NON-FACTORISATION OF ARF-KERVAIRE CLASSES THROUGH $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$

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Abstract. As an application of the upper triangular technology method of [8] it is shown that there do not exist stable homotopy classes of $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$ in dimension $2^s+1 - 2$ with $s \geq 2$ whose composition with the Hopf map to $\mathbb{R}P^\infty$ followed by the Kahn-Priddy map gives an element in the stable homotopy of spheres of Arf-Kervaire invariant one.

1. Introduction

1.1. For $n > 0$ let $\pi_n(\Sigma^\infty S^0)$ denote the $n$-th stable homotopy group of $S^0$, the 0-dimensional sphere. Via the Pontrjagin-Thom construction an element of this group corresponds to a framed bordism class of an $n$-dimensional framed manifold. The Arf-Kervaire invariant problem concerns whether or not there exists such a framed manifold possessing a Kervaire surgery invariant which is non-zero (modulo 2). In [4] it is shown that this can happen only when $n = 2^s+1 - 2$ for some $s \geq 1$. Resolving this existence problem is an important unsolved problem in homotopy theory (see [8] for a historical account of the problem together with new proofs of all that was known up to 2008). Recently important progress has made ([5]; see also [2], [3]) which shows that $n = 126$ is the only remaining possibility for existence (more details may be found in the survey article [9]).

In view of the renewed interest in the Arf-Kervaire invariant problem it may be of interest to describe a related non-existence result. An equivalence formulation (see [8], §1.8) is that there exists a stable homotopy class $\Theta : \Sigma^\infty S^{2^s+1-2} \to \Sigma^\infty \mathbb{R}P^\infty$ with mapping cone $\text{Cone}(\Theta)$ such that the Steenrod operation

$$Sq^{2^s} : H^{2^s-1}(\text{Cone}(\Theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \to H^{2^s+1-1}(\text{Cone}(\Theta); \mathbb{Z}/2)$$

is non-trivial. Using the upper triangular technology (UTT) of [8] we shall prove the following result:

Theorem 1.2.

Let $H : \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \to \Sigma^\infty \mathbb{R}P^\infty$ denote the map obtained by applying the Hopf construction to the multiplication on $\mathbb{R}P^\infty$. Then, if $s \geq 2$, there does not exist a stable homotopy class

$$\tilde{\Theta} : \Sigma^\infty S^{2^s+1-2} \to \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$$

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such that the composition $\Theta = H \cdot \tilde{\Theta}$ is detected by a non-trivial $Sq^{2^r}$ as in §1.1.

In §2.2 this result will be derived as a simple consequence of the UTT relations ([8] Chapter Eight). The basics of the UTT method are sketched in §2.1. Doubtless there are other ways to prove Theorem 1.2 (for example, from the results of [10]; see also [8] Chapter Two) but it provides an elegant application of UTT.

2. Upper triangular technology (UTT)

2.1. Let $F_{2n}(\Omega^2 S^3)$ denote the $2^n$-th filtration of the combinatorial model for $\Omega^2 S^3 \simeq W \times S^1$. Let $F_{2n}(W)$ denote the induced filtration on $W$ and let $B(n)$ be the Thom spectrum of the canonical bundle induced by $f_n: \Omega^2 S^3 \rightarrow BO$, where $B(0) = S^0$ by convention. From [7] one has a 2-local, left $bu$-module homotopy equivalence of the form $\bigvee_{n \geq 0} bu \wedge \Sigma^{4n} B(n) \xrightarrow{\sim} bu \wedge bo$.

Therefore, if $\Theta$ is as in §1.1 then

$$(bu \wedge bo)_* (\text{Cone}(\Theta)) \cong \bigoplus_{n \geq 0} (bu_*(\text{Cone}(\Theta) \wedge \Sigma^{4n} B(n)).$$

Let $\alpha(k)$ denote the number of 1’s in the dyadic expansion of the positive integer $k$. For $1 \leq k \leq 2^{s-1} - 1$ and $2^s \geq 4k - \alpha(k) + 1$ there are isomorphisms of the form ([8] Chapter Eight §4)

$$bu_{2^{s+1}-1}(C(\Theta) \wedge \Sigma^{4k} B(k)) \cong bu_{2^{s+1}-1}(\mathbb{RP}^\infty \wedge \Sigma^{4k} B(k)) \cong V_k \oplus \mathbb{Z}/2^{2s-4k+\alpha(k)}$$

where $V_k$ is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e. in filtration zero, represented on the $s = 0$ line) in the mod 2 Adams spectral sequence. The map $1 \wedge \psi^3 \wedge 1$ on $bu \wedge bo \wedge C(\Theta)$ acts on the direct sum decomposition like the upper triangular matrix

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \ldots \\
0 & 9 & 1 & 0 & 0 & \ldots \\
0 & 0 & 9^2 & 1 & 0 & \ldots \\
0 & 0 & 0 & 9^3 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}.$$

\[\text{In [8] and related papers I consistently forgot what I had written in my 1998 McMaster University notes “On } bu_*(BD_8)\text{”. Namely, in the description of Mahowald’s result I stated that } \Sigma^{4n} B(n) \text{ was equal to the decomposition factor } F_{in}/F_{in-1} \text{ in the Snaith splitting of } \Omega^2 S^3. \text{ Although this is rather embarrassing, I got the homology correct so that the results remain correct upon replacing } F_{in}/F_{in-1} \text{ by } \Sigma^{4n} B(n) \text{ throughout! I have seen errors like this in the World Snooker Championship where the no.1 player misses an easy pot by concentrating on positioning the cue-ball. In mathematics such errors are inexcusable whereas in snooker they only cost one the World Championship.}\]
In other words \((1 \wedge \psi^3 \wedge 1)_*\) sends the \(k\)-th summand to itself by multiplication by \(9^{k-1}\) and sends the \((k - 1)\)-th summand to the \((k - 2)\)-th by a map

\[
(t_{k,k-1})_* : V_k \oplus \mathbb{Z}/2^{2^s-4k+\alpha(k)} \rightarrow V_{k-1} \oplus \mathbb{Z}/2^{2^s-4k+4+\alpha(k-1)}
\]

for \(2 \leq k \leq 2^{s-1} - 1\) and \(2^s \geq 4k - \alpha(k) + 1\). The right-hand component of this map is injective on the summand \(\mathbb{Z}/2^{2^s-4k+\alpha(k)}\) and annihilates \(V_k\).

It is shown in [6] (also proved by UTT in [8] Chapter Eight when \(s \geq 2\)) that \(\Theta\) corresponds to a stable homotopy class of Arf-kervaire invariant one if and only if it is detected by the Adams operation \(\psi^3\) on \(i \in \text{bu}_{2s+1-1}(\text{Cone}(\Theta))\), an element of infinite order.

From these properties and the formula for \(\psi^3(i)\) one easily obtains a series of equations ([8] \$8.4.3\$) for the components of \((\eta \wedge 1 \wedge 1)_*(i)\) where \(\eta : S^0 \rightarrow \text{bu}\) is the unit of \(\text{bu}\)-spectrum. Here we have used the isomorphism \(\text{bu}_{2s+1-1}(C(\Theta)) \cong \text{bo}_{2s+1-1}(C(\Theta))\) since, strictly speaking, the latter group is the domain of \((\eta \wedge 1 \wedge 1)_*\). It is shown in ([8] Theorem 8.4.7) that this series of equations implies that the \(\text{bu}_{2s+1-1}(C(\Theta) \wedge \Sigma^2 B(2^{s-2}))\)-component of \((\eta \wedge 1 \wedge 1)_*(i)\) is non-trivial and gives some information on the identity of this non-trivial element.

It is this information which we shall now use to prove Theorem 1.2.

2.2. Proof of Theorem 1.2

Suppose, for a contradiction, that \(\Theta\) and \(\tilde{\Theta}\) exist. We must assume that \(s \geq 2\) because the UTT results of ([8] Theorem 8.4.7) are only claimed for this range.

The mod 2 cohomology of \(\Sigma^2 B(2^{s-2})\) is given by the \(\mathbb{F}_2\)-vector space with basis \(\{z_{2^j}, 0 \leq j \leq 2^{s-1} - 2; z_{2^j+2}, 0 \leq k \leq 2^{s-1} - 2\}\) on which the left action by \(S^1 q^{0,1} = S^1 q^1 s + S^2 q^2 s\) are given by \(S^1(z_{2^j}) = z_{2^j+1}\) for \(1 \leq j \leq 2^{s-1} - 1\) and \(S^1(z_{2^j}) = z_{2^j+2}\) for \(0 \leq j \leq 2^{s-1} - 2\) and \(S^0, S^1, S^0,1\) are zero otherwise. This cohomology module is the \(\mathbb{F}_2\)-dual of the “lightning flash” module depicted in ([1] p.341).

Now consider the two 2-local Adams spectral sequences

\[
E_2^{s,t} = \text{Ext}_B^{s,t}(H^*(C(\Theta); \mathbb{Z}/2) \otimes H^*(\Sigma^2 B(2^{s-2}; \mathbb{Z}/2)), \mathbb{Z}/2)
\]

which collapses and

\[
\tilde{E}_2^{s,t} = \text{Ext}_B^{s,t}(H^*(\tilde{C}(\Theta); \mathbb{Z}/2) \otimes H^*(\Sigma^2 B(2^{s-2}; \mathbb{Z}/2)), \mathbb{Z}/2)
\]

where \(B\) is the exterior subalgebra of the mod 2 Steenrod algebra generated by \(S^1 q^1\) and \(S^0,1\).

To fit in with the notation of ([8] Theorem 8.4.7) set \(s = q + 2\) in Theorem 1.2. As mentioned in §2.1 it is shown in ([8] Theorem 8.4.7) that the
component of $(\eta \wedge 1 \wedge 1)_*(t)$ lying in
\[ bu_{2g+3-1}(C(\Theta) \wedge \Sigma^2 P(B(2^s-2))) \]
\[ \cong bu_{2g+3-1}(\mathbb{R}P^\infty \wedge \Sigma^2 P(B(2^s-2))) \]
\[ \cong \text{Ext}^0_B(2^{g+3}-1)(H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes H^*(\Sigma^2 P(B(2^s-2); \mathbb{Z}/2), \mathbb{Z}/2) \]
\[ \subseteq \text{Hom}(\oplus_{u+v=2g+3-1} H^u(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes H^v(\Sigma^2 P(B(2^s-2); \mathbb{Z}/2), \mathbb{Z}/2) \]
corresponds to a homomorphism $f$ such that $f(x^{2^g+2-1} \otimes z_{2^g+2})$ is non-trivial.

The factorisation $\Theta = H \cdot \hat{\Theta}$ implies that there exists $h \in \tilde{E}_2^{0,2g+3-1} \subseteq \tilde{E}_2^{0,2g+3-1}$ such that $H_*(h) = f$. On the other hand
\[ \tilde{E}_2^{0,2g+3-1} \cong \text{Ext}^0_B(2^{g+3}-1)(H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty; \mathbb{Z}/2) \otimes H^*(\Sigma^2 P(B(2^s-2); \mathbb{Z}/2), \mathbb{Z}/2). \]
Therefore the homomorphism
\[ \text{Ext}^0_B(2^{g+3}-1)(H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty; \mathbb{Z}/2) \otimes H^*(F_{2g+2}/F_{2g+2-1}; \mathbb{Z}/2), \mathbb{Z}/2) \]
\[ (H \wedge 1)_* \downarrow \]
\[ \text{Ext}^0_B(2^{g+3}-1)(H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes H^*(F_{2g+2}/F_{2g+2-1}; \mathbb{Z}/2), \mathbb{Z}/2) \]
satisfies $(H \wedge 1)_*(h)(x^{2^{g+2}-1} \otimes z_{2^{g+2}}) = f(x^{2^{g+2}-1} \otimes z_{2^{g+2}}) \neq 0$. However
\[ (H \wedge 1)_*(h)(x^{2^{g+2}-1} \otimes z_{2^{g+2}}) \]
\[ = h(\sum_{a=1}^{2^{g+2}-2} x^a \otimes x^{2^{g+2}-a-1} \otimes z_{2^{g+2}}). \]

On the other hand
\[ Sq^1(x^a \otimes x^{2^{g+2}-2-a} \otimes z_{2^{g+2}}) \]
\[ = \alpha(x^a \otimes x^{2^{g+2}-2-a} \otimes z_{2^{g+2}} + x^{a+1} \otimes x^{2^{g+2}-2-a} \otimes z_{2^{g+2}}) \]
\[ + x^a \otimes x^{2^{g+2}-2-a} \otimes Sq^1(z_{2^{g+2}}) \]
\[ = \alpha(x^a \otimes x^{2^{g+2}-2-a} \otimes z_{2^{g+2}} + x^{a+1} \otimes x^{2^{g+2}-2-a} \otimes z_{2^{g+2}}) \]
since $Sq^1(z_{2^{g+2}})$ is trivial. Therefore
\[ f(x^{2^{g+2}-1} \otimes z_{2^{g+2}}) \in h(\text{Im}(Sq^1) \equiv 0 \]
because $h$ is a $B$-module homomorphism and $Sq^1$ is trivial on $\mathbb{Z}/2$. \[ \square \]

**Remark 2.3.** When $s = 2, 3$ in the situation of Theorem 2.2 there is a map $\alpha : \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$ but it is just not equal to $H$! In the loopspace structure of $Q\mathbb{R}P^\infty$ form the product minus the two projections to give a map $\mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow Q\mathbb{R}P^\infty$ which factors through the smash
product. The adjoint of this factorisation is $\alpha$. Then the smash product of two copies of a map of Hopf invariant one $\Sigma^\infty S^{2s-1} \to \Sigma^\infty \mathbb{R}P^\infty$ composed with $\alpha$ is detected by $Sq^{2s}$ on its mapping cone (see [10]).

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