Noise robustness in the detection of non-separable random unitary maps

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Abstract
We briefly review in this paper a recently proposed method to detect the properties of quantum noise processes and quantum channels. We illustrate in detail the method for detecting non-separable random unitary channels and consider, in particular, explicit examples of the CNOT and C-Z gates. We also analyse their robustness in the presence of noise for several quantum noise models.

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1. Introduction
Quantum noisy channels, and in general quantum noise processes, can be measured by means of complete process tomography [1]. Tomography does not need any a priori knowledge about the quantum process under consideration, but at the same time it requires a large number of measurement settings when it has to be implemented experimentally (which goes as \(d^4\), where \(d\) is the dimension of the quantum system on which the channel acts). In many realistic implementations, however, some a priori information in the form of a quantum channel, or a quantum noise process, is available and it is of great interest to determine experimentally with the minimum number of measurement settings whether or not the channel has a certain property (e.g. being entanglement breaking or non-separable random unitary). In this work, we review a recently proposed efficient method for quantum channel detection [2] avoiding complete quantum process tomography, and apply it to non-separable random unitary channels (RU). In particular, we study in detail its robustness in the presence of noise.

This paper is organized as follows. In section 2, we recall some preliminary notions that represent the main ingredients to develop the proposed quantum channel detection method, namely the Choi–Jamolkowski isomorphism and the entanglement witnesses. In section 3, we illustrate the method in the case of detection of non-separable random unitary maps. In section 4, we study in detail the robustness of the method in the presence of noise for depolarizing, dephasing, bit flip and amplitude damping noise. In section 5, we finally summarize the main results.

2. Preliminaries
Quantum channels, and in general quantum noise processes, are described by completely positive and trace preserving (CPT) maps \(\mathcal{M}\), which can be expressed in the Kraus form [3] as
\[
\mathcal{M}[\rho] = \sum_k A_k \rho A_k^\dagger,
\]
where \(\rho\) is the density operator of the quantum system on which the channel acts and the Kraus operators \(\{A_k\}\) fulfil the constraint \(\sum_k A_k^\dagger A_k = 1\).

In order to develop the detection method proposed, we will use the Choi–Jamolkowski isomorphism [4, 5], which gives a one-to-one correspondence between CPT maps acting on \(\mathcal{D}(\mathcal{H})\) (the set of density operators on \(\mathcal{H}\)) and bipartite density operators \(C_{\mathcal{M}}\) on \(\mathcal{H} \otimes \mathcal{H}\). This isomorphism can be described as
\[
\mathcal{M} \iff C_{\mathcal{M}} = \mathcal{M} \otimes \mathcal{I} [|\alpha\rangle\langle\alpha|],
\]
where \(\mathcal{I}\) is the identity map and \(|\alpha\rangle\) is the maximally entangled state with respect to the bipartite space \(\mathcal{H} \otimes \mathcal{H}\), i.e.
\[
|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d |k\rangle |k\rangle.
\]
(we consider here quantum channels acting on systems with finite dimension \(d\)).

By exploiting the above isomorphism, we are able to link some specific properties of quantum channels to the properties of the corresponding Choi states \(C_{\mathcal{M}}\). In particular, we find a connection between quantum channel properties and (multipartite) entanglement properties of the corresponding Choi states. The method works when we consider properties that are based on a convex structure of the quantum channels.

The second main ingredient employed is the concept of entanglement detection via witness operators [6]. We then
briefly recall here that a state $\rho$ is entangled if and only if there exists a Hermitian operator $W$ such that $\text{Tr}[W \rho] < 0$ and $\text{Tr}[W \rho_{\text{sep}}] \geq 0$ for all separable states. The correspondence we exploit is between the Choi states of the considered set of quantum channels and the set of separable states. Both represent convex subsets of the sets of all quantum channels acting on density operators on $\mathcal{H}$ and all bipartite density operators on $\mathcal{H} \otimes \mathcal{H}$, respectively, as shown in Figure 1.

3. Non-separable random unitary maps

We will now illustrate explicitly how the channel detection method works in the case of separable random unitary maps. Let us first recall the concept of random unitary channels (RU). These are defined as

$$\mathcal{W}[\rho] = \sum_k p_k U_k \rho U_k^\dagger, \quad (3)$$

where $U_k$ are unitary operators and $p_k \geq 0$ with $\sum_k p_k = 1$. Note that this kind of maps includes several interesting models of quantum noisy channels, such as the depolarizing channel or the phase damping channel and the bit flip channel [1].

Let us now assume that the system on which the random unitary channel acts is a bipartite system $\rho_{AB}$ (composed of systems $A$ and $B$). We can then identify a class of random unitary maps that is separable, namely that which can be written in the form

$$\mathcal{V}[\rho_{AB}] = \sum_k p_k (V_{k,A} \otimes W_{k,B}) \rho_{AB} (V_{k,A}^\dagger \otimes W_{k,B}^\dagger), \quad (4)$$

where $\rho_{AB}$ is a bipartite system, and both $V_{k,A}$ and $W_{k,B}$ are unitary operators for all $k$, acting on systems $A$ and $B$, respectively. Quantum channels of the above form are named separable random unitaries (SRU) and form a convex subset in the set of all CPT maps acting on bipartite systems $\rho_{AB}$. Interesting examples of channels of this form are given by Pauli memory channels [7].

When considering quantum channels acting on bipartite systems, the Choi state is a four-partite density operators as $S_{\text{SRU}}$

$$(U_A \otimes 1_C)(\alpha)_{AC} \otimes (U_B \otimes 1_D)(\alpha)_{BD}. \quad (5)$$

We can now detect non-separable RU maps (which correspond to the Choi states that are entangled in the bipartition AC–BD) by designing suitable witness operators that detect the corresponding Choi state with respect to biseparable in AC–BD states belonging to $S_{\text{SRU}}$.

We illustrate this procedure with a simple example. Consider the case of detecting a non-separable unitary operation $U$ acting on a bipartite system $AB$. A suitable detection operator can be constructed as

$$W_U = \beta 1 - C_U, \quad (6)$$

where the coefficient $\beta$ is the squared overlap between the closest biseparable state in the set $S_{\text{SRU}}$ and the entangled state $C_U$, namely

$$\beta = \max_{|\phi\rangle \in S_{\text{SRU}}} \langle \phi | C_U |\phi\rangle. \quad (7)$$

Note that since the maximum of a linear function over a convex set is always achieved on the extremal points, the maximum above can be always calculated by maximizing over the pure biseparable states (5).

We will specify the above construction to the particular case of the CNOT gate acting on two qubits. The corresponding Choi state has the form

$$C_{\text{CNOT}} = (\text{CNOT} \otimes 1) |\alpha\rangle \langle \alpha| (\text{CNOT} \otimes 1), \quad (8)$$

where the CNOT operation is given by

$$\text{CNOT} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \quad (9)$$

with 1 representing the $2 \times 2$ identity matrix and $X$ the Pauli operator $\sigma_x$.

The optimal coefficient $\beta$ equals 1/2 and the detection operator $W_{\text{CNOT}}$ can be decomposed into a linear combination of local operators as follows [2]:

$$W_{\text{CNOT}} = \frac{1}{2^{12}} (311111 - 11XX - XXX1 - X1XX - ZZ1Z + ZY1Y + YYXZ + YZXY - Z11Z - ZXZZ + YYXY + YXY1 + Y1X - 1ZZZ + 1YZY + XXXY + XZZY), \quad (10)$$

where for the simplicity of notation, $X$, $Y$ and $Z$ represent the Pauli operators and the tensor product symbol has been omitted. As we can see from the above form, the CNOT can be detected by using nine different local measurement settings [8]. Following [9, 10], it can be also easily proved that the above form is optimal in the sense that it involves the smallest number of measurement settings. From the point of view of implementations, the optimal detection procedure then works as follows: prepare a four-partite qubit system in the state $|\alpha\rangle = |\alpha\rangle_{AC} |\alpha\rangle_{BD}$, input qubits $A$ and $B$ to the quantum channel and finally make the set of

The states given by (5) correspond to the generating points of the set of SRU channels, which contains the extremal points.
nine local measurements reported above on the four-partite system $ABCD$ in order to measure the operator (10). If the resulting average value is negative, then the quantum channel is detected as a non-separable random unitary map.

As a second significant example consider the C-Z operation, which also represents an important two-qubit gate in quantum computation [1]. This operation is defined as

$$C-Z = \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} :$$

namely it has the same structure as the CNOT gate, with X replaced by Z. This case can be connected to the detection procedure for the CNOT gate by exploiting the following relation between the CNOT and C-Z gates:

$$C-Z = (I \otimes H)\text{CNOT}(I \otimes H),$$

where $H$ is the Hadamard gate, defined as $H = \frac{1}{\sqrt{2}}(X + Z)$. Since the two-gate operations differ only by a local unitary transformation, the maximization performed in equation (7) leads to the same value for $\beta$. The corresponding detection operator $W_{C-Z}$ can then be written in the form

$$W_{C-Z} = \frac{1}{d^2}(311111 - 1Z1Z - Z1Z1 - ZZZZZ$$

$$- ZXIX + ZY1Y - 1XXZ + Y1YZ$$

$$- XZX1 - X1XZ + YZY1 + Y1YZ$$

$$- Y1XX - YXXY - XXYY - XXYY),$$

which again corresponds to a set of nine local measurements.

4. Noise robustness

We will now study the robustness of the method in the presence of additional noise, which can influence the operation of the quantum channel. The situation we have in mind is the following. Suppose that we are given a witness $W_U$ of the form (6) to detect a unitary transformation $U$ acting on two qubits. Suppose also that the experimental implementation of $U$ leads to a new map $\mathcal{M}$, which is close to the original $U$ by construction but not exactly $U$ due to the presence of noise. Does the witness $W_U$ still detect the map $\mathcal{M}$ as a non-SRU map? To answer this question we have to check whether the expectation value of the witness $W_U$ on the map $\mathcal{M}$ is still negative.

Starting from the definition (6), the expectation value of $W_U$ on $C_{\mathcal{M}}$ can be expressed as

$$\text{Tr}[W_U C_{\mathcal{M}}] = \beta - \text{Tr}[C_{\mathcal{M}} C_{\mathcal{U}}].$$

By exploiting the Choi–Jamolkowski isomorphism, the overlap between two states $C_{\mathcal{L}}$ and $C_{\mathcal{M}}$ corresponding to the maps $\mathcal{L}$ and $\mathcal{M}$ can be generally written as

$$\text{Tr}[C_{\mathcal{M}} C_{\mathcal{L}}] = \frac{1}{d^2} \sum_{i,j=1}^{d} \text{Tr}[\mathcal{M}(\langle i | \langle j | \mathcal{L}(\langle j | \langle i |))].$$

where $\langle | |$ represents the computational basis for a $d^2$-dimensional bipartite system. In terms of the Kraus operators $\{A_k\}$ and $\{B_l\}$ of the maps $\mathcal{M}$ and $\mathcal{L}$, respectively, the above expression can be written as

$$\text{Tr}[C_{\mathcal{M}} C_{\mathcal{L}}] = \frac{1}{d^2} \sum_{k,l} |\text{Tr}[A_k^\dagger B_l]|^2,$$

where the double summation is over the Kraus operators and the absolute value comes from the identity $\text{Tr}[A^\dagger] = \text{Tr}[A^*]$. In the present case, with $d = 4$ and $\mathcal{L}$ given by a unitary operation $U$, the above expression takes the form

$$\text{Tr}[C_{\mathcal{M}} C_{\mathcal{U}}] = \frac{1}{16} \sum_k |\text{Tr}[A_k^\dagger U]|^2,$$

where the summation is now performed just over the Kraus operators $\{A_k\}$ of $\mathcal{M}$. The expectation value for the witness $W_U$ detecting the gate $U$ can then be rewritten as

$$\text{Tr}[W_U C_{\mathcal{M}}] = \beta - \frac{1}{16} \sum_k |\text{Tr}[A_k^\dagger U]|^2.$$
resulting channel shown above is still a random unitary channel.

We will first start from the detection of noisy CNOT gate via the witness operator \( W_{\text{CNOT}} \). From equation (17), we can compute the overlap between the noiseless Choi state \( \mathcal{C}_{\text{CNOT}} \) and the noisy case \( \mathcal{M}_{D, X} \), where \( \mathcal{M}_{D, X} \) is the composite map given by (20) with \( U_i = X \), as

\[
\text{Tr}[C_{\mathcal{M}_{D, X}}C_{\text{CNOT}}] = \frac{1}{16} \sum_{k, l} |\text{Tr}[D_1^k D_2^l C_{\text{CNOT}} D_1^k D_2^l]|^2,
\]

(21)

where \( \{D_1^k\} \) and \( \{D_2^l\} \) are the Kraus sets of \( \mathcal{D}_1 \otimes \mathcal{D}_1 \) and \( \mathcal{D}_2 \otimes \mathcal{D}_2 \), respectively. By performing the calculation explicitly and remembering that, apart from the parameters \( q_i \), the term on the right-hand side above is a symmetric matrix in \( k, l \), we arrive at the following expression for the expectation value:

\[
\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_{D, X}}] = \frac{1}{8} \sum_{i=1}^{4} (16q_i^2 q_i^2 + 2q_1 \bar{q}_i q_2 \bar{q}_2 + q_1 q_2 \bar{q}_1 \bar{q}_2 + q_1 \bar{q}_1 q_2 \bar{q}_2 + q_2 \bar{q}_2 q_1 \bar{q}_1 + q_2 \bar{q}_1 q_2 \bar{q}_1 + q_1 q_2 \bar{q}_1 \bar{q}_2 + q_1 \bar{q}_2 q_2 \bar{q}_2) \frac{1}{8}
\]

(22)

with the definition \( \bar{q}_i = 1 - \frac{3q_i}{2} \) for \( i = 1, 2 \).

Let us now study some special cases of the above situation. Suppose first that \( q_2 = 0 \), so that the noise affects the channel only before the CNOT. In this case the expectation value becomes

\[
\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_{D, X}}] = \frac{1}{8} \left( 1 - \bar{q}_1^2 \right)
\]

(23)

which is negative for \( q_1 < \frac{4 - 2\sqrt{2}}{4} \simeq 0.39 \). Therefore, the values of \( q_1 \) below this threshold lead to detection of the CNOT gate as a non-separable random unitary. Since the situation is symmetric, the same obviously holds when \( q_1 = 0 \) and we are looking at \( q_2 \), namely the action of the depolarizing channel either before or after the CNOT gate leads to the same noise threshold. Another interesting situation is when both the channels before and after the CNOT gate introduce the same level of noise, namely when \( q_1 = q_2 = q \). In this case we obtain the following expression for the expectation value:

\[
\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_{D, X}}] = \frac{1}{8} \left( 1 - \bar{q}_1^2 \right) (q - 2)^2 (5q^2 - 8q + 4)
\]

(24)

The CNOT gate is thus detected as a non-separable random unitary map when \( q < 0.21 \). Note that the threshold in this case is not as high as the one we obtained before, since the situation is much noisier because there are two sources of noise.

We will now consider the case of the C-Z gate. The detection of noisy C-Z gate via the witness \( W_{\text{C-Z}} \) turns out to give the same threshold of noise as for the CNOT gate. This is basically due to the symmetry properties of the depolarizing noise, which acts isotropically along the three directions of the Pauli matrices. It is then straightforward to find that the expectation value of \( W_{\text{C-Z}} \) on \( C_{\mathcal{M}_{D, X}} \), namely \( \text{Tr}[W_{\text{C-Z}}C_{\mathcal{M}_{D, X}}] \), is exactly given by equation (22). Hence, the analysis we performed in that case still holds for the C-Z gate.

As we can see, the presence of local depolarizing noise thus affects the CNOT and C-Z operations in such a way that, beyond a certain amount of noise, the noisy CNOT and C-Z operations become separable and are no longer detected by our method.

4.2. Dephasing noise

Let us now assume that phase damping noise is present, acting independently on the two qubits \( A \) and \( B \) in general both before and after the operation we want to detect (either CNOT or C-Z), as for the case of the depolarizing noise considered above. Phase damping noise \( \mathcal{P} \) is described by a CPT map of the form (19) where the probabilities are given by \( p_0 = 1 - q \), \( p_1 = p_2 = 0 \) and \( p_3 = q \). Note that also in this case the global resulting channel is still a random unitary channel.

In order to quantify the noise robustness of the witness \( W_{\text{CNOT}} \) with respect to phase damping noise, we calculate the expectation value of \( W_{\text{CNOT}} \) given by (6) (with \( \beta = 1/2 \)) with respect to the state \( C_{\mathcal{M}_{D, X}} \), i.e. the Choi state corresponding to the composite map \( \mathcal{M}_{P, X} = (\mathcal{P}_2 \otimes \mathcal{P}_2)C_{\text{CNOT}}(\mathcal{P}_1 \otimes \mathcal{P}_1) \). The problem thus reduces to evaluating the overlap between the Choi states \( C_{\text{CNOT}} \) and \( C_{\mathcal{M}_{D, X}} \). By using equation (18), this procedure leads to

\[
\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_{D, X}}] = \frac{1}{8} - (1 - q_1 - q_2 + 2q_1 q_2)^2
\]

(25)

From the above expression, we can see that \( \text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_{D, X}}] < 0 \) for certain intervals of the noise parameters \( q_1 \) and \( q_2 \). From the symmetry of the above expression, the action of dephasing noise either before or after the CNOT gate leads to the same result. In this case, namely \( q_2 = 0 \), the expectation value of \( W_{\text{CNOT}} \) is negative for \( q_1 < 1 - \frac{1}{\sqrt{2}} \simeq 0.29 \). When the dephasing channels introduce the same level of noise \( q_1 = q_2 = q \) the expectation value of \( W_{\text{CNOT}} \) turns out to be negative for \( q < 0.17 \) and therefore the CNOT operation can be detected in this range.

Regarding the robustness of the witness operator \( W_{\text{C-Z}} \), we need to compute the expectation value of \( W_{\text{C-Z}} \) with respect to the Choi state \( C_{\mathcal{M}_{D, X}} \), representing the noisy implementation of the C-Z gate, i.e. \( \mathcal{M}_{P, Z} = (\mathcal{P}_2 \otimes \mathcal{P}_2)C_{\text{C-Z}}(\mathcal{P}_1 \otimes \mathcal{P}_1) \). Following the same calculation as before, we obtain

\[
\text{Tr}[W_{\text{C-Z}}C_{\mathcal{M}_{D, X}}] = \frac{1}{8} - (1 - q_1 - q_2 + 2q_1 q_2)^2
\]

(26)

which differs from the expectation value calculated for the CNOT gate, see equation (25).

Also in this case, if the noise is present just before or after the gate, namely \( q_2 = 0 \) or \( q_1 = 0 \), respectively, our method detects the noisy C-Z as a non-separable random unitary map if \( q_1 < 1 - \frac{1}{\sqrt{2}} \simeq 0.29 \) (or \( q_2 < 1 - \frac{1}{\sqrt{2}} \simeq 0.29 \)). This threshold is exactly the same as the one found for \( W_{\text{CNOT}} \), thus the witness for detecting the C-Z turns out to be as robust against dephasing noise as \( W_{\text{CNOT}} \), revealing CNOT. If the two sources of noise have the same strength, i.e. \( q_1 = q_2 = q \), then the expectation value turns out to be negative if the noise level is \( q < \frac{3}{4}(1 - \sqrt{1 - \frac{1}{\sqrt{2}}} \simeq 0.18 \) or \( q > \frac{3}{4}(1 + \sqrt{1 - \frac{1}{\sqrt{2}}} \simeq 0.82 \). This behaviour may seem to be very surprising, since it follows that the witness \( W_{\text{C-Z}} \) can tolerate not only low levels of noise but also high levels. The only regime where it fails is when the noise has a medium strength. This effect can be explained by noting that dephasing noise always commutes with the C-Z gate; thus the noise can be thought to be applied twice before the regarded gate. For a high noise level \( q \), the action of two consecutive dephasing processes leads almost to
the identical map, since $Z^2 = 1$, and so the scenario can be thought to be noiseless. We wish to stress that this result is completely different from the one obtained for $W_{\text{CNOT}}$, since there only a low amount of noise was tolerated.

4.3. Bit flip noise

Another interesting model of noise is given by bit flip noise $\mathcal{P}$, defined as a CPT map of the form (19) with probabilities $p_0 = 1 - q$, $p_1 = q$ and $p_2 = p_3 = 0$. As before, we consider the situation in which the noise acts independently on the two qubits both before and after the controlled operation (either CNOT or C-Z) we aim to detect.

Let us first focus on the detection of the CNOT gate by the operator $W_{\text{CNOT}}$. By exploiting equation (18), where the composite map is now given by $\mathcal{M}_{B,X} = (\mathcal{B}_2 \otimes \mathcal{B}_2) \text{CNOT} (\mathcal{P}_1 \otimes \mathcal{P}_1)$, we arrive at the following expectation value of $W_{\text{CNOT}}$ over its noisy implementation $\mathcal{M}_{B,X}$:

$$\text{Tr}[W_{\text{CNOT}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - (1 - q_1)^2 (1 - q_2)^2 + q_1 q_2 (1 - q_1 q_2).$$

(27)

This turns out to be the same expectation value as for the case of dephasing noise; therefore the discussion already given below equation (25) still holds.

In order to study the robustness of $W_{\text{C-Z}}$ to detect C-Z with additional bit flip noise, we have to evaluate the quantity $\text{Tr}[W_{\text{C-Z}} C_{\mathcal{M}_{a,z}}]$ with $\mathcal{M}_{B,Z} = (\mathcal{B}_2 \otimes \mathcal{B}_2) \text{C-Z} (\mathcal{P}_1 \otimes \mathcal{P}_1)$. By using equation (18), we obtain

$$\text{Tr}[W_{\text{C-Z}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - (1 - q_1)^2 (1 - q_2)^2,$$

(28)

which allows us to derive different thresholds for the noise tolerance of C-Z. If noise is neglected either after ($q_2 = 0$) or before ($q_1 = 0$) the C-Z gate, then the method is able to tolerate a level of noise up to $1 - \frac{1}{\sqrt{2}}$, i.e. either $q_1 < 1 - \frac{1}{\sqrt{2}}$ or $q_2 < 1 - \frac{1}{\sqrt{2}}$. In the case where both the noise sources show the same amount of noise, namely $q_1 = q_2 = q$, it follows that the C-Z gate is detected as long as $q < 0.16$.

4.4. Amplitude damping noise

As a last noise model we consider the amplitude damping channel $\mathcal{A}$, which is not a random unitary noise, and it is described by the following Kraus operators acting on a qubit state:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

(29)

where $\gamma$ is the parameter characterizing the amount of damping.

In the case of $W_{\text{CNOT}}$, by following the same procedure described above and by considering now the composite map $\mathcal{M}_{A,X} = (\mathcal{A}_2 \otimes \mathcal{A}_2) \text{CNOT} (\mathcal{A}_1 \otimes \mathcal{A}_1)$, we have

$$\text{Tr}[W_{\text{CNOT}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - \frac{1}{p_2} \left\{(1 + \sqrt{\gamma_1}) (1 + \sqrt{\gamma_2}) + \gamma_1 \gamma_2 \gamma_3 \right\},$$

(30)

where we have defined $\gamma = 1 - \gamma$. As in the previous cases the above expression is symmetric under exchange of $\gamma_1$ and $\gamma_2$.

When noise acts only either before or after the CNOT gate, e.g. $\gamma_2 = 0$, the above expression is negative for $\gamma_1 < 0.53$. For the particular case of $\gamma_1 = \gamma_2 = \gamma$ we have that the above expression reduces to

$$\text{Tr}[W_{\text{CNOT}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - \frac{1}{p_2} \left\{(1 + \sqrt{\gamma})^2 + \gamma^2 \right\},$$

(31)

which is negative for $\gamma < 0.31$. Therefore the composite map can be detected as a non-separable random unitary in this range of noise parameter $\gamma$.

The noise robustness of $W_{\text{C-Z}}$ with respect to the amplitude damping noise can be studied starting from the expectation value of $W_{\text{C-Z}}$ over $\mathcal{M}_{A,Z} = (\mathcal{A}_2 \otimes \mathcal{A}_2) \text{C-Z} (\mathcal{A}_1 \otimes \mathcal{A}_1)$, which is given by

$$\text{Tr}[W_{\text{C-Z}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - \frac{1}{p_2} (1 + \sqrt{\gamma})(\gamma).$$

(32)

As we can see from the above expression, when noise is present only before the C-Z gate, i.e. $\gamma_2 = 0$, a negative result is found for $\gamma_1 < 0.53$, exactly as for $W_{\text{CNOT}}$. Note that, since the above expectation value is still invariant under exchange of $\gamma_1$ and $\gamma_2$, the same holds if noise acts just after the controlled gate. When noise before and after the C-Z gate is the same, i.e. $\gamma_1 = \gamma_2 = \gamma$, it is easy to show that

$$\text{Tr}[W_{\text{C-Z}} C_{\mathcal{M}_{a,z}}] = \frac{1}{2} - \frac{1}{p_2} (1 + \gamma)^2.$$ (33)

Thus the witness operator $W_{\text{C-Z}}$ detects the noisy C-Z as a non-random unitary map only if $\gamma < 0.31$. We would like to stress that this value is the same as before only because we truncate the root of equation (33) at the second digit.

5. Conclusions

In summary, we have reviewed an experimentally feasible method to detect specific properties of noisy quantum channels and we have analysed, in particular, the case of detection of non-separable random unitary maps. The advantage of the present method over standard quantum process tomography is that a much smaller number of measurement settings are needed in an experimental implementation. Moreover, the proposed scheme relies on the implementation of local measurements and is achievable with current technology, for example in a quantum optical setup [11]. We have also studied in detail the robustness of the method in the presence of noise and imperfections in the channel operation for the case of a unitary channel, considering explicit examples of CNOT and C-Z gates. We have discussed, in particular, four realistic noise models, namely the depolarizing, the dephasing, the bit flip and the amplitude damping noise, and derived the corresponding noise intervals in which the method works.

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