A Capacity Scaling Law for Artificial Neural Networks

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Abstract

By assuming an ideal neural network with gating functions handling the worst case data, we derive the calculation of two critical numbers predicting the behavior of perceptron networks. First, we derive the calculation of what we call the lossless memory (LM) dimension. The LM dimension is a generalization of the Vapnik–Chervonenkis (VC) dimension that avoids structured data and therefore provides an upper bound for perfectly fitting any training data. Second, we derive what we call the MacKay (MK) dimension. This limit indicates necessary forgetting, that is, the lower limit for most generalization uses of the network. Our derivations are performed by embedding the ideal network into Shannon’s communication model which allows to interpret the two points as capacities measured in bits. We validate our upper bounds with repeatable experiments using different network configurations, diverse implementations, varying activation functions, and several learning algorithms. The bottom line is that the two capacity points scale strictly linear with the number of weights.

1 Introduction

In this paper, we show that feed-forward neural networks, just like individual perceptrons or hopfield networks, can be best understood as associative memory and one way to understand challenges like overfitting and forgetting is to measure the capacity of that memory. Furthermore, a closer look at the error as a function of capacity then reveals that perceptrons go through two phase transitions, as indicated earlier by Wolfgang Kinzel [23], similar to the ones observed in the self-assembly of matter or the Ising model of ferromagnetism. As a result, it is impossible to make an artificial neural network that is sensitive to, but not disrupted by, new information once a certain threshold is reached. Likewise, too many input dimensions for too few data points will always result in overfitting and the network will only behave as associative memory.

Our theoretical derivation, backed up by repeatable empirical evidence, shows the scaling of the capacity of a neural network based on two critical points, which we call lossless-memory (LM) dimension and MacKay (MK) dimension, respectively. The LM dimension defines the point of guaranteed operation as memory and the MK dimension defines the point of guaranteed forgetting (generalization), even for very high dimensional networks. Both points scale strictly linear with the number of weights, exemplified as follows.

For a binary classifier, consisting of $h$ threshold perceptrons in the hidden layer with a bias, $k$ weighted inputs, and one binary output perceptron (a configuration commonly referred to as 3-layer multilayer perceptron (MLP)), we show that when we classify $n$ points of input in chaotic position (see Section 4), the following behavior will be observed.

• The LM dimension is bounded by
  \[ D_{LM} = h(k + 1) + (h + 1) \]  
  for $k > 1$. Having $n \leq D_{LM}$ samples implies perfect labeling in an ideal network.

• The MK dimension is bounded by
  \[ D_{MK} = 2 \times D_{LM} \]  
  for $k > 1$. Having $n \geq D_{MK}$ samples implies a $\geq 50\%$ chance that a labeling cannot be learned even with perfect training and an ideal network.

• For any connected perceptron network, the maximum capacity $C$ (MK dimension) of each perceptron is $C = 2k$ and scales linearly in $k$, the number of weights per perceptron (including bias).

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Using an embedding into Shannon’s communication model (Fig. 1), the capacity can be measured in bits and connected to other information-theoretic methods.

The article is structured as follows: Section 2 starts with the discussion of related work on bounds for artificial neural networks before we summarize the main parts of MacKay’s single perceptron proof [30] in Section 3. Section 4 then generalizes the result for arbitrary networks, including a discussion on how the capacities can be interpreted in bits and some convenient consequences of that. We have validated the two bounds experimentally using standard implementations of neural networks. These results are presented in Section 5 together with actual source code for reproducibility in Appendix A. Section 6 finally presents direct practical applications and discusses the limits of our results before Section 7 concludes the paper with future work.

2 Related Work

Understanding machine learning, as opposed to using it as a black box, requires insights into the training and testing data, the available hypothesis space of a chosen algorithm, the convergence and other properties of the optimization algorithm, and the effect of generalization and loss terms in the optimization problem formulation.

2.1 The Perceptron

One of the core questions that machine learning theory focuses on is the complexity of the hypothesis space and what functions can be modeled. For artificial neural networks, this question has recently become relevant again as deep learning seems to outperform shallow learning. For deep learning, single perceptrons with nonlinear, continuous gating function are concatenated in a layered fashion. Techniques like convolutional filters, drop out, early stopping, regularization, etc., are used to tune performance, leading to a variety of claims about the capabilities and limits of each of these algorithms (see for example [51]). Even though artificial neural networks have been popular for decades, understanding of the processes underlying them is usually based solely on empirical evidence in a particular application domain or task (see for example [32]).

In fact, the perceptron was introduced in 1958 [38] and since then has been extended in many variants, including but not limited to as described in [12, 13, 27, 28]. The perceptron uses a k-dimensional input and generates the output by applying a linear function to the input, followed by a gating function. The gating function is typically the identity function, the sign function, a sigmoid function, or the rectified linear unit (ReLU) [22, 33]. Motivated by brain research [15], perceptrons are stacked together to networks and usually trained by chain rule (backpropagation) [39, 40]. Even though perceptrons have been utilized for a long time, its capacities have been rarely explored beyond discussion of linear separability versus the XOR function and general mentions of overfitting, generalization and catastrophic forgetting. Moreover, overfitting and catastrophic forgetting have so far not been explained satisfactorily.

Catastrophic forgetting [31, 35] describes the effect that when the net is first trained on one set of labels and then on another set of labels, it very quickly loses its capability to classify the first set of labels. Recently, an approach was introduced to overcome catastrophic forgetting by avoiding too large weight changes for weights that are associated with the previously learned labels [24]. Our interpretation is that this approach is valid as small, alternating weight changes will give better results in case the total number of weight updates exceed the capacity of the network. Hence, we suggest future algorithms should avoid catastrophic forgetting by determining the capacity and avoid overflow.
2.2 VC Dimension

One of the largest contribution to machine learning theory comes from Vladimir N. Vapnik and Alexey Ya. Chervonenkis [47], including the Vapnik-Chervonenkis (VC) dimension. The VC dimension has been well known for decades [48]. It is defined as the largest natural number of samples in a dataset that can be shattered by a hypothesis space. This means that for a hypothesis space having VC dimension $D_{VC}$, there exists a dataset with $D_{VC}$ samples such that for any binary labeling ($2^{D_{VC}}$ possibilities) there exists a perfect classifier $f$ in the hypothesis space, that is, $f$ maps the samples perfectly to the labels. Due to perfect memorizing, it holds $D_{VC} = \infty$ for 1-nearest neighbor. Tight bounds have so far been computed for linear classifiers ($k + 1$) as well as decision trees [3].

The definition of VC dimension comes with two major drawbacks, however. First, it considers only the potential hypothesis space but not other aspects like the optimization algorithm, or loss and regularization function that effect the choice of the hypothesis [2]. Second, it is sufficient to provide only one example of a dataset to match the VC dimension. So given a more complex structure of the hypothesis space, the chosen data can take advantage of this structure. As a result, shatterability can be increased by increasing the structure of the data. While these aspects don’t matter much for simple algorithms, it is a major point for deep neural networks.

In [49], Vapnik et al. suggest to determine the VC dimension empirically, but state in their conclusion that the described approach does not apply to neural networks as they are “beyond theory”. So far, the VC dimension has only been approximated for neural networks. For example, Abu Mostafa argued loosely that the capacity must be bounded by $N^2$ with $N$ being the number of perceptrons [1]. Recently, [42] determined in their book that for a sigmoid activation function and a limited amount of bits for the weights, the loose upper bound of the VC dimension is $O(|E|)$ where $E$ is the set of edges and consequently $|E|$ the number of nonzero weights. Extensions of the boundaries have been derived for example for recurrent neural networks [26] and networks with piecewise polynomials [4] and piecewise linear [21] gating functions. Another article [25] describes a quadratic VC dimension for a very special case. The authors use a regular grid of $n$ times $n$ points in the two dimensional space and tailor their multilayer perceptron directly to this structure to use only $3n$ gates and $8n$ weights.

2.3 Other Measures

One measure that handles the properties of given data is the Rademacher complexity [5]. For understanding the properties of large neural networks, Zhang et al. [51] recently performed randomization tests. They show that their observed networks can memorize the data as well as the noise. This is proven by evaluating that their neural networks perfectly learn with random labels or with random data. This shows that the VC dimension of the analyzed networks is above the size of the used dataset. But it is not clear what the full capacity of the networks is. This observation also gives a good reason for why smaller size networks can outperform larger networks even though they have a lower capacity. Their capacity is still large enough to memorize the labeling of the data. A more elaborate extension of this evaluation has been provided by Arpit et al. [2]. Our paper indicates the lower limit for the size of the network.

A different approach using information theory comes from the group around Naftali Tishby [45]. They use the information bottleneck principle to analyze deep learning. For each layer, the previous layers are treated as an encoder that compresses the data $X$ to some better representation $T$ which is then decoded to the labels $Y$ by the consecutive layers. By calculating the respective mutual information $I(X, T)$ and $I(T, Y)$ for each layer they analyze networks and their behavior during training or when changing the amount of training data. A relevant statement to our work is then presented in [45]: “An immediate consequence of the DPI (Data Processing Inequality) is that information about $Y$ that is lost in one layer cannot be recovered in higher layers.” Thereby describing the learning capabilities of one layer, one can go to the minimum over all layers to describe the overall capabilities of a network. His results seem to be consistent with our experimental outcomes for second and higher deeper layers (see Section 5).

However, we describe the learning capabilities of neural networks using a different information theoretic view, namely the interpretation of neurons as memory cells. Apart from speculations in Section 6 that deeper hidden layers are able to serve as error correcting codes for previous layers, we assume an ideal network and do not investigate concrete architectures.

2.4 Memory Capacity

We are aware of recent questioning of the approach of discussing the memory capacity of neural networks [2, 51]. However, Occam’s razor [6] dictates to follow the path of least as-
3 Capacity of a Perceptron

In this section, we summarize the proof appearing in [30], Chapter 40.

3.1 Definitions

The following definitions will be required.

**Definition 3.1** (VC Dimension [47]). The VC dimension $D_{VC}$ of a hypothesis space $f$ is the maximum integer $D = D_{VC}$ such that some dataset of cardinality $D$ can be shattered by $f$. Shattered by $f$ means that any arbitrary labeling can be represented by a hypothesis in $f$ (or as in our context learned by a perceptron). If there is no maximum, it holds $D_{VC} = \infty$.

**Definition 3.2** (General Position [30]). “A set of points $\{x_n\}$ in K-dimensional space are in general position if any subset of size $\leq K$ is linearly independent, and no $K + 1$ of them lie in a $(K - 1)$-dimensional plane.”

3.2 Summary of Proof and Interpretation

MacKay interprets a perceptron as an encoder in a Shannon communication model [43] (compatible to our interpretation in Fig. 1). The input of the encoder are $n$ points in general position and a random labeling. The output of the encoder are the weights of a perceptron. The decoder receives the (perfectly learned) weights over a lossless channel. The question is then: Given the received set of weights and the knowledge of the data, can the decoder reconstruct the original labels of the points? In other words, the perceptron is interpreted as memory that stores a labeling of $n$ points relative to the data and the question is how much information can be stored by training a perceptron. In other words, we ask about the memory capacity of a perceptron.

This communication definition not only has the advantage that the mathematical framework of information theory can be applied to machine learning, it also allows to predict and measure neural network capacity in the actual unit of information, bits (more on this in Section 4.4).

The functionality of a perceptron is typically explained by the XOR example (i.e., showing that a perceptron with 2 input variables, which can have 4 states, can only model 14 of the 16 possible output functions). XOR and its negation cannot be linearly separated by a single threshold function of two variables and a bias. For an example of this explanation, see [36], section 3.2.2. MacKay effectively changes the computability question to a labeling question by asking: Given $n$ points, how many of the $2^n$ possible labelings in $\{0, 1\}^n$ can be learned by the model without an error (rather than computing binary functions of $k$ variables). Just as done by [11, 36], MacKay uses the relationship between the input dimensionality of the data $k$ and the number of inputs $n$ to the perceptron, which is denoted by a function $T(n, k)$ that indicates the number of “distinct threshold func-
tions” (separating hyperplanes) of \( n \) points in general position in \( k \) dimensions. The original function was derived by [41]. It can be calculated recursively as:

\[
T(n, k) = T(n - 1, k) + T(n - 1, k - 1), \quad (3)
\]

where \( T(n, 1) = T(1, k) = 2 \) or iteratively:

\[
T(n, k) = 2^{\sum_{l=0}^{k-1} \binom{n-1}{l}} \quad (4)
\]

Table 1 shows the \( T(n, k) \) function for small \( n \) and \( k \). It turns out that \( T(n, k) \) has very interesting properties. Namely,

\[
T(n, k) = 2^n \text{ for } k \geq n. \quad (5)
\]

This allows to derive the VC dimension for the case \( k = n \) where the number of possible binary labelings for \( n \) points is \( 2^n \). Since \( k = n \) and \( T(n, n) = 2^n \), all possible labelings of the input can be realized.

When \( k < n \), the \( T(n, k) \) function follows a calculation scheme based on the Pascal Triangle [10], which means that the bit loss due to incomplete shattering is still highly predictable. MacKay uses an error function based on the cumulative distribution of the standard Gaussian to perform that prediction and approximate the resulting distribution. More importantly, he defines a second point, which we call MK dimension. The MK dimension describes the largest number of samples such that typically only about 50% of all possible labelings can be separated by the binary classifier. He proofs this point to be a \( n = 2k \) for large \( k \) and illustrates that there is a sharp continuous drop in performance at this point. Since the sum of two independent normally distributed random variables is normal, with its mean being the sum of the two means, and its variance being the sum of the two variances, it is only natural that we will see in the following section that the MacKay point is linearly additive in the best case.

The quick analytic proof for what is shown in [30] is that \( T(2k, 2k) = 2^{2k} = 2^{\sum_{l=0}^{2k-1} \binom{2k-1}{l}} \) and that it is a symmetric sum where the last summand is equal to the first, etc. Hence, if we just take half of the summands, we get half of the values of the sum which means \( T(2k, k) = 2^{2k}/2 \). This means that a perceptron with \( k \) inputs can handle half of all possible labelings of \( 2k \) samples. Note that this direct proof actually does not require large \( n \) and that the transition is sharp in the sense that it is exactly 50% and with more samples the “performance” largely decreases and with less samples it largely increases because the summands in the middle are the largest.

MacKay concludes that the capacity of a perceptron is therefore \( 2k \) as the error before that point is small. We follow Kinzel’s physical interpretation [23] and understand that the perceptron error function undergoes two phase transitions: A first order transition at the VC dimension and a continuous one at the MK dimension. Based on this interpretation, we predict that the different phases will play a role on structuring and explaining machine learning algorithms. We will therefore, throughout this paper, discuss the two points separately.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\( n \times k \) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 2 & 6 & 8 & 8 & 8 & 8 \\
4 & 2 & 8 & 14 & 16 & 16 & 16 & 16 & 16 \\
5 & 2 & 10 & 22 & 30 & 32 & 32 & 32 & 32 \\
6 & 2 & 12 & 32 & 52 & 62 & 64 & 64 & 64 \\
7 & 2 & 14 & 44 & 84 & 114 & 126 & 128 & 128 \\
8 & 2 & 16 & 58 & 128 & 198 & 240 & 254 & 256 \\
\hline
\end{tabular}
\caption{Some values of the \( T(n, k) \) function indicating the number of distinct threshold functions on \( n \) points in general position in \( k \) dimensions as defined by [30].}
\end{table}

4 Capacity of a Perceptron Network

The following further definitions will be useful.

4.1 Definitions

For the remainder of this article, we will assume a feed-forward network. The weights are assumed to be real-valued and each unit has a bias, which counts as a weight. Note that no further assumptions about the architecture are required.

The definition of general position used in the previous section is typically used in linear algebra and is the most general case needed for a perceptron that uses a hyperplane for linear separation (see also Table 1 in [11]). For neural networks, a stricter setting is required since neural networks can implement arbitrary non-linear separations.

Definition 4.1 (Chaotic Position). A set of points \( \{x_n\} \) in \( K \)-dimensional space is in chaotic position, if and only if from any subset of size
< n it is not possible to infer anything about the positions of the remaining points.

Note that chaotic position implies general position, which was only excluding linear inference. Bear in mind that slightly distorted grid settings, as for example used in [25], are in general position but not in chaotic position. Chaotic position is equivalent of saying that no inference is possible about the structure of the data and the only thing a machine learner can do is memorize. The only distribution that satisfies this constraint is the uniform distribution [19].

As explained in Section 2, it is possible to achieve very high VC dimension by the choice of very special datasets. This has not been an issue yet for learning theory but from a practitioner perspective, this has been criticized [44, 51, 2]. To avoid the reported problems and consistent with our embedding into the Shannon communication model, we therefore propose a generalization of the VC dimension which we call lossless memory dimension.

**Definition 4.2 (Lossless Memory Dimension).** The lossless-memory dimension $D_{LM}$ is the maximum integer number $D_{LM}$ such that for any dataset with cardinality $n \leq D_{LM}$ and points in chaotic position, all possible labelings of this dataset can be represented with a function in the hypothesis space, that is, learned even with perfect training.

Note that for a single perceptron $D_{LM} = D_{VC}$ because chaotic position implies general position. As explained in Section 2, we will name the corresponding point where loss is guaranteed MacKay dimension.

**Definition 4.3 (MacKay Dimension).** The MacKay dimension $D_{MK}$ is the maximum integer $D_{MK}$ such that for any dataset with cardinality $n \leq D_{MK}$ and points in chaotic position at least 50% of all possible labelings of these datasets can be represented with a function in the hypothesis space, that is, learned with perfect training [30] (100% accuracy).

Consequently, a higher cardinality than $D_{MK}$ implies less than 50% of the labelings can be represented. We will show that for an ideal perceptron network the limit is exactly 50%.

### 4.2 Capacity Scaling Law

We will now extend MacKay’s capacity proof from a single perceptron to neural networks of many perceptrons. Generalizing to a neural network that is able to losslessly memorize the input labels means the network as a whole needs to be lossless, regardless of individual neurons being lossy. In fact, it is tempting to call $H_p = \log_2(T(n, k))$ something like the entropy of a perceptron as it indicates the expected number of bits of labeling a perceptron can assign to a set of points. However, empirically it can be easily observed that $H_p$ is not additive as it leads to an overestimation of the capabilities. Analytically, it is also not clear how individual hyperplanes are intersecting when perceptrons are connected in various ways. More particularly, it seems hard to assume a completely disjoint labeling of points when perceptrons that are able to implement the same functions are stacked. Instead, our proof abstracts from a concrete network architecture and assumes the ideal network, handling the worst case data. The two critical points described in Section 3 were obtained by describing the single perceptron as a memory cell. In the generalization, we interpret an artificial neural network as an ideal combination of these memory cells.

We can now show that the critical points $D_{LM}$ and $D_{MK}$ of any perceptron network scale linearly.

**Theorem 4.1 (Capacity Scaling Law).**

$$\sum_{j=1}^{I} C(P_j) = C \left( \sum_{j=1}^{I} P_j \right)$$

where $P_j$ is an arbitrary perceptron with $n_j$ inputs including a potential offset weight. The capacity is either $C = D_{MK}$ or $C = D_{LM}$ depending on the targeted phase. $\sum P_j$ denotes a neural network that combines the respective perceptrons perfectly and the data points are assumed to be in chaotic position.

**Proof.** Each neuron with weights $k$ (including bias) is able to implement exactly $T(n, k)$ different binary threshold functions over $n$ sample points. With the maximum number of binary labelings of $n$ points being $2^n$, it follows that the perceptron is at $D_{LE}$ when $T(n, k) = 2^n$. It is then able to store $n$ bits. Adding lossless memory cells with capacity $n$ and $n$ increases their capacity linearly in the amount of bits. Lossy memory cells are bit-additive as well. In fact, if perceptrons are combined in the best possible way (i.e., they label disjoint sets of sample points correctly), two perceptrons with $k_1$ and $k_2$ weights can maximally label $2^{k_1} \times 2^{k_2}$ points. The transitivity of addition implies that $D_{LM}$ is additive for any perceptron or network. In other words, it holds that $\sum_{j=1}^{I} D_{LM}(P_j) = D_{LM} \left( \sum_{j=1}^{I} P_j \right)$ in the case
that perceptrons are either lossless, or lossy but combined in the best possible way.

To proof the linear scalability of the MK dimension, we take a look at a ideal network at $D_{LM}$, where the number of weights $k$ equals the number of sample points $n$. It follows that each neuron is exactly responsible for its fraction of the labeling of the dataset thus $T(n, k) = 2^n$. If we increased $n$ without increasing $k$ and there was a hard cut-off (e.g., conventional transistor memory), we would not be able guarantee the assignment of any labels to new points as we would have to erase the original configuration bit-by-bit for each new label. However, as we discussed in Section 3, there is no hard cut-off as $T(2k, k) = \frac{1}{2}T(2k, 2k)$. Assuming each perceptron equally contributes to the labeling of more incoming points as needed, doubling $n$ for a fixed $k$ should result in each neuron being able to memorize the labeling of half of all points. Therefore $D_{MK} = 2D_{LM}$ generalizes to an ideal neural network.

In consequence, both $D_{LM}$ and $D_{MK}$ scale linearly, or $\sum_{j=1}^{n} C(P_j) = C\left(\sum_{j=1}^{n} P_j\right)$ for a perceptron $P_j$ with $u_j$ weights and $C = D_{MK}$ or $C = D_{LM}$ assuming an ideal network and points in chaotic position.

Practically, Equation 6 is an inequality “≥” when the data is not in chaotic position because the network should be able to exploit redundancies.

Please allow us to provide the following intuition regarding the MK dimension. Each perceptron reduces the amount of possible labelings by half but it is a joint reduction in the best case. It is not a separate reduction for each perceptron which would result in a reduction by $2^p$, with $p$ being the number of perceptrons. This can be understood by taking another look at the the $T(n,k)$ function. It is composed of the sum of two $T(n,k)$ functions of $n-1$: one that uses $k$ weights and one that uses $k-1$ weights. So when $n$ is increased, the new function space is composed of a space of functions that requires $k$ parameters (let’s call them “difficult”), and a space of functions that uses $k-1$ parameters (let’s call them “easy”). We could therefore argue that a network at MK dimension can only handle the easier labelings, which turn out to be exactly 50% of the functions modeled at $D_{LM}$ if we had twice the amount of edges. We do not know how a network of perceptrons behaves between $D_{LM}$ and $D_{MK}$ and after $D_{MK}$ as we cannot assume $T(n,k)$ to hold for non-linear separation functions. However, we can again assume an ideal network where the number of weights $k$ equals the number of sample points $n$ and each neuron is exactly responsible for its fraction of the labeling of the dataset. In other words, there are no “free labels to borrow” from anywhere. since $T(n,k)$ is monotonically falling for an increasing $\frac{k}{n}$, we can assume that there are no other extreme points between $D_{LM}$ and $D_{MK}$. Furthermore, we have to assume that capacity is asymptotically approaching 0 for high values of $\frac{k}{n}$. We can therefore assume that the phase transitions observed in a single perceptron generalize to a network of perceptrons.

### 4.3 Consequences

In conclusion, in an ideal network, all we need to do is add up the weights until the number of input points is reached to guarantee lossless memorization. Halving the number of weights, results in the MK dimension. Generalizing to a neural network that is able to losslessly memorize the input labels means the network as a whole needs to be lossless, regardless of individual neurons being lossy. Note that capacities in information theory are upper bounds, the measurable LM and MK dimension of a given concrete network structure is therefore most likely lower. It can be assumed that two lossy memory cells are more likely to complement each other when a global learning function is used. The learning function will therefore influence an empirically measured capacity, so does the gating function. In the extreme case of using the identity as gating function in an MLP, the capacities are limited to the first perceptron of the feed forward step, and we get again $k + 1$ and $2k + 2$ as dimensions because the overall binary separation function is still linear no matter how the perceptrons are combined.

The following formulas for the capacity of a network might come in handy in practice. If we count the bias weights of each perceptron as an edge of a graph, the number of incoming edges $|E|$ and the number of weights $P$ are identical and the following holds:

$$D_{MK} = 2|E|.$$  \hphantom{7}

(7)

As of the previous section, each perceptron has MK dimension $D_{MK} = 2|E|$. Hence, $h$ unconnected hidden units would have capacity

$$D_{MK} = h \cdot |E| \text{ and } D_{MK} = 2 \cdot D_{LM}.$$  \hphantom{7}

(8)

Connecting the hidden units with an output unit results in two observations. First, the output unit is a perceptron with $h$ inputs and a threshold and has the task of encoding $2^h$ possible labelings. The maximum capacity that this unit can add is therefore $D_{MK} = 2|E| = 2h$ bits.
Second, since we treat a perceptron as a memory unit, we do not care that the output unit looks at a restricted space compared to the input space, the capacity of the unit itself doesn’t change.

Hence, for a 3-layer Multilayer Perceptron (MLP) with \( h \) hidden units, \( k \) inputs, a single output node, and respective offsets we get:

\[
D_{MK} = 2\left(h(k+1) + h + 1\right). \tag{9}
\]

The LM dimension \( D_{LM} \) is consequently

\[
D_{LM} = |E| \tag{10}
\]

and respectively

\[
D_{LM} = (h(k+1) + h + 1) \tag{11}
\]

for the MLP.

A graphical illustration of the scaling law can be seen in Figure 2. The scaling law is also strongly supported by our empirical results (see Section 5).

One can generalize to more than binary classification as follows: If we assume a binary classification network at \( D_{LM} \), it means that any binary labeling can be implemented with one output unit. Adding a second output unit of the same capacity as the first one must work for a specific second binary labeling as well. Just imagine the first output unit away. A multi-classifier MK dimension gives every new output node a 50\% chance that it can implement a particular function.

4.4 Unit of Capacity

Measuring the capacities \( C \) of a neural network in bits follows directly from MacKay’s definition of a perceptron being an encoder/decoder in Shannon’s communication model. The intuition is that a binary labeling of \( d \) random points is a bit vector with dimension \( d \). If the capacity allows, the weights can label all the points as targeted by a training method. In other words, if the network has the capacity to store the labeling of the \( d \) points, there is no error and all the labeling bits are restored at the output. As shown in the previous sections, this is guaranteed at the LM dimension, which is the case for \( d = k + 1 = C(P) = D_{LM} \). The guaranteed capacity is therefore \( d \) bits. For the MK dimension, the capacity is therefore \( d/2 \) bits.

As a consequence, the LM and MK dimensions can be interpreted using information theoretic measures, such as the KL Divergence which allows predicting the minimally expected error for a given neural network, given some input data. It is also worth noting that the KL divergence is closely related to the cross entropy, which in turn has recently been proposed as an optimization criterion for deep learning algorithms [20].

5 Capacity Measurements

This section describes how we evaluated LM and MK dimension with empirical means. We observe that our theoretical capacities are indeed upper limits.

5.1 Experimental Setup

The basic principle for our empirical evaluation is to obtain samples from randomly generated data and increase the number of input points to the network step-by-step to test if the network can learn all possible labelings for the LM di-
mension or half of the possible labelings for the MK dimension.

Obviously, we expect our empirical measurements to be lower than the theoretical capacities. Practically, neither the ideal network nor the perfect training algorithm exists. Furthermore, for higher dimensions, we were only able to sample from the hypothesis space and could not test all labelings exhaustively. Therefore our goal was to create the best conditions possible and give the network the highest chance of reaching optimal capacity without violating the constraints of the theoretical framework. Thereby some practical workarounds are required for speedup and some limitations arise due to the exponential increase of the search space.

We mainly used the MLP implementation in scikit-learn [34] with L-BFGS [29] as optimizer. Our code is provided in the Appendix A. To control the randomness and ensure consistent results, we seed the randomizers with the respective index of the repetition. In case the optimizer does not fit the training data, we repeat its training up to 20 times. Our data was randomly generated by sampling from a normal distribution. We repeated evaluations with up to 20 different datasets if a labeling could not be fitted in the case of the LM dimension or if 50% of the labelings could not be fitted in the case of the MK dimension. The processing time of the latter is much higher for two reasons. First, a larger amount of samples has to be analyzed since at least 50% of all labelings have to be evaluated every time. Second, with more data the convergence of the MLP takes more iterations.

For completeness, every labeling would have to be tested. Due to symmetry in the class handling by the MLP, a minor speedup was achieved by testing only labelings where the last sample was labeled with a "0" and not a "1". This was not possible for large LM dimensions. Testing more than 2\(^{15}\) labelings was computationally too expensive for us. Hence, for more than 15 samples, we tested only a random selection of 2\(^{15}\) labelings. Due to this approximation, results might be above the true values for the given structure. The processing effort of the MK dimension is even worse and required to have a limit of 2\(^{10}\) samples. Given more resources, one could imagine a better approach where multiple random samplings are tested and the median result for the MK dimension and the worst result for the LM dimension is taken. We leave this as future work.

The number of tested labels also limits the possible dimensions of the MLP. We analyzed input dimensions: \([1, 2, 3, 4, 1]\) did not provide reliable results. For the number of hidden nodes, we looked at \([1, 2, 3, 4, 5, 6, 7, 8]\). Our implementation does not consider the difficulties of an MLP with class imbalance or redundancies. Here, higher empirical dimensions due to oversampling might be achievable.

### 5.2 Tuning the Implementation

Apart from the aforementioned implementation, we tested all other optimizers like “Adam” and “SGD” as well as the Keras library [8]. In most cases, the net was not able to fit the data in contrast to using L-BFGS. Hence the measured dimensions were very low. This could be interpreted as generalization capability of Adam and SGD because the optimizer is avoiding overfitting. Note, that L-BFGS approximates the second order derivative which makes it more accurate but also computationally more expensive and prone to get stuck in local minima.

We also tested different gating functions. Using the identity function, the network mostly behaved like a single perceptron as expected. For tanh and logistic function, results looked similar to the ReLU function but needed more repetitions and processing time.

As expected, the generation of the data had a significant impact on the results. Originally, we tested with uniformly sampled data. Changing it to sampling from a normal distribution improved our results dramatically (i.e., the empirically measured upper bound came closer to the theoretical). The number of different tested datasets using the same distribution had only a minor effect on when the empirical calculation reached its limit in LM or MK dimension. The testing of more than one dataset was solely to capture the randomness in the training algorithm and had no significant impact on the empirical results.

### 5.3 One Hidden Layer

The results with our described setting for the LM dimension are depicted in Figure 3.

Using just one hidden neuron behaved always like a single perceptron with LM dimension \(k + 1\) and MK dimension \(2(k + 1)\). The predicted linear relationship in \(h\) as well as in \(k\) for both dimensions can also be observed.

The comparison between theoretical and empirical dimensions shows a similar linear behavior. For the larger LM dimensions, the differences get smaller but this is probably due to sampling error. For the VC dimensions, it is more important to test all labelings because a single misclassification has an impact, whereas for the MK dimension this effect is less severe.
This could be improved in the future with more processing power.

We observed that the empirical MK dimension is extremely close to twice the empirical VC dimension. This is expected from the theoretical derivations but considering the aforementioned practical shortcuts, the clarity of this result increases our confidence in the validation experiments.

The empirical values for the VC dimension come quite close to $hk + 1$ for small numbers which is $2h$ off from the optimal value. By increasing the number of iterations and tested datasets, we also detected three special cases that are worth pointing out here.

For an MLP with 2 hidden nodes and input dimensions of 3, 4, or 5, we found a dataset example of 9, 11, or 13 samples respectively that could be shattered. In those cases, we tested all labelings. Those sample values are exactly one sample higher than $h(k + 1)$ and therefore above the storage capabilities of the hidden layer. Hence, the output neuron is making a significant contribution to the resulting learning capabilities, as predicted by the memory capacity formulation in this paper.

5.4 Two Hidden Layers

We also performed experiments with going deeper than one layer and, as expected, there was no more than linear increase in the capacity of the network. In fact, in case of using small $k$ and $h$, the obtained results were better by just one sample compared to the respective LM dimension with a one hidden layer architecture. In most cases, we observed that the LM dimension was actually far below the empirical values of a respective network with one hidden layer.

6 Interpretation and Utility

A perceptron can be interpreted as either a unit that models a function of $n$ input variables or, as we have done in this paper, as a memory unit that tries to label $n$ sample points with binary digits. The choice of interpretation makes a difference. For example, interpreting the neuron as a binary functional unit, one perceptron has to be able to represent $2^{2^n}$ different functions of $n$ input variables with $k + 1$ weights. A task that it is not able to achieve as the number of functions grows too quickly (see also [36], Section 6.3). Interpreting the perceptron as mem-
Insignificant difference allows for one thing that in [30] (Figure 4.9). However, this seemingly insignificant difference for one thing that a memory stick (which perfectly stores any 2^n labelings with n bits) is not able to do: generalization. Assume points form two clusters, the promise is that very few hyperplanes can be used to parametrize the difference between the two clusters. So interpreting perceptrons as memory units does not exclude generalization but at the same time allows us to understand their limits better. We therefore like to concur with the early assumptions in [1] that understanding neural networks as memory is indeed beneficial.

However, it is important to understand that the capacities, we derived for neural networks, are upper limits and that optimization methods, target functions, feature extraction, preprocessing, initialization methods, specialized architectures, and other such work will still be of practical value. Furthermore, as explained above, our proof is not constructive. Just like in Claude Shannon’s original work [43], we are not discussing the structure of the encoder. We are only quantifying maximum capabilities. Additionally, the main use of neural networks is generalization, not memorization. Therefore it might be acceptable to lose bits as long as they are the right ones.

While our work is network architecture agnostic and therefore implying that deep and flat networks have the same capacity, this should not be interpreted as them having necessarily the same performance. The first hidden layer has complete exposure to the input data and further layers only have exposure to the information-reduced output of the previous layers. Then again, deeper layers have the same capacity as earlier layers (assuming the same amount of weights) while looking at a smaller input space. So deep layers have more free capacity. This is countered by results [37] that neural networks with a so-called shortcut (i.e., where all the hidden layers are directly connected to the input) need overall less weights to learn the same labeling (still bounded by our capacity law). However, it is also easy to show that a deeper hidden layer can serve as an error correcting code to the previous layers. For example, the output of the previous hidden layer can be interpreted as repetition code. Practically speaking, the concrete architecture will be most dependent on the application, in that the variation of the linear separability of the input data and the learning function for that will dominate the quality of the prediction of the network.

Our result has at least three practical applications.

First, an overflow in capacity of all, some, or even a single perceptron serves as one definite explanation for (potentially catastrophic) forgetting.

Second, our experimental methodology, including the code in Appendix A, serves as a benchmark for the evaluation of neural network implementations. Using points in general position, one can test any learning algorithm and network architecture against the theoretical limit both for performance and efficiency (convergence rate).

Third, statistically speaking we should be able to compare networks with each other by simple bit counting of edges and comparing it to the total number of samples. For example, for a binary function of D variables, our analysis says that we will need a network structure with more than 2^D edges. For a 3-layer MLP this means, that we need more than \( \frac{2^{D-1}}{D+2} \) hidden neurons. Having less than \( 2^D - 1 \) edges, we know from the MK Dimension, that the chance of a labeling not being learned is even higher than 50%.

7 Conclusion

Using an information theoretic proof and empirical evidence, we show that the capacities of perceptrons in a neural network scale linearly with the number of weights. In fact, for guaranteed perfect memorization accuracy, a neural network can learn one bit per weight. The goal can be stretched to 2 bits per weight without too much error. After that, the error is guaranteed to be very large. While our work is an extension of the initial work by David MacKay, this article is the first to generalize the critical points to multiple perceptrons and derive a concrete scaling law. We show using repeatable experiments (see Appendix A) with a widely used open source framework that linear scaling holds and our theoretical bounds are actual upper bounds. This holds for all threshold-like activation function tested, including sigmoid and ReLU.

Future work should be dedicated to the phase between LM and MK dimension, which could be physically interpreted as the “liquid phase”, because in that phase it is clear that the network will make errors (i.e., lose bits compared to the input) but the error is log-proportional to the network complexity (i.e., the number of possi-
The use of neural networks is typically not that of memory but that of abstraction (assumed as learning) and qualifying this information loss for concrete data would therefore allow to reason about the generalization capabilities. For example, if only insignificant bits (noise) were lost as part of the error of the network, it would back up the hypothesis that one should learn in a way to avoid overfitting the data.

It is highly conceivable that weight values are not uniformly distributed, especially for deeply layered nets. This would mean that the capacity of individual perceptrons within a network can overflow as well. As a consequence, it will forget some previously learned information based on learning more. That individual overflow can cause interesting effects in the dynamics of the overall system and gives one possible cause for why the theoretical upper limit is not always reached. Investigating methods to detect such a local overflow and redirecting further input to other neurons could be interesting, especially when neural networks are trained in a parallelized fashion.

More challenges include understanding the deeper hidden layers as error correcting codes, generalizing these results to networks with more complex activation functions (such as in Radial Basis Function networks), and finding the bounds for convolutional networks. Last but not least, it remains to provide more constructive proofs of the capabilities depending on the gating function and the network architecture. Unfortunately, in general, finding an explicit formula that shows if a certain function is implemented by a network or not is very hard. Even just the case of binary functions would reduce to the problem of boolean satisfiability.

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A Code

```python
# Variable definitions
N = 120  # Maximum number of samples
K = [1, 2, 3, 4]  # Analyzed dimensions
H = [1, 2, 3, 4, 5, 6, 7, 8]  # Analyzed numbers of hidden layers
# Maximum number of samples, max_l = 10  # for random labelings
# Imports
import itertools
import numpy
import random
from sklearn.neural_network import MLPClassifier

print('n', 'k', 'h', 'correct', 'rate')

for k in K:  # input dimension
    numpy.random.seed(0)
    for h in H:  # number of hidden layers
        numpy.random.seed(0)
        for n in range(N):  # dataset size
            n += 1  # We start with one sample.
            data_res = []  # Good results
            if max_l < n:
                continue  # shortcut
            for i in range(20):  # 20 different datasets
                r_data = None  # random datasets
                for r_data in range(20):  # repeated runs till converged
                    data = numpy.random.normal(size=[N, k])
                    numpy.random.seed(0)
                    label_int = 0
                    for label_int in range(2**l_len):
                        if max_l < n:
                            label_int = \n                    random.randint(0, 2**n-1)
                    labels = int(i) for i in bin(label_int)[2+2**(N+2)]
                    d = data[:n]
                    converged = False
                    for r_mlp in range(20):
                        clf = MLPClassifier(hidden_layer_sizes=(h,)
                                             random_state=r_mlp,
                                             activation='relu',
                                             solver='lbfgs', alpha=0)
                        clf.fit(d, labels)
                        p = clf.predict(d)
                        if (p == labels).all():
                            true_results += 1
                            converged = True
                            break  # short converged
                    if not converged:
                        break  # shortcut after miss
                    data_res.append(true_results)
                    if true_results == 2**l_len:
                        break
                    true_results = max(data_res)
                    print(n, k, h, true_results, max_l=1.0/2**l_len)
                    if true_results*1.0/2**l_len < 0.95:
                        break

    true_results = 0
    for r_data in range(20):
        data = numpy.random.normal(size=[N, k])
        numpy.random.seed(0)
        true_results = 0
        for label_int in range(2**l_len):
            index = label_int
            if max_l < n:
                label_int = \n                    random.randint(0, 2**(n-1))
        labels = int(i) for i in bin(label_int)[2+2**(N+2)]
        d = data[:n]
        converged = False
        for r_mlp in range(20):
            clf = MLPClassifier(hidden_layer_sizes=(h,)
                                 random_state=r_mlp,
                                 activation='relu',
                                 solver='lbfgs', alpha=0)
            clf.fit(d, labels)
            p = clf.predict(d)
            if (p == labels).all():
                true_results += 1
                converged = True
            break  # short converged
        if not converged:
            break  # shortcut after miss
        data_res.append(true_results)
        if true_results == 2**l_len:
            break
        true_results = max(data_res)
        print(n, k, h, true_results, max_l=1.0/2**l_len)
        if true_results*1.0/2**l_len < 0.45:
            break
```

Figure 4: Python 2.7 code used for measuring the LM dimension with one hidden layer.

Figure 5: Python 2.7 code used for measuring the MK Dimension with one hidden layer.