Abstract. We examine some issues that arise in the $q$-deformation of a gauge theory. If the deformation is carried out by replacing the equal time commutators of free fields by the corresponding $q$-commutators, the resulting propagators are not very much different from those of the undeformed theory as long as one is dealing with weak fields; but the theory still violates causality. If one postulates a $q$-deformed $S$ matrix, the corresponding $q$-causal commutator has 2 poles of different strength and the result again amounts to a deformation of the Lorentz group.
1. Introduction.

A quantum field theory rests jointly on the underlying geometry and the dynamical laws operating in this geometry. The particular way the structure of the full theory is distributed between these two foundations is somewhat arbitrary. In recent years, beginning with the discoveries of supersymmetry and supergravity, the emphasis has been on the investigation of different geometries. Here we shall pursue the alternative option of modifying the dynamics and more specifically of modifying the quantization procedure. This alternative touches on another open and less explored subject, the rigidity of quantum mechanics.

There is an interesting deformation of quantum mechanics obtained by altering the Dirac prescription so as to replace classical Poisson brackets of dynamically conjugate variables by the $q$-commutators of the corresponding quantum observables. As far as this proposal has been investigated there seems to be no obstruction to the formulation of a $q$-quantum mechanics for finite systems;\(^1\) and if $q$ is close enough to unity to be compatible with present experiment, these theories may even be regarded as realistic. One may try to implement the same idea in field theory by working with fields and their dynamically conjugate fields. This is the first point we shall investigate here. There is an arbitrariness in this formulation however, and depending on which questions are asked, the resulting theory may differ either little or greatly from the $q = 1$ formalism. We shall also examine a second approach in which one postulates a “$q$-time ordering” of the $S$ matrix. In the latter case there is a violation of special relativity. This is an example in which the altered dynamics is inconsistent with the original geometry and therefore is not independent of an explicit $q$-deformation of that geometry—an avenue that has been much explored.\(^2\)

The effect of replacing the usual commutators (or anticommutators) of the field oscillators by $q$-commutators is to replace the occupation number $n$ by the corresponding basic number

$$\langle n \rangle = \frac{q^n - 1}{q - 1}.$$  

Then one might anticipate that the Einstein relation leading to the Planck law, namely:

$$\frac{n + 1}{n} = e^{\hbar \nu / kT}$$
would be replaced by

\[
\frac{\langle n + 1 \rangle}{\langle n \rangle} = e^{\hbar \nu / kT}
\]

which is equivalent to

\[
\langle n \rangle = \frac{1}{e^{\hbar \nu / kT} - q}.
\]

This last relation is in fact correct in both the E.B. \((q = 1)\) and F.D. \((q = -1)\) limits.

It is perhaps of interest to examine the \(q\)-formulation of other basic relations holding for the two kinds of statistics. We shall here discuss the perturbation sector of \(q\)-electrodynamics.

2. Quantization.

Quantization may be imposed via the field oscillators as follows. Denote the expansion of an arbitrary field by

\[
\psi_\alpha(x) = \sum_\rho [a_\rho f_\alpha(\rho, x) + \bar{b}_\rho g_\alpha(\rho, x)]
\]  \hspace{1cm} (2.1)

where

\[
\sum_\rho = \sum_r \int \! d\vec{p}
\]  \hspace{1cm} (2.2)

\[
f_\alpha(\rho, x) = \left( \frac{1}{2\pi} \right)^{3/2} \frac{u_\alpha(\vec{p}, r)}{(2p_\rho)^{1/2}} e^{-ipx}
\]  \hspace{1cm} (2.3)

\[
g_\alpha(\rho, x) = \left( \frac{1}{2\pi} \right)^{3/2} \frac{v_\alpha(\vec{p}, r)}{(2p_\rho)^{1/2}} e^{ipx}.
\]  \hspace{1cm} (2.4)

Here \(a(\bar{a})\) and \(b(\bar{b})\) are the absorption (emission) operators of particles and antiparticles respectively. The \(\rho\) sum is an integration over momentum and a sum over spin. The particle and antiparticle parts of the sum are related by complex conjugation of the exponentials and by charge conjugation of the spin dependent functions, according to the following relations:
The quantization of the oscillators may be described by the following equations:

\[ v(\rho, \vec{p}) = C\bar{u}^T = C(\gamma^o)^T u^* \] (2.5)
\[ u(\rho, \vec{p}) = C\bar{v}^T = C(\gamma^o)^T v^* . \] (2.6)

Introduce the operator \( \mathcal{C} \) in Hilbert space which takes a particle state into an antiparticle state:

\[ \mathcal{C} \bar{a}(\rho)\Psi_o = \epsilon^* \bar{b}(\rho)\Psi_o \] (2.11)

where \( |\epsilon| = 1 \) and \( \Psi_o \) is the vacuum state for which we assume

\[ \mathcal{C} \Psi_o = \Psi_o . \] (2.12)

Then

\[ \mathcal{C} \bar{a} \mathcal{C}^{-1} = \epsilon^* \bar{b} \] (2.13)
\[ \mathcal{C} a \mathcal{C}^{-1} = \epsilon b \] (2.14)

if we also assume

\[ \bar{\mathcal{C}} = \mathcal{C}^{-1} . \] (2.15)

Then

\[ \mathcal{C} (a(\rho)\bar{a}(\rho') - q\bar{a}(\rho')a(\rho)) \mathcal{C}^{-1} = \delta(\rho, \rho') \] (2.16)
implies
\[ b(\rho)b(\rho') - qb(\rho'b(\rho) = \delta(\rho, \rho'). \]

(2.17)

If we try to add the following \(q\)-commutators,
\[ (a(\rho), a(\rho'))_{q} = 0 \]
\[ (b(\rho), b(\rho'))_{q} = 0 \]

(2.18)
(2.19)

it is clear that in these relations the only permitted value of \(q\) is \(\pm 1\). We take \(q = +1\) and \(-1\) for “bose” and “fermi” particles respectively, in (2.18) and (2.19).

3. Expansion of the \(S\)-Matrix.

In expanding the \(S\)-matrix one encounters the causal propagator, \(\Delta_F\). Although the oscillators are now \(q\)-quantized, it is still easy to calculate \(\Delta_F\) in the absence of background fields; for the commutator may be expressed in terms of the \(q\)-commutator as follows:
\[ (a(\rho), \bar{a}(\rho'))_{q} = (a(\rho), \bar{a}(\rho')) + (q - 1)\bar{a}a \]

(3.1)

and therefore the vacuum expectation value is
\[ \langle 0 | (a(\rho), \bar{a}(\rho')) | 0 \rangle = \langle 0 | (a(\rho), \bar{a}(\rho'))_{q} | 0 \rangle = \delta(\rho, \rho') . \]

(3.2)

Likewise the anticommutator is
\[ \{ a(\rho), \bar{a}(\rho') \} = (a(\rho), \bar{a}(\rho')) + (q + 1)\bar{a}a \]

(3.3)

and
\[ \langle 0 | \{ a(\rho), \bar{a}(\rho') \} | 0 \rangle = \langle 0 | (a(\rho), \bar{a}(\rho'))_{q} | 0 \rangle = \delta(\rho, \rho') . \]

(3.4)

Therefore in the absence of a background field, the usual Feynman propagators are unchanged. Under these conditions the remaining effects of the \(q\)-commutators are relatively slight. On the other hand it is easy to show by calculating spacelike field commutators that this theory violates causality.\(^3\)
In a $q$-quantized theory, however, it is perhaps also natural to consider the $q$-time ordered product

\[ T_q (\psi(x) \psi(x')) = \psi(x) \psi(x') \quad t > t' \]

\[ = q \psi(x') \psi(x) \quad t < t' \]

(3.5a)

(3.5b)

This is obviously independent of the previous modification of the canonical commutators. When $q = -1$, (3.5) describes the usual $T$-product for Fermi fields.

One may express the $q$-time ordered product as

\[ T_q (\psi(x) \psi(x')) = \frac{1}{2} \left[ \{ \psi(x), \psi(x') \}_q + \epsilon(t - t') (\psi(x), \psi(x'))_q \right] \]

(3.6)

\[ \epsilon(t - t') = 1 \quad t > t' \]

\[ = -1 \quad t < t' \]

(3.7)

The naturalness of this product suggests that we examine the $q$ modified $S$ matrix:

\[ S^{(q)} = T_q (e^{i \int \mathcal{L}(x) d^4 x}) \]  

(3.8)

In the remainder of this paper we shall discuss some features of this quite different theory.

4. Normal Products.

To decompose (3.8) by Wick’s theorem we begin by describing normal products.

We first define the elementary normal products:

\[ N(1) = 0 \]

\[ N(\psi) = \psi \]  

(4.1)

(4.2)

Then by (2.7)

\[ N (a(\rho) \bar{a}(\rho') - q \bar{a}(\rho') a(\rho)) = N \delta(\rho, \rho') = 0 \]

(4.3)

and
The passage from (4.4) to (4.5) illustrates the general rule that one must move all absorption operators to the right in order to form a normal product. At the same time one picks up a power of $q$.

To get a string of absorption and emission operators into normal form there is no need to permute emission or absorption operators among themselves; but every permutation of $a$ and $\bar{a}$ produces $q$. Therefore instead of the usual parity factor one now has $q^n$.

One also has the usual decomposition into $+$ and $-$ frequency parts since these are associated with emission and absorption operators. Thus

$$N(AB) = N(A^+ + A^-, B^+ + B^-) = A^+ B^+ + A^- B^- + A^+ B^- + qB^+ A^-$$

or

$$N(AB) = AB - (A^-, B^+)_q$$

and

$$AB = N(AB) + \langle AB \rangle$$

since $\langle AB \rangle$, the vacuum expectation value of $AB$, is

$$\langle AB \rangle = \langle 0|A^- B^+|0 \rangle = \langle 0|(A^-, B^+)_q|0 \rangle.$$  

Eq. (4.7) may be rewritten by (4.2)

$$N(A)B = N(AB) + \langle AB \rangle.$$ 

This relation may be generalized by induction to

$$N(A_1 \ldots A_n)B = N(A_1 \ldots A_n B) + \sum_{1 \leq k \leq n} N(A_1 \ldots \overbrace{A_k \ldots A_n} B)$$
where \( A_1 \ldots A_n B \) are individually either absorption or emission operators and

\[
N(A_1 \ldots A_k \ldots A_n B) = \eta \langle A_k B \rangle N(A_1 \ldots A_{k-1} A_{k+1} \ldots A_n).
\]  

(4.11)

Here \( \eta = q^p \) results from the permutation of order between the left and the right sides of \((4.11)\). \( \langle A_k B \rangle \) is called a pairing.

The proof of \((4.10)\) follows along exactly the same lines as the usual proof, since the only difference in the two situations arises from the fact that usually \( q = \pm 1 \) and we finish with \((\pm 1)^p\) while here we finish with \( q^p \).

Wick’s theorem for normal products follows immediately from \((4.10)\) and states that any product may be decomposed into a sum of normal products with all possible pairings (including no pairings), namely:

\[
A_1 \ldots A_n = N(A_1 \ldots A_n) + N(A_1 A_2 \ldots A_n) + \ldots N(A_1 \ldots A_{n-1} A_n) + N(A_1 \ldots A_n) + \ldots .
\]  

(4.12)

This result is established by first assuming that it holds for \( n \), next multiplying by \( A_{n+1} \) and using \((4.10)\) to show that it holds for \( n + 1 \) and therefore by induction for all \( n \).

To expand \((3.8)\) one needs the \( q \)-chronological product \((3.5)\) and the corresponding pairing or vacuum expectation value

\[
\overline{\psi_1(x_1)\psi_2(x_2)} = \langle 0|T_q(\psi_1(x_1)\psi_2(x_2))|0 \rangle .
\]  

(4.13)

Now

\[
T_q(\psi_1(x_1)\psi_2(x_2)) = N(\psi_1(x_1)\psi_2(x_2) + \overline{\psi_1(x_1)\psi_2(x_2)}
\]  

(4.14)

and

\[
\psi_1(x_1)\psi_2(x_2) = \overline{\psi_1(x_1)\psi_2(x_2)} \quad x_1^o > x_2^o
\]  

(4.15a)

\[
= q \psi_2(x_2)\psi_1(x_1) \quad x_2^o > x_1^o
\]  

(4.15b)
Wick’s theorem as applied to $q$-chronological products now states that the $T_q$ product of a system of $n$ linear operators is equal to the sum of their normal products with all possible $q$-chronological pairings. These pairings are the $q$-causal propagators.

5. The $\Delta_q$ and $\Delta_{q,1}$ Functions.

By (3.6) the $q$-causal propagator is

$$\Delta_q F(x - x') = \langle 0 | T_q (\psi(x), \psi(x')) | 0 \rangle \quad (5.1)$$

$$= \frac{1}{2} [\Delta_q^1(x - x') + \epsilon(t - t') \Delta_q (x - x')] \quad (5.2)$$

where

$$\Delta_q^1(x - x') = \langle 0 | \{ \psi(x), \psi(x') \}_q | 0 \rangle \quad (5.3)$$

and

$$\Delta_q (x - x') = \langle 0 | (\psi(x), \psi(x'))_q | 0 \rangle \quad . \quad (5.4)$$

In this section we shall calculate $\Delta_q$ and $\Delta_{q,1}$.

Let us adopt the representation (2.1). Then to compute $\Delta_q$ and $\Delta_{q,1}$ we need

$$\langle 0 | (a(\rho), \bar{a}(\rho'))_q | 0 \rangle = \langle 0 | a(\rho) \bar{a}(\rho') - q \bar{a}(\rho') a(\rho) | 0 \rangle$$

$$= \delta(\rho, \rho') \quad (5.5)$$

$$\langle 0 | (\bar{b}(\rho), b(\rho'))_q | 0 \rangle = -q \delta(\rho, \rho') \quad (5.6)$$

$$\langle 0 | \{ a(\rho), \bar{a}(\rho') \}_q | 0 \rangle = \delta(\rho, \rho') \quad (5.7)$$

$$\langle 0 | \{ b(\rho), b(\rho') \}_q | 0 \rangle = q \delta(\rho, \rho') \quad (5.8)$$

By (2.1)

$$\langle 0 | (\psi_\alpha(x), \bar{\psi}_\beta(x'))_q | 0 \rangle = i (\Theta^{(+)}_{\alpha\beta}(p) \Delta_+(x - x') - q \Theta^{(-)}_{\alpha\beta} \Delta_-(x - x')) \quad (5.9)$$

where
\[ i \Delta_{\pm}(x) = \left( \frac{1}{2\pi} \right)^3 \int \frac{d\vec{p}}{2p_0} e^{\mp i\vec{p} \cdot \vec{x}} \]  
(5.10)

\[ \Theta_{\alpha\beta}^{(+)}(\vec{p}) = \sum_r u_\alpha(\vec{p}, r) \bar{u}_\beta(\vec{p}, r) \]  
(5.11)

\[ \Theta_{\alpha\beta}^{(-)}(\vec{p}) = \sum_r v_\alpha(\vec{p}, r) \bar{v}_\beta(\vec{p}, r) \]  
(5.12)

The \( r \)-sum is over spin states. For scalars there is no spin and therefore \( \Theta_{\pm} = 1 \). For spinors \( u \) and \( v \) refer to positive and negative energy states. (The bar means the Dirac adjoint for spinors and the complex conjugate for scalars.) Then for scalars

\[ \langle 0 | \psi(x), \bar{\psi}(x') \rangle_q | 0 \rangle = i \Delta_q^{-}(x - x') \]  
(5.13)

where

\[ \Delta_q^{-}(x) = \Delta_{+}(x) - q\Delta_{-}(x) . \]  
(5.14)

For spinors

\[ \Theta_{\alpha\beta}^{\pm}(\vec{p}) = \frac{1}{2m}(\pm m + \vec{p})_{\alpha\beta} . \]  
(5.15)

Then

\[ \langle 0 | \psi_\alpha(x), \bar{\psi}_\beta(x') \rangle_q | 0 \rangle = \frac{1}{2m}(m + i\partial)_{\alpha\beta} \Delta_{+}(x) - q\frac{1}{2m}(-m - i\partial)_{\alpha\beta} \Delta_{-}(x) \]  
(5.16)

where

\[ \Delta_q^{+}(x) = \Delta_{+}(x) + q\Delta_{-}(x) . \]  
(5.17)

But \( q = \pm 1 \) for scalars and spinors respectively. Hence

\[ \Delta_1^{-}(x) = \Delta_{-1}^{+}(x) . \]  
(5.18)

Therefore in the limit, \(|q| = 1\), both fields are causal.
For neutral vector fields with no antiparticles we have

\[ A_\alpha(x) = \left( \frac{1}{2\pi} \right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_{r=1}^{3} [a(\vec{p}, r)e^{-ipx} + \bar{a}(\vec{p}, r)e^{ipx}] e_\alpha(\vec{p}, r) \]  

(5.19)

where \( e_\alpha(\vec{p}, r) \) is the polarization vector. Then

\[ \langle 0 | (A_\alpha(x), A_\beta(x'))_q | 0 \rangle = i \Theta_{\alpha\beta}(\vec{p}) \Delta^-(x - x') \]  

(5.20)

since

\[ \langle 0 | (\bar{a}(\rho), a(\rho'))_q | 0 \rangle = -q \delta(\rho, \rho') \]  

(5.21)

Here

\[ \Theta^{(\pm)}_{\alpha\beta}(\vec{p}) = \Theta_{\alpha\beta}(\vec{p}) = \sum_{r=1}^{3} e_\alpha(r, \vec{p}) e_\beta(r, \vec{p}) \]  

(5.22)

or

\[ \Theta_{\alpha\beta} = g_{\alpha\beta} - \frac{p_\alpha p_\beta}{m^2} \]  

(5.23)

and

\[ \langle 0 | (A_\alpha(x), A_\beta(x'))_q | 0 \rangle = i \left( g_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{m^2} \right) \Delta^-(x - x') \]  

(5.24)

for a massive vector, while

\[ \langle 0 | (A_i^{tr}(x), A_j^{tr}(x'))_q | 0 \rangle = i \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) D_q^-(x - x') \]  

(5.25)

for the massless case. Here

\[ D(x) = \Delta(x; m = 0) \]  

(5.26)

and

\[ D_q^- = D^+ - qD^- \]  

(5.27)
Similarly the field anticommutators are found to have the following vacuum expectation values:

a) scalar

\[
\langle 0 | \{ \psi(x), \bar{\psi}(x') \}_q | 0 \rangle = \Delta^+_q (x - x') \tag{5.28}
\]

b) vector

\[
\langle 0 | \{ A^\text{tr}_i(x), A^\text{tr}_j(x') \}_q | 0 \rangle = i \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) D^+_q (x - x') \tag{5.29}
\]

with a similar expression for the massive case.

c) spinor

\[
\langle 0 | \{ \psi_\alpha(x), \bar{\psi}_\beta(x') \}_q | 0 \rangle = \frac{1}{2m} (m + i\theta)_{\alpha\beta} \Delta^-_q (x - x') \ . \tag{5.30}
\]

In the boson examples the transition from commutator to anti-commutator requires a change from \( \Delta^- \) and \( D^- \) to \( \Delta^+ \) and \( D^+ \) while in the fermionic case the change is from \( \Delta^+ \) to \( \Delta^- \).

6. The Causal Propagator.

By (5.2) and the results of the previous section for \( \Delta_q \) and \( \Delta_{q1} \), the \( q \)-causal propagators in the three cases are found to be the following:

a) scalar

\[
\Delta_{F_q}(x - x') = \frac{i}{2} \left[ \Delta^+_q (x - x') + \epsilon(t - t') \Delta^-_q (x - x') \right] \tag{6.1}
\]

b) massive vector

\[
(\Delta_{F_q})_{\alpha\beta}(x - x') = \left( g_{\alpha\beta} - \frac{\partial_{\alpha} \partial_{\beta}}{m^2} \right) \frac{i}{2} \left[ \Delta^+_q (x - x') + \epsilon(t - t') \Delta^-_q (x - x') \right] = \left( g_{\alpha\beta} - \frac{\partial_{\alpha} \partial_{\beta}}{m^2} \right) \Delta_{F_q}(x - x') \tag{6.2}
\]

c) spinor
\[(S_{F_q}(x - x'))_{\alpha\beta} = \frac{1}{2m}(m + i\partial)_{\alpha\beta} \frac{i}{2} [\Delta_{-}^{-}(x - x') + \epsilon(t - t')\Delta_{+}^{+}(x - x')] . \quad (6.3)\]

In the boson examples we have

\[
\Delta_{F_q} = \frac{1}{2} [(\Delta^{(+)}) + q\Delta^{(-)}) + \epsilon(t)(\Delta^{(+)}) - q\Delta^{(-)})] \]
\[
= \frac{1}{2} [(1 + \epsilon(t))\Delta^{(+)}) + q(1 - \epsilon(t))\Delta^{(-)}] \quad (6.4)\]
\[
= \Delta^{(+)}) \quad t > 0 \]
\[
= q\Delta^{(-)} \quad t < 0 \quad (6.5)\]

or, by (6.10),

\[
i \Delta_{F_q} = \left(\frac{1}{2\pi}\right)^3 \int_{H} \frac{dk}{2k_o} e^{-ikx} \quad t > 0 \quad (6.6a)\]
\[
= q \left(\frac{1}{2\pi}\right)^3 \int_{H} \frac{dk}{2k_o} e^{ikx} \quad t < 0 \quad (6.6b)\]

Here \(H\) indicates integration over the mass hyperboloid

\[
k_o = \omega = \sqrt{k^2 + m^2}^{1/2} . \quad (6.7)\]

These expressions are equivalent to

\[
\Delta_{F_q}(x) = \left(\frac{1}{2\pi}\right)^4 \int_{F_q} \frac{1}{k^2 - m^2} e^{-ikx} d^4k \quad (6.8)\]

where the \(F_q\) is the Feynman contour but the left hand pole is of strength \(q\), since (6.8) may be rewritten as follows:

\[
\Delta_{F_q}(x) = \left(\frac{1}{2\pi}\right)^3 \int \frac{dk}{2\pi i} e^{ikx} \frac{1}{2\pi i} \int_{C_+} e^{-ik_o t} \frac{1}{k_o^2 - \omega^2} \, dk_o \quad t > 0 \quad (6.9a)\]
\[
= \left(\frac{1}{2\pi}\right)^3 \int \frac{dk}{2\pi i} e^{ikx} \frac{q}{2\pi i} \int_{C_-} e^{-ik_o t} \frac{1}{k_o^2 - \omega^2} \, dk_o \quad t < 0 \quad (6.9b)\]
Here $C_+$ and $C_-$ are clockwise contours in the complex $k_o$ plane about the two points $\omega = \pm(k^2 + m^2)^{1/2}$ on the real axis. The preceding equation (6.9) is equivalent to (6.6).

An alternative way to write (6.8) is

$$\Delta F_q(x) = \left(\frac{1}{2\pi}\right)^4 \int_F \tilde{\Delta} F_q(k) e^{-ikx} d^4k \quad (6.10)$$

where $F$ is again the Feynman contour and the Fourier transform is

$$\tilde{\Delta} F_q = \frac{1}{2\omega} \left(\frac{1}{k_o - \omega} - \frac{q}{k_o + \omega}\right) = \frac{1}{2} \left(\frac{1 + q}{k^2 - m^2} + \frac{(1 - q)(k_o/\omega)}{k^2 - m^2}\right). \quad (6.11)$$

$\tilde{\Delta} F_q$ is the propagator in momentum space for the scalar field.

The corresponding propagators in momentum space for the vector and spinor may be obtained from (6.2) and (6.3). Except for spin (6.1) and (6.2) are the same but (6.3) differs because there the $\Delta^+_q$ and $\Delta^-_q$ functions are interchanged. Therefore the spinor propagator in momentum space is obtained from (6.12) and (6.3) by replacing $q$ by $-q$ as follows:

$$\frac{1}{2m}(m + \not{p}) \frac{1}{2} \left(\frac{1 - q}{p^2 - m^2} + \frac{(1 + q)(p_o/\omega)}{p^2 - m^2}\right). \quad (6.13)$$

In the limit $|q| = 1$ the scalar propagator becomes

$$\frac{1}{k^2 - m^2} \quad (6.14)$$

while the spinor propagator becomes

$$\frac{1}{2m}(m + \not{p}) \frac{1}{p^2 - m^2}. \quad (6.15)$$

The $q$-modified propagators take the usual form not only when $|q| = 1$ but also when $|q| \neq 1$ and $k$ lies on the mass hyperboloid ($k_o = \omega$). Therefore, if $|q| \neq 1$, the dependence on $q$ becomes significant only for internal lines. For the photon field we may write

$$D_{\mu\lambda}(x) = g_{\mu\lambda} \left(\frac{1}{2\pi}\right)^4 \left(\frac{1 + q}{2}\right) \int e^{-ikx} \left(\frac{1}{k^2 - m^2} + \frac{1 - q}{k^2 - m^2}\right) d^4k \quad (6.16)$$
and for the spinor field

\[ S_{\alpha\beta}(x) = (i\partial + m)_{\alpha\beta} \left( \frac{1}{2\pi} \right)^4 \int e^{-ipx} \frac{1}{2} \left( \frac{1 - q}{p^2 - m^2} + \frac{(1 + q)(p_\rho/\omega)}{p^2 - m^2} \right) d^4p . \]  

(6.17)

7. Tests.

It is of course possible to examine the effects of these altered propagators. We consider two examples from QED:

a) Electron-Electron Scattering.

Let electrons in states \((A, B)\) scatter into states \((C, D)\):

\[ A + B \rightarrow C + D . \]

To lowest order the matrix element in configuration space is

\[
\langle CD|S_q|AB\rangle = \frac{1}{2} \left( \frac{e}{\hbar c} \right)^2 \times \int \int d^4x_1 d^4x_2 \langle CD|N(J_\mu(x_1)J^\lambda(x_2))|AB\rangle \langle 0|T_qA_\mu(x_1)A_\lambda(x_2)|0\rangle
\]

(7.1)

where

\[
\langle 0|T_q(A_\mu(x_1)A_\lambda(x_2))|0\rangle = D^{q}_{\mu\lambda}(x_1, x_2)
\]

(7.2)

and

\[ J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \]

(7.3)

with

\[
\psi(x) = \sum(a(\rho)u(\rho, x) + \bar{b}(\rho)v(\rho, x)) .
\]

(7.4)

One finds

\[
\langle CD|N(J_\mu(x_1)J^\lambda(x_2))|AB\rangle = q(1 + qP_{AB})(1 + qP_{CD})\langle D|J_\mu(x_1)|B\rangle\langle C|J^\lambda(x_2)|A\rangle
\]

(7.5)
where

\[ \langle D|J^\mu(x)|B \rangle = \bar{u}_D(x)\gamma^\mu u_B(x) . \]  

(7.6)

Then

\[ \langle CD|S_q|AB \rangle = -\frac{i}{2} \left( \frac{e}{\hbar c} \right)^2 q \times \int \int d^4x_1 d^4x_2 (1 + q P_{AB}) (1 + q P_{CD}) (D|J^\mu(x_1)|B) \times (C|J^\lambda(x_2)|A) D_{\mu\lambda}^q(x_1, x_2) . \]  

(7.7)

If \( q = -1 \) for the initial and final particles

\[ D_{\mu\lambda}^{(1)}(x_1, x_2) = D_{\mu\lambda}^{(1)}(|x_1 - x_2|) = D_{\mu\lambda}^{(1)}(x_2, x_1) . \]  

(7.8)

Then

\[ \langle CD|S_q|AB \rangle = -i q \left( \frac{e}{\hbar c} \right)^2 q \times \int \int d^4x_1 d^4x_2 (1 - P_{CD}) (D|J^\mu(x_1)|B) (C|J^\lambda(x_2)|A) D_{\mu\lambda}^q(x_1 - x_2) . \]  

(7.9)

If \( q \neq 1 \),

\[ D_{\mu\lambda}^q(x_1, x_2) \neq D_{\mu\lambda}^q(x_2, x_1) . \]  

(7.10)

As a consequence there are four diagrams since the incoming as well as the outgoing lines are crossed in order to describe the \( q \)-antisymmetrization of the initial as well as the final state.

Let us consider just the contributions of the final state. In momentum space this is

\[ \langle CD|S^2|AB \rangle \sim q \left[ \langle P_C|J^\mu(0)|P_A \rangle \Delta_{CA} \langle P_D|J^\mu(0)|P_B \rangle - \langle P_D|J^\mu(0)|P_A \rangle \Delta_{DA} \langle P_C|J^\mu(0)|P_B \rangle \right] \]  

(7.11)

where, by (6.16),
\[ \Delta_{CA} = \frac{1}{|P_C - P_A|^2} F_{CA} \] (7.12)
\[ \Delta_{DA} = \frac{1}{|P_B - P_A|^2} F_{DA} \] (7.13)

and

\[ F_{CA} = \frac{1 + q}{2} \left( 1 + \frac{1 - q}{1 + q} \frac{E_A - E_C}{|\vec{P}_A - \vec{P}_C|} \right) \] (7.14)
\[ F_{DA} = \frac{1 + q}{2} \left( 1 + \frac{1 - q}{1 + q} \frac{E_A - E_D}{|\vec{P}_A - \vec{P}_D|} \right) \] (7.15)

If \( q = 1 \), for the intermediate boson the correction factors \( F_{CA} \) and \( F_{DA} \) are not present and one gets the usual formula for Moller scattering. If \( q \neq 1 \), these factors depend on frame. In the center of mass system there is no effect.

b) Electron Positron Annihilation.

We consider

\[ e^+ + e^- \rightarrow \gamma_1 + \gamma_2 \] (7.16)

The conservation of 4-momentum now reads

\[ p_+ + p_- = k_1 + k_2 \] (7.17)

In the previous example there was an internal photon line. In this example there is an internal electron line. By (6.17) the usual matrix element for \( e^+e^- \) annihilation is modified by the same factors that appeared in the modified Moller formula, namely:

\[ F_{e_+k_1} = \left( \frac{1 - q}{2} \right) \left( 1 + \frac{1 + q}{1 - q} \frac{E_+ - \epsilon_1}{|\vec{p}_+ - \vec{k}_1|} \right) \] (7.18)
\[ F_{e_+k_2} = \left( \frac{1 - q}{2} \right) \left( 1 + \frac{1 + q}{1 - q} \frac{E_+ - \epsilon_2}{|\vec{p}_+ - \vec{k}_2|} \right) \] (7.19)

In the center of mass system the numerators again vanish but again the result is dependent on frame. Therefore, although we have assumed relativistic kinematics for the free particles,
there remains a non-relativistic frame dependence in the final result, coming from the form of the internal propagators. That is, the $q$-dynamics explicitly violates relativity.

8. Remarks.

The preceding work has been based on two changes in the postulational basis of QED: first the use of $q$-commutators at equal times and second, the use of $q$-causal propagators. At weak field strengths or in the absence of background fields the use of $q$-commutators at equal times ordinarily makes very little difference if $q$ is sufficiently close to unity; but the theory is still non-causal. On the other hand, the use of $q$-causal propagators leads to non-relativistic results and thus merges with an alternative approach in which the Poincaré group is explicitly deformed. In both cases deformation of dynamical law is not independent of deformation of the geometry.

References.

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