HARMONIC ANALYSIS ON THE TWISTED FINITE POINCARÉ UPPER HALF-PLANE

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Abstract. We prove that the induced representation from a non trivial character of the Coxeter torus of GL(2, F), for a finite field F, is multiplicity-free; we give an explicit description of the corresponding (twisted) spherical functions and a version of the Heisenberg Uncertainty Principle.

1. Introduction

Let $F$ be a finite field, with $q$ elements, and $E$ be its unique quadratic extension. Put $G = GL(2, F)$ and denote by $K$ the Coxeter torus of $G$, realized as the subgroup of all matrices $m_z (z \in E^\times)$ of the maps $w \mapsto zw (w \in E)$ with respect to a fixed $F$-basis of $E$. Recall that the finite homogeneous space $\mathcal{H} = G/K$ may be looked upon as the finite analogue of (the double cover of) the classical Poincaré Upper Half Plane (see [4]). Harmonic analysis on $\mathcal{H}$ amounts to decompose the induced representation $\text{Ind}_K^G 1$ from the unit character $1$ of $K$ to $G$. We are interested here in the “twisted” version of this, i.e., the decomposition of the induced representation $\text{Ind}_K^G \Phi$ from a non (necessarily) trivial character $\Phi$ of $K$ to $G$. The real analogue of this case has been considered in [1]. We prove that this representation is multiplicity-free, taking advantage of the fact that this is so for $\text{Ind}_K^G 1$ (see [3]) and reducing the computation of the multiplicities in $\text{Ind}_K^G \Phi$ to the ones in $\text{Ind}_K^G 1$. We also give an explicit description of the corresponding (twisted) spherical functions. Finally, we give a version of the Heisenberg Uncertainty Principle.

2. The Multiplicity One Theorem for $\text{Ind}_K^G \Phi$.

2.1. The case $\Phi = 1$. We consider first the special case $\Phi = 1$ in which the multiplicity one theorem follows from a geometric argument. In fact, we have

$$\text{Ind}_K^G 1 \simeq (L^2(\mathcal{H}), \tau),$$

where $L^2(\mathcal{H})$ stands for the space of all complex functions on $\mathcal{H}$ endowed with the usual canonical scalar product, and $\tau$ denotes the natural representation of $G$ in

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$L^2(\mathcal{H})$, defined by $(\tau_g f)(z) = f(g^{-1}.z)$, where $z \mapsto g.z$ is the homographic action of $G$ on $\mathcal{H}$, given by $g.z = \frac{az + b}{cz + d}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in \mathcal{H}$.

**Definition 1.** For all $z, w \in \mathcal{H}$, we put $D(z, w) = \frac{N(z - w)}{N(z - \bar{w})}$ with the convention that $D(z, w) = \infty$ if $w = \bar{z}$.

**Proposition 1.** $D$ is an orbit classifying invariant function for the homographic action of $G$ in $\mathcal{H} \times \mathcal{H}$.

**Corollary 1.** The commuting algebra of $(L^2(\mathcal{H}), \tau)$ is commutative.

This follows from the fact that, the classifying invariant $D$ being symmetric, the $G$-orbits in $\mathcal{H} \times \mathcal{H}$ are also symmetric. \qed

### 2.2. The case of general $\Phi$.

Let’s denote by $\phi$ the restriction of $\Phi$ to $F \times$. We will prove that every twisting of an irreducible representation $\pi_\theta^d$ of $G$ (where the superscript $d$ denotes the dimension of $\pi$ and $\theta$ its character parameter) by the character $(\Phi + \Phi^q)$ is isomorphic to a representation of the form $\pi_{\phi}^{d'} + \pi_{\phi'}^{d''}$, when restricted to $K$. In fact we will work with the characters $\chi_\theta^d$ of the irreducible representations $\pi_\theta^d$ of $G$, for which we keep the notations of [5] or [2].

**Lemma 1.** On $K$ we have
\begin{align*}
(\Phi + \Phi^q)\chi_{\alpha,\alpha}^q &= \chi_{\Phi(\alpha N)}^{q-1} + \chi_{\phi(\alpha N)}^{q+1}, \\
(\Phi + \Phi^q)\chi_{\alpha,\beta}^{q+1} &= \chi_{\phi(\alpha N)}^{q+1} - \chi_{\Phi(\alpha N)}^{q-1}, \\
(\Phi + \Phi^q)\chi_{\alpha,\beta}^{q-1} &= \chi_{\Phi(\alpha N)}^{q-1} - \chi_{\phi(\alpha N)}^{q+1}, \\
(\Phi + \Phi^q)\chi_\Lambda^q &= \chi_{\Phi(\alpha N)}^{q-1} + \chi_{\phi(\alpha N)}^{q+1}.
\end{align*}
\qed

Now for a character $\chi$ of $G$, we have $\chi \circ \text{Frob} = \chi$ on $K$, as it follows from the character table. Therefore $\sum_K \Phi(k^q)\chi(k) = \sum_K \Phi(k)\chi(k)$ because Frob is an involutive automorphism.

Hence, the multiplicity of $\pi$ in $\text{Ind}_K^G \Phi$ equals $\frac{1}{2} \sum_K (\Phi + \Phi^q)(k)\pi(k)$ and so it is just the average of the multiplicities in $\text{Ind}_K^G 1$ of two representations of $G$ (one of which may be virtual!)

**Remark 1.** Put $\pi_{\alpha,\alpha}^{q+1} = \pi_\alpha^q + \pi_\alpha^1$ and $\pi_{\alpha,\alpha}^{q-1} = \pi_\alpha^q - \pi_\alpha^1$ for every $\alpha \in (F^\times)^\wedge$. It is easy to check than in the degenerate cases $\alpha = \beta$ (for $\pi = \pi_{\alpha,\beta}^{q+1}$ ) and $\Lambda = \Lambda^q$ (for $\pi = \pi_{\Lambda,\Lambda}^{q-1}$) we find for the multiplicities $m_1(\pi)$
\begin{equation}
m_1(\pi_{\alpha,\alpha}^{q+1}) = 1 \quad (\alpha \in (F^\times)^\wedge) (1)
\end{equation}
and
\[ m_1(\pi_{\alpha \circ N}^{q-1}) = -\delta_{\alpha,1} \quad (\alpha \in (F^\times)^\wedge) \] (2)

Using the fact that the multiplicities of the irreducible representations of \( G \) in \( \text{Ind}_K^G \Phi \) are at most one and also equations (1) and (2), we get that the multiplicities are also at most one in the more general case of \( \text{Ind}_K^G \Phi \).

\[ \square \]

2.3. The multiplicities \( m_{\Theta,d}(\Phi) \) of \( \pi_{\Theta}^d \) in \( \text{Ind}_K^G \Phi \) for general \( \Phi \in (E^\times)^\wedge \). In Table 1 below, \( \pi_{\Theta}^d \) denotes an irreducible representation of \( G \), of dimension \( d \) and parameter \( \Theta \). Then \( d \in \{1, q, q + 1, q - 1\} \) and \( \Theta \) is of the form \( \{\alpha, \beta\} \) with \( \alpha, \beta \in (F^\times)^\wedge \) or \( \{\Lambda, \Lambda^q\} \) with \( \Lambda \in (E^\times)^\wedge \).

| \( \pi_{\Theta}^d \) | \( m_{\Theta,d}(\Phi) \) |
|-----------------|-----------------|
| \( \pi_{\alpha,\alpha}^d \) | \( \delta_{\alpha^2,\phi} \) |
| \( \pi_{\alpha,\alpha}^{q+1} \) | \( \delta_{\alpha^2,\phi} - \delta_{\alpha \circ N,\Phi} \) |
| \( \pi_{\alpha,\beta}^{q+1} \) | \( \delta_{\alpha,\beta,\phi} \) |
| \( \pi_{\Lambda,\Lambda^q}^{q-1} \) | \( \delta_{\Lambda,\phi} - \delta_{\Lambda,\Phi} - \delta_{\Lambda^q,\Phi} \) |

Table 1. The multiplicities \( m_{\Theta,d}(\Phi) \)

NOTATIONS. Here \( \alpha, \beta \in (F^\times)^\wedge \) with \( \alpha \neq \beta \) and \( \Phi, \Lambda \in (E^\times)^\wedge \) with \( \Lambda \neq \Lambda^q \), and \( \lambda \) (resp. \( \phi \)) denotes the restriction of the character \( \Lambda \) (resp. \( \Phi \)) to \( (F^\times)^\wedge \).

3. The twisted spherical functions

3.1. The averaging construction. In this section \( G \) denotes an arbitrary finite group, \( K \) a subgroup of \( G \) and \( \Phi \) a one dimensional representation of \( K \). We notice that the spherical functions for the representation \( \text{Ind}_K^G \Phi \) are obtained as weighted averages of the characters of \( G \). More precisely:

**Definition 2.** Let \( L^1(G) \) be the group algebra of \( G \), realized as the convolution algebra of all complex functions of \( G \) and let \( L^1_K(G, K) \) be the convolution algebra of all complex functions \( f \) on \( G \) such that
\[ f(kgk') = \Phi(k)f(g)\Phi(k') \]
for all \( g \in G, k, k' \in K \). For any \( f \in L^1(G) \) put
\[ (P_{\Phi} f)(g) = \frac{1}{|K|} \sum_{k \in K} \Phi^{-1}(k)f(kg) \]
for all \( g \in G \).
Notice that the operator \( P_\Phi \) is just convolution with the idempotent function \( \varepsilon_K^\Phi \in L^1G \) which coincides with \(|K|^{-1} \Phi \) on \( K \) and vanishes elsewhere. Moreover \( L^1_K(G, K) \) may be written as \( \varepsilon_K^\Phi \ast L^1G \ast \varepsilon_K^\Phi \) and its elements \( f \) are characterized by the properties

\[
\varepsilon_K^\Phi \ast f = f = f \ast \varepsilon_K^\Phi.
\]

**Lemma 2.** Let \( \chi \) be the character of an irreducible representation \( \pi \) of \( G \). Then \( P_\Phi(\chi)(e) \neq 0 \) iff \( \pi \) appears in \( \text{Ind}_K^G \Phi \). \( \square \)

**Lemma 3.** \( P_\Phi(\chi) \) is a non-zero function iff it doesn’t vanish for \( g = e \). \( \square \)

**Proposition 2.** The mapping \( P_\Phi \) is an algebra epimorphism from the center \( Z(L^1G) \) of the convolution algebra \( L^1G \) onto the center \( Z(L^1_\Phi(G, K)) \) of the convolution algebra \( L^1_\Phi(G, K) \).

**Proof:** We have

\[
P_\Phi(f_1 \ast f_2) = \varepsilon_K^\Phi \ast (f_1 \ast f_2) = (f_1 \ast \varepsilon_K^\Phi) \ast f_2 = (f_1 \ast \varepsilon_K^\Phi \ast \varepsilon_K^\Phi) \ast f_2 = (\varepsilon_K^\Phi \ast f_1) \ast (\varepsilon_K^\Phi \ast f_2) = P_\Phi f_1 \ast P_\Phi f_2.
\]

since \( f_1 \) is central and \( \varepsilon_K^\Phi \) is idempotent. Moreover the dimension \( d \) of the image of \( Z(L^1G) \) under \( P_\Phi \) is the number of irreducible characters \( \chi \) of \( G \) such that \( P_\Phi(\chi) \neq 0 \); but \( P_\Phi(\chi) \neq 0 \) iff \( (P_\Phi(\chi))(e) \neq 0 \) and, the number \( (P_\Phi(\chi))(e) \) being the multiplicity in \( \text{Ind}_K^G \Phi \) of the representation \( \pi \) of \( G \) whose character is \( \chi \), we see that \( d \) is just the number of irreducible representations \( \pi \) of \( G \) appearing in \( \text{Ind}_K^G \Phi \), i.e. the dimension of the center of \( L^1_\Phi(G, K) \). \( \square \)

**Corollary 2.** The nonzero functions that satisfy the functional equation

\[
h(x)h(y) = \int_K \Phi(k)h(xky) \, dk
\]

linearly span the center of the algebra \( L^1_\Phi(G, K) \).

**Proof:** The functions \( h \) that satisfy the above functional equation are exactly the complex multiples of the functions \( P_\Phi(\chi) \); for a proof (see [3]). Therefore the corollary follows. \( \square \)

### 3.2. Explicit formulae for the twisted spherical functions.

Define

\[
S_\Lambda^\Phi(a) = -(q^2 - 1)^{-1} \sum_{(z, w) \in \Gamma_a} \Phi^{-1}(z) \Lambda(w)
\]

for \( \Lambda \in (E^\times)^\wedge \) and \( a \in F^\times \), where \( \Gamma_a \) denotes the set of all \( (z, w) \in E^\times \times E^\times \) such that \( N(w) = aN(z) \) and \( Tr(w) = 2(a + 1)^{-1}Tr(z) \).
Then the spherical function $\zeta_\Lambda^\Phi$ of $G$ associated to the cuspidal character $\chi_{q^{-1}}^\Lambda$ of $G$ is given on the representatives $d(a,1) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $(a \in F^\times)$ for the $K$-double cosets in $G$, by

$$\zeta_\Lambda^\Phi(d(a,1)) = S_\Lambda^\Phi(a) + q(q+1)^{-1}\delta_{a,1}\delta_{\lambda,\phi},$$

where $\lambda$ (resp. $\phi$) denotes the restriction of the character $\Lambda$ (resp. $\Phi$) of $E^\times$ to $F^\times$.

Notice that $a = 1$ corresponds to the origin in $H$ and $a = -1$ corresponds to the antipode of the origin in $H$. It is not difficult to check that these formulae for the spherical functions are equivalent to the ones given in [4] for the case $\Phi = 1$.

3.3. A new form for the cuspidal spherical functions for $\Phi = 1$ (char $F \neq 2$).

For $a \neq 1$, one has the following new expression for the spherical functions estimated in [3]

$$\zeta_\Lambda^\Phi(a) = (q+1)^{-1}\sum_{u \in U} \varepsilon(Tr(u) - (a + a^{-1}))(\varepsilon \omega)(u),$$

for $a \neq 1$, where $\varepsilon$ denotes the sign character of $F^\times$.

4. Heisenberg Uncertainty Principle

For this section, $G$ denotes an arbitrary finite group, $K$ any subgroup of $G$ and $\Phi$ any linear character of $K$.

Let $\hat{G}^\Phi$ be set of all the equivalence classes of irreducible representations of $G$ that contain the character $\Phi$ when restricted to $K$. For each equivalence class we choose, once and for all, a representative $(\pi, V_\pi)$. As usual, for each $f$ in $L^1(G)$, the Fourier Transform $\mathcal{F}(f)$, valued in the class $(\pi, V_\pi)$, is the linear operator $\mathcal{F}$ in $V_\pi$ defined by

$$\mathcal{F}(f)(\pi) := \pi(f) := \frac{1}{|G|} \int_G f(g)\pi(g^{-1})dg := \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g^{-1}).$$

We recall the statement of the Plancherel theorem for a function $f \in L^1_b(G,K)$

$$f(g) = \frac{1}{|G|} \sum_{\pi \in \hat{G}^\Phi} d_\pi \text{trace}(\pi(f)\pi(g));$$

here $g \in G$ arbitrary and $d_\pi := \text{dim } V_\pi$.

For any complex valued function $f$ on $G$, let $|\text{supp}(f)|$ denote the number of elements of the support of $f$. That is, the number of points of $G$ where $f$ takes nonzero values.

**Proposition 3 (Heisenberg Uncertainty Principle).** For any nonzero function $f \in L^1_b(G,K)$ we have

$$|\text{supp}(f)| (\sum_{\pi \in \text{supp}(\mathcal{F}(f))} d_\pi) \geq |G|.$$ 

Here $\text{supp}(\mathcal{F}(f))$ is the subset of $\hat{G}^\Phi$ where $\mathcal{F}(f)$ does not vanish.
Proof: For any function $f$ on $G$ we recall that
\[ \|f\|_2^2 = \sum_{x \in G} |f(x)|^2; \quad \|f\|_\infty = \max_{x \in G} |f(x)|; \quad \|f\|_2^2 \leq \|f\|_\infty^2 |\text{supp}(f)| \quad (*) \]

From now on, we fix a $G$–invariant inner product on $V_\pi$. Then $T^*$ denotes the adjoint of a linear operator $T$ on $V_\pi$ with respect to this inner product. Also $\|T\|$ denotes the Hilbert-Schmidt norm on $\text{End} V_\pi$ defined by $\text{trace}(TS^*)$, for $S, T \in \text{End} V_\pi$.

Since $f \in L_1^1(G, K)$, as we pointed out before, the Plancherel Theorem says that we have that $\text{supp}(f)$ is contained in $\hat{G}^\Phi$ and that
\[ f(x) = \frac{1}{G} \sum_{\pi \in \hat{G}^\Phi} d_\pi \text{trace}(\pi(f)\pi(x)). \]

The Cauchy–Schwarz inequality applied to the Hilbert-Schmidt inner product says that the first of the two following inequalities is true,
\[ \text{trace}(\pi(f)\pi(x)) \leq \|\pi(f)\|\|\pi(x)\| \leq \|\pi(f)\|, \]
the second inequality follows from the fact that $\|T\| = 1$ for a unitary operator.

Putting together the last two statements we get
\[ \|f\|_\infty \leq \frac{1}{G} \sum_{\pi \in \hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\| \]

The classical Cauchy–Schwarz inequality and the fact that $d_\pi = d_{\pi^*}^{\frac{1}{2}} d_{\pi^*}^{\frac{1}{2}}$ imply that
\[ \|f\|_\infty^2 \leq \frac{1}{|G|^2} \sum_{\pi \in \hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\|^2 \sum_{\pi \in \text{supp}(\mathcal{F}(f))} d_\pi. \]

Now the $L^2$–version of Plancherel Theorem says that
\[ \|f\|_2^2 = \frac{1}{|G|} \sum_{\pi \in \hat{G}^\Phi} d_\pi \|\mathcal{F}(f)(\pi)\|^2. \]

Therefore,
\[ \|f\|_\infty^2 \leq \frac{1}{|G|} \|f\|_2^2 \sum_{\pi \in \hat{G}^\Phi} d_\pi. \]
Since $f$ is nonzero, we apply (*) to the above inequality and get the desired result. \(\square\)

References

[1] Galina, E. and Vargas, J., Eigenvalues and eigenspaces for the twisted Dirac operator over $SU(n, 1)$ and $Spin(2n, 1)$, Trans. Amer. Math. Soc., 345 (1994), 97-113.
[2] Helversen-Pasotto, A., Repr´esentation de Gelfand-Graev et identit´es de Barnes, Enseign. Math. 32 (1986), 57-77.
[3] Katz, N., Estimates for Soto-Andrade sums, J. reine angew. Math. 438 (1993), 143-161.
[4] Soto-Andrade, J., Geometrical Gel’fand Models, tensor quotients and Weil representations, Proc. Symp. Pure Maths., 47, AMS, Providence, 1987, 305-316.
[5] Soto-Andrade, J., Répresentations de certains groupes symplectiques, Mém. 55-56, Soc. Math. France, 1975.
[6] Varadarajan, Spherical functions, Springer, Berlin.

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