Discrete random walk with barriers on a locally infinite graph

Theo van Uem

Amsterdam School of Technology, Weesperzijde 190, 1097 DZ Amsterdam, The Netherlands.

Email: t.j.van.uem@hva.nl

Abstract

We obtain expected number of arrivals, absorption probabilities and expected time before absorption for an asymmetric discrete random walk on a locally infinite graph in the presence of multiple function barriers. On each edge of the graph and in each barrier there are specific probabilities defined. Examples of multiple function barrier graphs and applications on the integers are given.

Keywords: Random walk, absorption, reflection, barrier, graph

2000 Mathematics Subject Classification: Primary 60G50; Secondary 60J05

1. Introduction

Random walk can be used in various disciplines: in economics to model share prices and their derivatives, in medicine and biology where absorbing barriers give a natural model for a wide variety of phenomena, in physics as a simplified model of Brownian motion, in ecology to describe individual animal movements and population dynamics, in statistics to analyze sequential test procedures, in computer science to estimate the size of the World Wide Web using randomized algorithms. Burioni and Cassi (2005) give a review of random walks on graphs, where the generalization of the concept of dimension to inhomogeneous structures, using infinite graphs, is considered. Durhuus, Jonsson and Wheater (2006) develop techniques to obtain rigorous bounds on the behaviour of random walks on combs. Using these bounds they calculate the spectral dimension of random combs with infinite teeth at random positions or teeth with random but finite length. Random walks have been studied for decades on regular structures such as lattices. We now give a brief historical review of the use of barriers in a one-dimensional discrete random walk. Weesakul (1961) discussed the classical problem of random walk restricted between a reflecting and an absorbing barrier. Using generating functions he obtains explicit expressions for the probability of absorption. Lehner (1963) studies one-dimensional random walk with a partially reflecting barrier using combinatorial methods. Gupta (1966) introduces the concept of a multiple function barrier (mfb): a state that can absorb, reflect, let through or hold for a moment. Dua, Khadilkar and Sen (1976) find the bivariate generating functions of the probabilities of a particle reaching a certain state under different conditions. Percus (1985) considers an asymmetric random walk, with one or two boundaries, on a one-dimensional lattice. At the boundaries, the walker is either absorbed or reflected back to the system. Using generating functions the probability distribution of being at position m after n steps is obtained, as well as the mean number of steps before absorption. El-Shehawey (2000) obtains absorption probabilities at the boundaries for a random walk between one or two partially absorbing boundaries as well as the conditional mean for the number of steps before stopping given the absorption at a specified barrier, using conditional probabilities.

In this paper we obtain expected number of arrivals, absorption probabilities and expected time before absorption for an asymmetric discrete random walk with multiple function barriers, expanding the theory in two directions:

A. the state space is no longer a one-dimensional lattice; our domain is a locally infinite graph.
B. one or two absorbing or reflecting barriers are replaced by a finite set of multiple functions barriers.
We choose a structure of our graph that can handle with the well known lattice structure as well as
structures like star graphs (finite and infinite) and cycle graphs.
Our graph consists of multiple function barriers (vertices), states on the edges between the mfb’s and
an infinite set of half lines, each with an infinite number of states, starting in each barrier. Each half
line has a topological end. On each edge of the graph a random walk with its own internal states and
jumping probabilities is introduced. When the walker reaches a multiple function barrier a random
walk is activated according to an extra set of probabilities, or the particle is absorbed in the barrier.
Each barrier and each edge has its own probability parameters. In section 2 we use generating
functions to find the expected number of arrivals to any state, the probability of absorption and the
expected time before absorption. In section 3 we analyze some examples of graphs with multiple
function barriers: two star graphs and a cycle graph. In the last section we apply our theory in a classic
situation: the (sub)set of integers. In appendices A and B we give proves of some results in section 2.

2. A graph with multiple function barriers

2.1. Description of the random walk with multiple function barriers

In a graph we have vertices M[0], M[1], M[2],….,M[N] representing the mfb’s. Between M[i] and
M[j] there is a random walk with a finite number of states n[i,j], which we number 1,2,3,…,n[i,j] in
the direction from M[i] to M[j] when i<j. We will use the abbreviation [i,j] for the edge between M[i]
and M[j]. Each random walk from M[i] to M[j] has its own parameters p=p[i,j], q=q[i,j] and r=r[i,j],
where p is the one-step forward probability, q one-step backward and r=1-p-q the probability to stay
for a moment in the same position. We demand p[i,j], ,q[i,j] >0 for each i and j. It is also possible to
move from M[i] along half lines with states 1,2,3,….
A half line starting in M[i] is labelled [i,k) and has end \( \Omega_{i,k} \) (i=0,1,…,N; k=1,2,…).

In M[i] there is probability \( p_{i,j}^* \) to move one step in the direction of M[j], probability \( p_{i,j}^* \) to stay
for a moment in M[i] , probability \( s_i > 0 \) for immediate absorption in M[i] and probability \( p_{i,k}^* \) to
move one step in the direction \( \Omega_{i,k} \), where (see also fig. 1)
\[
\sum_{j=0}^{N} p_{i,j}^* + \sum_{k=1}^{\infty} p_{i,k}^* + s_i = 1 \quad (i = 0,1,2,......N)
\]

![Diagram of the random walk](image)

Figure 1. Description of the random walk
Discrete random walk with barriers

We start either in \(i_0\) \((0 \leq i_0 \leq n[c,d]+1)\) on edge \([c,d]\) \((c<d)\), where \(i_0=0\) means: start in \(M[c]\) and a start in \(M[d]\) is represented by \(i_0=n[c,d]+1\), or we start in \(i_0 \geq 0\) on edge \([c,h]\), where \(i_0=0\) means: start in \(M[c]\).

2.2. Expected number of arrivals

We are interested in the expected number of arrivals in the mfb’s as well as the expected number of arrivals in the other states of our graph.

We define:

- \(P_k^{ij}=P(\text{system is in state } j \text{ after } k \text{ steps} | \text{start in } i)\)
- \(X_j = X_j(z) = X_{i,j}(z) = \sum_{k=0}^{\infty} p_{ij}^k z^k \) (if \(j\) is not a mfb)
- \(Y_j = Y_j(z) = Y_{i,j}(z) = \sum_{k=0}^{\infty} p_{i,M[j]}^k z^k \) for mfb \(M[j]\)
- \(x_j = x_{i,j} = X_j(1) = \text{expected number of arrivals in } j; \text{start in } i\)
- \(y_j = y_{i,j} = Y_j(1) = \text{expected number of arrivals in } M[j]; \text{start in } i\)

On an interval \([i,j]\):

- \(\rho = \rho(i, j) = \frac{p[i, j]}{q[i, j]} \) and \(n = n[i, j]\);

On a half line \([i,j)\):

- \(\rho = \rho(i, j) = \frac{p[i, j]}{q[i, j]} \)

We start in \(i_0\) on the interval \([c,d]\) or on the half line \([c,h]\).

**Theorem 2.1.** If \(\rho \neq 1\) then: \(y_k\) \((k=0,1,2,...,N)\) is the unique solution of

\[
\sum_{j=0}^{N} u_{ij} y_j = Q_i \quad (i=0,1,......,N)
\]

where

\[
u_{ij} = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{n+1}} \cdot p_{[i,j]}^* \quad \text{if } j < i \\ \frac{1-\rho}{1-\rho^{n+1}} \cdot p_{[i,j]}^* \quad \text{if } j > i \end{cases}
\]

\[
u_{ii} = p_{[i,i]}^* - 1 + \sum_{j<i} \left[ \frac{\rho(1-\rho^n)}{1-\rho^{n+1}} \cdot p_{[i,j]}^* + \sum_{j>i} \left( \frac{(1-\rho^n)}{1-\rho^{n+1}} \cdot p_{[i,j]}^* + \sum_{j=1}^{\infty} \rho^{n+1-j} p_{[i,j]}^* + \sum_{j=1}^{\infty} p_{[i,j]}^* \right) \right]
\]

Starting in \(i_0\) on interval \([c,d]\): \(Q_i = [\delta(i,d) - \delta(i,c)] \left( \frac{1-\rho^{n+1-i_0}}{1-\rho^{n+1}} \right) - \delta(i,d)\)

Starting in \(i_0\) on half line \([c,h]\): \(Q_i = \begin{cases} -\delta(i,c) \rho^{-i_0} \quad \text{if } \rho[c,h] > 1 \\ -\delta(i,c) \quad \text{if } \rho[c,h] \leq 1 \end{cases}\)

On interval \([i,j]\) we have:

If \((i,j)=(c,d)\) then:
\[
x_k = \begin{cases}
(1 - \rho^k) \frac{p_{[i,j]}^*}{q} y_j + (\rho^k - \rho^{n+1}) \frac{p_{[i,j]}^*}{p} y_i + (\rho^k - 1)(\frac{1 - \rho^{n+1-i_0}}{p-q}) & (1 \leq k \leq i_0) \\
(1 - \rho^{n+1}) & (i_0 \leq k \leq n)
\end{cases}
\]

If \((i, j) \neq (c, d)\) then:
\[
x_k = \frac{(1 - \rho^k) \frac{p_{[i,j]}^*}{q} y_j + (\rho^k - \rho^{n+1}) \frac{p_{[i,j]}^*}{p} y_i}{(1 - \rho^{n+1})}
\]

On half line \([i, j)\) we have:
If \((i, j) = (c, h)\) then:
\[
x_k = \begin{cases}
\left(\frac{\rho^{-i_0}}{p-q}\right)(\rho^k - 1) + \frac{p_{(c,h)}^*}{p} y_c & (1 \leq k \leq i_0, \ \rho = \rho(c, h) > 1) \\
\frac{1 - \rho^{-i_0}}{p-q} + \frac{p_{(c,h)}^*}{p} y_c & (k \geq i_0, \ \rho > 1) \\
\frac{p_{(c,h)}^*}{p} \rho^k y_c + \frac{\rho^k - 1}{p-q} & (1 \leq k \leq i_0, \ \rho < 1) \\
\frac{p_{(c,h)}^*}{p} \rho^k y_c + \frac{\rho^k (1 - \rho^{-i_0})}{p-q} & (k \geq i_0, \ \rho < 1)
\end{cases}
\]

If \((i, j) \neq (c, h)\) then:
\[
x_k = \begin{cases}
\frac{p_{(i,j)}^*}{p} y_i & \rho = \rho(i, j) > 1 \\
\frac{p_{(i,j)}^*}{p} \rho^k y_i & \rho \leq 1
\end{cases}
\]

If \(\rho = 1\) then we can obtain similar results by applying l'Hospital's rule.

**Proof.** See Appendix A

### 2.3. Probability of absorption

We are interested in the probability of absorption in a mfb as well as in the ends of the half lines.
We define: \(p_{(i_0, M[i])}^{(k)} = P\text{(system is in mfb M[i] after k steps | start in i_0)}\)

The probability of absorption in mfb M[i] is given by \(\sum_{k=0}^{\infty} p_{(i_0, M[i])}^{(k)} s_i = s_i y_i \quad (i = 0, 1, ..., N)\)

Another way to determine the probability of absorption in any mfb is given in the next corollary of Theorem 2.1. (Applications of this corollary can be found in sections 3.2.2 and 4.2)
Corollary 2.1. \( P(\text{absorption in any mfb}) = \)

\[
\begin{cases}
1 - \sum_{j=0}^{N} y_j \sum_{k=1}^{\infty} (1 - \rho_{j,k}^{-1}) \rho_{k,j}^* & \text{if we start on a half line with } \rho \leq 1 \text{ or on an interval} \\
\rho^{-i_0} - \sum_{j=0}^{N} y_j \sum_{k=1}^{\infty} (1 - \rho_{j,k}^{-1}) \rho_{k,j}^* & \text{if we start on a half line with } \rho > 1
\end{cases}
\]

Proof. Using Theorem 2.1 we get: \( \sum_{i=0}^{N} Q_i = -\rho^{-i_0} \) if we start on a half line with \( \rho > 1 \); otherwise

\[
\sum_{i=0}^{N} Q_i = -1.
\]

The same theorem gives:

\[
\sum_{i=0}^{N} \sum_{j=0}^{N} u_{ij} y_j = \sum_{j=0}^{N} y_j \sum_{i=0}^{N} u_{ij} = \sum_{j=0}^{N} y_j \sum_{i=0}^{N} [s_j - \sum_{k=1}^{\infty} (1 - \rho_{j,k}^{-1}) \rho_{k,j}^*] = \sum_{i=0}^{N} Q_i
\]

Lemma 1. Given a \([p,q,r]\) random walk on the non-negative integers with \( p > q \), a partial reflecting barrier in \( 0 \), start in \( i_0 \) and end \( \Omega \) we have:

\( P(\text{absorption in } \Omega) = (p - q)^*[\text{expected number of arrivals in } k \geq i_0] \)

Proof. In 0 we take \( \alpha \) as reflecting probability and \( 1 - \alpha \) as absorption probability. Solving the relevant difference equations we find:

\[
x_k = \begin{cases}
\left( \frac{p}{q} \right)^{k-i_0} - \frac{p(1-\alpha)(\frac{p}{q})^{-i_0}}{p-q} & (1 \leq k \leq i_0) \\
\frac{1}{p-q} - \frac{p(1-\alpha)(\frac{p}{q})^{-i_0}}{(p-q)(p-\alpha q)} & (k \geq i_0)
\end{cases}
\]

Using this result we also get

\[
P(\text{absorption in } \Omega) = 1 - (1 - \alpha) x_0 = 1 - (1 - \alpha) q x_1 = 1 - \frac{p(1-\alpha)(\frac{p}{q})^{-i_0}}{(p-\alpha q)}
\]

We are now ready for:

Corollary 2.2. If on a half line \([i,j]\) \((i=0,1,\ldots,N, j=1,2,\ldots)\) with end \( \Omega_{i,j} \) we have \( \rho(i,j) > 1 \), then:

\[
P(\text{absorption in } \Omega_{i,j}) = \begin{cases}
1 - \rho^{-i_0} + (1 - \rho^{-1}) \rho_{(c,h)}^* y_c & \text{if } (i,j) = (c,h) \\
(1 - \rho^{-1}) \rho_{(i,j)}^* y_i & \text{if } (i,j) \neq (c,h)
\end{cases}
\]

Proof. Use Theorem 2.1 and Lemma 1, where state 0 in Lemma 1 is now considered as the one point contraction of our complete graph minus our half line \([i,j]\).

We will use this corollary in sections 3.2.2 and 4.2

Remark 1. Corollary 2.1 is a consequence of corollary 2.2
2.4. Expected time before absorption

We are interested in the expected time before absorption. We define:

\( m_k[i,j] \) = expected time before absorption when starting in \( k \) (\( k=1,2,\ldots,n[i,j] \)) on interval \([i,j]\).

\( m_k[i,j] \) = expected time before absorption when starting in \( k \) (\( k=1,2,\ldots \)) on half line \([i,j]\).

\( n_k \) = expected time before absorption when starting in \( M[k] \) (\( k=0,1,2,\ldots,N \)).

**Theorem 2.2.** If on each half line \([i,j) (i=0,1,\ldots,N; j=1,2,\ldots) \) with \( p_{(i,j)}^* > 0 \) we have \( \rho(i,j) < 1 \) then:

\[ n_j (j=0,1,2,\ldots,N) \text{ is the unique solution of } \sum_{j=0}^N v_{ij} n_j = \Lambda_j \quad (i=0,1,\ldots,N) \]

where:

\[ v_{ij} = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{n+1}} p_{(i,j)}^* & \text{if } j > i \text{ and } \rho \neq 1 \\ \frac{1-\rho}{1-\rho^{n+1}} p_{(i,j)}^* & \text{if } j < i \text{ and } \rho \neq 1 \\ \frac{p_{(i,j)}}{n+1} & \text{if } j \neq i \text{ and } \rho = 1 \end{cases} \]

\[ \Lambda_j = -(1-s_j) + \sum_{j<i} p_{(i,j)}^* \left( \frac{1+n\rho^{n+1}-(n+1)\rho^n}{(p-q)(1-\rho^{n+1})} \right) + \sum_{j<i} p_{(i,j)}^* \left( \frac{n-(n+1)\rho+\rho^{n+1}}{(p-q)(1-\rho^{n+1})} \right) - \sum_{j<i} \frac{n p_{(i,j)}^*}{2p} + \sum_{j<i} \frac{p_{(i,j)}^*}{p-q} \]

On interval \([i,j]\):

\[ m_k = \begin{cases} \frac{(1-\rho^{n+1-k})n_i + (\rho^{n+1-k} - \rho^{n+1})n_j}{1-\rho^{n+1}} + \frac{n+1}{p-q} \left( \frac{p^{n+1-k} - \rho^{n+1}}{1-\rho^{n+1}} \right) - \frac{k}{p-q} & (\rho \neq 1) \\ \frac{(n+1-k)n_i + kn_j}{n+1} + \frac{k(n+1-k)}{2p} & (\rho = 1) \end{cases} \quad (i<j, \ k=0,1,\ldots,n[i,j]+1) \]

On half line \([i,j)\):

\[ m_k = n_i + \frac{k}{q-p} \quad (\rho < 1) \quad (k=0,1,\ldots) \]

**Proof.** See appendix B.
3. Examples of multiple function barrier graphs

3.1. A finite star graph

We consider a star graph with M[0] in the centre and all edges are of the interval type; we start in $i_0$ on edge [M[0],M[1]].

In a finite star graph we have:

$$p^*_{i,j} = 0 \text{ if } i, j \in \{1,2,\ldots,N\} \text{ and } i \neq j$$

Notation: $p_i = p[0,i], q_i = q[0,i], r_i = r[0,i]; \rho_i = \frac{p_i}{q_i} (i = 1,2,\ldots,N)$

We have:

$$p^*_{i,0} + p^*_{i,i} + s_i = 1 \quad (i = 1,2,\ldots,N); \sum_{i=0}^{N} p^*_{0,i} + s_0 = 1$$

3.1.1. Expected number of arrivals.

To obtain the expected number of arrivals in our finite star graph, we use Theorem 2.1:

$$\sum_{j=0}^{N} u_{ij} y_j = Q_i \quad (i = 0,1,\ldots,N)$$

Instead of the equation with i=0 we will use: $\sum_{i=0}^{N} s_i y_j = 1$. This equation is easily derived from

$$\sum_{i=0}^{N} \sum_{j=0}^{N} u_{ij} y_j = \sum_{i=0}^{N} Q_i \text{ (see also corollary 2.1) or by observing that we deal with the probability of absorption in our finite graph.}$$

If $\rho_i \neq 1$ we find:

$$y_i = \varsigma_i y_0 \quad (i = 2,3,\ldots,N) \quad \text{with} \quad \varsigma_i = \frac{(1-\rho_i) p^*_i}{s_i (1-\rho_i^{1+n[0,i]}) + (1-\rho_i) p^*_{i,0}} \quad (i = 1,2,3,\ldots,N);$$

$$\varsigma_0 = 1; \quad y_1 = \varsigma_1 y_0 + \alpha_1 \quad \text{where} \quad \alpha_1 = \frac{\rho_1^{1+n[0,1]} (\rho_1^{-\rho_1} - 1)}{s_1 (1-\rho_1^{1+n[0,1]}) + (1-\rho_1) p^*_{1,0}}$$

(19)
If $\rho_i = 1$, 

\[ y_i = \zeta_i \cdot y_0 \quad (i = 2, 3, \ldots, N) \] 
with 

\[ \zeta_i = \left[ \frac{p_{[0, i]}^*}{(1 + n[0, i])s_i + p_{[1, 0]}^*} \right] \quad (i = 1, 2, \ldots, N); \quad \zeta_0 = 1 \]

\[ y_1 = \zeta_1 \cdot y_0 + \alpha_1 ; \quad \alpha_1 = \frac{i_0}{(1 + n[0, 1])s_1 + p_{[1, 0]}^*} \]

In both cases we have $\sum_{i=0}^{N} s_i y_i = 1$ so:

\[ y_0 = \frac{1 - s_1 \alpha_1}{\sum_{i=0}^{N} s_i \zeta_i}; \quad y_1 = \zeta_1 \cdot y_0 + \alpha_1; \quad y_i = \frac{\zeta_i \cdot s_i}{\sum_{i=0}^{N} s_i \zeta_i} \quad (i = 2, \ldots, N) \]

If we start in the center $M[0]$ of the star graph we have:

\[ y_i = \frac{\zeta_i \cdot s_i}{\sum_{i=0}^{N} s_i \zeta_i} \quad (i = 0, 1, \ldots, N) \]

### 3.1.2 Mean absorption time in the presence of absorbing states in the endpoints.

We now consider a star graph with $M[0]$ in the centre and absorbing states in the endpoints $M[1], M[2], \ldots, M[N]$:

\[ p_{[i, j]}^* = 0 \quad \text{if} \quad i, j \in \{1, 2, \ldots, N\}; \quad s_i = 1 \quad (i = 1, 2, \ldots, N); \quad \sum_{i=0}^{N} p_{[0, i]}^* + s_0 = 1; \]

We start in $M[0]$. Notation: $p_i = p[0, i]$, $q_i = q[0, i]$, $r_i = r[0, i]$, $n_i = n[0, i]$; $\rho_i = \frac{p_i}{q_i} (i = 1, 2, \ldots, N)$

We define for $i = 1, 2, \ldots, N$:

\[ R_i = \begin{cases} 
1 - \rho_i^{n[0, i]} & \rho_i \neq 1 \\
1 - \rho_i^{n[0, i]+1} & \rho_i = 1
\end{cases} \]

\[ W_i = \begin{cases} 
\frac{1 + n[0, i] \rho_i^{n[0, i]+1} - (1 + n[0, i]) \rho_i^{n[0, i]}}{(p_i - q_i)(1 - \rho_i^{n[0, i]+1})} & \rho_i \neq 1 \\
-\frac{n[0, i]}{2p_i} & \rho_i = 1
\end{cases} \]

Theorem 2.2 gives now:

\[ n_0 = \frac{1 - s_0}{\sum_{i=0}^{N} W_i p_{[0, i]}^*} \]

\[ \nu_0 = \frac{1 - \rho_0^*}{\sum_{i=0}^{N} R_i p_{[0, i]}^*} \]
3.1.3. A special case.

We get a special finite star graph when we take N=2 and \( i_0 = 0 \).

Using section 3.1.1 we get the expected number of arrivals:

\[
y_i = \frac{s_i}{\sum_{i=0}^{2} s_i s_i} \quad (i = 0, 1, 2)
\]

Interpretation: a random walk on \([-A, B]\) with different probabilities 'left' and 'right' from the starting point 0 and three mfb's: 0, -A and B (\( A = n_2 + 1 \); \( B = n_1 + 1 \)).

Using section 3.1.2 we obtain the mean absorption time, starting in \( M[0] \), in the presence of absorbing states in the endpoints:

\[
n_0 = \frac{(1 - s_0) - \sum_{i=1}^{2} W_i P_{[0,i]}^*}{1 - P_{[0,0]}^* - \sum_{i=1}^{2} R_i P_{[0,i]}^*}
\]

Interpretation: a random walk on \([-A, B]\) with different probabilities 'left' and 'right' from the starting point 0 (mfb) and two absorbing barriers: -A and B (\( A = n_2 + 1 \); \( B = n_1 + 1 \)).

**Remark 2.** If \( p_1 = q_1 = p_2 = q_2 = p_{[0,1]}^* = p_{[0,2]}^* = \frac{1}{2} \); \( s_1 = s_2 = 1 \) we get the well known result:

\( n_0 = AB \), see Feller, p. 349.

3.2. An infinite star graph

We consider a star graph with a single mfb \( M[0] \) in the centre and all edges are of the half line type \([0, k) \) (\( k=1,2,\ldots \)); we start in \( i_0 \) on the edge \([0, 1)\).

Notation: \( p_i = p[0, i] \), \( q_i = q[0, i] \), \( r_i = r[0, i] \); \( \rho_i = \rho[0, i] = \frac{p_i}{q_i} \); \( p_i^* = p_{[0, i]}^* \) (\( i = 1, 2, \ldots \))
3.2.1. Expected number of arrivals and probability of arrival

To obtain the expected number arrivals and probability of arrival, we apply Theorem 2.1:

$$u_{00}y_0 = \begin{cases} -\rho_i^{-i_0} & \text{if } \rho_i > 1 \\ -1 & \text{if } \rho_i \leq 1 \end{cases}$$

where

$$u_{00} = p^n_{[0,0]} - 1 + \sum_{j=1}^{\infty} \rho^{-1}_j p^*_j + \sum_{j=1}^{\infty} \rho^{-1}_j p^*_j$$

which leads us to:

$$y_0 = \begin{cases} \frac{1}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j} & \text{if } \rho_i \leq 1 \\ \frac{\rho_i^{-i_0}}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j} & \text{if } \rho_i > 1 \end{cases}$$

(20)

Again using Theorem 2.1 we find:

On half line [0,1):

$$x_k = \begin{cases} \frac{\rho_i^{-i_0}}{p_i - q_i} \left( \rho_i^{-1} - 1 \right) + \frac{p_i^*}{p_i} y_0 & (1 \leq k \leq i_0, \rho_i > 1) \\ \frac{1 - \rho_i^{-i_0}}{p_i - q_i} + \frac{p_i^*}{p_i} y_0 & (k > i_0, \rho_i > 1) \\ \frac{p_i^* \rho_i^k}{p_i} y_0 + \frac{\rho_i^k - 1}{p_i - q_i} & (1 \leq k \leq i_0, \rho_i < 1) \\ \frac{p_i^* \rho_i^k}{p_i} y_0 + \frac{\rho_i^{-i_0}(1 - \rho_i^{-1})}{p_i - q_i} & (k > i_0, \rho_i < 1) \\ \frac{k}{p_i} + \frac{p_i^*}{p_i} y_0 & (1 \leq k \leq i_0, \rho_i = 1) \\ \frac{i_0}{p_i} + \frac{p_i^*}{p_i} y_0 & (k \geq i_0, \rho_i = 1) \end{cases}$$

On half line [0,m) (m=2,3,…..):

$$x_k = \begin{cases} \frac{p_m^*}{p_m} y_0 & \text{if } \rho_m > 1 \\ \frac{p_m^*}{p_m} y_0 & \text{if } \rho_m < 1 \end{cases}$$

We define $f_{ij}$ = probability to visit j when starting in i.

A well known result is: $f_{ij} = \frac{x_{ij}}{x_{jj}}$ (i $\neq$ j)

If j on halfline [0,m) and $\rho_m = 1$:

$$f_{0j} = \frac{p_m^*}{j(s_0 + \sum_{\rho_i > 1} (1 - \rho_i^{-1}) p_k^* + p_m^*)} (j = 1,2,.....)$$
3.2.2. Absorption probabilities

Besides absorption in \( M[0] \) there is the possibility of absorption in \( \Omega_{0,j} \) when \( \rho_j > 1 \).

When \( \rho_k \leq 1 \) \( (k = 0, 1, \ldots) \) (20) gives: \( y_0 = \frac{1}{s_0} \), so: \( P(\text{absorption in } M[0]) = 1 \).

Now we suppose that there is at least one \( k \) with: \( \rho_k > 1 \).

Using (19) with absorbing states in \( M[k] \) and taking \( n[0,k] \to \infty \) \( (k=1,2,\ldots,N) \) we find for \( m=1,2,\ldots,N \)

\[
\text{if } \rho_m > 1 \text{ then } \zeta_m = (1 - \rho_m^{-1}) p_m^* \quad \text{and} \quad \alpha_i = \begin{cases} 
0 & \text{if } \rho_i \leq 1 \\
1 - \rho_i^{-1} & \text{if } \rho_i > 1 
\end{cases}
\]

We first consider the case \( \rho_1 \leq 1 \):

\[
P(\text{absorption in } M[0]) = s_0 y_0 = \frac{s_0}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j^*}
\]

This leads us to: if \( \rho_m > 1 \) then:

\[
P(\text{absorption in } \Omega_{0,m}) = \frac{(1 - \rho_m^{-1}) p_m^*}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j^*} \quad (m=2,3,\ldots)
\]

We have: \( P(\text{absorption in } M[0]) + \sum_{\rho_m > 1} P(\text{absorption in } \Omega_{0,m}) = 1 \).

Next we analyze the case \( \rho_1 > 1 \):

\[
P(\text{absorption in } M[0]) = s_0 y_0 = \frac{s_0 \rho_1^{-1}}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j}
\]

This and again using (19) with absorbing states leads us to:

\[
P(\text{absorption in } \Omega_{0,1}) = \frac{(1 - \rho_1^{-1}) p_1^* \rho_1^{-1}}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j} + (1 - \rho_1^{-1})
\]

and if \( \rho_m > 1 \):

\[
P(\text{absorption in } \Omega_{0,m}) = \frac{(1 - \rho_m^{-1}) p_m^* \rho_m^{-1}}{s_0 + \sum_{\rho_j > 1} (1 - \rho_j^{-1}) p_j^*} \quad (m=2,3,\ldots)
\]

Here we also have: \( P(\text{absorption in } M[0]) + \sum_{\rho_m > 1} P(\text{absorption in } \Omega_{0,m}) = 1 \).

All results in this section can also be verified by applying corollary 2.1 in the mfb case and corollary 2.2 in the \( \Omega \) situation.
3.2.3. Mean absorption time

Applying Theorem 2.2 we obtain the mean absorption time in mfb $M[0]$, when starting in $M[0]$. If $\rho_j < 1 \ (j = 1,2,\ldots)$:

$$n_0 = \frac{1-s_0}{s_0} + \frac{1}{s_0} \sum_{j=1}^{\infty} \rho_j^{-1} p_j^{-1}$$

The summation can be divergent.

For the mean absorption time in state $k$ on half line $[0,j)$ we find:

$$m_k = n_0 + \frac{k}{q_j - p_j} \quad (k = 0,1,2,\ldots)$$

3.2.4. Special case: two half lines starting in mfb 0.

As a consequence we have now also analyzed a random walk on $(-\infty, \infty)$ with a single mfb in 0 and two parameters $\rho_1$ and $\rho_2$ right and left from the origin: take two edges, renumber the states of the second edge in -1,-2,\ldots and use parameter $\rho_1$ on the first and $\rho_2^{-1}$ on the second edge.

3.3. A Positive Oriented Cycle Graph

We have $N+1$ barriers in a cycle graph: $M[0], M[1], \ldots, M[N]$. We start in $M[0]$ and define an artificial barrier $M[N+1]$, which is the same as barrier $M[0]$. A positive oriented cycle graph is defined by:

$$p_{i,i}^* + p_{i,i+1}^* + s_i = 1 \quad (i = 0,1,\ldots,N)$$

Notation: $\rho_i = \rho[i,i+1]

![Diagram of a cycle graph]

3.3.1. Expected number of arrivals and probability of arrival

Theorem 2.1 gives now:

$$y_i = \frac{-u_{i,i-1}}{u_{ii}} y_{i-1} \quad (i = 1,2,\ldots,N)$$

Figure 5. A cycle graph
Put for i=1,2,…,N:

\[ M_i = \frac{-u_{i,i-1}}{u_{ii}} = \frac{p^*_{[i-1,i]}}{\alpha_i s_i + p^*_{[i,i+1]}} \]

where \( \alpha_i = \begin{cases} 1 - \rho_i n[i,i+1] + 1 \quad (\rho_i \neq 1) \\ (1 - \rho_i) \rho_i n[i,i+1] \quad (\rho_i = 1) \end{cases} \)

We now have:

\[ y_k = (\prod_{i=1}^{k} M_i) y_0 \quad (k = 0,1,\ldots,N) \quad \text{where} \quad \prod_{i=1}^{0} M_i = 1 \]

In a finite graph we have: \( \sum_{k=0}^{N} s_k y_k = 1 \), so:

\[ y_k = \frac{\prod_{i=1}^{k} M_i}{\sum_{n=0}^{N} s_n \prod_{i=1}^{n} M_i} \quad (k = 0,1,\ldots,N) \]

For \( i = 0,1,\ldots,N \) we have: if \( k \) on \( [i,i+1] \) then

\[ x_k = \begin{cases} \frac{(\rho^k - \rho n[i,i+1]) p^*_{[i,i+1]}}{(1 - \rho n[i,i+1])} y_i \quad (k = 0,1,\ldots,N); \quad \rho = \rho_i \neq 1 \\ \frac{(n[i,i+1] + 1 - k) p^*_{[i,i+1]} y_i}{(n[i,i+1] + 1) p[i,i+1]} \quad (k = 0,1,\ldots,N); \quad \rho = \rho_i = 1 \end{cases} \]

We also have:

\[ f_{M[0],M[0]} = 1 - y_0^{-1} = 1 - \sum_{n=0}^{N} s_n \prod_{i=1}^{n} M_i \]

3.3.2. Mean absorption time.

Using Theorem 2.2 we obtain:

\[ v_{i,i+1} = \frac{p^*_{[i,i+1]}}{n[i,i+1]}; \quad v_{i,i} = -s_i - \frac{p^*_{[i,i+1]}}{n[i,i+1]}; \quad \lambda_i = -(1 - s_i) - \frac{n[i,i+1]}{2 p_i} p^*_{[i,i+1]} \]

\[ n_{i+1} = \frac{\lambda_i - v_{i,i} n_i}{v_{i,i+1}} = \lambda_i n_i + \mu_i; \quad \lambda_i = \frac{-v_{i,i}}{v_{i,i+1}}; \quad \mu_i = \frac{\lambda_i}{v_{i,i+1}}; \quad (i = 0,1,2,\ldots,N) \]

\[ n_{k+1} = n_0 + \sum_{j=i+1}^{k} \lambda_j \]

\[ n_{N+1} = n_0; \quad \text{so:} \quad n_0 = \frac{\sum_{j=0}^{N} \prod_{i=0}^{j} \lambda_i}{1 - \prod_{i=0}^{N} \lambda_i} \]
4. Applications of multiple function barrier graphs

4.1. Introduction

Theorems 2.1 and 2.2 can also be used in analyzing random walks with multiple function barriers on (sub)set of the integers. E.g. a random walk on a half line \([0, \infty)\) with a partially reflecting barrier in 0 can be analyzed by using a simple network with one mfb \(M[0]\) and one end \(\Omega_{0,1}\).

In the next section we work out another application of our theory.

4.2: Random walk on the integers with two mfb’s.

We consider a discrete random walk on the integers of the p-q-r type \((p+q+r=1)\) with mfb’s in 0 and in \(N\) (\(N>0\)). In mfb 0 we have probabilities \(p_0, q_0, r_0, s_0\) \((p_0 + q_0 + r_0 + s_0 = 1, p_0q_0s_0 > 0)\) and in mfb \(N\) we have probabilities \(p_N, q_N, r_N, s_N\) \((p_N + q_N + r_N + s_N = 1, p_Nq_Ns_N > 0)\).

We can describe this random walk as a graph with three components: the first one is a finite interval \(M[0], M[1]\), where the states are numbered from \(M[0]\) to \(M[1]\): 0, 1, ..., \(N\), with \(N=n[0,1]+1\). The second component is a half line starting in \(M[1]\); states are numbered 0, 1, 2, ..., on this component, this corresponds with \(N, N+1, ...,\) in our original state space. The last component is a half line starting in \(M[0]\); states are numbered 0, 1, ..., on this half line, which corresponds with 0, -1, ... in the integers. On the interval and the first half line we have parameter \(\rho = \frac{p}{q}\), on the last half line we use parameter \(\rho^{-1}\) to get the desired random walk.

\[\Omega_{0,1} \xrightarrow{\rho^{-1}} M[0] \xrightarrow{\rho} M[1] \xrightarrow{\rho} \Omega_{1,1}\]

Figure 6. Random walk on the integers with two mfb’s.

First we handle with \(\rho \neq 1\).

We use Theorem 2.1 when starting in \(i_0\) \((0 < i_0 < N)\). Solving the two equations we find:

\[
y_0 = x_0 = \frac{(1 - \rho^{N-i_0})[s_N + p_N(1 - \rho^{-1})] + q_N(1 - \rho)}{[s_0(1 - \rho^N) + p_0(\rho^{N-1} - \rho^N)][s_N + p_N(1 - \rho^{-1})] + s_0q_N(1 - \rho)} = s_0x_0; \quad P(\text{absorption in } 0) = s_0x_0;
\]

\[
y_1 = x_N = \frac{s_0(\rho^{N-i_0} - \rho^N) + p_0(\rho^{N-1} - \rho^N)}{[s_0(1 - \rho^N) + p_0(\rho^{N-1} - \rho^N)][s_N + p_N(1 - \rho^{-1})] + s_0q_N(1 - \rho)} = s_Nx_N; \quad P(\text{absorption in } N) = s_Nx_N
\]

Using corollaries 2.1 and 2.2:

if \(\rho > 1\): \(P(\text{absorption in a mfb}) = 1 - p_N(1 - \rho^{-1})x_N\); \(P(\text{absorption in } \Omega_{1,1}) = p_N(1 - \rho^{-1})x_N\)

if \(\rho < 1\): \(P(\text{absorption in a mfb}) = 1 - q_0(1 - \rho)x_0\); \(P(\text{absorption in } \Omega_{0,1}) = q_0(1 - \rho)x_0\)
The second part of Theorem 2.1 gives:

\[
x_k = \begin{cases} 
(1 - \rho^k) \frac{q_N}{q} y_1 + (\rho^k - \rho^N) \frac{P_0}{p} y_0 + (\rho^k - 1) \frac{1 - \rho^{N-i_0}}{p-q} & (1 \leq k \leq i_0) \\
(1 - \rho^N) & (i_0 \leq k \leq N - 1) 
\end{cases}
\]

\[
x_k = \begin{cases} 
\frac{p_N}{p} y_1 & (k = N + 1, N + 2, \ldots) \\
\frac{p_N}{p} \rho^{k-N} y_1 & (k = N + 1, N + 2, \ldots) \\
\frac{q_0}{p} y_0 & (k = -1, -2, \ldots) \\
\frac{q_0 \rho^k}{p} y_0 & (k = -1, -2, \ldots) 
\end{cases}
\]

(\text{the last four lines are adapted on behalf of a different numbering of the state space and parameter } \rho^{-1} \text{ on the negative integers}).

Next we use Theorem 2.1 when starting in \(i_0 > N\); (original state space)

We study the case \(\rho > 1\) (\(\rho < 1\) proceeds along the same lines) and find:

\[
y_0 = x_0 = \frac{q_N (1 - \rho) \rho^{-i_0}}{[s_0 (1 - \rho^N) + p_0 (\rho^{N-1} - \rho^N)] [s_N + p_N (1 - \rho^{-1})] + s_0 q_N (1 - \rho)}
\]

\[
y_1 = x_N = \frac{[s_0 (1 - \rho^N) + p_0 (\rho^{N-1} - \rho^N)] [s_N + p_N (1 - \rho^{-1})] + s_0 q_N (1 - \rho)}{[s_0 (1 - \rho^N) + p_0 (\rho^{N-1} - \rho^N)] [s_N + p_N (1 - \rho^{-1})] + s_0 q_N (1 - \rho)}
\]

Now we can obtain the probability of arrival \(f_{ij}\).

If \(0 < i < N \text{ and } j > N\) then:

\[
f_{ij} = \frac{x_{ij}}{x_{ji}} = \frac{\frac{p^{i,j}_s}{p} y_1 [0 < i_0 < N]}{\left(1 - \rho^{-j}\right) + \frac{p^{i,j}_s}{p} y_1 [i_0 > N]} =
\]

\[
\frac{p_N (\rho - 1) [s_0 (\rho^{N-i} - \rho^N) + p_0 (\rho^{N-1} - \rho^N)]}{[s_N \rho (1 - \rho^{-j}) + p_N (\rho - 1) (2 - \rho^{-j}) [s_0 (1 - \rho^N) + p_0 (\rho^{N-1} - \rho^N)] + s_0 q_N (1 - \rho) \rho (1 - \rho^{-j})]
\]

This is the formula in graph language; for the original state space we need to change \(j\) in \(j-N\). Now we proceed with the case \(\rho = 1\).

We use Theorem 2.1 when starting in \(i_0\) (\(0 < i_0 < N\)); solving the two equations we find:

\[
y_0 = x_0 = \frac{q_N + (N - i_0) s_N}{p_0 s_N + s_0 (q_N + N s_N)}; \quad y_1 = x_N = \frac{(p_0 + s_0 i_0)}{p_0 s_N + s_0 (q_N + N s_N)}
\]
P(absorption in 0) = s_0 x_0; \quad P(absorption in N) = s_N x_N

\begin{align*}
x_k &= \begin{cases}
  \frac{q_0}{p} y_0 & (k = -1, -2, \ldots) \\
  \frac{(N-k)p_0y_0 + kq_Ny_1 + k(N-i_0)}{pn} & (1 \leq k \leq i_0) \\
  \frac{(N-k)p_0y_0 + kq_Ny_1 + (N-k)i_0}{pn} & (i_0 \leq k \leq N-1) \\
  \frac{p_N}{p} y_1 & (k = N+1, \ldots)
\end{cases}
\end{align*}

Next we use Theorem 2.1 when starting in $i_0 > N$ (original state space). We get:

\[ y_0 = x_0 = \frac{q_N}{p_0s_N + s_0(q_N + Ns_N)}; \quad y_1 = x_N = \frac{(p_0 + Ns_0)}{p_0s_N + s_0(q_N + Ns_N)} \]

If $0 < i < N$ and $j > N$ then:

\[ f_{ij} = \frac{x_{ij}}{x_{ji}} = \frac{p_{1(j)}y_1[0 < i_0 < N]}{p_{1(i)}y_1[i_0 > N]} = \frac{p_N(p_0 + s_0i)}{p_N(p_0 + Ns_0) + j[p_0s_N + s_0(q_N + Ns_N)]} \]

This is the formula in graph language; for the original state space we have to change $j$ in $j-N$.

Because of the fact that not both $\rho$ and $\rho^{-1}$ can be less than 1, we have: $m_k = \infty \ (k \in \mathbb{Z})$

5. Conclusion

Using generating functions we obtained expected number of arrivals, absorption probabilities and expected time before absorption for an asymmetric discrete random walk on a locally infinite graph in the presence of multiple function barriers (section 2). Explicit solutions were obtained for an oriented cycle graph, a finite and an infinite star graph (section 3) We also got results in the field of one-dimensional random walk: on the integers with two barriers (section 4), on an interval with three barriers and different probabilities between the barriers (section 3.1.3) and on the integers with one barrier and different probabilities left and right from the barrier (section 3.2.4).
Appendix A

A.1. Proof of theorem 2.1.

Case 1: \( z \neq 1 \) or \( p \neq q \)

The random walk between \( M[i] \) and \( M[j] \) (i<j and \( (i,j) \neq (c,d) \)) and the random walk on half line \( [i,j) \) (\( (i,j) \neq (c,h) \)) can be described by the difference equations:

\[
(1-rz)X_k = p z X_{k-1} + q z X_{k+1}
\]

with characteristic equation:

\[
q z \lambda^2 - (1-rz) \lambda + p z = 0
\]

Because of \( z \neq 1 \) or \( p \neq q \) we have:

\[
X_k = a \lambda_1^k + b \lambda_2^k \quad \text{with} \quad \lambda_1 \neq \lambda_2
\]

The random walk between \( M[c] \) and \( M[d] \) and the random walk on \([c,h)\) can be described by the difference equations:

\[
(1-rz)X_k = \delta(i_0,k) + p z X_{k-1} + q z X_{k+1}
\]

with solution:

\[
X_k = \begin{cases} 
\lambda_1^{k-i_0} + a_0 \lambda_1^k + b_0 \lambda_2^k & (0 \leq k \leq i_0) \\
\lambda_2^{k-i_0} + a_0 \lambda_1^k + b_0 \lambda_2^k & (k \geq i_0)
\end{cases}
\]

We first look at the interval case.

By focussing on states 1 and \( n=n[i,j] \) between \( M[i] \) and \( M[j] \) (i<j and \( (i,j) \neq (c,d) \)) we get:

\[
X_1 = p_{[i,j]} Y_1 + q z X_2 + rz X_1; \quad X_n = p z X_{n-1} + p_{[j,i]} z Y_j + rz X_n
\]

Using (2) and (4) we can express \( a \) and \( b \) in \( Y_i \) and \( Y_j \):

\[
(\lambda_2^{n+1} - \lambda_1^{n+1})a = \lambda_2^{n+1} \frac{p_{[i,j]} Y_1}{p} - \frac{p_{[j,i]} Y_j}{q} Y_1; \quad (\lambda_2^{n+1} - \lambda_1^{n+1})b = \frac{p_{[j,i]} Y_j}{q} - \frac{p_{[i,j]} Y_1}{p} Y_j
\]

Proceeding along the same lines but now between \( M[c] \) and \( M[d] \) gives:

\[
(\lambda_2^{n+1} - \lambda_1^{n+1})a_0 = \lambda_2^{n+1} \frac{p_{[i,j]} Y_1}{p} - \frac{p_{[j,i]} Y_j}{q} Y_1 + \frac{\lambda_2^{n+1} (\lambda_1^{-i_0} - \lambda_1^{-i_0})}{\sqrt{(1-rz)^2 - 4 p q z^2}}
\]

\[
(\lambda_2^{n+1} - \lambda_1^{n+1})b_0 = \frac{p_{[j,i]} Y_j}{q} - \frac{p_{[i,j]} Y_1}{p} Y_1 + \frac{\lambda_2^{n+1} (\lambda_1^{-i_0} - \lambda_1^{-i_0})}{\sqrt{(1-rz)^2 - 4 p q z^2}}
\]

We now consider a half line \([i,j)\) with \( (i,j) \neq (c,h)\):

Using (2) with \( a=0 \) and \( X_1 = p_{[i,j]} Y_1 + q z X_2 + rz X_1 \), we get:

\[
X_k = \lambda_2^k \frac{p_{[i,j]} Y_1}{p}
\]

After some calculation we obtain on halfline \([c,h)\):
18 Theo van Uem

\[
X_k = \left\{ \begin{array} {l}
\frac{\lambda_i^{-b} (\lambda_i^k - \lambda_i^k)}{\sqrt{(1-rz)^2 - 4pYz^2}} + \lambda_i^k \frac{p(i,j)Y_i}{p} & (0 \leq k \leq i_0) \\
\frac{\lambda_i^{-b} (\lambda_i^k - \lambda_i^k)}{\sqrt{(1-rz)^2 - 4pYz^2}} + \lambda_i^k \frac{p(i,j)Y_i}{p} & (k \geq i_0) 
\end{array} \right. 
\]  

(9)

We now focus on \( M[i] \) and its neighbours:

\[
Y_i = p_{[i,j]}^* z Y_i + \sum_{j<i} q(i,j)z X_j + \sum_{j<i} p(i,j)z X_n[i,j] + \sum_{j=1}^\infty q(i,j)z X_1[i,j] 
\]  

(10)

First we handle the interval \([i,j]\).

When \( i \neq c, i \neq d \) then (use (2)):

\[
X_1 = a \lambda_1 + b \lambda_2 \\
x_n = a \lambda_n^* + b \lambda_n^* 
\]

Substituting the formulae we found for \( a \) and \( b \) in (5) we get:

\[
q(\lambda_i^{n+1} - \lambda_i^n)X_1 = (\lambda_2 - \lambda_1)p_{[i,j]}^* Y_j + (\lambda_i^n - \lambda_i^n)p_{[i,j]}^* Y_i \quad (i < j) 
\]

and:

\[
p(\lambda_i^{n+1} - \lambda_i^n)X_n = (\lambda_2 - \lambda_1)(\lambda_i^n)(\lambda_i^n)p_{[i,j]}^* Y_j + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[i,j]}^* Y_i \quad (i < j) 
\]

Both formulae are valid for \( i < j \), but we need the last one with \( j < i \); interchanging \( i \) and \( j \) gives:

\[
p(\lambda_i^{n+1} - \lambda_i^n)X_n = (\lambda_2 - \lambda_1)(\lambda_i^n)(\lambda_i^n)p_{[i,j]}^* Y_j + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[i,j]}^* Y_i \quad (j < i) 
\]

(12)

When \( i = c \), using (3), (6) and (7):

\[
q(\lambda_i^{n+1} - \lambda_i^n)X_1 = (\lambda_2 - \lambda_1)p_{[j,i]}^* Y_j + (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_i + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_j + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_i 
\]

(13)

When \( i = d \):

\[
p(\lambda_i^{n+1} - \lambda_i^n)X_n = (\lambda_2 - \lambda_1)(\lambda_i^n)(\lambda_i^n)p_{[j,i]}^* Y_j + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_j + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_i + \lambda_i^n \lambda_2 (\lambda_i^n - \lambda_i^n)p_{[j,i]}^* Y_i 
\]

(14)

For the half line \([i,j)\) with \( i \neq c \), using (8):

\[
X_1 = \lambda_2 \frac{p_{[i,j]}^*}{p} Y_i 
\]

(15)

Half line \([i,j)\) with \( i = c \), using (9):

\[
X_1 = \lambda_2 \frac{p_{[i,j]}^* Y_i + \lambda_i^{-b}}{qz} 
\]

(16)

We are now ready for the final part.

For \( i \neq c, i \neq d \) we have, using (11) (12) and (15):

\[
Y_i = p_{[i,j]}^* z Y_i + \sum_{j<i} q(i,j)z X_j + \sum_{j<i} p(i,j)z X_n[i,j] + \sum_{j=1}^\infty q(i,j)z X_1[i,j] = 
\]

\[
p_{[i,j]}^* z Y_i + \left\{ \sum_{j<i} \left[ \frac{\lambda_2 (\lambda_i^n - \lambda_i^n)}{\lambda_i^{n+1} - \lambda_i^n} \right] p_{[i,j]}^* \right\} z Y_i + \sum_{j<i} \left[ \frac{(\lambda_2 - \lambda_1)(\lambda_i^n)(\lambda_i^n)}{\lambda_i^{n+1} - \lambda_i^n} \right] p_{[i,j]}^* \left\{ \sum_{j=1}^\infty q(i,j)z Y_j + \sum_{j=1}^\infty \left[ \frac{\lambda_i^n - \lambda_i^n}{\lambda_i^{n+1} - \lambda_i^n} \right] p_{[j,i]}^* \right\} z Y_j + 
\]

\[
\sum_{j<i} \left[ \frac{(\lambda_2 - \lambda_1)(\lambda_i^n)(\lambda_i^n)}{\lambda_i^{n+1} - \lambda_i^n} \right] p_{[j,i]}^* \left\{ \sum_{j=1}^\infty q(i,j)z Y_j + \sum_{j=1}^\infty \left[ \frac{\lambda_i^n - \lambda_i^n}{\lambda_i^{n+1} - \lambda_i^n} \right] p_{[j,i]}^* \right\} z Y_j 
\]
In the interval case we get an additional term \(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1} - \lambda_1}\) when \(i=c\) (use (13)) and
\[
\frac{\lambda_i^n \lambda_j^n (\lambda_{i-1} - \lambda_{i-1})}{\lambda_{n+1} - \lambda_1} \quad \text{when } i=d \text{ (use (14)).}
\]

When \(i=c\) we get in the half line case an additional term \(\lambda_i\) (use (16)).

We get the result of theorem 2.1 by taking \(z=1\) and noting that:

\[
\begin{align*}
\text{If } z=1 \text{ then } \\
\lambda_1 = \frac{p}{q} ; \lambda_2 = 1 \quad (p > q) \\
\lambda_1 = 1 ; \lambda_2 = \frac{p}{q} \quad (p < q)
\end{align*}
\]

Case 2: \(z=1\) and \(p=q\)

Using the same method as in case 1, but now with \(x\) and \(y\) instead of \(X\) and \(Y\) we find on interval \([i,j]\):

If \((i, j) \neq (c, d)\) then:

\[
x_k = ak + b, \quad (n+1)a = -\frac{p_{i,i}}{p} y_i + \frac{p_{i,j}}{p} y_j; \quad b = \frac{p_{i,i}}{p} y_i
\]

If \((i, j) = (c, d)\) then:

\[
x_k = \begin{cases} 
    a_0 k + b_0 & (0 \leq k \leq i_0) \\
    a_0 k + b_0 + \frac{i_0-k}{p} & (i_0 \leq k \leq n)
\end{cases}
\]

where

\[
(n+1)a_0 = -\frac{p_{i,j}}{p} y_i + \frac{p_{i,j}}{p} y_j + \frac{n+1-i_0}{p}; \quad b_0 = \frac{p_{i,j}}{p} y_i
\]

On half line \([i,j]\) we have:

If \((i, j) \neq (c, d)\) then:

\[
x_k = ak + b, \quad a = 0; \quad b = \frac{p_{i,j}}{p} y_i
\]

If \((i, j) = (c, d)\) then:

\[
x_k = \begin{cases} 
    a_0 k + b_0 & (0 \leq k \leq i_0) \\
    a_0 k + b_0 + \frac{i_0-k}{p} & (k \geq i_0)
\end{cases}
\]

where

\[
a_0 = \frac{1}{p} ; \quad b_0 = \frac{p_{i,j}}{p} y_j
\]

For \(i \neq c\) we have:

\[
y_i = p_{i,i} y_i + y_i \sum_{j \neq i} \frac{np_{i,j}}{n+1} + \sum_{j \neq i} \frac{p_{i,j}}{n+1} y_j + y_i \sum_{j=1}^n p_{i,j}
\]

We get the same answer using \(z=1\) and l’Hospital’s rule in case 1.

When starting in \(c\) or \(d\) on the interval \([c,d]\) we get the same results with an additional term

\[
(1 - \frac{i_0}{n+1}) \text{ when } i=c \text{ and } \left(\frac{i_0}{n+1}\right) \text{ when } i=d.
\]

Starting in \(c\) on \([c,h)\) we get an additional term 1.
Appendix B

B1. Proof of Theorem 2.2.

The random walk between M[c] and M[d] and the random walk on [c,h) can be described by the difference equations:

\[(1-r)m_k = pm_{k+1} + qm_{k-1} + 1\]  \hspace{1cm} (17)

First we discuss the interval part:

Interval [i,j], case \( \rho \neq 1 \)

A solution of (17) is given by:

\[m_k = a\rho^{-k} + b - \frac{k}{p-q} \quad (k = 0,1,\ldots,n[i, j] + 1)\]  \hspace{1cm} (18)

Using (18) with \( k=0 \) and \( k=n+1 \), we can express \( a \) and \( b \) in \( n_i \) and \( n_j \).

Using that expressions we get:

\[m_i = \frac{(1-\rho^n)n_i + (\rho^n - \rho^{n+1})n_j}{1-\rho^{n+1}} + \left(\frac{n+1}{p-q}\right)\left(\frac{\rho^n - \rho^{n+1}}{1-\rho^{n+1}}\right) - \frac{1}{p-q} \quad (i < j)\]

\[m_n = \frac{(1-\rho)n_i + (\rho - \rho^{n+1})n_j}{1-\rho^{n+1}} + \left(\frac{n+1}{p-q}\right)\left(\frac{\rho - \rho^{n+1}}{1-\rho^{n+1}}\right) - \frac{n}{p-q} \quad (i < j)\]

We use the last formula when \( i>j \), so we have to interchange \( i \) and \( j \) in the formula:

\[m_n = \frac{(1-\rho)n_i + (\rho - \rho^{n+1})n_j}{1-\rho^{n+1}} + \left(\frac{n+1}{p-q}\right)\left(\frac{\rho - \rho^{n+1}}{1-\rho^{n+1}}\right) - \frac{n}{p-q} \quad (i > j)\]

Interval [i,j], case \( \rho = 1 \)

\[m_k = ak + b - \frac{k^2}{2p} \quad (k = 0,1,\ldots,n[i, j] + 1)\]

Following the same method as in case \( \rho \neq 1 \) we get:

\[m_i = \frac{nm_i + n_j}{n+1} + \frac{n}{2p} \quad (i < j)\]

\[m_n = \frac{nm_i + n_j}{n+1} + \frac{n}{2p} \quad (i > j)\]

The same formulae are found by applying l'Hôpital's rule twice in the interval case \( \rho \neq 1 \).

Next we discuss a half line [i,j).

For a half line [i,j) with \( \rho(i, j)<1 \) we get:

\[m_k = n_i + \frac{k}{q-p} \quad (k = 0,1,2,\ldots)\]

Finally we use: \( n_i = p_{[i,i]}n_i + (1-s_i) \sum_{j>i} p_{[i,j]}m_j[i, j] + \sum_{j<i} p_{[i,j]}m_n[i, j] + \sum_{j=1}^{\infty} p_{[i,j]} m_j[i, j] \)
References

[1] Feller W 1968 *An Introduction to probability theory and its applications*, (third edition) Vol. 1, John Wiley, New York

[2] Weesakul B 1961 The random walk between a reflecting and an absorbing barrier, *Ann. Math. Statist.*, 32, 765-769

[3] Lehner G 1963 One-dimensional random walk with a partially reflecting barrier, *Ann. Math. Stat.*, 34, 405-412

[4] Gupta H C 1966 Random walk in the presence of a multiple function barrier, *Journ. Math Sci*, 1, 18-29

[5] Dua S, Khadilkar S and Sen K 1976 A modified random walk in the presence of partially reflecting barriers, *J. Appl. Prob.*, 13, 169-175

[6] Percus O E 1985 Phase transition in one-dimensional random walk with partially reflecting boundaries, *Adv. Appl. Prob.*. 17, 594-606

[7] El-Shehawey M A 2000 Absorption probabilities for a random walk between two partially absorbing boundaries. I, *J. Phys. A: Math. Gen.*, 33, 9005-9013.

[8] Burioni R and Cassi D 2005 Random walks on graphs: ideas, techniques and results, *J. Phys. A: Math. Gen.*, 38, R45-R78.

[9] Durhuus B, Jonsson T and Wheater J 2006 Random walks on combs, *J. Phys. A: Math. Gen.*, 39, 1009-1038