Moments of a single entry of circular orthogonal ensembles and Weingarten calculus

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Abstract

Consider a symmetric unitary random matrix \( V = (v_{ij})_{1 \leq i,j \leq N} \) from a circular orthogonal ensemble. In this paper, we study moments of a single entry \( v_{ij} \). For a diagonal entry \( v_{ii} \) we give the explicit values of the moments, and for an off-diagonal entry \( v_{ij} \) we give leading and subleading terms in the asymptotic expansion with respect to a large matrix size \( N \). Our technique is to apply the Weingarten calculus for a Haar-distributed unitary matrix.

Keywords: circular orthogonal ensemble, Weingarten calculus, random matrix, moments

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1 Introduction and results

In random matrix theory, given a random matrix \( X = (x_{ij})_{1 \leq i,j \leq N} \), the study of distributions for its eigenvalues is the main theme. Another typical theme is to study joint moments for matrix entries \( \mathbb{E}[x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_n,j_n}] \) or

\[
\mathbb{E}[x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_n,j_n} x_{i_1',j_1'} x_{i_2',j_2'} \cdots x_{i_m',j_m'}].
\]

(1.1)

When \( x_{ij} \) are Gaussian random variables, the technique of the calculation for the joint moments is called the Wick calculus. For a Wishart matrix and inverted Wishart matrix, such moments are calculated in [4, 5, 8, 10]. For a Haar-distributed matrix from classical Lie groups, the corresponding technique is called Weingarten calculus and developed in

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In those calculations, we employ deep combinatorics of permutations and matchings and representation theory of symmetric groups and hyperoctahedral groups.

There are three much-studied circular ensembles: circular orthogonal ensembles (COEs), circular unitary ensembles (CUEs), and circular symplectic ensembles (CSEs). The density function for their eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$ is proportional to $\prod_{1 \leq i < j \leq N} |\Lambda_i - \Lambda_j|^\beta$ with $\beta = 1$ (COE), $\beta = 2$ (CUE), and $\beta = 4$ (CSE). The CUE is nothing but the unitary group equipped with its Haar probability measure. See [14] for details.

In this paper, we consider the COE, which is the probability space of symmetric unitary matrices by the property of being invariant under automorphisms

$$V \rightarrow tU_0V U_0^*,$$

where $U_0$ is a unitary matrix. It is also a realization of the symmetric space $U(N)/O(N)$, where $U(N)$ and $O(N)$ is the unitary and real orthogonal group of degree $N$, respectively. If $U$ is a Haar-distributed unitary matrix from the unitary group $U(N)$, i.e., if $U$ is a CUE matrix, a random matrix $V$ from the COE can be given by $V = tUU^*$. In other words, if $V = (v_{ij})$ and $U = (u_{ij})$ then

$$v_{ij} = \sum_{k=1}^{N} u_{ki}u_{kj}.$$ \hfill (1.2)

Therefore a joint moment of the form (1.1) with $X = V$ can be written as a sum of joint moments of the forms (1.1) with $X = U$. Since the joint moments for a CUE matrix $U$ can be computed by the Weingarten calculus [1, 2, 13, 17], we can compute the joint moments for the COE matrix $V$, in principle. However, in general, their computations seem quite complicated.

In the light of that situation, in this paper we focus on only moments $\mathbb{E}[|v_{ij}|^{2n}]$ for a single matrix entry $v_{ij}$. Diagonal entries $v_{ii}$ and off-diagonal entries $v_{ij}$ have different distributions. For a diagonal entry, we give the explicit expression for $\mathbb{E}[|v_{ii}|^{2n}]$ as follows.

**Theorem 1.1.** Let $V = (v_{ij})_{1 \leq i,j \leq N}$ be an $N \times N$ COE matrix. For positive integers $i, N, n$ with $1 \leq i \leq N$,

$$\mathbb{E}[|v_{ii}|^{2n}] = \frac{2^n n!}{(N+1)(N+3) \cdots (N+2n-1)}.$$  

Unfortunately, we could not obtain a similar closed expression for an off-diagonal entry $v_{ij}$ ($i \neq j$). However, we give the leading and sub-leading terms in the asymptotic expansion of $\mathbb{E}[|v_{ij}|^{2n}]$ as $N \rightarrow \infty$.

**Theorem 1.2.** Fix positive integers $i, j, n$ with $i \neq j$. Let $V^{(N)} = (v_{ij}^{(N)})$, $N \geq 1$, be a sequence of $N \times N$ COE matrices. As the matrix size $N$ goes to the infinity,

$$\mathbb{E}[|v_{ij}^{(N)}|^{2n}] = n! \left( N^{-n} - \frac{n(n+1)}{2} N^{-n-1} \right) + O(N^{-n-2}).$$
Note that Theorem 1.1 implies
\[ \mathbb{E}[|v_{ii}^{(N)}|^{2n}] = 2^n n! (N^{-n} - n^2 N^{-n-1}) + O(N^{-n-2}). \]

The following result is obtained in [6, Corollary 1.1] in an analytic approach. We have its new algebraic proof via CUEs from Theorem 1.1 and 1.2.

**Corollary 1.3.** Fix positive integers \( i, j, n \) with \( i \neq j \). Let \( V^{(N)} = (v_{ij}^{(N)}) \), \( N \geq 1 \), be a sequence of \( N \times N \) COE matrices. Then, as \( N \to \infty \), both \( \sqrt{N/2} v_{ii}^{(N)} \) and \( \sqrt{N} v_{ij}^{(N)} \) converge to a standard complex Gaussian random variable.

This paper is organized as follows. In Section 2, we review the Weingarten calculus for the unitary group. In Section 3, the proofs for main theorems are given.

As a closing statement, we give facts for the moment \( \mathbb{E}[|x_{ij}|^{2n}] \), where \( x_{ij} \) is a matrix entry of a Haar-distributed unitary or orthogonal matrix. Compare with our theorems.

Let \( 1 \leq i, j \leq N \). For a Haar-distributed unitary matrix \( U = (u_{ij}) \) of size \( N \),
\[ \mathbb{E}[|u_{ij}|^{2n}] = \frac{n!}{N(N+1) \cdots (N+n-1)}; \]
For a Haar-distributed orthogonal matrix \( O = (o_{ij}) \) of size \( N \),
\[ \mathbb{E}[o_{ij}^{2n}] = \frac{(2n-1)!!}{N(N+2) \cdots (N+2n-2)}. \]
They are obtained in [15, 2].

**Remark 1.** After the submission of the present article, the author obtained stronger results in [12], which includes the theorems of this article.

## 2 Weingarten calculus for unitary groups

### 2.1 Weingarten functions

A *partition* \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of a positive integer \( n \) is a weakly-decreasing sequence of nonnegative integers satisfying \( n = |\lambda| := \sum_{i \geq 1} \lambda_i \). Then we write \( \lambda \vdash n \). The number of nonzero \( \lambda_i \) is called the length of \( \lambda \) and written as \( \ell(\lambda) \): \( \ell(\lambda) = |\{i \geq 1 \mid \lambda_i > 0\}| \). Put \( \lambda! = \lambda_1! \lambda_2! \cdots \).

The *Weingarten function* for the unitary group \( U(N) \) is a class function on the symmetric group \( S_n \) and given by
\[ Wg_{n}^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} \prod_{(i,j) \in \lambda} \frac{f^{\lambda}}{(N + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in S_n). \]
Here $\chi^\lambda$ is the irreducible character of $S_n$ associated with $\lambda$ and $f^\lambda$ is its degree (i.e., $f^\lambda$ is the value of $\chi^\lambda$ at the identity permutation $\text{id}_n$). The product $\prod_{i,j=1}^{\lambda} \prod_{i,j=1}^{\lambda}$ stands for $\prod_{i,j=1}^{\lambda}$.

We note that $W_{g_n}^{U(N)}(\sigma^{-1}) = W_{g_n}^{U(N)}(\sigma)$ and $W_{g_n}^{U(N)}(\sigma\tau) = W_{g_n}^{U(N)}(\tau\sigma)$ for all $\sigma, \tau \in S_n$.

**Lemma 2.1** ([1]). Fix a positive integer $n$ and permutation $\sigma \in S_n$. As $N \to \infty$, the Weingarten function $W_{g_n}^{U(N)}(\sigma)$ has the following asymptotic properties.

1. If $\sigma$ is the identity $\text{id}_n$, then $W_{g_n}^{U(N)}(\text{id}_n) = N^{-n} + O(N^{-n-2})$;

2. If $\sigma$ is a transposition, then $W_{g_n}^{U(N)}(\sigma) = -N^{-n-1} + O(N^{-n-3})$;

3. Otherwise, $W_{g_n}^{U(N)}(\sigma) = O(N^{-n-2})$.

For a finite set $I$ of positive integers, denote by $S_I$ the symmetric group acting on $I$. The **Young subgroup** $S_\lambda$ associated with $\lambda \vdash n$ is the subgroup of $S_n$ defined by

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots.$$ 

In particular, $S_{(n)} = S_n$ and $S_{(n^k)} = \{\text{id}_n\}$. The cardinality of $S_\lambda$ is $\lambda!$.

**Lemma 2.2.** Let $\mu \vdash n$ and let $S_\mu$ be its corresponding Young subgroup. Then we have

$$\sum_{\sigma \in S_\mu} W_{g_n}^{U(N)}(\sigma) = \frac{\mu!}{n!} \sum_{\ell(\lambda) \leq N}^{\lambda^\mu} f^\lambda K_{\lambda\mu} \prod_{(i,j) \in \lambda} (N + j - i).$$

Here $K_{\lambda\mu}$ is the Kostka number (i.e. the number of semi-standard tableaux of shape $\lambda$ and weight $\mu$; see [2, Chapter I]).

**Proof.** From the definition of the Weingarten function we have

$$\sum_{\sigma \in S_\mu} W_{g_n}^{U(N)}(\sigma) = \frac{|S_\mu|}{n!} \sum_{\ell(\lambda) \leq N}^{\lambda^\mu} f^\lambda \langle \text{res}_\mu^{S_n} \chi^\lambda, \xi_\mu \rangle_{S_n},$$

where $\xi_\mu$ is the trivial character of $S_\mu$ and $\langle \cdot, \cdot \rangle_{S_\mu}$ is the scalar product on $\mathbb{C}[S_\mu]$. It follows from the Frobenius reciprocity that $\langle \text{res}_\mu^{S_n} \chi^\lambda, \xi_\mu \rangle_{S_n} = \langle \chi^\lambda, \text{ind}_\mu^{S_n} \xi_\mu \rangle_{S_n}$, which coincides with the scalar product $\langle s_\lambda, h_\mu \rangle$ in the algebra of symmetric functions (see [3, Chapter I.7]), where $s_\lambda$ and $h_\lambda$ are the Schur and complete symmetric function, respectively. Hence, since $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$ (see [3, Chapter I (5.14)]), we have $\langle \text{res}_\mu^{S_n} \chi^\lambda, \xi_\mu \rangle_{S_n} = K_{\lambda\mu}$. \qed
2.2 Some identities for unitary matrix integrals

For a permutation \( \sigma \in S_n \) and a sequence \( i = (i_1, \ldots, i_n) \in [N]^n \), we put

\[
i^{\sigma} = (i_{\sigma(1)}, \ldots, i_{\sigma(n)}).
\]

This gives the action of \( S_n \) on \([N]^n\) from the right.

For two sequences \( i = (i_1, \ldots, i_n) \) and \( j = (j_1, \ldots, j_n) \) in \([N]^n\), we write \( i \sim j \) if \( j \) is a permutation of \( i \), so that there exists a permutation \( \sigma \in S_n \) satisfying \( j = i^{\sigma} \). For example, \((2, 3, 2, 1, 3) \sim (3, 3, 2, 2, 1)\).

Given \( U = (u_{ij}) \in U(N) \) and four sequences \( i = (i_1, i_2, \ldots, i_n) \), \( j = (j_1, \ldots, j_n) \), \( i' = (i'_1, \ldots, i'_m) \), \( j' = (j_1, \ldots, j'_m) \) of indices in \([N]\), we put

\[
U(i, j|i', j') = u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_m j'_m}.
\]

**Lemma 2.3** (\( \Pi \)). Let \( U \) be an \( N \times N \) Haar-distributed unitary matrix and let \( i = (i_1, i_2, \ldots, i_n) \), \( j = (j_1, \ldots, j_n) \), \( i' = (i'_1, \ldots, i'_m) \), \( j' = (j_1, \ldots, j'_m) \) be four sequences of indices in \([N]\). Then \( \mathbb{E}[U(i, j|i', j')] \) vanishes unless \( n = m \), \( i \sim i' \), and \( j \sim j' \). In this case,

\[
\mathbb{E}[U(i, j|i', j')] = \sum_{\sigma \in S_n} \sum_{\tau \in S_n} W_{\sigma} U(N)(\sigma \tau^{-1}).
\]

A sequence \( i \in [N]^n \) can be written as a rearrangement of a sequence of the form

\[
(l_1, l_1, \ldots, l_1, l_2, \ldots, l_2, \ldots, l_k, \ldots, l_k),
\]

where \( l_1, \ldots, l_k \in [N] \) are all distinct and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) is a partition of \( n \). Then we call \( \lambda \) the type of \( i \). For example, both \((7, 7, 7, 2, 2, 2, 3) \) and \((2, 5, 4, 2, 5, 5, 2) \) are of type \((3, 3, 1) \) \( \sim 7 \). Two sequences \( i \) and \( j \) have the same type if \( i \sim j \).

**Proposition 2.4.** Let \( U = (u_{ij})_{1 \leq i, j \leq N} \) be a Haar-distributed unitary matrix from \( U(N) \) and let \( i \in [N]^n \) be a sequence of type \( \mu \). Put \((1^n) := (1, 1, \ldots, 1)\) with \( n \) times. Then

\[
\mathbb{E}[|u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_l j_l}|^2] = \mathbb{E}[U((1^n), i)|(1^n), i)] = \frac{\mu!}{N(N + 1) \cdots (N + n - 1)}.
\]

**Proof.** Let \( i_\mu \) be the sequence in \([N]^n\) given by

\[
i_\mu = (\underbrace{1, 1, \ldots, 1}_{\mu_1}, 2, 2, \ldots, 2, \underbrace{l, l, \ldots, l}_{\mu_l}), \quad l = \ell(\mu).
\]

For each \( \sigma \in S_n \), \((i_\mu)^{\sigma} = i_\mu \) if and only if \( \sigma \in S_\mu \). Therefore it follows from Lemma 2.3 that

\[
\mathbb{E}[U((1^n), i)|(1^n), i)] = \mathbb{E}[U((1^n), i_\mu)|(1^n), i_\mu)]
\]

\[
= \sum_{\sigma \in S_n} \sum_{\tau \in S_\mu} W_{\sigma} U(N)(\sigma \tau^{-1}) = |S_\mu| \sum_{\sigma \in S_n} W_{\sigma} U(N)(\sigma).
\]
Since $K_{\lambda,(n)} = \delta_{\lambda,(n)}$ and $f^{(n)} = 1$, Lemma \[2.2\] at $\mu = (n)$ shows

$$\sum_{\sigma \in S_n} W_{n}^{U(N)}(\sigma) = \frac{1}{N(N + 1)(N + 2) \cdots (N + n - 1)}$$

from which the result follows.

We are not going to apply the following proposition. However, it is important because it gives a joint moment for diagonal entries of a CUE matrix $U$.

**Proposition 2.5.** Let $U = (u_{ij})_{1 \leq i, j \leq N}$ be a Haar-distributed unitary matrix from $U(N)$ and let $\mathbf{i} = (i_1, \ldots, i_n) \in [N]^n$ be a sequence of type $\mu$. Then

$$\mathbb{E}[|u_{i_1i_1}u_{i_2i_2} \cdots u_{i_ni_n}|^2] = \mathbb{E}[U(\mathbf{i}, \mathbf{i}^t, \mathbf{i}, \mathbf{i}^t)] = \frac{\langle \mu \rangle^2}{n!} \sum_{\lambda \vdash n} f^{\lambda} K_{\lambda, \mu}$$

**Proof.** The discussion similar to the previous proposition gives

$$\mathbb{E}[U(\mathbf{i}, \mathbf{i}^t, \mathbf{i}, \mathbf{i}^t)] = \sum_{\sigma \in S_n} \mathbb{E}[W_{\mu}^{U(N)}(\sigma \tau^{-1})] = |S_{\mu}| \sum_{\sigma \in S_{\mu}} W_{\mu}^{U(N)}(\sigma)$$

from which the statement follows by Lemma \[2.2\].

## 3 Proofs of main theorems

### 3.1 General properties

We use $X \overset{d}{=} Y$ to denote that random variables (or random matrices) $X$ and $Y$ have the same distribution. Let $V = (v_{ij})_{1 \leq i, j \leq N}$ be a COE matrix. Let $M(\pi) = (\delta_{i, \pi(j)})_{1 \leq i, j \leq N}$ be the permutation matrix associated to $\pi \in S_N$. Since $V \overset{d}{=} t^i M(\pi) V M(\pi)$ by the invariant property for the COE, we have $V \overset{d}{=} (v_{\pi(i), \pi(j)})_{1 \leq i, j \leq N}$. In particular, if $1 \leq i \neq j \leq N$, then $v_{ii} \overset{d}{=} v_{11}$ and $v_{ij} \overset{d}{=} v_{12}$.

Using a Haar-distributed unitary matrix $U = (u_{ij})_{1 \leq i, j \leq N}$, we can write $V = t^i U$. Let $\mathbf{j} = (j_1, j_2, \ldots, j_{2n}) \in [N]^{2n}$ and $\mathbf{j}' = (j'_1, j'_2, \ldots, j'_{2m}) \in [N]^{2m}$. Since $v_{ij} = \sum_k u_{ki} u_{kj}$, we have

$$\mathbb{E}[v_{j_1j_2}v_{j_3j_4} \cdots v_{j_{2m-1}j_{2m}} v_{j_1'j_2'}v_{j_3'j_4'} \cdots v_{j_{2m-1}'j_{2m}'}] = \sum_{k \in [N]^n} \sum_{k' \in [N]^m} \mathbb{E}[U(\mathbf{k}, \mathbf{j}^t ; \mathbf{k}', \mathbf{j}')]$$

where

$$\mathbf{k} = (k_1, k_2, \ldots, k_n), \quad \tilde{\mathbf{k}} = (k_1, k_2, \ldots, k_n), \quad \mathbf{k}' = (k'_1, k'_2, \ldots, k'_m), \quad \tilde{\mathbf{k}}' = (k'_1, k'_2, \ldots, k'_m).$$

Hence, by Lemma \[2.3\] we have
Lemma 3.1. $\mathbb{E}[v_{j_1,j_2}v_{j_3,j_4} \cdots v_{j_{2n-1},j_{2n}}v_{j_1,j_2}v_{j_3,j_4} \cdots v_{j_{2m-1},j_{2m}}] \text{ vanishes unless } n = m \text{ and } j \sim j'.$

We now suppose $j \sim j'.$ By Lemma 2.3 again, $\mathbb{E}[U(\tilde{k}, j; \tilde{k}, j')]$ vanishes unless $\tilde{k} \sim \tilde{k}'$, so that unless $k \sim k'$. Therefore

$$\mathbb{E}[v_{j_1,j_2}v_{j_3,j_4} \cdots v_{j_{2n-1},j_{2n}}v_{j_1,j_2}v_{j_3,j_4} \cdots v_{j_{2m-1},j_{2m}}]$$

$$= \sum_{k \in [N]^n} \sum_{k' \in [N]^n} \mathbb{E}[U(\tilde{k}, j; \tilde{k}', j')]$$

$$= \sum_{k \in [N]^n} \mathbb{E}[U(\tilde{k}, j; \tilde{k}, j') \prod_{\ell < k} \mathbb{E}[W_{2n}(\sigma_{\tau^{-1}})]]$$

3.2 Diagonal entry

Theorem 3.2. (see Theorem 1.1.) Let $V = (v_{ij})_{1 \leq i, j \leq N}$ be an $N \times N$ COE matrix. For positive integers $i, N, n$ with $1 \leq i \leq N$,

$$\mathbb{E}[|v_{ii}|^{2n}] = \frac{2^n n!}{(N + 1)(N + 3) \cdots (N + 2n - 1)}.$$

Proof. The equation (3.1) gives

$$\mathbb{E}[|v_{ii}|^{2n}] = \sum_{k \in [N]^n} \sum_{k' \in [N]^n} \mathbb{E}[U(\tilde{k}, (i^{2n})|\tilde{k}', (i^{2n}))] = \sum_{k \in [N]^n} \mathbb{E}[U(\tilde{k}, (i^{2n})|\tilde{k}, (i^{2n}))] \sum_{k' \sim k} 1.$$

Given a sequence $k$ of type $\mu$, the number of sequences $k'$ in $[N]^n$ with $k \sim k'$ is $\frac{n!}{\mu!}$, and the type of $\tilde{k}$ is $2\mu$. Hence, by Proposition 2.4 the last equation equals

$$\frac{n!}{N(N + 1)(N + 2) \cdots (N + 2n - 1)} \sum_{\mu \in \mathbb{N}} \sum_{k \in [N]^n} \frac{(2\mu)!}{\mu!} \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!!.$$

To end the proof of the theorem, it is enough to show the identity

$$\sum_{\mu \in \mathbb{N}} \sum_{k \in [N]^n} \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!! = N(N + 2)(N + 4) \cdots (N + 2n - 2),$$

which will be proved in the next subsection. \qed
Corollary 3.3. Let $V^{(N)} = (v_{ij}^{(N)})$, $N \geq 1$, be a sequence of COE matrices. Fix positive integers $i$ and $n$. As $N \to \infty$,
\[
E[|v_{ii}^{(N)}|^2] = 2^n n!(N^{-n} - n^2 N^{-n-1}) + O(N^{-n-2}).
\]

Corollary 3.4. (see Theorem 1.3.) Let $V^{(N)} = (v_{ij}^{(N)})$, $N \geq 1$, be a sequence of COE matrices. Fix a positive integer $i$. As $N \to \infty$, the random variable $\sqrt{N/2}v_{ii}^{(N)}$ converges to a standard complex Gaussian random variable in distribution.

Proof. Recall that a random variable $Z$ distributed to a standard complex normal distribution satisfies $E[Z^n \overline{Z^m}] = \delta_{n,m} n!$ with any positive integers $n, m$. Consider a random variable $Z_N = \sqrt{N/2}v_{ii}^{(N)}$. From the last corollary and Lemma 3.1 we have $E((Z_N)^m(\overline{Z_N})^n) = 0$ ($m \neq n$) and $\lim_{N \to \infty} E((Z_N)^n(\overline{Z_N})^n) = n!$. This implies that random variables $\{Z_N\}_{N \geq 1}$ converge in distribution to a standard complex normal variable (see, e.g., the proof of Theorem 3.1 in [16]).

3.3 Combinatorial lemmas

The purpose in this subsection is to give a remaining proof of (3.2).

Given a non-empty finite set $I$ of positive integers, denote by $M(I)$ the set of all unordered pairings on $\bigcup_{i \in I} \{2i-1, 2i\}$. For example, $M(\{1, 3\})$ consists of three pairings $\{\{1, 2\}, \{5, 6\}\}$, $\{\{1, 5\}, \{2, 6\}\}$, $\{\{1, 6\}, \{2, 5\}\}$.

Given $m \in M([n])$, we attach a graph $\Gamma(m)$ with vertices $1, 2, \ldots, 2n$ and with the edge set
\[
\{\{2k-1, 2k\} | k \in [n]\} \cup m.
\]
Denote by $\kappa(m)$ the number of connected components of $\Gamma(m)$.

Lemma 3.5. For positive integers $n, N$, we have
\[
\sum_{m \in M([n])} N^{\kappa(m)} = N(N+2)(N+4) \cdots (N+2n-2).
\]

Proof. We show it by induction on $n$. When $n = 1$, we have $\sum_{m \in M([1])} N^{\kappa(m)} = N$, so that the claim holds true.

Suppose the statement at $n$ holds true and consider the $n+1$ case. For $k = 1, 2, \ldots, 2n+1$, we set
\[
\mathcal{M}_k([n+1]) = \{m \in M([n+1]) | \{k, 2n+2\} \in m\}.
\]
We construct a bijection $\mathcal{M}_k([n+1]) \ni m \mapsto m' \in M([n])$ as follows. If $k = 2n+1$ and $m \in \mathcal{M}_k([n+1])$, we define $m'$ to be the pairing obtained by removing $\{2n+1, 2n+2\}$ from $m$. If $1 \leq k \leq 2n$ and $m \in \mathcal{M}_k([n+1])$, we define $m'$ to be the pairing obtained by
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removing \{k, 2n + 2\} and a pair \{i, 2n + 1\} (with some \(i\)) from \(m\) and by adding \(\{i, k\}\).

It is easy to see that the map \(\mathcal{M}_k([n+1]) \ni m \mapsto m' \in \mathcal{M}([n])\) is bijective and that

\[
\kappa(m) = \begin{cases} 
\kappa(m') + 1 & \text{if } m \in \mathcal{M}_{2n+1}([n+1]), \\
\kappa(m') & \text{if } m \in \mathcal{M}_k([n+1]) \text{ and } k = 1, 2, \ldots, 2n.
\end{cases}
\]

Hence it follows from the induction assumption that

\[
\sum_{m \in \mathcal{M}([n+1])} N^{\kappa(m)} = \sum_{m \in \mathcal{M}_{2n+1}([n+1])} N^{\kappa(m)} + \sum_{k=1}^{2n} \sum_{m \in \mathcal{M}_k([n+1])} N^{\kappa(m)}
\]

\[
= \sum_{n \in \mathcal{M}([n])} N^{\kappa(n)+1} + \sum_{k=1}^{2n} \sum_{n \in \mathcal{M}([n])} N^{\kappa(n)}
\]

\[
= N(N + 1) \cdots (N + 2n - 2)(N + 2n),
\]

from which the \(n + 1\) case follows.

\[\square\]

**Lemma 3.6.** Let \(\mu\) be a partition of \(n\) and let \((i_1, \ldots, i_{2n}) \in [N]^{2n}\) be a sequence of type \(2\mu = (2\mu_1, 2\mu_2, \ldots)\). Then

\[
\sum_{m \in \mathcal{M}([n])} \left( \prod_{(p,q) \in m} \delta_{i_p,i_q} \right) = \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!!. 
\]

**Proof.** Given a partition \(\mu\) of \(n\), we define \(\mathcal{M}(\mu)\) to be the set of all pairings \(m\) in \(\mathcal{M}([n])\) of the form

\[
m = m^{(1)} \sqcup m^{(2)} \sqcup \cdots,
\]

where \(m^{(1)} \in \mathcal{M}(\{1, 2, \ldots, \mu_1\})\), \(m^{(2)} \in \mathcal{M}(\{\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2\})\), and so on.

In order to prove the lemma, we can suppose

\[
(i_1, \ldots, i_{2n}) = i_{2\mu} = (1,1,\ldots,1,\underbrace{2,2,\ldots,2}_{2\mu_1},\ldots,\underbrace{1,1,\ldots,1}_{2\mu_l})
\]

because of

\[
\sum_{m \in \mathcal{M}([n])} \left( \prod_{(p,q) \in m} \delta_{i_p,i_q} \right) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \prod_{j=1}^{n} \delta_{i_{\sigma(2j-1)},i_{\sigma(2j)}}
\]

In that case, for each \(m \in \mathcal{M}([n])\), the product \(\prod_{(p,q) \in m} \delta_{i_p,i_q}\) does not vanish if and only if \(m \in \mathcal{M}(\mu)\). It follows the result since \(|\mathcal{M}(\mu)| = \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!!\).  

\[\square\]
Proof of Equation \((3.2)\). From Lemma 3.6

\[
\sum_{\mu \vdash n} \sum_{k \in [N]^n \text{(type of } k) = \mu} \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!! = \sum_{\mu \vdash n} \sum_{m \in \mathcal{M}([n])} \prod_{\{p,q\} \in m} \delta_{k_p, k_q}
\]

\[
= \sum_{m \in \mathcal{M}([n])} \sum_{k \in [N]^n \{p,q\} \in m} \prod \delta_{k_p, k_q}.
\]

Since it is immediate to see

\[
\sum_{k \in [N]^n \{p,q\} \in m} \prod \delta_{k_p, k_q} = N^{\kappa(m)},
\]

it follows from Lemma 3.5 that

\[
\sum_{\mu \vdash n} \sum_{k \in [N]^n \text{(type of } k) = \mu} \prod_{j=1}^{\ell(\mu)} (2\mu_j - 1)!! = \sum_{m \in \mathcal{M}([n])} N^{\kappa(m)} = N(N + 2) \cdots (N + 2n - 2),
\]

so that \((3.2)\) has been proved. It completes the proof of Theorem 1.1.

3.4 Off-diagonal entry

Let \(1 \leq i \neq j \leq N\). Put \(i = (i, j, i, j, \ldots, i, j) \in [N]^{2n}\). In a similar way to a diagonal-entry case, we have

\[
\mathbb{E}[|v_{ij}|^{2n}] = \sum_{\mu \vdash n} \frac{n!}{\mu!} \sum_{k \in [N]^n \text{(type of } k) = \mu} \mathbb{E}[U(k, i, k, i)]
\]

\[
= \sum_{\mu \vdash n} \frac{n!}{\mu!} \sum_{k \in [N]^n \text{(type of } k) = \mu} \sum_{\sigma \in S_{2\mu}} \sum_{\tau \in S_{2n}^*} W_{g_{\mathcal{U}(N)}}(\sigma \tau).
\]

where

\[
S_{2n}^* = S_{\{1,3,5,\ldots,2n-1\}} \times S_{\{2,4,6,\ldots,2n\}}
\]

is the stabilizer subgroup for \(i\). Since the number of sequences in \([N]^n\) of type \(\mu\) is

\[
\frac{n!}{\mu! \prod_{k \geq 1} m_k(\mu)! \cdot (N - \ell(\mu))!},
\]

we have
Lemma 3.8. Let $V = (v_{ij})_{1 \leq i,j \leq N}$ be a COE matrix. For positive integers $i, j, N, n$ with $1 \leq i \neq j \leq N$,

\begin{equation}
\mathbb{E}[|v_{ij}|^{2n}] = \sum_{\mu \vdash n} \frac{(n!)^2}{(\mu!)^2} \prod_{k \geq 1} m_k(\mu)! N(N - 1) \cdots (N - \ell(\mu) + 1) W(\mu, N)
\end{equation}

where

\begin{equation}
W(\mu, N) = \sum_{\sigma \in S_{2\mu}} \sum_{\tau \in S_{2n}} W_{\mu}^{(N)}(\sigma \tau).
\end{equation}

To see the asymptotics for $\mathbb{E}[|v_{ij}|^{2n}]$, we focus on $W(\mu, N)$ in the limit $N \to \infty$.

Lemma 3.8. Let $\mu \vdash n$. As $N \to \infty$,

$$W(\mu, N) = \begin{cases} N^{-2n} - n^2 N^{-2n-1} + O(N^{-2n-2}) & \text{if } \mu = (1^n) \\ 4N^{-2n} + O(N^{-2n-1}) & \text{if } \mu = (2, 1^{n-2}) \\ O(N^{-2n}) & \text{otherwise.} \end{cases}$$

Proof. We first consider the case where $\mu = (1^n)$. Let $\sigma \in S_{(2^n)} = S_{(1,2)} \times S_{(3,4)} \times \cdots \times S_{(2n-1,2n)}$ and $\tau \in S_{2n}^*$. It is easy to see that if $\sigma \tau = \text{id}_{2n}$ then $\sigma = \tau = \text{id}_{2n}$, and that if $\sigma \tau$ is a transposition then either $\sigma = \text{id}_{2n}$ or $\tau = \text{id}_{2n}$. Hence, from Lemma 2.1 it follows that $W((1^n), N) = N^{-2n} - cN^{-2n-1} + O(N^{-2n-2})$, where

$$c = |\{\sigma \in S_{(2^n)} \mid \text{transpositions}\}| + |\{\tau \in S_{2n}^* \mid \text{transpositions}\}| = n + 2 \binom{n}{2} = n^2,$$

so that the result at $\mu = (1^n)$ follows.

We next consider the case where $\mu = (2, 1^{n-2})$. Let $\sigma \in S_{2\mu} = S_{(1,2,3,4)} \times S_{(5,6)} \times \cdots \times S_{(2n-1,2n)}$ and $\tau \in S_{2n}$. Since $S_{2\mu} \cap S_{2n} = \{\text{id}_{2n}, (1 \, 3), (2 \, 4), (1 \, 3)(2 \, 4)\}$, the number of pairs $(\sigma, \tau)$ satisfying $\sigma \tau = \text{id}_{2n}$ is equal to 4. Hence, from Lemma 2.1 we have $W((2, 1^{n-2}), N) = 4N^{-2n} + O(N^{-2n-1})$.

The case where $\mu \neq (1^n), (2, 1^{n-2})$ is clear from Lemma 2.1.

We have the following leading and subleading terms in $\mathbb{E}[|v_{ij}|^{2n}]$ in the limit $N \to \infty$.

Theorem 3.9. (See Theorem 1.2.) Fix positive integers $i, j, n$ with $i \neq j$. Let $V^{(N)} = (v_{ij}^{(N)})$, $N \geq 1$, be a sequence of COE matrices. As $N \to \infty$,

$$\mathbb{E}[|v_{ij}^{(N)}|^{2n}] = n!N^{-n} - n! \frac{n(n+1)}{2} N^{-n-1} + O(N^{-n-2}).$$
Proof. We divide the sum in (3.3) into $\mu = (1^n)$, $\mu = (2, 1^{n-2})$, and other $\mu$’s. Since
$$N(N-1) \cdots (N-\ell(\mu)+1) = \begin{cases} N^n - \binom{n}{2} N^{n-1} + O(N^{n-2}) & \text{if } \mu = (1^n), \\ N^{n-1} + O(N^{n-2}) & \text{if } \mu = (2, 1^{n-2}), \\ O(N^{n-2}) & \text{otherwise}, \end{cases}$$

it follows from Lemma 3.8 that
$$\mathbb{E}[|v_{ij}^{(N)}|^{2n}] = n! \left( N^n - \binom{n}{2} N^{n-1} + O(N^{n-2}) \right) (N^{-2n} - n^2 N^{-2n-1} + O(N^{-2n-2}))$$
$$+ \frac{n!}{4} n(n-1)(N^{-n-1} + O(N^{-2})) (4N^{-2n} + O(N^{-2n-1}))$$
$$+ \sum_{\mu \neq (1^n),(2,1^{n-2})} O(N^{n-2}) \cdot O(N^{-2n}),$$
from which a straightforward calculation gives the result. \(\square\)

Corollary 3.10. (see Theorem 1.3.) Fix $i \neq j$. As $N \to \infty$, a sequence of the random variables $\{\sqrt{N}v_{ij}^{(N)}\}_{N \geq 1}$ converges to a standard complex Gaussian random variable in distribution.

Proof. The proof is similar to that of Corollary 3.4. \(\square\)

A Appendix: Examples of lower degrees

Let $V = (v_{ij})_{1 \leq i, j \leq N}$ be a COE matrix. We give examples of joint moments of small degrees.

A.1 Degree 1

Example 1. Let $j_1, j_2, j'_1, j'_2 \in [N]$. Consider the average $\mathbb{E}[v_{j_1j_2} v_{j'_1j'_2}]$.

1. $\mathbb{E}[|v_{ii}|^2] = \frac{2}{N+1}$ for each $1 \leq i \leq N$;

2. $\mathbb{E}[|v_{ij}|^2] = \frac{1}{N+1}$ for $1 \leq i \neq j \leq N$;

3. $\mathbb{E}[v_{j_1j_2} v_{j'_1j'_2}] = 0$ unless $(j_1, j_2) \sim (j'_1, j'_2)$.\footnote{The first case follows from Theorem 1.1. The second case follows from Proposition 3.7.}

The first case follows from Theorem 1.1. The second case follows from Proposition 3.7. The third case follows from Lemma 3.1. \(\square\)
A.2 Degree 2

Let $1 \leq i \neq j \leq N$ and let $j_1, j_2, j_3, j_4 \in [N]$ be distinct.

\begin{align*}
\text{E}[|v_{ii}|^4] &= \text{E}[|v_{11}|^4] = \frac{8}{(N + 1)(N + 3)}, \\
\text{E}[|v_{ij}|^4] &= \text{E}[|v_{12}|^4] = \frac{2}{N(N + 3)}, \\
\text{E}[|v_{ii}v_{jj}|^2] &= \text{E}[|v_{11}v_{22}|^2] = \frac{4(N + 2)}{N(N + 1)(N + 3)}, \\
\text{E}[v_{ij}^2v_{jj}^2] &= \text{E}[v_{12}^2v_{11}^2] = \frac{-4}{N(N + 1)(N + 3)}, \\
\text{E}[|v_{ij}v_{j_3j_4}|^2] &= \text{E}[|v_{12}v_{34}|^2] = \frac{N + 2}{N(N + 1)(N + 3)}.
\end{align*}

The equation (A.1) follows from Theorem 1.1.

For each partition $\mu$ of $n$, put $W_{g_4}(\mu) = W_{g_4}(\sigma)$, where $\sigma$ is a permutation in $S_n$ of cycle-type $\mu$. We use the following explicit expressions:

\begin{align*}
W_{g_4}(\mu) &= \frac{-5}{N(N^2 - 1)(N^2 - 4)(N^2 - 9)}; \\
W_{g_4}(\mu) &= \frac{2N^2 - 3}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}; \\
W_{g_4}(\mu) &= \frac{N^2 + 6}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}; \\
W_{g_4}(\mu) &= \frac{-1}{N(N^2 - 1)(N^2 - 9)}; \\
W_{g_4}(\mu) &= \frac{N^4 - 8N^2 + 6}{N^2(N^2 - 1)(N^2 - 4)(N^2 - 9)}.
\end{align*}

\textbf{Proof of (A.2).} We apply Proposition 3.7. Since

\begin{align*}
W((1^2), N) &= \sum_{\sigma \in <(1 2), (3 4)>} \sum_{\tau \in <(1 3), (2 4)>} W_{g_4}(\sigma \tau) \\
&= W_{g_4}(\mu) + 4W_{g_4}(\mu) + 3W_{g_4}(\mu) + 4W_{g_4}(\mu) + W_{g_4}(\mu) \\
&= \frac{N^2 + N + 2}{N^2(N - 1)(N + 1)(N + 2)(N + 3)}
\end{align*}

and

\begin{align*}
W((2), N) &= \sum_{\sigma \in S_4} \sum_{\tau \in S_4^*} W_{g_4}(\sigma \tau) = \frac{4}{N(N + 1)(N + 2)(N + 3)},
\end{align*}
We note that Proposition 3.7 gives
\[ \mathbb{E}[|v_j|^4] = 2N(N - 1) \cdot W((1^2), N) + N \cdot W((2), N) = \frac{2}{N(N + 3)}. \]

Consider the average \( \mathbb{E}[v_{j1} v_{j2} v_{j3} v_{j4} \frac{v_{j'1} v_{j'2} v_{j'3} v_{j'4}}{v_{j'1} v_{j'2} v_{j'3} v_{j'4}}] \), where \( j = (j_1, j_2, j_3, j_4) \sim j' = (j_1', j_2', j_3', j_4'). \) The equation (3.1) and a direct calculation give
\[ \text{(A.6)} \quad \mathbb{E}[v_{j1} v_{j2} v_{j3} v_{j4} \frac{v_{j'1} v_{j'2} v_{j'3} v_{j'4}}{v_{j'1} v_{j'2} v_{j'3} v_{j'4}}] 
= \sum_{i' \in \mathbb{N}^2} \sum_{k' \in \mathbb{N}^2} \sum_{\nu \in \mathbb{N}} \mathbb{E}[U(\tilde{k}, \nu | \tilde{k}', j, j')] 
= N(N - 1) \{ \mathbb{E}[U((1, 1, 2, 2), j|(1, 1, 2, 2), j')] + \mathbb{E}[U((1, 1, 2, 2), j|(2, 2, 1, 1), j')] \} + N \cdot \mathbb{E}[U((1^4), j|(1^4), j)]. \]

We note that \( \mathbb{E}[U((1^4), j|(1^4), j)] = \frac{4}{N(N + 1)(N + 2)(N + 3)} \) if \( j \) is of type \( \mu = 4. \)

**Proof of (A.3).** From Lemma 2.3, we have
\[ \mathbb{E}[U((1, 1, 2, 2), (1, 1, 2, 2)|(1, 1, 2, 2), (1, 1, 2, 2)) = 4 \sum_{\sigma \in S(2, 2)} W_{g_4}^{U(N)}(\sigma) = 4 \{ W_{g_4}^{U(N)}((1^4)) + 2W_{g_4}^{U(N)}((2,1,1)) + W_{g_4}^{U(N)}((2,2)) \} = \frac{4}{N^2(N - 1)(N + 3)} \]
and
\[ \mathbb{E}[U((1, 1, 2, 2), (1, 1, 2, 2)|(2, 2, 1, 1), (1, 1, 2, 2)) = \sum_{\sigma, \tau \in S(2, 2)} W_{g_4}^{U(N)}(\sigma(13)(24)\tau^{-1}) = 4 \sum_{\sigma \in S(2, 2)} W_{g_4}^{U(N)}((13)(24)\sigma) = 4 \{ 2W_{g_4}^{U(N)}((2,2)) + 2W_{g_4}^{U(N)}((4)) \} = \frac{8}{N^2(N^2 - 1)(N + 2)(N + 3)}. \]

Hence the equation (A.6) gives
\[ \mathbb{E}[|v_i v_j|^2] = N(N - 1) \cdot \frac{4}{N^2(N - 1)(N + 3)} + N(N - 1) \cdot \frac{8}{N^2(N^2 - 1)(N + 2)(N + 3)} + N \cdot \frac{4}{N(N + 1)(N + 2)(N + 3)} \]
\[ = \frac{N(N + 1)(N + 3)}{4(N + 2)}. \]
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Proof of (A.4). From Lemma [2.3] we have

\[
E[U((1, 1, 2, 2), (1, 2, 1, 2)|(1, 1, 2, 2), (1, 1, 2, 2))]
= \sum_{\sigma, \tau \in S_{(2, 2)}} W g_4^{U(N)}(\sigma((2 3)\tau)^{-1})
= 4 \sum_{\sigma \in S_{(2, 2)}} W g_4^{U(N)}(\sigma(2 3))
= 4 \left\{ W g_4^{U(N)}((2, 1, 1)) + 2 W g_4^{U(N)}((3, 1)) + W g_4^{U(N)}((4)) \right\}
= \frac{-4}{N^2(N - 1)(N + 2)(N + 3)}.
\]

Similarly, we have

\[
E[U((1, 1, 2, 2), (1, 2, 1, 2)|(2, 2, 1, 1), (1, 1, 2, 2))]
= \frac{-4}{N^2(N - 1)(N + 2)(N + 3)}.
\]

Hence the equation (A.6) gives

\[
E[v_{ij}^2 v_{ji} v_{jj}^2] = 2N(N - 1) \cdot \frac{-4}{N^2(N - 1)(N + 2)(N + 3)} + N \cdot \frac{4}{N(N + 1)(N + 2)(N + 3)}
= \frac{-4}{N(N + 1)(N + 3)}.
\]

Proof of (A.5). In a similar way to the proof of (A.3), we have

\[
E[U((1, 1, 2, 2), (1, 2, 3, 4)|(1, 1, 2, 2), (1, 2, 3, 4))]
= \sum_{\sigma \in S_{(2, 2)}} W g_4^{U(N)}(\sigma) = \frac{1}{N^2(N - 1)(N + 3)}
\]

and

\[
E[U((1, 1, 2, 2), (1, 2, 3, 4)|(2, 2, 1, 1), (1, 2, 3, 4))]
= \sum_{\sigma \in S_{(2, 2)}} W g_4^{U(N)}(\sigma(1 3)(2 4)) = \frac{2}{N^2(N^2 - 1)(N + 2)(N + 3)}.
\]

Hence the equation (A.6) gives

\[
E[v_{j_1j_2} v_{j_3j_4}] = E[v_{12} v_{34}] = 1 \cdot N(N - 1) \cdot \frac{1}{N^2(N - 1)(N + 3)} + N(N - 1) \cdot \frac{2}{N^2(N^2 - 1)(N + 2)(N + 3)}
+ N \cdot \frac{1}{N(N + 1)(N + 2)(N + 3)}
+ \frac{N + 2}{N(N + 1)(N + 3)}.
\]
A.3 Moments of traces

\[ \mathbb{E}[|\text{tr}(V)|^4] = \frac{8(N^2 + 2N - 2)}{(N + 1)(N + 3)}; \]
\[ \mathbb{E}[|\text{tr}(V^2)|^2] = \frac{4(N^2 + 2N - 1)}{(N + 1)(N + 3)}; \]
\[ \mathbb{E}[\text{tr}(V^2)\text{tr}(V)^2] = \frac{8}{(N + 1)(N + 3)}.\]

which are obtained in [2] Appendix from symmetric function theory. If we use results in the previous subsection, another proof is given as follows.

\[ \mathbb{E}[|\text{tr}(V)|^4] = \sum_{i_1, i_2, i_3, i_4} \mathbb{E}[v_{i_1i_1}v_{i_2i_2}v_{i_3i_3}v_{i_4i_4}] \]
\[ = N\mathbb{E}[|v_{11}|^4] + 2N(N - 1)\mathbb{E}[|v_{11}v_{22}|^2] = \frac{8(N^2 + 2N - 2)}{(N + 1)(N + 3)}; \]

\[ \mathbb{E}[|\text{tr}(V^2)|^2] = \sum_{i_1, i_2, i_3, i_4} \mathbb{E}[v_{i_1i_2}^2v_{i_3i_4}^2] \]
\[ = N\mathbb{E}[|v_{11}|^4] + 2N(N - 1)\mathbb{E}[|v_{12}|^4] = \frac{4(N^2 + 2N - 1)}{(N + 1)(N + 3)}; \]

\[ \mathbb{E}[\text{tr}(V^2)\text{tr}(V)^2] = \sum_{i_1, i_2, i_3, i_4} \mathbb{E}[v_{i_1i_2}^2v_{i_3i_4}^2v_{i_3i_4}] \]
\[ = N\mathbb{E}[|v_{11}|^4] + 2N(N - 1)\mathbb{E}[v_{12}v_{11}v_{22}] = \frac{8}{(N + 1)(N + 3)}.\]

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