Sound from rotors in non-uniform flow

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Abstract
An analysis is presented for the evaluation of the acoustic field of a rotating source in a non-uniform potential flow. Other than the restriction to low flow Mach numbers, the method is exact and general. The variation in radiation properties with source angle is handled by representation as a Fourier series in source angle, giving rise to an asymmetrically varying acoustic field evaluated by summation of the series. The method is used to develop an exact solution for the model problem of a rotor operating near a cylinder in cross-flow and sample calculations demonstrate the accuracy of the technique when compared to full numerical evaluation. The calculations show changes of greater than one decibel in the acoustic field due to flow non-uniformity at a flow Mach number of 0.15, a typical speed for propeller aircraft at takeoff.

1 Introduction
The prediction and analysis of sound from rotating sources is a long-standing problem in acoustics, for its theoretical interest and for its importance in many industrial applications. The problem can be viewed as made up of the determination of the source terms, typically related to the fluid dynamics of the system, and to the propagation of the resulting sound. In this paper, we consider propagation of sound from a known source, on the assumption that the source can be determined by some other means.

Even on linear theory, the calculation of the sound radiated by a rotating source, here modeled as a sinusoidally-varying source distributed on a ring,
contains many features which complicate the calculation. The development of prediction methods dates back some hundred years to the early studies of noise from aircraft propellers [1]. In this work, the source is assumed to be at rest and to radiate into a stationary fluid. The next extension to the theory was the inclusion of flow effects in the form of a uniform flow parallel to the axis of the source [2][3], corresponding to a propeller in forward flight, or to a propeller fixed in a mean flow, as in a wind tunnel. The introduction of flow generates effects not present in the stationary fluid case, because of changes in radiation efficiency caused by the increased speed of the source relative to the fluid, and convective effects on retarded time. The acoustic field, however, remains symmetric about the axis of rotation of the source, because of the axial symmetry of the flow.

The next advance in the theoretical treatment of the problem was the introduction of flow which is not symmetric about the source axis. For a uniform flow his arises when a propeller is inclined at some angle to the incoming flow, for example during take-off [4][5][6][7]. In this case, although the flow is uniform, because it is not aligned with the axis of rotation of the source, the radiation efficiency varies as a function of source angle. Early studies concentrated on the flow-generated unsteady source terms, which were known to be efficient radiators of sound and only later were asymmetric convection effects considered. Even if we consider only a steady source, of constant strength in its frame of reference, this variation in radiation efficiency gives rise to an asymmetric field. The asymmetry can be modeled in a number of ways. In this paper, we adopt the viewpoint of Hanson [6] who predicts the acoustic field as a Fourier series in azimuthal angle about the source axis, with the source also decomposed into a Fourier series or azimuthal modes. When there is no flow, or the flow is not inclined with respect to the rotor axis, each source mode gives rise to one field mode of the same order. When the flow is inclined with respect to the source axis, a single source mode generates multiple field modes, whose magnitudes depend on the source characteristics, and on the angle and speed of the flow.

In this paper, we extend the analysis of radiation from ring sources to the case where the source radiates into a non-uniform flow, a generalization from the existing theories for uniform flow. The only limitation is that the theory used is correct to first order in flow Mach number. In underwater applications this is not a significant restriction, but it does mean that the full range of speeds relevant to aeronautical applications is not covered. In practice, however, noise from aircraft is most troublesome during take-off
and landing when aircraft are near the ground, and in these cases aircraft are typically operating at low flight speed, so the restriction to low Mach number is not as limiting in practice as it might appear.

2 Analysis

Analysis of the problem of radiation from rotating, or ring, sources is handled by treating the source as having sinusoidal variation in azimuthal angle around the ring. Each such term gives rise to an integral which can be expressed exactly by a series expansion and, if necessary, the contribution from terms can be summed to give the overall field at any given frequency. In this paper, we follow the same procedure, but we must take account of the azimuthal variation in radiation properties caused by the non-uniform flow. This is done by representing the variation in flow potential as a set of azimuthal modes which convert the problem of radiation in non-uniform flow into a sum of modes radiating into a static fluid.

In the rest of the paper, we do not consider the determination of the magnitude of source terms, typically found from the aerodynamics of the problem, nor do we extend the analysis to include interaction of the radiated field with bodies in the flow. This is consistent with previous analyses [4, 5, 6] which have looked only at the effects of flow on direct radiation from the source, without considering the secondary effect of scattering by nearby bodies.
2.1 Sound from ring sources

The first step in evaluating the field from a ring source is a standard integral which has been studied extensively for many years \cite{2, 8, 4, 6, 9}. Figure 1 shows the geometry and notation. We adopt cylindrical coordinates \((r, \psi, x)\) with the source lying on a ring of radius \(a_0\) in the plane \(x = 0\). The source has strength \(s(\psi) \exp[-j\omega t]\), with the time variation suppressed from this point onwards. The acoustic potential at a field point \((r, \psi, x)\) is given by integration on the source ring,

\[
\phi(r, \psi, x) = \int_0^{2\pi} \frac{e^{ikR}}{4\pi R} s(\psi_1) \, d\psi,
\]

\[
R^2 = r^2 + x^2 + a_0^2 - 2a_0r \cos(\psi - \psi_1),
\]

with \(c\) speed of sound and wavenumber \(k = \omega/c\). Subscript 1 denotes a variable of integration. Decomposing the source term into a Fourier series in azimuthal angle \(\psi_1\),

\[
s(\psi_1) = \sum_{n=-\infty}^{\infty} s_n e^{in\psi_1},
\]

the potential is rewritten as a sum of terms

\[
\phi(r, \psi, x) = \sum_{n=-\infty}^{\infty} s_n e^{in\psi} I_n(k, a_0; r, x).
\]

The integral \(I_n(k, a_0; r, x)\) is the acoustic potential radiated by a single azimuthal mode of the source,

\[
I_n(k, a_0; r, x) = \int_0^{2\pi} \frac{e^{i(kR+n\psi_1)}}{4\pi R} \, d\psi_1, \tag{1}
\]

\[
R^2 = x^2 + r^2 - 2a_0r \cos \psi_1 + a_0^2,
\]

and is taken as the quantity of interest in the rest of the paper. The integral has an exact series representation which will be used later to calculate the
The acoustic field \( I_n(k, a_0; r, x) \) is given by

\[
I_n(k, a_0; r, x) = \frac{j^{2n+1}}{2} \sum_{q=0}^{\infty} v_q h_{n+2q}^{(1)}(k\rho) j_{n+2q}(k a_0) P_{n+2q}^n(x/\rho),
\]

where \( v_q = (-1)^q (2n + 4q + 1) (2q - 1)!! / (2n + 2q)!! \), \( \rho^2 = x^2 + r^2 \), and \( (\cdot)!! \) is the double factorial, \( h_{n}^{(1)}(\cdot) \) is the spherical Hankel function of the first kind, \( j_{n}(\cdot) \) is the spherical Bessel function, and \( P_{n}^{m}(\cdot) \) is the associated Legendre function.

The form of the acoustic field generated by the ring source is controlled by the parameter \( M_t = ka_0/n \). For a steady source rotating in a uniform flow, this is the Mach number associated with the rotation of the source, and arises in the analysis of propeller noise where it corresponds to the velocity at the “tip” of a rotor blade. When asymmetry is introduced into the system, this gives rise to modes of varying order \( n \), at constant wavenumber \( k \). The parameter also arises in Tyler and Sofrin’s classic study of propagation in ducts \[10\] where the effect of changing the mode order is described in terms of changes to \( M_t \). In particular, the propagation properties change when \( M_t > 1 \), the case of a “supersonic mode”. This happens when a non-uniformity of some kind generates extra modes of order \( n \pm m \), \( m = 1, 2, 3, \ldots \). Should there be a value of \( m \) such that \( m \geq n - ka_0 \), \( M_t \geq 1 \) and the \( n - m \) mode can radiate efficiently. If the amplitude of this mode is large enough, this can lead to large changes in the acoustic field.

The remainder of this paper extends the analysis for a ring source to the problem of radiation into non-uniform flow by converting it into a sum of integrals of the form of Equation 1, using two principal tools, Taylor’s transformation for propagation in low Mach number potential flow, and Panchekha’s formula for expansion of an exponential.

### 2.2 Sound propagation in non-uniform flows

Taylor’s transformation \[11, 12, 13\] is a method for converting problems of propagation in non-uniform flows at low Mach number into equivalent problems in a stationary fluid, by modifying the phase term of the Green’s function as a function of the flow potential. The flow is specified as a velocity...
potential $cM\Phi(x)$ where $M$ is the flow Mach number and $\Phi(x)$ is the velocity potential for unit velocity. To first order in Mach number, the acoustic potential at a point $x$ due to a point source at $x_0$ is given by

$$\phi(x) = \frac{e^{jkR}}{4\pi R} e^{jkM\Phi(x_0)} e^{-jkM\Phi(x)}.$$  \hspace{1cm} (3)

The prediction of sound in a low Mach number non-uniform flow thus becomes a phase-shifted version of the problem in a stationary fluid. The difficulty of the ring source problem lies in representing this phase shift in the radiation integral of Equation 1 in a manner amenable to analysis.

### 2.3 Sound from ring sources in non-uniform flows

Inserting Taylor’s transformation, Equation 3, into Equation 1 yields an integral representation for the acoustic potential in a general potential flow at low Mach number. The integral can be evaluated numerically but this may be time-consuming and yields no physical insight. We now show how the phase term of the Taylor transformation can be represented as a set of azimuthal modes converting the integral of Equation 1 into a sum of modes radiating into a stationary fluid.

Upon substitution of the Taylor transformation into Equation 1, the radiation integral becomes

$$\int_{0}^{2\pi} e^{j(kR+n\psi_1)} \frac{e^{jkM\Phi(\psi_1)}}{4\pi R} \, d\psi_1.$$  \hspace{1cm} (4)

Our task now is to rewrite the exponential containing the flow potential in a form suitable for analytical treatment.

The method used is based on the result of Panchekha [14] for series expansion of the exponential of a Taylor series:

$$\exp(a_0 + a_1 x + a_2 x^2 + \ldots) = e^{a_0} \sum_{n} A_n x^n,$$  \hspace{1cm} (5)

$$A_n = \sum_{p+n} \prod_{k,l \in p} \frac{a_k}{k!},$$  \hspace{1cm} (6)

where $p \vdash n$ denotes partitions $p$ of $n$, where $p$ is a set of terms $k$ and $l$ such that $k_1l_1 + k_2l_2 + \ldots = n$. 
In our case, we require an expansion for the exponential of a function represented as a Fourier series. Writing \( \mu = \exp[j\psi] \),

\[
  f(\psi) = \sum_{q=\infty}^{\infty} F_q e^{j\psi} = \sum_{q=\infty}^{\infty} F_q \mu^q.
\]  

(7)

To develop a Fourier series which can be used for the Taylor transformation, we write

\[
  f(\psi) = F_0 + f_+(\mu) + f_-(\mu),
\]

\[
  f_+(\mu) = \sum_{q=1}^{\infty} F_q \mu^q,
\]

\[
  f_-(\mu) = \sum_{q=1}^{\infty} F_{-q} \mu^{-q},
\]

and applying (5) yields

\[
  e^{\alpha f(\psi)} = e^{\alpha F_0} \left( \sum_{q=0}^{\infty} A_0^{(+)} \mu^q \right) \left( \sum_{q=0}^{\infty} A_0^{(-)} \mu^{-q} \right),
\]  

(8)

\[
  A_0^{(+)} = \sum_{p+q, k,l \in \mathbb{Z}} (\alpha F_l)^k \frac{1}{k!}, \quad A_0^{(-)} = \sum_{p+q, k,l \in \mathbb{Z}} (\alpha F_{-l})^k \frac{1}{k!},
\]

(9)

\[
  A_0^{(+)} = A_0^{(-)} = 0,
\]  

(10)

and the Fourier expansion of the exponential is given by

\[
  e^{\alpha f(\psi)} = \sum_{n=-\infty}^{\infty} c_n e^{j\psi},
\]  

(11)

\[
  c_0 = e^{\alpha F_0} \sum_{u=0}^{\infty} A_0^{(+)} A_u^{(-)},
\]

(12)

\[
  c_q = e^{\alpha F_0} \sum_{u=1}^{\infty} A_{u+q}^{(+)} A_u^{(-)},
\]

(13)

\[
  c_{-q} = e^{\alpha F_0} \sum_{u=q+1}^{\infty} A_{u-q}^{(+)} A_u^{(-)}.
\]  

(14)
Given a source of azimuthal order \( n \), the acoustic potential at some field point \( x \) is then

\[
e^{-jkM\Phi(x)}e^{jn\psi} \int_0^{2\pi} e^{j(kR+n\psi_1)} \frac{e^{jkM\Phi(\psi_1)}}{4\pi R} d\psi_1 = e^{-jkM\Phi(x)} \sum_{q=-\infty}^{\infty} c_q e^{j(n+q)\psi} \int_0^{2\pi} e^{j(kR+(n+q)\psi_1)} \frac{e^{jkM\Phi(\psi_1)}}{4\pi R} d\psi_1, \tag{15}
\]

where the coefficients \( c_n \) are determined using the previous results with \( \alpha = jkM \) and \( F_q \) the coefficients of the Fourier expansion of the flow potential on the source ring. The integrals on the right hand side can be determined using the exact series expression of Equation 1.

### 2.4 Summary of method

The calculation method for the problem of a ring source of order \( n \) radiating into a non-uniform flow of Mach number \( M \) can be summarized:

1. compute the flow potential \( \Phi(x) \) for unit velocity;
2. compute the coefficients of the Fourier series of \( \Phi(x) \) on the source ring;
3. apply the method of Section 2.3 to determine the coefficients \( c_q \);
4. use the series of Equation 2 and the summation of Equation 15 to determine the acoustic field.

In the general case, the Fourier coefficients of \( \Phi(x) \) may have to be determined numerically, but in the next section we present a model problem for which the required terms can be evaluated analytically.

### 3 Model problem

The method as presented so far is quite general as long as the Fourier expansion of the flow potential can be determined. In practice, this will have to be done numerically for many problems, but in the case of a source interacting with a cylinder, a model problem for a rotor near a wing, the Fourier coefficients can be found analytically. The problem contains the essential features of the modification of an acoustic field by a realistic potential flow and it will be seen that the flow gives rise to noticeable effects on the overall field.
3.1 Geometry and notation

The model problem is shown in Figure 2. A wing is represented by an infinite cylinder of radius $a$ lying along the $z$ axis. The source of radius $a_0$ is placed with its axis parallel to the $x$ axis and its center at $(x_0, y_0)$. In calculating the flow potential, cylindrical coordinates $(u, \theta, z)$ are used. In this coordinate system, the source center lies at $(u_0, \theta_0)$ and points on the ring are given by $(x_0, y_0 + a_0 \sin \psi_1, a_0 \cos \psi_1)$.

The solution for potential flow over a cylinder of radius $a$ can be found in elementary fluid mechanics textbooks and in the notation of this paper is

$$\Phi(x) = x \left(1 + \frac{a^2}{u^2}\right),$$

for flow of unit velocity in the positive $x$ direction. Figure 3 shows the streamlines for flow over a cylinder. The non-uniformity of the flow is clear near the cylinder, and it is also clear that the flow is nearly uniform more than about one cylinder radius from the cylinder surface.

In order to evaluate the effect of this flow on radiation from the ring source, we expand $\Phi$ in a Fourier series in $\psi_1$, the angular coordinate on the ring source of Figure 2:

$$\Phi(\psi_1) = x_0 \left(1 + \frac{a^2}{u_0^2 + 2a_0y_0 \sin \psi_1 + a_0^2 \sin^2 \psi_1}\right),$$

$$= \sum_{q=-\infty}^{\infty} \Phi_q e^{i q \psi_1},$$

$$\Phi_q = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\psi_1) e^{-i q \psi_1} \, d\psi_1.$$
Figure 3: Streamlines for flow in positive $x$ direction around a cylinder of radius $a$.

Coefficients of the Fourier series can be evaluated using tabulated integrals [15],

$$
\Phi_0 = x_0 + \frac{a^2}{B a_0} \cos(\gamma - \theta_0),
\Phi_q = (-j)^q \frac{a^2}{B a_0} (B^2 - 2 BS \cos \gamma + S^2)^{q/2} \cos \left[ (q - 1)(\theta_0 - \delta) + \gamma - \delta - q\pi/2 \right], \quad q > 0,
\Phi_{-q} = \Phi_q^*,
$$

where

$$
B = (S^4 + 2S^2 \cos 2\theta_0 + 1)^{1/4},
S = u_0/a_0,
\gamma = \frac{1}{2} \tan^{-1} \frac{\sin 2\theta_0}{S^2 + 2 \cos 2\theta_0},
\delta = \tan^{-1} \frac{B \sin \gamma}{B \cos \gamma - S}.
$$

The exponential containing the flow potential can then be written using
Equation 11 with $\alpha = jkM$ and $F_q = \Phi_q$,

$$e^{jkM\Phi(\psi_1)} = c_0 + \sum_{q=1}^{\infty} \left( c_q e^{jq\psi_1} + c_q^* e^{-jq\psi_1} \right),$$

with

$$c_q = \sum_{p=0}^{\infty} A_{p+q} A_{-p},$$
$$A_n = \sum_{p+n} \prod_{k,l \in p} \frac{(jkM\Phi_l)^k}{k!},$$
$$A_{-n} = \sum_{p+n} \prod_{k,l \in p} \frac{(jkM\Phi_l^*)^k}{k!}.$$

The field can then be computed as

$$\phi(x) = \sum_{q=-\infty}^{\infty} c_q e^{j(n+q)\psi - kM\Phi(x)} I_{n+q}(k; a_0; r, x),$$

reducing the general problem of radiation into a non-uniform flow to a summation of elementary fields generated by individual azimuthal modes.

3.2 Results

To apply the results of the paper to a representative problem, parameters have been estimated for the ATR class of turboprop aircraft. The propeller diameter and speed are 3.93m and 1200rpm respectively, so that the rotation Mach number for the source is 0.73. The cylinder diameter has been set equal to twice the maximum thickness of the ATR wing, with $a = 0.29m$. Flow Mach number $M = 0.15$, a typical value for takeoff of a turboprop aircraft. Results have been calculated for a baseline case of $n = 6$, representing the lowest harmonic of the tonal noise from a six-bladed propeller. Figure 4 shows the relative positions of the rotor and cylinder, with silhouettes of the three configurations considered. The source is placed at negative $x_0$ to correspond to a rotor ahead (upstream) of a wing leading edge.

Sample calculations for the problem are presented in Figure 5, in the form of the magnitude of acoustic potential with flow, scaled on the no-flow
Figure 4: Rotor positions for parametric study: left-hand side, notation; right-hand side, relative positions of rotor and cylinder. Case a: \((x_0, y_0) = (-a_0, -a)\); b: \((x_0, y_0) = (-5a/2, -a_0/2)\); c: \((x_0, y_0) = (-5a/4, -a_0/2)\).

case. As a check on the method, the integral of Equation 4 was evaluated numerically and its magnitude is also plotted. Results are shown for \(n = 6\) in each of the configurations of Figure 4 and for \(n = 8\) in the third configuration. In each case the field is calculated in the source plane on a circle of radius \(r = 2a_0\). In considering the magnitude of flow effects on the radiated noise, it is useful to note that Hanson found that a flow of Mach number \(M = 0.2\) at \(10^\circ\) to the rotor axis altered the sound level by about 1.5dB. In our case, the flow is nominally at 0° and all asymmetry in the system arises from the flow non-uniformity.

The first plot, for a rotor some distance upstream of the cylinder, shows a very small change in the acoustic potential. This is as might be expected from Figure 3 since the rotor lies in a region of flow which is almost uniform and very little asymmetry will be introduced by the changes in potential around the source ring. Moving the source closer to the cylinder, however, generates quite large changes in the acoustic field which, we emphasize, are purely the result of changes in the radiation properties of the source, which is held constant throughout the calculations. In the second plot, with the source brought within two radii of the cylinder surface, the field is modified by up to 0.3dB. In the third plot, moving the source closer still, a quarter radius from the cylinder surface, generates a larger change in the acoustic field, of some 0.6dB. The largest change shown, however, comes with the rotor in the same position, close to the cylinder, but with \(n\) increased from 6 to 8, approximating the replacement of a six-bladed propeller with an eight-
Figure 5: Magnitude of computed acoustic potential with $M = 0.15$ flow, scaled on no-flow result. From top to bottom: case $a$, $b$, $c$, and $c$ with $n = 8$. Solid line analytical result, dashed line numerical evaluation.
bladed. The change in the acoustic field is now greater than 1dB, comparable to the effect of a flow at an angle of 10° in Hanson’s study of incidence effects. This effect is caused by the introduction of new modes with higher $M_t$ radiating efficiently into the field.

4 Conclusions

An analysis has been developed for the prediction of sound from rotating sources in a low Mach number non-uniform potential flow, giving an exact formulation for the radiated field as a sum of azimuthal modes whose amplitude is determined by the local flow. The theory has been applied to the model problem of a rotor operating in the vicinity of a cylinder, for which the required coefficients can be found analytically. Sample calculations compare well to numerical evaluation of the acoustic field and the results show changes in the field which vary strongly with rotor position relative to the cylinder. The effect of flow non-uniformity on radiation properties has been found to be comparable in magnitude to the effect of incidence in propeller acoustics. Future work will consider disk sources, which have finite radial extent modeling propeller blades of finite span, and the effects of wing lift on the radiated noise, as finite circulation on the cylinder introduces changes in the flow potential. An important open question remains that of the relationship between the model problem and realistic systems which require computational aeroacoustics methods for their prediction.

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