Minimum supports of functions on the Hamming graphs with spectral constraints

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Abstract
We study functions defined on the vertices of the Hamming graphs $H(n,q)$. The adjacency matrix of $H(n,q)$ has $n+1$ distinct eigenvalues $n(q-1)-q \cdot i$ with corresponding eigenspaces $U_i(n,q)$ for $0 \leq i \leq n$. In this work, we consider the problem of finding the minimum possible support (the number of nonzeros) of functions belonging to a direct sum $U_i(n,q) \oplus U_{i+1}(n,q) \oplus \ldots \oplus U_j(n,q)$ for $0 \leq i \leq j \leq n$. For the case $i+j \leq n$ and $q \geq 3$ we find the minimum cardinality of the support of such functions and obtain a characterization of functions with the minimum cardinality of the support. In the case $i+j > n$ and $q \geq 4$ we also find the minimum cardinality of the support of functions, and obtain a characterization of functions with the minimum cardinality of the support for cases $i = j$, $i > \frac{n}{2}$ and $q \geq 5$. In particular, we characterize eigenfunctions from the eigenspace $U_i(n,q)$ with the minimum cardinality of the support for cases $i \leq \frac{n}{2}$, $q \geq 3$ and $i > \frac{n}{2}$, $q \geq 5$.

1 Introduction
Eigenspaces of graphs play an important role in algebraic graph theory (for example, see book [4]). This work is devoted to some extremal properties of eigenspaces of the Hamming graphs. We consider the problem of finding the minimum cardinality of the support of eigenfunctions of the Hamming graph $H(n,q)$. This problem is directly related to the problem of finding the minimum possible difference of two combinatorial objects and to the problem of finding the minimum cardinality of the bitrades. Bitrades are widely known subject and there are series of papers on Steiner bitrades [6], bitrades in ordered sets [3], Latin bitrades [2, 13] and bitrades in coding theory [11, 14].

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more details, connections between bitrades and eigenfunctions are described in [7, 8, 9, 10]. The problem of finding the minimum size of the support of eigenfunctions was studied for the Johnson graphs in [17], for the Doob graphs in [1], for the cubic distance-regular graphs in [15] and for the Paley graphs in [5]. The problem of finding the minimum cardinality of the support of eigenfunctions of the Hamming graphs $H(n, q)$ was completely solved for $q = 2$ in [9] based on ideas from [12]. In [16] this problem was solved for the second largest eigenvalue and arbitrary $q$.

In this work we find the minimum cardinality of the support of functions from the space $U_{i,j}(n, q)$ (a direct sum of eigenspaces of $H(n, q)$ corresponding to consecutive eigenvalues from $(q - 1)n - qi$ to $(q - 1)n - qj$) and give a characterization of functions with the minimum cardinality of the support for $i + j \leq n$, $q \geq 3$ and for $i = j$, $i > \frac{n}{2}$, $q \geq 5$. In particular, we find the minimum cardinality of the support of eigenfunctions of the Hamming graphs $H(n, q)$.

The paper is organized as follows. In Section 2, we introduce basic definitions and notations. In Section 3, we define two families of functions that have the minimum size of the support in the space $U_{i,j}(n, q)$ for $i + j \leq n$ and for $i + j > n$ respectively. In Section 4, we present auxiliary statements. In Section 5, we find the minimum size of the support of functions from the space $U_{i,j}(n, q)$ for $i + j \leq n$ and give a characterization of functions with the minimum cardinality of the support. In Section 6, we find the minimum size of the support of functions from the space $U_{i,j}(n, q)$ for $i + j > n$. In Section 7, we provide several curious examples and discuss further problems.

## 2 Basic definitions

Let $G = (V, E)$ be a graph. The set of neighbors of a vertex $x$ is denoted by $N(x)$. A real–valued function $f : V \rightarrow \mathbb{R}$ is called a $\lambda$–eigenfunction of $G$ if the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any $x \in V$. Note that the vector of values of a $\lambda$–eigenfunction is an eigenvector of the adjacency matrix of $G$ with eigenvalue $\lambda$ or the all-zero vector. The set of all $\lambda$–eigenfunctions of $G$ is called a $\lambda$–eigenspace of $G$. The support of a real–valued function $f$ is the set of nonzeros of $f$. The cardinality of the support of $f$ is denoted by $|f|$.

Let $\Sigma_q = \{0, 1, \ldots, q - 1\}$. The Hamming distance $d(x, y)$ between vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from $\Sigma_q^n$ is the number of positions $i$ such that $x_i \neq y_i$. The vertex set of the Hamming graph $H(n, q)$ is $\Sigma_q^n$ and two vertices are adjacent if the Hamming distance between them equals 1. It is well known that the set of eigenvalues of the adjacency matrix of $H(n, q)$ is $\{\lambda_i(n, q) = (n(q-1) - q \cdot i \mid i = 0, 1, \ldots, n\}$.

Denote by $U_i(n, q)$ the $\lambda_i(n, q)$–eigenspace of $H(n, q)$. The direct sum of subspaces $U_i(n, q) \oplus U_{i+1}(n, q) \oplus \ldots \oplus U_j(n, q)$
We define the function $a_{i,j}(n, q)$.

The Cartesian product $G \square H$ of graphs $G$ and $H$ is a graph with the vertex set $V(G) \times V(H)$; and any two vertices $(u, u')$ and $(v, v')$ are adjacent if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$. Let $G = G_1 \square G_2$, $f_1 : V(G_1) \to \mathbb{R}$ and $f_2 : V(G_2) \to \mathbb{R}$. Define the tensor product $f_1 \cdot f_2$ by the following rule: $(f_1 \cdot f_2)(x,y) = f_1(x)f_2(y)$ for $(x, y) \in V(G)$.

Let us take two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. The tensor product $x \otimes y$ is the vector $(x_1y_1, x_1y_m, x_2y_1, \ldots, x_ny_m)$ of length $nm$.

Let $y = (y_1, \ldots, y_{n-1})$ be a vertex of $H(n-1, q)$, $k \in \Sigma_q$ and $r \in \{1, 2, \ldots, n\}$. We consider the vector $x = (y_1, \ldots, y_{r-1}, k, y_r, \ldots, y_{n-1})$ of length $n$. Given a function $f : \Sigma_q^n \to \mathbb{R}$, we define the function $f_k^r : \Sigma_q^{n-1} \to \mathbb{R}$ by the rule $f_k^r(y) = f(x)$. We note that $f_k^r = f|_{x_r=k}$.

A function $f : \Sigma_q^n \to \mathbb{R}$ is called uniform if for any $r \in \{1, 2, \ldots, n\}$ there exists $l(r) \in \Sigma_q$ such that $f_k^r = f_m^r$ for all $k, m \in \Sigma_q \setminus \{l(r)\}$.

Recall that $S_n$ is the set of all permutations of length $n$. Let $f(x_1, x_2, \ldots, x_n)$ be a function and $\sigma \in S_n$. We define the function $f_\sigma$ by the following rule: $f_\sigma(x) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

### 3 Constructions of functions with the minimum size of the support

We define the function $a_1(k, m) : \Sigma_q^2 \to \mathbb{R}$ for $k, m \in \Sigma_q$ by the following rule:

$$a_1(k, m)(x, y) = \begin{cases} 1, & \text{if } x = k \text{ and } y \neq m; \\ -1, & \text{if } y = m \text{ and } x \neq k; \\ 0, & \text{otherwise}. \end{cases}$$

We note that $|a_1(k, m)| = 2(q - 1)$ for $k, m \in \Sigma_q$. The set of functions $a_1(k, m)$ where $k, m \in \Sigma_q$ is denoted by $A_1$.

We define the function $a_2(k, m) : \Sigma_q \to \mathbb{R}$ for $k, m \in \Sigma_q$ and $k \neq m$ by the rule:

$$a_2(k, m)(x) = \begin{cases} 1, & \text{if } x = k; \\ -1, & \text{if } x = m; \\ 0, & \text{otherwise}. \end{cases}$$

The set of functions $a_2(k, m)$ where $k, m \in \Sigma_q$ and $k \neq m$ is denoted by $A_2$.

Let $A_3 = \{f : \Sigma_q \to \mathbb{R} \mid f \equiv 1\}$. By the definition of an eigenfunction we see that $A_1 \subset U_1(2, q)$, $A_2 \subset U_1(1, q)$ and $A_3 \subset U_0(1, q)$.

We define the function $a_4(m) : \Sigma_q \to \mathbb{R}$ for $m \in \Sigma_q$ by the rule:

$$a_4(m)(x) = \begin{cases} 1, & \text{if } x = m; \\ 0, & \text{otherwise}. \end{cases}$$
The set of functions $a_4(m)$ where $m \in \Sigma_q$ is denoted by $A_4$. Note that $A_4 \subset U_{[0,1]}(1,q)$.

Figure 1: Function $a_4(1,1)$ for $q = 3$.

Figure 2: Function $a_4(0,2)$ for $q = 4$. 
The following lemma is a particular case of well known result for so-called NEPS of graphs (see [3], Theorem 2.3.4):

**Lemma 1.** Let $G_1$ and $G_2$ be two graphs, let $\lambda$ and $\mu$ be eigenvalues of $G_1$ and $G_2$ respectively, and let $x$ and $y$ be eigenvectors for $\lambda$ and $\mu$. Then the graph $G_1 \square G_2$ has the eigenvalue $\lambda + \mu$ and $x \otimes y$ is an eigenvector corresponding to $\lambda + \mu$.

Since $\lambda_i(m,q) + \lambda_j(n,q) = \lambda_{i+j}(m + n, q)$ and $H(m + n, q) = H(m, q) \square H(n, q)$, by Lemma 1 we immediately obtain the following corollary:

**Corollary 1.** Let $f_1 \in U_i(m,q)$ and $f_2 \in U_j(n,q)$. Then $f_1 \cdot f_2 \in U_{i+j}(m+n,q)$.

Let $i + j \leq n$. We say that a function $f : \Sigma^n_q \rightarrow \mathbb{R}$ belongs to the class $F_1(n,q,i,j)$ if

$$f = c \cdot \prod_{k=1}^{i} g_k \cdot \prod_{k=1}^{n-i-j} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where $c$ is a constant, $g_k \in A_1$ for $k \in [1,i]$, $h_k \in A_3$ for $k \in [1,n-i-j]$ and $v_k \in A_4$ for $k \in [1,j-i]$.

Let $i + j > n$. We say that a function $f : \Sigma^n_q \rightarrow \mathbb{R}$ belongs to the class $F_2(n,q,i,j)$ if

$$f = c \cdot \prod_{k=1}^{n-j} g_k \cdot \prod_{k=1}^{i+j-n} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where $c$ is a constant, $g_k \in A_1$ for $k \in [1,n-j]$, $h_k \in A_2$ for $k \in [1,i+j-n]$ and $v_k \in A_4$ for $k \in [1,j-i]$.

**Lemma 2.** The following statements are true:

1. Let $i + j \leq n$ and $f \in F_1(n,q,i,j)$. Then $f \in U_{[i,j]}(n,q)$ and $|f| = 2^i(q-1)^i q^{n-i-j}$.
2. Let \( i + j > n \) and \( f \in F_2(n,q,i,j) \). Then \( f \in U_{[i,j]}(n,q) \) and \(|f| = 2^j(q - 1)^{n-j} \).

Proof. As we noted above \( A_1 \subset U_1(2,q) \), \( A_2 \subset U_1(1,q) \), \( A_3 \subset U_0(1,q) \) and \( A_4 \subset U_{[0,1]}(1,q) \). Then using Corollary \( \square \) and the fact that \(|f_1 \cdot f_2| = |f_2| \cdot |f_2| \), we obtain the statement of this lemma. \( \square \)

In what follows, we prove that functions from \( F_1(n,q,i,j) \) and \( F_2(n,q,i,j) \) have the minimum size of the support in the subspace \( U_{[i,j]}(n,q) \) for \( i + j \leq n \), \( q \geq 3 \) and for \( i + j > n, q \geq 4 \) respectively.

4 Reduction Lemma

In this section we describe a connection between eigenspaces of the Hamming graphs \( H(n,q) \) and \( H(n-1,q) \).

Lemma 3. Let \( f \in U_i(n,q) \) and \( r \in \{1,2,\ldots,n\} \). Then the following statements are true:

1. \( f_k^r - f_m^r \in U_{i-1}(n-1,q) \) for \( k,m \in \Sigma_q \).
2. \( \sum_{k=0}^{q-1} f_k^r \in U_i(n-1,q) \).
3. \( f_k^r \in U_{i-1}(n-1,q) \oplus U_i(n-1,q) \) for \( k \in \Sigma_q \).

Proof. 1. The first case of this lemma was proved in [10] (Lemma 1).
2. Let \( t = (t_1,t_2,\ldots,t_n) \) be a vertex of \( H(n,q) \). Let
\[
x_r(m) = (t_1,\ldots,t_{r-1},m,t_{r+1},\ldots,t_n)
\]
for \( m \in \Sigma_q \) and
\[
x_{i,r}(a,m) = (t_1,\ldots,t_{i-1},a,t_{i+1},\ldots,t_{r-1},m,t_{r+1},\ldots,t_n)
\]
for \( a,m \in \Sigma_q \) and \( i \in \{1,2,\ldots,n\} \setminus \{r\} \). The set of neighbors \( z = (z_1,\ldots,z_n) \) of \( x_r(m) \) such that \( z_r = m \) is denoted by \( N(m,r) \). We see that \( N(m,r) = \{x_{i,r}(a,m) | i \neq r, a \neq t_i \} \). We note that
\[
N(x_r(m)) = \{(x_r(0),x_r(1),\ldots,x_r(q-1)) \setminus \{x_r(m)\}\} \cup N(m,r).
\]
Since \( f \) is an eigenfunction, we have
\[
\lambda_i(n,q) \cdot f(x_r(m)) = \sum_{i \neq r,a \neq t_i} f(x_{i,r}(a,m)) + \sum_{i=0}^{q-1} f(x_r(i)) - f(x_r(m)).
\]
Hence we obtain that
\[
(\lambda_i(n,q) - (q - 1)) \cdot \sum_{m=0}^{q-1} f(x_r(m)) = \sum_{i \neq r,a \neq t_i} \sum_{m=0}^{q-1} f(x_{i,r}(a,m)).
\]
Let \( y_r \) and \( y_i(a) \) be the vectors obtained by removing the \( r \)th coordinate in \( x_r(m) \) and \( x_{i,r}(a, m) \) respectively. Then

\[
\lambda_i(n - 1, q) \cdot (\sum_{m=0}^{q-1} f^r_m)(y_r) = \sum_{i \neq r, a \neq t_i} (\sum_{m=0}^{q-1} f^r_m)(y_i(a)).
\]

Since \( y_r \) has neighbors \( y_i(a) \) for \( i \neq r \) and \( a \neq t_i \) in \( H(n - 1, q) \), we prove that \( \sum_{m=0}^{q-1} f^r_m \) is a \( \lambda_i(n - 1, q) \)-eigenfunction of \( H(n - 1, q) \).

3. By the first case of this lemma we have that \( f^r_k - f^r_m \in U_{i-1}(n - 1, q) \) for \( m \neq k \). Hence

\[
(q - 1)f^r_k - \sum_{t \in \Sigma_q, t \neq k} f^r_t \in U_{i-1}(n - 1, q).
\]

The second case of this lemma implies that

\[
\sum_{t=0}^{q-1} f^r_t \in U_i(n - 1, q).
\]

Hence \( q \cdot f^r_k \in U_{i-1}(n - 1, q) \oplus U_i(n - 1, q) \). Thus \( f^r_k \in U_{i-1}(n - 1, q) \oplus U_i(n - 1, q) \).

Using the previous lemma for \( U_k(n, q) \), where \( i \leq k \leq j \), we obtain the following result:

**Lemma 4.** Let \( f \in U_{[i,j]}(n, q) \) and \( r \in \{1, 2, \ldots, n\} \). Then the following statements are true:

1. \( f^r_k - f^r_m \in U_{[i-1,j-1]}(n - 1, q) \) for \( k, m \in \Sigma_q \).
2. \( \sum_{k=0}^{q-1} f^r_k \in U_{[i,j]}(n - 1, q) \).
3. \( f^r_k \in U_{[i-1,j]}(n - 1, q) \) for \( k \in \Sigma_q \).

**Lemma 5.** Let \( f \in U_{[i,j]}(n, q) \), let \( r \in \{1, 2, \ldots, n\} \), and let \( m \in \Sigma_q \). If \( f^r_k \equiv 0 \) for any \( k \in \Sigma_q \setminus \{m\} \), then \( f^r_m \in U_{[i,j-1]}(n, q) \).

**Proof.** For \( k \neq m \) we obtain \( f^r_k - f^r_m \in U_{[i-1,j-1]}(n - 1, q) \) due to Lemma 4(1). Hence \( f^r_m \in U_{[i-1,j-1]}(n - 1, q) \). Lemma 4(2) implies that \( f^r_m \in U_{[i,j]}(n - 1, q) \). Then \( f^r_m \in U_{[i,j-1]}(n - 1, q) \).

**Lemma 6.** Let \( f : \Sigma_q^n \rightarrow \mathbb{R} \), let \( r \in \{1, 2, \ldots, n\} \) and let \( f^r_0 = f^r_1 = \ldots = f^r_{q-2} \). Then

\[
|f| \geq (q - 2)|f^r_0| + |f^r_{q-2} - f^r_{q-1}|
\]

**Proof.** We have

\[
|f| = \sum_{k=0}^{q-1} |f^r_k| = (q - 2)|f^r_0| + |f^r_{q-2} - f^r_{q-1}| \geq (q - 2)|f^r_0| + |f^r_{q-2} - f^r_{q-1}|.
\]

In Sections 5 and 6 we will use the main results of this section for inductive arguments.
5 Case $i + j \leq n$

In this section we prove the first main theorem of this paper:

**Theorem 1.** Let $f \in U_{i,j}(n,q)$, $i + j \leq n$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq 2^i(q-1)^i q^{n-i-j}$. Moreover, the equality $|f| = 2^i(q-1)^i q^{n-i-j}$ holds if and only if $f_\sigma \in F_1(n,q,i,j)$ for some permutation $\sigma \in S_n$.

**Proof.** Lemma 2 implies that if $f \in F_1(n,q,i,j)$, then $f \in U_{i,j}(n,q)$ and $|f| = 2^i(q-1)^i q^{n-i-j}$.

In what follows, in this theorem we assume that $|f| \leq 2^i(q-1)^i q^{n-i-j}$. Let us prove the theorem by induction on $n$, $i$ and $j$. Suppose that $f$ is a constant. Then $f$ is a $\lambda_0(n,q)$-eigenfunction of $H(n,q)$, i.e., $f \in U_0(n,q)$. In this case $|f| = q^n$, and the claim of the theorem holds. So, we can assume that $f_k \not= f_m$ for some $k,m \in \Sigma_q$ and $r \in \{1,2,\ldots,n\}$. Without loss of generality, we assume that $r = n$. For the function $f_k^n$ in the proof of this theorem we will use the more convenient notation $f_k$.

Now we prove the theorem for $i = 0$.

**Lemma 7.** Let $f \in U_{0,j}(n,q)$, $j \leq n$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq q^{n-j}$. Moreover $|f| = q^{n-j}$ if and only if $f_\sigma \in F_1(n,q,0,j)$ for some permutation $\sigma \in S_n$.

**Proof.** We assume that $|f| \leq q^{n-j}$. Let us prove this lemma by induction on $n$ and $j$. For $j = 0$, we have $f$ is a constant. So $|f| = q^n$ and the claim of the lemma holds. If $n = 1$ and $j > 0$, then $j = 1$. In this case the claim of the theorem also holds. So, in this lemma we can assume that $n \geq 2$. Let us prove the induction step. As we noted above there exist numbers $k$ and $m$ such that $f_k \not= f_m$ and $f_m \not\equiv 0$. Without loss of generality, we can assume that $k = q-2$ and $m = q-1$.

Lemma 4 implies that $f_{q-2} - f_{q-1} \in U_{0,j-1}(n-1,q)$. By the induction assumption, $|f_{q-2} - f_{q-1}| \geq q^{n-j}$. So,

$$|f| \geq |f_{q-2}| + |f_{q-1}| \geq |f_{q-2} - f_{q-1}| \geq q^{n-j}.$$  

On the other hand, we supposed that $|f| \leq q^{n-j}$. So, we have $f_k \equiv 0$ for every $k < q-2$. In particular, $f_0 \equiv 0$ because $q \geq 3$. In particular, $f_0 \not= f_{q-1}$, and considering $k = 0$ and $m = q-1$ similarly as above, we find that $f_{q-2} \equiv 0$ too. Thus,

$$f(x_1,\ldots,x_n) = f_{q-1}(x_1,\ldots,x_{n-1}) \cdot a_4(q-1)(x_n),$$

and the statement of the lemma follows from the induction assumption applied to $f_{q-1}$.

Further we will prove the theorem for $i \geq 1$. We note that if $n \leq 2$, $i + j \leq n$ and $i \geq 1$, then $n = 2$ and $i = j = 1$. In this case the statement of Theorem 1 was proved in [16] (Theorem 3). In what follows, in the proof of the theorem we assume that $n \geq 3$. 

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Lemma 8. Let $f$ be a non uniform function from $U_{i,j}(n,q)$, where $i + j \leq n$, $i \geq 1$ and $q \geq 3$. Then $|f| > 2^i(q-1)^i q^{n-i-j}$.

Proof. Case $q > 3$. Since $f$ is not a uniform function, there exist a number $r$ and distinct numbers $k$, $m$, $s$ and $t$ such that $f_k^r \neq f_m^r$ and $f_s^r \neq f_t^r$. Denote $\tilde{f}_k = f_k^r$ for $k \in \Sigma_q$. Lemma 4(1) implies that $\tilde{f}_k - f_m \in U_{i-1,j-1}(n-1,q)$ and $\tilde{f}_s - \tilde{f}_t \in U_{i-1,j-1}(n-1,q)$. By the induction assumption we have

$$|\tilde{f}_k - \tilde{f}_m| \geq 2^{i-1}(q-1)^i q^{n-i-j+1}$$

and

$$|\tilde{f}_s - \tilde{f}_t| \geq 2^{i-1}(q-1)^i q^{n-i-j+1}.$$

Then

$$|f| = \sum_{p=0}^{q-1} |\tilde{f}_p| \geq |\tilde{f}_k| + |\tilde{f}_m| + |\tilde{f}_s| + |\tilde{f}_t| \geq |\tilde{f}_k - \tilde{f}_m| + |\tilde{f}_s - \tilde{f}_t| \geq 2^i(q-1)^i q^{n-i-j+1} > 2^i(q-1)^i q^{n-i-j}.$$

Case $q = 3$. Since $f$ is not a uniform function, there exists a number $r$ such that $f_0^r \neq f_1^r$, $f_1^r \neq f_2^r$, and $f_0^r \neq f_2^r$. Denote $\tilde{f}_k = f_k^r$ for $k \in \Sigma_q$. Lemma 4(1) implies that $\tilde{f}_k - f_m \in U_{i-1,j-1}(n-1,q)$ for $k \neq m$. Then by the induction assumption we obtain that

$$|\tilde{f}_k - \tilde{f}_m| \geq 2^{2i-2} \cdot 3^{n-i-j+1}$$

for $k, m \in \Sigma_3$ and $k \neq m$. We note that

$$|\tilde{f}_0| + |\tilde{f}_1| + |\tilde{f}_2| = \frac{1}{2}(|\tilde{f}_0| + |\tilde{f}_1|) + \frac{1}{2}(|\tilde{f}_1| + |\tilde{f}_2|) + \frac{1}{2}(|\tilde{f}_0| + |\tilde{f}_2|).$$

Using inequalities $|\tilde{f}_k| + |\tilde{f}_m| \geq |\tilde{f}_k - \tilde{f}_m|$ for $k, m \in \Sigma_3$ and $k \neq m$, we obtain that

$$|\tilde{f}_0| + |\tilde{f}_1| + |\tilde{f}_2| \geq 2^{2i-3} \cdot 3^{n-i-j+2}.$$

Then

$$|f| \geq 2^{2i-3} \cdot 3^{n-i-j+2} > 2^{2i} \cdot 3^{n-i-j}.$$

Lemma 9. Let $f \in U_{i,j}(n,q)$, $r \in \{1, 2, \ldots, n\}$, $f_0^r = f_1^r = \ldots = f_{q-2}^r$, $f_i^r \neq f_{i-1}^r$, $i + j \leq n$, $i \geq 1$ and $q \geq 3$. Let $|f_0^r| > 2^{i-1}(q-1)^i q^{n-i-j}$. Then $|f| > 2^i(q-1)^i q^{n-i-j}$.

Proof. Lemma 4(1) implies that $f_{q-2}^r - f_{q-1}^r \in U_{i-1,j-1}(n-1,q)$. Hence by the induction assumption we obtain that

$$|f_{q-2}^r - f_{q-1}^r| \geq 2^{i-1}(q-1)^i q^{n-i-j+1}.$$

By Lemma 8, we have

$$|f| \geq (q-2)|f_0^r| + |f_{q-2}^r - f_{q-1}^r|.$$
Hence

\[ |f| > (q - 2)2^{i-1}(q - 1)^{i-1}q^{n-i-j} + 2^{i-1}(q - 1)^{i-1}q^{n-i-j+1} = 2^i(q - 1)^i q^{n-i-j}. \]

We continue the proof of Theorem 1. Recall that we assume \( |f| \leq 2^i(q - 1)^i q^{n-i-j} \). Then using Lemma 8 we obtain that \( f \) is a uniform function. Hence, without loss of generality, we can assume that \( f_0 = f_1 = \ldots = f_{q-2} \). Lemma 9 implies that \( |f_0| \leq 2^i(q - 1)^i q^{n-i-j} \). We have \( f_0 \in U_{[i-1,j]}(n-1,q) \) due to Lemma 4(3). Then by the induction assumption there are two cases: \( f_0 \equiv 0 \) or
\[ |f_0| = 2^i(q - 1)^i q^{n-i-j}. \]

Consider the case \( f_0 \equiv 0 \). Lemma 3 implies that \( f_{q-1} \in U_{[i,j-1]}(n-1,q) \). If \( i = j \), then \( f_{q-1} \in U_{[i,i-1]}(n-1,q) \) and we have that \( f_{q-1} \equiv 0 \). Hence \( f \equiv 0 \) for \( i = j \). So, we can assume that \( i < j \). By the induction assumption we obtain \( |f_{q-1}| \geq 2^{j}(q - 1)^j q^{n-i-j} \). Then
\[ |f| = \sum_{k=0}^{q-1} |f_k| = |f_{q-1}| \geq 2^i(q - 1)^i q^{n-i-j}. \]

Moreover, if \( |f| = 2^{j}(q - 1)^j q^{n-i-j} \), then \( |f_{q-1}| = 2^{j}(q - 1)^j q^{n-i-j} \). Then by the induction assumption for \( f_{q-1} \) we obtain that

\( (f_{q-1})_n \in F_i(n-1,q,i,j-1) \)

for some permutation \( \pi \in S_{n-1} \). Since \( f_0 = f_1 = \ldots = f_{q-2} \equiv 0 \), we have \( f = f_{q-1} \cdot a_d(q - 1) \) and \( a_d(q - 1) \in A_d \). So, we prove the theorem in this case.

Consider the case \( |f_0| = 2^{i-1}(q - 1)^{i-1}q^{n-i-j} \). Lemma 4(3) implies that \( f_0 \in U_{[i-1,j]}(n-1,q) \). Then by the induction assumption for \( f_0 \) we obtain that

\( (f_0)_{n} \in F_1(n-1,q,i-1,j) \)

for some permutation \( \pi \in S_{n-1} \). Without loss of generality one can take \( \pi \) equal the identity permutation, so \( f_0 \in F_1(n-1,q,i-1,j) \). Hence \( f_0 = g \cdot a_d(m) \) for some \( m \in S_q \). Without loss of generality, we assume that \( m = q - 1 \). Therefore, we have \( f_k = g \cdot a_d(q - 1) \) for any \( k < q - 1 \). Then \( f|_{x_{n-1}=a,x_n=b} \equiv 0 \) for \( a \in [0,q-2] \) and \( b \in [0,q-2] \) and \( f|_{x_{n-1}=q-1,x_n=c} = g \) for any \( c < q - 1 \). We also note that
\[ |g| = |f_0| = 2^{q-1}(q - 1)^{q-1} q^{n-i-j}. \]

Let us consider the functions \( f_0^{n-1}, f_1^{n-1}, \ldots, f_{q-1}^{n-1} \). Since \( f_{k}^{n-1}|_{x_{n-1}=0} \equiv 0 \) for any \( k < q - 1 \) and \( f_{q-1}^{n-1}|_{x_{n-1}=0} = g \), we see that \( f_k^{n-1} \neq f_{q-1}^{n-1} \) for any \( k < q - 1 \). On the other hand, \( f \) is uniform. Hence \( f_0^{n-1} = \ldots = f_{q-2}^{n-1} \). If \( f_0^{n-1} \equiv 0 \), then we have the case that we considered above (we can consider \( f_0^{n-1} \) instead of \( f_0 \)). So \( f_0^{n-1} \not\equiv 0 \). By Lemma 4(3) we have \( f_0^{n-1} \in U_{[i-1,j]}(n-1,q) \). Then by the induction assumption we obtain
\[ |f_0^{n-1}| \geq 2^{i-1}(q - 1)^i q^{n-i-j}. \]
Denote \( h = f_0^{n-1} |_{x_n = q-1} \). Then
\[
|h| = |f_0^{n-1}| \geq 2^{i-1}(q - 1)^{i-1}q^{n-i-j}.
\]

We also note that \( f_k^{n-1} |_{x_n = q-1} = h \) for any \( k < q - 1 \). Denote \( \varphi = f |_{x_n = q-1, x_n = q-1} \).

Recall that \(|f| \leq 2^i(q - 1)^i q^{n-i-j} \). On the other hand, we have
\[
|f| = (q - 1)(|g| + |h|) + |\varphi|,
\]
\[
|g| = 2^{i-1}(q - 1)^{i-1}q^{n-i-j} \text{ and } |h| \geq 2^{i-1}(q - 1)^{i-1}q^{n-i-j}. \text{ So } |h| = 2^{i-1}(q - 1)^{i-1}q^{n-i-j} \text{ and } \varphi \equiv 0.
\]

Let us prove that \( g + h \in U_{[i-1,j-2]}(n - 2, q) \). Since \( f_0 \in U_{[i-1,j]}(n - 1, q) \), by Lemma 3 we have \( g \in U_{[i-1,j-1]}(n - 2, q) \). Similarly we obtain that \( h \in U_{[i-1,j-1]}(n - 2, q) \). Consequently, we have \( g + h \in U_{[i-1,j-1]}(n - 2, q) \). On the other hand, Lemma 4(1) implies that \( f_{q-2} - f_{q-1} \in U_{[i-1,j-1]}(n - 1, q) \).

Applying Lemma 4(1) for \( f_{q-1} - f_{q-2} \), we obtain that \( g + h \in U_{[i-2,j-2]}(n - 2, q) \).

Therefore \( g + h \in U_{[i-1,j-2]}(n - 2, q) \).

Let us prove that \( g + h \equiv 0 \). Suppose \( g + h \not\equiv 0 \). Then by the induction assumption we have
\[
|g + h| \geq 2^{i-1}(q - 1)^{i-1}q^{n-i-j+1}.
\]

On the other hand, \(|g| = |h| = 2^{i-1}(q - 1)^{i-1}q^{n-i-j} \). So, we have
\[
2^i(q - 1)^{i-1}q^{n-i-j} = |g| + |h| \geq |g + h| \geq 2^{i-1}(q - 1)^{i-1}q^{n-i-j+1}.
\]

Therefore \( q \leq 2 \) and we have a contradiction. Thus \( g + h \equiv 0 \). Then \( f = g \cdot a_1(q - 1, q - 1) \). Applying the induction assumption for \( g \), we finish the proof of the theorem.

\[\square\]

**Corollary 2.** Let \( f \in U_i(n, q), i \leq \lfloor \frac{n}{2} \rfloor, q \geq 3 \) and \( f \not\equiv 0 \). Then \(|f| \geq 2^i(q - 1)^i q^{n-2i} \). Moreover, the equality \(|f| = 2^i(q - 1)^i q^{n-2i} \) holds if and only if \( f_\sigma \in F_i(n, q, i, i) \) for some permutation \( \sigma \in S_n \).

### 6 Case \( i + j > n \)

In this section we prove the second main result of this work. We find the minimum size of the support of functions from \( U_{[i,j]}(n, q) \) for \( i + j > n \). Firstly, we solve the problem for the uniform functions:

**Theorem 2.** Let \( f \) be a uniform function from \( U_{[i,j]}(n, q) \), where \( i + j \geq n, q \geq 3 \) and \( f \not\equiv 0 \). Then \(|f| \geq 2^{n-j}(q - 1)^{n-j} q^{i+j-n} \).

**Proof.** We note that the statement of the theorem was proved in Theorem 1 for \( n = i + j \). So, we can assume that \( i + j > n \). Let us prove this theorem by induction on \( n, i \) and \( j \). Consider the functions \( f_k^j \) for \( k \in \Sigma_q \). For the function
implies that

By Lemma 4(3), we have

Since $f|q$, we can assume that $f_0 = f_1 = \ldots = f_{q-2}$. We note that $f_k$ is uniform for any $k \in \Sigma_q$.

Firstly, we prove the theorem for $j = n$.

**Lemma 10.** Let $f$ be a uniform function from $U_{[i,n]}(n,q)$, where $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq q^3$.

**Proof.** Let us prove this lemma by induction on $n$ and $i$. For $i = 0$ and arbitrary $n$, we see that $|f| \geq 1$.

Let us prove the induction step. Suppose $f_0 \equiv 0$. Lemma 5 implies that $f_{q-1} \in U_{[i,n-1]}(n-1,q)$. Then by the induction assumption we obtain $|f_{q-1}| \geq q^3$. Hence $|f| = |f_{q-1}| \geq q^i$.

Consider the case $f_{q-1} \equiv 0$. Using Lemma 4(2), we have $f_0 \in U_{[i,n-1]}(n-1,q)$. Then by the induction assumption we obtain $|f_0| \geq q^i$. Therefore $|f| > q^i$.

Suppose $f_0 \not\equiv 0$ and $f_{q-1} \not\equiv 0$. We have $f_k \in U_{[i-1,n-1]}(n-1,q)$ for $k \in \Sigma_q$ due to Lemma 4(3). By the induction assumption we have $|f_k| \geq q^{i-1}$ for $k \in \Sigma_q$. Hence $|f| \geq q^3$.

Now we prove the theorem for $j < n$. We consider two cases.

**Case** $f_0 \equiv 0$. Since $f \not\equiv 0$, we have $f_{q-1} \not\equiv 0$. Lemma 5 implies that $f_{q-1} \in U_{[i,j-1]}(n-1,q)$. Then by the induction assumption we obtain $|f_{q-1}| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n}$. Hence $|f| = |f_{q-1}| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n}$.

**Case** $f_0 \not\equiv 0$. Lemma 4(2) implies that

$$\sum_{p=0}^{q-1} f_p = (q-1) \cdot f_0 + f_{q-1} \in U_{[i,j]}(n-1,q).$$

If $(q-1)f_0 + f_{q-1} \equiv 0$, then $f_0 - f_{q-1} = q \cdot f_0$. Then $f_k \in U_{[i-1,j-1]}(n-1,q)$ for $k \in \Sigma_q$ due to Lemma 4(1). Recall that in the beginning of the proof we assumed $i + j > n$. By the induction assumption we have

$$|f_k| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n-1}.$$ Hence $|f| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n}$.

Suppose $(q-1)f_0 + f_{q-1} \not\equiv 0$. We note that $(q-1)f_0 + f_{q-1}$ is uniform. By the induction assumption we have

$$|(q-1)f_0 + f_{q-1}| \geq 2^{n-j-1}(q-1)^{n-j-1}q^{i+j-n+1}.$$ Since $|f_0| + |f_{q-1}| \geq |(q-1) \cdot f_0 + f_{q-1}|$, we obtain

$$|f_0| + |f_{q-1}| \geq 2^{n-j-1}(q-1)^{n-j-1}q^{i+j-n+1}.$$ By Lemma 4(3), we have $f_0 \in U_{[i,j]}(n-1,q)$, and the induction assumption implies that $|f_0| \geq 2^{n-j-1}(q-1)^{n-j-1}q^{i+j-n}$.
Using the equality
\[ |f| = \sum_{p=0}^{q-1} |f_p| = (q-2) \cdot |f_0| + |f_0| + |f_{q-1}|, \]
we obtain \( |f| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n} \).

Now we prove the main theorem of this section.

**Theorem 3.** Let \( f \in U_{[i,j]}(n, q) \), \( i + j > n \), \( q \geq 4 \) and \( f \neq 0 \). Then \( |f| \geq 2^i(q-1)^{n-i} \). Moreover, for \( i = j \) and \( q \geq 5 \) the equality \( |f| = 2^i(q-1)^{n-i} \) holds if and only if \( f \in F_2(n, q, i, i) \) for some permutation \( \sigma \in S_n \).

**Proof.** Lemma 4 implies that if \( f \in F_2(n, q, i, j) \), then \( f \in U_{[i,j]}(n, q) \) and \( |f| = 2^i(q-1)^{n-i} \).

Let us prove this theorem by induction on \( n \), \( i \) and \( j \). Since \( i + j > n \), we have that \( i \geq 1 \). Suppose that there exist numbers \( k \) and \( r \) such that \( f_k^r \equiv 0 \). Without loss of generality, we assume that \( k = q - 1 \) and \( r = n \). For the function \( f_k^r \) we will use the more convenient notation \( f_k \). Lemma 1 implies that \( f_m - f_q^{-1} \in U_{[i-1,j-1]}(n-1, q) \) for \( m < q - 1 \). Therefore \( f_m \in U_{[i-1,j-1]}(n-1, q) \) for \( m < q - 1 \). So, if \( f_m \neq 0 \), then using the induction assumption for \( i + j > n + 1 \) and Theorem 2 in the case \( i + j = n + 1 \), we have \( |f_m| \geq 2^{n-1}(q-1)^{n-j} \). Since \( |f| = \sum_{p=0}^{q-1} |f_p| \), the number of \( k \) such that \( f_k \neq 0 \) is at most two. There are two variants.

In the first case there exists only one \( k \) such that \( f_k \neq 0 \). Without loss of generality, we assume that \( k = 0 \). We have \( f_0 \in U_{[i,j-1]}(n-1, q) \) due to Lemma 5. If \( i = j \), then \( f_0 \in U_{[i,i-1]}(n-1, q) \) and \( f \equiv 0 \). For \( i < j \) by the induction assumption we obtain \( |f_0| \geq 2^i(q-1)^{n-j} \). So \( |f| \geq 2^i(q-1)^{n-j} \).

In the second case there exist two numbers \( k \) and \( m \) such that \( f_k \neq 0 \) and \( f_m \neq 0 \). Without loss of generality, we assume that \( k = 0 \) and \( m = 1 \). As we noted above \( |f_0| \geq 2^{i-1}(q-1)^{n-j} \) and \( |f_1| \geq 2^{i-1}(q-1)^{n-j} \). So \( |f| = |f_0| + |f_1| \geq 2^i(q-1)^{n-j} \). Suppose that \( i = j \), \( q \geq 5 \) and the equality \( |f| = 2^i(q-1)^{n-i} \) holds. By Lemma 2 we obtain that \( f_0 + f_1 \in U_i(n-1, q) \). Since \( f_0 \in U_{i-1}(n-1, q) \) and \( f_1 \in U_{i-1}(n-1, q) \), we see that \( f_0 + f_1 \in U_{i-1}(n-1, q) \). Consequently \( f_0 + f_1 \equiv 0 \). Hence \( f = f_0 \cdot a_2(0,1) \). Since \( |f| = 2^i(q-1)^{n-i} \), we have \( |f_0| = 2^{i-1}(q-1)^{n-i} \). Applying the induction assumption for \( f_0 \) we prove this theorem.

Thus, in what follows in the proof of this theorem we can assume that \( f_k^r \neq 0 \) for any \( k \in \Sigma_q \) and \( v \in \{1, 2, \ldots, n\} \).

We need the following lemma.

**Lemma 11.** Let \( f \) be a non uniform function from \( U_{[i,j]}(n, q) \), where \( i + j > n \), \( f_k^r \neq 0 \) for \( k \in \Sigma_q \) and \( v \in \{1, 2, \ldots, n\} \), \( i \geq 1 \) and \( q \geq 4 \). Then \( |f| > 2^i(q-1)^{n-j} \) for \( q > 4 \) and \( |f| \geq 2^i(q-1)^{n-j} \) for \( q = 4 \).
Proof. Since $f$ is not a uniform function, there exist numbers $r$ and distinct
numbers $k$, $m$, $s$ and $t$ such that $f_k^r \neq f_m^r$ and $f_s^r \neq f_t^r$. Denote $\tilde{f}_k = f_k^r$ for
in $\Sigma_q$. Lemma [4] implies that $\tilde{f}_k - \tilde{f}_m \in U_{i-1,j-1}(n-1,q)$ and $\tilde{f}_s - \tilde{f}_t \in
U_{i-1,j-1}(n-1,q)$. Then using the induction assumption for $i + j > n + 1$ and
Theorem [1] in the case $i + j = n + 1$, we obtain that

$$|\tilde{f}_k - \tilde{f}_m| \geq 2^{i-1}(q-1)^{n-j}$$

and

$$|\tilde{f}_s - \tilde{f}_t| \geq 2^{i-1}(q-1)^{n-j}.$$

Therefore, we have

$$|f| = \sum_{p=0}^{q-1} |f_p| \geq |\tilde{f}_k| + |\tilde{f}_m| + |\tilde{f}_s| + |\tilde{f}_t| \geq |\tilde{f}_k - \tilde{f}_m| + |\tilde{f}_s - \tilde{f}_t| \geq 2^i(q-1)^{n-j}.$$

If $q > 4$, then there exists $d$ such that $d \notin \{k,m,s,t\}$ and $\tilde{f}_d \neq 0$. So

$$|f| > 2^i(q-1)^{n-j} \text{ for } q > 4.$$

Now we finish the proof of this theorem. Suppose that $f$ is not a uniform
function. Since $f_k^r \neq 0$ for any $k$ and $v$, by Lemma [1] we obtain that $|f| > 2^i(q-1)^{n-j}$ for $q > 4$ and $|f| \geq 2^i(q-1)^{n-j}$ for $q = 4$. So, we can assume that

$f$ is a uniform function. Then $|f| \geq 2^{n-j}(q-1)^{n-j}q^{i+j-n}$ due to Theorem [2].

Since $q > 3$, we obtain $|f| > 2^i(q-1)^{n-j}$.

Thus, if $f_k^r \neq 0$ for any $k \in \Sigma_q$ and $v \in \{1,2,\ldots,n\}$ and $q \geq 5$, then

$$|f| > 2^i(q-1)^{n-j}.$$ \hspace{1cm} \square

Corollary 3. Let $f \in U_i(n,q)$, $i > \lfloor \frac{n}{2} \rfloor$, $q \geq 4$ and $f \neq 0$. Then $|f| \geq 2^i(q-1)^{n-i}$. Moreover, for $q \geq 5$ the equality $|f| = 2^i(q-1)^{n-i}$ holds if and

only if $f_\sigma \in F_2(n,q,i,i)$ for some permutation $\sigma \in S_n$.

7 Discussion

The initial problem of finding functions from $U_{i,j}(n,q)$ with minimum size of
the support is formulated for arbitrary real-valued functions from corresponding
subspace. Surprisingly, Theorems [1] and [3] imply that such functions take only 3
distinct values. Moreover, such functions are equal to a tensor product of several
elementary eigenfunctions of the Hamming graphs of dimensions not greater than
2 after some permutation of coordinate positions. These elementary functions
belong to $A_1 \cup A_3 \cup A_4$ and $A_1 \cup A_2 \cup A_4$ for the cases $i + j \leq n$ and $i + j > n$
respectively.

One may notice, that bounds for the size of a support and corresponding
characterizations obtained in Theorems [1] and [3] require some lower bounds for
$q$, and in the case $i + j > n$ for $i \neq j$ there is no characterization. Further we
provide several examples explaining difficulties of characterisation for the case
$i + j > n$ and for small values of $q$. 

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Remark 1. In Theorem 2 we prove that $|f| \geq 2^i(q-1)^{n-j}$ for $f \in U_{i,j}(n,q)$, $i+j > n$ and $q \geq 4$. On the other hand, if $f \in F_2(n,q,i,j)$ and $i+j > n$, then $|f| = 2^i(q-1)^{n-j}$ and $f \in U_{i,j}(n,q)$ due to Lemma 2. We note that in general case for $f \in U_{i,j}(n,q)$ and $i+j > n$ the equality $|f| = 2^i(q-1)^{n-j}$ does not imply that $f_\sigma \in F_2(n,q,i,j)$ for some permutation $\sigma \in S_n$. Consider the following example:

**Example.** We define the function $g : \Sigma_q^2 \rightarrow \mathbb{R}$ by the following rule:

$$g(x, y) = \begin{cases} 
1, & \text{if } x = y = 0; \\
-1, & \text{if } x = y = q - 1; \\
0, & \text{otherwise.}
\end{cases}$$

Denote $g'(x, y) = g(y, x)$. We see that $|g| = 2$. We note that $g = a_2(0, q - 1) \cdot a_4(0) + a_4(q - 1) \cdot a_2(0, q - 1)$. Consequently, $g \in U_{1,2}(2, q)$ due to Corollary 1. Thus $g(x, y)$ has the minimum size of the support in $U_{1,2}(2, q)$ but $g \notin F_2(2, q, 1, 2)$ and $g' \notin F_2(2, q, 1, 2)$. Similar function can be also constructed for arbitrary $n > 2$. Therefore, a possible characterization of functions from $U_{i,j}(n, q)$ for $i+j > n$ and $i \neq j$ in terms of tensor products of some elementary functions may contain an infinite set of different elementary functions.

Remark 2. By the Corollary 3 for $f \in U_i(n, q)$, $i \geq \lceil \frac{q}{2} \rceil$ and $q \geq 5$ the equality $|f| = 2^i(q-1)^{n-i}$ holds if and only if $f_\sigma \in F_2(n,q,i,i)$ for some permutation $\sigma \in S_n$. The following example shows that for $f \in U_i(n, q)$, $i \geq \lceil \frac{q}{2} \rceil$ and $q=4$ the equality $|f| = 2^i(q-1)^{n-i}$ does not imply that $f_\sigma \in F_2(n,q,i,i)$ for some permutation $\sigma \in S_n$.

**Example.** We define the functions $h_1, h_2 : \Sigma^2_4 \rightarrow \mathbb{R}$ by the following rules:

$$h_1(x, y) = \begin{cases} 
-1, & \text{if } x = y = 0; \\
1, & \text{if } x = y = 2; \\
0, & \text{otherwise}
\end{cases}$$

and

$$h_2(x, y) = \begin{cases} 
1, & \text{if } x = 0 \text{ and } y \in \{1, 3\}; \\
-1, & \text{if } y = 2 \text{ and } x \in \{1, 3\}; \\
0, & \text{otherwise.}
\end{cases}$$

We define the function $h : \Sigma^3_4 \rightarrow \mathbb{R}$ by the following rule:

$$h(x, y, z) = \begin{cases} 
h_1(x, y), & \text{if } z = 0 \text{ or } z = 1; \\
h_2(x, y), & \text{if } z = 2; \\
h_2(y, x), & \text{if } z = 3.
\end{cases}$$

We note that $|h| = 12$. By the definition of an eigenfunction one can check that $h \in U_2(3, 4)$. Thus $h$ has the minimum size of the support in $U_2(3, 4)$ but $h_\sigma \notin F_2(3, 4, 2, 2)$ for any permutation $\sigma \in S_3$. 

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Remark 3. We note that Theorem 3 does not hold for $q = 3$. Let us consider the following example for $n = 3$ and $i = j = 2$.

Example. We define the function $v_1 : \Sigma_3^2 \rightarrow \mathbb{R}$ by the following rule:

$$v_1(x, y) = \begin{cases} 
1, & \text{if } x = y = 0; \\
-1, & \text{if } x = 1 \text{ and } y = 2; \\
0, & \text{otherwise}. 
\end{cases}$$

For $a, b \in \Sigma_3$ denote by $a \oplus b$ the sum of $a$ and $b$ modulo 3. We define the function $v : \Sigma_3^3 \rightarrow \mathbb{R}$ by the following rule:

$$v(x, y, z) = \begin{cases} 
v_1(x, y), & \text{if } z = 0; \\
v_1(x \oplus 1, y \oplus 1), & \text{if } z = 1; \\
v_1(x \oplus 2, y \oplus 2), & \text{if } z = 2. 
\end{cases}$$

We note that $|v| = 6$. By the definition of an eigenfunction one can check that $v \in U_2(3, 3)$. So $|v| < 8$ and Theorem 3 does not hold in this case.

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References

[1] E. A. Bespalov, On the minimum supports of some eigenfunctions in the Doob graphs, Siberian Electronic Mathematical Reports 15 (2018) 258–266.

[2] N. J. Cavenagh, The theory and application of latin bitrades: A survey, Mathematica Slovaca 58(6) (2008) 691–718.

[3] S. Cho, On the support size of null designs of finite ranked posets, Combinatorica 19(4) (1999) 589–595.

[4] D. Cvetkovic, P. Rowlinson and S. K. Simic, Eigenspaces of graphs, Encyclopedia of Mathematics and its Applications 66, Cambridge University Press, Cambridge, 1997.

[5] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, On eigenfunctions and maximal cliques of Paley graphs of square order, Finite Fields and Their Applications 52 (2018) 361–369.

[6] A. S. Hedayat, G. B. Khosrovshahi, Trades. In C. J. Colbourn and J. H. Dinitz, editors, Handbook of Combinatorial Designs, Discrete Mathematics and Its Applications, Chapman Hall/CRC, Boca Raton, London, New York, second edition, 644–648, 2006.
[7] D. S. Krotov, The extended 1-perfect trades in small hypercubes, Discrete Mathematics 340(10) (2017) 2559–2572.

[8] D. S. Krotov, The minimum volume of subspace trades, Discrete Mathematics 340(12) (2017) 2723–2731.

[9] D. S. Krotov, Trades in the combinatorial configurations, XII International Seminar Discrete Mathematics and its Applications, Moscow, 20–25 June 2016, 84–96 (in Russian).

[10] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, To the theory of q-ary Steiner and other-type trades, Discrete Mathematics 339(3) (2016) 1150–1157.

[11] P. R. J. Östergård, Switching codes and designs, Discrete Mathematics 312(3) (2012) 621–632.

[12] V. N. Potapov, On perfect 2-colorings of the q-ary n-cube, Discrete Mathematics 312(6) (2012) 1269–1272.

[13] V. N. Potapov, Multidimensional latin bitrade, Siberian Mathematical Journal 54(2) (2013) 317–324.

[14] V. N. Potapov, Cardinality spectra of components of correlation immune functions, bent functions, perfect colorings, and codes, Problems of Information Transmission 48(1) (2012) 47–55.

[15] E. V. Sotnikova, Eigenfunctions supports of minimum cardinality in cubical distance-regular graphs, Siberian Electronic Mathematical Reports 15 (2018) 223–245.

[16] A. Valyuzhenich, Minimum supports of eigenfunctions of Hamming graphs, Discrete Mathematics 340(5) (2017) 1064–1068.

[17] K. Vorob’ev, I. Mogilnykh, A. Valyuzhenich, Minimum supports of eigenfunctions of Johnson graphs, Discrete Mathematics 341(8) (2018) 2151–2158.

[18] K. V. Vorobev, D. S. Krotov, Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146.