PORTFOLIO OPTIMIZATION FOR JUMP-DIFFUSION RISKY ASSETS WITH REGIME SWITCHING: A TIME-CONSISTENT APPROACH

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Abstract. In this paper, an optimal portfolio selection problem with mean-variance utility is considered for a financial market consisting of one risk-free asset and two risky assets, whose price processes are modulated by jump-diffusion model, the two jump number processes are correlated through a common shock, and the Brownian motions are supposed to be dependent. Moreover, it is assumed that not only the risk aversion coefficient but also the market parameters such as the appreciation and volatility rates as well as the jump amplitude depend on a Markov chain with finite states. In addition, short selling is supposed to be prohibited. Using the technique of stochastic control theory and the corresponding extended Hamilton-Jacobi-Bellman equation, the explicit expressions of the optimal strategies and value function are obtained within a game theoretic framework, and the existence and uniqueness of the solutions are proved as well. In the end, some numerical examples are presented to show the impact of the parameters on the optimal strategies, and some further discussions on the case of \( n \geq 3 \) risky assets are given to demonstrate the important effect of the correlation coefficient of the Brownian motions on the optimal results.

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1. Introduction. Portfolio optimization selection aims to seek a best asset allocation which maximizes the terminal profit or minimizes the risk. Mean-variance criterion, as one of the popular criteria proposed by Markowitz [18], has become one of the milestones in mathematical finance. In recent twenty years, there are numerous works on the mean-variance problem and its extension in finance. See for example, Li and Ng [12] developed an embedding technique to change the originally mean-variance problem into a stochastic linear-quadratic (LQ) control problem in a discrete-time setting. This technique was extended in Zhou and Li [31], along with an indefinite stochastic LQ control approach, to the continuous time case. Zhou and Yin [32] studied the Markovitz’s mean-variance portfolio problem for the continuous time model with Markov regime switching and the optimal portfolio and efficient frontier were derived. Further extensions and improvements in insurance and finance applications are carried out, see for example, Chen et al. [9], Bi and Guo [5], Bi et al. [6], Liang et al. [14], Ming et al. [19], Zhou et al. [30], Sun and Guo [22], and the references therein.

However, it is well known that the mean-variance criterion lacks of iterated-expectation property, which gives rise to time-inconsistent strategy in the sense that Bellman optimality principle is not available any more. One of the basic ways of handling time inconsistency in optimal control problems is to study pre-committed problem, where “optimal” is interpreted as “optimal from the point of view of time zero”, see Richardson [21], Li and Ng [12], Lim and Zhou [17]; Another way is to take the time inconsistency more seriously and look for a time consistent strategy, see Forsyth and Wang [11], Bensoussan et al. [2, 3] and Wei et al. [24]. Recently, Björk and Murgoci [7] first proposed the case where the risk aversion depended dynamically on current wealth, and that the objective function was to maximize the mean-variance utility

\[
E_{t,x}[R_T^\pi] - \frac{\gamma(x)}{2} \text{Var}_{t,x}[R_T^\pi],
\]

in which \(\gamma(x)\) is the so-called state-dependent risk aversion. Following this work, Björk et al. [8] studied the optimal portfolio problem with one risk-free asset and one risky asset modulated by geometric Brownian motion (GBM); Zhang and Liang [28] extended it to the optimal portfolio problem with jump-diffusion risky assets, and Bi and Cai [4] investigated an optimal investment-reinsurance strategies for an insurer under the Value-at-Risk constraints.

In practice, there exists a number of market regimes such as bullish and bearish market regimes to describe the state of the financial market. Markov regime-switching models, being more realistic, have attracted much attention in finance and related fields. In this situation, the market state can take values in one of a finite number of regimes, and then, various market parameters such as stocks’ appreciation and volatility rates, will take different values under different market modes. For its applications in stochastic optimal control problems, see Elliot and Hoek [10] for asset allocation problem; Zhou and Yin [32] for a mean-variance portfolio selection problem; Wei et al. [25] for reinsurance and dividend problem; Zhang and Siu [29] for proportional reinsurance and investment problem with no short selling. In addition, Wei et al. [23] studied asset-liability management problem with Markov regime-switching, where both the prices process of the stocks and the liability are governed by GBM and the objective function adopted is to maximize

\[
E_{t,x}[R_T^\pi] - \frac{\gamma(t,i)}{2} \text{Var}_{t,x}[R_T^\pi].
\]
Replaced liability by reinsurance, Liang and Song [13] considered an optimal investment and reinsurance problem, in which the surplus process is described by classical Lundberg model, and appreciation rate and risk aversion coefficient are affected by a Markov chain.

This paper extends their work to the jump-diffusion risky asset model, where the counting processes are correlated through a common shock and the Brownian motions are supposed to be dependent. This kind of model is more reasonable for the real financial market, since the information often comes as a surprise which usually leads to a jump in the price of stock, and some of the information can cause a common shock to both of risky assets, such as the fluctuation of the interest rate or oil price. For more detail of the common shock dependent risk model, see Liang and Yuen [15], Yuen et al. [26], Bi et al. [6], Liang et al. [14], Liang et al. [16]. In addition, not only the risk aversion coefficient but also the market parameters such as the appreciation and volatility rates as well as the jump amplitude depend on a Markov chain with finite states. Moreover, unlike the model in Wei et al. [23] and Liang and Song [13], we require that short selling is not allowed. Under the criterion of maximizing the mean-variance utility of the terminal wealth, by the technique of stochastic control theory and the corresponding extended Hamilton-Jacobi-Bellman (HJB) system, the optimal strategies and value function are derived within a game theoretic framework, and the existence and uniqueness of the solutions are proved as well. In particular, some further discussion on the case of $n \geq 3$ risky assets are given to show the important effect of the correlation coefficient of the Brownian motions on the optimal results.

The main contributions of the present paper have five folds: Firstly, we add jumps into the price process of the risky assets and consider the common shock dependence between the two jump-diffusion risky assets. This is different from the existing literature such as Björk et al. [8], Liang and Song [13] and Zeng et al. [27], in which either geometric Brownian motion or independent risk model is discussed. We find that the jump-diffusion and common shock model would reduce the investor’s losses and risks compared with the investment strategy obtained under the geometric Brownian motion; Secondly, we consider the Markov regime switching in various market parameters and the mean-variance criterion depending not only on the time but also on the market state. It generalizes the risk model from a diffusion process to a Markov regime switching model. Different from Wei et al. [23] and Liang and Song [13], the present paper studies a jump-diffusion process, and both the jump amplitudes and the jump number processes are also dependent on the market states, which reflects the underlying market environment better and makes the analysis more complicated; Thirdly, short selling is not allowed in current paper, and we find that the investment constraint will decrease the value function; Fourthly, we provide two important lemmas which play key roles in discussing the optimal strategies and value function. It shows that the optimal strategy for a risk-averse does not depend on the wealth process if a candidate strategy satisfies a certain form. In addition, we also prove the existence and uniqueness of the solutions to a system of differential equations, and derive the explicit expressions of the optimal results; The last but not the least, in Section 5 we further extend our work to the case of a financial market consisting of $n \geq 3$ risky assets and obtain corresponding optimal results under some additional assumptions.

The rest of the paper is organized as follows. In Section 2, the models and problems are formulated. In Section 3, we investigate the existence and uniqueness
of the solutions to a system of differential equations, and derive the closed-form expressions of the optimal results according to the corresponding extended HJB equation. Some numerical examples are carried out to show the impacts of some model parameters on the optimal strategy in Section 4. In Section 5, we extend the results with $n = 2$ risky assets to the case of $n \geq 3$, and some further results are obtained. Finally, we conclude the paper in Section 6.

2. Model and problem formulation.

2.1. Model and assumptions. We consider a continuous time, right-continuous stationary Markov chain $\{\alpha(t) : t \in [0, T]\}$ taking values in a finite state space $\mathcal{M} = \{e_1, e_2, \ldots, e_l\}$ on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{P})$, where $e_i \in \mathbb{R}^l$ and the $j$th component of $e_i$ is the Kronecker delta $\delta_{ij}$. Assume that the chain is homogeneous and irreducible, and has a generator $Q = (q_{ij})_{1 \times l}$ with $\sum_{j=1}^{l} q_{ij} = 0$ for any $e_i \in \mathcal{M}$. Throughout this paper, the notation $M'$ represents the transpose of any vector or matrix $M$.

Suppose that a financial market consists of a risk-free asset (bond) and two risky assets (stocks). The price process of the bond $S_0(t)$ is modeled by

$$dS_0(t) = r_0(t)S_0(t)dt, \quad t \in [0, T],$$

where $r_0(t) (> 0)$ represents the risk-free interest rate on $[0, T]$. Here, we assume that the risk-free rate is independent of the Markov chain. This assumption is reasonable since the risk-free rate is generally regulated by the government and the change of the risk-free rate is not as great as the fluctuation of the risky assets’ prices. In addition, this assumption is also advantageous to deal with the problem.

The price processes of the other two stocks $S_k(t) (k = 1, 2)$ satisfy the following stochastic differential equations

$$dS_k(t) = S_k(t-)[r_k(t, \alpha(t-))dt + \sigma_k(t, \alpha(t-))dW_k(t) + \int_{\mathbb{R}_0} \eta_k(t, \alpha(t-), z_k)M_{k,\alpha(t-)}(dz_k, dt)]. \quad (2.1)$$

Here $W_1(t)$ and $W_2(t)$ are standard Brownian motions and their correlation coefficient is denoted by $\rho$. The two jump number processes $M_{1,\alpha(t-)}(dz_1, dt)$ and $M_{2,\alpha(t-)}(dz_2, dt)$ are correlated in the way that

$$M_{1,\alpha(t)}(dz_1, dt) = N_{1,\alpha(t)}(dz_1, dt) + N_{0,\alpha(t)}(dz_1, dt),$$
$$M_{2,\alpha(t)}(dz_2, dt) = N_{2,\alpha(t)}(dz_2, dt) + N_{0,\alpha(t)}(dz_2, dt),$$

where $N_{1,\alpha(t)}(\cdot, dt)$, $N_{2,\alpha(t)}(\cdot, dt)$ and $N_{0,\alpha(t)}(\cdot, dt)$ are independent Poisson random measures with parameters $\lambda_{1,\alpha(t)}$, $\lambda_{2,\alpha(t)}$ and $\lambda_{0,\alpha(t)}$, respectively, and have the compensators as follows

$$\tilde{N}_{1,\alpha(t)}(\cdot, dt) = N_{1,\alpha(t)}(\cdot, dt) - \nu_{1,\alpha(t)}(\cdot)dt,$$
$$\tilde{N}_{2,\alpha(t)}(\cdot, dt) = N_{2,\alpha(t)}(\cdot, dt) - \nu_{2,\alpha(t)}(\cdot)dt,$$
$$\tilde{N}_{0,\alpha(t)}(\cdot, dt) = N_{0,\alpha(t)}(\cdot, dt) - \nu_{0,\alpha(t)}(\cdot)dt.$$
is the total wealth of the investor at time \( t \) corresponding to the strategy \( u \). If \( u_k(t) < 0, k = 1, 2 \), it implies that the \( k \)th stock is sold short. In this paper, an important constraint is the prohibition of short selling, which means that \( u_1(t), u_2(t) \) must be satisfied with conditions \( u_1(t) \geq 0 \) and \( u_2(t) \geq 0 \). With the condition of self-financing, we can write the wealth process as follows

\[
dR^u(t) = \frac{u_1(t)}{S_1(t)}dS_1(t) + \frac{u_2(t)}{S_2(t)}dS_2(t) + \frac{u_0(t)}{S_0(t)}dS_0(t)
\]

\[
= [r_0(t)R^u(t) + (r_1(t, \alpha(t)) - r_0(t))u_1(t) + (r_2(t, \alpha(t)) - r_0(t))u_2(t)]dt
\]

\[
+ \sigma_1(t, \alpha(t))u_1(t)dW_1(t) + \sigma_2(t, \alpha(t))u_2(t)dW_2(t)
\]

\[
+ \int_{D_1} \eta_1(t, \alpha(t), z_1)u_1(t)N_{1, \alpha(t)}(dz_1, dt)
\]

\[
+ \int_{D_2} \eta_2(t, \alpha(t), z_2)u_2(t)N_{2, \alpha(t)}(dz_2, dt)
\]

\[
+ \int\int_{D_3} [\eta_1(t, \alpha(t), z_1)u_1(t) + \eta_2(t, \alpha(t), z_2)u_2(t)]N_{0, \alpha(t)}(dz_1, dz_2, dt).
\]

(2.2)

where \( D_1, D_2 \) are measurable subsets of \( \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \) and \( D_3 \) is a measurable subset of \( \mathbb{R}^2 \setminus E \) with

\[
E = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}.
\]

Throughout this paper, we impose the following assumptions:

**Assumption 2.1.** We assume that

(A1) the two risky assets can be traded continuously over time \([t, T]\), and there are no transaction costs and taxes in trading;

(A2) \( r_0(t), r_1(t, e_i), r_2(t, e_i), \sigma_k(t, e_i), \eta_k(t, e_i, z_k), k = 1, 2 \) are measurable and uniformly bounded in \( t \);

(A3) for any state \( e_i \in \mathcal{M} \),

\[
\begin{aligned}
& r_1(t, e_i) + (\lambda_{1,e_i} + \lambda_{0,e_i})E[\eta_1(t, e_i, X_{j,e_i})] > r_0(t), \\
& r_2(t, e_i) + (\lambda_{2,e_i} + \lambda_{0,e_i})E[\eta_2(t, e_i, Y_{j,e_i})] > r_0(t),
\end{aligned}
\]

where \( X_{j,e_i} \) and \( Y_{j,e_i} \) are the jump amplitudes of risky asset \( S_1(t) \) and \( S_2(t) \) at state \( e_i \), respectively, and \( \{X_{j,e_i}, j \geq 1\} \{Y_{j,e_i}, j \geq 1\} \) have the common distribution with \( X_{e_i}(Y_{e_i}) \). The hypothesis (A3) is reasonable, since for an investor, the expected return of the risky asset in the financial market would be larger than the risk-free interest rate. In addition, we assume that \( W_1(t)(W_2(t)) \), \( \{X_{j,e_i}, j \geq 1\} \{Y_{j,e_i}, j \geq 1\} \), \( N_{1,e_i}(\cdot, dt) \), \( N_{2,e_i}(\cdot, dt) \) and \( N_{0,e_i}(\cdot, dt) \) are mutually independent for any state \( e_i \in \mathcal{M} \).

**Remark 2.1.** A strategy \( u(t) = (u_1(t), u_2(t))' \) is said to be admissible if \( u(\cdot) \) satisfies the following conditions:

(a) \( u(t) = (u_1(t), u_2(t))' \) is \( \mathcal{F}(t) \)-predictable;

(b) \( u_1(t) \geq 0 \) and \( u_2(t) \geq 0 \);

(c) \( E\int_0^T (|u_1(t)|^2 + |u_2(t)|^2)dt < +\infty \);

(d) equation (2.2) has a unique solution \( R(\cdot) \) corresponding to \( u(\cdot) \).

The set of all admissible strategies is denoted by \( \mathcal{U} \).
2.2. Problem formulation. The objective in this paper is to maximize the expected utility of terminal wealth, where the utility is of mean-variance form, i.e., the objective function which we want to maximize is

\[
J(t, x, e_i, u) = E_{t,x,e_i}[R^u(T)] - \frac{\gamma(t, e_i)}{2} \text{Var}_{t,x,e_i}[R^u(T)]
\]

\[
:= E_{t,x,e_i}[F(t, e_i, R^u(T)) + G(t, e_i, E_{t,x,e_i}[R^u(T)])],
\]

where

\[
\begin{align*}
F(t, e_i, x) &= x - \frac{\gamma(t, e_i)}{2} x^2, \\
G(t, e_i, x) &= \frac{\gamma(t, e_i)}{2} x^2,
\end{align*}
\]

\[
E_{t,x,e_i}[\cdot], \text{Var}_{t,x,e_i}[\cdot]
\]

are conditional expectation and variance with initial capital \(x\) and state \(e_i\) at time \(t\), respectively, and risk aversion process \(\gamma(t, e_i) > 0\).

Based on Björk and Murgoci [7], this problem is time-inconsistent since there is a form of expectation \(E[\cdot]\) in the function \(G(\cdot)\), which implies that Bellman optimality principle does not hold any more. To get a time-inconsistent strategy, we will present the following definition of an equilibrium control.

**Definition 2.1.** Given a control law \(u^*\), construct a control law \(u_h\) by

\[
u_h(s, y) = \begin{cases} u(s, y), & t \leq s < t + h, \quad y \in \mathbb{R}, \\
u^*(s, y), & t + h \leq s \leq T, \quad y \in \mathbb{R}, \end{cases}
\]

where \(u \in \mathcal{U}\) and \(h > 0\). If

\[
\liminf_{h \to 0} \frac{J(t, x, e_i, u^*) - J(t, x, e_i, u_h)}{h} \geq 0
\]

for any \(u \in \mathcal{U}\) and \((t, x, e_i) \in [0, T] \times \mathbb{R} \times \mathcal{M}\), we say that \(u^*\) is an equilibrium control law. The value function \(V\) is defined by

\[
V(t, x, e_i) = J(t, x, e_i, u^*).
\]

Let \(C^{1,2}([0, T] \times \mathbb{R})\) denote the space of \(\varphi(t, x, e_i)\) such that \(\varphi(t, x, e_i)\) and its derivatives \(\varphi_t(t, x, e_i), \varphi_x(t, x, e_i), \varphi_{xx}(t, x, e_i)\) are continuous on \([0, T] \times \mathbb{R}\). For any function \(\varphi(t, x, e_i) \in C^{1,2}\) and any fixed \(u \in \mathcal{U}\), the usual infinitesimal generator \(\mathcal{A}^u\) for jump-diffusion model (2.2) is

\[
\mathcal{A}^u \varphi(t, x, e_i)
\]

\[
= \varphi_t(t, x, e_i) + [r_0(t)x + (r_1(t, e_i) - r_0(t))u_1 + (r_2(t, e_i) - r_0(t))u_2] \varphi_x(t, x, e_i)
\]

\[
+ \frac{1}{2} \sigma^2(t, e_i)u_1^2 + \sigma^2(t, e_i)u_2^2 + 2\rho \sigma_1(t, e_i) \sigma_2(t, e_i) u_1 u_2 \varphi_{xx}(t, x, e_i)
\]

\[
+ \int_{D_1} \varphi(t, x + \eta_1(t, e_i, z_1)u_1, e_i) - \varphi(t, x, e_i) \nu_{e_i}(dz_1)
\]

\[
+ \int_{D_2} \varphi(t, x + \eta_2(t, e_i, z_2)u_2, e_i) - \varphi(t, x, e_i) \nu_{e_i}(dz_2)
\]

\[
+ \sum_{j=1}^{J} q_{ij} \varphi(t, x, e_j) - \varphi(t, x, e_i).
\]

Now we will define the extended Hamilton-Jacobi-Bellman equation, which is similar to Björk and Murgoci [7].
Definition 2.2. The extended HJB equation for the Nash equilibrium problem is defined as follows:

\[
\begin{align*}
\sup_{u \in U} \left\{ A^u V(t, x, e_i) - A^u f(t, x, e_i) + A^u f^{0, e_i}(t, x, e_i) \right. \\
- A^u (G \circ g)(t, x, e_i) + \mathcal{H}^u g(t, x, e_i) \left. \right\} = 0, \quad 0 \leq t \leq T, \\
A^u f^{0, e_i}(t, x, e_i) = 0, \quad 0 \leq t \leq T, \\
A^u g(t, x, e_i) = 0, \quad 0 \leq t \leq T, \\
V(T, x, e_i) = F(T, e_i, x) + G(T, e_i, x), \\
f^{0, e_i}(T, x, e_i) = F(s, e_i, x), \\
g(T, x, e_i) = x, \\
\end{align*}
\]

(2.5)

where functions \( f \) and \( g \) are defined by

\[
\begin{align*}
f(t, x, e_i, s, e_j) &:= E_{t, x, e_i}[F(s, e_j, R^{u^*}(T))], \\
g(t, x, e_i) &:= E_{t, x, e_i}[R^{u^*}(T)], \\
\end{align*}
\]

(2.6)

(2.7)

\[
u^* = \arg \sup_{u \in U} \left\{ A^u \phi(t, x, e_i) - A^u f(t, x, e_i, t, e_i) + A^u f^{0, e_i}(t, x, e_i) \right. \\
- A^u (G \circ g)(t, x, e_i) + \mathcal{H}^u g(t, x, e_i) \left. \right\},
\]

and \( f^{0, e_i}, G \circ g, \mathcal{H}^u g \) are taking the following forms

\[
\begin{align*}
f^{0, e_i}(t, x, e_i) &= f(t, x, e_i, s, e_j), \\
(G \circ g)(t, x, e_i) &= G(t, e_i, g(t, x, e_i)), \\
\mathcal{H}^u g(t, x, e_i) &= G_t(g(t, x, e_i))A^u g(t, x, e_i).
\end{align*}
\]

The following theorem shows a connection between the solutions of the extended HJB equation and optimal values.

Theorem 2.3. (Verification Theorem) Assume that \((\phi, f, g)\) is a solution of the extended system in Definition 2.2, and \(u^*\) is the control strategy which realizes the supremum in the first equation of (2.5). Then \(u^*\) is the equilibrium strategy, and \(\phi\) coincides with the value function \(V(t, x, e_i)\), that is, \(\phi(t, x, e_i) = V(t, x, e_i)\). Furthermore, \(f\) and \(g\) can be interpreted according to (2.6) and (2.7), respectively.

Proof. The proof is similar to Zhang and Liang [28], so we omit it here.

3. Solutions for the extended HJB equation. In this section, we will try to solve the extended HJB equation (2.5). From the probabilistic interpretations (2.6) and (2.7), it is easy to see that

\[
V(t, x, e_i) = f(t, x, e_i, t, e_i) + G(t, e_i, g(t, x, e_i)),
\]

(3.1)

which implies

\[
\begin{align*}
A^u V(t, x, e_i) &= A^u (f(t, x, e_i, t, e_i) + G(t, e_i, g(t, x, e_i))) \\
&= A^u f(t, x, e_i, t, e_i) + A^u G(t, e_i, g(t, x, e_i)).
\end{align*}
\]
Taking these functions \( f(t, x, e_i) \), we obtain the following theorem.

\[
\left\{ \begin{array}{l}
\sup_{u \in \mathcal{U}} \left\{ A^u f^{t, e_i}(t, x, e_i) + \gamma(t, e_i) g(t, x, e_i) A^u g(t, x, e_i) \right\} = 0, \\
A^u f^{s, e_i}(t, x, e_i) = 0, \\
A^u g(t, x, e_i) = 0, \\
f^{s, e_i}(T, x, e_i) = x - \frac{\gamma(s, e_i)}{2} x^2, \\
g(T, x, e_i) = x.
\end{array} \right.
\]

For simplicity, we denote \( r_0(t), r_1(t, e_i), r_2(t, e_i), \sigma_1(t, e_i), \sigma_2(t, e_i), \eta_1(t, e_i, z_1), \eta_2(t, e_i, z_2) \) by \( r_0, r_{1,i}, r_{2,i}, \sigma_{1,i}, \sigma_{2,i}, \eta_{1,i}, \eta_{2,i} \) respectively. Formula (3.1) tells us that if we want to derive value function \( V(t, x, e_i) \), we just need to obtain the values of functions \( f(t, x, e_i, t, e_i) \) and \( g(t, x, e_i) \). Therefore, in the following context, we tend to solve the extended HJB system (3.2). Notice that both \( f \) and \( g \) are functions of \( E_{t,x,e_i}[R^u(t)] \) and \( E_{t,x,e_i}[(R^u(t))^2] \). According to Björk et al. \[8\] where the risky asset is described by a GBM, we find that the expressions of \( E_{t,x}[R^u(t)] \) and \( E_{t,x}[(R^u(t))^2] \) are linear function and quadratic function w.r.t \( x \), respectively. Inspired by these results, we assume that

\[
\left\{ \begin{array}{l}
E_{t,x,e_i}[R^u(t)] = P_1(t, e_i)x + Q_1(t, e_i), \\
E_{t,x,e_i}[(R^u(t))^2] = K(t, e_i)x^2 + P_2(t, e_i)x + Q_2(t, e_i).
\end{array} \right.
\]

Then functions \( f \) and \( g \) have the following forms

\[
\left\{ \begin{array}{l}
g(t, x, e_i) = P_1(t, e_i)x + Q_1(t, e_i), \\
f^{t, e_i}(t, x, e_i) = P_1(t, e_i)x + Q_1(t, e_i) - \frac{\gamma(t, e_i)}{2} [K(t, e_i)x^2 + P_2(t, e_i)x + Q_2(t, e_i)].
\end{array} \right.
\]

and the value function becomes

\[
V(t, x, e_i) = \frac{\gamma(t, e_i)}{2} (P_1^2(t, e_i) - K(t, e_i))x^2 \\
+ \left[ P_1(t, e_i) + \frac{\gamma(t, e_i)}{2} (2P_1(t, e_i)Q_1(t, e_i) - P_2(t, e_i)) \right] x \\
+ \left[ Q_1(t, e_i) + \frac{\gamma(t, e_i)}{2} (Q_1^2(t, e_i) - Q_2(t, e_i)) \right].
\]

For convenience, we denote

\[
\left\{ \begin{array}{l}
\mu_{1i} = E[\eta_1(t, e_i, X_{e_i})], \\
\beta_{1i} = E[\eta_1^2(t, e_i, X_{e_i})], \\
\mu_{2i} = E[\eta_2(t, e_i, Y_{e_i})], \\
\beta_{2i} = E[\eta_2^2(t, e_i, Y_{e_i})],
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
a_{1i} = (\lambda_{1e_i} + \lambda_{0e_i}) \mu_{1i}, \\
a_{2i} = (\lambda_{2e_i} + \lambda_{0e_i}) \mu_{2i}, \\
\xi_{1i}^2 = (\lambda_{1e_i} + \lambda_{0e_i}) \beta_{1i}, \\
\xi_{2i}^2 = (\lambda_{2e_i} + \lambda_{0e_i}) \beta_{2i}.
\end{array} \right.
\]

Taking these functions \( f^{t, e_i}(t, x, e_i) \), \( g(t, x, e_i) \) as well as their derivatives back into (3.2), we obtain following theorem.
Theorem 3.1. The extended HJB system for problem (2.3) and (2.4) is given as follows

\[
\begin{align*}
\sup_{u_1,u_2} \left\{ \theta_1(t,e_i) + \mathcal{L}_1(t,e_i) + \sum_{j=1}^{l} q_{ij} \varphi_1(t,e_i,e_j) \right\} &= 0, \\
\theta_2(t,e_i) + \mathcal{L}_2(t,e_i) + \sum_{j=1}^{l} q_{ij} \varphi_2(t,e_i,e_j) &= 0, \\
\theta_3(t,e_i) + \mathcal{L}_3(t,e_i) + \sum_{j=1}^{l} q_{ij} \varphi_3(t,e_i,e_j) &= 0,
\end{align*}
\]

where

\[
\begin{align*}
\theta_1(t,e_i) &= \dot{P}_1(t,e_i)x + \dot{Q}_1(t,e_i) - \frac{\gamma(t,e_i)}{2} [\dot{K}(t,e_i)x^2 + \dot{P}_2(t,e_i)x + \dot{Q}_2(t,e_i)] \\
&+ \gamma(t,e_i)[P_1(t,e_i)\dot{P}_1(t,e_i)x^2 \\
&+ (P_1(t,e_i)\dot{Q}_1(t,e_i) + \dot{P}_1(t,e_i)Q_1(t,e_i))x + Q_1(t,e_i)\dot{Q}_1(t,e_i)], \\
\theta_2(t,e_i) &= \dot{P}_1(t,e_i)x + \dot{Q}_1(t,e_i) - \frac{\gamma(s,e_k)}{2}[K(t,e_i)x^2 + P_2(t,e_i)x + Q_2(t,e_i)], \\
\theta_3(t,e_i) &= \dot{P}_1(t,e_i)x + \dot{Q}_1(t,e_i); \\
\mathcal{L}_1(t,e_i) &= [r_0x + \Gamma_1(u_1,u_2)]\Delta(t,e_i) + \frac{1}{2}[\Gamma_2(u_1,u_2)][-\gamma(t,e_i)K(t,e_i)], \\
\mathcal{L}_2(t,e_i) &= [r_0x + \Gamma_1(u_1,u_2)][P_1(t,e_i) - \gamma(s,e_k)K(t,e_i)x - \frac{\gamma(s,e_k)}{2}P_2(t,e_i)] \\
&+ \frac{1}{2}[\Gamma_2(u_1^*,u_2^*)][-\gamma(s,e_k)K(t,e_i)], \\
\mathcal{L}_3(t,e_i) &= [r_0x + \Gamma_1(u_1^*,u_2^*)]P_1(t,e_i); \\
\varphi_1(t,e_i,e_j) &= P_1(t,e_j)x + Q_1(t,e_j) \\
&- \frac{\gamma(t,e_i)}{2}[K(t,e_j)x^2 + P_2(t,e_j)x + Q_2(t,e_j)] \\
&- [P_1(t,e_i)x + Q_1(t,e_i) - \frac{\gamma(t,e_i)}{2}(K(t,e_i)x^2 + P_2(t,e_i)x + Q_2(t,e_i)))] \\
&+ [\gamma(t,e_i)P_1(t,e_i)x + \gamma(t,e_i)Q_1(t,e_i)] \\
&\times [P_1(t,e_j)x + Q_1(t,e_j) - P_1(t,e_i)x - Q_1(t,e_i)], \\
\varphi_2(t,e_i,e_j) &= P_1(t,e_j)x + Q_1(t,e_j) \\
&- \frac{\gamma(s,e_k)}{2}[K(t,e_j)x^2 + P_2(t,e_j)x + Q_2(t,e_j)] \\
&- [P_1(t,e_i)x + Q_1(t,e_i) - \frac{\gamma(s,e_k)}{2}(K(t,e_i)x^2 + P_2(t,e_i)x + Q_2(t,e_i))]; \\
\varphi_3(t,e_i,e_j) &= P_1(t,e_j)x + Q_1(t,e_j) - [P_1(t,e_i)x + Q_1(t,e_i)] \\
&\text{with}
\end{align*}
\]

\[
\begin{align*}
\Gamma_1(U_1,U_2) &= (r_{ii} - r_0 + a_{ii})U_1 + (r_{ji} - r_0 + a_{ji})U_2, \\
\Gamma_2(U_1,U_2) &= (\sigma_1^2 + \xi_1^2)U_1^2 + (\sigma_2^2 + \xi_2^2)U_2^2 + 2(\lambda_{1i} \mu_{1i} \mu_{2i} + \rho \sigma_{1i} \sigma_{2i})U_1U_2, \\
\Delta(t,e_i) &= P_1(t,e_i) - \gamma(t,e_i)K(t,e_i)x + \frac{1}{2}P_2(t,e_i) \\
&- P_1^2(t,e_i)x - P_1(t,e_i)Q_1(t,e_i).
\end{align*}
\]
Let 
\[ h(u_1, u_2) = \theta_1(t, e_i) + \mathcal{L}\zeta_1(t, e_i) + \sum_{j=1}^{l} q_{ij}\varphi_j(t, e_i, e_j). \] (3.8)

Differentiating \( h(u_1, u_2) \) w.r.t \( u_1 \) and \( u_2 \), respectively, we have

\[
\begin{align*}
\frac{\partial h}{\partial u_1} &= (r_{1i} - r_0 + a_{1i})\Delta(t, e_i) \\
&\quad - \left[ (\sigma_{1i}^2 + \xi_{1i}^2)u_1 + (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_2)\right] \gamma(t, e_i) K(t, e_i), \\
\frac{\partial h}{\partial u_2} &= (r_{2i} - r_0 + a_{2i})\Delta(t, e_i) \\
&\quad - \left[ (\sigma_{2i}^2 + \xi_{2i}^2)u_2 + (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_1)\right] \gamma(t, e_i) K(t, e_i), \\
\frac{\partial^2 h}{\partial u_1^2} &= -\left( (\sigma_{1i}^2 + \xi_{1i}^2) + \gamma(t, e_i) K(t, e_i) \right), \\
\frac{\partial^2 h}{\partial u_2^2} &= -\left( (\sigma_{2i}^2 + \xi_{2i}^2) + \gamma(t, e_i) K(t, e_i) \right), \\
\frac{\partial^2 h}{\partial u_1 \partial u_2} &= -(\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2}) \gamma(t, e_i) K(t, e_i).
\end{align*}
\]

Obviously, the Hessian matrix can be written as

\[
H = \begin{pmatrix}
\frac{\partial^2 h}{\partial u_1^2} & \frac{\partial^2 h}{\partial u_1 \partial u_2} \\
\frac{\partial^2 h}{\partial u_2 \partial u_1} & \frac{\partial^2 h}{\partial u_2^2}
\end{pmatrix}
= \begin{pmatrix}
(\sigma_{1i}^2 + \xi_{1i}^2) + \lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} & \gamma(t, e_i) K(t, e_i) \\
\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} & (\sigma_{2i}^2 + \xi_{2i}^2) + \gamma(t, e_i) K(t, e_i)
\end{pmatrix}.\]

For each fixed state \( e_i \), by Cauchy-Schwartz inequality, it is not difficult to prove that the inequality

\[
(\sigma_{1i}^2 + \xi_{1i}^2)(\sigma_{2i}^2 + \xi_{2i}^2) - (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2})^2 > 0
\]

holds and later we will see that \( K(t, e_i) > 0 \). Therefore, \( H \) is a negative define matrix which means that the maximizer \( \{\hat{u}_1, \hat{u}_2\} \) of the \( h(u_1, u_2) \) is the solution of the following equations

\[
\begin{align*}
\{ & (r_{1i} - r_0 + a_{1i})\Delta(t, e_i) \\
&\quad - [ (\sigma_{1i}^2 + \xi_{1i}^2)u_1 + (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_2)\gamma(t, e_i) K(t, e_i) = 0, \\
\{ & (r_{2i} - r_0 + a_{2i})\Delta(t, e_i) \\
&\quad - [ (\sigma_{2i}^2 + \xi_{2i}^2)u_2 + (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_1)\gamma(t, e_i) K(t, e_i) = 0,
\end{align*}
\]

which yields

\[
\begin{cases}
\hat{u}_1(t, x, e_i) = m_{1i} q, \\
\hat{u}_2(t, x, e_i) = m_{2i} q,
\end{cases}
(3.9)
\]

where

\[
m_{1i} = \frac{(\sigma_{2i}^2 + \xi_{2i}^2)(r_{1i} - r_0 + a_{1i}) - (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_2)(r_{2i} - r_0 + a_{2i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(\sigma_{2i}^2 + \xi_{2i}^2) - (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2})^2},
\]

\[
m_{2i} = \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i}) - (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2} u_1)(r_{1i} - r_0 + a_{1i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(\sigma_{2i}^2 + \xi_{2i}^2) - (\lambda_{0e_i} \mu_{1i} \mu_{2i} + \rho_{\sigma_1 \sigma_2})^2},
\]

\[
q = \frac{\Delta(t, e_i)}{\gamma(t, e_i) K(t, e_i)}. \quad (3.10)
\]

From the expressions of \( \hat{u}_1(t, x, e_i) \) and \( \hat{u}_2(t, x, e_i) \), we can’t determine whether \( \hat{u}_1(t, x, e_i) \) and \( \hat{u}_2(t, x, e_i) \) are non-negative or not since we have not confirmed whether \( m_{1i} \), \( m_{2i} \) and \( q \) are non-negative. In order to find an optimal strategy in \( \mathcal{U} \), we need to analyse this problem for several cases depending on the values of \( m_{1i} \), \( m_{2i} \) and \( q \). Before investigating each case, we will give two important lemmas first, which play key roles in this paper.
Denote
\[
\begin{align*}
\delta_{1i} &= \min\left\{ \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{1i})}{r_{1i} - r_0 + a_{2i}}, \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{1i})}{r_{1i} - r_0 + a_{1i}} \right\}, \\
\delta_{2i} &= \max\left\{ \frac{(\sigma_{2i}^2 + \xi_{2i}^2)(r_{1i} - r_0 + a_{1i})}{r_{2i} - r_0 + a_{2i}}, \frac{(\sigma_{2i}^2 + \xi_{2i}^2)(r_{2i} - r_0 + a_{1i})}{r_{1i} - r_0 + a_{1i}} \right\},
\end{align*}
\]

The values of \(\delta_{1i}\) and \(\delta_{2i}\) are depending on the market parameters. For discussing conveniently, we assume that
\[
\begin{align*}
\delta_{1i} &= \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{2i})}{r_{1i} - r_0 + a_{1i}}, \\
\delta_{2i} &= \frac{(\sigma_{2i}^2 + \xi_{2i}^2)(r_{1i} - r_0 + a_{1i})}{r_{2i} - r_0 + a_{2i}}.
\end{align*}
\]

**Lemma 3.2.** Under the assumption \((A_3)\), we have
\[
\mu_{1i}\mu_{2i} \leq \frac{\delta_{2i} - \rho\sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}}.
\]

**Proof.** From the assumption \((A_3)\), we can easily get \(\delta_{1i} > 0\) and \(\delta_{2i} > 0\). If we suppose that
\[
\mu_{1i}\mu_{2i} > \frac{\delta_{2i} - \rho\sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}},
\]
then we have
\[
\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i} > \delta_{2i} > 0,
\]
and thus we obtain
\[
\begin{align*}
\left\{ \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{1i})}{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})} < 0, \\
\frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{1i} - r_0 + a_{1i})} < 0.
\end{align*}
\]
Solving these inequalities with respect to \((r_{1i} - r_0 + a_{1i})\) yields
\[
\begin{align*}
r_{1i} - r_0 + a_{1i} &< \frac{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}, \\
r_{1i} - r_0 + a_{1i} &> \frac{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{1i})},
\end{align*}
\]
that is,
\[
\frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})} < r_{1i} - r_0 + a_{1i}
\]
\[
< \frac{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{1i})}.
\]

However, by Cauchy-Schwartz inequality, we have the following fact
\[
\frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{2i} - r_0 + a_{2i})} / \frac{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})(r_{1i} - r_0 + a_{1i})}{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{1i} - r_0 + a_{1i})} = \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}{(\lambda_{0e_i}\mu_{1i}\mu_{2i} + \rho\sigma_{1i}\sigma_{2i})^2} > 1,
\]
which is contrary to (3.11). In other words, the case \(\mu_{1i}\mu_{2i} > \frac{\delta_{2i} - \rho\sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}}\) wouldn’t be existed. Therefore, we complete the proof. □
Lemma 3.3. Suppose that the investor is risk-averse. For any strategy \((u_1, u_2) \in \mathbb{R} \times \mathbb{R}\), if \((u_1, u_2)\) satisfies
\[
\begin{align*}
\begin{cases}
u_1 &= A_1(t, e_i) \frac{p^2_t(t, e_i) - K(t, e_i)}{K(t, e_i)} x + B_1(t, e_i), \\
u_2 &= A_2(t, e_i) \frac{p^2_t(t, e_i) - K(t, e_i)}{K(t, e_i)} x + B_2(t, e_i),
\end{cases}
\end{align*}
\]
where \(A_1(t, e_i), A_2(t, e_i), B_1(t, e_i), B_2(t, e_i)\) are some continuous functions w.r.t. \(t\) at state \(e_i\), and \(A_1(t, e_i), A_2(t, e_i)\) are irrelevant with \(P_1(t, e_i), K(t, e_i)\). Then we have \(\rho > 0\) for any \(t \in [0, T]\).

Proof. Taking \(u_1^*\) and \(u_2^*\) with the form of (3.12) back into (3.4) and (3.5), separating the variables with and without \(x^2\) and \(x\), and simplifying, we have
\[
\begin{align}
&\dot{K}(t, e_i) + \left[2(\gamma_0 - \Gamma_1(A_1(t, e_i), A_2(t, e_i))) + \Gamma_2(A_1(t, e_i), A_2(t, e_i))\right] K(t, e_i) \\
&+ \left[2\Gamma_1(A_1(t, e_i), A_2(t, e_i)) - 2\Gamma_2(A_1(t, e_i), A_2(t, e_i))\right] P_2(t, e_i) \\
&+ \Gamma_2(A_1(t, e_i), A_2(t, e_i)) \frac{p^4_t(t, e_i) - K(t, e_i)}{K(t, e_i)} + \sum_{j=1}^l q_{ij} [K(t, e_j) - K(t, e_i)] = 0, \\
K(T, e_i) &= 1,
\end{align}
\]
(3.13)
\[
\begin{align}
&\dot{P}_1(t, e_i) + \left[r_0 - \Gamma_1(A_1(t, e_i), A_2(t, e_i))\right] P_1(t, e_i) \\
&+ \Gamma_1(A_1(t, e_i), A_2(t, e_i)) \frac{p^4_t(t, e_i)}{K(t, e_i)} + \sum_{j=1}^l q_{ij} [P_1(t, e_j) - P_1(t, e_i)] = 0, \\
P_1(T, e_i) &= 1,
\end{align}
\]
(3.14)
\[
\begin{align}
&\dot{P}_2(t, e_i) + \left[r_0 + \Gamma_1(A_1(t, e_i), A_2(t, e_i))\right] \frac{p^4_t(t, e_i)}{K(t, e_i)} - 1\right] P_2(t, e_i) \\
&+ \Gamma_1(B_1(t, e_i), B_2(t, e_i)) \frac{p^4_t(t, e_i)}{K(t, e_i)} + \left[(\sigma_1^2 + \xi_1^2) A_1(t, e_i) B_1(t, e_i) \\
&+ (\sigma_2^2 + \xi_2^2) A_2(t, e_i) B_2(t, e_i) + \lambda_{\alpha_1} \mu_1 \mu_2 + \rho \sigma_1 \sigma_2\right] \\
&\times (A_1(t, e_i) B_2(t, e_i) + A_2(t, e_i) B_1(t, e_i))\right] \frac{p^4_t(t, e_i)}{K(t, e_i)} - 1\right] K(t, e_i) \\
&+ \sum_{j=1}^l q_{ij} [P_2(t, e_j) - P_2(t, e_i)] = 0, \\
P_2(T, e_i) &= 0,
\end{align}
\]
(3.15)
\[
\begin{align}
&\dot{Q}_1(t, e_i) + \Gamma_1(B_1(t, e_i), B_2(t, e_i)) P_1(t, e_i) + \sum_{j=1}^l q_{ij} [Q_1(t, e_j) - Q_1(t, e_i)] = 0, \\
Q_1(T, e_i) &= 0,
\end{align}
\]
(3.16)
\[
\begin{align}
&\dot{Q}_2(t, e_i) + \Gamma_1(B_1(t, e_i), B_2(t, e_i)) P_2(t, e_i) \\
&+ \Gamma_2(B_1(t, e_i), B_2(t, e_i)) K(t, e_i) + \sum_{j=1}^l q_{ij} [Q_2(t, e_j) - Q_2(t, e_i)] = 0, \\
Q_2(T, e_i) &= 0,
\end{align}
\]
(3.17)
Firstly, combined with equations (3.13) and (3.14), we find that
\[
K(t, e_i) = e^{\int_0^T 2\gamma_0(s)ds}
\]
and
\[
P_1(t, e_i) = e^{\int_0^T \gamma_0(s)ds}
\]
Since we assume that the investor is risk-averse, we have \( \gamma \). We only need to prove
\[
\text{Proof.}
\]
Based on values of \( K(t, e_i) \), \( P_1(t, e_i) \) and the relationship between \( P_2(t, e_i) \), \( P_1(t, e_i) \) and \( Q_1(t, e_i) \), taking them back into (3.10), we obtain
\[
\varrho = \frac{1}{\gamma(t, e_i)} e^{-\int_t^T r_0(s) ds}.
\]
Since we assume that the investor is risk-averse, we have \( \gamma(t, e_i) > 0 \). Therefore, \( \varrho > 0 \) for any \( t \in [0, T] \). So we complete the proof.

**Theorem 3.4.** The differential equations (3.13) \( \sim (3.17) \) admit a unique solution \( K(t, e_i) \), \( P_1(t, e_i) \), \( P_2(t, e_i) \), \( Q_1(t, e_i) \) and \( Q_2(t, e_i) \) in \( t \in [0, T] \), respectively.

**Proof.** Since \( P_2(t, e_i) \), \( Q_1(t, e_i) \) and \( Q_2(t, e_i) \) are functions of \( K(t, e_i) \) and \( P_1(t, e_i) \), we only need to prove \( K(t, e_i) \) and \( P_1(t, e_i) \) are unique, respectively. In the following context, we will give the proof in detail for \( P_1(t, e_i) \). Along the same lines, we can prove the uniqueness for \( K(t, e_i) \).

Using Itô’s Lemma to \( P_1(s, o(s)) \), we obtain that
\[
P_1(t, e_i) = 1 - \int_t^T \left[ \dot{P}_1(s, o(s)) + \sum_{j=1}^l \dot{q}_j(s)(P_1(s, e_j) - P_1(s, o(s))) \right] ds.
\]
Replacing \( \dot{P}_1(s, o(s)) \) by
\[
-\zeta_1(s, o(s))P_1(s, o(s)) - \zeta_2(s, o(s)) \frac{P^3(s, o(s))}{K(s, o(s))} - \sum_{j=1}^l q_j(s)(P_1(s, e_j) - P_1(s, o(s)))
\]
yields
\[
P_1(t, e_i) = 1 + \int_t^T \left[ \zeta_1(s, o(s))P_1(s, o(s)) + \zeta_2(s, o(s)) \frac{P^3(s, o(s))}{K(s, o(s))} \right] ds,
\]
where
\[
\begin{align*}
\zeta_1(t, e_i) &= r_0 - (r_1 - r_0 + a_{11})A_1(t, e_i) - (r_2 - r_0 + a_{21})A_2(t, e_i), \\
\zeta_2(t, e_i) &= (r_1 - r_0 + a_{11})A_1(t, e_i) + (r_2 - r_0 + a_{21})A_2(t, e_i), \\
q_j(t) &= \sum_{i=1, i \neq j}^l q_{ij} \int_0^t \alpha(s-), e_i ds.
\end{align*}
\]
Assume that \( P_1(t, e_i) \) and \( P_1^*(t, e_i) \) are two solutions to (3.14), we have
\[
\begin{align*}
|P_1(t, e_i) - P_1^*(t, e_i)| &= \int_t^T \left[ \zeta_1(s, o(s))(P_1(s, o(s)) - P_1^*(s, o(s))) \\
&+ \zeta_2(s, o(s)) \frac{P^3(s, o(s)) - P^3_1(s, o(s))}{K(s, o(s))} \right] ds \\
&\leq M_1 \int_t^T \left[ P_1(s, o(s)) - P_1^*(s, o(s)) \right] ds, \text{ a.s.}
\end{align*}
\]
where $M_1$ is a positive constant. The last inequality follows from the assumptions $(A_1)$ and $(A_2)$. Then the Gronwall inequality implies that $P_1(t, e_i) = P^*_1(t, e_i)$. Therefore, $P_1(t, e_i)$ is a unique solution to (3.14).

**Remark 3.1.** From the results of Lemma 3.3, we can rewrite (3.9) in a simpler way, that is,

\[
\begin{align*}
\hat{u}_1(t, x, e_i) &= \frac{m_1}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}, \\
\hat{u}_2(t, x, e_i) &= \frac{m_2}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}.
\end{align*}
\]

From (3.18), we know that the symbols of the strategies $\hat{u}_1(t, x, e_i)$ and $\hat{u}_2(t, x, e_i)$ are determined by $m_{1i}$ and $m_{2i}$, respectively.

Now we will solve the original problem. For the admissible strategy’s condition that $(u_1^*, u_2^*) \in U$ and by the result of Lemma 3.2, we need to discuss the following two cases:

**Case 1:** $\mu_1 \mu_2 < \frac{\delta_1 - \rho \sigma_1 \sigma_2}{\Lambda_{0_1}}$

**Case 2:** $\frac{\delta_1 - \rho \sigma_1 \sigma_2}{\Lambda_{0_1}} \leq \mu_1 \mu_2 \leq \frac{\delta_2 - \rho \sigma_1 \sigma_2}{\Lambda_{0_1}}$.

In the following, we will only give the detail analysis for Case 1, the results for Case 2 can be derived by the same way.

**Case 1:** $\mu_1 \mu_2 < \frac{\delta_1 - \rho \sigma_1 \sigma_2}{\Lambda_{0_1}}$

In this case, $m_{1i} > 0$ and $m_{2i} > 0$, and then the optimal strategies are given by (3.18). That is,

\[
\begin{align*}
u_1^*(t, x, e_i) &= \frac{m_{1i}}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}, \\
u_2^*(t, x, e_i) &= \frac{m_{2i}}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}.
\end{align*}
\]

Comparing with Lemma 3.3, in this case, $A_1(t, e_i), A_2(t, e_i), B_1(t, e_i)$ and $B_2(t, e_i)$ are taking values as follows

\[
\begin{align*}
A_1(t, e_i) &= 0, \\
A_2(t, e_i) &= 0, \\
B_1(t, e_i) &= \frac{m_{1i}}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}, \\
B_2(t, e_i) &= \frac{m_{2i}}{\gamma(t, e_i)} e^{-\int_t^T r_u(s) ds}.
\end{align*}
\]

Then we can directly drive

\[
\begin{align*}
K(t, e_i) &= e^{\int_t^T 2r_u(s) ds}, \\
P_1(t, e_i) &= e^{\int_t^T r_u(s) ds}, \\
P_2(t, e_i) &= 2P_1(t, e_i)Q_1(t, e_i),
\end{align*}
\]

and

\[
\begin{align*}
Q_1(t, e_i) &= E_{t, e_i} \left\{ e^{\int_t^T \frac{n_{2\alpha}(s)}{\gamma(s, \alpha(s))} ds} \right\}, \quad (3.19) \\
Q_2(t, e_i) &= E_{t, e_i} \left\{ e^{\int_t^T \left[ \frac{2n_{2\alpha}(s)}{\gamma(s, \alpha(s))} Q_1(s, \alpha(s)) + \frac{n_{1\alpha}(s)}{2\gamma^2(s, \alpha(s))} \right] ds} \right\}, \quad (3.20)
\end{align*}
\]

where

\[
\begin{align*}
n_{1i} &= (\sigma_{1i}^2 + \xi_{1i}^2) m_{1i}^2 + (\sigma_{2i}^2 + \xi_{2i}^2) m_{2i}^2 + 2(\lambda_{0i} \sigma_{1i} \mu_{2i} + \rho \sigma_{1i} \sigma_{2i}) m_{1i} m_{2i}, \\
n_{2i} &= (r_{1i} - r_0 + \alpha_{1i}) m_{1i} + (r_{2i} - r_0 + \alpha_{2i}) m_{2i}.
\end{align*}
\]
Here $Q_1(t, e_i)$ and $Q_2(t, e_i)$ are derived respectively by the standard procedure to the Feynman-Kac representation of the solution to a system of differential equations

$$
\begin{cases}
\dot{Q}_1(t, e_i) + \frac{n_i}{\gamma(t, e_i)} + \sum_{j=1}^{f} q_{ij}[Q_1(t, e_j) - Q_1(t, e_i)] = 0, \\
Q_1(T, e_i) = 0,
\end{cases}
$$

and

$$
\begin{cases}
\dot{Q}_2(t, e_i) + \frac{n_i}{\gamma(t, e_i)} Q_1(t, e_i) + \sum_{j=1}^{f} q_{ij}[Q_2(t, e_j) - Q_2(t, e_i)] = 0, \\
Q_2(T, e_i) = 0.
\end{cases}
$$

These results are summarized in the following proposition.

**Proposition 3.1.** When $\mu_{11} \mu_{21} < \frac{\delta_{11} - \rho \sigma_1 \sigma_2}{\lambda_{0i}}$, the optimal strategy for problem (2.3) and (2.4) is

$$
u^*(t, x, e_i) = (\hat{u}_1(t, x, e_i), \hat{u}_2(t, x, e_i)),
$$

where

$$
\begin{cases}
\hat{u}_1(t, x, e_i) = \frac{(\sigma_1^2 + \xi_1^2)(r_i - r_0 + a_{1i}) - (\lambda_{0i}, \mu_{11}, \mu_{21} + \rho \sigma_1 \sigma_2, i)(r_i - r_0 + a_{2i})}{(\sigma_1^2 + \xi_1^2)(\sigma_1^2 + \xi_1^2)} e^{-\int_t^T r_0(s)ds}, \\
\hat{u}_2(t, x, e_i) = \frac{(\sigma_1^2 + \xi_1^2)(r_i - r_0 + a_{2i}) - (\lambda_{0i}, \mu_{11}, \mu_{21} + \rho \sigma_1 \sigma_2, i)(r_i - r_0 + a_{1i})}{(\sigma_1^2 + \xi_1^2)(\sigma_1^2 + \xi_1^2)} e^{-\int_t^T r_0(s)ds}.
\end{cases}
$$

The value function is

$$V(t, x, e_i) = e^{\int_t^T r_0(s)ds} q(t, e_i) + \frac{\gamma(t, e_i)}{2}(Q_1(t, e_i) - Q_2(t, e_i))$$

with $Q_1(t, e_i), Q_2(t, e_i)$ are given by (3.19) and (3.20), respectively.

**Case 2:** $\frac{\delta_{11} - \rho \sigma_1 \sigma_2}{\lambda_{0i}} \leq \mu_{11} \mu_{21} \leq \frac{\delta_{11} - \rho \sigma_1 \sigma_2}{\lambda_{0i}}$

In this case, we have $m_{1i} \geq 0$ and $m_{2i} \leq 0$ so that $u_2^* = 0$. Plugging $u_2^* = 0$ back into (3.8) yields

$$\tilde{h}(u_1) = [r_0 x + (r_{1i} - r_0 + a_{1i})u_1]\Delta(t, e_i) - \frac{1}{2}(\sigma_1^2 + \xi_1^2)u_1^2 \gamma(t, e_i) K(t, e_i).$$

By the first order condition of $\tilde{h}(u_1)$, we have

$$(r_{1i} - r_0 + a_{1i})\Delta(t, e_i) - (\sigma_1^2 + \xi_1^2)u_1 \gamma(t, e_i) K(t, e_i) = 0,$$

and thus we obtain the maximizer $\hat{u}_1(t, x, e_i)$ of $\tilde{h}(u_1)$. That is

$$\hat{u}_1(t, x, e_i) = \frac{r_{1i} - r_0 + a_{1i}}{(\sigma_1^2 + \xi_1^2)} \frac{\Delta(t, e_i)}{\gamma(t, e_i) K(t, e_i)} - \frac{r_{1i} - r_0 + a_{1i}}{(\sigma_1^2 + \xi_1^2)} \gamma(t, e_i) = \frac{r_{1i} - r_0 + a_{1i}}{(\sigma_1^2 + \xi_1^2)} \gamma(t, e_i) e^{-\int_t^T r_0(s)ds} > 0.$$
Theorem 3.5. When \( \frac{\delta_{11} - \rho \sigma_{11} \sigma_{21}}{\lambda_{0e_1}} \leq \mu_{11} \mu_{21} \leq \frac{\delta_{21} - \rho \sigma_{11} \sigma_{21}}{\lambda_{0e_1}} \), the optimal strategy for problem (2.3) and (2.4) is

\[
 u^*(t, x, e_i) = \begin{cases} 
 r(t, e_i) + a_1(t, e_i) \left[ \frac{r(t, e_i) - r_0 + a_1(t, e_i)}{\gamma(\sigma_{11}^2 + \xi^2)} \right] e^{-f^T r_0(s) ds}, & 0 < f < 1 \\
 r(t, e_i) + a_1(t, e_i) \left[ \frac{r(t, e_i) - r_0 + a_1(t, e_i)}{\gamma(\sigma_{11}^2 + \xi^2)} \right] e^{-f^T r_0(s) ds}, & f \geq 1
\end{cases}
\]

The value function is

\[
 V(t, x, e_i) = e^{f^T r_0(s) ds} x + \frac{1}{2} \langle \delta(t, e_i), \psi(t, e_i) \rangle - Q_1(t, e_i) - Q_2(t, e_i),
\]

where \( Q_1(t, e_i) \) and \( Q_2(t, e_i) \) are given by (3.19), (3.20), (3.21) and (3.22), respectively.
Remark 3.2. Denote by $\mathcal{U}_R$ the set of all $u(t)$ satisfying Remark 2.1 except (b) and the corresponding value function is denoted by $V_R(t, x, e_i)$, which implies that short selling is allowed in this situation. From Theorem 3.5, we can see that the optimal strategies are the same for the cases with and without constraint under the condition of

$$\mu_{1i}\mu_{2i} < \frac{\delta_{1i} - \rho \sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}},$$

and thus the value functions are identical either.

However, when

$$\frac{\delta_{1i} - \rho \sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}} \leq \mu_{1i}\mu_{2i} \leq \frac{\delta_{2i} - \rho \sigma_{1i}\sigma_{2i}}{\lambda_{0e_i}},$$

the optimal strategies are different. For this case, we give a numerical comparison to illustrate the effects of no short selling constraint on the value function. Without loss of generality, assume that there are two regimes in the financial market, that is, ‘bullish’ and ‘bearish’. We set the values of parameters in this case as follows.

| parameters | $T$ | $x$ | $\gamma_i$ | $r_0$ | $r_{1i}$ | $r_{2i}$ | $\rho$ | $\sigma_{1i}$ | $\sigma_{2i}$ | $\mu_{1i}$ | $\mu_{2i}$ | $\beta_{1i}$ | $\beta_{2i}$ | $\lambda_{1i}$ | $\lambda_{2i}$ | $\lambda_{0i}$ |
|------------|-----|-----|----------|------|--------|--------|------|-------------|-------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $e_1$ (bullish) | 5   | 2   | 0.5      | 0.3  | 0.60   | 0.30   | 0.2  | 0.3         | 0.8         | 0.06      | 0.08      | 0.02      | 0.055     | 2         | 0.5       | 1         |
| $e_2$ (bearish) | 5   | 2   | 1        | 0.3  | 0.25   | 0.07   | 0.1  | 0.4         | 0.5         | 0.03      | 0.04      | 0.03      | 0.032     | 1         | 0.3       | 0.6       |

Suppose that the financial market is in state $e_1$ at beginning, then it jumps to the state $e_2$ at time $t = 3$ and keeps in this state until the terminal time. From Figure 1, we can see that $V_R(t, x, e_i)$ is always greater than $V(t, x, e_i)$, which means that it will reduce the mean-variance utility of the investors when the no short selling constraint is considered in this situation.

![Figure 1. The comparison of value functions with and without constraint.](image)

Remark 3.3. In financial market, if

$$\begin{cases}
\delta_{2i} = \frac{(\sigma_{1i}^2 + \xi_{1i}^2)(r_{2i} - r_0 + a_{2i})}{r_{2i} - r_0 + a_{2i}}, \\
\delta_{1i} = \frac{(\sigma_{2i}^2 + \xi_{2i}^2)(r_{1i} - r_0 + a_{1i})}{r_{1i} - r_0 + a_{2i}}.
\end{cases}$$

the optimal results in Theorem 3.5 are still feasible. The only difference is that in Case 2, we will have $u^*(t, x, e_i) = (0, \tilde{u}_2(t, x, e_i))$ and the corresponding parameters of risky asset $S_1(t)$ will be replaced by those of $S_2(t)$. 


Remark 3.4. In Case 2, we just consider a finance market with one risk-free asset and one risky asset since we have no amount invested into another risky asset.

(i): If we suppose that \( \eta_1(t, \alpha(t^-), z_1) = 0 \) and \( \eta_2(t, \alpha(t^-), z_2) = 0 \) in (2.1), i.e., we just discuss diffusion model of the risky assets without jump. In this way, \( \mu_1, \mu_2, \beta_1, \) and \( \beta_2 \) are zero so that \( \alpha_1, \alpha_2, \xi_1^1, \) and \( \xi_2^2 \) are zero as well. Then, the corresponding optimal strategy and value function of Proposition 3.2 are given by:

\[
\begin{align*}
&\tilde{u}_0(t, x, e_i) = \frac{r_1-r_0}{\gamma(t, e, \sigma_1^2)} e^{-\int_t^T r_0(s) ds}, \\
&\hat{V}(t, x, e_i) = e^\int_t^T r_0(s) ds x + \tilde{Q}_1(t, e_i) + \frac{\gamma(t, e)}{2} \left( \tilde{Q}_1^2(t, e_i) - \tilde{Q}_2(t, e_i) \right),
\end{align*}
\]

where \( \tilde{Q}_1(t, e_i) \) and \( \tilde{Q}_2(t, e_i) \) satisfy

\[
\begin{align*}
\dot{\tilde{Q}}_1(t, e_i) + \frac{(r_1-r_0)^2}{\gamma(t, e, \sigma_1^2)} + \sum_{j=1}^l q_{ij} \tilde{Q}_1(t, e_j) &= 0, \\
\dot{\tilde{Q}}_1(T, e_i) &= 0, \\
\dot{\tilde{Q}}_2(t, e_i) + \frac{(r_1-r_0)^2}{\gamma(t, e, \sigma_1^2)} \left( \frac{1}{\gamma(t, e)} + 2\tilde{Q}_1(t, e_i) \right) + \sum_{j=1}^l q_{ij} \tilde{Q}_2(t, e_j) &= 0, \\
\dot{\tilde{Q}}_2(T, e_i) &= 0.
\end{align*}
\]

These results are coincident with those in Wei et al. [23] if there is no liability in their paper.

(ii): (Comparing with pre-commitment strategy) Assume that only one risky asset was considered in Zhou and Yin [32]. According to the Theorem 6.1 in Zhou and Yin [32], the optimal pre-commitment strategy \( u^*(t, x, i) \) is

\[
u^*(t, x, i) = \frac{-r_1(t, e_i) - r_0(t)}{\sigma_1^2(t, e_i)} [x + (\lambda^* - z) e^{-\int_t^T r_0(s) ds}] - \gamma(t, e) e^{-\int_t^T r_0(s) ds}
\]

From (3.23), we can observe that the time-consistent investment strategy is a linear function with respect to the optimal pre-commitment investment strategy. However, it is difficult to decide whether the value of the time-consistent strategy is larger or not, which depends on the parameters of the financial market and the initial wealth of the investor.

(iii): Furthermore, if we do not consider the regime switching in Case 2 with \( \eta_1(t, \alpha(t^-), z_1) = 0 \) and \( \eta_2(t, \alpha(t^-), z_2) = 0 \), and at the same time, \( \gamma(t), r_0(t), r_1(t), r_2(t), \sigma_1(t), \sigma_2(t) \) are constants, then the results of (i) would be

\[
\begin{align*}
&\tilde{u}_0(t, x) = \frac{r_1-r_0}{\sigma_1^2} e^{-r_0(T-t)} , \\
&\hat{V}(t, x) = e^{r_0(T-t)} x + \tilde{Q}_1(t) + \frac{\gamma(t, e)}{2} \left( \tilde{Q}_1^2(t) - \tilde{Q}_2(t) \right),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{Q}_1(t) &= \frac{(r_1-r_0)^2}{\gamma(t, e)} (T-t) , \\
\tilde{Q}_2(t) &= \frac{(r_1-r_0)^2}{\gamma(t, e)} (T-t) + \left( \frac{(r_1-r_0)^2}{\gamma(t, e)} (T-t) \right)^2.
\end{align*}
\]

These results are equivalent to those of Proposition 10.1 in Björk and Murgoci [7] or Remark 5.2 in Wei et al. [23].

(iv): (Comparing with dynamically optimal strategy) Based on the case (B) in Theorem 3 of Pedersen and Peskír [20], the optimal amount \( u^*(t, x) \) invested into
the risky asset is given as
\[
    u^*(t, x) := u^d_0(t, x) = \frac{r_1 - r_0}{\gamma \sigma^2_1} e^{\frac{(r_1 - r_0)^2}{\sigma^2_1}(T-t)}
\]

(3.24)

with \( c \) in Pedersen and Peskir [20] replaced by \( \frac{1}{2} \). Equation (3.24) states that the time-consistent investment strategy is positive proportion to the dynamically optimal investment strategy. Besides, compared with a time-consistent investor, the dynamically investor will invest more into the risky asset for a same financial market, and thus she or he will obtain more benefits while take on much more risk.

4. Numerical examples. In this section, we perform a numerical analysis to illustrate the effect of different parameters on optimal strategies. Assume that \( l = 2 \), i.e., the financial market is divided into two regimes as ‘bullish’ and ‘bearish’, which correspond to regime 1 and regime 2 in our examples, respectively. For convenience, we assume that \( \gamma(t, e_i) := \gamma(e_i) \), \( \eta_1(t, e_i, X_{e_i}) := X_{e_i} \) and \( \eta_2(t, e_i, Y_{e_i}) := Y_{e_i} \). Denote \( \lambda_{0e_i} \) and \( \gamma(e_i) \) by \( \lambda_{0i} \) and \( \gamma_i \), respectively. In the following examples, unless stated otherwise, the values of these parameters are given as in Table 2.

| parameters | \( T \) | \( x \) | \( \gamma_i \) | \( r_0 \) | \( r_{1i} \) | \( r_{2i} \) | \( \rho \) | \( \sigma_{1i} \) | \( \sigma_{2i} \) | \( \mu_{1i} \) | \( \mu_{2i} \) | \( \beta_{1i} \) | \( \beta_{2i} \) | \( \lambda_{1i} \) | \( \lambda_{2i} \) | \( \lambda_{0i} \) |
|-----------|-------|-------|--------|--------|--------|--------|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( e_1 \) (bullish) | 5 | 2 | 0.5 | 0.3 | 0.6 | 0.65 | 0.2 | 0.7 | 0.8 | 0.06 | 0.08 | 0.04 | 0.045 | 2 | 0.5 | 1 |
| \( e_2 \) (bearish) | 5 | 2 | 1 | 0.3 | 0.35 | 0.37 | 0.1 | 0.4 | 0.5 | 0.03 | 0.04 | 0.03 | 0.032 | 1 | 0.3 | 0.6 |

We can see from Table 2 that the parameters’ values of the risky assets, such as appreciation rate \( r_{ki} \) and the expect value of jump amplitude \( \mu_{ki} \), in bullish state are larger than those in bearish case, which is reasonable and realistic for the real financial market. Besides, larger return means greater risk. Therefore, in bullish state, volatility rate \( \sigma_{ki} \) and second moment \( \beta_{ki} \) are larger than those in bearish market as well.

Figure 2 shows that the optimal portfolio strategies \( u^*_1 \) and \( u^*_2 \) increase as \( t \) increases. It is easy to see that \( u^*_1 \) and \( u^*_2 \) are smaller in ‘bearish’ economic environment than those in ‘bullish’ case, respectively. It’s reasonable since ‘bearish’
economic environment means bad economic situation so that the investor would rather invest less money to the risky asset. In addition, this figure also reveals us that under the same economic environment (for example, bullish), the investor would invest more to the risky asset $S_1(t)$. This is to be expected since risky asset $S_1(t)$ has smaller risk and the investor is risk averse.

Figure 3 investigates the influence of common shock dependence parameter $\lambda_{01}$ on optimal strategies $u_1^*$ and $u_2^*$ under the regime 1. In figures (a) and (b) with $\mu_{11} = 0.06$ and $\mu_{21} = 0.08$, we can see that $u_1^*$ and $u_2^*$ increase as $\lambda_{01}$ increases, while we draw the opposite conclusion in figures (c) and (d) with $\mu_{11} = -0.06$ and $\mu_{21} = -0.08$. This is also to be expected. A greater value of $\lambda_{01}$ implies a greater value of expected jump number, and thus the investor would invest more to the risky asset when $\mu_{11}$ and $\mu_{21}$ are positive and invest less when $\mu_{11}$ and $\mu_{21}$ are negative.

![Figure 3](image_url)

**Figure 3.** The effect of parameter $\lambda_{01}$ on optimal strategies with regime 1.

We can also observe that the investor would lose benefit when the market is good (positive jump) and bear more risks when the market is bad (negative jump) if she adopts such an investment strategy obtained under the no common shock dependence (see the solid line in Figure 3). That is, when the market is good, if an investor uses the solid line investment strategy, which means that he/she invests less, he/she would lose some benefits since the risky assets in this situation have higher...
returns; while, when the market is bad, if an investor uses the solid line investment strategy as well, implying that he/she invests more, he/she would bear more risks because the risky assets in this case have lower returns and higher risks. From this perspective, a common shock dependent risk model reflects the financial market more realistic and reasonable, and actually it has reduced the investor’s losses and risks in some degree. Therefore, it is necessary and meaningful to investigate the common shock dependent risk model for a financial market.

Figure 4 shows that, compared with the pure diffusion market, the investor would invest more to the risky asset when $\mu_{11}$ and $\mu_{21}$ are positive and invest less when $\mu_{11}$ and $\mu_{21}$ are negative in the jump-diffusion market. This is a natural consequence in the practical financial market.

Moreover, for the realistic jump-diffusion market, it is worth noting that the investor would lose benefit when the jump is positive and face more risks when the jump is negative if she adopts the investment strategy obtained under the geometric Brownian motion. In other words, the jump-diffusion asset model in fact reduces the investor’s losses and risks. Thus, it is significant to consider a jump-diffusion risky asset model.

Figure 5 discusses the effects of the risk-free rate on optimal strategies. It shows that the optimal strategies $u_1^*$ and $u_2^*$ decrease as risk-free rate $r_0$ increases. This
Further results. In this section, we may extend our work to the case of a financial market consisting of \( n \geq 3 \) risky assets, and the dollar amounts invested in the \( n \) stocks are \( u(t) = (u_1(t), u_2(t), \ldots, u_n(t)) \). Similar to equation (2.1), assume that the price process of the stock is governed by

\[
dS_k(t) = S_k(t^-) \left[ r_k(t, \alpha(t^-)) dt + \sigma_k(t, \alpha(t^-)) dW_k(t) \right. \\
+ \left. \int_{\mathbb{R}_0} \eta_k(t, \alpha(t^-), z_k) M_{k, \alpha(t^-)}(dz_k, dt) \right], \quad k = 1, 2, \ldots, n,
\]

where

\[
M_{k, \alpha(t^-)}(dz_k, dt) = N_{k, \alpha(t^-)}(dz_k, dt) + N_{0, \alpha(t^-)}(dz_k, dt).
\]

Suppose that \( N_{1, \alpha(t^-)}(\cdot, dt), N_{2, \alpha(t^-)}(\cdot, dt), \ldots, N_{n, \alpha(t^-)}(\cdot, dt) \) and \( N_{0, \alpha(t^-)}(\cdot, dt) \) are mutually independent, and the correlation coefficient of \( W_k(t) \) and \( W_j(t) \) is denoted by \( \rho_{kj} \). Obviously, we have \( \rho_{kj} = \rho_{jk} \) and \( \rho_{kk} = 1 \), for \( k, j = 1, 2, \ldots, n \).

Then the state process with investment strategy \( u(t) \) becomes

\[
dR^u(t) = \left[ r_0(t)R^u(t) + \sum_{k=1}^{n} (r_k(t, \alpha(t^-)) - r_0(t))u_k(t) \right] dt \\
+ \sum_{k=1}^{n} \sigma_k(t, \alpha(t^-))u_k(t) dW_k(t) \\
+ \sum_{k=1}^{n} \int_{\mathbb{R}_0} \eta_k(t, \alpha(t^-), z_k)u_k(t) N_{k, \alpha(t^-)}(dz_k, dt) \\
+ \int \cdots \int \sum_{k=1}^{n} \eta_k(t, \alpha(t^-), z_k)u_k(t) N_{0, \alpha(t^-)}(dz_1, dz_2, \ldots, dz_n, dt).
\]

Under the same objective function and criterion as in (2.3) and (2.4), with the same approach as in Section 3, it is straightforward to derive the corresponding Hessian matrix as

\[
H = -\gamma(t, e_i)K(t, e_i)A,
\]

where

\[
K(t, e_i) = e^{\int_0^T 2r_0(s) ds} > 0,
\]

and

\[
A = \begin{pmatrix}
(\sigma^2_{i1} + \xi^2_{1i}) & \varsigma_{12} & \cdots & \varsigma_{1j} & \cdots & \varsigma_{1n} \\
\varsigma_{21} & (\sigma^2_{21} + \xi^2_{21}) & \cdots & \varsigma_{2j} & \cdots & \varsigma_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\varsigma_{ni} & \varsigma_{n2} & \cdots & \varsigma_{nj} & \cdots & (\sigma^2_{ni} + \xi^2_{ni})
\end{pmatrix}
\]

(5.1)

with \( \varsigma_{kj} = \varsigma_{jk} = (\rho_{kj} \sigma_{ki} \sigma_{ji} + \lambda_{0i} \mu_{ki} \mu_{ji}) \).
Then for any vector \( C = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n \), we have

\[
CAC^T = \sum_{j=1}^{n} c_j^2 (\sigma_{ji}^2 + \xi_{ji}^2) + \sum_{j=1}^{n} \sum_{k \neq j}^n c_j c_k (\rho_{kj} \sigma_{ki} \sigma_{ji} + \lambda_{oi} \mu_{ki} \mu_{ji} )
\]

\[
> \lambda_{oi} \left( \sum_{j=1}^{n} c_j \mu_{ji} \right)^2 + \sum_{j=1}^{n} \left( c_j^2 \sigma_{ji}^2 + \sum_{k \neq j}^n c_j c_k \rho_{kj} \sigma_{ki} \sigma_{ji} \right)
\]

(5.2)

From the above inequality of (5.2), it is not difficult to find that matrix \( A \) is a positive definite matrix if \( \rho_{kj} = 0 \) or \( \rho_{kj} = 1 \) for any \( k, j = 1, 2, \cdots, n \), so that the Hessian matrix \( H \) is a negative definite matrix.

However, if \( \rho_{kj} = 0 \) or \( \rho_{kj} = 1 \) does not hold for any \( k, j = 1, 2, \cdots, n \), the matrix \( A \) is not necessary positive definite for \( n \geq 3 \). For example, let \( n = 3 \), and the value of some parameters are given as follows:

| Parameters-set | \( \sigma_{11} \) | \( \sigma_{21} \) | \( \sigma_{31} \) | \( \lambda_{01} \) | \( \lambda_{11} \) | \( \lambda_{21} \) | \( \lambda_{31} \) | \( \mu_{11} \) | \( \mu_{21} \) | \( \mu_{31} \) | \( \beta_{11} \) | \( \beta_{21} \) | \( \beta_{31} \) | \( \rho_{12} \) | \( \rho_{13} \) | \( \rho_{23} \) |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | 1.3             | 1.5             | 2.5             | 0.1             | 0.6             | 0.2             | 0.4             | 0.2             | 0.5             | 0.3             | 0.30            | 0.22            | 0.85            | 0.40            | 1.0             | 0.22            | 0.85            | 0.40            | 1.0             |

By some calculation, we get \( \det(A) = -0.1189 < 0 \). According to the properties of positive definite matrix, we know that matrix \( A \) is not a positive definite matrix and then the Hessian matrix \( H \) is not negative definite. Therefore, there is no maximizer point existed for the control problem in this case. That is to say, for \( n \geq 3 \), whether the Hessian matrix \( H \) is negative definite or not strongly depends on the values of correlation coefficient. But for the positive definite matrix \( A \), we can get the following results.

**Proposition 5.1.** For the problem (2.3) and (2.4) with \( n \geq 3 \) risky assets, let \( A \) be a matrix given by (5.1). Suppose that \( A \) is a positive definite matrix, then the optimal investment strategy

\[
u^*(t) = (u_1^*(t), u_2^*(t), \cdots, u_n^*(t))
\]

without no-short-selling constraint is

\[
u_k^*(t) = \frac{1}{\gamma(t, e_i)} \left| A_k \right| A \left| e^{-\int_t^T \gamma_0(s)ds} \right|, \quad k = 1, 2, \cdots, n,
\]

where

\[
A_k = \begin{pmatrix}
(\sigma_{11}^2 + \xi_{11}^2) & \varsigma_{12} & \cdots & \varsigma_{1,k-1} & (r_{11} - r_0 + a_{11}) & \varsigma_{1,k+1} & \cdots & \varsigma_{1n} \\
\varsigma_{21} & (\sigma_{21}^2 + \xi_{21}^2) & \cdots & \varsigma_{2,k-1} & (r_{21} - r_0 + a_{21}) & \varsigma_{2,k+1} & \cdots & \varsigma_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\varsigma_{n1} & \varsigma_{n2} & \cdots & \varsigma_{n,k-1} & (r_{ni} - r_0 + a_{ni}) & \varsigma_{n,k+1} & \cdots & (\sigma_{ni}^2 + \xi_{ni}^2)
\end{pmatrix}
\]

with \( \varsigma_{kj} = \varsigma_{jk} = (\rho_{kj} \sigma_{ki} \sigma_{ji} + \lambda_{0i} \mu_{ki} \mu_{ji}) \).

6. **Conclusion.** We first recap the main results of the paper. We consider a mean-variance optimal portfolio problem for an investor, whose risky asset price processes are modulated by jump-diffusion process, where the jump number processes are correlative through common shock dependence and the Brownian motions are supposed to be dependent. Moreover, we assume that short selling is prohibited and the market state can take values in one of a finite number of regimes, which lead to the problem more complicated, and thus we have to discuss it with two cases.
Under the criterion of maximizing the expected mean-variance utility of the terminal wealth, using the technique of stochastic control theory, we derive the explicit expression of the optimal control strategies and value function within a game theoretic framework. We also give some further discussion for the case of a financial market consisting of \( n \geq 3 \) risky assets, the optimal investment strategies without constraint are derived.

For the future research, one can extend the objective criterion of this paper to the mean-variance utility with risk aversion coefficient which depends not only on the market state but also on initial capital; one can also consider the case with the transaction costs during the assets’ transaction, either for fixed transaction cost or for proportional transaction cost. In addition, it is meaningful to study the scaled mean-variance optimization. All these problems will be more challenging.

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REFERENCES

[1] N. Bäuerle, Benchmark and mean-variance problems for insurers, Mathematical Methods of Operations Research, 62 (2005), 159–165.

[2] A. Bensoussan, K. C. Wong, S. C. P. Yam and S. P. Yung, Time-consistent portfolio selection under short-selling prohibition: From discrete to continuous setting, SIAM Journal on Financial Mathematics, 5 (2014), 153–190.

[3] A. Bensoussan, K. C. Wong and S. C. P. Yam, A paradox in time-consistency in the mean-variance problem, Finance and Stochastics, 23 (2019), 173–207.

[4] J. Bi and J. Cai, Optimal investment-reinsurance strategies with state dependent risk aversion and VaR constraints in correlated markets, Insurance: Mathematics and Economics, 85 (2019), 1–14.

[5] J. Bi and J. Guo, Optimal mean-variance problem with constrained controls in a jump-diffusion financial market for an insurer, Journal of Optimization Theory and Applications, 157 (2013), 252–275.

[6] J. Bi, Z. Liang and F. Xu, Optimal mean-variance investment and reinsurance problems for the risk model with common shock dependence, Insurance: Mathematics and Economics, 70 (2016), 245–258.

[7] T. Björk and A. Murgoci, A General Theory of Markovian Time Inconsistent Stochastic Control Problems, Working Paper, Stockholm School of Economics, Sweden, 2010.

[8] T. Björk, A. Murgoci and X. Y. Zhou, Mean-variance portfolio optimization with state dependent risk aversion, Mathematical Finance, 24 (2014), 1–24.

[9] P. Chen, H. Yang and G. Yin, Markowitz’s mean-variance asset-liability management with regime switching: A continuous-time model, Insurance: Mathematics and Economics, 43 (2008), 456–465.

[10] R. J. Elliott and J. Hoek, An application of hidden Markov models to asset allocation problems, Finance and Stochastics, 1 (1997), 229–238.

[11] P. A. Forsyth and J. Wang, Continuous time mean variance asset allocation: A time-consistent strategy, European Journal of Operational Research, 209 (2011), 184–201.

[12] D. Li and W. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, Mathematical Finance, 10 (2000), 387–406.

[13] Z. Liang and M. Song, Time-consistent reinsurance and investment strategies for mean-variance insurer under partial information, Insurance: Mathematics and Economics, 65 (2015), 66–76.

[14] Z. Liang, J. Bi, K. C. Yuen and C. Zhang, Optimal mean-variance reinsurance and investment in a jump-diffusion financial market with common shock dependence, Mathematical Methods of Operations Research, 84 (2016), 155–181.

[15] Z. Liang and K. C. Yuen, Optimal dynamic reinsurance with dependent risks: Variance premium principle, Scandinavian Actuarial Journal, 2016 (2016), 18–36.
[16] Z. Liang, K. C. Yuen and C. Zhang, Optimal reinsurance and investment in a jump-diffusion financial market with common shock dependence, Journal of Applied Mathematics and Computing, 56 (2018), 637–664.

[17] A. E. B. Lim and X. Y. Zhou, Quadratic hedging and mean-variance portfolio selection with random parameters in a complete market, Mathematics of Operations Research, 27 (2002), 101–120.

[18] H. M. Markowitz, Portfolio Selection, John Wiley & Sons, Inc., New York, Chapman & Hall Ltd., London, 1959.

[19] Z. Ming, Z. Liang and C. Zhang, Optimal mean-variance reinsurance with dependent risks, ANZIAM Journal, 58 (2016), 162–181.

[20] J. L. Pedersen and G. Peskir, Optimal mean-variance portfolio selection, Mathematics and Financial Economics, 11 (2017), 137–160.

[21] H. R. Richardson, A minimum variance result in continuous trading portfolio optimization, Management Science, 35 (1989), 1045–1055.

[22] Z. Sun and J. Guo, Optimal mean-variance investment and reinsurance problem for an insurer with stochastic volatility, Mathematical Methods of Operations Research, 88 (2018), 59–79.

[23] J. Wei, K. C. Wong, S. C. P. Yam and S. P. Yung, Markowitz’s mean-variance asset-liability management with regime switching: A time-consistent approach, Insurance: Mathematics and Economics, 53 (2013), 281–291.

[24] J. Wei, D. Li and Y. Zeng, Robust optimal consumption-investment strategy with nonexponential discounting, Journal of Industrial and Management Optimization, 16 (2020), 207–230.

[25] J. Wei, H. Yang and R. Wang, Classical and impulse control for the optimization of dividend and proportional reinsurance policies with regime switching, Journal of Optimization Theory and Applications, 147 (2010), 358–377.

[26] K. C. Yuen, Z. Liang and M. Zhou, Optimal proportional reinsurance with common shock dependence, Insurance: Mathematics and Economics, 64 (2015), 1–13.

[27] Y. Zeng, Z. Li and Y. Lai, Time-consistent investment and reinsurance strategies for mean-variance insurers with jumps, Insurance: Mathematics and Economics, 52 (2013), 498–507.

[28] C. Zhang and Z. Liang, Portfolio optimization for jump-diffusion risky assets with common shock dependence and state dependent risk aversion, Optimal Control Applications and Methods, 38 (2017), 229–246.

[29] X. Zhang and T. K. Siu, On optimal proportional reinsurance and investment in a Markovian regime-switching economy, Acta Mathematica Sinica English Series, 28 (2012), 67–82.

[30] J. Zhou, X. Yang and J. Guo, Portfolio selection and risk control for an insurer in the Lévy market under mean-variance criterion, Statistics and Probability Letters, 126 (2017), 139–149.

[31] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, Applied Mathematics and Optimization, 42 (2000), 19–33.

[32] X. Y. Zhou and G. Yin, Markowitz’s mean-variance portfolio selection with regime switching: A continuous-time model, SIAM Journal on Control and Optimization, 42 (2003), 1466–1482.

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