1 Introduction

The purpose here is to give a direct computation of the zeta-function curvature for the determinant line bundle of a family of APS-type boundary value problems.

Here is the sort of computation we have in mind.

1.0.1 Example: \( \zeta \)-curvature on \( \mathbb{C}P^1 \)

Consider the simplest case, \( D = id/dx \) over \([0, 2\pi]\) with Laplacian \( \Delta = -d^2/dx^2 \). Global boundary conditions for \( D \) are parameterized by \( \mathbb{C}P^1 \). Specifically, over the dense open subset of \( \mathbb{C}P^1 \) parameterizing complex lines \( l_z \subset \mathbb{C}^2 \) given by the homogeneous coordinates \([1, z]\) for \( z \in \mathbb{C} \) the orthogonal projection \( P_z = \frac{1}{1+|z|^2} \left( \frac{1}{z} \bar{z} \right) \) onto \( l_z \) parametrizes the boundary condition

\[ P_z \begin{pmatrix} \psi(0) \\ \psi(2\pi) \end{pmatrix} = 0; \]

that is, \( \psi(0) = -\bar{z} \psi(2\pi) \). Let \( D_{P_z} \) denote \( D \) with domain restricted to functions satisfying this boundary condition. The adjoint boundary problem is \( D_{P_z}^* \) with projection \( P_z^* = \frac{1}{1+|z|^2} \left( \frac{|z|^2}{-z} \right) \) corresponding to \( -z \phi(0) = \phi(2\pi) \). Then \( \Delta_{P_z} \) has discrete spectrum

\[ \{(n + \alpha)^2, (n - \alpha)^2 : n \in \mathbb{N} \}, \]

where \( u = e^{2\pi i \alpha} \) satisfies \( u^2(1 + |z|^2) + 2u(z + \bar{z}) + (1 + |z|^2) = 0 \).

The zeta determinant of \( \Delta_{P_z} \) is therefore

\[ \det \zeta \Delta_{P_z} = 4 \sin^2 \pi \alpha = \frac{2(1 + |z|^2)}{1 + |z|^2}, \quad [1, z] \in \mathbb{C}P^1. \]  

(1.0.1)

The Quillen metric evaluated on the holomorphic section identified with the abstract determinant \( z \mapsto \det D_{P_z} \) is \( \|\det D_{P_z}\|^2 = \det \zeta \Delta_{P_z} \) and hence the canonical curvature (1,1) -form of the determinant line bundle is

\[ \overline{\partial \partial} \log \det \zeta \Delta_{P_z} = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \mathrm{Kahler \ form \ on \ } \mathbb{C}P^1. \]  

(1.0.2)

1.0.2 \( c_1 \) of the determinant

More generally, determinant bundles arise in geometric analysis, in the representation theory of loop groups, and in the construction of conformal field theories. In a general sense, they facilitate the construction of projective representations from the bordism category to categories of graded rings. The basic invariant of a determinant bundle which one aims to compute is its Chern class.

1.0.3 Example: closed surfaces

A well known instance of that is for a family of compact boundaryless surfaces \( \{\Sigma_y \mid y \in Y\} \) parametrized by a smooth manifold \( Y \). Let \( M = \bigcup_{y \in Y} \Sigma_y \) and \( \pi : M \to Y \) the projection map.
Let $T_y$ be the tangent bundle to $\Sigma_y$, and $T := T(M/Y) = \bigcup_{y \in Y} T_y \to M$ the tangent bundle along the fibres. The index bundle $\Ind \overline{\partial}_{(m)}$ of the family of D-bar operators $\overline{\partial}_{(m)} = \{ \overline{\partial}_y \mid y \in Y \}$ acting on sections of $T^{\otimes m}$ is the element $f_!(T^{\otimes m})$ of $K(Y)$, and the Grothendieck-Riemann-Roch theorem says

$$\text{ch}(f_!(T^{\otimes m})) = f_*(\text{ch}(T^{\otimes m}) \text{ Todd}(T)),$$

where $f_* : H^i(M) \to H^{i-2}(Y)$ is integration over the fibres. That is, with $\xi = c_1(T)$

$$\text{ch}(\Ind \overline{\partial}_{(m)}) = f_* \left( e^{m\xi} \cdot \frac{\xi}{1 - e^{-\xi}} \right) = f_* \left( 1 + (m + 1/2)\xi + \frac{1}{2}(m^2 + m + 1/6)\xi^2 + \ldots \right).$$

Hence $c_1$ of the determinant line bundle $\Det \overline{\partial}_{(m)}$ is

$$c_1(\Det \overline{\partial}_{(m)}) = \frac{1}{12} (6m^2 + 6m + 1) f_*(\xi^2) \in H^2(Y). \tag{1.0.3}$$

### 1.0.4 Quillen on the curvature formula

More refined formulae may be sought at the level of smooth invariants. The fundamental result in this direction was obtained by Quillen in 1984 in a very beautiful four page article \cite{Quillen1984} in which the zeta function regularized curvature of the determinant line bundle $\Det \Sigma_Y$ of a family of Cauchy Riemann operators $\Sigma_Y = \{ D : \Omega(\Sigma, E) \to \Omega^{0,1}(\Sigma, E) \}$ acting on sections of a complex vector bundle $E$ over a closed Riemann surface $\Sigma$ was computed to be

$$F_\zeta(\Sigma_Y) = \text{ Kahler form on } Y \tag{1.0.4}$$

where in this case $Y = \Omega^{0,1}(\Sigma, \text{End } E)$.

### 1.0.5 Bismut on Quillen

Following Quillen’s idea of constructing a superconnection on the index bundle \cite{Quillen1984}, Bismut \cite{Bismut1984} proved in a tour de force a local index theorem for a general family $D$ of Dirac-type operators associated to a geometric fibration $\pi : M \to Y$ with fibre a compact boundaryless manifold and, furthermore, with $F_\zeta(D) \in \Omega^2(Y)$ the curvature of the $\zeta$-connection on the determinant line bundle $\Det D$, extended \cite{Bismut1984} to

$$F_\zeta(D) = \text{ ind}_{[2]}, \tag{1.0.5}$$

where $\text{ ind } \in \Omega^*(Y)$ is the family index density, equal to $\int_{M/Y} \hat{A}(M/Y) \text{ ch}(V)$ in the case of a family of twisted Dirac operators, and the subscript indicates the 2-form component \cite{Bismut1984}.

It is worth emphasizing here the geometric naturality of the formulae; in each of the above cases, including the example of \cite{Quillen1984}, the $\zeta$-curvature hits the index form ‘on the nose’ — any other connection will have curvature differing from this by an exact 2-form.

### 1.0.6 Melrose and Piazza on Bismut

That naturality persists to the analysis of families of APS boundary problems $D_P$ for which the fibre of $\pi : M \to Y$ is a compact manifold with boundary and $\partial M \neq \emptyset$, and $P = \{ P_y \}$ is a smooth family of $\psi$do projections on the space of boundary sections which is pointwise (w.r.t. $Y$) commensurable with the APS projection.

The principal contribution in this direction is the Chern character formula of Melrose-Piazza \cite{Melrose-Piazza1991} proved using $b$-calculus and generalizing Bismut-Cheeger \cite{Bismut-Cheeger1982}. From this Piazza \cite{Piazza1991} inferred the $b$ zeta-curvature function formula on the $b$ determinant bundle $\Det^b(D_P)$ to be

$$F_\zeta^b(D_P) = \text{ ind}_{[2]} + \left| \overline{\eta}_P \right|_{(2)},$$
where \( \tilde{\eta}_P := \pi^{-1/2} \int_0^\infty \text{Tr}(\mathcal{B}_t e^{-B_t^2}) \, dt \) is an eta-form of a \( t \)-rescaled superconnection \( \mathcal{B}_t = \mathcal{B}_t(P) \) twisted by \( P \) for the family of Dirac operators on the boundary \( \partial M \).

### 1.0.7 A direct computation

On the other hand, \( D_P \) is already a smooth family of Dirac-Fredholm operators and it is natural to seek a direct computation of the \( \zeta \)-curvature formula for the determinant line bundle \( \text{Det} D_P \), along the lines of example of [1.0.1, without use of \( b \)-calculus or other completions. It turns out, indeed, that there is a canonical \( \zeta \)-function connection on \( \text{Det} D_P \) and one has:

**Theorem 1.1** Let \( F_\zeta(D_P) \) be the curvature 2-form of the \( \zeta \)-connection on \( \text{Det} D_P \). Then

\[
F_\zeta(D_P) = F_\zeta(D_{P(D)}) + R^{\kappa, \nu} \quad \text{in } \Omega^2(Y) \tag{1.0.6}
\]

with \( R^{\kappa, \nu} \) the 2-form component of a relative \( \eta \)-form depending only on boundary data; the fibration of closed boundary manifolds and on \( \text{ran}(P) = W \) and on \( \text{ran}(P(D)) = K \). Here, \( P(D) \) is the family of Calderón projections defined by \( D \), equal at \( y \in Y \) to the projection onto the (infinite dimensional) subspace equal to the restriction of \( \text{Ker} \; D_y \) to the boundary. The determinant bundle \( \text{Det} D_{P(D)} \) is trivial. Its \( \zeta \)-curvature is canonically exact; there is a preferred 1-form \( \beta_\zeta(D) \in \Omega^1(Y) \) such that

\[
F_\zeta(D_{P(D)}) = d\beta_\zeta(D). \tag{1.0.7}
\]

The definition of \( R^{\kappa, \nu} \), which is simple and completely canonical, and why it is a ‘relative eta form’, is given in [1. The formula (1.0.6) is extremely ‘clean’, in so far as it is the simplest relation that might exist between \( F_\zeta(D_P) \) and \( R^{\kappa, \nu} \), both of which represent \( c_1(\text{Det} D_P) \). It extends to geometric families of boundary problems the principle of ‘reduction to the boundary’ present in the analysis of Grubb and Seeley [12], [10] and Bruening and Lesch [5] of resolvent and zeta traces of pseudodifferential boundary problems, also in Booss-Wojechowski [7], and in the zeta determinant formulae in joint work with Krzysztof Wojciechowski [24] and in [2].

### 1.0.8 Example: surfaces

For a real compact surface \( \Sigma \) with boundary \( S^1 \) our conclusions generalize the example of [1.0.1 (and [1.0.3) as follows. A choice of conformal structure \( \tau \in \text{Conf}(\Sigma) \) turns \( \Sigma \) into a Riemann surface with a D-bar operator \( \overline{\partial}_\tau : \Omega^0(\Sigma) \rightarrow \Omega^{0,1}(\Sigma) \). Since \( P(\overline{\partial}_\tau) \) differs from the APS projection \( \Pi \geq \) by only a smoothing operator \( \mathcal{B}_t \) a suitable parameter space of well-posed boundary conditions is the smooth Grassmannian \( \text{Gr} \) of pseudodifferential operator \( (\psi \text{do}) \) projections \( P \) with \( P = P(\overline{\partial}_\tau) \) smoothing. We obtain in this way the family of APS boundary problems

\[
\overline{\partial}_P := (\overline{\partial}_\tau)_P : \text{dom}(\overline{\partial}_P) = \text{Ker}(P \circ \gamma) \rightarrow \Omega^{0,1}(\Sigma)
\]

parametrized by \( P \in \text{Gr} \). In this case \( F_\zeta(\overline{\partial}_P) = 0 \) and (1.0.6) is

\[
F_\zeta(\overline{\partial}_P) = \text{Tr}(PdPdP) = \text{Kahler form on } \text{Gr}. \tag{1.0.8}
\]

The restriction of \( \text{Det} \overline{\partial}_P \) to the loop group via the embedding \( \text{LG} \hookrightarrow \text{Gr} \) based at \( P(\overline{\partial}_\tau) \) is the central extension of \( \text{LG} \) (Segal [23]), while \( F_\zeta(\overline{\partial}_P)|_{\text{LG}} \) is the 2-cocycle of the extension. On the other hand, one may consider the opposite situation of the family of D-bar operators on \( \Sigma \)

\[
\overline{\partial}_{P, m} = \{ (\overline{\partial}_\tau)|_{\Sigma} \mid \tau \in \text{Conf}(\Sigma) \}
\]

parametrized by \( Y = \text{Conf}(\Sigma) \) and with fixed boundary condition \( \Pi \geq \) acting on sections of \( T^\infty \Sigma \). \( \text{Det} \overline{\partial}_{P, m} \) pushes-down to the moduli space \( \mathcal{M}(\Sigma) = \text{Conf}(\Sigma)/\text{Diff}(\Sigma, \partial \Sigma) \) by the group

\[
\pi_2(\text{Conf}(\Sigma)) = \{ (\overline{\partial}_\tau)|_{\Sigma} \mid \tau \in \text{Conf}(\Sigma) \}.
\]
of diffeomorphisms of $\Sigma$ equal to the identity on the boundary. In particular, for the unit disc $D$ then $\mathcal{M}(D) = \text{Diff}^+ S^1/\text{PSU}_{1,1}$. By the functoriality of our constructions and the computations of \cite{13} we obtain that the $\zeta$-curvature of the determinant line bundle over $\text{Diff}^+ S^1/\text{PSU}_{1,1}$ is

$$F_\zeta(\partial_{\nu^{-1}}) = F_\zeta(\partial_{\nu^{-1}}) + \frac{1}{12} (6m^2 + 6m + 1) \pi_*(\nu) - \frac{1}{12} e,$$

where $\pi_*(\nu)$ is integration over the fibre of a Godbillon-Vey form, $e$ an Euler form \cite{15}, and $P(\nu^{-1})$ the family of Calderón boundary conditions.

\section{Fibrations of Manifolds}

Let $\pi : M \overset{X}{\rightarrow} Y$ be a smooth fibration of manifolds with fibre diffeomorphic to a compact connected manifold $X$ of dimension $n$ with boundary $\partial X \neq \emptyset$. The total space $M$ is itself a manifold with boundary $\partial M$ and there is a boundary fibration $\partial \pi : \partial M \overset{\partial \pi}{\rightarrow} Y$ of closed manifolds of dimension $n - 1$. For example, for a fibration of surfaces over $Y = S^1$ then $\partial M$ is a disjoint union of 2-tori fibred by the circle.

We assume there exists a collar neighbourhood $U \subset M$ of $\partial M$ with a diffeomorphism

$$U \cong [0, 1) \times \partial M,$$

(2.0.9)
corresponding fibrewise to a collar neighbourhood $[0, 1) \times \partial X$ of each fibre $X_y := \pi^{-1}(y)$.

\subsection{Bundles over fibrations}

A smooth family of vector bundles associated to $\pi : M \overset{X}{\rightarrow} Y$ is defined to be a finite-rank $C^\infty$ vector bundle $E \rightarrow M$. Formally, we may then consider the infinite-dimensional bundle $\mathcal{H}(E) \rightarrow Y$ whose fibre at $y \in Y$ is the space $\mathcal{H}_y(E) := \Gamma(X_y, E|_{X_y})$ of $C^\infty$ sections of $E$ over $X_y$. Concretely, a section of $\mathcal{H}(E)$ is defined to be a section of $E$ over $M$,

$$\Gamma(Y, \mathcal{H}(E)) := \Gamma(M, E).$$

(2.1.1)

Thus, in practise one works with the right-side of (2.1.1), as indicated below.

$\Gamma(Y, \mathcal{H}(E))$ is then a $C^\infty(Y)$-module via

$$C^\infty(Y) \times \Gamma(Y, \mathcal{H}(E)) \rightarrow \Gamma(Y, \mathcal{H}(E)), \quad (f, s) \mapsto f \cdot s := \pi^*(f)s,$$

(2.1.2)

that is, $f \cdot s(m) = f(\pi(m))s(m)$.

The restriction map to boundary sections

$$\gamma : \Gamma(Y, \mathcal{H}(E)) \rightarrow \Gamma(Y, \mathcal{H}(E|_{\partial M}))$$

(2.1.3)
is defined by the restriction map to the boundary on the total space

$$\gamma : \Gamma(M, E) \rightarrow \Gamma(\partial M, E|_{\partial M})$$

(2.1.4)

with $E|_{\partial M} = \cup_{m \in \partial M} E_m$ the bundle $E$ along $\partial M$. Relative to (2.0.9)

$$E|_{U} = \gamma^*(E|_{\partial M})$$

(2.1.5)

and $\Gamma(U, E) \cong C^\infty([0, 1)) \otimes \Gamma(M, E|_{\partial M})$. Here, $\text{rank}(E|_{\partial M}) = \text{rank}(E)$, so, for example, $TM|_{\partial M}$ is not the same thing as $T(\partial M)$, whose sections are vector fields along the boundary, while a section of $TM|_{\partial M}$ includes vector fields which point out of the boundary; one has $TM|_{\partial M} \cong \mathbb{R} \oplus T(\partial M)$.
The vertical tangent bundle $T(M/Y)$ (resp. $T(\partial M/Y)$) is the subbundle of $TM$ (resp. $T\partial M$) whose fibre at $m \in M$ (resp. $m \in \partial M$) is the tangent space to the fibre $X_{\pi(m)}$ (resp. $\partial X_{\pi(m)}$). $\pi^*(TY)$ is the pull-back subbundle from the base. Likewise, there is the dual bundle $T^*M$ with subbundle $T^*(M/Y)$, whose sections are vertical forms along $M$, and $\pi^*(\wedge T^*Y)$. More generally, the de-Rham algebra on $Y$ with values in $H(E)$ is the direct sum of the

$$A^k(Y, H(E)) = \Gamma(M, \pi^*(\wedge^k T^*Y) \otimes E \otimes |\wedge^\pi|^{1/2}). \quad (2.1.6)$$

The line bundle of vertical densities $|\wedge^\pi|$ is included to facilitate integration along the fibre.

### 2.2 Connections

A connection (or covariant derivative) on $\mathcal{H}(E)$ is specified by a fibration ‘connection’ on $M$

$$TM \cong T(M/Y) \oplus T_H M, \quad (2.2.1)$$

and a vector bundle connection on $E$

$$\nabla : \Gamma(M, E) \to \Gamma(M, E \otimes T^*M), \quad (2.2.2)$$

which are compatible with the induced boundary connections.

The fibration connection is a complementary subbundle to $T(M/Y)$, specifying an isomorphism $\pi^*(TY) \cong T_H M$ and hence a lift of vector fields from the base to horizontal vector fields on $M$

$$\Gamma(Y, TY) \cong \Gamma(M, T_H M), \quad \xi \mapsto \xi_H. \quad (2.2.3)$$

A connection

$$\nabla^M : A^0(Y, \mathcal{H}(E)) \to A^1(Y, \mathcal{H}(E)) \quad (2.2.4)$$

is then defined by

$$\nabla^M_\xi s = \bar{\nabla}_\xi_H s, \quad s \in \Gamma(M, E), \xi \in C^\infty(Y, TY). \quad (2.2.5)$$

Compatibility with the boundary means, first, that in the collar $U$

$$\bar{\nabla}|_U = \gamma^* \bar{\nabla}^{\partial M} = \partial_u du + \bar{\nabla}^{\partial M}$$

where $u \in [0, 1)$ is the normal coordinate to $\partial M$ and $\bar{\nabla}^{\partial M} : \Gamma(\partial M, E_{\partial M}) \to \Gamma(\partial M, E_{\partial M} \otimes T^*M)$ is the induced connection on $E_{\partial M}$, defining $\nabla^{\partial M} : A^0(Y, \mathcal{H}(E_{\partial M})) \to A^1(Y, \mathcal{H}(E_{\partial M}))$ by

$$\nabla^{\partial M}_\xi s = \bar{\nabla}^{\partial M}_\xi_H s, \quad s \in \Gamma(\partial M, E_{\partial M}). \quad (2.2.6)$$

Secondly, that with respect to the boundary splitting

$$T(\partial M) \cong T(\partial M/Y) \oplus T_H \partial M$$

induced by

$$TU \cong \mathbb{R} \oplus T\partial M \quad (2.2.7)$$

and the splitting (2.2.1), one has for $\xi \in C^\infty(Y, TY)$ that

$$(\xi_H)|_U \in C^\infty(U, T_H(\partial M)),$$

that is,

$$du(\xi_H) = 0,$$

where $du$ is extended from $U$ to $M$ by zero. One then has from (2.2)
Lemma 2.1
\[ \gamma \circ \tilde{\nabla}_{\xi_H} = \tilde{\nabla}^\partial M \circ \gamma, \quad \xi \in C^\infty(Y, TY), \]  
(2.2.8)
as maps \( \Gamma(M, E) \rightarrow \Gamma(\partial M, E_{\omega M}) \).

The curvature of the connection (2.2.4) evaluated on \( \xi, \eta \in C^\infty(Y, TY) \)
\[ R(\xi, \eta) \in \Gamma(Y, \text{End}(\mathcal{H}(E))) \]
is the smooth family of first-order differential operators (as in [3] Prop(1.11))
\[ R(\xi, \eta) := \tilde{\nabla}_{\xi_H} \tilde{\nabla}_{\eta_H} - \tilde{\nabla}_{\eta_H} \tilde{\nabla}_{\xi_H} - \tilde{\nabla}_{[\xi, \eta]_H} = \tilde{R}(\xi, \eta) + \tilde{\nabla}_{[\xi, \eta]_H - [\xi_H, \eta_H]} \]
where \( \tilde{R}(\xi, \eta) \in \Gamma(M, \text{End} E) \) is the curvature of \( \tilde{\nabla} \). The above compatibility assumptions state that \( \gamma_*(\xi_H), \gamma_*(\eta_H) \in C^\infty(\partial M, T_H(\partial M)) \) and
\[ R(\xi_H, \eta_H) \circ \gamma = R^{\omega M}(\gamma_*(\xi_H), \gamma_*(\eta_H)) \in \Gamma(Y, \text{End}(\mathcal{H}(E_{\omega M}))), \]
(2.2.10)
where \( R^{\omega M}(\alpha, \beta) \) is the curvature of (2.2.6).

2.2.1 Example: spin connection

For our purposes, here, it is not necessary to specify which particular connection on \( E \) is being used, as the constructions are functorial. However, to compute the local index form curvature for \( \tilde{M} \), \( \tilde{E} \) of compact \( \tilde{E} \) is being twisted over \( M \) with boundary the pseudodifferential boundary operator \( (\psi/dbo) \) calculus as developed by Grubb [10], generalizing the Boutet de Monvel algebra, may be applied to define a vertical calculus of operators with oscillatory integral kernels along the fibres comprising trace operators from interior to boundary sections, vertical Poisson operators taking sections over the boundary \( \partial M \) into the interior, and restricted \( \psi/dbo \) and singular Green’s operators over the interior of \( M \). This vertical \( \psi/dbo \) algebra is denoted
\[ \Gamma(Y, \Psi_\nu(E^+, E^-)) = \Psi^\nu_{\text{vert}}(N, E^+, E^-). \]
The algebras $A \in \Gamma(Y, \Psi^{\pm}(E^+, E^-))$ (see \cite{22}) and $\Psi_{\text{vert}, b}(M, E^+, E^-)$ of generalized $\psi$dos are described in more detail in the Appendix.

For a local trivialization of the fibration and of $E$ one may locally identify a vertical $\psi$do $A$ with a single $\psi$do (or $\psi$dbo) $A_y$ acting on a fixed space and depending on a local parameter $y$ in $Y$.

### 3.1 Families of Dirac-type operators

Let $\mathcal{D}$ be a family of Dirac-type operators associated to the fibration $\pi : M \to Y$ of compact manifolds with boundary with vector bundles $E^\pm \to M$, such that in $\mathcal{U}$

$$
\mathcal{D}|_{\mathcal{U}} = \mathcal{Y}\left(\frac{\partial}{\partial x_n} + \mathcal{D}_{\partial M}\right),
$$

(3.1.1)

where $\mathcal{D}_{\partial M} \in \Psi_{\text{vert}}(\partial M, E_{\partial M})$ a family of Dirac-type operators associated to the boundary fibration of closed manifolds, and $\mathcal{Y} \in \Gamma(\partial M, \text{End}(E_{\partial M}))$ is a bundle isomorphism.

#### 3.1.1 Vertical Poisson and Calderón operators

Let $\hat{\mathcal{M}} = M \cup_{\partial M} (-M) \to Y$ be the fibration of compact boundaryless manifolds with fibre the double manifold $\hat{X}_y = X_y \cup_{\partial X_y} (-X_y)$. With the product structure (2.0.3), $\mathcal{D}$ extends by the proof for a single operator, as in \cite{7} Chap.9, to an invertible vertical first-order differential operator $\hat{\mathcal{D}} \in \Psi^1(\hat{\mathcal{M}}, \hat{E}^+, \hat{E}^-)$, where $\hat{E}_M^+ = E^+$ and $r^+\hat{D} e^+ = \mathcal{D}$. As indicated in Appendix (A.1) and accounted for in detail in \cite{22}, there is therefore a smooth family of resolvent $\psi$dos of order $-1$ $\hat{\mathcal{D}}^{-1} \in \Psi_{\text{vert}}^1(\hat{\mathcal{M}}, \hat{E}^-, \hat{E}^+)$. Define

$$
\mathcal{D}_+^{-1} := r^+\hat{\mathcal{D}}^{-1} e^+ \in \Psi_{\text{vert}, b}^1(M, E^-, E^+).
$$

Since $\hat{\mathcal{D}}\mathcal{D}_+^{-1} = 1$ on $\Gamma(\hat{\mathcal{M}}, \hat{\mathcal{E}})$, with $1$ the vertical identity operator, and since $\mathcal{D}$ is local

$$
\mathcal{D}\mathcal{D}_+^{-1} = 1 \quad \text{on} \quad \Gamma(M, E^-).
$$

Thus there is a short exact sequence $0 \longrightarrow \text{Ker}(\mathcal{D}) \longrightarrow \Gamma(M, E^+) \xrightarrow{\mathcal{D}} \Gamma(M, E^-) \longrightarrow 0$, where

$$
\text{Ker}(\mathcal{D}) = \{ s \in \Gamma(M, E^+) \mid \mathcal{D}s = 0 \quad \text{in} \quad M \setminus \partial M \}.
$$

(3.1.3)

On the other hand, $\mathcal{D}_+^{-1}$ is not a left-inverse but (by an obvious modification of \cite{22}, \cite{24}, \cite{7} §12)

$$
\mathcal{D}_+^{-1} \mathcal{D} = 1 - K\gamma \quad \text{on} \quad \Gamma(M, E^+),
$$

(3.1.4)

where $\gamma$ is the restriction operator (2.1.4) and the **vertical Poisson operator associated to $\mathcal{D}$** is

$$
K = \mathcal{D}_+^{-1} \gamma \mathcal{Y},
$$

(3.1.5)

with $\gamma$ as in (3.2.4). Composing with boundary restriction defines the **vertical Calderón projection** (3, \cite{22}, \cite{26}, \cite{7})

$$
\mathcal{P}(\mathcal{D}) := \gamma \circ K \in \Gamma(Y, \Psi_{\text{vert}}^0(E_{\partial M})) := \Psi_{\text{vert}}^0(\partial M, E_{\partial M})
$$

(3.1.6)

with range the space of vertical Cauchy data

$$
\text{ran}(\mathcal{P}(\mathcal{D})) = \gamma \text{Ker}(\mathcal{D}) = \{ f \in \Gamma(\partial M, E_{\partial M}) \mid f = \gamma s, \ s \in \text{Ker}(\mathcal{D}) \}.
$$

(3.1.7)

This may be formally characterized as the space of sections of the infinite-dimensional subbundle $\mathcal{K}(\mathcal{D}) \subset \mathcal{H}(E_{\partial M})$ with fibre $\mathcal{K}(D_y) = \gamma \text{Ker}(D_y)$ at $y \in Y$ (and, likewise, $\text{Ker}\mathcal{D}$ as the space of
sections of the formal subbundle of $\mathcal{H}(E^+)$ with fibre $\ker D_y$. However, as with $\mathcal{H}(E_{\partial M})$ in (2.1), concretely one only works with the space of sections of $K(D)$

$$\Gamma(Y, K(D)) := \{ f \in \Gamma(\partial M, E_{\partial M}) \mid f = \gamma s, \ s \in \ker (D) \} = \text{ran}(P(D)).$$

(3.1.8)

(Note, on the other hand, $K(D)$ is not the space of sections of a subbundle of $E_{\partial M}$.)

By the fibrewise Unique Continuation property, restriction $\gamma : \ker D \rightarrow \Gamma(Y, K(D))$ defines a canonical isomorphism with right-inverse

$$K : \Gamma(Y, K(D)) \xrightarrow{\sim} \ker (D).$$

(3.1.9)

### 3.2 Well-posed boundary problems for $D$

The vertical Calderón projection (3.1.6) provides the reference $\psi do$ on boundary sections with respect to which is defined any vertical well-posed boundary condition for $D$.

#### 3.2.1 Smooth families of boundary $\psi do$ projections

We consider smooth families of $\psi dos$ on $\Gamma(Y, \mathcal{H}(E_{\partial M}))$ which are perturbations of the Calderón projection of the form

$$P = P(D) + S \in \Psi^0_{\text{vert}}(\partial M, E_{\partial M}),$$

(3.2.1)

where

$$S \in \Psi^{-\infty}_{\text{vert}}(\partial M, E_{\partial M})$$

is a vertical smoothing operator (smooth family of smoothing operators), cf. Appendix. From Birman-Solomyak (11), Seeley (12) (see also 3.2) may be replaced by the projection onto $\text{ran}(P)$ to define an equivalent boundary problem. So we may assume $P^2 = P$ and $P^* = P$, where the adjoint is with respect to the Sobolev completions and vertical inner-product defined by metric on $E_{\partial M}$ and the choice of vertical density $d_{\partial M/Y} = \Gamma(\partial M, |^{-1} T^* (\partial M/Y))$.

The family APS projection $\Pi_y = \{ \Pi_{y} \mid y \in Y \}$ is only smooth in $y$ when $\dim \ker (D_{\partial M})_y$ is constant (11). Nevertheless, we refer to (3.2.1) as a vertical $\psi do$ of APS-type.

The choice of $P$ in (3.2.1) distinguishes the subspace of the space of boundary sections

$$\Gamma(Y, W) := \text{ran}(P) = \{ P f \mid f \in \Gamma(\partial M, E_{\partial M}) \} \subset \Gamma(\partial M, E_{\partial M}) := \Gamma(Y, \mathcal{H}(E_{\partial M})).$$

(3.2.2)

Here, $W$ is the formal infinite-rank subbundle of $\mathcal{H}(E_{\partial M})$ with fibre $W_y = \text{ran}P_y \subset \Gamma(\partial X_y, (E_{\partial M})_y)$, whose local bundle structure follows from the invertibility of the operators $P_y, P_y : W_y \rightarrow W_y$ for $y$ near $y$. Analytically, though, just as with $K(D)$, one works in practise with (1.2.2).

Given any two choices $P, P'$ of the form (3.2.1), one has the smooth family of Fredholm operators

$$P' \circ P : \Gamma(Y, W) \rightarrow \Gamma(Y, W')$$

(3.2.3)

where $\Gamma(\partial M, W) := \text{ran}(P)$. We may write this as a section of the formal bundle $\text{Hom}(W, W')$ in so far as we declare the sections of the latter to precisely be the subspace of $\Psi_{\text{vert}}(\partial M, E_{\partial M})$

$$\Gamma(Y, \text{Hom}(W, W')) := \{ P' \circ A \circ P \mid A \in \Psi_{\text{vert}}(\partial M, E_{\partial M}) \}.$$

(3.2.4)

Note here that

$$P' \circ P \in \Psi^0_{\text{vert}}(\partial M, E_{\partial M})$$

is a smooth vertical $\psi do$ on boundary sections. The reference to it as a ‘smooth family of Fredholm operators’ means additionally that there is smooth vertical $\psi do$ on boundary sections

$$Q_{p, p'} \in \Psi^0_{\text{vert}}(\partial M, E_{\partial M})$$
such that
\[ Q_{p,p'} \circ (P' \circ P) = P + PS', \quad S' \in \Psi^-_{\text{vert}}(\partial M, E_{\partial M}), \quad (3.2.5) \]

and hence that \( Q_{p,p'} \) is a parametrix for (3.2.3); that is, restricted to \( \Gamma(Y, W) \) (3.2.5) is
\[ (Q_{p,p'} \circ (P' \circ P))_{|W} = I_w + PS'P, \quad (3.2.6) \]

where \( I_w \) denotes the identity on \( \Gamma(Y, W) \). Indeed, we may take, for example, \( Q_{p,p'} = P \circ P' \).

### 3.2.2 Vertical APS-type boundary problems

The choice of \( P \) in (3.2.1) additionally distinguishes the subspace of interior sections on the total space of the fibration (which is not itself the space of sections of some subbundle of \( E^+ \))
\[ \Gamma(Y, H_p(E^+)) := \ker (P \circ \gamma) = \{ s \in \Gamma(M, E^+) \mid P\gamma s = 0 \} \subset \Gamma(M, E^+) := \Gamma(Y, H(E^+)). \quad (3.2.7) \]

We may consider the infinite-dimensional bundle \( H_p(E) \to Y \) with fibre at \( y \in Y \) the space of \( C^\infty \) sections of \( E^+ \) over \( X_y \) which lie in \( \ker (P_y \circ \gamma) \), related to \( W \) via the exact sequence
\[ 0 \to H_p(E^+) \to H(E^+) \xrightarrow{P\gamma} W \to 0. \]

Concretely, however, one works in practise with (3.2.7).

A smooth family of APS-type boundary problems is the restriction of \( D \) to the subspace \( \Gamma(Y, W) \).

\[ D_p := D : \ker (P \circ \gamma) = \Gamma(Y, H_p(E^+)) \to \Gamma(M, E^{-}). \quad (3.2.8) \]

\( D_p \) restricts over \( X_y \) to \( D_{p_y} := (D_y)_{p_y} : \text{dom}(D_{p_y}) \to \Gamma(X_y, E_y^-) \) in a local trivialization of the fibration of manifolds, an APS boundary problem in the usual single operator sense.

The existence of the Poisson operator (3.2.3) reduces the construction of a vertical parametrix for \( D_p \) to the construction of a parametrix for the operator (3.2.1) on boundary sections
\[ S(P) := P \circ P(D) : \Gamma(Y, K(D)) \to \Gamma(Y, W). \quad (3.2.9) \]

Explicitly, let \( U \subset Y \) be the open subset of points in \( Y \) where \( S(P) \) is invertible. That is, relative to any local trivialization of the geometric fibration \( M \to Y \) and bundles at \( y \in Y \) the Fredholm family \( S(P) \) parametrizes an operator \( S_y(P_y) = P_y \circ P(D_y) : K(D_y) \to \text{ran}(P_y) \) in the usual single operator sense; \( y \in U \) if \( S_y(P_y) \) is invertible. Over \( U \) we define
\[ K(P)_{|U} := K \circ P(D)S(P)_{|U}^{-1}P : \Gamma(\pi_{\alpha}^{-1}(U), E_{\partial M}) \to \Gamma(\pi_{\alpha}^{-1}(U), E^+), \quad (3.2.10) \]

where \( \pi_{\alpha} : \partial M \to Y \) is the boundary fibration. Then Green’s theorem for the vertical densities along the fibres locally refines (3.1.4) to
\[ (D_{p})_{|U}^{-1}D_{|U} = I_{|U} - K(P)_{|U} \gamma : \Gamma(\pi_{\alpha}^{-1}(U), E^+) \to \Gamma(\pi_{\alpha}^{-1}(U), E^+). \quad (3.2.11) \]

Moreover, if \( D_{p'} \) is also invertible over \( U \)
\[ (D_{p'})_{|U}^{-1} = (D_{p})_{|U}^{-1}D_{(p')_{|U}}^{-1} = D_{p'}^{-1} - K(P)_{|U}P\gamma D_{p'}^{-1} : \Gamma(\pi_{\alpha}^{-1}(U), E^-) \to \Gamma(\pi_{\alpha}^{-1}(U), E^+), \quad (3.2.12) \]

We note, globally on \( M \), that:

**Proposition 3.1** With the above assumptions the relative inverse is a vertical smoothing operator
\[ (D_{p})_{|U}^{-1} - (D_{p'})_{|U}^{-1} \in \Gamma(U, \Psi^{-\infty}_{\text{vert}, \beta}(E_{\pi_{\alpha}^{-1}(U)})). \quad (3.2.13) \]
More generally, for a general APS-type vertical $\psi$do projection $P \in \Psi^0_{\text{vert}}(\partial M, E_{\partial M})$ a global parametrix for the smooth family of APS-type boundary problems $D_P : \text{Ker}(P \circ \gamma) \to \Gamma(M, E^-)$ is given by

$$D^-_1 - K Q_{P, P(\partial Y)}^1 \gamma D^-_1 \in \Gamma(Y, \Psi^{-\infty}_{\text{vert}, P}(E_{\partial M})),$$

(3.2.14)

where $Q_{P, P(\partial Y)}$ is any parametrix as in (3.2.12) for $S(P)$, for example $Q_{P, P(\partial Y)} = P(D) \circ P$.

**Proof.** We have $P = P(D) + S$, $P' = P(D) + S'$ for vertical smoothing operators $S$, $S' \in \Psi^{-\infty}_{\text{vert}}(\partial M, E_{\partial M})$.

Hence

$$P - P' \in \Psi^{-\infty}_{\text{vert}}(\partial M, E_{\partial M})$$

(3.2.15)

and

$$P(1 - P') = -P S' \in \Psi^{-\infty}_{\text{vert}}(\partial M, E_{\partial M})$$

(3.2.16)

are vertical smoothing operator operators. By (3.2.12)

$$(D_P)|_U - (D_{P'})|_U = -K(P)P \gamma(D_{P'})|_U^{-1} = -K(P)|_U P(1 - P') \gamma(D_{P'})|_U^{-1} \quad \text{over} \quad M_U = \pi^{-1}(U)$$

(3.2.17)

which by (3.2.16) and the composition rules of the $\psi$dbo calculus (cf. §A.2) is smoothing.

The assertion that (3.2.14) is a parametrix is an obvious slight modification of the argument leading to (3.2.12).

\[\square\]

## 4 The Determinant Line Bundle

From Proposition 3.1 the choice of $P$ restricts $D$ to a family $D_P$ of Fredholm operators. It also has the consequence that the kernels of the restricted operators no longer define a vector bundle (formally 3.1.3 does), rather they define a virtual bundle $\text{Ind} D_P \in K(Y)$. Likewise, from §3.2.1, $S(P) : \Gamma(Y, K((D)) \to \Gamma(Y, \mathcal{W})$ is a smooth Fredholm family defining an element $\text{Ind} S(P) \in K(Y)$. The determinant line bundles $\text{Det} D_P$ and $\text{Det} S(P)$ are the top exterior powers of these elements, at least in K-theory. To make sense of them as smooth complex line bundles we use the following trivializations, with respect to which the zeta connection will be constructed.

### 4.0.3 Determinant lines

The determinant of a Fredholm operator $T : H \to H'$ exists abstractly not as a number but as an element $\det T$ of a complex line $\text{Det} T$. A point of $\text{Det} T$ is an equivalence class $[S, \lambda]$ of pairs $(S, \lambda)$, where $S : H \to H'$ differs from $T$ by a trace-class operator and relative to the equivalence relation $(Sq, \lambda) \sim (S, \lambda \det_F q)$ for $q : H \to H$ of Fredholm-determinant class. Scalar multiplication on $\text{Det} T$ is $\mu[S, \lambda] = [S, \mu \lambda]$. The determinant $\det T := [T, 1]$ is non-zero if and only if $T$ is invertible, and there is a canonical isomorphism

$$\text{Det} T \cong \bigwedge^{\text{max}} \text{Ker} T^* \otimes \bigwedge^{\text{max}} \text{Cok} T.$$  

(4.0.18)

For Fredholm operators $T_1, T_2 : H \to H'$ with $T_i - T$ trace class and $T_2$ invertible

$$\frac{\det T_1}{\det T_2} = \det_F (T_1 T_2^{-1}),$$

(4.0.19)

where the quotient on the left side is taken in $\text{Det} T$ and $\det_F$ on the right-side in $H'$. 
4.0.4 The line bundle Det $S(P)$

For each smooth family of smoothing operators $\sigma = \{\sigma_y\} \in \Gamma(Y, \Psi^{-\infty}_\text{vert}(E_{\partial M})) = \Psi^{-\infty}_\text{vert}(\partial M, E_{\partial M})$

\[ P_\sigma = P + P_\sigma P \in \Gamma(Y, \Psi^0_\text{vert}(E_{\partial M})) = \Psi^0_\text{vert}(\partial M, E_{\partial M}) \]  

(4.0.20)

and the open subset of $Y$

\[ U_\sigma := \{y \in Y \mid S(P_\sigma)_y := (P_y + P_\sigma \sigma_y P_y) \circ P(D_y) : K(D_y) \longrightarrow \text{ran}(P_y) \text{ invertible} \}. \]  

(4.0.21)

Over $U_\sigma$ one has the canonical trivialization

\[ U_\sigma \longrightarrow \text{Det } S(P)|_{U_\sigma} = \bigcup_{y \in U_\sigma} \text{Det } S(P)_y, \quad y \longmapsto \text{det } S(P_\sigma)_y := [(P_\sigma \circ P(D))_y], \]  

(4.0.22)

where $S(P)_y := [P_y \circ P(D_y) : K(D_y) \longrightarrow \text{ran}(P_y)]$. Note that $\text{det } S(P_\sigma)_y \neq 0$, and that $S(P_\sigma)_y - S(P)_y$ is the restriction of a smoothing operator so that

\[ \text{det } S(P_\sigma)_y \in \text{Det } S(P)_y \setminus \{0\}. \]  

(4.0.23)

Over the intersection $U_\sigma \cap U_{\sigma'} \neq \emptyset$ the transition function by (4.0.13) is the function

\[ U_\sigma \cap U_{\sigma'} \longrightarrow \mathbb{C}^*, \quad y \longmapsto \text{det}_F \left( S(P_\sigma)_y \circ S(P_{\sigma'})^{-1}_y \right), \]  

(4.0.24)

where the Fredholm determinant is taken on $\text{ran}(P_y)$ and varies holomorphically with $y$.

4.0.5 The line bundle Det $D_P$

The bundle structure of Det $D_P$ is defined by perturbing $D_{P_\sigma}$ to an invertible operator. It is crucial for the construction of the $\zeta$ connection to do so by perturbing the $\psi do P_y$, not $D_y$.

To do this we mediate the local trivializations of Det $D_P$ through those of Det $S(P)$ in 4.0.4.

Precisely, the family of $\psi dos$ $P_\sigma$ in (4.0.20) is of APS-type

\[ P_\sigma - P(D) \in \Psi^{-\infty}_\text{vert}(\partial M, E_{\partial M}), \]  

(4.0.25)

defining the vertical boundary problem $D_{P_\sigma} : \Gamma(Y, \mathcal{H}_{P_\sigma}(E^+)) \longrightarrow \Gamma(Y, \mathcal{H}(E^-))$. From (3.2.2)

\[ U_\sigma := \{y \in Y \mid (D_{P_\sigma})_y : \text{dom}((D_{P_\sigma})_y) \longrightarrow \Gamma(X_y, E_y^-) \text{ invertible} \}, \]  

(4.0.26)

over which there is the local trivialization

\[ U_\sigma \longrightarrow \text{Det } D_{P_\sigma}|_{U_\sigma}, \quad y \longmapsto \text{det } ((D_{P_\sigma})_y) = [(D_{P_\sigma})_y], \]  

(4.0.27)

The equivalence of (4.0.22) and (4.0.26) is the identification for any APS-type $\tilde{P} \in \Psi^0_\text{vert}(\partial M, E_{\partial M})$ of the kernel of $(D_{\tilde{P}})_y$ with that of $S(P)_y$ defined by the Poisson operator $K_y$, and likewise of the cokernels. It follows that there is a canonical isomorphism

\[ \text{Det } (D_{\tilde{P}})_y \cong \text{Det } S(\tilde{P})_y \text{ with } \text{det } (D_{\tilde{P}})_y \longmapsto \text{det } S(\tilde{P})_y. \]  

(4.0.28)

The local trivialization of Det $D_P$ is then defined through the canonical isomorphisms of complex lines applied to (4.0.27)

\[ \text{Det } (D_{P_\sigma})_y \cong \text{Det } S(P_\sigma)_y = \text{Det } S(P)_y \cong \text{Det } (D_P)_y, \]  

(4.0.29)

where the central equality is from (4.0.4). By construction the transition functions for Det $D_P$ are precisely (4.0.24); that is, as functions of $y \in U_\sigma \cap U_{\sigma'}$

\[ \text{det } (D_{P_\sigma})_y = \text{det}_F \left( S(P_\sigma)_y \circ S(P_{\sigma'})^{-1}_y \right) \text{det } (D_{P_{\sigma'}})_y \text{ in } \text{Det } (D_P)_y. \]  

(4.0.30)
Thus the bundle structure of $\text{Det} \mathcal{D}_P$ is constructed using that of $\text{Det} \mathcal{S}(P)$, as with all other spectral invariants of $\mathcal{D}_P$ owing to the facts in §4.2.2.

With respect to smooth families of boundary conditions $P, \tilde{P} \in \Psi^0_{\text{vert}}(\partial M, E_{\partial M})$

$$\text{Det} \mathcal{D}_P \cong \text{Det} \mathcal{D}_{\tilde{P}} \otimes \text{Det} (P \circ \tilde{P}), \quad (4.0.31)$$

which may be viewed as a smooth version of the K-theory identity

$$\text{Ind} \mathcal{D}_P = \text{Ind} \mathcal{D}_{\tilde{P}} + \text{Ind} (P \circ \tilde{P}). \quad (4.0.32)$$

These are a consequence of the following general (useful) identifications.

**Theorem 4.1** Let $A_1 : \mathcal{H}^l \to \mathcal{H}^{l'}$, $A_2 : \mathcal{H} \to \mathcal{H}^l$ be smooth (resp. continuous) families of Fredholm operators acting between Frechét bundles over a compact manifold $Y$. Then there is a canonical isomorphism of $C^\infty$ (resp $C^0$) line bundles

$$\text{Det} A_1 A_2 \cong \text{Det} A_1 \otimes \text{Det} A_2$$

with $\det A_1 A_2 \longleftrightarrow \det A_1 \otimes \det A_2$. In $K(Y)$ one has

$$\text{Ind} A_1 A_2 = \text{Ind} A_1 + \text{Ind} A_2 \quad (4.0.33)$$

For a proof of Theorem 4.1 see [23].

5 **Hermitian Structure**

The (Quillen) $\zeta$-metric on $\text{Det} \mathcal{D}_P$ is defined over $U_\sigma$ by evaluating it on the non-vanishing section $\det \mathcal{D}_P,\sigma$

$$|| \det(\mathcal{D}_P,\sigma)y||_\zeta^2 = \det_\zeta(\Delta P,\sigma)_y, \quad (5.0.34)$$

where the right-side is the $\zeta$-determinant of the vertical Laplacian boundary problem for an APS-type $\psi$do $P$

$$\Delta P = \Delta := D^* D : \text{dom}(\Delta P) \to \Gamma(M, E^-) \quad (5.0.35)$$

with $\text{dom}(\Delta P) = \{ s \in \Gamma(M, E^+) \mid P \gamma s = 0, \ P^* \gamma D s = 0 \}$ and $P^* := \Upsilon(I - P_y) \Upsilon^*$ the adjoint vertical boundary condition.

From [21] Thm(4.2) we know that

$$|| \det(\mathcal{D}_P,\sigma)y||_\zeta^2 = \frac{\det_F (S(P,\sigma)^y S(P,\sigma)_y)}{\det_F (S(P,\sigma')^y S(P,\sigma')_y)} || \det(\mathcal{D}_P,\sigma')_y||_\zeta^2, \quad (5.0.36)$$

which is the patching condition with respect to the transition functions (4.0.30) for (5.0.34) to define a global metric on the determinant line bundle $\text{Det} \mathcal{D}_P$.

6 **Connections on DetD_P**

There are two natural ways to put a connection on the determinant bundle $\text{Det} \mathcal{D}_P$. The first of these is associated to the boundary fibration and its curvature may be viewed as a relative $\eta$-form. The second, is the $\zeta$-function connection, the object of primary interest here.
6.1 A connection on \( \text{Det} \, S(P) \)

The first connection is defined on \( \text{Det} \, S(P) \), which defines a connection on \( \text{Det} \, D_P \) via the isomorphism (by construction) between these line bundles.

The endomorphism bundle \( \text{End} \, (\mathcal{H}(E_{\partial M})) \) whose sections are the boundary vertical \( \psi \)dos

\[
\Gamma(Y, \text{End} \, (\mathcal{H}(E_{\partial M}))) := \Psi^*_\text{vert} \, (\partial M, E_{\partial M})
\]

has an induced connection (also denoted \( \nabla^{\partial M} \)) from \( \nabla^{\partial M} \) on \( \Gamma(Y, \mathcal{H}(E_{\partial M})) \) in \([2.26]\) by

\[
\nabla^{\partial M} A := [\nabla^{\partial M}_\xi, A] \in \Psi^*_\text{vert} \, (\partial M, E_{\partial M}),
\]

where \( \xi \in C^\infty(Y, TY) \). That is,

\[
(\nabla^{\partial M}_\xi A) f = \nabla^{\partial M}_\xi (A f) - A(\nabla^{\partial M}_\xi f), \quad f \in \Gamma(\partial M, E_{\partial M}). \quad (6.1.1)
\]

Let \( P(D) \in \Psi^0(\partial M, E_{\partial M}) \) be the Calderón vertical \( \psi \)do projection, and let \( P \in \Psi^0(\partial M, E_{\partial M}) \) be any other vertical APS-type boundary condition \([3.2.1]\). Then there are induced connections

\[
\nabla^{\psi} = P \cdot \nabla^{\partial M} \cdot P, \quad \nabla^\kappa = P(D) \cdot \nabla^{\partial M} \cdot P(D)
\]

defined on the Fréchet bundles \( W \) and \( \mathcal{K}(D) \), in the sense that

\[
\nabla^{\psi}_\xi : \Gamma(Y, W) = \{ Ps \mid s \in \Gamma(\partial M, E_{\partial M}) \} \rightarrow \Gamma(Y, W)
\]

with

\[
\nabla^{\psi}_\xi s = P \nabla^{\partial M}_\xi (Ps), \quad s \in \Gamma(Y, W),
\]

satisfies the Leibnitz rule, and likewise for \( \nabla^\kappa \). We therefore have the induced connection \( \nabla^{\psi, W} \) on the restricted hom-bundle \( \text{Hom}(\mathcal{K}(D), W) \), where, as in \([3.2.4]\),

\[
\Gamma(Y, \text{Hom}(\mathcal{K}(D), W)) := \{ P \circ C \circ P(D) \mid C \in \Psi^*_\text{vert} \, (\partial M, E_{\partial M}) \}, \quad (6.1.2)
\]

defined by

\[
(\nabla^{\psi, W}_\xi A) s = \nabla^{\psi}_\xi (As) - A(\nabla^{\psi}_\xi s), \quad s \in \Gamma(\partial M, E_{\partial M}), \quad A \in \Gamma(Y, \text{Hom}(\mathcal{K}, W)). \quad (6.1.3)
\]

One then has a connection on \( \text{Det} \, S(P) \) by setting over \( U_\sigma \)

\[
\nabla^{\psi}_{U_\sigma} \text{det} \, S(P_\sigma) = \omega^{s(p, \sigma)} \text{det} \, S(P_\sigma)
\]

where the locally defined 1-form in \( \Omega^1(U_\sigma) \) is

\[
\omega^{s(p, \sigma)} = \text{Tr} \, (S(P_\sigma)^{-1} \nabla^{\psi, W}_{\xi} S(P_\sigma)),
\]

with \( \Gamma(Y, W_\sigma) = \text{ran}(P_\sigma) \). The trace on the right-side of \([1.1.3]\) is the usual vertical trace (along the fibres, as recalled in the Appendix), by construction taken over \( \Gamma(Y, \mathcal{K}(D)) \subset \Gamma(\partial M, E_{\partial M}) \).

Notice, here, that \( A^{-1} \nabla^{\psi, W}_{\xi} A \) will not be a \( \text{trace-class} \) family of \( \psi \)dos for a general invertible vertical \( \psi \)do \( A \in \Gamma(Y, \text{Hom}(\mathcal{K}, W)) \subset \Psi^*_\text{vert} \, (\partial M, E_{\partial M}) \). That this is nevertheless the case when \( A = S(P) \), so that the right-side of \([6.1.3]\) is well defined, is immediate from \([4.0.23]\) and \([6.1.3]\).

The local 1-forms define a global connection with respect to \([4.0.24]\) by the identity in \( \Omega^1(U_{\sigma} \cap U_{\sigma'}) \)

\[
d_\xi \text{det}_F (S(P_\sigma) \circ S(P_{\sigma'})^{-1}) = \text{Tr} \, (S(P_\sigma)^{-1} \nabla^{\psi, W}_{\xi} S(P_{\sigma})) - \text{Tr} \, (S(P_{\sigma'})^{-1} \nabla^{\psi, W}_{\xi} S(P_{\sigma'})).
\]

which is a standard Fredholm determinant identity \( d_\xi \text{det}_F C = \text{Tr} \, (C^{-1} \nabla^{\psi} C) \) for a smooth family of Fredholm-determinant class operators \( y \mapsto C(y) \).
6.1.1 Curvature of $\nabla^{S(P)}$

The curvature of the connection $\nabla^{S(P)}$ on the complex line bundle $\text{Det} S(P) \to Y$ is the globally defined 2-form

$$R^{\kappa, \psi} = (\nabla^{S(P)})^2 \in \Omega^2(Y)$$

(6.1.6)
determined by

$$R^{\kappa, \psi}_{|U_\sigma} = d\omega^{S(P)} \in \Omega^2(U_\sigma).$$

(6.1.7)

**Remark.** No use is made of the interpretation of $\psi$ as a ‘Frechet bundle’. The 2-form $R^{\kappa, \psi}$ is constructed concretely as the vertical trace of a vertical $\psi$do-valued form on $M$ (cf. Appendix).

6.1.2 Why $R^{\kappa, \psi}$ is a relative eta form

The APS $\eta$-invariant of a single invertible Dirac-type operator $\partial$ over a closed manifold $N$ is

$$\eta(\partial) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-1/2} \text{Tr} (\partial e^{-t\partial^2}) \, dt = \text{Tr} (\partial|\partial|^{-s-1})_{s=0|}^{\text{mer}},$$

the superscript indicating the meromorphically continued trace evaluated at $s = 0$. Equivalently,

$$\eta(\partial) = \text{Tr} ((\Pi_0^2 - \Pi_\sigma^2)|\partial|^{-s})_{s=0|}^{\text{mer}}$$

(6.1.8)
is the zeta function quasi-trace of the involution $\Pi_0^2 - \Pi_\sigma^2$ defined by the order zero $\psi$do projections $\Pi_0^2 = \frac{1}{2}(I + \partial|\partial|^{-1})$ and $\Pi_\sigma^2 = \frac{1}{2}(I - \partial|\partial|^{-1}) = (\Pi_\sigma^2)^{\perp}$ onto the positive and negative spectral subspaces of $\partial$.

Consider $\psi$do projections $P, P'$ with $P - \Pi_\sigma$ a and $P' - \Pi_\sigma$ smoothing operators. Since $P - P'$ is smoothing the relative variant of (6.1.8) exists without regularization

$$\eta(P, P') = \text{Tr} \left( (P - P^\perp) - (P' - (P')^\perp) \right).$$

(6.1.9)

One then has $\eta(\Pi_0^2, \Pi_\sigma^2) = \eta(\partial) - \eta(\partial')$ for $\partial - \partial'$ a finite-rank $\psi$do, and the relative index formula

$$\frac{\eta(P, P')}{2} = \text{ind} (\partial_P) - \text{ind} (\partial_{P'}),$$

(6.1.10)

which is the pointwise content of (6.0.32). This is the form degree zero in the boundary Chern character form $\eta(P', P)$ whose component in $\Omega^{2k}(Y)$ is up to a constant the vertical trace

$$\eta(P, P')_{|2k} = \text{Tr} \left( (\nabla^{\psi})^{2k} - (\nabla^{\psi'})^{2k} \right).$$

In particular, $R^{\kappa, \psi} = \eta(P(D), P)_{|2}$. 

6.2 The zeta function connection on $\text{Det} D_P$

The $\zeta$-connection on $\text{Det} D_P$ is defined locally on $U_\sigma$ by

$$\nabla^{\zeta, \psi} \text{det} D_P = \omega^{\zeta, \psi} \text{det} D_P$$

(6.2.1)

for

$$\omega^{\zeta, \psi} = - \text{Tr} (\Delta_{P(D)}^{-s} D_P \nabla P^{-1})_{s=0|}^{\text{mer}} \in \Omega^1(U_\sigma),$$

(6.2.2)

where $\text{Tr} : \Gamma(Y, \Psi^{-\infty}_b(E)) = \Psi_{\text{vert}, b}^{\infty}(M, E) \to \mathcal{C}^{\infty}(Y)$ is the vertical trace (integral over the fibres, see Appendix).
Here, the notation $\text{Tr} (Q(s))_{s=0}^{\text{mer}}$ for a family of operators $Q(s)$ depending holomorphically on $s$ and of trace-class for $\Re(s) >> 0$, means the constant term around $s = 0$ (the ‘finite part’) in the Laurent expansion of the meromorphic extension $\text{Tr} (Q(s))_{s=0}^{\text{mer}}$ of the trace of $Q(s)$ from $\Re(s) >> 0$ to all of $\mathbb{C}$, assuming this is defined.

The definition of $\omega_{s, \psi}$ has particular features which make it work (and be essentially canonical choice). These are as follows.

The operator $D_{P_\nu}$ on the right-side of (6.2.2) means that

$$(\nabla^r_\xi D^{-1}_{P_\nu}) s \in \text{dom}(D_{P_\nu}), \quad s \in \Gamma(M, E^-).$$

(6.2.3)

Ensuring that (6.2.3) holds is the job of the connection $\nabla^r$, which is constructed in §6.2.1 (this issue is not present in the case of boundaryless manifolds). That is,

$$\omega^{\nabla^r, \nu} = - \text{Tr} (\Delta^s_{P(D)} D^{\nabla^r} D^{-1}_{P_\nu})|_{s=0}^{\text{mer}} \in \Omega^1(U_\sigma)$$

(6.2.4)

while the additional subscript in (6.2.2) indicates (6.2.3). $\nabla^r$ has also to be such that the local 1-forms (6.2.2) patch together to define a global $\zeta$-connection on $\text{Det} D_P$.

The regularized trace of $D_{P_\nu} \nabla^r D^{-1}_{P_\nu}$ in (6.2.2) is defined for any vertical APS $\psi$do projection $P$ using the complex power $\Delta^s_{P(D)}$ of the Calderón Laplacian $\Delta_{P(D)} = DD^*$ (cf. (6.0.3)).

This differs from the case of boundaryless manifolds which, recall, works as follows. Suppose $D$ is a smooth family of Dirac-type operators associated to a fibration $\pi : N \rightarrow Y$ of compact boundaryless manifolds. Then the determinant line bundle $\text{Det} D$ may be constructed with respect to local charts $U_s = \{y \in Y \mid D + s \text{ invertible} \}$ with $s \in \Psi^{-\infty}(N, E^+, E^-)$ a vertical smoothing operator. Over $U_s$ one has the trivialization $y \rightarrow \det(D_s + s_y) \in \text{Det}(D_s + s_y)$ and the $\zeta$-connection 1-form is $- \text{Tr} (\Delta^s_{\nu}(D + s) \nabla(D + s)^{-1})|_{s=0}^{\text{mer}}$, where $\Delta_s$ is the Laplacian of $D + s$.

What makes the patching work in this case is

$$\text{Tr} ((\Delta^s_{\nu} - \Delta^s_{\nu'})(D + s) \nabla(D + s)^{-1})|_{s=0}^{\text{mer}} = 0 \quad \text{for} \ s, s' \in \Psi^{-\infty}(N, E^+, E^-).$$

(6.2.5)

This is easily seen; for example, from the precise formulae of [17]. This might suggest that the local $\zeta$-connection form on $\text{Det} D_P$ be defined as $\text{Tr} (\Delta^s_{\nu'} D^{\nabla^r} D^{-1}_{P_\nu})|_{s=0}^{\text{mer}}$. But these forms do not patch together, because the analogue of the left-side of (6.2.3) does not vanish. This one knows from the pole structure of the meromorphic continuation of the trace to all of $\mathbb{C}$, from $[12, 13, 14, 15]$ the constant term in the Laurent expansion at zero depends on $P_\nu, P_{\nu'}$.

In contrast, the connection forms $\omega^{\nabla^r, \nu}$ do patch together (Theorem 5.2).

This carries a certain naturality, the family of vertical APS boundary problems $D_{P(D)}$ is distinguished by the fact that it is invertible (at all points $y \in Y$), and thus so is $\Delta_{P(D)}$, providing a global regularizing operator not available in the case of general family $D$ over boundaryless manifolds. In general, changing the regularizing family $Q(s)$ of elliptic $\psi$dos used to define the connection form $\text{Tr} (Q(s)(D + s) \nabla(D + s)^{-1})|_{s=0}^{\text{mer}}$ results in additional residue term classes.

### 6.2.1 A connection on $\text{Hom}(\mathcal{H}(E^-), \mathcal{H}_P(E^+))$

To define a connection on $\text{Det} D_P$ requires a connection on the bundle $\text{Hom}(\mathcal{H}(E^-), \mathcal{H}_P(E^+))$ whose sections are the subspace of vertical $\psi$dos with range in $\text{Ker}(P \circ \gamma) = \text{dom}(D_P)$

$$\Gamma (M, \text{Hom}(\mathcal{H}(E^-), \mathcal{H}_P(E^+))) := \{ \alpha \in \Psi_{\text{vert}, \beta}(M, E^-, E^+) \mid P_{\gamma} \alpha s = 0, \ s \in \Gamma(M, E^-) \}.$$  

All that that requires is a natural connection $\tilde{\nabla}^\nu$ on $\mathcal{H}_P(E^+)$, meaning a connection $\tilde{\nabla}^\nu$ on $\Gamma(Y, \mathcal{H}(E^+)) = \Gamma(M, E^+)$ which preserves $\text{Ker}(P \circ \gamma) = \text{dom}(D_P)$. That is, such that

$$P_{\gamma} \tilde{\nabla}^\nu s := P_{\gamma} \tilde{\nabla}_{\xi}^\nu s = 0 \quad \text{for} \ s \in \Gamma(M, E^+) \quad \text{with} \ P_{\gamma} s = 0.$$  

(6.2.6)
For, then, there is the induced connection (also denoted $\nabla^\rho$) on $\Hom(\mathcal{H}(E^-), \mathcal{H}_P(E^+))$

$$\nabla^\rho_\xi : \Gamma(M, \Hom(\mathcal{H}(E^-), \mathcal{H}_P(E^+))) \rightarrow \Gamma(M, \Hom(\mathcal{H}(E^-), \mathcal{H}_P(E^+))),$$

$$(\nabla^\rho_\xi A)_s : = \nabla^\rho_\xi (A s) - A(\nabla^\rho_M (s)) : = \bar{\nabla}^\rho_\xi (A s) - A(\bar{\nabla}^\rho_M s) \quad (6.2.7)$$

where $\bar{\nabla}$ is the connection $\text{(2.2.6)}$ and $\xi \in \Gamma(Y, TY)$. We then evidently have

$$P_\gamma(\nabla^\rho_\xi A)_s = 0 \quad \text{for } s \in \Gamma(M, E^-). \quad (6.2.8)$$

This is how $\nabla^\rho D_P^{-1}$ in $\text{(5.2.3)}$ is defined, and why $D\nabla^\rho D_P^{-1} = D_P \nabla^\rho D_P^{-1}$.

The task, then, is to define the connection $\nabla^\rho$ in $\text{(6.2.8)}$. The connection $\text{(2.2.6)}$, $\nabla^\rho_M$ on $\mathcal{H}(E^+)$ does not restrict to a connection on $\mathcal{H}_P(E^+)$ (except when $P$ is constant in $y \in Y$ as in the example of $\text{(6.0.8)}$, i.e. $\text{(6.2.6)}$ does not hold for $\nabla$. We define $\nabla^\rho$ by adding a correction term to $\nabla^\rho_M$ in an essentially canonical way, as follows.

First, for an APS-type vertical boundary $\psi$ do $P \in \Gamma(Y, \Psi^0_{\text{vert}, \psi}(E_{\partial M})) : = \Psi^0_{\text{vert}}(\partial M, E_{\partial M})$ we have its covariant derivative

$$\nabla^\rho_{\partial M} P \in \in \Gamma(Y, \Psi^0_{\text{vert}, \psi}(E_{\partial M})),$$

where $\nabla^\rho_{\partial M}$ is the connection $\text{(6.1.1)}$. Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a smooth function with $\phi(u) = 1$ for $0 \leq u < 1/4$ and $\phi(u) = 0$ for $u \geq 3/4$. Define

$$m_\phi : M \rightarrow \mathbb{R}$$

with support in the collar neighbourhood $\mathcal{U}$ of $\partial M$ by

$$m_\phi(x) = \begin{cases} 0, & x \in M \backslash \mathcal{U}, \\ \phi(u), & x = (u, z) \in \mathcal{U} = [0, 1) \times \partial M. \end{cases}$$

Then we define

$$\nabla^\rho : = \nabla^\rho_M + m_\phi P(\nabla^\rho_{\partial M} P)_\gamma. \quad (6.2.9)$$

Thus for $\xi \in \Gamma(Y, TY)$ and $s \in \Gamma(M, E^+)$

$$\nabla^\rho_\xi s : = \bar{\nabla}^\rho_\xi s : = \bar{\nabla}^\rho_{\xi M}s + m_\phi P(\bar{\nabla}^\rho_{\xi M} P)_\gamma s \quad (6.2.10)$$

and the second (endomorphism) term acts by

$$(m_\phi P(\bar{\nabla}^\rho_{\xi M} P)_\gamma s)_x = \begin{cases} 0, & x \in M \backslash \mathcal{U}, \\ \phi(u) P(\bar{\nabla}^\rho_{\xi M} P)(s(0, z)), & x = (u, z) \in \mathcal{U}. \end{cases} \quad (6.2.11)$$

Because of the restriction map $\gamma$ the Leibnitz property does not hold for $\nabla^\rho$ on $\Gamma(M, E^+)$ as a $C^\infty(M)$ module. It does hold, however, for $\Gamma(M, E^+)$ as a $C^\infty(Y)$ module $\text{(2.1.2)}$, which is exactly what we need; that is, for the $C^\infty(Y)$ multiplication $\text{(2.1.3)}$

$$\nabla^\rho f \cdot s = df \cdot s + f \cdot \nabla^\rho s \quad \text{for } f \in C^\infty(Y), \quad s \in \Gamma(Y, \mathcal{H}_P(E^+)). \quad (6.2.12)$$

**Proposition 6.1** $\nabla^\rho$ defines a connection on $\mathcal{H}_P(E^+)$. That is, $\text{(5.2.4)}$ holds so that

$$\nabla^\rho_\xi : \Gamma(Y, \mathcal{H}_P(E^+)) \rightarrow \Gamma(Y, \mathcal{H}_P(E^+)) \quad (6.2.13)$$

and satisfies the Leibnitz property $\text{(5.2.12)}$. 

Proof The Leibnitz property of the first term (2.2.5) of $\nabla^p$ is standard

$$\nabla^p_\xi (f \cdot s) = \nabla^p_{\xi_H} ((f \circ \pi) s) = \xi_H (f \circ \pi) \cdot s + (f \circ \pi) \nabla^p_{\xi_H} s = df(\xi) \cdot s + f \cdot \nabla^p_\xi s$$

using the Leibnitz property of $\nabla$ for the second equality and the chain rule for the third. Thus (6.2.12) is equivalent to the linearity for $f \in C^\infty(Y)$ and $s \in \Gamma(M, E^\ast)$

$$m_\phi P(\nabla^{\ast M}_\xi P) \gamma(f \cdot s) = f \cdot m_\phi P(\nabla^{\ast M}_\xi P) \gamma s,$$

and this holds because $f$ acts as a constant on each fibre $X_y$ of $M$, by definition (2.1.3). Precisely, we may assume $x = (u, z) \in U$, the expressions being zero otherwise, and then from (6.2.11)

$$m_\phi P(\nabla^{\ast M}_\xi P) \gamma(f \cdot s)(u, z) = \phi(u) P(\nabla^{\ast M}_\xi P) (f(\pi(0, z))s(0, z))$$

$$= f(\pi(0, z)) \phi(u) P(\nabla^{\ast M}_\xi P)(s(0, z))$$

$$= f(\pi(u, z)) \phi(u) P(\nabla^{\ast M}_\xi P)(s(0, z))$$

$$= (f \cdot m_\phi P(\nabla^{\ast M}_\xi P) \gamma s)(u, z)$$

which is (6.2.14). To see (6.2.6), we have applying $P \circ \gamma$ to (6.2.10)

$$P \gamma \nabla^p_\xi s = P \gamma \nabla^{\xi_H}_s + P(\nabla^{\ast M}_\xi P)(\gamma s)$$

$$= P \nabla^{\ast M}_\xi (\gamma s) + P(\nabla^{\ast M}_\xi P)(\gamma s),$$

using (2.2.8) for the second equality. From (6.1.1)

$$P(\nabla^{\ast M}_\xi P)(h) = P \nabla^{\ast M}_{\xi_H} (Ph) - P \nabla^{\ast M}_{\xi_H} h, \quad h \in \Gamma(\partial M, E_{\ast M}).$$

So with $h = \gamma s$ and the assumption of (6.2.6)

$$P(\nabla^{\ast M}_\xi P)(\gamma s) = - P \nabla^{\ast M}_{\xi_H} (\gamma s), \quad h \in \Gamma(\partial M, E_{\ast M}),$$

and hence (6.2.15) vanishes.

\[\square\]

6.2.2 Curvature of $\nabla^{\zeta, p}$

The curvature of the connection $\nabla^{\zeta, p}$ on the complex line bundle $\text{Det} D_p \to Y$ is the globally defined two form

$$F_\zeta(D_p) = (\nabla^{\zeta, p})^2 \in \Omega^2(Y)$$

(6.2.16)

determined locally by

$$F_\zeta(D_p)|_{U_\sigma} = d \omega^{\zeta, p_\sigma} \in \Omega^2(U_\sigma).$$

(6.2.17)

Theorem 6.2 The locally defined $\zeta$ 1-forms (6.2.3) define a connection on the determinant line bundle $\text{Det} D_p$ with curvature

$$F_\zeta(D_p) = F_\zeta(D_{P(D_p)}) + R^{\zeta, \omega}. $$

(6.2.18)

$F_\zeta(D_{P(D_p)})$ is canonically exact, precisely

$$\beta_\zeta := \text{Tr}(\Delta_{P(D_p)}^D) D \nabla^{P(D_p)} D_{P(D_p)}^{-1})_{z=0}^{\text{mer}} \in \Omega^1(Y)$$

(6.2.19)

is a globally defined 1-form and

$$F_\zeta(D_{P(D_p)}) = d \beta_\zeta. $$

(6.2.20)
6.3 Proof of Theorem 6.2

For the patching of the connection forms, the issue is that there are two candidates for the local connection over $U_\sigma \cap U_{\sigma'}$ defined by (6.2.1). Let $l$ be a smooth section of $\text{Det} D_p$ over $U_\sigma \cap U_{\sigma'}$. Then

$$l = f_\sigma \cdot \det D_{p_\sigma} = f_{\sigma'} \cdot \det D_{p_{\sigma'}}$$

for smooth functions $f_\sigma, f_{\sigma'} : U_\sigma \cap U_{\sigma'} \to \mathbb{C}$. The covariant derivative of $l$ is therefore

$$\nabla \cdot \cdot \cdot (f_\sigma \cdot \det D_{p_\sigma}) = d f_\sigma \cdot \det D_{p_\sigma} + f_\sigma \cdot \omega \cdot \cdot \cdot \cdot \cdot \cdot$$

and also

$$\nabla \cdot \cdot \cdot (f_{\sigma'} \cdot \det D_{p_{\sigma'}}) = d f_{\sigma'} \cdot \det D_{p_{\sigma'}} + f_{\sigma'} \cdot \omega \cdot \cdot \cdot \cdot \cdot \cdot$$

and these must coincide. From (4.0.30)

$$f_{\sigma'} = \det F \left( S(P_\sigma) \circ S(P_{\sigma'})^{-1} \right) f_\sigma \quad \text{on} \quad U_\sigma \cap U_{\sigma'}.$$  

Hence, using (6.1), the patching condition for the locally defined connection forms is

$$\omega \cdot \cdot \cdot (f_\sigma \cdot \det D_{p_\sigma}) - \omega \cdot \cdot \cdot (f_{\sigma'} \cdot \det D_{p_{\sigma'}}) = \omega \cdot \cdot \cdot (f_\sigma) - \omega \cdot \cdot \cdot (f_{\sigma'}) \quad \text{on} \quad U_\sigma \cap U_{\sigma'}.$$  

(6.3.1)

We will prove a slightly more general statement, which also captures (6.2.18). Let $P, P' \in \Psi_\text{vert}^0(\partial M, E_{\partial \mathcal{M}})$ be any two vertical $\text{vdo}$ APS projections and let $U$ be the open subset of $Y$ where both $(D_p)_y$ and $(D_{p'})_y$ are invertible. Then

$$\omega \cdot \cdot \cdot - \omega \cdot \cdot \cdot = \omega \cdot \cdot \cdot - \omega \cdot \cdot \cdot$$

on $U$.  

(6.3.2)

Here,

$$\omega \cdot \cdot \cdot = - \left( \Delta^{-s}_{P(D)} D_p \nabla \cdot \cdot \cdot D^{-1}_{p1} \right) \bigg|_{s=0}$$

and

$$\omega \cdot \cdot \cdot = \text{Tr}(S(P)^{-1} \nabla \cdot \cdot \cdot S(P)).$$

From (6.1.3), $\omega \cdot \cdot \cdot = 0$ and hence, using (6.1.7) and (6.2.17), (6.3.2) also proves (6.2.18) globally in $\Omega^2(Y)$; note that the right-hand side of (6.1.7) and (6.2.14) are independent of the choice of $\sigma$, i.e. $dw_\cdot \cdot \cdot = dw_\cdot \cdot \cdot = P_\sigma'(D_p)|_{U_\sigma \cap U_{\sigma'}}$ and likewise for $\omega_\cdot \cdot \cdot$. Clearly, establishing (6.3.2) de facto proves the identity for the perturbations of $P$ and $P'$ on each chart $U_\sigma$, and hence shows (6.2.18) globally.

To see (6.3.2), since the vertical trace defining the zeta form is taken on $\Gamma(M, E^-)$ (or, rather, $L^2(M, E^-)$) we have

$$- (\omega \cdot \cdot \cdot - \omega_\cdot \cdot \cdot) = \text{Tr} \left( \Delta^{-s}_{P(D)} \left( D_p \nabla \cdot \cdot \cdot D^{-1}_{p1} - D_{p'} \nabla_\cdot \cdot \cdot D^{-1}_{p1} \right) \right) \bigg|_{s=0}.$$  

(6.3.3)

(Not that

$$\text{Tr} \left( \Delta^{-s}_{P(D)} D_p \nabla \cdot \cdot \cdot D^{-1}_{p1} \right) - \text{Tr} \left( \Delta^{-s}_{P(D)} D_{p'} \nabla_\cdot \cdot \cdot D^{-1}_{p1} \right) = \text{Tr} \left( \Delta^{-s}_{P(D)} \left( D_p \nabla \cdot \cdot \cdot D^{-1}_{p1} - D_{p'} \nabla_\cdot \cdot \cdot D^{-1}_{p1} \right) \right)$$

for large Re($s$), and by the uniqueness of continuation this extends to all of $\mathbb{C}$.) From (3.2.15) and (6.1.1)

$$\nabla \cdot \cdot \cdot - \nabla_\cdot \cdot \cdot = m_\phi \left( P(\nabla \cdot \cdot \cdot P) - P'(\nabla \cdot \cdot \cdot P') \right)$$

$\in \Gamma(Y, \Psi_{\text{vert, v}}^\infty(E)),$

(6.3.4)

and hence using Proposition (3.2.13)

$$\nabla_\cdot \cdot \cdot D^{-1}_{p1} - \nabla_\cdot \cdot \cdot D^{-1}_{p1} \left( \nabla_\cdot \cdot \cdot - \nabla_\cdot \cdot \cdot \right) D^{-1}_{p'} + \nabla_\cdot \cdot \cdot \left( D^{-1}_{p} - D^{-1}_{p'} \right)$$

$\in \Gamma(Y, \Psi_{\text{vert, v}}^\infty(E)).$  

(6.3.5)
Hence
\[ D_p \nabla \partial^s - D_p' \nabla \partial^{s'} D_p^{-1} = D \left( \nabla \partial^s D_p^{-1} - \nabla \partial^{s'} D_p^{-1} \right) \in \Gamma(Y; \Psi_{\text{vert, } \partial}^{\infty}(E)) \]
is also a smooth family of smoothing operators (with $C^\infty$ kernel). It follows that we may swap the order of the operators inside the trace on the right-side of (6.3.3) to obtain
\[ -(\omega \circ, r - \omega \circ, r') = \text{Tr} \left( \left( D_p \nabla \partial^s D_p^{-1} - D_p' \nabla \partial^{s'} D_p^{-1} \right) \Delta\gamma^{s} \right)_{s=0} \]
\[ = \text{Tr} \left( D \left( \nabla \partial^s D_p^{-1} - \nabla \partial^{s'} D_p^{-1} \right) \Delta\gamma^{s} \right)_{s=0} \]
\[ = \text{Tr} \left( D \left( \nabla \partial^s D_p^{-1} - \nabla \partial^{s'} D_p^{-1} \right) \Delta\gamma^{s} \right)_{s=0} \]
\[ = \text{Tr} \left( D \left( \nabla \partial^s D_p^{-1} - \nabla \partial^{s'} D_p^{-1} \right) \right) \]
using that $\Delta\gamma^{s} \gamma^{s-1}$ is vertically norm continuous for $\Re(s) > -1$ and, in particular, at $s = 0$, and hence that we may take $s$ down to zero without continuation of the vertical trace.

Using (6.3.4), (6.3.5) and (3.2.12), we therefore have
\[ \omega \circ, r - \omega \circ, r' = \text{Tr} \left( D \nabla \gamma^{s} \left( K(P) \gamma D_p^{s} \right) \right) \]
\[ - \text{Tr} \left( D \gamma \left( P (\nabla \partial^m P) - P' (\nabla \partial^m P') \right) \gamma D_p^{-1} \right) \]
\[ + \text{Tr} \left( D \gamma \left( P (\nabla \partial^m P) P(D) S(P)^{-1} \gamma D_p^{-1} \right) \right) \]
using the fact that each term is a vertical smoothing operator, as in the proof of Proposition 3.2.13 for terms (I) and (III). We will deal with these terms in reverse order.

**Term (III):**
Again, in view of (6.3.4) we may permute the order of operators in the trace to obtain
\[ \text{Term (III)} = \text{Tr} \left( P \gamma D_p^{-1} \gamma \left( P (\nabla \partial^m P) P(D) S(P)^{-1} \right) \right) \]
\[ \overset{\text{(3.2.11)}}{=} \text{Tr} \left( P \gamma (1 - K(P') \gamma) \left( P (\nabla \partial^m P) P(D) S(P)^{-1} \right) \right) \]
\[ \overset{\text{(6.1.4), (6.2.10)}}{=} \text{Tr} \left( P (1 - P(D) S(P')^{-1} P') P (\nabla \partial^m P) P(D) S(P)^{-1} \right) \]
\[ = \text{Tr} \left( P(D) \left( S(P)^{-1} P - S(P')^{-1} P' \right) P (\nabla \partial^m P) P(D) \right), \]
circling the operator $P(D) S(P)^{-1} P = P(D) \circ P(D) S(P)^{-1} P$ around for the final equality.

**Term (II):**
Since $P \gamma D_p^{-1} = P \circ P \gamma (1 - P') D_p^{-1}$ is a composition of vertically smoothing and $L^2$-bounded operators we may cycle the operators in the trace to obtain
\[ \text{Term (II)} = - \text{Tr} \left( \gamma D_p^{-1} \gamma \left( P (\nabla \partial^m P) - P' (\nabla \partial^m P') \right) \right) \]
\[ \overset{\text{(6.2.11)}}{=} - \text{Tr} \left( \gamma (1 - K(P') \gamma) \left( P (\nabla \partial^m P) - P' (\nabla \partial^m P') \right) \right) \]
\[ = - \text{Tr} \left( P (\nabla \partial^m P) - P' (\nabla \partial^m P') - P(D) S(P')^{-1} P' \left( P (\nabla \partial^m P) - P' (\nabla \partial^m P') \right) \right). \]
Hence from \((3.1.9)\):
\[
\text{D} \nabla^M \left( K(P)P \gamma D_p^{-1} \right) = \text{D} \nabla^M (K) \circ P(D)S(P)^{-1}P \gamma D_p^{-1}
\]
and therefore
\[
\text{Term (I)} = \text{Tr} \left( \text{D} \nabla^M (K) \circ P(D)S(P)^{-1}P \gamma D_p^{-1} \right)
\]
\[
= \text{Tr} \left( P(D)S(P)^{-1}P \gamma D_p^{-1}D \nabla^M (K) P(D) \right)
\]
\[
= \text{Tr} \left( P(D)S(P)^{-1}P \gamma (I - K(P)P' \gamma) \nabla^M (K) P(D) \right)
\]
\[
= \text{Tr} \left( \left( P(D)S(P)^{-1}P - P(D)S(P')^{-1}P' \right) \nabla^\omega (P(D)) P(D) \right).
\]

Summing the expression for terms (I), (II) and (III),
\[
\omega^{\gamma,p} - \omega^{\gamma,p'} = \text{Tr} \left( P(D)S(P)^{-1}P(\nabla^\omega P) P(D) - P(D)S(P')^{-1}P' (\nabla^\omega P') P(D) \right)
\]
\[
+ P(D)S(P)^{-1}P(\nabla^\omega P(D)) P(D) - P(D)S(P')^{-1}P' (\nabla^\omega P(D)) P(D) \right)
\]
\[
+ \text{Tr} \left( P(\nabla^\omega P) - P' (\nabla^\omega P') \right).
\]

From
\[
S(P)^{-1} \nabla^{\gamma,\omega} S(P) = P(D)S(P)^{-1}P(\nabla^\omega P) P(D) + P(D)S(P)^{-1}P(\nabla^\omega P(D)) P(D)
\]
we are therefore left with
\[
\omega^{\gamma,p} - \omega^{\gamma,p'} = \omega^{s(p)} - \omega^{s(p')} + \text{Tr} \left( P(\nabla^\omega P) - P' (\nabla^\omega P') \right).
\]

Since \(P \in \Psi^0_\text{vert}(\partial M, E_{\omega M})\) is an indempotent we have
\[
\nabla^\omega P = \nabla^\omega (P^2) = P \nabla^\omega P + (\nabla^\omega P) \circ P
\]
and hence composing with \(P\) on the left that
\[
P(\nabla^\omega P) \circ P = 0.
\]
Hence
\[
\text{Tr} \left( P(\nabla^\omega P) - P' (\nabla^\omega P) \right) = \text{Tr} \left( P(\nabla^\omega P) \circ P - P' (\nabla^\omega P') \circ (P')^\perp \right)
\]
with $P^\perp = 1 - P$. Writing the operator inside the trace as
\[ P\nabla^{\alpha\beta} P \circ (P^\perp - (P')^\perp) + (P\nabla^{\alpha\beta} P - P'\nabla^{\alpha\beta} P') \circ (P')^\perp, \]
the bracketed vertical $\psi$dos are smoothing and we may cycle the operators through the trace leaving
\[ \text{Tr} \left( P(\nabla^{\alpha\beta} P) - P'(\nabla^{\alpha\beta} P') \right) = \text{Tr} \left( P^\perp P\nabla^{\alpha\beta} P \circ P^\perp - (P')^\perp P'\nabla^{\alpha\beta} P' \circ (P')^\perp \right) = 0. \qedhere \]

\section*{Appendix}

\section{Vertical Pseudodifferential Operators}

\subsection{Families of $\psi$dos on a closed manifold}

A smooth family of $\psi$dos of constant order $\mu$ associated to a fibration $\pi : N \rightarrow Y$ of compact boundaryless manifolds, with $\dim(X) = n$, with vector bundle $E \rightarrow N$ means a classical $\psi$do
\[ A : \Gamma(N, E^+) \rightarrow \Gamma(N, E^-) \]
with Schwartz kernel $k_A \in D'(N \times \pi^*, N, E \boxtimes E)$ a vertical distribution, where the fibre product $N \times \pi N$ consists of pairs $(x, x') \in N \times N$ which lie in the same fibre, i.e. $\pi(x) = \pi(x')$, such that in any local trivialization $k_A$ is an oscillatory integral with vertical symbol $a \in S^\nu_{\text{vert}}(N/Y)$ of order $\nu$. Here, $\xi$ is restricted to the vertical momentum space, along the fibre. We refer to $A$ as a vertical $\psi$do associated to the fibration and denote this subalgebra of $\psi$dos on $N$ by
\[ \Gamma(Y, \Psi^\nu(E^+, E^-)) = \Psi^\nu_{\text{vert}}(N, E^+, E^-). \]

Thus in any local trivialization $N|_{U_Y} \cong U_Y \times X_y$ over an open subset $U_Y \subset Y$ with $y \in U_Y$, and a trivialization $E \cong U_Y \times V_y \times \mathbb{R}^N$ of $E$ with $V_y$ an open subset of $X_y$, a vertical amplitude of constant (in $y$) order $\nu$ is an element
\[ a = a(y, x, x', \xi) \in \Gamma \left( U_Y \times (V_y \times V_{y'}) \times \mathbb{R}^n \setminus \{0\}, \text{End}(\mathbb{R}^N) \right) \quad (A.1.1) \]
satisfying the estimate on compact subsets $K \subset N$
\[ |\partial_x^\alpha \partial_{x'}^\beta \partial_y^\gamma \partial_\xi^\delta a| < C_{\alpha,\gamma,\beta,\delta,K} (1 + |\xi|^\nu-|\beta|). \quad (A.1.2) \]

We denote this as $a \in S^\nu_{\text{vert}}(N/Y)$. Here, $\xi$ may be identified with an element of the vertical (or fibre) cotangent space $T^*_y(N/Y)$. The kernel of $A$ is then locally written on $U_Y \times V_y$ as the distribution with singular support along the diagonal
\[ k_A(y, x, x') = \int_{\mathbb{R}^n} e^{i(x-x').\xi} a(y, x, x', \xi) \, d\xi. \]

If $A$ has order $\nu < -n$ this integral is convergent and with respect to a vertical volume form $d_{N/Y}$ the trace $\text{Tr} A$ is the smooth function on $Y$
\[ (\text{Tr} A)(y) = \int_{N/Y} \text{tr} (k_A(y, x, x)) \, d_{N/Y} x \in C^\infty(Y), \quad \nu < -n. \]

If $w \in S^-_{\text{vert}}(N/Y) = \bigcap_{\nu} S^\nu_{\text{vert}}(N/Y)$ then the kernel is an element
\[ k_w \in \Gamma(N \times \pi N, E^* \boxtimes E) \]
and defines a vertical smoothing operator (smooth family of smoothing operators)

\[ W \in \Gamma(Y, \Psi^{-\infty}(E)). \]

Any vertical \( \psi \)do of order \( \nu \) may be written in the form \( A = \text{OP}(a) + W \) with \( a = a(y, x, \xi) \) a vertical symbol and \( W \) a vertical smoothing operator. Assuming this representation, we will consider here only classical vertical \( \psi \)dos, meaning that the symbol has an asymptotic expansion \( a \sim \sum_{j \geq 0} a_j \) with \( a_j \) positively homogeneous in \( \xi \) of degree \( \nu - j \). The leading symbol \( a_0 \) has an invariant realization as a smooth section \( a_0 \in \Gamma(T^*(N/Y), \varphi^*(\text{End}(E))) \),

where \( \varphi : T^*(N/Y) \rightarrow N \). If \( a_0 \) is an invertible bundle map then \( A \in \Gamma(Y, \Psi(E)) \) is said to be an elliptic family. If there exists \( \theta \) such that \( a_0 - \lambda I \) is invertible for each \( \lambda \in R_\theta = \{re^{i\theta} \mid r > 0\} \), where \( I \) is the identity bundle operator, then \( A \) is elliptic with principal angle \( \theta \). In the latter case one has the resolvent family

\[ (A - \lambda)^{-1} \in \Gamma(Y, \Psi^{-\nu}(E)) \quad (A.1.3) \]

and the complex powers

\[ A_\theta^z := \frac{i}{2\pi} \int_C \lambda_\theta^z (A - \lambda)^{-1} d\lambda \in \Gamma(Y, \Psi^{z\nu}(E)), \quad (A.1.4) \]

where \( C \) is a contour running in along \( R_\theta \) from infinity to a small circle around the origin, clockwise around the circle, then back out to infinity along \( R_\theta \), as accounted for in detail in [22]. A principal angle, and hence the complex powers, can only exist if the pointwise index is zero.

For example, if \( p \in \mathcal{S}_{\text{vert}}(N/Y) \) is a polynomial of order \( m \in \mathbb{N} \) in \( \xi \) and elliptic, then the corresponding vertical \( \psi \)do \( D \in \Gamma(Y, \Psi^m(E^+, E^-)) \) is a smooth family of elliptic differential operators of order \( m \). Specifically, this is the case for a geometric fibration of Riemannian spin manifolds [2,2] with associated smooth family of twisted Dirac operators [3,3]. (The space \( \Gamma(Y, \Psi(E^+, E^-)) \) of vertical \( \psi \)dos between different bundles \( E^\pm \) is defined by a trivial elaboration of the above.)

### A.2 Families of pseudodifferential boundary operators

Let \( \pi : M \rightarrow Y \) be a smooth fibration of compact manifolds with boundary with vector bundle \( E \rightarrow M \) and let

\[ \tilde{\pi} : \tilde{M} \rightarrow Y \]

be a smooth fibration of compact boundaryless manifolds with vector bundle \( \tilde{E} \rightarrow M \) such that

\[ M \subset \tilde{M}, \quad \tilde{E}|_M = E, \quad \text{and} \quad \tilde{\pi}|_M = \pi. \]

We consider the following vertical families of pseudodifferential boundary operators (vertical \( \psi \)dos) as defined in the single operator case by Grubb [3], elaborating the algebra of Boutet de Monvel. First, one has the truncated, or restricted, \( \psi \)dos

\[ A_+: \Gamma(M, E) \rightarrow \Gamma(M, E), \quad A_+ = r^+Ae^+, \quad (A.2.1) \]

where

\[ A \in \Psi_{\text{vert}}(\tilde{M}, \tilde{E}) \]

is a vertical \( \psi \)do associated to the fibration of closed manifolds, and

\[ r^+: \Gamma(\tilde{M}, \tilde{E}) \rightarrow \Gamma(M, E), \quad e^+: \Gamma(M \setminus \partial M, E) \rightarrow \Gamma(\tilde{M}, \tilde{E}) \]
where

$$\frac{\partial^j}{\partial \xi^j} a_j(0, w, -\xi_n, 0) = (-1)^{\nu-j-|\alpha|} |\partial^j_x \partial^{\alpha}_\xi a_j(0, w, \xi_n, 0)| \quad \text{for } |\xi_n| \geq 1,$$

where $\xi = (\xi_n, \xi_w)$ relative to (2.2.7).

More generally, one considers

$$A_+ + G : \Gamma(M, E) \longrightarrow \Gamma(M, E)$$

with $G$ a vertical singular Green’s operator (vertical sgo), which we return to in a moment.

A vertical trace operator of order $\mu \in \mathbb{R}$ and class $r \in \mathbb{N}$ is an operator from interior to boundary sections of the form

$$T : \Gamma(M, E) \longrightarrow \Gamma(\partial M, E_{ol})$$

$$T = \sum_{0 \leq j < r} S_j \gamma_j + T'$$

where the $S_j \in \Psi_{vert}(\partial M, E_{ol})$ are vertical psido on the boundary fibration of closed manifolds, as in §4, while $\gamma_j s(x_n, w) = \partial_x x_n s(0, w)$ are the restriction maps to the boundary. The additional term is an operator of the form $T' = \gamma A_e$ for some restricted psido (A.2.1).

A vertical Poisson operator of order $\mu \in \mathbb{R}$ here will be an operator from boundary to interior sections of the form

$$K : \Gamma(\partial M, E_{ol}) \longrightarrow \Gamma(M, E)$$

$$K = s^* B \gamma^* C, \quad \gamma : \Gamma(\tilde{M}, \tilde{E}) \rightarrow \Gamma(\partial M, E_{ol})$$

where $B \in \Psi_{vert}^{r-1}(\tilde{M}, \tilde{E})$ is a family of psido in the sense of §4 while $C \in \Psi_{vert}^m(\partial M, E_{ol})$ is a vertical differential operator on the boundary fibration of order $m$; however, $m = 0$ in the following. Note, the restriction map $\gamma$ is here coming from $\tilde{M}$, rather than $M$ (same notation).

To make the composition rules work one includes vertical sgo operators in (A.2.2) of order $\nu$ and class $r \in \mathbb{N}$, these have the form

$$G = \sum_{0 \leq j < r} K_j \gamma_j + G'$$

where $K_j$ is a vertical Poisson operator of order $\nu - j$, and $G'$ is defined in local coordinates near $\partial M$ by an oscillatory integral on a sgo symbol $g$ satisfying standard estimates in $\xi$ [10].

As with closed manifolds if $A + G$ in (A.2.2) has order $\nu < -n$ and we assume the order of $C$ in (A.2.4) is $m = 0$ then the ‘distribution kernel’ is continuous and the trace $\text{Tr} A$ is the smooth function, or differential form for de Rham valued symbols, on $Y$

$$\text{Tr} (A + G)(y) = \int_{M/Y} k_{A+G}(y, x, x) d_{M/Y} x \in \mathcal{A}(Y), \quad (\text{order}(A + G) < < 0).$$

For each of the above classes of psido one considers the subclass of operators defined by polyhomogeneous symbols, appropriately formulated [10]. We denote the resulting algebra by

$$\Gamma(Y, \Psi_\beta(E)) = \Psi_{vert, b}(M, E).$$

When the kernel is an element

$$k_{A+G} \in \Gamma(M \times_{\pi} M, E^* \boxtimes E)$$

then the operator defines a vertical smoothing operator (smooth family of smoothing operators)

$$A + G \in \Gamma(Y, \Psi^{-\infty}_s(E)) = \Psi^{-\infty}_{vert, b}(M, E).$$

We refer to [11] and references therein for a precise account of the pseudodifferential boundary operator calculus, which extends to the case of vertical operators in a similar way to the case for compact boundaryless manifolds [22].
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