Families of diffeomorphisms and concordances detected by trivalent graphs

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Abstract
We study families of diffeomorphisms detected by trivalent graphs via the Kontsevich classes. We specify some recent results and constructions of the second named author to show that those non-trivial elements in homotopy groups \(\pi_*(\text{BDiff}_\partial(D^d)) \otimes \mathbb{Q}\) are lifted to homotopy groups of the moduli space of \(h\)-cobordisms \(\pi_*(\text{BDiff}_\sqcup(D^d \times I)) \otimes \mathbb{Q}\). As a geometrical application, we show that those elements in \(\pi_*(\text{BDiff}_\partial(D^d)) \otimes \mathbb{Q}\) for \(d \geq 4\) are also lifted to the rational homotopy groups \(\pi_*(\mathcal{M}_{2,\text{psc}}(D^d)) \otimes \mathbb{Q}\) of the moduli space of positive scalar curvature metrics. Moreover, we show that the same elements come from the homotopy groups \(\pi_*(\mathcal{M}_{2,\text{psc}}(D^d \times I; g_0)) \otimes \mathbb{Q}\) of moduli space of concordances of positive scalar curvature metrics on \(D^d\) with fixed-round metric \(h_0\) on the boundary \(S^{d-1}\).

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1 | RESULTS

1.1 | Extension of graph surgery to concordance

Let $\text{Diff}_3(D^d)$ be the group of diffeomorphisms $\phi: D^d \to D^d$ which restrict to the identity near the boundary $\partial D^d = S^{d-1}$.

Recently, the second author obtained the following theorem.

**Theorem 1.1** [18–21]. Let $d \geq 4$. For each $k \geq 2$, the evaluation of Kontsevich’s characteristic classes on $D^d$-bundles gives an epimorphism

$$\pi_{k(d-3)} B\text{Diff}_3(D^d) \otimes \mathbb{Q} \to \mathcal{A}^{\text{even/odd}}_k$$

to the space of $\mathcal{A}^{\text{even/odd}}_k$ of trivalent graphs. For $k = 1$, the same result holds for the group $\pi_{2n-2} B\text{Diff}_3(D^{2n+1}) \otimes \mathbb{Q}$ for many odd integers $d = 2n + 1 \geq 5$ satisfying some technical condition.$^*$

Theorem 1.1 was proved by evaluating Kontsevich’s characteristic classes ([12]) on elements constructed by surgery on trivalent graphs embedded in $D^d$.

Here we recall the definition of the spaces $\mathcal{A}^{\text{even/odd}}_k$ of connected trivalent graphs, which are the trivalent parts of Kontsevich’s graph homology [12]. In general, trivalent graph has even number of vertices, and if it is $2k$, then the number of edges is $3k$. Let $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of a trivalent graph $\Gamma$, respectively. Labelings of a trivalent graph $\Gamma$ are given by bijections $V(\Gamma) \to \{1, 2, \ldots, 2k\}$, $E(\Gamma) \to \{1, 2, \ldots, 3k\}$. Let $\mathcal{G}_k$ be the vector space over $\mathbb{Q}$ spanned by the set $\mathcal{G}_k^0$ of all labeled connected trivalent graphs with $2k$ vertices modulo isomorphisms of labeled graphs. The version $\mathcal{A}^{\text{even}}_k$, which works for even-dimensional manifolds, is defined by

$$\mathcal{A}^{\text{even}}_k = \mathcal{G}_k / \text{IHX, label change},$$

$^*$ $d = 5, 7, 9, 11, 15, 19, 23, 24, 25, \ldots$, checked by non-integrality of some rational numbers involving the Bernoulli numbers in [18]. Actually, this holds for all $d \geq 5$ odd ([13]). See also Remark 1.7
where the IHX relation is given in Figure 1 and the label change relation is generated by the following relations:

\[ \Gamma' \sim -\Gamma, \quad \Gamma'' \sim \Gamma. \]

Here, \( \Gamma' \) is the graph obtained from \( \Gamma \) by exchanging labels of two edges, and \( \Gamma'' \) is the graph obtained from \( \Gamma \) by exchanging labels of two vertices. The version \( \mathcal{A}^{\text{odd}}_k \), which works for odd-dimensional manifolds, can be defined similarly as \( \mathcal{A}^{\text{even}}_k \) except a small modification in the orientation convention. Namely, let \( \tilde{\mathcal{G}}_k \) be the vector space over \( \mathbb{Q} \) spanned by the set \( \tilde{\mathcal{G}}_k^0 \) of all pairs \( (\Gamma, o) \) of labeled connected trivalent graphs \( \Gamma \) with \( 2k \) vertices modulo isomorphisms of labeled graphs and orientations \( o \) of the real vector space \( H^1(\Gamma; \mathbb{R}) \). Then we define

\[ \mathcal{A}^{\text{odd}}_k = \tilde{\mathcal{G}}_k / \text{IHX, label change, orientation reversal}, \]

where the IHX and the label change relation is the same as above, and the orientation reversal is the following:

\[ (\Gamma, -o) \sim -(\Gamma, o). \]

Let \( X \) be a \( d \)-dimensional path-connected smooth manifold with non-empty boundary. We also let \( \text{Diff}_\cup(X \times I) := \text{Diff}(X \times I, X \times \{0\} \cup \partial X \times I) \) be the group of pseudoisotopies. There is a natural fiber sequence

\[ \text{Diff}_\delta(X \times I) \xrightarrow{i} \text{Diff}_\cup(X \times I) \xrightarrow{\partial} \text{Diff}_\delta(X \times \{1\}), \tag{1} \]

where \( i : \text{Diff}_\delta(X \times I) \rightarrow \text{Diff}_\cup(X \times I) \) is the inclusion, and \( \partial : \text{Diff}_\cup(X \times I) \rightarrow \text{Diff}_\delta(X \times \{1\}) \) restricts a diffeomorphism \( \psi : X \times I \rightarrow X \times I \) to the top part of the boundary \( \psi|_{X \times \{1\}} \). This gives a corresponding fiber sequence of the classifying spaces

\[ B\text{Diff}_\delta(X \times I) \xrightarrow{i} B\text{Diff}_\cup(X \times I) \xrightarrow{\partial} B\text{Diff}_\delta(X \times \{1\}). \tag{2} \]

Remark 1.2. The group of pseudoisotopies \( \text{Diff}_\cup(X \times I) \) is often denoted as \( C_\delta(X) \).

The first main result of this paper is the following.

**Theorem 1.3** (Theorem 2.3). Let \( d \geq 4 \). All the elements given by surgery on trivalent graphs with \( 2k \) vertices, \( k \geq 1 \), are in the image of the homomorphism

\[ \partial_* : \pi_{k(d-3)}B\text{Diff}_\cup(X \times I) \rightarrow \pi_{k(d-3)}B\text{Diff}_\delta(X). \]
Let \( m := 1 \) if \( d \) even and \( m := (d - 1)/2 \) for \( d \) odd. Furthermore, each element in the group \( \pi_{k(d-3)}BDiff_{\partial}(X) \) constructed by surgery on a trivalent graph embedded in \( X \) has a lift in the group \( \pi_{k(d-3)}BDiff_{\partial}(X \times I) \) represented by a smooth \((X \times I)\)-bundle over \( S^{k(d-3)} \) that admits a fiberwise handle decomposition with a single handle pair in each fiber, of indices \( m \) and \( m + 1 \). Moreover, each pair is geometrically canceling (up to an appropriate isotopy).

Remark 1.4. The reader who is familiar with Cerf’s graphic ([3, 8, 9], etc.) would find that this condition is equivalent to that the family admits a graphic consisting of a single ‘lens’.

Remark 1.5. A version of this theorem for bordism group was pointed out to the second author by Peter Teichner [22, Theorem 9.3]. We would like to emphasize the following new features in Theorem 1.3.

(1) The crucial feature that will be used in our applications to rational homotopy groups of the moduli spaces of metrics of positive scalar curvature is that our handlebodies are built using a single \( m \)-handle and an \( S^{k(d-3)} \)-family of \( (m + 1) \)-handles that geometrically cancel it (up to an appropriate isotopy).

(2) We describe in this paper the details about the interpretation of the graph surgery in terms of spherical modifications along framed Hopf links, which were sketched in [22, §9]. This could also be applied to constructions of families of embeddings in a manifold.

Corollary 1.6. Let \( d \geq 4 \) and \( k \geq 2 \). If \( d \) is even (respectively, if \( d \) is odd), then \( \pi_{k(d-3)}BDiff_{\partial}(D^d \times I) \otimes \mathbb{Q} \) is non-trivial whenever \( \mathscr{A}_{k}^{\text{even}} \) (respectively, \( \mathscr{A}_{k}^{\text{odd}} \)) is non-trivial. For \( k = 1 \), \( \pi_{2n-2}BDiff_{\partial}(D^{2n+1} \times I) \otimes \mathbb{Q} \) is non-trivial for many odd integers \( d = 2n + 1 \geq 5 \) satisfying the same technical condition as in Theorem 1.1.

Remark 1.7.

(1) Note that this includes results for pseudoisotopies of \( D^4 \). It was proved in [20] that \( \pi_2BDiff_{\partial}(D^4 \times I) \otimes \mathbb{Q} \) is non-zero. Theorem 1.3 shows that \( \pi_kBDiff_{\partial}(D^d \times I) \otimes \mathbb{Q} \) are non-trivial for many \( k > 2 \). This is new result.

(2) Recently, A. Kupers and O. Randal-Williams ([14]) and M. Krannich and O. Randal-Williams ([13]) computed the rational homotopy groups of \( BDiff_{\partial}(D^d) \) in some wide range of dimensions surprisingly completely. In particular, it follows from their results that for \( n > 5 \), the natural map

\[
\pi_{2n-2}BDiff_{\partial}(D^{2n+1} \times I) \otimes \mathbb{Q} \to \pi_{2n-2}BDiff_{\partial}(D^{2n+1}) \otimes \mathbb{Q}
\]

is an isomorphism and both terms are isomorphic to \( \mathbb{Q} \oplus (K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q}) \). In particular, Corollary 1.6 for \( d = 2n + 1 > 11 \) and \( k = 1 \) follows from their results.

1.2 Application to the moduli space of psc-metrics

Let \( h_{0} \) be the standard round metric on \( S^{d-1} = \partial D^{d} \), and \( \mathcal{R}_{\partial}(D^{d})_{h_{0}} \) be the space of Riemannian metrics \( g \) on the disk \( D^{d} \) which have a form \( h_{0} + dt^{2} \) near the boundary \( S^{d-1} \). The group \( Diff_{\partial}(D^{d}) \) acts on \( \mathcal{R}_{\partial}(D^{d})_{h_{0}} \) by pulling a metric back: \( g \cdot \phi \mapsto \phi^{*}g \). It is easy to see that this action is free,
and, since the space $\mathcal{R}_\partial(D^d)_{h_0}$ is contractible, there is a homotopy equivalence

$$BDiff_\partial(D^d) \sim \mathcal{M}_\partial(D^d)_{h_0} := \mathcal{R}_\partial(D^d)_{h_0}/Diff_\partial(D^d).$$

Thus the moduli space $\mathcal{M}_\partial(D^d)_{h_0}$ could be thought as a geometrical model of the classifying space $BDiff_\partial(D^d)$. Below we identify the spaces $\mathcal{M}_\partial(D^d)_{h_0}$ and $BDiff_\partial(D^d)$. Let $\mathcal{R}^{\text{psc}}(D^d)_{h_0} \subset \mathcal{R}_\partial(D^d)_{h_0}$ be a subspace of metrics with positive scalar curvature (which will abbreviated as ‘psc-metrics’). We have the following diagram of principal $Diff_\partial(D^d)$-fiber bundles:

$$\begin{array}{ccc}
\mathcal{R}^{\text{psc}}(D^d)_{h_0} & \overset{i}{\to} & \mathcal{R}_\partial(D^d)_{h_0} \\
p \downarrow & & \downarrow \bar{p} \\
\mathcal{M}^{\text{psc}}(D^d)_{h_0} & \overset{i}{\to} & \mathcal{M}_\partial(D^d)_{h_0},
\end{array}$$

Here $\mathcal{M}^{\text{psc}}(D^d)_{h_0} := \mathcal{R}^{\text{psc}}(D^d)_{h_0}/Diff_\partial(D^d)$ is the moduli space of psc-metrics.

**Theorem 1.8.** Let $d \geq 4$ be an integer. All classes given by surgery on trivalent graphs are in the image of the induced map

$$\iota_* : \pi_q \mathcal{M}^{\text{psc}}(D^d)_{h_0} \otimes \mathbb{Q} \to \pi_q BDiff_\partial(D^d) \otimes \mathbb{Q}.$$ 

Hence, all non-trivial elements of $\pi_q BDiff_\partial(D^d) \otimes \mathbb{Q}$ given by surgery on trivalent graphs lift to non-trivial elements of $\pi_q \mathcal{M}^{\text{psc}}(D^d)_{h_0} \otimes \mathbb{Q}$.

**Remark 1.9.** For $d \geq 6$, Theorem 1.8 follows also from [4, Theorem F]. We give a geometrical proof of Theorem 1.10 which, in particular, proves Theorem 1.8. In fact, our proof proves a stronger statement for the existence of the lift in the moduli space. Namely, all classes given by surgery on trivalent graphs are in the image of the induced map $\iota_* : \pi_q \mathcal{M}^{\text{psc}}(X)_{h_0} \to \pi_q BDiff_\partial(X)$ for an arbitrary smooth manifold $X$ of dimension $d \geq 4$ having a psc metric $h_0$. For $d = 4$, Theorem 1.8 is in contrast to that the classes in $\pi_q BDiff(X)$ detected by Seiberg–Witten theory do not admit fiberwise psc-metrics [11, 17].

Next, we fix some geometrical data. Consider the subset $(D^d \times \{0\}) \cup (S^{d-1} \times I) \subset D^d \times I$ and fix a psc-metric $g_0 \in \mathcal{R}^{\text{psc}}(D^d \times \{0\})_{h_0}$. We view the cylinder $D^d \times I$ as a manifold with corners. Let $U$ be a collar of $(D^d \times \{0\}) \cup (S^{d-1} \times I)$; we assume that $U$ is parametrized by $(x, t, s)$ near the corner $S^{d-1} \times \{0\}$, as it is shown in Figure 2, where $x \in S^{d-1} \times \{0\}$.

We consider a subspace $\mathcal{R}_\cup(D^d \times I; g_0) \subset \mathcal{R}(D^d \times I)$ of Riemannian metrics $\tilde{g}$ which restrict to

$$\begin{cases}
g_0 + ds^2 & \text{near } D^d \times \{0\} \\
h_0 + ds^2 + dt^2 & \text{near } S^{d-1} \times I \\
g + ds^2 & \text{near } D^d \times \{1\} \text{ for some } g \in \mathcal{R}(D^d \times \{1\})_{h_0},
\end{cases}$$

(3)
Let $\mathcal{R}^{\text{psc}}(D^d \times I; g_0)_{h_0} \subset \mathcal{R}(D^d \times I; g_0)_{h_0}$ be a corresponding subspace of psc-metrics. Again, we notice that the group $\text{Diff}(D^d \times I)$ acts freely on a contractible space $\mathcal{R}(D^d \times I; g_0)_{h_0}$. In particular, we have homotopy equivalence

$$BDiff(D^d \times I) \sim \mathcal{M}(D^d \times I; g_0)_{h_0} := \mathcal{R}(D^d \times I; g_0)_{h_0} / \text{Diff}(D^d \times I).$$

Again we have the following diagram of principal $\text{Diff}(D^d \times I)$-fiber bundles:

$$\xymatrix{
\mathcal{R}^{\text{psc}}(D^d \times I; g_0)_{h_0} \ar[r]^i \ar[d]^p & \mathcal{R}(D^d \times I; g_0)_{h_0} \ar[d]^p \\
\mathcal{M}^{\text{psc}}(D^d \times I; g_0)_{h_0} \ar[r] & \mathcal{M}(D^d \times I; g_0)_{h_0}.
}$$

We also notice that the restriction map

$$\mathcal{R}^{\text{psc}}(D^d \times I; g_0)_{h_0} \to \mathcal{R}^{\text{psc}}_2(D^d)_{h_0}, \quad \tilde{g} \mapsto g = \tilde{g}|_{D^d \times \{1\}},$$

where $g$ is given in (3), induces a map of corresponding moduli spaces:

$$\vartheta^{\text{psc}} : \mathcal{M}^{\text{psc}}(D^d \times I; g_0)_{h_0} \to \mathcal{M}^{\text{psc}}_2(D^d)_{h_0}.$$

**Theorem 1.10.** Let $d \geq 4$ be an integer. All lifts in $\pi_q \mathcal{M}^{\text{psc}}(D^d)_{h_0} \otimes \mathbb{Q}$ found in Theorem 1.8 are in the image of the homomorphism

$$\vartheta^{\text{psc}}_* : \pi_{k(d-3)+m} \mathcal{M}^{\text{psc}}(D^d \times I; g_0)_{h_0} \otimes \mathbb{Q} \to \pi_{k(d-3)+m} \mathcal{M}^{\text{psc}}_2(D^d)_{h_0} \otimes \mathbb{Q}.$$

Hence, any non-trivial elements of $\pi_k \text{Diff}_d(D^d) \otimes \mathbb{Q}$ given by surgery on trivalent graphs lift to non-trivial elements of $\pi_{k(d-3)} \mathcal{M}^{\text{psc}}(D^d \times I; g_0)_{h_0} \otimes \mathbb{Q}$.

### 1.3 Conventions

- A $(W, \partial W)$-bundle is a smooth $W$-bundle with structure group $\text{Diff}_3(W)$. We say that two $(W, \partial W)$-bundles $\pi_i : E_i \to B$ $(i = 0, 1)$ are concordant if the disjoint union, the $(W, \partial W)$-
bundle \( \pi_0 \sqcup \pi_1 : E_0 \sqcup E_1 \rightarrow B \sqcup B \) over disjoint copies of \( B \), is the restriction of a \((W, \partial W)\)-bundle \( \tilde{\pi} : \tilde{E} \rightarrow [0,1] \times B \) to \( \tilde{\pi}^{-1}([0,1] \times B) \). If \( W = X \times I \) for a compact manifold \( X \), we denote by \( \partial_\cup W = X \times \{0\} \cup \partial X \times I \). Then a \((W, \partial_\cup W)\)-bundle is a smooth \( W \)-bundle with structure group \( \text{Diff}_\cup(W) \). Concordance between two \((W, \partial_\cup W)\)-bundles is defined similarly as above.

- A **framed embedding** (or a framed link) consists of an embedding \( \varphi : S \rightarrow X \) between smooth manifolds and a choice of a normal framing \( \tau \) on \( \varphi(S) \), where by a normal framing we mean a trivialization \( \nu(\varphi(S)) \cong \varphi(S) \times \mathbb{R}^{\text{codim} \varphi(S)} \) of the normal bundle.
- We will often say ‘a framed embedding \( \varphi \)’ or ‘a framed link \( \varphi \)’, instead of \((\varphi, \tau)\).
- We consider links as submanifolds equipped with parametrizations. Thus in this paper links are embeddings. Also, we assume that families of links are smoothly parametrized.
- We will consider **trivialities** of families or bundles in several different meanings. Instead of saying just ‘trivial bundle’, we will say that a bundle/family is **trivialized** if it is equipped with a trivialization. If it admits at least one trivialization, we say that it is **trivializable**. A given family \( \{\varphi_s\} \) of some objects \( \varphi_s \) is **strictly trivial** if \( \varphi_s \) does not depend on \( s \), that is, \( \varphi_s = \varphi_{s_0} \) for some \( s_0 \). It seems usual to say that a bundle is trivial if it is trivializable.

## 2 | GRAPH SURGERY

We take an embedding \( \Gamma \rightarrow \text{Int} X \) of a labeled, edge-oriented trivalent graph \( \Gamma \). We put a Hopf link of the spheres \( S^{d-2} \) and \( S^1 \) at the middle of each edge, as in Figure 3.

Then every vertex of \( \Gamma \) gives a \( Y \)-shaped component \( Y\text{-graph} \) of Type I or and II, see Figure 4 below, that is, an \( Y \)-graph is a vertex together with framed spheres \( S^{d-2} \) and \( S^1 \).
attached. We call the attached spheres *leaves* of a Y-graph. This construction transforms the graph $\Gamma$ into $2k$ components Y-graphs. We take small closed tubular neighborhoods of those Y-graphs, namely, the disjoint union of the $\epsilon$-tubular neighborhoods of the leaves and the trivalent vertex (a point) connected by $\epsilon/2$-tubular neighborhoods of the edges for some small $\epsilon$, and denote them by $V^{(1)}, V^{(2)}, ..., V^{(2k)}$. They form a disjoint union of handlebodies embedded in $\text{Int} X$. A Type I Y-graph gives a handlebody (of a Type I) which is diffeomorphic to the handlebody obtained from a $d$-ball by attaching two 1-handles and one $(d-2)$-handle in a standard way, namely, along unknotted unlinked standard attaching spheres in the boundary of $D^d$. A Type II Y-graph gives a handlebody (of a Type II) which is diffeomorphic to the handlebody obtained from a $d$-ball by attaching one 1-handle and two $(d-2)$-handles in a standard way.

Let $V = V^{(i)}$ be one of the Type I handlebodies and let $\alpha_I : S^0 \to \text{Diff}(\partial V), S^0 = \{-1, 1\}$, be the map defined by $\alpha_i(-1) = \emptyset$, and by setting $\alpha_i(1)$ as the ‘Borromean twist’ corresponding to the Borromean string link $D^d-2 \cup D^d-2 \cup D^1 \hookrightarrow D^d$. The detailed definition of $\alpha_I$ can be found in [20, §4.5].

Let $V = V^{(i)}$ be one of the Type II handlebodies and let $\alpha_{II} : S^{d-3} \to \text{Diff}(\partial V)$ be the map defined by comparing the trivializations of the family of complements of an $S^{d-3}$-family of embeddings $D^{d-2} \cup D^1 \cup D^1 \hookrightarrow D^d$ obtained by parametrizing the second component in the Borromean string link $D^{d-2} \cup D^{d-2} \cup D^1 \hookrightarrow D^d$ with that of the trivial family of $\partial V$. The detailed definition of $\alpha_{II}$ can be found in [20, §4.6].

The Borromean string link (see Figure 5) has the following important property, which will be used later.

Property 2.1. If one of the three components in the Borromean string link $D^{d-2} \cup D^{d-2} \cup D^1 \hookrightarrow D^d$ is deleted, then the string link given by the remaining two components is isotopic relative to the boundary to the standard inclusion of disks.

For each $i$th vertex of $\Gamma$ we let $K_i = S^0$ or $S^{d-3}$ depending on whether this vertex is of Type I or II. Accordingly, let $\alpha_i : K_i \to \text{Diff}(\partial V^{(i)})$ be $\alpha_I$ or $\alpha_{II}$. Let $B_\Gamma = K_1 \times \cdots \times K_{2k}$. By using the
families of twists above, we define

\[ E^\Gamma = (B^\Gamma \times (X - \text{Int} (V(1) \cup \cdots \cup V(2k)))) \cup_\partial (B^\Gamma \times (V(1) \cup \cdots \cup V(2k))), \]

where the gluing map is given by

\[ \psi : B^\Gamma \times (\partial V(1) \cup \cdots \cup \partial V(2k)) \to B^\Gamma \times (\partial V(1) \cup \cdots \cup \partial V(2k)) \]

\[ \psi(t_1, \ldots, t_{2k}, x) = (t_1, \ldots, t_{2k}, \alpha_i(t_i)(x)) \quad (\text{for } x \in \partial V(i)). \]

**Proposition 2.2** [20]. Let \( X \) be a \( d \)-dimensional compact manifold having a framing \( \tau_0 \). The natural projection \( \pi^\Gamma : E^\Gamma \to B^\Gamma \) is an \((X, \partial)\)-bundle, and it admits a vertical framing that is compatible with the surgery and that agrees with \( \tau_0 \) near the boundary, and it gives an element of

\[ \Omega^{SO}_{(d-3)k}(\text{BDiff}(X, \partial)), \]

where \( \text{BDiff}(X, \partial) \) is the classifying space for framed \((X, \partial)\)-bundles.

The following is a most precise statement of our first main result, Theorem 1.3.

**Theorem 2.3.** Let \( d \geq 4 \) be an integer. Let \((m_1, m_2) = (1, d-2)\) if \( d \) is even and let \( m_1 = m_2 = (d-1)/2 \) if \( d \) is odd.

1. The \((X, \partial)\)-bundle \( \pi^\Gamma : E^\Gamma \to B^\Gamma \) for an embedding \( \phi : \Gamma \to \text{Int} X \) is related by an \((X, \partial)\)-bundle bordism to an \((X, \partial)\)-bundle \( \varpi^\Gamma : E^\Gamma \to S^{k(d-3)} \) obtained from the product bundle

\[ S^{k(d-3)} \times X \to S^{k(d-3)} \]

by fiberwise surgeries along a \( S^{k(d-3)} \)-family of framed links \( h_s : S^{m_1} \cup S^{m_2} \to \text{Int} X, s \in S^{k(d-3)}, \) which satisfies the following conditions.

(a) \( h_s \) is isotopic to the Hopf link for each \( s \).

(b) The restriction of \( h_s \) to \( S^{m_2} \) is a constant \( S^{k(d-3)} \)-family.

(c) There is a small neighborhood \( N \) of \( \text{Im} \phi \) such that the image of \( h_s \) is included in \( N \) for all \( s \in S^{k(d-3)} \).

2. There exists an \((X \times I, \partial_\infty (X \times I))\)-bundle \( \Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)} \) such that

(a) the fiberwise restriction of \( \Pi^\Gamma \) to \( X \times \{1\} \) is \( \varpi^\Gamma_h \),

(b) the manifold \( W^\Gamma_h \) is obtained by attaching \( S^{k(d-3)} \)-families of \((d+1)\)-dimensional \( m_1 \)- and \((m_1+1)\)-handles to the product \((X \times I)\)-bundle \( S^{k(d-3)} \times (X \times I) \to S^{k(d-3)} \) at the top part of the cylinder \( S^{k(d-3)} \times (X \times \{1\}) \).

For concreteness, we prove Theorem 2.3 in the case of even \( d \) in Corollary 5.4 and Proposition 5.5. The case of odd \( d \) is completely analogous, with the following replacements.

- Y-surgery is given by an \( S^{m-1} \)-family of embeddings \( D^m \cup D^m \cup D^m \to D^{2m+1} \) obtained by parametrizing a Borromean string link \( D^{2m-1} \cup D^m \cup D^m \to D^{2m+1} \). We take this at each trivalent vertex, so \( B^\Gamma = S^{m-1} \times S^{m-1} \times \cdots \times S^{m-1} \) (2k factors).
- In the proof of an analog of Proposition 5.5, we replace a trivial family of \((m+1)\)-handles with that of \( m \)-handles.
3 | ALTERNATIVE DEFINITION OF Y-SURGERY BY FRAMED LINKS

3.1 | Framed link for Type I surgery

Let $d \geq 4$. Let $K_1, K_2, K_3$ be the unknotted spheres in $\text{Int} X$ that are parallel to the cores of the handles of Type I handlebody $V$ of indices $1, 1, d - 2$, respectively. Let $c_i$ be a small unknotted sphere in $\text{Int} V$ that links with $K_i$ with the linking number 1. Let $L'_1 \cup L'_2 \cup L'_3$ be a Borromean rings of dimensions $d - 2, d - 2, 1$ embedded in a small ball in $\text{Int} V$ that is disjoint from

$$K_1 \cup K_2 \cup K_3 \cup c_1 \cup c_2 \cup c_3.$$

For each $i = 1, 2, 3$, let $L_i$ be a knotted sphere in $\text{Int} V$ obtained by connect summing $c_i$ and $L'_i$ along an embedded arc that is disjoint from the cocores of the 1-handles and from other components, so that functions of $L_i$ are mutually disjoint. Then $K_i \cup L_i$ is a Hopf link in $d$-dimension. If $K_i$ is null in $X$ for $i = 1, 2, 3$, namely, $K_i$ bounds an embedded disk in $X$, then each component of the six-component link $\bigcup_{i=1}^3 (K_i \cup L_i)$ is an unknot in $X$, and we may consider it as a framed link by canonical framings induced from the standard sphere by the isotopies along the spanning disks (Figure 6, $V^{(1)}$). The following is a framed link definition of Type I surgery.

**Definition 3.1** (Y-surgery of Type I). We define the Type I surgery on $V$ to be the surgery along the six-component framed link $\bigcup_{i=1}^3 (K_i \cup L_i)$ in $V$.

We will see in Section 3.4 (Remark 3.8) that this definition is equivalent to that we have given in Section 2. For $d = 3$, this equivalence was proven by S. Matveev in [15]. Our proof is an analog of [7, §2] or [5, Lemma 2.1].
3.2 Family of framed links for Type II surgery

Similarly, let $K_1, K_2, K_3$ be the unknotted spheres in $\text{Int} X$ that are parallel to the cores of the handles of Type II handlebody $V$ of indices $1, d - 2, d - 2$, respectively. Let $c_i$ be a small framed unknotted sphere in $\text{Int} V$ that links with $K_i$ with linking number 1. Let $L'_{1,s} \cup L_{2,s} \cup L'_{3,s}$ (where $s \in S^{d-3}$) be a $(d - 3)$-parameter family of three component framed links of dimensions $d - 2, 1, 1$ with only (isotopically) unknotted components embedded in a small ball in $\text{Int} V$ disjoint from $K_1 \cup K_2 \cup K_3 \cup c_1 \cup c_2 \cup c_3$ such that $L'_{1,s}, L'_{2,s}, L'_{3,s}$ are unknotted components in $V$ that do not depend on $s$, and the union of the locus of $L'_{2,s}$ and $L'_{1,s} \cup L'_{3,s}$ forms a closure of the Borromean string link of dimensions $d - 2, d - 2, 1$.

For each $i = 1, 2, 3$, let $L_i,s$ be a knotted sphere in $\text{Int} V$ obtained by connect summing $c_i$ and $L'_{i,s}$ along an embedded arc that is disjoint from the cocores of the 1-handles and from other components, so that functions of $L_{i,s}$ are mutually disjoint. Then $K_i \cup L_{i,s}$ is a Hopf link in $d$-dimension. If $K_i$ is null in $X$ for $i = 1, 2, 3$, then each component of the six-component link $\bigcup_{i=1}^{3} (K_i \cup L_{i,s})$ is fiberwise isotopic to a constant family of an unknotted in $X$, and we may consider it as a family of framed links by canonical framings (Figure 6, $V^{(2)}$). The following is a framed link definition of Type II surgery.

**Definition 3.2 (Y-surgery of Type II).** We define the Type II surgery on $V$ to be the $S^{d-3}$-family of surgeries along the family of the six-component framed link $\bigcup_{i=1}^{3} (K_i \cup L_{i,s})$, $s \in S^{d-3}$, in $V$, which produces a $(V, \partial)$-bundle over $S^{d-3}$.

We will see in Section 3.5 (Remark 3.11) that this definition is equivalent to that we have given in Section 2.

3.3 Hopf link surgery for links

We would like to describe the effect of a Y-surgery of Type I or II when a link in the complement of the Y-graph is present. Since a Y-surgery consists of surgeries of three Hopf links, we shall first consider the effect of a single Hopf link surgery.

3.3.1 Surgery of $X$ on a framed link $L$

We shall recall the definition of surgery on a framed link $L$ in $\text{Int} X$. Let $W = X \times I$ and let $W^L$ be a $(d + 1)$-dimensional cobordism obtained from $W$ by attaching disjoint handles along $L \times \{1\}$ in $X \times \{1\}$. In more detail, for each framed embedding $\ell : S^1 \hookrightarrow L \times \{1\}$, we attach $(d + 1)$-dimensional $(1 + 1)$-handle along a small tubular neighborhood of $\ell$. The handle attachments can be done disjointly and simultaneously, and gives a $(d + 1)$-dimensional cobordism $W^L$ between $X \times \{0\}$ and some $d$-manifold $X^L$, see Figure 7. We say that $W^L$ is obtained from $X \times I$ by surgery along $L$, or by attaching handles along $L$. Let $\partial_- W^L = X \times \{0\} \cup \partial X \times I$, $\partial_+ W^L = X \times \{0\}$, and $\partial_+ W^L = \partial_- W^L \setminus \text{Int} \partial_- W^L = X^L$. Here, $W$ may be more general cobordism. Namely, let $W$ be a relative cobordism between $\partial_- W = X \times \{0\}$ and some another manifold $\partial_+ W$ such that

$$\partial W = \partial_+ W \cup_{\partial X \times \{1\}} (\partial X \times I) \cup_{\partial X \times \{0\}} \partial_- W.$$

(4)
Let $\partial_+ W = (\partial X \times I) \cup_{\partial X \times \{0\}} \partial_- W$. For a framed link $L$ in $\partial_+ W$, we define the manifold $W^L$ as a $(d + 1)$-dimensional cobordism obtained from $W$ by attaching disjoint handles along $L$ in $\partial_+ W$. A $(W, \partial_+ W)$-bundle is defined as a $W$-bundle with structure group $\text{Diff}(W, \partial_+ W)$. Concordance between two $(W, \partial_+ W)$-bundles can be defined similarly as for $W = X \times I$.

### 3.3.2 Concordance of a cobordism

When a link $c$ in $X$ is present, surgery on a framed link $L$ in $X \setminus c$ changes the pair $(X, c)$. It may happen that surgeries for two choices $(c, L)$ and $(c', L')$ of the links in $X$ should be considered equivalent. Here, we consider the notion of concordance between two such data, defined as follows.

**Definition 3.3.** Let $W$ be a relative cobordisms between $\partial_+ W = X \times \{0\}$ and $\partial_+ W$ satisfying (4) above, and let $W'$ be another such relative cobordism such that $W$ and $W'$ are concordant as $(W, \partial_+ W)$-bundles over points. For framed links $L$ and $L'$ in $\partial_+ W$ and $\partial_+ W'$, respectively, we say that the pairs $(W, L)$ and $(W', L')$ are concordant if $L \sqcup L'$ is the restriction of a trivialized (fiberwise normally) framed subbundle $\tilde{L}$ of the top boundary of a concordance between the cobordisms $W$ and $W'$, see Figure 8.

**Remark 3.4.** The definition of concordance for manifold pair is not as usual. Usually, the projection $\text{proj}_q|_{\tilde{L}} : \tilde{L} \to I$ for the concordance $\tilde{L}$ may not be level-preserving, whereas we assume so. It is evident from definition that if $(W, L)$ and $(W', L')$ are concordant, the cobordisms $W^L$ and $W'^{L'}$ are concordant.
3.3.3 Hopf link surgery for links

Suppose that a $d$-manifold $X$ is equipped with some embedded objects inside, such as links or Y-links. By a small Hopf link in $X$, we mean a Hopf link in a $d$-ball $b$ in $X$ with sufficiently small radius so that $b$ is disjoint from the given embedded objects in $X$.

Let $K, L$ be the components of a Hopf link in $\text{Int} X$ of dimensions $1, d-2$ with standard framing and with spanning disks $d_1, d_2$ in $\text{Int} X$, respectively. Let $c_1, c_2$ be framed spheres of dimensions $d-2, 1$, respectively, in $\text{Int} X$ such that $d_1$ (respectively, $d_2$) intersects $c_1$ (respectively, $c_2$) transversally by one point and does not intersect other component in $c_1 \cup c_2$ nor $K \cup L$ (see Figure 9, left). Let $N_{d_1 \cup d_2}$ be a small closed neighborhood of $d_1 \cup d_2$. Let $c_1' \cup c_2'$ be a framed link in $\text{Int} X$ obtained from $c_1 \cup c_2$ by component-wise connect-summing a small Hopf link in $N_{d_1 \cup d_2}$. Let $K' \cup L'$ be another framed Hopf link in $N_{d_1 \cup d_2}$ that is small and disjoint from $c_1' \cup c_2'$ (See Figure 9, right.)

The following lemma is an analog of [7, Proposition 2.2].

**Lemma 3.5** (Hopf link surgery). Let $W = X \times [a, b]$, where we identify $X \times \{b\}$ with $X$. Then the pairs $(W^{K \cup L}, c_1 \cup c_2)$ and $(W^{K' \cup L'}, c_1' \cup c_2')$ are concordant. Moreover, we may assume that the concordance is strictly trivial on $(X \setminus \text{Int} N_{d_1 \cup d_2}) \times [a, b]$.

This lemma can also be applied to a general cobordism $W$ with $\partial_+ W = X$ by considering a collar neighborhood of $\partial_+ W$ as $X \times [a, b]$.

**Proof.** Let $L_1 = L$ and $L_2 = K$. Before going to the proof, we define band-sums $c_i \# L_i$. We choose an embedded path $\gamma_i$ in $d_i$ that goes from $d_i \cap c_i$ to $d_i \cap L_i$. Then we may connect-sum $c_i$ with $L_i$ along $\gamma_i$ so that the result is disjoint from $d_i$. More precisely, the restriction of the normal bundle of $d_i$ on $\gamma_i$ is an $\mathbb{R}^{\dim c_i}$-bundle. Thus $\gamma_i$ can be thickened to a $\dim c_i$-disk bundle in $N_{d_1 \cup d_2}$ that is perpendicular to $d_i$ and its restriction on the endpoints is $\dim c_i$-disks in $c_i$ and $L_i$. The disk bundle is a $(\dim c_i + 1)$-dimensional 1-handle attached to $c_i \cup L_i$ along which surgery can be performed. This surgery produces the connected sum $c_i \# L_i$ along $\gamma_i$ whose result is disjoint from $d_i$. See Figure 10, left.
Now let us return to the framed link $K \cup L \cup c_1 \cup c_2$. We perform surgeries on the link $K \cup L = L_2 \cup L_1$, and then the component $c_i$ can be slid over $\gamma_i$ and the $(\dim L_i + 1)$-handle attached to $L_i$. The result of the handle slide is $c_i \# \overline{L_i}$ defined as in the previous paragraph, where $\overline{L_i}$ is a parallel copy of $L_i$ obtained from $L_i$ by slightly pushing off by one direction of the framing on $L_i$. We denote by $c_i''$ the resulting framed sphere $c_i \# \overline{L_i}$. We assume that $c_i'' \setminus c_i$ is included in $N_{d_1 \cup d_2}$ and agrees with $c_i$ outside $N_{d_1 \cup d_2}$. The link $c_i'' \cup c_2''$ is obtained from $c_1 \cup c_2$ by component-wise connect-summing Hopf links in $N_{d_1 \cup d_2}$, which is realized by handle slides. Thus

\[
(W^{K \cup L}, c_1 \cup c_2) \text{ and } (W^{K \cup L}, c_1'' \cup c_2'')
\]

are concordant.

We need to show that $(c_1'' \cup c_2'') \cup (K \cup L)$ is isotopic in $N_{d_1 \cup d_2}$ to $(c_1' \cup c_2') \cup (K' \cup L')$. Since $c_1''$ is disjoint from $d_1$, the component $K$ can be shrunk along $d_1$ to a small sphere $K'$ in a small $d$-disk $b$ around the point $d_1 \cap L$ without intersecting $c_1'' \cup c_2''$ as in Figure 10. Similarly, since $c_2''$ is disjoint from $d_2$, the component $L$ can be shrunk along $d_2$ to a small sphere $L'$ in $b$, without intersecting $c_1'' \cup c_2''$, so that $K' \cup L'$ is a small Hopf link in $b$.

Then similar isotopy can be performed for $c_1'' \cup c_2''$ in $N_{d_1 \cup d_2}$ so that the part parallel to $L_1$ and $L_2$ is shrunk to a small Hopf link with bands. We may assume that this isotopy is disjoint from $b$. The result of the deformation is $(c_1' \cup c_2') \cup (K' \cup L')$. Thus, the deformations performed so far give a desired concordance.

Remark 3.6. The framed Hopf link $K \cup L$ may be replaced by some ‘smooth family’ of framed Hopf links. More precisely, let $K_s \cup L_s$ be a smooth family of framed Hopf links parametrized over a compact connected manifold $B$ with a base point $s_0$, such that

(a) $L_s = L$, and hence $L_s$ bounds $d_2$,
(b) $d_2$ intersects 1-dimensional arc in $c_2$ transversally by one point,
(c) $K_s$ bounds a smooth family of disks $d_{1,s}$ in $\text{Int} X$ such that for each $s$, $L_s$ intersects $d_{1,s}$ transversally by one point, and $K_s$ intersects $d_s$ transversally by one point,
(d) $d_{1,s}$ and $d_{1,s_0}$ agree on a neighborhood of the arc $d_{1,s_0} \cap d_2$.

Then surgery on the family $K_s \cup L_s$ gives a family of cobordisms that is concordant (in the sense of Definition 3.9) to the strictly trivial family of cobordisms with a non-trivial family of spheres $c_2', s$ on the top, which is obtained from $c_2$ by connected-summing with parallel copies of $K_s$, see Figure 11.

For example, if $B = S^{d-3}$, then the family $K_s$ may be chosen so that the associated map $B \times D^2 \to B \times \text{Int} X$ for the spanning disks $d_{1,s}$ intersects $B \times c_1$ transversally by one point in

![Figure 11](image-url)
B × IntX). Such a family of framed Hopf links surgery will play an important role in the framed link description of the Type II surgery in Lemma 3.10.

3.4 Type I Y-surgery for links

We say that a leaf $\ell$ of a Y-graph $T$ is simple relative to a submanifold $c$ in IntX with $\dim \ell + \dim c = d - 1$, if the following conditions are satisfied.

(1) The leaf $\ell$ bounds a disk $m$ in IntX.
(2) The disk $m$ intersects $c$ transversally by one point.

See Figure 12(a). We say that a Y-graph $T$ with leaves $\ell_1, \ell_2, \ell_3$ is simple relative to a three-component link $c_1 \cup c_2 \cup c_3$ in IntX with $\dim \ell_i + \dim c_i = d - 1$, if the following conditions are satisfied.

(1) The leaves $\ell_1, \ell_2, \ell_3$ bound disjoint disks $m_1, m_2, m_3$ in IntX, respectively.
(2) For each $i$, the disk $m_i$ intersects $c_i$ transversally by one point and does not intersect other components in $c_1 \cup c_2 \cup c_3$.

See Figure 12(b). In this case, we take a small closed neighborhood of $T \cup m_1 \cup m_2 \cup m_3$ that is a $d$-disk and denote it by $N(T)$.

Let $L_T$ be the framed link associated to $T$, as in Definition 3.1. We define $W^T$ as $W^{L_T}$ in the sense of Section 3.3.1.

Lemma 3.7 (Type I surgery). Let $W = X \times [a, b]$, where we identify $X \times \{b\}$ with $X$. Suppose that the leaves of a Y-graph $T$ of Type I in IntX of dimensions 1, 1, $d - 2$ are linked to framed submanifolds $c_1, c_2, c_3$ of dimensions $d - 2, d - 2, 1$, respectively, and that $T$ is simple relative to $c_1 \cup c_2 \cup c_3$. Let $c_1' \cup c_2' \cup c_3'$ be a framed link that is obtained from $c_1 \cup c_2 \cup c_3$ by component-wise connect-summing Borromean rings in $N(T)$.

(1) There are three disjoint small Hopf links $h_1, h_2, h_3$ in $N(T) \setminus (c_1' \cup c_2' \cup c_3')$ and a concordance between the pairs $(W^T, c_1 \cup c_2 \cup c_3)$ and $(W^{h_1 \cup h_2 \cup h_3}, c_1' \cup c_2' \cup c_3')$ that is strictly trivial on the complement $(X \setminus \text{Int} N(T)) \times [a, b]$.
(2) Moreover, if we consider up to isotopy, we may assume that two of the components of the link $c_1' \cup c_2' \cup c_3'$ agree as subsets of IntX with those of $c_1 \cup c_2 \cup c_3$. 

![Figure 12](image-url) Simple Y-graphs.
Remark 3.8. Lemma 3.7 shows that the two definitions of Type I surgeries: ‘the complement of thickened string link’ given in Section 2 and ‘framed link surgery’ in Definition 3.1 are equivalent. Namely, let $L$ be the six-component framed link of Definition 3.1 in $V$. Then the latter definition is given by surgery along $L$ in $V$. According to Lemma 3.7 and if we consider modulo small Hopf links, this surgery replaces $V$ with another one that is obtained by taking the complements of the Borromean string link. The relative diffeomorphism type of the resulting manifold is determined uniquely and agrees with the former definition of Type I surgery.

Proof of Lemma 3.7. Lemma 3.7 is obtained by iterated applications of the concordance deformations of Lemma 3.5. Namely, by Definition 3.1, the surgery on $T$ is given by the surgery on a six-component framed link $\bigcup_{i=1}^3(K_i \cup L_i)$. Since $T$ is simple relative to $c_1 \cup c_2 \cup c_3$, the component $K_i$ bounds a disk $d_i$ in $N(T)$. After relabeling if necessary, we may assume that for each $i$, the intersections $d_i \cap c_i$ and $d_i \cap L_i$ are both one point and orthogonal, and $d_i$ does not have other intersections with other link components. (See Figure 13d.) We choose an embedded path $\gamma_i$ in $d_i$ that goes from $d_i \cap c_i$ to $d_i \cap L_i$. Then we may define the band sum $c_i \# L_i$ along $\gamma_i$ so that the result is disjoint from $d_i$, as in the proof of Lemma 3.5.

After performing surgeries on the framed link $\bigcup_{i=1}^3(K_i \cup L_i)$, the component $c_i$ can be slid over $\gamma_i$ and then over the $(\dim L_i + 1)$-handle attached to $L_i$. The result of the handle slide is $c_i \# L_i$, where $L_i$ is a parallel copy of $L_i$ obtained from $L_i$ by slightly pushing off by one direction of the framing on $L_i$. We define $c'_i$ as the resulting framed sphere $c_i \# L_i$. We assume that $L_i$ is included in $N(T)$ and that $c'_i$ agrees with $c_i$ outside $N(T)$. Now a framed link $c'_1 \cup c'_2 \cup c'_3$ has been obtained from $c_1 \cup c_2 \cup c_3$ by sliding components over the handles attached to $\bigcup_{i=1}^3(K_i \cup L_i)$, and can also be obtained by component-wise connect-summing Borromean rings in $N(T)$. (See Figure 13e.)

We need to show that the Hopf links $K_i \cup L_i$ can be deformed into a small Hopf link $h_i$. Since $c'_1$ is disjoint from $d_1$, the component $K_1$ can be shrunked along $d_1$ to a small sphere $K'_1$ in a small $d$-disk around the point $d_1 \cap L_1$, without intersecting $c'_1 \cup c'_2 \cup c'_3$ during the shrinking isotopy. Then by sliding other components $K_j \cup L_j$ over $K'_1$ for $j \neq 1$, the component $L_1$ can be made unlinked from $K_j \cup L_j$. This slide does not change the isotopy type of

$$(K_2 \cup L_2) \cup (K_3 \cup L_3) \cup c'_1 \cup c'_2 \cup c'_3$$

in $N(T)$, though does change that in $N(T) \setminus (K'_1 \cup L_1)$. Now the Hopf link $K'_1 \cup L_1$ can be shrunk into a small Hopf link $h_1$ without affecting other components. After that, similar slidings can be performed for the Hopf links $K_2 \cup L_2$ and $K_3 \cup L_3$ so that they can be separated and shrunk into disjoint small Hopf links $h_2, h_3$, respectively. Thus the deformations performed so far consist of

\[\text{FIGURE 13 Type I surgery on a } Y\text{-graph } T.\]
isotopy and slides over handles, which give a desired concordance as in (1). The condition (2) follows from Property 2.1.

3.5 Type II Y-surgery for links

We shall give an analog of Lemma 3.7 for Type II Y-surgery, which is Lemma 3.10.

**Definition 3.9.** Let $W$ be as in Definition 3.3 and let $\pi_i : E_i \to B (i = 0, 1)$ be $(W, \partial W)$-bundles. Let $\widetilde{L}_0$ and $\widetilde{L}_1$ be fiberwise framed trivialized subbundles of the top boundaries of $E_0$ and $E_1$, respectively. We say that the pairs $(\pi_0, \widetilde{L}_0)$ and $(\pi_1, \widetilde{L}_1)$ are concordant if $\widetilde{L}_0 \cup \widetilde{L}_1$ is the restriction of a trivialized (fiberwise normally) framed subbundle $\widetilde{L}_{01}$ of the top boundary of a concordance between $\pi_0$ and $\pi_1$. Let $\pi_i^{\widetilde{L}_0} : E_i^{\widetilde{L}_0} \to B$ be the $(W^{\widetilde{L}_0}, \partial W^{\widetilde{L}_0})$-bundle obtained by attaching a trivialized $B$-family of handles along $\widetilde{L}_0$, where $L_0$ is a fiber of $\widetilde{L}_0$.

**Lemma 3.10** (Type II surgery). Let $W = X \times [a, b]$, where we identify $X \times \{b\}$ with $X$, and let $\pi_0 : S^{d-3} \times W \to S^{d-3}$ be the product $W$-bundle. Suppose that the leaves of a Y-graph $T$ of Type II in Int $X$ of dimensions $1, d-2, d-2$ are linked to framed submanifolds $c_1, c_2, c_3$ of dimensions $d-2, 1, 1$, respectively, and that $T$ is simple relative to $c_1 \cup c_2 \cup c_3$. Let $c'_1 \cup c'_2 \cup c'_3$ be a framed trivialized subbundle of $S^{d-3} \times \text{Int} X \to S^{d-3}$ that is obtained from $S^{d-3} \times (c_1 \cup c_2 \cup c_3) \simeq S^{d-3} \times \text{Int} X$ by fiberwise component-wise connect-summing a framed $S^{d-3}$-family of framed links $S^{d-2} \cup S^1 \cup S^1 \to N(T)$ that defines the Type II surgery.

(1) There is a concordance between the pairs of $(W, \partial W)$-bundles over $S^{d-3}

\[(\pi_0^T, S^{d-3} \times (c_1 \cup c_2 \cup c_3)) \text{ and } (\pi_0^{h_1 \cup h_2 \cup h_3}, c'_1 \cup c'_2 \cup c'_3),\]

which is strictly trivial on $(X - \text{Int} N(T)) \times [a, b]$.

(2) We may assume that two of the components of $c'_1 \cup c'_2 \cup c'_3$ agree with those of the inclusion $S^{d-3} \times (c_1 \cup c_2 \cup c_3) \simeq S^{d-3} \times \text{Int} X$.

**Remark 3.11.** Lemma 3.10 shows that the two definitions of Type II surgeries given in Section 2 and Definition 3.2 are equivalent, as in Remark 3.8.

**Proof of Lemma 3.10.** Proof is analogous to that of Lemma 3.7 and can be done by iterated applications of the concordance deformations of Remark 3.6. We only need to replace $L_i$ in Lemma 3.7 with a family of links $L_{i,s}$ in $N(T)$, $s \in S^{d-3}$. By Definition 3.2, the surgery on $T$ is given by the surgery on a six-component link

\[\bigcup_{i=1}^3 (K_i \cup L_{i,s})\]

in each fiber over $s \in S^{d-3}$. We assume that for all $s$, $L_{i,s}$ agrees with $L_{i,s_0}$ near the base point of $L_{i,s_0}$. Then $K_i \cup L_{i,s}$ satisfies the conditions (a)–(d) of Remark 3.6. Then $c'_1 \cup c'_2 \cup c'_3$ is defined by fiberwise component-wise connected-summing the family $L_{1,s} \cup L_{2,s} \cup L_{3,s}$ of framed links to $S^{d-3} \times (c_1 \cup c_2 \cup c_3)$. The proofs of (1) and (2) are parallel to those for Lemma 3.7.

The following lemma is an analog of Habiro’s move 10 ([7, Proposition 2.7]).
Lemma 3.12 (Y-graph with Null-leaf). Let $W = X \times [a, b]$, where we identify $X \times \{b\}$ with $X$. Suppose that the leaves $\ell_1, \ell_2, \ell_3$ of a Y-graph $T$ of Type I or II in $\text{Int} X$ bound disjoint disks $m_1, m_2, m_3$ in $\text{Int} X$, respectively. Suppose that there are disjoint submanifolds $c_1, c_2$ in $\text{Int} X \setminus T$ such that $\ell_i$ is simple relative to $c_i$ for $i = 1, 2$, and that $c_1 \cup c_2$ is disjoint from $m_3$. (See Figure 14.) Then there are three disjoint small Hopf links $h_1, h_2, h_3$ in $N(T) \setminus (c_1 \cup c_2)$ such that

1. if $T$ is of type I, there is a concordance between the pairs $(W^T, c_1 \cup c_2)$ and $(W^{h_1 \cup h_2 \cup h_3}, c_1 \cup c_2)$ that is strictly trivial on $(X \setminus \text{Int} N(T)) \times [a, b]$, and
2. if $T$ is of type II, there is a concordance between the pairs

$$((S^{d-3} \times W)^T, S^{d-3} \times (c_1 \cup c_2)) \text{ and } (S^{d-3} \times W^{h_1 \cup h_2 \cup h_3}, S^{d-3} \times (c_1 \cup c_2)),$$

which is strictly trivial on the product $(S^{d-3} \times (X \setminus \text{Int} N(T)) \times [a, b])$.

Proof. Case (1) is a corollary of Lemma 3.7. It suffices to delete $c_3$ and $c'_3$ in Lemma 3.7. By Property 2.1 of Borromean rings, $(c'_1 \cup c'_2) \cap N(T)$ in Lemma 3.7 is isotopic to $(c_1 \cup c_2) \cap N(T)$ fixing the boundary. Case (2) can be proven similarly by using Lemma 3.10 instead of Lemma 3.7. □

4 | FAMILY OF FRAMED LINKS FOR GRAPH SURGERY

4.1 | Surgery on a collection of Y-graphs for a link

Let $G$ be a connected uni-trivalent graph embedded in $\text{Int} X$ such that

1. $G$ has $r$ trivalent vertices and at least one univalent vertex,
(2) edges are oriented in a way that the orientations of edges at each trivalent vertex are the same as that of Y-graph of Type I or II,
(3) the univalent vertices of $G$ are on components of some spherical link $L$ in $\text{Int } X$ consisting of 1- and $(d-2)$-spheres,
(4) $L \cap \text{Int } G = \emptyset$, where $\text{Int } G$ is the complement of the union of univalent vertices in $G$,
(5) each univalent vertex of $G$ that is ‘inward’ to $G$ is attached to a $(d-2)$-sphere in $L$,
(6) each univalent vertex of $G$ that is ‘outward’ from $G$ is attached to a 1-sphere in $L$.

We shall describe the effect of the surgery on $G$ in Proposition 4.1. To state Proposition 4.1, we introduce some notations. We take a small closed neighborhood $N(G)$ of $G$ such that its intersection with $L$ consists of a small neighborhood of a univalent vertex of $G$ in a component of $L$. As before, we may construct a Y-link $G_1 \cup \cdots \cup G_r$ inside $N(G)$ by putting a framed Hopf link at each edge of $G$ between trivalent vertices and by replacing each univalent vertex with a leaf that bounds a disk in $N(G)$ transversally intersecting $L$ at a point. We call such a leaf a simple leaf of $G$ relative to $L$. Then we define surgery on $G$ by the surgery on the Y-link $G_1 \cup \cdots \cup G_r$.

Let $\mathbf{b}$ be a small $d$-disk and let $\mathbf{w}$ be the relative cobordism obtained from $\mathbf{b} \times I$ by surgery along a small Hopf link $h$ in $\text{Int } \mathbf{b} \times \{1\}$. For a relative cobordism $W$ between $\partial_-W = X \times \{0\}$ and $\partial_+W \cong X$ such that $\partial W = \partial_+W \cup_{\partial X \times [1]} (\partial X \times I) \cup_{\partial X \times \{0\}} \partial_-W$, let $W_N$ denote the boundary connected sum of $W$ and $N$ copies of $\mathbf{w}$ along disjoint union of disks $D^{d-1} \times I \subset \partial X \times I$. (See Figure 16.)

Let $B_G = S^{a_1} \times S^{a_2} \times \cdots \times S^{a_r}$, where $a_i = 0$ or $d-3$ depending on whether $G_i$ is of Type I or II, respectively. Let $p^G : W_G \to B_G$ be the $(W_3r, \partial_+W_3r)$-bundle obtained from the $(X \times I)$-bundle (trivialized) over $B_G$ by surgery along the associated set of families of framed Hopf links in $X \times \{1\}$ for $G$ as in Definitions 3.1 and 3.2.

**Proposition 4.1.** Let $W = X \times [a, b]$, where we identify $X \times \{b\}$ with $X$. Let $G$ be a connected univalent graph with $r$ trivalent vertices, embedded in $\text{Int } X$, and attached to some link $L$ as above. Let $b_1, \ldots, b_3r$ be disjoint small $d$-balls in $N(G) \setminus (L \cup G_1 \cup \cdots \cup G_r)$ and let $h_1, \ldots, h_3r$ be small Hopf links in $N(G) \setminus (L \cup G_1 \cup \cdots \cup G_r)$ such that $h_i \subset \text{Int } b_i$. (See Figure 17.) Let $L_{N(G)} = L \cap N(G)$. Then there is a $B_G$-family of embeddings of the union of disks $L_{N(G)}$ into $N(G) \setminus (b_1 \cup \cdots \cup b_3r)$:

$$\Phi_s : L_{N(G)} \to N(G) \setminus (b_1 \cup \cdots \cup b_3r) \quad (s \in B_G),$$

which agree with the inclusion near $\partial N(G)$ such that there is a concordance between the pairs

$$(W^G, B_G \times L) \text{ and } (B_G \times W^{h_1 \cup \cdots \cup h_3r}, \text{Int } L).$$
THE neighborhood $N(G)$ of the Y-link from Figure 15 is shaded, and each small ball $b_i$ is embedded near the $i$th edge of $G$.

where $\bar{L} = \bigcup_{s \in B_G} L_s$, which is a trivialized subbundle of $B_G \times \partial_s W_{3r} = B_G \times X$, and $L_s$ is obtained from $L$ by replacing $L_{N(G)}$ by $\Phi_s$. Moreover, we may assume that the concordance is strictly trivial on $B_G \times ((X \setminus \text{Int} N(G)) \times [a, b])$, and for any choice of a component $\ell$ of $L_{N(G)}$, we may assume that the restriction of $\Phi_s$ to all the components in $L_{N(G)} \setminus \ell$ does not depend on the parameter $s$, after a fiberwise isotopy, which depends on the choice of $\ell$.

Proof. We prove this by induction on $r$. The case $r = 1$ has been proved in Lemma 3.7 or 3.10.

For general $r$, we assume that the result holds true for connected uni-trivalent graphs with at most $r - 1$ trivalent vertices. Let $G$ be a connected uni-trivalent graph with $r$ trivalent vertices as in the statement. Then by assumption (condition (1) above) $G$ has a trivalent vertex $v_r$ that is incident to a univalent vertex by a single edge. We decompose $G$ into two parts by cutting the edges incident to $v_r$ but not incident to univalent vertices, and apply the induction hypothesis on the part with $r - 1$ trivalent vertices. More precisely, we put a framed Hopf link at each edge of $G$ incident to $v_r$ but not incident to univalent vertices according to the edge orientation by the rule of Figure 3, and then replace each univalent vertex of $G$ with a simple leaf relative to $L$. We may assume that this process yields a disjoint union of two connected objects: one is a connected uni-trivalent graph with $r - 1$ trivalent vertices with some spheres attached to univalent vertices, and another is a Y-graph. We denote the two components by $G'$ and $G''$, respectively. We consider closed neighborhoods $N'$ and $N''$ of $G'$ and $G''$, respectively, given as follows. Let $\mu$ be the union of spanning disks of the simple leaves of $G'$ each of which intersects $L$ transversally by one point. Here, we assume that each leaf is a round sphere about a point with small radius so that it has a canonical flat spanning disk and we assume that $\mu$ is the union of such flat disks. Now we take a small closed neighborhood $N'$ of $G' \cup \mu$. Also, let $N'' = N(G'')$. We assume that $N'$ is disjoint from $G''$. On the other hand, $N''$ intersects $G'$ according to the definition of $N(G'')$.

Let $\delta'$ be the union of $3(r - 1)$ disjoint small Hopf links in $N' \setminus G'$. By induction hypothesis, we see that there are $B_{G'}$-family of disks from $D' = L \cap N'$ inside $N'$:

$$\varphi_{s'} : D' \to N' \quad (s' \in B_{G'})$$

and a concordance between

$$(W^{G'}, B_{G'} \times L) \text{ and } (B_{G'} \times W^{s'}, \bar{L}')$$

where $\bar{L}' = \bigcup_{s' \in B_{G'}} L_{s'}$ and $L_{s'}$ is obtained from $L$ by replacing $D'$ by $\varphi_{s'}$. Moreover, we may assume that the restriction of $\varphi_{s'}$ to the components of $D'$ except one is a strictly trivial family.
We next consider the effect of the surgery on $G''$. Let $\lambda$ be the disjoint union of all the leaves of $G'$ that intersect $N''$, and let $L'' = L \cup \lambda$. Let $\delta''$ be the union of three disjoint small Hopf links in $N'' \setminus G''$. By Lemma 3.7 or 3.10, we see that there are $S^a_r$-families of leaves $\tilde{\lambda} = \bigcup_{s'' \in S^a_r} \lambda_{s''}$ in $N' \cup N''$ and a concordance between

$$(W^{G''}, S^a_r \times L'') \text{ and } (S^a_r \times W^{\delta''}, \tilde{L}'').$$

where $\tilde{L}'' = \tilde{\lambda} \cup (S^a_r \times L)$, such that for each $s''$, $\lambda_{s''}$ agrees with $\lambda$ near the basepoint of the leaf. The replacement of $\lambda$ with $\lambda_{s''}$ gives a family of embeddings of $G'$ in $N(G) \setminus L$:

$$\varphi_{s''} : G' \to N(G) \setminus L \quad (s'' \in S^a_r).$$

By isotopy extension, the family $\varphi_{s''}$ can be extended to a family of embeddings $\varphi'_{s''} : N' \to N(G)$ ($s'' \in S^a_r$).

Now we combine the two surgeries for $G'$ and $G''$. Let $\tilde{L}$ be the trivialized subbundle of $B_G \times X$ obtained from $B_G \times L$ by replacing $B_G \times L_{N''}$ by the composition

$$\varphi'_{s''} \circ \varphi_{s} : D' \to N(G) \quad (s', s'') \in B_G.$$

By the results of the previous paragraphs, we see that there is a concordance between the pairs $(W^G, B_G \times L)$ and $(B_G \times W^{\delta' \cup \delta''}, \tilde{L})$. Note that the restriction of $\tilde{L}$ to the components of $L_{N(G)}$ that intersect $N'$ is a strictly trivial family except one component, and also the restriction of $\tilde{L}$ to the components of $I_{N(G)}$ that intersects $N''$ is a strictly trivial family. This completes the induction.

□

4.2 | Replacing a trivalent graph with a family of Hopf links

Now we shall nearly complete the proof of Theorem 2.3 (1), by proving the corresponding statement for $B_{1_1}$-family instead of $S^{k(d-3)}$-family.

**Proposition 4.2.** Let $\Gamma$ be a labeled edge-oriented trivalent graph as in Section 2 with $2k$ vertices. The $(X, \partial)$-bundle $\pi^\Gamma : E^\Gamma \to B^\Gamma$ for an embedding $\phi : \Gamma \to \text{Int} X$ is concordant to a $(X, \partial)$-bundle obtained from the product bundle $S^{k(d-3)} \to S^{k(d-3)} \times X$ by fiberwise surgeries along a $B_{1_1}$-family of framed links $h_s : S^1 \cup S^{d-2} \to \text{Int} X$, $s \in B_\Gamma$, which satisfies the following conditions.

(1) $h_s$ is isotopic to the Hopf link for each $s$.
(2) The restriction of $h_s$ to $S^{d-2}$ component is a constant $B_{1_1}$-family.
(3) There is a small neighborhood $N$ of $\text{Im} \phi$ such that the image of $h_s$ is included in $N$ for all $s \in B_{1_1}$.

We will prove this by trying to construct a concordance between the families of cobordisms for the two surgeries and by restriction to the top faces. Of course, there is no such concordance in the obvious sense since the numbers of components of the framed links for $\Gamma$-surgery and surgery along a family of Hopf-links are different. We modify the assumption slightly so that a concordance between the two families of cobordisms will make sense.

Now we set $W = X \times I$ and let $p^\Gamma : W^\Gamma \to B_{1_1}$ be the $(W^{(6k)}_6, \partial)\cup W^{6k})$-bundle obtained from the trivial $W$-bundle by surgery along the associated family of $(6k \times 2 = 12k$ component) framed links
in \(X \times \{1\}\) for the \(\Gamma\)-surgery. The restriction of this bundle to the top face gives the former \((X, \partial)\)-bundle \(\pi^\Gamma : E^\Gamma \to B^\Gamma\). The number \(6k\) is because there are \(2k\) Y-graphs for the \(\Gamma\)-surgery each gives rise to three Hopf links. On the other hand, the latter \((X, \partial)\)-bundle of Proposition 4.2 is the top face of a \((W_1, \partial_0 W_1)\)-bundle over \(B^\Gamma\).

We add to \(W^\Gamma\) one more Hopf link surgery without changing the \((X, \partial)\)-bundle on the top face, as follows. Let \(G_1 \cup \cdots \cup G_{2k}\) be the Y-link for the embedding \(\phi\) of \(\Gamma\). Let \(a_1 \cup b_1\) be the framed Hopf link for the first edge of \(\Gamma\) as in Figure 3, which are leaves of some Y-graphs. We replace \(a_1 \cup b_1\) by a framed ‘Hopf chain’ \(c_1 \cup c_2 \cup c_3 \cup c_4\) such that

- \(\dim c_1 = \dim c_3 = \dim a_1, \dim c_2 = \dim c_4 = \dim b_1\),
- \(c_i \cup c_{i+1}\) is a Hopf link for \(i = 1, 2, 3\), see Figure 18.

Then the leaves \(a_1\) and \(b_1\) are replaced by \(c_1\) and \(c_4\), respectively, and \(G_1 \cup \cdots \cup G_{2k}\) becomes a Y-link \(G'_1 \cup \cdots \cup G'_{2k}\) that is linked to the Hopf link \(c_2 \cup c_3\). The Y-link \(G'_1 \cup \cdots \cup G'_{2k}\) is the one obtained from a uni-trivalent graph \(G\) attached to the link \(L = c_2 \cup c_3\), as in Proposition 4.1 below. By Lemma 3.5, this replacement does not change the concordance class of the pair up to small Hopf links. Namely, there are a small Hopf link \(h\) in \(\partial W\) that is disjoint from the Y-link \(G_1 \cup \cdots \cup G_{2k}\) and \(L\), and a family of concordances between the pairs

\[(W^\Gamma, B^\Gamma \times h) \quad \text{and} \quad (W^\Gamma_{G'_1 \cup \cdots \cup G'_{2k}}, B^\Gamma \times (c_2 \cup c_3))\]

parameterized by \(B^\Gamma\). Let \(p^\Gamma_1 : W^\Gamma_1 \to B^\Gamma_1\) be the \((W_{6k+1}, \partial_0 W_{6k+1})\)-bundle given by fiberwise surgery

\[(W^\Gamma_{G'_1 \cup \cdots \cup G'_{2k}}, B^\Gamma \times (c_2 \cup c_3)).\]

The number \(6k + 1\) is due to the addition of \(c_2 \cup c_3\). The newly added Hopf link \(c_2 \cup c_3\) will serve as the family of Hopf links \(h_s\) of Proposition 4.2.

Proposition 4.2 is an immediate corollary of the following lemma, which gives an extension of Proposition 4.2 to cobordisms.

**Lemma 4.3.** Let \(\Gamma\) be as in Proposition 4.2. The above \((W_{6k+1}, \partial_0 W_{6k+1})\)-bundle \(p^\Gamma_1 : W^\Gamma_1 \to B^\Gamma_1\) determined by an embedding \(\phi : \Gamma \to \text{Int} X \times \{1\}\) is concordant to a \((W_{6k+1}, \partial_0 W_{6k+1})\)-bundle that is obtained from the product \(W\)-bundle \(B^\Gamma \times W \to B^\Gamma\) by fiberwise handle attachments along some \(B^\Gamma\)-family of framed links \(h_s : S^1 \cup S^{d-2} \to \text{Int} X \times \{1\}, s \in B^\Gamma\), and fiberwise boundary-connected sums with \(6k\) copies of the trivial \((w, \partial_0 w)\)-bundle \(p_0 : B^\Gamma \times w \to B^\Gamma\), where \(h_s\) satisfies the conditions (1)–(3) of Proposition 4.2.

**Proof.** We assume without loss of generality that \(\dim c_2 = 1\) and \(\dim c_3 = d - 2\). Applying Proposition 4.1 for the Y-link \(G'_1 \cup \cdots \cup G'_{2k}\) and \(L = c_2 \cup c_3\), we see that surgery on \(G'_1 \cup \cdots \cup G'_{2k}\) produces
a $B_{Γ}$-family of embeddings of $L \cap N(G)$ into $N(G)$, whose restriction to $c_3 \cap N(G)$ is a trivial family. This gives the desired family of framed Hopf links.

5 | **BORDISM MODIFICATION TO AN $S^{k(d-3)}$-FAMILY OF SURGERIES**

5.1 | **From a $B_{Γ}$-family to an $S^{k(d-3)}$-family**

We shall complete the proof of Theorem 2.3 (1).

**Proposition 5.1.** Let $G$ be a uni-trivalent graph attached to a framed link $L$, as in Proposition 4.1. The $B_{Γ} = S^{a_1} \times \cdots \times S^{a_r}$-family of framed embeddings of disks $L_N(G) = L \cap N(G)$ in $N(G)$ of Proposition 4.1 can be deformed into an $S^{a_1+\cdots+a_r}$-family by an oriented bordism in the space $Emb^f_3(L_N(G), N(G))$.

To prove Proposition 5.1, we shall instead prove the following stronger lemma.

**Lemma 5.2.** The map $B_{Γ} \to Emb^f_3(L_N(G), N(G))$ for the $B_{Γ}$-family of Proposition 4.1 factors up to homotopy over a map $B_{Γ} \to S^{a_1+\cdots+a_r}$ of degree 1.

**Proof.** We prove this by induction on $r$. The case $r = 1$ is obvious. Assume that the map

$$g_{r-1} : B_{Γ'} = S^{a_1} \times \cdots \times S^{a_{r-1}} \to Emb^f_3(L_N(G), N(G))$$

for a Y-link $G_1 \cup \cdots \cup G_{r-1}$ that corresponds to a connected uni-trivalent graph $G'$ factors up to homotopy into a degree 1 map $S^{a_1} \times \cdots \times S^{a_{r-1}} \to S^{a_1+\cdots+a_{r-1}}$ and a map

$$\bar{g}_{r-1} : S^{a_1+\cdots+a_{r-1}} \to Emb^f_3(L_N(G'), N(G')).$$

Since $g_{r-1}$ is null-homotopic, one may apply Lemma 5.3 below, and the map $\bar{g}_{r-1}$ is null-homotopic.

Adding one more Y-graph $G_r$ so that $G_1 \cup \cdots \cup G_r$ corresponds to a connected uni-trivalent graph $G$, we obtain a map $g_r : B_{Γ'} \times S^{a_r} \to Emb^f_3(L_N(G), N(G))$ that factors up to homotopy over a degree 1 map $B_{Γ'} \times S^{a_r} \to S^{a_1+\cdots+a_{r-1}} \times S^{a_r}$.

The restrictions of the induced map

$$\bar{g}_r : S^{a_1+\cdots+a_{r-1}} \times S^{a_r} \to Emb^f_3(L_N(G), N(G))$$

to the subspaces $S^{a_1+\cdots+a_{r-1}} \times \{\ast\}$ and $\{\ast\} \times S^{a_r}$ of $S^{a_1+\cdots+a_{r-1}} \times S^{a_r}$ are pointed null-homotopic in $Emb^f_3(L_N(G), N(G))$ by Lemma 3.12 and by the nullity of $\bar{g}_{r-1}$ in

$$Emb^f_3(L_N(G'), N(G')).$$

Thus the map $\bar{g}_r$ factors up to homotopy over a degree 1 map $S^{a_1+\cdots+a_{r-1}} \times S^{a_r} \to S^{a_1+\cdots+a_r}$. □
Lemma 5.3 [19, Proof of Lemma B]. Let \( B = S^{a_1} \times S^{a_2} \times \cdots \times S^{a_s} \) and let

\[
A = (\{\ast\} \times S^{a_2} \times \cdots \times S^{a_s}) \cup (S^{a_1} \times \{\ast\} \times \cdots \times S^{a_s}) \cup \cdots \cup (S^{a_1} \times S^{a_2} \times \cdots \times \{\ast\}).
\]

For a space \( Y \), suppose that we have a pointed null-homotopy of a pointed map \( g : B \to Y \) and another pointed null-homotopy of the restriction \( g|_A : A \to Y \). Then \( g \) can be factored up to homotopy into a pointed map \( B \to B/A \approx S^{a_1+\cdots+a_s} \) and a null-homotopic map \( B/A \to Y \).

Corollary 5.4. The \( B_{Γ} \)-family of framed links \( h_s : S^1 \cup S^{d-2} \to \text{Int} X, s \in B_{Γ} \), in Proposition 4.2 can be deformed by a bordism in the space of embeddings, into a \( S^{k(d-3)} \)-family of framed embeddings \( S^1 \cup S^{d-2} \to \text{Int} X \) that satisfies the following conditions.

1. \( h_s \) is isotopic to the Hopf link for each \( s \).
2. The restriction of \( h_s \) to \( S^{d-2} \) component is a constant \( S^{k(d-3)} \)-family.
3. There is a small neighborhood \( N \) of \( \text{Im} \phi \) such that the image of \( h_s \) is included in \( N \) for all \( s \in S^{k(d-3)} \).

Hence, fiberwise handle attachments along the family of embeddings over the bordism gives a bundle bordism of cobordism bundles \( p_{Γ}^1 : W_{Γ}^1 \to B_{Γ} \) to a \( (W_{6k+1}, \partial \cup W_{6k+1}) \)-bundle over \( S^{k(d-3)} \), which restricts on the top face to a \( (X, \partial) \)-bundle bordism between \( π_{Γ} : E_{Γ}^1 \to B_{Γ} \) and a \( (X, \partial) \)-bundle \( \varpi_{Γ} : E_{Γ}^1 \to S^{k(d-3)} \).

5.2 Modification into a family of \( h \)-cobordisms

We prove Theorem 2.3 (2).

Proposition 5.5. There exists a \( (X \times I, \partial_{\downarrow}, (X \times I)) \) bundle \( \Pi_{Γ}^1 : W_{Γ}^1 \to S^{k(d-3)} \) such that

1. the fiberwise restriction of \( \Pi_{Γ}^1 \) to \( X \times \{1\} \) is \( \varpi_{Γ} \),
2. \( W_{Γ}^1 \) is obtained by attaching \( S^{k(d-3)} \)-families of 1- and 2-handles to the product \( X \times I \)-bundle \( S^{k(d-3)} \times (X \times I) \to S^{k(d-3)} \times (X \times \{1\}) \).

Proof. By Corollary 5.4, there is a cobordism bundle \( E \to S^{k(d-3)} \) which is obtained from the trivialized \( (X \times I) \)-bundle by attaching families of 2- and \( (d-1) \)-handles along \( h_s \) and whose restriction to \( \partial_{\downarrow} E \) agrees with \( \varpi_{Γ} \).

Let \( E_1 \) be the family of handlebodies obtained by attaching only the family of \( (d-1) \)-handles to the strictly trivial \( (X \times I) \)-bundle by \( h_s|_{S^{d-2}} \). Since the attaching map \( h_s|_{S^{d-2}} \) of the family of \( (d-1) \)-handles is a strictly trivial family by Corollary 5.4, the family \( E_1 \) is a strictly trivial bundle, on the top of which the 2-handle may be attached along the attaching sphere induced by \( h_s|_{S^{1}} \) on \( \partial_{\downarrow} E_1 \) that may not be strictly trivial.

Attaching a \( (d-1) \)-handle to \( X \times I \) along an unknotted framed \( (d-2) \)-sphere on \( X \times \{1\} \) turns the top face into \( X\#(S^{d-1} \times S^1) \). Also, the same manifold can be obtained by attaching a 1-handle along a framed 0-sphere on \( X \times \{1\} \) instead of a \( (d-1) \)-handle. Thus, we may replace the strictly trivial bundle \( E_1 \) by another family \( E'_1 \) of handlebodies that is obtained by attaching strictly trivial
family of 1-handles to $X \times I$, without changing the manifold

$$\partial_+ E_1 = S^{k(d-3)} \times (X \# (S^{d-1} \times S^1)).$$

Then we attach a family of 2-handles to $E'_1$ along the attaching spheres induced by $h_s|_{S^1}$ on $\partial_+ E'_1 = \partial_+ E_1$, see Figure 19. The resulting bundle $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ is a $(X \times I)$-bundle, since the two handles are in a canceling position in a fiber, namely, the descending disk of the 2-handle and the ascending disk of the 1-handle intersects transversally in one point in $\partial_+ E_1$. Then by M. Morse’s result [16] (see also [10, Theorem 5.4 (First Cancellation Theorem)]), the pair of two handles can be eliminated and the cobordism can be modified into the trivial $h$-cobordism. By construction, $\partial_+ W^\Gamma_h = \overline{E}^\Gamma$ and $\pi^\Gamma$ restricts to $\varpi^\Gamma$. □

Remark 5.6. We notice that by construction, the bundle $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ admits a fiberwise Morse function $f : W^\Gamma_h \to \mathbb{R}$ and a fiberwise gradient-like vector field $\xi$ for $f$ such that the family of handle decompositions for $\xi$ agrees with that of the 1- and 2-handles in Proposition 5.5. Such a family of Morse functions can be constructed by applying [10, Theorem 3.12] for the families of handles, which is possible since the families of handles are given by families of attaching maps, and the construction of the Morse function in the proof of [10, Theorem 3.12] for the surgery $\chi(V, \varphi)$ depends smoothly on the attaching maps $\varphi$.

Remark 5.7. A similar trick to turn the constant $(d-1)$-handle upside down into a 1-handle was recently used in a similar setting by David Gay in [6].

6 | PROOF OF THEOREMS 1.8 AND 1.10

As noted in the introduction, Theorem 1.8 is an immediate corollary of Theorem 1.10, which we now prove. We will use a result from [2] and the following definition.

Definition 6.1 (cf. [2, Definition 2.7]). Let $\pi : E \to B$ be a smooth bundle of $(d + 1)$-dimensional cobordisms. Denote by $V_b$ a fiber over a point $b \in B$, where $V := V_b$ is a relative cobordism between $\partial_0 V$ and $\partial_1 V$ such that $\partial V = \partial_0 V \cup \bar{\partial} V \cup \partial_1 V$. We assume that a structure group of the bundle $\pi : E \to B$ is $\text{Diff}_{\text{loc}}(V)$, that is, of those diffeomorphisms which restrict to the identity near $\partial_0 V \cup \bar{\partial} V$. We denote by $E_0$, $E_3$, and $E_1$ a restriction of the bundle $E$ to the fibers $\partial_0 V$, $\bar{\partial} V$, and $\partial_1 V$, respectively.
respectively. A smooth map $F : E \to B \times I$ is said to be an admissible family of Morse functions or admissible with fold singularities with respect to $\pi$ if it satisfies the following conditions.

(1) The diagram

\[
\begin{array}{ccc}
E & \xrightarrow{F} & B \times I \\
\pi \downarrow & & \downarrow p_1 \\
B & \xrightarrow{p_1} & 
\end{array}
\]

commutes. Here $p_1 : B \times I \to B$ is projection on the first factor.

(2) The pre-images $F^{-1}(B \times \{0\})$ and $F^{-1}(B \times \{1\})$ coincide with the submanifolds $E_0$ and $E_1$, respectively.

(3) The set $\text{Cr}(F) \subset E$ of critical points of $F$ is contained in $E \setminus (E_0 \cup E_3 \cup E_1)$ and near each critical point of $F$ the bundle $\pi$ is equivalent to the trivial bundle $\mathbb{R}^k \times \mathbb{R}^{d+1} \xrightarrow{p_1} \mathbb{R}^k$ so that with respect to these coordinates on $E$ and on $B$ the map $F$ is a standard map $\mathbb{R}^k \times \mathbb{R}^{d+1} \to \mathbb{R}^k \times \mathbb{R}$ with a fold singularity.

(4) For each $z \in B$ the restriction

\[
f_b = F|_{V_b} : V_b \to \{b\} \times I \xrightarrow{p_2} I
\]

is an admissible Morse function, that is, its critical points have indices $\leq d - 2$.

**Lemma 6.2.** Let $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ be as in Proposition 5.5. Then each $(X \times I, \partial_\cup (X \times I))$-bundle $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ of $h$-cobordisms has an admissible family of Morse functions as above provided $d \geq 4$.

**Proof.** Let $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ be a $(X \times I, \partial_\cup (X \times I))$-bundle of $h$-cobordisms as above. We just noticed in Remark 5.6 that there is a fiberwise Morse function $f : W^\Gamma_h \to \mathbb{R}$ which agrees with that of the 1- and 2-handles (in the case when $d$ is even). In the case when $d$ is odd we deal with $m$- and $(m + 1)$-handles, where $m = (d - 1)/2$. In particular, the codimension of the handles (inside the $h$-cobordisms) is at least three for all $d \geq 4$. 

**Proof of Theorem 1.10.** We consider bundles of $h$-cobordisms we have constructed. In both cases, when $d$ is even or odd, Lemma 6.2 guarantees that such a bundle satisfies the conditions of [2, Theorem 2.9] since $\Pi^\Gamma : W^\Gamma_h \to S^{k(d-3)}$ has only admissible fold singularities. Thus we obtain that every fiber has a psc-metric. This proves Theorems 1.8 and 1.10. 

**Remark 6.3.** We should admit that [2, Theorem 2.9] assumes that the structure group is $\text{Diff}_\partial(V)$; however, it is easy to see that the same proof works in more general situation, in particular when the structure group is $\text{Diff}_\cup(V)$.

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