On global models for finite rotating objects in equilibrium in cosmological backgrounds

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The studies in general relativity of rotating finite objects in equilibrium have usually focused on the case when they are truly isolated, this is, the models to describe finite objects are embedded in an asymptotically flat exterior vacuum. Known results ensure the uniqueness of the vacuum exterior field by using the boundary data for the exterior field given at the surface of the object plus the decay of the exterior field at infinity. The final aim of the present work is to study the consequences on the interior models by changing the boundary condition at infinity to one accounting for the embedding of the object in a cosmological background. Considering first the FLRW standard cosmological backgrounds, we are studying the general matching of FLRW with stationary axisymmetric spacetimes in order to find the new boundary condition for the vacuum region. Here we present the first results.

1 Introduction

The studies in General Relativity of rotating finite objects (or local systems), i.e. astrophysical objects as stars, planet orbits, galaxies, clusters, etc, in equilibrium have usually focused on the case when they are truly isolated, this is, so that the exterior field tend to zero as moving away from the object. For account for that, the models constructed to describe the finite object are embedded in an asymptotically flat exterior vacuum region.

Most of the work produced along these lines have followed the theoretical approach based on the construction of global models by means of the matching of spacetimes: the whole configuration is composed by two regions, one spacetime describing the interior of the object and another to describe the vacuum exterior, which have been matched across the surface of the object at all times $\Sigma$. To account for the equilibrium state of the rotating configuration, the whole matched spacetime, and hence the interior and exterior regions, are to be strictly stationary. In addition, it has been usually naturally assumed that the model is axially symmetric.

The stationary and axisymmetric exterior vacuum region is described by two functions $(U, \Omega)$, that satisfy an elliptic system of partial differential equations, known as the Ernst equations \([1]\). The boundary conditions for the problem will come determined by the object, on its surface ($\Sigma$), plus the asymptotic flat behaviour at infinity. The boundary data for the pair $(U, \Omega)$ on $\Sigma$, which is determined from the matching conditions with a given interior, consists of the values of the pair of functions on $\Sigma$, up to a constant additive factor for $\Omega$, plus the values of their normal derivatives to $\Sigma$. In other words, the boundary data is given by $\{U|_{\Sigma}, \Omega|_{\Sigma} + c_\Omega, n(U)|_{\Sigma}, n(\Omega)|_{\Sigma}\}$, where $c_\Omega$ is an arbitrary (real) constant and $n$ is the vector normal to $\Sigma$. This constitutes a set of Cauchy data for an elliptic problem, and therefore the problem is overdetermined (although not unique because of $c_\Omega$).
Now, using the fall off for $U$ and $\Omega$ that asymptotic flatness requires, known results ensure the uniqueness of the exterior field given the object $[2, 3]$. In fact, the necessary conditions on the Cauchy data for the existence of the exterior field have been also found $[4]$. This fact determines that not every model for the interior of a finite object can describe an isolated finite object.

Nevertheless, in a more realistic situation for any astrophysical object, moving away from the object one should eventually reach a large scale region which ought to be not flat but described by a dynamical cosmological model. In the framework mentioned above, this means that we have to change the boundary conditions at infinity implied by the asymptotically flat behaviour, to some others accounting for the embedding of the object into a cosmological background. To that end, we consider a vacuum stationary axisymmetric region (in which a compact object could reside) matched to a cosmological background.

To begin with, the cosmological backgrounds we consider are the Friedman-Lemaître-Robertson-Walker (FLRW) models. The results presented here basically state that first, the slicing of the matching hypersurface at FLRW side given by constant values of the cosmological time coincide with that of constant value of an intrinsically defined time (see below) at the stationary side. More importantly, the surfaces defined by the slicing must be spheres.

### 2 Summary on junction of spacetimes

The formalism for matching two $C^2$ spacetimes$^1$ $(W^\pm, g^\pm)$ with respective boundaries $\Sigma^\pm$ of arbitrary character, even changing from point to point, was presented in $[5]$. The starting point for a further development of the matching conditions by subdividing them into constraint and evolution equations by using a $2+1$ decomposition was introduced in $[6]$ (see also $[7]$). For completeness, this section is devoted for a summary of the formalisms. We refer to $[5, 6, 7]$ for further details.

Gluing $(W^+, g^+, \Sigma^+)$ to $(W^-, g^-, \Sigma^-)$ across their boundaries consists in constructing a manifold $\mathcal{V} = W^+ \cup W^-$ identifying both the points and the tangent spaces of $\Sigma^+$ and $\Sigma^-$. This is equivalent to the existence of an abstract three-dimensional $C^3$ manifold $\sigma$ and two $C^3$ embeddings $\Psi_\pm : \sigma \rightarrow W^\pm$ can be constructed such that $\Psi_\pm(\sigma) = \Sigma^\pm$. Points on $\Sigma^+$ and $\Sigma^-$ are identified by the diffeomorphism $\Psi_+ \circ \Psi_-^{-1}$. We denote by $\Sigma(\subset \mathcal{V}) = \Sigma^+ = \Sigma^-$ the identified matching hypersurface. The conditions that ensure the existence of a continuous metric $g$ in $\mathcal{V}$, such that $g = g^\pm$ in $\mathcal{V} \cap W^\pm$ are the so-called preliminary junction conditions and require first the equivalence of the induced metrics on $\Sigma^\pm$, i.e. $\Psi_+^*(g^+) = \Psi_-^*(g^-)$, where $\Psi^*$ denotes the pull-back of $\Psi$. Secondly, one requires the existence of two $C^2$ vector fields $L_\pm$ defined over $\Sigma^\pm$, transverse everywhere to $\Sigma^\pm$, with different relative orientation ($L_+$ points $W^+$ inwards whereas $L_-$ points $W^-$ outwards) and satisfying $\Psi_+^*(L_+) = \Psi_-^*(L_-), \Psi_+^*(L_+(L_+)) = \Psi_-^*(L_-(L_-))$, where $L_\pm = g^\pm(L_\pm, \cdot)$. The existence of these so-called rigging vector fields is not ensured when the boundaries have null points $[8]$.

Now, the Riemann tensor in $(\mathcal{V}, g)$ can be defined in a distributional form (see $[5]$). In order to avoid singular terms in the Riemann tensor on $\Sigma$, a second set of conditions must be imposed. This second set demands the equality of the so-called generalized second fundamental forms with respect of the rigging one-forms, and can be expressed as

$$\Psi_+^*(\nabla^+ L_+) = \Psi_-^*(\nabla^- L_-),$$

(1)

where $\nabla^\pm$ denotes the Levi-Civita covariant derivative in $(W^\pm, g^\pm)$. If (1) are satisfied for one choice of pair of riggings, then they do not depend on the choice of riggings $[8]$.

$^1$ A $C^m$ spacetime is a paracompact, Hausdorff, connected $C^{m+1}$ manifold with a $C^m$ Lorentzian metric (convention $\{-1, 1, 1, 1\}$).
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Once the whole set of matching conditions hold, the finite one-side limits of the Riemann tensor of \((\mathcal{V}, g)\) on \(\Sigma\), and in any \(C^1\) coordinate system covering \(\Sigma\) (or part thereof), satisfy the following relation

\[
R^+_{\alpha\beta\mu\nu}|_{\Sigma} = R^-_{\alpha\beta\mu\nu} + n_\alpha n_\mu B_{\beta\nu} - n_\beta n_\nu B_{\alpha\mu} - n_\alpha n_\nu B_{\beta\mu} + n_\beta n_\mu B_{\alpha\nu} |_{\Sigma},
\]

where \(R^+_{\alpha\beta\mu\nu}, \) are the Riemann tensors of \((W^\pm, g^\pm)\), respectively, \(n\) is the normal one-form to \(\Sigma\), and \(B_{\alpha\beta}\) is a symmetric tensor which is defined up to the transformation \(B_{\alpha\beta} \rightarrow B_{\alpha\beta} + X_\alpha n_\beta + X_\beta n_\alpha,\) for arbitrary one-form \(X\).

Following [6,7], the 2+1 splitting of the matching conditions starts by foliating \((\sigma, \Psi^+(g^-))\) with a set of spacelike \(C^3\) two-surfaces \(\sigma_\tau\), where \(\tau \in \mathbb{R}\). Let \(i_\tau : \sigma_\tau \rightarrow \sigma\) be the inclusion map of \(\sigma_\tau\) into \(\sigma\). The compositions \(\Psi_{\tau} \equiv \Psi_\pm \circ i_\tau\) define embeddings of \(\sigma_\tau\) into \((W^\pm, g^\pm)\), and the images \(\mathcal{S}_\tau^\pm \equiv \Psi_{\tau} \circ (\sigma_\tau)\) are spacelike two-surfaces lying on \(\Sigma^\pm\) by construction. Clearly, the identification of \(\Sigma^+\) with \(\Sigma^-\) through \(\Psi^- \circ \Psi^+\) induces the identification of \(\Psi_{\tau}^+\) with \(\mathcal{S}_\tau^-\) by the diffeomorphism \(\Psi_{\tau}^+ \circ (\Psi_{\tau}^-)^{-1}\). The identified surfaces will be denoted by \(S_\tau \equiv S_\tau^+ = S_\tau^-\), and thus \(S_\tau \subset \Sigma\). For any given point \(x \in S_\tau\), let us denote by \(N_x S_\tau^\pm\) the two-dimensional Lorentzian vector space, subset of the cotangent space \(T^*_{x} W^\pm\), spanned by the normal one-forms of \(\mathcal{S}_\tau^\pm\) at \(x\). The (normal) bundle with fibers \(N_x S_\tau^\pm\) and base \(\mathcal{S}_\tau^\pm\) will be denoted by \(\mathcal{NS}_\tau^\pm\).

The matching conditions impose restrictions on \(S_\tau\) for each value of \(\tau\). These are called the constraint matching conditions and consist of two parts. First, the restriction of the preliminary junction conditions on \(S_\tau\) imposes the isometry of \(\mathcal{S}_\tau^+\) and \(\mathcal{S}_\tau^-\), i.e.

\[
\Psi_{\tau}^+(g^+) = \Psi_{\tau}^-(g^-). \tag{3}
\]

Secondly, and in order to ensure the identification of the tangent spaces in \(\Sigma^\pm\), for every \(x \in S_\tau\) there must exist a linear and isometric map

\[
f_{\tau}^\pm : N_x S_\tau^+ \rightarrow N_x S_\tau^- \tag{4}
\]

with the following property, inherited by [11]: the second fundamental form of \(\mathcal{S}_\tau^\pm\) with respect to any \(n \in \mathcal{NS}_\tau^\pm\), denoted by \(K_{\tau}^+(n) \equiv \Psi_{\tau}^+(\nabla^+ n)\), and the corresponding image through \(f_{\tau}\), i.e. the one-form field \(f_{\tau}(n)\) to \(\mathcal{S}_\tau^-\), will have to coincide, i.e.

\[
K_{\tau}^+(n) = K_{\tau}^-(f_{\tau}(n)), \quad \forall n \in \mathcal{NS}_\tau^+. \tag{5}
\]

For further details and more explicit forms of the above expressions we refer to [11].

3 Matching FLRW with stationary and axisymmetric spacetimes

Regarding the FLRW spacetime, and since we will follow the procedures used in [12], let us review some notation and conventions.

**Definition 1.** Let \((\mathcal{M}, g_{\mathcal{M}})\) be a complete, simply connected, three-dimensional Riemannian manifold of constant curvature and \(I \subset \mathbb{R}\) an open interval. A FLRW spacetime \((\mathcal{V}^{RW}, g^{RW})\) is the manifold \(\mathcal{V}^{RW} = I \times \mathcal{M}\) endowed with the metric \(g^{RW} = -dt^2 + a^2(t) g_{\mathcal{M}}\), where the so-called scale factor \(a(t)\) is a positive \(C^3\) function on \(I\), and such that

1. The energy density \(\rho\) and the pressure of the cosmological flow \(p\) satisfy \(\rho \geq 0, \rho + p \neq 0\),

2. the expansion \(\dot{a}/a\) vanishes nowhere on \(I\) (dot denotes \(dt/dt\)).

**Definition 2.** To start with, no specific matter content in the stationary and axisymmetric region will be assumed, although the corresponding \(G_2\) on \(T_2\) (necessarily) Abelian group
will be assumed to act orthogonally transitively (OT). Therefore, there exist coordinates \(\{T, \Phi, x^M\}\) \(M, N, \ldots = 2, 3\) such that the metric \(g^{xx}\) in the OT stationary and axisymmetric region \(W^{xx}\) reads \(\ddots\)

\[
ds_{xx}^2 = -e^{2U} (dT + A d\Phi)^2 + e^{-2U} W^2 d\Phi^2 + g_{MN} dx^M dx^N,
\]

where \(U, A, W\) and \(g_{MN}\) are functions of \(x^M\), the axial Killing vector field is given by \(\eta = \partial_\Phi\), and a timelike Killing vector field is given by \(\xi = \partial_T\).

Special attention is given to the one-form \(\zeta = -dT\) and its corresponding vector field \(\mathcal{Z}\), which are orthogonal to the hypersurfaces of constant \(T\), as well as to \(\eta\) everywhere. In fact, \(\mathcal{Z}\) is intrinsically defined as the future-pointing timelike vector field tangent to the orbits of the \(G_2\) group and also to the axial Killing vector field \(\mathcal{E}_M\) (see \(\mathcal{II}\)). Note that \(\mathcal{Z}\) is hypersurface orthogonal, but it is not a Killing vector field. The norm of \(\mathcal{Z}\) is given by \(\zeta^\alpha \zeta_\alpha = -F^2\) with \(F \equiv e^U (A^2 W^2 - e^{-2U})^{1/2}\).

We will denote by \(\{E_M\}\) any two linearly independent vector fields spanning the surfaces orthogonal to the orbits of the \(G_2\) group, so that the set \(\{\zeta, \eta, E_M\}\) constitute a basis of the tangent spaces at every point in \((W^{xx}, g^{xx})\). In the coordinate system used in \(\mathcal{II}\) one could simply take the choice \(E_M = \partial_{x^M}\).

The only assumption made on \(\Sigma\) is that of being generic, i.e. such that the function given by the values of the cosmological time on \(\Sigma^{RW}\), say \(\chi\) has no local maximum or minimum \(\mathcal{II}\). The domain \(\Sigma^{RW}_0 \subset \Sigma^{RW}\) defined as those points with regular values of \(\chi\) is then dense in \(\Sigma^{RW}\) \(\mathcal{II}\). Nevertheless, this assumption is easily shown to be a property when the stationary region is vacuum, which is the case in which we will be interested in eventually.

**Proposition 1.** Let \((\mathcal{V}, g)\) be the matching spacetime between a FLRW region \((W^{RW}, g^{RW})\) and an OT stationary and axisymmetric region \((W^{xx}, g^{xx})\) across a connected, generic matching hypersurface \(\Sigma\) preserving the symmetry. Let \(S^{RW}_r\) be the natural foliation in FLRW given by \(\Sigma^{RW} \cap \{t = \tau\}\). Then, the following geometrical properties hold:

1. \(\mathcal{Z}\) is orthogonal to each surface \(S_\tau\), and hence \(S^{RW}_r\), at any point \(p \in S_\tau\).
2. Each connected component of \(S_\tau\), and hence \(S^{RW}_r\) and \(S^{xx}_r\) is a two-sphere with the standard metric and it is an umbilical submanifold in \((\mathcal{V}, g)\). Furthermore, there exists a spherically symmetric coordinate system \(\{t, r, \theta, \phi\}\) in \((W^{RW}, g^{RW})\) such that this surface corresponds to \(r = \text{const.}\) and \(t = \text{const.}\).

**Proof:** We start by fixing a regular value \(\tau_0\) of \(\chi\) (see above) and the corresponding surface \(S^{RW}_{\tau_0}\). For any point \(p \in S^{RW}_{\tau_0}\) consider an open neighbourhood \(U \subset \Sigma^{RW}_{\tau_0}\) of \(p\). Let us denote by \(e_A\) \((A, B, C = 1, 2)\) a pair of vector fields on \(U\) (restricting the size of \(U\) if necessary) which are linearly independent at every point and tangent to the foliation \(\{S^{RW}_r\}\), and define \(h_{AB} = g(e_A, e_B)|_U\), where \(g(\cdot, \cdot)\) represents the scalar product in the matched spacetime \((\mathcal{V}, g)\). To complete the basis of \(T_p \mathcal{V}\) for every \(q \in U\) we take the restriction of fluid velocity vector on \(U\), \(u|_U\), and the vector field \(s\), defined as the unit normal vector of \(S^{RW}_r\) which is tangent to \(\{t = \tau\}\) (and points inwards in \(W^{RW}\)). The vector \(s\) is thus spacelike and transverse to \(\Sigma^{RW}\) at non-critical values of \(\chi\), i.e., it is transverse to \(\Sigma^{RW}_{\tau_0}\) and thus \(n(s) \neq 0\) in \(U\). By construction, \(u|_U\) and \(s\) are mutually orthogonal and also orthogonal to \(e_A\). By the identification of \(\Sigma^{xx}\) and \(\Sigma^{RW}\) in \(\mathcal{V}\), the vector field \(\zeta\) at any \(q \in U\) can be expressed in the basis \(\{u|_U, s, e_A\}\) as

\[
\zeta|_U = F \cosh \beta u - F \sinh \beta \cos \alpha s + c^A e_A|_U,
\]

where \(\alpha, \beta, c^A\) are scalar functions on \(U\) and \(c^A c^B h_{AB} = F^2 \sin^2 \beta \sin^2 \alpha\).

\(^2\) For the sake of simplicity in the following expressions, vectors (and functions) \(v\) defined only on \(U\) will appear as either \(v\) or the redundant expression \(v|_U\).
Because of the preservation of the axial symmetry \( \mathbb{I} \), the restriction to \( \Sigma \) of \( \eta \) will have to be tangent to \( \Sigma \) and tangent to the restriction to \( \Sigma \) of a Killing vector field in \( (\mathcal{V}_{\text{RW}}, g^{\text{RW}}) \), say \( \eta_{\text{RW}} \), which in turn will be also tangent to the foliation \( \{ S_\tau \} \). This means \( \eta|_U = \eta_{\text{RW}}|_U = \eta^A e_A \), for some functions \( \eta^A \) defined on \( U \). The mutual orthogonality of \( \zeta \) and \( \eta \) demands that

\[
c^A \eta^B h_{AB} = 0. \tag{8}
\]

It can be easily checked that the following two vector fields defined on \( U \),

\[
v_A = [n(s - \tanh \beta \cos \alpha u)] e_A + \frac{h_{AB}c^B}{F \cosh \beta} \left[ n(s) u - n(u) s \right]_U,
\]

are tangent to \( \Sigma \) and orthogonal to \( \zeta \). From \( g(v_A, \eta)|_U = [n(s - \tanh \beta \cos \alpha u)] |_U h_{AB} \eta^B \) we see that the vector \( v \equiv c^A v_A \) on \( U \), apart from being tangent to \( \Sigma \) and orthogonal to \( \zeta|_U \) by construction, is also orthogonal to \( \eta|_U \) by virtue of \( \mathbb{S} \). Therefore, there exist two functions \( a^M \) on \( U \) such that \( v = a^M e_M|_U \), as follows from \( \mathbb{S} \).

The Riemann tensor in the FLRW region reads

\[
R^B_{\alpha \beta \mu \nu} = \frac{\theta + p}{2} \left[ u_{\alpha u_{\mu}} g^B_{\beta \nu} - u_{\alpha u_{\nu}} g^B_{\beta \mu} + u_{\beta u_{\mu}} g^B_{\alpha \nu} - u_{\beta u_{\nu}} g^B_{\alpha \mu} \right] + \frac{\theta}{3} \left[ u_{\mu u_{\nu}} g^B_{\beta \alpha} - g^B_{\alpha \nu} g^B_{\beta \mu} \right].
\]

Due to the orthogonal transitivity in the stationary and axisymmetric region, we have \( R^B_{\alpha \beta \mu \nu} \eta^C \eta^D E^\nu_M|_U = 0 \) for \( M = 2, 3 \). As a result, the contraction of \( \mathbb{S} \) with \( \zeta^C|_U, \eta^D|_U, \eta^B|_U \) and \( e^C \) leads to \( 0 = -(q + p)/2 g(u, \zeta) g(u, v_C) h_{AB} \eta^A \eta^B \eta^C|_U \), which by virtue of \( \mathbb{S} \) and \( \mathbb{I} \) can be expressed as

\[
0 = \frac{\theta + p}{2} n(s) h_{CD} c^C c^D h_{AB} \eta^A \eta^B|_U. \tag{10}
\]

Using \( \theta + p \neq 0 \), the fact that \( h_{AB} \) is positive definite and that \( \eta|_U = e_A e_A \) only vanishes at points in the axis, one has that \( e^A \) vanish on a dense subset of \( U \), hence \( e^A = 0 \) for \( A = 1, 2 \) by continuity. Therefore \( \mathbb{I} \) becomes

\[
\zeta|_U = F \cosh \beta u - F \sinh \beta s|_U, \tag{11}
\]

with a change in sign in \( \beta \) if necessary, and conclusion (1) follows.

Expression \( \mathbb{I} \) implies \( \zeta|_q \in N_q S^x_{\tau_0} \) for every \( q \in U \), and it can be reexpressed by \( \mathbb{I} \), using \( g^{\text{RW}} \) to lower the indices of \( u \) and \( s \), as

\[
f^2 \zeta|_q = F \cosh \beta u - F \sinh \beta s|_q \tag{12}
\]

for every \( q \in U \). It is now convenient to introduce the vector field \( \lambda \) on \( U \) defined as \( \lambda = e^{AB} h_{BC} \eta^C e_A \equiv \lambda^A e_A \), where \( e^{AB} = -e^{BA}, e^{12} = 1 \), which is tangent to the foliation \( \{ S^x_\tau \} \) and orthogonal to \( \eta|U \). Since it is also orthogonal to \( \zeta|_U \), then it will have the form \( \lambda = \lambda^M M|_U \) as seen from \( \Sigma^{x \xi} \). The components of the second fundamental form of \( S^x_{\tau_0} \) with respect to \( \zeta \), \( K^z_{S^x_{\tau_0}} \left( \zeta|_{S^x_{\tau_0}} \right)_{AB} \), which is symmetric, can be computed and used to obtain

\[
K^z_{S^x_{\tau_0}} \left( \zeta|_{S^x_{\tau_0}} \right)_{AB} \lambda^A \lambda^B|_x = K^z_{S^x_{\tau_0}} \left( \zeta|_{\psi^{x}(x)} \right)_{AB} \eta^A \eta^B|_x = 0,
\]

\[
K^z_{S^x_{\tau_0}} \left( \zeta|_{\psi^{x}(x)} \right)_{AB} \eta^A \lambda^B|_x = \frac{[g(\eta, \eta)]^2}{2W^2} \lambda^M M^P E^P_M \partial_{\alpha} \left( \frac{e^{2U}}{2W^2} \right)_{|_{\psi^{x}(x)}},
\]

for every \( x \in S^x_{\tau_0} \). On the other hand, for \( u \) one gets
Regarding $\mathbf{s}$, the crucial point here is that, because of the preservation of one isometry across $\Sigma$, the second fundamental form $K_{S_{\tau_0}}^{RW}(\mathbf{s}_{\psi_{\tau_0}^{RW}})$ is diagonal in the basis $\{\mathbf{\lambda}_{S_{\tau_0}^{RW}}, \mathbf{\eta}_{S_{\tau_0}^{RW}}\}$. Indeed, since $\{\mathbf{\lambda}_{S_{\tau_0}^{RW}}, \mathbf{\eta}_{S_{\tau_0}^{RW}}\}$ span the surfaces $S_{\tau_0}^{RW}$, orthogonal to $\mathbf{s}_{S_{\tau_0}^{RW}}$, and hence $g_{S_{\tau_0}^{RW}}^{RW}(\mathbf{s}, \mathbf{[\mathbf{\lambda}, \mathbf{\eta}]})|_{S_{\tau_0}^{RW}} = 0$, and due to the fact that $\mathbf{\eta}_{S_{\tau_0}^{RW}}$ is a Killing vector field in $(\mathcal{V}^{RW}, g^{RW})$, one also has $g^{RW}(\mathbf{[\mathbf{\lambda}, \mathbf{s}_{S_{\tau_0}^{RW}]}]) = 0$. Therefore, the following chain of identities hold:

$$K_{S_{\tau_0}}^{RW}(\mathbf{s}_{\psi_{\tau_0}^{RW}})_{AB} \lambda^a \eta^b |_x = \lambda^a \eta^b_{RW} \nabla^RW_{\alpha} s_{|\psi_{\tau_0}^{RW}(x)} =$$

$$= \frac{1}{2} \left( \eta^\alpha_{RW} S^\beta_{\alpha} \nabla^RW_{\alpha} \lambda^\beta - \lambda^\alpha s^\beta \nabla^RW_{\alpha} \eta^\beta_{RW} \right) |_{\psi_{\tau_0}^{RW}(x)} = \frac{1}{2} \lambda^\alpha \mathbf{s}_{\psi_{\tau_0}^{RW}} |_{\psi_{\tau_0}^{RW}(x)} = 0. \quad (14)$$

We are now ready to apply the constraint matching equations (15) to $\zeta_{S_{\tau_0}^{RW}}$ using (14). The non diagonal part of (15), as follows from the above, leads to the vanishing of (13) for every $x \in S_{\tau_0}^{RW}$, which implies $\lambda^\alpha \partial_\alpha \left( e^{2\alpha A/g(\mathbf{\eta}, \mathbf{\eta})} \right) |_{S_{\tau_0}^{RW}} = 0$. In short, the constraint matching conditions lead us to $K_{S_{\tau_0}}^{xx}(\zeta_{S_{\tau_0}^{RW}}) = 0$. By virtue of (14), one finally has $K_{S_{\tau_0}}^{RW}(\mathbf{u}_{S_{\tau_0}^{RW}}) - \tanh \beta K_{S_{\tau_0}}^{RW}(\mathbf{s}_{S_{\tau_0}^{RW}}) = 0$. Now, since $\dot{\alpha}$ is nowhere zero by assumption, $\beta$ is nowhere zero on $S_{\tau_0}^{RW}$, and therefore

$$K_{S_{\tau_0}}^{xx}(\mathbf{m}_{S_{\tau_0}^{RW}}) = \frac{\dot{\alpha}}{\alpha} \mathbf{m} (u - \coth \beta \mathbf{s}) |_{S_{\tau_0}^{RW}} h|_{\tau_0} \quad (15)$$

for every normal one-form $\mathbf{m}$ to $S_{\tau_0}^{RW}$, where $h|_{\tau_0}$ is the induced metric on $S_{\tau_0}^{RW}$. Equation (14), in particular, tells us that $S_{\tau_0}^{RW}$ is umbilical in FLRW, and hence in the resulting matched spacetime $(\mathcal{V}, g)$. At this point, the rest of the proof showing points (2) and (3) follows strictly the proof of Proposition 1 and its Corollary 1 in [7].

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References

1. H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselaers and E. Herlt Exact solutions of Einstein’s field equations. Second Edition, Cambridge University Press, Cambridge 2003
2. M. Mars and J.M.M. Senovilla, Mod. Phys. Lett. A13 (1998) 1509
3. R. Vera, Class. Quantum Grav. 20 (2003) 2785
4. M. Mars In preparation
5. M. Mars and J.M.M. Senovilla, Class. Quantum Grav. 10 (1993) 1865
6. M. Mars, Phys. Rev. D 57 (1998) 3389
7. M. Mars, Class. Quantum Grav. 18 (2001) 3645
8. M. Mars, J.M.M. Senovilla and R. Vera In preparation
9. B. Carter, Commun. Math. Phys. 17 (1970) 233
10. J.M. Bardeen, Ap. J. 162 (1970) 71
11. R. Vera, Class. Quantum Grav. 19 (2002) 5249