On stabilization of nonlinear systems with drift by time-varying feedback laws *

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Abstract

This paper deals with the stabilization problem for nonlinear control-affine systems with the use of oscillating feedback controls. We assume that the local controllability around the origin is guaranteed by the rank condition with Lie brackets of length up to 3. This class of systems includes, in particular, mathematical models of rotating rigid bodies. We propose an explicit control design scheme with time-varying trigonometric polynomials whose coefficients depend on the state of the system. The above coefficients are computed in terms of the inversion of the matrix appearing in the controllability condition. It is shown that the proposed controllers can be used to solve the stabilization problem by exploiting the Chen–Fliess expansion of solutions of the closed-loop system. We also present results of numerical simulations for controlled Euler’s equations and a mathematical model of underwater vehicle to illustrate the efficiency of the obtained controllers.

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1 INTRODUCTION

The stabilization of underactuated mechanical systems with uncontrollable linearization is a challenging mathematical problem related to the theory of critical cases in the sense of Lyapunov. This problem is of great importance in robotics as the motion of nonholonomic mobile robots and underactuated manipulators is generically described by systems of ordinary differential equations whose linear approximation does not satisfy Kalman’s rank condition at a reference configuration.

On the one hand, it is a well-known fact that even relatively simple models of nonholonomic systems of the form
\[ \dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \] (1)
do not admit “regular” stabilizers in the class of time-invariant feedback laws \( u = k(x) \) if
\[ \text{rank}(f_1(0), f_2(0), ..., f_m(0)) = m < n, \]
see [1]. On the other hand, each nonlinear control system
\[ \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad f(0, 0) = 0 \] (2)
adopts a continuous time-varying feedback \( u = k(t, x) \) that stabilizes the trivial equilibrium, provided that small-time local controllability conditions are satisfied at \( x = 0 \) together with some regularity assumptions [2]. However, the problem of constructing a stabilizing feedback law for an arbitrary controllable system (2) (or even (1) under higher-order controllability conditions) is far from being satisfactory solved. It is hardly possible to mention all the achievements in this area due to lack of space, so we just refer to [2] for a survey of essentially nonlinear design tools.

The goal of this paper is to develop an effective approach for constructing time-varying stabilizers for the class of nonlinear control-affine systems \( \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i \), assuming that the local controllability at \( x = 0 \) is ensured by a rank condition with the vector fields \( f_1, ..., f_m \) and their Lie brackets of the form \( [f_{i_1}, f_{i_2}](x), [f_{j_1}, [f_{j_2}, f_{j_3}]](x), [f_l, f_0](x), \) and \( [f_{i_1}, [f_{i_2}, f_0]](x) \). Our study is motivated by control problems in nonholonomic mechanics and rigid body dynamics, where such type of controllability conditions naturally appears.
The main idea of our construction is summarized in Section II. A family of time-periodic feedback controls with coefficients depending on the state vector will be presented explicitly to stabilize the equilibrium of the considered class of systems. The proposed control algorithm extends our previous design schemes \cite{3,4} for the case of systems with drift, i.e. for $f_0(x) \neq 0$. We will also discuss applications of this control algorithm for the stabilization of a rotating satellite and underwater vehicle in Section III.

1.1 Notations, definitions, and auxiliary results

**Definition 1** We say that there is a resonance of order $N \in \mathbb{N}$ between pairwise distinct numbers $k_1, \ldots, k_n$, if there exist relatively prime integers $c_1, \ldots, c_q$ such that $|c_1| + \ldots + |c_q| = N$ and $c_1 k_1 + \ldots + c_n k_n = 0$.

For a given $\varepsilon > 0$, define a partition $\pi_\varepsilon$ of $\mathbb{R}^+ = [0, +\infty)$ into the intervals $[t_j, t_{j+1})$, $t_j = \varepsilon j$, $j = 0, 1, 2, \ldots$.

**Definition 2** Given a time-varying feedback law $u = h(t, x)$, $h : \mathbb{R}^+ \times D \to \mathbb{R}^n$, $\varepsilon > 0$, and $x^0 \in \mathbb{R}^n$, a $\pi_\varepsilon$-solution of system (2) corresponding to $x^0 \in D$ and $h(t, x)$ is an absolutely continuous function $x(t) \in D$, defined for $t \in \mathbb{R}^+$, such that $x(0) = x^0$ and

$$\dot{x}(t) = f(x(t), h(t, x(t_j))), \quad t \in [t_j, t_{j+1}),$$

for each $j = 0, 1, 2, \ldots$

Throughout this paper, $\|a\|$ denotes the Euclidean norm of a vector $a \in \mathbb{R}^n$, and the norm of an $n \times n$-matrix $F$ is defined as $\|F\| = \sup_{\|y\|=1} \|Fy\|$.

For a $\delta > 0$, $B_\delta(x^*)$ denotes the $\delta$-neighborhood of $x^* \in \mathbb{R}^n$, and $\overline{B_\delta(x^*)}$ is the closure of $B_\delta(x^*)$. For $h \in C^1(\mathbb{R}^n; \mathbb{R})$ and $\xi \in \mathbb{R}^n$, we define the column $\nabla h(\xi) := \left( \frac{\partial h(x)}{\partial x} \right)^\top |_{x=\xi}$. For a function $f : \mathbb{R} \to \mathbb{R}$, $f(z) = O(z)$ as $z \to 0$ means that there is a $c > 0$ such that $|f(z)| \leq c|z|$ in some neighborhood of 0. For $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the Lie derivative is denoted as $L_g f(x) = \lim_{s \to 0} \frac{f(x+sg(x))-f(x)}{s}$, and $[f, g](x) = L_g f(x) - L_f g(x)$ stands for the Lie bracket.

We will also use the following result.
Lemma 1 ([5,6]) Let the vector fields $f_i$ be Lipschitz continuous in a domain $D \subseteq \mathbb{R}^n$, and $f_i \in C^{\nu+1}(D \setminus \Xi; \mathbb{R})$, where $\Xi = \{ x \in D : f_i(x) = 0 \text{ for all } 1 \leq i \leq \ell \}$, $\nu \geq 1$. Assume, moreover, that $L_{f_j} f_i L_{f_l} f_i \in C(D; \mathbb{R}^n)$, for all $i, j, l = 1, \ell$. Let $x(t) \in D$, $t \in [0, \tau]$, be a solution of the system
\[
\dot{x} = \sum_{i=1}^{\ell} f_i(x) w_i(t) \quad \text{with} \quad u \in C([0, \tau]; \mathbb{R}^m) \quad \text{and} \quad x(0) = x^0 \in D.
\]
Then, for any $t \in [0, \tau]$, $x(t)$ can be represented by the $\nu$-th order Chen–Fliess series:
\[
x(t) = x^0 + \sum_{i=1}^\nu \sum_{j_1, \ldots, j_1=1}^{\ell} L_{f_{j_1}} L_{f_{j_{i-1}}} \cdots f_{j_i}(x^0) \times \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} w_{j_1}(s_1) \cdots w_{j_i}(s_i) ds_i \cdots ds_1 + r(t)
\]
where
\[
r(t) = \sum_{j_1, \ldots, j_{\nu+1}=1}^{\ell} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{\nu}} L_{f_{j_{\nu+1}}} L_{f_{j_{\nu}}} \cdots f_{j_1}(x(s_{\nu+1})) \times w_{i_1}(s_1) \cdots w_{j_{\nu+1}}(s_{\nu+1}) ds_{\nu+1} \cdots ds_1
\]
is the remainder.

Note that the above Chen–Fliess expansion can be obtained from the Volterra series [5].

2 Main results

In this section, we consider a class of nonlinear control systems of the form
\[
\dot{x} = f_0(x) + \sum_{k=1}^{m} f_k(x) u_k
\]
where $x = (x_1, \ldots, x_n)^T \in D$ is the state vector, $D \subset \mathbb{R}^n$ is a domain, $0 \in D$, $u = (u_1, \ldots, u_m)^T$ is the control, and the vector fields $f_0, f_1, \ldots, f_m \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the following rank condition: for all $x \in D$,
\[
\text{span}\left\{ f_i(x), [f_i, f_j](x), [f_j, [f_j, f_j]](x), [f_i, f_0](x), [f_i, [f_j, f_0]](x) \right\} = \mathbb{R}^n,
\]
where $i \in S_1$, $(i_1, i_2) \in S_2$, $(j_1, j_2, j_3) \in S_3$, $l \in S_{10}$, $(l_1, l_2) \in S_{20}$, with some set of indices $S_1, S_{10} \subseteq \{1, 2, \ldots, m\}$, $S_2, S_{20} \subseteq \{1, 2, \ldots, m\}^2$, $S_3 \subseteq \{1, 2, \ldots, m\}^3$ such that $|S_1| + |S_2| + |S_3| + |S_{10}| + |S_{20}| = n$. In [7], we have considered local approximate steering and local path-following problems for the case $S_2 = \emptyset$, $S_3 = \emptyset$, and $S_{20} = \emptyset$. However, the results of the above paper do not guarantee any stability properties. Below we propose novel stabilizability results for system (4) satisfying (5) and introduce explicit formulas for the control design.

To simplify the presentation, we will first consider two cases:

1) $|S_1| + |S_{10}| = n$, i.e. the vector fields of the system satisfy the rank condition

$$\text{span} \{ f_i(x), [f_i, f_0](x) \} = \mathbb{R}^n; \quad (6)$$

2) $|S_1| + |S_{20}| = n$, i.e. the vector fields of system satisfy the rank condition

$$\text{span}\{ f_i(x), [f_i, [f_{i_1}, f_{i_2}], f_0] \}(x) = \mathbb{R}^n, \quad (7)$$

Then we will show how to combine the above particular cases with the controls from [8] in case when the rank condition (5) is satisfied.

To stabilize system (4) at $x = 0$, we use a parameterized family of control functions $u_k = u_k^\varepsilon(t, x)$ consisting of several terms, each of which is aimed to implement the motion in the direction of a certain vector field from the rank condition for small values of the parameter $\varepsilon > 0$.

2.1 Stabilization of system (4) satisfying the rank condition (6)

For the case 1), our proposed control design is as follows:

$$u_k^\varepsilon(t, x) = \sum_{i \in S_1} h_i^k(x) + \frac{1}{\varepsilon} \sum_{l \in S_{10}} h_{l0}^k(t, x) \quad (8)$$

with

$$h_i^k(x) = \delta_{ki} a_i(x),$$

$$h_{l0}^k(t, x) = \delta_{l0} 2\pi k_{l0} a_{l0}(x) \sin\left(\frac{2\pi k_{l0} t}{\varepsilon}\right), \quad k = 1, \ldots, m,$$

where $\delta_{ij}$ is the Kronecker delta, $k_{l0}$ are pairwise distinct positive integers. The real-valued coefficient functions $a_i(x)$ and $a_{l0}(x)$ are defined to ensure
the descent of a certain potential function \( V \in C^2(\mathbb{R}^n; \mathbb{R}^+) \):

\[
\left( (a_i)_{i \in S_1} (a_{i0})_{i \in S_{i0}} \right) ^\top = a^{(1)}(x) = -F_1^{-1}(x)(\gamma \nabla V(x) + f_0(x)),
\]

(9)

where \( \gamma > 0 \) plays the role of control gain, and \( F_1^{-1}(x) \) is the inverse matrix for

\[ \mathcal{F}_1(x) = \left( (f_i(x))_{i \in S_1} ([f_i, f_0](x))_{i \in S_{i0}} \right). \]

Obviously, \( F_1^{-1}(x) \) exists whenever the rank condition (5) holds.

Note that the proposed control formulas are much simpler compared to the ones used in [7].

The main idea behind our control design approach is the approximation of trajectories of the auxiliary system

\[
\dot{x} = -\gamma \nabla V(x), \quad \bar{x} \in D, \quad \bar{x}(0) = x(0)
\]

(10)

by the trajectories of (4). Indeed, under the proposed choice of control functions, the representation (3) yields

\[
x(\varepsilon) = x^0 + \varepsilon \left( f_0(x^0) + \mathcal{F}_1(x^0) a^{(1)}(x^0) \right) + \Omega_1(x^0, \varepsilon) + R(\varepsilon),
\]

(11)

where the expression for \( \Omega_1(x^0, \varepsilon) \) is given in the Appendix, and the remainder \( R_1(\varepsilon) \) is calculated as \( r(\varepsilon) \) in Lemma 1 with the summation indices \( j_1, j_2, j_3 \in \{0, 1, \ldots, m\} \) and \( w_0(t) \equiv 1 \).

In this paper, we take \( V(x) = \frac{1}{2} \|x\|^2 \) to stabilize system (4) at the origin. However, more general classes of potential function can be considered, similarly to [8]. We suppose that

\[
f_0(0) = L_{f_0} f_0(0) = 0 \quad (A1)
\]

and apply formula (9) to (11). Then, for each compact set \( D_0 \subseteq D, 0 \in D_0 \), there exist constants \( \sigma_i \geq 0 \) such that, for any \( x^0 \in D_0 \),

\[
\|x(\varepsilon) - x^0\| \leq \|x^0\|(1 - \varepsilon \gamma) + \varepsilon \sigma_1 \|x^0\|^2 + \varepsilon^2 \sigma_2 \|x^0\| + \|R_1(\varepsilon)\|.
\]

Based on the obtained estimate, the following result can be proved.
Proposition 1 Let $D \subseteq \mathbb{R}^n$, $f_i \in C^3(D; \mathbb{R}^n)$, $i = 0, \ldots, m$. Suppose that assumption (A1) holds and, furthermore, there exists an $\alpha > 0$ such that $\|F^{-1}(x)\| \leq \alpha$ for all $x \in D$. If the functions $u_k = u_k^\varepsilon(t, x)$, $k = 1, \ldots, m$, are defined by (8)–(9), then there exist $\gamma, \delta, \bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, each $\pi_\varepsilon$-solution of system (4) with the initial data $x(0) = x^0 \in B_\delta(0)$ is well-defined on $t \in \mathbb{R}^+$ and $\|x(t)\| \to 0$ as $t \to \infty$.

The proof of the above proposition is based on subtle estimates of the remainder $R_1(\varepsilon)$ and the techniques from [3, 8]. In particular, it goes along the same line as the proofs of [3, Theorem 2.2] and [8, Theorem 1] with taking into account the drift term $f_0(x)$. The detailed proof of Proposition 1, including description of connections between $\varepsilon, \gamma, \delta$, will be presented in the extended version of the paper.

Remark 1 If the vector fields of system (4) fail to satisfy (A1), a weaker result on practical stabilizability of system (4) can be deduced using the techniques proposed, e.g., in [9]. We will illustrate this case with an example in Section III.B.

2.2 Stabilization of system (4) satisfying the rank condition (7)

For the case 2), we propose the following controls:

$$u_k^\varepsilon(t, x) = \sum_{i \in S_1} h_{ki}^\varepsilon(x) + \frac{1}{\varepsilon} \sum_{(i_1, i_2) \in S_20} h_{k_{i_1}i_20}(t, x), \quad (12)$$

where

$$h_{ki}^\varepsilon(x) = \delta_{ki} a_i(x),$$

$$h_{k_{i_1}i_20}(t, x) = 4\pi \kappa_{i_1i_20} \sqrt{|a_{i_1i_20}(x)|} \cos \left(\frac{2\pi \kappa_{i_1i_20} t}{\varepsilon}\right) \times \left(\delta_{ki_1} + \delta_{ki_2} \text{sign}(a_{i_1i_20}(x))\right),$$

for $k = 1, \ldots, m$, and pairwise distinct positive integers $\kappa_{i_1i_20}$. Moreover, we assume that there are no resonances of order 2 between $\kappa_{i_1i_20}$. Similarly to the previous subsection, we define the state-dependent coefficients $a_i(x)$ and $a_{i_1i_20}(x)$ according to the formula

$$\begin{pmatrix} (a_i)_{i \in S_1} (a_{i_1i_20})_{(i_1, i_2) \in S_20} \end{pmatrix}^\top = a^{(2)}(x) = -\mathcal{F}_2^{-1}(x)(\gamma \nabla V(x) + f_0(x)), \quad (13)$$
where $\gamma > 0$, and $\mathcal{F}^{-1}_2(x)$ is the inverse matrix for

$$
\mathcal{F}_2(x) = \left( (f_i(x))_{i \in S_1}, \left( [f_{i_1}, [f_{i_2}, f_0]](x) + [f_{i_2}, [f_{i_1}, f_0]](x) \right)_{i \in S_{20}} \right).
$$

In this case, the expansion (3) takes the form

$$
x(\epsilon) = x^0 + \epsilon (f_0(x^0) + \mathcal{F}_2(x^0)a^{(2)}(x^0)) + \Omega_2(x^0, \epsilon) + R_2(\epsilon). \quad (14)
$$

The explicit formula for $\Omega_2(x^0, \epsilon)$ is given in the Appendix, and the remainder $R_2(\epsilon)$ is calculated according to Lemma 1. To ensure that the function $V(x) = \frac{1}{2} \|x\|^2$ decays over the time period $\epsilon$ along the trajectories of system (4) with controls (12)–(13), we suppose that assumption (A1) holds together with the following properties:

$$
\begin{align*}
L_{f_0}L_{f_0}f_0(0) &= [f_0, [f_0, f_k]](0) = 0, \\
[f_{i_1}, [f_{i_1}, f_0]](x) + [f_{i_2}, [f_{i_2}, f_0]](x) &= O(\|x\|^\mu), \\
[f_{i_1}, [f_{i_2}, f_0]](x) &= O(\|x\|^\mu) \text{ as } \|x\| \to 0 \text{ with some } \mu > 0,
\end{align*}
$$

(A2)

for any $(l_1, l_2) \in S_{20}$ and any $k : (l_1, k) \in S_{20}$ or $(k, l_2) \in S_{20}$.

Then, for each compact set $D_0 \subseteq D$, $0 \in D_0$, there exist constants $\tilde{\sigma}_i \geq 0$ such that, for any $x^0 \in D_0$,

$$
\|x(\epsilon) - x^0\| \leq \|x^0\| \left( 1 - \epsilon \gamma \right) + \epsilon \|x^0\|^{1+\mu} \\
+ \epsilon^2 \|x^0\| + \|R_2(\epsilon)\|.
$$

Again, estimating the remainder $R_2(\epsilon)$ and using the techniques from [3, 8], we can state the following result.

**Proposition 2** Let $D \subseteq \mathbb{R}^n$, $f_i \in C^4(D; \mathbb{R}^n)$, $i = 0, \ldots, m$. Suppose that assumptions (A1) and (A2) hold and, furthermore, there exists an $\alpha > 0$ such that $\|\mathcal{F}^{-1}_2(x)\| \leq \alpha$ for all $x \in D$. If the functions $u_k = u_k^\epsilon(t, x)$, $k = 1, \ldots, m$, are defined by (12)–(13), then there exist $\gamma, \delta, \tilde{\epsilon} > 0$ such that, for any $\epsilon \in (0, \tilde{\epsilon}]$, each $\pi_\epsilon$-solution of system (4) with the initial data $x(0) = x^0 \in B_\delta(0)$ is well-defined on $t \in \mathbb{R}^+$ and $\|x(t)\| \to 0$ as $t \to \infty$.  

2.3 Stabilization of system [4] satisfying the rank condition [5]

If the rank condition [5] involves Lie brackets of the type
\[ [f_{i_1}, f_{i_2}](x), \ [f_{j_1}, [f_{j_2}, f_{j_3}]}(x), \ [f_{i_1}, f_{i_2}](x), \ [f_{i_1}, [f_{i_2}, f_{i_0}]}(x), \ [f_{i_1}, [f_{i_2}, f_{j_0}]}(x), \]
then stabilizing controllers can be constructed on the basis of formulas [8], [12], and the scheme from [8] in the following way:

\[ u_k^i(t, x) = \sum_{i \in S_1} h^k_i(x) + \frac{1}{\sqrt{\varepsilon}} \sum_{(i_1, i_2) \in S_2} h^k_{i_1i_2}(t, x) \]
\[ + \frac{1}{\sqrt{\varepsilon}^2} \sum_{(j_1, j_2, j_3) \in S_3} h^k_{j_1j_2j_3}(t, x) \]
\[ + \frac{1}{\varepsilon} \sum_{i \in S_{10}} h^k_{i0}(t, x) + \frac{1}{\varepsilon} \sum_{(i_1, j_2) \in S_{20}} h^k_{i_1j_20}(t, x), \]

where \( h^k_i(x), \ h^k_{i0}(t, x), \ h^k_{i_1j_20}(t, x) \) are constructed as in Sections II.A, II.B, and

\[ h^k_{i_1i_2}(x) = 2\sqrt{\pi \kappa_{i_1i_2}} a_{i_1i_2} \left( \delta_{ki_1} \text{sign}(a_{i_1i_2}(x)) \cos \left( \frac{2\pi \kappa_{i_1i_2} t}{\varepsilon} \right) \right) \]
\[ + \delta_{ki_2} \sin \left( \frac{2\pi \kappa_{i_1i_2} t}{\varepsilon} \right), \]
\[ h^k_{j_1j_2j_3}(t, x) = 2 \sqrt{2\pi^2 (\kappa_{j_1j_2j_3}^2 - \kappa_{j_2j_1j_3}^2) a_{j_1j_2j_3}(x)} \]
\[ \times \left( \sin \left( \frac{2\pi \kappa_{j_1j_2j_3} t}{\varepsilon} \right) \right) \left( \delta_{kj_1} \cos \left( \frac{2\pi \kappa_{j_2j_1j_3} t}{\varepsilon} \right) + \delta_{kj_2} \right) \]
\[ + \delta_{kj_3} \cos \left( \frac{2\pi \kappa_{j_2j_1j_3} t}{\varepsilon} \right). \]

In this case, the conditions of Propositions [1] and [2] should be satisfied with the corresponding components \( h^k_i(x), \ h^k_{i0}(t, x), \) and \( h^k_{i_1j_20}(t, x). \) The integers \( \kappa_{i_0}, \ k_{i_1i_2}, \ k_{i_1j_{j_2j_3}}, \ k_{j_1j_{j_2j_3}}, \ k_{j_1j_{j_2j_3}} = \kappa_{j_1j_{j_2j_3}} + \kappa_{j_2j_{j_1j_3}}, \ k_{j_1j_{j_2j_3}} = \kappa_{j_1j_{j_2j_3}} - \kappa_{j_2j_{j_1j_3}}, \) have to be positive and pairwise distinct. Moreover, if \( S_{20} \neq \emptyset, \) then we assume that there are no resonances of order 2 between the above-listed frequencies. The column vector of state-dependent coefficients

\[ a(x) = \left( (a_i)_{i \in S_1}, (a_{i_1i_2})_{(i_1, i_2) \in S_2}, (a_{j_1j_2j_3})_{(j_1, j_2, j_3) \in S_3}, \right) \]
\[ (a_{i_0})_{i \in S_{10}}, (a_{i_1j_20})_{(i_1, j_2) \in S_{20}}}^\top. \]
is defined as
\[ a(x) = -\mathcal{F}^{-1}(x) \left( \gamma \nabla V(x) + f_0(x) \right), \]
where \( \gamma > 0 \), and \( \mathcal{F}^{-1}(x) \) is the inverse matrix for
\[
\mathcal{F}(x) = \left( (f_i(x))_{i \in S_1} \left( ([f_{i_1}, f_{i_2}])_{i_1, i_2} \in S_2 \right) \left( [f_{j_1}, [f_{j_2}, f_{j_3}]](x)_{(j_1, j_2, j_3)} \in S_3 \right) \left( [f_{l_1}, f_{l_2}](x) \right)_{l \in S_10} \right) \left( [f_{l_1}, [f_{l_2}, f_0]](x) + [f_{l_2}, [f_{l_1}, f_0]](x) \right)_{l \in S_20}. \]
Stabilizability conditions for this case will be formulated in the extended version of this work.

3  EXAMPLES

3.1  Stabilization of a rotating rigid body

Consider Euler's equations for a rigid body rotating around its center of mass:
\[
\begin{align*}
\dot{x}_1 &= \alpha_1 x_2 x_3 + u_1, \\
\dot{x}_2 &= \alpha_2 x_1 x_3 + u_2, \\
\dot{x}_3 &= \alpha_3 x_1 x_2,
\end{align*}
\tag{16}
\]
where \( x_1, x_2, x_3 \) are projections of the angular velocity vector on the principal axes of inertia of the body, and \( u_1, u_2 \) correspond to the control torques with respect to the first and the second principal axis, respectively. The parameters \( \alpha_i \) are related to the central moments of inertia of the rigid body \( J_1, J_2, J_3 \) as
\[
\alpha_1 = \frac{J_2 - J_3}{J_1}, \quad \alpha_2 = \frac{J_3 - J_1}{J_2}, \quad \alpha_3 = \frac{J_1 - J_2}{J_3}.
\]
In the sequel, we assume that \( \alpha_3 \neq 0 \). System (16) represents the angular motion of a spacecraft as an absolutely rigid body without moving masses. The control torques \( u_1 \) and \( u_2 \) can be generated by jet engines, and the change of mass due to the operation of engines is neglected.

The stabilization problem for Euler's equation with time-invariant feedback controls has been already thoroughly studied by several authors. The orientation of a satellite along a given direction (uniaxial stabilization) was considered in [13] for Euler's equations coupled with Poisson's equations using a three-dimensional control (fully actuated case). It was also shown in [13]
Figure 1: Time-plots of the trajectories of system (16) with controls (20) (red) and controls from [10] (green), [11] (dark blue), and [12] (light blue). Figure a): $x(0) = (3, 2, 1)^\top$; figure b): $x(0) = (0, 0, 2)^\top$.

Figure 2: Left: time-plot of the function $\|u(t)\|$, where $u(t)$ is the control given by (20) (red) and from [10] (green), [11] (dark blue), [12] (light blue). Right: time-plot of the function $\frac{1}{t}\ln\|x(t)\|$, where $x(t)$ is the solution of system (16) with controls (20). In both figures, $x(0) = (3, 2, 1)^\top$. 
that the stabilization of two given directions is possible in the fully actuated case.

By applying the feedback transformation \( y_1 = x_1, y_2 = x_2, y_3 = x_3/\alpha_3, \)
\( w_1 = \alpha_1 x_2 x_3 + u_1, w_2 = \alpha_2 x_1 x_3 + u_2, \) system (16) takes the form
\[
\dot{y}_1 = w_1, \quad \dot{y}_2 = w_2, \quad \dot{y}_3 = y_1 y_2.
\] (17)

As it was proved in [1] and [10], the equilibrium \( y = 0 \) of (17) is asymptotically stabilizable by a smooth time-invariant feedback law. However, the analysis in [10] is based on the center manifold approach, and the resulting reduced system \( \dot{z} = -\kappa z^5 + o(|z|^5) \) does not exhibit exponential decay rate as \( t \to +\infty \). The trivial equilibrium of system (17) is shown to be stabilizable in finite time by means of discontinuous state feedback controls [14]. The problem of robust stabilization of system (16) with external disturbances was addressed in [15].

Note that the trivial equilibrium of Euler’s equations is stabilizable by a one-dimensional control acting along a “skewed” direction, provided that the rigid body is asymmetric [16]. This result was extended in [17] to the body with two identical principal moments of inertia. It was also shown there that the body with a spherical tensor of inertia cannot be stabilized by a one-dimensional control.

In this section, we will illustrate the behavior of trajectories of system (16) with controls (12). We will present simulation results to show that the solutions of the closed-loop system decay exponentially with time.

Let us rewrite control system (16) in the vector form (4):
\[
\dot{x} = f_0(x) + u_1 f_1(x) + u_2 f_2(x), \quad x \in \mathbb{R}^3, \; u \in \mathbb{R}^2,
\] (18)
with
\[
f_0 = \begin{pmatrix} \alpha_1 x_2 x_3 \\ \alpha_2 x_1 x_3 \\ \alpha_3 x_1 x_2 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

It can be shown that system (18) satisfies the controllability condition
\[
\text{span}\{f_1, f_2, [f_1, [f_2, f_0]]\} = \mathbb{R}^3 \text{ for all } x \in \mathbb{R}^3,
\] (19)
provided that \( \alpha_3 \neq 0 \), i.e. \( J_1 \neq J_2 \). Thus, it is easy to see that the assumptions (A1) and (A2) are satisfied. In particular,
\[
[f_1, [f_1, f_0]](x) \equiv 0, \quad [f_2, [f_2, f_0]](x) \equiv 0, \quad [f_0, [f_1, f_2]](x) \equiv 0,
\]
and, for \( \|x\| \to 0 \),
\[
[f_0, [f_0, f_1]](x) = \left( \alpha_1(\alpha_2 x_3^2 + \alpha_3 x_2^2) \ 0 \ 0 \right)^\top = O(\|x\|^2),
\]
\[
[f_0, [f_0, f_2]](x) = \left( 0 \ \alpha_2(\alpha_1 x_3^2 + \alpha_3 x_1^2) \ 0 \right)^\top = O(\|x\|^2).
\]
Furthermore, since
\[
[f_1, [f_2, f_0]](x) \equiv [f_2, [f_1, f_0]](x) \equiv (0 \ 0 \alpha_3)^\top,
\]
the matrix
\[
\mathcal{F}_2(x) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2\alpha_3
\end{pmatrix}
\]
is obviously nonsingular and satisfy the conditions of Proposition 2 with \( \alpha = \left(2 + \frac{1}{4\alpha_3}\right)^{1/2} \).

Thus, we can apply the stabilization scheme proposed in Section 2.B with \( S_1 = \{1, 2\} \) and \( S_{20} = \{(1, 2)\} \). With \( \kappa_{120} = 1 \), stabilizing controllers take the form:
\[
\begin{align*}
u^1_\varepsilon(t, x) &= a_1(x) + \frac{4\pi\sqrt{|a_{120}|}}{\varepsilon} \cos \left(\frac{2\pi t}{\varepsilon}\right), \\
u^2_\varepsilon(t, x) &= a_2(x) + \frac{4\pi\sqrt{|a_{120}|}}{\varepsilon} \operatorname{sign}(a_{120}) \cos \left(\frac{2\pi t}{\varepsilon}\right),
\end{align*}
\]
where
\[
\begin{pmatrix}
a_1(x) \\
a_2(x) \\
a_{120}(x)
\end{pmatrix} = -\mathcal{F}_2^{-1}(x) \left( \gamma x + f_0(x) \right)
\]
\[
= - \begin{pmatrix}
\gamma x_1 + \alpha_1 x_2 x_3 \\
\gamma x_2 + \alpha_3 x_1 x_3 \\
\frac{\gamma}{2\alpha_3} x_1 + \frac{1}{2} x_1 x_2
\end{pmatrix}.
\]
Similarly to the results of [3], it can be shown that the trajectories of system (16) with controls (20) exponentially converge to the origin.

For the numerical simulation, we put
\( \alpha_1 = 3, \alpha_2 = 2, \alpha_3 = 1, \gamma = 5, \varepsilon = 1 \).
The resulting time-plots of \(x_1(t), x_2(t), x_3(t)\) for two sets of initial conditions are presented in Fig. 1 (in red). To illustrate other stabilizing strategies for a rotating rigid body with two control torques, we also show simulations results with smooth time-invariant controls from [10, p. 293] (in green), discontinuous time-invariant controls from [11, Eq. (22)] (in dark green), and time-varying controls from [12, Eqs. (24)–(25)] (in light blue). Note that the controls [20] and [12, Eqs. (24)–(25)] ensure exponential convergence to zero for the initial data from an entire neighborhood of the origin, while the controls [11, Eq. (22)] are not applicable if \(x_1(0) = x_2(0) = 0\), and the controls from [10, p. 293] ensure only asymptotic (but not exponential) convergence. The time-plots of all mentioned control functions are presented in Fig. 2, left. The plot of the function \(\frac{1}{2} \ln \|x(t)\|\) illustrates the exponential decay rate of the solutions of (16) (see Fig. 2, right).

### 3.2 Stabilization of an underwater vehicle

In this subsection, we consider a nonlinear system with a non-vanishing drift term. Namely, we consider the equations of motion of an autonomous underwater vehicle studied, e.g., in [18]. As in [7], assume that one of the components of the angular velocity is uncontrolled and remains constant. Then the dynamics can be represented in the following control-affine form:

\[
\dot{x} = f_0(x) + \sum_{k=1}^{3} f_k(x)u_k, \tag{21}
\]

where \((x_1, x_2, x_3)\) denote the coordinates of the center of mass, and \((x_4, x_5, x_6)\) specify the vehicle orientation (Euler’s angles),

\[
f_0(x) = \omega \left( 0 \ 0 \ 0 \ \cos(x_4) \tan(x_5) - \sin(x_4) \cos(x_4) \sec(x_5) \right)^\top,
\]

\[
f_1(x) = \left( \cos(x_5) \cos(x_6) \cos(x_5) \sin(x_6) - \sin(x_5), 0, 0, 0 \right)^\top,
\]

\[
f_2(x) = \left( 0 \ 0 \ 0 \ 1 \ 0 \ 0 \right)^\top,
\]

\[
f_3(x) = \left( 0 \ 0 \ 0 \ \sin(x_4) \tan(x_5) \cos(x_4) \sin(x_4) \sec(x_5) \right)^\top.
\]

The control \(u_1\) represents the translational velocity along the \(x_1\)-axis, \((u_2, u_3)\) control the two components of the angular velocity, while the third component \(\omega\) is assumed to be constant. Straightforward computations show that,
for all $x \in D = \{ x \in \mathbb{R}^6 : -\frac{\pi}{2} < x_5 < \frac{\pi}{2} \}$, system (21) satisfies the rank condition (5) with $S_1 = \{ 1, 2, 3 \}$, $S_2 = \{ (1, 3), (2, 3) \}$, $S_{10} = \{ 1 \}$, $S_{20} = \emptyset$:

$$\text{span}\{ f_1(x), f_2(x), f_3(x), [f_1, f_3](x), [f_2, f_3](x), [f_1, f_0](x) \} = \mathbb{R}^n \text{ for all } x \in D.$$ 

Following the control design algorithm described in Section II.C, we take the following control functions:

$$u_1^\varepsilon(t, x) = a_1(x) + 2\sqrt{\frac{\pi \kappa_{13} |a_{13}(x)|}{\varepsilon}} \text{sign}(a_{13}(x)) \cos\left(\frac{2\pi \kappa_{13} t}{\varepsilon}\right)$$
$$+ 2\sqrt{\frac{\pi \kappa_{10} a_{10}(x)}{\varepsilon}} \sin\left(\frac{2\pi \kappa_{01} t}{\varepsilon}\right),$$

$$u_2^\varepsilon(t, x) = a_2(x) + 2\sqrt{\frac{\pi \kappa_{23} |a_{23}(x)|}{\varepsilon}} \text{sign}(a_{23}(x)) \cos\left(\frac{2\pi \kappa_{23} t}{\varepsilon}\right),$$

$$u_3^\varepsilon(t, x) = a_3(x) + 2\sqrt{\frac{\pi \kappa_{13} |a_{13}(x)|}{\varepsilon}} \sin\left(\frac{2\pi \kappa_{13} t}{\varepsilon}\right)$$
$$+ 2\sqrt{\frac{\pi \kappa_{23} |a_{23}(x)|}{\varepsilon}} \sin\left(\frac{2\pi \kappa_{23} t}{\varepsilon}\right),$$

(22)

where $\kappa_{12}, \kappa_{23}, \kappa_{10}$ are pairwise distinct integers, and

$$(a_1(x) \ a_2(x) \ a_{13}(x) \ a_{23}(x) \ a_{10}(x))^\top = -\mathcal{F}^{-1}(x) \left( \gamma x + f_0(x) \right)$$

with $\gamma > 0$, 

$$\mathcal{F}(x) = (f_1(x) \ f_2(x) \ f_3(x) \ [f_1, f_3](x) \ [f_2, f_3](x) \ [f_1, f_0](x)) .$$

For the numerical simulation, we take 

$$\omega = 2, \ \kappa_{13} = 1, \ \kappa_{23} = 2, \ \kappa_{10} = 3, \ \gamma = 5, \ \varepsilon = 1,$$

Note that the drift term $f_0$ does not satisfy assumption (A1), which means that $x^* = 0 \in \mathbb{R}^6$ is not an equilibrium of the corresponding closed-loop system. However, as it is mentioned in Remark $\square$, it is still possible to ensure practical convergence to $x^*$ (i.e. convergence to some $\Delta$-neighborhood of $x^*$), as it is demonstrated in Fig. $\blacksquare$. In the future work, we expect to thoroughly analyze the relation between control parameters and the radius $\Delta$ of a neighborhood that can be achieved by the trajectories of the corresponding closed-loop system.
Figure 3: Time-plots of the trajectories for system (21)–(22) with the initial condition $x(0) = (-1, 1, 1, \frac{3\pi}{2}, \frac{3\pi}{8}, \pi)^T$. 
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APPENDIX

The explicit formulas for $\Omega_1, \Omega_2$ in (11) and (14) are as follows:

$$
\Omega_1(x^0, \varepsilon) = \varepsilon \sum_{i \in S_1, l \in S_{10}} a_i(x^0) a_{l0}(x^0) [f_i, f_l](x^0)
$$

$$
+ \frac{\varepsilon^2}{2} \left\{ L_{f0}f_0(x^0) + \sum_{i \in S_1} a_i(x^0) \left( L_{f0}f_i(x^0) + L_{fi}f_0(x^0) \right) \right. 
$$

$$
+ \left. \sum_{i_1, i_2 \in S_1} L_{fi_1}f_{i_2}(x^0) a_{i_1}(x^0) a_{i_2}(x^0) \right\}
$$

and

$$
\Omega_2(x^0, \varepsilon) = \varepsilon \left\{ \sum_{(l_1, l_2) \in S_{20}} |a_{l_1l_20}(x^0)| \left( [f_{l_1}, [f_{l_1}, f_0]](x^0) ight. 
$$

$$
+ \left. [f_{l_2}, [f_{l_2}, f_0]](x^0) \right) + 4 \sum_{k_1, k_2=1}^{m} \sum_{(l_1, l_2) \in S_{20}, (j_1, j_2) \in S_{20}} \zeta_{k_1l_1l_2} \zeta_{k_2l_2l_2} 
$$

$$
\times \frac{k_{l_1l_20}k_{j_1j_20}}{k_{j_1j_20}^2 - k_{l_1l_20}^2} \sqrt{|a_{l_1l_20}| |a_{j_1j_20}|} \left( [f_0, [f_{l_1}, f_{l_2}]](x^0) ight. 
$$

$$
+ \left. \sum_{i \in S_1} a_i(x^0) [f_i, [f_{l_1}, f_{l_2}]](x^0) \right) + \sum_{i \in S_1} \sum_{(l_1, l_2) \in S_{20}} a_i(x^0) 
$$

$$
\times |a_{l_1l_20}(x^0)| \left( [f_{l_1}, [f_{l_1}, f_i]](x^0) + [f_{l_2}, [f_{l_2}, f_i]](x^0) \right) + \text{sign}(a_{l_1l_20}(x^0)) \left( [f_{l_1}, [f_{l_1}, f_i]](x^0) + [f_{l_2}, [f_{l_2}, f_i]](x^0) \right) \right\} 
$$

$$
+ \frac{\varepsilon^2}{2} \left\{ L_{f0}f_0(x^0) + \sum_{i \in S_1} a_i(x^0) \left( L_{f0}f_i(x^0) + L_{fi}f_0(x^0) \right) \right\}
$$

19
\[
+ \sum_{i_1, i_2 \in S_1} L_{f_i} f_{i_2}(x^0) a_{i_1}(x^0) a_{i_2}(x^0) \}
+ \frac{\varepsilon^2}{\pi} \{ \sum_{k=1}^{m} \sum_{(l_1, l_2) \in S_{20}} \zeta_{k l_1 l_2} \sqrt{|a_{i_1 i_2 0}(x^0)|} \left( [f_0, [f_0, f_1]](x^0) \right.
+ \sum_{i \in S_1} a_i(x^0) \left( [f_0, [f_i, f_0]](x^0) + [f_i, [f_0, f_0]](x^0) \right)
+ \sum_{i_1, i_2 \in S_1} a_{i_1}(x^0) a_{i_2}(x^0) [f_{i_1}, [f_{i_2}, f_0]](x^0) \bigg) \}
+ \frac{\varepsilon^3}{6} \left\{ L_{f_0} L_{f_0} f_0(x^0) + \sum_{i \in S_1} a_i(x^0) \left( L_{f_0} L_{f_0} f_i(x^0) \right.ight.
+ L_{f_0} L_{f_i} f_0(x^0) + L_{f_i} L_{f_0} f_0(x^0) \bigg)
+ \sum_{i_1, i_2 \in S_1} a_{i_1}(x^0) a_{i_2}(x^0) \left( L_{f_0} L_{f_{i_1}} f_{i_2}(x^0) + L_{f_{i_1}} L_{f_0} f_{i_2}(x^0) \right)
+ L_{f_{i_1}} L_{f_{i_2}} f_0(x^0) \bigg)
+ \left. \sum_{i_1, i_2, i_3 \in S_1} a_{i_1}(x^0) a_{i_2}(x^0) a_{i_3}(x^0) L_{f_{i_1}} L_{f_{i_2}} f_{i_3}(x^0) \right\}.
\]

Here \( \zeta_{k l_1 l_2} = \delta_{k l_1} + \delta_{k l_2} \text{sign}(a_{k l_1 l_2}(x^0)). \)