NONLOCALITY IN NONLINEAR QUANTUM MECHANICS

W. LÜCKE

Arnold Sommerfeld Institute for Mathematical Physics, Technical University
Clausthal, Leibnizstr. 10, D-38678 Clausthal, GERMANY
E-mail: aswl@pt.tu-clausthal.de

A general method for testing essential nonlocality of nonlinear modifications of quantum mechanics is presented and applied to show the inconsistency of I. Bialynicki-Birula’s and J. Mycielski’s nonlinear quantum theory.

1 Nonlinear Modifications of Quantum Mechanics

Many authors considered nonlinear Schrödinger equations of the form

\[ i\hbar \frac{\partial}{\partial t} \Psi_t(\vec{x}) = \left( -\frac{\hbar^2}{2m} \Delta + V(\vec{x}, t) \right) \Psi_t(\vec{x}) + F[\Psi_t(\vec{x})] \Psi_t(\vec{x}) , \]

where \( F[\Psi] = R[\Psi] \Psi \) is a local nonlinear Functional of \( \Psi \). The essential point is that (1) is not interpreted as an equation for some field operator but as a classical evolution equation for the quantum mechanical wave function:

\[ \int_{B} |\Psi_t(\vec{x})|^2 \, d\vec{x} = \text{probability for location within } B \]

(at time \( t \) for normalized \( \Psi_t \)). Special cases were even tested experimentally.

All these efforts seemed useless according to N. Gisin’s claim:

“All deterministic nonlinear Schrödinger equations are irrelevant.”

By this Gisin meant the following:

Consider a Bell-like situation as sketched in Figure 1. Then, if the source produces entangled 2-particle states, there is always a physical observable for the particle sent to the right, the probability distribution of which is instantaneously (substantially) changed by...
suitable measurements (involving only low energy transfer) on the other particle ‘behind the moon’.

Actually, Gisin assumed the following:

(G1): If the observable $A$ resp. $B$ of the particle ‘behind the moon’ is measured at time $t = 0$ the partial state of the other particle at times $t \geq 0$ is given by a density matrix $\rho_{\text{right}}(t)$ of the form

$$
\sum_{\alpha} x_{\alpha} P_{\Psi}(\alpha, t) \quad \text{resp.} \quad \sum_{\beta} x_{\beta} P_{\Psi}(\beta, t),
$$

where the $P_{\Psi}(\alpha, t)$ resp. $P_{\Psi}(\beta, t)$ are pure states evolving according to the corresponding 1-particle equation.

(G2): All self-adjoint (bounded) operators correspond to observables.

That Gisin’s claim is wrong since Gisin’s assumption (G1), the projection postulate, is unjustified in nonlinear quantum mechanics has already been pointed out by Polchinski, who determined conditions which are sufficient for the absence of essential nonlocality.

Accepting Gisin’s assumption (G2), Polchinski concluded that his conditions – violated for prominent examples of nonlinear Schrödinger equations – are also necessary to avoid essential nonlocality. However, as explained

\[d\]His justification: “So far we have only used linear quantum mechanics”.

\[e\]This especially implies $\sum_{\alpha} x_{\alpha} P_{\Psi}(\alpha, t) = \sum_{\beta} x_{\beta} P_{\Psi}(\beta, t)$ for $t = 0$ but — as realized by Gisin — not generally for $t > 0$.

\[f\]See also for examples.
already in [126], also assumption (G2) is unjustified in nonlinear quantum mechanics and definitely wrong for the situation reconsidered in the next section. This is why valid proofs of essential nonlocality have to be more involved.

2 What we can Learn from Nonlinear Gauge Transformations

Consider the well-defined special case

\[(N_D(\Psi))(\vec{x}) \overset{\text{def}}{=} e^{\frac{-2mD}{\hbar} \ln |\Psi(\vec{x})|} \Psi(\vec{x}) , \quad D \in \mathbb{R} ,\]

of the ‘nonlinear gauge transformations’ exploited by H.-D. Doebner et al. [13] If \(\Psi_t'(\vec{x})\) is a solution of (1) for \(R = 0\) then straightforward calculation shows that

\[\Psi_t(\vec{x}) = (N_D(\Psi_t'))(\vec{x})\]

is a solution of (1) for

\[R[\Psi_t] = \hbar D \left( \frac{i}{2} \frac{\Delta \rho_t}{\rho_t} \Psi_t + c_1 \frac{\vec{\nabla} \cdot \vec{J}_t}{\rho_t} + c_2 \frac{\Delta \rho_t}{\rho_t} + c_3 \frac{\vec{J}_t}{\rho_t} + + c_4 \frac{\vec{J}_t \cdot \vec{\nabla} \rho_t}{(\rho_t)^2} + c_5 \frac{(\vec{\nabla} \rho_t)^2}{(\rho_t)^2} \right) \tag{4}\]

with

\[c_1 = 1 , \quad c_2 = -2c_5 = -mD/\hbar , \quad c_3 = 0 , \quad c_4 = -1 , \quad (5)\]

where

\[\rho_t \overset{\text{def}}{=} |\Psi_t|^2 , \quad \vec{J}_t \overset{\text{def}}{=} \frac{1}{2i} \left( \overline{\Psi_t} \overline{\nabla} \Psi_t - \Psi_t \overline{\nabla} \overline{\Psi_t} \right) .\]

This way we get a deterministic nonlinear Schrödinger equation which, interpreted by [8], describes the same physics as the corresponding \((R = 0)\) linear Schrödinger equation, since

\[|\Psi_t(\vec{x})| = |\Psi_t'(\vec{x})| .\]

That, contrary to Gisin’s as well as Polchinski’s claim, there is no real problem with locality is no surprise since now nonlinear projection operators

\[E = N_D \circ E' \circ N_D^{-1} ,\]

instead of linear projectors \(E'\), have to be used [13] to get the correct probabilities

\[\|E(\Psi_t)\|^2 = \|E'(\Psi_t')\|^2 = \langle \Psi_t' | E' \Psi_t' \rangle .\]

\[^9\text{The general Doebner-Goldin equation is given by (1) and (4) without the restriction on the parameters } c_\nu \in \mathbb{R} .\]
Hence assumption \( (G2) \) is obviously wrong in nonlinear quantum mechanics.
Moreover, w.r.t. the nonlinear \( E \), density matrices are inadequate for the description of classical mixtures:

\[
\sum_{\alpha} x_\alpha \| E(\Psi_\alpha) \|^2 = \text{trace} \left( E' \sum_{\alpha} x_\alpha P_{\Psi_\alpha} \right) \neq \text{trace} \left( E \sum_{\alpha} x_\alpha P_{\Psi_\alpha} \right).
\]

Therefore also assumption \((G1)\) turns out to be quite inadequate for nonlinear modifications of quantum mechanics.

The simple example \((1)/(4)/(5)\) tells us that essential nonlocality should to be checked by using nothing else than the evolution equation together with its basic interpretation \((2)\).

### 3 The Doebner-Goldin Equation Interpreted as 2-Particle Equation

Let us interpret \((1)\) as a two-particle equation:

\[
\vec{x} = (\vec{x}_1, \vec{x}_2), \quad \vec{x}_j = \text{position of particle } j.
\]

Moreover assume that the potential is of the form

\[
V(\vec{x}) = V_2(\vec{x}_2 - \vec{x}_0), \quad \vec{x}_0 \text{ fixed},
\]

and that \( \Psi_t(\vec{x}) \) is the solution of \((1)\) fulfilling the initial condition

\[
\Psi_0(\vec{x}_1, \vec{x}_2) = f(\vec{x}_1, \vec{x}_2 - \vec{x}_0).
\]

Then, obviously, we have an unacceptable nonlocality, if the position probability density

\[
\rho_1(\vec{x}_1, t) \overset{\text{def}}{=} \int \left| \Psi_t(\vec{x}_1, \vec{x}_2) \right|^2 d\vec{x}_2 \tag{6}
\]

for particle 1 depends on \( V_2 \).

That the latter happens for certain cases of the general Doebner-Goldin equation, if interpreted as 2-particle equation, was first proved by Werner.\(^{20}\)

Inspired by E. Nelson\(^{23}\) he considered pairs of 1-dimensional particles, i.e. \( \vec{x} = (x_1, x_2) \in \mathbb{R}^2 \), with

\[
V_2(x_2) = \lambda (x_2)^2, \quad \lambda \in \mathbb{R}, \tag{7}
\]

\(^{h}\)Note that \((1)\) does not depend on \( \vec{x}_0 \). Therefore the effect on particle 1, if any, can be produced by acting on particle 2 as far away as one likes.
and entangled initial conditions of Gaussian type. The corresponding solutions are of the form

$$\Psi_t(x_1, x_2) = e^{i(t)} e^{-\sum_{j,k=1}^{2} x_j x_k / 2} C_{jk}(t), \quad C_{jk} = C_{kj},$$

where the $C_{jk}(t)$ fulfill a simple system of first order ordinary differential equations that can be used to determine their time derivatives at $t = 0$ and thus

$$\left( \partial_t^n \int (x_1)^2 \rho_1(x_1, t) dx_1 \right)_{|t=0}$$

for arbitrary $n \in \mathbb{Z}_+$. Werner found that (8) depends on $\lambda$ for $n = 3$ and suitable $C_{jk}(0)$ unless

$$c_3 = c_1 + c_4 = 0. \quad (9)$$

In principle, using (1) directly, one may calculate

$$\partial \left( \partial_n \int |\Psi_t(x, y)|^2 dy \right)_{|t=0}$$

as a functional of $\Psi_0$ and $V$ for given $R$. This shows that (8) varies also with strictly localized changes of $V_2$. However, for $n > 3$ the calculation becomes too involved and could not even be managed by use of computer algebra. On the other hand, Werner’s Ansatz turned out to be too special to uncover essential nonlocality of the Galilei covariant cases of the Doebner-Goldin 2-particle equation. This is why

$$\left( \partial_t^n \int x_1 \rho_1(x_1, t) dx_1 \right)_{|t=0}$$

was checked for $n = 4$ in showing that the Doebner-Goldin equation is essentially nonlocal, when interpreted as a 2-particle equation, unless the parameters $c_\nu$ are chosen such that (1) is (formally) linearizable by some nonlinear gauge transformation.

4 Inconsistency of Bialynicki-Birula’s and Mycielski’s Theory

For the nonlinear Schrödinger equation of Bialynicki-Birula and Mycielski, given by

$$R(\Psi)(\vec{x}) = -2b \ln |\Psi(\vec{x})|, \quad b \in \mathbb{R}, \quad (12)$$

\footnote{Note that (12) is equivalent to Galilei covariance of the Doebner Goldin equation.}

\footnote{Equation (12) was already considered by H. Košíak with $b < 0$.}
testing (11) is of no use, since the Ehrenfest relations hold\[2\] In such cases one should check

\[ S_{k,n}[\Psi_0, V_2] \defeq \partial_\lambda \left( \partial_\mu^n \int e^{ikx} |\Psi_t(x,y)|^2 \, dx \, dy \right) \bigg|_{t=0} \tag{13} \]

for \( k \in \mathbb{R} \) and \( n \in \mathbb{Z}_+ \). In principle this is equivalent to testing (10) but has the advantage of testing (11): The integral over \( x \) allows for partial integrations which simplify the resulting expressions considerably.

For simplicity, let \( \frac{\hbar^2}{2m} = 1 \) and assume \( R \) to be real valued\[3\] as in (12):

\[ (R[\Psi])(\vec{x}) = (R[\Psi])(\vec{x}) \tag{14} \]

Then, defining

\[ T_{k,\nu}(t) \defeq \int e^{ikx} \overline{\Psi_t} \partial_\nu \Psi_t \, dx \, dy, \]

\[ D_{k,\mu,\nu}(t) \defeq \int e^{ikx} (\partial_\mu R[\Psi_0]) \overline{\Psi_t} \partial_\nu \Psi_t \, dx \, dy, \]

we get from (1) by partial integration

\[ i\frac{d}{dt} T_{k,\nu} = -k^2 T_{k,\nu} + 2ik T_{k,\nu+1} + \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} D_{k,\mu,\nu-\mu} \]

for \( \vec{x} = (x_1, x_2) \in \mathbb{R}^2 \) and \( V(\vec{x}) = V_2(x_2) \). Iteration of this gives

\[ \left( i\frac{d}{dt} \right)^3 T_{k,0} = -k^6 T_{k,0} + 6ik^5 T_{k,1} + 12k^4 T_{k,2} - 8ik^3 T_{k,3} - (4ik^3 D_{k,1,0} + 8k^2 D_{k,1,1} + 4k^2 D_{k,2,0} + 2k D_{k,1,1}) \] .

Hence, e.g.,

\[ \left( \partial_\nu^{n+3} \int x^2 |\Psi_t(x,y)|^2 \, dx \, dy \right) \bigg|_{t=0} = - \left( \partial_\nu^{n+3} \partial_\nu (\partial_\nu T_{k,0}(t)) \bigg|_{t=0} \right) = 8i \partial_\nu^n (2 D_{0,1,1} + D_{0,2,0}) . \]

For the special case

\[ m = 1, \quad V_2(y) = \lambda y^2, \quad \Psi_0(x,y) = \frac{e^{-x^2-y^2-xy}}{\int e^{-2x^2-2y^2-2xy} \, dx \, dy} . \]

\[ ^k\text{For the Doebner-Goldin case (4) the latter can always be achieved by some nonlinear gauge transformation of the type considered in Section 4.} \]
running a simple computer algebra program (see appendix) shows that (1) and (12) imply
\[ \partial_\lambda \left( (i\partial_t)^3 (2D_{0,1,1}(t) + D_{0,2,0}(t)) \right)_{|t=0} = 32b. \]
This means that (1)/(12) is essential nonlocal - against the basic philosophy of Bialynicki-Birula’s and J. Mycielski’s theory.

5 Identical Particles

Up to now we tacitly assumed that the two particles (with equal masses) considered in Figure 1 can be distinguished. Therefore one might still hope that Bialynicki-Birula’s and J. Mycielski’s theory is consistent for identical particles. However, even for 2-particles states which are symmetric or anti-symmetric w.r.t. exchange of the particles essential nonlocality is unavoidable.

To show this denote by \( \Psi_{g,U}^t \) the solution of
\[ i\partial_t \Psi_{g,U}^t = \left( -\Delta + \lambda U - 2b \ln \left| \Psi_{g,U}^t \right| \right) \Psi_{g,U}^t \]
fulfilling the initial condition
\[ \Psi_{g,U}^0(x,y) = g(x,y). \]
For fixed \( f(x,y) \) and \( \sigma \in \{+1, -1\} \) define
\[ U^{(d)}(x,y) \overset{\text{def}}{=} V(y-d) + V(x-d), \]
\[ \Psi_0^{(d)}(x,y) \overset{\text{def}}{=} f(x,y-d) + \sigma f(y,x-d), \]
\[ \chi_0^{(d)}(x,y) \overset{\text{def}}{=} f(x,y-d), \quad \phi_0^{(d)}(x,y) \overset{\text{def}}{=} f(y,x-d). \]
and
\[ \chi_t^{(d)} \overset{\text{def}}{=} \Psi_{t,U^{(d)}}^0, \quad \phi_t^{(d)} \overset{\text{def}}{=} \Psi_{t,U^{(d)}}^0. \]
Obviously, if \( V \in D(\mathbb{R}) \) and \( f \in S(\mathbb{R}^2) \),
\[ \partial_\lambda \left( \tilde{\partial}_G \left( \left| \Psi_t^{(d)}(x,y) \right|^2 - \left| \chi_t^{(d)}(x,y) \right|^2 - \left| \phi_t^{(d)}(x,y) \right|^2 \right) \right) \xrightarrow{d \to \infty} 0 \]
holds for every region \( G \subset \mathbb{R}^3 \times \mathbb{R}^3. \)

\footnote{This result was confirmed by R. Werner using his method described in Section 4.}
Moreover,

\[
\lim_{d \to \infty} \partial_\lambda \left( \partial_t^6 \int_G \left| \phi^{(d)}_\lambda(x, y) \right|^2 \, dx \, dy \right)_{t=0}
\]

\[
= \lim_{d \to \infty} \partial_\lambda \left( \partial_t^6 \int_G \left| \chi^{(d)}_\lambda(x, y) \right|^2 \, dx \, dy \right)_{t=0}
\]

\[
= c^{f,U}_\lambda \overset{\text{def}}{=} \partial_\lambda \left( \partial_t^6 \int_{x \in \mathcal{O}} \left| \Psi^{f,U}_t(x, y) \right|^2 \, dx \, dy \right)_{t=0}
\]

holds for

\[\mathcal{G} = \{(x, y) \in \mathbb{R}^2 : x \in \mathcal{O} \vee y \in \mathcal{O}\}, \quad \mathcal{O} \text{ bounded}, \quad U(x, y) = V(y).\]

Therefore, under these conditions,

\[
\partial_\lambda \left( \partial_t^6 \int_{x \in \mathcal{O}} \left| \Psi^{(d)}_t(x, y) \right|^2 \, dx \, dy \right)_{t=0} \overset{d \to \infty}{\to} = c^{f,U}_\lambda.
\]

Since, as shown in Section 4, \(c^{f,U}_\lambda\) can be arranged to be nonzero we conclude:

The postulate of symmetry or antisymmetry of the wave function w.r.t. to exchange of particles does not prevent essential nonlocality.

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Appendix: Maple V (Release 4) Session

PROCEDURES:
\[
> \text{del} := \text{proc}(f) \text{global} \ x, y, t; \text{option operator; unapply}(\text{diff}(f(x,y,t),x^2)+\text{diff}(f(x,y,t),y^2),(x,y,t));\end{\text{proc}}.
\]

> pot := \text{proc}(f) \text{global} \ x, y, t; \text{option operator; unapply}(V(y) \ast f(x,y,t),(x,y,t));\end{\text{proc}}.

> Idot := \text{proc}(f) \text{global} \ x, y, t; \text{option operator; unapply(simplify(subs(diff(P(x,y,t),t)=-\text{del}(P(x,y,t))+\text{pot}(P(x,y,t)),}...\text{)}),...\text{)}.

8
diff(PB(x,y,t),t)=(del(PB)(x,y,t)-pot(PB)(x,y,t)), \text{diff}(f(x,y,t), t))), (x,y,t));

> end:
>
EVALUATION:
> term0 := (x,y,t) \rightarrow 2 \cdot \text{diff}(\ln PB(x,y,t) \cdot P(x,y,t), x) \cdot PB(x,y,t) \cdot \text{diff}(P(x,y,t), x) + \text{diff}(\ln PB(x,y,t) \cdot P(x,y,t), x^2) \cdot PB(x,y,t) \cdot P(x,y,t);
> term1 := (x,y,t) \rightarrow \text{simplify}(Idot(term0)(x,y,t));
> term2 := (x,y,t) \rightarrow \text{simplify}(Idot(term1)(x,y,t));
> term3 := (x,y,t) \rightarrow \text{simplify}(Idot(term2)(x,y,t));

SPECIAL CASE:
> spec := proc(f)
> global x,y,t;
> option operator;
> unapply( subs(V(y)=lambda * y^2, P(x,y,t)=exp(-x^2 -y^2 -x*y),
PB(x,y,t)=exp(-x^2 -y^2 -x*y), f(x,y,t)), (x,y,t));
> end:
>
RESULT:
> \int(\int(\text{simplify}(diff(spec(term3)(x,y,t),lambda)), x=-\infty..\infty),
y=-\infty..\infty);

\[ \frac{1}{2} - \frac{32}{3} \pi \frac{3}{3} \]

> \int(\int(exp(-2*x^2 -2*y^2 -2*x*y), x=-\infty..\infty),
y=-\infty..\infty);

\[ \frac{1}{2} \]

\[ \frac{1}{3} \pi \frac{3}{3} \]

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