Discrete continuous mathematical models in the dynamics of horizontal vibrations of buildings

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Abstract. In the design practice of design, dynamic models are used, based on the discretization of structures whose inertial parameters are always distributed. The use of discretized models leads to the formation of large-dimensional problems, and the estimation of discretization errors is extremely complicated by the presence of constructive irregularities and heterogeneities of the boundary conditions of the calculated objects. In view of the foregoing, the proposed work is devoted to the formation of mathematical models that allow combining elements with concentrated inertial parameters (discrete elements) with deformable elements whose inertial parameters are distributed (continuous elements). The need for the formation of such models arises, for example, when calculating buildings for horizontal dynamic effects and also when solving the problems of dynamic analysis of building structures with the presence of vibroactive technological equipment.

Currently, the tasks of assessing the dynamic properties and residual seismic resistance of low-rise buildings are relevant. In the Irkutsk Region alone, there are more than one thousand two hundred (4-5) story buildings of the 335 series, the assessment of the residual seismic resistance of which must be determined. When solving such problems, one inevitably has to simulate the dynamics of buildings. The formation of dynamic models is accompanied by questions of the choice of modeling methods and a description of the nature of the distribution of the inertial parameters of the building.

With horizontal seismic effects, it is advisable to consider the mass of the overlap as concentrated, since taking into account the deformability of the overlap in the plane in this case does not exceed the necessary refinements in the formation of an adequate model, and the complexity of forming the model is great and absolutely not justified. Similarly, the situation is formed when deciding on the need to take into account the deformability of bearing vertical structures, if they are represented by wall element-diaphragms.

Thus, when describing the dynamic model of cross-wall buildings, it is advisable to use models in which the dynamic parameters of vertical structures are distributed vertically, and the nodes of intersection of vertical and horizontal elements have concentrated inertial elements due to the presence of overlappings.

The description of the elastic properties of vertical elements requires solving the question of the need to take into account various types of deformations that determine with some accuracy the displacement of the nodes of the dynamic model. It is known that the defining types of deformations for bar bearing elements are bending deformations, the account of which for Euler beams [6] gives an accuracy of more than 95% when ignoring shear deformations.
For low-rise wall structures (4-5 floors), the ratio of the sizes of the sections to the longitudinal ones does not fit into the conditions that determine the category of Euler beams, since the \( l/n \) ratio in this case is close to unity. Here \( l \) is the height of the vertical bearing element, \( h \) is the section height of this element. The design diagram of such an element can be depicted as a short beam of length \( l \).

Let us consider the displacement of the node \( B \) when exposed to a unit horizontal force \( P \) applied to this node, taking into account various options for the deformation of the element.

The movement carried out due to bending:

\[
\Delta_u = \int_0^l \frac{\overline{MN}}{EI} \, dy = \int_0^l y \cdot y \cdot 12 \frac{1}{Eb} \, dy = \frac{4l^3}{Eb} h^3
\]

Let be \( l = hn \), then \( \Delta_u = \frac{4n^3}{Eb} \).

Shifting movement:

\[
\Delta_v = \int_0^l \frac{1}{Gh} \, dy = \int_0^l \frac{1}{Gbh} \, dy = \frac{3l}{Eb} \frac{3n}{Gb} \quad \text{here } G = E/3
\]

Thus, the deformability of the supporting structures of rectangular sections is largely determined by bending deformations.

To solve the problems of dynamics (direct and inverse), it is inevitable to determine the frequency characteristics of the structure. Consider the definition of the frequency parameters of the structure, taking into account the bending nature of the deformation of the bearing elements.

First, we consider the possibility of determining the natural frequencies of the system with inertial parameters distributed along the length of the supporting structures. To enable the further inclusion of inertial elements with lumped parameters, which can be represented in the form of overlappings, it is advisable to use the harmonic element method [7-9]. Obviously, the direct purpose of the harmonic element method is to design structures for forced harmonic influences. Determination of frequency characteristics - using the method of dynamic compliance and the method of harmonic elements (GAE), it is possible to carry out by taking into account the conditions of resonance manifestations arising from harmonic influences. This approach was implemented for low-rise structures that took into account shear deformations.

To take into account the manifestations of bending deformations in vertical structures, we consider a feature of the formation of harmonic elements using the dynamic displacement method [10].

The development of the dynamic displacement method allows the use of both distributed and concentrated parameters in harmonically loaded structural elements. In this case, it is possible to simulate harmonic oscillations with heterogeneity of boundary conditions and an irregular distribution of physical parameters. The determination of the dynamic properties of mathematical models is possible based on the formation of matrices of an ensemble of such elements.

Some similarity in the construction of solutions allows us to speak about the manifestation of the properties of finite elements, but in contrast to the classical versions specialized for solving stationary dynamics problems and containing as parameters a constant frequency of monoharmonic oscillations and parameters of distributed and concentrated masses.

Given these features, such dynamic elements are called harmonic elements (GaE) [7-9, 11]. The nodal matching of the harmonic solutions of the equations of the dynamic state of the elements obtained in the form of amplitudes of harmonic reactions in the imposed bonds allows for the analytical representation of vibrational forms in the form of functions of amplitudes on the coordinates of the axis of rigidity of the bent element. The latter circumstance greatly simplifies the analysis and generalization of the solution results and allows for non-traditional methods of vibration isolation based on the formation of vibrational forms with predetermined properties [1, 2].

Let us consider forced steady-state harmonic oscillations with a frequency \( \omega \) of a beam of length \( a \) with a linear mass \( p \) uniformly distributed along the length and bending stiffness \( EJ \) when it is also subjected to a constant longitudinal force \( N \).

Oscillations of the beam are carried out under harmonic influences with unit amplitudes applied at some points (nodes) of the beam. In the absence of interstitial dynamic action, the equation of dynamic equilibrium of the elementary section of the beam in the span between the nodes is described by the
Euler–Bernoulli equation [10], where $V$ is the displacement of the axis of rigidity of the beam along the $y$ axis.

$$EJv''_{xx} - NV_{xx} + \rho V_{tt} = 0.$$  
(1)

For a given frequency of external action $\omega$, the vibrational forms of the beam are uniquely determined by the amplitude vector of nodal harmonic displacements

$$Y = \begin{pmatrix} Y(0), Y_{1x}, Y(l), Y_{xx} \end{pmatrix}^T,$$  
(2)

in which the components are arranged in the order of numbering of the corresponding bonds. The vector $Y_{\sin \omega}$ defines the boundary conditions for equation (1). Let us imagine a solution of the equation of beam oscillation in the expansion in forced vibrational forms from single harmonic displacements (with unit amplitudes) of the bonds that ensure the fastening of the nodes.

A single harmonic movement of each bond with number $i$ in its direction causes reactions in all superimposed bonds. The amplitudes of the magnitudes of the bonds of these reactions ordered by bond numbers form a certain vector $R_i$ of the amplitudes of the harmonic reactions. Alternate, in the order of numbering, harmonic movements of bonds form a matrix of amplitudes of dynamic reactions $R = \{R_1, R_2, R_3 \}$

In the general case, the number of bonds can be equal to two to four (Fig. 1) and depends on the boundary conditions (methods of fixing the boundary nodes).

Fig. 1. Design schemes of beams with various options for fastenings;
1, 2, 3, 4 - link numbers

The rules for signs of movements and turns in the directions of bonds, as well as signs of the magnitudes of reactions are shown in Fig. 2

Fig. 2 The rule of signs of displacements and reactions in nodal connections
To determine the coefficients of the matrix $R$ and the function $Y(x)$ of the amplitudes of the forced vibrations of the beam, we use the method of separation of variables \[12, 13\]. To do this, imagine the solution in the form:

$$V(x) = Y(x) \sin(\omega t)$$

Substitute the desired solution in equation (2). Reducing the resulting equation by $\sin(\omega t)$, we obtain the ordinary differential equation,

$$Y''(x) - \frac{NY''(x)}{EJ} - Y(x) \frac{\omega^2 \rho}{EJ} = 0 \quad (3)$$

The characteristic equation of which is of the form:

$$\mu^4 - \frac{N}{EJ} \mu^2 - \frac{\omega^2 \rho}{EJ} = 0.$$ 

Performing a change of variables in the characteristic equation $\mu^2 = q$, form a quadratic equation:

$$q^2 - \frac{N}{EJ} q - \frac{\omega^2 \rho}{EJ} = 0.$$ 

Having determined the roots of this equation, we obtain:

$$q_{1,2} = \frac{N}{2EJ} \pm \sqrt{\frac{N^2}{4(EJ)^2} + \frac{\omega^2 \rho}{EJ}}.$$ 

Passing to the original variable, we have solutions of this equation in the following forms:

$\mu_1 = q, \quad \mu_2 = -q, \quad \mu_3 = is, \quad \mu_4 = -is,$

where

$$q = \sqrt{\frac{N}{2EJ} + \sqrt{\frac{N^2}{4(EJ)^2} + \frac{\omega^2 \rho}{EJ}}},$$

$$s = \sqrt{\frac{N}{2EJ} - \sqrt{\frac{N^2}{4(EJ)^2} + \frac{\omega^2 \rho}{EJ}}}.$$ 

Common decision $Y(x)$ equation (3) can be represented as a linear combination

$$Y(x) = H(x) \hat{C} \quad (4)$$

four linearly independent partial solutions forming a basis vector function

$$H(x) = (e^{qx}, e^{-qx}, \sin(sx), \cos(sx)),$$

where $\hat{C}$ is a vector column of coefficients of a linear combination of the right side of the expression (4) \[14\].

The boundary conditions for solving equation (3) can be specified as the values of the components of the vector (2) of the amplitudes of displacements along the directions of the bonds.

Consider oscillations of a non-deformable in the plane of the floor disk pivotally supported by a certain number of bent vertical elements having a distributed mass $\rho$ (Fig. 3).

![Fig. 3. The design scheme of horizontal vibrations of an undeformable disk on massive bending elements](image)

$m$ - the mass of the overlap. $\rho$ - distributed mass of racks
Such a constructive scheme allows us to analyze the simplest version of the influence of the distributed masses of the supporting vertical structures on the dynamic properties of a system containing a concentrated mass \( m \) and distributed masses of bent elements.

For a given circuit \( Y_{\alpha x}\big|_{\omega=0}=0 \). When determining \( Y(x) \), the coefficients of linear combinations of the fundamental solution are determined in the form of a matrix \( C \) by alternately setting the unit amplitudes of displacements along the directions of the bonds.

The vectors of amplitudes of unit displacements ordered by bond numbers form the matrix \( L \). The matrices \( C \) and \( L \) have the following form:

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33} \\
C_{41} & C_{42} & C_{43}
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

Elements of the matrix \( C \) are determined from the solution of the system of equations:

\[
AC = L,
\]

where \( A \) – matrix formed from the basis vector of the function \( H(x) \) as follows:

\[
A = \begin{bmatrix}
H(O) \\
H'(x) |_{x=0} \\
H''(x) |_{x=0} \\
H_\alpha(x) |_{x=0}
\end{bmatrix}.
\]

The matrix of amplitudes of reactions in bonds with alternating unit harmonic movements of bonds in the order of their numbering (matrix of harmonic reactions) is defined as:

\[
R = -EFCH,
\]

where

\[
H = [H^T_{\alpha x}(x) |_{x=0}, H^T_{\alpha x}(x) |_{x=0}, H^T_{\alpha x}(x) |_{x=0}].
\]

The vector column \( Y \) of the amplitudes of the nodal displacements corresponding to the vector \( Y \) of the amplitudes of the harmonic actions along the directions of the bonds with frequency \( \omega \) is determined from the solution of the system of equations:

\[
RY = F, \quad \text{or} \quad -EF(A^{-1}L)^T Y = F.
\]

\[
Y(x) = Y^T C^T H(x).
\]

Taking into account the boundary conditions allows us to obtain the matrix \( A \), which in this case has the form:

\[
A = \begin{bmatrix}
1 & 1 & 0 & 1 \\
q & -q & s & 0 \\
\exp(a \cdot q) & \exp(-a \cdot q) & \sin(a \cdot s) & \cos(a \cdot s) \\
\exp(a \cdot q) \cdot q^2 & \exp(-a \cdot q) \cdot q^2 & -\sin(a \cdot s) \cdot s^2 & -\cos(a \cdot s) \cdot s^2
\end{bmatrix}.
\]

Having determined matrix \( C \) from equation (5) and substituting it into expression (6), we determine the amplitudes of the dynamic reactions arranged in the form of the matrix \( R \). The magnitude of the amplitude of the dynamic reaction in connection with number 3 has the form:

\[
r_{33} = EF \cdot \frac{(s^2 \cdot \cos(a \cdot s) + q^2 \cdot \cosh(a \cdot q)) \cdot (q^3 \cdot s + q \cdot s^3)}{[s^2 + q^2] \cdot (q \cdot \sin(a \cdot s) \cdot \cosh(a \cdot q) - s \cdot \cos(a \cdot s) \cdot \sin(a \cdot q)]}.
\]

The addition of a discrete mass during harmonic action with a frequency \( \omega \) forms the amplitude of the dynamic reaction in the form of:

\[
r = r_{33} - m \cdot \omega^2.
\]

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