Alternative approach to gravity and MOND

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ABSTRACT
The classical gravitational two-body problem is generalized in order to be applicable also to weak gravitational fields. The equation of motion holds both for terrestrial and large cosmic scales, the Newtonian gravitational law represents a mathematical limit of the generalized form. Motivation comes from observational results on rotation curves of galaxies. Existence of a dark matter is not assumed.

The crucial laws of physics hold and also the potential energy of the system is symmetric with respect to masses of the two bodies. Shortcomings of the results published for decades, including MOND theories and false-yet-familiar approaches, are overcome.

The impact on searching for a fundamental physical theory is stressed. Some of the conventional ideas of the past centuries do not hold for the zone of small accelerations, e.g., the principle of least action using the Lagrangian density of potentials and fields does not work. We may look forward to great changes in our understanding of the evolution of the Universe.

Key words: gravitation, galaxies, dark matter

1 INTRODUCTION

Dark matter is standardly considered to be an important component of the Universe. The existence of the dark matter is generally accepted for about four decades, although arguments in favor of the existence of the invisible matter appeared in the early 1920s (Kapteyn 1922, Oort 1932, Zwicky 1933). The conventional approach to observational results, e.g., flat rotation curves of galaxies, states that the masses of nearby spiral galaxies are dominated by the invisible dark matter (Swinbank 2017, Genzel et al. 2017).

As a consequence, observational data on decreasing rotation curves of distant galaxies are interpreted as ‘distant galaxies lack dark matter’, or, ‘Surprisingly, galaxies in the distant Universe seem to contain comparatively little of it.’ (Swinbank 2017).

Dark matter has not been detected directly, despite the best efforts of physicists. This suggests the possibility that dark matter does not exist. A modification of the Newton’s laws of motion or gravitational law is considered as a possibility of understanding the astronomical observations (see, e.g., Milgrom 1983; Famaey and McGaugh 2012, McGaugh et al. 2016, Rossenfelder and McGaugh 2018). These approaches are conventionally entitled as the MOND or MOG theories (MOdified Newtonian Dynamics, MOdified Gravity). The situation seems, partially, analogous to that in the second half of the 19-th century, when an attempt to modify the Newtonian gravitational law was motivated by the explanation of the advance of the perihelion of Mercury. The corresponding modification of the Newton’s gravitational law is well-known. It was elaborated by Einstein in 1915. Does there exist another important modification of the Newtonian gravitational law? The modification which is not incorporated in the general theory of relativity?

Newton succeeded in finding the gravitational law by dealing with the summarization of the observational data. The qualitative and quantitative summarization of the data was done by Kepler in 1609 and 1619. The summarization is known as the Kepler’s laws. The Newton’s law of universal gravitation is known from 1686. Similarly, we may try to find the physical modification of the gravitational law if we take into account some relevant observational results.

As a starting point we will consider the summarization of the observations presented by McGaugh et al. (2016). The authors offer a simple formula describing acceleration acting on a body moving on circular orbit in a spiral galaxy. The observed centripetal acceleration is simply related to the acceleration generated by the visible galactic mass. The simple relation reads

\[ g_{\text{obs}} = \frac{g_{\text{bar}}}{1 - \exp\left(-\sqrt{g_{\text{bar}}/g_+}\right)}, \]

where \( g_{\text{obs}} \) and \( g_{\text{bar}} \) correspond to the observed and baryonic gravitational accelerations, \( g_+ = 1.2 \times 10^{-10} \text{ m s}^{-2} \). Some other forms, instead of Eq. (1), are presented in lit-
2 GENERALIZATION

We generalize Eq. (1) into the vectorial form

\[ \vec{g}_{\text{obs}} = \frac{\vec{g}_{\text{bar}}}{1 - \exp\left(-\sqrt{|\vec{g}_{\text{bar}}|/g_+}\right)} \quad (2) \]

and the equation of motion of a body is

\[ \vec{v} = \frac{\vec{g}_{\text{bar}}}{1 - \exp\left(-\sqrt{|\vec{g}_{\text{bar}}|/g_+}\right)}, \quad (3) \]

where the dot denotes differentiation with respect to time and \( \vec{g}_{\text{bar}} \) denotes the “classical gravitational acceleration acting on the body”, the acceleration without dark matter. The quotation marks warn us that the statement is not exactly correct. For the purpose of this paper we can say that the gravitational acceleration \( \vec{g}_{\text{bar}} \) is the acceleration acting between the two bodies, see Secs. 3.1 and 3.2.

The equation of motion respects both the Newton second law and the astronomical observations. The presented equation of motion explains the astronomical observations on the large scales and it reduces to the well-known results of the classical physics when accelerations of the moving bodies are large in comparison with \( g_+ \).

At first, we will be interested in the case \( g_{\text{bar}} \ll g_+ \), in what follows. Eq. (3) reduces to

\[ \vec{v} = \sqrt{g_+} \frac{\vec{g}_{\text{bar}}}{|\vec{g}_{\text{bar}}|}. \quad (4) \]

3 2-BODIES AND SMALL ACCELERATIONS

Let us consider two point bodies of masses \( m_1 \) and \( m_2 \) at positions \( \vec{r}_1 \) and \( \vec{r}_2 \) in an inertial frame of reference when \( g_{\text{bar}} \ll g_+ \). Currently we do not express \( g_{\text{bar}} \) through \( m_1, m_2, \vec{r}_1 \) and \( \vec{r}_2 \). Finding the relation for \( g_{\text{bar}} \) requires some effort and the relation will be specified later on in this section, see Eqs. (14).

We will treat two approaches. The first considers validity of Eq. (4) in an inertial frame of reference. The second approach treats Eq. (4) as an equation of motion describing relative motion of bodies.

3.1 First approach

The two-body problem follows the equations of motion, in an inertial frame of reference,

\[ \dot{\vec{v}}_1 = - \sqrt{G g_+ \frac{m_2}{|\vec{r}_1 - \vec{r}_2|^2}} (\vec{r}_1 - \vec{r}_2), \]

\[ \dot{\vec{v}}_2 = + \sqrt{G g_+ \frac{m_1}{|\vec{r}_1 - \vec{r}_2|^2}} (\vec{r}_1 - \vec{r}_2), \quad (5) \]

where \( G \) is the gravitational constant and \( \vec{r}_1, \vec{r}_2 \) are position vectors of the bodies in the inertial frame of reference.

Eqs. (5) do not enable a conservation of energy. Moreover, the relation \( m_2 \dot{v}_1 + m_1 \dot{v}_2 = 0 \) does not hold. This approach corresponds to that presented by, e.g., Felten (1984), Famaey and McGaugh (2012 - p. 42). We want to avoid the problems.

3.2 Second approach

In this section we consider Eq. (4) as an equation of motion valid for relative motion. Thus, we treat the relative motion at first. Then discussion on motions in inertial frames follows.

3.2.1 Relative motion

According to Eq. (4), the relative motion of two bodies is

\[ \dot{\vec{v}} = - \sqrt{G g_+ \frac{m_1 + m_2}{|\vec{r}|^2}} \vec{r}, \quad (6) \]

since \( \vec{g}_{\text{bar}} = - G (m_1 + m_2) \vec{r}/|\vec{r}|^3 \).

3.2.2 Motion in an inertial frame

In order to find equations of motion for the two bodies, we are interested in \( \dot{\vec{v}}_1 \) and \( \dot{\vec{v}}_2 \). The expressions for the two quantities can be uniquely found from the relations \( \dot{\vec{v}}_1 - \dot{\vec{v}}_2 = \vec{v}, m_1 \dot{v}_1 + m_2 \dot{v}_2 = 0 \) and Eq. (6).

The two-body problem obtains the following equations of motion, in an inertial frame of reference,

\[ m_1 \dot{v}_1 = - \sqrt{G g_+ \frac{m_1 m_2}{|m_1 + m_2|} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^2}} \]

\[ = - G m_1 m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^2}, \]

\[ m_2 \dot{v}_2 = + \sqrt{G g_+ \frac{m_1 m_2}{|m_1 + m_2|} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^2}} \]
The first of Eqs. (5) is based on Eq. (4) and
\[ E_{\text{m}} = \frac{G m_1 m_2}{r_1 - r_2} \],
\[ L \equiv \sqrt{G (m_1 + m_2) / g_+} \].

Eqs. (7) can be written also in the form
\[ m_1 \ddot{v}_1 = - \frac{\partial E_{\text{pot}}}{\partial r_1}, \]
\[ m_2 \ddot{v}_2 = - \frac{\partial E_{\text{pot}}}{\partial r_2}, \]
\[ E_{\text{pot}} = \frac{G m_1 m_2}{L} \ln \left( \frac{|\vec{r}_1 - \vec{r}_2|}{L} \right), \]
\[ L \equiv \sqrt{G (m_1 + m_2) / g_+} \].

Eqs. (8) can be written as the conservation of energy:
\[ \frac{dE}{dt} = 0, \]
\[ E = E_{\text{kin}} + E_{\text{pot}}, \]
\[ E_{\text{kin}} = \frac{1}{2} m_1 (\dot{r}_1)^2 + \frac{1}{2} m_2 (\dot{r}_2)^2, \]
\[ E_{\text{pot}} = \frac{G m_1 m_2}{L} \ln \left( \frac{|\vec{r}_1 - \vec{r}_2|}{L} \right), \]
\[ L \equiv \sqrt{G (m_1 + m_2) / g_+} \],
where \( E_{\text{kin}} \) and \( E_{\text{pot}} \) are the kinetic and potential energies, \( E \) is the total energy.

3.3 Discussion

The first of Eqs. (5) is based on Eq. (4) and
\[ \ddot{g}_{\text{bar}} = - \frac{G m_2}{|\vec{r}_1 - \vec{r}_2|^2} (\vec{r}_1 - \vec{r}_2), \]
which is based on the Newtonian equation of motion
\[ m_1 \ddot{v}_1 = - \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2} (\vec{r}_1 - \vec{r}_2). \]

The second of Eqs. (5) is based on Eq. (4) and
\[ \ddot{g}_{\text{bar}} = + \frac{G m_1}{|\vec{r}_1 - \vec{r}_2|^2} (\vec{r}_1 - \vec{r}_2), \]
which is based on the Newtonian equation of motion
\[ m_2 \ddot{v}_2 = + \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2} (\vec{r}_1 - \vec{r}_2). \]

Eqs. (10) and (12) are not consistent in the magnitude: the right-hand-side of Eq. (10) contains \( m_2 \), but the right-hand-side of Eq. (12) contains \( m_1 \). This explains the violation of the total momentum.

Eqs. (11) and (13) contain the same mass-terms on the left-hand-sides and the right-hand-sides. However, the accelerations do not depend on the corresponding masses, \( \ddot{v}_j \) does not depend on \( m_j, j = 1, 2 \). This is well-known as the equivalence between the inertial and gravitational masses. This result is used as the crucial fact in the relativistic theory of gravity, the general theory of relativity.

In the zone of weak fields the situation differs from the classical case. Observations and the requirement of the conservation of the total energy and momentum lead to Eqs. (7), if accelerations fulfill the condition \( g_{\text{bar}} \ll g_+ \),

\[ g_{\text{bar}} \ll g_+, \]
\[ g_{\text{bar}} = G (m_1 + m_2) / |\vec{r}_1 - \vec{r}_2|^2. \]

Both of Eqs. (7) show that the acceleration \( \ddot{v}_j \) depends also on \( m_j, j = 1, 2 \). The real relativistic theory of gravity has to take into account this important fact. As for the large cosmic scales, the real relativistic theory of gravity differs from the Einstein’s general theory of relativity.

4 THE TWO-BODY PROBLEM FOR ARBITRARY DISTANCE

Considerations presented in Secs. 3.2.1 and 3.2.2 correspond to Eq. (1). We are interested in the general form corresponding to Eq. (3).

4.1 Relative motion

We can write for the relative motion
\[ \ddot{\vec{v}} = \ddot{\vec{r}} = - \frac{G (m_1 + m_2)}{1 - \exp \left( - L / |\vec{r}| \right)} \frac{\vec{r}}{|\vec{r}|^3}, \]
\[ L = \sqrt{G (m_1 + m_2) / g_+}, \]
where \( \vec{r} \) is the relative position vector of the bodies of masses \( m_1 \) and \( m_2 \), \( \vec{r} = \vec{r}_1 - \vec{r}_2 \) or, \( \vec{r} = \vec{r}_2 - \vec{r}_1 \).

The acceleration between the two bodies of the masses \( m_1 \) and \( m_2 \) depends on the sum of the masses \( m_1 + m_2 \). If one of the masses is dominant, then the acceleration practically does not depend on the mass of the other body. This result is a generalization of the Galileo Galilei’s observations of the free fall: the acceleration of an object falling on the Earth does not depend on the object’s mass.

Eqs. (15) can be rewritten to the form
\[ \ddot{\vec{r}} = - \frac{\partial \Phi_p}{\partial \vec{r}}. \]
\[ \Phi_p = - \frac{G (m_1 + m_2)}{L} \ln \left[ \exp \left( \frac{L}{|\vec{r}|} \right) - 1 \right], \]
\[ L = \sqrt{G (m_1 + m_2) / g_+}. \]

Eq. (10) leads to the conservation of energy
\[ \frac{dE}{dt} = 0, \]
\[ \varepsilon = \varepsilon_{\text{kin}} + \varepsilon_{\text{pot}}, \]
\[ \varepsilon_{\text{kin}} = \frac{1}{2} \left( \frac{m_1 m_2}{L} \right) (\vec{v})^2, \]
\[ \varepsilon_{\text{pot}} = \frac{G m_1 m_2}{L} \Phi_p, \]
\[ \Phi_p = - \frac{G (m_1 + m_2)}{L} \ln \left[ \exp \left( \frac{L}{|\vec{r}|} \right) - 1 \right], \]
\[ L = \sqrt{G (m_1 + m_2) / g_+}, \]
where \( \varepsilon_{\text{kin}} \) and \( \varepsilon_{\text{pot}} \) are the kinetic and potential energies, \( \varepsilon \) is the total energy.

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4.2 Motion in an inertial frame

The equation of motion in an inertial frame of reference is

\[ \begin{align*}
   m_1 \ddot{\vec{v}}_1 &= - \frac{G m_1 m_2}{1 - \exp\left(-L/|\vec{r}_1 - \vec{r}_2|\right)} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}, \\
   m_2 \ddot{\vec{v}}_2 &= + \frac{G m_1 m_2}{1 - \exp\left(-L/|\vec{r}_1 - \vec{r}_2|\right)} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}, \\
   L &= \sqrt{G (m_1 + m_2)/g_+}. \quad (18)
\end{align*} \]

Eqs. (18) immediately show the conservation of the total linear momentum, \( m_1 \dot{\vec{v}}_1 + m_2 \dot{\vec{v}}_2 = 0 \).

Without any loss of generality, let us concentrate on the first of Eqs. (18). The classical case, \( g_+ \to 0 \), would yield the acceleration \( \dot{\vec{v}}_1 \) independent on the mass of \( m_1 \). However, in our case \( \dot{\vec{v}}_1 \) is independent on \( m_1 \) only when \( m_1 \ll m_2 \), i.e., as if the body of negligible mass \( m_1 \) would move in a relatively strong gravitational field. This conclusion corresponds to the conclusion valid for the relative motion described by Eqs. (15). There is some kind of unification between Eqs. (15) and (18), as for the dependence of the acceleration on the masses.

Accelerations \( \dot{\vec{v}}_1 \) and \( \dot{\vec{v}}_2 \) given by Eqs. (15) depend on both masses, \( m_1 \) and \( m_2 \). The acceleration of a body is not given only by a source field. The acceleration of the body depends also on the mass of the body.

Eqs. (18) can be rewritten to the form

\[ \begin{align*}
   m_1 \ddot{\vec{v}}_1 &= - \frac{\partial E_{\text{pot}}}{\partial \vec{r}_1}, \\
   m_2 \ddot{\vec{v}}_2 &= - \frac{\partial E_{\text{pot}}}{\partial \vec{r}_2}, \\
   E_{\text{pot}} &= - \frac{G m_1 m_2}{L} \ln \left\{ \exp \left( \frac{L}{|\vec{r}_1 - \vec{r}_2|} \right) - 1 \right\}, \\
   L &= \sqrt{G (m_1 + m_2)/g_+}. \quad (19)
\end{align*} \]

The conservation of energy reads

\[ \frac{dE}{dt} = 0, \quad E = E_{\text{kin}} + E_{\text{pot}}, \]

\[ \begin{align*}
   E_{\text{kin}} &= \frac{1}{2} m_1 (\vec{v}_1)^2 + \frac{1}{2} m_2 (\vec{v}_2)^2, \\
   E_{\text{pot}} &= - \frac{G m_1 m_2}{L} \ln \left\{ \exp \left( \frac{L}{|\vec{r}_1 - \vec{r}_2|} \right) - 1 \right\}, \\
   L &\equiv \sqrt{G (m_1 + m_2)/g_+}. \quad (20)
\end{align*} \]

where \( E_{\text{kin}} \) and \( E_{\text{pot}} \) are the kinetic and potential energies, \( E \) is the total energy.

4.3 Discussion - Newtonian limit

Eqs. (15)- (17), and, Eqs. (18)- (20) are new equations and they are more general than the Newtonian equations of motion for the gravitational action. The Newtonian results can be obtained from the new equations in the limiting case \( g_+ \to 0 \), e.g.,

\[ E_{\text{pot}}(\text{Newtonian}) = \lim_{g_+ \to 0} E_{\text{pot}}, \quad (21) \]

where the potential energy \( E_{\text{pot}} \) is given in Eqs. (20). Similarly, the limit \( g_+ \to 0 \) reduces Eqs. (15)- (20) to the equations of classical physics.

Only bounded orbits exist for finite total energy and \( g_+ \neq 0 \). This result differs from the two-body problem in classical physics, \( g_+ = 0 \), when \( \varepsilon < 0 \) characterizes bounded orbits and \( \varepsilon \geq 0 \) corresponds to the unbounded orbits (the parabolic orbit is sometimes called to be marginally bounded, e.g., Fitzpatrick 2012, p. 45).

4.4 Discussion - conventional physics

The conventional approach in physics, not only in gravitational physics, is the usage of the terms ‘intensity of the field’ and ‘potential’. The gravitational mass \( m_* \) at the position \( \vec{r}_* \) generates the intensity \( E_{\text{c}} \) and the potential \( \Phi_{\text{c}} \). In the Newtonian gravity

\[ \begin{align*}
   \dot{\vec{v}} &= \vec{E}_{\text{c}}, \\
   \vec{E}_{\text{c}} &= - G m_* \frac{\vec{r} - \vec{r}_*}{|\vec{r} - \vec{r}_*|^3}, \\
   \Phi_{\text{c}} &= - \frac{G m_*}{|\vec{r} - \vec{r}_*|}. \quad (22)
\end{align*} \]

The important property is

\[ \begin{align*}
   \Phi_{\text{n}} &= \Phi_{\text{c}} (m_*, |\vec{r} - \vec{r}_*|) , \\
   \vec{E}_{\text{n}} &= \vec{E}_{\text{c}} (m_*, \vec{r} - \vec{r}_*). \quad (23)
\end{align*} \]

The potential \( \Phi_{\text{n}} \) does not depend on the mass \( m \) of the test particle. Similarly, the intensity of the gravitational field \( \vec{E}_{\text{n}} \) does not depend on the mass \( m \) of the test particle. This is closely connected with the equivalence between the inertial and gravitational masses which corresponds to the equivalence principle in the general theory of relativity.

4.5 Discussion - new physics

On the basis of Eq. (19) we can write

\[ \begin{align*}
   \dot{\vec{v}} &= - \frac{\partial \Phi_{\text{n}}}{\partial \vec{r}}, \\
   \Phi_{\text{n}} &= - \frac{G m_*}{L} \ln \left\{ \exp \left( \frac{L}{|\vec{r} - \vec{r}_*|} \right) - 1 \right\}, \\
   L &\equiv \sqrt{G (m_1 + m_2)/g_+}. \quad (24)
\end{align*} \]

The important property is

\[ \Phi_{\text{n}} \equiv \Phi_{\text{n}} (m, m_*, |\vec{r} - \vec{r}_*|). \quad (25) \]

The ‘potential’ \( \Phi_{\text{n}} \) depends not only on the source mass \( m_* \), but also on the mass of the test particle \( m \).

The potential energy \( U_{\text{n}} \) of the system is

\[ \begin{align*}
   U_{\text{n}} &= - \frac{G m_*}{L} \ln \left\{ \exp \left( \frac{L}{|\vec{r} - \vec{r}_*|} \right) - 1 \right\}, \\
   L &\equiv \sqrt{G (m_1 + m_2)/g_+}. \quad (26)
\end{align*} \]

see Eqs. (19), (20), or, Eq. (21) with \( U = m \Phi_{\text{n}} \). We want to stress the symmetry between the masses \( m \) and \( m_* \), or, between the pairs \((m, \vec{r})\) and \((m_*, \vec{r}_*)\).
4.6 Discussion - comparison of the conventional and new approaches

The result represented by Eq. 26 differs from the conventional physical approach represented by Eqs. 25. Eq. 26 is more general and it reduces to 23 in the limiting case $\Phi_c = \lim_{g \to 0} \Phi_n$. (27)

4.7 Some other approaches to weak fields

The previous Secs. 4.4, 4.5 and 4.6 point out that the conventional physical approaches to gravitational physics probably hold only in the limiting case $g \to 0$. This suggests that also theoretical approaches to weak gravitational fields may not be correct. We will discuss the situation in this section.

4.7.1 QUMOND - theory

The idea of QUMOND (see, e.g., Famaey and McGaugh 2012, 46-48 pp., Milgrom 2010) is the preservation of the ‘matter action’ $S_{kin} + S_{in} = \int \rho (\dot{r}^2/2 - \Phi_Q) \, d^3 \vec{x} \, dt$ and the gravitational action is modified in the following way

$$ S_{gr} = \int L_{gr} \, d^3 \vec{x} \, dt , $$

$$ L_{gr} = - \frac{1}{8\pi G} \left[ 2 \nabla \Phi_Q \cdot \nabla \Phi_R - g_+^2 \frac{\left( \nabla \Phi_R \right)^2}{g_+^2} \right] , (28) $$

where $L_{gr}$ is the Lagrangian density of the gravitational action, $Q$ represents dimensionless function and $\Phi_N$ is the Newtonian potential.

Variation of the total action $S = S_{kin} + S_{in} + S_{gr}$ with respect to the configuration space coordinates yields the equation of motion

$$ \ddot{r} = - \nabla \Phi_Q . \tag{29} $$

Variation with respect to $\Phi_Q$ yields

$$ \Delta \Phi_N = 4\pi G \rho \tag{30} $$

and variation with respect to $\Phi_R$ yields

$$ \Delta \Phi_R = \nabla \cdot \left[ \nu \left( \frac{\left| \nabla \Phi_R \right|^2}{g_+^2} \right) \nabla \Phi_R \right] , \tag{31} $$

or,

$$ \Delta \Phi_R = [\nabla \nu (y)] \cdot [\nabla \Phi_R] + \nu (y) \Delta \Phi_R \tag{32} $$

where

$$ \nu (y) = \frac{dQ(z)}{dz} , \quad z = y^2 \tag{33} $$

and

$$ Q(z) \to z \quad \text{for} \quad z \gg 1 , $$

$$ Q(z) \to (4/3) z^{3/4} \quad \text{for} \quad z \ll 1 . \tag{34} $$

4.7.2 QUMOND - two-body problem

Considering a source of the mass $m_*$, the mass density $\rho$ and the potential $\Phi_N$ are

$$ \rho = m_* \delta (\vec{r} - \vec{r}_* ) \tag{35} , $$

$$ \Phi_N = - \frac{G m_*}{|\vec{r} - \vec{r}_*|} \tag{35} , $$

In the case $z \ll 1$ we obtain

$$ \nabla \Phi_N = G m_* \frac{\vec{r} - \vec{r}_*}{|\vec{r} - \vec{r}_*|} , $$

$$ \Delta \Phi_N = 4\pi G m_* \delta (\vec{r} - \vec{r}_*) = 0 , $$

$$ \nu (y) = \frac{1}{\sqrt{g_+}} \tag{36} $$

$$ \nu \left( \frac{\left| \nabla \Phi_N \right|^2}{g_+} \right) = \sqrt{\frac{g_+}{g_+}} = \sqrt{\frac{g_+}{G m_* |\vec{r} - \vec{r}_*|}} \, . \tag{37} $$

Consequently, Eqs. 32 and 36 give

$$ \Delta U_Q = \sqrt{g_+ G m_* |\vec{r} - \vec{r}_*|^{-2}} \tag{38} $$

for the considered zone, far from the source.

On the basis of Eqs. 29 and 37 we can write for the potential energy $U_Q = m_\Phi_Q$

$$ \Delta U_Q = m \sqrt{g_+ G m_* |\vec{r} - \vec{r}_*|^{-2}} \tag{38} $$

when $G m_* |\vec{r} - \vec{r}_*|^{-2} / g_+ \ll 1$.

Eq. 33 immediately shows that the QUMOND theory is not a physical theory. The potential energy is not symmetric in masses $m$ and $m_*$.

We remind that the new approach fulfills the symmetry between the masses $m$ and $m_*$, see Eqs. 26.

4.7.3 Bekenstein-Milgrom MOND - theory

The idea of the Bekenstein-Milgrom MOND - theory (see, e.g., Famaey and McGaugh 2012, p. 44, Bekenstein and Milgrom 1984) is the preservation of the ‘matter action’ $S_{kin} + S_{in} = \int \rho (\dot{r}^2/2 - \Phi_{BM}) \, d^3 \vec{x} \, dt$ and the gravitational action is modified in the following way

$$ S_{gr} = \int L_{gr} \, d^3 \vec{x} \, dt , $$

$$ L_{gr} = - \frac{1}{8\pi G} \left[ 2 \nabla \Phi_{BM} \cdot \nabla \Phi_{BM} - g_+^2 \frac{\left( \nabla \Phi_{BM} \right)^2}{g_+^2} \right] , \tag{39} $$

where $L_{gr}$ is the Lagrangian density of the gravitational action and $F$ can be any dimensionless function.

The equation of motion is

$$ \ddot{r} = - \nabla \Phi_{BM} \tag{40} $$

and the variation of the total action $S = S_{kin} + S_{in} + S_{gr}$ with respect to $\Phi_{BM}$ leads to a non-linear generalization of the Newtonian Poisson equation

$$ \nabla \cdot \left[ \mu \left( \frac{\left| \nabla \Phi_{BM} \right|^2}{g_+^2} \right) \nabla \Phi_{BM} \right] = 4\pi G \rho \tag{41} $$

where

$$ \mu (x) = \frac{dF(z)}{dz} , \quad z = x^2 \tag{42} $$

and

$$ F(z) \to z \quad \text{for} \quad z \gg 1 , $$

$$ F(z) \to (2/3) z^{3/2} \quad \text{for} \quad z \ll 1 . \tag{43} $$
4.7.4 Bekenstein-Milgrom MOND - two-body problem

Considering a source of the mass $m_*$, we can write for the mass density $\rho$ and for the Laplace operator the well-known relations

$$\rho = m_\star \delta(r - r_\star) ,$$

$$\nabla \left( \frac{1}{r - r_\star} \right) = -4\pi \delta(r - r_\star) . \tag{44}$$

Using Eqs. (43) and (44), we obtain

$$\mu \left( \nabla \Phi_{BM} \right) / g_+ \nabla \Phi_{BM} = G m_\star \frac{\vec{r} - \vec{r}_\star}{|\vec{r} - \vec{r}_\star|^3}$$

$$- \nabla \times \left( \frac{\vec{w}}{g_+} \right) , \tag{45}$$

where $\vec{w}$ is an arbitrary vectorial function. This follows from the simple identity $\nabla \cdot \left( \nabla \times (\vec{w}/g_+) \right) = 0$. We will not need any specification of $\vec{w}$ in this paper, see Eq. (43) and the text below it.

Eqs. (42), (43) and (45) yield for the zone far from the source

$$\frac{\nabla \Phi_{BM}}{g_+} \nabla \Phi_{BM} = G m_\star \frac{\vec{r} - \vec{r}_\star}{|\vec{r} - \vec{r}_\star|^3}$$

$$- \nabla \times \left( \frac{\vec{w}}{g_+} \right) , \tag{46}$$

or,

$$\nabla \Phi_{BM} = \sqrt{g_+ G m_\star} \left| \vec{r} - \vec{r}_\star \right|^{-1} \nabla \times \left( \frac{\vec{r} - \vec{r}_\star}{g_+ G m_\star} \nabla \times \vec{w} \right) \left| \vec{r} - \vec{r}_\star \right|^{1/2} . \tag{47}$$

Eq. (47) yields

$$\nabla U_{BM} = m \sqrt{g_+ G m_\star} \left| \vec{r} - \vec{r}_\star \right|^{-1} \nabla \times \left( \frac{\vec{r} - \vec{r}_\star}{g_+ G m_\star} \nabla \times \vec{w} \right) \left| \vec{r} - \vec{r}_\star \right|^{1/2} \tag{48}$$

for the potential energy $U_{BM} = m \Phi_{BM}$ when $G m_\star / \left| \vec{r} - \vec{r}_\star \right|^2 / g_+ \ll 1$.

Eq. (47) immediately shows that the Bekenstein-Milgrom MOND theory is not a physical theory. The potential energy is not symmetric in masses $m$ and $m_\star$. We remind that the new approach fulfills the symmetry between the masses $m$ and $m_\star$, see Eqs. (29).

One can easily verify that $\Phi_{BM}$ given by Eqs. (24) is not a solution of Eq. (41).

4.8 Discussion on weak field theories

All published theories are based on the Lagrangian formulation of the extremal action and the Lagrangian density. The Lagrangian density assumes that some kind of gravitational potential exists. As we have discussed in Secs. 4.4 [15] and 4.6, these approaches are not consistent with the important property presented by Eq. (29).

Moreover, the results for the potential energies of the system of two bodies with masses $m$ and $m_\star$ are not symmetric between the masses $m$ and $m_\star$, or, between the pairs $(m, \vec{r})$ and $(m_\star, \vec{r}_\star)$, see Eqs. (38) and (18). The new approach fulfills the symmetry between the masses $m$ and $m_\star$, see Eqs. (29).

5 GENERAL CASE - VARIOUS FITS

This section treats various fits corresponding to the case presented by Eq. (39). The fits may represent a better approximation to reality than the function used in Eq. (11), see Famaey and McGaugh (2012).

5.1 Equation of motion - various fits

On the basis of Eq. (40) we can make a generalization

$$\vec{v} = f \left( \frac{\vec{g}_{\bar{m}}}{g_+} \right) \vec{g}_{\bar{m}} , \tag{49}$$

where the dot denotes differentiation with respect to time and $\vec{g}_{\bar{m}}$ denotes the classical gravitational acceleration. As a function $f$ the following functions fitting the observational data may be used:

$$f(x) = \frac{1}{1 - \exp \left( -\sqrt{x} \right) } , \tag{50}$$

or,

$$f(x) = \left( 1 + \sqrt{1 + 4 x^{n/2}} \right)^{1/n} , \tag{51}$$

or,

$$f(x) = \left[ 1 - \exp \left( -x^{n/2} \right) \right]^{-1/n} , \tag{52}$$

or,

$$f(x) = \left[ 1 - \exp \left( -x^{n} \right) \right]^{-1/(2n)} + \left[ 1 - 1/(2n) \right] \exp (-x^n) , \tag{53}$$

Although the presented functions are not identical, their limits for the case $x \ll 1$ correspond to the case treated in Sec. 3.

If a modification of the Newton’s law of gravity is real, then the correct theory should offer the right form of the function $f$.

5.2 Two bodies: Relative motion

On the basis of Eq. (40) and $\vec{g}_{\bar{m}} = -G(m_1 + m_2)\vec{r}/r^3$ we can write

$$\vec{v} = -f \left( \frac{\vec{g}_{\bar{m}}}{g_+} \right) \frac{G(m_1 + m_2)}{r^3} \vec{r} ;$$

$$\sqrt{\left| \vec{g}_{\bar{m}} \right| / g_+} = \frac{L}{r} ,$$

$$L \equiv \sqrt{G(m_1 + m_2)/g_+} . \tag{54}$$

The functions $f$ are defined by Eqs. (50)-(53).

The limiting case $|\vec{g}_{\bar{m}}|/g_+ \ll 1$ corresponds to the case discussed in Sec. 3. See also Sec. 5.2.1.

5.3 Two bodies: Motion in an inertial frame

Relations $\vec{v}_1 - \vec{v}_2 = \vec{v}, m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0$ and Eqs. (42) enable to write equations of motion in an inertial frame of reference:

$$m_1 \vec{v}_1 = \frac{m_1 m_2}{m_1 + m_2} f \left( \frac{\vec{g}_{\bar{m}}}{g_+} \right) \vec{g}_{\bar{m}}.$$
\[
\begin{align*}
    m_2 \ddot{r}_2 &= - \frac{m_1 m_2}{m_1 + m_2} f \left( \frac{g_{\text{bar}}}{g_+} \right) g_{\text{bar}}, \\
    g_{\text{bar}} &= - G (m_1 + m_2) \frac{\dot{r}_1 - \dot{r}_2}{|\dot{r}_1 - \dot{r}_2|}, \\
    \sqrt{\frac{|g_{\text{bar}}|}{g_+}} &= L \frac{1}{|\dot{r}_1 - \dot{r}_2|}. \\
    L &= \sqrt{G (m_1 + m_2) / g_+}.
\end{align*}
\]

and the function \( f \) is defined, e.g., by one of Eqs. (20)–(23).

Eqs. (55) fulfill both the conservation of the total momentum and the energy of the system.

The limiting case \( |g_{\text{bar}}| / g_+ \ll 1 \) corresponds to the case treated in Sec. 3, see mainly Sec. 3.2.2.

6 CONCLUSION

The paper presents generalized two-body problem overcoming the shortcomings of the results presented by, e.g., Felten (1984), Bekenstein and Milgrom (1984), Milgrom (2010), Famaey and McGaugh (2012), i.e., the MOND results. Our formulation fulfills the standard laws of physics, the Newton’s laws of motion and the conservations of energy and momentum.

The classical two-body problem is generalized for the case of small gravitational accelerations when a new gravitational constant \( g_+ \approx 1.2 \times 10^{-10} \text{ m s}^{-2} \) plays an important role. The generalized equation of motion leads to the results consistent with observations of rotation curves of galaxies without any assumption on the existence of dark matter.

The generalized equation of motion reduces to the classical two-body problem in a mathematical limit \( g_+ \to 0 \). The physical laws of the conservation of energy, linear and angular momenta hold. The potential energy of the system is symmetric with respect to masses of the two bodies, compare Eq. (19). This important property is violated in the equations of motion presented since the 1980ies in attempts of generalization of the Newtonian theory. The inconsistency of the theories is inherently connected with the methods of physics used from the 19-th century.

The equation of motion derived in this paper leads not only to a new generalized gravitational physics. The found result has a crucial impact on searching for fundamental physical theories. The conventionally used ideas about potential and intensity of the gravitational field do not hold for the zones of small accelerations. The real ‘potential’ and ‘intensity’ depend not only on the source mass of the gravitational field, but also on the test particle mass, compare Eq. (19) and discussion in Sec. 3.5. The principle of least action, the Hamilton’s principle, in the form \( \delta S = 0 \), where \( S = \int L \, d^4 \vec{s} \, dt \) and \( L \) is the Lagrangian density depending on potentials and fields, does not work in the zones of small gravitational accelerations. These fundamental changes in the understanding of the physical Nature would not exist if one could prove that the conventional description of the gravitation used for more than a hundred years is correct. In that case the existence of the dark matter is inevitable. In the opposite case we have to await great changes in our understanding of the evolution of the Universe, the cosmology.

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