Kinetic impedance and depairing in thin and narrow superconducting films

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We use both Eilenberger-Usadel and Ginzburg-Landau (GL) theory to calculate the superfluid’s temperature-dependent kinetic inductance for all currents up to the depairing current in thin and narrow superconducting films. The calculations apply to BCS weak-coupling superconductors with isotropic gaps and transport mean-free paths much less than the BCS coherence length. The kinetic inductance is calculated for the response to a small alternating current when the film is carrying a dc bias current. In the slow-experiment/fast-relaxation limit, in which the superconducting order parameter quasistatically follows the time-dependent current, the kinetic inductance diverges as the bias current approaches the depairing value. However, in the fast-experiment/slow-relaxation limit, in which the the superconducting order parameter remains fixed at a value corresponding to the dc bias current, the kinetic inductance rises to a finite value at the depairing current. We then use time-dependent GL theory to calculate the kinetic impedance of the superfluid, which includes not only the kinetic reactance but also the kinetic resistance of the superfluid arising from dissipation due to order-parameter relaxation. The kinetic resistance is largest for angular frequencies $\omega$ obeying $\omega \tau_s > 1$, where $\tau_s$ is the order-parameter relaxation time, and for bias currents close to the depairing current. We also include the normal fluid’s contribution to dissipation in deriving an expression for the total kinetic impedance. The Appendices contain many details about the temperature-dependent behavior of superconductors carrying current up to the depairing value.

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I. INTRODUCTION

The kinetic inductance, arising chiefly from the kinetic energy of the superfluid, plays an important role in superconducting devices fabricated using thin and narrow superconducting films.\(^1\)–\(^3\) In such cases the kinetic inductance is generally much larger than the geometric inductance arising from stored magnetic energy.\(^4\)–\(^6\) For example, the kinetic inductance plays a prominent role in determining the reset time of superconducting single-photon detectors (SSPDs) fabricated with meandering superconducting lines.\(^7\)–\(^9\) Various calculations of the kinetic inductance, relevant to the performance of microstrip resonators\(^10\) and microwave kinetic inductance detectors (MKIDs),\(^11\) have been carried out using (a) the London equations neglecting the current-induced suppression of the order parameter,\(^4\)–\(^6,12\) (b) the Ginzburg-Landau (GL) equations,\(^2,13,14\) (c) a GL-inspired London-equation approach accounting for the current-induced suppression of the order parameter,\(^15,16\) and (d) the BCS theory.\(^2,10\) Our goal in this paper is to present theoretical calculations of the kinetic inductance for all temperatures and for all currents up to the depairing current for sample dimensions and properties applicable to present experimental studies of SSPDs\(^7,17\) and micro-resonators.\(^2\) Because these studies have used thin high-resistance films of NbN,\(^1,2,7\)–\(^9,18\) Nb,\(^2\) NbTiN,\(^19\) and TaN\(^20,21\) in the dirty limit, we adopt an isotropic s-wave BCS description, although many of our results can be extended to apply under more general assumptions. We consider thin ($d \ll \lambda_0$) superconducting films of width $W$ much less than the two-dimensional screening length ($\Lambda$ from Pearl length\(^22\) $\Lambda = 2\lambda_0^2/d$, where $\lambda_0$ is the temperature-dependent weak-field London penetration depth and $d$ is the film thickness. The condition $W \ll \Lambda$ guarantees that the self-field generated by the current has a negligible effect upon the current density $j$, which therefore flows with the same spatial distribution as in the normal state.\(^21\) Moreover, this condition also guarantees that the inductance $L = L_m + L_k$ is dominated by the kinetic inductance of the superfluid $L_k$, which is typically larger than the geometric inductance $L_m$ (associated with the energy stored in the magnetic field) by a factor of order $\Lambda/W$.\(^4\) We focus on the calculation of the superfluid’s kinetic inductivity $L_k$. For a long strip of length $L$, width $W$, and thickness $d$, the kinetic inductance is 

$$L_k = \frac{L_k L}{W}$$

Where the superconductor carries such a low current that the superconducting order parameter is not significantly suppressed, the electromagnetic behavior is well described by the London equation, and the kinetic energy density of the superfluid can be expressed as

$$U_k = \frac{1}{2} n_s m v_s^2 = \frac{1}{2} \left( \frac{m}{n_s e V_s} \right) j_s^2 = \frac{1}{2} \mu_0 \lambda_0^2 j_s^2 = \frac{1}{2} \frac{L_k \lambda_0^2 j_s^2}{\mu_0},$$

(1)

where $n_s$ is the superfluid density, $m$ the electron mass, $v_s$ the superfluid velocity component in the $x$ direction, $j_s = -n_s e v_s$, the supercurrent density component in the $x$ direction, and $-e$ the electron charge, such that the kinetic inductivity of the superfluid obeys

$$L_k = \frac{L_k \lambda_0^2 (T)}{\mu_0} = m n_s e v_s^2.$$  

(2)

The subscripts $0$ on $n_s$, $\lambda_0$, and $L_k$ are a reminder that these quantities apply in the limit as $j_s \to 0$. The simple relationship given in Eq. (2) has made possible the determination of $\lambda_0(T)$ vs $T$ in YBa$_2$Cu$_3$O$_{7-\delta}$ from kinetic-inductance measurements.\(^20\)
When the superconductor carries high currents, however, calculation of the superfluid’s kinetic inductance becomes more complicated, especially when the current density approaches the depairing value $j_d$. In high currents it is no longer possible to define the kinetic inductance by considering only the stored kinetic energy density as in Eq. (1), because, as examined in detail in Appendix D, increasing $j_s$ to large values also affects the superconducting condensation energy by suppressing the superconducting order parameter. To account for this effect, we take advantage of Maki’s recognition that the current-induced suppression of the order parameter in a thin film can be treated using a pair-breaking parameter exactly analogous to that used by Abrikosov and Gor’kov in their study of the effect of paramagnetic impurities upon superconductivity. The fusion of the theories for these two problems has resulted in a large body of related work, much of which we summarize for the benefit of the reader in Sec. II and the Appendices. As noted by previous authors, a convenient starting point for this purpose is the mean-field Eilenberger-Usadel theory.

An additional complication in calculating the kinetic inductance is that $n_s$, which depends on the superconducting order parameter, can change only on a time scale slower than a variety of difficult-to-determine internal relaxation times, which we here represent crudely by a single relaxation time $\tau_s$. As a consequence, $n_s$ may or may not be able to follow the changes in $j_s$ and $v_s$. Calculations of the current dependence of $L_k$ are simplified in two limiting cases: (a) slow experiments (fast relaxation, Sec. III), in which $j_s$ and $v_s$ vary on an experimental time scale $\tau_{\exp}$ much longer than the relaxation time $\tau_s$, such that the order parameter and the superfluid density $n_s$ quasistatically follow $j_s$ and $v_s$, and (b) fast experiments (slow relaxation, Sec. IV), in which $j_s$ and $v_s$ change so rapidly about their time averages $\bar{j}_s$ and $\bar{v}_s$ (on a time scale $\tau_{\exp}$ much shorter than $\tau_s$) that the order parameter and $n_s$ cannot track the time dependence, and $n_s$ remains very close to the value corresponding to $\bar{j}_s$ and $\bar{v}_s$. To provide an approximation to the transition between these two limiting cases, in Sec. V we employ a simplified phenomenological model based on the time-dependent GL (TDGL) equations to calculate the complex impedance due to the superfluid. In Sec. VI we include the normal-fluid’s resistive contribution to the total complex impedance, and in Sec. VII we provide a brief summary and discussion of our results. Various details of the calculation are included in Appendices A-D.

II. SUPERFLUID VELOCITY DEPENDENCE OF THE SUPERCURRENT DENSITY AND THE DEPAIRING CURRENT DENSITY

The purpose of this section is to explain clearly how to calculate the many effects of the current-induced suppression of the order parameter that have been obtained by previous authors. We need a formalism that allows us to calculate the depairing current density $j_d(T)$ in superconductors with a short normal-state mean-free path at all temperatures in the superconducting state. A compact way of doing this is to employ the quasiclassical Eilenberger theory as formulated by Usadel for the dirty limit.

Consider a superconducting strip extending along the $x$ direction when the current is uniform. Let $j_s$ and $A_s = mv_s/e$ denote the $x$ components of the supercurrent density $j_s$ and the gauge-invariant vector potential $A_s = A + (\phi_0/2\pi)\nabla \gamma$, where $A$ is the gauge-dependent vector potential, $\phi_0 = h/2e$ the superconducting flux quantum, and $\gamma$ the gauge-dependent phase of the superconducting order parameter.

The supercurrent density can always be expressed as

$$j_s = -n_e e v_s,$$

but in general $n_s$ is a function of the superfluid velocity $v_s$ and has the values $n_s = 0$ when $v_s = 0$ but decreases monotonically to zero as $|v_s|$ increases. For positive $j_s$ and negative values of $v_s$, the supercurrent density $j_s = n_s e |v_s|$ initially increases linearly as a function of $|v_s|$, reaches a maximum $j_d(T)$ (the depairing or pair-breaking current density) at $|v_s| = v_m(T)$, then decreases to zero at $|v_s| = v_m(T)$. When the superconductor is current-biased, only the portion of the curve $j_s$ vs $|v_s|$ for $0 \leq |v_s| \leq v_m(T)$ is accessible. On the other hand, following a suggestion by Fulde and Ferrell, Bhatnagar and Stern showed that it is possible to probe experimentally the shape of $j_s$ vs $|v_s|$ even for $v_m(T) \leq |v_s| \leq v_m(T)$ using a multiply-connected sample geometry. In this paper we first examine the behavior of $j_s$ over the full range of values of $v_s$, but later in applying these results to study the kinetic inductance we limit our attention to the current-biased case in which $j_s$ is a single-valued function of $v_s$ in the range $0 \leq j_s \leq j_d$.

A. Depairing current density calculated from the Usadel equations

For the problem at hand the Usadel equations can be written as

$$-hD(GF' - FG')' = 2\Delta G - 2h\omega_n F, \quad (3)$$

$$G^2 + |F|^2 = 1, \quad (4)$$

$$\Delta \ln \frac{T_{c0}}{T} = 2\pi k_B T \sum_{n=0}^{\infty} \left( \frac{\Delta}{h\omega_n} - F \right), \quad (5)$$

$$j_s = -4\pi e N(0) D k_B T \sum_{n=0}^{\infty} \text{Im} F^* F', \quad (6)$$

where $\Delta$ is the superconducting order parameter, $D = v_F^2 \tau/3 = v_F \ell/3$ is the diffusivity, $v_F$ is the average velocity of electrons at the Fermi surface, $\tau$ is the normal-state transport lifetime, $\ell$ is the mean-free path, $h\omega_n = (2n + 1)\pi k_B T$ is the Matsubara frequency, $N(0)$ is the density of Bloch states of one spin at the Fermi level, $T$...
is the temperature, and $T_{c0}$ is the zero-current transition temperature. The primes in Eq. (3) denote differentiation with respect to $x$. These equations describe supercurrent flow in a superconductor with an s-wave isotropic gap in the weak-coupling limit of the BCS theory.$^{50}$

However, this mean-field theory does not account for the possibility that one- or two-dimensional fluctuations could grow to produce phase slips or vortex crossings.

Since $W \ll \Lambda$, we can neglect the self-field of the current and choose a gauge for which we may replace the gauge-invariant gradient $\nabla + 2\pi i A/\phi_0$ by $\hat{x}\partial_x$. Looking for solutions of the form $\Delta = \Delta_q e^{iqx}$, $F = F_{nq} e^{iqx}$, and $G = G_{nq}$, where $q$ is the gradient of the phase of the order parameter, we find that Eqs. (3) and (4) become

$$Q F_{nq} G_{nq} = \Delta_q G_{nq} - \hbar \omega_n F_{nq}, \quad (7)$$
$$G^2_{nq} + F^2_{nq} = 1, \quad (8)$$

where $Q = \hbar D q^2/2$.

Throughout this paper, the symbol $q$ appears frequently; it can be regarded as a compact abbreviation for the gauge-invariant vector potential $A_s$, the superfluid velocity $v_s$, or the gradient of the phase $\gamma$ of the order parameter, since all these quantities are related via $q = 2\pi A_s/\phi_0 = 2m v_s/\hbar = \gamma/x$. The presence of a subscript $q$ indicates that the subscripted quantity is a function of $q$. We show later that as $q$ increases, the pair-breaking effect reduces the superconducting transition temperature $T_{c0}$, the order parameter $\Delta_q$ (Figs. 1 and 5) and the superfluid density $n_{sq}$ (Fig. 2), and increases the penetration depth $\lambda_q$ [Eq. (17)]. We show here that these $q$ dependencies combine to produce the current dependence of the kinetic inductance shown in Figs. 7 and 8. However, the $q$ dependence of $\lambda_q$ could be shown somewhat more directly in sensitive measurements of the penetration depth in the Meissner state as a function of applied ac and dc magnetic fields.

As noted by Maki,$^{37}$ Eqs. (7) and (8) are equivalent to those of the Abrikosov-Gor’kov (AG) theory$^{30}$ for pair-breaking scattering, except for the replacement of the AG spin-flip scattering rate $1/\tau_m$ by $D q^2/2$. Our results therefore share many properties with those of the AG theory. For example, we show that the transition temperature $T_{c0}$ depends upon $q$ and decreases monotonically from its value $T_{c0}$ at $q = 0$ to zero at a critical value of $q$ given by $q_{m}(0) = 1/\xi(0) = (\pi \xi_0 \ell/3)^{1/2}$, where $\xi_0 = \hbar v_F/\pi \Delta_0(0)$ is the BCS coherence length.$^{50}$ For a fixed value of $q$, the order parameter $\Delta_q(T)$ is nonzero only for temperatures $T$ less than $T_{c0}$; equivalently, for a fixed temperature $T$, the order parameter $\Delta_q(T)$ is nonzero only for values of $q$ less than $q_{m}(T) = 1/\xi(T)$.

Introducing $n_{sq} = G_{nq}/F_{nq}$, we find that Eqs. (7) and (8) can be written as

$$\eta / \epsilon = u_{sq} \left( 1 - \frac{\zeta}{\sqrt{1 + u_{sq}^2}} \right), \quad (9)$$

and

$$F_{nq} = \frac{1}{\sqrt{1 + u_{sq}^2}}. \quad (10)$$

where $\eta = n + 1/2$, $\epsilon = \Delta_q/2\pi k_B T$ and $\zeta = Q/\Delta_q$. $u_{sq}$ (which depends implicitly upon $T$) can be obtained for arbitrary values of $\eta$, $\epsilon$, and $\zeta$ by solving Eq. (9) as a quartic equation [see Appendix A].

With the introduction of $u_{sq}$, two equations remain to be solved. The self-consistency equation (5) in the presence of the current becomes, for general values of $\Delta_q$, $\omega$, $Q$, and $T$,

$$\ln \frac{1}{t} = \sum_{n=0}^{\infty} \left( \frac{1}{n+1/2} - \frac{1}{\epsilon \sqrt{1 + u_{sq}^2}} \right). \quad (11)$$

In general, $\Delta_q(T)$ must be obtained by numerically solving Eq. (11) using Eq. (A2), but the results can be checked against analytic results obtainable in the limits of $q \to 0$ and $q \to q_{m}(T)$. Figure 1 shows $[\Delta_q(T)/\Delta_0(0)]^2$ as a function of $q/q_m(0)$ for a series of values of the reduced temperature $t = T/T_{c0}$. (Closely related plots were given as Fig. 1 in Ref. 32 and Fig. 4 in Ref. 33.) The sums for $t \geq 0.1$ were evaluated by summing $n$ from 0 to 500, but for $t = 0$ we used the analytic results in Eqs. (23)-(25).

From the current equation (6) we find that when the superfluid velocity $v_s$ is in the $x$ direction, the supercurrent density in that direction is$^{25}$

$$j_{sq}(T) = -n_{sq}(T) c v_s. \quad (12)$$

![FIG. 1: $[\Delta_q(T)/\Delta_0(0)]^2$ vs $q/q_m(0)$ obtained from Eqs. (11) and (A2) for various values of $t = T/T_{c0}$. The dotted lines show the linear behavior as $[\Delta_q(T)/\Delta_0(0)]^2 \to 0$ in the limit as $q \to q_m(T)$.](image-url)
From Eqs. (6), (10) and (12) we obtain a general expression for the $q$-dependent superfluid density,

$$n_s(T) = \frac{8\pi m N(0) D k_B T}{\hbar} \sum_{n=0}^{\infty} \frac{1}{1 + u_{nq}^2}.$$  \hfill (13)

When $q \to 0$, we have $u_{n0} = 2\pi k_B T(n + 1/2)/\Delta_0(T)$ [see Eq. (9)], and when this is used in Eq. (13), evaluation of the sum yields

$$n_{s0}(T) = \frac{2\pi m N(0) D \Delta_0(T)}{\hbar} \tanh \left( \frac{\Delta_0(T)}{2k_B T} \right),$$  \hfill (14)

such that

$$n_{s0}(0) = \frac{2\pi m N(0) D \Delta_0(0)}{\hbar}$$  \hfill (15)

and

$$n_s(T)/n_{s0}(0) = \frac{4k_B T}{\Delta_0(0)} \sum_{n=0}^{\infty} \frac{1}{1 + u_{nq}^2}. \hfill (16)$$

In general, $n_s(T)/n_{s0}(0)$ must be obtained by numerically solving Eqs. (11) and (16) using Eq. (A2). Figure 2 shows $n_s(T)/n_{s0}(0)$ as a function of $q/q_m(0)$ for a series of values of the reduced temperature $t = T/T_{c0}$. The sums for $t \geq 0.1$ were evaluated by summing $n$ from 0 to 500, but for $t = 0$ we used the analytic results in Eqs. (26)-(27).

As shown in Fig. 2, $n_s(T)/n_{s0}(0)$ depends upon $q$ and vanishes at $q = q_m(T)$ (see Appendix C). The corresponding $q$-dependent penetration depth $\lambda_q(T)$ can be obtained from

$$\frac{n_s(T)}{n_{s0}(0)} = \frac{\lambda_q^2(0)}{\lambda_q^2(T)}. \hfill (17)$$

Note, however, that current-biased experiments can access values of $q$ only up to $q_m(T)$, where the magnitude of the current density reaches the depairing limit $j_d(T)$.

The general expression for the supercurrent density is

$$j_s(T) = -\frac{n_s(T)e^2 A_q}{m} = -\frac{A_q}{\mu_0 \lambda_q^2(T)}. \hfill (18)$$

From Eq. (12) or (18) we see that, because $j_s(T)$ is the product of $n_s(T)$ (a monotonically decreasing function of $q$) and $e v_x = e^2 A_q/m = e h q/2 m$, the magnitude of $j_s(T)$ reaches a maximum, called the depairing current density $j_d(T)$, when $q = q_m(T)$, where $0 < q_m(T) < q_m(T)$. We define $\tilde{j}_q(T)$ as the magnitude of $j_s(T)$ normalized to $n_{s0}(0) v_m(0) = \langle n_{s0}(0) \rangle = 2\pi m N(0) \lambda_0(0)^2 \xi(0)$, such that

$$\tilde{j}_q(T) = \frac{n_s(T)}{n_{s0}(0)} \frac{q}{q_m(0)}. \hfill (19)$$

The maximum value of $\tilde{j}_q(T)$ vs $q$ is the normalized depairing current density $\tilde{j}_d(T)$.

In general, $\tilde{j}_q(T)$ must be obtained numerically from Eqs. (11), (16), (19), and (A2), but the results can be checked against analytic results obtainable in the limits $t \to 0$ and $t \to 1$, to be discussed in more detail later in Secs. II B and II C. Figure 3 shows the general behavior of $j_q(T)$ as a function of $q/q_m(0)$ for a series of values of the reduced temperature $t = T/T_{c0}$. (Plots similar to Fig. 2 were shown as Fig. 1 in Ref. 28 and Fig. 2 in Ref. 39.) The points label the values of $j_d$ and $q$ corresponding to the depairing current density $j_d$ and $q_d$. The solid line in Fig. 4 shows $[j_d(T)/j_d(0)]^{2/3} = [\tilde{j}_d(T)/\tilde{j}_d(0)]^{2/3}$ as a function of $t = T/T_{c0}$, and the dotted line illustrates how $j_c(T)$ approaches the $(1-t)^{2/3}$ behavior in the GL regime close to $T_{c0}$.
For all temperatures below $T_{c0}$, estimates of $j_d(T)$ can be obtained from

$$j_d(T) = p_d(T) \frac{\phi_0}{2\pi \mu_0 \lambda_0^2(T)} \xi(T),$$

where $p_d(T)$ is a dimensionless function defined by Eq. (20). The dashed curve in Fig. 4 shows $p_d(T)$, which is obtained numerically from all the other quantities in Eq. (20), varies from 0.475 at $T = 0$ [Eq. (30)] to 0.385 as $T \rightarrow T_{c0}$ [Eq. (42)] with a maximum of 0.483 at $t = T/T_{c0} = 0.17$.

Similarly, estimates of the depairing current density also can be obtained for all temperatures from

$$j_d(T) = k_d(T) H_c(T) / \lambda_0(T),$$

where the dimensionless quantity $k_d(T)$, defined by Eq. (21) and shown by the dot-dashed curve in Fig. 4, varies from 0.595 at $T = 0$ [Eq. (31)] to 0.544 as $T \rightarrow T_{c0}$ [Eq. (42)] with a maximum of 0.608 at $t = T/T_{c0} = 0.21$. The values of $k_d$ shown in Fig. 4 were obtained from

$$k_d(T) = 0.595 \frac{j_d(T)}{H_c(T)} \frac{\lambda_0(T)}{\lambda_0(0)}$$

via Eqs. (32), (D16) and (B2), where $H_c(T)$ is the temperature-dependent bulk thermodynamic critical field (see Appendix D). Plots similar to Fig. 4 were given as Fig. 9 in Ref. 38 and Fig. 4 in Ref. 39.

B. Depairing current density at zero temperature

At $T = 0$, the $q$ dependence of $\Delta_q(0)$ can be obtained by converting the sum in Eq. (11) to an integral over $\nu_{nq}$.

The result is

$$\frac{\Delta_q(0)}{\Delta_0(0)} = \exp(-\pi \zeta_0/4), \quad 0 \leq \zeta_0 \leq 1,$$

$$= \exp[-(\zeta_0 \sin^{-1} \zeta_0 - \sqrt{1 - \zeta_0^2})/2 - \cosh^{-1} \zeta_0], \quad \zeta_0 \geq 1,$$

where

$$\zeta_0 = \frac{h D q^2}{2 \Delta_q(0)} = \frac{1}{2} \left( \frac{q}{\nu_{m}(0)} \right)^2 \frac{\Delta_0(0)}{\Delta_q(0)}$$

Figure 5 shows $[\Delta_q(0)/\Delta_0(0)]^2$ vs $q/\nu_{m}(0)$ obtained from Eqs. (23)-(25). The dashed line shows the linear behavior $6[1 - q/\nu_{m}(0)]$ as $q \rightarrow q_{m}(0)$.
The result is \[ \frac{n_{sq}(0)}{n_{so}(0)} = \exp(-\pi \zeta_0/4)(1 - 4\zeta_0/3\pi), \quad 0 \leq \zeta_0 \leq 1, \] \[ = \exp(-\pi \zeta_0/4)\left\{ \frac{2}{3\pi \zeta_0^3} \left[ (1 + 2\zeta_0^2) \sqrt{\zeta_0^2 - 1 - 2\zeta_0^3} \right] + \frac{2}{\pi} \sin^{-1} \zeta_0^{-1} \right\}, \quad \zeta_0 \geq 1, \] where \( \zeta_0 \) is given by Eq. (25). \( n_{sq}(0) \) is shown as the dashed curve in Fig. 6. The \( q \)-dependent penetration depth at zero temperature \( \xi_q(0) \) can be obtained from

\[ \frac{n_{sq}(0)}{n_{so}(0)} = \frac{\lambda_0^2(0)}{\lambda_0^2(q)} \]. \tag{28} \]

To obtain the depairing current density, consider the \( q \)-dependent (or \( \nu_s \)-dependent) supercurrent density at \( T = 0 \) [Eq. (12)], normalized to \( -n_{so}(0) \epsilon_n(0) = -\phi_0/2\mu_0 \lambda_0(0)^2 \xi(0) \),

\[ \tilde{j}_q(0) = \frac{n_{sq}(0)}{n_{so}(0)} \left( \frac{q}{\xi_m(0)} \right), \tag{29} \]

shown as the solid curve in Fig. 6. (Plots similar to Fig. 6 were shown as Fig. 1 in Ref. 28 and Fig. 2 in Ref. 39.) The point and dotted lines show the maximum \( j_q(0) \), the normalized depairing supercurrent density \( j_\lambda(0) = 0.475 \), which occurs at \( q_m(0)/q_m(0) = 0.689 \) and \( \zeta_0 = 0.300, \tag{30} \)

The resulting zero-temperature depairing supercurrent density can be expressed in several ways:

\[ j_\lambda(0) = 0.475 \frac{\phi_0}{2\pi \mu_0 \lambda_0^2(0)} \xi(0), \tag{31} \]

\[ = 1.491 N(0) \epsilon(\Delta(0))^{1/2} \sqrt{D/H}, \tag{32} \]

Equation (31) coincides with the result given for the depairing supercurrent density in Ref. 38. Here \( H_e(T) \) is the temperature-dependent bulk thermodynamic critical field (see Appendix D), and \( H_e(0) = \Delta(0)N(0)/\mu_0 \).

### C. Depairing current density in the GL regime

As shown by Gor’tkov, \( \lambda^2 \) the GL theory of superconductivity is \( \lambda^2 \) derivable from the microscopic BCS theory \( \lambda^2 \) at temperatures \( T \) very close to the transition temperature \( T_{c0} \). This assures us that we also can apply the Usadel equations to recover the GL results. We present the GL depairing-current results here for completeness, even though they are so well known that they appear in textbooks.\(^{24} \) For \( T \) close to \( T_{eq} \) or, equivalently, for \( q \) close to \( q_m(T) \), the \( q \)-dependent order parameter \( \Delta_q(T) \) becomes small, and it is useful to expand \( u_{nq} \) and \( 1/\epsilon \sqrt{1 + u_{nq}^2} \) in powers of \( \epsilon = \Delta_q(T)/2\pi k_B T \):

\[ \frac{1}{\epsilon \sqrt{1 + u_{nq}^2}} = \frac{1}{\eta + \alpha} - \frac{\epsilon^2}{2} \left[ \frac{1}{(\eta + \alpha)^3} - \frac{\alpha}{(\eta + \alpha)^4} \right] \right) + \mathcal{O}(\epsilon^4), \tag{33} \]

where \( \alpha = e^{-\gamma} q_m(T) \). \tag{34} \]

Substituting this into Eq. (11) and keeping only the lowest order terms, we know that \( \epsilon = \Delta_q(T)/2\pi k_B T \ll 1 \) when \( 1 - t \ll 1 \), where \( t = T/T_{c0} \), we obtain

\[ \ln \frac{T_{c0}}{T} = \sum_{n=0}^{\infty} \left\{ \frac{1}{n+1/2} - \frac{1}{n+1/2+\alpha} \right\} + \left[ \frac{1}{(n+1/2+\alpha)^3} - \frac{\alpha}{(n+1/2+\alpha)^4} \right] \frac{\epsilon^2}{2}, \tag{35} \]

The sums can be expressed in terms of digamma functions and their derivatives. When \( 1 - t \ll 1 \), Eq. (35) has solutions only for \( \alpha \ll 1 \) and can be expanded as

\[ 1 - t = \frac{\pi^2 e^{-\gamma} q^2}{8q_m^2(0)} + \frac{7\zeta(3) \epsilon^2}{2}. \tag{36} \]

Solving for \( \epsilon^2 \), dividing by \( \epsilon^2 = 2(1 - t)/7\zeta(3) \), and making use of \( q_m^2(T) = 8e^{-\gamma}(1 - t)/\pi^2 q_m^2(0) \) (see Appendix C), we obtain

\[ \frac{\Delta_q^2(T)}{\Delta_q^2(0)} = 1 - \frac{q^2}{q_m^2(T)}, \tag{37} \]

where \( q_m(T) = 1/\xi(T) \).

Substituting the expansion of Eq. (33) into Eq. (16), keeping only the lowest order terms, we obtain

\[ \frac{n_{sq}(T)}{n_{so}(0)} = \frac{\lambda_0^2(q)}{\lambda_0^2(0)} = \frac{\Delta_q^2(T)}{\Delta_q^2(0)} = f^2 = 1 - \frac{q^2}{q_m^2(T)}, \tag{38} \]

and

\[ \frac{n_{sq}(T)}{n_{so}(0)} = \frac{\lambda_0^2(q)}{\lambda_0^2(T)} = \frac{4\pi e^{-\gamma}}{7\zeta(3)} \left( 1 - \frac{q^2}{q_m^2(T)} \right)(1 - t). \tag{39} \]

The reduced \( q \)-dependent supercurrent density becomes

\[ \tilde{j}_q(T) = \frac{4\pi e^{-\gamma}}{7\zeta(3)} q_m(T) \left( 1 - \frac{q^2}{q_m^2(T)} \right) q_m(T)(1 - t), \tag{40} \]

whose maximum occurs at \( q_d(T)/q_m(T) = 1/\sqrt{3} \), such that (see Appendix C) the reduced depairing current density is

\[ \tilde{j}_d(T) = \frac{16\sqrt{2}e^{3\gamma/2}}{21\sqrt{3}\zeta(3)}(1 - t)^{3/2} = 1.230(1 - t)^{3/2}. \tag{41} \]
Thus, in the GL regime the depairing current density can be expressed as:

\[ j_d(T) = \frac{\phi_0}{2\pi \mu_0 \lambda_0^2(T) \xi(T)} \tag{42} \]

where \( \phi_0 = 2/3 \),

\[ j_d(T) = 3.865 N(0)e[\Delta_0(0)]^{3/2} \sqrt{D/H}(1-t)^{-3/2} \tag{43} \]

or, since \( \sqrt{2}H_c = \phi_0 / 2\pi \mu_0 \lambda_0 \xi \) in the GL theory,

\[ j_d(T) = 0.544 H_c(T) \lambda_0(T) \tag{44} \]

where \( 0.544 = (2/3)^{3/2} \).

To simplify calculations in the GL limit later in Secs. III and IV we introduce the parameter \( \phi \) (0 \( \leq \phi \leq \pi/2 \)), such that

\[ |j_q|/j_d = \sin \phi \tag{45} \]

\[ q/q_m(T) = (2/\sqrt{3}) \sin(\phi/3) \tag{46} \]

\[ f^2 = [1 + 2\cos(2\phi/3)]/3 \tag{47} \]

### III. SUPERFLUID KINETIC INDUCTIVITY IN SLOW EXPERIMENTS (FAST RELAXATION)

In Sec. II we have summarized the results of previous authors and provided details of how to account quantitatively for the current-induced suppression of the order parameter in dirty thin-film superconductors at all temperatures in the superconducting state. We are now in a position to calculate the kinetic inductivity of the superfluid measured in slow, low-frequency (or, equivalently, fast relaxation) current-biased experiments, in which both \( j_s \) \((|j_s| \leq j_d(T)) \) and \( v_s \) \((|v_s| \leq v_d(T)) \) vary on a time scale \( \tau_{exp} \) much longer than the relaxation time \( \tau_s \) required for the superconducting order parameter to change.13,24,45

In a one-dimensional conductor carrying a uniform current the gauge-invariant electric potential \( \Phi = \int (\phi_0/2\pi) d\gamma / dt \) is zero,55 and the electric field along the conductor is \( E = -dA_s / dt = L_k(q,T) d\tilde{j}_q(T) / dt \).56

Since from Eq. (18) we have \( A_s = (\phi_0 / 2\pi) q = -j_q(T) \mu_0 \lambda_0^2(T) \), taking the time derivative and using \( d\tilde{j}_q(T) / dt = (dF / dq)dq / dt \) we obtain the kinetic inductivity of the superfluid for slow experiments,

\[ L_k(q,T) = \mu_0 \left| \frac{d}{dq} \left( \frac{q}{\lambda_0^2(T)} \right) \right|^{-1} = \left| \frac{dj_q(T)}{dA_s} \right|^{-1} = \mu_0 \lambda_0^2(T) F_s \left( \frac{|j_s|}{j_d(T)} \right) \tag{48} \]

where the slow-experiment function \( F_s \) is simply \( L_k(q,T) / \mu_0 \lambda_0^2(T) \) but expressed as a function of the normalized current density \( |j_s|/j_d(T) \) rather than as a function of \( q \). While \( q \) is a convenient theoretical variable, \( j_s \) is a more convenient variable for the current-biased case.

For \( |q| < q_d(T) \) and \( |j_s| < j_d(T) \), \( j_s \) is a single-valued function of \( q \), shown in Figs. 3 and 6.

In the limit of small currents, when \( q \rightarrow 0 \), \( L_k(q,T) \) reduces to \( L_k(0,T) = L_{k0}(T) \) [Eq. (2)]. However, as can be seen from Figs. 3 and 6, \( |dj_q(T)/dq| \) decreases monotonically for increasing values of \( q \) and becomes zero at the depairing value. Accordingly, as \( q \) increases, \( L_k(q,T) \) starts from \( L_{k0}(T) \), increases monotonically, and diverges at \( q = q_d(T) \), where \( |j_s| = j_d(T) \). Because \( L_k(q,T)/L_{k0}(T) \) diverges as \( |j_s| \rightarrow j_d(T) \), we show in Fig. 7 the typical dependence of the inverse, \( L_{k0}(T)/L_k(q,T) = 1/F_s(j_s/j_d(T)) \), vs \( |j_s|/j_d(T) \). This figure was obtained by (a) evaluating \( j_q(T) \) [Eq. (19)] and \( dj_q(T)/dq \) numerically for \( t = T/T_0 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, \) and 0.9, and analytically [see Eq. (40)] in the GL limit \( t \rightarrow 1 \), (b) calculating

\[ \frac{L_{k0}(T)}{L_k(q,T)} = \frac{n_{s0}(0)}{n_{s0}(T)} \frac{d\tilde{j}_q(T)}{dq} \tag{49} \]

using Eq. (14) to evaluate \( n_{s0}(0)/n_{s0}(T) \), and (c) making a parametric plot of \( L_{k0}(T)/L_k(q,T) \) vs \( |j_s|/j_d(T) \). As shown by the solid curve for \( t = 0 \), the dotted curve for \( t = 0.3 \), and the dashed curve for \( t \rightarrow 1 \), the behavior of \( L_{k0}(T)/L_k(q,T) \) vs \( |j_s|/j_d(T) \) is not monotonic as the temperature changes, but the curves for all other temperatures (not shown) lie in a narrow band between the dotted and dashed curves. As \( |j_s|/j_d \rightarrow 1 \), all the curves have an inverse-square-root dependence close to that in the GL limit \( t \rightarrow 1 \),

\[ L_{kGL}(T)/L_{kGL}(q,T) = (2\sqrt{3}/3)[1 - |j_s|/j_d(T)]^{-1/2} \tag{50} \]

The curves shown in Fig. 7 can be represented by

\[ y_{sn}(x) = (1 - x^n)^{1/n} \] (not shown in Fig. 7), which fits the calculated values of \( y = L_{k0}(T)/L_k(q,T) \) vs \( x = |j_s|/j_d(T) \) for \( 0 \leq x < 0.97 \) with 1\% accuracy for \( n = (2.21, 0.1, 0.2, 0.27, 0.2, 0.3, 0.28, 0.4, 0.25, 0.5, 0.22, 0.6, 0.18, 0.7, 0.16, 0.8, 0.13, 0.9, \) and 2.11, \( t \rightarrow 1 \)).

To calculate the kinetic inductivity in the GL limit shown by the dashed curve in Fig. 7, it is convenient to use the parametric relations \( x = |j_s|/j_d(T) \) and \( y = L_{kGL}(T)/L_{kGL}(q,T) = 2\cos(2\phi/3) - 1 \), where \( 0 \leq \phi \leq \pi/2 \) [see Eqs. (45)-(47)]. The slow-experiment kinetic inductivity of the superfluid in the GL limit is

\[ L_{kGL}(q,T) = \mu_0 \lambda_0^2(T) F_{sGL} \left( \frac{|j_s|}{j_d(T)} \right) \tag{51} \]

where

\[ F_{sGL}(x) = \frac{1}{2\cos(2\phi/3) - 1} \tag{52} \]

and \( \phi = \sin^{-1} x \). For small values of \( x \),

\[ F_{sGL}(x) = 1 + \frac{4}{9} x^2 + \frac{80}{243} x^4 + O(x^6), \tag{53} \]

and \( F_{sGL}(x) \) diverges at \( x = 1 \), as noted in Ref. 13. (See also the upper solid curve in Fig. 10.)
IV. SUPERFLUID KINETIC INDUCTIVITY IN FAST EXPERIMENTS (SLOW RELAXATION)

We next consider fast experiments (or, equivalently, slow relaxation), in which the current density \( j_s(t) = j_s + j_s(t) \) as a function of the time \( t \) changes rapidly about its time average \( j_s \), on a time scale \( \tau_{exp} \) much shorter than the relaxation time \( \tau_s \). In this case neither the order parameter \( \Delta_q \) nor the \( q \)-dependent penetration depth \( \lambda_q \) can follow the time dependence of the current, but instead they remain frozen to their values at \( q = \bar{q} \) given by \( j_s = \bar{j}_{s\bar{q}} = -\bar{A}_s/\mu_0\lambda_q^3 \), where \( \bar{A}_s = m\bar{v}_s/c = h\bar{q}/2c \). From \( j_s(t) = -\bar{A}_s/\mu_0\lambda_q^3 \), \( A_s(t) = \bar{A}_s + A_s(t) \), and \( E(t) = -dA_s(t)/dt = L_k^G(q,T)j_s(t)/dt \), we find that the kinetic inductivity of the superfluid in fast experiments is

\[
L_k^G(q,T) = \mu_0\lambda_q^3(T)F_f^G\left(\frac{\bar{j}_{s\bar{q}}}{J_d(T)}\right),
\]

which can be evaluated numerically using Eq. (16) as

\[
\frac{L_k(q,T)}{L_k^{0}(T)} = \frac{n_{s\bar{q}}(T)}{n_{s\bar{q}}} = \sum_{n=0}^{\infty} \frac{1}{1 + u_n^2} \sum_{n=0}^{\infty} \frac{1}{1 + v_n^2}. \tag{55}
\]

(See Fig. 2.) Here the fast-experiment function \( F_f \) is simply \( L_k^G(q,T)/\mu_0\lambda_q^3(T) \) but expressed as a function of the normalized current density \( \bar{j}_{s\bar{q}}/J_d(T) \) rather than as a function of \( q \). When \( \bar{q} \to 0 \), \( L_k^G(q,T) \) reduces to \( L_k(0,T) = L_k^{0}(T) \) [Eq. (2)].

Shown in Fig. 8 is the typical dependence of \( L_k(q,T)/L_k^{0}(T) \) vs \( \bar{j}_{s\bar{q}}/J_d(T) \). This figure was obtained by (a) evaluating \( j_s(T) \) and \( L_k(q,T)/L_k^{0}(T) \) numerically for \( t = T/T_s = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, \) and 0.9, and analytically in the GL limit \( t \to 1 \) [see Eqs. (45) and (47)], and (b) making a parametric plot of \( L_k(q,T)/L_k^{0}(T) \) vs \( \bar{j}_{s\bar{q}}/J_d(T) \). As shown by the solid curve for \( t = 0 \), the dotted curve for \( t = 0.3 \), and the dashed curve for \( t \to 1 \), the behavior of \( L_k(q,T)/L_k^{0}(T) \) vs \( \bar{j}_{s\bar{q}}/J_d(T) \) is not monotonic as the temperature changes, but the curves for all other temperatures (not shown) lie in a narrow band between the dotted and dashed curves. As \( \bar{j}_{s\bar{q}}/J_d(T) \), all the curves approach their limiting values in the range 1.41 - 1.50 (solid symbols in Fig. 8) with infinite slope.

The curves shown in Fig. 8 can be represented by \( y_{fr}(x) = y_0 - (y_0 - 1)(1 - x^n)^{1/\alpha} \) (not shown in Fig. 8), which fits the calculated values of \( y = L_k(q,T)/L_k^{0}(T) \) for \( x = \bar{j}_{s\bar{q}}/J_d(t) \), where \( y_0 \) is the value of \( L_k(q,T)/L_k^{0}(T) \) at \( x = 1 \), for \( 0 \leq x < 0.97 \) within 0.5% for \( (y_0, n, \alpha) = (1.451, 2.48, 0), (1.448, 2.47, 0.1), (1.422, 2.45, 0.2), (1.412, 2.46, 0.3), (1.417, 2.50, 0.4), (1.432, 2.50, 0.5), (1.448, 2.50, 0.6), (1.463, 2.50, 0.7), (1.477, 2.50, 0.8), (1.490, 2.50, 0.9), \) and \( (1.500, 2.50, t \to 1) \).

To calculate \( L_k(q,T) \) in the GL limit shown by the dashed curve in Fig. 8, we used Eqs. (45)-(47) The kinetic inductivity of the superfluid for fast experiments in the GL limit is

\[
L_k^{GL}(q,T) = \mu_0\lambda_q^3(T)F_f^{GL}\left(\frac{\bar{j}_{s\bar{q}}}{J_d(T)}\right), \tag{56}
\]

where

\[
F_f^{GL}(x) = \frac{1}{f(x)} = \frac{3}{1 + 2\cos(2\phi/3)}, \tag{57}
\]

and \( \phi = \sin^{-1} x \). As noted in Ref. 13, for small values of \( x \),

\[
F_f^{GL}(x) = 1 + \frac{4}{27}x^2 + \frac{16}{243}x^4 + O(x^6), \tag{58}
\]

and \( F_f^{GL}(x) \) approaches 3/2 with infinite slope as \( x \to 1 \). (See also the lower solid curve in Fig. 10.)

V. KINETIC IMPEDIVITY OF THE SUPERFLUID \( Z_{ks} \)

In the above sections we discussed the situations when \( \tau_{exp}/\tau_s \) is large or small. To describe in detail the transition between these two limits is well beyond the scope of this paper, because this topic involves nonequilibrium processes with numerous relaxation times. However, we present here an oversimplified procedure for approximating the transition between the two limits, although the actual transition is likely to be much more complicated.

For the moment we restrict our attention to the GL regime and employ a phenomenological model assuming that the time dependence of \( f \) is determined by simplest version of the time-dependent GL (TDGL) equation,

\[
\tau_s df/dt = f - f^3 - A_s^2 f, \tag{59}
\]
where \( A'_s = A_s / (\phi_0 / 2\pi \xi) \). An important caution here is that we are using this equation in the gapped state, even though a nonlinear TDGL equation has been rigorously justified only in a gapless superconductor,\(^{57}\) where near \( T_c \) and at frequencies \( \omega \tau_s \ll 1 \), \( \tau_s = \pi \hbar / 8k_B(T_c - T) \).\(^{24}\)

In the GL regime \( [T_c - T] \ll T_c \), the supercurrent density [Eq. (18)] becomes in dimensionless quantities

\[
 j_s' = f^2 A'_s \tag{60}
\]

where \( j_s' = j_s / (\phi_0 / 2\pi \mu_0 \xi^2) \). We now consider experiments in which the supercurrent density \( j_s'(t) \) as a function of the time \( t \) changes about its time average on a time scale \( \tau_{exp} \) comparable with the relaxation time \( \tau_s \). In particular, we consider the linear response of the superconducting strip to a time-dependent supercurrent density given by \( j_s'(t) = j_s0 + j_s1 e^{i\omega t} \), where \( j_s0 \), the bias current density, is fixed to be in the range \( 0 \leq |j_s0| < j_s' \), and \( j_s1 \), the amplitude of the ac current density, obeys \( j_s1 \ll j_s0 \). In this section we assume that the frequencies are sufficiently low that normal-fluid currents are not excited such that the current is all supercurrent. To analyze the linear response of the reduced order parameter to the ac current, we substitute \( j_s' = j_{s0}' + j_{s1}' e^{i\omega t} \) and \( f = f_0 + f_1 e^{i\omega t} \) into Eq. (59), where \( f_0 \) (\( \sqrt{2}/3 \leq f_0 \leq 1 \)) is the solution of Eq. (59) in the time-independent case when \( j_s' = j_{s0}' \). We then linearize Eq. (59) by neglecting terms of order \( j_{s1}'^2 \) and \( f_1^2 \). The solution is

\[
 f_1 = -\frac{2j_{s0}'j_{s1}'}{f_0(6f_0^2 - 4 + i\omega \tau_s)} \tag{61}
\]

(Note that this result is obtained for sinusoidal variation of the supercurrent around a fixed value of \( j_{s0}' \). A different result would be obtained for sinusoidal variation of the gauge-invariant vector potential around a fixed value of \( A'_{s0} \).

From \( E = -dA_s / dt \), Eq. (60), and \( j_{s0}' = f_0^2(1 - f_0^2) \) we obtain the electric field in the linear-response approximation:

\[
 E = \mu_0 \lambda_0^2 \left( \frac{2f_0^2 + i\omega \tau_s}{f_0^2(6f_0^2 - 4 + i\omega \tau_s)} \right) \frac{dj_{s1}}{dt} \tag{62}
\]

\[
 = \frac{Z_{ks}j_{s1} = (R_{ks} + i\lambda_{ks})j_{s1}}{\tau_{eff}} \tag{63}
\]

\[
 = (R_{ks} + i\lambda_{ks})j_{s1}. \tag{64}
\]

The coefficient of \( dj_{s1}/dt \) on the right-hand side of Eq. (62) reduces to the slow-experiment inducivity \( L_k^{GL}(q, T) \) [Eq. (51)] in the limit \( \omega \tau_s \to 0 \) and to the fast-experiment inducivity \( L_k^{GL}(q, T) \) [Eq. (56)] in the limit \( \omega \tau_s \to \infty \).

The complex kinetic impedance (specific impedance or complex resistivity) of the superfluid \( Z_{ks} \), here evaluated in the GL regime, can be conveniently expressed in terms of \( F_s^{GL} \) [Eq. (52)] and \( F_s^{GL} \) [Eq. (57)] as

\[
 Z_{ks} = i\omega \mu_0 \lambda_0^2 \left( \frac{F_s^{GL} + F_s^{GL} i\omega \tau_{eff}}{1 + i\omega \tau_{eff}} \right), \tag{65}
\]

where

\[
 \tau_{eff} = F_s^{GL} \omega \tau_s / 2. \tag{66}
\]

(For a long strip of length \( L \), width \( W \), and thickness \( d \), the complex kinetic impedance is \( Z_{ks} = Z_{ks} \ell / Wd \).)

The real part \( R_{ks} \) of the superfluid kinetic impedance is the frequency-dependent resistivity of the superfluid due to order-parameter relaxation. Using the the parametric relation \( x = |f_s0| / f_0 = \sin \phi \) as above, we obtain

\[
 R_{ks} = \frac{\mu_0 \lambda_0^2 G(x, \omega \tau_s)}{f_s0}, \tag{67}
\]

\[
 G(x, \omega \tau_s) = \frac{\beta(\phi)(\omega \tau_s)^2}{\alpha(\phi)^2 + (\omega \tau_s)^2}, \tag{68}
\]

\[
 \alpha(\phi) = 4 \cos(2\phi/3) - 2, \tag{69}
\]

\[
 \beta(\phi) = \frac{16 \sin^2(\phi/3)}{1 + 2 \cos(2\phi/3)}. \tag{70}
\]

\[
 G(x, \omega \tau_s) \text{ is shown in Fig. 9 as a function of } x \text{ for various values of } \omega \tau_s. \text{ In limiting cases, we have}
\]

\[
 G(x, \omega \tau_s) \begin{cases} 
 = \frac{16}{27} x^2 + \frac{64}{243} x^4 + O(x^6), & \omega \tau_s \gg 1, \tag{71} \\
 = \left[ \frac{4}{27} x^2 + \frac{16}{81} x^4 + O(x^6) \right] (\omega \tau_s)^2, & \omega \tau_s \ll 1, \tag{72}
\end{cases}
\]

and \( G(x, \omega \tau_s) \) approaches 2 with infinite slope as \( x \to 1 \).

Although the superconducting strip has zero dc electrical resistivity, under ac conditions order-parameter relaxation contributes to dissipation of energy in a manner similar to the way it contributes to flux-flow
dissipation.\textsuperscript{24,58–61} The time-averaged rate of energy dissipation per unit volume via order-parameter relaxation is \((1/2)\mathcal{R}_{ks}j_d^2\), which also can be calculated using the dissipation function discussed in Refs. 59 and 60.

The superfluid’s kinetic reactivity is \(\chi_{ks} = \omega \mathcal{L}_k\), where the superfluid’s kinetic inductivity \(\mathcal{L}_k\) is

\[
\mathcal{L}_k = \mu_0 \lambda_0^2 H \left( \frac{|j_{a0}|}{j_d}, \omega \tau_s \right),
\]

\[
H(x, \omega \tau_s) = \frac{1}{1 + 2 \cos(2\phi/3)} \left[ 3 + \frac{16\alpha(\phi) \sin^2(\phi/3)}{\alpha^2(\phi) + (\omega \tau_s)^2} \right].
\]

Shown in Fig. 10 are plots of \(H(x, \omega \tau_s)\) vs \(x\) for various values of \(\omega \tau_s\). As expected, \(H(x, \omega \tau_s)\) approaches the slow-experiment result \(F_{GL}^G(x)\) when \(\omega \tau_s \rightarrow 0\) and the fast-experiment result \(F_{GL}^L(x)\) as \(\omega \tau_s \rightarrow \infty\).

In the above we have used the relatively simple formalism of the time-dependent GL theory, bearing in mind that there are questions of whether this theory can legitimately be used for gapped superconductors and how \(\tau_s\) can be determined. A reasonable starting point for an approximate phenomenological theory of the complex kinetic impedance of the superfluid \(Z_{ks}\) at lower temperatures outside the GL regime would be to replace the quantities \(F_{GL}^G\) and \(F_{GL}^L\) in Eqs. (65) and (66) by the more general expressions \(F_s\) and \(F_{G}\) given in Eqs. (48) and (54). Note from Figs. 7 and 8 that \(F_s = \mathcal{L}_k(qT)/\mu_0 \lambda_0^2(T)\) and \(F_{G} = \mathcal{L}_k(q, T)/\mu_0 \lambda_0^2(T)\) as functions of \(|j_{a0}]/j_d\) and \([j_{a0}]/j_d\) do not differ greatly from their GL counterparts, \(F_{GL}^G\) and \(F_{GL}^L\). Although the theory we have presented here is not rigorous, our results suggest that when \(\omega \tau_s \gg 1\) but when \(\omega\) is well below the superconducting gap frequency \(2\Delta_0(T)/\hbar\), order-parameter relaxation gives rise to a current- and frequency-dependent contribution to the ac resistivity separate from that due to the normal fluid (thermally excited quasiparticles). However, as discussed above, remaining unknown is how to calculate order-parameter relaxation and how to determine the relaxation times that should replace \(\tau_s\) in a more complete theory.\textsuperscript{45}

VI. KINETIC IMPEDANCE INCLUDING THE NORMAL-FLUID RESPONSE

We examine next the current dependence of the dissipation arising from the flow of thermally excited quasiparticles at frequencies \(\omega\) well below the superconducting gap frequency \(2\Delta_0(T)/\hbar\). This is the frequency regime where a two-fluid approach is generally applicable.\textsuperscript{24} However, the two-fluid terminology needs to be used with caution, because, as explained below, coherence-factor effects can produce dissipation greater than in the normal state.\textsuperscript{62} Although the quasiparticles have their own kinetic inductivity,\textsuperscript{5} their reactive contribution to the total kinetic impedance is negligible at the frequencies of interest here. The quasiparticles’ only significant contribution to the ac normal-fluid current density is therefore

\[
j_{n1} = \sigma_1 E,
\]

where \(\sigma_1\) corresponds to the real part of the complex conductivity \(\sigma = \sigma_1 - i\sigma_2\), the linear response function connecting \(\mathbf{j}\) and \(E\) calculated by Mattis and Bardeen.\textsuperscript{63}

We begin by reexpressing \(\sigma_1\) for the BCS case as\textsuperscript{64}

\[
\sigma_1 = \frac{\sigma n}{\omega} \int_{-\infty}^{\infty} d\omega' [f(\omega' - \frac{\omega}{2}) - f(\omega' + \frac{\omega}{2})]
\]

\[
\times [n_1(\omega' - \frac{\omega}{2})n_1(\omega' + \frac{\omega}{2}) + p_1(\omega' - \frac{\omega}{2})p_1(\omega' + \frac{\omega}{2})],
\]

where \(\omega\) is expressed in energy units (\(\hbar = 1\)) and \(f(\omega) = \cdots\)
$1/(1 + e^{\beta \omega})$ is the Fermi function. For the BCS case,
\begin{align}
n_1(\omega) &= \text{Re} \frac{\omega}{\sqrt{\omega^2 - \Delta^2}}, \quad (77) \\
p_1(\omega) &= \text{Re} \frac{-\Delta}{\sqrt{\omega^2 - \Delta^2}}, \quad (78)
\end{align}
n_1(\omega) = p_1(\omega) = 0 \text{ for } |\omega| < \Delta(T), \text{ and the signs of the square roots are chosen such that } n_1(\omega) \text{ is an even function and } p_1(\omega) \text{ an odd function of } \omega. \text{ At } T = 0, \text{ the Fermi functions freeze out all contributions to } \sigma_n \text{ for } \omega < 2\Delta(0), \text{ such that } \sigma_n \text{ is nonvanishing only for } \omega > 2\Delta(0), \text{ when the integral of Eq. (76) can be expressed in terms of complete elliptic integrals.}^{24,63} \text{ For } T > 0 \text{ and frequencies obeying } \omega \ll 2\Delta(0), \sigma_1/\sigma_n \text{ plotted as a function of temperature is found theoretically in dirty superconductors}^{62} \text{ to be very small at low temperatures, rising to a maximum at which } \sigma_1/\sigma_n > 1 \text{ (for example,}\sigma_{1\text{max}}/\sigma_n = 2.17 \text{ at } t = 0.864 \text{ when } \omega/\Delta(0) = 0.02) \text{ and returning to } 1 \text{ at } t = T/T_{c0} = 1. \text{ A similar temperature dependence has been seen experimentally in several materials}.^{55-68}

Because the coherence-factor terms $n_1$ and $p_1$ in Eq. (76) yield a logarithmic divergence for $\omega = 0$, $\sigma_1$ can be evaluated approximately at low temperatures [$\Delta(T)/k_BT > 2$] and low frequencies [$\omega \ll 2\Delta(0)$] by introducing a cutoff energy $\epsilon \sim \omega$. The leading term in the result is
\begin{equation}
\sigma_1 \approx 2\sigma_n \left[ \frac{\Delta(T)}{k_BT} \right] \exp \left[-\frac{\Delta(T)}{k_BT} \right] \ln \left(\frac{k_BT}{\epsilon}\right). \quad (79)
\end{equation}

The expressions for $\sigma_1$ derived by Nam$^{34-36}$ for strong-coupling superconductors also can be put into the form of Eq. (76), except that $\Delta$ in Eqs. (77) and (78) must then be replaced by the complex gap function $\Delta(\omega)$, which contains additional phonon-related $\omega$ dependence due to the electron-phonon interaction.$^{69}$

For the pair-breaking theory of superconductors with paramagnetic impurities$^{32-36}$ or, as in the case of interest here, current-carrying thin films,$^{27-29,31,37,70,71}$ Eqs. (77) and (78) are replaced by
\begin{align}
n_1(\omega) &= \text{Re} \frac{u}{\sqrt{u^2 - 1}}, \quad (80) \\
p_1(\omega) &= \text{Re} \frac{1}{\sqrt{u^2 - 1}}, \quad (81)
\end{align}

where $u$ is given by Eq. (D3), $n_1(\omega) = p_1(\omega) = 0$ for $|\omega| < \omega_g$ with $\omega_g/\Delta_q = (1 - \zeta^2)^{3/2}$, and the signs of the square roots are chosen such that $n_1(\omega)$ is an even function and $p_1(\omega)$ an odd function of $\omega$. (See, for example, Fig. 4 in Ref. 32 or Fig. 6 in Ref. 33.) Numerical evaluation of Eq. (76) using Eqs. (80) and (81) in the limit as $\zeta \to 0$ yield $\sigma_1(\omega)$ values in agreement with the dirty-limit Mattis-Bardeen results.$^{24,63}$

Figure 11 shows $\sigma_1/\sigma_n$ vs $|j_s|/j_d$ for $\omega/\Delta_0(0) = 0.02$ and several values of $t = T/T_{c0}$. $\sigma_1/\sigma_n$ was first calculated from Eq. (76) as a function of $q/q_m(0)$ up to $q_d(T)/q_m(0)$, accounting for the $q$ and temperature dependence of $\zeta = (q^2/2q_m^2)\Delta_0(0)/\Delta_q(T)$. (See Fig. 1.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11.pdf}
\caption{$\sigma_1/\sigma_n$ vs $|j_s|/j_d$ calculated from Eqs. (76), (80), and (81) for $\omega/\Delta_0(0) = 0.02$ and $t = T/T_{c0} = 0.2, 0.4, 0.6, \text{ and } 0.8$.}
\end{figure}

Figure 11 was then constructed as a parametric plot using $|j_s|/j_d$ vs $q/q_m(0)$ up to $q_d(T)/q_m(0)$ obtained as in Fig. 3. The main features of the dependence of $\sigma_1/\sigma_n$ can be understood from Eq. (79), where $\Delta(T)$ is replaced by $\Delta_q(T)$ (see Fig. 1) and the cutoff $\epsilon$ is replaced by the larger of $\epsilon_q \sim \omega$ or the peak width in $\sigma_1(\omega)$, $\epsilon_q \sim \Delta_q - \omega_q \approx (3/2)\Delta_q\zeta^{2/3}$. The exponential term in Eq. (79) tends to make $\sigma_1$ increase as $q$ and $|j_s|$ (see Figs. 3 and 6) increase, as seen in Fig. 11 for small values of $|j_s|/j_d$, where $\epsilon_q > \epsilon_j$ and the $\ln(k_BT/\epsilon_j)$ term is a constant. Maxima occur when $\epsilon_j \approx \epsilon_q$. To the right of the maxima, $\epsilon_j > \epsilon_q$, and the term $\ln(k_BT/\epsilon_j)$, which is a decreasing function of $q$ and $|j_s|$, plays a stronger role.

The current density carried by the normal fluid, accounting for the effects of the coherence factor, is
\begin{equation}
j_{n1} = \sigma_1 E. \quad (82)
\end{equation}

We also can express the normal-fluid response in terms of the normal-fluid’s kinetic impedivity $Z_{kn} = 1/\sigma_1$. Because the total ac current density carried by the strip is $j_1 = j_{s1} + j_{n1}$, and $E = Z_{ks}s_j$, the overall kinetic impedivity $Z_k = R_k + i\lambda_k = R_k + i\omega_L$ of the strip, including the normal-fluid response, is the impedances-in-parallel combination,
\begin{equation}
Z_k = (Z_{kn}^{-1} + Z_{kn}^{-1})^{-1}. \quad (83)
\end{equation}

Note that as $T \to T_{cq}$, $|Z_k|$ diverges, $Z_{kn} \to 1/\sigma_n$, and $Z_k$ approaches the normal-state resistivity.

In the limit as $j_{s0} \to 0$, our results reduce to the well-known two-fluid description$^{24}$ when the impedivity can be expressed in terms of the complex conductivity $\sigma = \sigma_1 - i\sigma_2$, the linear response function connecting $j$ and $E$ calculated by Mattis and Bardeen,$^{63}$ as $j_{s0} \to 0$, $R_{ks} \to 0$, $X_{ks} \to \omega\lambda_k = \mu_0\omega\lambda_0^2 = 1/\sigma_2$, $\sigma_{n1} \to \sigma_1; Z_{ks} \to i/\sigma_2$, $Z_{kn} \to 1/\sigma_1$, and $Z_k \to (\sigma_1 - i\sigma_2)^{-1} = \sigma^{-1}$. At temperatures above $T_{cq}$, the normal-state impedance is $Z = R_n + i\omega(L_k + L_m)$, where $L_m$
is the geometric inductance associated with stored magnetic energy and, for a long strip of conduction-electron density $n_c$, total length $\ell$, width $W$, and thickness $d$, $L_k = (m/n_c e^2)(\ell/Wd)$ is the normal-state kinetic inductance.$^5$ The impedance is usually dominated by the normal-state resistance $R_n = \rho_n \ell/Wd$ except at very high frequencies.$^5$

**VII. DISCUSSION**

In this paper we have presented fundamental theoretical calculations of the kinetic impedance of thin and narrow impure superconducting films for all temperatures and for all currents up to the depairing current. Our results should be applicable to ongoing experimental studies of small-scale superconducting devices in which the kinetic inductance plays an important role. Our calculations have shown examples of how the kinetic inductance and the normal-fluid dissipation depend upon the dc applied current. However, experiments examining the in-phase and out-of-phase third and higher harmonics might provide a more sensitive means of revealing the influence of the nonlinearities implied by these current dependencies.$^{72}$

Our results in the GL regime for the bias-current dependence of the kinetic inductance are in agreement with those of Anlage et al.$^{13}$ and Annunziata et al.$^2$ for the slow-experiment case and with the result of Anlage et al.$^{13}$ for the fast-experiment case. However, Annunziata et al.$^2$ in examining the case of $T = 0$ and noting correctly that $\lambda_0^2(0)$ is inversely proportional to $\Delta_0(0)$ (in our notation) [see Eq. (B3)], assumed that $\lambda_0^2(0)$ is inversely proportional to $\Delta_2(0)$. However, this assumption is incorrect, as can be seen from Eqs. (23), (26), and (28). As a consequence, their prediction for the current dependence of the kinetic inductance does not agree with our results for either slow or fast experiments.

In thin and narrow strips with sharp corners, current crowding leads to suppression of the order parameter in the immediate vicinity of sharp inner corners, and this can cause the critical current in such devices to be considerably lower than the depairing value.$^{19,23,73-76}$ By optimally rounding the inner corners, one should be able to raise the critical current to values close to the depairing value.$^{19,23,75}$

As discussed in Sec. II, our calculation of the critical depairing current density $j_d$ has been carried out within a mean-field approach disregarding fluctuations. However, the experimental critical current density $j_c$ could be somewhat smaller than $j_d$ as a result of thermal or quantum fluctuations, which can initiate phase slips in 1D wires$^{77,78}$ or vortex nucleation at the edges of strips.$^{15,16,23,79}$ when an energy barrier is overcome or suppressed to zero. The fluctuation-limited $j_c$ therefore may prevent the observation of both the predicted divergence of the slow-experiment kinetic inductivity at $j_d$ and the approach to the maximum values of the fast-experiment kinetic inductivity (shown by the filled symbols in Fig. 8). In fact, previous experimental observations of a relatively small kinetic inductance rising to a peak and then rapidly dropping to smaller values with increasing applied current$^{2,14,80,81}$ are explainable in terms of the growth of high-resistance normal regions as the current density rises above $j_c$.

As discussed in Secs. III-V, relaxation of the superfluid, characterized approximately here by the relaxation time $\tau_s$, plays an important role in determining both the real and imaginary parts of the kinetic impedance. To examine the physics of relaxation dynamics is beyond the scope of this paper, and for further discussion we refer the reader to Refs. 24 and 45 and references therein. Nevertheless, Figs. 9 and 10 suggest means by which $\tau_s$ could be estimated from experimental determinations of the superfluid’s ac resistivity and inductivity. At the very least, such experiments should be able to reveal whether they are in the slow- or fast-experiment limit, and experiments carried out at different frequencies and temperatures might be able to show the transition between these two limits.

In this paper we have used the concept of the $q$-dependent penetration depth, which increases as the current density increases. This occurs because an increase of the current density causes a decrease in the magnitude of the superconducting order parameter. This concept is the basis of the effective field-dependent penetration depth in type-II superconductors $\lambda(B, T)$, which diverges as $B \rightarrow B_{c2}(T)$. This quantity has been introduced to understand such phenomena as small-angle neutron-scattering form factors,$^{82}$ magnetic coupling of vortex lattices in dc superconducting transformers,$^{83,84}$ magnetization curves in low-pinning superconductors,$^{85}$ elastic properties of the vortex lattice,$^{86}$ and $\mu$SR measurements in the mixed state.$^{87,88}$

Here we have used the wavevector $q = 2\pi A_s/\phi_0$ to discuss the nonlinear response of the supercurrent to the gauge-invariant vector potential $A_s$ in the local limit. It is important not to confuse our $q$ with the $q$ used by Tinkham$^{24}$ in his discussion of the nonlocal electrodynamics of superconductors, where it arises from Fourier transforming the convolution integral relating the linear response of the current density $j(r)$ to the vector potential $A(r')$, as calculated by Mattis and Bardeen.$^{63}$

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Appendix A: \( u_{nq} \)

The function \( u_{nq}(\eta, \epsilon, \zeta) \) is the solution of the quartic equation
\[
e^2 u^4 - 2\eta \epsilon u^3 + (\eta^2 + \epsilon^2 - 2\zeta^2)u^2 - 2\eta \epsilon u - \eta^2 = 0, \tag{A1}
\]
obtained from Eq. (9). The desired solution of the quartic equation is
\[
u_{nq} = (f + g + \eta/\epsilon)/2, \tag{A2}
\]
where
\[
egin{align}
a &= -2[\epsilon^6(\zeta^2 - 1)^3 - 3\epsilon^4(1 + 16\zeta^2 + \zeta^4)\eta^2 + 3\epsilon^2(\zeta^2 - 1)\eta^4 - \eta^6], \tag{A3} \\
b &= \sqrt[-4]{\epsilon^6 + a^2}, \tag{A4} \\
c &= \epsilon^2(\zeta^2 - 1) - \eta^2, \tag{A5} \\
d &= (a + b)^{1/3}, \tag{A6} \\
e &= \frac{2^{2/3}c^2 + d^2}{2^{1/3}c^2d}, \tag{A7} \\
f &= \sqrt{\frac{2(\zeta^2 - 1) + \eta^2}{3\epsilon^2} + e}, \tag{A8} \\
g &= \sqrt{\frac{2(\zeta^2 - 1) + \eta^2}{3\epsilon^2} - e + \frac{2(\zeta^2 + 1)\eta}{\epsilon f}}. \tag{A9}
\end{align}
\]

Similar solutions of the quartic equation that arises in the closely related problem of the density of states in the Abrikosov-Gor’kov theory were obtained in Refs. 35 and 89.

Appendix B: Zero-current limit

In the limit of zero current, \( q = 0 \) \((v_s \to 0)\), \( u_{n0} = \eta/\epsilon \), and we obtain from Eq. (11) after defining \( t = T/T_{c0} \) and \( \delta_{0}(T) = \Delta_{0}(T)/2\pi k_B T_{c0} \),
\[
\ln \left\{ \frac{1}{t} \sum_{n=0}^{\infty} \left( \frac{1}{n+1/2} - \frac{1}{\sqrt{(n+1/2)^2+\delta_{0}^2(T)/\epsilon^2}} \right) \right\} \tag{B1}
\]
which yields the temperature dependence of \( \Delta_{0}(T) \) for all temperatures between 0 and \( T_{c0} \), the transition temperature for \( q = 0 \). An analysis of this equation as \( t \to 0 \) reveals that \( 4\pi \sigma_{n0}(0) = 2\Delta_{0}(0)/k_B T_{c0} = 2\pi e^{-\gamma} = 3.528 \), which is consistent with the fact that the above form of the Usadel theory coincides with the weak-coupling BCS theory for an s-wave isotropic gap on a spherical Fermi surface. Values of \( \Delta_{0}(T)/\Delta_{0}(0) \) vs \( t = T/T_{c0} \), which reproduce well-known results, can be obtained by numerically carrying out the sum in Eq. (B1), and it can be shown analytically that \( [\Delta_{0}(T)/\Delta_{0}(0)]^2 \to (8e^{2}\gamma/7\zeta(3))(1 - t) \) as \( t \to 1 \).

In the zero-q limit, the sum in Eq. (16) can be evaluated analytically as shown in Eq. (14), such that, as
\[
\frac{n_{n0}(T)}{n_{n0}(0)} = \frac{\lambda_{0}^{2}(T)}{\lambda_{0}^{2}(0)} \tag{B2}
\]
previously discussed in Refs. 91 and 24,
\[
\frac{n_{s0}(T)}{n_{s0}(0)} = \frac{\lambda_{0}^{2}(T)}{\lambda_{0}^{2}(0)} = \frac{\Delta_{0}(T)}{\Delta_{0}(0)} \tanh \left[ \frac{\Delta_{0}(T)}{2k_B T} \right]. \tag{B3}
\]

From Eq. (2), \( \xi_{0} = h\nu_{F}/\pi \Delta_{0}(0) \), and the normal-state conductivity \( \sigma_{n} = 2e^{2}N(0)D \), we obtain
\[
\frac{1}{\mu_{0}\lambda_{0}^2(T)} = \frac{n_{s0}(T)e^{2}}{m} = \frac{\pi\sigma_{n}\Delta_{0}(T)}{h} \tanh \left[ \frac{\Delta_{0}(T)}{2k_B T} \right]. \tag{B4}
\]

Figure 12 exhibits the temperature dependence of \( n_{s0}(T) \). As \( t \to 1 \), \( n_{s0}(T)/n_{s0}(0) \to [4\pi e^{2}/7\zeta(3)](1 - t) = 2.660(1 - t) \).

At \( T = 0 \), we can write
\[
\frac{1}{\mu_{0}\lambda_{0}^2(0)} = \frac{2}{3} N(0)e^{2}v_{F}^{2} \left( \frac{\ell}{\xi_{0}} \right), \tag{B4}
\]
but because the zero-temperature London penetration depth \( \lambda_{L}(0) \) can be expressed as
\[
\frac{1}{\mu_{0}\lambda_{L}^2(0)} = \frac{2}{3} N(0)e^{2}v_{F}^{2}, \tag{B5}
\]
we see that \( \lambda_{0}(0) = \lambda_{L}(0)(\xi_{0}/\ell)^{1/2} \). (Recall that \( \ell \ll \xi_{0} \) in the dirty limit under consideration here.)

Appendix C: Zero-gap limit

To find the boundary in the \( t-q \) plane where \( \Delta_{q}(T) \) is reduced to zero, note from Eq. (9) that \( \epsilon \sqrt{1 + u_{nq}^2} \to n + 1/2 + Q/2\pi k_B T \) in the limit \( \epsilon = \Delta_{q}(T)/2\pi k_B T \to 0 \), such that Eq. (11) then yields the \( q \)-dependent transition temperature. Defining \( t_{eq} = T_{cT}/T_{c0} \), \( P = \)
A similar procedure can be used to determine \( q_m(T) \), the value of \( q \) that drives \( \Delta_q(T) \) to zero for a given value of \( t \). Equation (11) then yields

\[
\ln \frac{1}{t_{eq}} = \sum_{n=0}^{\infty} \left( \frac{1}{n + 1/2} - \frac{1}{n + 1/2 + \alpha} \right), \tag{C1}
\]

where \( \psi \) is the digamma function and \( P/P_m = [q/q_m(0)]^2 = [\nu/\nu_m(0)]^2 \). Figure 13 shows \( t_{eq} \) as a function of \( q/q_m(0) = \nu/\nu_m(0) \). Expansions of \( t_{eq} \) about \( q = 0 \) and \( q = q_m(0) \) yield, respectively, the approximations

\[
t_{eq} = 1 - \frac{\pi^2 e^{-\gamma}}{8} \frac{q^2}{q_m(0)} = 1 - 0.693 \frac{q^2}{q_m(0)}, \tag{C3}
\]

\[
= \sqrt{3} e^{-\gamma} \sqrt{1 - \frac{q}{q_m(0)}} = 0.972 \sqrt{1 - \frac{q}{q_m(0)}}, \tag{C4}
\]

which are shown as the dashed and dotted curves in Fig. 13.

A similar procedure can be used to determine \( q_m(T) \), shown in Fig. 13. Expansions of \( q_m(T)/q_m(0) \) about \( t = 0 \) and \( t = 1 \) yield, respectively, the approximations

\[
\frac{q_m(T)}{q_m(0)} = 1 - \frac{2e^2}{3} t^2 = 1 - 1.057t^2, \tag{C7}
\]

\[
= \sqrt{\frac{8e^2}{\pi^2}} e^{-\gamma} \sqrt{1 - \frac{t}{1.202}} = 1.202 e^{-\gamma} \sqrt{1 - \frac{t}{1.202}}, \tag{C8}
\]

which correspond to the dotted and dashed curves in Fig. 13. Equations (C7) and (C8) are most easily obtained by making the replacements \( q \to q_m(T) \) and \( t_{eq} \to t \) in Eqs. (C4) and (C3).

The upper critical field is related to the temperature-dependent coherence length via \( B_{c2}(T) = \phi_0/2\pi \xi^2(T) \), and, as discussed in Ref. 92, in the dirty limit \( B_{c2} \) can be obtained from the equation

\[
\ln \left( \frac{1}{t} \right) = \psi \left( \frac{1}{2} + \rho \right) - \psi \left( \frac{1}{2} \right), \tag{C9}
\]

where \( t = T/T_{c0} \) and

\[
\rho = \left( \frac{e^{-\gamma}}{4} \right) \frac{B_{c2}(T)}{B_{c2}(0)} t, \tag{C10}
\]

where \( B_{c2}(0) = \phi_0/2\pi \xi^2(0) \) and \( \xi(0) = (\pi \xi_0 \ell/3)^{1/2} \).

As \( t \to 0 \), \( B_{c2}(T)/B_{c2}(0) \approx 1 - (2e^2/3)^2 t^2 \).

As \( t \to 1 \), \( B_{c2}(T)/B_{c2}(0) \to (8e^2/\pi^2)(1 - t) = 1.444(1 - t) \). Note that this result is consistent with what Helfand and Werthamer found for their normalized field \( h^*(0) = B_{c2}(0)/(dB_{c2}/dT)_{t=0} = 0.69 = 1/1.444 \) in the dirty limit. Since \( \xi(T) \to \sqrt{\pi^2 e^{-\gamma}/24\xi_0^2} \sqrt{1 - t} = 0.852\sqrt{\xi_0^2/\ell} \), we have in this limit \( \lambda_0(T)/\xi(T) = \kappa = \sqrt{42\xi(3)/\pi^4} \lambda_0(0)/\xi = 0.720\lambda_0(0)/\xi \).

A plot of \( B_{c2}(T)/B_{c2}(0) \) is equivalent to a plot of \( [\xi(0)/\xi(T)]^2 \), as shown in Fig. 12. Moreover, comparing Eqs. (C9) and (C9) with Eqs. (C5)-(34), we see that \( q_m(T) = 1/\xi(T) \) and \( q_m(0) = 1/\xi(0), \) and for all temperatures we have

\[
\frac{B_{c2}(T)}{B_{c2}(0)} = \frac{\xi^2(0)}{\xi^2(T)} = \frac{q_m(T)}{q_m(0)}. \tag{C11}
\]

Appendix D: Work done and free-energy changes resulting from current changes

It is of interest to examine the changes in energy as the current increases from zero to some final value. The work done per unit volume is

\[
W_v = \int_0^q \int_0^t J_{sq} Edt' = \left( \frac{\phi_0}{2\pi} \right) \int_0^q J_{sq} dq', \tag{D1}
\]

which is equal to the change in the free-energy density, as we show below.
The free-energy density $\Omega_q(T) = F_S(T) - F_N(T)$ of a current-carrying superconductor relative to the energy density of the normal state can be obtained by taking advantage of the theoretical similarities to the problem of superconducting alloys containing paramagnetic impurities. The expression for $\Omega_q(T) = F_S(T) - F_N(T)$ for the latter case obtained by Slaski et al.\cite{32} in their Eq. (5.6) can be rewritten compactly for the current-carrying superconductor as

$$\Omega_q = N(0) \int_0^\infty \text{Re} \left[ 2 \nu \pi + \frac{\Delta_q}{\sqrt{u^2 - 1}} \right] \tanh \left( \frac{\beta \omega}{2} \right) d\omega, \quad (D2)$$

using the replacements and changes in notation $\Gamma \to Q$, $\Delta(T, \Gamma) \to \Delta_q(T)$, $N_0 \to N(0)$, and $\omega' \to 0$ (weak-coupling limit). Here $\beta = 1/k_B T$,

$$\omega = u \left( 1 - i \frac{\zeta}{\sqrt{u^2 - 1}} \right), \quad (D3)$$

and

$$p(\omega) = -\int_0^\infty \left[ \frac{u'}{\sqrt{u'^2 - 1}} - 1 \right] d\omega' \quad (D4)$$

$$= \Delta_q \left[ \left( 1 - i \frac{\zeta}{\sqrt{u^2 - 1}} \right) \left( \sqrt{u^2 - 1} - u \right) \right. - \left. \frac{i \zeta}{2(u^2 - 1)} \right] \quad (D5)$$

is a quantity arising from partial integration of the term proportional to $\ln(1 + e^{-\omega})$ in the entropy contribution to $\Omega_q(T)$. Some useful relations are $\text{Re}[p(0)] = 0$ and

$$\int_0^\infty \text{Re} \left[ \frac{p(\omega) + \omega}{(1 + \omega)^{1/2}} \right] = -\frac{\Delta_q^2}{4}. \quad (D6)$$

The quantity within the brackets in the integrand of Eq. (D2) can be expressed in terms of $\eta$ as

$$\left( 2 \frac{\sqrt{u^2 - 1}}{u} - 2 + \frac{1}{u \sqrt{u^2 - 1}} \right), \quad (D7)$$

and the integral in Eq. (D2) can be evaluated using contour integration around the boundaries of the first quadrant of the complex $\omega$ plane, taking into account the poles along the imaginary axis at the Matsubara frequencies $i \omega_n = i2\pi k_B T(n + 1/2) = i2\pi k_B T \eta$. The result is

$$\Omega_q(T) = -N(0)(2\pi k_B T)^2 \sum_{n=0}^\infty \eta \left[ 2 \left( \frac{u_n}{1 + u_n^2} \right) - 1 \right] + \frac{1}{u_n \sqrt{1 + u_n^2}}. \quad (D8)$$

Differention of Eqs. (11) and (D8) with respect to $q$ yields the general result that

$$\frac{d\Omega_q(T)}{dq} = N(0)(2\pi k_B T)^2 \sum_{n=0}^\infty \frac{1}{1 + u_n^2} \quad (D9)$$

$$= -\left( \frac{\phi_0}{2\pi} \right) j_q(T). \quad (D10)$$

This result agrees with Eq. (D1).

In the zero-temperature limit, the sum in Eq. (D8) can be converted to an integral over $\eta$ using Eq. (9), with the result

$$\Omega_q(0) = -\frac{N(0)\Delta_q^2}{2} \left( 1 - \frac{\pi \zeta^2}{\Delta_q^2} + \frac{2\zeta^2}{3} \right), \quad (D10)$$

$$\Omega_q(0) = -\frac{N(0)\Delta_q^2}{2} \left( 1 - \frac{\pi \zeta^2}{\Delta_q^2} + \frac{2\zeta^2}{3} \right) + \frac{2\zeta^2}{3} \zeta^2 - 1, \quad (D11)$$

where $\Delta_q(0)$ is given by Eqs. (23) and (24), and $\zeta_0$ by Eq. (25).

It is possible to write the free-energy density as the sum of two terms, $\Omega_q(T) = F_{cq}(T) + F_{kq}(T)$, where we identify $F_{cq}(T)$ as the condensation-energy density and $F_{kq}(T)$ as the kinetic-energy density:

$$F_{cq}(T) = -N(0)(2\pi k_B T)^2 \sum_{n=0}^\infty \frac{\nu_n \left( \frac{u_n}{1 + u_n^2} \right)}{\sqrt{1 + u_n^2}} \quad (D13)$$

$$F_{kq}(T) = N(0)(2\pi k_B T) \sum_{n=0}^\infty \frac{1}{1 + u_n^2}. \quad (D14)$$

Note, however, that both terms depend upon $q$ and interact. As $q$ increases from zero, $F_{kq}(T)$ initially decreases from zero and the magnitude of $F_{cq}(T)$ decreases, but their sum $\Omega_q(T)$ decreases to zero as $q \to q_m(T)$. Figure 14 shows the $q$ dependence of the energy densities $\Omega_q$, $F_{cq}$, and $F_{kq}$ at $T = 0$.

It is important to note that $F_{kq}(T)$ is not equal to $L_k(q, T) j_q^2 / 2$ except in the limit as $q \to 0$ [see Eq. (1)].
The bulk thermodynamic critical field $H_c(T)$ is defined via the superconducting condensation energy at $q = 0$:

$$\frac{1}{2} n_0 H_c^2(T) = -\Omega_0(T)$$

$$= N(0)(2\pi k_B T)^2 \sum_{n=0}^{\infty} \left( 2\sqrt{\eta^2 + \epsilon_0^2} + 2\eta - \frac{\epsilon_0^2}{\sqrt{\eta^2 + \epsilon_0^2}} \right),$$

where $\eta = n + 1/2$ and $\epsilon_0 = \Delta_0(T)/2\pi k_B T$. Equation (D16), which follows from Eq. (D8) when $q \to 0$ and $u_{nq} \to u_{n0} = \eta/\epsilon_0$ [see Eq. (9)], reproduces the BCS temperature dependence of $H_c(T)$.

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When $\nabla \cdot j_s = 0$, the conditions $P = 0$ and $E = -dA_s/dt$ hold within the generalized time-dependent GL theory, and we assume that these conditions also apply here.

This definition of the inductivity is the natural extension of the definition of the inductance $L$ via $V = LdI/dt$ relating the voltage drop $V$ across an inductor to the current $I$.

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