GROUP ACTION ON POLISH SPACES

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Abstract. In this paper we investigate the action of Polish groups (not necessary abelian) on an uncountable Polish spaces. We consider two main situations. First, when the orbits given by group action are small and the second when the family of orbits are at most countable. We have found some subgroups which are not measurable with respect to a given \( \sigma \)-ideals on the group and the action on some subsets gives a completely nonmeasurable sets with respect to some \( \sigma \)-ideals with a Borel base on the Polish space. In most cases the general results are consistent with ZFC theory and are strictly connected with cardinal coefficients. We give some suitable examples, namely the subgroup of isometries of the Cantor space where the orbits are sufficiently small. In a opposite case we give an example of the group of the homeomorphisms of a Polish space in which there is a large orbit and we have found the subgroup without Baire property and a subset of the mentioned space such that the action of this subgroup on this set is completely nonmeasurable set with respect to the \( \sigma \)-ideal of the subsets of first category.

1. Notation and Terminology

In this paper we will use standard set theoretic notation following [5]. In particular, for any set \( X \) and any cardinal \( \kappa \) a set \( [X]^{<\kappa} \) denotes the family of all subsets of \( X \) with size less than \( \kappa \). Moreover, \( \mathcal{P}(X) \) denotes the \( \sigma \)-algebra of all Borel subsets of \( X \).

Let \( X \) be any uncountable Polish space. By \( \mathcal{B}(X) \) we denote the \( \sigma \)-algebra of all Borel subsets of \( X \).

\( I \subseteq \mathcal{P}(X) \) is an ideal on \( X \) if \( I \) is closed under taking subsets and under finite unions. We assume that \( I \) is nontrivial meaning that \( I \neq \emptyset \) and \( X \notin I \). If for every sequence \( (A_n)_{n \in \omega} \) of elements of \( I \) we have that \( \bigcup_{n \in \omega} A_n \in I \) then \( I \) is called \( \sigma \)-ideal.

Let us recall that \( \mathcal{F} \subseteq I \) is a base of \( I \) iff

\[(\forall Y \in I)(\exists B \in \mathcal{F}) \ Y \subseteq B.\]
If, additionally $\mathcal{F}$ consists of Borel sets then we say that $\mathcal{F}$ is a Borel base of $I$.

We will use the following cardinal coefficients:

**Definition 1.1.** Let $I \subset \mathcal{P}(X)$ be a $\sigma$-ideal on a Polish space $X$. Assume that $I$ has Borel base. Set

$$cov(I) = \min\{|\mathcal{A}| : \mathcal{A} \subset I \land \bigcup \mathcal{A} = X\},$$

$$cov_h(I) = \min\{|\mathcal{A}| : (\mathcal{A} \subset I) \land (\exists B \in \mathcal{B}(X) \setminus I) (\bigcup \mathcal{A} = B)\},$$

$$cof(I) = \min\{|\mathcal{A}| : \mathcal{A} \subset I \land \mathcal{A} \text{ is a Borel base of } I\}.$$

**Definition 1.2.** We say that a pair $(X, I)$ is a Polish ideal space iff

- $X$ is uncountable Polish space,
- $I \subset \mathcal{P}(X)$ is a $\sigma$-ideal containing singletons and having Borel base.

Let us remark that a classical example of Polish ideal space is $(X, M)$, where $X$ is an uncountable Polish space without isolated points and $M$ stands for $\sigma$-ideal of meager sets i.e. $\sigma$-ideal generated by closed nowhere dense sets.

The cardinal coefficients connected to $M$ does not depend on $X$ (see e.g. [6]). Moreover $cov(M) = cov_h(M)$.

By $\mathcal{B}_I^+(X)$ we denote the set $\mathcal{B}(X) \setminus I$ of all $I$-positive Borel subsets of the space $X$.

Let us notice that similarly as in the context of ideals we can define a base of $\mathcal{B}_I^+(X)$. In this case we say that $\mathcal{F} \subseteq \mathcal{B}_I^+(X)$ is a base if

$$(\forall A \in \mathcal{B}_I^+(X))(\exists B \in \mathcal{F})(B \subseteq A).$$

The base of $\mathcal{B}_I^+(X)$ consists of Borel sets, so it is always of size smaller or equal to continuum.

**Definition 1.3.** Let $(X, I)$ be Polish ideal space. Assume that $C \subseteq D \subseteq X$. We say that $C$ is completely $I$-nonmeasurable in $D$ iff

$$(\forall B \in \mathcal{B}_I^+(X))(B \cap D \notin I \rightarrow (B \cap C \notin I \land B \cap (D \setminus C) \notin I)).$$

Let us remark that this notion was studied e.g. in papers [10], [11], [2], [9], [12].

**Definition 1.4.** We say that a pair $(G, \cdot)$ is a Polish group if $(G, \cdot)$ is a topological group and $G$ with its topology is a Polish space. Neutral element of the group $(G, \cdot)$ will be denoted by $e$. If $X$ is any set and $(G, \cdot)$ is a group then a mapping $F : G \times X \rightarrow X$ is called an action of a group $G$ on $X$ iff the following conditions are fulfilled:

1. $(\forall x \in X)F(e, x) = x$,
2. $(\forall x \in X)(\forall g, h \in G) F(g, F(h, x)) = F(gh, x)$.

For the reader’s convenience $F(g, x)$ is denoted by $gx$ and $(G, \cdot)$ by $G$. 

In the same fashion as in Definition 1.2 we define a notion of Polish ideal group.

**Definition 1.5.** We say that a triple \((G, \cdot, J)\) is a Polish ideal group iff \((G, J)\) is Polish ideal space and \((G, \cdot)\) is a Polish group.

**Definition 1.6.** Let \(G\) be any group and \(X\) be any set. Let \(A \subset X\) be any subset of the \(X\) then the set \(GA = \{gx \in X : (g, x) \in G \times A\}\) is called an orbit of the set \(A\) by the group \(G\) and whenever \(A = \{x\}\) is singleton then we will write \(Gx\) instead of \(G\{x\}\) for convenience.

Of course the family \(\{Gx : x \in X\}\) forms a partition of \(X\). Each set of the form \(Gx\) can be viewed as an abstract class with respect to the following equivalence relation:

\[(x \sim y) \iff ((\exists g \in G) (y = gx))\]

Let \(A \subset G\) be any subset of the group \(G\) then \(\langle A \rangle_G\) denotes the subgroup of \(G\) generated by the set \(A\). Let us observe that for any nonempty subset \(A \subset G\) we have

\[\langle A \rangle_G = \left\{ \prod_{i \in \omega} f(i)^{g(i)} \in G : n \in \omega \land f \in A^n \land g \in \mathbb{Z}^n \right\}.\]

Here \(h^0 = e \in G\) whenever \(h \in G\).

## 2. Results

First, let us consider the situation when the set of orbits of singletons is quite large.

**Theorem 2.1.** Let \((X, I)\) be Polish ideal space and \((G, \cdot)\) be any Polish group acting on \(X\). Assume that the following condition is fulfilled:

\[(\forall B \in \mathcal{B}^+_I(X)) \text{ cof}(I) \leq |\{Gb : b \in B\}|.\]

Then there exists a subgroup \(H \leq G\) and a subset \(A \subset X\) such that \(A\) and \(HA\) are completely \(I\)-nonmeasurable subsets of \(X\). Moreover if \((G, J)\) is a Polish space and there exists Borel bases \(\mathcal{B}_G \subset \mathcal{B}^+_I(G)\) and \(\mathcal{B}_X \subset \mathcal{B}^+_I(X)\) with

\[|\mathcal{B}_G| = |\mathcal{B}_X| \leq |\{Gb : b \in B\}|,\]

then \(H\) is completely \(J\)-nonmeasurable in the group \(G\).

**Proof.** Let us enumerate bases \(\mathcal{B}_G = \{C_\alpha : \alpha < \lambda\}\) and \(\mathcal{B}_X = \{B_\alpha : \alpha < \lambda\}\) where \(\lambda \leq |\{Gb : b \in B\}|.\) Now let assume that we are in \(\alpha < \lambda\) step construction with the following transfinite sequence:

\[(\{a_\xi, d_\xi, h_\xi, e_\xi\} \in B_\xi \times B_\xi \times C_\xi \times C_\xi : \xi < \alpha}\]

with the following conditions:
(1) the collection of orbits \( \{ Ga_\xi : \xi < \alpha \} \cup \{ Gd_\xi : \xi < \alpha \} \) is pairwise disjoint,
(2) \( \langle h_\xi : \xi < \alpha \rangle_G \cap \{ c_\xi : \xi < \alpha \} = \emptyset \).

By assumption we can find 
\( a_\alpha, d_\alpha \in B_\alpha \setminus \bigcup \{ \{ Ga_\xi : \xi < \alpha \} \cup \{ Gd_\xi : \xi < \alpha \} \} \)
and \( h_\alpha \in C_\alpha \setminus \langle h_\xi : \xi < \alpha \rangle_G \) because \( |\langle Z \rangle_G| \leq \aleph_0 \cdot |Z| \) for any set \( Z \subset G \). Then \( \alpha \) step is finished.

Now let us take the following sets: 
\( H = \langle h_\alpha \in G : \alpha < \lambda \rangle_G, A = \{ a_\alpha \in X : \alpha < \lambda \} \) and \( D = \{ d_\alpha \in X : \alpha < \lambda \} \). Then by our transfinite construction \( H \) is completely \( J \)-nonmeasurable subgroup of \( G \), \( A \) and \( D \) are completely \( I \)-nonmeasurable subsets of the Polish space \( X \). Moreover by the following inclusion
\( A \subset HA \subset D^c \)
we see that \( HA \) is completely \( I \)-nonmeasurable subset of \( X \). \( \square \)

By a simple modification of the proof of the previous Theorem (by double transfinite induction) we can derive a stronger result:

**Theorem 2.2.** Let \( (X, I) \) be Polish ideal space and \( (G, \cdot) \) be any group acting on \( X \). If the following condition is fulfilled:
\[ (\forall B \in \mathcal{B}_I^+(X)) \text{ cof}(I) \leq |\{ Gb : b \in B \}|. \]
Then there exists subgroup \( H \leq B \) and the pairwise disjoint family \( \{ A_\alpha : \alpha < \text{ cof}(I) \} \subset \mathcal{P}(X) \) such that:
\( (1) (\forall \alpha < \text{ cof}(I)) A_\alpha, HA_\alpha \) are completely \( I \)-nonmeasurable in \( X \),
\( (2) (\forall \alpha, \beta \alpha < \beta < \text{ cof}(I) \longrightarrow HA_\alpha \cap HA_\beta = \emptyset. \)
Moreover if \( (G, J) \) is a Polish space and there exists Borel bases \( \mathcal{B}_G \subset \mathcal{B}_I^+(G) \) and \( \mathcal{B}_X \subset \mathcal{B}_I^+(X) \) with
\[ |\mathcal{B}_G| = |\mathcal{B}_X| \leq |\{ Gb : b \in B \}|, \]
then \( H \) is completely \( J \)-nonmeasurable in the group \( G \).

Now, let us now consider the situation when, in contrast to previous theorems, there exists only one orbit.

**Theorem 2.3.** Let \( (G, \cdot) \) be any group and let \( (X, I) \) be a Polish ideal space. If for some (every) \( x \in X \) \( Gx = X \) and
\[ (\exists \lambda < 2^\omega)(\forall x, y \in X) x \neq y \longrightarrow |G_{x,y}| \leq \lambda \]
where \( G_{x,y} = \{ g \in G : y = gx \} \) then there exists a subgroup \( H \leq G \) and a subset \( A \subset X \) such that \( A \) and \( HA \) are complete \( I \)-nonmeasurable sets in \( X \). Moreover, if \( (G, J) \) forms Polish ideal space then our \( H \) is completely \( J \)-nonmeasurable in \( G \).
Let us enumerate the Borel bases in the group $G$, $\mathcal{B}_G = \{C_\alpha : \alpha < 2^\omega\}$ and $\mathcal{B}_X = \{B_\alpha : \alpha < 2^\omega\}$. Let us suppose that we are in the $\alpha$-step of recursive construction:

$$\langle (a_\xi, d_\xi, h_\xi, c_\xi) \in B_\xi \times B_\xi \times C_\xi \times C_\xi : \xi < \alpha \rangle$$

with the following conditions:

$$H_\alpha A_\alpha \cap D_\alpha = \emptyset, \text{ and } H_\alpha \cap F_\alpha = \emptyset,$$

where $H_\alpha = \langle \{h_\xi : \xi < \alpha\}\rangle_G$, $F_\alpha = \{c_\xi : \xi < \alpha\}$, $A_\alpha = \{a_\xi : \xi < \alpha\}$ and $D_\alpha = \{d_\xi : \xi < \alpha\}$.

Now let us choose any element $a_\alpha \in B_\alpha \setminus D_\alpha$. Set $W_\alpha$ to be the following set

$$\{h \in G : \exists \xi < \alpha \exists n \in \omega \exists f \in \langle \{h\} \cup H_\alpha\rangle^n \exists g \in \mathbb{Z}^\omega d_\xi = \prod_{i \in n} f(i)^{g(i)} a_\alpha\}$$

By our assumption $W_\alpha$ has the cardinality less or equal to $\aleph_0|H_\alpha|$. Let us choose any $h_\alpha \in C_\alpha \setminus W_\alpha$. It is possible because $C_\alpha$ contains some uncountable perfect set. Now choose any element $c_\alpha \in C_\alpha \setminus \langle \{h_\alpha\} \cup H_\alpha\rangle_G$ by the same argument. Finally we can choose any element $d_\alpha \in B_\alpha \setminus \langle \{h_\alpha\} \cup H_\alpha\rangle_{\langle a_\alpha\rangle \cup A_\alpha}$, what is possible because we have the following cardinal inequality:

$$|\langle \{h_\alpha\} \cup H_\alpha\rangle_{\langle a_\alpha\rangle \cup A_\alpha}| \leq |\langle \{h_\alpha\} \cup H_\alpha\rangle \cup \{a_\alpha\} \cup A_\alpha| \leq |\xi|\aleph_0 < 2^\omega.$$  

Then our construction is done by transfinite induction Theorem. Let us observe that the following sets

$$A = \{a_\alpha : \alpha < 2^\omega\}, D = \{d_\alpha : \alpha < 2^\omega\}, H = \langle h_\alpha : \alpha < 2^\omega\rangle_G = \bigcup_{\alpha < 2^\omega} H_\alpha,$$

fulfills assertion of this Theorem because $A \subset HA \subset D^c$. So, the proof is finished.

With a little more afford we can prove the analogous theorem in the case when the set of all orbits is small and those orbits are $I$-measurable subsets of $X$ (which means that belong to the $\sigma$-algebra $\mathcal{B}[I]$ generated by sets which are Borel or belongs to the ideal $I$). Namely, we have the following result.

**Theorem 2.4.** Let $(X, I)$ be a Polish ideal space and $(G, \cdot)$ be any group acting on $X$. Assume that $X = \bigcup\{Gx_n : n \in \omega\}$ is a union of the countable many $I$-positive and $I$-measurable orbits. Suppose that

$$(\exists \lambda < 2^\omega)(\forall x, y \in X) x \neq y \rightarrow |G_{x,y}| \leq \lambda,$$

where $G_{x,y} = \{g \in G : y = gx\}$. Then there exists a subgroup $H \leq G$ and countably many families $\mathcal{A}_n$ such that for every $n \in \omega \mathcal{A}_n = \{A^n_\alpha : \alpha < 2^\omega\} n \in \omega$ is a family of continuum many pairwise disjoint subsets of $X$ with the following conditions:

$$(\forall n \in \omega)(\forall \alpha < 2^\omega)A^n_\alpha, HA^n_\alpha \text{ is completely } I \text{ -- nonmeasurable in } Gx_n.$$
Moreover, if \((G, J)\) forms Polish ideal space then our \(H\) is completely \(J\)-nonmeasurable in \(G\).

**Theorem 2.5.** Let \((G, \cdot, J)\) be a Polish ideal group which acts on the Polish ideal space \((X, I)\). Let us assume that

1. \(\text{cov}_h(J) = \text{cov}_h(I) = \text{cof}(I) = \text{cof}(J)\),
2. for any \(n \in \omega, s \in \mathbb{Z}^n\) there exists \(G' \subseteq G\) such that \(G \setminus G' \in J\) and for every \(g \in G'\), \(a \in G^m\) the following condition holds
   \[
   \{h \in G : \prod_{i \in n} a_i \cdot h^n = g\} \in J.
   \]

Then there is a completely \(J\)-nonmeasurable subgroup \(H\) in \(G\) and completely \(I\)-nonmeasurable subset \(A \subseteq X\) such that \(HA\) is completely \(I\)-nonmeasurable in the space \(X\).

**Proof.** Let us enumerate the collections of \(I\) positive Borel bases in \(G\) and \(X\) \(\{P_\xi : \xi < \text{cof}(J)\}\), \(\{B_\xi : \xi < \text{cof}(I)\}\) respectively. Then by transfinite recursion we define sequence of the length \(\kappa = \text{cof}(I)\):

\[
\langle (H_\xi, A_\xi, g_\xi, d_\xi) : \xi < \kappa\rangle
\]

such that for any \(\xi, \eta < \kappa\), we have

1. \(H_\xi \leq G\) and \(H_\xi \cap P_\xi \neq \emptyset\),
2. \(A_\xi \in \mathcal{P}(X)\) and \(A_\xi \cap B_\xi \neq \emptyset\),
3. if \(\xi \leq \eta\) then \(H_\xi \subseteq H_\eta\) and \(A_\xi \subseteq A_\eta\),
4. whenever \(\xi \leq \eta\) then \(g_\xi \notin H_\eta\) and \(d_\xi \notin H_\eta A_\eta\).

The sets \(H = \bigcup_{\xi < \kappa} H_\xi\) and \(A = \bigcup_{\xi < \kappa} A_\xi\) gives required assertion.

Let us consider \(\alpha\)-th setp of our transfinite induction. Let \(H = \bigcup_{\xi < \alpha} H_\xi\), the sets \(\{g_\xi : \xi < \alpha\}\) and \(\{d_\xi : \xi < \alpha\}\) have size less than \(\text{cov}_h(J) = \text{cov}_h(I)\). Then by the second assumption of the our theorem we can find some elements \(h, g \in P_\alpha \cap G'\) such that \(\langle H \cup \{h\} \rangle \cap (\{g_\xi : \xi < \alpha\} \cup \{g\}) = \emptyset\) and \(\langle H \cup \{h\} \rangle (A \cup \{a\}) \cap \{d_\xi : \xi < \alpha\} = \emptyset\) where \(A = \bigcup_{\xi < \alpha} A_\xi\).

Then we can find some \(a, d \in B_\alpha\) such that \(\langle H \cup \{h\} \rangle (A \cup \{a\}) \cap (\{d_\xi : \xi < \alpha\} \cup \{d\}) = \emptyset\). Let us set \(H_\alpha = \langle H \cup \{h\} \rangle\), \(A_\alpha = A \cup \{a\}\), \(g_\alpha = g\) and \(d_\alpha = d\), what finishes construction at \(\alpha\)-th step of induction. \(\Box\)

The condition (2) from the previous Theorem looks artificially. However, we can consider a natural situation, where \(J\) is the ideal of meager sets \(\mathcal{M}\). In such case we have the following proposition

**Proposition 2.1.** Let \((G, \cdot)\) be a Polish space. Fix \(n \in \omega, s \in \mathbb{Z}^n\). Then there exists comeager \(G' \subseteq G\) such that for every \(g \in G'\), \(a \in G^m\) the following set

\[
\{h \in G : \prod_{i \in n} a_i \cdot h^n = g\}
\]

is meager.
Proof. Let \( M = \{(h, g, a_0, \ldots, a_{n-1}) \in G^{n+2} : \prod_{i \in n} a_i \cdot h^i = g\} \). \( M \) is a closed subset of \( G^{n+2} \). Moreover, its interior is empty. So, \( M \) is a closed nowhere dense set. By Kuratowski-Ulam theorem comeager many first sections of \( M \) are meager. Using Kuratowski Ulam theorem, a set \( \{h \in G : \prod_{i \in n} a_i \cdot h^i = g\} \) is meager for \( (g, a_0, \ldots, a_{n-1}) \in C \), where \( C \subseteq \) is comeager. Using Kuratowski-Ulam theorem \( n \)-many times we can find \( G' \) – comeager subset of \( G \) such that \( G^{n+1} \subseteq C \). \( G' \) is the required set.

\[ \square \]

**Theorem 2.6.** Let \((X, I)\) be a Polish ideal space and \( \text{non}(I) \leq \text{cov}_h(I) \). Assume that \((G, \cdot)\) is a group which acts on \( X \). If \( H \leq G \) and \( A \in I \) are such that \( HA \) contains a Borel set \( B \notin I \) then there is a subgroup \( H' \leq H \) such that \( H'A \) is completely \( I \)-nonmeasurable in some \( I \)-positive Borel set.

Proof. Let \( B \in \mathcal{B}_I^+(X) \) be a \( I \)-positive Borel set such that \( B \subseteq HA \). Let us find a set \( T \subseteq B \) witness of \( \text{non}(I) \). Then let \( F : T \to H \) be such that \( t \in F(t)A \) holds for any \( t \in T \). Let \( H' = \langle F[T] \rangle \) be a subgroup of \( H \) generated by \( F[T] \). Then

\[ |H'| = |F[T]| \leq |T| = \text{non}(I) < \text{cov}_h(I). \]

We have that \( T \subseteq F[T]A \subseteq H'A \). So, \( H'A \notin I \). Notice that \( H'A = \bigcup \mathcal{F} \), where family \( \mathcal{F} = \{hA : h \in H'\} \subseteq I \) of sets from \( \sigma \)-ideal \( I \) has size less than \( \text{cov}_h(I) \). It shows that any \( I \)-positive Borel set can not be covered by the family \( \mathcal{F} \).

\[ \square \]

3. **Applications**

In investigations of topological spaces \( X \) the crucial role is played by the space of all homeomorphisms \( \mathcal{H}(X) \) on the space \( X \). \( \mathcal{H}(X) \) is endowed with so-called compact-open topology which is generated by the following subbase

\[ \{V(K, U) : K \subseteq X \text{ is compact and } U \subseteq X \text{ is open in } X\}, \]

where

\[ V(K, U) = \{f \in \mathcal{H} : f[K] \subseteq U\}. \]

In the case when \( X \) is a compact metric space, \( \mathcal{H}(X) \) is also metrizable. A metric on \( \mathcal{H}(X) \) can be defined for every \( f, g \in \mathcal{H}(X) \) by the following formula

\[ d(f, g) = \sup_{x \in X} \{d(f(x), g(x))\} + \sup_{y \in X} d(f^{-1}(y), g^{-1}(y)). \]

When \( X \) is compact Polish space then \( \mathcal{H}(X) \) is also Polish one, see [4] for example.

Let us now consider a natural context, when a Polish group \( G \) acts on itself by left shifts. As a consequence of Theorem 2.3 we get the following result.
Corollary 3.1. Let \((G, \cdot, J)\) be a Polish ideal group. Then there exists \(H < G\) and \(A \subseteq G\) such that \(H, A, HA\) are completely \(J\)-nonmeasurable.

Proof. It is enough to check that the assumptions of Theorem 2.3 are fulfilled.

Naturally \(Ge = \{ge : g \in G\} = G\).

Notice that if \(x \neq y, x, y \in G\) then
\[G_{x,y} = \{g \in G : y = gx\} = \{yx^{-1}\},\]
so, it has size 1. \(\square\)

Now, let us consider a situation when a group of homeomorphisms \(\mathcal{H}(X)\) acts in a natural way on a compact Polish space \(X\).

Corollary 3.2. Assume that \(\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})\). Let \(X\) be a compact Polish space without isolated points. Then there exist a completely \(\mathcal{M}\)-nonmeasurable subgroup \(H < \mathcal{H}(X)\) and a completely \(\mathcal{M}\)-nonmeasurable subset \(A \subseteq X\) such that \(HA\) is completely \(\mathcal{M}\)-nonmeasurable.

Proof. We will use Theorem 2.5. The assumptions are fulfilled, what follows from Proposition 2.1. \(\square\)

From the other side the following example is a simple corollary from Theorem 2.2 when we can find many different orbits.

Corollary 3.3. If \(G\) is a subgroup of the group of all isometries on the Cantor space \(2^\omega\) defined as follows
\[G = \{T_X : X \in \mathcal{P}\{n \in \omega : n \equiv 0 \mod 2\}\}\]
where for any \(x \in 2^\omega\) and \(n \in \omega\)
\[T_X(x)(n) = \begin{cases} x(n) & \text{when } n \notin X \\ 1 - x(n) & \text{when } n \in X. \end{cases}\]
Then there is a subgroup \(H\) of \(G\) and uncountable many pairwise disjoint subsets \(\{A_\alpha \subset 2^\omega : \alpha < \text{cof}(\mathcal{M})\}\) such that \(HA_\alpha\) are completely \(\mathcal{M}\)-nonmeasurable in the Cantor space \(2^\omega\) for any \(\alpha < \text{cof}(\mathcal{M})\). Moreover, \(\{HA_\alpha : \alpha < \text{cof}(\mathcal{M})\}\) forms a pairwise disjoint family of subsets of the Cantor space.

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