Dependence on a collection of Poisson random variables

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Abstract

We propose two novel ways of introducing dependence among Poisson counts through the use of latent variables in a three levels hierarchical model. Marginal distributions of the random variables of interest are Poisson with strict stationarity as special case. Order-$p$ dependence is described in detail for a temporal sequence of random variables, however spatial or spatio-temporal dependencies are also possible. A full Bayesian inference of the models is described and performance of the models is illustrated with a numerical analysis of maternal mortality in Mexico.

Keywords: Autoregressive process, integer-valued time series, latent variables, moving average process, stationary process.

1 Introduction

Time series models are mainly discrete time stationary processes. The support of the random variables involved is usually continuous and unbounded (e.g. Box and Jenkins 1970). The study of discrete time stationary processes with discrete marginal distributions is less common, however there have been some proposals (e.g. McKenzie 1985).

In this article we define discrete time stochastic processes with Poisson marginal distributions. Construction of our proposal is based on the use of latent variables, through hierarchical models, which allows us to define different orders of dependence in space and time.
To define the main building block of our proposal we first introduce some notation:

$\text{Ber}(\alpha)$ denotes a Bernoulli density with success probability $\alpha$; $\text{Bin}(n, \alpha)$ denotes a binomial density with $n$ Bernoulli trials and success probability $\alpha$; $\text{Po}(\mu)$ denotes a Poisson density with mean (rate) $\mu$; $\text{Mul}(n, \alpha)$ denotes a multinomial density with $n$ number of trials and vector of probabilities $\alpha$. Then it is straightforward to show that:

If $X \sim \text{Po}(\mu)$ and $Y \mid X \sim \text{Bin}(x, \alpha) \iff Y \sim \text{Po}(\mu \alpha)$ and $X - y \mid Y \sim \text{Po}(\mu (1 - \alpha))$. (1)

One of the first proposals in the literature is the integer-valued first order autoregressive process, INAR(1), which for a process $\{X_t\}$ is defined as (McKenzie, 1985; Al-Osh and Alzaid, 1987)

$$X_t = \alpha \circ X_{t-1} + \epsilon_t,$$  \hspace{1cm} (2)

where “$\circ$” denotes the binomial thinning operator defined as $\alpha \circ X = \sum_{j=1}^X B_j$ with $B_j \sim \text{Ber}(\alpha)$. In other words $\alpha \circ X \mid X \sim \text{Bin}(x, \alpha)$. If we denote $Y_t = \alpha \circ X_{t-1}$ in (2), then $Y_t \mid X_{t-1} \sim \text{Bin}(x_{t-1}, \alpha)$. Moreover, if the innovations are Poisson distributed, $\epsilon_t \sim \text{Po}(\mu(1 - \alpha))$, then $X_t - y_t \mid Y_t \sim \text{Po}(\mu(1 - \alpha))$. Thus if $X_{t-1} \sim \text{Po}(\mu)$, result [1] implies that marginally $X_t \sim \text{Po}(\mu)$. The autocorrelation function of (2) can be obtained analytically and has the form $\text{Corr}(X_t, X_{t+s}) = \rho(s) = \alpha^s$ for $s \geq 0$.

Later, McKenzie (1988) generalized the INAR(1) process to the ARMA type. For instance, the Poisson MA($q$) process is defined as

$$X_t = Z_t + \beta_1 \circ Z_{t-1} + \cdots + \beta_q \circ Z_{t-q},$$  \hspace{1cm} (3)

where $\beta_i \in (0, 1)$ for all $i$, and $Z_t \sim \text{Po}(\mu/\beta)$ with $\beta = \sum_{i=0}^{q} \beta_i$ and $\beta_0 = 1$. Denoting by $Y_i = \beta_i \circ Z_{t-i}$ then $Y_i \mid Z_{t-i} \sim \text{Bin}(z_{t-i}, \beta_i)$ and from [1] $Y_i \sim \text{Po}(\mu\beta_i/\beta)$ marginally. Now, using the additive property of independent Poisson random variables, it becomes that $X_t \sim \text{Po}(\mu)$. The autocorrelation function of (3) is given by $\rho(s) = \sum_{i=0}^{q-s} \beta_i \beta_{i+s} / \sum_{i=0}^{q} \beta_i$ for $s \leq q$, and zero otherwise.
Another generalization of INAR(1) process is that of Alzaid and Al-Osh (1990), who proposed the INAR(p) process as follows

\[ X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \epsilon_t, \]  

(4)

where \( \alpha_i > 0 \) for all \( i \) with \( \sum_{i=1}^{p} \alpha_i < 1 \), and the conditional distribution of the vector \( (\alpha_1 \circ X_t, \alpha_2 \circ X_t, \ldots, \alpha_p \circ X_t) \mid X_t \sim \text{Mul}(x_t, \alpha) \) with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \). Even if the distribution for the innovations \( \epsilon_t \) in (4) is Poisson, the marginal distribution of \( X_t \) is not Poisson.

The description of the rest of the paper is as follows: In Section 2 we describe the construction of two dependent Poisson sequences in time and characterise its marginal distribution and correlation induced. Bayesian inference of model parameters is described in Section 3. Section 4 reports a numerical study of integer-valued time series of maternal mortality in Mexico. Section 5 presents some extensions to more general dependencies, as seasonal, periodic and spatial. We conclude with some remarks in Section 6.

2 Temporal dependence

Let \( \{X_t\} \) be a temporal sequence of random variables. For each \( t \) we require a set of two latent variables, say \( (Y_t, W_t) \), and define a three level hierarchical model to induce a temporal dependence of order \( p \). We propose two ways of defining dependence among the \( X_t \)'s by either, linking the variables of the second level with those of the third level across times (type A), or linking the variables of the first level to those of the second level across times (type B). Figure 1 illustrates these two types, where the dependence shown is of order \( p = 1 \).

In general, the \( W_t \)'s will be independent Poisson random variables and the \( Y_t \)'s will be a binomial thinning of the \( W_t \)'s. Additionally, \( Y_t \) and/or \( W_t \) will exist for \( t = -p, -p+1, \ldots, T \).
2.1 Type A dependence

The first construction is defined by the following hierarchical representation

\[
W_t \overset{\text{iid}}{\sim} \text{Po}(\mu),
\]

\[
Y_t \mid W_t \overset{\text{ind}}{\sim} \text{Bin}(w_t, \alpha_t),
\]

\[
X_t = \sum_{i=0}^{p} y_{t-i} \mid Y \overset{\text{ind}}{\sim} \text{Po} \left( \mu \left( 1 - \sum_{i=0}^{p} \alpha_{t-i} \right) \right)
\]

where \( \mu > 0, \alpha_t > 0 \) and \( \sum_{i=0}^{p} \alpha_{t-i} < 1 \).

Properties of this type A construction are given in Proposition 1.

**Proposition 1** Let \( \{X_t\} \) be defined by equations (5). Then the marginal distribution of \( X_t \) is \( \text{Po}(\mu) \), and the autocorrelation between \( X_t \) and \( X_{t+s} \) is given by

\[
\text{Corr}(X_t, X_{t+s}) = \sum_{i=0}^{p-s} \alpha_{t-i},
\]

for \( 1 \leq s \leq p \) and zero for \( s > p \).

**Proof** We note that the first level can be marginalised to keep only levels two and three. After doing so, \( Y_t \sim \text{Po}(\mu \alpha_t) \) marginally and they are all independent across \( t \). Then,
\[ \sum_{i=0}^{p} Y_{t-i} \sim \text{Po}(\mu \sum_{i=0}^{p} \alpha_{t-i}). \] Finally, from (1), we obtain \( X_{t} \sim \text{Po}(\mu) \) marginally for \( t = 1, \ldots, T \). To obtain the correlation, we use conditional independence properties and the iterative covariance formula to obtain that \( \text{Cov}(X_{t}, X_{t+s}) = \text{Var}(\sum_{i=0}^{p-s} Y_{t-i}) \). Computing this value and dividing by the variance we obtain the result. \( \diamond \)

The process (5) becomes strictly stationary when \( \alpha_{t} = \alpha \) for all \( t \), and the correlation induced can be computed explicitly as given in Proposition 1. Correlation expression is a function of the thinning probabilities of the shared elements in the definition of \( X_{t} \) and \( X_{t+s} \).

To see the similarities with previous proposals, we can re-write construction (5) as

\[ X_{t} = \sum_{i=0}^{p} Y_{t-i} + \epsilon_{t} = \sum_{i=0}^{p} \alpha_{t-i} \circ W_{t-i} + \epsilon_{t}, \]

where \( \epsilon_{t} \sim \text{Po}(\mu (1 - \sum_{i=0}^{p} \alpha_{t-i})). \) As such, it would resemble the Poisson MA(q) given in (3) but with \( p \) instead of \( q \) and with an extra innovation term. However, the most important difference are the “coefficients” or thinning probabilities \( \alpha_{t} \), which in our proposal they move along \( t \), whereas in the MA(q) they are fixed for any \( t \).

### 2.2 Type B dependence

The second construction is defined by the following hierarchical representation

\[ W_{t} \overset{\text{iid}}{\sim} \text{Po}\left( \frac{\mu}{p+1} \right), \]

\[ Y_{t} \mid W \overset{\text{ind}}{\sim} \text{Bin}\left( \sum_{i=0}^{p} w_{t-i}, \alpha_{t} \right), \]

\[ X_{t} - y_{t} \mid Y_{t} \overset{\text{ind}}{\sim} \text{Po}(\mu (1 - \alpha_{t})), \]

where \( \mu > 0 \) and \( \alpha_{t} \in (0, 1) \).

Properties of this type B construction are given in Proposition 2.

**Proposition 2** Let \( \{X_{t}\} \) be defined by equations (6). Then the marginal distribution of \( X_{t} \) is \( \text{Po}(\mu) \), and the autocorrelation between \( X_{t} \) and \( X_{t+s} \) is given by

\[ \text{Corr}(X_{t}, X_{t+s}) = \alpha_{t} \alpha_{t+s} \left( \frac{p+1-s}{p+1} \right), \]
for $1 \leq s \leq p$ and zero for $s > p$.

**Proof** Using the additive property of independent Poisson variables, we obtain that \( \sum_{i=0}^{p} W_{t-i} \sim \text{Po}(\mu) \). Now, from [1], the marginal distribution of the latent variables in the second level becomes $Y_t \sim \text{Po}(\mu \alpha_t)$. Finally, we obtain that $X_t \sim \text{Po}(\mu)$ marginally for $t = 1, \ldots, T$. Now for the correlation, we use the iterative covariance formula and apply conditional independence properties twice, we first obtain that \( \text{Cov}(X_t, X_{t+s}) = \alpha_t \alpha_{t+s} \times \text{Var} \left( \sum_{i=0}^{p-s} W_{t-i} \right) \). Computing this value and dividing by the variance we obtain the result.

\( \diamond \)

Again, the process [6] becomes strictly stationary when $\alpha_t = \alpha$ for all $t$, and the correlation induced can also be computed explicitly as given in Proposition [2]. However, this correlation is a function of the thinning probabilities of times $t$ and $t + s$, and the number of shared elements in the definition of $Y_t$ and $Y_{t+s}$.

We note that the marginal distribution of the latent $Y_t$'s variables, in both type A and type B constructions, are the same, $Y_t \sim \text{Po}(\mu \alpha_t)$. However in [5] they are independent, whereas in [6] they are dependent.

Re-writing model [6] into an additive form we have

$$X_t = Y_t + \epsilon_t = \alpha_t \circ \sum_{i=0}^{p} W_{t-i} + \epsilon_t,$$

where $\epsilon_t \sim \text{Po}(\mu (1 - \alpha_t))$. This expression looks like a MA(0) process with innovation term, or like an INAR(1) process where the thinning operates over the sum of latent variables \( \sum_{i=0}^{p} W_{t-i} \) instead of over the lagged variable $X_{t-1}$.

### 3 Bayesian inference

Let $X_1, \ldots, X_T$ be an observable finite time series of integer-valued random variables. We assume that the law describing the sequence is one of the previously defined type A or type B models. The idea is to produce inference about the unknown parameters of the
models $\alpha = \{\alpha_t, t = 1, \ldots, T\}$ and $\mu$, and for that we propose a Bayesian approach. We define our prior knowledge through beta and gamma distributions, respectively. That is $\alpha_t \sim \text{Be}(a_\alpha, b_\alpha)$, independently for $t = 1, \ldots, T$ and $\mu \sim \text{Ga}(a_\mu, b_\mu)$.

Since latent variables $Y$ and $W$ are not observable, we treat them as missing data and define an augmented likelihood (e.g. Tanner, 1991). For type A model the extended likelihood has the form

$$f(x, y \mid \alpha, \mu) = \prod_{t=1}^{T} \text{Po} \left( x_t - \sum_{i=0}^{p} y_{t-i} \mid \mu \left( 1 - \sum_{i=0}^{p} \alpha_{t-i} \right) \right) \text{Po}(y_t \mid \mu \alpha_t)$$

and for type B model this has the form

$$f(x, y, w \mid \alpha, \mu) = \prod_{t=1}^{T} \text{Po} \left( x_t - y_t \mid \mu (1 - \alpha_t) \right) \text{Bin} \left( y_t \mid \sum_{i=0}^{p} w_{t-i}, \alpha_t \right) \text{Po} \left( w_t \mid \frac{\mu}{p+1} \right).$$

Posterior distributions will be characterised through their full conditional distributions, which have been included in the Appendix for both types of models. Distributions (i)–(iii) correspond to type A model, whereas distributions (iv)–(vii) correspond to type B model. Posterior inference is therefore obtained through the implementation of a Gibbs sampler (Smith and Roberts, 1993).  

### 4 Numerical analysis

Unfortunately maternal mortality is still an important public health problem in Mexico. According to the World Health Organization maternal mortality is defined as a death from preventable causes related to pregnancy and childbirth. The Mexican National Institute of Geography and Statistics reports the annual number of maternal deaths for the 32 political states of Mexico (https://www.inegi.org.mx/sistemas/olap/proyectos/bd/continuas/mortalidad/mortalidadgeneral.asp). Information is available from 1990 until 2018, that is, a total of $T = 29$ years.

We analysed the 32 time series with both types of models. To define the prior distributions we took $a_\alpha = b_\alpha = a_\mu = b_\mu = 0.01$. For $p$ we took a set of different values to compare, say
\( p \in \{0, 1, 2, 3, 4, 5, 6\}. \) A Gibbs sampler was implemented in Fortran with 16,000 iterations, a burn-in period of 1,000 and kept one of every 5\( \text{th} \) iteration, after burn-in, to produce posterior summaries. Convergence of the chain was assessed informally by looking at the trace plots, ergodic means and autocorrelation functions of the chains.

To assess model fit we computed the L-measure which is a predictive statistic that summarises variance and mean square error (bias) of the posterior predictive distribution of each \( X_t \). This is defined as \((Ibrahim and Laud, 1994)\)

\[
L(\nu) = \frac{1}{T} \sum_{t=1}^{T} \operatorname{Var}(X_t^F | x) + \frac{\nu}{T} \sum_{t=1}^{T} \{E(X_t^F | x) - x_t\}^2,
\]

where \(X_t^F\) and \(x_t\) denote the predictive and observed value of \(X_t\), respectively.

Table 1 reports the values of the L-measure with \(\nu = 1/2\), obtained when fitting models of types A and B to the 32 time series of the maternal mortality dataset, for \(p = 0, 1, \ldots, 6\). For each type of model, the value of \(p\) with the smallest L-measure is highlighted in bold. Apart from Aguascalientes and Zacatecas, where the best fitting is achieved for \(p = 0\) (independence) in one of the two types of models, for the rest of the states the best fitting model is obtained for \(p > 0\), which implies a temporal dependence. Now, comparing the best fitting from the two types, for 31 of the 32 states, type A model outperforms type B model. The only state where type B model is slightly better is Colima with an L-measure of 2.81 as compared to 2.84 obtained by the best type A model.

Figures 2, 3 and 4 show the performance of best fitting models for type A (left panel) and type B (right panel) for Baja California, Coahuila and CDMX (Mexico City), respectively. In these figures type A model shows a better fitting than type B model with more accurate predictions and narrower credible bands. On the other hand, Figure 5 displays model performance for the state of Colima. This is the only case where type B model slightly outperforms type A model.

Finally, to place our two proposals in context, we fitted the most commonly used model
for integer valued time series, which is the INAR(1) model \(2\). We also carried out Bayesian inference of this model and took prior distributions \(\alpha \sim \text{Be}(0.01, 0.01)\) and \(\mu \sim \text{Ga}(0.01, 0.01)\) for the model parameters. We implemented a Gibbs sampler with the same specifications as above and computed the L-measure \(7\) with \(\nu = 1/2\). This goodness of fit statistic is also reported in the last column of Table \(1\) for the 32 states of Mexico. Interestingly, for 26 of the 32 states our best fitting type A model outperforms the INAR(1) model.

5 Extensions

Considering Figure \(1\), we note that the processes are still well defined if any of the diagonal arrows are removed. So in general, we can make \(X_t\) to be defined in terms of \(Y_{t-i}\), in type A construction, or \(Y_t\) to be defined in terms of \(W_{t-i}\), in type B construction, for any \(i\) not necessarily consecutive. Therefore we can define more general seasonal \(\text{[Nabeya, 2001]}\) or periodic \(\text{[McLeod, 1994]}\) dependent models.

If the seasonality of the process is \(s\), a dependent model of order \(p\) would have the form

\[
X_t - \sum_{i=0}^{p} y_{t-si} \mid Y \overset{\text{ind}}{\sim} \text{Po} \left( \mu \left( 1 - \sum_{i=0}^{p} \alpha_{t-si} \right) \right),
\]

for a type A construction with levels 1 and 2 as in \(5\) and with \(\sum_{i=0}^{p} \alpha_{t-si} < 1\), and

\[
Y_t \mid W \overset{\text{ind}}{\sim} \text{Bin} \left( \sum_{i=0}^{p} w_{t-si}, \alpha_t \right),
\]

for a type B construction with levels 1 and 3 as in \(6\). In both types we obtain \(X_t \sim \text{Po}(\mu)\) marginally.

Now, if we re-write the time index as \(t = t(r,m) = (r - 1)s + m\), for \(r = 1, 2, \ldots\) and \(m = 1, \ldots, s\) we can define a periodic dependent model of orders \((p_1, \ldots, p_s)\). For instance, for monthly data, \(s = 12\) and \(r\) and \(m\) denote the year and month, respectively. The model would be

\[
X_t - \sum_{i=0}^{pm} y_{t(r,m)-i} \mid Y \overset{\text{ind}}{\sim} \text{Po} \left( \mu \left( 1 - \sum_{i=0}^{pm} \alpha_{t(r,m)-i} \right) \right).
\]
for a type A construction with levels 1 and 2 as in (5) and with $\sum_{i=0}^{p_m} \alpha_{t(r,m)-i} < 1$, and

$$Y_t \mid \mathbf{W} \sim \text{Bin} \left( \sum_{i=0}^{p_m} w_{t(r,m)-i}, \alpha_t \right)$$

for a type B construction with levels 1 and 3 as in (6). Here only type A obtains $X_t \sim \text{Po}(\mu)$ marginally, whereas type B obtains $X_t \sim \text{Po} \left( \mu \left\{ 1 - \alpha_t + \alpha_t(p_m + 1)/(p + 1) \right\} \right)$ marginally.

Alternatively, both constructions can also be suitably defined for a spatial setting. Let us assume that the index $t$ denotes spatial location instead of time, and consider $\partial_t$ to be the set of neighbours of location $t$. Then, a spatial dependence model would be

$$X_t - \sum_{i \in \partial_t} y_i \mid \mathbf{Y} \sim \text{Po} \left( \mu \left( 1 - \sum_{i \in \partial_t} \alpha_i \right) \right),$$

for a type A construction with levels 1 and 2 as in (5) and with $\sum_{i \in \partial_t} \alpha_i < 1$, and

$$Y_t \mid \mathbf{W} \sim \text{Bin} \left( \sum_{i \in \partial_t} w_i, \alpha_t \right),$$

for a type B construction with levels 1 and 3 as in (6). Again, only type A construction would obtain $X_t \sim \text{Po}(\mu)$ marginally.

Furthermore, combinations of any temporal with spatial dependences are also possible by an appropriate definition of the sums.

6 Concluding remarks

We have introduced two novel ways of defining dependence, in space and time, among Poisson random variables. Our proposal relies on the use of latent variables in a three levels hierarchical model. Both constructions have shown a good performance when modelling real datasets, with an advantage for type A model over type B model, for the specific maternal mortality dataset analysed here. Additionally, our models outperformed the most commonly used INAR(1) model in the maternal mortality dataset.
When using our proposals for modelling purposes, one has to be aware of their different features. Type B construction induces a correlation, given in Proposition 2, that only depends on two parameters. On the other hand, type A construction induces a more flexible autocorrelation, see Proposition 1, in the sense that it could be based on several more parameters.

Finally, these constructions are flexible enough to be used in different contexts. For the maternal mortality dataset, our models were used as sampling models to describe the law of the data. However, they can also be used as prior distributions for discrete functional integer-valued parameters, in a Bayesian nonparametric analysis.

Appendix

Full conditional distributions for model parameters and latent variables to perform posterior inference for type A and type B models. For simplicity we assume that $Y_t = 0$, $W_t = 0$ and $\alpha_t = 0$ for $t \leq 0$.

For type A model, distributions are:

i) For $Y_t$, $t = 1, \ldots, T$

$$f(y_t \mid \text{rest}) \propto \frac{\alpha_t \mu^{-p} \left\{ \prod_{j=0}^{p} \left(1 - \sum_{i=0}^{p} \alpha_{t+j-i}\right) \right\}^{-1}}{y_t!} \prod_{j=0}^{p} (x_{t+j} - \sum_{i=0}^{p} y_{t+j-i})! I_{\{0, \ldots, c_t\}}(y_t),$$

with $c_t = \min_{j=0, \ldots, p} \{x_{t+j} - \sum_{i=0, i \neq j}^{p} y_{t+j-i}\}$

ii) For $\alpha_t$, $t = 1, \ldots, T$

$$f(\alpha_t \mid \text{rest}) \propto \alpha_t^{\alpha_t+y_t-1} (1 - \alpha_t)^{b_n-1} e^{p \mu \alpha_t} \prod_{j=0}^{p} \left(1 - \sum_{i=0}^{p} \alpha_{t+j-i}\right)^{x_{t+j} - \sum_{i=0}^{p} y_{t+j-i}} I_{\{0, d_t\}}(\alpha_t)$$

where $d_t = \min_{j=0, \ldots, p} \left\{1 - \sum_{i=0, i \neq j}^{p} \alpha_{t+j-i}\right\}$
iii) For $\mu$

$$f(\mu \mid \text{rest}) = \text{Ga} \left( \mu \mid a_\mu + \sum_{t=1}^{T} x_t - \sum_{t=1}^{T} \sum_{i=1}^{p} y_{t-i}, b_\mu + T + \sum_{t=1}^{T} \sum_{i=1}^{p} \alpha_{t-i} \right)$$

For type B model, distributions are:

iv) For $Y_t$, $t = 1, \ldots, T$

$$f(y_t \mid \text{rest}) \propto \frac{\{\alpha_t \mu^{-1}(1 - \alpha_t)^{-2}\}^{y_t}}{y_t! (\sum_{i=0}^{p} w_{t-i} - y_t)!} \cdot I_{\{0, \ldots, m_t\}}(y_t),$$

with $m_t = \min\{x_t, \sum_{i=0}^{p} w_{t-i}\}$

v) For $W_t$, $t = 1, \ldots, T$

$$f(w_t \mid \text{rest}) \propto \left\{ \prod_{j=0}^{p} \left( \frac{\sum_{i=0}^{p} w_{t+j-i}}{y_{t+j}} \right) \right\} \left\{ \frac{\mu}{p+1} \prod_{j=0}^{p} (1 - \alpha_{t+j}) \right\}^{w_t} \frac{1}{w_t!} \cdot I_{\{h_t, h_t+1 \ldots\}}(w_t),$$

where $h_t = \max_{j=0, \ldots, p} \{y_{t+j} - \sum_{i=0, i \neq j}^{p} w_{t+j-i}\}$

vi) For $\alpha_t$, $t = 1, \ldots, T$

$$f(\alpha_t \mid \text{rest}) \propto \alpha_t^{\alpha_t+y_t-1} (1 - \alpha_t)^{b_\alpha+x_t+\sum_{i=0}^{p} w_{t-i} - 2y_t - 1} \cdot e^{\mu \alpha_t} \cdot I_{(0,1)}(\alpha_t)$$

vii) For $\mu$

$$f(\mu \mid \text{rest}) = \text{Ga} \left( \mu \mid a_\mu + \sum_{t=1}^{T} (x_t + w_t - y_t), b_\mu + T \left( \frac{p+2}{p+1} \right) - \sum_{t=1}^{T} \alpha_t \right)$$

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Table 1: L-measure with $\nu = 1/2$ when fitting models of types A and B for $p = 0, \ldots, 6$ to maternal mortality data for the 32 states of Mexico. Smallest value, within each model type is shown in bold. L-measure for INAR(1) model is also included in the last column.
Figure 2: Maternal mortality for Baja California state. Best fitting models. Type A with $p = 1$ (left) and type B with $p = 4$ (right). Observed data (solid grey), point prediction (thick dotted red) and 95% credible interval (dotted red).

Figure 3: Maternal mortality for Coahuila state. Best fitting models. Type A with $p = 1$ (left) and type B with $p = 4$ (right). Observed data (solid grey), point prediction (thick dotted red) and 95% credible interval (dotted red).
Figure 4: Maternal mortality for CDMX (Mexico City) state. Best fitting models. Type A with $p = 1$ (left) and type B with $p = 4$ (right). Observed data (solid grey), point prediction (thick dotted red) and 95% credible interval (dotted red).

Figure 5: Maternal mortality for Colima state. Best fitting models. Type A with $p = 1$ (left) and type B with $p = 4$ (right). Observed data (solid grey), point prediction (thick dotted red) and 95% credible interval (dotted red).