\textbf{$L^2$-concentration phenomenon for Zakharov system below energy norm*}

Daoyuan Fang, Sijia Zhong
Department of Mathematics, Zhejiang University,
Hangzhou 310027, China

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\textbf{Abstract}

In this paper, we’ll prove a $L^2$-concentration result of Zakharov system in space dimension two, with radial initial data $(u_0, n_0, n_1) \in H^s \times L^2 \times H^{-1}$ ($\frac{1}{16} < s < 1$), when blow up of the solution happens by $I$-method. In additional to that we find a blow up character of this system. Furthermore, we improve the global existence result of Bourgain’s to above spaces.

\textit{Keywords:} Zakharov system in space dimension two; $L^2$-concentration; blow up; global existence

\section{1 Introduction}

In this paper, we consider the following Zakharov system in space dimension two:

\begin{equation}
\begin{aligned}
    iu_t + \Delta u &= nu, \\
    \Box n &= \partial_t n - \Delta n = \Delta |u|^2, \\
    u(0, x) &= u_0(x), \quad n(0, x) = n_0(x), \quad n_t(0, x) = n_1(x), 
\end{aligned}
\end{equation}

where $\Delta$ is the Laplacian operator in $\mathbb{R}^2$, $u : [0, T) \times \mathbb{R}^2 \to \mathbb{C}$, $n : [0, T) \times \mathbb{R}^2 \to \mathbb{R}$, and $u_0, n_0, n_1$ are the initial data. We consider the Hamiltonian case, that is, we assume that there is a $w_0 : \mathbb{R}^2 \to \mathbb{R}$ such that $n_t(0) = n_1 = -\Delta w_0$. Then for any $t$, there is a $w(t)$ such that $n_t(t) = -\Delta w(t) = -\nabla \cdot v(t)$, where $v(t) = \nabla w(t)$. In this case, (1.1) can be written in the form

\begin{equation}
\begin{aligned}
    iu_t + \Delta u &= nu, \\
    n_t &= -\nabla \cdot v, \\
    v_t &= -\nabla n - \nabla |u|^2, \\
    u(0, x) &= u_0(x), \quad n(0, x) = n_0(x), \quad v(0, x) = v_0(x), 
\end{aligned}
\end{equation}

The Zakharov system was introduced in [18] to describe the long wave Langmuir turbulence in a plasma. The function $u$ represents the slowly varying envelope of the rapidly oscillating electric

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field, and the function \( n \) denotes the deviation of the ion density from its mean value. We usually place the initial data \( u_0 \in H^k \), the initial position \( n_0 \in H^l \) and the initial velocity \( n_1 \in H^{l-1} \) for some real \( k, l \).

It is well-known that the Schrödinger equation is invariant under the dilation transformation

\[
    u(t, x) \rightarrow u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x),
\]

while the wave equation is invariant with the following transformation

\[
    n(t, x) \rightarrow n_\lambda(t, x) = \lambda n(\lambda t, \lambda x).
\]

However, the Zakharov system doesn’t have a true scale invariance because the two relevant dilation transformations are incompatible. Nevertheless the critical regularity is \((k, l) = \left(\frac{-1}{2}, 0\right)\).

For the local existence theory about this system. From [11], one can see that when \( d = 2 \), the Cauchy problem (1.1) with \((u_0, n_0, n_1) \in H^k \times H^l \times H^{l-1} \) is local well posed if \( l \geq 0 \) and \( 2k - (l + 1) \geq 0 \). Therefore the lowest allowed values of \((k, l)\) is \((\frac{1}{2}, 0)\).

On the other hand, if we replace \( \Box \) in (1.1) by \( \Box_c = c^{-2} \partial_t^2 - \Delta \), i.e. introducing explicitly the ion sound velocity, then considering the limit \( c \rightarrow \infty \), the system (1.1) reduces formally to the nonlinear Schrödinger equation

\[
    iu_t + \Delta u = -|u|^2 u,
\]

which is just the \( L^2 \)-critical focusing case for \( d = 2 \).

As for this Schrödinger equation, the results in [13], [14] and so on for \( s = 1 \), and [8], [10], for \( 1 > s > \frac{1 + \sqrt{11}}{5} \), tell us that there is some \( L^2 \)-concentration phenomenon for finite time blow up solutions, i.e.

\[
    \limsup_{t \uparrow T^*} \sup_{B \subset \mathbb{R}^2} \int_B |u(t)|^2 \geq \|Q\|_{L^2}^2.
\]

Here, \( Q \) is the ground state for Schrödinger equation, that is, the unique positive solution (up to translations) of

\[
    \Delta Q - Q + |Q|^2 Q = 0.
\]

In [2], [15] and [16] the convergence of the solutions of the \( c \) dependent Zakharov system to those of NLS equation when \( c \rightarrow \infty \) was studied, which implies that the \( L^2 \)-concentration phenomenon like \( L^2 \)-critical focusing Schrödinger equations may also happen. Glangetas and Merle in [12] proved this phenomenon for \((k, l) = (1, 0)\) which is the energy case.

We are interested here in the \( L^2 \)-concentration phenomenon for \( s < 1 \) when blow up occurs of Zakharov system as well. What we want to show is for some \( 0 < k < 1 \) this phenomenon also holds true:
Theorem 1.1. For \((u_0, n_0, n_1) \in H^s \times L^2 \times H^{-1}\), radial, \(\frac{16}{17} < s < 1\), if \((u, n)\) is a blow-up solution to equation (1.1), i.e. \(T^* < \infty\) is its maximum existing time, then there is a constant \(m_n > 0\) depending on the initial data such that the following properties are true: \(\forall R > 0\),

\[
\limsup_{t \to T^*} \|u(t, x)\|_{L^2(|x| \leq R)} \geq \|Q\|_{L^2}, \quad (1.5)
\]

and

\[
\limsup_{t \to T^*} \|n(t, x)\|_{L^1(|x| \leq R)} \geq m_n. \quad (1.6)
\]

Remark 1.2. We can’t remove the radial requirement because of the endpoint Strichartz estimate for Schrödinger equation we needed.

As a quick result of the above theorem, and by the conservation of \(L^2\)-norm of \(u\), one has:

Corollary 1.3. For \((u_0, n_0, n_1) \in H^s \times L^2 \times H^{-1}\), radial, \(\frac{16}{17} < s < 1\), if \(\|u_0\|_{L^2} < \|Q\|_{L^2}\), then the corresponding solution to (1.1) is global, i.e. \(T^* = \infty\).

In fact, the global well posedness for \(k = l + 1 \geq 3\) and small data is considered in [1]. Then Bourgain [3] [4] introduced a new method to study the Cauchy problem for nonlinear dispersive evolution equation, and applied it in [5] to prove well posedness (both local and global) for finite energy solutions namely for \(k = l + 1 = 1\) (also with small initial data). Therefore, the above result is an improvement of the former result.

Now, let’s briefly state about the proofs to Theorem 1.1.

As we consider the Hamiltonian case, there are two conservations: mass and energy (if exists). \(\forall t \in [0, T^*)\),

\[
\int_{\mathbb{R}^2} |u(t, x)|^2 \, dx = \int_{\mathbb{R}^2} |u_0(x)|^2 \, dx, \quad (1.7)
\]

and

\[
H(t) = H(u(t), n(t), v(t)) = H(u_0, n_0, v_0) = H_0, \quad (1.8)
\]

where

\[
H(u, n, v) = \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 + n(t, x)|u(t, x)|^2 + \frac{1}{2}n^2(t, x) + \frac{1}{2}|v(t, x)|^2 \, dx \quad (1.9)
\]

and \(v\) has been defined before.

First, we split \(n\) into its positive and negative frequency parts according to

\[
n_{\pm} = n \pm i\Lambda^{-1} \partial_t n, \quad (1.10)
\]

where \(\Lambda = \sqrt{-\Delta}\). Thus \(n = \frac{n_+ + n_-}{2}\), \(n_+ = \bar{n}_-\), and equation (1.1) equals to

\[
\begin{cases}
  iu_t = -\Delta u + \frac{n_+ + n_-}{2}u \\
  (i\partial_t + \Lambda)n_{\pm} = \mp\Lambda^{-1}\Box n = \pm\Lambda|u|^2 \\
  u(0) = u_0, \; n_{\pm}(0) = n_{\pm 0} = n_0 \pm i\Lambda^{-1}n_1.
\end{cases} \quad (1.11)
\]
It is obvious that \((u_0, n_{\pm}) \in H^s \times L^2\) by the regularity of \(u_0\), \(n_0\) and \(n_1\).

Then the expression of energy (or Hamiltonian) above is

\[
H(t) = H(u, n_{\pm})(t) = \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|n_{\pm}\|_{L^2}^2 + \frac{1}{2}\int (n_{\pm} + \bar{n}_{\pm})|u|^2 \, dx.
\] (1.12)

The purpose of us is to imitate the \(H^1\) argument with the energy. But the energy is infinite in the \(H^s \times L^2\)-setting, thus we applying a smoothing operator to make \(u\) and \(n_{\pm}\) in \(H^1 \times H^{1-s}\) and define the usual energy of this new object. However, the energy is not conserved any more, so the crucial point here is to estimate the growth of the modified total energy. The main difficult of this step is the low regularity of \(n_{\pm}\). In the other hand the wave equation doesn’t possess Strichartz estimates \([9]\) as good as Schrödinger, and we need some endpoint Strichartz estimates, which leads to the requirement of radial condition.

During the proof, we find a character of finite time blow up of Zakharov system, i.e. when \(t \to T^* < \infty\), \(\|Iu(t)\|_{H^1}\) would go to infinite and \(\liminf_{t\to T^*} \|In_{\pm}(t)\|_{H^{1-s}} > 0\). In fact, from the local existence theory, we can get \(\|u(t)\|_{H^s} + \|n_{\pm}\|_{L^2} \to \infty\) as \(t \to T^*\), so \(\|Iu(t)\|_{H^1} + \|In_{\pm}(t)\|_{H^{1-s}} \to \infty\). Then we prove this character by another view of the local existence result and the particular form of the system, that the nonlinear term of the second equation is independent on \(n_{\pm}\).

In Section 2, we’ll give some notations, norms and estimates. Then in Section 3, the local existence theory will be studied while in Section 4, we’ll estimate the change of the modified energy which is the main part of the paper. In Section 5, the proof for Theorem 1.1 is given.

## 2 Notations, Norms and Estimates

\(A \lesssim B\) means there is a universal constant \(c > 0\), such that \(A \lesssim cB\), and \(A \sim B\) when both \(A \lesssim B\) and \(B \lesssim A\).

\(<\xi> = (1 + |\xi|^2)^{\frac{1}{2}}\).

c+ means \(c + \epsilon\) while \(c-\) means \(c - \epsilon\), for some \(\epsilon > 0\) small enough.

For given \(N >> 1\), define smoothing operators \(I_N:\nabla\)

\[
\hat{I_N}f(\xi) = m_N(\xi)\hat{f}(\xi),
\] (2.1)

where

\[
m_N(\xi) = \begin{cases} 
1, & \xi \leq N \\
(\xi/N)^{s-1}, & \xi \geq 3N,
\end{cases}
\] (2.2)

and \(m_N(\xi)\) is smoothing, radial, nonnegative, and monotone in \(|\xi|\). We drop \(N\) from the notation for short when there is no confusion.
By computation, we have

$$\|f\|_{X^{i_{0},b}} \leq \|If\|_{X^{i_{0}+1,b}} \leq N^{1-s}\|f\|_{X^{i_{0},b}}$$  \hspace{1cm} (2.3)

for any $i_{0} \geq 0$, $b \in \mathbb{R}$. Here, we used the $X^{m,b}$-space which is defined as follows: for an equation of the form $if_{t} - \varphi(-i\nabla)f = 0$, where $\varphi$ is a measurable function, let $X^{m,b}_{\varphi}$ be the completion of $S(\mathbb{R} \times \mathbb{R}^{2})$ with respect to

$$\|f\|_{X^{m,b}_{\varphi}} := \|<\xi>^{m} <\tau>^{b} \mathcal{F}(e^{-it\varphi(-i\partial_{x})}f(t,x))\|_{L^{2}_{\tau,\xi}}.$$ \hspace{1cm} (2.4)

We denote Fourier transform w.r.t both $x$ and $t$ by $\hat{\cdot}$, while only w.r.t $x$ or $t$ by $\tilde{\cdot}$.

For a given time interval $I$, we define

$$\|f\|_{X^{m,b}_{\varphi,I}} = \inf_{g |_{I} = f} \|g\|_{X^{m,b}_{\varphi}},$$

and also omit $I$ if there is no confusion.

For $\varphi(\xi) = \pm|\xi|$, we use the notation $X^{m,b}_{\pm}$, while for $\varphi(\xi) = -|\xi|^{2}$ simply $X^{m,b}$.

Now, we are listing some well-known estimates for these norms.

1. If $u$ is a solution of $iu_{t} - \varphi(-i\partial_{x})u = 0$ with $u(0) = f$ and $\psi$ is a cut off function in $C_{0}^{\infty}(\mathbb{R})$ with $\text{supp}\psi \subset (-2,2)$, $\psi \equiv 1$ on $[-1,1]$, $\psi(t) = \psi(-t)$, $\psi(t) \geq 0$, $\psi_{\delta}(t) := \psi(\frac{t}{\delta})$, $0 < \delta \leq 1$, we have for $b > 0$,

$$\|\psi_{1}u\|_{X^{m,b}_{\varphi}} \leq c\|f\|_{H^{m}}.$$ \hspace{1cm} (2.5)

If $v$ is a solution of problem $iv_{t} - \varphi(-i\partial_{x})v = F$, $v(0) = 0$, we have for $b' + 1 \geq b \geq b' > -\frac{1}{2}$

$$\|\psi_{\delta}v\|_{X^{m,b}_{\varphi}} \leq c\delta^{b' - b}\|F\|_{X^{m,b}_{\varphi}}.$$ \hspace{1cm} (2.6)

The proofs for these two estimates could be found in [11].

2. For $\frac{1}{2} > b > b' \geq 0$, $0 < \delta \leq 1$, $m \in \mathbb{R}$

$$\|\psi_{\delta}f\|_{X^{m,b}_{\varphi}} \leq c\delta^{b' - b}\|f\|_{X^{m,b}_{\varphi}}.$$ \hspace{1cm} (2.7)

3. Strichartz estimates.

For $\frac{2}{q} = 1 - \frac{1}{r}$, $q \geq 2$ and $u$ is radial,

$$\|u\|_{L^{q}_{t}L^{r}_{x}} \leq c\|u\|_{X^{0,\frac{1}{2}+}}.$$ \hspace{1cm} (2.8)

and for $\frac{1}{q} = 1 - \frac{2}{r}$ and $q > 2$,

$$\|v\|_{L^{q}_{t}L^{r}_{x}} \leq c\|v\|_{X^{0,\frac{1}{2}+}}.$$ \hspace{1cm} (2.9)
4. From [6], one has for $|\xi_i| \sim N_i$, $i = 1, 2$, $N_1 \leq N_2$,
\[
\|u_1 u_2\|_{L^2([0,\delta] \times \mathbb{R}^2)} \leq c \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|u_1\|_{X^0,\frac{1}{2}^+} \|u_2\|_{X^0,\frac{1}{2}^+}. \tag{2.10}
\]

5. For $s_1 \leq s_2$
\[
\|f\|_{X^{s_1,b}} \leq c \|f\|_{X^{s_2,b}}, \tag{2.11}
\]
and for $b_1 \leq b_2$
\[
\|f\|_{X^{b_1}} \leq c \|f\|_{X^{b_2}}. \tag{2.12}
\]

Finally, we give the sharp Gagliardo-Nirenberg inequality for $\mathbb{R}^2$, which could been found in [17].
\[
\frac{1}{2} \|u\|^4_{L^4} \leq \frac{\|u\|^2_{L^2}}{\|Q\|_{L^2}^2} \|\nabla u\|_{L^2}^2, \quad \text{for } u \in H^1, \text{ and } u \neq 0. \tag{2.13}
\]

3 Local Existence Theory

The existence and uniqueness for system (1.11) holds by the results of [11] for $(u_0, n_{\pm}) \in H^s \times L^2$, $s \geq \frac{1}{2}$.

If we apply operator $I$ to the system (1.11), we have
\[
\begin{aligned}
&i\partial_t (Iu) + \Delta Iu = I(n_+ + n_-)u \\
&(i\partial_t \mp \Lambda)In_{\pm} = \pm \Lambda I(|u|^2)
\end{aligned}
\]
\[Iu(0) = Iu_0, \quad In_{\pm}(0) = In_{\pm 0}. \tag{3.1}
\]

**Proposition 3.1.** Assume $(u_0, n_{\pm 0}) \in H^s \times L^2$, and $1 > s \geq \frac{1}{2}$. Then there exists a positive number $\delta = \min\{c \frac{1}{\|Iu_0\|_{H^{1-s}}^2 + 17\epsilon}, c \frac{1}{\|Iu_0\|_{H^{1-s}}^2 + 17\epsilon}\}$, with that $\epsilon > 0$ is a small enough parameter, such that system (3.1) has a unique local solution $(Iu, In_{\pm})$ in the time interval $[0, \delta]$ with the property:
\[
\|Iu\|_{X^1,\frac{1}{2}^+} \lesssim \|Iu_0\|_{H^1}, \quad \|In_{\pm}\|_{X^{1-s,\frac{1}{2}^+}} \lesssim \|In_{\pm 0}\|_{H^{1-s}}. \tag{3.2}
\]

**Proof.** Let
\[
E = \{ (Iu, In_{\pm}) \|Iu\|_{X^1,\frac{1}{2}^+} \lesssim \|Iu_0\|_{H^1}, \quad \|In_{\pm}\|_{X^{1-s,\frac{1}{2}+}} \lesssim \|In_{\pm 0}\|_{H^{1-s}} \},
\]
and $(S_0, S_1, S_1)$ defined on $E$ as
\[
\begin{aligned}
S_0(Iu) &= \psi_1 e^{it\Delta} Iu_0 - \frac{i}{2} \psi_1 \int_0^t e^{i(t-s)\Delta} \psi_{\delta} I((n_+ + n_-)u) ds, \\
S_1(In_{\pm}) &= \psi_1 e^{it\Lambda} In_{\pm 0} \mp i \psi_1 \int_0^t e^{i(t-s)\Lambda} \psi_{\delta} \Lambda I(|u|^2) ds,
\end{aligned}
\]
where $\psi_1$ and $\psi_\delta$ are defined before for $0 < \delta \leq 1$.

Then, taking $b' = -\frac{1}{2} +$ and $b = \frac{1}{2} +$ in (2.5) and (2.6), it exists
\[
\|S_0(Iu)\|_{X^{1,1}} \leq c\|Iu_0\|_{H^1} + c\|I((n_+ + n_-)u)\|_{X^{1,1}},
\] (3.3)
and
\[
\|S_1(In_\pm)\|_{X^{1,1}} \leq c\|In_{\pm0}\|_{H^{1,-s}} + c\|\Lambda I(u^2)\|_{X^{1,1}},
\] (3.4)
Next, we use Lemma 3.4 of [11] and [7] to get
\[
\|I((n_+ + n_-)u)\|_{X^{1,1}} \leq c\delta^{\frac{1}{2} - 4\epsilon}\|In_{\pm}\|_{X^{1,1}},
\] (3.5)
We also use Lemma 3.5 of [11] and [7] to get
\[
\|\Lambda I(u^2)\|_{X^{1,1}} \leq c\delta^{\frac{1}{2} - 4\epsilon}\|Iu\|_{X^{1,1}},
\] (3.6)
Combining these estimates together and because $n_+ = n_-$, there exists
\[
\|S_0(Iu)\|_{X^{1,1}} \leq c\|Iu_0\|_{H^1} + c\delta^{\frac{1}{2} - 4\epsilon}\|In_{\pm}\|_{X^{1,1}},
\] (3.7)
and
\[
\|S_1(In_{\pm})\|_{X^{1,1}} \leq c\|In_{\pm0}\|_{H^{1,-s}} + c\delta^{\frac{1}{2} - 4\epsilon}\|Iu\|_{X^{1,1}},
\] (3.8)
Letting $\delta \sim \min\{\frac{1}{\|In_{\pm0}\|_{H^{1,-s}}^{2+17\epsilon}}, \frac{\|In_{\pm0}\|_{H^{1,-s}}^{2+17\epsilon}}{\|Iu_0\|_{H^1}}\}$, such that $\delta^{\frac{1}{2} - 4\epsilon}\|In_{\pm0}\|_{H^{1,-s}} \lesssim 1$ and $\delta^{\frac{1}{2} - 4\epsilon}\|Iu_0\|_{H^1} \lesssim \|In_{\pm0}\|_{H^{1,-s}}$, then we have
\[
(S_0(Iu), S_1(In_{\pm})) \in E,
\] (3.9)
hence $(S_0, S_1, S_1) : E \to E$.

One can prove $(S_0, S_1, S_1)$ is a contraction map with the same method. Thus by the standard fixed point theory, we get the local existence of (3.1). And the uniqueness follows in the same way.

**Proposition 3.2.** Assume $(u_0, n_{\pm0}) \in H^s \times L^2$, with $1 > s \geq \frac{1}{2}$, then there exists a positive number $\delta = \frac{\epsilon}{M^{2+17\epsilon}}$, with $M = (\|u_0\|_{H^s} + \|n_{\pm0}\|_{L^2})$, such that system (1.11) has a unique local solution in the time interval $[0, \tilde{\delta}]$ with the property:
\[
\|u\|_{X^s, \frac{1}{2}} + \|n_{\pm}\|_{X^{0, \frac{1}{2}}} \lesssim M.
\] (3.10)

**Remark 3.3.** The proof is almost the same as Proposition 3.1 except some small changes.
Proof. Let $E = \{(u, n) \mid \|u\|_{X^s} + \|n\|_{X^s} \leq M\}$ and also define $(S_0, S_1, S_2)$ as

\[
S_0(u) = \psi_1 e^{i\alpha} u_0 - \frac{i}{2} \psi_1 \int_0^t e^{i(t-s)\Lambda} \psi_3 (n_+ + n_-) u s, \\
S_1(n) = \psi_1 e^{i\alpha} n_0 + i \psi_1 \int_0^t e^{i(t-s)\Lambda} \psi_3 (|u|^2) ds,
\]

where $\psi_1$ and $\psi_3$ are defined as before.

Then, like Proposition 3.1, we have

\[
\|S_0(u)\|_{X^s} \leq c \|u\|_{H^s} + c \delta^{1/2 - 4\epsilon} \|n_+\|_{X^s} \|u\|_{X^s},
\]

and

\[
\|S_1(n)\|_{X^s} \leq c \|n\|_{L^2} + c \delta^{1/2 - 4\epsilon} \|u\|_{X^s}^2.
\]

Thus we just need to take $\delta = \frac{c}{M^{1/2 - 4\epsilon}}$, then the result of the proposition follows.

From the above Proposition we can see that,

**Corollary 3.4.** If $(u(t), n(t))$ is a finite time blow up solution in $H^s \times L^2$, $\frac{1}{2} \leq s < 1$, with the initial data as above, then $\|u(t)\|_{H^s} + \|n(t)\|_{L^2} \to \infty$, as $t \to T^*$ where $[0, T^*)$ is the maximum life span, which is also equivalent to $\|u(t)\|_{H^s} + \|n(t)\|_{L^2} \to \infty$ as $t \to T^*$ because of $n_+ = n_-.$

**Corollary 3.5.** If $(u(t), n(t))$ is a finite time blow up solution in $H^s \times L^2$, $\frac{1}{2} \leq s < 1$, then

\[
\|Iu(t)\|_{H^1} \to \infty, \text{ as } t \to T^*,
\]

and

\[
\liminf_{t \to T^*} \|Iu(t)\|_{H^{1-s}} > 0,
\]

i.e. there is a $c > 0$ such that $\|Iu(t)\|_{H^{1-s}} \geq c$.

**Proof.** As $\|Iu\|_{H^1} \succeq \|u\|_{H^s}$ and $\|Iu\|_{H^{1-s}} \succeq \|u\|_{L^2}$, by Corollary 3.1, we have

\[
\|Iu\|_{H^1} + \|Iu\|_{H^{1-s}} \to \infty, \text{ as } t \to T^*, \text{ for fixed } N >> 1.
\]

On the other hand, from the proof of Proposition 3.1, one can find that if replacing $\psi_1$ with $\psi_{T^*}$, the estimates also hold. Thus, for $T < T^*$,

\[
\|Iu\|_{X^s, \frac{1}{2}} \leq c \|Iu\|_{H^{1-s}} + c T^{\frac{1}{2} - 4\epsilon} \|Iu\|_{X^{1, \frac{1}{2}}},
\]

and

\[
\|Iu\|_{X^s, \frac{1}{2}} \leq c \|Iu_0\|_{H^1} + c T^{\frac{1}{2} - 4\epsilon} \|Iu_0\|_{X^{1, \frac{1}{2}} + \|Iu\|_{X^{1, \frac{1}{2}}}}.
\]
Hence, if $\|Iu(t)\|_{H^1} \to \infty$, as $t \to T^*$, i.e. $\|Iu\|_{L^\infty([0,T^*),H^1)} \leq A$, for some $A < \infty$, then it has
\[
\|In^+\|_{H^{1-s}} \lesssim \|In^+_0\|_{X^{1-s,\frac{1}{2}+}} + \|In^+_0\|_{H^{1-s}} + T\frac{1}{2} - 4\epsilon \|Iu\|^{2}_{X^{1,\frac{1}{2}+}} \lesssim N^{1-s}\|n^+_0\|_{L^2} + T^s\frac{1}{2} - 4\epsilon \|Iu\|^{2}_{L^\infty([0,T^*),H^1)} \lesssim N^{1-s}\|n^+_0\|_{L^2} + T^s\frac{1}{2} - 4\epsilon A^2 < \infty,
\] (3.18)
for fixed $N >> 1$, by Proposition 3.1 (3.16) and $T^* < \infty$, which contradicts to (3.15). This proves (3.13).

Next, if $\liminf_{t \to T^*} \|In^+ (t)\|_{H^{1-s}} = 0$, then there would be a subsequence $\{t_n\}$, $t_n \to T^*$ as $n \to \infty$, such that $\lim_{n \to \infty} \|In^+(t_n)\|_{H^{1-s}} = 0$, so from (3.17) we have,
\[
\|Iu(t_n)\|_{X^{1,\frac{1}{2}+}} \leq c\|Iu_0\|_{H^1} + cT_n^{\frac{1}{2} - 4\epsilon} \|In^+(t_n)\|_{X^{1-s,\frac{1}{2}+}} \|Iu(t_n)\|_{X^{1,\frac{1}{2}+}} \leq cN^{1-s}\|u_0\|_{H^s} + cT^s\frac{1}{2} - 4\epsilon \|In^+(\tilde{t}_n)\|_{H^{1-s}} \|Iu(t_n)\|_{X^{1,\frac{1}{2}+}},
\]
for some $\tilde{t}_n$, which satisfies $|\tilde{t}_n - t_n| \lesssim \delta$ by the local existence theory Proposition 3.1. Hence, since $T^* \leq \infty$, for $n \to \infty$,
\[
\|Iu(t_n)\|_{X^{1,\frac{1}{2}+}} \lesssim N^{1-s}\|u_0\|_{H^s}.
\] (3.19)

(3.19) gives
\[
\|Iu(t_n)\|_{H^1} < c < \infty,
\]
for fixed $N >> 1$, which contradicts to (3.13).

\[
\square
\]

4 Estimates for the Modified Energy

In this section we’ll get the exact control of the increment of the modified energy.

As the modified energy is $H(t) = H(Iu, In^+) = \|\nabla Iu\|_{L^2}^2 + \frac{1}{2}\|In^+\|_{L^2}^2 + \frac{1}{2} \int I(n^+ + \tilde{n}^+)\|Iu\|_{L^2}^2 dx$, and it is not conserved any more, we have to control its growth. The following is the main proposition of the paper:

**Proposition 4.1.** Let $(Iu, In^\pm)$ be a solution of (3.1) on $[0,\delta]$ in the sense of Proposition 3.1. Then the following estimate holds $(N >> 1)$:
\[
|H(\delta) - H(0)| \leq cN^{-2+s+} \delta^{0+}\|In^+\|_{X^{1-s,\frac{1}{2}+}} \|Iu\|_{X^{1,\frac{1}{2}+}}^2 + cN^{-2+s+}\|In^+\|_{X^{1-s,\frac{1}{2}+}}^2 \|Iu\|_{X^{1,\frac{1}{2}+}}^2. \quad (4.1)
\]
Proof.

\[
\frac{dH(t)}{dt} = 2Re \int \nabla u \nabla u_t dx + Re \int \overline{I_n} I_{n+t} + \frac{1}{2} \int (\overline{(I_{n+})_t} + (I_{n-})_t) |u|^2 + Re \int (I_{n+} + I_{n-}) \overline{I_n} u_t
\]

\[
= -Im \int \Delta \overline{u} ((n_+ + n_-) u) - (I_{n+} + I_{n-}) I_u
\]

\[
+ \frac{1}{2} Im \int \overline{I((n_+ + n_-) u)} (I((n_+ + n_-) u) - (I_{n+} + I_{n-}) I_u)
\]

\[
- Im \int \overline{I_{n+}} \Lambda (|u|^2) - |I_u|^2)
\]

Integrate by \( t \) on \([0, \delta]\), it has

\[
|H(\delta) - H(0)| \leq |\int_0^\delta \int_{\mathbb{R}^2} \Delta \overline{u} ((n_+ + n_-) u) - (I_{n+} + I_{n-}) I_u dx dt|
\]

\[
+ \frac{1}{2} \int_0^\delta \int_{\mathbb{R}^2} \overline{I((n_+ + n_-) u)} (I((n_+ + n_-) u) - (I_{n+} + I_{n-}) I_u) dx dt|
\]

\[
+ |\int_0^\delta \int_{\mathbb{R}^2} \overline{I_{n+}} \Lambda (|u|^2) - |I_u|^2) dx dt|
\]

\[
= I + II + III.
\]

(4.2)

To prove Proposition 4.1 we have to control \( I, II \) and \( III \) in \((4.2)\) respectively.

First for \( I \), it has,

**Lemma 4.2.** \( I \lesssim (N^{-2+s+\delta^0} + N^{-\frac{1}{2}+s+2\delta^1}) ||I_{n+}||_{X_{\frac{1}{2}}^{1-s}}^2 ||I_u||_{X_{\frac{1}{2}}^{1-s}}^2 \).

**Proof.** As

\[
I = |\int_0^\delta \int_{\mathbb{R}^2} \Delta \overline{u} ((n_+ + n_-) u) - (I_{n+} + I_{n-}) I_u dx dt|
\]

\[
\sim |\int_0^\delta \int_{\xi} |\xi|^2 m(\xi) \hat{u}(\xi) \frac{(m(\xi_2 + \xi_3) - m(\xi_2) m(\xi_3)) m(\xi_2) \hat{n}_+ (\xi_2) m(\xi_3) \hat{u}(\xi_3)) d\xi dt|
\]

here * denotes integration over the set \( \{ \sum_{i=1}^3 \xi_i = 0 \} \) (or \( \{ \sum_{i=1}^4 \xi_i = 0 \} \)).

We break the function \( u \) and \( n_+ \) into a sum of dyadic constituents, each with frequency support \(< \xi_i > \sim 2^j, j = 0, \cdots \) and denote \( u_i = P_{N_i} u, n_+ = P_{N_i} n_+ \).

In the following, let’s note \( m_i = m(\xi_i), |\xi_i| = N_i, N_{max} = \max_{1 \leq i \leq 3} N_i \) (or \( N_{max} = \max_{1 \leq i \leq 4} N_i \)).

Remark also that w.l.o.g. \( \hat{n}_1, \hat{n}_2, \hat{n}_3 \geq 0 \).

As if both \( N_2 \) and \( N_3 \ll N \), then \( m_2 = m_3 = 1 \) such that \( \frac{m(\xi_2 + \xi_3) - m(\xi_2) m(\xi_3)}{m(\xi_2) m(\xi_3)} = 0 \), the left hand side of the inequality becomes 0, which is trivial.
Inclusively, we just need to prove

\[ I' := N_1^2 \int_0^\delta \int_s^\delta \frac{m(\xi_2 + \xi_3) - m_2m_3}{m_2m_3} |u_1 n + 2 u_3| \lesssim N^{-2+s+\delta_0'} \|N_n\|_{X^{1-s, \frac{1}{2}+}} \|Iu\|_{X^{1, \frac{1}{2}+}^{\geq}}, \quad (4.3) \]

with the assumption that at least one of \( N_2 \) and \( N_3 \gtrsim N \).

**Case 1.** \( N_{\text{max}} \sim N_2 \sim N_3 \gtrsim N \)

In this case, \( \frac{m(\xi_2 + \xi_3) - m_2m_3}{m_2m_3} \lesssim \frac{m_1}{m_2m_3} \sim \frac{1}{m_2m_3} \lesssim \left( \frac{N_2}{N} \right)^{2(1-s)} \), by \( \sum_{i=1}^3 \xi_i = 0 \) and the definition of \( m \).

- **a.** \( N_1 \lesssim N_3^\xi \)

  It exists,

  \[ I' \lesssim N_1^2 \left( \frac{N_2}{N} \right)^{2(1-s)} \|n_{+2}\|_{L_{t,x}^2} \|u_1 u_3\|_{L_{t,x}^2} \]

  \[ \lesssim N_1^2 \left( \frac{N_2}{N} \right)^{2(1-s)} \delta^\frac{1}{2} \|n_{+2}\|_{X^{0, \frac{1}{2}+}_{+}} \|u_1\|_{X^{0, \frac{1}{2}+}_{+}} \|u_3\|_{X^{0, \frac{1}{2}+}_{+}} \]

  \[ \lesssim N_1^2 \left( \frac{N_2}{N} \right)^{2(1-s)} \delta^\frac{1}{2} \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \|n_{+2}\|_{X^{1-s, \frac{1}{2}+}_{+}} \|u_1\|_{X^{1, \frac{1}{2}+}} \|u_3\|_{X^{1, \frac{1}{2}+}} \]

  \[ \lesssim N_{\text{max}}^{-\frac{s}{2}+s+2\xi} \delta^\frac{1}{2} \|N_n\|_{X^{1-s, \frac{1}{2}+}_{+}} \|Iu\|_{X^{1, \frac{1}{2}+}^{\geq}}, \quad (4.4) \]

by the definition of \( X_{\phi}^{m,b} \)-space, (2.7), (2.10) and (2.11).

- **b.** \( N_2 \sim N_1 \sim N_3 \)

  We have to take the fourier transform of \( t \) into account in this case, and w.l.o.g \( \hat{u}_1, \hat{n}_{+2}, \hat{u}_3 > 0 \).

  \[ I' \lesssim N_1^2 \left( \frac{N_2}{N} \right)^{2(1-s)} \int_{**} \hat{u}_1(\tau_1)\hat{n}_{+2}(\tau_2)\hat{u}_3(\tau_3) \hat{\phi}(\tau_0) d\xi d\tau, \quad (4.5) \]

here ** denotes integration over \( \sum_{i=1}^3 \xi_i = \sum_{i=0}^3 \tau_i = 0 \), and \( \phi(t) \) is the characteristic function of the time interval \([0, \delta]\).

  It is known that \( \hat{\phi}(\tau) = \frac{1}{\sqrt{2\pi}} e^{i\tau \phi - \frac{1}{2}\tau} \in L^1_{t} \) but not in \( L^1_{t}^{\geq} \).

  To deal with this case, we need the following algebraic inequality.

  \[ |\xi_1| \lesssim < \tau_1 + |\xi_1| > ^{\frac{1}{2}} + < \tau_2 + |\xi_2| > ^{\frac{1}{2}} + < \tau_3 + |\xi_3| > ^{\frac{1}{2}} + |\tau_0| ^{\frac{1}{2}}, \quad (4.6) \]

and consider every 4 cases according to which terms on the r.h.s is dominant.

**Subcase 1.** \( < \tau_1 + |\xi_1| > ^{\frac{1}{2}} \) dominant.
for the same reason as subcase 1.

\( I' \lesssim \langle N \rangle^{2(1-s)} N^{-\frac{1}{4}} \int_{\Sigma^s} <\tau_1 + |\xi|> |\phi| \frac{d\xi}{d\tau} \)

by Hölder inequality, Berstein inequality, (2.A), (2.M), and Hausdorff-Young, which gives

\[ \|F^{-1}(<\phi>)\|_{L^\infty_u} \lesssim \|\phi\|_{L^{1+}_t} \lesssim \delta^{0+}. \]  

**Subcase 2.** \(<\tau_2 + |\xi|> \frac{1}{4}$$ dominant.

\( I' \lesssim \langle N \rangle^{2(1-s)} N^{-\frac{1}{2}} \int_{\Sigma^s} <\tau_2 + |\xi|> \frac{1}{4} |\phi(\tau_0)| \hat{u}_1(\tau_0) \hat{u}_2(\tau_2) \hat{u}_3(\tau_3) d\xi d\tau \)

**Subcase 3.** \(<\tau_3 + |\xi|> \frac{1}{4}$$ dominant.

Almost the same as subcase 1.

**Subcase 4.** \(|\tau_0| \frac{1}{2}$$ dominant.

\( I' \lesssim \langle N \rangle^{2(1-s)} N^{-\frac{1}{2}} \int_{\Sigma^s} |\tau_0| \frac{1}{2} |\phi(\tau_0)| \hat{u}_1 \hat{u}_2 \hat{u}_3 d\xi d\tau \)

\[ \lesssim \langle N \rangle^{2(1-s)} N^{-\frac{1}{2}} \int_{\Sigma^s} |\tau| \frac{1}{2} |\phi| \hat{u}_1 \hat{u}_2 \hat{u}_3 \|L^2_{\xi_1} L^1_{\tau_1}. \]
The first factor is estimated as follows by Hölder w.r.t. $\tau_1$:

\[
\|\tilde{u}_1\|_{L^2_{\xi_1} L^1_{r_1}} = \|\tilde{u}_1 < \tau_1 + |\xi_1|^2 > \frac{1}{2} + \tau_1 + |\xi_1|^2 > -\frac{1}{2} \|_{L^2_{\xi_1} L^1_{r_1}} \\
\lesssim \|\tilde{u}_1 < \tau_1 + |\xi_1|^2 > \frac{1}{2} + \|_{L^2_{\xi_1} L^1_{r_1}} < \tau_1 + |\xi_1|^2 > -\frac{1}{2} \|
\lesssim \frac{1}{N_1}\|u_1\|_{X^1_{0, \frac{1}{2}+}},
\]

(4.11)

since $(-\frac{1}{2} - \epsilon)^{2(1+\epsilon)} - 1 < -1$ which ensure the integrable condition at infinite for $\tau_1$.

The second factor is bounded by Young's inequality by

\[
\|\hat{\phi} \ast \tilde{n}_{n+2} \ast \tilde{u}_3\|_{L^2_{\xi} L^1_{r}} \lesssim \|\tau|^{\frac{1}{2}} |\hat{\phi}|\|_{L^2_{\xi} L^1_{r}} \|\tilde{u}_3 \ast \tilde{n}_{n+2}\|_{L^2_{\xi} L^2_{n+2}} \\
\lesssim \delta^{0+} \|\tilde{u}_3\|_{L^1_{\xi} L^{2+n+2}_{r}} \|\tilde{n}_{n+2}\|_{L^2_{\xi} L^1_{r}},
\]

(4.12)

here we use the bound $|||\tau|^{\frac{1}{2}} |\hat{\phi}|||_{L^{2+n+2}_{\xi} L^1_{r}} \lesssim \delta^{0+}$.

Because

\[
\|\tilde{u}_3\|_{L^1_{\xi} L^{2+n+2}_{r}} \lesssim N^{\frac{7}{2}}_3 \|\tilde{u}_3 < \xi_3 > < \tau_3 + |\xi_3|^2 > \frac{1}{2} + \xi_3 > -\frac{1}{2} \|_{L^{2-n+2}_{\xi} L^1_{r}} \\
\lesssim N^{\frac{7}{2}}_3 \|\tilde{u}_3 < \xi_3 > < \tau_3 + |\xi_3|^2 > \frac{1}{2} + \|_{L^2_{\xi} L^2_{r}} \| < \xi_3 > -\frac{1}{2} \|_{L^2_{\xi} L^1_{r}} \| \xi_3|^2 > -\frac{1}{2} \|_{L^{2-n+2}_{\xi} L^1_{r}} \\
\lesssim N^{\frac{7}{2}}_3 \|u_3\|_{X_{0, \frac{1}{2}+}} \lesssim N^{\frac{7}{2}-1}_3 \|u_3\|_{X_{0, \frac{1}{2}+}},
\]

(4.13)

by $\frac{2(1-\epsilon)}{\epsilon}(-\frac{1}{2} - \epsilon) < -1$ and $2(-\frac{1}{2} - \epsilon) + 1 < -1$.

And

\[
\|\tilde{n}_{n+2}\|_{L^1_{\xi} L^1_{r}} = \|\tilde{n}_{n+2} < \tau_2 + |\xi_2|^2 > \frac{1}{2} + \tau_2 + |\xi_2|^2 > -\frac{1}{2} \|_{L^2_{\xi} L^1_{r}} \\
\lesssim \|\tilde{n}_{n+2} < \tau_2 + |\xi_2|^2 > \frac{1}{2} + \|_{L^2_{\xi} L^2_{r}} \| < \tau_2 + |\xi_2|^2 > -\frac{1}{2} \|_{L^{2-n+2}_{\xi} L^1_{r}} \\
\lesssim \|n+2\|_{X_{0, \frac{1}{2}+}} \lesssim \frac{1}{N^{1-s}_2} \|n+2\|_{X_{1-s, \frac{1}{2}+}},
\]

(4.14)

since $2(-\frac{1}{2} - \epsilon) < -1$.

Hence,

\[
(4.12) \lesssim N^{\frac{7}{2}-1}_3 N^{(1-s)}_2 \|u_3\|_{X_{0, \frac{1}{2}+}} \|n+2\|_{X_{1-s, \frac{1}{2}+}},
\]

(4.15)

then with (4.11), it has

\[
I' \leq \left(\frac{N^{1-s}_2}{N}\right) N^{\frac{7}{2}-1}_3 \frac{1}{N^{1-s}_2} \|I_{n+2}\|_{X_{1-s, \frac{1}{2}+}} \|Iu\|_{X_{1-s, \frac{1}{2}+}}^2 \lesssim N^{\frac{7}{2}-1}_2 \delta^{0+} \|I_{n+2}\|_{X_{1-s, \frac{1}{2}+}} \|Iu\|_{X_{1-s, \frac{1}{2}+}}^2.
\]

(4.16)
c. \( N_3^\epsilon \leq N_1 \leq N_3 \sim N_2 \).

Deal with this situation like case b, hence, for subcase 1,

\[
I' \lesssim \left( \frac{N_2}{N} \right)^{2(1-s)} \frac{1}{N_1} \frac{1}{N_2^{1-s}} \frac{1}{N_3} N_3^2 \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}},
\]

\[
\lesssim N_{\text{max}}^N N^{-2+s+\epsilon} N_1^{-\epsilon} N_3^{-\frac{\epsilon}{2}} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}},
\]

\[
\lesssim N_{\text{max}}^N N^{-2+s+\epsilon} N_3^{-\frac{\epsilon}{2}} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}},
\]

\[
\lesssim N_{\text{max}}^N N^{-2+s+\epsilon} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}},
\]

And the other three subcases could be dealt with in the same way.

**Case 2** \( N_{\text{max}} \sim N_2 \sim N_1 \gg N \).

a. \( N_1 \sim N_2 \sim N_3 \).

This case is the same as Case 1b.

b. \( N_1 \sim N_2 \gg N_3 \gg N \).

Then \( \left| \frac{m(\xi_2 + \xi_3) - m_2}{m_2 m_3} \right| \lesssim \frac{m(\xi_2 + \xi_3)}{m_2 m_3} \lesssim \frac{m_3}{m_2 m_3} = \frac{1}{m_3} \lesssim \left( \frac{N}{N} \right)^{1-s} \), and with the same assumption and argument as Case 1b gives

\[
I' \lesssim \left( \frac{N_3}{N} \right)^{1-s} N_1^2 \int_{s,s} \tilde{u}_1(\tau_1) \tilde{n}_2(\tau_2) \tilde{u}_3(\tau_3) |\phi(\tau_0)| d\xi d\tau.
\]  \quad (4.17)

We also divide this case into 4 subcases as before.

As the main part is almost the same, we only show some difference in the follow.

**Subcase 1** \( \tau_1 + |\xi_1|^2 > \frac{7}{2} \) dominant.

\[
I' \lesssim \left( \frac{N_3}{N} \right)^{1-s} N_1^2 \int_{s,s} \tilde{u}_1(\tau_1) \tilde{n}_2(\tau_2) \tilde{u}_3(\tau_3) |\phi(\tau_0)| d\xi d\tau.
\]  \quad (4.18)

**Subcase 2** \( \tau_2 + |\xi_2| > \frac{5}{2} \) dominant.
\[ I' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \frac{1}{N_1} \delta^{0+} \frac{1}{N_2} \frac{1}{N_3} N_3 \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} \]
\[ \lesssim N_{max}^{-\frac{s}{2}} N^{-2+s} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} . \]

**Subcase 3** \( \tau_3 + |\xi| > \frac{1}{2} \) dominant.

\[ I' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \frac{1}{N_1} \delta^{0+} \frac{1}{N_2} \frac{1}{N_3} N_1 N_3 \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} \]
\[ \lesssim N_{max}^{-\frac{s}{2}} N^{-2+s} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} . \]

**Subcase 4** \( |\tau_0| > \frac{1}{2} \) dominant.

\[ I' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \frac{1}{N_1} N_3^{\frac{s}{2}} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} \]
\[ \lesssim N_{max}^{-\frac{s}{2}} N^{-2+s} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} . \]

\( c \) \( N_1 \sim N_2 \gg N > N_3 \).

\[ |m(\xi_2 + \eta_3) - m_2| \lesssim |m(\xi_2 + \eta_3) - m_2| \lesssim \frac{|N_3|}{N_2} \lesssim |N_3| \frac{|N_3|}{N_2} \]. We also deal with it in 4 subcases.

**Subcase 1** \( \tau_1 + |\xi_1| > \frac{1}{2} \) dominant.

\[ I' \lesssim N_3 \frac{1}{N_2} \frac{1}{N_1} \frac{1}{N_3} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} \lesssim N_{max}^{-\frac{s}{2}} N^{-2+s} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} . \]

(4.19)

by Hölder inequality, Berstein inequality, (2.8), (2.9) and (4.8).

**Subcase 2** \( \tau_2 + |\xi_2| > \frac{1}{2} \) dominant.

\[ I' \lesssim N_3 \frac{1}{N_2} \frac{1}{N_1} \delta^{0+} \frac{1}{N_3} \frac{1}{N_2} \frac{1}{N_3} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} \lesssim N_{max}^{-\frac{s}{2}} N^{-2+s} \delta^{0+} \| I n_+ \|_{X^{1-s, \frac{1}{2}}} \| I u \|_{X^{1, \frac{1}{2}}} . \]

(4.20)

also by Hölder inequality, Berstein inequality, (2.8), (2.9) and (4.8).

**Subcase 3** \( \tau_3 + |\xi_3| > \frac{1}{2} \) dominant.

The same as subcase 1.

**Subcase 4** \( |\tau_0| > \frac{1}{2} \) dominant.
By Hölder inequality, (4.11) and (4.15), we have

$$I' \lesssim \frac{N_3}{N_2} < N_3 > \frac{1}{N_2^{1-s}} \delta^0 \| \nu_+ \|_{X^{1-s, \frac{3}{2}}} \| u \|_{X^{1, \frac{3}{2}}} \lesssim N_{max} N^{-2+s} \delta^0 \| \nu_+ \|_{X^{1-s, \frac{3}{2}}} \| u \|_{X^{1, \frac{3}{2}}}.$$  \hfill (4.21)

**Case 3** $N_{max} \sim N_3 \sim N_1 \gtrsim N$.

a. $N_3 \sim N_1 \sim N_3$.

The same as Case 1b.

b. $N_3 \sim N_1 \gg N_2 \gtrsim N$.

$$| \frac{m(\xi_2 + \xi_3) - m_2 m_3}{m_3 m_2} | \lesssim \frac{m(\xi_2 + \xi_3) - m_2 m_3}{m_3 m_2} \lesssim \frac{m_3}{m_2} \lesssim \left( \frac{N_3}{N} \right)^{1-s}. $$

Argue as before, we have:

$$I' \lesssim \left( \frac{N_2}{N} \right)^{1-s} (\frac{1}{N_1 N_2^{1-s} N_3} \sum_1 \frac{1}{N_1} + \frac{1}{N_1 N_2^{1-s} N_3} \sum_1 \frac{1}{N_1} + N_3^{\frac{s}{2} - 1} N_1^{\frac{s}{2} - 1} N_3^{\frac{s}{2} - 1} \sum_1 \| \nu_+ \|_{X^{1-s, \frac{3}{2}}} \| u \|_{X^{1, \frac{3}{2}}}.$$  \hfill (4.22)

$$I' \lesssim N_{max} N^{-2+s} \delta^0 \| \nu_+ \|_{X^{1-s, \frac{3}{2}}} \| u \|_{X^{1, \frac{3}{2}}}.$$  \hfill (4.23)

Taking all the above estimates into account, the result of Lemma 4.2 holds.

Now, let's deal with $II$.

**Lemma 4.3.**

$$II \lesssim N^{-2+s} \| \nu_+ \|_{X^{1-s, \frac{3}{2}}}^2 \| u \|_{X^{1, \frac{3}{2}}}^2. $$  \hfill (4.22)

**Proof.** Like part $I$, to prove the estimate for $II$, we just need to prove

$$II' := \int_0^\delta \int_\epsilon \left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \left| \frac{m(\xi_3 + \xi_4) - m_3 m_4}{m_3 m_4} \right| \nu_1 \nu_2 \nu_3 \nu_4 \lesssim N^{-2+s} \| \nu_+ \|_{X^{1-s, \frac{3}{2}}}^2 \| u \|_{X^{1, \frac{3}{2}}}^2. $$  \hfill (4.23)

with the same notations and assumptions in Lemma 4.2.
One can also easily to see that if both $N_3$ and $N_4 << N$, l.h.s would be zero, which is trivial, so we can suppose at least one of $N_3$ and $N_4 \gtrsim N$.

**Case 1** $N_{\text{max}} \sim N_3 \sim N_4 \gtrsim N$.

So $|\frac{m(\xi_1 + \xi_2) - m_{m^4}}{m_{m^4}}| \leq |\frac{m(\xi_1 + \xi_2)}{m_{m^4}}| \lesssim \frac{1}{m_{m^4}} \lesssim \left(\frac{N}{N}\right)^{2(1-s)}$.

a $N_1, N_2 << N$.

In this case, $\frac{m(\xi_1 + \xi_2)}{m_{m^2}} \sim 1$, and

$$II' \lesssim \frac{N_3}{N} \frac{2(1-s)}{||n_1+u_2||_{L_{t,x}^2} ||n_2+u_4||_{L_{t,x}^2}}$$

$$\lesssim \frac{N_3}{N} \frac{2(1-s)}{||n_1||_{L_2^\infty L_2^\infty} ||u_2||_{L_2^2 L_2^\infty} ||n_3||_{L_2^\infty L_2^\infty} ||u_4||_{L_2^2 L_2^\infty}}$$

$$\lesssim \frac{N_3}{N} \frac{2(1-s)}{||n_1||_{X_+^{0,\frac{3}{4}}} ||u_2||_{X_+^{0,\frac{3}{4}}} ||n_3||_{X_+^{0,\frac{3}{4}}} ||u_4||_{X_+^{0,\frac{3}{4}}}}$$

$$\lesssim \frac{N_3}{N} \frac{2(1-s)}{<N_1>^{1-s} <N_2> N_3^{-1-s} N_4} ||n_1||_{X_+^{1-s,\frac{3}{4}}} ||u_2||_{X_+^{1-s,\frac{3}{4}}} ||n_3||_{X_+^{1-s,\frac{3}{4}}} ||u_4||_{X_+^{1-s,\frac{3}{4}}}$$

$$\lesssim N_{\text{max}}^{-\epsilon} N^{-2+s} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}}$$

(4.24)

by Hölder inequality, (2.8) and (2.9).

b $N_1 << N, N_2 \gtrsim N$.

Then, $\frac{m(\xi_1 + \xi_2)}{m_{m^2}} \sim 1$, and

$$II' \lesssim ||n_1+u_2||_{L_{t,x}^2} ||n_2+u_4||_{L_{t,x}^2}$$

$$\lesssim \frac{1}{<N_1>^{1-s} N_2 N_3^{-1-s} N_4} ||n_1||_{X_+^{1-s,\frac{3}{4}}} ||u_2||_{X_+^{1-s,\frac{3}{4}}} ||n_3||_{X_+^{1-s,\frac{3}{4}}} ||u_4||_{X_+^{1-s,\frac{3}{4}}}$$

$$\lesssim N_{\text{max}}^{-\epsilon} N^{-4+s} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}}$$

(4.25)

c $N_1 \gtrsim N, N_2 << N$.

$\frac{m(\xi_1 + \xi_2)}{m_{m^2}}$ is also $\sim 1$.

$$II' \lesssim ||n_1+u_2||_{L_{t,x}^2} ||n_2+u_4||_{L_{t,x}^2}$$

$$\lesssim \frac{1}{N_1^{-1-s} <N_2> N_3^{-1-s} N_4} ||n_1||_{X_+^{1-s,\frac{3}{4}}} ||u_2||_{X_+^{1-s,\frac{3}{4}}} ||n_3||_{X_+^{1-s,\frac{3}{4}}} ||u_4||_{X_+^{1-s,\frac{3}{4}}}$$

$$\lesssim N_{\text{max}}^{-\epsilon} N^{-3+2s} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}} ||Iu||^2_{X_+^{1-s,\frac{3}{4}}}$$

(4.26)

d $N_1, N_2 \gtrsim N$. 

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It exists \( \frac{m(\xi_1+\xi_2)}{m_1m_2} \lesssim \frac{1}{m_1m_2} \lesssim \left( \frac{N}{N_3} \right)^{1-s} (N_3)^{1-s} \), thus

\[
II' \lesssim \left( \frac{N}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} ||n_1u_2||_{L^2_t, x} ||n_3u_4||_{L^2_t, x} \\
\lesssim \left( \frac{N}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \frac{1}{N_1^{1-s}N_2N_3^{1-s}N_4} ||n_1||_{X^{1-s, \frac{1}{2}+}} ||u_2||_{X^{1-s, \frac{1}{2}+}} ||n_3||_{X^{1-s, \frac{1}{2}+}} ||u_4||_{X^{1, \frac{1}{2}+}} \\
\lesssim N^{-\epsilon}N^{-4+2s+} ||In_+||_{X^{1-s, \frac{1}{2}+}}^2 ||In_+||_{X^{1-s, \frac{1}{2}+}}^2.\tag{4.27}
\]

**Case 2** \( N_{\text{max}} \sim N_1 \sim N_3 \gtrsim N \).

a) \( N_2, N_1 << N \).

In this case, \( \frac{m_1+\xi_2}{m_1m_2} \lesssim \frac{1}{m_1} \sim 1 \), and \( \frac{m(\xi_1+\xi_2)-m_3m_4}{m_3m_4} \lesssim \frac{1}{m_3m_4} \lesssim \frac{N_4}{N} \).

Therefore,

\[
II' \lesssim \frac{N}{N_3} ||n_1u_2||_{L^2_t, x} ||n_3u_4||_{L^2_t, x} \\
\lesssim \frac{N}{N_3} \frac{1}{N_1^{1-s}N_2N_3^{1-s}N_4} ||n_1||_{X^{1-s, \frac{1}{2}+}} ||u_2||_{X^{1-s, \frac{1}{2}+}} ||n_3||_{X^{1-s, \frac{1}{2}+}} ||u_4||_{X^{1, \frac{1}{2}+}} \\
\lesssim N^{-\epsilon}N^{-3+2s+} ||In_+||_{X^{1-s, \frac{1}{2}+}}^2 ||u||_{X^{1, \frac{1}{2}+}}.\tag{4.28}
\]

b) \( N_2 << N, N_4 \gtrsim N \).

So \( \frac{m(\xi_1+\xi_2)}{m_1m_2} \sim 1 \), \( \frac{m(\xi_1+\xi_2)-m_3m_4}{m_3m_4} \lesssim \frac{m_3m_4}{m_3m_4} \sim \frac{1}{m_3m_4} \lesssim \frac{N_4}{N} \sim (N/N)^{1-s} \), and

\[
II' \lesssim \left( \frac{N}{N} \right)^{1-s} ||n_1u_2||_{L^2_t, x} ||n_3u_4||_{L^2_t, x} \\
\lesssim \left( \frac{N}{N} \right)^{1-s} \frac{1}{N_1^{1-s}N_2N_3^{1-s}N_4} ||n_1||_{X^{1-s, \frac{1}{2}+}} ||u_2||_{X^{1-s, \frac{1}{2}+}} ||n_3||_{X^{1-s, \frac{1}{2}+}} ||u_4||_{X^{1, \frac{1}{2}+}} \\
\lesssim N^{-\epsilon}N^{-3+2s+} ||In_+||_{X^{1-s, \frac{1}{2}+}}^2 ||In_+||_{X^{1-s, \frac{1}{2}+}}^2.\tag{4.29}
\]

c) \( N_2 \gtrsim N, N_4 << N \).

Then \( \frac{m(\xi_1+\xi_2)}{m_1m_2} \sim \frac{m_3m_4}{m_1m_2} \sim \frac{1}{m_2} \lesssim \left( \frac{N_4}{N} \right)^{1-s} \), \( \frac{m(\xi_1+\xi_2)-m_3m_4}{m_3m_4} \lesssim \frac{N_4}{N_3} \), and

\[
II' \lesssim \left( \frac{N_2}{N} \right)^{1-s} \frac{N_4}{N_3} ||n_1u_2||_{L^2_t, x} ||n_3u_4||_{L^2_t, x} \\
\lesssim \left( \frac{N_2}{N} \right)^{1-s} \frac{N_4}{N_3} \frac{1}{N_1^{1-s}N_2N_3^{1-s}N_4} ||n_1||_{X^{1-s, \frac{1}{2}+}} ||u_2||_{X^{1-s, \frac{1}{2}+}} ||n_3||_{X^{1-s, \frac{1}{2}+}} ||u_4||_{X^{1, \frac{1}{2}+}} \\
\lesssim N^{-\epsilon}N^{-4+2s+} ||In_+||_{X^{1-s, \frac{1}{2}+}}^2 ||In_+||_{X^{1-s, \frac{1}{2}+}}^2.\tag{4.30}
\]

d) \( N_2, N_4 \gtrsim N \).

d1) At least one of \( N_2 \) and \( N_4 \sim N_1 \sim N_3 \), w.l.o.g we suppose \( N_2 \sim N_1 \sim N_3 \).
Hence, \( \left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \lesssim \frac{1}{m_1 m_2} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \) and \( \left| \frac{m(\xi_3 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \frac{m(\xi_1 + \xi_4)}{m_{3m_4}} \lesssim \frac{1}{m_{3m_4}} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \), and

\[
II' \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \left( \frac{N_3}{N} \right)^{1-s} \left( \frac{N_4}{N} \right)^{1-s} \frac{1}{N_1^{1-s} N_2 N_3^{-s} N_4} \| I m_+ \|_{X_t^{1-s, \frac{1}{2}+}}^2 \| I u \|_{X_t^{1-s, \frac{1}{2}+}}^2.
\]

\[d2 \ N_2, N_4 << N_1 \sim N_3.\]

Then \( \left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \lesssim \frac{1}{m_1 m_2} \lesssim \left( \frac{N_2}{N} \right)^{1-s} \) and \( \left| \frac{m(\xi_3 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \frac{m(\xi_1 + \xi_4)}{m_{3m_4}} \lesssim \frac{m_3}{m_{3m_4}} \lesssim \frac{1}{m_4} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \).

\[
II' \lesssim \left( \frac{N_2}{N} \right)^{1-s} \left( \frac{N_4}{N} \right)^{1-s} \frac{1}{N_1^{1-s} N_2 N_3^{-s} N_4} \| I m_+ \|_{X_t^{1-s, \frac{1}{2}+}}^2 \| I u \|_{X_t^{1-s, \frac{1}{2}+}}^2.
\]

Case 3 \( N_{max} \sim N_2 \sim N_3 \gtrsim N.\)

\(a \ N_1, N_4 << N.\)

\[
\left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \sim \frac{m_2}{m_2} \sim 1, \quad \left| \frac{m(\xi_1 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \frac{N_4}{N_3}, \quad \text{and}
\]

\[
II' \lesssim \frac{N_4}{N_3} \| n_{+1} u_2 \|_{L_t^2} \| n_{+3} u_4 \|_{L_t^2},
\]

\[
\lesssim \frac{N_4}{N_3} \frac{1}{N_1^{1-s} N_2 N_3^{-s} N_4} \| n_{+1} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_2 \|_{X_t^{1-s, \frac{1}{2}+}} \| n_{+3} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_4 \|_{X_t^{1-s, \frac{1}{2}+}} \lesssim N_{max}^{-\epsilon} N^{-3+s} \| I m_+ \|_{X_t^{1-s, \frac{1}{2}+}}^2 \| I u \|_{X_t^{1-s, \frac{1}{2}+}}^2.
\]

\[m(\xi_1 + \xi_2) \lesssim \frac{m_2}{m_2} \sim 1, \quad \left| \frac{m(\xi_1 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \left( \frac{N_4}{N_3} \right)^{1-s} \quad \text{and}
\]

\[
II' \lesssim \left( \frac{N_4}{N} \right)^{1-s} \| n_{+1} u_2 \|_{L_t^2} \| n_{+3} u_4 \|_{L_t^2},
\]

\[
\lesssim \left( \frac{N_4}{N} \right)^{1-s} \frac{1}{N_1^{1-s} N_2 N_3^{-s} N_4} \| n_{+1} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_2 \|_{X_t^{1-s, \frac{1}{2}+}} \| n_{+3} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_4 \|_{X_t^{1-s, \frac{1}{2}+}} \lesssim N_{max}^{-\epsilon} N^{-3+s} \| I m_+ \|_{X_t^{1-s, \frac{1}{2}+}}^2 \| I u \|_{X_t^{1-s, \frac{1}{2}+}}^2.
\]

\(b \ N_1, N_4 \lesssim N.\)

\[
\left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \sim \frac{m_2}{m_2} \sim 1, \quad \left| \frac{m(\xi_3 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \left( \frac{N_4}{N_3} \right)^{1-s} \quad \text{and}
\]

\[
II' \lesssim \left( \frac{N_4}{N} \right)^{1-s} \| n_{+1} u_2 \|_{L_t^2} \| n_{+3} u_4 \|_{L_t^2},
\]

\[
\lesssim \left( \frac{N_4}{N} \right)^{1-s} \frac{1}{N_1^{1-s} N_2 N_3^{-s} N_4} \| n_{+1} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_2 \|_{X_t^{1-s, \frac{1}{2}+}} \| n_{+3} \|_{X_t^{1-s, \frac{1}{2}+}} \| u_4 \|_{X_t^{1-s, \frac{1}{2}+}} \lesssim N_{max}^{-\epsilon} N^{-3+s} \| I m_+ \|_{X_t^{1-s, \frac{1}{2}+}}^2 \| I u \|_{X_t^{1-s, \frac{1}{2}+}}^2.
\]

\(c \ N_1 \geq N, N_4 << N.\)

\[
\left| \frac{m(\xi_1 + \xi_2)}{m_1 m_2} \right| \sim \left| \frac{m(\xi_3 + \xi_4)}{m_1 m_2} \right| \lesssim \frac{m_3}{m_2} \sim \frac{1}{m_1} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \quad \text{and}
\]

\[
\left| \frac{m(\xi_3 + \xi_4) - m_{3m_4}}{m_{3m_4}} \right| \lesssim \left| \frac{\nabla m_4 \xi_4}{m_3} \right| \lesssim \frac{N_4}{N_1}.
\]
Then

\[
II' \lesssim \left( \frac{N_1}{N} \right)^{1-s} N_1^{1-s} N_2^{1-s} N_3^{1-s} \lesssim N_4 > \|n+1\|_{X_+^{1-s,\frac{1}{2}+}} \|u_2\|_{X_+^{1,\frac{1}{2}+}} \|n+3\|_{X_+^{1-s,\frac{1}{2}+}} \|u_4\|_{X_+^{1,\frac{1}{2}+}}
\]

\[
\lesssim N_{\text{max}}^{-\varepsilon} N^{-4+2s+} \|I_{3+1}\|_{X_+^{1-s,\frac{1}{2}+}}^2 \|I u\|_{X_+^{1,\frac{1}{2}+}}^2.
\]

**d** \(N_1, N_4 \gtrsim N\).

\[
\frac{m(\xi + \xi)_{m_1 m_2}}{m_1 m_2} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_4}{N} \right)^{1-s}, \frac{m(\xi + \xi - m_3 m_4)}{m_3 m_4} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_4}{N} \right)^{1-s}, \text{ and}
\]

\[
II' \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \left( \frac{N_3}{N} \right)^{1-s} \left( \frac{N_4}{N} \right)^{1-s} \|n+1\|_{X_+^{1-s,\frac{1}{2}+}} \|u_2\|_{X_+^{1,\frac{1}{2}+}} \|n+3\|_{X_+^{1-s,\frac{1}{2}+}} \|u_4\|_{X_+^{1,\frac{1}{2}+}}
\]

\[
\lesssim N_{\text{max}}^{-\varepsilon} N^{-4+2s+} \|I_{3+1}\|_{X_+^{1-s,\frac{1}{2}+}}^2 \|I u\|_{X_+^{1,\frac{1}{2}+}}^2.
\]

**Case 4** \(N_{\text{max}} \sim N_1 \sim N_4 \gtrsim N\).

**a** \(N_2, N_3 \ll N\).

\[
\frac{m(\xi + \xi)_{m_1 m_2}}{m_1 m_2} \sim \frac{m}{m_1} \sim 1, \frac{m(\xi + \xi - m_3 m_4)}{m_3 m_4} \sim \frac{\|u_4\|_{m_4}}{m_4} \lesssim \frac{N}{N_4}.
\]

Thus

\[
II' \lesssim \frac{N_2}{N_4} \|n+1\|_{L_{t,x}^2} \|n+3\|_{L_{t,x}^2}
\]

\[
\lesssim \frac{N_2}{N_4} \frac{1}{N_1^{1-s}} < N_2 > < \frac{N_3}{N_4} \|n+1\|_{X_+^{1-s,\frac{1}{2}+}} \|u_2\|_{X_+^{1,\frac{1}{2}+}} \|n+3\|_{X_+^{1-s,\frac{1}{2}+}} \|u_4\|_{X_+^{1,\frac{1}{2}+}}
\]

\[
\lesssim N_{\text{max}}^{-\varepsilon} N^{-3+2s+} \|I_{3+1}\|_{X_+^{1-s,\frac{1}{2}+}}^2 \|I u\|_{X_+^{1,\frac{1}{2}+}}^2.
\]

**b** \(N_2 \ll N, N_3 \gtrsim N\).

\[
\frac{m(\xi + \xi)_{m_1 m_2}}{m_1 m_2} \sim 1, \frac{m(\xi + \xi - m_3 m_4)}{m_3 m_4} \sim \frac{m}{m_3 m_4} \sim \frac{1}{m_3} \lesssim \left( \frac{N_1}{N} \right)^{-1-s}, \text{ and}
\]

\[
II' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \|n+1\|_{L_{t,x}^2} \|n+3\|_{L_{t,x}^2}
\]

\[
\lesssim \left( \frac{N_3}{N} \right)^{1-s} \frac{1}{N_1^{1-s}} < N_2 > \frac{N_3}{N_4} \|n+1\|_{X_+^{1-s,\frac{1}{2}+}} \|u_2\|_{X_+^{1,\frac{1}{2}+}} \|n+3\|_{X_+^{1-s,\frac{1}{2}+}} \|u_4\|_{X_+^{1,\frac{1}{2}+}}
\]

\[
\lesssim N_{\text{max}}^{-\varepsilon} N^{-3+2s+} \|I_{3+1}\|_{X_+^{1-s,\frac{1}{2}+}}^2 \|I u\|_{X_+^{1,\frac{1}{2}+}}^2.
\]

**c** \(N_2 \gtrsim N, N_3 \ll N\).

\[
\frac{m(\xi + \xi)_{m_1 m_2}}{m_1 m_2} \sim \frac{m}{m_1 m_2} \sim \frac{m}{m_2} \lesssim \left( \frac{N_2}{N} \right)^{1-s}, \frac{m(\xi + \xi - m_3 m_4)}{m_3 m_4} \lesssim \frac{N_3}{N_4}.
\]
Then
\[
II' \lesssim \left( \frac{N_2}{N} \right)^{1-s} \frac{N_3}{N_4} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}} + \|n_{s+1}u_1\|_{L^2_{t,x}} \|n_{s+3}u_4\|_{L^2_{t,x}}
\]
\[
\lesssim \left( \frac{N_2}{N} \right)^{1-s} \frac{N_3}{N_4} \frac{1}{N_1^{1-s}N_2 < N_3 > 1-s N_4} \|n_{s+1}u_1\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} + \|n_{s+3}u_4\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}
\]
\[
\lesssim \frac{N_2}{N} N^{-4+2s+} \|In_{s+1}\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} \|Iu\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}.
\]

4.37

\[
\text{d } N_2, N_3 \gtrsim N.
\]

The same as Case 3(d).

Case 5 \(N_{max} \sim N_2 \sim N_4 \gtrsim N\).

a \(N_1, N_3 < N\).
\[
\left| \frac{m(\xi_j + \xi_k)}{m_1 m_2} \right| \lesssim 1, \left| \frac{m(\xi_j + \xi_k) - m_3 m_4}{m_3 m_4} \right| \lesssim \frac{N_4}{N_3}, \text{ and}
\]
\[
II' \lesssim \frac{N_3}{N_4} \|n_{s+1}u_2\|_{L^2_{t,x}} \|n_{s+3}u_4\|_{L^2_{t,x}}
\]
\[
\lesssim \frac{N_3}{N_4} < N_1 > 1-s N_2 < N_3 > 1-s N_4 \|n_{s+1}u_2\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} + \|n_{s+3}u_4\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}
\]
\[
\lesssim \frac{N_3}{N_4} N^{-3+2s+} \|In_{s+1}\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} \|Iu\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}.
\]

4.38

b \(N_1 < N, N_3 \gtrsim N\).
\[
\left| \frac{m(\xi_j + \xi_k)}{m_1 m_2} \right| \lesssim 1, \left| \frac{m(\xi_j + \xi_k) - m_3 m_4}{m_3 m_4} \right| \lesssim \frac{m(\xi_j + \xi_k)}{m_3 m_4} \approx \frac{m_2}{m_3} \approx \frac{1}{m} \lesssim \left( \frac{N_4}{N_3} \right)^{1-s}, \text{ and}
\]
\[
II' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \|n_{s+1}u_2\|_{L^2_{t,x}} \|n_{s+3}u_4\|_{L^2_{t,x}}
\]
\[
\lesssim \left( \frac{N_3}{N} \right)^{1-s} < N_1 > 1-s N_2 N_3 \|n_{s+1}u_2\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} + \|n_{s+3}u_4\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}
\]
\[
\lesssim \frac{N_3}{N_4} N^{-3+2s+} \|In_{s+1}\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} \|Iu\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}.
\]

4.39

c \(N_1 \gtrsim N, N_3 < N\).
\[
\left| \frac{m(\xi_j + \xi_k)}{m_1 m_2} \right| \lesssim \frac{m(\xi_j + \xi_k)}{m_1 m_2} \lesssim \frac{m_4}{m_1} \lesssim \left( \frac{N_3}{N} \right)^{1-s}, \left| \frac{m(\xi_j + \xi_k) - m_3 m_4}{m_3 m_4} \right| \lesssim \frac{N_3}{N_4}, \text{ and}
\]
\[
II' \lesssim \left( \frac{N_3}{N} \right)^{1-s} \|n_{s+1}u_2\|_{L^2_{t,x}} \|n_{s+3}u_4\|_{L^2_{t,x}}
\]
\[
\lesssim \left( \frac{N_3}{N} \right)^{1-s} < N_1 > 1-s N_2 N_3 \|n_{s+1}u_2\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} + \|n_{s+3}u_4\|_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}
\]
\[
\lesssim \frac{N_3}{N} N^{-4+2s+} \|In_{s+1}\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}} \|Iu\|^2_{X^{s-\frac{1}{2}, \frac{1}{2}}_{t,x}}.
\]

4.40

d \(N_1, N_3 \gtrsim N\).
Lemma 4.4.

\[ III' \lesssim N^{-2+s} \delta^{\frac{1}{2}} \| Iu \|_{X^{s-\frac{1}{2}+}}^2. \]  

(4.41)

Proof. With the same notations and argument as before, to prove this lemma, we just need the following estimate:

\[ III' = N_1 \int_0^\delta \int_s^\delta \frac{|m(\xi_2 + \xi_3) - m_2m_3|}{m_2m_3} |\dot{n}_+ u_2 \dot{u}_3 \lesssim N^{-2+s} \delta^{\frac{1}{2}} \| Iu \|_{X^{s-\frac{1}{2}+}}^2. \]  

(4.42)

If both \( N_2 \) and \( N_3 \ll N \), l.h.s would be zero, and the inequality holds.

On the other hand, w.l.o.g we assume \( N_3 \ll N_2 \).

So let’s suppose \( N_2 \gtrsim N \). Since \( \sum_{i=1}^3 \xi_i = 0 \), \( N_1 \lesssim N_2 \).

Now, we’ll discuss in two subcases.

Case 1 \( N_2 \gtrsim N >> N_3 \).

As \( \sum_{i=1}^3 \xi_i = 0 \), then \( N_1 \sim N_2 \), and \( \frac{|m(\xi_2 + \xi_3) - m_2m_3|}{m_2m_3} \sim \frac{|m(\xi_2 + \xi_3) - m_2|}{m_2} \lesssim \frac{\sum m_2 \xi_3}{m_2} \lesssim \frac{N_3}{N_2} \).

\[ III' \lesssim \frac{N_3}{N_2} \| n+1 \|_{L^2_{t,x}} \| u_2 u_3 \|_{L^2_{t,x}} \]

\[ \lesssim \frac{N_3}{N_2} \| n+1 \|_{X^{0,0}(\frac{N_3}{N_2})^\frac{1}{2}} \| u_2 \|_{X^{0,\frac{1}{2}+}} \| u_3 \|_{X^{0,\frac{1}{2}+}} \]

\[ \lesssim \frac{N_3}{N_2} \left( \frac{N_3}{N_2} \right)^{\frac{1}{2}} \frac{1}{N_{1^{-\frac{1}{2}}}} \frac{1}{N_{1^{-\frac{1}{2}}}} \delta^{\frac{1}{2}} \| n+1 \|_{X^{1-\frac{1}{2},0}} \| u_2 \|_{X^{1,\frac{1}{2}+}} \| u_3 \|_{X^{1,\frac{1}{2}+}} \]

\[ \lesssim N_3 \frac{N_2}{N_3} N_2^{-2+s} \delta^{\frac{1}{2}} \| Iu \|_{X^{s-\frac{1}{2}+}}^2. \]  

(4.43)

by Hölder inequality, \( (2.7) \) and \( (2.10) \).

Case 2 \( N_2 \gtrsim N_3 \gtrsim N \).

Subcase a \( N_1 \sim N_2 \gtrsim N_3 \gtrsim N \).

Finally, let’s consider \( III \).

The same as Case 2(d).
\[
\frac{|m(\xi_2 + \xi_3) - m_2 m_3|}{m_2 m_3} \lesssim \frac{m(\xi_2 + \xi_3)}{m_2 m_3} \sim \frac{m_1}{m_3} \lesssim \left(\frac{N_1}{N}\right)^{1-s}, \text{ then}
\]

\[
III' \lesssim N_1 \left(\frac{N_3}{N}\right)^{1-s} \left\| n_{+1} \right\|_{L^2_{l,x}} \left\| u_{2} u_3 \right\|_{L^2_{l,x}}
\]

\[
\lesssim N_1 \left(\frac{N_3}{N}\right)^{1-s} \left\| n_{+1} \right\|_{X^{0,0}\left(\frac{N_3}{N_2}\right)^{1/2}} \left\| u_2 \right\|_{X^{0,1/2}} \left\| u_3 \right\|_{X^{0,1/2}}
\]

\[
\lesssim N_1 \left(\frac{N_3}{N}\right)^{1-s} \left(\frac{N_3}{N_2}\right)^{3/2} \frac{1}{N_1^{1-s}} \frac{1}{N_2} \frac{1}{N_3} \delta^{1/2} \left\| n_{+1} \right\|_{X^{1-s,0}} \left\| u_2 \right\|_{X^{1,1/2}} \left\| u_3 \right\|_{X^{1,1/2}}
\]

\[
\lesssim N^{-\varepsilon}_m N^{-2+s+2}\delta^{1/2} \left\| I_n u \right\|_{X^{1-s,1/2}} \left\| I u \right\|_{X^{1,1/2}}, \quad (4.44)
\]

with the same reason as case 1.

**Subcase b** \( N_2 \geq N_3 \geq N, \ N_2 >> N_1 \).

So \( N_2 \sim N_3 \) and \( \frac{|m(\xi_2 + \xi_3) - m_2 m_3|}{m_2 m_3} \lesssim \frac{m(\xi_2 + \xi_3)}{m_2 m_3} \sim \frac{1}{m_2} \lesssim \left(\frac{N_3}{N}\right)^{2(1-s)} \).

\[
III' \lesssim N_1 \left(\frac{N_3}{N}\right)^{2(1-s)} \left\| n_{+1} \right\|_{L^2_{l,x}} \left\| u_{2} u_3 \right\|_{L^2_{l,x}}
\]

\[
\lesssim N_1 \left(\frac{N_3}{N}\right)^{2(1-s)} \left\| n_{+1} \right\|_{X^{0,0}\left(\frac{N_3}{N_2}\right)^{1/2}} \left\| u_2 \right\|_{X^{0,1/2}} \left\| u_3 \right\|_{X^{0,1/2}}
\]

\[
\lesssim N_1 \left(\frac{N_3}{N}\right)^{2(1-s)} \left(\frac{N_3}{N_2}\right)^{3/2} \frac{1}{N_1^{1-s}} \frac{1}{N_2} \frac{1}{N_3} \delta^{1/2} \left\| n_{+1} \right\|_{X^{1-s,0}} \left\| u_2 \right\|_{X^{1,1/2}} \left\| u_3 \right\|_{X^{1,1/2}}
\]

\[
\lesssim N^{-\varepsilon}_m N^{-2+s+2}\delta^{1/2} \left\| I_n u \right\|_{X^{1-s,1/2}} \left\| I u \right\|_{X^{1,1/2}}, \quad (4.45)
\]

Now, combing the results of three lemmas above, we can get Proposition 4.11 easily.

\[\square\]

5 **Proof of Theorem 1.1**

Let

\[
\Sigma_u(t) = \sup_{0 \leq \tau \leq t} \left\| I_N \right\|_{L^2_{l,x}} \quad (5.1)
\]

\[
\Sigma_{n_+}(t) = \sup_{0 \leq \tau \leq t} \left\| I_N n_+ \right\|_{L^2_{l,x}} \quad (5.2)
\]

\[
\bar{\Sigma}_u(t) = \sup_{0 \leq \tau \leq t} \left\| I_N u(\tau) \right\|_{X^{1,1/2}} \quad (5.3)
\]

\[
\bar{\Sigma}_{n_+}(t) = \sup_{0 \leq \tau \leq t} \left\| I_N n_+(\tau) \right\|_{X^{1-s,1/2}} \quad (5.4)
\]

and

\[
\Lambda(t) = \sup_{0 \leq \tau \leq t} \left\| I_N \right\|_{L^2_{l,x}} \quad (5.5)
\]

First of all, we’ll prove the following proposition.
Proposition 5.1. With the condition of Theorem 1.1 for \( 1 > s > \frac{16}{17}, \forall T < T^* < \infty \) and close to \( T^* \) enough,

\[
|H(T)| = |H(Iu(T), In_+ (T))| \lesssim N^{p(s)},
\]

where \( N \sim \Lambda(T) \frac{10+34 \epsilon}{16s-6(35-34 \epsilon)} \), \( \epsilon \) small enough such that \( 0 < \epsilon < \frac{17-16s}{69-68s} \) and \( p(s) < 2 \).

Proof. From Proposition 4.1 and the condition of Theorem 1.1 there exists

\[
|H(\delta) - H(0)| \lesssim N^{-2+s+\delta^0+} \| In_+ \|_{X^1 \frac{1}{\delta^2} +} \| Iu \|_{X^1 \frac{1}{\delta^2} +} + N^{-2+s+} \| In_+ \|_{X^1 \frac{1}{\delta^2} +} \| Iu \|_{X^1 \frac{1}{\delta^2} +}
\]

\[
\lesssim N^{-2+s+} \| In_+ \|_{X^1 \frac{1}{\delta^2} +} \| Iu \|_{X^1 \frac{1}{\delta^2} +},
\]

(5.7)

since \( \| In_+ \|_{X^1 \frac{1}{\delta^2} +} \| Iu \|_{X^1 \frac{1}{\delta^2} +} \gtrsim \| In_+ \|_{H^1-s} + \| Iu \|_{H^1} \to \infty \) for \( t \to T^* \).

On the other hand, by Proposition 3.1 we choose \( \delta^{-1} \sim \sum u(T)^{2+17 \epsilon} + (sup_{0 \leq t \leq T} \frac{\| Iu(t) \|^2_{H^1}}{\| Iu(t) \|^2_{H^1}})^{2+17 \epsilon} \).

As from (3.14), it has \( \| In_+(t) \|_{H^1-s} \geq c > 0 \), as \( t \to T^* \). w.l.o.g we suppose \( \| In_+(t) \| \geq c \)

for \( 0 \leq t < T^* \), otherwise, we just need to calculate from \( H(t^*) \) for some \( t^* < T^* \). Thus \( \delta^{-1} \lesssim \sum n_+(T)^{2+17 \epsilon} + \sum u(T)^{2+17 \epsilon} \), and the number of iteration steps to reach the given time \( T \) is \( \frac{T}{\delta} \lesssim T(\sum n_+(T)^{2+17 \epsilon} + \sum u(T)^{2+17 \epsilon}) \).

Combining these estimates with (5.7), the whole increment of energy is

\[
T(\sum n_+(T)^{2+17 \epsilon} + \sum u(T)^{2+17 \epsilon}) N^{-2+s+\epsilon} \sum n_+(T)^2 \sum u(T)^2
\]

\[
\lesssim N^{-2+s+\epsilon} (\sum n_+(T)^{4+17 \epsilon} \sum u(T)^2 + \sum n_+(T)^2 \sum u(T)^{6+34 \epsilon}).
\]

(5.8)

Then, from (3.16) for \( T < T^* \),

\[
\| (In_+) \|_{X^1 \frac{1}{\delta^2} +} \leq c \| In_+0 \|_{H^1-s} + cT \frac{1}{\delta^2} \| Iu \|_{X^1 \frac{1}{\delta^2} +}.
\]

(5.9)

Hence,

\[
\sum n_+(T) \lesssim N^{1-s} \| n_+0 \|_{L^2} + \sum u(T)^2 \lesssim N^{1-s} + \sum u(T)^2,
\]

(5.10)

then put it into (5.8), and by the relationship (3.2), then

\[
N^{-2+s+\epsilon} (N^{6+34 \epsilon}) \sum u(T)^2 + \sum u(T)^{10+34 \epsilon} + N^{2(1-s)} \sum u(T)^{6+34 \epsilon} + \sum u(T)^{10+34 \epsilon})
\]

\[
\lesssim N^{-2+s+\epsilon} (N^{17 \epsilon}) (1-s) \sum u(T)^2 + \sum u(T)^{10+34 \epsilon} + N^{2(1-s)} \sum u(T)^{6+34 \epsilon} + \sum u(T)^{10+34 \epsilon})
\]

\[
\lesssim N^{-2+s+\epsilon} (N^{17 \epsilon}) (1-s) N^{2(1-s)} \Lambda(T)^2 + N^{2(1-s)} N^{6+34 \epsilon} \Lambda(T)^{6+34 \epsilon} + N^{10+34 \epsilon} \Lambda(T)^{10+34 \epsilon})
\]

\[
\lesssim N^{4-5s+(18-17s) \epsilon} \Lambda(T)^2 + N^{6-7s+(35-34s) \epsilon} \Lambda(T)^{6+34 \epsilon} + N^{8-9s+(35-34s) \epsilon} \Lambda(T)^{10+34 \epsilon}.
\]

(5.11)
On the other hand,
\[ |H(0)| = |H(Iu_0, In_{n+0})| = \|\nabla Iu_0\|_{L^2}^2 + \frac{1}{2} \|In_{n+0}\|_{L^2}^2 + \frac{1}{2} \int (In_{n+0} + \overline{Tn_{n+0}}) |Iu|^2 dx \]
\[ \lesssim \|\nabla Iu_0\|_{L^2}^2 + \|In_{n+0}\|_{L^2}^2 + \|In_{n+0}\|_{L^2} \|Iu_0\|_{L^2}^2 \]
\[ \lesssim \|\nabla Iu_0\|_{L^2}^2 + \|In_{n+0}\|_{L^2}^2 + \|In_{n+0}\|_{L^2} \|\nabla Iu_0\|_{L^2}^2 \]
\[ \lesssim \|\nabla Iu_0\|_{L^2}^2 + \|In_{n+0}\|_{L^2}^2 + \|\nabla Iu_0\|_{L^2}^2 \]
\[ \lesssim N^{2(1-s)} \|u_0\|_{H^1}^2 + \|n_{n+0}\|_{L^2}^2 \lesssim N^{2(1-s)}, \quad (5.12) \]

Hence,
\[ |H(T)| \lesssim |H(0)| + |H(T) - T(0)| \]
\[ \lesssim N^{2(1-s)} + N^{4-5s+(18-17s)\epsilon} \Lambda(T)^2 + N^{6-7s+(35-34s)\epsilon} \Lambda(T)^{6+34\epsilon} \]
\[ + N^{8-9s+(35-34s)\epsilon} \Lambda(T)^{10+34\epsilon}. \quad (5.13) \]

Then, choose \( N = \Lambda^{\frac{10+34\epsilon}{t s - 6 - (35-34s)\epsilon}} \), so that the first and fourth terms in (5.13) give comparable contributions. A calculation reveals that the second and third terms in (5.13) produces a smaller correction. Thus
\[ p(s) = 2(1-s) \frac{10 + 34\epsilon}{t s - 6 - (35-34s)\epsilon} < 2 \iff s > \frac{16}{17} \quad \text{and} \quad 0 < \epsilon < \frac{17s - 16}{69 - 68s}. \quad (5.14) \]

Now we turn to prove Theorem 1.1.

**Proof.** Let \( \{t_k\}_{k=1}^\infty \) be a sequence such that \( t_k \uparrow T^* \) as \( k \to \infty \), and for each \( t_k \),
\[ \|u(t_k)\|_{H^s} = \Lambda(t_k). \]

By the result of Corollary 3.5, that \( \|u(t)\|_{H^s} \to \infty \), it’s achievable.

Denote \( u_k = u(t_k) \), and \( Iu_k = I_{N(t_k)}u(t_k) \), with \( N(t_k) \) taken as in Proposition 5.1.

Then, let \( \lambda_k = \|Iu_k\|_{H^s} \geq \Lambda(t_k) \). Do the scaling as follows:
\[ \tilde{u}_k = \lambda_k^{-1}Iu(t_k, x\lambda_k^{-1}) \quad (5.15) \]
\[ \tilde{n}_k = \lambda_k^{-2}In(t_k, x\lambda_k^{-1}). \quad (5.16) \]
and by direct calculations, we have
\[ \| \tilde{u}_k \|_{L^2} = \| Iu_k \|_{L^2} \leq \| u_k \|_{L^2} = \| u_0 \|_{L^2}, \] (5.17)
\[ \| \nabla \tilde{u}_k \|_{L^2} \leq 1, \] (5.18)
\[ \lim_{k \to \infty} \| \tilde{u}_k \|_{L^2} = 1, \] (5.19)
and
\[ \lim_{k \to \infty} \| \nabla \tilde{u}_k \|_{L^2} = 1 \] (5.20)
since \( N(t_k) \to \infty \) for \( t \to T^* \) by Proposition 5.1.

Thus, \( \{ \tilde{u}_k \}_{k=1}^\infty \) is a bounded sequence in \( H^1 \) and has a weakly convergent subsequence, which we still denote as \( \{ \tilde{u}_k \} \), and \( \tilde{u} \in H^1 \), such that
\[ \tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in} \quad H^1. \] (5.21)

Then, as \( u \) is radial, then by Radial Compactness Lemma, it exists
\[ \tilde{u}_k \to \tilde{u} \quad \text{in} \quad L^4. \] (5.22)

On the other hand, let
\[ E(Iu) = \| \nabla Iu(t) \|_{L^2}^2 - \frac{1}{2} \| Iu(t) \|_{L^4}^4, \] (5.23)
and
\[ H_1(Iu, In) = \| \nabla Iu(t) \|_{L^2}^2 + \frac{1}{2} \| In \|_{L^2}^2 + \int In |Iu|^2 = E(Iu) + \frac{1}{2} \int (In + |Iu|^2)^2. \] (5.24)
Hence
\[ H(t) = H_1(Iu, In) + \frac{1}{2} \| Iv \|_{L^2}^2. \] (5.25)

Combing all the above estimates together, we have
\[ E(\tilde{u}_k) = \lambda_k^{-2} E(Iu_k) \leq \lambda_k^{-2} H_1(Iu_k, In_k) \leq \lambda_k^{-2} H(Iu_k, In_k) \leq c\lambda_k^{-2} \Lambda^{p(s)}(t_k) \leq c\Lambda_k^{(s)-2} \to 0, \]
as \( k \to \infty \), by Proposition 5.1 and the definition of \( t_k \).
Thus,
\[ \limsup_{k \to \infty} E(\tilde{u}_k) \leq 0, \] (5.26)
and
\[ \limsup_{k \to \infty} H_1(\tilde{u}_k, \tilde{n}_k) \leq 0. \] (5.27)

Therefore,
\[ \liminf_{k \to \infty} \|\tilde{u}_k\|_4^4 = 2 \liminf_{k \to \infty} \|\nabla \tilde{u}_k\|_{L^2}^2 - 2E(\tilde{u}_k) \geq 2, \] (5.28)

and
\[ 0 \geq \limsup_{k \to \infty} H_1(\tilde{u}_k, \tilde{n}_k) \geq \frac{1}{2} \limsup_{k \to \infty} \int (\tilde{n}_k + |\tilde{u}_k|^2)^2 - \|\tilde{u}_k\|_4^4, \] (5.29)
or in other words,
\[ 0 \geq \limsup_{k \to \infty} H_1(\tilde{u}_k, \tilde{n}_k) \geq \limsup_{k \to \infty} \left( \frac{1}{2} \|\tilde{n}_k\|_{L^2}^2 + \int |\tilde{n}_k| |\tilde{u}_k|^2 \right) \geq \limsup_{k \to \infty} \left( \frac{1}{2} \|\tilde{n}_k\|_{L^2}^2 - \frac{1}{4} \|\tilde{n}_k\|_{L^2}^2 - \|\tilde{u}_k\|_4^4 \right), \]

i.e.
\[ \limsup_{k \to \infty} \|\tilde{n}_k\|_{L^2}^2 \leq 4 \liminf_{k \to \infty} \|\tilde{u}_k\|_4^4 \leq c, \] (5.30)

by Sobolev embedding theory, (5.17) and (5.18).

Claim 5.2. \( \forall R > 0, \)
\[ \liminf_{k \to \infty} \|Iu_k\|_{L^2(B(0,R))} \geq \|Q\|_{L^2} \] (5.31)

and
\[ \liminf_{k \to \infty} \|In_k\|_{L^1(B(0,R))} \geq m_n, \] (5.32)

where \( m_n > 0 \) depending on the initial data.

Proof. If the claim doesn’t exist, then there is a subsequence of \( \{t_k\} \), (still denote it as \( \{t_k\} \)), such that
\[ \limsup_{k \to \infty} \int_{|x| < R_0} |Iu_k|^2 \leq \|Q\|_{L^2}^2 - \delta_0, \] (5.33)
or
\[ \limsup_{k \to \infty} \int_{|x| < R_0} |In_k| = 0, \] (5.34)

for some \( R_0 > 0, \) and \( \delta_0 > 0. \)

Then by scaling, \( \forall R > 0, \)
\[ \limsup_{k \to \infty} \int_{|x| < R} |\tilde{u}_k|^2 \leq \|Q\|_{L^2}^2 - \delta_0, \] (5.35)
or
\[ \limsup_{k \to \infty} \int_{|x| < R} |\tilde{n}_k| = 0, \] (5.36)
as \( \lambda_k \to \infty \) for \( k \to \infty. \)
From (5.22) and (5.28), there exists
\[ \| \tilde{u} \|^4_{L^4} \geq 2. \] (5.37)

And also by (5.21) and (5.35), we have
\[ \int_{|x|<R} |\tilde{u}|^2 \leq \liminf_{k \to \infty} \int_{|x|<R} |\tilde{u}_k|^2 \leq \| Q \|^2_{L^2} - \delta_0, \]
for any \( R > 0 \). Hence by letting \( R \to \infty \),
\[ \| \tilde{u} \|^2_{L^2} \leq \| Q \|^2_{L^2} - \delta_0. \] (5.38)

On the other hand, from (5.30), we can see that \( \{ \tilde{n}_k \} \) is bounded in \( L^2 \), hence there is \( \tilde{n} \), such that
\[ \tilde{n}_k \rightharpoonup \tilde{n} \quad \text{in} \quad L^2. \] (5.39)

From (5.39) and (5.36)
\[ \int_{|x|<R} |\tilde{n}| \leq cR \frac{2}{\delta^*} \left( \int_{|x|<R} |\tilde{n}|^2 \right)^{\frac{1}{2}} \leq cR \frac{2}{\delta^*} \liminf_{k \to \infty} \left( \int_{|x|<R} |\tilde{n}_k|^2 \right)^{\frac{1}{2}} = 0. \]
i.e.
\[ \tilde{n} = 0, \quad \text{a.e.} \] (5.40)
by letting \( R \to \infty \).

Therefore,
\[ \| \tilde{u} \|^2_{L^2} \leq \| Q \|^2_{L^2} - \delta_0 \quad \text{or} \quad \tilde{n} = 0. \] (5.41)

Furthermore, since \( \tilde{u}_k^2 \to \tilde{u}^2 \) and \( \tilde{n}_k \rightharpoonup \tilde{n} \) in \( L^2 \), we have
\[ \int \tilde{n}_k |\tilde{u}_k|^2 \to \int \tilde{n} |\tilde{u}|^2, \quad \text{as} \quad k \to \infty. \] (5.42)

Therefore,
\[ H_1(\tilde{u}, \tilde{n}) = \| \nabla \tilde{u} \|^2_{L^2} + \frac{1}{2} \| \tilde{n} \|^2_{L^2} + \int \tilde{n} |\tilde{u}|^2 \leq \liminf_{k \to \infty} (\| \nabla \tilde{u}_k \|^2_{L^2} + \frac{1}{2} \| \tilde{n}_k \|^2_{L^2} + \int \tilde{n}_k |\tilde{u}_k|^2) = \liminf_{k \to \infty} H_1(\tilde{u}_k, \tilde{n}_k) \leq 0, \] (5.43)
or equivalently,
\[ E(\tilde{u}) + \frac{1}{2} \int (\tilde{n} + |\tilde{u}|^2)^2 \leq 0. \] (5.44)

**Case 1** If \( \| \tilde{u} \|^2_{L^2} \leq \| Q \|^2_{L^2} - \delta_0. \)


Then, by (5.44) and sharp Gagliardo-Nirenberg (2.13), we have

\[ 0 \geq E(\tilde{u}) = \|\nabla \tilde{u}\|_{L^2}^2 - \frac{1}{2} \|\tilde{u}\|_{L^4}^4, \]

\[ \geq \|\nabla \tilde{u}\|_{L^2}^2 - \frac{1}{2} \|Q\|_{L^2}^2 \|\nabla \tilde{u}\|_{L^2}^2 \]

\[ \geq (1 - \frac{\|Q\|_{L^2}^2 - \delta_0}{\|Q\|_{L^2}^2}) \|\nabla \tilde{u}\|_{L^2}^2 \]

\[ = \frac{\delta_0}{\|Q\|_{L^2}^2} \|\nabla \tilde{u}\|_{L^2}^2. \] (5.45)

Because of (5.37), \|\nabla \tilde{u}\|_{L^2}^2 \neq 0, which is a contradiction.

**Case 2** If \( \tilde{n} = 0 \).

Then

\[ 0 \geq H_1(\tilde{u}, \tilde{n}) = \|\nabla \tilde{u}\|_{L^2}^2, \] (5.46)

which is also a contradiction. □

With Claim 5.2 we can get the result of the Theorem quickly.

That is,

\[ \|Q\|_{L^2} \leq \liminf_{k \to \infty} \|Iu_k\|_{L^2(B(0,R))} \leq \liminf_{k \to \infty} \|u_k\|_{L^2(B(0,R))} \leq \limsup_{t \to T^*} \|u(t)\|_{L^2(B(0,R))}, \] (5.47)

and

\[ m_n \leq \liminf_{k \to \infty} \|In_k\|_{L^1(B(0,R))} \leq \liminf_{k \to \infty} \|n_k\|_{L^1(B(0,R))} \leq \limsup_{t \to T^*} \|n(t)\|_{L^1(B(0,R))}. \] (5.48)

□

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