NUMERICAL METHODS OF OPTIMAL ACCURACY FOR WEAKLY SINGULAR VOLterra INTEGRAL EQUATIONS

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Abstract. Weakly singular Volterra integral equations of the different types are considered. The construction of accuracy-optimal numerical methods for one-dimensional and multidimensional equations is discussed. Since this question is closely related with the optimal approximation problem, the orders of the Babenko and Kolmogorov $n^{-\infty}$ widths of compact sets from some classes of functions have been evaluated. In conclusion we adduce some numerical illustrations for 2-D Volterra equations.

1. Introduction. Definitions and auxiliary statements

Volterra integral equations have numerous applications in economy, ecology, medicine [2, 8]. For a detailed study of approximate methods for Volterra integral equations (VIEs) including Abel–Volterra equations we refer to, e.g., [2, 6, 8, 9, 10, 15, 12, 13, 16] and the references therein. In the most of these papers authors deal with the one-dimensional equations. In numerical analysis of VIEs there are several common approaches based on the application of special nonuniform grids and the smoothing transformations. In this paper we summarize our results concerning the construction of numerical methods of optimal accuracy for multidimensional weakly singular VIEs. Some of these results are published for the first time.

The paper is organized as follows. In Section 1, we introduce the classes of functions being used and prove some statements concerning the smoothness of
exact solutions of VIEs. Section 2 is dedicated to evaluation of the Babenko and Kolmogorov widths of compact sets from introduced classes of functions. There are also constructed the special local splines realizing the optimal estimates. In Section 3, we describe the projective method for multidimensional VIEs based on the approximation of the exact solutions by these splines. The numerical example for 2-D VIE is given in Section 4.

1.1. Classes of functions. Let \( \Omega = [0, T] \) where \( l = 1, 2, \ldots \) is a dimension of the problem, \( t = (t_1, \ldots, t_l) \), \( m = (m_1, \ldots, m_l) \), \( |m| = m_1 + \cdots + m_l \); \( D^m f \equiv \frac{\partial^{|m|} f(t_1, \ldots, t_l)}{\partial t_1^{m_1} \cdots \partial t_l^{m_l}} \).

**Definition 1.1.** Let \( u \) be a positive integer and \( \gamma \) be a non-integer. Let \( Q_{r, \gamma}^u(\Omega, M) \) be a class of functions \( f(t) \) defined on \( \Omega \) and satisfying the following conditions:

\[
|D^m f| \leq M, \ 0 \leq |m| \leq r,
|D^m f| \leq M(1 + \ln^u \rho(t, \Gamma_0)) \left( \frac{1}{\rho(t, \Gamma_0)} \right)^{|m| - r - \gamma}, \ \ r < |m| \leq s, \ t \in (\Omega \setminus \Gamma_0),
\]

where \( M \) is some constant, \( 0 < M < \infty, s = r + [\gamma] + 1, \gamma = [\gamma] + \mu, 0 < \mu < 1, \zeta = 1 - \mu; \Gamma_0 \) is an intersection of the boundary \( \Gamma \) of domain \( \Omega \) with the union of coordinate planes; \( \rho(t, \Gamma_0) = \min_{i} |t_i| \);

**Definition 1.2.** Let \( u \) be a positive integer and \( \gamma \) be a non-integer. Let \( Q_{r, \gamma}^u(\Omega, M) \) be a class of functions \( f(t) \) defined on \( \Omega \) and satisfying the following conditions:

\[
|D^m f| \leq M, \ 0 \leq |m| \leq r,
|D^m f| \leq M(1 + \ln^u \rho(t, 0)) \left( \frac{1}{\rho(t, 0)} \right)^{|m| - r - \gamma}, \ \ r < |m| \leq s, \ t \neq 0,
\]

where \( M \) is some constant; \( s = r + [\gamma] + 1, \gamma = [\gamma] + \mu, 0 < \mu < 1, \zeta = 1 - \mu; \rho(t, 0) = \min_{k=1,\ldots,l} |t_k| \).

**Definition 1.3.** Let \( r = 1, 2, \ldots, 0 < \gamma \leq 1 \). Function \( f(t) \) belongs to the class \( B_{r, \gamma}^u(\Omega, A) \) if the following inequalities hold:

\[
|f(t_1, \ldots, t_l)| \leq A, \quad |D^m f| \leq A^{|m|}|m|^{|m|}, \ 0 < |m| \leq r, \ t \in \Omega,
|D^m f| \leq A^{|m|}|m|^{|m|}(1 + \ln^u \rho(t, \Gamma_0)) \left( \frac{1}{\rho(t, \Gamma_0)} \right)^{|m| - r - 1 + \gamma}, \ t \in (\Omega \setminus \Gamma_0), \ r < |m| < \infty,
\]

where \( A \) is a constant independent of \( |m| \); \( \Gamma_0 \) is an intersection of the boundary \( \Gamma \) of domain \( \Omega \) with the union of coordinate planes.

**Definition 1.4.** Let \( r \) be a positive integer and \( \gamma \) be a non-integer. Let \( Q_{r, \gamma}(\Omega, M) \) be a class of functions \( f(t) \) defined on \( \Omega \) and satisfying the following conditions:

\[
|D^m f| \leq M, \ 0 \leq |m| \leq r,
|D^m f| \leq \frac{M}{\left( \rho(t, \Gamma) \right)^{|m| - r - \gamma}}, \ \ r < |m| \leq s, \ t \in (\Omega \setminus \Gamma),
\]

where \( M \) is some constant, \( 0 < M < \infty, s = r + [\gamma] + 1, \gamma = [\gamma] + \mu, 0 < \mu < 1, \zeta = 1 - \mu; \Gamma \) is a boundary of domain \( \Omega \).
1.2. **Smoothness of solutions.** For a simplicity of the presentation let us consider the one-dimensional integral equation

\[ x(t) - \int_0^t H(t, \tau)x(\tau)d\tau = f(t), \quad 0 \leq t \leq T. \]  (1.1)

We introduce the following lemmas concerning the smoothness of an exact solution of (1.1):

**Lemma 1.5.**

1. Let \( H(t, \tau) \in Q^{u*}_{r;\gamma}([0, T], 1) \) with respect to each variable separately, \( f(t) \in Q^{u*}_{r;\gamma}([0, T], 1) \). Then the unique solution \( x(t) \) of equation (1.1) belongs to the class \( Q^{u*}_{r;\gamma}([0, T], M) \).

2. Let \( H(t, \tau) \in B^{u*}_{r;\gamma}([0, T], A_1) \) with respect to each variable separately, \( f(t) \in B^{u*}_{r;\gamma}([0, T], A_2) \). Then there exist a unique solution \( x(t) \) of equation (1.1) such that \( x(t) \in B^{u*}_{r;\gamma}([0, T], A) \).

**Proof.** It is well-known that the exact solution of (1.1) is a continuous function.

Let \( C_0 = \max_{t \in [0, T]} |x(t)|, \ F_k = \max_{t \in [0, T]} |f^{(k)}(t)|, \ H_k = \max_{k_1, k_2 \leq k, \ k_1 + k_2 = k, \ 1 \leq k \leq r} \frac{\partial^k H(t, \tau)}{\partial \tau^{k_1} \partial t^{k_2}}, \ 0 \leq k_1, k_2 \leq k \leq k_1 + k_2 = k \). Let also \( M_k \) be constants depending only on the order \( k \).

In order to prove the assertions of this lemma we differentiate formally the expression \( x(t) = f(t) + \int_0^t H(t, \tau)x(\tau)d\tau: \)

\[ x'(t) = f'(t) + \int_0^t H'_t(t, \tau)x(\tau)d\tau + H(t, t)x(t). \]  (1.2)

Since the right hand of this formula is a continuous function, there exists the continuous derivative \( x'(t) \) of the exact solution \( x(t) \) which can be estimated as follows \( |x'(t)| \leq F_1 + H_0C_0 + H_1C_0T = M_1. \)

Differentiating (1.2) one more time, we have

\[ x''(t) = f''(t) + (H'_t + H''_t)x(t) + H(t, t)x'(t) + \int_0^t H''_t(t, \tau)x(\tau)d\tau + H'_t(t, t)x(t). \]

Taking into account that

\[ H_k, F_k = \begin{cases} O(1), & k \leq r; \\ O\left(\frac{1 + |\ln u(t)|}{t^{k-r}}\right), & k > r, \end{cases} \]

and using the previous estimate for \( |x'(t)| \) we obtain

\[ |x''(t)| \leq F_2 + 2H_1C_0 + H_0(F_1 + H_0C_0 + H_1C_0T) + H_2C_0T + H_1C_0 = F_2 + H_2C_0T + 3H_1C_0 + H_0(F_1 + H_0C_0 + H_1C_0T) = M_2. \]

Further,

\[ |x'''(t)| \leq F_3 + 7H_2C_0 + 5H_1|x'| + H_0|x''| + H_3C_0T = M_3. \]
Thus, differentiating (1.1) \( k \)-times \( k \leq s \) we can conclude that there exists \( x^{(k)}(t) \) and the estimate holds

\[
|x^{(k)}(t)| \leq \left\{ \begin{array}{ll}
M_k, & k \leq r; \\
\frac{M_k(1+|\ln u(t)|)}{t^{k-r-\gamma}}, & r < k \leq s.
\end{array} \right.
\]

Therefore \( x(t) \in Q^{u*}_{r,\gamma}([0, T], M) \) with \( M = \max_{k=1,s}\{M_k, 1\} \) and the first statement of Lemma is proved.

Consider now the class \( B^{u*}_{r,\gamma}([0, T]) \). It’s easy to see that \( |x^{(r)}(t)| \) is bounded. The estimate for derivative of order \( m, r < m < \infty \), has the following form

\[
|x^{(m)}(t)| \leq \frac{C_1^m m^m (1 + |\ln u(t)|)}{t^{m-r-1+\gamma}} + \frac{C_2^m m^m (1 + |\ln u(t)|) C_0 T}{t^{m-r-1+\gamma}} + \frac{A m^m m^m (1 + |\ln u(t)|)}{t^{m-r-2+\gamma}} \leq \frac{A m^m m^m (1 + |\ln u(t)|)}{t^{m-r-1+\gamma}},
\]

where the constants \( C_1, C_2, C_3 \) depend only on \( A_1, A_2 \).

Hence the exact solution \( x(t) \in B^{u*}_{r,\gamma}([0, T], A) \).

The statements concerning the smoothness of the exact solution of weakly singular VIEs are distributed to a case of the multidimensional integral equations of the following form

\[
(I - K)x \equiv x(t) - \int_0^{t_1} \cdots \int_0^{t_1} h(t, \tau) g(t - \tau) x(\tau) d\tau = f(t),
\]  

(1.3)

where \( t = (t_1, \ldots, t_l), \tau = (\tau_1, \ldots, \tau_l), 0 \leq t_1, \ldots, t_l \leq T; \) weakly singular kernels \( g(t - \tau) \) may have the form

\[
g(t_1, \ldots, t_l) = t_1^{r+\alpha} \cdots t_l^{r+\alpha}, \quad r \geq 0, \quad \alpha > -1
\]

(1.4)

or

\[
g(t_1, \ldots, t_l) = (t_1^2 + \cdots + t_l^2)^{r+\alpha}, \quad r \geq 0, \quad \alpha > -1.
\]

(1.5)

Applying the analogous technique with much more complicated computations, we have:

(1) If \( g(t_1, \ldots, t_l) \) has the form (1.4), \( h(t, \tau) \) has the continuous partial derivatives up to order \( s = r + [\gamma] + 1 \) and \( f(t) \in Q^{u*}_{r,\gamma}(\Omega, 1) \), then the exact solution \( x(t) \) of equation (1.3) belongs to \( Q^{u*}_{r,\gamma}(\Omega, M) \) with \( \gamma = s - r - \alpha \). If \( h(t, \tau) \) is an analytical function, \( f(t) \in B^{u*}_{r,\gamma}(\Omega, 1) \), then \( x(t) \in B^{u*}_{r,\gamma}(\Omega, A) \).

(2) If \( g(t_1, \ldots, t_l) \) has the form (1.5), \( h(t, \tau) \) has the continuous partial derivatives up to order \( s = r + [\gamma] + 1 \) and \( f(t) \in Q^{u*}_{r,\gamma}(\Omega, 1) \), then the exact solution \( x(t) \) of equation (1.3) belongs to \( Q^{u*}_{r,\gamma}(\Omega, M) \) with \( \gamma = s - r - \alpha \).

2. THE OPTIMAL RECONSTRUCTION OF FUNCTIONS FROM \( Q^{u*}_{r,\gamma}(\Omega, M) \), \( Q^{u**}_{r,\gamma}(\Omega, M) \) AND \( B^{u*}_{r,\gamma}(\Omega, A) \)

In order to construct methods of optimal accuracy for numerical solution of VIEs we need in optimal methods for the approximation of functions from the classes \( Q^{u*}_{r,\gamma}(\Omega, M) \), \( Q^{u**}_{r,\gamma}(\Omega, M) \) and \( B^{u*}_{r,\gamma}(\Omega, A) \).
For this purpose the Babenko and Kolmogorov n-widths of compact sets from these classes are evaluated and local splines (continuous and discontinuous) are constructed. The error orders of these splines coincide with the magnitudes of the widths.

The splines are composed of interpolating polynomials constructed with the use of values of approximating functions. Such construction allows us to avoid the computation of any derivatives of functions which are approximated. In Section 3 we use these splines in numerical schemes.

The obtained assertions are diffusion of results of the papers [4, 7, 3].

Let \( B \) be a Banach space, \( X \subset B \) be a compact set, and \( \Pi : X \to \bar{X} \) be a mapping of \( X \subset B \) onto a finite-dimensional space \( \bar{X} \).

**Definition 2.1.** [1] Let \( L^a \) be \( n \)-dimensional subspaces of the linear space \( B \). The Kolmogorov \( n \)-width \( d_n(X, B) \) is defined by
\[
d_n(X, B) = \inf_{L^a} \sup_{x \in X} \inf_{u \in L^a} \| x - u \|, \tag{2.1}
\]
where the external infimum is calculated over all \( n \)-dimensional subspaces of \( L^a \).

**Definition 2.2.** [1] The Babenko \( n \)-width \( \delta_n(X) \) is defined by the expression
\[
\delta_n(X) = \inf_{\Pi : X \to R^n} \sup_{x \in X} \operatorname{diam} \Pi^{-1}(x),
\]
where the infimum is calculated over all continuous mappings \( \Pi : X \to R^n \).

If the infimum in (2.1) is attained for some \( L^a \), this subspace is called an extremal subspace.

The widths play the important role in the numerical analysis and approximation theory since they have close relations to many optimal problems such as \( \varepsilon \)-complexity of integration and approximation, optimal differentiation, and optimal approximation of solutions for the operator equations.

A detailed study of these problems in view of general theory of optimal algorithms is given in [14].

Throughout this paper \( A \) and \( A_k, \ k = 1, 2, \ldots \), denote some positive constants that do not depend on \( N \). Let \( f \in W^r(1) \), \( t \in [a, b] \), \( c \in [a, b] \). We denote by \( T_{r-1}(f, [a, b], c) \) the Taylor polynomial of \( f \) of order \( r - 1 \) with respect to the point \( c \); i.e. \( T_{r-1}(f, [a, b], c) = \sum_{j=0}^{r-1} (f^{(j)}(c)/j!)(t - c)^j \). For \( f(t_1, t_2) \in C^2(1) \), \( t = (t_1, t_2) \in G = [a_1, b_1; a_2, b_2] \), \( v = (v_1, v_2) \in G \) we can re-write its Taylor polynomial of order \( r - 1 \) as \( T_{r-1}(f, G, v) = \sum_{j=0}^{r-1} d_j(f, v) / j! \), where \( d_j(f, v) \) is a polynomial of order \( j \) given by \( d_j(f, v)(t) = \sum_{i=0}^{j} C_j^i (\partial^i f(v) / \partial t_1^i \partial t_2^{j-i})(t_1 - v_1)^i (t_2 - v_2)^{j-i} \). The Taylor polynomials for functions of \( l \geq 2 \) variables can be re-written similarly.

**Theorem 2.3.** Let \( \Omega = [0, 1] \). Let \( r, u \) be positive integers, and \( \gamma \) be a positive non-integer. The estimate \( \delta_n(Q^{u*}_{r,\gamma}(\Omega, 1)) \asymp d_n(Q^{u*}_{r,\gamma}(\Omega, 1), C) \asymp n^{-s} \) holds, where \( s = r + [\gamma] + 1 \).

**Proof.** First we estimate the infimum for \( \delta_n(Q^{u*}_{r,\gamma}(\Omega, 1)) \). Note that \( Q_{r,\gamma}(\Omega, 1) \subset Q^{u*}_{r,\gamma}(\Omega, 1) \). We know [4, 7] that \( \delta_n(Q_{r,\gamma}(\Omega, 1)) \asymp n^{-s} \). Therefore
\[
\delta_n(Q^{u*}_{r,\gamma}(\Omega, 1)) \geq \delta_n(Q_{r,\gamma}(\Omega, 1)) \asymp n^{-s}.
\]
To construct a continuous local spline with \( n \) parameters that approximates the functions of \( Q_{r,\gamma}^s(\Omega, 1) \) with the accuracy \( An^{-s} \), we divide the interval \([0, 1]\) into \( N \) subintervals \( \Delta_k = [t_k, t_{k+1}], k = 0, 1, \ldots, N-1 \), by the points \( t_k = (k/N)^v \) \( k = 0, 1, \ldots, N, v = s/(s-\gamma) \).

Put \( M_0 = [\ln u/(r+1-\mu)] + 1 \), \( M_k = [\ln u/s N] + 1 \), \( k = 1, \ldots, N-1, \mu = 1 - \zeta \). Divide each \( \Delta_k \) into \( M_k \), \( k = 0, 1, \ldots, N-1 \), equal subintervals and denote the latter ones by \( \Delta_{k,j}, j = 0, 1, \ldots, M_k - 1, k = 0, 1, \ldots, N-1 \).

For each interval \([a, b]\) we now choose a polynomial \( P_s(f, [a, b]) \) interpolating \( f(t) \in Q_{r,\gamma}^s(\Omega, 1) \) at the endpoints \( a \) and \( b \) in the following way. Denote the zeros of the Chebyshev polynomial of the first kind of degree \( s \) by \( \zeta_k, k = 1, 2, \ldots, s \). Map \([\zeta_1, \zeta_s] \subset [-1, 1] \) with an affine-linear transformation onto \([a, b]\) so that the points \( \zeta_1 \) and \( \zeta_s \) are mapped onto \( a \) and \( b \) respectively. Denote the images of the points \( \zeta_i \) under this mapping by \( \zeta_i^*, i = 1, 2, \ldots, s \).

We then denote by \( P_s(f, [a, b]) \) the interpolation polynomial of degree \( s - 1 \) with respect to the nodes \( \zeta_i^*, i = 1, 2, \ldots, s \).

Define on \([0, 1]\) the function \( f_N \) composed of the piecewise polynomials

\[
P_s(f, [t_{k,j}, t_{k+1,j}]], \quad j = 0, 1, \ldots, M_k - 1, k = 1, 2, \ldots, N - 1.
\]

Estimating the error \( \|f - f_N\| \) we obtain for \( j = 0, \ldots, M_k - 1, 1 \leq k \leq N - 1 \)

\[
\|f - f_N\|_{C(\Delta_{k,j})} \leq \frac{c(t_{k+1} - t_k)^s}{M_k s!} \left( \frac{N}{k} \right)^{v+1} \left( 1 + \left| \ln u \left( \frac{k}{N} \right)^v \right| \right) \leq \frac{c}{N^s}.
\]

For \( k = 0 \), the following estimate is true: \( \|f - f_N\|_{C(\Delta_{0,0})} \leq E_{s-1}(f, \Delta_{0,0})(1 + \lambda_s) \), where \( \lambda_s \) is the Lebesgue constant with respect to the nodes used for \( P_s(f, \Delta_k^1) \).

Using the Taylor expansion \( T_r(f, \Delta_{0,0}, 0) \) we have

\[
E_{s-1}(f, \Delta_{0,0}) \leq \|f - T_r(f, \Delta_{0,0}, 0)\|_{C(\Delta_{0,0})} \leq \frac{1}{r!} \max_{t \in \Delta_{0,0}} \int_0^t \frac{(1 + |\ln u \tau|)}{\tau^\mu} (t - \tau)^r d\tau \leq \frac{c}{r!} h_{0,0}^r \max_{t \in \Delta_{0,0}} \int_0^t \frac{|\ln u \tau|}{\tau^\mu} d\tau \leq ch_{0,0}^{r+1-\mu} |\ln u h_{0,0}| \leq \frac{c}{N^s}.
\]

Recall \( h_{0,0} = N^{-v}/M_0 \leq cN^{-v} (\ln N)^{-u/(r+1-\mu)} \). Hence, \( \|f - f_N\|_{C(\Delta_{0,0})} \leq cN^{-s} \).

One can estimate the norms \( \|f - f_N\|_{C(\Delta_{0,j})}, j = 0, 1, \ldots, M_0 - 1 \) in a similar way. Thus, we have obtained \( \|f - f_N\|_{C(\Omega)} \leq cN^{-s} \).

It remains to estimate the number \( n \) of nodes of the local spline \( f_N \). It is easy to see that \( n \geq N \). Thus, we have \( \|f - f_N\|_{C(\Omega)} \leq cn^{-s} \) and \( d_n(Q_{r,\gamma}^s(\Omega, 1), C) \leq cn^{-s} \).

Comparing the last inequality with the estimate \( \delta_n(Q_{r,\gamma}^s(\Omega, 1)) \geq cn^{-s} \) and taking into account the inequality \( \delta_n \leq 2d_n \) we complete the proof of Theorem.

Now we estimate the Kolmogorov and Babenko widths for the functional class \( Q_{r,\gamma}^s(\Omega, 1) \), \( \Omega = [0, 1]^l, l = 2, 3, \ldots \).
Theorem 2.4. Let $\Omega = [0, 1]^l$, $l \geq 2$, $u = 1, 2, \ldots, v = s/(s - \gamma)$. Then

$$\delta_n(Q_{r,\gamma}^{u*}(\Omega, 1)) \simeq d_n(Q_{r,\gamma}^{u*}(\Omega, 1), C) \simeq \left(\frac{1}{n}\right)^{s/l} \text{if } v < l/(l - 1);$$

(2.2)

$$c\frac{(\ln n)^{(ul+s)/l}}{n^{s/l}} \leq \delta_n(Q_{r,\gamma}^{u*}(\Omega, 1)) \leq 2d_n(Q_{r,\gamma}^{u*}(\Omega, 1), C) \leq c\frac{(\ln n)^{(us/(r+1-\mu))}}{n^{s/l}}$$

(2.3)

for $lu/(r + 1 - \mu) \geq ul/s + 1$,

$$\delta_n(Q_{r,\gamma}^{u*}(\Omega, 1)) \simeq d_n(Q_{r,\gamma}^{u*}(\Omega, 1), C) \simeq \frac{(\ln n)^{(ul+s)/l}}{n^{s/l}}$$

(2.4)

for $lu/(r + 1 - \mu) < ul/s + 1$, if $v = l/(l - 1)$;

$$\delta_n(Q_{r,\gamma}^{u*}(\Omega, 1)) \simeq d_n(Q_{r,\gamma}^{u*}(\Omega, 1), C) \simeq \frac{\ln^n n}{n^{(s-\gamma)/(l-1)}} \text{ if } v > l/(l - 1).$$

(2.5)

We start to estimate the Kolmogorov widths for the case when $v \leq l/(l - 1)$. First, we construct a local spline not necessarily continuous which approximates the functions of the classes $Q_{r,\gamma}^{u*}(\Omega, 1)$ for $v \leq l/(l - 1)$ and has the error given in the right-hand side of (2.2)–(2.4). Afterwards we construct a continuous local spline having the same error of approximation. This requires some modifications which we indicate below.

Let $\Delta^k$ denote the set

$$\Delta^k = \left\{ t \in \Omega : \left(\frac{k}{N}\right)^v \leq d(t, \Gamma) \leq \left(\frac{k+1}{N}\right)^v, \; k = 0, 1, \ldots, N - 1 \right\}.$$

We now partition the domains $\Delta^k$, $k = 0, \ldots, N - 1$, in the following way. Decompose each $\Delta^k$ into cubes and parallelepipeds $\Delta_{i_1,\ldots,i_l}^k$ with their edges parallel to the axes. The lengths of edges are not less than the value $h_k$ and less than $2h_k$, $h_k = ((k + 1)/N)^v - (k/N)^v$, $k = 0, 1, \ldots, N - 1$.

Let $M_0 = [(\ln N)^{(u+1-\mu)/l}] + 1$, $M_k = [(\ln N/k)^{u/s}] + 1$, $k = 1, 2, \ldots, N - 1$. Dividing each edge of $\Delta_{i_1,\ldots,i_l}^k$ into $M_k$ equal subintervals and passing the planes parallel to the coordinate planes through the points of division we partition $\Delta_{i_1,\ldots,i_l}^k$ into $\Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k$.

Earlier we used the interpolating polynomial $P_s(f, [a, b])$ for functions $f$ of one variable. As a next step we consider a possible multivariate counterpart. More precisely, for a function $f(t_1, \ldots, t_l)$ of $l$ variables on $[a_1, b_1; \ldots; a_l, b_l]$ we define the interpolating polynomial $P_{s,\ldots,s}(f, [a_1, b_1; \ldots; a_l, b_l])$ iteratively:

$$P_{s,\ldots,s}(f, [a_1, b_1; \ldots; a_l, b_l]) = P_{s_1}(P_{s_2}(\cdots P_{s_l}(f; [a_1, b_1]; [a_{l-1}, b_{l-1}]); \cdots; [a_1, b_1])).$$

The polynomial $P_{s,\ldots,s}(f, \Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k)$ interpolates $f(t_1, \ldots, t_l)$ in $\Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k$.

We piece together the interpolating polynomials $P_{s,\ldots,s}(f, \Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k)$ and construct a local spline $f_N$. Next we estimate the approximation of $f \in Q_{r,\gamma}^{u*}(\Omega, 1)$ by $f_N$.

Remark 2.5. We will use polynomials $P_{s,\ldots,s}(f, \Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k)$ when $s \geq r + 2$, and polynomials $P_{s+1,\ldots,s+1}(f, \Delta_{i_1,\ldots,i_l; j_1,\ldots,j_l}^k)$ when $s = r + 1$. Without loss of generality we demonstrate our computations when $s \geq r + 2$. 
Estimating an approximation \( f_N \) to \( f \in Q^u_{\gamma} (\Omega, 1) \) we obtain for \( 1 \leq k \leq N-1 \)

\[
\| f - P_{s_1,\ldots,s} (f, \Delta^k_{i_1,\ldots,i_l}) \|_{C(\Delta^k_{i_1,\ldots,i_l})} \leq c \left( \left( \frac{k+1}{N} \right)^v + \left( \frac{k}{N} \right)^v \right)^s \frac{1}{(\ln \frac{N}{k})^{u/s}} \leq \frac{c}{N^s} . \tag{2.6}
\]

If \( k = 0 \), then

\[
\| f - P_{s_1,\ldots,s} (f, \Delta^0_{i_1,\ldots,i_l,j_1}) \|_{C(\Delta^0_{i_1,\ldots,i_l,j_1})} \leq c E_{r+1,\ldots,r+1} (f, \Delta^0_{i_1,\ldots,i_l,j_1}) \chi_s^l,
\]

where \( E_{r+1,\ldots,r+1} (f; \Delta^0_{i_1,\ldots,i_l,j_1}) \) is the best approximation to a function \( f \) in the space \( C \) by a polynomial of degree \( r \) in each variable in \( \Delta^0_{i_1,\ldots,i_l,j_1} \).

Using Taylor’s expansion with the remainder in integral form we have

\[
E_{r+1,\ldots,r+1} (f, \Delta^0_{i_1,\ldots,i_l,j_1}) \leq ch_{00}^{r+1-\mu} \int_0^1 (1 - \tau)^{r-1} (1 + \ln^u (\tau h_00)) d\tau \leq ch_{00}^{r+1-\mu} \ln^u h_00 \leq c \left( \frac{1}{N} \right)^{v(r+1-\mu)} = c \left( \frac{1}{N} \right)^s ,
\]

where \( h_00 = h_0/M_0, h_0 = (1/N)^v \). Hence,

\[
\| f - P_{s_1,\ldots,s} (f, \Delta^0_{i_1,\ldots,i_l,j_1}) \|_{C(\Delta^0_{i_1,\ldots,i_l,j_1})} \leq c \left( \frac{1}{N} \right)^s . \tag{2.7}
\]

Combining (2.6) – (2.7) gives

\[
\| f - f_N \| \leq c \left( \frac{1}{N} \right)^s . \tag{2.8}
\]

Now we estimate the number of nodes used in constructing \( f_N \). We study two cases i) \( v < l/(l-1) \) and ii) \( v = l/(l-1) \).

i). Let \( v < l/(l-1) \). Let also \( m \) be the number of faces of \( \Delta^k_{i_1,\ldots,i_l} \). The estimate follows immediately from the chain of inequalities

\[
n \geq m \sum_{k=1}^{N-1} \left( \frac{2 - 2(k/N)^v}{(k+1/N)^v - (k/N)^v} \right)^{l-1} M_k + 2mN^{v(l-1)} ([\ln N] + 1)^{l/(r+1-\mu)} \asymp N^l . \tag{2.9}
\]

The inequalities (2.8) – (2.9) yield \( \| f - f_N \| \leq cn^{-s/l} \), where \( n \) is the number of nodes of the local spline.

ii). Let \( v = l/(l-1) \). As we have already derived for \( v < l/(l-1) \), the bound follows immediately from the chain of equalities

\[
n \geq m \sum_{k=1}^{N-1} \left( \frac{2 - 2(k/N)^v}{(k+1/N)^v - (k/N)^v} \right)^{l-1} M_k + 2mN^{v(l-1)} ([\ln N] + 1)^{l/(r+1-\mu)} \asymp cN^l ([\ln N]^{l/(r+1-\mu)} + cN^l (\ln N)^{(ul/s)+1} . \tag{2.10}
\]

It remains to express \( N \) in terms of \( n \). It is necessary to study two cases:

i). The estimate \( N \leq n^{1/l} ([\ln n]^{l/(r+1-\mu)} \) holds, if \( lu/(r + 1 - \mu) \geq ul/s + 1 \);

ii). \( N \leq n^{1/l} ([\ln n]^{(ul/s)/(dl)}) \) holds, if \( lu/(r + 1 - \mu) < ul/s + 1 \).
From the obtained estimates for \( N \) and inequality (2.8) we conclude that the right parts of estimates (2.3) and (2.4) hold.

Now we describe the necessary modifications to the computations above in order to construct a continuous local spline approximating \( Q_{r,\gamma}^{*}(\Omega, 1) \) for \( v \leq l/(l - 1) \) and having the estimate given in (2.2) - (2.4). Let \( \Delta^{k} \) be defined as above and \( h_{k} = ((k + 1)/N)^{v} - (k/N)^{v}, \) \( h_{k}^{*} = h_{k}/M_{k}, \) \( k = 0, 1, \ldots, N - 1. \) We then shall use the following modified partitioning of \( \Omega. \) As before decompose \( \Delta^{N-2} \) into cubes or parallelepipeds \( \Delta_{i_{1},\ldots,i_{t}}^{N-2} \) with edges parallel to the axes, and whose lengths is not less than \( h_{N-2} \) and does not exceed then \( 2h_{N-2}, \) but now choose the partition in such a way that the vertices of \( \Delta^{N-1} \) are contained in the set consisting of all vertices of the subdomains \( \Delta_{i_{1},\ldots,i_{t}}^{N-2} \).

Next, partition each \( \Delta_{i_{1},\ldots,i_{t}}^{N-2} \) into cubes or parallelepipeds \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-2} \) with edges parallel to the axes. We divide each edge of the subdomain \( \Delta_{i_{1},\ldots,i_{t}}^{N-2} \) into \( M_{N-2} \) equal subintervals and pass the planes parallel to the coordinate planes through the points of division. Finally we decomposed \( \Delta^{N-2} \) into \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-2} \) and pass the planes parallel to the coordinate planes through the vertices of \( \Delta^{N-2} \cap \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-2} \) \( \Delta^{N-3} \) with \( \Delta^{N-2} \) which are located on a common face of the hyperplane \( \Delta_{i_{1},\ldots,i_{t}}^{N-2} \). Denote the obtained subdomains by \( g_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-3} \) \( h_{N-3}^{*} = h_{N-3}/M_{N-3}. \) Consider \( g_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-3} = [a_{1}, b_{1}; \ldots; a_{t}, b_{t}] \) if the length of the edge \( (a_{k}, b_{k}) \) exceeds \( 2h_{N-3}^{*}, \) we divide \( (a_{k}, b_{k}) \) into \( [b_{k} - a_{k}/h_{N-3}^{*}] \) equal subintervals and pass the planes parallel to the coordinate planes through the points of division. We shall refer to the result of this procedure as \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-3} \). This way we have \( \Delta^{N-3} \) decomposed into \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{N-3} \). Continuing this process we partition the domain \( \Omega \) into subdomains \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{k} \), \( k = 0, 1, \ldots, N - 2. \)

One estimates the total number of \( \Delta_{i_{1},\ldots,i_{j_{1}},\ldots,j_{t}}^{k} \) in \( \Omega \) using (2.9), (2.10).

Now we construct the continuous spline \( f_{N}^{*} \) approximating a function \( f \) of \( l \) variables.

The polynomial \( P_{a_{i_{1}},b_{i_{1}}}(f, \Delta^{N-1}) \) interpolates \( f \) in \( \Delta^{N-1} \) and \( P_{a_{i_{1}},b_{i_{1}}}(\tilde{f}, \Delta^{N-2}_{i_{1},\ldots,j_{1},\ldots,j_{t}}) \) interpolates \( \tilde{f} \) in \( \Delta^{N-2}_{i_{1},\ldots,j_{1},\ldots,j_{t}}. \) We say that the function \( \tilde{f} \) equals to \( f \) at all points of interpolation except for those located on the hypersurface \( \Delta^{N-1} \cap \Delta^{N-2}. \) We then work with the interpolating polynomials \( P_{a_{i_{1}},b_{i_{1}}}(\tilde{f}, \Delta^{k}_{i_{1},\ldots,j_{1},\ldots,j_{t}}), k = N - 3, \ldots, 1, 0. \)

Next we piece together all the interpolating polynomials \( P_{a_{i_{1}},b_{i_{1}}}(f, \Delta^{N-1}), \) \( P_{a_{i_{1}},b_{i_{1}}}(\tilde{f}, \Delta^{k}_{i_{1},\ldots,j_{1},\ldots,j_{t}}), k = 0, 1, 2, \ldots, N - 2, \) that interpolate \( f \) within each \( \Delta^{N-1}, \) \( \Delta^{k}_{i_{1},\ldots,j_{1},\ldots,j_{t}} \) and construct the continuous local spline \( f_{N}^{*} \).

Repeating the above computations for a non-continuous local spline we obtain

\[
\|f - f_{N}^{*}\|_{C(\Omega)} \leq cN^{-s}.
\] (2.11)

Using the inequalities (2.9), (2.10), (2.11) we have proved justice of right side of expressions (2.2) - (2.4).

Let \( v > l/(l - 1) \). First, we construct a local spline not necessary continuous which approximates the functions of \( Q_{r,\gamma}^{*}(\Omega, 1) \) for \( v > l/(l - 1) \) and has the error
not exceeding $c(ln^u n)n^{-(s-\gamma)/(l-1)}$. Afterwards we construct a continuous local spline having the same error of approximation.

When constructing a local spline we employ the same process as used before (see the part of constructing a not necessarily continuous local spline). We define the domains $\Delta^k$ and partition them into $\Delta^k_{i_1, \ldots, i_l}$, $k = 0, 1, \ldots, N-2$, in a similar way we did for $v \leq l/(l-1)$.

Clearly, the number $n$ of $\Delta^k_{i_1, \ldots, i_l}$ is estimated by

$$n \simeq N^{u/(l-1)}. \tag{2.12}$$

The polynomial $P_{s,\ldots,s}(f; \Delta^k_{i_1, \ldots, i_l})$ interpolates $f$ in $\Delta^k_{i_1, \ldots, i_l}$, $k = 0, 1, \ldots, N-1$. Hence the local spline $f_N$ is composed of the polynomials $P_{s,\ldots,s}(f; \Delta^k_{i_1, \ldots, i_l})$, $k = 0, 1, \ldots, N-1$.

It is easy to see that for $1 \leq k \leq N-1$ the following estimate holds

$$\|f - f_N\|_{C(\Delta^k_{i_1, \ldots, i_l})} \leq cN^{-s}(\ln N)^u.$$ 

Let $k = 0$. Without loss of generality we demonstrate our computations in $\Delta^0_{0,\ldots,0} = [0, t_1; 0, t_1; \ldots; 0, t_1]$, where $t_1 = \left(\frac{1}{N}\right)^u$. Using Taylor’s expansion we obtain

$$\|f - f_N\|_{C(\Delta^0_{0,\ldots,0})} \leq cN^sE_{r,\ldots,r}(f, \Delta^0_{0,\ldots,0}) \leq c\max_{\tau \in \Delta^0_{0,\ldots,0}} \sum_{|k|=r+1} \frac{1}{k!} \int_0^1 (1 - \tau)^r \tau_k \left(1 + \frac{\ln^u d(\tau k, \Gamma)}{d(\tau k, \Gamma)^{1-\zeta}}\right) d\tau \leq c\frac{\ln^u N}{N^s}.$$ 

From the previous estimate and the equality (2.12) we have

$$\|f - f_N\|_{C(\Omega)} \leq c\frac{\ln^u N}{N^s} \leq c\frac{\ln^u n}{n^{(s-\gamma)/(l-1)}}.$$ 

To construct the continuous local spline $f^*_N$ approximating $Q^u_{r,\gamma}(\Omega, 1)$ for $v \leq l/(l-1)$ with the error $c(ln^u n)n^{-(s-\gamma)/(l-1)}$, we employ all above constructions for the continuous local spline $f^*_N$ approximating $Q^u_{r,\gamma}(\Omega, 1)$ when $v \leq l/(l-1)$.

From the previous estimates and the equality (2.12) we have

$$d_n(Q^u_{r,\gamma}(\Omega, 1), C) \leq \|f - f^*_N\|_{C(\Omega)} \leq c\frac{\ln^u N}{N^s} \leq c\frac{\ln^u n}{n^{(s-\gamma)/(l-1)}}.$$ 

Let estimate $\delta_n(Q^u_{r,\gamma}(\Omega, 1))$ for $v = s/(s-\gamma), v < l/(l-1)$.

It is easy to see that $Q^u_{r,\gamma}(\Omega, 1) \subset Q^s_{r,\gamma}(\Omega, 1)$. The estimate $\delta_n(Q^s_{r,\gamma}(\Omega, 1)) \geq cn^{-s/l}$ was proved in [4]. Therefore $\delta_n(Q^u_{r,\gamma}(\Omega, 1)) \geq cn^{-s/l}$.

Let estimate $\delta_n(Q^u_{r,\gamma}(\Omega, 1))$ for $v = s/(s-\gamma), v = l/(l-1)$.

We decompose the domain $\Omega$ into subdomains $\Delta^k_{i_1, \ldots, i_l}$, $k = 0, 1, \ldots, N-1$ following the procedure which was described above (see the part of constructing a not necessarily continuous local spline).

Now, we introduce $M_0 = [(\ln N)^u/s]+1$, $M_k = [(\ln N^k)^u/s]+1$, $k = 1, 2, \ldots, N-1$. 
Let estimate the number $\Delta_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k$, $k = 0, 1, \ldots, N-1$. Clearly

$$n \asymp m \sum_{k=1}^{N-1} \left( \frac{2 - 2 \left( \frac{k}{N} \right)^v}{\left( \frac{k+1}{N} \right)^v - \left( \frac{k}{N} \right)^v} \right)^{l-1} M_k^l + 2mN^{v(l-1)}(\ln N)^{\nu/s} \asymp N^l(\ln N)^{1+\nu/s}.$$  

Let $\Delta_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k = [b_{i_1,j_1}, b_{i_1,j_1+1}; \ldots; b_{i_l,j_l}, b_{i_l,j_l+1}]$. Introduce the functions

$$\varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k(t) = \begin{cases} A_k \frac{(t_1-b_{i_1,j_1})(b_{i_1,j_1+1}-t_1)-(t_l-b_{i_l,j_l})(b_{i_l,j_l+1}-t_l))^s}{(h_k/M_k)^{2(v-1)}(k+1/N)^v}, & t \in \Delta_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k, \\ 0, & t \in \Omega \setminus \Delta_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k, \end{cases}$$

where $h_k = ((k+1)/N)^v - (k/N)^v$, $k = 0, 1, \ldots, N-1$. Constants $A_k$, $k = 0, 1, \ldots, N-1$, are chosen such that

$$|D^s \varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k| \leq \frac{1}{((k+1)/N)^v} \left( 1 + \left| \ln \left( \frac{k+1}{N} \right)^v \right| \right).$$

Obviously, such constants exist and do not depend on $N, u, \gamma$.

Let estimate the maximum values of $\varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k(t)$.

Clearly,

$$\varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k(t) \geq A_k \left( \frac{h_k}{M_k} \right)^s \left( \frac{N}{k+1} \right)^{v}\left(1 + \left| \ln \left( \frac{k+1}{N} \right)^v \right| \right) = \frac{c}{N^s}$$

for $k = 1, 2, \ldots, N-1$;

$$\varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^0(t) \geq A_0 \left( \frac{h_0}{M_0} \right)^s \left( \frac{N}{1} \right)^{v}\left(1 + \left| \ln \left( \frac{1}{N} \right)^v \right| \right) \geq \frac{c}{N^s}.$$  

Let $\xi(t)$ be a linear combination

$$\xi(t) = \sum_{k,i_1,\ldots,i_l,j_1,\ldots,j_l} C_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k \varphi_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k(t),$$

where $|C_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k| \leq 1$. Here the summation is taken over all domains $\Delta_{i_1,\ldots,i_l,j_1,\ldots,j_l}^k$ of $\Omega$.

Repeating the arguments presented in [1, 4] we have

$$\delta_n(Q_{r,\gamma}^u, (\Omega, 1)) \geq c \frac{(\ln n)^{u+s/t}}{n^{s/t}}.$$  

Let estimate $\delta_n(Q_{r,\gamma}^u, (\Omega, 1))$ for $v = s/(s-\gamma)$, $v > l/(l-1)$.

Let $\Delta^0$ be the set $\Delta^0 = \{ t \in \Omega : 0 \leq d(t, \Gamma) \leq (1/N^v) = \rho_0 \}.$

Let $\Delta^k$ be the set $\Delta^k = \{ t \in \Omega : \rho_{k-1} \leq d(t, \Gamma) \leq \rho_k \leq 1 \}$, where $\rho_k$ is defined by $h_k/\rho_k = N^{-s} \ln u N$, and $h_k = \rho_k - \rho_{k-1}$, $k = 1, 2, \ldots, m$. Here $m$ is the largest integer value when $\rho_m \leq 1$. If $\rho_m = 1$, then $\Omega$ is decomposed into $\Delta^k$, $k = 0, 1, \ldots, m$. If $\rho_m < 1$, then $\Delta^{m+1}$ is the set $\Delta^{m+1} = \{ t \in \Omega : \rho_m \leq d(t, \Gamma) \leq 1 \}$.

Without loss of generality we demonstrate our computations for $\rho_m = 1$. Now we show that the equations $h_k/\rho_k = N^{-s} \ln u N$ are solvable.

Let $p_k^* = (k/N)^v$, $k = 0, 1, \ldots, N$, $h_k^* = \rho_k - \rho_{k-1}$, $k = 1, \ldots, N$. Then $h_k^*/\rho_1^* = 1/N^s$ if $k = 1$.  

For $k = 2, \ldots, N$, we have
\[
\frac{h_k^{s+}}{\rho_k^s} = \frac{(k^\nu - (k - 1)^{\nu})^s}{(k/N)^{\nu} s} = \left(\frac{v(\nu)}{k^\nu}\right)^s \frac{1}{N^s} \geq \left(\frac{k - 1}{k}\right)^{\nu \gamma} \frac{1}{N^s} \geq \left(\frac{1}{2}\right)^{\nu \gamma} \frac{1}{N^s}.
\]

Thus there exists a sequence $\rho_k^* = \left(\frac{k}{N}\right)^{\nu}$, $k = 0, 1, \ldots, N$, such that
\[
\frac{h_k^{s+}}{\rho_k^s} \geq \left(\frac{1}{2}\right)^{\nu \gamma} \frac{1}{N^s} = \frac{c}{N^s}, \quad h_k^* = \rho_k^* - \rho_{k-1}^*.
\]

On the other hand, $\varphi(\rho) = \frac{(\rho - \rho_{k-1}^*)^s}{\rho^{s+}}$ is an increasing function if $\rho > \rho_{k-1}$ for any $\rho_{k-1}$.

Therefore there exists a sequence $\rho_k$ such that $\frac{(\rho_k - \rho_{k-1})^s}{\rho^{s+}} \geq N^{s} \ln^u N$, moreover $h_k = \rho_k - \rho_{k-1} > \rho_k^* - \rho_{k-1}^*, k = 1, \ldots, m$.

Hence the number $m$ of $\Delta^k, k = 0, 1, \ldots, m$ is less than $N$. We decompose each $\Delta^k$ into cubes or parallelepipeds $\Delta_{i_1, i_2, \ldots, i_l}^k$ in a way described above (see the part of constructing a not necessarily continuous local spline). Clearly, the total number of $\Delta_{i_1, i_2, \ldots, i_l}^k, k = 0, 1, \ldots, m$ is equal to $n \simeq n_0 \simeq N^{\nu\gamma(l-1)}$, where $n_0$ is the number of $\Delta_{i_1, i_2, \ldots, i_l}^0$.

Let $\Delta_{i_1, i_2, \ldots, i_l}^0 = [b_{i_1}, b_{i_1+1}; \ldots; b_{i_l}, b_{i_l+1}], k = 0, 1, \ldots, m$.

Introduce the functions
\[
\varphi_{i_1, i_2, \ldots, i_l}^0(t) = \begin{cases} A_0^{((t_1-b_{i_1}^0)(t_{i+1}^0-b_{i_1}^0)) \cdots (t_l-b_{i_l}^0)(t_{i+1}^0-b_{i_l}^0))^{s+} N^{\nu\gamma} \ln^u N, & t \in \Delta_{i_1, i_2, \ldots, i_l}^0, \\ 0, & t \in \Omega \setminus \Delta_{i_1, i_2, \ldots, i_l}^0. \end{cases}
\]
\[
\varphi_{i_1, i_2, \ldots, i_l}^k(t_1, \ldots, t_l) = \begin{cases} A_k^{((t_1-b_{i_1}^k)(b_{i_1+1}^k-t_{i_1}) \cdots (t_l-b_{i_l}^k)(b_{i_l+1}^k-t_{i_l}))^{s+} N^{\nu\gamma} \ln^u N, & t \in \Delta_{i_1, i_2, \ldots, i_l}^k, \\ 0, & t \in \Omega \setminus \Delta_{i_1, i_2, \ldots, i_l}^k. \end{cases}
\]

$k = 1, 2, \ldots, m$. Constants $A_k, k = 0, 1, \ldots, m$, are chosen such that $|D^\nu \varphi_{i_1, i_2, \ldots, i_l}^0| \leq N^{\nu\gamma} \ln^u N, |D^\nu \varphi_{i_1, i_2, \ldots, i_l}^k| \leq \frac{1}{\rho_k^s}$.

Obviously, such constants exist and do not depend on $N, u, \gamma$.

Let estimate the maximum values of $\varphi_{i_1, i_2, \ldots, i_l}^k(t), k = 0, 1, \ldots, m$.

Clearly,
\[
\varphi_{i_1, i_2, \ldots, i_l}^0(t) \geq c h_0^s N^{\nu\gamma} \ln^u N = c \left(\frac{1}{N}\right)^{\nu\gamma} \ln^u N = c \frac{\ln^u N}{N^s},
\]
\[
\varphi_{i_1, i_2, \ldots, i_l}^k(t) \geq c \frac{h_k^s}{\rho_k^s} = c \frac{\ln^u N}{N^s}, h_k = \rho_k - \rho_{k-1}, k = 1, 2, \ldots, m.
\]

Let $\xi(t)$ be a linear combination $\xi(t) = \sum_{k, i_1, i_2, \ldots, i_l} C^k_{i_1, i_2, \ldots, i_l} \varphi_{i_1, i_2, \ldots, i_l}^k(t)$, where $|C_{i_1, i_2, \ldots, i_l}^k| \leq 1$. Here the summation is taken over all domains $\Delta_{i_1, i_2, \ldots, i_l}^k$ of $\Omega$. Repeating the arguments presented in [1, 4], we have
\[
\delta_n(Q_{1, \gamma}^{u, v}(\Omega, 1)) \geq c \frac{\ln^u N}{N^s} = c \frac{\ln^u n}{n^{(s-\gamma)(l-1)}}.
\]

From the inequalities in the right sides of (2.2) – (2.5), the inequality $\delta_n \leq 2d_n$ and estimates of the Babenko widths we complete the proof of Theorem.
Theorem 2.6. Let $\Omega = [0, T]$. The estimate holds
$$d_n(B^{0,k}_{r,\gamma}(\Omega, A), C(\Omega)) \leq c2^{-\sqrt{n}(r+1-\gamma)}. \quad (2.13)$$

Proof. At first we construct a continuous local spline realizing the estimate (2.13). It will allow us to obtain an upper bound estimate for the Kolmogorov $n$—width $d_n(B^{0,k}_{r,\gamma}(\Omega, A), C(\Omega))$.

The interval $[0, T]$ is divided into $N+1$ segments $\Delta_k = [v_k, v_{k+1}]$, $k = 0, \ldots, N$, with the knots $v_0 = 0$, $v_k = 2^{k-1-N}T$, $k = 1, \ldots, N + 1$.

Let
$$\xi^k_j = \frac{v_{k+1} + v_k}{2} + \frac{v_{k+1} - v_k}{2}y_j, \quad j = 1, 2, \ldots, m_k - 2; \xi^k_0 = v_k, \xi^k_{m_k - 1} = v_{k+1}; \quad k = 0, 1, \ldots, N,$$

where $y_j$ are the roots of the first kind Chebyshev polynomials of degree $m_k - 2$; $m_0 = r$, $m_k = \left[\frac{10}{9}k(r + 1 - \gamma)AT\right] + 1$, $k = 1, 2, \ldots, N$.

Denote as earlier by $P_k(f, \Delta_k)$ the operator interpolating the function $f(t)$, $t \in \Delta_k$, with the polynomial of degree $m_k - 1$ constructed at the nodes $\xi^k_k$, $k = 0, N$.

Denote also $h_k = v_{k+1} - v_k$, $k = 0, 1, \ldots, N$.

Let then $f_N(t)$ be a local spline defined in $[0, T]$ and composed of polynomials $P_k(f, \Delta_k)$, $k = 0, 0, 1, \ldots, N$

It is easy to see that the approximation error at $t \in \Delta_0$ is
$$\|f(t) - f_N(t)\|_{C[\Delta_0]} \leq A_1 \ln rh_0^{r+1-\gamma} = A_2 \left(\frac{T}{2N}\right)^{r+1-\gamma} = \frac{A_3}{2N(r+1-\gamma)}.$$
where
\[ \Phi_{m_k-1,i}(t) = \frac{(t - t_1) \cdots (t - t_{i-1})(t - t_{i+1}) \cdots (t - t_{m_k-2})}{(t_i - t_1) \cdots (t_i - t_{i-1})(t_i - t_{i+1}) \cdots (t_i - t_{m_k-2})} , \]
\[ \varphi_{m_k-1,i}(t) = \frac{(t - t_0)(t - t_{m_k-1})}{(t_i - t_0)(t_i - t_{m_k-1})} . \]

It is obvious that
\[
\max_{t \in [-a,a]} |\varphi_{m_k-1,i}(t)| = \frac{a^2}{|t_i + a)(t_i - a)|} \leq \frac{a^2}{a^2 - t_i^2} = \frac{a^2}{a^2 - a^2 \cos^2 \frac{1}{2(m_k-2)}} = \\
= \frac{1}{1 - \cos^2 \frac{1}{2(m_k-2)}} = \frac{1}{\sin^2 \frac{1}{2(m_k-2)}} \sim m_k^2, \forall i .
\]

Taking into account that
\[
\max_{t \in [-a,a]} |\psi_{m_k-1,0}(t)| \geq \max_{t \in [-a,a]} |\psi_{m_k-1,m_k-1}(t)| \sim m_k ,
\]
we have
\[
\lambda_{m_k} \leq A_4 m_k^2 \sum_{i=1}^{m_k-2} |\Phi_{m_k-1,i}(t)| + |\psi_{m_k-1,0}(t)| + |\psi_{m_k-1,m_k-1}(t)| = O(m_k^2 \ln m_k) .
\]

Therefore, the approximation error \( \|f(t) - f_N(t)\|_{C[\Delta_k]} \), \( k = 1, N - 1 \) can be estimated as (see, e.g., [11]):
\[
\|f(t) - f_N(t)\|_{C[\Delta_k]} \leq A_5 \lambda_{m_k} \left( \frac{v_{k+1} - v_k}{2m_k} \right)^q A^q q^q \frac{T^q}{v_k^{q-r-1+\gamma}} ,
\]
where \( q = \left[ \frac{5(r+1-\gamma)k}{9} \right] + 1 \) is the maximal order of the derivatives used for the estimation of an error.

Continuing the previous inequality we have
\[
\|f(t) - f_N(t)\|_{C[\Delta_k]} \leq A_6 m_k^2 \ln m_k \left( \frac{T^{2k-1-N}}{2m_k} \right)^q A^q q^q \frac{T^q}{v_k^{q-r-1+\gamma}} = \\
= A_7 k^2 \ln k T^q A^q q^q \frac{2^{N+2-k}q^{2(k-1-N)(q-r-1+\gamma)}}{2^{N-k}} = \\
A_8 k^2 \ln k T^q A^q q^q \left( \frac{10}{9} k(r + 1 - \gamma) \right)^q A^q T^q 2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)} = \\
A_9 k^2 \ln k q^q \frac{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}}{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}} = \\
= A_9 k^2 \ln k \frac{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}}{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}} \leq A_9 k^2 \ln k \frac{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}}{2^{N(r+1-\gamma)} 2^{q-(r+1-\gamma)(k-1)}} .
\]

Hence, for all sufficiently large \( N \) the estimate holds
\[
\|f(t) - f_N(t)\|_{C[\Delta_k]} \leq \frac{A_{10}}{2^{N(r+1-\gamma)}} .
\]
Therefore for the whole segment \([0, T]\) we have

\[
\|f(t) - f_N(t)\|_{C[0,T]} \leq \frac{A_{11}}{2^{N(r+1-\gamma)}}. \tag{2.15}
\]

The total number \(n\) of the functionals used for the construction of a spline can be estimated as

\[
n = \sum_{k=0}^{N} m_k = \sum_{k=0}^{N} \frac{10}{9} k(r + 1 - \gamma)AT \asymp N^2.
\]

Inequality (2.15) allows us to define an upper bound of the Kolmogorov \(n\)-width

\[
d_n(B_{r,\gamma}^{0*}(\Omega, A), C(\Omega)) \leq \frac{A_{13}}{2^{\sqrt{n(r+1-\gamma)}}}.
\]

Note that instead of (2.14) we can also use the another system of the nodes:

\[
\xi_j^k = \frac{v_{k+1} + v_k}{2} + \frac{v_{k+1} - v_k}{2} y_j, \quad j = 0, 2, \ldots, m_k - 1; \quad k = 0, 1, \ldots, N,
\]

where \(y_j\) are the roots of the first kind Chebyshev polynomials of degree \(m_k - 1\); \(m_0 = r, m_k = \left[k(r + 1 - \gamma)AT\right] + 1, k = 1, 2, \ldots, N\).

This allows us to eliminate the additional multiplier \(m_k^2\) in the estimate of the Lebesgue constant \(\lambda_{m_k}\). However, the closed system of nodes (2.14) is more suitable in practice for the numerical solution of VIEs by the projective method described in Section 3.

**Theorem 2.7.** Let \(\Omega = [0, T]^l\), \(l = 2, 3, \ldots\). Then the estimates hold

\[
d_n(Q_{r,\gamma}^{0*}(\Omega, M), C(\Omega)) \asymp d_n(Q_{r,\gamma}^{0*}(\Omega, M), C(\Omega)) \asymp n^{-s/l}.
\]

**Proof.** Let estimate \(d_n(Q_{r,\gamma}^{0*}(\Omega, M), C(\Omega))\). It is easy to see that \(Q_{r,\gamma}^{0*}(\Omega, M) \subset Q_{r,\gamma}^{0*}(\Omega, M)\). The estimate \(d_n(Q_{r,\gamma}^{0*}(\Omega, 1), C(\Omega)) \geq cn^{-s/l}\) was proved in [4]. Therefore \(d_n(Q_{r,\gamma}^{0*}(\Omega, M), C(\Omega)) \geq cn^{-s/l}\).

In order to estimate \(d_n(Q_{r,\gamma}^{0*}(\Omega, M), C(\Omega))\) we cover domain \(\Omega\) with cubes as follows. The cube \(\Delta^1 = \Delta^1_{1,\ldots,1}\) is an intersection of domains

\[
\left(0 \leq t_1 \leq \left(\frac{1}{N}\right)^v T\right) \cap \cdots \cap \left(0 \leq t_l \leq \left(\frac{1}{N}\right)^v T\right),
\]

\(v = s/(s - \gamma)\) if \(\gamma\) is an integer, \(v = s/(s - \lceil \gamma \rceil - 1)\) if \(\gamma\) is a non-integer.

The domain \(\Delta^2\) is then defined as \(\Delta^2 = \Delta^2_2 \setminus \Delta^2_1\), where

\[
\Delta^2_k = \left\{(t_1, \ldots, t_l) : 0 \leq t_1, \ldots, t_l \leq \left(\frac{k}{N}\right)^v T\right\},
\]

\[
\Delta^2_0 = \left\{(t_1, \ldots, t_l) : 0 \leq t_1, \ldots, t_l < \left(\frac{k}{N}\right)^v T\right\}.
\]

This domain is covered with cubes and parallelepipeds \(\Delta^2_{i_1,\ldots,i_l}\) which edges are parallel to the axes of coordinates and do not exceed \(h_1 = \left(\frac{1}{N}\right)^v T - \left(\frac{1}{N}\right)^v T\). The further construction is carried out by analogy.
Each domain $\Delta^k = \Delta^l_1 \setminus \Delta^l_{k-1}$, $k = 3, \ldots, N - 1$, is covered with cubes and parallelepipeds $\Delta^k_{i_1, \ldots, i_l}$ with edges not exceeding $h_{k-1} = (\frac{k}{N})^v T - (\frac{k-1}{N})^v T$.

Let us define the number $n$ of parallelepipeds $\Delta^k_{i_1, \ldots, i_l}$. It is easy to see that

$$n \asymp \sum_{k=1}^{N-1} \left[ \frac{(k+1)^v}{(k+\theta)^v} \right]^{l-1} \asymp \sum_{k=1}^{N-1} \left[ \frac{(k+1)^v}{(k)^v} \right]^{l-1} \asymp \sum_{k=1}^{N-1} k^{l-1} \asymp N^l.$$  

The construction of the local spline $f^*_N(t_1, \ldots, t_l)$ and further argumentation are carried out by analogy to the proof of Theorem 2.4. □

**Theorem 2.8.** Let $\Omega = [0, T]^l$, $l = 2, 3, \ldots$, $0 < \gamma \leq 1$. Then the estimates hold

$$\delta_n(B_{r,\gamma}^{0r}(\Omega, A), C(\Omega)) \asymp d_n(B_{r,\gamma}^{0r}(\Omega, A), C(\Omega)) \asymp \frac{1}{n^{(r+1-\gamma)/l-1}}. \quad (2.16)$$

**Proof.** Let $\Delta_0$ be a set of points $t \in \Omega$ such that $0 \leq \rho(t, \Gamma_0) \leq 2^{-N} T$, and $\Delta_k$, $k = 1, 2, \ldots, N$, be a set of points $t \in \Omega$ such that

$$\frac{2^{k-1}}{2^{N}} T \leq \rho(t, \Gamma_0) \leq \frac{2^k}{2^{N}} T.$$

Let us cover each domain $\Delta_k$, $k = 0, 1, \ldots, N$, with cubes $\Delta^k_{i_1, \ldots, i_l}$. The edges of these cubes are parallel to the edges of $\Omega$. These edges are not less than $h_k$ and not more than $2h_k$, where $h_k = \frac{2^{k-1}}{2^{N}} T$, $k = 0, 1, \ldots, N - 1$.

Now we estimate a general number of elements $\Delta^k_{i_1, \ldots, i_l}$ covering domain $\Omega$. It is obvious that

$$n \asymp \sum_{k=1}^{N} \left[ 1 - \frac{2^k}{2^{N}} \right]^{l-1} \asymp \sum_{k=1}^{N} \left( 2^{N-k+1} - 1 \right)^{l-1} \asymp \sum_{k=1}^{N} \frac{2^{(N+1)(l-1)}}{2^{k(l-1)}} =$$

$$= \frac{1}{2^{l-1} - 1} \left( 2^{(N+1)(l-1)} - 2^{l-1} \right).$$

Thus, $n \asymp 2^{N(l-1)}$. Repeating the arguments of the paper [7] we obtain the estimate $\delta_n(B_{r,\gamma}^{0r}(\Omega, A), C(\Omega)) \asymp 2^{-N(r+1-\gamma)}$ and conclude

$$\delta_n(B_{r,\gamma}^{0r}(\Omega, A), C(\Omega)) \asymp \frac{1}{n^{(r+1-\gamma)/l-1}}.$$  

The construction of a continuous local spline realizing estimate (2.16) is similar to construction given in the Theorem 2.4. Here the parameter $s$ is equal to $s = \left[ \frac{10}{9} N (r + 1 - \gamma) AT \right] + 1$.

Taking into account well known inequality $\delta_n \leq 2d_n$ connecting the Babenko and Kolmogorov $n$–widths, we finish the proof. □

### 3. Approximate Solution of Multidimensional VIEs

In this section we consider the multidimensional VIEs of the form

$$(I - K)x \equiv x(t) - \int_0^{t_1} \cdots \int_0^{t_1} h(t, \tau)g(t - \tau)x(\tau)d\tau = f(t), \quad (3.1)$$
where \( t = (t_1, \ldots, t_l), \tau = (\tau_1, \ldots, \tau_l), 0 \leq t_1, \ldots, t_l \leq T; \) weakly singular kernel \( g(t - \tau) \) has the form
\[
g(t_1, \ldots, t_l) = t_1^{r_1} \cdots t_l^{r_l}, \quad r \geq 0, \quad \alpha > -1.
\]

3.1. **Numerical scheme.** We look for an approximate solution of (3.1) as the spline \( x_N^*(t_1, \ldots, t_l) \) with the unknown values \( x_N^*(\xi_{1i}, \ldots, \xi_{ki}), (\xi_{1i}, \ldots, \xi_{ki}) \in \Delta_{i_1, \ldots, i_l}, k = 0, 1, \ldots, N - 1, \) at the knots of the grid.

The construction of the spline \( x_N^*(t_1, \ldots, t_l) \) is described in Section 2 and depends on the considered class of function.

The values \( x_N^*(\xi_{1i}, \ldots, \xi_{ki}) \) in each cube \( \Delta_{i_1, \ldots, i_l}, k = 0, 1, \ldots, N - 1, \) are determined step-by-step by the spline-collocation technique from the systems of linear equations
\[
(I - K) P_N[x(t), \Delta_{i_1, \ldots, i_l}] \equiv P_N[x(t), \Delta_{i_1, \ldots, i_l}] 
- P_N \left[ \int_{\Delta_{i_1, \ldots, i_l}} \cdots \int_{\Delta_{i_1, \ldots, i_l}} P_N^r[h(t, \tau)] g(t - \tau) P_N[x(\tau), \Delta_{i_1, \ldots, i_l}] d\tau \right] = (3.2)
= P_N[f_{i_1, \ldots, i_l}(t), \Delta_{i_1, \ldots, i_l}].
\]

Here \( P_N \) is an operator of projection on the set of the local splines of the form \( x_N^*(t_1, \ldots, t_l); f_{i_1, \ldots, i_l}^r(t_1, \ldots, t_l) \) is a new right part of equation (3.1) including the integrals over domains \( \Delta_{i_1, \ldots, i_l}, j = 0, 1, \ldots, k \), processed at the previous steps (in these domains the spline values are already known).

All the integrals in (3.2) are calculated using the Gauss-type cubature formulas. There are several ways for choosing the numeration of the subdomains \( \Delta_{i_1, \ldots, i_l}, k = 0, 1, \ldots, N - 1. \) One of such ways allowing the parallelization of the computing process we indicate in [5].

3.2. **Convergence substantiation.** Let us rewrite equation (3.1) and projective method (3.2) in the operator form:
\[
x - Kx = f, \quad K : X \rightarrow X, \quad X \subset C(\Omega), \quad \Omega = [0, T]^l, \quad l = 2, 3, \ldots,
\]
\[
x_N - P_N K x_N = P_N f, \quad P_N : X \rightarrow X_N, \quad X_N \subset C(\Omega), \quad (3.3)
\]
where \( X \) is one of the sets \( Q_{r,\gamma}^\mu(\Omega, M) \) or \( B_{r,\gamma}^\mu(\Omega, A); X_N \) are the sets of corresponding local splines.

Since the homogenous Volterra integral equation \( x - Kx = 0 \) has only the trivial solution, the operator \( I - K \) is injective. Hence, the operator \( I - K \) has the bounded inverse operator \( (I - K)^{-1} : X \rightarrow X. \) For all sufficiently large \( N \) we have the estimates
\[
\|(I - P_N K)^{-1}\|_{C(\Omega)} = \|(I - K) + (K - P_N K)\|^{-1}_{C(\Omega)} \leq \leq \frac{\|(I - K)^{-1}\|_{C(\Omega)}}{1 - \|(I - K)^{-1}\|_{C(\Omega)}\|K - P_N K\|_{C(\Omega)}} \leq 2\|(I - K)^{-1}\|_{C(\Omega)} = A (const)
\]
if
\[
\|K - P_N K\|_{C(\Omega)} \leq \frac{1}{2\|(I - K)^{-1}\|_{C(\Omega)}}.
\]
Let us show that the last estimate holds for all sufficiently large \( N \). Since 
\[ y(t) \equiv (Kx)(t) \in X \] and \( X \) is a dense set in \( C(\Omega) \) (this is valid for \( Q_{r,\gamma}^{u}(\Omega, M) \) and \( B_{r,\gamma}^{u}(\Omega, A) \)), we have 
\[
\| K - P_{N}K \|_{C(\Omega)} = \sup_{x \in X, \| x \| \leq 1} \max_{t \in \Omega} |x(t) - P_{N}x(t)| \leq \varepsilon_{N},
\]
where \( \varepsilon_{N} \to 0 \) as \( N \to \infty \). Therefore, 
\[
\| K - P_{N}K \|_{C(\Omega)} \leq \frac{1}{2\| (I - K)^{-1} \|} \text{ starting with sufficiently large } N.
\]
Thus, the operators \((I - P_{N}K)^{-1}\) are exist and uniformly bounded and equation (3.4) has a unique solution for all sufficiently large \( N \). Taking into account that \( P_{N}x \to x \) as \( N \to \infty \) for all \( x \in X \), we apply the projection operator \( P_{N} \) both to the left and the right parts of equation (3.3): 
\[
x - P_{N}Kx = P_{N}f + x - P_{N}x.
\]
Subtracting this equation from (3.4), we obtain 
\[
(I - P_{N}K)(x_{N} - x) = P_{N}x - x,
\]
\[
(x_{N} - x) = (I - P_{N}K)^{-1}(P_{N}x - x).
\]
This implies 
\[
\| x_{N} - x \|_{C} \leq A\| P_{N}x - x \|_{C} \leq \varepsilon_{N}(X). \tag{3.5}
\]
Thus, the accuracy of the approximate solution obtained via projective method (3.4) is determined by the accuracy \( \varepsilon_{N}(X) \) of the approximation of functions from \( X \) by the local splines.

On the other hand, it follows from the Theorems given in Section 2 that for the functions from \( X \) the order of estimate (3.5) cannot be improved (see Theorems 2.1 and 2.2). Hence, we conclude that algorithm (3.2) is of optimal accuracy order on the classes \( Q_{r,\gamma}^{u}(\Omega, M) \) and \( B_{r,\gamma}^{u}(\Omega, A) \).

It was proved in paper [15] that such numerical methods for VIEs are also optimal with respect to complexity order.

The numerical solution of multidimensional VIEs with the optimal accuracy requires a huge number of arithmetical operations. In [5], employing the 2-D VIE case as an example, we investigate the problem of accelerating the computing process by using multiprocessor computers.

**Remark 3.1.** The suggested algorithm can be also applied to numerical solution of multidimensional VIEs of the form 
\[
x(t) - \int_{0}^{t_{2}} \cdots \int_{0}^{t_{2}} \int_{0}^{t_{2}} h(t, \tau) \left[ (t_{1} - \tau_{1})^{2} + \cdots + (t_{l} - \tau_{l})^{2} \right]^{r+\alpha} x(\tau)d\tau_{1} \cdots d\tau_{l} = f(t),
\]
\[
t = (t_{1}, \ldots, t_{l}), \quad \tau = (\tau_{1}, \ldots, \tau_{l}), \quad t \in [0, T],
\]
with coefficients from the class \( Q_{r,\gamma}^{u}(\Omega, M) \).
Example 4.1. Consider first the following integral equation

\[ x(t_1, t_2) - \int_0^{t_1} \int_0^{t_2} \frac{x(\tau_1, \tau_2) d\tau_1 d\tau_2}{\sqrt{t_1 - \tau_1} \sqrt{t_2 - \tau_2}} = \int_0^{t_1} \int_0^{t_2} \frac{25\pi^2 t_1^6 t_2^6}{1048576}, \]  

(4.1)

\[ (t_1, t_2) \in \Omega = [0, 1]^2. \]

The kernel and the right part of equation (4.1) belong to the class \( B_{2,0.5}(\Omega, A) \) and simultaneously to the class \( Q_{2,\gamma}(\Omega, M) \) for any \( \gamma = n + 0.5, \) where \( n = 0, 1, \ldots \). The exact solution of (4.1) is \( x(t_1, t_2) = (t_1 t_2)^{2.5}. \)

Two different algorithms of approximation have been applied to (4.1) for each of these classes (see their description in Section 2). The results are given below.

Here \( N \) is the number of subdomains of the main partition for \( \Omega; \)

\[ \begin{array}{|c|cccccccc|}
\hline
N & 1 & 2 & 3 & 5 & 10 & 15 & 20 \\
\hline
\varepsilon_1 & 7.17e - 3 & 1.12e - 5 & 2.17e - 7 & 4.89e - 8 & 6.13e - 9 & 6.32e - 11 & 5.89e - 13 \\
\hline
\varepsilon_2 & 0.011 & 6.15e - 4 & 9.05e - 6 & 6.73e - 7 & 2.39e - 8 & 6.84e - 10 & 2.87e - 11 \\
\hline
\end{array} 

Table 1. The error on the class \( Q_{2,2.5}(\Omega, M) \).

\[ \begin{array}{|c|cccccccc|}
\hline
N & 1 & 2 & 3 & 5 & 10 & 15 & 20 \\
\hline
\varepsilon_1 & 6.25e - 4 & 1.04e - 7 & 3.10e - 8 & 5.99e - 9 & 5.23e - 10 & 7.31e - 13 & 6.19e - 15 \\
\hline
\varepsilon_2 & 0.002 & 7.45e - 6 & 9.73e - 7 & 7.62e - 8 & 1.48e - 9 & 6.35e - 12 & 3.17e - 13 \\
\hline
\end{array} 

Table 2. The error on the class \( B_{2,0.5}(\Omega, A) \).

\[ \varepsilon_1 = \max_{t_j} |x(t_i, t_j) - x_N(t_i, t_j)| \]

is the error at the nodes of the grid;

\[ \varepsilon_2 = \|x(t_1, t_2) - x_N(t_1, t_2)\|_{C(\Omega)} \]

is the error in \( \Omega. \)

Example 4.2. In the second example we construct the numerical solution of the equation with standard weakly singular kernel:

\[ x(t_1, t_2) - \int_0^{t_1} \int_0^{t_2} \frac{x(\tau_1, \tau_2) d\tau_1 d\tau_2}{\sqrt{t_1 - \tau_1} \sqrt{t_2 - \tau_2}} = \sqrt{t_1 t_2} - \frac{\pi^2}{4} t_1 t_2, \quad (t_1, t_2) \in \Omega = [0, 1]^2. \]

(4.2)

The exact solution of (4.2) is \( x(t_1, t_2) = \sqrt{t_1 t_2}. \) Applying to (4.2) the similar spline-collocation algorithm (with grading exponent of nonuniform mesh \( q = 8 \) and the spline order \( m = 4 \)) we obtain the following error results

\[ \begin{array}{|c|cccccccc|}
\hline
N & 1 & 2 & 3 & 5 & 10 & 15 & 20 \\
\hline
\varepsilon_1 & 0.043 & 1.6e - 3 & 2.11e - 4 & 1.72e - 5 & 6.93e - 7 & 6.58e - 8 & 9.56e - 9 \\
\hline
\varepsilon_2 & 0.051 & 2.1e - 3 & 4.43e - 4 & 3.91e - 5 & 1.17e - 6 & 1.23e - 7 & 3.68e - 8 \\
\hline
\end{array} 

Table 3. The error results for (4.2).
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