On the Symmetry and Recovery of Steady Continuously Stratified Periodic Water Waves

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Abstract. This paper considers two-dimensional steady continuously stratified periodic water waves. We first use the moving plane method to establish that the stratified water waves must be symmetric provided streamlines in a neighborhood of the trough line are monotone. Further, we prove that all streamlines are real analytic (including the free surface) by applying the Schauder estimates on the uniform oblique derivative problem. Finally we provide an analytic expansion method to recover the water waves from relative horizontal velocity on the axis of symmetry and wave height. All results in this paper hold for stratified waves of large (or small) amplitude.

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1. Introduction

The phenomenon of stratification in the oceanography and geophysical fluid has attracted a great deal of scholars to investigate its mechanism. It’s known that the two-dimensional steady stratified periodic water waves problem can be reduced to a general class of nonlinear elliptic boundary value problems. The first part of our work is devoted to the symmetry of steady stratified periodic waves by using the method of moving planes, which was initially developed by Alexandrov and then was used extensively by Serrin [1], Gidas [2], Berestycki and Nirenberg [3] and others. In the wake of the symmetry result, we provide an analytic expansion method to recover the stratified waves from relative horizontal velocity on the axis of symmetry.

To better explain our aims, we begin with a brief description of available results. The existence of two-dimensional steady stratified periodic gravity waves was established by Walsh in [4], who also studied the stratified waves with surface tension in [5,6]. Recently, the existence and qualitative theory for stratified solitary water waves were considered in [7] and the continuous depends on the density for stratified steady water waves was established in [30]. The symmetry of steady stratified periodic waves of small amplitude was discussed in [8] based on some important works [9–12]. Hence it is natural to expect that the monotone stratified waves of large amplitude are necessarily symmetric. In addition, recovery of water waves from given data has also developed gradually as a significant subject. In [13–15], the authors reconstructed the steady periodic wave profiles from pressure measurements at the bed. In [16,17], pressure transfer functions were used to determine the profiles of solitary water waves. Meanwhile, a power series reconstruction formula from horizontal velocity on vertical symmetry axis for Stokes waves was derived in [18]. For the recovery of stratified steady water waves, there are some new results in [19], where authors uniquely determine the stratified waves from pressure. In fact, the precise pressure measurements at the flat bed are not easily obtained due to the complex conditions in sea and defects of measuring tools. Thus one may ask how to reconstruct the steady stratified water waves from other data.
In this paper, we aim to solve these problems and our results are summarized as follows. In Sect. 2, we recall the governing equations for two-dimensional steady stratified periodic water waves in three equivalent formulations. In Sect. 3, we deduce symmetry by assuming the streamlines are monotone in a neighborhood of the trough line. It’s worth noting that there are no any restrictions on the elevation of stratified water waves in our work, which is an improvement to the results in [8]. In Sect. 4, we first prove the analyticity of each streamline (see [20–23] for unstratified waves), then we provide an analytic expansion method to recover the stratified waves from relative horizontal velocity on the axis of symmetry.

2. Equivalent Formulations of Stratified Water Waves

2.1. Governing Equations in Velocity Field Formulation

Firstly, let’s recall the governing equations for two-dimensional steady stratified periodic water waves in [4]. These are the Euler’s equations and the equation of mass conservation. Choose Cartesian coordinates \((X,Y)\) such that \(X\)-axis points to the horizontal and \(Y\)-axis points to the vertical. Assume that the free surface is given by \(Y = \eta(t,X)\), \((u(t,X,Y),v(t,X,Y))\) is the velocity field of the flow over the flat bed \(Y = -d\) with \(d > 0\), \(\rho = \rho(t,X,Y) > 0\) is the density and \(P = P(t,X,Y)\) is the pressure. All of these functions depend on \((X-ct)\) and \(Y\) in steady periodic travelling waves, where \(c\) represents the speed of wave. For convenience, let \(x = X - ct, y = Y\) and consider the problem in \(\Omega = \{(x,y)|-\pi < x < \pi, -d < y < \eta(x)\}\).

The incompressibility gives that the vector field \((u,v)\) is divergence free
\[ u_x + v_y = 0. \tag{2.1} \]
Conservation of mass implies
\[ -c\rho_x + (\rho u)_x + (\rho v)_y = 0. \tag{2.2} \]
Hence (2.1) and (2.2) give
\[ (u - c)\rho_x + v\rho_y = 0. \tag{2.3} \]
Taking the conservation of momentum and the boundary conditions into consideration, then the governing equations in velocity field formulation can be expressed by the following nonlinear problem (see [4])
\[
\begin{align*}
\frac{u}{x} + v\frac{y}{y} &= 0 & \text{for} & -d \leq y \leq \eta(x), \\
(u - c)\rho_x + v\rho_y &= 0 & \text{for} & -d \leq y \leq \eta(x), \\
\rho(u - c)u_x + \rho uv_y &= -P_x & \text{for} & -d \leq y \leq \eta(x), \\
\rho(u - c)v_x + \rho vv_y &= -P_y - gp & \text{for} & -d \leq y \leq \eta(x), \\
v &= 0 & \text{for} & y = -d, \\
v &= (u - c)\eta_x & \text{on} & y = \eta(x), \\
P &= P_{\text{atm}} & \text{on} & y = \eta(x),
\end{align*}
\tag{2.4}
\]
where \(P_{\text{atm}}\) is the constant atmosphere pressure, and \(g = 9.8m/s^2\) is the (constant) gravitational acceleration at the Earth’s surface. The solutions we consider are required to be periodic in the variable \(x\), namely \((u,v,P,\eta)\) are all \(2\pi\)-periodic in \(x\). We assume that \(\eta\) oscillates around the line \(y = 0\) such that
\[ \int_{-\pi}^{\pi} \eta(x)dx = 0. \tag{2.5} \]
In addition, we suppose
\[ u < c \tag{2.6} \]
throughout the fluid, which excludes the presence of stagnation points in the flow (see [24] for details).
2.2. Governing Equation in Stream Function Formulation

To reformulate the problem (2.4) into a simpler one, we may introduce a (relative) pseudo-stream function \( \psi = \psi(x, y) \) satisfying
\[
\psi_x = -\sqrt{\rho}v, \quad \psi_y = \sqrt{\rho}(u - c) < 0. \tag{2.7}
\]

Here we add a factor of \( \rho \) to the typical definition of the stream function for incompressible fluid (see [24]), which takes into account the effects of stratification.

The level sets of \( \psi \) can be regarded as streamlines of the flow and (2.7) implies
\[
\psi(x, \eta(x)) - \psi(x, -d) = \int_{-d}^{\eta(x)} \sqrt{\rho(x, y)}(u(x, y) - c)dy.
\]

Define the relative pseudo mass flux (relative to the uniform flow at speed \( c \)) as
\[
p_0 = \psi(x, \eta(x)) - \psi(x, -d) = \int_{-d}^{\eta(x)} \sqrt{\rho(x, y)}(u(x, y) - c)dy, \tag{2.8}
\]
which is less than 0 due to (2.6). The boundary conditions in (2.4) give
\[
\frac{dp_0}{dx} = \eta_x(\sqrt{\rho(x, \eta(x))}(u(x, \eta(x)) - c)) + \int_{-d}^{\eta(x)} \eta_x(\sqrt{\rho(x, y)}(u(x, y) - c))dy
= \sqrt{\rho(x, \eta(x))}v(x, \eta(x)) - \int_{-d}^{\eta(x)} \eta_y(\sqrt{\rho(x, y)}v)dy = 0,
\]
which indicates that \( p_0, \psi(x, \eta(x)) \) and \( \psi(x, -d) \) are constants. Without loss of generality, we choose \( \psi \equiv 0 \) on the free boundary \( y = \eta(x) \), which forces \( \psi = -p_0 \) on \( y = -d \). Since \( \rho \) is transported, it must be constant on the streamlines. Then we let streamline density function \( \rho \in C^{1,\alpha}([p_0,0]; R^+) \) be given by
\[
\rho(x, y) = \rho(-\psi(x, y)) \tag{2.9}
\]
throughout the fluid. For the water waves to be physically relevant and stably stratified, the streamline density function \( \rho \) must be nondecreasing as depth increases, i.e. \( \rho' \leq 0 \).

From the Euler equation (2.4) we obtain Bernoulli’s law, which states that
\[
E = \frac{\rho}{2}((u - c)^2 + v^2) + gy\rho + P, \tag{2.10}
\]
where \( E \) is the hydraulic head and it's a constant along each streamline. Under the assumption (2.6), there exists a function \( \beta \in C^{1,\alpha}([0,|p_0|]; R) \) such that
\[
\frac{dE}{d\psi} = -\beta(\psi), \tag{2.11}
\]
where \( \beta \) is called the Bernoulli function (see [4]). Physically it describes the variation of specific energy between the streamlines. It is worth noting that when \( \rho \) is a constant, \( \beta \) reduces to the vorticity function. Combining (2.10) with (2.11), we have
\[
\frac{dE}{d\psi} = \Delta \psi - gy\rho'(-\psi) = -\beta(\psi(x, y)). \tag{2.12}
\]

Evaluating Bernoulli’s law on the free surface \( y = \eta(x) \), we obtain
\[
2E|_\eta = 2P_{atm} + |\nabla \psi|^2 + 2g\eta\rho(-\psi) \text{ on } y = \eta(x). \tag{2.13}
\]
Summarizing the above considerations gives
\[
\begin{cases}
\Delta \psi - g y \rho'(-\psi) = -\beta(\psi) & \text{in } -d < y < \eta(x), \\
|\nabla \psi|^2 + 2g p(-\psi)(y + d) = Q & \text{on } y = \eta(x), \\
\psi = 0 & \text{on } y = \eta(x), \\
\psi = -p_0 & \text{on } y = -d,
\end{cases}
\]
(2.14)
where \( Q = 2(E|\eta - P_{atm} + g \rho|_n d). \)

2.3. Governing Equation in Height Function Formulation

The system (2.14) is also difficult to handle due to its nonlinear character and the unknown surface \( \eta(x) \). However, the assumption (2.6) enables us to introduce the Dubreil-Jacotin’s transformation (see [25]) by
\[
q = x, \quad p = -\psi(x, y),
\]
(2.15)
which transforms the fluid domain
\[
\Omega = \{ (x, y) : x \in (-\pi, \pi), \quad -d < y < \eta(x) \}
\]
into rectangular domain
\[
D = \{ (q, p) : -\pi < q < \pi, \quad p_0 < p < 0 \}.
\]
Let \( \overline{D} \) represent its closure. Then the function \( \beta \) in (2.9) and \( \rho \) in (2.11) can be written as
\[
\beta = \beta(-p), \quad \rho = \rho(p).
\]
(2.16)
Define the height function by
\[
h(q, p) := y + d.
\]
(2.17)
It’s easy to deduce
\[
\psi_y = -\frac{1}{h_p}, \quad \psi_x = \frac{h_q}{h_p},
\]
(2.18)
\[
\partial_q = \partial_x + h_q \partial_y, \quad \partial_p = h_p \partial_y.
\]
(2.19)
Taking the mean of (2.17) along the free surface yields
\[
d = \int_{-\pi}^{\pi} h(q, 0) dq.
\]
(2.20)
A simple computation gives
\[
h_q = \frac{v}{u - c}, \quad h_p = \frac{1}{\sqrt{\rho(c - u)}} > 0,
\]
(2.21)
and
\[
u = c - \frac{1}{\sqrt{\rho h_p}}, \quad v = -\frac{h_q}{\sqrt{\rho h_p}}.
\]
(2.22)
Consequently, we can rewrite the governing equations in terms of the height function \( h \) by
\[
\begin{cases}
(1 + h_q^2) h_{pp} - 2h_q h_p h_{qp} + h_p^2 h_{qq} + [\beta - g(h - d)\rho']h_p^3 = 0 & \text{in } p_0 < p < 0, \\
1 + h_q^2 + h_p^2(2g h - Q) = 0 & \text{on } p = 0, \\
h = 0 & \text{on } p = p_0.
\end{cases}
\]
(2.23)
Now let’s define the nonlinear differential operators
\[
A(h)\varphi = (1 + h_q^2)\varphi_{pp} - 2h_q h_p \varphi_{qp} + h_p^2 \varphi_{qq}
\]
(2.24)
and

$$B(h)\varphi = h_\eta \varphi_\eta + (2gh - Q)h_\rho \varphi_\rho + gh_\rho^2 \varphi.$$  \hspace{1cm} (2.25)

The operator $A(h)$ is uniformly elliptic because there exists a constant $\lambda > 0$ such that

$$(1 + h_\eta^2)\xi_1^2 - 2h_\eta h_\rho \xi_1 \xi_2 + h_\rho^2 \xi_2^2 \geq \lambda (\xi_1^2 + \xi_2^2), \text{ for } \xi_1, \xi_2 \in R.$$  

The operator $B(h)$ is uniformly oblique due to

$$(2gh - Q)h_\rho = -\frac{1 + h_\eta^2}{h_\rho} \leq -\frac{1}{\max_{\partial\Omega} h_\rho} < 0.$$  

3. Symmetry of Each Streamline in Stratified Water Waves

In this section, we show the symmetry result by using elliptic maximum principles and the moving plane method (see [1,9,10]), which can be stated as follows.

**Theorem 1.** Steady continuously stratified periodic water waves without internal stagnation points are symmetric if

1. the wave profile $y = \eta(x)$ is monotone between the trough and the crest;
2. other streamlines are monotone in a neighborhood of the trough line $x = -\pi$.

**Remark 1.** (a) The existence of stratified water waves has been rigorously proved in [4].
(b) To the best of our knowledge, (2.6) in our analysis plays a key role and it’s not expected that the symmetry result holds again if there is internal stagnation point in the flow.
(c) The assumption (2) in Theorem 1 aims to start the moving planes method, which can ensure that

$$h(-\pi, p) = h(\pi, p) \leq h(q, p) \text{ for } -\pi < q < -\pi + \varepsilon, \text{ } p_0 < p \leq 0 \hspace{1cm} (3.1)$$

and

$$h(q_1, p) \leq h(q_2, p) \text{ for } -\pi < q_1 < q_2 < -\pi + \varepsilon, \text{ } p_0 < p \leq 0, \hspace{1cm} (3.2)$$

for $\varepsilon > 0$ small.

Before giving the proof, we first introduce the maximum principles and narrow region principle.

**Lemma 1** (see Theorem 2.13, Theorem 2.15 and Remark 2.16 in [26]).

Suppose $D \subset \mathbb{R}^2$ be an open rectangle and $\omega \in C^2(D) \cap C(\overline{D})$ satisfy $L\omega \leq 0$ for the uniformly elliptic operator $L = a_{ij} \partial_{ij} \omega + b_i \partial_i \omega + c \omega$ with $a_{ij}, b_i, c \in C(\overline{D})$. If $\omega \geq 0$ in $D$, then the followings hold

1. The weak maximum principle: $\omega$ attains its minimum on $\partial D$.
2. The strong maximum principle: If $\omega$ attains its minimum in $D$, then $\omega$ is constant in $\overline{D}$.
3. Hopf’s principle: Let $Q$ be a point on $\partial D$, different from the corners of the rectangle $\overline{D}$. If $\omega(Q) < \omega(X)$ for all $X \in D$, then $\partial_n \omega(X) < 0$ where $\nu$ is the outward normal direction at $Q$.

**Lemma 2** (Narrow region principle, see Proposition 1.1 in [27]).

Suppose that

1. $D_\delta \subset \mathbb{R}^2$ is a bounded narrow region with diam $\delta$ small enough and any function $\omega \in C^2(D_\delta) \cap C(\overline{D_\delta})$;
2. $L\omega = a_{ij} \partial_{ij} \omega + b_i \partial_i \omega + c \omega \geq 0$ (or $\leq 0$) where $L$ is uniformly elliptic operator with $a_{ij}, b_i, c \in C(D_\delta)$;
3. $u|_{\partial D_\delta} \leq 0$ (or $\geq 0$);
   Then $u \leq 0$ (or $\geq 0$) on $\overline{D_\delta}$.

**Remark 2.** (d) In general, Lemma 1 holds for the uniformly elliptic operator $L$ with a zeroth order term $c(x) < 0$, but the additional condition $\omega \geq 0$ in $D$ would remove the restriction for the sign of $c(x)$. Indeed, let $c(x) = c^+(x) - c^-(x)$, where $c^+ := \max\{0, c\}$ and $c^- := \max\{0, -c\}$, then

$L\omega := a_{ij} \partial_{ij} \omega + b_i \partial_i \omega - c^- \omega = L\omega - c^+ w \leq 0$ in $D$ provided $L\omega \leq 0, \omega \geq 0$ in $D$. In the following, $\omega \geq 0$ in $D$ always holds due to the definition of $\lambda_0$ below.
(e) There is no requirement for the sign of \(c(x)\) in narrow region Lemma 2.

Now let’s turn to prove the Theorem 1.

Proof. Fix that \(h, \tilde{h}\) with \(\int_{-\pi}^{\pi} h dq = \int_{-\pi}^{\pi} \tilde{h} dq\) are solutions to (2.23) and define

\[
L := A(h) + B_1(h, \tilde{h}) \partial_q + B_2(h, \tilde{h}) \partial_p + C, \tag{3.3}
\]

where operator \(A\) is defined by (2.24) and operators \(B_1, B_2, C\) are given by

\[
\begin{align*}
B_1(h, \tilde{h}) &= \tilde{h}_{pp}(h_q + \tilde{h}_q) - 2h_p \tilde{h}_{qp}, \\
B_2(h, \tilde{h}) &= \tilde{h}_{qq}(h_p + \tilde{h}_p) - 2h_q \tilde{h}_{qp} + [\beta - g \rho'(\tilde{h} - d)](h^2_p + h_p \tilde{h}_p + \tilde{h}^2_p), \\
C &= -g \rho' h^3_p.
\end{align*}
\tag{3.4}
\]

Let \(v = \tilde{h} - h\), it’s easy to verify that \(v\) satisfies

\[
\begin{align*}
Lv &= 0 \\
(h_q + \tilde{h}_q)v_q + (h_p + \tilde{h}_p)(2\rho q \tilde{h} - Q)v_p + 2h^2_p q g v &= 0 \quad \text{in } p_0 < p < 0, \\
v &= 0 \quad \text{on } p = 0, \\
&= 0 \quad \text{on } p = p_0.
\end{align*}
\tag{3.5}
\]

For a reflection parameter \(\lambda \in (-\pi, 0)\), the reflection of \(q\) about \(\lambda\) is given by \(2\lambda - q\) and let

\[
D^\lambda = \{(q, p) \in D : -\pi < q < \lambda, \ p_0 < p < 0\}.
\]

Fix a point \((q, p) \in D^0\), the associated reflection function is defined by

\[
\omega(q, p; \lambda) = h(2\lambda - q, p) - h(q, p).
\]

It’s obvious that \(L\omega = 0\) and \(\omega(q, p; \lambda)\) satisfies

\[
\begin{align*}
\omega(\lambda, p; \lambda) &= 0 \quad \text{for } p \in [p_0, 0], \\
\omega(q, p_0; \lambda) &= 0 \quad \text{for } q \in [-\pi, \lambda],
\end{align*}
\tag{3.6}
\]

where the first equality is immediate from the definition of \(\omega(q, p; \lambda)\) and the second follows from the boundary condition \(h = 0\) on \(p = p_0\). Since (3.1) and (3.2) indicate that \(\omega \geq 0\) in \(\tilde{D}^\lambda\) for \(-\pi < \lambda < -\pi + \frac{\pi}{2}\), where \(\tilde{D}^\lambda = D^\lambda \cup \{(q, p) | -\pi \leq q < \lambda, p = 0\} \cup \{(q, p) | q = -\pi, p_0 < p < 0\}\), there exists an extremal position \(\lambda_0\) for \(\lambda\), which is defined as

\[
\lambda_0 = \sup\{\lambda \in (-\pi, 0] : \omega(q, p; \lambda) \geq 0 \text{ in } \tilde{D}^\lambda\}.
\]

One of the following two alternatives must occur:

Case 1 \(\lambda_0 = 0\) (see Fig. 1);

Case 2 \(\lambda_0 \in (-\pi, 0)\) (see Fig. 2).

Step 1 Suppose that Case 1 occurs , (3.6) and the definition of \(\lambda_0\) imply that the reflection function \(\omega\) satisfies the following boundary conditions

\[
\begin{align*}
\omega(-\pi, p; 0) &\geq 0 \quad \text{for } p \in (p_0, 0), \\
\omega(0, p; 0) &= 0 \quad \text{for } p \in [p_0, 0], \\
\omega(q, 0; 0) &\geq 0 \quad \text{for } q \in [-\pi, 0), \\
\omega(q, p_0; 0) &= 0 \quad \text{for } q \in [-\pi, 0].
\end{align*}
\tag{3.7}
\]

In addition, the definition of \(\lambda_0\) implies

\[
\omega \geq 0 \quad \text{in } D^0.
\]

Then the strong maximum lemma shows that the minimum of \(\omega\) can be attained only on \(\partial D^0\) unless \(\omega\) is a constant. Considering the boundary conditions in (3.7), we have

\[
\omega(q, p; 0) > 0 \text{ in } D^0 \text{ or } \omega(q, p; 0) \equiv 0 \text{ in } \overline{D^0}.
\tag{3.8}
\]

If \(\omega\) vanishes throughout \(\overline{D^0}\), the symmetry is obtained.
If \( \omega(q,p;0) > 0 \) for \((q,p) \in D^0\), we can deduce a contradiction by using Hopf lemma. Since \( q = -\pi \) and \( q = \pi \) are trough lines, it’s easy to see \( h_q(\pm\pi, \tilde{p}) = 0 \) for \( p_0 < \tilde{p} < 0 \). Let \( A = (-\pi, \tilde{p}) \) (see Fig. 1), then \( \omega_q(A) = \omega_q(-\pi, \tilde{p}; 0) = h_q(\pi, \tilde{p}) - h_q(-\pi, \tilde{p}) = 0 \). Besides, we also have that \( \omega(A) = 0 \) and \( \omega(A) < \omega(q,p;0) \) for \((q,p) \in D^0\). Then the Hopf lemma gives that \( \omega_q(A) = -\partial_\nu \omega(A) > 0 \), which is a contradiction.

Step 2 Assume Case 2 takes place, we have \( \lambda_0 \in (-\pi, 0) \). In general, one may require that the reflected surface is tangent at some point \((q_0,0)\) in this case. However, here we take a different tack due to the new definition of \( \lambda_0 \). As in Case 1, (3.6) and the definition of \( \lambda_0 \) imply that

\[
\begin{align*}
\begin{cases}
\omega(-\pi, p; \lambda_0) \geq 0 & \text{for } p \in (p_0, 0), \\
\omega(\lambda_0, p; \lambda_0) = 0 & \text{for } p \in [p_0, 0], \\
\omega(q_0, 0; \lambda_0) \geq 0 & \text{for } q \in [-\pi, \lambda_0), \\
\omega(q_0, p_0; \lambda_0) = 0 & \text{for } q \in [-\pi, \lambda_0].
\end{cases}
\end{align*}
\]

(3.9)

The definition of \( \lambda_0 \) yields

\[
\omega(q,p;\lambda_0) \geq 0 \text{ in } D^{\lambda_0}.
\]

Using the strong maximum again, we have that \( \omega \) attains its minimum only on \( \partial D^{\lambda_0} \) unless \( \omega \) is a constant. In view of the boundary conditions (3.9), one of the following two cases holds

\[
\omega(q,p;\lambda_0) > 0 \text{ in } D^{\lambda_0} \text{ or } \omega(q,p;\lambda_0) \equiv 0 \text{ in } D^{\lambda_0}. \tag{3.10}
\]

If

\[
\omega(q,p;\lambda_0) > 0 \text{ in } D^{\lambda_0}, \tag{3.11}
\]

we show a contradiction with the definition of \( \lambda_0 \) by using narrow region principle. Let \( \delta_0 \) be sufficiently small and define \( \omega(q,p;\lambda) : = \omega^{\lambda}(q,p) \), we are going to work on the narrow region \( D_{\delta_0} = D^{\lambda_0+\delta_0} \setminus D^{\lambda_0-\delta_0} \) (see Fig. 2). It’s easy to see that \( L \omega^{\lambda_0+\delta_0} = 0 \) and \( \omega^{\lambda_0+\delta_0}(q,p) \) satisfies the following boundary conditions

\[
\begin{align*}
\begin{cases}
\omega^{\lambda_0+\delta_0}(\lambda_0 + \delta_0, p) = 0 & \text{for } p \in [p_0, 0], \\
\omega^{\lambda_0+\delta_0}(\lambda_0 - \delta_0, p) \geq 0 & \text{for } p \in (p_0, 0), \\
\omega^{\lambda_0+\delta_0}(q,p_0) = 0 & \text{for } q \in [\lambda_0 - \delta_0, \lambda_0 + \delta_0].
\end{cases}
\end{align*}
\]

(3.12)

The first equality in (3.12) is immediate from the definition of \( \omega \), and the last one follows from the boundary condition \( h = 0 \) on \( p = p_0 \). The second inequality in (3.12) follows from the following arguments. From (3.11), we have

\[
\omega^{\lambda_0}(\lambda_0 - \delta_0, p) = h(\lambda_0 + \delta_0, p) - h(\lambda_0 - \delta_0, p) > 0 \text{ for } p \in (p_0, 0). \tag{3.13}
\]

For small enough \( \delta_0 \), the continuity of \( h \) with respect to \( q \) in (3.13) shows

\[
\omega^{\lambda_0+\delta_0}(\lambda_0 - \delta_0, p) = h(\lambda_0 + 3\delta_0, p) - h(\lambda_0 - \delta_0, p) \geq 0 \text{ for } p \in (p_0, 0).
\]
In this section, we discuss the recovery of the continuously stratified water waves from relative horizontal velocity on axis of symmetry and wave height, which can be summarized by the following theorem.

4. Recovery of the Waves from Relative Horizontal Velocity on Axis of Symmetry and Wave Height

In this section, we discuss the recovery of the continuously stratified water waves from relative horizontal velocity on axis of symmetry and wave height, which can be summarized by the following theorem.

Now let’s look at the values of $\omega^{\lambda_0+\delta_0}$ on the surface and we claim that

$$\omega^{\lambda_0+\delta_0}(q,0) \geq 0 \text{ for } q \in [\lambda_0 - \delta_0, \lambda_0 + \delta_0].$$

(3.14)

For a clear presentation, let’s divide this process into the following three steps. Firstly, we prove that

$$\omega^{\lambda_0}(q,0) > 0 \text{ for } q \in (-\pi, \lambda_0).$$

If it’s wrong, then there exists $q_0 \in (-\pi, 0)$ for which $\omega(q_0,0; \lambda_0) = 0$. Moreover, the mapping $q \mapsto h(q,0)$ and $q \mapsto h(2\lambda_0 - q,0)$ are tangent to each other at $(q_0,0)$, i.e $\partial_q \omega^{\lambda_0}(q_0,0) = 0$ (see Fig. 2). Evaluating the Bernoulli boundary conditions $1 + h_q^2 + h_p^2(2\rho gh - Q) = 0$ where $\rho$ is constant density along the free surface, we deduce that $h_p(q_0,0) = h_p(2\lambda_0 - q_0,0)$ (i.e $\partial_p \omega^{\lambda_0}(q_0,0) = 0$). This will lead to a contradiction by applying the Hopf lemma at $(q_0,0)$. Secondly, we show that

$$\partial_q \omega^{\lambda_0}(\lambda_0,0) < 0.$$  

(3.15)

The arguments above imply $\omega^{\lambda_0}(q,0) > 0$ for $q \in (-\pi, \lambda_0)$ and $\omega^{\lambda_0}(\lambda_0,0) = 0$. As $q \in (\lambda_0, \lambda_0 + \epsilon_0)$ for any small $\epsilon_0 > 0$, then $2\lambda_0 - q \in (\lambda_0 - \epsilon_0, \lambda_0)$. It’s easy to deduce that $\omega^{\lambda_0}(q_0,0) = h(2\lambda_0 - q,0) - h(q,0) < 0$ (see Fig. 2). Thus (3.15) follows from the definition of the derivative. Finally, we finish the proof of this claim. Since $\partial_q \omega^{\lambda}(q,0)$ is a continuous function of $q$ and $\lambda$, (3.15) implies

$$\partial_q \omega^{\lambda_0+\delta_0}(q,0) < 0,$$

(3.16)

provided $|q - \lambda_0| \leq \delta_0$. For $\lambda_0 - \delta_0 \leq q \leq \lambda_0 + \delta_0$, the fact $\omega^{\lambda_0+\delta_0}(\lambda_0 + \delta_0,0) = 0$ and the monotonicity in (3.16) yield (3.14).

Now we apply the narrow region lemma on the narrow region $D_{\delta_0}$ and (3.12) and (3.14) imply $

\omega^{\lambda_0+\delta_0}(q,p) \geq 0 \text{ for } (q,p) \in \overline{D_{\delta_0}}, \text{ hence } \omega^{\lambda_0+\delta_0}(q,p) \geq 0 \text{ for } (q,p) \in \overline{D^{\lambda_0+\delta_0}}.$

This is in contradiction with the definition of $\lambda_0$.

Thus

$$\omega(q,p; \lambda_0) \equiv 0 \text{ in } \overline{D^{\lambda_0}},$$

which indicates that $h(-\pi,p) = h(2\lambda_0 + \pi,p) = h(\pi,p) \text{ for } p \in [p_0,0]$. Considering the assumption (1) in Theorem 1, we have that $2\lambda_0 + \pi$ lies to the right of the wave crest, which forces the map $q \mapsto h(q,0)$ is nonincreasing for $2\lambda_0 + \pi \leq q \leq \pi$. Namely, $q = \lambda_0$ must be the location of the crest line (see Fig. 2). In this case, the water waves are also symmetric about $q = \lambda_0$ due to the periodicity. □
Theorem 2. Consider two-dimensional steady continuously stratified periodic water waves with given analytic stratified density \( \rho \in C^\infty([p_0, 0]; R^+) \) and Benoulli’s function \( \beta \in C^\infty([0, |p_0|]; R) \), and assume that the relative horizontal velocity on the axis of symmetry \( u(0, y) = -c \) and wave height \( \eta(0) \) are known, then we can recover the wave profile \( \eta \), the velocity field \((u, v)\) and the pressure \(P\) within the fluid.

Remark 3. (f) It’s known that the Eq. (2.4) with solutions \((u, v, P, \eta)\) are equivalent to the Eq. (2.14) with solutions \((\eta, \psi)\) (see [4]), hence it’s sufficient to recover \((\eta, \psi)\).

(g) The analyticity of functions \(\rho(p)\) and \(\beta(-p)\) ensure that they can be expanded in form of power series at some point with given coefficients.

Before proving Theorem 2, let’s first show that the streamlines are analytic.

Lemma 3 (see Lemma 3.2 in [21]). Let \( l = 1 \) or 2 and let \( \| \cdot \| \) denote some Hölder norm \( \| \cdot \|_{0, \alpha} \) or \( \| \cdot \|_{1, \alpha} \). Suppose that \( k_0 \) is an integer with \( k_0 \geq l + 1 \), and \( \partial^k u_j \in C^{0, \alpha}(\overline{D}) \) for all \( k \leq k_0, j = 1, 2, 3 \). If there exists a constant \( M \geq 1 \) such that

\[
\forall l + 1 \leq k \leq k_0, \| \partial^k u_j \| \leq M^{k-l}(k-l-1)!, \quad j = 1, 2, 3,
\]

then there is a positive constant \( C_s \) depending only on \( l \) such that

\[
\forall l + 1 \leq k \leq k_0, \| \partial^k q(u_1 u_2 u_3) \| \leq C_s \sum_{j=1}^{3} \| u_j \|_{l+1, \alpha} + 1^6 M^{k-l}(k-l-1)!.
\]

Lemma 4. Assume that \( \rho \in C^{1, \alpha}([p_0, 0]; R^+) \), \( \beta \in C^{0, \alpha}([0, |p_0|]; R) \) are given and \( h \in C^{2, \alpha}_{\text{per}}(\overline{D}) \) is the solution to (2.23). Then, the mapping \( q \mapsto h(q, p) \), for any fixed \( p \in [p_0, 0] \), is analytic in \( R \).

Proof. We divide the process of proof into following three steps.

Step 1 Show \( \partial^q h \in C^{2, \alpha}_{\text{per}}(\overline{D}) \) for any \( n \in N \).

Taking the derivative with respect to \( q \) on both sides of (2.23), we have

\[
\begin{cases}
A(h) h_q = f_1 + f_2 & \text{in } p_0 < p < 0, \\
B(h) h_q = 0 & \text{on } p = 0, \\
h_q = 0 & \text{on } p = p_0,
\end{cases}
\]

where operators \( A(h) \) and \( B(h) \) are given by (2.24) and (2.25), and \( f_1, f_2 \) are expressed by

\[
\begin{align*}
f_1 &= -(\partial_q h^2_q) h_{pp} + 2(\partial_q (h_p h_q)) h_{qp} - (\partial_q h^2_p) h_{qq}, \\
f_2 &= -\beta + g \rho (\partial_q h^3_p) + g \rho^2 \partial_q (h h^3_p).
\end{align*}
\]

Since \( h \in C^{2, \alpha}_{\text{per}}(\overline{D}) \), the coefficients of operators \( A(h) \) and \( B(h) \) belong to \( C^{1, \alpha}_{\text{per}}(\overline{D}) \). Similarly, \( \rho \in C^{1, \alpha}([p_0, 0]; R^+) \) and \( \beta \in C^{0, \alpha}([0, |p_0|]; R) \) imply that \( f_1 \) and \( f_2 \) are in \( C^{0, \alpha}(\overline{D}) \). Therefore, the standard Schauder estimate (see Theorem 6.30 in [28]) yields \( h_q \in C^{2, \alpha}_{\text{per}}(\overline{D}) \) with

\[
\| h_q \|_{C^{2, \alpha}} \leq C(\| h_q \|_{C^0} + \| f_1 \|_{C^{0, \alpha}} + \| f_2 \|_{C^{0, \alpha}}),
\]

where \( C = C(\alpha, \| h \|_{C^{2, \alpha}}) \).

Then we repeat this process by taking the derivative with respect to \( q \) up to 2 order, 3 order until \( n \) order on both sides of (2.23). The Leibnitz formula is used to yield

\[
\begin{cases}
A(h) \partial^q h = -F_1 - F_2 & \text{in } p_0 < p < 0, \\
B(h) \partial^q h = -\Phi_1 - \Phi_2 & \text{on } p = 0, \\
\partial^q h = 0 & \text{on } p = p_0,
\end{cases}
\]
with
\[
\begin{align*}
F_1 &= \sum_{k=1}^n C_n^k (\partial_q^k h_2^3)(\partial_q^{n-k} h_{pp}) - 2(\partial_q^k (h_p h_q))(\partial_q^{n-k} h_{qp}) + (\partial_q^k h_2^3)(\partial_q^{n-k} h_{qq}), \\
F_2 &= (\beta + g\rho^p)(\partial_q h_3^3) - g\rho^p \sum_{k=0}^n C_n^k (\partial_q h_3^3), \\
\Phi_1 &= \frac{1}{2} \sum_{k=1}^{n-1} C_n^k (\partial_q^k h_q)(\partial_q^{n-k} h_q) + \frac{1}{2}(2g\rho h - Q) \sum_{k=1}^{n-1} C_n^k (\partial_q^k h_p)(\partial_q^{n-k} h_p), \\
\Phi_2 &= g\rho \sum_{k=1}^{n-1} C_n^k (\partial_q^k h)(\partial_q^{n-k} h^2_p),
\end{align*}
\]
where \( C_n^k = \frac{n!}{k!(n-k)!} \). Using the Schauder estimate based on standard interaction, we could get \( \partial_q^n h \in C^2_{\text{per}}(\bar{D}) \) with
\[
\|\partial_q^n h\|_{C^{2,\alpha}} \leq C(\|\partial_q^n h\|_{C^0} + \sum_{i=1}^2 \|F_i\|_{C^{0,\alpha}} + \sum_{i=1}^2 \|\Phi_i\|_{C^{1,\alpha}}).
\] (4.4)

Step 2 Show \( \|\partial_q^n h\|_{C^{2,\alpha}} \leq C^n n! \) with \( C > 0 \) independent of \( n \).

For convenience, here we apply directly the estimates (Lemma 3.5–3.8) in [21] to show that there is a constant \( C(\alpha, \|h\|_{C^4,\alpha}, \|\beta\|_{C^0,\alpha}, \|\rho\|_{C^{1,\alpha}}) \) such that
\[
\|\partial_q^n h\|_{C^{2,\alpha}} \leq C^n n!
\] (4.5)
provided that the new term \( g\rho^p \sum_{k=0}^n C_n^k (\partial_q^k h)(\partial_q^{n-k} h_3^3) \) in \( F_2 \) satisfies
\[
\|g\rho^p \sum_{k=0}^n C_n^k (\partial_q^k h)(\partial_q^{n-k} h_3^3)\|_{C^{0,\alpha}} \leq C^n n!
\] (4.6)
where \( C_1 \), independent of \( n \), is a constant. Thanks to \( \rho \in C^{1,\alpha}([p_0, 0]; R^+), \beta \in C^{0,\alpha}([0, |p_0|]; R) \) and
\[
\|\sum_{k=0}^n C_n^k (\partial_q^k h)(\partial_q^{n-k} h_3^3)\|_{C^{0,\alpha}} \leq \sum_{k=0}^n C_n^k \|\partial_q^k h\|_{C^0,\alpha} \|\partial_q^{n-k} h_3^3\|_{C^{0,\alpha}},
\]
then (4.6) follows from the skillful Lemma 3 (taking \( l = 2, k_0 = n, M = C_1 \) and \( u_1 = u_2 = u_3 = h_p \)).

Step 3 Show \( q \mapsto h(q, p) \) is analytic in \( R \).

From Step 2, we have
\[
\max_{(q, p) \in \bar{D}} |\partial_q^n h(q, p)| \leq \|\partial_q^n h\|_{C^{2,\alpha}} \leq C^n n!
\] (4.7)
For any \( p \in [p_0, 0] \), the periodicity of variable \( q \) implies
\[
\max_{q \in R} |\partial_q^n h(q, p)| \leq C^n n!, \text{ for } n \in N.
\] (4.8)
Thus we can write \( h(q, p) \) into the form of Taylor series at some point \( (q_0, p) \in \bar{D} \), and the remainder term is
\[
\frac{\partial_q^{n+1} h(q_0, p)}{(n+1)!} (q - q_0)^{n+1}.
\] (4.9)

From (4.5), it’s easy to see
\[
|\frac{\partial_q^{n+1} h(q_0, p)}{(n+1)!} (q - q_0)^{n+1}| \leq \frac{|q - q_0|^{n+1}}{C} \rightarrow 0, \text{ } n \rightarrow \infty
\] (4.10)
for \( q_0 - \frac{1}{C} < q < q_0 + \frac{1}{C} \), hence the mapping \( q \mapsto h(q, p) \) is analytic in \( R \).

\[ \square \]

Lemma 5. If the mapping \( q \mapsto h(q, p) \) is analytic in \( R \), then each streamline \( y = y(x) \) is a real-analytic curve. Moreover, the mapping \( x \mapsto \psi(x, y) \) is analytic in \( R \).

Proof. The streamline \( \psi(x, y) = p \), with fixed \( p \in [p_0, 0] \), can be described by the graph of some function \( y = y(x) (y = \eta(x) \text{ if } p = 0) \) due to (2.7). The analyticity of \( x \mapsto y(x) \) follows at once from the analyticity of \( q \mapsto h(q, p) \) obtained in Lemma 4 because of the partial hodograph change of variables (2.15). On the other hand, the explicit analytic function \( y = y(x) \) is determined by the implicit function \( \psi(x, y) = p \).
where $p$ is a fixed constant. Therefore, the analyticity of $x \mapsto \psi(x,y)$ can be obtained due to the analyticity of $y(x)$ (see [29]).

Now we are in the position to prove the Theorem 2.

**Proof.** For each point $(x_0, y) \in \overline{D}$, Lemma 5 implies

$$|\psi(x, y) - \sum_{k=0}^{n} \frac{\partial_x^k \psi(x_0, y)}{k!} (x-x_0)^k| \leq \varepsilon, \ n \to \infty$$

(4.11)

for $|x-x_0| < \delta$. Because analyticity is local property, the radius of convergence of above Taylor series is independent of the point $(x_0, y)$. So we can extend the function $\psi$ to the whole area $\overline{D}$. Without loss of generality, here we only determine $\psi(x, y)$ in $\{(x, y)| -\delta \leq x \leq \delta, -d \leq y \leq \eta(x)\}$, where $\delta$ is enough small. By the symmetry and evenness of function $\psi$ with respect to $x$, we have

$$\partial_x^{2n+1} \psi(0, y) = 0, \ \text{for} \ n \in \mathbb{N}, \ y \in [-d, \eta(x)].$$

(4.12)

Then the analytic function $\psi(x, y)$ at $x = 0$ can be expressed by

$$\psi(x, y) = \sum_{n=0}^{\infty} a_{2n}(y)x^{2n}, \ \text{for} \ |x| < \delta, \ y \in [-d, \eta(x)],$$

(4.13)

where $a_{2n}(y) = \frac{\partial_x^{2n} \psi(0, y)}{(2n)!}$. It’s sufficient to complete the proof by determining the coefficients $a_{2n}(y)$. By argument on $p_0$ defined in (2.8), we know that $p_0$ is independent of $x$. Then, choosing $x = 0$ in (2.8) yields

$$p_0 = \int_{-d}^{\eta(0)} \sqrt{\rho(0, y)}[u(0, y) - c]dy,$$

(4.14)

which means $p_0$ is determined by given $\eta(0), u(0, y) - c$ and $\rho$. In addition, the definition of pseudo-stream function $\psi(x, y)$ implies

$$\psi(x, y) = -p_0 + \int_{-d}^{y} \sqrt{\rho(x, s)}[u(x, s) - c]ds,$$

(4.15)

then

$$a_0(y) = \psi(0, y) = -p_0 + \int_{-d}^{y} \sqrt{\rho(0, s)}[u(0, s) - c]ds$$

(4.16)

is also determined.

Taking (4.13) into the first equation of (2.14), we have

$$\sum_{n=1}^{\infty} (2n)(2n-1) \frac{\partial_x^{2n} \psi(0, y)}{(2n)!} x^{2n-2} + \sum_{n=0}^{\infty} \frac{\partial_x^2 \partial_x^{2n} \psi(0, y)}{(2n)!} x^{2n} = gy\rho' - \beta.$$

(4.17)

Since $p = -\psi(x, y)$ and the mappings $x \mapsto \psi(x, y), p \mapsto \rho'(p)$ are analytic, if denote $\rho'(p) = \rho'(\psi(x, y))$, it’s obvious that the mapping $x \mapsto \rho' \circ (\psi(x, y))$ is analytic for $x \in R$. At the same time, this mapping is even because $x \mapsto \psi(x, y)$ is even. Similarly it also holds for the mapping $x \mapsto \beta \circ (\psi(x, y))$. Thus, we can express $\rho'$ and $\beta$ by

$$\rho' = \sum_{n=0}^{\infty} b_{2n}(y)x^{2n},$$

(4.18)

$$\beta = \sum_{n=0}^{\infty} c_{2n}(y)x^{2n},$$

(4.19)
in \( \{(x,y) | -\delta \leq x \leq \delta, -d \leq y \leq \eta(x) \} \). It’s worth noting that \( b_{2n}(y) \) and \( c_{2n}(y) \) have been given (see Remark 3). Equations (4.17), (4.18) and (4.19) yield

\[
\begin{cases}
2a_2(y) + a_0''(y) = gyb_0(y) + c_0(y), \\
12a_4(y) + a_2''(y) = gyb_2(y) + c_2(y), \\
\vdots \\
(2n)(2n-1)a_{2n}(y) + a_{2n-2}''(y) = gyb_{2n-2}(y) + c_{2n-2}(y).
\end{cases}
\] (4.20)

From the first equation in (4.20), \( a_2(y) \) can be determined. Then, \( a_4(y) \) also can be confirmed according to the second equation in (4.20). All the coefficients \( a_{2n}(y) \) would be obtained by repeating above process, i.e., the pseudo-stream function \( \psi = \psi(x, y) \) is determined.

Finally, it remains to recover surface profile \( \eta(x) \). From the second equation of (2.14), we have

\[
\eta(x) = \frac{Q - |\nabla \psi|^2}{2g\rho|\eta|} - d,
\] (4.21)

where

\[
Q = 2(E|_{\eta} - P_{atm} + g\rho|_{\eta}d) = 2\left(\frac{(u(0, \eta(0)) - c)^2}{2} + g\rho|_{\eta}(0) + g\rho|_{\eta}d\right).
\] (4.22)

Combining (4.21) with (4.22), we can obtain

\[
\eta(x) = \frac{(u(0, \eta(0)) - c)^2 - |\nabla \psi|^2}{2g\rho|\eta|} + \eta(0).
\] (4.23)

\[\square\]

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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**References**

[1] Serrin, J.: A symmetry problem in potential theory. Arch. Ration. Mech. Anal. **43**, 304–318 (1971)
[2] Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979)
[3] Berestycki, H., Nirenberg, L.: Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations. J. Geom. Phys. **5**, 237–275 (1988)
[4] Walsh, S.: Stratified steady periodic water waves. SIAM J. Math. Anal. **41**, 1054–1105 (2009)
[5] Walsh, S.: Steady stratified periodic gravity waves with surface tension I: local bifurcation. Discrete Contin. Dyn. Syst. Ser. A. **34**, 3287–3315 (2014)
[6] Walsh, S.: Steady stratified periodic gravity waves with surface tension II: global bifurcation. Discrete Contin. Dyn. Syst. Ser. A. **34**, 3241–3285 (2014)
[7] Chen, R.M., Walsh, S., Wheeler, M.H.: On the existence and qualitative theory for stratified solitary water waves. C. R. Acad. Sci. Paris Sér. I Math. **354**, 601–605 (2016)
[8] Walsh, S.: Some criteria for the symmetry of stratified water waves. Wave Motion **46**, 350–362 (2009)
[9] Constantin, A., Ehrnström, M., Wahlén, E.: Symmetry of steady periodic gravity water waves with vorticity. Duke Math. J. **140**(3), 591–603 (2007)
[10] Hur, V.H.: Symmetry of steady periodic water waves with vorticity. Phil. Trans. R. Soc. A **365**, 2203–2214 (2007)
[11] Constantin, A., Escher, J.: Symmetry of steady periodic surface water waves with vorticity. J. Fluid Mech. 498, 171–181 (2004)
[12] Constantin, A., Escher, J.: Symmetry of steady deep-water waves with vorticity. Eur. J. Appl. Math. 15, 755–768 (2004)
[13] Clamond, D.: New exact solutions for easy recovery of steady wave profiles from bottom pressure measurements. J. Fluid Mech. 726, 547–558 (2013)
[14] Clamond, D., Constantin, A.: Recovery of steady periodic wave profiles from pressure measurements at the bed. J. Fluid Mech. 714, 463–475 (2013)
[15] Escher, J., Schlurmann, T.: On the recovery of the free surface from the pressure within periodic traveling water waves. J. Nonlinear Math. Phys. 15, 50–57 (2008)
[16] Henry, D.: On the pressure transfer function for solitary water waves with vorticity. Math. Ann. 357, 23–30 (2013)
[17] Constantin, A.: On the recovery of solitary wave profiles from pressure measurements. J. Fluid Mech. 699, 376–384 (2012)
[18] Matioc, B.V.: Recovery of Stokes waves from velocity measurements on an axis of symmetry. J. Phys. A 48, 255501 (2015)
[19] Chen, R.M., Walsh, S.: Unique determination of stratified steady water waves from pressure. J. Differ. Equ. 264, 115–133 (2018)
[20] Constantin, A., Escher, J.: Analyticity of periodic travelling free surface water waves with vorticity. Ann. Math. 173, 559–568 (2011)
[21] Chen, H., Li, W.X., Wang, L.J.: Regularity of traveling free surface water waves with vorticity. J. Nonlinear Sci. 23, 1111–1142 (2013)
[22] Henry, D.: Analyticity of the streamlines for periodic travelling free surface capillary-gravity water waves with vorticity. SIAM J. Math. Anal. 42(6), 3103–3111 (2010)
[23] Craig, W., Matei, A.M.: On the regularity of the Neumann problem for free surfaces with surface tension. Proc. Am. Math. Soc. 135(8), 2497–2504 (2007)
[24] Constantin, A.: Nonlinear water waves with applications to wave-current interactions and tsunamis. In: CBMS-NSF Conference Series in Applied Mathematics, vol. 81, SIAM, Philadelphia (2011)
[25] Jactin, M.L.: Sur la détermination rigoureuse des ondes permanentes périodiques d’ampleur finite. J. Math. Pures Appl. 13, 217–291 (1934)
[26] Fraenkel, L.E.: An Introduction to Maximum Principles and Symmetry in Elliptic Problems. Cambridge University Press, Cambridge (2000)
[27] Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bol. Soc. Bras. Mat. 22, 1–37 (1991)
[28] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin (2001)
[29] Buffoni, B., Toland, J.: Analytic Theory of Global Bifurcation. Princeton University Press (2003)
[30] Chen, R.M., Walsh, S.: Continuous dependence on the density for stratified steady water waves. Arch. Ration. Mech. Anal. 219, 741–792 (2016)

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