ON THE FITTING HEIGHT AND INSOLUBLE LENGTH OF FINITE GROUPS

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Abstract. We prove two conjectures of E. Khukhro and P. Shumyatsky concerning the Fitting height and insoluble length of finite groups. As a by-product of our methods, we also prove a generalization of a result of Flavell, which itself generalizes Wielandt’s Zipper Lemma and provides a characterization of subgroups contained in a unique maximal subgroup. We also derive a number of consequences of our theorems, including some applications to the set of odd order elements of a finite group inverted by an involutory automorphism.

1. Introduction

A classical result of R. Baer [1] states that an element \( x \) of a finite group \( G \) is contained in the Fitting subgroup \( F(G) \) of \( G \) if and only if \( x \) is a left Engel element of \( G \). That is, \( x \in F(G) \) if and only if there exists a positive integer \( k \) such that \([g, k \; x] := [g, x, \ldots, x] \) (with \( x \) appearing \( k \) times) is trivial for all \( g \in G \). (In this paper, we use left normed commutators, so that \([x_1, x_2, x_3, \ldots, x_k] := [[[x_1, x_2], x_3], \ldots, x_k] \)). The result was generalized by E. Khukhro and P. Shumyatsky in [6] via an analysis of the sets \( E_{G,k}(x) := \{[g, k \; x] : g \in G \} \).

In this notation, Baer’s Theorem states that \( x \in F(G) \) if and only if \( E_{G,k}(x) = \{1\} \) for some positive integer \( k \). The generalization of Khukhro and Shumyatsky takes three directions. First, if \( G \) is soluble then a complete generalization is obtained: [6, Theorem 1.1] proves that if the Fitting height of the subgroup \( \langle E_{G,k}(x) \rangle \) is \( h \), then \( x \) is contained in \( F_{h+1}(G) \) - the \((h + 1)\)-st Fitting subgroup of \( G \).

Secondly, they also discuss analogous results for insoluble groups: For a finite group \( G \), write \( F_1^*(G) \) for the \( i \)-th generalized Fitting subgroup of \( G \). That is, \( F_1^*(G) := F^*(G) \) is the generalized Fitting subgroup of \( G \), and \( F_i^*(G) \) is the inverse image of \( F^*(G/F_{i-1}^*(G)) \) in \( G \) for \( i \geq 2 \). Thus, in particular, \( F_i(G) = F_i^*(G) \) when \( G \) is soluble. It is proven in [6, Theorem 1.2] that if \( \langle E_{G,k}(x) \rangle \) has generalized Fitting height \( h \), then \( x \) is contained in \( F_{f(x,h)}^*(G) \) for a certain function \( f \) defined in terms of \( h \) and the number of prime divisors of the order of \( x \) (counting multiplicities). The authors conjecture in [6, Conjecture 7.1] that in this case, \( x \) is in fact contained in \( F_{h+1}^*(G) \). The first main result of this paper is a proof of this conjecture.

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Theorem 1.1. Let \( G \) be a finite group, and let \( x \) be an element of \( G \). Suppose that \( \langle E_{G,k}(G) \rangle \) has generalized Fitting height \( h \) for some positive integer \( k \). Then \( x \) is contained in \( F_{h+1}^*(G) \).

Thirdly, another length parameter for finite groups is discussed. For a finite group \( G \), write \( \lambda(G) \) for the insoluble length of \( G \). That is, \( \lambda(G) \) is the minimum number of insoluble factors in a normal series for \( G \) each of whose factors is either soluble or a direct product of non-abelian simple groups. In particular, a group is soluble if and only if \( \lambda(G) = 0 \). The group \( R_0(G) \) is defined to be the soluble radical of \( G \), while \( R_i(G) \) is defined to be the largest normal subgroup of \( G \) with insoluble length \( i \), for \( i \geq 1 \). The series \( 1 \leq R_0(G) \leq \ldots \leq R_i(G) \leq \ldots \leq G \) is called the upper insoluble series for \( G \), and \([6, \text{Theorem 1.3}]\) shows that if \( \langle E_{G,k}(x) \rangle \) has insoluble length \( h \), then \( x \) is contained in \( R_{r(x,h)}(G) \) for a certain function \( r \) defined in terms of \( h \) and the number of prime divisors of \( x \) (again counting multiplicities). Khukhro and Shumyatsky conjectured in \([6, \text{Conjecture 7.2}]\) that we should have \( x \in R_h(G) \) in this case, and our next result proves their conjecture.

Theorem 1.2. Let \( G \) be a finite group, and let \( x \) be an element of \( G \). Suppose that \( \langle E_{G,k}(\alpha) \rangle \) has insoluble length \( h \) for some positive integer \( k \). Then \( x \) is contained in \( R_h(G) \).

Bounds on the insoluble length and the Fitting height of a finite group have proved to be powerful tools in both finite and profinite group theory. In particular, such bounds were crucial in the reduction of the Restricted Burnside Problem to soluble and nilpotent groups due to P. Hall and G. Higman \([4]\). J. Wilson also used such bounds when reducing the problem of proving that periodic profinite groups are locally finite to pro-\( p \)-groups \([7]\). E. Zelmanov then solved both of these problems in his famous papers \([8, 9, 10]\).

If \( A \) is a subgroup of \( H \), let \( A^H \) denote the subgroup \( H \) generated by all the conjugates of \( A \) in \( H \). Theorems 1.1 and 1.2 can in fact be deduced from the following general result.

Theorem 1.3. Let \( G \) be a group which satisfies the max condition on subgroups, and the min condition on subnormal subgroups, and let \( A \) be a subgroup of \( G \) with \( A^G = G \). Set \( Y := Y_G(A) = \langle H < G | A \leq H = A^H \rangle \). Then one of the following holds.

(1) \( Y = G \).

(2) \( A \) is contained in a unique maximal subgroup of \( G \).

Theorem 1.3 will follow from a generalization of a result of Flavell, which itself generalizes Wielandt’s Zipper Lemma.

As a by-product of our methods, we also obtain strong results concerning the sets \( E_{G,k}(x) \).

Theorem 1.4. Let \( G \) be a finite group, and let \( \alpha \) be an element of \( \text{Aut}(G) \) with the property that \( [G, \alpha] = G \). Then \( G = \langle E_{G,k}(\alpha) \rangle \) for all positive integers \( k \).
In Theorem 1.4, the commutators \([g, k, \alpha]\) are understood to be computed in \(\langle G, \alpha \rangle\).

An easy corollary of Theorem 1.4 is the following:

**Corollary 1.5.** Let \(G\) be a finite group, and let \(x\) be an element of \(G\). Let \(H\) be the final term in the subnormal series \(G \geq [G, x] \geq [G, x, x] \geq \ldots\)

(1) \(\langle E_{G, k}(\alpha) \rangle\) is subnormal in \(G\) for all \(k\); and
(2) \(H = K\).

In the special case where the automorphism \(\alpha\) in Theorem 1.4 is an involution, a stronger result is available. First, in this case define the set \(J_G(\alpha) := \{ g \in G : g\) has odd order and \(g^\alpha = g^{-1}\}\).

We then have the following.

**Theorem 1.6.** Let \(1 \neq G\) be a finite group, and let \(\alpha \in \text{Aut}(G)\) be an involution. Suppose that \([G, \alpha] = G\). Then \(J_G(\alpha) = E_{G, k+1}(\alpha)\), where \(k\) is maximal with the property that \(G\) has an element \(g\) inverted by \(\alpha\) with \(g^{2^k}\) of odd order. In particular, \(G = \langle J_G(\alpha) \rangle\).

We also record a corollary of Theorem 1.6 which may be of independent interest.

**Corollary 1.7.** Let \(G\) be a finite group, and let \(\alpha \in \text{Aut}(G)\) be an involution. Then \(|G/R(G)|\) can be bounded in terms of \(|J_G(\alpha)|\), where \(R(G)\) denotes the soluble radical of \(G\).

We will write \(Z(G)\), \(F(G)\), and \([G,G]\) to denote the centre, Fitting subgroup, and derived subgroup of \(G\), respectively. For an element \(g\) of \(G\), we will denote the order of \(g\) by \(|g|\).

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2. Proofs of the main theorems

We begin this section by showing that Theorems 1.1, 1.2, 1.4 and 1.6 together with Corollary 1.5, follow from Theorem 1.3.

**Proposition 2.1.** Assume that Theorem 1.3 holds. Then Theorems 1.1, 1.2, 1.4 and 1.6 hold.

**Proof.** We first prove that Theorem 1.3 holds. So assume that \(G\) is a finite group, \(\alpha\) is an automorphism of \(G\), and \([G, \alpha] = G\). Suppose that \(k\) is minimal with the property that \(\langle E_{G, k}(\alpha) \rangle \neq G\). Note that if \(H\) is a subgroup of \(G\), then \([H, \alpha] = H\) if and only if \(\langle \alpha^H \rangle = \langle H, \alpha \rangle\). Set \(X := \langle G, \alpha \rangle\). Thus, if \(Y := Y_X(\alpha) = X\), then \(X = \langle \alpha^H : H < G \rangle\) and the result
follows by induction on the order of $G$. Thus, by Theorem [1.3] we may assume that $\alpha$ is contained in a unique maximal subgroup $K$ of $X$. Since $\langle G, \alpha \rangle$ is generated by conjugates of $\alpha$, $K$ is self normalizing.

Now, by the minimality of $k$ we may choose an element $h := [g, k_{-1}^\alpha]$ of $X \setminus K$. Then $[h, \alpha] \in K$, so $\alpha^h \in K$. Whence $\alpha \in K^{h^{-1}} \neq K$, contrary to assumption. This completes the proof of Theorem [1.4].

We next prove Corollary [1.5]. Let $N$ be a minimal normal subgroup of $A$. Let $R = \langle E_{G,k}(\alpha) \rangle$. We need to show that $R$ is subnormal in $A$. By induction $RN$ is subnormal in $A$ and so it suffices to prove that $N$ normalizes $R$ or $RN = RC_N(\alpha)$ (since $C_N(\alpha)$ normalizes $R$).

First suppose that $N$ is elementary abelian. Then since $N$ acts trivially on $N$, $R$ acts completely reducibly on $N$. Let $M$ be an irreducible $R$-submodule of $N$ with $M \cap R = 1$ (or equivalently $M$ is not contained in $H$). Then $M_0 := [M, \alpha, \ldots, \alpha]$ (the $k$-fold commutator) is contained in $M \cap R = 1$. Thus $C_M(\alpha) \neq 1$ and so $C_M(\alpha) \cap M \neq 1$. Since $M$ is an irreducible $R$-module, this implies that $M \leq C_M(\alpha)R$. Since this is true for any irreducible submodule, $N \leq C_N(\alpha)R$.

So we may assume that $N$ is a direct product of $t \geq 1$ components. By Theorem [1.4] $R \cap N$ contains $[N, \alpha]$, whence $N = (N \cap R)C_N(\alpha)$ and as above, $N$ normalizes $H$. This proves (1)) of Corollary [1.5]. Since $H = [H, \alpha]$, it follows by Theorem [1.4] that $H = \langle E_{H,m}(\alpha) \rangle \leq \langle E_{G,m}(\alpha) \rangle$ for all $m$, whence $H \leq K$. Obviously $K \leq H$, whence $H = K$.

We now prove Theorem [1.6]. So assume that $\alpha$ is an involution. Let $k$ be maximal with the property that there exists $g \in G$ with $g$ not of 2-power order, and $|g|^2 = 2^k$. Then since $[g, k^\alpha] = g^{(-2)^k}$, we have that $E_{G,k+1}(\alpha) = E_{G,k+2}(\alpha) = \ldots = E_{G,s}(\alpha) = \ldots$, and each element of $E_{G,k+1}(\alpha)$ is inverted by $\alpha$. Thus, $E_{G,k}(\alpha) \subseteq J_G(\alpha)$, and the result follows from Theorem [1.4]. Note that $J_G(\alpha)$ is contained in $E_{G,j}$ for all $j$, whence $J_G(\alpha) = E_{G,j}(\alpha)$ for $j > k$.

Next, we prove that Theorems [1.1] and [1.2] follow from Theorem [1.4].

Indeed, suppose that $G$ is a finite group, $\alpha$ is an element of $G$, and $k$ is a positive integer. Let $H$ be the stable term in the subnormal series $G \geq [G, \alpha] \geq [G, \alpha, \alpha] \geq \ldots$. Then $H = [H, \alpha]$, so $H = \langle E_{H,k}(\alpha) \rangle \leq \langle E_{G,k}(\alpha) \rangle$ by Theorem [1.4]. Now, an easy exercise shows that the generalized Fitting height [respectively insoluble length] of $H^G$ coincides with the generalized Fitting height [resp. insoluble length] of $H$, since $H$ is subnormal in $G$. Moreover, the generalized Fitting height [resp. insoluble length] of $H$ is at most the generalized Fitting height [resp. insoluble length] of $\langle E_{G,k}(\alpha) \rangle$. Let $h$ be the generalized Fitting height [resp. insoluble length] of $H$. Then $H^G$, being a normal subgroup of $G$ with generalized Fitting height [resp. insoluble length] $h$, is contained in $F^h_h(G)$ [resp. $R_h(G)$]. Whence, $F^h_h(G)\alpha$
Thus, by Proposition 2.1 we just need to prove Theorem 1.3 and Corollary 1.7. To this end, for a proper subgroup $A$ of a group $H$, we define the normal closure descending series for $A$ in $H$ as follows. Let $H_0 = H$ and let $H_{i+1} = \langle AH_i \rangle$. We define $F(A, H) := \cap_{i \geq 0} H_i$.

We first note the following trivial facts:

**Lemma 2.2.** Let $H, A, H_i$ and $F(A, H)$ be as defined above, and assume that $H$ satisfies the min condition on subnormal subgroups. Then

(i) $H_{i+1}$ is normal in $H_i$ and $H_j$ is subnormal in $H$ for all $j$.
(ii) $\langle A^{F(A, H)} \rangle = F(A, H)$.
(iii) If $A \leq L \leq H$ and $\langle A^L \rangle = L$, then $L \leq F(A, H)$.

**Proof.** Parts (i) and (ii) follow immediately from the definition of the series $H = H_0 \geq H_1 \geq \ldots$, since $F(A, H) = H_m$ is a member of the series in this case. So assume that $A \leq L \leq H$ and that $\langle A^L \rangle = L$. Then $H_0 = LH_0$, so $H_1 = \langle A^{LH_0} \rangle \geq \langle A^L \rangle = L$. Extending this argument inductively yields $L \leq H_i$ for all $i$. Whence, $L \leq H_m = F(A, H)$. □

Next, we prove a generalization of a result of Flavell, which states that if $G$ is finite and $A$ is a proper subgroup of $G$ which is contained in at least two maximal subgroups and is subnormal in all but at most one of the maximal subgroups in which it is contained, then $A$ is contained in a proper normal subgroup of $G$. Write $\mathcal{M}(A)$ for the set of maximal subgroups of $G$ containing $A$. We remark that Wielandt’s Zipper Lemma [5, Theorem 2.9], which is usually stated for finite groups, holds in the more general case where $G$ satisfies the max condition for subgroups, while his Join Lemma [5, Theorem 2.5], holds whenever $G$ satisfies the max condition for subnormal subgroups. Whence, from [2, proof of Main Theorem] we can see that Flavell’s Theorem holds in the more general case where $G$ satisfies the max condition on subgroups.

Our generalization can now be given as follows.

**Lemma 2.3.** Let $G$ be a group satisfying the max condition on subgroups, and the min condition on subnormal subgroups. Let $A$ be a proper subgroup of $G$ satisfying the following:

(a) $A$ is contained in at least two maximal subgroups of $G$.
(b) The set $\{F(A, H) : H \in \mathcal{M}(A)\}$ has a unique maximal element.

Then $A$ is contained in a proper normal subgroup of $G$. 
Proof. Clearly, we may assume that \( \langle A^G \rangle = G \). Denote by \( \Omega(A, G) \) the set of subgroups \( H \) of \( G \) with the property that \( \langle A^H \rangle = H \), and let \( Y \) be the unique maximal element of the set \( \{ F(A, H) : H \in \mathcal{M}(A) \} \). Also, let \( M \) be a maximal subgroup of \( G \) containing \( Y \).

Now, choose \( X \in \Omega(A, G) \) maximal with respect to \( X \) being contained in at least two maximal subgroups of \( G \). This set is not empty since \( A \) has this property. If \( L \neq M \) is any maximal subgroup of \( G \) containing \( X \), observe that \( X \) is the stable term in the normal closure series for \( A \) in \( L \) (by part (iii) of the previous lemma, \( X \) is contained in the stable term, which in turn is contained in \( Y \leq M \) and by maximality, it is the stable term).

Thus, \( X \) is subnormal in all but at most one of the maximal subgroups in which it is contained. Flavell’s Theorem [2] then implies that \( X \) is contained in a proper normal subgroup of \( G \). Since \( A \leq X \), this completes the proof. \( \square \)

We remark that this does indeed generalize the theorem of Flavell mentioned above. To see this, suppose that \( G \) is finite, \( A < G \), \( |\mathcal{M}(A)| \geq 2 \), and \( A \) is subnormal in all but at most one member, say \( M \), of \( \mathcal{A} \). We claim that \( A \) is contained in a proper normal subgroup of \( G \). Clearly, we may assume that \( \langle A^G \rangle = G \). We first prove that \( A = F(A, L) \) for any maximal subgroup \( L \neq M \) containing \( A \). Let \( A = H_{d+1} < H_d < \ldots < H_1 < H_0 = L \) be a subnormal chain for \( A < L \), where \( H_i \leq H_{i+1} \). Consider the normal closure descending series \( F(A, L) = L_m < \ldots < L_1 < L_0 = L \) for \( A \) in \( L \) as defined above. Then \( L_1 = A^L \leq H_1 \), and it follows via an easy inductive argument that \( L_i \leq H_i \) for all \( i \). Thus, \( A \leq F(A, L) \leq L_{d+1} = H_{d+1} = A \), so \( A = F(A, L) \), as claimed. It follows that the set \( \{ F(A, H) : H \in \mathcal{M}(A) \} \) has a unique maximal element. Whence the claim.

We are now ready to prove Theorem \[1.3\].

Proof of Theorem \[1.3\] Let \( \Omega(A, G) \) be as in the proof of Lemma \[2.3\] and let \( Y = \langle H | H \in \Omega(A, G) \rangle \). Assume that \( Y \neq G \), and let \( M \) be a maximal subgroup of \( G \) containing \( Y \). Then \( Y = F(A, M) \).

Now, every subgroup in \( \Omega(A, G) \) is contained in \( Y \) (and so in \( M \)). Suppose that \( B \) is contained in some maximal subgroup \( K \neq M \). Then \( F(A, K) \in \Omega(A, G) \), so \( B \leq F(A, K) \leq M \). Whence, \( F(A, K) \leq Y \) by Lemma \[2.2\] part (iii). It follows that the set \( \{ F(A, H) : H \in \mathcal{M}(A) \} \) has a unique maximal element. Thus, \( A \) is contained in a proper normal subgroup of \( G \), by Lemma \[2.3\] This contradicts \( \langle A^G \rangle = G \), and completes the proof. \( \square \)

Finally, we prove Corollary \[1.7\].

Proof of Corollary \[1.7\] Recall that \( G \) is a finite group, \( \alpha \) is an involutory automorphism of \( G \), \( X := [G, \alpha] \), and \( R := R(X) \) is the soluble radical of \( X \). We need to prove that \( |X/R| \) can be bounded in terms of \( |J| \), where \( J := J_G(\alpha) \). Writing bars to denote reduction modulo \( R \),
we have that $|\mathcal{X}|$ can be bounded in terms of $|\mathcal{F}|$, where $F$ is the subgroup of $G$ containing $R$ with $\mathcal{F} = F^*(\mathcal{X})$. Clearly, we may assume that $\mathcal{X} > 1$. Now, by construction we have $\mathcal{F} = [F, \alpha]$. Let $H$ be the stable term in the subnormal series $F \geq [F, \alpha] \geq [F, \alpha, \alpha] \geq \ldots$. Then $\mathcal{H} = \mathcal{F}$, and $[H, \alpha] = H$. Then $J_1 := J_H(\alpha) \subseteq J$ generates $F$ modulo $R$, and the set $\mathcal{J}_1$ is normalized by $C_{\mathcal{F}}(\alpha)$. Since $\mathcal{F}$ is a direct product of non-abelian simple groups, we have $Z((\mathcal{F}, \alpha)) = 1$. Hence, $|C_{\mathcal{F}}(\alpha)| \leq |J|! \leq |J|!$. Finally, the main theorem in [3] implies that $\mathcal{F}$ has a nilpotent normal subgroup of index bounded by a function of $|C_{\mathcal{F}}(\alpha)|$. Thus, $|\mathcal{F}|$ can be bounded in terms of $|C_{\mathcal{F}}(\alpha)|$, and this completes the proof. 

\[ \square \]

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