Direct integral method, complete discrimination system for polynomial and applications to classifications of all single traveling wave solutions to nonlinear differential equations: a survey

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Abstract

Complete discrimination system for polynomial and direct integral method were discussed systematically. In particularly, we pointed out some mistaken viewpoints. Combining with complete discrimination system for polynomial, direct integral method was developed to become a powerful method and was applied to a lot of nonlinear mathematical physics equations. All single traveling wave solutions to these equations can be obtained. As examples, we gave all traveling wave solutions to some equations such as mKdV equation, Sine-Gordon equation, Double Sine-Gordon equation, triple Sine-Gordon equation, Fujimoto-Watanabe equation, coupled Harry-Dym equation and coupled KdV equation and so on. At the end, we give the partial answers to Fan's a problem.

Keywords: complete discrimination system for polynomial, direct integral method, traveling wave solution, nonlinear differential equation

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1 Introduction

In order to solve the traveling wave solutions to some nonlinear evolution equations, sometimes these equations are reduced to an ordinary differential equations such as

\[ u'(
\xi
) = G(u, \theta_1, \cdots, \theta_m), \quad (1) \]
where $\theta_1, \ldots, \theta_m$ are parameters, its solutions can be wrote by integral form

$$\xi - \xi_0 = \int \frac{du}{G(u, \theta_1, \ldots, \theta_m)}.$$  

(2)

Thus what we need is only to solve the integral (2). According to different parameters, we will give different solution to the integral. This is so called direct integral method. Although direct integral method is a routine method, but to decide the parameter's scope is still rather difficult. Thus the most important steps are to decide the parameter's scopes and the corresponding integral solutions. A new mathematical tool named complete discrimination system for polynomial([1-5]) is applied to this problem so that the parameter's scopes is solved.

It is just because of combining with complete discrimination system that direct integral method becomes a powerful and efficient method. There are a lot of nonlinear mathematical physics equations such as KdV equation, Hirota equation, breaking soliton equation, Coupled mKdV equation, Chen-lee-Liu equation, Kundu equation, cubic Schrödinger equation, Sine-Gordon equation, double Sine-Gordon equation, Sinh-Gordon equation, double Sinh-Gordon equation, triple Sinh-Gordon equation, Combine KdV-mKdV equation and nonlinear Klein-Gordon equation and so on, their all traveling wave solutions can be obtained by using direct integral method. These complete results can't be obtained by other any indirect methods (see [6-19]) such as trial equation method([19]), Jacobi elliptic function expanse method ([12]), sub-equation method([15,16] and homogenous balance method and so on.

Although direct integral method is the most basic method, but there are two obvious mathematical facts about the integral(2) to be ignored. One fact is that different variable transformation don't offer to a different solution, different variable transformation only give different representation of solution. Another fact is that it is just different parameter's scope to offer to different solution. Some authors didn't notice the fact one([20-23]), they still think that a new variable transformation can bring new solutions. We must point out that this is a mistaken viewpoint. In fact, the elementary calculus tell us that a defined function has infinite primitive functions which are different each other by a constant, and that a new variable transformation only offer to a new representation of the primitive function.

In the present paper, using complete discrimination system for polynomial and direct integral method, we give the classifications of all single traveling wave solutions to a lot of nonlinear differential equations such as mKdV equation and Sine-Gordon equation, Fujimoto-Watanabe equation, coupled Harry-Dym equation and coupled KdV equation and so on. An the end, we give the partial answer to Fan's a problem.

This paper is organized as follows. In section 2, we deal with some equations whose reduced ODE's take the form of $(u')^2 = a_3 u^3 + a_2 u^2 + a_1 u + a_0$, and obtain the classifications of all single traveling wave solutions to those equations.
In section 3, we deal with some equations whose reduced ODE’s take the form of \((u')^2 = a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0\), and obtain the classifications of all single traveling wave solutions to those equations. In section 4, we give some applications of the complete discrimination system for the fifth order polynomial. In section 5, we deal with two equations with rational form of reduced equations and obtain classifications of their all single traveling wave solutions. In section 6, we use a one-parameter’s thick obtain the classifications of all single traveling wave solutions to some equations such as coupled KdV equation and so on. In section 7, we discuss Fan’s two problems, and give an answer to the first problem.

2 Applications of the complete discrimination system for the third order polynomial

There are a lot of nonlinear evolution equations such as KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0; \]  \hspace{1cm} (3)

BBM equation

\[ u_t + u_x + uu_x + \alpha u_{xxt} = 0; \]  \hspace{1cm} (4)

Klein-Gordon equation

\[ u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0; \]  \hspace{1cm} (5)

Landau-Ginburg-Higgs equation

\[ u_{tt} - u_{xx} - m^2u + g^2u^3 = 0; \]  \hspace{1cm} (6)

Sine-Gordon equation

\[ u_{xt} = a \sin u; \]  \hspace{1cm} (7)

Dodd-Bullough-Mikhailov equation

\[ u_{xt} + p \exp(u) + q \exp(-2u) = 0; \]  \hspace{1cm} (8)

Zhiber-Shabaut equation

\[ u_{xt} + p \exp(u) + q \exp(-u) + r \exp(-2u) = 0; \]  \hspace{1cm} (9)

Sinh-Gordon equation

\[ u_{xt} = a \sinh u; \]  \hspace{1cm} (10)

Joseph-Egri equation Sine-Gordon equion

\[ u_t + u_x + uu_x + \beta u_{xtt} = 0; \]  \hspace{1cm} (11)

Kundu equation

\[ uu_t - u_{xx} - 2i(2\alpha - 1)|u|^2u_x* + \alpha(4\alpha - 1)|u|^4u = 0; \]  \hspace{1cm} (12)
derivation Shrodinger equation
\[ u_t = i u_{xx} + i(c_3|u|^2 + c_5|u|^4)u + s_2(|u|^2 u)_x = 0; \quad (13) \]

Ablowitz equation
\[ i u_t = u_{xx} - 4i(|u|^2 u_x \ast + 8|u|^4 u); \quad (14) \]

and so on. They all can be reduced to the following third order integrable ODE
\[ (u')^2 = a_3 u^3 + a_2 u^2 + a_1 u + a_0, \quad (15) \]

So if we give the classification of all solutions of the Eq.(15), then all single traveling wave solutions to those equations (3)-(14) can be classified. According to the Ref.[19], we list those results for convenience in the following. Let
\[ w = (a_3)^{\frac{1}{2}} u, \quad d_2 = a_2 (a_3)^{-\frac{2}{3}}, \quad d_1 = a_1 (a_3)^{-\frac{4}{3}}, \quad d_0 = a_0. \quad (16) \]

Then Eq.(219) becomes
\[ \pm (a_3)^{\frac{1}{2}} (\xi - \xi_0) = \int \frac{1}{\sqrt{w^3 + d_2 w^2 + d_1 w + d_0}} \, dw. \quad (17) \]

Denote
\[ F(w) = w^3 + d_2 w^2 + d_1 w + d_0. \quad (18) \]

\[ \Delta = -27 \left( \frac{2a_3^3}{27} + d_0 - \frac{d_1 d_2}{3} \right)^2 - 4 \left( d_1 - \frac{d_2^2}{3} \right)^3, \quad (19) \]
\[ D_1 = d_1 - \frac{d_2^2}{3}, \quad (20) \]

where \( \Delta \) and \( D_1 \) make up a complete discrimination system for \( F(w) \). There are the following four cases to be discussed(19):

Case 1: \( \Delta = 0, D_1 < 0 \), then we have \( F(w) = (w - \alpha)^2 (w - \beta), \alpha \neq \beta \). If \( w > \beta \), the solutions are as follows:
\[ u = (a_3)^{-\frac{1}{2}} [(a - \beta) \tan^2 \left( \frac{\sqrt{-\alpha - \beta}}{2} (a_3)^{\frac{1}{4}} (\xi - \xi_0) \right) + \beta], \quad \alpha > \beta; \quad (21) \]
\[ u = (a_3)^{-\frac{1}{2}} [(a - \beta) \coth^2 \left( \frac{\sqrt{-\alpha - \beta}}{2} (a_3)^{\frac{1}{4}} (\xi - \xi_0) \right) + \beta], \quad \alpha > \beta; \quad (22) \]
\[ u = (a_3)^{-\frac{1}{2}} [(-a + \beta) \sec^2 \left( \frac{\sqrt{-\alpha + \beta}}{2} (a_3)^{\frac{1}{4}} (\xi - \xi_0) \right) + \beta], \quad \alpha < \beta. \quad (23) \]
Case 2: $\Delta = 0$, $D_1 = 0$, then we have $F(w) = (w - \alpha)^3$, the solution is as follows:
\[ u = 4(a_3)^{-\frac{2}{3}}(\xi - \xi_0)^{-2} + \alpha. \] (24)

Case 3: $\Delta > 0$, $D_1 < 0$, then $F(w) = (w - \alpha)(w - \beta)(w - \gamma)$, we suppose that $\alpha < \beta < \gamma$. When $\alpha < w < \beta$, we have
\[ u = (a_3)^{-\frac{1}{4}}[\alpha + (\beta - \alpha) \text{sn}^2(\frac{\sqrt{\gamma - \alpha}}{2}(a_3)^{\frac{1}{3}}(\xi - \xi_0)), m)]. \] (25)

When $w > \gamma$, we have
\[ u = (a_3)^{-\frac{1}{4}}[\gamma - \beta \text{sn}^2(\frac{\sqrt{\gamma - \alpha}}{2}(a_3)^{\frac{1}{3}}(\xi - \xi_0)), m]) \text{cn}^2(\frac{\sqrt{\gamma - \alpha}}{2}(a_3)^{\frac{1}{3}}(\xi - \xi_0)), m), \] (26)
where $m^2 = \frac{\beta - \alpha}{\gamma - \alpha}$.

Case 4: $\Delta < 0$, then we have $F(w) = (w - \alpha)(w^2 + pw + q)$, $p^2 - 4q < 0$. We have
\[ u = (a_3)^{-\frac{1}{4}}[\alpha - \sqrt{\alpha^2 + p\alpha + q} + \frac{2\sqrt{\alpha^2 + p\alpha + q}}{1 + \text{cn}((\alpha^2 + p\alpha + q)^{\frac{1}{3}}(\xi - \xi_0)), m)], \] (27)
where $m^2 = \frac{1}{2}(1 - \frac{\alpha + p}{\sqrt{\alpha^2 + p\alpha + q}})$.

We take only three equations as examples to illustrate this method, other equations can be dealt with similarly.

**Example 1.** Classification of all single traveling wave solutions to KdV equation

Although KdV equation is a classical equation, its single traveling wave solutions had been discussed many years ago. But its classification hasn’t been reported, some author also try to give its new single traveling wave solutions, so we take it as an example and give its all single traveling wave solutions. In fact under the traveling wave transformation, KdV equation can be reduced to the following ODE:
\[ (u')^2 = -\frac{2}{k^2}u^3 - \frac{\omega}{k^4}u^2 + c_1 u + c_0. \] (28)

Thus it is easy to give the classification of all solutions to ODE 28 according to the former results. For briefly we omit them.
Example 2. Classification of all single traveling wave solutions to Sine-Gordon equation.

Sine-Gordon equation reads

\[ u_{xt} = a \sin u, \tag{29} \]

which widely applied in physics and engineering. Its so-called new traveling wave solutions have been obtained in different functional forms by different methods([25-30]). Among these some method([30]) is very complex but not essence and natural. But we must point out that it is very easy to give all traveling wave solutions to Sine-Gordon equation. In the following, we only need elementary integral method to do this thing. Instituting traveling wave transformation \( u = u(\xi), \xi = kx + \omega t \) into Eq. (29) and integrating once yield

\[ \pm(\xi - \xi_0) = \int \frac{du}{\sqrt{-\frac{a}{k\omega}(\cos u - c)}}, \tag{30} \]

where \( c \) is an arbitrary constant. Take variable transformation

\[ u = \arccos w, \tag{31} \]

then Eq. (30) becomes

\[ \pm(\xi - \xi_0) = \int \frac{dw}{\sqrt{\frac{a}{k\omega}(w - 1)(w + 1)(w - c)}}. \tag{32} \]

Firstly we assume \( \frac{a}{k\omega} > 0 \). There are the following three cases to be discussed:

Case 1: \( c = 1 \), then we have

\[ u = \arcsin\{2 \tanh^2\left(\frac{1}{2} \frac{\sqrt{2a}}{k\omega}(\xi - \xi_0)\right) - 1\}; \tag{33} \]

\[ u = \arcsin\{2 \coth^2\left(\frac{1}{2} \frac{2a}{k\omega}(\xi - \xi_0)\right) - 1\}. \tag{34} \]

Case 2: \( c = -1 \), then we have

\[ u = \arcsin\left(-2 \sec^2\left(\frac{1}{2} \frac{2a}{k\omega}(\xi - \xi_0)\right) + 1\right). \tag{35} \]

Case 3: \( c \neq \pm 1 \), we reorder 1, -1, \( c \) and denote them by \( \alpha < \beta < \gamma \). In fact there are three case such as \( 1 > c > -1, c > 1 \), and \( -1 > c \). When \( \alpha < w < \beta \), we have

\[ u = \arcsin\{\alpha + (\beta - \alpha) \text{sn}^2\left(\frac{\sqrt{\gamma - \alpha}}{2} \frac{2a}{k\omega}(\xi - \xi_0), m\right)\}. \tag{36} \]
When $w > \gamma$, we have
\begin{align*}
  u &= \arcsin\left\{ \frac{\gamma - \beta \sin^2\left( \frac{\sqrt{\gamma - \alpha}}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0) \right), m)}{\operatorname{cn}^2\left( \frac{\sqrt{\gamma - \alpha}}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0), m \right)} \right\}, \tag{37}
\end{align*}
where $m^2 = \frac{\beta - \alpha}{\gamma - \alpha}$.

If $\frac{w}{k\omega} < 0$, we take $w = -v$, then Eq.\(32\) becomes
\begin{align*}
  \pm (\xi - \xi_0) &= \int \frac{dv}{\sqrt{-\frac{2a}{k\omega} (v - 1)(v + 1)(v + c)}} \tag{38}
\end{align*}

We have similar three cases:

Case 1: $c = 1$, then we have
\begin{align*}
  u &= -\arcsin\left\{ 2 \tanh^2\left( \frac{1}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0) \right) - 1 \right\}; \tag{39}
\end{align*}
\begin{align*}
  u &= -\arcsin\left\{ 2 \coth^2\left( \frac{1}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0) \right) - 1 \right\}. \tag{40}
\end{align*}

Case 2: $c = -1$, then we have
\begin{align*}
  u &= -\arcsin\left\{ -2 \sec^2\left( \frac{1}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0) \right) + 1 \right\}. \tag{41}
\end{align*}

Case 3: $c \neq \pm 1$, we reorder $1, -1, c$ and denote them by $\alpha < \beta < \gamma$. When $\alpha < w < \beta$, we have
\begin{align*}
  u &= -\arcsin\left\{ \alpha + (\beta - \alpha) \sin^2\left( \frac{\sqrt{\gamma - \alpha}}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0) \right), m \right\}. \tag{42}
\end{align*}

When $w > \gamma$, we have
\begin{align*}
  u &= -\arcsin\left\{ \frac{\gamma - \beta \sin^2\left( \frac{\sqrt{\gamma - \alpha}}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0), m \right)}{\operatorname{cn}^2\left( \frac{\sqrt{\gamma - \alpha}}{2} \sqrt{-\frac{2a}{k\omega}} (\xi - \xi_0), m \right)} \right\}, \tag{43}
\end{align*}
where $m^2 = \frac{\beta - \alpha}{\gamma - \alpha}$.

**Example 3.** Classification of all single envelope traveling wave solutions to Hirota equation

Hirota equation reads
\begin{align*}
  iu_t + u_{xx} + 2|u|^2 u + i\alpha u_{xxx} + 6i\alpha |u|^2 u_x = 0, \tag{44}
\end{align*}
when $\alpha = 0$, it becomes Schrodinger equation. Take envelope travelling wave transformation $u = u(\xi) \exp(\eta)$, $\xi = x + \omega t$, $\eta = px + qt$, then separating the real part and the imagine part of Eq.\(44\) yields

\[(\omega + 2p - 3\alpha p^2)u' + 6\alpha u^2 u' + \alpha u''' = 0, \quad (45)\]

\[(\alpha p^3 - q - p^2)u + 2(1 - 3\alpha p)u^2 + (1 - 3\alpha p)u'' = 0. \quad (46)\]

In order to make the Eq.\(45\) and the Eq.\(46\) consistence, the parameters must satisfy the following condition:

\[\frac{\omega + 2p - 3\alpha p^2}{\alpha p^3 - q - p^2} = \frac{\alpha}{1 - 3\alpha p}. \quad (47)\]

Under the condition \(47\), the above two equations become one equation

\[(u')^2 = -u^4 + \frac{\alpha p^3 - q - p^2}{3\alpha p - 1} u^2 + c \quad (48)\]

where $c$ is an arbitrary constant. The corresponding integral form is

\[\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{-u^4 + \frac{\alpha p^3 - q - p^2}{3\alpha p - 1} u^2 + c}}. \quad (49)\]

in order to solve the above integral, we take the change of variable $w = u^2$, then the integral becomes

\[\pm 2(\xi - \xi_0) = \int \frac{dw}{\sqrt{w(-w^2 + \frac{\alpha p^3 - q - p^2}{3\alpha p - 1} w + c)}}. \quad (50)\]

Therefore according to the classification of all solutions of the Eq.\(15\), it is easy to give classification of all single envelope traveling wave solutions of the Hirota equation. We omit concrete expressions for briefly.

3 Applications of the complete discrimination system for the fourth order polynomial

We first write the complete discrimination system for the fourth order polynomial $f(w) = w^4 + pw^2 + qw + r$ as follows\(\text{(1)}\):

\[D_1 = 4, D_2 = -p, D_3 = 8rp - 2p^3 - 9q^2, \quad D_4 = 4p^4r - p^3q^2 + 36prq^2 - 32r^2p^2 - \frac{27}{4}q^4 + 64r^3, \quad F_2 = 9p^2 - 32pr. \quad (51)\]
Then we consider the following ODE

\[(w'(\xi))^2 = \epsilon(w^4 + pw^2 + qw + r), \quad (52)\]

where \(\epsilon = \pm 1\). Rewrite Eq. (52) by integral form as follows:

\[
\pm(\xi - \xi_0) = \int \frac{dw}{\sqrt{\epsilon(w^4 + pw^2 + qw + r)}}, \quad (53)
\]

Although Eq.(52) has been studied extensively, some solutions were obtained under some special parameter’s conditions. But we will give in the following the classification of its all solutions. According to the complete discrimination system for polynomial \(F(w) = w^4 + pw^2 + qw + r\), there are nine cases to be discussed\[3\]), we list those results for convenience in the following.

Case 1: \(D_4 = 0, D_3 = 0, D_2 < 0\). Then we have

\[F(w) = ((w - l_1)^2 + s_1^2)^2, \quad (54)\]

where \(l_1, s_1\) are real numbers, \(s_1 > 0\). When \(\epsilon = 1\), we have

\[w = s_1 \tan[s_1(\xi - \xi_0)] + l_1. \quad (55)\]

Case 2: \(D_4 = 0, D_3 = 0, D_2 = 0\). Then we have

\[F(w) = w^4. \quad (56)\]

When \(\epsilon = 1\), we have

\[w = -\frac{1}{\xi - \xi_0}. \quad (57)\]

Case 3: \(D_4 = 0, D_3 = 0, D_2 > 0, E_2 = 0\). Then we have

\[F(w) = (w - \alpha)^2(w - \beta)^2, \quad (58)\]

where \(\alpha, \beta\) are real numbers, \(\alpha > \beta\). If \(\epsilon = 1\), when \(w > \alpha\) or \(w < \beta\), we have

\[w = \frac{\beta - \alpha}{2}[\coth \frac{\alpha - \beta}{2}(\xi - \xi_0) - 1] + \beta. \quad (59)\]

When \(\alpha > w > \beta\), we have

\[w = \frac{\beta - \alpha}{2}[\tanh \frac{\alpha - \beta}{2}(\xi - \xi_0) - 1] + \beta. \quad (60)\]

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Case 4: $D_4 = 0, D_3 > 0, D_2 > 0$. Then we have

$$F(w) = (w - \alpha)^2(w - \beta)(w - \gamma),$$  \hspace{1cm} (61)

where $\alpha, \beta, \gamma$ are real numbers, and $\beta > \gamma$. If $\epsilon = 1$, when $\alpha > \beta$ and $w > \beta$, or when $\alpha < \gamma$ and $w < \gamma$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\alpha - \beta)(\alpha - \gamma)}} \ln \left[ \frac{\sqrt{(w - \beta)(\alpha - \gamma)} - \sqrt{(\alpha - \beta)(w - \gamma)}}{w - \alpha} \right].$$  \hspace{1cm} (62)

When $\alpha > \beta$ and $w < \gamma$, or when $\alpha < \gamma$ and $w < \beta$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\alpha - \beta)(\alpha - \gamma)}} \ln \left[ \frac{\sqrt{(w - \gamma - \alpha)(\alpha - \gamma)} - \sqrt{(\beta - \alpha)(w - \gamma)}}{w - \alpha} \right].$$  \hspace{1cm} (63)

When $\beta > \alpha > \gamma$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\beta - \alpha)(\alpha - \gamma)}} \arcsin \left( \frac{(w - \beta)(\alpha - \gamma) + (\alpha - \beta)(w - \gamma)}{|(w - \alpha)(\beta - \gamma)|} \right).$$  \hspace{1cm} (64)

If $\epsilon = -1$, when $\alpha > \beta$ and $w > \beta$, or when $\alpha < \gamma$ and $w < \gamma$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\alpha - \beta)(\alpha - \gamma)}} \ln \left[ \frac{\sqrt{(w - \beta)(\alpha - \gamma)} - \sqrt{(\alpha - \beta)(w - \gamma)}}{w - \alpha} \right].$$  \hspace{1cm} (65)

When $\alpha > \beta$ and $w < \gamma$, or when $\alpha < \gamma$ and $w < \beta$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\alpha - \beta)(\alpha - \gamma)}} \ln \left[ \frac{\sqrt{(w - \gamma - \alpha)(\alpha - \gamma)} - \sqrt{(\beta - \alpha)(w - \gamma)}}{w - \alpha} \right].$$  \hspace{1cm} (66)

When $\beta > \alpha > \gamma$, we have

$$\pm (\xi - \xi_0) = \frac{1}{\sqrt{(\beta - \alpha)(\alpha - \gamma)}} \arcsin \left( \frac{(w - \beta)(\alpha - \gamma) + (\alpha - \beta)(w - \gamma)}{|(w - \alpha)(\beta - \gamma)|} \right).$$  \hspace{1cm} (67)

Case 5: $D_4 = 0, D_3 = 0, D_2 > 0, E_2 = 0$. Then we have

$$F(w) = (w - \alpha)^3(w - \beta),$$  \hspace{1cm} (68)

where $\alpha, \beta$ are real numbers. If $\epsilon = 1$, when $w > \alpha, w > \beta$, or $w < \alpha, w < \beta$, we have

$$w = \frac{4(\alpha - \beta)}{(\alpha - \beta)^2(\xi - \xi_0)^2 - 4}. \hspace{1cm} (69)$$

If $\epsilon = -1$, when $w > \alpha, w < \beta$; or $w < \alpha, w > \beta$, we have

$$w = \frac{4(\beta - \alpha)}{4 + (\alpha - \beta)^2(\xi - \xi_0)^2}. \hspace{1cm} (70)$$
Case 6: $D_4 = 0, D_2D_3 < 0$. Then we have
\[ F(w) = (w - \alpha)^2((w - l_1)^2 + s_1^2), \] where $\alpha, l_1$ and $s_1$ are real numbers. If $\epsilon = 1$, we have
\[ w = \frac{\exp \left[ \pm \sqrt{\alpha - l_1}^2 + s_1^2 (\xi - \xi_0) \right] - \gamma + \sqrt{\alpha - l_1}^2 + s_1^2)}{\exp \left[ \pm \sqrt{\alpha - l_1}^2 + s_1^2 (\xi - \xi_0) \right] - \gamma} - 1, \] where
\[ \gamma = \frac{\alpha - 2l_1}{\sqrt{\alpha - l_1}^2 + s_1^2}. \] 

Case 7: $D_4 > 0, D_3 > 0, D_1 > 0$. Then we have
\[ F(w) = (w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4), \] where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers, and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. If $\epsilon = 1$, when $w > \alpha_1$ or $w < \alpha_4$, we have
\[ w = \frac{\alpha_2(\alpha_1 - \alpha_4) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - \alpha_1(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - (\alpha_2 - \alpha_4)}. \] 

When $\alpha_2 > w > \alpha_3$, we have
\[ w = \frac{\alpha_4(\alpha_2 - \alpha_3) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - \alpha_3(\alpha_2 - \alpha_4)}{(\alpha_2 - \alpha_3) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - (\alpha_2 - \alpha_4)}. \] 

If $\epsilon = -1$, when $\alpha_1 > w > \alpha_2$, we have
\[ w = \frac{\alpha_3(\alpha_1 - \alpha_2) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - \alpha_2(\alpha_1 - \alpha_3)}{(\alpha_1 - \alpha_2) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - (\alpha_1 - \alpha_3)}. \] 

When $\alpha_3 > w > \alpha_4$, we have
\[ w = \frac{\alpha_4(\alpha_3 - \alpha_4) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - \alpha_4(\alpha_3 - \alpha_1)}{(\alpha_3 - \alpha_4) \sin^2 \left( \frac{\sqrt{\alpha_1 - \alpha_3}(\alpha_2 - \alpha_4)}{2} (\xi - \xi_0), m \right) - (\alpha_3 - \alpha_1)}, \] where $m^2 = \frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}$.

Case 8: $D_4 < 0, D_2D_3 > 0$. Then we have
\[ F(w) = (w - \alpha)(w - \beta)((w - l_1)^2 + s_1^2), \]
where \( \alpha, \beta, l_1 \) and \( s_1 \) are real numbers, and \( \alpha > \beta > s_1 > 0 \), we have (here positive sign for \( \epsilon = -1 \), negative for \( \epsilon = 1 \)).

\[
w = \frac{a \cn \left( \frac{\sqrt{-2s_1m_1(\alpha-\beta)}}{2mm_1} (\xi - \xi_0), m \right) + b}{c \cn \left( \frac{\sqrt{-2s_1m_1(\alpha-\beta)}}{2mm_1} (\xi - \xi_0), m \right) + d},
\]

where

\[
c = \alpha - l_1 - \frac{s_1}{m_1}, \quad d = \alpha - l_1 - s_1m_1,
\]

\[
a = \frac{1}{2} [(\alpha + \beta)c - (\alpha - \beta)d], \quad b = \frac{1}{2} [(\alpha + \beta)d - (\alpha - \beta)c],
\]

\[
E = \frac{s_1^2 + (\alpha - l_1)(\beta - l_1)}{s_1(\alpha - \beta)}, \quad m_1 = E \pm \sqrt{E^2 + 1}, \quad m^2 = \frac{1}{1 + m_1^2},
\]

we choose \( m_1 \) such that \( \epsilon m_1 < 0 \).

Case 9: \( D_4 > 0, D_2D_3 \leq 0 \), then we have

\[
F(w) = ((w - l_1)^2 + s_1^2)(w - l_2)^2 + s_2^2),
\]

where \( l_1, l_2, s_1 \) and \( s_2 \) are real numbers, and \( s_1 > s_2 > 0 \). If \( \epsilon = 1 \), we have

\[
w = \frac{a \sn (\eta(\xi - \xi_0), m) + b \cn (\eta(\xi - \xi_0), m)}{c \sn (\eta(\xi - \xi_0), m) + d \cn (\eta(\xi - \xi_0), m)}
\]

where

\[
c = -s_1 - \frac{s_2}{m_1}, \quad d = l_1 - l_2, \quad a = l_1c + s_1d, \quad b = l_1d - s_1c,
\]

\[
E = \frac{s_1^2 + s_2^2 + (l_1 - l_2)^2}{2s_1s_2}, \quad m_1 = E + \sqrt{E^2 - 1},
\]

\[
m^2 = 1 - \frac{1}{m_1^2}, \quad \eta = s_2 \sqrt{\frac{m_1^2c^2 + d^2}{c^2 + d^2}}.
\]

**Remark 1:** For case 7 and case 8, we can use another transformation to obtain more simple representations. For example, for case 7, we consider only \( \epsilon = 1 \), as an example, we take a transformation as follows:

\[
w_1 = \frac{A^+}{w - \alpha_1},
\]

where \( A = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) \), then corresponding integral becomes

\[
\pm a^+ (\xi - \xi_0) = \int \frac{dw_1}{\sqrt{(w_1 - \beta_1)(w_1 - \beta_2)(w_1 - \beta_3)}}.
\]
where \( \beta_1 = \frac{A_1^2}{a_3 - a_1} \), \( \beta_2 = \frac{A_2^4}{a_3 - a_1} \), \( \beta_3 = \frac{A_3^4}{a_3 - a_1} \). When \( \beta_1 > w_1 > \beta_2 \), we have

\[
w_1 = \beta_3 + (\beta_2 - \beta_3) \text{sn}^2 \left( \frac{\sqrt{\beta_1 - \beta_3}}{2} (\xi - \xi_0), m \right).
\]  
(87)

When \( w_1 > \beta_1 \), we have

\[
w_1 = \frac{\beta_1 - \beta_2}{\beta_3 - \beta_2} \text{sn} \left( \frac{\sqrt{\beta_1 - \beta_3}}{2} (\xi - \xi_0), m \right) \text{cn} \left( \frac{\sqrt{\beta_1 - \beta_3}}{2} (\xi - \xi_0), m \right).
\]  
(88)

where \( m = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3} \).

For the case of \( \epsilon = -1 \) and the case 8, we can deal with them similarly.

**Remark 2**: All above integrals can be found in Ref. [24].

**Example 1**: Classification of all single travelling wave solutions to mKdV equation.

mKdV equation reads

\[
u_t + au^2 u_x + bu_{xxx} = 0,
\]  
(89)

Its exact travelling wave solutions have been studied extensively ([20-23]). But its all travelling wave solutions haven’t been reported. Using direct integral method and complete discrimination system for the fourth order polynomial, we give in the following all travelling wave solutions to mKdV equation.

Take travelling wave transformation as follows:

\[
u = u(\xi), \xi = kx + \omega t.
\]  
(90)

Instituting Eq. (90) into Eq. (89) and integrating twice yield

\[
(u')^2 = A(u^4 + pu^2 + qu + r),
\]  
(91)

where

\[
A = -\frac{a}{6bk^2}, \quad p = \frac{ak}{6\omega},
\]  
(92)

and \( q, r \) are two arbitrary constants. Write Eq. (91) by integral form

\[
\pm \sqrt{\epsilon} A(\xi - \xi_0) = \int \frac{du}{\sqrt{\epsilon (u^4 + pu^2 + qu + r)}}.
\]  
(93)

where \( \epsilon = \pm 1 \). If \( A > 0 \), we take \( \epsilon = 1 \). If \( A < 0 \), we take \( \epsilon = -1 \). According to the former results, we can give the classification of all single travelling wave solutions to MKdV equation, we omit them for briefly.
Example 2. Classification of all single traveling solutions to combined KdV-MKdV equation\((10,11)\) which reads

\[ u_t + auu_x + bu^2u_x + u_{xxxx} = 0, \quad (94) \]

where \(a\) and \(b\) are physical parameters. Taking traveling wave transformation

\[ u = u(\xi_1), \xi_1 = kx - \omega t, \]

and instituting it into Eq.\((94)\) and integrating once yields:

\[ (u')^2 = a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0, \quad (95) \]

where

\[ a_4 = -\frac{a}{6k^2}, \quad a_3 = -\frac{b}{3k^2}, \quad a_2 = \frac{\omega}{k^3}, \quad (96) \]

and \(a_1, a_0\) are arbitrary constants. If \(a < 0\), we take transformations as follows:

\[ w = a_4^{\frac{1}{4}}(u + \frac{a_3}{4a_4}), \quad \xi = a_4^{\frac{1}{4}}\xi_1. \quad (97) \]

Then Eq.\((95)\) becomes

\[ (w')^2 = w^4 + pw^2 + qw + r, \quad (98) \]

where

\[ p = \frac{a_2}{\sqrt{a_4}}, \quad q = a_4^{\frac{1}{4}}\left(\frac{a_3}{8a_4} + \frac{a_2a_3}{2a_4} + a_1\right), \quad r = a_0 - \frac{a_1a_3}{4a_4} + \frac{a_2a_3^2}{16a_4^2} - \frac{3a_4^4}{256a_4^3}. \quad (99) \]

If \(a > 0\), we take transformations as follows:

\[ w = (-a_4)^{\frac{1}{4}}(u + \frac{a_3}{4a_4}), \quad \xi = (-a_4)^{\frac{1}{4}}\xi_1 \quad (100) \]

Then Eq.\((95)\) becomes

\[ (w')^2 = -(w^4 + pw^2 + qw + r), \quad (101) \]

where

\[ p = -\frac{a_2}{\sqrt{-a_4}}, \quad q = (-a_4)^{-\frac{1}{4}}\left(-\frac{a_3}{8a_4} + \frac{a_2a_3}{2a_4} - a_1\right), \quad r = -a_0 - \frac{a_1a_3}{4a_4} - \frac{a_2a_3^2}{16a_4^2} + \frac{3a_4^4}{256a_4^3}. \quad (102) \]

According to the classification of solutions to ODE \((w')^2 = \pm(w^4 + pw^2 + qw + r)\), we can give all traveling wave solutions to combined KdV-MKdV equation. We omit them for simplicity.
Example 3. Classification of all single traveling wave solutions to Konopelchenko-Dubrovsky equation ([31, 32]) which reads
\[ u_t = u_{xxx} - 6\beta uu_x + \frac{3}{2}\alpha^2 u^2 u_x - 3w_y + 3\alpha uu_x = 0, \]  
\[ w_x = u_y. \]  
(103)  
(104)

Taking traveling wave transformation \( u = u(\xi), \xi = k_1 x + k_2 y + \omega t \), substituting it into Eq. (103) and integrating twice yield the same form equation with Eq. (109):
\[ (u')^2 = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0, \]  
(105)

where
\[ a_4 = \frac{\alpha^2}{4k_1^4}, \quad a_3 = \frac{\alpha - 2\beta}{2k_1^2}, \quad a_2 = \frac{2(k_1 \omega - 3k_2^2)}{k_1^4}, \]  
(106)

and \( a_1, a_0 \) are arbitrary constants. Notice \( a_4 > 0 \). According to the example 1, it is easy to write all traveling wave solutions to KD equation, we omit them for briefly. All nine cases under the condition \( a_4 > 0 \) in section 4 are possible.

Example 4: Classification of all single traveling solutions to double Sine-Gordon equation ([42, 43]) which reads
\[ u_{xx} - u_{tt} = \sin au + \lambda \sin a^2 u, \]  
(107)

where \( a \) and \( \lambda \) are nonzero constants. Its some exact traveling wave solutions have been obtained. We give in the following its all traveling wave solutions.

Take traveling wave transformation
\[ u = u(\xi_1), \quad \xi_1 = k x - \omega t, \quad k \neq \pm \omega. \]  
(108)

Instituting Eq. (108) into Eq. (107) and integrating once yields
\[ \pm(\xi_1 - \xi_{10}) = \int \frac{du}{\sqrt{c_2 - c_1 \cos au - c_0 \cos \frac{a}{2} u}}. \]  
(109)

where
\[ c_1 = \frac{2}{a(k_2 - \omega^2)}, \quad c_0 = \frac{4\lambda}{a(k_2 - \omega^2)}. \]  
(110)

In order to solve above integral (109), we take transformation as follows:
\[ u = \frac{2}{a} \arccos v. \]  
(111)
Instituting Eq. (111) into integral (109), we have

$$\pm(\xi_1 - \xi_{10}) = \int \frac{2 \, dv}{\sqrt{(v - 1)(v + 1)(2c_1v^2 + c_0v + c_1 - c_2)}}. \quad (112)$$

If $c_1 > 0$, let

$$w = \sqrt{c_1}v + \frac{c_0}{4c_1}, \quad \xi = \frac{a}{\sqrt{2c_1}}\xi_1. \quad (113)$$

Thus Eq. (112) becomes respectively

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{(w - \alpha)(w - \beta)(w^2 + d_0)}}. \quad (114)$$

where

$$\alpha = \frac{c_0}{4c_1} + \sqrt{c_1}, \quad \beta = \frac{c_0}{4c_1} - \sqrt{c_1}, \quad d_0 = \frac{c_1 - c_2}{2} - \frac{c_0^2}{14c_1^2}. \quad (115)$$

If $c_1 < 0$, let

$$w = \sqrt{-c_1}v - \frac{c_0}{4c_1}, \quad \xi = \frac{a}{\sqrt{2c_1}}\xi_1. \quad (116)$$

Thus Eq. (112) becomes respectively

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{-(w - \alpha)(w - \beta)(w^2 + d_0)}}. \quad (117)$$

where

$$\alpha = -\frac{c_0}{4c_1} + \sqrt{-c_1}, \quad \beta = -\frac{c_0}{4c_1} - \sqrt{-c_1}, \quad d_0 = \frac{c_1 - c_2}{2} - \frac{c_0^2}{14c_1^2}. \quad (118)$$

Denote

$$F(w) = (w - \alpha)(w - \beta)(w^2 + d_0), \quad (119)$$

and rewrite it as follows:

$$F(w) = w^4 + pw^2 + qw + r, \quad (120)$$

where

$$p = d_0 - (\alpha + \beta), \quad r = -d_0(\alpha + \beta), \quad s = d_0\alpha\beta. \quad (121)$$

But because of special form of $F(w)$ (see Eq. (119)), we know that the case 1,
case 2, case 6 and case 9 in section 3 are impossible to occur in the case of double Sine-Gordon equation. Other cases are possible. Thus we can give all traveling solutions to double Sine-Gordon equation according to the cases 3-5 and cases 7-8 in section 3. We omit them for briefly.

**Example 5.** Classification of all single traveling wave solutions to Pochhammer-Chree Equation (35) which reads

\[ u_{tt} - \alpha u_{xx} + \beta u + \gamma u^3 + \delta u^5. \]  
(122)

Taking traveling wave transformation \( u = u(\xi), \xi = kx + \omega t \), instituting it into Eq. (122) and integrating once yields

\[ (u')^2 = a_3 u^6 + a_2 u^4 + a_1 u^2 + a_0, \]  
(123)

where

\[ a_3 = -\frac{\delta}{3(\omega^2 - \alpha k^2)}, \quad a_2 = -\frac{\gamma}{2(\omega^2 - \alpha k^2)}, \quad a_1 = -\frac{\beta}{\omega^2 - \alpha k^2}, \]  
(124)

and \( a_0 \) is an arbitrary constant. Let \( v = u^2 \), then Eq. (123) becomes

\[ (v')^2 = 2v(a_3 v^3 + a_2 v^2 + a_1 v + a_0). \]  
(125)

Similar to the example 3, we can give all traveling wave solutions to Pochhammer-Chree equation (122). Because of special form of the right polynomial of Eq. (125), the case 1 and case 9 in section 3 are impossible to occur, all other cases 2-8 are possible. We omit them for simplicity.

**Example 6:** Classification of all single traveling wave solutions to classical Bousinesq system

\[ \eta_t + [(1 + \eta)u]_x = -\frac{1}{4}u_{xxx}, \]  
(126)

\[ u_t + uu_x\eta_x = 0. \]  
(127)

Take traveling wave transformation \( \eta = \eta(\xi), u = u(\xi), \xi = kx + \omega t \), instituting it into above system and integrating once yield

\[ c_0 + \omega \eta + k(1 + \eta)u = -\frac{k^3}{4}u'', \]  
(128)

\[ \eta = c_1 - \frac{\omega}{k} u - \frac{1}{2}u^2. \]  
(129)

Instituting Eq. (129) into Eq. (128) and integrating once, we have

\[ (u')^2 = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0, \]  
(130)
where
\begin{align*}
a_4 &= \frac{1}{k^2}, \quad a_3 = \frac{8\omega}{3k^3}, \quad a_2 = \frac{4\omega^2 - 4k^2(1 + c_1)}{k^4}, \quad a_1 = c_0 + c_1\omega, \quad (131)
\end{align*}
and \(a_0\) is an arbitrary constant. Notice \(a_4 > 0\) and according to the example 1, we can easily to obtain all traveling wave solutions to classical Boussinesq system. For briefly we omit them.

4 Applications of the complete discrimination system for the fifth order polynomial

We consider the following ODE
\begin{equation}
(w'(\xi))^2 = F(w) = w^5 + pw^3 + qw^2 + rw + s. \quad (132)
\end{equation}
We write its complete discrimination system as follows (see [1, 3]):
\begin{align*}
D_2 &= -p, \quad D_3 = 40rp - 12p^3 - 45q^2, \\
D_4 &= 12p^4r - 4p^3q^2 + 117pqr^2 - 88r^2p^2 \\
&\quad - 40qsp^2 - 27q^4 - 300qrs + 160r^3, \\
D_5 &= -1600qsr^3 - 3750qps^3 + 2000ps^2r^2 - 4p^3q^3r^2 + 16p^3q^3s \\
&\quad - 900r^2p^3 + 825pq^2r^3 + 2250q^2r^2s^2 + 16p^3r^3 \\
&\quad + 108p^5s^2 - 128q^4r^2 - 27r^2q^4 + 108sq^5 + 256r^5 + 3125s^4 \\
&\quad - 72rsq^3 + 560sqr^2p^2 - 630pqrs^3, \\
E_2 &= 160r^2p^3 + 900q^2r^2 - 48rp^5 + 60r^2p^2 + 150pqrs + 16q^2p^4 \\
&\quad - 1100qsp^3 + 625s^2p^2 - 3375sq^3, \\
F_2 &= 3q^2 - 8rp. \quad (133)
\end{align*}
According to the complete discrimination system for polynomial \(f(w)\), there are twelve cases to be discussed.

Case 1: \(D_5 = 0, D_4 = 0, D_3 > 0, E_2 \neq 0\). Then we have
\begin{equation}
F(w) = (w - \alpha)^2(w - \beta)^2(w - \gamma), \quad (134)
\end{equation}
where \(\alpha, \beta, \gamma\) are real numbers, \(\alpha \neq \beta \neq \gamma\). When \(w > \gamma\), we have
\begin{align*}
\pm \frac{\alpha - \beta}{2}(\xi - \xi_0) &= \sqrt{\gamma - \alpha} \arctan \frac{\sqrt{w - \gamma}}{\sqrt{\gamma - \alpha}} - \sqrt{\gamma - \beta} \arctan \frac{\sqrt{w - \gamma}}{\sqrt{\gamma - \beta}}, \\
&\quad \gamma > \alpha, \gamma > \beta; \quad (135)
\end{align*}
\begin{align*}
\pm \frac{\alpha - \beta}{2}(\xi - \xi_0) &= \sqrt{\gamma - \alpha} \arctan \frac{\sqrt{w - \gamma}}{\sqrt{\gamma - \alpha}} - \frac{1}{\sqrt{\gamma - \beta}}, \\
&\quad \gamma > \alpha, \gamma > \beta.
\end{align*}
\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\sqrt{\gamma - \beta} \arctan \frac{\sqrt{w - \gamma}}{\sqrt{\gamma - \beta}} + \frac{1}{2\sqrt{\alpha - \gamma}} \times \ln \left| \frac{\sqrt{w - \gamma} - \sqrt{\alpha - \gamma}}{\sqrt{w - \gamma} + \sqrt{\alpha - \gamma}} \right|, \gamma > \alpha, \gamma < \beta; \tag{136}
\]

\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\frac{1}{2\sqrt{\alpha - \gamma}} \ln \left| \frac{\sqrt{w - \gamma} - \sqrt{\alpha - \gamma}}{\sqrt{w - \gamma} + \sqrt{\alpha - \gamma}} \right| - \frac{1}{2\sqrt{\beta - \gamma}} \times \ln \left| \frac{\sqrt{w - \gamma} - \sqrt{\beta - \gamma}}{\sqrt{w - \gamma} + \sqrt{\beta - \gamma}} \right|, \gamma < \alpha, \gamma < \beta. \tag{137}
\]

Case 2: \(D_5 = 0, D_4 = 0, D_3 = 0, D_2 \neq 0, F_2 \neq 0\). Then we have
\[
F(w) = (w - \alpha)^3 (w - \beta)^2, \tag{139}
\]
where \(\alpha, \beta\), are real numbers, \(\alpha \neq \beta\). When \(w > \alpha\), we have
\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\frac{1}{\sqrt{w - \alpha}} - \sqrt{\alpha - \beta} \arctan \frac{\sqrt{w - \alpha}}{\sqrt{\alpha - \beta}}, \alpha > \beta; \tag{140}
\]

\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\frac{1}{2\sqrt{\beta - \alpha}} \ln \left| \frac{\sqrt{w - \beta} - \sqrt{\alpha - \beta}}{\sqrt{w - \beta} + \sqrt{\alpha - \beta}} \right|, \alpha < \beta. \tag{141}
\]

Case 3: \(D_5 = 0, D_4 = 0, D_3 = 0, D_2 \neq 0, F_2 = 0\). Then we have
\[
F(w) = (w - \alpha)^4 (w - \beta), \tag{142}
\]
where \(\alpha, \beta\), are real numbers, \(\alpha \neq \beta\). When \(w > \alpha\), we have
\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\frac{1}{2\sqrt{w - \alpha}} - \frac{1}{2\sqrt{\beta - \alpha}} \arctan \frac{\sqrt{w - \beta}}{\sqrt{\beta - \alpha}}, \alpha < \beta; \tag{143}
\]

\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = -\frac{1}{4\sqrt{\alpha - \beta}} \ln \left| \frac{\sqrt{w - \alpha} - \sqrt{\beta - \alpha}}{\sqrt{w - \alpha} + \sqrt{\beta - \alpha}} \right|, \alpha > \beta. \tag{144}
\]

Case 4: \(D_5 = 0, D_4 = 0, D_3 = 0, D_2 = 0\). Then we have
\[
F(w) = (w - \alpha)^5, \tag{145}
\]
where \( \alpha \) is real number. When \( w > \alpha \), we have
\[
\pm (\xi - \xi_0) = -\frac{2}{3} (w - \alpha)^{-\frac{2}{3}}.
\] (146)

Case 5: \( D_5 = 0, D_4 = 0, D_3 < 0, E_2 \neq 0 \). Then we have
\[
F(w) = (w - \alpha)(w^2 + rw + s)^2,
\] (147)
where \( \alpha \) is real number, \( r^2 - 4s < 0 \). When \( w > \alpha \), we have
\[
\pm (\xi - \xi_0) = -\frac{2}{\rho \sqrt{4s - r^2}} \left( \cos \varphi \arctan \frac{2 \rho \sin \varphi \sqrt{w - \alpha}}{w - \alpha - \rho^2} + \frac{\sin \varphi}{2} \right)
\times \ln \left| \frac{w - \alpha - \rho^2 - 2 \rho \cos \varphi \sqrt{w - \alpha}}{w - \alpha - \rho^2 + 2 \rho \cos \varphi \sqrt{w - \alpha}} \right|,
\] (148)
where
\[
\rho = (\alpha^2 + r \alpha + s)^{\frac{1}{4}}, \varphi = \frac{1}{2} \arctan \frac{\sqrt{4s - r^2}}{2 \alpha - \rho}.
\] (149)

Case 6: \( D_5 = 0, D_4 > 0 \). Then we have
\[
F(w) = (w - \alpha)^2(w - \alpha_1)(w - \alpha_2)(w - \alpha_3),
\] (150)
where \( \alpha, \alpha_1, \alpha_2, \alpha_3 \) are real numbers, \( \alpha_1 > \alpha_2 > \alpha_3 \). We have
\[
\pm (\xi - \xi_0) = -\frac{2}{(\alpha - \alpha_2)\sqrt{\alpha_2 - \alpha_3}} \left\{ F(\varphi, k) - \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha} \Pi(\varphi, \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha}, k) \right\},
\] (151)
where \( \alpha \neq \alpha_1, \alpha \neq \alpha_2, \alpha \neq \alpha_3 \) and
\[
F(\varphi, k) = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}
\] (152)
\[
\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\varphi}{(1 + n \sin^2 \varphi) \sqrt{1 - k^2 \sin^2 \varphi}}.
\] (153)

Case 7: \( D_5 = 0, D_4 = 0, D_3 < 0, E_2 = 0 \). Then we have
\[
F(w) = (w - \alpha)^3((w - l_1)^2 + s_1^2),
\] (154)
where \( \alpha, l_1 \) and \( s_1 \) are real numbers. When \( w > \alpha \), if \( \alpha \neq l_1 + s_1 \), we have
\[
\pm (\xi - \xi_0) = \frac{\tan \theta + \cot \theta}{2(s_1 \tan \theta - l_1 - \alpha) \sqrt{s_1^2}} F(\varphi, k) - \frac{s_1 \tan \theta + s_1 \cot \theta}{s_1 \cot \theta + l_1 + \alpha}
\]
\[
\times \left\{ \frac{\tan \theta + l_1 + \alpha}{(s_1 \cot \theta + l_1 - \alpha) \sin \varphi} \sqrt{1 - k^2 \sin^2 \varphi + F(\varphi, k) - E(\varphi, k)} \right\}; \tag{155}
\]

if \( \alpha = l_1 + s_1 \), we have
\[
\pm (\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4 s^3}} \left( \frac{1}{k} \arcsin(k \sin \varphi) - F(\varphi, k) \right), \tag{156}
\]

where
\[
\tan 2\theta = \frac{s_1}{\alpha - l_1}, \quad k = \sin \theta, \quad 0 < \theta < \frac{\pi}{2}, \quad E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \psi} \, d\psi. \tag{157}
\]

Case 8: \( D_5 = 0, D_4 < 0 \). Then we have
\[
F(w) = (w - \alpha)^2 (w - \beta)((w - l_1)^2 + s_1^2), \tag{158}
\]

where \( \alpha, l_1 \) and \( s_1 \) are real numbers. When \( w > \beta \), if \( \alpha \neq l_1 - s_1 \tan \theta, \alpha \neq l_1 + s_1 \cot \theta \), we have
\[
\pm (\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4 s^3}} \left( \frac{1}{k} \arcsin(k \sin \varphi) - F(\varphi, k) \right); \tag{159}
\]

if \( \alpha = l_1 - s_1 \tan \theta \) we have
\[
\pm (\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4 s^3}} \left( \frac{1}{k} \arcsin(k \sin \varphi) - F(\varphi, k) \right); \tag{160}
\]

if \( \alpha = l_1 + s_1 \cot \theta \), we have
\[
\pm (\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4 s^3}} (F(\varphi, k) - \frac{1}{\sqrt{1 - k^2}} \ln \sqrt{1 - k^2 \sin^2 \varphi + \sqrt{1 - k^2} \sin \varphi}); \tag{161}
\]

where
\[
\tan 2\theta = \frac{s_1}{\beta - l_1}, \quad k = \sin \theta, \quad 0 < \theta < \frac{\pi}{2}. \tag{162}
\]

Case 9: \( D_5 = 0, D_4 = 0, D_3 > 0, E_2 = 0 \). Then we have
\[
F(w) = (w - \alpha)^3 (w - \beta)(w - \gamma), \tag{163}
\]

where \( \alpha, \beta \) and \( \gamma \) are real numbers. When \( w > \alpha > \beta > \gamma \), we have
\[
\pm \frac{\alpha - \beta}{2} (\xi - \xi_0) = \frac{1}{\sqrt{\alpha - \gamma}} E(\text{arcsin} \sqrt{\frac{\alpha - \gamma}{w - \gamma} \sqrt{\frac{\beta - \gamma}{\alpha - \gamma}}}). \tag{157}
\]
\[ -\sqrt{\frac{w - \beta}{(w - \alpha)(w - \gamma)}}. \]  

(164)

In other cases such as \( w > \beta > \alpha > \gamma \) and so on, we can give the corresponding solutions similarly. We omit them for simplicity.

Case 10: \( D_5 > 0, D_4 > 0, D_3 > 0, D_2 > 0 \), we have
\[ F(w) = (w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4)(w - \alpha_5). \]  

(165)

Then the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral such as
\[ \pm(\xi - \xi_0) = \int \frac{dw}{\sqrt{(w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4)(w - \alpha_5)}}. \]  

(166)

Case 11: \( D_5 < 0 \), we have
\[ F(w) = (w - \alpha_1)(w - \alpha_2)(w - \alpha_3)((w - l_1)^2 + s_1^2). \]  

(167)

Then the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral such as
\[ \pm(\xi - \xi_0) = \int \frac{dw}{\sqrt{(w - \alpha_1)(w - \alpha_2)(w - \alpha_3)((w - l_1)^2 + s_1^2)}}. \]  

(168)

Case 12: \( D_5 > 0 \land (D_4 \leq 0 \lor D_3 \leq 0 \lor D_2 \leq 0) \), where \( \land \) means "and", \( \lor \) means "or". Then we have
\[ F(w) = (w - \alpha)((w - l_1)^2 + s_1^2)((w - l_2)^2 + s_2^2). \]  

(169)

Then the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral such as
\[ \pm(\xi - \xi_0) = \int \frac{dw}{\sqrt{(w - \alpha)((w - l_1)^2 + s_1^2)((w - l_2)^2 + s_2^2)}}. \]  

(170)

Example 1: All traveling solutions to triple Sine-Gordon equation([3],[42-45])
\[ u_{xx} - u_{tt} = c \sin u + \frac{1}{3}a \sin \frac{1}{3}u + \frac{2}{3}b \sin \frac{2}{3}u, \]  

(171)

where \( a, b \) and \( c \) are nonzero constants. Its some exact traveling wave solutions have been obtained. In Ref.[3], we used the similar idea to deal with it, but we
didn’t recognize that its all single traveling wave solutions can be obtained. We give in the following its all traveling wave solutions.

Take traveling wave transformation

\[ \xi_1 = kx - \omega t, \quad k \neq \pm \omega. \]  \hspace{1cm} (172)

Instituting Eq. (172) into Eq. (171) and integrating once yields

\[ \pm (\xi_1 - \xi_{10}) = \int \frac{du}{\sqrt{c_2 - 2c_1 \cos u - 2a_1 \cos \frac{1}{3} u - 2b_1 \cos \frac{2}{3} u}}, \]  \hspace{1cm} (173)

where

\[ a_1 = \frac{a}{k_2 - \omega^2}, \quad b_1 = \frac{b}{k_2 - \omega^2}, \quad c_1 = \frac{c}{k_2 - \omega^2}. \]  \hspace{1cm} (174)

In order to solve above integral (173), we take transformation as follows:

\[ u = 3 \arccos v. \]  \hspace{1cm} (175)

Instituting Eq. (175) into integral (173), we have

\[ \pm (\xi_1 - \xi_{10}) = \int \frac{3dv}{\sqrt{(v - 1)(v + 1)(8c_1v^3 + 4b_1v^2 + (2a_1 - 6c_1)v - 2b_1 - c_2)}}. \]  \hspace{1cm} (176)

Let

\[ v = (8c_1)^{-\frac{1}{3}} w - \frac{b_1}{6c_1}, \quad \xi = \frac{1}{3} (8c_1)^{\frac{1}{3}} \xi_1. \]  \hspace{1cm} (177)

Thus Eq. (176) becomes

\[ \pm (\xi - \xi_0) = \int \frac{dv}{\sqrt{(w - \alpha)(w - \beta)(w^3 + d_1 w + d_0)}}, \]  \hspace{1cm} (178)

where

\[ \alpha = (8c_1)^{\frac{1}{3}} \left( \frac{b_1}{6c_1} + 1 \right), \quad \beta = (8c_1)^{\frac{1}{3}} \left( \frac{b_1}{6c_1} - 1 \right); \]  \hspace{1cm} (179)

\[ d_1 = (8c_1)^{\frac{1}{3}} \left( \frac{a_1}{4c_1} - \frac{b_1^2}{12c_1^2} - \frac{3}{4} \right), \quad d_0 = 8c_1^{\frac{1}{3}} \left( \frac{b_1^3}{108c_1^3} - \frac{b_1a_1}{24c_1^2} - \frac{(b_1 + c_2)}{8c_1} \right). \]  \hspace{1cm} (180)

Denote

\[ F(w) = (w - \alpha)(w - \beta)(w^3 + d_1 w + d_0). \]  \hspace{1cm} (181)
Rewrite it as follows:

\[ F(w) = w^5 + pw^3 + qw^2 + rw + s, \quad (182) \]

where

\[ p = \alpha \beta - d_1(\alpha + \beta), \quad q = d_0 - d_1(\alpha + \beta), \quad r = \alpha \beta d_1 - d_0(\alpha + \beta), \quad s = d_0 \alpha \beta. \quad (183) \]

But because of special form of \( F(w) \) (see Eq. (181)), we know that the case 4, case 5, case 7 and case 12 in section 4 are impossible to occur in the case of triple Sine-Gordon equation. Other cases are possible. Thus we can give all traveling solutions to triple Sine-Gordon equation according to the cases 1-3, case 6 and cases 8-11 in section 4. We omit them for briefly.

Example 2. All traveling wave solutions to \( D + 1 \) dimensional Klein-Gordon equation with fourth order nonlinear term

\[ \sum_{i=1}^{D} u_{x_i x_i} - u_{tt} = \sum_{j=0}^{4} b_j w^j. \quad (184) \]

Take traveling wave transformation \( u = u(\xi_1), \quad \xi_1 = \sum_{i=1}^{D} k_i x_i + \omega t. \) Then Eq. (184) becomes

\[ (u')^2 = a_0 + \sum_{i=1}^{5} a_i u^i, \quad (185) \]

where \( a_0 \) is integral constant and

\[ a_i = \frac{2b_i}{(i + 1)(\sum_{i=1}^{D+1} k_i^2 - \omega^2)}. \quad (186) \]

In order to reduce Eq. (185) to the form (132), we take transformations as follows:

\[ w = a_5^{\frac{1}{5}} u + \frac{4}{5} a_4 a_5^{-\frac{4}{5}}, \quad \xi = a_5^{\frac{1}{5}} \xi_1. \quad (187) \]

Then Eq. (185) becomes

\[ \pm (\xi - \xi_0) = \int \frac{dw}{\sqrt{w^5 + pw^3 + qw^2 + rw + s}}, \quad (188) \]

where

\[ p = \frac{16}{5} a_7 a_5^{-\frac{4}{5}} + a_3 a_5^{-\frac{2}{5}}, \quad q = \frac{32}{25} a_3 a_5^{-\frac{4}{5}} + (a_2 - 3a_3) a_5^{-\frac{2}{5}}. \]
According to the cases 1-12 in section 4, it is easy to give the classification of all traveling wave solutions to Eq. (184). We omit them for briefly.

5 The solutions of irrational integrals and applications

There are some mathematical physics equations whose reduced ODE’s have the form of \((u')^2 = \frac{g(u)}{f(u)}\), where \(g(u)\) and \(f(u)\) are two polynomials. So the classifications of their all traveling wave solutions are more complex.

Example 1. Classification of all single traveling wave solutions to Fujimoto-Watanabe equation

Fujimoto-Watanabe equation \([37]\) reads

\[
u_t = u^3u_{xxx} + 3u^2u_xu_xx + 3\varepsilon u^2u_x, \tag{190}\]

under the traveling wave transformation \(u = u(\xi), \xi = kx + \omega t\), the corresponding reduced ODE is as follows:

\[
\pm (\xi - \xi_0) = \int \frac{udu}{\sqrt{a_3u^3 + a_2u^2 + a_1u + a_0}}, \tag{191}\]

where \(a_3 = -\frac{2\varepsilon}{k}, a_1 = -\frac{2\varepsilon}{k^2}, a_2\) and \(a_0\) are two arbitrary constants. For simplicity, we take the change of variable \(w = a_3^{-\frac{1}{3}}u\), then the Eq. (191) becomes

\[
\pm a_3^\frac{2}{3}(\xi - \xi_0) = \int \frac{wdw}{\sqrt{w^3 + d_2w^2 + d_1w + d_0}}, \tag{192}\]

where \(d_2 = -a_3^{-\frac{4}{3}}a_2, \quad d_1 = a_3^{-\frac{1}{3}}a_1, \quad d_0 = a_0\). Denote \(F(w) = w^3 + d_2w^2 + d_1w + d_0\), its complete discrimination system is \(\Delta = -27(\frac{2d_2^2}{d_1} + d_0 - \frac{d_1d_2}{d_1^2})^2 - 4(d_1 - \frac{d_2^2}{d_1})^3\) and \(D_1 = d_1 - \frac{d_2^3}{d_1^2}\). There are the following four cases to be discussed:

Case 1: \(\Delta = 0, D_1 < 0\), then we have \(F(w) = (w - \alpha)^2(w - \beta), \alpha \neq \beta\). If \(w > \beta\), the solutions are as follows:

\[
\pm a_3^\frac{2}{3}(\xi - \xi_0) = 2\sqrt{a_3^\frac{1}{3}u - \beta} + \frac{\alpha}{\sqrt{\beta - \alpha}} \arctan \frac{\sqrt{a_3^\frac{1}{3}u - \beta}}{\sqrt{\beta - \alpha}}, \quad \alpha > \beta; \tag{193}\]
\( \pm a_3^2 (\xi - \xi_0) = 2 \sqrt{\frac{\alpha}{a_3^3} u - \beta} + \frac{\alpha}{2 \sqrt{\alpha - \beta}} \ln \sqrt{\frac{a_3^3 u - \beta - \sqrt{\alpha - \beta}}{a_3^3 u - \beta + \sqrt{\alpha - \beta}}} \quad \alpha > \beta; \) (194)

Case 2: \( \Delta = 0, \ D_1 = 0, \) then we have \( F(w) = (w - \alpha)^3, \) the solution is as follows:

\[
\pm a_3^2 (\xi - \xi_0) = 2 \sqrt{\frac{\alpha}{a_3^3} u - \beta} - \frac{\alpha}{\sqrt{\frac{\alpha}{a_3^3} u - \beta}}. \quad (195)
\]

Case 3: \( \Delta > 0, \ D_1 < 0, \) then \( F(w) = (w - \alpha)(w - \beta)(w - \gamma), \) we suppose that \( \alpha < \beta < \gamma. \) It is easy to see the corresponding integral can be expressed by the second kind of elliptic integrals.

Case 4: \( \Delta < 0, \) then we have \( F(w) = (w - \alpha)(w^2 + pw + q), \ p^2 - 4q < 0. \) Then the corresponding integral can be expressed by the second kind of elliptic integrals.

**Example 2.** The classification of all single traveling wave solutions to coupled Harry-Dym equation([38]):

\[
u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x + v_x + \varepsilon u_x, \quad (196)
\]

\[
v_t = -u x v - \frac{1}{2} u v_x + \varepsilon v_x. \quad (197)
\]

Through traveling wave transformation and integrations, we obtain

\[
v = \frac{c_0}{(\frac{1}{4} ku + \omega - k\varepsilon)^2}, \quad (198)
\]

\[
(u')^2 = \frac{2}{k^2} \frac{(u + r_0)^4 + a_3(u + r_0)^3 + a_2(u + r_0)^2 + a_1(u + r_0) + a_0}{u + r_0}, \quad (199)
\]

where \( r_0 = \frac{2\varepsilon}{k} - 2\varepsilon, \) \( a_3 = -4(\frac{\omega}{k} - \varepsilon), \) \( a_0 = \frac{k^2}{2}c_1. \) Denote \( F(u) = (u + r_0)^4 + a_3(u + r_0)^3 + a_2(u + r_0)^2 + a_1(u + r_0) + a_0, \) according to the section 4, the complete discrimination system for the fourth order polynomial \( F(u) \) is as follows:

\[
D_1 = 4, \ D_2 = -p, \ D_3 = 8rp - 2p^3 - 9q^2, \quad D_4 = 4p^4r - p^3q^2 + 36prq^2 - 32r^2p^2 - \frac{27}{4}q^4 + 64r^3, \quad F_2 = 9p^2 - 32pr, \quad (200)
\]

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where \( p = a_2, \ q = (\triangle^3 - \frac{a_1 a_3}{4} + a_1), \ r = a_0 - \frac{a_1 a_3}{4} + \frac{a_2 a_3^2}{16} - \frac{3a_1^4}{256} \). There are the following cases to be discussed.

Case 1: \( D_4 = 0, D_3 = 0, D_2 < 0 \). Then we have

\[
F(u) = (u + r_0)^2 + l(u + r_0) + s^2,
\]

where \( l, s \) are real numbers, \( l^2 - 4s^2 < 0 \). We have

\[
\pm(\xi - \xi_0) = \frac{1}{2\sqrt{2s - l}} \ln \frac{u + r_0 + \sqrt{2s - l}\sqrt{u + r_0} + s}{u + r_0 - \sqrt{2s - l}\sqrt{u + r_0} + s}
\]

\[
+ \frac{1}{\sqrt{2s + l}} \left\{ \arctan \frac{4\sqrt{u + r_0} \pm \sqrt{2s - l}}{2\sqrt{2s + l}} + \arctan \frac{4\sqrt{u + r_0} + \sqrt{2s - l}}{2\sqrt{2s + l}} \right\}.
\]  

Case 2: \( D_4 = 0, D_3 = 0, D_2 = 0 \). Then we have

\[
F(u) = (u + r_0 - \alpha)^2.
\]

If \( \alpha > 0 \), we have

\[
\pm(\xi - \xi_0) = \frac{1}{2\sqrt{\alpha}} \ln \frac{\sqrt{u + r_0} - \sqrt{\alpha}}{\sqrt{u + r_0} + \sqrt{\alpha}} - \frac{\sqrt{u + r_0}}{u + r_0 - \alpha}.
\]  

If \( \alpha < 0 \), we have

\[
\pm(\xi - \xi_0) = \frac{1}{\sqrt{-\alpha}} \arctan \frac{\sqrt{u + r_0}}{\sqrt{-\alpha}} - \frac{\sqrt{u + r_0}}{u + r_0 - \alpha}.
\]

Case 3: \( D_4 = 0, D_3 = 0, D_2 > 0, E_2 = 0 \). Then we have

\[
F(u) = (u + r_0 - \alpha)^2(u + r_0 - \beta)^2,
\]

where \( \alpha, \beta \) are real numbers, \( \alpha > \beta \). If \( \alpha > \beta > 0 \), we have

\[
\pm(\alpha + \beta)(\xi - \xi_0) = \sqrt{\alpha} \ln \frac{\sqrt{u + r_0} - \sqrt{\alpha}}{\sqrt{u + r_0} + \sqrt{\alpha}} - \sqrt{\beta} \ln \frac{\sqrt{u + r_0} - \sqrt{\beta}}{\sqrt{u + r_0} + \sqrt{\beta}}.
\]  

If \( 0 > \alpha > \beta \), we have

\[
\pm(\alpha + \beta)(\xi - \xi_0) = 2\sqrt{-\alpha} \arctan \frac{\sqrt{u + r_0}}{\sqrt{-\alpha}} - 2\sqrt{-\beta} \arctan \frac{\sqrt{u + r_0}}{\sqrt{-\beta}}.
\]  

If \( \alpha > 0 > \beta \), we have

\[
\pm(\alpha + \beta)(\xi - \xi_0) = \sqrt{\alpha} \ln \frac{\sqrt{u + r_0} - \sqrt{\alpha}}{\sqrt{u + r_0} + \sqrt{\alpha}} - 2\sqrt{-\beta} \arctan \frac{\sqrt{u + r_0}}{\sqrt{-\beta}}.
\]

Case 4. In all other cases such as \( F(u) = (u - \alpha)^2(u - \beta)(u - \gamma) \) and so on, the corresponding solutions can be expressed with elliptic functions and hyper-elliptic functions. We omit them for briefly.
6 Classifications of solutions to some more general nonlinear differential equations

There are many nonlinear differential equations which can be reduced to an integrable ODE as follows

\[ u''(\xi) + p(u)(u'(\xi))^2 + q(u) = 0. \] (210)

These equations include High order KdV equation \([50]\)

\[ u_t + u_x + \alpha uu_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha \beta (\rho_2 uu_{xxx} + \rho_3 u_x u_{xx}) = 0; \] (211)

Coupled KdV equation \([51]\)

\[ u_t + \varepsilon_1 v_x + \varepsilon_2 u^2 u_x + \varepsilon_3 u_{xxx} = 0, \] (212)
\[ v_t + \eta_1(uv)_x + \eta_2 vv_x = 0; \] (213)

Coupled mKdV equation \([52]\)

\[ u_t + \varepsilon_1 vv_x + \varepsilon_2 uu_x + \varepsilon_3 u_{xxx} = 0, \] (214)
\[ v_t + \eta_1(uv)_x + \eta_2 vv_x = 0, \] (215)

and so on. We give the general solution of the Eq. \((210)\) in the following:

**Lemma** \([53]\): The general solution of ODE \((210)\) is as follows:

\[ \pm (\xi - \xi_0) = \int \frac{du}{\sqrt{\exp(-2 \int p(u)du)[c - 2 \int q(u) \exp(2 \int p(u)du)du]}}, \] (216)

where \(c\) and \(\xi_0\) are two arbitrary constants.

We take the coupled KdV equation as an example to illustrate our method for shortly. Other equations can be dealt with similarly.

**Example**: Classification of all single traveling wave solutions to coupled KdV equation.

Take the traveling wave transformation as \(u = u(\xi), v = v(\xi), \ \xi = kx + \omega t\), its reduced equations are as follows:

\[ \gamma k^3 u'' + \frac{k \beta}{2} u^2 - \frac{k \alpha}{2} v^2 + \omega u + c_0 = 0, \] (217)

\[ u = -\frac{\delta}{2 \rho} v - \frac{c_1}{\rho k} - \frac{\omega}{\rho k}, \] (218)

where \(c_0\) and \(c_1\) are two integral constants. There are two cases to be discussed.
Case 1. \( c_1 = 0 \). We have \( u = -\frac{\delta}{\rho} v - \frac{\omega}{\rho k} \), and \( v \) satisfies the following equation

\[
(v')^2 = b_3 v^3 + b_2 v^2 + b_1 v + b_0,
\]

where

\[
b_3 = \frac{2\rho}{\gamma k^2 \delta} \left( \frac{\alpha k}{3} + \frac{\beta k \delta^2}{12 \rho^2} \right), \quad b_2 = \frac{\beta \delta \omega - \delta \omega \rho}{\gamma k^2 \rho^2}, \quad b_1 = \frac{2\rho}{\gamma k^2 \delta} \left\{ \frac{\beta \omega^2}{k \rho^2} - \frac{2\omega^2}{k \rho} + 2c_0 \right\},
\]

and \( b_0 \) is an arbitrary constant. Denote \( d_2 = \frac{b_2}{b_3}, d_1 = \frac{b_1}{b_3}, d_0 = \frac{b_0}{b_3} \). \( F(v) = v^3 + d_2 v^2 + d_1 v + d_0 \), and \( \Delta = -27d_0^2 - 4(-\frac{c_0}{3})^3(-\frac{c_0}{3})^{-1} \), \( D_1 = -\frac{2\alpha}{3}(-\frac{2\alpha}{3})^{-\frac{1}{2}} \).

Here \( \Delta \) and \( D_1 \) make up a complete discrimination system for \( F(v) \). We give the classification of all solutions to the Eq. (219) as follows:

Case 1.1: If \( \Delta = 0, D_1 < 0 \), then we have

\[
F(v) = (v - \alpha)^2(v - \beta), \alpha \neq \beta.
\]

If \( v > \beta \), the solutions are as follows:

\[
v = (\alpha - \beta) \tanh^2 \left( \frac{b_3 \sqrt{\alpha - \beta}}{2} (\xi - \xi_0) \right) + \beta, \quad \alpha > \beta; \quad (221)
\]

\[
v = (\alpha - \beta) \coth^2 \left( \frac{b_3 \sqrt{\alpha - \beta}}{2} (\xi - \xi_0) \right) + \beta, \quad \alpha > \beta; \quad (222)
\]

\[
v = (\beta - \alpha) \sec^2 \left( \frac{b_3 \sqrt{\beta - \alpha}}{2} (\xi - \xi_0) \right) + \alpha, \quad \alpha < \beta. \quad (223)
\]

Case 1.2: If \( \Delta = 0, D_1 = 0 \), then we have \( F(v) = (v - \alpha)^3 \), the solution is as follows:

\[
v = 4b_3 (\xi - \xi_0)^{-2} + \alpha. \quad (224)
\]

Case 1.3: If \( \Delta > 0, D_1 < 0 \), then \( F(v) = (v - \alpha)(v - \beta)(v - \gamma) \), we suppose that \( \alpha < \beta < \gamma \). When \( \alpha < v < \beta \), we have

\[
v = \alpha + (\beta - \alpha) \sinh^2 \left( \frac{b_3 \sqrt{\gamma - \alpha}}{2} (\xi - \xi_0) \right), m; \quad (225)
\]

when \( v > \gamma \), we have

\[
v = \frac{\gamma - \beta \sinh^2 \left( \frac{b_3 \sqrt{\gamma - \alpha}}{2} (\xi - \xi_0) \right), m)}{\cosh^2 \left( \frac{b_3 \sqrt{\gamma - \alpha}}{2} (\xi - \xi_0) \right), m)}, \quad (226)
\]

where \( m^2 = \frac{\beta - \alpha}{\gamma - \alpha} \).
Case 1.4: If $\Delta < 0$, then $F(v) = (v - \alpha)(v^2 + pv + q), \quad p^2 - 4q < 0$. We have

$$v = \alpha - \sqrt{\alpha^2 + p\alpha + q} + \frac{2\sqrt{\alpha^2 + p\alpha + q}}{1 + \cn((\alpha^2 + p\alpha + q)^{\frac{1}{4}}((\xi - \xi_0), m)}],$$

(227)

where $m^2 = \frac{1}{2}(1 - \frac{\alpha^2 + \frac{p^2}{4}}{\sqrt{\alpha^2 + p\alpha + q}})$.

Case 2. $c_1 \neq 0$. According to the lemma, then $v$ satisfies the following integral form:

$$\pm(\xi - \xi_0) = \int \frac{(u^2 - \frac{4\alpha}{\sqrt{\alpha^2 + p\alpha + q}})dv}{v\sqrt{a_8v^8 + \cdots + a_1v + a_0}},$$

(228)

where $a_1, \cdots, a_8$ can be expressed by the corresponding parameters. Denote $F(v) = a_8v^8 + \cdots + a_1v + a_0$, its complete discrimination system is too complex to write. We omit these solutions for briefly.

7 Discussions and conclusions

In Ref. [16], Fan proposed an expanse method to seek for the single traveling wave solutions to nonlinear partial differential equations. His key idea is to assume that the solution has a polynomial form

$$u = a_n\phi^n + \cdots + a_1\phi + a_0$$

of a function $\phi$ which satisfies an ODE as follows:

$$\phi' = \pm \sqrt{c_r\phi^r + \cdots + c_1\phi + c_0},$$

(229)

According to the balance principle, a relation of $n$ and $r$ can be obtained. In his paper, Fan take a special case of $r = 4$ as an example to apply his method. In the end of his paper, he said "we only have investigated a special case when $r = 4$. The details for those cases will be investigated in our future works". In fact, there are two problems need to answer. One problem is whether or not there exist new solutions in the cases of $r > 4$ for some equations. The second problem is how to prove some solutions are new in the cases of $r > 4$ if the equation considered can’t be directly reduced to an integral form.

According to the above analysis to direct integral method, we can make a conclusion that there aren’t new solutions in the cases of $r > 4$ for those equations which can be directly reduced to an integral form. Thus we have answered the Fan’s first problem.

Direct integral method is very natural and simple for solving traveling wave solutions to nonlinear partial differential equations. But it is just complete discrimination system for polynomial make direct integral method becomes an efficient and powerful method. A lot of nonlinear mathematical physics equations can be dealt with by direct integral method, their all traveling wave solutions can be classified. These complete results can’t be obtained by using of
other methods. On the other hand, a lot of nonlinear evolution equations such as (1+1)-dimensional Boussinesq equation and (3+1)-dimensional KP equation and so on, can be reduced to integral forms by setting some integral constants to zeroes. For those equations, we can not give their all traveling wave solutions, but we can still obtain their abundant exact solutions. For example, we consider the (2+1)-dimensional boussinesq equation

\[ u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} - u_{yy} = 0. \]  

(230)

Instituting traveling wave transformation \( u = u(\xi), \xi = k_1 x + k_2 y + \omega t \) into Eq. (230) and integrating twice yield

\[ k_1^4 u'' = (\omega^2 - k_1^2 - k_2^2)u + 3k_1^2 u^2 + c_2 \xi + c_1. \]  

(231)

where \( c_1 \) and \( c_2 \) are two arbitrary constants. If we take \( c_2 = 0 \), then we can apply the direct integral method to it. In fact, integrating Eq. (231) once yields

\[ \pm k_1^2 (\xi - \xi_0) = \int \frac{du}{\sqrt{k_1^2 u^3 + (\omega^2 - k_1^2 - k_2^2)u^2 + c_1 u + c_0}}, \]  

(232)

where \( c_0 \) is an arbitrary constant. According to Ref.[19], we can easily to give classification of all solutions to Eq. (232). Thus it is easy to obtain abundant traveling wave solutions to Eq. (230). For briefly, we omit them.

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