A CHARACTERIZATION OF $X$ FOR WHICH SPACES $C_p(X)$ ARE DISTINGUISHED AND ITS APPLICATIONS

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Abstract. We prove that the locally convex space $C_p(X)$ of continuous real-valued functions on a Tychonoff space $X$ equipped with the topology of pointwise convergence is distinguished if and only if $X$ is a $\Delta$-space in the sense of Knight in [Trans. Amer. Math. Soc. 339 (1993), pp. 45–60]. As an application of this characterization theorem we obtain the following results:

1) If $X$ is a Čech-complete (in particular, compact) space such that $C_p(X)$ is distinguished, then $X$ is scattered.
2) For every separable compact space of the Isbell–Mrówka type $X$, the space $C_p(X)$ is distinguished.
3) If $X$ is the compact space of ordinals $[0, \omega_1]$, then $C_p(X)$ is not distinguished.

We observe that the existence of an uncountable separable metrizable space $X$ such that $C_p(X)$ is distinguished, is independent of ZFC. We also explore the question to which extent the class of $\Delta$-spaces is invariant under basic topological operations.

1. Introduction

Following J. Dieudonné and L. Schwartz [9] a locally convex space (lcs) $E$ is called distinguished if every bounded subset of the bidual of $E$ in the weak*–topology is contained in the closure of the weak*–topology of some bounded subset of $E$. Equivalently, a lcs $E$ is distinguished if and only if the strong dual of $E$ endowed with the strong topology is barrelled, (see [19 8.7.1]). A. Grothendieck [17] proved that a metrizable lcs $E$ is distinguished if and only if its strong dual is bornological. We refer the reader to survey articles [6] and [2] which present several more modern results about distinguished metrizable and Fréchet lcs.

Throughout the article, all topological spaces are assumed to be Tychonoff and infinite. By $C_p(X)$ and $C_k(X)$ we mean the spaces of all real-valued continuous functions on $X$.
functions on a Tychonoff space $X$ endowed with the topology of pointwise convergence and the compact-open topology, respectively. By a bounded set in a topological vector space (in particular, $C_p(X)$) we understand any set which is absorbed by every 0-neighbourhood.

For spaces $C_p(X)$ we proved in [14] the following theorem (the equivalence (1) ⇔ (4) has been obtained in [12]).

**Theorem 1.1.** For a Tychonoff space $X$, the following conditions are equivalent:

1. $C_p(X)$ is distinguished.
2. $C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e. for every bounded set $A$ in $\mathbb{R}^X$ there exists a bounded set $B$ in $C_p(X)$ such that $A \subset \text{cl}_{\mathbb{R}^X}(B)$.
3. For every $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in \text{cl}_{\mathbb{R}^X}(B)$.
4. The strong dual of the space $C_p(X)$ carries the finest locally convex topology.

Several examples of $C_p(X)$ with(out) distinguished property have been provided in papers [12], [13] and [14]. The aim of this research is to continue our initial work on distinguished spaces $C_p(X)$.

The following concept plays a key role in our paper. We show its applicability for the studying of distinguished spaces $C_p(X)$.

**Definition 1.2** ([21]). A topological space $X$ is said to be a $\Delta$-space if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of $X$ with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of $X$, also with empty intersection, and such that $D_n \subset V_n$ for every $n \in \omega$.

We should mention that R. W. Knight [21] called all topological spaces $X$ satisfying the above Definition 1.2 by $\Delta$-sets. The original definition of a $\Delta$-set of the real line $\mathbb{R}$ is due to G. M. Reed and E. K. van Douwen (see [28]). In this paper, for general topological spaces satisfying Definition 1.2 we reserve the term $\Delta$-space. The class of all $\Delta$-spaces is denoted by $\Delta$.

One of the main results of our paper, Theorem 2.1 says that $X$ is a $\Delta$-space if and only if $C_p(X)$ is a distinguished space. This characterization theorem has been applied systematically for obtaining a range of results from our paper.

Our main result in Section 3 states that a Čech-complete (in particular, compact) $X \in \Delta$ must be scattered. A very natural question arises about what those scattered compact spaces $X \in \Delta$ are. In view of Theorem 2.1 it is known that a Corson compact $X$ belongs to the class $\Delta$ if and only if $X$ is a scattered Eberlein compact space [14]. With the help of Theorems 2.1 and 6.8 we show that the class $\Delta$ contains also all separable compact spaces of the Isbell–Mrówka type. Nevertheless, as we demonstrate in Section 3 there are compact scattered spaces $X \notin \Delta$ (for example, the compact space $[0, \omega_1]$).

Section 4 deals with the questions about metrizable spaces $X \in \Delta$. We notice that every $\sigma$-scattered metrizable space $X$ belongs to the class $\Delta$. For separable metrizable spaces $X$, our analysis reveals a tight connection between distinguished $C_p(X)$ and well-known set-theoretic problems about special subsets of the real line $\mathbb{R}$. We observe that the existence of an uncountable separable metrizable space $X$ such that $C_p(X)$ is distinguished is independent of ZFC and it is equivalent to the existence of a separable countably paracompact nonnormal Moore space. We refer readers to [24] for the history of the normal Moore problem.
In Section 5 we study whether the class $\Delta$ is invariant under the basic topological operations: subspaces, (quotient) continuous images, finite/countable unions and finite products. We pose several new open problems.

2. Characterization theorem

In this section we provide a characterization of distinguished spaces $C_p(X)$ in terms of topological properties of the space $X$. For the reader’s convenience we recall some relevant terminology.

(a) A disjoint cover $\{X_\gamma : \gamma \in \Gamma\}$ of $X$ is called a partition of $X$.

(b) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an expansion of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in $X$ if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.

(c) A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called point-finite if no point belongs to infinitely many $U_\gamma$-s.

**Theorem 2.1.** For a Tychonoff space $X$, the following conditions are equivalent:

1. $C_p(X)$ is distinguished.
2. Any countable partition of $X$ admits a point-finite open expansion in $X$.
3. Any countable disjoint collection of subsets of $X$ admits a point-finite open expansion in $X$.
4. $X$ is a $\Delta$-space.

**Proof.** Observe that every collection of pairwise disjoint subsets of $X$, $\{X_\gamma : \gamma \in \Gamma\}$ can be extended to a partition by adding a single set $X_* = X \setminus \bigcup\{X_\gamma : \gamma \in \Gamma\}$. If the obtained partition admits a point-finite open expansion in $X$, then removing one open set we get a point-finite open expansion of the original disjoint collection. This shows evidently the equivalence (2) $\iff$ (3).

Assume now that (3) holds. Let $\{D_n : n \in \omega\}$ be a decreasing sequence subsets of $X$ with empty intersection. Define $X_n = D_n \setminus D_{n+1}$ for each $n \in \omega$. By assumption, a disjoint collection $\{X_n : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in $X$. Then $\{V_n = \bigcup\{U_i : i \geq n\} : n \in \omega\}$ is an open decreasing expansion in $X$ with empty intersection. This proves the implication (3) $\Rightarrow$ (4).

Next we show (4) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of $X$. Define $D_0 = X$ and $D_n = X \setminus \bigcup\{X_i : i < n\}$ for each $n \in \omega$. Then $X_n \subseteq D_n$ for every $n$, the sequence $\{D_n : n \in \omega\}$ is decreasing and its intersection is empty. Assuming (4), we find an open decreasing expansion $\{U_n : n \in \omega\}$ of $\{D_n : n \in \omega\}$ in $X$ such that $\bigcap\{U_n : n \in \omega\} = \emptyset$. For every $x \in X$ there is $n$ such that $x \notin U_m$ for each $m > n$, it means that $\{U_n : n \in \omega\}$ is a point-finite expansion of $\{X_n : n \in \omega\}$ in $X$. This finishes the proof (3) $\Rightarrow$ (4) $\Rightarrow$ (2) $\iff$ (3).

Now we prove the implication (1) $\Rightarrow$ (2). Let $\{X_n : n \in \omega\}$ be any countable partition of $X$. Fix any function $f \in \mathbb{R}^X$ which satisfies the following conditions: for each $n \in \omega$ and every $x \in X_n$ the value of $f(x)$ is greater than $n$. By assumption, there is a bounded subset $B$ of $C_p(X)$ such that $f \in dcl_{\mathbb{R}^X}(B)$. Hence, for every $n \in \omega$ and every point $x \in X_n$, there exists $f_x \in B$ such that $f_x(x) > n$. But $f_x$ is a continuous function, therefore there is an open neighbourhood $U_x \subseteq X$ of $x$ such that $f_x(y) > n$ for every $y \in U_x$. We define an open set $U_n \subseteq X$ as follows: $U_n = \bigcup\{U_x : x \in X_n\}$. Evidently, $X_n \subseteq U_n$ for each $n \in \omega$. If we assume that the open expansion $\{U_n : n \in \omega\}$ is not point-finite, then there exists a point $y \in X$ such that there are infinitely many numbers $n$ with $y \in U_{x_n}$ for some $x_n \in X_n$. This means that $\sup\{g(y) : g \in B\} = \infty$, which contradicts the boundedness of $B$. 


It remains to prove (2) \(\Rightarrow\) (1). By Theorem 1.1, we need to show that for every mapping \(f \in \mathbb{R}^X\) there is a bounded set \(B \subset C_p(X)\) such that \(f \in cl_{\mathbb{R}^X}(B)\). If there exists a constant \(r > 0\) such that \(\sup\{|f(x)| : x \in X\} < r\), then we take \(B = \{h \in C(X) : \sup\{|h(x)| : x \in X\} < r\}\). It is easy to see that \(B\) is as required.

Let \(f \in \mathbb{R}^X\) be unbounded. Denote by \(Y_0 = \emptyset\) and \(Y_n = \{x \in X : n - 1 \leq |f(x)| < n\}\) for each non-zero \(n \in \omega\). Define \(\varphi : X \to \omega\) by the rule: if \(Y_n \neq \emptyset\) then \(\varphi(x) = n\) for every \(x \in Y_n\). So, \(|f| < \varphi\). Put \(X_n = \varphi^{-1}(n)\) for each \(n \in \omega\). Note that some sets \(X_n\) might happen to be empty, but the collection \(\{X_n : n \in \omega\}\) is a partition of \(X\) with countably many nonempty \(X_n\)-s. By our assumption, there exists a point-finite open expansion \(\{U_n : n \in \omega\}\) of the partition \(\{X_n : n \in \omega\}\). Define \(F : X \to \omega\) by \(F(x) = \max\{n : x \in U_n\}\). Obviously, \(f < F\). Finally, we define \(B = \{h \in C_p(X) : |h| \leq F\}\). Then \(f \in cl_{\mathbb{R}^X}(B)\), because for every finite subset \(K \subset X\) there is a function \(h \in B\) such that \(f \mid_K = h \mid_K\). Indeed, given a finite subset \(K \subset X\), let \(\{V_x : x \in K\}\) be the family of pairwise disjoint open sets such that \(x \in V_x \subset U_{\varphi(x)}\) for every \(x \in K\). For each \(x \in K\), fix a continuous function \(h_x : X \to [-\varphi(x), \varphi(x)]\) such that \(h_x(x) = f(x)\) and \(h_x\) is equal to the constant value 0 on the closed set \(X \setminus V_x\). One can verify that \(h = \sum_{x \in K} h_x \in B\) is as required.

Below we present a straightforward application of Theorem 2.1.

**Corollary 2.2 (14).** Let \(Z\) be any subspace of \(X\). If \(Y\) belongs to the class \(\Delta\), then \(Z\) also belongs to the class \(\Delta\).

**Proof.** If \(\{Z_\gamma : \gamma \in \Gamma\}\) is any collection of pairwise disjoint subsets of \(Z\) and there exists a point-finite open expansion \(\{U_\gamma : \gamma \in \Gamma\}\) in \(X\), then obviously \(\{U_\gamma \cap Z : \gamma \in \Gamma\}\) is a point-finite expansion consisting of the sets relatively open in \(Z\). It remains to apply Theorem 2.1. \(\square\)

The last result can be reversed, assuming that \(X \setminus Z\) is finite.

**Proposition 2.3.** Let \(Z\) be a subspace of \(X\) such that \(Y = X \setminus Z\) is finite. If \(Z\) belongs to the class \(\Delta\), then \(X\) belongs to \(\Delta\) as well.

**Proof.** Let \(\{X_n : n \in \omega\}\) be any countable collection of pairwise disjoint subsets of \(X\). Denote by \(F\) the set of those \(n \in \omega\) such that \(X_n \cap Y \neq \emptyset\). There might be only finitely many \(X_n\)-s which intersect the finite set \(Y\), hence \(F \subset \omega\) is finite. If \(n \in F\), then we simply declare that \(U_n\) is equal to \(X\). Consider the subcollection \(\{X_n : n \in \omega \setminus F\}\). It is a countable collection of pairwise disjoint subsets of \(Z\). Since \(Z \in \Delta\), by Theorem 2.1 there is a point-finite open expansion \(\{U_n : n \in \omega \setminus F\}\) in \(Z\). Observe that \(Z\) is open in \(X\), therefore all those \(U_n\)-s remain open in \(X\). Bringing all \(U_n\)-s of both sorts together we obtain a point-finite open expansion \(\{U_n : n \in \omega\}\) in \(X\). Finally, \(X \in \Delta\), by Theorem 2.1. \(\square\)

**Remark 2.4.** The following applicable concept has been re-introduced in [14]. A family \(\{N_x : x \in X\}\) of subsets of a Tychonoff space \(X\) is called a *scant cover* for \(X\) if each \(N_x\) is an open neighbourhood of \(x\) and for each \(u \in X\) the set \(X_u = \{x \in X : u \in N_x\}\) is finite.\(^1\)

Our Theorem 2.1 generalizes one of the results obtained in [14] stating that if \(X\) admits a scant cover \(\{N_x : x \in X\}\) then \(C_p(X)\) is distinguished. Indeed, let

\(^1\)The referee kindly informed the authors that this notion also is known in the literature under the name *the point-finite neighbourhood assignment.*
\{X_{\gamma} : \gamma \in \Gamma\} be any collection of pairwise disjoint subsets of \(X\). Define \(U_{\gamma} = \bigcup \{X_{\nu} : \nu \in X_{\gamma}\}\). It is easily seen that \(\{U_{\gamma} : \gamma \in \Gamma\}\) is a point-finite open expansion in \(X\), by definition of a scant cover. Applying Theorem 2.1, we conclude that \(C_{p}(X)\) is distinguished.

3. Applications to compact spaces \(X \in \Delta\)

First we recall a few definitions and facts (probably well-known) which will be used in the sequel. A space \(X\) is said to be scattered if every nonempty subset \(A\) of \(X\) has an isolated point in \(A\). Denote by \(A^{(1)}\) the set of all non-isolated (in \(A\)) points of \(A \subset X\). For ordinal numbers \(\alpha\), the \(\alpha\)-th derivative of a topological space \(X\) is defined by transfinite induction as follows.

\[
X^{(0)} = X; \quad X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}; \quad X^{(\gamma)} = \bigcap_{\alpha < \gamma} X^{(\alpha)} \text{ for limit ordinals } \gamma.
\]

For a scattered space \(X\), the smallest ordinal \(\alpha\) such that \(X^{(\alpha)} = \emptyset\) is called the scattered height of \(X\) and is denoted by \(ht(X)\). For instance, \(X\) is discrete if and only if \(ht(X) = 1\).

The following classical theorem is due to A. Pelczyński and Z. Semadeni.

**Theorem 3.1** ([29, Theorem 8.5.4]). A compact space \(X\) is scattered if and only if there is no continuous mapping of \(X\) onto the segment \([0, 1]\).

A continuous surjection \(\pi : X \to Y\) is called irreducible (see [29, Definition 7.1.11]) if for every closed subset \(F\) of \(X\) the condition \(\pi(F) = Y\) implies \(F = X\).

**Proposition 3.2** ([29, Proposition 7.1.13]). Let \(X\) be a compact space and let \(\pi : X \to Y\) be a continuous surjection. Then there exists a closed subset \(F\) of \(X\) such that \(\pi(F) = Y\) and the restriction \(\pi|_{F} : F \to Y\) is irreducible.

**Proposition 3.3** ([29, Proposition 25.2.1]). Let \(X\) be a compact space and let \(\pi : X \to Y\) be a continuous surjection. Then \(\pi\) is irreducible if and only if whenever \(E \subset X\) and \(\pi(E)\) is dense in \(Y\), then \(E\) is dense in \(X\).

Recall that a Tychonoff space \(X\) is Čech-complete if \(X\) is a \(G_{\delta}\)-set in some (equivalently, any) compactification of \(X\), (see [10], 3.9.1). It is well known that every locally compact space and every completely metrizable space is Čech-complete. Next statement resolves an open question posed in [14].

**Theorem 3.4.** Every Čech-complete (in particular, compact) \(\Delta\)-space is scattered.

**Proof.**

**Step 1** (\(X\) is compact). On the contrary, assume that \(X\) is not scattered. First, by Theorem 3.1 there is a continuous mapping \(\pi\) from \(X\) onto the segment \([0, 1]\). Second, by Proposition 3.2 there exists a closed subset \(F\) of \(X\) such that \(\pi(F) = [0, 1]\) and the restriction \(\pi|_{F} : F \to [0, 1]\) is irreducible. Since \(X \in \Delta\) the compact space \(F\) also belongs to \(\Delta\), by Corollary 2.2. For simplicity, without loss of generality we may assume that \(F\) is \(X\) itself and \(\pi : X \to [0, 1]\) is irreducible.

Let \(\{X_{n} : n \in \omega\}\) be a partition of \([0, 1]\) into dense sets. Put \(Y_{n} = \bigcup_{k \geq n} X_{k}\), and \(Z_{n} = \pi^{-1}(Y_{n})\) for all \(n \in \omega\). Then all sets \(Z_{n}\) are dense in \(X\) by Proposition 3.3 and the intersection \(\bigcap_{n \in \omega} Z_{n}\) is empty. Every compact space \(X\) is a Baire space, i.e. the Baire category theorem holds in \(X\), hence if \(\{U_{n} : n \in \omega\}\) is any open expansion of \(\{Z_{n} : n \in \omega\}\), then the intersection \(\bigcap_{n \in \omega} U_{n}\) is dense in \(X\). In view of our Theorem 2.1 this conclusion contradicts the assumption \(X \in \Delta\), and the proof follows.
Step 2 (X is any Čech-complete space). By the first step we deduce that every compact subset of X is scattered. But any Čech-complete space X is scattered if and only if every compact subset of X is scattered. A detailed proof of this probably folklore statement can be found in [30]. □

**Proposition 3.5.** If X is a first-countable compact space, then X ∈ Δ if and only if X is countable.

**Proof.** If X ∈ Δ, then X is scattered, by Theorem 3.4. By the classical theorem of S. Mazurkiewicz and W. Sierpiński [29, Theorem 8.6.10], a first-countable compact space is scattered if and only if it is countable. This proves (i) ⇒ (ii). The converse is known [14] and follows from the fact that any countable space X = {x_n : n ∈ ω} admits a scant cover. Indeed, define X_n = {x_i : i ≥ n}. Then the family {X_n : n ∈ ω} is a scant cover of X. Now it suffices to mention Remark 2.4. □

**Remark 3.6.** Theorem 3.4 extends also a well-known result of B. Knaster and K. Urbanik stating that every countable Čech-complete space is scattered [20]. It is easy to see that a countable Baire space contains a dense subset of isolated points, but in general does not have to be scattered. We don’t know whether every Baire Δ-space must have isolated points.

Recall that an Eberlein compact is a compact space homeomorphic to a subset of a Banach space with the weak topology. A compact space is said to be a Corson compact space if it can be embedded in a Σ-product of the real lines. Every Eberlein compact is Corson, but not vice versa. However, every scattered Corson compact space is a scattered Eberlein compact space [1].

**Theorem 3.7 ([14]).** A Corson compact space X belongs to the class Δ if and only if X is a scattered Eberlein compact space.

Bearing in mind Theorem 3.4 to show Theorem 3.7 it suffices to use the fact that every scattered Eberlein compact space admits a scant cover (the latter follows from the proof of [5, Lemma 1.1]) and then apply Remark 2.4.

Being motivated by the previous results one can ask if there exist scattered compact spaces X ∈ Δ which are not Eberlein compact. The next question is also crucial: Does a compact scattered space X /∈ Δ exist? Below we answer both questions positively.

We need the following somewhat technical

**Theorem 3.8.** Let Z = C_0 ∪ C_1 be a space such that

1. C_0 ∩ C_1 = ∅,
2. C_0 is an open $F_σ$ subset of Z,
3. both C_0 and C_1 belong to the class Δ.

Then Z also belongs to the class Δ.

**Proof.** By assumption, C_0 = \bigcup \{F_n : n ∈ ω\}, where each F_n is closed in Z. Let \{X_n : n ∈ ω\} be any countable collection of pairwise disjoint subsets of Z. Our target is to define open sets $U_n ⊇ X_n$, n ∈ ω in such a way that the collection \{U_n : n ∈ ω\} is point-finite. We decompose the sets X_n = X^0_n ∪ X^1_n, where X^0_n = X_n ∩ C_0 and X^1_n = X_n ∩ C_1. By Theorem 2.1 the collection \{X^0_n : n ∈ ω\} expands to a point-finite open collection \{U^0_n : n ∈ ω\} in C_0. The set C_0 is open in Z, therefore $U^0_n$ are open in Z as well.
Now we consider the disjoint collection \( \{ X^1_n : n \in \omega \} \) in \( C_1 \). By assumption, \( C_1 \in \Delta \), therefore applying Theorem \ref{t:main} once more, we find a point-finite expansion \( \{ V^1_n : n \in \omega \} \) in \( C_1 \) consisting of sets which are open in \( C_1 \). Every set \( V^1_n \) is a trace of some set \( W^1_n \), which is open in \( Z \), i.e. \( V^1_n = W^1_n \cap C_1 \), and every \( W^1_n \) is open in \( Z \). We refine the sets \( W^1_n \) by the formula \( U^1_n = W^1_n \setminus \bigcup \{ F_i : i \leq n \} \). Since all sets \( F_i \) are closed in \( Z \), the sets \( U^1_n \) remain open in \( Z \). Since all sets \( F_i \) are disjoint with \( C_1 \), the collection \( \{ U^1_n : n \in \omega \} \) remains to be an expansion of \( \{ X^1_n : n \in \omega \} \). Furthermore, the collection \( \{ U^1_n : n \in \omega \} \) is point-finite, because \( \{ V^1_n : n \in \omega \} \) is point-finite, and every point \( z \in C_0 \) belongs to some \( F_n \), hence \( z \notin U^1_m \) for every \( m \geq n \). Finally, we define \( U_n = U^1_n \cup U^1_n \). The collection \( \{ U_n : n \in \omega \} \) is a point-finite open expansion of \( \{ X_n : n \in \omega \} \), and the proof is complete. \( \square \)

This yields the following

**Corollary 3.9.** Let \( Z \) be any separable scattered space such that its scattered height \( ht(Z) \) is equal to 2. Then \( Z \in \Delta \).

**Proof.** The structure of \( Z \) is the following. \( Z = C_0 \cup C_1 \), where \( C_0 \) is a countable dense in \( Z \) set consisting of isolated in \( Z \) points and \( C_1 \) consists of all accumulation points. Moreover, the space \( C_1 \) with the topology induced from \( Z \) is discrete. All conditions of Theorem \ref{t:main} are satisfied, and the result follows. \( \square \)

Our first example will be the one-point compactification of an Isbell–Mrówka space \( \Psi(A) \). We recall the construction and basic properties of \( \Psi(A) \). Let \( A \) be an almost disjoint family of subsets of the set of natural numbers \( \mathbb{N} \) and let \( \Psi(A) \) be the set \( \mathbb{N} \cup A \) equipped with the topology defined as follows. For each \( n \in \mathbb{N} \), the singleton \( \{ n \} \) is open, and for each \( A \in A \), a base of neighbourhoods of \( A \) is the collection of all sets of the form \( \{ A \} \cup B \), where \( B \subset A \) and \( |A \setminus B| < \omega \). The space \( \Psi(A) \) is then a first-countable separable locally compact Tychonoff space. If \( A \) is a maximal almost disjoint (MAD) family, then the corresponding Isbell–Mrówka space \( \Psi(A) \) would be in addition pseudocompact. (Readers are advised to consult \cite{18} which surveys various topological properties of these spaces).

**Theorem 3.10.** There exists a separable scattered compact space \( X \) with the following properties:

- (a) The scattered height of \( X \) is equal to 3.
- (b) \( X \in \Delta \).
- (c) \( X \) is not an Eberlein compact space.

**Proof.** Let \( A \) be any uncountable almost disjoint (in particular, MAD) family of subsets of \( \mathbb{N} \) and let \( Z \) be the corresponding first-countable separable locally compact Isbell–Mrówka space \( \Psi(A) \). It is easy to see that \( Z = \Psi(A) \) satisfies the assumptions of Corollary \ref{c:main}. Hence, \( Z \in \Delta \). Now, denote by \( X \) the one-point compactification of the separable locally compact space \( Z \). Then the scattered height of \( X \) is equal to 3. Note that \( X \in \Delta \) by Proposition \ref{p:main}. Moreover, \( X \) is not an Eberlein compact space, since every separable Eberlein compact space is metrizable, while \( \Psi(A) \) is metrizable if and only if \( A \) is countable. \( \square \)

Now we show that there do exist scattered compact spaces which are not in the class \( \Delta \). We will use the classical Pressing Down Lemma. Let \( [0, \omega_1) \) be the set of all countable ordinals equipped with the order topology. For simplicity, we identify \( [0, \omega_1) \) with \( \omega_1 \). A subset \( S \) of \( \omega_1 \) is called a stationary subset if \( S \) has nonempty
intersection with every closed and unbounded set in $\omega_1$. A mapping $\varphi : S \to \omega_1$ is called regressive if $\varphi(\alpha) < \alpha$ for each $\alpha \in S$. The proof of the following fundamental statement can be found for instance in \cite{22}.

**Theorem 3.11** (Pressing Down Lemma). Let $\varphi : S \to \omega_1$ be a regressive mapping, where $S$ is a stationary subset of $\omega_1$. Then for some $\gamma < \omega_1$, $\varphi^{-1}\{\gamma\}$ is a stationary subset of $\omega_1$.

It is known that there are plenty of stationary subsets of $\omega_1$. In particular, every stationary set can be partitioned into countably many pairwise disjoint stationary sets \cite{22}. Note that $\omega_1$ is a scattered locally compact and first-countable space. Next statement resolves an open question posed in \cite{14}.

**Theorem 3.12.** The compact scattered space $[0, \omega_1]$ is not in the class $\Delta$.

*Proof.* It suffices to show that $\omega_1$ does not belong to the class $\Delta$. Assume, on the contrary, that $\omega_1 \in \Delta$. Denote by $L$ the set of all countable limit ordinals. Evidently, $L$ is a closed unbounded set in $\omega_1$. Take any representation of $L$ as the union of countably many pairwise disjoint stationary sets $\{S_n : n \in \omega\}$. By Theorem 2.1 there exists a point-finite open expansion $\{U_n : n \in \omega\}$ in $\omega_1$.

For every $\alpha \in U_n$ there is an ordinal $\beta(\alpha) < \alpha$ such that $[\beta(\alpha), \alpha) \subset U_n$. In fact, for every $n \in \omega$ we can define a regressive mapping $\varphi_n : S_n \to \omega_1$ by the formula: $\varphi_n(\alpha) = \beta(\alpha)$. Since $S_n$ is a stationary set for every $n$, we can apply to $\varphi_n$ the Pressing Down Lemma. Hence, for each such $n$ there are a countable ordinal $\gamma_n$ and an uncountable subset $T_n \subset S_n$ with the following property: $[\gamma_n, \alpha) \subset U_n$ for every $\alpha \in T_n$. Denote $\gamma = \sup\{\gamma_n : n \in \omega\} \in \omega_1$. Because all $T_n$ are unbounded, for all natural $n$ we have an ordinal $\alpha_n \in T_n$ such that $\gamma < \alpha_n$ and $[\gamma_n, \alpha_n) \subset U_n$. This implies that $\gamma \in U_n$ for every $n \in \omega$. However, a collection $\{U_n : n \in \omega\}$ is point-finite. The obtained contradiction finishes the proof. \qed

The function space $C_k(X)$ is called Asplund if every separable vector subspace of $C_k(X)$ isomorphic to a Banach space, has the separable dual.

**Proposition 3.13.** If $X \in \Delta$, then the space $C_k(X)$ is Asplund. The converse conclusion fails in general.

*Proof.* Let $K(X)$ be the family of all compact subset of $X$. By the assumption and Corollary 2.22 each $K \in K(X)$ belongs to the class $\Delta$. Clearly, $C_k(X)$ is isomorphic to a linear subspace of the product $\prod_{K \in K(X)} C_k(K)$ of Banach spaces $C_k(K)$. Assume that $E$ is a separable vector subspace of $C_k(X)$ isomorphic to a Banach space. Observe that $E$ is isomorphic to a subspace of the finite product $\prod_{j \in F} C_k(K_j)$ for $K_j \in K(X)$ and $j \in F$. Indeed, let $B$ be the unit (bounded) ball of the normed space $E$. Then there exists a finite set $F$ such that $\pi_F$ is a natural projections from $E$ onto $C_k(K_j)$. Let $\pi_F$ be the (continuous) projection from $F$ onto $\prod_{j \in F} C_k(K_j)$. Then $\pi_F \mid E$ is an injective projection from $E$ onto $\prod_{j \in F} C_k(K_j)$. The injectivity of $\pi_F \mid E$ follows from the fact that $B$ is a bounded neighbourhood of zero in $E$. It is easy to see that the image of $\pi_F \mid E(B)$ is an open neighbourhood of zero in $\prod_{j \in F} C_k(K_j)$. On the other hand, $\prod_{j \in F} C_k(K_j)$ is isomorphic to the space $C_k(\bigoplus_{j \in F} K_j)$ and the compact space $\bigoplus_{j \in F} K_j$ is scattered. By the classical \cite{14} Theorem 12.29 $E$ must have the separable dual $E^*$. Hence, $C_k(X)$ is Asplund. The converse fails, as Theorem 3.12 shows for $X = [0, \omega_1]$. \qed
Since every infinite compact scattered space $X$ contains a nontrivial converging sequence, for such $X$ the Banach space $C(X)$ is not a Grothendieck space, (see [8]).

**Corollary 3.14.** If $X$ is an infinite compact and $X \in \Delta$, then the Banach space $C(X)$ is not a Grothendieck space. The converse fails, as $X = [0, \omega_1]$ applies.

For non-scattered spaces $X$ Theorem 3.4 implies immediately the following

**Corollary 3.15.** If $X$ is a non-scattered space, the Stone-Čech compactification $\beta X$ is not in the class $\Delta$.

**Proposition 3.16.** Let $X = \beta Z \setminus Z$, where $Z$ is any infinite discrete space. Then $X$ is not in the class $\Delta$.

**Proof.** $\beta Z \setminus Z$ does not have isolated points for any infinite discrete space $Z$. □

It is known that $X = [0, \omega_1]$ is the Stone-Čech compactification of $[0, \omega_1)$. We showed that $X \notin \Delta$. Also, $\beta Z \notin \Delta$ for any infinite discrete space $Z$. Every scattered Eberlein compact space belongs to the class $\Delta$ by Theorem 3.7; however, no Eberlein compact $X$ can be the Stone-Čech compactification $\beta Z$ for any proper subset $Z$ of $X$ by the Preiss–Simon theorem (see [2, Theorem IV.5.8]). All of these facts provide a motivation for the following result.

**Example 3.17.** There exists an Isbell–Mrówka space $Z$ which is almost compact in the sense that the one-point compactification of $Z$ coincides with $\beta Z$ (see [18, Theorem 8.6.1]). Define $X = \beta Z$. Then $X \in \Delta$, by Theorem 3.10.

4. **Metrizable spaces** $X \in \Delta$

In this section we try to describe constructively the structure of nontrivial metrizable spaces $X \in \Delta$. Note first that every scattered metrizable $X$ is in the class $\Delta$ since every such space $X$ homeomorphically embeds into a scattered Eberlein compact $[3]$, and then Theorem 3.7 and Corollary 2.2 apply. We extend this result as follows.

A topological space $X$ is said to be $\sigma$-scattered if $X$ can be represented as a countable union of scattered subspaces and $X$ is called strongly $\sigma$-discrete if it is a union of countably many of its closed discrete subspaces. Strongly $\sigma$-discreteness of $X$ implies that $X$ is $\sigma$-scattered, for any topological space. For metrizable $X$, by the classical result of A. H. Stone [31], these two properties are equivalent.

**Proposition 4.1.** Any $\sigma$-scattered metrizable space belongs to the class $\Delta$.

**Proof.** In view of aforementioned equivalence, every subset of $X$ is $F_\sigma$. If every subset of $X$ is $F_\sigma$, then $X \in \Delta$. This fact apparently is well-known (see also a comment after Claim 4.2). For the sake of completeness we include a direct argument. We show that $X$ satisfies the condition (2) of Theorem 2.1. Let $\{X_n : n \in \omega\}$ be any countable disjoint partition of $X$. Denote $X_n = \bigcup\{F_{n,m} : m \in \omega\}$, where each $F_{n,m}$ is closed in $X$. Define open sets $U_n$ as follows: $U_0 = X$ and $U_n = X \setminus \bigcup\{F_{k,m} : k < n, m < n\}$ for $n \geq 1$. Then $\{U_n : n \in \omega\}$ is a point-finite open expansion of $\{X_n : n \in \omega\}$ in $X$. □

A metrizable space $A$ is called an absolutely analytic if $A$ is homeomorphic to a Souslin subspace of a complete metric space $X$ (of an arbitrary weight), i.e. $A$ is expressible as $A = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} A_{\sigma|n}$, where each $A_{\sigma|n}$ is a closed subset of $X$. It is
known that every absolutely analytic metrizable space $X$ (in particular, every Borel subspace of a complete metric space) either contains a homeomorphic copy of the Cantor set or it is strongly $\sigma$-discrete. Therefore, for absolutely analytic metrizable space $X$ the converse is true: $X \in \Delta$ implies that $X$ is strongly $\sigma$-discrete [14].

However, the last structural result can not be proved in general for all (separable) metrizable spaces without extra set-theoretic assumptions. Let us recall several definitions of special subsets of the real line $\mathbb{R}$ (see [23], [28]).

(a) A $Q$-set $X$ is a subset of $\mathbb{R}$ such that each subset of $X$ is $F_\sigma$, or, equivalently, each subset of $X$ is $G_\delta$ in $X$.

(b) A $\lambda$-set $X$ is a subset of $\mathbb{R}$ such that each countable $A \subset X$ is $G_\delta$ in $X$.

(c) A $\Delta$-set $X$ is a subset of $\mathbb{R}$ such that for every decreasing sequence $\{D_n : n \in \omega\}$ subsets of $X$ with empty intersection there is a decreasing expansion $\{V_n : n \in \omega\}$ consisting of open subsets of $X$ with empty intersection.

Claim 4.2. The existence of an uncountable separable metrizable $\Delta$-space is equivalent to the existence of an uncountable $\Delta$-set.

Proof. Note that every separable metrizable space homeomorphically embeds into a Polish space $\mathbb{R}^\omega$ and the latter space is a one-to-one continuous image of the set of irrationals $\mathbb{P}$. Therefore, if $M$ is an uncountable separable metrizable space, then there exist an uncountable set $X \subset \mathbb{R}$ and a one-to-one continuous mapping from $X$ onto $M$. It is easy to see that $X$ is a $\Delta$-set provided $M$ is a $\Delta$-space. □

Note that in the original definition of a $\Delta$-set, G. M. Reed used $G_\delta$-sets instead of open sets and E. van Douwen observed that these two versions are equivalent [28]. From the original definition it is obvious that each $Q$-set must be a $\Delta$-set. The fact that every $\Delta$-set is a $\lambda$-set is known as well. K. Kuratowski showed that in ZFC there exist uncountable $\lambda$-sets. The existence of an uncountable $Q$-set is one of the fundamental set-theoretical problems considered by many authors. F. Hausdorff showed that the cardinality of an uncountable $Q$-set $X$ has to be strictly smaller than the continuum $\mathfrak{c} = 2^{\aleph_0}$, so in models of ZFC plus the Continuum Hypothesis (CH) there are no uncountable $Q$-sets. Let us outline several of the most relevant known facts.

1. Martin’s Axiom plus the negation of the Continuum Hypothesis (MA + ¬CH) implies that every subset $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a $Q$-set (see [16]).

2. It is consistent that there is a $Q$-set $X$ such that its square $X^2$ is not a $Q$-set [15].

3. The existence of an uncountable $Q$-set is equivalent to the existence of an uncountable strong $Q$-set, i.e. a $Q$-set all finite powers of which are $Q$-sets [26].

4. No $\Delta$-set $X$ can have cardinality $\mathfrak{c}$ [27]. Hence, under MA, every subset of $\mathbb{R}$ that is a $\Delta$-set is also a $Q$-set. Recently we proved the following claim: If $X$ has a countable network and $|X| = \mathfrak{c}$, then $C_p(X)$ is not distinguished [14]. In view of our Theorem 2.1 this fact means that no $\Delta$-space $X$ with a countable network can have cardinality $\mathfrak{c}$ [2].

5. It is consistent that there exists a $\Delta$-set $X$ that is not a $Q$-set [21]. Of course, there are plenty of nonmetrizable $\Delta$-spaces with non-$G_\delta$ subsets, in ZFC.

6. An uncountable $\Delta$-set exists if and only if there exists a separable countably paracompact nonnormal Moore space (see [33] and [27]).

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²The referee kindly informed the authors that the last result can be derived easily from the actual argument of [27].
Summarizing, the following conclusion is an immediate consequence of our Theorem 2.1 and the known facts about Δ-sets listed above.

**Corollary 4.3.**

1. The existence of an uncountable separable metrizable space such that $C_p(X)$ is distinguished, is independent of ZFC.
2. There exists an uncountable separable metrizable space $X$ such that $C_p(X)$ is distinguished, if and only if there exists a separable countably paracompact nonnormal Moore space.

5. **Basic operations in Δ and open problems**

In this section we consider the question whether the class Δ is invariant under the following basic topological operations: subspaces, continuous images, quotient continuous images, finite/countable unions, finite products.

1. **Subspaces.** Trivial because of Corollary 2.2.

2. **(Quotient) continuous images.** Evidently, every topological space is a continuous image of a discrete one. The following assertion in fact has been remarked for the first time in [25].

**Proposition 5.1** ([25]). There exists in ZFC a MAD family $A$ on $\mathbb{N}$ such that the corresponding Isbell–Mrówka space $\Psi(A)$ admits a continuous mapping onto the closed interval $[0,1]$.

Detailed constructions of such MAD families $\mathcal{M}$ can be found in [4], [32].

Thus, the class Δ is not invariant under continuous images even for first-countable separable locally compact spaces. However, a continuous mapping in Proposition 5.1 cannot be quotient.

**Proposition 5.2.** Every quotient continuous image of any Isbell–Mrówka space is a Δ-space.

**Proof.** We observe that by construction any Isbell–Mrówka space $\Psi(A) = \mathbb{N} \cup A$ satisfies the following property: every subset of $A$ is closed in $\Psi(A)$ and $\mathbb{N}$ is obviously countable. Let $Y$ be the image of $\Psi(A)$ under a quotient continuous mapping $f$. We show that $Y$ enjoys the same property. Indeed, denote by $M$ the image $f(\mathbb{N})$ and put $F = Y \setminus M$. Evidently, $M$ is at most countable and for every subset $B$ of $F$ the preimage $f^{-1}(B)$ is closed in $\Psi(A)$ as a subset of $A$, therefore $B$ is closed in $Y$. It follows that $Y$ is $Q$-set space, i.e. every subset of $Y$ is $F_\sigma$. We noticed in the proof of Proposition 4.1 that the latter property implies that $Y \in \Delta$. \qed

Note also that the class of scattered Eberlein compact spaces preserves continuous images. We were unable to resolve the following major open problem.

**Problem 5.3.** Let $X$ be any compact Δ-space and $Y$ be a continuous image of $X$. Is $Y$ a Δ-space?

Even a more general question is open.

**Problem 5.4.** Let $X$ be any Δ-space and $Y$ be a quotient continuous image of $X$. Is $Y$ a Δ-space?

Towards a solution of these problems we obtained several partial positive results.
Proposition 5.5. Let $X$ be any $\Delta$-space and $\varphi : X \to Y$ be a quotient continuous surjection with only finitely many nontrivial fibers. Then $Y$ is also a $\Delta$-space.

Proof. By assumption, there exists a closed subset $K \subset X$ such that $\varphi(K)$ is finite and $\varphi \mid_{X \setminus K} : X \setminus K \to Y \setminus \varphi(K)$ is a one-to-one mapping. Both sets $X \setminus K$ and $Y \setminus \varphi(K)$ are open in $X$ and $Y$, respectively. Since $\varphi$ is a quotient continuous mapping, it is easy to see that $\varphi \mid_{X \setminus K}$ is a homeomorphism. $X \setminus K$ is a $\Delta$-space, hence $Y \setminus \varphi(K)$ is also a $\Delta$-space. Finally, $Y$ is a $\Delta$-space, by Proposition 5.6. □

Proposition 5.6. Let $X$ be any $\Delta$-space and $\varphi : X \to Y$ be a closed continuous surjection with finite fibers. Then $Y$ is also a $\Delta$-space.

Proof. Let $\{Y_n : n \in \omega\}$ be a partition of $Y$. By assumption, the partition $\{\varphi^{-1}(Y_n) : n \in \omega\}$ admits a point-finite open expansion $\{U_n : n \in \omega\}$ in $X$. Clearly, $\varphi(X \setminus U_n)$ are closed sets in $Y$. Define $V_n = Y \setminus \varphi(X \setminus U_n)$ for each $n \in \omega$. We have that $\{V_n : n \in \omega\}$ is an open expansion of $\{Y_n : n \in \omega\}$ in $Y$. It remains to verify that the family $\{V_n : n \in \omega\}$ is point-finite. Indeed, let $y \in Y$ be any point. Each point in the fiber $\varphi^{-1}(y)$ belongs to a finite number of sets $U_n$. Since the fiber $\varphi^{-1}(y)$ is finite, $y$ is contained only in a finite number of sets $V_n$ which finishes the proof. □

3. Finite/countable unions.

Proposition 5.7. Assume that $X$ is a finite union of closed subsets $X_i$, where each $X_i$ belongs to the class $\Delta$. Then $X$ also belongs to $\Delta$. In particular, a finite union of compact $\Delta$-spaces is also a $\Delta$-space.

Proof. Denote by $Z$ the discrete finite union of $\Delta$-spaces $X_i$. Obviously, $Z$ is a $\Delta$-space which admits a natural closed continuous mapping onto $X$. Since all fibers of this mapping are finite, the result follows from Proposition 5.6. □

We recall a definition of the Michael line. The Michael line $X$ is the refinement of the real line $\mathbb{R}$ obtained by isolating all irrational points. So, $X$ can be represented as a countable disjoint union of singletons (rationals) and an open discrete set. Nevertheless, the Michael line $X$ is not in $\Delta$ [14]. This example and Proposition 5.7 justify the following

Problem 5.8. Let $X$ be a countable union of compact subspaces $X_i$ such that each $X_i$ belongs to the class $\Delta$. Does $X$ belong to the class $\Delta$?

4. Finite products. We already mentioned earlier that the existence of a $Q$-set $X \subset \mathbb{R}$ such that its square $X^2$ is not a $Q$-set, is consistent with ZFC.

Problem 5.9. Is the existence of a $\Delta$-set $X \subset \mathbb{R}$ such that its square $X^2$ is not a $\Delta$-set, consistent with ZFC?

It is known that the finite product of scattered Eberlein compact spaces is a scattered Eberlein compact.

Problem 5.10. Let $X$ be the product of two compact spaces $X_1$ and $X_2$ such that each $X_i$ belongs to the class $\Delta$. Does $X$ belong to the class $\Delta$?

Our last problem is inspired by Theorem 5.10.

Problem 5.11. Let $X$ be any scattered compact space with a finite scattered height. Does $X$ belong to the class $\Delta$?
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