Non-local dispersal equations with almost periodic dependence.
II. Asymptotic dynamics of Fisher-KPP equations

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Dedicated to Professor Jibin Li on the occasion of his 80th Birthday

Abstract. This series of two papers is devoted to the study of the principal spectral theory of nonlocal dispersal operators with almost periodic dependence and the study of the asymptotic dynamics of nonlinear nonlocal dispersal equations with almost periodic dependence. In the first part of the series, we investigated the principal spectral theory of nonlocal dispersal operators from two aspects: top Lyapunov exponents and generalized principal eigenvalues. Among others, we provided various characterizations of the top Lyapunov exponents and generalized principal eigenvalues, established the relations between them, and studied the effect of time and space variations on them. In this second part of the series, we study the asymptotic dynamics of nonlinear nonlocal dispersal equations with almost periodic dependence applying the principal spectral theory developed in the first part. In particular, we study the existence, uniqueness, and stability of strictly positive almost periodic solutions of Fisher KPP equations with nonlocal dispersal and almost periodic dependence. By the properties of the asymptotic dynamics of nonlocal dispersal Fisher-KPP equations, we also establish a new property of the generalized principal eigenvalues of nonlocal dispersal operators in this paper.

Key words. Nonlocal dispersal, top Lyapunov exponents, generalized principal eigenvalue, Fisher-KPP equations, almost periodic solutions.

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1 Introduction

This paper is devoted to the study of the asymptotic dynamics of the following nonlinear nonlocal dispersal equation,

$$\partial_t u = \int_D \kappa(y-x)u(t,y)dy + uf(t,x,u), \quad x \in \bar{D},$$  \hspace{1cm} (1.1)

where $D \subset \mathbb{R}^N$ is a bounded domain or $D = \mathbb{R}^N$, and $\kappa(\cdot)$ and $f(\cdot, \cdot, \cdot)$ satisfy

(H1) $\kappa(\cdot) \in C^1(\mathbb{R}^N, [0, \infty))$, $\kappa(0) > 0$, $\int_{\mathbb{R}^N} \kappa(x)dx = 1$, and there are $\mu, M > 0$ such that $\kappa(x) \leq e^{-\mu|x|}$ and $|\nabla \kappa| \leq e^{-\mu|x|}$ for $|x| \geq M$.

(H2) $f(t, x, u)$ is $C^1$ in $u$; $f(t, x, u)$ and $f_u(t, x, u)$ are uniformly continuous and bounded on $(\mathbb{R} \times \bar{D} \times E)$ for any bounded set $E \subset \mathbb{R}$; $f(t, x, u)$ is almost periodic in $t$ uniformly with respect to $x \in \bar{D}$ and $u$ in bounded sets of $\mathbb{R}$; $f(t, x, u)$ is also almost periodic in $x$ uniformly with respect to $t \in \mathbb{R}$ and $u$ in bounded sets when $\bar{D} = \mathbb{R}^N$; $f(t, x, u) + 1 < 0$ for all $(t, x) \in \mathbb{R} \times \bar{D}$ and $u \geq 1$; and $\sup_{t \in \mathbb{R}, x \in \bar{D}} f_u(t, x, u) < 0$ for each $u \geq 0$.

Typical examples of the kernel function $\kappa(\cdot)$ satisfying (H1) include the probability density function of the normal distribution $\kappa(x) = \frac{1}{\sqrt{(2\pi)^N}} e^{-|x|^2}$ and any $C^1$ convolution kernel function supported on a bounded ball $B(0, r) = \{x \in \mathbb{R}^N \mid |x| < r\}$. A prototype of $f(t, x, u)$ satisfying (H2) is $f(t, x, u) = a(t, x) - b(t, x)u$, where $a(t, x)$ and $b(t, x)$ are bounded and uniformly continuous in $(t, x) \in \mathbb{R} \times \bar{D}$; are almost periodic in $t$ uniformly with respect to $x \in \bar{D}$; are also almost periodic in $x$ uniformly with respect to $t \in \mathbb{R}$ when $\bar{D} = \mathbb{R}^N$; and $\inf_{t \in \mathbb{R}, x \in \bar{D}} b(t, x) > 0$.

Dispersal, the mechanism by which a species expands the distribution of its population, is a central topic in biology and ecology. Most continuous models related to dispersal are based upon reaction-diffusion equations such as

$$\begin{cases}
    u_t = \Delta u + ug(t, x, u), & x \in \Omega \\
    u = 0, \quad x \in \partial\Omega,
\end{cases} \hspace{1cm} (1.2)$$

$$\begin{cases}
    u_t = \Delta u + ug(t, x, u), & x \in \Omega \\
    \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega,
\end{cases} \hspace{1cm} (1.3)$$

where $\Omega$ is a bounded smooth domain, or

$$u_t = \Delta u + ug(t, x, u), \quad x \in \mathbb{R}^N. \hspace{1cm} (1.4)$$

In such equations, the dispersal is represented by the Laplacian and is governed by random walk. It is referred to as random dispersal and is essentially a local behavior describing the movement of cells or organisms between adjacent spatial locations.

In reality, the movements of some organisms can occur between non-adjacent spatial locations. For such a model species, one can think of trees of which seeds and pollens are disseminated on a
wide range. Reaction-diffusion equations are not proper to model such dispersal. The following nonlocal dispersal equations are commonly used models to integrate the long range dispersal for populations having a long range dispersal strategy (see [11, 14, 15, 21, 34], etc):

\[
\partial_t u = \int_\Omega \kappa(y-x)u(t,y)dy - u(t,x) + ug(t,x,u), \quad x \in \bar{\Omega},
\]

(1.5)

\[
\partial_t u = \int_\Omega \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u), \quad x \in \bar{\Omega},
\]

(1.6)

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain, and

\[
\partial_t u = \int_{\mathbb{R}^N} \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u), \quad x \in \mathbb{R}^N.
\]

(1.7)

In equations (1.5), (1.6), and (1.7), the dispersal kernel \(\kappa(\cdot)\) describes the probability to jump from one location to another and the support of \(\kappa(\cdot)\) can be thought of as the range of dispersion of the cells.

Observe that (1.5) can be written as

\[
\partial_t u = \int_{\mathbb{R}^N} \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u), \quad x \in \bar{\Omega}
\]

(1.8)

complemented with the following Dirichlet-type boundary condition

\[
u(t,x) = 0, \quad x \in \mathbb{R}^N \setminus \bar{\Omega},
\]

(1.9)

and (1.6) can be written as

\[
\partial_t u = \int_{\mathbb{R}^N} \kappa(y-x)[u(t,y) - u(t,x)]dy + ug(t,x,u), \quad x \in \bar{\Omega}
\]

(1.10)

complemented with the following Neumann-type boundary condition

\[
\int_{\mathbb{R}^N \setminus \bar{\Omega}} \kappa(y-x)u(t,y)dy = \int_{\mathbb{R}^N \setminus \bar{\Omega}} \kappa(y-x)u(t,x)dy, \quad x \in \bar{\Omega}.
\]

(1.11)

The reader is referred to [29] for the relation between (1.8)+(1.9) and the reaction diffusion equation (1.2) with Dirichlet boundary condition, and the relation between (1.10)+(1.11) and the reaction diffusion equation (1.3) with Neumann boundary condition.

Observe also that (1.5) (respectively (1.6), or (1.7)) can be written as (1.1) with \(D = \Omega\) and \(f(t,x,u) = -1 + g(t,x,u)\) (respectively \(D = \Omega\) and \(f(t,x,u) = -\int_D \kappa(y-x)dy + g(t,x,u)\), or \(D = \mathbb{R}^N\) and \(f(t,x,u) = -1 + g(t,x,u)\)). Hence the theory on the dynamics of (1.1) to be developed in this paper can be applied to (1.5), (1.6), and (1.7).

Considering a population model, among the fundamental dynamical issues are asymptotic behavior of solutions with strictly positive initials, propagation phenomena of solutions with compact supported or front-like initials when the underlying environment is unbounded, and the effects of dispersal strategy and spatial-temporal variations on the population dynamics. These
dynamical issues have been extensively studied for population models described by reaction
diffusion equations and are quite well understood in many cases. Recently there has also been
extensive investigation on these dynamical issues for nonlocal dispersal population models (see
[1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 16, 18, 20, 26, 27, 30, 31, 32, 33, 35, 36], etc.). However,
the understanding of these issues for nonlocal dispersal equations is much less, and, to our
knowledge they have been essentially investigated in specific situations such as time and space
periodic media or time independent and space heterogeneous media.

The objective of this paper is to study the asymptotic dynamics of solutions of (1.1) with
strictly positive initials, which would provide some foundation for the study of the propagation
dynamics of positive solutions of (1.1) with compact supported or front-like initials (see Remark
1.5). We point out that, in contrast to the Laplacian, the integral operator in (1.1) is not a local
operator. The mathematical analysis of (1.1) appears to be difficult even though the dispersal
is represented by a bounded integral operator. Unlike the case of reaction-diffusion equations,
the forward flow associated with (1.1) does not have a regularizing effect.

Note that \( u(t, x) \equiv 0 \) is a solution of (1.1), which is referred to as the trivial solution of (1.1).
If \( a(t, x) = f(t, x, 0) \), then the following nonlocal linear equation

\[
\partial_t u = \int_D \kappa(y - x) u(t, y) dy + a(t, x) u, \quad x \in \bar{D},
\]

is the linearization of (1.1) at this trivial solution. Hence the principal spectral theory established
for (1.12) in [23] has its own interests and also plays an important role in the study of the
asymptotic dynamics of (1.1) in this paper.

To state the main results on the asymptotic dynamics of strictly positive solutions of (1.1)
and some new properties of the generalized principal eigenvalues of (1.12) to be established in
this paper, we first recall some of the results on the principal spectrum of (1.12) established in
[23].

1.1 Principal spectral theory of (1.12)

In this subsection, we recall some of the results on the principal spectrum of (1.12) established
in [23].

Let

\[
X(D) = C^h(\bar{D}) = \{ u \in C(\bar{D}) \mid u \text{ is uniformly continuous and bounded} \} \tag{1.13}
\]

with norm \( \| u \| = \sup_{x \in D} |u(x)| \). If no confusion will occur, we may put

\[
X = X(D),
\]

\[
X^+ = \{ u \in X \mid u(x) \geq 0, \quad x \in \bar{D} \},
\]

and

\[
X^{++} = \{ u \in X^+ \mid \inf_{x \in \bar{D}} u(x) > 0 \}.
\]
Throughout the rest of this subsection, we assume that $a(t, x)$ satisfies the following (H3).

**(H3)** $a(t, x)$ is bounded and uniformly continuous in $(t, x) \in \mathbb{R} \times \overline{D}$, and is almost periodic in $t$ uniformly with respect to $x \in \overline{D}$, and is also almost periodic in $x$ uniformly with respect to $t \in \mathbb{R}$ when $D = \mathbb{R}^N$.

Sometimes, we may also assume that $a(t, x)$ satisfies the following (H3)'.

**(H3)'** $a(t, x)$ is limiting almost periodic in $t$ uniformly with respect to $x \in \overline{D}$ and is also limiting almost periodic in $x$ when $D = \mathbb{R}^N$ (see Definition 2.1(2)).

For any $s \in \mathbb{R}$ and $u_0 \in X$, let $u(t, x; s, u_0)$ be the unique solution of (1.12) with $u(s, x; s, u_0) = u_0(x)$ (the existence and uniqueness of solutions of (1.12) with given initial function $u_0 \in X$ follow from the general semigroup theory, see [25]). Let $\Psi(t, s; a, D)$ be the solution operator of (1.12) on $X$, that is,

$$\Psi(t, s; a, D)u_0 = u(t, \cdot; s, u_0).$$

**Definition 1.1.** Let

$$\lambda_{PL}(a, D) = \limsup_{t-s \to \infty} \frac{\ln \|\Psi(t, s; a, D)\|}{t-s}, \quad \lambda'_{PL}(a, D) = \liminf_{t-s \to \infty} \frac{\ln \|\Psi(t, s; a, D)\|}{t-s}.$$  

$\lambda_{PL}(a, D)$ and $\lambda'_{PL}(a, D)$ are called the top Lyapunov exponents of (1.12).

Let

$$\mathcal{X}(D) = C^b_{\text{unif}}(\mathbb{R} \times \overline{D}) := \{u \in C(\mathbb{R} \times \overline{D} | u \text{ is uniformly continuous and bounded}\}$$  

with the norm $\|u\| = \sup_{(t, x) \in \mathbb{R} \times \overline{D}} |u(t, x)|$. In the absence of possible confusion, we may write

$$\mathcal{X} = \mathcal{X}(D),$$

$$\mathcal{X}^+ = \{u \in \mathcal{X} | u(t, x) \geq 0, \ t \in \mathbb{R}, \ x \in \overline{D}\},$$

and

$$\mathcal{X}^{++} = \{u \in \mathcal{X}^+ | \inf_{t \in \mathbb{R}, x \in \overline{D}} u(t, x) > 0\}.$$  

Let $L(a) : \mathcal{D}(L(a)) \subset \mathcal{X} \to \mathcal{X}$ be defined as follows,

$$(L(a)u)(t, x) = -\partial_t u(t, x) + \int_D \kappa(y-x)u(t, y)dy + a(t, x)u(t, x).$$

Let

$$\Lambda_{PE}(a, D) = \{\lambda \in \mathbb{R} | \exists \phi \in \mathcal{X}, \inf_{t \in \mathbb{R}} \phi(t, x) \geq 0, \text{ for each } x \in \overline{D}, \ \phi(\cdot, x) \in W^{1,1}_{\text{loc}}(\mathbb{R}) \text{ and } (L(a)\phi)(t, x) \geq \lambda \phi(t, x) \text{ for a.e. } t \in \mathbb{R}\}$$  

(1.15)
\[ \Lambda_{PE}(a, D) = \{ \lambda \in \mathbb{R} \mid \exists \phi \in \mathcal{X}, \inf_{t \in \mathbb{R}, x \in D} \phi(t, x) > 0, \text{ for each } x \in D, \phi(\cdot, x) \in W^{1,1}_0(\mathbb{R}) \text{ and } \\
(L(a)\phi)(t, x) \leq \lambda \phi(t, x) \text{ for } a.e. t \in \mathbb{R} \}. \tag{1.16} \]

We point out that the condition \( \phi(\cdot, x) \in W^{1,1}_0(\mathbb{R}) \) for each \( x \in D \) on the test function \( \phi \in \mathcal{X} \) in (1.15) and (1.16) is needed for the comparison principle (see the proof of Proposition 2.3). In the definition of the sets \( \Lambda_{PE}(a, D) \) and \( \Lambda'_{PE}(a, D) \) in [23], this condition was absent, which is not because it was not needed, but was missed.

**Definition 1.2.** Define 

\[ \lambda_{PE}(a, D) = \sup \{ \lambda \mid \lambda \in \Lambda_{PE}(a, D) \} \]

and 

\[ \lambda'_{PE}(a, D) = \inf \{ \lambda \mid \lambda \in \Lambda'_{PE}(a, D) \}. \]

Both \( \lambda_{PE}(a, D) \) and \( \lambda'_{PE}(a, D) \) are called generalized principal eigenvalues of (1.12).

Let 

\[ \hat{a}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T a(t, x) dt \tag{1.17} \]

(see Proposition 2.1 for the existence of \( \hat{a}(\cdot) \)). Let 

\[ \bar{a} = \frac{1}{|D|} \int_D \hat{a}(x) dx \tag{1.18} \]

when \( D \) is bounded, and 

\[ \bar{a} = \lim_{q_1, q_2, \ldots, q_N \to \infty} \frac{1}{q_1 q_2 \cdots q_N} \int_0^{q_N} \cdots \int_0^{q_2} \int_0^{q_1} \hat{a}(x_1, x_2, \ldots, x_N) dx_1 dx_2 \cdots dx_N \tag{1.19} \]

when \( D = \mathbb{R}^N \) and \( a(t, x) \) is almost periodic in \( x \) uniformly with respect to \( t \in \mathbb{R} \) (see Proposition 2.1 for the existence of \( \bar{a} \)). Note that \( \hat{a}(x) \) is the time average of \( a(t, x) \), and \( \bar{a} \) is the space average of \( \hat{a}(x) \).

If no confusion occurs, we may put \( \lambda_{PL}(a) = \lambda_{PL}(a, D), \lambda'_{PL}(a) = \lambda'_{PL}(a, D), \lambda_{PE}(a) = \lambda_{PE}(a, D), \) and \( \lambda'_{PE}(a) = \lambda'_{PE}(a, D) \). Among others, we proved the following results in [23].

**Proposition 1.1.** Assume (H3).

1. (Theorem 1.1 in [23])
   \[ \lambda'_{PL}(a) = \lambda_{PL}(a) = \lim_{t-s \to \infty} \frac{\ln \| \Psi(t, s; a) u_0 \|}{t-s} = \lim_{t-s \to \infty} \frac{\ln \| \Psi(t, s; a) \|}{t-s} \]
   for any \( u_0 \in X \) with \( \inf_{x \in D} u_0(x) > 0 \).

2. (Theorem 1.2 in [23]) \( \lambda_{PE}(a) \leq \lambda'_{PE}(a) = \lambda_{PL}(a) \). If \( a(t, x) \) satisfies (H3)' , then \( \lambda_{PE}(a) = \lambda'_{PE}(a) \).
(3) (Theorem 1.4 in [23, 24]) \( \lambda_{PL}(a) \geq \lambda_{PL}(\bar{a}) \geq \sup_{x \in D} \bar{a}(x) \).

(4) (Theorem 1.3 in [23, 24]) \( \lambda_{PE}(a) \geq \sup_{x \in D} \bar{a}(x) \). If \( a(t, x) \) satisfies (H3)', then \( \lambda_{PE}(a) \geq \lambda_{PE}(\bar{a}) \geq \sup_{x \in D} \bar{a}(x) \).

(5) (Theorem 1.3 in [23, 24]) If \( D \) is bounded, \( a(t, x) \equiv a(x) \), and \( \kappa(\cdot) \) is symmetric, then

\[
\lambda_{PE}(a) \geq \bar{a} + \frac{1}{|D|} \int_{D} \int_{D} \kappa(y - x)dydx,
\]

where \( |D| \) is the Lebesgue measure of \( D \).

(6) (Theorem 1.3 in [23, 24]) If \( D = \mathbb{R}^{N} \), \( a(t, x) \equiv a(x) \) is almost periodic in \( x \), and \( \kappa(\cdot) \) is symmetric, then

\[
\lambda_{PE}(a) \geq \bar{a} + 1.
\]

(7) (Theorem 1.5(1) in [23, 24]) If \( a(t, x) \equiv a(x) \) and satisfies (H3)', then

\[
\lambda_{PE}(a) = \sup\{\lambda | \lambda \in \tilde{\Lambda}_{PE}(a)\} = \inf\{\lambda | \lambda \in \Lambda_{PE}(a)\} = \lambda'_{PE}(a),
\]

where

\[
\tilde{\Lambda}_{PE}(a) = \{\lambda \in \mathbb{R} | \exists \phi \in X, \phi(x) \geq 0, \int_{D} \kappa(y - x)\phi(y)dy + a(x)\phi(x) \geq \lambda \phi(x) \forall x \in \bar{D}\}
\]

and

\[
\Lambda_{PE}(a) = \{\lambda \in \mathbb{R} | \exists \phi \in X, \inf_{x \in D} \phi(x) > 0, \int_{D} \kappa(y - x)\phi(y)dy + a(x)\phi(x) \leq \lambda \phi(x) \forall x \in \bar{D}\}.
\]

We conclude this subsection with some remark on the generalized principal eigenvalues and top Lyapunov exponents of \( \{1,12\} \).

**Remark 1.1.**

(1) It remains open whether \( \lambda_{PE}(a) = \lambda'_{PE}(a) \) for any \( a(\cdot, \cdot) \) satisfying (H3).

(2) The test function \( \phi \) in the definition of \( \lambda'_{PE}(a) \) and \( \lambda_{PE}(a) \) is not required to be almost periodic in \( t \). The definition of \( \lambda_{PL}(a) \), \( \lambda'_{PL}(a) \), \( \lambda_{PE}(a) \), and \( \lambda'_{PE}(a) \) applies to the case where \( a(t, x) \) is bounded and uniformly continuous in \( t \in \mathbb{R} \) and \( x \in \bar{D} \). For such general \( a(t, x) \), by the arguments of Theorem 1.2 in [23], we have the following relations between \( \lambda_{PL}(a) \), \( \lambda'_{PL}(a) \), \( \lambda_{PE}(a) \), and \( \lambda'_{PE}(a) \),

\[
\lambda_{PE}(a) \leq \lambda'_{PL}(a) \leq \lambda_{PL}(a) \leq \lambda'_{PE}(a).
\]

(3) Assume that \( a(t, x) \) satisfy (H3) with \( D = \mathbb{R}^{N} \). By the definition of \( \lambda'_{PE}(a, D) \), it is easy to see that

\[
\lambda'_{PE}(a, D_1) \leq \lambda'_{PE}(a, D_2)
\]

for any \( D_1 \subset D_2 \subset \mathbb{R}^{N} \). But due to the requirement of the continuity of the test functions in \( \Lambda_{PE}(a, D) \), it is not clear whether \( \lambda_{PE}(a, D_1) \leq \lambda_{PE}(a, D_2) \) also holds for any \( D_1 \subset D_2 \). In this paper, we will prove this also holds by applying the criteria for the existence of strictly positive entire solutions of \( \{1,1\} \) (see Theorem 1.3).
1.2 Main results

In this subsection, we state the main results of this paper. Throughout this subsection, we assume (H1) and (H2).

Observe that a function \( u(t,x) \) satisfying \((1.1)\) need not be continuous in \( x \). In this paper, unless specified otherwise, when we say that \( u(t,x) \) is a solution of \((1.1)\) on an interval \( I \), it means that, for each \( t \in I \), \( u(t,\cdot) \in X \), and the mapping \( I \ni t \mapsto u(t,\cdot) \in X \) is differentiable. Such a solution \( u(t,x) \) is clearly differentiable in \( t \) and is continuous in both \( t \) and \( x \).

Note that, by general semigroup theory (see [25]), for any \( s \in \mathbb{R} \) and \( u_0 \in X \), \((1.1)\) has a unique (local) solution \( u(t,x; s, u_0) \) with \( u(s,x; s,u_0) = u_0(x) \). Moreover, for any \( u_0 \in X^+ \), \( u(t,x; s,u_0) \) exists globally, that is, \( u(t,x; s,u_0) \) exists for all \( t \geq s \) (see the comparison principle, Proposition 2.3 (2)). A solution \( u(t,x) \) of \((1.1)\) defined for all \( t \in \mathbb{R} \) is called an entire solution. An entire solution \( u(t,x) \) of \((1.1)\) is said to be positive if \( u(t,x) > 0 \) for any \( (t,x) \in \mathbb{R} \times \overline{D} \) and strictly positive if \( \inf_{t \in \mathbb{R}, x \in D} u(t,x) > 0 \). A strictly positive entire solution \( u(t,x) \) of \((1.1)\) is called an almost periodic solution if it is almost periodic in \( t \) uniformly with respect to \( x \in \overline{D} \) in the case that \( D \) is bounded and is almost periodic in both \( t \) and \( x \) in the case that \( D = \mathbb{R}^N \).

In the rest of the paper, \( u(t,x; s, u_0) \) always denotes the solution of \((1.1)\) with \( u(s,\cdot; s, u_0) = u_0 \in X \), unless specified otherwise. Among others, we prove

**Theorem 1.1.** (a) (Uniqueness) There is at most one strictly positive bounded entire solution of \((1.1)\).

(b) (Almost periodicity) Any strictly positive bounded entire solution of \((1.1)\) is almost periodic.

(c) (Stability) If \( u^*(t,x) \) is a strictly positive bounded almost periodic solution of \((1.1)\), then for any \( u_0 \in X^+ \),

\[
\lim_{t \to \infty} \| u(t,\cdot; t_0, u_0) - u^*(t,\cdot) \|_\infty = 0.
\]

(d) (Frequency module) If \( u^*(t,x) \) is a strictly positive bounded almost periodic solution of \((1.1)\), then

\[
\mathcal{M}(u^*) \subset \mathcal{M}(f),
\]

where \( \mathcal{M}(\cdot) \) denotes the frequency module of an almost periodic function.

**Theorem 1.2.** Let \( a(t,x) = f(t,x,0) \).

(a) (Existence) Equation \((1.1)\) has a strictly positive bounded almost periodic solution if and only if \( \lambda_{PE}(a) > 0 \).

(b) (None existence) If \( \lambda_{PL}(a) < 0 \), then the trivial solution \( u \equiv 0 \) of \((1.1)\) is globally asymptotically stable in the sense that for any \( u_0 \in X^+ \),

\[
\| u(t,\cdot; 0, u_0) \|_X \to 0 \text{ as } t \to \infty.
\]
Corollary 1.1. Let \( a(t, x) = f(t, x, 0) \).

(a) If \( \sup_{x \in D} \hat{a}(x) > 0 \), then equation (1.1) has a strictly positive almost periodic solution.

(b) If \( \kappa(\cdot) \) is symmetric, \( a(t, x) \equiv a(x) \), and \( \bar{a} > -\frac{1}{|D|} \int_D \int_D \kappa(y - x) dy dx \) when \( D \) is bounded and \( \bar{a} > -1 \) when \( D = \mathbb{R}^N \), then equation (1.1) has a strictly positive almost periodic solution.

Proof. (a) By [23, Theorem 1.3(1)], \( \lambda_{PE}(a) \geq \sup_{x \in D} \hat{a}(x) \); (a) then follows from Theorems 1.1 and 1.2.

(b) By [23, Theorem 1.3(2),(3)], \( \lambda_{PE}(a) \geq \bar{a} + \frac{1}{|D|} \int_D \int_D \kappa(y - x) dy dx \) when \( D \) is bounded and \( \lambda_{PE}(a) \geq \bar{a} + 1 \) when \( D = \mathbb{R}^N \); (b) then follows from Theorems 1.1 and 1.2.

We also establish a new property of the generalized principal eigenvalues of (1.12) on the domain \( D \).

Theorem 1.3. Suppose that \( a(t, x) \) satisfies (H3) with \( D = \mathbb{R}^N \). For any \( D_1 \subset D_2 \), there holds

\[
\lambda_{PE}(a, D_1) \leq \lambda_{PE}(a, D_2),
\]

(1.21)

where \( D_1 \) is bounded and \( D_2 \) is bounded or \( D_2 = \mathbb{R}^N \).

We point out that in this paper as well as [23], we consider \( \lambda_{PE}(a, D) \) with \( D \) being either bounded or the whole space \( \mathbb{R}^N \). Hence it is assumed that \( D_1 \) is bounded in Theorem 1.3. For otherwise, if \( D_1 = \mathbb{R}^N \), then \( D_2 = \mathbb{R}^N \) and nothing needs to be proved.

1.3 Comments on the main results

In this subsection, we give some comments on the main results of this paper.

First, we give some comments on our results in some special cases.

Comment 1.1 (Extension of existing results in special cases).

(1) For the case that the function \( f(t, x, u) \) is time independent or time periodic and is periodic in \( x \) when \( D = \mathbb{R}^N \), similar results on the asymptotic dynamics of (1.1) as Theorems 1.1 and 1.2 have been obtained in [2, 3, 26, 30]. Our results recover those results in [2, 3, 26, 30].

(2) In [28], results similar to theorems 1.1 and 1.2 for the case \( D = \mathbb{R} \) were obtained for general time dependence under the condition \( \lim_{t-s \to \infty} \inf_{t-s \to \infty} \inf_{x \in \mathbb{R}} \int_{t-s}^t \int_{x \in \mathbb{R}} f(\tau, x, 0) d\tau > -1 \). Note that when \( D = \mathbb{R} \) and \( f(t, x, u) \) is almost periodic in \( t \) uniformly with respect to \( x \in \mathbb{R} \), \( a_{inf}(t) := \inf_{x \in \mathbb{R}} a(t, x) \) is also almost periodic in \( t \), where \( a(t, x) = f(t, x, 0) \). Note also that \( \lambda_{PE}(a) \geq \lambda_{PE}(a_{inf}) = 1 + \lim_{t-s \to \infty} \frac{1}{t-s} \int_{t-s}^t a_{inf}(\tau) d\tau \). Thus our results extend Theorem 2.1 of [28] in the case when \( D = \mathbb{R} \) and \( f(t, x, u) \) is almost periodic in \( t \).
(3) It should be pointed out that, in the case that $D = \mathbb{R}^N$ and $f(t, x, u) \equiv f(x, u)$ is not almost periodic in $x$, the existence, uniqueness, and stability of positive solutions when $\lambda_P(a) < 0$ were established in [4, Theorem 1.1], where $a(x) = f(x, 0)$ and

$$\lambda_p(a) := \sup \{ \lambda \in \mathbb{R} \mid \exists \phi \in C(\mathbb{R}^N), \phi > 0 \text{ s.t. } \int_{\mathbb{R}^N} \kappa(y - x)\phi(y)dy + a(x)\phi + \lambda\phi \leq 0 \}.$$

Note that the test function in the definition $\lambda_p(a)$ may not be uniformly continuous and bounded, which are required for the test functions in the definition of $\lambda'_{PE}(a)$. Hence,

$$\bar{\Lambda}_{PE}(a) \subset -\Lambda_p(a).$$

In the case that $a(x)$ is limiting almost periodic, by Proposition 1.1 (7), we then have

$$\lambda_{PE}(a) = \lambda'_{PE}(a) \geq -\lambda_p(a).$$

Hence, in such a case, $\lambda_p(a) < 0$ implies $\lambda_{PE}(a) > 0$ and our results improve [4, Theorem 1.1] in the sense that the positive solution we obtained is strictly positive and almost periodic.

Second, we give some comments on the time and space variations.

Comment 1.2 (Effects of time and space variations).

(1) If $a(t, x) = f(t, x, 0)$ is limiting almost periodic, then $\lambda_{PE}(a) \geq \lambda_{PE}(\hat{a})$ (see Proposition 1.1(4)), which shows that time variation does not reduce the generalized principal eigenvalue $\lambda_{PE}$. Thus Theorem 1.2(b) indicates that time variation may favor the persistence of species.

(2) If $a(t, x) = f(t, x, 0)$ is independent of $t$, $\kappa(\cdot)$ is symmetric, and $D = \mathbb{R}^N$, then $\lambda_{PE}(a) \geq \bar{a} + 1 = \lambda_{PE}(\bar{a})$ (see Proposition 1.1(6)). Theorem 1.2(b) then indicates that space variation may favor the persistence of species.

Third, we give some comments on the proofs of the main results.

Comment 1.3 (Difficulties in the proofs). By Theorem 1.2 $\lambda_{PE}(a) > 0$ is a necessary and sufficient condition for the existence of a unique strictly positive almost periodic solution of (1.1), where $a(t, x) = f(t, x, 0)$. Note that $\lambda_{PE}(a) > 0$ indicates that the trivial solution $u = 0$ of (1.1) is unstable. It is naturally expected that the instability of the trivial solution $u = 0$ implies the existence of a positive entire solution. In fact, this has been proved for the random dispersal counterpart of (1.1). However, thanks to the lack of the regularizing effect of the forward flow associated with (1.1) and the lack of Poincaré map in non-periodic time dependent case, it is very nontrivial to prove the existence of strictly positive almost periodic solutions of (1.1).

Fourth, we give some comments on the extension of the main results to more general cases.
Comment 1.4 (Extension of the main results to non-almost periodic cases). As mentioned in Remark 1.1, the definitions of \( \lambda_{PL}(a) \), \( \lambda'_{PL}(a) \), \( \lambda_{PE}(a) \), and \( \lambda'_{PE}(a) \) apply to general \( a(t,x) \) which is bounded and uniformly continuous. When \( f(t,x,u) \) is not assumed to be almost periodic in \( t \), if \( \lambda_{PE}(a) > 0 \) (\( a(t,x) = f(t,x,0) \)), we still have a positive continuous function \( u^*(t,x) \) which satisfies (1.1) for all \( t \in \mathbb{R} \) and \( x \in \bar{D} \). Moreover, if \( D \) is bounded, then \( u^*(t,x) \) is a strictly positive entire solution of (1.1) and is asymptotically stable with respect to positive perturbations. But in general, \( u^*(t,x) \) may not be strictly positive (see Remark 1.1).

Finally, we give some comments on the application of the main results to the study of propagation phenomena in (1.1) when \( D = \mathbb{R}^N \).

Comment 1.5 (Propagation dynamics). Suppose that \( D = \mathbb{R}^N \) and \( \lambda_{PE}(a) > 0 \), where \( a(t,x) = f(t,x,0) \). Then by Theorems 1.1 and 1.2, (1.1) has a unique strictly positive almost periodic solution \( u^*(t,x) \) that attract all solutions with strictly positive initials uniformly, but \( u^*(t,x) \) does not attract solutions with compactly supported or front-like initials uniformly. Biologically, such an initial indicates that the population initially resides in a bounded region or in one side of the whole space. Naturally, the population with such initial distribution will spread into the region where there is no population initially as time evolves. It is interesting to ask how fast the population spreads. Based on the investigation in the time independent or periodic case (see [27, 30]), it is equivalent to ask how fast the region where the solution is near \( u^*(t,x) \) grows. To be a little more precise, for a given compact supported initial \( u_0 \) (i.e. \( u_0(x) \geq 0 \) and \( \{x \in \mathbb{R}^N | u_0(x) > 0 \} \) is bounded and non-empty) or front-like initial \( u_0 \) (i.e. \( u_0(x) - u^*(x,0) \to 0 \) as \( x \cdot \xi \to -\infty \) and \( u_0(x) = 0 \) for \( x \cdot \xi \gg 1 \) for some unit vector \( \xi \in \mathbb{R}^N \)), and given \( 0 < \epsilon \ll 1 \), let

\[
D(t,u_0) = \{ x \in \mathbb{R}^N : |u(t,x;0,u_0) - u^*(t,x)| \leq \epsilon \}.
\]

By the stability of \( u^*(t,x) \) with respect to strictly positive initials, it is expected that \( D(t,u_0) \) grows as \( t \) increases. It is interesting to know how fast \( D(t,u_0) \) grows. We plan to study this problem somewhere else.

The rest of the paper is organised as follows. In section 2 we give the preliminary definitions and results to be used in the rest of the paper. Theorems 1.1 and 1.2 are proved in Sections 3 and 4 respectively. Section 5 is devoted to the proof of Theorem 1.3.

2 Preliminary

In this section, we present the preliminary definitions and results to be used in the rest of the paper.

2.1 Almost Periodic functions

In this subsection, we recall the definition and present some basic properties of almost periodic functions.

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Definition 2.1.  
(1) A bounded function \( f \in C(\mathbb{R}, \mathbb{R}) \) is said to be almost periodic if for any \( \epsilon > 0 \), the set 
\[ T(f, \epsilon) = \{ \tau \in \mathbb{R} \mid |f(t + \tau) - f(t)| < \epsilon \quad \forall t \in \mathbb{R} \} \]
is relatively dense in \( \mathbb{R} \).

(2) Let \( f(t, x) \in C(\mathbb{R} \times E, \mathbb{R}) \), where \( E \) is a subset of \( \mathbb{R}^N \), \( f(t, x) \) is said to be almost periodic in \( t \) uniformly with respect to \( x \in E \), if it is uniformly continuous on \( \mathbb{R} \times E \) and for any fixed \( x \in E \), \( f(t, x) \) is an almost periodic function of \( t \).

(3) Let \( E \subset \mathbb{R}^N \) and \( f \in C(\mathbb{R} \times E, \mathbb{R}) \). \( f \) is said to be limiting almost periodic in \( t \) uniformly with respect to \( x \in E \), if there is a sequence \( f_n(t, x) \) of uniformly continuous functions which are periodic in \( t \) such that
\[ \lim_{n \to \infty} f_n(t, x) = f(t, x) \]
uniformly in \( (t, x) \in \mathbb{R} \times E \).

(4) Let \( f \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \). \( f(t, x) \) is said to be almost periodic in \( x \) uniformly with respect to \( t \in \mathbb{R} \) if \( f \) is uniformly continuous in \( (t, x) \in \mathbb{R} \times \mathbb{R}^N \) and for each \( 1 \leq i \leq N \), \( f(t, x_1, x_2, \cdots, x_N) \) is almost periodic in \( x_i \).

(5) Let \( f(t, x) \in C(\mathbb{R} \times E, \mathbb{R}) \) be an almost periodic function in \( t \) uniformly with respect to \( x \in E \subset \mathbb{R}^N \). Let \( \Lambda \) be the set of real numbers \( \lambda \) such that
\[ a(x, \lambda, f) := \lim_{T \to \infty} \int_0^T f(t, x)e^{-i\lambda t} \, dt \]
is not identically zero for \( x \in E \). The set consisting of all real numbers which are linear combinations of elements of the set \( \Lambda \) with integer coefficients is called the frequency module of \( f(t, x) \), which we denote by \( M(f) \).

Lemma 2.1. A function \( f(t, x) \) is almost periodic in \( t \) uniformly with respect to \( x \in E \subset \mathbb{R}^K \) if and only if it is uniformly continuous on \( \mathbb{R} \times E \) and for every pair of sequences \( \{s_n\}_{n=1}^{\infty}, \{r_m\}_{m=1}^{\infty} \), there are subsequences \( \{s_n'\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}, \{r_m'\}_{m=1}^{\infty} \subset \{r_m\}_{m=1}^{\infty} \) such that for each \( (t, x) \in \mathbb{R} \times \mathbb{R}^K \),
\[ \lim_{m \to \infty} \lim_{n \to \infty} f(t + s_n' + r_m', x) = \lim_{n \to \infty} f(t + s_n' + r_m', x). \]
Proof. See [12, Theorems 1.17 and 2.10]. \[ \square \]

Definition 2.2. For an almost periodic function \( a(t, x) \) in \( t \), the value
\[ \hat{a}(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t a(t, x) \, dt. \]
is called the mean value of \( a \).
Proposition 2.1.  (1) If \( f(t,x) \) is almost periodic in \( t \) uniformly with respect to \( x \in E \), then for any sequence \( \{t_n\} \subset \mathbb{R} \), there is a subsequence \( \{t_{n_k}\} \) such that the limit \( \lim_{k \to \infty} f(t + t_{n_k}, x) \) exists uniformly in \( (t,x) \in \mathbb{R} \times E \).

(2) If \( f(t,x) \) is almost periodic in \( t \) uniformly with respect to \( x \in E \), then the limit

\[
\hat{f}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t,x) dt
\]

exists uniformly with respect to \( x \in E \). If \( E = \mathbb{R}^N \) and for each \( 1 \leq i \leq N, f(t,x_1,x_2,\cdots,x_N) \) is also almost periodic in \( x_i \) uniformly with respect to \( t \in \mathbb{R} \) and \( x_j \in \mathbb{R} \) for \( 1 \leq j \leq N, j \neq i \), then the limit

\[
\hat{f} := \lim_{q_1,q_2,\cdots,q_N \to \infty} \frac{1}{q_1q_2\cdots q_N} \int_0^{q_N} \cdots \int_0^{q_2} \int_0^{q_1} \hat{f}(x_1,x_2,\cdots,x_N) dx_1dx_2\cdots dx_N
\]

exists.

Proof. (1) It follows from [12, Theorem 4.5]

(2) It follows from [12, Theorem 3.1] \( \square \)

Proposition 2.2. Let \( f, g \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \) be two almost periodic functions in \( t \) uniformly with respect to \( x \) in bounded sets. \( \mathcal{M}(g) \subset \mathcal{M}(f) \) if and only if for any sequence \( \{t_n\} \subset \mathbb{R} \) if \( \lim_{n \to \infty} f(t+t_n,x) = f(t,x) \) uniformly for \( t \in \mathbb{R} \) and \( x \) in bounded sets, then there is \( \{t_{n_k}\} \) a subsequence of \( \{t_n\} \) such that \( \lim_{k \to \infty} g(t+t_{n_k},x) = g(t,x) \) uniformly for \( t \in \mathbb{R} \) and \( x \) in bounded sets.

Proof. See [12, Theorem 4.5] \( \square \)

2.2 Comparison principle

In this subsection, we introduce super- and sub-solutions of (1.1) in some general sense and present a comparison principle and some related properties for solutions of (1.1).

Recall that, for any \( s \in \mathbb{R} \) and \( u_0 \in X \), \( u(t,x;s,u_0) \) denotes the unique solution of (1.1) with \( u(s,x;s,u_0) = u_0(x) \). Let \( T_{\max}(s,u_0) \in (0,\infty) \) be the largest number such that \( u(t,x;s,u_0) \) exists on \([s,s + T_{\max}(s,u_0)) \). To indicate the dependence of \( u(t,x;s,u_0) \) on \( D \), we may write it as \( u(t,x;s,u_0,D) \).

Definition 2.3. A continuous function \( u(t,x) \) on \([t_0,t_0+\tau) \times \bar{D} \) is called a super-solution (or sub-solution) of (1.1) on \([t_0,t_0+\tau) \) if for any \( x \in \bar{D} \), \( u(\cdot,x) \in W^{1,1}(t_0,t_0+\tau), \) and satisfies,

\[
\frac{\partial u}{\partial t}(t,x) \geq (or \leq) \int_D \kappa(y-x)u(t,y)dy + u(t,x)f(t,x,u) \quad a.e. \ t \in (t_0,t_0+\tau).
\]
Note that, in literature, super-solutions (or sub-solutions) of (1.1) on \([t_0, t_0 + \tau]\) are defined to be functions \(u(\cdot, \cdot) \in C^{1,0}([t_0, t_0 + \tau] \times \bar{D})\) satisfying (2.1) for all \(t \in (t_0, t_0 + \tau)\) and \(x \in D\). Super-solutions (sub-solutions) of (1.1) defined in the above are more general. Nevertheless, we still have the following comparison principle.

**Proposition 2.3. (Comparison Principle)**

1. If \(u^1(t, x)\) and \(u^2(t, x)\) are bounded sub and super-solutions of (1.1) on \([0, \tau]\) and \(u^1(0, \cdot) \leq \ u^2(0, \cdot)\), then \(u^1(t, \cdot) \leq u^2(t, \cdot)\) for \(t \in [0, \tau]\).

2. For every \(u_0 \in X^+\), \(u(t, x; s, u_0)\) exists for all \(t \geq s\).

**Proof.** (1) Set \(v(t, x) = e^{\alpha t}(u^2(t, x) - u^1(t, x))\). Then for each \(x \in D\), \(v(t, x)\) satisfies

\[
\frac{\partial v}{\partial t} \geq \int_D \kappa(y-x)v(t, y)dy + p(t, x)v(t, x) \quad \text{for a.e. } t \in [0, \tau),
\]

where \(p(t, x) = a(t, x) + c\),

\[
a(t, x) = \int_0^1 \frac{\partial}{\partial s} \left((su^2(t, x) + (1 - s)u^1(t, x))f(t, x, su^2(t, x) + (1 - s)u^1(t, x))\right) ds,
\]

and \(c > 0\) is such that \(p(t, x) > 0\) for all \(t \in \mathbb{R}\) and \(x \in D\). Since \(u^i(\cdot, x) \in W^{1,1}(0, \tau)\) for each \(x \in D\), by [9] Theorem 2, Section 5.9], we have that

\[
v(t, x) - v(0, x) = \int_0^t v(t, s, x) ds \geq \int_0^t \left(\int_D \kappa(y-x)v(s, y)dy + p(s, x)v(s, x)\right) ds \quad \forall t \in (0, \tau), \ x \in \bar{D}.
\]

The rest of the proof follows from the arguments in Proposition 2.1 of [30].

(2) Note that \(u \equiv 0\) is an entire solution of (1.1) and \(u \equiv M\) is a super-solution of (1.1) when \(M \gg 1\). By (1),

\[
0 \leq u(t, x; s, u_0) \leq M \quad \forall t \in [s, s + T_{\text{max}}(s, u_0)), \ x \in \bar{D}, \ M \gg 1.
\]

This implies that \(T_{\text{max}}(s, u_0) = \infty\) and (2) follows. \(\square\)

**Proposition 2.4.** Let \(D_0 \subset D\). Then

\[
u(t, x; s, u_0|_{D_0}, D_0) \leq u(t, x; s, u_0, D) \quad \forall t \geq s, \ x \in \bar{D}_0,
\]

where \(u_0 \in C^b_{\text{unif}}(\bar{D}), u_0 \geq 0\).

**Proof.** Observe that \(u(t, x; s, u_0, D)\) solves

\[
u_t = \int_D \kappa(y-x)u(t, y)dy + u(t, x)f(t, x, u), \quad x \in \bar{D},
\]

\[
\geq \int_{D_0} \kappa(y-x)u(t, y)dy + u(t, x)f(t, x, u), \quad x \in \bar{D}_1.
\]

Since \(u_0|_{D_0} \leq u_0\) the inequality follows from Proposition 2.3. \(\square\)
For given $r > 0$ and $x_0 \in \mathbb{R}^N$, let

$$B_r(x_0) = \{ x \in \mathbb{R}^N \mid |x - x_0| < r \}.$$ 

**Proposition 2.5.** Let $0 < \delta_0 < 1$ and $r_0 > 0$ be given positive numbers. Suppose that (H1) holds. Then for any given positive integer $k$, there exist a positive number $\mu = \mu(r_0, \delta_0, k)$ and a positive integer $i = i(r_0, \delta_0, k)$ such that

$$\inf_{x \in B_{kr_0}(0)} \sum_{j=0}^{i} \frac{(K^j u)(x)}{j!} \geq \frac{\mu}{\kappa u = \kappa \ast u.} \text{ In particular}$$

$$(e^{K}u)(x) \geq \mu \quad \forall x \in B_{kr_0}(0).$$

**Proof.** From (H1), we know that $\kappa$ is continuous and $\kappa(0) > 0$ so we can find $0 < r < \frac{r_0}{2}$ such that $\kappa(x) \geq \frac{1}{2} \kappa(0)$ for every $x$ in $\bar{B}_r(0)$. Now let $u \in L^\infty(\mathbb{R}^N)$ be a nonnegative function satisfying $\int_{B_r(0)} u dx \geq \delta_0$. We claim that

$$\inf_{x \in B_{(m+1)r}(0) \setminus B_{mr}(0)} (K^{m+1}u)(x) \geq \frac{\delta_0 \kappa(0)^m}{2m+1} \prod_{i=1}^{m} \left| B_{r}(ire_1) \cap B_r((i-1)re_1) \right| \quad \forall m \geq 1$$

where $e_1$ is the unit vector $(1, 0, \ldots, 0) \in \mathbb{R}^N$.

Observe from the definition of $r$ that

$$(Ku)(x) \geq \int_{B_r(0)} \kappa(y-x)u(y)dy \geq \frac{1}{2} \kappa(0) \int_{B_r(0)} u(y)dy \geq \frac{1}{2} \kappa(0) \delta_0 \quad \forall x \in \bar{B}_r(0).$$

Hence

$$\inf_{x \in B_r(0)} (Ku)(x) \geq \frac{1}{2} \kappa(0) \delta_0.$$

We proceed by induction to show that (2.4) holds.

To this end, let us first show that the claim holds for $m = 1$. Observe that for every $r \leq |x| \leq 2r$ and $y \in B_r(\frac{r}{|x|})$, $|y - x| \leq |y - \frac{r}{|x|}x| + |x - \frac{r}{|x|}x| = |y - \frac{r}{|x|}x| + |x| - r < 2r$. Hence, by (2.5) for every $x \in \bar{B}_{2r}(0) \setminus B_r(0)$, we have

$$K^2 u(x) \geq \int_{B_r(\frac{r}{|x|})} \kappa(y-x)(Ku)(y)dy \geq \frac{1}{2} \kappa(0) \int_{B_r(\frac{r}{|x|})} (Ku)(y)dy \geq \frac{\kappa(0)^2}{2^2} \delta_0 \left| B_r(\frac{r}{|x|}x) \cap B_r(0) \right|.$$ 

Since the Lebesgue measure is rotation invariant and $0 < \delta_0 < 1$, we conclude from the last inequality that

$$\inf_{x \in B_{2r}(0) \setminus B_r(0)} K^2 u(x) \geq \frac{\kappa(0)^2}{2^2} \delta_0 \left| B_r(re_1) \cap B_r(0) \right| \geq \frac{\delta_0 \kappa(0)^2}{2^2} \left| B_r(re_1) \cap B_r(0) \right|$$

which proves (2.4) for $m = 1$. 


Next, suppose that (2.4) holds for some $m \geq 1$, we show that it also holds for $m+1$. Indeed, as in the previous case, observe that, as shown in the schematic below, we have the following:

$$|y - x| \leq |y - (m+1)r \frac{x}{|x|}| + |x - (m+1)r \frac{x}{|x|}| < 2r$$

for $(m+1)r \leq |x| \leq (m+2)r$ and $y \in B_r \left( \frac{(m+1)r x}{|x|} \right)$.

Observe also that

$$B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \left( \overline{B}_{(m+1)r}(0) \setminus B_{rm}(0) \right) = B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \overline{B}_{(m+1)r}(0) \quad \forall x \neq 0.$$ 

For notational convenience, let $B_{mr}(0) := B^m_0$ and $B_{(m+1)r}(0) := B^{m+1}_0$. Using the induction hypothesis and recalling the choice of $r$, we obtain for every $x \in \overline{B}_{(m+2)r}(0) \setminus B_{(m+1)r}(0)$ that

$$\mathcal{K}^{m+2} u(x) \geq \int_{B_r \left( \frac{(m+1)r x}{|x|} \right)} \kappa(y-x) \mathcal{K}^{m+1} u(y) dy$$

$$\geq \frac{\kappa(0)}{2} \delta_0 \int_{B_r \left( \frac{(m+1)r x}{|x|} \right)} \mathcal{K}^{m+1} u(y) dy$$

$$\geq \frac{\kappa(0)}{2} \delta_0 \left| B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \left( \overline{B}^{m+1}_0 \setminus B^{m}_0 \right) \right| \inf_{x \in B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \left( \overline{B}^{m+1}_0 \setminus B^{m}_0 \right)} \mathcal{K}^{m+1} u(x)$$

$$= \frac{\kappa(0)}{2} \delta_0 \left| B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \overline{B}^{m+1}_0 \right| \inf_{x \in B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \left( \overline{B}^{m+1}_0 \setminus B^{m}_0 \right)} \mathcal{K}^{m+1} u(x)$$

$$\geq \frac{\delta_0 \kappa(0)}{2^{m+2}} \left| B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \overline{B}^{m+1}_0 \right| \Pi_{i=1}^{m} \left| B_r (ire_1) \cap B_r ((i-1)re_1) \right|.$$ 

Again, since the Lebesgue measure is rotation invariant, then $\left| B_r \left( \frac{(m+1)r x}{|x|} \right) \cap \overline{B}_{(m+1)r}(0) \right| = \left| B_r ((m+1)re_1) \cap \overline{B}_{(m+1)r}(0) \right|$, which together with the last inequality show that the claim also holds for $m+1$. 

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We then deduce that the claim holds for every $m \geq 1$. Now, by choosing $m \gg 1$ such that $B_{kr_0}(0) \subset B_{mr}(0)$, we can derive from (2.4) that (2.3) holds with $i = m$. 

### 2.3 Part metric

In this subsection, we recall the decreasing property of part metric between two positive solutions of (1.1).

For given $u, v \in X^+$, the part metric between $u$ and $v$, denoted by $\rho(u, v)$, is defined by

$$\rho(u, v) = \inf \{\ln \alpha | \frac{1}{\alpha} u \leq v \leq \alpha u, \alpha \geq 1\}.$$

**Proposition 2.6.** (1) For any $u_1, u_2 \in X^+$ and $t > s$, $\rho(u(t, \cdot; s, u_1), u(t, \cdot; s, u_2)) \leq \rho(u_1, u_2)$.

(2) For any $\delta > 0$, $\sigma > 0$, $M > 0$ and $\tau > 0$ with $\delta < M$ and $\sigma \leq \ln \frac{M}{\sigma}$, there is $\tilde{\sigma} > 0$ such that for any $u_0$, $v_0 \in X^+$ with $\delta \leq u_0(x) \leq M$, $\delta \leq v_0(x) \leq M$ for $x \in \mathbb{R}^N$ and $\rho(u_0, v_0) \geq \sigma$, there holds

$$\rho(u(s + \tau, \cdot; s, u_0), u(s + \tau, \cdot; s, v_0)) \leq \rho(u_0, v_0) - \tilde{\sigma} \quad \forall s \in \mathbb{R}.$$

**Proof.** (1) See [26, Proposition 5.1].

(2) See [17, Proposition 3.4].

### 3 Uniqueness, stability, and frequency module of almost periodic solutions

In this section, we study the uniqueness, almost periodicity, and stability of a strictly positive bounded entire solution of (1.1) and prove Theorem 1.1.

We first prove two lemmas.

**Lemma 3.1.** Suppose that $g(t, x)$ is a uniformly continuous, bounded function in $t \in \mathbb{R}$ and $x \in \bar{D}$, with $g(t, x) > 0$ for all $(t, x) \in \mathbb{R} \times D$, and $f(t, x, u)$ satisfies (H2). Then for any fixed $x \in D$, the ODE

$$u_t = g(t, x) + uf(t, x, u) \quad (3.1)$$

has at most one strictly positive bounded entire solution $u^*(t)$.

**Proof.** It can be proved by properly modifying the arguments in [22, Theorem 2.1]. For completeness, we provide a proof in the following.

Fix $x \in D$. Suppose that (3.1) has two strictly positive bounded entire solutions $u_1^*(t)$ and $u_2^*(t)$, $u_1^*(t) \neq u_2^*(t)$. Without loss of generality, we may assume that $u_1^*(0) < u_2^*(0)$. Then by comparison principle for ODEs,

$$u_1^*(t) < u_2^*(t) \leq M \quad \forall t \in \mathbb{R}.$$
By (H2), there is \( \alpha > 0 \) such that
\[
\frac{d}{dt} \ln \left( \frac{u_1^*(t)}{u_2^*(t)} \right) = \frac{u_1^*(t)}{u_1^*(t)} - \frac{u_2^*(t)}{u_2^*(t)} = \frac{g(t, x)}{u_1^*(t)} - \frac{g(t, x)}{u_2^*(t)} + f(t, x, u_1^*(t)) - f(t, x, u_2^*(t)) > f(t, x, u_1^*(t)) - f(t, x, u_2^*(t)) \geq \alpha (u_2^*(t) - u_1^*(t)) \quad \forall t \in \mathbb{R}.
\] (3.2)

This implies that \( \ln \left( \frac{u_1^*(t)}{u_2^*(t)} \right) \) increases in \( \mathbb{R} \) and then there is some \( 0 < c < 1 \) such that
\[
\frac{u_1^*(t)}{u_2^*(t)} \leq \frac{u_1^*(0)}{u_2^*(0)} \leq c < 1 \quad \forall t \leq 0.
\]
Hence
\[
u_2^*(t) - u_1^*(t) = u_2^*(t) \left( 1 - \frac{u_1^*(t)}{u_2^*(t)} \right) \geq (1 - c)u_2^*(t) \quad \forall t \leq 0.
\]

This together with (3.2) implies that there is \( \beta > 0 \) such that
\[
\frac{d}{dt} \ln \left( \frac{u_1^*(t)}{u_2^*(t)} \right) \geq \beta \quad \forall t \leq 0.
\]
Integrating the above inequality from \( t \) to 0 for \( t \leq 0 \), we have
\[
\ln \left( \frac{u_1^*(t)}{u_2^*(t)} \right) \leq \ln \left( \frac{u_1^*(0)}{u_2^*(0)} \right) + \beta t \quad \forall t \leq 0
\]
and then
\[
\frac{u_1^*(t)}{u_2^*(t)} \leq \frac{u_1^*(0)}{u_2^*(0)} e^{\beta t} \quad \forall t \leq 0.
\]
Letting \( t \to -\infty \), we obtain
\[
\lim_{t \to -\infty} \frac{u_1^*(t)}{u_2^*(t)} = 0,
\]
which contradicts \( u_1^*(t) \) and \( u_2^*(t) \) being two strictly positive entire bounded solutions of (3.1). Hence (3.11) has at most one strictly positive bounded entire solution.

**Lemma 3.2.** Suppose that \( u^*(t, x) \) is a strictly positive and bounded measurable function on \( \mathbb{R} \times \bar{D} \), is differentiable in \( t \) for each \( x \in \bar{D} \), and satisfies (1.1) for \( t \in \mathbb{R} \) and \( x \in \bar{D} \), that is,
\[
\frac{\partial u^*}{\partial t} (t, x) = \int_D \kappa (y - x) u^*(t, y) dy + u^*(t, x) f(t, x, u^*(t, x)), \quad t \in \mathbb{R}, \ x \in \bar{D}.
\] (3.3)

Then \( u^*(t, x) \) is uniformly continuous in \( t \in \mathbb{R} \) and \( x \in \bar{D} \), and \( \mathbb{R} \ni t \mapsto u^*(t, \cdot) \in X \) is differentiable and hence \( u^*(t, x) \) is a strictly positive bounded solution of (1.1).
Proof. We first show that $u^*(t, x)$ is uniformly continuous in $t$ uniformly with respect to $x \in \bar{D}$ and is uniformly continuous in $x$ uniformly with respect to $t \in \mathbb{R}$, i.e., for any $\epsilon > 0$, there is $\delta > 0$ such that for any $t_1, t_2 \in \mathbb{R}$ and $x_1, x_2 \in \bar{D}$ with $|t_1 - t_2| < \delta$ and $|x_1 - x_2| < \delta$, there hold

$$
|u^*(t_1, x) - u^*(t_2, x)| < \epsilon \quad \forall x \in \bar{D}
$$

and

$$
|u^*(t_1) - u^*(t_2)| < \epsilon \quad \forall t \in \mathbb{R}.
$$

Observe that $u^*_t(t, x)$ is a bounded function of $t \in \mathbb{R}$ and $x \in \bar{D}$. This implies that $u^*(t, x)$ is uniformly continuous in $t$ uniformly with respect to $x \in \bar{D}$ and that

$$
g(t, x) := \int_D \kappa(y - x) u^*(t, y) dy
$$

is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$.

Assume that $u^*(t, x)$ is not uniformly continuous in $x \in \bar{D}$ uniformly with respect to $t \in \mathbb{R}$.

Then there is $\epsilon_0 > 0$, $t_n \in \mathbb{R}$, and $x_n, \bar{x}_n \in \bar{D}$ such that

$$
|x_n - \bar{x}_n| \leq \frac{1}{n} \quad \forall n \geq 1,
$$

and

$$
|u^*(t_n, x_n) - u^*(t_n, \bar{x}_n)| \geq \epsilon_0 \quad \forall n \geq 1.
$$

Let $u_n(t) = u^*(t + t_n, x_n)$ and $\bar{u}_n(t) = u^*(t + t_n, \bar{x}_n)$, then

$$
\frac{du_n(t)}{dt} = g(t + t_n, x_n) + u_n(t) f(t + t_n, x_n, u_n)
$$

and

$$
\frac{d\bar{u}_n(t)}{dt} = g(t + t_n, \bar{x}_n) + \bar{u}_n(t) f(t + t_n, \bar{x}_n, \bar{u}_n).
$$

Note that $u_n(t)$ and $\bar{u}_n(t)$ are uniformly continuous in $t \in \mathbb{R}$. Since $u^*(t, x)$ is strictly positive and bounded, there are $\delta_1 > 0$, $M \gg 1$ such that

$$
\delta_1 \leq u^*(t, x) \leq M \quad \forall t \in \mathbb{R}, x \in \bar{D}.
$$

This yields that $u_n(t)$ and $\bar{u}_n(t)$ are uniformly bounded. Furthermore, By (H2) and the uniform continuity of $g(t, x)$, we see that their derivatives are bounded, hence $u_n(t)$ and $\bar{u}_n(t)$ are equi-continuous. Therefore, using the usual diagonal argument and Arzela-Ascoli’s theorem, without loss of generality, we may assume that there are $u^*_1(t), u^*_2(t), g^*(t, x)$ and $f^*(t, x, u)$ such that

$$
\lim_{n \to \infty} u_n(t) = u^*_1(t), \quad \lim_{n \to \infty} \bar{u}_n(t) = u^*_2(t),
$$

$$
\lim_{n \to \infty} g(t + t_n, x + x_n) = g^*(t, x), \quad \lim_{n \to \infty} g(t + t_n, x + \bar{x}_n) = g^*(t, x),
$$

(3.8)
Proof of Theorem 1.1.

(a) Suppose that there are two strictly positive bounded entire solutions
locally uniformly in $t \in \mathbb{R}, x \in \bar{D}$, and $u \in \mathbb{R}$. By (3.6)-(3.10), $\frac{d u_n(t)}{d t}$ and $\frac{d \bar{u}_n(t)}{d t}$ also converge locally uniformly in $t \in \mathbb{R}$ as $n \to \infty$. It then follows that $u_1^*(t)$ and $u_2^*(t)$ are differentiable in $t$ and are two strictly positive bounded entire solutions of

$$u_t = g^*(t, 0) + u f^*(t, 0, u).$$

By Lemma 3.1, $u_1^*(t) \equiv u_2^*(t)$, in particular, $u_1^*(0) = u_2^*(0)$, which contradicts (3.5). Hence $u^*(t, x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$.

Next, we prove that $\mathbb{R} \ni t \mapsto u^*(t, \cdot) \in X$ is differentiable. By the uniform continuity of $u^*(t, x)$ in $t \in \mathbb{R}$ and $x \in \bar{D}$, $\mathbb{R} \ni t \mapsto u^*(t, \cdot) \in X$ is continuous. By (3.3), for each $x \in \bar{D}$, $u^*(\cdot, x) \in W^{1,1}_{loc}(\mathbb{R})$. Hence $u^*(t, x)$ is both super-solution and sub-solution of (1.1) on any interval $(a, b)$. Then, by Proposition 2.3, for any given $t_0 \in \mathbb{R},$

$$u^*(t, \cdot) = u(t, \cdot; t_0, u^*(t_0, \cdot)) \quad \forall t \geq t_0.$$  

This implies that $\mathbb{R} \ni t \mapsto u^*(t, \cdot) \in X$ is differentiable, and $u^*(t, x)$ is a strictly positive bounded entire solution of (1.1). \hfill \Box

Next, we prove Theorem 1.1.

**Proof of Theorem 1.1** (a) Suppose that there are two strictly positive bounded entire solutions $u_1^*$ and $u_2^*$ of (1.1). If $u_1^* \neq u_2^*$, then we can find $t_0 \in \mathbb{R}$ such that $u_1^*(t_0, \cdot) \neq u_2^*(t_0, \cdot)$. This implies that there is $\sigma > 0$ such that $\rho(u_1^*(t_0, \cdot), u_2^*(t_0, \cdot)) \geq \sigma$. By Proposition 2.6 (1),

$$\rho(u_1^*(t, \cdot), u_2^*(t, \cdot)) \geq \sigma \quad \forall t \leq t_0.$$  

Then by Proposition 2.6 (2), there is $\bar{\sigma} > 0$ such that

$$\rho(u_1^*(t_0, \cdot), u_2^*(t_0, \cdot)) \geq \rho(u_1^*(t_0 - k, \cdot), u_2^*(t_0 - k, \cdot)) - k\bar{\sigma} \quad \forall k = 1, 2, \ldots.$$  

(3.11)

Note that $\rho(u_1^*(t_0 - k, \cdot), u_2^*(t_0 - k, \cdot))$ is bounded for $k \in \mathbb{N}$. This together with (3.11) implies that

$$\rho(u_1^*(t_0, \cdot), u_2^*(t_0, \cdot)) \leq \rho(u_1^*(t_0 - k, \cdot), u_2^*(t_0 - k, \cdot)) - k\bar{\sigma} \to -\infty$$  

as $k \to \infty$, which is a contradiction. Therefore, a strictly positive bounded entire solution of (1.1) is unique.

(b) Suppose that $u^*(t, x)$ is a strictly positive bounded entire solution of (1.1). We show that $u^*(t, x)$ is almost periodic in $t$ uniformly with respect to $x \in \bar{D}$. By Lemma 3.2, $u^*(t, x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$. It then suffices to prove that for each $x \in \bar{D}$, $u^*(t, x)$ is almost periodic in $t$. To this end, let $\{t_n\}$ and $\{s_n\}$ be any two sequences of $\mathbb{R}$. By (H2)
and the uniform continuity of \( u^*(t, x) \), without loss of generality, we may assume that there are \( \tilde{f}(t, x, u), \check{f}(t, x, u), \hat{f}(t, x, u) \) satisfying (H2), and \( \bar{u}^*(t, x), \tilde{u}^*(t, x), \hat{u}^*(t, x) \) such that

\[
\lim_{n \to \infty} f(t+t_n, x, u) = \check{f}(t, x, u), \quad \lim_{m \to \infty} \tilde{f}(t+s_m, x, u) = \hat{f}(t, x, u), \quad \lim_{n \to \infty} f(t+t_n+s_n, x, u) = \hat{f}(t, x, u)
\]

locally uniformly in \((t, x, u) \in \mathbb{R} \times \bar{D} \times \mathbb{R}, and

\[
\lim_{n \to \infty} u^*(t+t_n, x) = \tilde{u}^*(t, x), \quad \lim_{m \to \infty} \check{u}^*(t+s_m, x) = \bar{u}^*(t, x), \quad \lim_{n \to \infty} u^*(t+t_n+s_n, x) = \hat{u}^*(t, x)
\]

locally uniformly in \((t, x) \in \mathbb{R} \times \check{D}. Moreover, using (1.1), \( \partial_t u^*(t+t_n, x) \) also converges locally uniformly in \((t, x) \in \mathbb{R} \times \check{D} \) as \( n \to \infty \), and then \( \tilde{u}^*(t, x) \) is differentiable in \( t \) and satisfies (1.1) with \( f \) being replaced by \( \check{f} \) for each \( t \in \mathbb{R} \) and \( x \in \check{D}. \) By Lemma 3.2, \( \tilde{u}^*(t, x) \) is a strictly positive bounded entire solution of (1.1) with \( f \) being replaced by \( \check{f}. \) Similarly, \( \check{u}^*(t, x) \) (resp. \( \hat{u}^*(t, x) \)) is a strictly positive bounded entire solution of (1.1) with \( f \) being replaced by \( \hat{f} \) (resp. \( \hat{f} \)). By Lemma 2.1, \( \tilde{f}(t, x, u) = \hat{f}(t, x, u) \). Then by (a), \( \tilde{u}^*(t, x) = \hat{u}^*(t, x) \). By Lemma 2.1 again, \( u^*(t, x) \) is almost periodic in \( t \).

By the arguments similar to the proof of almost periodicity of \( u^*(t, x) \) in \( t \), we have that \( u^*(t, x) \) is almost periodic in \( x \) when \( D = \mathbb{R}^N. \)

(c) Suppose that \( u^*(t, x) \) is a strictly positive bounded entire solution of (1.1). We prove that \( u^*(t, x) \) is asymptotically stable with respect to strictly positive perturbation. First note that there are \( \delta_1 > 0, M \gg 1 \) such that

\[
\delta_1 \leq u^*(t, x) \leq M \quad \forall \ t \in \mathbb{R}, x \in D.
\] 

For given \( u_0 \in X^+ \) and \( t_0 \in \mathbb{R}, let \( u(t, x; t_0, u_0) \) be the solution to (1.1) with \( u(t_0, x; t_0, u_0) = u_0(x). \) Observe that, for some \( 0 < b \ll 1, bu^*(t, x) \) is a subsolution of (1.1), and \( u \equiv M \) is a supersolution of (1.1) when \( M \gg 1 \). Therefore, we can find \( 0 < b \ll 1 \) and \( M \gg 1 \) such that

\[
bu^*(t_0, x) \leq u_0(x) \leq M \quad \forall \ x \in \check{D}.
\]

By Proposition 2.3

\[
bu^*(t, x) \leq u(t, x; t_0, u_0) \leq M \quad \forall \ t \geq t_0, \ x \in \check{D}.
\]

Let \( \rho(t; t_0) = \rho(u(t + t_0, \cdot; u_0), u^*(t + t_0, \cdot)) \) for every \( t \geq 0 \). We claim that

\[
\limsup_{t \to \infty} \sup_{t_0 \in \mathbb{R}} \rho(t; t_0) = 0.
\] 

Suppose on the contrary that (3.14) is false. Then we can find sequences \( \{t_{0,n}\}_{n \geq 1} \) and \( \{t_n\}_{n \geq 1} \) with \( t_n \geq 1 + n \) for each \( n \geq 1 \) such that

\[
\sigma_0 := \inf_{n \geq 1} \rho(t_n; t_{0,n}) > 0.
\]
By proposition 2.6(1), we know that $\rho(t; t_{0,n}) \geq \rho(t; t_{0,n}) \geq \sigma_0$ for every $n \geq 1$ and $0 \leq t \leq t_n$. Thus, by (3.12), (3.13) and proposition 2.6(2), there is $\tilde{\delta} > 0$ such that

$$
\rho(t + 1; t_{0,n}) \leq \rho(t; t_{0,n}) - \tilde{\delta} \quad \forall \ n \geq 1, \ 0 \leq t < t_n.
$$

In particular, since $n < t_n$ for each $n \geq 1$,

$$
\rho(n + 1; t_{0,n}) \leq \rho(n; t_{0,n}) - \tilde{\delta} \leq \cdots \leq \rho(0; t_{0,n}) - (n + 1)\tilde{\delta} \quad \forall \ n \geq 1.
$$

Hence we have

$$
0 < \sigma_0 \leq \rho(t_{0,n}; t_{0,n}) \leq \rho(n + 1; t_{0,n}) \leq \rho(0; t_{0,n}) - (n + 1)\tilde{\delta} \quad \forall \ n \geq 1.
$$

This yields a contradiction since $\rho(0; t_{0,n}) = \rho(u^*(t_{0,n}, \cdot), u_0) \leq \ln(M) \rho$ for all $n \geq 1$. Hence we conclude that (3.14) must hold. Now, (3.14) implies that

$$
\lim_{t \to \infty} \sup_{t_0 \in \mathbb{R}} \| u^*(t + t_0, \cdot) - u(t + t_0, \cdot; t_0, u_0) \|_\infty = 0.
$$

This establishes the asymptotic stability of $u^*(t, x)$ with respect to strictly positive perturbations.

(d) Suppose that $u^*(t, x)$ is a strictly positive bounded entire solution of (1.1). We prove that $\mathcal{M}(u^*) \subset \mathcal{M}(f)$. For any given sequence $\{t_n\}$ in $\mathbb{R}$, suppose that $f(t + t_n, x, u) \to f(t, x, u)$ uniformly on bounded sets. By (a) and (b), there is a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that, $u^*(t + t_{n_k}, x) \to u^*(t, x)$ uniformly on bounded sets, as $k \to \infty$. Similarly, for any given sequence $\{x_n\}$ in $\mathbb{R}^N$, if $f(t, x + x_n, u) \to f(t, x, u)$ uniformly on bounded sets, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $u^*(t, x + x_{n_k}) \to u^*(t, x)$ as $k \to \infty$ locally uniformly. It then follows from Proposition 2.2 that $\mathcal{M}(u^*) \subset \mathcal{M}(f)$.

4 Existence and nonexistence of positive bounded entire solutions

In this section, we study the existence of a strictly positive bounded entire solution of (1.1) and prove Theorem 1.2.

Proof of Theorem 1.2 (a) First, suppose that (1.1) has a strictly positive bounded entire solution $u^*(t, x)$. By (H2), $f_{\text{inf}}(u) := \inf_{t \in t, x \in \bar{D}} f_u(t, x, u)$ is continuous in $u \geq 0$ and $f_{\text{inf}}(u) < 0$ for $u \geq 0$. Let $u_{\text{inf}}^* = \inf_{t \in t, x \in \bar{D}} u^*(t, x)$ and $u_{\text{sup}}^* = \sup_{t \in t, x \in \bar{D}} u^*(t, x)$. Then for any $0 < \lambda \leq -u_{\text{inf}}^* \cdot \sup_{u \in [0, u_{\text{sup}}^*]} f_{\text{inf}}(u)$, we have

$$
\begin{align*}
 f(t, x, u^*(t, x)) - f(t, x, 0) &= \int_0^1 \frac{d}{ds} f(t, x, su^*(t, x))ds \\
 &= u^*(t, x) \int_0^1 f_u(t, x, su^*(t, x))ds \\
 &\leq -\lambda \quad \forall t \in \mathbb{R}, \ x \in \bar{D}.
\end{align*}
$$

(4.1)
This implies that
\[
 u^*_t = \int_D \kappa(y-x)u^*(t,y)dy + u^*f(t, x, u^*(t, x)) \\
= \int_D \kappa(y-x)u^*(t,y)dy + u^*(f(t, x, 0) + f(t, x, u^*(t, x)) - f(t, x, 0)) \\
\leq \int_D \kappa(y-x)u^*(t,y)dy + u^*(f(t, x, 0) - \lambda) \quad \forall t \in \mathbb{R}, x \in \bar{D}.
\]

It then follows that \( \lambda_{PE}(a) \geq \lambda > 0 \), where \( a(t, x) = f(t, x, 0) \).

Next, suppose that \( \lambda_{PE}(a) > 0 \). Let \( M \gg 1 \). Then \( u(t, x) \equiv M \) is a supersolution of \( (1.1) \). By Proposition 2.3, \( u(t, x; -K, M) \leq M \). This implies that \( u(t, x; -K, M) \) decreases as \( K \) increases. Hence we can define
\[
(0 \leq) u^*(t, x) := \lim_{K \to \infty} u(t, x; -K, M)(\leq M) \quad \forall t \in \mathbb{R}, x \in \bar{D}.
\]
(4.2)

It is clear that \( u^*(t, x) \) is measurable in \( (t, x) \in \mathbb{R} \times \bar{D} \). Moreover, note that
\[
u_t(t, x; -K, M) = \int_D \kappa(y-x)u(t, y; -K, M)dy + u(t, x; -K, M)f(t, x, u(t, x; -K, M))
\]
for all \( t > -K \) and \( x \in \bar{D} \). This together with the Dominated Convergence Theorem implies that, for each fixed \( x \in \bar{D} \),
\[
u^*_t(t, x) = \int_D \kappa(y-x)u^*(t, y)dy + u^*f(t, x, u^*(t, x)) \quad \forall t \in \mathbb{R},
\]
and then \( u^*(t, x) \in W^{1,1}_{\text{loc}}(\mathbb{R}) \).

In the following, we prove that \( u^*(t, x) \) is strictly positive. We do so in two steps.

**Step 1.** In this step, we prove that there is \( r_x > 0 \) such that
\[
\inf_{t \in \mathbb{R}, y \in B_{r_x}(x) \cap D} u^*(t, y) > 0.
\]
(4.4)

Let \( \lambda \in \Lambda_{PE}(a) \) be such that \( 0 < \lambda < \lambda_{PE}, \lambda_{PE} - \lambda \ll 1 \). Let \( \phi \in X^+ \) satisfy \( \inf_{t \in \mathbb{R}} \phi(t, x) \geq 0, \| \phi \|_X = 1 \), for each \( x \in \bar{D}, \phi \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) and
\[
\lambda \phi(t, x) \leq L \phi(t, x) \quad \text{for a.e.} \ t \in \mathbb{R}.
\]

By (H2), \((f(t, x, 0) - f(t, x, b\phi) - \lambda) \phi(t, x) \leq 0 \) for \( 0 < b \ll 1 \). Thus for each \( x \in \bar{D}, u(t, x) = b\phi(t, x) \) solves
\[
\frac{\partial u(t, x)}{\partial t} \leq \int_D \kappa(y-x)u(t, y)dy + a(t, x)u(t, x) - \lambda u(t, x) \\
= \int_D \kappa(y-x)u(t, y)dy + u(t, x)f(t, x, u) \\
+ (f(t, x, 0) - f(t, x, u) - \lambda) u(t, x) \\
\leq \int_D \kappa(y-x)u(t, y)dy + u(t, x)f(t, x, u) \quad \text{for a.e.} \ t \in \mathbb{R}.
\]
Hence, $b\phi$ is a subsolution of \ref{1.1}. Therefore, by Proposition \ref{2.3}

$$u(t, x; -K, M) \geq u(t, x; -K, b\phi(-K, x)) \geq b\phi(t, x) \quad \forall t \geq -K, \ x \in \bar{D}. \quad (4.5)$$

Since $\inf_{t \in \mathbb{R}} \phi(t, x) \not\equiv 0$, we can find $x_0 \in D$ such that

$$\delta_1 := \inf_{t \in \mathbb{R}} b\phi(t, x_0) > 0.$$ 

Moreover, by the continuity of $\inf_{t \in \mathbb{R}} \phi(t, x)$ in $x$, we have

$$\inf_{t \in \mathbb{R}} b\phi(t, x) \geq \frac{\delta_1}{2} \text{ for } x \in D_0 := B_{r_0}(x_0) \cap D \text{ for some } r_0 > 0. \quad (4.6)$$

Observe that there is $m > 0$ such that $\|f(t, x, u(t, x; -K, M))\| \leq m$ for all $t \geq -K$ and $x \in D$. Thus $u(t, x; -K, M)$ solves

$$\partial_t u \geq \int_{D} \kappa(y - x)u(t, y)dy - mu(t, x) \quad \forall t > -K, \ x \in \bar{D}.$$ 

This together with \ref{4.5} implies that

$$u(t + 1, x; -K, M) \geq e^{-m} (e^{K} b\phi(t, \cdot))(x) \quad \forall t \geq -K, \ x \in \bar{D}, \quad (4.7)$$

where $K(u)(x) = \int_{D} \kappa(y - x)u(y)dy$ for $u \in X$. Hence

$$u^*(t, x) \geq e^{-m} (e^{K} b\phi(t, \cdot))(x) \quad \forall t \in \mathbb{R}, \ x \in D. \quad (4.8)$$

By the arguments of Proposition \ref{2.5} and \ref{4.6}, for each $x \in \bar{D}$, there are $r_x > 0$ and $\mu_x > 0$ such that

$$(e^{K} b\phi(t, \cdot))(y) \geq \mu_x \quad \forall t \in \mathbb{R}, \ y \in B_{r_x}(x) \cap D.$$ 

This together with \ref{4.8} implies \ref{4.4}.

**Step 2.** In this step, we prove that

$$\inf_{t \in \mathbb{R}, x \in D} u^*(t, x) > 0. \quad (4.9)$$

In the case that $D$ is bounded, \ref{4.9} follows from \ref{4.4}.

In the case that $D = \mathbb{R}^N$, by the almost periodicity of $a(t, x)$ in $x$, for any given $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that any ball of radius $r_\varepsilon$ contains some $\tilde{x} \in T_\varepsilon$, where

$$T_\varepsilon := \{ \tilde{x} : |a(t, x) - a(t, x + \tilde{x})| < \varepsilon \ \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \}.$$ 

For given $\varepsilon > 0$, we can find a sequence $\{\tilde{x}_n\}_{n \in \mathbb{N}} \in T_\varepsilon$ such that

$$\mathbb{R}^N = \bigcup_{n \in \mathbb{N}} B_{2r_\varepsilon}(\tilde{x}_n), \quad (4.10)$$

for given $\varepsilon > 0$, we can find a sequence $\{\tilde{x}_n\}_{n \in \mathbb{N}} \in T_\varepsilon$ such that

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for given $\varepsilon > 0$, we can find a sequence $\{\tilde{x}_n\}_{n \in \mathbb{N}} \in T_\varepsilon$ such that

$$\mathbb{R}^N = \bigcup_{n \in \mathbb{N}} B_{2r_\varepsilon}(\tilde{x}_n), \quad (4.10)$$
where \( B_{2r_\varepsilon}(\tilde{x}_n) := \{ x \in \mathbb{R}^N : \| x - \tilde{x}_n \| < 2r_\varepsilon \} \). Let \( \varepsilon = \frac{\lambda}{2} \). Then

\[
\frac{\partial (\phi(t,x))}{\partial t} \leq \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t, x + \tilde{x}_n)\phi(t,x) + (a(t, x) - a(t, x + \tilde{x}_n) - \lambda)\phi(t,x)
\]

\[
\leq \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t, x + \tilde{x}_n)\phi(t,x) + (\varepsilon - \lambda)\phi(t,x)
\]

\[
= \int_{\mathbb{R}^N} \kappa(y-x)\phi(t,y)dy + a(t, x + \tilde{x}_n)\phi(t,x) - \frac{\lambda}{2}\phi(t,x).
\]

Hence, for some \( 0 < \tilde{b} < 1 \), \( \tilde{b}\phi \) is a subsolution of

\[
\partial_t u(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy + u(t,x)f(t,x + \tilde{x}_n, u(t,x)), \quad x \in \mathbb{R}^N. \tag{4.11}
\]

By Proposition 2.3 we have

\[
\tilde{b}\phi(t,x) \leq u(t, x + \tilde{x}_n; -K, M) \quad \text{for } t \geq -K, \ x \in \mathbb{R}^N, \ \tilde{x}_n \in T_\varepsilon.
\]

By arguments similar to (4.7), we have

\[
u(t + 1, x + \tilde{x}_n; -K, M) \geq e^{-m}e^{K\tilde{b}\phi(t, \cdot)}, \quad \forall t \geq -K, \ x \in \mathbb{R}^N. \tag{4.12}
\]

Without loss of generality, we may assume \( x_0 = 0 \) in (4.6). Then by Proposition 2.5 (4.6), and (4.7), there is \( \tilde{\delta}_2 > 0 \) such that

\[
u(t + 1, x + \tilde{x}_n; -K, M) \geq \tilde{\delta}_2 \quad \forall t \geq -K, \ x \in \mathbb{B}_{2r_\varepsilon}(0).
\]

This together with (4.10) implies that

\[
u(t + 1, x; -K, M) \geq \tilde{\delta}_2 \quad \forall t \geq -K, \ x \in \mathbb{R}^N, \tag{4.13}
\]

which implies (4.9).

By (4.2), (4.3), (4.9), and Lemma 3.2, \( u^*(t,x) \) is a strictly positive bounded entire solution of (1.1).

(b) Assume that \( \lambda_{PL} < 0 \). For any \( u_0 \geq 0 \),

\[
u(t, x; 0, u_0) \leq \Phi(t, 0)u_0 \quad \forall t \geq 0, \ x \in D.
\]

Note that

\[
\limsup_{t \to \infty} \frac{\ln \| \Phi(t, 0)u_0 \|}{t} \leq \lambda_{PL} < 0.
\]

Hence

\[
0 \leq \limsup_{t \to \infty} \| u(t, \cdot; 0, u_0) \| \leq \limsup_{t \to \infty} \| \Phi(t, 0)u_0 \| = 0.
\]

The theorem thus follows.
Remark 4.1. As mentioned in Remark 1.1, the definitions of $\lambda_{PL}(a), \lambda_{PL}'(a), \lambda_{PE}(a),$ and $\lambda_{PE}'(a)$ apply to general $a(t,x)$ which is bounded and uniformly continuous. When $f(t,x,u)$ is not assumed to be almost periodic in $t$, if $\lambda_{PE}(a) > 0$, then $u^*(t,x)$ defined in (4.2) is bounded on $\mathbb{R} \times \bar{D}$, is differentiable in $t$ and $\inf_{t \in \mathbb{R}} u^*(t,x) > 0$ for each $x \in \bar{D}$, and satisfies (1.1) for each $t \in \mathbb{R}$ and $x \in \bar{D}$. Hence $\partial_t u^*(t,x)$ is bounded on $\mathbb{R} \times \bar{D}$. We can also prove that $u^*(t,x)$ is continuous in $x \in \bar{D}$. In fact, let $g^*(t,x) = \int_D \kappa(y-x) u^*(t,y) dy$. It is clear that $g^*(t,x)$ is uniformly continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$ for any $x_0 \in \bar{D}$ and $\{x_n\} \subset \bar{D}$ with $x_n \to x_0$, without loss of generality, we may assume that $u^*(t,x_n) \to \tilde{u}^*(t)$, $g(t,x_n) \to g(t,x_0)$, and $f(t,x_n,u^*(t,x_n)) \to f(t,x_0,\tilde{u}^*(t))$ as $n \to \infty$ locally uniformly in $t \in \mathbb{R}$. By (4.3), we have
\[
\tilde{u}^*_t = g(t,x_0) + \tilde{u}^*(t)f(t,x_0,\tilde{u}^*(t)) \quad \forall t \in \mathbb{R}
\]
and
\[
u^*_t(t,x_0) = g(t,x_0) + u^*(t,x_0)f(t,x_0,u^*(t,x_0)) \quad \forall t \in \mathbb{R}.
\]
By (4.4) and Lemma 3.1, $\tilde{u}^*(t) = u^*(t,x_0)$. It then follows that $u^*(t,x)$ is also continuous in $x \in \bar{D}$. But $u^*(t,x)$ may not be strictly positive. However, if $D$ is bounded, then $u^*(t,x)$ is a strictly positive entire solution of (1.1) and is asymptotically stable with respect to positive perturbations.

5 Monotonicity of $\lambda_{PE}(a,D)$ in $D$

In this section, we investigate the monotonicity of $\lambda_{PE}(a,D)$ in $D$ and prove Theorem 1.3

Proof of Theorem 1.3 Let $D_1 \subset D_2$ be given. Without loss of generality, we may assume that $\lambda_{PE}(a,D_2) = 0$. For otherwise, we can replace $a(t,x)$ by $a(t,x) - \lambda_{PE}(a,D_2)$. It then suffices to prove that $\lambda_{PE}(a,D_1) \leq 0$. We prove it by contradiction.

First, assume that $\lambda_{PE}(a,D_1) > 0$. Let $\delta > 0$ be such that $\lambda_{PE}(a-\delta,D_1) > 0$. By Theorem 1.2, there is a strictly positive bounded entire solution $u_1^*(t,x)$ of
\[
u_t = \int_{D_1} \kappa(y-x) u(t,y) dy + u(t,x)(a(t,x) - \delta - u(t,x)), \quad x \in \bar{D}_1.
\]
For given $M > 0$, let $u_2(t,x; -K,M)$ be the solution of
\[
u_t = \int_{D_2} \kappa(y-x) u(t,y) dy + u(t,x)(a(t,x) - \delta/2 - u(t,x)), \quad x \in \bar{D}_2
\]
with $u_2(-K,x; -K,M) = M$. By Propositions 2.3 and 2.4
\[
u_1^*(t,x) \leq u_2(t,x; -K,M) \quad \forall t \geq -K, x \in \bar{D}_1, \quad M \gg 1,
\]
and
\[
u_2(t,x; -K,M) \leq M \quad \forall t \geq -K, x \in \bar{D}_2, \quad M \gg 1.
\]
Fix $M \gg 1$. By the arguments of Theorem 1.2,

$$u^*_2(t, x) := \lim_{K \to \infty} u_2(t, x; -K, M) (\leq M), \quad t \in \mathbb{R}, \ x \in \bar{D}_2$$

is well defined, and satisfies (5.2) for all $t \in \mathbb{R}$ and $x \in \bar{D}_2$.

Next, we claim that $u^*_2(t, x)$ is strictly positive. We divide the proof of the claim into two cases.

**Case 1.** $D_2$ is bounded. Note that there is $m > 0$ such that

$$a(t, x) - \delta/2 - u_2(t, x; -K, M) \geq -m \quad \forall \ t \geq -K, \ x \in \bar{D}_2.$$

This together with (5.2) and Proposition 2.3 implies that

$$u_2(t, \cdot; -K, M) \geq e^{-m} e^{K_2 u_2(t - 1, \cdot; -K, M)} \quad \forall \ t \geq -K + 1,$$

where $K_2 u = \int_{D_2} \kappa(y - x) u(y) dy$ for $u \in C^b_{\text{unif}}(\bar{D}_2)$. By (5.3), there is $\delta_0 > 0$ such that

$$\int_{D_2} u_2(t - 1, x; -K, M) dx \geq \delta_0 \quad \forall \ t \geq -K + 1, \ x \in \bar{D}_2.$$

This together with the arguments of Proposition 2.5 implies that there is $\tilde{\delta}_0 > 0$ such that

$$u_2(t, x; -K, M) \geq \tilde{\delta}_0 \quad \forall \ t \geq -K + 1, \ x \in \bar{D}_2.$$

It then follows that

$$u^*_2(t, x) \geq \tilde{\delta}_0 \quad \forall \ t \in \mathbb{R}, \ x \in \bar{D}_2.$$

Hence the claim holds in the case that $D_2$ is bounded.

**Case 2.** $D_2 = \mathbb{R}^N$. By the almost periodicity of $a(t, x)$ in $x$, there are $\{x_n\} \subset \mathbb{R}^N$ and $r > 0$ such that

$$\mathbb{R}^N = \bigcup_{n=1}^{\infty} B_r(x_n),$$

and

$$|a(t, x + x_n) - a(t, x)| \leq \delta/2 \quad \forall \ t \in \mathbb{R}, \ x \in \mathbb{R}^N.$$

Then

$$\partial_t u^*_1(t, x) = \int_{D_1} \kappa(y - x) u^*_1(t, y) dy + u^*_1(t, x)(a(t, x) - \delta - u^*_1(t, x))$$

$$\leq \int_{D_1} \kappa(y - x) u^*_1(t, y) dy + u^*_1(t, x)(a(t, x + x_n) - \delta/2 - u^*_1(t, x)) \quad \forall \ t \in \mathbb{R}, \ x \in \bar{D}_1.$$

This together with Propositions 2.3 and 2.4 implies that

$$u_2(t, x + x_n; -K, M) \geq u^*_1(t, x) \quad \forall \ t \geq -K, \ x \in \bar{D}_1.$$
and then
\[ u_2^*(t, x) \geq u_1^*(t, x - x_n) \quad \forall t \in \mathbb{R}, \ x - x_n \in \bar{D}_1. \]

By the arguments in Case 1, there is \( \delta_0 > 0 \) such that
\[ u_2^*(t, x) \geq \delta_0 \quad \forall t \in \mathbb{R}, \ x \in B_r(x_n), \ n \geq 1. \]

Therefore, \( u^*(t, x) \) is strictly positive and the claim also holds in the case \( D_2 = \mathbb{R}^N \).

Now, by Lemma 3.2, \( u_2^*(t, x) \) is uniformly continuous in \( t \in \mathbb{R} \) and \( x \in \bar{D}_2 \). Hence \( u_2^*(t, x) \) can be used as a test function in the definition of \( \Lambda_{PE}(a, D_2) \).

\[-\frac{\partial u_2^*}{\partial t} + \int_{D_2} \kappa(y - x)u_2^*(t, y)dy + a(t, x)u_2^*(t, x) \geq \frac{\delta}{2}u_2^*(t, x), \quad t \in \mathbb{R}, \ x \in \bar{D}_2. \tag{5.6}\]

This implies that \( \lambda_{PE}(a, D_2) \geq \frac{\delta}{2} > 0 \), which is a contradiction. Hence \( \lambda_{PE}(a, D_1) \leq 0 \). The theorem is thus proved. \( \square \)

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