KPP reaction–diffusion system with a nonlinear loss inside a cylinder

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Abstract
We consider in this paper a reaction–diffusion system in the presence of a flow and under a KPP hypothesis. While the case of a single-equation has been extensively studied since the pioneering Kolmogorov–Petrovski–Piskunov paper, the study of the corresponding system with a Lewis number not equal to 1 is still quite open. Here, we will prove some results about the existence of travelling fronts and generalized travelling fronts solutions of such a system with the presence of a nonlinear space-dependent loss term inside the domain. In particular, we will point out the existence of a minimal speed, above which any real value is an admissible speed. We will also give some spreading results for initial conditions decaying exponentially at infinity.

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1. Introduction and main results

Background and aim of the paper. Reaction–diffusion equations and related spreading phenomena have been extensively studied in the past decades, due to the numerous applications in various fields of natural sciences, ranging from chemical and biological contexts [17] to combustion and many-particle systems [20]. In particular, existence of travelling front solutions has been a topic of high interest [4, 21], as well as flame quenching [9, 19] and influence of parameters on the speed of propagation [2, 3]. We refer to [5, 7, 22] for large reviews of this mathematical area.

This analysis has been recently extended to systems such as

\[
\begin{aligned}
T_t + u(y)T_x &= \Delta T + f(T)Y, \\
Y_t + u(y)Y_x &= L^{-1} \Delta Y - f(T)Y,
\end{aligned}
\]

(1.1)
both in one-dimensional [13, 18] and multi-dimensional [1, 14–16] settings. This problem is posed in a cylinder $\Omega = \mathbb{R}_+ \times \omega_1 \subset \mathbb{R}^d$ where $\omega$ is a smooth bounded domain of $\mathbb{R}^{d-1}$, with various boundary conditions. We also assume that $u \in C^{1,\alpha}(\overline{\omega})$ (for some $\alpha > 0$) is the first component of a divergence-free shear flow $(u(y), 0)$ with zero average:

$$
\int_{\omega} u(y)dy = 0. \tag{1.2}
$$

To fix the ideas, we will invoke the ‘combustion’ terminology and refer to the function $T$ as ‘temperature’ and to the function $Y$ as ‘concentration’.

This system is said to be of the KPP-type if $f \in C^1([0, +\infty); \mathbb{R})$ and $f(0) = 0 < f(s) \leq f'(0)s, \quad f'(s) \geq 0$ for all $s > 0$ and $f(+\infty) = +\infty$.

Under this hypothesis, system (1.1) has been studied both in the adiabatic case [16] and in the case of heat loss on the boundary [1, 15], that is with homogeneous Neumann and Robin boundary conditions. See also [14] for some extensions to periodic media. These papers presented sufficient conditions for the existence of travelling fronts as well as qualitative properties of such solutions. One of the tools the authors used is the study of a principal eigenvalue problem arising from the linearized system. Without the KPP hypothesis, the situation is much less clear. For instance, for nonlinearities $f(T)$ of the ignition type (that is, with an ignition temperature $\theta > 0$ such that $f(T) = 0$ for $T < \theta$), existence of travelling waves was established only for Lewis numbers close to 1 [10] or in dimension 1 [18].

In this paper, we will show similar results for the following system, still posed in the cylindric domain $\Omega$:

$$
\begin{cases}
T_t + u(y)T_x = \Delta T + f(y, T)Y - h(y, T),

Y_t + u(y)Y_x = \text{Le}^{-1}\Delta Y - f(y, T)Y,
\end{cases} \tag{1.3}
$$

with Neumann boundary conditions

$$
\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{1.4}
$$

where $n$ denotes the outward unit normal on $\partial \Omega$. Here, $f \in C^1(\overline{\omega} \times [0, +\infty); \mathbb{R})$ and we assume that there exists $s_0 > 0$ such that the set of functions $(f(y, .))_{y \in \overline{\omega}}$ is bounded in $C^{1,\alpha}([0, s_0]; \mathbb{R})$. Moreover, the function $f$ satisfies, by analogy with the KPP case,

$$
f(., 0) = 0 < f(., T) \leq \frac{\partial f}{\partial T}(., 0)T, \quad \frac{\partial f}{\partial T} \geq 0 \quad \text{for all } T > 0, \quad \text{and } f(., +\infty) = +\infty,
$$

where the last limit is assumed to be uniform with respect to $y \in \overline{\omega}$. Furthermore, $h \in C^1(\overline{\omega} \times [0, +\infty); \mathbb{R})$ denotes the heat loss, which takes place in the whole domain, and is such that $(h(y, .))_{y \in \overline{\omega}}$ is bounded in $C^{1,\alpha}([0, s_0]; \mathbb{R})$, along with the conditions

$$
\begin{cases}
h(., 0) = 0 \leq \frac{\partial h}{\partial T}(., 0)T \leq h(., T) \leq KT < +\infty \quad \text{for all } T \geq 0 \quad \text{and some } K > 0,

\int_{\omega} \frac{\partial h}{\partial T}(y, 0)dy > 0.
\end{cases} \tag{1.5}
$$

The condition on the integral over $\omega$ of $(\partial h/\partial T)(y, 0)$ means that the heat loss is non trivially equal to 0 in the domain. The bounds on $h$ are technical hypotheses: $-h(., T) \leq -(\partial h/\partial T)(., 0)T$ is similar to the KPP-condition on $f$ and will allow us to use comparisons with the linearized problem, while the boundedness of $\partial h/\partial T$ will allow us to use some standard estimates.

The space dependence of the heat loss allows us to question whether the solution of (1.3)–(1.4) converges to a solution of (1.1) with Robin boundary conditions when $h$ converges to a Dirac mass $\delta_\Omega$. This will be the subject of a forthcoming paper [12].
Here, we will follow two main axes. First, we will search for travelling front solutions, that is solutions of (1.3)–(1.4) of the form \( T(t, x, y) = \bar{T}(x - ct, y) \) and \( Y(t, x, y) = \bar{Y}(x - ct, y) \). We say that \((c, T, Y)\) is a travelling front solution of (1.3)–(1.4) if in the moving frame \( x' = x - ct \) (we drop the primes and the tildes immediately), \( T' \) and \( Y' \) satisfy

\[
\begin{align*}
\Delta T + (c - u(y))T_x + f(y, T)Y - h(y, T) &= 0 \\
Le^{-1} \Delta Y + (c - u(y))Y_x - f(y, T)Y &= 0
\end{align*}
\]  

(1.6)
in \( \Omega \),

\[
\begin{align*}
T(+\infty, .) &= 0, \quad Y(+\infty, .) = 1, \\
T_0(-\infty, .) &= Y_0(-\infty, .) = 0,
\end{align*}
\]  

(1.7)

where the limits are uniform with respect to \( y \in \overline{\omega} \). Conditions (1.7) mean that the right-hand side is a cold region with reactant concentration close to 1, with rather weak conditions behind the front, where the temperature and reactant density are not \textit{a priori} imposed. The relative concentration \( Y \) is assumed to range in \([0, 1]\) and not identically equal to 1. The temperature \( T \) is nonnegative and not identically equal to 0.

The other aim of this paper will be to establish criteria for flame blow-off, extinction and propagation. That is, we will consider the solution \((T, Y)\) of the Cauchy problem defined by (1.3)–(1.4) with an initial profile \((T_0, Y_0)\) such that

\begin{align*}
0 &\leq T_0, \quad T_0 \text{ is bounded}, \quad 0 \leq Y_0 \leq 1, \\
\exists \lambda &> 0, \quad \exists C_1, \quad C_2 > 0, \quad C_2 e^{-\lambda t} \leq T_0(x, y) \leq C_2 e^{-\lambda t} \quad \text{in} \quad \mathbb{R}^+ \times \overline{\omega}, \quad \lambda > 0, \quad \lambda > 0, \quad C_3 e^{-\lambda t} \leq 1 - Y_0(x, y) \leq C_3 e^{-\lambda t} \quad \text{in} \quad \mathbb{R}^+ \times \overline{\omega}.
\end{align*}

(1.8)

We say that the flame becomes extinct if \( \|T(t, \ldots)\|_{L^\infty(\Omega)} \to 0 \) as \( t \to +\infty \). The flame is blown-off if there exists a function \( \Phi(\xi) \) so that \( \Phi(\xi) \to 0 \) as \( \xi \to +\infty \), and \( T(t, x, y) \leq \Phi(x + ct) \) with some \( c > 0 \). Lastly, the flame propagates with speed \( c > 0 \) to the right if for any \( c' > c, \quad T(t, x + c't, y) \to 0 \) as \( t \to +\infty \) while for the speed \( c \) itself, one can find \( x_1 \in \mathbb{R} \) and \( \alpha(x_1, y) > 0 \) such that \( T(t, x_1 + ct, y) \geq \alpha(x_1, y) \) for all \( t \geq 1 \) and \( y \in \overline{\omega} \).

\textit{Some useful notations.} From the KPP hypothesis, we can expect the behaviour of system (1.6) to be determined by the linearized system with \( T = 0 \) and \( Y = 1 \), that is with the values ahead of the front. Let us introduce the following principal eigenvalue problem depending on a parameter \( \lambda \in \mathbb{R} \):

\[
\begin{align*}
\begin{cases}
-\Delta \phi_\lambda - \lambda u(y) \phi_\lambda + \left( \frac{\partial h}{\partial T}(y, 0) - \frac{\partial f}{\partial T}(y, 0) \right) \phi_\lambda &= \mu_{b,f}(\lambda) \phi_\lambda & \text{in} \ \omega, \\
\frac{\partial \phi_\lambda}{\partial n} &= 0 & \text{on} \ \partial \omega.
\end{cases}
\end{align*}
\]  

(1.9)

That is, \( \mu_{b,f}(\lambda) \) is the unique eigenvalue of (1.9) that corresponds to a positive eigenfunction \( \phi_\lambda(y) \). Nonnegative solutions of the form \( T(x, y) = \phi(y) e^{-\lambda t} \) of the linearized system only exist if \( \phi = \phi_\lambda \) and \( \mu_{b,f}(\lambda) = \lambda^2 - c \lambda \). For the conditions (1.7) ahead of the front to be satisfied, \( \lambda \) must be positive, thus the need of assumptions on \( \mu_{b,f} \). Throughout this paper, we will show with a more rigorous approach the role of those assumptions.

Let us first enounce some properties of the function \( \mu_{b,f} \). The eigenfunction \( \phi_\lambda \) can be normalized so that \( \|\phi_\lambda\|_{L^2(\omega)} = 1 \). With this normalization, one gets

\[
\mu_{b,f}(\lambda) = \int_\omega |\nabla \phi_\lambda(y)|^2 dy - \lambda \int_\omega u(y) \phi_\lambda^2(y) dy + \int_\omega \left( \frac{\partial h}{\partial T}(y, 0) - \frac{\partial f}{\partial T}(y, 0) \right) \phi_\lambda^2(y) dy.
\]  

(1.10)
It follows from (1.5) that $\mu_{h,0}(0) > 0$. Furthermore, by the variational principle, we have

$$
\mu_{h,f}(\lambda) = \min_{\psi \in H^1(\Omega), \|\psi\|_2 = 1} \left( \int_\Omega |\nabla \psi(y)|^2 dy - \lambda \int_\Omega u(y)\psi^2(y) dy \right.
$$

$$
+ \int_\Omega \left( \frac{\partial h}{\partial T}(y,0) - \frac{\partial f}{\partial T}(y,0) \right) \psi^2(y) dy \right),
$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm. This implies that $\mu_{h,f}(\lambda)$ is concave as an infimum of a family of affine functions. We now give one last property of $\mu_{h,f}$, which will allow us to discuss the conditions of our theorems later in this paper.

Remember first that $\lambda \mapsto \mu_{h,f}(\lambda)$ and $\lambda \mapsto \phi_\lambda$ are analytic functions of $\lambda$. When differentiating (1.9) with respect to $\lambda$, we obtain

$$
\Delta \phi_\lambda - \lambda u(y)\phi_\lambda' = -u(y)\phi_\lambda + \left( \frac{\partial h}{\partial T}(y,0) - \frac{\partial f}{\partial T}(y,0) \right) \phi_\lambda' = \mu_{h,f}(\lambda)\phi_\lambda + \mu_{h,f}(\lambda)\phi_\lambda',
$$

(1.11)

where the prime denotes derivative with respect to $\lambda$. By multiplying (1.9) by $\phi_\lambda'$, (1.11) by $\phi_\lambda$, subtracting one equation from the other and using the $L^2$-normalization of $\phi_\lambda$, we obtain that

$$
\mu_{h,f}(\lambda) = -\int_\Omega u(y)\phi_\lambda^2(y) dy.
$$

We also introduce the following principal eigenvalue problem:

$$
\begin{cases}
-\Delta \psi_\lambda - \lambda u(y)\psi_\lambda = \nu(\lambda)\psi_\lambda & \text{in } \Omega, \\
\frac{\partial \psi_\lambda}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.13)

That is, $\nu(\lambda)$ is the unique eigenvalue of (1.13) that corresponds to a positive eigenfunction $\psi_\lambda(y)$. In fact, this is the same principal eigenvalue problem with $h = f = 0$ (the purpose of its introduction is only to simplify some of our notations), which arises from the linearized $Y$-equation (1.6). In particular, the function $\nu(\lambda)$ is concave. Furthermore, as any positive constants is an eigenfunction of (1.13) with $\lambda = 0$, and from (1.2)–(1.12) with $h = f = 0$, we have that $\nu(0) = \nu'(0) = 0$. This in turn implies that $\nu(\lambda)$ is nonpositive for all $\lambda \in \mathbb{R}$.

Main results. The following theorem states qualitative properties of travelling front solutions:

**Theorem 1.** Let $(c, T, Y)$ be a solution of (1.6)–(1.7) and (1.4) such that $0 < T$ and $0 < Y < 1$. Then $T$ is bounded, $T(-\infty,.) = 0$, $Y(-\infty,.) = Y_\infty \in (0,1)$, $\mu_{h,f}(0) < 0$, $c > 0$ and $c \geq c^*$, where $c^*$ is then defined by

$$
c^* = \min\{c \in \mathbb{R}, \exists \lambda > 0, \mu_{h,f}(\lambda) = \lambda^2 - c\lambda\} = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda},
$$

(1.14)

and $k(\lambda) = \lambda^2 - \mu_{h,f}(\lambda)$.

Note that since $k$ is strictly convex and under the hypothesis $\mu_{h,f}(0) < 0$, it is straightforward to check that the equation $k(\lambda) = c\lambda$ has one positive solution $\lambda^*$ for $c = c^*$ and two positive solutions $\lambda_1, \lambda_2$ for $c > c^*$, with $\lambda_1 < \lambda^* < \lambda_2$.

Theorem 1 states that $c^*$ is a minimal speed. Conversely, we show under some conditions the existence of travelling front solutions for any speed $c \geq c^*$. 

Theorem 2. (a) Assume that $\mu_{h,f}(0) < 0$. For any $c > \max(0, c^*)$, there exists a solution
$(T, Y)$ of (1.6)–(1.7) and (1.4) such that $T$ is bounded, $T(-\infty, \cdot) = 0$, $T > 0$, $0 < Y < 1$
and $Y(-\infty, \cdot) = Y_\infty \in (0, 1)$.
(b) Assume that $\sup_{\lambda \in \mathbb{R}}(\mu_{h,f}(\lambda) - \lambda^2) < 0$. Then $c^* > 0$ and there exists a solution
$(T, Y)$ of (1.6)–(1.7) and (1.4) with minimal speed $c = c^*$, and such that $T$ is bounded,
$T(-\infty, \cdot) = 0$, $T > 0$, $0 < Y < 1$ and $Y(-\infty, \cdot) = Y_\infty \in (0, 1)$.

Remark 1. One can check that when $h$ and $f$ are independent of $y \in \Omega$ and under the
hypothesis $\mu_{h,f}(0) = h'(0) - f'(0) < 0$, both parts of theorem 2 are verified. That is, there
exists a non-trivial travelling front solution for any speed $c \geq c^* > 0$.

Lastly, the following theorem deals with the Cauchy problem:

Theorem 3. Let $(T, Y)$ be a solution of (1.3)–(1.4) with an initial profile $(T_0, Y_0)$ verifying
(1.8). Let $\lambda$ be the decay rate of $T_0$ as in (1.8).

(a) Extinction. If $\mu_{h,f}(0) > 0$, then $T(t, x, y) \leq Ce^{-\gamma t}$ for all $t \geq 0$ and $(x, y) \in \Omega$
where $\gamma = \mu_{h,f}(0) > 0$ and $C$ is a positive constant.
(b) Blow-off. Let us assume that there exists $0 < \eta \leq \lambda $ such that $\mu_{h,f}(\eta) - \eta^2 > 0$
Then $T(t, x, y) \leq Ce^{-\eta(x+y)\delta}$ for all $t \geq 0$ and $(x, y) \in \Omega$ with $C, \delta > 0$.
(c) Propagation. Let us assume that $\mu_{h,f}(0) < 0$, $\mu_{h,f}(\lambda) - \lambda^2 < 0$ and $\lambda < \lambda^*$. Then
c $c := k(\lambda)/\lambda > \max(0, c^*)$ and the solution propagates with speed $c$.

Parts (a) of theorems 2 and 3 reflect the fact that $\mu_{h,f}(0) < 0$ is a sufficient condition for the
existence of a travelling front solution, and show that it is also almost a necessary condition
for the propagation of the flame (the case $\mu_{h,f}(0) = 0$ is still open). It is also important to
note that part (c) of theorem 3 underlines the link between the speed of propagation and the
decay rate of temperature on the right.

Let us discuss the completeness of this theorem. If $\mu_{h,f}(0) > 0$, we can apply part (a).
If $\mu_{h,f}(0) < 0$, we first consider the case $c^* < 0$. Let then $\lambda_1 < \lambda_2$ be the solutions
of $\mu_{h,f}(s) - s^2 = 0$. We have $\lambda_1 < \lambda^* < \lambda_2$. Furthermore, $\mu_{h,f}(s) - s^2$ is negative
for $s \in (0, \lambda_1)$ and positive for $s \in (\lambda_1, \lambda_2)$. Thus, part (b) applies for $\lambda > \lambda_2$ and part (c) for
$\lambda < \lambda_1$. In the case $c^* \geq 0$, we have that $\mu_{h,f}(s) - s^2$ is negative for $s \in (0, \lambda^*)$ and nonpositive
everywhere. Thus, part (c) applies for $\lambda < \lambda^*$, but the problem is still open for $\lambda \geq \lambda^*$. In
the latter case, we may at least say that the solution cannot propagate with speed $c > c^*$, by
placing ourselves in a moving frame with speed $c$, and then using part (b) of our theorem.
This argument, along with the well-known fact of the propagation with minimal speed in the
single-equation case for a Heaviside initial condition [8], may allow us to conjecture that the
solution propagates with speed $c^*$ for $\lambda > \lambda^*$. Nevertheless, no significant result has been
made in this direction to our knowledge in the system case.

We prove theorem 1 in section 2. Section 3 is dedicated to the proof of part (a) of theorem 2
(existence of travelling front solutions for $c > c^*$), while part (b) is treated in section 4 (case
$c = c^*$). Lastly, section 5 deals with the Cauchy problem, that is theorem 3.

2. Qualitative properties of travelling fronts

Let $(c, T, Y)$ be a non-trivial solution of (1.6)–(1.7) and (1.4).
2.1. Boundedness of temperature

We first prove that $Y$ converges to a constant as $x \to -\infty$. To this end, we integrate equation (1.6) satisfied by $Y$ over the domain $(-N, N) \times \omega$ with $N > 0$. We obtain

$$\int_{\omega} \left[ \text{Le}^{-1}(Y_+(N, y) - Y_-(N, y)) + (c - u(y))(Y(N, y) - Y(-N, y)) \right] dy$$

$$= \int_{(-N, N) \times \omega} f(y, T(x, y)) Y(x, y) dx dy. \quad (2.1)$$

Recall that $Y_+(x, y) \to 0$ as $x \to -\infty$ from (1.7). Besides, as $T$ is bounded for $x > 0$ and $Y$ converges to 1 as $x \to +\infty$, it follows from standard elliptic estimates (see for instance the standard book [11]) that $Y_+(x, y) \to 0$ as $x \to +\infty$. Since $Y$ is bounded, we finally have that the left-hand side is bounded independently of $N$. Therefore, as $f(y, T)Y$ is a positive function, we conclude that

$$\int_{\Omega} f(y, T(x, y)) Y(x, y) dx dy < +\infty. \quad (2.2)$$

Furthermore, by multiplying equation (1.6) satisfied by $Y$ by itself and integrate over the domain $(-N, N) \times \omega$ with $N > 0$, we obtain

$$\int_{\omega} \left[ \text{Le}^{-1}(Y_+(N, y) - Y_-(N, y))Y(N, y) - Y_+(N, y) - \frac{1}{2}(c - u(y))(Y^2(N, y)) - Y^2(-N, y) \right] dy$$

$$= \int_{(-N, N) \times \omega} \left[ f(y, T(x, y)) Y^2(x, y) + \text{Le}^{-1}|\nabla Y|^2 \right] dx dy.$$

The left-hand side is again bounded independently of $N$. Since $0 < Y < 1$ and from (2.2)

$$\int_{\Omega} |\nabla Y(x, y)|^2 dx dy < +\infty.$$

Still, since the function $T$ is not known to be a priori bounded, we cannot use $W^{2,p}_{\text{loc}}$ estimates to prove the convergence of $Y$ to a constant as $x \to -\infty$. We use a method from [15] to overcome this difficulty. We fix $a \in \mathbb{R}$ and let $(x_k)_{k \in \mathbb{N}}$ be any sequence converging to $-\infty$ as $k \to +\infty$. We introduce the translate $Y_k(x, y) = Y(x_k + a + x, y)$. We have

$$\int_{(a+x_k, a+1+x_k) \times \omega} |\nabla Y|^2 dx dy = \int_{(0,1) \times \omega} |\nabla Y_k|^2 dx dy \to 0 \quad \text{as } k \to +\infty.$$

Hence, up to extraction of a subsequence, $(Y_k)_{k \in \mathbb{N}}$ converges in $H^1((0, 1) \times \omega)$ to a constant $Y_\omega^0 \in [0, 1]$. We then use (2.1) with $N = -x_k - a - \xi$ for $\xi \in (0, 1)$ ($k$ is chosen large enough so that $-x_k - a - 1 > 0$ and integrate over $\xi \in (0, 1)$. We obtain

$$\int_{(0,1) \times \omega} \text{Le}^{-1}(Y_k(-x_k - a - \xi, y) - Y_k(x_k + a + \xi, y)) d\xi dy$$

$$= \int_{(0,1) \times \omega} (c - u(y))(Y(-x_k - a - \xi, y) - Y(x_k + a + \xi, y)) dy$$

$$+ \int_{(0,1) \times \omega} f(y, T(x, y)) Y(x, y) dx dy \to 0 \quad \text{as } k \to +\infty.$$

Recall that $Y_+(\infty, \cdot) = Y_+(+\infty, \cdot) = 0$, $Y(+\infty, \cdot) = 1$, and that $u$ has mean zero (1.2). Thus, by passing to the limit and by the dominated convergence theorem, we obtain

$$\int_{\Omega} f(y, T(x, y)) Y(x, y) dx dy = (1 - Y_\omega^0) c|\omega| > 0, \quad (2.3)$$
and as a consequence, \( Y_{\infty}^2 \) < 1 does not depend on \( a \) nor on the sequence \((x_k)_{k\in\mathbb{N}}\). It also already implies that \( c > 0 \). We conclude that \( Y(x_k + \ldots) \) converges to \( Y_{\infty} \in [0,1) \) in \( H^1_{\text{loc}}(\Omega) \) as \( k \to +\infty \) for any sequence \((x_k)_{k\in\mathbb{N}} \to -\infty \).

Let us now prove that \( T \) is globally bounded. We proceed by contradiction. Since \( T(+\infty, \cdot) = 0 \), there has to exist a sequence \((x_k, y_k)_{k\in\mathbb{N}} \) in \( \mathbb{R} \times \mathbb{R} \) such that \( x_k \to -\infty \) and \( T(x_k, y_k) \to +\infty \) as \( k \to +\infty \). Since the functions \( Y, f(y, T)/T \) and \( h(y, T) / T \) are bounded in \( \Omega \), it follows from standard elliptic estimates and the Harnack inequality up to the boundary that \( |\nabla T|/T \) is also bounded in \( \Omega \). Thus, we also have

\[
\min_{(x,y)\in[x_k-1,x_k+1] \times \mathbb{R}} T(x,y) \to +\infty, \quad \min_{(x,y)\in[x_k-1,x_k+1] \times \mathbb{R}} f(y, T(x,y)) \to +\infty
\]  

(2.4)
as \( k \to +\infty \) (recall that \( f(y, +\infty) = +\infty \) uniformly in \( y \in \mathbb{R} \)). But

\[
\int_{\Omega} f(y, T(x,y)) Y(x,y) dx \, dy \geq \int_{(x_k-1,x_k+1) \times \mathbb{R}} f(y, T(x,y)) Y(x,y) dx \, dy
\]

\[
\geq \min_{(x,y)\in[x_k-1,x_k+1] \times \mathbb{R}} f(y, T(x,y)) \times \int_{(x_k-1,x_k+1) \times \mathbb{R}} Y(x,y) dx \, dy,
\]

and

\[
\int_{(x_k-1,x_k+1) \times \mathbb{R}} Y(x,y) dx \, dy \to 2|\Omega| Y_{\infty}
\]
as \( k \to +\infty \). We conclude that if \( T \) is unbounded, then \( Y_{\infty} = 0 \). Let us define

\[
T_k(x,y) = \frac{T(x_k + x, y)}{T(x_k, y_k)}
\]

These functions are locally bounded, satisfy Neumann boundary conditions and

\[
\Delta T_k + (c - u(y)) T_k \cdot (g_{1,k} - g_{2,k}) T_k = 0 \quad \text{in} \, \Omega,
\]

where

\[
g_{1,k}(x,y) = \frac{f(y, T(x_k + x, y))}{T(x_k + x, y)} Y(x_k + x, y), \quad g_{2,k}(x,y) = \frac{h(y, T(x_k + x, y))}{T(x_k + x, y)}.
\]

First, we have

\[
0 \leq g_{1,k}(x,y) \leq \frac{\partial f}{\partial T}(y, 0) Y(x_k + x, y) \leq \max_{\mathbb{R}} \frac{\partial f}{\partial T} (\cdot, 0),
\]

and \( g_{1,k} \to 0 \) in \( L^1_{\text{loc}}(\Omega) \) because \( Y_{\infty} = 0 \). On the other hand, from (1.5):

\[
0 \leq \frac{\partial h}{\partial T}(y, 0) \leq g_{2,k}(x,y) \leq K.
\]

Thus, the sequence \((g_{2,k})_{k\in\mathbb{N}}\) is bounded in \( L^\infty(\Omega) \) and converges weakly in \( L^1(\Omega) \) up to extraction of some subsequence to a function \( g \in L^\infty(\Omega) \).

Lastly, since the functions \( g_{1,k} \) and \( g_{2,k} \) are uniformly bounded in \( L^\infty(\Omega) \), the functions \( T_k \) are then bounded in \( W^{1,p}_{\text{loc}}(\Omega) \) for all \( 1 \leq p < +\infty \). Up to extraction of a subsequence, they converge weakly in \( W^{1,p}_{\text{loc}}(\Omega) \) for all \( 1 \leq p < +\infty \) and in \( C^{1,\beta}_{\text{loc}}(\Omega) \) for all \( 0 \leq \beta < 1 \), to a nonnegative solution \( T_{\infty} \) of

\[
\Delta T_{\infty} + (c - u(y)) T_{\infty} - g T_{\infty} = 0 \quad \text{in} \, \Omega
\]

with Neumann boundary conditions on \( \partial \Omega \). The elliptic regularity theory implies that the function \( T_{\infty} \) is actually of the class \( C^{2,0}_{\text{loc}}(\Omega) \) (remember that \( u \in C^{0,0}_{\text{loc}}(\Omega) \)). It follows from the condition \( T_{\infty}(-\infty, \cdot) \to 0 \) and from (2.4) that \( T_{\infty} \) is a function of \( y \) only, such that

\[
\Delta T_{\infty} - g T_{\infty} = 0 \quad \text{in} \, \Omega
\]

(2.5)
with Neumann boundary conditions on $\partial \Omega$. Furthermore, $T_k(0, y_k) = 1$ and one can assume, up to extraction of another subsequence, that the sequence $y_k$ converges to $y_\infty \in \overline{\Omega}$ as $k \to +\infty$. Therefore, $T_{\infty}(0, y_\infty) = 1$ and the strong maximum principle and the Hopf lemma imply that $T_{\infty}$ is positive in $\overline{\Omega}$. Here, recall that $g$ is the limit in $L^{1,\infty}(\overline{\Omega})$ of the sequence $(g_{2,k})_{k \in \mathbb{N}}$ where $g_{2,k}(x, y) \geq (\partial h / \partial T)(y, 0)$ for all $(x, y) \in \Omega$. Then, for any $N > 0$

$$
\int_{(N,N) \times \omega} g(x, y)T_{\infty}(x, y)dx \, dy \geq \int_{(N,N) \times \omega} \frac{\partial h}{\partial T}(y, 0)T_{\infty}(x, y)dx \, dy \\
\geq 2N \int_{\omega} \frac{\partial h}{\partial T}(y, 0) \, dy \times \min_{(N,N) \times \partial \Omega} T_{\infty} > 0.
$$

However, since $T_{\infty,x} = 0$ and because of the Neumann boundary conditions on $\partial \Omega$, integrating (2.5) over $(-N, N) \times \omega$ leads to

$$
\int_{(-N,N) \times \omega} g(x, y)T_{\infty}(x, y) \, dx \, dy = 0.
$$

This enters in contradiction with (2.6). We conclude that $T$ belongs to $L^{\infty}(\Omega)$.

2.2. The left limit for temperature

We now show that $T \to 0$ as $x \to -\infty$. We integrate equation (1.6) satisfied by $T$ over the domain $(-N, N) \times \omega$ with $N > 0$. We obtain

$$
\int_{\omega} \left[ T_x(\omega, y)T(\omega, y) - T_x(-N, y)T(-N, y) + (c - u(y)) \left( T(\omega, y) - T(-N, y) \right) \right] \, dy \\
= \int_{(-N,N) \times \omega} h(y,T(x,y))dx \, dy - \int_{(-N,N) \times \omega} f(y,T(x,y))Y(x,y) \, dx \, dy.
$$

It follows from (1.7) and standard elliptic estimates that $T_x(\pm \infty, y) = 0$. Since $T \in L^{\infty}(\Omega)$ and from (2.2), we deduce that

$$
\int_{\Omega} h(y,T(x,y)) \, dx \, dy < +\infty.
$$

We now multiply equation (1.6) satisfied by $T$ by $T$ itself and integrate over the domain $(-N, N) \times \omega$ with $N > 0$. We obtain

$$
\int_{\omega} \left[ T_x(\omega, y)T(\omega, y) - T_x(-N, y)T(-N, y) + \frac{1}{2}(c - u(y))(T^2(\omega, y) - T^2(-N, y)) \right] \, dy \\
= \int_{(-N,N) \times \omega} h(y,T)T \, dx \, dy - \int_{(-N,N) \times \omega} f(y,T)YT \, dx \, dy + \int_{(-N,N) \times \omega} |\nabla T|^2 \, dx \, dy.
$$

As before, the left-hand side is bounded independently of $N$ and we saw that the first two integrals of the right-hand side converge as $N \to +\infty$ ($T$ is bounded), whence

$$
\int_{\Omega} |\nabla T|^2 \, dx \, dy < +\infty.
$$

Let now $(x_k)_{k \in \mathbb{N}}$ be any sequence such that $x_k \to -\infty$ as $k \to +\infty$. We define the functions $T_{x_k}(x, y) = T(x + x_k, y)$ for each $k \in \mathbb{N}$. It follows from standard elliptic estimates that this sequence is bounded in $W^{2,p}_{\text{loc}}(\overline{\Omega})$ for all $1 \leq p < +\infty$. Therefore, up to extraction of a subsequence, it converges to a function $T_{\infty}$ in $C_{\text{loc}}^1(\overline{\Omega})$. Because of (2.8), we know that $T_{\infty}$ is a constant. Furthermore,

$$
2T_{\infty} \int_{\omega} \frac{\partial h}{\partial T}(y, 0) \, dy = \lim_{k} \int_{(x_k - 1, x_k + 1) \times \omega} \frac{\partial h}{\partial T}(y, 0)T(x, y) \, dx \, dy = 0.
$$
as $k \to +\infty$, whence $T_\infty = 0$ does not depend on the choice of the sequence $(x_k)_{k \in \mathbb{N}}$. We conclude that $T(x, y) \to 0$ when $x \to -\infty$ locally uniformly in $y \in \Omega$.

2.3. Proof of the inequality: $\mu_{h,f}(0) < 0$

Assume by contradiction that $\mu_{h,f}(0) \geq 0$. Let $\phi = \phi_0$ be a positive solution of \eqref{eq:1.9} with $\lambda = 0$. The function $\phi$ satisfies

$$
\Delta \phi + \left( \frac{\partial f}{\partial T}(y, 0) - \frac{\partial h}{\partial T}(y, 0) \right) \phi \leq \Delta \phi + \mu_{h,f}(0) \phi + \left( \frac{\partial f}{\partial T}(y, 0) - \frac{\partial h}{\partial T}(y, 0) \right) \phi = 0 \quad \text{in } \omega,
$$

with Neumann boundary conditions on $\partial \omega$. Since $T$ is globally bounded and $\phi$ positive on $\bar{\omega}$, there exists $\gamma > 0$ such that $T(x, y) \leq \gamma \phi(y)$ in $\bar{\Omega}$. Since $T > 0$ and $T(\pm\infty, \cdot) = 0$, there also exists $\gamma^* > 0$ such that $T(x, y) \leq \gamma^* \phi(y)$ in $\bar{\Omega}$ with equality somewhere. But since $T > 0$ and $Y < 1$, the function $T$ satisfies

$$
\Delta T + (c - u(y))T_x + \left( \frac{\partial f}{\partial T}(y, 0) - \frac{\partial h}{\partial T}(y, 0) \right) T
\geq \Delta T + (c - u(y))T_x + f(T)Y - h(y, T) = 0
$$

in $\Omega$ with Neumann boundary conditions on $\partial \Omega$. Let now $z(x, y) = T(x, y) - \gamma^* \phi(y)$. The function $z$ is nonpositive in $\bar{\Omega}$ and vanishes somewhere. Besides, $z$ satisfies

$$
\Delta z + (c - u(y))z_x + \left( \frac{\partial f}{\partial T}(y, 0) - \frac{\partial h}{\partial T}(y, 0) \right) z \geq 0
$$

in $\Omega$, together with Neumann boundary conditions on $\partial \Omega$. It then follows from the strong maximum principle and the Hopf lemma that $z = 0$, whence $T(x, y) = \gamma^* \phi(y)$ in $\Omega$. This is impossible since $\gamma^* > 0$ and $T(+\infty, \cdot) = 0$. We conclude that $\mu_{h,f}(0) < 0$.

2.4. A lower bound for the front speed

We saw in section 2.1 that $c > 0$. We now prove that $c \geq c^*$ where $c^*$ defined in section 1.

We use the lemma below, which holds without any assumption on $\mu_{h,f}$, and which describes the link between the linearized system and the behaviour of the solution near $+\infty$.

**Lemma 1.** Let $(c, T, Y)$ be a solution of \eqref{eq:1.6} and \eqref{eq:1.4} such that $0 < T$, $0 < Y < 1$, $T(+\infty, \cdot) = 0$ and $Y(+\infty, \cdot)$ exists. Then there exists $\Lambda \geq 0$ such that $\mu_{h,Y_\infty}(\Lambda) = \Lambda^2 - c\Lambda$ where $Y_\infty = Y(+\infty, \cdot)$ and $\mu_{h,Y_\infty}$ principal eigenvalue of \eqref{eq:1.9} with $f$ replaced by $Y_\infty f$.

**Proof.** By Harnack’s inequality, we know that $|\nabla T|/T$ is globally bounded. Let

$$
\Lambda = -\liminf_{x \to +\infty} \min_{y \in \bar{\Omega}} \frac{T_x(x, y)}{T(x, y)},
$$

and let us check that $\Lambda$ satisfies the conclusion of the lemma. First, since $T(+\infty, \cdot) = 0$ and $T > 0$, $\Lambda$ is nonnegative. Let $(x_k, y_k)_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{R} \times \bar{\omega}$ such that $x_k \to +\infty$ and $T(x_k, y_k)/T(x_k, y_k) \to \Lambda$ as $k \to +\infty$. Up to extraction of a subsequence, one can assume that $y_k \to y_\infty \in \bar{\omega}$ as $k \to +\infty$. Next, define the normalized and shifted temperature for all $k \in \mathbb{N}$ and $(x, y) \in \bar{\Omega}$:

$$
T_k(x, y) = \frac{T(x_k + x, y)}{T(x_k, y_k)}.
$$

Since $|\nabla T|/T$ is bounded in $\bar{\Omega}$, the sequence of functions $(T_k)_k$ is bounded in $L^\infty_{\text{loc}}(\bar{\Omega})$. We also have for each $k \in \mathbb{N}$ that $T_k$ satisfies Neumann boundary conditions on $\partial \Omega$, along with

$$
\Delta T_k + (c - u(y))T_{k,x} + \left( \frac{f(Y, T(x_k, y_k))}{T(x_k, y_k)} Y_k - h(y, T(x_k, y_k)) T_k \right) = 0
$$

in $\Omega$, where $Y_k(x, y) = Y(x + x_k, y)$ is the shifted concentration.
Recall that $T(x + x_k, y) \to 0$ and $Y(x + x_k, y) \to Y_\infty$ locally uniformly in $(x, y) \in \overline{\Omega}$ as $k \to +\infty$. It then follows from standard elliptic estimates that, up to extraction of a subsequence, the sequence $T_k$ converges weakly in $W^{1,p}_c(\Omega)$ for all $1 \leq p < +\infty$ and strongly in $C^{1,\beta}_c(\Omega)$ for all $0 \leq \beta < 1$, to a function $T_\infty$ which satisfies

$$\Delta T_\infty + (c - u(y))T_\infty, \Omega + \left( \frac{\partial f}{\partial T}(y, 0)Y_\infty - \frac{\partial h}{\partial T}(y, 0) \right)_T_\infty = 0 \quad \text{in} \ \Omega,$$

with Neumann boundary conditions on $\partial \Omega$. Since $T_k(x, y) \geq 0$ and $T_k(0, y_k_1) = 1$, we also have $T_\infty \geq 0$ in $\Omega$ and $\partial U_\infty(0, y_\infty) = 1$, whence $\partial T_\infty > 0$ in $\Omega$, as follows from the Hopf lemma and the strong maximum principle. We can then define $z = T_\infty, T_\infty$, which satisfies $z \geq -\Lambda$ in $\bar{\Omega}$ and $z(0, y_\infty) = -\Lambda$ owing to the definition of $\Lambda$ and the choice of the sequence $(x_k, y_k)$. Moreover, the function $z$ is a solution of the following elliptic equation:

$$\Delta z + 2\frac{\nabla T_\infty}{T_\infty} \nabla z + (c - u(y))z = 0 \quad \text{in} \ \Omega,$$

along with Neumann boundary conditions on $\partial \Omega$. It is then implied by Hopf lemma and the strong maximum principle that $z(x, y) = -\Lambda$ in $\Omega$, that is $T_\infty(x, y) = e^{-\Lambda x} \phi(y)$ in $\Omega$ where $\phi(y)$ is a positive solution of

$$-\Delta \phi - \Lambda u(y) \phi + \left( \frac{\partial h}{\partial T}(y, 0) - \frac{\partial f}{\partial T}(y, 0)Y_\infty \right) \phi = (\Lambda^2 - c\Lambda)\phi \quad \text{in} \ \Omega,$$

with Neumann boundary conditions on $\partial \Omega$. By uniqueness of the principal eigenvalue of (1.9), we conclude that $\mu_{h, Y_\infty}(\Lambda) = \Lambda^2 - c\Lambda$. The proof of the lemma is now complete. \hfill \Box

We apply lemma 1 with $Y_\infty = 1$ to non-trivial solutions of (1.6)–(1.7) and (1.4). Since $\mu_{h, f}(0) < 0$ and $\Lambda \geq 0$, it follows that $\Lambda > 0$, whence $c \geq c^*$ by definition of $c^*$ (see (1.14)).

### 2.5. The left limit for concentration

Lastly, we show the convergence of $Y$ as $x \to -\infty$ to a constant $Y_\infty \in (0, 1)$. We have shown in section 2.1 the existence of such a constant in $(0, 1)$. We now argue by contradiction and assume that $Y_\infty = 0$. First, since $c^* \geq c^*$, there exists $\lambda > 0$ such that $\mu_{h, f}(\lambda) = \lambda^2 - c\lambda$. Besides, we then have that

$$\lambda^2 - c\lambda = \mu_{h, f}(\lambda) \leq \mu_{h, 0}(\lambda) - \min_{y \in \bar{\Omega}} \frac{\partial f}{\partial T}(y, 0). \quad (2.9)$$

Since $T$ is bounded, there exists a constant $C_0 > 0$ such that $T(x, y) \leq C_0 e^{-\lambda x}$ for all $x \leq 0$ and $y \in \bar{\Omega}$. We then show that there exist $\gamma, \delta \geq 0$ such that $T(x, y) \leq e^{\delta x}$ for $x \leq 0$.

Indeed, let

$$\varepsilon = \min \left( \frac{\mu_{h, 0}(0)}{2}, \frac{1}{2} \min_{y \in \bar{\Omega}} \frac{\partial f}{\partial T}(y, 0) \right) > 0,$$

and $A \geq 0$ such that

$$\forall x \leq -A, \ \forall y \in \bar{\Omega}, \ \frac{\partial f}{\partial T}(y, 0)Y(x, y) \leq \varepsilon.$$

Such a $A$ exists since $Y(-\infty,) = Y_\infty = 0$. From (2.9), there exists $\Lambda > \lambda$ such that

$$-\mu_{h, 0}(\Lambda) - c\Lambda + \Lambda^2 < -\frac{1}{2} \min_{y \in \bar{\Omega}} \frac{\partial f}{\partial T}(y, 0) < -\varepsilon. \quad (2.10)$$

We denote by $U$ the positive function defined by $T(x, y) = U(x, y)e^{-\Lambda x}\phi_{0, \Lambda}(y)$, where $\phi_{0, \Lambda}$ solves (1.9) with the parameter $\Lambda$ and $f = 0$, normalized so that $\|\phi_{0, \Lambda}\|_{L^2(\Omega)} = 1$. One
has $U(-\infty,.) = 0$ as $T(x,y) \leq C_0 e^{-\lambda x}$ for all $x \leq 0$ and $\lambda < \Lambda$, and $\partial_y U = 0$ on $\partial \Omega$. Furthermore, it is straightforward to check that

$$
\Delta U + (c - u(y) - 2\Lambda)U_x + 2 \nabla \phi_0 \frac{\partial h}{\partial \Omega}(y,0) = \frac{\partial f}{\partial \Omega}(y,T(x,y)) - c\Lambda + g(x,y) \quad U = 0 \quad \text{in } \Omega,
$$

where

$$
g(x,y) = \frac{f(y,T(x,y))}{T(x,y)} \quad Y(x,y) \leq \frac{\partial f}{\partial \Omega}(y,0) Y(x,y) \leq \varepsilon
$$

for all $(x,y) \in (-\infty, -A] \times \omega$. Therefore, it follows from (1.5) that

$$
\Delta U + (c - u(y) - 2\Lambda)U_x + 2 \nabla \phi_0 \frac{\partial h}{\partial \Omega}(y,0) = \frac{\partial f}{\partial \Omega}(y,T(x,y)) - c\Lambda + g(x,y) \quad U = 0 \quad \text{in } \Omega,
$$

for all $(x,y) \in (-\infty, -A] \times \omega$. Because of (2.10), we shall now apply the maximum principle to the previous operator, and look for a suitable supersolution. Since $\varepsilon \leq \mu_{\text{h},0}(0)/2$, there exists $\delta > 0$ such that $\delta^2 + c\delta - \mu_{\text{h},0}(\delta) + \varepsilon < 0$. One can then check that the function

$$
U(x,y) = e^{(\Lambda + \delta)x} \times \frac{\phi_{\text{h},-\delta}(y)}{\phi_{\text{h},\Lambda}(y)},
$$

where $\phi_{\text{h},-\delta}$ solves (1.9) with the parameter $-\delta$ and $f = 0$, satisfies

$$
\Delta U + (c - u(y) - 2\Lambda)U_x + 2 \nabla \phi_0 \frac{\partial h}{\partial \Omega}(y,0) = \frac{\partial f}{\partial \Omega}(y,T(x,y)) - c\Lambda + g(x,y) \quad U(x,y) \leq 0 \quad \text{in } \Omega,
$$

with homogeneous Neumann boundary conditions. It follows from the maximum principle that the difference $U - \bar{U}$ cannot attain an interior negative minimum. Moreover, $\bar{U} \geq 0$ and one can normalize the function $\phi_{\text{h},-\delta}$ so that $U(-A, y) \leq \bar{U}(A, y)$ for all $y \in \overline{\Omega}$. Finally, both $U$ and $\bar{U}$ tend to 0 as $x \to -\infty$. We conclude that for all $x \leq -A$ and $y \in \overline{\Omega}$, we have $U(x,y) \leq \bar{U}(x,y)$. In other words, for any $y \geq \max_{y \in \overline{\Omega}} \phi_{\text{h},-\delta}(y)$ and $(x,y) \in (-\infty, A] \times \overline{\Omega}$, we have that $T(x,y) \leq e^{\delta x} \phi_{\text{h},-\delta}(y) \leq \gamma e^{\delta x}$. Since $\bar{T}$ is bounded, we can choose $\gamma$ such that this inequality is satisfied for all $x \leq 0$:

$$
T(x,y) \leq \gamma e^{\delta x}. \quad (2.11)
$$

We now claim that

$$
M := \lim_{x \to -\infty, y \in \overline{\Omega}} \frac{Y(x,y)}{T(x,y)} = 0. \quad (2.12)
$$

From the Harnack inequality and the fact that $f(y,T)$ is bounded, we know that $M$ is finite. Furthermore, $M \geq 0$ because $Y > 0 = Y(-\infty,.)$. Let now $(x_k, y_k)_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{R} \times \overline{\Omega}$ such that $x_k \to -\infty$ and

$$
\frac{Y(x_k, y_k)}{T(x_k, y_k)} \to M \quad \text{as } k \to +\infty.
$$

Up to extraction of some subsequence, one can assume that $y_k \to y_\infty \in \overline{\Omega}$ as $k \to +\infty$. Consider now the functions

$$
Y_k(x,y) = \frac{Y(x + x_k, y)}{Y(x_k, y_k)}.
$$

They are locally bounded in $\overline{\Omega}$ and satisfy

$$
\text{Le}^{-1} \Delta Y_k + (c - u(y)) Y_{k,x} = f(y, T(x + x_k, y)) Y_k \quad \text{in } \Omega
$$

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with Neumann boundary conditions. Moreover, $f(y, T(x + x_k, y)) \to 0$ locally uniformly in $\Omega$ as $k \to +\infty$ because $T(-\infty, \cdot) = 0$ and $f(y, 0) = 0$ for all $y \in \partial \Omega$. From standard elliptic estimates, up to extraction of some subsequence, the functions $Y_k$ converge weakly in $W_{\text{loc}}^1(\Omega)$ for all $1 \leq p < +\infty$ and strongly in $C_{\text{loc}}^1(\Omega)$ for $0 \leq \beta < 1$ to a solution $Z$ of

\[ \text{Le}^{-1} \Delta Z + (c - u(y))Z_x = 0 \quad \text{in } \Omega \]

with homogeneous Neumann boundary conditions. Furthermore, $Z(0, y_\infty) = 1$, $Z \geq 0$ and thus $Z > 0$ in $\Omega$ from the strong maximum principle and the Hopf lemma. We also have that $Z_x/Z \leq M$ in $\Omega$ and $Z_z(0, y_\infty)/Z(0, y_\infty) = M$, owing to the definition of $M$ and of the sequence $(x_k, y_k)$. However, the function $W(x, y) = Z_x(x, y)/Z(x, y)$ satisfies

\[ \text{Le}^{-1} \Delta W + 2\text{Le}^{-1} \frac{\nabla Z}{Z} \cdot \nabla W + (c - u(y))W_x = 0 \quad \text{in } \Omega \]

with homogeneous Neumann boundary conditions. Therefore, by the maximum principle and the Hopf lemma, $W(x, y) = M$ for all $(x, y) \in \partial \Omega$. In other words, $Z(x, y) = e^{Mx} \psi(y)$ where $\psi$ positive function in $\partial \Omega$ and verifies

\[ \text{Le}^{-1} \Delta \psi + \text{Le}^{-1} M^2 \psi + M(c - u(y))\psi = 0 \quad \text{in } \omega \]

with homogeneous Neumann boundary conditions. As a consequence, by uniqueness of the principal eigenvalue of $(1.13)$, $M^2 + cM \text{Le} = \nu(-\text{MLe})$. The left-hand side is nonnegative ($c$ is positive) while the right-hand side is nonpositive. As a conclusion, $M = 0$ and then $Z = \psi$ principal eigenfunction of $(1.13)$ with parameter $0$ and $\psi(y_\infty) = 1$, namely $\psi = 1$ in $\partial \Omega$. Thus, $Z = 1$ in $\Omega$.

Fix now $\beta > 0$ such that $\beta < \delta$ with $\delta$ as in $(2.11)$. It follows from $(2.12)$ that there exists $A \geq 0$ such that $Y(x, y) / Y(x, y) \leq \beta$ for all $x \leq -A$ and $y \in \partial \Omega$. It follows immediately, for $\kappa = e^{-\beta A} \text{min}_{\partial \Omega} \nu(-A, y) > 0$, that

\[ \forall x \leq -A, \quad Y(x, y) \geq \kappa e^{\beta x}. \tag{2.13} \]

As we have shown in the proof of $(2.12)$, there exists a sequence $(x_k, y_k)_{k \in \mathbb{N}}$ such that $x_k \to -\infty$ and the functions $(x, y) \to Y(x + x_k, y)/Y(x_k, y_k)$ converge to the constant 1 at least in $C_{\text{loc}}^1(\Omega)$ as $k \to +\infty$. Without loss of generality, one can assume that $x_k \leq -A \leq 0$ for all $k$. Now use the fact that $Y_\infty = Y(-\infty, \cdot) = Y_x(-\infty, \cdot) = 0$ and integrate equation $(1.6)$ satisfied by $Y$ over the domain $(-\infty, x_k) \times \omega$. One obtains

\[ \text{Le}^{-1} \int_{\omega} Y(x_k, y)dy + c \int_{\omega} Y(x_k, y)dy - \int_{\omega} u(y)Y(x_k, y)dy < \gamma |\omega| \delta^{-1} e^{\beta x_k} \times \max_{y \in \partial \Omega} \frac{\partial f}{\partial T}(y, 0), \tag{2.14} \]

because of $(2.11)$ and $f(T)Y \leq \partial f / \partial T(y, 0)T$. Furthermore, as $Y(x + x_k, y)/Y(x_k, y_k) \to 1$ in $C_{\text{loc}}^1(\Omega)$ and since $u(y)$ is bounded in $\omega$ and has mean zero, it follows that

\[ \int_{\omega} Y(x_k, y)dy \to 0 \quad \text{and} \quad \int_{\omega} u(y)Y(x_k, y)dy \to |\omega|^{-1} \int_{\partial \Omega} u(y)dy = 0 \]

as $k \to +\infty$. Putting that together with $(2.14)$, one gets that

\[ \frac{c}{2} \int_{\omega} Y(x_k, y)dy \leq \gamma |\omega| \delta^{-1} e^{\beta x_k} \times \max_{y \in \partial \Omega} \frac{\partial f}{\partial T}(y, 0), \]

for $k$ large enough, because $c > 0$. But $(2.13)$ together with $x_k \leq -A$ then yields

\[ \frac{c x_k |\omega|}{2} \delta^{\beta x_k} \leq \gamma |\omega| \delta^{-1} e^{\beta x_k} \times \max_{y \in \partial \Omega} \frac{\partial f}{\partial T}(y, 0) \]

for $k$ large enough. Since $0 < \beta < \delta$ and $x_k \to -\infty$, one gets a contradiction. Thus, $Y_\infty = 0$ is impossible and $Y_\infty \in (0, 1)$. The proof of theorem 1 is achieved. $\square$
3. Existence of fronts with non-minimal speeds

Here, we prove part (a) of theorem 2. We assume that \( \mu_{h,f}(0) < 0 \) and let \( c = \max(0, c^*) \). First, we construct sub- and supersolutions of (1.6). We then use a fixed point theorem on bounded cylinders to construct approximate solutions. Lastly, by passing to the limit of an infinite cylinder, we obtain a solution of (1.6) with the wanted qualitative properties. This standard procedure has already been applied to show the existence of fronts in [1, 6, 16].

3.1. Sub- and supersolutions in \( \Omega \)

Note first that the constant 1 is a supersolution for \( Y \).

Supersolution for \( T \). We construct a supersolution for the \( T \)-equation (1.6) with \( Y = 1 \). Since \( c > c^* \), let \( \lambda_1 \) be the smallest nonnegative root of \( k(\lambda) = c\lambda \). The real number \( \lambda_1 \) is in fact positive thanks to \( k(0) = -\mu_{h,f}(0) > 0 \). Then let \( T \) be the function defined in \( \Omega \) by

\[
T(x, y) = \phi_{\lambda_1}(y) e^{-\lambda_1 x},
\]

Here \( \phi_{\lambda_1} \) is the positive principal eigenfunction of (1.9) with \( \lambda = \lambda_1 \), normalized so that \( \| \phi_{\lambda_1} \|_{L^\infty(\Omega)} = 1 \). The function \( T \) satisfies the Neumann boundary conditions on \( \partial \Omega \), and is a supersolution for the equation on \( T \) in (1.6) with \( Y = 1 \), i.e.

\[
\Delta T + (c - u(y)) T_x + f(y, T) - h(y, T) \leq 0 \quad \text{in } \Omega.
\]

Sub-solution for \( Y \). Since \( v(0) = v'(0) = 0 < c \), one can choose \( \beta > 0 \) small enough so that

\[
\begin{align*}
0 < \beta < \lambda_1, \\
\nu(\beta \psi_{\beta \lambda}) - \beta^2 + c\beta \lambda &> 0,
\end{align*}
\]

and \( \gamma > 0 \) large enough so that

\[
\begin{align*}
\gamma \times \min_{y \in \Omega} \psi_{\beta \lambda} &\geq 1, \\
\gamma \lambda e^{-1}(v(\beta \lambda) - \beta^2 + c\beta \lambda) \times \min_{y \in \Omega} \psi_{\beta \lambda} &> \max_{y \in \Omega} \frac{\partial f}{\partial T}(y, 0),
\end{align*}
\]

where \( \psi_{\beta \lambda} \) is the positive eigenfunction of (1.13) with \( \lambda = \beta \lambda \), normalized in such a way that \( \| \psi_{\beta \lambda} \|_{L^\infty(\Omega)} = 1 \). Let \( Y \) be defined by

\[
Y(x, y) = \max \left( 0, 1 - \gamma \psi_{\beta \lambda}(y) e^{-\beta x} \right).
\]

Note that \( Y = 0 \) for \( x \leq 0 \). Let us check that \( Y \) is a sub-solution for (1.6) with \( T = \bar{T} \). Note first that \( Y \) satisfies the Neumann boundary conditions on \( \partial \Omega \). Moreover, when \( Y > 0 \), then \( x > 0 \) and

\[
Le^{-1} \Delta Y + (c - u(y)) Y_x - f(y, \bar{T}) Y
\]

\[
\geq \gamma La^{-1}(v(\beta \lambda) - \beta^2 + c\beta \lambda) \psi_{\beta \lambda}(y) e^{-\beta x} - \frac{\partial f}{\partial T}(y, 0) \phi_{\lambda_1}(y) e^{-\lambda_1 x} \left( 1 - \gamma \psi_{\beta \lambda}(y) e^{-\beta x} \right)
\]

\[
\geq \gamma Le^{-1}(v(\beta \lambda) - \beta^2 + c\beta \lambda) \psi_{\beta \lambda}(y) e^{-\beta x} - \frac{\partial f}{\partial T}(y, 0) e^{-\beta x} \geq 0,
\]

since \( f \) of the KPP-type, \( 0 < \phi_{\lambda_1} \leq 1 \) in \( \Omega \) and because of (3.1)–(3.2).
Sub-solution for $T$. Lastly, we will construct a sub-solution for $T$ with $Y = Y$. Recall that $k(\lambda_1) = c\lambda_1$. Since $k(0) > 0$ and $\lambda_1$ is the smallest positive root of $k(\lambda) = c\lambda$, we have $k'(\lambda_1) \leq c$. Furthermore, if $k'(\lambda_1) = c$, then $k(\lambda) \geq c\lambda$ for all $\lambda \in \mathbb{R}$ by convexity of $k$, whence $c^* \geq c$, which is impossible. We conclude that $k'(\lambda_1) < c$.

The above allows us to choose $\eta > 0$ small enough so that

$$0 < \eta < \min(\beta, \alpha \lambda_1),$$

$$\varepsilon := c(\lambda_1 + \eta) - k(\lambda_1 + \eta) > 0,$$

where $\alpha > 0$ such that $f(y, \cdot)$ and $h(y, \cdot)$ are of class $C^{1,\alpha}([0, s_0])$ for some $s_0 > 0$ uniformly in $y \in \Omega$. Let $M \geq 0$ be such that

$$f(y, s) \geq \frac{\partial f}{\partial T}(y, 0)s - Ms^{1+\alpha},$$

$$h(y, s) \leq \frac{\partial h}{\partial T}(y, 0)s + Ms^{1+\alpha},$$

for all $s \in [0, s_0]$ and for all $y \in \Omega$.}

Now take $x_0 \geq 0$ sufficiently large so that $Y(x, y) = 1 - y\psi_{\partial \Omega}(y)e^{-\beta y}$ for all $(x, y) \in (x_0, +\infty) \times \Omega$. Next, let $\delta > 0$ be large enough so that

$$\phi_{x_1}(y)e^{-\lambda_1x} - \delta\phi_{x_1+\eta}(y)e^{-(\lambda_1+\eta)x} \leq s_0 \text{ in } \Omega,$$

$$\phi_{x_1}(y)e^{-\lambda_1x} - \delta\phi_{x_1+\eta}(y)e^{-(\lambda_1+\eta)x} \leq 0 \text{ in } (-\infty, x_0] \times \Omega,$$

$$\delta \varepsilon \times \min \phi_{x_1+\eta} \geq \frac{\beta}{\gamma} \max \frac{\partial f}{\partial T}(y, 0) + 2M.$$  

Finally, we define, for all $(x, y) \in \Omega$,

$$\mathcal{T}(x, y) = \max(0, \phi_{x_1}(y)e^{-\lambda_1x} - \delta\phi_{x_1+\eta}(y)e^{-(\lambda_1+\eta)x}).$$

The function $T$ satisfies the Neumann boundary conditions on $\partial \Omega$. Let us now check that $T$ is a sub-solution to (1.6) with $Y = Y$. Note first that $0 \leq \mathcal{T} \leq s_0$ in $\Omega$. Moreover, if $\mathcal{T}(x, y) > 0$, then $x > x_0 \geq 0$ whence $0 \leq Y(x, y) = 1 - y\psi_{\partial \Omega}(y)e^{-\beta y}$.

Then, in that case:

$$\Delta \mathcal{T} + (c - u(y))\mathcal{T} + f(y, \mathcal{T} - h(y, \mathcal{T})

\geq \Delta \mathcal{T} + (c - u(y))\mathcal{T} + \left(\frac{\partial f}{\partial T}(y, 0)\mathcal{T} - M\mathcal{T}^{1+\alpha}\right) (1 - y\psi_{\partial \Omega}(y)e^{-\beta y})

\geq \frac{\partial h}{\partial T}(y, 0)\mathcal{T} - M\mathcal{T}^{1+\alpha}\n
\geq -\delta(k(\lambda_1 + \eta) - c(\lambda_1 + \eta))\phi_{x_1+\eta}(y)e^{-(\lambda_1+\eta)x}

\geq -\frac{\partial f}{\partial T}(y, 0)\gamma\psi_{\partial \Omega}(y) e^{-\beta y} - 2M\mathcal{T}^{1+\alpha}\n
\geq \delta \varepsilon \phi_{x_1+\eta}(y) e^{-(\lambda_1+\eta)x} \frac{\partial f}{\partial T}(y, 0)\gamma e^{-(\lambda_1+\beta)x} - 2Me^{-\lambda_1(1+u)x}

\geq \left(\delta \varepsilon \phi_{x_1+\eta}(y) - \frac{\partial f}{\partial T}(y, 0)\gamma - 2M\right) e^{-(\lambda_1+\eta)x} \geq 0,$$

because of (3.3), (3.4), (3.5) and since $0 < \phi_{x_1+\eta}(y), 0 < \psi_{\partial \Omega}(y) \leq 1$ in $\Omega$.

3.2. The finite cylinder problem

Here, we construct a solution of (1.6) in a finite cylinder $\Omega_a = (-a, a) \times \omega$ with $a > 0$. Let $C(\overline{\Omega_a})$ denote the space of continuous functions in $\overline{\Omega_a}$, with the usual sup-norm. Observe that $0 \leq \mathcal{T} \leq \mathcal{T}$ and $0 \leq Y \leq 1$ in $\overline{\Omega_a}$. We denote by $E_a$ the set

$$E_a = \{(T, Y) \in C(\overline{\Omega_a}; \mathbb{R}^2), \mathcal{T} \leq \mathcal{T} \leq \mathcal{T} \text{ and } Y \leq Y \leq 1 \text{ in } \overline{\Omega_a}\}.$$

The set $E_a$ is a convex closed bounded subset of the Banach space $C(\overline{\Omega_a}; \mathbb{R}^2)$.
We now consider a fixed point problem for an approximation of the travelling front solution in $\Omega_a$. For any pair $(T_0, Y_0) \in E_a$, let $(T, Y) = \Phi_a(T_0, Y_0)$ be the unique solution of

\[
\begin{align*}
\Delta T + (c - u(y)) T_x - K_a T &= -f(y, T_0) Y_0 + h(y, T_0) - K_a T_0 & \text{in } \Omega_a, \\
L^{-1}\Delta Y + (c - u(y)) Y_x - f(y, T_0) Y &= 0 & \text{in } \Omega_a,
\end{align*}
\]

together with the boundary conditions

\[
\begin{align*}
T(\pm a, y) &= T(\pm a, y), & Y(\pm a, y) &= Y(\pm a, y) & \text{for } y \in \partial \Omega,
\end{align*}
\]

\[
\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 & \text{on } [-a, a] \times \partial \Omega.
\]

Since $h \in C^1(\overline{\Omega}) \cap [0, +\infty) \cap \mathbb{R}$, we can assume that $K_a$ is positive and such that $\forall y \in \overline{\Omega}$:

\[
s \in \left[0, \sup_{\Omega_a} T\right] \rightarrow h(y, s) - K_a s \text{ is decreasing.}
\]

Such a solution $(T, Y)$ exists, belongs to $C(\overline{\Omega_a}; \mathbb{R}^2)$ and it is unique (see [4, 7]). To show that $\Phi_a$ has a fixed point, we show that it is compact and that it leaves $E_a$ invariant.

$E_a$ is invariant by $\Phi_a$. Let $(T_0, Y_0)$ be any element of $E_a$, and $(T, Y) = \Phi_a(T_0, Y_0)$. One can check that $\overline{T}$ satisfies

\[
\Delta \overline{T} + (c - u(y)) \overline{T}_x - K_a \overline{T} \geq -f(y, \overline{T}) Y + h(y, \overline{T}) - K_a \overline{T}_0,
\]

where the last inequality follows from (3.7) and the monotonicity of $f$. Furthermore, $\overline{T}$ satisfies the same boundary conditions as $T$ on the boundary of $\Omega_a$. The weak maximum principle implies that $\overline{T} \leq T$ in $\overline{\Omega_a}$. The inequalities $\mathcal{T} \leq \overline{T}$, $Y \geq Y$ and $Y \leq 1$ in $\overline{\Omega_a}$ can be checked similarly. We conclude that $\Phi_a$ leaves $E_a$ invariant.

The map $\Phi_a$ is compact. We introduce $(k_1, j_1) = \Phi_a(\overline{T}, 1)$ and $(k_2, j_2) = \Phi_a(\overline{T}, 1)$. For any pair $(T_0, Y_0) \in E_a$ and $(T, Y) = \Phi_a(T_0, Y_0)$, one has

\[
\Delta k_1 + (c - u(y)) k_{1,x} - K_a k_1 = -f(y, \overline{T}) Y + h(y, \overline{T}) - K_a \overline{T}_0
\]

and thus $T \leq k_1$ in $\overline{\Omega_a}$. Similarly, one can check that $Y \leq j_2$ in $\overline{\Omega_a}$. Thus,

\[
\begin{align*}
\frac{T - T}{\bar{T}} &\leq k_1 & \leq \overline{T} & \leq T & \leq k_2 & \leq j_2 & \leq Y & \leq \overline{Y} & \leq Y & \leq \overline{Y} & \leq 1 & \text{in } \overline{\Omega_a},
\end{align*}
\]

(3.8) for any pair $(T_0, Y_0) \in E_a$ and $(T, Y) = \Phi_a(T_0, Y_0)$.

Let $(T_0^n, Y_0^n)$ be a sequence in $E_a$ and $(T^n, Y^n) = \Phi_a(T_0^n, Y_0^n)$. By standard elliptic estimates up to the boundary, the sequence $(T^n, Y^n)$ is bounded in $C^1(D; \mathbb{R}^2)$ norm for any compact subset $D \subset \Sigma_a = \overline{\Omega_a} \setminus \{\pm a\} \times \partial \Omega$. Therefore, by a diagonal extraction process, there exists a subsequence which converges locally uniformly in $\Sigma_a$ to a pair $(T, Y)$ of continuous functions in $\Sigma_a$. Since each $(T^n, Y^n)$ satisfies (3.8), it follows that $(T, Y)$ satisfies (3.8) in $\Sigma_a$. Furthermore, as we have $k_1(\pm a, y) = T(\pm a, y)$ and $j_2(\pm a, y) = Y(\pm a, y)$ in $\overline{\Omega_a}$, and since $\overline{T}$, $k_1$ and $j_2$ are continuous in $\overline{\Omega_a}$, the functions $(T, Y)$ can be extended in $\overline{\Omega_a}$ by two continuous functions, still denoted by $(T, Y)$, satisfying (3.8) in $\overline{\Omega_a}$.

For any $\varepsilon > 0$, there exists $\kappa > 0$ such that

\[
\begin{align*}
0 \leq k_1 - \overline{T} &\leq \varepsilon & \text{in } [-a - a + \kappa] \times \overline{\Omega_a} \cup [a - \kappa, a] \times \overline{\Omega_a}, \\
0 \leq j_2 - \overline{Y} &\leq \varepsilon & \text{in } [a - a + \kappa] \times \overline{\Omega_a} \cup [-a - \kappa, -a] \times \overline{\Omega_a},
\end{align*}
\]

and thus $|T^n - T| \leq \varepsilon$ and $|Y^n - T| \leq \varepsilon$ in the same sets, for all $n$. On the other hand, the sequence $(T^n, Y^n)$ converges uniformly to $(T, Y)$ in $[-a - \kappa, a - \kappa] \times \overline{\Omega_a}$. Therefore, $(T^n, Y^n)$ converges uniformly to $(T, Y)$ in $\overline{\Omega_a} = [-a, a] \times \overline{\Omega_a}$ and thus the map $\Phi_a$ is compact.
A fixed point of \( \Phi_n \). One then concludes from the Schauder fixed point theorem that \( \Phi_n \) has a fixed point in \( E_a \). In other words, there exists a solution \( (T_a, Y_a) \in E_a \) of system (1.6) in \( \Omega_a \), together with the boundary conditions (3.6).

3.3. Passage to the infinite cylinder

Let now \((a_n)_{n \in \mathbb{N}}\) be an increasing sequence of positive numbers such that \( a_n \to +\infty \) as \( n \to +\infty \). Let \((T_{a_n}, Y_{a_n})_{n \in \mathbb{N}}\) be a sequence of solutions of (1.6) in \( \Omega_{a_n} \) and (3.6) with \( a = a_n \).

From standard elliptic estimates up to the boundary, the sequence \((T_{a_n}, Y_{a_n})_{n \in \mathbb{N}}\) is bounded in \( C^{2,1}(\overline{\Omega}) \). Up to extraction of a subsequence, the functions \((T_{a_n}, Y_{a_n})_{n \in \mathbb{N}}\) then converge in \( C^2(\overline{\Omega}) \) to a pair \((T, Y)\) in \( C^2(\overline{\Omega}) \) of solutions of (1.6)–(1.4) such that

\[
0 \leq T \leq \bar{T}, \quad 0 \leq Y \leq 1 \quad \text{in} \ \overline{\Omega}.
\]

In particular, we have \( T(+\infty, y) = 0 \) and \( Y(+\infty, y) = 1 \) uniformly in \( y \in \partial \Omega \). Furthermore, since \( Y(x, y) \) and \( T(x, y) \) are positive for large \( x \), the strong maximum principle implies that \( Y > 0 \) and \( T > 0 \) in \( \Omega \). Moreover, since \( f(y, T) > 0 \), the function \( Y \) cannot be identically equal to 1, whence \( Y < 1 \) in \( \Omega \) from the strong maximum principle.

It now remains to be shown that \( T \) is bounded, and that the functions \((T, Y)\) satisfy the right conditions at \(-\infty\).

**Boundedness of \( T \).** Assume by contradiction that \( T \) is not in \( L^\infty(\Omega) \). Since \( 0 \leq T \leq \bar{T} \) in \( \overline{\Omega} \), the only possibility for the function \( T \) to grow is on the left. Let then a sequence \((x_n, y_n)_{n \in \mathbb{N}}\) in \( \mathbb{R} \times \partial \Omega \) such that \( T(x_n, y_n) \to +\infty \) and \( x_n \to -\infty \) as \( n \to +\infty \). It follows from the Harnack inequality that for each \( R > 0 \),

\[
\min_{(x, y) \in [x_n - R, x_n + R] \times \partial \Omega} T(x, y) \to +\infty
\]

as \( n \to +\infty \). Let also \( m = \min_{\partial \Omega} f(y, 1) > 0 \). By concavity of the function \( \nu \) defined in (1.13) and since, \( \nu(0) = 0 \), there exist exactly two real numbers \( \rho_- < 0 < \rho_+ \) such that

\[
Le^{-1} \nu(-\rho_+ Le) = Le^{-1} \rho_+^2 + e^\rho_+ - m.
\]

We denote by \( \psi_{\pm} \) the two principal eigenfunctions of the problem (1.13) with the values \( \lambda = -\rho_{\pm} Le \), normalized so that, say, \( \min_{\partial \Omega} \psi_{\pm} = 1 \). The functions \( u_{\pm}(x, y) = e^{\rho_{\pm} y} \psi_{\pm}(y) \) then satisfy

\[
Le^{-1} \Delta u_{\pm} + (c - u(y))u_{\pm, y} - mu_{\pm} = 0 \quad \text{in} \ \Omega,
\]

with Neumann boundary conditions. Fix now any \( R > 0 \) and choose \( N \in \mathbb{N} \) so that

\[
\min_{(x, y) \in [x_n - R, x_n + R] \times \partial \Omega} T(x, y) \geq 1
\]

for all \( n \geq N \). Then, as the function \( f(y, T) \) is increasing in the variable \( T \), we have that \( f(y, T) \geq f(y, 1) \geq m \) in \( [x_n - R, x_n + R] \times \partial \Omega \) for all \( n \geq N \). Whence, on the same domain,

\[
Le^{-1} \Delta Y + (c - u(y))Y_y - mY \geq 0.
\]

The function \( Y \) also satisfies the Neumann boundary conditions on \( \partial \Omega \). Furthermore, \( Y \leq 1 \) in \( \Omega \). It then follows from the weak maximum principle that

\[
\forall (x, y) \in [x_n - R, x_n + R] \times \partial \Omega, \ Y(x, y) \leq e^{\rho_-(x-x_n-R)} \psi_+(y) + e^{\rho_-(x-x_n+R)} \psi_-(y).
\]
Therefore, along the section \( x = x_n \), the function \( Y \) is small:

\[
\limsup_{n \to +\infty} \max_{Y(x_n, y)} Y(x_n, y) \leq \max_{\lambda} \left( \max_{\Sigma} \psi_+, \max_{\Sigma} \psi_- \right) \times (e^{-\rho_R} + e^{\rho_R}).
\]

Since \( R > 0 \) can be chosen arbitrary, we conclude that \( Y(x_n, \cdot) \to 0 \) uniformly in \( \bar{\omega} \) as \( n \to +\infty \). Let now \( \epsilon > 0 \) be any positive real number, and \( N \in \mathbb{N} \) such that \( Y(x_n, y) \leq \epsilon \) for all \( n \geq N \) and \( y \in \bar{\omega} \). Since the function \( Y \) satisfies

\[
\text{Le}^{-1} \Delta Y + (c - u(y))Y = f(y, T)Y \geq 0,
\]

it follows from the weak maximum principle that \( Y(x, y) \leq \epsilon \) for all \( (x, y) \in [x_n, x_N] \times \bar{\omega} \) and \( n \geq N \) such that \( x_n \leq x_N \). Since \( x_n \to -\infty \) as \( n \to +\infty \), one concludes that \( Y \leq \epsilon \) in \( (-\infty, x_N] \times \bar{\omega} \). As \( Y \geq 0 \), we finally obtain that \( Y(-\infty, \cdot) = 0 \) uniformly in \( Y \in \bar{\omega} \).

We now use the same arguments as in section 2.5. We have shown that \( Y(-\infty, \cdot) = 0 \). Furthermore, since \( T \leq \bar{T} \), we know that there exist \( C_0 > 0 \) and \( \lambda > 0 \) solution of \( k(\lambda) = c\lambda \) such that \( T \leq C_0 e^{-\lambda s} \). As already shown in section 2.5, it then implies that there exist \( A, \gamma, \delta > 0 \) such that \( T \leq y^\delta \) for all \( x \leq -A \), which is in contradiction with \( T(x_n, y_n) \to +\infty \) and \( x_n \to -\infty \) as \( n \to +\infty \). We conclude that \( T \) is bounded.

**Behaviour of the solution on the left.** It now only remains to show that \( T_L(-\infty, \cdot) = Y_L(-\infty, \cdot) = 0 \). In fact, we have that \( T \) and \( Y \) converge to constants as \( x \to -\infty \). Since \( T \) and \( Y \) are globally bounded, standard elliptic estimates and Harnack inequality imply that \( \nabla T \) and \( \nabla Y \) are globally bounded as well. One can then proceed as in section 2.1 and conclude that \( Y \) converges to a constant \( Y_{\infty} \) when \( x \to -\infty \), hence \( Y_L(-\infty, \cdot) = 0 \).

As in section 2.2, we show that \( \int_{\Omega} (\partial h / \partial T)(y, 0)T(x, y) \, dx \, dy < +\infty \) and \( \int_{\Omega} |\nabla T|^2 \, dx \, dy < +\infty \). Hence \( T(-\infty, \cdot) = Y_L(-\infty, \cdot) = 0 \). The proof of part (a) of theorem 2 is complete. \( \square \)

**4. Existence of fronts with minimal speed**

This section is dedicated to the proof of part (b) of theorem 2. Here, we will assume that

\[
\sup_{\lambda \in \mathbb{R}} (\mu_{h, f}(\lambda) - \lambda^2) < 0.
\]

Let us compare (4.1) with the condition \( \mu_{h, f}(0) < 0 \). As we said in remark 1, those hypotheses are equivalent in the case \( f \) and \( h \) independent of \( y \). Otherwise, it depends on the flow \( u \). Indeed, let first \( h \) be of the form \( h(T) = aT \) with \( a \in \mathbb{R}^+ \) such that \( \mu_{h, f}(0) = 0 \). Such a \( h \) exists because, as one can easily check, \( \mu_{0, f}(0) < 0 \) and \( \mu_{aT, f}(0) = \mu_{0, f}(0) + a \) for all \( a \in \mathbb{R}^+ \). Furthermore, from section 1, we know that

\[
\mu'_{h, f}(0) = -\int_0^1 u(y)\phi_0'(y) \, dy,
\]

where \( \phi_0 \) is a solution of (1.9) with \( \lambda = 0 \) and \( L^2(\omega) \) norm equal to 1. Note that \( \phi_0 \) is independent of \( u \). Thus, if \( \phi_0 \) is not constant, which is equivalent to say that \( (\partial f / \partial T)(y, 0) \) is not constant, a suitable choice of \( u \) allows us to obtain any value for \( \mu'_{h, f}(0) \). For instance, we can choose \( u \) so that \( \mu'_{h, f}(0) > 0 \), and then there exists \( \lambda > 0 \) such that \( \mu_{h, f}(\lambda) - \lambda^2 > 0 \). Besides, let the sequence \( \{h_n\}_{n \in \mathbb{N}} \) defined by \( h_n(T) = h(T) - \frac{1}{n} T = (a - \frac{1}{n})T \) for \( n \in \mathbb{N} \) be large enough so that \( h_n \) satisfies (1.5). Then, for a sufficiently large \( n \), one can check that \( \mu_{h_n, f}(0) < 0 \) but \( \mu_{h_n, f}(\lambda) - \lambda^2 > 0 \), and those two conditions are not equivalent.
4.1. Boundedness of a sequence of solutions for different speeds

We first show the following general lemma, which holds without any hypothesis on $\mu, h, f$:

**Lemma 2.** Let $(c_n, T_n, Y_n)$ be a sequence of solutions of (1.6)–(1.7) and (1.4) such that $0 < T_n$ and $0 < Y_n < 1$ in $\Omega$ for each $n \in \mathbb{N}$, and $\sup_n c_n < +\infty$. Then

$$\sup_n \|T_n\|_{L^\infty(\Omega)} < +\infty.$$ 

**Proof.** Under those hypotheses, since $c_n \geq c^*$ and $c_n > 0$ for each $n \in \mathbb{N}$ by theorem 1, we have that the sequence $c_n$ is bounded. Thus, up to extraction of a subsequence, one can assume that $c_n \to c_\infty \in [\max(c^*, 0), +\infty)$ as $n \to +\infty$. Furthermore, theorem 1 also implies that for each $n \in \mathbb{N}$, the function $T_n$ is globally bounded. Assume now, for the sake of a contradiction, that the sequence $(\|T_n\|_{L^\infty(\Omega)})_{n \in \mathbb{N}}$ is not bounded. Up to extraction of a subsequence, one can assume that $\|T_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$.

From the boundary conditions (1.7) and theorem 1, we know that each pair $T_n$ satisfies $T_n(-\infty, \cdot) = T_n(+\infty, \cdot) = 0$. Thus, each $T_n$ attains a maximum inside the cylinder $\Omega$, and there exists a sequence of points $(x_n, y_n) \in \Omega$ such that $T_n(x_n, y_n) = \max_{\Omega}T_n \to +\infty$ as $n \to +\infty$. Up to extraction of another subsequence, we may assume that $y_n \to y_\infty \in \partial\Omega$ as $n \to +\infty$.

Define now the normalized shifts

$$U_n(x, y) = \frac{T_n(x + x_n, y)}{T_n(x_n, y_n)}.$$ 

Each function $U_n$ satisfies $0 < U_n \leq 1$ in $\Omega$, Neumann boundary conditions on $\partial\Omega$ and

$$\Delta U_n + (c_n - u(y))U_{n,x} + \frac{f(y, T_n(x_n, y_n)U_n)}{T_n(x_n, y_n)}Z_n - g_n U_n = 0 \quad \text{in } \Omega,$$

where $Z_n(x, y) = Y_n(x + x_n, y)$ is the shifted concentration, and

$$g_n = \frac{h(y, T_n(x + x_n, y))}{T_n(x + x_n, y)}.$$ 

We already saw in section 2.1 that the sequence $(g_n)_{n \in \mathbb{N}}$ converges, up to extraction of a subsequence, to a function $g$ weakly in $L^{1,\ast}(\Omega)$ as $n \to +\infty$. Since $g_n(x, y) \geq (\partial h/\partial T)(y, 0)$, we have $g$ nonnegative and positive on a set of positive measure.

In order to pass to the limit as $n \to +\infty$, we now claim that

$$\forall R > 0, \quad \max_{-R \leq x \leq R} Z_n(x, y) \to 0 \quad \text{as } n \to +\infty. \quad (4.2)$$

The proof is similar to the one in section 3.3, and relies on the construction of a supersolution for $Z_n$ using principal eigenfunctions of (1.13).

Lastly, we know that the functions $U_n$ are uniformly bounded (by 1) in $L^\infty(\Omega)$. It follows from standard elliptic estimates that up to extraction of some subsequence, the functions $U_n$ converge as $n \to +\infty$ in $W^{2,p}_{\text{loc}}(\Omega)$ weak for all $1 < p < +\infty$ and strongly in $C^1_{\text{loc}}(\Omega)$ to a function $U_\infty$ which satisfies

$$\Delta U_\infty + (c_\infty - u(y))U_{\infty,x} - g U_\infty = 0 \quad \text{in } \Omega$$

with Neumann boundary conditions. Furthermore, $0 \leq U_\infty \leq 1$ and $U_\infty(0, y_\infty) = 1$. The strong maximum principle and the Hopf lemma then imply that $U_\infty = 1$ in $\Omega$. This is a contradiction, since $g$ is positive on a set of positive measure. The lemma is proved. \qed
4.2. Upper bound on $Y(-\infty, .)$

Here, we give a corollary of lemma 1:

**Corollary 1.** Let $(c, T, Y)$ be a solution of (1.6) and (1.4) such that $0 < T$ and $0 < Y < 1$. Then $Y_\infty = Y(-\infty, .)$ exists and

$$Y_\infty \leq a^* := 1 + \frac{\mu_{h,f}(0)}{\min_{y \in \mathbb{R}} \partial_T(y, 0)} < 1. \quad (4.3)$$

**Proof.** Note that by theorem 1, we already know that $Y_\infty = Y(-\infty, .)$ exists and that $T(-\infty, .) = 0$. Therefore, we can apply lemma 1 to $(-c, \tilde{T}, \tilde{Y})$ solution of (1.6) with $u$ replaced by $-u$, where $\tilde{T}(x, .) = T(-x, .)$ and $\tilde{Y}(x, .) = Y(-x, .)$. We obtain that there exists $\Lambda > 0$ such that $\mu_{h,Y_n}(\Lambda) = c\Lambda + \Lambda^2$. Let now $\phi_{-\Lambda}$ be the principal eigenfunction of (1.9) normalized so that $\|\phi_{-\Lambda}\|_{L^2(\mathbb{R})} = 1$. One can easily check that

$$\mu_{h,Y_n}(\Lambda) \leq \mu_{h,f}(\Lambda) + \int_\omega (1 - Y_\infty) \frac{\partial f}{\partial T}(y, 0) \phi_{-\Lambda}^2(y) dy,$$

and thus

$$Y_\infty \int_\omega \frac{\partial f}{\partial T}(y, 0) \phi_{-\Lambda}^2(y) dy \leq \mu_{h,f}(\Lambda) - c\Lambda - \Lambda^2 + \int_\omega \frac{\partial f}{\partial T}(y, 0) \phi_{-\Lambda}^2(y) dy \leq \mu_{h,f}(0) + \int_\omega \frac{\partial f}{\partial T}(y, 0) \phi_{-\Lambda}^2(y) dy,$$

where the last inequality follows from the concavity of $\mu_{h,f}$: indeed, for $c \geq c^*$, we have that $c \geq \mu_{h,f}(0)$, and thus $\mu_{h,f}(\lambda) - (\lambda^2 - c\lambda) \leq \mu_{h,f}(0)$ for all $\lambda \leq 0$. Since $\mu_{h,f}(0) < 0$ from theorem 1, we have obtained (4.3). \qed

4.3. Proof of part (b) of theorem 2

We assume that $\sup_{\lambda \in \mathbb{R}} (\mu_{h,f}(\lambda) - \lambda^2) < 0$. It then follows immediately from the definition of $c^*$ that $c^* > 0$. To prove the existence of a non-trivial travelling front solution with speed $c^*$, we use an approximation by a sequence of fronts with speeds larger than $c^*$.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of speeds such that $c_n > c^*$ for all $n$ and $c_n \to c^*$ as $n \to +\infty$. For each $n$, there exists a bounded solution $(T_n, Y_n)$ of (1.6)–(1.7) and (1.4) with the speed $c = c_n$, such that $T_n > 0$ and $0 < Y_n < 1$ in $\mathbb{R}$. According to (1.7) and theorem 1, we have

$$T_n(\infty, .) = 0 \quad \text{and} \quad Y_n(\infty, .) = 1,$$

$$T_n(-\infty, .) = 0 \quad \text{and} \quad Y_n(-\infty, .) = Y_{n,\infty} \in (0, 1).$$

It also follows from lemma 2 that there exists a constant $M > 0$ such that

$$\forall n \in \mathbb{N}, \forall (x, y) \in \mathbb{R}, 0 < T_n(x, y) \leq M. \quad (4.4)$$

As we have mentioned, our strategy is to pass to the limit as $n \to +\infty$, in order to get a solution of (1.6)–(1.7) and (1.4) with the speed $c = c^*$. Any shift of the travelling wave $(T_n, Y_n)$ in the variable $x$ along the cylinder is, of course, also a travelling wave, and the main technical difficulty here is to shift suitably the functions $(T_n, Y_n)$ so that the limit pair is non-trivial and satisfies the correct limiting conditions at infinity. For that we have to identify a region where both $T_n$ and $Y_n$ are uniformly not very flat.
Locating the interface. Let \(a^*\) be as defined in (4.3). For each \(a \in (a^*, 1)\), and \(n \in \mathbb{N}\), we define
\[
x_n^a = \min\{x \in \mathbb{R}, Y_a \geq a \in [x, +\infty) \times \bar{\Omega}\}.
\]
Since the functions \(Y_a\) are continuous in \(\bar{\Omega}\), satisfy \(Y_a(+:\infty) = 1\) and \(Y_a(-\infty) = a^*\), the real numbers \(x_n^a\) are well defined. Moreover, \(x_n^a\) is nondecreasing in \(a \in (a^*, 1)\) for each fixed \(n\). Observe that, also,
\[
Y_n \geq a \in [x_n^a, +\infty) \times \bar{\Omega} \quad \text{and} \quad \min_{\bar{\Omega}} Y_n(x_n^a, \cdot) = a.
\]
Since \(Y_n(+:\infty) = 1\), we have \(\|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \bar{\Omega})} > 0\). Furthermore, since \(|\nabla Y_n(x, y)| \to 0\) as \(x \to +\infty\) uniformly in \(y \in \bar{\Omega}\), we can define the points
\[
\tilde{x}_n^a = \min\{x \in [x_n^a, +\infty), \exists y \in \bar{\Omega}, |\nabla Y_n(x, y)| = \|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \bar{\Omega})}\}.
\]
We now introduce the following lemma, that shows that to the right of \(x_n^a\), there are regions where \(Y_a\) are uniformly not too flat.

**Lemma 3.** For all \(a \in (a^*, 1)\), we have
\[
\inf_{x_n^a} \|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \bar{\Omega})} > 0.
\]
The proof of this lemma is postponed until the end of the section.

Normalization of \((T_n, Y_n)\) and passage to the limit. Let us now complete the proof of the existence of a non-trivial bounded solution \((T, Y)\) of (1.6)–(1.7) and (1.4) with the speed \(c = c^*\). Choose any \(a \in (a^*, 1)\), and let \(\tilde{y}_n^a\) be a sequence of points in \(\bar{\Omega}\) such that \(|\nabla Y_n(\tilde{x}_n^a, \tilde{y}_n^a)| = \|\nabla Y_n\|_{L^\infty([x_n^a, +\infty) \times \bar{\Omega})}\) for all \(n \in \mathbb{N}\). Up to extraction of a subsequence, one can assume that the sequence \(\tilde{y}_n^a\) converges to a point \(\tilde{y}_a^* \in \bar{\Omega}\). Lemma 3 implies that
\[
\inf_n |\nabla Y_n(\tilde{x}_n^a, \tilde{y}_n^a)| > 0.
\]
(4.5)
For each \(n\) and \((x, y) \in \bar{\Omega}\), define the shifted functions
\[
T_n^a(x, y) = T_a(x + \tilde{x}_n^a, y), \quad Y_n^a(x, y) = Y_a(x + \tilde{x}_n^a, y).
\]
Recall that both \(T_n^a\) and \(Y_n^a\) are uniformly bounded in \(\bar{\Omega}\), independently of \(n\) (that is (4.4)). By standard elliptic estimates up to the boundary, these functions, as well as the shifts \(T_n^a\) and \(Y_n^a\) are also bounded in \(C^{2,\alpha}(\bar{\Omega})\), uniformly in \(n\). Up to extraction of a subsequence, one can assume that the sequence \((T_n^a, Y_n^a)\) converges to a function \((T^a, Y^a)\) in \(C^{2,\alpha}(\bar{\Omega})\) as \(n \to +\infty\). Passing to the limit, we conclude that the pair \((T^a, Y^a)\) satisfies (1.6) and (1.4) with \(c = c^*\). They obey the uniform bounds \(0 < T^a \leq M\) and \(0 \leq Y^a \leq 1\) in \(\bar{\Omega}\). Furthermore, (4.5) implies that
\[
|\nabla Y^a(0, \tilde{y}_a^*)| > 0.
\]
(4.6)
Thus, \(Y^a\) is not a constant. By the strong maximum principle and Hopf lemma, we can conclude that \(0 < Y^a < 1\) in \(\bar{\Omega}\), and \(Y^a\) is non-trivial.

Let us now check that \(T^a > 0\). Otherwise, if \(T^a\) vanishes somewhere in \(\bar{\Omega}\), then it is identically equal to 0 by the strong maximum principle and Hopf Lemma. In that case, the function \(Y^a\) would satisfy
\[
Lc^{-1} \Delta Y^a + (c^* - u(y))Y_x^a = 0 \quad \text{in } \Omega
\]
with Neumann boundary conditions. We apply now the same method as in section 2.1 to obtain that, for a sequence \(A_n \to +\infty\), the shifted functions \(Y^a(\pm A_n + x, y)\) converge to two constant \(Y^a_{\pm}\) such that
\[
c^*(Y^a_+ - Y^a_-) = \frac{1}{2}\int_{\Omega} f(T^a)Y^a = 0.
\]
that is \( Y^a_n = Y^a \). Lastly, multiplying (4.3) by \( Y^a \), integrating over the cylinder \((-A_n, A_n) \times \omega\) and passing to the limit as \( n \to +\infty \) imply that
\[
\int_{\Omega} |\nabla Y^a|^2 = 0,
\]
which contradicts (4.6). We conclude that \( T^a > 0 \) in \( \Omega \).

**The limits at infinity.** One can proceed as in sections 2.1 and 2.2 to conclude that \( Y(-\infty, .), \ Y(+\infty, .), T(-\infty, .) \) and \( T(+\infty, .) \) exist. Moreover, we have that \( T(+\infty, .) = 0 \) from (2.7) and \( Y_i(-\infty, .) = T_i(-\infty, .) = 0 \) by standard elliptic estimates.

We will now show that \( Y_a = Y(+\infty, .) = 1 \). We claim that the sequence \( z^a_n = \tilde{x}^a_n - x^a_n \) is bounded. Otherwise, up to extraction of another subsequence, we would have \( z^a_n \to +\infty \) as \( n \to +\infty \). Thus, for each \((x, y) \in \Omega\), we would have \( x + \tilde{x}^a_n \geq x^a_n \) for sufficiently large \( n \), and then \( Y^a_n(x, y) = Y_n(x + \tilde{x}^a_n, y) \geq a \). This would imply that \( Y^a(x, y) \geq a > a^* \) in \( \Omega \). But \( Y^a(-\infty, .) \leq a^* \) from the calculations in section 4.2, which leads to a contradiction.

Let now \( b \) be any real number in \((a, 1)\). As in the previous argument, the shifted functions \( T^b_n(x, y) = T_n(x + \tilde{x}^b_n, y) \) and \( Y^b_n(x, y) = Y_n(x + \tilde{x}^b_n, y) \) converge in \( C^2_{\text{loc}}(\Omega) \) as \( n \to +\infty \), up to extraction of a subsequence, to a pair \((T^b, Y^b)\) of solutions of (1.6) and (1.4) with \( c^* = c^* \), such that \( T^b(-\infty, .) \leq a^* \). We claim that the sequence \( (x^b_n - \tilde{x}^b_n)_{n \in \mathbb{N}} \) is bounded. Indeed, as we know that the sequence \( x^a_n = \tilde{x}^a_n - x^a_n \) is bounded, if the sequence \( (x^a_n - \tilde{x}^a_n) \) is unbounded, then the sequence of nonnegative numbers \( \tilde{x}^b_n - x^a_n \) would be unbounded, which would imply that \( Y^b(-\infty, .) \geq a > a^* \). This is a contradiction.

Therefore, the sequence \( (x^b_n - \tilde{x}^b_n) \) is also bounded, and there exists \( A^b_n \geq 0 \) (which depends on \( a \) and \( b \) but not on \( n \)) such that \( x^b_n - \tilde{x}^b_n \leq A^b_n \) for all \( n \). However, for each \((x, y) \in [A^b_n, +\infty) \times \overline{\omega}\), we have then \( x + \tilde{x}^b_n \geq x^b_n \) and thus
\[
Y^b_n(x, y) = Y_n(x + \tilde{x}^b_n, y) \geq b
\]
for all \( n \). As a consequence, we have that \( Y^a(+\infty, .) \geq b \). Since \( b \) was arbitrarily chosen in \((a, 1)\), we conclude that \( Y^a(+\infty, .) = 1 \). This completes the proof of theorem 2. \( \square \)

**4.4. Proof of lemma 3**

Assume by contradiction that the conclusion of lemma 3 does not hold for a real number \( a \in (a^*, 1) \). Up to extraction of a subsequence, one can then assume that
\[
\|\nabla Y_n\|_{L^\infty([x^a_n, +\infty) \times \omega)} \to 0 \quad \text{as} \quad n \to +\infty.
\] (4.7)

**Temperature is small on the right.** We first claim that in this case, the ‘temperature interface’ is located far to the left of the ‘concentration interface’, that is, we have
\[
\|T_n\|_{L^\infty([x^a_n, +\infty) \times \omega)} \to 0 \quad \text{as} \quad n \to +\infty.
\] (4.8)

Indeed, assume now that (4.7) holds and (4.8) does not. Then there exist \( \varepsilon > 0 \) and a sequence \((x_n, y_n)_{n \in \mathbb{N}} \) in \( \Omega \) such that \( x_n \geq x^a_n \) and \( T_n(x_n, y_n) \geq \varepsilon \) for all \( n \in \mathbb{N} \). Up to extraction of a subsequence, we can assume that \( y_n \to y_\infty \in \overline{\omega} \) as \( n \to +\infty \). The standard elliptic estimates imply that the sequence of shifted functions \( T_n(x + x_n, y) \) and \( Y_n(x + x_n, y) \) converge in \( C^2_{\text{loc}}(\Omega) \), up to extraction of some subsequence, to a pair \((T, Y)\) of solutions of (1.6) with \( c^* = c^* \). Furthermore, \( T \) and \( Y \) satisfy \( T(0, y_\infty) \geq \varepsilon \) and
\[
0 \leq Y \leq 1, \quad 0 \leq T \leq M \quad \text{in} \quad \Omega,
\]
\[
Y \geq a > 0, \quad |\nabla Y| = 0 \quad \text{in} \quad [0, +\infty) \times \overline{\omega}.
\]
The strong maximum principle and Hopf lemma imply that \( T > 0 \) and \( Y > 0 \) in \( \overline{\Omega} \). This is a contradiction because \( Y \) is a constant in \([0, +\infty) \times \overline{\omega}\) and thus has to satisfy \( f(y, T)Y = 0 \) in the same domain.

**Temperature decays exponentially on the right.** We then claim that under assumptions (4.7) and hence (4.8), \( T_n \) decays exponentially uniformly to the right of \( x_n^a \), that is there exist a positive number \( \lambda > 0 \), an integer \( N \) and \( A \geq 0 \) so that for all \( n \geq N \) and all \((x, y) \in [x_n^a + A, +\infty) \times \overline{\omega}\) we have

\[
\frac{T_{n,x}(x, y)}{T_n(x, y)} \leq -\lambda. \tag{4.9}
\]

As \( T_n > 0 \) and since the speeds \( c_n \) are bounded, it follows from standard elliptic estimates and the Harnack inequality that the functions \( |\nabla T_n|/T_n \) are uniformly bounded in \( \Omega \) and independently of \( n \). Assume now that (4.9) does not hold. Then, after extraction of a subsequence, there exists a sequence of points \((x_n, y_n)\) such that

\[
\lim_{n \to +\infty} (x_n - x_n^a) = +\infty \tag{4.10}
\]

and

\[
\lim_{n \to +\infty} \frac{T_{n,x}(x_n, y_n)}{T_n(x_n, y_n)} \geq 0. \tag{4.11}
\]

Set the normalized and shifted temperature

\[
U_n(x, y) = \frac{T_n(x + x_n, y)}{T_n(x_n, y_n)}
\]

for all \( n \) and \((x, y) \in \overline{\Omega}\). We now proceed similarly to the proof of lemma 1. Up to extraction of a subsequence, one can assume that \( y_n \to y_\infty \in \overline{\omega} \) as \( n \to +\infty \), and that the sequence of the shifted concentrations \( Z_n(x, y) = Y_n(x + x_n, y) \) converges to a function \( Z \) in \( C^{2}_{\text{loc}}(\overline{\Omega}) \) as \( n \to +\infty \). But (4.7) and (4.10) imply that \( Z \) is a constant and belongs to \([a, 1] \) (recall that \( a \leq Y_n \leq 1 \) in \([x_n^a, +\infty) \times \overline{\omega}\)). As a consequence, up to extraction of another subsequence, the positive functions \( U_n \) converge in all \( W_{\text{loc}}^{2,p}(\overline{\Omega}) \) weak (for \( 1 < p < +\infty \)) to a classical nonnegative solution \( U \) of

\[
\Delta U + (c^* \, u(y))U_x + \frac{\partial f}{\partial T}(y, 0)ZU - \frac{\partial h}{\partial T}(y, 0)U = 0 \quad \text{in } \Omega,
\]

with Neumann boundary conditions on \( \partial \Omega \). Furthermore, we have that \( U(0, y_\infty) = 1 \) while (4.11) implies

\[
\frac{U_x(0, y_\infty)}{U(0, y_\infty)} \geq 0.
\]

It follows from the strong maximum principle and the Hopf lemma that \( U > 0 \) in \( \overline{\Omega} \) and it follows from standard elliptic estimates and the Harnack inequality that the function \(|\nabla U|/U\) is bounded in \( \Omega \). Let \((x'_n, y'_n)_{n \in \mathbb{N}} \) be a sequence of points in \( \overline{\Omega} \) such that

\[
\frac{U_x(x'_n, y'_n)}{U(x'_n, y'_n)} \to \sup_\Omega \frac{U_x}{U} =: \overline{M} \geq 0 \quad \text{as } n \to +\infty.
\]

Up to extraction of a subsequence, one can assume that \( y'_n \to y'_\infty \) as \( n \to +\infty \). Next, with the same arguments as above, the functions

\[
V_n(x, y) = \frac{U(x + x_n^a, y)}{U(x_n^a, y_n^a)}
\]
are bounded in $C^{\alpha,\gamma}_{\text{loc}}(\Omega)$ independently of $n$ and converge in $C^2_{\text{loc}}(\Omega)$, up to extraction of some subsequence, to a nonnegative function $V$ solving the same linear equation as $U$. Furthermore, we have that $V(0, y'_0) = 1$. Therefore, by the strong maximum principle and the Hopf lemma, $V$ is positive in $\Omega$. Moreover, we have

$$\frac{V_t}{V} \leq M \text{ in } \Omega \quad \text{and} \quad \frac{V_t(0, y'_0)}{V(0, y'_0)} = M.$$ 

However, one can easily check that the function $V_t/V$ satisfies a linear elliptic equation in $\Omega$ without the zeroth-order term, together with the Neumann boundary condition on $\partial \Omega$. Since $V_t/V$ attains its maximum at the point $(0, y'_0)$, the maximum principle implies that $V_t/V$ is identically equal to $M$ in $\Omega$. In other words, there exists a positive function $\phi(y)$ such that $V(x, y) = e^{M_y \phi(y)}$ in $\Omega$. It follows that $\phi$ is a principal eigenfunction of (1.9) (with $Zf$ instead of $f$) and that, by uniqueness of the principal eigenvalue, $\mu_{b, Zf}(-M) = c^* M + \overline{M}^2$. Recall that $M \geq 0$ and $c^* > 0$. Hence, as in section 4.2, it implies that $Z \leq a^*$. But this contradicts the fact that $Z \geq a > a^*$, thus (4.9) must hold.

**A sub-solution for $Y_n$.** We have just shown that for all $n \geq N$ and $(x, y) \in [x_n^d + A, +\infty) \times \Omega$, we have

$$0 < T_n(x, y) \leq T_n(x_n^d + A, y) e^{-\lambda(x - x_n^d - A)} \leq M e^{-\lambda(x - x_n^d - A)}.$$ 

The last inequality above follows from (4.4). On the other hand, for all $x \in [x_n^d, x_n^d + A]$ we have that $e^{-\lambda(x - x_n^d - A)} \geq 1$. Then the above inequality holds in the whole half-strip $x \geq x_n^d$. We apply the same strategy as in section 3.1: we use the above exponential bound for temperature to create a sub-solution for $Y_n$. We choose $\beta \geq 0$ as in (3.1) (with $c = c^*$ and $\lambda_1$ replaced by $\lambda$), and $\gamma > 0$ large enough so that

$$\gamma \times \min_{\overline{\Omega}} \psi_{\beta \lambda} \geq 1,$$

$$\gamma \lambda e^{-1}(v(y) - \beta^2 + c^* \beta \lambda e) \times \min_{\overline{\Omega}} \psi_{\beta \lambda} \geq \max_{y \in \overline{\Omega}} \frac{\partial f}{\partial T}(y, 0) Me^{\lambda A}.$$ 

For each $n \geq N$ and all $(x, y) \in \Omega$, we define $Y_n(x, y) = \max(0, 1 - \gamma \psi_{\beta \lambda}(y) e^{-\beta(x - x_n^d)})$.

As in section 3.1, each function $Y_n$ is a sub-solution for the $Y$-equation (1.6) with $c = c_n \geq c^*$ and $T = T_n$. As $f(y, T_n) \geq 0$, it then follows from the weak maximum principle that

$$\forall n \geq N, V_n(x, y) \in [x_n^d, +\infty) \times \Omega, Y_n(x, y) \geq Y_n(x, y) \geq 1 - \gamma \psi_{\beta \lambda}(y) e^{-\beta(x - x_n^d)}.$$ 

In particular, there exists $L_0 > 0$ independent of $n$ so that we have $Y_n(x_n^d + L_0, y) \geq (1+a)/2$ for all $y \in \partial \Omega$. However, since $\min_{\Omega} Y_n(x_n^d, y) = a < 1$ for all $n$, we finally reach a contradiction to our assumption (4.7). This completes the proof of lemma 3.

5. Criteria for flame extinction, blow-off or propagation

This section deals with the proof of theorem 3 and is divided in two parts. The first part treats of both flame extinction and blow-off and relies on the search for a suitable supersolution for temperature. The case of flame propagation is treated separately and uses the same method as in section 3 to also construct a sub-solution.

5.1. Flame extinction and blow-off

Let $(T, Y)$ be the solution of the Cauchy problem defined by (1.3)–(1.4) with an initial profile $(T_0, Y_0)$ verifying (1.8), and let $\lambda > 0$, $C > 0$ such that $T_0(x, y) \leq Ce^{-\lambda x}$ in $\mathbb{R}^+ \times \partial \Omega$. 


Moreover, we have that $T_0$ is bounded. Therefore, by increasing $C$, we can assume without loss of generality that we also have $T_0 \leq C$ in the entire domain $\Omega$.

We then observe that $0 \leq T(t, x, y)$ and $0 \leq Y(t, x, y) \leq 1$ for all $t \geq 0$ and $(x, y) \in \overline{\Omega}$ as follows from the maximum principle. It is also straightforward to check that $0 \leq T(t, x, y) \leq \Phi(t, x, y)$ for all $t \geq 0$ and $(x, y) \in \Omega$, where $\Phi$ is any solution of

$$\Phi_t + u(x)\Phi_x \geq \Delta \Phi + \frac{\partial f}{\partial T}(y, 0)\Phi - \frac{\partial h}{\partial T}(y, 0)\Phi \quad \text{for } t \geq 0 \text{ and } (x, y) \in \Omega \quad (5.1)$$

with Neumann boundary conditions on $[t \geq 0] \times \partial \Omega$, provided that $T_0(x, y) \leq \Phi(0, x, y)$ for all $(x, y) \in \overline{\Omega}$.

If $\mu_{h,f}(0) > 0$, we choose the function

$$\Phi(t, x, y) = Ce^{-\mu_{h,f}(0)t} \phi_0(y),$$

where $\phi_0$ is the positive eigenfunction of (1.9) with parameter 0, normalized so that $\min \phi_0 = 1$. Such a $\Phi$ indeed satisfies (5.1) with Neumann boundary conditions, and $T_0 \leq C \leq \Phi(0, y)$, which proves part (a) of theorem 3. Note that here, we only used the fact that $T_0$ is bounded.

We now assume that there exists $\gamma \in (0, \lambda]$ such that $\mu_{h,f}(\gamma) - \eta^2 > 0$. Let us look for the supersolution $\Phi$ in the form

$$\Phi(t, x, y) = Ce^{-\eta(x+\gamma t)} \phi_\gamma(y),$$

where $\gamma > 0$ is to be determined. Note that the fact that $\eta \leq \lambda$ and the above bounds on $T_0$ guarantees that $T_0(x, y) \leq \Phi(0, x, y)$ for all $(x, y) \in \overline{\Omega}$ regardless of the choice of $\gamma$. Insert now the expression of $\Phi$ in (5.1) and obtain that we need

$$-\eta\gamma \phi_\gamma - \eta u(y)\phi_\gamma \geq \eta^2 \phi_\gamma + \Delta \phi_\gamma + \frac{\partial f}{\partial T}(y, 0)\phi_\gamma - \frac{\partial h}{\partial T}(y, 0)\phi_\gamma \quad \text{in } \overline{\Omega}.$$ 

This holds if and only if $\eta \gamma \leq \mu_{h,f}(\gamma) - \eta^2$. Since the right-hand side is positive, this inequality holds for some small $\gamma > 0$, which concludes the proof of part (b) of theorem 3.

### 5.2. Propagation

Let $(T, Y)$ be the solution of the Cauchy problem defined by (1.3)–(1.4) with an initial profile $(T_0, Y_0)$ verifying (1.8). We assume that $\mu_{h,f}(0) < 0$ (thus $c^*$ and $\lambda^*$ are well defined) and that $k(\lambda) = \lambda^2 - \mu_{h,f}(\lambda) > 0$, which implies that $c := k(\lambda)/\lambda > 0$. Lastly, it follows from $\lambda \leq \lambda^* \leq \lambda^c$ that $c > c^*$ (recall that $k(s) = cs$ has no positive solution for $c < c^*$ and only one for $c = c^*$, which is $\lambda^*$), and that $\lambda$ is the smallest positive root of $k(s) = cs$ (recall that $k(s) = cs$ has two positive solutions $\lambda_1, \lambda_2$ for $c > c^*$, with $\lambda_1 < \lambda^* < \lambda_2$).

As before, the maximum principle implies that $0 \leq T(t, x, y)$ and $0 \leq Y(t, x, y) \leq 1$ for all $t \geq 0$ and $(x, y) \in \overline{\Omega}$. We now proceed as in section 3 in order to construct sub- and supersolutions for $T$ that both move with speed $c$. First, we define the function

$$\mathcal{T}(t, x, y) = Ce^{-\lambda(x-ct)} \phi_\gamma(y) > 0,$$

where $C > 0$ is chosen large enough so that $T_0(x, y) \leq \mathcal{T}(0, x, y)$ for all $(x, y) \in \overline{\Omega}$, and $\phi_\gamma$ is normalized so that $\|\phi_\gamma\|_{L^\infty(0,\infty)} = 1$. The function $\mathcal{T}$ is a supersolution for the $T$-equation (1.3) with $Y \leq 1$. It follows that $T(t, x, y) \leq \mathcal{T}(t, x, y)$ for all $t \geq 0$ and $(x, y) \in \overline{\Omega}$. This
implies in particular that for \((x, y) \in \overline{\mathbb{R}}^2\) and \(c' > c\), we have
\[ T(t, x + c't, y) \leq Ce^{-\lambda(x+ct)} \rightarrow 0 \quad \text{as} \ t \rightarrow +\infty. \]

It now remains to find \(x_1 \in \mathbb{R}\) and \(\alpha(x_1, y) > 0\) such that \(T(t, x_1 + ct, y) \geq \alpha(x_1, y)\) for all \(t \geq 1\) and \(y \in \overline{\mathbb{R}}\). To do this, we search for a suitable sub-solution. Let first \(\beta\) and \(\gamma\) be as in (3.1) and (3.2) in section 3.1, and define
\[ \sum(t, x, y) = \max(0, 1 - C\gamma\psi_{\beta\lambda}(y)e^{-\beta(x-ct)}). \]

This is a sub-solution for the \(Y\)-equation (1.3) with \(T \leq T\), and the maximum principle implies that \(Y(t, x, y) \geq \sum(t, x, y)\) for \(t \geq 0\) and \((x, y) \in \overline{\mathbb{R}}^2\), provided that \(Y_0(x, y) \geq 1 - C\gamma\psi_{\beta\lambda}(y)e^{-\beta x}\). This indeed holds for \(\beta \leq \gamma'\) and \(C\gamma \times \min_{\overline{\mathbb{R}}} \psi_{\beta\lambda} \geq C_3\) (this is possible since \(\beta\) could be chosen arbitrarily small and \(\gamma\) arbitrarily large).

Lastly, we choose \(\eta > 0\) and \(\delta > 0\) as in (3.3) and (3.5) with \(x_0\) such that \(Y > 0\) for \(x - ct \geq x_0\). Then
\[ \sum(t, x, y) = C' \max(0, \phi_\eta(y)e^{-\lambda(x-ct)} - \delta\phi_{x_0\eta}(y)e^{-\lambda(x-ct)}), \]
where \(C' \in (0, 1)\) to be determined, is a sub-solution for the \(T\)-equation (1.3) with \(Y \geq Y\).

In order to apply the maximum principle, it remains to check that \(T_0(x, y) \geq \sum(0, x, y)\) for all \((x, y) \in \overline{\mathbb{R}}^2\). Indeed, for \(x \leq x_0\), then \(\sum(0, x, y) = 0 \leq T_0(x, y)\). On the other hand, for \(x > x_0 \geq 0\), we have that \(T_0(x, y) \geq C_1e^{-\lambda x} \geq C'\phi_\eta(y)e^{-\lambda x} \geq \sum(0, x, y)\), provided that \(C' \leq C_1\). Therefore, it follows from the maximum principle that \(T(t, x, y) \geq \sum(t, x, y)\) for all \(t \geq 0\) and \((x, y) \in \overline{\mathbb{R}}^2\). Let now \(x_1 \in \mathbb{R}\) such that
\[ \alpha(x_1, y) := \phi_\eta(y)e^{-\lambda x_1} - \delta\phi_{x_0\eta}(y)e^{-\lambda x_1} > 0 \]
for all \(y \in \overline{\mathbb{R}}\). We then have that \(T(t, x_1 + ct, y) \geq \sum(t, x_1 + ct, y) = \alpha(x_1, y) > 0\), which concludes the proof of part (c) of theorem 3.

\[ \square \]

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