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1 Introduction

1.1 Main results

The purpose of this work is to prove the self duality of Hitchin’s integrable system: Hitchin’s system for a complex reductive Lie group $G$ is dual to Hitchin’s system for the Langlands dual group $L^G$. This statement can be interpreted at several levels.

- To start with, there is an isomorphism (depending on the choice of an invariant bilinear pairing) between the bases of the Hitchin systems for $G$ and $L^G$, interchanging the discriminant divisors.

- The general fiber of the neutral connected component $\text{Higgs}_0$ of Hitchin’s system for $G$ is an abelian variety. We show that it is dual to the corresponding fiber of the neutral connected component $L^G\text{Higgs}_0$ of the Hitchin system for $L^G$.

- The non-neutral connected components $\text{Higgs}_\alpha$ form torsors over $\text{Higgs}_0$. According to the general philosophy of [DP08], these are dual to certain gerbes. In our case, we identify these duals as natural gerbes over $L^G\text{Higgs}_0$. The gerbe $\text{Higgs}$ of $G$-Higgs bundles was introduced and analyzed in [DG02]. This serves as a universal object: we show that the gerbes involved in the duals of the non-neutral connected components $\text{Higgs}_\alpha$ are induced by $\text{Higgs}$.

- More generally, we establish a duality over the complement of the discriminant between the gerbe $\text{Higgs}$ of $G$-Higgs bundles and the gerbe $L^G\text{Higgs}$ of $L^G$-Higgs bundles, which incorporates all the previous dualities.

- Finally, the duality of the integrable systems lifts to an equivalence of the derived categories of $\text{Higgs}$ and $L^G\text{Higgs}$. A striking corollary is the construction of eigensheaves for the natural Hecke operators on Higgs bundles. These can be viewed as ”abelianized” versions, or classical limits, of the Hecke eigensheaves predicted by the Geometric Langlands correspondence.

Several special cases of our result are already known. The $GL(n)$ case can be traced back to Hitchin’s original work [Hit87], which also includes some speculation that Langlands duality may explain the nature of the Prym varieties obtained for the other classical groups. Hausel and Thaddeus [HT03] considered the case $G = SL(n)$, $L^G = PGL(n)$. They also showed the equality of stringy Hodge numbers for these Langlands-dual Hitchin systems and discussed the relationship to mirror symmetry for hyper-Kähler manifolds. The general duality of Hitchin systems is the starting point of Arinkin’s approach [Ari02] to the quasi-classical geometric Langlands correspondence. This approach was recently utilized by Bezrukavnikov and Braverman [BeBra07] who proved the geometric Langlands correspondence for curves over finite fields for $G = L^G = GL_n$. As explained in [BJSV95], [KW07], the duality of the
gerbes $Higgs$ and $L^{\text{Higgs}}$ proven in Theorem $[3]$ was expected to hold on physical grounds. The construction of the abelianized Hecke eigen-sheaves, mostly in the case $G = \text{GL}(n)$, was discussed by one of us (R.D.) in several talks circa 1990, but has not previously appeared in print.

### 1.2 Outline

The Hitchin system $h : Higgs \to B$ for the group $G$ and a curve $C$ [Hit87] is an integrable system whose total space is the moduli space of semistable $K_C$-valued principal $G$-Higgs bundles on $C$. The base $B$ parametrizes $K_C$-valued cameral covers, which are certain covers $p : \tilde{C}_b \to C$ with an action of $W$, the Weyl group of $G$ (we recall the definition of cameral covers in Definition $[1]$). The cameral cover over a general $b \in B$ is a $W$-Galois cover. For classical simple groups $G$, the base $B$ also parametrizes appropriate spectral covers $\overline{p} : \overline{C}_b \to C$. The Hitchin fiber $h^{-1}(b)$ can be described quite precisely, [Hit87, Fal93, Don93, Don95, Sco98, DG02]. There is a natural discriminant divisor $\Delta \subset B$ (see section $[1.4.2]$) such that for $b \in B - \Delta$, the connected component of $h^{-1}(b)$ is isomorphic to a certain Abelian variety $P_b$ which can be described as a generalized Prym variety of $\tilde{C}_b$ (or of $\tilde{C}_b$) over $C$.

#### 1.2.1 The base

Our basic duality is stated in Theorem $[1]$ below and in section $[3]$. The Hitchin base $B$ and the universal cameral cover $\overline{C} \to C \times B$ depend on the group $G$ only through its Lie algebra $\mathfrak{g}$. As a first step towards the duality between the Hitchin system $h : Higgs \to B$ for $G$ and the Hitchin system $L^{\text{Higgs}} \to L^B$ for $L^G$, we note that the bases are isomorphic and the isomorphism preserves discriminants. The choice of a $G$-invariant bilinear form on $\mathfrak{g}$ determines an isomorphism $l^{\text{base}} : B \to L^B$ between the Hitchin bases for the Langlands-dual algebras $\mathfrak{g}$, $\mathfrak{l}^G$. This isomorphism lifts to an isomorphism $l^{\text{cam}}$ of the corresponding universal cameral covers. (These isomorphisms are unique up to automorphisms of $\overline{C} \to C \times B$:

There is a natural action of $\mathbb{C}^\times$ on $B$ which also lifts to an action on $\overline{C} \to C \times B$. The apparent ambiguity we get in the choice of the isomorphisms $l^{\text{base}}, l^{\text{cam}}$ is eliminated by these automorphisms.)

Under these isomorphisms, cameral covers of type $\mathcal{B}$ are interchanged with those of type $C$. For the remaining simple algebras, of types $\text{ADEFG}$, we can choose an isomorphism of the Lie algebras $\mathfrak{g}$ and $\mathfrak{l}^{\mathfrak{g}}$. We can also choose the Cartan subalgebras to match under this isomorphism. This induces identifications: $\mathfrak{g} = \mathfrak{l}^{\mathfrak{g}}, B = \mathfrak{l}^B, \overline{C} = \mathfrak{l}^{\overline{C}}$, so we can view the isomorphisms $l^{\text{base}}, l^{\text{cam}}$ as automorphisms. For the simply laced Lie algebras (of types $\text{ADE}$) we can then take $l^{\text{base}}, l^{\text{cam}}$ to be the identity. But for the Lie algebras of types $\mathfrak{F}$, $\mathfrak{G}$, the natural isomorphism $l^{\text{base}}$ is not the identity: it takes one cameral cover to another, interchanging short and long roots. This phenomenon was recently noted in the Kapustin-Witten work [KW07] on the geometric Langlands Correspondence, and was used in the Argyres-Kapustin-Seiberg work [AKS05] on $S$-duality in $N = 4$ gauge theories.

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1.2.2 The fibers

The remainder of Theorem A concerns the fiberwise duality. We need to show that the connected component \( P_b \) of the Hitchin fiber \( h^{-1}(b) \) over \( b \in B - \Delta \) is dual (as a polarized abelian variety) to the connected component \( L_{P_{\text{base}}(b)} \) of the corresponding fiber for the Langlands-dual system. This is achieved by analyzing the cohomology of three group schemes \( T \supset T \supset T_0 \) over \( C \) attached to a group \( G \). Of these, the first two were introduced in [DG02], where it was shown that \( h^{-1}(b) \) is a torsor over \( H^1(C, T) \). We recall the definitions of these two group schemes and add the third, \( T_0 \), which is simply their maximal subgroup scheme all of whose fibers are connected. (These fibers are the connected components of the original fibers.) It was noted in [DG02] that \( T = T_0 \) except when \( G = SO(2r + 1) \) for \( r \geq 1 \). Dually, we note here that \( T = T_0 \) except for \( G = Sp(r) \), \( r \geq 1 \). In fact, it turns out that the connected components of \( H^1(T_0) \) and \( H^1(T) \) are dual to the connected components of \( H^1(LT), H^1(LT_0) \), and we are able to identify the intermediate objects \( H^1(T), H^1(LT) \) with enough precision to deduce that they are indeed dual to each other.

Altogether we get the following theorem whose proof is discussed in section 3.

**Theorem A** Let \( G \) be a simple complex group, \( L G \) the Langlands dual complex group, and \( C \) a smooth, connected, compact curve of genus \( g > 0 \).

1. There is an isomorphism \( l_{\text{base}} : B \to L B \), from the base of the \( G \)-Hitchin system to the base of the \( L G \)-Hitchin system, which is uniquely determined up to overall scalar, and is such that:
   - \( l_{\text{base}} \) preserves discriminants: \( l_{\text{base}}(\Delta) = L \Delta \).
   - \( l_{\text{base}} \) lifts to an isomorphism \( l_{\text{cam}} : \tilde{C} \to L \tilde{C} \) between the universal cameral covers of \( C \).

2. For \( b \in B - \Delta \), the corresponding \( G \) and \( L G \) Hitchin fibers are dual. The duality is given by an isomorphism of polarized abelian varieties

\[
l_b : P_b \xrightarrow{\cong} \left( L P_{\text{base}}(b) \right)^D,
\]

where \( P^D \) denotes the dual abelian variety of \( P \). The isomorphism \( l_b \) is the restriction of a global duality \( l \) of \( \text{Higgs}_0 \) and \( L \text{Higgs}_0 \) over \( B - \Delta \).

Several topological results that are needed in our proof are collected in section 6. The main result of that section is an explicit formula for the cocharacters of the Hitchin Prym.
1.2.3 Other components and duality of gerbes

The connected components of \( \text{Higgs} \) are indexed by the fundamental group \( \pi_1(G) \). The component \( \text{Higgs}_0 \) corresponding to the neutral element parametrizes \( G \)-Higgs bundles which are induced from \( G_{\text{sc}} \)-Higgs bundles on \( C \), where \( G_{\text{sc}} \) is the universal cover group of \( G \). As shown by Hitchin [Hit92] and reviewed here, the restriction of \( h \) to this neutral component always admits a section (determined by the choice of a theta characteristic, or spin structure, on the curve \( C \)). In 3.5 we show that the group of connected components of a Hitchin fiber \( h^{-1}(b) \) is \( \pi_1(G) \). In particular, the connected components of \( h^{-1}(b) \) are its intersections with the connected components of \( \text{Higgs} \) itself.

We extend the basic duality to the non-neutral components in section 4.3. The non-canonical isomorphism from non-neutral components of the Hitchin fiber to \( \text{P}^{\text{base}} \) can result in the absence of a section, i.e. in a non-trivial torsor structure [HT03, DP08]. In general, the duality between a family of abelian varieties \( A \rightarrow B \) over a base \( B \) and its dual family \( A^{\vee} \rightarrow B \) is given by a Poincare sheaf which induces a Fourier-Mukai equivalence of derived categories. It is well known [DP08, BeBra07, BB09] that the Fourier-Mukai transform of an \( A \)-torsor \( A_{\alpha} \) is an \( \mathcal{O}^* \)-gerbe \( \alpha \mathcal{O}_{A^{\vee}} \) on \( A^{\vee} \).

In our case there is indeed a natural stack mapping to \( \text{Higgs} \), namely the moduli stack \( \text{Higgs}^{\text{ss}} \) of semistable \( G \)-Higgs bundles on \( C \). Over the locus of stable bundles, the stabilizers of this stack are isomorphic to the center \( Z(G) \) of \( G \) and so over the stable locus \( \text{Higgs}^{\text{ss}} \) is a gerbe. The stack \( \text{Higgs}^{\text{ss}} \) is an open substack in the stack \( \text{Higgs} \) of all (not necessarily semistable) Higgs bundles. Another important stack is the stack \( \text{Higgs}^{\text{reg}} \) of regularized Higgs bundles which parametrizes Higgs bundles together with a choice of a sheaf of regular centralizers for the Higgs field. There is a forgetful map \( \text{Higgs}^{\text{reg}} \rightarrow \text{Higgs} \). There are natural analogues of the Hitchin map for these stacks. The stack \( \text{Higgs}^{\text{reg}} \) was introduced and analyzed in [DG02] and the fibers of the relevant Hitchin map were completely described in terms of spectral data. The analysis in [DG02] shows that over \( B - \Delta \) all these stacks coincide. Since we will always work over \( B - \Delta \) we will use the clean notation \( \text{Higgs} \) rather than the more cumbersome \( \text{Higgs}^{\text{reg}} \).

From [DP08] we know that every pair \( \alpha \in \pi_0(\text{Higgs}) = \pi_1(G), \beta \in \pi_1(\mathcal{L}G) = Z(G)^{\vee} \) defines a \( U(1) \)-gerbe \( \mathfrak{g}_{\text{Higgs}} \) on the connected component \( \text{Higgs}_{\alpha} \) and that there is a Fourier-Mukai equivalence of categories \( D^b(\mathfrak{g}_{\text{Higgs}}^\beta) \cong D^b(\mathfrak{l}_{\text{Higgs}}^\alpha) \). In our case we find that all the \( U(1) \)-gerbes \( \mathfrak{l}_{\text{Higgs}}^\beta \) are induced from the single \( Z(G) \)-gerbe \( \text{Higgs} \), restricted to component \( \text{Higgs}_{\alpha} \) via the homomorphisms \( \mathfrak{l} : Z(G) \rightarrow U(1) \). These results culminate in Theorem [3] which gives a duality between the Higgs gerbes \( \text{Higgs} \) and \( L_{\text{Higgs}} \). We also note that the gerbe \( \mathfrak{h}_{\text{Higgs}} \) measures the obstruction to lifting the universal \( G_{\text{ad}} \)-Higgs bundle to a universal \( G \)-Higgs bundle.

In summary we get the following theorem whose proof is given in section 4.3.

**Theorem** [3] Let \( \text{Higgs} \) be the stack of \( G \)-Higgs bundles on a curve \( C \) and let \( L_{\text{Higgs}} \) be the stack of \( L \)-Higgs bundles on \( C \). Use the isomorphism \( \mathfrak{l}_{\text{base}} : B \rightarrow L \) from Theorem [A](1)
to identify $B - \Delta$ with $^L B - ^L \Delta$. Under this identification one has a canonical isomorphism

$$\ell : \mathcal{Higgs}_{B - \Delta} \overset{\sim}{\rightarrow} (^L \mathcal{Higgs}_{B - \Delta})^D$$

of commutative group stacks over $B - \Delta$. The isomorphism $\ell$ intertwines the action of the translation operators $\text{Trans}^{\lambda, \tilde{x}}$ on $\mathcal{Higgs}_{B - \Delta}$ with the action of the tensorization operators $\text{Tens}^{\lambda, \tilde{x}}$ on $(^L \mathcal{Higgs}_{B - \Delta})^D$.

Here $(^L \mathcal{Higgs}_{B - \Delta})^D$ denotes the dual of $^L \mathcal{Higgs}_{B - \Delta}$ viewed as a family of commutative group stacks over $B - \Delta$, i.e. $(^L \mathcal{Higgs}_{B - \Delta})^D$ is the stack of all commutative group stack homomorphisms from $^L \mathcal{Higgs}_{B - \Delta}$ to the commutative group stack $BG_m$ over $B - \Delta$.

The key to the proof of Theorem B is in our ability to move freely among the components of $\mathcal{Higgs}$ via the abelianized Hecke correspondences. These abelianized Hecke correspondences are carefully introduced in the Appendix, following a review of the cameral cover yoga of [DG02]. Note that the abelianized Hecke correspondences appear twice in this work. In section 4.3 they act on $\mathcal{Higgs}$ and are used for tying the components together. On the other hand, they also occur in the Classical Limit Conjecture 2.5 where they act on the Langlands dual spaces $^L \mathcal{Higgs}$. These are the same correspondences, except that the group is $G$ in one case and $^L G$ in the other. The Appendix discusses $^L \mathcal{Higgs}$; for the purposes of section 4.3, the same discussion applies but the $^L (\bullet)$ superscript needs to be dropped.

Combining Theorem B with the abelian Fourier-Mukai duality gives the main result of this work, an extension of Theorem B to the case of reductive groups:

**Theorem C** Let $G$ be a connected complex reductive group, let $^L G$ be the Langlands dual reductive group, and let $C$ be a smooth compact complex curve. Write $\mathcal{Higgs}_G$ and $\mathcal{Higgs}_{(^L G)}$ for the stacks of $K_C$-valued Higgs bundles on $C$ with structure group $G$ and $^L G$ respectively. Then there is an isomorphism $[\text{base}] : B \rightarrow [\mathcal{Higgs}_G]$ of the respective Hitchin bases which gives an identification $B - \Delta \cong ^L B - ^L \Delta$. Under this identification one has an isomorphism

$$\mathcal{Higgs}_G \cong (\mathcal{Higgs}_{(^L G)})^D$$

of commutative group stacks over $B - \Delta$, intertwining the action of translation and tensorization operators.

The proof of this theorem is discussed in section 5.1.

1.2.4 Derived categories and Hecke eigensheaves

Finally, our Theorem D allows one to view the Fourier-Mukai duality in Theorems B and C as a classical limit of the geometric Langlands correspondence: under this duality, the structure sheaves of gerby points on $\mathcal{Higgs}_0$ are transformed into coherent sheaves on the space
$L^\text{Higgs}$ (or equivalently Higgs sheaves on $L^\text{Bun}$) which are eigensheaves for the abelianized Hecke correspondences. These abelianized Hecke correspondences (or translation operators) and their action on $L^\text{Higgs}$ are introduced in the Appendix.

**Theorem D** A topologically trivial $G$-Higgs bundle $(V, \varphi)$ on $C$ determines an eigensheaf for the abelianized Hecke operators. Explicitly let $p : \tilde{C} \to C$ be a cameral cover corresponding to a point in $B - \Delta$, and let $\mathcal{T}_{\tilde{C}}$ be the corresponding sheaf of regular centralizers on $C$. The choice of $(V, \varphi)$ gives:

- A $T$-torsor $\mathcal{L}_{(V, \varphi)}$ on $\tilde{C}$.
- A representable structure morphism $\iota : B \text{Aut}((V, \varphi)) \to \text{Higgs}_0$.

Write $\mathcal{O}_{(V, \varphi)} := \iota_* \mathcal{O}_{B \text{Aut}((V, \varphi))}$ for the corresponding sheaf on $\text{Higgs}_0$. Then for every character $\mu \in \Lambda^\vee$ we have a functorial isomorphism

$$L^\text{ab} \mathbb{H}^u \left( c_0(\mathcal{O}_{(V, \varphi)}) \right) \cong c_0(\mathcal{O}_{(V, \varphi)}) \boxtimes \mu \left( \mathcal{L}_{(V, \varphi)} \right),$$

i.e. $c_0(\mathcal{O}_{(V, \varphi)})$ is an abelianized Hecke eigensheaf with eigenvalue $\mathcal{L}_{(V, \varphi)}$.

### 1.3 Open problems and loose ends

Our argument is non algebraic, in that we use cohomology of constructible sheaves and Hodge theory to prove the duality between families of abelian varieties. It would be nice to have a purely algebraic argument which is local and universal in nature.

Our work deals with smooth cameral covers, establishing the Hitchin duality over the complement of the discriminant. A major step forward would be to formulate and prove the extension to the entire base. The heart of the matter would presumably be an understanding of what happens over the nilpotent cones of the dual systems.

A finer understanding of the classical limit of the Geometric Langlands Conjecture and its relation to the abelianized version proved here would be desirable. We discuss some of the relevant issues, somewhat informally, in section 2. The rest of the paper does not depend on that section.

### 1.4 Review and notation

#### 1.4.1. Let $G$ be a simple complex algebraic group and let $L^G$ be the Langlads dual complex group. The Lie algebras of $G$ and $L^G$ will be denoted by $\mathfrak{g}$ and $L^G$. We fix maximal tori $T \subset G$ and $L^T \subset L^G$ and denote the corresponding Cartan subalgebras by $\mathfrak{t} \subset \mathfrak{g}$ and $L^\mathfrak{t} \subset L^\mathfrak{g}$. We will also write $T_\mathbb{R} \subset G_\mathbb{R}$ and $L^T_\mathbb{R} \subset L^G_\mathbb{R}$ for the compact real forms of the complex groups and $\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ and $L^\mathfrak{t}_\mathbb{R} \subset L^\mathfrak{g}_\mathbb{R}$ will denote the corresponding real Lie algebras. We denote the space of $\mathbb{C}$-linear functions on $\mathfrak{t}$ by $\mathfrak{t}^\vee$, and the space of $\mathbb{R}$-linear functions on $\mathfrak{t}_\mathbb{R}$ by $\mathfrak{t}_\mathbb{R}^\vee$. Langlands duality gives an isomorphism $\mathfrak{t}^\vee = L^\mathfrak{t}$ which is compatible with the real
structure. We fix this isomorphism once and for all. We will also write $W$ for the isomorphic Weyl groups of $G$ and $^L G$.

We denote the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^\vee$ by $(\cdot, \cdot) : \mathfrak{t}^\vee \otimes \mathfrak{t} \to \mathbb{C}$, while we write $\langle \cdot, \cdot \rangle : \mathfrak{t} \otimes \mathfrak{t} \to \mathbb{C}$ for the Killing form on $\mathfrak{t}$. For any group $G$ we have a natural collection of lattices

$$\text{root}_g \subset \text{char}_G \subset \text{weight}_g \subset \mathfrak{t}^\vee \quad \text{coroot}_g \subset \text{cochar}_G \subset \text{coweight}_g \subset \mathfrak{t}.$$ 

Here $\text{root}_g \subset \text{weight}_g \subset \mathfrak{t}^\vee$ are the root and weight lattice corresponding to the root system on $\mathfrak{g}$ and $\text{char}_G = \text{Hom}(T, \mathbb{C}^\times) = \text{Hom}(T_{\mathbb{R}}, S^1)$ is the character lattice of $G$. Analogously,

$$\text{coroot}_g = \{ x \in \mathfrak{t} | (\text{weight}_g, x) \subset \mathbb{Z} \} \cong \text{weight}^{\vee}_g$$

$$\text{coweight}_g = \{ x \in \mathfrak{t} | (\text{root}_g, x) \subset \mathbb{Z} \} \cong \text{root}^{\vee}_g$$

are the coroot and coweight lattices of $\mathfrak{g}$, and

$$\text{cochar}_G = \text{Hom}((\mathbb{C}^\times)^\times, T) = \text{Hom}(S^1, T_{\mathbb{R}}) = \{ x \in \mathfrak{t} | (\text{char}_G, x) \subset \mathbb{Z} \} \cong \text{char}^{\vee}_G$$

is the cocharacter lattice of $G$.

The Langlands duality isomorphism $^L \mathfrak{t}^\vee = \mathfrak{t}$ identifies $\text{root}_{^L \mathfrak{g}} = \text{coroot}_{\mathfrak{g}}$, $\text{char}_{^L \mathfrak{g}} = \text{cochar}_G$, and $\text{weight}_{^L \mathfrak{g}} = \text{coweight}_g$. To every root $\alpha \in \text{root}_g$ of $\mathfrak{g}$ one associates in a standard way a coroot $\alpha^\vee \in \text{coroot}_g$, given by the formula $(\cdot, \alpha^\vee) := 2(\alpha, \cdot)/\langle \alpha, \alpha \rangle$. Under the identification $\text{root}_{^L \mathfrak{g}} = \text{coroot}_{\mathfrak{g}}$ the root system of $^L \mathfrak{g}$ is mapped to the system of coroots of $\mathfrak{g}$ so that the short and long roots get exchanged.

1.4.2. Let $C$ be a smooth compact complex curve of genus $g > 0$. Recall \cite{Hit87} that the total space $\text{Higgs}$ of Hitchin’s system parametrizes semistable $K_C$-valued principal $G$-Higgs bundles on $C$, i.e. pairs $(V, \varphi)$ where $V$ is a principal $G$-bundle on $C$, $\text{ad}(V)$ is the vector bundle associated to $V$ by the adjoint representation, and $\varphi$ is a global section of $\text{ad}(V) \otimes K_C$. We recall that a $G$-bundle $V$ on $C$ is semistable if for any parabolic subgroup $P \subset G$ and any $P$-subbundle $V' \subset V$, the degree of $V'$ is $\leq 0$. Similarly, a Higgs bundle $(V, \varphi)$ is semistable if for any parabolic subgroup $P \subset G$ and any $P$-Higgs subbundle $(V', \varphi')$ of $(V, \varphi)$, the degree of $V'$ is $\leq 0$.

Let $h : \text{Higgs} \to B$ and $^L h : ^L \text{Higgs} \to ^L B$ denote the Hitchin integrable systems for $C$ and $G$ and $^L G$ respectively. The base $B$ can be identified with the space of sections $H^0(C, (K_C \otimes \mathfrak{t})/W)$. Its points $b \in B$ parametrize $K_C$-valued cameral covers $\tilde{C}_b \to C$ of $C$ \cite{Fal93} \cite{Don93} \cite{Don95} \cite{DG02}.

**Definition 1.1** (see \cite{DG02}) A cameral cover of $C$ is a scheme $\tilde{C}$ together with a morphism $p : \tilde{C} \to C$ and a $W$-action along the fibers of $p$ satisfying:

- $p$ is finite and flat over $C$;
- as an $\mathcal{O}_C$-module with $W$ action, $p_*(\mathcal{O}_{\tilde{C}})$ is locally isomorphic to $\mathcal{O}_C \otimes \mathbb{C}[W]$;

\[8\]
• locally with respect to the etale (or analytic) topology on $C$, $\tilde{C}$ is a pull-back of the $W$-cover $t \to t/W$.

For any line bundle $L$ on $C$, the $L$-valued cameral covers are those parametrized by $H^0(C, (L \otimes t)/W)$.

We will say that a cameral cover $p_b : \tilde{C}_b \to C$ has simple Galois ramification if all ramification points $x \in D_b \subset \tilde{C}_b$ of $p$ have ramification index one. The ramification divisor $D_b \subset \tilde{C}_b$ of a cameral cover $p_b : \tilde{C}_b \to C$ with simple Galois ramification is a disjoint union $D_b = \bigsqcup \alpha D_b^\alpha$ of subdivisors labeled by the roots of $g$ [DG02]. We will denote the universal cameral cover $\tilde{C} \to B \times C$. The discriminant $\Delta \subset B$ is the locus of all $b$ for which $p_b : \tilde{C}_b \to C$ does not have simple Galois ramification.

The Hitchin fiber $h^{-1}(b)$ for $b \in B - \Delta$ is, in general, disconnected but all of its connected components are torsors over a generalized Prym variety $P_b$ naturally associated with the cover $p_b : \tilde{C}_b \to C$ [Fal93, Don93, DG02, DDP07]. If $G$ is a classical group the generalized Prym variety can also be attached to a (non-Galois) spectral cover $\tilde{p}_b : \tilde{C}_b \to C$ [Hit87].

The connected components of the space $\text{Higgs}$ are labeled by the topological types of Higgs bundles, which in turn are labeled by elements in $H^2(C, \pi_1(G)) = \pi_1(G)$. The component $\text{Higgs}_0$ corresponding to the neutral element parametrizes $G$-Higgs bundles which are induced from $G_{sc}$-Higgs bundles on $C$, where $G_{sc}$ is the universal covering group of $G$.

The restriction of $h$ to this neutral component always admits a section, determined by the choice of a theta characteristic on the curve $C$ and called the Hitchin section [Hit92]. The construction of the Hitchin section is also reviewed in the proof of Lemma 4.1.

1.4.3. For a finitely generated abelian group $H$ we will write $H_{tors} \subset H$ for the torsion subgroup of $H$; $H_{tf} := H/H_{tors}$ for the maximal torsion free quotient of $H$.

Throughout the paper we will frequently use several duality transformations. The most important ones are as follows:

$(\bullet)^\vee$: will denote the duality operation $\text{Hom}_Z(\bullet, Z)$ which we will be applying to free abelian groups of finite rank.

$(\bullet)^\wedge$: depending on the context will denote the Pontryagin duality operation $\text{Hom}(\bullet, S^1)$ on locally compact topological abelian groups or the Cartier duality operation $\text{Hom}(\bullet, G_m)$ on (complexes of) flat abelian group sheaves over a base scheme.

$(\bullet)^D$: depending on the context will denote the duality operation $R \text{Hom}(\bullet, O^\times[1])$ on complexes of abelian groups or the duality operation $\text{Hom}_{\text{grp-stack}}(\bullet, B G_m)$ on commutative group stacks over a base;

$L(\bullet)$: depending on the context will denote the Langlands duality on reductive groups or Lie algebras, the Langlands duality on maximal tori or Cartan algebras, or the induced Langlands duality operation on the various sheaves of regular centralizers.
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2 The classical limit

Our duality of the Hitchin integrable systems for a pair of Langlands dual groups can be interpreted as a specialization or a “classical limit” of the Deligne, Laumon, Beilinson-Drinfeld geometric version of the Langlands conjecture [BD03]. In this section we will explain how this interpretation works. We will suppress the technicalities required to make the statements precise. The material of this section is the motivation behind much of what we do, but it will not be needed explicitly anywhere in the paper. The proof of the main results begins in section 3.

First we will need to introduce some notation. To avoid subtleties requiring rigidification or derived structures we will only discuss the case of semisimple groups $G$ and $^L G$. Let as before $^L \Bun$ denote the moduli stack of principal $^L G$-bundles on $C$. We will also let $\Loc$ denote the moduli stack of algebraic $G$-local systems on $C$, i.e. the moduli stack of pairs $(V, \nabla)$ where $V$ is a principal algebraic $G$-bundle on $C$ and $\nabla$ is a flat algebraic connection on $C$. For a sheaf of algebras $\mathcal{A}$ on an algebraic stack $X$ we will write $D_{\text{coh}}(X, \mathcal{A})$ for the derived category of complexes of $\mathcal{A}$-modules whose cohomology sheaves are coherent $\mathcal{A}$-modules. The sheaves of algebras $\mathcal{A}$ that we will be primarily interested in will be $\mathcal{A} = \mathcal{O}_X$ - the structure sheaf of $X$, or $\mathcal{A} = \mathcal{D}_X$ - the sheaf of algebraic differential operators on $X$, or $\mathcal{A} = \text{Sym}^* T_X$ - the symmetric algebra on the tangent sheaf of $X$. According to [BD03] a form of the following conjecture must hold:

**Conjecture 2.1** There exists a canonical equivalence of categories (the geometric Langlands correspondence):

$$\zeta : D_{\text{coh}}(\Loc, \mathcal{O}) \xrightarrow{\cong} D_{\text{coh}}(^L \Bun, \mathcal{D}),$$

which intertwines the action of the tensorization functors on $D_{\text{coh}}(\Loc, \mathcal{O})$ with the action of the Hecke functors on $D_{\text{coh}}(^L \Bun, \mathcal{D})$. 
The tensorization functors \( W^\mu,x : \text{D}_{\text{coh}}(\text{Loc}, \mathcal{O}) \to \text{D}_{\text{coh}}(\text{Loc}, \mathcal{O}) \), and the Hecke functors \( L^H^\mu,x : \text{D}_{\text{coh}}(\text{Bun}, \mathcal{D}) \to \text{D}_{\text{coh}}(\text{Bun}, \mathcal{D}) \), are endofunctors of the respective categories of sheaves labeled by the same data: pairs \((x, \mu)\), where \( x \in C \) is a closed point and \( \mu \in \text{cochar}^+(L^G) = \text{char}^+(G) \) is a dominant cocharacter for \( L^G \), or equivalently a dominant character for \( G \).

Given such a pair \((x, \mu)\) one defines the tensorization functor \( W^\mu,x \) as
\[
W^\mu,x : \text{D}_{\text{coh}}(\text{Loc}, \mathcal{O}) \to \text{D}_{\text{coh}}(\text{Loc}, \mathcal{O})
\]
\[
\mathcal{F} \mapsto \mathcal{F} \otimes \rho^\mu (\mathcal{V}(\text{Loc} \times \{x\}))
\]
and the Hecke functor \( L^H^\mu,x \) as
\[
L^H^\mu,x : \text{D}_{\text{coh}}(\text{Bun}, \mathcal{D}) \to \text{D}_{\text{coh}}(\text{Bun}, \mathcal{D})
\]
\[
\mathcal{M} \mapsto \mathcal{M}^\mu,x((p^\mu,x)^* \mathcal{M} \otimes L^I^\mu,x)
\]
i.e. as the integral transform on \( \mathcal{D} \)-modules with kernel \( L^I^\mu,x \in \text{D}_{\text{coh}}(\text{Hecke}^\mu,x, \mathcal{D}) \). Here:

- \( \rho^\mu \) is the irreducible representation of \( G \) with highest weight \( \mu \), \( \mathcal{V} \to \text{Loc} \times C \) is the principal \( G \)-bundle underlying the universal local system, and \( \rho^\mu (\mathcal{V}(\text{Loc} \times \{x\})) \) is the vector bundle on \( \text{Loc} \) associated with \( \mathcal{V}(\text{Loc} \times \{x\}) \) via the representation \( \rho^\mu \).

- \( \text{Hecke}^\mu,x \) is the moduli stack of triples \((V, V', \beta)\), where \( V \) and \( V' \) are principal \( L^G \)-bundles on \( C \), and \( \beta \) is an isomorphism of \( V|_{C-\{x\}} \) with \( V'|_{C-\{x\}} \), such that for every irreducible representation \( \rho \) of \( L^G \), the isomorphism:
\[
\rho(\beta) : \rho(V)|_{C-\{x\}} \cong \rho(V')|_{C-\{x\}}
\]
of vector bundles away from \( x \in C \) extends to an injection of locally free sheaves on \( C \) with pole at \( x \) of order bounded by \( \mu \). More precisely, on \( C \) we have
\[
\rho(\beta) : \rho(V) \hookrightarrow \rho(V') (\langle \mu, \lambda^\rho \rangle \cdot x),
\]
where \( \lambda^\rho \in \text{char}^+(L^G) \) denotes the highest weight of \( \rho \).

- The stack \( \text{Hecke}^\mu,x \) is equipped with two projections
\[
\begin{array}{c}
\text{Bun} \\
\downarrow p^\mu,x \\
\text{Hecke}^\mu,x \\
\downarrow q^\mu,x \\
\text{Bun}
\end{array}
\]
where \( p^\mu,x((V, V', \beta)) := V \) and \( q^\mu,x((V, V', \beta)) := V' \). Both maps \( p^\mu,x \) and \( q^\mu,x \) are proper representable locally trivial fibrations.


**L**$H_{\mu,x}$ is the Goresky-MacPherson middle perversity extension $j_* \left( \mathbb{C} \left[ \dim L_{\text{Hecke}}^{\mu,x} \right] \right)$ of the trivial rank one local system on the smooth part $j : (L_{\text{Hecke}}^{\mu,x})^{\text{smooth}} \hookrightarrow L_{\text{Hecke}}^{\mu,x}$ of the Hecke stack.

**Remark 2.2** In the recent work of Kapustin-Witten [KW07] the geometric Langlands correspondence $c$ is interpreted physically in two different ways. On one hand it is argued that Conjecture 2.1 is a mirror symmetry statement relating the A and B-type branes on the hyper-Kähler moduli spaces of Higgs bundles. On the other hand Kapustin and Witten use a gauge theory/string duality to show that Conjecture 2.1 can be thought of as an electric-magnetic duality between supersymmetric four-dimensional Gauge theories with structure groups $G$ and $L_G$ respectively. In this interpretation the functors $W^{\mu,x}$ and $L_{H_{\mu,x}}$ are viewed as natural symmetry operations in gauge theory associated to closed loops in an appropriately chosen four-manifold. In this context $W^{\mu,x}$ appear as the so called Wilson loop operators, and $L_{H_{\mu,x}}$ as the 'tHooft loop operators.

**Remark 2.3** The derived category $D_{\text{coh}}(\mathcal{L}oc, \mathcal{O})$ has a natural orthogonal spanning class of objects: the structure sheaves of all closed (stacky) points. Each of these skyscraper sheaves is an eigensheaf for all of the tensorization functors: if $V = (V, \nabla)$ is a $G$-local system on $C$ and if $\mathcal{O}_V$ is the structure sheaf of the corresponding stacky point of $\mathcal{L}oc$, then $W^{\mu,x}(\mathcal{O}_V) = \mathcal{O}_V \otimes \rho(\mu)(V)_x$. The geometric Langlands correspondence $c$ in Conjecture 2.1 therefore sends structure sheaves of points on $\mathcal{L}oc$ to Hecke eigen-$D$-module on $L_{\text{Bun}}$: for every $G$-local system $V$,

$$L_{H_{\mu,x}} (c(\mathcal{O}_V)) = c(\mathcal{O}_V) \otimes \rho_{\mu}(V)_x.$$  

The Hecke eigen-$D$-module on $L_{\text{Bun}}$ for $G = GL_n$ and an irreducible local system $V$, was constructed by Drinfeld [Dr83] for $n = 2$ and by Frenkel, Gaitsgory and Vilonen [FGV02, G02] for all $n$. The general form of the conjecture and the case of other groups is still open.

An interesting feature of Conjecture 2.1 is that it specializes naturally in a one parameter family. The special fiber of the specialization is (an appropriate version of) the stack of $K_C$-valued Higgs bundles on $C$ and so is naturally related to the Hitchin system. Recall [Hit87] that a $K_C$-valued $G$-Higgs bundle on $C$ is a pair $(V, \theta)$, where $V$ is a principal $G$-bundle and $\theta \in H^0(C, \text{ad}(V) \otimes K_C)$ is a $K_C$-valued section of the adjoint bundle of $V$.

The specialization of the geometric Langlands conjecture has a different nature on the two sides of the conjecture. On the left hand side it comes from a geometric specialization of the stack $\mathcal{L}oc$, whereas on the right side it comes from specializing the filtered sheaf of non-commutative rings $D$ to its associated sheaf of graded commutative rings:

- On the left hand side of the conjecture there is a natural geometric one parameter jump deformation of the stack $\mathcal{L}oc$ of $G$-local systems to the stack $\mathcal{H}iggs$ of $K_C$-valued $G$-Higgs bundles [Sim97]. More precisely there is a family of stacks $\mathcal{H} \to \mathbb{C}$
parametrized by the affine line $\mathbb{C}$ such that $\text{Loc}$ is isomorphic to the general fiber, and the fiber over $0 \in \mathbb{C}$ is equal to $\text{Higgs}$. Explicitly $[\text{Sim97}]$ $\mathcal{H}$ is the moduli stack of Deligne’s $z$-connections on $C$, i.e. $\mathcal{H}$ parametrizes triples $(V, \nabla, z)$, where $\pi : V \to C$ is a principal $G$-bundle on $C$, $z \in \mathbb{C}$ is a complex number, and $\nabla$ is a differential operator satisfying the Leibnitz rule up to a factor of $z$. Equivalently, $\nabla$ is a $z$-splitting of the Atiyah sequence for $V$:

$$0 \to \text{ad}(V) \to \mathcal{E}(V) \xrightarrow{\sigma} T_C \to \nabla \to 0.$$ 

Here $\text{ad}(V) = V \times_{\text{ad}G}$ is the adjoint bundle of $V$, $\mathcal{E}(V) = (\pi_* T_V)^G$ is the Atiyah algebra of $V$, $\sigma : \mathcal{E}(V) \to T_C$ is the map induced from $d\pi : T_V \to \pi^* T_C$, and $\nabla$ is a map of vector bundles satisfying $\sigma \circ \nabla = z \cdot \text{id}_{T_C}$.

The map $\mathcal{H} \to \mathbb{C}$ assigns to $(V, \nabla, z)$ the complex number $z$ and is equivariant under the action of $\mathbb{C}^\times$ which rescales $z$. This $\mathbb{C}^\times$-action trivializes $\mathcal{H}|_{\mathbb{C}^\times}$ and so we have a specialization family:

$$\text{Higgs} \subset \mathcal{H} \supset \mathcal{H}|_{\mathbb{C}^\times} \cong \text{Loc} \times \mathbb{C}^\times$$

Passing to derived categories of coherent sheaves one gets a one parameter deformation of $D_{\text{coh}}(\text{Loc}, \mathcal{O})$ to $D_{\text{coh}}(\text{Higgs}, \mathcal{O})$ of the left hand side category in Conjecture 2.1.

- On the right hand side we have a natural one parameter jump deformation of the sheaf $\mathcal{D}$ of rings of differential operators on $L\text{Bun}$ to the symmetric algebra of the tangent sheaf on $L\text{Bun}$. More precisely, the filtration by order of differential operators on $\mathcal{D}$ gives rise to an associated Rees sheaf $\mathcal{R} \to L\text{Bun} \times \mathbb{C}$ - a sheaf of rings, flat over the affine line $\mathbb{C}$ yielding specializations $\mathcal{R}|_{L\text{Bun} \times \{z\}} \cong \mathcal{D}$ for $z \neq 0$ $\mathcal{R}|_{L\text{Bun} \times \{0\}} \cong \text{Sym}^\bullet T$

Explicitly let $p_1 : L\text{Bun} \times \mathbb{C}^\times \to L\text{Bun}$ be the projection on the first factor. Define a subsheaf $\mathcal{R} \subset p_1^* \mathcal{D}$ as follows. A section of $p_1^* \mathcal{D}$ is of the form $\sum z^i P_i$ for $P_i \in \mathcal{D}$; by definition, the section is in $\mathcal{R}$ if and only if the degree of $P_i$ is at most $i$, i.e. $P_i \in \mathcal{D}^{\leq i}$.

Passing to derived categories of modules we get a one parameter deformation of $D_{\text{coh}}(L\text{Bun}, \mathcal{D})$ to $D_{\text{coh}}(L\text{Bun}, \text{Sym}^\bullet T)$ of the right hand side category in Conjecture 2.1. Furthermore the abelian category of coherent $\text{Sym}^\bullet T$-modules on $L\text{Bun}$ is naturally equivalent to the category of coherent sheaves of $\mathcal{O}$-modules on $\text{Spec}(\text{Sym}^\bullet T)$, i.e. on the total space of the cotangent bundle of the smooth algebraic stack $L\text{Bun}$. It is well known (see e.g. [Hit87, BD03]) that this total space can be identified naturally
with the moduli stack $L\mathcal{H}iggs$ of all $K_C$-valued $L G$-Higgs bundles on $C$. Using this identification we can recast the deformation of $D_{\text{coh}}(L\mathcal{B}un, \mathcal{D})$ to $D_{\text{coh}}(L\mathcal{B}un, \text{Sym}^* T)$ as a one parameter deformation of $D_{\text{coh}}(L\mathcal{B}un, \mathcal{D})$ to $D_{\text{coh}}(L\mathcal{H}iggs, \mathcal{O})$.

One expects that the geometric Langlands correspondence $\mathfrak{c}$ also deforms along with the above of the two sides of Conjecture 2.1. In other words we expect to have a classical limit correspondence $\mathfrak{c}_\text{cl} : D_{\text{coh}}(\mathfrak{H}iggs, \mathcal{O}) \to D_{\text{coh}}(L\mathcal{H}iggs, \mathcal{O})$ which is an equivalence of categories and is a one parameter deformation of $\mathfrak{c} : D_{\text{coh}}(\mathcal{L}oc, \mathcal{O}) \to D_{\text{coh}}(L\mathcal{B}un, \mathcal{D})$ Here the deformation from $D_{\text{coh}}(\mathcal{L}oc, \mathcal{O})$ to $D_{\text{coh}}(\mathfrak{H}iggs, \mathcal{O})$ should be thought of as a specialization from the general to the closed fiber in the stack of dg enhanced derived categories of quasi-coherent sheaves along the fibers of the family of spaces $\mathcal{H} \to \mathbb{C}$. Similarly, the deformation from $D_{\text{coh}}(L\mathcal{B}un, \mathcal{D})$ to $D_{\text{coh}}(L\mathcal{H}iggs, \mathcal{O})$ should be viewed as a specialization from the general to the closed fiber in the stack of dg enhanced derived categories of complexes of $\mathcal{R}$-modules along the fibers of the family $L\mathcal{B}un \times \mathbb{C}$.

It is also expected that the classical limit correspondence $\mathfrak{c}_\text{cl}$ should intertwine suitably defined specializations of the tensorization and Hecke functors. The specialization of the tensorization functors is easy to describe: for each $x \in C$ and $\mu \in \text{char}^+(G)$ as before we define a classical limit tensorization functor as

$$W^{\mu,x} : D_{\text{coh}}(\mathfrak{H}iggs, \mathcal{O}) \to D_{\text{coh}}(\mathfrak{H}iggs, \mathcal{O})$$

where $\mathcal{V} \to \mathfrak{H}iggs \times \mathbb{C}$ is the principal $G$-bundle underlying the universal Higgs bundle $(\mathcal{V}, \vartheta) \in \Gamma(\text{ad}(\mathcal{V}) \otimes p_C^* K_C)$.

The passage to the classical limit for Hecke functors is more involved. First notice that the spectral correspondence (see e.g. [Don95]) gives an equivalence of the abelian category of quasi-coherent sheaves on $L\mathcal{H}iggs = \text{tot}(T_{\mathcal{V}un})$ with the abelian category of $\Omega^1$-valued quasi-coherent Higgs sheaves on $L\mathcal{B}un$, that is with the abelian category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a quasi-coherent sheaf on $L\mathcal{B}un$ and $\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$ is an $\mathcal{O}$-linear map satisfying $\varphi \wedge \varphi = 0$. In particular we can view $D_{\text{coh}}(L\mathfrak{H}iggs, \mathcal{O})$ as a full subcategory of the derived category $D_{\mathfrak{H}iggs}(L\mathcal{B}un)$ of quasi-coherent Higgs sheaves on $L\mathcal{B}un$.

Since the Hecke functors were defined as integral transforms for $\mathcal{D}$ modules on $L\mathcal{B}un$ we can use the Higgs sheaf interpretation of $D_{\text{coh}}(L\mathfrak{H}iggs, \mathcal{O})$ and define the classical limit of the Hecke functor as an integral transform for Higgs sheaves. There is one missing ingredient for such a definition however: we must specify a specialization of the kernel $\mathcal{D}$-module $Lf^{\mu,x} \mathcal{H}ecke$ to a quasi-coherent Higgs sheaf on the Hecke stack $L\mathcal{H}ecke^{\mu,x}$. For this one can use the same process that we used to define the classical limit of the right hand side of Conjecture 2.1 namely the Rees deformation of a filtered object to its associated graded.

More precisely, suppose that we can find a quasi-coherent sheaf $Lf^{\mu,x} \mathcal{H}ecke$ on $L\mathcal{H}ecke^{\mu,x} \times \mathbb{C}$ so that:
• $L^\mu,x$ is a module over the Rees sheaf for the sheaf of differential operators on $L^\text{Hecke}^\mu,x$;

• The restriction of $L^\mu,x$ to $L^\text{Hecke}^\mu,x \times \{1\}$ is isomorphic to $L^I\mu,x$ as a $D$-module.

Typically such an extension $L^\mu,x$ of $L^I\mu,x$ will come from choosing a good filtration on $L^I\mu,x$, since for any good filtration we can take $L^\mu,x$ to be the Rees module associated with the filtration. Thus one strategy for finding the classical limit will be to equip $L^I\mu,x$ with a functorial good filtration.

The restriction $L^\mu,x := L^\mu,x / z \cdot L^\mu,x$ of $L^\mu,x$ to $L^\text{Hecke}^\mu,x \times \{0\}$ is then naturally a module over the associated graded ring $\text{gr} D (\cong \text{Sym}^* T)$, i.e. it is a Higgs sheaf on $L^\text{Hecke}^\mu,x$ which can be viewed as the classical limit Hecke kernel. This immediately gives rise to a classical limit Hecke functor

$$(3) \quad L^\text{Higgs}^\mu,x : \ D_\text{coh} (L^{Higgs}, \mathcal{O}) \rightarrow D_\text{coh} (L^{Higgs}, \mathcal{O})$$

In general one expects that the correct filtration on the intersection cohomology sheaf $L^I\mu,x$ comes from mixed Hodge theory. By definition $L^I\mu,x$ is the middle perversity extension of the trivial rank one local system from the smooth locus of $L^{Bun}$. In particular, Saito’s theory [Sai90] implies that $L^I\mu,x$ has a canonical structure of a mixed Hodge module. It is natural to expect that the Hodge filtration of this mixed Hodge module will provide the correct classical limit of $L^I\mu,x$. This is trivially the case for minuscule $\mu$’s and suggests the following

**Definition 2.4** The classical limit Hecke kernel $L^\mu,x\mathcal{J}$ is the associated graded of $L^I\mu,x$ with respect to the Hodge filtration in Saito’s mixed Hodge module structure on $L^I\mu,x$.

Using the classical limit Hecke kernel we can now define the classical limit Hecke functor by formula (3). With these definition of $L^\mu,xW$ and $L^\mu,xH$ we can now formulate the full version of the classical limit geometric Langlands conjecture:

**Conjecture 2.5** There exists a canonical equivalence of categories (the geometric Langlands correspondence):

$$\text{cl} : D_{\text{coh}} (Higgs, \mathcal{O}) \cong D_{\text{coh}} (L^{Higgs}, \mathcal{O}),$$

which intertwines the action of the classical limit tensorization functors $L^\mu,xW$ with the action of the classical limit Hecke functors $L^\mu,xH$.

This conjecture is one of the principal motivations for our results. We will prove a version of this conjecture in section 5.4. In the Appendix we review the abelianization
of the stack of Higgs bundles \cite{Hit87, DG02} and introduce the abelianized Hecke functors on $D_{\text{coh}}(Higgs, \mathcal{O})$ which again generate a commutative algebra of endofunctors. In section \ref{sec:abelianized-hecke} we show that, away from the discriminant, there exists a Fourier-Mukai kernel on $Higgs \times (LHiggs)$ which gives an equivalence of $D_{\text{coh}}(Higgs, \mathcal{O})$ with $D_{\text{coh}}(LHiggs, \mathcal{O})$ and intertwines the algebra of $\mathfrak{W}^{\mu,x}$'s and the algebra of abelianized Hecke functors. The precise comparison of the classical limit Hecke functors and our abelianized Hecke functors is somewhat subtle and will be addressed in a forthcoming work of Arinkin and Bezrukavnikov. Away from the discriminant they identify the algebras of classical and abelianized Hecke functors. This lends support to \ref{sec:abelianized-hecke} as the correct definition of the classical limit.

3 Duality for cameral Prym varieties

In this section we formulate and prove the main duality result for Hitchin Pryms.

**Theorem A** Let $G$ be a simple complex group, $L^G$ the Langlands dual complex group, and $C$ a smooth, connected, compact curve of genus $g > 0$.

1. There is an isomorphism $l_{\text{base}} : B \to L^G B$, from the base of the $G$-Hitchin system to the base of the $L^G$-Hitchin system, which is uniquely determined up to overall scalar, and is such that:
   - $l_{\text{base}}$ preserves discriminants: $l_{\text{base}}(\Delta) = L^G \Delta$.
   - $l_{\text{base}}$ lifts to an isomorphism $l_{\text{cam}} : \tilde{C} \to L^G \tilde{C}$ between the universal cameral covers of $C$.

2. For $b \in B - \Delta$, the corresponding $G$ and $L^G$ Hitchin fibers are dual. The duality is given by an isomorphism of polarized abelian varieties

   $$l_b : P_b \cong (L P_{l_{\text{base}}(b)})^D,$$

   where $P^D$ denotes the dual abelian variety of $P$. The isomorphism $l_b$ is the restriction of a global duality $l$ of $Higgs_0$ and $LHiggs_0$ over $B - \Delta$.

**Remark 3.1** Given two isomorphic Lie algebras $\mathfrak{g}, \mathfrak{g}'$, there is a canonical isomorphism $W \cong W'$ between their Weyl groups and an isomorphism $\mathfrak{t} \cong \mathfrak{t}'$ between their Cartan subalgebras, taking roots to roots and intertwining the Weyl actions. This isomorphism is unique up to the action of $W$. For Langlands self-dual algebras, this gives a canonical choice of an invariant bilinear form such that the composition $\mathfrak{t} \to \mathfrak{t}^\vee = L^G \mathfrak{t} \cong \mathfrak{t}$ sends short roots to
long roots. The resulting automorphism of $t$ is in $W$ if $g$ is simply laced (i.e. of type ADE) but not otherwise (types FG), since it sends long roots to a multiple (greater than 1) of the short roots. The induced automorphism of the base $B$ of the Hitchin system then will be the identity for types ADE but not for types FG. The action of these non-trivial automorphisms of the Hitchin space was recently identified [AKS05] as an $S$-duality transformation in $N = 4$ gauge theories compatible with a $T$-duality transformation upon embedding in string theory.

Proof of Theorem A (1) Recall that $B = H^0(C, (K_C \otimes t)/W)$ and similarly $L^B = H^0(C, (K_C \otimes L^t)/W)$. The choice of an invariant scalar product gives an isomorphism $\kappa : t \xrightarrow{\sim} t^\vee = L^t$ compatible with the $W$-action and taking reflection hyperplanes in $t$ to reflection hyperplanes in $L^t$. Therefore the isomorphism $\Gamma^{\text{base}} : H^0(C, (K_C \otimes t)/W) \to H^0(C, (K_C \otimes L^t)/W)$ induced from $\kappa$ will preserve discriminants.

The isomorphism $\kappa$ globalizes to a commutative diagram of bundles over $C$:

$$
\begin{array}{ccc}
K_C \otimes t & \overset{id_{K_C} \otimes \kappa}{\longrightarrow} & K_C \otimes L^t \\
\downarrow & & \downarrow \\
(K_C \otimes t)/W & \overset{id_{K_C} \otimes \kappa}{\longrightarrow} & (K_C \otimes L^t)/W
\end{array}
$$

The universal cameral covers $\tilde{\mathcal{C}}$ and $L^C\tilde{\mathcal{C}}$ are the pullbacks of the columns of this diagram by the natural evaluation maps

$$
H^0(C, (K_C \otimes t)/W) \times C \longrightarrow \text{tot}(K_C \otimes t)/W
$$

$$
H^0(C, (K_C \otimes L^t)/W) \times C \longrightarrow \text{tot}(K_C \otimes L^t)/W.
$$

The isomorphism $\Gamma^{\text{cam}} : \tilde{\mathcal{C}} \to L^C\tilde{\mathcal{C}}$ is then induced by the isomorphism in the top row of the diagram.

(2) Very roughly our argument will proceed as follows. As a corollary of Claim 3.6 below we establish an isomorphism of lattices $\text{cochar}(P) \cong \text{cochar}(L^P D)$. Tensoring with $S^1$ we get a diffeomorphism of the underlying real tori. From the Leray spectral sequence we get a compatible identification of the universal covers of $P$ and $L^P D$ showing that this diffeomorphism is a complex analytic isomorphism. Some direct topological calculations in section 6 then show that this isomorphism respects the natural polarizations.

In order to avoid too many exceptional cases in the exposition, we will from now on exclude the case when $G$ is of type $A_1$. This case is well understood and recorded in the literature, see e.g. [HT03], [DDD+06].

Let $b \in B - \Delta$ and let $C = \mathcal{C}_b$ be the corresponding cameral cover of $C$, which from now on we identify with the cameral cover $L^C\mathcal{C}_{\text{base}(b)}^{\text{base}}$ via the isomorphism $\Gamma^{\text{cam}}$. We will

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denote the covering map by \( p : \tilde{C} \to C \). Let \( T \subset G \) be the maximal torus of \( G \) and let 
\[ \Lambda := \text{cochar}_G = \text{Hom}(C^\times, T) \] be the corresponding cocharacter lattice.

The two sheaves of commutative groups 
\[ T = p_*(\Lambda \otimes O_C^\times)^W \]
\[ \mathcal{T} = \left\{ t \in \mathcal{T} \middle| \text{for every root } \alpha \text{ of } \mathfrak{g} \text{ we have } \alpha(t)_{|D^\alpha} = 1 \right\} \]
were introduced in [DG02]. In the above formula we identify \( \Lambda \otimes C^\times \) with \( T \) and we view 
a root \( \alpha \) as a homomorphism \( \alpha : T \to C^\times \). The divisor \( D^\alpha \subset \tilde{C} \) is the fixed divisor for the reflection \( \rho^\alpha \in W \) corresponding to \( \alpha \). To these we now add a third group scheme \( T^o \), the connected component of \( T \). By definition, this is the maximal group subscheme of \( T \) all of whose fibers are connected.

It will be convenient to introduce real forms \( T^o_R, T_R, \) and \( \mathcal{T}_R \) which are defined in the same way but with the holomorphic sheaf \( O_C^\times \) replaced by the constant real sheaf \( S^1 \).

By definition we have sheaf inclusions \( T^o_R \subset T_R \subset \mathcal{T}_R \). At any \( x \in C \), which is not a branch point of \( p \), the fibers of the three sheaves are equal to each other and non-canonically isomorphic to the compact torus \( T_R := \Lambda \otimes S^1 \). At a simple branch point \( s \in C \) sitting under a ramification point in \( D^\alpha \subset \tilde{C} \), the fibers are:

\[
T_{R,s}^o = \{ \lambda \otimes z \mid \alpha^\vee (z^{(\alpha,\lambda)}) = 1 \text{ in } T_R \}
\]
\[
T_{R,s} = \{ \lambda \otimes z \mid z^{(\alpha,\lambda)} = 1 \text{ in } S^1 \}
\]
\[
T^o_{R,s} = \{ \lambda \otimes z \mid (\alpha, \lambda) = 0 \text{ in } \mathbb{Z} \}
\]

One of the two main results in [DG02] was that the Hitchin fiber over \( b \) is a torsor over the group \( H^1(C, \mathcal{T}) \) (computed in the etale or in the analytic topology). To carry out the comparison with the Hitchin fiber for \( L^G \), we will also make use of the complex algebraic groups \( H^1(C, T^o) \) and \( H^1(C, \mathcal{T}) \). We do not know how to complete the argument algebraically, so we resort to a topological argument which assures us that the sheaves \( T^o, \mathcal{T}, \mathcal{T} \) have the same first cohomology as their real forms \( T^o_R, \mathcal{T}_R, \) and \( \mathcal{T}_R \). We briefly recall the argument, which was first observed in [DDP07].

**Lemma 3.2** The natural inclusion maps of sheaves \( T^o_R \subset T^o, T_R \subset \mathcal{T}, \mathcal{T}_R \subset \mathcal{T} \) induce isomorphisms on first cohomology (in the analytic topology).

**Proof.** The inclusion of groups \( S^1 \subset C^\times \) induces a natural inclusion of sheaves
\[
\nu : \Lambda \otimes S^1 \hookrightarrow \Lambda \otimes O_C^\times.
\]
We claim that \( \nu \) induces an isomorphism of commutative Lie groups
\[
h^1(\nu) : H^1(C, (p_*(\Lambda \otimes S^1))^W) \cong H^1(C, (p_*(\Lambda \otimes O_C^\times))^W).
\]
\[
\begin{array}{ccc}
H^1(C, T^o_R) & \cong & H^1(C, T^o) \\
\cong & & \cong \\
H^1(C, \mathcal{T}_R) & \cong & H^1(C, \mathcal{T}).
\end{array}
\]

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Indeed, observe that $H^1(C, (p_*(\Lambda \otimes S^1))^W)$ is isogenous to $H^1(\tilde{C}, \Lambda \otimes S^1)^W$ and similarly $H^1(C, (p_*(\Lambda \otimes O^\times_{\tilde{C}}))^W)$ is isogenous to $H^1(\tilde{C}, \Lambda \otimes O^\times_{\tilde{C}})^W$. Under these isogenies the map $h^1(\nu)$ is compatible with the map

$$H^1(\tilde{C}, \Lambda \otimes S^1)^W \to H^1(\tilde{C}, \Lambda \otimes O^\times_{\tilde{C}})^W$$

and so $h^1(\nu)$ has at most a finite kernel and a finitely generated cokernel.

Let $Q$ be the cokernel of the injective map of sheaves $[5]$. Since the constant sheaf $C^\times_{\tilde{C}}$ has a resolution $C^\times_{\tilde{C}} \to O^\times_{\tilde{C}} \to \Omega^1_{\tilde{C}}$, and since $C^\times = S^1 \times \mathbb{R}$, it follows from the snake lemma that $Q$ is isomorphic to a sheaf of $\mathbb{R}$-vector spaces on $\tilde{C}$ which is an extension of $\Lambda \otimes \Omega^1_{\tilde{C}}$ (considered as a sheaf of $\mathbb{R}$-vector spaces) by the constant sheaf $\Lambda \otimes \mathbb{R}$. Consider the push-forward of the exact sequence of $W$-equivariant abelian sheaves

$$1 \to \Lambda \otimes S^1 \to \Lambda \otimes O^\times_{\tilde{C}} \to Q \to 0$$

by the finite map $p : \tilde{C} \to C$. We get a short exact sequence

$$1 \to p_*(\Lambda \otimes S^1) \to p_*(\Lambda \otimes O^\times_{\tilde{C}}) \to p_*Q \to 0$$

of $W$-equivariant sheaves on $C$. Taking the derived functors of $W$ invariants with coefficients in these sheaves we get a long exact sequence of cohomology sheaves:

$$0 \longrightarrow (p_*(\Lambda \otimes S^1))^W \longrightarrow (p_*(\Lambda \otimes O^\times_{\tilde{C}}))^W \longrightarrow (p_*Q)^W$$

$$\mathcal{H}^1(W, p_*(\Lambda \otimes S^1)) \longrightarrow \mathcal{H}^1(W, p_*(\Lambda \otimes O^\times_{\tilde{C}}))$$

Since in the analytic topology $W$ acts properly discontinuously on $\tilde{C}$ and since $p : \tilde{C} \to C$ is assumed to have simple Galois ramification, it follows that the sheaves $\mathcal{H}^1(W, p_*(\Lambda \otimes S^1))$ and $\mathcal{H}^1(W, p_*(\Lambda \otimes O^\times_{\tilde{C}}))$ are supported at branch points of $p : \tilde{C} \to C$ and that their stalks at a branch point are the first cohomologies of $\mathbb{Z}/2$ with coefficients in $T_\mathbb{R}$ and $T$ respectively (see e.g. [Gro57, Theorem 5.3.1]). However $T$ is the product of $T_\mathbb{R}$ with an $\mathbb{R}$-vector space and so the natural map $H^1(\mathbb{Z}/2, T_\mathbb{R}) \to H^1(\mathbb{Z}/2, T)$ is injective (in fact is an isomorphism). Therefore the natural map $\mathcal{H}^1(W, p_*(\Lambda \otimes S^1)) \to \mathcal{H}^1(W, p_*(\Lambda \otimes O^\times_{\tilde{C}}))$ is also injective and so we have a short exact sequence

$$1 \to \mathcal{T}_\mathbb{R} \to \mathcal{T} \to (p_*Q)^W \to 0$$

of sheaves on $C$. From the associated long exact sequence in cohomology

$$0 \longrightarrow H^0(C, \mathcal{T}_\mathbb{R}) \longrightarrow H^0(C, \mathcal{T}) \longrightarrow H^0(C, (p_*Q)^W)$$

$$H^1(C, \mathcal{T}_\mathbb{R}) \xrightarrow{h^1(\nu)} H^1(C, \mathcal{T}) \longrightarrow H^1(C, (p_*Q)^W)$$

$$H^2(C, \mathcal{T}_\mathbb{R}) \rightarrow \cdots$$
Lemma 3.3 (a)  This is standard and in fact the statement for $\varepsilon$ is also yields the identification $H^2(C, \mathcal{T}_{\mathbb{R}}) \cong (\Lambda \otimes S^1)^W$. Since $G$ is assumed simple it follows that $H^1(C, \mathcal{T}_{\mathbb{R}})$ is finite and since $H^1(C, (p_s Q)^W)$ is an $\mathbb{R}$-vector space we get that $\ker h = 0$. Also, note that $H^2(C, \mathcal{T}_{\mathbb{R}}) = H^2(C, (p_s (\Lambda \otimes S^1))^W)$ is isogenous to $H^2(C, p_s (\Lambda \otimes S^1))^W = H^2(\tilde{C}, \Lambda \otimes S^1)^W \cong (\Lambda \otimes S^1)^W$. Since $G$ is assumed simple it follows that $H^2(C, \mathcal{T}_{\mathbb{R}})$ is finite and since $H^1(C, (p_s Q)^W)$ is an $\mathbb{R}$-vector space we get that $H^1(C, (p_s Q)^W) \to H^2(C, \mathcal{T}_{\mathbb{R}})$ is the zero map. But from the long exact sequence this kernel is equal to the cokernel of $h^1(\nu)$ which is finitely generated as an abelian group. Hence $\text{coker}(h^1(\nu)) = 0$ and $h^1(\nu)$ is an isomorphism.

Next note that by our assumption of simple Galois ramification for $p : \tilde{C} \to C$ and from the definitions of $\mathcal{T}$ and $\mathcal{T}_{\mathbb{R}}$, it follows that $\mathcal{T}/\mathcal{T}$ is a sheaf of groups, which is supported at the branch points of $p$, and whose stalk at a branch point $s$ is representable by the finite group $\mathcal{T}_{\mathbb{R},s}/\mathcal{T}_{\mathbb{R},s}$. Using this fact and the isomorphism $h^1(\nu)$, we can compare the long exact cohomology sequences associated with $0 \to \mathcal{T} \to \mathcal{T} \to \mathcal{T}/\mathcal{T} \to 0$ and $0 \to \mathcal{T}_{\mathbb{R}} \to \mathcal{T}_{\mathbb{R}} \to \mathcal{T}_{\mathbb{R}}/\mathcal{T}_{\mathbb{R}} \to 0$, to conclude that $H^1(C, \mathcal{T}_{\mathbb{R}}) \cong H^1(C, \mathcal{T})$. The same reasoning also yields the identification $H^1(C, \mathcal{T}_{\mathbb{R}}^*) \cong H^1(C, \mathcal{T}^*)$. \hfill $\square$

Recall that a root $\alpha$ for $\mathfrak{g}$ determines a homomorphism $(\alpha, \bullet) : \Lambda_G \to \mathbb{Z}$. We let $\varepsilon = \varepsilon_{\alpha, G}$ be the positive generator of the image. We also define $\varepsilon^\vee = \varepsilon_{\alpha, G}^\vee := \varepsilon_{\alpha, G}^\vee$. 

Lemma 3.3 (a) $\varepsilon_{\alpha, G} = 2$ when $G = \text{Sp}(r)$ and $\alpha$ is a long root, and $\varepsilon_{\alpha, G} = 1$ in all other cases. Dually $\varepsilon_{\alpha, G}^\vee = 2$ when $G = \text{SO}(2r + 1)$ and $\alpha$ is a short root and $\varepsilon_{\alpha, G}^\vee = 1$ in all other cases.

(b) $\varepsilon_{\alpha, G}$ is characterized by the property that $\alpha/\varepsilon_{\alpha, G}$ is a primitive vector in $\Lambda^\vee$.

Proof. (a) This is standard and in fact the statement for $\varepsilon_{\alpha, G}^\vee$ was already noted in [DG02]. The explicit argument goes as follows. Without loss of generality we may assume that $\alpha$ is a simple root. For any group $G$ we have $\Lambda_G \supset \text{coroot}_g$, so for any root $\beta$ we get $(\alpha, \beta^\vee) \in (\alpha, \Lambda_G)$. When $\beta$ is simple we can read this number from the Dynkin diagram:

$$
(\alpha, \beta^\vee) = \frac{2(\alpha, \beta)}{\langle \beta, \beta \rangle} = \begin{cases} 
2 & \text{when } \alpha = \beta, \\
-n & \text{when } \beta \text{ is short and } n \text{ edges connect } \alpha \text{ and } \beta, \\
-1 & \text{when } \beta \text{ is long and connected to } \alpha, \\
0 & \text{otherwise.}
\end{cases}
$$

This shows that $\varepsilon_{\alpha, G} = 1$ unless all roots $\beta$ connected to $\alpha$ are short and connect to $\alpha$ by an even number of edges. This happens only when $\alpha$ is a long root and $\mathfrak{g}$ is of type $C_r$. In the latter case we compute that $(\alpha, \text{coroot}_g) = 2\mathbb{Z}$, while $(\alpha, \text{coweight}_g) = \mathbb{Z}$.

(b) Clearly if $k$ is an integer and $\alpha/k \in \Lambda^\vee$, then $k$ divides $\varepsilon_{\alpha, G}$. So we only need to check that for a long root $\alpha$ of $\text{Sp}(r)$ we have that $\alpha/2$ is in the weight lattice. But the root lattice for type $C_r$ has generators $e_1 - e_2, \ldots, e_{r-1} - e_r, 2e_r$ and the weight lattice has generators
Consider the cover \( p : \tilde{C} \to C \). We will denote the branch locus of this cover by \( S = \{s_1, \ldots, s_b\} \subset C \). For each \( i = 1, \ldots, b \) we will write \( \alpha_i \) for the root of \( g \) determined (up to \( W \) action) by \( s_i \). Let \( \varepsilon_i := \varepsilon_{\alpha_i, G} \) and \( \varepsilon_i^\vee := \varepsilon_{\alpha_i, G}^\vee \). We write \( j : U \to C \) for the inclusion of the complement, and \( p^o : p^{-1}(U) \to U \) for the unramified part of \( p \). Define a local system \( A \) on \( U \) by \( A := (p_\varepsilon^o \Lambda)^W \). Note that the fibers of \( A \) are non-canonically isomorphic to \( \Lambda \).

We can also consider \( L A = (p_\varepsilon^o (\varepsilon \Lambda))^W \). The canonical identification \( L \Lambda = \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z}) \) gives also an identification \( L A = A^\vee = \text{Hom}(\Lambda, \mathbb{Z}) \).

**Lemma 3.4** There are natural isomorphisms of sheaves \( \mathcal{T}^o_R \cong (j_* A) \otimes S^1 \) and \( \mathcal{O}_R \cong j_* (A \otimes S^1) \), while \( \mathcal{O}_R \) is determined by the commutative diagram:

\[
\begin{array}{c}
0 \\
0 \to \mathcal{T}^o_R \to \mathcal{T}_R \to \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \\
0 \to \mathcal{O}_R \to \mathcal{T}_R \to \mathbb{Z}/\varepsilon_i^\vee \\
0 \to \mathcal{T}^o_R \to \mathcal{T}_R \to \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \\
0 \to \mathcal{O}_R \to \mathcal{T}_R \to \mathbb{Z}/\varepsilon_i^\vee \\
0 \to 0
\end{array}
\]

**Proof.** Clearly the sheaves \( \mathcal{T}^o_R, \mathcal{O}_R, (j_* A) \otimes S^1 \) and \( j_* (A \otimes S^1) \) coincide on \( U \). Since \( A \) is a local system we have (see Section 6.1) \( (j_* A)_{s_i} \cong \Lambda^{\alpha_i} \), where \( \rho_i(\lambda) = \lambda - (\alpha_i, \lambda) \alpha_i^\vee \) is the reflection corresponding to \( \alpha_i \). Similarly \( (j_* (A \otimes S^1))_{s_i} \cong (\Lambda \otimes S^1)^{\alpha_i} \). The formula for the reflection \( \rho_i \) and ([1]) now imply that \( \mathcal{T}^o_{R, s_i} = (j_* A)_{s_i} \otimes S^1 \) and \( \mathcal{O}_{R, s_i} = (j_* (A \otimes S^1))_{s_i} \).

On the stalk at \( s_i \) the map \( \xi \) is given by

\[\xi(\lambda \otimes z) := z^{(\alpha_i/\varepsilon_i, \lambda)} \in \mu_{\varepsilon_i^\vee} \subset S^1.\]

Here \( \mu_{\varepsilon_i^\vee} \subset S^1 \) denotes the roots of unity of order \( \varepsilon_i^\vee \). Since \( \varepsilon_i^\vee \) divides 2, we have a natural identification \( \mu_{\varepsilon_i^\vee} = \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee \).

From ([1]) we now deduce that \( \mathcal{T}^o_{\varepsilon_i, s_i} = \ker(\xi) \), and that \( \mathcal{T}_R = \ker(\varepsilon^\vee \circ \xi) \), where

\[\varepsilon_i : \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i^\vee \to \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \]

is the map which multiplies the \( i \)-th summand by \( \varepsilon_i^\vee. \)

**Claim 3.5** (i) The connected components \( P^o, P, \mathcal{O}_R \) of \( H^1(C, \mathcal{T}^o), H^1(C, \mathcal{T}), H^1(C, \mathcal{O}_R) \) are abelian varieties. The natural maps \( H^1(C, \mathcal{T}^o) \to H^1(C, \mathcal{T}) \to H^1(C, \mathcal{O}_R) \) and \( P^o \to P \to \mathcal{O}_R \) are surjective.
(ii) The group of connected components of $H^1(C, \mathcal{T})$ is $\mathbb{Z}/2$ for $G = \text{Sp}(r)$ and is $\pi_1(G)$ otherwise.

(iii) The group of connected components of $H^1(C, \mathcal{T})$ is always $\pi_1(G)$, so the components of the fiber of $h : \text{Higgs} \to B$ are in one-to-one correspondence with the components of the $G$-Hitchin Higgs system itself.

Proof. (i) We already noted that the connected component of $H^1(C, \mathcal{T})$ is an abelian variety, and that $\mathcal{T}/\mathcal{T}$ and $\mathcal{T}/\mathcal{T}^o$ have finite supports and fibers which are finite groups. It follows that $H^1(C, \mathcal{T}^o)$ and $H^1(C, \mathcal{T})$ map to $H^1(C, \mathcal{T})$ surjectively with finite kernels. In particular the connected components of $H^1(C, \mathcal{T}^o)$ and $H^1(C, \mathcal{T})$ are also abelian varieties.

(ii) The case $g = 1$ is elementary: the cameral cover is a product of $C$ with a finite scheme, so there is no ramification, $U = C$, and the result is straightforward. So assume $g > 1$.

We consider the exponential sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$ of constant sheaves on $C$.

Tensoring with $j_*A$ gives

$$\text{Tor}_1(j_*A, S^1) \to j_*A \to (j_*A) \otimes \mathbb{R} \to \mathcal{T}_\mathbb{R}^o \to 0.$$ 

The sheaf $\text{Tor}_1(j_*A, S^1)$ is supported on $S$ while $j_*A$ has no compactly supported sections. Therefore $\partial = 0$ and we get a short exact sequence

$$0 \to H^1(j_*A \otimes \mathbb{R})/H^1(j_*A) \to H^1(C, \mathcal{T}_\mathbb{R}^o) \to H^2(C, j_*A) \to 0.$$

The group of connected components of $H^1(C, \mathcal{T}_\mathbb{R}^o)$ is therefore $H^2(C, j_*A)$, which can be identified (see Lemma 6.3) with $H^1(U, A^\vee)_{\text{tor}}$.

The calculation performed in Corollary 6.6 below gives that:

$$H^1(U, A^\vee)_{\text{tor}} = \left( \frac{(\Lambda^\vee)^b}{(1 - \rho_1, 1 - \rho_2, \ldots, 1 - \rho_b)\Lambda^\vee} \right)_{\text{tor}}.$$

Now for any inclusion of lattices $N \subset M$, the torsion in $M/N$ is equal to the quotient $N'/N$ where $N' := \{m \in M | k \cdot m \in N \text{ for some } k \neq 0 \in \mathbb{Z} \}$ is the saturation of $N$ in $M$. In our case $N = \Lambda^\vee$, while the saturation is

$$N' = \{\xi \in t^\vee \mid (\xi, \alpha^\vee) \cdot \alpha \in \Lambda^\vee \text{ for every root } \alpha \}.$$

This holds since our genericity assumption on $\tilde{C}$ implies that $D^\alpha \subset \tilde{C}$ is non-empty for every root $\alpha$. Using the characterization of $\varepsilon_{\alpha,G}$ in Lemma 6.3(b), we see that $\xi \in N'$ if and only if $\varepsilon_{\alpha,G}(\xi, \alpha^\vee) \in \mathbb{Z}$ for all roots $\alpha$. In case $G = \text{Sp}(r)$ we see that $N'$ contains the weight lattice as a sublattice of index two. Explicitly the weight lattice is generated by $e_1, \ldots, e_r$ and $N'$.
is spanned by the $e_i$'s and the additional vector $\frac{1}{2} \sum_{i=1}^r e_i$. For all other $G$, all $\varepsilon$'s are 1, so $N'$ is the weight lattice. We conclude that

$$H^1(U, A^\vee)_{\text{tor}} = \begin{cases} \mathbb{Z}/2 & \text{when } G = \text{Sp}(r) \\ \text{wts}_g / A^\vee_G = \pi_1(G)^\wedge & \text{for all other } G. \end{cases}$$

This completes the proof of (ii).

(iii) As we saw in Lemma 3.4 we have $\mathcal{T}^0 = \mathcal{T}$ except when $G = \text{Sp}(r)$. In the latter case the fiber of the Hitchin map was shown to be connected in [Hit87], using an interpretation via spectral covers. \hfill \square

In general for any compact torus $H$ we define the cocharacter lattice $\text{cochar}(H)$ as the lattice of homomorphisms from the circle $S^1$ to $H$. We recover $H$ as $\text{cochar}(H) \otimes S^1$.

**Claim 3.6**

(i) There is a natural isomorphism $\text{cochar}(P^o) = H^1(C, j_* A)_{\text{tf}}$.

(ii) There is a natural isomorphism $\text{cochar}(\mathcal{P}) = H^1(C, j_* A^\vee)^\vee$.

(iii) The map $\zeta : \text{cochar}(\mathcal{P}) \to \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee$ induced from the map $\xi$ in Lemma 3.4 satisfies

$$\ker(\zeta) = \text{cochar}(P^o)$$
$$\ker(\varepsilon^\vee \circ \zeta) = \text{cochar}(P).$$

**Proof.** (i) By Lemma 3.4 we know that $\mathcal{T}_R^o = (j_* A) \otimes S^1$. As in the proof of Claim 3.5 (ii), we tensor the exponential sequence for $S^1$ by $j_* A$ and we get

$$\overline{\text{Tor}}_1(j_* A, S^1) \xrightarrow{\partial} j_* A \xrightarrow{j_* A \otimes \mathbb{R}} \mathcal{T}_R^o \longrightarrow 0.$$ 

Again the sheaf $\overline{\text{Tor}}_1(j_* A, S^1)$ is supported on $S$ while $j_* A$ has no compactly supported sections. Therefore $\partial = 0$ and we get a short exact sequence

$$0 \longrightarrow H^1(j_* A) \otimes S^1 \longrightarrow H^1(C, \mathcal{T}^o) \longrightarrow H^2(C, j_* A) \longrightarrow 0.$$ 

Since $H^2(C, j_* A)$ is finite and $H^1(j_* A) \otimes S^1$ is connected, it follows that $P^o = H^1(j_* A) \otimes S^1$, or equivalently $\text{cochar}(P^o) = H^1(C, j_* A)_{\text{tf}}$.

(ii) Start with the Leray spectral sequence (aka Mayer-Vietoris) for the inclusion $j : U \subset C$ and the sheaf $A$. It gives

$$0 \to H^1(C, j_* A) \to H^1(U, A) \to Q \to 0,$$
where $Q = \ker(H^0(R^1 j_* A) \to H^2(j_* A))$ (see Section 6.1 for details). We tensor this sequence with $S^1$ and map to the Leray sequence for $j$ and $A \otimes S^1$:

$$
\begin{array}{c}
Q_{\text{tor}} \longrightarrow H^1(C, j_* A) \otimes S^1 \longrightarrow H^1(U, A) \otimes S^1 \longrightarrow Q \otimes S^1 \longrightarrow 0 \\
0 \longrightarrow H^1(C, \overline{T}_R) \longrightarrow H^1(U, A \otimes S^1) \longrightarrow Q \otimes S^1 \longrightarrow 0.
\end{array}
$$

Recall that $\overline{T}_R$ is the connected component of $H^1(C, \overline{T}_R)$ and that by Lemma 3.4 we have $\overline{T}_R = j_*(A \otimes S^1)$. It follows that $\overline{T}_R$ can be identified with the image $\text{im} [P^o \to H^1(U, A) \otimes S^1]$. In particular, on character lattices we get

$$
\text{char} \overline{T}_R = \text{im} \left[ H^1(U, A)^\vee \to \text{char} P^o \right] = H^1(C, j_* A^\vee)^{\text{tf}},
$$

where the last equality follows from Corollary 6.4.

(iii) This is immediate from parts (i) and (ii) and the commutative diagram of sheaves in Lemma 3.4. \qed

The statement of the previous claim can be organized in a diagram:

$$
\begin{array}{c}
\text{char}(P) \subset H^1(C, j_* A)^{\text{tf}} \subset H^1(C, j_* A^\vee)^{\text{tf}} \\
\downarrow \quad \quad \quad \downarrow \\
0 \subset \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i^\vee \subset \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee.
\end{array}
$$

Writing the analogous diagram for $L^G$ and dualizing gives

$$
\begin{array}{c}
\text{char}(L^G) \subset H^1(C, j_* A)^{\text{tf}} \subset H^1(C, j_* A^\vee)^{\text{tf}} \\
\downarrow \quad \quad \quad \downarrow \\
0 \subset \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i^\vee \subset \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee
\end{array}
$$

This gives the desired isomorphism of lattices: $\text{cochar}(P) \cong \text{cochar}(L^G)^{\text{tf}}$. Tensoring with $S^1$, we get a diffeomorphism of the underlying real manifolds. The Leray spectral sequence for $p : \tilde{C} \to C$ allows us to identify the universal covers of $P$ and $L^G$ with the complex vector spaces $H^1(\tilde{C}, \Lambda \otimes \mathcal{O}_{\tilde{C}})^W$ and $H^1(\tilde{C}, \Lambda^\vee \otimes \mathcal{O}_{\tilde{C}})^W$, showing that the diffeomorphism is an isomorphism of complex manifolds. Finally, we need to check that the two polarizations correspond to each other. This amounts to the compatibility of our isomorphism with the Poincare duality map for the cohomologies of $A$ and $A^\vee$ on $U$. We defer this calculation to Lemma 6.3 and Corollary 6.4 in section 6. We have thus produced an isomorphism between the polarized abelian varieties $P$ and $L^G$. This completes the proof of Theorem A. \qed

4 Duality for Higgs gerbes

In this section we extend the duality of cameral Pryms established in Theorem A to a more general duality for the stacks of Higgs bundles, considered as families of stacky groups over the Hitchin base.
4.1 Triviality

The moduli stack of $G$-Higgs bundles on $C$ was defined and studied in [DG02]. We briefly recall the highlights of that discussion. Let

$$
\begin{array}{c}
\tilde{\mathcal{C}} \\
\downarrow \hat{\pi} \\
B \\
\downarrow \pi \\
B \times C \\
p \\
\end{array}
$$

denote the universal cameral cover. A sheaf $\mathcal{T}$ of abelian groups on $B \times C$ was introduced in [DG02]. It was defined as:

$$
\mathcal{T}(U) = \left\{ t \in \Gamma(p^{-1}(U), \Lambda \otimes \mathcal{O}_{\tilde{\mathcal{C}}}^\times)^W \left| \text{for every root } \alpha \text{ of } \mathfrak{g} \text{ we have } \alpha(t)|_{D^{\alpha}} = 1 \right. \right\}.
$$

It was shown in [DG02] that relatively over the Hitchin base the stack $\mathcal{Higgs}$ of Higgs bundles is a banded $\mathcal{T}$-gerbe. Informally, this means that the sheaf of groups for which $\mathcal{Higgs}$ is a gerbe is $\mathcal{T}$ itself rather than a more general sheaf of groups which is only locally isomorphic to $\mathcal{T}$. Equivalently, when viewed as a stack over the Hitchin base, $\mathcal{Higgs}$ is a torsor over the commutative group stack $\mathcal{T}_{\text{tors}}$ parametrizing $\mathcal{T}$-torsors along the fibers of $\pi : B \times C \to B$. This description is valid for Higgs bundles over a base variety of arbitrary dimension. When the base is a compact curve, the picture can be made even more precise.

**Lemma 4.1** Let $C$ be a smooth compact curve and let $\mathcal{Higgs}$ be the moduli stack of $G$-Higgs bundles on $C$.

(a) The commutative group stack $\mathcal{T}_{\text{tors}}$ parametrizing the $\mathcal{T}$-torsors along the fibers of $\pi$ is isomorphic to the Picard stack associated (see [SGA, Section 1.4 of Exposé XVIII] and [Lau96, Section 6]) with the amplitude one complex $R^\bullet \pi_\ast \mathcal{T}[1]$ of abelian sheaves on $B$.

(b) There exists an isomorphism $\mathcal{Higgs} \cong \mathcal{T}_{\text{tors}}$ of stacks over $B$.

**Proof.** (a) This follows from the fact that $\pi : B \times C \to B$ is smooth of relative dimension one, the standard description of torsors in terms of Čech cocycles, and the definition (see [SGA, Section 1.4 of Exposé XVIII] and [Lau96, Section 6]) of a Picard stack associated with an amplitude one complex of abelian sheaves.

(b) By [DG02, Theorem 4.4] the stack $\mathcal{Higgs}$ is a torsor over the commutative group stack $\mathcal{T}_{\text{tors}}$. Thus to get the isomorphism $\mathcal{Higgs} \cong \mathcal{T}_{\text{tors}}$, it suffices to show that the stacky Hitchin fibration $\mathcal{h} : \mathcal{Higgs} \to B$ admits a section. This is due to Hitchin who in [Hit92] constructed a family of holomorphic sections of $\mathcal{h} : \mathcal{Higgs} \to B$ induced from a Kostant section of the Chevalley map $\mathfrak{g} \to \mathfrak{g}/G \cong \mathfrak{t}/W$. Since these sections play a prominent role in what follows, we briefly recall Hitchin’s construction.
Let \( \{e, f, g\} \subset \mathfrak{g} \) be any principal \( \mathfrak{sl}_2 \) triple in \( \mathfrak{g} \). This means that \( e, f, g \) span a Lie subalgebra in \( \mathfrak{g} \) isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \), and that \( e \) and \( f \) are regular nilpotent elements of \( \mathfrak{g} \). Let \( \mathcal{C}(e) \subset \mathfrak{g} \) be the centralizer of the element \( e \) in the algebra \( \mathfrak{g} \). Consider the linear coset \( f + \mathcal{C}(e) = \{f + x \mid x \in \mathcal{C}(e)\} \). In [Kos66] Kostant showed that the map \( \mathfrak{g} \to \mathfrak{t}/W \) becomes an isomorphism, when restricted to \( f + \mathcal{C}(e) \). Thus \( f + \mathcal{C}(e) \) is a section for the Chevalley projection. By construction this section consists of regular elements in \( \mathfrak{g} \) and is a generalization of the rational canonical form of a matrix.

Fix a Kostant section \( k: \mathfrak{t}/W \to \mathfrak{g} \), corresponding to an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \). The inclusion of the \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) induces a group homomorphism \( \text{diag}: \text{SL}_2(\mathbb{C}) \to \mathbb{G} \). Let \( \zeta \in \text{Pic}^{g-1}(\mathbb{C}) \), \( \zeta^2 = K \) be a theta characteristic on \( \mathbb{C} \). Consider the frame bundle \( \text{Isom}(\zeta \oplus \zeta^{-1}, O^{\oplus 2}) \) of the vector bundle \( \zeta \oplus \zeta^{-1} \) on \( \mathbb{C} \). This is a principal \( \text{SL}_2(\mathbb{C}) \)-bundle which via \( \text{diag} \) gives rise to an associated principal \( \mathbb{G} \) bundle \( P := \text{Isom}(\zeta \oplus \zeta^{-1}, O^{\oplus 2}) \times_{\text{diag}} \mathbb{G} \) on \( \mathbb{C} \). Recall that the Hitchin base \( B \) is the space of sections of the bundle \( (K \otimes \mathfrak{t})/W \) on \( \mathbb{C} \). Let \( U \) denote the total space of the bundle \( (K \otimes \mathfrak{t})/W \), and let \( u: U \to \mathbb{C} \) be the natural projection. We have

\[
\text{ad}(u^*P) = u^* \text{ad}(P) = u^* \text{Isom}(\zeta \oplus \zeta^{-1}, O^{\oplus 2}) \times_{\text{ad(diag)}} \mathfrak{g}.
\]

In [Hit92] Hitchin checked that the Kostant section \( k: \mathfrak{t}/W \to \mathfrak{g} \) induces a well defined section \( \varphi \in H^0(U, \text{ad}(u^*P) \otimes u^*K) \) and hence a \( u^*K \)-valued Higgs bundle \( (u^*P, \varphi) \) on \( U \). Pulling back this Higgs bundle by the sections \( b \in B = H^0(C, U) \), one gets a family of Higgs bundles on \( \mathbb{C} \), parametrized by \( B \). We will call the resulting section of \( h: \text{Higgs} \to B \) the Hitchin section and denote it by \( v: B \to \text{Higgs} \).

\[\square\]

### 4.2 Stabilizers, components, and universal bundles

From now on, we restrict our attention to the open substack of \( \text{Higgs} \) consisting of stable \( \mathbb{G} \)-Higgs bundles whose automorphism group is the minimal possible, i.e. coincides with the center of \( \mathbb{G} \). The Hitchin fiber for a cameral cover in \( B - \Delta \) consists only of stable Higgs bundles, and in fact each Higgs bundle in such fiber has minimal automorphism group:

**Lemma 4.2**

(i) \( \text{Higgs}_{|B - \Delta} \) is a smooth Deligne-Mumford stack with a coarse moduli space \( \text{Higgs}_{|B - \Delta} \). If we view \( \text{Higgs}_{|B - \Delta} \) as a group stack, then the group of connected components, as well as the connected components of each fiber of \( h: \text{Higgs}_{|B - \Delta} \to (B - \Delta) \) are canonically isomorphic to \( \pi_1(G) \).

(ii) \( \text{Higgs}_{|B - \Delta} \) is a banded \( \mathbb{Z}(G) \)-gerbe over \( \text{Higgs}_{|B - \Delta} \) which is locally trivial over \( B - \Delta \). In particular the restriction of \( \text{Higgs}_{|B - \Delta} \) to a Hitchin fiber is a trivial gerbe.

(iii) The gerbe \( \text{Higgs}_{|B - \Delta} \to \text{Higgs}_{|B - \Delta} \) measures the obstruction to lifting the universal \( \mathbb{G}_{\text{ad}} \)-Higgs bundle to a universal \( \mathbb{G} \)-Higgs bundle.

**Proof.** (i) It is well known [Sim94, Sim95] that the stack \( \text{Higgs} \) of \( \mathbb{G} \)-Higgs bundles is an Artin algebraic stack with an affine diagonal which is locally of finite type. The
As explained in Lemma 4.1, the automorphism group of any object in the groupoid which is smooth at all points with finite stabilizers. Therefore it suffices to show that \( \operatorname{Higgs}_{|B - \Delta} \) parametrizes Higgs bundles with minimal automorphism group. In [Fal93, Theorem III.2] Faltings showed that \( B \) contains a Zariski open and dense subset \( B^0 \subset B \), such that \( \operatorname{Higgs}_{|B^0} \) parametrizes only stable Higgs bundles with automorphism group \( Z(G) \). We will give a direct argument for this over \( B - \Delta \), i.e. we will show that \( B^0 \supset B - \Delta \).

Fix a point \( b \in (B - \Delta) \) and let \( p : \tilde{C} \to C \) be the corresponding cameral cover. We must show that every object in the groupoid \( \operatorname{Higgs}_{\tilde{C}} := h^{-1}(b) \) has automorphism group \( Z(G) \). As explained in Lemma 6.6, let \( \operatorname{Higgs}_{\tilde{C}} \) is the groupoid of \( T_{\tilde{C}} := T_{\{b\} \times C} \)-torsors, and hence the automorphism group of any object in \( \operatorname{Higgs}_{\tilde{C}} \) is isomorphic to the cohomology group \( H^0(C, T_{\tilde{C}}) \). By the argument we used in the proof of Theorem 2.2 we have isomorphisms \( H^0(C, T_{\tilde{C}}) \cong H^0(C, T_{\tilde{C}, \mathbb{R}}) \), \( H^0(C, T_{\tilde{C}}^0) \cong H^0(C, T_{\tilde{C}, \mathbb{R}}^0) \), \( H^0(C, \nabla_{\tilde{C}}) \cong H^0(C, \nabla_{\tilde{C}, \mathbb{R}}) \). Thus it suffices to compute \( H^0(C, T_{\tilde{C}, \mathbb{R}}^0) \). We start by calculating the global sections of \( T_{\mathbb{R}}^0 = (j_* A) \otimes S^1 \). As in the proof of Claim 3.5(ii) we get a short exact sequence of sheaves on \( \tilde{C} \):

\[
0 \to j_* A \to (j_* A) \otimes \mathbb{R} \to (j_* A) \otimes S^1 \to 0.
\]

Passing to cohomology, and taking into account that \( H^0(C, j_* A) = 0 \) and \( H^0(C, (j_* A) \otimes \mathbb{R}) = H^0(C, j_* A) \otimes \mathbb{R} = 0 \), we get that

\[
H^0(C, T_{\tilde{C}, \mathbb{R}}^0) = \ker \left[ H^1(C, j_* A) \to H^1(C, j_* A \otimes \mathbb{R}) \right] = H^1(C, j_* A)_{\text{tor}}.
\]

The latter group can be calculated explicitly from Corollary 6.6. In the notation of Corollary 6.6 let \( N \) denote the saturation of \( (1 - \rho_1, \ldots, 1 - \rho_b) \Lambda \) inside \( \oplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \). Then

\[
H^1(C, j_* A)_{\text{tor}} = N/(1 - \rho_1, \ldots, 1 - \rho_b) \Lambda = \{ \xi \in \mathfrak{t} \mid (\xi, \alpha) \in \varepsilon_{\alpha, G} \mathbb{Z} \text{ for every root } \alpha \} / \Lambda
\]

\[
= \begin{cases} Z(G) & \text{if } G \neq \text{Sp}(r) \\ 0 & \text{if } G = \text{Sp}(r). \end{cases}
\]

From Lemma 3.4 we know that as long as \( G \neq \text{Sp}(r) \) we have \( T_{\tilde{C}, \mathbb{R}}^0 = T_{\tilde{C}, \mathbb{R}} \). This proves our claim for \( G \neq \text{Sp}(r) \). For \( G = \text{Sp}(r) \), Lemma 3.4 gives a short exact sequence

\[
0 \to T_{\tilde{C}, \mathbb{R}}^0 \to T_{\tilde{C}, \mathbb{R}} \to \oplus_{i=1}^b \mathbb{Z} / \varepsilon_i \to 0,
\]

and after passing to cohomology we get

\[
H^0(C, T_{\tilde{C}, \mathbb{R}}) = \ker \left[ \oplus_{i=1}^b \mathbb{Z} / \varepsilon_i \to H^1(C, T_{\mathbb{R}}^0) \right] = Z(\text{Sp}(r)) \cong \mathbb{Z}/2,
\]

27
where $Z(\text{Sp}(r)) \cong \mathbb{Z}/2$ maps diagonally in $\bigoplus_{i=1}^{b} \mathbb{Z}/2_i$. This proves our assertion about the automorphisms of objects in $\text{Higgs}_C$ and finishes the proof of (i).

(ii) As we saw above, the $\mathcal{T}$ torsors in $\text{Higgs}_{|B-\Delta}$ all have automorphism groups isomorphic to $Z(G)$, and so $\text{Higgs}_{|B-\Delta}$ is a $Z(G)$-gerbe on $\text{Higgs}_{|B-\Delta}$. In particular $R^0\pi_*\mathcal{T}$ is a local system on $B - \Delta$ with fiber $Z(G)$. However $Z(G) = T^W \subset T$ and so, by the definition of $\mathcal{T}$ we have a canonical inclusion of the constant sheaf $Z(G)$ into $\mathcal{T}$. Thus, every element in $Z(G)$ gives rise to a global section of $\mathcal{T}$ on $B \times C$, and hence to a global section of $R^0\pi_*\mathcal{T}$ on $B - \Delta$. This shows that $R^0\pi_*\mathcal{T}$ is the constant sheaf and so $\text{Higgs}_{|B-\Delta}$ is banded as a gerbe over $\text{Higgs}_{|B-\Delta}$. Finally, note that locally over $B - \Delta$, the universal cameral cover $\tilde{C}$ admits a section. The stack of $\mathcal{T}$-torsors which are framed along such a section is isomorphic to the space $\text{Higgs}$, which shows that the gerbe $\text{Higgs}_{|B-\Delta}$ is locally trivial over $(B - \Delta)$.

(iii) This is completely analogous to the $\text{PSL}(r)$ argument in [HT03]. Let $G_{\text{ad}}$ be the adjoint form of $G$. From part (ii) it follows that the stack of $G_{\text{ad}}$-Higgs bundles that have cameral cover in $B - \Delta$ is actually a space, i.e. $\text{Higgs}_{G_{\text{ad}}|\{(B-\Delta)\}} = \text{Higgs}_{G_{\text{ad}}|\{(B-\Delta)\}}$. Since the stack always has a universal bundle, we have a universal $G_{\text{ad}}$-Higgs bundle $(\mathcal{V}, \varphi)$ on $\text{Higgs}_{G_{\text{ad}}|\{(B-\Delta)\}} \times C$. The natural map $G \to G_{\text{ad}}$ induces a morphism of spaces $q : \text{Higgs}_{|B-\Delta} \to \text{Higgs}_{G_{\text{ad}}|\{(B-\Delta)\}}$ and we can consider the pullback $G_{\text{ad}}$-Higgs bundle $q^*(\mathcal{V}, \varphi)$ on $\text{Higgs}_{|B-\Delta}$. Since ker$[G \to G_{\text{ad}}] = Z(G)$, it follows that the obstruction to lifting $q^*(\mathcal{V}, \varphi)$ to a $G$-Higgs bundle is simply the obstruction to the existence of an universal $G$-Higgs bundle on $\text{Higgs}_{|\{(B-\Delta)\}} \times C$, i.e it is the gerbe $\text{Higgs}_{|\{(B-\Delta)\}} \times C$. In particular, restricting to $\text{Higgs}_{|\{(B-\Delta)\}} \times \{\text{pt}\}$ we get the statement (iii). \hfill $\square$

### 4.3 Global duality

We are now ready to state the main result of this section. For any commutative group stack $h : \mathcal{X} \to S$ over a scheme $S$ (for us $S$ will always be $B - \Delta$) with zero section $0 : S \to \mathcal{X}$, and group law $a : \mathcal{X} \times_S \mathcal{X} \to \mathcal{X}$, we define the dual commutative group stack as the stack of homomorphisms of commutative group stacks from $\mathcal{X}$ to $BG_m$:

$$\mathcal{X}^D := \text{Hom}_{\text{grp-stack}}(\mathcal{X}, O^x_S)[1]w.$$ 

Geometrically $\mathcal{X}^D$ is the stack of group extensions of $\mathcal{X}$ by $O^x$, or equivalently, the stack parametrizing triples $(\mathcal{L}, m, f)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}$, and $(m, f)$ is a 'theorem of the square structure' on $\mathcal{L}$ [Mum03, Br87]. Concretely the square structure consists of an isomorphism of line bundles $m : p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \to a^*\mathcal{L}$ on $\mathcal{X} \times_S \mathcal{X}$ and a framing $f : v^*\mathcal{L} \cong \mathcal{O}_S$ which satisfy the normalizations $(0 \circ h \times \text{id})^*m = h^*e \otimes \text{id}_S$ and $(\text{id} \times 0 \circ h)^*m = \text{id}_S \otimes h^*e$ on $\mathcal{X}$ and the cocycle condition $(p_{1+2} \times p_3)^*m \circ (p_{12}^*m \otimes \text{id}) = (p_1 \times p_{2+3})^*m \circ (\text{id} \otimes p_{23}^*m)$ on the triple product $\mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X}$.

Note that if $\mathcal{X}$ is an abelian scheme over $S$, then $\mathcal{X}^D$ is the usual dual abelian scheme. Also for well behaved group stacks $\mathcal{X}$ the duality operation $(\bullet)^D$ converts disconnectedness
into gerbiness and vice versa \cite{DP08}. In particular if the relative group \(\pi_0(\mathcal{X}/S)\) of connected components is a finite flat group scheme over \(S\), then \(\mathcal{X}^D\) will be a banded \(\pi_0(\mathcal{X}/S)^\wedge\)-gerbe. Here as usual \(\pi_0(\mathcal{X}/S)^\wedge\) denotes the Pontryagin dual group \(\text{Hom}_S(\pi_0(\mathcal{X}/S), \mathcal{O}_S^\times)\).

With this notation we now have:

**Theorem B** Let \(\mathcal{Higgs}\) be the stack of \(G\) Higgs bundles on a curve \(C\) and let \(\mathcal{LHiggs}\) be the stack of \(L^G\) Higgs bundles on \(C\). Use the isomorphism \(\text{base} : B \rightarrow L^B\) from **Theorem A(1)** to identify \(B - \Delta\) with \(L B - L \Delta\). Under this identification one has a canonical isomorphism

\[
\ell : \mathcal{Higgs}|_{B-\Delta} \xrightarrow{\approx} (\mathcal{LHiggs}|_{B-\Delta})^D
\]

of commutative group stacks over \(B-\Delta\). The isomorphism \(\ell\) intertwines the action of the translation operators \(\text{Trans}^{\lambda, \tilde{\varpi}}\) on \(\mathcal{Higgs}|_{B-\Delta}\) with the action of the tensorization operators \(\text{Tens}^{\lambda, \tilde{\varpi}}\) on \((\mathcal{LHiggs}|_{B-\Delta})^D\).

**Remark 4.3** Here we only consider the parts of the stacks of Higgs bundles sitting over \(B - \Delta\). To simplify notation throughout the proof we will write \(\mathcal{Higgs}\) and \(\mathcal{LHiggs}\) instead of \(\mathcal{Higgs}|_{B-\Delta}\) and \(\mathcal{LHiggs}|_{B-\Delta}\).

**Remark 4.4** It will be clear form the proof of **Theorem B** that the canonical isomorphism we construct is functorial with respect to isogenies \(G \rightarrow G'\) between simple groups.

### 4.4 Strategy of the proof

Before we present any details of the proof we outline the strategy that we will follow. Fix a cameral cover \(\tilde{C}\) corresponding to a point in \(B - \Delta\). We want to extend our isomorphism

\[
\nabla_{\tilde{C}} : P_{\tilde{C}} \xrightarrow{\approx} \mathcal{L}P_{\tilde{C}} = LP_{\tilde{C}}^D
\]

from **Theorem A(2)** to a natural isomorphism of group stacks

\[
\mathcal{Higgs}_{\tilde{C}} \xrightarrow{\approx} (\mathcal{LHiggs}_{\tilde{C}})^D.
\]

Naturality will imply in particular that the isomorphisms \((9)\) of individual Hitchin fibers will globalize to an isomorphism

\[
\mathcal{Higgs} \xrightarrow{\approx} (\mathcal{LHiggs})^D
\]

over \(B - \Delta\). We will construct \((9)\) in several steps.
(i) We will construct actions of the algebra of abelianized Hecke operators $\text{Trans}^{\lambda, \tilde{x}}$ on $\mathcal{Higgs}_{\tilde{C}}$ and of the abelianized tensorization operators $\text{Tens}^{\lambda, \tilde{x}}$ on $(L\mathcal{Higgs})^{D}_{\tilde{C}}$. Both types of operators are labeled by a point $\tilde{x} \in \tilde{C}$ and a cocharacter $\lambda \in \Lambda = \text{cochar} T$. We will use the abelianized Hecke operators to construct a Higgs version of the Abel-Jacobi map for cameral covers.

(ii) The stacks in (9) are (possibly disconnected) commutative group stacks. In Section 4.2 we saw that the groups of connected components of both $\mathcal{Higgs}_{\tilde{C}}$ and $(L\mathcal{Higgs})^{D}_{\tilde{C}}$ are naturally identified with $\pi_{1}(G) = \Lambda / \text{coroot}_{g}$. The actions of $\text{Trans}^{\lambda, \tilde{x}}$ and $\text{Tens}^{\lambda, \tilde{x}}$ on $\pi_{1}(G)$ are induced from the translation action of $\lambda$ on $\Lambda$ and so the abelianized Hecke and tensorization operators permute transitively the components of $\mathcal{Higgs}_{\tilde{C}}$ and $(L\mathcal{Higgs})^{D}_{\tilde{C}}$.

Thus to construct the map (9) it suffices to construct an isomorphism

$$\mathcal{Higgs}_{0, \tilde{C}} \xrightarrow{\sim} (L\mathcal{Higgs})^{D}_{\tilde{C}}$$

of the connected components of the identity of our stacks so that:

- the isomorphism (10) extends the isomorphism of abelian varieties (8).
- the isomorphism (10) intertwines the action of the Hecke and tensorization operators that preserve the connected components of the identity. That is, for every $\tilde{x} \in \tilde{C}$, $\lambda \in \text{coroot}_{g}$, (10) the action of $\text{Trans}^{\lambda, \tilde{x}}$ on $\mathcal{Higgs}_{0, \tilde{C}}$ with the action of $\text{Tens}^{\lambda, \tilde{x}}$ on $(L\mathcal{Higgs})^{D}_{\tilde{C}}$;

Indeed an isomorphism (10) with these properties will extend uniquely to an isomorphism (9) by Hecke equivariance.

(iii) The stacks $\mathcal{Higgs}_{0, \tilde{C}}$ and $(L\mathcal{Higgs})^{D}_{\tilde{C}}$ are connected commutative group stacks with moduli spaces $P_{\tilde{C}}$ and $LP^{D}_{\tilde{C}}$ respectively. We construct (10) by exhibiting explicit groupoid scheme presentations for these stacks and showing that the isomorphism $l_{\tilde{C}}$ extends to an isomorphism of the presentations. We also check that the abelianized Hecke and tensorization operators are induced from automorphisms of the presentations, and that the lift of $l_{\tilde{C}}$ intertwines these automorphisms. To achieve this we study the moduli spaces of neutralizations of $\mathcal{Higgs}_{0}$ and $(L\mathcal{Higgs})^{D}$ along each Hitchin fiber.

4.5 Abelianized Hecke operators and the Abel-Jacobi map

We will use the abelianization of Higgs bundles described in [DG02, Section 6] to define Hecke operators on the moduli stack of Higgs bundles.
4.5.1. For every \( \lambda \in \Lambda := \text{cochar}_G = \text{Hom}(\mathbb{C}^\times, T) \) and \( \tilde{x} \in \tilde{C} \subset \tilde{G} \) we will construct a canonical Hecke automorphism \( \text{Trans}^{\lambda, \tilde{x}} : \text{Higgs}_{\tilde{C}} \to \text{Higgs}_{\tilde{C}} \). These automorphisms can be combined into a single map of stacks

\[
\text{Trans} : \text{Higgs} \times_{(B - \Delta)} \tilde{C} \times \Lambda \to \text{Higgs},
\]

which we will call the abelianized Hecke correspondence. This map induces a natural map on coarse moduli spaces which we will denote again by \( \text{Trans} \).

Let \( (V, \varphi) \) be a \( K_C \)-valued \( G \)-Higgs bundle with cameral cover \( p : \tilde{C} \to C \), and let \( \tilde{x} \in \tilde{C}, \lambda \in \Lambda \). Informally \( \text{Trans} \) takes the data \( ((V, \varphi), \tilde{x}, \lambda) \) to a new Higgs bundle \( (V', \varphi') \) having the same cameral cover \( p : \tilde{C} \to C \), an underlying \( G \)-bundle \( V' \) which is a modification of \( V \) at \( p(\tilde{x}) \) in the direction \( \lambda \), and a Higgs field \( \varphi' \) which agrees with the original \( \varphi \) on \( C - \{ p(\tilde{x}) \} \).

More formally, [DG02, Theorem 6.4] establishes an equivalence between the groupoid of \( G \)-Higgs bundles on \( C \) with cameral cover \( \tilde{C} \) and the groupoid of \( G \)-spectral data on \( \tilde{C} \). Spectral data are collections \( (\mathcal{L}, \mathfrak{i}, \mathfrak{b}) \) consisting of a ramification twisted \( W \)-invariant \( T \)-bundle \( \mathcal{L} = \mathcal{L}_{(V, \varphi)} \) on \( \tilde{C} \) equipped with additional twisting and framing structures \( \mathfrak{i} \) and \( \mathfrak{b} \) satisfying compatibility conditions. For the convenience of the reader we review the precise description of \( G \)-spectral data and the abelianized Hecke correspondences \( \text{Trans}^{\lambda, \tilde{x}} \) in Appendix A (Note that the Appendix discusses Hecke for \( ^G \!	ext{L} \) while here, dually, we need Hecke for \( G \)).

As explained in Appendix A the abelianized Hecke correspondence \( \text{Trans}^{\lambda, \tilde{x}} \) is an automorphism of the stack \( \text{Higgs}_{\tilde{C}} \) defined by tensoring with a particular \( T_{\tilde{C}} \) torsor \( S^{\lambda, \tilde{x}} \) on \( C \). In the spectral picture \( \text{Trans}^{\lambda, \tilde{x}} \) modifies the bundle \( \mathcal{L} \) and the framing \( \mathfrak{b} \) in the spectral datum. Specifically (see Appendix A) \( \text{Trans}^{\lambda, \tilde{x}}(\mathcal{L}, \mathfrak{i}, \mathfrak{b}) = (\mathcal{L} \otimes S^{\lambda, \tilde{x}}, \mathfrak{i}, \mathfrak{b} \otimes s^{\lambda, \tilde{x}}) \), where \( S^{\lambda, \tilde{x}} \) is the \( W \)-equivariant \( T \)-bundle:

\[
S^{\lambda, \tilde{x}} := \bigotimes_{w \in W} (w \lambda) \left( \mathcal{O}_{\tilde{C}}(w\tilde{x}) \right).
\]

The necessary compatibilities are automatic as long as \( \tilde{x} \in \tilde{C} \) is not a ramification point of \( p : \tilde{C} \to C \). Since these compatibilities are closed conditions, they hold on all of \( \tilde{C} \) (or \( \tilde{G} \)).

4.5.2. By construction the abelianized Hecke operator \( \text{Trans}^{\lambda, \tilde{x}} : \text{Higgs}_{\tilde{C}} \to \text{Higgs}_{\tilde{C}} \) shifts the components of \( \text{Higgs}_{\tilde{C}} \) by the image of \( \lambda \) in \( \pi_1(G) = \Lambda/\text{coroot}_\Lambda \). In particular, any component of \( \text{Higgs}_{\tilde{C}} \) can be reached from the neutral one by applying a suitable Hecke operator. As a side note, observe that the \( T \)-bundles \( S^{\lambda, \tilde{x}} \) all have the same topological type and so the tensorization by these bundles preserves components of \( \text{Bun}_T \):

**Lemma 4.5** For every \( \tilde{x} \in \tilde{C} \) and every \( \lambda \in \Lambda \) the \( T \)-bundle \( S^{\lambda, \tilde{x}} \) is topologically trivial.

**Proof.** The topological type of a \( T \)-bundle on \( \tilde{C} \) is classified by a characteristic class in the lattice \( H^2(\tilde{C}, \pi_1(T)) = H^2(\tilde{C}, \Lambda) \cong \Lambda \). From (12) it follows that the characteristic class of \( S^{\lambda, \tilde{x}} \) is a \( W \)-invariant element in \( \Lambda \). Since \( G \) is assumed semisimple we have that the group of \( W \)-invariants \( \Lambda^W \) is trivial, which proves the lemma. \( \square \)
4.5.3. Let now \( v : (B - \Delta) \to \mathcal{Higgs} \) be a Hitchin section. By applying the Hecke map \( \text{Trans} \) to \( v \) we get a Higgs bundle version of the Abel-Jacobi map:

\[
\tilde{C} \times \Lambda \xrightarrow{\alpha^G} \mathcal{Higgs} \\
(\tilde{x}, \lambda) \xrightarrow{\text{Trans}^{\lambda, \tilde{x}}(h(\tilde{x}))}.
\]

Composing this map with the map \( \mathcal{Higgs} \to \mathcal{Higgs} \) to the coarse moduli space we get a moduli space version of the Abel-Jacobi map:

\[
\tilde{C} \times \Lambda \xrightarrow{\alpha^G} \mathcal{Higgs}.
\]

For the construction of the groupoid presentations of \((\mathcal{Higgs})^D\) and \(\mathcal{Higgs}\) we will also need the Langlands dual version of these maps. Fix a Hitchin section \( L \mathcal{v} : (B - \Delta) \to L\mathcal{Higgs} \) and let

\[
\tilde{C} \times L\Lambda \xrightarrow{\alpha^L} L\mathcal{Higgs}.
\]

be the corresponding Abel-Jacobi maps. The restrictions of these maps to slices of the form \( \{\tilde{x}\} \times L\Lambda \) yields group homomorphisms

\[
\Lambda \xrightarrow{\alpha^{L, \tilde{x}}} L\mathcal{Higgs}.
\]

For future reference we introduce the shortcut notation \( e_{\tilde{x}} : \Lambda \to \mathcal{Higgs}_{\tilde{C}} \) and \( L e_{\tilde{x}} : L\Lambda \to L\mathcal{Higgs}_{\tilde{C}} \) for the homomorphisms \( \alpha^G(\tilde{x}, -) \) and \( \alpha^{L, G}(\tilde{x}, -) \). We will also write \( e'_{\tilde{x}} : \text{coroot}_{g} \to P_{\tilde{C}} \) and \( L e'_{\tilde{x}} : \text{coroot}_{Lg} \to L P_{\tilde{C}} \) for the restrictions of \( \alpha^G(\tilde{x}, -) \) and \( \alpha^{L, G}(\tilde{x}, -) \) to \( \text{coroot}_{g} \) and \( \text{coroot}_{Lg} \) respectively.

4.6 Abelianized tensorization operators

Since \((\mathcal{Higgs})^D\) is the group stack of line bundles with square structures on \( L\mathcal{Higgs} \) it follows that every line bundle \( L \mathcal{L} \) with square structure on \( L\mathcal{Higgs} \) will give rise to a tensorization automorphism \( (-) \otimes L \mathcal{L} \) of \((\mathcal{Higgs})^D\).
The abelianized tensorization operator $\text{Tens}^{\lambda, \tilde{x}} : (L^\text{Higgs}_{\tilde{C}})^D \to (L^\text{Higgs}_{\tilde{C}})^D$ will be given concretely as the tensorization $\text{Tens}^{\lambda, \tilde{x}}(-) := (-) \otimes L^{\lambda, \tilde{x}}$ with a particular line bundle with square structure $L^{\lambda, \tilde{x}}$ which we construct next.

To ensure the additivity in $\lambda$ of the operators $\text{Tens}^{\lambda, \tilde{x}}$ we will construct the line bundle $L^{\lambda, \tilde{x}}$ on $L^\text{Higgs}_{\tilde{C}}$ with square structure on $L^\text{Higgs}_{\tilde{C}} \times L^\Lambda$, interpreted as a $T$-bundle on $\tilde{C}$, will equal $S^{\lambda, \tilde{x}}$.

Note that by Lemma 4.1, the choice of the Hitchin section $L^v$ identifies $L^\text{Higgs}$ with the group stack over $B - \Delta$ associated with $R^\bullet \pi_*(L^T)[1]$, and similarly identifies $L^\text{Higgs}$ with the commutative group scheme on $B - \Delta$ representing the sheaf $R^1 \pi_*(L^T)$. The sheaf inclusion $L^T \subset L^\nabla$ induces natural maps of complexes of abelian sheaves on $B - \Delta$:

\begin{align*}
R^* \pi_*(L^T) &\longrightarrow R^* \pi_*(L^\nabla) \\
Pic(\tilde{C}/(B - \Delta)) \otimes L^\Lambda \\
\text{Leray}
\end{align*}

The map labeled “Leray” comes from the Leray spectral sequence for $p : \tilde{C} \to (B \times C)$ and is an isomorphism because $p$ is finite.

The composite map $L^\lambda : R^* \pi_*(L^T) \to Pic(\tilde{C}/B - \Delta) \otimes L^\Lambda$ induces a morphism of stacks $L^\lambda : L^\text{Higgs} \to Pic(\tilde{C}/(B - \Delta)) \otimes L^\Lambda = \text{Bun}_{L^T}$, where $\text{Bun}_{L^T}$ denotes the stack parametrizing $L^T$-bundles along the fibers of $\tilde{\pi} : \tilde{C} \to (B - \Delta)$.

Similarly, if in diagram (13) we replace $R^* \pi_*$ with $R^1 \pi_*$ we get a composite map $L^\lambda : R^1 \pi_*(L^T) \to Pic(\tilde{C}/B - \Delta) \otimes L^\Lambda$ which induces a morphism of spaces $L^\lambda : L^\text{Higgs} \to Pic(\tilde{C}/(B - \Delta)) \otimes L^\Lambda = \text{Bun}_{L^T}$.

Combining these maps with the Abel-Jacobi maps for Higgs bundles we get commutative
Here as usual $\text{diag}_w(\bullet) = w^*((\bullet) \times_w L^T)$ is the diagonal action of $w$ on $L^T$-bundles and $\text{aj} : \widetilde{C} \to \mathcal{P}ic(\mathcal{C}/(B - \Delta))$ denotes the classical Abel-Jacobi map, sending a point $\tilde{x} \in \widetilde{C}$ to the line bundle $\mathcal{O}_C(\tilde{x})$ of degree one on $\widetilde{C}$.

We are now ready to construct the line bundles $L^\lambda_{\tilde{x}}$ on $L_{\text{Higgs}}$ given by $\tilde{x} \in \widetilde{C}$, $\lambda \in \Lambda$. For this we will use the well known fact [Lau96] that the Picard gerbe on any smooth family of curves is self-dual. Note that this is precisely our Theorem $\text{B}$ in the abelian case.

More precisely, for any smooth compact complex curve $\Sigma$, there exists a Poincare sheaf on $\mathcal{P}ic(\Sigma) \times \mathcal{P}ic(\Sigma)$ which induces a canonical isomorphism $(\mathcal{P}ic(\Sigma))^D = \mathcal{P}ic(\Sigma)$. In fact this isomorphism is induced by the classical Abel-Jacobi map $\text{aj} : \Sigma \to \mathcal{P}ic(\Sigma)$:

$$\text{aj}^* : (\mathcal{P}ic(\Sigma))^D \to \mathcal{P}ic(\Sigma).$$

We apply this to the curve $\widetilde{C}$ and tensor with $L^\Lambda$ to get an induced isomorphism

$$(\text{aj} \times \text{id})^* : (\mathcal{P}ic(\widetilde{C}) \otimes L^\Lambda)^D \overset{\cong}{\to} \mathcal{B}un_T(\widetilde{C}).$$

In particular, for every $\tilde{x} \in \widetilde{C}$ and any $\lambda \in \Lambda$ we can find a canonical line bundle $L^\lambda_{\tilde{x}}$ on $\mathcal{P}ic(\widetilde{C}) \times L^\Lambda$ such that $(\text{aj} \times \text{id})^* L^\lambda_{\tilde{x}} = \lambda(\mathcal{O}_C(\tilde{x}))$. To finish the construction we set

$$L^\lambda_{\tilde{x}} := a^* L^\lambda_{\tilde{x}}$$

and invoke the commutative diagram (14) to get the desired identity

$$(15) \quad (a)^{L_{\text{Higgs}}} L^\lambda_{\tilde{x}} = (a)^{L_{\text{Higgs}}} (a)^* L^\lambda_{\tilde{x}} = \left( \sum_{w \in W} \text{diag}_w \right)^* (\text{aj} \times \text{id})^* L^\lambda_{\tilde{x}} = S^\lambda_{\tilde{x}}.$$

This concludes the construction of $L^\lambda_{\tilde{x}}$ and $\text{Tens}^\lambda_{\tilde{x}}$. Again by construction the operators $\text{Tens}^\lambda_{\tilde{x}}$ will act transitively on the connected components of $(L_{\text{Higgs}} C)^D$. For future reference we will write $e^D_{\tilde{x}} : \Lambda \to (L_{\text{Higgs}} C)^D$ for the homomorphism given by $\lambda \mapsto L^\lambda_{\tilde{x}}$. Similarly we will write $L e^D_{\tilde{x}} : L^\Lambda \to (L_{\text{Higgs}} C)^D$ for the analogous homomorphism associated with the Langlands dual group. Finally we will denote the restrictions of these homomorphisms to $\text{coroot}_g \subset \Lambda$ by $e^D_{\tilde{x}}' : \text{coroot}_g \to (L_{\text{Higgs}} C)^D$ and $L e^D_{\tilde{x}}' : \text{coroot}_L g \to (L_{\text{Higgs}} C)^D$ respectively.
Remark 4.6 Note that the operation of tensoring with the line bundle $L^λ, ˜x$ gives rise to an abelianized tensorization (or Wilson) auto equivalence

$$L_{ab} W^λ, ˜x : D_{coh}(L^Higgs_{C′}, O) \xrightarrow{\cong} D_{coh}(L^Higgs_{C′}, O).$$

Similarly to the Hecke operators, the algebra of these auto equivalences can be related to the algebra of the classical limit tensorization functors $L^W λ, x$ discussed in Section 2. As with the Hecke action we will not discuss the precise relationship here but will work with the abelianized versions only.

4.7 Proof of global duality

From the analysis in section 4.2 we know that the group of connected components of $Higgs$ is naturally $π_1(G)$ while the group of connected components of $(L^Higgs)^D$ is $Z(LG)^\wedge = \text{Hom}(Z(LG), \mathbb{C}^\times)$. Thus the groups of connected components of $Higgs$ and $(L^Higgs)^D$ are naturally identified by the isomorphism $LΛ \cong Λ^\vee$. In the previous section we defined the Hecke action on the stack $Higgs$ and the tensorization action on the stack $(L^Higgs)^D$, and we noted that these actions induce transitive actions on sets of connected components of $Higgs$. Furthermore, the compatibility (15) guarantees that the two actions on groups of connected components match under the identification $π_1(G) \cong Z(LG)^\wedge$. The transitivity reduces the problem of constructing the isomorphism (9) to the problem of constructing a canonical isomorphism of connected $Z(G)$-gerbes

$$Higgs_0 \cong (L^Higgs)^D,$$

which intertwines the actions of all abelianized Hecke and tensorization operators that preserve these connected components.

Indeed, the isomorphism (16) extends automatically to the desired isomorphism (9) by Hecke equivariance. The operators $\text{Trans}^λ, ˜x$ and $\text{Tens}^λ, ˜x$ are labeled by $( ˜x, λ) ∈ ˜G × Λ$. The operators preserving the connected component of the identity on either side are the Hecke or the tensorization operators labeled by $( ˜x, λ) ∈ ˜G × \text{coroot}_g$. Therefore to finish the proof of Theorem B we must construct a lift of $l_{C′} : P_{C′} \rightarrow LP^D_{C′}$ from Theorem A to an isomorphism

$$Higgs_{0, ˜C} \cong (L^Higgs_{C′})^D,$$

which intertwines these Hecke and tensorization operators. Explicitly this means that for every $ ˜x ∈ ˜C$ the isomorphism (17) fits in a commutative diagram of group stacks

$$Higgs_{0, ˜C} \xrightarrow{l_{C′}} (L^Higgs_{C′})^D.$$
Here we fixed the cameral cover \( \tilde{C} \) to simplify the exposition. However we will construct the isomorphism (17) in such a way that the construction will automatically globalize over \( B - \Delta \). The idea is to construct groupoid presentations of the gerbes \( \mathcal{Higgs}_{0,\tilde{C}} \) and \( (L \mathcal{Higgs}_{\tilde{C}})^D \) and then argue that these presentations are naturally isomorphic.

To carry this out recall \( \mathcal{Higgs}_{0,\tilde{C}} \) and \( (L \mathcal{Higgs}_{\tilde{C}})^D \) are neutralizable \( Z(G) \)-gerbes with coarse moduli spaces \( P_{\tilde{C}} \) and \( L P^D_{\tilde{C}} \) respectively. Our strategy is to build atlases for the gerbes as moduli spaces of neutralizations and then identify explicitly these moduli spaces with each other by matching the respective geometric data.

First let us look at the right hand side of (17).

4.7.1. A groupoid presentation of \( (L \mathcal{Higgs}_{\tilde{C}})^D \): The Hitchin fiber \( L \mathcal{Higgs}_{\tilde{C}} \) is an algebraic group which fits in a short exact sequence of abelian groups

\[
0 \rightarrow L P_{\tilde{C}} \rightarrow L \mathcal{Higgs}_{\tilde{C}} \rightarrow \pi_1(LG) \rightarrow 0.
\]

Since \( L G \) is a simple Lie group we have that \( \pi_1(LG) \) is finite, and since \( L P_{\tilde{C}} \) is divisible, it follows that the sequence (19) is split.

Passing to duals we see that the dual gerbe now fits in a short exact sequence of commutative group stacks

\[
0 \rightarrow \pi_1(LG)^D \rightarrow (L \mathcal{Higgs}_{\tilde{C}})^D \rightarrow L P^D_{\tilde{C}} \rightarrow 0.
\]

Since \( L P^D_{\tilde{C}} \) is the coarse moduli space of \( (L \mathcal{Higgs}_{\tilde{C}})^D \) it follows that every splitting of (19) will induce a neutralization of the gerbe \( (L \mathcal{Higgs}_{\tilde{C}})^D \), i.e. a map from \( (L \mathcal{Higgs}_{\tilde{C}})^D \) to the gerbe \( [L P^D/\pi_1(LG)] \) (with \( Z(G) \) acting trivially) which for every test scheme \( S \) gives rise to an equivalence \( (L \mathcal{Higgs}_{\tilde{C}})^D(S) \rightarrow [L P^D/\pi_1(LG)](S) \) between the associated groupoids of sections. In fact a choice of a splitting of (19) is equivalent to choosing a neutralization which is compatible with the group structures.

As we saw in sections 4.5.2 and 4.5.3 the homomorphism \( L e_\tilde{x} : L \Lambda \rightarrow L \mathcal{Higgs}_{\tilde{C}} \) induces a map of short exact sequences of abelian groups:

\[
0 \rightarrow L P_{\tilde{C}} \rightarrow L \mathcal{Higgs}_{\tilde{C}} \rightarrow \pi_1(LG) \rightarrow 0 \text{,}
\]

where \( L e_\tilde{x} \) is the restriction of \( L e_\tilde{x} \) to \( \text{coroot}_{Lg} \). Now every choice of a homomorphism

\[
\phi : L \Lambda \rightarrow L P_{\tilde{C}}
\]

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which lifts $^t\mathbf{e}_\xi^k : \mathrm{coroot}_{L_0} \to L P_C$ will give a splitting of (19). Moreover the set of all such lifts is in bijection with the set of all splittings of (19).

Thus the choice of a pair $(\tilde{x}, \phi)$ gives a splitting of (19) and a neutralization of the gerbe $(L_{\text{Higgs}}^c_{\tilde{C}})^D$ compatible with the group structure. The space parametrizing such pairs can be used to build an atlas of the gerbe $(L_{\text{Higgs}}^c)^D$. To avoid complications with branching we will only use pairs $(\tilde{x}, \phi)$ in which $\tilde{x}$ is not a ramification point for the cover $\tilde{C} \to C$. Specifically let $\tilde{C}^0 \subset \tilde{C}$ be the complement of the ramification divisor of the map $\tilde{C} \to C$. Define a cover $\tilde{C}_{RHS}^0 \to \tilde{C}^0$ parametrizing all pairs $(\tilde{x}, \phi)$ as above:

$$\tilde{C}_{RHS}^0 := \left\{ (\tilde{x}, \phi) \mid \tilde{x} \in \tilde{C}^0, \phi : L_{\Lambda} \to \Lambda_{P_C}, \text{ so that } \phi|_{\text{coroot}_{L_0}} = ^t\mathbf{e}_\xi^k. \right\}.$$

Denote by $\tilde{C}_{RHS}^0 \to (B - \Delta)$ the family of all $\tilde{C}_{RHS}^0$ viewed as a cover of an open subset of the universal cameral cover. With this notation we get

**Lemma 4.7** The variety

$$U_{RHS} := \tilde{C}_{RHS}^0 \times_{B - \Delta} (L_{\text{Higgs}}^c_{\tilde{C}})^D$$

is an atlas for the $Z(G)$-gerbe $(L_{\text{Higgs}}^c)^D$.

**Proof.** The statement is relative over the Hitchin base so we can argue fiber by fiber. By definition $\tilde{C}_{RHS}^0$ is a etale Galois cover of $\tilde{C}^0$ with Galois group $\text{Hom}(\pi_1(L_{\Gamma}), L P_C)$. The assignment $(\tilde{x}, \phi) \to \phi$ gives a map from $\tilde{C}_{RHS}^0$ to the space of splittings of (19) and so the pull back of the gerbe $(L_{\text{Higgs}}^c_{\tilde{C}})^D$ to $\tilde{C}_{RHS}^0 \times L P_C^D$ is equipped with a universal neutralization. Thus $\tilde{C}_{RHS}^0 \times L P_C^D$ is a smooth atlas for $(L_{\text{Higgs}}^c_{\tilde{C}})^D$ with a structure morphism $\tilde{C}_{RHS}^0 \times L P_C^D \to (L_{\text{Higgs}}^c_{\tilde{C}})^D$ given by the universal neutralization. $\square$

The structure map $U_{RHS} \to (L_{\text{Higgs}}^c)^D$ gives rise to a groupoid presentation

$$U_{RHS} \times (L_{\text{Higgs}}^c)^D U_{RHS} \rightrightarrows U_{RHS}$$

of the gerbe $(L_{\text{Higgs}}^c)^D$.

Denote the space $U_{RHS} \times (L_{\text{Higgs}}^c)^D U_{RHS}$ by $R_{RHS}$. We can describe the space $R_{RHS}$ explicitly. The gerbe $(L_{\text{Higgs}}^c)^D$ has a coarse moduli space $(L_{\text{Higgs}}^c_{\tilde{C}})^D$ and so $R_{RHS}$ will be a $Z(G)$-torsor over

$$U_{RHS} \times (L_{\text{Higgs}}^c_{\tilde{C}})^D U_{RHS} = \tilde{C}_{RHS}^0 \times_{B - \Delta} \tilde{C}_{RHS}^0 \times_{B - \Delta} (L_{\text{Higgs}}^c_{\tilde{C}})^D.$$

A point in this space mapping to $\tilde{C} \in B - \Delta$ consists of the data $((\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2), \mathbb{L})$ where $(\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2)$ are in the cover $\tilde{C}_{RHS} \to \tilde{C}^0$, and $\mathbb{L}$ is a point of $L P_C^D$. 37
To any such point we can attach group homomorphisms $L e_{\tilde{x}_1}, L e_{\tilde{x}_2}: L \Lambda \to L \text{Higgs}_C$, and $\phi_1, \phi_2 : L \Lambda \to L P_C$. The fact that $\tilde{C} \in B - \Delta$ implies that $\tilde{C}$ is smooth and irreducible. Hence $\tilde{C}^0$ is connected and so $L e_{\tilde{x}_1} - L e_{\tilde{x}_2}$ will map $L \Lambda$ to the connected component of the identity of the group $L \text{Higgs}_C$. Since $\phi_i$ agree with $L e_{\tilde{x}_i}$ on coroots it follows that the homomorphism

$$(L e_{\tilde{x}_1} - L e_{\tilde{x}_2}) - (\phi_1 - \phi_2) : L \Lambda \to L P_C$$

will factor through a homomorphism

$$(\nu((\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2)) : \pi_1(L G) \to L P_C)$$

Now since $L P^D_C = \text{Hom}_{\text{gp-stack}}(L P_C, B G_m)$ we can view $L$ as an abelian group extension

$$0 \to \mathbb{C}^\times \to L \to L P_C \to 0.$$ 

Pulling back this extension by the map $\nu((\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2))$ gives an extension of $\pi_1(L G)$ by $\mathbb{C}^\times$. Since $\pi_1(L G)$ is finite and $\mathbb{C}^\times$ is divisible this extension is necessarily split. We have obtained the following description of $\mathcal{R}_{\text{RHS}}$:

**Lemma 4.8** $\mathcal{R}_{\text{RHS}}$ is a $Z(G)$-torsor over $U_{\text{RHS}} \times (t_{\text{Higgs}_0})^\nu U_{\text{RHS}}$ whose fiber over a point is the set of all splittings of the group extension

$$0 \to \mathbb{C}^\times \to \nu((\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2))^* L \to \pi_1(L G) \to 0,$$

where $\nu((\tilde{x}_1, \phi_1), (\tilde{x}_2, \phi_2))$ is defined in formula (20).

### 4.7.2. A groupoid presentation of $\text{Higgs}_{0, \tilde{C}}$:

The gerbe $\text{Higgs}_{0, \tilde{C}}$ fits in a short exact sequence of commutative group stacks

$$(21) \quad 0 \to B Z(G) \to \text{Higgs}_{0, \tilde{C}} \to P_{\tilde{C}} \to 0.$$ 

Following the same pattern as above we will construct an atlas for $\text{Higgs}_{0, \tilde{C}}$ from the moduli space of all splittings of (21). To construct sections of $\text{pr}$ we will again use points of the cameral cover $\tilde{C}$.

Since by [DG02, Section 6] $\text{Higgs}_{0, \tilde{C}}$ can be identified with the moduli stack of (topologically trivial) $G$-spectral data $(\mathcal{L}, i, b)$ we have a universal family of spectral data

$$(\mathcal{L}, i, b) \to \text{Higgs}_{0, \tilde{C}} \times \tilde{C}.$$ 

Let now $\tilde{x} \in \tilde{C}^0$ be a point away from ramification. Restricting $\mathcal{L}$ to $\text{Higgs}_{0, \tilde{C}} \times \{\tilde{x}\}$ we get a $T$-bundle $\mathcal{L}_{\tilde{x}}$ with square structure, i.e. an extension

$$0 \to T \to \mathcal{L}_{\tilde{x}} \to \text{Higgs}_{0, \tilde{C}} \to 0$$

of commutative group stacks.
**Remark 4.9** This square structure is encoded in the interpretation of $\mathcal{Higgs}_{0,\tilde{C}}$ as moduli of spectral data. To see this suppose that $S$ is a test scheme. A section $\xi : S \to \mathcal{Higgs}_{0,\tilde{C}}$ of the Higgs stack over $S$ is given by a family of spectral data $(\mathcal{L}_{x}, i^{x}, b^{x})$ on $S \times \tilde{C}$. Since $\xi^{*}(\mathcal{L}_{x}, i^{x}, b^{x}) = (\mathcal{L}_{\tilde{x}}, i^{\tilde{x}}, b^{\tilde{x}})$ it follows that $\xi^{*}\mathcal{L}_{\tilde{x}} = \mathcal{L}_{\tilde{x}}^{\xi} = \mathcal{L}_{\tilde{x}}^{\xi}$. If now $\xi_{1}$ and $\xi_{2}$ are two sections of $\mathcal{Higgs}_{0,\tilde{C}}$ over $S$, then we can add $\xi_{1}$ and $\xi_{2}$ in the group structure on $\mathcal{Higgs}_{0,\tilde{C}}$ to get a new section $\xi_{1} + \xi_{2} : S \to \mathcal{Higgs}_{0,\tilde{C}}$. But in terms of spectral data the group structure is given by tensoring the corresponding $T$-torsors, twists, and framings. Thus

$$(\xi_{1} + \xi_{2})^{*}(\mathcal{L}_{x}, i^{x}, b^{x}) = (\mathcal{L}_{\tilde{x}}, i^{\tilde{x}}, b^{\tilde{x}}) \cdot (\mathcal{L}_{\tilde{x}}, i^{\tilde{x}}, b^{\tilde{x}}) = (\mathcal{L}_{\tilde{x}}, i^{\tilde{x}} \otimes i^{\tilde{x}} + i^{\tilde{2}} \otimes id + id \otimes b^{\tilde{2}}, b^{\tilde{1}} \otimes id + id \otimes b^{\tilde{2}}),$$

and so

$$(\xi_{1} + \xi_{2})^{*}\mathcal{L}_{\tilde{x}} = \mathcal{L}_{\tilde{x}}^{\xi_{1} + \xi_{2}} = \mathcal{L}_{\tilde{x}}^{\xi_{1}} \otimes \mathcal{L}_{\tilde{x}}^{\xi_{2}} = (\xi_{1}^{*}\mathcal{L}_{\tilde{x}}) \otimes (\xi_{2}^{*}\mathcal{L}_{\tilde{x}}).$$

This identification gives the multiplication isomorphism in the square structure. The identity isomorphism in the square structure is defined analogously.

**Lemma 4.10** The following data are canonically equivalent:

(a) A neutralization of the $Z(G)$-gerbe $\mathcal{Higgs}_{0,\tilde{C}} \to P_{\tilde{C}}$, compatible with the group structures;

(b) A splitting of \([21]\);

(c) A lift of $\mathcal{L}_{\tilde{x}}'$ to a $T$-torsor (with square structure) on $P_{\tilde{C}}$.

**Proof.** The data (a) and (b) are tautologically the same.

To identify data (b) and (c) note that by construction, the stabilizer $Z(G)$ of any section $\xi : S \to \mathcal{Higgs}_{0,\tilde{C}}$ acts on $\mathcal{L}_{\tilde{x}}$ via the canonical inclusion $Z(G) \subset T$. In other words we can view $\mathcal{L}_{\tilde{x}}$ as a neutralization of the $T$-gerbe on $P_{\tilde{C}}$ induced from the $Z(G)$-gerbe $\mathcal{Higgs}_{0,\tilde{C}} \to P_{\tilde{C}}$. To obtain a neutralization of the $Z(G)$-gerbe we will have to choose additional data. Specifically, the natural map from $\mathcal{Higgs}_{0,\tilde{C}}$ to the induced $T$-gerbe $\mathcal{Higgs}_{0,\tilde{C}} \times_{BZ(G)} BT$ is a $T/Z(G)$-torsor. Pulling back this $T/Z(G)$-torsor to $P_{\tilde{C}}$ via the map $\sigma_{\tilde{x}} : P_{\tilde{C}} \to \mathcal{Higgs}_{0,\tilde{C}} \times_{BZ(G)} BT$ corresponding to $\mathcal{L}_{\tilde{x}}$ gives a $T/Z(G)$-torsor $\mathcal{L}_{\tilde{x}}'$ (with square structure) on $P_{\tilde{C}}$:

$$
\begin{array}{ccc}
\mathcal{L}_{\tilde{x}}' & \overset{\Box}{\longrightarrow} & \mathcal{Higgs}_{0,\tilde{C}} \\
\downarrow & & \downarrow \\
P_{\tilde{C}} & \overset{\sigma_{\tilde{x}}}{\longrightarrow} & \mathcal{Higgs}_{0,\tilde{C}} \times_{BZ(G)} BT
\end{array}
$$
In this way we obtain an identification of $Higgs_{0,\tilde{C}}$ with the gerbe of all lifts of $\mathcal{L}'_{\tilde{x}}$ to a $T$-torsor with square structure on $P_{\tilde{C}}$. Therefore a splitting of (21) is the same thing as a choice of such a lift. □

To package all the above data in the most efficient way, note first that the fact that the stabilizer action on $\mathcal{L}_{\tilde{x}}$ is tautological implies that the $T/Z(G)$-torsor with square structure induced from $\mathcal{L}_{\tilde{x}}$ will descend to a $T/Z(G)$-torzor on $P_{\tilde{C}}$. In terms of group extensions this means that the induced extension

$$0 \rightarrow T/Z(G) \rightarrow \mathcal{L}_{\tilde{x}}' \rightarrow Higgs_{0,\tilde{C}} \rightarrow 0$$

is canonically a pullback of an extension

$$0 \rightarrow T/Z(G) \rightarrow \mathcal{L}_{\tilde{x}}' \rightarrow P_{\tilde{C}} \rightarrow 0$$

via the structure homomorphism $pr : Higgs_{0,\tilde{C}} \rightarrow P_{\tilde{C}}$. This extension is precisely the $T/Z(G)$ torsor we described before the statement of Lemma 4.10 and so a splitting of (21) corresponds to a pair $(\tilde{x}, \Phi)$ where $\Phi$ is an extension

$$0 \rightarrow T \rightarrow \Phi \rightarrow P_{\tilde{C}} \rightarrow 0$$

together with a choice of a group isomorphism $\Phi / Z(G) \cong \mathcal{L}_{\tilde{x}}'$ inducing an isomorphism of extensions:

$$0 \rightarrow T/Z(G) \rightarrow \Phi / Z(G) \rightarrow P_{\tilde{C}} \rightarrow 0 \cong 0 \rightarrow T/Z(G) \rightarrow \mathcal{L}_{\tilde{x}}' \rightarrow P_{\tilde{C}} \rightarrow 0.$$

Note that by the five lemma the isomorphism $\Phi / Z(G) \cong \mathcal{L}_{\tilde{x}}'$ will be unique if it exists. So the choice of such an isomorphism will not be extra data.

The space parametrizing pairs $(\tilde{x}, \Phi)$ can be used to build an atlas for the gerbe $Higgs_{0,\tilde{C}}$. To that end define an etale cover $\tilde{C}_{LHS}^0 \rightarrow \tilde{C}$ parametrizing all pairs $(\tilde{x}, \Phi)$, i.e.

$$\tilde{C}_{LHS}^0 := \left\{ (\tilde{x}, \Phi) \mid \tilde{x} \in \tilde{C}^0, \Phi \text{ is an extension of } P_{\tilde{C}} \text{ by } T, \text{ so that the extension} \right\}$$

$$0 \rightarrow T/Z(G) \rightarrow \Phi / Z(G) \rightarrow P_{\tilde{C}} \rightarrow 0 \text{ is isomorphic to}$$

$$0 \rightarrow T/Z(G) \rightarrow \mathcal{L}_{\tilde{x}}' \rightarrow P_{\tilde{C}} \rightarrow 0.$$

Denote by $\tilde{C}_{LHS}^0 \rightarrow (B - \Delta)$ the family of all $\tilde{C}_{LHS}^0$ viewed as a cover of the universal cameral cover. With this notation we now have
Lemma 4.11 The variety
\[ \mathcal{U}_{LHS} := \tilde{\mathcal{C}}_{LHS}^0 \times_{B - \Delta} \text{Higgs}_0 \]
is an atlas for the \( Z(G) \)-gerbe \( \text{Higgs}_{0, \tilde{C}} \).

Proof. Again the statement is relative over the Hitchin base and we can argue fiber by fiber. The assignment \( (\tilde{x}, \Phi) \to \Phi \) gives a map from \( \tilde{\mathcal{C}}_{LHS}^0 \) to the space of splittings of \( \text{Higgs}_0 \) and so the pull back of the gerbe \( \text{Higgs}_{0, \tilde{C}} \) to \( \tilde{\mathcal{C}}_{LHS}^0 \times P_{\tilde{C}} \) is equipped with a universal neutralization. Thus \( \tilde{\mathcal{C}}_{LHS}^0 \times P_{\tilde{C}} \) is a smooth atlas for \( \text{Higgs}_{0, \tilde{C}} \) with a structure morphism \( \tilde{\mathcal{C}}_{LHS}^0 \times P_{\tilde{C}} \to \text{Higgs}_{0, \tilde{C}} \) given by the universal neutralization.

As usual the structure map \( \mathcal{U}_{LHS} \to \text{Higgs}_{0, \tilde{C}} \) gives rise to a groupoid presentation
\[ \mathcal{U}_{LHS} \times_{\text{Higgs}_{0, \tilde{C}}} \mathcal{U}_{LHS} \]
of the gerbe \( \text{Higgs}_{0, \tilde{C}} \). The relations
\[ \mathcal{R}_{LHS} := \mathcal{U}_{LHS} \times_{\text{Higgs}_{0, \tilde{C}}} \mathcal{U}_{LHS} \]
for this presentation form a variety, which is a \( Z(G) \)-torsor over the space
\[ \mathcal{U}_{LHS} \times_{\text{Higgs}_0} \mathcal{U}_{LHS} = \tilde{\mathcal{C}}_{LHS}^0 \times_{B - \Delta} \tilde{\mathcal{C}}_{LHS}^0 \times_{B - \Delta} \text{Higgs}_0. \]
A point in this space sitting over a cameral cover \( \tilde{C} \in B - \Delta \) consists of the data \( ((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2), L) \) where \( (\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2) \) are in the cover \( \tilde{\mathcal{C}}_{LHS}^0 \to \tilde{C}^0 \), and \( L \) is a point of \( P_{\tilde{C}} \).

Since the stabilizer stack of \( \text{Higgs}_{0, \tilde{C}} \) acts tautologically on each of the \( Z(G) \)-torsors \( \mathcal{L}_{\tilde{x}_1} \) and \( \mathcal{L}_{\tilde{x}_2} \), it follows that the \( T \)-torsor \( \mathcal{L}_{\tilde{x}_1} \otimes \mathcal{L}_{\tilde{x}_2}^{-1} \) fits in a group stack extension
\[ 0 \to T \to \mathcal{L}_{\tilde{x}_1} \otimes \mathcal{L}_{\tilde{x}_2}^{-1} \to \text{Higgs}_{0, \tilde{C}} \to 0 \]
which is a pullback of an extension
\[ 0 \to T \to \mathcal{M}_{\tilde{x}_1, \tilde{x}_2} \to P_{\tilde{C}} \to 0 \]
via the map \( \text{pr} : \text{Higgs}_{0, P_{\tilde{C}}} \to P_{\tilde{C}} \). Therefore the \( T \)-torsor
\[ N ((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2)) := \mathcal{M}_{\tilde{x}_1, \tilde{x}_2} \otimes \Phi_1^{-1} \otimes \Phi_2 \]
fits in a group extension
\[ 0 \to T \to N ((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2)) \to P_{\tilde{C}} \to 0. \]
Since the \( T/Z(G) \)-torsors induced from \( \mathcal{M}_{\tilde{x}_1, \tilde{x}_2} \) and \( \Phi_1 \otimes \Phi_2^{-1} \) are both equal to \( \mathcal{L}_{\tilde{x}_1} \otimes \mathcal{L}_{\tilde{x}_2} \)
it follows that the extension
\[ 0 \to T/Z(G) \to N ((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2))/Z(G) \to P_{\tilde{C}} \to 0. \]
is split. Furthermore, since \( P_C \) is an abelian variety, and \( T/Z(G) \) is affine it follows that \( (\tilde{\tau}) \) will have a unique splitting \( \sigma : P_C \to N((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2))/Z(G) \) Pulling back the \( Z(G) \)-torsor \( N((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2))/Z(G) \) by \( \sigma \) gives a \( Z(G) \)-torsor on \( P_C \). Now the definition of the relations \( \mathcal{R}_{LHS} \) immediately gives the following

**Lemma 4.12** \( \mathcal{R}_{LHS} \) is a \( Z(G) \)-torsor over \( U_{LHS} \times_{\text{Higgs}_0} U_{LHS} \) whose fiber over a point \( ((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2), L) \) is the fiber of the \( Z(G) \)-torsor

\[
\sigma^* N((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2))
\]
at \( L \in P_C \).

### 4.7.3. The construction of the isomorphism:

The duality of Higgs stacks will follow if we can show that our isomorphism of Prym varieties lifts to an isomorphism between the groupoid presentations we described in Sections 4.7.2 and 4.7.1. Thus the existence of the isomorphism (17) will follow from the following:

**Theorem 4.13** (a) For all \( \tilde{x} \in \tilde{C} \), \( \lambda \in \text{coroot}_g \), the action of the Hecke operators \( \text{Trans}^{\lambda, \tilde{x}} \), on \( \text{Higgs}_0 \) and the action of the tensorization operators \( \text{Tens}^{\lambda, \tilde{x}} \) on \( (L\text{Higgs})^D \) lift to actions on the groupoids \( \mathcal{R}_{LHS} \rightarrow U_{LHS} \) and \( \mathcal{R}_{RHS} \rightarrow U_{RHS} \) respectively.

(b) The isomorphism

\[
1 : \text{Higgs}_0 \cong (L\text{Higgs}_0)^D
\]
of abelian schemes over \( B - \Delta \) lifts to a canonical isomorphism of groupoids

\[
\begin{array}{ccc}
\mathcal{R}_{LHS} & \xrightarrow{=} & \mathcal{R}_{RHS} \\
\downarrow & & \downarrow \\
U_{LHS} & \xrightarrow{=} & U_{RHS} \\
\downarrow & & \downarrow \\
\text{Higgs}_0 & \xrightarrow{=} & (L\text{Higgs}_0)^D
\end{array}
\]

which intertwines \( \text{Trans}^{\lambda, \tilde{x}} \) with \( \text{Tens}^{\lambda, \tilde{x}} \).

**Proof.** The statement is relative over the Hitchin base and so it suffices to construct the isomorphism of groupoids canonically on every Hitchin fiber.

The lifts of the Hecke and the tensorization operators claimed in part (a) are pre-built in the constructions of the two groupoid presenations. So the only thing to do is to construct the isomorphisms \( u \) and \( r \) in part (b).
Fix $\tilde{C} \in B - \Delta$. Write $U_{\text{LHS}, \tilde{C}}$, $R_{\text{LHS}, \tilde{C}}$ for the restriction of the atlas and relations on the Hitchin fiber over $\tilde{C}$. We want to construct an isomorphism of groupoids

$$
\begin{array}{c}
R_{\text{LHS}, \tilde{C}} \xrightarrow{\iota_{\tilde{C}}} R_{\text{RHS}, \tilde{C}} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
U_{\text{LHS}, \tilde{C}} \xrightarrow{\iota_{\tilde{C}}} U_{\text{RHS}, \tilde{C}} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
P_{\tilde{C}} \xrightarrow{\iota_{\tilde{C}}} L P_{\tilde{C}}^D
\end{array}
$$

which lifts $\iota_{\tilde{C}}$ and is Hecke equivariant.

Recall that $U_{\text{LHS}, \tilde{C}} = \tilde{C}_{\text{LHS}}^0 \times P_{\tilde{C}}$ and $U_{\text{RHS}, \tilde{C}} = \tilde{C}_{\text{RHS}}^0 \times L P_{\tilde{C}}^D$. We will define the isomorphism of atlases $u_{\tilde{C}} : U_{\text{LHS}, \tilde{C}} \to U_{\text{RHS}, \tilde{C}}$ as a product $u_{\tilde{C}} = a_{\tilde{C}} \times \iota_{\tilde{C}}$, where

$$a_{\tilde{C}} : \tilde{C}_{\text{LHS}}^0 \to \tilde{C}_{\text{RHS}}^0$$

is the isomorphism of $\text{Hom}(\pi_1(\text{L}G), L P_{\tilde{C}})$-Galois covers of $\tilde{C}^0$ defined as follows.

Given a point $(\tilde{x}, \Phi) \in \tilde{C}_{\text{LHS}}^0$ and an element $\mu \in L \Lambda = \text{char}(T)$ we get an associated element $\mu \ast \Phi \in \text{Ext}^1(P_{\tilde{C}}, \mathbb{C}^\times) = P_{\tilde{C}}^D$:

$$
\begin{array}{cccc}
0 & \to & T & \to & \Phi & \to & P_{\tilde{C}} & \to & 0 \\
0 & \to & \mathbb{C}^\times & \to & \mu \ast \Phi & \to & P_{\tilde{C}} & \to & 0.
\end{array}
$$

Combined with the isomorphism $(l_{\tilde{C}}^D)^{-1} : P_{\tilde{C}}^D \to L P_{\tilde{C}}$ this construction gives rise to a homomorphism

$$
\phi_\Phi : L \Lambda \to L P_{\tilde{C}}, \quad \mu \mapsto (l_{\tilde{C}}^D)^{-1}(\mu \ast \Phi).
$$

We set

$$a_{\tilde{C}}^{-1}(\tilde{x}, \Phi) := (\tilde{x}, \phi_\Phi).$$

It is straightforward to check that this map is an isomorphism but for future reference it will be useful to write the inverse map explicitly.

Let $(\tilde{x}, \phi) \in \tilde{C}_{\text{RHS}}^0$. Let $\mathcal{P} \to P_{\tilde{C}} \times L P_{\tilde{C}}$ be the Poincare $\mathbb{C}^\times$-torsor corresponding to the isomorphism $l_{\tilde{C}}$. Consider the pullback $(\text{id} \times \phi)^* \mathcal{P}$. It is a $\mathbb{C}^\times$-torsor on $P_{\tilde{C}} \times L \Lambda$, and since $\mathcal{P}$ is a biextension, it follows that for all $\mu \in L \Lambda$ the restriction $((\text{id} \times \phi)^* \mathcal{P})|_{P_{\tilde{C}} \times \{\mu\}}$ is a $\mathbb{C}^\times$-torsor with square structure and that the map

$$0 \to \mathbb{G}_m \to ((\text{id} \times \phi)^* \mathcal{P}) \to P_{\tilde{C}} \times L \Lambda \to 0,$$

is a short exact sequence of commutative group schemes over $P_{\tilde{C}}$. Equivalently this means that $((\text{id} \times \phi)^* \mathcal{P}$ is a $\text{Hom}(L \Lambda, \mathbb{C}^\times) = T$ torsor on $P_{\tilde{C}}$ with square structure. It is now immediate to see that

$$a_{\tilde{C}}^{-1}(\tilde{x}, \phi) = (\tilde{x}, ((\text{id} \times \phi)^* \mathcal{P}).$$
Next we need to lift the isomorphism \( u_C \) to an isomorphism of relations. To that end, note that if \((\tilde{x}_1, \Phi_1)\) and \((\tilde{x}_2, \Phi_2)\) are two points of \( \tilde{C}_{LHS}^0 \), then tautologically we have

\[
\Phi_N((\tilde{x}_1, \Phi_1); (\tilde{x}_2, \Phi_2)) = (L e_{\tilde{x}_1} - L e_{\tilde{x}_2}) - (\Phi_1 - \Phi_2) = \nu((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2)) \circ Lq
\]

(24)

where \( N((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2)) \) and \( \nu((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2)) \) are defined by (22) and (20) respectively, and \( Lq \) is the natural projection \( Lq : L\Lambda \to \pi_1(LG) \).

In view of the identification (24) and Lemma 4.12 and Lemma 4.8 the construction of the isomorphism \( u_C \) now reduces to a general statement about abelian varieties which we formulate next.

Suppose that \( P \) is a polarized abelian variety and let \( P^D \) be the dual abelian variety. Let

\[
0 \to T \to N \to P \to 0
\]

be a short exact sequence of commutative algebraic groups and suppose that the induced sequence

(25)

\[
0 \to T/Z(G) \to N/Z(G) \to P \to 0
\]

is split. Let \( \mathbb{L} \in P \) be a point. The data \((N, \mathbb{L})\) gives rise to two \( Z(G) \)-torsors (over a point):

(i) The splittings of the sequence (25) form a torsor over the algebraic group homomorphisms \( \text{Hom}(P, T/Z(G)) \). Since \( T/Z(G) \) is affine it follows that we have a unique splitting of (25) which we will denote by \( \sigma : P \to N/Z(G) \). The group \( N \) is a \( Z(G) \)-torsor over \( N/Z(G) \) and so the pullback \( \sigma^*N \) is a \( Z(G) \)-torsor over \( P \). The fiber \((\sigma^*N)_\mathbb{L}\) of this torsor at \( \mathbb{L} \) is a \( Z(G) \)-torsor over a point. We denote this torsor by \( LHS_{N,\mathbb{L}} \).

(ii) Let \( \phi_N : L\Lambda \to P^D \) be the homomorphism which to each \( \mu \in L\Lambda \cong \text{char}(T) \) assigns the induced \( \mathbb{C}^x \) extension \( \mu_\ast N \). Regarding \( \mathbb{L} \) as a group extension

\[
0 \to \mathbb{C}^x \to \mathbb{L} \to P^D \to 0
\]

we get a \( Z(G) \)-torsor over a point defined as

\[
RHS_{N,\mathbb{L}} := \{ \tilde{\nu} : L\Lambda \to \mathbb{L} \mid \tilde{\nu} \text{ lifts } \nu \},
\]

where \( \nu : \pi_1(LG) \to P^D \) is the unique homomorphism for which \( \nu \circ Lq = \phi_N \).

Now the construction of \( u_C \) follows from

**Lemma 4.14** The \( Z(G) \)-torsors \( LHS_{N,\mathbb{L}} \) and \( RHS_{N,\mathbb{L}} \) are canonically isomorphic.
Proof. $LHS_{N,L} \text{ and } RHS_{N,L}$ are the fibers at $L$ of two torsors $LHS_N$ and $RHS_N$ on $P$. So to prove the lemma it suffices to identify the corresponding group extensions canonically. As explained above $LHS_{N,L}$ is the fiber at $L$ of the group extension

$$0 \to Z(G) \to \sigma^*N \to P \to 0,$$

and so $LHS_N = \sigma^*N$.

Next consider the Poincare $\mathbb{C}^\times$-torsor $\mathcal{P} \to P \times P^D$. From the biextension property it follows that we can view $\mathcal{P}$ as an extension of commutative groups schemes on $P$:

$$0 \to \mathbb{G}_m \to \mathcal{P} \to P^D \to 0,$$

where $P^D$ denotes the constant group scheme on $P$ with fiber $P^D$. At the same time, the extension $N$ gives a homomorphism $\phi_N : L\Lambda \to P^D$, $\phi_N(\mu) := \mu \cdot N$ and the condition that the sequence \eqref{26} is split implies that $\phi_N$ factors through the quotient $L\mathfrak{q} : L\Lambda \to \pi_1(LG)$. In other words there is a unique homomorphism $\nu : \pi_1(LG) \to P^D$ satisfying $\phi_N = \nu \circ L\mathfrak{q}$. Pulling back the Poincare extension \eqref{26} by $\nu$ gives an extension of commutative group schemes on $P$:

$$0 \to \mathbb{G}_m \to \nu^* \mathcal{P} \to \pi_1(LG) \to 0.$$

The sequence of group schemes \eqref{27} is split locally on $P$ but in general is not globally split. The sheaf of local splittings of \eqref{27} is representable by a space $RHS_N$ which is a principal $\text{Hom}(\pi_1(LG), \mathbb{C}^\times) = Z(G)$-bundle on $P$. By definition $RHS_{N,L}$ is the fiber of $RHS_N$ at $L \in P$.

To compare $LHS_N$ and $RHS_N$ as $Z(G)$-bundles on $P$ note first that they are classified by the same element in $H^1(P, Z(G))$. Indeed, the sequence $0 \to T \to N \to P \to 0$ corresponds to an element in $\text{Ext}^1(P, T)$. Under the natural identifications

$$\text{Ext}^1(P, T) = \text{Ext}^1(P, \text{Hom}(L\Lambda, \mathbb{C}^\times)) \xrightarrow{(\dagger)} \text{Hom}(L\Lambda, \text{Ext}^1(P, \mathbb{C}^\times)) \xrightarrow{(\ddagger)} \text{Hom}(L\Lambda, P^D)$$

this element is just $\phi_N \in \text{Hom}(L\Lambda, P^D)$. The identification $(\dagger)$ is just the adjunction isomorphism, and the identification $(\ddagger)$ is the contraction with the Poincare biextension class $e_{\mathcal{P}}(\bullet, \bullet)$ for $P \times P^D$. In other words, the sequence $0 \to T \to N \to P \to 0$ corresponds to the class $e_{\mathcal{P}}(\bullet, \phi_N)$. By the same token the sequence $0 \to Z(G) \to \sigma^*N \to P \to 0$ corresponds to the class $e_{\mathcal{P}}(\bullet, \nu) \in \text{Ext}^1(P, Z(G))$, i.e. $LHS_N$ is classified by $e_{\mathcal{P}}(\bullet, \nu)$.

Similarly, the class of the torsor $RHS_N$ can be computed from its definition. The local-to-global spectral sequence identifies the group $H^1(P, \mathcal{H}\text{om}(\pi_1(LG), \mathbb{G}_m))$ with the subgroup of $\text{Ext}^1_p(\pi_1(LG), \mathbb{G}_m))$, consisting of extension classes of locally split extensions. Since $RHS_N$ is the torsor of local splittings of \eqref{27}, its class in $H^1(P, Z(G)) = H^1(P, \mathcal{H}\text{om}(\pi_1(LG), \mathbb{G}_m))$ will be precisely the extension class of \eqref{27} viewed as an element in the subgroup

$$H^1(P, \mathcal{H}\text{om}(\pi_1(LG), \mathbb{G}_m)) \subset \text{Ext}^1_p(\pi_1(LG), \mathbb{G}_m)).$$
But the class of (27) is given by $e_{\mathcal{P}}(\bullet, \nu)$, and so $RHS_N$ is also classified by the element $e_{\mathcal{P}}(\bullet, \nu)$. Since $LHS_N$ and $RHS_N$ are classified by the same element of $H^1(P, Z(G))$ it follows that they are isomorphic as $Z(G)$ torsors.

To exhibit a canonical isomorphism between $LHS_N$ and $RHS_N$ as $Z(G)$-extensions of $P$ it now suffices to identify the fibers $LHS_N$ and $RHS_N$ at some point of $P$. Let $o \in P$ be the origin. Then from the descriptions (i) and (ii) we see that $LHS_{N,o}$ and $RHS_{N,o}$ are both canonically isomorphic to $Z(G)$. Since $LHS_N$ and $RHS_N$ are isomorphic as covering spaces of $P$ this isomorphism of fibers extends to a unique canonical isomorphism of $LHS_N$ and $RHS_N$ as $Z(G)$-extensions of $P$.

Applying the previous lemma to $P = P_{\tilde{C}}$, and $N = N((\tilde{x}_1, \Phi_1), (\tilde{x}_2, \Phi_2))$ yields the desired isomorphism $\tau_{\tilde{C}}$. The intertwining property of the induced isomorphism of gerbes $\mathcal{Higgs}_{\tilde{C}} \cong (^L\mathcal{Higgs})^D$ follows tautologically from the construction. This completes the proof of Theorem 4.13.

Remark 4.15 The proof of Lemma 4.14 can be streamlined somewhat and the isomorphism $u_{\tilde{C}}$ can be constructed without computation of extension classes. As we explained above, specifying $N$ is the same thing as specifying an element $\nu \in \text{Hom}(\pi_1(L_G), P^D)$. Varying $N$ we see that $LHS_N$ and $RHS_N$ fit together in $Z(G)$-torsors $LHS$ and $RHS$ on $\text{Hom}(\pi_1(L_G), P^D) \times P$. In the proof of Lemma 4.14 we argued that these torsors are canonically isomorphic by first showing that they are abstractly isomorphic and then noticing that $LHS|_{\text{Hom}(\pi_1(L_G), P^D) \times \{o\}}$ and $RHS|_{\text{Hom}(\pi_1(L_G), P^D) \times \{o\}}$ are canonically trivial.

We can instead show directly that the torsors are canonically isomorphic by using the see-saw theorem. Indeed if $A$ and $B$ are connected projective algebraic groups, and if we have two extensions of $A \times B$ by $Z(G)$ whose restrictions on $\{o\} \times B$ and $A \times \{o\}$ are identified, then the see-saw theorem implies that the extensions themselves are identified by a unique isomorphism. Consider $A = \text{Hom}(\pi_1(L_G), P^D)$ and $B = P$. The restrictions of $LHS$ and $RHS$ to the slices $\text{Hom}(\pi_1(L_G), P^D) \times \{o\}$ and $\{o\} \times P$ are identified in an obvious manner from the definitions but unfortunately the see-saw theorem does not apply immediately since $\text{Hom}(\pi_1(L_G), P^D)$ is a finite (and hence disconnected) group. We can remedy this by embedding the finite group $\text{Hom}(\pi_1(L_G), P^D)$ in an abelian variety on which the see-saw theorem does apply. We pull back the extension

$$0 \to \text{coroot}_g \to \Lambda \to \pi_1(L_G) \to 0$$
via some surjective homomorphism $L \to \pi_1(LG)$:

$$
\begin{array}{ccc}
0 & \rightarrow & \text{coroot}_L \\
\downarrow & & \downarrow \\
L & \rightarrow & \pi_1(LG) \\
\downarrow & & \downarrow \\
L & \rightarrow & \pi_1(LG) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{coroot}_L \\
\end{array}
$$

(28)

In the same way we defined $LHS$ and $RHS$ we can use the second row of this diagram to define two $\text{Hom}(L\Lambda, C^\times) = T$-torsors $\hat{LHS}$ and $\hat{RHS}$ on $\text{Hom}(L\Lambda, P^D) \times P$. Note that $\text{Hom}(\pi_1(LG), P^D) \subset \text{Hom}(L\Lambda, P^D)$ and by construction we have tautological identifications

$$
\begin{align*}
\hat{LHS}|_{\text{Hom}(\pi_1(LG), P^D) \times P} &= LHS \times Z(G)T, \\
\hat{RHS}|_{\text{Hom}(\pi_1(LG), P^D) \times P} &= RHS \times Z(G)T.
\end{align*}
$$

(29)

In particular the restrictions of $\hat{LHS}/Z$ and $\hat{RHS}/Z$ to $\text{Hom}(\pi_1(LG), P^D) \times P$ are naturally trivialized, hence naturally isomorphic. Thus the restriction of the $T/Z$ torsor of isomorphisms $\mathcal{I}som(\hat{LHS}/Z, \hat{RHS}/Z)$ to $\text{Hom}(\pi_1(LG), P^D) \times P$ admits a canonical trivialization $\text{triv}_0$.

Now $\text{Hom}(L\Lambda, P^D)$ is connected and the see-saw argument provides a canonical identification $\hat{LHS} \cong \hat{RHS}$ and hence a canonical trivialization $\text{triv}$ of the $T$-torsor of isomorphisms $\mathcal{I}som(\hat{LHS}, \hat{RHS})$. From the identifications (29) it then follows that the $Z(G)$-torsor of isomorphisms $\mathcal{I}som(\hat{LHS}, \hat{RHS})$ is naturally identified with the torsor of all trivializations of $\mathcal{I}som(\hat{LHS}, \hat{RHS})|_{\text{Hom}(\pi_1(LG), P^D) \times P}$ that lift the trivialization $\text{triv}_0$ of the torsor $\mathcal{I}som(\hat{LHS}/Z, \hat{RHS}/Z)|_{\text{Hom}(\pi_1(LG), P^D) \times P}$. But diagram (29) implies $\text{triv}$ and $\text{triv}_0$ are compatible trivializations, and so $\text{triv}$ naturally trivializes $\mathcal{I}som(\hat{LHS}, \hat{RHS})|_{\text{Hom}(\pi_1(LG), P^D) \times P}$ as a $Z(G)$-torsor over $\mathcal{I}som(\hat{LHS}/Z, \hat{RHS}/Z)|_{\text{Hom}(\pi_1(LG), P^D) \times P}$.

\section{Extensions and refinements}

In this section we extend Theorem 3 from simple groups to all reductive groups. We also discuss some additional structures related to the weight filtration. Finally, we draw the main geometric corollaries of the duality: existence of a Fourier-Mukai equivalence, and the construction of Hecke eigensheaves.

\subsection{Extension to reductive groups}

Our main duality result extends to general reductive groups.
Theorem C  Let $\mathbb{G}$ be a connected complex reductive group, let $L^G$ be the Langlands dual reductive group, and let $C$ be a smooth compact complex curve. Write $\text{Higgs}_{\mathbb{G}}$ and $\text{Higgs}_{(L^G)}$ for the stacks of $K_C$-valued Higgs bundles on $C$ with structure group $\mathbb{G}$ and $L^G$ respectively. Then there is an isomorphism $\text{Higgs}_{\mathbb{G}} \cong (\text{Higgs}_{(L^G)})^L$ of commutative group stacks over $B - \Delta$, intertwining the action of translation and tensorization operators.

Remark  Since in this proof we need to work with the moduli stacks of Higgs bundles for different isogenous groups we will deviate from our normal notation and will label the corresponding stacks with the group as a subscript. Thus we will write $\text{Higgs}_{\mathbb{G}}, \text{Higgs}_{L^G}$, etc..

Proof.  The proof is somewhat involved. We start by presenting what seems to us to be a very natural homological approach. However, this runs into technical issues, which we explain in Remark 5.1. Rather than settling these issues, we find it easier to change tack and give a separate argument in subsections 5.1.1, 5.1.2, 5.1.3. This argument is a modification of the proof of Theorem B.

Since $\mathbb{G}$ is connected and reductive, we can always fit $\mathbb{G}$ in a short exact sequence

$$1 \to K \to G \times H \to \mathbb{G} \to 1,$$

where $K$ is a finite subgroup in the center $Z(G \times H)$ of $G \times H$, $G = \prod_{i=1}^n G_i$ is a product of complex simple groups, and $H \cong (\mathbb{C}^*)^b$ is an affine complex torus. Passing to Langlands duals gives the sequence

$$1 \to K^\wedge \to L^G \to L G \times L H \to 1.$$

Next observe that the construction of the Hitchin base, the formation of the moduli stack of Higgs bundles, the definition of the sheaf $T$, as well as the operations $^L(\bullet)$ and $(\bullet)^D$, all respect the operation of taking products of groups. Combined with Theorems A and B and with the standard selfduality of $\text{Pic}$ of a smooth curve, we get an identification of the Hitchin bases for $G \times H$ and $L G \times L H$, as well as a global duality

$$\text{Higgs}_{(L G \times L H)}^D \cong \text{Higgs}_{G \times H}.$$

Also, since the Hitchin base depends only on the Lie algebra, and not on the Lie group, it follows that the identification of the Hitchin bases for $G \times H$ and $L G \times L H$ can be interpreted as the desired isomorphism $B \cong L B$. 

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Furthermore, the definition of $\mathcal{T}$ (see (3)) gives short exact sequences of abelian sheaves on $B \times C$:

\[
\begin{align*}
0 & \longrightarrow K \longrightarrow \mathcal{T}_G \oplus H \longrightarrow \mathcal{T}_G \longrightarrow 0 \\
0 & \longrightarrow K^\wedge \longrightarrow \mathcal{T}_{LG} \longrightarrow \mathcal{T}_{LG} \oplus {}^L H \longrightarrow 0.
\end{align*}
\]

Applying $R\pi^*[1]$ to the first sequence we get a distinguished triangle in $D^b(B)$:

\[
\begin{align*}
R\pi^*K[1] & \longrightarrow R\pi^*\mathcal{T}_G[1] \oplus R\pi^*H[1] \longrightarrow R\pi^*\mathcal{T}_G[1] \longrightarrow R\pi^*K[2].
\end{align*}
\]

Applying $R\text{Hom}(R\pi^*(\cdot), \mathcal{O}^\times)$ to the second sequence, and taking into account the isomorphism (32) and the isomorphism $\mathcal{Higgs}_G \cong R\pi^*\mathcal{T}_G[1]$ from Lemma 4.1, we get another distinguished triangle

\[
\begin{align*}
R\text{Hom}(R\pi^*K^[1], \mathcal{O}^\times) & \longrightarrow R\pi^*\mathcal{T}_G[1] \oplus R\pi^*H[1] \longrightarrow (R\pi^*\mathcal{T}_{LG}[1])^D \longrightarrow R\text{Hom}(R\pi^*K^\wedge, \mathcal{O}^\times).
\end{align*}
\]

We wish to show that these two triangles are isomorphic and so we have a quasi-isomorphism $R\pi^*\mathcal{T}_G[1] \cong (R\pi^*\mathcal{T}_{LG}[1])^D$. Now Poincare duality on a smooth curve $C$ identifies the cohomology of $C$ with coefficients in $K$ with the Pontryagin dual of the cohomology of $C$ with coefficients in $K^\wedge$. In particular, it induces an isomorphism of complexes

\[
\text{PD} : R\pi^*K[1] \cong \longrightarrow R\text{Hom}(R\pi^*K^[1], \mathcal{O}^\times)
\]

and what we have to show is that it intertwines the maps $g$ and $^Lg$. So, we must show that the diagram:

\[
\begin{align*}
\begin{array}{ccc}
R\pi^*K[1] & \xrightarrow{g} & R\pi^*\mathcal{T}_G[1] \oplus R\pi^*H[1] \\
\text{PD} & & \text{PD} \\
R\text{Hom}(R\pi^*K^[1], \mathcal{O}^\times) & \xrightarrow{^Lg} & \end{array}
\end{align*}
\]

commutes in the derived category.

Let $K_G, K_H$ be the projections to $G, H$ respectively of (the image in $G \times H$ of) $K$. These are central subgroups. The short exact sequence (30) extends to a commutative diagram with exact rows:

\[
\begin{array}{ccc}
1 & \longrightarrow & K \longrightarrow G \times H \longrightarrow G \longrightarrow 1 \\
& \downarrow & \downarrow & \downarrow & \\
1 & \longrightarrow & K_G \times K_H \longrightarrow G \times H \longrightarrow G \times H \longrightarrow 1.
\end{array}
\]
Now diagram (36) factors:

\[
\begin{array}{ccc}
R\pi_*K[1] & \longrightarrow & R\pi_*K_G[1] \oplus R\pi_*K_H[1] \\
\downarrow \text{PD} & & \downarrow \text{PD} \\
R\text{Hom}(R\pi_*K^\wedge[1], \mathcal{O}_\wedge) & \longrightarrow & R\text{Hom}(R\pi_*(K_G^\wedge \oplus K_H^\wedge)[1], \mathcal{O}_\wedge)
\end{array}
\]

Commutativity of the square follows from functoriality of Poincare duality. The triangle part of this diagram is just (36) but for the subgroup \(K_G \times K_H\) instead of \(K\). Commutativity for the \(G\)-factor of the triangle is discussed below. Commutativity for the torus \(H\) is obvious.

To understand better the \(G\)-factor of the triangle we have to examine more closely the relationship between Poincare duality on \(C\) and the canonical isomorphism \(\ell: \mathfrak{Higgs}_G \to (\mathfrak{Higgs}_{L_G})^D\) from Theorem [1]. Let us examine this commutativity statement for a fixed cameral cover \(\tilde{\mathcal{C}} \in B - \Delta\). Since \(K_G \subset Z(G) \subset G\), we again have short exact sequences of sheaves of abelian groups on \(C\)

\[
\begin{array}{ccc}
1 & \longrightarrow & K_G \longrightarrow \mathcal{T}_G \longrightarrow \mathcal{T}_{G/K} \longrightarrow 1,
\end{array}
\]

and

\[
\begin{array}{ccc}
1 & \longrightarrow & K_G^\wedge \longrightarrow \mathcal{T}_{l(G/K)} \longrightarrow \mathcal{T}_{l_G} \longrightarrow 1,
\end{array}
\]

where now \(\mathcal{T}_G, \mathcal{T}_{G/K}, \mathcal{T}_{l(G/K)}, \mathcal{T}_{l_G}\) are the sheaves corresponding to the cover \(\tilde{\mathcal{C}} \to C\).

Passing to cohomology in (37) yields a map

\[g_G: R\Gamma(C, K_G)[1] \to R\Gamma(C, \mathcal{T}_G)[1].\]

Similarly, passing to cohomology in (38) yields a map

\[\partial_G: R\Gamma(C, \mathcal{T}_{l_G}) \to R\Gamma(C, K_G^\wedge)[1].\]

Dualize \(\partial_G\) by applying \(R\text{Hom}(\bullet, \mathbb{C}^\times)\) to obtain a map

\[\partial_G^\wedge: R\text{Hom}(R\Gamma(C, K_G^\wedge)[1], \mathbb{C}^\times) \to R\text{Hom}(R\Gamma(C, \mathcal{T}_{l_G}), \mathbb{C}^\times).\]

With this notation the commutativity of the \(G\)-part of the triangle (for a fixed cameral cover \(\tilde{\mathcal{C}}\)) becomes the statement that the diagram

\[
\begin{array}{ccc}
R\Gamma(C, K_G)[1] & \longrightarrow & R\Gamma(C, \mathcal{T}_G)[1] \\
\downarrow \text{PD} & & \downarrow \ell_{\tilde{\mathcal{C}}} \\
R\text{Hom}(R\Gamma(C, K_G^\wedge)[1], \mathbb{C}^\times) & \longrightarrow & R\text{Hom}(R\Gamma(C, \mathcal{T}_{l_G}), \mathbb{C}^\times) = (\mathfrak{Higgs}_{L_G, \tilde{\mathcal{C}}})^D
\end{array}
\]
commutes in the derived category of abelian groups. Note also that the commutativity of (36) on the level of cohomology is a statement about sheaves and so can be checked locally on the Hitchin base. In other words the commutativity of (39) on the level of cohomology implies the commutativity of (36) on the level of cohomology. (This will be relevant for the discussion in subsection 5.1.1.)

In summary, if we could establish the commutativity of (39) and (36), this would imply that the distinguished triangles (34) and (35) are isomorphic and so we would have a quasi-isomorphism

\[(R\pi_*\mathcal{T}_L)[1])^D \cong R\pi_*\mathcal{T}_G[1].\]

Passing to the associated stacks we would obtain the statement of Theorem C.

**Remark 5.1** Despite its streamlined form and homological appeal the above argument has a couple of drawbacks. First, the duality isomorphism (40) is obtained by using the cone-filling axiom in triangulated categories and so it is unclear whether this construction can be made canonical. Second, the whole construction is based on showing that (39) and (36) commute. The commutativity of (39) and (36) is hard to verify, mainly because our duality isomorphism \(\ell : \text{Higgs}_G \rightarrow (\text{Higgs}_{L,G})^D\) was constructed geometrically rather than homologically.

Instead, we will follow a slightly different approach, which is a combination of the homological and geometric arguments.

1. We will use the above ideas to analyze the connected components of \(\text{Higgs}_G\) and \((\text{Higgs}_{L,G})^D\) and to extend the duality of Prym varieties \(P_{G,\tilde{C}}\) and \(P_{L,G,\tilde{C}}\) to a canonical duality between the Prym varieties \(P_{G,\tilde{C}}\) and \(P_{L,G,\tilde{C}}\). We will see that this boils down to the case of a simple group, treated in Theorem A, plus the commutativity of (39) and of (36) on the level of cohomology, which is much easier than the commutativity in the derived category.

2. The proof of Theorem C (or Theorem B for reductive groups) is reduced to the connected case, via a computation showing that: \(\pi_0(\text{Higgs}_G) = \pi_0(\text{Higgs}_{L,G}) = \pi_1(G)\).

3. The isomorphism for stacky connected components is almost identical to the corresponding part of Theorem B except that the splitting of some sequences needs to be checked more carefully, as the quotient groups involved are finitely generated rather than finite.

We complete the argument over the next three subsections.
5.1.1. First observe that in the long exact sequences of cohomology associated with
the distinguished triangles (34) and (35) the groups $H^1(C, K)$ and $H^1(C, K^\wedge)^\wedge$ embed in the
connected components of the groups $\text{Higgs}_{G \times H, \tilde{C}}$ and $\left(\left(\text{Higgs}_{L \times L, \tilde{C}}\right)_0^D\right)^D$. This gives
canonical identifications

$$P_{G, \tilde{C}} \cong \left(\left(P_{G, \tilde{C}} \times H^1(C, H)\right)_0 \right) / H^1(C, K)$$

(41)

$$(P_{L, G, \tilde{C}})^D \cong \left(\left(P_{L, G, \tilde{C}}^D \times \left(H^1(C, L H)\right)_0^D\right) / H^1(C, K^\wedge)^\wedge, \right.$$ where $H^1(C, H)_0$ and $H^1(C, L H)_0$ denote the connected components of $H^1(C, H)$ and $H^1(C, L H)$
respectively.

Now, in order to show that the isomorphisms of abelian varieties

$$I_{G, \tilde{C}} : \quad P_{G, \tilde{C}} \xrightarrow{\cong} (P_{L, G, \tilde{C}})^D$$

$$I_H : \quad (H^1(C, H))_0 \xrightarrow{\cong} (H^1(C, L H)_0)^D$$

induce a canonical isomorphism of abelian varieties

$$I_{G, \tilde{C}} : \quad P_{G, \tilde{C}} \xrightarrow{\cong} (P_{L, G, \tilde{C}})^D$$

it suffices to check the commutativity of the diagram one obtains from (36) after passing to
degree 0 cohomology. This is much more manageable. In fact we have the following slightly
stronger statement:

**Lemma 5.2** The diagrams (39) and (36) commute on the level of cohomology.

**Proof.** As we noted above the commutativity of (36) on the level of cohomology is a
statement about sheaves. This can be checked locally on the Hitchin base and so it is
enough to check the statement for (39) on the level of cohomology. The complexes in the
left column of (39) are concentrated in degrees $(-1)$, 0, and 1, while the complexes in the
right column are concentrated in degrees $(-1)$ and 0. Computing the cohomologies in degree
$(-1)$ we get:

$$\begin{array}{ccccccc}
H^{-1} : & H^0(C, K_G) & \xrightarrow{h^{-1}(\partial_G)} & H^0(C, T_G) & \xrightarrow{Z(G)} & Z(G) \\
\text{PD} \downarrow & \downarrow & & & & \\
H^2(C, K_G^\wedge)^\wedge & \xrightarrow{h^{-1}(\partial_G^\wedge)} & \text{Hom}(H^1(C, T_L G), \mathbb{C}^\times) & \xrightarrow{h^{-1}(\ell_G)} & \pi_1(L G)^\wedge & \xrightarrow{Z(G)} & \\
\text{Hom}(\pi_0(H^1(C, T_L G)), \mathbb{C}^\times) & & & & & & \\
\end{array}$$
Since $C$ is a smooth compact curve we have canonical identifications $H^0(C, K_G) = K_G$ and $H^2(C, K_G^\wedge)^\wedge = K_G^{\wedge\wedge} = K_G$, and so the diagram of cohomologies in degree $(-1)$ becomes

$$
\begin{array}{ccc}
K_G & \xrightarrow{\text{id}} & Z(G) \\
\downarrow & & \downarrow \\
K_G & \xrightarrow{h^{-1}(\ell_C)} & Z(G)
\end{array}
$$

The two horizontal arrows here are simply the inclusion of $K_G$ in $Z(G)$ and so to show that this diagram commutes we only need to check that $h^{-1}(\ell_C)$ induces the identity when restricted to $K_G$. This however is automatic since the map of presentations in Theorem 4.13 is $Z(G)$-equivariant.

Similarly, the diagram of cohomologies in degree 0 reads:

$$
\begin{array}{ccc}
H^1(C, K_G) & \xrightarrow{h^0(gG)} & H^1(C, T_G) \\
\downarrow & & \downarrow \\
H^1(C, K_G^\wedge) & \xrightarrow{h^0(\partial g^\wedge)} & R^0 \text{Hom}(R\Gamma(C, T_LG), \mathbb{C}^\times)
\end{array} = \left(\left(\text{Higgs}_{L_G, \tilde{C}}\right)_0\right)^D
$$

The top horizontal map is the map of inducing a $T_G$ torsor from a $K_G$ torsor via the inclusion $K_G \subset T_G$ and thus lands in the connected component of the identity $P_\tilde{C}$ of the abelian group $\text{Higgs}_{G, \tilde{C}}$. Similarly the bottom horizontal map is the dual of the natural map $\left(\text{Higgs}_{L_G, \tilde{C}}\right)_0 \rightarrow L^P \rightarrow H^1(C, K_G^\wedge)[1]$ and thus also factors through the connected component of the identity $L^P_{\tilde{C}}$ of $\left(\left(\text{Higgs}_{L_G, \tilde{C}}\right)_0\right)^D$. The right vertical map is the map of commutative group schemes induced from $\ell_C$, i.e. coincides with the map $l_{\tilde{C}}$ from Theorem $\mathbb{A}$. In other words the commutativity of the diagram of degree 0 cohomologies reduces to showing that the diagram

$$
\begin{array}{ccc}
H^1(C, K_G) & \xrightarrow{\text{PD}} & P_\tilde{C} \\
\downarrow & & \downarrow \\
H^1(C, K_G^\wedge) & \xrightarrow{L^P_{\tilde{C}}} & \mathbb{C}^\times
\end{array}
$$

commutes.

Passing to cocharacter lattices for the abelian varieties $P_\tilde{C}$ and $L^P_{\tilde{C}}$ we see that the commutativity of (42) is equivalent to the commutativity of the following diagram of finitely generated abelian groups:

$$
\begin{array}{ccc}
\text{cochar}(P_\tilde{C}) & \xrightarrow{h^0(\ell_C)} & H^1(C, K_G)^\wedge \\
\downarrow & & \downarrow \\
\text{cochar}(L^P_{\tilde{C}}) & \xrightarrow{h^0(\ell_C)} & H^1(C, K_G)^\wedge
\end{array}
$$
However in the proof of Theorem A we constructed the isomorphism $\text{cochar}(P_\tilde{C})^\vee \to \text{cochar}(L_{P_\tilde{C}})$ by sandwiching both groups between the two isogeneous lattices $H^1(C, J_\ast A^\vee) \subset H^1(C, J_\ast A)^\vee_{tf}$. Furthermore Lemma 6.3, Corollary 6.4, and Claim 3.6 imply that the sandwiching maps commute with the Poincare duality isomorphisms for the cohomologies of $A$ and $A^\vee$ on $U$. This shows that (43) commutes.

Finally, the diagram of cohomologies in degree 1 reads

$$
\begin{array}{ccc}
H^1 : & H^2(C, K_G) & \to 0 \\
PD & \downarrow & \downarrow \\
H^0(C, K_G^\wedge) & \to & 0
\end{array}
$$

and so is automatically commutative.

This completes the proof of the lemma. $\square$

It is immediate from the lemma that the diagram

$$
\begin{array}{ccc}
H^1(C, K) & \to P_{G, \tilde{C}} \times H^1(C, H)_{0} \\
PD & \downarrow & \downarrow P_{G, \tilde{C}} \times l_H \\
H^1(C, K^\wedge)^{\wedge} & \to & (H^1(C, L_H)_{0})^D
\end{array}
$$

commutes, and so the isomorphism $l_{G, \tilde{C}} \times l_H$ induces a natural isomorphism of the quotients for the top and bottom inclusions. By (41) this gives the desired isomorphism:

$$
l_{G, \tilde{C}} : P_{G, \tilde{C}} \xrightarrow{\cong} (P_{L_{G, \tilde{C}}})^D.
$$

5.1.2. Having the isomorphism $l_{G, \tilde{C}}$ of abelian varieties at our disposal we can proceed to construct the isomorphism of Higgs stacks as in Section 4.7. The construction of the presentations of the two stacks and the isomorphism of presentations follows almost verbatim the constructions in the case of a simple $G$ described in Section 4.7. We will not repeat the steps in the proof of Theorem B but will only indicate the changes necessary to make the arguments work for a general reductive $G$.

First note that the abelianized Hecke and tensorization operators were constructed in Section 4.5 and Section 4.6 by using the Abel-Jacobi map and the abelianization procedure from [DG02] which works for arbitrary reductive groups. In particular the reasoning in Section 4.5 and Section 4.6 applies directly to the case of reductive groups and again gives Hecke and tensorization automorphisms

$$
\begin{array}{ccc}
\text{Trans}^{\lambda, \tilde{x}} : & \text{Higgs}_{G, \tilde{C}} & \xrightarrow{\cong} \text{Higgs}_{G, \tilde{C}} \\
\text{Tens}^{\lambda, \tilde{x}} : & (\text{Higgs}_{L_{G, \tilde{C}}})^D & \xrightarrow{\cong} (\text{Higgs}_{L_{G, \tilde{C}}})^D
\end{array}
$$

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labeled by characters $\lambda$ of the maximal torus of $G$ and points $\tilde{x}$ of $\tilde{C}$. This will be done in Section 5.1.3.

In particular, if we can check that the group of connected components of $\text{Higgs}_G$ (or $\text{Higgs}_L$) is naturally isomorphic to $\pi_1(G)$, we can proceed as in the beginning of Section 4.7 and reduce the problem of constructing the duality between $\text{Higgs}_G$ and $\text{Higgs}_L$ to constructing an isomorphism of connected $Z(G)$-gerbes

$$\text{Higgs}_{G,0} \cong (\text{Higgs}_{LG})^{D}$$

which intertwines the action of the Hecke and tensorization operators labeled by $(\lambda, \tilde{x}) \in \text{coroot}(\text{Lie}(G)) \times \tilde{C}$.

To show that we indeed have a natural identification $\pi_0(\text{Higgs}_G) = \pi_1(G)$ we will again use the homological description of the Higgs moduli space. The short exact sequence of groups (30) induces a short exact sequence of fundamental groups:

$$0 \rightarrow \pi_1(G \times H) \rightarrow \pi_1(G) \rightarrow K \rightarrow 0.$$  

On the other hand if we take the first short exact sequence of abelian sheaves in (33) and pass to cohomology, then we will get a (piece of a) long exact sequence

$$0 \longrightarrow H^1(C, K) \longrightarrow H^1(C, T_G \oplus H) \longrightarrow H^1(C, T_G) \longrightarrow H^2(C, K) \longrightarrow 0.$$  

Here the injectivity at $H^1(C, K)$ follows because the long exact sequence in cohomology begins as

$$0 \rightarrow K \rightarrow Z(G) \times H \rightarrow Z(G) \rightarrow H^1(K) \rightarrow \cdots$$

and because the map $Z(G) \times H \rightarrow Z(G)$ is surjective. Similarly the surjectivity at $H^2(C, K)$ follows because the sheaf $T_G \oplus H$ has cohomologies only in degrees 0 and 1. For $T_G$ this is proven in Section 3 and for $H$ this is obvious since $H$ is a product of several copies of $G_m$.

Separating the groups in sequence (46) into their connected and disconnected parts gives a commutative diagram with exact rows and columns:

$$\begin{array}{ccc}
0 & \rightarrow & H^1(C, K) \\
\downarrow & & \downarrow \\
0 & \rightarrow & P_{G, \tilde{C}} \times H^1(C, H_0) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1(C, \tilde{K}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \pi_1(G \times H) \\
\downarrow & & \downarrow \\
0 & \leftarrow & 0 \\
\end{array}$$
We have already discussed the top row of this diagram - it gives the top isomorphism in (41). For the present discussion the important part is the bottom row. The first Chern class gives a natural map from the bottom row of (47) to the sequence (45). This map is the identity on the two outside groups, and so the map from \( \pi_0(\text{Higgs}_{G,\tilde{C}}) \) to \( \pi_1(G) \) must be an isomorphism.

5.1.3. Next we can proceed to construct the isomorphism (44) by following the steps in Section 4.7: construct atlases for the two gerbes and then show that the corresponding groupoid presentations are isomorphic. This works without any modifications but the details of the geometry are slightly altered by the fact that \( \pi_1(G) \) is no longer finite but rather is a finitely generated abelian group. Again the atlases are constructed as moduli spaces of splittings of extensions but in the reductive case these are extensions of finitely generated abelian groups by abelian varieties. Such extensions are again split since the finitely generated abelian groups are products of free abelian groups and finite abelian groups and the abelian varieties are divisible groups. This fact allows us to carry out the Section 4.7 constructions of atlases and relations for the Higgs stacks in the reductive case. The resulting atlases are no longer finite Galois covers of the cameral curves but rather are families of abelian varieties over finite Galois covers. This does not affect the rest of the arguments in any way. Finally for the construction of the isomorphism of groupoid presentations we note that the isomorphism argument given in Remark 4.15 applies directly to the reductive setting.

This completes the proof of Theorem C.

\[ \square \]

5.2 Generalized 1-motives

In this section we point out two minor refinements of our results. These refinements will not be used elsewhere in the paper.

The duality isomorphisms in Theorem B and Theorem C respect all the additional structures on the stacks of Higgs bundles. For instance if \( G \) is semisimple we can view the stacks of Higgs bundles as generalized 1-motives in the sense of [Lau96]. We claim that the duality isomorphisms respect the weight filtrations of these 1-motives:

The isomorphism of group stacks \( (L\text{Higgs})^D \cong \text{Higgs} \) in the Theorem B is compatible with the isomorphism of abelian schemes

\[ (L\text{Higgs}_0)^D \cong L^P \cong P = \text{Higgs}_0, \]

constructed in the proof of Theorem A(2). More precisely, if we use \( \text{base} \) to identify \( B - \Delta \) with \( L^B - L^\Delta \), then the Hitchin fibrations allow us to view \( \text{Higgs} \) and \( L\text{Higgs} \) as Beilinson 1-motives over \( B - \Delta \). This means that \( \text{Higgs} \) and \( L\text{Higgs} \) are commutative group stacks over \( B - \Delta \), which are naturally filtered, with graded pieces which are either abelian varieties,
or finite abelian groups, or classifying stacks of finite abelian groups. The filtrations are given as

\[
\begin{align*}
W_0 \mathcal{H}iggs & \supset W_{-1} \mathcal{H}iggs \supset W_{-2} \mathcal{H}iggs \supset 0 \\
\mathcal{H}iggs & \supset \mathcal{H}iggs_0 & BZ(G)
\end{align*}
\]

respectively

\[
\begin{align*}
W_0(L\mathcal{H}iggs) & \supset W_{-1}(L\mathcal{H}iggs) \supset W_{-2}(L\mathcal{H}iggs) \supset 0 \\
L\mathcal{H}iggs & \supset L\mathcal{H}iggs_0 & BZ(LG)
\end{align*}
\]

and the duality operation \((\bullet)^D := \text{Hom}_{\text{sp}}(\bullet, \mathcal{O}_B^\times[1])\), is compatible with these filtrations:

**Lemma 5.3** The duality \((\bullet)^D\) transforms each filtered commutative group stack into a stack of the same type, and the isomorphism \((L\mathcal{H}iggs)^D \cong \mathcal{H}iggs\) respects the filtrations.

**Proof.** The above filtrations give rise to short exact sequences of commutative group stacks over \(B - \Delta\):

\[
\begin{align*}
0 & \longrightarrow \mathcal{H}iggs_0 \longrightarrow \mathcal{H}iggs \longrightarrow \pi_1(G) \longrightarrow 0 \\
0 & \longrightarrow BZ(G) \longrightarrow \mathcal{H}iggs_0 \longrightarrow \mathcal{H}iggs_0 \longrightarrow 0
\end{align*}
\]

and

\[
\begin{align*}
0 & \longrightarrow \mathcal{H}iggs_0 \longrightarrow \mathcal{H}iggs \longrightarrow \pi_1(G) \longrightarrow 0 \\
0 & \longrightarrow BZ(G) \longrightarrow \mathcal{H}iggs \longrightarrow \mathcal{H}iggs \longrightarrow 0.
\end{align*}
\]

Writing the same sequences for \(L\mathcal{H}iggs\) and applying \((\bullet)^D\) we get

\[
\begin{align*}
0 & \longrightarrow (L\mathcal{H}iggs_0)^D \longrightarrow (L\mathcal{H}iggs)^D \longrightarrow Z(LG)^\wedge \longrightarrow 0 \\
0 & \longrightarrow B\pi_1(LG)^\wedge \longrightarrow (L\mathcal{H}iggs)^D \longrightarrow (L\mathcal{H}iggs_0)^D \longrightarrow 0
\end{align*}
\]

and

\[
\begin{align*}
0 & \longrightarrow (L\mathcal{H}iggs)^D \longrightarrow (L\mathcal{H}iggs)^D \longrightarrow Z(LG)^\wedge \longrightarrow 0 \\
0 & \longrightarrow B\pi_1(LG)^\wedge \longrightarrow (L\mathcal{H}iggs)^D \longrightarrow (L\mathcal{H}iggs_0)^D \longrightarrow 0
\end{align*}
\]

The fact that the isomorphism \((7)\) in Theorem \(B\) respects the filtrations is equivalent to showing that \((7)\) induces an identification of short exact sequences \((\ast) \cong (L\ast D)\) (equivalently \((\ast\ast) \cong (L\ast D)\)). This follows by considering the compatible isomorphisms of commutative group stacks (or spaces) that we obtained in the proof of Theorem \(B\).
where in the top row we have the isomorphism \((7)\) from Theorem \(B\) and in the bottom row we have the isomorphism from Theorem \(A(2)\).

The proof of Theorem \(A\) and the calculation in the proof of Lemma \(4.2(\text{i})\) suggest that the duality of Hitchin systems proven in Theorem \(B\) admits a refinement in the case when the simple group \(G\) is of type \(B\) or \(C\). In general, the inclusion of sheaves \(\mathcal{T}^0 \subset \mathcal{T} \subset \mathcal{T}\) gives rise to three stacky integrable systems over the Hitchin base \(B\):

\[
\begin{array}{ccc}
\mathcal{H}^0_G & \xrightarrow{\mathbb{H}^0_G \to} & \mathcal{H}^{\text{Higgs}}_G \xrightarrow{\mathbb{H}^{\text{Higgs}}_G \to} \mathcal{H}^0_G \\
\mathcal{T} & \xrightarrow{\text{Tors} \mathcal{T} \to} & \mathcal{T} & \xrightarrow{\text{Tors} \mathcal{T} \to} \mathcal{T}
\end{array}
\]

**Lemma 5.4** The duality statement \((\mathcal{H}^{\text{Higgs}}_G)^D \cong \mathcal{H}^{\text{Higgs}}_G\) extends to a duality \((\mathcal{H}^0_G)^D \cong \mathcal{H}^0_G\).

**Proof.** The corresponding coarse moduli spaces admit cohomological interpretations as \(H^1(\mathcal{T}^0_G), H^1(\mathcal{T}_G),\) and \(H^1(\mathcal{T}_G)\) respectively. These integrable systems coincide for all simple groups \(G \neq \text{Sp}(r), \text{SO}(2r + 1)\), and

\[
\begin{align*}
\mathcal{H}^{\text{Higgs}}_{\text{Sp}(r)} & \cong \mathcal{H}^{\text{Higgs}}_{\text{Sp}(r)} \\
\mathcal{H}^0_{\text{SO}(2r + 1)} & \cong \mathcal{H}^{\text{Higgs}}_{\text{SO}(2r + 1)}.
\end{align*}
\]

Now, the calculations in Claim \(3.5(\text{ii})\) and Lemma \(4.2(\text{i})\) give the following values for the stabilizer groups and the groups of connected components of these group stacks:

| \(G\)       | \(H^0(\mathcal{T}^0_G)\) | \(H^0(\mathcal{T}_G)\) | \(H^0(\mathcal{T}_G)\) |
|-------------|--------------------------|--------------------------|--------------------------|
| \(\text{Sp}(r)\) | 0                        | \(Z(G) = \mathbb{Z}/2\)  | \(Z(G) = \mathbb{Z}/2\)  |
| \(\text{SO}(2r + 1)\) | \(Z(G) = 0\)             | \(Z(G) = 0\)             | \(\mathbb{Z}/2\)         |

and

| \(G\)       | \(\pi_0(H^1(\mathcal{T}^0_G))\) | \(\pi_0(H^1(\mathcal{T}_G))\) | \(\pi_0(H^1(\mathcal{T}_G))\) |
|-------------|--------------------------|--------------------------|--------------------------|
| \(\text{Sp}(r)\) | \(\mathbb{Z}/2\)        | \(\pi_1(G) = 0\)        | \(\pi_1(G) = 0\)        |
| \(\text{SO}(2r + 1)\) | \(\pi_1(G) = \mathbb{Z}/2\) | \(\pi_1(G) = \mathbb{Z}/2\) | \(0\)                  |

The above tables and the calculation of the cocharacter lattices of the Prym varieties \(P^0\) and \(P\) in Claim \(3.6\) show that the isomorphism \((\mathcal{H}^0_G)^D \cong \mathcal{H}^0_G\) holds for the graded pieces with respect to the weight filtrations. The full duality of filtered objects follows from the argument in the proof of Theorem \(4.13\) \(\square\)
5.3 Equivalence of derived categories

Theorem B has some immediate corollaries. First, we get a categorical equivalence

**Corollary 5.5** Over \( B - \Delta \), there is a Fourier-Mukai type equivalence of derived categories

\[
c : D_c^b(\mathcal{Higgs}) \cong D_c^b(L\mathcal{Higgs}).
\]

Moreover, for every \( \alpha \in \pi_0(\mathcal{Higgs}) = \pi_1(G) = Z(LG)^\wedge \), and every \( \beta \in \pi_0(L\mathcal{Higgs}) = \pi_1(LG) = Z(G)^\wedge \), the functor \( c \) gives rise to a Fourier-Mukai equivalence

\[
D_c^b(\beta \mathcal{Higgs}_\alpha) \cong D_c^b(\alpha L\mathcal{Higgs}_\beta)
\]

for the derived categories of the induced \( O^X \)-gerbes.

**Proof.** The isomorphism (7) implies that the \( O^X \)-gerbes \( \alpha L\mathcal{Higgs}_\beta \) and \( \beta \mathcal{Higgs}_\alpha \) are compatible, in the sense of [DP08]. In particular, the categorical equivalence statement from [DP08] implies the equivalence of derived categories \( D_c^b(\mathcal{Higgs}_0) = D_c^b(L\mathcal{Higgs}) \). To get the full categorical duality \( D_c^b(L\mathcal{Higgs}) \cong D_c^b(\mathcal{Higgs}) \), one can combine (7) with the duality for representations of commutative group stacks described in Arinkin’s appendix to [DP08] (see also [BeBra07]), or invoke the recent result [BB09] of O. Ben-Bassat. In fact, Ben-Bassat’s proof works in a much more general context and will imply the full categorical duality even over the discriminant \( \Delta \), as long as one can show that the Poincaré sheaf on the cameral Pryms extends across \( \Delta \).

\( \Box \)

5.4 Hecke eigensheaves

As observed in Lemma 5.3, the duality in Theorem B and C respects the weight filtrations on \( \mathcal{Higgs} \) and \( L\mathcal{Higgs} \). In particular we have \( \mathcal{Higgs}_0 \cong (L\mathcal{Higgs})^D \) and so \( c \) restricts to a well defined equivalence

\[
c_0 : D_c^b(\mathcal{Higgs}_0) \cong D_c^b(L\mathcal{Higgs}).
\]

Finally, we have that the natural orthogonal spanning class of the category \( D_c^b(\mathcal{Higgs}_0) \) is transformed by \( c \) into the class of automorphic sheaves on \( L\mathcal{Higgs} \). This is precisely the sense in which the categorical equivalence \( c \) can be thought of as a classical limit of the geometric Langlands correspondence. To spell this out, recall that in the proof of Theorem B (see also Appendix A), we introduced abelianized Hecke maps

\[
L Trans^\mu : L \mathcal{Higgs}_{\tilde{C}} \times \tilde{C} \to L \mathcal{Higgs}_{\tilde{C}}
\]

labeled by characters \( \mu \in L\Lambda = \Lambda^\vee = char(T) \) of \( T \). These maps were constructed at the stack level. Here we use specifically the induced maps on the moduli spaces. Recall that the map \( L Trans^\mu \) gives rise to an abelianized Hecke operator \( L_{ab} H^\mu := (L Trans^\mu)^* : D_c^b(L \mathcal{Higgs}_{\tilde{C}}) \to D_c^b(L \mathcal{Higgs}_{\tilde{C}} \times \tilde{C}) \).
**Theorem D** A topologically trivial $G$-Higgs bundle $(V, \varphi)$ on $C$ determines an eigensheaf for the abelianized Hecke operators. Explicitly let $p : \tilde{C} \to C$ be a cameral cover corresponding to a point in $B - \Delta$, and let $T_{\tilde{C}}$ be the corresponding sheaf of regular centralizers on $C$. The choice of $(V, \varphi)$ gives:

- A $T$-torsor $\mathcal{L}_{(V,\varphi)}$ on $\tilde{C}$.
- A representable structure morphism $\iota : B \text{Aut}((V, \varphi)) \to \text{Higgs}_0$.

Write $o_{(V,\varphi)} := \iota^* \mathcal{O}_{\text{Aut}((V, \varphi))}$ for the corresponding sheaf on $\text{Higgs}_0$. (This is nothing but the structure sheaf of the stacky point of $\text{Higgs}_0$ corresponding to $(V, \varphi)$.) Then for every character $\mu \in \Lambda^\vee$ we have a functorial isomorphism

$$^L_{ab}H^\mu \left( o_{(V,\varphi)} \right) \cong o_{(V,\varphi)} \boxtimes \mu \left( \mathcal{L}_{(V,\varphi)} \right),$$

i.e. $o_{(V,\varphi)}$ is an abelianized Hecke eigensheaf with eigenvalue $\mathcal{L}_{(V,\varphi)}$.

**Proof.** This is automatic from the definition of the Hecke correspondences, the abelianization procedure of [DG02, Theorem 6.4], and the fact that the categorical equivalence $c$ is compatible with the usual Fourier-Mukai equivalence of $\text{Higgs}_0$ and $^L\text{Higgs}_0$ as discussed in Theorem 4.13. \hfill \Box

### 6 The topological structure of a cameral Prym

In this section we discuss the cohomology groups describing the cocharacter lattices of cameral Prym varieties and the behavior of those groups under Poincaré duality. Most of the material in section 6.1 is well known, but we couldn’t find it in the literature, in the form needed for the proof of Theorem A. We include here the necessary statements; the proofs are left to the reader (or can be found at arXiv:math/0604617 v1). The results in Section 6.2 are new. We give an explicit description of the cocharacter lattice of a cameral Prym in terms of the local monodromies of a cameral cover. We used this result in the proof of Claim 3.5 to analyze the connected components of the Hitchin fiber, but it may also be of independent interest.

#### 6.1 Remarks on local system cohomology

Let $C$ be a smooth compact complex curve of genus $g$ and let $S = \{s_1, \ldots, s_b\} \subset C$ be a finite set of points. We write $U := C - S$ for the complement of $S$ and denote by $\iota : S \hookrightarrow C$ and $j : U \hookrightarrow C$ the corresponding closed and open inclusions. We will also fix a base point $o \in U$.

Let $A$ be a local system on $U$ of free abelian groups of rank $r$. Let $A^\vee := \text{Hom}_{\mathcal{O}_U}(A, \mathbb{Z}_U)$ denote the dual local system. We want to understand the cohomology of $j_* A$ in concrete terms and to find the precise relationship between the cohomology of $j_* A$ and $j_*(A^\vee)$. This
is all standard for local systems of vector spaces, see e.g. [Loo97], but it requires some care for local system of free abelian groups.

Suppose \( s_i \in S \) and let \( s_i \in D_i \subset C \) be a small disc centered at \( s_i \) and not containing any other point of \( S \). Fix a point \( \sigma_i \in \partial D_i \) and let \( c_i \) denote the loop starting and ending at \( \sigma_i \) and traversing \( \partial D_i \) once in the positive direction. Write \( \text{mon}(c_i) : A_{\sigma_i} \to A_{\sigma_i} \) for the monodromy operator associated with \( c_i \). Now, from the definition of the direct image and the fact that \( A \) is locally constant we get the following description of the stalk of \( j_\ast A \) at \( s_i \):

\[
(j_\ast A)_{s_i} = \lim_{\leftarrow} \left\{ H^0(V \cap U, A) \mid s_i \in V, \ V \subset C - \text{open} \right\} \\
= H^0(D_i - \{s_i\}, A) \\
= (A_{\sigma_i})^{\text{mon}(c_i)}.
\]

Here, as usual \( (A_{\sigma_i})^{\text{mon}(c_i)} := \{ a \in A_{\sigma_i} \mid \text{mon}(c_i)(a) = a \} \) denotes the invariants of the \( \text{mon}(c_i) \)-action.

To organize things better, we choose an ordered system of arcs \( \{a_i\}_{i=1}^b \) in \( C - \bigcup_{i=1}^b D_i \) which connect the base point \( \sigma \) with each of the points \( \sigma_i \) as in Figure 1.

![Figure 1: An arc system for \( S \subset C \).](image)

A choice of an arc system yields a collection of elements \( \gamma_i \in \pi_1(U, \sigma) \). Geometrically \( \gamma_i \) is the \( \sigma \)-based loop in \( U \) obtained by tracing \( a_i \), followed by tracing \( c_i \) and then tracing back \( a_i \) in the opposite direction. Since parallel transport along \( a_i \) identifies the stalks \( A_\sigma \) and \( A_{\sigma_i} \) and conjugates the monodromy transformation \( \text{mon}(\gamma_i) \) into the monodromy transformation \( \text{mon}(c_i) \), it follows that we also have the identification

\[
(j_\ast A)_{s_i} = (A_\sigma)^{\text{mon}(\gamma_i)}
\]
for all $s_i \in S$. To simplify notation we set $\rho_i := \text{mon}(\gamma_i)$.

With this notation in place we are ready to analyze the cohomology of the sheaves $A$ and $j_*A$. First note that $U$ is a smooth 2-manifold and so has homological dimension 2 with respect to compactly supported cohomology [Ive86, Section III.9] or [Dim04, Section 3.1]. Also, since $A$ is a locally constant sheaf, it cannot have any compactly supported sections and so the only compactly supported cohomology groups of $A$ that can be potentially non-zero are $H^1_c(U, A)$ and $H^2_c(U, A)$. On the other hand, since $A$ is a local system, its cohomology is homotopy invariant. Taking into account the fact that $U$ is homotopy equivalent to a bouquet of circles, we conclude that the only cohomology groups of $A$ that are potentially non-zero are $H^0(U, A)$ and $H^1(U, A)$. Furthermore the following version of Poincare duality holds for these groups:

**Lemma 6.1** The cup product pairing

$$(\text{cup}) \quad H^k_c(U, A) \otimes H^{2-k}(U, A^\vee) \xrightarrow{\cup} H^2_c(U, A \otimes A^\vee) \xrightarrow{\text{tr}} \mathbb{Z},$$

induces a perfect pairing between the free abelian groups $H^1(U, A)_{\text{tf}}$ and $H^1_c(U, A^\vee)_{\text{tf}}$. Moreover $H^0(U, A)$ has no torsion and the cup product pairing $\text{(cup)}$ induces a perfect pairing between $H^0(U, A)$ and $H^2_c(U, A^\vee)_{\text{tf}}$.

The proof, using Verdier duality and the universal coefficient theorem, is omitted.

Next we will use this information to compute the cohomology of $A$ and $j_*A$ explicitly in terms of the monodromy. We begin with a standard lemma:

**Lemma 6.2** The direct image sheaves $R^k j_* A$ can be described as follows:

(a) The sheaf $j_* A$ fits in a short exact sequence

$$0 \rightarrow j^! A \rightarrow j_* A \rightarrow \oplus_{i=1}^b A^\rho_i \rightarrow 0,$$

where $j^!$ denotes the pushforward with compact supports.

(b) The sheaf $R^1 j_* A$ satisfies

$$R^1 j_* A = \oplus_{i=1}^b (A^\rho_i),$$

where $(A^\rho_i) := A^\rho_i/(1 - \rho_i) A^\rho_i$ denotes the group of coinvariants of the $\rho_i$-action on $A^\rho_i$.

(c) $R^k j_* A = 0$ for all $k \geq 2$.

**Proof.** One can easily compute $H^k(c_i, A)$ via group cohomology. Since $c_i \cong S^1 = K(\mathbb{Z}, 1)$ we have

$$H^k(c_i, A) = H^k(\pi, M),$$

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where \( \pi := \pi_1(c, o_1) \cong \mathbb{Z} \), and \( M \) denotes the \( \pi \) module \((\mathfrak{A}, \mathfrak{r})\). Now the group ring \( \mathbb{Z}[\pi] = \mathbb{Z}[t] \) is a polynomial ring in one variable over \( \mathbb{Z} \) and so \( \mathbb{Z} \) has a two step resolution

\[
0 \rightarrow \mathbb{Z}[t] \xrightarrow{\partial} \mathbb{Z}[t] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0
\]

by free \( \mathbb{Z}[\pi] \)-modules. Here \( \varepsilon : \mathbb{Z}[t] \rightarrow \mathbb{Z} \) is the augmentation map \( \varepsilon(p(t)) := p(0) \), and \( \partial : \mathbb{Z}[t] \rightarrow \mathbb{Z}[t] \) is the map \( \partial p(t) := (t - 1)p(t) \) of multiplication by \( t - 1 \). In particular, for any \( \pi \) module \( M = (\mathfrak{A}, \mathfrak{r}) \) we can compute \( H^\bullet(\pi, M) \) as the cohomology of the complex

\[
\begin{array}{c}
\text{(degree 0)} \\
\text{(degree 1)}
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}_\pi(\mathbb{Z}[t], M) & \xrightarrow{\delta} & \text{Hom}_\pi(\mathbb{Z}[t], M) \\
\text{\hspace{1cm} } & \text{\hspace{1cm} } & \text{\hspace{1cm} } \\
\text{\hspace{1cm} } & \text{\hspace{1cm} } & \text{\hspace{1cm} } \\
\mathfrak{A} & \xrightarrow{\tau^{-1}} & \mathfrak{A}.
\end{array}
\]

In other words \( H^0(\pi, M) = \mathfrak{A}^\tau \), \( H^1(\pi, M) = \mathfrak{A}_t \), and \( H^k(\pi, M) = 0 \) for \( k \geq 2 \). We leave the remaining details to the reader.

Alternatively, one can give a very elementary argument based on the usual gluing \cite[Section 1.4]{BBD82}, \cite[Chapter 4, Exercise 3]{GM03} for the inclusions \( U \hookrightarrow C \leftarrow S \), which yields a distinguished triangle

\[
j_*j^{-1}F^\bullet \rightarrow F^\bullet \rightarrow j_*t^{-1}F^\bullet \rightarrow j_*F^\bullet[1]
\]

defined for every \( F^\bullet \in D(\mathbb{Z}_U - \text{mod}) \). Our lemma follows by taking \( F^\bullet := Rj_*A \). \( \square \)

As a consequence of the calculation of \( R^k j_*A \) one immediately gets the following:

**Lemma 6.3** The cohomology of the sheaf \( j_*A \) satisfies:

\[
\begin{align*}
H^0(C, j_*A) &= A^\pi_1(U, o) \\
H^1(C, j_*A) &= \text{im} \left[ H^1_c(U, A) \rightarrow H^1(U, A) \right] \\
H^2(C, j_*A)_{\text{tf}} &= \left( (A^\vee_0)^\pi_1(U, o) \right)^\vee \\
H^2(C, j_*A)_{\text{tor}} &= H^1(U, A^\vee)^\wedge_{\text{tor}}.
\end{align*}
\]

**Corollary 6.4** For any local system \( A \) of finite rank free abelian groups on \( U \) we have a natural identification

\[
H^1(C, j_*A^\vee)_{\text{tf}} = \text{im} \left[ H^1(U, A)^\vee \rightarrow H^1(C, j_*A)^\vee \right]
\]
Proof. By the previous lemma we have an identification
\[ H^1(C, j_* A^\vee) = \text{im} \left[ H^1_c(U, A^\vee) \to H^1(U, A^\vee) \right]. \]
In particular we have \( H^1(C, j_* A^\vee)_{\text{ff}} = \text{im} \left[ H^1_c(U, A^\vee)_{\text{ff}} \to H^1(U, A^\vee)_{\text{ff}} \right] \). On the other hand, by Lemma 6.3 the natural map \( H^1_c(U, A^\vee)_{\text{ff}} \to H^1(U, A^\vee)_{\text{ff}} \) is equal to minus the transpose of the map \( H^1_c(U, A)_{\text{ff}} \to H^1(U, A)_{\text{ff}} \). Since by Lemma 6.3 we have
\[ H^1(C, j_* A) = \text{im} \left[ H^1_c(U, A) \to H^1(U, A) \right], \]
we get that
\[ \text{im} \left[ H^1(U, A)^\vee \to H^1_c(U, A)^\vee \right] = \text{im} \left[ H^1(U, A)^\vee \to H^1(C, j_* A)^\vee \right], \]
which yields the lemma. \( \square \)

6.2 The cocharacters of a cameral Prym

Let \( p : \tilde{C} \to C \) be a generic Galois cameral cover as in the proof of Theorem A. Let \( \{s_1, \ldots, s_b\} \subset C \) be the branch points of this cover. We write \( j : U \hookrightarrow C \) for the inclusion of the complement, and \( p^\circ : p^{-1}(U) \to U \) for the unramified part of \( p \). Define a local system \( A \) on \( U \) by \( A := (p^\circ_! \Lambda)^W \). The canonical identification \( L \Lambda = \Lambda^\vee = \text{Hom}(\Lambda, \Lambda) \) gives also an identification \( L A = A^\vee = \text{Hom}(\Lambda, \Lambda) \).

Fix a base point \( \sigma \in U \) and choose an arc system as in the previous section. We choose once and for all an identification \( A_o \cong \Lambda \). By definition, the monodromy \( \text{mon}_\sigma : \pi_1(U, \sigma) \to GL(\Lambda) \) factors through \( W \subset SO(\Lambda, \langle \bullet, \bullet \rangle) \subset GL(\Lambda) \). By the genericity assumption on \( p : \tilde{C} \to C \), it follows that the monodromy image of each generator \( \gamma_i \in \pi_1(U, \sigma) \) is a reflection \( \rho_i : \Lambda \to \Lambda \) corresponding to some root \( \alpha_i \) of \( \mathfrak{g} \). Explicitly we have \( \rho_i(\lambda) = \lambda - (\alpha_i, \lambda) \cdot \alpha_i^\vee \), where \( \alpha_i^\vee \in \Lambda \) is the coroot corresponding to \( \alpha_i \). Since for each root \( \alpha \) the divisor \( D^\alpha \subset \text{tot}(K_C \otimes t) \) is ample, it follows that the collection of roots \( \{\alpha_1, \ldots, \alpha_b\} \) contains both long and short roots of \( \mathfrak{g} \). Let \( \varepsilon_i := \varepsilon_{\alpha_i, C} \) and \( \varepsilon_i^\vee := \varepsilon_{\alpha_i, C}^\vee \).

In the proof of Theorem A we described the cocharacter lattice of the cameral Prym \( P \) corresponding to \( p : \tilde{C} \to C \) and \( G \) in terms of the first cohomology of the sheaves \( j_* A \) and \( j_* A^\vee \) on \( C \). We now give explicit formulas for these cohomology groups. Note that without a loss of generality we may assume that \( S \), the arcs \( a_i \) and the discs \( D_i \) are all contained in the interior of a disc \( D \subset C \) for which \( \sigma \in \partial D \). In particular we can choose a collection of \( \sigma \) based loops \( \delta_1, \ldots, \delta_{2g} \subset C - D \) which intersect only at \( \sigma \) and form a system of standard \( a \cdot b \) generators for the fundamental group \( \pi_1(C, \sigma) \) of the compact curve \( C \). Choosing the orientation of the loops \( \delta_j \) and \( \gamma_i \) appropriately we get a presentation of the fundamental group of \( U \):
\[ \pi_1(U, \sigma) = \left\langle \delta_1, \ldots, \delta_{2g}, \gamma_1, \ldots, \gamma_b \mid \prod_{j=1}^g [\delta_i, \delta_{g+i}] \prod_{i=1}^b \gamma_i = 1 \right\rangle. \]
To simplify notation we set \( w_i := \text{mon}_\sigma(\delta_i) \in W \). Finally, it will be convenient to add to \( S \) an extra point \( s_0 \neq \sigma \in U \) and a loop \( \gamma_0 \) around \( s_0 \), oriented so that the relation defining
Note also that the deletion of $s_0$ from $U$ will not affect our interpretation of $H^1(C, j_* A)$ as a kernel of a homomorphism. That is, we still have

$$H^1(C, j_* A) = \ker \left[ H^1(U, A) \to \bigoplus_{i=1}^b \Lambda / (1 - \rho_i) \Lambda \right] = \ker \left[ H^1(U - \{s_0\}, A) \to \bigoplus_{i=0}^b \Lambda / (1 - \rho_i) \Lambda \right],$$

since the deletion adds a copy of $\Lambda$ to both $H^1(\bullet, A)$ and $R^1 j_* A$.

**Proposition 6.5** \(\text{(a)}\) **There is a natural isomorphism**

$$H^1(U - \{s_0\}, A) \cong \frac{\Lambda^{2g+b}}{(1 - w_1, \ldots, 1 - w_{2g}, 1 - \rho_1, \ldots, 1 - \rho_b) \Lambda},$$

which depends only on the choice of an arc system, the $a - b$ loops $\delta_j$, and the identification $A_\rho \cong \Lambda$. In addition, there is a non-canonical isomorphism

$$H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus \frac{\Lambda^b}{(1 - \rho_1, \ldots, 1 - \rho_b) \Lambda}.$$

**Proposition 6.5** \(\text{(b)}\) **Under the isomorphism** $H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus (\Lambda^b / (1 - \rho_1, \ldots, 1 - \rho_b) \Lambda)$, **the subgroup** $H^1(C, j_* A) \subset H^1(U, A)$ **can be identified as**

$$H^1(C, j_* A) = H^1(C, \Lambda) \oplus \frac{\ker \left[ \sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \bigoplus_{i=1}^b \mathbb{Z} e_i \alpha_i^\gamma \to \Lambda \right]}{(1 - \rho_1, \ldots, 1 - \rho_b) \Lambda}.$$

**Proof.** \(\text{(a)}\) The surface $U - \{s_0\} = C - \{s_0, s_1, \ldots, s_b\}$ is homotopy equivalent to the bouquet consisting of the $2g + b$ oriented circles $\delta_1, \ldots, \delta_{2g}, \gamma_1, \ldots, \gamma_b$, where all circles are attached to each other at the point $\bullet$. The fundamental group $\pi$ of this bouquet of circles is a free group on the generators $\delta_1, \ldots, \delta_{2g}, \gamma_1, \ldots, \gamma_b$, and the local system $A$ corresponds to the action on of this free group on $\Lambda$ specified by the monodromy transformations $(w_1, \ldots, w_{2g}, \rho_1, \ldots, \rho_b) \in W^{2g+b}$. As in the proof of Lemma 6.2, the trivial $\pi$-module $\mathbb{Z}$ has a free $\mathbb{Z}[\pi]$-resolution given by

$$0 \to (\bigoplus_{j=1}^{2g} \mathbb{Z}[\pi] e_{\delta_j}) \oplus (\bigoplus_{i=1}^b \mathbb{Z}[\pi] e_{\gamma_i}) \overset{\partial}{\to} \mathbb{Z}[\pi] \to \mathbb{Z} \to 0,$$

where $(\bigoplus_{j=1}^{2g} \mathbb{Z}[\pi] e_{\delta_j}) \oplus (\bigoplus_{i=1}^b \mathbb{Z}[\pi] e_{\gamma_i})$ is the free $\mathbb{Z}[\pi]$ module on generators $e_{\delta_j}, e_{\gamma_i}$ and $\partial e_{\delta_j} = 1 - \delta_j, \partial e_{\gamma_i} = 1 - \gamma_i$.

Applying $\text{Hom}_{\mathbb{Z}[\pi]}(\bullet, \Lambda)$ and computing cohomology we get the identification

$$H^1(U - \{s_0\}, A) = H^1(\pi, \Lambda) \cong \frac{\Lambda^{2g+b}}{(1 - w_1, \ldots, 1 - w_{2g}, 1 - \rho_1, \ldots, 1 - \rho_b) \Lambda}.$$

Next we claim that by making appropriate choices, the topological description description of the camera cover can be brought into a particularaly simple form. Recall [DDP07], that there is a natural inclusion

$$\left( H^0(C, K_C) \otimes 1 \right) / W \hookrightarrow B.$$
A cameral cover \( p_b : \tilde{C}_b \to C \) corresponding to a generic point \( b \) in the image of (48) is reduced but completely reducible:

\[
\tilde{C}_b = \bigcup_{w \in W} \tilde{C}_{b,w},
\]

with each irreducible component \( \tilde{C}_{b,w} \) isomorphic to \( C \). Let \( D \subset C \) be a disc containing the image of all the singular points (= intersection of components) of \( \tilde{C}_b \). We get that \( p^{-1}(C-D) \) is completely disconnected:

\[
(49) \quad p^{-1}(C-D) = \coprod_{w \in W} [C-D]_w, \quad [C-D]_w := \tilde{C}_{b,w} \cap p^{-1}(C-D),
\]

with each connected components \( [C-D]_w \) isomorphic to \( C-D \).

A general cameral cover \( \tilde{C}_{b'} \) with \( b' \in B - \Delta \) near \( b \in B \) will be smooth and will still satisfy (49). By taking all the \( \gamma_i \) in \( D \) and all the \( \delta_j \) in \( C-D \) we have \( w_j = 1, j = 1, \ldots, 2g \).

Consequently

\[
(50) \quad H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus \frac{\Lambda^b}{(1 - \rho_1, \ldots, 1 - \rho_b)},
\]

for the cover \( \tilde{C}_{b'} \). Since \( B - \Delta \) is connected, it follows that (50) will hold for any \( \tilde{C}_{b''}, b'' \in B - \Delta \) and an appropriate choice of \( \gamma_i \)'s and \( \delta_j \)'s.

(b) As argued in the previous section, the group \( H^1(C, j_*A) \) is the kernel of the natural map \( H^1(U - \{s_0\}, A) \to H^0(C, R^1 j_* A) = \bigoplus_{i=0}^b \Lambda/(1 - \rho_i) \Lambda \). Under the identification (50), it is immediate that \( H^1(C, \Lambda) \) is contained in the kernel of this map and that the restriction of the map to the summand \( \Lambda^b/(1 - \rho_1, \ldots, 1 - \rho_b) \) is given by

\[
(51) \quad (\lambda_1, \ldots, \lambda_b) \mapsto (\varphi(\lambda_1, \ldots, \lambda_b), \lambda_1 + (1 - \rho_1) \Lambda, \ldots, \lambda_b + (1 - \rho_b) \Lambda),
\]

where the map \( \varphi : \Lambda^b \to \Lambda/(1 - \rho_0) \Lambda = \Lambda \) corresponds to the relation \( \prod_{\delta_j = \frac{2g}{j+1}\gamma_i} \prod_{i=1}^{b} \gamma_i = \gamma_0 \).

In general, suppose that we are given a bouquet of circles \( V = c_1 \vee \ldots \vee c_n \) and suppose we have a word \( d = \prod_{i=1}^{n} d_i \) in \( \pi := \pi_1(V) \) for which all \( d_i \)'s are in \( \{c_1, \ldots, c_n, c_1^{-1}, \ldots, c_n^{-1}\} \subset \pi \).

Consider the cyclic subgroup in \( \pi \) generated by \( d \) and let \( M \) be some \( \pi \)-module. The inclusion \( \langle d \rangle \subset \pi \) induces a map on cohomology \( H^1(\pi, M) \to H^1(\langle d \rangle, M) \) which can be explicitly calculated. For this we only need to consider the tree which is the universal cover of \( V \) and follow the branches of that tree labeled by the letters \( d_i \) in the word \( d \). More invariantly this corresponds to a map of resolutions

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{Z}[[d]]e_d & \xrightarrow{\partial} & \mathbb{Z}[[d]] & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow{(\ast\ast)} & & \downarrow{(\ast)} & & \downarrow{(\ast)} & & \downarrow{(\ast)} & & \\
0 & \rightarrow & \bigoplus_{i=1}^{n} \mathbb{Z}[\pi]e_{c_i} & \xrightarrow{\partial} & \mathbb{Z}[\pi] & \rightarrow & \mathbb{Z} & \rightarrow & 0,
\end{array}
\]
where \((\ast)\) is the natural inclusion and \((\ast\ast)\) sends \(e_d\) to the sum \(\sum_{j=1}^p \text{sgn}(d_j) \left( \prod_{i=1}^{j-1} d_i \right) e_{d_j}\), where \(\text{sgn}(d_j) = \pm 1\) depending on whether \(d_j\) is one of the \(c_i\)'s or one of the \(c_i^{-1}\)'s.

Combining this formula with the observation that \((1 - \rho_i)(\lambda) = (\alpha_i, \lambda)\alpha_i^\vee\), we see that the kernel of (51) is precisely \(\ker \left[ \sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \oplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \to \Lambda \right]\). \(\Box\)

In particular, for the torsion subgroups of \(H^1(C, j_* A)\) and \(H^1(U, A)_{\text{tor}}\) we get

**Corollary 6.6**

\[
H^1(C, j_* A)_{\text{tor}} = \left( \frac{\bigoplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee}{(1 - \rho_1, \ldots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}
\]

\[
H^1(U, A)_{\text{tor}} = H^1(U - \{s_0\}, A)_{\text{tor}} = \left( \frac{\Lambda^b}{(1 - \rho_1, \ldots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}.
\]

**Proof.** Since \(H^1(C, \Lambda)\) is torsion free, Proposition 6.5 implies that

\[
H^1(C, j_* A)_{\text{tor}} = \left( \frac{\ker \left[ \sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \bigoplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \to \Lambda \right]}{(1 - \rho_1, \ldots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}.
\]

The corollary now follows by noticing that the saturation of \((1 - \rho_1, \ldots, 1 - \rho_b)\Lambda\) inside the lattice \(\bigoplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee\) is the same as the saturation of \((1 - \rho_1, \ldots, 1 - \rho_b)\Lambda\) inside the lattice \(\ker \left[ \sum_{i=1}^b \prod_{k=0}^{i-1} \rho_k : \bigoplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \to \Lambda \right]\), since the latter lattice is a kernel to a map to \(\Lambda\) which is torsion free.

Finally, \(H^1(U, A)_{\text{tor}} = H^1(U - \{s_0\}, A)_{\text{tor}}\) since the deletion of \(s_0\) adds a copy of \(\Lambda\) as a direct summand. \(\Box\)

**Appendix A: Hecke functors and spectral data**

In section 2 we defined the classical limit Hecke functors

\[
L_{H}^{\mu, x} : D^b_{\text{qcoh}}(L^{\text{Higgs}}, \mathcal{O}) \to D^b_{\text{qcoh}}(L^{\text{Higgs}}, \mathcal{O}),
\]

which were labeled by pairs \((\mu, x)\) with \(\mu \in \text{char}(G)\) and \(x \in C\). On the other hand the derived category of quasi-coherent sheaves on \(L^{\text{Higgs}}\) is equipped with another collection of endo-functors: the abelianized Hecke functors \(L_{\text{ab}, H}^{\mu, x}\). Here again \(\mu \in \text{char}(G)\), but \(\bar{x}\) is a point in some cameral cover \(\widetilde{C} \to C\) which we will assume smooth, i.e. \(\widetilde{C} = \widetilde{C}_b\), where
\( b \in B - \Delta \). By definition the functor \( L_{ab} \mathbb{H}^{\mu, \bar{x}} \) is the integral transform on quasi-coherent sheaves on \( L^{Higgs} \) whose kernel is the structure sheaf of the abelianized Hecke correspondence

\[
L_{ab} \mathcal{H} \rightarrow L_{ab} \mathcal{H}^{\mu, \bar{x}}
\]

Here \( L^{Higgs}_{\widetilde{C}} \) denotes the moduli stack of \( LG \)-Higgs bundles on \( C \) with cameral cover \( \widetilde{C} \), i.e. the fiber of the (stacky) Hitchin map over \( \widetilde{C} \). The correspondence \( L_{ab} \mathcal{H}^{\mu, \bar{x}}_{\widetilde{C}} \) is the graph of a specific translation isomorphism

\[
L^{\text{Trans}}_{\mu, \bar{x}} : L^{Higgs}_{\widetilde{C}} \rightarrow L^{Higgs}_{\widetilde{C}}
\]

that we describe next. The essential ingredient in the definition of \( L^{\text{Trans}}_{\mu, \bar{x}} \) is the description \([DG02, \text{Theorem 6.4}]\) of the fiber of the Hitchin map \( L^{Higgs}_{\widetilde{C}} \) in terms of geometric data on \( \widetilde{C} \): the so called spectral data.

To make things explicit we recall this description next.

**A.1 Spectral data for principal Higgs bundles**

**A.1.1. Bundles and cocycles.** Suppose \( \widetilde{C} \rightarrow C \) is a fixed abstract cameral cover for \( LG \). For every root \( \alpha \) of \( LG \) denote by \( D_{\alpha} \subset \widetilde{C} \) the divisor fixed by the reflection \( \rho_{\alpha} \in W \) corresponding to \( \alpha \). Consider the following principal bundles:

- \( R_{\alpha} \): the \( C^\times \)-bundle corresponding to \( D_{\alpha} \), i.e. \( R_{\alpha} := \mathcal{O}_{\widetilde{C}}(D_{\alpha})^\times \).
- \( \mathcal{R}_{\rho_{\alpha}} \): the \( LT \) bundle defined by \( \mathcal{R}_{\rho_{\alpha}} := \alpha^\vee(R_{\alpha}) \in \text{Bun}_{\widetilde{C}, LT} \). Here \( \alpha^\vee \) is the coroot corresponding to \( \alpha \), viewed as a cocharacter \( \alpha^\vee : \mathbb{C}^\times \rightarrow LT \).

In \([DG02, \text{Lemma I.5.4, Proposition I.5.5}]\) it is shown that the assignment \( \alpha \mapsto \mathcal{R}_{\rho_{\alpha}} \) extends uniquely to a map \( \mathcal{R} : W \rightarrow \text{Bun}_{\widetilde{C}, LT} \) which is multiplicative in the sense that for all \( w, w' \in W \) we have a canonical isomorphism

\[
\varpi(w, w') : \mathcal{R}_{-w, w'} \cong \mathcal{R}_{w, w} \otimes \mathcal{R}_{-w, w'}.
\]

Here \( \otimes \) denotes the natural tensor product of \( LT \)-bundles on \( \widetilde{C} \), \( \mathcal{R}_{w, w'} := w^* \left( [\bullet] \times w LT \right) \) denotes the action of \( w \) on the groupoid of \( LT \)-bundles which is the combination of the pullback \( w^*([\bullet]) \) of \( LT \)-bundles via the automorphism \( w : \widetilde{C} \rightarrow \widetilde{C} \), and the pushout \( [\bullet] \times w LT \) of \( LT \)-bundles via the automorphism \( w \in W \subset \text{Aut}(LT) \).

**Remark A.1** (i) Consider the commutative group stack \( \text{Bun}_{\widetilde{C}, LT} \) of all \( LT \)-bundles on \( \widetilde{C} \). The group \( W \) acts on \( \text{Bun}_{\widetilde{C}, LT} \), i.e. a \( w \in W \) acts on \( \text{Bun}_{\widetilde{C}, LT} \) via \( \ell \rightarrow \mathcal{R}_{w}(\ell) \) for any.
$LT$-bundle $\ell$. In these terms $R$ is just a 1-cocycle of $W$ with values in the stacky $W$-module $\text{Bun}_{\tilde{C},LT}$.

(ii) In our case $\tilde{C}$ is not just an abstract cameral cover but is a cameral cover for $K_C$-valued Higgs bundles. For such $\tilde{C}$ we can describe the line bundles $R_\alpha$ and the cocycle $R$ more explicitly.

In this case we have a natural map $\tilde{C} \to \text{tot}(t \otimes K_C)$ and we can use this map to describe the divisor $D^\alpha$. Indeed, since $\alpha$ is a linear functional on $t$, we can view it as a map

$$t \otimes K_C \to K_C$$

of vector bundles on $C$ or equivalently, if we write $p : \text{tot}(t \otimes K_C) \to C$ for the natural projection, we can view $\alpha$ as a section

$$\alpha \in H^0(\text{tot}(t \otimes K_C), p^*K_C).$$

Restricting this section to $\tilde{C}$ we get a section $\alpha_{\tilde{C}} \in H^0(\tilde{C}, \pi^*K_C)$ whose divisor is precisely $D^\alpha$. In particular $\alpha_{\tilde{C}}$ gives an isomorphism

$$\alpha_{\tilde{C}} : R_\alpha \tilde{\to} \pi^*K_C.$$

Similarly we get identifications

$$R_{\rho_\alpha} = \alpha^\vee(\pi^*K_C)$$

and more generally $R_w = (\rho^\vee - w\rho^\vee)(\pi^*K_C)$, where $\rho^\vee$ denotes the half-sum of all positive coroots.

A.1.2. Twisted automorphisms. Next suppose we are given a $LT$-bundle $\mathcal{L}$. We can use the cocycle $R$ to define the group of $R$-twisted automorphisms of $\mathcal{L}$ covering the action of $W$ on $\tilde{C}$. Namely we set

$$\text{Aut}_R(\mathcal{L}) := \left\{ (w, i) \mid w \in W \atop i : w^*(\mathcal{L}^w) \tilde{\to} L \otimes R_{-1} \right\}$$

with the group law given by ordinary composition in $W$ and the isomorphisms $\varpi(w, w') : R_{w} \otimes w^*(L_w) \tilde{\to} L \otimes R_{w}^{-1}$ which are part of the cocycle data.

Note also that for every $\mathcal{L}$ we get a complex of groups

(A.1) \quad $LT \to \text{Aut}_R(\mathcal{L}) \to W$.

As explained in [DG02, Section I.6] this complex is exact for a connected compact $\tilde{C}$ but this need not be the case in general.

A.1.3. A $\mathbb{C}^\times$-torsor from a principal $SL_2$. Suppose now that $\alpha$ is a simple root for $L_G$, and let $M^\alpha = P^\alpha/\text{Rad}_u P^\alpha$ be the maximal reductive quotient of the minimal parabolic $P^\alpha$.
given by \( \alpha \). Let \( G_\alpha \to [M^\alpha, M^\alpha] \) be the universal cover of the semi-simple part of \( M^\alpha \). Note that \( G_\alpha \cong SL_2(\mathbb{C}) \), and that

\[
[M^\alpha, M^\alpha] = \begin{cases} 
G_\alpha, & \text{if } \alpha^\vee \text{ is primitive,} \\
G_\alpha/(\pm 1), & \text{if } \alpha^\vee \text{ is not primitive.}
\end{cases}
\]

Also, if we take \( T_\alpha \subset G_\alpha \) to be the maximal torus which is the preimage of the torus (image of \( L^T \) in \( M^\alpha \)) \( \cap [M^\alpha, M^\alpha] \), then the coroot map \( \alpha^\vee : \mathbb{C}^\times \to L^T \) lifts uniquely to an isomorphism \( \mathbb{C}^\times \to T_\alpha \) which by abuse of notation we will also denote by \( \alpha^\vee \).

Now consider the normalizer subgroup \( N(T_\alpha) \subset G_\alpha \) of \( T_\alpha \) in \( G_\alpha \) and set

\[ L_\alpha := N(T_\alpha) - T_\alpha. \]

By construction \( L_\alpha \) is naturally a \( T_\alpha \)-torsor and hence a \( \mathbb{C}^\times \)-torsor via the identification \( \alpha^\vee : \mathbb{C}^\times \to T_\alpha \).

A.1.4. Spectral data. With all this at hand we can now define the notion of spectral data which is adapted to a given cameral cover. Let \( \pi : \tilde{C} \to C \) be an abstract cameral cover for \( L^G \), \( R := \{ R_\alpha \}_\alpha \) denote the collection of line bundles given by the divisors \( D^\alpha \subset \tilde{C} \), and let \( \mathcal{R} \in Z^1(\tilde{W}, \text{Bun}_{\tilde{C},L^T}) \) be the corresponding cocycle.

**Definition A.2** An \( L^G \) spectral datum of type \( (R, \mathcal{R}) \) is a triple \( (L, i, b) \), where

- **(bundle)** \( L \) is a principal \( L^T \)-bundle on \( \tilde{C} \);

- **(twist)** \( i : N(L^T) \to \text{Aut}_{\mathcal{R}}(L) \) is a homomorphism from the normalizer of \( L^T \) in \( L^G \) to the group of \( \mathcal{R} \)-twisted endomorphisms of \( L \). The homomorphism \( i \) should fit in a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & L^T & \longrightarrow & N(L^T) & \longrightarrow & W & \longrightarrow & 0 \\
& & \text{id} & & i & & \text{id} & & \\
& & L^T & \longrightarrow & \text{Aut}_{\mathcal{R}}(L) & \longrightarrow & W,
\end{array}
\]

where the second row is the complex of groups \((A.1)\).

- **(framing)** \( b = \{ b_\alpha \}_{\alpha \in \Delta(L^G)^\text{simple}} \) is a collection of isomorphisms

\[
b : \alpha(L)|_{D^\alpha} \sim R_{\alpha|D^\alpha} \otimes L_\alpha^{-1}
\]

of principal \( \mathbb{C}^\times \)-bundles on \( D^\alpha \). Here as above \( L_\alpha \) is viewed as a \( \mathbb{C}^\times \)-torsor (on a point) via the isomorphism \( \alpha^\vee : \mathbb{C}^\times \to T_\alpha \), and \( \otimes \) denotes the group law on \( \mathbb{C}^\times \)-torsors.
The triple \((\mathcal{L}, \mathbf{i}, \mathbf{b})\) should satisfy the following compatibility condition. Let \(n \in L_\alpha\), let \(\pi \in N(L^T)\) denote the image of \(n\) under the map \(L_\alpha \subset N_\alpha \rightarrow N(L^T)\), and let \(\mathbf{i}(\pi) : \rho^*_\alpha (\mathcal{L}^{\rho_\alpha}) \sim \mathcal{L} \otimes R^{-1}_{\rho_\alpha}\) be the corresponding isomorphism of \(L^T\)-bundles. When restricted to \(D^\alpha\) this gives an isomorphism \(\mathbf{i}(\pi)|_{D^\alpha} : \mathcal{L}^{\rho_\alpha}|_{D^\alpha} \sim \mathcal{L}|_{D^\alpha} \otimes R^{-1}_{\rho_\alpha}|_{D^\alpha}\), which can be rewritten as an isomorphism \(\mathbf{j}(\pi, D^\alpha)\):

\[
\begin{align*}
(\mathcal{L}^{\rho_\alpha} \otimes \mathcal{L}^{-1})|_{D^\alpha} & \xrightarrow{\cong} R^{-1}_{\rho_\alpha}|_{D^\alpha} \\
\| & \\
-\alpha^\vee (\alpha(\mathcal{L})|_{D^\alpha}) & \xrightarrow{\mathbf{j}(\pi, D^\alpha)} \alpha^\vee (R_{\rho_\alpha}|_{D^\alpha}).
\end{align*}
\]

Then the compatibility condition on \((\mathcal{L}, \mathbf{i}, \mathbf{b})\) is

\([C]\) For every simple root \(\alpha\) of \(L^G\) and every \(n \in L_\alpha\) we have

\[\mathbf{j}(\pi, D^\alpha) = -\alpha^\vee (\mathbf{b}(n)).\]

The main fact we will use is [DG02, Theorem 6.4] which identifies the Hitchin fiber \(L^\text{Higgs}_{\mathcal{C}}\) with the moduli stack of \(L^G\) spectral data of type \((R, \mathcal{R})\).

### A.2 Abelianized Hecke functors

We want to define the abelianized Hecke functors

\[(A.2) \quad L^\text{Trans}^{\mu, \bar{x}} := (-) \otimes S^{\mu, \bar{x}} : L^\text{Higgs}_{\mathcal{C}} \rightarrow L^\text{Higgs}_{\mathcal{C}}\]

for \(\mu \in \text{char}(G)\) and \(\bar{x} \in \mathcal{C}\). They are translations which send a given Higgs bundle, or equivalently a spectral datum \((\mathcal{L}, \mathbf{i}, \mathbf{b}) \in L^\text{Higgs}_{\mathcal{C}}\), to its tensor product with a certain \(L^T\)-torsor \(S^{\mu, \bar{x}}\). ([DG02, Theorem 4.4] states that \(L^\text{Higgs}_{\mathcal{C}}\) is a gerbe over \(\text{Tors}_{\mathcal{C}}, L^T\). So a Higgs bundle can be tensored with a \(T\)-torsor.) To define \(S^{\mu, \bar{x}}\) we will use the norm map:

\[\text{Nm} : \text{Tors}_{\mathcal{C}}, L^T \rightarrow \text{Tors}_{\mathcal{C}}, L^T\]

associating a natural \(L^T\)-torsor on \(C\) with each \(L^T\)-torsor on \(\mathcal{C}\). The norm map \(\text{Nm}\) is defined as follows. Averaging over \(w \in W\) sends a \(L^T\)-bundle \(F\) to a \(W\)-equivariant \(L^T\)-bundle \(S = \otimes_{w \in W} \text{diag}_w(F)\) on \(\mathcal{C}\). Consider the subsheaf of sections in \(S\) whose value at each point in \(\mathcal{C}\) is fixed under the stabilizer of that point. This subsheaf automatically descends to a sheaf of \(L^\mathcal{C}\)-modules on \(C\). This sheaf is representable by a \(L^\mathcal{C}\)-torsor \(\mathcal{S}\). The associated line bundle \(\alpha(S)\) comes with a preferred frame along \(D_\alpha\). The subsheaf of sections in \(S\) which over \(D_\alpha\) map to the the preferred frame in \(\alpha(S)\) and are fixed under the reflection corresponding to \(\alpha\) will descend to a sheaf of \(L^\mathcal{T}\)-modules on \(C\). This sheaf is representable by \(L^\mathcal{T}\)-torsor which we define to be \(\text{Nm}(F)\).
Now starting with the $L^T\text{-torsor }\mu(\mathcal{O}_{\tilde{C}}(\tilde{x}))$ on $\tilde{C}$, we can apply $Nm$ to obtain a $L^T$-torsor $\mathcal{M}$. This provides the definition of our abelianized Hecke functors $L^T \text{Trans}^{\mu,\tilde{x}}$. (We thank the referee for suggesting improvements to an earlier version of this discussion.)

We can also rewrite the operation (A.2) as an operation on spectral data. For instance, given $(\mathcal{L}, i, b) \in L^\text{Higgs}_{\tilde{C}}$ the action of $L^T \text{Trans}^{\mu,\tilde{x}}$ on $(\mathcal{L}, i, b)$ results in a new spectral datum $(\mathcal{L} \otimes S^{\mu,\tilde{x}}, i, b \otimes s^{\mu,\tilde{x}})$ where $S^{\mu,\tilde{x}}$ is a $W$-equivariant $L^T$-bundle on $\tilde{C}$ given by:

$$S^{\mu,\tilde{x}} := \bigotimes_{w \in W} (w \mu)(\mathcal{O}_{\tilde{C}}(w \tilde{x})).$$

More precisely, note that the notion of spectral data defined above depends on the collection of line bundles $R$ on the $D^\alpha$'s and on the cocycle $\mathcal{S}$. But the same definition makes sense for any pair $(S, \mathcal{S})$, where $S$ is a collection of line bundles on the $D^\alpha$'s and $\mathcal{S}$ is a $\text{Bun}_{\tilde{C}, L^T}$-valued cocycle for $W$ compatible with $S$, i.e. is equipped with $L^T$-torsor isomorphisms $\mathcal{S}_{\rho|D^\alpha} = \alpha^\vee(S_\rho)$ for all $\alpha$. In particular we can take the pair $(\mathcal{O}^x, 0)$ to consist of the trivial line bundles and the zero cocycle. A spectral datum for this pair will be a triple $(S, i, s)$, where $S$ is just a $W$-equivariant bundle and $i$ is the identity.

From the definition it is clear that the collection of spectral data of type $(R, \mathcal{S})$ is a torsor over the collection of spectral data of type $(\mathcal{O}^x, 0)$. In particular $S^{\mu,\tilde{x}}$ will correspond to spectral data $(S^{\mu,\tilde{x}}, id, s^{\mu,\tilde{x}})$ and so $L^T \text{Trans}^{\mu,\tilde{x}}$ will be given as

$$L^T \text{Trans}^{\mu,\tilde{x}}(\mathcal{L}, i, b) := (\mathcal{L} \otimes S^{\mu,\tilde{x}}, i, b \otimes s^{\mu,\tilde{x}}).$$

The associated abelianized Hecke functor is the integral transform corresponding to the structure sheaf of the graph of $L^T \text{Trans}^{\mu,\tilde{x}}$, i.e. we have

$$L^T_{\text{ab}} \mathbb{H}^{\mu,\tilde{x}}_{\tilde{C}} := \left( L^T \text{Trans}^{\mu,\tilde{x}} \right)^* : D^b_{\text{qcoh}}(L^\text{Higgs}_{\tilde{C}}, \mathcal{O}) \to D^b_{\text{qcoh}}(L^\text{Higgs}_{\tilde{C}}, \mathcal{O}).$$

These abelianized Hecke functors appear in the proof of Theorem 3 and in the statements in Section 5.4. Their Langlands dual versions $\text{Trans}^{\lambda,\tilde{x}}_{\text{ab}} \mathbb{H}^{\lambda,\tilde{x}}_{\tilde{C}}$ are used in section 4.3.

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