A sharp bound on RIC in generalized orthogonal matching pursuit

Wengu Chen and Huanmin Ge ‡

March 7, 2018

Abstract

Generalized orthogonal matching pursuit (gOMP) algorithm has received much attention in recent years as a natural extension of orthogonal matching pursuit. It is used to recover sparse signals in compressive sensing. In this paper, a new bound is obtained for the exact reconstruction of every $K$-sparse signal via the gOMP algorithm in the noiseless case. That is, if the restricted isometry constant (RIC) $\delta_{NK+1}$ of the sensing matrix $A$ satisfies

$$\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}},$$

then the gOMP can perfectly recover every $K$-sparse signal $x$ from $y = Ax$. Furthermore, the bound is proved to be sharp in the following sense. For any given positive integer $K$, we construct a matrix $A$ with the RIC

$$\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{N} + 1}}$$

such that the gOMP may fail to recover some $K$-sparse signal $x$. In the noise case, an extra condition on the minimum magnitude of the nonzero components of every $K$-sparse signal combining with the above bound on RIC of the sensing matrix $A$ is sufficient to recover the true support of every $K$-sparse signal by the gOMP.

---

* W. Chen is with Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China, e-mail: chenwg@iapcm.ac.cn.
† H. Ge is with Graduate School, China Academy of Engineering Physics, Beijing, 100088, China, e-mail: gehuanmin@163.com.
‡ This work was supported by the NSF of China (Nos.11271050, 11371183).
Keywords: Sensing matrix, Generalized orthogonal matching pursuit, Restricted isometry constant, Sparse signal.

1 Introduction

It is well known that compressive sensing acquires sparse signals at a rate greatly below Nyquist rate. It has attracted growing attention in recent years [1]-[7]. The main aim of compressive sensing is to reconstruct signal from inaccurate and incomplete measurements.

One consider the following compressive sensing model:

\[ y = Ax + e, \]

where \( y \in \mathbb{R}^m \) is a measurement vector, the matrix \( A \in \mathbb{R}^{m \times n} (m \ll n) \) is a sensing matrix, the vector \( x \in \mathbb{R}^n \) is an unknown \( K \)-sparse signal \((K \ll n)\) and \( e \in \mathbb{R}^m \) is a measurement error vector. The goal is to recover unknown signal \( x \) based on \( y \) and \( A \). In this paper, denote by \( A_i (i = 1, 2, \cdots, n) \) the \( i \)-th column of \( A \) and all columns of \( A \) are normalized, i.e., \( \| A_i \|_2 = 1 \) for \( i = 1, 2, \cdots, n \). Define the support of the vector \( x \) by \( T = \text{supp}(x) = \{ i | x_i \neq 0 \} \) and the size of its support with \( |T| = |\text{supp}(x)| \). For a signal \( x \), if \( |\text{supp}(x)| \leq K \), \( x \) is called \( K \)-sparse.

For the recovery of the \( K \)-sparse signal \( x \), the most intuitive approach is to solve the following optimization problem

\[
\min_x \| x \|_0 \quad \text{subject to} \ Ax - y \in \mathcal{B},
\]

where \( \| x \|_0 \) denotes the \( l_0 \) norm of \( x \), i.e., the number of nonzero coordinates, \( \mathcal{B} \) is a bounded error set, i.e., \( \mathcal{B} = \{ e \in \mathbb{R}^m \mid \| e \|_2 \leq \varepsilon \} \). Particularly, in the noiseless case, \( \mathcal{B} = \{ 0 \} \). Unfortunately, it is well-known that the above optimization problem is NP-hard. Therefore, researchers seek computationally efficient methods to approximate the sparse signal \( x \), such as \( l_1 \) minimization [8], \( l_p \) \((0 < p < 1)\) minimization [9], greedy algorithm [10] and so on.

To ensure that the \( K \)-sparse solution is unique, we shall need the restricted isometry property (RIP) introduced by Candès and Tao in [8]. A matrix \( A \) satisfies the restricted isometry property of order \( K \) if there exists a constant \( \delta_K \) such that

\[
(1 - \delta_K) \| x \|_2^2 \leq \| Ax \|_2^2 \leq (1 + \delta_K) \| x \|_2^2
\]
holds for all $K$-sparse signals $x$. And the smallest constant $\delta_K$ is called as the restricted isometry constant (RIC). Candès and Tao also proposed that if $\delta_{2K} < 1$, the above optimization problem has a unique $K$-sparse solution [3]. Candès showed that if $\delta_{2K} < \sqrt{2} - 1$ then the above optimization problem (1.1) is equivalent to the $l_1$ minimization problem in [2]. Up to now, there are many results improving the bound on the RIC such as [4], [7] and [11]-[13].

Recently, there is a family of iterative greedy algorithms which have attracted significant attention to recover sparse signals including orthogonal least square (OLS) [14], orthogonal matching pursuit (OMP) [15], generalized orthogonal matching pursuit (gOMP) [16], regularized orthogonal matching pursuit (ROMP) [17], orthogonal multi-matching pursuit (OMMP) [18], stagewise orthogonal matching pursuit (StOMP) [19], subspace pursuit (SP) [20] and compressive sampling matching pursuit (CoSaMP) [21].

Specifically, OMP algorithm is one of the most effective algorithm in sparse signals recovery due to its implementation simplicity and competitive recovery performance. In the noiseless case, many efforts have been made to find out sufficient conditions based on RIC for OMP to exactly reconstruct every $K$-sparse signal $x$ within K iterations. Davenport and Wakin demonstrated that OMP can recover exactly the $K$-sparse signal $x$ under $\delta_{K+1} < \frac{1}{3\sqrt{K}}$ [22]. Since then, there are many papers to improve the condition in [23]-[28]. Recently, Mo improved the sufficient condition to $\delta_{K+1} < \frac{1}{\sqrt{K+1}}$, and proved this condition is sharp [28]. In the presence of noise, Shen and Li proved that OMP can exactly recover the support of the $K$-sparse signal $x$ under $\delta_{K+1} < \frac{1}{\sqrt{K+3}}$ and some assumption on the minimum magnitude of the nonzero elements of $x$ in [29]. Later, these sufficient conditions on RIC upper bound and minimum magnitude of the nonzero elements of $K$-sparse signal $x$ have been improved in [30] and [31].

Wang, Kwon and Shim introduced generalized orthogonal matching pursuit [16], which is a natural extension of OMP. It is well known that OMP algorithm only selects one correct index at each iteration. However the gOMP algorithm selects $N$ ($N \geq 1$) indices which contain at least one correct index from the support of $x$ in each iteration. Therefore the number of iteration for the gOMP algorithm is much smaller comparing with OMP algorithm. Wang, Kwon and Shim obtained that a sufficient condition

$$\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K} + 3\sqrt{N}}$$
can ensure the reconstruction of any $K$-sparse signals [16]. Later, Satpathi et al. improved the sufficient condition to $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K + 2} \sqrt{N}}$ in [32]. They also refined the bound further to
\[
\delta_{NK+1} < \frac{\sqrt{N}}{\sqrt{K+1} + \sqrt{N}},
\]
which is $\delta_{K+1} < \frac{1}{\sqrt{K+1}}$ of OMP in [24] and [25] for $N = 1$.

Motivated by the mentioned papers, we further investigate the recovery of any $K$-sparse signals by the gOMP. In this paper, we demonstrate that the condition
\[
\delta_{NK+1} < \frac{1}{\sqrt{N} + 1}
\]
is sufficient to perfectly reconstruct any $K$-sparse signals via the gOMP in the noiseless case. As $N = 1$, the sufficient condition is $\delta_{K+1} < \frac{1}{\sqrt{K+1}}$ which is a sharp bound for OMP [28]. Moreover, for any given $K \in \mathbb{N}^+$, we construct a matrix $A$ satisfying
\[
\delta_{NK+1} = \frac{1}{\sqrt{N} + 1}
\]
such that the gOMP may fail to recover some $K$-sparse signal $x$. That is, the above bound $\delta_{NK+1} < \frac{1}{\sqrt{N} + 1}$ is sharp for the gOMP. In noise case, we also show $\delta_{NK+1} < \frac{1}{\sqrt{N} + 1}$ together with a minimum magnitude of the nonzero elements of the $K$-sparse signal $x$ can ensure the reconstruction of the support of $x$ via the gOMP.

The frame of the gOMP is listed in the table 1.

**TABLE 1**

The gOMP algorithm

| Input | measurements $y \in \mathbb{R}^m$, sensing matrix $A \in \mathbb{R}^{m \times n}$, sparse level $K$, number of indices for each selection $N$ ($N \leq K$ and $N \leq \frac{m}{K}$). |
|-------|--------------------------------------------------------------------------------------------------|
| Initialize | iteration count $k = 0$, residual vector $r^0 = y$, estimated support set $\Lambda^0 = \emptyset$. |
| While | $\|r^k\|_2 > \epsilon$ and $k < \min\{K, \frac{m}{K}\}$ do $k = k + 1$. |
| (Identification step) | Select indices set $T^k$ corresponding to $N$ largest (in magnitude) in $A' r^{k-1}$. |
| (Augmentation step) | $\Lambda^k = \Lambda^{k-1} \cup T^k$. |
| (Estimation step) | $\hat{x}_{\Lambda^k} = \arg\min_u \|y - A_{\Lambda^k} u\|_2$. |
| (Residual Update step) | $r^{k} = y - A_{\Lambda^k} \hat{x}_{\Lambda^k}$. |

End
Output  the estimated signal $\hat{x} = \arg \min_{u: \text{supp}(u) = \Lambda^k} \|y - Au\|_2$

The rest of the paper is organized as follows. In Section 2, we give some notations and prove some basic lemmas that will be used. The main results and their proofs are given in Section 3.

2 Notations and preliminaries

Throughout this paper, let $\Gamma$ be an index set and $\Gamma^c$ be the complementary set of $\Gamma$. The standard notation $\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$ denotes the $l_\infty$-norm of the vector $x \in \mathbb{R}^n$. $x_\Gamma \in \mathbb{R}^{|\Gamma|}$ denotes the vector composed of components of $x \in \mathbb{R}^n$ indexed by $i \in \Gamma$, i.e., $(x_\Gamma)_i = x_i$ ($i \in \Gamma$). Define $\tilde{x}_\Gamma \in \mathbb{R}^n$ by

$$(\tilde{x}_\Gamma)_i = \begin{cases} x_i, & i \in \Gamma; \\ 0, & \text{others,} \end{cases}$$

where $i = 1, 2, \ldots, n$. Denote by $A_\Gamma$ a submatrix of $A$ corresponding to $\Gamma$ which consists of all columns with index $i \in \Gamma$ of $A$ and the usual inner product of $\mathbb{R}^n$ with $\langle \cdot, \cdot \rangle$. Let $e_i \in \mathbb{R}^n$ be the $i$-th coordinate unit vector.

Let $\alpha_{N}^{k+1}$ be the $N$-th largest correlation in magnitude between $r^k$ and $A_i$ ($i \in (T \cup \Lambda^k)^c$) and $\beta_{1}^{k+1}$ be the largest correlation in magnitude between $r^k$ and $A_i$ ($i \in (T - \Lambda^k)$) in the $(k + 1)$-th iteration of the gOMP algorithm. Let $W_{k+1} \subseteq (T \cup \Lambda^k)^c$ be the set of $N$ indices which correspond to $N$ largest correlation in magnitude between $r^k$ and $A_i$ ($i \in (T \cup \Lambda^k)^c$).

$A_{\Lambda^k}^\dagger$ represents the pseudo-inverse of $A_{\Lambda^k}$ when $A_{\Lambda^k}$ is full column rank ($|\Lambda^k| \leq m$), i.e., $A_{\Lambda^k}^\dagger = (A_{\Lambda^k}' A_{\Lambda^k})^{-1} A_{\Lambda^k}'$. Moreover, $P_{\Lambda^k} = A_{\Lambda^k} A_{\Lambda^k}^\dagger$ and $P_{\Lambda^k}^\perp = I - P_{\Lambda^k}$ denote two orthogonal projection operators which project a given vector orthogonally onto the spanned space by all columns of $A_{\Lambda^k}$ and onto its orthogonal complement respectively.

First, we recall the following lemma, that is, the monotonicity of the restricted isometry constant in [8], [20].

**Lemma 2.1.** For any $K_1 \leq K_2$, if the sensing matrix $A$ satisfies the RIP of order $K_2$, then $\delta_{K_1} \leq \delta_{K_2}$.

Next, we show the main lemma that is very useful during our analysis.
Lemma 2.2. For any $S, C > 0$, let $t = \pm \sqrt{\frac{S + 1}{S} - 1}$ and

$$
t_i = \begin{cases} 
-\frac{C}{2}(1 - t^2), & \langle Ax, Ae_i \rangle \geq 0; \\
+\frac{C}{2}(1 - t^2), & \langle Ax, Ae_i \rangle < 0,
\end{cases}
$$

where $i \in W \subseteq \{1, 2, \cdots, n\}$ that is a nonempty subset. Then we have $t^2 < 1$ and

$$
\|A(x + \sum_{i \in W} t_i e_i)\|^2 - \|A(t^2 x - \sum_{i \in W} t_i e_i)\|^2 = (1 - t^4) \left( \langle Ax, Ax \rangle - C \sum_{i \in W} |\langle Ax, Ae_i \rangle| \right).
$$

Proof. For $t = \pm \sqrt{\frac{S + 1}{S} - 1}$, we have that

$$
t^2 = \left(\frac{\sqrt{S + 1} - 1}{S}\right)^2 = \frac{\sqrt{S + 1} - 1}{\sqrt{S + 1} + 1} < 1.
$$

The result in the lemma is established by the following chain of equalities and the definition of $t_i$ ($i \in W$):

$$
\begin{align*}
\|A(x + \sum_{i \in W} t_i e_i)\|^2 - \|A(t^2 x - \sum_{i \in W} t_i e_i)\|^2 & = \langle Ax, Ax \rangle + 2 \sum_{i \in W} t_i \langle Ax, Ae_i \rangle + 2 \sum_{i,j \in W, i \neq j} t_i t_j \langle Ae_i, Ae_j \rangle + \sum_{i \in W} t_i^2 \langle Ae_i, Ae_i \rangle \\
& - \left( t^4 \langle Ax, Ax \rangle - 2t^2 \sum_{i \in W} t_i \langle Ax, Ae_i \rangle + 2 \sum_{i,j \in W, i \neq j} t_i t_j \langle Ae_i, Ae_j \rangle + \sum_{i \in W} t_i^2 \langle Ae_i, Ae_i \rangle \right) \\
& = (1 - t^4) \langle Ax, Ax \rangle + 2(1 + t^2) \sum_{i \in W} t_i \langle Ax, Ae_i \rangle \\
& = (1 - t^4) \left( \langle Ax, Ax \rangle - \frac{2}{1 - t^2} \sum_{i \in W} |t_i||\langle Ax, Ae_i \rangle| \right) \\
& = (1 - t^4) \left( \langle Ax, Ax \rangle - \frac{2}{1 - t^2}(1 - t^2)C \sum_{i \in W} |\langle Ax, Ae_i \rangle| \right) \\
& = (1 - t^4) \left( \langle Ax, Ax \rangle - C \sum_{i \in W} |\langle Ax, Ae_i \rangle| \right).
\end{align*}
$$

We have already completed the proof of the Lemma 2.2. \qed

Remark 1. The Lemma 2.2 is a generalization of Lemma II.1 in [28]. The main idea of its proof is from the idea of Lemma II.1.

Remark 2. If $x$ is replaced by $cx$ with non-zero scalar $c$, the results of Lemma 2.2 keep unchanged.
3 Main results

3.1 Noiseless case

It is well known that if at least one index of \( N \) indices selected is correct in every iteration, the gOMP makes a success, i.e., in each iteration, there exists \( \beta^k_1 > \alpha^k_N \) (1 \( \leq k \leq K \)). The following theorems show a sufficient condition guarantees the gOMP algorithm success. The proof of these theorems mainly uses Lemmas 2.1 and 2.2. By Remark 2 we assume \( \|x\|_2 = 1 \) in the proof of Theorem 3.1 and \( \|\tilde{\omega}_{T \cup \Lambda^k}\|_2 = 1 \) in the proof of Theorem 3.2.

**Theorem 3.1.** Suppose \( x \) is a \( K \)-sparse signal and the restricted isometry constant \( \delta_{K+N} \) of the sensing matrix \( A \) satisfies

\[
\delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N}} + 1},
\]

Then the gOMP algorithm makes a success in the first iteration.

**Remark 3.** In [16], authors proved that

\[
\delta_{K+N} < \frac{\sqrt{N}}{\sqrt{K} + \sqrt{N}}
\]

is sufficient to make a success in the first iteration of the gOMP. It is clear that

\[
\delta_{K+N} < \frac{\sqrt{N}}{\sqrt{K} + \sqrt{N}} < \frac{1}{\sqrt{\frac{K}{N}} + 1},
\]

i.e., the sufficient condition (3.1) is weaker than that in [16].

**Proof.** In the first iteration, by the definition of \( \alpha^1_N \), it satisfies

\[
\alpha^1_N = \min \{\|\langle Ae_i, Ax \rangle\|_2 \mid i \in W_1\} \leq \frac{\sum_{i \in W_1} |\langle Ae_i, Ax \rangle|}{N},
\]

(3.2)

where \( W_1 \subseteq T^c \).
For $\beta_1$ which is the largest correlation in magnitude in $A'_T Ax$, we have

\[
\langle Ax, Ax \rangle = \langle A \sum_{i \in T} x_i e_i, Ax \rangle \\
= \sum_{i \in T} x_i \langle A e_i, Ax \rangle \\
\leq \sum_{i \in T} |x_i| |\langle A e_i, Ax \rangle| \\
\leq \beta_1 \| x \|_1 \\
\leq \beta_1 \sqrt{K} \| x \|_2 \\
\leq \beta_1 \sqrt{K}.
\] (3.3)

Let $t = -\frac{\sqrt{K} + 1 - 1}{\sqrt{K}}$ and

\[
t_i = \begin{cases} 
-\frac{\sqrt{K}}{2N}(1 - t^2), & \langle Ax, A e_i \rangle \geq 0; \\
+\frac{\sqrt{K}}{2N}(1 - t^2), & \langle Ax, A e_i \rangle < 0,
\end{cases}
\]

where $i \in W_1 \subseteq T^c$ with $|W_1| = N$, then we have that

\[
t^2 = \frac{\sqrt{K} + 1 - 1}{\sqrt{K} + 1 + 1} < 1
\]

and

\[
\sum_{i \in W_1} t_i^2 = \left( \frac{\sqrt{K}}{2N}(1 - t^2) \right)^2 N \\
= \frac{K}{4N} (1 - t^2)^2 \\
= \frac{K}{4N} \left( 1 - \frac{\sqrt{K} + 1 - 1}{\sqrt{K} + 1 + 1} \right)^2 \\
= \frac{K}{N} \left( \frac{1}{\sqrt{K} + 1 + 1} \right)^2 \\
= \frac{\sqrt{K} + 1 - 1}{\sqrt{K} + 1 + 1} = t^2.
\] (3.4)

By (3.2), (3.3) and Lemma 2.2, we obtain

\[
(1 - t^4) \sqrt{K} (\beta_1 - \alpha_N) \geq (1 - t^4) \left( \langle Ax, Ax \rangle - \sqrt{K} \sum_{i \in W_1} |\langle A e_i, Ax \rangle| \right) \\
= \| A(x + \sum_{i \in W_1} t_i e_i) \|_2^2 - \| A(t^2 x - \sum_{i \in W_1} t_i e_i) \|_2^2.
\] (3.5)
Because the sensing matrix $A$ satisfies the RIP of order $K+N$ with $\delta_{K+N}$, $\|x\|_2 = 1$ with $\text{supp}(x) \subseteq T$, $W_1 \subseteq T^c$, it follows from (3.4) that

$$
\|A(x + \sum_{i \in W_1} t_i e_i)\|_2^2 - \|A(t^2 x - \sum_{i \in W_1} t_i e_i)\|_2^2 \\
\geq (1 - \delta_{K+N}) \left( \|x + \sum_{i \in W_1} t_i e_i\|_2^2 \right) - (1 + \delta_{K+N}) \left( \|t^2 x - \sum_{i \in W_1} t_i e_i\|_2^2 \right) \\
= (1 - \delta_{K+N}) (\|x\|_2^2 + \sum_{i \in W_1} t_i^2) - (1 + \delta_{K+N}) (t^4 \|x\|_2^2 + \sum_{i \in W_1} t_i^2) \\
= (1 - \delta_{K+N}) (1 + t^2) - (1 + \delta_{K+N})(t^4 + t^2) \\
= (1 - t^4) - \delta_{K+N}(1 + t^2)^2 \\
= (1 + t^2)^2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{K+N} \right).
$$

It follows from the definition of $t$ that

$$
\frac{1 - t^2}{1 + t^2} = \frac{1 - \frac{\sqrt{K} + 1 - 1}{\sqrt{K} + 1 + 1}}{1 + \frac{\sqrt{K} + 1 - 1}{\sqrt{K} + 1 + 1}} = \frac{1}{\sqrt{\frac{K}{N} + 1}}.
$$

Therefore by the condition $\delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$, we obtain

$$
(1 - t^4) \sqrt{K}(\beta_1^1 - \alpha_N^1) \geq (1 + t^2)^2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{K+N} \right) \\
\geq (1 + t^2)^2 \left( \frac{1}{\sqrt{\frac{K}{N} + 1}} - \delta_{K+N} \right) \\
> 0,
$$

i.e., $\beta_1^1 > \alpha_N^1$ which represents the gOMP selects at least one index from the support $T$.

As mentioned, if $\delta_{K+N} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$, then the gOMP algorithm makes a success in the first iteration.

**Theorem 3.2.** If the gOMP algorithm has performed $k$ iterations successfully, where $1 \leq k < K$. And the sensing matrix $A$ satisfies the RIP of order $NK + 1$ with RIC $\delta_{NK+1}$
fulfilling
\[ \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}. \]

Then in the \((k+1)\)-th iteration, the gOMP will make a success.

**Proof.** For the gOMP algorithm, \( r^k = P_{A^k}^\perp y \) is orthogonal to each column of \( A_{\Lambda^k} \) then

\[
\begin{align*}
r^k &= P_{A^k}^\perp y \\
    &= P_{A^k}^\perp A_T x_T \\
    &= P_{A^k}^\perp (A_{T - \Lambda^k} x_{T - \Lambda^k} + A_{T \cap \Lambda^k} x_{T \cap \Lambda^k}) \\
    &= P_{A^k}^\perp A_{T - \Lambda^k} x_{T - \Lambda^k} \\
    &= A_{T - \Lambda^k} x_{T - \Lambda^k} - P_{A^k} A_{T - \Lambda^k} x_{T - \Lambda^k} \\
    &= A_{T - \Lambda^k} x_{T - \Lambda^k} - A_{\Lambda^k} z_{\Lambda^k} \\
    &= A_{T \cup A^k} \omega_{T \cup A^k},
\end{align*}
\]

where we used the fact that \( P_{A^k} A_{T - \Lambda^k} x_{T - \Lambda^k} \in \text{span}(A_{\Lambda^k}) \), so \( P_{A^k} A_{T - \Lambda^k} x_{T - \Lambda^k} \) can be written as \( A_{\Lambda^k} z_{\Lambda^k} \) for some \( z_{\Lambda^k} \in \mathbb{R}^{\Lambda^k} \) and \( \omega_{T \cup A^k} \) is given by

\[ \omega_{T \cup A^k} = \begin{pmatrix} x_{T - \Lambda^k} \\ -z_{\Lambda^k} \end{pmatrix}. \]

By the definition of \( \alpha_{N+1}^{k+1} \) and \( \beta_{1}^{k+1} \), we have that

\[
\begin{align*}
\alpha_{N+1}^{k+1} &= \min \{ \langle A e_i, r^k \rangle : i \in W_{k+1}, W_{k+1} \subseteq (T \cup \Lambda^k)^c \} \\
    &\leq \frac{\sum_{i \in W_{k+1}} \langle A e_i, r^k \rangle}{N} \\
    &= \frac{\sum_{i \in W_{k+1}} \langle A e_i, A_{T \cup A^k} \omega_{T \cup A^k} \rangle}{N} \\
    &= \frac{\sum_{i \in W_{k+1}} \langle A e_i, A_{\tilde{\omega}_{T \cup A^k}} \rangle}{N}. \tag{3.6}
\end{align*}
\]

and

\[
\begin{align*}
\beta_{1}^{k+1} &= \| A_{T - \Lambda^k}^{\perp} r^k \|_\infty \\
    &= \| [A_{T - \Lambda^k} A_{T \cap \Lambda^k}^\perp A_{T \cup A^k} \omega_{T \cup A^k}] \|_\infty \\
    &= \| A_{T - \Lambda^k}^{\perp} A_{T \cup A^k} \omega_{T \cup A^k} \|_\infty \\
    &= \| A_{T - \Lambda^k}^{\perp} A_{T \cup A^k} \omega_{T \cup A^k} \|_\infty \\
    &= \| A_{T \cup A^k}^{\perp} A_{T \cup A^k} \omega_{T \cup A^k} \|_\infty. \tag{3.7}
\end{align*}
\]
Notice the fact that
\[
\|A_T^*A_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k\|_\infty \\
\geq \frac{1}{\sqrt{K}}\|A_T^*A_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k\|_2 \\
= \frac{1}{\sqrt{K}}\|A_{T \cup \Lambda}^kA_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k\|_2.
\] (3.8)

By the hypothesis of \(\|\omega_{T \cup \Lambda}^k\|_2 = 1,\) (3.7) and (3.8), it follows that
\[
\langle A\tilde{\omega}_{T \cup \Lambda}^k, A\tilde{\omega}_{T \cup \Lambda}^k \rangle = \langle A_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k, A_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k \rangle \\
= \langle A_T^*A_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k, \omega_{T \cup \Lambda}^k \rangle \\
\leq \|A_{T \cup \Lambda}^kA_{T \cup \Lambda}^k\omega_{T \cup \Lambda}^k\|_2\|\omega_{T \cup \Lambda}^k\|_2 \\
\leq \sqrt{K}\beta_1^{k+1}\|\omega_{T \cup \Lambda}^k\|_2 \\
= \sqrt{K}\beta_1^{k+1}.
\] (3.9)

As in the proof of Theorem 3.1, let \(t = -\sqrt{\frac{K}{N}}(1 - t^2)\) and
\[
ts_i = \begin{cases} 
-\frac{\sqrt{K}}{2N}(1 - t^2), & \langle Ax, Ae_i \rangle \geq 0; \\
+\frac{\sqrt{K}}{2N}(1 - t^2), & \langle Ax, Ae_i \rangle < 0,
\end{cases}
\]
where \(i \in W_{k+1} \subseteq (\Lambda^k \cup T)^c.\) By (3.6), (3.9) and Lemma 2.2, we obtain
\[
(1 - t^4)\sqrt{K}(\beta_1^{k+1} - \alpha_N^{k+1}) \\
\geq (1 - t^4)\left(\langle A\tilde{\omega}_{T \cup \Lambda}^k, A\tilde{\omega}_{T \cup \Lambda}^k \rangle - \sqrt{K}\sum_{i \in W_{k+1}}|\langle Ae_i, A\tilde{\omega}_{T \cup \Lambda}^k \rangle|\right) \\
= \|A(\tilde{\omega}_{T \cup \Lambda}^k + \sum_{i \in W_{k+1}} t_ie_i)\|_2^2 - \|A(t^2\tilde{\omega}_{T \cup \Lambda}^k - \sum_{i \in W_{k+1}} t_ie_i)\|_2^2.
\] (3.10)

Let \(l = |T \cap \Lambda^k|,\) then \(k \leq l \leq K\) and \(Nk + K - l + N \leq NK + 1.\) Since \(A\) satisfies RIP of order \(NK + 1\) with \(\delta_{NK+1},\) \(\|\tilde{\omega}_{T \cup \Lambda}^k\|_2 = 1\) with \(\text{supp}(\tilde{\omega}_{T \cup \Lambda}^k) \subseteq T \cup T_{\Lambda}^k, W_{k+1} \subseteq (T \cup \Lambda^k)^c,\)
it follows from Lemma 2.1 that

\[ \| A(\tilde{\omega}_{T \cup \Lambda} + \sum_{i \in W_{k+1}} t_i e_i) \|_2^2 - \| A(t^2 \tilde{\omega}_{T \cup \Lambda} - \sum_{i \in W_{k+1}} t_i e_i) \|_2^2 \]

\[ \geq (1 - \delta_{N+K-1+L+N}) \left( \| \tilde{\omega}_{T \cup \Lambda} + \sum_{i \in W_{k+1}} t_i e_i \|_2^2 \right) \]

\[ - (1 + \delta_{N+K-1+L+N}) \left( \| t^2 \tilde{\omega}_{T \cup \Lambda} - \sum_{i \in W_{k+1}} t_i e_i \|_2^2 \right) \]

\[ = (1 - \delta_{N+K-1+L+N}) \left( \| \tilde{\omega}_{T \cup \Lambda} \|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right) \]

\[ - (1 + \delta_{N+K-1+L+N}) \left( t^4 \tilde{\omega}_{T \cup \Lambda} \|_2^2 + \sum_{i \in W_{k+1}} t_i^2 \right) \]

\[ = (1 - \delta_{N+K-1+L+N})(1 + t^2) - (1 + \delta_{N+K-1+L+N})(t^4 + t^2) \]

\[ = (1 + t^2)^2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{N+K-1+L+N} \right) \]

\[ \geq (1 + t^2)^2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) . \]

Since

\[ \frac{1 - t^2}{1 + t^2} = \frac{1}{\sqrt{\frac{K}{N} + 1}} \]

and the condition \( \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}} \), we obtain

\[ (1 - t^4) \sqrt{K}(\beta_{1}^{k+1} - \alpha_{N}^{k+1}) \geq (1 + t^2)^2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) \]

\[ \geq (1 + t^2)^2 \left( \frac{1}{\sqrt{\frac{K}{N} + 1}} - \delta_{NK+1} \right) \]

\[ > 0, \]

i.e., \( \beta_{1}^{k+1} > \alpha_{N}^{k+1} \) which ensures that the set \( \Lambda^{k+1} \) contains at least one correct index in the \((k + 1)\)-th iteration of the gOMP algorithm.

As mentioned, we have completed the proof of the theorem.

Now combining the condition for success in the first iteration in Theorem 3.1 with that in non-initial iterations in Theorem 3.2, we obtain overall sufficient condition of the gOMP algorithm guaranteeing the perfect recovery of \( K \)-sparse signals in the following theorem.
Theorem 3.3. Suppose $x$ is a $K$-sparse signal and the sensing matrix $A$ satisfies RIP of order $KN + 1$ with the RIC $\delta_{NK+1}$ fulfilling

$$\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}.$$ 

Then the gOMP algorithm can recover the signal $x$ exactly.

Proof. For $N \geq 1$, $K \geq 1$ and $N \leq \min\{K, \frac{m}{N}\}$, then $K + N \leq NK + 1$. It follows Lemma 2.1 that

$$\delta_{K+N} \leq \delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}.$$ 

By Theorems 3.1 and 3.2, the gOMP algorithm can recover perfectly any $K$-sparse signals under the sufficient condition $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$ from $y = Ax$. \qed

Remark 4. The condition $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$ is weaker than the sufficient condition $\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$ in [30].

Remark 5. If $N = 1$, this sufficient condition is consistent with the sharp condition $\delta_{K+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}$ of OMP in [28].

In the following theorem, we show that the proposed bound $\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{N} + 1}}$ is optimal.

Theorem 3.4. For any given $K \in \mathbb{N}^+$, there are a $K$-sparse signal $x$ and a matrix $A$ satisfying

$$\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{N} + 1}}$$

such that the gOMP may fail.
Proof. For any given positive integer $K$, let $A \in \mathbb{R}^{(NK+1) \times (NK+1)}$ be

$$A = \begin{pmatrix}
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\sqrt{\frac{K}{K+N}} I_K & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\frac{1}{1} & \cdots & \frac{1}{1} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0
\end{pmatrix},$$

where $b = \sqrt{K(K + N)}$. Then we have that

$$A'^{T} A = \begin{pmatrix}
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\frac{K}{K+N} I_K & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \frac{1}{K+N} & \cdots & \frac{1}{K+N} \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0 \\
\frac{1}{K+N} & \cdots & \frac{1}{K+N} & 0 & \cdots & 0
\end{pmatrix},$$

Moreover, by direct calculation, we obtain that

$$|A'^{T} A - \lambda I| = (1 - \lambda)^{NK-K} \left( \frac{K}{K+N} - \lambda \right)^{K-1} \left( \lambda^2 - 2\lambda + \frac{K}{K+N} \right).$$

It is clear that $\frac{K}{K+N}$ and 1 are eigenvalue of $A'^{T} A$ with multiplicity of $K - 1$ and $NK - K$ respectively. $1 \pm \frac{1}{\sqrt{K+N+1}}$ also are eigenvalue of $A'^{T} A$. Therefore we have

$$\delta_{NK+1} = \frac{1}{\sqrt{\frac{K}{K+N} + 1}}.$$ 

Consider $K$-sparse signal $x = (1, 1, \cdots, 1, 0 \cdots, 0)' \in \mathbb{R}^{NK+1}$, i.e., $T = \text{supp}(x) = \{1, 2, \cdots, K\}$. As $i \in T$, we have

$$|\langle Ae_i, y \rangle| = |\langle Ae_i, Ax \rangle| = |\langle A'^{T} A e_i, x \rangle| = \frac{K}{K+N}.$$
For $i \in \{K + 1, \ldots, NK + 1 - N\}$, it follows immediately that

$$|\langle Ae_i, y \rangle| = |\langle Ae_i, Ax \rangle| = |\langle A'Ae_i, x \rangle| = 0$$

If $i \in \{NK + 2 - N, \ldots, NK + 1\}$, we have

$$|\langle Ae_i, y \rangle| = |\langle Ae_i, Ax \rangle| = |\langle A'Ae_i, x \rangle| = \frac{K}{K + N}.$$ 

Therefore, we have $\beta_1^1 = \frac{K}{K + N}$ and $\alpha_N^1 = \frac{K}{K + N}$ by the definitions of $\beta_1^1$ and $\alpha_N^1$, that is, $\beta_1^1 = \alpha_N^1$. This implies the gOMP may fail to identify at least one correct index in the first iteration. So the gOMP algorithm may fail for the given matrix $A$ and the $K$-sparse signal $x$.

\[\square\]

### 3.2 Noise case

In the subsection, we show a sufficient condition guarantees exact support identification by the gOMP algorithm from $y = Ax + e$. This sufficient condition is in terms of the RIC $\delta_{NK+1}$ and the minimum magnitude of the nonzero entries of $K$-sparse signal $x$. Here, we only consider $l_2$ bounded noise, i.e., $\|e\|_2 \leq \varepsilon$.

**Theorem 3.5.** Suppose $\|e\|_2 \leq \varepsilon$ and the sensing matrix $A$ satisfies

$$\delta_{NK+1} < \frac{1}{\sqrt{\frac{K}{N} + 1}}. \tag{3.11}$$

Moreover, assume all the nonzero components $x_i$ satisfy

$$|x_i| > \frac{\frac{2\sqrt{K}\varepsilon}{\sqrt{\frac{K}{N} + 1}}}{\frac{1}{\sqrt{\frac{K}{N} + 1}} - \delta_{NK+1}}. \tag{3.12}$$

Then the gOMP algorithm with the stopping rule $\|r_k\|_2 \leq \varepsilon$ recovers the correct support of any $K$-sparse signals $x$.

**Proof.** Use mathematical induction method to prove the theorem. Suppose the gOMP performed $k$ iterations successfully. Consider the $(k + 1)$-th iteration. Firstly, we observe that

$$r_k = P_{A_k}^1 y = P_{A_k}^1 A_T x_T + P_{A_k}^1 e = A_{T \cup A_k} \omega_{T \cup A} + (I - P_{A_k}) e$$

15
for some $\omega_{T \cup \Lambda^k}$ as in the proof of Theorem 3.2. Consider the following two cases to prove the theorem.

- Case 1: $T - \Lambda_k = \emptyset$

In this case, there is $T \subseteq \Lambda_k$. Then the correct support $T$ of the original $K$-sparse signal $x$ has already been recovered.

- Case 2: $T - \Lambda_k \neq \emptyset$, i.e., $|T - \Lambda_k| \geq 1$

By the definitions of $\alpha_{N}^{k+1}$ and $\beta_{1}^{k+1}$, we obtain that

$$
\alpha_{N}^{k+1} = \min \{ \langle Ae_i, r^k \rangle : i \in W_{k+1}, W_{k+1} \subseteq (T \cup \Lambda^k)^c \} \\
\leq \frac{\sum_{i \in W_{k+1}} |\langle Ae_i, r^k \rangle|}{N} \\
\leq \frac{\sum_{i \in W_{k+1}} |\langle Ae_i, A_{T \cup \Lambda^k} \omega_{T \cup \Lambda^k} \rangle| + \sum_{i \in W_{k+1}} |\langle Ae_i, (I - P_{\Lambda^k}) e \rangle|}{N} \\
= \frac{\sum_{i \in W_{k+1}} |\langle Ae_i, A_{T \cup \Lambda^k} \omega_{T \cup \Lambda^k} \rangle| + \sum_{i \in W_{k+1}} |\langle Ae_i, (I - P_{\Lambda^k}) e \rangle|}{N}
$$

and

$$
\beta_{1}^{k+1} = \| A_{T - \Lambda^k} r^k \|_{\infty} \\
= \| A_{T} r^k \|_{\infty} \\
= \| A_{T \cup \Lambda^k} r^k \|_{\infty} \\
\geq \| A_{T \cup \Lambda^k} A_{T \cup \Lambda^k} \omega_{T \cup \Lambda^k} \|_{\infty} - \| A_{T \cup \Lambda^k} (I - P_{\Lambda^k}) e \|_{\infty}.
$$

Let $t = -\frac{\sqrt{2N}}{\sqrt{K} + 1}$ and

$$
t_i = \begin{cases} 
-\frac{\sqrt{K}}{2N} (1 - t^2) \| \omega_{T \cup \Lambda^k} \|_2, & \langle Ax, Ae_i \rangle \geq 0; \\
+\frac{\sqrt{K}}{2N} (1 - t^2) \| \omega_{T \cup \Lambda^k} \|_2, & \langle Ax, Ae_i \rangle < 0,
\end{cases}
$$

where $i \in W_{k+1} \subseteq (\Lambda^k \cup T)^c$. Then we have

$$
\sum_{i \in W_{k+1}} t_i^2 = t^2 \| \omega_{T \cup \Lambda^k} \|_2^2.
$$
It follows from (3.8), (3.9), (3.13) and (3.14) that

\[(1 - t^4)\sqrt{K}\|w_{T\cup A^k}\|_2(\beta_N^{k+1} - \alpha_N^{k+1})\]
\[\geq (1 - t^4)\left(\langle A\tilde{w}_{T\cup A^k}, A\tilde{w}_{T\cup A^k}\rangle - \sqrt{K}\|w_{T\cup A^k}\|_2\|A'_{T\cup A^k}(I - P_{\Lambda^k})e\|_\infty\right.\]
\[\left. - \sqrt{K}\|w_{T\cup A^k}\|_2(\sum_{i \in W_{k+1}}|\langle Ae_i, A\tilde{w}_{T\cup A^k}\rangle| + \sum_{i \in W_{k+1}}|\langle Ae_i, (I - P_{\Lambda^k})e\rangle|)\right)\]
\[= A(\tilde{w}_{T\cup A^k} + \sum_{i \in W_{k+1}}t_ie_i)\|_2^2 - A(t^2\tilde{w}_{T\cup A^k} - \sum_{i \in W_{k+1}}t_ie_i)\|_2^2\]
\[-(1 - t^4)\sqrt{K}\|w_{T\cup A^k}\|_2\left(\|A'_{T\cup A^k}(I - P_{\Lambda^k})e\|_\infty + \frac{\sum_{i \in W_{k+1}}|\langle Ae_i, (I - P_{\Lambda^k})e\rangle|}{N}\right).\]

As in the proof of Theorem 3.2, \(l = |T \cap \Lambda^k|\) then \(Nk + K - l + N \leq NK + 1\). Because \(A\) satisfies RIP of order \(NK + 1\) with \(\delta_{NK+1}, supp(\tilde{w}_{T\cup A^k}) \subseteq T \cup \Lambda^k, W_{k+1} \subseteq (T \cup \Lambda^k)^c\), it follows from (3.15) and Lemma 2.1 that

\[\|A(\tilde{w}_{T\cup A^k} + \sum_{i \in W_{k+1}}t_ie_i)\|_2^2 - A(t^2\tilde{w}_{T\cup A^k} - \sum_{i \in W_{k+1}}t_ie_i)\|_2^2\]
\[\geq (1 - \delta_{NK+K-l+N})\left(\|\tilde{w}_{T\cup A^k} + \sum_{i \in W_{k+1}}t_ie_i\|_2^2\right)\]
\[-(1 + \delta_{NK+K-l+N})\left(t^2\tilde{w}_{T\cup A^k} - \sum_{i \in W_{k+1}}t_ie_i\|_2^2\right)\]
\[= (1 - \delta_{NK+K-l+N})\left(\|\tilde{w}_{T\cup A^k}\|_2^2 + \sum_{i \in W_{k+1}}t_i^2\right)\]
\[-(1 + \delta_{NK+K-l+N})\left(t^4\|\tilde{w}_{T\cup A^k}\|_2^2 + \sum_{i \in W_{k+1}}t_i^2\right)\]
\[= (1 - \delta_{NK+K-l+N})\|\tilde{w}_{T\cup A^k}\|_2^2(1 + t^2) - (1 + \delta_{NK+K-l+N})\|\tilde{w}_{T\cup A^k}\|_2^2(t^4 + t^2)\]
\[= (1 - t^4)\|\tilde{w}_{T\cup A^k}\|_2^2 - \delta_{NK+K-l+N}\|\tilde{w}_{T\cup A^k}\|_2^2(1 + t^2)^2\]
\[= (1 + t^2)^2\|\tilde{w}_{T\cup A^k}\|_2^2\left(\frac{1 - t^2}{1 + t^2} - \delta_{NK+K-l+N}\right)\]
\[\geq (1 + t^2)^2\|\tilde{w}_{T\cup A^k}\|_2^2\left(\frac{1 - t^2}{1 + t^2} - \delta_{NK+1}\right).\]

Moreover, notice the fact that

\[\|A'(I - P_{\Lambda^k})e\|_\infty = \max_i|\langle Ae_i, (I - P_{\Lambda^k})e\rangle| \leq \|Ae_i\|_2\|(I - P_{\Lambda^k})e\|_2 \leq \|e\|_2 \leq \varepsilon.\]
By the above three inequalities, (3.11) and (3.12), it follows that

\[
(1 - t^4) \sqrt{K} \| \omega_{T \cup \Lambda_k} \|_2 (\beta_{k+1}^N - \alpha_{k+1}^N) \\
\geq (1 + t^2)^2 \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) - (1 - t^4) \sqrt{K} \| \omega_{T \cup \Lambda_k} \|_2 \\
\left( \| A'_{T \cup \Lambda_k} (I - P_{\Lambda_k}) e \|_\infty + \frac{\sum_{i \in W_{k+1}} | \langle A e_i, (I - P_{\Lambda_k}) e \rangle |}{N} \right) \\
\geq (1 + t^2)^2 \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) - (1 - t^4) \sqrt{K} \| \omega_{T \cup \Lambda_k} \|_2 \left( \varepsilon + \frac{N \varepsilon}{N} \right) \\
= (1 + t^2)^2 \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 - 2 \sqrt{K} \varepsilon \frac{1 - t^2}{1 + t^2} \\
\geq (1 + t^2)^2 \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 \left( \frac{1 - t^2}{1 + t^2} - \delta_{NK+1} \right) \| x_{T - \Lambda_k} \|_2 - 2 \sqrt{K} \varepsilon \frac{1 - t^2}{1 + t^2} \\
\geq (1 + t^2)^2 \| \tilde{\omega}_{T \cup \Lambda_k} \|_2 \left( \frac{1}{\sqrt{\frac{K}{N} + 1}} - \delta_{NK+1} \right) \| T - \Lambda_k \|_{\infty} \min_{i \in T - \Lambda_k} | x_i | - \frac{2 \sqrt{K} \varepsilon}{\sqrt{\frac{K}{N} + 1}} \\
> 0,
\]

i.e., \( \beta_{k+1}^N > \alpha_{k+1}^N \) which guarantees at least one index selected from the correct support in the \( (k + 1) \)-th iteration.

Therefore the gOMP with the stopping rule \( \| r_k \|_2 \leq \varepsilon \) recovers the correct support of any \( K \)-sparse signal \( x \) under conditions (3.11) and (3.12).

\[ \square \]

4 Acknowledgements

This work was supported by the NSF of China (Nos.11271050, 11371183).

References

[1] D. L. Donoho, Compressed sensing, IEEE Trans. Inf. Theory, vol. 56, no. 4, pp. 1289-1306, 2006.

[2] E. J. Candès, The restricted isometry property and its implications for compressed sensing, Comptes Rendus Mathematique, vol. 346, no. 9-10, pp. 589-592, 2008.

[3] R. Baraniuk, M. Davenport, R. DeVore and M. Wakin, A simple proof of the restricted isometry property for random matrices, Construct. Approx., vol. 28, no. 3, pp. 253-263, 2008.
[4] T. T. Cai, L. Wang and G. Xu, Shifting inequality and recovery of sparse signals, IEEE Trans. Signal Process., vol. 58, pp. 1300-1308, Mar. 2010.

[5] E. Liu and V. N. Temlyakov, The orthogonal super greedy algorithm and applications in compressed sensing, IEEE Trans. Inf. Theory, vol. 58, no. 4, pp. 2040-2047, 2012.

[6] T. T. Cai and A. R. Zhang, Compressed sensing and affine rank minimization under restricted isometry, IEEE Trans. Signal Process., vol. 61, no. 13, pp. 3279-3290, Apr. 2013.

[7] T. T. Cai and A. R. Zhang, Sparse representation of a polytope and recovery of sparse signals and low-rank matrices, IEEE Trans. Inform. Theory, vol. 60, no. 1, January 2014.

[8] E. Candès and T. Tao, Decoding by linear programming, IEEE Trans. Inf. Theory, vol. 51, no. 12, pp. 4203-4215, Dec. 2005.

[9] R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, IEEE Signal Process. Lett, vol. 14, no. 10, pp. 707-710, 2007.

[10] J. A. Tropp, Greed is good: Algorithmic results for sparse approximation, IEEE Trans. Inf. Theory, vol. 50, pp. 2231-2242, 2004.

[11] Q. Mo and S. Li, New bounds on the restricted isometry constant $\delta_{2k}$, Appl. Comput. Harmon. Anal., vol. 31, no. 3, pp. 460-468, 2011.

[12] T. T. Cai, L. Wang and G. Xu, New bounds for restricted isometry constants, IEEE Trans. Inf. Theory, vol. 56, no. 9 pp. 4388-4394, Sep. 2010.

[13] T. T. Cai and A. R. Zhang, Sharp RIP bound for sparse signal and low rank matrix recovery, Appl. Comput. Harmon. Anal., vol. 35, pp. 74-93, Jan. 2013.

[14] S. Chen, S. A. Billings and W. Luo, Orthogonal least squares methods and their application to non-linear system identification, Int. J. Contr., vol. 50, no. 5, pp. 1873-1896, 1989.

[15] J. A. Tropp and A. C. Gilbert, Signal recovery from random measurements via orthogonal matching pursuit, IEEE Trans. Inf. Theory, vol. 53, no. 12, pp. 4655-4666, 2007.
[16] J. Wang, S. Kwon and B. Shim, Generalized orthogonal matching pursuit, IEEE Transactions on Signal Processing, vol. 60, no. 12, pp. 6202-6216, Dec. 2012.

[17] D. Needell and R. Vershynin, Signal recovery from incomplete and inaccurate measurements via regularized orthogonal matching pursuit, IEEE J. Sel. Topics Signal Process., vol. 4, no. 2, pp. 310-316, 2010.

[18] Z. Q. Xu, The performance of orthogonal multi-matching pursuit under RIP, Arxiv: 1210.5323v2, 2012.

[19] D. L. Donoho, I. Drori, Y. Tsaig and J. L. Starck, Sparse solution of underdetermined linear equations by stagewise orthogonal matching pursuit, IEEE Trans. Inf. Theory, vol. 58, no. 2, pp. 1094-1121, 2012.

[20] W. Dai and O. Milenkovic, Subspace pursuit for compressive sensing signal reconstruction, IEEE Trans. Inf. Theory, vol. 55, no. 5, pp. 2230-2249, May 2009.

[21] D. Needell and J. A. Troop, CoSaMP: Iterative signal recovery from incomplete and inaccurate samples, Appl. Comput. Harmon. Anal., vol. 26, no. 3, pp. 301-321, Mar. 2009.

[22] M. A. Davenport and M. B. Wakin, Analysis of orthogonal matching pursuit using the restricted isometry property, IEEE Trans. Inf. Theory, vol. 56, pp. 4395-4401, 2010.

[23] S. Huang and J. Zhu, Recovery of sparse signals using OMP and its variants: Convergence analysis based on RIP, Inverse problems, vol. 27, no. 3, p. 035003, Mar. 2011.

[24] J. Wang and B. Shim, On the recovery limit of sparse signals using orthogonal matching pursuit, IEEE Trans. signal process., vol. 60, no. 9, pp. 4973-4976, Sep. 2012.

[25] Q. Mo and Y. Shen, A remark on the restricted isometry property in orthogonal matching pursuit, IEEE Trans. Inf. Theory, vol. 58, no. 6, pp. 3654-3656, June 2012.

[26] J. Wen, X. Zhu and D. Li, Improved bounds on restricted isometry constant for orthogonal matching pursuit, Electronic letters, vol. 49, no. 23, pp. 1487-1489, 2013.

[27] W. Dan and R. H. Wang, Robustness of orthogonal matching pursuit under restricted isometry property, Science china Mathematics, vol. 57, no. 3, pp. 627-634, 2014.
[28] Q. Mo, A sharp restricted isometry constant bound of orthogonal matching pursuit, arXiv:1501.01708 v1[cs.IT] 8 Jan 2015.

[29] Y. Shen and S. Li, Sparse signals recovery from noise measurements by orthogonal matching pursuit, Inverse Problems and Imaging, vol. 9, no. 1, 2011.

[30] R. Wu, W. Huang and D. R. Chen, The exact support recovery of sparse signals with noise via orthogonal matching pursuit, IEEE signal processing letters, vol. 20, no. 4, April 2013.

[31] L. H. Chang and J. Y. Wu, An improved RIP-based performance guarantee for sparse signal recovery via orthogonal matching pursuit, IEEE Trans. Inf. Theory, vol. 60, no. 9, September 2014.

[32] S. Satpathi, R. L. Das and M. Chakraborty, Improving the bound on the RIP constant in generalized orthogonal matching pursuit, IEEE Signal Processing Letters, vol. 20, no. 20, pp. 1074-1077, 2013.