On the X-ray transform of planar symmetric 2-tensors

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ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC 2-TENSORS

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ABSTRACT. In this paper we study the attenuated X-ray transform of 2-tensors supported in strictly convex bounded subsets in the Euclidean plane. We characterize its range and reconstruct all possible 2-tensors yielding identical X-ray data. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

This paper concerns the range characterization of the attenuated X-ray transform of symmetric 2-tensors in the plane. Range characterization of the non-attenuated X-ray transform of functions (0-tensors) in the Euclidean space has been long known [10, 11, 19], whereas in the case of a constant attenuation some range conditions can be inferred from [17, 1, 2]. For a varying attenuation the two dimensional case has been particularly interesting with inversion formulas requiring new analytical tools: the theory of A-analytic maps originally employed in [3], and ideas from inverse scattering in [24]. Constraints on the range for the two dimensional X-ray transform of functions were given in [25, 4], and a range characterization based on Bukhgeim’s theory of A-analytic maps was given in [30].

Inversion of the X-ray transform of higher order tensors has been formulated directly in the setting of Riemannian manifolds with boundary [32]. The case of 2-tensors appears in the linearization of the boundary rigidity problem. It is easy to see that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors. For two dimensional simple manifolds with boundary, injectivity with in the solenoidal tensor fields has been established fairly recent: in the non-attenuated case for 0- and 1-tensors we mention the breakthrough result in [29], and in the attenuated case in [34]; see also [13] for a more general weighted transform. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher can be found in [27], with a range characterization in [28]. In the Euclidean
case we mention an earlier inversion of the attenuated X-ray transform of solenoidal tensors in [16]; however this work does not address range characterization.

Different from the recent characterization in terms of the scattering relation in [28], in this paper the range conditions are in terms of the Hilbert-transform for $A$-analytic maps introduced in [30, 31]. Our characterization can be understood as an explicit description of the scattering relation in [26, 27, 28] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible 2-tensors yielding identical X-ray data; see (30) for the non-attenuated case and (82) for the attenuated case.

For a real symmetric 2-tensor $F \in L^1(\mathbb{R}^2; \mathbb{R}^{2 \times 2})$,

$$ F(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2, $$

and a real valued function $a \in L^1(\mathbb{R}^2)$, the $a$-attenuated X-ray transform of $F$ is defined by

$$ X_a F(x, \theta) := \int_{-\tau(x, \theta)}^{\tau(x, \theta)} \langle F(x + t\theta), \theta, \theta \rangle \exp \left\{ -\int_t^{\tau(x, \theta)} a(x + s\theta) ds \right\} dt, $$

where $\theta$ is a direction in the unit sphere $S^1$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$. For the non attenuated case $a \equiv 0$ we use the notation $XF$.

In this paper, we consider $F$ be defined on a strictly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary $\Gamma$; further regularity and the order of vanishing will be specified in the theorems. In the attenuated case we assume $a > 0$ in $\overline{\Omega}$.

For any $(x, \theta) \in \overline{\Omega} \times S^1$ let $\tau(x, \theta)$ be length of the chord in the direction of $\theta$ passing through $x$. Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$ \Gamma_\pm := \{(x, \theta) \in \Gamma \times S^1 : \pm \theta \cdot n(x) > 0\}, $$

and the variety

$$ \Gamma_0 := \{(x, \theta) \in \Gamma \times S^1 : \theta \cdot n(x) = 0\}, $$

where $n(x)$ denotes outer normal.

The $a$-attenuated X-ray transform of $F$ is realized as a function on $\Gamma_+$ by

$$ X_a F(x, \theta) = \int_{-\tau(x, \theta)}^{0} \langle F(x + t\theta), \theta, \theta \rangle e^{-\int_t^{\tau(x, \theta)} a(x + s\theta) ds} dt, \quad (x, \theta) \in \Gamma_+. $$

We approach the range characterization through its connection with the transport model as follows: The boundary value problem

$$ \theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \langle F(x)\theta, \theta \rangle \quad (x, \theta) \in \Omega \times S^1, $$

$$ u|_{\Gamma_-} = 0 $$
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Figure 1. Definition of $\Gamma_{\pm}$

has a unique solution in $\Omega \times S^1$ and

$$u|_{\Gamma_+}(x, \theta) = X_\theta F(x, \theta), \quad (x, \theta) \in \Gamma_+.$$  \hspace{1cm} (8)

The X-ray transform of 2-tensors occurs in the linearization of the boundary rigidity problem [32]: For $\epsilon > 0$ small, let

$$g^\epsilon(x) := I + \epsilon F(x) + o(\epsilon), \quad x \in \Omega,$$

be a family of metrics perturbations from the Euclidean, where $I$ is the identity matrix and $F$ is as in (1). For an arbitrary pair of boundary points $x, y \in \Gamma$ let $d^\epsilon(x, y)$ denote their distance in the metric $g^\epsilon$. The boundary rigidity problem asks for the recovery of the metric $g^\epsilon$ from knowledge of $d^\epsilon(x, y)$ for all $x, y \in \Gamma$. In the linearized case one seeks to recover $F(x)$ from $\frac{d}{d\epsilon}|_{\epsilon=0} d^2_\epsilon(x, y)$. Taking into account the length minimizing property of geodesic one can show that

$$\frac{1}{|x-y|} \frac{d}{d\epsilon}|_{\epsilon=0} d^2_\epsilon(x, y) = \int_{-|x-y|}^0 \langle F(x + t\theta)\theta, \theta \rangle dt = X F(x, \theta),$$

where $\theta := \frac{x-y}{|x-y|} \in S^1$.

2. Preliminaries

In this section we briefly introduce the properties of Bukhgeim’s $A$-analytic maps [7] needed later.

For $z = x_1 + ix_2$, we consider the Cauchy-Riemann operators

$$\overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2.$$  \hspace{1cm} (9)
Let $l_\infty, l_1$ be the space of bounded (, respectively summable) sequences, $L : l_\infty \to l_\infty$ be the left shift

$$L \langle u_{-1}, u_{-2}, \ldots \rangle = \langle u_{-2}, u_{-3}, u_{-4}, \ldots \rangle.$$

**Definition 2.1.** A sequence valued map

$$z \mapsto u(z) := \langle u_{-1}(z), u_{-2}(z), u_{-3}(z), \ldots \rangle$$

is called $L$-analytic, if $u \in C(\bar{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ and

$$\overline{\partial}u(z) + L \partial u(z) = 0, \quad z \in \bar{\Omega}.$$  

For $0 < \alpha < 1$ and $k = 1, 2$, we recall the Banach spaces in [30]:

$$l^{1,k}_\infty(\Gamma) := \left\{ u = \langle u_{-1}, u_{-2}, \ldots \rangle : \sup_{\zeta \in \Gamma} \sum_{j=1}^{\infty} j^k |u_{-j}(\zeta)| < \infty \right\},$$

$$C^\alpha(\Gamma; l_1) := \left\{ u : \sup_{\xi \in \Gamma} \|u(\xi)\|_{l_1} + \sup_{\xi, \eta \in \Gamma} \frac{\|u(\xi) - u(\eta)\|_{l_1}}{|\xi - \eta|^\alpha} < \infty \right\}.$$

By replacing $\Gamma$ with $\bar{\Omega}$ and $l_1$ with $l_\infty$ in (12) we similarly define $C^\alpha(\bar{\Omega}; l_1)$, respectively, $C^\alpha(\bar{\Omega}; l_\infty)$.

At the heart of the theory of $A$-analytic maps lies a Cauchy-like integral formula introduced by Bukhgeim in [7]. The explicit variant (13) appeared first in Finch [8]. The formula below is restated in terms of $L$-analytic maps as in [31].

**Theorem 2.1.** [31, Theorem 2.1] For some $g = \langle g_{-1}, g_{-2}, g_{-3}, \ldots \rangle \in l^{1,1}_\infty(\Gamma) \cap C^\alpha(\Gamma; l_1)$ define the Bukhgeim-Cauchy operator $B$ acting on $g$,

$$\Omega \ni z \mapsto \langle (Bg)_{-1}(z), (Bg)_{-2}(z), (Bg)_{-3}(z), \ldots \rangle,$$

by

$$B^g_{-n}(z) := \frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{\Gamma} \frac{g_{-n-j}(\zeta)(\zeta - z)^j}{(\zeta - z)^{j+1}} d\zeta$$

$$- \frac{1}{2\pi i} \sum_{j=1}^{\infty} \int_{\Gamma} \frac{g_{-n-j}(\zeta)(\zeta - z)^{j-1}}{(\zeta - z)^j} d\zeta, \quad n = 1, 2, 3, \ldots$$

Then $B^g \in C^{1,\alpha}(\Omega; l_\infty) \cap C(\bar{\Omega}; l_\infty)$ and it is also $L$-analytic.

For our purposes further regularity in $B^g$ will be required. Such smoothness is obtained by increasing the assumptions on the rate of decay of the
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terms in \( g \) as explicit below. For \( 0 < \alpha < 1 \), let us recall the Banach space \( Y_\alpha \) in [30]:

\[
Y_\alpha = \left\{ g \in l^{1,2}_\infty(\Gamma) : \sup_{\xi,\mu \in \Gamma, \xi \neq \mu} \sum_{j=1}^{\infty} j |g_{-j}(\xi) - g_{-j}(\mu)|^\alpha |\xi - \mu| \right\}.
\]

**Proposition 2.1.** [31, Proposition 2.1] If \( g \in Y_\alpha, \alpha > 1/2 \), then

\[
Bg \in C^{1,\alpha}(\Omega; l^1_{1}) \cap C^{\alpha}(\Omega; l^1_{1}) \cap C^{2}(\Omega; l^\infty).
\]

The Hilbert transform associated with boundary of \( \mathcal{L} \)-analytic maps is defined below.

**Definition 2.2.** For \( g = (g_{-1}, g_{-2}, g_{-3}, \ldots) \in l^{1,1}_\infty(\Gamma) \cap C^{\alpha}(\Gamma; l^1_{1}) \), we define the Hilbert transform \( Hg \) componentwise for \( n \geq 1 \) by

\[
(Hg)_n(\xi) = \frac{1}{\pi} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - \xi} d\zeta
\]

\[
+ \frac{1}{\pi} \int_{\Gamma} \left( \frac{d\zeta}{\zeta - \xi} - \frac{d\zeta}{\zeta - \xi} \right) \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left( \frac{\zeta - \xi}{\zeta - \xi} \right)^j, \xi \in \Gamma.
\]

The following result justifies the name of the transform \( \mathcal{H} \). For its proof we refer to [30, Theorem 3.2].

**Theorem 2.2.** For \( 0 < \alpha < 1 \), let \( g \in l^{1,1}_\infty(\Gamma) \cap C^{\alpha}(\Gamma; l^1_{1}) \). For \( g \) to be boundary value of an \( \mathcal{L} \)-analytic function it is necessary and sufficient that

\[
(I + i\mathcal{H})g = 0,
\]

where \( \mathcal{H} \) is as in (16).

**3. THE NON-ATTENUATED CASE**

In this section we assume \( a \equiv 0 \). We establish necessary and sufficient conditions for a sufficiently smooth function on \( \Gamma \times S^1 \) to be the X-ray data of some sufficiently smooth real valued symmetric 2-tensor \( F \). For \( \theta = (\cos \varphi, \sin \varphi) \in S^1 \), a calculation shows that

\[
\langle F(x)\theta, \theta \rangle = f_0(x) + f_2(x)e^{2i\varphi} + f_2(x)e^{-2i\varphi},
\]

where

\[
f_0(x) = \frac{f_{11}(x) + f_{22}(x)}{2}, \quad \text{and} \quad f_2(x) = \frac{f_{11}(x) - f_{22}(x)}{4} + \frac{f_{12}(x)}{2}.
\]

The transport equation in (6) becomes

\[
\theta \cdot \nabla u(x, \theta) = f_0(x) + f_2(x)e^{2i\varphi} + f_2(x)e^{-2i\varphi}, \quad x \in \Omega.
\]
For \( z = x_1 + ix_2 \in \Omega \), we consider the Fourier expansions of \( u(z, \cdot) \) in the angular variable \( \theta = (\cos \varphi, \sin \varphi) \):

\[
u(z, \theta) = \sum_{-\infty}^{\infty} u_n(z)e^{in\varphi}.
\]

Since \( u \) is real valued its Fourier modes occur in conjugates,

\[
u_{-n}(z) = \overline{u_n(z)}, \quad n \geq 0, \quad z \in \Omega.
\]

With the Cauchy-Riemann operators defined in (9) the advection operator becomes

\[
\theta \cdot \nabla = e^{-i\varphi} \overline{\partial} + e^{i\varphi} \partial.
\]

Provided appropriate convergence of the series (given by smoothness in the angular variable) we see that if \( u \) solves (20) then its Fourier modes solve the system

\[
\begin{align*}
\overline{\partial}u_1(z) + \partial u_{-1}(z) &= f_0(z), \\
\overline{\partial}u_{-1}(z) + \partial u_{-3}(z) &= f_2(z), \\
\overline{\partial}u_{2n}(z) + \partial u_{2n-2}(z) &= 0, \quad n \leq 0, \\
\overline{\partial}u_{2n-1}(z) + \partial u_{2n-3}(z) &= 0, \quad n \leq -1,
\end{align*}
\]

The range characterization is given in terms of the trace

\[
g := u|_{\Gamma \times S^1} = \left\{ \begin{array}{ll}
XF(x, \theta), & (x, \theta) \in \Gamma_+, \\
0, & (x, \theta) \in \Gamma_\pm \cup \Gamma_0.
\end{array} \right.
\]

More precisely, in terms of its Fourier modes in the angular variables:

\[
g(\zeta, \theta) = \sum_{-\infty}^{\infty} g_n(\zeta)e^{in\varphi}, \quad \zeta \in \Gamma.
\]

Since the trace \( g \) is also real valued, its Fourier modes will satisfy

\[
g_{-n}(\zeta) = \overline{g_n(\zeta)}, \quad n \geq 0, \quad \zeta \in \Gamma.
\]

From the negative even modes, we built the sequence

\[
g^{\text{even}} := \langle g_0, g_{-2}, g_{-4}, \ldots \rangle.
\]

From the negative odd modes starting from mode \(-3\), we built the sequence

\[
g^{\text{odd}} := \langle g_{-3}, g_{-5}, g_{-7}, \ldots \rangle.
\]

Next we characterize the data \( g \) in terms of the Hilbert Transform \( \mathcal{H} \) in (16). We will construct simultaneously the right hand side of the transport equation (20) and the solution \( u \) whose trace matches the boundary data \( g \). Construction of \( u \) is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation.
Except from negative one mode \( u_{-1} \) all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (13) using boundary data. Other then having the trace \( g_{-1} \) on the boundary \( u_{-1} \) is unconstrained. It is chosen arbitrarily from the class of functions

\[
\psi \in C^1(\overline{\Omega}; \mathbb{C}) : \psi|_{\Gamma} = g_{-1}
\]

(30)

**Theorem 3.1** (Range characterization in the non-attenuated case). Let \( \alpha > 1/2 \).

(i) Let \( F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2\times 2}) \). For \( g := \left\{ \begin{array}{ll} XF(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_0 \end{array} \right. \), consider the corresponding sequences \( g^{\text{even}} \) as in (28) and \( g^{\text{odd}} \) as in (29). Then \( g^{\text{even}}, g^{\text{odd}} \in l^{1,1}_\infty(\Gamma) \cap C^\alpha(\Gamma; l_1) \) satisfy

\[
[I + i\mathcal{H}]g^{\text{even}} = 0,
\]

(31)

\[
[I + i\mathcal{H}]g^{\text{odd}} = 0,
\]

(32)

where the operator \( \mathcal{H} \) is the Hilbert transform in (16).

(ii) Let \( g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1)) \) be real valued with \( g|_{\Gamma_\infty} = 0 \). If the corresponding sequence \( g^{\text{even}}, g^{\text{odd}} \in Y_\alpha \) satisfies (31) and (32), then there exists a real valued symmetric 2-tensor \( F \in C(\Omega; \mathbb{R}^{2\times 2}) \), such that \( g|_{\Gamma_+} = XF \). Moreover for each \( \psi \in \Psi_g \) in (30), there is a unique real valued symmetric 2-tensor \( F_\psi \) such that \( g|_{\Gamma_+} = XF_\psi \).

**Proof.** (i) **Necessity**

Let \( F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2\times 2}) \). Since \( F \) is compactly supported inside \( \Omega \), for any point at the boundary there is a cone of lines which do not meet the support. Thus \( g \equiv 0 \) in the neighborhood of the variety \( \Gamma_0 \) which yields \( g \in C^{1,\alpha}(\Gamma \times S^1) \). Moreover, \( g \) is the trace on \( \Gamma \times S^1 \) of a solution \( u \in C^{1,\alpha}(\overline{\Omega} \times S^1) \) of the transport equation (20). By [30, Proposition 4.1] \( g^{\text{even}}, g^{\text{odd}} \in l^{1,1}_\infty(\Gamma) \cap C^\alpha(\Gamma; l_1) \).

If \( u \) solves (20) then its Fourier modes satisfy (21), (22), (23) and (24). Since the negative even Fourier modes \( u_{2n} \) of \( u \) satisfies the system (23) for \( n \leq 0 \), then

\[
z \mapsto u^{\text{even}}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \cdots \rangle
\]

is \( \mathcal{L} \)-analytic in \( \Omega \) and the necessity part in Theorem 2.2 yields (31).

The equation (24) for negative odd Fourier modes \( u_{2n-1} \) starting from mode \(-3\) yield that the sequence valued map

\[
z \mapsto u^{\text{odd}}(z) := \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \cdots \rangle
\]

is \( \mathcal{L} \)-analytic in \( \Omega \) and the necessity part in Theorem 2.2 yields (32).

(ii) **Sufficiency**
To prove the sufficiency we will construct a real valued symmetric 2-tensor $F$ in $\Omega$ and a real valued function $u \in C^1(\Omega \times S^1) \cap C(\overline{\Omega} \times S^1)$ such that $u|_{\Gamma \times S^1} = g$ and $u$ solves (20) in $\Omega$. The construction of such $u$ is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1: The construction of negative even modes $u_{2n}$ for $n \leq 0$.**

Let $g \in C^{\alpha}(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$ be real valued with $g|_{\Gamma_\cup \Gamma_0} = 0$. Let the corresponding sequences $g^{even}$ and $g^{odd}$ satisfying (31) and (32). By [30, Proposition 4.1(ii)] $g^{even}, g^{odd} \in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to construct the negative even Fourier modes:

$$
\langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \ldots \rangle := B_{g^{even}}(z), \quad z \in \Omega.
$$

By Theorem 2.1, the sequence valued map

$$
z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), \ldots \rangle,
$$

is $\mathcal{C}$-analytic in $\Omega$, thus the equations

$$
\overline{\partial}u_{-2k} + \partial u_{-2k-2} = 0,
$$

are satisfied for all $k \geq 0$. Moreover, the hypothesis (31) and the sufficiency part of Theorem 2.2 yields that they extend continuously to $\Gamma$ and

$$
u_{-2k}|_{\Gamma} = g_{-2k}, \quad k \geq 0.
$$

**Step 2: The construction of positive even modes $u_{2n}$ for $n \geq 1$.**

All of the positive even Fourier modes are constructed by conjugation:

$$u_{2k} := \overline{u_{-2k}}, \quad k \geq 1.
$$

By conjugating (34) we note that the positive even Fourier modes also satisfy

$$
\overline{\partial}u_{2k+2} + \partial u_{2k} = 0, \quad k \geq 0.
$$

Moreover, they extend continuously to $\Gamma$ and

$$
u_{2k}|_{\Gamma} = \overline{u_{-2k}}|_{\Gamma} = \overline{g_{-2k}} = g_{2k}, \quad k \geq 1.
$$

Thus, as a summary, we have shown that

$$
\overline{\partial}u_{2k} + \partial u_{2k-2} = 0, \quad \forall k \in \mathbb{Z},
$$

$$u_{2k}|_{\Gamma} = g_{2k}, \quad \forall k \in \mathbb{Z}.
$$

**Step 3: The construction of modes $u_{-1}$ and $u_1$.**

Let $\psi \in \Psi_g$ as in (30). We define

$$
u_{-1} := \psi, \quad \text{and} \quad u_1 := \overline{\psi}.
$$

Since $g$ is real valued, we have

$$u_1|_{\Gamma} = \overline{g_{-1}} = g_1.$$
Step 4: The construction of negative odd modes $u_{2n-1}$ for $n \leq -1$.

Use the Bukhgeim-Cauchy Integral formula (13) to construct the other odd negative Fourier modes:

$$\langle u_{-3}(z), u_{-5}(z), \cdots \rangle := \mathcal{B}g^{\text{odd}}(z), \quad z \in \Omega.$$  \hspace{1cm} (43)

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \ldots \rangle,$$

is $\mathcal{L}$-analytic in $\Omega$, thus the equations

$$\overline{\partial}u_{2k-1} + \partial u_{2k-3} = 0,$$

are satisfied for all $k \leq -1$. Moreover, the hypothesis (32) and the sufficiency part of Theorem 2.2 yields that they extend continuously to $\Gamma$ and

$$u_{2k-1}|_{\Gamma} = g_{2k-1}, \quad \forall k \leq -1.$$  \hspace{1cm} (45)

Step 5: The construction of positive odd modes $u_{2n+1}$ for $n \geq 1$.

All of the positive odd Fourier modes are constructed by conjugation:

$$u_{2k+3} := \overline{u_{-(2k+3)}}, \quad k \geq 0.$$  \hspace{1cm} (46)

By conjugating (44) we note that the positive odd Fourier modes also satisfy

$$\overline{\partial}u_{2k+3} + \partial u_{2k+1} = 0, \quad \forall k \geq 1.$$  \hspace{1cm} (47)

Moreover, they extend continuously to $\Gamma$ and

$$u_{2k+3}|_{\Gamma} = \overline{u_{-(2k+3)}}|_{\Gamma} = \overline{g_{-(2k+3)}} = g_{2k+3}, \quad k \geq 0.$$  \hspace{1cm} (48)

Step 6: The construction of the tensor field $F_\psi$ whose X-ray data is $g$.

We define the 2-tensor field

$$F_\psi := \left( \begin{array}{cc} f_0 + 2\Re f_2 & 2\Im f_2 \\ 2\Im f_2 & f_0 - 2\Re f_2 \end{array} \right),$$

where

$$f_0 = 2\Re(\partial \psi), \quad \text{and} \quad f_2 = \overline{\partial \psi} + \partial u_{-3}.$$  \hspace{1cm} (50)

In order to show $g|_{\Gamma_+} = X F_\psi$ with $F_\psi$ as in (49), we define the real valued function $u$ via its Fourier modes

$$u(z, \theta) := u_0(z) + \psi(z)e^{-i\varphi} + \overline{\psi(z)}e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z)e^{in\varphi},$$

and check that it has the trace $g$ on $\Gamma$ and satisfies the transport equation (20).

Since $g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$, we use [30, Corollary 4.1] and [30, Proposition 4.1 (iii)] to conclude that $u$ defined in (51) belongs to
$C^{1,\alpha}(\Omega \times S^1) \cap C^{\alpha}(\overline{\Omega} \times S^1)$. In particular $u(\cdot, \theta)$ for $\theta = (\cos \varphi, \sin \varphi)$ extends to the boundary and its trace satisfies

$$u(\cdot, \theta)|_{\Gamma} = \left( u_0 + \psi e^{-i\varphi} + \overline{\psi} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right)|_{\Gamma},$$

where in the third equality above we used (40), (45), (48), (42) and definition of $\psi \in \Psi_g$ in (30).

Since $u \in C^{1,\alpha}(\Omega \times S^1) \cap C^{\alpha}(\overline{\Omega} \times S^1)$, the following calculation is also justified:

$$\theta \cdot \nabla u = e^{-2i\varphi}(\overline{\partial} u_0 + \partial u_{-3}) + e^{2i\varphi}(\partial \overline{u}_0 + \overline{\partial} u_3) + \overline{\partial} \psi + \partial \psi + e^{2i\varphi} \overline{\partial} \psi$$

$$+ e^{-2i\varphi} (\partial u_0 + \overline{\partial} u_{-2}) + e^{2i\varphi} (\partial \overline{u}_0 + \partial u_2)$$

$$+ \sum_{n=1}^{\infty} (\partial u_{-n} + \overline{\partial} u_{n-2}) e^{-i(n+1)\varphi} + \sum_{n=1}^{\infty} (\overline{\partial} u_{n+2} + \partial u_n) e^{i(n+1)\varphi}.$$
only in terms of \( g \) on \( \Gamma_+ \), where \( g := u|_{\Gamma \times S^1} \). More precisely, let \( \tilde{u} \) be the solution of the boundary value problem

\[
\theta \cdot \nabla \tilde{u}(x, \theta) = \langle F(x) \theta, \theta \rangle, \quad x \in \Omega,
\]

\[
\tilde{u}(z, \theta) = -\frac{1}{2} g|_{\Gamma_+}(z, -\theta), \quad (z, \theta) \in \Gamma_-.
\]

Then one can see that

\[
\tilde{u}|_{\Gamma_+} = \frac{1}{2} g|_{\Gamma_+},
\]

and therefore \( \tilde{u}|_{\Gamma \times S^1} \) is an odd function of \( \theta \). This shows that we can work with the following odd extension:

\[
\tilde{g}(z, \theta) := \frac{g(z, \theta) - g(z, -\theta)}{2}, \quad (z, \theta) \in (\Gamma \times S^1) \setminus \Gamma_0,
\]

and \( \tilde{g} = 0 \) on \( \Gamma_0 \). Note that \( \tilde{g} \) is the trace of \( \tilde{u} \) on \( \Gamma \times S^1 \).

The range characterization can be given now in terms of the odd Fourier modes of \( \tilde{g} \), namely in terms of

\[
\tilde{g} := \langle \tilde{g}_{-3}, \tilde{g}_{-5}, \tilde{g}_{-7}, \ldots \rangle.
\]

**Corollary 3.1.** Let \( \alpha > 1/2 \).

(i) Let \( F \in C^{1,\alpha}_{0}(\Omega; \mathbb{R}^{2 \times 2}) \), \( \tilde{u} \) be the solution of \( (52) \) and \( \tilde{g} \) as in \( (55) \). Then \( \tilde{g} \in L^{1,1}_{\infty}(\Gamma) \cap C^{\alpha}(\Gamma; l_1) \) and

\[
[I + i\mathcal{H}] \tilde{g} = 0,
\]

where the operator \( \mathcal{H} \) is the Hilbert transform in \( (16) \).

(ii) Let \( g \in C^{\alpha}(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1)) \) be real valued with \( g|_{\Gamma_+ \cup \Gamma_0} = 0 \). Let \( \tilde{g} \) be its odd extension as in \( (54) \) and the corresponding \( \tilde{g} \) as in \( (55) \). If \( \tilde{g} \) satisfies \( (56) \), then there exists a real valued symmetric \( 2 \)-tensor \( F \in C(\Omega; \mathbb{R}^{2 \times 2}) \), such that \( g|_{\Gamma_+} = X^{\alpha}F \). Moreover for each \( \psi \in \Psi_g \) in \( (30) \), there is a unique real valued symmetric \( 2 \)-tensor \( F_{\psi} \) such that \( g|_{\Gamma_+} = X^{\alpha}F_{\psi} \).

### 4. The Attenuated Case

In this section we assume an attenuation \( a \in C^{2,\alpha}(\overline{\Omega}), \alpha > 1/2 \) with

\[
\min_{\overline{\Omega}} a > 0.
\]

We establish necessary and sufficient conditions for a sufficiently smooth function \( g \) on \( \Gamma \times S^1 \) to be the attenuated X-ray data, with attenuation \( a \), of some sufficiently smooth real symmetric \( 2 \)-tensor, i.e. \( g \) is the trace on \( \Gamma \times S^1 \) of some solution \( u \) of

\[
\theta \cdot \nabla u(x, \theta) + a(x) u(x, \theta) = \langle F(x) \theta, \theta \rangle, \quad (x, \theta) \in \Gamma \times S^1.
\]
Different from 1-tensor case in [31] (where there is uniqueness), in the 2-tensor case there is non-uniqueness: see the class of function in (82).

As in [30] we start by the reduction to the non-attenuated case via the special integrating factor $e^{-h}$, where $h$ is explicitly defined in terms of $a$ by

$$h(z, \theta) := Da(z, \theta) - \frac{1}{2} (I - iH) Ra(z \cdot \theta^\perp, \theta),$$

where $\theta^\perp$ is orthogonal to $\theta$, $Da(z, \theta) = \int_0^\infty a(z + t\theta)dt$ is the divergence beam transform of the attenuation $a$, $Ra(s, \theta) = \int_{-\infty}^\infty a(s\theta^\perp + t\theta)dt$ is the Radon transform of the attenuation $a$, and the classical Hilbert transform $Hh(s) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{h(t)}{s-t}dt$ is taken in the first variable and evaluated at $s = z \cdot \theta^\perp$. The function $h$ was first considered in the work of Natterer [21]; see also [8], and [6] for elegant arguments that show how $h$ extends from $S^1$ inside the disk as an analytic map.

The lemma 4.1 and lemma 4.2 below were proven in [31] for a vanishing at the boundary, $a \in C^{1,\alpha}_0(\overline{\Omega})$, $\alpha > 1/2$. We explain here why the vanishing assumption is not necessary: we extend $a$ in a neighbourhood $\tilde{\Omega}$ of $\Omega$ with compact support, $\tilde{a} \in C^{1,\alpha}_0(\tilde{\Omega})$. We apply the results [31, Lemma 4.1 and Lemma 4.2] for the extension $\tilde{a}$ and use it on $\Omega$.

**Lemma 4.1.** [31, Lemma 4.1] Assume $a \in C^{p,\alpha}(\overline{\Omega})$, $p = 1, 2$, $\alpha > 1/2$, and $h$ defined in (58). Then $h \in C^{p,\alpha}(\overline{\Omega} \times S^1)$ and the following hold

(i) $h$ satisfies

$$\theta \cdot \nabla h(z, \theta) = -a(z), \ (z, \theta) \in \Omega \times S^1. \quad (59)$$

(ii) $h$ has vanishing negative Fourier modes yielding the expansions

$$e^{-h(z, \theta)} := \sum_{k=0}^\infty \alpha_k(z) e^{ik\varphi}, \quad e^{h(z, \theta)} := \sum_{k=0}^\infty \beta_k(z) e^{ik\varphi}, \ (z, \theta) \in \overline{\Omega} \times S^1, \quad (60)$$

with

(iii)

(61) $z \mapsto (\alpha_1(z), \alpha_2(z), \alpha_3(z), ...,) \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1),$

(62) $z \mapsto (\beta_1(z), \beta_2(z), \beta_3(z), ...,) \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1).$
(iv) For any \( z \in \Omega \)

\[
\frac{\partial}{\partial \beta_0}(z) = 0,
\]

\[
\frac{\partial}{\partial \beta_1}(z) = -a(z)\beta_0(z),
\]

\[
\frac{\partial}{\partial \beta_{k+2}}(z) + \partial \beta_k(z) + a(z)\beta_{k+1}(z) = 0, \quad k \geq 0.
\]

(v) For any \( z \in \Omega \)

\[
\frac{\partial}{\partial \alpha_0}(z) = 0,
\]

\[
\frac{\partial}{\partial \alpha_1}(z) = a(z)\alpha_0(z),
\]

\[
\frac{\partial}{\partial \alpha_{k+2}}(z) + \partial \alpha_k(z) + a(z)\alpha_{k+1}(z) = 0, \quad k \geq 0.
\]

(vi) The Fourier modes \( \alpha_k, \beta_k, k \geq 0 \) satisfy

\[
\alpha_0\beta_0 = 1, \quad \sum_{m=0}^{k} \alpha_m\beta_{k-m} = 0, \quad k \geq 1.
\]

From (59) it is easy to see that \( u \) solves (57) if and only if \( v := e^{-h}u \) solves

\[
\theta \cdot \nabla v(z, \theta) = \langle F(z)\theta, \theta \rangle e^{-h(z, \theta)}.
\]

If \( u(z, \theta) = \sum_{n=-\infty}^{\infty} u_n(z) e^{in\varphi} \) solves (57), then its Fourier modes satisfy

\[
\frac{\partial}{\partial u_1}(z) + \partial u_{-1}(z) + a(z)u_0(z) = f_0(z),
\]

\[
\frac{\partial}{\partial u_0}(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = 0,
\]

\[
\frac{\partial}{\partial u_{-1}}(z) + \partial u_{-3}(z) + a(z)u_{-2}(z) = f_2(z),
\]

\[
\frac{\partial}{\partial u_{n}}(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -2,
\]

where \( f_0, f_2 \) as defined in (19).

Also, if \( v := e^{-h} = \sum_{n=-\infty}^{\infty} v_n(z) e^{in\varphi} \) solves (70), then its Fourier modes satisfy

\[
\frac{\partial}{\partial v_1}(z) + \partial v_{-1}(z) = \alpha_0(z)f_0(z) + \alpha_2(z)f_2(z),
\]

\[
\frac{\partial}{\partial v_0}(z) + \partial v_{-2}(z) = \alpha_1(z)f_2(z),
\]

\[
\frac{\partial}{\partial v_{-1}}(z) + \partial v_{-3}(z) = \alpha_0(z)f_2(z),
\]

\[
\frac{\partial}{\partial v_{n}}(z) + \partial v_{n-2}(z) = 0, \quad n \leq -2,
\]

where \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are the Fourier modes in (60), and \( f_0, f_2 \) as defined in (19).

The following result shows that the equivalence between (74) and (75) is intrinsic to negative Fourier modes only.
Lemma 4.2. [31, Lemma 4.2] Assume $a \in C^{1,\alpha} (\Omega), \alpha > 1/2$.

(i) Let $v = \langle v_{-2}, v_{-3}, \ldots \rangle \in C^1 (\Omega, l_1)$ satisfy (75), and $u = \langle u_{-2}, u_{-3}, \ldots \rangle$ be defined componentwise by the convolution

$$u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq -2,$$

where $\beta_j$’s are the Fourier modes in (60). Then $u$ solves (74) in $\Omega$.

(ii) Conversely, let $u = \langle u_{-2}, u_{-3}, \ldots \rangle \in C^1 (\Omega, l_1)$ satisfy (74), and $v = \langle v_{-2}, v_{-3}, \ldots \rangle$ be defined componentwise by the convolution

$$v_n := \sum_{j=0}^{\infty} \alpha_j u_{n-j}, \quad n \leq -2,$$

where $\alpha_j$’s are the Fourier modes in (60). Then $v$ solves (75) in $\Omega$.

The operators $\partial, \overline{\partial}$ in (9) can be rewritten in terms of the derivative in tangential direction $\partial_\tau$ and derivative in normal direction $\partial_n$,

$$\partial_n = \cos \eta \partial_{x_1} + \sin \eta \partial_{x_2},$$
$$\partial_\tau = -\sin \eta \partial_{x_1} + \cos \eta \partial_{x_2},$$

where $\eta$ is the angle made by the normal to the boundary with $x_1$ direction (Since the boundary $\Gamma$ is known, $\eta$ is a known function on the boundary). In these coordinates

$$\partial = \frac{e^{-i\eta}}{2} (\partial_n - i \partial_\tau), \quad \overline{\partial} = \frac{e^{i\eta}}{2} (\partial_n + i \partial_\tau).$$

Next we characterize the attenuated X-ray data $g$ in terms of its Fourier modes $g_0, g_{-1}$ and the negative index modes $\gamma_{-2}, \gamma_{-3}, \gamma_{-4} \ldots$ of

$$e^{-h(\zeta, \theta)} g(\zeta, \theta) = \sum_{k=-\infty}^{\infty} \gamma_k (\zeta) e^{ik\varphi}, \quad \zeta \in \Gamma.$$

To simplify the statement, let

$$g_h := \langle \gamma_{-2}, \gamma_{-3}, \gamma_{-4}, \ldots \rangle,$$

and from the negative even, respectively, negative odd Fourier modes, we built the sequences

$$g_h^{\text{even}} = \langle \gamma_{-2}, \gamma_{-4}, \ldots \rangle, \quad \text{and} \quad g_h^{\text{odd}} = \langle \gamma_{-3}, \gamma_{-5}, \ldots \rangle.$$

Note that $\gamma_{-1}$ is not included in the $g_h^{\text{odd}}$ definition. As before we construct simultaneously the right hand side of the transport equation (57) together with the solution $u$. Construction of $u$ is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. Apart from zeroth mode $u_0$ and negative one mode $u_{-1}$, all Fourier modes are constructed uniquely from the data $g_h^{\text{even}}, g_h^{\text{odd}}$. The
mode \( u_0 \) will be chosen arbitrarily from the class \( \Psi^a \) with prescribed trace and gradient on the boundary \( \Gamma \) defined as

\[
\Psi^a_g := \left\{ \psi \in C^2(\Omega; \mathbb{R}) : \psi|_\Gamma = g_0, \quad \partial_n \psi|_\Gamma = -2 \Re e^{-i\eta} \left( \partial \sum_{j=0}^\infty \beta_j (\mathcal{B}g_h)_{-2-j} \right|_\Gamma + a|_\Gamma g_{-1} \right\},
\]

where \( \mathcal{B} \) be the Bukhgeim-Cauchy operator in (13), \( \beta_j \)'s are the Fourier modes in (60) and \( g_h \) in (80). The mode \( u_{-1} \) is defined in terms of \( u_0 \), see (99).

Recall the Hilbert transform \( \mathcal{H} \) in (16).

**Theorem 4.1** (Range characterization in the attenuated case). Let \( \alpha \in C^{2,\alpha}(\Omega) \), \( \alpha > 1/2 \) with \( \min \alpha > 0 \).

(i) Let \( F \) be a real valued symmetric 2-tensor \( \psi \) such that \( F(x,\theta) \) is compactly supported inside \( \Omega \), then the negative Fourier modes \( g^\text{odd} \) in (80) and \( g_h \) in (80).

(ii) Let \( g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1)) \) be real valued with \( g|_{\Gamma_+ \cup \Gamma_0} = 0 \). If the corresponding sequences \( g_{h-}^\text{even}, g_{h-}^\text{odd} \) satisfy (83) and (84) then there exists a symmetric 2-tensor \( \mathcal{F} \in C(\Omega; \mathbb{R}^{2 \times 2}) \), such that \( g|_{\Gamma_+} = X_\alpha \mathcal{F} \). Moreover for each \( \psi \in \Psi^a \) in (82), there is a unique real valued symmetric 2-tensor \( \mathcal{F}_{\psi} \) such that \( g|_{\Gamma_+} = X_\alpha \mathcal{F}_{\psi} \).

**Proof.** (i) **Necessity**

Let \( \mathcal{F} \in C^{1,\alpha}_{0;\Omega} \). Since \( \mathcal{F} \) is compactly supported inside \( \Omega \), for any point at the boundary there is a cone of lines which do not meet the support. Thus \( g \equiv 0 \) in the neighborhood of the variety \( \Gamma_0 \) which yields \( g \in C^{1,\alpha}(\Gamma \times S^1) \). Moreover, \( g \) is the trace on \( \Gamma \times S^1 \) of a solution \( u \in C^{1,\alpha}(\Omega \times S^1) \). By [30, Proposition 4.1] \( g_{h-}^\text{even}, g_{h-}^\text{odd} \in l_{\infty}^{1,\alpha}(\Gamma \cap C(\Gamma;l_1)) \).

Let \( v := e^{-h}u = \sum_{n=-\infty}^{\infty} v_n(z) e^{i\eta z} \), then the negative Fourier modes of \( v \) satisfy (75). In particular its negative odd subsequence \( \langle v_{-3}, v_{-5}, \ldots \rangle \) and negative even subsequence \( \langle v_{-2}, v_{-4}, \ldots \rangle \) are \( \mathcal{L} \)-analytic with traces \( g_{h-}^\text{odd} \).
respectively $g_h^{\text{even}}$. The necessity part of Theorem 2.2 yields (83):

$$[I + i\mathcal{H}]g_h^{\text{odd}} = 0, \quad [I + i\mathcal{H}]g_h^{\text{even}} = 0.$$ 

If $u$ solves (57), then its Fourier modes satisfy (71), (72), (73), and (74). The negative Fourier modes of $u$ and $v$ are related by

$$u_n = \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq 0,$$

where $\beta_j$’s are the Fourier modes in (60). The restriction of (72) to the boundary yields

$$\partial u_0 |_\Gamma = - \partial u_{-2} |_\Gamma - (au_{-1}) |_\Gamma.$$ 

Expressing $\partial$ in the above equation in terms of $\partial_r$ and $\partial_n$ as in (78) yields

$$\frac{e^{in}}{2} (\partial_n + i\partial_r) u_0 |_\Gamma = - \partial u_{-2} |_\Gamma - a g_{-1}.$$ 

Simplifying the above expression and using $\partial_r u_0 |_\Gamma = \partial_r g_0$, yields

$$\partial_n u_0 |_\Gamma + i\partial_r g_0 = -2e^{-in} (\partial u_{-2} |_\Gamma + a g_{-1}).$$ 

The imaginary part of the above equation yields (84). This proves part (i) of the theorem.

(ii) **Sufficiency**

To prove the sufficiency we will construct a real valued symmetric 2-tensor $F$ in $\Omega$ and a real valued function $u \in C^1(\Omega \times S^1) \cap C(\overline{\Omega} \times S^1)$ such that $u|_{\Gamma \times S^1} = g$ and $u$ solves (57) in $\Omega$. The construction of such $u$ is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1: The construction of negative modes $u_n$ for $n \leq -2$.**

Let $g \in C^\alpha (\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$ be real valued with $g|_{\Gamma_0} = 0$. Let the corresponding sequences $g_h^{\text{even}}, g_h^{\text{odd}}$ as in (81) satisfying (83) and (84). By [30, Proposition 4.1(ii)] and [30, Proposition 5.2(iii)] $g_h^{\text{even}}, g_h^{\text{odd}} \in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to define the $\mathcal{L}$-analytic maps

$$v^{\text{even}}(z) = \langle v_{-2}(z), v_{-4}(z), \ldots \rangle := B g_h^{\text{even}}(z), \quad z \in \Omega,$n

$$v^{\text{odd}}(z) = \langle v_{-3}(z), v_{-5}(z), \ldots \rangle := B g_h^{\text{odd}}(z), \quad z \in \Omega.$$ 

By intertwining let also define

$$v(z) := \langle v_{-2}(z), v_{-3}(z), \ldots \rangle, \quad z \in \Omega.$$ 

By Proposition 2.1

$$v^{\text{even}}, v^{\text{odd}}, v \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty).$$
Moreover, since \( g_{h}^{\text{even}}, g_{h}^{\text{odd}} \) satisfy the hypothesis (83), by Theorem 2.2 we have

\[
v^{\text{even}}|_{r} = g_{h}^{\text{even}} \quad \text{and} \quad v^{\text{odd}}|_{r} = g_{h}^{\text{odd}}.
\]

In particular

\[
v_{n}|_{r} = \sum_{k=0}^{\infty} (\alpha_{k}|_{r}) g_{n-k}, \quad n \leq -2.
\]

For each \( n \leq -2 \), we use the convolution formula below to construct

\[
u_{n} := \sum_{j=0}^{\infty} \beta_{j} v_{n-j}.
\]

Since \( a \in C^{2,\alpha}(\Omega) \), by (62), the sequence \( z \mapsto \langle \beta_{0}(z), \beta_{1}(z), \beta_{2}(z), \ldots \rangle \) is in \( C^{2,\alpha}(\Omega; l_{1}) \cap C^{\alpha}(\Omega; l_{1}) \). Since convolution preserves \( l_{1} \), the map is in

\[
z \mapsto \langle u_{-2}(z), u_{-3}(z), \ldots \rangle \in C^{1,\alpha}(\Omega; l_{1}) \cap C^{\alpha}(\Omega; l_{1}).
\]

Moreover, since \( v \in C^{2}(\Omega; l_{\infty}) \) as in (88), we also conclude from convolution that

\[
z \mapsto \langle u_{-2}(z), u_{-3}(z), \ldots \rangle \in C^{2}(\Omega; l_{\infty}).
\]

The property (91) justifies the calculation of traces \( u_{n}|_{r} \) for each \( n \leq -2 \):

\[
u_{n}|_{r} = \sum_{j=0}^{\infty} \beta_{j}|_{r} (v_{n-j}|_{r}).
\]

Using (89) in the above equation gives

\[
u_{n}|_{r} = \sum_{j=0}^{\infty} \beta_{j}|_{r} \sum_{k=0}^{\infty} \alpha_{k}|_{r} g_{n-j-k}.
\]

A change of index \( m = j + k \), simplifies the above equation

\[
u_{n}|_{r} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \alpha_{k} \beta_{m-k} g_{n-m},
\]

\[
= \alpha_{0} \beta_{0} g_{n} + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \alpha_{k} \beta_{m-k} g_{n-m}.
\]

Using Lemma 4.1 (vi) yields

\[
u_{n}|_{r} = g_{n}, \quad n \leq -2.
\]

From the Lemma 4.2, the constructed \( u_{n} \) in (90) satisfy

\[
\bar{\partial} u_{n} + \partial u_{n-2} + au_{n-1} = 0, \quad n \leq -2.
\]

**Step 2: The construction of positive modes \( u_{n} \) for \( n \geq 2 \).**
All of the positive Fourier modes are constructed by conjugation:

\[ u_n := \overline{u_{-n}}, \quad n \geq 2. \] (95)

Moreover using (93), the traces \( u_n|_\Gamma \) for each \( n \geq 2 \):

\[ u_n|_\Gamma = \overline{u_{-n}|_\Gamma} = \overline{g_{-n}} = g_n, \quad n \geq 2. \] (96)

By conjugating (94) we note that the positive Fourier modes also satisfy

\[ \overline{\partial} u_{n+2} + \partial u_n + a u_{n+1} = 0, \quad n \geq 2. \] (97)

**Step 3: The construction of modes \( u_0, u_{-1} \) and \( u_1 \).**

Let \( \psi \in \Psi_g^a \) as in (82) and define

\[ u_0 := \psi, \] (98)

and

\[ u_{-1} := \frac{-\overline{\partial} \psi - \partial u_{-2}}{a}, \quad u_1 := \overline{u_{-1}}. \] (99)

By the construction \( u_0 \in C^2(\Omega; l_\infty) \) and \( u_{-1} \in C^1(\Omega; l_\infty) \), and

\[ \overline{\partial} u_0 + \partial u_{-2} + a u_{-1} = 0 \] (100)

is satisfied. Furthermore, by conjugating (100) yields

\[ \partial u_0 + \partial u_2 + a u_1 = 0. \] (101)

Since \( \psi \in \Psi_g^a \), the trace of \( u_0 \) satisfies

\[ u_0|_\Gamma = g_0. \] (102)

We check next that the trace of \( u_{-1} \) is \( g_{-1} \):

\[ u_{-1}|_\Gamma = \left. \frac{-\overline{\partial} \psi - \partial u_{-2}}{a} \right|_\Gamma \]

\[ = -\frac{1}{a} \left. \frac{e^{in}}{2} (\partial_n + i \partial_\tau) \psi \right|_\Gamma - \frac{1}{a} \left. \partial u_{-2} \right|_\Gamma \]

\[ = -\frac{1}{2a} \left. e^{in} \{ \partial_n \psi + i \partial_\tau \psi \} \right|_\Gamma + 2e^{-in} \partial u_{-2}|_\Gamma \]

\[ = g_{-1}, \] (103)

where the last equality uses (84) and the condition in class (82).

**Step 4: The construction of the tensor field \( F_\psi \) whose attenuated X-ray data is \( g \).**

We define the 2-tensor

\[ F_\psi := \begin{pmatrix} f_0 + 2 \Re f_2 & 2 \Im f_2 \\ 2 \Im f_2 & f_0 - 2 \Re f_2 \end{pmatrix}, \] (104)
where

\begin{align}
(105) 
  f_0 &= -2 \Re \left( \frac{\partial \psi + \partial u_{-2}}{a} \right) + a \psi, \text{ and } \\
(106) 
  f_2 &= -a \partial \left( \frac{\partial \psi + \partial u_{-2}}{a} \right) + \partial u_{-3} + au_{-2}.
\end{align}

Note that \(f_2\) is well defined as \(u_{-2} \in C^2(\Omega; l_\infty)\) from (92).

In order to show \(g|_{\Gamma} = X_\alpha F\psi\) with \(F\psi\) as in (104), we define the real valued function \(u\) via its Fourier modes

\begin{equation}
(107) 
  u(z, \theta) := u_0(z) + u_{-1} e^{-i\varphi} + \bar{u}_{-1}(z) e^{i\varphi} \\
  + \sum_{n=2}^{\infty} u_{-n}(z) e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z) e^{in\varphi}.
\end{equation}

We check below that \(u\) is well defined, has the trace \(g\) on \(\Gamma\) and satisfies the transport equation (57).

For convenience consider the intertwining sequence

\[ u(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), u_{-3}(z), ... \rangle, \quad z \in \Omega. \]

Since \(u \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1)\), by [30, Proposition 4.1 (iii)] we conclude that \(u\) is well defined by (107) and as a function in \(C^{1,\alpha}(\Omega \times S^1) \cap C^{\alpha}(\overline{\Omega} \times S^1)\). In particular \(u(\cdot, \theta)\) for \(\theta = (\cos \varphi, \sin \varphi)\) extends to the boundary and its trace satisfies

\[ u(\cdot, \theta)|_{\Gamma} = \left( u_0 + u_{-1} e^{-i\varphi} + \bar{u}_{-1} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right)|_{\Gamma} \]

\[ = u_0|_{\Gamma} + u_{-1}|_{\Gamma} e^{-i\varphi} + \bar{u}_{-1}|_{\Gamma} e^{i\varphi} + \sum_{n=2}^{\infty} (u_{-n}|_{\Gamma}) e^{-in\varphi} + \sum_{n=2}^{\infty} (u_n|_{\Gamma}) e^{in\varphi} \]

\[ = g_0 + g_{-1} e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \]

\[ = g(\cdot, \theta), \]

where is the third equality we have used (93), (96), (102), and (103).
Since \( u \in C^{1,\alpha}(\Omega \times S^1) \cap C^\alpha(\Omega \times S^1) \), the following calculation is also justified:

\[
\theta \cdot \nabla u + au = e^{-i\varphi} \overline{\partial u_0} + e^{i\varphi} \partial u_0 + e^{-2i\varphi} \overline{\partial u_{-1}} + \partial u_1 + \partial u_{-1} + e^{2i\varphi} \partial u_1 \\
+ \sum_{n=2}^{\infty} \overline{\partial u_{-n}} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi} \\
+ \sum_{n=2}^{\infty} \overline{\partial u_n} e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi} \\
+ au_0 + au_{-1} e^{-i\varphi} + au_1 e^{i\varphi} + \sum_{n=2}^{\infty} au_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} au_n e^{in\varphi}.
\]

Rearranging the modes in the above equation yields

\[
\theta \cdot \nabla u + au = e^{-2i\varphi}(\overline{\partial u_{-1}} + \partial u_{-3} + au_{-2}) + e^{2i\varphi}(\partial u_1 + \overline{\partial u_3} + au_2) \\
+ e^{-i\varphi}(\overline{\partial u_0} + \partial u_{-2} + au_{-1}) + e^{i\varphi}(\partial u_0 + \overline{\partial u_2} + au_1) \\
+ \overline{\partial u_1} + \partial u_{-1} + au_0 + \sum_{n=2}^{\infty} (\overline{\partial u_{n+2}} + \partial u_n + au_{n+1}) e^{i(n+1)\varphi} \\
+ \sum_{n=2}^{\infty} (\overline{\partial u_{-n}} + \partial u_{-n+2} + au_{-n-1}) e^{-i(n+1)\varphi}.
\]

Using (94), (97), (100) and (101) simplifies the above equation

\[
\theta \cdot \nabla u + au = e^{-2i\varphi}(\overline{\partial u_{-1}} + \partial u_{-3} + au_{-2}) + e^{2i\varphi}(\partial u_1 + \overline{\partial u_3} + au_2) \\
+ \overline{\partial u_1} + \partial u_{-1} + au_0.
\]

Now using (105) and (106), we conclude (57)

\[
\theta \cdot \nabla u + au = e^{-2i\varphi}f_2 + e^{2i\varphi} \overline{f_2} + f_0 = \langle F_\psi \theta, \theta \rangle.
\]

\[
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