The Perfect Quark-Gluon Vertex Function

K. Orginos, W. Bietenholz, R. Brower, S. Chandrasekharan and U.-J. Wiese

a Dept. of Physics, Brown University, Providence, RI 02912, USA
b HLRZ c/o Forschungszentrum Jülich, 52425 Jülich, Germany
c Dept. of Physics, Boston University, Boston, MA 02215, USA
d CTP-LNS, MIT, Cambridge, MA 02139, USA

We evaluate a perfect quark-gluon vertex function for QCD in coordinate space and truncate it to a short range. We present preliminary results for the charmonium spectrum using this quasi-perfect action.

Approximately perfect lattice actions have a potential to suppress lattice artifacts much more than the \( O(a) \) improvement, which is very fashionable at this conference. However, it is still unclear how well this more sophisticated improvement program can be applied to QCD. Here we discuss our recent progress in the construction of a quasi-perfect action for QCD, and show preliminary results of its application to heavy quarks.

For free fermions we have derived extremely local perfect lattice actions of the form,

\[
S[\bar{\Psi}, \Psi] = \sum_{x,y} \bar{\Psi}_x \left( \gamma_\mu \rho_\mu (y-x) + \lambda (y-x) \right) \Psi_y.
\]

The couplings decay exponentially (in \( d > 1 \)). We truncate them to a unit hypercube by means of 3-periodic boundary conditions. The resulting “hypercube fermion” has still excellent spectral and thermodynamic properties. As an example, we give the couplings for \( m = 1 \) in Table 1 and show the dispersion relation in Fig. 1. It is very close to the continuum dispersion and approximates rotational symmetry very well.

Expansion to \( O(gA_\mu) \) for full QCD yields

\[
S[\bar{\Psi}, \Psi, A_\mu] = S[\bar{\Psi}, \Psi] + S[A_\mu] + gV[\bar{\Psi}, \Psi, A_\mu] + \frac{1}{(2\pi)^2} \int_{B^2} dp dq \Psi^\dagger (-p)V_\mu(p, q) A_\mu^a(p-q)\lambda^a_{ij} \Psi^j(q).
\]

For our first experiments we impose a tough truncation and arrive at the following parameterization of \( V_\mu(x, y) \):

| \( y-x \) | \( \rho_1(y-x) \) | \( \lambda(y-x) \) |
|---|---|---|
| (0000) | 0 | 1.26885069540 |
| (1000) | 0.05457967484 | -0.03008271460 |
| (1100) | -0.01101007028 | -0.01082956270 |
| (1110) | -0.00325481234 | -0.00471575763 |
| (1111) | -0.00120632489 | -0.00221240767 |

Table 1

The “hypercube fermion” couplings at \( m = 1 \).

The perfect action for free gluons, \( S[A_\mu] \), has also been discussed in [1]. Moreover, we found an explicit but complicated expression for the perfect quark-gluon vertex function \( V_\mu(p, q) \).

Numerically the vertex function can be evaluated and transformed to c-space, where it is represented by a set of link couplings, which depend on the fermion positions.

We truncate the \( O(gA_\mu) \) perfect action and parameterize it in a gauge invariant form:

1) Take the mean value of the link couplings over short lattice paths connecting \( \Psi \) and \( \bar{\Psi} \).

2) Re-scale the path and plaquette couplings such that their sum amounts to \( \lambda(r) \) for the scalar terms (\( \propto 1 \)), \( \rho_\mu(r) \) for the vector terms (\( \propto \gamma_\mu \)), \( s(m) = (m/\hat{m})^2[1/\hat{m} - 1/m] \), \( \hat{m} = e^m - 1 \) for the plaquette couplings (\( \propto \sigma_{\mu\nu} \)) and a value \( s_1(m) \) – obtained from a low \( \vec{p} \) expansion – for the terms \( \propto \gamma_\mu \gamma_\nu \gamma_\rho \).

For our first experiments we impose a tough truncation and arrive at the following parameterization of \( V_\mu(x, y) \):

*Based on a poster presented by K. Orginos at LAT97.
the form

\[ H = m_s + \frac{1}{2m_{\text{kin}}} \vec{p}^2 + \frac{1}{2m_B} \vec{B} + \ldots \]

where \( \vec{B} \) is assumed const. in time, and \( \Sigma_k = \epsilon_{ijk} \sigma_{ij} / 2 \). Here \( m_s \) is the static, \( m_{\text{kin}} \) the kinetic and \( m_B \) the magnetic mass. In the continuum they all coincide, and for the hypercube fermion \( m_s = m \).

We saw before that \( m_{\text{kin}}(m_s) \) is drastically improved for the hypercube fermion, compared to the Wilson fermion \( \bar{B} \). Of particular interest for the hyperfine splitting is \( m_B(m_s) \), see Fig. 2.

For the scalar term, \( L_1 \) is a path consisting of one link, \( L_2 \) is a staple and \( L_3, L_4 \) are the shortest paths connecting \( \Psi, \Psi \) separated by a 2d, 3d, 4d diagonal in the unit hypercube, respectively. The same holds for the vector term, if the fermions are separated in the \( \mu \) direction. The standard clover-like plaquette coupling is called \( c_0 \), and \( c_1 \) refers to the case where \( \Psi, \Psi \) are separated by one lattice spacing; it is the coupling to the plaquettes attached to their connecting link. Finally, if the fermions are separated by \( \hat{\rho} \), then we couple them to the plaquettes in the perpendicular \( \mu, \nu \) planes (touching the fermions), and to the connecting link, by \( s_1 \).

A low \( \vec{p} \) expansion leads to a Hamiltonian of

\[ H = m_s + \frac{1}{2m_{\text{kin}}} \vec{p}^2 + \ldots \]

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We now present preliminary results on charmonium spectroscopy using the fermionic action presented above and the Wilson gauge action. We simulate quenched at \( \beta = 5.6 \), the lattice spacing is 0.24 fm and the size \( 8^3 \times 16 \). We include 30 lattices and consider the meson dispersion relation as well as the spectrum of the low lying states. The latter was evaluated at \( m = 0 \) and \( m = 1 \). We then interpolate to the mass of \( \eta_c \) (2.98 GeV), which corresponds to \( m \approx 0.9 \).

Regarding the dispersion relation we compute the “speed of light” from \( cp = \sqrt{E^2 - M^2} \), see

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The dispersion relation of the hypercube fermion at \( m = 1 \) in various directions.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The magnetic vs. static mass. Remarkably, the Wilson fermion can be corrected up to the quartic order by the parameters we introduced before: pure Wilson fermion: \( m_B = m_s + m_3^2 + \ldots \). Sheikholeslami-Wohlert action (\( c_0 = 1 \)): \( m_B = m_s + \frac{2}{3} m_3^2 + \ldots \) (and \( m_B = m_{\text{kin}} \)). Split clover term into \( c_0 = \frac{2}{3}, c_1 = \frac{1}{3} \): \( m_B = m_s - \frac{4}{15} m_3^2 + \ldots \).}
\end{figure}
Fig. 3, and obtain \( c = 1.04 \pm 0.05 \). The mesonic dispersion is impressively accurate even for large momenta.

![Figure 3. The mesonic dispersion relation.](image)

In Fig. 4 we show the charmonium spectrum, including the 1s and 2s states of \( \eta_c \) and \( J/\psi \). The 2s-1s splitting looks fine, but the hyperfine splitting is clearly too small (10.3 MeV). We also note that the entirely new term \( \propto \gamma_\mu \gamma_\nu \gamma_\rho \) has a positive influence on the spectrum. Without of it, both types of splitting decrease by about one fourth.

![Figure 4. The charmonium spectrum. The experimental values are dashed, and only the ground state of \( \eta_c \) is fitted.](image)

Finally we consider one more point, \( \beta = 5.5 \), where \( \eta_c \) corresponds to \( m \simeq 1.1 \). Now the lattice is even coarser, \( a = 0.28 \) fm. The hyperfine splitting decreases to 7.4 MeV, such that its slope for decreasing lattice spacing points up, as it should. It is hardly justified to extrapolate the connection of these two points by a straight line down to the continuum limit. If one does so nevertheless one obtains \( M_{J/\psi} - M_{\eta_c} \sim 28 \) MeV. The reference value here is perhaps not so much the experimental value of \( 118 \pm 2 \) MeV, but rather the relativistic quenched results \( \sim 70 \) MeV; for a review see 4. NRQCD reported a larger value for some time, but more recent observations indicate that the NR expansion is not really applicable to charmonium 5.

To summarize our first experience with the hypercube fermion plus truncated vertex function, we observed that properties related to the restoration of Lorentz symmetry are strongly improved: fermionic dispersion, kinetic mass and mesonic dispersion. However, properties related to the magnetic interaction are only successful in part: \( m_B \) is significantly improved, but the hyperfine splitting is too small. The 2s-1s splitting looks fine. On the other hand, the mass is still strongly renormalized.

Since renormalization due to \( O(A^2) \) is not incorporated yet, we expect tadpole improvement to amplify the hyperfine splitting and to reduce the dependence on the lattice spacing. We also hope to further improve the parameterization.

But even if this action should not be satisfactory in a direct application, we expect it to be a good starting point for the non-perturbative multigrid improvement program described in 2.

Furthermore it helped to identify the significant non-standard terms, in particular the \( c_1 \) and the \( s_1 \) term. Their dominating role in the improvement is a reliable observation, even if the exact coefficients may still change.

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