3D time-domain spectral elements for stress waves modelling

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Abstract. Elastic stress waves induced by piezoelectric transducers are extensively used for damage detection purposes. Induced high frequency impulse signals cause that stress wave modelling by the finite element method is inefficient. Instead, numerical model based on the time-domain spectral element method has been developed to simulate stress wave propagation in metallic structures induced by the piezoelectric transducers. The model solves the coupled electromechanical field equations simultaneously in three-dimensional case. Visualisation of the propagating elastic waves generated by the actuator of different shapes and properties has been performed.

1. Introduction
The research area of smart materials has attracted many researchers, because of promising potential in applications such as: morphing, energy harvesting, vibration control and damage detection. Taking advantages of piezoelectric effect in devices such as piezotransducers or recently Macro Fibre Composites (MFC) embedded into structure it is possible to built structures which merge superb mechanical properties of composites with tuning opportunity of static and dynamic response. Moreover, piezo-actuators and piezo-sensors can be used for Structural Health Monitoring. The analysis of piezoelectric composite structures such as piezoelectric laminated plates or structural elements activated by piezo-actuators requires efficient and accurate electromechanical modelling of both the mechanical and electric responses such as mechanical displacements and electric potentials. Exact 3D analytical solutions have been presented for the piezoelectric response of simply supported flat panels and rectangular plates in [1–4]. Since the exact 3D analytical solutions are available only for some regular shapes with specified simple boundary conditions, it is desirable to use approximate method for more complex structures (i.e., the finite element method, the finite difference method, the boundary element method).

Modelling of mechanical displacement field by the full 3D finite elements [5; 6] typically results in huge number of degrees of freedom and high computational cost. For this reason, many 2D models with different assumptions made on the through-the-thickness mechanical displacement field displacement have been proposed including layerwise approximation [7–9] or multi-layer modelling [10]. More recently, method of sublayers has been proposed [11] which combines a 2D single-layer representation model for the mechanical displacement field with a 3D layerwise-like approximation for the electric potential field. However, these models are rarely used in stress waves problems.
Furthermore, the through-the-thickness electric potential field distribution can be approximated by various models: linear [12] for thin actuators or nonlinear [10; 13] or trigonometric [7]. Kapuria [14] even proposed an electromechanical model that combines the displacement field approximations of the third order zigzag theory with a layerwise approximation for the electric potential. Each electric potential representation model has its own advantages and disadvantages in terms of accuracy and ease of use and computational cost. Moreover, in the case of elastic wave propagation, the major numerical challenge is to minimize numerical dispersion error [15] (error of phase and group velocities). Minimisation of the error requires sufficient discrete spatial resolution per minimum wavelength. In order to obtain solution with a less than 1% dispersive error, explicit finite element method requires more than 20 elements per minimum wavelength [15]. Thus, application of 3D finite elements for wave propagation modelling is impractical.

The spectral element method (SEM) in the time domain proposed by Patera [16] is an alternative to the existing numerical methods in the field of stress waves modelling. This method originates from the use of spectral series for the solution of partial differential equations [17]. The idea of SEM is very similar to FEM except for the specific approximation functions it uses. Elemental interpolation nodes are located at points corresponding to zeros of an appropriate family of orthogonal polynomials (Legendre or Chebyshev). A set of local shape functions consisting of Lagrange polynomials, which are spanned on these points, are built and used. As a consequence of this, as well as the use of the Gauss-Lobatto-Legendre integration rule, a diagonal form of the mass matrix is obtained. In this way the cost of numerical calculations is much less expensive than in the case of any classic FE approach. Moreover, the numerical errors decrease faster than any power of \(1/p\) (so called spectral convergence), where \(p\) is the order of the applied polynomial.

The main fields of application of SEM nowadays include: fluid dynamics [18], heat transfer [19], acoustics [20; 21], seismology [22; 23], etc. The application of SEM for problems of propagating waves in anisotropic crystals has been shown in [24].

The first attempt to the use of SEM for problems of propagation of elastic waves in 2D structural elements with crack has been done by Zak et al. [25]. 36-node spectral membrane element with two degrees of freedom per node has been developed. The crack has been modelled by simple splitting of the nodes between appropriate spectral elements. This approach has been extended to isotropic and composite plates [25–27].

The simplified actuator-induced Lamb wave propagation analysis using 3D SEM has been performed in [28].

The aim of this paper is to develop higher order 3D spectral element, which assures fast convergence of mechanical displacement field as well as electric potential field. Both actuation and sensing is taken into account. Moreover, presented special time integration algorithm based on the central difference scheme and element by element approach overcomes problem of cost of memory resources.

2. Spectral element formulation
Spectral element is derived in the same manner as finite element based on weight residual method. The SEM approximates the field variables in elements using higher-order one-dimensional Lagrange polynomial and its tensor product. The displacement field can be approximated as follows:

\[
\begin{bmatrix}
  u^e(\xi, \eta, \zeta) \\
  v^e(\xi, \eta, \zeta) \\
  w^e(\xi, \eta, \zeta)
\end{bmatrix}
= N^e \hat{u}^e = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{i=0}^{n} N_i^e(\xi) N_j^e(\eta) N_k^e(\zeta) I_3
\begin{bmatrix}
  \hat{u}^e(\xi_i, \eta_j, \zeta_k) \\
  \hat{v}^e(\xi_i, \eta_j, \zeta_k) \\
  \hat{w}^e(\xi_i, \eta_j, \zeta_k)
\end{bmatrix}
\]
where $N_i^n$ is the one-dimensional shape function (nth-order 1D Lagrange interpolation function at $n + 1$ Gauss-Lobatto-Legendre points), $\mathbf{u}^e$ — nodal degrees of freedom, $\mathbf{I}_3$ — unit matrix of the size 3x3. Approximation of potential and geometry of the element can be assumed similarly. Distribution of the nodes in the fifth-order 3D spectral element is shown in Fig. 1. The element has 216 nodes with three mechanical degrees of freedom ($\hat{u}, \hat{v}, \hat{w}$) and one electrical degree of freedom ($\hat{\phi}$) per node.

In this paper spectral elements are used for modelling propagating waves in isotropic or orthotropic plate excited by piezotransducer bonded to the upper surface of the plate (see Fig. 1). However, laminates can be considered too by using layers of spectral elements of various properties in each lamina.

2.1. Linear piezoelectric constitutive equations

The linear piezoelectric constitutive equations in Voigt notation can be expressed as:

$$\sigma = c^E \varepsilon - e^T E$$  \hspace{1cm} (2)

$$D = e \varepsilon + g E$$  \hspace{1cm} (3)

where $\varepsilon$ is the strain vector, $D$ the electric displacement vector, $\sigma$ the stress vector, $c^E$ the elasticity matrix under constant electric field, $e$ the piezoelectric constant matrix, $E$ the electric field vector, $g$ the dielectric constant matrix.

The electric field vector $\mathbf{E}$ is related to the electric potential field $\phi$ by using a gradient vector $\nabla$ [5]:

$$\mathbf{E} = -\nabla \phi$$  \hspace{1cm} (4)

2.2. Approximations of the field variables

Taking into account small deformations of any point in the solid linear strains can be approximated as:

$$\varepsilon^e(\xi, \eta, \zeta) = B^e \mathbf{u}^e$$  \hspace{1cm} (5)
where $B'_{\text{u}}$ is strain–nodal displacement matrix calculated as:

$$B'_{\text{u}} = L \mathbf{N}'(\xi, \eta, \zeta)$$

$$L = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix},
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix}
= J^{-1} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial \zeta}
\end{bmatrix}$$

(6)

where $J$ is the Jacobian matrix. Using Eq. (4) and electric field approximation similar to Eq. (1) electric field vector is given by equation:

$$\mathbf{E} = -B_{\phi} \hat{\phi}$$

(7)

where $B_{\phi}$ is electric field–nodal potential matrix calculated as:

$$B_{\phi} = \begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix} \mathbf{N}'(\xi, \eta, \zeta)$$

(8)

2.3. Elementary governing equations of motion

The governing equations of motion of piezoelectric solid element can be derived by using Hamilton’s variational principle in which the total work done by the external mechanical and electrical forces is taken into account. Finally equations of motion of piezoelectric element can be written in the matrix form:

$$\mathbf{M}^e \ddot{\mathbf{u}}^e + \mathbf{K}_{uu}^e \mathbf{u}^e - \mathbf{K}_{u\phi}^e \dot{\phi}^e = \mathbf{f}^e$$

(9)

$$\mathbf{K}_{\phi u}^e \dot{\mathbf{u}}^e + \mathbf{K}_{\phi\phi}^e \dot{\phi}^e = \mathbf{q}^e$$

(10)

where $\mathbf{M}^e$ denotes the elementary mass matrix, $\mathbf{K}_{uu}^e$ the elementary stiffness matrix, $\mathbf{K}_{u\phi}^e$ and $\mathbf{K}_{\phi u}^e$ the piezoelectric coupling matrices, $\mathbf{K}_{\phi\phi}^e$ the dielectric permittivity matrix, $\dot{\phi}^e$ the elementary electric potential vector, $\mathbf{f}^e$ the nodal external force vector, $\mathbf{q}^e$ the nodal externally
applied charge vector. Particular matrices and vectors from Eq. (10) are defined as:

\[
M_e = \int_{V_e} N^T \rho N dV_e \tag{11}
\]

\[
K_{uu}^e = \int_{V_e} B_u^T c B_u dV_e \tag{12}
\]

\[
K_{\alpha \phi}^e = -\int_{V_e} B_u^T \alpha \phi dV_e \tag{13}
\]

\[
K_{\phi u}^e = (K_{u \phi}^e)^T \tag{14}
\]

\[
K_{\phi \phi}^e = \int_{V_e} B_{\phi}^T g B_{\phi} dV_e \tag{15}
\]

\[
f^e = \int_{V_e} N^T P_{\theta} dV_e + \int_{\Gamma_{S}} N^T P_{S} d\Gamma_{S} + P^e \tag{16}
\]

\[
q^e = \int_{\Gamma_{\phi}} N_{\phi}^T q_{\theta} d\Gamma_{\phi} \tag{17}
\]

The governing equation of motions (9–10) can be further condensed in such a way that the unknown potentials are sacrificed in favor of the unknown displacements:

\[
M \ddot{u} + (K_{uu} + K^S) \dot{u} = f + f^A \tag{18}
\]

where \( K^S \) denotes stiffness matrix induced by electromechanical coupling which depends on electric boundary condition (open circuit, closed circuit, actuator) and is obtained as:

\[
K^S = K_{\phi u}^T K_{\phi u}^{-1} K_{\phi u} \tag{19}
\]

It should be noticed that the elementary dielectric permittivity matrix is not positive definite. To make static condensation (18) possible, electric boundary conditions must be applied. Imposed electrical boundary conditions are applied to matrices \( K_{\phi u} \) and \( K_{\phi \phi} \).

2.4. Actuation

Actuation is accomplished by the equivalent mechanical force vector \( f^A \) of the applied voltage of the piezoelectric actuator which can be expressed as follows:

\[
f^A = K_{u \phi}^A \hat{\theta}^{(A)} \tag{20}
\]

In case of excitation \( V_1 \) applied at the top surface of the piezoelectric electrode and assuming that the electric potential is zero at the bottom surface of the actuator, the induced potential distribution can be calculated using formula:

\[
\hat{\theta}^{(A)} = - (K_{\phi \phi}^A)^{-1} K_{\phi \phi,1} V_1 \tag{21}
\]

in which \( K_{\phi \phi}^A \) and \( K_{\phi \phi,1} \) are the submatrices of the matrix \( K_{\phi \phi} \) corresponding to electrical boundary conditions. For example in case of 3rd-order 3D spectral element \( K_{\phi \phi}^A \) matrix correspond to set of node numbers \((I, I)\), \( I \in [n \cdot n + 1 : n \cdot n \cdot n - n \cdot n] \) whereas \( K_{\phi \phi,1} \) matrix correspond to set of node numbers \((I, I_1)\), \( I_1 \in [1 : n \cdot n, n \cdot n \cdot n - n \cdot n + 1 : n \cdot n \cdot n] \), where colon denotes range of natural numbers.
2.5. Sensing

Sensing is performed according to the equation:

\[ \hat{\phi}^{(S)}(S) = -(K_{\phi\phi}^{S})^{-1}K_{\phi u}^{S} \hat{u} \]  \hspace{1cm} (22)

where \( K_{\phi\phi}^{S} \) and \( K_{\phi u}^{S} \) are the corresponding submatrices of \( K_{\phi\phi} \) and \( K_{\phi u} \), respectively (open circuit is considered here).

2.6. Solving equation of motion

Assuming notation \( F_{t} = f + f^{A} \), \( K = K_{uu} + K^{S} \) and using central difference scheme, the Eq. (18) can be rearranged into the form:

\[
\begin{pmatrix}
\frac{1}{\Delta t^2} M_{0} & \hat{F} \\
\frac{1}{\Delta t^2} M_{0} & -\frac{2}{\Delta t^2} M_{2}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_{t+\Delta t} \\
\mathbf{u}_{t-\Delta t}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{u}_{t} - \Delta t \\
\mathbf{u}_{t} + \Delta t
\end{pmatrix} - \begin{pmatrix}
\hat{F} \\
\hat{F}
\end{pmatrix}
\]  \hspace{1cm} (23)

In Eq. (23) vector \( \hat{F} \) can be calculated element by element without necessity of global stiffness matrix assembly. Proposed algorithm for explicit integration of equation of motion is as follows:

✦ Loop over elements \( e \), \( e \in [1, n_{el}] \)
  ✦ Calculation of characteristic mass matrix \( m^{e} \)
  ✦ Assembly of the diagonal mass matrix into the global vector form:
    \[
    \mathbf{M} = \sum_{e=1}^{n_{el}} A_{e} \text{diag}(m^{e}),
    \]
    where \( A_{e} \) denotes assembly function
  ✦ End of the loop over elements \( e \)
  ✦ Calculation of constants \( a_{0} = 1/\Delta t^2 \), \( a_{2} = 2a_{0} \) and auxiliary vectors \( \mathbf{M}_{0} = a_{0} \mathbf{M}, \mathbf{M}_{2} = a_{2} \mathbf{M}, \mathbf{M}^{a} = 1/\mathbf{M}_{0}^{a} \)
  ✦ Calculation of displacement vector \( \mathbf{u}_{t-\Delta t} \) or setting equal zero
  ✦ Initialisation \( t = t_{0} \)
  ✦ Loop over time steps \( t \)
    ✦ Loop over elements \( e \)
      ✦ Calculation of characteristic stiffness matrix \( k^{e} \) for each element
      ✦ Multiplication \( \hat{F} = k^{e} \mathbf{u}_{I}^{e} \), where \( I \) denotes degrees of freedom corresponding to element \( e \)
      ✦ Assembly of global vector \( \hat{F} = \sum_{e=1}^{n_{el}} A_{e} (\hat{F}^{e}) \)
    ✦ End of loop over elements \( e \)
    ✦ Calculation of the effective force vector \( \hat{R} = F_{t} - \hat{F} + M_{0}^{a} \mathbf{u}_{t}^{a} - M_{2}^{a} \mathbf{u}_{t-\Delta t}^{a} \), where indices \( \alpha \) denotes multiplication without summation (vectorisation)
    ✦ Solution at the time \( t + \Delta t \) using vectorisation \( \mathbf{u}_{t+\Delta t}^{a} = \hat{M}^{a} \hat{R}^{a} \)
    ✦ \( \mathbf{u}_{t-\Delta t} = \mathbf{u}_{t}, \mathbf{u}_{t} = \mathbf{u}_{t+\Delta t}, \ t = t + \Delta t \)
    ✦ End of the loop over time steps \( t \)

3. Numerical calculations

Numerical calculations have been carried out for two different problems. The first is static analysis of a piezoelectric bimorph which was performed in order to validate model. Numerical results of the bimorph consisted of 48 spectral elements have been compared with various
approaches given in [11]. The error was less than 2% for maximum deflection \((L/h = 10)\)
and 0% for electric potential in comparison to full 3D finite element model (ANSYS SOLID5
9600 elements).

The second problem is actuation and reception of elastic waves in aluminium plate of
dimensions 315x315x1 mm\(^3\) by circular and rectangular PZT-4 piezoelectric ceramics perfectly
bonded to the surface of the plate. Piezoelectric material constants can be found in [11], whereas
for aluminium it has been assumed: Young modulus \(E = 71\text{ GPa}\), Poisson ratio 0.33, mass
density \(\rho = 2700 \text{ kg/m}^3\). It has been assumed that excitation signal is five tone burst sine
modulated by Hanning window with the carrier frequency of 100 kHz. According to Lamb’s
wave dispersion equation [29], at this frequency-thickness value, there are only fundamental
symmetric and antisymmetric modes. The more dispersive antisymmetric wave mode has a
wavelength of 9.6 mm, and the symmetric wave mode has a wavelength of 54.3 mm. In order to
assure accurate solution about ten nodes of spectral elements per minimum wavelength should
be used [28].

![Figure 2. Snapshot of guided elastic wave propagation excited by piezotransducer (transverse
displacement after the time 70\(\mu\)s)](image)

Exemplary results of numerical simulations for circular piezotransducer of diameter 5 mm
and rectangular piezotransducer of the width 5 mm and length 25 mm are shown in Fig. 2a and
Fig. 2b, respectively.

4. Conclusions
In spite of long calculation time, 3D spectral elements are excellent for modelling of wave
propagation phenomena including electromechanical coupling. Both mechanical and electrical
displacements converge very fast with increasing mesh density. Hence, accuracy the same or
better as in case of finite elements can be obtained in much less time.

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