Relativistic Spectral Properties of Landau Operator with \(\delta\)– and \(\delta'\)– cylinder interactions

G. Honnouvo and M. N. Hounkonnou

Unité de Recherche en Physique Théorique (URPT)
Institut de Mathématiques et de Sciences Physiques (IMSP)
01 B.P. 2628 Porto-Novo, BENIN,
International Chair in Mathematical Physics and Applications (ICMPA)
01 BP 2628 Porto-Novo, BENIN,
g_honnouvo@yahoo.fr, hounkonnou1@yahoo.fr

Abstract

Using the theory of self-adjoint extensions, we study the relativistic spectral properties of the Landau operator with \(\delta\) and \(\delta'\) interactions on a cylinder of radius \(R\) for a charged spin particle system, formally given by the Hamiltonian\
\[ H^G_B = (p - A)^2 \mathbb{1} + \sigma \cdot B + G V(r), \]
acting in \(L^2(\mathbb{R}^2) \otimes \mathbb{C}^2\). \(G\) a scalar \(2 \times 2\) real matrix. \(\mathbb{1}\) is the identity matrix. The potential vector has the form \(A = (B/2)(-y, x)\) and \(B > 0\).

1 Introduction

Over the last decades, there have been major research efforts in studying Schrödinger operators to describe properties of charged particles in magnetic systems (see \(^1\), \(^2\) and references therein). Gesztesy et al.\(^3\) have devoted a communication to the point interaction in magnetic field systems in the nonrelativistic case. More recently \(^4\), we have studied the nonrelativistic case of this model. In this paper, we deal with the relativistic models for a two-dimensional quantum Hamiltonian describing a charged spin particle in a constant and uniform magnetic field superimposed to \(\delta\) and \(\delta'\)-interaction on a cylinder of radius \(R\).

We consider an electron confined in the \(x-y\) plane and subjected to a uniform magnetic field perpendicular to the plane, using the symmetric gauge vector potential \(A = (B/2)(-y, x)\), \(B > 0\).

We provide a complete spectral analysis of the given operator and deduce helpful properties when this operator is perturbed by \(\delta\) or \(\delta'\) interaction on the cylinder of radius \(R\). We recover the results obtained in \(^3\), \(^4\) for the point interaction as a particular case of our study.
First, using the von Neumann theory of self-adjoint (s.a.) extension of linear symmetric operators\textsuperscript{5}, we characterized the self-adjoint extension of this operator. The relativistic type Hamiltonian is formally expressed as:
\[
\hat{H}_B^G = (p - A)^2 \mathbb{I} + \begin{pmatrix} B + \alpha V(r) & 0 \\ 0 & -B + \beta V(r) \end{pmatrix},
\] (1.1)
where
\[
V(r) = \begin{cases} \xi \delta(r - R), \\ \xi' \delta'(r - R), \end{cases} \quad \text{with } \xi \in \mathbb{R}, \ R > 0.
\] (1.2)

The paper is organized as follows: In section 2, we investigate the relativistic model of $\delta$. In section 3, the relativistic model of $\delta'$. Finally, in section 4, we conclude with some remarks.

## 2 The model of the relativistic $\delta$-cylinder interaction

In this section, we derive the properties of the relativistic quantum Hamiltonian describing a charged spin particle in a constant magnetic field $\mathbf{B}$ coupled with $\delta$ interaction on the cylinder of radius $R$, given by
\[
\hat{H}_B^G = (p - A)^2 \mathbb{I} + \sigma \cdot \mathbf{B} + G \delta(R - r),
\] (2.1)
where
\[
\sigma = (\sigma^1, \sigma^2, \sigma^3),
\] (2.2)
and the $\sigma^i$ are the Pauli matrices defined by
\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.
\] (2.3)
\alpha and $\beta$ are real numbers.

Let us consider the operator
\[
\hat{H}_B = (p - A)^2 \mathbb{I} + \sigma \cdot \mathbf{B},
\] (2.4)
and the closed symmetric operator
\[
\hat{\mathcal{H}}_B = \begin{pmatrix} (p - A)^2 + B & 0 \\ 0 & (p - A)^2 - B \end{pmatrix},
\] (2.5)
with the domain
\[ \mathcal{D}(\hat{H}_B) = \{ \psi \in H^{2,2}(\mathbb{R}^2) \otimes \mathbb{C}^2, \psi(S^R) = 0; \hat{H}_B \psi \in L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \}, \] (2.6)
where \( S^R = \{ x \in \mathbb{R}^2, |x| = R \} \) is a circle of radius \( R \) centered at the origin in \( \mathbb{R}^2 \), and \( H^{k,p}(\Omega) \) is the Sobolev space of indices \((k,p)\).

Let us now decompose the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \) as follows
\[ \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 = \left( L^2([0,\infty]) \otimes L^2(S^1) \right) \otimes \mathbb{C}^2, \] (2.7)
\( S^1 \) being the unit circle in \( \mathbb{R}^2 \).

Similarly to the nonrelativistic study, the following isometry is introduced in order to remove the weight factor \( r \) from the measure:
\[ \tilde{U} : \left\{ \begin{array}{l} L^2((0,\infty); r dr) \longrightarrow L^2((0,\infty); dr) \equiv L^2((0,\infty)) \\ f \mapsto (\tilde{U}f)(r) = \sqrt{r}f(r). \end{array} \right. \] (2.8)
Then, we get the following decomposition of the Hilbert space \( \mathcal{H} \):
\[ \mathcal{H} = \bigoplus_{m=-\infty}^{m=+\infty} \bigoplus_{m'=-\infty}^{m'=-\infty} \left[ \left( \tilde{U}^{-1}(L^2([0,\infty])) \otimes \left[ e^{im\phi} \over \sqrt{2\pi} \right] \right) \bigoplus \left( \tilde{U}^{-1}(L^2([0,\infty])) \otimes \left[ e^{im'\theta} \over \sqrt{2\pi} \right] \right) \right], \] (2.9)
so that the operator \( \hat{H}_B \) writes
\[ \hat{H}_B = \bigoplus_{m=-\infty}^{m=+\infty} \bigoplus_{m'=-\infty}^{m'=+\infty} \tilde{U}^{-1} \hat{H}_{B,m,m'} \tilde{U} \otimes \mathbb{I}, \] (2.10)
where the radial part \( \hat{H}_{B,m,m'} \) is defined by
\[ \hat{H}_{B,m,m'} = \begin{pmatrix} h^1_{B,m} & 0 \\ 0 & h^2_{B,m'} \end{pmatrix}, \] (2.11)
\[ h^1_{B,m} = -\frac{d^2}{dr^2} + \left( \frac{m}{r} + \frac{B}{2r} \right)^2 - \frac{1}{4r^2} + B \]
and
\[ h^2_{B,m'} = -\frac{d^2}{dr^2} + \left( \frac{m'}{r} + \frac{B}{2r} \right)^2 - \frac{1}{4r^2} - B \]
with the domain
\[ \mathcal{D}(\hat{H}_{B,m,m'}) = \left\{ f \in \left( L^2([0,\infty]; dr) \cap H^{2,2}_{\text{loc}}([0,\infty]) \right) \otimes \mathbb{C}^2; f(R_\pm) = 0; \right. \left. \hat{H}_{B,m,m'} f \in L^2((0,\infty)) \otimes \mathbb{C}^2 \right\}, \quad m \in \mathbb{Z}, m' \in \mathbb{Z}. \] (2.12)
The adjoint operator $\hat{H}_{B,m,m'}^*$ of $\hat{H}_{B,m,m'}$ is defined by (2.11) in the domain

$$D(\hat{H}_{B,m,m'}^*) = \left\{ f \in \left( L^2([0, \infty[, dr) \cap H^{2,2}_{loc}([0, \infty[ \setminus \{ R \}) \right) \otimes \mathbb{C}^2 : f(R_+) = f(R_-) \equiv f(R); \right.$$

$$\hat{H}_{B,m,m'} f \in L^2((0, \infty)) \otimes \mathbb{C}^2 \right\}, \quad m \in \mathbb{Z}, \ m' \in \mathbb{Z} \tag{2.13}$$

The indicial equation reads

$$\hat{H}_{B,m,m'}^* \tilde{\psi}_{B,m,m'}(k, r) = k \tilde{\psi}_{B,m,m'}(k, r),$$

or equivalently

$$\hat{H}_{B,m,m'}^* \begin{pmatrix} \phi^1_{B,m,m'} \\ \phi^2_{B,m,m'} \end{pmatrix} = k \begin{pmatrix} \phi^1_{B,m,m'} \\ \phi^2_{B,m,m'} \end{pmatrix}, \tag{2.14}$$

$$k \in \mathbb{C} \setminus \mathbb{R},$$

which has two solutions:

$$\tilde{\psi}^1_{B,m,m'} = \begin{pmatrix} \phi^1_{B,m,m'} \\ 0 \end{pmatrix}, \tag{2.15}$$

and

$$\tilde{\psi}^2_{B,m,m'} = \begin{pmatrix} 0 \\ \phi^2_{B,m,m'} \end{pmatrix}, \tag{2.16}$$

where

$$\phi^1_{B,m,m'}(k, r) = \begin{cases} N^1_{B,m} G^{(1)}_{B,m}(k, R) \times F^{(1)}_{B,m}(k, r); & r \leq R, \\ N^1_{B,m} F^{(1)}_{B,m}(k, R) \times G^{(1)}_{B,m}(k, r); & r \geq R, \end{cases} \tag{2.17}$$

$$\phi^2_{B,m,m'}(k, r) = \begin{cases} N^2_{B,m} G^{(2)}_{B,m}(k, R) \times F^{(2)}_{B,m}(k, r); & r \leq R, \\ N^2_{B,m} F^{(2)}_{B,m}(k, R) \times G^{(2)}_{B,m}(k, r); & r \geq R, \end{cases} \tag{2.18}$$

$$F^{(1)}_{B,m}(k, r) = r^{1/2 + |m|} e^{-\frac{1}{4} B r^2} F_1 \left( \frac{1}{2}(|m| + m + 1 - \frac{k-B}{B} |m| + 1; \frac{B}{2} r^2 \right), \tag{2.19}$$

$$G^{(1)}_{B,m}(k, r) = r^{1/2 + |m|} e^{-\frac{1}{4} B r^2} U \left( \frac{1}{2}(|m| + m + 1 - \frac{k-B}{B} |m| + 1; \frac{B}{2} r^2 \right), \tag{2.19}$$

$$F^{(2)}_{B,m}(k, r) = r^{1/2 + |m'|} e^{-\frac{1}{4} B r^2} F_1 \left( \frac{1}{2}(|m'| + m' + 1 - \frac{k+B}{B} |m'| + 1; \frac{B}{2} r^2 \right), \tag{2.20}$$

$$G^{(2)}_{B,m}(k, r) = r^{1/2 + |m'|} e^{-\frac{1}{4} B r^2} U \left( \frac{1}{2}(|m'| + m' + 1 - \frac{k+B}{B} |m'| + 1; \frac{B}{2} r^2 \right), \tag{2.20}$$

$$N^1_{B,m} = \left( ||P^1_{B,m}(k)||_{L^2([0, \infty[)} \right)^{-1}, \tag{2.21}$$
\[ N_{B,m'}^2 = \left( \| P_{B,m'}(k) \|_{L^2[0,\infty)} \right)^{-1}, \]  

(2.22)

\[ P_{B,m}(k,r) = \begin{cases} G_{B,m}(k,R) \times F_{B,m}(k,r) ; & r \leq R, \\ F_{B,m}(k,R) \times G_{B,m}(k,r) ; & r \geq R, \end{cases} \]  

(2.23)

and

\[ P_{B,m'}(k,r) = \begin{cases} G_{B,m'}(k,R) \times F_{B,m'}(k,r) ; & r \leq R, \\ F_{B,m'}(k,R) \times G_{B,m'}(k,r) ; & r \geq R. \end{cases} \]  

(2.24)

Since the indicial equation admits two solutions, \( \hat{H}_{B,m,m'} \) has deficiency indices \((2, 2)\) and, consequently, all self-adjoint (s.a) extensions of \( \hat{H}_{B,m,m'} \) are given by a 4-parameter family of (s.a.) operators. In particular, we define here

\[ H_{B,m,m'}^{G,m,m'} = \begin{pmatrix} h_{B,m}^1 & 0 \\ 0 & h_{B,m'}^2 \end{pmatrix} \]  

(2.25)

with the domain

\[ \mathcal{D}(H_{B,m,m'}^{G,m,m'}) = \left\{ f \in \left( L^2([0, \infty), [0, \infty) \right] \right. \bigotimes \mathbb{C}^2 ; f(R+) = f(R-) \equiv f(R); \\ f'(R+) - f'(R-) = G_{m,m'}f(R); H_{B,m,m'}^{G,m,m'}f \in L^2((0, \infty)) \bigotimes \mathbb{C}^2 \right\}, \]  

(2.26)

and

\[ G_{m,m'} = \begin{pmatrix} \alpha_m & 0 \\ 0 & \beta_{m'} \end{pmatrix}. \]  

(2.27)

The case \( G_{m,m'} = 0 \) coincides with the free kinetic energy Hamiltonian \( H_{B,m,m'}^G \) for fixed quantum numbers \( m, m' \).

Let \( G = \{ G_{m,m'} \}_{m, m' \in \mathbb{Z}} \) and introduce in \( L^2(\mathbb{R}^2) \bigotimes \mathbb{C}^2 \) the operator

\[ H_B^G = \bigoplus_{m=-\infty}^{m=+\infty} \bigoplus_{m'=-\infty}^{m'=+\infty} \tilde{U}^{-1} H_{B,m,m'}^{G,m,m'} \tilde{U} \bigotimes \mathbb{1}. \]  

(2.28)

By definition, \( H_B^G \) is the rigorous mathematical formulation of the formal expression (2.1). Actually, it provides a slight generalization of (2.1), since \( G \) may depend on \( m \in \mathbb{Z} \) and \( m' \in \mathbb{Z} \).
Theorem 2.1 (i) The resolvent of $H_{B,m,m'}^{G,m,m'}$ is given by

$$(H_{B,m,m'}^{G,m,m'} - k)^{-1} = (H_{B,m,m'}^{O} - k)^{-1} + \sum_{i,j=1}^{2} \mu_{i,j}(k) \left( \bar{\Psi}_{B,m,m'}^{j}(k), \bar{\Psi}_{B,m,m'}^{i}(k) \right),$$

where

$$k \in \rho(H_{B,m,m'}^{G,m,m'}) \cap \rho(H_{B,m,m'}^{O}), \quad m \in \mathbb{Z}, \quad m' \in \mathbb{Z},$$

(2.29)

and

$$\mu_{1,1}(k) = \frac{\alpha_{m}^{1}}{N_{B,m}^{1} \left( G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - \alpha_{m}G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) \right)},$$

(2.30)

$$\mu_{1,2}(k) = 0,$$

(2.31)

$$\mu_{2,1}(k) = 0$$

(2.32)

and

$$\mu_{2,2}(k) = \frac{\beta_{m'}}{N_{B,m'}^{2} \left( G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - \beta_{m'}G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) \right)},$$

(2.33)

The Green function $G_{B,m,m'}^{O}(k,r,r')$ of $H_{B,m,m'}^{O}$ has the form

$$(H_{B,m,m'}^{O} - k)^{-1} = G_{B,m,m'}^{O}(k,r,r') = \begin{pmatrix} g_{m,m'}^{1,1}(k,r,r') & g_{m,m'}^{1,2}(k,r,r') \\ g_{m,m'}^{2,1}(k,r,r') & g_{m,m'}^{2,2}(k,r,r') \end{pmatrix},$$

(2.34)

where

$$g_{m,m'}^{1,2}(k,r,r')(r,r') = g_{m,m'}^{2,1}(k,r,r') = 0,$$

(2.35)

$$g_{m,m'}^{1,1}(k,r,r') = \begin{cases} N_{B,m}^{1}G_{B,m}^{(1)}(k,r) \times F_{B,m}^{(1)}(k,r') & \text{if } r' \leq r, \\
N_{B,m}^{1}F_{B,m}^{(1)}(k,r) \times G_{B,m}^{(1)}(k,r') & \text{if } r' \geq r \end{cases}$$

(2.36)

and

$$g_{m,m'}^{2,2}(k,r,r') = \begin{cases} N_{B,m'}^{2}G_{B,m'}^{(2)}(k,r) \times F_{B,m'}^{(2)}(k,r') & \text{if } r' \leq r, \\
N_{B,m'}^{2}F_{B,m'}^{(2)}(k,r) \times G_{B,m'}^{(2)}(k,r') & \text{if } r' \geq r \end{cases}$$

(2.37)

We note that $g_{m,m'}^{1,1}(k,R,r) = \phi_{B,m}^{1}(k,r)$ and $g_{m,m'}^{2,2}(k,R,r) = \phi_{B,m}^{2}(k,r)$.

(ii) The resolvent of $H_{B}^{O}$ is given by
\[
(H^G_B - k)^{-1} = (H^O_B - k)^{-1} + \bigoplus_{m=-\infty}^{m=+\infty} \bigoplus_{m'=-\infty}^{m=+\infty} 2 \sum_{i,j=1}^{\mu_{i,j}(k)} \left( |.\tilde{\Psi}_{B,m,m'}(k)| \right)^{-1} \tilde{\Psi}_{B,m,m'}(k),
\]

\[k \in \rho(H^G_B) \cap \rho(H^O_B), \ m \in \mathbb{Z}, \ m' \in \mathbb{Z}. \tag{2.38}\]

\textbf{Proof:} Since \( \hat{H}_{B,m,m'} \) has deficiency indices (2.2), it follows from Krein’s formula \(^6\) that the resolvent of \( H^{G,m,m'}_{B,m,m'} \) is given by

\[
(H^{G,m,m'}_{B,m,m'} - k)^{-1} = (H^0_{B,m,m'} - k)^{-1} + \sum_{i,j=1}^{\mu_{i,j}(k)} \left( \tilde{\Psi}_{B,m,m'}^{i}(k), . \right) \tilde{\Psi}_{B,m,m'}^{j}(k),
\]

\[k \in \rho(H^{G,m,m'}_{B,m,m'}) \cap \rho(H^O_{B,m,m'}), \ m \in \mathbb{Z}, \ m' \in \mathbb{Z}. \tag{2.39}\]

Since \( G^{G,m,m'}_{B,m,m'}(k, r, r') \) must satisfy the following equation:

\[
(H^O_{B,m,m'} - k)G^{G,m,m'}_{B,m,m'}(k, r, r') = \begin{pmatrix}
\delta(r - r') & 0 \\
0 & \delta(r - r')
\end{pmatrix}, \tag{2.40}
\]

one has

\[
(h^1_{B,m} - z)g^{1,1}_{m,m'}(k, r, r') = \delta(r - r'), \tag{2.41}
\]

\[
g^{1,2}_{m,m'}(k, r, r') = g^{2,1}_{m,m'}(k, r, r') = 0, \tag{2.42}
\]

\[
(h^2_{B,m} - z)g^{2,2}_{m,m'}(k, r, r') = \delta(r - r'). \tag{2.43}
\]

which implies that

\[
g^{1,1}_{m,m'}(k, r, r') = \begin{cases} N^1_{B,m} G^{(1)}_{B,m}(k, r) \times F^{(1)}_{B,m}(k, r') & ; r' \leq r, \\
N^1_{B,m} F^{(1)}_{B,m}(k, r) \times G^{(1)}_{B,m}(k, r') & ; r' \geq r, \end{cases} \tag{2.44}
\]

and

\[
g^{2,2}_{m,m'}(k, r, r') = \begin{cases} N^2_{B,m} G^{(2)}_{B,m'}(k, r) \times F^{(2)}_{B,m'}(k, r') & ; r' \leq r, \\
N^2_{B,m'} F^{(2)}_{B,m'}(k, r) \times G^{(2)}_{B,m'}(k, r') & ; r' \geq r. \end{cases} \tag{2.45}
\]

For the determination of \( \mu_{i,j}(k) \), we proceed as follows. Let \( g \in L^2([0, \infty]) \) and define the function

\[
\chi_{m,m'}(k, r) = \left( (H^{G,m,m'}_{B,m,m'} - k)^{-1} g \right)(r).
\]
Since $\chi_{m,m'} \in D(H^{G_{m,m'}}_{B,m,m'})$, it follows that $\chi_{m,m'}$ should satisfy the boundary conditions of $D(H^{G_{m,m'}}_{B,m,m'})$, the implementation of which gives

$$\mu_{1,1}(k) = \frac{\alpha_m}{N_{B,m}^{1}\left(G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - \alpha_m G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R)\right)},$$

(2.46)

$$\mu_{1,2}(k) = 0,$$

(2.47)

$$\mu_{2,1}(k) = 0$$

(2.48)

and

$$\mu_{2,2}(k) = \frac{\beta_{m'}}{N_{B,m'}^{2}\left(G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - \beta_{m'} G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R)\right)}.$$

(2.49)

Inserting (2.46), (2.47) (2.48) and (2.49) into ((2.39)), we deduce the expression (2.29). Eq. ((2.38) follows from (2.28) and ((2.29) .

Spectral properties of $H^{G_{m,m'}}_{B,m,m'}$ are provided by the following theorem where $\sigma_{ess}(\cdot)$, $\sigma_{sc}(\cdot)$ and $\sigma_{p}(\cdot)$ represent the same properties as mentioned.

**Theorem 2.2 :** For all $\alpha_m \in (-\infty, \infty)$ and $\beta_{m'} \in (-\infty, \infty)$, we have the following results

$$\sigma_{ess}(H^{G_{m,m'}}_{B,m,m'}) = \emptyset,$$

(2.50)

$$\sigma_{sc}(H^{G_{m,m'}}_{B,m,m'}) = \emptyset,$$

(2.51)

$$\sigma_{p}(H^{G_{m,m'}}_{B,m,m'}) = \left\{ E \in \mathbb{R} / G_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R) - G_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R) - \alpha_m G_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R) = 0 \right\}.$$

(2.52)

The negative bound states are related to the eigenvalues of $H^{G_{m,m'}}_{B,m,m'}$ obtained from the equation

$$\text{det}(\mu_{i,j}(E))^{-1} = 0; \ E < 0,$$

(2.53)

which has at most two solutions $E_0 < 0$. 

8
Proof: We know that \( \sigma_{ess}(H^{O}_{B,m,m'}) = \emptyset \). Using Weyl’s theorem 8, we have
\[
\sigma_{ess}(H_{B,m,m'}^{G,m'}) = \sigma_{ess}(H_{B,m,m'}^{O}) = \emptyset.
\] (2.54)

Eq.(2.38) implies that the point spectrum is determined by
\[
\sigma_{p}(H_{B,m,m'}^{G,m'}) = \left\{ E \in \mathbb{R} \bigg/ \text{det}(\mu_{i,j}(E))^{-1} = 0 \right\}
\]
\[
= \left\{ E \in \mathbb{R} / G^{(1)}_{B,m}(E,R)F^{(1)}_{B,m}(E,R) - G^{(1)}_{B,m}(E,R)F^{(1)}_{B,m}(E,R) - \alpha_{m}G^{(1)}_{B,m}(E,R)F^{(1)}_{B,m}(E,R) = 0 \right\}
\]
or \[
G^{(2)}_{B,m}(E,R)F^{(2)}_{B,m}(E,R) - G^{(2)}_{B,m}(E,R)F^{(2)}_{B,m}(E,R) - \beta_{m}G^{(2)}_{B,m}(E,R)F^{(2)}_{B,m}(E,R) = 0 \right\}.
\]
which proves (2.52). The last part of this theorem follows from the statement of \( \tilde{5} \) [theorem 1, page 116].

3 The model of relativistic \( \delta' \)- cylinder interaction

In this section, we deal with the relativistic quantum Hamiltonian of a charged spin particle in a constant uniform magnetic field \( B \) coupled with \( \delta' \) interaction on the cylinder of radius \( R \), given by
\[
H_{B}^{G} = (p - A)^{2} \mathbbm{1} + \sigma \cdot B + G\delta'(R - r).
\] (3.1)

Let us consider the closed symmetric operator, formally defined by
\[
\hat{H}_{B} = (p - A)^{2} \mathbbm{1} + \sigma \cdot B,
\] (3.2)

with the domain
\[
\mathcal{D}(\hat{H}_{B}) = \{ \psi \in H^{2,2}(\mathbb{R}^{2}) \otimes \mathcal{C}^{2}, \psi'(S^{R}) = 0; \hat{H}_{B}\psi \in L^{2}(\mathbb{R}^{2}) \otimes \mathcal{C}^{2} \}.
\] (3.3)

Performing the same decomposition (2.9) and (2.10) for the Hilbert space \( \mathcal{H} \) and for the operator \( \hat{H}_{B} \), we readily recover Eq.(2.11) for the operator \( \hat{H}_{B,m,m'} \):
\[
\hat{H}_{B,m,m'} = \begin{pmatrix}
h^{1}_{B,m} & 0 \\
0 & h^{2}_{B,m'}
\end{pmatrix}
\] (3.4)

with
$$D(\hat{H}_{B,m,m'}) = \left\{ f \in \left( L^2([0, \infty[\times dr) \cap H^2_0([0, \infty[) \right) \otimes \mathbb{C}^2 ; f'(R_{\pm}) = 0 ; \hat{H}_{B,m,m'} f \in L^2((0, \infty)) \otimes \mathbb{C}^2 \right\}, m, m' \in \mathbb{Z}. \right.$$  

Here the boundary conditions require the derivative \( f' \) to vanish at the circle of radius \( R \) for the \( \delta' \)-interaction instead of the continuity of the function \( f \) at \( r = R \) in the case of \( \delta \)-interaction (see Eq. (2.12)).

The adjoint operator \( \hat{H}^*_B,m,m' \) of \( \hat{H}_{B,m,m'} \) is defined by

$$\hat{H}^*_B,m,m' = \begin{pmatrix} h^1_{B,m} & 0 \\ 0 & h^2_{B,m'} \end{pmatrix} \quad (3.6)$$

with the domain

$$D(\hat{H}^*_B,m) = \left\{ f \in \left( L^2([0, \infty[\times dr) \cap H^2_0([0, \infty[\setminus\{R\}) \right) \otimes \mathbb{C}^2 ; f'(R_{\pm}) = f'(R) ; \hat{H}_{B,m,m'} f \in L^2((0, \infty)) \otimes \mathbb{C}^2 \right\}, m, m' \in \mathbb{Z}. \right.$$  

The indicial equation reads

$$\hat{H}^*_B,m,m' \tilde{\psi}^1_{B,m,m'}(k,r) = k \tilde{\psi}^1_{B,m,m'}(k,r),$$

or equivalently

$$\hat{H}^*_B,m,m' \begin{pmatrix} \phi^1_{B,m} \\ \phi^2_{B,m'} \end{pmatrix} = k \begin{pmatrix} \phi^1_{B,m} \\ \phi^2_{B,m'} \end{pmatrix}, \quad (3.8)$$

\( k \in \mathbb{C} \setminus \mathbb{R} \), which has two solutions:

$$\tilde{\psi}^1 = \begin{pmatrix} \phi^1_{B,m} \\ 0 \end{pmatrix} \quad (3.9)$$

and

$$\tilde{\psi}^2 = \begin{pmatrix} 0 \\ \phi^2_{B,m'} \end{pmatrix} \quad (3.10)$$

where

$$\phi^1_{B,m}(k,r) = \begin{cases} M^1_{B,m} [G^1_{B,m}(k,r)]'_{r=R} \times F^1_{B,m}(k,r) ; & r < R , \\ M^1_{B,m} [F^1_{B,m}(k,r)]'_{r=R} \times G^1_{B,m}(k,r) ; & r > R , \end{cases} \quad (3.11)$$

$$\phi^2_{B,m'}(k,r) = \begin{cases} M^2_{B,m'} [G^2_{B,m'}(k,r)]'_{r=R} \times F^2_{B,m'}(k,r) ; & r < R , \\ M^2_{B,m'} [F^2_{B,m'}(k,r)]'_{r=R} \times G^2_{B,m'}(k,r) ; & r > R , \end{cases} \quad (3.12)$$
\[ F_{B,m}^{(1)}(k, r) = r^{1/2+|m|}e^{-\frac{1}{2}Br^2} F_1 \left( \frac{1}{2}(|m| + m + 1 - \frac{k-B}{B}), |m| + 1; \frac{B}{2}r^2 \right), \]  
\[ G_{B,m}^{(1)}(k, r) = r^{1/2+|m|}e^{-\frac{1}{2}Br^2} U \left( \frac{1}{2}(|m| + m + 1 - \frac{k-B}{B}), |m| + 1; \frac{B}{2}r^2 \right), \]  
\[ F_{B,m'}^{(2)}(k, r) = r^{1/2+|m'|}e^{-\frac{1}{2}Br^2} F_1 \left( \frac{1}{2}(|m'| + m' + 1 - \frac{k+B}{B}), |m'| + 1; \frac{B}{2}r^2 \right), \]  
\[ G_{B,m'}^{(2)}(k, r) = r^{1/2+|m'|}e^{-\frac{1}{2}Br^2} U \left( \frac{1}{2}(|m'| + m' + 1 - \frac{k+B}{B}), |m'| + 1; \frac{B}{2}r^2 \right), \]  
\[ M_{B,m}^1 = \left( ||P_{B,m}^1(k)||_{L^2([0,\infty))} \right)^{-1}, \]  
\[ M_{B,m'}^2 = \left( ||P_{B,m'}^2(k)||_{L^2([0,\infty))} \right)^{-1}, \]  
with
\[ P_{B,m}^1(k, r) = \begin{cases} [G_{B,m}^{(1)}(k, r)]_{r=R} \times F_{B,m}^{(1)}(k, r); & r < R, \\ [F_{B,m}^{(1)}(k, r)]_{r=R} \times G_{B,m}^{(1)}(k, r); & r > R, \end{cases} \]  
and
\[ P_{B,m'}^2(k, r) = \begin{cases} [G_{B,m'}^{(2)}(k, r)]_{r=R} \times F_{B,m'}^{(2)}(k, r); & r < R, \\ [F_{B,m'}^{(2)}(k, r)]_{r=R} \times G_{B,m'}^{(2)}(k, r); & r > R. \end{cases} \]

The indicial equation also admits two solutions and \( \hat{H}_{B,m,m'} \) has deficiency indices \((2, 2)\). Consequently, all self-adjoint (s.a) extensions of \( \hat{H}_{B,m,m'} \) are also given by a 4-parameter family of (s.a) operators. In particular, we define here
\[ H_{B,m,m'}^{G,m'} = \begin{pmatrix} h_{B,m}^1 & 0 \\ 0 & h_{B,m'}^2 \end{pmatrix} \]  
with the domain
\[ \mathcal{D}(H_{B,m,m'}^{G,m'}) = \left\{ f \in \left( L^2([0,\infty[, dr) \cap H_{loc}^{2,2}([0,\infty[-\{R}\right) \right) \bigotimes \mathbb{C}^2; \ f'(R+) = f'(R_-) \equiv f'(R); \right. \\
\left. f(R+) - f(R_-) = G_{m,m'}f'(R); \ \hat{H}_{B,m,m'}f \in L^2((0,\infty)) \bigotimes \mathbb{C}^2 \right\}, \]
\[ m, m' \in \mathbb{Z}, \ G_{m,m'} \text{ defined as previously}. \]  

As expected, the continuity conditions are required here for the derivative function \( f' \) at the point \( r = R \). This result respects the situation encountered in the case of the nonrelativistic \( \delta' \) cylinder interaction.
The case $G_{m,m'} = O$ also coincides with the free kinetic energy Hamiltonian $H_{B,m,m'}^O$ for fixed quantum numbers $m, m'$.

Let $G = \{G_{m,m'}\}_{m,m' \in \mathbb{Z}}$ and introduce in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ the operator

$$H_B^G = \bigoplus_{m=+\infty}^{m=-\infty} \bigoplus_{m'=+\infty}^{m'=-\infty} U^{-1}H_{B,m,m'}^O U \otimes \mathbb{1}.$$  

(3.21)

By definition, $H_B^G$ is the rigorous mathematical formulation of the formal expression (3.1). Actually, it provides a slight generalization of (3.1), since $G$ may depend on $m, m' \in \mathbb{Z}$.

**Theorem 3.1** (i) The resolvent of $H_{B,m,m'}^G$ is given by

$$(H_{B,m,m'}^G - k)^{-1} = (H_{B,m,m'}^O - k)^{-1} + \sum_{i,j=1}^{2} \mu_{i,j}(k) \left( \tilde{\Psi}_{B,m,m'}^i(k), \cdot \right) \tilde{\Psi}_{B,m,m'}^j(k),$$

$$k \in \rho(H_{B,m,m'}^G) \cap \rho(H_{B,m,m'}^O), \ m, m' \in \mathbb{Z},$$

(3.22)

where

$$\mu_{1,1}(k) = \frac{\alpha_m}{M_{B,m}^1 \left( G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - \alpha_m G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) \right)},$$

(3.23)

$$\mu_{1,2}(k) = 0,$$

(3.24)

$$\mu_{2,1}(k) = 0$$

(3.25)

and

$$\mu_{2,2}(k) = \frac{\beta_{m'}}{M_{B,m'}^2 \left( G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - \beta_{m'} G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) \right)}.$$

(3.26)

The Green function $G_{B,m,m'}^O(k,r,r')$ of $H_{B,m,m'}^O$ has the form

$$G_{B,m,m'}^O(k,r,r') = \begin{pmatrix} g_{m,m'}^{1,1}(k,r,r') & g_{m,m'}^{1,2}(k,r,r') \\ g_{m,m'}^{2,1}(k,r,r') & g_{m,m'}^{2,2}(k,r,r') \end{pmatrix}.$$  

(3.27)

Since $G_{B,m,m'}^O(k,r,r')$ must satisfy the following equation:

$$(H_{B,m,m'}^O - k)G_{B,m,m'}^O(k,r,r') = \begin{pmatrix} \delta(r - r') & 0 \\ 0 & \delta(r - r') \end{pmatrix},$$

(3.28)

one has
\[(h_{B,m}^2 - z)g_{m,m'}^{1,1}(k, r, r') = \delta(r - r'), \quad (3.29)\]

\[g_{m,m'}^{1,2}(k, r, r') = g_{m,m'}^{2,1}(k, r, r') = 0, \quad (3.30)\]

\[(h_{B,m'}^2 - z)g_{m,m'}^{2,2}(k, r, r') = \delta(r - r'), \quad (3.31)\]

and we have

\[
g_{m,m'}^{1,1}(k, r, r') = \begin{cases} M_1 B,m G_{B,m}(k, r) \times F_{B,m}(k, r') & ; r' < r, \\ M_2 B,m F_{B,m}(k, r) \times G_{B,m}(k, r') & ; r' > r. \end{cases} \quad (3.32)\]

and

\[
g_{m,m'}^{2,2}(k, r, r') = \begin{cases} M_1 B,m G_{B,m'}(k, r) \times F_{B,m'}(k, r') & ; r' < r, \\ M_2 B,m F_{B,m'}(k, r) \times G_{B,m'}(k, r') & ; r' > r. \end{cases} \quad (3.33)\]

We note that \(g_{m,m}(k, R, r) = \phi_{B,m,m'}^1(k, r)\) and \(g_{m,m'}(k, r) = \phi_{B,m,m'}^2(k, r)\).

(ii) The resolvent of \(H_B^G\) is given by

\[
(H_B^G - k^2)^{-1} = (H_B^O - k^2)^{-1} + \bigoplus_{m=-\infty}^{m=+\infty} \bigoplus_{m'=-\infty}^{m'=+\infty} \sum_{i,j=1}^{2} \mu_{i,j}(k) \left(|.|^{-1} \tilde{\Psi}_{B,m,m'}^i(k), .| \right) |.-1 \tilde{\Psi}_{B,m,m'}^j(k), \\
k \in \rho(H_B^G) \cap \rho(H_B^O), \quad m, m' \in \mathbb{Z}. \quad (3.34)\]

**Proof:** Since \(H_{B,m,m'}\) has deficiency indices \((2.2)\), it follows from Krein’s formula \(^6\) that the resolvent of \(H_{B,m,m'}^G\) is given by

\[
(H_{B,m,m'}^G - k)^{-1} = (H_{B,m,m'}^O - k)^{-1} + \sum_{i,j=1}^{2} \mu_{i,j}(k) \left(\tilde{\Psi}_{B,m,m'}^i(k), .\right) \tilde{\Psi}_{B,m,m'}^j(k), \\
k \in \rho(H_{B,m,m'}^G) \cap \rho(H_{B,m,m'}^O), \quad m, m' \in \mathbb{Z}. \quad (3.35)\]

For the determination of \(\mu_{i,j}(k)\), we proceed as follows. Let \(g \in L^2([0, \infty])\) and define the function

\[
\chi_{m,m'}(k, r) = \left((H_{B,m,m'}^G - k)^{-1} g\right)(r). \]

Since \(\chi_{m,m'} \in D(H_{B,m,m'}^G)\), it follows that \(\chi_{m,m'}\) should satisfy the boundary conditions of \(D(H_{B,m,m'}^G)\). The implementation of these boundary conditions gives
\[
\mu_{1,1}(k) = \frac{\alpha_m}{M_{B,m}^{1}\left(G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - G_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R) - \alpha_mG_{B,m}^{(1)}(k,R)F_{B,m}^{(1)}(k,R)\right)},
\]

(3.36)

\[
\mu_{1,2}(k) = 0,
\]

(3.37)

\[
\mu_{2,1}(k) = 0
\]

(3.38)

and

\[
\mu_{2,2}(k) = \frac{\beta_{m'}}{M_{B,m'}^{2}\left(G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R) - \beta_{m'}G_{B,m'}^{(2)}(k,R)F_{B,m'}^{(2)}(k,R)\right)}.
\]

(3.39)

Inserting (3.36), (3.37), (3.38) and (3.39) into (3.35), we deduce the expression (3.22). Eq. (3.34) follows from (3.21) and (3.35).

Spectral properties of \(H_{B,m,m'}^{G_{m,m'}}\) are provided by the following theorem.

**Theorem 3.2**: For all \(\alpha_m \in (-\infty, \infty)\) and \(\beta_{m'} \in (-\infty, \infty)\), we have the following results

\[
\sigma_{ess}(H_{B,m,m'}^{G_{m,m'}}) = \emptyset,
\]

(3.40)

\[
\sigma_{sc}(H_{B,m,m'}^{G_{m,m'}}) = \emptyset
\]

(3.41)

\[
\sigma_p(H_{B,m,m'}^{G_{m,m'}}) = \left\{ E \in \mathbb{R} / G_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R) - G_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R)
\right. \\
- \alpha_mG_{B,m}^{(1)}(E,R)F_{B,m}^{(1)}(E,R) = 0 \\
\left. or \ G_{B,m'}^{(2)}(E,R)F_{B,m'}^{(2)}(E,R) - G_{B,m'}^{(2)}(E,R)F_{B,m'}^{(2)}(E,R)
\right.
- \beta_{m'}G_{B,m'}^{(2)}(E,R)F_{B,m'}^{(2)}(E,R) = 0 \right\}.
\]

(3.42)

The negative eigenvalues of \(H_{B,m,m'}^{G_{m,m'}}\) are obtained from the equation

\[
\text{det}(\mu_{i,j}(E))^{-1} = 0; \ E < 0,
\]

(3.43)

which has at most two solutions \(E_0 < 0\).

**Proof:**

Similar to the proof of Theorem 2.2.
4 Remarks

Let us point out that the nonrelativistic results can be trivially deduced from the relativistic one setting the scalar parameter $\beta = 0$.

The results for the point interaction at the origin appear as a particular case of the cylinder interaction investigated here. Indeed, when $R \to 0$ in (2.1) and (3.1), we recover the boundary conditions corresponding to the nonrelativistic point interaction investigated in $^3$. Finally, let us recall that the properties for point interaction placed at any point $x$ could be found using the transformation relation $t_x H_\alpha t_{-x}$ given in $^6$, where $t_x$ is a translation application of vector $x$, $H_\alpha$ being the Hamiltonian perturbed by point interaction at the origin $r = 0$.

Acknowledgments: The authors thank the Belgian cooperation CUD - CIUF/UAC - IMSP, the Abdus Salam International Center for Theoretical Physics and the Conseil Regional Provence - Alpes - Cote d’Azur (France) for their financial support.
References

[1] V. A. Geiler, St. Petersburg Math. J. 3, 489-531, (1992).

[2] B. S. Pavlov, Uspekhi Mat. Nauk B 42, 99-131, (1987).

[3] F. Gesztesy, H. Holden and P. Šeba, “On point interactions in magnetic field systems”, Proceedings of the Schrödinger Operators, Standard and Non-standard, Dubna, USSR, edited by Pavel Exner and Petr Šeba (World Scientific, 1988).

[4] G. Honnouvo and M. N. Hounkonnou, Spectral Properties of Landau Operator with δ-interaction, J. Phys. A (2003, in print).

[5] N. I. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space, Vol. 2, (Pitman, Boston, 1981).

[6] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, Solvable Models in Quantum Mechanics, Texts and Monographs in Physics (Springer Verlag, Berlin, 1988).

[7] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972).

[8] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. 4, Analysis of Operators (Academic, New York, 1978).