Non-perturbative Results on Universal Quantities of Statistical Mechanics Models

G. Mussardo$^{a,b,c}$

$^a$International School for Advanced Studies, Via Beirut 2-4, 34013 Trieste, Italy
$^b$Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
$^c$International Centre of Theoretical Physics
Strada Costiera 12, 34014 Trieste

Abstract

Exact calculations of some universal quantities of two-dimensional statistical models in the vicinity of their fixed points are illustrated.

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One of the most important successes of Quantum Field Theory (QFT) in recent years consists of the quantitative analysis of the universality classes of two-dimensional statistical mechanics models close to their second order phase transition points. Right at the critical points, where the correlation length $\xi$ diverges, methods of Conformal Field Theories (CFT) are particularly powerful to determine exactly the spectrum of anomalous dimensions, structure constants of the OPE algebra, correlation functions etc. i.e. all relevant universal quantities which characterise the strong interactions of the order parameter fluctuations [1]. The hypothesis of conformal invariance led to the explicit solution of a large number of important systems in statistical mechanics as conformally invariant field theories, among which the Ising model, the Ising model with annealed vacancies at its tricritical point, models with $S_q$ and $O(n)$ ($n \leq 2$) symmetries, particular limits thereof (percolation and Self–Avoiding Walks), Yang–Lee Edge Singularity, Ashkin–Teller model, RSOS and WZW models, to name a few (see, for instance [2]).

For many scopes, however, the data provided by CFT do not exhaust all physical information relative to the phase transitions. In a real sample, conformal invariance may be broken by the presence of impurities, by finite–size effects or, simply, by an imperfect fine–tuning of the experimental knobs. It is in any case necessary to slightly move the systems away from criticality in order to study their responses to external fields. These responses are generally encoded in a set of different susceptibilities and scaling functions of the statistical models. On a more theoretical level, a perturbation of the conformal action is required to investigate the space of the coupling constants and its topology consisting of the location of the fixed points and the Renormalization Group flows which connect them. As we will briefly illustrate in the following, the study of the statistical models away from criticality is a very rich subject, on which a lot of progress has been recently achieved. Since the breaking of scale invariance is associated to the appeareance of some mass scales into the problem, a preliminary important question regards the degree of universality exhibited by the off–critical systems. Although this quality is in general spoiled — each model presenting in fact some sensitivity to the specific lattice realization, to the presence of sub–leading interactions, etc. — a certain universal behaviour may be however observed if one keeps sufficiently close to the critical point, in the so–called scaling region where the correlation length $\xi$ is finite but large enough to still discard the microscopic details of the models. Hence, as far as the attention is restricted to the scaling region, QFT can be safely applied to characterise the off–critical massive exitations and their dynamics. For the very nature of its methods, QFT should provide one of the most efficient ways to compute universal ratios, i.e. pure numbers characterising the scaling region of each universality class. Some examples will do more than an abstract definition to make their meaning clear.
Consider the Self-Avoiding-Walk problem. The mean square end-to-end distance of \( N \)-step walks and the mean square radius of gyration of loops scale as \( \langle R^2_e \rangle_N \sim CN^{2\nu} \) and \( \langle R^2_g \rangle_N \sim DN^{2\nu} \), respectively (\( N \) is the number of monomers) (Figure 1). The critical exponent is fixed by CFT (in two dimensions \( \nu = \frac{3}{4} \)) whereas \( C \) and \( D \) are amplitudes. By themselves, they are not universal but their ratio \( C/D \) is free of all scales and therefore universal. This number has been theoretically predicted by QFT methods in Ref. [3]: its value, \( C/D = 13.70 \) is in excellent agreement with its numerical determination by series expansions on different lattices [4]. Note that this ratio depends of course on the class of universality: for the different class of universality represented by the Random Walk problem, the above ratio assumes in fact the value \( C/D = 12 \).

Consider the large distance behaviour of the two-point correlation function of some order parameter \( \Phi \),

\[
\langle \Phi(x)\Phi(0) \rangle \sim A_1 e^{-m_1 r} + A_2 e^{-m_2 r} + A_3 e^{-m_3 r} + \cdots
\]

where \( r = |x| \). The amplitudes \( A_i \) and the mass scales \( m_i \) entering the exponential falling-off of the correlator are not universal: the former depend, in particular, on the normalization of the order parameter \( \Phi \), the latter on the lattice space and/or the value of external fields which spoil the conformal invariance. However, their ratios

\[
\frac{A_2}{A_1}, \frac{A_3}{A_1}, \cdots
\]

\[
\frac{m_2}{m_1}, \frac{m_3}{m_1}, \cdots
\]

are universal numbers and, as such, computable by QFT approach. For the Ising model in an external magnetic field — a long-standing problem of statistical mechanics —, the above mass ratios (as well as the remaining \( m_4/m_1, \ldots, m_8/m_1 \) coming from the exact solution of the model) have been computed by Zamolodchikov [3], with the result \( m_2/m_1 = 2 \cos \frac{\pi}{5} = 1.618..; m_3/m_1 = 2 \cos \frac{\pi}{30} = 1.989.., \) etc. while the amplitude ratios and the scaling form of the two-point function

\[ 
\text{Figure 1.a: Open } N \text{-step configuration.} \\
\text{Figure 1.b: } N \text{-step loop configuration.}
\]
of the magnetization operator $\sigma(x)$ have been computed in Ref. [8] and compared successfully with numerical data (see the figure below, where the two-point correlation function $\langle \sigma(x)\sigma(0) \rangle$ is plotted versus lattice space distances. The point on the graph represent numerical data while the continuum curve is the theoretical estimate). Similar calculations have been performed for many other statistical models as well: for the Tricritical Ising Model, for instance, the above quantities were calculated in [9] and [10], respectively; for the Yang-Lee model can be found in Refs. [11] and [12].

![Figure 2, from Ref. [8]](image)

- Consider the finite–size effects present in the scaling region of a statistical model defined on a cylinder of length $L$ and width $R$, with $L/R \gg 1$. Its propagation along the axes $L$ is ruled by the Hamiltonian $\mathcal{H}(R)$. The eigenvalues of this Hamiltonian — as functions of $R$ — (Figure 3) turn out to contain quite a large number of relevant information. First of all, they have the scaling form

$$\mathcal{E}_i(R) = \frac{2\pi}{R} e_i(m_1 R) , \quad i = 0, 1, 2 \ldots .$$

At very short distance scales, the critical fluctuations are expected to dominate so that the spectrum must coincide with that of the conformal point given by

$$\mathcal{E}_i \simeq \frac{2\pi}{R} \left( 2\Delta_i - \frac{c}{12} \right) , \quad m_1 R \ll 1 ,$$

where $c$ denotes the central charge and $\Delta_i$ the conformal dimensions of the scaling fields in the underlying conformal theory. In the infrared limit, on the other hand,
one should recover the spectrum of the massive theory and therefore the energy levels are given by

\[ E_i \simeq E_{\text{vac}} + M_i, \quad m_1 R \gg 1 \]  

(5)

where the first term takes into account the vacuum bulk energy contribution and \( M_i \) denotes the mass-gap of the i-th level. Therefore, mass ratios and other universal numbers, as for instance the combination \( E_{\text{vac}}/m_1^2 \), can be directly extracted from such kind of numerical data. By considering once again the Ising model in a magnetic field as an example, it is easy to check that the determination of the first three mass ratios indeed confirms the aforementioned Zamolodchikov’s prediction for the spectrum and moreover that \( E_{\text{vac}}/m_1^2 = -1/\left( \sin \frac{\pi}{5} \sin \frac{\pi}{3} \sin \frac{\pi}{30} \right) = -0.0617.. \) (Figure 3.a) (for a lattice determination of this universal ratio see [15]). It should be mentioned that a detailed theory of the approaching of the energy levels to their asymptotic infinite–volume values can also be developed, and that a direct measurement of the elastic scattering \( S \)-matrix can be performed too [16].

Finally, a few words on level crossings: they are expected to occur when the off–critical dynamics is ruled by integrable QFT. On the contrary, phenomena of level repulsions of the energy lines are expected to be observed when we are in presence of non–integrable theories. This seems to be indeed the case, as illustrated for instance by the comparison of Figure 3.a with the Figure 3.b, the latter showing the first energy levels of the Ising model, deformed by both thermal and magnetic

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**Figure 3.a:** First energy levels of the Ising model at \( T = T_c \) in a magnetic field versus the width \( R \) of the cylinder: integrable case.

**Figure 3.b:** First energy levels of the Ising model at \( T \neq T_c \) in a magnetic field versus the width \( R \) of the cylinder: non–integrable case.
deformations. This double deformation of the Ising model defines a non–integrable QFT which has been extensively analysed in [17] and [21].

After a certain familiarity with the topics of interest in this paper has been hopefully achieved with the help of the above examples, let us now address the main question: how can the scaling region of the statistical models be efficiently controlled? The key idea is to consider such models as deformations of the corresponding CFT [5], in such a way that their action can be written as

\[ A = A_{CFT} + \sum_i \lambda_i \int \varphi_i(x) d^2x . \]  

(6)

\( \varphi_i \) are scaling fields of conformal dimension \( \Delta_i \) and, correspondingly, \( \lambda_i \) are dimensionful coupling constants \( \lambda_i \sim \xi^{2(\Delta_i - 1)} \). Within this setting, Conformal Field Theory provides a complete description of the short–distance (ultra–violet) properties and the problem is then to extract the large–distance behaviour of the field theories (3), i.e. to study their infrared region. In this respect, of all the possible deformations the so–called integrable ones play a very special role: they possess an infinite number of conserved quantities which gives the possibility of solving them along non–perturbative methods. First of all, their on–shell characterisation is rather simple because, in virtue of the infinite number of conserved charges, all their scattering processes are completly elastic and factorizable \[4, 5\]. The whole set of scattering amplitudes (and consequently, the exact spectrum of the excitations) can be very often explicitly computed as solutions of managable functional equations, expressing general conditions, as unitarity, crossing symmetry and the bootstrap equivalence of all excitations, i.e. the absence of distinction between bound states and asymptotic particles (for a review, see Ref. [7]).

One of the simplest examples of such bootstrap theories is provided by the scattering theory of the off–critical Yang–Lee model \[11\]: the massive theory consists in this case of a single massive excitation \( A \), which may be regarded as bound state of itself. The exact two–body \( S \)–matrix of the model satisfies in this case the functional equations

\[ S(\beta)S(-\beta) = 1 ; \]

\[ S(\beta) = S(i\pi - \beta) ; \]

\[ S(\beta) = S \left( \beta - i\frac{\pi}{3} \right) S \left( \beta + i\frac{\pi}{3} \right) . \]  

(7)

where the standard parameterisation of two-dimensional on mass–shell momenta in terms of rapidities is adopted: \( p^0 = m \cosh \beta_1, \ p^1 = m \sinh \beta_1, \ \beta \equiv \beta_1 - \beta_2 \). The minimal

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1Some criteria can be given to select the integrable deformations. For a detailed discussion on this point, see \[4, 5\].

2In the following we will only refer to massive theories but massless integrable theories can be studied as well, by means of a suitable extension of the formalism, see \[22, 23\].
solution of the above equations is then given by

\[ S(\beta) = \frac{\tanh(\beta + \frac{2\pi i}{3})}{\tanh(\beta - \frac{2\pi i}{3})}. \] (8)

Other examples of exactly solvable scattering theories can be found in [7].

However, for the scope of statistical mechanics, the most important feature of the two-dimensional integrable models is that all the Form Factors (FF) can be exactly determined. These are the matrix elements of the local operators \( O(x) \) on the asymptotic states of the theory

\[ F_{a_1...a_n}^{O}(\beta'_1, \ldots, \beta'_m|\beta_1, \ldots, \beta_n) \equiv \langle b_1...b_m | A_{b_1}(\beta'_1) \ldots A_{b_m}(\beta'_m) | O(0) | A_{a_1}(\beta_1) \ldots A_{a_n}(\beta_n) \rangle_{0}^{\text{out}}. \] (9)

Their computation can be performed once the exact \( S \)-matrix, the bound state structure and the asymptotic behaviour of the matrix elements (this being fixed by CFT, though) are known: the FF possess, in fact, branch cut singularities dictated by the unitarity conditions of the \( S \)-matrix and moreover satisfy an infinite number of recursive equations originating from kinematical and bound state poles\[8, 12, 24, 25\]. Interestingly enough, for many statistical models close solutions of the above conditions satisfied by the FF have been given in terms of elegant mathematical formulas, consisting of elementary symmetric polynomials and determinant expressions thereof (see, for instance [12, 13, 25]).

The knowledge of the Form Factors of the theory has two important consequences. The first is that we can investigate the off-shell behaviour of the theory, i.e. we can compute the two-point (as well as higher-point) correlation functions of the model by means of the spectral representations obtained through the unitarity sum (Figure 4)

\[ \langle \Phi(x)\Phi(0) \rangle = \sum_{n=0}^{\infty} \int_{\beta_1 > \beta_2 \ldots \beta_n} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \langle 0|\Phi(0)|A_{a_1}(\beta_1) \cdots A_{a_n}(\beta_n)\rangle^2 e^{-|x|\sum_{k=1}^{n} m_k \cosh \beta_k}. \] (10)

![Figure 4.](image)

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3 The reader is referred to the original literature for comprehensive studies on this subject.
A theoretical argument as well as a long practise with spectral representations of two-dimensional models have shown a quality of the foremost practical importance, namely their very fast rate of convergence for all distance scales (see, for instance [3, 8, 10, 12, 23]). In view of this property, correlation functions of many statistical models have been determined with remarkable accuracy by means of limited mathematical efforts.

The second consequence is that the knowledge of the FF permits to test non-integrable aspects of statistical models. Non-integrable models are of course the rule rather than the exception and many statistical systems fall, in fact, in this class of models: the Ising model at a generic point of its phase diagram, for instance, or the Ashkin–Teller in presence of additional magnetic couplings. Other examples of non-integrable systems are provided by spin wave propagation in anisotropic magnetic liquids or ultra–short optical pulses propagating in resonance degenerate medium [19] and — in field theory context — by the massive Schwinger model [20]. The approach proposed in [17, 18] to study non-integrable models consists of viewing them as deformations of the integrable ones. In this way, corrections to the masses and to the scattering amplitudes can be computed in terms of the FF of the operator(s) which spoils integrability, in much the same way as done in perturbation theory in quantum mechanics [17]. However, the breaking of integrability has the largest effect on non-trivial topological sectors of the statistical model [18]. This is the case of the kinks of the Ising in its low temperature phase or the solitons present in those models described by equations of Sine-Gordon type. Phenomena of confinement and resonance states are also usually observed as a consequence of non-integrable perturbations.

In conclusion, a thorough understanding of integrable statistical models and field theories has been achieved in the last decade, with the determination of their exact spectra, correlation functions, Renormalization Group flows and cross-over phenomena, etc. Some progress has also been recently achieved in the study of non-integrable models although a more detailed analysis is still needed for these models. The complete characterisation of their dynamics may be in fact regarded as one of the most important open problems of the subject.

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