On the Bogoliubov theory: Casimir effect in a single weakly interacting Bose gas at zero-temperature with Neumann boundary condition

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Abstract

Developing Bogoliubov theory of weakly interacting Bose gas in uncompacted three-dimension space [1], quantum fluctuation energy of one component dilute gas of Bose-Einstein condensate (BEC) confined to two parallel plates investigated at zero-temperature in grand canonical ensemble (GCE) with Neumann boundary condition (BC). The Casimir force considered in comparison to the one with Robin BC, Dirichlet BC and periodic BC.

1 Introduction

Following Casimir’s original calculation [2], D.C. Robert and Y. Pomeau [3, 4] studied of Casimir effect in a dilute homogeneous BEC gas restricted between two very large parallel plates separated by a distance ℓ. In which, Casimir energy defined by subtracting excitation energy corresponds to continuous momentum spectrum from excitation energy corresponds to discrete momentum spectrum. The well known relation (to leading-order) of Casimir pressure is $p_c = -\frac{\pi^2 \hbar v_s}{30\ell^4}$, in which $\bar{\hbar}$ is Planck’s constant, $v_s$ is speed of sound in the system. By this way, the author of Ref.[5] proved that $p_c = -\frac{\pi^2 \hbar v_s}{180\ell^4}$ with periodic BC. These author also developed multidimensional cut-off technique to solve Casimir effect problem [6]. Later, the result in Ref.[5] recovered within framework of quantum field theory in one-loop approximation and series expansion of excitation energy about small value of $\xi/\ell$ [7], $\xi$ being healing length of condensate. In Ref.[8], based on the Hamiltonian formalism, the authors once again proved that $p_c = -\frac{\pi^2 \hbar v_s}{180\ell^4}$, moreover higher terms of Casimir pressure calculated based on zeta-functional regularization. Also using quantum field theory combine cut-off momentum, Thu N.V. [9] obtained the result coincides exactly with the one given in Refs.[3, 4]. Recently, we not only obtained a similar result [10] but also developed the calculating to the double-bubble approximation within framework of Cornwall - Jackiw - Tomboulis (CJT) effective potential in improved Hartree - Fock approximation (IHF) [11]. In one-loop approximation, Casimir effect in two-component Bose-Einstein condensates attained by Thu N.V. et al in Ref.[12].

Different from above calculations, we now deal with Casimir effect by calculating excitation term of ground state energy given in Bogoliubov theory with imposing of Neumann BC. The Casimir energy appears in finite terms after Euler-Maclaurin formula employed. This paper is organized as follows. In section 2, quantum fluctuation energy and Casimir force of a single weakly interacting Bose gas are investigated. Discussions and Conclusions are given in section 3.

2 Quantum fluctuation energy and Casimir force of a single weakly interacting Bose gas

Firstly, we consider the quantum fluctuation energy of a weakly interacting Bose gas in uncompacted three-dimension space, which derived from Bogoliubov theory reads as [1]

$$E^q(k) = \frac{1}{2} \sum_{k \neq 0} \left[ \epsilon(k) - g n_0 - \frac{\hbar^2 k^2}{2m} + \frac{m (g n_0)^2}{\hbar^2 k^2} \right].$$  (1)
In which $\vec{k}$ being the wave vector; $g = 4\pi\hbar^2a_s/m > 0$ determines the strength of repulsive intraspecies interaction; $a_s$ being $s$-wave scattering length; $m$ is atomic mass; $n_0$ is atoms density, in dilute (weakly interacting) Bose gas the diluteness condition is required, this means that $n_0a_s^3 \ll 1$:

$$
\varepsilon(k) = \sqrt{\frac{\hbar^2k^2}{2m} \left( \frac{\hbar^2k^2}{2m} + 2gn_0 \right)}
$$

(2)

defines elementary excitation of the system, which firstly proposed by N.N. Bogoliubov in 1947, so-called Bogoliubov dispersion relation. Recently, Eq.(2) also found out by different methods, for example within perturbative framework of quantum field theory [15], CJT formalism [16, 11]. Assume that our system in the box $V = L_x \times L_y \times L_z$, from now on one notes $\ell \equiv L_z$. The wave vector satisfies Neumann BC has the form [17]

$$
k = \sqrt{k_x^2 + k_y^2 + k_z^2},
$$

(3)

$$
k_x = \frac{\pi n_x}{L_x}, \quad k_y = \frac{\pi n_y}{L_y}, \quad k_y = \frac{\pi n_y}{L_y}, \quad n_x, n_y, n_z = 0, 1, 2, ...
$$

(4)

Plugging (2), (3) and (4) into (1) yields

\[
E^a(k) = \frac{1}{2} \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty} \left[ \sqrt{\frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2} \right)} \right] + 2gn_0 \right]

(5)

In uncompact space, namely $(L_x, L_y, \ell) \to \infty$, triple sums in Eq.(5) be replaced by triple integrals, its rewritten as

\[
E^a(k) = \frac{1}{2} \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \left[ \sqrt{\frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2} \right)} \right] + 2gn_0 \right]

(6)

Changing of variables by setting $x = \frac{\pi n_x}{L_x}, y = \frac{\pi n_y}{L_y}, z = \frac{\pi n_z}{L_z}$ one has

\[
E^a(k) = \frac{1}{2} \int_{\frac{L_x}{2}}^\infty \int_{\frac{L_y}{2}}^\infty \int_{\frac{L_z}{2}}^\infty dx \int_0^\infty dy \int_0^\infty dz \left[ \sqrt{\frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2} \right)} \right] + 2gn_0 \right]

(7)

In spherical coordinate Eq.(7) has the form

\[
E^a(k) = \frac{1}{2} \int_{\frac{L_x}{2}}^\infty \int_{\frac{L_y}{2}}^\infty \int_{\frac{L_z}{2}}^\infty dr^2 \left[ \sqrt{\frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2} \right)} \right] + 2gn_0 \right]

(8)

where $r^2 = x^2 + y^2 + z^2$.

Perform above integral over $r \in [0, \Lambda]$, and takes limit $\Lambda \to \infty$ one finds

\[
E^a(k) = \frac{1}{2} \int_{\frac{L_x}{2}}^\infty \int_{\frac{L_y}{2}}^\infty \int_{\frac{L_z}{2}}^\infty dr^2 \left[ \sqrt{\frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2} \right)} \right] + 2gn_0 \right]

(9)

where $N = n_0V$ is total atoms of the system. Eq.(9) exactly coincides with the result in Ref.[1] where periodic BC is imposed, also the same as the result derived by T.D.Lee et all in Refs.[13, 14].
In order to investigate Casimir effect, we assume that the system confined to two parallel plates of area \( L_x \times L_y \), they are perpendicular to \( O_z \)-axis and separated by a distance \( \ell \). In the following, one only discusses finite-size effect along \( z \)-direction, this means that \( \ell^2 \ll L_x \times L_y \). Furthermore, distance between plates has to enough for manifestation of quantum fluctuation \([5]\), namely \( \ell/\xi \gg 1 \), where \( \xi = h/\sqrt{2mg_0\hbar} \). Notes that the system under considered in GCE, it’s connected to a bulk reservoir of condensation with condensation density equal to \( n_0 \).

Owing to compactification of \( z \)-direction, Eq.\((5)\) become

\[
E^q(k) = \frac{1}{2} \int_0^\infty dx \int_0^\infty dy \sum_{n_z=0}^{\infty} \left[ \sqrt{h^2(x^2 + y^2 + \pi^2 n_z^2/\ell^2)} \left( \frac{h^2(x^2 + y^2 + \pi^2 n_z^2/\ell^2)}{2m} + 2g_0n_z \right) \right]
\]

\[
- g_0 - \frac{h^2(x^2 + y^2 + \pi^2 n_z^2/\ell^2)}{2m} + \frac{m(g_0)^2}{h^2(x^2 + y^2 + \pi^2 n_z^2/\ell^2)}.
\]

Converting integrals to polar coordinate, Eq.\((10)\) read as

\[
E^q(k) = \frac{1}{2} \int_0^\infty dr \int_0^\infty d\theta \sum_{n_z=0}^{\infty} \left[ \sqrt{h^2(r^2 + \pi^2 n_z^2/\ell^2)} \left( \frac{h^2(r^2 + \pi^2 n_z^2/\ell^2)}{2m} + 2g_0n_z \right) \right]
\]

\[
- g_0 - \frac{h^2(r^2 + \pi^2 n_z^2/\ell^2)}{2m} + \frac{m(g_0)^2}{h^2(r^2 + \pi^2 n_z^2/\ell^2)}.
\]

in which \( r^2 = x^2 + y^2 \).

Integrating over \( r \in [0, \Lambda] \) of \((11)\), and take limit \( \Lambda \to \infty \) yield

\[
E^q(k) = \frac{1}{2} \int_0^\infty \pi g_0 n_z \sqrt{\pi^2 n_z^2/\ell^2} + \frac{4m(g_0)^2}{8m\ell^3} - \pi^2 h^2 n_z^2 \sqrt{\pi^2 n_z^2/\ell^2} + \frac{4m(g_0)^2}{8m\ell^3} + \frac{m(g_0)^2}{h^2} \left( \frac{1}{4} + \frac{\ell}{2\pi n_z} \sqrt{\pi^2 n_z^2/\ell^2} + \frac{4m(g_0)^2}{8m\ell^3} \right).
\]

\[
(12)
\]

Employing Euler-Maclaurin formula in the from \([18]\)

\[
\sum_{n=0}^{\infty} f[n] = \int_0^\infty f[x]dx + \frac{1}{2}(f[0]+f[\infty])
\]

\[
+ \frac{1}{12}(f^{(1)}[\infty] - f^{(1)}[0]) - \frac{1}{720}(f^{(3)}[\infty] - f^{(3)}[0]) + \frac{1}{30240}(f^{(5)}[\infty] - f^{(5)}[0]) + (13)
\]

with \( f[n_z] \equiv E^q(k) \), and keeping the finite terms after perform calculations on the right hand side of Eq.\((13)\), one finds

\[
E^q(k) = (L_x \times L_y \times \ell) \frac{1}{2} g_0^2 \frac{128}{15 \pi} \sqrt{n_0 a_s^3} + (L_x \times L_y) \frac{\sqrt{\pi h \sqrt{g_0^3}}}{4 \pi} \sqrt{n_0 a_s^3} - (L_x \times L_y) \frac{\pi^2 h v_s}{480 \sqrt{m\ell^3}}.
\]

\[
(14)
\]

Above formula can be rewritten as

\[
E^q(k) = (L_x \times L_y \times \ell) \frac{1}{2} g_0^2 \frac{128}{15 \pi} \sqrt{n_0 a_s^3} + (L_x \times L_y) \frac{\sqrt{\pi h n_0 v_s}}{4} \sqrt{n_0 a_s^3} - (L_x \times L_y) \frac{\pi^2 h v_s}{480 \ell^3}.
\]

\[
(15)
\]

in which \( v_s = \sqrt{g_0/\hbar} \) is speed of sound. If the plates doesn’t appear, this means that the uncompacted condition satisfied, the two last terms on the right hand side of Eq.\((15)\) canceled out, then the quantum fluctuation energy term of the system in uncompacted space Eq.\((9)\) recovered.

The terms on the right hand side of Eq.\((15)\) analyzed as following. The first term has the form of Eq.\((9)\), its defines bulk component of quantum fluctuation energy, from which the density of excitation energy evaluated equal to \( \frac{1}{2} g_0^2 \frac{128}{15 \pi} \sqrt{n_0 a_s^3} \), this also is excitation energy density of bulk condensation reservoir. The second term in proportion to area of plates, from which one can find out contributing of quantum fluctuation to surface tension is \( \frac{\sqrt{\pi h n_0 v_s}}{4} \sqrt{n_0 a_s^3} \). Easily recognize
that the last term is Casimir effect manifestation, Casimir energy per unit area of plates defined by

\[ E_C(\ell) = -\frac{\pi^2 \hbar v_s}{480 \ell^3}. \]  

(16)

The Casimir force acts on per unit of plates obtained from derivative of the Casimir energy with respect to the distance between two plates is

\[ F_C(\ell) = -\frac{\partial E_C}{\partial \ell} = \frac{\pi^2 \hbar v_s}{160 \ell^4}. \]  

(17)

In order to compare our result with the one in Refs.[4, 5, 7, 9], Eq.(17) scaled by \( F_0 = \hbar v_s / \xi^4 \) as following

\[ F_C(\ell) = -F_0 \frac{\pi^2}{160} \frac{1}{L^4}. \]  

(18)

where \( L = \ell / \xi \) is dimensionless distance between plates.

By this way, the results correspond to Dirichlet BC, Robin BC in Refs.[4, 9], respectively, read

\[ F_C(\ell) = -F_0 \frac{\pi^2}{480 L^3}, \]  

\[ F_C(\ell) = -F_0 \frac{\pi^2}{480} \frac{1}{(L + 1/\sqrt{2})^4}. \]  

(19)

and the result corresponds to periodic BC in Refs.[5, 7] is

\[ F_C(\ell) = -F_0 \frac{\pi^2}{30 L^4}. \]  

(20)

Using Eqs.(18), (19) and (20) we plotted Fig.1, which represent the evolution of Casimir force over distance between two plates with different BCs. One recognize that the force corresponds to periodic BC much greater than the one corresponds to other BCs, Casimir force is minimum when Robin BC imposed.

3 Discussions and Conclusions

In this paper, quantum fluctuation energy and Casimir force of one component weakly interacting Bose gas with Neumann BC has been investigated at zero-temperature in GCE. In the system
confined between two parallel plates, contributions to the excitation energy include bulk term, surface term and especially is Casimir term. In which, density of bulk excitation energy equal to the one of bulk condensation reservoir. Contributing of quantum fluctuation to surface tension in proportion to speed of sound, it’s independent of the distance between two plates. With Neumann BC, Casimir force caused by spatial restriction also is short-ranged and attractive force, its depends on sound speed and decays as a power law of the distance between two plates. In our calculation, Casimir force acts on the plates greater than the one with Robin BC and Dirichlet BC, but smaller than the one with periodic BC. In addition, we recovered quantum fluctuation energy term in uncompacted spatial case.

The divergence problem of the integral over the wave vector solved by cancel out infinite components in employing progress of Euler-Maclaurin formula.

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