HOMOTOPY ON SPATIAL GRAPHS AND GENERALIZED SATO-LEVINE INVARIANTS

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Abstract. Edge-homotopy and vertex-homotopy are equivalence relations on spatial graphs which are generalizations of Milnor’s link-homotopy. Fleming and the author introduced some edge (resp. vertex)-homotopy invariants of spatial graphs by applying the Sato-Levine invariant for the constituent 2-component algebraically split links. In this paper, we construct some new edge (resp. vertex)-homotopy invariants of spatial graphs without any restriction of linking numbers of the constituent 2-component links by applying the generalized Sato-Levine invariant.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a finite graph. An embedding $f$ of $G$ into the 3-sphere $S^3$ is called a spatial embedding of $G$ or simply a spatial graph. We call the image of $f$ restricted on a cycle (resp. mutually disjoint cycles) in $G$ a constituent knot (resp. constituent link) of $f$, where a cycle is a graph homeomorphic to a circle. A spatial embedding of a planar graph is said to be trivial if it is ambient isotopic to an embedding of the graph into a 2-sphere in $S^3$. A spatial embedding $f$ of $G$ is said to be split if there exists a 2-sphere $S$ in $S^3$ such that $S \cap f(G) = \emptyset$ and each connected component of $S^3 - S$ has intersection with $f(G)$, and otherwise $f$ is said to be non-splittable.

Two spatial embeddings of $G$ are said to be edge-homotopic if they are transformed into each other by self crossing changes and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge, and vertex-homotopic if they are transformed into each other by crossing changes on two adjacent spatial edges and ambient isotopies. These equivalence relations were introduced by Taniyama [17] as generalizations of Milnor’s link-homotopy on oriented links [12], namely if $G$ is a mutually disjoint union of cycles then these are none other than link-homotopy. It is known that edge (resp. vertex)-homotopy on spatial graphs behaves quite differently than link-homotopy on oriented links. Taniyama introduced the $\alpha$-invariant of spatial graphs by taking a weighted sum of the second coefficient of the Conway polynomial of the constituent knots [16]. By applying the $\alpha$-invariant, it is shown that the spatial embedding of $K_4$ as illustrated in Fig. 1.1 (1) is not trivial up to edge-homotopy, and two spatial embeddings of $K_{3,3}$ as illustrated in Fig. 1.1 (2) and (3) are not vertex-homotopic. Note that each of these spatial graphs does not have a constituent link. On the other hand, some invariants of spatial graphs defined by taking a weighted sum of the third coefficient of the Conway polynomial of the constituent 2-component links were introduced.

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by Taniyama as $\mathbb{Z}_2$-valued invariants if the linking numbers are even \cite{18}, and by Fleming and the author as integer-valued invariants if the linking numbers vanish \cite{4}. By applying these invariants, it is shown that each of the spatial graphs as illustrated in Fig. 1.2 (1) and (2) is non-splittable up to edge-homotopy, and the spatial graph as illustrated in Fig. 1.2 (3) is non-splittable up to vertex-homotopy.

Note that each of these spatial graphs does not contain a constituent link which is not trivial up to link-homotopy.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Figure 1.1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Figure 1.2.}
\end{figure}

Our purpose in this paper is to construct some new edge (resp. vertex)-homotopy invariants of spatial graphs without any restriction of linking numbers of the constituent 2-component links by applying a weighted sum of the generalized Sato-Levine invariant. Here the generalized Sato-Levine invariant $\tilde{\beta}(L) = \tilde{\beta}(K_1, K_2)$ is an ambient isotopy invariant of an oriented 2-component link $L = K_1 \cup K_2$ which appears in various ways independently \cite{1}, \cite{2}, \cite{10}, \cite{11}, \cite{9}, \cite{13} and can be calculated by

\[ \tilde{\beta}(L) = a_3(L) - \text{lk}(L) \{ a_2(K_1) + a_2(K_2) \}, \]

where $a_i$ denotes the $i$-th coefficient of the Conway polynomial and $\text{lk}(L) = \text{lk}(K_1, K_2)$ denotes the linking number of $L$. It is known that the original Sato-Levine invariant $\beta(L) \mid_{14}$ coincides with $a_3(L)$ if $\text{lk}(L) = 0$ \cite{3}, \cite{15}. Thus in this case we have that $\tilde{\beta}(L) = \beta(L)$. As a consequence, our invariants are generalizations of Fleming and the author’s homotopy invariants of spatial graphs defined in \cite{4}.

This paper is organized as follows. In the next section, we show some formulas about the generalized Sato-Levine invariant of oriented 2-component links needed later. In section 3, we give the definitions of our invariants and state their invariance up to edge (resp. vertex)-homotopy. In section 4, we give some examples.
2. Some formulas about the generalized Sato-Levine invariant

We first show the change in the generalized Sato-Levine invariant of oriented 2-component links which differ by a single self crossing change.

**Lemma 2.1.** Let \( L_+ = J_1 \cup K \) and \( L_- = J_2 \cup K \) be two oriented 2-component links and \( L_0 = J_1 \cup J_2 \cup K \) an oriented 3-component link which are identical except inside the depicted regions as illustrated in Fig. 2.1. Suppose that \( \text{lk}(L_+) = \text{lk}(L_-) = m \).

Then it holds that

\[
\tilde{\beta}(L_+) - \tilde{\beta}(L_-) = \text{lk}(K, J_i) \{ m - \text{lk}(K, J_i) \} \quad (i = 1, 2).
\]

We remark here that this formula has already known (see [11, Theorem 8.7] for example), but we give the proof again for readers’ convenience.

**Proof.** By the skein relation of the Conway polynomial and well-known formulas for the first coefficient of the Conway polynomial of an oriented 2-component link (cf. [8]) and for the second coefficient of the Conway polynomial of an oriented 3-component link (cf. [6], [5], [7]), we have that

\[
\begin{align*}
(2.1) \quad a_2(J_+) - a_2(J_-) & = \text{lk}(J_1, J_2), \\
(2.2) \quad a_3(L_+) - a_3(L_-) & = \text{lk}(J_1, J_2) \text{lk}(J_2, K) + \text{lk}(J_2, K) \text{lk}(J_1, K) \\
& + \text{lk}(J_1, K) \text{lk}(J_1, J_2).
\end{align*}
\]

Note that

\[
(2.3) \quad \text{lk}(J_1, K) + \text{lk}(J_2, K) = m.
\]

Thus by (2.1), (2.2) and (2.3), we have that

\[
\begin{align*}
\tilde{\beta}(L_+) - \tilde{\beta}(L_-) & = a_3(L_+) - m \{ a_2(J_+) + a_2(K) \} - a_3(L_-) - m \{ a_2(J_-) + a_2(K) \} \\
& = a_3(L_+) - a_3(L_-) - m \{ a_2(J_+) - a_2(J_-) \} \\
& = \text{lk}(J_1, J_2) \{ m - \text{lk}(J_1, K) \} + \text{lk}(J_2, K) \text{lk}(J_1, K) \\
& + \text{lk}(J_1, K) \text{lk}(J_1, J_2) - m \text{lk}(J_1, J_2) \\
& = \text{lk}(J_2, K) \text{lk}(J_1, K).
\end{align*}
\]

Therefore by (2.3) we have the result. \( \square \)
Next we investigate the change in the generalized Sato-Levine invariant of oriented 2-component links which differ by inverting the orientation on one of the components. The original definition implies that the value of the Sato-Levine invariant does not depend on the orientation of each component. But the value of the generalized Sato-Levine invariant depends on it in general.

**Theorem 2.2.** Let $L = J_1 \cup J_2$ be an oriented 2-component link with $\text{lk}(L) = m$. Let $L' = (−J_1) \cup J_2$ be the oriented 2-component link obtained from $L$ by inverting the orientation of $J_1$. Then it holds that

$$\tilde{\beta}(L) - \tilde{\beta}(L') = \frac{1}{6}(m^3 - m).$$

**Proof.** Let $T_m$ and $T'_m$ be two oriented 2-component links as illustrated in Fig. 2.2.

By a direct calculation we have that $\text{lk}(T_m) = m$, $\text{lk}(T'_m) = −m$ and

\begin{align*}
    a_3(T_m) &= \frac{1}{6}(m^3 - m), \\
    a_3(T'_m) &= 0.
\end{align*}

![Figure 2.2](image-url)

By the classification of oriented 2-component links up to link-homotopy [12], we have that $L$ and $T_m$ are transformed into each other by self crossing changes and ambient isotopies. Namely there exists a sequence of oriented 2-component links $L = L_0, L_1, \ldots, L_{k−1}, L_k = T_m$ such that $L_i$ is obtained from $L_{i−1}$ by a single self crossing change $c_i$ ($i = 1, 2, \ldots, k$). Let $M_i = K_0^{(i)} \cup K_1^{(i)} \cup K_2^{(i)}$ be the oriented 3-component link obtained from $L_{i−1}$ by smoothing it on $c_i$, where $K_1^{(i)}$ and $K_2^{(i)}$ are actual separated parts ($i = 1, 2, \ldots, k$). Namely we have the upper part of a skein tree as follows.

\begin{align*}
    L & \xrightarrow{c_1} L_1 & \xrightarrow{c_2} L_2 & \xrightarrow{c_3} \cdots & \xrightarrow{c_{k−1}} L_{k−1} & \xrightarrow{c_k} T_m \\
    M_1 & \xrightarrow{c_1} M_2 & \xrightarrow{c_2} \cdots & \xrightarrow{c_{k−1}} M_{k−1} & \xrightarrow{c_k} M_k
\end{align*}

We define a sign of $c_i$ as follows: $\varepsilon(c_i) = 1$ if $c_i$ changes a positive crossing into a negative crossing and $−1$ if $c_i$ changes a negative crossing into a positive crossing.
Since \( \text{lk}(L_i) = m \) \((i = 1, \ldots, k)\), we have that
\[
(2.7) \quad \text{lk}(K_0^{(i)}, K_1^{(i)}) + \text{lk}(K_0^{(i)}, K_2^{(i)}) = m
\]
for \(i = 1, 2, \ldots, k\). Then by (2.4), (2.6) and (2.7), we have that
\[
a_3(L) = a_3(T_m) + \sum_{i=1}^{k} \varepsilon(c_i) a_2(M_i)
\]
\[
= \frac{1}{6}(m^3 - m) + \sum_{i=1}^{k} \varepsilon(c_i) \left\{ \text{lk}(K_0^{(i)}, K_1^{(i)})\text{lk}(K_1^{(i)}, K_2^{(i)}) + \text{lk}(K_1^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_1^{(i)}) \right\}
\]
\[
+ \text{lk}(K_1^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_2^{(i)}) + \text{lk}(K_0^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_1^{(i)}) \}
\]
\[
= \frac{1}{6}(m^3 - m) + \sum_{i=1}^{k} \varepsilon(c_i) \left\{ m - \text{lk}(K_0^{(i)}, K_2^{(i)}) \right\} \text{lk}(K_1^{(i)}, K_2^{(i)})
\]
\[
+ \text{lk}(K_1^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_2^{(i)}) + \text{lk}(K_0^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_1^{(i)}) \}
\]
\[
(2.8) \quad \frac{1}{6}(m^3 - m) + \sum_{i=1}^{k} \varepsilon(c_i) \left\{ m\text{lk}(K_1^{(i)}, K_2^{(i)}) + \text{lk}(K_0^{(i)}, K_2^{(i)})\text{lk}(K_0^{(i)}, K_1^{(i)}) \right\}.
\]

In the same way as above, we have that \( L' \) and \( T'_m \) are transformed into each other by self crossing changes and ambient isotopies and obtain the following skein tree from (2.6) immediately:

\[
(2.9) \quad L' \xrightarrow{c_1'} L_1' \xrightarrow{c_2'} L_2' \xrightarrow{c_3'} \cdots \xrightarrow{c_{k-1}'} L_{k-1}' \xrightarrow{c_k'} T'_m
\]

We also denote \( M'_i = K_0^{r(i)} \cup K_1^{r(i)} \cup K_2^{r(i)} \), where \( K_1^{r(i)} \) and \( K_2^{r(i)} \) are actual separated parts obtained by smoothing \( L_{i-1}' \) on \( c_i' \) \((i = 1, 2, \ldots, k)\). Note that
\[
(2.10) \quad \varepsilon(c_i) = \varepsilon(c_i'),
\]
\[
(2.11) \quad \text{lk}(K_1^{r(i)}, K_2^{r(i)}) = \text{lk}(K'_1^{r(i)}),
\]
\[
(2.12) \quad \text{lk}(K_0^{r(i)}, K_1^{r(i)})\text{lk}(K_0^{r(i)}, K_2^{r(i)}) = \text{lk}(K'_0^{r(i)}, K'_1^{r(i)})\text{lk}(K'_0^{r(i)}, K'_2^{r(i)})
\]
for \(i = 1, 2, \ldots, k\). Then by (2.5), (2.9), (2.10), (2.11) and (2.12), we have that
\[
a_3(L') = a_3(T'_m) + \sum_{i=1}^{k} \varepsilon(c_i') a_2(M'_i)
\]
\[
= \sum_{i=1}^{k} \varepsilon(c_i') \left\{ -\text{mlk}(K'_1^{r(i)}), K'_2^{r(i)}) + \text{lk}(K'_0^{r(i)}, K'_2^{r(i)})\text{lk}(K'_0^{r(i)}, K'_1^{r(i)}) \right\}
\]
\[
(2.13) \quad = \sum_{i=1}^{k} \varepsilon(c_i) \left\{ -\text{mlk}(K_1^{r(i)}, K_2^{r(i)}) + \text{lk}(K_0^{r(i)}, K_2^{r(i)})\text{lk}(K_0^{r(i)}, K_1^{r(i)}) \right\}.
\]
On the other hand, by (2.6) we can see easily that

\[ a_2(J_1) + a_2(J_2) = \sum_{i=1}^{k} \varepsilon(c_i) \text{lk}(K^{(i)}_1, K^{(i)}_2). \]

(2.14)

By combining (2.8), (2.13) and (2.14), we have that

\[ a_3(L) - a_3(L') = \frac{1}{6}(m^3 - m) + 2m \sum_{i=1}^{k} \varepsilon(c_i) \text{lk}(K^{(i)}_1, K^{(i)}_2) \]

(2.15)

Then by (2.14), we have that

\[
\hat{\beta}(L) - \hat{\beta}(L') = a_3(L) - a_3(L') - m \{a_2(J_1) + a_2(J_2)\} - a_3(L') - 2m \{a_2(-J_1) + a_2(J_2)\}
\]

\[ = \frac{1}{6}(m^3 - m). \]

This completes the proof.

\[\square\]

**Remark 2.3.** Let \( f \) be a spatial embedding of a graph \( G \) and \( \gamma, \gamma' \) two disjoint cycles of \( G \). By Theorem 2.2 if \( \text{lk}(f(\gamma), f(\gamma')) = 0, \pm 1 \) then the value of \( \hat{\beta}(f(\gamma), f(\gamma')) \) does not depend on the orientation of \( f(\gamma) \) and \( f(\gamma') \), namely it is well-defined. But if \( \text{lk}(f(\gamma), f(\gamma')) \neq 0 \), then Theorem 2.2 implies that the value of \( \hat{\beta}(f(\gamma), f(\gamma')) \) have the indeterminacy arisen from a choice of the orientations of \( f(\gamma) \) and \( f(\gamma') \).

3. **Definitions of invariants**

From now onward, we assume that a graph \( G \) is oriented, namely an orientation is given for each edge of \( G \). For a subgraph \( H \) of \( G \), we denote the set of all cycles of \( H \) by \( \Gamma(H) \). For an edge \( e \) of \( H \), we denote the set of all oriented cycles of \( H \) which contain the edge \( e \) and have the orientation induced by the orientation of \( e \) by \( \Gamma_e(H) \). For a pair of two adjacent edges \( e \) and \( e' \) of \( H \), we denote the set of all oriented cycles of \( H \) which contain the edges \( e \) and \( e' \) and have the orientation induced by the orientation of \( e \) by \( \Gamma_{e,e'}(H) \). We set \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) for a positive integer \( n \) and \( \mathbb{Z}_0 = \mathbb{Z} \). We call a map \( \omega : \Gamma(H) \rightarrow \mathbb{Z}_n \) a **weight on \( \Gamma(H) \)** over \( \mathbb{Z}_n \). Then we say that a weight \( \omega \) on \( \Gamma(H) \) over \( \mathbb{Z}_n \) is **weakly balanced on an edge** \( e \) if

\[
\sum_{\gamma \in \Gamma_e(H)} \omega(\gamma) \equiv 0 \pmod{n},
\]

and **weakly balanced on a pair of adjacent edges** \( e \) and \( e' \) if

\[
\sum_{\gamma \in \Gamma_{e,e'}(H)} \omega(\gamma) \equiv 0 \pmod{n}.
\]

Let \( G = G_1 \cup G_2 \) be a disjoint union of two graphs, \( \omega_i \) a weight on \( \Gamma(G_i) \) over \( \mathbb{Z}_n \) \((i = 1, 2)\) and \( f \) a spatial embedding of \( G \). Then we say that a weight \( \omega_i \) is...
null-homologous on an edge $e$ of $G_i$ with respect to $f$ and $\omega_j$ $(i \neq j)$ if

$$\text{lk} \left( \sum_{\gamma \in \Gamma_e(G_i)} \omega_1(\gamma)f(\gamma), f(\gamma') \right) \equiv 0 \pmod{n}$$

for any $\gamma' \in \Gamma(G_j)$ with $\omega_j(\gamma') \neq 0$, and null-homologous on a pair of adjacent edges $e$ and $e'$ of $G_i$ with respect to $f$ and $\omega_j$ $(i \neq j)$ if

$$\text{lk} \left( \sum_{\gamma \in \Gamma_{e,e'}(G_i)} \omega_1(\gamma)f(\gamma), f(\gamma') \right) \equiv 0 \pmod{n}$$

for any $\gamma' \in \Gamma(G_j)$ with $\omega_j(\gamma') \neq 0$.

**Example 3.1.** Let $G = G_1 \cup G_2$ be the graph as illustrated in Fig. 3.1. We denote the cycle $e_i \cup e_j$ of $G_1$ by $\gamma_{ij}$. Let $\omega_1$ be the weight on $\Gamma(G_1)$ over $\mathbb{Z}$ defined by

$$\omega_1(\gamma) = \begin{cases} 1 & (\gamma = \gamma_{12}, \gamma_{34}) \\ -1 & (\gamma = \gamma_{23}, \gamma_{14}) \\ 0 & \text{(otherwise)} \end{cases}$$

and $\omega_2$ the weight on $\Gamma(G_2)$ over $\mathbb{Z}$ defined by $\omega_2(\gamma') = 1$. Let $f$ be the spatial embedding of $G$ as illustrated in Fig. 3.1. Note that

$$\Gamma_{e_1}(G_1) = \{\gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{e_1 + e_2, e_1 - e_3, e_1 + e_4\}$$

and

$$\sum_{\gamma \in \Gamma_{e_1}(G_1)} \omega_1(\gamma)\gamma = (e_1 + e_2) - (e_1 + e_4) = e_2 - e_4.$$

Then we have that

$$\text{lk} \left( \sum_{\gamma \in \Gamma_{e_1}(G_1)} \omega_1(\gamma)f(\gamma), f(\gamma') \right) = \text{lk} (f(e_2 - e_4), f(\gamma')) = 0.$$

Therefore $\omega_1$ is null-homologous on $e_1$ with respect to $f$ and $\omega_2$. 

**Figure 3.1.**
Example 3.2. Let $G = G_1 \cup G_2$ be the graph as illustrated in Fig. 3.2. We denote the cycle of $G_1$ which contains $e_i$ and $e_j$ ($1 \leq i, j \leq 4$) by $\gamma_{ij}$. Let $\omega_1$ be the weight on $\Gamma(G_1)$ over $\mathbb{Z}$ defined by

$$\omega_1(\gamma) = \begin{cases} 
1 & (\gamma = \gamma_{14}, \gamma_{23}) \\
-1 & (\gamma = \gamma_{13}, \gamma_{24}) \\
0 & \text{(otherwise)},
\end{cases}$$

and $\omega_2$ the weight on $\Gamma(G_2)$ over $\mathbb{Z}$ defined by $\omega_2(\gamma') = 1$. Let $f$ be the spatial embedding of $G$ as illustrated in Fig. 3.2. Note that

$$\Gamma_{e_1,e_5}(G_1) = \{\gamma_{13}, \gamma_{14}\} = \{e_1 + e_5 - e_3 - e_6, e_1 + e_5 - e_4 - e_6\}$$

and

$$\sum_{\gamma \in \Gamma_{e_1,e_5}(G_1)} \omega_1(\gamma)\gamma = -(e_1 + e_5 - e_3 - e_6) + (e_1 + e_5 - e_4 - e_6) = e_3 - e_4.$$

Then we have that

$$\text{lk} \left( \sum_{\gamma \in \Gamma_{e_1,e_5}(G_1)} \omega_1(\gamma)f(\gamma), f(\gamma') \right) = \text{lk} (f(e_3 - e_4), f(\gamma')) = 0.$$

Therefore $\omega_1$ is null-homologous on a pair of adjacent edges $e_1$ and $e_5$ with respect to $f$ and $\omega_2$.

Now let $G = G_1 \cup G_2$ be a disjoint union of two graphs, $\omega_i$ a weight on $\Gamma(G_i)$ over $\mathbb{Z}_n$ ($i = 1, 2$) and $f$ a spatial embedding of $G$. For $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$, we put

$$\eta(f(\gamma), f(\gamma')) = \frac{1}{6}(m^3 - m)$$

where $m = \text{lk}(f(\gamma), f(\gamma'))$ under arbitrary orientations of $\gamma$ and $\gamma'$. Then we define that

$$\tilde{\eta}_{\omega_1,\omega_2}(f) = \gcd \{ \eta(f(\gamma), f(\gamma')) | \gamma \in \Gamma(G_1), \gamma' \in \Gamma(G_2), \omega_1(\gamma)\omega_2(\gamma') \neq 0 \ (\text{mod } n) \},$$

where $\gcd$ means the greatest common divisor. Note that $\tilde{\eta}_{\omega_1,\omega_2}(f)$ is a well-defined non-negative integer which does not depends on the choice of orientations of each...
pair of disjoint cycles. Then we define \( \tilde{\beta}_{\omega_1, \omega_2}(f) \in \mathbb{Z}_n \) by

\[
\tilde{\beta}_{\omega_1, \omega_2}(f) \equiv \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \omega_2(\gamma') \tilde{\beta}(f(\gamma), f(\gamma')) \pmod{\gcd\{n, \tilde{\eta}_{\omega_1, \omega_2}(f)\}}.
\]

Here we may calculate \( \tilde{\beta}(f(\gamma), f(\gamma')) \) under arbitrary orientations of \( \gamma \) and \( \gamma' \).

**Remark 3.3.** (1) For an oriented 2-component link \( L \), \( \tilde{\beta}(L) \) is not a link-homotopy invariant of \( L \). Thus \( \tilde{\beta}_{\omega_1, \omega_2}(f) \) may be not an edge (resp. vertex)-homotopy invariant of \( f \) as it is. See also Remark 4.6.

(2) By Theorem 2.2, the value of \( \tilde{\beta}(f(\gamma), f(\gamma')) \) is well-defined modulo \( \eta(f(\gamma), f(\gamma')) \). This is the reason why we consider the modulo \( \tilde{\eta}_{\omega_1, \omega_2}(f) \) reduction.

Then, let us state the invariance of \( \tilde{\beta}_{\omega_1, \omega_2} \) up to edge (resp. vertex)-homotopy under some conditions on graphs and its spatial embeddings.

**Theorem 3.4.** (1) If \( \omega_i \) is weakly balanced on any edge of \( G_i \) and null-homologous on any edge of \( G_i \) with respect to \( f \) and \( \omega_j \) (\( i = 1, 2, i \neq j \)), then \( \tilde{\beta}_{\omega_1, \omega_2}(f) \) is an edge-homotopy invariant of \( f \).

(2) If \( \omega_i \) is weakly balanced on any pair of adjacent edges of \( G_i \) and null-homologous on any pair of adjacent edges of \( G_i \) with respect to \( f \) and \( \omega_j \) (\( i = 1, 2, i \neq j \)), then \( \tilde{\beta}_{\omega_1, \omega_2}(f) \) is a vertex-homotopy invariant of \( f \).

**Proof.** (1) Let \( f \) and \( g \) be two spatial embeddings of \( G \) such that \( g \) is edge-homotopic to \( f \). Then it holds that

\[
\tilde{\eta}_{\omega_1, \omega_2}(f) = \tilde{\eta}_{\omega_1, \omega_2}(g)
\]

because the linking number of a constituent 2-component link of a spatial graph is an edge-homotopy invariant. First we show that if \( f \) is transformed into \( g \) by self crossing changes on \( f(G_1) \) and ambient isotopies then \( \tilde{\beta}_{\omega_1, \omega_2}(f) = \tilde{\beta}_{\omega_1, \omega_2}(g) \).

It is clear that any link invariant of a constituent link of a spatial graph is also an ambient isotopy invariant of the spatial graph. Thus we may assume that \( g \) is obtained from \( f \) by a single crossing change on \( f(e) \) for an edge \( e \) of \( G_1 \) as illustrated in Fig. 3.3. Moreover, by smoothing on this crossing point we can obtain the spatial embedding \( h \) of \( G \) and the knot \( J_h \) as illustrated in Fig. 3.3. Then by (3.1), Lemma
and the assumptions for $\omega_1$, we have that
\[
\tilde{\beta}_{\omega_1, \omega_2}(f) - \tilde{\beta}_{\omega_1, \omega_2}(g) = \\
\sum_{\gamma \in \Gamma(G_2)} \omega_1(\gamma) \omega_2(\gamma') \left\{ \tilde{\beta}(f(\gamma), f(\gamma')) - \tilde{\beta}(g(\gamma), g(\gamma')) \right\}
\]
\[
= \sum_{\gamma \in \Gamma(G_2)} \omega_2(\gamma') \left\{ \text{lk}(h(\gamma'), J_h) \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \text{lk}(f(\gamma), f(\gamma')) - \text{lk}(h(\gamma'), J_h) \right\}
\]
\[
= \sum_{\gamma \in \Gamma(G_2)} \omega_2(\gamma') \left\{ \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \text{lk}(f(\gamma), f(\gamma')) - \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \text{lk}(h(\gamma'), J_h) \right\}
\]
\[
\equiv 0 \pmod{n, \tilde{\eta}_{\omega_1, \omega_2}(f)}.
\]
Therefore we have that $\tilde{\beta}_{\omega_1, \omega_2}(f) = \tilde{\beta}_{\omega_1, \omega_2}(g)$. In the same way we can show that if $f$ is transformed into $g$ by self crossing changes on $f(G_2)$ and ambient isotopies then $\tilde{\beta}_{\omega_1, \omega_2}(f) = \tilde{\beta}_{\omega_1, \omega_2}(g)$. Thus we have that $\tilde{\beta}_{\omega_1, \omega_2}(f)$ is an edge-homotopy invariant of $f$.

![Figure 3.3.](image)

(2) By considering the triple of spatial embeddings as illustrated in Fig. 3.4, we can prove (2) in a similar way as the proof of (1). We omit the details. \[\square\]
Remark 3.5. In particular, if it holds that
\[ \omega_1(\gamma) \omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0 \]
for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \), then \( \tilde{\beta}_{\omega_1, \omega_2}(f) \) coincides with Fleming and the author’s invariant \( \tilde{\beta}_{\omega_1, \omega_2}(f) \) defined in [4].

4. Examples

Let \( G \) be a planar graph which is not a cycle. An embedding \( p : G \to S^2 \) is said to be cellular if the closure of each of the connected components of \( S^2 - p(G) \) on \( S^2 \) is homeomorphic to the disk. Then we regard the set of the boundaries of all of the connected components of \( S^2 - p(G) \) as a subset of \( \Gamma(G) \) and denote it by \( \Gamma_p(G) \). We say that \( G \) admits a checkerboard coloring on \( S^2 \) if there exists a cellular embedding \( p : G \to S^2 \) such that we can color all of the connected components of \( S^2 - p(G) \) by two colors (black and white) so that any of the two components which are adjacent by at least one edge have distinct colors. We denote the subset of \( \Gamma_p(G) \) which corresponds to the black (resp. white) colored components by \( \Gamma_p^b(G) \) (resp. \( \Gamma_p^w(G) \)). Then, for any edge \( e \) of \( G \), there exist exactly two cycles \( \gamma \in \Gamma_p^b(G) \) and \( \gamma' \in \Gamma_p^w(G) \) such that \( e \subset \gamma \) and \( e \subset \gamma' \). Thus we have the following immediately.

Proposition 4.1. Let \( G \) be a planar graph which is not a cycle and admits a checkerboard coloring on \( S^2 \) with respect to a cellular embedding \( p : G \to S^2 \). Let \( \omega_p \) be the weight on \( \Gamma(G) \) over \( \mathbb{Z}_n \) defined by

\[
\omega_p(\gamma) = \begin{cases} 
1 & (\gamma \in \Gamma_p^b(G)) \\
n - 1 & (\gamma \in \Gamma_p^w(G)) \\
0 & (\gamma \in \Gamma(G) - \Gamma_p(G)). 
\end{cases}
\]

Then \( \omega_p \) is weakly balanced on any edge of \( G \).

We call the weight \( \omega_p \) in Proposition 4.1 a checkerboard weight. Moreover, by giving the counter clockwise orientation to each \( p(\gamma) \) for \( \gamma \in \Gamma_p^b(G) \) and the clockwise orientation to each \( p(\gamma) \) for \( \gamma \in \Gamma_p^w(G) \) with respect to the orientation of \( S^2 \), an orientation is given for each edge of \( G \) naturally. We call this orientation of \( G \) a checkerboard orientation over the checkerboard coloring. Since the orientation of
each edge $e$ is coherent with the orientation of each cycle $\gamma \in \Gamma_p(G)$ which contains $e$, by Theorem 4.4 we have the following.

**Theorem 4.2.** Let $G = G_1 \cup G_2$ be a disjoint union of two planar graphs such that $G_i$ is not a cycle and admits a checkerboard coloring on $S^2$ with respect to a cellular embedding $p_i : G \to S^2$ ($i = 1, 2$). Let $\omega_{p_i}$ be the checkerboard weight on $\Gamma(G_i)$ over $\mathbb{Z}_n$ ($i = 1, 2$). We orient $G$ by the checkerboard orientation of $G_i$ over the checkerboard coloring ($i = 1, 2$). Then, for a spatial embedding $f$ of $G$, if $\omega_i$ is null-homologous on any edge of $G_i$ with respect to $f$ and $\omega_j$ ($i = 1, 2$, $i \neq j$), then $\hat{\beta}_{\omega_1, \omega_2}(f) \pmod{n}$ is an edge-homotopy invariant of $f$.

**Example 4.3.** Let $G = G_1 \cup G_2$ be a disjoint union of two planar graphs as in Theorem 4.2 and $f$ a spatial embedding of $G$. Let $\omega_{p_i} : \Gamma(G_i) \to \mathbb{Z}_n$ be the checkerboard weight ($i = 1, 2$), where

$$n = \gcd \{\text{lk}(f(\gamma), f(\gamma')) \mid \gamma \in \Gamma_{p_1}(G_1), \gamma' \in \Gamma_{p_2}(G_2)\}.$$ 

Then, for any edge $e$ of $G_i$ and any $\gamma' \in \Gamma_{p_j}(G_j)$ ($i \neq j$), we have that

$$\text{lk} \left( \sum_{\gamma \in \Gamma_{(G_i)}} \omega_i(\gamma)f(\gamma), f(\gamma') \right) = \sum_{\gamma \in \Gamma_{(G_i)}} \omega_i(\gamma)\text{lk}(f(\gamma), f(\gamma')) \equiv 0 \pmod{n}.$$ 

Thus we have that $\omega_i$ is null-homologous on any edge of $G_i$ with respect to $f$ and $\omega_j$ ($i = 1, 2$, $i \neq j$). Therefore we have that $\hat{\beta}_{\omega_{p_1}, \omega_{p_2}}(f) \pmod{n}$ is an edge-homotopy invariant of $f$.

For example, let $\Theta_4$ be the graph with two vertices $u$ and $v$ and 4 edges $e_1, e_2, e_3, e_4$ each of which joins $u$ and $v$. We denote the cycle of $\Theta_4$ consists of two edges $e_i$ and $e_j$ by $\gamma_{ij}$. Let $p : \Theta_4 \to S^2$ be the cellular embedding as illustrated in the left-hand side of Fig. 4.1. It is clear that $\Theta_4$ admits the checkerboard coloring on $S^2$ with respect to $p$ as illustrated in the center of Fig. 4.1. The right-hand side of Fig. 4.1 shows the checkerboard orientation of $\Theta_4$ over the checkerboard coloring.

![Figure 4.1](image)

Let $G = \Theta_4^1 \cup \Theta_4^2$ be a disjoint union of two copies of $\Theta_4$. For a non-negative integer $m$, let $f_m$ and $g_m$ be two spatial embeddings of $G$ as illustrated in Fig. 4.2. Note that

$$\text{lk}(f_m(\gamma), f_m(\gamma')) = \text{lk}(g_m(\gamma), g_m(\gamma')) = 0 \text{ or } m$$
for any $\gamma \in \Gamma(\Theta_1^4)$ and $\gamma' \in \Gamma(\Theta_2^4)$. So we have that $n = m$. Let $\omega_1 : \Gamma(\Theta_1^4) \to \mathbb{Z}_m$ be the checkerboard weight ($i = 1, 2$). Then, by a direct calculation we can see that the constituent 2-component link of $f_m$ which has a non-zero generalized Sato-Levine invariant is only $L = f_m(\gamma_{14} \cup \gamma'_{14})$ and $\tilde{\beta}(L) = 2$. Actually the other constituent 2-component link $f_m(\gamma \cup \gamma')$ for $\gamma \in \Gamma_p(\Theta_1^4)$ and $\gamma' \in \Gamma_p(\Theta_2^4)$ is a trivial 2-component link or $T_m'$ as illustrated in Fig. [4.2] Thus we have that $\tilde{\beta}_{\omega_1, \omega_2}(f_m) \equiv 2 \pmod{m}$. On the other hand, we can see that each constituent 2-component link $g_m(\gamma \cup \gamma')$ for $\gamma \in \Gamma_p(\Theta_1^4)$ and $\gamma' \in \Gamma_p(\Theta_2^4)$ is a trivial 2-component link or $T_m'$. Thus we have that $\tilde{\beta}_{\omega_1, \omega_2}(g_m) \equiv 0 \pmod{m}$. Therefore we have that $f_m$ and $g_m$ are not edge-homotopic if $m \neq 1, 2$. We remark here that the case of $m = 0$ has already shown by Fleming and the author in [4 Example 4.3].

\[ \begin{align*}
\sum_{\gamma \in \Gamma_{e_1}(\Theta_1^4)} \omega_1(\gamma) \gamma &= e_2 - e_4, \\
\sum_{\gamma \in \Gamma_{e_2}(\Theta_1^4)} \omega_1(\gamma) \gamma &= e_1 - e_3, \\
\sum_{\gamma \in \Gamma_{e_3}(\Theta_1^4)} \omega_1(\gamma) \gamma &= e_4 - e_2, \\
\sum_{\gamma \in \Gamma_{e_4}(\Theta_1^4)} \omega_1(\gamma) \gamma &= e_3 - e_1.
\end{align*} \]

This implies that $\omega_1$ is null-homologous on any edge of $G_1$ with respect to a spatial embedding $f$ of $G$ and $\omega_2$ if and only if

\[ \text{lk}(f(\gamma_{13}), f(\gamma')) = \text{lk}(f(\gamma_{24}), f(\gamma')) = 0 \] (4.1)

for any $\gamma' \in \Gamma_p(\Theta_2^4)$. The same condition can be said of $\omega_2$. For an integer $m$, let $f_m$ be the spatial embedding of $G$ as illustrated in Fig. [4.3] Note that

\[ \text{lk}(f_k(\gamma), f_k(\gamma')) = \text{lk}(f_i(\gamma), f_i(\gamma')) = 0 \text{ or } 1 \text{ } (k \neq l) \]

for any $\gamma \in \Gamma(\Theta_1^4)$ and $\gamma' \in \Gamma(\Theta_2^4)$. Since we can see that $\omega_i$ satisfies (4.1), we have that $\omega_i$ is null-homologous on any edge of $G_i$ with respect to $f_m$ and $\omega_j$ ($i = 1, 2$, $i \neq j$). Namely $\tilde{\beta}_{\omega_1, \omega_2}(f_m)$ is an integer-valued edge-homotopy invariant of $f_m$.
Then, by a direct calculation we can see that the constituent 2-component link of $f_m$ which has a non-zero generalized Sato-Levine invariant is only $L = f_m(\gamma_{14} \cup \gamma'_{14})$ and $\tilde{\beta}(L) = 2m$. Actually the other constituent 2-component link $f_m(\gamma \cup \gamma')$ for $\gamma \in \Gamma_p(\Theta^1_4)$ and $\gamma' \in \Gamma_p(\Theta^2_4)$ is a Hopf link. Thus we have that $\tilde{\beta}_{\omega_1,\omega_2}(f_m) = 2m$. Therefore we have that $f_k$ and $f_l$ are not edge-homotopic for $k \neq l$.

![Figure 4.3.](image)

**Example 4.5.** Let $H$ be the same oriented graph as $G_1$ in Fig. 3.2. We denote the cycle of $H$ which contains $e_i$ and $e_j$ by $\gamma_{ij}$. Let $G = H^1 \cup H^2$ be a disjoint union of two copies of $H$ and $f_m$ the spatial embedding of $G$ as illustrated in Fig. 4.4. This spatial embedding $f_m$ contains exactly one constituent 4-component link $L_m = f_m(\gamma_{12} \cup \gamma_{34} \cup \gamma'_{12} \cup \gamma'_{34})$. By calculating Milnor’s $\mu$-invariant $[12]$ of length 4 of $L_m$, it can be shown that $f_k$ and $f_l$ are not vertex-homotopic for $k \neq l$. But we can also prove this fact by our vertex-homotopy invariant as follows. Let $\omega$ be the same weight on $\Gamma(H)$ over $\mathbb{Z}$ as $\omega_1$ in Example 3.2. We define the weight $\omega_i$ of $\Gamma(H^i)$ over $\mathbb{Z}$ in the same way as $\omega_i (i = 1, 2)$. It is easy to see that $\omega_i$ is weakly balanced on any pair of adjacent edges of $H^i (i = 1, 2)$. Moreover, we have that

$$
\sum_{\gamma \in \Gamma_{e_1,e_5}(H)} \omega(\gamma)\gamma = e_3 - e_4, \quad \sum_{\gamma \in \Gamma_{e_4,e_5}(H)} \omega(\gamma)\gamma = 0,
$$

$$
\sum_{\gamma \in \Gamma_{e_1,e_6}(H)} \omega(\gamma)\gamma = e_3 - e_4, \quad \sum_{\gamma \in \Gamma_{e_2,e_5}(H)} \omega(\gamma)\gamma = e_4 - e_3,
$$

$$
\sum_{\gamma \in \Gamma_{e_2,e_6}(H)} \omega(\gamma)\gamma = e_4 - e_3, \quad \sum_{\gamma \in \Gamma_{e_3,e_5}(H)} \omega(\gamma)\gamma = e_1 - e_2,
$$

$$
\sum_{\gamma \in \Gamma_{e_3,e_6}(H)} \omega(\gamma)\gamma = e_1 - e_2, \quad \sum_{\gamma \in \Gamma_{e_4,e_5}(H)} \omega(\gamma)\gamma = 0,
$$

$$
\sum_{\gamma \in \Gamma_{e_4,e_6}(H)} \omega(\gamma)\gamma = e_2 - e_1, \quad \sum_{\gamma \in \Gamma_{e_4,e_5}(H)} \omega(\gamma)\gamma = e_2 - e_1.
$$
This implies that $\omega_1$ is null-homologous on any pair of adjacent edges of $H^1$ with respect to a spatial embedding $f$ of $G$ and $\omega_2$ if and only if

$$\text{lk}(f(\gamma_1), f(\gamma')) = \text{lk}(f(\gamma_3), f(\gamma')) = 0$$

for any $\gamma' \in \Gamma(H^2)$ and $\omega_2(\gamma') \neq 0$. The same condition can be said of $\omega_2$. Then we can see that $\omega_i$ satisfies (4.2) for $f_m$, namely $\omega_i$ is null-homologous on any edge of $G_i$ with respect to $f_m$ and $\omega_i$ ($i = 1, 2, i \neq j$). Namely $\tilde{\beta}_{\omega_1, \omega_2}(f_m)$ is a vertex-homotopy invariant of $f_m$. Since the linking number of each constituent 2-component link of $f_m$ is 0 or $\pm 1$, we have that $\tilde{\eta}_{\omega_1, \omega_2}(f_m) = 0$. Namely $\tilde{\beta}_{\omega_1, \omega_2}(f_m)$ is integer-valued. By a direct calculation we have that $\tilde{\beta}_{\omega_1, \omega_2}(f_m) = 2m$ in the same way as Example 4.4. Therefore we have that $f_k$ and $f_l$ are not vertex-homotopic for $k \neq l$.

**Remark 4.6.** In Theorems 3.4 and 4.2, the condition “$\omega_i$ is null-homologous on any edge of $G_i$ with respect to $f$ and $\omega_j$ ($i = 1, 2, i \neq j$)” is essential. Let $G = \Theta_1 \cup \Theta_1^2$ be a disjoint union of two copies of $\Theta_1$ oriented in the same way as Example 4.4 and $\omega_i : \Gamma(\Theta_1) \to \mathbb{Z}$ the checkerboard weight ($i = 1, 2$). Let $f$ and $g$ be two spatial embeddings of $G$ as illustrated in Fig. 4.5. Note that $f$ and $g$ are edge-homotopic. But by a direct calculation we have that $\tilde{\beta}_{\omega_1, \omega_2}(f) = -1$ and $\tilde{\beta}_{\omega_1, \omega_2}(g) = 0$, namely $\tilde{\beta}_{\omega_1, \omega_2}(f)$ is not an edge-homotopy invariant of $f$. Actually $\omega_1$ is not null-homologous on $e_4$ with respect to $f$ and $\omega_2$.

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Figure 4.5.

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