PUZZLES AND (EQUIVARIANT) COHOMOLOGY OF GRASSMANNIANS

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ABSTRACT. The product of two Schubert cohomology classes on a Grassmannian $Gr_k(\mathbb{C}^n)$ has long been known to be a positive combination of other Schubert classes, and many manifestly positive formulae are now available for computing such a product (e.g. the Littlewood-Richardson rule, or the more symmetric puzzle rule from [Hon2]). Recently in [C] it was shown, nonconstructively, that a similar positivity statement holds for $T$-equivariant cohomology (where the coefficients are polynomials). We give the first manifestly positive formula for these coefficients, in terms of puzzles using an “equivariant puzzle piece”.

The proof of the formula is mostly combinatorial, but requires no prior combinatorics, and only a modicum of equivariant cohomology (which we include). As a by-product the argument gives a new proof of the puzzle (or Littlewood-Richardson) rule in the ordinary-cohomology case, but this proof requires the equivariant generalization in an essential way, as it inducts backwards from the “most equivariant” case.

This formula is closely related to the one in [MS] for multiplying factorial Schur functions in three sets of variables, although their rule does not give a positive formula in the sense of [C]. We include a cohomological interpretation of this problem, and a puzzle formulation for it.

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1. Summary of results

In [Hon2] we introduced a new rule (the puzzle rule) for computing Schubert calculus (intersection theory on Grassmannians $Gr_k(C^n)$), and proved it by reduction to the honeycombs of $[Hon1]$. This reduction implicitly involved the somewhat tricky relation between this cohomology ring and the representation ring of the general linear group $GL_n(C)$, and so our derivation of the puzzle rule was somewhat indirect. In this paper we give an independent and nearly self-contained proof of the puzzle rule. The proof is mainly combinatorial; the only non-combinatorial aspects are a small amount of equivariant cohomology theory and the Pieri rule $S_{\lambda'} \cdot S_{\lambda} = \sum_{\lambda \geq \lambda'} S_{\lambda'}$. (For completeness, we include a combinatorial proof of the Pieri rule in an appendix.) In particular, we avoid any use of the Littlewood-Richardson rule.

In the course of our argument we also give a formula for equivariant Schubert calculus on Grassmannians, that is manifestly positive in the sense of $[G]$ (which shall be defined in a moment); to our knowledge this is the first such formula.

1.1. The puzzle rule for $H^*(Gr_k(C^n))$. We first recall the definition of Schubert calculus, and the puzzle rule from [Hon2] which computes this calculus.

Let $0 \leq k \leq n$ be fixed integers. Abusing notation, we let $[n]_k$ denote the set of strings $\lambda = \lambda_1 \ldots \lambda_n$ consisting of $k$ ones and $n-k$ zeroes in arbitrary order, e.g. $[3]_2 = \{110, 101, 011\}$. This set $[n]_k$ has an obvious left action of the permutation group $S_n$. In particular, the long word $w_0$ acts on $[n]_k$ by reversal, e.g. $w_0 \cdot 01101 = 10110$. If $\lambda \in [n]_k$, we define an inversion of $\lambda$ to be a pair $1 \leq i < j \leq n$ with $1 = \lambda_i > \lambda_j = 0$. We denote the set of inversions by $\text{inv}(\lambda)$ and the number of inversions by $l(\lambda) = |\text{inv}(\lambda)|$. Observe that in $[n]_k$ there is a unique string $\text{id} := 0^{n-k}1^k$ with no inversions, a unique string $\text{div} := 0^{n-k-1}101^{k-1}$ (assuming $0 < k < n$) with one inversion, and a unique string $w_0 \cdot \text{id} := 1^k0^{n-k}$ with the maximal number $k(n-k) = \dim_C(Gr_k(C^n))$ of inversions.

If $\lambda \in [n]_k$ is a string, we let $C^\lambda := \bigoplus_{i=1}^n C^{\lambda_i}$ denote the corresponding coordinate $k$-plane in $C^n$, and let $X_\lambda$ be the Schubert cycle in $Gr_k(C^n)$ defined as

$$X_\lambda := \{ V_k \in Gr_k(C^n) : \dim(V_k \cap F_i) \geq \dim(C^{\lambda_i} \cap F_i), \ \forall i \in [1, n] \}$$

Indeed, one can use the results in this paper, together with the correspondences in [Hon2] and [BZ12], to prove that the Littlewood-Richardson rule computes Schubert calculus, though this is not the most direct derivation of this fact.

Readers familiar with Schubert classes on flag manifolds may wish to think of $[n]_k$ as $S_n$ quotiented by the right-action of $S_{n-k} \times S_k$. There are many suggestive hints that the arguments in this paper should extend from Grassmannians to flag manifolds, but we have so far been unable to extend them.
where $F_i := \mathbb{C}^{n-i+1}$ is the anti-standard $i$-plane. Equivalently, $X_\lambda$ is the closure of the set 
$$\{V_k \in \text{Gr}_k(\mathbb{C}^n) : \lambda_i = \dim((V_k \cap F_i)/(V_k \cap F_{i-1})) \text{ for all } i = 1, \ldots, n\}.$$ 

The **Schubert class** $S_\lambda \in H^*(\text{Gr}_k(\mathbb{C}^n))$ is the Poincaré dual of the cycle $X_\lambda$. In particular, the degree of $S_\lambda$ is $2l(\lambda)$. These classes are well-known to give a basis (over $\mathbb{Z}$) for the cohomology ring $H^*(\text{Gr}_k(\mathbb{C}^n))$, and as such we can expand uniquely the product $S_\lambda S_\mu$ of any two classes as a sum over the basis $\{S_\nu\}$, weighted by the structure constants $c^\nu_{\lambda \mu}$ of the multiplication. These integers $c^\nu_{\lambda \mu}$ are the concern of (ordinary) “Schubert calculus”.

Schubert calculus can be computed by many combinatorial rules, most famously the Littlewood-Richardson rule; we however shall use the more symmetric puzzle rule from [Hon2], which we now recall.

Define an (ordinary) **puzzle piece** as one of the following three plane figures with labeled edges:

1. a unit triangle with all edges labeled 0
2. a unit triangle with all edges labeled 1
3. a unit rhombus (two unit triangles glued together along an edge), the two edges clockwise of acute vertices labeled 0, the other two labeled 1.

![Figure 1](image)

**Figure 1.** The three puzzle pieces, in all their lattice orientations. From left-to-right we have an upward 0-triangle, a downward 0-triangle, an upward 1-triangle, a downward 1-triangle, a N-S rhombus, a NW-SE rhombus, and a SW-NE rhombus.

Note that the set of puzzle pieces is closed under rotation but not reflection (the reflection of a rhombus puzzle piece is not again a puzzle piece). See figure 1.

Define an (ordinary) **puzzle** as a decomposition of an equilateral triangle into triangles and rhombi, all edges labeled 0 or 1, such that each region is a puzzle piece. (Alternately, one can speak of attaching puzzle pieces together, with edges required to match up as in a jigsaw puzzle.) We will always align our puzzles to have a South side, Northwest side and Northeast side; this forces the edges of puzzle pieces to be oriented E-W, NW-SE, or NE-SW, the triangles to be oriented upward or downward, and the rhombi to be oriented N-S, NW-SE, or SW-NE. Some examples of puzzles are pictured in figure 2.

Define a **labeled equilateral triangle** to be an upward-pointing equilateral triangle of some integer side-length $n$, with the $3n$ unit edges on the boundary labeled either 0 or 1. Clearly every puzzle $P$ induces a labeled equilateral triangle $\partial P$, which we refer to as the **boundary** of $P$.

Given any three strings $\lambda, \mu, \nu \in \binom{n}{k}$, we let $\Delta_{\lambda \mu \nu}$ denote the labeled equilateral triangle with NW side labeled $\lambda$, NE side labeled $\mu$, and S side labeled $\nu$ (all read clockwise). We

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3 Our definition of northwest will be at a 60° angle to north, rather than 45°. Similarly for southwest, etc.
FIGURE 2. Some examples of puzzles. If $P$ denotes the right-most puzzle, then $\partial P = \Delta^{1010}_{1001,0101,0101} = \Delta^{1010}_{1001,0101}$. 

also let $\Delta^\nu_{\lambda \mu} = \Delta^{w_0 \cdot \mu \cdot \nu}_{\lambda}$ denote the labeled equilateral triangle with NW side $\lambda$, NE side $\mu$, and S side $\nu$, all read left-to-right. If $P$ is a puzzle, we let $P_{NW}, P_{NE}, P_S$ be the three strings of labels of $\partial P$ read clockwise, thus

$$\partial P = \Delta_{P_{NW} P_{NE} P_S} = \Delta_{P_{NW} P_{NE}}^{w_0 \cdot P_S}.$$ 

We will also call a puzzle with boundary $\Delta^\nu_{\lambda \mu}$ a $\Delta^\nu_{\lambda \mu}$-puzzle.

Our first main result shall be a new, and essentially self-contained, proof of the following theorem.

**Theorem 1 (Puzzles compute Schubert calculus).** [Hon2] Let $0 \leq k \leq n$, and let $\lambda, \mu, \nu$ be three elements of $\binom{[n]}{k}$ indexing Schubert classes $S_\lambda, S_\mu, S_\nu$ in $H^*(\text{Gr}_k(\mathbb{C}^n))$. Then the following (equivalent) statements hold:

1. The intersection number $\int_{\text{Gr}_k(\mathbb{C}^n)} S_\lambda S_\mu S_\nu$ is equal to the number of puzzles $P$ with $\partial P = \Delta^\nu_{\lambda \mu \nu}$.
2. The structure constant $c^\nu_{\lambda \mu}$ is equal to the number of puzzles with $\partial P = \Delta^\nu_{\lambda \mu}$.
3. 

$$S_\lambda S_\mu = \sum_{\text{puzzles } P: P_{NW} = \lambda, P_{NE} = \mu} S_{w_0 \cdot P_S}.$$ 

This first formulation, in terms of Schubert intersection numbers, realizes several symmetries evident in that problem. Note that the $120^\circ$ rotation of a puzzle is again a puzzle, corresponding to the fact that

$$\int_{\text{Gr}_k(\mathbb{C}^n)} S_\lambda S_\mu S_\nu = \int_{\text{Gr}_k(\mathbb{C}^n)} S_\mu S_\lambda S_\nu = \int_{\text{Gr}_k(\mathbb{C}^n)} S_\nu S_\lambda S_\mu.$$ 

We include here the standard proof that these integrals are a priori positive. They are visibly computing the number of signed intersection points of three Schubert cycles, perturbed to be transverse. It turns out to be possible to achieve this perturbation by replacing the standard flag $(F_i)$ by two other generic flags, which means the three transverse cycles are again complex subvarieties. Then the intersection points all have positive sign. Unfortunately this simple proof, which generalizes to arbitrary flag manifolds for arbitrary groups $G$, does not provide a formula (under most people’s notions of “formula”).

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4One would also expect a similar geometric interpretation of the commutativity property $S_\lambda S_\mu = S_\mu S_\lambda$; the puzzle rule can indeed be shown to be commutative but this turns out to be much more non-trivial.
From degree considerations we see that the structure constants $c_{\lambda \mu}^\nu$ vanish unless $l(\lambda) + l(\mu) = l(\nu)$. We invite the reader to see how this simple fact can also be deduced from the puzzle rule.

If $\lambda \in \binom{n}{k}$, define the **dual string** $\lambda^* \in \binom{n}{n-k}$ to be the string $w_0 \cdot \lambda$ with all 0s and 1s exchanged; thus for instance $011001 = 11001$. The dual string $\lambda^*$ gives a Schubert class $S_{\lambda^*}$ on the dual Grassmannian $Gr_{n-k}(\mathbb{C}^n)$. Similarly, given a puzzle $P$, we can define the **dual puzzle** $P^*$ by reflecting $P$ left-right and exchanging 1s and 0s everywhere. Observe that this gives a puzzle-theoretic proof of the equation

\[
\int_{Gr_k(\mathbb{C}^n)} S_{\lambda} S_{\mu} S_{\nu} = \int_{Gr_{n-k}(\mathbb{C}^n)} S_{\mu^*} S_{\lambda^*} S_{\nu^*},
\]

Grassmann duality $Gr_k(\mathbb{C}^n) \cong Gr_{n-k}(\mathbb{C}^n)$ gives a geometric proof of this identity, as follows. It takes a $k$-plane to its perpendicular $(n-k)$-plane (with respect to the standard Hermitian form on $\mathbb{C}^n$), and the Schubert variety $X_\lambda$ to the *opposite* Schubert variety $w_0 \cdot X_{\lambda^*}$ (thinking of $w_0$ as the antidiagonal permutation matrix). Since the transformation $w_0$ is deformable to the identity transformation, $w_0 \cdot X_{\lambda^*}$ again represents the Schubert class $S_{\lambda^*}$.

The third formulation in theorem 1 is very suitable for computations; an example is in figure 3.

![Figure 3. The four puzzles P with P_{NW} = P_{NE} = 010101, computing S_{010101} = S_{110001} + 2 S_{101010} + S_{011100} in H^*(Gr_3(\mathbb{C}^6)).](image)

**1.2. A new puzzle piece, for $H^*_T(Gr_k(\mathbb{C}^n))$.** To prove theorem 1 we shall generalize the result so that it computes the answer to a harder question, namely the product structure in the $T$-equivariant cohomology of Grassmannians. In section 2 we recall the (very few) necessary facts about $T$-equivariant cohomology we need to set up this question. For now, we need only four:

- the equivariant cohomology ring $H^*_T(Gr_k(\mathbb{C}^n))$ is naturally a graded module over the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$ (itself the equivariant cohomology of a point);
- $H^*_T(Gr_k(\mathbb{C}^n))$ has a natural basis of “equivariant Schubert classes” $\{S_\lambda\}$, with $S_\lambda$ having degree $2l(\lambda)$;
- there is a natural forgetful map $H^*_T(Gr_k(\mathbb{C}^n)) \to H^*(Gr_k(\mathbb{C}^n))$ to ordinary cohomology, which consists of setting all the $y_i$ to 0;
- this forgetful map takes each equivariant Schubert class $S_\lambda$ to the corresponding ordinary Schubert class $S_\lambda$.

In particular, one can speak of “equivariant Schubert calculus”, which concerns the structure constants $c_{\lambda \mu}^\nu \in \mathbb{Z}[y_1, \ldots, y_n]$ in the product expansion $S_\lambda S_\mu = \sum_\nu c_{\lambda \mu}^\nu S_\nu$. By

\[5\text{In this paper, all summations over Greek indices shall range over } \binom{n}{k}.\]
degree considerations in this graded ring, we know \( \deg c_{\nu}^{\lambda \mu} \) is a homogeneous polynomial of degree \( l(\lambda) + l(\mu) - l(\nu) \). In particular, \( c_{\nu}^{\lambda \mu} \) vanishes when \( l(\lambda) + l(\mu) < l(\nu) \), and agrees with the ordinary structure constants \( c_{\nu}^{\lambda \mu} \) when \( l(\lambda) + l(\mu) = l(\nu) \) (which is why we can safely use the same notation for both).

It is not hard to show that the equivariant structure constants \( c_{\nu}^{\lambda \mu} \) actually live in the subring \( \mathbb{Z}[y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}] \). In it is proven that written as polynomials in these differences, the structure constants have positive integer coefficients (this was first conjectured by Dale Peterson). As in the non-equivariant case, the proof does not directly give a formula for the \( c_{\nu}^{\lambda \mu} \).

To compute these \( c_{\nu}^{\lambda \mu} \), we need to generalize our notion of puzzle a bit. We introduce the **equivariant puzzle piece**: this is the same as the N-S rhombus puzzle piece but with the 1s and 0s interchanged. A puzzle using some equivariant pieces is given in figure 4.

![Figure 4](image_url)

**Figure 4.** A puzzle with two equivariant pieces, which are shaded. The left equivariant piece has weight \( y_4 - y_1 \), the right \( y_5 - y_4 \), so this puzzle contributes \((y_4 - y_1)(y_5 - y_4)\) to the calculation of \( c_{101011,101010}^{110101,100100} \).

To each equivariant piece \( p \) in a puzzle, we associate a **weight** \( \text{wt}(p) \), which we compute by dropping lines SW and SE from the piece until they poke out the \( i \)th and \( j \)th place on the South side and then setting \( \text{wt}(p) := y_j - y_i \). See figure 5. The weights of the pieces in figure 4 are given as example. We can then associate a **weight** \( \text{wt}(P) \) to every puzzle \( P \) by defining \( \text{wt}(P) = \prod_p \text{wt}(p) \), where \( p \) ranges over the equivariant pieces of \( P \). (An empty product is taken to be 1, of course.)

![Figure 5](image_url)

**Figure 5.** Locations of some equivariant puzzle pieces, and their corresponding weights.

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\(^6\)Since equivariant Schubert calculus generalizes ordinary, we can safely call these again “puzzles” and need not introduce a term “equivariant puzzles”. Rather, one might call a puzzle **ordinary** if one wanted to emphasize that it happens to contain no equivariant pieces.
Observe that we necessarily have \( i < j \) in the above definition of \( \text{wt}(p) \). In particular, the weight \( \text{wt}(P) \) of a puzzle can be expressed as a positive combination of \( y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1} \).

The main result of this paper is

**Theorem 2** (Puzzles compute equivariant Schubert calculus). Let \( 0 \leq k \leq n \), and let \( \lambda, \mu, \nu \) be three elements of \( \binom{n}{k} \), indexing equivariant Schubert classes \( \tilde{S}_\lambda \tilde{S}_\mu \tilde{S}_\nu \) in \( H^*_T(Gr_k(\mathbb{C}^n)) \). Then the following (equivalent) statements hold:

1. The structure constant \( c^\nu_{\lambda\mu} \) is equal to the sum of the weights of all puzzles \( P \) with \( \partial P = \Delta^\nu_{\lambda\mu} \). In particular, we explicitly demonstrate for \( Gr_k(\mathbb{C}^n) \) the abstract positivity result in \( [G] \).

2. \( \tilde{S}_\lambda \tilde{S}_\mu = \sum_{ \text{puzzles } P: P_{NW} = \lambda, P_N = \mu } \text{wt}(P) S_{w_0 \cdot P} \)

This obviously implies the second and third formulations of theorem 1. There is no close analogue of the first formulation. In ordinary cohomology the three formulations could be equated via the formula

\[
\int_{Gr_k(\mathbb{C}^n)} S_\lambda S_\mu S_\nu = \delta_{\lambda,w_0 \cdot \mu} \sum_{\rho} c^\rho_{\lambda\mu} c^{w_0 \cdot \id}_{\rho\nu}.
\]

So a positive formula for the structure constants gives a positive formula for the integrals. The converse does not seem to be obviously true.

This paper is organized as follows. In the “geometric” part (sections 2–3, plus an optional Appendix) of the paper we set up the machinery from equivariant cohomology which we will need, culminating in the equivariant Pieri identities for Schubert classes and structure constants. In the “combinatorial” part (sections 4–5) we show that the equivariant puzzle rule obeys these Pieri identities, which will imply theorem 2 (and hence theorem 1) by an induction argument. We remark that this induction only seems

\[7\]
to be available in the equivariant setting, so we cannot give a completely non-equivariant proof of theorem \[\square\] by these techniques.

Finally, in section \[\square\] we compare the results here with those in \[\text{[MS]}\] for multiplying factorial Schur functions (which are nothing other than double (i.e. equivariant) Schubert polynomials for Grassmannian permutations). They solve a different problem, which also reduces to ordinary Schubert calculus when \(l(\nu) = l(\lambda) + l(\mu)\). We introduce cohomological formulations of their problem, and a reformulation of their rule in terms of “MS-puzzles”.

We have had many useful conversations with Chris Woodward, our coauthor on \([\text{Hon}2]\). Our approach to ordinary Schubert calculus by inducting from the equivariant counterpart was inspired by \([\text{MS}]\). We are very grateful to Anda Degeratu for suggesting the name “puzzle”.

2. Equivariant cohomology, especially of Grassmannians

Fix \(0 \leq k \leq n\). In this section we give a combinatorial definition of the equivariant cohomology ring \(H^*_T(\text{Gr}_k(\mathbb{C}^n))\), which we interpret as lists of polynomials indexed by \(\binom{n}{k}\) satisfying some congruence conditions. We then invoke some standard facts about equivariant cohomology to determine that this ring is indeed the ring of equivariant cohomology classes on the Grassmannian, and is equipped with a basis of equivariant Schubert classes which map to ordinary Schubert classes under the forgetful map. Our reference for combinatorial properties of equivariant cohomology is \([\text{GZ}]\).

2.1. A combinatorial description of \(H^*_T(\text{Gr}_k(\mathbb{C}^n))\). Begin by defining \(H^*_T(\text{pt})\) to be the polynomial ring \(\mathbb{Z}[y_1, \ldots, y_n]\) in \(n\) variables (without yet worrying what “\(H^*_T\)” means in general). Define \(\bigoplus_{\binom{n}{k}} H^*_T(\text{pt})\) to be the space of all lists of polynomials \(\alpha = (\alpha|_\lambda)\), indexed by elements \(\lambda \in \binom{n}{k}\). This is clearly a commutative ring, and a \(H^*_T(\text{pt})\)-module (where \(H^*_T(\text{pt})\) acts diagonally on each term \(\alpha|_\lambda\) of the list).

Suppose that \(\alpha \in \bigoplus_{\binom{n}{k}} H^*_T(\text{pt})\). Call \(\alpha\) a class if it satisfies the GKM conditions:

For each pair \(\lambda, \lambda' \in \binom{n}{k}\) differing only in places \(i\) and \(j\), the difference \(\alpha|_\lambda - \alpha|_{\lambda'}\) should be a multiple of \(y_i - y_j\).

Examples. The list \(\alpha|_\lambda := 1\) is a class, since all the relevant differences are 0. The list \(\alpha|_\lambda := \sum_{i=1}^n \lambda_i y_i\) is also a class where the multiples are all 1. For each \(\mu\), the list \(\alpha|_\lambda := \delta_{\lambda,\mu} \prod_{i<j} (y_i - y_j)\) is also a class. In figure \[\square\] are a list of some very special classes in \(H^*(\text{Gr}_n(\mathbb{C}^4))\).

Now define \(H^*_T(\text{Gr}_k(\mathbb{C}^n)) \subseteq \bigoplus_{\binom{n}{k}} H^*_T(\text{pt})\) to be the set of all classes. It is obviously a subring and a submodule of \(\bigoplus_{\binom{n}{k}} H^*_T(\text{pt})\).

We chose this rather odd approach to \(H^*_T(\text{Gr}_k(\mathbb{C}^n))\) to emphasize the point that one does not actually need much equivariant cohomology theory to prove Theorem \[\square\] and one could in fact just think of \(H^*_T(\text{Gr}_k(\mathbb{C}^n))\) as an abstract ring of lists of polynomials to be manipulated combinatorially without ever having to understand what the functor \(H^*_T\) means.

Equivalently, we have \(\lambda' = (i \leftrightarrow j)\lambda\), where \((i \leftrightarrow j) \in S_n\) is the transposition of \(i\) and \(j\).

Topologically, this class \(\alpha\) arises as the equivariant first Chern class of the \(k\)th exterior power of the tautological \(k\)-plane bundle on the Grassmannian.
Define the support $\text{supp}(\alpha)$ of a class $\alpha$ as the set $\lambda \in \binom{n}{k}$ such that $\alpha|\lambda \neq 0$. Recall that $\binom{n}{k}$ is a lattice, where the partial order is given by $\lambda' \geq \lambda$ if one has $\sum_{i=1}^{j} \lambda'_i \geq \sum_{i=1}^{j} \lambda_i$ for all $j = 1, \ldots, n$. Say that $\alpha$ is supported above $\lambda$ if one has $\lambda' \geq \lambda$ for all $\lambda' \in \text{supp}(\alpha)$.

Let $\lambda \in \binom{n}{k}$, and let $\alpha$ be a class supported above $\lambda$. This forces $\alpha|\lambda$ to be a multiple of $\prod_{(i,j) \in \text{inv}(\lambda)} (y_j - y_i)$. If we have the stronger relationship $\alpha|\lambda = \prod_{(i,j) \in \text{inv}(\lambda)} (y_j - y_i)$, and also $\alpha|\mu$ is homogeneous of degree $l(\lambda)$ for all $\mu \in \binom{n}{k}$, we call $\alpha$ a Schubert class corresponding to $\lambda$. For some examples, see the Schubert classes in $H^*_T(\text{Gr}_2(\mathbb{C}^4))$ in figure 7.

**Lemma 1** (Schubert classes are unique). For each $\lambda \in \binom{n}{k}$ there is at most one Schubert class $\alpha$ corresponding to $\lambda$.

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11In this paper we give each generator $y_i$ a degree of 1, although from the cohomological considerations below it could be argued that the $y_i$ really deserve to have degree 2. We believe however that setting $\text{deg}(y_i) = 2$ here would be too confusing. We remark that if we replaced the equivariant cohomology ring with the equivariant Chow ring (which is equivalent for $\text{Gr}_k(\mathbb{C}^n)$) then the $y_i$ genuinely do have degree 1.
Proof. Suppose for contradiction that there were two distinct Schubert classes \( \alpha, \alpha' \) corresponding to \( \lambda \). Let \( \mu \) be a minimal element of the support of the class \( \alpha - \alpha' \). Since \( \alpha, \alpha' \) agree on \( \lambda \) and are supported above \( \lambda \), we have \( \mu > \lambda \). By the GKM conditions this forces \( (\alpha - \alpha')|_{\mu} \) to be a multiple of \( \prod_{(i,j) \in \text{inv}(\mu)} (y_j - y_i) \). But this contradicts the fact that \( \alpha - \alpha' \) is homogeneous of degree \( l(\lambda) < l(\mu) \).

To prove existence of Schubert classes is a little trickier. We now give a topological proof that there exists a Schubert class \( \tilde{S}_\lambda \) for each \( \lambda \in \binom{n}{k} \). We also give a purely combinatorial proof in the Appendix.

2.2. \( T \)-equivariant cohomology. Let \( T := (S^1)^n \) be a torus, and \( T - \text{Top} \) the category of topological spaces with a continuous \( T \)-action, the morphisms being equivariant maps. Then \( T \)-equivariant cohomology is a contravariant functor from \( T - \text{Top} \) to supercommutative\(^{12}\) graded rings. We do not define it here, as all we need are a few of its properties:

1. \( H^*_T(\text{pt}) \) is (as promised) a polynomial ring \( \mathbb{Z}[y_1, \ldots, y_n] \), whose generators are given formal degree 2, and correspond to a basis of the dual of the Lie algebra of \( T \). In particular we can think of weights of \( T \) as giving elements of \( H^*_T(\text{pt}) \) (i.e. the weights are linear combinations of the \( \{y_i\} \)).

2. If \( X \) is a \( T \)-invariant oriented cycle in a compact oriented manifold \( Y \), then \( X \) determines\(^{13}\) an equivariant cohomology class on \( Y \), which we will denote \([X]\). The degree of \([X]\) is the codimension of \( X \) in \( Y \). If \( f \in Y^T \) is not in \( X \), then the pullback of \([X]\) to \([f]\) is zero.

3. There is a natural “forgetful” map from \( H^*_T(Y) \to H^*(Y) \), which factors as

\[
H^*_T(Y) \to H^*_T(Y) \otimes_{H^*_T(\text{pt})} \mathbb{Z} \to H^*(Y).
\]

It takes the equivariant class \([X] \in H^*_T(Y)\) associated to a \( T \)-invariant cycle to the ordinary Poincaré dual of \( X \) in \( H^*(Y) \).

4. Given a \( T \)-space \( Y \), there are two natural equivariant maps associated, \( Y^T \hookrightarrow Y \to \text{pt} \), where \( Y^T \) is the set of fixed points. These induce ring homomorphisms backwards in equivariant cohomology, \( H^*_T(\text{pt}) \to H^*_T(Y) \to H^*_T(Y^T) \cong H^*(Y^T) \otimes_{H^*_T(\text{pt})} \). In other words, the functor \( H^*_T \) takes values in the category of algebras over \( H^*_T(\text{pt}) \).

Now assume that \( Y \) is a smooth projective variety, and \( T \) acts on \( Y \) algebraically with isolated fixed points. (In particular, \( Y \) could be the Grassmannian \( \text{Gr}_k(\mathbb{C}^n) \)). Then much more is true (see [GKM]):

1. \( H^*_T(Y) \) is a free module over \( H^*_T(\text{pt}) \).

2. \( Y \) has a cell decomposition by complex cells \( X_f \) (see [BB]), corresponding to the fixed points \( f \in Y^T \), whose closures give a basis of equivariant cohomology as a module over \( H^*_T(\text{pt}) \). The restriction \([X_f]|_f\) of a class \([X_f]\) to the point \( f \) is the product of the weights in the normal bundle at the point \( f \).

3. The forgetful map \( H^*_T(Y) \otimes_{H^*_T(\text{pt})} \mathbb{Q} \to H^*(Y) \) is an isomorphism on rational cohomology.

\(^{12}\)Elements of odd degree anticommute with one another, as in ordinary cohomology. In the case of interest, the Grassmannian, there are none of these anyway and the ring is therefore commutative.

\(^{13}\)In ordinary cohomology \([X]\) can be defined via Poincaré duality, but that is not available in equivariant cohomology. Nonetheless this should be the intuition.
4. The natural map \( H^*_T(Y) \to H^*_T(Y^T) \cong \bigoplus_{Y^T} H^*_T(pt) \) is injective.\footnote{This is very odd from the point of view of ordinary cohomology – we are restricting cohomology classes to individual points, which would seem very forgetful, but the above assertion says that we actually lose no information. The intuition should be that the equivariant cohomology of a point is very big, being a polynomial ring.}

These last two statements are the most combinatorially interesting: they say that we can do all our calculations with lists of polynomials indexed by the fixed points \( Y^T \), and then once we understand the ring \( H^*_T(Y) \), we can (if we wish) set all the generators of \( H^*_T(pt) \) to zero and recover ordinary cohomology.

It remains to understand the image of \( H^*_T(Y) \) inside \( \bigoplus_{Y^T} H^*_T(pt) \). Given a class \( \alpha \in H^*_T(Y) \) and a fixed point \( f \in Y^T \), let \( \alpha_f \in H^*_T(pt) \) denote the restriction of \( \alpha \) to \( f \), so that this map is \( \alpha \mapsto (\alpha_f)_{f \in Y^T} \). As it turns out, this image can be characterized by the T-invariant copies of \( \mathbb{C}P^1 \) inside \( Y \).

**Theorem.** [GKM] Let \( T \) act on a smooth projective variety \( Y \), with \( Y^T \) finite, and let \( Z \) be a T-invariant \( \mathbb{C}P^1 \) inside \( Y \). Then \( Z^T \) consists of two points \( Z_{\text{north}} \) and \( Z_{\text{south}} \). If \( w_Z \) is the weight of the \( T \)-action on the one-dimensional tangent space \( T_{Z_{\text{south}}}Z \), then we have the “GKM condition”

\[
\alpha|_{Z_{\text{south}}} - \alpha|_{Z_{\text{north}}} \text{ is a multiple of } w_Z
\]

for all classes \( \alpha \in H^*_T(Y) \).

Conversely, suppose that there are only finitely many \( T \)-invariant \( \mathbb{C}P^1 \)'s, and \( (\alpha|_{\lambda}) \) is an element of \( \bigoplus_{Y^T} H^*_T(pt) \) which obeys the GKM condition for every \( T \)-invariant \( \mathbb{C}P^1 \). Then \( (\alpha|_{\lambda}) \) lies in the image of \( H^*_T(Y) \) inside \( \bigoplus_{Y^T} H^*_T(pt) \).

The first statement can be proven by applying the functor \( H^*_T \) to the inclusions

\[
(Z_{\text{north}}, Z_{\text{south}}) \hookrightarrow Z \hookrightarrow Y.
\]

The converse is deeper; see [GKM].

### 2.3. Grassmannians.

We now apply this technology to the case of the Grassmannian \( \text{Gr}_k(\mathbb{C}^n) \), in order to verify our claimed combinatorial description of \( H^*_T(\text{Gr}_k(\mathbb{C}^n)) \).

The torus \( T \) acting in this case is the \( n \)-dimensional torus \( T = U(1)^n \), so our base ring \( H^*_T(pt) \) is \( \mathbb{Z}[y_1, \ldots, y_n] \). This torus acts by the diagonal action on \( \mathbb{C}^n \), and thus also acts on \( \text{Gr}_k(\mathbb{C}^n) \). The fixed points \( \text{Gr}_k(\mathbb{C}^n)^T \) are just the coordinate \( k \)-planes \( \{ C^\lambda : \lambda \in \binom{n}{k} \} \); we shall abuse notation and refer to the fixed point \( C^\lambda \) simply as \( \lambda \).

Two fixed points \( \lambda, \lambda' \) are connected by a \( T \)-invariant \( \mathbb{C}P^1 \) if and only if \( \lambda = (i \leftrightarrow j)\lambda' \) for some \( 1 \leq i < j \leq n \). If \( Z \) is such a \( \mathbb{C}P^1 \), then the action of \( T \) on \( T_{\lambda}Z \) has weight \( \pm(y_j - y_i) \), and similarly for \( T_{\lambda'}Z \).

From the GKM theorem we thus see that the equivariant cohomology ring \( H^*_T(\text{Gr}_k(\mathbb{C}^n)) \) is isomorphic (both as a ring and as a \( H^*_T(pt) \)-module) to the ring \( H^*_T(\text{Gr}_k(\mathbb{C}^n)) \) defined earlier combinatorially, and we shall no longer bother to distinguish the two rings.

For each \( \lambda \in \binom{n}{k} \), the Schubert cycle \( X_\lambda \) defined in the introduction is oriented and \( T \)-invariant, so it induces an equivariant cohomology class \( \tilde{S}_\lambda := [X_\lambda] \). These cycles are the closures of a cell decomposition of \( \text{Gr}_k(\mathbb{C}^n) \) into complex cells. At the fixed point \( \lambda \), the weights of the \( T \) action on the normal bundle of \( X_\lambda \) are given by \( \{ y_j - y_i : (i, j) \in \text{inv}(\lambda) \} \), so we have \( \tilde{S}_\lambda \bigwedge (y_j - y_i). \) The only fixed points in \( X_\lambda \) are those corresponding...
to strings above \( \lambda \), so \( \tilde{S}_\lambda \) is supported above \( \lambda \). From degree considerations we see that \( \deg(\tilde{S}_\lambda) = 1(\lambda) \) (treating the \( y_i \) as having degree 1), and from the first half of the GKM theorem we see that \( \tilde{S}_\lambda \) obeys the GKM conditions. Combining this with lemma \( \square \) we see that \( \tilde{S}_\lambda \) is indeed the Schubert class corresponding to \( \lambda \). Furthermore, we see that \( \tilde{S}_\lambda \) maps to the ordinary cohomology class \( S_\lambda \) under the forgetful map to ordinary cohomology.

This concludes our construction of the equivariant Schubert classes \( \tilde{S}_\lambda \). In section \( \S 6 \) we will also relate these classes to factorial Schur functions, which are polynomials in many more variables\( \square \).

2.4. Schubert classes form a basis. Having used topological considerations to construct the equivariant Schubert classes \( \tilde{S}_\lambda \), we shall use a simple combinatorial argument to show that they form a basis for \( H^*_T(Gr_k(\mathbb{C}^n)) \).

**Proposition 1.** The \( \tilde{S}_\lambda \) form a \( H^*_T(pt) \)-basis for \( H^*_T(Gr_k(\mathbb{C}^n)) \). More specifically, any class \( \alpha \in H^*_T(Gr_k(\mathbb{C}^n)) \) can be written uniquely as an \( H^*_T(pt) \)-linear combination of \( \tilde{S}_\lambda \) using only those \( \lambda \) such that \( \lambda \geq \mu \) for some \( \mu \in \text{supp}(\alpha) \).

**Proof.** This essentially follows from the upper-triangularity of the Schubert classes with respect to the order on \( \binom{n}{k} \). We now give the details.

First, we show the \( \tilde{S}_\lambda \) are linearly independent. Suppose for contradiction that \( \sum_\lambda Y_\lambda \tilde{S}_\lambda = 0 \) for some \( Y_\lambda \in H^*_T(pt) \) which are not identically zero. Among all \( \mu \in \binom{n}{k} \) with \( Y_\mu \neq 0 \), we pick a \( \mu \) which is minimal in the lattice \( \binom{n}{k} \). But then the restriction of \( \sum_\lambda Y_\lambda \tilde{S}_\lambda \) to \( \mu \) is \( Y_\mu \tilde{S}_\mu|_\mu \neq 0 \), contradiction.

To see that the \( \tilde{S}_\lambda \) span, let \( \alpha \) be a class one is attempting to write as a \( H^*_T(pt) \)-linear combination of some classes \( \{ \tilde{S}_\lambda \} \) satisfying those conditions. Let \( \mu \) be a minimal element of the support of \( \alpha \). From the GKM conditions we see that \( \alpha|_\mu \) must be a multiple \( \beta \) of \( \prod_{(i,j) \in \text{inv}(\mu)} (y_j - y_i) \tilde{S}_\mu|_\mu \). Subtracting \( \beta \tilde{S}_\mu \) we can inductively reduce the support of \( \alpha \) upwards until it is gone. This only uses those \( \tilde{S}_\lambda \) for which \( \lambda \geq \mu \) for some \( \mu \in \text{supp}(\alpha) \).

**Example.** Consider the class \( \tilde{S}_{0101}\tilde{S}_{1010} \), which is supported above 1010 (refer back to figure \( \S7 \) to see these classes). Following the algorithm given in proposition \( \S3 \) to write this in the Schubert basis, we first subtract off a multiple of \( \tilde{S}_{1010} \) itself, the multiple being \( \tilde{S}_{0101} \cdot 1010 = y_4 - y_1 \). The remainder is supported at 1100, and is in fact 1 times \( \tilde{S}_{1100} \). In all \( \tilde{S}_{0101}\tilde{S}_{1010} = (y_4 - y_1)\tilde{S}_{1010} + \tilde{S}_{1100} \). This is an example of the “equivariant Pieri rule” proved in proposition \( \S2 \) We verify the puzzle rule in this example, in figure \( \S8 \).

Since the \( \tilde{S}_\lambda \) form a \( H^*_T(pt) \)-basis of the ring \( H^*_T(Gr_k(\mathbb{C}^n)) \), we can define structure constants \( c_{\lambda \mu}^\nu \in H^*_T(pt) \) for all \( \lambda, \mu, \nu \in \binom{n}{k} \) by the formula

\[
\tilde{S}_\lambda \tilde{S}_\mu = \sum_\nu c_{\lambda \mu}^\nu \tilde{S}_\nu.
\]

\(^{15}\)It is remarkable that these lists of polynomials can be wrapped up into individual polynomials. This can also be traced to a geometrical fact, which is that Grassmannians can be constructed as symplectic quotients of affine space. Then the Kirwan map from equivariant cohomology of affine space (a polynomial ring) maps onto the equivariant cohomology of the Grassmannian. Since we will always work with classes, rather than factorial Schur functions, we do not go into the details of this argument.
Since we obtain the conditions \( \nu \geq \lambda, \mu \) and \( l(\nu) \leq l(\lambda) + l(\mu) \). In the special case \( l(\nu) = l(\lambda) + l(\mu) \), the structure constants are integers and agree with those from ordinary Schubert calculus.

At the other extreme, when \( \lambda = \nu \) we have \( c_{\lambda \mu}^\lambda = \tilde{S}_{\mu | \lambda} \).

Proof. The first claim follows since each \( \tilde{S}_\lambda \) has degree \( l(\lambda) \). In particular, \( c_{\lambda \mu}^\lambda \) vanishes when \( l(\nu) > l(\lambda) + l(\mu) \). The class \( \tilde{S}_\lambda \tilde{S}_\mu \) is supported above \( \lambda \) and above \( \mu \), so by proposition we obtain the conditions \( \nu \geq \lambda \). If we apply the forgetful map from equivariant cohomology to ordinary cohomology, then the structure constants of non-zero degree all vanish, leaving only those with \( l(\nu) = l(\lambda) + l(\mu) \), which explains the last claim in the first paragraph.

Finally, if we restrict \( \tilde{S}_\lambda \tilde{S}_\mu \) to \( \lambda \), we obtain

\[
\tilde{S}_\lambda|_{\lambda}\tilde{S}_\mu|_{\lambda} = c_{\lambda \mu}^\lambda \tilde{S}_\lambda|_{\lambda} + \sum_{\nu; \nu \neq \lambda} c_{\lambda \mu}^\nu \tilde{S}_\nu|_{\lambda}.
\]

Since \( c_{\lambda \mu}^\nu \) vanishes unless \( \nu \geq \lambda \), and \( \tilde{S}_\nu \) is supported above \( \nu \), we see that all the terms in the summation vanish. The claim then follows since \( \tilde{S}_\lambda|_{\lambda} = \prod_{(i,j) \in \text{inv}(\lambda)} (y_j - y_i) \) is non-zero.

From the above lemma we see that the equivariant structure constants \( c_{\lambda \mu}^\nu \) compute Schubert classes when \( l(\nu) = l(\lambda) \), and ordinary structure constants when \( l(\nu) = l(\lambda) + l(\mu) \). In the next section we prove a Pieri rule which bridges the gap between these two extreme cases.

3. Pieri-based recurrence relations

In this section we assume that \( 0 < k < n \), since the Schubert calculus for the \( k = 0 \) and \( k = n \) cases are trivial.

Let \( \text{div} := 000 \ldots 010111 \ldots 1 \) denote the unique element of \( \binom{n}{k} \) with one inversion. The corresponding Schubert class \( \tilde{S}_\text{div} \) is the only one of degree 1, coming from the unique Schubert divisor (hence the name). With \( \tilde{S}_\text{div} \), and the associativity of the equivariant cohomology ring, we will establish recurrence relations on the Schubert classes \( \{\tilde{S}_\lambda\} \) and the equivariant structure constants \( c_{\lambda \mu}^\nu \).
Lemma 3. The Schubert divisor class \( \tilde{S}_{\text{div}} \) is given by
\[
\tilde{S}_{\text{div}}|_{\lambda} := \sum_{j=1}^{n} \lambda_j y_j - \sum_{i=1}^{n} y_i.
\]

Proof. The right-hand side is clearly homogeneous of degree 1 = \( \text{deg}(\text{div}) \), supported above \( \text{div} \), and equals \( y_{k+1} - y_k = \prod_{(j,i) \in \text{div}(y_j - y_i)} \) when restricted to \( \lambda = \text{div} \). It can easily be shown to also obey the GKM conditions. The claim then follows from lemma [1].

Write \( \lambda' \to \lambda \) if \( \lambda' > \lambda \) and \( l(\lambda') = l(\lambda) + 1 \); this is the covering relation in the lattice \( \binom{n}{k} \). Equivalently, \( \lambda' \to \lambda \) if \( \lambda' = \alpha 10 \beta \) and \( \lambda = \alpha 01 \beta \) for some strings \( \alpha, \beta \). Thus for instance 110101 \( \to \) 101101, 110011.

Proposition 2 (The equivariant Pieri rule).
\[
\tilde{S}_{\text{div}} \tilde{S}_\lambda = (\tilde{S}_{\text{div}}|_{\lambda}) \tilde{S}_\lambda + \sum_{\lambda' : \lambda' \to \lambda} \tilde{S}_{\lambda'}.
\]

Proof. From lemma [2] and the fact that \( \text{deg}(\text{div}) = 1 \) we have
\[
\tilde{S}_{\text{div}} \tilde{S}_\lambda = (\tilde{S}_{\text{div}}|_{\lambda}) \tilde{S}_\lambda + \sum_{\lambda' : \lambda' \to \lambda} c_{\text{div},\lambda}^{\lambda'} \tilde{S}_{\lambda'},
\]
where the \( c_{\text{div},\lambda}^{\lambda'} \) are the structure constants for ordinary Schubert calculus. The claim then follows from the ordinary-cohomology Pieri rule.\(^{10}\) \( \tilde{S}_{\text{div}} S_\lambda = \sum_{\lambda' : \lambda' \to \lambda} S_{\lambda'} \) (as proved in [1]).

In the appendix we shall give an alternative proof of proposition [2] which does not go through the ordinary Pieri rule.

The equivariant Pieri rule gives a recurrence relation on the structure constants \( c_{\lambda\mu}^{\nu} \):

Theorem 3. [MS] For any \( \lambda, \mu, \nu \) we have the recurrence relation
\[
(\tilde{S}_{\text{div}}|_{\nu} - \tilde{S}_{\text{div}}|_{\lambda}) c_{\lambda\mu}^{\nu} = \left( \sum_{\lambda' : \lambda' \to \lambda} c_{\lambda\mu}^{\nu'} - \sum_{\nu' : \nu' \to \nu'} c_{\lambda\mu}^{\nu'} \right).
\]

The above recurrence was proven in [MS] by a different argument; it had also been observed by A. Okounkov.

Proof. We use associativity of multiplication in \( H_\ast^T(\text{Gr}_k(\mathbb{C}^n)) \) and the equivariant Pieri rule to expand \( \tilde{S}_{\text{div}} \tilde{S}_\lambda \tilde{S}_\mu \) in two different ways:
\[
(\tilde{S}_{\text{div}} \tilde{S}_\lambda) \tilde{S}_\mu = ((\tilde{S}_{\text{div}}|_{\lambda}) \tilde{S}_\lambda + \sum_{\lambda' : \lambda' \to \lambda} \tilde{S}_{\lambda'}) \tilde{S}_\mu = (\tilde{S}_{\text{div}}|_{\lambda}) \sum_{\rho} c_{\lambda\mu}^{\rho} \tilde{S}_{\rho} + \sum_{\lambda' : \lambda' \to \lambda} \sum_{\rho} c_{\lambda\mu}^{\rho} \tilde{S}_{\rho}
\]
and
\[
\tilde{S}_{\text{div}} (\tilde{S}_\lambda \tilde{S}_\mu) = \tilde{S}_{\text{div}} \sum_{\rho} c_{\lambda\mu}^{\rho} \tilde{S}_{\rho} = \sum_{\rho} c_{\lambda\mu}^{\rho} (\tilde{S}_{\text{div}}|_{\rho}) \tilde{S}_{\rho} + \sum_{\rho, \rho' \to \rho} \tilde{S}_{\rho'}.
\]
\(^{10}\)The “Pieri rule” sometimes refers to a more general rule than we need here, for multiplying by \( S_\lambda \) where \( \lambda = 0 \ldots 0 1 0 \ldots 0 1 \ldots 1 \). The equivariant version of this rule was recently formulated \([\mathbb{C}]\)-positively in \([\mathbb{R}]\), for flag manifolds (not just Grassmannians).
Comparing coefficients of $\tilde{S}_\nu$, we get
\[ \tilde{S}_{\text{div}|\lambda} c^\nu_{\lambda \mu} + \sum_{\lambda' : \lambda' \to \lambda} c^\nu_{\lambda' \mu} = c^\nu_{\lambda \mu} \tilde{S}_{\text{div}|\nu} + \sum_{\nu' : \nu' \to \nu} c^\nu_{\nu' \mu} \]
as desired. \qed

The above recurrence gives us a purely combinatorial way to verify that a putative formula for equivariant structure constants indeed works:

**Corollary 1.** Let $0 \leq k \leq n$. Suppose that we have an assignment $(\lambda, \mu, \nu) \mapsto d^\lambda_{\lambda \mu}$ from $\binom{n}{k}$ to $H^*_T(\text{pt})$ obeying the following identities:

- For any $\lambda \in \binom{n}{k}$, we have
  \[ d^\lambda_{\lambda \lambda} = \prod_{(i, j) \in \text{inv}(\lambda)} (y_j - y_i). \]

- For any $\lambda, \mu \in \binom{n}{k}$, we have
  \[ (\tilde{S}_{\text{div}|\lambda} - \tilde{S}_{\text{div}|\mu}) d^\lambda_{\lambda \mu} = \sum_{\mu' : \mu' \to \mu} d^\lambda_{\lambda \mu'}. \]

- For any $\lambda, \mu, \nu \in \binom{n}{k}$ we have
  \[ (\tilde{S}_{\text{div}|\nu} - \tilde{S}_{\text{div}|\lambda}) d^\nu_{\lambda \mu} = \sum_{\lambda' : \lambda' \to \lambda} d^\nu_{\lambda' \mu} - \sum_{\nu' : \nu' \to \nu'} d^\nu_{\nu' \mu'}. \]

Then $c^\nu_{\lambda \mu} = d^\nu_{\lambda \mu}$ for all $\lambda, \mu, \nu$.

The identity $\text{(1)}$ thus involves only $\Delta^\lambda_{\lambda \lambda}$-puzzles, while $\text{(2)}$ involves $\Delta^\lambda_{\lambda \mu}$-puzzles and $\text{(3)}$ involves general $\Delta^\lambda_{\lambda \mu}$-puzzles.

**Proof.** If $k = 0$ or $k = n$ then we must have $\lambda = \mu = \nu$, and the claim follows from $\text{(1)}$, Lemma $\text{2}$ and the definition of $\tilde{S}_\lambda$. So we assume $0 < k < n$.

To begin with, we use the first two properties of $d$ to show that $d^\lambda_{\lambda \mu} = \tilde{S}_{\mu|\lambda}$ (which we already knew to be equal to $c^\lambda_{\lambda \mu}$). We induct on the quantity $l(\lambda) - l(\mu)$, which is clearly bounded from below. If $\lambda \neq \mu$, then $\tilde{S}_{\text{div}|\lambda} - \tilde{S}_{\text{div}|\mu}$ is non-zero, and the claim follows from $\text{(2)}$, proposition $\text{2}$ and the induction hypothesis$^{17}$ (observing that $l(\lambda) - l(\mu') = l(\lambda) - l(\mu) - 1$). In the base case $\lambda = \mu$ we instead use $\text{(1)}$ and our definition of the class $\tilde{S}_\mu$.

Now we show that $c^\nu_{\lambda \mu} = d^\nu_{\lambda \mu}$ in general. We induct on the quantity $l(\nu) - l(\lambda)$, which is also bounded from below. The base case $\nu = \lambda$ follows from lemma $\text{2}$ and the previous paragraph. In all other cases $\tilde{S}_{\text{div}|\nu} - \tilde{S}_{\text{div}|\lambda}$ is non-zero, and we can use $\text{(3)}$, theorem $\text{3}$ and the observation that $l(\nu') - l(\lambda) = l(\nu) - l(\lambda') = l(\nu) - l(\lambda) - 1$. \qed

We can tighten this further using the duality operation $P \mapsto P^*$ on puzzles, which takes $\partial P = \Delta^\nu_{\lambda \lambda}$ to $\partial P^* = \Delta^\nu_{\lambda' \lambda'}$. Also, we have $\text{wt}(P^*) = \overline{\text{wt}(P)}$, where $x \mapsto \overline{x}$ is the involution

$^{17}$Observe that this induction argument implies that $d^\lambda_{\lambda \mu}$ vanishes unless $\lambda \geq \mu$, which is of course consistent with the support properties of $\tilde{S}_\mu$. Similarly, the argument in the next paragraph shows that $d^\nu_{\lambda \mu}$ vanishes unless $\nu \geq \lambda$.  

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on $H^*_T(\text{pt})$ defined by $\overline{y_i} := -y_{n+1-i}$ for $i = 1, \ldots, n$. From these observations and the definition of $d_{\lambda\mu}^\nu$ we see that

$$d_{\lambda\mu}^\nu = \overline{d_{\mu\lambda}^\nu}.$$

**Lemma 4.** Let $\{d_{\lambda\mu}^\nu \in H^*_T(\text{pt})\}$ be a family satisfying (1), (3) and $d_{\lambda\mu}^\nu = \overline{d_{\mu\lambda}^\nu}$. Also assume the vanishing condition

$$d_{\lambda\mu}^\nu = 0 \quad \text{unless } \nu \geq \lambda, \mu. \quad (4)$$

Then (2) follows automatically (and so by the corollary, $c_{\lambda\mu}^\nu = d_{\lambda\mu}^\nu$).

**Proof.** If we thus apply (3) with $\lambda, \mu, \nu$ replaced by $\mu^*, \lambda^*, \nu^*$ and apply the involution, we obtain

$$d_{\lambda\mu}^\nu = \sum_{\mu' : (\mu')^* \to \mu^*} d_{\lambda\mu'}^\nu - \sum_{\nu' : \nu^* \to (\nu')^*} d_{\lambda\mu'}^\nu.$$

From lemma 3 we have $\tilde{S}_{\text{div}|\nu^*} - \tilde{S}_{\text{div}|\lambda^*} = \tilde{S}_{\text{div}|\nu} - \tilde{S}_{\text{div}|\lambda}$ while from the definitions we see that $(\mu')^* \to \mu^*$ is equivalent to $\mu' \to \mu$. We obtain

$$d_{\lambda\mu}^\nu = \sum_{\mu' : \mu' \to \mu} d_{\lambda\mu'}^\nu - \sum_{\nu' : \nu \to \nu'} d_{\lambda\mu'}^\nu.$$

We now specialize this to the case $\nu = \lambda$, and use (4) to see that $d_{\lambda\mu}^\lambda = 0$ when $\lambda \to \lambda'$. The claim (2) follows.

We can now outline the proof of theorem 2. We will assume $0 < k < n$ as the $k = 0$, $k = n$ cases are trivial. To prove the first conclusion of theorem 2, it will suffice by the above corollary to show that the quantity

$$d_{\lambda\mu}^\nu := \sum_{P : \partial P = \Delta_{\lambda\mu}^\nu} \text{wt}(P)$$

obeys the identities (1), (3), and (4). These will be proven in the next two sections.

### 4. SW-NE Rhombi, and the Proofs of (1) and (4)

In this section we prove the identities (1) and (4).

We give first a “Green’s theorem” argument to constrain the interior of a puzzle from its boundary. Suppose that $p$ is a SW-NE rhombus. If we drop lines SE from $p$, they will poke out of the $j$th and $(j+1)$st place of the South side of the puzzle for some $1 \leq j < n$. We then define the discrepancy of $p$ to be $\text{disc}(p) := y_{j+1} - y_j$.

**Lemma 5.** Let $P$ be a $\Delta_{\lambda\mu}^\nu$-puzzle. Then

$$\sum_{p \text{ is a SW-NE rhombus of } P} \text{disc}(p) = \tilde{S}_{\text{div}|\nu} - \tilde{S}_{\text{div}|\lambda}.$$

Note that the edges of a SW-NE rhombus are parallel to the $\lambda$ and $\nu$ sides of the puzzle.

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18 This could also have been obtained from (3) and the commutativity property $d_{\lambda\mu}^\nu = d_{\mu\lambda}^\nu$. This commutativity property is of course true (since it manifestly holds for the $c_{\lambda\mu}^\nu$) but is non-trivial to prove using puzzles, so we rely instead on duality.
Proof. Let \( p \) be any puzzle piece of \( P \), and let \( e \) be an edge of \( p \). We give the pair \((p,e)\) a “flux” \( \text{flux}(p,e) \) as follows. If \( e \) is a 0-edge, or a NW-SE 1-edge, we set \( \text{flux}(p,e) := 0 \). Otherwise we drop a line SE from \( e \) until it pokes out of the \( j \)th place on the South side, and set \( \text{flux}(p,e) := \pm y_j \), where the sign \( \pm \) is positive if \( e \) is on the SW, SE, or S side of \( p \), and negative if \( e \) is on the N, NW, or NE side.

Now compute the total flux \( \sum_e \text{flux}(p,e) \) of a puzzle piece \( p \). By checking each case from Figure 1 (and the equivariant piece) in turn, we see that \( p \) has total flux zero unless \( p \) is a NW-SE rhombus, in which case the total flux is \( \text{disc}(p) \).

Finally, add up the flux of all the puzzle pieces in \( P \). At each internal edge, the contributions from the two pieces containing that edge cancel one another. So the total flux reduces to a sum over the edges on the boundary \( \partial P \) of \( P \), which can be computed as

\[
\sum_{i=1}^{n} \nu_i y_i - \sum_{i=1}^{n} \lambda_i y_i = \tilde{S}_{\text{div} | \nu} - \tilde{S}_{\text{div} | \lambda}.
\]

Combining this with the previous paragraph we obtain the lemma. \( \square \)

This has some very pleasant corollaries:

**Corollary 2.** Let \( P \) be a \( \Delta_{\nu}^{\mu} \)-puzzle.

Then \( \lambda, \mu, \nu \) all have the same number of 1’s (they are elements of the same \( \binom{n}{k} \)).

Also, the number of rhombi in \( P \) with edges parallel to the \( \lambda \) and \( \nu \) sides is \( l(\nu) - l(\lambda) \). Similarly when \( \lambda \) is replaced with \( \mu \) throughout.

**Proof.** For the first, specialize at \( y_i \equiv 1 \). (This argument can be presented much more simply than we have done here!) For the second, specialize at \( y_i \equiv i \). \( \square \)

Another consequence is

**Corollary 3.** Let \( P \) be a \( \Delta_{\nu}^{\mu} \)-puzzle. Then:

- We must have \( \nu \geq \lambda \) and \( \nu \geq \mu \) in the partial order on \( \binom{n}{k} \). (This is \( 4 \).)
- If \( \lambda = \nu \), there can be no SW-NE rhombi.
- If \( \lambda = \mu = \nu \), the can be no SW-NE or NW-SE rhombi.

**Proof.** Since the discrepancies \( \text{disc}(p) \) are all positive in the sense of \( C \), we see from the previous lemma that \( \tilde{S}_{\text{div} | \nu} - \tilde{S}_{\text{div} | \lambda} \) is non-negative. But this is equivalent to \( \nu \geq \lambda \). Furthermore, if \( \nu = \mu \), then there cannot be any SW-NE rhombi, since \( \tilde{S}_{\text{div} | \nu} - \tilde{S}_{\text{div} | \lambda} \) would then be strictly positive, a contradiction.

To obtain the corresponding statements concerning \( \mu \), we replace \( P \) by the dual puzzle \( P^* \) defined in the introduction. (Alternatively, one can “dualize” the proof of lemma 5 by the appropriate reflection and swapping of 0-edges and 1-edges.) \( \square \)

We now prove \( 1 \), in the form of

**Proposition 3.** There exists a unique \( \Delta_{\lambda}^{\mu} \)-puzzle \( P \), and its weight is \( \prod_{(i,j) \in \text{inv} (\lambda)} (y_j - y_i) \).

**Proof.** Define a “diamond” in a puzzle to be any of the following objects:

- A N-S rhombus piece;
• An equivariant puzzle piece;
• Two triangular puzzle pieces joined by an E-W edge.

Note that the NW label on a diamond matches that on the SE, likewise the NE and SW labels match.

Let $P$ be a $\Delta_{\lambda_1, \ldots, \lambda_n}$-puzzle. By the third conclusion of Corollary 3, $P$ contains no SW-NE or NW-SE rhombi. Thus we can cut $P$ along all NW-SE and NE-SW lines without slicing through any rhombi. Except for the triangles attached to the South side, the sliced-up $P$ falls into diamonds.

We analyze $P$ starting from the bottom. First, attach the isolated triangles. Then in each trough, fill in the unique diamond that fits. We give the example of $\lambda = 1001$.

Layer by layer, this creates the only puzzle with $\lambda$ edge (read left to right) that uses no NW-SE or NE-SW rhombi. By the matching properties of diamonds, the NW and NE edges also end up labeled $\lambda$. This shows the existence and uniqueness.

An equivariant piece comes whenever the trough to be filled has a 0 on the SE and 1 on the SW, coming from an inverted 0 and 1 in $\lambda$. This shows that the weight is as advertised.

It remains to prove $R_3$. This will be done in the next section, at the end of which we give the proof of theorem $\mathfrak{2}$.

5. GASHED PUZZLES: THE PROOF OF $R_3$

We give first the crucial definition, and then a rough indication of the argument.

**Definition.** We define a gashed puzzle $(P, g)$ as a decomposition of a labeled equilateral triangle $\partial P$ into a collection $P$ of puzzle pieces, along with a line segment $g$ in the triangular lattice (which we refer to as the gash), such that

• The gash $g$ is contained in the equilateral triangle (either on the boundary $\partial P$ or in the interior), and is oriented either E-W or SW-NE;
• every edge not on the gash has at most one label (as in a non-gashed puzzle)
• if the gash is oriented SW-NE, then it is length 2, and the labels on each side are a 0 then a 1 (read clockwise)
• if the gash is oriented E-W, then it is length at least 2, with all but the first and last edge passing through the short diagonals of some equivariant rhombi. The labels on each side are a 0, then the short diagonals of some equivariant rhombi, then 1 (read clockwise).

Some examples of gashed puzzles appear in figure 9. As with non-gashed puzzles, we can define the weight $\text{wt}(P, g)$ of a gashed puzzle to be the product of the weights of all the equivariant pieces $p \in P$. Thus for instance the second puzzle in figure 9 has weight $(y_3 - y_2)(y_4 - y_2)$. 

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We now give an extremely rough indication of the argument, which decomposes (3) into the four identities (5)-(8) to come. Recall that equation (3) computes $c_{\nu}^{\lambda \mu}$ from $\{c_{\lambda' \mu}'\}$ for $\lambda' \rightarrow \lambda$ and $\{c_{\lambda' \mu}'\}$ for $\nu' \rightarrow \nu$. We will take puzzles with boundary $\Delta_{\lambda' \mu}'$ and attach a gash on their NW side, changing the boundary labels to $\Delta_{\lambda \mu}$. (This will eventually give equation (5).)

Then we will use some local rules for propagating a gash through a gashed puzzle (the map $\phi$ in proposition 4, giving equation (8)), preserving the weight. The gash will usually come out on the S side, and when removed it leaves a puzzle with boundary $\Delta_{\nu}^{\lambda} \mu'$ (equation (6)). If the gash always makes it through, then $\sum_{\lambda' \rightarrow \lambda} d_{\lambda' \mu}^{\nu} - \sum_{\nu' \rightarrow \nu} d_{\lambda \mu}^{\nu'}$ (the right-hand side of (3)) will be zero. This occurs in the $c_{1100,1010}$ example given in figure 10 and therefore $c_{1100,1010} = 0$.

![Figure 9](image)

**Figure 9.** Three gashed puzzles. The third has boundary $\Delta_{1100,1010}$ even though without the gash it would have been a puzzle with boundary $\Delta_{1100,0110}$.

![Figure 10](image)

**Figure 10.** Gash propagation in gashed puzzles with boundary $\Delta_{1100,1010}$. In the third case the gash goes through an equivariant piece. Since there are no other gashed puzzles with this boundary, the left-hand side of (3) vanishes, and $c_{1100,1010,0110} = 0$. 

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Frequently though, an equivariant piece can cause a gash to heal (or appear) on its own, and (by (7)) these extra terms will give the left-hand side \((\tilde{S}_{\text{div}}|_{\nu} - \tilde{S}_{\text{div}}|_{\lambda})d_{\lambda\mu}^{\nu}\) of equation (3). The \(c_{001,010} = 1\) example appears in figure 11.

**Figure 11.** The gash propagates to an equivariant piece, and they cancel one another, leaving a “scab” (defined in subsection 5.1).

Fix \(\lambda, \mu, \nu \in \binom{n}{k}\). We define the set \(G\) to be the collection of all gashed puzzles \((P, g)\) with \(\partial P = \Delta_{\lambda\mu}^{\nu}\). We define four subsets of \(G\) (local pictures given in figure 12):

\[
G_{\text{ext}}^{\text{left}} := \{(P, g) \in G : g \text{ lies in the NW boundary } P_{\text{NW}} = \lambda \text{ of } P\}
\]

\[
G_{\text{ext}}^{\text{left}} := \{(P, g) \in G : g \text{ contains the SE edge of an equivariant piece}\}
\]

\[
G_{\text{ext}}^{\text{right}} := \{(P, g) \in G : g \text{ lies in the S boundary } P_{\text{S}} = w_{0} \cdot \nu \text{ of } P\}
\]

\[
G_{\text{int}}^{\text{right}} := \{(P, g) \in G : g \text{ contains the NW edge of an equivariant piece}\}
\]

**Figure 12.** Local pictures in gashed puzzles in \(G_{\text{ext}}^{\text{left}}, G_{\text{ext}}^{\text{right}}, G_{\text{int}}^{\text{left}}, G_{\text{int}}^{\text{right}}\).

We define \(G_{\text{ext}}^{\text{left}}\) to be the union of the (obviously disjoint) sets \(G_{\text{ext}}^{\text{left}}, G_{\text{int}}^{\text{left}}, G_{\text{ext}}^{\text{right}}\), and \(G_{\text{ext}}^{\text{right}}\) to be the union of the (obviously disjoint) sets \(G_{\text{int}}^{\text{right}}, G_{\text{ext}}^{\text{right}}\). (The sets \(G_{\text{ext}}^{\text{left}}\) and \(G_{\text{ext}}^{\text{right}}\) may intersect.)

Our proof of (3) will come down to four identities. Two are very simple:

\[
(5) \quad \sum_{(P, g) \in G_{\text{ext}}^{\text{left}}} \text{wt}(P, g) = \sum_{\lambda' : \lambda' \to \lambda} \sum_{P' : \partial P' = \Delta_{\lambda'\mu}^{\nu}} \text{wt}(P')
\]

\[
(6) \quad \sum_{(P, g) \in G_{\text{right}}^{\text{right}}} \text{wt}(P, g) = \sum_{\nu' : \nu' \to \nu} \sum_{P' : \partial P' = \Delta_{\lambda'\mu}^{\nu'}} \text{wt}(P')
\]

We begin with (5). Let \((P, g)\) be an arbitrary element of \(G_{\text{ext}}^{\text{left}}\). Then \(g\) lies on a pair of edges where \(P_{\text{NW}} = \lambda\) reads 10. If one removes the gash, one obtains a non-gashed puzzle \(P'\) with boundary \(\partial P' = \Delta_{\lambda'\mu}^{\nu}\) where \(\lambda'\) is equal to \(\lambda\) but with 10 replaced by 01. In particular, we have \(\lambda' \to \lambda\). This argument can be reversed; given any \(\Delta_{\lambda'\mu}^{\nu}\)-puzzle \(P'\)
with $\lambda' \to \lambda$, we can take the two edges where $\lambda$ and $\lambda'$ disagree, and swap them to create a gashed puzzle $(P, g)$. Since this affects no equivariant pieces, one has $\mathfrak{wt}(P, g) = \mathfrak{wt}(P')$, and (5) follows.

The proof of (6) is exactly the same, except for one minor subtlety: observe that if $g \subseteq P_S$ then the gash $g$ must have length 2, since $P_S$ does not contain the short diagonal of any equivariant rhombus pieces.

The other two identities

\begin{equation}
\sum_{(P, g) \in G_{\text{left}}^{\gamma}} \mathfrak{wt}(P, g) - \sum_{(P, g) \in G_{\text{int}}^{\gamma}} \mathfrak{wt}(P, g) = \sum_{P' : \partial P' = \Delta_{\lambda}^{\gamma}} \mathfrak{wt}(P') (\tilde{S}_{\text{div}}|_{\gamma} - \tilde{S}_{\text{div}}|_{\lambda})
\end{equation}

\begin{equation}
\sum_{(P, g) \in G_{\text{right}}^{\gamma}} \mathfrak{wt}(P, g) = \sum_{(P, g) \in G_{\text{left}}^{\gamma}} \mathfrak{wt}(P, g)
\end{equation}

are more subtle and will be proved in the next two subsections.

5.1. **Proof of (7).** To prove (7) we need to introduce the notion of a **scab**. Let $P'$ be a $\Delta_{\lambda}^{\gamma}$ puzzle. We define a **left-scab** of $P'$ to be any pair $\kappa$ of puzzle pieces in $P'$ consisting of an SW-NE rhombus sitting atop a downward 1-triangle. Similarly define a **right-scab** of $P'$ to be any pair $\kappa$ of puzzle pieces in $P'$ consisting of an upward 1-triangle sitting atop a SW-NE rhombus.

We define the weight $\mathfrak{wt}(\kappa)$ of the scab by $\mathfrak{wt}(\kappa) := \mathfrak{wt}(p)$, where $p$ is the unique equivariant piece which can fit inside the region occupied by $\kappa$.

**Lemma.** We have

\[ \sum_{(P, g) \in G_{\text{left}}^{\gamma}} \mathfrak{wt}(P, g) = \sum_{P' : \partial P' = \Delta_{\lambda}^{\gamma}} \mathfrak{wt}(P') \mathfrak{wt}(\kappa) \]

and

\[ \sum_{(P, g) \in G_{\text{right}}^{\gamma}} \mathfrak{wt}(P, g) = \sum_{P' : \partial P' = \Delta_{\lambda}^{\gamma}} \mathfrak{wt}(P') \mathfrak{wt}(\kappa). \]

**Proof.** Let $(P, g)$ be a gashed puzzle in $G_{\text{left}}^{\gamma}$. Then $g$ must be a NE-SW line segment of length 2, whose SW edge is the SE edge of an equivariant piece $p$. From Figure 12 we thus see that there must be a downward 1-triangle $t$ between $p$ and the NE edge of $g$, as in figure 12. Observe that if we replace $p$ and $t$ with a left-scab $\kappa$, we obtain an ungashed puzzle $P'$ with $\partial P' = \partial p = \Delta_{\lambda}^{\gamma}$ and $\mathfrak{wt}(P') \mathfrak{wt}(\kappa) = \mathfrak{wt}(P') \mathfrak{wt}(p) = \mathfrak{wt}(P)$.

This procedure can be reversed; given any ungashed puzzle $P'$ with $\partial p = \Delta_{\lambda}^{\gamma}$ and given any left-scab $\kappa$ of $P'$, we can replace the scab $\kappa$ with an equivariant piece $p$ and a downward 1-triangle $t$, creating a gashed puzzle $P$ with $\partial P = \partial P' = \Delta_{\lambda}^{\gamma}$ and $\mathfrak{wt}(P) = \mathfrak{wt}(P') \mathfrak{wt}(p) = \mathfrak{wt}(P') \mathfrak{wt}(\kappa)$. This proves the first claim.

The second claim is similar (indeed, it is essentially a $180^\circ$ rotation of the first claim) and is left to the reader.
The argument above motivates the terminology; when a gash closes up it leaves a scab, and conversely, a scab can come off producing a new gash.

From the above lemma, we see that to prove (7) it will suffice to show

**Lemma.** Let $P'$ be a $\Delta^\chi_{\lambda\mu}$-puzzle. Then

$$
\sum_{\kappa: \kappa \text{ is a right-scab of } P'} \wt(\kappa) - \sum_{\kappa: \kappa \text{ is a left-scab of } P'} \wt(\kappa) = \tilde{S}_{\text{div}}|_{\lambda} - \tilde{S}_{\text{div}}|_{\nu}.
$$

**Proof.** This is another Green’s theorem argument. Let $p$ be any puzzle piece of $P'$, and let $e$ be an edge of $p$. We give the pair $(p, e)$ a “flux” $\text{flux}(p, e)$ as follows. If $e$ is a 0-edge, or a NW-SE 1-edge, we set $\text{flux}(p, e) := 0$. If $e$ is an E-W 1-edge, we drop a line SW from $e$ until it pokes out of the $i$th place on the South side, and set $\text{flux}(p, e) := +y_i$ if $e$ is on the N side of $p$, and $\text{flux}(p, e) := -y_i$ if $e$ is on the S side of $p$. Similarly, if $e$ is a SW-NE 1-edge, we drop a line SE from $e$ until it pokes out of the $j$th place on the South side, and set $\text{flux}(p, e) := +y_j$ if $e$ is on the NW side of $p$, and $\text{flux}(p, e) := -y_j$ if $e$ is on the SE side of $p$.

Now compute the total flux $\sum_e \text{flux}(p, e)$ of a puzzle piece $p$. By checking each case from Figure 1 (and the equivariant piece) in turn, we see that $p$ has total flux zero unless $p$ is a 1-triangle. Furthermore, if $p$ is an upward 1-triangle sitting atop the south boundary $P_S$, then $p$ also has total flux 0. Finally, if $p$ is an upward 1-triangle sitting atop a downward 1-triangle $p'$, then the total flux of $p$ and $p'$ is zero.

Thus the only upward 1-triangles $p$ which have non-zero flux are those which sit atop SW-NE rhombi. But in that case $p$ belongs to a right-scab $\kappa$, and the total flux of $p$ can be easily computed to equal $\wt(\kappa)$. Similarly, only the downward 1-triangles $p$ which have non-zero flux are those which sit below SW-NE rhombi, so they belong to a left-scab $\kappa$, and the total flux of $p$ can be easily computed to be $-\wt(\kappa)$.

Finally, add up the flux of all the puzzle pieces in $P$. At each internal edge, the contributions from the two pieces containing that edge cancel one another. So the total flux reduces to a sum over the edges on the boundary $\partial P$ of $P$, which can be computed as

$$
-\sum_{i=1}^n \nu_i y_i + \sum_{i=1}^n \lambda_i y_i = \tilde{S}_{\text{div}}|_{\lambda} - \tilde{S}_{\text{div}}|_{\nu}.
$$

Combining this with the previous paragraph we obtain the lemma. □

**Proof of (7).** This follows from the two lemmata just proven:

$$
\sum_{(P, g) \in G^\text{right}_{nt}} \wt(P, g) - \sum_{(P, g) \in G^\text{left}_{nt}} \wt(P, g)
$$

$$
= \sum_{p': \partial p' = \Delta^\chi_{\lambda\mu}} \wt(p') \left( \sum_{\kappa: \kappa \text{ is a left-scab of } p'} \wt(\kappa) - \sum_{\kappa: \kappa \text{ is a right-scab of } p'} \wt(\kappa) \right)
$$

$$
= \sum_{p': \partial p' = \Delta^\chi_{\lambda\mu}} \wt(p') (\tilde{S}_{\text{div}}|_{\nu} - \tilde{S}_{\text{div}}|_{\lambda}).
$$

□
5.2. Proof of (8). Equation (8) is equivalent to

\[ \sum_{(P, g) \in G \setminus G_{\text{right}}} \text{wt}(P, g) = \sum_{(P, g) \in G \setminus G_{\text{left}}} \text{wt}(P, g). \]

This shall be an immediate consequence of

**Proposition 4.** There exists a weight-preserving bijection \( \phi \) from \( G \setminus G_{\text{right}} \) to \( G \setminus G_{\text{left}} \).

**Proof.** Let \((P, g)\) be an element of \( G \setminus G_{\text{right}} \). We shall construct an element \((P', g') = \phi(P, g)\) of \( G \setminus G_{\text{left}} \) for which \( \text{wt}(P', g') = \text{wt}(P, g) \). This will require only a local surgery on \( P \), in which some pieces are replaced and the gash moves.\[19\]

Suppose first that \( g \) is a SW-NE gash, and consider the pieces to its right, with a vertex on the center of the gash. Since \((P, g) \not\in G_{\text{right}}\)\[19\], there cannot be an equivariant piece immediately to the right of \( g \), which leaves three possibilities:

\[ \begin{array}{c}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{c}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \]

In each of these cases, we remove the pieces and gash, and replace them as follows:

\[ \begin{array}{c}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{c}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \]

In each case, the labels on the boundary do not change, which means that the new set of pieces and gash match fit into the puzzle where the old ones were. So this creates a new gashed puzzle \((P', g')\), and this is how we define \( \phi(P, g) \).

Now take the case that \( g \) is an E-W gash. Since \((P, g) \not\in G_{\text{right}}\)\[19\] we see that \( g \) is not on the S edge of the puzzle.

Suppose first that \( g \) has length 2, and consider the pieces below \( g \) with a vertex on the center of the gash. There are four possibilities, which we give below, along with their replacements in \((P', g') = \phi(P, g)\):

\[ \begin{array}{c}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{c}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \]

\[ \begin{array}{c}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{c}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \]

\[ \begin{array}{c}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{c}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \quad \begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \]

It is worth noting, for readers cognizant of the “gentle path” technology of [Hon2], that the center of the gash always moves along a gentle path. Indeed, the loop-breathing in [Hon2] can be interpreted as introducing a “double gash” crossing the gentle loop, propagating one gash around the loop, and once it gets back removing them both.
Now suppose that $g$ has length $l > 2$. Then the two extreme edges of $g$ are separated by $l - 2$ equivariant pieces as shown below (with $l = 5$):

Observe that between and below any two equivariant pieces on the gash there must be another equivariant piece (since nothing else can fit in that space). That leaves $2^2 = 4$ possibilities for the ends, depending on whether there are more equivariant pieces in that row below. In each case we move the gash down one step, possibly stretching it or shrinking it by length 1:

(In these pictures the gash begins length 4, and ends length 5, 4, 4, or 3.) This creates a new gashed puzzle $(P', g')$, with which we define $\phi(P, g)$.

We have now defined $\phi(P, g) = (P', g')$ for all $(P, g) \in G \setminus G_{left}$. A simple examination of all cases verifies that $(P', g')$ is a gashed puzzle with $\partial P' = \partial P = \Delta_{\lambda \mu}$, and also $(P', g') \notin G_{right}$. It is obvious that $wt(P', g') = wt(P, g)$ since no equivariant pieces are created, destroyed, or moved.

If we rotate these local-replacement recipes by $180^\circ$, we get a similar map $\phi'$ from $G \setminus G_{right}$ to $G \setminus G_{left}$. This is easily checked to be the inverse of $\phi$, which is therefore a bijection.
One can use the same rules to define a correspondence between $G_{\text{left}}$ and $G_{\text{right}}$, but they must be iterated. This was the viewpoint of the examples in figures 10 and 11 at the beginning of the section.

\textbf{Proof of theorem 2}. Combining (5), (6), (7), (8), we obtain

$$
\sum_{P', \partial P' = \Delta^\nu_{\lambda \mu}} \text{wt}(P')(\tilde{S}_{\text{div}}|_{\nu} - \tilde{S}_{\text{div}}|_{\lambda})
$$

$$
= \sum_{(P,g) \in G_{\text{right}}^{\text{left}}} \text{wt}(P, g) - \sum_{(P,g) \in G_{\text{left}}^{\text{left}}} \text{wt}(P, g) \quad \text{by (7)}
$$

$$
= \sum_{(P,g) \in G_{\text{left}}^{\text{left}}} \text{wt}(P, g) - \sum_{(P,g) \in G_{\text{right}}^{\text{right}}} \text{wt}(P, g) \quad \text{by (8)}
$$

$$
= \sum_{\lambda' : \lambda' \rightarrow \lambda} \sum_{P' : \partial P' = \Delta^\nu_{\lambda' \mu}} \text{wt}(P') - \sum_{\nu' : \nu' \rightarrow \nu'} \sum_{P' : \partial P' = \Delta^\nu_{\lambda' \mu}} \text{wt}(P') \quad \text{by (5) and (6)}
$$

and this is (3).

Then by lemma 4 and proposition 3 we obtain the first statement of theorem 2.

There is one foolish subtlety in obtaining the second statement of the theorem: the first statement (and the recurrences (2)-(3)) only constrain $d_{\nu \lambda \mu}$ for $\lambda, \mu, \nu$ all having the same number of 1s, and so a priori we might worry that the product $\tilde{S}_{\lambda} \tilde{S}_{\mu}$ might be miscalculated to have some extra terms in which $\nu$ has a different number of 1s. But by corollary 2 above, the number of 1s is the same on all sides of a puzzle. The second statement follows. \qed

\section{6. The Molev-Sagan Problem}

In this section we compare the results of this paper with the earlier work in [MS], which was a major source of inspiration for this paper. We also give a homological (or “geometrical”) interpretation of the structure constants computed in [MS].

The paper [MS] is concerned with multiplying “factorial Schur functions” $s_{\lambda}(x|y)$ for $\lambda \in \binom{n}{k}$. These functions are polynomials in two sets of variables $\{x_1, \ldots, x_k\}, \{y_1, \ldots y_n\}$, and are related to the classes $\tilde{S}_{\lambda}$ by

\textbf{Lemma.} [O], [MS] For any $\lambda, \mu \in \binom{n}{k}$ we have $\tilde{S}_{\lambda}|_{\mu} = s_{\lambda}(y_{i_1} \mid y_{i_k})$ where $y_{i_1} := \{y_{i_1}, \ldots, y_{i_k}\}$ and $n \geq i_1 \ldots i_k \geq 1$ are the $k$ integers $\{1 \leq i \leq n : \mu_i = 1\}$ in decreasing order.

\textbf{Proof.} It is easy to check the GKM conditions, and these $s_{\lambda}$ have the right vanishing and normalization conditions (as proved in [O] and repeated in [MS]). Then apply lemma \ref{lem:GKM} \qed

The problem solved in [MS] is more general than the one we have stated: they consider the mixed structure constants $e_{\mu}(y, z)$, which are polynomials in variables $y$ and $z$, given by the product expansion

$$s_{\theta}(x|z) s_{\mu}(x|y) = \sum_{\nu} e_{\nu}(y, z) s_{\nu}(x|y).$$
The $e_{\nu\lambda}^\gamma$ reduce to the structure constants for equivariant cohomology of Grassmannians at the specialization $y \equiv z$ (and to ordinary cohomology at $y \equiv z \equiv 0$).

The formula in [MS] writes $e_{\nu\lambda}^\gamma(y, z)$ as a sum over “barred tableaux,” each one contributing a certain product $\prod (y_i - z_i)$. In that sense their formula is positive (and reduces to the Littlewood-Richardson rule, in the case that $l(\nu) = l(\theta) + l(\mu)$). Unfortunately, many of their terms have $(y_i - z_j)$ factors with $i \leq j$, as the example

$$s_{10}(x|z) s_{01}(x|y) = (x_1 - z_1) \cdot 1 = (x_1 - y_1) + (y_1 - z_1) \cdot 1 = s_{10}(x|y) + (y_1 - z_1)s_{01}(x|y)$$

already shows. For this reason, the computation of Molev-Sagan structure constants is too general a setting for finding a formula (as in theorem 2) for equivariant Schubert calculus that is manifestly positive in the sense of [G].

We now give a cohomological interpretation of the structure constants $e_{\nu\mu}^\gamma$, which we christen “triple Schubert calculus”, and sketch how one can also compute these coefficients using “MS-puzzles”. (Cohomology does not explicitly appear in [MS] – they consider the computation of the $(e_{\nu\mu}^\gamma)$ purely as a combinatorial question.)

6.1. Double Schubert calculus vs. equivariant Schubert calculus. In this subsection we recall the (well-known) connection between double Schubert calculus and equivariant Schubert calculus. In a nutshell, the connection is that $H^r_t(X) \to H^*(\text{Flags}(\mathbb{C}^n) \times X)$ for any partial flag manifold $X$, taking equivariant Schubert classes to “double Schubert classes”. We begin by recalling some standard material on double Schubert calculus (originally defined in [S]), and its geometric interpretation.

Let $\text{Flags}(\mathbb{C}^n)$ denote the space of flags (i.e. maximal chains of subspaces) in $\mathbb{C}^n$. This has a transitive action of $GL_n(\mathbb{C})$ induced from its action on $\mathbb{C}^n$, and the stabilizer of the standard flag $(F_1)$ is the upper triangular matrices $B$, so $\text{Flags}(\mathbb{C}^n) \cong GL_n(\mathbb{C})/B$.

If we denote the lower triangular matrices by $B_-$, then the Schubert cells $X_{\lambda}$ on the Grassmannian $Gr_k(\mathbb{C}^n)$ are exactly the $B_-$ orbits, whose Poincaré duals gave us the Schubert basis $S_{\lambda}$ of ordinary cohomology $H^*(Gr_k(\mathbb{C}^n))$. These were indexed by patterns $\lambda \in \binom{n}{k}$ recording the intersection of the $k$-plane with the anti-standard flag.

Analogously, we can consider closures of the $GL_n(\mathbb{C})$-orbits on $\text{Flags}(\mathbb{C}^n) \times Gr_k(\mathbb{C}^n)$, which are again indexed by $\binom{n}{k}$ (recording the intersection of the $k$-plane with the flag, which is now varying). The Poincaré duals $D_{\lambda} \in H^*(\text{Flags}(\mathbb{C}^n) \times Gr_k(\mathbb{C}^n))$ of these $GL_n(\mathbb{C})$-orbit closures then form an $H^*(\text{Flags}(\mathbb{C}^n))$-basis for $H^*(\text{Flags}(\mathbb{C}^n) \times Gr_k(\mathbb{C}^n))$ as $\lambda$ varies over $\binom{n}{k}$.

Since $H^*(\text{Flags}(\mathbb{C}^n) \times Gr_k(\mathbb{C}^n))$ is a ring as well as an $H^*(\text{Flags}(\mathbb{C}^n))$-module, we can define structure constants $\{f_{\lambda\mu}^\nu\} \in H^*(\text{Flags}(\mathbb{C}^n))$ for the multiplication:

$$D_{\lambda}D_{\mu} = \sum_{\nu} f_{\lambda\mu}^\nu D_{\nu}.$$  

The computation of the $f_{\lambda\mu}^\nu$ is the concern of double Schubert calculus, and has the following homological interpretation. Fix a generic element $g$ of $GL_n(\mathbb{C})$. The class $D_{\lambda}D_{\mu}$ corresponds to an irreducible cycle of pairs $(F, V) \in \text{Flags}(\mathbb{C}^n) \times Gr_k(\mathbb{C}^n)$ where $V$ satisfies two intersection conditions with $F$: $V$ intersects $F$ $\lambda$-much, and intersects $gF$ $\mu$-much. The class $\sum_{\nu} f_{\lambda\mu}^\nu D_{\nu}$ corresponds to a union of cycles, each of which put only one condition

\footnote{In [MS] they also permit $s_\lambda$ to be a skew Schur function, not just a Schur function, but we have not been able to find any cohomological interpretation of these.}
on $V$ (that it intersect $F$ $v$-much), while also requiring that $F$ live in a cycle Poincaré dual to $f_{\lambda \mu}^\nu$. So the equation requiring these two to be homologous is somehow splitting the double burden on $V$ to a single burden on $V$ and a single burden on $F$.

Restricting $\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n)$ to $\text{Flags}(\mathbb{C}^n) \times \text{pt}$, each $D_\lambda$ maps to $S_\lambda$, which shows that these $f_{\lambda \mu}^\nu$ generalize the structure constants $c_{\lambda \mu}^\nu$ of ordinary Schubert calculus. This was also true of the structure constants of equivariant Schubert calculus, and like them, the $f_{\lambda \mu}^\nu$ carry a degree.\footnote{This degree is the degree of a cohomology class rather than a polynomial, and to be precise it is $2(l(\lambda) + l(\mu) - l(\nu))$ rather than $l(\lambda) + l(\mu) - l(\nu)$.}

We can connect double Schubert calculus with equivariant Schubert calculus using the following property of equivariant cohomology: if $f$ are Schubert calculus constants $c_{\lambda \mu}^\nu$ form a $\lambda \mu$-basis for $\text{Gr}_k(\mathbb{C}^n) \equiv \text{Gr}_n(\mathbb{C})/P$, we thus have

$$H^*_T(\text{Gr}_k(\mathbb{C}^n)) \cong H^*_{\text{pt}}(\text{GL}_n(\mathbb{C}))$$

$$\cong H^*_{T \times \text{GL}_n(\mathbb{C}) \times P}(\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \cong H^*_T(\text{Gr}_n(\mathbb{C}) \times \text{Gr}_n(\mathbb{C})/P)$$

$$\rightarrow H^*(T \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})/P) \cong H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n)),$$

where we have used the fact that $\text{GL}_n(\mathbb{C})$ is isomorphic to $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ quotiented by the diagonal action of $\text{GL}_n(\mathbb{C})$. Thus the structure constants $c_{\lambda \mu}^\nu$ for equivariant Schubert calculus live in $H^*_{\text{GL}_n(\mathbb{C})}(\text{Flags}(\mathbb{C}^n)) \cong H^*_{\text{GL}_n(\mathbb{C}) \times \text{Gr}_k(\mathbb{C}^n)) \cong H^*_{\text{pt}}(\text{pt}) \equiv H^*_T(\text{pt})$, and the forgetful map from $H^*_{\text{GL}_n(\mathbb{C})}(\text{Flags}(\mathbb{C}^n))$ to $H^*(\text{Flags}(\mathbb{C}^n))$ maps these constants to the double Schubert calculus constants $f_{\lambda \mu}^\nu$.

To summarize the above discussion, while equivariant Schubert calculus essentially lacks a definition in terms of intersecting cycles, one is provided by double Schubert calculus, of which equivariant Schubert calculus is a refinement.\footnote{One can show that it is the only such refinement satisfying some natural stability properties in the limit $n \rightarrow \infty$. This is perhaps a bad way to see things, though, since such a limit can only be defined for classical Lie groups, whereas double and equivariant Schubert calculus can be defined for arbitrary Lie groups.}

6.2. Triple Schubert calculus. Consider the cohomology ring $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$ as a module over the cohomology ring $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$ of the first and third factors. Since the classes $D_w$ form a basis of $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n))$, we see that the classes $[D_w \otimes 1]$ form a $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$-basis for $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$. One could compute the structure constants for multiplication in this basis, but one would just obtain the double Schubert constants $f_{\lambda \mu}^\nu$ again (or to be pedantic, we would obtain $f_{\lambda \mu}^\nu \otimes 1$).

Since $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n)) \cong H^*(\text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$, the classes $D_w$ induce a corresponding basis $D_w^I$ for $H^*(\text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$. The classes $1 \otimes D_w^I$ thus form another $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$-basis for $H^*(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$. Again, the structure constants for this basis are no richer than for double Schubert calculus.

In [MS] the authors (implicitly) considered the hybrid problem of computing the structure constants $e''_{\theta \mu}^\nu \in H^*(\text{Flags}(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n))$ in the expansion

$$(1 \otimes D_{\theta}^I)(D_{\mu} \otimes 1) = \sum_{\nu} e''_{\theta \mu}^\nu (D_{\nu} \otimes 1).$$
This has the following homological interpretation. We are now looking for triples \((F_1, V, F_2) \in \text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n) \times \text{Flags}(\mathbb{C}^n)\). The left side of the equation says that \(V\) intersects \(F_1\) \(\mu\)-much and also intersects \(F_2\) \(\theta\)-much. The right side is a union of cycles, in each of which \(V\) intersects \(F_1\) \(\nu\)-much, has no condition directly relating \(V\) to \(F_2\), and instead \(F_1\) and \(F_2\) are related by a condition Poincaré dual to \(e^{\nu}_\theta\). Again, the equation is shifting the double burden on \(V\) to a single burden on \(V\) and a single burden on the pair \((F_1, F_2)\).

In analogy with double Schubert calculus we feel it is appropriate to dub the computation of the \(e^{\nu}_\theta\) triple Schubert calculus. Note that these \(e\)’s are not symmetric in \(\theta\) and \(\mu\), and only become so when restricted to the flag manifold sitting diagonally in the first and third factor.

It is worth noting that triple Schubert calculus has many extensions – e.g. K-theory, replacing the Grassmannian by a flag manifold, or using groups other than \(\text{GL}_n(\mathbb{C})\).\(^{24}\) In this way, one can view \([MS]\) as establishing a positivity result for triple Schubert calculus on Grassmannians.\(^{23}\)

### 6.3. An alternate interpretation: equivariant double Schubert calculus

For completeness, we use the connection between double Schubert calculus and equivariant Schubert calculus discussed in subsection 6.1 to recast triple Schubert calculus as “equivariant double Schubert calculus”.

Inside \(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n)\), we have two interesting families of subvarieties parameterized by \(\binom{n}{k}\): the diagonal-\(\text{GL}_n(\mathbb{C})\) orbit closures, and the varieties \(\text{Flags}(\mathbb{C}^n) \times \{\lambda\}\) corresponding to the Schubert cycles \(\{\lambda\}\). Both families are invariant under the diagonal action of the torus, and so define families of equivariant cohomology classes \(\{D_\lambda\}, \{1 \otimes S_\mu\}\) in \(H^*_T(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n))\). Either family gives a \(H^*_T(\text{Flags}(\mathbb{C}^n))\)-basis of \(H^*_T(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n))\). Therefore we can expand the product

\[
D_\theta(1 \otimes S_\mu) = \sum_{\nu} e^{\nu}_\theta[1 \otimes S_\nu]
\]

where the coefficients \(e^{\nu}_\theta\) live in \(H^*_T(\text{Flags}(\mathbb{C}^n))\). Following a similar analysis as in subsection 6.1, one can show that these coefficients \(e^{\nu}_\theta \in H^*_T(\text{Flags}(\mathbb{C}^n))\) refine the coefficients \(e^{\nu}_\theta\) in triple Schubert calculus.

The results of [MS] have the rather surprising consequence that the constants \(e^{\nu}_\theta\) and \(e^{\nu}_\theta\) can be lifted beyond their respective rings \(H^*(\text{Flags}(\mathbb{C}^n))\) and \(H^*_T(\text{Flags}(\mathbb{C}^n))\) to actual double polynomials \(e^{\nu}_\theta \in H^*_T(\text{Flags}(\mathbb{C}^n))\); this would suggest (speaking loosely) that one should be able to replace the cohomology ring \(H^*_T(\text{Flags}(\mathbb{C}^n) \times \text{Gr}_k(\mathbb{C}^n))\) with the “doubly equivariant” ring \(H^*_T(\text{Flags}(\mathbb{C}^n))\) (using an ineffective action of \(T \times T\) on \(\text{Gr}_k(\mathbb{C}^n)\)). While this can indeed be done, the homological perspective is lost, because the classes being multiplied are no longer representable by subvarieties.

\(^{23}\)It seems difficult to formulate the notion of positivity for groups other than \(\text{GL}_n(\mathbb{C})\); in [MS] the roots \(y_1 - y_j\) of \(\text{GL}_n(\mathbb{C})\) are implicitly seen as a specialization of \(y_1 - z_j\), and it is unclear how to extend this to arbitrary root systems. For many of the other extensions, no satisfactory explicit combinatorial formula for the structure constants is known.

\(^{24}\)Much as [Bu] did for K-theoretic Schubert calculus on Grassmannians. In both cases, they gave a formula directly rather than an abstract reason for positivity such as the one given in [C] for equivariant Schubert calculus.
6.4. MS-puzzles solve the Molev-Sagan problem. Define an MS-puzzle as an arrangement of puzzle pieces forming a diamond of size $n$, looking something like a very large equivariant piece. They therefore have the usual NW and NE boundaries, and now SW and SE boundaries as well. We will require the labels on the NE boundary (read clockwise) to be the inversion-free string $id = 0^{n-k}1^k$. Some examples are in figure 13.

Define the MS-weight of an equivariant piece as $y_j - z_i$, where $i$ measures the distance from the SE side and $j$ from the SW side, starting from 1, and the weight of an MS-puzzle as the product of the MS-weights of its equivariant pieces.

**Theorem.** The Molev-Sagan structure constant $e^v_{\mu \nu}$ is equal to the sum of the weights of the MS-puzzles with $\theta$ on the SW side, $\mu$ on the NW side, and $\nu$ on the SE side, all read bottom-to-top. (The NE side has all 0's, then all 1's, read clockwise.)

We will not prove the theorem here, except to say that the Molev-Sagan structure constants satisfy a recurrence similar to that in corollary 1 and the MS-puzzle formula can be shown to satisfy this recurrence through a slight variant of the arguments in section 5.

Of course, similar recurrences were proven in [MS] using barred tableaux. It is possible, though quite unpleasant, to establish a weight-preserving bijection between MS-puzzles and Molev-Sagan barred tableaux, but we will not present one here.

An example of MS-puzzles in action is in figure 13 demonstrating the equality

$$s_{0101}(x|z) \cdot s_{0101}(x|y) = s_{1001}(x|y) + s_{0110}(x|y) + ((y_3 - z_1) + (y_1 - z_2)) s_{0101}(x|y) + s_{0110}(x|y).$$

![Figure 13](image)

**Figure 13.** The MS-puzzles computing $s_{0101}(x|z) \cdot s_{0101}(x|y) = ((y_3 - z_1) + (y_1 - z_2)) s_{0101}(x|y) + s_{1001}(x|y) + s_{0110}(x|y)$.

It is interesting to compare this calculation to that of $S^2_{0101} = (y_3 - y_2) \tilde{S}_{0101} + \tilde{S}_{1001} + \tilde{S}_{0110}$ using (non-MS) puzzles, as done in figure 14, which only uses three puzzles.

---

25 This definition of weight does not reduce to the definition we needed for (non-MS) puzzles, in the case that the equivariant piece is in the top half. The definition that would do that would be $y_i - y_j$ for $i$ the distance from the SW side and $j$ the distance from the NW side.
The fact that equivariant Schubert calculus and Molev-Sagan structure constants both reduce to ordinary Schubert calculus, in the case \( l(\nu) = l(\lambda) + l(\mu) \), is reflected in the fact that the two ordinary puzzles in the second calculation occur as the lower halves of the corresponding MS-puzzles (rotated 60°).

7. APPENDIX: EXISTENCE OF SCHUBERT CLASSES, AND THE EQUIVARIANT PIERI RULE

In this appendix we extend the standard combinatorial proof of existence of Schubert classes (via divided difference operators) to equivariant Schubert classes.\(^2\) Recall that, before, we established the existence of equivariant Schubert classes by direct topological means, but this did not give a formula for restrictions to fixed points. As a corollary of the formula we get a direct proof of the equivariant Pieri rule (which then implies ordinary Pieri rule as a corollary).

The permutation group \( S_n \) acts on \( \binom{n}{k} \) and on \( H^*_T(\text{pt}) = \mathbb{Z}[y_1, \ldots, y_n] \) in obvious ways. If \( \alpha \) is a class, and \( w \in S_n \), put these actions together to define \( w \cdot \alpha \) by

\[
\{w \cdot \alpha\}_\mu := w \cdot \{\alpha\}_{w^{-1}\mu}
\]

which is easily seen to again be a class (i.e. satisfies the GKM divisibility conditions). We will care most about the case \( w = s_i := (i \leftrightarrow i + 1) \).

We now define the divided difference operators \( \{\partial_i\} \). If \( \alpha \) is a class, define \( \partial_i \alpha \) by

\[
\partial_i \alpha := (\alpha - s_i \cdot \alpha)/(y_{i+1} - y_i).
\]

A priori, this is just a list of rational functions. But in fact these \( \{\partial_i\} \) turn out to define endomorphisms of \( H^*_T(\text{Gr}_k(\mathbb{C}^n)) \) (as a vector space):

**Lemma.** If \( \alpha \) is a class, then \( \partial_i \alpha \) is also a class.

**Proof.** From the GKM conditions we see that \( \partial_i \alpha \in \bigoplus_{\binom{n}{k}} H^*_T(\text{pt}) \). We want to know that \( \partial_i \alpha \) itself satisfies the GKM conditions, i.e. that \( \{\partial_i \alpha\}_\mu - \{\partial_i \alpha\}_{\mu'} \) is a multiple of \( y_j - y_k \) if \( \mu, \mu' \) differ in only the \( j, k \) positions. Plainly this is true if \( j = i, k = i + 1 \) (or vice versa) since then the difference is zero. Otherwise, the division by \( y_{i+1} - y_i \) is irrelevant since its GCD with \( y_j - y_k \) is one, and then the divisibility follows from the fact that \( \alpha \) and \( s_i \cdot \alpha \) are both classes. \( \square \)

Recall that in section\(^2\) we gave a topological proof of the existence of Schubert classes (which we already knew by lemma\([\text{[ }\) to be unique). The first conclusion in the following lemma gives a combinatorial proof, using divided difference operators, and the second conclusion will be used in the proof of equivariant Pieri.

\(^2\)This essentially follows Demazure’s work [\text{[ D, which was implicitly a calculation in equivariant K-theory localized at the fixed points of the flag manifold.}
Lemma 6. Fix \( i \in \{1, \ldots, n-1\} \) and \( \lambda \in \binom{n}{k} \).

If \( \lambda_i > \lambda_{i+1} \) (i.e. \( s_i \cdot \lambda < \lambda \)), then \( \partial_i \tilde{S}_\lambda = \tilde{S}_{s_i \cdot \lambda} \).

If however \( \lambda_i \leq \lambda_{i+1} \) (i.e. \( s_i \cdot \lambda \geq \lambda \)), then \( \partial_i \tilde{S}_\lambda = 0 \).

In particular, one can construct the Schubert class \( \tilde{S}_\lambda \) by starting with the class \( \tilde{S}_{w_0 \cdot \text{id}} \) (which is trivial to compute) and applying successive divided difference operators.

Proof. The class \( \tilde{S}_\lambda \) is supported above \( \lambda \), which implies that \( \partial_i \tilde{S}_\lambda \) is supported inside \( \{s_i \cdot \lambda\} \cup \{\mu \in \binom{n}{k} : l(\mu) \geq l(\lambda)\} \). On the other hand, from degree considerations \( \partial_i \tilde{S}_\lambda \) is a linear combination of Schubert classes of degree at most \( l(\lambda) - 1 \). From these two facts and Proposition \( \mathbb{P} \) we see that \( \partial_i \tilde{S}_\lambda \) must vanish if \( s_i \cdot \lambda \geq \lambda \), and is an integer multiple of \( \tilde{S}_{s_i \cdot \lambda} \) if \( s_i \cdot \lambda < \lambda \). In the latter case, we can show this multiple is 1 by the straightforward computation

\[
\partial_i \tilde{S}_\lambda|_{s_i \cdot \lambda} = -s_i \cdot \tilde{S}_\lambda/(y_{i+1} - y_i) = \prod_{(j,k) \in \text{inv}(s_i \cdot \lambda)} (y_k - y_j) = \tilde{S}_{s_i \cdot \lambda}|_{s_i \cdot \lambda}.
\]

We can now prove the equivariant Pieri rule directly.

Proposition (The equivariant Pieri rule).

\[
\tilde{S}_{\text{div}} \tilde{S}_\lambda = (\tilde{S}_{\text{div}}|_{\lambda}) \tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} \tilde{S}_{\lambda'}.
\]

Proof. From Lemma \( \mathbb{P} \) we have

\[
\tilde{S}_{\text{div}} \tilde{S}_\lambda = (\tilde{S}_{\text{div}}|_{\lambda}) \tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} c_{\text{div}, \lambda}^{\lambda'} \tilde{S}_{\lambda'}
\]

for some integers \( c_{\text{div}, \lambda}^{\lambda'} \); our task is to show that \( c_{\text{div}, \lambda}^{\lambda'} = 1 \).

If \( \lambda' \rightarrow \lambda \), then they must differ in only two spots \( i, i+1 \), where \( \lambda \) has 01 and \( \lambda' \) has 10. Applying \( \partial_i \) we get

\[
\partial_i (\tilde{S}_{\text{div}} \tilde{S}_\lambda) = (\tilde{S}_{\text{div}}|_{\lambda}) \partial_i \tilde{S}_\lambda + c_{\text{div}, \lambda}^{\lambda'} \partial_i \tilde{S}_{\lambda'} + \sum_{\mu \in \lambda \rightarrow \lambda, \mu \neq \lambda'} c_{\text{div}, \lambda}^{\mu} \partial_i \tilde{S}_\mu
\]

By Lemma \( \mathbb{P} \) we have \( \partial_i \tilde{S}_\lambda = 0 \), hence \( s_i \cdot \tilde{S}_\lambda = \tilde{S}_\lambda \), and

\[
\partial_i (\tilde{S}_{\text{div}} \tilde{S}_\lambda) = \frac{\tilde{S}_{\text{div}} \tilde{S}_\lambda - s_i \cdot (\tilde{S}_{\text{div}} \tilde{S}_\lambda)}{y_{i+1} - y_i} = \frac{\tilde{S}_{\text{div}} - s_i \cdot \tilde{S}_{\text{div}}}{y_{i+1} - y_i} \tilde{S}_\lambda = (\partial_i \tilde{S}_{\text{div}}) \tilde{S}_\lambda = \tilde{S}_\lambda.
\]

Also we have \( \partial_i \tilde{S}_{\lambda'} = \tilde{S}_\lambda \) and \( \partial_i \tilde{S}_\mu = 0 \) in the above summation. The claim \( c_{\text{div}, \lambda}^{\lambda'} = 1 \) follows.

Applying the forgetful map to ordinary cohomology we recover the ordinary Pieri rule

\[
S_{\text{div}} S_\lambda = \sum_{\lambda' \rightarrow \lambda} S_{\lambda'}.
\]
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