Corrigenda to "$L^p$ estimates and asymptotic behavior for finite energy solutions of extremals to Hardy-Sobolev inequalities", Trans. Amer. Math. Soc. 363 (2011), no. 1, 37–62

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The claim and the proof of [3, Theorem 2.9] are not correct as stated. We give a correction in Section 1.6. In addition, we correct several typos in the text and strengthen slightly some results in [3].

1. Corrigenda

Throughout we use the notation of [3].

1.1. Except for the introduction of [3], throughout the paper the assumption $0 \leq s \leq p$ should be $0 \leq s < p$. While the case $s = p$ is included in the Hardy-Sobolev inequality recalled in [3, Theorem 1.1], the problems considered in the rest of the paper assumed tacitly that $s < p$. The latter condition was implied in some, but not all, of the statements in [3] by some of the made assumptions.

1.2. The following corrections should be made in the statement of [3, Theorem 2.1]. The assumption on the exponent $t$ should have been $0 \leq t < \min\{p, k\}$ rather than the more restrictive $0 \leq t < \min\{p, s\}$. The same correction applies to the fourth line on [3, p. 41].

1.3. In identity [3, (2.21)] the norms of $u$ should not be raised to the power $q$.

1.4. In the proof of [3, Theorem 2.4], the formula for $r'$ should be $r' = p^*/(p^*(s) - p)$, noting that $p^* = p^*(0)$, and we use Sobolev’s inequality (i.e., Hardy-Sobolev’ inequality with $s = 0$) in order to see that $V \in L^{r'}(\Omega)$.

1.5. The exponents $p$ and $p'$ appearing in the denominators of the first line of [3, (2.51)] should be switched. Thus, the correct form of [3, (2.51)] is

$$\begin{align*}
|\nabla u|^{p-2} & \alpha p^{-1} F(u) \nabla u \cdot \nabla \alpha \leq \frac{\epsilon}{p'} \alpha p |\nabla \alpha|^{p-1} F(u) + \frac{\epsilon^{-p/p'} p'}{p} |\nabla \alpha| p |F(u) u|^{p/p'} \\
& \leq \frac{\epsilon}{p} \alpha p |\nabla G|^{p} + C \frac{\epsilon^{-(p-1)}}{p} q^{p-1} |\nabla \alpha|^{p} G^{p}
\end{align*}$$

with $C = C(p)$. Therefore, the correct form of the inequality following [3, (2.51)] is

$$\text{LHS} \geq (1 - (p - 1)\epsilon) \int \alpha p |\nabla G|^{p} dz - C \epsilon^{-(p-1)} q^{p-1} \int |\nabla \alpha|^{p} G^{p} dz$$

and then we fix $\epsilon = \frac{1}{2(p-1)}$ so that $(1 - (p - 1)\epsilon) = 1/2$.

1.6. Here, we correct the statement of [3, Theorem 2.9] by adding an additional assumption whose origin can be traced to [2, Lemma (4.2)] and then in section 1.7 we give the necessary modification of the proof.

Additional Assumptions in [3, Theorem 2.9]. In addition to the assumptions of [3, Theorem 2.9], suppose that $V \in L^{t_o}$ for some $t_o > r'$ and there are constants $R_o$ and $K_o$ such that we have

$$\int_{R^n \setminus B(0, R)} |V|^{t_o} dz \leq \frac{K_o}{R^{(p-s)t_o - n}},$$

for all $R \geq R_o$. 

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It is worth pointing the following example from [1] Example 3, p. 51] where it is observed that
\[ u(z) = |z|^{2-n} \ln |z| \]
has finite energy outside the unit ball and satisfies the equation
\[ \Delta u = -V u, \quad \text{with} \quad V(z) = (n-2) \frac{1}{|z|^{2} \ln |z|}. \]
However, \( u \) does not have the fast decay at infinity, i.e., the same as the fundamental solution of the Laplacian.
Furthermore, \( V \in L^{\infty} \) for any \( t_{o} \geq n/2 \). On the other hand, we note that \( V \) does not satisfy (2) since
\[ \int_{|z| > R} \frac{V^{t_{o}}}{dz} \approx \frac{(\ln R)^{t_{o}-1}}{(t_{o} - 1) R^{2t_{o} - n} + \ldots}. \]
Therefore, the claim of [3, Theorem 2.9] does not hold with the assumptions of the paper. On the other hand, as pointed above, this counterexample is excluded by the added assumption (2).

1.7. Proof of [3, Theorem 2.9] with the extra assumption (2). The argument in the proof of [3, Theorem 2.9] is incomplete due to the dependence on the constant \( C \) in the first integral in the right-hand-side of equation [3, (2.52)], which follows from equation [3, (2.16)]. The correct form of [3, (2.52)] is with \( Cq^{-1} \) in the first term in the right-hand-side of equation [3, (2.52)], i.e.,
\[ \int_{|z| > R} |\nabla (\alpha G)^{p} d\nu| \leq C q^{-1} \left( \int_{\Omega} \frac{|\alpha G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/t_{o}} \left( \int_{\Omega} \frac{1}{|x|^{\kappa}} d\nu \right)^{1/ \kappa}, \]
where \( C = C(p) \) is a constant independent of \( q \). In particular, we cannot do infinitely many iterations using [3, (2.53)] and obtain the claimed estimate.

With the added additional assumptions on \( V \) stated in section 1.6 the proof follows from the original argument in [3, Theorem 2.9] with the following corrections. The general outline is that to handle the term involving the potential, which will be kept in the right-hand side, we will rely on the new assumption (2) and several inequalities rather than the original use of [3, (2.54)]. Starting with (3) where \( \alpha \) is a smooth cut-off function to be specified in a moment, the first integral in the right-hand-side of equation [3, (2.52)], i.e.,
\[ \int_{\Omega} \frac{1}{|x|^{\kappa}} d\nu \leq C q^{-1} \left( \int_{\Omega} \frac{|\alpha G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/t_{o}} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/ \kappa}, \]
where \( C = C(p) \) is a constant independent of \( q \). In particular, we cannot do infinitely many iterations using [3, (2.53)] and obtain the claimed estimate.

The last identities together with the assumptions \( t_{o} > r' = \frac{n}{p-s} \) and \( s < p < n \), imply that we have
\[ t = \frac{p t_{o}}{n - p + s t_{o}} > 1, \quad \kappa = \frac{p t_{o}}{(p - s) t_{o} - n} > 1, \quad \kappa' = \frac{tp}{p'(st)}, \quad \text{and} \quad 0 \leq st < p. \]
Therefore, with \( \epsilon > 0 \) the inequality \( ab \leq \epsilon \frac{a^{\kappa'}}{\kappa'} + \epsilon^{-\left(\kappa-1\right)} \frac{b^{\kappa}}{\kappa} \), the Hardy-Sobolev inequality and the identity \( \kappa' / t = p/p'(st) \) give
\[ \int_{\Omega} \frac{1}{|x|^{\kappa}} d\nu \leq \epsilon \frac{a^{\kappa'}}{\kappa'} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/t_{o}} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/ \kappa'} + \epsilon^{-\left(\kappa-1\right)} \frac{b^{\kappa}}{\kappa} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{\kappa/ \kappa'}, \]
and for some constant \( C_{1} = C_{1}(p, n) \) we have
\[ \int_{\Omega} \frac{1}{|x|^{\kappa}} d\nu \leq C_{1} q^{\left(\kappa-1\right)(p-1)} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/t_{o}} \left( \int_{\Omega} \frac{|G|^{p} |x|^{\kappa}}{|x|^{\kappa}} d\nu \right)^{1/ \kappa'} + C_{1} q^{-1} \left( \int_{\Omega} |\nabla (\alpha G)^{p} d\nu| \right). \]
with $\epsilon$ chosen so that

$$C_0 q^{p-1} \frac{\epsilon}{\kappa T} s^{p,q}_{p, st} = 1/2.$$ 

With the choice of the cut-off function $\alpha$ made after [3, (2.57)] we have that $\alpha \in C^{\infty}_o (B(z, r))$ with $\alpha = 1$ on $B(z, \rho)$ where $0 < \rho < r < R = |z|/2$, so that the gradient satisfies $|\nabla \alpha| \leq 2/(r - \rho)$. This will be used to estimate the second integral in the righthand side of (6). On the other hand, the new extra assumption $B$ on the choice of the cut-off function $\alpha$ in the proof of [3, Theorem 2.9] we can use

$$\sum_{j=0}^{\infty} 1/q_j = p - 1$$

with a constant $C_3 = C_3(s, p, n, K_0, R_0)$. The last inequality is the analogue of inequality [3, (2.58)] where, now, we have full control of the constant $C$. At this point the Moser iteration argument described after [3, (2.58)] gives the claimed inequalities [3, (2.47)] and [3, (2.48)] noting that since $q_j = q_0 \delta^j$, $j = 0, 1, \ldots$, we have

$$p \sum_{j=0}^{\infty} 1/q_j = p - 1$$

This completes the (correction) to the proof of [3, Theorem 2.9].

1.8. **Addition to [3, Section 2]**. The following Corollary strengthens [3, Theorem 2.9] in the case of an exterior domain.

**Corollary 1.1.** Let all of the assumptions of [3, Theorem 2.9] (including the Additional Assumption $B$) hold. If the complement of $\Omega$ is a compact subset of $\mathbb{R}^n$, then the claim of [3, Theorem 2.9] is valid for any $q_0 > p - 1$ rather than $q_0 \geq p$.

**Proof.** The proof follows the steps of the above proof of [3, Theorem 2.9], so we only indicate the differences. Since $V \in L^{t_0}$ for some $t_0 > r'$ from [3, Theorem 2.1] we have that $u$ is a bounded function. In particular, in the proof of [3, Theorem 2.9] we can use

$$G(t) = t^{q/p} \quad \text{and} \quad F(t) = \left( \frac{q}{p} \right)^q t^{q/p+1} - q/p + 1$$

by fixing an $l$ sufficiently large in [3, (2.15)] so that $u \leq l$. Notice that, now, $F$ and $G$ satisfy $|G(u)|^p = u^q$ and $F(u) = u |G'(u)|^p$ hence we have again [3, (2.16)] and [3, (2.49)]. Furthermore, in the case when the complement of $\Omega$ is a compact subset of $\mathbb{R}^n$, the cut-off function $\alpha$ used in the proof of [3, Theorem 2.9], recalled here above inequality (7), will be a function with compact support in $\Omega$ for all $R$ sufficiently large. Thus, for any $q > p - 1$ and $\epsilon > 0$ a positive constant we have that $\alpha^p F(u + \epsilon) \in D^{1,p}(\Omega)$ can be taken as a test function in the beginning of the proof of [3, Theorem 2.9]. The proof continues as in section [1.7] leading to (8) in which now we have $G = G(u + \epsilon)$. Letting $\epsilon \to 0$ we obtain [3] for any $q > p - 1$ such that $u \in L^q$. \qed
1.9. **Correction to** [3, Theorem 2.10]. First, the statement of [3, Theorem 2.10] is about equation [3, (2.5)] as stated before the theorem. Thus, in the statement of [3, Theorem 2.10] we assume [3, (2.24)] with \( V_0 = 0 \), i.e., \( u \) is a weak non-negative solution of
\[
- \text{div}(|\nabla u|^{p-2}\nabla u) \leq R(z)\frac{u^{p-1}}{|x|^s} \quad \text{in } \Omega
\]
with \( R \in L^{r'} \cap L^\infty \) for some \( t_o > r' \). Furthermore, since [3, Theorem 2.10] is stated as a direct corollary of [3, Theorem 2.1, Theorem 2.5 and Theorem 2.9] and the latter required the additional assumption as described earlier, we need to add the extra assumption that the potential \( R \) satisfies the condition \( \text{(2)} \) (with \( V \) replaced by \( R \)). Finally, the domain \( \Omega \) should be assumed to have a compact complement for reasons we explain below.

The condition that \( \Omega \) is an exterior domain (the complement is compact) is imposed due to the fact that the decay rate \( u(z) \leq C|z|^{n/n_q} \) given by [3, Theorem 2.9] is valid for \( q_0 \geq p \) while the decay of the fundamental solution corresponds to the limiting exponent \( q_0 = p^*/p' = n(n-p)/(n-p) \) which is less than \( p \). However, in the case of a domain with a compact complement Corollary 1.1 can be applied since \( p^*/p' > p - 1 \), hence \( q_0 \) can be taken as close to \( p^*/p' \) as we wish using [3, Theorem 2.5] as in the original argument.

1.10. **Correction and Addition to** [3, Section 3]. The statement of [3, Theorem 3.1] should be corrected by requiring that \( \Omega = \mathbb{R}^n \), i.e., we are considering entire solutions.

The correction is needed because the proof of [3, Theorem 3.1] uses [3, (3.2)], where \( p = 2 \), which implies [3, (3.6)]. However, [3, (3.2)] relies on the decay of the solution implied by [3, Theorem 2.10], thus we need to verify the additional assumption on the potential from section 1.6 which in this case is \( ru^{2^*(s)-2} \) with \( R \in L^\infty \). The verification detailed below uses the scaling invariance of the equation when the domain is the entire space. In fact, we give the following general result valid not only for solutions to the equations modeled on the scalar curvature equations, but also for entire non-negative finite energy solutions of equations modeled on the Euler-Lagrange equation of the extremals of the considered Hardy-Sobolev inequality.

**Lemma 1.2.** Let \( R \) be a bounded function, which is non-negative when \( p \neq 2 \). If \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \) is a non-negative weak solution of the inequality
\[
- \text{div}(|\nabla u|^{p-2}\nabla u) \leq R(z)\frac{u^{p^*(s)-1}}{|x|^s} \quad \text{on } \mathbb{R}^n,
\]
then for any \( 0 < \theta < 1 \) there exists a constant \( C_\theta > 0 \), such that,
\[
u(z) \leq \frac{C_\theta}{1 + |z|^\theta|p|^{s-1}}\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)},
\]

**Proof.** First, exploiting the scale invariance we will show that \( u \) has the slow decay at infinity, see (13) below. Second, we will show that \( V(z) = R(z)u^{p^*(s)-p} \) satisfies the additional assumptions listed in section 1.6. These two observations suffice for the proof of (12). Indeed, the bound (12) follows directly from [3, Theorem 2.5] and Corollary 1.1. We only need to notice that we can take \( q_o \) in [3, Theorem 2.9] as close to \( p^*/p' \) as we want, since for \( p > 1 \) we have \( p^*/p' = (np - n)/(n - p) > p - 1 \). We turn to the detailed proofs of the two claims.

As mentioned in the preceding paragraph, in the first step of the proof we will show that for all \( |z|/2 > R_o \) any solution of (11) has the following "slow decay" property
\[
u(z) \leq \frac{C}{|z|^{(n-p)/p}},
\]
with a constant \( C \) depending on \( p, n \) and \( \|u\|_{\mathcal{D}^{1,p}(\Omega)} \), see [4] for the case of the Yamabe equation.

In fact, if \( u \) is a solution of (11) then for any \( \lambda > 0 \) the function
\[
v(w) = \lambda^{(n-p)/p}u(\lambda w), \quad w = \lambda w,
\]

\footnote{The author thanks Professor Annunziata Loidusic for pointing the fact that the given argument requires an extra assumption on the domain.}
is also a finite energy solution of (11) with $R(z)$ replaced with $R_{\lambda}(w) = R(\lambda w)$. Let us notice that
\[ \|R_{\lambda}\|_{L^\infty(\mathbb{R}^n)} = \|R\|_{L^\infty(\mathbb{R}^n)} \]
and also the $D^{1,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ norms are invariant under this scaling, $\|u\|_{D^{1,p}(\Omega)} = \|v\|_{D^{1,p}(\Omega)}$.

In order to show the slow decay, it is then enough to show that there exist constants $R_\alpha$ and $C$, depending only on $p$, $n$, and the invariant under the scaling norms, such that for all $z_0$ with $\lambda = |z_0|/2 > R_\alpha$ we have on the ball $B(w_\alpha, 1)$, $w_\alpha = z_0/\lambda$, the estimate
\[ (15) \quad \max_{w \in B(w_\alpha, 1)} v(w) \leq C. \]
Indeed, (15) implies
\[ \left(\frac{|z_0|}{2}\right)^{(n-p)/p} u(z_0) \leq \left(\frac{|z_0|}{2}\right)^{(n-p)/p} \sup_{|z-z_0| < |z_0|/2} u(z) = \sup_{|w-z_0| < \lambda} \lambda^{(n-p)/p} u(\lambda w) = \sup_{|w-w_0| < 1} v(w) \leq C, \]
which gives (15).

Let us observe that the desired bound (15) is suggested by the local version of [3, Theorem 2.1] stated in [3, Remark 2.3]. However, we have to make sure that the local supremum bound is independent of $\lambda$, therefore we provide the details of the argument.

Let us show that for any fixed $t_\alpha > r'$ we have $V(w) \equiv V_{\lambda}(w) = R(\lambda w)\alpha^\gamma - p(w) \in L^\infty$ near the point $w_\alpha = z_\alpha/\lambda$, $\lambda = |z_\alpha|/2$, with a uniform bound on the norm as stated in (15) below. For this we follow essentially the argument of the proof of [3, Theorem 2.9] after [3, (2.53)] (corrected here by [3]), but now using a smooth bump function $\alpha$ depending only on the distance to the point $w_\alpha$ with support in $B(w_\alpha, \rho_2)$ with $\alpha \equiv 1$ on $B(w_\alpha, \rho_1)$ for $1/2 < \rho_1 < \rho_2 < 1$; furthermore, we take the function $G$ as in the proof of [3, Theroem 2.9] but defined using $v$ instead of $u$.

Therefore, from [3], Hölder’ and Hardy-Sobolev’ inequalities ( $pr = p^*(rs)!$ ), we have
\[ (16) \quad \|\nabla(\alpha G)\|_{L^p} \leq C_0 q^{q-1} \|V\|_{L^{p'}(\text{supp } \alpha)} \|\nabla(\alpha G)\|_{L^p} + C_0 q^{q-1} \int |\nabla \alpha| G \, dz. \]
Notice that the function $V$ satisfies
\[ \|V\|_{L^{p'}(\text{supp } \alpha)} \leq \|R\|_{L^\infty} \int_{B(w_\alpha, 1)} \frac{v(w)^{p'} \, dw}{\|R\|_{L^\infty}} = \|R\|_{L^\infty} \int_{B(z_\alpha, |z_\alpha|/2)} \frac{u(z)^{p'} \, dz}{\|R\|_{L^\infty}} \leq \|R\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(0, |z_\alpha|/2)} u(z)^{p'} \, dz. \]
Hence, for any fixed $q_1 > p$, there exists $R_\alpha = R_\alpha(q_1)$ sufficiently large so that
\[ (17) \quad C q_1^{q_1-1} \|V\|_{L^{p'}(\text{supp } \alpha)} \leq 1/2 \]
for $\lambda = |z_\alpha|/2 \geq R_\alpha$. Trivially, (17) holds also for all $q$, such that, $p \leq q \leq q_1$. This shows the validity of [3, (2.56)] with the above choice of $\alpha$ and $G$, hence we also have [3, (2.57)] with $u$ replaced by $v$ and then [3, (2.58)] which in our case reads
\[ (18) \quad \left(\frac{1}{|B(w_\alpha, \rho_1)|} \int_{B(w_\alpha, \rho_1)} v^{\delta q} \, dw\right)^{\frac{1}{\delta q}} \leq C \left(\frac{p/q}{p-1} \right)^{1/p} \left(\frac{1}{|B(w_\alpha, \rho_2)|} \int_{B(w_\alpha, \rho_2)} v^q \, dw\right)^{\frac{1}{q}}, \]
where $\delta = p^*/p > 1$. After finitely many Moser’s iterations of the obtained inequality starting with $q = p^*$ we see that for any fixed $t_\alpha > r'$ we have $v \in L^{p^*}(p^*(s)-p)$ and the function $V$ satisfies
\[ (19) \quad \|V\|_{L^{p^*}(B_{1/2})} \leq C, \]
unformly for all $\lambda = |z_\alpha|/2 \geq R_\alpha$ using again that the $L^p(\mathbb{R}^n)$ norm is invariant under the scaling.

Next, we follow the proof of [3, Theorem 2.1b] with $G = G \circ v$, but in this local case, we use for $1/4 < \rho_1 < \rho_2 < 1/2$ a smooth bump function $\alpha$ depending on the distance to $w_\alpha$ such that $\alpha$ is supported in $B(w_\alpha, \rho_2)$ with $\alpha \equiv 1$ on $B(w_\alpha, \rho_1)$. We use again [3]. The left-hand side is estimates from below by the Hardy-Sobolev inequality, taking into account the properties of the bump function. On the other hand,
the first terms in the right-hand side of (3) is estimated from above by Hölder’s inequality using (19). The second term is estimated with the help of the inequalities

\[ |\nabla \alpha| \leq 4/(\rho_2 - \rho_1) \quad \text{and} \quad |x| \leq |z| \leq |z - \omega_0| + |\omega_0| \leq \rho_2 + 2 = 5/2 \]

on \( B(\omega_0, \rho_2) \setminus B(\omega_0, \rho_1) \) which contains the support of \(|\nabla \alpha|\). Hence, we obtain

\[
\frac{1}{S_{p,sr}} \left( \int_{B(\omega_0, \rho_1)} G_{p,sr}(\alpha_\omega)^{p/(sr)} \, dz \right)^{1/p} \leq \int_{\mathbb{R}^n} |\nabla (\alpha G)|^p \, dz \leq C_0 q^{p-1} \left( \int_{\mathbb{R}^n} V^{(\alpha G)^p} \, dz + \int_{\mathbb{R}^n} |\nabla \alpha|^p G^p \, dz \right) \]

\[
\leq C_1 q^{p-1} \left[ \|V\|_{L^{\infty}(B(\omega_0, \rho_2))} \left( \int_{B(\omega_0, \rho_2)} |G_{pt_0}^{p,sr}| \, dz \right)^{1/t_0'} \right] + \left( \frac{2}{3} \right)^s \frac{1}{(\rho_2 - \rho_1)^p} \int_{B(\omega_0, \rho_1)} G^p \, dz \]

\[
\leq C_1 (1 + \|V\|_{L^{\infty}(B(\omega_0, \rho_2))}) \frac{q^{p-1}}{(\rho_2 - \rho_1)^p} \left[ \left( \int_{B(\omega_0, \rho_2)} |G_{pt_0}^{p,sr}| \, dz \right)^{1/t_0'} \right] \]

after using also \( 4 < 1/(\rho_2 - \rho_1) \), the uniform bound (19), and finally Hölder’s inequality taking into account that \( 1 < t_0' < r \).

Therefore, using \( p^*(rs) = pr \), as noted in [3] (2.3), and Fatou’s lemma show the existence of a constant \( C \), independent of \( \lambda = |z_\omega| / 2 \geq R_0 \), such that,

\[
\left( \int_{B_{p_1}} |u|^q r \, dz \right)^{1/r} \leq C \frac{q^{p-1}}{(\rho_2 - \rho_1)^p} \left( \int_{B_{p_2}} |u|^q r \, dz \right)^{1/t_0'}
\]

with \( 0 < t_0' s < rs \). An iteration of the above inequality starting with \( q = p \), exactly as in the proof of [3] Theorem 2.1b] gives the claimed bound (13) using that \( rs < k \) by the assumption [3] (2.4).

At this point we can verify the second claim made at the beginning of the proof, i.e., that (13) imply the added extra assumptions described in Section 1.6. Indeed, since \( V = R(z) u^{p(s)-p} \in L^r(\mathbb{R}^n) \), [3] Theorem 2.4] implies that \( V \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), while from the slow decay (13) for \( |z| > 2R_0 \) we have

\[
V(z) \leq \frac{C p^*(s)-p}{|z|^{p^*(s)-p}} = \frac{C p^*(s)-p}{|z|^{p^*(s)-p}}
\]

which implies (2).

This completes the proof of Lemma 1.2.

\[ \square \]

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