Integrability from Point Symmetries in a family of Cosmological Horndeski Lagrangians

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For a family of Horndeski theories, formulated in terms of a generalized Galileon model, we study the integrability of the field equations in a Friedmann-Lemaître-Robertson-Walker spacetime. We are interested in point transformations which leave invariant the field equations. Noether’s theorem is applied to determine the conservation laws for a family of models that belong to the same general class. The cosmological scenarios with or without an extra perfect fluid with constant equation of state parameter are the two important cases of our study. The De Sitter universe and ideal gas solutions are derived by using the invariant functions of the symmetry generators as a demonstration of our result. Furthermore, we discuss the connection of the different models under conformal transformations while we show that when the Horndeski theory reduces to a canonical field the same holds for the conformal equivalent theory. Finally we discuss how singular solutions provides nonsingular universes in a different frame and vice versa.

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1. INTRODUCTION

The plethora of phenomena which have been discovered the last few years have led to the consideration of alternative/modified gravitational theories [1]. These extended theories of gravity provide additional terms in the field equations which - in conjunction to those of General Relativity - can explain the various phases of the universe. In physical science the existence of a fundamental axiom, such as Hamilton’s principle, is of paramount importance. Among the theories that are generated by an action integral, those that are of second-order in respect to their equations of motion possess a distinguished role; from Newtonian Mechanics to General Relativity.

A particular family of theories which has drawn attention are the so-called Horndeski theories. Horndeski, in 1974 [2] derived the most general action for a scalar field in a four-dimensional Riemannian space in which the Euler-Lagrange equations are at most of second-order. All relevant scalar-tensor theories with this property, such as the Brans-Dicke [3,4], the Galileon [5,6] and the generalized Galileon [7] belong to the general family of Horndeski theories. Although the latter are of second-order, they provide non-canonical nonlinear field equations which - even for the simplest line element of the underlying space - might not be able to lead to solutions in terms of closed-form expressions. Due to the high level of nonlinearity, numerical methods are applied in order to approximate the evolution of the system. However, whether a solution actually exists is not always known. For that reason, in this work, we are motivated to study the integrability of the field equations for a class of families of Horndeski theories (or equivalently those of the generalized Galileon Lagrangian [8]).

There are various methods to study the integrability of a system of differential equations. Two of the most famous are: a) the existence of invariant transformations, i.e. symmetries and b) the singularity analysis; for a recent discussion and comparison of these two methods see [10]. Both of these have been applied widely in gravitational theories [11–17]. In several cases the gravitational field equations, after a specific ansatz for the line element is adopted, can be derived by point-like Lagrangians [18]. The application of Noether’s theorem over the corresponding mini-superspace action has been utilized for the determination of conservation laws/analytical solutions in various models both at the classical as well as in the quantum level, for instance see [19–30] and references therein. Noether’s Theorem is the main mathematical tool that we use in this study.

From the various different classes of generators of the invariant transformations we consider the most simple; the one corresponding to the so-called point symmetries. Point symmetries are the generators of the invariant transformations

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in the base manifold in which the dynamical system is defined. The application of Noether’s theorem for point symmetries leads to conservation laws linear in the momentum. Well known conservation laws of this type from classical mechanics are those of the momentum and the angular momentum. The plan of the paper it follows.

In Section 2 we present our model which is a special consideration of the Horndeski Lagrangian and can be seen as a first generalization of the canonical scalar-tensor theories. Moreover we consider the cosmological scenario of an isotropic and homogeneous universe in which an extra fluid exists with constant equation of state parameter. For that model the field equations are calculated while the minisuperspace Lagrangian is discussed. Furthermore our model admits four unknown functions which define the specific form of the cosmological model. In Section 3, we apply Noether’s theorem in the minisuperspace Lagrangian in order to specify the unknown form of the functions which define the model and derive the corresponding Noetherian conservation laws. For the completeness of our analysis we consider separately the cases with or without an ideal gas and the cases with zero or nonzero spatial curvature for the underlying spacetime. In order to demonstrate the usefulness of our results we present some closed-form (special) solutions in Section 4 which are derived from the invariant functions of the admitted symmetry vectors. Finally in Section 5 we discuss our results and specifically we discuss the relation between our models under conformal transformations and we show how a singular universe is mapped to a nonsingular universe under the change of the frame.

2. THE MODEL

The gravitational action integral that we consider is of the form

$$S = \int \sqrt{-g} \left[ h(\phi)R - \frac{\omega(\phi)}{2} \phi^{\mu} \phi_{,\mu} - V(\phi) - \frac{g(\phi)}{2} \phi^{\mu} \phi_{,\mu} \Delta \phi \right] d^4x + S_m, \quad (1)$$

that falls into the class of a generalized Galileon (or Horndeski) model \[9, 31\], where \(\phi\) inherits the symmetries of the FLRW spacetime (2). As we can see, when \(g(\phi) \rightarrow 0\), an action integral of that type, including the Brans-Dicke case, is recovered.

Our goal is to derive all admissible gravitation models in (1) that possess an integral of motion of the form previously described. In that respect, note that the existence of \(\omega(\phi)\) is not trivial since its absorption with a reparameterization, \(\phi = f(\varphi) = \int \omega(\varphi)^{-1/2} d\varphi\), leads to the transformation of the Laplaci of \(\phi\) as (primes denote differentiation with respect to the argument)

$$\Delta \phi = f'(\varphi) \Delta \varphi + f''(\varphi) \phi^{\mu} \phi_{,\mu}, \quad (3)$$

introducing a new kinetic energy squared term in action (1). This is in contrast to what happens in the case of a scalar field whose action does not involve a coupling with derivatives of \(\phi\) over its kinetic term, i.e. when in our case \(g(\phi) = 0\), where without loss of generality we can always select \(\omega(\phi)\) to be a constant.

The mini-superspace Lagrangian that we can derive with the help of the gravitation plus scalar field part of (1) and (2) by integrating out the spatial degrees of freedom reads

$$L \left( N, a, \dot{a}, \phi, \dot{\phi} \right) = \frac{a^2 g(\phi) \dot{\phi}^3}{N^3} - \frac{a^3 g(\phi) \dot{\phi}^4}{6N^3} + \frac{6a h(\phi) \dot{\phi}}{N} - \frac{6ah(\phi) \dot{\phi}^2}{2N} + \frac{a^3 \omega(\phi) \dot{\phi}^2}{2N} + a^3 NV(\phi), \quad (4)$$

where the dot stands for differentiation with respect to time \(t\).

In what regards the matter part, we start our investigation by considering the \(S_m = 2 \int L_m d^4x\) contribution in (1) to be that of a perfect fluid obeying the barotropic equation of state \(P = \gamma \rho\), with \(P(t)\) and \(\rho(t)\) the pressure and energy density respectively. In order to perform the variation of \(L_m = \sqrt{-g} \rho\) with respect to the metric we need to

\[1\] We assume that the field \(\phi\) inherits the symmetries of the FLRW spacetime \[2\].
a priori define a continuity equation\[32\]. We choose the following well known relation for an ideal gas in a FLRW space-time:

$$\frac{\dot{\rho}}{\rho} = -3(1 + \gamma)\frac{\dot{a}}{a} \rightarrow \rho(t) = \rho_0 a^{-3(\gamma + 1)},$$

where $\rho_0$ is a constant of integration. Thus, the relevant addition to the mini-superspace Lagrangian is

$$L_m(N, a) = \sqrt{-g} \rho(t) = N \rho_0 a^{-3\gamma}.\tag{6}$$

As a result the total Lagrangian reads

$$L_{tot} = L + 2L_m = a^2 g(\phi) \dot{\phi}^2 + \frac{a^3 g(\phi)}{6N^3} + \frac{6a h(\phi) \dot{a} \dot{\phi}}{N} + \frac{6a h(\phi) \dot{a} \dot{\phi}}{N} - \frac{a^2 \omega(\phi) \dot{\phi}^2}{2N} + a^3 N V(\phi) + 2N \rho_0 a^{-3\gamma}.\tag{7}$$

It can be easily verified that the three Euler-Lagrange equations

$$E_N = \frac{\partial L_{tot}}{\partial N} = 0,\tag{8a}$$

$$E_q = \frac{\partial L_{tot}}{\partial q} - \frac{d}{dt} \left( \frac{\partial L_{tot}}{\partial \dot{q}} \right) = 0, \quad q = (a, \phi),\tag{8b}$$

are completely equivalent to the field equations of motion of (1) for the metric

$$h(\phi)G_{\mu\nu} = \frac{\omega}{2} \phi,_{\mu}\phi,_{\nu} - \frac{1}{2} g_{\mu\nu} \left( \frac{\omega'}{2} \phi^{,\kappa} \phi,_{\kappa} + V(\phi) \right) - \frac{1}{2} g_{\mu\nu} G^{(1)}(1) \phi^{,\kappa} G^{(1)}(1) \phi,_{\nu}$$

$$- \frac{1}{2} G_{X \phi} \phi_{,\mu} \phi_{,\nu} \phi_{,\kappa} + T^{(m)}_{\mu\nu} \tag{9}$$

and the scalar field

$$h'(\phi) R + (\omega(\phi) \phi^{,\kappa} )_{,\kappa} - \frac{\omega'}{2} \phi^{,\kappa} \phi^{,\kappa} - V'(\phi) + \left( G^{(1)}(1) \phi,_{\kappa} G^{(1)}(1) \phi,_{\kappa} \right)_{,\kappa} + G^{(1)}(1) \phi,_{\kappa} \phi_{,\kappa} = 0, \tag{10}$$

whenever the line-element (2) is substituted and the isometries of the space-time are inherited by the matter $\phi = \phi(t)$. The various quantities that appear in these equations are: $G^{(1)}(1) = g(\phi) X$, $X = -\frac{1}{2} \phi^{,\kappa} \phi_{,\kappa}$ and

$$T^{(m)}_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + pg_{\mu\nu}.\tag{11}$$

for the energy-momentum tensor of the fluid with $u^\mu = (\frac{1}{\sqrt{a}}, 0, 0, 0)$ the comoving velocity.

### 3. SYMMETRIES AND CONSERVATION LAWS

By considering the mini-superspace action to be form invariant under point transformations generated by\[2\]

$$Y = \chi(t, a, \phi, N) \frac{\partial}{\partial t} + \xi_1(t, a, \phi, N) \frac{\partial}{\partial a} + \xi_2(t, a, \phi, N) \frac{\partial}{\partial \phi} + \xi_3(t, a, \phi, N) \frac{\partial}{\partial N},\tag{12}$$

one is naturally led to the well known infinitesimal criterion

$$\text{pr}^{(1)}(L_{tot}) + L_{tot} \frac{d\lambda}{dt} = \frac{dF}{dt},\tag{13}$$

where $\text{pr}^{(1)}Y$ is the first prolongation of $Y$, i.e. the extension of the generator in the first jet space spanned by $(t, a, \phi, N, \dot{a}, \dot{\phi}, \dot{N})$. Clearly, since $\dot{N}$ does not appear in (11), we can disregard the relevant term and just write

$$\text{pr}^{(1)}Y = Y + \Phi_1 \frac{\partial}{\partial a} + \Phi_2 \frac{\partial}{\partial \phi},\tag{14}$$

\[2\] For details on the Noether’s theorem we refer the reader to [33].
where $\Phi_i = \frac{\partial L}{\partial q^i} - g^i \frac{\partial g^i}{\partial t}$, $q = (a, \phi)$, $i = 1, 2$.

Application of (14) leads to an overdetermined system of partial differential equations to be solved for $\chi(t, a, \phi, N)$, $\xi_i(t, a, \phi, N)$, $i = 1, \ldots, 3$, and $F(t, a, \phi, N)$. The former is formed by gathering the coefficients of all terms involving the derivatives of $a$, $\phi$ and $N$, to which the functions entering the generator (12) have no dependence. By gradually integrating the equations, we obtain the following results (the basic steps to the solution can be found in appendix A):

- An infinite dimensional symmetry group generated by

$$Y_\infty = \chi(t) \frac{\partial}{\partial t} - N \chi(t) \frac{\partial}{\partial N},$$

with $\chi(t)$ an arbitrary function of time. Its appearance reflects the fact that the mini-superspace action for (11) is form invariant under arbitrary time transformations. Its existence leads, through Noether’s second theorem, to a differential identity between the Euler-Lagrange equations of motion (8). The latter implies the well known fact that not all of them are independent, i.e. $L_{tot}$ is a constrained (or singular) Lagrangian. It is well known that when an infinite dimensional symmetry group is present (i.e. Noether’s second theorem is applicable) the system is necessarily singular, however the inverse is not true [34]. In our case this particular symmetry is a remnant of a more general group, the four dimensional diffeomorphism, $\chi(x)^\mu \frac{\partial}{\partial x^\mu}$, under which the full gravitational action (11) is form invariant.

- Additionally to the previous group - which always exists for time reparameterization invariant Lagrangians [35] - we obtain a symmetry generator if $h(\phi)$, $\omega(\phi)$, $g(\phi)$ and $V(\phi)$ satisfy certain criteria. In particular, we see that if $\gamma \neq \frac{1}{3}$, then

$$Y = \frac{a}{3 - 1} \frac{\partial}{\partial \phi} - \frac{\phi}{\lambda} \frac{\partial}{\partial \phi} + N \frac{3 \gamma}{3 \gamma - 1} \frac{\partial}{\partial N},$$

satisfies (13) whenever

$$h(\phi) = \phi \frac{2(\gamma - 1) \lambda}{3 \gamma - 1}, \quad g(\phi) = g_0 \phi^{3(\lambda - 1)}, \quad V(\phi) = V_0 \phi^{3(\gamma - 1) \lambda - 3 \lambda}, \quad \omega(\phi) = \omega_0 \phi^{2(\gamma - 1) \lambda - 2}$$

in which $\lambda$ is a constant. Note that the corresponding gauge function appearing in (13) is trivial in the calculation, i.e. $F(t, a, \phi, N) =$const. and this remains that way for every case that we examine later on in the analysis. Note also, that (18) and (19) describe a particular solution of (13) in the special case when $\gamma = -1$. When the fluid contribution plays the role of a cosmological constant, it can be absorbed inside the potential $V(\phi)$ (as can be seen by the form of $L_{tot}$) and the general solution of the $\gamma = -1$ case is the same as the one that we get if we completely omit the fluid.

On the other hand, for the special case where the perfect fluid describes radiation, i.e. $\gamma = 1/3$ the situation changes and the symmetry generator assumes the general form

$$Y = \frac{a}{2g'(\phi)^2} \frac{\partial}{\partial a} + \frac{g(\phi)}{g'(\phi)} \frac{\partial}{\partial \phi} + N \frac{g(\phi)g''(\phi)}{2g'(\phi)^2} \frac{\partial}{\partial N},$$

while the functions entering the action need to be

$$h(\phi) = \frac{1}{g'(\phi)}, \quad g(\phi) = g_0 \frac{g'(\phi)^3}{g(\phi)^3}, \quad V(\phi) = \frac{V_0}{g'(\phi)^2}, \quad \omega(\phi) = \omega_0 \frac{g'(\phi)}{g(\phi)^2} - \frac{3g''(\phi)}{g'(\phi)^3},$$

where $g(\phi)$ is an arbitrary non-constant function. As a result, for $\gamma = 1/3$, there exists an infinite family of models belonging to the general class of actions of the form (11) that possesses an integral of motion of the type we are investigating.

### 3.1. Particular case: $\rho_0 = 0$

Let us see how the situation alters if we remove the ideal gas from our considerations. In other words let us use the Lagrangian given by (1), in the criterion (13) instead of the Lagrangian $L_{tot}$. Then, in addition to the diffeomorphism...
group characterized by $\gamma = 1$ - which always exists for time reparameterization invariant Lagrangians $\gamma$ - we obtain a symmetry generator

$$Y = a \frac{(\lambda + 3)\dot{g}^{2}(\phi) - 3\ddot{g}(\phi)\dot{g}''(\phi) \partial}{6\dot{g}'(\phi)^{2}} - \ddot{g}(\phi) \partial_{\phi} + N \frac{(\lambda + 3)\dot{g}'(\phi)^{2} - \ddot{g}(\phi)\dot{g}(\phi) \partial}{2\dot{g}'(\phi)^{2}} \partial N,$$

that satisfies (13) when

$$h(\phi) = \frac{1}{\dot{g}'(\phi)}, \quad g(\phi) = g_{0} \frac{\dot{g}'(\phi)^{3}}{\dot{g}(\phi)^{\beta}}, \quad V(\phi) = V_{0} \frac{\dot{g}(\phi)^{\lambda+3}}{\dot{g}'(\phi)^{2}}, \quad \omega(\phi) = \omega_{0} \frac{\dot{g}'(\phi) - 3\ddot{g}'(\phi)^{2}}{\dot{g}'(\phi)^{3}}$$

with $\lambda$ being a constant and $\ddot{g}(\phi)$ again an arbitrary (non-constant) function of $\phi$. Once more, we have an infinite set of physically different models that are characterized by a function $\ddot{g}(\phi)$. We can see that, in comparison to the $\gamma = 1/3$ case, result (21) is identical to (19) in the special case when $\lambda = -3$.

It is useful to study how the inclusion of spatial curvature (either positive or negative) may alter the conditions under which a symmetry generator appears. If we consider the line element

$$ds^{2} = -N^{2}dt^{2} + \frac{a(t)^{2}}{1 + k r^{2}} (dx^{2} + dy^{2} + dz^{2})$$

where $r^{2} = x^{2} + y^{2} + z^{2}$, Lagrangian (14) is modified by the addition of an extra term in the potential and reads

$$L_{k} = \frac{a^{2}g(\phi)\dot{\phi}^{2}}{N^{3}} + \frac{a^{3}g(\phi)\dot{\phi}^{4}}{6N^{3}} + 6a^{2}h(\phi)\dot{\phi}^{2} + \frac{6a^{2}h(\phi)\dot{\phi}^{2}}{N} + \frac{a^{3}\omega(\phi)\dot{\phi}^{2}}{2N} + a^{3}NV(\phi) - 6kaNh(\phi).$$

Thus, for the same class of point transformations we considered earlier and with the application of the infinitesimal symmetry criterion (13), it is straightforward to derive the result given by (20) and (21) under the condition that $\lambda = -3$ or equivalently result (13) and (19) corresponding to the radiation fluid case where $\gamma = 1/3$. In other words when $k \neq 0$ we have the same situation as in the $\gamma = 1/3$ case. The function $\ddot{g}(\phi)$ once more remains arbitrary, under the restriction of course of not being constant. As we can see, even though the non vanishing of $k$ imposes a restriction ($\lambda = -3$) in comparison to the $k = 0$ case, there still exists an infinite number of models admitting an integral of motion of the type we consider in this work.

Moreover, for the combined case where we have both a non-zero spatial curvature and a perfect fluid, in other words when the Lagrangian is being given by $L_{k} + 2L_{m}$, it comes as no surprise that the existence of an integral of motion implies that $\gamma = 1/3$. In fact the result is once more exactly the same with what we see in (18) and (19).

Having derived the previous results, we can use Noether’s second theorem and derive the integrals of motion corresponding to each case. It can be easily verified that

$$\frac{dI}{dt} = -\xi_{3}E_{0} + A'E_{i}, \quad i = 1, 2$$

where the conserved charge is

$$I = \xi_{1} \frac{\partial L}{\partial a} + \xi_{2} \frac{\partial L}{\partial \phi}$$

while $A'E_{i}$ denotes a linear combination of the two spatial equations of motion. Of course, in place of $L$ there can be either $L_{tot}$ of (1), $L$ of (1) or $L_{k}$ of (23), depending on the generator that we use and the case that we examine. In every situation, we have on mass shell the corresponding conserved quantity (25) for each symmetry vector field $Y$. The fact that in every case but the generic fluid with $\gamma \neq 1/3$ an arbitrary function $\ddot{g}(\phi)$ is involved in the generator as well as in the action itself, implies that we possess an infinite collection of models - corresponding to different sets of functions $h(\phi), g(\phi), V(\phi)$ and $\omega(\phi)$ - which admit at least one integral of motion of this type; hence being integrable. Note here, that the fact that we are led to an infinite number of models through the arbitrariness of $\ddot{g}(\phi)$ is owed to the use of the reparameterization invariant Lagrangian (14). Had we chosen to adopt the gauge $N = 1$ at the Lagrangian level, then only a particular case would have emerged where $\ddot{g}(\phi) = \phi^{\nu}$, with the corresponding $\chi(t)$ for this symmetry being $\chi(t) \propto t$. Of course, due to the system being autonomous, $\partial_{t}$ also exists; the latter being the only remnant of (15) after fixing the gauge. The difference appearing here in the arbitrariness of $\ddot{g}(\phi)$ lies in the consideration of Lagrangian (14), which naturally generates the constraint equation $E_{0} = \frac{\partial}{\partial N} = 0$ in (13) and (24), allowing the derivation of a larger class of symmetries.

The existence of an integral of motion of the form (25) is of great interest in the search of solutions. The system at hand possesses two degrees of freedom that are bound by a constraint equation. Thus, the existence of $I$ implies
that, in principle, we need only to solve the constraint equation together with \( I = \text{const.} \) in order to fully integrate the system. As a result, our problem is immediately reduced to one that involves only first order differential equations. The usefulness of \( I = \text{const.} \) also lies in the fact that it is linear in \( \dot{a} \), an advantage which the constraint equation does not have since it is non-linear in both velocities \( \dot{a} \) and \( \dot{\phi} \). Nevertheless, even for the simplest of models defined by \( g(\phi) \), the situation can be highly complicated. In most of cases it may be more useful if the result is used in a way to distinguish any existing solutions that are invariant under the action of the generator, i.e. derive the characteristic of \( Y \) for a given model \( g(\phi) \) so as to deduce a possible relation between \( a \) and \( \phi \) and use the latter together with the integral of motion and the constraint equation.

4. INVARIANT SOLUTIONS

In what follows, and in order to demonstrate the importance of our results, we present a few illustrative applications of invariant solutions that we derive in the gauge \( N = 1 \) with and without a perfect fluid. For a treatment of how the conserved quantity can be utilized to lead to more general solutions expressed in an arbitrary gauge we refer the reader to appendix [13]. Note that the solutions which we present here are derived with a mentality of keeping all terms inside the action, i.e. we do not express any additional solutions which may have a vanishing \( V_0, g_0 \) or \( \omega_0 \).

1. Perfect fluid, \( \gamma \neq 1/3 \) case. Let as consider the FLRW spacetime with zero spatial curvature, i.e. \( k = 0 \). As is known invariant solutions can be constructed with the help of the symmetry generator. For example, in the case of a generic perfect fluid with \( \gamma \neq 1/3 \), generator (16) implies an invariant relation of the form \( a^{\frac{1}{3-\gamma}} \phi = \text{const.} \). Hence, it is easy to derive the solution

\[
a(t) = t^{\gamma/3}, \quad \phi(t) = \phi_0 t^{\frac{3-\gamma}{1-\gamma}},
\]

with

\[
\begin{align*}
g_0 &= \frac{3\gamma^2 \lambda^2 \phi_0^{\frac{6+\lambda}{3-\gamma}}}{6\lambda^2 \left(3\gamma \left(2\gamma \rho_0 \phi_0^{\frac{3(\gamma+1)\lambda}{3-\gamma}} + \gamma V_0 \phi_0^{\frac{6+\lambda}{3-\gamma}} - 2\right) + 4\right) + (1-\gamma)^2 \omega_0} \\
\omega_0 &= \frac{6\lambda^2 \left[3\gamma \left(\gamma(1-3\gamma)\rho_0 \phi_0^{\frac{3(\gamma-1)\lambda}{3-\gamma}} + 2\gamma V_0 \phi_0^{\frac{6+\lambda}{3-\gamma}} + 2\right) - 4\right]}{(1-3\gamma)^2}.
\end{align*}
\]

The on mass shell value of the conserved quantity \( I \) for solution (26) is \( I = \frac{6\gamma(\gamma+1)\rho_0}{3\gamma-1} \).

It is interesting to note what occurs when the theory assumes a Brans-Dicke-like form, i.e. when \( h(\phi) = \phi \) and \( \omega(\phi) = \omega_0 \phi^{-1} \). This happens when \( \lambda = \frac{3\gamma-1}{3-\gamma} \) and it leads to a particular solution with \( \gamma = -1 \) (cosmological constant contribution in the action) that reads

\[
a(t) = t^{-1/3}, \quad \phi(t) = \phi_0 t^2
\]

where now \( \omega_0 = -5/3, V_0 = -2\rho_0 \), while \( g_0 \) is kept arbitrary. We can observe that solution (27) satisfies the same condition as the vacuum Brans-Dicke solution (for a spatially flat universe), namely that \( a(t)^3 \phi(t) \propto t \) (in the gauge \( N = 1 \)). This is explained by the fact that for \( \gamma = -1 \) the potential in (17) becomes a constant with the value \( V_0 = -2\rho_0 \) so that it effectively cancels the cosmological constant contribution of the fluid in (17).

Nevertheless, for an arbitrary \( \gamma \), we can see how the symmetry that is present in the Brans-Dicke model - satisfying \( a^2 \phi = \text{constant} \) - is generalized in the class of models we are considering here. In particular we encounter the more general relation \( a^{\frac{1}{3-\gamma}} \phi = \text{constant} \), that is implied by the existing symmetry generator. Of course in our case, (26) is not a general solution of the equations of motion but a particular one, which however, as a power-law, is of special cosmological interest.

2. In the absence of a fluid, \( g(\phi) = e^{-\mu \phi} \) case. For a spatially flat FLRW line-element, we choose the function appearing in (21) to be \( \tilde{g}(\phi) = e^{-\mu \phi} \). After an appropriate redefinition of the constants \( \omega_0, g_0 \) and \( V_0 \), we derive the corresponding model to be

\[
h(\phi) = e^{\mu \phi}, \quad \omega(\phi) = \omega_0 e^{\mu \phi}, \quad g(\phi) = g_0 e^{\mu(\lambda+3)\phi}, \quad V(\phi) = V_0 e^{-\mu(\lambda+1)\phi}.
\]
A special solution in terms of a power-law for the scale factor exists
\[ a(t) = t^\sigma, \quad \phi(t) = \frac{2}{\mu(\lambda + 2)} \ln(\phi_0 t), \quad \lambda \neq -2, \]
when the constants \( \omega_0 \) and \( V_0 \) are given by
\[
\omega_0 = \frac{2g_0\phi_0^2((\lambda + 2)(3\sigma - 1) - 2)}{\mu(\lambda + 2)^2} + \mu^2((\lambda + 2)(\lambda + 3)\sigma + \lambda),
\]
\[
V_0 = \frac{4g_0\phi_0^4(3\sigma - 1) + 2\mu^3\phi_0^2(\lambda + 2)((\lambda + 2)\sigma + 1)(3(\lambda + 2)\sigma - \lambda)}{\mu^3(\lambda + 2)^3},
\]
while the rest parameters remain free. It can be seen that on this solution the conserved quantity \( I \) (the Noetherian conservation law) becomes zero.

3. Case \( \tilde{\phi}(\phi) = \phi^{-\mu} \), without a perfect fluid. Again in (21), we make the choice \( \tilde{\phi}(\phi) = \phi^{-\mu} \) and obtain (once more with an appropriate redefinition of the constants \( V_0, g_0, \omega_0 \) in the action and \( \lambda \to \lambda/\mu \)) the subsequent set of functions entering the action
\[
h(\phi) = \phi^{\mu + 1}, \quad \omega(\phi) = \omega_0 \phi^{-\mu - 1}, \quad g(\phi) = g_0 \phi^{3(\mu - 1) + \lambda}, \quad V(\phi) = V_0 \phi^{2 - \mu - \lambda}.
\]

Two solutions for which the conserved quantity is again zero can be easily derived for this model. The first solution is again a power-law
\[ a(t) = t^\sigma, \quad \phi(t) = \phi_0 t^\kappa, \]
where
\[
\lambda = \frac{2}{\kappa} - 2\mu + 1,
\]
\[
\omega_0 = -g_0\kappa^2(\mu - 1)\phi_0^{2/\kappa} + g_0\kappa(3\sigma - 1)\phi_0^{2/\kappa} + \frac{4\sigma}{\kappa^2} + \frac{2(\mu + 1)(\sigma + 1)}{\kappa} - 2(\mu + 1)^2,
\]
\[
V_0 = \frac{1}{2}\phi_0^{2/\kappa} \left( g_0\kappa^4\phi_0^{4/\kappa} + g_0\kappa^3(3\sigma - 1)\phi_0^{2/\kappa} + 2\kappa^2(\mu + 1)^2 + 3\kappa(\mu + 1)(5\sigma - 1) + 4\sigma(3\sigma - 1) \right),
\]
are the constants appearing in (30).

Once more we can also distinguish a Brans-Dicke sub-case that appears when \( \mu = 0 \). It is interesting to observe that condition \( a(t)^3\phi(t) \propto t \) of the vacuum Brans-Dicke model does not apply here, although it was still relevant in the corresponding model of case 1. As we can see, the power \( \sigma \) in the scale factor expression is not connected to the power of time in the \( \phi(t) \). Only if we choose to turn the action into the pure Brans-Dicke form does (31) with \( \mu = 0 \) satisfy the aforementioned condition: By setting \( g_0 = 0 \) and \( V_0 = 0 \), which immediately results in \( \kappa = 1 - 3\sigma \) or \( \kappa = -2\sigma \).

Another solution we can derive for the \( \tilde{\phi}(\phi) = \phi^{-\mu} \) model leads to a space-time of a constant scalar Ricci curvature, a de Sitter universe that is:
\[ a(t) = e^{\sigma t}, \quad \phi(t) = \phi_0 e^{\kappa t} \]
when the following relations hold
\[
\lambda = 1 - 2\mu,
\]
\[
\omega_0 = g_0\kappa (\kappa(1 - \mu) + 3\sigma) - \frac{2(\mu + 1)(\kappa(\mu + 1) - \sigma)}{\kappa},
\]
\[
V_0 = \frac{g_0\kappa^4}{2} + \frac{3}{2}g_0\kappa^3\sigma + \kappa^2(\mu + 1)^2 + 5\kappa(\mu + 1)\sigma + 6\sigma^2.
\]

4. Case \( k \neq 1, \tilde{\phi}(\phi) = e^{-\mu\phi} \), without a perfect fluid. Let us furthermore consider the case with nonzero spatial curvature \( k \neq 0 \). Then, from the two particular types of solutions we examined above in cases 2 and 3, only for \( \tilde{\phi}(\phi) = e^{-\mu\phi} \) arises the following configuration:
\[
a(t) = t, \quad \phi(t) = -\frac{2}{\mu} \ln(\phi_0 t),
\]
\[
\omega_0 = k - 8g_0\mu\phi_0^2, \quad V_0 = 4\mu\phi_0^2(2g_0\mu\phi_0^2 - k),
\]
which, as long as the scale factor is concerned, it corresponds to the $k = -1$ solution of Einstein’s gravity in vacuum, that is the Milne solution. Here, it is given for any sign of $k$ with the difference being carried in the coupling (through $\omega_0$ and $V_0$) that is necessary for its existence. The integral of motion - calculated with the help of [24], [23] and [18] - assumes on mass shell the value $\bar{I} = \frac{2k}{\mu_0^2}$.

As is obvious, we are able - just by choosing a particular function $\tilde{g}(\phi)$ - to find through (21) or (19) the corresponding class of integrable models that admit a conserved quantity of the aforementioned form. As we already stated, it is due to the existence of the constraint equation [30] that one can in principle obtain the general solution by considering only the first order system $E_0 = \frac{\partial I}{\partial \Phi} = 0$ and $\bar{I} = \text{const}$. However, because of the complexity of the equations this is a highly nontrivial task.

5. DISCUSSION

We remark that for the cases without the extra matter fluid where we have obtained an infinite set of models owed to the arbitrariness of $\tilde{g}(\phi)$, there exists an interesting connection among them. These models can be mapped to each other by conformal transformations. Take for example two different models whose action is characterized by two different scalar field functions $\tilde{g}_1(\phi_1)$ and $\tilde{g}_2(\phi_2)$. Then, the transformation

$$\phi_1 = \tilde{g}_1^{-1} \circ \tilde{g}_2(\phi_2), \quad \tilde{g}_{\mu
u}^{(1)} = \frac{1}{(\tilde{g}_1^{-1} \circ \tilde{g}_2)'(\phi_2)} \tilde{g}_{\mu
u}^{(2)},$$

(34)

where $\tilde{g}_{\mu
u}^{(1)}$ and $\tilde{g}_{\mu
u}^{(2)}$ are the two corresponding spacetimes, maps the one action to the other. What is more, if we try to map the relevant form of the action to the Einstein frame we can see that an interesting “degeneration” occurs:

The different models that we get for the various $\tilde{g}(\phi)$ all collapse to a single action. Take for example the case where the initial action is characterized by the functions of $\phi$ given by (21). It can be seen to be mapped into an action of the form

$$\bar{S} = \int \sqrt{-\bar{g}} \left[ \frac{1}{2} \bar{R} - \frac{1}{2} \bar{\Phi} \bar{\Phi} + V_1 e^{\sigma \Phi} - e^{-\sigma \Phi} \left( \omega_1 \bar{\Phi} + \omega_2 \left( \bar{\Phi}^2 \right)^2 \right) \right] d^4x$$

(35)

by a conformal transformation in the metric and a re-parametrization of $\phi(\Phi)$. The new to the old variables are related by the expressions

$$\bar{g}_{\mu\nu} = \frac{2}{\tilde{g}'(\phi)} g_{\mu\nu}, \quad \Phi = \frac{1}{\sigma} \ln \left( \frac{V_0 \tilde{g}(\phi)^{\lambda+3}}{4V_1} \right)$$

(36)

with $V_1$ a constant and $\omega_1$ and $\omega_2$ in [35] being related to the initial constants $g_0$, $V_0$, $\omega_0$ and $\lambda$. We can see that the difference of this action lies in the existence of a term that is quadratic to the kinetic energy of the scalar field, while in [11] we considered a theory that has at most linear expressions in $X = -\frac{1}{2} \phi^\mu \phi_\mu$.

However, what is also interesting here is that when $V(\Phi) \simeq e^{\sigma \Phi}$ dominates then the contribution of the terms $e^{-\sigma \Phi} \left( \omega_1 \bar{\Phi} + \omega_2 \left( \bar{\Phi}^2 \right)^2 \right)$ is negligible, which means that [35] reduces to the Action Integral of a minimally coupled scalar field. The latter is exactly the limit which relays a canonical scalar field between the Jordan and the Einstein frames.

In general, great care needs to be taken when using conformal transformations. The fact that all the different models for the various choices of $\tilde{g}(\phi)$ can be mapped to [35] does not make them equivalent to the latter, or even to each other (due to [34]) for that matter. Two actions in order to be physically equivalent, they need to be mapped by gauge transformations of the theories under consideration. In the case of gravitational actions, those are the four dimensional diffeomorphism of space-time. As we know, conformal transformations cannot always be attributed to coordinate changes. As a result, the gravitational space-time arising in each situation is generally different.

In order to see that consider the nonsingular solution [32] corresponding to a de Sitter universe, which came out of a model characterized by $\tilde{g}(\phi) = \phi^{-\mu}$. If we choose to map this model to [35] then, by virtue of [36] we obtain

$$\bar{N}(t) = -\frac{2\phi(t)^{\mu+1}}{\mu} = c_1 e^{\sqrt{\mu} t}, \quad \bar{a}(t) = -\frac{2\phi(t)^{\mu+1}}{\mu} e^{\sigma t} = c_1 e^{\sqrt{\mu} t},$$

(37)

\footnote{A similar situation can be seen to arise in the result of [21], where again infinite models in a family of scalar tensor theories was recovered. By means of conformal transformations they can be mapped in the Einstein frame to a case of a minimally coupled scalar field with exponential potential.}
where $\bar{N}$, $\bar{a}$ are the new lapse and scale factor respectively, while $c_1$ is a constant. It is easy to see that if you go from this metric
\[ ds^2 = -\bar{N}^2 dt^2 + \bar{a}^2 (dx^2 + dy^2 + dz^2) \] (38)
to the gauge where $\bar{N} = 1$ (so as to compare with what we have in (32)) by performing the time transformation
\[ \bar{N}(t) dt = d\tau \Rightarrow t = \ln \left( \frac{\kappa (\mu + 1) \tau}{c_1} \right) \] (39)
we derive, with the appropriate scalings in $x$, $y$ and $z$, the line element
\[ ds^2 = -d\tau^2 + \tau \psi(dx^2 + dy^2 + dz^2), \] (40)
where $\psi$ is a constant. What was an exponential solution with a constant Ricci scalar in the theory we are investigating, has now become a power law with $R(\tau) \propto \tau^{-2}$ and a space-time with curvature singularity at $\tau = 0$ in the system described by action (35). Hence we can see that, whenever the conformal transformation does not correspond to a general coordinate transformation, the gravitational properties of the system are bound to change and the solutions represent different geometries. For discussions on the relation between analytical solutions and physical quantities between the Jordan and the Einstein frames see [37–39] and references therein.

Additionally to the previous geometrical considerations, extra care needs to be taken when a fluid is used as a matter source. This is owed to the fact that you need to pre-define a continuity equation in order to derive a rule for the variation of the energy density $\rho$ with respect to the metric $g_{\mu\nu}$. In the beginning of our analysis we considered a perfect fluid that is characterized by continuity equation (5). It is a well known fact that the latter is not conformally invariant, while its solution is necessarily utilized at the level of the minisuperspace Lagrangian. Henceforth, after a conformal transformation is being made, one needs to take into account a different fluid, which is now interacting with the scalar field, in order to make such a correspondence possible. This also results in a change of the physical behaviour of the system, since the properties of the matter source need to be altered.

In a future work we plan to study the integrability of other Horndeski theories by including more terms in the action integral and for other kind of transformations which leaves the field equations invariants, such as the generalized symmetries as also to investigate the effects of the conformal transformations in Hordenski theories.

Appendix A: Calculation of the symmetry generator

As indicated in the main text, application of (13) leads to an overdetermined system of partial differential equations. The latter is formed by gathering and demanding that are zero the coefficients multiplying terms involving derivatives of $a$, $\phi$ and $N$. For Lagrangian $L_{tot}$ as written in (7) we infer from the coefficients of $\dot{a}^3$, $\dot{a}^2 \dot{N}$ and $\dot{a}^2 \dot{\phi}$ that we need to set respectively
\[
\begin{cases}
\partial_a \chi = 0 \\
\partial_N \chi = 0 \Rightarrow \chi = \chi(t). \\
\partial_\phi \chi = 0
\end{cases}
\] (A1)
The fourth order coefficients $\dot{a}^2 \dot{\phi}^2$ and $\dot{a} \dot{\phi}^2 \dot{N}$ each imply
\[
\partial_a \xi_2 = 0 \quad \text{and} \quad \partial_N \xi_2 = 0,
\] (A2)
which means that $\xi_2 = \xi_2(t, \phi)$. However, with the help of $\chi = \chi(t)$, we can get a further restriction from the coefficient of $\dot{a} \dot{\phi}^2$ that leads to
\[
\partial_t \xi_2 = 0 \Rightarrow \xi_2 = \xi_2(\phi).
\] (A3)
Due to (A1) and (A3), the terms involving $\dot{\phi}^3 \dot{N}$ and $\dot{\phi}^3$ bring about the conditions
\[
\begin{cases}
\partial_N \xi_1 = 0 \\
\partial_t \xi_1 = 0 \Rightarrow \xi_1 = \xi_1(a, \phi).
\end{cases}
\] (A4)
Thanks to the above restrictions (A1), (A3) and (A4) we also get
\[
\begin{align*}
\partial_N F &= 0 \\
\partial_{\phi} F &= 0 \Rightarrow F = F(t) \\
\partial_a F &= 0
\end{align*}
\] (A5)
from $\dot{N}$, $\dot{\phi}$ and $\dot{a}$ respectively.

The equation emanating from the coefficient of $\dot{a}^2$ can be solved algebraically with respect to $\dot{\xi}_3(t,a,\phi,N)$
\[
\dot{\xi}_3(t,a,\phi,N) = N \left( 2\partial_a \xi_1(t,a,\phi) + \frac{\xi_1(t,a,\phi)}{a} + \frac{\xi_2(t,\phi)h'(\phi)}{h(\phi)} - \dot{\chi}(t) \right)
\] (A6)
the latter being a consequence of the fact that no derivative of $N$ enters in the Lagrangian. After this step one can see that the equation extracted from the coefficient of $\dot{\phi}^3$ is
\[
ah(\phi)\xi_2(t,\phi)g'(\phi) - g(\phi) [h(\phi) (5a\partial_a \xi_1(t,a,\phi) + \xi_1(t,a,\phi) - 3a\xi_2(t,\phi)) + 3a\xi 211(\phi)h'(\phi)] = 0
\] (A7)
and it can be immediately integrated to yield
\[
\xi_1(t,a,\phi) = \frac{1}{6} a \left( \frac{\xi_2(t,\phi)g'(\phi)}{g(\phi)} - \frac{3\xi_2(t,\phi)h'(\phi)}{h(\phi)} + 3\xi_2(t,\phi) \right) + \frac{\xi_0(t,\phi)}{a^{1/2}}.
\] (A8)

With the dependence with respect to $a$ being now completely specified, the relation produced by the coefficient of $\dot{\phi}^3$ involves only unknown functions of $\phi$. Hence, we can now start gathering coefficients with respect to powers of $a$ that appear inside it. With the help of a useful reparametrization $g(\phi) = g_1(\phi)^3$ one can straightforwardly obtain:
\[
\begin{align*}
\dot{\xi}_2(\phi) &= \frac{c_1 + c_2g_1(\phi)}{g_1'(\phi)} \\
\dot{\xi}_0(\phi) &= \frac{c_3}{h(\phi)^{3/2}}.
\end{align*}
\] (A9, A10)
where the $c_i$’s indicate constants of integration. The only appearance of time in the remaining coefficient is inside the zero-th order component (that multiplies no derivative of $\dot{a}$, $\dot{\phi}$ or $\dot{N}$) and it implies that
\[
F(t) = \text{const.}
\] (A11)

Thus, the gauge function becomes trivial.

With the results up to here all dependence of the functions of the generator with respect to $a$, $t$ and $N$ is specified. Inside each of the remaining equations we can gather coefficients with respect to $a$, since now the unknown functions involve only $\phi$. However, due to the existence of $\gamma$ the way that the coefficients are to be gathered depends on its value. We can distinguish two main cases:

- **Case $\gamma \neq 1/3$.** The equation produced by the zero-th order coefficient leads to
\[
V(\phi) = \frac{c_4 h(\phi)^2}{(c_1 + c_2g_1(\phi))^{1/3}}
\] (A12)
\[
h(\phi) = c_5 (c_1 + c_2g_1(\phi))^{\frac{1}{3(\gamma-1)}}
\] (A13)
\[
c_5 = 0.
\]

We note that, since we want the most general result, we avoid any special solution that leads to vanishing of any of the functions involved in our starting action. The $\dot{\phi}^2$ component leads to
\[
\omega(\phi) = c_6 (c_1 + c_2g_1(\phi))^{\frac{1+3\gamma}{1-3\gamma}} g_1'^2
\] (A14)
while the $\dot{\phi}^4$ gives rise to the third order equation
\[
g_1''g_1'g_1 - 2(g_1'')^2g_1 + g_1''g_1^2 = 0
\] (A15)
which under a transformation $g_1 = \exp \left( \int g_2(\phi) d\phi \right)$ becomes
\[
g_2''g_2 - 2(g_2')^2 = 0
\] (A16)
with general solution $g_2 = \frac{c_3}{\phi + c_3}$. Thus, the final function needed so that all equations are satisfied is

$$g_1(\phi) = c_7(\phi + c_9)^{c_8}. \quad (A17)$$

At this point the system of equations is completely satisfied. By absorbing the trivial constant $c_9$ inside $\phi$ with a translation and with an appropriate parametrization of the rest of the constants we obtain result (16), (17). Additionally, we can observe that the function $\chi(t)$ remained arbitrary through the calculation and this explains the existence of (15).

- Case $\gamma = 1/3$. With this choice of $\gamma$ the zero-th order equation implies

$$V(\phi) = \frac{c_4 h(\phi)^2}{(c_1 + c_2 g_1(\phi))^3},$$

$$c_2 = c_3 = 0. \quad (A18)$$

The relations from the $\dot{g}^2$ and $\dot{g}^4$ coefficients respectively lead to

$$\omega(\phi) = c_3 h(\phi) (g'_1)^3 - \frac{h'(\phi)^2}{h(\phi)} \quad (A19)$$

$$h(\phi) = c_7 \frac{e^{c_6 g_1(\phi)}}{g'_1(\phi)}. \quad (A20)$$

It is a matter of reparametrizing the constants of integration and an introduction of a new function $\tilde{\gamma}(\phi)$ as $g_1(\phi) = \frac{1}{\tilde{\gamma}(\phi)}$ to obtain result (18), (19) and of course the same comments as in the previous case, to the arbitrariness of $\chi(t)$, hold.

For the sake of completeness we have to point out that apart from the case $\gamma = 1/3$ there are also other values of $\gamma$ that can lead to a different gathering of terms in the coefficients of powers of $a$; namely $\gamma = -1$ and $\gamma = -7/5$. The first gives the same result as the case without fluid (see eqs. (20), (21)) with the sole difference of a potential that is $V(\phi) = V_0 \frac{g' \hat{\gamma}^{\lambda}}{\hat{g}} - 2\rho_0$ in place of the one appearing in (21). That is one that cancels the cosmological constant role of the fluid. The second case, $\gamma = -7/5$, after appropriate reparametrizations leads to the exact same result as the generic $\gamma \neq 1/3$ result and that is why we do not make a separate presentation of it here.

**Appendix B: More general solutions**

In section IV we presented some invariant solutions where the functional dependence between $a$ and $\phi$ can be extracted with the help of the infinitesimal symmetry generator. Although these are particular solutions which can be derived in a rather simple manner, they are the most cosmologically interesting. Even though we have proven the existence of an integral of motion $I$, thus reducing the problem to solving two first order differential equations: the constraint and $I = \text{const}$. The acquisition of the general solution is still a very difficult task, especially due to the non-linearity of the constraint equation in both derivatives involved $\dot{a}$ and $\dot{\phi}$.

Here, we want to exhibit a method with which the integral of motion $I$ can be used to derive the solution for the spatially flat case in the absence of a perfect fluid, when $I = 0$. Some of the solution we derived in section IV lead to $I = 0$ but still they are not the general solution of this case, but rather particular solutions of it. Although this method does not lead to the full solution (where $I = \text{const}$.) of the system, it can be applied for an arbitrary function $\tilde{\gamma}(\phi)$. Thus, giving in closed form the full solution with the property $I = 0$ for any model of the integrable type we are considering.

At first let us construct the conserved quantity $I$ of (25) by using the $\xi_1$ and $\xi_2$ of (20)

$$\xi_1 = a \frac{\lambda (\lambda + 3) \dot{\tilde{\gamma}}(\phi)^2 - 3 \tilde{\gamma}(\phi) \dot{\tilde{\gamma}}''(\phi)}{6 \tilde{\gamma}(\phi)^2}, \quad \xi_2 = -\frac{\ddot{\tilde{\gamma}}(\phi)}{\dot{\tilde{\gamma}}(\phi)} \quad (B1)$$

and make the transformation $a(t) = \exp \left( \int u(t) dt \right)$. Then relation $I = 0$ reduces to

$$u \left( 4(\lambda + 3) N^2 \tilde{g}(\phi)^{\lambda + 6} \tilde{g}'(\phi) - 6g_0 \tilde{g}(\phi) \dot{\phi}^2 \tilde{g}'(\phi)^4 \right) + \phi \left[ \dot{\tilde{g}}(\phi)^2 \left( 2\omega_0 N^2 \tilde{g}(\phi)^{\lambda + 5} - g_0 (\lambda + 7) \dot{\phi}^2 \tilde{g}'(\phi)^3 \right) + \ddot{\tilde{g}}(\phi) \ddot{\tilde{g}}''(\phi) \left( 3g_0 \dot{\phi}^2 \tilde{g}'(\phi)^3 - 2(\lambda + 3) N^2 \tilde{g}(\phi)^{\lambda + 5} \right) \right] = 0, \quad (B2)$$
which can be algebraically solved with respect to the function \( u(t) \). Substitution of the latter inside the constraint equation \( \frac{\partial \lambda}{\partial \phi} = 0 \) leads to

\[
8(\lambda + 3)^2 V_0 \tilde{g}(\phi)^{4(\lambda + 5)} N^8 - 4\phi^2 (3\omega_0^2 - \omega_0(\lambda + 3)^2 + 6g_0(\lambda + 3)V_0) \tilde{g}(\phi)^{3(\lambda + 5)} \tilde{g}'(\phi)^3 N^6
+ 2g_0\phi^4 (6\omega_0(\lambda + 7) + 9g_0V_0 - 2(\lambda + 6)(\lambda + 3)^2) \tilde{g}(\phi)^{2(\lambda + 5)} \tilde{g}'(\phi)^5 N^4
+ 3g_0(-3\omega_0 + \lambda(\lambda + 2) - 19)\phi^6 \tilde{g}(\phi)^{\lambda + 5} \tilde{g}'(\phi(t))^9 N^2 + 9g_0^3 \phi^8 \tilde{g}'(\phi)^{12} = 0,
\]

which is an eighth order algebraic equation with respect to \( N(t) \). However, only even powers of \( N \) appear. So, by making a substitution \( N(t) = \sqrt{\tilde{n}(t)} \) in the latter, the latter reduces to a fourth order polynomial equation and can be solved analytically with respect to \( n(t) \). We refrain from giving the expression here, but general formulas can be found in the bibliography [40]. As a result we are able, in the case of \( I = 0 \), to derive the solution of the system in a purely algebraic manner for every admissible function \( \tilde{g}(\phi) \) without even fixing the gauge, since we have not set \( N = 1 \) or any of the rest two degrees of freedom to be a specific function of \( t \). We only satisfied dynamical equations, even though it was just for the special case \( I = 0 \). The solution is expressed through the relations

\[
N(t) = \sqrt{\tilde{n}(\phi, \dot{\phi})}, \quad a(t) = e^{\int n(\phi, \dot{\phi}) dt}
\]

in terms of an arbitrary function \( \phi(t) \), i.e. the scalar field plays effectively the role of time.

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