Hodograph solutions for the generalized dKP equation

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Abstract

We investigate the integrable (2 + 1)-dimensional generalized dispersionless KP (GdKP) equation (or Manakov-Santini system) from the Lax-Sato form. Several particular three-component reductions are considered so that the GdKP equation can be reduced to hydrodynamic systems. Then one can construct infinite exact solutions of GdKP by the generalized hodograph method.

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1 Introduction

Recently, in [1, 2, 3] Manakov and Santini proposed an inverse scattering transform for the multidimensional Hamiltonian vector fields, and analyzed the Cauchy problem for the following system of PDEs in 2+1 dimensions, the so-called Manakov-Santini equation [1, 2, 3, 4]

\[
\begin{align*}
\frac{1}{3} u_{1,xt} &= \frac{1}{4} u_{1,yy} + (u_1u_{1,x})_x + \frac{1}{2} v_{1,x}u_{1,xy} - \frac{1}{2} u_{1,xx}v_{1,y}, \\
\frac{1}{3} v_{1,xt} &= \frac{1}{4} v_{1,yy} + u_1v_{1,xx} + \frac{1}{2} v_{1,x}v_{1,xy} - \frac{1}{2} v_{1,xx}v_{1,y},
\end{align*}
\]

(1)

where \( u_1 = u_1(x,y,t) \) and \( v_1 = v_1(x,y,t) \) are two distinct field variables. From the second equation, one can see that \( u_1 \) can be expressed as differential polynomial of \( v_1 \). After plugging it into the first equation, we obtain the nonlinear fourth order (2+1)-dimensional PDEs for \( v_1 \). It is noticed that for \( v_1 = 0 \) reduction, the system reduces to the famous dKP equation

\[
\frac{1}{3} u_{1,xt} = \frac{1}{4} u_{1,yy} + (u_1u_{1,x})_x,
\]

(2)

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while \( u_1 = 0 \) reduction gives the equation associated with Einstein-Weyl space \([5, 6, 7, 8]\)
\[
\frac{1}{3}v_{1,xt} = \frac{1}{4}v_{1,yy} + \frac{1}{2}v_{1,xv_{1,xy}} - \frac{1}{2}v_{1,xx}v_{1,y}. \tag{3}
\]

When \( u_1 = v_{1y}/2 = \Phi_{yy}/2 \) we can obtain the dispersionless limit of the discrete KP equation \([9]\):
\[
\frac{1}{3}\Phi_{xt} = \frac{1}{4}\Phi_{yy} + \frac{1}{4}(\Phi_{xy})^2,
\]
or by setting \( \Phi_{xy} = \tilde{\Phi} \),
\[
\frac{1}{3}\tilde{\Phi}_{xt} = \frac{1}{4}\tilde{\Phi}_{yy} + \frac{1}{4}(\tilde{\Phi}^2)_{xy}.
\]

One remarks that we also have the differential reduction \( u_1 = v_{1x} \) associated with the Einstein-Weyl space \([10]\):
\[
\frac{1}{3}v_{1,xt} = \frac{1}{4}v_{1,yy} + \frac{1}{2}v_{1,xv_{1,xy}} - \frac{1}{2}(v_{1,y} - v_{1x})v_{1,xx}.
\]

In \([11]\), the hierarchy of GdKP is constructed using the Lax-Sato formulation. According to these results, one way to investigate the Manakov-Santini equation is to generalize it to the GdKP hierarchy, so that its integrability can be established such as infinite symmetries, finite-dimensional reductions, etc. The main purpose of the work is to find the exact solutions of Manakov-Santini equation \([11]\) by generalized hodograph method via some suitable three-dimensional reductions of hydrodynamic type.

This paper is organized as follows. In section 2, we use the GdKP hierarchy of Lax-Sato representation, from which we reproduce the Manakov-Santini equation in 2+1 dimensions. In section 3, relating to the Orlov operator we impose a special generating function as a starting point of reduction for the GdKP hierarchy. We then use the two-dimensional reductions of Lax operators arising from the ordinary dKP hierarchy to construct systems in hydrodynamic form. In section 4, we briefly recall the Hodograph method \([14]\) which transforms the hydrodynamic form into the linear one. In particular, we solve four examples that are described in section 3 and obtain hodograph solutions of the Manakov-Santini equation in rational type. Section 5 is devoted to the concluding remarks.

## 2 Generalized dKP hierarchy

The generalized dKP (GdKP or Manakov-Santini) hierarchy was constructed by the works \([11, 12]\) which can be defined by the Lax-Sato equations
\[
\frac{\partial \psi}{\partial t_n} = A_n \frac{\partial \psi}{\partial x} - B_n \frac{\partial \psi}{\partial p}, \quad \psi = \left( \mathcal{L} \mathcal{M} \right), \tag{4}
\]
or, equivalently, by the generating equation
\[
(J_0^{-1}d\mathcal{L} \wedge d\mathcal{M})_- = 0, \tag{5}
\]
where \( A_n = (J_0^{-1}\partial \mathcal{L}^n/\partial p)_+ \), \( B_n = (J_0^{-1}\partial \mathcal{L}^n/\partial x)_+ \) and the Lax and Orlov operators \( \mathcal{L}(p), \mathcal{M}(p) \) are the Laurent series
\[
\mathcal{L} = p + \sum_{n=1}^{\infty} u_n(x)p^{-n}, \tag{6}
\]
\[ \mathcal{M} = \sum_{n=1}^{\infty} u_n \mathcal{L}^{n-1} + \sum_{n=1}^{\infty} v_n(x) \mathcal{L}^{-n}. \]  

Here \((\cdot \cdot \cdot)_{+} (\cdot \cdot \cdot)_{-}\) denote respectively the projection on the polynomial part (negative powers), and \(J_0\) is defined by the Poisson bracket

\[ J_0 = \{ \mathcal{L}, \mathcal{M} \} = \frac{\partial \mathcal{L}}{\partial p} \frac{\partial \mathcal{M}}{\partial x} - \frac{\partial \mathcal{L}}{\partial x} \frac{\partial \mathcal{M}}{\partial p} = \frac{\partial \mathcal{L}}{\partial p} \left( \frac{\partial \mathcal{M}}{\partial x} \Big|_{\mathcal{L}_{\text{fixed}}} \right), \]

\[ = 1 + v_{1x} p^{-1} + (v_{2x} - u_1) p^{-2} + (v_{3x} - 2u_2 - 2v_{1x}u_1) p^{-3} + \ldots. \]

Note that in dKP case \(J_0 = 1\). Some of \(A_n\) and \(B_n\) are given by

\[ A_1 = 1, \]
\[ A_2 = 2p - 2v_{1x}, \]
\[ A_3 = 3p^2 - 3v_{1x}p + 6u_1 + 3(v_{1x})^2 - 3v_{2x}, \]
\[ A_4 = 4p^3 - 4v_{1x}p^2 + (12u_1 + 4(v_{1x})^2 - 4v_{2x}) p + 12u_2 - 4v_{3x} + 8v_{1x}v_{2x} - 4(v_{1x})^3 - 8u_1v_{1x}, \]

and

\[ B_1 = 0, \]
\[ B_2 = 2u_{1x}, \]
\[ B_3 = 3u_{1x}p - 3u_{1x}v_{1x} + 3u_{2x}, \]
\[ B_4 = 4u_{1x}p^2 + (4u_{2x} - 4v_{1x}v_{1x}) p + 4u_{1x} (4u_1 + (v_{1x})^2 - v_{2x}) - 4u_{2x}v_{1x} + 4u_{3x}. \]

From the Lax equation of \(\mathcal{L}\) in (4), the evolution equations of \(u_k\) with respect to \(t_2\) and \(t_3\) are derived respectively, by

\[ \partial_{t_2} u_k = 2 \left( u_{k+1,x} - v_{1x}u_{k,x} + (k - 1)u_{k-1}v_{1x} \right), \]
\[ \partial_{t_3} u_k = 3 \left( u_{k+2,x} + ku_{k}v_{1x} - v_{1x}u_{k+1,x} + (2u_1 + (v_{1x})^2 - v_{2x}) u_{k,x} - (k - 1)u_{k-1}u_{1x} + (k - 1)u_{k-1}u_{2x} \right), \]

where \(k \geq 1\). The first few nontrivial flows of \(u_1, u_2\) and \(u_3\) are read off as

\[ \partial_{t_2} u_1 = 2(u_{2,x} - v_{1x}u_{1,x}), \]
\[ \partial_{t_2} u_2 = 2(u_{3,x} - v_{1x}u_{2,x} + u_{1}u_{1,x}), \]
\[ \partial_{t_2} u_3 = 2(u_{4,x} - v_{1x}u_{3,x} + 2u_{2}u_{1,x}), \]
\[ \partial_{t_3} u_1 = 3(u_{3,x} + 3u_{1}u_{1,x} - v_{1x}u_{2,x} - v_{2,x}u_{1,x} + (v_{1x})^2 u_{1,x}), \]
\[ \partial_{t_3} u_2 = 3(u_{4,x} - v_{1x}u_{3,x} + 3u_{1}u_{2,x} + 2u_{1}u_{2} - v_{1x}u_{1}u_{1,x} + (v_{1x})^2 u_{2,x} - v_{2,x}u_{2,x}). \]

Alternatively, it has been shown \([4]\) that the Lax equations for \(\mathcal{L}\) in (4) is equivalent to

\[ \frac{\partial p(\mathcal{L})}{\partial t_n} \bigg|_{\mathcal{L}_{\text{fixed}}} = A_n(p(\mathcal{L})) \frac{\partial p(\mathcal{L})}{\partial x} \bigg|_{\mathcal{L}_{\text{fixed}}} + B_n(p(\mathcal{L})), \]
where
\[ p(L) = L - \sum_{n=1}^{\infty} \tilde{u}_n L^{-n} \]
is the inverse map of \( L(p) \) with relations
\[ \tilde{u}_1 = u_1, \quad \tilde{u}_2 = u_2, \quad \tilde{u}_3 = u_1^2 + u_3, \quad \tilde{u}_4 = 3u_1 u_2 + u_4, \ldots \] (14)
Substituting the expression of \( p(L) \) into (13) and using the formula \( \text{res}(L^k dL) = \delta_{k-1} \) we can formally obtain the evolution equations for \( \tilde{u}_k \):
\[ \partial_t \tilde{u}_k = \text{res} \left( A_n(p(L)) \sum_{m=1}^{\infty} \tilde{u}_{m,x} L^{k-m-1} - B_n(p(L)) L^{k-1} \right), \quad k \geq 1. \]
For the evolution equations of \( v_n \) we notice that the Lax equation for \( M \) in (4) can be expressed as the following way:
\[ \frac{\partial M}{\partial t} \bigg|_{L \text{ fixed}} + \frac{\partial M}{\partial x} \frac{\partial L}{\partial t} = A_n(p(L)) \left( \frac{\partial M}{\partial x} \bigg|_{L \text{ fixed}} + \frac{\partial M}{\partial x} \frac{\partial L}{\partial t} \right) - B_n(p(L)) \frac{\partial M}{\partial x} \frac{\partial L}{\partial t}. \]
After eliminating the Lax equation of \( L \) we have
\[ \frac{\partial M}{\partial t} \bigg|_{L \text{ fixed}} = A_n(p(L)) \frac{\partial M}{\partial x} \bigg|_{L \text{ fixed}}. \] (15)
Substituting the expression (7) into the above and again, using \( \text{res}(L^k dL) = \delta_{k-1} \) it follows that
\[ \partial_t v_k = \text{res} \left( A_n(p(L)) \left( \frac{\partial_x M}{\partial L} \bigg|_{L \text{ fixed}} \right) L^{k-1} \right), \quad n, k \geq 1. \]
For example, for \( n = 2, 3 \) we have
\[ \begin{align*}
\partial_2 v_k &= 2 \left( v_{k+1,x} - v_{1,x} v_{k,x} - \tilde{u}_k - \sum_{m=1}^{k-1} \tilde{u}_{k-m} v_{k,x} \right), \\
\partial_3 v_k &= 3 \left( v_{k+2,x} - v_{1,x} v_{k+1,x} + \tilde{u}_k v_{1,x} - 2\tilde{u}_{k+1} + (2u_1 + (v_{1,x})^2 - v_{2,x}) v_{k,x} \right. \\
&\quad \left. - 2 \sum_{m=1}^{k} \tilde{u}_{k-m+1} v_{m,x} + \sum_{m=1}^{k} \tilde{u}_{k-m} (\tilde{u}_m + v_{1,x} v_{m,x}) + \sum_{l=2}^{k-1} \tilde{u}_{k-l} v_{k-l,x} \right) \\
\end{align*} \]
We read off few of them as
\[ \begin{align*}
\partial_2 v_1 &= 2(v_{2,x} - (v_{1,x})^2 - u_1), \\
\partial_2 v_2 &= 2(v_{3,x} - v_{1,x} v_{2,x} - v_{1,x} u_1 - u_2), \\
\partial_2 v_3 &= 2(v_{4,x} - v_{1,x} v_{3,x} - v_{1,x} u_2 - v_{2,x} u_1 - u_1^2 - u_3), \\
\partial_3 v_1 &= 3 \left( v_{3,x} + v_{1,x} u_1 - 2v_{1,x} v_{2,x} + (v_{1,x})^3 - 2u_2 \right), \\
\partial_3 v_2 &= 3 \left( v_{4,x} - (v_{2,x})^2 - v_{1,x} (v_{3,x} + u_2) + (v_{1,x})^2 (v_{2,x} + u_1) - u_1^2 - 2u_3 \right),
\end{align*} \] (16) (17) (18)
where Eq. (14) has been used. We remark here that (15) can be written as the following form of conservation equations
\[ \frac{\partial}{\partial t} \left( \frac{\partial M}{\partial x} \bigg|_{L \text{ fixed}} \right) = \frac{\partial}{\partial x} \left( A_n \frac{\partial M}{\partial x} \bigg|_{L \text{ fixed}} \right), \] (19)
which means that the coefficients \( \{v_{n,x}\}_{n=1}^{\infty} \) are just the infinitely many conserved quantities of (11). To get (2+1)-dimensional equation of the GdKP hierarchy, we start from Eq. (12) and eliminate \( u_{3,x} \) and \( v_{2,x} \) by Eqs. (11) and (16) respectively to obtain

\[
\frac{1}{3} \partial_{t_3} u_1 = \frac{1}{2} \partial_{t_2} u_2 - \frac{1}{2} u_{1,x} \partial_{t_2} v_1 + u_1 u_{1,x}. \tag{20}
\]

In the similar way, Eq. (18) together with (16), (17) obtain

\[
\frac{1}{3} \partial_{t_3} v_1 = \frac{1}{2} \partial_{t_2} v_2 + v_{1,x} u_1 - \frac{1}{2} v_{1,x} \partial_{t_2} v_1 - u_2. \tag{21}
\]

Now differentiating Eqs. (20) and (21) respectively with respect to \( x \) and using \( u_{2,x} \) and \( v_{2,x} \) by Eqs. (10) and (16), we have the Manakov-Santini equation (1).

**Remark:** As mentioned before, when \( v_1 = 0 \) the GdKP hierarchy reduces to the dKP hierarchy:

\[
L = p + \sum_{n=1}^{\infty} u_n p^{-n}, \quad M = \sum_{n=1}^{\infty} n t_n L^{n-1} + \sum_{n=2}^{\infty} v_n L^{-n},
\]

\[
\partial_{t_n} \Psi = \{ \Omega_n, \Psi \}, \quad \Omega_n \equiv (L^n)_+, \quad \Psi \equiv \left( \begin{array}{c} L \\ M \end{array} \right)
\]

which means the Poisson bracket \( \{ L, M \} = 1 \) is canonical. From this, it is easy to see that those \( v_{n,x} \) \( (n \geq 2) \) in \( \partial_x M \) (for \( L \) fixed) are nothing but the conserved quantities \( H_n \) (up to a scaling constant) generated by the inverse of \( L \), i.e.,

\[
p = L - \sum_{k=1}^{\infty} H_k L^{-k}.
\]

Consequently, in contrast to the generalized dKP, the conservation equations in dKP case

\[
\partial_{t_n} \left( M_x \big|_{L \text{ fixed}} \right) = \partial_x \left( \partial_{p} \Omega_n (p(L)) \left( M_x \big|_{L \text{ fixed}} \right) \right)
\]

is equivalent to

\[
\partial_{t_n} p(L) = \partial_x \Omega_n (p(L)).
\]

### 3 Three-dimensional reductions

Reductions of the GdKP is unlike the dKP system. Except for the Lax reductions of the dKP hierarchy, the complex relations between \( \{ v_{n,x} \}_{n=1}^{\infty} \) and \( \{ u_n \}_{n=1}^{\infty} \) have to be considered to construct systems of finite-dimensional reductions. To solve Manakov-Santini equation (1), we only need three-dimensional reductions. From these reductions, one can reduce (1) to hydrodynamic form and then obtain infinite exact solutions from the theory of generalized hodograph theory.

To this end, let us introduce the generating function

\[
Q := \left( \partial_x M \big|_{L} \right)^{-1} \tag{22}
\]
then equation (19) is equivalent to
\begin{equation}
\partial_t Q = A_n(\partial_x Q) - (\partial_x A_n)Q := \langle A_n, Q \rangle.
\end{equation}

Next we consider the particular reduction of (23) by the solutions $Q = Q(p(\mathcal{L}), \mathbf{V})$, where $\mathbf{V} = (V_1, \ldots, V_m)$. Then taking into account (13), equation (23) reduces to
\begin{equation}
\partial_t Q\big|_p = A_n\left(\partial_x Q\big|_p\right) - (\partial_x A_n)Q - B_n(\partial_p Q).
\end{equation}

In what follows, we discuss the simplest case for $m = 1$ with $V_1 := v$. For simplicity, one sets $v_2 = 0$. Then we assume the following ansatz
\begin{equation}
Q = \frac{p}{p-v},
\end{equation}
where $v = -v_{1,x}$, which can be seen by comparing the both sides of (22) with fixed $\mathcal{L}$. Substituting (25) into (24) we derive
\begin{equation}
\partial_t v = A_n(\partial_x v) - (\partial_x A_n)(p-v) + \frac{v}{p}B_n.
\end{equation}

By the result, evaluated at the point $p = v$, the root of $Q^{-1}$ yields the evolution equations of $v$
\begin{equation}
\partial_t v = A_n(p=v)\partial_x v + B_n(p=v).
\end{equation}

With the ansatz (25) we see that Eq. (13) for $n=2,3$ can be read off as
\begin{align*}
\partial_t p &= 2(p+u)p_x + 2u_{1,x}, \\
\partial_t p &= 3(p^2 + vp + v^2)p_x + 3(p+u)u_{1,x} + 3u_{2,x}.
\end{align*}

Then the compatibility condition $\partial_t \partial_t p = \partial_t \partial_t p$ with independent variables $p^0, p, p_x, pp_x$, implies
\begin{align*}
6vu_{2,x} + 6vv_x u_{1,x} - 3u_{2,xt} - 6(u_{1,x})^2 + \\
-12u_1u_{1,xx} - 3v_yu_{1,x} - 3vu_{1,xt} + 2u_{1,xt} &= 0, \\
2u_{2,xx} + 2(vu_{1,x}) - u_{1,xt} &= 0, \\
3u_{2,x} + v_t + 6vu_{1,x} + 3v^2v_x - 3vv_t - 6u_1v_x - 3u_{1,t} &= 0, \\
2u_{1,x} + 4vv_x - v_t &= 0.
\end{align*}

By equations (28), (30) we obtain
\begin{align*}
\partial_t u_1 &= 2vu_{1,x} + 2u_{2,x}, \\
\partial_t v &= 4vv_x + 2u_{1,x}.
\end{align*}

Substituting into (27), (29) we have also
\begin{align*}
\partial_t u_1 &= (3/2)u_{2,y} + 6u_{1,ux} + 3v^2u_{1,x}, \\
\partial_t v &= 9v^2v_x + 3u_{2,x} + 6(vu_{1,x}).
\end{align*}

It is easy to see that based on the ansatz (25) eqs. (32), (34) are nothing but the evolution equations of $v$ given by (26) for $n=2,3$ and moreover; (31), (33) are related to equations...
Consequently, the ansatz (25) is admissible with the general expansion of Lax operator (6) to have commuting flows, at least, up to $t_3$. Nevertheless, it is feasible to apply (25) with some suitable Lax reductions to construct explicit solutions to the Manakov-Santini equations in 2+1 dimensions ($t_1 = x, t_2, t_3$).

Below, under the ansatz (25), we shall use the Lax reductions arising from the dKP hierarchy such as $n$th-KdV, Zakharov as well as the waterbag type reductions, i.e.,

\[
\mathcal{L}^{n+1} = p^{n+1} + w_1 p^{n-1} + w_2 p^{n-2} + \cdots + w_{n-1} + w_n,
\]

\[
\mathcal{L} = p + \sum_{i=1}^{n-1} \frac{w_i}{p - w_i},
\]

\[
\mathcal{L} = p + \sum_{i=1}^{n} \epsilon_i \log(p - w_i), \quad \epsilon_1 + \cdots + \epsilon_n = 0
\]

to construct finite-dimensional hydrodynamical systems in two and three components. Four examples shall be discussed as follows.

- $(n,m) = (1,1)$-reduction (associated with the dKdV reduction),
- $(n,m) = (2,1)$-reduction (associated with the dBoussinesq reduction),
- $(n,m) = (2,1)$-reduction (associated with the Zakharov reduction).
- $(n,m) = (2,1)$-reduction (associated with the Waterbag reduction).

Notice that they are different from that of ordinary dKdV, dBoussinesq, Zakharov and waterbag type reductions. In GdKP system, a new variable $v$ is involved.

### 3.1 $(1,1)$-reduction: dKdV type

In this case $\mathcal{L}^2 = p^2 + u$. Comparing to the expression of (6) we have $u_1 = u/2, u_3 = -u^2/8, u_5 = u^5/16, \ldots$ and $u_n = 0$ for $n \in \text{even}$ etc. Using the first few $A_n$ and $B_n$

\[
A_1 = 1, \quad A_2 = 2(p + v), \quad A_3 = 3(p^2 + vp + u + v^2),
\]

\[
B_1 = 0, \quad B_2 = u_x, \quad B_3 = \frac{3}{2}(p + v)u_x,
\]

and from Eqs.(4) for $\mathcal{L}$ and (26), we obtain the first few nontrivial equations

\[
\begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} 2v & 0 \\ 1 & 4v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x,
\]

\[
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 3u + 3v^2 & 0 \\ 3v & 3u + 9v^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x,
\]

\[
\begin{pmatrix} u \\ v \end{pmatrix}_{t_4} = \begin{pmatrix} 6uv + 4v^3 & 0 \\ 3u + 6v^2 & 12uv + 16v^3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x,
\]

where $t_1 = x, t_2 = y, t_3 = t$. They satisfy not only the compatibility conditions $\partial_y \partial_t u = \partial_t \partial_y u$ and $\partial_y \partial_t v = \partial_t \partial_y v$, but also community of flows for whole hierarchy. On the other
hand, by the relation \[ \text{(22)} \], we derive some of the conserved quantities

\[ v_{1,x} = -v, \quad v_{3,x} = -\frac{1}{2}uv, \quad v_{5,x} = -\frac{3}{8}u^2v, \quad v_{7,x} = -\frac{5}{16}u^3v, \quad \ldots \]

\[ v_{n,x} = 0, \quad n \in \text{even}. \]

Indeed, we list some conservation laws as follows:

\[
\begin{align*}
\partial_y(v_{1,x}) &= -v_y = -\partial_x(u + 2v^2), \\
\partial_t(v_{1,x}) &= -v_t = -3\partial_x(uv + v^3), \\
\partial_y(v_{3,x}) &= -\frac{1}{2}(uv)_y = -\frac{1}{4}\partial_x(u^2 + 4uv^2), \\
\partial_t(v_{3,x}) &= -\frac{1}{2}(uv)_t = -\frac{3}{2}\partial_x(u^2v + uv^3), \\
\partial_y(v_{5,x}) &= -\frac{3}{8}(u^2v)_y = -\frac{1}{8}\partial_x(u^3 + 6u^2v^2), \\
\partial_t(v_{5,x}) &= -\frac{3}{8}(u^2v)_t = -\frac{9}{8}\partial_x(u^3v + u^2v^3).
\end{align*}
\]

Note that, using the relations \( u = 2u_1, v = -v_{1,x} \) and the fact that \( v_{1,y} = -(u + 2v^2) \), we see that the compatibility \( \partial_y\partial_y p = \partial_y\partial_x p \) with independent variables \( p, \partial_x p, p\partial_x p \) implies \( \text{(36)} \) as well as the Manakov-Santini equation \( \text{(1)} \).

### 3.2 (2,1)-reduction: dBoussinesq type

In this case, \( \mathcal{L}^3 = p^3 + up + w \). Comparing to \( \text{(6)} \) we have \( u_1 = u/3, u_2 = w/3, u_3 = -u^2/9, u_4 = -2uw/9, \) etc. Some of \( A_n \) and \( B_n \) are given by

\[
\begin{align*}
A_1 &= 1, \quad A_2 = 2(p + v), \quad A_3 = 3(p^2 + vp + \frac{2}{3}u + v^2), \\
B_1 &= 0, \quad B_2 = \frac{2}{3}u_x, \quad B_3 = (p + v)u_x + w_x.
\end{align*}
\]

Then from Eqs.\( \text{(1)} \) for \( \mathcal{L} \) and \( \text{(26)} \) we derive the first few nontrivial equations

\[
\begin{align*}
\begin{pmatrix} u \\ w \\ v \end{pmatrix}_y &= \begin{pmatrix} 2v & 2 & 0 \\ -2u/3 & 2v & 0 \\ 2/3 & 0 & 4v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x, \\
\begin{pmatrix} u \\ w \\ v \end{pmatrix}_t &= \begin{pmatrix} u + 3v^2 & 3v & 0 \\ -uv & u + 3v^2 & 0 \\ 2v & 1 & 2u + 9v^2 \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x, \\
\begin{pmatrix} u \\ w \\ v \end{pmatrix}_{t_4} &= 4 \begin{pmatrix} 2uv/3 + v^3 + w & 2u/3 + v^2 & 0 \\ -w^2/3 - 2u^2/9 & v^3 + 2uw/3 + w & 0 \\ 2u/9 + v^2 & 2v/3 & 2uv + w + 4v^3 \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x,
\end{align*}
\]

where \( t_1 = x, t_2 = y, t_3 = t \). They satisfy the compatibility conditions \( \partial_{t_l}\partial_{t_m}u = \partial_{t_m}\partial_{t_l}u, \partial_{t_l}\partial_{t_m}w = \partial_{t_m}\partial_{t_l}w \) and \( \partial_{t_l}\partial_{t_m}v = \partial_{t_m}\partial_{t_l}v \) for \( l, m = 1, 2, 3 \). They are NOT compatible.
with the $t_4$-flow; however, this does not affect the construction of the exact solutions in 2+1 dimensions: $(x,y,t)$. Also, the relation (22) yields

$$v_{1,x} = -v, \quad v_{2,x} = 0, \quad v_{3,x} = -\frac{1}{3}uv, \quad v_{4,x} = -\frac{1}{3}wv,$$

$$v_{5,x} = -\frac{1}{9}u^2v, \quad v_{6,x} = -\frac{1}{3}uwv,$$

e etc. In particular, $v_{1,x}$ satisfies

$$\partial_y(v_{1,x}) = -2\partial_x(u/3 + v^2), \quad \partial_t(v_{1,x}) = -\partial_x(2w + w + 3v^3). \quad (40)$$

### 3.3 (2,1)-reductions: Zakharov type

In this case, $\mathcal{L} = p + u/(p-w)$. Comparing to (21) we have $u_1 = u, u_2 = uw, u_3 = uw^2, u_4 = uw^3$, etc. Some of $A_n$ and $B_n$ are given by

$$A_1 = 1, \quad A_2 = 2(p + v), \quad A_3 = 3(p^2 + vp + 2u + v^2),$$
$$B_1 = 0, \quad B_2 = 2u, \quad B_3 = 3(p + v)u + 3uw_x.$$

Then from Eqs.(24) for $\mathcal{L}$ and (26) we derive the first two nontrivial members as

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_y = \begin{pmatrix} 2v + 2w & 2u & 0 \\ 2 & 2v + 2w & 0 \\ 2 & 0 & 4v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x,$$

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_t = \begin{pmatrix} 9u + 3w^2 + 3uv + 3v^2 & 6uw + 3uv & 0 \\ 6w + 3v & 9u + 3w^2 + 3uv + 3v^2 & 0 \\ 3w + 6v & 3u & 6u + 9v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x,$$

where $t_1 = x, t_2 = y, t_3 = t$. They satisfy the compatibility conditions $\partial_t \partial_{t_m} u = \partial_{t_m} \partial_t u$, $\partial_t \partial_{t_m} w = \partial_{t_m} \partial_t w$ and $\partial_t \partial_{t_m} v = \partial_{t_m} \partial_t v$ for $l, m = 1, 2, 3$. Also, they are NOT compatible with the $t_4$-flow.

### 3.4 (2,1)-reductions: Waterbag type

In this case,

$$\mathcal{L} = p + \epsilon \log \frac{p-u}{p-w} = p + \epsilon \sum_{n=1}^{\infty} \frac{1}{n!} (w^n - u^n)p^{-n},$$

and some of $A_n$ and $B_n$ are

$$A_1 = 1, \quad A_2 = 2(p + v), \quad A_3 = 3(p^2 + vp - 2\epsilon u + 2\epsilon w + v^2),$$
$$B_1 = 0, \quad B_2 = 2\epsilon(-u_x + w_x), \quad B_3 = 3\epsilon(-(p + u + v)u_x + (p + w + v)w_x).$$

The $t_2$- and $t_3$-flows for $u, w, v$ can be derived by Eqs.(21) and (26) as

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_y = \begin{pmatrix} 2u + 2v - 2\epsilon & 2\epsilon & 0 \\ -2\epsilon & 2w + 2v + 2\epsilon & 0 \\ -2\epsilon & 2\epsilon & 4v \end{pmatrix} \begin{pmatrix} u \\ w \\ v \end{pmatrix}_x.$$
\[
\begin{pmatrix}
  u \\
w \\
v
\end{pmatrix}_y = \begin{pmatrix}
  3(u^2 + uv + v^2 + -4wu + 2v - ev) & 3\epsilon(u + w + v) & 0 \\
-3\epsilon(u + w + v) & 3(u^2 + uv + v^2 + +4w - 2\epsilon u + ev) & 0 \\
-3\epsilon - 6ev & 3\epsilon w + 6ev & -6\epsilon u + 6\epsilon w + 9v^2
\end{pmatrix}\begin{pmatrix}
  u \\
w \\
v
\end{pmatrix}_x,
\]

(42)

where \(t_1 = x, t_2 = y, t_3 = t\). They satisfy the compatibility conditions \(\partial_t \partial_t u = \partial_t \partial_t u, \partial_t \partial_m w = \partial_t \partial_t w\) and \(\partial_t \partial_m v = \partial_t \partial_t v\) for \(t, m = 1, 2, 3\). Likewise, they are NOT compatible with the \(t_4\)-flow.

### 4 Hodograph solutions

Having setup several reductions of the GdKP hierarchy to quasilinear (hydrodynamic) systems, we want to find their exact solutions by using the generalized hodograph method. In [14], Gibbons and Kodama developed an systematic way to generalize the hodograph transformation such that it can be used to solve hydrodynamic system with enough symmetries. Now the generalized hodograph transformation means the interchanging of the role of the dependent and independent variables: \((t_1, \ldots, t_N) \leftrightarrow (u_1, \ldots, u_N)\). In this section, following [14], we briefly recall the method as follows.

The quasilinear system is defined by

\[
\partial_t u_i = \sum_{j=1}^{N} a_{ij} \partial_x u_j, \quad i, l = 1, 2, \ldots, N.
\]

(43)

where \(a_{ij}\) are functions of \((u_1, \ldots, u_N)\) and \(a_{ij} = \delta_{ij}\). Defining the \((N-1)\)-form

\[
\Psi_l^{(N-1)} = (-1)^{l+1} dt_1 \wedge \cdots \wedge \widehat{dt_l} \wedge \cdots \wedge dt_N, \quad l = 1, \ldots, N
\]

where the caret means the \(l\)th term is omitted. The equation (43) can be rewritten as the differential forms

\[
du_i \wedge \Psi_l^{(N-1)} = \sum_{j=1}^{N} a_{ij} du_j \wedge \Psi_l^{(N-1)}.\]

(44)

and then one has

\[
\Psi_l^{(N-1)} = \sum_{j=1}^{N} f_j^l \Phi_j^{(N-1)}, \quad l = 1, \ldots, N,
\]

(45)

where \(\Phi_j^{(N-1)}\) are the \((N-1)\)-form

\[
\Phi_j^{(N-1)} = (-1)^{j+1} du_1 \wedge \cdots \wedge \widehat{du_j} \wedge \cdots \wedge du_N,
\]

and \(f_j^l\) are cofactors of the Jacobian \(J = \partial(t_1, \ldots, t_N)/\partial(u_1, \ldots, u_N)\) with non-vanishing \(J\). Substituting (13) into (14), equation (13) can be reduced to the following nonlinear PDEs, called hodograph equations for \((t_1, \ldots, t_N)\):

\[
f_j^l = \sum_{k=1}^{N} a_{jk} f_k^l, \quad l = 2, 3, \ldots, N.
\]

(46)
It has been shown [14] that the integrability conditions for the exact forms $\Psi^{(N-1)}_l$, i.e. $d\Psi^{(N-1)}_l = 0$, can determine $f^l_j$ from a linear system of the defining equations

$$\sum_{j=1}^{N} \partial_j f^l_j = \sum_{k=1}^{N} \sum_{j=1}^{N} \partial_j (a^l_{jk} f^l_k) = 0, \quad l = 1, \ldots, N. \quad (47)$$

where $\partial_j := \partial/\partial u_j$. Using the fact that

$$\sum_{j=1}^{N} (\partial_j t_n) f^l_j = \delta_n \frac{\partial (t_1, \ldots, t_N)}{\partial (u_1, \ldots, u_N)}$$

and the hodograph equation (46),

$$\sum_{k=1}^{N} \left[ \delta_{lk} \partial_k t_1 - \sum_{j=1}^{N} (\partial_j t_n) a^l_{jk} \right] f^l_k = 0. \quad (48)$$

From this equation, one obtains the following linear system as the dual equations of the hodograph equation

$$\delta_{ln} \partial_k t_1 - \sum_{j=1}^{N} (\partial_j t_n) a^l_{jk} = \sum_{r=2}^{N} \phi^l_{nr} \partial_k t_r, \quad l = 2, \ldots, N, \quad n, k = 1, \ldots, N, \quad (48)$$

where $\phi^l_{nr} = \phi^l_{nr} (u_1, \ldots, u_N)$ are functions to be determined by a particular solution with the lowest scaling weight (see below). We then show that, by the dual system (48), we can further construct polynomial solutions of higher scaling weights without using the hodograph equation (43).

As examples shown below, we study the cases for (1,1)- and (2,1)-reductions with matrix elements $a^l_{jk}$ given by Sections 3.1–3.4. The explicit solutions of Manakov-Santini equation can be solved in classes of rational as well as polynomial type.

(I) dKdV type. In (1,1)-reduction, system (35) is of the quasilinear form with the following $2 \times 2$ matrices corresponding to $t_2$- and $t_3$-flows,

$$a^2 = \begin{pmatrix} 2v & 0 \\ 1 & 4v \end{pmatrix}, \quad a^3 = \begin{pmatrix} 3u + 3v^2 & 0 \\ 3v & 3u + 9v^2 \end{pmatrix}. \quad (49)$$

The compatibility conditions (47) for $f^l_1 = \partial_v y$ and $f^l_2 = -\partial_u y$ ($\partial_1 := \partial_u, \partial_2 := \partial_v$) are

$$\begin{pmatrix} \partial_u & \partial_v \\ 2v \partial_u + \partial_v & 4\partial_v \partial_v \end{pmatrix} \begin{pmatrix} f^l_1 \\ f^l_2 \end{pmatrix} = 0, \quad (49)$$

which admit the polynomial type solutions of the following form with scaling weights $[u] = 2$ and $[v] = 1$:

$$f^l_1 = \sum_{2l_1+l_2=K-1} \alpha_{l_1,l_2} u^{l_1} v^{l_2}, \quad f^l_2 = \sum_{2l_1+l_2=K-2} \beta_{l_1,l_2} u^{l_1} v^{l_2},$$

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where $\alpha_{l_1,t_2}$ and $\beta_{l_1,t_2}$ are some constants to be determined and $K = 1, 2, \ldots$. For instance, for $K = 1$ we have $f_1^1 = \alpha, f_2^1 = \beta$. Substituting into (49), the hodograph equation (46) yields

$$y = \alpha v, \quad x = \alpha(u - v^2),$$

where $\alpha$ is an arbitrary constant. Using solutions (50), we see that the correction terms $\phi_{n_2}$ of the dual linear equation (48) are determined by $\phi_1^2 = 8v^2$ and $\phi_2^2 = -6v$. Thus, equation (48) provides

$$0 = \sum_{j=1}^{2} (\partial_j x) a_{jk}^2 + 8v^2 \partial_k y, \quad \partial_k x = \sum_{j=1}^{2} (\partial_j y) a_{jk}^2 - 6v \partial_k y,$$

which allows a direct way to solve the higher weight of polynomial solutions. In practice, let us look for the next solutions, $K = 2$. We have $x = c_1 uv + c_2 v^3$ and $y = c_3 u + c_4 v^2$. Making use of (51), we solve $c_1 = 0, \ c_3 = -3c_2/8, \ c_4 = -3c_2/4$, thus

$$x = c_2 v^3 \quad y = -\frac{3}{8} c_2 (u + 2v^2),$$

where $c_2$ is an arbitrary constant. Similar procedures can be done for finding other polynomial type solutions. We list first few of them with fixed scaling constant in Table 1.

| $K$ | $f_1^1$ | $f_1^2$ | $x$ | $y$ |
|-----|---------|---------|-----|-----|
| 1   | 1       | 0       | $u - v^2$ | $v$ |
| 2   | $12v$   | $-3$    | $-8v^3$ | $3u + 6v^2$ |
| 3   | $u + 3v^2$ | $-v$  | $\frac{1}{2}u^2 - uv^2 - \frac{3}{2}v^4$ | $uv + v^3$ |
| 4   | $uv + \frac{4}{3}v^3$ | $-\frac{1}{3}u - \frac{1}{2}v^2$ | $-\frac{2}{3}uv^3 - \frac{8}{15}v^5$ | $\frac{1}{3}u^2 + \frac{1}{2}uv^2 + \frac{1}{3}v^4$ |

We remark here that the above solutions of $x$ and $y$ can be treated as the initial value at $t = 0 \ [15, 16]$. In the present system, the remaining task is finding the exact solutions in 2+1 dimensions $(x,y,t)$. It was stated \[17, 18\] that the dependence of the time variable $t$ may be found according to the relation of $a_2$ and $a_3$, i.e.,

$$a_3 = \frac{3}{4}(a_2)^2 - \frac{3}{2}va^2 + 3(u + v^2)I,$$

where $I$ is the $2 \times 2$ identity matrix. After changing $(x, y, t) \rightarrow (u, v, t)$ with the dependent variables $x = x(u, v, t)$ and $y = y(u, v, t)$, the hodograph equation corresponding to the $t$-flow in (36) (in addition to (46)) is given by

$$\left( \begin{array}{c} \partial(x, y)/\partial(v, t) \\ -\partial(x, y)/\partial(u, t) \end{array} \right) = a_3 \left( \begin{array}{c} \partial_v y \\ -\partial_u y \end{array} \right),$$

where $\partial(x, y)/\partial(v, t)$ and $\partial(x, y)/\partial(u, t)$ are the Jacobian of $(x, y)$ with respect to $(v, t)$ and $(u, t)$, respectively. One can show that, combining the above and (16) with the relation
and requiring that \( \partial_u y \) and \( \partial_x y \) are independent variables, the implicit hodograph equations have the following form (string equation)

\[
x + \left( 3(u + v^2) - \frac{3}{4} \det(a^2) \right) t = t_1^0,
\]

\[
y + \left( \frac{3}{4} \mathrm{tr}(a^2) - \frac{3}{2} v \right) t = t_2^0,
\]

where \( t_1^0 = t_1^0(u,v) \) and \( t_2^0 = t_2^0(u,v) \) are initial values at \( t = 0 \) and can be found in Table I. Choosing, for example, \( K = 1 \), equation (53) becomes

\[
x + 3(u - v^2)t = u - v^2,
\]

\[
y + 3vt = v.
\]

Then we solve the hodograph solution as

\[
u(x,y,t) = \frac{x - 3xt + y^2}{9t^2 - 6t + 1}, \quad v(x,y,t) = \frac{-y}{3t - 1}.
\]

One can verify that \( u(x,y,t) \) and \( v(x,y,t) \) satisfy the \( y \) and \( t \)-flows of (36). Now using \( u_1 = u/2 \), \( v_{1,x} = -v \) and the conservation equations (37), (38), i.e., \( v_{1,y} = -(u + 2v^2) \), \( v_{1,t} = -3(uv + v^3) \), we obtain a set of rational solution to the Manakov-Santini equation

\[
u_1(x,y,t) = \frac{1}{2} \frac{x - 3xt + y^2}{(3t - 1)^2}, \quad v_1(x,y,t) = \frac{-y(x - 3xt + y^2)}{(3t - 1)^2}.
\]

(II) dBoussinesq type. For the case of \((2,1)\)-reduction in Section 3.2 we have, for the \( t_2 \)- and \( t_3 \)-flows, the quasilinear system (39) can be characterized by

\[
(a^2)_{3 \times 3} = \begin{pmatrix} 2v & 2 & 0 \\ -2u/3 & 2v & 0 \\ 2/3 & 4v & 0 \end{pmatrix}, \quad (a^3)_{3 \times 3} = \begin{pmatrix} 3v^2 + u & 3v & 0 \\ -uv & 3v^2 + u & 0 \\ 2v & 1 & 9v^2 + 2u \end{pmatrix}.
\]

The compatibility conditions (47) for \( f_1^1 = (\partial_y y)(\partial_y t) - (\partial_x y)(\partial_x t) \), \( f_2^1 = -(\partial_y y)(\partial_y t) + (\partial_y y)(\partial_x t) \) and \( f_3^1 = (\partial_y y)(\partial_x t) - (\partial_x y)(\partial_x t) \) (\( \partial_1 := \partial_u, \partial_2 := \partial_u, \partial_3 := \partial_y \)) are

\[
\begin{pmatrix}
2v \partial_u - \frac{2}{3} uv \partial_w + \frac{2}{3} \partial_v \\
\partial_u u + 3v^2 \partial_u - uv \partial_w + 2 \partial_v v \\
v \partial_u + 3v^2 \partial_u + (u + 3v^2) \partial_w + \partial_v - 2u \partial_v + 9 \partial_v v^2
\end{pmatrix}
\begin{pmatrix}
\partial_u \\
\partial_w \\
\partial_v
\end{pmatrix}
= 0.
\]

The polynomial type solutions admit the following form with scaling weights \([u] = 2\), \([w] = 3\) and \([v] = 1\):

\[
f_1^1 = \sum_{2l_1 + 2l_2 + l_3 = 2K - 3} \alpha_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3},
\]

\[
f_2^1 = \sum_{2l_1 + 2l_2 + l_3 = 2K - 2} \beta_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3},
\]

\[
f_3^1 = \sum_{2l_1 + 2l_2 + l_3 = 2K - 4} \gamma_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3},
\]
where $\alpha, \beta, \gamma$ and $\delta_{l,c}$ are constants to be determined and $K = 1, 2, \ldots$. To begin with, for $K = 1$, we have $f_1^1 = \alpha, f_1^2 = \beta$ and $f_1^3 = \gamma$. Plugging into (46), we obtain $\alpha = \gamma = 0$. Then by the definitions of $f_k^1$ and noticing that the scaling weights of $u, w$ and $v$, one has
\[ y = c_1 u + c_2 v^2, \quad t = c_3 v, \]
where $c_1, c_2, c_3$ are constants with $c_1c_3 = -\beta$. Now substituting $y$ and $t$ into the hodograph equation (46) for $l = 2$, we obtain the expression of $x$, i.e., $x = -2c_1(uv - w) + h(v)$, where $h(v)$ is a function of $v$. Then for $l = 3$ one finds that $c_2 = -3c_1/2, c_3 = 2c_1$ and $h(v)$ has to be a constant. To summarize, we solve the polynomial solutions as
\[ x = -2c_1(uv - w), \quad y = c_1(u - 3v^2/2), \quad t = 2c_1v, \] (57)
where $c_1$ is an arbitrary constant. Similarly, in this case the dual hodograph equation (given by (48))
\[ \delta_{2n,2n} \partial_x x = \sum_{j=1}^{3} (\partial_j t_n) \phi_{j2}^2 + \sum_{r=2}^{3} \phi_{n2}^2 \partial_r t_r, \]
\[ \delta_{3n,3n} \partial_x x = \sum_{j=1}^{3} (\partial_j t_n) \phi_{j3}^3 + \sum_{r=1}^{3} \phi_{n3}^3 \partial_r t_r, \quad n, k = 1, 2, 3, \] (58)
with unknown functions $\phi_{nr}^2, \phi_{nr}^3$, can be found by using the simplest solutions (57), i.e.,
\[ \phi_{12}^2 = 4v^2 + (8/3)u, \quad \phi_{13}^2 = 6v^3 + 8uv, \quad \phi_{22}^2 = -2v, \quad \phi_{23}^2 = 3v^2 - u, \]
\[ \phi_{12}^3 = -4/3, \quad \phi_{23}^3 = -6v, \]
\[ \phi_{12}^2 = 6v^3 + 8uv, \quad \phi_{13}^3 = 9v^4 + 21uv^2 + 2u^2, \quad \phi_{22}^3 = 3v^2 - u, \]
\[ \phi_{23}^3 = 18v^3 + (3/2)uv, \quad \phi_{32}^3 = -6v, \quad \phi_{33}^3 = -9v^2 - 3u. \]
Hence the equation (58) provides useful formulas for determining the polynomial type solutions in higher weights. For example, we derive the next set of polynomial solution for $K = 2$. The weights for $x, y$ and $t$ are now 4, 3 and 2, respectively and have the following general expressions
\[ x = c_1 u^2 + c_2 uv^2 + c_3 wv + c_4 v^4, \]
\[ y = c_5 uv + c_6 w + c_7 v^3, \]
\[ t = c_8 u + c_9 v^2. \]
Then substituting the above into (58), we find $c_2 = c_7 = -c_9 = 3c_1, \quad c_3 = c_4 = c_5 = 0, \quad c_6 = -3c_1/2, \quad c_8 = -c_1$. Therefore, we have
\[ x = c_1(u^2 + 3uv^2), \quad y = -\frac{3}{2}c_1(w - 2v^3), \quad t = -c_1(u + 3v^3), \]
where $c_1$ is an arbitrary constant. To find the (2+1)-dimensional equations involving $(x, y, t)$ that satisfy (59), we choose, for example, the expression of (57) and obtain the explicit hodograph solution
\[ u(x, y, t) = \frac{1}{c_1} y + \frac{3}{8c_1^2} t^2, \]
\[ w(x, y, t) = \frac{1}{2c_1} x + \frac{1}{2c_1^2} yt + \frac{3}{16c_1^3} t^3, \]
\[ v(x, y, t) = \frac{1}{2c_1} t. \]
Finally, by the relations \( u_1 = u/3 \) and \( v_{1,x} = -v \) and the conservation laws (10) one can easily solve the solution satisfying the Manakov-Santini equation as

\[
\begin{align*}
  u_1(x, y, t) &= \frac{1}{3c_1}y + \frac{1}{8c_1^2}t^2, \\
  v_1(x, y, t) &= -\frac{1}{2c_1}xt - \frac{1}{3c_1}y^2 - \frac{3}{4c_1^2}yt^2 - \frac{15}{64c_1^4}t^4.
\end{align*}
\]

(III) Zakharov type. In Section 3.3 for the case of (2,1)-reduction, we have

\[
(a^2)_{3\times3} = \begin{pmatrix}
  2v + 2w & 2u & 0 \\
  2 & 2v + 2w & 0 \\
  2 & 0 & 4v
\end{pmatrix},
\]

\[
(a^3)_{3\times3} = \begin{pmatrix}
  3v^2 + 9u + 3w^2 + 3vw & 3vu + 6uw & 0 \\
  3v + 6u & 3v^2 + 9u + 3w^2 + 3vw & 0 \\
  6v + 3w & 3u & 9v^2 + 6u
\end{pmatrix}.
\]

The compatibility conditions (47) show that \( f_1^1, f_2^1 \) and \( f_3^1 \) have polynomial type solutions as the following expressions with scaling weights \([u] = 2, [w] = 1 \) and \([v] = 1\):

\[
\begin{align*}
  f_1^1 &= \sum_{2l_1 + l_2 + l_3 = 2K-1} \alpha_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3}, \\
  f_2^1 &= \sum_{2l_1 + l_2 + l_3 = 2K-2} \beta_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3}, \\
  f_3^1 &= \sum_{2l_1 + l_2 + l_3 = 2K-2} \gamma_{l_1,l_2,l_3} u^{l_1} w^{l_2} v^{l_3},
\end{align*}
\]

where \( \alpha_{l_1,l_2,l_3}, \beta_{l_1,l_2,l_3} \) and \( \gamma_{l_1,l_2,l_3} \) are unknown constants. By the similar method, we list the first two set of polynomial type solutions as follows. For \( K = 1 \), we have

\[
\begin{align*}
  x &= 12c(ww + 3uv - w^2v - uv^2), \\
  y &= -3c(4u - w^2 - 4uv - v^2), \\
  t &= -4c(w + v).
\end{align*}
\]

Then for \( K = 2 \), one has

\[
\begin{align*}
  x &= 6c'(-5u^2 - 4uw^2 - 12uwv - 9v^2 + 4w^3v + 7w^2v^2 + 4uv^3), \\
  y &= 3c'(5uw - 2w^3 - 8w^2v - 8wv^2 - 2v^3), \\
  t &= 2c'(5u + 3w^2 + 4uv + 3v^2),
\end{align*}
\]

where \( c \) and \( c' \) are arbitrary constants. Here we have used the same dual hodograph equations (58) in the previous case, with \( \phi_{1l}^l, l = 2, 3 \) being given by

\[
\begin{align*}
  \phi_{12}^2 &= 8u + 4v^2, \\
  \phi_{13}^2 &= 24uw + 48uv - 12w^2v - 12wv^2 + 6v^3, \\
  \phi_{22}^2 &= -2v, \\
  \phi_{23}^2 &= -9u + 3w^2 + 12uw + 3v^2,
\end{align*}
\]
\( \phi_{22}^2 = -4/3, \)
\( \phi_{23}^3 = -4w - 6v, \)
\( \phi_{12}^3 = 24uw + 48uv - 12w^2v - 12w^2 - 6v^3, \)
\( \phi_{13}^3 = 54u^2 + 54uw^2 + 144wuv + 153w^2v^2 - 36w^3v - 81w^2v^2 - 54wv^3 + 9v^4, \)
\( \phi_{22}^3 = -9u + 3w^2 + 12wv + 3v^2, \)
\( \phi_{23}^3 = -9uw - (9/2)uv + 9w^3 + (81/2)w^2v + 54wv^2 + 18v^3, \)
\( \phi_{32}^3 = -4w - 6v, \)
\( \phi_{33}^3 = -15u - 9w^2 - 18wv - 18v^2. \)

These quantities can be used to determine the polynomial type solutions in higher weights.

(IV) **Waterbag type.** From Section [5.4] we have

\[
(a^2)_{3\times3} = \begin{pmatrix} 2u + 2v - 2\varepsilon & 2\varepsilon & 0 \\
-2\varepsilon & 2w + 2v + 2\varepsilon & 0 \\
-2\varepsilon & 2\varepsilon & 4v \end{pmatrix},
\]

\[
(a^3)_{3\times3} = \begin{pmatrix} 3(u^2 + uv + v^2 + 4\varepsilon u + 2\varepsilon w - \varepsilon v) & 3(u + w + v) & 0 \\
-3\varepsilon(u + w + v) & 3(w^2 + vw + v^2 + 4\varepsilon w - 2\varepsilon u + \varepsilon v) & 0 \\
-3u - 6\varepsilon v & 3\varepsilon w + 6\varepsilon v & -6\varepsilon u + 6\varepsilon w + 9\varepsilon^2 \end{pmatrix}.
\]

The compatibility conditions (47) show that \( f_1^1, f_2^1 \) and \( f_3^1 \) have the following polynomial type solutions with scaling weights \([u] = [w] = [v] = [\varepsilon] = 1:\)

\[
f_j^1 = \sum \alpha_{l_0,l_1,l_2,l_3}^j \varepsilon^{l_0} u^{l_1} w^{l_2} v^{l_3} \quad \text{with} \quad l_0 + l_1 + l_2 + l_3 = 2K - 1,
\]

where \( K = 1, 2, \ldots \) and \( \alpha_{l_0,l_1,l_2,l_3}^j \) for \( j = 1, 2, 3 \) are unknown constants. By the similar calculations, we derive the first two polynomial solutions as follows. For \( K = 1 \), we have

\[
x &= 6C(\varepsilon u^2 + 2uuv + 6uv + uv^2 - \varepsilon w^2 - 6\varepsilon wv + wv^2), \\
y &= -3C(uv + 2uv + 4\varepsilon u + 2wv - 4\varepsilon w + v^2), \\
t &= 2C(u + w + 2v),
\]

while for \( K = 2 \)

\[
x &= 4C'(4\varepsilon u^3 + 6uv w - 15\varepsilon^2 u^2 + 18\varepsilon uv^2 + 3v^2 u^2 + 27v^2 \varepsilon u + 6v^3 u + 15wv^2 u + 30w^2 u + 6w^2 v - 18w^2 v^2 - 4w^3 v^2) + 27w^2 \varepsilon uv + 6w^3 v - 15w^2 v^2 + 3v^2 w^2), \\
y &= -3C'(5\varepsilon u^2 + 4u^2 v + 2u^2 w + 2w^2 v + 8w^2 v + 18uvv + 4v^3 + 8w^2 v^2 - 5w^2 v + 4w^2 v), \\
t &= 4C'(u^2 + w^2 + 3v^2 + uv + 2uv + 2wv - 5u v + 5w v),
\]

where \( C \) and \( C' \) are arbitrary constants. Here we have used the dual hodograph equations (58), with \( \phi_{l_2}^l, l = 2, 3 \) being given by

\[
\phi_{12}^2 = 8\varepsilon w - 8\varepsilon u + 4\varepsilon v.
\]
\[ \phi_{13}^2 = 12e^2 + 48vew - 48veu - 6v^2w - 6v^3 - 12eu^2 - 6u^2v - 12uw, \]
\[ \phi_{22}^2 = -2v, \]
\[ \phi_{23}^2 = 6wv + 3v^2 + 9eu + 3uw - 9ew + 6uw, \]
\[ \phi_{32}^2 = -4/3, \]
\[ \phi_{33}^2 = -2u - 6v - 2w, \]
\[ \phi_{12}^3 = 12e^2 + 48vew - 48veu - 6v^2w - 6v^3 - 12eu^2 - 6u^2v - 12uw, \]
\[ \phi_{13}^3 = 72e^2v - 63v^2wu - 18vwu^2 + 9v^4 + 54e^2w^2 + 18ew^3 - 9v^2w^2 - 18e^3 + 54e^2u^2 \]
\[ -9v^2u^2 - 27v^3u - 27v^3w - 108e^2uw - 153euw^2 - 18vw^2 + 153e^2ew - 72e^2v, \]
\[ \phi_{22}^3 = 6wv + 3v^2 + 9eu + 3uw - 9ew + 6uw, \]
\[ \phi_{23}^3 = (9/2)ev^2 + 27v^2u - (9/2)e^2v^2 + 27v^2w + 18v^3 + (9/2)evu + 9vw^2 + (9/2)wu^2 \]
\[ +9vw^2 + (9/2)w^2u + (45/2)vwu - (9/2)evw, \]
\[ \phi_{32}^3 = -2u - 6v - 2w, \]
\[ \phi_{33}^3 = -3u^2 - 9vw + 15eu - 18v^2 - 3wu - 9ww - 15ew - 3w^2. \]

Likewise, these quantities are useful to determine the polynomial type solutions in higher weights.

### 5 Concluding remarks

In this paper we have studied the Manakov-Santini equation from the Lax-sato formulation (GdKP hierarchy). By suitable three-component reductions, one can reduce the Manakov-Santini equation to hydrodynamic (quasi-linear) systems. Using the compatibility conditions \( \partial_t \partial_m u = \partial_m \partial_t u, \partial_t \partial_m w = \partial_m \partial_t w \) and \( \partial_t \partial_m v = \partial_m \partial_t v \) for \( l, m = 1, 2, 3 \), one can find infinite exact solutions by the generalized hodograph method. But in all the three-component reductions, the commuting flows are only up to \( t_3 \), NOT to \( t_4 \). Therefore, it is quite natural to pose the following question:

- How can we find the hydrodynamic reductions for the generalized dispersionless KP hierarchy?

This issue will be published elsewhere.

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