EXTRIANGULATED IDEAL QUOTIENTS AND GABRIEL-ZISMAN LOCALIZATIONS

YU LIU AND PANYUE ZHOU*

Abstract. Let \((B, E, s)\) be an extriangulated category and \(S\) be an extension closed subcategory of \(B\). In this article, we prove that the Gabriel-Zisman localization \(B/S\) can be realized as an ideal quotient inside \(B\) when \(S\) satisfies some mild conditions. The ideal quotient is an extriangulated category. We show that the equivalence between the ideal quotient and the localization preserves the extriangulated category structure. We also discuss the relations of our results with Hovey twin cotorsion pairs and Verdier quotients.

1. Introduction

Triangulated categories were introduced in the mid 1960’s by Verdier [V]. Having their origins in algebraic geometry and algebraic topology, triangulated categories have by now become indispensable in many different areas of mathematics. The Verdier quotient \(T/S\) of a triangulated category \(T\) by a triangulated subcategory \(S\) is defined by a universal property with respect to triangulated functors out of \(T\). On the other hand, \(T/S\) is in fact a localization of \(T\), which means it is obtained from \(T\) by formally inverting a class of morphisms. For example, derived categories are certain Verdier quotients of homotopy categories. Localization is a process of adding formal inverses to an algebraic structure known as Gabriel-Zisman localization [GZ], morphisms in the new category can be regarded as compositions of the original morphisms and the formal inverses that were added. This makes Verdier quotients a bit hard to understand since taking Verdier quotients drastically change morphisms.

Iyama and Yang [IYa2] gave a sufficient condition for a Verdier quotient \(T/S\) of a triangulated category \(T\) by a thick subcategory \(S\) to be realized inside of \(T\) as an ideal quotient. Concretely speaking, they assume that \(T\) and \(S\) satisfy the following conditions:

(T0) \(T\) is a triangulated category with a shift functor [1], and \(S\) is a thick subcategory of \(T\). Denote by \(U := T/S\) the Verdier quotient (which is a triangulated category).

(T1) \(S\) has a torsion pair \((X, Y)\).

(T2) \((X, X^\perp)\) and \((Y, Y^\perp)\) form two torsion pairs in \(T\), where \(X^\perp =: \{ T \in T \mid \text{Hom}_T(X, T) = 0 \}\) and \(Y^\perp =: \{ T \in T \mid \text{Hom}_T(T, Y) = 0 \}\).

Define two full subcategories of \(T\):

\[ Z := X^\perp \cap Y^\perp[1] \quad \text{and} \quad M := X[1] \cap Y. \]

Denote by \(Z/[M]\) the ideal quotient category of \(Z\) by \(M\). Iyama and Yang [IYa2] realized the Verdier quotient \(U = T/S\) as the ideal quotient \(Z/[M]\).

**Theorem 1.1.** [IYa2, Theorem 1.1] With the assumptions (T0), (T1) and (T2) as above, the composition \(Z \hookrightarrow T \twoheadrightarrow U\) of natural functors induces an equivalence of additive categories: \(Z/[M] \xrightarrow{\cong} U\).

When the condition (T1) is replaced by a special case:

(T1') \((X, Y)\) is a co-t-structure in \(S\),

the equivalence above becomes a triangle equivalence, see [IYa2, Theorem 1.2]. There are many examples of this realization, such as Buchweitz’s equivalence [Bu, KV, R] between the singularity category of...
an Iwanaga-Gorenstein ring and the stable category of Cohen-Macaulay modules over the ring. More related researches can be found in [Am, BOJ, C, CZ, G, IYa1, K1, OPS, ZH].

Recently, the notion of an extriangulated category was introduced by Nakaoka and Palu [NP] as a simultaneous generalization of exact categories and triangulated categories. Exact categories and extension closed subcategories of a triangulated category are extriangulated categories, hence many results on exact categories and triangulated categories can be unified in the framework of an extriangulated category. There are also plenty of examples of extriangulated categories which are neither exact nor triangulated, see [NP, ZZ, HZZ, ZhZ].

In [NP], Nakaoka and Palu gave a bijection between Hovey twin cotorsion pairs and admissible model structures. Let \((\mathcal{B}, \mathcal{E}, \mathcal{s})\) be an extriangulated category satisfying some mild conditions, and \(((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))\) be a Hovey twin cotorsion pair in \(\mathcal{B}\). Define two full subcategories of \(\mathcal{B}\):

\[
\mathcal{M} := \mathcal{U} \cap \mathcal{V} \quad \text{and} \quad \mathcal{Z} := \mathcal{T} \cap \mathcal{U}.
\]

Consider the following classes of morphisms:

- \(w\text{Fib} := \) the class of deflations \(f\) with \(\text{CoCone}(f) \in \mathcal{V}\);
- \(w\text{Cof} := \) the class of inflations \(g\) with \(\text{Cone}(g) \in \mathcal{S}\);
- \(\mathcal{W} := \) \(w\text{Fib} \circ w\text{Cof}\).

They showed the following theorem.

**Theorem 1.2.** [NP, Corollary 5.25, Theorem 6.20] The composition of the canonical inclusion \(i: \mathcal{Z} \hookrightarrow \mathcal{B}\) and the Gabriel-Zisman localization \(\ell: \mathcal{B} \to \mathcal{B}[\mathcal{W}^{-1}]\) induces an equivalence \(\bar{\ell}: \mathcal{Z}/[\mathcal{M}] \xrightarrow{\cong} \mathcal{B}[\mathcal{W}^{-1}],\) which is depicted as follows:

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{i} & \mathcal{B} \\
\downarrow{\pi} & & \downarrow{\ell} \\
\mathcal{Z}/[\mathcal{M}] & \xrightarrow{\bar{\ell}} & \mathcal{B}[\mathcal{W}^{-1}].
\end{array}
\]

Moreover, \(\mathcal{B}[\mathcal{W}^{-1}]\) is a triangulated category.

Note that \((\mathcal{X}, \mathcal{Y})\) is a torsion pair in a triangulated category \(\mathcal{T}\) in the sense of Iyama and Yoshino [IYo] if and only if \((\mathcal{X}[1], \mathcal{Y})\) is a cotorsion pair in \(\mathcal{T}\) in the sense of Nakaoka [N]. In this article, by using cotorsion pairs, we will develop a theory which shows that similar equivalences between localizations and ideal quotients exist under a more general setting. Let \((\mathcal{B}, \mathcal{E}, \mathcal{s})\) be an extriangulated category. This paper is dedicated to the Gabriel-Zisman localization \(\mathcal{B}/\mathcal{S}\) of an extriangulated category \((\mathcal{B}, \mathcal{E}, \mathcal{s})\) with respect to an extension closed subcategory \(\mathcal{S}\) to be realized as an idea quotient inside \(\mathcal{B}\). Our main result is the following.

**Theorem 1.3.** (see Theorem 3.8 and Sections 3 and 4 for details) Let \(k\) be a field, \((\mathcal{B}, \mathcal{E}, \mathcal{s})\) be a Krull-Schmidt, Hom-finite, \(k\)-linear extriangulated category satisfying condition (WIC), and \(\mathcal{S}\) be an extension closed full subcategory of \(\mathcal{B}\). Let \(\mathcal{X}\) and \(\mathcal{Y}\) be full subcategories of \(\mathcal{B}\) satisfying the following conditions:

1. \((\mathcal{X}, \mathcal{X}^{\perp 1}), (\mathcal{Y}^{\perp 1}, \mathcal{Y}))\) is a twin cotorsion pair in \(\mathcal{B}\), where \(\mathcal{X}^{\perp 1} := \{E \in \mathcal{B} \mid \mathcal{E}(\mathcal{X}, E) = 0\}\) and \(\mathcal{Y}^{\perp 1} := \{E \in \mathcal{B} \mid \mathcal{E}(E, \mathcal{Y}) = 0\}\).

2. \((\mathcal{X}, \mathcal{Y})\) is a cotorsion pair in \(\mathcal{S}\).

Let

\[
\mathcal{Z} : = \mathcal{X}^{\perp 1} \cap \mathcal{Y}^{\perp 1} \quad \text{and} \quad \mathcal{W} : = \mathcal{X} \cap \mathcal{Y}.
\]

Assume that \(\mathcal{S}\) is closed under taking cones, which means

for any \(\mathcal{E}\)-triangle \(A \to B \to C \to \), if \(A, B \in \mathcal{S}\), then we have \(C \in \mathcal{S}\).

Then the Gabriel-Zisman localization \(\mathcal{B}/\mathcal{S}\) (see Definition 3.7 for details) can be realized as the ideal quotient \(\mathcal{Z}/[\mathcal{W}]\), that is, there exists an equivalence \(\bar{F}: \mathcal{Z}/[\mathcal{W}] \xrightarrow{\cong} \mathcal{B}/\mathcal{S}\). Moreover, when \(\mathcal{B}\) has enough projectives and enough injectives,

1. if \(\mathcal{S}\) is a thick subcategory of \(\mathcal{B}\), then \(F\) becomes an extriangle equivalence (see Definition 4.1 for details);
2. if \((\mathcal{X}, \mathcal{X}^{\perp 1}), (\mathcal{Y}^{\perp 1}, \mathcal{Y})\) are hereditary cotorsion pairs, then \(F\) becomes a triangle equivalence.
Remark 1.4. In Theorem 1.1 and Theorem 1.2, the equivalences between ideal quotients and localizations are additive, although the localizations are triangulated and the ideal quotients are extriangulated. Theorem 1.3 points out that the equivalences have better properties.

This article is organized as follows. In Section 2, we review some elementary concepts and properties of extriangulated categories. In Section 3, we show the main result of this article: the existence of equivalences between localizations and quotient categories. In Section 4, we show that under certain conditions, the equivalence has better properties.

2. Preliminaries

Let us briefly recall the definition and some basic properties of extriangulated categories. For more details, see [NP, Section 2.3].

Let \( \mathcal{B} \) be an additive category equipped with an additive bifunctor
\[
\mathcal{E}: \mathcal{B}^{op} \times \mathcal{B} \to \text{Ab},
\]
where Ab is the category of abelian groups. For any objects \( A, C \in \mathcal{B} \), an element \( \delta \in \mathcal{E}(C, A) \) is called an \( \mathcal{E} \)-extension. Let \( \mathfrak{s} \) be a correspondence which associates with an equivalence class of sequences
\[
\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]
\]
to any \( \mathcal{E} \)-extension \( \delta \in \mathcal{E}(C, A) \). This \( \mathfrak{s} \) is called a realization of \( \mathcal{E} \), if it makes the diagrams in [NP, Definition 2.9] commutative. A triplet \((\mathcal{B}, \mathcal{E}, \mathfrak{s})\) is called an extriangulated category if:

(1) \( \mathcal{E}: \mathcal{B}^{op} \times \mathcal{B} \to \text{Ab} \) is an additive bifunctor.
(2) \( \mathfrak{s} \) is an additive realization of \( \mathcal{E} \).
(3) \( \mathcal{E} \) and \( \mathfrak{s} \) satisfy some ‘additivity’ and ‘compatibility’ conditions in [NP, Definition 2.12].

We collect some basic concepts which will be used later.

Definition 2.1. Let \((\mathcal{B}, \mathcal{E}, \mathfrak{s})\) be an extriangulated category.

(1) If a sequence \( A \xrightarrow{x} B \xrightarrow{y} C \) realizes \( \delta \in \mathcal{E}(C, A) \), we call the pair \((A \xrightarrow{x} B \xrightarrow{y} C, \delta)\) an \( \mathcal{E} \)-triangle, and write it in the following way:
\[
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}.
\]
We usually do not write this “\( \delta \)” if it is not used in the argument.

(2) An object \( P \in \mathcal{B} \) is called projective if for any \( \mathcal{E} \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) and any morphism \( c: P \to C \), there exists \( b: P \to B \) satisfying \( by = c \). We denote the subcategory of projective objects by \( \mathcal{P} \). Dually, the subcategory of injective objects is denoted by \( \mathcal{I} \).

(3) We say that \( \mathcal{B} \) has enough projectives if for any object \( C \in \mathcal{B} \), there exists an \( \mathcal{E} \)-triangle
\[
A \xrightarrow{x} P \xrightarrow{y} C
\]
with \( P \in \mathcal{P} \). Dually we can define having enough injectives.

(4) Let \( \mathcal{S} \) be a subcategory of \( \mathcal{B} \). We say \( \mathcal{S} \) is extension closed if in any \( \mathcal{E} \)-triangle
\[
A \xrightarrow{x} B \xrightarrow{y} C
\]
with \( A, C \in \mathcal{S} \), we have \( B \in \mathcal{S} \).

(5) Any \( \mathcal{E} \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) induces the following long exact sequences:
\[
\text{Hom}_\mathcal{B}(X, A) \to \text{Hom}_\mathcal{B}(X, B) \to \text{Hom}_\mathcal{B}(X, C) \to \mathcal{E}(X, A) \to \mathcal{E}(X, B) \to \mathcal{E}(X, C);
\]
\[
\text{Hom}_\mathcal{B}(C, X) \to \text{Hom}_\mathcal{B}(B, X) \to \text{Hom}_\mathcal{B}(A, X) \to \mathcal{E}(C, X) \to \mathcal{E}(B, X) \to \mathcal{E}(A, X)
\]
where \( X \) is an arbitrary object in \( \mathcal{B} \).

Remark 2.2. Any extension closed subcategory \( \mathcal{M} \) of an extriangulated category \((\mathcal{B}, \mathcal{E}, \mathfrak{s})\) has a natural extriangulated category structure \((\mathcal{M}, \mathcal{E}|_\mathcal{M}, \mathfrak{s}|_\mathcal{M})\) which inherits from \((\mathcal{B}, \mathcal{E}, \mathfrak{s})\), where \( \mathcal{E}|_\mathcal{M} \) is the restriction of \( \mathcal{E} \) onto \( \mathcal{M}^{op} \times \mathcal{M} \) and \( \mathfrak{s}|_\mathcal{M} \) is the restriction of \( \mathfrak{s} \).

In this paper, let \( k \) be a field and \((\mathcal{B}, \mathcal{E}, \mathfrak{s})\) be a Krull-Schmidt, Hom-finite, \( k \)-linear extriangulated category. Let \( \mathcal{P} \) (resp. \( \mathcal{I} \)) be the subcategory of projective (resp. injective) objects. When we say that \( \mathcal{C} \) is a subcategory of \( \mathcal{B} \), we always assume that \( \mathcal{C} \) is full and closed under isomorphisms.

In this paper, the cotorsion pair is the main tool we use.

Definition 2.3. [NP, Definition 2.1] Let \( \mathcal{U} \) and \( \mathcal{V} \) be two subcategories of \( \mathcal{B} \) which are closed under direct summands. We call \((\mathcal{U}, \mathcal{V})\) a cotorsion pair if it satisfies the following conditions:
(a) \( E(\mathcal{U}, \mathcal{V}) = 0 \).

(b) For any object \( B \in \mathcal{B} \), there exist two \( E \)-triangles

\[
V^B \to U^B \to B \to \to \quad B \to V_B \to U_B \to \to
\]

satisfying \( U_B, U^B \in \mathcal{U} \) and \( V_B, V^B \in \mathcal{V} \).

**Remark 2.4.** For an extension closed subcategory \( \mathcal{M} \), we say a pair of subcategories \((\mathcal{X}, \mathcal{Y})\) is a cotorsion pair in \( \mathcal{M} \) if \( \mathcal{X} \subseteq \mathcal{M} \), \( \mathcal{Y} \subseteq \mathcal{M} \) and \((\mathcal{X}, \mathcal{Y})\) is a cotorsion pair in the extriangulated category \((\mathcal{M}, E|_\mathcal{M}, s|_\mathcal{M})\).

By the definition of a cotorsion pair, we can immediately conclude the following result.

**Lemma 2.5.** Let \((\mathcal{U}, \mathcal{V})\) be a cotorsion pair in \( \mathcal{B} \).

(a) \( \mathcal{V} = \mathcal{U}^{-1} := \{ X \in \mathcal{B} \mid E(\mathcal{U}, X) = 0 \} \).

(b) \( \mathcal{U} = \mathcal{V}^{-1} := \{ Y \in \mathcal{B} \mid E(\mathcal{Y}, Y) = 0 \} \).

(c) \( \mathcal{U} \) and \( \mathcal{V} \) are closed under extensions and direct sums.

(d) \( I \subseteq \mathcal{V} \) and \( P \subseteq \mathcal{U} \).

**Proof.** (a) Since \( E(\mathcal{U}, \mathcal{V}) = 0 \), we have \( \mathcal{V} \subseteq \mathcal{U}^{-1} \). Conversely, for any \( M \in \mathcal{U}^{-1} \), since \((\mathcal{U}, \mathcal{V})\) is a cotorsion pair, there exists an \( E \)-triangle

\[
M \xrightarrow{x} V_M \xrightarrow{y} U_M \xrightarrow{\delta} \to
\]

where \( U_M \in \mathcal{U} \) and \( V_M \in \mathcal{V} \). Since \( M \in \mathcal{U}^{-1} \), we have \( E(\mathcal{U}, M) = 0 \). It follows that \( \delta = 0 \) and then \( x \) is a section. This shows that \( M \) is a direct summand of \( V_M \). Since \( \mathcal{V} \) is closed under direct summands, we have \( M \in \mathcal{V} \).

Dually we can show (b).

(c) By (a) and (b), \( \mathcal{U} \) and \( \mathcal{V} \) are closed under direct sums. Now we prove that \( \mathcal{U} \) is closed under extensions. Similarly, we can show that \( \mathcal{V} \) is closed under extensions.

Assume that

\[
A \xrightarrow{f} B \xrightarrow{g} C \to
\]

is an \( E \)-triangle where \( A, C \in \mathcal{U} \). By [NP, Proposition 3.11], we have the following exact sequence:

\[
E(C, \mathcal{V}) \to E(B, \mathcal{V}) \to E(A, \mathcal{V}).
\]

Since \( \mathcal{U} = \mathcal{V}^{-1} \) by (a) and \( A, C \in \mathcal{U} \), we have \( E(\mathcal{A}, \mathcal{V}) = 0 \) and \( E(\mathcal{C}, \mathcal{V}) = 0 \). It follows that \( E(\mathcal{B}, \mathcal{V}) = 0 \) which implies \( B \in \mathcal{V}^{-1} = \mathcal{U} \).

(d) By [NP, Proposition 3.24] and its dual, we have \( E(\mathcal{P}, \mathcal{V}) = 0 \) and \( E(\mathcal{U}, \mathcal{I}) = 0 \). By (a) and (b), we obtain that \( \mathcal{P} \subseteq \mathcal{U} \) and \( \mathcal{I} \subseteq \mathcal{V} \).

**Remark 2.6.** There is no simple relation between torsion and cotorsion pairs in the exact categories as in the triangulated categories. For example, assume that \( \mathcal{A} \) is an exact category with enough injectives \( \mathcal{I} \), then \((\mathcal{A}, \mathcal{I})\) is a cotorsion pair in \( \mathcal{A} \), but for any non-zero subcategory \( \mathcal{C} \subseteq \mathcal{A} \), \((\mathcal{C}, \mathcal{I})\) is not a torsion pair.

We introduce a kind of important cotorsion pairs, which can be regarded as a generalization of co-t-structures.

**Definition 2.7.** A cotorsion pair \((\mathcal{U}, \mathcal{V})\) is called **hereditary** if it satisfies the following conditions:

(a) For any \( E \)-triangle \( A \to B \to C \to \to \), \( B, C \in \mathcal{U} \) implies \( A \in \mathcal{U} \).

(b) For any \( E \)-triangle \( A \to B \to C \to \to \), \( A, B \in \mathcal{V} \) implies \( C \in \mathcal{V} \).

When \( \mathcal{B} \) has enough projectives and enough injectives, we can define higher extensions \( E^i(\mathcal{P}, -), i \geq 1 \) \((E^1 := E)\) of the bifunctor \( E \) (see [LN, Section 5.1] for details). Any \( E \)-triangle \( A \to B \to C \to \to \) induces the following long exact sequences:

\[
\begin{align*}
E(X, A) & \to E(X, B) \to E(X, C) \to E^2(X, A) \to E^2(X, B) \to E^2(X, C) \to \cdots; \\
E(C, X) & \to E(B, X) \to E(A, X) \to E^2(C, X) \to E^2(B, X) \to E^2(A, X) \to \cdots
\end{align*}
\]

where \( X \) is an arbitrary object in \( \mathcal{B} \). Moreover, we have \( E^i(\mathcal{P}, -) = 0 \) and \( E^i(\mathcal{I}, -) = 0 \) for any positive integer \( i \). We have the following proposition.
**Proposition 2.8.** When $\mathcal{B}$ has enough projectives and enough injectives, for any cotorsion pair $(\mathcal{U}, \mathcal{V})$, the following conditions are equivalent:

(i) $\mathcal{E}^2(\mathcal{U}, \mathcal{V}) = 0$.

(ii) For any $\mathcal{E}$-triangle $A \to B \to C \to$, $B, C \in \mathcal{U}$ implies $A \in \mathcal{U}$.

(iii) For any $\mathcal{E}$-triangle $A \to B \to C \to$, $A, B \in \mathcal{V}$ implies $C \in \mathcal{V}$.

**Proof.** We show that (i)$\iff$(ii), (i)$\iff$(iii) is by dual.

(i)$\Rightarrow$(ii): For any $\mathcal{E}$-triangle $A \to B \to C \to$ with $B, C \in \mathcal{U}$, we have an exact sequence:

$$0 = \mathcal{E}(B, V) \to \mathcal{E}(A, V) \to \mathcal{E}^2(C, V) = 0$$

for any $V \in \mathcal{V}$, which implies that $\mathcal{E}(A, V) = 0$. By Lemma 2.5, we obtain that $A \in \mathcal{U}$.

(ii)$\Rightarrow$(i): Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $\mathcal{B}$ has enough projectives, $U$ admits an $\mathcal{E}$-triangle $U' \to P \to U \to$ with $P \in \mathcal{P}$. By Lemma 2.5, $P \in \mathcal{U}$, hence $U' \in \mathcal{U}$. Then we have an exact sequence:

$$0 = \mathcal{E}(U', V) \to \mathcal{E}^2(U, V) \to \mathcal{E}^2(P, V) = 0$$

which implies $\mathcal{E}^2(U, V) = 0$. □

Let $(\mathcal{U}_1, \mathcal{V}_1), (\mathcal{U}_2, \mathcal{V}_2)$ be two cotorsion pairs. By Lemma 2.5, we can find that

$$U_1 \subseteq U_2 \iff V_2 \subseteq V_1 \iff \mathcal{E}(U_1, \mathcal{V}_2) = 0.$$

**Definition 2.9.** A pair of cotorsion pairs $((\mathcal{U}_1, \mathcal{V}_1); (\mathcal{U}_2, \mathcal{V}_2))$ is called a *twin cotorsion pair* if $U_1 \subseteq U_2$.

In the rest of this paper, let $((\mathcal{X}, \mathcal{Y}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair. For convenience, we put $\mathcal{W} := \mathcal{X} \cap \mathcal{Y}$.

Denote by $[\mathcal{W}](A, B)$ the subgroup of $\mathcal{E}(A, B)$ consisting of the morphisms $f$ factoring through objects in $\mathcal{W}$. We denote by $\mathcal{B}/[\mathcal{W}]$ (or $\overline{\mathcal{B}}$ for short) the category which has the same objects as $\mathcal{B}$, and

$$\mathcal{Hom}_{\mathcal{B}}(A, B) = \mathcal{Hom}_{\mathcal{B}}(A, B)/[\mathcal{W}](A, B)$$

for any $A, B \in \mathcal{B}$. For any morphism $f \in \mathcal{Hom}_{\mathcal{B}}(A, B)$, we denote its image in $\mathcal{Hom}_{\overline{\mathcal{B}}}(A, B)$ by $\overline{f}$.

**Lemma 2.10.** If $\mathcal{X} \cap \mathcal{V} = \mathcal{U} \cap \mathcal{X}$, then $\mathcal{Hom}_{\overline{\mathcal{B}}}(\mathcal{U}, \mathcal{Y}) = 0$ and $\mathcal{Hom}_{\overline{\mathcal{B}}}(\mathcal{X}, \mathcal{Y}) = 0$.

**Proof.** If $\mathcal{X} \cap \mathcal{V} = \mathcal{U} \cap \mathcal{X}$, then $\mathcal{X} \cap \mathcal{V} = \mathcal{W} = \mathcal{U} \cap \mathcal{Y}$. Let $u : U \to Y$ be a morphism such that $U \in \mathcal{U}$ and $Y \in \mathcal{Y}$. $Y$ admits an $\mathcal{E}$-triangle $Y' \to U' \to Y''$ where $U' \in \mathcal{Y} \cap \mathcal{U} = \mathcal{W}$ and $Y'' \in \mathcal{Y}$. Since $\mathcal{E}(U, Y'') = 0$, there is a morphism $u' : U \to U'$ such that $yu = u$. Hence $\pi = 0$, which implies $\mathcal{Hom}_{\overline{\mathcal{B}}}(\mathcal{U}, \mathcal{Y}) = 0$.

Dually, we can show that $\mathcal{Hom}_{\overline{\mathcal{B}}}(\mathcal{X}, \mathcal{Y}) = 0$. □

Let

$$S_L = \{ B \in \mathcal{B} \mid \exists \text{ E-triangle } Y \to X \to B \to \text{ with } X \in \mathcal{X} \text{ and } Y \in \mathcal{Y} \},$$

$$S_R = \{ B \in \mathcal{B} \mid \exists \text{ E-triangle } B \to Y' \to X' \to \text{ with } X' \in \mathcal{X} \text{ and } Y' \in \mathcal{Y} \}.$$

**Lemma 2.11.** If $\mathcal{X} \cap \mathcal{V} = \mathcal{U} \cap \mathcal{X}$, then $S_L$ and $S_R$ are closed under direct summands.

**Proof.** We show that $S_R$ is closed under direct summands, and the other half is by dual.

Let $A_1 \oplus A_2 \in S_R$. Then it admits an $\mathcal{E}$-triangle $A_1 \oplus A_2 \to Y_0 \to X_0 \to$ with $Y_0 \in \mathcal{Y}$ and $X_0 \in \mathcal{X}$. For $i \in \{1, 2\}$, $A_i$ admits an $\mathcal{E}$-triangle $A_1 \to Y_i \to U_i \to$ with $Y_i \in \mathcal{Y}$ and $U_i \in \mathcal{U}$. Hence we have the following commutative diagrams:

\[
\begin{array}{ccc}
A_1 \oplus A_2 & \longrightarrow & Y_1 \oplus Y_2 \\
\downarrow \alpha & & \downarrow \beta \\
A_1 \oplus A_2 & \longrightarrow & Y_0 \\
\downarrow \alpha & & \downarrow \beta \\
A_1 \oplus A_2 & \longrightarrow & Y_1 \oplus Y_2 \\
\end{array}
\]

By [NP, Corollary 3.5], the right diagram implies that $1 - \beta \alpha$ factors through $Y_1 \oplus Y_2$. This implies $U_1 \oplus U_2$ is a direct summand of $X_0 \oplus Y_1 \oplus Y_2$. By Lemma 2.10, any morphism from $U_1 \oplus U_2$ to $Y_1 \oplus Y_2$ factors through $\mathcal{W}$, hence any indecomposable object of $U_1 \oplus U_2$ is isomorphic to an object in $\mathcal{X}$. Hence $U_1 \in \mathcal{X}$, which implies $A_i \in S_R$. Then $S_R$ is closed under direct summands. □
Remark 2.12. By the proof of Lemma 2.10, if \( X \cap V = U \cap Y \), then

(a) any object \( X \in X \) admits an \( E \)-triangle \( X \to W_X \to X' \to \) with \( W_X \in W \) and \( X' \in X \);
(b) any object \( Y \in \mathcal{Y} \) admits an \( E \)-triangle \( Y' \to W_Y \to Y \to \) with \( W_Y \in W \) and \( Y' \in \mathcal{Y} \).

Hence \( X, \mathcal{Y} \subseteq S_L \cap S_R \).

We also have the following observation.

Lemma 2.13. Let \( S \) be an extension closed subcategory of \( B \) such that \((X, \mathcal{Y})\) is a cotorsion pair in \( S \). Then \( X \cap V = \mathcal{Y} \cap U \). Moreover, \( S \) is closed under direct summands.

Proof. By definition, we have \( X \cap V \subseteq X \cap V \). Let \( Z \in X \cap V \). Since \( X \subseteq S \), \((X, \mathcal{Y})\) is a cotorsion pair in \( S \), \( Z \) admits an \( E \)-triangle \( Z \to W \to X \to \) with \( W \in X \) and \( W \in X \cap V \). This \( E \)-triangle splits, which implies \( Z \) is a direct summand of \( W \in W \), hence \( Z \in W \). Dually we can show that \( Y \cap U = W \).

Since \( S \subseteq S_R \cap S_L \), by Lemma 2.11, we get that \( S \) is closed under direct summands. \( \square \)

Definition 2.14. A subcategory \( M \) of \( B \) is called a thick subcategory in \( B \) provided that it is closed under direct summands and for any \( E \)-triangle

\[ A \to B \to C \to \]

in \( M \), if any two objects of \( A, B \) and \( C \) belong to \( M \), then so is the third one.

Lemma 2.15. If \((X, \mathcal{Y})\) is a cotorsion pair in a thick subcategory \( S \), then \( S_R = S_L = S \).

Proof. By definition, we have \( S \subseteq S_R \cap S_L \). But on the other hand, since \( S \) is thick and \( X \subseteq S \), \( \mathcal{Y} \subseteq S \), by definition we have \( S_R \subseteq S \) and \( S_L \subseteq S \). Hence \( S_R = S_L = S \). \( \square \)

Remark 2.16. If \((X, \mathcal{Y}, (U, \mathcal{Y}))\) satisfies the condition \( S_R = S_L \), then it is called a Hovey twin cotorsion pair (see [NP, Definition 5.1] for more details). Hence if \((X, \mathcal{Y})\) satisfies the condition in Lemma 2.15, \((X, \mathcal{Y}, (U, \mathcal{Y}))\) is a Hovey twin cotorsion pair. Note that if \((X, \mathcal{Y}, (U, \mathcal{Y}))\) is Hovey, we always have \( X \cap V = U \cap Y \) by [NP, Remark 4.17, Remark 5.2].

3. Localizations and Quotient Subcategories

In an \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \to \). \( x \) is called an inflation and \( y \) is called a deflation. From now on, we also assume \( B \) satisfies condition \( (WIC) \) ([NP, Condition 5.8]):

- If we have a deflation \( h : A \xrightarrow{h} B \xrightarrow{\partial} C \), then \( g \) is also a deflation.
- If we have an inflation \( h : A \xrightarrow{h} B \xrightarrow{\partial} C \), then \( f \) is also an inflation.

Note that this condition automatically holds on triangulated categories and Krull-Schmidt exact categories.

Under this condition, we can show the following lemma.

Lemma 3.1. Let \( C \) be a subcategory of \( B \) which is closed under direct summands.

(a) If \( f : C \to B \) is a right \( C \)-approximation of \( B \) and a deflation, then there is a deflation \( f_1 : C_1 \to B \) which is a minimal right \( C \)-approximation.
(b) If \( f' : B \to C' \) is a left \( C \)-approximation of \( B \) and an inflation, then there is an inflation \( f_1' : B \to C_1' \) which is a minimal left \( C \)-approximation.

Proof. We only need to prove (a). Dually we can show (b).

If \( f : C \to B \) is a deflation, since \( B \) is Krull-Schmidt, there exists a decomposition \( C = C_1 \oplus C_2 \) such that \( f = (f_1 \circ 0) : C_1 \oplus C_2 \to B \) where \( f_1 \) is right minimal. Since \( (f_1 \circ 0) = f_1 \circ (\partial \circ 0) \), by condition \( (WIC) \), we get that \( f_1 \) is also a deflation. Let \( C_0 \) be any object in \( C \) and \( g \in \text{Hom}_B(C_0, B) \). Since \( f \) is a right \( C \)-approximation of \( B \), there is a morphism \( (h_1^C) : C_0 \to C_1 \oplus C_2 \) such that \( g = f \circ (h_1^C) = f_1 h_1 \). Hence \( f_1 \) is a right \( C \)-approximation of \( B \). \( \square \)

3.1. Extriangulated quotient categories. In the rest of the paper, let \( Z = U \cap \mathcal{Y} \). Since \( U \) and \( \mathcal{Y} \) are closed under extensions, \( Z \) is also closed under extensions, hence it is an extriangulated subcategory of \( B \). Note that \( E(W, Z) = 0 = E(\mathcal{Z}, W) \), by definition, \( W \) is a subcategory of projective-injective objects in \( Z \). Then by [NP, Proposition 3.30], \( Z/\mathcal{W} \) has an extriangulated structure induced by \( Z \).

Moreover, we have the following proposition.
Proposition 3.2. If \((X, \mathcal{Y})\) and \((U, \mathcal{Y})\) are hereditary cotorsion pairs, then \(Z\) is a Frobenius subcategory in which \(\mathcal{W}\) is the subcategory of enough projective-injective objects, which implies that \(Z/[\mathcal{W}]\) is a triangulated category.

Proof. According to [NP, Corollary 7.4], we only need to show that \(\mathcal{W}\) is the subcategory of enough projective-injective objects. Let \(Z \in \mathcal{Z}\) be any object. It admits an \(\mathcal{E}\)-triangle \(V \rightarrow X \rightarrow Z \rightarrow \) where \(X \in \mathcal{X}\) and \(V \in \mathcal{V}\). Since \(Z, X \in U\), by definition, we have \(V \in U \cap \mathcal{V} = Z\). Moreover, since \(V, Z \in V\), we have \(X \in V\), hence \(X \in \mathcal{X} \cap \mathcal{V} = \mathcal{W}\) by Lemma 2.13. Dually we can show that \(Z\) admits an \(\mathcal{E}\)-triangle \(Z \rightarrow Y \rightarrow U \rightarrow \) where \(Y \in \mathcal{W}\) and \(U \in Z\). Thus \(\mathcal{Z}\) is a Frobenius subcategory in which \(\mathcal{W}\) is the subcategory of projective-injective objects, which implies that \(Z/[\mathcal{W}]\) is a triangulated category by [ZZ, Theorem 3.13].

If \((X, \mathcal{Y})\) and \((U, \mathcal{Y})\) are hereditary cotorsion pairs, then the triangulated category structure on \(Z/[\mathcal{W}]\) is the following:
- the suspension functor
  \((1) : A \mapsto A(1), a \mapsto a(1)\)
- and distinguished triangles
  \[
  A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A(1)
  \]
  are given by the following commutative diagram:

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A(1) \\
\downarrow \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
W_A \xrightarrow{\alpha} A(1) \xrightarrow{\alpha(1)} W_D \xrightarrow{\beta} D(1)
\end{array}
\]

with \(W_A, W_D \in \mathcal{W}\). Here \(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A(1)\) is an arbitrary \(\mathcal{E}\)-triangle in \(Z\) and \(a : A \rightarrow D\) is an arbitrary morphism in \(Z\).

3.2. Hovey twin cotorsion pairs. In [NP, Section 5], Nakaoka-Palu showed that if we have a Hovey twin cotorsion pair \(((X, \mathcal{Y}), (U, \mathcal{Y}))\), then we can get an equivalence between \(Z/[\mathcal{W}]\) and a localization with respect to \(((X, \mathcal{Y}), (U, \mathcal{Y}))\). By Lemma 2.15, we can find that the torsion pairs in [IYa2, Theorem 1.1] induces a Hovey twin cotorsion pair, and this theorem becomes a special case of the results in [NP, Section 5].

A question is that given a Hovey twin cotorsion pair \(((X, \mathcal{Y}), (U, \mathcal{Y}))\), can we find an extension closed subcategory \(\mathcal{S}\) in which \((X, \mathcal{Y})\) is a cotorsion pair. We can answer this question by the following proposition.

Proposition 3.3. Let \(((X, \mathcal{Y}), (U, \mathcal{Y}))\) be a Hovey twin cotorsion pair (which means \(\mathcal{S}_L = \mathcal{S}_R\)). Then \(\mathcal{S} := \mathcal{S}_L(= \mathcal{S}_R)\) is a thick subcategory in which \((X, \mathcal{Y})\) is a cotorsion pair.

Proof. We only need to show that \(\mathcal{S}\) is a thick subcategory, then by definition we know that \((X, \mathcal{Y})\) is a cotorsion pair in \(\mathcal{S}\).

(1) By Lemma 2.11, \(\mathcal{S}\) is closed under direct summands.
  Let \(A \xrightarrow{z} B \xrightarrow{y} C \rightarrow \) be an \(\mathcal{E}\)-triangle.
(2) If \(A, C \in \mathcal{S}\), we show that \(B \in \mathcal{S}\). \(A\) admits an \(\mathcal{E}\)-triangle \(A \rightarrow Y_A \rightarrow X_A \rightarrow \) with \(Y_A \in \mathcal{Y}\) and \(X_A \in \mathcal{X}\), and \(C\) admits an \(\mathcal{E}\)-triangle \(Y_C \rightarrow X_C \rightarrow C \rightarrow \) with \(Y_C \in \mathcal{Y}\) and \(X_C \in \mathcal{X}\), then we have
the following commutative diagrams

Since \(((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y}))\) is a Hovey twin cotorsion pair, we always have \(\mathcal{X} \cap \mathcal{V} = \mathcal{W} = \mathcal{U} \cap \mathcal{Y}\) by Remark 2.16, then \(Y_A\) admits an \(E\)-triangle \(Y_1 \rightarrow W_1 \rightarrow Y_A \rightarrow \) with \(Y_1 \in \mathcal{Y}\) and \(W_1 \in \mathcal{W}\) by the proof of Lemma 2.10. We have the following commutative diagram

with \(Y_2 \in \mathcal{Y}\), hence \(D \in S\). Then \(D\) admits an \(E\)-triangle \(D \rightarrow Y_D \rightarrow X_D \rightarrow \) with \(Y_D \in \mathcal{Y}\) and \(X_D \in \mathcal{X}\). We have the following commutative diagrams:

with \(X' \in \mathcal{X}\), hence \(B \in S\).

(3) If \(A, B \in S\), we show that \(C \in S\). In the previous argument, we know that \(C\) admits a commutative diagram

\[
\begin{array}{c}
A \rightarrow B \rightarrow C \\
\downarrow \quad \downarrow \quad \downarrow \\
Y_A \rightarrow D \rightarrow C \\
\downarrow \quad \downarrow \\
X_A = X_A
\end{array}
\]
Since \( B \in \mathcal{S} \) and \( \mathcal{S} \) is closed under extensions, we get that \( D \in \mathcal{S} \). Then \( D \) admits an \( \mathcal{E} \)-triangle \( Y^D \to X^D \to D \to \) with \( Y^D \in \mathcal{Y} \) and \( X^D \in \mathcal{X} \). We get the following commutative diagram

\[
\begin{array}{ccc}
Y^D & \longrightarrow & Y^D \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & X^D \longrightarrow C \to \\
\downarrow & & \downarrow \\
Y_A & \longrightarrow & D \longrightarrow C \to \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow
\end{array}
\]

with \( Y' \in \mathcal{Y} \), hence \( C \in \mathcal{S} \).

(4) Dually we can show that \( B, C \in \mathcal{S} \) implies \( A \in \mathcal{S} \).

Hence \( \mathcal{S} \) is a thick subcategory. \( \square \)

The following proposition gives a sufficient condition when \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) becomes a Hovey twin cotorsion pair.

**Proposition 3.4.** Let \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) be a twin cotorsion pair. If

(a) \( \mathcal{X} \cap \mathcal{V} = \mathcal{U} \cap \mathcal{Y} \);

(b) \( (\mathcal{X}, \mathcal{V}) \) and \( (\mathcal{U}, \mathcal{Y}) \) are hereditary cotorsion pairs,

then \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) is a Hovey twin cotorsion pair. Moreover, if \( \mathcal{M} \) is an extension closed subcategory in which \( (\mathcal{X}, \mathcal{Y}) \) is a cotorsion pair, then \( \mathcal{M} = \mathcal{S}_L = \mathcal{S}_R \).

**Proof.** Let \( \mathcal{S} = \mathcal{S}_L \cap \mathcal{S}_R \). By the condition (a), we know that \( \mathcal{X}, \mathcal{Y} \subseteq \mathcal{S} \). Then by Lemma 2.15 and Remark 2.16, we only need to show that \( \mathcal{S} \) is a thick subcategory.

(1) By Lemma 2.11, \( \mathcal{S} \) is closed under direct summands.

Let \( A \xrightarrow{f} B \xrightarrow{g} C \to \) be an \( \mathcal{E} \)-triangle.

(2) If \( A, C \in \mathcal{S} \), we show that \( B \in \mathcal{S} \). \( A \) admits an \( \mathcal{E} \)-triangle \( A \to Y_A \to X_{A} \to \) with \( Y_A \in \mathcal{Y} \) and \( X_{A} \in \mathcal{X} \), and \( C \) admits an \( \mathcal{E} \)-triangle \( Y^C \to X^C \to C \to \) with \( Y^C \in \mathcal{Y} \) and \( X^C \in \mathcal{X} \), then we have the following commutative diagrams

\[
\begin{array}{ccc}
A & \longrightarrow & B \longrightarrow C \to \\
\downarrow & & \downarrow \\
Y_A & \longrightarrow & D \longrightarrow C \to \\
\downarrow & & \downarrow \\
X_A & \longrightarrow & X_A \to \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow
\end{array}
\quad
\begin{array}{ccc}
Y^C & \longrightarrow & Y^C \\
\downarrow & & \downarrow \\
Y_A \oplus X^C & \longrightarrow & Y_A \oplus X^C \longrightarrow X^C \to \\
\downarrow & & \downarrow \\
Y_A \oplus X^C & \longrightarrow & D \longrightarrow C \to \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow
\end{array}
\]

Since by the proof of Lemma 2.10, \( Y_A \) admits an \( \mathcal{E} \)-triangle \( Y_1 \to W_1 \to Y_A \to \) with \( Y_1 \in \mathcal{Y} \) and \( W_1 \in \mathcal{W} \), and \( X^C \) admits an \( \mathcal{E} \)-triangle \( X^C \to W_2 \to X_2 \to \) with \( X_2 \in \mathcal{X} \) and \( W_2 \in \mathcal{W} \), we have the following commutative diagrams

\[
\begin{array}{ccc}
Y_C & \longrightarrow & Y_A \oplus X^C \longrightarrow D \to \\
\downarrow & & \downarrow \\
Y_C & \longrightarrow & Y_A \oplus W_2 \longrightarrow Y' \to \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_2 \to \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow
\end{array}
\quad
\begin{array}{ccc}
Y_1 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
Y_2 \oplus X^C & \longrightarrow & W_1 \oplus X^C \longrightarrow D \to \\
\downarrow & & \downarrow \\
Y_2 \oplus X^C & \longrightarrow & D \longrightarrow C \to \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow
\end{array}
\]
with $Y_2 \in \mathcal{Y}$. Since $(U, \mathcal{Y})$ is hereditary, by definition we have $Y' \in \mathcal{Y}$. Then we have the following commutative diagrams:

\[
\begin{array}{ccc}
B & \rightarrow & D \\
\downarrow & & \downarrow \\
B & \rightarrow & X_A \\
\end{array}
\quad
\begin{array}{ccc}
Y_2 & \rightarrow & Y_2 \\
\downarrow & & \downarrow \\
X'' & \rightarrow & X_A \\
\end{array}
\]

with $X' \in \mathcal{X}$. Since $(X, Y)$ is hereditary, by definition we have $X'' \in \mathcal{X}$. Hence $B \in \mathcal{S}$ and $\mathcal{S}$ is closed under extensions.

(3) If $A, B \in \mathcal{S}$, we show that $C \in \mathcal{S}$. In the previous argument, we know that $C$ admits a commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
Y_A & \rightarrow & D \\
\downarrow & & \downarrow \\
X_A & \rightarrow & X_A \\
\end{array}
\quad
\begin{array}{ccc}
Y_2 & \rightarrow & Y_2 \\
\downarrow & & \downarrow \\
X'' & \rightarrow & X_A \\
\end{array}
\]

Since $B \in \mathcal{S}$, and $\mathcal{S}$ is closed under extensions, we get that $D \in \mathcal{S}$. Then $D$ admits an $\mathcal{E}$-triangle $D \rightarrow Y_D \rightarrow X_D \rightarrow$, with $Y_D \in \mathcal{Y}$ and $X_D \in \mathcal{X}$. We have the following commutative diagram

\[
\begin{array}{ccc}
Y_A & \rightarrow & D \\
\downarrow & & \downarrow \\
Y_A & \rightarrow & Y_D \\
\downarrow & & \downarrow \\
X_D & \rightarrow & X_D \\
\end{array}
\quad
\begin{array}{ccc}
D & \rightarrow & C \\
\downarrow & & \downarrow \\
Y_D & \rightarrow & Y_0 \\
\downarrow & & \downarrow \\
X_D & \rightarrow & X_D \\
\end{array}
\]

Since $(U, \mathcal{Y})$ is hereditary, by definition we have $Y_0 \in \mathcal{Y}$. By [LN, Proposition 1.20], we can choose a morphism $y_0 : Y_D \rightarrow Y_0$ to make an $\mathcal{E}$-triangle $D \rightarrow Y_D \rightarrow X_D \rightarrow$, with $Y_D \in \mathcal{Y}$ and $X_D \in \mathcal{X}$. We have the following commutative diagram
following commutative diagram
\[
\begin{array}{ccc}
Y^S & \longrightarrow & X^S \\
\downarrow & & \downarrow \\
Y^S & \longrightarrow & W_0 \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & X_0 \\
\end{array}
\]

Since \((\mathcal{U}, \mathcal{Y})\) is hereditary, by definition we have \(Y \in \mathcal{Y}\). By the similar argument as above, we can get an \(\mathcal{E}\)-triangle \(X^S \rightarrow S \oplus W_0 \rightarrow Y \rightarrow \).

3.3. A localization of \(B\) realized by \(Z/[W]\). We first establish a functor \(G\) from \(B\) to \(Z/[W]\). For any object \(B\) of \(B\), we fix two \(\mathcal{E}\)-triangles
\[
B \xrightarrow{v_B} V_B \longrightarrow X_B \rightarrow \quad Y^B \xrightarrow{z_B} Z_B \oplus V_B \rightarrow Y \rightarrow 
\]
where \(X_B \in \mathcal{X}\), \(Y^B \in \mathcal{Y}\), \(v_B\) is a minimal left \(V\)-approximation and \(z_B\) is a minimal right \(U\)-approximation. We have \(Z_B \in Z\). Note that by Lemma 3.1, we can take these minimal approximations.

Let \(f : B \rightarrow C\) be any morphism in \(B\). Then we have the following commutative diagrams:
\[
\begin{array}{ccc}
B & \xrightarrow{v_B} & V_B \longrightarrow X_B \\
\downarrow & & \downarrow \\
C & \xrightarrow{v_C} & V_C \longrightarrow X_C \\
\end{array} \quad \begin{array}{ccc}
Y^B & \xrightarrow{z_B} & Z_B \oplus V_B \\
\downarrow & & \downarrow \\
Y^C & \xrightarrow{z_C} & Z_C \oplus V_C \\
\end{array}
\]

Lemma 3.5. If we have \(v'_f : V_B \rightarrow V_C\) and \(z'_f : Z_B \rightarrow Z_C\) such that the following diagrams commute:
\[
\begin{array}{ccc}
B & \xrightarrow{v_B} & V_B \longrightarrow X_B \\
\downarrow & & \downarrow \\
C & \xrightarrow{v_C} & V_C \longrightarrow X_C \\
\end{array} \quad \begin{array}{ccc}
Y^B & \xrightarrow{z_B} & Z_B \oplus V_B \\
\downarrow & & \downarrow \\
Y^C & \xrightarrow{z_C} & Z_C \oplus V_C \\
\end{array}
\]
then \(\overline{v}_f = \overline{v'}_f\) and \(\overline{z}_f = \overline{z'}_f\).

Proof. If we have the commutative diagrams above, then \(v_f - v'_f : V_B \rightarrow V_C\) factors through \(X_B\), hence by Lemma 2.13 and Lemma 2.10 factors through \(W\). Then \(z_C \circ (z_f - z'_f) = w_2 \circ w_1\) with \(w_1 : Z_B \rightarrow W\) and \(w_2 : W \rightarrow V_C\). Then there is a morphism \(w_3 : W \rightarrow Z_C\) such that \(w_2 = z_C \circ w_3\). Thus \(z_C \circ ((z_f - z'_f) - w_3 w_1) = 0\), then \((z_f - z'_f) - w_3 w_1\) factors through \(Y^C\), hence by Lemma 2.10 factors through \(W\). Then \(\overline{v}_f = \overline{v'}_f\). \(\square\)

Now we can define a functor \(G\) from \(B\) to \(Z/[W]\), acting as follows:
\[
G(B) = Z_B, \quad G(f) = \overline{v}_f.
\]
Remark 3.6. By the construction of $G$, we have $G(v_B) = G(z_B) = T_{Z_B}$ and $G(S) = 0$. Moreover, $G$ is additive.

Definition 3.7. Denote by $\mathcal{R}$ the following class of morphisms:

$$\{ f : B \to C \mid \exists \text{E-triangle } B \xrightarrow{f} C \xrightarrow{S} S \to \text{ with } S \in S \}.$$

Denote by $B/S$ the Gabriel-Zisman localization of $B$ with respect to $\mathcal{R}$ (see [GZ, Section I.2] or [K2, Section 2.2] for more details of such localization).

In this localization, any morphism $f \in \mathcal{R}$ becomes invertible. For any morphism $g$, we denote its image in $B/S$ by $\bar{g}$.

We will show the following theorem.

Theorem 3.8. Assume $S$ is closed under taking cones, which means in any E-triangle $A \to B \to C \to S \to S$, $A, B \in S$ implies that $C \in S$. Then the Gabriel-Zisman localization $B/S$ is equivalent to $Z/\mathcal{W}$. We first show an important property of functor $G$.

Proposition 3.9. $G(f)$ is an isomorphism for any morphism $f : B \to C$ in $\mathcal{R}$.

Proof. Morphism $f$ admits an E-triangle $B \xrightarrow{f} C \xrightarrow{S} S \to$ with $S \in S$. Then we have the following commutative diagram

```
\begin{array}{c}
B \xrightarrow{v_B} V_B \xrightarrow{x_B} X_B \xrightarrow{S} \\
\downarrow f \downarrow \uparrow d_1 \downarrow \uparrow d_2 \\
C \xrightarrow{e'} C' \xrightarrow{x'} X_B \xrightarrow{S} \\
\downarrow \downarrow \uparrow \uparrow \\
S \xrightarrow{S} S
\end{array}
```

Since $V_C \in \mathcal{V}$, we have $E(X_B, V_C) = 0$, then we get the following commutative diagram

```
\begin{array}{c}
B \xrightarrow{v_B} V_B \xrightarrow{x_B} X_B \xrightarrow{S} \\
\downarrow f \downarrow d_1 \downarrow \uparrow \uparrow \\
C \xrightarrow{e'} C' \xrightarrow{x'} X_B \xrightarrow{S} \\
\downarrow \downarrow \uparrow \uparrow \\
V_C \xrightarrow{(d_2 \; \; \; d_1)} V_C \oplus X_B \xrightarrow{X_B} X_B \xrightarrow{S} \\
\downarrow \downarrow \downarrow \downarrow \\
X_C \xrightarrow{X_C}
\end{array}
```

such that $d_2d_1 = \overline{f}$ by the proof of Lemma 3.5. Then we have a commutative diagram

```
\begin{array}{c}
V_B \xrightarrow{d_1} C' \xrightarrow{(d_2 \; \; \; d_1)} S \xrightarrow{S} \\
\downarrow \downarrow \downarrow \\
V_B \xrightarrow{(d_2 \; \; \; d_1)} V_C \oplus X_B \xrightarrow{S_1} S_1 \xrightarrow{S_1} \\
\downarrow \downarrow \downarrow \\
X_C \xrightarrow{X_C}
\end{array}
```
with $S_1 \in \mathcal{S}$. Since $X_B$ admits an $E$-triangle $X_B \xrightarrow{w} W \rightarrow X_1 \rightarrow$ where $W \in \mathcal{W}$ and $X_1 \in \mathcal{X}$, we have the following commutative diagram

\[
\begin{array}{c}
\Vdash \xrightarrow{(d_{2d_1} z_B \oplus)} \xrightarrow{(z_B \oplus X_B)} S_1 \xrightarrow{\sim} \\
V_B \xrightarrow{\sim} \xrightarrow{(d_{2d_1} \oplus)} \xrightarrow{(z_B \oplus X_B)} S_2 \xrightarrow{\sim} \\
X_1 \xrightarrow{\sim} X_1
\end{array}
\]

with $S_2 \in \mathcal{S}$. $S_2$ admits an $E$-triangle $Y_2 \rightarrow X_2 \rightarrow S_2 \rightarrow \cdots$ with $Y_2 \in \mathcal{Y}$ and $X_2 \in \mathcal{X}$, then we have the following commutative diagram

\[
\begin{array}{c}
\xrightarrow{Y_2 \oplus X_2 \rightarrow} S_1 \xrightarrow{\sim} \\
V_B \xrightarrow{\sim} \xrightarrow{(d_{2d_1} \oplus)} \xrightarrow{(z_B \oplus X_B)} S_2 \xrightarrow{\sim} .
\end{array}
\]

Since $Y_2, V_C \oplus W \in \mathcal{V}$, we get that $X_2 \in \mathcal{X} \cap \mathcal{V} = \mathcal{W}$. Now we obtain the following commutative diagrams

\[
\begin{array}{c}
Y_B \xrightarrow{Y_B} \\
Y_3 \xrightarrow{Z_B \oplus X_2 \rightarrow} V_C \oplus W \rightarrow \cdots \\
\quad \xrightarrow{(z_B \oplus X_2 \oplus)} \xrightarrow{(Y_C \oplus Y_C)} Z_C \oplus W \rightarrow \cdots \\
\end{array}
\]

with $Y_3 \in \mathcal{P}$. Since $E(Z_C \oplus W, Y_3) = 0$, we have the following commutative diagram:

\[
\begin{array}{c}
Y_3 \xrightarrow{Z_B \oplus X_2 \oplus Y_C \rightarrow} Z_C \oplus W \rightarrow \cdots \\
\quad \xrightarrow{(z_B \oplus X_2 \oplus)} \xrightarrow{(Y_C \oplus Y_C)} Z_C \oplus W \rightarrow \cdots \\
Y_3 \xrightarrow{(0_0 0)} \xrightarrow{(Z_C \oplus W \oplus Y_3 \rightarrow)} Z_C \oplus W \rightarrow \cdots .
\end{array}
\]

By Lemma 2.10, $\text{Hom}_\mathcal{S}(Z_B, Y_3) = 0$, hence $\mathfrak{z}$ is a section and $Z_B$ is a direct summand of $Z_C$ in $\mathcal{B}$. On the other hand, if an indecomposable object is a direct summand of both $Y_C$ and $Z_C$, then it has to lie in $\mathcal{W}$. Hence $Y_C$ is a direct summand of $Y_3$ in $\mathcal{B}$. Since $Z_B \oplus Y_C \simeq Z_C \oplus Y_3$ in $\mathcal{B}$, we obtain that $\mathfrak{z}$ is an isomorphism. Since $d_{2d_1} z_B = w f_{z_B} = \varpi z_B$, by the proof of Lemma 3.5, we get that $\mathfrak{z} = \varpi f$. Hence $\varpi f$ is an isomorphism.
By Proposition 3.9 and the universal property of the localization functor $L_R : B \to B/S$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
B & \xrightarrow{G} & Z/[W], \\
\downarrow{L_R} & & \downarrow{H} \\
B/S & \xrightarrow{} & 
\end{array}
$$

The following lemma is useful. The proof is an analogue of [BM, Lemma 3.5], so we omit it.

**Lemma 3.10.**

1. Let $X$ be any object in $B$ and $S \in S$. Then $X \oplus S \xrightarrow{(1_x, 0)} X$ is invertible in $B/S$, and its inverse is $X \xrightarrow{(1_x, 0)} X \oplus S$.

2. Let $B, C$ be any objects in $B$ and $f, f' \in \text{Hom}_B(B, C)$. If $\overline{f} = 0$, then $f + f' = f$ in $B/S$.

In the rest of this subsection, we assume that $S$ is closed under taking cones.

**Corollary 3.11.** For any $\mathbb{E}$-triangle

$$
\begin{array}{ccc}
Y & \xrightarrow{u} & U \xrightarrow{a} A \xrightarrow{} \\
\downarrow{w} & & \downarrow{v} \\
Y & \xrightarrow{s} & W \xrightarrow{\text{--}} S \xrightarrow{\text{--}} \\
\downarrow{w} & & \downarrow{v} \\
U' & \xrightarrow{} & U' \\
\downarrow{w} & & \downarrow{v} \\
\end{array}
$$

with $Y \in \mathcal{Y}$, $U \in U$, $u$ is invertible in $B/S$.

**Proof.** Since $U$ admits an $\mathbb{E}$-triangle $U \xrightarrow{\text{--}} W \xrightarrow{\text{--}} U' \xrightarrow{\text{--}}$ with $W \in W$ and $U' \in U$, we have the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{u} & U \xrightarrow{a} A \xrightarrow{} \\
\downarrow{w} & & \downarrow{v} \\
Y & \xrightarrow{s} & W \xrightarrow{\text{--}} S \xrightarrow{\text{--}} \\
\downarrow{w} & & \downarrow{v} \\
U' & \xrightarrow{} & U' \\
\downarrow{w} & & \downarrow{v} \\
\end{array}
$$

with $S \in S$, since $S$ is closed under taking cones. By [LN, Proposition 1.20], we can choose a morphism $s' : W \to S$ to make an $\mathbb{E}$-triangle $U \xrightarrow{\text{--}} W \xrightarrow{(a, s')} S \xrightarrow{\text{--}}$. Hence $U \xrightarrow{(-u, w)} A \oplus W$ is invertible in $B/S$. By Lemma 3.10, $-u = (1) \circ (-u) : U \to A$ is invertible in $B/S$, so is $u$. \[\square\]

**Remark 3.12.** Let $f : B \to C$ be any morphism in $\mathcal{R}$. In the following commutative diagrams

$$
\begin{array}{ccc}
B & \xrightarrow{\nu_B} & V_B \xrightarrow{X_B} \xrightarrow{} \\
\downarrow{f} & & \downarrow{v_f} \\
C & \xrightarrow{\nu_C} & V_C \xrightarrow{X_C} \xrightarrow{} \\
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{\nu_B} & Z_B \xrightarrow{z_B} \xrightarrow{} \\
\downarrow{f} & & \downarrow{v_f} \\
Y & \xrightarrow{\nu_C} & Z_C \xrightarrow{z_C} \xrightarrow{} \\
\end{array}
$$

$v_f$ is invertible. Moreover, since $z_B$ and $z_C$ are invertible by Corollary 3.11, $z_f$ is also invertible in $B/S$.

**Lemma 3.13.** Let $f : B \to C$ be any morphism in $\mathcal{R}$. Then we have the following commutative diagram in $B/S$

$$
\begin{array}{ccc}
C & \xrightarrow{\nu_C} & V_C \xrightarrow{z_C^{-1}} \xrightarrow{} \\
\downarrow{L^{-1}} & & \downarrow{L^{-1}} \\
B & \xrightarrow{\nu_B} & V_B \xrightarrow{z_B^{-1}} \xrightarrow{} \\
\end{array}
\quad
\begin{array}{ccc}
Z_B & \xrightarrow{z_B} & \xrightarrow{} \\
\downarrow{z} & & \downarrow{z} \\
Z_C & \xrightarrow{z_C} & \xrightarrow{} \\
\end{array}
$$

where $z : Z_C \to Z_B$ is a morphism in $Z$ such that $\overline{z} = G(f)^{-1}$. 

Proof. By Proposition 3.9, \( G(f) = \varpi_f \) is an isomorphism, let \( \varpi = G(f)^{-1} \). Then \( 1_{Z_C} = z_f z \) factors through \( W \). So by Lemma 3.10, we have \( 1_{Z_C} = z_f z \) in \( B/S \). By the following commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\nu_f} & V_B \\
\downarrow{f} & & \downarrow{v_f} \\
C & \xrightarrow{v_C} & Z_C
\end{array}
\]

we have \( v_f z_B z = z_C z_f z \). By applying \( L_R \) to this equation, we get \( v_f z_B z = z_C z_f z \), then \( z_f z^{-1} = z_B z_f z^{-1} \). Hence we obtain the desired commutative diagram.

We now give the proof of Theorem 3.8.

Proof of Theorem 3.8. We show that \( H \) is an equivalence. Since \( G|_Z \) is identical on the objects, we know that \( H \) is dense.

We show that \( H \) is faithful. Let \( \alpha : B \to C \) be any morphism in \( B/S \). It has the form \( B \xrightarrow{\beta_0} D_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{n-1}} D_n \xrightarrow{\beta_n} C \) where \( \beta_i \) is a morphism \( f_\alpha \) or a morphism \( g_\beta^{-1} \) with \( g_\beta \in R \). We have a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\beta_0} & D_1 \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
Z_1 & \xrightarrow{\zeta} & Z_1 \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
\cdots & & \cdots \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
D_n & \xrightarrow{\zeta} & D_n \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
Z_C & \xrightarrow{\zeta} & Z_C
\end{array}
\]

where \( Z_i \in Z \) and \( z_1 \) are morphisms in \( Z \) by Lemma 3.13. Denote \( z_n z_{n-1} \cdots z_1 z_0 \) by \( \zeta \), we have

\[
\alpha = \nu_C^{-1} \zeta C^{-1} \zeta C^{-1} \nu_B.
\]

If there exists a morphism \( \alpha' : B \to C \) in \( B/S \) such that \( H(\alpha) = H(\alpha') \), then we also have \( \alpha' = \nu_C^{-1} \zeta C^{-1} \zeta C^{-1} \nu_B \) with some \( \zeta' \in \text{Hom}_B(Z_B, Z_C) \). Since \( H(\nu_C) = H(\nu_B) = T_Z \nu_C \) and \( H(\nu_B) = H(\nu_B) = T_Z \nu_B \) by Remark 3.6, we can get that \( \zeta = H(\zeta') = H(\zeta') = \zeta' \). Hence \( \zeta - \zeta' \) factors through \( W \), which implies \( \zeta = \zeta' \). Thus \( \alpha = \alpha' \).

Finally we show that \( H \) is full. Let \( \gamma : H(B) \to H(C) \) be any morphism. By a similar argument as above, we can get that \( \gamma = \varpi \) where \( z \) is a morphism in \( Z \). Since we have the following commutative diagram in \( B/S \):

\[
\begin{array}{ccc}
B & \xrightarrow{\alpha} & V_B \\
\downarrow{\alpha} & \downarrow{\nu_C} & \downarrow{\nu_C} \\
C & \xrightarrow{\nu_C} & Z_C
\end{array}
\]

we have \( H(\alpha) = H(\nu_C)^{-1} H(\nu_B) H(\nu_B)^{-1} H(\nu_B) \), then \( H(\alpha) = H(\varpi) = \varpi = \gamma \), hence \( H \) is full.

Since we have the following commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & B \\
\downarrow{\pi} & \downarrow{L_H} & \downarrow{F} \\
Z/W & \xrightarrow{\pi} & B/S
\end{array}
\]

where \( i \) is the embedding functor and \( \pi \) is the canonical quotient functor, we can get the following proposition.

Proposition 3.14. \( F \) is a quasi-inverse of \( H \).

Proof. By the proof of Theorem 3.8 we can obtain \( HF = \text{Id}_{Z/[W]} \). Let \( \alpha : B \to C \) be any morphism in \( B/S \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
F H(B) & \xrightarrow{\nu_B^{-1}} & B \\
\downarrow{F H(\alpha)} & & \downarrow{\alpha} \\
F H(C) & \xrightarrow{\nu_C^{-1}} & C
\end{array}
\]
which implies $FH \cong \text{Id}_{B/S}$.

**Remark 3.15.** Assume that $\mathcal{B}$ has enough injectives. Let $\mathcal{M}$ be a contravariantly finite subcategory which contains all the injective objects and is closed under extensions, direct summands and taking cones. Then $(\mathcal{M}, \mathcal{T})$ is a cotorsion pair in $\mathcal{M}$. Moreover, we have a twin cotorsion pair $((\mathcal{M}, \mathcal{M}^{\perp 1}), (\mathcal{B}, \mathcal{T}))$ in $\mathcal{B}$. By Theorem 3.8, $\mathcal{M}^{\perp 1}/[\mathcal{T}] \simeq \mathcal{B}/\mathcal{M}$.

Dually, assume that $\mathcal{B}$ has enough projectives. Let $\mathcal{N}$ be a covariantly finite subcategory which contains all the projective objects and is closed under extensions, direct summands and taking cocones. Then $(\mathcal{P}, \mathcal{N})$ is a cotorsion pair in $\mathcal{N}$. Moreover, we have a twin cotorsion pair $((\mathcal{P}, \mathcal{B}), (\mathcal{N}^{\perp 1}, \mathcal{N}))$ in $\mathcal{B}$ and $\mathcal{N}^{\perp 1}/[\mathcal{P}] \simeq \mathcal{B}/\mathcal{N}$.

In particular, if $\mathcal{B}$ is a triangulated category and $\mathcal{M}$ is a contravariantly (resp. covariantly) finite thick subcategory, then the Verdier quotient $\mathcal{B}/\mathcal{M}$ is a triangle equivalent to $\mathcal{M}^{\perp 1}$ (resp. $\mathcal{N}^{\perp 1}$), which is a thick subcategory of $\mathcal{B}$.

**Remark 3.16.** There are many examples of extriangulated subcategories which have enough injectives. For example, any right triangulated category with a right semi-equivalence is an extriangulated category with enough injectives, see [T, Corollary 3.13].

### 3.4. Other localizations

In this subsection, we assume that $\mathcal{S}$ is closed under taking cones. We first show a useful proposition.

**Proposition 3.17.** For any $\mathcal{E}$-triangle $Y \to A \xrightarrow{f} B \to \to$ with $Y \in \mathcal{Y}$, $G(f)$ is an isomorphism.

**Proof.** Since $A$ admits an $\mathcal{E}$-triangle $A \to Y_A \to U_A \to \to$ with $Y_A \in \mathcal{Y}$ and $U_A \in \mathcal{U}$, we have the following commutative diagram

\[
\begin{array}{c}
Y & \xrightarrow{f} & A & \xrightarrow{B} & B \to \to \\
\downarrow & & \downarrow & & \\
Y & \xrightarrow{Y_A} & S & \xrightarrow{S} & S \to \to \\
\downarrow & & \downarrow & & \\
U_A & = & U_A & = & U_A \\
\end{array}
\]

Since $\mathcal{S}$ is closed under taking cones, we have $S \in \mathcal{S}$. By [LN, Proposition 1.20], there is an $\mathcal{E}$-triangle $A \xrightarrow{(\sim f)} B \oplus Y_A \to S \to$. By Proposition 3.9, we get that $G(f)$ is an isomorphism. \hfill $\square$

Denote by $\mathcal{W}_1$ the following class of morphisms:

\[
\{ f : A \to B \mid \exists \text{ $\mathcal{E}$-triangle } A \xrightarrow{f} B \xrightarrow{X} X \to \text{ with } X \in \mathcal{X} \}.
\]

Denote by $\mathcal{W}_2$ the following class of morphisms:

\[
\{ g : C \to D \mid \exists \text{ $\mathcal{E}$-triangle } Y \xrightarrow{C} C \xrightarrow{D} D \to \text{ with } Y \in \mathcal{Y} \}.
\]

Denote by $\mathcal{W}$ the following class of morphisms:

\[
\{ h = g \circ f \mid f \in \mathcal{W}_1 \text{ and } g \in \mathcal{W}_2 \}.
\]

Denote by $\mathcal{B}/[\mathcal{W}^{-1}]$ the Gabriel-Zisman localization of $\mathcal{B}$ with respect to $\mathcal{W}$, and by $L : \mathcal{B} \to \mathcal{B}/[\mathcal{W}^{-1}]$ the localization functor.

**Remark 3.18.** If $\mathcal{S}$ is a thick subcategory, then by Lemma 2.15 we have $\mathcal{S}_R = \mathcal{S}_L$ and $((\mathcal{X}, \mathcal{Y}), (\mathcal{U}, \mathcal{Y}))$ is a Hovey twin cotorsion pair. $\mathcal{B}/[\mathcal{W}^{-1}]$ becomes the localization discussed in [NP, Section 5].

Since by Propositions 3.9 and 3.17, morphisms in $\mathcal{W}_1$ and $\mathcal{W}_2$ become invertible in $\mathcal{B}/\mathcal{S}$, there exists a unique functor $F_1 : \mathcal{B}/[\mathcal{W}^{-1}] \to \mathcal{B}/\mathcal{S}$ such that $F_1 L = L_R$. On the other hand, we have the following proposition.

**Proposition 3.19.** Morphisms in $\mathcal{R}$ are invertible in $\mathcal{B}/[\mathcal{W}^{-1}]$. 

Proof. Let \( f : A \to B \) be a morphism in \( \mathcal{R} \). Then it admits an \( \mathbb{E} \)-triangle \( A \xrightarrow{f} B \to S \to \) with \( S \in \mathcal{S} \). Since \( \mathcal{S} \) admits an \( \mathbb{E} \)-triangle \( Y \to X \to S \to \) with \( Y \in \mathcal{Y} \) and \( X \in \mathcal{X} \), we have the following commutative diagram:

\[
\begin{array}{ccc}
Y & \to & Y \\
\downarrow & & \downarrow \\
A & \xrightarrow{d_1} & D & \xrightarrow{d_2} & X \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & S
\end{array}
\]

which implies \( f \in \mathcal{W} \). Hence \( \mathcal{R} \subseteq \mathcal{W} \) and morphisms in \( \mathcal{R} \) are invertible in \( \mathcal{B}[\mathcal{W}^{-1}] \).

By this proposition, there exists a unique functor \( F_2 : \mathcal{B}/\mathcal{S} \to \mathcal{B}[\mathcal{W}^{-1}] \) such that \( F_2 L_\mathcal{R} = L \). Hence \( L = F_2 F_1 L \) and \( L_\mathcal{R} = F_1 F_2 L_\mathcal{R} \). By the universal property of the localization functors, we have \( F_2 F_1 = \text{Id}_{[\mathcal{W}^{-1}]} \) and \( F_2 F_1 = \text{Id}_{[\mathcal{S}]} \). This means \( \mathcal{B}/\mathcal{S} \) is equivalent to \( \mathcal{B}[\mathcal{W}^{-1}] \).

We claim that when \( \mathcal{S} \) is closed under taking cones, \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) is a generalized Hovey twin cotorsion pair in the sense of \([O, \text{Definition 3.9}]\). By \([O, \text{Definition 3.9 and Theorem 3.10}]\), we only need to check the following fact:

For an \( \mathbb{E} \)-triangle \( Y \to U \xrightarrow{\Delta} B \to \) with \( Y \in \mathcal{Y}, U \in \mathcal{U}, G(u) \) is an isomorphism in \( \mathbb{Z}/[\mathcal{W}] \).

This is just followed by Proposition 3.17.

Denote by \( \mathcal{V}_1 \) the following class of morphisms:

\[ \{ f : A \to V \mid \exists \ \text{\( \mathbb{E} \)-triangle} \ A \xrightarrow{f} V \to X \to \ \text{with} \ V \in \mathcal{V}, X \in \mathcal{X} \} \]

Denote by \( \mathcal{V}_2 \) the following class of morphisms:

\[ \{ g : U \to B \mid \exists \ \text{\( \mathbb{E} \)-triangle} \ Y \xrightarrow{g} U \to B \to \ \text{with} \ Y \in \mathcal{Y}, U \in \mathcal{U} \} \]

Denote by \( \mathcal{V} \) the following class of morphisms:

\[ \{ h = g \circ f \mid f \in \mathcal{V}_1 \text{ and } g \in \mathcal{V}_2 \} \]

Denote by \( \mathcal{B}[\mathcal{V}^{-1}] \) the Gabriel-Zisman localization of \( \mathcal{B} \) with respect to \( \mathcal{V} \). Ogawa \([O, \text{Theorem 3.10}]\) showed that there is an equivalence: \( \Phi : \mathbb{Z}/[\mathcal{W}] \to \mathcal{B}[\mathcal{V}^{-1}] \). Then

\[ \Phi \circ H : \mathcal{B}/\mathcal{S} \to \mathcal{B}[\mathcal{V}^{-1}] \quad \text{and} \quad \Phi \circ H \circ F_1 : \mathcal{B}[\mathcal{W}^{-1}] \to \mathcal{B}[\mathcal{V}^{-1}] \]

are equivalences between localizations.

In summary, we have the following proposition.

**Proposition 3.20.** Let \( \mathcal{S} \) be an extension closed subcategory which is also closed under taking cones. Let \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) be a twin cotorsion pair such that \( (\mathcal{X}, \mathcal{Y}) \) is a cotorsion pair in \( \mathcal{S} \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{B}[\mathcal{W}^{-1}] & \xrightarrow{F_1} & \mathcal{B} \\
\downarrow \Phi & & \downarrow L_\mathcal{R} \circ G \\
\mathbb{Z}/[\mathcal{W}] & \xrightarrow{\Delta} & \mathcal{B}[\mathcal{V}^{-1}]
\end{array}
\]

**Remark 3.21.** According to \([O, \text{Theorem 3.14}]\), if \( ((\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})) \) is a Hovey twin cotorsion pair, then we always have an equivalence between \( \mathcal{B}[\mathcal{W}^{-1}] \) and \( \mathcal{B}[\mathcal{V}^{-1}] \). Note that in general we do not have such equivalence.

In the following example, the twin cotorsion pair is not Hovey.
Example 3.22. Let $Q: 1 \to 2 \to 3 \to 4$ be the quiver of type $A_4$ and $B := D^b(kQ)$ the bounded derived category of $kQ$ whose Auslander-Reiten quiver is the following:

\[ \cdots \]
\[ \cdots \]
\[ \cdots \]

Let $\mathcal{X}$ be the subcategory whose indecomposable objects are marked by $\bullet$ in the diagram. Let $\mathcal{Y}$ be the subcategory whose indecomposable objects are marked by $\heartsuit$ in the diagram ($\heartsuit$ will continue to appear on the right side of the diagram):

\[ \cdots \]
\[ \cdots \]
\[ \cdots \]

Let $\mathcal{S}$ be the smallest full subcategory of $B$ that contains both $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathcal{S}$ is an extension closed subcategory such that $\mathcal{S}[1] \subseteq \mathcal{S}$ (hence $\mathcal{S}$ is closed under taking cones), and $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\mathcal{S}$. $\mathcal{Y} = \mathcal{X}^{\perp_1}$ is the subcategory whose indecomposable objects are marked by $\spadesuit$ in the diagram: ($\spadesuit$ will continue to appear on the both sides of the diagram):

\[ \cdots \]
\[ \cdots \]
\[ \cdots \]

$\mathcal{U} = \perp \mathcal{Y}$ is the subcategory whose indecomposable objects are marked by $\heartsuit$ in the diagram: ($\heartsuit$ will continue to appear on the left side of the diagram):

\[ \cdots \]
\[ \cdots \]
\[ \cdots \]

The indecomposable objects in $Z/[W]$ are marked by $\clubsuit$ in the diagram: ($\clubsuit$ will continue to appear on the left side of the diagram):

\[ \cdots \]
\[ \cdots \]
\[ \cdots \]

Note that the indecomposable objects marked by $\star$ in the diagram become zero objects in $B/\mathcal{S}$. The twin cotorsion pair in this example is NOT a Hovey twin cotorsion pair, since the objects marked by $\star$ lie in $\mathcal{S}_R$, but do not lie in $\mathcal{S}_L$.

3.5. Some observations. Observe that we replace the assumption “$\mathcal{S}$ is closed under taking cones” by “$(\mathcal{U}, \mathcal{Y})$ is a hereditary cotorsion pair”, all the results still hold. In fact, we have the following proposition.

Proposition 3.23. If $(\mathcal{U}, \mathcal{Y})$ is a hereditary cotorsion pair, then $\mathcal{S}$ is closed under taking cones. Moreover, $\mathcal{S} = \mathcal{S}_L$. 
Proof. Let $S_1 \rightarrow S_2 \rightarrow S \rightarrow$ be an $E$-triangle with $S_1, S_2 \in S$. Since $S_1$ admits an $E$-triangle $S_1 \rightarrow Y_1 \rightarrow X_1$ with $Y_1 \in \mathcal{Y}$ and $X_1 \in \mathcal{X}$, we have the following commutative diagram

\[
\begin{array}{ccc}
S_1 & \rightarrow & S_2 \\
\downarrow & & \downarrow \\
Y_1 & \rightarrow & S_3 \\
\downarrow & & \downarrow \\
X_1 & \rightarrow & X_1
\end{array}
\]

with $S_3 \in S$. Since $S_3$ admits an $E$-triangle $S_3 \rightarrow Y_3 \rightarrow X_3$ with $Y_3 \in \mathcal{Y}$ and $X_3 \in \mathcal{X}$, we have the following commutative diagram

\[
\begin{array}{ccc}
Y_1 & \rightarrow & S_3 \\
\downarrow & & \downarrow \\
Y_1 & \rightarrow & Y_3 \\
\downarrow & & \downarrow \\
X_1 & \rightarrow & X_3
\end{array}
\]

with $Y \in \mathcal{Y}$. Then we can get an $E$-triangle $S_3 \rightarrow S \oplus Y_1 \rightarrow Y \rightarrow$. Since $S$ is closed under extensions and direct summands, we have $S \in S$. By definition, $S \subseteq S_L$. Let $A \in S_L$. Then $A$ admits an $E$-triangle $Y^A \rightarrow X^A \rightarrow A$ with $Y^A \in \mathcal{Y}$ and $X^A \in \mathcal{X}$. Since $X^A$ admits an $E$-triangle $X^A \rightarrow W \rightarrow X$ with $W \in \mathcal{W}$ and $X \in \mathcal{X}$, we have the following commutative diagram

\[
\begin{array}{ccc}
Y^A & \rightarrow & X^A \\
\downarrow & & \downarrow \\
Y^A & \rightarrow & W \\
\downarrow & & \downarrow \\
X & \rightarrow & X
\end{array}
\]

with $Y' \in \mathcal{Y}$. Then we can get an $E$-triangle $X^A \rightarrow A \oplus W \rightarrow Y' \rightarrow$, which implies that $A \in S$. Hence $S = S_L$. \qed

At the end of this section, we drop the assumption that “$S$ is an extension closed subcategory in which $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair”. We give a sufficient condition when $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in an extension closed subcategory which is closed under taking cones.

We first show a useful lemma.

**Lemma 3.24.** Let $\mathcal{C}$ and $\mathcal{D}$ be two subcategories of $\mathcal{B}$ which are closed under extensions. Denote by $\mathcal{C} \ast \mathcal{D}$ the following subcategory:

\[
\{ B \in \mathcal{B} \mid \exists E\text{-triangle } C \rightarrow B \rightarrow D \rightarrow \text{ with } C \in \mathcal{C} \text{ and } D \in \mathcal{D} \}.
\]

If $E(\mathcal{C}, \mathcal{D}) = 0$, then $\mathcal{C} \ast \mathcal{D}$ is closed under extensions.

**Proof.** For convenience, denote $\mathcal{C} \ast \mathcal{D}$ by $\mathcal{K}$. Let $K_1 \rightarrow K \rightarrow K_2 \rightarrow$ be an $E$-triangle with $K_1, K_2 \in \mathcal{K}$. Since $K_2$ admits an $E$-triangle $C_2 \rightarrow K_2 \rightarrow D_2$ with $C_2 \in \mathcal{C}$ and $D_2 \in \mathcal{D}$, we get the following
Proof. Since $M$ is closed under taking cones, then $M$ have $\text{Hom}_D$ closed under taking cones. By Proposition 3.25. Let $\mathcal{B}$ be an triangulated category with shift functor $[1]$. Assume that the twin cotorsion pair $((\mathcal{X}, \mathcal{Y}), (\mathcal{U}, \mathcal{Y}))$ satisfies the following conditions:

1. $\mathcal{X} \cap \mathcal{V} = \mathcal{U} \cap \mathcal{V}$;
2. $(\mathcal{U}, \mathcal{Y})$ is a hereditary cotorsion pair.

Then $\mathcal{M} = S_L$ is the only extension closed subcategory in which $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair. Moreover, $\mathcal{M}$ is closed under taking cones.

Proof. Since $\mathcal{B}$ is triangulated, we have $S_R = \mathcal{X}[-1]*\mathcal{Y}$ and $S_L = \mathcal{X}*\mathcal{Y}[1]$. Since $(\mathcal{U}, \mathcal{Y})$ is hereditary, we have $\text{Hom}_\mathcal{B}(\mathcal{X}, \mathcal{Y}[2]) = 0$. By Lemma 3.24, $S_L$ and $S_R$ are closed under extensions. Hence $\mathcal{M} = S_L \cap S_R$ is an extriangulated subcategory of $\mathcal{B}$ in which $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair. By Proposition 3.23, we have $\mathcal{M} = S_L \subseteq S_R$. Since $\mathcal{Y}[1] \subseteq \mathcal{Y} \subseteq \mathcal{M}$, we have $\mathcal{M}[1] \subseteq S_R[1] = S_L = \mathcal{M}$, which implies that $\mathcal{M}$ is closed under taking cones.

If there is another extension closed subcategory $\mathcal{M}_1$ in which $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair, then we immediately get $\mathcal{M}_1 \subseteq \mathcal{M}$. Any object $M \in \mathcal{M}$ admits an triangle $Y_M \to X_M \to M \to Y_M[1]$ with $Y_M \in \mathcal{Y}$ and $X_M \in \mathcal{X}$, since $Y_M[1] \in \mathcal{Y}$ and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$, we have $M \in \mathcal{M}_1$. Hence $\mathcal{M}_1 = \mathcal{M}$. 

4. Extriangle equivalences

Definition 4.1. Let $(\mathcal{B}_1, \mathcal{E}_1, \sigma_1)$ and $(\mathcal{B}_2, \mathcal{E}_2, \sigma_2)$ be two extriangulated categories and $\sigma : \mathcal{B}_1 \to \mathcal{B}_2$ be an equivalent functor. $\sigma$ is called an extriangle equivalence if the following conditions are satisfied:
(1) \( \sigma \) preserves \( E \)-triangles, which means for any \( E_1 \)-triangle \( A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C_1 \longrightarrow \), there exists an \( E_2 \)-extension \( \delta \in E_2(\sigma(C_1), \sigma(A_1)) \) such that \( \sigma(A_1) \xrightarrow{\sigma(x_1)} \sigma(B_1) \xrightarrow{\sigma(y_1)} \sigma(C_1) \xrightarrow{\delta} \) is an \( E_2 \)-triangle.

(2) For any \( E_2 \)-triangle \( A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C_2 \longrightarrow \) in \( B_2 \), there exists an \( E_1 \)-triangle \( A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C_1 \longrightarrow \) in \( B_1 \) admitting an isomorphism of \( E_2 \)-triangles

\[
\begin{array}{ccc}
A_2 & \xrightarrow{x_2} & B_2 \\
\downarrow \cong & & \downarrow \cong \\
\sigma(A_1) & \xrightarrow{\sigma(x_1)} & \sigma(B_1) \\
\downarrow \cong & & \downarrow \cong \\
\sigma(C_1) & \xrightarrow{\sigma(y_1)} & \sigma(C_2) \\
\end{array}
\]

**Remark 4.2.** If \( B_1 \) and \( B_2 \) are exact categories, then any extriangle equivalence is an exact equivalence (see [B, Definition 5.1]). If \( B_i \) is a triangulated category with shift functor \([1]_{B_i} (i = 1, 2)\), then an extriangle equivalence \( \sigma : B_1 \rightarrow B_2 \) is a “weak” triangle equivalence in the sense that we do not know if there is a natural isomorphism between \( \sigma \circ [1]_{B_1} \) and \([1]_{B_2} \circ \sigma\).

**Remark 4.3.** In [B-TS, Definition 2.32], a functor called \( n \)-exangulated functor was defined between \( n \)-exangulated categories. When \( n = 1 \), such functor is called an extriangulated functor (between extriangulated categories, see also [NOS, Definition 2.11]). According to [B-TS, Theorem 2.33], our definition is slightly weak than [B-TS, Definition 2.32] when \( n = 1 \), since we do not assume the existence of a natural transformation from \( E_1 ((-), (-)) \) to \( E_2(\sigma^\text{op}, \sigma(-)) \).

In this section, we assume that \( B \) has enough enough projectives and enough injectives, and \( S \) is a thick subcategory of \( B \). We show that \( F : \mathbb{Z}/[W] \rightarrow B/S \) is an extriangle equivalence.

Since \((\mathcal{A}, \mathcal{Y}), (\mathcal{U}, \mathcal{Y})\) is a Hovey twin cotorsion pair by Lemma 2.15, by the discussion in Subsection 3.4, we can use the results in [NP, Section 6].

Let \( f : A \rightarrow B \) be any morphism in \( B \). We have the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{y_A} & U_A \\
\downarrow f & & \downarrow u_f \\
B & \xrightarrow{y_B} & U_B \\
\end{array}
\]

where \( U_A, U_B \in \mathcal{U}, y_A, y_B \) are left minimal \( \mathcal{Y} \)-approximations. Since \( B/S \) and \( B[\mathcal{W}^{-1}] \) are equivalent to each other, by the results in [NP, Section 6], we can define an auto-equivalence \([1] \) on \( B/S \) such that \( A[1] = U_A \) and \( f[1] = u_f \). Moreover, the following commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 \\
\downarrow \alpha_1 & & \downarrow b_1 & & \downarrow c_1 \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 \\
\end{array}
\]

induces a commutative diagram in \( B/S \):

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{\alpha_1} & A_1[1] \\
\downarrow \alpha_2 & & \downarrow b_1 & & \downarrow c_1 & & \downarrow \alpha_2 \\
A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{\alpha_2} & A_2[1]. \\
\end{array}
\]

The rows are called the **standard triangles** in \( B/S \). The **distinguished triangles** are the sequences which are isomorphic to the standard triangles. By [NP, Theorem 6.20], \( B/S \) is a triangulated category with distinguished triangles and the shift functor \([1] \). Note that any triangulated category can be viewed as an extriangulated category (see [NP, Proposition 3.22] for details).
Lemma 4.4. Let \( A \xrightarrow{x} B \xrightarrow{y} C \to \) be any \( \mathcal{E} \)-triangle in \( \mathcal{B} \). Then there exists an isomorphism of triangles

\[
\begin{array}{c}
A' \xleftarrow{x'} B \xrightarrow{y'} C' \to A'[1] \\
\cong \quad \cong \quad \cong \\
A \xleftarrow{x} B \xrightarrow{y} C \to A[1]
\end{array}
\]

in \( \mathcal{B}/\mathcal{S} \) such that the first row admits an \( \mathcal{E} \)-triangle \( A' \xrightarrow{(x')} B \oplus I \xrightarrow{(y')} C' \to \) with \( A' \in \mathcal{U} \) and \( I \in \mathcal{I} \).

Proof. Since \((\mathcal{U}, \mathcal{Y})\) is a cotorsion pair, \( A \) admits an \( \mathcal{E} \)-triangle \( Y \xrightarrow{a} A' \xrightarrow{a} A \to \) with \( A' \in \mathcal{U} \) and \( Y \in \mathcal{Y} \). \( Y \) admits an \( \mathcal{E} \)-triangle \( Y \xrightarrow{I} C' \xrightarrow{S} \) with \( I \in \mathcal{I} \) and \( S \in \mathcal{S} \).

By Lemma 3.10, \( a : A' \to A \) is invertible. \( S' \) admits an \( \mathcal{E} \)-triangle \( S' \to I' \to S'' \to \) with \( I' \in \mathcal{I} \) and \( S'' \in \mathcal{S} \), then we have a commutative diagram

\[
\begin{array}{c}
Y \xrightarrow{a} A' \xrightarrow{a} A \xrightarrow{a} \\
I \xrightarrow{(a)} A \oplus I \xrightarrow{(y')^1} C' \xrightarrow{(y')} C
\end{array}
\]

By Lemma 3.10, \( c : C' \to C \) is invertible. Hence we have the following commutative diagram

\[
\begin{array}{c}
S' \xrightarrow{c} C' \xrightarrow{c} C \xrightarrow{c} \\
I' \xrightarrow{(c)} C \oplus I' \xrightarrow{(1,0)} C \xrightarrow{c}
\end{array}
\]

which induces an isomorphism of triangles

\[
\begin{array}{c}
A' \xleftarrow{x'} B \xrightarrow{y'} C' \to A'[1] \\
\cong \quad \cong \quad \cong \\
A \xleftarrow{x} B \xrightarrow{y} C \to A[1]
\end{array}
\]

Dually, we have the following lemma:
Lemma 4.5. Let \( A \xrightarrow{x} B \xrightarrow{y} C \rightarrow \) be any \( E \)-triangle in \( B \). Then there exists an isomorphism of triangles

\[
\begin{array}{ccc}
A & \xrightarrow{z} & B \\
\downarrow & & \downarrow \\
A'' & \xrightarrow{z''} & B''
\end{array}
\begin{array}{ccc}
& & C \\
\downarrow & & \downarrow \\
& & A'[1]
\end{array}
\]

in \( B/S \) such that the second row admits an \( E \)-triangle \( A'' \xrightarrow{(z'')} B \oplus P \xrightarrow{y'' *} C'' \rightarrow \) with \( C'' \in \mathcal{V} \) and \( P \in \mathcal{P} \).

Then following proposition shows that \( F \) is an extriangle equivalence.

Proposition 4.6. Let \( A \xrightarrow{x} B \xrightarrow{y} C \rightarrow \) be any \( E \)-triangle in \( B \). There is an isomorphism between \( E \)-triangles

\[
\begin{array}{ccc}
A & \xrightarrow{z} & B \\
\downarrow & & \downarrow \\
Z^A & \xrightarrow{z^A} & Z^B
\end{array}
\begin{array}{ccc}
& & C \\
\downarrow & & \downarrow \\
& & Z^[A][1]
\end{array}
\]

in \( B/S \), where \( Z^A \xrightarrow{z^A} Z^B \xrightarrow{z^B} Z^C \rightarrow \) is an \( E \)-triangle in \( Z \).

**Proof.** By Lemma 4.4 and Lemma 4.5, we can assume that \( A \in \mathcal{U} \) and \( C \in \mathcal{V} \). Since \( A \) admits an \( E \)-triangle \( A \xrightarrow{z^A} Z^A \xrightarrow{X^A} C \rightarrow \) with \( Z^A \in \mathcal{Z} \) and \( X^A \in \mathcal{X} \), we have the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow & & \downarrow \\
Z^A & \xrightarrow{B^A} & C
\end{array}
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow \\
X^A & \xrightarrow{X^A} & C
\end{array}
\]

Since \( C \) admits an \( E \)-triangle \( Y^C \xrightarrow{z^C} Z^C \xrightarrow{C} \rightarrow \) with \( Z^C \in \mathcal{Z} \) and \( Y^C \in \mathcal{Y} \), we have the following commutative diagram

\[
\begin{array}{ccc}
Y^C & \xrightarrow{z^C} & Z^C \\
\downarrow & & \downarrow \\
Z^A & \xrightarrow{z^A} & Z^B
\end{array}
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow \\
Z^A & \xrightarrow{B^A} & C
\end{array}
\]

with \( Z^B \in \mathcal{Z} \). Then we get an isomorphism of triangles

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow & & \downarrow \\
Z^A & \xrightarrow{z^A} & Z^B
\end{array}
\begin{array}{ccc}
& & C \\
\downarrow & & \downarrow \\
& & Z^A[1]
\end{array}
\]
Hence there exists an isomorphism of $\mathcal{E}$-triangles in $\mathcal{B}/\mathcal{S}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\Delta} & B \\
\downarrow \cong & & \downarrow \cong \\
Z^A & \xrightarrow{\Delta} & Z^B \\
\end{array}
\quad \text{and}
\begin{array}{ccc}
C \\
\downarrow \cong & & \downarrow \cong \\
Z^C \\
\end{array}
$$

\[ Z^A[1] = \mathcal{W} \]

\[ A \rightarrow B \rightarrow C \rightarrow A \]

\[ Z \rightarrow \mathcal{W} \rightarrow \mathcal{Z} \]

Remark 4.7. We can only get that $Z^A[1]$ lies in $\mathcal{U}$ (not necessarily in $\mathcal{Z}$).

**Theorem 4.8.** Assume that $(\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})$ are hereditary cotorsion pairs. Then $F : Z/[\mathcal{W}] \rightarrow \mathcal{B}/\mathcal{S}$ becomes a triangle equivalence.

**Proof.** By Proposition 3.2, we know that $Z/[\mathcal{W}]$ is a triangulated category with shift functor (1). Moreover, by Proposition 3.4, we can find that $\mathcal{S} = \mathcal{S}_R = \mathcal{S}_L$ is a thick subcategory, and $(\mathcal{X}, \mathcal{V}), (\mathcal{U}, \mathcal{Y})$ is a Hovey twin cotorsion pair. By definition, if $Z \in \mathcal{Z}$, then $Z[1] = Z(1)$ in $\mathcal{B}/\mathcal{S}$. For an $\mathcal{E}$-triangle $A \xrightarrow{\ell} B \xrightarrow{\mu} C \rightarrow \rightarrow$ in $\mathcal{Z}$, the image of the induced triangle $A \xrightarrow{\ell} B \xrightarrow{\mu} C \rightarrow A(1)$ by equivalent functor $F$ becomes a standard triangle $A \xrightarrow{\ell} B \xrightarrow{\mu} C \xrightarrow{\Delta} A[1]$ in $\mathcal{B}/\mathcal{S}$. Moreover, $F \circ (1) = [1] \circ F$, this shows that $F : Z/[\mathcal{W}] \rightarrow \mathcal{B}/\mathcal{S}$ is a triangle equivalence.

Remark 4.9. We can find similar results in [S, Theorem 6.21(2), Proposition 7.14] in the context of Hovey’s abelian model structures. Although the terminology twin cotorsion pair was not used, but hereditary twin cotorsion pairs were actually considered as well, see also [S, Definition 6.19]. A connection to ideal quotients is also considered in [S, Lemma 6.16, Theorem 6.7(1)]. In fact, Theorem 1.3 strengthens Šťovíček’s result even for abelian cases.

We give an example of our main results.

**Example 4.10.** Let $Q$ be the following infinite quiver:

$$
\cdots \xrightarrow{x_{-5}} -4 \xrightarrow{x_{-4}} -3 \xrightarrow{x_{-3}} -2 \xrightarrow{x_{-2}} -1 \xrightarrow{x_0} 0 \xrightarrow{x_1} 1 \xrightarrow{x_2} 2 \xrightarrow{x_3} 3 \xrightarrow{x_4} \cdots
$$

Let $\Lambda = kQ/(x_i x_{i+1} x_{i+2}, i \neq 4k, x_{4k} x_{4k+1})$. Then the AR-quiver of $\mathcal{B} = \text{mod}\Lambda$ is the following:

\[ \text{Diagram for AR-quiver}\]

The additive closure of the indecomposable objects denoted by $\star$ and $\diamond$ form a thick subcategory in $\mathcal{B}$, which is denoted by $\mathcal{S}$. We denote the additive closure of the indecomposable objects in $\diamond$ by $\mathcal{X}$, it is the subcategory of all the projective objects in $\mathcal{B}$. Then $(\mathcal{X}, \mathcal{S})$ is a cotorsion pair in $\mathcal{S}$, and we also have two hereditary cotorsion pairs $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{S}^\perp, \mathcal{S})$. In this case, $\mathcal{Z} = \mathcal{S}^\perp$ and $\mathcal{W} = \mathcal{X}$. The indecomposable objects in $(\mathcal{S}^\perp)/[\mathcal{X}]$ are denoted by $\blacklozenge$ in the diagram:

\[ \text{Diagram for indecomposables in (S^\perp)/[X]}\]
The indecomposable objects denoted by $\blacklozenge$ are non-zero objects in $B/S$ which do not lie in $\mathcal{S}$, they are isomorphic to the objects in $\mathcal{S}$:

$$
\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge
$$

**ACKNOWLEDGMENTS**

The authors would like to thank the reviewers for their valuable suggestions on mathematical content and English expressions.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Conflict of Interests** The authors declare that they have no conflicts of interest to this work.

**References**

[Am] C. Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. Ann. Inst. Fourier (Grenoble) 59 (2009), no. 6, 2525–2590.

[Bu] R. Buchweitz. Maximal Cohen-Macaulay modules and Tate cohomology. With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz. Mathematical Surveys and Monographs, 262. American Mathematical Society, Providence, RI, 2021.

[B] T. Bühler. Exact categories. Expo. Math. 28(2010), 1–69.

[BM] A. Buan, R. Marsh. From triangulated categories to module categories via localisations. Trans. Amer. Math. Soc. 365 (2013), no. 6, 2845–2861.

[B-TS] R. Bennett-Tennenhaus, A. Shah. Transport of structure in higher homological algebra. J. Algebra 574 (2021), 514–549.

[BOJ] P. Bergh, S. Oppermann, D. Jørgensen. The Gorenstein defect category. Q. J. Math. 66 (2015), no. 2, 459–471.

[C] X. Chen. Relative singularity categories and Gorenstein-projective modules. Math. Nachr. 284 (2011), no. 2-3, 199–212.

[CZ] X. Chen, P. Zhang. Quotient triangulated categories. Manuscripta Math. 123 (2007), no. 2, 167–183.

[G] L. Guo. Cluster tilting objects in generalized higher cluster categories. J. Pure Appl. Algebra 215 (2011), no. 9, 2055–2071.

[GZ] P. Gabriel, M. Zisman. Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc., New York, 1967.

[HZZ] J. Hu, D. Zhang, P. Zhou. Proper classes and Gorensteinness in extriangulated categories. J. Algebra 551 (2020), 23–60.

[IYa1] O. Iyama, D. Yang. Silting reduction and Calabi-Yau reduction of triangulated categories. Trans. Amer. Math. Soc. 370 (2018), no. 11, 7861–7898.

[IYa2] O. Iyama, D. Yang. Quotients of triangulated categories and equivalences of Buchweitz, Orlov, and Amiot-Guo-Keller. Amer. J. Math. 142 (2020), no. 5, 1641–1659.

[IYo] O. Iyama, Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math. 172 (2008), no. 1, 117–168.

[K1] H. Krause. The stable derived category of a Noetherian scheme. Compos. Math. 141 (2005), no. 5, 1128–1162.

[K2] H. Krause. Localization theory for triangulated categories. Triangulated categories, 161–235, London Math. Soc. Lecture Note Ser., 375, Cambridge Univ. Press, Cambridge, 2010.
[KV] B. Keller, D. Vossieck. Sous les catégories dérivées. C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 6, 225–228.

[LN] Y. Liu, H. Nakaoka. Hearts of twin cotorsion pairs on extriangulated categories. J. Algebra 528 (2019), 96–149.

[N] H. Nakaoka. General heart construction on a triangulated category (I): Unifying t-structures and cluster tilting subcategories. Appl. Categ. Structures 19 (2011), no. 6, 879–899.

[NP] H. Nakaoka, Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Différ. Catég. 60 (2019), no. 2, 117–193.

[NOS] H. Nakaoka, Y. Ogawa, A. Sakai, Localization of extriangulated categories. J. Algebra 611 (2022), 341–398.

[O] Y. Ogawa. Abelian categories from triangulated categories via Nakaoka-Palu’s localization. Appl. Categ. Structures 30 (2022), no. 4, 611–639.

[OPS] S. Oppermann, C. Psaroudakis, T. Stai. Change of rings and singularity categories. Adv. Math. 350 (2019), 190–241.

[R] J. Rickard. Derived categories and stable equivalence. J. Pure Appl. Algebra 61 (1989), no. 3, 303–317.

[S] J. Štovíček. Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves. Advances in representation theory of algebras, 297–367. EMS Ser. Congr. Rep. European Mathematical Society (EMS), Zürich, 2013.

[T] A. Tattar. Right triangulated categories: As extriangulated categories, aisles and co-aisles. arXiv: 2106.09107, 2021.

[V] J. Verdier. Des catégories dérivées des catégories abéliennes. With a preface by Luc Illusie. Edited and with a note by Georges Maltsiniotis. Astérisque No. 239, 1996.

[W] J. Wei. Relative singularity categories, Gorenstein objects and silting theory. J. Pure Appl. Algebra 222 (2018), no. 8, 2310–2322.

[ZH] Y. Zheng, Z. Huang. Triangulated equivalences involving Gorenstein projective modules. Canad. Math. Bull. 60 (2017), no. 4, 879–890.

[ZZ] P. Zhou, B. Zhu. Triangulated quotient categories revisited. J. Algebra 502 (2018), 196–232.

[ZhZ] B. Zhu, X. Zhuang. Tilting subcategories in extriangulated categories. Front. Math. China 15 (2020), no. 1, 225–253.