Incoherence-Optimal Matrix Completion

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Abstract

This paper considers the matrix completion problem. We show that it is not necessary to assume joint incoherence, which is a standard but unintuitive and restrictive condition that is imposed by previous studies. This leads to a sample complexity bound that is order-wise optimal with respect to the incoherence parameter (as well as to the rank $r$ and the matrix dimension $n$ up to a log factor). As a consequence, we improve the sample complexity of recovering a semidefinite matrix from $O(nr^2 \log^2 n)$ to $O(nr \log^2 n)$, and the highest allowable rank from $\Theta(\sqrt{n}/\log n)$ to $\Theta(n/\log^2 n)$. The key step in proof is to obtain new bounds on the $\ell_\infty, \ell_2$-norm, defined as the maximum of the row and column norms of a matrix. To illustrate the applicability of our techniques, we discuss extensions to SVD projection, structured matrix completion and semi-supervised clustering, for which we provide order-wise improvements over existing results. Finally, we turn to the closely-related problem of low-rank-plus-sparse matrix decomposition. We show that the joint incoherence condition is unavoidable here for polynomial-time algorithms conditioned on the Planted Clique conjecture. This means it is intractable in general to separate a rank-$\omega(\sqrt{n})$ positive semidefinite matrix and a sparse matrix. Interestingly, our results show that the standard and joint incoherence conditions are associated respectively with the information (statistical) and computational aspects of the matrix decomposition problem.

1 Introduction

The matrix completion problem concerns recovering a low-rank matrix from an observed subset of its entries. Recent research [11, 32, 20, 26, 24] has demonstrated the following remarkable fact: if a rank-$r$ $n \times n$ matrix satisfies certain incoherence properties, then it is possible to exactly reconstruct the matrix with high probability from $nr \log \log(n) \ll n^2$ uniformly sampled entries using efficient polynomial-time algorithms.

In previous work, the sample complexity $\Theta(nr \log(n))$ is achieved only for matrices that satisfy two types of incoherence conditions with constant parameters. The first condition, known as standard incoherence, is a natural and necessary requirement; it prevents the information of the row and column spaces of the matrix from being too concentrated in a few rows or columns. A second condition, called joint incoherence (or strong incoherence), is also needed. It requires the left and right singular vectors of the matrix to be unaligned with each other. This condition is quite unintuitive, and does not seem to have a natural interpretation. As we demonstrate later, this condition is often restrictive and precludes a large class of otherwise well-conditioned matrices. For example, positive semidefinite matrices have a non-constant joint incoherence parameter on the order of $\Omega(r)$, and previous results thus require the number of observations to be proportional to $nr^2$ instead of $nr$. In several applications of matrix completion discussed later, the joint incoherence condition leads to artificial and undesired constraints. In contrast, numerical experiments suggest that this condition is not needed.

In this paper, we prove that the joint incoherence condition is not necessary and can be completely eliminated. With $\Omega(nr \log^2 n)$ uniformly sampled entries, one can recover a matrix that satisfies the standard incoherence condition (with a constant parameter) but is not jointly incoherent (e.g., a
positive semidefinite matrix). As we show in Section 2, our sample complexity bounds are order-wise optimal with respect to not only the matrix dimensions $n$ and $r$ but also to its incoherence parameters except for a log $n$ factor. As a consequence, we improve the sample complexity of recovering a positive semidefinite matrix from $O(nr^2 \log^2 n)$ to $O(nr \log^2 n)$, and the highest allowable rank from $\Theta(\sqrt{n}/ \log n)$ to $\Theta(n/ \log^2 n)$.

Our results apply to the standard nuclear norm minimization approach to matrix completion. The improvements are achieved by a new analysis based on bounds involving the $\ell_{\infty,2}$ matrix norm, defined as the maximum of the row and column norms of the matrix. This differs from previous approaches that use $\ell_\infty$ bounds. We show that this technique can be extended to obtain strong theoretical guarantees for the following two problems:

1. the analysis of a Singular Value Decomposition (SVD) projection algorithm for matrix completion;
2. structured matrix completion and semi-supervised clustering with side information.

In both problems we achieve order-wise improvements over existing results. We believe the $\ell_{\infty,2}$ norm is useful more broadly. For example, in the follow-up work [14], a weighted version of the $\ell_{\infty,2}$ norm plays a crucial role in the analysis of general low-rank matrices that violate the standard incoherence condition.

Finally, we turn to the closely related problem of matrix decomposition, where one is asked to recover a low-rank matrix and a sparse matrix from their sum. We show that the joint incoherence condition is necessary in this setting based on the computational complexity assumption of the Planted Clique problem. In particular, any decomposition algorithm that does not require the joint incoherence condition would solve Planted Clique with clique size $o(\sqrt{n})$, a problem that has been extensively studied and is widely believed to be intractable in polynomial time. This implies that it is computationally hard in general to separate a rank-$\omega(\sqrt{n})$ positive semidefinite matrix and a sparse matrix. Interestingly, our results show that the standard incoherence condition is inherently associated the information-theoretic (or statistical) aspect of the problem, whereas the joint incoherence condition reflects the computational aspect.

Related work We briefly survey existing related work; detailed comparisons with these results are provided after we present our theorems. Matrix completion is first studied in [10], which initiates the use of the nuclear norm minimization approach. The work in [11, 32, 20, 26] provides state-of-the-art theoretical guarantees on exact completion. Alternative algorithms for matrix completion are considered in [24, 26, 8]. All these works require the joint incoherence condition (or a sample complexity that is at least quadratic in $r$). Our extensions to SVD projection, structured matrix completion and semi-supervised clustering are inspired by the work in [26, 37]; we improve upon their results. The low-rank and sparse matrix decomposition problem is considered first in [12, 9] and subsequently in [15, 28, 1] 23. The work in [9, 15, 28] prove the success of specific algorithms assuming the standard and joint incoherence conditions. Our results show that these two incoherence conditions are in fact necessary for all algorithms, or all polynomial-time algorithms, due to statistical and computational reasons. The seminal work in [6, 5] is the first to use the Planted Clique problem to establish statistical limits under computational constraints; they consider the problem of sparse Principal Component Analysis (PCA). A similar approach is taken in [30] for the submatrix detection problem.

Organization In Section 2, we present our main result and show that the joint incoherence condition is not needed in matrix completion. We discuss extensions to SVD projection and structured matrix completion in Section 3. In Section 4, we turn to the matrix decomposition problem and show that the joint incoherence condition is unavoidable there. We prove our main theorem in Section 5 with some technical aspects of the proofs deferred to the appendix. The paper is concluded with a discussion in Section 6.
Notation  Lower case bold letters (e.g., \( z \)) denote vectors, while capital bold face letters (e.g., \( Z \)) denote matrices. For a matrix \( Z, Z_{ij} \) and \( (Z)_{ij} \) both denote its \((i,j)\)-th entry. By with high probability (w.h.p.) we mean with probability at least \( 1 - c_1 (n_1 + n_2)^{-c_2} \) for some universal constants \( c_1, c_2 > 0 \), where \( n_1 \) and \( n_2 \) are dimensions of the low-rank matrix. \( \| \cdot \|_2 \) denotes the vector \( \ell_2 \) norm. \( \| \cdot \|_F, \| \cdot \|_* \) and \( \| \cdot \| \) denote the Frobenius, nuclear and spectral norms for matrices, respectively.

2 Main Results

We now define the matrix completion problem. Suppose \( M \in \mathbb{R}^{n_1 \times n_2} \) is an unknown matrix with rank \( r \). For each \((i,j)\), \( M_{ij} \) is observed with probability \( p \) independent of all others. Let \( \Omega \) be the set of the indices of the observed entries. The matrix completion problem asks for recovering \( M \) from the observations \( \{M_{ij}, (i,j) \in \Omega \} \). The standard and arguably the most popular approach to matrix completion is the nuclear norm minimization method \( [10] \):

\[
\min_{X} \| X \|_* \quad \text{s.t.} \quad X_{ij} = M_{ij} \text{ for } (i,j) \in \Omega,
\]

where \( \| X \|_* \) is the nuclear norm of the matrix \( X \), defined as the sum of its singular values. Our goal is to obtain sufficient conditions under which the optimal solution to the problem (1) is unique and equal to \( M \) with high probability.

It is observed in [10] that if \( M \) is equal to zero in nearly all of rows or columns, then it is impossible to complete \( M \) unless all of its entries of are observed. To avoid such pathological situations, it has become standard to assume \( M \) to have additional properties known as incoherence. Suppose the rank-\( r \) SVD of \( M \) is \( U \Sigma V^\top \). \( M \) is said to satisfy the standard incoherence condition with parameter \( \mu_0 \) if

\[
\max_{1 \leq i \leq n_1} \| U^\top e_i \|_2 \leq \sqrt{\frac{\mu_0 r}{n_1}},
\]

\[
\max_{1 \leq j \leq n_2} \| V^\top e_j \|_2 \leq \sqrt{\frac{\mu_0 r}{n_2}},
\]

where \( e_i \) are the \( i \)-th standard basis with appropriate dimension. Note that \( 1 \leq \mu_0 \leq \frac{\min(n_1,n_2)}{r} \).

Previous work also requires \( M \) to satisfy an additional joint incoherence (or strong incoherence) condition with parameter \( \mu_1 \), defined as

\[
\max_{i,j} \| (UV^\top)_{ij} \| \leq \sqrt{\frac{\mu_1 r}{n_1 n_2}}.
\]

Under these two conditions, existing results require \( p \geq \max\{\mu_0, \mu_1\} r \text{polylog}(n)/n \) to recover \( M \in \mathbb{R}^{n \times n} \). If we let \( \mu_0 \) and \( \mu_1 \) to be the smallest numbers that satisfy (2) and (3), then we have \( \mu_1 \geq \mu_0 \) as can be seen from the relations \( \sum_i (UV^\top)^2_{ij} = \| U^\top e_j \|^2 \) and \( \sum_j (UV^\top)^2_{ij} = \| U^\top e_i \|^2 \). Therefore, the joint incoherence parameter \( \mu_1 \) is the dominant factor in these previous bounds. As will be discussed in Section 2.2, while the standard incoherence (2) is a natural condition, the joint incoherence condition (3) is restrictive and unintuitive. In several important settings, \( \mu_1 \) is as large as \( \mu_0^2 r \), so previous results require \( O(nr^2 \text{polylog}(n)) \) observations even if \( \mu_0 = O(1) \).

In the following main theorem of the paper, we show that the joint incoherence is not necessary. The theorem only requires the standard incoherence condition.

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\(^1\)This is known as the Bernoulli model \([9]\). Other widely used models include the sampling with/without replacement models \([20, 10, 21, 32]\). Recovery guarantees for one model can be easily translated to others with only a change in constant factors \([11, 21]\).
Theorem 1. Suppose $M$ satisfies the standard incoherence condition (2) with parameter $\mu_0$. There exist universal constants $c_0, c_1, c_2 > 0$ such that if

$$p \geq c_0 \mu_0 r \log^2 (n_1 + n_2) \min\{n_1, n_2\},$$

then $M$ is the unique optimal solution to (1) with probability at least $1 - c_1 (n_1 + n_2)^{-c_2}$.

We provide comments and discussion in the next two sub-sections.

2.1 Optimality of Theorem 1

Candès and Tao [11] prove the following lower-bound on the sample complexity of matrix completion.

Proposition 1 ([11], Theorem 1.7). Suppose $n_1 = n_2 = n$ and $\Omega$ is sampled as above. If we do not have the condition

$$p \geq \frac{1}{2} \mu_0 r \log(2n) \frac{n}{n},$$

and the RHS above is less than 1, then with probability at least $\frac{1}{4}$, there exist infinitely many pairs of distinct matrices $M' \neq M''$ of rank at most $r$ and obeying the standard incoherence condition (2) with parameter $\mu_0$ such that $M'_{ij} = M''_{ij}$ for all $(i, j) \in \Omega$.

This shows that that $p \gtrsim \mu_0 r \log(n)/n$ is necessary for any method to determine $M$ (even if one knows $r$ and $\mu_0$ ahead of time). With an additional $c' \log(n)$ factor, Theorem 1 matches this lower bound. In particular, it is optimal in terms of its scaling with the incoherence parameter $\mu_0$.

We note that the condition in Proposition 1 is an information/statistical lower-bound: when the value of $p$ is below this bound, there is not enough information in the observed entries to uniquely determine an rank-$r$, $\mu_0$-incoherent matrix even if one has infinite computational power. In Section 4, we show that in the closely related problem of matrix decomposition, the incoherence parameters are associated with both information and computational lower bounds.

2.2 Consequences and Comparison with Prior Work

The previous best result for exact matrix completion is given in [32, 20]. They show that $M \in \mathbb{R}^{n \times n}$ can be recovered by the nuclear minimization approach if the sampling probabilities satisfy

$$p \gtrsim \frac{\max\{\mu_0, \mu_1\} r \log^2 n}{n}.$$ 

Using an alternative algorithm, Keshavan et al. [20] show that recovery can be achieved with

$$p \gtrsim \frac{\max\{\mu_0 r \log n, \mu_1^2 r^2\}}{n}.$$ 

Similar results are given in [24], which also requires the sample complexity to be proportional to $\mu_1$ (or equivalently, quadratic in $r$). In light of Proposition 1, these results are not optimal with respect to the incoherence parameters due to the dependence on the joint incoherence $\mu_1$. Theorem 1 eliminates this extra dependence.

The improvement in Theorem 1 is significant both qualitatively and quantitatively. The standard incoherence condition (2) is natural and necessary. A small standard incoherence parameter $\mu_0$ ensures that the information of the row and column spaces of $M$ is not too concentrated on a small number of rows/columns. In contrast, the joint incoherence assumption (3), which requires the matrices $U$ and $V$ containing the left and right singular vectors to be “unaligned” with each other, does not have a natural explanation. In applications, the quantity $\mu_0$ often has clear physical meanings while $\mu_1$ does not. For example, in the application to recovering the affinity matrices between clustered objects from
Figure 1: The minimum observation probability $p$ for recovering a $900 \times 900$ rank-$r$ matrix with $\mu_0 = 1$ and $\mu_1 = r$. We use the IALM method in [29] to solve the nuclear minimization problem [1]. For each $r$ and $p$, we run the simulation for 20 trials. The $Y$-axis shows the smallest value of $p$ for which the normalized recovery error $\|\hat{M} - M\|_F / \|M\|_F$ is smaller than $10^{-4}$ in at least 19 trials.

partial observations [35, 37] (discussed in Section 3). $\mu_0$ is a function of the minimum cluster size, but a bound on $\mu_1$ bears no natural motivation. As another example, the work in [16] uses Hankel matrix completion to recover spectrally sparse signals obeying two types of conditions. The first condition can be traced to standard incoherence and is equivalent to (the natural requirement of) the supporting frequencies being spread out. On the other hand, the second set of conditions, which resemble joint incoherence, cannot be reduced to a property of only the frequencies. The manuscript [17], which appeared after this paper was posted online [13], removes these second set of conditions using similar techniques as in Theorem 1.

Quantitatively, the joint incoherence condition is much more restrictive than standard incoherence. By Cauchy-Schwarz inequality we always have $\mu_1 r \leq \mu_0 r^2$. The equality $\mu_1 r = \mu_0 r^2$ holds in the important setting where the matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite (psd) and thus has $U = V$. In this case, applying previous guarantees would require $p \gtrsim \frac{\mu_0 r^2 \log^2 n}{r}$. This translates to $\mu_0 r$ times more observations than guaranteed by Theorem 1. In particular, these previous bounds are quadratic in $r$, and thus they require $r = o(\sqrt{n})$ regardless of $p$. This is clearly unnecessary, since any matrix can be completed regardless of its rank given a sufficiently large $p$. We verify this fact by simulation. We construct $M$ as a $0 - 1$ block diagonal matrix with $r$ diagonal blocks of size $\frac{n}{r} \times \frac{n}{r}$. It is easy to see that $M$ is positive semidefinite with $\mu_0 = 1$ and $\mu_1 = r$. Figure 1 shows the minimum values of $p$ needed to recover $M$ for different $r$ in the simulation. It can be seen that $p$ indeed scales linearly in $r$ as predicted by Theorem 1. In particular, we recover PSD matrices with rank well over $\sqrt{n}$, which would not be possible if the joint incoherence condition were necessary.

3 Extensions

As we mention in the introduction, the proof of Theorem 1 crucially relies on the use of the matrix $\ell_{\infty, 2}$-norm. In this section, we present two extensions of this idea to the analysis of an SVD-projection algorithm, and to structured matrix completion and semi-supervised clustering.

3.1 Error Bound for SVD Projection

Our first example is the derivation of error bounds for an SVD-projection algorithm for matrix completion. Let $\mathcal{P}_M M$ be the matrix obtained from $M$ by setting all the unobserved entries to zero.
Given the partial observation $\mathcal{P}_\Omega M$, Keshavan et al. [26] propose the following two-step algorithm for approximating $M$. Step 1: Set to zero all columns and rows in $\mathcal{P}_\Omega M$ with degrees larger that $2pn$, where the degree of a column or row is the number of non-zero entries of this column/row. Let $\tilde{M}_\Omega$ be the output. Step 2: Compute the SVD of $\tilde{M}_\Omega$

$$\tilde{M}_\Omega = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^\top$$

and return the re-scaled rank-$r$ projection $T_r(\tilde{M}_\Omega) := \frac{1}{\sqrt{p}} \sum_{i=1}^{r} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_j^\top$. Theorem 1.1 in [26] provides the following bound on the approximation error

$$\|M - T_r(\tilde{M}_\Omega)\|_F \leq c \sqrt{\frac{rn}{p}} \|M\|_\infty, \text{ w.h.p.}, \quad (4)$$

where $\|M\|_\infty := \max_{i,j} |M_{ij}|$ is the matrix $\ell_\infty$ norm. This result is proved using a combination of tools from measure concentration and random graph theory.

As a simple corollary of our Lemma 2 in Section 5, we obtain a new error bound stated in terms of $\|M\|_\infty$ and $\|M\|_{\infty,2}$, where $\|M\|_{\infty,2}$ is the matrix $\ell_{\infty,2}$ norm of $M$, defined as

$$\|Z\|_{\infty,2} := \max_i \left\{ \max_b \left\| \sum_j Z^2_{ib} \right\|, \max_j \left\| \sum_a Z^2_{ai} \right\| \right\}; \quad (5)$$

that is, $\|M\|_{\infty,2}$ is the maximum of the row and column norms of $M$.

**Corollary 1.** Suppose $p \geq c_0 \log n$. With high probability, we have

$$\|M - T_r(\tilde{M}_\Omega)\|_F \leq c' \left( \frac{\sqrt{r} \log n}{p} \|M\|_\infty + \sqrt{\frac{r \log n}{p} \|M\|_{\infty,2}} \right).$$

We prove the corollary in Appendix C. The corollary improves upon (4) whenever $p \geq \log^2 n$ and $\|M\|_{\infty,2} < \sqrt{n \log n} \|M\|_\infty$. Note that for a general matrix $M$, $\|M\|_{\infty,2}$ is always no more than $\sqrt{n} \|M\|_\infty$, and can be much smaller. An example of such a matrix is an affinity matrix with a block-diagonal structure, which is discussed in the next sub-section. Here the improvement is again due to using the $\ell_{\infty,2}$ norm.

### 3.2 Structured Matrix Completion and Semi-Supervised Clustering

We next consider the extension to the **structured matrix completion** problem. In several applications of matrix completion including semi-supervised clustering [37] (which we shall discuss in more details) and multi-label learning [39], one has access to additional side information about the column/row spaces of the low-rank matrix $M$. In particular, one aims to complete an unknown rank-$r$ matrix $M = U\Sigma V^\top \in \mathbb{R}^{n \times n}$ from the partial observations $\mathcal{P}_\Omega M$, but is given the structural information that the column (row, resp.) space of $M$ lie in a known $\tilde{r}$-dimensional subspace of $\mathbb{R}^n$ spanned by the columns of $\tilde{U} \in \mathbb{R}^{n \times \tilde{r}} \ (\tilde{V} \in \mathbb{R}^{n \times \tilde{r}}, \text{resp.});$ here $\tilde{r}$ may be smaller than the ambient dimension $n$. In other words, we know $\text{col}(U) \subseteq \text{col}(\tilde{U})$ and $\text{col}(V) \subseteq \text{col}(\tilde{V})$, where $\text{col}(\cdot)$ denotes the column space. Without loss of generality, we may assume each of $\tilde{U}$ and $\tilde{V}$ has orthogonal columns with unit norms.

Given $\mathcal{P}_\Omega M$, $\tilde{U}$ and $\tilde{V}$, we formulate the following modified nuclear norm minimization problem:

$$\min_X \|X\|_* \quad \text{s.t.} \quad \mathcal{P}_\Omega(\tilde{U}XX^\top) = \mathcal{P}_\Omega M. \quad (6)$$

For this formulation we have the following guarantee.
Theorem 2. Suppose $U$ and $V$ satisfy the standard incoherence condition \( \mu \) with parameter $\mu_0$, and $\tilde{U}$ and $\tilde{V}$ satisfies $\tilde{\mu}$ with parameter $\tilde{\mu}_0$. For some universal constants $c_0, c_1$ and $c_2$, $X^* := U^\top M \tilde{V}$ is the unique optimal solution to the program (6) with probability at least $1 - c_1 n^{-c_2}$ provided

$$p \geq c_0 \frac{\max\{\mu_0, \tilde{\mu}_0\} \max\{\mu_0, \tilde{\mu}_0, \mu_1\} \bar{r} \bar{r} \log \bar{r} \log n}{n^2}.$$ 

Given $X^*$, we can recover $M$ by $M = \tilde{U} X^* \tilde{V}^\top$ since $\tilde{U} \tilde{U}^\top U = U$ and $\tilde{V} \tilde{V}^\top \tilde{V} = \tilde{V}$. We prove this theorem in Appendix D.

Theorem 2 shows that with the knowledge of the $\bar{r}$-dimensional subspaces $\text{col}(\tilde{U})$ and $\text{col}(\tilde{V})$, the number of observations needed to complete $M$ is on the order of $pn^2 \approx \mu_0 \bar{r} \log(\bar{\mu}_0 \bar{r}) \log(\bar{\mu}_0 \bar{r}) \log n$, which is $\Theta(\bar{r} \bar{r} \log \bar{r} \log n)$ for constant $\mu_0$ and $\bar{\mu}_0$. If $\bar{r} \ll n$, meaning that we have strong structural information, then this number is much smaller than the usual requirement $\Theta(n \log^2 n)$. On the other hand, setting $\bar{r} = n$ recovers Theorem 4 for standard matrix completion where there is no additional structural information. We note that we assume $M$ is a square matrix here for simplicity; the results can be trivially extended to general rectangular matrices.

Near the completion of the writing of this paper, an independent study \([30]\) on structured matrix completion was made available. There they require among other things the following condition:\( \text{ii} \)

$$p \geq c_{10} \frac{\max\{\mu_0, \tilde{\mu}_0\} \max\{\mu_0, \tilde{\mu}_0, \mu_1\} \bar{r} \bar{r} \log \bar{r} \log n}{n^2}, \tag{7}$$

where $\mu_1$ is the joint incoherence parameter of $U$ and $V$ defined in \([3]\). Theorem 2 is better than the result in (7) in two ways. First, Theorem 2 avoids the superfluous dependence on the joint incoherence parameter $\mu_1$, which can be as large as $\mu_0^2 \bar{r}$ as previously discussed. Second, even in the ideal setting with $\mu_1 = \mu_0$, the bound in (7) requires $p$ to scale with $\max\{\mu_0, \tilde{\mu}_0\}$, whereas the bound in Theorem 2 scales with $\mu_0 \tilde{\mu}_0$, which is strictly smaller whenever $\mu_0 \neq \tilde{\mu}_0$. We note that \([30]\) discusses a nice application of structured matrix completion to the multi-label learning problem.

### 3.2.1 Applications to semi-supervised clustering

Another interesting application of structured matrix completion is presented in \([37]\). There they consider the semi-supervised clustering problem, where the goal is to partition a set of $n$ objects into $r$ clusters given the objects’ feature vectors $w_i \in \mathbb{R}^d, i = 1, \ldots, n$ and some must-link and cannot-link constraints $M$ and $\mathcal{C}$. In particular, $(i, j) \in M$ means objects $i$ and $j$ must be assigned to the same cluster, and $(i, j) \in \mathcal{C}$ means they cannot. Let $M \in \{0, 1\}^{n \times n}$ be the true affinity matrix, with $M_{ij} = 1$ if and only if objects $i$ and $j$ are in the same cluster. Note that $M$ has a block-diagonal structure, with the number of blocks equal to the number of clusters $r$ and thus $\text{rank}(M) = r$. Suppose the rank-$r$ SVD of $M$ is $M = U \Sigma U^\top$. The authors of \([37]\) make the important observation that in practice, the columns of $U$ often (approximately) lie in the space spanned by first $\bar{r}$ singular vectors $\tilde{U} \in \mathbb{R}^{n \times \bar{r}}$ of the input features $W = [w_1, w_2, \ldots, w_n] \in \mathbb{R}^{n \times d}$ for some $\bar{r} \ll n$. In this case, one can use the extra information from the features $W$ to improve clustering performance. The task of recovering the affinity matrix $M$ given $\tilde{U}$ and the must-link/cannot-link constraints $\Omega := M \cup \mathcal{C}$ is precisely a structured matrix completion problem.

Specifically, \([37]\) considers the setup where the set of observed entries $\Omega$ are distributed according to the Bernoulli model with probability $p_\Omega$ the smallest cluster size is $n_{\text{min}}$, and $\tilde{U}$ has standard incoherence parameter $\bar{\mu}$ as defined in \([3]\). Note that the standard incoherence parameter of $U$ is $n/(rn_{\text{min}})$ due to the block diagonal structure of the affinity matrix $M$. Using previous techniques

\( ^3 \)In \([30]\) they consider the sampling without replacement model for the observed entries. Their results can be translated to the Bernoulli model considered in this paper, as we have done here. See also footnote 4.

\( ^4 \)To be precise, the diagonal entries $M_{ii} = 1$ are known; clearly having more observations cannot decrease the probability that the program (9) outputs the correct solution. Moreover, since the affinity matrix satisfies $M_{ij} = M_{ji}$, each observation is a pair of entries of $M$. This technicality can be easily handled, and we omit the details here.
in matrix completion, it is shown in [37] that \( X^* := U^T M \bar{U} \) is the unique optimal solution to the program [6] w.h.p. provided

\[
p \gtrsim \frac{\mu_0 \bar{r} \log^2 n}{n_{\min}^2}.
\]

(8)

Note the quadratic term \( n_{\min}^2 \) on the RHS, which is due to the joint incoherence parameter of \( U \) taking the value \( n_{\min}^2 / (r n_{\min}^2) \). Suppose \( \bar{r} = n \); a consequence of (7) is that, even if \( M \) is fully observed \((p = 1)\), the cluster size must be at least \( n_{\min} = \Theta(\sqrt{n}) \) and thus the possible number of clusters \( r \) cannot exceed \( n / n_{\min} = \Theta(\sqrt{n}) \). These restrictions are undesirable, and clearly unnecessary when \( p = 1 \).

Using Theorem 2, we can eliminate these \( \sqrt{n} \) restrictions and significantly reduce the sample complexity. Plugging \( \mu_0 = n / (r n_{\min}) \) into the theorem, we obtain that the program (6) succeeds with high probability provided

\[
p \gtrsim \frac{\mu_0 \bar{r} \log(\mu_0 \bar{r}) \log n}{n_{\min}}.
\]

The last RHS is order-wise smaller than the RHS of the previous bound (8) by a multiplicative factor of \( \frac{n_{\min} \log(\mu_0 \bar{r})}{\log n} \). In particular, when \( \bar{r} = n \) and ignoring logarithm factors, we allow the size of the clusters to be as small as \( n_{\min} = \Theta(1) \) and the number of clusters be as large as \( r = \Theta(n) \). These significantly improve over the results in [37] which require \( n_{\min} = \Omega(\sqrt{n}) \) and \( r = O(\sqrt{n}) \). Moreover, if \( n_{\min} = \sqrt{n} \), then our result require \( n / n_{\min} = \sqrt{n} \) times fewer observations than the previous bound [8].

4 Incoherence in Matrix Decomposition: Information and Computational Lower Bounds

Having shown that the joint incoherence is not needed in matrix completion, we now turn to a closely related problem, namely low-rank and sparse matrix decomposition [12, 9]. In contrast to matrix completion, we show that the joint incoherence condition is unavoidable in matrix decomposition, at least if one asks for polynomial-time algorithms.

Suppose \( L^* \in \mathbb{R}^{n \times n} \) is a symmetric rank-\( r \) matrix obeying the standard and joint incoherence conditions [2] and [3] with parameters \( \mu_0 \) and \( \mu_1 \), respectively, and \( S^* \in \mathbb{R}^{n \times n} \) is a symmetric matrix where each pair of entries \( S^*_{ij} = S^*_{ji} \) is non-zero with probability \( \tau \), independent of all others. The matrix decomposition problem concerns with recovering \((L^*, S^*)\) given the sum \( A = L^* + S^* \). A now standard approach is to solve the following convex program [12, 9]:

\[
\min_{L, S} ||L||_\infty + \lambda ||S||_1
\]

s.t. \( L + S = A \),

(9)

where \( ||S||_1 := \sum_{i,j} |S_{ij}| \) is the matrix \( \ell_1 \) norm. Under the above setting, it has been shown in [9, 28, 15] that \( (L^*, S^*) \) is the unique optimal solution to (9) for a suitable \( \lambda \) with probability at least \( 1 - n^{-10} \) provided that \( \tau < c_0 \) for any constant \( c_0 < \frac{1}{2} \) and

\[
c_1 \max \{ \mu_0, \mu_1 \} r \log^2 n \leq 1
\]

(10)

for some constant \( c_1 \) that might depend on \( c_0 \); cf. Theorems 1 and 2 in [15, 8]. Note the dependence on \( \mu_1 \) above. Consequently, when \( L^* \) is positive semi-definite with \( \mu_1 = \mu_0^2 \bar{r} \), the condition (10) requires \( r = o(\sqrt{n}) \). Unlike the matrix completion setting which does not have a natural motivation for the \( \ell_\infty \)-type requirement in the joint incoherence condition [3], the \( \ell_\infty \) norm arises naturally in the matrix decomposition problem as it is the dual norm of the \( \ell_1 \)-norm in the formulation [9].

In [9, 28, 15], the sufficiency of (10) is proved for non-symmetric matrices, but it is straightforward to show that the same holds for the symmetric case considered here.
In fact, we show that the joint incoherence condition is not specific to the formulation (9), but is in fact required by all polynomial-time algorithms under a widely-believed computational complexity assumption. We prove this by connecting the matrix decomposition problem to the Planted Clique problem, defined as follows. A graph on \( n \) nodes is generated by connecting each pairs of nodes independently with probability \( \frac{1}{2} \), and then randomly picking a subset of \( n_{\min} \) nodes and making them fully connected (hence a clique). The goal is to find the planted clique given the graph. The Planted Clique problem has been extensively studied; cf. [3,2] for an overview of the known results. In the regime of \( n_{\min} = o(\sqrt{n}) \), there is no known polynomial-time algorithm for this problem despite years of effort. In fact, this regime is widely believed to be intractable in polynomial time. The average case hardness of this regime has been proved under certain computational models [33,19], and has been utilized in cryptography [4,25] and other applications [2,22,27]. The work [5] is the first to use this hardness assumption to obtain bounds on statistical accuracy of sparse PCA given computational constraints, and a similar approach is taken in [30] for submatrix detection. We therefore adopt the following computational assumption on the Planted Clique problem, where we recall that a size \( n_{\min} \) clique is planted in an Erdos-Renyi random graph \( G(n, \frac{1}{2}) \) with \( n \) nodes and edge probability \( \frac{1}{2} \).

For any constant \( \epsilon > 0 \), there is no algorithm with running time polynomial in \( n \) that, for all \( n \) and with probability at least \( \frac{1}{2} \), finds the planted clique with size \( n_{\min} \leq n^{\frac{1}{2} - \epsilon} \) given the random graph.

This version of the assumption is similar to Conjecture 4.3 in [2].

The following theorem provides necessary conditions for the success of matrix decomposition algorithms. The proof is given in Appendix E.

**Theorem 3.** The following two statements are true for the matrix decomposition problem with \( \tau = 1/3 \).

1. Suppose \( r = 1 \) and the assumption \( A1 \) holds. For any constant \( \epsilon' > 0 \), there is no algorithm with running time polynomial in \( n \) that, for all \( n \) and with probability at least \( \frac{1}{2} \), solves the matrix decomposition problem with
   \[
   \frac{\mu_1^{1-\epsilon'}}{n} \geq 1.
   \]

2. Suppose \( \mu_0 \geq 2 \). There is no algorithm that, for all \( n \) and with probability at least \( \frac{1}{2} \), solves the matrix decomposition problem with
   \[
   \frac{1}{12} \cdot \frac{\mu_0 r \log n}{n} \geq 1.
   \]

If we modify the assumption \( A1 \) by assuming that the Planted \( r \)-Clique problem [31] with \( r \) disjoint planted cliques of size \( o(\sqrt{n}) \) is intractable in polynomial time, then the first part of the theorem holds with
\[
\frac{\mu_1^{1-\epsilon'} r}{n} \geq 1.
\]
Together with the second part of the theorem, this result shows that, under the planted clique assumption, the standard and joint incoherence conditions are both necessary for solving matrix decomposition in polynomial time. Therefore, the bound in (10) is unlikely to be improvable (up to a polylog factor) using polynomial-time algorithms. In particular, this implies that the matrix decomposition problem is intractable in general for positive semidefinite matrices with rank \( r = \omega(\sqrt{n}) \) since in this case \( \mu_1 r = \mu_0 r^2 \geq r^2 \).

We note that the first part of Theorem 3 is a computational limit. It is proved by showing that if there is a matrix decomposition algorithm that does not require the joint incoherence condition,
then the algorithm would solve the computationally hard problem of finding a planted clique with size \( n_{\text{min}} = o(\sqrt{n}) \). On the other hand, the second part of the theorem is an information/statistical limit applicable to all algorithms regardless of their computational complexity, and is proved by an information-theoretic argument. Interestingly, Theorem 3 shows that the standard incoherence and the joint incoherence are associated with the statistical and computational aspects of the matrix decomposition problem, respectively.

5 Proof of Theorem 1

We prove the our main result Theorem 1 in this section. While Theorem 1 can be derived from the more general Theorem 2, we choose to provide a separate proof of Theorem 1 in order to highlight the main innovation (the use of the \( \ell_{\infty,2} \) norm) of the analysis. The general setting of Theorem 2 requires several additional technical steps.

The high level roadmap of the proof is a standard one: by convex analysis, to show that \( M \) is the unique optimal solution to the program (1), it suffices to construct a dual certificate \( Y \) obeying several subgradient-type conditions. One of the conditions requires the spectral norm \( \|Y\| \) to be small. Previous work bounds \( \|Y\| \) by the \( \ell_{\infty} \) norm \( \|Z\|_{\infty} := \max_{i,j} |Z_{ij}| \) of a certain matrix \( Z \), which ultimately links to \( \|UV^T\|_{\infty} \) and thus leads to the joint incoherence condition in [9]. Here we derive a new bound using the \( \ell_{\infty,2} \) norm \( \|Z\|_{\infty,2} \) as defined in [9]. Note that \( \|Z\|_{\infty,2} \) is always no greater than \( \sqrt{\max\{n_1,n_2\}} \|Z\|_{\infty} \) for any \( Z \in \mathbb{R}^{n_1 \times n_2} \). In our setting, there is a significant gap between \( \|UV^T\|_{\infty,2} \leq \frac{\sqrt{\mu_0}}{\min\{n_1,n_2\}} \) and \( \sqrt{\max\{n_1,n_2\}} \|UV^T\|_{\infty} \leq \frac{\mu r}{\sqrt{\min\{n_1,n_2\}}} \). This leads to a tighter bound of \( \|Y\| \) and hence less restrictive incoherence conditions.

We now turn to the details. To simplify notion, we prove the results for square matrices \( (n_1 = n_2 = n) \); the results for non-square matrices are proven in exactly the same fashion. Some additional notation is needed. We use \( c \) and its derivatives \( (c', c'', \text{ etc.}) \) for universal positive constants. By \textit{with high probability} (w.h.p.) we mean with probability at least \( 1 - c_1 n^{-c_2} \) for some constants \( c_1, c_2 > 0 \) independent of the problem parameters \((n, r, p, \mu_0, \mu_1)\). Throughout the proof the constant \( c_2 \) can be made arbitrarily large by choosing the constant \( c_0 \) in Theorem 1 sufficiently large. The proof below involves \( 80 \log n + 1 \) random events, each of which is shown to occur with high probability. By the union bound their intersection also occurs with high probability.

A few additional notations are needed. The inner product between two matrices is given by \( \langle X, Z \rangle := \text{trace}(X^T Z) \). The projections \( P_T \) and \( P_{T^\perp} \) are given by

\[
P_T(Z) := UU^T Z + ZVV^T - UU^T ZV V^T.
\]

and \( P_{T^\perp}(Z) := Z - P_T(Z) \). \( P_{\Omega}(Z) \) denotes the matrix given by \((P_{\Omega}(Z))_{ij} = Z_{ij} \) if \((i, j) \in \Omega \) and zero otherwise. We use \( I \) to denote the identity matrix for matrices. For \( 1 \leq i, j \leq n \), we define the random variable \( \gamma_{ij} := \mathbb{I}(i, j) \in \Omega \), where \( \mathbb{I}(\cdot) \) is the indicator function. The projection \( R_{\Omega} \) is given by

\[
R_{\Omega} := \frac{1}{p} P_{\Omega} Z = \sum_{i,j} \frac{1}{p} \gamma_{ij} Z_{ij} e_i e_j^T.
\]

As usual, \( \|z\|_2 \) is the \( \ell_2 \) norm of the vector \( z \), and \( \|Z\|_{\text{F}} \) and \( \|Z\| \) are the Frobenius norm and spectral norm of the matrix \( Z \), respectively. For an operator \( A \) on matrices, its operator norm is defined as \( \|A\|_{\text{op}} := \sup_{Z \in \mathbb{R}^{n \times n}} \|A(Z)\|_{\text{F}} / \|Z\|_{\text{F}} \).

Subgradient Optimality Condition Following our proof roadmap, we now state a sufficient condition for \( M \) to be the unique optimal solution to the optimization problem (1).

**Proposition 2.** Suppose \( p \geq \frac{1}{n} \). The matrix \( M \) is the unique optimal solution to (1) if the following conditions hold:
1. $\|P_T R_\Omega P_T - P_T\|_{op} \leq \frac{1}{2}$.

2. There exists a dual certificate $Y \in \mathbb{R}^{n \times n}$ which satisfies $P_\Omega (Y) = Y$ and
   
   (a) $\|P_{T^\perp} (Y)\| \leq \frac{1}{2},$
   
   (b) $\|P_T (Y) - UV^\top\|_F \leq \frac{1}{4n}.$

A somewhat different version of the proposition appears in [32, 20]. We prove the proposition in Appendix A.

Approximate Isometry The requirement $p \geq \frac{1}{n}$ in Proposition 2 clearly holds under the conditions of Theorem 1. The following standard result shows that the approximate isometry in Condition 1 is also satisfied.

Lemma 1 (Theorem 4.1 in [10]; Lemma 11 in [15]). If $p \geq c_0 \frac{\mu \log n}{n}$ for some $c_0$ sufficiently large, then w.h.p.

$$\|P_T R_\Omega P_T - P_T\| \leq \frac{1}{2}.$$ 

Constructing the Dual Certificate We now construct a dual certificate $Y$ that satisfies Condition 2 in Proposition 2. We do this using the Golfing Scheme [20, 9]. Set $k_0 := 20 \log n$. Assume for now the set $\Omega$ of observed entries is generated from $\Omega = \bigcup_{k=1}^{k_0} \Omega_k$, where for each $k$ and matrix index $(i, j)$, $P[(i, j) \in \Omega_k] = q := 1 - (1 - p)^{1/k_0}$ and is independent of all others. Clearly this $\Omega$ has the same distribution as the original model. Let $W_0 := 0$ and for $k = 1, \ldots, k_0$, define

$$W_k := W_{k-1} + R_{\Omega_k} P_T (UV^\top - P_T W_{k-1}),$$

where the operator $R_{\Omega_k}$ is defined analogously to $R_\Omega$ as $R_{\Omega_k}(Z) := \sum_{i,j} \frac{1}{q} 1_{((i, j) \in \Omega_k)} Z_{ij} e_i e_j^\top$. The dual certificate is given by $Y := W_{k_0}$. We have $P_\Omega (Y) = Y$ by construction. The proof of Theorem 1 is completed if we show that $Y$ satisfies Conditions 2(a) and 2(b) in Proposition 2 w.h.p.

Lemmas on Matrix Norms The key step of our proof is to show that $Y$ satisfies Condition 2(a) in Proposition 2 i.e., we need to bound $\|P_{T^\perp} (Y)\|$. Here our proof departs from existing work – we establish bounds on this quantity in terms of the $\ell_\infty$ norm. This is done with the help of two lemmas. The first one bounds the spectral norm of $(R_\Omega - I) Z$ in terms of the $\ell_\infty$ and $\ell_2$ norms of $Z$. This gives tighter bounds than previous approaches [11, 21, 32, 26] that use solely the $\ell_\infty$ norm of $Z$.

Lemma 2. Suppose $Z$ is a fixed $n \times n$ matrix. For a universal constant $c > 1$, we have w.h.p.

$$\| (R_\Omega - I) Z \| \leq c \left( \frac{\log n}{p} \|Z\|_\infty + \sqrt{\frac{\log n}{p}} \|Z\|_{\infty, 2} \right).$$

The second lemma further controls the $\ell_{\infty, 2}$ norm.

Lemma 3. Suppose $Z$ is a fixed matrix. If $p \geq c_0 \frac{\mu \log n}{n}$ for some $c_0$ sufficiently large, then w.h.p.

$$\| (P_T R_\Omega - P_T) Z \|_{\infty, 2} \leq \frac{1}{2} \left( \frac{n}{\mu_0 r} \|Z\|_\infty + \frac{1}{2} \|Z\|_{\infty, 2} \right).$$

We prove Lemmas 2 and 3 in Appendix B. We also need a standard result that controls the $\ell_\infty$ norm.

Lemma 4 (Lemma 3.1 in [9]; Lemma 13 in [15]). Suppose $Z$ is a fixed $n \times n$ matrix in $T$. If $p \geq c_0 \frac{\mu \log n}{n}$ for some $c_0$ sufficiently large, then w.h.p.

$$\| (P_T R_\Omega P_T - P_T) Z \|_\infty \leq \frac{1}{2} \|Z\|_\infty.$$

Equipped with the lemmas above, we are ready to validate Condition 2 in Proposition 2. 

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Validating Condition 2(b) Set $D_k := UV^\top - P_T(W_k)$ for $k = 0, \ldots, k_0$. By definition of $W_k$, we have $D_0 = UV^\top$ and

$$D_k = (P_T - P_T R_{\Omega_k} P_T) D_{k-1}. \tag{13}$$

Note that $\Omega_k$ is independent of $D_{k-1}$ and $q \geq p/k_0 \geq c_0 \mu q \log(n)/n$ under the conditions in Theorem 1. Applying Lemma 3 with $\Omega$ replaced by $\Omega_k$, we obtain that w.h.p.

$$\|D_k\|_F \leq \|P_T - P_T R_{\Omega_k} P_T\| \|D_{k-1}\|_F \leq \frac{1}{2} \|D_{k-1}\|_F$$

for each $k$. Applying the above inequality recursively with $k = k_0, k_0 - 1, \ldots, 1$ gives

$$\|P_T(Y) - UV^\top\|_F = \|D_{k_0}\|_F \leq \left(\frac{1}{2}\right)^{k_0} \|UV^\top\|_F \leq \frac{1}{4n^2} \cdot \sqrt{r} \leq \frac{1}{4n}.$$ 

Validating Condition 2(a) Note that $Y = \sum_{k=1}^{k_0} R_{\Omega_k} P_T (D_{k-1})$ by construction. We therefore have

$$\|P_{T^\perp}(Y)\| \leq \sum_{k=1}^{k_0} \|P_{T^\perp} (R_{\Omega_k} P_T - P_T) (D_{k-1})\| \leq \sum_{k=1}^{k_0} \|(R_{\Omega_k} - I) P_T (D_{k-1})\|.$$ 

Applying Lemma 2 with $\Omega$ replaced by $\Omega_k$ to each summand of the last R.H.S., we get that w.h.p.

$$\|P_{T^\perp}(Y)\| \leq c \sum_{k=1}^{k_0} \left(\frac{\log n}{q} \|D_{k-1}\|_\infty + \sqrt{\frac{\log n}{q} \|D_{k-1}\|_{\infty, 2}}\right) \leq \frac{c}{\sqrt{c_0}} \sum_{k=1}^{k_0} \left(\frac{n}{\mu_0} \|D_{k-1}\|_\infty + \sqrt{\frac{n}{\mu_0} \|D_{k-1}\|_{\infty, 2}}\right), \tag{14}$$

where the last inequality follows from $q \geq c_0 \mu q \log(n)/n$. We proceed by bounding $\|D_{k-1}\|_\infty$ and $\|D_{k-1}\|_{\infty, 2}$. Using (13), and repeatedly applying Lemma 4 with $\Omega$ replaced by $\Omega_k$, we obtain that w.h.p.

$$\|D_{k-1}\|_\infty = \|(P_T - P_T R_{\Omega_k-1} P_T) \cdots (P_T - P_T R_{\Omega_1} P_T) D_0\|_\infty \leq \left(\frac{1}{2}\right)^{k-1} \|UV^\top\|_\infty.$$ 

By Lemma 3 with $\Omega$ replaced by $\Omega_k$, we obtain that w.h.p.

$$\|D_{k-1}\|_{\infty, 2} = \|(P_T - P_T R_{\Omega_k-1} P_T) D_{k-2}\|_{\infty, 2} \leq \frac{1}{2} \sqrt{\frac{n}{\mu^p}} \|D_{k-2}\|_\infty + \frac{1}{2} \|D_{k-2}\|_{\infty, 2}.$$ 

Using (13) and combining the last two display equations gives w.h.p.

$$\|D_{k-1}\|_{\infty, 2} \leq k \left(\frac{1}{2}\right)^{k-1} \sqrt{\frac{n}{\mu^p}} \|UV^\top\|_\infty + \left(\frac{1}{2}\right)^{k-1} \|UV^\top\|_{\infty, 2}.$$ 

Substituting back to (14), we get w.h.p.

$$\|P_{T^\perp}(Y)\| \leq \frac{c}{\sqrt{c_0}} \frac{n}{\mu_0} \|UV^\top\|_\infty \sum_{k=1}^{k_0} (k+1) \left(\frac{1}{2}\right)^{k-1} + \frac{c}{\sqrt{c_0}} \sqrt{\frac{n}{\mu_0}} \|UV^\top\|_{\infty, 2} \sum_{k=1}^{k_0} \left(\frac{1}{2}\right)^{k-1} \leq \frac{6c}{\sqrt{c_0}} \frac{n}{\mu_0} \|UV^\top\|_\infty + 2c \sqrt{\frac{n}{\mu_0}} \|UV^\top\|_{\infty, 2}.$$
But the standard incoherence condition (2) implies that
\[
\|UV\|_\infty \leq \max_{i,j} \|U^T e_i\|_2 \|V^T e_j\|_2 \leq \frac{\mu_0 r}{n},
\]
\[
\|UV\|_{\infty,2} \leq \max \left\{ \max_i \|e_i^T UV\|_2^2, \max_j \|UV^T e_j\|_2^2 \right\} \leq \sqrt{\frac{\mu_0 r}{n}}.
\]
It follows that w.h.p.
\[
\|P_T (Y)\| \leq \frac{6c}{\sqrt{c_0}} + \frac{2c}{\sqrt{c_0}} \leq \frac{1}{2}
\]
provided \(c_0\) is sufficiently large. This completes the proof of Theorem 1.

6 Discussion

In this paper, we consider exact matrix completion and show that the joint incoherence condition imposed by all previous work is in fact not necessary. We discuss two extensions of this result, namely in bounding the approximation errors of SVD projection, and in structured matrix completion and semi-supervised clustering. We then show that the joint incoherence condition is unavoidable in the apparently similar problem of low-rank and sparse matrix decomposition based on the computational hardness assumption of the Planted Clique problem.

The improvements in the matrix completion problem are achieved via the use of \(\ell_{\infty,2}\)-type bounds. The \(\ell_{\infty,2}\) norm seems to be natural in the context of low-rank matrices as it captures the relative importance of the rows and columns. It is interesting to see if the techniques in this paper are relevant more generally.

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Appendices

A Proof of Proposition 2

Consider any feasible solution \(X\) to (1) with \(P_\Omega (X) = P_\Omega (M)\). Let \(G\) be an \(n \times n\) matrix which satisfies \(\|P_T G\| = 1\) and \(\langle P_T G, P_T (X - M) \rangle = \|P_T (X - M)\|_s\). Such \(G\) always exists by duality between the nuclear norm and the spectral norm. Because \(UV^T + P_T G\) is a sub-gradient of \(\|Z\|_*\) at \(Z = M\), we get
\[
\|X\|_* - \|M\|_* \geq \langle UV^T + P_T G, X - M \rangle.
\]
We also have \(\langle Y, X - M \rangle = \langle P_\Omega (Y), P_\Omega (X - M) \rangle = 0\) since \(P_\Omega (Y) = Y\). It follows that
\[
\|X\|_* - \|M\|_* \geq \langle UV^T + P_T G - Y, X - M \rangle = \|P_T (X - M)\|_s + \langle UV^T - P_T Y, X - M \rangle - \langle P_T Y, X - M \rangle \geq \|P_T (X - M)\|_s - \|UV^T - P_T Y\|_F \|P_T (X - M)\|_F - \|P_T Y\| \|P_T (X - M)\|_* \geq \frac{1}{2} \|P_T (X - M)\|_* - \frac{1}{4n^3} \|P_T (X - M)\|_F .
\]
Lemma 5. If \( \|X\|_* - \|M\|_* \geq \frac{1}{2} \|P_{T^\perp}(X - M)\|_* - \frac{1}{4n^2} \cdot \sqrt{2n^5} \|P_{T^\perp}(X - M)\|_* > \frac{1}{8} \|P_{T^\perp}(X - M)\|_* \).

The RHS is strictly positive for all \( X \) with \( P_{\Omega}(X - M) = 0 \) and \( X \neq M \). Otherwise we must have \( P_T(X - M) = X - M \) and \( P_T P_{\Omega} P_T (X - M) = 0 \), contradicting the assumption \( \|P_T P_{\Omega} P_T - P_T\|_{op} \leq \frac{1}{2} \). This proves that \( M \) is the unique optimum.

**Proof.** Observe that

\[
\|\sqrt{\beta \mathcal{R}_\Omega P_T(Z)}\|_F = \sqrt{\langle (P_T \mathcal{R}_\Omega P_T - P_T) Z, P_T(Z) \rangle + \langle P_T(Z), P_T(Z) \rangle} \\
\geq \sqrt{\|P_T(Z)\|_F^2 - \|P_T \mathcal{R}_\Omega P_T - P_T\| \|P_T(Z)\|_F^2} \geq \frac{1}{\sqrt{\beta}} \|P_T(Z)\|_F,
\]

where the last inequality follows from the assumption \( \|P_T \mathcal{R}_\Omega P_T - P_T\|_{op} \leq \frac{1}{2} \). On the other hand, \( P_{\Omega}(Z) = 0 \) implies \( \mathcal{R}_\Omega(Z) = 0 \) and thus

\[
\|\sqrt{\beta \mathcal{R}_\Omega P_T(Z)}\|_F = \|\sqrt{\beta \mathcal{R}_\Omega P_{T^\perp}(Z)}\|_F \leq \frac{1}{\sqrt{\beta}} \|P_{T^\perp}(Z)\|_F \leq n^5 \|P_{T^\perp}(Z)\|_F.
\]

Combining the last two display equations gives

\[
\|P_T(Z)\|_F \leq \sqrt{2n^5} \|P_{T^\perp}(Z)\|_F \leq \sqrt{2n^5} \|P_{T^\perp}(Z)\|_*.
\]

\( \square \)

### B  Proofs of Technical Lemmas in Section 5

We prove the technical lemmas that are used in the proof of Theorem 1. The proofs use the matrix Bernstein inequality, restated below.

**Theorem 4** (54). Let \( X_1, \ldots, X_N \in \mathbb{R}^n \) be independent zero mean random matrices. Suppose

\[
\max \left\{ \left\| \mathbb{E} \sum_{k=1}^N X_k X_k^\top \right\|, \left\| \mathbb{E} \sum_{k=1}^N X_k \right\| \right\} \leq \sigma^2
\]

and \( \|X_k\| \leq B \) almost surely for all \( k \). Then for any \( c > 1 \), we have

\[
\left\| \sum_{k=1}^N X_k \right\| \leq \sqrt{4c\sigma^2 \log(2n) + cB \log(2n)}.
\]

with probability at least \( 1 - (2n)^{-(c-1)} \).

We also make use of the following facts: for all \( i \) and \( j \), we have

\[
\|P_T(e_i e_j^\top)\|_F^2 \leq \frac{2 \mu q r}{n}.
\]

This follows from the definition of \( P_T \) and the standard incoherence condition (2).
B.1 Proof of Lemma 2
We may write
\[(R_\Omega - I) Z = \sum_{i,j} S_{(ij)} := \sum_{i,j} \left( \frac{1}{p} \gamma_{ij} - 1 \right) Z_{ij} e_i e_j^\top,\]
where \(\{S_{(ij)}\}\) are independent matrices satisfying \(\mathbb{E}[S_{(ij)}] = 0\) and \(\|S_{(ij)}\|_2 \leq \frac{1}{p} \|Z\|_\infty\). Moreover, we have
\[\mathbb{E} \sum_{i,j} S_{(ij)}^\top S_{(ij)} = \sum_{i,j} Z_{ij}^2 e_i e_j^\top e_i e_j^\top \mathbb{E} \left( \frac{1}{p} \gamma_{ij} - 1 \right)^2 = \sum_{i,j} \frac{1-p}{p} Z_{ij}^2 e_i e_j^\top\]
and thus
\[\left\| \mathbb{E} \sum_{i,j} S_{(ij)}^\top S_{(ij)} \right\| \leq \frac{1}{p} \max_j \left[ \sum_{i=1}^n Z_{ij}^2 \right] \leq \frac{1}{p} \|Z\|_{2,\infty}^2.\]
In a similar way we can bound \(\left\| \mathbb{E} \sum_{i,j} S_{(ij)} S_{(ij)}^\top \right\|\) by the same quantity. Applying the matrix Bernstein inequality in Theorem 4 proves the lemma.

B.2 Proof of Lemma 3
Fix \(b \in [n]\). The \(b\)-th column of the matrix \((P_T R_\Omega - P_T) Z\) can be written as
\[((P_T R_\Omega - P_T) Z) e_b = \sum_{i,j} s_{(ij)} := \sum_{i,j} \left( \frac{1}{p} \gamma_{ij} - 1 \right) Z_{ij} P_T (e_i e_j^\top) e_b,\]
where \(\{s_{(ij)}\}\) are independent column vectors in \(\mathbb{R}^n\). Note that \(\mathbb{E} [s_{(ij)}] = 0\) and
\[\|s_{(ij)}\|_2 \leq \frac{1}{p} \sqrt{\frac{\mu_0 r}{n}} \|Z\|_\infty \leq \frac{1}{c_0 \log n} \sqrt{\frac{\mu_0 r}{n}} \|Z\|_\infty,\]
where the last inequality follows from the assumption of \(p\) in the statement of the lemma. We also have
\[\left\| \mathbb{E} \sum_{i,j} s_{(ij)}^\top s_{(ij)} \right\| = \left\| \sum_{i,j} \mathbb{E} \left( \left( \frac{1}{p} \gamma_{ij} - 1 \right) \right) Z_{ij}^2 \|P_T (e_i e_j^\top) e_b\|_2 \right\|^2 = \frac{1-p}{p} \sum_{i,j} Z_{ij}^2 \|P_T (e_i e_j^\top) e_b\|_2^2.\]
Observe that
\[\|P_T (e_i e_j^\top) e_b\|_2 = \|UU^\top e_i e_j^\top e_b + (I -UU^\top) e_i e_j^\top VV^\top e_b\|_2 \leq \sqrt{\frac{\mu_0 r}{n}} |e_j^\top e_b| + |e_j^\top VV^\top e_b|\]
using the incoherence condition (2). It follows that
\[\left\| \mathbb{E} \sum_{i,j} s_{(ij)}^\top s_{(ij)} \right\| \leq \frac{2}{p} \sum_{i,j} Z_{ij}^2 \frac{\mu_0 r}{n} |e_j^\top e_b|^2 + \frac{2}{p} \sum_{i,j} Z_{ij}^2 |e_j^\top VV^\top e_b|^2\]
\[= \frac{2\mu_0 r}{pn} \sum_i Z_{ib}^2 + \frac{2}{p} \sum_j |e_j^\top VV^\top e_b|^2 \sum_i Z_{ij}^2 \leq \frac{2\mu_0 r}{p} \|Z\|_{2,\infty}^2 + \frac{2}{p} \left\|VV^\top e_b\right\|^2 \|Z\|_{2,\infty}^2 \leq \frac{4\mu_0 r}{pn} \|Z\|_{2,\infty}^2 \leq \frac{4}{c_0 \log n} \|Z\|_{2,\infty}^2.\]
We can bound \(\|E\left[\sum_{i,j} s(ij)s(ij)^\top\right]\|\) by the same quantity in a similar manner. Treating \(\{s(ij)\}\) as \(n \times 1\) matrices and applying the matrix Bernstein inequality in Theorem 4 gives that w.h.p.

\[
\|(P_T R_{\Omega} - P_T) Z\|_2 \leq \frac{1}{2} \sqrt{\frac{n}{\mu_0^p}} \|Z\|_\infty + \frac{1}{2} \|Z\|_\infty,2
\]

provided \(c_0\) in the lemma statement is large enough. In a similar fashion we prove that \(\|e_a^\top ((P_T R_{\Omega} - P_T) Z)\|\) is bounded by the same quantity w.h.p. The lemma follows from a union bound over all \((a, b) \in [n] \times [n]\).

C  Proofs of Corollary 1

When \(p \geq \frac{\log^2 n}{n}\), the standard Bernstein inequality and a union bound implies that w.h.p. the degrees (i.e., the number of observed entries) of the rows and columns of \(P_{\Omega} M\) are bounded by \(2pn\). This means \(\frac{1}{p} \tilde{M}\Omega = \frac{1}{p} P_{\Omega} M = R_{\Omega} M\). By Lemma 4 we have

\[
\left\|\frac{1}{p} \tilde{M}\Omega - M\right\| \leq c \left(\frac{1}{p} \|M\|_\infty \log n + \sqrt{\frac{1}{p} \log n \|M\|_\infty,2}\right).
\]

(16)

Let \(\sigma_i\) be the \(i\)-th singular value of \(M\) (with \(\sigma_i = 0\) for \(i > r\)), and recall that \(\tilde{\sigma}_i\) is the \(i\)-th singular values of \(\tilde{M}\Omega\). By Weyl’s inequality \([7]\), we obtain that for \(i = r + 1, \ldots, n\),

\[
\frac{1}{p} \tilde{\sigma}_i = \left|\frac{1}{p} \tilde{\sigma}_i - \sigma_i\right| \leq \left\|\frac{1}{p} \tilde{M}\Omega - M\right\|.
\]

(17)

It follows that

\[
\left\|M - T_r(\tilde{M}\Omega)\right\| \leq \left\|M - \frac{1}{p} \tilde{M}\Omega\right\| + \left\|\frac{1}{p} \tilde{M}\Omega - T_r(\tilde{M}\Omega)\right\|
\]

\[
= \left\|M - \frac{1}{p} \tilde{M}\Omega\right\| + \max_{i=r+1, \ldots, n} \frac{1}{p} \tilde{\sigma}_i
\]

\[
\leq 2c \left(\frac{1}{p} \|M\|_\infty \log n + \sqrt{\frac{1}{p} \log n \|M\|_\infty,2}\right),
\]

where we use (16) and (17) in the last inequality. Since the rank of \(M - T_r(\tilde{M}\Omega)\) is at most \(r\), we have \(\left\|M - T_r(\tilde{M}\Omega)\right\|_F \leq \sqrt{r} \left\|M - T_r(\tilde{M}\Omega)\right\|\) and the corollary follows.

D  Proofs of Theorem 2

The proof is similar to that of Theorem 1 and we shall point out where they differ. We use the same notations as in the proof of Theorem 1 except that throughout this section we re-define the two projections:

\[
P_T Z := UU^T ZV V^T + \tilde{U} \tilde{U}^T Z \tilde{V} \tilde{V}^T - UU^T Z \tilde{V} \tilde{V}^T,
\]

\[
P_{T,1} Z := (UU^T - UU^T) Z (\tilde{V} \tilde{V}^T - VV^T).\]

Note that \(P_T Z + P_{T,1} Z = \tilde{U} \tilde{U}^T Z \tilde{V} \tilde{V}^T\). Since \(\text{col}(U) \subseteq \text{col}(\tilde{U})\) and \(\frac{\mu_x}{n} \leq \frac{\tilde{\mu}_x}{n}\), one can verify that under the incoherence assumption on \(U\) and \(\tilde{U}\) in the theorem statement, we have for all \(i, j, b \in [n]\),

\[
\left\|P_T (e_i e_j^\top)\right\|_F^2 = \left\|U^T e_i e_j^\top \tilde{V}\right\|_F^2 + \left\|U^T e_i e_j^\top \tilde{V}\right\|_F^2 \leq \frac{2 \mu_{0r}}{n} + \frac{\mu_{0r}}{n}.
\]

(18)

\[
\left\|\left(P_{T,1} e_i e_j^\top\right)^\top\right\|_F = \left\|\left(U^T - UU^T\right) e_i e_j^\top \tilde{V} \tilde{V}^T - UU^T \tilde{V} \tilde{V}^T\right\|_F \leq \frac{\mu_{0r}}{n}
\]

(19)

\[
\left\|P_T (e_i e_j^\top) e_b\right\|_2^2 \leq \frac{2 \mu_{0r} \tilde{\mu}_{0r}}{n}.
\]

(20)
We have the following subgradient optimality condition.

**Proposition 3.** $X^* := \bar{U}^T M \bar{V}$ is the unique optimal solution to the program (6) if the following conditions are satisfied: 1. $\|P_T - P_T R_\Omega P_T\|_\infty \leq \frac{1}{2}$ and $\frac{1}{\sqrt{p}} \|P_\Omega P_T\|_F \leq \frac{\sqrt{2\mu_0 r}}{\mu_0 r}$; 2. there exist a dual certificate $Y$ with $P_\Omega Y = Y$ and obeys (a) $\|P_T Y - UV^T\|_F \leq \sqrt{\frac{\mu_0 r}{2\mu_0 r}}$ and (b) $\|P_T^\perp Y\| \leq \frac{1}{2}$.

**Proof.** Consider any feasible solution $X$ to (6). Let $\Delta := \bar{U}^T X^T V - M$ and $G \in \mathbb{R}^{p \times n}$ be such that $\|P_T^\perp G\| = 1$ and $\langle P_T^\perp G, P_T^\perp \Delta \rangle = \|P_T^\perp \Delta\|_*$. Note that $P_\Omega \Delta = 0$, $\|X\|_* = \|\bar{U}^T X^T V\|_*$ and $\|\bar{U}^T M \bar{V}\|_* = \|M\|_*$. Similarly to the proof of Proposition 2, we have

$$\|X\|_* - \|\bar{U}^T M \bar{V}\|_* \geq \langle UV^T - P_T Y + P_T^\perp G - P_T^\perp Y, \Delta \rangle \geq \frac{1}{2} \|P_T^\perp \Delta\|_* - \sqrt{\frac{\mu_0 r}{32 \mu_0 r}} \|P_T \Delta\|_F.$$  

On the other hand, since $\|P_T R_\Omega P_T - P_T\|_\infty \leq \frac{1}{2}$ by assumption, we have

$$\frac{1}{\sqrt{p}} \|P_T \Delta\|_F \leq \sqrt{\langle (P_T R_\Omega P_T - P_T) \Delta, P_T \Delta \rangle + \langle P_T \Delta, P_T \Delta \rangle} = \frac{1}{\sqrt{p}} \|P_\Omega P_T \Delta\|_F.$$  

Because $\bar{U}^T \Delta \bar{V}^T = \Delta$, we have $0 = P_\Omega (\Delta) = P_\Omega (P_T + P_T^\perp) \Delta$ and thus

$$\frac{1}{\sqrt{p}} \|P_\Omega P_T \Delta\|_F = \frac{1}{\sqrt{p}} \|P_\Omega P_T^\perp \Delta\|_F \leq \frac{\sqrt{2 \mu_0 r}}{\mu_0 r} \|P_T^\perp \Delta\|_F,$$

where the last inequality follows from Condition 1 in the statement of the proposition. Combining the last two display equations gives $\|P_T \Delta\|_F \leq \sqrt{\frac{\mu_0 r}{\mu_0 r}} \|P_T^\perp \Delta\|_*$. It follows from [12] that

$$\|X\|_* - \|\bar{U}^T M \bar{V}\|_* \geq \frac{1}{2} \|P_T^\perp \Delta\|_* - \frac{1}{4} \|P_T^\perp \Delta\|_* \geq \frac{1}{4} \|P_T^\perp \Delta\|_*.$$  

The last RHS is strictly positive for all $\Delta$ with $P_\Omega \Delta = 0$ and $\Delta \neq 0$; otherwise we would have $P_T \Delta = (P_T + P_T^\perp) \Delta = \Delta$ and thus $P_T R_\Omega P_T \Delta = 0$, contradicting $\|P_T R_\Omega P_T - P_T\|_\infty \leq \frac{1}{2}$. This proves that $X^* := \bar{U}^T M \bar{V}$ is the unique optimal solution to (6). □

We proceed by showing that Condition 1 in Proposition 3 is satisfied w.h.p. under the conditions of Theorem 2. This is done in the lemma below, which is proved in Section D.1 to follow.

**Lemma 6.** If $p \geq c_0 \frac{\mu_0 r}{n^2} \log n$ for some sufficiently large constant $c_0$, then w.h.p. we have

$$\|P_T R_\Omega P_T - P_T\|_\infty \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{\sqrt{p}} \|P_\Omega P_T^\perp\|_\infty \leq \frac{\sqrt{2 \mu_0 r}}{\mu_0 r}.$$  

We now construct a dual certificate $Y$ using the golfing scheme. This is done similarly as before; in particular, we let $k_0 := 20 \log(32 \mu_0 r)$, $q := 1 - (1 - p)^{1/k_0} \geq \frac{p}{k_0}$, $W_k$ be given by (12) (with the re-defined $P_T$) and $Y := W_{k_0}$. Clearly $P_\Omega (Y) = Y$ by construction. Note that for $k \in [k_0]$, the matrix $D_k := UV^T - P_T (W_k)$ again satisfies (13). It follows that $\|D_k\|_F \leq \frac{1}{2} \|D_{k-1}\|_F$ by the first inequality in Lemma 6 and thus

$$\|P_T Y - UV^T\|_F = \|D_{k_0}\|_F \leq \left(\frac{1}{2}\right)^{k_0} \|D_0\|_F \leq \sqrt{\frac{r}{32 \mu_0 r}} \leq \sqrt{\frac{\mu_0 r}{32 \mu_0 r}},$$

proving Condition 2(a) in Proposition 3. To prove Condition 2(b), we need three lemmas which are analogues of Lemmas 2, 3 and 4 in the proof of Theorem 1.
Lemma 7. Suppose $Z$ is a fixed $n \times n$ matrix. For some universal constant $c > 1$, we have w.h.p.
\[
\|P_{T}^\perp (R_\Omega - I) Z\| \leq c \left( \frac{\mu_0 r \log n}{p n} \|Z\|_\infty + \sqrt{\frac{\mu_0 r \log n}{p n}} \|Z\|_{\infty,2} \right).
\]

Lemma 8. Suppose $Z$ is a fixed matrix. If $p \geq c_0 \frac{\mu_0 r \log n}{n}$ for some $c_0$ sufficiently large, then w.h.p.
\[
\|(P_T R_\Omega - P_T) Z\|_{\infty,2} \leq \frac{1}{2} \sqrt{n \mu_0 r} \|Z\|_\infty + \frac{1}{2} \|Z\|_{\infty,2}
\]

Lemma 9. Suppose $Z$ is a fixed $n \times n$ matrix. If $p \geq c_0 \frac{\mu_0 r \log n}{n}$ for some $c_0$ sufficiently large, then w.h.p.
\[
\|(P_T R_\Omega - P_T) Z\|_\infty \leq \frac{1}{2} \|Z\|_\infty.
\]

We prove these lemmas in Sections D.2–D.4 to follow. Following the same lines as in the proof of Theorem 1, we obtain
\[
\|P_{T}^\perp Y\| \leq \sum_{k=1}^{k_0} \|P_{T}^\perp (R_{\Omega_k} P_T - P_T) D_{k-1}\|.
\]

Applying Lemma 7 with $\Omega$ replaced by $\Omega_k$ to each summand of the last R.H.S, we get w.h.p.
\[
\|P_{T}^\perp Y\| \leq c \frac{\mu_0 r \log p}{q n} \sum_{k=1}^{k_0} \|D_{k-1}\|_\infty + c \sqrt{\frac{\mu_0 r \log p}{q n}} \sum_{k=1}^{k_0} \|D_{k-1}\|_{\infty,2}
\]
\[
\leq c' \frac{\mu_0 r}{\sqrt{c_0} \mu_0 r} \sum_{k=1}^{k_0} \|D_{k-1}\|_\infty + c' \sqrt{\frac{n \mu_0 r}{\sqrt{c_0}}} \sum_{k=1}^{k_0} \|D_{k-1}\|_{\infty,2}
\]

where the last inequality follows from $q \geq \frac{n}{20 \log (32 \mu_0 r)} \geq c_0 \frac{\mu_0 r \log n}{20n}$. Again following the same lines as in the proof of Theorem 1, but using the new Lemmas 7 and 8, we can bound the two terms above as
\[
\|D_{k-1}\|_\infty \leq \left( \frac{1}{2} \right)^{k-1} \|UV^\top\|_\infty,
\]
\[
\|D_{k-1}\|_{\infty,2} \leq k \left( \frac{1}{2} \right)^{k-1} \sqrt{\frac{n \mu_0 r}{\mu_0 r}} \|UV^\top\|_\infty + \left( \frac{1}{2} \right)^{k-1} \|UV^\top\|_{\infty,2}.
\]

It follows that
\[
\|P_{T}^\perp Y\| \leq c'' \frac{\mu_0 r}{\sqrt{c_0} \mu_0 r} \|UV^\top\|_\infty + \sqrt{\frac{n \mu_0 r}{\mu_0 r}} \|UV^\top\|_{\infty,2}.
\]

The inequality $\|P_{T}^\perp (Y)\| \leq \frac{1}{2}$ then follows from the incoherence conditions (2) of $U$ and $V$ provided $c_0$ is sufficiently large. This proves Condition 2(b) in Proposition 3 and hence completes the proof of Theorem 2.

D.1 Proof of Lemma 6

The proof of the first inequality is identical to that of Lemma 1 except that we use (18) instead of (15) (cf. Theorem 4.1 in [10] and Lemma 11 in [15]).

To prove the second inequality, recall that $\gamma_{ij} := 1((i, j) \in \Omega)$ is the indicator variable. For $(i, j) \in [n] \times [n]$, let $S_{(ij)}$ be the operator that maps $Z \in \mathbb{R}^{n \times n}$ to $\left( \frac{1}{p} \gamma_{ij} - 1 \right) \langle P_{T}^\perp Z, e_i e_j^\top \rangle P_{T}^\perp (e_i e_j^\top)$. Observe that $\{S_{(ij)}\}$ are independent zero-mean self-adjoint operators, and
\[
\frac{1}{p} S_{(ij)\perp} P_T P_{T}^\perp - P_{T}^\perp = \sum_{i,j} S_{(ij)}.
\]
By (19), we know that for any $Z$, 
$$\|S_{(i)} Z\|_F \leq \frac{1}{p} \|Z\|_F \|P_{T^\perp} (e_i e_j^\top)\|_F \leq \frac{\hat{\mu}_0 r^2}{p n^2} \|Z\|_F,$$
and
$$\left\| \mathbb{E} \sum_{i,j} S_{(i)}^2 Z \right\|_F = \left\| \mathbb{E} \sum_{i,j} \left( \frac{1}{p} \gamma_{ij} - 1 \right)^2 (P_{T^\perp} Z, e_i e_j^\top) \|P_{T^\perp} (e_i e_j^\top)\|_F^2 \right\|_F \leq \frac{1}{p} \left( \max_{i,j} \|P_{T^\perp} (e_i e_j^\top)\|_F^2 \right) \left\| \sum_{i,j} (P_{T^\perp} Z, e_i e_j^\top) e_i e_j^\top \right\|_F \leq \frac{\hat{\mu}_0 r^2}{p n^2} \|P_{T^\perp} Z\|_F.$$
This means
$$\|S_{(i)}\|_{op} \leq \frac{\hat{\mu}_0 r^2}{p n^2} \quad \text{and} \quad \left\| \mathbb{E} \sum_{i,j} S_{(i)}^2 \right\|_{op} \leq \frac{\hat{\mu}_0 r^2}{p n^2}.$$
Applying the matrix Bernstein inequality in Theorem 4, we obtain w.h.p.
$$\left\| \frac{1}{p} P_{T^\perp} \mathbb{E} P_{T^\perp} - P_{T^\perp} \right\|_{op} \leq c \frac{\hat{\mu}_0 r^2}{p n^2} \log(2n) + \frac{\hat{\mu}_0 r^2}{\sqrt{pn} \sqrt{4c \log(2n)}} \leq c' \frac{\hat{\mu}_0 r^2 \sqrt{\log(2n)}}{\sqrt{\mu_0 r n}}$$
for some constant $c'$, where the last inequality follows from $\mu_0 r \leq \hat{\mu}_0 r$ and the assumption $p \geq c_0 \frac{\mu_0 r^2}{n^2} \log(2n)$. It follows that
$$\frac{1}{\sqrt{p}} \left\| \mathbb{E} P_{T^\perp} \right\|_{op} \leq \sqrt{\frac{1}{p} \left\| P_{T^\perp} \mathbb{E} P_{T^\perp} - P_{T^\perp} \right\|_{op} + \left\| P_{T^\perp} \right\|_{op}} \leq \sqrt{c' \frac{\hat{\mu}_0 r^3 \log(2n)^3}{p \mu_0 r n}} + 1 \leq \sqrt{\frac{2 \hat{\mu}_0 r}{\mu_0 r}},$$
where in the last inequality we again use the assumption $p \geq c_0 \frac{\mu_0 r^2}{n^2} \log(2n)$.

D.2 Proof of Lemma 7

We can write
$$P_{T^\perp} (R_{\Omega} - I) Z = \sum_{i,j} S_{(i)} := \sum_{i,j} \left( \frac{1}{p} \gamma_{ij} - 1 \right) Z_{ij} P_{T^\perp} (e_i e_j^\top),$$
where $\{S_{(i)}\}$ are independent $n \times n$ matrices satisfying $\mathbb{E}[S_{(i)}] = 0$ and
$$\|S_{(i)}\| \leq \frac{1}{p} \|Z_{ij}\| \|P_{T^\perp} (e_i e_j^\top)\|_F \leq \frac{\hat{\mu}_0 r}{p n} \|Z\|_\infty.$$ 
by (19). Moreover, since $\text{col}(U) \subseteq \text{col}({\bar{U}})$, $\text{col}(V) \subseteq \text{col}(V)$ and $\bar{U} U^\top - U U^\top$, $V V^\top - V V^\top$ are projections, we have
$$(P_{T^\perp} (e_i e_j^\top))^\top P_{T^\perp} (e_i e_j^\top) = |e_i^\top (U U^\top - U U^\top) e_i| (V V^\top - V V^\top) e_j e_j^\top (V V^\top - V V^\top)$$
and thus
$$\left\| \mathbb{E} \sum_{i,j} S_{(i)} S_{(i)}^\top \right\| = \frac{1}{p} \left\| \sum_j (V V^\top - V V^\top) e_j e_j^\top (V V^\top - V V^\top) \sum_i |e_i^\top (U U^\top - U U^\top) e_i| Z_{ij}^2 \right\| \leq \frac{1}{p} \left\| \sum_j e_j e_j^\top \sum_i |e_i^\top (U U^\top - U U^\top) e_i| Z_{ij} \right\| \leq \frac{\hat{\mu}_0 r}{p n} \|Z\|_\infty^2.$$ 

We can bound $\left\| \mathbb{E} \sum_{i,j} S_{(i)} S_{(i)}^\top \right\|$ by the same quantity in a similar manner. Applying the matrix Bernstein inequality in Theorem 4 proves the lemma.
D.3 Proof of Lemma 8

Fix $b \in [n]$. The $b$-th column of the matrix $(P_\Omega R - P_T) Z$ can be written as

$$(P_\Omega R - P_T) Z e_b = \sum_{i,j} s_{(ij)} := \sum_{i,j} \left(\frac{1}{p} \gamma_{ij} - 1\right) Z_{ij} P_T(e_i e_j^\top) e_b,$$

where $\{s_{(ij)}\}$ are independent column vectors in $\mathbb{R}^n$. Note that $\mathbb{E} [s_{(ij)}] = 0$ and

$$\|s_{(ij)}\|_2 \leq \frac{1}{p} |Z_{ij}| \|P_T(e_i e_j^\top) e_b\|_2 \leq \frac{2 \mu_0 \rho}{\sqrt{n}} \|Z\|_\infty \leq \frac{2 \sqrt{n}}{c_0 \log n} \|Z\|_\infty$$

by [20] and the assumption on $p$. We also have

$$\mathbb{E} \sum_{i,j} s_{(ij)} \|s_{(ij)}\|^2 = \sum_{i,j} \mathbb{E} \left[ \left(\frac{1}{p} \gamma_{ij} - 1\right)^2 \right] Z_{ij}^2 \|P_T(e_i e_j^\top) e_b\|_2^2 = \frac{1}{p} \sum_{i,j} Z_{ij}^2 \|P_T(e_i e_j^\top) e_b\|_2^2.$$

Because

$$\|P_T(e_i e_j^\top) e_b\|_2 = \|UU^T e_i e_j^\top V V^\top e_b + (UU^T - UU^T) e_i e_j^\top V V^\top e_b\|_2$$

$$\leq \sqrt{\frac{\mu_0 \rho}{n}} |e_j^\top V V^\top e_b| + \sqrt{\frac{\mu_0 \rho}{n}} |e_j^\top V V^\top e_b|,$$

it follows that

$$\mathbb{E} \sum_{i,j} s_{(ij)} \|s_{(ij)}\| \leq \frac{2 \mu_0 \rho}{\sqrt{n}} \sum_{i,j} Z_{ij}^2 \frac{\mu_0 \rho}{n} \|e_j^\top V V^\top e_b\|_2^2$$

$$\leq \frac{2 \mu_0 \rho}{\sqrt{n}} \|Z\|_\infty,2 \|V V^\top e_b\|_2^2 + \frac{2 \mu_0 \rho}{\sqrt{n}} \|Z\|_\infty,2 \|V V^\top e_b\|_2^2.$$

We can bound $\|\mathbb{E} \sum_{i,j} s_{(ij)} \|s_{(ij)}\|\|$ in a similar manner. Treating $\{S_{(ij)}\}$ as $n \times 1$ matrices and applying the Matrix Bernstein inequality in Theorem 3, we get

$$\|(P_\Omega R - P_T) Z e_b\|_2 \leq \sqrt{\frac{n}{\mu_0 \rho}} \|Z\|_\infty + \frac{1}{2} \|Z\|_\infty,2, \text{ w.h.p.}$$

provided $c_0$ in the statement of the lemma is sufficiently large. In a similar fashion we can prove that the $\|e_i^\top ((P_\Omega R - P_T) Z)\|_2$ is bounded by the same quantity w.h.p. The lemma follows from a union bound over all $(a, b) \in [n] \times [n]$.

D.4 Proof of Lemma 9

Fix $(a, b) \in [n] \times [n]$. We can write the $(a, b)$ entry of the matrix $(P_\Omega R - P_T) Z$ as

$$(P_\Omega R - P_T) Z_{ab} = \sum_{i,j} s_{ij} := \sum_{i,j} \left(\frac{1}{p} \gamma_{ij} - 1\right) Z_{ij} \langle e_i e_j^\top, P_T(e_a e_b^\top) \rangle,$$
where \( s_{ij} \in \mathbb{R} \) are independent zero-mean random variables. By \([18]\) and the assumption on \( p \), we have
\[
|s_{ij}| \leq \frac{1}{p} |Z_{ij}| \left\| P_T(e_i e_j^\top) \right\|_F \left\| P_T(e_a e_b^\top) \right\|_F \leq \frac{1}{2c_0 \log n} \| Z \|_\infty.
\]
and
\[
\left| \mathbb{E} \sum_{i,j} s_{ij}^2 \right| = \sum_{i,j} \mathbb{E} \left[ \left( \frac{1}{p} \gamma_{ij} - 1 \right)^2 \right] Z_{ij}^2 \langle e_i e_j^\top, P_T(e_a e_b^\top) \rangle^2 \leq \frac{1}{p} \| Z \|_\infty^2 \sum_{i,j} \langle e_i e_j^\top, P_T(e_a e_b^\top) \rangle^2 \leq \frac{1}{2c_0 \log n} \| Z \|_\infty^2.
\]

Applying the Bernstein inequality in Theorem 4, we conclude that w.h.p. \( \| (P_T \mathcal{R}_\Omega P_T - P_T) Z \|_{ab} \leq \frac{1}{2} \| Z \|_{\infty} \) for \( c_0 \) sufficiently large. The lemma follows from a union bound over all \( a, b \in [n] \times [n] \).

**E  Proof of Theorem 3**

**E.1  Part 1 of the theorem**

We first describe an equivalent formulation of the planted clique problem. Let \( \bar{A} \in \mathbb{R}^{n \times n} \) be the adjacency matrix of the graph, and \( L^* \in \{0,1\}^{n \times n} \) be the matrix with \( L^*_{ij} = 1 \) if and only if the nodes \( i \) and \( j \) are both in the clique. Let \( S^* := \bar{A} - L^* \). Note that for each \((i,j) \notin \text{support}(L^*) = \{ (i,j) : L^*_{ij} = 1 \}, \) the pair \( S^*_{ij} = S^*_{ji} \) is non-zero with probability \( 1/2 \); for each \((i,j) \in \text{support}(L^*) \), we always have \( S^*_{ij} = S^*_{ji} = 0 \).

We reduce the planted problem above to the matrix decomposition problem using subsampling. Given the matrix \( \bar{A} \), we set each \( A_{ij} \) to zero with probability \( \frac{1}{2} \) independently, and let \( A \) be the resulting matrix. If we let \( S := \bar{A} - L \), then each pair \( S_{ij} = S_{ji} \) is non-zero with probability \( \tau = \frac{1}{2} \). Moreover, the matrix \( L \) has rank 1 and satisfies the standard and joint incoherence conditions \([2]\) and \([3]\) with parameters \( \mu_0 = 1/n_{\min} \) and \( \mu_1 = n^2/n_{\min}^2 \). Hence recovering \( (L^*, S^*) \) from \( A \) is a special case of the matrix decomposition problem. If there exists a polynomial-time algorithm that, for all \( n \), finds \( L^* \) given \( A \) with probability at least \( \frac{1}{2} \) when
\[
\frac{\mu_1^{1-\epsilon'} - \epsilon'}{n} \geq 1,
\]
then it means this algorithm recovers the planted clique with \( n_{\min} \leq n^{\frac{1}{2} - \frac{\epsilon'}{2(1-\epsilon')}} \) from \( \bar{A} \), which violates the assumption \( A_1 \).

**E.2  Part 2 of the theorem**

For simplicity, we assume \( K := \frac{n}{2p} \) and \( \frac{n}{2} \) are both integers. Let \( M := n/2 \). Suppose \( L^* \) takes value uniformly at random from a set \( L = \{ L^{(1)}, L^{(2)}, \ldots, L^{(M)} \} \subseteq \mathbb{R}^{n \times n} \) which we now define. Let \( L^{(0)} \) be the symmetric block-diagonal matrix with \( r \) contingent blocks of size \( K \times K \), where \( L^{(0)}_{ij} = 1 \) inside the blocks and 0 otherwise, and the blocks are in the first \( rK \) columns. Note that \( \mu_0 \geq 2 \) by assumption, so \( rK \leq \frac{1}{2} \) and thus the last \( \frac{1}{2} \) rows and columns of \( L^{(0)} \) are all zeros. For \( l = 1, \ldots, M \), let \( L^{(l)} \) be the matrix obtained from \( L^{(0)} \) by swapping the first row and column with the \( (n/2 + l) \)-th row and column, respectively. In other words, the first block of \( L^{(l)} \) corresponds to the rows and columns with indices \( \{ 2, 3, \ldots, K, n/2 + 1 \} \), and the other \( r - 1 \) blocks are the same as those in \( L^{(0)} \). It is easy to check that each \( L^{(l)} \) has rank \( r \) and satisfies the standard incoherence condition \([2]\) with parameter
μ₀. We further assume that conditioned on \( \mathbf{L}^* \), the matrix \( \mathbf{S}^* \) is distributed as follows: \( S^*_{ij} \) equals -1 with probability \( \tau = 1/3 \) and 0 otherwise for \((i,j) \in \text{support}(\mathbf{L}^*)\), and \( S^*_{ij} \) equals 1 with probability \( \tau = 1/3 \) and 0 otherwise for \((i,j) \notin \text{support}(\mathbf{L}^*)\). Finally, recall that \( \mathbf{A} = \mathbf{L}^* + \mathbf{S}^* \).

We now compute an upper bound on the mutual information \( I(\mathbf{L}^*; \mathbf{A}) \). Let \( \mathbb{P}(\ell) \) be the distribution of \( \mathbf{A} \) conditioned on \( \mathbf{L}^* = \mathbf{L}^{(\ell)} \), and we use \( D(\mathbb{P}(\ell) \| \mathbb{P}(\ell')) \) to denote the Kullback-Leibler (KL) divergence between \( \mathbb{P}(\ell) \) and \( \mathbb{P}(\ell') \). By definition of the mutual information and the convexity of the KL divergence, we have

\[
I(\mathbf{L}^*; \mathbf{A}) = \frac{1}{M} \sum_{\ell=1}^{M} D \left( \mathbb{P}(\ell) \| \frac{1}{M} \sum_{\ell' = 1}^{M} \mathbb{P}(\ell') \right) \leq \frac{1}{M^2} \sum_{\ell, \ell'} D \left( \mathbb{P}(\ell) \| \mathbb{P}(\ell') \right).
\]

With slight abuse of notation, we use \( D(q_1 \| q_2) := q_1 \log \frac{q_1}{q_2} + (1-q_1) \log \frac{1-q_1}{1-q_2} \) to denote the KL divergence between two Bernoulli distributions with parameters \( q_1 \) and \( q_2 \). Direct computation gives

\[
D \left( \mathbb{P}(\ell) \| \mathbb{P}(\ell') \right) = (K+1)D \left( \frac{2}{3} \| \frac{1}{3} \right) + (K+1)D \left( \frac{1}{3} \| \frac{2}{3} \right) \leq K + 1
\]

for all \( \ell, \ell' = 1, \ldots, M \), where the inequality above follows from \( \log x \leq x - 1 \). It follows that \( I(\mathbf{L}^*; \mathbf{A}) \leq K + 1 \).

We now apply the Fano’s inequality \cite{Fan} to obtain that for any measurable function \( \hat{\mathbf{L}} \) of \( \mathbf{A} \),

\[
\mathbb{P} \left( \hat{\mathbf{L}} \neq \mathbf{L}^* \right) \geq 1 - \frac{I(\mathbf{L}^*; \mathbf{A}) + \log 2}{\log M} \geq 1 - \frac{K + 1 + \log 2}{\log(n/2)} \geq \frac{1}{2},
\]

where the probability is with respect to the randomness of \( \mathbf{L}^* \) and \( \mathbf{S}^* \), and the last inequality holds when \( \frac{\log n}{12K} = \frac{\mu^* \log n}{12n} \geq 1 \) and \( n \geq 10 \). Because the supremum is lower bounded by the average, we obtain

\[
\sup_{\mathbf{L}^* \in \mathcal{L}} \mathbb{P} \left( \hat{\mathbf{L}} \neq \mathbf{L}^* \right) \geq \frac{1}{2},
\]

where the probability is with respect to the randomness of \( \mathbf{S}^* \).

**References**

[1] Alekh Agarwal, Sahand Negahban, and Martin J Wainwright. Noisy matrix decomposition via convex relaxation: Optimal rates in high dimensions. *The Annals of Statistics*, 40(2):1171–1197, 2012.

[2] Noga Alon, Alexandr Andoni, Tali Kaufman, Kevin Matulef, Ronitt Rubinfeld, and Ning Xie. Testing k-wise and almost k-wise independence. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 496–505. ACM, 2007.

[3] Noga Alon, Michael Krivelevich, and Benny Sudakov. Finding a large hidden clique in a random graph. *Random Structures and Algorithms*, 13(3-4):457–466, 1998.

[4] Benny Applebaum, Boaz Barak, and Avi Wigderson. Public-key cryptography from different assumptions. In *Proceedings of the 42nd ACM Symposium on Theory of Computing*, pages 171–180. ACM, 2010.

[5] Quentin Berthet and Philippe Rigollet. Complexity theoretic lower bounds for sparse principal component detection. *Journal of Machine Learning Research: Workshop and Conference Proceedings*, 30:1046–1066, 2013.

[6] Quentin Berthet and Philippe Rigollet. Optimal detection of sparse principal components in high dimension. *Annals of Statistics*, 41(1):1780–1815, 2013.
[7] Rajendra Bhatia. *Perturbation Bounds for Matrix Eigenvalues*. Longman, Harlow, 1987.

[8] Jian-Feng Cai, Emmanuel J. Candès, and Zuowei Shen. A singular value thresholding algorithm for matrix completion. *SIAM Journal on Optimization*, 20(4):1956–1982, 2010.

[9] Emmanuel J. Candès, Xiaodong Li, Yi Ma, and John Wright. Robust principal component analysis? *Journal of the ACM*, 58(3):11, 2011.

[10] Emmanuel J. Candès and Benjamin Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772, 2009.

[11] Venkat Chandrasekaran, Sujay Sanghavi, Pablo Parrilo, and Alan Willsky. Rank-sparsity incoherence for matrix decomposition. *SIAM Journal on Optimization*, 21(2):572–596, 2011.

[12] Yudong Chen. Incoherence-optimal matrix completion. *arXiv preprint arXiv:1310.0154*, 2013.

[13] David Gross. Recovering low-rank matrices from few coefficients in any basis. *IEEE Transactions on Information Theory*, 57(3):1548–1566, 2011.

[14] Vitaly Feldman, Elena Grigorescu, Lev Reyzin, Santosh Vempala, and Ying Xiao. Statistical algorithms and a lower bound for detecting planted cliques. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing*, pages 655–664. ACM, 2013.

[15] Ari Juels and Marcus Peinado. Hiding cliques for cryptographic security. *Designs, Codes and Cryptography*, 20(3):269–280, 2000.

[16] Prateek Jain, Praneeth Netrapalli, and Sujay Sanghavi. Low-rank matrix completion using alternating minimization. *Proceedings of the 45th Annual ACM Symposium on Theory of Computing*, pages 665–674. ACM, 2013.
[27] Pascal Koiran and Anastasios Zouzias. Hidden cliques and the certification of the restricted isometry property. *arXiv preprint arXiv:1211.0665*, 2012.

[28] Xiaodong Li. Compressed sensing and matrix completion with constant proportion of corruptions. *Constructive Approximation*, 37(1):73–99, 2013.

[29] Zhouchen Lin, Minming Chen, Leqin Wu, and Yi Ma. The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices. *UIUC Technical Report UILU-ENG-09-2215*, 2009.

[30] Zongming Ma and Yihong Wu. Computational barriers in minimax submatrix detection. *arXiv preprint arXiv:1309.5914*, 2013.

[31] Frank McSherry. Spectral partitioning of random graphs. In *Proceedings of 42nd IEEE Symposium on Foundations of Computer Science*, pages 529–537, 2001.

[32] Benjamin Recht. A simpler approach to matrix completion. *Journal of Machine Learning Research*, 12:3413–3430, 2011.

[33] Benjamin Rossman. *Average-case complexity of detecting cliques*. PhD thesis, Massachusetts Institute of Technology, 2010.

[34] Joel A. Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of Computational Mathematics*, 12(4):389–434, 2012.

[35] Jiaming Xu, Rui Wu, Kai Zhu, Bruce Hajek, R. Srikant, and Lei Ying. Jointly clustering rows and columns of binary matrices: Algorithms and trade-offs. In *The 2014 ACM International Conference on Measurement and Modeling of Computer Systems*, SIGMETRICS ’14, pages 29–41, 2014.

[36] Miao Xu, Rong Jin, and Zhi-Hua Zhou. Speedup matrix completion with side information: Application to multi-label learning. In *Advances in Neural Information Processing Systems*, pages 2301–2309, 2013.

[37] Jinfeng Yi, Lijun Zhang, Rong Jin, Qi Qian, and Anil Jain. Semi-supervised clustering by input pattern assisted pairwise similarity matrix completion. In *Proceedings of The 30th International Conference on Machine Learning*, pages 1400–1408, 2013.