Symmetrization, factorization and arithmetic of quasi-Banach function spaces

Paweł Kolwicz*, Karol Leśnik* and Lech Maligranda

Abstract

We investigate relations between symmetrizations of quasi-Banach function spaces and constructions such as Calderón–Lozanovskii spaces, pointwise product spaces and pointwise multipliers. We show that under reasonable assumptions the symmetrization commutes with these operations. We determine also the spaces of pointwise multipliers between Lorentz spaces and Cesàro spaces. Developed methods may be regarded as an arithmetic of quasi-Banach function spaces and proofs of Theorems 3, 4 and 6 give a kind of tutorial for these methods. Finally, the above results will be used in proofs of some factorization results.

1 Introduction and preliminaries

The functional $x \mapsto \|x\|$ on a given vector space $X$ is called a quasi-norm if the following three conditions are satisfied: $\|x\| = 0$ iff $x = 0$; $\|ax\| = |a| \|x\|$, $x \in X$, $a \in \mathbb{R}$; there exists $C = C_X \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$. We call $\| \cdot \|$ a $p$-norm where $0 < p \leq 1$ if, in addition, it is $p$-subadditive, that is, $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

A very important result here is the Aoki–Rolewicz theorem (cf. [27, Theorem 1.3 on p. 7], [46, p. 86], [47, pp. 6–8]): if $0 < p \leq 1$ is given by $C = 2^{1/p - 1}$, then there exists an equivalent $p$-norm $\| \cdot \|_p$ so that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \quad \text{and} \quad \|x\|_1 \leq \|x\| \leq 2C\|x\|_1$$

for all $x, y \in X$. Precisely,

$$\|x\|_1 = \inf\{\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p} : x = \sum_{k=1}^{n} x_k, x_1, x_2, \ldots, x_n \in X, n = 1, 2, \ldots\}$$

*Research partially supported by Ministry of Science and Higher Education of Poland, Grant number 04/43/DSPB/0094.

2010 Mathematics Subject Classification: 46E30, 46B20, 46B42.

Key words and phrases: Banach ideal spaces, quasi-Banach ideal spaces, symmetrization operation, Calderón spaces, Calderón–Lozanovskii spaces, symmetric spaces, pointwise multipliers, pointwise multiplication, product spaces, Cesàro spaces, factorization.
defines such a $p$-norm on $X$. The quasi-norm $\| \cdot \|$ induces a metric topology on $X$: in fact a metric can be defined by $d(x, y) = \| x - y \|^p_p$, when the quasi-norm $\| \cdot \|$ is $p$-subadditive. We say that $X = (X, \| \cdot \|)$ is a quasi-Banach space if it is complete for this metric.

A quasi-normed or normed space $E = (E, \| \cdot \|_E)$ is said to be a quasi-normed ideal (function) space or normed ideal (function) space on $I$, where $I = (0, 1)$ or $I = (0, \infty)$ with the Lebesgue measure $m$, if $E$ is a linear subspace of $L^0(I)$ and satisfies the so-called ideal property, which means that if $y \in E, x \in L^0$ and $|x(t)| \leq |y(t)|$ for almost all $t \in I$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$. If, in addition, $E$ is a complete space, then we say that $E$ is a quasi-Banach ideal space or a Banach ideal space (a quasi-Banach function space or a Banach function space), respectively. We assume that $E$ has a weak unit, i.e., it has a function $x$ in $E$ which is positive a.e. on $I$ (see [32] and [45]).

A quasi-normed ideal space $(E, \| \cdot \|_E)$ is called normable if there exists on $E$ a norm $\| \cdot \|_1$ equivalent to $\| \cdot \|_E$, that is there are constants $A, B > 0$ such that $A\|x\|_1 \leq \|x\|_E \leq B\|x\|_1$ for all $x \in E$.

Recall that a quasi-normed ideal space $E$ has the Fatou property if $0 \leq x_n \uparrow x \in L^0$ with $x_n \in E$ and $\sup_{n \in \N} \|x_n\|_E < \infty$ imply that $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$. Recall also that $E$ is order continuous if for every $x \in E$ and any $x_n \to 0$ a.e. with $0 \leq x_n \leq \|x\|$ we have $\|x_n\|_E \to 0$.

The Köthe dual (or associated space) $E'$ to a quasi-normed ideal space $E$ on $I$ is the space of all $x \in L^0(I)$ such that 

$$
\|x\|_{E'} = \sup \left\{ \int_I |x(t)y(t)| \, dt : \|y\|_E \leq 1 \right\} < \infty. \tag{1}
$$

It may happen that $E' = \{0\}$ but if $E' \neq \{0\}$ (for example, when $E$ is a Banach ideal space), then $(E', \| \cdot \|_{E'})$ is a Banach ideal space. Observe that $E'$ has the Fatou property and if $E$ is a Banach ideal space, then $E$ has the Fatou property if and only if $E'' = E$ (cf. [42] p. 30 and [57]). For $0 < p \leq \infty$ we define the conjugate number $p'$ by

$$p' := \begin{cases} 1 & \text{if } p = \infty, \\ p/(p-1) & \text{if } 1 < p < \infty, \\ \infty & \text{if } 0 < p \leq 1. \end{cases} \tag{2}
$$

The weighted quasi-normed ideal space $E(w)$, where $w: I \to (0, \infty)$ is a measurable function (weight on $I$), is defined by the norm $\|x\|_{E(w)} = \|xw\|_E$.

By a symmetric space on $I$ we mean a (quasi-)normed ideal space $E = (E, \| \cdot \|_E)$ with the additional property that for any two equimeasurable functions $x \sim y, x, y \in L^0(I)$ (that is, they have the same distribution functions $d_x = d_y$, where $d_x(\lambda) = m(\{t \in I : |x(t)| > \lambda\}), \lambda \geq 0$) and $x \in E$ we have that $y \in E$ and $\|x\|_E = \|y\|_E$. In particular, $\|x\|_E = \|x^*\|_E$, where $x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}, t \geq 0$.

A symmetric space $E$ has the majorant property if $y \in E, \int_0^t x^*(s) \, ds \leq \int_0^t y^*(s) \, ds$ for all $t \in I$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$. For example, a symmetric normed space $E$ with the Fatou property or being order continuous has the majorant property (cf. [37] p. 105).

The dilation operator $D_s, s > 0$, is defined by $D_sx(t) = x(t/s)$ for $t \in I = (0, \infty)$ and

$$D_sx(t) = \begin{cases} x(t/s) & \text{if } t < \min\{1, s\}, \\ 0 & \text{if } s \leq t < 1, \end{cases}$$

2
for \( t \in I = (0, 1) \). This operator is bounded in any symmetric quasi-normed space \( E \) on \( I \) (and \( \|s\|_{E \to E} \leq \max(1, s) \) for symmetric normed spaces, see \([37]\) pp. 96–98 for \( I = (0, \infty) \) and \([42]\) p. 130) for both cases) and in some nonsymmetric quasi-normed function spaces.

For two ideal (quasi-) normed spaces on \( I \) the symbol \( E \hookrightarrow F \) means that the inclusion \( E \subset F \) is continuous with a norm which is not bigger than \( C \), i.e., \( \|x\|_F \leq C\|x\|_E \) for all \( x \in E \). In the case when the embedding \( E \hookrightarrow F \) holds with some (unknown) constant \( C > 0 \) we simply write \( E \hookrightarrow F \). Moreover, \( E = F \) (and \( E \equiv F \)) means that the spaces are the same and the norms are equivalent (equal).

More information about normed or Banach ideal spaces and symmetric spaces can be found, for example, in the books \([4], [32], [37] \) and \([42]\). Moreover, information on quasi-normed spaces, quasi-normed function spaces and symmetric spaces we can find, for example, in the books \([27], [53] \) and the papers \([24], [31], [46], [51]\).

By \( \mathcal{P} \) we denote the set of concave nondecreasing functions \( \rho_0: [0, \infty) \to [0, \infty) \) which are 0 only at 0 and we identify \( \mathcal{P} \) with sets of functions \( \rho: [0, \infty) \times [0, \infty) \to [0, \infty) \) by putting \( \rho(s, t) = s\rho_0(t/s) \) for \( s > 0 \) and 0 for \( s = 0 \).

For two normed ideal spaces \( E, F \) on \( I \) and \( \rho \in \mathcal{P} \) the Calderón–Lozanovskii space (construction) \( \rho(E, F) \) is defined as the set of all \( x \in L^0(I) \) such that for some \( x_0 \in E, x_1 \in F \) with \( \|x_0\|_E \leq 1, \|x_1\|_F \leq 1 \) and for some \( \lambda > 0 \) we have \( |x| \leq \lambda \rho(|x_0|, |x_1|) \) a.e. on \( I \). The norm \( \|x\|_\rho = \|x\|_{\rho(E, F)} \) of an element \( x \in \rho(E, F) \) is defined as the infimum values of \( \lambda \) for which the above inequality holds. It can be shown that

\[
\rho(E, F) = \{ x \in L^0(I) : |x| \leq \rho(|x_0|, |x_1|) \text{ a.e. on } I \text{ for some } x_0 \in E, x_1 \in F \}
\]

and

\[
\|x\|_{\rho(E, F)} = \inf \{ \max \{\|x_0\|_E, \|x_1\|_F\} : |x| \leq \rho(|x_0|, |x_1|) \text{ a.e. on } I, x_0 \in E, x_1 \in F \}.
\]

If \( \rho(u, v) = u^\theta v^{1-\theta} \) with \( 0 < \theta < 1 \) we write \( E^\theta F^{1-\theta} \) instead of \( \rho(E, F) \) and these are Calderón spaces (Calderón product) defined already in 1964 in \([10]\). Another important situation, investigated by Calderón and independently by Lozanovskii in 1964, appears when we put \( F \equiv L^\infty \) (see \([10], [43], [44]\)). We can see that they are generalizations of Orlicz spaces. Moreover, the \( p \)-convexification \( E^{(p)} \) of \( E \), for \( 1 < p < \infty \), is a special case of Calderón product

\[
E^{1/p}(L^\infty)^{1-1/p} = E^{(p)} = \{ x \in L^0 : |x|^p \in E \} \text{ and } \|x\|_{E^{(p)}} = \|\|x\|^p\|_{E^p}^{1/p}.
\]

More information on the Calderón–Lozanovskii spaces can be found in the books \([37], [45]\).

For two quasi-normed ideal spaces \( E, F \) on \( I \) we can define similarly the Calderón–Lozanovskii space (construction) \( \rho(E, F) \) obtaining the quasi-normed ideal space (cf. \([29]\) and \([52]\)). Also the definition of \( p \)-convexification makes sense even for \( 0 < p < \infty \). Note also that \( E^{(p)} \) may be just a quasi-normed ideal space for \( 0 < p < 1 \) even if \( E \) is a normed ideal space.

Consider the Hardy operator \( H \) and its formal Köthe dual \( H^* \) defined for \( x \in L^0(I) \) by

\[
Hx(t) = \frac{1}{t} \int_0^t x(s) \, ds, \quad H^*x(t) = \int_t^l \frac{x(s)}{s} \, ds \text{ with } l = m(I), \ t \in I.
\]
Note that if $0 < p < 1$, then neither $H$ nor $H^*$ are bounded on $L^p(w)$ spaces for any weight $w$ (cf. p. 41]), therefore we need to consider their “r-convexifications” for $0 < r < \infty$, which are defined by

$$H_r x = [H(|x|^r)]^{1/r} \quad \text{and} \quad H_r^* x = [H^* (|x|^r)]^{1/r},$$

provided the corresponding integrals are finite. These operators are not linear but they are $\alpha$-sublinear, that is,

$$H_r(\lambda x) = |\lambda| H_r x \quad \text{and} \quad H_r(x + y) \leq c (H_r x + H_r y)$$

and similarly for the operator $H_r^*$, where $c = \max(1, 2^{1/r-1})$.

In the case $w(t) = t^\alpha$, $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$ it is easy to prove that if $r \leq p$ and $r(\alpha + 1/p) < 1$, then $H_r$ is bounded on $L^p(w)$ with the norm $\leq (1 - \alpha r - r/p)^{-1/r}$ (see Theorem 2(i)). Also if $r \leq p$ and $\alpha + 1/p > 0$, then $H_r^*$ is bounded on $L^p(w)$ with the norm $\leq (\alpha r + r/p)^{-1/r}$.

Using the Fubini theorem we obtain the following equality

$$H_r H_r^* x(t) = [H_r x(t)^r + H_r^* x(t)^r]^{1/r}, \quad \text{for} \quad t \in I. \quad (3)$$

In fact,

$$H_r H_r^* x(t)^r = \frac{1}{t} \int_0^t H_r^* x(s)^r \, ds = \frac{1}{t} \int_0^t \left( \int_s^t \frac{|x(u)|^r}{u} \, du \right) ds$$

$$= \frac{1}{t} \int_0^t \left( \int_0^u \frac{|x(u)|^r}{u} \, du \right) ds + \frac{1}{t} \int_0^t \left( \int_0^t \frac{|x(u)|^r}{u} \, du \right) ds$$

$$= \frac{1}{t} \int_0^t |x(u)|^r \, du + \int_0^t \frac{|x(u)|^r}{u} \, du = H_r x(t)^r + H_r^* x(t)^r.$$

For two quasi-normed ideal spaces $E$, $F$ on $I$ the product space $E \odot F$ is

$$E \odot F = \{ u \in L^0(I) : u = x \cdot y \quad \text{for some} \quad x \in E \quad \text{and} \quad y \in F \},$$

and for $u \in E \odot F$ we put

$$\| u \|_{E \odot F} = \inf \{ \| x \|_E \| y \|_F : u = x \cdot y, x \in E, y \in F \}.$$

First note that the product $E \odot F$ is a linear space thanks to the ideal property of $E$ and $F$ (see (36)). The space $(E \odot F, \| \cdot \|_{E \odot F})$ is a quasi-normed ideal space on $I$ (even if $E, F$ are normed spaces). More about product spaces with some computations can be found in (38) and (36) (see also (6,7) for the case of sequence spaces).

The space of (pointwise) multipliers $M(E,F)$ is defined as

$$M(E,F) = \{ x \in L^0 : xy \in F \quad \text{for each} \quad y \in E \}$$

with the operator norm

$$\| x \|_{M(E,F)} = \sup_{\| y \|_E = 1} \| xy \|_F.$$

Properties and several examples of above constructions are presented in (35), (36), (48), (51).

We collect below several simple and useful facts.
Remark 1. Let $E$ be a quasi-Banach ideal space.

(i) If $D_2$ is bounded on $E$, then $D_2$ is bounded on $E^{(p)}$ for each $p > 0$ and $\|D_2\|_{E^{(p)} \to E^{(p)}} \leq \|D_2\|^{1/p}_{E \to E'}$.

(ii) If $D_2$ is bounded on $E, F$, then $D_2$ is bounded on $E \odot F$ and $\|D_2\|_{E \odot F \to E \odot F} \leq \|D_2\|_{E \to E} \|D_2\|_{F \to F'}$.

(iii) If $H$ is bounded on $E$, then $H$ is bounded on $E^{(p)}$ for all $p > 1$ and $\|H\|_{E^{(p)} \to E^{(p)}} \leq \|H\|^{1/p}_{E \to E'}$.

In the case when $E$ is a Banach ideal space, then we also have

(iv) $H$ is bounded on $E$ if and only if $H^*$ is bounded on $E'$ and $\|H\|_{E \to E'} = \|H^*\|_{E' \to E'}$.

(v) $D_p$ is bounded on $E$ if and only if $D_{1/p}$ is bounded on $E'$ for each $p > 0$ and $\|D_p\|_{E \to E} = \|D_{1/p}\|_{E' \to E'}$.

Proof. (i) It follows by the definition. (ii) By the assumption, $D_2$ is bounded on $E^{1/2}F^{1/2}$ (see [35, Theorem 15.13, p.190]). Since $E \odot F = (E^{1/2}F^{1/2})^{(1/2)}$ (see [36, Theorem 1]), the conclusion follows by (i). (iii) Since $H$ is bounded on $L^\infty$, it is enough to apply the equality $E^{(p)} = E^{1/p}(L^\infty)^{1-1/p}$. (iv) The necessity follows from the equality $\int (H^*x)y = \int xHy$ for each $x \in E'$ and $y \in E$. (v) The proof comes from the definition of the Köthe dual. □

The paper is organized as follows: In Section 2 we define symmetrization $E^\ast$ of a quasi-normed ideal space $E$ on $I = (0,1)$ or $I = (0, \infty)$ and collect some preliminary properties.

In Section 3 we investigate a commutativity property of symmetrization operation $E \mapsto E^\ast$ with some known constructions, like the sum of the spaces $E + F$, the Calderón-Lozanovskiĭ construction $\rho(E, F)$, the pointwise product $E \odot F$ and the Köthe duality $E'$. In Theorem 1 we found conditions under which $\rho(E, F)^\ast = \rho(E^\ast, F^\ast)$, in particular, when $(E + F)^\ast = E^\ast + F^\ast$. Then, in Theorem 2 we prove that $(E')^\ast = (E^\ast)'$ under additional assumption on $E$, which is essential (see Example 3), that is, the Köthe duality does not commute with symmetrization, in general for Banach ideal spaces.

In Section 4 we give sufficient conditions under which the space of pointwise multipliers $M(E, F)$ commutes with the symmetrization operation $E \mapsto E^\ast$, that is $M(E, F)^\ast = M(E^\ast, F^\ast)$ (Theorem 3). We also fully identify the space of pointwise multipliers for classical Lorentz spaces $M(L^{p_1, q_1}, L^{p_2, q_2})$ (Theorem 4).

In Section 5 the notion of the explicit factorization for product space $E \odot F$ is introduced. In Theorem 5 we proved that under some assumptions on quasi-Banach ideal spaces $E, F$ from the explicit factorization for $G = E \odot F$ it follows equality for symmetrizations $G^\ast = E^\ast \odot F^\ast$ and the explicit factorization holds.

In section 6 we prove that, under some assumptions, from the factorization $F = E \odot M(E, F)$ we can conclude the factorization of respective symmetrizations $F^\ast = E^\ast \odot M(E^\ast, F^\ast)$.

Finally, in Section 7, the space of multipliers and factorization of Cesàro spaces is presented in Theorem 6 and Corollary 4.
2 Symmetrization of quasi-normed ideal spaces

Let $E = (E, \| \cdot \|_E)$ be a quasi-normed ideal space on $I$. The symmetrization $E^\ast$ of $E$ is defined as

$$E^\ast = \{ x \in L^0(I) : x^* \in E \}$$

with the functional $\| x \|_{E^\ast} = \| x^* \|_E$. For two quasi-normed ideal spaces $E, F$ on $I$ it follows directly from the definition that:

$$E \subseteq F \implies E^\ast \subseteq F^\ast \quad \text{and} \quad (E \cap F)^\ast = E^\ast \cap F^\ast.$$  \hspace{1cm}  (5)

It may happen that symmetrization is trivial, that is, $E^\ast = \{0\}$, as it is for example if $E = L^1(1/t)$ or $E = L^\infty(1/t)$. It is easy to see that $E^\ast \neq \{0\}$ if and only if $\chi_{(0, a)} \in E$ for some $a > 0$.

A. Kamińska and Y. Raynaud proved in [31, Lemma 1.4] that the functional $\| \cdot \|_{E^\ast}$ is a quasi-norm if and only if there is a constant $1 \leq A < \infty$ such that

$$\| D_2 x^* \|_{E^\ast} \leq A \| x^* \|_E$$

for all $x^* \in E$,  \hspace{1cm}  (6)

and then $(E^\ast, \| \cdot \|_{E^\ast})$ is a quasi-normed symmetric space. The smallest possible constant $A$ in (6) we denote by $A_E$. Furthermore, if $(E, \| \cdot \|_E)$ is a quasi-Banach space, then $(E^\ast, \| \cdot \|_{E^\ast})$ is quasi-Banach too (see [31, Lemma 1.4]).

Condition (6) means that the dilation operator $D_2$ is bounded on the cone of nonnegative nonincreasing elements $x = x^* \in E$. It implies in particular that $E^\ast$ is a linear space which follows immediately from the inequality $(x + y)^* (t) \leq x^* (t/2) + y^* (t/2)$. However, it is not obvious whether the linearity of $E^\ast$ gives the condition (6). We show that these conditions are in fact equivalent. Denote by

$$E^\downarrow = \text{the cone of nonnegative and nonincreasing elements } x = x^* \in E.$$

**Lemma 1.** Let $E$ be a quasi-normed ideal space on $I$. Then $E^\ast$ is a linear space if and only if $D_2 x \in E^\downarrow$ for each $x \in E^\downarrow$.

**Proof.** The sufficiency follows from the inequality $(x + y)^* (t) \leq x^* (t/2) + y^* (t/2)$. We prove the necessity. Suppose there is an element $x \in E^\downarrow$ such that $D_2 x \notin E^\downarrow$. Thus, $x = x^* \in E$ and $(D_2 x)^* = D_2 x$, whence $D_2 x \notin E$. We consider two cases.

(1) Let $I = (0, \infty)$. Set $A = \bigcup_{k=0}^\infty (2k, 2k + 1]$ and $A' = (0, \infty) \setminus A$. Let $\sigma_1 : A \to (0, \infty)$ and $\sigma_2 : A' \to (0, \infty)$ be measure preserving transformations. Define

$$x_1 = x \circ \sigma_1 \quad \text{and} \quad x_2 = x \circ \sigma_2.$$

Then $x_1, x_2 \in E^\ast$, because $x_1^* = x_2^* = x^* = x \in E$. Moreover, since $x_1 \perp x_2$, so $d_{x_1 + x_2} = 2d_x$ and $(x_1 + x_2)^* (t) = x^* (t/2) = D_2 x \notin E$. It means that $E^\ast$ is not a linear space.

(2) Assume that $I = [0, 1)$. Let $x_1 = D_{1/2} D_2 x$. Then supp $x_1 \subset (0, 1/2)$ and $x_1 = x_1^* \leq x \in E$. Moreover, $D_2 x_1 = D_2 D_{1/2} D_2 x = D_2 x \notin E$. Take

$$x_2 (t) = \begin{cases} x_1 (t - 1/2) & \text{for } t \in (1/2, 1), \\ 0 & \text{if } t \in (0, 1/2). \end{cases}$$

Thus $x_2^* = x_1^* \in E$ and consequently $x_1, x_2 \in E^\ast$. On the other hand, $(x_1 + x_2)^* (t) = x^* (t/2) = D_2 x \notin E$. Thus again $E^\ast$ is not a linear space. \hfill $\Box$
Lemma 2. Assume that $E$ is a quasi-normed ideal space on $I$. If $D_2x \in E^\perp$ for each $x \in E^\perp$, then there is a constant $1 \leq A < \infty$ such that $\|D_2x\|_E \leq A \|x\|_E$ for all $x \in E^\perp$.

Proof. Suppose the condition is not satisfied, that is, we find a sequence $(x_n)$ in $E^\perp$ such that $\|x_n\|_E = 1$ and $\|D_2x_n\|_E \geq n (2C)^n$, where the constant $C$ is from the quasi-triangle inequality of $E$. Let $y = \sum_{n=1}^{\infty} (2C)^{-n} x_n$. Since $\sum_{n=1}^{\infty} C^n \|(2C)^{-n} x_n\| < \infty$, by Theorem 1.1 from [46], we conclude that $y \in E$. Obviously, $y \in E^\perp$. Furthermore,

$$D_2y = D_2 \left( \sum_{n=1}^{\infty} (2C)^{-n} x_n \right) \geq D_2 \left( (2C)^{-n} x_n \right) = (2C)^{-n} D_2x_n$$

for each $n$. Thus $\|D_2y\|_E \geq (2C)^{-n} \|D_2x_n\|_E \geq (2C)^{-n} n (2C)^n = n$ for each $n$. Since $(D_2y)^* = D_2y$, it follows that $D_2y \notin E^\perp$. \hfill \Box

Now we are ready to conclude a stronger and more complete characterization than it has been presented in [31, Lemma 1.4].

Corollary 1. Let $E$ be a quasi-normed ideal space on $I$. The following statements are equivalent:

(i) $E^\ast$ is a linear space.

(ii) For each $x \in E^\perp$ we have $D_2x \in E^\perp$.

(iii) There is a constant $1 \leq A < \infty$ such that $\|D_2x\|_E \leq A \|x\|_E$ for all $x \in E^\perp$.

(iv) $(E^\ast, \| \cdot \|_{E^\ast})$ is a quasi-normed space.

Proof. The equivalences $(i) \iff (ii) \iff (iii)$ come from the above two lemmas. The last equivalence $(iii) \iff (iv)$ has been proved in [31, Lemma 1.4]. \hfill \Box

It is worth to mention that the equivalence $(i) \iff (iv)$ for the Lorentz spaces $\Lambda_{p,w^n}$ is already known. Namely, each of conditions $(i)$, $(iv)$ is equivalent to $W \in \Delta_2$ which has been proved in [15] and [28], respectively (see the discussion following Problem 1 in Section 4 for the respective definitions).

Of course, if $E^\ast = \{0\}$, then condition (i) is satisfied trivially. We present some examples when $E^\ast \neq \{0\}$ and condition (i) does not hold (equivalently none of conditions from Corollary 1 is satisfied).

Example 1. (a) Consider $E = L^\infty \left( \frac{1}{1-t} \right)$ on $I = (0,1)$. Then, taking $x = \chi_{(0,1/2)}$ we have $x = x^* \in E$, whence $E^\ast \neq \{0\}$. Moreover, $D_2x = \chi_{(0,1)} \notin E$.

(b) For $a > 0$ let

$$w_a(t) = \chi_{(0,a)}(t) + \frac{1}{t-a} \chi_{(a,\infty)}(t), \quad t > 0.$$  

Consider the weighted Banach ideal space $E = L^1(w_a)$ on $I = (0,\infty)$ and its symmetrization $E^\ast = \left[ L^1(w_a) \right]^\ast = \Lambda_{1,w_a}$. Of course, $E^\ast \neq \{0\}$ since $\chi_{(0,a)} \in E^\ast$ with $\|\chi_{(0,a)}\|_{E^\ast} = a$. On the other hand,

$$\|D_2\chi_{(0,a)}\|_E = a + \int_{a}^{2a} \frac{1}{t-a} dt = \infty,$$  

7
which means that $\| \cdot \|_{E^{(s)}}$ is not a quasi-norm (see [31, Lemma 1.4]), equivalently, none of conditions from Corollary 1 is satisfied. For example, $W_a(t) = \int_0^t w_a(s)ds = t$ for $0 < t < a$ and $W_a(t) = \infty$ for $t > a$. Thus $W_a \notin \Delta_2$ and by Remark 1.3 in [15] we get that $E^{(s)} = \Lambda_{1,w_a}$ is not a linear space. In fact, the symmetrization was recently (cf. [18], [19], [30], [31], [33], [34]). The Lorentz, Marcinkiewicz and Orlicz–Lorentz spaces are particular cases of this construction. In fact, the symmetrization which means that $\| \cdot \|_{L^\infty}$ of the Musielak–Orlicz spaces is not a linear space. In fact, the symmetrization which was investigated in [15], [28], the symmetrization $[L^\infty(w)]^{(*)}$ of the weighted space $L^\infty(w)$ is the Marcinkiewicz space $M_w$, and the symmetrization $(L^p)^{(*)}$ of the Musielak–Orlicz spaces $L^p$ with $\Phi(t,u) = \varphi(u)w(t)$ is the Orlicz–Lorentz space $\Lambda_{\varphi,w}$, which structure was and still is investigated in many papers (cf. [31] and literature therein).

The symmetrization $E^{(s)}$ of a Banach ideal space $E$ has been intensively studied recently (cf. [18], [19], [30], [31], [33], [34]). The Lorentz, Marcinkiewicz and Orlicz–Lorentz spaces are particular cases of this construction. In fact, the symmetrization $[L^p(w)]^{(*)}$ of weighted Lebesgue spaces $L^p(w)$ even for $0 < p < \infty$ is the Lorentz space $\Lambda_{p,w}$, which structure was investigated in [15], [28], the symmetrization $[L^\infty(w)]^{(*)}$ of the weighted space $L^\infty(w)$ is the Marcinkiewicz space $M_w$, and the symmetrization $(L^p)^{(*)}$ of the Musielak–Orlicz spaces $L^p$ with $\Phi(t,u) = \varphi(u)w(t)$ is the Orlicz–Lorentz space $\Lambda_{\varphi,w}$, which structure was and still is investigated in many papers (cf. [31] and literature therein).

The following lemma completes the above discussion of the case $E^{(s)} \neq \{0\}$.

**Lemma 3.** Let $E$ be a quasi-Banach ideal space $E$ on $I$ such that the dilation operator $D_2$ is bounded on $E^\downarrow$.

(i) If $E^{(s)} \neq \{0\}$, then $\chi_{(0,a)} \in E$ for each $a > 0$ and $E^{(s)}$ has a weak unit.

(ii) Let $I = (0,1)$. Then $E^{(s)} \neq \{0\}$ if and only if $L^\infty \hookrightarrow E$.

(iii) Let $I = (0,\infty)$. Then $E^{(s)} \neq \{0\}$ if and only if $L^\infty_0 \hookrightarrow E$, where $L^\infty_0$ is the linear subspace of $L^\infty$ consisting of all essentially bounded functions with bounded support.

**Proof.** (i) The weak unit in $E^{(s)}$ can be given by $x_0 = \sum_{n=1}^{\infty} x_n$ with $x_n = \frac{\chi_{(n-1,n)}}{b_n\|\chi_{(n-1,n)}\|_E}$, where $b_n$’s are chosen so that the sequence $\{b_n\|\chi_{(n-1,n)}\|_E\}$ is increasing and $\sum_{n=1}^{\infty} 1/b_n < \infty$ (cf. [33]).

In both cases (ii) and (iii) only the necessity need to be proved.

(ii) By the assumption, there is $a > 0$ with $x = \chi_{(0,a)} \in E$. Consequently, we find $k \in \mathbb{N}$ such that $\chi_{(0,1)} = D_2^k x \in E$, which proves the conclusion.

(iii) Similarly as above we conclude that $\chi_{(0,b)} \in E$ for each $b > 0$ and we are done. □
Remark 2. Let $E$ be a quasi-Banach ideal space on $I$ and $E^{(s)} \neq \{0\}$. If $\|H_{r}x^{s}\|_{E} \leq C \|x^{s}\|_{E}$ for all $x^{s} \in E$, then $\|D_{2}x^{s}\|_{E} \leq 2^{1/r}C \|x^{s}\|_{E}$ for all $x^{s} \in E$. In particular, if the operator $H_{r}$ is bounded on $E$, then (6) holds with $A_{E} \leq 2^{1/r}\|H_{r}\|_{E \rightarrow E}$.

Proof. Indeed, for $t \in I$, we have

$$H_{r}x^{s}(t) = \left( \int_{0}^{1} x^{s}(st)^{r}ds \right)^{1/r} \geq \left( \int_{0}^{1/2} x^{s}(st)^{r}ds \right)^{1/r} \geq x^{s}(t/2)2^{-1/r},$$

and so $\|H_{r}x^{s}\|_{E} \geq 2^{-1/r}\|D_{2}x^{s}\|_{E}$. \qed

Below we can see that a similar result with the operator $H_{r}^{*}$ is not true.

Example 2. We show that $H_{r}^{*}$ is bounded on $E = L^{r}(w)$ with $0 < r < \infty$, $w(t) = e^{t}$ on $I = [0, \infty)$, but estimate (6) does not hold. Namely, we have

$$\|H_{r}^{*}x\|_{L^{r}(w)}^{r} = \int_{0}^{\infty} \left( \int_{t}^{\infty} \frac{|x(s)|^{r}}{s}ds \right) e^{tr}dt \int_{0}^{\infty} \left( \int_{0}^{s} e^{tr}dt \right) \frac{|x(s)|^{r}}{s}ds = \int_{0}^{\infty} \frac{e^{s}r - 1}{sr} |x(s)|^{r}ds \leq \int_{0}^{\infty} e^{sr} |x(s)|^{r}ds = \|x\|_{L^{r}(w)}$$

and

$$\frac{\|D_{2}x\|_{L^{r}(w)}^{r}}{\|x\|_{L^{r}(w)}^{r}} = \frac{e^{2ar} - 1}{e^{ar} - 1} \rightarrow \infty \text{ as } a \rightarrow \infty.$$

Remark 3. The operator $H_{r}$ (or $H_{r}^{*}$) is bounded on a quasi-Banach ideal space $E$ if and only if the operator $H$ (or $H^{*}$) is bounded on $E^{(1/r)}$. Moreover,

$$\|H_{r}\|_{E \rightarrow E^{(1/r)}} = \|H\|_{E^{(1/r)} \rightarrow E^{(1/r)}}^{1/r} \quad \text{and} \quad \|H_{r}^{*}\|_{E \rightarrow E^{(1/r)}} = \|H^{*}\|_{E^{(1/r)} \rightarrow E^{(1/r)}}^{1/r}.$$

Remark 4. Let $E$ be a quasi-Banach ideal space. If $H$ is bounded on $E$, then $H$ is bounded on $E^{(*)}$ and $E^{(*)}$ has the majorant property.

Proof. Since $H$ is bounded on $E$, by Remark 2 and Corollary 1 we conclude that $E^{(*)}$ is a quasi-normed space. Let $x \in E^{(*)}$. By the assumption we obtain

$$\|Hx\|_{E^{(*)}} = \|(Hx)^{s}\|_{E} \leq \|(Hx)^{s}\|_{E} = \|Hx^{s}\|_{E} \leq C \|x^{s}\|_{E} = C \|x\|_{E^{(*)}}.$$

We prove the majorant property. Let $x \in E^{0}$, $y \in E^{(*)}$ and $Hx^{s} \leq Hy^{s}$. We need to prove that $x \in E^{(*)}$ or equivalently $x^{s} \in E$. But this follows from $x^{s} \leq Hx^{s} \leq Hy^{s} \in E$. \qed

3 Symmetrization of some known constructions

Consider two quasi-Banach ideal spaces $E, F$ on $I$. The symmetrization commutes with the intersection $(E \cap F)^{(*)} = E^{(*)} \cap F^{(*)}$ and from (5) we conclude that

$$E^{(*)} + F^{(*)} \leftrightarrow (E + F)^{(*)}.$$
We can then ask what about commutativity of the symmetrization operation $E \mapsto E^{(s)}$ with the sum or more general with the Calderón–Lozanovskii construction?

Let us therefore investigate the symmetrization of the Calderón–Lozanovskii construction.

First, note that for two symmetric spaces $E$ and $F$ the Calderón–Lozanovskii space $\rho(E, F)$ for any $\rho \in P$ is also a symmetric space up to an equivalence of quasi-norms (using Lemma 4.3 from [37, p. 93] and $\rho$ for positive operators – cf. [45, Theorem 15.13 on p. 190]). Consequently,

$$\rho(E, F)^{(s)} = \rho(E, F) = \rho(E^{(s)}, F^{(s)})$$

In [36] it has been proved that the Calderón construction $E^\theta F^{1-\theta}$ commutes with the symmetrization operation $E \mapsto E^{(s)}$ for Banach ideal spaces $E, F$ (see [36, Lemma 4]).

Now, we generalize this result to the case of Calderón–Lozanovskii construction $\rho(E, F)$ and for quasi-Banach ideal spaces $E, F$. This also shows that for the Calderón construction $E^\theta F^{1-\theta}$ we may use much weaker assumptions about the spaces than in [36].

In the following theorem we consider two constructions. Below the assumptions on $D_2$ operator imply that space $\rho(E^{(s)}, F^{(s)})$ is a linear, quasi-normed space. By Corollary [1] the space $\rho(E, F)^{(s)}$ is linear if and only if the operator $D_2$ is bounded on $\rho(E, F)^{\downarrow}$, which is not a “nice” assumption. This is a motivation and a reason for the formulation below.

First we prove only inclusions between sets $\rho(E^{(s)}, F^{(s)})$ and $\rho(E, F)^{(s)}$ and the respective inequalities (7) and (8). Next, we conclude the equality between sets $\rho(E^{(s)}, F^{(s)}) = \rho(E, F)^{(s)}$, which implies that the space $\rho(E, F)^{(s)}$ is in fact a linear, quasi-normed space. For the same reasons we formulate in a special way Corollary [2] and Theorem [2].

**Theorem 1.** Let $E$ and $F$ be quasi-Banach ideal spaces such that $E^{(s)} \neq \{0\}$, $F^{(s)} \neq \{0\}$ and the operator $D_2$ is bounded both on $E^{\downarrow}$ and $F^{\downarrow}$ – see Corollary [7] for equivalent conditions. Then:

(i) The Calderón–Lozanovskii construction $\rho(E^{(s)}, F^{(s)}) \neq \{0\}$, $\rho(E^{(s)}, F^{(s)}) \subset \rho(E, F)^{(s)}$ and

$$\|x\|_{\rho(E,F)^{(s)}} \leq C_1 \|x\|_{\rho(E^{(s)}, F^{(s)})} \text{ for all } x \in \rho(E^{(s)}, F^{(s)})$$

with $C_1 \leq \max(A_E, A_F)$, where $A_E, A_F$ are the best constants in (6).

(ii) If, additionally, the operator $H_r^*$ is bounded on the spaces $E, F$ for some $r > 0$, then $\rho(E, F)^{(s)} \subset \rho(E^{(s)}, F^{(s)})$ and

$$\|x\|_{\rho(E^{(s)}, F^{(s)})} \leq C_2 \|x\|_{\rho(E,F)^{(s)}} \text{ for all } x \in \rho(E, F)^{(s)}$$

with $C_2 \leq 2^{1/r} \max(1, 2^{1/r-1}) \cdot \max(A_E \|H_r^*\|_{E \mapsto E}, A_F \|H_r^*\|_{F \mapsto F}).$

In particular, the inequalities (7) and (8) imply that the functional $\|\cdot\|_{\rho(E,F)^{(s)}}$ is a quasi-norm on the space $\rho(E, F)^{(s)}$ and

$$\rho(E, F)^{(s)} = \rho(E^{(s)}, F^{(s)}).$$
Proof. (i) Since $E^{(s)} \neq \{0\}$ and $F^{(s)} \neq \{0\}$ are quasi-normed spaces it follows, by the Aoki–Rolewicz theorem, that there are $p_0$-norm $\| \cdot \|_1$ on $E^{(s)}$ and $p_1$-norm on $F^{(s)}$. But $E^{(s)}, F^{(s)}$ are symmetric $p_1$-normed spaces, which gives inclusions $L^{p_0} \cap L^{\infty} \hookrightarrow E^{(s)}, L^{p_1} \cap L^{\infty} \hookrightarrow F^{(s)}$, with $C_3 \leq 2^{1/p_0} \| \chi_{[0,1]} \|_E$ and $C_4 \leq 2^{1/p_1} \| \chi_{[0,1]} \|_F$ (see [2, Theorem 1]).

Clearly, if $E_1 \hookrightarrow E_2$ with $B \geq 1$, then $\rho(E_1, F) \hookrightarrow B \rho(E_2, F)$ and similarly for the inclusion with respect to the second variable. Since for $p = \min(p_0, p_1)$ we have $L^p \cap L^{\infty} \hookrightarrow L^{p_i} \cap L^{\infty}$, $(i = 0, 1)$, thus putting all above information together we obtain

$$L^p \cap L^{\infty} = \rho(L^p \cap L^{\infty}, L^p \cap L^{\infty}) \hookrightarrow \rho(L^{p_0} \cap L^{\infty}, L^{p_1} \cap L^{\infty}) \hookrightarrow \rho(E^{(s)}, F^{(s)}),$$

which means that the last space is nontrivial.

We will prove the inclusion $\rho(E^{(s)}, F^{(s)}) \subset \rho(E, F)^{(s)}$. Let $x \in \rho(E^{(s)}, F^{(s)})$. Then $|x| \leq \lambda \rho(|x_0|, |x_1|)$ for some $\lambda > 0$ and $\|x_0\|_{E^{(s)}} \leq 1, \|x_1\|_{F^{(s)}} \leq 1$. Recall that for a given function $\rho \in \mathcal{P}$ the function $\tilde{\rho}$ defined by

$$\tilde{\rho}(a, b) = \inf_{u, v > 0} \frac{au + bv}{\rho(u, v)}$$

for all $a, b \geq 0$ belongs to $\mathcal{P}$ and this operation is an involution on $\mathcal{P}$, that is, $\tilde{\tilde{\rho}} = \rho$ (see [44, Lemma 2] and also [45, Lemma 15.8]). Since

$$\rho(|x_0(t)|, |x_1(t)|) \leq \frac{|x_0(t)| u + |x_1(t)| v}{\tilde{\rho}(u, v)}$$

for all $u, v > 0$, it follows that

$$\rho(|x_0|, |x_1|)^*(t) \leq \frac{x_0^*(t/2) u + x_1^*(t/2) v}{\tilde{\rho}(u, v)}.$$

Taking infimum over all $u, v > 0$, we get

$$\rho(|x_0|, |x_1|)^*(t) \leq \tilde{\rho}(x_0^*(t/2), x_1^*(t/2)) = \rho(x_0^*(t/2), x_1^*(t/2)).$$

Consequently,

$$x^*(t) \leq \lambda \rho(x_0^*(t/2), x_1^*(t/2)) \leq \lambda \max(A_E, A_F) \rho\left(\frac{x_0^*(t/2)}{A_E}, \frac{x_1^*(t/2)}{A_F}\right),$$

where $A_E, A_F$ are the smallest constants in [45] for $E, F$, respectively. This means that $x \in \rho(E, F)^{(s)}$ with the norm $\leq \lambda \max(A_E, A_F)$. Thus, $\rho(E^{(s)}, F^{(s)}) \subset \rho(E, F)^{(s)}$ and inequality (7) is proved.

(ii) Assume now that $H_r^*$ is bounded on the spaces $E, F$ for some $r > 0$. Then it is bounded on $\rho(E, F)$ since this construction is an interpolation space between $E$ and $F$ for positive operators (see [45, Theorem 15.13 on p. 190]).

Now, we will prove the reverse inclusion $\rho(E, F)^{(s)} \subset \rho(E^{(s)}, F^{(s)})$. Let $x \in \rho(E, F)^{(s)}$. Then $x^* \in \rho(E, F)$ and $x^* \leq \lambda \rho(|x_0|, |x_1|)$ for some $\lambda > 0$ and $\|x_0\|_E \leq 1, \|x_1\|_F \leq 1$. Note that

$$x^*(t) \leq 2^{1/r} \left(\int_{t/2}^t x^*(s)^r \frac{ds}{s}\right)^{1/r} \leq 2^{1/r} \left(\int_{t/2}^\infty x^*(s)^r \frac{ds}{s}\right)^{1/r}.$$
Consequently,
\[ x^* \leq 2^{1/r} \lambda D_2 H_r^* (\rho (|x_0|, |x_1|)) . \]

On the other hand, applying the definition of \( \widehat{\rho} \) and the equality \( \widehat{\rho} = \rho \) (see the proof of (i)), we get
\[
H_r^* \rho (|x_0|, |x_1|) \leq H_r^* \left( \frac{|x_0| u + |x_1| v}{\widehat{\rho}(u, v)} \right) = \max (1, 2^{1/r-1}) \frac{(H_r^*|x_0|)u + (H_r^*|x_1|)v}{\widehat{\rho}(u, v)}
\]
for every \( u, v > 0 \). Taking infimum over all \( u, v > 0 \) we obtain
\[
H_r^* \rho (|x_0|, |x_1|) \leq \max (1, 2^{1/r-1}) \widehat{\rho}((H_r^*|x_0|), (H_r^*|x_1|)) = \max (1, 2^{1/r-1}) \rho ((H_r^*|x_0|), (H_r^*|x_1|)).
\]

Thus,
\[
x^* \leq \lambda 2^{1/r} \max (1, 2^{1/r-1}) D_2 \rho (H_r^*|x_0|, H_r^*|x_1|) = \lambda 2^{1/r} \max (1, 2^{1/r-1}) \rho (D_2 H_r^*|x_0|, D_2 H_r^*|x_1|).
\]

By Ryff’s theorem there exists a measure preserving transformation:
(I) \( \omega : I \to I \) such that \( x^* \circ \omega = |x| \) a.e. when \( m(\text{supp} \ x) < \infty \),
(II) \( \omega : \text{supp} \ x \to (0, \infty) \) such that \( x^* \circ \omega = |x| \) a.e. on \( \text{supp} \ x \) when \( m(\text{supp} \ x) = \infty \)
under the additional assumption that \( x^*(\infty) = 0 \) (see [4], Theorem 7.5 for \( I = (0, 1) \) or Corollary 7.6 for \( I = (0, \infty) \)). Note that our assumptions imply that \( x^*(\infty) = 0 \) in the same way as in the proof of Lemma 4 from [36]. Therefore,
\[
|x| = x^*(\omega) \leq \lambda 2^{1/r} \max (1, 2^{1/r-1}) \rho ((D_2 H_r^*|x_0|)(\omega), (D_2 H_r^*|x_1|)(\omega))
\]
with \( u_0 = (D_2 H_r^*|x_0|)(\omega) \) and \( u_1 = (D_2 H_r^*|x_1|)(\omega) \). Let us see that \( u_0 \in E^{(*)} \) and \( u_1 \in F^{(*)} \)
with \( \|u_0\|_{E^{(*)}} \leq A_E \|H_r^*\|_{E \to E} \) and \( \|u_1\|_{F^{(*)}} \leq A_F \|H_r^*\|_{F \to F} \). In fact, similarly as in the
proof of Lemma 4 in [36], \( H_r^*|x_i| \) is a nonincreasing function and so is \( D_2 H_r^*|x_i| \) (\( i = 0, 1 \)),
which gives that
\[
D_2 H_r^*|x_i| = [(D_2 H_r^*|x_i|)(\omega)]^* \quad \text{for} \quad i = 0, 1.
\]
Hence,
\[
\|u_0\|_{E^{(*)}} = \|u_0^*\|_E = \|[D_2 H_r^*|x_0|](\omega)]^*\|_E = \|D_2 H_r^*|x_0|\|_E \\
\leq A_E \|H_r^*|x_0|\|_E \leq A_E \|H_r^*\|_{E \to E} \|x_0\|_E \leq A_E \|H_r^*\|_{E \to E}
\]
and
\[
\|u_1\|_{F^{(*)}} = \|u_1^*\|_F = \|[D_2 H_r^*|x_1|](\omega)]^*\|_F = \|D_2 H_r^*|x_1|\|_F \\
\leq A_F \|H_r^*|x_1|\|_F \leq A_F \|H_r^*\|_{F \to F} \|x_1\|_F \leq A_F \|H_r^*\|_{F \to F},
\]

12
which means that \( x \in \rho(E^{(s)}, F^{(s)}) \) with the norm \( \leq \lambda C_2 \), where

\[
C_2 \leq 2^{1/r} \max(1, 2^{1/r-1}) \cdot \max(A_E\|H^*_F\|_{E \to E}, A_F\|H^*_F\|_{F \to F}).
\]

Thus, \( \rho(E, F)^{(s)} \subset \rho(E^{(s)}, F^{(s)}) \) and inequality (8) is proved. Summing up cases (i) and (ii) we conclude that the functional \( \| \cdot \|_{\rho(E, F)^{(s)}} \) is a quasi-norm on the space \( \rho(E, F)^{(s)} \) and equality (9) holds.

Without additional assumptions, like those in Theorem 1(ii), we can have that \( E^{(s)} + F^{(s)} \not\subset (E + F)^{(s)} \) even for Banach ideal spaces \( E, F \). Examples below are inspired by [11] Examples 1 and 3] and [12] Example 3].

**Example 3.** (a) For \( 0 < a < b \) let \( E = L^1(w_a) \) and \( F = L^1(w_b) \), where weights \( w_a, w_b \) are as in Example 1(b). Then \( E + F = L^1(\min(w_a, w_b)) \), \( (E + F)^{(s)} = \Lambda_{1, \min(w_a, w_b)} \) and \( \chi_{(0,c)} \in (E + F)^{(s)} \) for any \( c > 0 \) since \( \min(w_a, w_b) \leq \max(1, \frac{1}{b-a}) \).

If \( c < b \), then \( \chi_{(0,c)} \in E^{(s)} + F^{(s)} \) since for the decomposition \( \chi_{(0,c)} = \chi_{(0,a)} + \chi_{(a,c)} \) we have \( \chi_{(0,a)} \in E^{(s)} \) and \( \chi_{(a,c)} \in F^{(s)} \). The last fact follows from the observation that \( \chi_{(0,c)} = \chi_{(0,c-a)} \leq \chi_{(0,b)} \in L^1(w_b) \).

If \( c > b \), then \( \chi_{(0,c)} \notin E^{(s)} + F^{(s)} \) since any decomposition of \( \chi_{(0,c)} \) into decreasing functions has the form \( \alpha \chi_{(0,c)} + (1-a)\chi_{(0,c)} \) with \( 0 \leq a \leq 1 \) and \( \chi_{(0,c)} \notin E^{(s)}, \chi_{(0,c)} \notin F^{(s)} \).

Therefore, \( E^{(s)} + F^{(s)} \not\subseteq (E + F)^{(s)} \), but unfortunately, by Remark 3.1 in [15], the spaces \( E^{(s)} \) and \( F^{(s)} \) are not linear since \( W_a(t) = \infty \) for \( t > a \) and \( W_b(t) = \infty \) for \( t > b \). Moreover, \( W(t) = \int_0^t \min(w_a(s), w_b(s)) ds < \infty \) for \( t > 0 \).

(b) We give now examples of linear spaces \( E^{(s)}, F^{(s)}, (E + F)^{(s)} \) for which we still have only proper inclusion \( E^{(s)} + F^{(s)} \not\subseteq (E + F)^{(s)} \). For \( 0 < w \in L^1(0, \infty) \), \( w \) decreasing, continuous and \( w(t) \leq 1 \) for all \( t > 0 \), let

\[
w_0 = \sum_{n=0}^{\infty} w\chi(2n,2n+1) + \sum_{n=0}^{\infty} \chi(2n+1,2n+2) \quad \text{and} \quad w_1 = \sum_{n=0}^{\infty} \chi(2n,2n+1) + \sum_{n=0}^{\infty} w\chi(2n+1,2n+2).
\]

Take \( E = L^1(w_0), F = L^1(w_1) \) and \( x = \chi_{(0,\infty)} \). Then \( x \in (E + F)^{(s)} \), but \( x \notin E^{(s)} + F^{(s)} \) because for any decomposition \( x = x_0 + x_1 \) we have \( m(A_i) = \infty \) for at least one \( i \), where \( A_i = \{ t \in I: x_i(t) \geq 1/2 \} \). Thus, \( E^{(s)} + F^{(s)} \not\subseteq (E + F)^{(s)} \).

On the other hand, the functions \( W_i(t) = \int_0^t w_i(s) ds \) satisfy the \( \Delta_2 \)-condition \( W_0(2t) \leq (2 + \frac{1}{w(1)})W_0(t) \) and \( W_1(2t) \leq 3W_1(t) \) for all \( t > 0 \) (we skip the detailed calculations). Thus, by observation in [28], we have that \( E^{(s)} = L^1(w_0)^{(s)} = \Lambda_{1,w_0} \) and \( F^{(s)} = L^1(w_1)^{(s)} = \Lambda_{1,w_1} \) are quasi-Banach spaces. Since \( E + F = L^1(w) \) it follows that the space \( (E + F)^{(s)} = L^1(w)^{(s)} = \Lambda_{1,w} \), as \( w \) is a decreasing function, is even a Banach space.

Note that for any \( 0 < r < \infty \) the operator \( H^*_F \) is not bounded on \( E = L^1(w_0) \) neither on \( F = L^1(w_1) \). Namely, if \( x_0 = \chi_{\cup_{n=0}(2n,2n+1)} \), then \( x_0 \in E \) since

\[
\int_0^\infty x_0(t)w_0(t) dt = \sum_{n=0}^{\infty} \int_{2n}^{2n+1} w(t) dt \leq \int_0^\infty w(t) dt < \infty.
\]
However,

$$H_r^* x_0(2n)^r = \int_{2n}^{\infty} \frac{|x_0(s)|^r}{s} ds = \sum_{k=n}^{\infty} \int_{2k}^{2k+1} \frac{|x_0(s)|^r}{s} ds$$

$$= \sum_{k=n}^{\infty} \int_{2k}^{2k+1} \frac{1}{s} ds \geq \sum_{k=n}^{\infty} \frac{1}{2k+1} = \infty$$

for any $n \in \mathbb{N}$. Thus $H_r^* x_0(2n) = \infty$. Similarly, if $x_1 = \chi_{\cup_{n=0}^{\infty}(2n+1,2n+2)}$, then $x_1 \in F$ and $H_r^* x_1(2n+1)^r = \infty$ for any $n \in \mathbb{N}$.

Moreover, for any $r \geq 1$ the operator $H_r$ is not bounded on the cone of nonnegative nonincreasing functions in $E$ and $F$. In fact, $x = x^* = \chi_{(0,1)} \in E \cap F = L^1(w_0) \cap L^1(w_1)$ but

$$\|H_r x\|_E \geq \int_1^{\infty} \left( \frac{1}{t} \int_0^t |x(s)|^r ds \right)^{1/r} w_0(t) dt = \int_1^{\infty} t^{-1/r} w_0(t) dt$$

$$\geq \sum_{n=0}^{\infty} \int_{2n+1}^{2n+2} t^{-1/r} w_0(t) dt = \sum_{n=0}^{\infty} \int_{2n+1}^{2n+2} t^{-1/r} dt$$

$$\geq \sum_{n=0}^{\infty} \frac{1}{(2n+2)^{1/r}} = \infty.$$}

Similarly, the operator $H_r$ is not bounded on $F^\perp$.

Let us consider now the problem of commutativity of the symmetrization operation $E \mapsto E^{(s)}$ with the pointwise product and the Köthe duality. The first result is a simple consequence of the known results and the second has been already concluded in [30, Corollary 1.6], but we give a direct proof.

**Corollary 2.** Let $E$ and $F$ be quasi-Banach ideal spaces such that $E^{(s)} \not= \{0\}$ and $F^{(s)} \not= \{0\}$. If the operator $D_2$ is bounded both on $E^\perp$ and $F^\perp$, then $E^{(s)} \odot F^{(s)} \not= \{0\}$, $E^{(s)} \odot F^{(s)} \subset (E \odot F)^{(s)}$ and

$$\|x\|_{(E \odot F)^{(s)}} \leq (C_1)^2 \|x\|_{E^{(s)} \odot F^{(s)}} \quad \text{for all } x \in E^{(s)} \odot F^{(s)}, \quad (10)$$

where $C_1$ is the constant from Theorem [7] with the function $\rho(s,t) = s^{1/2} t^{1/2}$. If, additionally, the operator $H_r^\perp$ is bounded on the spaces $E, F$ for some $r > 0$, then $(E \odot F)^{(s)} \subset E^{(s)} \odot F^{(s)}$ and

$$\|x\|_{E^{(s)} \odot F^{(s)}} \leq (C_2)^2 \|x\|_{(E \odot F)^{(s)}} \quad \text{for all } x \in (E \odot F)^{(s)}, \quad (11)$$

where $C_2$ is the constant from Theorem [7] with the function $\rho(s,t) = s^{1/2} t^{1/2}$. In particular, the inequalities (10) and (11) imply that the functional $\|\cdot\|_{(E \odot F)^{(s)}}$ is a quasi-norm on the space $(E \odot F)^{(s)}$ and

$$(E \odot F)^{(s)} = E^{(s)} \odot F^{(s)}.$$
Proof. Applying Theorem 1(iv) from [36], Theorem 1(i), commutativity of the p-convexification with the symmetrization, that is, the equality \((E^{(s)})^{(p)} \equiv (E^{(p)})^{(s)}\) and again Theorem 1(iv) from [36] we get immediately
\[
\|x\|_{\|E_{2}\|^{(s)}} = \|x\|_{\|E_{2}\|^{(1/2)F(1/2)}^{(1/2)(s)}} = \|x\|_{\|E_{2}\|^{(1/2)F(1/2)}^{(1/2)}} \leq C_{1}^{2} \|x\|_{\|E_{2}\|^{(1/2)F(1/2)}^{(1/2)}}^{(1/2)} = C_{1}^{2} \|x\|_{\|E_{2}\|^{(s)}}. 
\]
This establishes inequality (11) with the equality when assumptions from Theorem 1(ii) are satisfied.

The commutativity of the symmetrization operation \(E \mapsto E^{(s)}\) with the Köthe duality operation has been proved by Kamińska and Mastyło [30, p. 231]. We will give, however, a direct proof and also show that the assumption on boundedness of \(H^{*}\) on \(E\) is essential.

**Theorem 2.** Let \(E\) be a quasi-Banach ideal space such that \((E^{(s)}) \neq \{0\}\), the operator \(D_{2}\) is bounded on \(E_{1}\) and \((E')^{(s)} \neq \{0\}\). Then \((E')^{(s)} \subset (E^{(s)})'\) and
\[
\|x\|_{(E^{(s)})'} \leq \|x\|_{(E')^{(s)}} \quad \text{for all } x \in (E')^{(s)}. \tag{12}
\]
If, additionally, the operator \(H^{*}\) is bounded on the space \(E\), then \((E^{(s)})' \subset (E')^{(s)}\) and
\[
\|x\|_{(E')^{(s)}} \leq \|H^{*}\|_{E \rightarrow E} \|x\|_{(E^{(s)})'} \quad \text{for all } x \in (E')^{(s)}. \tag{13}
\]
In particular, the inequalities (12) and (13) imply that the functional \(\|\cdot\|_{(E')^{(s)}}\) is a quasi-norm on the space \((E')^{(s)}\) and \((E')^{(s)} = (E^{(s)})'\).

Proof. Clearly, \(x \in (E')^{(s)}\) if and only if \(x^{*} \in E'\) and
\[
\|x^{*}\|_{E'} = \sup_{\|y\|_{E} \leq 1} \int_{I} x^{*}(t)|y(t)| dt < \infty.
\]
Moreover, \(x \in (E^{(s)})'\) if and only if \(x^{*} \in (E^{(s)})'\) and
\[
\|x^{*}\|_{(E^{(s)})'} = \sup_{\|y\|_{E^{(s)}} \leq 1} \int_{I} x^{*}(t)|y(t)| dt = \sup_{\|y\|_{E} \leq 1, y^{*} = y} \int_{I} x^{*}(t)|y^{*}(t)| dt < \infty.
\]
Thus, \((E')^{(s)} \subset (E^{(s)})'\) and inequality (12) is proved. To finish the proof we need to show the inequality
\[
\sup_{\|y\|_{E} \leq 1} \int_{I} x^{*}(t)|y(t)| dt \leq C \sup_{\|y\|_{E} \leq 1, y^{*} = y} \int_{I} x^{*}(t)|y^{*}(t)| dt \text{ for } x \in (E^{(s)})',
\]
where \(C = \|H^{*}\|_{E \rightarrow E}\). Since \(Hx^{*} \geq x^{*}\) and using the duality of \(H\) and \(H^{*}\) we get
\[
\sup_{\|y\|_{E} \leq 1} \int_{I} x^{*}(t)|y(t)| dt \leq \sup_{\|y\|_{E} \leq 1} \int_{I} (Hx^{*}(t))|y(t)| dt = \sup_{\|y\|_{E} \leq 1} \int_{I} x^{*}(t)H^{*}(|y|)(t) dt \\
\leq \sup_{\|h\|_{E} \leq C, h = h^{*}} \int_{I} x^{*}(t)h(t) dt \\
\leq C \sup_{\|y\|_{E} \leq 1, y^{*} \geq y} \int_{I} x^{*}(t)y(t) dt.
\]
We proved inequalities (12) and (13). This gives that the functional \( \| \cdot \|_{(E')^{'(s)}} \) is a quasi-norm on the space \((E')^{(s)}\) and \((E')^{(s)} = (E^{(s)})'\).

**Example 4.** We will give an example of a Banach ideal space \(E\) on \((0, 1)\) such that \(H^*\) is not bounded on \(E\) and \((E')^{(s)} \neq (E^{(s)})'\).

For \(A_n = (2^{-n}, 2^{-n+1})\), \(B_n = (2^{-n}, 3 \times 2^{-n-1})\), \(C_n = (3 \times 2^{-n-1}, 2^{-n+1})\) and \(a_n = (3/2)^n\) with \(n = 1, 2, 3, \ldots\) define two weights

\[
w = \sum_{n=1}^{\infty} (\chi_{B_n} + a_n \chi_{C_n}) \quad \text{and} \quad v = \sum_{n=1}^{\infty} a_n \chi_{A_n}.
\]

Let \(E = L^1(w)\) on \(I = (0, 1)\). We will show that \((E')^{(s)} \neq (E^{(s)})'\) and \(H^*\) is not bounded on \(E\).

We have \(E' = L^\infty(1/w)\) and consequently \((E')^{(s)} = L^\infty\). In fact, the inclusion \(L^\infty \to (E')^{(s)}\) is evident, since \(1/w \leq 1\). On the other hand, if \(x = x^*\) with \(x(0^+) = \infty\), then choosing arbitrary \(t_n \in B_n\) we have \(x(t_n) \to \infty\), so that \(x \notin L^\infty(1/w)\).

On the other hand, it is easy to see that \(E^{(s)} \neq L^1\) (further we will even identify this space). In fact, let \(x = \sum_{n=1}^{\infty} a_n \chi_{A_n}\). Then \(x = x^*\) and \(x \in L^1\). Moreover,

\[
\int_0^1 x(t)w(t) dt \geq \int_{\bigcup_{n=1}^{\infty} C_n} x(t)w(t) dt = \sum_{n=1}^{\infty} a_n^2 2^{-n-1} = 2^{-1} \sum_{n=1}^{\infty} (9/8)^n = \infty,
\]

and so \((E^{(s)})' \neq L^\infty\), as claimed.

Moreover, \(D_2\) is bounded on \(E\). In fact, we have \(w(2t) \leq w(t)\) for \(0 \leq t \leq 1/2\), because if \(t \in B_n\) then \(2t \in B_{n-1}\) and if \(t \in C_n\) then \(2t \in C_{n-1}\). Therefore,

\[
\|D_2x\|_E = \int_0^1 x(t/2)w(t) dt = 2 \int_0^{1/2} x(t)w(2t) dt \leq 2 \int_0^{1/2} x(t)w(t) dt \leq 2 \|x\|_E.
\]

Now we identify \(E^{(s)}\) by showing that \(E^{(s)} = L^1(v)^{(s)} = \Lambda_{1,v}\).

In fact, the inclusion \(L^1(v)^{(s)} \subset E^{(s)}\) is obvious, since \(v \geq w\) and so \(L^1(v) \subset L^1(w) = E\). To see the second inclusion, let \(x = x^* \in L^1(w)\) and notice that \(x(t) \leq x(t/2)\) for each \(t \in (0, 1)\). Put \(y(t) = x(t/2)\). Then also \(y \in L^1(w)\), since \(D_2\) is bounded on the cone of nonincreasing nonnegative elements of \(E\). Moreover, we can see that

\[
x(t) \leq y(t + 2^{-n-1}),
\]

when \(t \in B_n\), i.e., \(t + 2^{-n-1} \in C_n\) (we may say that \(x\chi_{B_n}\) is dominated by \(y\chi_{C_n}\) shifted by \(2^{-n-1}\) into the right – the best is to see it on the picture). Consequently,

\[
\int_0^1 x(t)v(t) dt = \int_{\bigcup_{n=1}^{\infty} B_n} x(t)v(t) dt + \int_{\bigcup_{n=1}^{\infty} C_n} x(t)v(t) dt
\]

\[
= \sum_{n=1}^{\infty} a_n \int_{B_n} x(t) dt + \int_{\bigcup_{n=1}^{\infty} C_n} x(t)w(t) dt
\]

\[
\leq \sum_{n=1}^{\infty} a_n \int_{C_n} y(t) dt + \int_{\bigcup_{n=1}^{\infty} C_n} y(t)w(t) dt
\]

\[
= 2 \int_{\bigcup_{n=1}^{\infty} C_n} y(t)w(t) dt \leq 2 \|y\|_E \leq 4 \|x\|_E.
\]
Then \(x \in L^1(v)\) and so \(E^{(*)} = L^1(w)^{(*)} = L^1(v)^{(*)} = \Lambda_{1,v}\).

At the end it may be instructive to see that the operator \(H^*\) is not bounded on \(E\). In order to prove this, we will show that the operator \(H\) is not bounded on \(E' = L^\infty(1/w)\). Evidently, \(w \in L^1(1/w)\). For \(2^{-n} < t < 2^{-n+1}\) we have

\[
Hw(t) = \frac{1}{t} \int_0^t w(s)ds \geq 2^{-n} \int_0^{2^{-n}} w(s)ds = 2^{-n} \sum_{k=n+1}^\infty \int_{2^{-k+1}}^{2^{-k}} w(s)ds
\]

\[
\geq 2^{-n} \sum_{k=n+1}^\infty \int_{3 \cdot 2^{-k+1}}^{2 \cdot 2^{-k+1}} w(s)ds \geq 2^{-n} \sum_{k=n+1}^\infty \left(\frac{3}{2}\right)^{k-1} \cdot \frac{3}{4}
\]

\[
= 2^{-n} \frac{1}{2} \sum_{k=n+1}^\infty \left(\frac{3}{4}\right)^k = 2^{-n} \left(\frac{3}{4}\right)^{n+1}, 4 = \frac{1}{2} \left(\frac{3}{2}\right)^{n+1},
\]

and so

\[
\sup_{t \in [0,1]} \frac{Hw(t)}{w(t)} = \max_{n \in N} \sup_{2^{-n} < t < 2^{-n+1}} \frac{Hw(t)}{w(t)} \geq \frac{1}{2} \max_{n \in N} (3/2)^{n+1} \sup_{2^{-n} < t < 2^{-n+1}} \frac{1}{w(t)} = \frac{1}{2} \max_{n \in N} (3/2)^{n+1} = \infty,
\]

that is, \(Hw \notin L^\infty(1/w)\). It means that the operator \(H\) is not bounded on \(E' = L^\infty(1/w)\).

**Remark 5.** If \(E\) is a quasi-Banach ideal space such that \(E^{(*)} = \{0\}\), then

\[(E^{(*)})' = \{x \in L^0 : xy \in L^1 \text{ for all } y \in E^{(*)}\} = L^0\]

and this is a trivial case as well as \((E')^{(*)} = \{0\}\).

### 4 Symmetrization of pointwise multipliers

The next problem of our interest is the commutativity of the symmetrization operation \(E \mapsto E^{(*)}\) with the space of pointwise multipliers. If \(E, F\) are nontrivial symmetric spaces on \(I\), then \(M(E, F)\) is also symmetric space (see [35, Theorem 2.2(i)]) and we obtain

\[M(E^{(*)}, F^{(*)}) = M(E, F) = M(E, F)^{(*)}.\]

We want to have the same result for Banach ideal spaces. We will prove it with the help of the “arithmetic of function spaces”, which use our Theorem [24] and some results from the paper [36]. Recall that \(E^{(*)}\) is normable if there is a norm \(\|\cdot\|_0\) on \(E^{(*)}\) which is equivalent to \(\|\cdot\|_{E^{(*)}}\).

**Theorem 3.** Let \(E, F\) be Banach ideal spaces on \(I\) such that \(F\) has the Fatou property, \(E^{(*)} \neq \{0\}, F^{(*)} \neq \{0\}\) are normable spaces, the operator \(D_2\) is bounded on \(F^{1}\), \((F^{(*)})^{(*)} \neq \{0\}\) and \(((E \odot F^{(*)})^{(*)})^{(*)} \neq \{0\}\). Assume that the following conditions hold:

(i) The operator \(H^*\) is bounded on the spaces \(F\) and \(E \odot F^{(*)}\).
(ii) For some \( r > 0 \), the operator \( H^*_r \) is bounded on \( E, F' \), \( \|H_r x^*\|_E \leq C_E \|x^*\|_E \) for all \( x^* \in E \) and \( \|H_r x^*\|_{F'} \leq C_{F'} \|x^*\|_{F'} \) for all \( x^* \in F' \).

Then
\[
M(E^{(s)}, F^{(s)}) = M(E, F)^{(s)}.
\]

**Proof.** Applying in the subsequent steps several results from [36], we obtain
\[
M(E^{(s)}, F^{(s)}) = M(E^{(s)} \odot (F^{(s)})', F^{(s)} \odot (F^{(s)})')
\]
[by Theorem 4 from [36] with \( G = F^{(s)} \) a Banach space]
\[
= M(E^{(s)} \odot (F^{(s)})', \ell^1)
\] [by the Lozanovskii factorization theorem]
\[
= [E^{(s)} \odot (F^{(s)})']' = [E^{(s)} \odot (F^{(s)})']'
\] [by Theorem 2, since \( H^* \) is bounded on \( F \)]
\[
= [(E \odot F')^{(s)}]'
\] [by Corollary 2, using the assumptions on \( H^*_r \) and \( H_r \)]
\[
= [(E \odot F')^{(s)}]^{(s)} \quad \text{[by Theorem 2 since \( H^* \) is bounded on \( E \odot F' \)]}
\]
\[
= M(E, F'^{(s)})^{(s)} \quad \text{[by Corollary 3 from [36]]}
\]
\[
= M(E, F)^{(s)} \quad \text{[by the Fatou property of \( F \)].}
\]

\[ \square \]

Note that the assumptions on \( H^*_r \) and \( H_r \) in (ii) allow us to apply Corollary 2 (see also Remark 2 and condition (ii)).

Let us comment assumptions of Theorem 3.

(a) \((F')^{(s)} \neq \{0\}\) and \(H^*\) is bounded on \( F \) to get equality \((F^{(s)})' = (F')^{(s)}\) and \( \|\cdot\|_{(F'^{(s)})'} \sim \|\cdot\|_{(F^{(s)})'} \) (see Theorem 2). In particular, since \((F^{(s)})' \neq \{0\}\) is a Banach space, the functional \( \|\cdot\|_{(F'^{(s)})'} \) is a quasi-norm.

If we don’t have boundedness of \( H^* \) on \( F \), then under assumption that \((F')^{(s)} \neq \{0\}\) we obtain only the inclusion \((F')^{(s)} \xrightarrow{1} (F^{(s)})' \) (see Example 4), and in consequence only the inclusion \([E^{(s)} \odot (F^{(s)})']' \xrightarrow{1} [E^{(s)} \odot (F'^{(s)})']\).

(b) the operator \( H^*_r \) is bounded on \( E, F' \) and we have estimates of \( H_r \) in \( E, F' \) on the cone of nonnegative nonincreasing elements.

If we don’t have these assumptions but only \( E^{(s)} \odot (F')^{(s)} \neq \{0\}\), then \( E^{(s)} \odot (F')^{(s)} \overset{C_1}{\xrightarrow{1}} (E \odot F')^{(s)} \) with the constant \( C_1 \) from Theorem 1 (see also Corollary 2), and so only inclusion \([E \odot (F')^{(s)}]' \xrightarrow{1/C_2} [E^{(s)} \odot (F')^{(s)}]'\) is valid. Finally, since the operator \( H_r \) is bounded on \( E^{(s)} \), \((F')^{(s)}\) is a quasi-normed space by Remark 2 and Corollary 4.

(c) \( H^* \) bounded on \( E \odot F' \). Without this assumption we will have only the inclusion \([E \odot (F')]'^{(s)} \xrightarrow{1} [E \odot (F')]'^{(s)}\).

Note also that for the spaces \( E \odot F', (E \odot F')' \) we need to have quasi-normed spaces after taking the symmetrizations \((E \odot F')^{(s)}, [(E \odot F')]'^{(s)}\). Notice that our assumptions
imply that there are constants $A, B > 0$ such that

$$A \|x\|_{(E \odot F')^{(*)}} \leq \|x\|_{E^{(*)} \odot (F')^{(*)}} \leq B \|x\|_{(E \odot F')^{(*)}}$$

(14)

for all $x \in (E \odot F')^{(*)}$ (see Corollary 2). Moreover, the functional $\|\cdot\|_{(E^{(*)} \odot (F')^{(*)})}$ is a quasi-norm because for quasi-normed spaces $X, Y$ the Calderón space $X^{1/2}Y^{1/2}$ is quasi-normed and so is $X \odot Y = (X^{1/2}Y^{1/2})^{(1/2)}$ (cf. Corollary 1 from [36]). Consequently, by (14), the functional $\|\cdot\|_{(E \odot F')^{(*)}}$ is also a quasi-norm.

The assumption $\{(E \odot F')^{(*)}\} \neq \{0\}$ is necessary, because we apply Theorem 2 for the space $E \odot F'$. Finally, applying Theorem 2 for the space $E \odot F'$, we get $\{(E \odot F')^{(*)}\} = \{(E \odot F')^{(*)}\}'$ with equivalent respective functionals $\|\cdot\|_{(E \odot F')^{(*)}} \sim \|\cdot\|_{(E \odot F')^{(*)}}'$, whence in particular $\{(E \odot F')^{(*)}\}' \neq \{0\}$. Thus $(E \odot F')^{(*)}'$ is a Banach space. Consequently, the functional on $(E \odot F')^{(*)}$ is a quasi-norm.

**Remark 6.** In Theorem 5 we need to have that the spaces $E^{(*)} \neq \{0\}, F^{(*)} \neq \{0\}$ are normable spaces, to be able to use Theorem 4 from [36]. Note that the condition $E^{(*)} \neq \{0\}$ has been discussed in Lemma 6.

We are coming here to an interesting question.

**Problem 1.** Characterize quasi-normed or normed ideal spaces $E$ for which $E^{(*)}$ is normable.

It may happen that $E$ is a quasi-normed and $E^{(*)}$ is normable. Indeed, if we take $E = L^1(0,1/2) \oplus L^{1/2}[1/2,1]$ then $E^{(*)} = L^1(0,1)$.

Moreover, if $E = L^p(w)$ with $0 < p < \infty$ and with the weight $w: (0, \infty) \to (0, \infty)$, then $E^{(*)} = (L^p(w))^{(*)} = \Lambda_{p,w}$ is the Lorentz space with the quasi-norm

$$\|x\|_{\Lambda_{p,w}} = \left( \int_0^\infty x^*(t)^p w(t)^p \, dt \right)^{1/p}.$$ 

Assume that a weight function $w$ is locally integrable and satisfies the following conditions:

- $W_p(t) = \int_0^t w(s)^p \, ds < \infty$ for all $t > 0$ (this gives $\Lambda_{p,w} \neq \{0\}$),
- $W_p \in \Delta_2$, that is, there is a constant $C > 0$ such that $W_p(2t) \leq CW_p(t)$ for all $t > 0$ (then $\Lambda_{p,w}$ is a linear space [15, Remark 1.3 and Theorem 1.4])
- $W_p(\infty) = \int_0^\infty w(s)^p \, ds = \infty$, otherwise, $\Lambda_{p,w}$ is not separable.

Note that $\Lambda_{p,w} \neq \{0\}$ if and only if $\int_0^t w(s)^p \, ds < \infty$ for some $t > 0$. It is known that for $1 < p < \infty$ the Lorentz space $\Lambda_{p,w}$ is normable if and only if $\int_0^\infty s^{-p}w(s)^p \, ds \leq C^{-p} \int_0^\infty w(s)^p \, ds$ for all $t > 0$ (see [28, Theorem A], [38, Theorem 12]). Moreover, $\Lambda_{1,w}$ is normable if and only $W_1(t)/t$ is a pseudo-decreasing function, that is, a decreasing with a constant (see [28, Theorem 4], [38, p. 104]). If $0 < p < 1$, then $\Lambda_{p,w}$ is not normable since it contains copy of $l^p$ (see [28, Theorem 1]).

If $E = L^\infty(w)$ with the weight $w: (0, \infty) \to (0, \infty)$, then $E^{(*)} = (L^\infty(w))^{(*)} = M_w$ is the Marcinkiewicz space generated by the functional $\|x\|_{M_w} = \sup_{t>0} w(t) x^*(t)$. We do not exclude the case $M_w = \{0\}$. We assume that the fundamental function

$$\tilde{w}(t) := \|\chi_{(0,t)}\|_{M_w} = \sup_{s>0} w(s) \chi_{(0,t)}(s) = \sup_{0<s<t} w(s)$$
satisfies \( \tilde{w}(t) < \infty \) for each \( t > 0 \). This function is increasing and
\[
\sup_{t>0} \tilde{w}(t) x^*(t) = \sup_{t>0} w(t) x^*(t).
\]
A nontrivial part of the proof of the last equality is the estimate
\[
\sup_{t>0} \tilde{w}(t) x^*(t) = \sup_{t>0} \left[ \sup_{0<s \leq t} w(s) x^*(t) \right] \leq \sup_{s>0} \left[ \sup_{0<s \leq t} w(s) x^*(s) \right] \leq \sup_{s>0} w(s) x^*(s).
\]
The functional \( \| \cdot \|_{M_w} \) is a quasi-norm if and only if \( \tilde{w} \in \Delta_2 \), that is, there exists a constant \( D \geq 1 \) such that \( \tilde{w}(2t) \leq D \tilde{w}(t) \) for all \( t > 0 \) (see Haaker [22, Theorem 1.1]). The Marcinkiewicz space \( M_w \) is normable if and only if there exists a constant \( B \geq 1 \) such that \( \int_0^t \frac{1}{\tilde{w}(s)} ds \leq B \frac{t}{\tilde{w}(t)} \) for all \( t > 0 \) (see Haaker [22, Theorem 2.4]).

**Example 5.** We will apply Theorem 3 with \( E = L^p(t^a), F = L^q(t^b) \) and \( a, b \in R \). We need to check the respective assumptions.
(a) Suppose \( 1 < q < p < \infty \). Then:
1° \( E^{(s)} \neq \{0\}, F^{(s)} \neq \{0\} \) are normable spaces if and only if \( -1/p < a < 1 - 1/p, -1/q < b < 1 - 1/q \) and conditions (15) hold, where \( a \neq b \) requires in particular that \( L^u(t^a-b) \) if and only if \( (b-a)u' > 1 \). Thus \( a \neq b \) requires in particular that \( L^u(t^a-b) \) if and only if \( (b-a)u' > 1 \). Thus \( (b-a)u' > 1 \).

Moreover, for the cases 4°–7° see section 1, the discussion above inequality (3),
4°. \( H^s \) is bounded on \( F = L^q(t^b) \) if and only if:
(i) \( 1 - a/b - 1/u = -1/p + 1/q \) if and only if:
(ii) \( 1 \leq u = 1/p + 1/q \), whence \( 1/q - 1/p \geq 0 \); this condition is satisfied automatically by \( q < p \).
6°. \( H^s \) is bounded on \( E, F' \) if and only if \( a > -1/p, b < 1 - 1/q \) (independent of \( r \)),
7°. \( H_\ast \) is bounded on \( E, F' \) if and only if \( r(a + 1/p) < 1 \) and \( r(-b + 1/q) < 1 \). For small \( r > 0 \) the last two estimates are valid, because \( a + 1/p \in (0, 1) \) and \( b + 1/q \in (0, 1) \) by 1°. Summing up, the assumptions on \( a, b \) are the following
\[
-1/p < a < 1 - 1/p, -1/q < b < 1 - 1/q \text{ and } (b-a) \frac{pq}{p-q} > -1. \tag{15}
\]

Theorem 3 gives us that if \( 1 < q < p < \infty \) and conditions (15) hold, then
\[
M(\Lambda_{p,t^p}, \Lambda_{q,t^q}) = [M(L^p(t^a), L^q(t^b))]^{(s)} = [L^s(t^{b-a})]^{(s)} = \Lambda_{s,t(b-a)+}, \tag{16}
\]
where \( 1/s = 1/q - 1/p \).

In the following cases we analogously check the required assumptions.
(b) For $q = 1 < p < \infty$. For the respective condition 3$''$ we have 
\[ ((E \circ F')^\prime)^{(s)} = (L^p(t^{b-a}))^{(s)} \neq \{0\} \]
holds if and only if $(b-a)\frac{p}{p-1} > -1$. We check others assumptions
similarly getting the identification \[10\] with $1/s = 1 - 1/p$ under the assumption
\[-1/p < a < 1 - 1/p, -1 < b < 0 \text{ and } (b-a)\frac{p}{p-1} > -1.

(c) For $q = 1, p = \infty$ we have
\[ M(M_{p,a}, \Lambda_{1,t^b}) = M(L^{\infty}(t^a), L^1(t^b))^{(s)} = \Lambda_{1,t^{b-a}}, \]
whenever $-1 < b \leq 0 \leq a < 1$ and $b-a > -1$ (the last inequality comes from
the assumption 3$''$).

(d) For $1 \leq p < q < \infty$ it holds
\[ M(\Lambda_{p,t^{ap}}, \Lambda_{p,t^{bp}}) = M(L^p(t^a), L^p(t^b))^{(s)} = L^\infty(t^{b-a})^{(s)} = M_{p,b-a}, \]
whenever $-1/p < a, b < 1 - 1/p$ and $0 \leq b-a < 1/p$ (the last inequality comes from
the assumption 3$''$ and 5$''$).

(e) For $p = q = \infty$ it holds
\[ M(M_{t^a}, M_{t^b}) = M(L^{\infty}(t^a), L^{\infty}(t^b))^{(s)} = L^\infty(t^{b-a})^{(s)} = M_{t,b-a}, \]
whenever $a, b \in (0, 1)$ and $b-a \geq 0$ (the last inequality comes from the assumption 3$''$). Note
that for $r \in \left(0, \min\{\frac{1}{a}, \frac{1}{1+b}\}\right)$ the assumption (ii) of Theorem \[2\] is satisfied. Note
also that $M_{t,b-a} = \{0\}$ if $b < a$.

The above result can be applied to describe the space of multipliers between classical
Lorentz spaces $L^{p,q}$, which particular cases were proved in \[49\]–\[51\].

For $0 < p, q \leq \infty$ consider the classical Lorentz function spaces $L^{p,q} = L^{p,q}(I)$ on
$I = (0, 1)$ or $I = (0, \infty)$ defined by the quasi-norms
\[ \|x\|_{p,q} = \begin{cases} \left( \frac{m(I)}{0 \int t^{1/p}x^\prime(t)\frac{dt}{t}}} \right)^{1/q}, & \text{for } 0 < p \leq \infty, 0 < q < \infty, \\ \sup_{0<t<m(I)} t^{1/p}x^\prime(t), & \text{for } 0 < p \leq \infty, q = \infty. \end{cases} \]

Note that $L^{p,p} \equiv L^p$ for $0 < p \leq \infty$ and $L^{\infty,q} = \{0\}$ for $0 < q < \infty$. Consequently, for
$p = \infty$ we consider only the case $q = \infty$. Recall that $(L^{p,q})^{(r)} = L^{p,rq}$, where $(L^{p,q})^{(r)}$ is
the $r$-convexification ($1 < r < \infty$) of the Lorentz space $L^{p,q}$.

We characterize below all multipliers $M(L^{p_1,q_1}, L^{p_2,q_2})$. First, we describe cases when
one of spaces $L^{p_1,q_1}$ or $L^{p_2,q_2}$ is equal to $L^\infty$ because this limit cases do not suit to
the formal model of the below theorem.

**Remark 7.** If $p_1 = q_1 = \infty$, then $M(L^{p_1,q_1}, L^{p_2,q_2}) = L^{p_2,q_2}$ for all $p_2, q_2 > 0$. If $p_2 =
q_2 = \infty$ we need only to consider the case $0 < p_1 < \infty$ in which $M(L^{p_1,q_1}, L^{p_2,q_2}) =
M(L^{p_1,q_1}, L^\infty) = \{0\}$ for each $q_1 > 0$ by Proposition 2.3(ii),(iv) in \[35\].
Theorem 4. Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $I = (0, 1)$ or $I = (0, \infty)$.

(i) If either $p_1 < p_2$ or $p_1 = p_2$ and $q_1 > q_2$, then $M(L^{p_1,q_1}, L^{p_2,q_2}) = \{0\}$.

(ii) If either $p_1 > p_2$ or $p_1 = p_2$ and $q_1 \leq q_2$, then

$$M(L^{p_1,q_1}, L^{p_2,q_2}) = L^{p_3,q_3},$$

where

$$\frac{1}{p_3} = \frac{1}{p_2} - \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q_3} = \begin{cases} \frac{1}{q_2} - \frac{1}{q_1} & \text{if } q_1 > q_2, \\ 0 & \text{if } q_1 \leq q_2. \end{cases}$$

Proof. (i) It is enough to apply Proposition 2.3(ii),(iv) in [35] and the respective strict embeddings between Lorentz spaces $L^{p,q}$ which are well known.

We present, however, also a direct proof which idea has been taken from [38] Theorem 2 (see also [35] Proposition 2.3). We have two inclusions: $L^{p,q} \hookrightarrow L^{p_2,q_2}$ for $0 < q_1 < q_2 \leq \infty$ and for any $0 < p < \infty$ (see [4] Proposition 4.2) or [20] Proposition 1.4.10); also if $0 < m(A) < \infty$, then $L^{p_2,\infty}(A) \hookrightarrow L^{p_1,q}(A)$ for $0 < p_1 < p_2 < \infty$ and for any $0 < q \leq \infty$ since for $0 < q < \infty$

$$\|x\|_{p_1,q} = \left( \int_0^{m(A)} [t^{1/p_1}x^* (t)]^{q/p} \frac{dt}{t} \right)^{1/q} \leq \sup_{0 < t < m(A)} t^{1/p_2}x^* (t) \left( \int_0^{m(A)} t^{(1/p_1 - 1/p_2)q} \frac{dt}{t} \right)^{1/q} = C \|x\|_{p_2,\infty},$$

where $C = C(p_1, p_2, q, m(A))$. If $q = \infty$, then

$$\|x\|_{p_1,\infty} \leq \|x\|_{p_2,\infty} \sup_{0 < t < m(A)} t^{1/p_1 - 1/p_2} = m(A)^{1/p_1 - 1/p_2} \|x\|_{p_2,\infty}.$$

Now, if there exists $0 \neq x \in M(L^{p_1,q_1}, L^{p_2,q_2})$, then $|x(t)| > 0$ for almost all $t \in A$ with $0 < m(A) < \infty$. Let $A_n = \{ t \in A : \frac{n}{n+1} \leq |x(t)| \leq n \}$, $n = 1, 2, \ldots$ Then $A_n \nearrow A$ and so $m(A_n) > 0$ for $n \geq n_0$. If $y \in L^{p_1,q_1}(A_n)$, then $y\chi_{A_n} \in L^{p_1,q_1}(I)$ and

$$\frac{1}{n} y\chi_{A_n} \leq x y\chi_{A_n} \in L^{p_2,q_2}(A_n).$$

Hence, $L^{p_1,q_1}(A_n) \hookrightarrow L^{p_2,q_2}(A_n)$ for $n \geq n_0$. This is, of course, not possible for $p_1 < p_2$ since then it will be $L^{p_2,q_2}(A_n) \hookrightarrow L^{p_2,\infty}(A_n) \hookrightarrow L^{p_1,q_1}(A_n)$ and consequently $L^{p_1,q_1}(A_n) = L^{p_2,q_2}(A_n)$, which is not possible because $m(A_n) > 0$ for $n \geq n_0$.

Also it is not possible for $p_1 = p_2$ and $q_2 < q_1$ since again $L^{p_2,q_2}(A_n) \hookrightarrow L^{p_1,q_2}(A_n) \hookrightarrow L^{p_1,q_1}(A_n)$ and we get a contradiction.

(ii) Assume first that $1 < p_1, p_2, q_1, q_2$. Note that $L^{p,q}$ is normable for $p, q > 1$. We divide the proof into three parts.

Case A. Suppose that $1 < p_2 < p_1 < \infty$ and $1 < q_2 \leq q_1 \leq \infty$. It means that $1/p_3 = 1/p_2 - 1/p_1 > 0$ and $1/q_3 = 1/q_2 - 1/q_1 \geq 0$. Thus we can write

$$L^{p_3,q_3} \odot L^{p_1,q_1} = L^{p_2,q_2},$$
according to [9] (cf. [36]). Applying cancellation law from [36, Theorem 4] we see that
\[ M(L^{p_1,q_1}, L^{p_2,q_2}) = M(L^{p_1,q_1} \odot L^\infty, L^{p_3,q_3} \odot L^{p_1,q_1}) = M(L^\infty, L^{p_3,q_3}) = L^{p_3,q_3}. \]

We also present an independent proof basing on Example [5]. Let \( 1 < p_2 < p_1 \) and \( 1 < q_2 < q_1 < \infty \). Then, from Example [5](a) and (b) with \( a = 1/p_1 - 1/q_1, b = 1/p_2 - 1/q_2 \), we obtain
\[ M(L^{p_1,q_1}, L^{p_2,q_2}) = M(\Lambda_{q_1, t^{q_1/p_1 - 1}}, \Lambda_{q_2, t^{q_2/p_2 - 1}}) = \Lambda_{q_3, t^{(b-a)q_3}} = \Lambda_{q_3, t^{q_3/p_3 - 1}} = L^{p_3,q_3}. \]

Case B. Let \( 1 < p_2 < p_1 < \infty \) and \( 1 < q_1 < q_2 \leq \infty \). Then it is easy to see that assumptions of Theorem 2.2 in [35] are satisfied. Consequently, the fundamental function of \( M(L^{p_1,q_1}, L^{p_2,q_2}) \) is equivalent to \( t^{1/p_2 - 1/p_1} \). Applying the maximality of Marcinkiewicz space (see [4] and condition (29) in [36]) we get
\[ M(L^{p_1,q_1}, L^{p_2,q_2}) \subset M_{1/p_2 - 1/p_1} = L^{p_3,\infty}. \]

On the other hand, using once again [9] we can write
\[ L^{p_3,\infty} \odot L^{p_1,q_1} = L^{p_2,q_1} \subset L^{p_2,q_2}. \]

Thus
\[ L^{p_3,\infty} \subset M(L^{p_1,q_1}, L^{p_2,q_1}) \subset M(L^{p_1,q_1}, L^{p_2,q_2}). \]

Case C. Suppose \( 1 < p_1 = p_2 < \infty \) and \( 1 < q_1 \leq q_2 \). Note that both spaces \( L^{p_2,1} \) and \( L^{p_2,\infty} \) have the same fundamental functions. Moreover, \( L^{p_2,1} \subset L^{p_2,q_1} \) and \( L^{p_2,q_2} \subset L^{p_2,\infty} \) (see the proof of (i)). In consequence,
\[ L^\infty \subset M(L^{p_2,q_1}, L^{p_2,q_2}) \subset M(L^{p_2,1}, L^{p_2,\infty}) \subset L^\infty, \]

where the last inclusion follows from [36], Proposition 3. In fact, we have the equality \( M(L^{p_2,1}, L^{p_2,\infty}) = L^\infty \), because the reverse embedding \( L^\infty \subset M(L^{p_2,1}, L^{p_2,\infty}) \) is clear by \( L^{p_2,1} \subset L^{p_2,\infty} \).

In order to remove assumption \( 1 < p_2, p_1, q_2, q_1 \) we need to notice that
\[ (L^{p_2,q_2})^r = L^{p_2r,q_2r} \]
and
\[ M(E^r, F^r) = M(E, F)^r \]
for arbitrary \( r > 0 \). Since multiplying by \( r \) does not damage relations between \( p_2, p_1, q_2 \) and \( q_1 \), we may choose \( r \) so that \( 1 < rp_2, rp_1, rq_2, rq_1 \) and apply the previous part of theorem together with above equalities. \( \square \)

In particular, for \( p_2 = q_2 = 1 \) we obtain from Theorem [4] the well-known results on Köthe duality (see [13, 14, 17, 23] and also Grafakos book [20, pp. 52–55]):

(a) If \( 0 < p < 1 \) and \( 0 < q < \infty \), then \( (L^p,q)^r = \{0\} \) (Hunt [23, p. 262]).
(b) If \( 0 < p < 1 \), then \( (L^p,\infty)^r = \{0\} \) (Haaker [22, Theorem 4.2], Cwikel [13, Theorem 1]).
(c) If $1 < q \leq \infty$, then $(L^{1,q})' = \{0\}$ (Hunt \cite{23} pp. 262–263; observe that Cwikel-Sagher \cite{17} proved that the dual space of $(L^{1,\infty})^* \neq \{0\}$ but there is no its exact description).

(d) If $1 < p < \infty$, $1 \leq q \leq \infty$, then $(L^{p,q})' = L^{p',q'}$ (Hunt \cite{23} Theorem 2.7).

(e) If $1 < p < \infty$, $0 < q < 1$, then $(L^{p,q})' = L^{p,\infty}$ (Hunt \cite{23} Theorem 2.7).

Moreover, for example, for $q_1 = p_2 = 1$ we obtain from Theorem \cite{4} some probably new results on pointwise multipliers:

(f) If $0 < p < 1$ and $0 < q < \infty$, then $M(L^{p,1},L^{1,q}) = \{0\}$.

(g) If $p = 1$ and $1 \leq q < \infty$, then $M(L^{1},L^{1,q}) = L^{\infty}$. Moreover, $M(L^{1},L^{1,q}) = \{0\}$ if $0 < q < 1$.

(h) If $1 < p < \infty$ and $1 \leq q < \infty$, then $M(L^{p,1},L^{1,q}) = L^{p',\infty}$.

## 5 Explicit factorization of product spaces

A factorization of function or sequence spaces is a powerful tool which found applications in interpolation theory, geometry of Banach spaces (for example the idea of indicator function from \cite{25,20}) and operator theory (for example the proof of Nehari theorem in \cite{54}). Usually it is enough to know that for each $f \in G$ there exist $g \in E, h \in F$ satisfying $f = gh$ with $\|f\|_G \approx \|g\|_E \|h\|_F$, i.e. $G = E \odot F$. However, in some cases (see for example \cite{9}), the existence is not enough and one prefers to know explicit formulas which for a given $f$ produce $g$ and $h$ as above.

It is evident how to factorize $f \in L^p$ in order to get $f = gh$ satisfying $g \in L^q, h \in L^r$ and $\|f\|_p = \|g\|_q \|h\|_r$, i.e. $g = |f|^{p/q}$ and $h = |f|^{p/r}\text{sgn}f$, where $1/p = 1/q + 1/r$ (see Example \ref{example} below). Similar explicit formulas follow directly from the respective factorization theorems for Orlicz spaces \cite{15} Theorem 10.1(b) and Calderón–Lozanovskii spaces \cite{36} Theorem 5, or for Lorentz spaces \cite{9}. In this section we explain how to derive explicit formulas for factorization of symmetrized space, once we know the respective formulas for initial space.

**Definition 1.** Let $G = E \odot F$. We will say that the explicit factorization for $G = E \odot F$ holds if we have explicit formulas for maps $\varphi : G \to E$ and $\psi : G \to F$ such that each $x \in G$ can be written as

$$x = \varphi(x)\psi(x) \text{ and } \|x\|_G \approx \|\varphi(x)\|_E \|\psi(x)\|_F.$$ 

**Example 6.** Suppose $E$ and $F$ are quasi–Banach ideal spaces and $w, w_0, w_1$ are positive weights such that $w = w_0 w_1$. If $G = E \odot F$ and the explicit factorization holds, then $G(w) = E(w_0) \odot F(w_1)$ and the explicit factorization holds.

In particular, $L^p(w) = L^{p_0}(w_0) \odot L^{p_1}(w_1)$ with $1/p = 1/p_0 + 1/p_1$, $w = w_0 w_1$ and the maps

$$\varphi(x) = \left( |x| w \right)^{p/p_0} \text{sgn}x / w_0 \text{ and } \psi(x) = \left( |x| w \right)^{p/p_1} / w_1,$$

satisfy conditions of Definition \ref{definition}.

**Proof.** We will show the respective equalities and also maps, which give that explicit factorizations.
Firstly, we show that \( G = E \circ F \) implies \( G(w) = E(w_0) \circ F(w_1) \) with \( w = w_0 w_1 \). In fact, if \( x \in E(w_0) \) and \( y \in F(w_1) \), then \( x w_0 \in E, y w_1 \in F \), and by assumption \( x w_0 y w_1 \in G \), whence \( x y \in G(w) \), that is, \( E(w_0) \circ F(w_1) \to G(w) \).

Conversely, if \( x \in G(w) \), then \( x w \in G \) and, by the assumption that for \( G \) the explicit factorization holds, there are maps \( \varphi: G \to E \) and \( \psi: G \to F \) such that

\[
x w = \varphi(x w) \psi(x w) \quad \text{and} \quad \|x w\|_E \approx \|\varphi(x w)\|_E \|\psi(x w)\|_F.
\]

Taking

\[
\varphi_w(x) = \varphi(x w)/w_0 \quad \text{and} \quad \psi_w(x) = \psi(x w)/w_1
\]

we obtain \( \varphi_w(x) \psi_w(x) = \frac{\varphi(x w) \psi(x w)}{w_0 w_1} = \frac{\varphi(x w) \psi(x w)}{w} = x \) and

\[
\|\varphi_w(x)\|_{E(w_0)} \|\psi_w(x)\|_{F(w_1)} = \|\varphi(x w)\|_E \|\psi(x w)\|_F \approx \|x w\|_E = \|x\|_{G(w)}.
\]

Therefore, \( G(w) \to E(w_0) \circ F(w_1) \) and (17) is the explicit factorization. \( \square \)

**Theorem 5.** Suppose \( E \) and \( F \) are quasi-Banach ideal spaces such that the operators \( H_r \) and \( H_r^* \) are bounded on \( E^{1/2}, F^{1/2} \) for some \( r > 0 \). If for \( G = E \circ F \) the explicit factorization holds, then \( G^{(*)} = E^{(*)} \circ F^{(*)} \) and the explicit factorization holds.

**Remark 8.** It is easy to prove that the following conditions are equivalent: (a) the operator \( H_r \) is bounded on \( E^{1/2}, F^{1/2} \), (b) the operator \( H_r \) is bounded on \( E^{1/2r}, F^{1/2r} \), (c) the operator \( H_{2r} \) is bounded on \( E \).

**Proof of Theorem 5.** Note that, by the assumption on the operator \( H_r \) and Remark 2, the condition (6) is satisfied for the spaces \( E, F \). First, we will prove the inclusion \( E^{(*)} \circ F^{(*)} \to G^{(*)} \), where \( C = A_E A_F \), where \( A_E, A_F \) are the best constants in (6). Suppose \( x \in E^{(*)}, y \in F^{(*)} \). Then \( x^* \in E, y^* \in F \) and by the assumption, \( u = x^* y^* \in G \). We need to show that \( x y \in G^{(*)} \) or equivalently \((x y)^* \in G \). But

\[
(x y)^*(t) \leq x^*(t/2) y^*(t/2) = D_2 x^*(t) D_2 y^*(t).
\]

Moreover, \( D_2 x^* \in E, D_2 y^* \in F \), by the assumption which finishes the proof of the first inclusion.

Now, we want to prove the reverse inclusion \( G^{(*)} \to E^{(*)} \circ F^{(*)} \). Let \( x \in G^{(*)} \). Then \( x^* \in G \) and, by the assumption that the explicit factorization holds,

\[
x^* = \varphi(x^*) \psi(x^*) \quad \text{with} \quad \|x^*\|_E \approx \|\varphi(x^*)\|_E \|\psi(x^*)\|_F.
\]

For \( \varphi_1(x) = \varphi(x^* \right)^2 \) and \( \psi_1(x) = \psi(x^*)^2 \) we have

\[
x^* = \varphi_1(x)^{1/2} \psi_1(x)^{1/2}
\]

and, applying twice the Hölder–Rogers inequality and the equality (3), we get

\[
x^* \leq H_r(x^*) \leq [H_r(x^*)^r + H_r^*(x^*)^r]^{1/r} = H_r H_r^*(x^*) = H_r H_r^*(\varphi_1(x)^{1/2} \psi_1(x)^{1/2})
\]

\[
\leq H_r [H_r^*(\varphi_1(x))]^{1/2} [H_r^*(\psi_1(x))]^{1/2} = [H_r H_r^*(\varphi_1(x))]^{1/2} [H_r H_r^*(\psi_1(x))]^{1/2}.
\]

25
Define \( \varphi_2(x) = [H_r H_r^*(\varphi_1(x))]^{1/2} \) and \( \psi_2(x) = \frac{x^*}{\varphi_2(x)} \).

Clearly, \( x^* = \varphi_2(x) \psi_2(x) \). Denote \( C_0 = \|H_r\|_{E^{(1/2)} \rightarrow E^{(1/2)}} \), \( C_1 = \|H_r\|_{E^{(1/2)} \rightarrow F^{(1/2)}} \), \( D_0 = \|H_r^*\|_{E^{(1/2)} \rightarrow E^{(1/2)}} \) and \( D_1 = \|H_r^*\|_{E^{(1/2)} \rightarrow F^{(1/2)}} \). Note that \( \varphi_1(x) \in E^{(1/2)} \), \( \psi_1(x) \in F^{(1/2)} \). Then \( \varphi_2(x) \in E \) because

\[
\|\varphi_2(x)\|_E = \frac{\|H_r H_r^*(\varphi_1(x))\|^{1/2}}{E} = \|H_r H_r^*(\varphi_1(x))\|^{1/2}_{E^{(1/2)}} \\
\leq C_0^{1/2} D_0^{1/2} \|\varphi_1(x)\|^{1/2}_{E^{(1/2)}} = C_0^{1/2} D_0^{1/2} \|\varphi(x^*)\|_E.
\]

Since similarly we have \( \psi_2(x) \leq [H_r H_r^*(\psi_1(x))]^{1/2} \in F \), \( x^* \in E \odot F \).

Recall that the rank function \( r_x \) of the function \( x \) is defined by the formula

\[
r_x(t) = m\{s : |x(s)| > |x(t)| \text{ or } |x(s)| = |x(t)| \text{ and } s \leq t\}.
\]

It is known that \( r_x \) preserves measure and \( |x| = x^* \circ r_x \) a.e. provided \( x^*(\infty) = 0 \) (see Proposition 2 and 3 in [55]).

Since \( \varphi_2(x) \) is decreasing it follows that \((\varphi_2(x) \circ r_x) = (\varphi_2(x))^* \in E \), whence \( \varphi_2(x) \circ r_x \in E^{(*)} \). Similarly, we have \( \psi_2(x) \circ r_x \leq [H_r H_r^*(\psi_1(x))]^{1/2} \circ r_x \in F^{(*)} \). Thus, \( x \in E^{(*)} \odot F^{(*)} \) and \( G^{(*)} \hookrightarrow E^{(*)} \odot F^{(*)} \). Moreover, the explicit factorization for \( |x| \) is given by the formula

\[
\varphi_3(x) = [H_r H_r^*(\varphi_1(x))]^{1/2} \circ (r_x) \quad \text{and} \quad \psi_3(x) = \frac{|x|}{[H_r H_r^*(\varphi_1(x))]^{1/2} \circ (r_x)},
\]

where \( x^* = \varphi(x^*) \psi(x^*) \) is the respective explicit factorization of \( x^* \) in \( E \odot F \), \( \varphi_1(x) = \varphi(x^*)^2 \) and \( \psi_1(x) = \psi(x^*)^2 \).

We need only to prove that \( x^*(\infty) = 0 \) if \( I = (0, \infty) \). Suppose \( x^*(\infty) = a > 0 \), then, by equality (18), \( \varphi_1(x)^{1/2} \psi_1(x)^{1/2} \geq a \) for almost all \( t > 0 \) and considering the sets \( A = \{t > 0 : \varphi(x^*) \geq \sqrt{a}\} \), \( B = \{t > 0 : \psi(x^*) \geq \sqrt{a}\} \) we obtain \( A \cup B = (0, \infty) \) up to a set of measure zero. Then

\[
H_r^*(\varphi_1(x))(t) = \left( \int_t^\infty \varphi_1(x)(s)^r \frac{ds}{s} \right)^{1/r} \geq \left( \int_{A \cap (t, \infty)} \frac{a^{r/2}}{s} ds \right)^{1/r}
\]

and

\[
H_r^*(\psi_1(x))(t) = \left( \int_t^\infty \psi_1(x)(s)^r \frac{ds}{s} \right)^{1/r} \geq \left( \int_{B \cap (t, \infty)} \frac{a^{r/2}}{s} ds \right)^{1/r},
\]

which means \( H_r^*(\varphi_1(x))(t) + H_r^*(\psi_1(x))(t) = +\infty \) for all \( t > 0 \). Since

\[
(0, \infty) = \{t > 0 : H_r^*(\varphi_1(x))(t) = \infty\} \cup \{t > 0 : H_r^*(\psi_1(x))(t) = \infty\}
\]

(maybe except a set of measure zero) it follows that \( H_r^*(\varphi_1(x)) \notin E^{(1/2)} \) or \( H_r^*(\psi_1(x)) \notin F^{(1/2)} \), which is a contradiction because \( \varphi_1(x) \in E^{(1/2)} \) and \( \psi_1(x) \in F^{(1/2)} \). □
6 On factorization of symmetrized spaces

The classical factorization theorem of Lozanovskii states that for any Banach ideal space \( E \) the space \( L^1 \) has a factorization \( L^1 = E \odot E' \). The natural generalization has been investigated in [36] (see also [52], [56]): for Banach ideal spaces \( E \) and \( F \), when it is possible to factorize \( F \) through \( E \), i.e., when the equality \( F = E \odot M(E, F) \) holds?

Of course, such a natural generalization is not true without additional assumptions on the spaces, as we can see on examples presented in [36, Section 6]. In particular:

\[
L^2,1 \odot M(L^2,1, L^2) \not\equiv L^2,1 \odot M(L^2, L^2, \infty) \not\equiv L^2 \odot L^\infty \equiv L^2 \not\hookrightarrow L^2,\infty
\]

(see [36, Example 2]).

It is easy to see that if for Banach ideal spaces \( E \) and \( F \) the space \( F \) has a factorization through \( E \), i.e., \( F = E \odot M(E, F) \), then the corresponding weighted factorization holds, that is

\[
F(w_1) = E(w_0) \odot M(E(w_0), F(x_1)).
\] (19)

In fact, applying Example 6(b) and property (x) from [35] we get

\[
E(w_0) \odot M(E(w_0), F(x_1)) = E(w_0) \odot M(E, F)(x_1/w_0)
= (E \odot M(E, F))(x_1) = F(x_1).
\]

Applying Theorem 3 and Corollary 2 we get a result on factorization of symmetrizations.

**Corollary 3.** Suppose that assumptions from the Theorem 3 for the spaces \( E, F \) and the assumptions from Corollary 2 for the spaces \( E, M(E, F) \) are satisfied. If \( F \) factorizes through \( E \), i.e., \( F = E \odot M(E, F) \), then the symmetrization \( F^{(*)} \) factorizes through the symmetrization \( E^{(*)} \), that is,

\[
F^{(*)} = E^{(*)} \odot M(E^{(*)}, F^{(*)}).
\] (20)

**Proof.** We have

\[
E^{(*)} \odot M(E^{(*)}, F^{(*)}) = E^{(*)} \odot M(E^{(*)}, F^{(*)})
= [E \odot M(E, F)]^{(*)}
= F^{(*)}
\]

[by Theorem 3]

[by Corollary 2]

[by assumption].

\[
\]

Consequently, from (19) and Corollary 3 we can get that, under some assumptions on Banach ideal spaces \( E, F \), if \( F = E \odot M(E, F) \), then

\[
E(w_0)^{(x)} \odot M(E(w_0)^{(x)}, F(x_1)^{(x)}) = F(x_1)^{(x)}.
\]

Then the factorization of classical spaces of the type \( E^{(*)} \) like Lorentz and Marcinkiewicz spaces (see [36, Examples 3 and 4]) comes also from the known factorization of a respective weighted \( L^p \)-spaces or just \( L^p \) spaces.
Theorem 1 (iii)], $E \circ F = (E^{(p)} \circ F^{(p)})^{1/p}$ and similarly for $E \circ G$ we conclude by the assumption that

$$(E^{(p)} \circ F^{(p)})^{1/p} = (E^{(p)} \circ G^{(p)})^{1/p} \quad \text{and so} \quad E^{(p)} \circ F^{(p)} = E^{(p)} \circ G^{(p)}.$$

For Banach ideal spaces $E, F$, however, we have by [36, Theorem 1 (iv)] that the product space is $1/2$-convexification of the Calderón product: $E \circ F = (E^{1/2} F^{1/2})^{1/2}$ and we obtain

$$[(E^{(p)})^{1/2}(F^{(p)})^{1/2}]^{(1/2)} = [(E^{(p)})^{1/2}(G^{(p)})^{1/2}]^{(1/2)},$$

which gives $(E^{(p)})^{1/2}(F^{(p)})^{1/2} = (E^{(p)})^{1/2}(G^{(p)})^{1/2}$ and by uniqueness of Calderón–Lozanovskii construction [5, Corollary 1] (see also [16, Theorem 3.5] with a direct proof) we get $F^{(p)} = G^{(p)}$ or $F = G$.

(ii) If $F = E \circ M(E, F)$, then by the Lozanovskii factorization theorem

$$F' \circ M(E, F) \circ E = F' \circ F = L^1 = E' \circ E. \quad (21)$$

Observe that, by property (xi) in [35] the space $M(E, F)$ has the Fatou property, and by Corollary 1 (ii) in [36] the space $F' \circ M(E, F)$ has the Fatou property. Thus, by (i) above, we get $E' = F' \circ M(E, F)$ and finally, by property (vii) in [35] saying that $M(E, F) = M(F', E')$ we get $E' = F' \circ M(F', E')$. \hfill \Box

Now we discuss the factorization of Cesàro spaces. Recall that for a Banach ideal space $E$ on $I$ the Cesàro function space $CE = CE(I)$ is defined as

$$CE = \{ f \in L^0(I) : H|f| \in E \} \quad \text{with the norm} \quad \| f \|_{CE} = \| H|f| \|_E; \quad (22)$$

and the Tandori function space $\tilde{E} = \tilde{E}(I)$ as

$$\tilde{E} = \{ f \in L^0(I) : \tilde{f} \in E \} \quad \text{with the norm} \quad \| f \|_{\tilde{E}} = \| \tilde{f} \|_E, \quad (23)$$

where $H$ is a Hardy operator and $\tilde{f}(x) = \text{ess sup}_{t \in I, t \geq x} |f(t)|$ (cf. [11, 39, 40]). For example, if $E = L^p(I)$ the respective space $CL^p(I)$ is the classical Cesàro function space denoted usually by $Ces_p(I)$. Similarly, in the sequence case $E = l^p$ we have $ces_p := CL^p.$
Theorem 6. Let $E, F$ be symmetric Banach function spaces on $I = (0, \infty)$ (or symmetric Banach sequence spaces) with the Fatou property such that the operator $H$ is bounded on $E$ and on $F$. Assume that $F$ factorizes through $E$, that is, $F = E \odot M(E, F)$. Then

$$M(CE, CF) = \widetilde{M(CE, CF)}. \quad (24)$$

Proof. Let $E, F$ be symmetric Banach function spaces on $I = (0, \infty)$. Note that $CE \neq \{0\}, CF \neq \{0\}$ and $(CE)' = \widetilde{E}', (CF)' = \widetilde{F}'$ (see [39, Theorem 1 and 2]). Denote $G = M(E, F)$. The space of multipliers $G$ is a symmetric space (see [35, Theorem 2.2(i)]) and $G = M(F', E')$ (cf. [35, property (vii)]).

First, we show that

$$\widetilde{F'} \circ G = F' \circ \widetilde{G}. \quad \text{(25)}$$

In fact, applying Theorem 1(iv) from [36], Theorem 4 from [40] (since $F'$ and $G$ are symmetric), the equality $\widetilde{E'(\nu)} = (E^{(\nu)})$ and again Theorem 1(iv) from [36] we obtain

$$\widetilde{F'} \circ G = \left[(\widetilde{F'})^{1/2} \widetilde{G}^{1/2}\right]^{1/2} = \left[(F')^{1/2}G^{1/2}\right]^{1/2} = F' \circ \widetilde{G}. \quad \text{(26)}$$

Second, since $F = E \odot M(E, F)$ it follows by the Lozanovskii factorization theorem that

$$[F' \odot M(E, F)] \circ E = F' \circ [M(E, F) \circ E] = F' \circ F = L^1 = E' \circ E,$$

and by Lemma [4.i) we get $E' = F' \circ M(E, F)$. Thus,

$$\widetilde{F'} \circ \widetilde{G} = F' \circ \widetilde{G} = \widetilde{E'}. \quad \text{(27)}$$

Using the last equality, Theorem 2 from [39] on the Köthe duality of abstract Cesàro spaces $(CE)' = (\widetilde{E}')$ and the Lozanovskii factorization theorem we obtain

$$CE \circ \widetilde{F'} \circ \widetilde{G} = CE \circ \widetilde{E'} = CE \circ (CE)' = L^1. \quad \text{(28)}$$

Taking $L^1 = (\widetilde{G}')' \circ \widetilde{G}$ in (24) and applying Lemma [4.i), we obtain $CE \circ \widetilde{F'} = (\widetilde{G})'$, whence

$$(CE \circ \widetilde{F'})' = (\widetilde{G})'' = \widetilde{G} = M(\widetilde{E}, F). \quad \text{(29)}$$

Applying Theorem 4 from [36], Theorem 2 from [39], the Lozanovskii factorization theorem, the Köthe duality $(CF)' = (\widetilde{F'})'$ and the identification (29) we obtain

$$M(CE, CF) = M(CE, (CF)', CF \circ (CF)') = M(CE \circ (CF)', L^1) = [CE \circ (CF)']' = (CE \circ \widetilde{F'})' = M(E, F). \quad \text{(30)}$$

The proof is the same for symmetric Banach sequence spaces, applying Theorem 6 instead of Theorem 2 from [39].

Remark 9. Note that the above theorem for Banach function spaces on $I = (0, \infty)$ is also true with some different set of assumptions. Namely, if $E, F$ are Banach ideal spaces on $I = (0, \infty)$ with the Fatou property such that both the operators $H, H^*$ and $D_\tau$ are bounded on $E$ and on $F$, for some $\tau \in (0, 1)$, then it is enough to apply in the proof Theorem 3 instead of Theorem 2 from [39].
Example 7. Let $1 < q \leq p \leq \infty$. Set $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then
\[ M(Ces_p(I), Ces_q(I)) = M(\tilde{L}^p(I), \tilde{L}^q(I)) = \tilde{L}^r(I) \quad \text{with} \quad I = (0, \infty) \] (27)
and
\[ M(ces_p, ces_q) = \tilde{l}^r, \]
where $\tilde{l}^r = \{x = (x_n) : (\sum_{n=1}^{\infty} \sup_{k \geq n} |x_k|)^{1/r} < \infty\}$.

Proof. Since for $1 < q \leq p \leq \infty$ we have $M(L^p(I), L^q(I)) \equiv L^r(I)$ (cf. [18, Proposition 3]) and $L^p(I) \otimes L^q(I) = L^r(I)$ (cf. [51, p. 1373] and [36, Example 1(a)]), where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then using Theorem 6 with necessary restrictions on $p, q$ we obtain
\[ M(Ces_p(I), Ces_q(I)) = M[L^p(I), L^q(I)] = \tilde{L}^r(I). \]
Also for $I = (0, \infty)$ we have
\[ M(\tilde{L}^p, \tilde{L}^q) = M[(\tilde{L}^q)' , (\tilde{L}^p)'] = M(Ces_{q'}, Ces_{p'}) = \tilde{l}^r, \]
since $1/p' - 1/q' = 1/q - 1/p = 1/r$. For the sequence case the proof is the same. \qed

Note that C. Bennett proved the above result $M(ces_p, ces_q) = \tilde{l}^r$ for Cesàro sequence spaces in [3].

Problem 2. Prove an analogous result to Theorem 6 for $I = (0, 1)$.

We need to assume that $E, F$ are symmetric Banach function spaces with the Fatou property such that the operators $H$ and $H^*$ are bounded on $E$ and on $F$. Then, by Corollary 13 from [39] on the Köthe duality of abstract Cesàro spaces on $I = (0, 1)$, we have $(CE)' = (E'(1/w))$ for $w(t) = 1 - t, t \in I$. Suppose we try to prove this result similarly as for $I = (0, \infty)$. Unfortunately, we are not able to apply Theorem 4 from [40] because the respective space $F'(1/w)$ is not symmetric. Thus, for Theorem 4 from [40], we need to assume that $H, H^*$ are bounded on $F'(1/w)$ and $M(E, F)$ which do not seem to be reasonable.

Note that we can not apply Theorem 4 in the case when $M(E, F) = L^\infty$ with $E \neq F$ or $M(E, F) = \{0\}$, because the factorization assumption is not satisfied. However, for $1 < p \leq q < \infty$ we have $M(l^p, l^q) = l^\infty$ and $M(L^p(I), L^q(I)) = \{0\}$. Consequently, it is natural to find descriptions of $M(CE, CF)$ in this cases. Note that C. Bennett [3] proved that if $1 < p \leq q < \infty$, then $M(ces_p, ces_q) = \{x = (x_n) : \sup_{n \in N} n^{1/q - 1/p} |x_n| < \infty\}$.

Corollary 4. Suppose the assumptions of Theorem 6 are satisfied. Then Cesàro function (sequence) space $CF$ can be factorized by another Cesàro function (sequence) space $CE$, that is,
\[ CF = CE \odot M(CE, CF). \]
Proof. (i) Let $E, F$ be symmetric Banach function spaces on $I = (0, \infty)$. Applying equality (25) we get $CE \circ \tilde{F} \circ M(E, F) = L^1$. By our Theorem 6 and Theorem 2 from [39] we conclude that

$$CE \circ M(CE, CF) \circ \tilde{F} = L^1 = CF \circ (CF)' = CF \circ \tilde{F},$$

whence, by Lemma 4(i),

$$CE \circ M(CE, CF) = CF.$$

The proof in the sequence case is the same.\qed

Since factorization of Lebesgue spaces (Example 7) and Orlicz spaces (Theorem 2 in [41], cf. also Theorem 9 and Corollary 8 in [36]) is known, it is easy to conclude the respective factorization of Cesàro function spaces $Ces_p$, Cesàro–Orlicz function spaces $Ces_\varphi$. Note that we may consider also different weighted Cesàro function spaces

$$Ces_p (w) = CL^p (w) = \{ x \in L^0 : xw \in CL^p \} = \{ x \in L^0 : H |xw| \in L^p \}$$

or

$$C (L^p (w)) = \{ x \in L^0 : H |x| \in L^p (w) \} = \{ x \in L^0 : wH |x| \in L^p \}.$$

Then applying our results one can conclude the respective factorization of spaces $Ces_p (w)$ and $C (L^p (t^a))$.

8 Acknowledgements

The author Paweł Kolwicz and Karol Leśniki are supported by the Ministry of Science and Higher Education of Poland, grant number 04/43/DSPB/0094.

References

[1] S. V. Astashkin and L. Maligranda, Structure of Cesàro function spaces, Indag. Math. (N.S.) 20 (2009), no. 3, 329–379.

[2] J. Bastero, H. Hudzik and A. M. Steinberg, On smallest and largest spaces among rearrangement-invariant $p$-Banach function spaces $(0 < p < 1)$, Indag. Math. (N.S.) 2 (1991), no. 3, 283–288.

[3] G. Bennett, Factorizing the Classical Inequalities, Mem. Amer. Math. Soc. 120, AMS, Providence 1996.

[4] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, Boston 1988.

[5] E. I. Berezhnoi and L. Maligranda, Representation of Banach ideal spaces and factorization of operators, Canad. J. Math. 57 (2005), no. 5, 897–940.

[6] M. Buntinas, Products of sequence spaces, Analysis 7 (1987), 293–304.
[7] M. Buntinas and G. Goes, *Products of sequence spaces and multipliers*, Radovi Mat. 3 (1987), 287–300.

[8] F. Cabello Sánchez, *Pointwise tensor products of function spaces*, J. Math. Anal. Appl. 418 (2014), 317–335.

[9] F. Cabello Sánchez, *Factorization in Lorentz spaces, with an application to centralizers*, J. Math. Anal. Appl. 446 (2017), no. 2, 1372–1392.

[10] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113–190.

[11] J. Cerdà and J. Martín, *Interpolation of operators on decreasing functions*, Math. Scand. 78 (1996), 233–245.

[12] J. Cerdà and J. Martín, *Interpolation of some cones of function spaces*, in: Interaction Between Functional Analysis, Harmonic Analysis, and Probability (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math. 175, Dekker, New York 1996, 145–151.

[13] M. Cwikel, *On the conjugate of some function spaces*, Studia Math. 45 (1973), 49–55.

[14] M. Cwikel, *The dual of Weak L¹*, Ann. Inst. Fourier (Grenoble), 25 (1975), 81–125.

[15] M. Cwikel, A. Kamińska, L. Maligranda and L. Pick, *Are generalized Lorentz “spaces” really spaces?*, Proc. Amer. Math. Soc. 132 (2004), no. 12, 3615–3625.

[16] M. Cwikel, P. G. Nilsson and G. Schechtman, *Interpolation of weighted Banach lattices. A characterization of relatively decomposable Banach lattices*, Mem. Amer. Math. Soc. 165 (2003), vi+127 pp.

[17] M. Cwikel and Y. Sagher, *L(p, ∞)*, Indiana Univ. Math. J. 21 (1972), no. 9, 781–786.

[18] P. Foralewski, *Some fundamental geometric and topological properties of generalized Orlicz–Lorentz function spaces*, Math. Nachr. 284 (2011), no. 8–9, 1003–1023.

[19] P. Foralewski, *On some geometric properties of generalized Orlicz–Lorentz function spaces*, Nonlinear Anal. 75 (2012), no. 17, 6217–6236.

[20] L. Grafakos, *Classical Fourier Analysis*, 2nd Edition, Springer, New York 2008.

[21] K.-G. Grosse-Erdmann, *The Blocking Technique, Weighted Mean Operators and Hardy’s Inequality*, Lecture Notes in Math. 1679, Springer–Verlag, Berlin 1998.

[22] A. Haaker, *On the conjugate space of Lorentz space*, Technical Report, Lund 1970, 1–23; reprinted as: A. Sparr, *On the conjugate space of the Lorentz space L(φ, q)*, Contemporary Math. 445 (2007), 313–336.

[23] R. A. Hunt, *On L(p,q) spaces*, Enseignement Math. 12 (1966), no.2, 249–276.

[24] N. J. Kalton, *Convexity conditions for non-locally convex lattices*, Glasgow Math. J. 25 (1984), 141–152.
[25] N. J. Kalton, *Differentials of complex interpolation processes for Köthe function spaces*, Trans. Amer. Math. Soc. 333 (1992), 479–529.

[26] N. J. Kalton, *The basic sequence problem*, Studia Math. 116 (1995), no. 2, 167–187.

[27] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Space Sampler*, London Math. Society Lecture Note Series 89, Cambridge Univ. Press, Cambridge 1984.

[28] A. Kamińska and L. Maligranda, *Order convexity and concavity of Lorentz spaces \( \Lambda_{p,w} \), \( 0 < p < \infty \)*, Studia Math. 160 (2004), no. 3, 267–286.

[29] A. Kamińska, L. Maligranda and L. E. Persson, *Indices, convexity and concavity of Calderón–Lozanovskii spaces*, Math. Scand. 92 (2003), 141–160.

[30] A. Kamińska and M. Mastyło, *Abstract duality Sawyer formula and its applications*, Monatsh. Math. 151 (2007), no. 3, 223–245.

[31] A. Kamińska and Y. Raynaud, *Isomorphic copies in the lattice \( E \) and its symmetrization \( E^{(*)} \) with applications to Orlicz–Lorentz spaces*, J. Funct. Anal. 257 (2009), no. 1, 271–331.

[32] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Nauka, Moscow 1977 (Russian); English transl. Pergamon Press, Oxford–Elmsford, New York 1982.

[33] P. Kolwicz, *Local structure of generalized Orlicz–Lorentz function spaces*, Comment. Math. 55 (2015), no. 2, 211–227.

[34] P. Kolwicz, *Local structure of symmetrizations \( E^{(*)} \) with applications*, J. Math. Anal. Appl. 440 (2016), no. 2, 810–822.

[35] P. Kolwicz, K. Leśniki and L. Maligranda, *Pointwise multipliers of Calderón–Lozanovskii spaces*, Math. Nachr. 286 (2013), no. 8–9, 876–907.

[36] P. Kolwicz, K. Leśniki and L. Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 266 (2014), no. 2, 616–659.

[37] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc., Providence 1982 [Russian version Nauka, Moscow 1978].

[38] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelský Servis, Plzeň 2007.

[39] K. Leśniki and L. Maligranda, *Abstract Cesàro spaces: Duality*, J. Math. Anal. Appl. 424 (2015), no. 2, 932–951.

[40] K. Leśniki and L. Maligranda, *Interpolation of abstract Cesàro, Copson and Tandori spaces*, Indag. Math. 27 (2016), 764–785.

[41] K. Leśniki and J. Tomaszewski, *Pointwise multipliers of Orlicz function spaces and factorization*, Positivity 21 (2017), 1563–1573.
[42] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*, Springer-Verlag, Berlin–New York 1979.

[43] G. Ja. Lozanovski˘ı, *Certain Banach lattices. IV*, Sibirsk. Mat. Zh. 14 (1973), 140–155 (in Russian); English transl. in: Siberian. Math. J. 4 (1973), 97–108.

[44] G. Ja. Lozanovski˘ı, *Transformations of ideal Banach spaces by means of concave functions*, in: “Qualitative and Approximate Methods for the Investigation of Operator Equations”, no. 3, Yaroslav. Gos. Univ., Yaroslavl 1978, 122–148 (Russian).

[45] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics 5, University of Campinas, Campinas SP, Brazil 1989.

[46] L. Maligranda, *Type, cotype and convexity properties of quasi-Banach spaces*, in: “Banach and Function Spaces”, Proc. of the Internat. Symp. on Banach and Function Spaces (Oct. 2–4, 2003, Kitakyushu-Japan), Editors M. Kato and L. Maligranda, Yokohama Publ. 2004, 83–120.

[47] L. Maligranda, *Tosio Aoki (1910–1989)*, in: “Banach and Function Spaces”, Proc. of the Internat. Symp. on Banach and Function Spaces (14–17 Sept. 2006, Kitakyushu-Japan), Editors M. Kato and L. Maligranda, Yokohama Publ. 2008, 1–23.

[48] L. Maligranda and L. E. Persson, *Generalized duality of some Banach function spaces*, Indag. Math. 51 (1989), no. 3, 323–338.

[49] E. Nakai, *Pointwise multipliers*, Memoirs of The Akashi College of Technology 37 (1995), 85–94.

[50] E. Nakai, *Pointwise multipliers on the Lorentz spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. 45 (1996), 1–7.

[51] E. Nakai, *Pointwise multipliers on several function spaces – a survey*, Linear and Nonlinear Analysis, Special Issue on ISBFS 2015, 3 (2017), no. 1, 27–59.

[52] P. Nilsson, *Interpolation of Banach lattices*, Studia Math. 82 (1985), no. 2, 135–154.

[53] S. Okada, W. J. Ricker and E. A. Sánchez Pérez, *Optimal Domain and Integral Extension of Operators. Acting in Function Spaces*, Birkhäuser Verlag, Basel 2008.

[54] V. V. Peller, *Hankel Operators and Their Applications*, Springer, Berlin 2003.

[55] J. V. Ryff, *Measure preserving transformation and rearrangements*, J. Math. Anal. Appl. 31 (1970), 449–458.

[56] A. R. Schep, *Products and factors of Banach function spaces*, Positivity 14 (2010), 301–319.

[57] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam 1983.
