Global Solutions of the Equations of 3D Compressible Magnetohydrodynamics with Zero Resistivity

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To my family and my wife Candy

Abstract. We prove the global-in-time existence of $H^2$ solutions of the equations of compressible magnetohydrodynamics with zero magnetic resistivity in three space dimensions. Initial data are taken to be small in $H^2$ modulo a constant state and initial densities are positive and essentially bounded. The present work generalizes the results obtained by Kawashima in [14].

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1. Introduction

We prove the global existence of $H^2$ solutions of the equations of barotropic, compressible magnetohydrodynamics (MHD) in three space dimensions:

$$
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u^j)_t + \text{div}(\rho u u) + P(\rho)_{xj} + \left(\frac{1}{2}|B|^2\right)_{xj} - \text{div}(B^j B) = \mu \Delta u^j + \lambda \text{div} u_{xj}, \\
B^j_t + \text{div}(B^j u - u^j B) = 0, \\
\text{div} B = 0.
$$

Here $x \in \mathbb{R}^3$ is the spatial coordinate, $t \geq 0$ is time, $\rho, u = (u^1, u^2, u^3)$ and $B = (B^1, B^2, B^3)$ are the unknown functions of $x$ and $t$ representing the density, velocity, and (scaled) magnetic field in a compressible ionized fluid, $P = P(\rho)$ is the pressure, $\mu$, and $\lambda$ are positive viscosity constants, div and $\Delta$ are the usual spatial divergence and Laplace operators.

The above system (1.1)-(1.4) is obtained by combining the Navier-Stokes equations for compressible barotropic flow with Maxwell’s equations in free space and the ideal Ohm’s law. We briefly outline the derivation (see Cabannes [2] for details) Using Ampere’s law, we have

$$
\varepsilon_0 \frac{\partial E}{\partial t} + J = \mu_0^{-1} \text{curl} B,
$$

where $E, J, \text{ and } B$ are the electric field, current density and magnetic field, and $\varepsilon_0, \mu_0$ are constants. The first term on the left is taken to be negligible in the given application, and so by the Biot-Savart law, the magnetic force per volume of fluid is

$$
J \times B = \mu_0^{-1} \text{curl} B \times B = \frac{\text{div}(BB^T)}{\mu_0} - \nabla \left( \frac{|B|^2}{2\mu_0} \right),
$$
and it gives the forcing terms following the pressure in (1.2) after scaling out the factor $\mu_0$. On the other hand, we combine the ideal Ohm’s law $E + u \times B = 0$ with Faraday’s law so that
\[
\frac{\partial B}{\partial t} = -\text{curl}E = \text{curl}(u \times B),
\]
and together with the Gauss’s law for magnetism (1.3), we obtain equation (1.3).

The system (1.1)-(1.4) is solved subject to initial conditions
\[
(\rho(\cdot, 0), u(\cdot, 0), B(\cdot, 0)) = (\rho_0, u_0, B_0),
\]
where $\rho_0$ is bounded above and below away from zero, $\text{div}B_0 = 0$ in a suitable sense and modulo constants, $(\rho_0, u_0, B_0)$ is small in $H^2(\mathbb{R}^3)$.

The subject of MHD was first initiated by Alfvén [1] in 1940’s, since then it has become one of the most challenging topics in fluid dynamics. In the fully viscous case, we further have the resistivity term $\nu B_j$ on the right side of (1.3), so that the system becomes
\[
\rho_t + \text{div}(\rho u) = 0,
\]
\[
(\rho u^j)_t + \text{div}(\rho u^j u) + P(\rho)_{x_j} + \left(\frac{1}{2}|B|^2\right)_{x_j} - \text{div}(B^j B) = \mu \Delta u^j + \lambda \text{div}u_{x_j},
\]
\[
B^j_t + \text{div}(B^j u - u^j B) = \nu B^j,
\]
\[
\text{div}B = 0,
\]
where $\nu > 0$ is the resistivity constant. A large variety of solutions to (1.6)-(1.9) can be obtained for different initial data stated in (1.5). When $(\rho_0, u_0, B_0)$ is taken to be close to a constant in $H^3(\mathbb{R}^3)$, Kawashima [14] proved global existence of “small-smooth” solutions to the system (1.1)-(1.4), and such solutions remain close to the same constant in $H^3(\mathbb{R}^3)$. On the other hand, for initial data with arbitrary large energy and nonnegative density, Sart [20] and also Hu and Wang [12]-[13] showed global existence of “large-weak” solutions to (1.1)-(1.4) based on a method introduced by Lions [16] and Feireisl [4]-[5]. Solutions in this general class possess very little regularity, and some of those solutions may be even non-physical (see [9] and [10]). In between those two types of solutions just mentioned, there is another type called “intermediate-class” solutions which is first introduced by Hoff [6] for Navier-Stokes equations and later extended by Suen and Hoff [23] for MHD. In this case, initial data is taken to be small in $L^2(\mathbb{R}^3)$, and initial densities are assumed to be nonnegative and essentially bounded. Solutions possess just enough regularity which makes it possible for one to develop uniqueness and continuous dependence theory (see Hoff and Santos [11] for Navier-Stokes equations).

On the other hand, for the case when resistivity $\nu$ becomes zero (system (1.1)-(1.4)), there is not much known result in the literature. In Kawashima [14], assuming initial data in Sobolev spaces $H^s$ with higher regularity ($s > 5/2$), both local and global existence theory of smooth solutions for the Cauchy problem of (1.1)-(1.4) were established. The goal of the present paper is thus to generalize Kawashima’s result by establishing global existence of smooth solutions with small initial $H^2$ data. It seems to be less trivial than one could expect due to the absence of the damping mechanism in the transport equation of $B$. We use a kind of standard energy estimate similar to that which is used for the incompressible viscoelastic fluids suggested by Lei-Liu-Zhou [15]. It enables us to extract some weak dissipation from the system which is crucial in our analysis.

We introduce two variables associated with the system (1.1)-(1.4) which are important to our analysis. The first one is the usual vorticity matrix $\omega = \omega^{j,k} = u^j_{x_k} - u^k_{x_j}$, while the other one is the effective viscous flux $F$ given by
\[
F = (\mu + \lambda)\text{div}u - P(\rho) + P(\bar{\rho}).
\]
where $\bar{\rho}$ is a positive constant density. By adding and subtracting terms, we can rewrite the momentum equation (1.2) in terms of $F$ and $\omega$:
\[
\rho u^j_t = F_{x_j} + \mu \omega^{j,k}_{x_k} - \left(\frac{1}{2}|B|^2\right)_{x_j} + \text{div}(B^j B).
\]
The decomposition \((1.11)\) also implies that
\[
\Delta F = \text{div}(g),
\]
where \(g^j = \rho \dot{w}^j + \left(\frac{1}{2}|B^j|^2\right)_{x_j} - \text{div}(B^j B)\). We refer to Hoff [9] for a more detailed discussion of \(F\).

We now give a precise formulation of our results. First concerning the pressure \(P\) we assume that
\[
P \in C([0, \infty)) \cap C^3((0, \infty)), P(0) = 0 \text{ and } P(\rho) > 0 \text{ for } \rho > 0;
\]
and there are positive densities \(\rho'' > \rho'\) such that \(P\) is increasing on \([0, \rho']\) and on \([\rho'', \infty)\), \(P(\rho) > P(\rho')\) for \(\rho > \rho'\), and \(P(\rho) > P(\rho'')\) for \(\rho > \rho''\).

Next we fix a positive reference density \(\bar{\rho}\) and then choose positive bounding densities \(\underline{\rho} \) and \(\bar{\rho}\) satisfying
\[
\underline{\rho} < \min\{\bar{\rho}, \rho'\} \text{ and } \max\{\bar{\rho}, \rho''\} < \bar{\rho},
\]
and finally we define a positive number \(\delta\) by
\[
\delta = \min\{\min\{\bar{\rho}, \rho'\} - \underline{\rho}, \bar{\rho} - \max\{\bar{\rho}, \rho''\}\}, \quad \frac{1}{2}(\bar{\rho} - \underline{\rho}).
\]
(Notice that \(\delta\) need not be “small” in the usual sense.) Concerning the diffusion coefficients \(\mu, \lambda\), we assume that
\[
\mu, \lambda > 0.
\]
Concerning the initial data \((\rho_0, u_0, B_0)\) we assume that there is a positive number \(d < \delta\) such that
\[
\underline{\rho} + d < \text{ess inf } \rho_0 \leq \text{ess sup } \rho_0 < \bar{\rho} - d.
\]
We assume also that
\[
\text{div} B_0 = 0
\]
and we write
\[
C_0 = \|\rho_0 - \bar{\rho}\|_{H^2} + \|u_0\|_{H^2} + \|B_0\|_{H^2}. \tag{1.19}
\]

Weak solutions are defined in the usual way; we say that \((\rho, u, B)\) is a weak solution of \((1.1)-(1.5)\) provided that \((\rho - \bar{\rho}, \rho u, B) \in C([0, \infty); H^{-1}(\mathbb{R}^3))\) with \((\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0)\), \(\nabla u, \nabla B \in L^2(\mathbb{R}^3 \times (0, \infty)), \text{div} B(\cdot, t) = 0\) in \(\mathcal{D}'(\mathbb{R}^3)\) for \(t > 0\), and the following identities hold for times \(t_2 \geq t_1 \geq 0\) and \(C^1\) test functions \(\varphi\) having uniformly bounded support in \(x\) for \(t \in [t_1, t_2]\):

\[
\int_{\mathbb{R}^3} \rho(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi) dx dt, \tag{1.20}
\]

\[
\int_{\mathbb{R}^3} \rho u^j(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [\rho u^j \varphi_t + \rho u^j u \cdot \nabla \varphi + P(\rho) \varphi_{x_j}] dx dt
\]
\[+ \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[\frac{1}{2} |B|^2 \varphi_{x_j} - B^j B \cdot \nabla \varphi \right] dx dt
\]
\[- \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \left[\mu \nabla u^j \cdot \nabla \varphi + \lambda (\text{div } u) \varphi_{x_j} \right] dx dt, \tag{1.21}
\]
and
\[
\int_{\mathbb{R}^3} B^j(x, \cdot) \varphi(x, \cdot) dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [(B^j u - u^j B) \cdot \nabla \varphi] dx dt. \tag{1.22}
\]

We denote the material derivative of a given function \(v\) by \(\dot{v} = v_t + \nabla v \cdot u\), and if \(X\) is a Banach space we will abbreviate \(X^3\) by \(X\) when convenient. Finally if \(I \subset [0, \infty)\) is an interval, \(C^1(I; X)\) will be the elements \(v \in C(I; X)\) such that the distribution derivative \(v_t \in \mathcal{D}'(\mathbb{R}^3 \times \text{int } I)\) is realized as an element of \(C(I; X)\).

The following is the main result of this paper:
Theorem 1.1 Assume that the system parameters \( P, \mu, \lambda \) in (1.1)-(1.4) satisfy the conditions in (1.13)-(1.16) and let \( \delta \) be as defined in (1.15). Let positive numbers \( N \) and \( d < \delta \) be given. Then there are positive constants \( a, C \) and \( \theta \) depending on the parameters and assumptions in (1.13)-(1.16), on \( N \), and on a positive lower bound for \( d \), such that, if initial data \((\rho_0, u_0, B_0)\) is given satisfying (1.17)-(1.19) with
\[
C_0 < a, \tag{1.23}
\]
then there is a solution \((\rho, u, B)\) to (1.1)-(1.5) in the sense of (1.20)-(1.22) on all of \( \mathbb{R}^3 \times [0, \infty) \). The solution satisfies the following:
\[
\partial_t^j D_x^\alpha u \in L^\infty([0, \infty); H^{2j-|\alpha|}(\mathbb{R}^3)), \quad \partial_t^j D_x^\alpha \rho, \partial_t^j D_x^\alpha B \in L^\infty([0, \infty); H^{2j-|\alpha|}(\mathbb{R}^3)), \tag{1.24}
\]
for all \( j, \alpha \) satisfying \( 2j + |\alpha| \leq 2 \), with
\[
\rho \leq \rho(x, t) \leq \bar{\rho} \text{ a.e. on } \mathbb{R}^3 \times [0, \infty), \tag{1.26}
\]
and
\[
sup_{0 < s \leq t} \left( \| (\rho - \bar{\rho}, u, B)(\cdot, s) \|_{L^2}^2 + \| (\rho_t, u_t, B_t)(\cdot, s) \|_{L^2}^2 \right) + \int_0^t \left( \| \nabla u(\cdot, s) \|_{L^2}^2 + \| u_t(\cdot, s) \|_{L^1}^2 \right) ds \leq CC_0^6. \tag{1.27}
\]

The proof is given below in sections 2–5 and begins with a number of \textit{a priori} bounds for local-in-time smooth solutions. The existence of such smooth solutions is therefore crucial, and for this we rely on the following result of Kawashima [14], pp. 34–35 and pp. 52–53:

**Theorem (Kawashima)** Assume that \( \mu \) and \( \lambda \) are strictly positive and that the pressure \( P \in C^3((0, \infty)) \). Then given \( \bar{\rho} > \rho > 0 \) and \( C_3 > 0 \), there is a positive time \( T \) depending on \( \bar{\rho}, \rho, \) and \( C_3 \) and on the system parameters \( \mu, \lambda \) and \( P \), such that, if initial data \((\rho_0 - \bar{\rho}, u_0, B_0)\) is given satisfying
\[
\| (\rho_0 - \bar{\rho}, u_0, B_0) \|_{H^3(\mathbb{R}^3)} < C_3, \]
\[
\inf \rho_0 \geq \underline{\rho}, \text{ and } \text{div} B_0 = 0, \text{ then there is a solution } (\rho, u, B) \text{ to (1.1)-(1.5) defined on } \mathbb{R}^3 \times [0, T] \text{ satisfying}
\]
\[
\rho - \bar{\rho}, B \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3)) \tag{1.28}
\]
and
\[
u \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^1(\mathbb{R}^3)) \cap L^2([0, T]; H^4(\mathbb{R}^3)). \tag{1.29}
\]
The equations in (1.1)-(1.4) are satisfied in the sense of equality of weak derivatives on \( \mathbb{R}^3 \times (0, T) \), each of these weak derivatives realized as an element of \( C([0, T]; H^1(\mathbb{R}^3)) \), and the weak forms (1.20)-(1.22) hold.

Furthermore, there is a positive number \( a \) depending on \( \mu, \nu, \xi, \) and \( P \) such that, if the above hypotheses hold with \( C_3 < a \), then the solution exists on all of \( \mathbb{R}^3 \times [0, \infty) \).

As a by-product of our analysis we show that when the initial data \((\rho_0, u_0, B_0)\) satisfies the smallness assumption (1.23) and is in \( H^3 \), but is not necessarily small in \( H^3 \), then the corresponding smooth solution in fact exists for all time. Since the weak solutions of Theorem 1.1 are constructed as limits of such global smooth solutions, we conclude that small-energy smooth solutions are dense in the set of small-energy weak solutions. The result is as follows:

**Theorem 1.2** Assume that the system parameters \( P, \mu, \lambda \) in (1.1)-(1.4) satisfy the conditions in (1.13)-(1.16) and let \( \delta \) be as defined in (1.15). Let positive numbers \( N \) and \( d < \delta \) be given. Then given initial data \((\rho_0, u_0, B_0)\) satisfying (1.17)-(1.19), the smallness condition (1.23), and the regularity condition \((\rho_0 - \bar{\rho}, u_0, B_0) \in H^3(\mathbb{R}^3) \), the corresponding smooth solution \((\rho, u, B)\) described in Kawashima’s theorem above exists on all of \( \mathbb{R}^3 \times [0, \infty) \).
This paper is organized as follows. We begin the proofs of Theorem 1.1 in section 2 with a number of \textit{a priori} bounds for local-in-time smooth solutions. In section 3 we derive the necessary bounds for density by applying the estimates in Theorem 2.1 in a maximum principle argument along particle trajectories of the velocity, making important use of the monotonicity of $P$ as described in (1.13). The small-energy assumption (1.23) then enables us to close these arguments to show in Theorem 3.1 that both the pointwise bounds for density and the \textit{a priori} energy bounds of Theorem 3.1 do hold as long as the smooth solution exists. Finally section 4 we apply these now uncontingent \textit{a priori} bounds to show that the $H^3$ norms of low-energy smooth solutions cannot blow up in finite time, and as a consequence that such solutions can be extended to all of $\{t > 0\}$, thereby proving Theorem 1.1 and 1.2.

We make use of the following standard facts (see Ziemer \cite{Ziemer} Theorem 2.1.4, Remark 2.4.3, and Theorem 2.4.4, for example). First, given $r \in [2, 6]$ there is a constant $C(r)$ such that for $w \in H^1(\mathbb{R}^3)$,
\begin{equation}
\|w\|_{L^r(\mathbb{R}^3)} \leq C(r) \left( \|w\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla w\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r} \right). \tag{1.30}
\end{equation}
Next, for any $r \in (1, \infty)$ there is a constant $C(r)$ such that for $w \in W^{1,r}(\mathbb{R}^3)$,
\begin{equation}
\|w\|_{L^\infty(\mathbb{R}^3)} \leq C(r) \|w\|_{W^{1,r}(\mathbb{R}^3)} \tag{1.31}
\end{equation}
and
\begin{equation}
\langle w \rangle_{\mathbb{R}^3}^{\alpha} \leq C(r) \|\nabla w\|_{L^r(\mathbb{R}^3)}, \tag{1.32}
\end{equation}
where $\alpha = 1 - 3/r$. Finally if $\Gamma$ is the fundamental solution of the Laplace operator on $\mathbb{R}^3$, then there is a constant $C$ such that for any $f \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$,
\begin{equation}
\|\Gamma_x \ast f\|_{L^\infty(\mathbb{R}^3)} \leq C \left( \|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^4(\mathbb{R}^3)} \right). \tag{1.33}
\end{equation}
2. Energy Estimates

In this section we derive a priori bounds for smooth solutions \((\rho - \tilde{\rho}, u, B) \in C([0, T]; H^3(\mathbb{R}^3))\) as described in Kawashima’s theorem in section 1. These bounds will depend only on the quantities \(C_0\) and \(d\) appearing in (1.17)-(1.19) and will be independent of the initial regularity and the time of existence. Specifically, we define a functional \(A(t)\) for a given solution by

\[
A(t) = \sup_{0 < s \leq t} \left( ||(\rho - \tilde{\rho}, u, B)(\cdot, s)||_{H^2}^2 + ||(\rho_t, u_t, B_t)(\cdot, s)||_{L^2}^2 \right) + \int_0^t \left( ||\nabla u(\cdot, s)||_{H^2}^2 + ||u_t(\cdot, s)||_{H^1}^2 \right) ds,
\]

and we obtain the following a priori bound for \(A(t)\) under the assumptions that the initial energy \(C_0\) in (1.19) is small and that the density remains bounded above and below away from zero:

**Theorem 2.1** Assume that the system parameters in (1.1)-(1.4) satisfy the conditions in (1.13)-(1.16) and let \(\theta\) be as defined in (1.15). Let positive numbers \(N\) and \(d < \delta\) be given. Then there are positive constants \(a, M, \theta\) depending on the parameters and assumptions in (1.13)-(1.16) and on a positive lower bound for \(d\), such that: if \((\rho, u, B)\) is a solution of (1.1)-(1.4) on \(\mathbb{R}^3 \times (0, T)\) in the sense of Kawashima’s theorem with initial data \((\rho_0 - \tilde{\rho}, u_0, B_0) \in H^3(\mathbb{R}^3)\) satisfying (1.17)-(1.19) with \(C_0 < a\), and if

\[
0 \leq \rho(x, t) \leq \tilde{\rho} \text{ on } \mathbb{R}^3 \times [0, T],
\]

then

\[
A(T) \leq MC_0^\theta.
\]

The proof will be given in a sequence of lemmas. It will be seen that the assumed regularity (1.28)-(1.29) suffices to justify the estimates that follow. The most subtle part in the analysis is to extract the dissipative structure of the system, which can be partially accomplished by introducing auxiliary variable functions \(w\) and \(v\) in Lemma 2.4. The methods we apply here are inspired by those of Lei-Liu-Zhou [15] for incompressible viscoelastic fluids.

We begin with the following \(L^2\) energy estimate:

**Lemma 2.2** Assume that the hypotheses and notations of Theorem 2.1 are in force. Then

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( G(\rho) + \frac{1}{2} \rho |u|^2 + \frac{1}{2} |B|^2 \right) dx + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + \lambda (\text{div } u)^2) dx = 0.
\]

**Proof.** We multiply the momentum equation by \(u^j\), sum over \(j\) and integrate to obtain that for \(0 \leq t \leq T\),

\[
\int_{\mathbb{R}^3} \left( \frac{1}{2} \rho |u|^2 dx \right) _0^t dx + \int_{0}^t \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + \lambda (\text{div } u)^2 \right) dx ds
\]

\[
+ \int_{0}^t \int_{\mathbb{R}^3} u \cdot \nabla P dx ds + \int_{0}^t \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + (\xi - \mu)(\text{div } u)^2 \right) dx ds = 0,
\]

where the divergence of a matrix is taken row-wise. Next we define

\[
G = \rho \int_{\tilde{\rho}}^\rho \frac{1}{s^2}(P(s) - \tilde{P}) ds
\]

where \(\tilde{P} = P(\tilde{\rho})\), so that by the mass equation,

\[
G(\rho_t) + \text{div}(G(\rho)u) + (P(\rho) - \tilde{P})\text{div } u = 0.
\]
Integrating and adding the result to (2.3) we then get
\[
\int_{\mathbb{R}^3} \left[ \frac{1}{2} \rho |u|^2 + G(\rho) \right] dx|_0^t + \int_0^t \int_{\mathbb{R}^3} u \cdot \text{div} \left[ \left( \frac{1}{2} |B|^2 \right) I_{3 \times 3} - BB^T \right] dxds \\
+ \int_0^t \int_{\mathbb{R}^3} [\mu |\nabla u|^2 + (\xi - \mu) (\text{div} u)^2] dxds = 0. \tag{2.4}
\]
Finally we multiply the magnetic field equation by \( B \) and integrate to get
\[
\int_{\mathbb{R}^3} \frac{1}{2} |B|^2 dx|_0^t + \int_0^t \int_{\mathbb{R}^3} B \cdot \text{div} \left( Bu^T - uB^T \right) dxds = 0. \tag{2.5}
\]
We then obtain (2.2) by adding (2.5) to (2.4) and using the fact that
\[
\int_0^t \int_{\mathbb{R}^3} [u \cdot \text{div} \left( \frac{1}{2} |B|^2 I_{3 \times 3} - BB^T \right) + B \cdot \text{div}(Bu^T - uB^T)] dxds = 0.
\]

Next we derive preliminary bounds for \( u, B \) and \( \rho \) in \( L^\infty([0,T];H^2(\mathbb{R}^3)) \):

**Lemma 2.3** Assume that the hypotheses and notations of Theorem 2.1 are in force. There is a positive number \( \theta_1 \) depends only on \( \mu \) and \( \lambda \) such that for \( t \in (0,T) \),
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + \rho |\dot{u}|^2 + |\nabla B|^2 + |\Delta B|^2 + |B_t|^2 + |\nabla \rho|^2 + |\Delta \rho|^2 + |\rho_t|^2) dx \right] + \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |\nabla \dot{u}|^2) dx \\
\leq M \left\| (\rho - \bar{\rho}, u, B)(\cdot, t) \right\|_{H^2}^{\theta_1} \left[ \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla \dot{u}|^2 + |\dot{u}|^2 + |\Delta \dot{u}|^2 + |\rho_t|^2) dx \right] \\
+ M \int_{\mathbb{R}^3} |\nabla u|^2 dx. \tag{2.6}
\]

**Proof.** We multiply the momentum equation by \( \dot{u}^j \), sum over \( j \) and integrate to obtain
\[
\int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx = \int_{\mathbb{R}^3} \left[ -\ddot{u} \nabla P + \mu \dot{u} \cdot \Delta u + \lambda \dot{u} \cdot \nabla (\text{div} u) - \dot{u} \cdot \nabla \left( \frac{1}{2} |B|^2 \right) \right] dx.
\]
Integrating by parts and absorbing terms,
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} |\nabla u|^2 dx \right] + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx \leq M \left[ \int_{\mathbb{R}^3} (|\nabla P|^2 + |\nabla u|^2 + |\nabla u|^3 + |B|^4) dx \right]. \tag{2.7}
\]
Next, we take the material derivative on the momentum equation, multiply it by \( \dot{u}^j \) and integrate
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx = -\int_{\mathbb{R}^3} \dot{u}^j \left[ P_{ij} + \text{div}(P_{j} u) \right] dx \\
+ \int_{\mathbb{R}^3} \mu \dot{u}^j \left[ \Delta u^j_t + \text{div}(\Delta u^j) \right] dx + \int_{\mathbb{R}^3} \lambda \sigma^5 \dot{u}^j \left[ \text{div}(u)_{j} + \text{div}(\text{div}(u)_j) \right] dx \\
+ \int_{\mathbb{R}^3} \dot{u}^j \left[ -(\frac{1}{2} |B|^2)_{j} + \text{div}((\frac{1}{2} |B|^2)_{j}) \right] dx + \int_{\mathbb{R}^3} \dot{u}^j \left[ \text{div}(BB^j), \text{div}(\text{div}(BB^j)u) \right] dx.
\]
Integrating by parts and absorbing terms,
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dx \right] + \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx \leq M \left[ \int_{\mathbb{R}^3} (\rho^2 |\nabla u|^2 + |B_t|^2 |B|^2 + |\nabla u|^4) dx \right],
\]
\[
\leq M \left[ \int_{\mathbb{R}^3} \rho^2 |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^4 dx \right] \\
+ M \|B(\cdot, t)\|_{L^\infty}^2 \left( \int_{\mathbb{R}^3} (|\nabla u|^2 |B|^2 + |\nabla B|^2 |u|^2) dx \right) \tag{2.8}
\]
where the last inequality follows from the induction equation in (1.1)-(1.4).
For the term $\frac{d}{dt} \left[ \int_{\mathbb{R}^3} (|\rho|^2 + |B|^2) \right]$, we differentiate the mass equation with respect to $t$ and integrate to obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\rho|^2 \, dx \leq \int_{\mathbb{R}^3} |(\rho u)| \, \nabla \rho \, dx$$

$$\leq \int_{\mathbb{R}^3} (|\text{div}(\rho u)| |u| + |\rho||\dot{u}| + |\rho||\nabla u||u|) \, \nabla \rho \, dx \quad (2.9)$$

For the terms $\int_{\mathbb{R}^3} (|\nabla B|^2 + |\Delta B|^2) \, dx$ and $\int_{\mathbb{R}^3} (|\nabla \rho|^2 + |\Delta \rho|^2) \, dx$, we differentiate the mass equation with respect to $x$ and integrate

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla \rho|^2 + |\Delta \rho|^2) \, dx \leq \int_{\mathbb{R}^3} |\nabla u||\nabla \rho|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u||\rho||\nabla \rho| \, dx \quad (2.10)$$

$$+ \int_{\mathbb{R}^3} |D_x^3 u||\rho||\nabla \rho| \, dx + \int_{\mathbb{R}^3} |\Delta u||\nabla \rho||\Delta \rho| \, dx + \int_{\mathbb{R}^3} |\nabla u||\Delta \rho|^2 \, dx$$

$$\leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|^2_{H^2} \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta u|^2 + |D_x^3 u|^2 + |\Delta \rho|^2) \, dx \right].$$

Similarly, we differentiate the induction equation and integrate to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (|\nabla B|^2 + |\Delta B|^2) \, dx \leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|^2_{H^2} \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta u|^2 + |D_x^3 u|^2 + |\Delta \rho|^2) \, dx \right]. \quad (2.11)$$

Adding (2.7)–(2.11),

$$\frac{d}{dt} \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + \rho|\dot{u}|^2 + |\nabla B|^2 + |\Delta B|^2 + |\nabla \rho|^2 + |\Delta \rho|^2) \, dx \right] + \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla \dot{u}|^2) \, dx \quad (2.12)$$

$$\leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|^2_{H^2} \left[ \int_{\mathbb{R}^3} (|\Delta u|^2 + |D_x^3 u|^2 + |\nabla u|^2 + |\Delta B|^2 + |\Delta \rho|^2) \, dx \right] + M \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

It remains to estimate the terms $\int_{\mathbb{R}^3} |\Delta u|^2 \, dx$ and $\int_{\mathbb{R}^3} |D_x^3 u|^2 \, dx$. Using the definition of $F$ and the momentum equation,

$$(\mu + \lambda) \Delta u^j = F_{x_j} + (\mu + \lambda) \omega_{x_j}^{i,k} + P_{x_j}, \quad (2.13)$$

$$\Delta F = \text{div}(g), \quad (2.14)$$

where $g^j = \rho \dot{u}^j + \frac{1}{2} |B|^2 \delta_{x_j} - \text{div}(B^3 B)$. Therefore, by standard elliptic theory we obtain

$$||\Delta u(\cdot, t)||^2_{L^2} \leq M \left[ ||\nabla F(\cdot, t)||^2_{L^2} + ||\nabla \omega(\cdot, t)||^2_{L^2} + ||\nabla P(\cdot, t)||^2_{L^2} + ||\nabla B||B(\cdot, t)||^2_{L^2} \right]; \quad (2.15)$$

$$||D_x^3 u(\cdot, t)||^2_{L^2} \leq M \left[ ||\Delta F(\cdot, t)||^2_{L^2} + ||\Delta \omega(\cdot, t)||^2_{L^2} + ||\Delta P(\cdot, t)||^2_{L^2} + ||\nabla(|\nabla B||B)(\cdot, t)||^2_{L^2} \right]; \quad (2.16)$$

$$||\nabla F(\cdot, t)||^2_{L^2} \leq M \left[ ||\rho \dot{u}(\cdot, t)||^2_{L^2} + ||\nabla B||B(\cdot, t)||^2_{L^2} \right]. \quad (2.17)$$

Using (2.15), (2.17) on (2.12), (2.6) follows. \qed

We extract the dissipative structure of the system by introducing auxiliary variable functions $w$ and $v^{(j)}$ as follows:
Lemma 2.4 Assume that the hypotheses and notations of Theorem 2.1 are in force. There is a positive number $\theta_2$ depends only on $\mu$ and $\lambda$ such that for $t \in (0, T]$,

\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} (|w|^2 + \sum_{j=1}^3 |v^{(j)}|^2) \, dx \right] + \int_{\mathbb{R}^3} (|\nabla w|^2 + \sum_{j=1}^3 |\nabla v^{(j)}|^2) \, dx \\
\leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|_{H^2}^2 \left[ \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla u|^2 + |\Delta B|^2 + |\Delta \rho|^2) \, dx \right]. \tag{2.18}
\]

where $w = \rho^2 \dot{u} + \nabla \rho$ and $v^{(j)} = \rho^2 \dot{u} + \nabla B^j$ for $j = 1, 2, 3$.

**Proof.** We give the proof for $w$, the argument for $v^{(j)}$ is similar. By the definition of $w$,

\[ w_t = \Delta w - \Delta (\rho^2 \dot{u} + \nabla \rho) + (\rho^2 \dot{u} + \nabla \rho)_t, \]

we then multiply the above by $w$ and integrate to obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \int_{\mathbb{R}^3} |w|^2 \, dx \right] + \int_{\mathbb{R}^3} |\nabla w|^2 \, dx = - \int_{\mathbb{R}^3} \Delta (\rho^2 \dot{u} + \nabla \rho) \cdot w \, dx + \int_{\mathbb{R}^3} (\rho^2 \dot{u} + \nabla \rho)_t \cdot w. \tag{2.19}
\]

The first term in the right side of (2.19) is bounded by

\[
\int_{\mathbb{R}^3} |\nabla w| |\nabla (\rho^2 \dot{u} + \nabla \rho)| \, dx \leq M \left( \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} (\rho^4 |\nabla \dot{u}|^2 + |\rho|^2 |\nabla \rho|^2 |\dot{u}|^2 + |\Delta \rho|^2) \, dx \right)^{\frac{1}{2}}.
\]

and the second term on the right side of (2.19) is bounded by

\[
\int_{\mathbb{R}^3} |\rho_t| |\rho \dot{u}| |w| + \int_{\mathbb{R}^3} (\rho \dot{u})_t \rho \, dx + \int_{\mathbb{R}^3} |\rho_t| |\nabla w| \, dx \\
\leq \int_{\mathbb{R}^3} |\text{div}(\rho u)| |\rho \dot{u}| |w| + \int_{\mathbb{R}^3} |\nabla (\rho u)| (|P_t| + |\nabla u| + |B_t| |B|) \, dx + \left( \int_{\mathbb{R}^3} |\rho_t|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \right)^{\frac{1}{2}},
\]

where the last inequality follows by the mass equation and the momentum equation. Using (2.20), (2.21) on (2.19) and absorbing terms,

\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} |w|^2 \, dx \right] + \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \\
\leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|_{H^2}^2 \left[ \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla u|^2 + |\Delta B|^2 + |\Delta \rho|^2) \, dx \right].
\]

\(\square\)

**Proof of Theorem 2.1:** First by the definition of $w$ and $v^{(j)}$,

\[
\int_{\mathbb{R}^3} (|\Delta B|^2 + |\Delta \rho|^2) \, dx \leq \int_{\mathbb{R}^3} (\rho^4 |\nabla \dot{u}|^2 + |\rho|^2 |\nabla \rho|^2 |\dot{u}|^2) \, dx + \int_{\mathbb{R}^3} (|\nabla w|^2 + \sum_{j=1}^3 |\nabla v^{(j)}|^2) \, dx.
\]
Therefore we add (2.22), (2.6), (2.18) together and use the assumption \( \underline{\rho} \leq \rho \leq \bar{\rho} \) to obtain
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} (|\rho - \bar{\rho}|^2 + |u|^2 + |B|^2 + |\nabla \rho|^2 + |\nabla u|^2 + |\nabla B|^2 + |\Delta B|^2 + |\Delta \rho|^2) dx \right]
+ \int_{\mathbb{R}^3} (|\nabla u|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla w|^2 + \sum_{j=1}^{3} |\nabla v^{(j)}|^2) dx
\leq M \|(\rho - \bar{\rho}, u, B)(\cdot, t)\|_{L^2}^2 \left[ \int_{\mathbb{R}^3} (|\nabla u|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla w|^2 + \sum_{j=1}^{3} |\nabla v^{(j)}|^2) dx \right] + M \int_{\mathbb{R}^3} |\nabla u|^2 dx,
\]
for some \( M > 0 \) and \( \theta > 0 \). We define a functional \( X(t) \) by
\[
X(t) = \sup_{0 < s \leq t} \int_{\mathbb{R}^3} (|\rho - \bar{\rho}|^2 + |u|^2 + |B|^2 + |\nabla \rho|^2 + |\nabla u|^2 + |\nabla B|^2 + |\dot{u}|^2 + |\Delta B|^2 + |\Delta \rho|^2)(x, s) dx
+ \int_{0}^{t} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\dot{u}|^2 + |\nabla \dot{u}|^2 + |\nabla w|^2 + \sum_{j=1}^{3} |\nabla v^{(j)}|^2)(x, s) dx ds,
\]
then
\[
\sup_{0 < s \leq t} \|(\rho - \bar{\rho}, u, B)(\cdot, s)\|_{L^2}^2 \leq X(t),
\]
and so by integrating (2.22) from 0 to \( t \), we get
\[
X(t) \leq MX(t)^{\theta \mu + 1} + X(0) + M \int_{0}^{t} |\nabla u|^2 dx
\leq MX(t)^{\theta \mu + 1} + X(0) + M \mu^{-1} C_0,
\]
where the last inequality follows from (2.2). Using the fact that \( X(t) \) is continuous in time, there are positive constants \( a, M \), and \( \theta \) such that if \( C_0 < a \), then
\[
A(T) \leq X(T) \leq MC_0^{\theta},
\]
which finishes the proof of Theorem 2.1. \( \square \)

3. Pointwise bounds for the density

In this section we derive pointwise bounds for the density \( \rho \), bounds which are independent both of time and of initial smoothness. This will then close the estimates of Theorem 2.1 to give an uncontingent estimate for the functional \( A \) defined in (2.1). The result is as follows:

**Theorem 3.1** Assume that the system parameters in (1.1)-(1.4) satisfy the conditions in (1.7)-(1.16) and let positive numbers \( d < \delta \) be given. Then there are positive constants \( a, M \), and \( \theta \) depending on the parameters and assumptions in (1.7)-(1.16) and on a positive lower bound for \( d \), such that: if \((\rho_0, u, B)\) is a solution of (1.1)-(1.4) on \( \mathbb{R}^3 \times [0, T] \) in the sense of Kawashima’s theorem with initial data \((\rho_0 - \bar{\rho}, u_0, B_0) \in H^3(\mathbb{R}^3)\) satisfying (1.17)-(1.19) with \( C_0 < a \) and \( \rho(x, t) > 0 \) on \( \mathbb{R}^3 \times [0, T] \), then in fact
\[
\underline{\rho} \leq \rho(x, t) \leq \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T]
\]
and
\[
A(T) \leq MC_0^{\theta}.
\]
The proof of Theorem 3.1 consists of a maximum-principle argument applied along integral curves of the velocity field. We begin with two auxiliary propositions:

**Lemma 3.2** Let $(\rho, u, B)$ be as in Theorem 3.1 and suppose that $0 < c_2 \leq \rho \leq c_1$ on $\mathbb{R}^3 \times [0, T]$. Fix $t_0 \geq 0$ and define the particle trajectories $x : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ by

\[
\begin{cases}
\dot{x}(t, y) = u(x(t, y), t) \\
x(t_0, y) = y.
\end{cases}
\]

Then there is a constant $C$ depending only on $c_1$ and $c_2$ such that if $g \in L^1(\mathbb{R}^3)$ is nonnegative and $t \in [0, T]$, then each of the integrals $\int_{\mathbb{R}^3} g(x(t, y))dy$ and $\int_{\mathbb{R}^3} g(x)dx$ is bounded by $C$ times the other.

**Proof.** The proof is exactly as in Hoff [6], pg. 27. \hfill \square

**Proof of Theorem 3.1:** First we choose positive numbers $b$ and $b'$ satisfying

\[ \bar{\rho} < b < \underline{\rho} + d < \bar{\rho} - d < b' < \bar{\rho}. \]

Recall that $\rho_0$ takes values in $[\underline{\rho} + d, \bar{\rho} - d]$, so that $\rho \in [\underline{\rho}, \bar{\rho}]$ on $\mathbb{R}^3 \times [0, \tau]$ for some positive $\tau$ by the time regularity (1.28). It then follows from Theorem 2.1 that $A(\tau) \leq MC_0^g$, where $M$ is now fixed. We shall show that if $C_0$ is further restricted, then in fact $b < \rho < b'$ on $\mathbb{R}^3 \times [0, \tau]$, and therefore by a simple open-closed argument that $b < \rho < b'$ on all of $\mathbb{R}^3 \times [0, T]$, and therefore that $A(T) \leq MC_0^g$ as well. We shall prove the required lower bound, the proof of the upper bound being similar.

Fix $y \in \mathbb{R}^3$ and define the corresponding particle path $x(t)$ by

\[
\begin{cases}
\dot{x}(t) = u(x(t), t) \\
x(0) = y.
\end{cases}
\]

Suppose that there is a time $t_1 \leq \tau$ such that $\rho(x(t_1), t_1) = b$. We may take $t_1$ minimal and then choose $t_0 < t_1$ maximal such that $\rho(x(t_0), t_0) = \underline{\rho} + d$. Thus $\rho(x(t), t) \in [b, \underline{\rho} + d]$ for $t \in [t_0, t_1]$. We consider two cases:

Case 1: $t_0 < t_1 \leq T \wedge 1$.

We have from the definition (1.12) of $F$ and the mass equation that

\[
\mu \frac{d}{dt} [\log \rho(x(t), t) - \log(\bar{\rho})] + P(\rho(x(t), t)) - P = -F(x(t), t).
\]

Integrating from $t_0$ to $t_1$ and abbreviating $\rho(x(t), t)$ by $\rho(t)$, etc., we then obtain

\[
\mu [\log \rho(s) - \log(\bar{\rho})]_{t_0}^{t_1} + \int_{t_0}^{t_1} [P(s) - P]ds = -\int_{t_0}^{t_1} F(s)ds. \tag{3.3}
\]

We shall show that

\[
\int_{t_0}^{t_1} F(s)ds \leq \tilde{M}C_0^g \tag{3.4}
\]

for a constant $\tilde{M}$ which depends on the same quantities as the $M$ from Theorem 2.1 (which has been fixed). If so, then from (3.3),

\[
\mu [\log b - \log(\underline{\rho} + d)] \geq -\int_{t_0}^{t_1} [P(s) - P]ds - \tilde{M}C_0^g \geq -\tilde{M}C_0^g, \tag{3.5}
\]

where the last inequality holds because $\rho(t)$ takes values in $[b, (\underline{\rho} + d)] \subset [\rho, \min\{\bar{\rho}, \rho\}']$ for $t \in [t_0, t_1]$, and $P$ is increasing on $[\rho, \rho']$ (see (1.7)-(1.15)). But (3.5) cannot hold if $C_0$ is small depending on $\tilde{M}, b$, and $\underline{\rho} + d$. Stipulating this smallness condition, we therefore conclude that there is no time $t_1$ such that $\rho(t_1) = \rho(x(t_1), t_1) = b$. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $b < \rho$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed. The proof that $\rho < b'$ is similar.
To prove (3.8) we recall the definition of $F$ and apply Theorem 2.1 to obtain
\[
\int_{t_0}^{t_1} F(s) ds \leq \int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty} ds \\
\leq \int_{t_0}^{t_1} \left[ (\mu + \lambda)\|\text{div}u(\cdot, s)\|_{L^\infty} + \|(P - \tilde{P})(\cdot, s)\|_{L^\infty} \right] ds \\
\leq \int_{t_0}^{t_1} \left[ (\mu + \lambda)\|\Delta u(\cdot, s)\|_{L^4} + \|\nabla u(\cdot, s)\|_{L^4} + \|(P - \tilde{P})(\cdot, s)\|_{L^4} + \|\nabla P(\cdot, s)\|_{L^4} \right] ds \\
\leq \tilde{M}A(T) \leq \tilde{M}C_0^\theta.
\]

Case 2: $1 \leq t_0 < t_1$.
Again by the mass equation and the definition (1.12) of $F$,
\[
\frac{d}{dt}(\rho(t) - \tilde{\rho}) + \mu^{-1}\rho(t)(P(t) - P(\tilde{\rho})) = -\mu^{-1}\rho(t)F(t).
\]
Multiplying by $(\rho(t) - \tilde{\rho})$ we get
\[
\frac{1}{2} \frac{d}{dt}(\rho(t) - \tilde{\rho})^2 + \mu^{-1}f(t)\rho(t)(\rho(t) - \tilde{\rho})^2 = -\mu^{-1}\rho(t)(\rho(t) - \tilde{\rho})F(t)
\]
where $f(t) = (P(t) - P(\tilde{\rho}))\rho(t) - \tilde{\rho})^{-1}$. Since $f(t) \geq f(t_0) > 0$ on $[t_0, t_1]$, we have that
\[
\mu^{-1}f(t)\rho(t)(\rho(t) - \tilde{\rho})^2 \geq \mu^{-1}f(t_0)\rho(t_0)(\rho(t) - \tilde{\rho})^2
\]
for $t \in [t_0, t_1]$ and therefore from (3.6) that
\[
(\rho(t_1) - \tilde{\rho})^2 - (\rho(t_0) - \tilde{\rho})^2 \leq \tilde{M} \int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty}^2 ds.
\]
We shall show that
\[
\int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty}^2 ds \leq \tilde{M}C_0^\theta,
\]
so that from (3.7),
\[
0 < (b - \tilde{\rho})^2 - (\rho + d - \tilde{\rho})^2 \leq \tilde{M}C_0^\theta.
\]
This cannot hold if $C_0$ is sufficiently small, however, so that, as in Case 1, there is no time $t_1$ such that
\[
\rho(t_1) = \rho(x(t_1), t_1) = b.
\]
Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $b < \rho$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed.

To prove (3.5) we apply (1.12) and (1.33) to get
\[
\int_{t_0}^{t_1} \|F(\cdot, s)\|_{L^\infty}^2 ds \leq \int_{t_0}^{t_1} \left[ \|g(\cdot, s)\|_{L^4}^2 + \|g(\cdot, s)\|_{L^2}^2 \right] ds.
\]
The first integral on the right side above is bounded as follows:
\[
\int_{t_0}^{t_1} \|g(\cdot, s)\|_{L^4}^2 ds \leq \int_{t_0}^{t_1} \left[ \|\dot{\rho}u(\cdot, s)\|_{L^4}^2 + \|\nabla B \cdot B(\cdot, s)\|_{L^4}^2 \right] ds \\
\leq \int_{t_0}^{t_1} \left[ \|\dot{u}(\cdot, s)\|_{L^2}^\frac{1}{2} \|\nabla \dot{u}(\cdot, s)\|_{L^4}^\frac{1}{2} + \|\nabla B \cdot B(\cdot, s)\|_{L^4}^\frac{1}{2} \|D_x(\nabla B \cdot B(\cdot, s))\|_{L^2}^\frac{1}{2} \right] ds \\
\leq \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\dot{u}|^2 dx ds \right)^\frac{1}{4} \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx ds \right)^\frac{3}{4} \\
+ \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\nabla B \cdot B|^2 dx ds \right)^\frac{1}{4} \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |D_x(\nabla B \cdot B)|^2 dx ds \right)^\frac{3}{4} \leq \tilde{M}C_0^\theta,
\]
where the last inequality follows from Theorem 2.1. The second integral on the right side of (3.9) can be bounded in a similar way, and (3.8) is proved.  

\qed
4. Global Existence of Smooth Solutions: Proof of Theorem 1.1 and 1.2

In this section we prove the global-in-time existence of smooth solutions with initial data satisfying
the condition \((1.23)\). Specifically we show in Theorem 4.1 below that, as a consequence of the
time-independent bounds \((3.1)\) and \((3.2)\) in Theorem 3.1, the \(H^3(\mathbb{R}^3)\) norms of such solutions remain finite in
finite time. Theorem 1.1 and 1.2 then follow immediately.

**Theorem 4.1** Assume that the system parameters in \((1.1)-(1.4)\) satisfy the assumptions made in
Theorem 1.1 and let \(d, \delta, a, \theta,\) and \(M\) be as described in Theorem 3.1. Then there is a continuous function
\(M'(T)\) such that, if \((\rho, u, B)\) is a smooth solution of \((1.1)-(1.4)\) on \(\mathbb{R}^3 \times [0, T]\) as described in Theorem
3.1 with initial data \((\rho_0 - \tilde{\rho}, u_0, B_0) \in H^3(\mathbb{R}^3)\) satisfying \((1.7)-(1.9)\) as well as the condition \((1.23)\), then

\[
\sup_{0 \leq s \leq T} \| (\rho - \tilde{\rho}, u, B)(\cdot, s) \|_{H^3} + \int_0^T \| u(\cdot, s) \|^2_{H^4} ds \leq M'(T). \tag{4.1}
\]

*Proof.* We give the proof in a sequence of steps which are similar to those given in Suen and Hoff \([23]\)
pp. 26–31. Throughout this proof we let \(M\) denote a generic function of \(T\) as described above.

**Step 1:** We record the following bounds for spatial derivatives of \(u\) in terms of its time derivative and
lower-order terms:

\[
\| D^2_{ij} u(\cdot, t) \|_{L^2} \leq M \left[ \| \nabla \rho \cdot \dot{u}(\cdot, t) \|_{L^2} + \| \rho \nabla \dot{u}(\cdot, t) \|_{L^2} + \| B \cdot D^2_{ij} B(\cdot, t) \|_{L^2} + \| \nabla B^2(\cdot, t) \|_{L^2} + \| D^2_{ij} P(\rho)(\cdot, t) \|_{L^2} \right]. \tag{4.2}
\]

These follow immediately from the momentum equation and the ellipticity of the Lamé operator
\(L = \mu \Delta + \lambda \nabla \text{div}\). \(\square\)

**Step 2:** The velocity vector satisfies the following bounds:

\[
\sup_{0 \leq t \leq T} \| \nabla \dot{u}(\cdot, t) \|_{L^2} + \int_0^T \int_{\mathbb{R}^3} |D^2_{ij} \dot{u}|^2 dx dt \leq M'(T) \tag{4.3}
\]

and

\[
\sup_{0 \leq s \leq T} \| u(\cdot, s) \|_{H^3} \leq M'(T). \tag{4.4}
\]

*Proof:* First for \(h > 0\) we define the forward difference quotient \(D^h_t\) by

\[
D^h_t(f)(t) = \frac{f(t + h) - f(t)}{h},
\]

and we let \(\phi^j = D^h_t(u^j) + u \cdot \nabla u^j\). Then by differentiating and differenting the momentum equation and
applying the bounds in \((3.1)-(3.2)\) in an elementary way, we obtain

\[
\int_{\mathbb{R}^3} \rho |\phi_{x_j}|^2 dx + \int_0^T \int_{\mathbb{R}^3} \left( |\nabla \phi_{x_j}|^2 + |D^h_t(\text{div}(u_{x_j})) + u \cdot \nabla (\text{div}(u_{x_j}))|^2 \right) dx ds
\]

\[
\leq M'(T) + \int_0^T \int_{\mathbb{R}^3} |\nabla \phi|^2 dx ds + O(h),
\]

where \(O(h) \to 0\) as \(h \to 0\) and \((4.3)\) follows immediately. We then substitute the results in \((3.1)-(3.2)\)
and \((4.3)\) into \((4.2)\) to obtain the \(H^3\) bounds in \((4.1)\). \(\square\)

**Step 3:** The following preliminary bounds hold for the \(L^2([0, T]; H^4(\mathbb{R}^3))\) norms of \(u\):

\[
\int_0^T \int_{\mathbb{R}^3} |D^h_2 u|^4 dx ds \leq M'(t) + M \int_0^T \int_{\mathbb{R}^3} (|D^2_2 \rho|^2 + |D^2_2 B|^2) dx ds. \tag{4.5}
\]

*Proof:* These bounds follow immediately by differentiating the momentum equation twice with respect
to space, expressing the above fourth derivatives of \(u\) in terms of second derivatives of \(\dot{u}, \nabla \rho, \nabla B,\) and
lower order terms via the ellipticity alluded to in Step 2, and then applying the bounds in (3.1)-(3.2) and (4.3).

**Step 4:** The following bounds for $\rho$, $u$ and $B$ hold:

$$
\sup_{0\leq t\leq T} (\|D^2_x\rho(\cdot,t)\|_{L^2} + \|D^2_xB(\cdot,t)\|_{L^2}) + \int_0^T \int_{\mathbb{R}^3} |D^4_xu|^2 \, dx \, ds \leq M'(T). \quad (4.6)
$$

**Proof:** We apply two space derivatives and one spatial difference operator $D^h_j$ to the mass equation to obtain an evolution equation for $D^h_jD_xD_{x_k}\rho$, then multiply by this same quantity and integrate. Proceeding exactly as in Step 4 and applying (4.5) we obtain

$$
\int_{\mathbb{R}^3} |D^h_jD_xD_{x_k}\rho|^2 \, dx \leq M'(T) + \int_0^T \int_{\mathbb{R}^3} |D^h_jD_xD_{x_k}\rho|^2 \, dx \, ds + M \int_0^T \int_{\mathbb{R}^3} |D^4_xu|^2 \, dx \, ds \leq M'(T) + M \int_0^T \int_{\mathbb{R}^3} |D^3_x\rho|^2 \, dx \, ds.
$$

We let $h \to 0$ and apply Gronwall’s inequality to obtain the required bound for the first term on the left in (4.6). The bound for $B$ can be obtained in a similar way. Then substituting into (4.5) we obtain a bound for the second. This proves (4.6) and together with (4.4) completes the proof of Theorem 4.1.

**Proof of Theorem 1.1 and 1.2:** It suffices to prove Theorem 1.2 since Theorem 1.1 follows immediately by standard compactness argument. The proof of Theorem 1.2 involves an open-closed argument on the time interval which is identical to the one given in Suen and Hoff [23] pp. 31. We omit the details here.

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