Saari’s homographic conjecture for a planar equal-mass three-body problem under the Newton gravity

Toshiaki Fujiwara¹, Hiroshi Fukuda¹, Hiroshi Ozaki² and Tetsuya Taniguchi¹

¹ College of Liberal Arts and Sciences, Kitasato University, 1-15-1 Kitasato, Sagamihara, Kanagawa 252-0329, Japan
² General Education Program Center, Tokai University, Shimizu Campus, 3-20-1, Orido, Shimizu, Shizuoka 424-8610, Japan

E-mail: fujiwara@kitasato-u.ac.jp, fukuda@kitasato-u.ac.jp, ozaki@tokai-u.jp and tetsuya@kitasato-u.ac.jp

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Abstract
Saari’s homographic conjecture in an N-body problem under the Newton gravity is as follows. Configurational measure \( \mu = \sqrt{I U} \), which is the product of the square root of the moment of inertia \( I = \sum m_k (\sum m_k)^{-1} \sum m_i m_j r_{ij}^2 \) and the potential function \( U = \sum m_i m_j / r_{ij} \), is constant if and only if the motion is homographic. Where \( m_k \) represents the mass of the body \( k \) and \( r_{ij} \) represents the distance between bodies \( i \) and \( j \). We prove this conjecture for the planar equal-mass three-body problem. In this work, we use three sets of shape variables. In the first step, we use \( \zeta = 3 q_3 / (2(q_2 - q_1)) \), where \( q_k \in \mathbb{C} \) represents the position of the body \( k \). Using \( r_1 = r_{23} / r_{12} \) and \( r_2 = r_{31} / r_{12} \) in the intermediate step, we finally use \( \mu \) itself and \( \rho = r_{12} r_{23} r_{31} \). The shape variables \( \mu \) and \( \rho \) make our proof simple.

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1. Saari’s homographic conjecture

In 2005, Donald Saari formulated his conjecture in the following form [10, 11]; in the N-body problem under the potential function

\[
U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{r_{ij}^{\alpha}}, \quad \alpha > 0,
\]

(1)

a motion has a constant configurational measure

\[
\mu = I^{\alpha/2} U
\]

(2)
if and only if the motion is homographic. Here, \( r_{ij} \) represents the mutual distance between the bodies \( i \) and \( j \), and \( I \) represents the moment of inertia

\[
I = \left( \sum_{1 \leq k \leq N} m_k \right)^{-1} \sum_{1 \leq i < j \leq N} m_i m_j r_{ij}^2. \tag{3}
\]

Diacu et al called this conjecture the ‘Saari homographic conjecture’ and partly proved this conjecture for some cases [2]. Recently, the present authors proved this conjecture for the planar equal-mass three-body problem for \( \alpha = 2 \) [3]. In this paper, we extend our proof to \( \alpha = 1 \), the Newton gravity.

In section 2, we derive the equations of motion for the size change, rotation and shape change. To do this, we use the shape variable \( \zeta \),

\[
\zeta = \frac{3}{2} \frac{q_3}{q_2 - q_1}, \tag{4}
\]

introduced by Moeckel and Montgomery [8]. Here, \( q_k \in \mathbb{C}, k = 1, 2, 3 \), represents the position of the body \( k \). Then, in section 3, we investigate motions with \( \mu = \text{constant} \) and non-homographic, and we derive a necessary condition that must be satisfied by such motion. The contents in sections 2 and 3 are the reviews of our previous paper [3], although we changed a few notations. To prove the Saari conjecture, we will show that no finite orbit satisfies the necessary condition. To attain this purpose, the expression of the necessary condition by \( \zeta \) is too complex. To simplify the expression, we will use another set of shape variables,

\[
r_1 = |\zeta - 1/2| = r_{23}/r_{12}, \quad r_2 = |\zeta + 1/2| = r_{31}/r_{12}. \tag{5}
\]

Then, using the invariance of the system under the permutations of \( \{q_1, q_2, q_3\} \), we rewrite the necessary condition in another set of shape variables \( \mu \) itself and \( \rho \),

\[
\mu = \frac{l^{1/2}}{r_{12}} \left( \frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}} \right), \quad \rho = \frac{l^{1/2}}{r_{12} r_{23} r_{31}}, \tag{6}
\]

that are manifestly invariant under the permutations. Since we are considering \( \mu = \text{constant} \) orbits, variables \( \mu \) and \( \rho \) make our proof easy. This expression is given in section 4. The proof of the Saari conjecture is given in section 5. In section 6, we give discussions.

### 2. Equations of motion

In this section, we summarize the equations of motion for \( \alpha = 1 \) in terms of size, rotation and shape. We do not assume \( \mu = \text{constant} \) in this section.

Let \( q_k \in \mathbb{C} \) be the position and mass \( m_k = 1 \) for \( k = 1, 2, 3 \). We take the center-of-mass frame \( \sum q_k = 0 \). The Lagrangian is given by

\[
L = \frac{1}{2} \sum \left| \frac{dq_k}{dt} \right|^2 + U. \tag{7}
\]

We take the shape variable \( \zeta \in \mathbb{C} \) in (4). This variable is invariant under the scaling and rotation, \( q_k \rightarrow \lambda e^{i\theta} q_k \) with \( \lambda, \theta \in \mathbb{R} \). Thus, \( \zeta \) depends only on shape. Let us define \( \xi_k = q_k/(q_2 - q_1) \). Then, we have

\[
\xi_1 = -\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_2 = \frac{1}{2} - \frac{\zeta}{3}, \quad \xi_3 = \frac{2\zeta}{3}. \tag{8}
\]

Since the triangles \( q_1 q_2 q_3 \) and \( \xi_1 \xi_2 \xi_3 \) are similar and have the same orientation, we have two variables \( I \geq 0 \) and \( \theta \in \mathbb{R} \), such that

\[
q_k = \sqrt{I} e^{i\theta} \frac{\xi_k}{\sqrt{\sum |\xi_k|^2}}. \tag{9}
\]

We take $I$, $\theta$ and $\zeta$ for dynamical variables. In the following, we identify $\zeta = x + iy$ and $x = (x, y) \in \mathbb{R}^2$. By direct calculations, we obtain the Lagrangian

$$L = \frac{\dot{I}^2}{8I} + \frac{I}{2} \left( \dot{\theta} + \frac{\frac{2}{3} \dot{x} \wedge \dot{x}}{1 + \frac{4}{3} |x|^2} \right)^2 + \frac{I}{2} \frac{\frac{2}{3} |x|^2}{(1 + \frac{4}{3} |x|^2)^2} + \frac{\mu(x)}{\sqrt{I}}. \quad (10)$$

Here, $\dot{}$ represents time derivative, $x \wedge \dot{x} = \dot{x} y - y \dot{x}$ and

$$\mu(x) = \frac{1}{\sqrt{\frac{2}{3} + \frac{2}{3} |x|^2}} \left( 1 + \frac{1}{\sqrt{(x - 1/2)^2 + y^2}} + \frac{1}{\sqrt{(x + 1/2)^2 + y^2}} \right). \quad (11)$$

Since, $\theta$ is cyclic, the angular momentum $C$ is constant of motion:

$$C = \frac{\partial L}{\partial \dot{\theta}} = I \left( \dot{\theta} + \frac{\frac{2}{3} \dot{x} \wedge \dot{x}}{1 + \frac{4}{3} |x|^2} \right). \quad (12)$$

Therefore, the total energy $E$ is given by

$$E = \frac{\dot{I}^2}{8I} + \frac{C^2}{2I} + \frac{I}{2} \frac{\frac{2}{3} |x|^2}{(1 + \frac{4}{3} |x|^2)^2} - \frac{\mu(x)}{\sqrt{I}}. \quad (13)$$

The three terms in the kinetic energy are kinetic energy for the size change for the rotation and for the shape change, respectively. The equation of motion for $I$ yields the Lagrange–Jacobi identity,

$$\ddot{I} = 4E + 2U. \quad (14)$$

From this equation, we obtain the following ‘Saari relation’ [10]:

$$\frac{d}{dt} \left( \frac{I^2}{2} \frac{\frac{4}{3} |x|^2}{(1 + \frac{4}{3} |x|^2)^2} \right) = \sqrt{I} \frac{d\mu}{dt}. \quad (15)$$

Using the ‘time’ variable $s$ defined by

$$\frac{ds}{dt} = \frac{1}{2I} \left( 1 + \frac{4}{3} |x|^2 \right), \quad (16)$$

the Saari relation is written as

$$\frac{d}{ds} \left( \frac{1}{6} \left| \frac{dx}{ds} \right|^2 \right) = \sqrt{I} \frac{d\mu}{ds}. \quad (17)$$

The equation of motion for $x$ in terms of $s$ is

$$\frac{d^2x}{ds^2} = 4C - \frac{\frac{2}{3} x \wedge \frac{dx}{ds}}{1 + \frac{4}{3} |x|^2} \left( \frac{dy}{ds}, -\frac{dx}{ds} \right) + 3\sqrt{I} \frac{d\mu}{dx}. \quad (18)$$

Up to here, we did not assume $\mu = \text{constant}$.

3. Necessary condition

Now, we consider a motion with $\mu = \text{constant}$. By the Saari relation (15), we have

$$\left| \frac{dx}{ds} \right| = v, \quad (19)$$

with constant $v \geq 0$.

For the case when $v = 0$, $dx/ds = 0$ and then $d^2x/ds^2 = 0$. The equation of motion (18) yields $\partial \mu / \partial x = 0$. Namely, the motion is homographic and the system remains one of the central configurations.

Let us examine the case $v > 0$. In this case, the point $x(s)$ moves on the curve $\mu(x)$ with a finite speed $v$. Since the number of points $\partial \mu / \partial x = 0$ are 5, we can always take a finite arc
on which \( \partial \mu / \partial x \neq 0 \). To satisfy \( d\mu / ds = 0 \), the velocity \( dx/ds \) must be orthogonal to \( \partial \mu / \partial x \), so we have

\[
\frac{dx}{ds} = \frac{\epsilon v}{|\partial \mu / \partial x|} \left( -\frac{\partial \mu}{\partial y} \frac{\partial \mu}{\partial x} \right).
\]

(18)

Here, \( \epsilon = \pm 1 \) determines the direction of the motion. Then, acceleration (16) is given by

\[
\frac{d^2 x}{ds^2} = \left( \frac{\epsilon v}{1 + 4|x|^2/3} \frac{\partial \mu}{\partial x} \right) \left( 4C - \frac{8\epsilon v}{3|\partial \mu / \partial x|} x \cdot \frac{\partial \mu}{\partial x} \right) + 3\sqrt{T} \frac{\partial \mu}{\partial x}.
\]

(19)

Thus, velocity (18) and acceleration (19) determine the curvature of this orbit

\[
\kappa = \frac{1}{1 + 4|x|^2/3} \left( \frac{4C}{v} + \frac{8\epsilon v}{3|\partial \mu / \partial x|} (x \cdot \frac{\partial \mu}{\partial x}) \right) - \frac{3\epsilon \sqrt{T}}{v^2} \left| \frac{\partial \mu}{\partial x} \right|.
\]

(20)

On the other hand, the curve \( \mu(x) = \) constant has its own curvature,

\[
\kappa = \frac{\epsilon}{|\partial \mu / \partial x|^3} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right).
\]

(21)

Equating the two expressions for \( \kappa \), we have a necessary condition for the motion

\[
\sqrt{T} = -\frac{4Cv}{3(1 + 4|x|^2/3)|\partial \mu / \partial x|} + \frac{8v^2}{9(1 + 4|x|^2/3)|\partial \mu / \partial x|^2} \left( x \cdot \frac{\partial \mu}{\partial x} \right)
\]

\[
-\frac{v^2}{3|\partial \mu / \partial x|^2} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right).
\]

(22)

This is the condition that any motion with \( \mu = \) constant and \( dx/dt \neq 0 \) must satisfy. The equation of motion is invariant under the scale transformation \( q_k \to \lambda q_k \) and \( t \to \lambda^{1/2} t \). This transformation makes \( \sqrt{T} \to \lambda \sqrt{T} \), \( C \to \lambda^{1/2} C \), \( x \to x \), \( s \to \lambda^{-1/2} s \), and \( v \to \lambda^{1/2} v \). Using this invariance, we can take \( v = \sqrt{3} \) without losing generality. We write \( C \) for \( \epsilon C \). Then, the necessary condition is

\[
\sqrt{T} = -\frac{4C}{\sqrt{3}(1 + 4|x|^2/3)|\partial \mu / \partial x|} + \frac{8}{3(1 + 4|x|^2/3)|\partial \mu / \partial x|^2} \left( x \cdot \frac{\partial \mu}{\partial x} \right)
\]

\[
-\frac{1}{|\partial \mu / \partial x|^4} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right).
\]

(23)

and the energy is given by

\[
E = \frac{1}{2} \left( \frac{d\sqrt{T}}{dt} \right)^2 + \frac{C^2 + 1}{2T} - \frac{\mu}{\sqrt{T}}
\]

(24)

Substituting \( d\sqrt{T}/dt = (\partial \sqrt{T}/\partial x) \cdot (dx/ds)(ds/dt) \), \( dx/ds \) in (18) and condition (23) into this expression for the energy, we will obtain the necessary condition expressed only by the shape variable \( x \). However, condition (23) in \( x \) turns out to be very complex to treat. In the following section, we will rewrite condition (23) in a concise form.

4. Invariance of the necessary condition

Since we are considering the equal mass case, the theory is invariant under the permutations of positions \( \{q_i\} \). The exchange of \( q_1 \) and \( q_2 \) makes \( \zeta \to -\zeta \) and \( x \to -x \). The invariance of the necessary condition (23) is manifest. On the other hand, the cyclic permutation \( q_1 \to q_2 \to q_3 \to q_1 \) makes

\[
\zeta \to \zeta' = \frac{3}{2} q_1 \frac{q_3 - q_2}{q_3 - q_2} = 1 \frac{3/2 + \zeta}{2 1/2 - \zeta}.
\]

(25)

The invariance of (23) under this transformation is not manifest. In this section, we will rewrite the necessary condition in a manifestly invariant form.
4.1. Invariants

Under the map (25), the Lagrange points $\zeta = \pm i \sqrt{3}/2$ are fixed and the Euler points $\zeta = -3/2, 0, 3/2$ are cyclically permuted. Let us define $\mu_k = \ell_l^{ij}/r_{ij}$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. Expressions by $\zeta$ are

$$\mu_1 = \frac{1}{|\zeta - 1/2|} \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}, \quad \mu_2 = \frac{1}{|\zeta + 1/2|} \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}, \quad \mu_3 = \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}. \quad (26)$$

These three $\mu_k$ are also cyclically permuted by (25). Note that the exchange $q_i \leftrightarrow q_j$ makes the exchange $\mu_i \leftrightarrow \mu_j$. Therefore, $\mu = \mu_1 + \mu_2 + \mu_3$ is invariant under the permutations of $q_i$.

The kinetic energy for the shape change must be invariant. Actually, we can easily check the invariance of

$$\frac{4}{3} \frac{|d\zeta|^2}{(1 + \frac{4}{3}|\zeta|^2)^2}. \quad (27)$$

So, it is natural to treat the space of $\zeta$ as a metric space whose distance is given by equation (27), and the map (25) is the isometric transformation. Actually, Hsiang and Straume [4, 5], Chenciner and Montgomery [1], Montgomery [9] and Mockel [7] showed that this metric space is the ‘shape sphere’ and the distance (27) is the distance on the shape sphere. Kuwabara and Tanikawa also noted that the shape sphere is useful to investigate the equal-mass free-fall problem [6, 12]. The map (25) makes the shape sphere $2\pi/3$ rotation around the axis that connects the two Lagrange points. The map $\zeta \rightarrow -\zeta$ makes the $\pi$ rotation around the axis that connects one of the Euler points (corresponds to $x = 0$) and one of the two-body collisions (corresponds to $x = \infty$).

Let us use the notations in the tensor analysis. We write $\zeta = x^1 + ix^2, x = (x, y) = (x^1, x^2)$ and $\partial_i = \partial/\partial x^i$. The metric tensor $g_{ij}$ and its inverse are

$$g_{ij} = \frac{4}{3} \frac{\delta_{ij}}{(1 + \frac{4}{3}|\zeta|^2)^2}, \quad (g_{ij})^{-1} = g^{ij} = \frac{3}{4} \left(1 + \frac{4}{3}|\zeta|^2\right)^2 \delta^{ij}, \quad (28)$$

where $\delta_{ij} = \delta^{ij}$ are the Kronecker delta

$$\delta_{ij} = \delta^{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \quad (29)$$

Let $|g|$ be the determinant of $g_{ij}$,

$$|g| = \text{det}(g_{ij}) = \frac{16}{9} \frac{1}{(1 + \frac{4}{3}|\zeta|^2)^3}. \quad (30)$$

As mentioned above, the configurational measure $\mu$ is invariant. One obvious invariant is the magnitude of the gradient vector of $\mu$. We write

$$|\nabla \mu|^2 = \sum_{i,j} g^{ij}(\partial_i \mu)(\partial_j \mu) = \frac{3}{4} \left(1 + \frac{4}{3}|\zeta|^2\right)^2 \left|\frac{\partial \mu}{\partial x}\right|^2. \quad (31)$$

Therefore, the first term of the right-hand side of the necessary condition (23) is simply $-2C/|\nabla \mu|$. The other obvious invariant is the Laplacian of $\mu$,

$$\Delta \mu = \sum_{ij} \frac{1}{|g|} \partial_i (g^{ij} \sqrt{|g|} \partial_j \mu) = \frac{3}{4} \left(1 + \frac{4}{3}|\zeta|^2\right)^2 \frac{\partial}{\partial x} \cdot \frac{\partial \mu}{\partial x}. \quad (32)$$
Now, let us consider the following invariant:

$$
\lambda = \sum_{ij} g^{ij} (\partial_i \mu) (\partial_j \nabla \mu)^2 = \frac{3}{4} \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right)^2 \frac{\partial \mu}{\partial \mathbf{x}} \cdot \frac{\partial \mu}{\partial \mathbf{x}} |\nabla \mu|^2. \tag{33}
$$

Explicitly performing the differentials, it yields

$$
\lambda = 3 \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right)^3 \left(\mathbf{x} \cdot \frac{\partial \mu}{\partial \mathbf{x}} \right) |\frac{\partial \mu}{\partial \mathbf{x}}|^2 + \frac{9}{16} \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right)^4 \frac{\partial \mu}{\partial \mathbf{x}} \cdot \frac{\partial \mu}{\partial \mathbf{x}} \frac{\partial \mu}{\partial \mathbf{x}}. \tag{34}
$$

Using this expression, the second and the third terms in the necessary condition (23) are simply expressed as \(\lambda / (2|\nabla \mu|^4) - \Delta \mu / |\nabla \mu|^2\). Thus, the necessary condition is expressed in the following invariant form:

$$
\sqrt{I} = -\frac{2C |\nabla \mu| + \lambda}{2|\nabla \mu|^3} - \frac{\Delta \mu}{|\nabla \mu|^2}. \tag{35}
$$

The last obvious invariant that we will use is

$$
D \phi = \frac{1}{\sqrt{|g|}} \sum_{i,j} \epsilon^{ij} (\partial_i \mu) (\partial_j \phi) = \frac{3}{4} \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right)^2 \frac{\partial \mu}{\partial \mathbf{x}} \wedge \frac{\partial \phi}{\partial \mathbf{x}}, \tag{36}
$$

for any invariant \(\phi\). Where, \(\epsilon^{ij}\) is Lévi-Civita’s anti-symmetric symbol,

$$
\epsilon^{ij} = \begin{cases} 
1 & \text{for } (i,j) = (1,2), \\
-1 & \text{for } (i,j) = (2,1), \\
0 & \text{for } i = j.
\end{cases} \tag{37}
$$

4.2. Invariant variables

For the Newton potential, it is natural to use the variables \(r_1\) and \(r_2\) defined by (5). Relations between \(\mu_4\) defined in (26) and \(r_1\) and \(r_2\) are

$$
\mu_1 = r_1^{-1} \mu_3, \quad \mu_2 = r_2^{-1} \mu_3, \quad \mu_3 = \sqrt{(1 + r_1^2 + r_2^2)/3}. \tag{38}
$$

Now, consider the expression for the above invariants \(|\nabla \mu|^2, \Delta \mu, \lambda\) in terms of \(r_1\) and \(r_2\). Let us write one of them \(\psi(r_1, r_2)\). It is composed by differentials of \(\mu\) by \(r_1\) or \(r_2\) and products of \(r_1\) and \(r_2\). Then, the result is composed of terms of rational function of \(\sqrt{(1 + r_1^2 + r_2^2)/3}, r_1\) and \(r_2\), namely \(\mu_3, \mu_3/\mu_1\) and \(\mu_3/\mu_2\). Then, \(\psi\) has the following form:

$$
\psi = f(r_1, r_2) + g(r_1, r_2) \sqrt{\frac{1 + r_1^2 + r_2^2}{3}} = f \left( \frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2} \right) + g \left( \frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2} \right) \mu_3. \tag{39}
$$

Here, \(f\) and \(g\) represent some rational functions. The function \(\psi\) is invariant under the permutation of \(q_i\), namely the permutation of \(\mu_i\). So, it must be a ratio of some symmetric polynomials of \(\mu_i\). Therefore, it must have the following expression:

$$
\psi = h(\mu, \nu, \rho), \tag{40}
$$

where \(h\) is a rational function of elementary symmetric polynomials

$$
\mu = \mu_1 + \mu_2 + \mu_3, \quad \nu = \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1, \quad \rho = \mu_1 \mu_2 \mu_3. \tag{41}
$$
The expression in terms of $\mu_1$, or $\mu, \nu, \rho$ is not unique, since, by relation (38), there is an identity $\mu_1^{-2} + \mu_2^{-2} + \mu_3^{-2} = 3$. Namely,

$$\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2 = 3\mu_1^2 \mu_2^2 \mu_3^2.$$  

(42)

Therefore, we can eliminate $\nu$, using

$$\nu = \sqrt{2\mu \rho + 3\rho^2}.$$  

(43)

The expression of $\psi = h(\mu, \sqrt{2\mu \rho + 3\rho^2}, \rho)$ is unique. Thus, the necessary condition will be expressed by a function of invariant shape variables $\mu$ and $\rho$.

Let us express $|\nabla \mu|^2$ by $\mu$ and $\rho$. In terms of $r_i$, it is

$$|\nabla \mu|^2 = \frac{(1 + r_1^2 + r_2^2)}{3} \left[ \left( \frac{\partial \mu}{\partial r_1} \right)^2 + \left( \frac{\partial \mu}{\partial r_2} \right)^2 + \frac{r_1^2 + r_2^2 - 1}{r_1 r_2} \frac{\partial \mu}{\partial r_1} \frac{\partial \mu}{\partial r_2} \right].$$  

(44)

By a direct calculation, we obtain

$$|\nabla \mu|^2 = \frac{1 + r_1^2 + r_2^2}{9r_1^2 r_2^2} \left( 2r_1^2 + r_2^2 \right)$$

$$+ r_1^2 \left( r_2^2 (r_1 + r_2) - r_1 r_2 (r_1^2 + r_2^2) - r_1^2 r_2 - 4r_1^2 r_2 (r_1 + r_2) \right)$$

$$+ (2r_1^2 + r_1^2 r_2 - 2r_1^2 r_2^2 - 4r_1^2 r_2^2 - r_1^2 r_2^2 + r_1^2 r_2^2 + 2r_2^2)$$

$$+ r_1 r_2 \left( r_1^2 + r_2^2 \right) + 2 \left( r_1^2 + r_2^2 \right) - r_1 r_2).$$  

(45)

Substituting $r_1 = \mu_3/\mu_1$ and $r_2 = \mu_3/\mu_2$, we obtain

$$|\nabla \mu|^2 = \frac{\left( \mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2 \right)}{2} \left( - \left( \mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2 \right) \right)$$

$$- \mu_1^4 \mu_2^4 \mu_3^4 \left( \mu_1^2 + \mu_2^2 + \mu_3^2 \right) - 4\mu_1^4 \mu_2^4 \mu_3^4 \left( \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1 \right)$$

$$+ 2 \left( \mu_1^2 \mu_2^2 \mu_3^2 \mu_1^2 + \ldots \right) + \left( \mu_1^2 \mu_2^2 \mu_3^2 + \ldots \right).$$  

(46)

In the last line, dots in the parentheses represent similar five terms of permutation of $\mu_1, \mu_2, \mu_3$.

Then expressing by $\mu, \nu, \rho$, we obtain the expression

$$|\nabla \mu|^2 = \frac{\nu^2}{9 \rho^6} \left[ - \nu^2 + 7 \mu \nu^5 \rho + 2 \mu^4 \nu^2 \rho^2 \right.$$

$$- 22 \mu^2 \nu^3 \rho^2 - 3 \nu^4 \rho^2 - 4 \mu^5 \rho^3 + 24 \mu \nu^2 \rho^3 + 18 \mu^2 \nu^2 \rho^3 - 27 \mu^2 \rho^4 \right].$$  

(47)

As mentioned above, expressions (46) and (47) are not unique due to identity (42). Eliminating $\nu$, we finally obtain the following unique expression:

$$|\nabla \mu|^2 = - \mu^2 + 2 \mu^4 + 6 \mu \rho - 9 \rho^2 - 3(2 \mu^2 - \mu \rho + 3 \rho^2) \sqrt{2 \mu \rho + 3 \rho^2}. $$  

(48)

Thus, we obtain the expression for $|\nabla \mu|^2$ in manifestly invariant variables $\mu$ and $\rho$.

By a similar way, $\Delta \mu$ in $(r_1, r_2)$ and $(\mu, \rho)$ are

$$\Delta \mu = \frac{1 + r_1^2 + r_2^2}{3} \left( \frac{\partial \mu}{\partial r_1} \left( \frac{\partial \mu}{\partial r_1} \right) + \frac{\partial \mu}{\partial r_2} \left( \frac{\partial \mu}{\partial r_2} \right) \right)$$

$$+ \frac{r_1^2 + r_2^2 - 1}{r_1 r_2} \frac{\partial \mu}{\partial r_1} \frac{\partial \mu}{\partial r_2} \right).$$  

(49)

$$\Delta \mu = \mu + 2 \mu^3 + 6 \rho - 6 \mu \sqrt{2 \mu \rho + 3 \rho^2}.$$  

Similarly, the expressions for $\lambda$ are

$$\lambda = \frac{1 + r_1^2 + r_2^2}{3} \left( \frac{\partial \mu}{\partial r_1} \frac{\partial \mu}{\partial r_1} + \frac{\partial \mu}{\partial r_2} \frac{\partial \mu}{\partial r_2} + \frac{r_1^2 + r_2^2 - 1}{2 r_1 r_2} \left( \frac{\partial \mu}{\partial r_1} \frac{\partial \mu}{\partial r_2} \frac{\partial \mu}{\partial r_1} \right) \right) \left| \nabla \mu \right|^2.$$  

(49)
\[ \lambda = \frac{1}{2}(4\mu^3 - 24\mu^5 + 32\mu^7 - 72\mu^9 + 660\mu^4\rho + 324\mu^2 \rho^2 \\
+ 36\mu^3 \rho^2 - 432\rho^3 + 891\mu^2 \rho^3 + 2349\mu^4 \rho^4 - 243\rho^6 \\
+ 3(24\mu^3 - 60\mu^5 - 156\mu^7 \rho + 28\mu^4 \rho^2 + 324\mu^2 \rho^2 \\
- 93\mu^3 \rho^2 - 216\rho^3 - 27\mu^4 \rho^3 + 81\mu^2 \rho^4)\sqrt{2\mu \rho + 3\rho^2}). \]  

Finally, \((D\rho)^2\) is also invariant under the exchange of \(q_i\); therefore, it has an expression by \(\mu\) and \(\rho\),

\[
(D\rho)^2 = \frac{1}{4} \left( \frac{1}{(\partial^2 T) \rho} \right)^2 \\
(D\rho)^2 = \frac{\rho^2(2\mu + 3\rho)}{4} \left( -2(2\mu + 3\rho)(4\mu^4 + 134\mu^2 \rho - 12\mu^3 \rho - 177\rho^2 + 9\mu^2 \rho^2) \\
+ 2(28\mu^3 + 108\rho - 36\mu^2 \rho - 45\mu^2 \rho^2 + 54\rho^3)\sqrt{2\mu \rho + 3\rho^2} \right). 
\]

### 5. Proof of the Saari conjecture

In the previous section, we find the expression for the necessary condition \((34)\) in terms of \(\mu\) and \(\rho\) by \((48), (49)\) and \((50)\). Since we are assuming \(\mu = \text{constant}\), the time-dependent variable is only \(\rho\). Therefore, \(\partial T/\partial t = (\partial T/\partial \rho)(\partial \rho/\partial t)\). Using \((37)\),

\[
\left( \frac{\partial T}{\partial t} \right) = \frac{1}{\rho^2} \frac{(D\rho)^2}{\partial \rho} \left( \frac{\partial T}{\partial \rho} \right)^2. 
\]

Substituting this expression and the necessary condition \((34)\) into the expression of the energy \((24)\), we obtain the necessary condition for \(\rho\) with three parameters \(E, C\) and \(\mu\),

\[
E = \frac{1}{2\rho^2} \frac{(D\rho)^2}{\partial \rho} \left( \frac{\partial T}{\partial \rho} \right)^2 + \frac{C^2 + 1}{2\rho} - \frac{\mu}{\sqrt{\rho}}. 
\]

If there is some finite motion with \(\mu = \text{constant}\) and non-homographic, then this condition must be satisfied by some finite range of \(\rho\). However, since the right-hand side of \((52)\) is the analytic function of \(\rho\), condition \((52)\) must be satisfied for all range of \(\rho\).

In the vicinity of \(\rho = 0\), we have the expansion of \((34)\):

\[
\sqrt{\rho} = a_0 + a_{1/2}\sqrt{\rho} + a_1 \rho + O(\rho^{3/2}), 
\]

with

\[
a_0 = \frac{2(1 - \mu^2 + C\sqrt{1 + 2\mu^2})}{\mu(1 - 2\mu^2)}, \\
a_{1/2} = \frac{3\sqrt{2}\left((-2 + \mu^2)\sqrt{-1 + 2\mu^2} - 2C(-1 + 2\mu^2)\right)}{(1 - 2\mu^2)^2\sqrt{\mu(-1 + 2\mu^2)}}, \\
a_1 = \frac{3\left((-2 + \mu^2)(1 + 6\mu^2) - 2C(1 + 7\mu^2)\sqrt{-1 + 2\mu^2}\right)}{\mu^2(-1 + 2\mu^2)^3}, 
\]

and

\[
|\nabla \rho|^2 = \mu^2(-1 + 2\mu^2) - 6\sqrt{2}\mu^5 \sqrt{\rho} + 6\mu \rho + O(\rho^{3/2}), \\
(D\rho)^2 = -4\mu^6 \rho^2 + O(\rho^{5/2}). 
\]

Then we obtain the power series expansion of \((52)\) by \(\sqrt{\rho}\) up to the order \(\rho\) at \(\rho = 0\).
The term of order $\rho^0$ in (52) determines $E$. Therefore, this order gives no information for $C$ and $\mu$. The coefficient of order $\sqrt{\rho}$ is

$$0 = \frac{-3\mu^2(1 + C\sqrt{-1 + 2\mu^2})^2((-2 + \mu^2) - 2C\sqrt{-1 + 2\mu^2})}{4\sqrt{2}(-1 + \mu^2 - C\sqrt{-1 + 2\mu^2})^3}. \quad (54)$$

The solutions $C$ of this equation are

$$C = -\frac{1}{\sqrt{-1 + 2\mu^2}}, \quad -\frac{2 + \mu^2}{2\sqrt{-1 + 2\mu^2}}. \quad (55)$$

For the case $C = -1/\sqrt{-1 + 2\mu^2}$,

$$\sqrt{I} = \frac{2\mu}{-1 + 2\mu^2} + \frac{3\sqrt{2}\mu^{3/2}}{(-1 + 2\mu^2)^{3/2}}\sqrt{\rho} + \frac{9(1 + 2\mu^2)}{(-1 + 2\mu^2)^3}\rho + O(\rho^{3/2}), \quad (56)$$

and the coefficient of order $\rho^1$ in equation (52) is

$$0 = -\frac{9\mu(-2 + \mu^2)}{16(-1 + 2\mu^2)^4}. \quad (57)$$

while the right-hand side is always negative since $\mu = \sqrt{(1 + r_1^2 + r_2^2)/3 (1 + 1/r_1 + 1/r_2)} \geq 3$. For the case $C = (-2 + \mu^2)/(2\sqrt{-1 + 2\mu^2})$, the coefficient $a_{1/2}$ vanishes,

$$\sqrt{I} = \frac{\mu}{-1 + 2\mu^2} - \frac{3(-2 + \mu^2)}{(-1 + 2\mu^2)^3}\rho + O(\rho^{3/2}), \quad (58)$$

and the coefficient of order $\rho^1$ in equation (52) is

$$0 = \frac{3\mu(-2 + \mu^2)}{4(-1 + 2\mu^2)^4}. \quad (59)$$

while the right-hand side is always positive for $\mu \geq 3$.

Thus, there are no parameters $C$ and $\mu$ that satisfy the necessary condition (52). This completes the proof for the Saari homographic conjecture.

6. Discussions

We have proved the Saari conjecture for the equal-mass planar three-body problem under the Newton gravity.

The symmetry under the permutation of the positions $\{q_1, q_2, q_3\}$ plays a crucial role in our method. For the equal mass and Newton potential case, the necessary condition (34) is a symmetric rational function of $\mu_1, \mu_2$ and $\mu_3$. Thus, it is a function of $\mu$ and $\rho$ as in equation (40). This makes our proof simple.

The next step will be the case with the general mass ratio and general homogeneous potential $U = \sum m_i m_j/\rho_{ij}^\alpha, \alpha > 0$. For this case, however, an invariant function under the permutation for the suffix of bodies will not have a simple form of manifestly invariant variables such as $\mu$ and $\rho$. We hope, some day, someone may find a proof for the conjecture for the general mass ratio under the Newton potential in some extension of our method. On the other hand, we are afraid that it is hard to extend our method to general $\alpha$. We would have to find a completely new method for general $\alpha$.

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