal, \(x \circ y \subseteq A_i\). Therefore, \(x \circ y \subseteq \bigcap_{i \in I} A_i\). Thus, \(\bigcap_{i \in I} A_i\) is a subsemihypergroup of \(H\). Next we show that \((\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i\). We have

\[
(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \\
\subseteq (A_i)^m \circ H \circ (A_i)^n \quad \text{(as } \bigcap_{i \in I} A_i \subseteq A_i, \forall i \in I) \\
\subseteq A_i \quad \text{(as } A_i\text{'s are } (m, n)\text{-hyperideals).}
\]

Thus \((\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i\). Finally, we show that \((\bigcap_{i \in I} A_i)^m \subseteq \bigcap_{i \in I} A_i\). Let \(a \in \bigcap_{i \in I} A_i\), \(h \in H\) such that \(h \leq a\). As \(a \in A_i\) for each \(i \in I\) and \(A_i\)'s
are \((m, n)\)-hyperideals, \(h \in A_i\) for each \(i \in I\). Therefore, \(h \in \bigcap_{i \in I} A_i\), as required.

**Theorem 2.10.** [20] Let \(H\) be an ordered semihypergroup. Then the following conditions hold:

(i) Let \(\{L_i \mid i \in I\}\) be a set of \((m, 0)\)-hyperideals of \(H\). If \(\bigcap_{i \in I} L_i \neq \emptyset\), then \(\bigcap_{i \in I} L_i\) is an \((m, 0)\)-hyperideal of \(H\).

(ii) Let \(\{R_i \mid i \in I\}\) be a set of \((0, n)\)-hyperideals of \(H\). If \(\bigcap_{i \in I} R_i \neq \emptyset\), then \(\bigcap_{i \in I} R_i\) is a \((0, n)\)-hyperideal of \(H\).

Let \(H\) be an ordered semihypergroup and \(A\) be any non-empty subset of \(H\). We denote \(P = \{J \mid J\) is an \((m, n)\)-hyperideal of \(H\) containing \(A\}\). Clearly, \(P \neq \emptyset\) since \(H \in P\). Let \([A]_{m,n} = \bigcap_{J \in P} J\). As \(A \subseteq J\) for each \(J \in P\), \([A]_{m,n} \neq \emptyset\). By Theorem 1.9, \([A]_{m,n}\) is an \((m, n)\)-hyperideal of \(H\) containing \(A\). The \((m, n)\)-hyperideal \([A]_{m,n}\) is called the \((m, n)\)-hyperideal of \(H\) generated by \(A\).

Similarly, \([A]_{m,0}\) and \([A]_{0,n}\) are called \((m, 0)\)-hyperideal and \((0, n)\)-hyperideal of \(H\) generated by \(A\), respectively.

**Theorem 2.11.** Let \(H\) be an ordered semihypergroup and \(A\) be a non-empty subset of \(H\). Then

\[
[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n]
\]

for any positive integers \(m, n\).

**Proof.** Clearly \((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n] \neq \emptyset\). Now we have

\[
\begin{align*}
&= (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \cup (\bigcup_{i=1}^{m+n} A^i) \circ A^m \circ H \circ A^n \cup (A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i) \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \cup (A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i) \\
&\subseteq (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \cup (A^m \circ H \circ A^n)
\end{align*}
\]

(1).

Let \(x \in (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i)\). Then, \(x \in z_1 \circ z_2\) for some \(z_1, z_2 \in \)
\[ \bigcup_{i=1}^{m+n} A^i. \] Then, \( z_1 = A^p, z_2 = A^q \) for some \( 1 < p, q \leq m + n \). There are two cases arising. If \( p + q \leq m + n \), then \( z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \). If \( m + n \leq p + q \), then \( z_1 \circ z_2 \subseteq A^m \circ H \circ A^n \). Therefore, in both cases \( z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \). As \( x \in z_1 \circ z_2, x \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \). Thus, \( (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \). Therefore, from (1), \( (\bigcup_{i=1}^{m+n} A^i) \cup A^m \circ H \circ A^n \circ (\bigcup_{i=1}^{m+n} A^i) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \circ A) \). Hence, \( (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \) is a subsemihypergroup of \( H \) containing \( A \). We have

\[
\left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right)^m \circ H \\
= ( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n )^{m-1} \circ \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \circ H \\
\subseteq ( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n )^{m-1} \circ \left( \bigcup_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H \right) \\
\subseteq ( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n )^{m-1} \circ (A \circ H) \\
= (A^m \circ H). 
\]

Similarly, \( H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \subseteq (H \circ A^n) \). Therefore, we have

\[
((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^m \circ H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \\
\subseteq (A^m \circ H \circ A^n) \\
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n). 
\]

Also \( (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \). Therefore, \( (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \) is an \( (m, n) \)-hyperideal of \( H \) containing \( A \). It follows that \( [a]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \). For the reverse inclusion, let \( x \in (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \), that is, there exist \( z \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \)
such that \( x \leq z \). If \( z \in \bigcup_{i=1}^{m+n} A^i \), then \( z = A^p \) for some \( 1 \leq p \leq m + n \).

Therefore, \( x \in [A]_{m,n} \). If \( z \in A^m \circ H \circ A^n \), then

\[
A^m \circ H \circ A^n \subseteq ([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n \subseteq [A]_{m,n}.
\]

Therefore, \( z \in [A]_{m,n} \) implies \( x \in [A]_{m,n} \). Hence, \( [A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \), as required.

**Theorem 2.12.** [20] Let \( H \) be an ordered semihypergroup and \( A \) be any non-empty subset of \( H \). Then:

(i) \([A]_{m,0} = (\bigcup_{i=1}^{m} A^i \cup A^m \circ H)\);

(ii) \([A]_{0,n} = (\bigcup_{i=1}^{n} A^i \cup H \circ A^n)\).

**Theorem 2.13.** Let \( H \) be an ordered semihypergroup and \( A \) be a non-empty subset of \( H \). Then

\[
(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n)
\]

for any positive integers \( m, n \).

**Proof.** We have

\[
([A]_{m,n})^m \circ H
= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^m \circ H
= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ H
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A \circ H)
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (A \circ H)
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \circ A \circ H \cup A^m \circ H \circ A^n \circ A \circ H
\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (A^2 \circ H)
= (A^m \circ H].

Similarly, \( H \circ ([A]_{m,n})^n \subseteq H \circ A^n \). Therefore, \( (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) \subseteq (A^m \circ H \circ A^n) \). The reverse inclusion is obvious, that is, \( (A^m \circ H \circ A^n) \subseteq (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) \). Hence, \( (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n) \).

\[\text{Theorem 2.14. } [20] \text{ Let } H \text{ be an ordered semihypergroup and } A \text{ be a non-empty subset of } H. \text{ Then}
\]

(i) \( ([A]_{m,0})^m \circ H = (A^m \circ H) \) for any positive integer \( m \).

(ii) \( H \circ ([A]_{0,n})^n = (H \circ A^n) \) for any positive integer \( n \).

3 \((m,n)\)-regularity in ordered semihypergroups

In this section, we characterize \((m,n)\)-regular, \((m,0)\)-regular and \((0,n)\)-regular ordered semihypergroup in terms of its \((m,n)\)-hyperideals, \((m,0)\)-hyperideals and \((0,n)\)-hyperideals.

**Definition 3.1.** Let \( H \) be an ordered semihypergroup and \( m, n \) be non-negative integers. An element \( a \) of \( H \) is said to be an \((m,n)\)-regular element if \( a \in (a^m \circ H \circ a^n) \). The ordered semihypergroup \( H \) is said to be \((m,n)\)-regular if each element of \( H \) is \((m,n)\)-regular, equivalently, for each subset \( A \) of \( H \) we have \( A \subseteq (A^m \circ H \circ A^n) \). Here, \( A^0 \circ H = H \circ A^0 = H \).

It is clear from Definition 3.1 that, for each non-negative integers \( m \) and \( n \) every \((m,n)\)-regular ordered semihypergroup is \((r,s)\)-regular \((r \leq m, s \leq n \) are non-negative integers\). In particular, for any positive integers \( m \) and \( n \), an \((m,n)\)-regular ordered semihypergroup is regular. Indeed, \( a \in (a^m \circ H \circ a^n) \subseteq (a \circ H \circ a) \). On the other hand, for each positive integer \( m \), an \((m,0)\)-regular ordered semihypergroup need not be a regular ordered semihypergroup.

**Proposition 3.2.** Let \( H \) be an \((m,n)\)-regular ordered semihypergroup and \( A \) be a generalized \((m,n)\)-hyperideal of \( H \) for any positive integers \( m, n \). Then \( A \) is an \((m,n)\)-hyperideal of \( H \).
Proof. Let \( a, b \in A \). Since \( H \) is an \((m, n)\)-regular ordered semihypergroup, there exist \( x, y \in H \) such that \( a \leq a^m \circ x \circ a^n, b \leq b^m \circ y \circ b^n \). Therefore, \( a \circ b \leq a^m \circ x \circ a^n \circ b^m \circ y \circ b^n = a^m \circ (x \circ a^n \circ b^m \circ y) \circ b^n \subseteq A^n \circ H \circ A^m \subseteq A \) whence \( a \circ b \subseteq (A] = A \). Thus \( A \) is a subsemihypergroup of \( H \). Hence, \( A \) is an \((m, n)\)-hyperideal of \( H \).

\( (m, n)\)-Hyperideals in ordered semihypergroups

Theorem 3.3. Let \( H \) be an ordered semihypergroup and \( m, n \) be non-negative integers. The set of all \((m, 0)\)-hyperideals, \((0, n)\)-hyperideals, and \((m, n)\)-hyperideals will be denoted by \( I_{(m, 0)} \), \( I_{(0, n)} \) and \( I_{(m, n)} \), respectively. Then, we have

(i) \( H \) is \((m, 0)\)-regular if and only if \( I_{(m, 0)} \) is \((m, 0)\)-regular;
(ii) \( H \) is \((0, n)\)-regular if and only if \( I_{(0, n)} \) is \((0, n)\)-regular;
(iii) \( H \) is \((m, n)\)-regular if and only if \( I_{(m, n)} \) is \((m, n)\)-regular.

Proof. (i) When \( m = 0 \), the statement holds trivially because \( H \) is the only \((0, 0)\)-hyperideal of \( H \). So, let \( m \neq 0 \) and \( A \in I_{(m, 0)} \). Therefore \( (A^m \circ H) \subseteq A \). As \( S \) is \((m, 0)\)-regular, \( A \subseteq (A^m \circ H) \). Thus, \( A = (A^m \circ H) \). Since \( H \in I_{(m, 0)} \), \( A \) is a \((m, 0)\)-regular element of \( I_{(m, 0)} \). Hence \( I_{(m, 0)} \) is \((m, 0)\)-regular. For the converse, assume that \( I_{(m, 0)} \) is \((m, 0)\)-regular. Take any \( a \in S \). As \([a]_{m, 0} \) is in \( I_{(m, 0)} \) and \( I_{(m, 0)} \) is \((m, 0)\)-regular, there exists \( B \in I_{(m, 0)} \) such that \([a]_{m, 0} = ([a]_{m, 0})^m \circ B \subseteq ([a]_{m, 0})^m \circ H \subseteq (([a]_{m, 0})^m \circ H) \). By Theorem 2.14, \( ([a]_{m, 0})^m \circ H = (a^m \circ H) \). As \( \{a\} \subseteq [a]_{m, 0} \), we have \( a \in (a^m \circ H) \). Hence \( H \) is \((m, 0)\)-regular.

(ii) On the similar lines to (i), we may prove (ii).

(iii) If \( m = n = 0 \), then the statement is true because \( I_{(0, 0)} = \{H\} \). If \( m \neq 0 \) and \( n = 0 \) or \( m = 0 \) and \( n \neq 0 \), then the statement follows by (i) and (ii), respectively. So, let \( m \neq 0, n \neq 0 \) and \( A \in I_{(m, n)} \). Therefore \( (A^m \circ H \circ A^n) \subseteq A \). As \( H \) is \((m, n)\)-regular, \( A \subseteq (A^m \circ H \circ A^n) \). Thus, \( A = (A^m \circ H \circ A^n) \). Since \( H \in I_{(m, n)} \), \( A \) is an \((m, n)\)-regular element of \( I_{(m, n)} \). Hence, \( I_{(m, n)} \) is \((m, n)\)-regular. For the converse, assume that \( I_{(m, n)} \) is \((m, n)\)-regular and \( a \in H \). As \([a]_{m, n} \) is in \( I_{(m, n)} \) and \( I_{(m, n)} \) is \((m, n)\)-regular, there exists \( B \in I_{(m, n)} \) such that \([a]_{m, n} = ([a]_{m, n})^m \circ B \circ ([a]_{m, n})^n \subseteq ([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n \subseteq (([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n) \). By Theorem 4.1, we have \( (([a]_{m, n})^m \circ H \circ ([a]_{m, n})^n) = (a^m \circ H \circ a^n) \). As \( \{a\} \subseteq [a]_{m, n} \), \( a \in (a^m \circ H \circ a^n) \). This implies that \( a \) is an \((m, n)\)-regular element of \( H \). Hence, \( H \) is \((m, n)\)-regular.
Lemma 3.4. [20] Let $H$ be an ordered semihypergroup. If the sets of all $(m,0)$-hyperideals and $(0,n)$-hyperideals are denoted by $I_{(m,0)}$ and $I_{(0,n)}$ respectively, then

(i) $H$ is $(m,0)$-regular if and only if $R = (R^m \circ H)$ (\forall R \in I_{(m,0)})$, where $m$ is any positive integer;

(ii) $H$ is $(0,n)$-regular if and only if $L = (H \circ L^n)$ (\forall L \in I_{(0,n)})$, where $n$ is any positive integer.

Theorem 3.5. Let $H$ be an ordered semihypergroup and $m, n$ be non-negative integers. The set of all $(m,n)$-hyperideals will be denoted by $I_{(m,n)}$. Then $H$ is $(m,n)$-regular if and only if $A = (A^m \circ H \circ A^n)$ for all $A \in I_{(m,n)}$.

Proof. If $m = n = 0$, then the statement is true because $I_{(0,0)} = \{H\}$. If $m \neq 0$ and $n = 0$ or $m = 0$ and $n \neq 0$, then the statement follows by Lemma 3.4. So, let $m \neq 0, n \neq 0$ and $A \in I_{(m,n)}$. Then, by definition of $(m,n)$-regularity, we have $A \subseteq (A^m \circ H \circ A^n)$ and, by definition of $(m,n)$-hyperideal, we have $(A^m \circ H \circ A^n) \subseteq (A) = A$. Hence, $A = (A^m \circ H \circ A^n)$.

For the converse, assume that $A = (A^m \circ H \circ A^n)$ for each $A \in I_{(m,n)}$. Take any $a \in H$, so $[a]_{m,n} \in I_{(m,n)}$. From Theorem 4.1 and by the assumption, $[a]_{m,n} = (([a]_{m,n})^m \circ H \circ [a]_{m,n}) = (a^m \circ H \circ a^n)$. As $\{a\} \subseteq [a]_{m,n}$, $a \in (a^m \circ H \circ a^n)$. Hence, $H$ is $(m,n)$-regular.

Theorem 3.6. Let $H$ be an ordered semihypergroup and $m, n$ be non-negative integers. Then, $H$ is $(m,n)$-regular if and only if $L \cap R = (R^m \circ L^n)$ for each $(0,0)$-hyperideal $R$ and for each $(0,n)$-hyperideal $L$ of $H$.

Proof. The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Lemma 3.4. So, let $m \neq 0, n \neq 0$, $R$ be any $(m,0)$-hyperideal and $L$ be any $(0,n)$-hyperideal of $H$. Therefore $R^m \circ L^n \subseteq (R^m \circ H) \subseteq (R) = R$ and $(R^m \circ L^n) \subseteq (H \circ L^n) \subseteq (L) = L$. Therefore, $(R^m \circ L^n) \subseteq R \cap L$. As $H$ is $(m,n)$-regular, we have

\[(R \cap L)^m \circ H \circ (R \cap L)^n \]
\[\subseteq (R^m \circ H \circ L^n)
\[\subseteq (R^m \circ H \circ L^{n-1} \circ (L^m \circ H \circ L^n)) \text{ (as } H \text{ is } (m,n)-\text{regular})
\[= (R^m \circ H \circ L^{n-1} \circ L^m \circ H \circ L^n) \text{ (by Lemma 1.1)}
\[\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ H \circ L^n) \text{ (as } H \text{ is } (m,n)-\text{regular})
\[\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ H \circ L^n)) \text{ (as } H \circ L^n \subseteq (H \circ L^n))\]
\( (m, n) \)-Hyperideals in ordered semihypergroups

\[ \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n \circ H \circ L^n)] \quad \text{(by Lemma 1.1)} \]
\[ \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n] \quad \text{(by Lemma 1.1)} \]
\[ \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n \circ L^m \circ H \circ L^n \circ H \circ L^n) \]
\[ \vdots \]
\[ \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^{m-1} \circ \cdots \circ L^{m-1} \circ (L^m \circ H \circ L^n) \]
\[ \circ (H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ \subseteq (R^m \circ H \circ L^{n-1} \circ (L^{m-1})^{n-1} \circ L^m \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ = (R^m \circ H \circ (L^{n-1} \circ L^{mn-m-n+1} \circ L^m) \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ = (R^m \circ (H \circ L^m) \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ \subseteq (R^m \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ \subseteq (R^m \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ \subseteq (R^m \circ H \circ L^n \circ H \circ L^n \circ \cdots \circ H \circ L^n) \]
\[ \subseteq (R^m \circ (H \circ L^n)^n) \]
\[ \subseteq (R^m \circ L^n). \]

Therefore, \( L \cap R = (R^m \circ L^n) \).

Conversely, assume that \( L \cap R = (R^m \circ L^n) \) for each \((m, 0)\)-hyperideal \( R \) and for each \((0, n)\)-hyperideal \( L \) of \( H \). Let \( a \in S \). As \([a]_{m,0}\) is an \((m, 0)\)-hyperideal and \( H \) is a \((0, n)\)-hyperideal of \( H \), we have

\[ [a]_{m,0} = [a]_{m,0} \cap H = \left( (\{a\}_{m,0})^m \circ H^n \right) \]
\[ \subseteq ((\{a\}_{m,0})^m \circ H) = (a^m \circ H) \quad \text{(by Theorem 2.14)} \]

Similarly, \([a]_{0,n} \subseteq (H \circ a^n)\). As \((a^m \circ H)\) and \((H \circ a^n)\) are an \((m, 0)\)-hyperideal and \((0, n)\)-hyperideal of \( H \), by hypothesis we get

\[ \{a\} \subseteq [a]_{m,0} \cap [a]_{0,n} \subseteq (a^m \circ H) \cap (H \circ a^n) \]
\[ = ((a^m \circ H)^m \circ (H \circ a^n)^n) \quad \text{(by hypothesis)} \]
\[ \subseteq (a^m \circ H \circ a^n). \]

Hence, \( H \) is \((m, n)\)-regular. \( \square \)

**Theorem 3.7.** Let \( H \) be an ordered semihypergroup and \( m, n \) be positive integers (either \( m \geq 2 \) or \( n \geq 2 \)). Then, the following are equivalent:
(i) Each \((m,n)\)-hyperideal of \(H\) is idempotent;
(ii) For each \((m,n)\)-hyperideals \(A, B\) of \(H\), \(A \cap B \subseteq (A^m \circ B^n)\);
(iii) \([a]_{m,n} \cap [b]_{m,n} \subseteq ([a]_{m,n})^m \circ ([b]_{m,n})^n \forall a, b \in H\);
(iv) \([a]_{m,n} \subseteq ([a]_{m,n})^m \circ ([a]_{m,n})^n \forall a \in H\);
(v) \(H\) is \((m,n)\)-regular.

**Proof.** (i) \(\Rightarrow\) (ii) Assume that each \((m,n)\)-hyperideal of \(H\) is idempotent. Let \(A\) and \(B\) be any \((m,n)\)-hyperideals of \(H\). As \(A \cap B\) is an \((m,n)\)-hyperideal of \(H\), we have

\[
A \cap B = ((A \cap B)^2) = ((A \cap B) \circ ((A \cap B))^2) = ((A \cap B)^3) = \cdots = ((A \cap B)^{m+n}) = ((A \cap B)^m \circ (A \cap B)^n) \subseteq (A^m \circ B^n).
\]

(ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (iv) are obvious.

(iv) \(\Rightarrow\) (v) Take any \((m,n)\)-hyperideal \(A\) of \(H\). As \(H\) is (\(m,n\))-regular, \(a \in A\). Then, by (iv), we have

\[
[a]_{m,n} \subseteq ([a]_{m,n})^m \circ ([a]_{m,n})^n \\
\subseteq ([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ ([a]_{m,n})^m \circ ([a]_{m,n})^n \\
= ([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ ([a]_{m,n})^m \circ ([a]_{m,n})^n \ (\text{by Lemma 1.1}) \\
\subseteq ([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n \\
= ([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n \ (\text{by Lemma 1.1}) \\
= (a^m \circ H) \circ ([a]_{m,n})^n \ (\text{by Theorem 4.1}) \\
= (a^m \circ (H \circ [a]_{m,n})) \ (\text{by Lemma 1.1}) \\
= (a^m \circ (H \circ a^n)) \ (\text{by Theorem 4.1}) \\
= (a^m \circ H \circ a^n) \ (\text{by Lemma 1.1})
\]

As \(\{a\} \subseteq [a]_{m,n}, a \in (a^m \circ H \circ a^n)\). Hence \(H\) is \((m,n)\)-regular.

(v) \(\Rightarrow\) (i) Take any \((m,n)\)-hyperideal \(A\) of \(H\). As \(H\) is \((m,n)\)-regular and \(A\) is an \((m,n)\)-hyperideal, \(A = (A^m \circ H \circ A^n)\). Now

\[
(A \circ A) = ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n)) \subseteq (A^m \circ H \circ A^n) = A
\]
and

\[
A = (A^m \circ H \circ A^n) = ((A^m \circ H \circ A^n))^m \circ H \circ A^n \\
= (A^m \circ H \circ A^n) \cdots \circ (A^m \circ H \circ A^n) \circ H \circ A^n \\
= (A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ \ldots \circ (A^m \circ H \circ A^n) \circ H \circ A^n
\]
\((A^m \circ H \circ A^n) \circ \cdots \circ (A^m \circ H \circ A^n) \circ H \circ A^n\)

\( \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ H \circ A^n) \)
\( \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ H \circ (H \circ A^n)) \)
\( \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ (H \circ A^n) \circ (H \circ A^n)) \)
\( \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ (H \circ A^n) \circ (H \circ A^n)) \)
\( = (A \circ A) \).

Therefore, \( A = (A \circ A) \). Hence, each \((m, n)\)-hyperideal of \( H \) is an idempotent.

The following example shows that the condition \( m \geq 2 \) or \( n \geq 2 \) in Theorem 3.7 is necessary.

**Example 3.8.** [24] Let \( H = \{a, b, c, d, e\} \). Define a hyperoperation \( \circ \) on \( H \) by the table

| \( \circ \) | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) |
|---|---|---|---|---|---|
| \( a \) | \{a\} | \{a\} | \{a\} | \{a\} | \{a\} |
| \( b \) | \{a\} | \{a, b\} | \{a\} | \{a, d\} | \{a\} |
| \( c \) | \{a\} | \{a, e\} | \{a, c\} | \{a, c\} | \{a, e\} |
| \( d \) | \{a\} | \{a, b\} | \{a, d\} | \{a, d\} | \{a, b\} |
| \( e \) | \{a\} | \{a, e\} | \{a\} | \{a, c\} | \{a\} |

and the order \( \leq \) on \( H \) as \( \leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e)\} \). The covering relation \( \prec \) and the figure of \( H \) are as

\( \prec := \{(a, b), (a, c), (a, d), (a, e)\} \)

Now, \( (H, \circ, \leq) \) is a regular ordered semihypergroup. One may easily check that \( A = \{a, e\} \) is a bi-hyperideal of \( H \), but \( A \neq (A^2) \).
4 Relations $\mathcal{I}_n$, $m\mathcal{I}$, $\mathcal{H}_m^n$ and $B_m^n$ on ordered semihypergroups

In this section, the relations $\mathcal{I}_n$, $m\mathcal{I}$, $\mathcal{H}_m^n$ and $B_m^n$ on an ordered semihypergroup are introduced. Then, some related properties of these relations are studied.

Definition 4.1. Let $H$ be an ordered semihypergroup and $m, n$ be positive integers. We define the relations $\mathcal{I}_n$, $m\mathcal{I}$, $\mathcal{H}_m^n$ and $B_m^n$ as

\[
\mathcal{I}_n = \{(a, b) \in S \times S \mid [a]_{0,n} = [b]_{0,n}\};
\]

\[
m\mathcal{I} = \{(a, b) \in S \times S \mid [a]_{m,0} = [b]_{m,0}\};
\]

\[
\mathcal{H}_m^n = m\mathcal{I} \cap \mathcal{I}_n;
\]

\[
B_m^n = \{(a, b) \in S \times S \mid [a]_{m,n} = [b]_{m,n}\}.
\]

Clearly, all the relations defined above are equivalence relations on $H$.

Lemma 4.2. Let $H$ be an ordered semihypergroup and $a, b \in H$ be $m\mathcal{I}$-related (respectively, $\mathcal{I}_n$-related). Then, $(a^m \circ H) = (b^m \circ H)$ (respectively, $(H \circ a^n) = (H \circ b^n)$).

Proof. Suppose that $(a, b) \in m\mathcal{I}$. Then, by definition, $[a]_{m,0} = [b]_{m,0}$, i.e. $(\bigcup_{i=1}^m a_i \cup a^m \circ H) = (\bigcup_{i=1}^m b_i \cup b^m \circ H)$. Therefore, $\{a\} \subseteq (\bigcup_{i=1}^m b_i \cup b^m \circ H)$ and $\{b\} \subseteq (\bigcup_{i=1}^m a_i \cup a^m \circ H)$. Thus, $(a^m \circ H) \subseteq ((\bigcup_{i=1}^m b_i \cup b^m \circ H)^m \circ H) = ((b^m \circ H)^m \circ H)$ (by Theorem 4.1). Similarly, from $\{b\} \subseteq (\bigcup_{i=1}^m a_i \cup a^m \circ H)$, we have $(b^m \circ H) \subseteq (a^m \circ H)$. Hence $(a^m \circ H) = (b^m \circ H)$. Similarly, we may show that $(a, b) \in \mathcal{I}_n$ implies $(H \circ a^n) = (H \circ b^n)$. \hfill \Box

Lemma 4.3. Let $H$ be an ordered semihypergroup and $a, b \in H$ be $\mathcal{H}_m^n$-related. Then, $(a^m \circ H) = (b^m \circ H)$, $(H \circ a^n) = (H \circ b^n)$ and $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$.

Proof. Suppose that $(a, b) \in \mathcal{H}_m^n$. Then, by definition, $(a, b) \in m\mathcal{I}$ and $(a, b) \in \mathcal{I}_n$. By Lemma 4.1, $(a^m \circ H) = (b^m \circ H)$ and $(H \circ a^n) = (H \circ b^n)$. Therefore, we have $(a^m \circ H \circ a^n) = ((a^m \circ H) \circ a^n) = ((b^m \circ H) \circ a^n) = (b^m \circ H \circ a^n) = (b^m \circ (H \circ a^n)) = (b^m \circ (H \circ b^n)) = (b^m \circ H \circ b^n)$.

\hfill \Box

Lemma 4.4. Let $H$ be an ordered semihypergroup. Then, $B_m^n \subseteq \mathcal{H}_m^n$. 

(m, n)-Hyperideals in ordered semihypergroups

Proof. Let \((a, b) \in B_m^n\). Then, \([a]_{m,n} = [b]_{m,n}\), i.e. \(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\) = \(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\). So \(a^i \subseteq \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) and \(b^i \subseteq \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\) for each \(i \in \{1, 2, \ldots, m + n\}\). It follows that \(\bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\) and \(\bigcup_{i=1}^{m+n} b^i \subseteq \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\).

Now \((a^m \circ H) \subseteq (((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n)^m \circ H) \cup (b^m \circ H)) \subseteq (((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n)^m \circ H) \cup (b^m \circ H))\). Therefore, by Theorem 4.1, \((a^m \circ H) \subseteq (b^m \circ H)\) and \((b^m \circ H) \subseteq (a^m \circ H)\). Now

\[
[a]_{m,0} = \left(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H\right)
\]

\[
\subseteq \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup a^m \circ H\] (since \(\bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\))

\[
\subseteq \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup (a^m \circ H)\] (as \(a^m \circ H \subseteq (a^m \circ H)\))

\[
\subseteq \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup (b^m \circ H)\] (as \((a^m \circ H) \subseteq (b^m \circ H)\))

\[
= \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup (b^m \circ H)\] (by Lemma 1.1)

\[
= \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup (b^m \circ H)\] (by Lemma 1.1)

\[
\subseteq \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n\right) \cup (b^m \circ H)\] (since \(\bigcup_{i=1}^{m+n} b^i \subseteq \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H\))

\[
= \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H\right)\] (as \(b^m \circ H \circ b^n \subseteq (b^m \circ H)\))

\[
= [b]_{m,0};
\]

and

\[
[b]_{m,0} = \left(\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H\right)
\]

\[
\subseteq \left(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\right) \cup (b^m \circ H)\] (since \(\bigcup_{i=1}^{m+n} b^i \subseteq \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\))

\[
\subseteq \left(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\right) \cup (b^m \circ H)\] (as \(b^m \circ H \subseteq (b^m \circ H)\))

\[
\subseteq \left(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n\right) \cup (a^m \circ H)\] (as \((b^m \circ H) \subseteq (a^m \circ H)\))
\[=(( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H) \] (by Lemma 1.1)
\[=( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H) \] (by Lemma 1.1)
\[\subseteq (\bigcup_{i=1}^{m} a^i \cup a^m \circ H \cup a^m \circ H \circ a^n \cup a^m \circ H) \] (since \[\bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^{m} a^i \cup a^m \circ H \]
\[\subseteq (\bigcup_{i=1}^{m} a^i \cup a^m \circ H) \] (as \[a^m \circ H \circ a^n \subseteq a^m \circ H \])
\[=[a]_{m,n} \] (as \[a \circ b \circ a \circ b \subseteq a \circ b \circ a \circ b \]).

Therefore, \([a]_{m,0} = [b]_{m,0} \). Similarly, one can show that \([a]_{0,n} = [b]_{0,n} \). Thus, \((a, b) \in \mathcal{H}_m \). Hence, \(\mathcal{B}_m^m \subseteq \mathcal{H}_m^m \).

**Theorem 4.5.** Let \(H \) be an \((m, n)\)-regular ordered semihypergroup. Then, \(\mathcal{B}_m^m = \mathcal{H}_m^m \).

**Proof.** Let \((a, b) \in \mathcal{H}_m^m \). Therefore, by Lemma 4.2, \((a^m \circ H \circ a^n) = (b^m \circ H \circ b^n) \). As \(S \) is \((m, n)\)-regular, \(a \in (a^m \circ H \circ a^m) \) and \(b \in (b^m \circ H \circ b^n) \). So \(a^i \subseteq (a^m \circ H \circ a^m) \) for each \(i \in \{1, 2, \ldots, m + n\} \), it follows that \(\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^m) \). Thus, \([a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m) = (a^m \circ H \circ a^m) \) and similarly \([b]_{m,n} = (b^m \circ H \circ b^n) \). Thus, \([a]_{m,n} = [b]_{m,n} \), i.e. \((a, b) \in \mathcal{B}_m^m \). This implies that \(\mathcal{H}_m^m \subseteq \mathcal{B}_m^m \). Hence, by Lemma 4.3, \(\mathcal{B}_m^m = \mathcal{H}_m^m \).

**Lemma 4.6.** If \(B_x \) and \(B_y \) are two \((m, n)\)-regular \(\mathcal{B}_m^n \)-classes contained in the same \(\mathcal{H}_m^n \)-class of ordered semihypergroup \(H \), then \(B_x = B_y \).

**Proof.** As \(x \) and \(y \) are \((m, n)\)-regular elements of \(H \), \(x \in (x^m \circ H \circ x^n) \) and \(y \in (y^m \circ H \circ y^n) \), \(\{x\}^i \subseteq (x^m \circ H \circ x^n) \) and \(\{y\}^i \subseteq (y^m \circ H \circ y^n) \) for each \(i \in \{1, 2, \ldots, m + n\} \). It follows that \(\bigcup_{i=1}^{m+n} x^i \subseteq (x^m \circ H \circ x^m) \) and \(\bigcup_{i=1}^{m+n} y^i \subseteq (y^m \circ H \circ y^n) \). Therefore, \([x]_{m,n} = (x^m \circ H \circ x^n) \) and \([y]_{m,n} = (y^m \circ H \circ y^n) \). Since \(x \) and \(y \) are contained in the same \(\mathcal{H}_m^n \)-class, by Lemma 4.2, \((x^m \circ H \circ x^n) = (y^m \circ H \circ y^n) \). So \([x]_{m,n} = [y]_{m,n} \). Therefore, \(xB_{m,y}^n \). Hence, \(B_x = B_y \).

5 \((m, 0)\)-regularity \([(0, n)\)-regularity\] and \((m, n)\)-right weakly regularity of a \(\mathcal{B}_m^n \)-class, \(\mathcal{Q}_m^n \)-class and \(\mathcal{H}_m^n \)-class

In this section, the \((m, 0)\)-regular, \((0, n)\)-regular, \((m, n)\)-regular and \((m, n)\)-right weakly regular class of the relations \(\mathcal{H}_m^n \) and \(\mathcal{B}_m^n \) are studied.
Lemma 5.1. An $\mathcal{H}^n_m$-class $H$ of an ordered semihypergroup is $(m,0)$-regular $[(0,n)]$-regular if it contains an $(m,0)$-regular $[(0,n)]$-regular element.

Proof. Let $a$ be an $(m,0)$-regular element and $c$ be an element of $\mathcal{H}^n_m$-class $H$. This implies $[b]_{m,0} = [a]_{m,0}$ and $a \in (a^m \circ H)$. Therefore, $\{a\}^i \subseteq (a^m \circ H)$ for each $i \in \{1,2,\ldots,m\}$. Then $\bigcup_{i=1}^m a^i \subseteq (a^m \circ H)$ implies $[\bigcup_{i=1}^m a^i] \subseteq (a^m \circ H) = (a^m \circ H)$. Thus, $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^m a^i \cup a^m \circ H) = (\bigcup_{i=1}^m a^i \cup (a^m \circ H) = (a^m \circ H)$. So $b$ is an $(m,0)$-regular element of $\mathcal{H}^n_m$-class $H$. Hence, the $\mathcal{H}^n_m$-class $H$ is $(m,0)$-regular. The dual statement follows on the similar lines.

Lemma 5.2. An $\mathcal{H}^n_m$-class $H$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.

Proof. The proof is similar to the proof of Lemma 5.1.

Lemma 5.3. A $\mathcal{B}^n_m$-class $B$ of an ordered semihypergroup is $(m,n)$-regular if it contains an $(m,n)$-regular element.

Proof. Let $a \in B$ be an $(m,n)$-regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n)$ so that $\{a\}^i \subseteq (a^m \circ H \circ a^n)$ for each $i \in \{1,2,\ldots,m+n\}$, so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n)$ implies $[\bigcup_{i=1}^{m+n} a^i] \subseteq ((a^m \circ H \circ a^n) = (a^m \circ H \circ a^n)$. Since $a,b \in B$, $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n) = (\bigcup_{i=1}^{m+n} a^i \cup (a^m \circ H \circ a^n) = (a^m \circ H \circ a^n)$. By Lemmas 4.3 and 4.2, we have $[a^m \circ H \circ a^n] = (b^m \circ H \circ b^n)$, so $b \in (b^m \circ H \circ b^n)$. Thus, $b$ is an $(m,n)$-regular element of $B$. Hence, $B$ is $(m,n)$-regular.

Definition 5.4. Let $H$ be an ordered semihypergroup and $m,n$ be positive integers. An element $a$ of $H$ is said to be an $(m,n)$-right weakly regular element if $a \in (a^m \circ H \circ a^n \circ H)$. The ordered semihypergroup $H$ is said to be $(m,n)$-right weakly regular if each element of $H$ is $(m,n)$-right weakly regular, equivalently, for each subset $A$ of $H$, $A \subseteq (A^m \circ H \circ A^n \circ H)$.

Lemma 5.5. A $\mathcal{B}^n_m$-class $B$ of an ordered semihypergroup $H$ is $(m,n)$-right weakly regular if it contains an $(m,n)$-right weakly regular element.

Proof. Let $a \in B$ be an $(m,n)$-right weakly regular element and $b \in B$. Then, $a \in (a^m \circ H \circ a^n \circ H)$. This implies that $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H)$.
for each $i \in \{1,2,\ldots,m+n\}$, so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H)$ implies
\[ (\bigcup_{i=1}^{m+n} a^i) \subseteq ((a^m \circ H \circ a^n \circ H)] = (a^m \circ H \circ a^n \circ H). \]
So, $(a^m \circ H \circ a^n] \subseteq ((a^m \circ H \circ a^n \circ H)] \circ H \circ ((a^m \circ H \circ a^n \circ H)] \subseteq (a^m \circ H \circ a^n \circ H).$

Since $a, b \in B$, $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n] = (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n] \subseteq (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n \circ H] = (a^m \circ H \circ a^n \circ H] \subseteq (\bigcup_{i=1}^{m+n} a^i] \cup (a^m \circ H \circ a^n \circ H] = (a^m \circ H \circ a^n \circ H] \subseteq (b^m \circ H \circ b^n \circ H] \subseteq (b^m \circ H \circ b^n \circ H].$ Therefore, $b$ is an $(m,n)$-right weakly regular element of $B$. Hence, $B$ is $(m,n)$-right weakly regular.

**Corollary 5.6.** An ordered semihypergroup $H$ is $(m,n)$-regular ($(m,n)$-right weakly regular) if and only if each $B^m_n$-class of $H$ contains an $(m,n)$-regular ($(m,n)$-right weakly regular) element.

**Lemma 5.7.** An $H^m_n$-class $H$ of an ordered semihypergroup is $(m,n)$-right weakly regular if it contains an $(m,n)$-right weakly regular element.

**Proof.** Let $a$ be an $(m,n)$-right weakly regular element and $b$ be an element of $H^m_n$-class $H$. Then, $a \in (a^m \circ H \circ a^n \circ H).$ This gives that $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H)$ for each $i \in \{1,2,\ldots,m+n\}$, and so $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H)$ implies $\bigcup_{i=1}^{m+n} a^i \subseteq ((a^m \circ H \circ a^n \circ H] = (a^m \circ H \circ a^n \circ H)$. Therefore, $(a^m \circ H) \subseteq ((a^m \circ H \circ a^n \circ H] \circ H = (a^m \circ H \circ a^n \circ H) \subseteq (a^m \circ H \circ a^n \circ H].$ Since $a, b \in H$, $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H] = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H] = (a^m \circ H) \subseteq (a^m \circ H \circ a^n \circ H].$ So, by Lemma 4.2, $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$. This implies that $[b]_{m,0} \subseteq (a^m \circ H \circ a^n \circ H] = (a^m \circ H \circ a^n \circ H] = (b^m \circ H \circ b^n] \circ H = (b^m \circ H \circ b^n] \circ H = (b^m \circ H \circ b^n \circ H].$ Therefore, $b \in (b^m \circ H \circ b^n \circ H]$ and thus, $b$ is an $(m,n)$-right weakly regular element of $H^m_n$-class $H$. Hence, $H$ is $(m,n)$-right weakly regular.

**Corollary 5.8.** An ordered semihypergroup $H$ is (respectively, $(m,0)$-regular, $(0,n)$-regular, $(m,n)$-regular) $(m,n)$-right weakly regular if and only if each $H^m_n$-class of $H$ contains a (respectively, $(m,0)$-regular, $(0,n)$-regular, $(m,n)$-regular) $(m,n)$-right weakly regular element.

### 6 Conclusion

The main purpose of the present paper is to introduce the equivalence relations $mI, nB^m_n$ and $H^m_n$ on an ordered semihypergroup and enhance the un-
understanding of different classes of ordered semihypergroups (\((m, n)\)-regular, \((m, 0)\)-regular, \((0, n)\)-regular, \((m, n)\)-right weakly regular) by considering the structural influence of the equivalence relations \(mI, I_n, B^n_m, \) and \(H^n_m\). In particular, if we take \(m = 1 = n\), the equivalence relations \(mI, I_n\) and \(H^n_m\) are reduced to the equivalence relations \(R, L\) and \(H\) in ordered semihypergroup, respectively, which mimic the definition of the usual Green’s relations \(R, L\) and \(H\) in plain semihypergroups [11]. Also when we take \(m = 1 = n\) in Theorems 1.9, 1.11, 4.1, 3.6, and 4.2, and Lemmas 4.1, 4.2, 4.3, 4.3, 5.1, and 5.2, then we obtain all the results for bi-hyperideals in an ordered semihypergroup and some characterizations of regular ordered semihypergroups, which is the main application of the results presented in this paper.

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