Cryptanalysis of the Hillery-Bužek-Berthiaume quantum secret-sharing protocol

Su-Juan Qin¹,², Fei Gao¹, Qiao-Yan Wen¹, and Fu-Chen Zhu³
(1. State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing, 100876, China)
(2. School of Science, Beijing University of Posts and Telecommunications, Beijing, 100876, China)
(3. National Laboratory for Modern Communications, P.O.Box 810, Chengdu, 610041, China)

Email: qsjuan@sohu.com
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The participant attack is the most serious threat for quantum secret-sharing protocols. We present a method to analyze the security of quantum secret-sharing protocols against this kind of attack taking the scheme of Hillery, Bužek, and Berthiaume (HBB) [Phys. Rev. A 59 1829 (1999)] as an example. By distinguishing between two mixed states, we derive the necessary and sufficient conditions under which a dishonest participant can attain all the information without introducing any error, which shows that the HBB protocol is insecure against dishonest participants. It is easy to verify that the attack scheme of Karlsson, Koashi, and Imoto [Phys. Rev. A 59, 162 (1999)] is a special example of our results. To demonstrate our results further, we construct an explicit attack scheme according to the necessary and sufficient conditions. Our work completes the security analysis of the HBB protocol, and the method presented may be useful for the analysis of other similar protocols.

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I. INTRODUCTION

Quantum cryptography is a technique which permits parties to communicate over an open channel in a secure way. Quantum secret sharing (QSS) is an important branch of quantum cryptography, which allows a secret to be shared among many participants in such a way that only the authorized groups can reconstruct it. In fact, there are two types in quantum secret sharing, that is, the sharing of classical secret and that of quantum information. The former was first proposed by Hillery, Bužek and Berthiaume (HBB) [1] (called HBB hereafter), and the latter was first presented by Cleve, Gottesman and Lo [2]. Since the above pioneering works appeared, QSS has attracted a great deal of attention (please see [3, 4] for the sharing of classical secret and [5] for that of quantum information).

As we know, the designing schemes and analyzing their security are two inherent directions of cryptography, which are opposite to but stimulate each other. Each of them is necessary to the development of cryptography. This is also the case in quantum cryptography [6, 7, 8, 9, 10, 11]. However, because the theory of quantum information remains still far from satisfactorily known, the development of quantum cryptanalysis is relatively slow, especially in QSS. In fact, it is complex to analyze the security of QSS protocols because multiple participants are involved and not all are honest, and therefore few results [12, 13, 14] have been obtained.

In this paper, we present a method to analyze the security of QSS protocols taking the HBB scheme [1] as an example. The security of HBB has been discussed from several aspects. Ref. [1] analyzed an intercept-resend attack by a dishonest participant and an entangle-measure attack by an external attacker. References [12, 13, 14] investigated the relation between security and the violation of some Bell’s inequalities by analyzing several eavesdropping scenarios. However, their analyses are incomplete because not all the individual attacks are covered. Reference [3] showed that the HBB scheme was insecure to a skillful attack, and gave a remedy; but this analysis is not systematic. Here, we consider the original HBB protocol and give a complete and systematic analysis of security against a participant attack. From our analysis we also get the same result as Ref. [3], and, moreover, we derive the necessary and sufficient (NAS) conditions for a successful attack, which is more important. From the NAS conditions, we can find many attack schemes easily (including the eavesdropping strategy in Ref. [3]), which will deal with the difficulty that breaking a protocol is unsystematic. Although the result is partly not new [3], the method (which is indeed our main aim) is. This method might be useful for the analysis of other protocols.

The paper is structured as follows. In Sec. II, we review the HBB protocol briefly. In Sec. III, we analyze general participant attack strategies, and derive the NAS conditions under which a dishonest participant attains the whole secret without introducing any error. In Sec. IV, we give a simple scheme to achieve the attack successfully. Finally, we give a conclusion and discussion in Sec. V. Cumbersome computations and formulas are summarized in the Appendix.

II. THE HBB PROTOCOL

Let us introduce the principle of the HBB scheme [1] first. The dealer Alice wants to divide her secret mes-
TABLE I: Correlations between Alice’s, Bob’s measurement results and Charlie’s results. Alice’s (Bob’s) measurement results are listed in the first column (line).

| Alice/Bob | $x^+$ | $x^-$ | $y^+$ | $y^-$ |
|-----------|--------|--------|--------|--------|
| $x^+$ | $x^+$ | $x^-$ | $y^+$ | $y^-$ |
| $x^-$ | $x^-$ | $x^+$ | $y^-$ | $y^+$ |
| $y^+$ | $y^-$ | $y^+$ | $x^-$ | $x^+$ |
| $y^-$ | $y^+$ | $y^-$ | $x^+$ | $x^-$ |

sage between her two agents, Bob, and Charlie. At the beginning, Alice prepares a sequence of GHZ triplets in the state $(1/\sqrt{2})(|000\rangle + |111\rangle)_{ABC}$, where the subscripts $A$, $B$ and $C$ denote the three particles for Alice, Bob and Charlie, respectively. For each triplet, Alice keeps particle $A$ and sends particle $B$ to Bob and $C$ to Charlie. As in the Bennett-Brassard 1984 scheme [15] scheme, all the three parties choose randomly the measuring basis (MB) $x$ or $y$ to measure their particles and then they publish their MBs. The announcement should be done in the following way: Bob and Charlie both send their MBs to Alice, who then sends all three MBs to Bob and Charlie [16]. Note that no one can learn other’s bases before having to reveal his, otherwise as pointed out in Ref. [1], he could cheat more successfully. When the number of the parties who choose $x$ is odd, the outcomes are useful. Thanks to the features of the GHZ state, Charlie and Bob can deduce the outcomes of Alice when they cooperate (see Table I [3]). To check for eavesdropping, Alice chooses randomly a large subset of the outcomes to analyze the error rate. That is, Alice requires Bob and Charlie to announce their outcomes of the samples in public. If the error rate is lower than a threshold value, they keep the remaining outcomes as secret key.

III. THE ATTACK ON THE HBB PROTOCOL

Now let us give a complete discussion of the security of the HBB scheme. As pointed out in Refs. [17, 18, 19], a participant generally has more advantages in an attack than an outside eavesdropper in the secret-sharing protocols. If a QSS protocol is secure for a dishonest participant, it is secure for any eavesdropper. Therefore, to analyze the security, we should concentrate our attention on participant attack. Without loss of generality, we assume the attacker is Charlie, denoted Charlie*. He seeks to learn Alice’s secret himself without introducing any error during the eavesdropping check. In order to take advantage of Alice’s and Bob’s delayed information about their MBs, a wise attack strategy for Charlie* is as follows. When the qubits $B$ and $C$ are sent out by Alice, he lets an ancilla, initially in some state $|\chi\rangle$, interact unitarily with them (the dimensionality of the ancilla is a free variable which causes no loss in generality). After the interaction, Charlie* sends qubit $B$ to Bob, stores qubit $C$ and his ancilla until Alice announces the MBs used by the three parties. Finally, Charlie measures the qubits at his site to achieve the secret according to Alice’s announcements.

We now describe the procedure in detail. After Alice sends out the two qubits, $B$ and $C$, Charlie* intercepts them and they interacts with his ancilla. After that, the state of the whole system may be written as

$$|\Psi\rangle_{ABCE} = \sum_{i,j=0}^{1} a_{ij} |ij\rangle_{AB} |\varepsilon_{ij}\rangle_{CE},$$  \hspace{1cm} (1)

where $|\varepsilon_{ij}\rangle$ refers to the state of Charlie* after the interaction and is normalized, and $a_{ij}$ is complex number that satisfies

$$\sum_{i,j=0}^{1} |a_{ij}|^2 = 1.$$

A. The conditions to escape detection

As mentioned above, to use the information about Alice’s and Bob’s MBs, Charlie* does not measure his qubits until Alice reveals them, and then he can choose different methods accordingly. Note that when Alice requires Charlie* to declare his MBs, Charlie* generates a random sequence of $x$ and $y$ to forge his MBs, actually he does not measure any qubit. If the MBs chosen by all the three parties satisfy the condition that the number of $x$ is odd, the results are kept, otherwise they are discarded. Therefore Charlie* knows Alice’s and Bob’s MBs for every useful triplet which can be utilized in the subsequent steps. When some triplets are chosen by Alice to detect eavesdropping, Charlie* then measures his corresponding qubits and announces outcomes according to Alice’s and Bob’s MBs. Now we explore the conditions they must be satisfied if Charlie* wants to escape from being detected.

Let us first consider the case where both Alice and Bob measure their qubits in $x$ direction, and of course, Charlie* declares $x$. The state of the whole system $|\Psi\rangle_{ABCE}$ can be rewritten as
We can see from Table 4 that without eavesdropping, if Alice’s and Bob’s results are $x^+x^+$ or $x^-x^-$, Charlie*’s announcement should be $x^+$, otherwise, his announcement should be $x^-$. In a convenient depiction, we denote Charlie*’s state as $|\varphi_{jkn}^\pm\rangle$ which is normalized, when Alice’s and Bob’s results are $j^m$ and $k^n$, where $j, k \in \{x, y\}$ and $m, n \in \{+, -\}$. To avoid being found out, Charlie* should have the ability to discriminate completely between the two sets $\{|\varphi_{x^+x^+}\rangle, |\varphi_{x^-x^-}\rangle\}$, $\{|\varphi_{x^+x^-}\rangle, |\varphi_{x^-x^+}\rangle\}$. As shown in Ref. 20, two sets $S_1$, $S_2$ can be perfectly discriminated if and only if the subspaces they span are orthogonal. So the scalar products of Charlie*’s states have to satisfy four constraints:

$$ \begin{align}
\langle \varphi_{x^+x^+} | \varphi_{x^-x^-} \rangle & = 0, \\
\langle \varphi_{x^+x^-} | \varphi_{x^-x^+} \rangle & = 0, \\
\langle \varphi_{x^-x^-} | \varphi_{x^-x^+} \rangle & = 0, \\
\langle \varphi_{x^-x^-} | \varphi_{x^+x^-} \rangle & = 0.
\end{align} \quad (4)$$

From Eqs. (3) and (4), we obtain

$$ \begin{align}
a_{00}^o a_{01} (|\varepsilon_{00}\rangle |\varepsilon_{01}\rangle - a_{01}^* a_{10} (|\varepsilon_{00}\rangle |\varepsilon_{10}\rangle) = 0, \\
a_{00}^o a_{01} (|\varepsilon_{00}\rangle |\varepsilon_{10}\rangle - a_{10}^* a_{01} (|\varepsilon_{01}\rangle |\varepsilon_{01}\rangle) = 0, \\
a_{01}^2 - a_{01}^* a_{01} (|\varepsilon_{01}\rangle |\varepsilon_{10}\rangle + a_{10} a_{01} (|\varepsilon_{10}\rangle |\varepsilon_{01}\rangle) - |a_{10}|^2 = 0, \\
|a_{00}|^2 - a_{00}^* a_{11} (|\varepsilon_{00}\rangle |\varepsilon_{11}\rangle + a_{11} a_{00} (|\varepsilon_{11}\rangle |\varepsilon_{00}\rangle) - |a_{11}|^2 = 0.
\end{align} \quad (5)$$

Similarly, the constraints are then found in the Appendix for other cases. Finally, we obtain results from Eqs. (5), (A.3), (A.6) and (A.9):

$$ \begin{align}
a_{00}^o a_{01} (|\varepsilon_{00}\rangle |\varepsilon_{01}\rangle) = a_{01}^* a_{10} (|\varepsilon_{00}\rangle |\varepsilon_{10}\rangle) = 0, \\
a_{00} a_{11} (|\varepsilon_{01}\rangle |\varepsilon_{11}\rangle) = a_{10} a_{01} (|\varepsilon_{00}\rangle |\varepsilon_{10}\rangle) = 0, \\
a_{01} a_{11} (|\varepsilon_{11}\rangle |\varepsilon_{01}\rangle) = a_{10} a_{01} (|\varepsilon_{10}\rangle |\varepsilon_{11}\rangle) = 0, \\
|a_{00}| = |a_{11}|, \\
|a_{01}| = |a_{10}|.
\end{align} \quad (6)$$

Obviously, Charlie* can succeed in escaping detection by Alice and Bob when his operations satisfy Eq. (6).

**B. The maximum information the attacker can attain**

After escaping from detection, Charlie* measures the remaining qubits to deduce Alice’s secret. Now let us compute the maximum information that Charlie* can gain. From Eqs. (3) and (6), we can see if Alice’s result is $x^+$, Charlie*’s state collapses to $|\varphi_{x^+x^+}\rangle$ or $|\varphi_{x^+x^-}\rangle$ with equal probability, otherwise collapses to $|\varphi_{x^-x^-}\rangle$ or $|\varphi_{x^-x^+}\rangle$ with equal probability. So to get information of Alice’s result, $x^+$ or $x^-$, Charlie* should distinguish between two mixed states $\rho_{x^+} = \frac{1}{2} |\varphi_{x^+x^+}\rangle \langle \varphi_{x^+x^+}| + \frac{1}{2} |\varphi_{x^+x^-}\rangle \langle \varphi_{x^+x^-}|$ and $\rho_{x^-} = \frac{1}{2} |\varphi_{x^-x^-}\rangle \langle \varphi_{x^-x^-}| + \frac{1}{2} |\varphi_{x^-x^+}\rangle \langle \varphi_{x^-x^+}|$ occurring with equal a priori probability. Generally, there are two ways to discriminate between two states, minimum error discrimination and unambiguous discrimination. In Ref. 21, the authors showed the minimum failure probability $Q_E$ attainable in unambiguous discrimination is always at least twice as large as the minimum-error probability $P_E$ in ambiguous discrimination for two arbitrary mixed quantum states. So we should take the ambiguous discrimination to get the maximum information.

Using the well-known result 22 that to discriminate between two mixed states $\rho_1$ and $\rho_2$ occurring with a priori probabilities $p_1$ and $p_2$, respectively, where $p_1 + p_2 = 1$, the minimum-error probability attainable is $P_E = \frac{1}{2} - \frac{1}{2} |p_2 \rho_2 - p_1 \rho_1|$, where $|\langle A| \rangle = \text{Tr} \sqrt{A A^\dagger}$, we get the minimum-error probability to discriminate between $\rho_{x^+}$ and $\rho_{x^-}$ under the constraints of Eq. (6)

$$ P_E = \frac{1}{2} (1 - 4|a_{00}| \cdot |a_{10}|). \quad (7)$$

Considering the other three cases (see the Appendix) with similar strategy, we get the same results as Eq. (7).

The mutual information between Alice and Charlie* in terms of Shannon entropy is

$$ I^{AC} = 1 + P_E \log P_E + (1 - P_E) \log (1 - P_E). \quad (8)$$

Now the task is maximizing $I^{AC}$ with the constraints of Eqs. (2) and (6). Using the Lagrange multiplier method, we attain the maximum $I^{AC}_{\text{max}} = 1$ under conditions

$$ \begin{align}
\langle \varepsilon_{00} | \varepsilon_{01} \rangle = \langle \varepsilon_{00} | \varepsilon_{10} \rangle = \langle \varepsilon_{00} | \varepsilon_{11} \rangle = 0, \\
\langle \varepsilon_{01} | \varepsilon_{10} \rangle = \langle \varepsilon_{01} | \varepsilon_{11} \rangle = \langle \varepsilon_{10} | \varepsilon_{11} \rangle = 0, \\
|a_{00}| = |a_{10}| = |a_{11}| = \frac{1}{2}.
\end{align} \quad (9)$$

Now, we have the NAS conditions for a dishonest participant to attack HBB successfully. Therefore the HBB protocol is insecure (in its original form). Obviously, $|\varepsilon_{00}\rangle$, $|\varepsilon_{01}\rangle$, $|\varepsilon_{10}\rangle$, and $|\varepsilon_{11}\rangle$ are orthogonal to each other, which indicates that a dishonest participant need prepare one additive qubit at least. It is easy to verify that the eavesdropping strategy in Ref. 3 is a special example of our results, where two additive qubits are used and $a_{00} |000\rangle = \frac{1}{2} |000\rangle$, $a_{01} |010\rangle = - \frac{1}{2} |001\rangle$, $a_{10} |101\rangle = \frac{1}{2} |110\rangle$, and $a_{11} |111\rangle = - \frac{1}{2} |111\rangle$. 

$$ |\psi\rangle_{ABCE} = \frac{1}{2} \left[ |x^+\rangle_A |x^+\rangle_B (a_{00} |\varepsilon_{00}\rangle + a_{01} |\varepsilon_{01}\rangle + a_{10} |\varepsilon_{10}\rangle + a_{11} |\varepsilon_{11}\rangle)_{CE} + |x^+\rangle_A |x^-\rangle_B (a_{00} |\varepsilon_{00}\rangle - a_{01} |\varepsilon_{01}\rangle - a_{10} |\varepsilon_{10}\rangle - a_{11} |\varepsilon_{11}\rangle)_{CE} + |x^-\rangle_A |x^+\rangle_B (a_{00} |\varepsilon_{00}\rangle + a_{01} |\varepsilon_{01}\rangle - a_{10} |\varepsilon_{10}\rangle - a_{11} |\varepsilon_{11}\rangle)_{CE} + |x^-\rangle_A |x^-\rangle_B (a_{00} |\varepsilon_{00}\rangle - a_{01} |\varepsilon_{01}\rangle + a_{10} |\varepsilon_{10}\rangle + a_{11} |\varepsilon_{11}\rangle)_{CE} \right].$$
IV. AN EXAMPLE OF SUCCESSFUL ATTACK

According to Eq. (9), we can construct some attack schemes easily. Here we give an even simpler scheme than Ref. 3 with only one additive qubit. Generally, the ancilla is the standard state $|0\rangle$. We choose $a_{00}|e_{00}\rangle = \frac{1}{2}|00\rangle$, $a_{01}|e_{01}\rangle = \frac{1}{2}|01\rangle$, $a_{10}|e_{10}\rangle = \frac{1}{2}|10\rangle$, and $a_{11}|e_{11}\rangle = -\frac{1}{2}|11\rangle$ which satisfy Eq. (9). Comparing the initial state with the state after interaction (see Eq. (1)), we can derive the operations performed by Charlie*.

Now we describe the attack orderly. Charlie* prepares the ancilla $E$ in state $|0\rangle$. After Alice sends out two qubits $B$ and $C$, Charlie* intercepts them, performs $H = (|0\rangle\langle 0| + |1\rangle\langle 1|)/\sqrt{2}$ on the qubit $B$ and CNOT operation on $B, E$ (see Fig. 1). The entangled state of Alice, Bob and Charlie* is converted from $|\Psi_0\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)_{ABC} \otimes |0\rangle_E$ to

$$|\Psi_1\rangle = \frac{1}{2}(|00\rangle_{AB}|0\rangle_{CE} + |01\rangle_{AB}|01\rangle_{CE} \quad (10)
+ |10\rangle_{AB}|10\rangle_{CE} - |11\rangle_{AB}|11\rangle_{CE}).$$

After Alice and Bob measure their qubits, the whole system is changed into $|\Psi_2\rangle$ (see Fig. 2 and Fig. 3) which varies according to their MBs. Let us describe all the cases in detail.

(i) If both Alice’s and Bob’s MBs are $x$, Charlie*’s state collapses to one of the four results

$$|\varphi_{x+x^+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)_{CE},$$
$$|\varphi_{x+x^-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)_{CE}, \quad (11)
|\varphi_{x-x^+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)_{CE},$$
$$|\varphi_{x-x^-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)_{CE}.$$ 

(ii) When Alice and Bob measure their qubits in $x, y$ basis, respectively, Charlie*’s state may be one of the four states

$$|\varphi_{x+y^+}\rangle = \frac{1}{2}(|00\rangle - i|01\rangle + |10\rangle + i|11\rangle)_{CE},$$
$$|\varphi_{x+y^-}\rangle = \frac{1}{2}(|00\rangle + i|01\rangle + |10\rangle - i|11\rangle)_{CE}, \quad (12)
|\varphi_{x-y^+}\rangle = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)_{CE},$$
$$|\varphi_{x-y^-}\rangle = \frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle + i|11\rangle)_{CE}.$$ 

(iii) When Alice and Bob measure their qubits in $y$ basis, respectively, Charlie*’s state may be one of the four states

$$|\varphi_{y+y^+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - i|10\rangle + i|11\rangle)_{CE},$$
$$|\varphi_{y+y^-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle - i|10\rangle - i|11\rangle)_{CE}, \quad (13)
|\varphi_{y-y^+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + i|10\rangle - i|11\rangle)_{CE},$$
$$|\varphi_{y-y^-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle + i|10\rangle + i|11\rangle)_{CE}.$$ 

(iv) When Alice’s and Bob’s MBs are $y$, Charlie*’s state collapses to one of the four results

$$|\varphi_{y+y^+}\rangle = \frac{1}{2}(|00\rangle - i|01\rangle - i|10\rangle + |11\rangle)_{CE},$$
$$|\varphi_{y+y^-}\rangle = \frac{1}{2}(|00\rangle + i|01\rangle - i|10\rangle - |11\rangle)_{CE}, \quad (14)
|\varphi_{y-y^+}\rangle = \frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle - |11\rangle)_{CE},$$
$$|\varphi_{y-y^-}\rangle = \frac{1}{2}(|00\rangle + i|01\rangle + i|10\rangle + |11\rangle)_{CE}.$$ 

It is easy to validate that the four states are orthogonal to each other in every case, which implies that they can be distinguished perfectly. Consequently, Charlie* can not only get the secret of Alice but also escape from
Alice and Bob measure their qubits, Charlie* measures qubit C in the same basis as Alice, and qubit E in computational basis. He can deduce Alice’s results from his measurement outcomes.

We take case (i) as an example to describe Charlie*’s operations. Let us first explain how Charlie* can escape from being detected when the qubits are chosen to check eavesdropping. Charlie* wants to deduce his proper decryption or distill information. Therefore there are some simple ways to fulfill Charlie*’s objective.

We now discuss how Charlie* can obtain the secret information from his qubits. He only needs distinguish between \( \{ |\varphi_{x^+}\rangle, |\varphi_{x^-}\rangle \} \) and \( \{ |\varphi_{x^+}\rangle, |\varphi_{x^-}\rangle \} \) to get Alice’s secret \( x^+ \) or \( x^- \). The circuit to achieve this task is illustrated in Fig. 2. After the \( U \) operation, the states in Eq. (11) are changed into

\[
|\varphi_{x^+}\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)_{CE},
|\varphi_{x^-}\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_{CE},
|\varphi_{x^-}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{CE},
|\varphi_{x^+}\rangle = \frac{1}{\sqrt{2}} (-|01\rangle + |10\rangle)_{CE}.
\]

Then Charlie* measures each qubit in computational basis. If the measurement results of C, E are 00 or 11, Charlie*’s announcement is 1 (corresponding to \( |1\rangle, |x^-\rangle \) or \( |y^-\rangle \) hereafter), otherwise his announcement is 0 (corresponding to \( |0\rangle, |x^+\rangle \) or \( |y^+\rangle \) hereafter). According to Table II we can see no error occurs, and therefore Charlie* can escape from being detected.

We now discuss how Charlie* can obtain the secret information from his qubits. He only needs distinguish between \( \{ |\varphi_{x^+}\rangle, |\varphi_{x^-}\rangle \} \) and \( \{ |\varphi_{x^+}\rangle, |\varphi_{x^-}\rangle \} \) to get Alice’s secret \( x^+ \) or \( x^- \). The circuit to achieve this task is illustrated in Fig. 3. After the \( W \) operation, the
paper is to give a method to analyze a participant attack in QSS. We introduce this method taking the HBB scheme \[1\] as an example. A dishonest participant intercepts all the qubits, they interact with his ancilla, and he then resends them out. He then measures his qubits after other participants reveal their useful information. By discriminating between two mixed states, we obtain the NAS conditions under which the dishonest participant can attain all the information without introducing any error. This result shows that the HBB protocol is insecure (in its original form). Finally, we give an example achieving the proposed attack to demonstrate our results further.

Although the result that the HBB scheme is insecure (in its original form) is not new, the method of analyzing the participant attack is, to our knowledge. The treatment we have presented appears to cover all individual participant attacks allowed by physical laws. This method can be applied to other similar QSS protocols with some modifications. We believe that this method would be useful in designing related schemes and analyzing their security. On the one hand, we can construct attack strategies easily according to the NAS conditions when a protocol has security loopholes. On the other hand, we can show that protocol is secure if the attack conditions cannot be reached. For example, applying this method to the enhanced protocol \[3\], we can show it is secure (Such analysis is beyond the scope of this paper).

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**APPENDIX A: CONSTRAINTS ON CHARLIE*'S PROBES**

In this appendix, we find the conditions which Charlie*'s operations need satisfy when no errors are to occur in the procedure of detection in other three cases.

(1) When Alice, Bob and Charlie* choose the MBs \(x, y, x\) respectively, the whole system \(|\Psi\rangle_{ABCE}\) can be rewritten as

\[
\begin{align*}
|\Psi\rangle_{ABCE} &= \frac{1}{2} \left[ |x^+ y^+\rangle (a_0 |00\rangle + ia_0 |01\rangle - a_1 |10\rangle - ia_1 |11\rangle) \\
&\quad + |x^+ y^-\rangle (a_0 |00\rangle + ia_0 |01\rangle + a_1 |10\rangle + ia_1 |11\rangle) \\
&\quad + |x^- y^+\rangle (a_0 |00\rangle - ia_0 |01\rangle - a_1 |10\rangle + ia_1 |11\rangle) \\
&\quad + |x^- y^-\rangle (a_0 |00\rangle + ia_0 |01\rangle - a_1 |10\rangle - ia_1 |11\rangle) \right].
\end{align*}
\]

(A1)

According to Table II, when Alice’s and Bob’s results are \(x^+ y^+\) or \(x^- y^-\), Charlie*’s announcement should be \(y^-\), otherwise, his announcement should be \(y^+\). Therefore, Charlie* should be capable of distinguishing between the two sets, \(|\varphi_{x^+ y^+}, \varphi_{x^- y^-}\rangle\) and \(|\varphi_{x^- y^-}, \varphi_{x^+ y^+}\rangle\), to avoid being detected. That is

\[
\begin{align*}
\langle \varphi_{x^+ y^+} | \varphi_{x^+ y^-} \rangle &= 0, \\
\langle \varphi_{x^+ y^-} | \varphi_{x^- y^+} \rangle &= 0, \\
\langle \varphi_{x^- y^-} | \varphi_{x^- y^+} \rangle &= 0, \\
\langle \varphi_{x^- y^+} | \varphi_{x^+ y^-} \rangle &= 0.
\end{align*}
\]

(A2)

Then we get

\[
\begin{align*}
a_{00} a_{01} (\varepsilon_{00} |01\rangle) + a_{11} (\varepsilon_{11} |10\rangle) &= 0, \\
a_{00} a_{10} (\varepsilon_{00} |10\rangle) - a_{11} a_{01} (\varepsilon_{11} |01\rangle) &= 0, \\
|a_{01}|^2 - ia_{01} a_{10} (\varepsilon_{10} |01\rangle) - ia_{10} a_{01} (\varepsilon_{10} |01\rangle) &= 0, \\
|a_{00}|^2 + ia_{00} a_{11} (\varepsilon_{00} |11\rangle) + ia_{11} a_{00} (\varepsilon_{11} |00\rangle) &= 0.
\end{align*}
\]

(A3)

(2) When Alice, Bob and Charlie* choose the MBs \(y, x, y\), respectively, \(|\Psi\rangle_{ABCE}\) can be rewritten as

\[
\begin{align*}
|\Psi\rangle_{ABCE} &= \frac{1}{2} \left[ |y^+ x^+\rangle (a_0 |00\rangle + a_0 |01\rangle - ia_1 |10\rangle - ia_1 |11\rangle) \\
&\quad + |y^+ x^-\rangle (a_0 |00\rangle - a_0 |01\rangle - ia_1 |10\rangle + ia_1 |11\rangle) \\
&\quad + |y^- x^+\rangle (a_0 |00\rangle + a_0 |01\rangle + ia_1 |10\rangle + ia_1 |11\rangle) \\
&\quad + |y^- x^-\rangle (a_0 |00\rangle - a_0 |01\rangle + ia_1 |10\rangle - ia_1 |11\rangle) \right].
\end{align*}
\]

(A4)

According to Table III the results, \(y^+ x^+\) or \(y^- x^-\), imply Charlie*’s announcement should be \(y^-\), and others imply \(y^+\). For the same reason, we let

\[
\begin{align*}
\langle \varphi_{y^+ x^+} | \varphi_{y^+ x^-} \rangle &= 0, \\
\langle \varphi_{y^+ x^-} | \varphi_{y^- x^+} \rangle &= 0, \\
\langle \varphi_{y^- x^-} | \varphi_{y^- x^+} \rangle &= 0, \\
\langle \varphi_{y^- x^+} | \varphi_{y^- x^-} \rangle &= 0.
\end{align*}
\]

(A5)

We then have

\[
\begin{align*}
a_{00} a_{01} (\varepsilon_{00} |01\rangle) - a_{11} a_{10} (\varepsilon_{11} |10\rangle) &= 0, \\
a_{00} a_{10} (\varepsilon_{00} |10\rangle) + a_{11} a_{01} (\varepsilon_{11} |01\rangle) &= 0, \\
|a_{01}|^2 + ia_{01} a_{10} (\varepsilon_{10} |01\rangle) + ia_{10} a_{01} (\varepsilon_{10} |01\rangle) &= 0, \\
|a_{00}|^2 + ia_{00} a_{11} (\varepsilon_{00} |11\rangle) + ia_{11} a_{00} (\varepsilon_{11} |00\rangle) &= 0.
\end{align*}
\]

(A6)
The results, $y^+y^+$ or $y^-y^-$, imply Charlie’s announcement should be $x^-$, and others imply $x^+$. For the same reason, we let

$$\begin{align*}
\langle \varphi_{y^+y^+} | \varphi_{y^+y^+} \rangle &= 0, \\
\langle \varphi_{y^+y^+} | \varphi_{y^-y^+} \rangle &= 0, \\
\langle \varphi_{y^-y^-} | \varphi_{y^-y^-} \rangle &= 0.
\end{align*}$$

(A7)

We then have

$$\begin{align*}
a_{00}a_{01}\langle \varphi_{\xi_{00}} | \varphi_{\xi_{01}} \rangle + a_{11}a_{10}\langle \varphi_{\xi_{11}} | \varphi_{\xi_{10}} \rangle &= 0, \\
|a_{00}|^2 + |a_{01}|^2 - |a_{10}|^2 &= 0,
\end{align*}$$

(A9)

\[|a_{11}|^2 - |a_{10}|^2 = 0.\]

(3) When Alice, Bob and Charlie* choose the MBs $y$, $y$, $x$, respectively, $\langle \Psi \rangle_{ABCE}$ can be rewritten as

$$\begin{aligned}
\langle \Psi \rangle_{ABCE} &= \\
&= \frac{1}{2}\{ |y^+y^+\rangle(a_{00}|\xi_{00}\rangle - ia_{01}|\xi_{01}\rangle - ia_{10}|\xi_{10}\rangle - a_{11}|\xi_{11}\rangle) \\
&\quad + |y^+y^-\rangle(a_{00}|\xi_{00}\rangle + ia_{01}|\xi_{01}\rangle - ia_{10}|\xi_{10}\rangle + a_{11}|\xi_{11}\rangle) \\
&\quad + |y^-y^+\rangle(a_{00}|\xi_{00}\rangle - ia_{01}|\xi_{01}\rangle + ia_{10}|\xi_{10}\rangle + a_{11}|\xi_{11}\rangle) \\
&\quad + |y^-y^-\rangle(a_{00}|\xi_{00}\rangle + ia_{01}|\xi_{01}\rangle + ia_{10}|\xi_{10}\rangle - a_{11}|\xi_{11}\rangle).\}
\end{aligned}$$

(8)

We then have

\[a_{00}a_{01}\langle \xi_{00} | \xi_{01} \rangle + a_{11}a_{10}\langle \xi_{11} | \xi_{10} \rangle = 0,\]

\[|a_{00}|^2 + |a_{01}|^2 - |a_{10}|^2 = 0,\]

\[|a_{11}|^2 - |a_{10}|^2 = 0.\]

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[16] Alice need not publicize her MBs; it suffices if she tells which instances should be used to generate a common key. However, this is equivalent for the participant attack because a dishonest participant can also deduce the others MBs in the following way: He first wiretaps the other agents MBs when they are transmitted to Alice and he knows that the useful instances satisfy the relation that the number of $x$ measurements is odd, so he can deduce Alices MBs for the useful instances according to his and the other agents MBs.
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