An Improved Second Order Poincaré Inequality for Functionals of Gaussian Fields

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Abstract

We present an improved version of the second order Gaussian Poincaré inequality, developed in Chatterjee (2009) and Nourdin, Peccati and Reinert (2009), which we use in order to bound distributional distances between functionals of Gaussian fields and a normal random variable. Some applications are developed, including a quantitative version of the Sinai-Soshnikov CLT and the Breuer-Major theorems, improving some previous findings in the literature.

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1 Introduction

The aim of this paper is to prove and apply some new forms of second order Poincaré inequalities for the normal approximation of non-linear functionals of Gaussian fields, thus significantly improving some of the findings from [Cha09] and [NPR09]. Our analysis is motivated by the fact that, in recent years, second order Poincaré inequalities, providing presumably optimal rates of convergence, have been established for Poisson and Rademacher functionals (see [LPS16] and [KRT17]). However, for functionals of Gaussian fields, the existing estimates often provide rates of convergence that are not sharp (see e.g. the discussion in Remark 4.3 of [NPR09]).

1.1 Previous work and plan of the paper

Let \( N \sim \mathcal{N}(0,1) \) be a standard Gaussian random variable. A classical result of stochastic analysis is the so-called Gaussian Poincaré inequality, which states that

\[
\var(f(N)) \leq E[f'(N)^2],
\]

for every differentiable function \( f : \mathbb{R} \to \mathbb{R} \). In particular, (1.1) becomes an equality if and only if \( f \) is affine. Such result was discovered by J. Nash in [Nas56], and then reproved by H. Chernoff in [Che81]. The estimate (1.1) implies that, if the random variable \( f'(N) \) has a small \( L^2 \) norm, then \( f(N) \) has small fluctuations. The Gaussian Poincaré inequality holds in the much more general setting of functionals of isonormal Gaussian processes (or functionals of Gaussian fields) and associated Malliavin operators. Indeed, the results proved in [HPA95] (which make use of the Malliavin calculus) allow one to recover the following version of (1.1): let \( X \) be an isonormal Gaussian process over some real separable Hilbert space \( \mathcal{H} \), and let \( F \in \mathcal{D}^{1,2} \) be a Malliavin-differentiable functional of \( X \) (see Section 1.2 for rigorous definitions); then, it holds that

\[
\var(F) \leq E[\|DF\|_{\mathcal{H}}^2],
\]

with equality if and only if \( F \) lives in the first Wiener chaos of \( X \). Note that if \( H = \mathbb{R}^d \) and \( F = f(X_1, \ldots, X_d) \) with \( (X_1, \ldots, X_d) \) a standard Gaussian vector, then (1.2) takes the following form

\[
\var(F) \leq E[\|\nabla f(X_1, \ldots, X_d)\|_{\mathbb{R}^d}^2],
\]

where \( \nabla f \) is the gradient of \( f \). In [Cha09], the author has showed that, under some adequate integrability assumptions on the operator norm of the \( d \times d \) Hessian matrix \( \nabla^2 f \), one can iterate (1.1) in order to assess the total variation distance \( d_{TV} \) between the law of \( F \) and the law of a Gaussian random variable with matching mean and variance. The precise result is the following (see Section 1.2 for the definition of total variation distance \( d_{TV} \)):
Theorem 1.1 (Second order Poincaré inequality). Let $X = (X_1, \ldots, X_d)$ be a standard Gaussian vector in $\mathbb{R}^d$. Take any $g \in C^2(\mathbb{R}^d)$ and let $\nabla g$ and $\nabla^2 g$ denote the gradient and Hessian of $g$. Suppose $W = g(X)$ has a finite fourth moment and let $\sigma^2 = \text{Var}(W)$. Let $Z \sim \mathcal{N}(E[W], \sigma^2)$, then

$$d_{TV}(W,Z) \leq \frac{2\sqrt{5}}{\sigma^2} \left( E \|\nabla g(X)\|_{\mathbb{R}^d}^4 \right)^{1/4} \left( E \|\nabla^2 g(X)\|_\infty^4 \right)^{1/4},$$

(1.4)

where $\|\cdot\|_\infty$ stands for the operator norm of $\nabla^2 g(X)$ regarded as a random $d \times d$ matrix.

Soon after [Cha09], the authors of [NPR09] pointed out that the finite-dimensional Stein-type inequalities leading to relation (1.4) are special instances of more general estimates, which can be obtained by combining Stein’s method and Malliavin calculus on an infinite-dimensional Gaussian space. In particular, in [NPR09] the following general version of (1.4) is obtained, involving functionals of arbitrary infinite-dimensional Gaussian fields (precise definitions of the Sobolev space $D^{2,4}$ and of insonormal Gaussian process will be given in Section 1.2).

Theorem 1.2 (Second order Poincaré inequality – infinite dimension). Let $X$ be an isonormal Gaussian process over some real separable Hilbert space $H$, and let $F \in D^{2,4}$. Assume that $E[F] = \mu$ and $\text{Var} F = \sigma^2$. Let $N \sim \mathcal{N}(\mu, \sigma^2)$. Then,

$$d_{TV}(F,N) \leq \frac{\sqrt{10}}{\sigma^2} \left( E \|DF\|_H^4 \right)^{1/4} \left( E \|D^2F\|_\infty^4 \right)^{1/4}.$$

(1.5)

As already discussed, the initial impetus for the present paper comes from the fact that (as described e.g. in Remark 4.3 of [NPR09]), once these inequalities are applied, they often give suboptimal rate of convergence. Indeed, since in most applications of interest it is not possible to compute directly the expectation involving the operator norm in both bounds (1.4) and (1.5), one is forced to move farther away from the distance in distribution and use bounds on the operator norm instead of computing it directly. Our strategy in order to overcome this difficulty is to adapt to the Gaussian setting an approach recently developed in [LPS16], in which the authors prove a second order Poincaré inequality for Gaussian approximation on the Poisson space, yielding presumably optimal rates in several geometric applications.

The next theorem contains one of the estimates developed in the present paper – see Theorem 2.1 below for a complete statement.

Theorem 1.3. Assume that $H = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a Polish space endowed with its Borel $\sigma$-field and $\mu$ is a positive, $\sigma$-finite and non-atomic measure and let $F \in D^{2,4}$ be such that $E[F] = 0$ and $E[F^2] = \sigma^2$. If $N \sim \mathcal{N}(0, \sigma^2)$, then

$$d_{TV}(F,N) \leq \frac{2\sqrt{3}}{\sigma^2} \left( \int_{A \times A} \left\{ E \left[ \left( (D^2 F \otimes_1 D^2 F) (x,y) \right)^2 \right] \right\}^{1/2} \times \left\{ E \left[ \left( D F(x) D F(y) \right)^2 \right] \right\}^{1/2} \, d\mu(x) d\mu(y) \right)^{1/2}.$$
Remark 1.1. The fact that $H$ is a $L^2$ space is fundamental for our proof. We will also see that our results are general enough, in order to imply explicit bounds for non-linear functionals of finite Gaussian vectors with arbitrary covariance matrices.

Our main applications concern three examples:

- The first example is related to the central limit theorem (CLT) for the trace of a power $p_n$ of a $n \times n$ Gaussian Wigner matrix, firstly introduced in the famous paper [SS98], where the authors proved a non-quantitative CLT. Then, using his formulation of a second order Poincaré inequality, the author of [Cha09] gave a quantitative CLT (QCLT) in the case that $p_n = o(\log n)$. We will prove a QCLT for $p_n = o(n^{1/15})$, which gives a Berry-Esseen bound for the distance in total variation when $p_n = o(n^{1/3})$.

- The second example, which was firstly introduced in [NPR09], is a QCLT for a non-linear functional of a stationary Gaussian field. Actually, [NPR09] already conjectured that their rate was sub-optimal and that they expected it to be as the one that we obtained. A related example was also treated in [FT16], in which the authors also reached the rate of our estimate, for some transformations whose Hermite expansion decays exponentially fast. However, our approach is completely different since it avoids the use of the Wiener chaos expansion of the functional (see Section 1.2 for definition of Wiener chaos expansion).

- The last example is a QCLT for a non-linear positive functional of a Brownian sheet on $\mathbb{R}^n$, exploding around singularities in the domain of integration. This is a generalization of limit theorems studied, with different techniques, in [NP05] and [NP09]. In the first paper, qualitative CLTs for quadratic functionals of a Brownian sheet on $\mathbb{R}^n$ are deduced; while in the second one, an exact quantitative version is determined.

Plan of the paper. Our paper is organized as follows: in the next section we explain the general setting, providing all the basic ingredients that we will use thorough the paper. In Section 2 we present our main results, while Section 3 contains the proofs. In Section 4 we present our main application to random matrices, i.e. we prove a QCLT for the trace of a power $p_n$ of a $n \times n$ Gaussian Wigner matrix (some technical proofs are contained in the Appendix); then we prove QCLTs for some non-linear functionals of Gaussian fields, in particular non-linear functionals of stationary Gaussian fields (this is connected to the Breuer-Major Theorem [BMS83]) and non-linear positive functionals of a Brownian sheet on $\mathbb{R}^n$.

1.2 General setting

Probability distances We will consider several notions of distances between the distributions of two random vectors $X, Y$ with values in $\mathbb{R}^m$, $m \geq 1$ (see [NP12, Appendix C] and the references therein for a complete discussion):
1. The Kolmogorov distance
\[
d_{\text{Kol}}(X,Y) = \sup_{z_1,\ldots,z_m \in \mathbb{R}} \left| P(X < (\infty, z_1] \times \cdots \times (\infty, z_m]) + P(Y < (\infty, z_1] \times \cdots \times (\infty, z_m]) \right|. \tag{1.6}
\]

2. The total variation distance
\[
d_{\text{TV}}(X,Y) = \sup_{B \in \mathcal{B}(\mathbb{R}^m)} \left| P(X \in B) - P(Y \in B) \right|. \tag{1.7}
\]

3. The Wasserstein distance
\[
d_W(X,Y) = \sup_{h \in \mathcal{H}} \left| E[h(X)] - E[h(Y)] \right|,
\]
where \(\mathcal{H}\) is the class of all functions \(h : \mathbb{R}^m \to \mathbb{R}\) such that \(\|h\|_{\text{Lip}} \leq 1\), with
\[
\|h\|_{\text{Lip}} = \sup_{x,y \in \mathbb{R}^m, x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_{\mathbb{R}^m}}. \tag{1.9}
\]

It is immediate to note that \(d_{\text{Kol}}(\cdot,\cdot) \leq d_{\text{TV}}(\cdot,\cdot)\). Moreover, if \(X\) is any real-valued random variable and \(N \sim \mathcal{N}(0,1)\), then \(d_{\text{Kol}}(X,N) \leq 2\sqrt{d_W(X,N)}\) (see, among others, [CGS11, Theorem 3.3] and more generally [APP16, Theorem 3.1]).

**Gaussian analysis and Malliavin calculus** We will now present the basic elements of Gaussian analysis and Malliavin calculus that are used in this paper. The reader is referred to the two monographs [Nua06] and [NP12] for further informations.

Let \(H\) be a real, separable Hilbert space with inner product \(\langle \cdot,\cdot \rangle_H\). An isonormal Gaussian process \(X = \{X(h) : h \in H\}\) over \(H\) is a centered Gaussian family defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(E[X(h)X(g)] = \langle g, h \rangle_H\) for every \(h, g \in H\). We will always assume \(\mathcal{F} = \sigma(X)\) and write \(L^2(\Omega)\) instead of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\).

Let \(S\) denote the set of all random variables of the form
\[
f(X(\phi_1),\ldots,X(\phi_m)), \tag{1.10}
\]
where \(m \geq 1\), \(f : \mathbb{R}^m \to \mathbb{R}\) is a \(C^\infty\)-function such that \(f\) and all its partial derivatives have at most polynomial growth at infinity, and \(\phi_i \in H, i = 1,\ldots,m\). Note that the space \(S\) is dense in \(L^q(\Omega)\) for every \(q \geq 1\). Let \(F \in S\) be of the form (1.10), and \(p \geq 1\) be an integer. The Malliavin derivative of \(F\) is the element of \(L^2(\Omega; H)\) defined by
\[
DF = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(X(\phi_1),\ldots,X(\phi_m))\phi_i; \tag{1.11}
\]
while the second Malliavin derivative of \(F\) is the element of \(L^2(\Omega; H^{\otimes 2})\) given by
\[
D^2F = \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(X(\phi_1),\ldots,X(\phi_m))\phi_i \otimes \phi_j, \tag{1.12}
\]
where $H^\otimes 2$ is the second symmetric tensor power of $H$.

For $\alpha = 1, 2$, the operator $D^\alpha$ is closable ($D^1 := D$), so we can extend the domain of $D^\alpha$ to the space $\mathbb{D}^{\alpha,2}$, which is defined as the closure of $S$ with respect to the norm

$$\|F\|_{\mathbb{D}^{\alpha,2}} = (E[|F|^2] + E[\|DF\|^2_H] + E[\|D^2F\|^2_{H^\otimes 2}]1_{\{\alpha=2\}})^{1/2}.$$ 

Plainly, $\mathbb{D}^{2,2} \subset \mathbb{D}^{1,2}$. We call $\mathbb{D}^{\alpha,2}$ the domain of $D^\alpha$ in $L^2(\Omega)$. The space $\mathbb{D}^{\alpha,2}$ is a Hilbert space with respect to the inner product

$$\langle F, G \rangle_{\mathbb{D}^{\alpha,2}} = E[FG] + E[(DF, DG)_H] + E[(D^2F, D^2G)_{H^\otimes 2}]1_{\{\alpha=2\}}.$$ 

Note that the Malliavin derivative satisfies the following chain rule. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, then if $F \in \mathbb{D}^{1,2}$, $\psi(F) \in \mathbb{D}^{1,2}$ and we have that

$$D\psi(F) = \psi'(F) DF.$$ 

For $n \in \mathbb{N} \cup \{0\}$, we call $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2})$ the $n$th Hermite polynomial. For each $n \geq 0$ we define

$$\mathcal{H}_n = \text{span} \{H_n(X(h)), h \in H, \|h\|_H = 1\}^{\|\cdot\|_{L^2(\Omega)}}.$$ 

The space $\mathcal{H}_n$ is called the $n$th Wiener chaos of $X$. Clearly, we have $\mathcal{H}_0 = \mathbb{R}$ and $\mathcal{H}_1 = X$. Moreover, it is well known that $\mathcal{H}_n \perp \mathcal{H}_m$ for every $n \neq m$ and thus that the sum $\bigoplus_{n=0}^\infty \mathcal{H}_n$ is direct in $L^2(\Omega)$. By the density of polynomial functions, this implies that every random variable $F \in L^2(\Omega)$ admits a unique expansion of the type $F = E[F] + \sum_{n=1}^\infty F_n$ where $F_n \in \mathcal{H}_n$ and the series converges in $L^2(\Omega)$.

The Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$ is defined for all $t \geq 0$ and $F \in L^2(\Omega)$ by $P_t(F) = \sum_{p=0}^\infty e^{-pt} J_p(F) \in L^2(\Omega)$, where $J_p(F) = \text{Proj}(F|H_p)$ stands for the orthogonal projection of $F$ onto the $p$-th Wiener chaos. One can prove that for every $t > 0$ and every $q \geq 1$, $P_t$ is a contraction on $L^q(\Omega)$, that is

$$E[|P_t(F)|^q] \leq \|F\|_{L^q(\Omega)}^q, \forall F \in L^q(\Omega).$$

Let $F \in L^1(\Omega)$, let $X'$ be an independent copy of $X$, and assume that $X$ and $X'$ are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. Since $F$ is measurable with respect to $X$, we can write $F = f(X)$ with $f : \mathbb{R}^H \to \mathbb{R}$ a measurable mapping determined $\mathbb{P} \circ X^{-1}$ a.s.. We have the so-called Mehler formula

$$P_tF = E\left[f(e^{-t}X + \sqrt{1-e^{-2t}}X')|X\right], \quad t \geq 0.$$ 

(1.14) The generator $L$ of the Ornstein-Uhlenbeck semigroup is defined as

$$LF = -\sum_{p=1}^\infty pJ_p(F)$$
with domain given by

\[
\text{Dom} \, L = \left\{ F \in L^2(\Omega) : \sum_{p=1}^{\infty} p^2 E \left[ J_p(F)^2 \right] < \infty \right\}.
\]

For any \( F \in L^2(\Omega) \) we define \( L^{-1}F = -\sum_{p=1}^{\infty} \frac{1}{p} J_p(F) \). The operator \( L^{-1} \) is called the pseudo-inverse of \( L \). The name of \( L^{-1} \) is justified by the fact that for any \( F \in L^2(\Omega) \), \( L^{-1}F \in \text{Dom} \, L \) and

\[
LL^{-1}F = F - E(F).
\]

Let \( F \in \mathbb{D}^{1,2} \) with \( E[F] = 0 \), then the following relation holds

\[
- DL^{-1}F = \int_0^\infty e^{-t} P_t DF dt = -(L - I)^{-1} DF. \tag{1.15}
\]

In the case where the real separable Hilbert space has the form \( H = L^2(A, \mathcal{A}, \mu) \), where \((A, \mathcal{A})\) is a Polish space endowed with its Borel \( \sigma \)-field and \( \mu \) is a positive, \( \sigma \)-finite and non-atomic measure, then, \( \forall F \in \mathbb{D}^{2,2} \) there exist two measurable processes \( Y : \Omega \times A \to \mathbb{R} \) and \( Z : \Omega \times A \times A \to \mathbb{R} \) such that for almost each \((\omega, a, b) \in \Omega \times A \times A\), \( DF(\omega, a) = Y(\omega, a) \) and \( D^2F(\omega, a, b) = Y(\omega, a, b) \) (for a detailed discussion see [Nua06, Section 1.2.1]); for the rest of the paper we will always identify \( DF \) and \( D^2F \) with \( Y \) and \( Z \), respectively.

Note that if \( L^2(A, \mathcal{A}, \mu) \) is as above and \( H \) is another real and separable Hilbert space, there exists an isomorphism

\[
i : H \to L^2(A, \mathcal{A}, \mu)
\]

\[
h \mapsto i(h)
\]

in such a way that \( \{X(h) : h \in H\} \overset{d}{=} \{W(i(h)) : h \in H\} \), where \( W \) is an isonormal Gaussian process over \( L^2(A, \mathcal{A}, \mu) \).

## 2 Main results

### 2.1 Main estimates

Let \( H = L^2(A, \mathcal{A}, \mu) \), where \((A, \mathcal{A})\) is a Polish space endowed with its Borel \( \sigma \)-field and \( \mu \) is a positive, \( \sigma \)-finite and non-atomic measure. Our main abstract result is the following.

**Theorem 2.1.** Let \( F \in \mathbb{D}^{2,4} \) be such that \( E[F] = 0 \) and \( E[F^2] = \sigma^2 \), and let \( N \sim \mathcal{N}(0, \sigma^2) \); then

\[
d_{TV}(F, N) \leq \frac{4}{\sigma^2} \left( \int_{A \times A} \left\{ E \left[ ((D^2F \otimes_1 D^2F)(x, y))^2 \right] \right\}^{1/2} \right.	imes
\]

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\[ d_{Kol}(F, N) \leq \frac{2}{\sigma^2} \left( \int_{A \times A} \left\{ E \left[ ((D^2 F \otimes_1 D^2 F) (x, y))^2 \right] \right\}^{1/2} \times \right. \\
\left. \times \left\{ E \left[ (DF(x)DF(y))^2 \right] \right\}^{1/2} d\mu(x)d\mu(y) \right)^{1/2}, \quad (2.1) \]

\[ d_W(F, N) \leq \sqrt{8 \sqrt{\frac{\sigma^2}{\sigma^2 \pi}}} \left( \int_{A \times A} \left\{ E \left[ ((D^2 F \otimes_1 D^2 F) (x, y))^2 \right] \right\}^{1/2} \times \right. \\
\left. \times \left\{ E \left[ (DF(x)DF(y))^2 \right] \right\}^{1/2} d\mu(x)d\mu(y) \right)^{1/2}. \quad (2.2) \]

2.2 Corollaries and extensions

Theorem 2.1 contains, as a special case, probabilistic approximations involving random variables of the form \( F = f(X_1, \ldots, X_d) \), where \((X_1, \ldots, X_d)^T\) is a standard Gaussian vector and \( f : \mathbb{R}^d \to \mathbb{R} \) is a \( C^2 \) function such that its partial derivatives have sub-exponential growth. Indeed, if \( A_1, \ldots, A_d \in \mathcal{A} \) are such that \( A_i \cap A_j = \emptyset \) for each \( i, j \) such that \( i \neq j \) and \( \mu(A_i) = 1 \ \forall \ i \), then we have that

\[ F \overset{d}{=} f(X(1_{A_1}), X(1_{A_2}), \ldots, X(1_{A_d})). \]

Moreover, in view of (1.11) and (1.12), we have that

\[ DF(x) = \sum_{i=1}^{d} \nabla_i f(X) 1_{A_i}(x), \]

\[ D^2 F(x, y) = \sum_{i, j=1}^{d} \nabla^2_{ij} f(X) 1_{A_i}(x) 1_{A_j}(y), \]

where \( \nabla_i f(X) \) is the \( i \)-th component of the gradient of \( f \) and \( \nabla^2_{ij} \) is the \( ij \)-th entry of the Hessian matrix of \( f \). This implies that

\[ D^2 F \otimes_1 D^2 F(x, y) = \int_A d\mu(w) \sum_{i, j=1}^{d} \nabla^2_{ij} f(X) 1_{A_i}(x) 1_{A_j}(w) \]
\[ \sum_{k, l=1}^{d} \nabla^2_{kl} f(X) 1_{A_k}(y) 1_{A_l}(w) \]
\[ = \sum_{i, k, l=1}^{d} \nabla^2_{il} f(X) \nabla^2_{kl} f(X) 1_{A_i}(x) 1_{A_k}(y) \int_A 1_{A_i}(w) d\mu(w) \]
Theorem 2.2. Let $f$ become, respectively, $\mathcal{N}(0, I_d \times d)$ and $F := f(X)$ for some $f \in C^2(\mathbb{R}^d)$ such that $E[F] = 0$ and $E[F^2] = \sigma^2$. Let $N \sim \mathcal{N}(0, \sigma^2)$, then

\[
\begin{align*}
    d_{TV}(F, N) &\leq \frac{4}{\sigma^2} \sqrt{\sum_{i,j=1}^d \left[ E \left( \left( \sum_{j=1}^d \nabla_{ij}^2 F \nabla_{ij}^2 F \right)^2 \right) \right]^{1/2}} \left[ E \left[ (\nabla_i F \nabla_j F)^2 \right] \right]^{1/2}, \\
    d_{KL}(F, N) &\leq \frac{2}{\sigma^2} \sqrt{\sum_{i,j=1}^d \left[ E \left( \left( \sum_{j=1}^d \nabla_{ij}^2 F \nabla_{ij}^2 F \right)^2 \right) \right]^{1/2}} \left[ E \left[ (\nabla_i F \nabla_j F)^2 \right] \right]^{1/2}.
\end{align*}
\]

In this case, the quantities on the right hand side of inequalities (2.1), (2.2) and (2.3) become, respectively,

\[
\begin{align*}
    \left\{ E \left[ (D^2 F \otimes_1 D^2 F)(x, y)^2 \right] \right\}^{1/2} &= \left\{ E \left[ \left( \sum_{i,k=1}^d \left( \sum_{l=1}^d \nabla^2_{il} f(X) \nabla^2_{kl} f(X) \right) \mathbb{1}_{A_i}(x) \mathbb{1}_{A_k}(y) \right)^2 \right] \right\}^{1/2} \\
    &= \left\{ \sum_{i,k=1}^d E \left[ \left( \sum_{l=1}^d \nabla^2_{il} f(X) \nabla^2_{kl} f(X) \right)^2 \mathbb{1}_{A_i}(x) \mathbb{1}_{A_k}(y) \right] \right\}^{1/2} \\
    &= \left\{ \sum_{i,k=1}^d \left\{ E \left[ \left( \sum_{l=1}^d \nabla^2_{il} f(X) \nabla^2_{kl} f(X) \right)^2 \right] \mathbb{1}_{A_i}(x) \mathbb{1}_{A_k}(y) \right\} \right\}^{1/2}
\end{align*}
\]

and

\[
\begin{align*}
    \left\{ E \left[ (DF(x)DF(y))^2 \right] \right\}^{1/2} &= \left\{ \sum_{i,k=1}^d \left\{ E \left[ (\nabla_i f(X)\nabla_k f(X)) (\mathbb{1}_{A_i}(x) \mathbb{1}_{A_k}(y)) \right] \right\} \right\}^{1/2} \\
    &= \left\{ \sum_{i,k=1}^d \left\{ E \left[ (\nabla_i f(X)\nabla_k f(X))^2 \right] \mathbb{1}_{A_i}(x) \mathbb{1}_{A_k}(y) \right\} \right\}^{1/2}
\end{align*}
\]

Hence, when $F = f(X_1, \ldots, X_d)$, with $(X_1, \ldots, X_d)$ a standard Gaussian vector, our main result takes the following form.

**Theorem 2.2.** Let $X = (X_1, \ldots, X_d) \sim \mathcal{N}(0, I_d \times d)$ and $F := f(X)$ for some $f \in C^2(\mathbb{R}^d)$ such that $E[F] = 0$ and $E[F^2] = \sigma^2$. Let $N \sim \mathcal{N}(0, \sigma^2)$, then

\[
\begin{align*}
    d_{TV}(F, N) &\leq \frac{4}{\sigma^2} \sqrt{\sum_{i,j=1}^d \left[ E \left( \left( \sum_{j=1}^d \nabla_{ij}^2 F \nabla_{ij}^2 F \right)^2 \right) \right]^{1/2}} \left[ E \left[ (\nabla_i F \nabla_j F)^2 \right] \right]^{1/2}, \\
    d_{KL}(F, N) &\leq \frac{2}{\sigma^2} \sqrt{\sum_{i,j=1}^d \left[ E \left( \left( \sum_{j=1}^d \nabla_{ij}^2 F \nabla_{ij}^2 F \right)^2 \right) \right]^{1/2}} \left[ E \left[ (\nabla_i F \nabla_j F)^2 \right] \right]^{1/2}.
\end{align*}
\]
and

\[ d_W(F, N) \leq \sqrt{\frac{8}{\sigma^2 \pi} \sum_{i=1}^{d} \left\{ E \left[ \left( \sum_{j=1}^{d} \nabla_i^2 F \nabla_j^2 F \right)^2 \right] \right\}^{1/2} \left\{ E \left[ (\nabla_i F \nabla_j F)^2 \right] \right\}^{1/2}} \]

where \( \nabla_i^2 F \) is the \( ij \)-th entry of the Hessian matrix of \( F = f(X) \) and \( \nabla_i F \) is the \( i \)-th element of the gradient of \( F \).

**Remark 2.1.** Note that Theorem 2.2 also applies to the case of a vector \( X \) with a general covariance, that is \( X \sim \mathcal{N}(0, B^2) \), where \( B^2 \) is a symmetric and positive definite matrix. Indeed, one has that \( F = f(X) = g(Z) \), where \( g = f \circ B \) and \( Z = (Z_1, \ldots, Z_d) \sim \mathcal{N}(0, I) \). Therefore we have that

\[ d_{TV}(F, N) \leq \frac{4}{\sigma^2} \left( \sum_{i,j=1}^{d} \left\{ E \left[ \left( \sum_{j=1}^{d} \nabla_i^2 g(Z) \nabla_j^2 g(Z) \right)^2 \right] \right\}^{1/2} \left\{ E \left[ (\nabla_i^2 g(Z) \nabla_i^2 g(Z))^2 \right] \right\}^{1/2} \right) \]

\[ = \frac{4}{\sigma^2} \left( \sum_{i,j=1}^{d} \left\{ E \left[ \left( \sum_{k,m,r,s=1} \sum_{j=1}^{d} b_{mk} b_{jr} \nabla^2_{km} F \nabla^2_{rs} F \right)^2 \right] \right\}^{1/2} \times \right) \]

\[ \times \left\{ E \left[ \left( \sum_{k,m=1}^{d} b_{km} \nabla_k F \nabla_m F \right)^2 \right] \right\}^{1/2} \]

with \( b_{ij} \) the \( ij \)-th entry of the matrix \( B \).

Using the multidimensional version of [NP12, Theorem 5.1.3] (which is one of the main ingredients of our main result’s proof, see Section 3), that is [NP12, Theorem 6.1.1], Theorem 2.1 can be easily extended to a multidimensional setting as follows:

**Theorem 2.3.** Let \( F = (F_1, \ldots, F_d) \), where, for each \( i = 1, \ldots, d \), \( F_i \in \mathbb{D}^{2,2} \) is such that \( E[F_i] = 0 \) and \( E[F_i F_j] = c_{ij} \). Let \( N \sim \mathcal{N}(0, C) \), where \( C = \{ c_{ij} \}_{i,j=1,\ldots,d} \) is a symmetric and positive definite matrix, then we have that

\[ d_{TV}(F, N) \leq \sqrt{d} \| C^{-1} \|_{op} \| C \|_{op} \times \]

\[ \times \left\{ \sum_{i,j=1}^{d} \int_{A \times A} 2 \left\{ E \left[ (D^2 F_i \otimes_1 D^2 F_i)(x,y) \right]^2 \right\}^{1/2} \left\{ E \left[ (DF_j(x)DF_j(y))^2 \right] \right\}^{1/2} + \right\} \]

\[ + 2 \left\{ E \left[ (D^2 F_j \otimes_1 D^2 F_j)(x,y) \right]^2 \right\}^{1/2} \left\{ E \left[ (DF_i(x)DF_i(y))^2 \right] \right\}^{1/2} d\mu(x) d\mu(y) \right\}^{1/2} . \]
3 Proof of Theorem 2.1

An important ingredient to prove our main result is a theorem proved in [NP12, Theorem 5.1.3] and, with slight more generality, in [Nou13, Theorem 5.2].

**Theorem 3.1.** Let \( F \in D^{1,2} \) with \( E[F] = 0 \) and \( E[F^2] = \sigma^2 \), and let \( N \sim N(0, \sigma^2) \).

Then,

\[
d_{W}(F, N) \leq \sqrt{\frac{2}{\sigma^2 \pi} E \left[ \left| 1 - \left< DF, -DL^{-1}F \right>_H \right| \right]},
\]

\[
d_{TV}(F, N) \leq \frac{2}{\sigma^2} E \left[ \left| 1 - \left< DF, -DL^{-1}F \right>_H \right| \right],
\]

and

\[
d_{Kol}(F, N) \leq \frac{1}{\sigma^2} E \left[ \left| 1 - \left< DF, -DL^{-1}F \right>_H \right| \right].
\]

We will also need the following proposition.

**Proposition 3.2.** Let \( F, G \in D^{2,4} \) such that \( E[F] = E[G] = 0 \). Then, it holds that

\[
E \left[ \left( \text{Cov}(F, G) - \left< DF, -DL^{-1}G \right>_{L^2(A, \mu)} \right)^2 \right] \leq 2 \int_{A \times A} \{ E \left[ \left( (D^2 F \otimes_1 D^2 F)(x, y) \right)^2 \right] \}^{1/2} \times \times \left\{ E \left[ \left( DG(x)DG(y) \right)^2 \right] \right\}^{1/2} d\mu(x)d\mu(y) + 2 \int_{A \times A} \{ E \left[ \left( DF(x)DF(y) \right)^2 \right] \}^{1/2} \times \times \left\{ E \left[ \left( (D^2 G \otimes_1 D^2 G)(x, y) \right)^2 \right] \right\}^{1/2} d\mu(x)d\mu(y).
\]

**Proof.** Using the fact that \( \text{Cov}(F, G) = E \left( (DF, -DL^{-1}G)_{L^2(A, \mu)} \right) \) and the Poincaré inequality \([1.2]\) (note that one needs \( F, G \in D^{2,4} \) for \( (DF, -DL^{-1}G)_{L^2(A, \mu)} \) to be in \( D^{1,2} \) and apply \([1.2]\), see [NPR09, Lemma 3.2]), we have

\[
E \left[ \left( \text{Cov}(F, G) - \left< DF, -DL^{-1}G \right>_{L^2(A, \mu)} \right)^2 \right] = \text{Var} \left( \left< DF, -DL^{-1}G \right>_{L^2(A, \mu)} \right)
\]

\[
\leq E \left( \left\| D(DF, -DL^{-1}G)_{L^2(A, \mu)} \right\|^2_{L^2(A, \mu)} \right) 
\]

\[
\leq 2E \left( \left\| (D^2 F, -DL^{-1}G)_{L^2(A, \mu)} \right\|^2_{A_1} \right) + 2E \left( \left\| DF, -D^2 L^{-1}G \right\|^2_{L^2(A, \mu)} \right),
\]

where the last inequality follows from the fact that (again, according to [NPR09, Lemma 3.2])

\[
D(DF, -DL^{-1}G)_{L^2(A, \mu)} = (D^2 F, -DL^{-1}G)_{L^2(A, \mu)} + (DF, -D^2 L^{-1}G)_{L^2(A, \mu)}.
\]
Let us first consider $A_1$: given the fact that

$$-DL^{-1}G = \int_0^\infty e^{-t}P_tDGdt$$

and using Mehler formula (1.14), we deduce that

$$\langle D^2F,-DL^{-1}G \rangle_{L^2(A,\mu)} = \langle D^2F, \int_0^\infty e^{-t}P_tDGdt \rangle_{L^2(A,\mu)} = \langle D^2F, \int_0^\infty e^{-t} \left( Dg \left( e^{-t}X + \sqrt{1-e^{-2t}X^t} \right) \right) dt \rangle_{L^2(A,\mu)} = \int_0^\infty e^{-t}E \left[ \left( D^2F, Dg \left( e^{-t}X + \sqrt{1-e^{-2t}X^t} \right) \right) X \right] dt.$$

Hence, Jensen inequality and Fubini theorem yield that

$$A_1 = \left\| \int_0^\infty e^{-t}E \left[ \left( D^2F, Dg \left( e^{-t}X + \sqrt{1-e^{-2t}X^t} \right) \right) X \right] dt \right\|^2_{L^2(A,\mu)}$$

$$\leq \int_0^\infty e^{-t}E \left[ \left\| \int_A (D^2F)(x,y)Dg(X_1)(x)d\mu(x) \right\|^2_{L^2(A,\mu)} X \right] dt$$

$$= \int_0^\infty e^{-t} \int_A \int_A D^2F(x,y)D^2F(z,y) \times$$

$$\times E \left[ Dg(X_1)(x)Dg(X_1)(z) \right] d\mu(x)d\mu(z)d\mu(y)dt$$

$$= \int_0^\infty e^{-t} \int_{A \times A} \int_A D^2F(x,y)D^2F(z,y)P_t(DG(x)DG(z))d\mu(x)d\mu(z)d\mu(y)dt.$$

Now we can use Cauchy-Schwarz inequality and the contractivity of $P_t$ to have

$$E(A_1) \leq \int_0^\infty e^{-t} \int_A \left\{ E \left[ \left( \int_A D^2F(x,y)D^2F(z,y)d\mu(y) \right)^2 \right] \right\}^{1/2} \times$$

$$\times \left\{ E \left[ \left( P_t(DG(x)DG(z)) \right)^2 \right] \right\}^{1/2} d\mu(x)d\mu(z)$$

$$\leq \int_A \left\{ E \left[ \left( D^2F \otimes_1 D^2F \right)(x,z) \right)^2 \right\}^{1/2} \times$$

$$\times \left\{ E \left[ \left( DG(x)DG(z) \right)^2 \right] \right\}^{1/2} d\mu(x)d\mu(z).$$

Now consider $A_2$ (which is treated similarly): given the fact that

$$-D^2L^{-1}G = \int_0^\infty e^{-2t}P_tD^2Gdt$$
and using Mehler formula \((3.1)\), we obtain
\[
\langle DF, -D^2 L^{-1} G \rangle_{L^2(A,\mu)} = \\
= \langle D^2 F, \int_0^\infty e^{-2t} P_t D^2 G dt \rangle_{L^2(A,\mu)} \\
= \langle DF, \int_0^\infty e^{-2t} E \left( D^2 g \left( e^{-t} X + \sqrt{1 - e^{-2t}} X' \right) \bigg| X \right) dt \rangle_{L^2(A,\mu)} \\
= \int_0^\infty e^{-2t} E \left[ \langle DF, D^2 g (X_t) \rangle_{L^2(A,\mu)} \right] dt.
\]

Hence, using Jensen inequality and Fubini theorem, we deduce
\[
A_2 = 4 \left\| \int_0^\infty e^{-2t} \frac{1}{2} E \left[ \langle DF, D^2 g (X_t) \rangle_{L^2(A,\mu)} \right] dt \right\|_{L^2(A,\mu)}^2 \\
\leq 2 \int_0^\infty e^{-2t} E \left[ \left\| \langle DF, D^2 g (X_t) \rangle_{L^2(A,\mu)} \right\|_{L^2(A,\mu)}^2 \right] dt \\
= 2 \int_0^\infty e^{-2t} E \left[ \left\| \int_A (DF)(x) D^2 g(X_t)(x, y) d\mu(x) \right\|_{L^2(A,\mu)}^2 \right] dt \\
= 2 \int_0^\infty e^{-2t} \int_{A^2} DF(x) DF(z) P_t \left( \int_A D^2 G(x, y) D^2 G(z, y) d\mu(y) \right) d\mu(x) d\mu(z) dt.
\]

Now we can use the Cauchy-Schwarz inequality and the contractivity of \(P_t\) to infer that
\[
E (A_2) \leq 2 \int_0^\infty e^{-2t} \int_{A \times A} \left\{ E \left[ P_t \left( \int_A D^2 G(x, y) D^2 G(z, y) d\mu(y) \right) \right]^2 \right\}^{1/2} \times \\
\times \left\{ E \left[ (DF(x) DF(y))^2 \right] \right\}^{1/2} d\mu(x) d\mu(y) \\
\leq \int_{A \times A} \left\{ E \left[ \left( D^2 G \otimes_1 D^2 G \right)(x, y) \right]^2 \right\}^{1/2} \times \\
\times \left\{ E \left[ (DF(x) DF(y))^2 \right] \right\}^{1/2} d\mu(x) d\mu(y).
\]

Finally,
\[
E \left[ \left( \text{Cov}(F, G) - \langle DF, -DL^{-1} G \rangle_{L^2(A,\mu)} \right)^2 \right] \leq 2E(A_1) + 2E(A_2),
\]
which gives the desired conclusion. \(\square\)

Taking \(G = F\) in Proposition \(3.2\) one has that
\[
E \left[ \left| 1 - \langle DF, -DL^{-1} F \rangle_{L^2(A,\mu)} \right| \right] \leq \sqrt{E \left[ \left( 1 - \langle DF, -DL^{-1} F \rangle_{H} \right)^2 \right]} \\
\leq 2 \int_{A \times A} \mu(dx) \mu(dy) \left\{ E \left[ (DF(x) DF(y))^2 \right] \right\}^{1/2} \left\{ E \left[ \left( D^2 F \otimes_1 D^2 F \right)(x, y) \right]^2 \right\}^{1/2}.
\]

As a consequence, combining \((3.1)\) with Theorem \(3.1\) we immediately obtain Theorem \(2.1\).
4 Applications

4.1 Traces of Wigner matrices

4.1.1 Main result

Let \( X = (X_{ij})_{1 \leq i \leq j \leq n} \) be a Gaussian vector with values in \( \mathbb{R}^{n(n+1)/2} \) and \( Y(X) = (Y_{ij}(X))_{1 \leq i,j \leq n} \) be the matrix whose \( ij \)-th entry is \( X_{ij} \) if \( i \leq j \) and \( X_{ji} \) if \( i > j \). The random matrix

\[
A(X) = \frac{1}{\sqrt{n}} Y(X), \quad n \geq 1,
\]

is called Wigner matrix of dimension \( n \times n \). Our aim in this section is to use our bound in order to assess the distance between the distribution of \( G_n = g(X) = \text{Tr}(A(X)^p) \) and a Gaussian distribution \( \mathcal{N} \), where \( X \sim \mathcal{N}(0, I_{d 	imes d}) \), and \( p = p_n \) is a numerical sequence possibly diverging to infinity. This problem was firstly tackled in [SS98], where the authors prove a qualitative CLT in the case of \( p = o\left(n^{1/2}\right) \) (then extended to \( p = o\left(n^{2/3}\right) \) in [SS99]). In particular they proved that \( \text{Tr}(A^p) - E[\text{Tr}(A^p)] \) converges in distribution towards a Gaussian random variable with zero expectation and variance equal to \( \frac{1}{\pi} \). This example has been further analyzed in [Cha09], where the author gives a quantitative CLT bounding the distance in total variation between a standard Gaussian random variable and \( \text{Tr}(A^p) - E[\text{Tr}(A^p)] \sqrt{\text{Var Tr}A^p} \). A necessary condition for the bound in [Cha09] to go to zero is that \( p = o\left(\log n\right) \); here, we obtain a quantitative CLT in total variation distance for \( \frac{\text{Tr}(A^p) - E[\text{Tr}(A^p)]}{\sqrt{\text{Var Tr}A^p}} \) when \( p = o\left(n^{4/15}\right) \). Our main result is the following.

**Theorem 4.1.** If \( p = o\left(n^{4/15}\right) \), then \( F_n := \frac{G_n - E[G_n]}{\sqrt{\text{Var} G_n}} \to \mathcal{N} \) in distribution as \( n \to \infty \) and there exists a universal constant \( C < \infty \) such that

\[
d_{TV}(F_n, N) \leq C \left( \frac{e^{3/4}}{(2 \pi)^{3/8}} \frac{p^{7/8}}{n^{1/4}} + \frac{2 e}{2^{1/8}} \frac{p^{15/8}}{\sqrt{n}} \right).
\]

Moreover, if \( p = o\left(n^{1/4}\right) \), we have that

\[
d_{TV}(F_n, N) = O \left( \frac{p^{15/8}}{\sqrt{n}} \right).
\]

**Remark 4.1.** Our proof shows that there exists a numerical sequence \( \eta_n \in (0, \infty) \) such that \( \forall n \)

\[
d_{TV}(F_n, N) \leq \eta_n \left( \frac{e^{3/4}}{(2 \pi)^{3/8}} \frac{p^{7/8}}{n^{1/4}} + \frac{2 e}{2^{1/8}} \frac{p^{15/8}}{\sqrt{n}} \right),
\]

and \( \eta_n \to 4\pi \) as \( n \to \infty \).

**Remark 4.2.** Note that such a result improves the findings from [Cha09], in which the author obtains a similar result under the condition of \( p_n = o\left(\log n\right) \); although it cannot achieve the level of generality of [SS98, SS99], where a qualitative CLT is reached for \( p_n = o\left(n^{2/3}\right) \).
4.1.2 First computations and sketch of the proof

Note that the following relations hold:

\[
\frac{\partial}{\partial a_{ij}} \text{Tr} (A^p) = p (A^{p-1})_{ji}
\]

and

\[
\frac{\partial^2}{\partial a_{ij} \partial a_{rs}} \text{Tr}(A^p) = p \sum_{q=0}^{p-2} \text{Tr} \left( \frac{\partial A^q}{\partial a_{ij}} \frac{\partial A^{p-2-q}}{\partial a_{rs}} \right) = p \sum_{q=0}^{p-2} \text{Tr} \left( E_{ij} A^q E_{rs} A^{p-2-q} \right) = p \sum_{q=0}^{p-2} (A^q)_{jr} (A^{p-2-q})_{is},
\]

where \( E_{ij} \) is the \( n \times n \) matrix whose entries are all zero except for the \( ij \)-th. Now, note that we can write \( X = \frac{1}{2} Z \), where \( Z \sim \mathcal{N}(0, I_{d \times d}) \). Then, \( g(x) = g(\frac{x}{2}) = f(z) \), and we have

\[
\frac{\partial f}{\partial z_{kl}} (z) = \sum_{i,j=1}^{n} \frac{\partial}{\partial a_{ij}} \text{Tr}(A^p) \frac{\partial a_{ij}}{\partial z_{kl}} (z) = \sum_{i,j=1}^{n} p (A^{p-1})_{ji} \frac{1}{2 \sqrt{n}} \left( 1_{(k,l)=(i,j)} + 1_{(k,l)=(j,i)} - 1_{(k,l)=(i,j)} \right) = \frac{p}{\sqrt{n}} (A^{p-1})_{kl} 1_{(k \neq l)} + \frac{p}{2 \sqrt{n}} (A^{p-1})_{kk} 1_{(k=l)}
\]

and

\[
\frac{\partial^2 f}{\partial z_{kl} \partial z_{hm}} (z) = \sum_{i,j=1}^{n} \frac{\partial}{\partial a_{ij}} \text{Tr}(A^p) \frac{\partial^2 a_{ij}}{\partial x_{kl} \partial x_{hm}} + \sum_{i,j,r,s=1}^{n} \frac{\partial^2}{\partial a_{ij} \partial a_{rs}} \text{Tr}(A^p) \frac{\partial a_{ij}}{\partial x_{kl}} \frac{\partial a_{rs}}{\partial x_{hm}} = \sum_{i,j,r,s=1}^{n} p \sum_{q=0}^{p-2} \left\{ (A^q)_{jr} (A^{p-2-q})_{is} \right\} \frac{\partial a_{ij}}{\partial x_{kl}} \frac{\partial a_{rs}}{\partial x_{hm}} = \sum_{i,j,r,s=1}^{n} p \sum_{q=0}^{p-2} (A^q)_{jr} (A^{p-2-q})_{is} \frac{1}{4n} \left( 1_{(i,j)=(k,l)} + 1_{(i,j)=(l,k)} \right) \left( 1_{(r,s)=(h,m)} + 1_{(r,s)=(m,h)} \right)
\]

\[
= \frac{p}{4n} \sum_{q=0}^{p-2} \left\{ (A^q)_{lh} (A^{p-2-q})_{mk} + (A^q)_{lm} (A^{p-2-q})_{hk} \right\}.
\]
We have that

\[ B \leq TV_F G B + n := 2 \left( \sum_{i,k,l,m=1}^n \left( \sum_{j,h=1}^p \nabla_{ik,jh}^2 g \nabla_{lm,jh}^2 g \right)^2 \right) \]

Plugging these relations into (2.4) we deduce that

\[
d_{TV}(F_n, N)^2 = d_{TV} \left( G_n - E[G_n], \mathcal{N}(0, \text{Var} G_n) \right)^2 \leq \frac{16}{(\text{Var} G_n)^2} \sum_{i,k,l,m=1}^n \left\{ E \left[ \left( \sum_{j,h=1}^p \nabla_{ik,jh}^2 g \nabla_{lm,jh}^2 g \right)^2 \right] \right\}^{1/2} \left\{ E \left[ (\nabla_{ik} g \nabla_{lm} g)^2 \right] \right\}^{1/2}
\]

(4.1)

\[
= \frac{1}{(\text{Var} G_n)^2} \sum_{i,k,l,m=1}^n \left\{ E \left[ \left( \sum_{j,h=1}^p \left( \frac{p}{n} \sum_{q=0}^{p-2} \left( (A^q)_{kj} (A^{p-2-q})_{hi} + (A^q)_{kh} (A^{p-2-q})_{ji} + (A^q)_{ij} (A^{p-2-q})_{hk} + (A^q)_{ih} (A^{p-2-q})_{jk} \right) \right) \right)^2 \right] \right\}^{1/2} \times \left\{ E \left[ \left( \frac{p}{\sqrt{n}} (A^{p-1})_{ik} \frac{p}{\sqrt{n}} (A^{p-1})_{im} \right)^2 \right] \right\}^{1/2}
\]

(4.2)

Now define

\[
B_{iklm} := \sum_{j,h=1}^n \left( \frac{p}{n} \sum_{q=0}^{p-2} \left( (A^q)_{kj} (A^{p-2-q})_{hi} + (A^q)_{kh} (A^{p-2-q})_{ji} + (A^q)_{ij} (A^{p-2-q})_{hk} + (A^q)_{ih} (A^{p-2-q})_{jk} \right) \right) \times \left( \frac{p}{n} \sum_{q=0}^{p-2} \left( (A^q)_{mj} (A^{p-2-q})_{hl} + (A^q)_{mh} (A^{p-2-q})_{j} + (A^q)_{lj} (A^{p-2-q})_{hm} + (A^q)_{lh} (A^{p-2-q})_{jm} \right) \right).
\]

We have that

\[
B_{iklm} = \frac{2p^2}{n^2} \sum_{q_1,q_2=0}^{p-2} \left\{ (A^{Q_1})_{km} (A^{2p-4-Q_1})_{il} + (A^{Q_1+p-2-q_2})_{kl} (A^{q_2+p-2-q_1})_{im} + (A^{Q_1})_{kl} (A^{2p-4-Q_1})_{im} + (A^{Q_1+p-2-q_2})_{km} (A^{q_2+p-2-q_1})_{il} + (A^{Q_1})_{im} (A^{2p-4-Q_1})_{kl} + (A^{Q_1+p-2-q_2})_{il} (A^{q_2+p-2-q_1})_{km} + (A^{Q_1})_{il} (A^{2p-4-Q_1})_{km} \right\}
\]

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\[
\begin{align*}
&= \frac{16 p^2}{n^2} \sum_{q_1, q_2=0}^{p-2} (A^{Q_1})_{k_m} (A^{2p-4-Q_1})_{i_l} \\
&= \frac{16 p^2}{n^2} \sum_{Q_1=0}^{2p-4} (Q_1 + 1) (A^{Q_1})_{k_m} (A^{2p-4-Q_1})_{i_l},
\end{align*}
\]

where \( Q_1 = q_1 + q_2 \). Hence

\[
B_{iklm}^2 = \frac{2^8 p^4}{n^4} \sum_{Q_1, Q_2=0}^{2p-4} (Q_1 + 1) (Q_2 + 1) \times \\
\times (A^{Q_1})_{k_m} (A^{2p-4-Q_1})_{i_l} (A^{Q_2})_{k_m} (A^{2p-4-Q_2})_{i_l}.
\]

It follows that the first term of the product in (4.1) has the form

\[
\mathcal{A}_1(i, k, l, m) := E \left[ B_{iklm}^2 \right] \\
= \frac{2^8 p^4}{n^4} \sum_{Q_1, Q_2=0}^{2p-4} (Q_1 + 1) (Q_2 + 1) \times \\
\times E \left[ (A^{Q_1})_{k_m} (A^{2p-4-Q_1})_{i_l} (A^{Q_2})_{k_m} (A^{2p-4-Q_2})_{i_l} \right].
\]

Hence, considering the fact that \( \text{Var} G_n \rightarrow \frac{1}{\pi} \) as \( n \rightarrow \infty \) (see Theorem A.2), estimate (4.1) becomes

\[
d_{TV}(F_n, N)^2 \leq \pi^2 \sum_{i,k,l,m=1}^{n} \left( \mathcal{A}_1(i, k, l, m) \right)^{1/2} \left( \mathcal{A}_2(i, k, l, m) \right)^{1/2}, \tag{4.3}
\]

where

\[
\mathcal{A}_2(i, k, l, m) := \frac{p^4}{n^4} E \left[ (A^{p-1})_{i_k} (A^{p-1})_{i_m} (A^{p-1})_{i_k} (A^{p-1})_{i_m} \right]. \tag{4.4}
\]

**Notation 4.1.** Given \( p = p_n \) such that \( \lim_{n \to \infty} p_n = \infty \), and sequences \( \{A(p, n), C(p, n) : n \geq 1\} \) such that \( A(p, n) \) possibly depends on indices \( i, k, l, m, Q_1, Q_2 \), we will write

\[
A(p, n) = o(C(p, n))
\]

to indicate the relation

\[
\frac{A(p, n)}{C(p, n)} \leq \varepsilon_n,
\]

where \( \varepsilon_n \to 0 \) as \( n \to \infty \) and \( \varepsilon_n \) does not depend on \( i, k, l, m, Q_1, Q_2 \).

The (quite technical) proofs of the forthcoming Propositions 4.2 and 4.3 are presented in detail in Appendix A.
Proposition 4.2. For fixed $i, k, l, m$, we have that

$$
\mathcal{A}(i, k, l, m) \leq \frac{4^4 p^4}{n^4} \sum_{Q_1, Q_2 = 0}^{2p-4} (Q_1 + 1) (Q_2 + 1) \times
\times \left\{ \frac{e}{2 \sqrt{2 \pi p^3}} 1_{\{i = l\}} + \frac{e}{2 n \sqrt{2 \pi p^3}} 1_{\{i \neq l\}} \right\} 1_{\{Q_1 = Q_2 = 0\}} +
+ 2 \left[ \frac{e^2}{\sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i = l, k = m\}} + \frac{2 e^2}{n \sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i \neq l, k = m\}} +
+ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i \neq l, k \neq m\}} \right] 1_{\{Q_1, Q_2 \text{ even}, Q_1 \neq 0\}} +
+ \left[ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \right] 1_{\{Q_1, Q_2 \text{ odd}\}} \right\} (1 + o(1)),
$$

where $2Q = Q_1 + Q_2$ and $o(1)$ indicates a numerical sequence converging to zero, as $n \uparrow \infty$.

Proposition 4.3. For fixed $i, k, l, m$, we have that

$$
\mathcal{A}_2(i, k, l, m) \leq \frac{p^4 e^2}{n^2 \pi} \left\{ \frac{1}{n^2 p^3} 1_{\{i \neq k, l \neq m\}} + \frac{1}{n p^3} 1_{\{i = k, l \neq m\}} + \frac{1}{p^3} 1_{\{i = k, l = m\}} \right\} (1 + o(1)),
$$

(4.5)

where $o(1)$ indicates a numerical sequence converging to zero, as $n \uparrow \infty$.

Proof of Theorem 4.1 assuming Proposition 4.2 and Proposition 4.3 Simply plugging the results from Proposition 4.2 and Proposition 4.3 into (4.3), we have that

$$
d(F_n, N)^2 \leq \pi^2 \sum_{i, k, l, m = 1}^{n} \left( \frac{4^4 p^4}{n^4} \sum_{Q_1, Q_2 = 0}^{2p-4} (Q_1 + 1) (Q_2 + 1) \times
\times \left\{ \frac{e}{2 \sqrt{2 \pi p^3}} 1_{\{i = l\}} + \frac{e}{2 n \sqrt{2 \pi p^3}} 1_{\{i \neq l\}} \right\} 1_{\{Q_1 = Q_2 = 0\}} +
+ 2 \left[ \frac{e^2}{\sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i = l, k = m\}} + \frac{2 e^2}{n \sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i \neq l, k = m\}} +
+ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} 1_{\{i \neq l, k \neq m\}} \right] 1_{\{Q_1, Q_2 \text{ even}, Q_1 \neq 0\}} +
+ \left[ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \right] 1_{\{Q_1, Q_2 \text{ odd}\}} \right\} \right)^{1/2} \times
\times \left( \frac{p^4 e^2}{n^2 \pi} \left\{ \frac{1}{n^2 p^3} 1_{\{i \neq k, l \neq m\}} + \frac{1}{n p^3} 1_{\{i = k, l \neq m\}} + \frac{1}{p^3} 1_{\{i = k, l = m\}} \right\} \right)^{1/2} (1 + o(1))
$$

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\[
\begin{align*}
&\leq 16\pi^2 \sum_{i,k,l,m=1}^{n} \left( \frac{p^4}{n^4} \frac{e}{2n\sqrt{2\pi p^3}} \mathbb{1}_{\{i\neq l\}} + \frac{4p^6}{n^4} \sum_{Q_1,Q_2=1}^{2p-4} \frac{e^2}{n^2\sqrt{2\pi^2 p^3}} \mathbb{1}_{\{i\neq k,l\neq m\}} \right)^{1/2} \\
&\quad \times \left( \frac{p^4 e^2}{n^2\pi n^2 p^3} \mathbb{1}_{\{i\neq k,l\neq m\}} \right)^{1/2} (1+o(1)) \\
&\leq 16\pi^2 \left( \frac{e^{3/2}}{(2\pi)^{3/4}} \frac{p^{7/4}}{\sqrt{n}} + \frac{4e^2}{2^{1/4}\pi} \frac{p^{15/4}}{n} \right)^{(1+o(1))}.
\end{align*}
\]

Now, it is straightforward to see that the bound goes to zero if \( p = o(n^{4/15}) \). Moreover, if \( p = o(n^{1/4}) \), then
\[
\frac{e^{3/4}}{(2\pi)^{3/8}} \frac{p^{7/8}}{n^{1/4}} = o \left( \frac{2e}{2^{1/8}\pi} \frac{p^{15/8}}{\sqrt{n}} \right),
\]
so the theorem is established.

\section*{4.2 Infinite dimension: Non-linear functional of an isonormal Gaussian process}

In this section, we use our results in order to assess the distance in distribution between a general non-linear integral functional of a stationary Gaussian process and a Gaussian distribution. This example was already studied in \cite[NPR09, Section 6]{npr09}, where the authors stressed that their rate of convergence was presumably suboptimal. Very recently, in \cite{ft16}, the authors obtained a result which is close in spirit to Proposition 4.4 below, focussing in particular on sequences of the type
\[
F_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(X_i) - E[f(X_i)], \quad n \geq 1.
\]

Observe that the techniques developed in \cite{ft16} only apply to functions \( f \) such that
\[
\sum_{q=m}^{\infty} \frac{c_q^2}{q!} (2+\varepsilon)^{2q} < \infty, \quad \text{for some } \varepsilon > 0,
\]

where \( f(x) = \sum_{q=m}^{\infty} \frac{c_q}{q!} H_q(x) \) is the Hermite expansion of \( f \), and therefore \( m \in \mathbb{N} \) is the Hermite rank of the function \( f \). As the reader will see, the very nature of our approach allows us to ignore the Wiener chaos decomposition of \( F_n \) and obtain a QCLT for \( F_n \) with a rate of convergence that is presumably optimal.

Our starting point is the following general setting, which is flexible enough for many applications.

Let \((A, \mathcal{B}(A), \mu)\) be a measure space where \( A \subset \mathbb{R} (\mu(A) > 0) \). Let \( X = \{X(h) : h \in H\} \) be an isonormal Gaussian process over a real separable Hilbert space \( H \) and let \( \{h_a : a \in A\} \subset H \) be such that the scalar product \( \langle h_a, h_b \rangle = \varrho(a-b) \), with \( \varrho(0) = 1 \), only depends on the difference \( a - b \), \( \forall a, b \in A \), with
\[
\int_{\mathbb{R}} |\varrho(a)| \, d\mu(a) < \infty.
\]
We define \( \{X_a = X(h_a) : a \in A\} \) and assume that the mapping \((\omega, a) \mapsto X(h_a)(\omega)\) is jointly measurable.

Let \( f : \mathbb{R} \to \mathbb{R} \) be a real function of class \( C^2 \) such that \( E|f(N)| < \infty \) and \( E|f''(N)|^4 < \infty \), with \( N \sim \mathcal{N}(0, 1) \) (which implies \( E|f(N)|^2, E|f'(N)|^4 < \infty \), via the classical Poincaré inequality). We can define the functional \( F \) of \( (X(h_a))_{a \in A} \) as follows

\[
F = \frac{1}{\sqrt{\mu(A)}} \int_A f(X(h_a)) - E[f(X(h_a))] \, d\mu(a).
\]

Since \( H \) is a real separable Hilbert space, we know that (see Section 1.2) there exists an isomorphism

\[
i : H \rightarrow L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx)
\]

\[
h_a \mapsto i(h_a) =: K_a
\]

and this obviously implies that

\[
F \overset{d}{=} \frac{1}{\sqrt{\mu(A)}} \int_A f(W(K_a)) - E[f(W(K_a))] \, d\mu(a),
\]

where \( W \) is an isonormal Gaussian process over \( L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx) \).

**Proposition 4.4.** Assume that

\[
|K_a(s)| \leq g(a - s),
\]

where \( g(t) \) is such that \( \int_\mathbb{R} g(t) \, d\mu(t) < \infty \). Then

\[
d_{TV}\left( \frac{F}{\sqrt{\text{Var} F}}, N \right) \leq \frac{C}{\sqrt{\mu(A)}},
\]

where \( N \sim \mathcal{N}(0, 1) \) and \( C \) is a constant that does not depend on \( \mu(A) \).

**Proof.** By definition of Malliavin derivatives with respect to \( W \) and thanks to the stochastic Fubini theorem (see [Ver12]), we have that

\[
DF = \frac{1}{\sqrt{\mu(A)}} \int_A f'(W_a) \, K_a(x) \, d\mu(a)
\]

and

\[
D^2 F_T = \frac{1}{\sqrt{\mu(A)}} \int_A f''(W_a) \, K_a(x) \, K_a(y) \, d\mu(a),
\]

where \( W_a := W(K_a) \).

Now, Theorem 2.1 yields that

\[
d_{TV}\left( \frac{F}{\sqrt{\text{Var} F}}, N \right) \leq 4 \left( \int_{\mathbb{R}^2} \sqrt{E[(D^2 F \otimes_1 D^2 F)(x, y)]^2} \times \right.
\]

\[
\left. \sqrt{E[(DF(x)DF(y))^2]} \, dx \, dy \right)^{1/2},
\]

\[
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\]
and one has to assess the quantities on the right hand side of the previous inequality. We have
\[
E \left[ ((D^2 F \otimes_1 D^2 F) (x, y))^2 \right] =
\leq E \left[ \frac{1}{\mu(A)^2} \left( \int_{A^2} f''(W_a) f''(W_b) \varrho(a - b) i (h_a) (x) i (h_b) (y) d\mu(a) d\mu(b) \right)^2 \right]
\leq \frac{E |f''(N)|^4}{\mu(A)^2} \left( \int_{A^2} |\varrho(a - b) K_a(x) K_v(y)| d\mu(a) d\mu(b) \right)^2.
\]

and
\[
E \left[ (DF(x)DF(y))^2 \right] =
\leq \frac{E |f''(N)|^4}{\mu(A)^2} \left( \int_{A^2} |K_a(x) K_v(y)| d\mu(a) d\mu(b) \right)^2.
\]

Consequently, we obtain that
\[
d_{TV} \left( \frac{F}{\sqrt{\text{Var} F}}, N \right) \leq \frac{4 (E |f''(N)|^4 E |f'(N)|^4)^{1/4}}{\mu(A)} \times
\times \left\{ \int_{A^4} |\varrho(a - b)| \int_{\mathbb{R}_+} |K_a(x) K_v(x)| dx \int_{\mathbb{R}_+} |K_b(y) K_d(y)| dy d\mu(a) d\mu(b) d\mu(c) d\mu(d) \right\}^{1/2}
\leq \frac{c}{\mu(A)} \left\{ \int_{A^2} |\varrho(a - b)| \int_{\mathbb{R}} g(a - x) \left( \int_{\mathbb{R}} g(c - x) d\mu(c) \right) dx \times \int_{\mathbb{R}} g(b - y) \left( \int_{\mathbb{R}} g(d - y) dr \right) dy d\mu(a) d\mu(b) \right\}^{1/2}
\leq \frac{c}{\mu(A)} \left\{ \int_{\mathbb{R}} g(w) d\mu(w) \right\}^{1/2}
\leq \frac{c}{\mu(A)} \left\{ \int_{\mathbb{R}} g(w) d\mu(w) \right\}^{1/2}
\leq \frac{c}{\sqrt{\mu(A)}}
\]

where
\[
c = 4 \left( E |f''(N)|^4 E |f'(N)|^4 \right)^{1/4} \left( \int_{\mathbb{R}} g(w) d\mu(w) \right)^{1/2} \left\{ \int_{\mathbb{R}} |\varrho(x)| d\mu(x) \right\}^{1/2}
\]

which is the desired result.

\[\square\]

In the next two sections we will see how this result can be applied to more concrete situations.
4.2.1 Non-linear functional of a continuous stationary Gaussian process

We will use Proposition 4.4 in order to estimate the rate of convergence of a non-linear functional of a stationary Gaussian process towards a Gaussian distribution.

Let $Y$ denote a centered stationary Gaussian process such that
\[ \int_{\mathbb{R}} |\varphi(x)| \, dx < \infty, \]
where $\varphi(u - v) := E[Y_u Y_v]$. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function of class $C^2$ such that $E|f(N)| < \infty$ and $E|f''(N)|^4 < \infty$, and $N \sim \mathcal{N}(0, 1)$. Fix $a < b$ in $\mathbb{R}$ and, for any $T > 0$, consider the integral functional
\[ F_T = \frac{1}{\sqrt{(a - b)T}} \int_{aT}^{bT} (f(Y_u) - E[f(N)]) \, du. \]

Here we take as the isonormal Gaussian process $X = \{X(h) : h \in H\}$, where $H$ is defined as the closure of the set of all step functions on $\mathbb{R}$ with respect to the inner product
\[ \langle \mathbbm{1}_{[0,s]}, \mathbbm{1}_{[0,t]} \rangle_H = E[Y_s Y_t]. \]
In particular, in this way one has that $Y_t = X\left(\mathbbm{1}_{[0,t]}\right)$. Moreover, we know that there always exists an isomorphism
\[ i : H \to L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx), \]
which implies that $Y_t$ can be also written as
\[ Y_t = W(K_t), \]
where $K_t := i\left(\mathbbm{1}_{[0,t]}\right)$ and $W$ is an isonormal Gaussian process over $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$. So here $A = [aT, bT]$, $\mu$ is the Lebesgue measure and Proposition 4.4 becomes:

**Proposition 4.5.** Assume that
\[ |K_t(x)| \leq g(t - x), \]
where $g(y)$ is a measurable function such that $\int_{\mathbb{R}} g(y) \, dy < \infty$. Then
\[ d_{TV}\left(\frac{F_T}{\sqrt{\text{Var} F_T}}, N\right) \leq C \sqrt{T}, \]
where $N \sim \mathcal{N}(0, 1)$ and $C$ is a constant that does not depend on $T$.

Hence, as $T \to \infty$ we obtain a quantitative central limit theorem.

**Remark 4.3.** In [NPR09, Theorem 6.1], the authors obtain a suboptimal rate of convergence for $F_T$, that is
\[ d_W\left(\frac{F_T}{\sqrt{\text{Var} F_T}}, N\right) \leq \frac{C}{T^{1/4}}. \]
This was partly due to the fact that the operator norm of $D^2 F_T$ cannot be directly computed, so the authors had to move farther away from the distance in distribution and bound $\|D^2 F_T\|_{\text{op}}^2$ by $\|D^2 F_T \otimes_1 D^2 F_T\|_{H^{\otimes 2}}^2$. In [FT16, Proposition 4.7], the authors gain a result which is similar to the one of Proposition 4.5, however only for an $f$ satisfying the condition (4.6), and in particular heavily relying on the Wiener chaos decomposition of $F_T$.

**Remark 4.4.** Proposition 4.5 trivially holds for both the case of the increment of a standard Brownian motion $\{B_t : t \geq 0\}$, that is for

$$Y_t = B_{t+1} - B_t = W(1_{(t,t+1)}) ,$$

and the case of a centred Ornstein-Uhlenbeck process, namely when

$$Y_t = W(\sigma e^{-\theta(t-x)} 1_{[0,t)}(x)) ,$$

where $W$ is an isonormal Gaussian process over $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$. Moreover, it holds for the increment of a fractional Brownian with Hurst parameter $H < 1/2$ motion and since this last example is less trivial it will be explained in detail in the following section.

**The increments of a fractional Brownian motion** We will now show that Proposition 4.5 applies to the case where the process $\{Y_t\}_{t \geq 0}$ is defined as the increment of a fractional Brownian motion with Hurst parameter $H < 1/2$, that is

$$Y_t := B_{t+1}^H - B_t^H ,$$

where $\{B_t^H : t \geq 0\}$ is a centred Gaussian process with covariance function

$$E[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) .$$

The fractional Brownian motion $B_t^H$ has more than one representation in terms of kernels and we take the following one (see [Nou12, Section 2.3])

$$B_t^H = \frac{1}{c_H} W\left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} 1_{(-\infty,0)}(u) + (t-u)^{H-\frac{1}{2}} 1_{[0,t)}(u)\right) ,$$

where

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du < \infty} .$$

Hence the kernel of $B_t^H$ is given by

$$\hat{K}_t(u) = \frac{1}{c_H} \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} 1_{u \in (-\infty,0)} + (t-u)^{H-\frac{1}{2}} 1_{u \in [0,t)}\right) ,$$

so the absolute value of the one of $Y_t$ is

$$|K_t(u)| = |\hat{K}_{t+1}(u) - \hat{K}_t(u)| .$$
Now, the function \( g \) satisfies conditions of Proposition 4.5 and we have that the following
\[
\int (x + 1) H^{-1/2} 1_{\{x \in (-1,0)\}} - x H^{-1/2} 1_{\{x \in (0,\infty)\}} dx = \frac{1}{c_H} (g_1(x) + g_2(x))
\]
Now,
\[
\int g_1(x) dx = \int_{-1}^{0} (x + 1) H^{-1/2} dx = \frac{1}{H + \frac{1}{2}} < \infty
\]
and
\[
\int g_2(x) dx = \int_{0}^{\infty} (x + 1) H^{-1/2} - x H^{-1/2} dx.
\]
Now, the function \( g_2 \) is integrable around 0 and, for \( N \) large enough,
\[
\int_{N}^{\infty} (x + 1) H^{-1/2} - x H^{-1/2} dx = \int_{N}^{\infty} x H^{-1/2} \left( \frac{1 + 1/x}{1/x} - 1 \right) dx
\]
\[
= \int_{N}^{\infty} x H^{-1/2} \left( \frac{1 + 1/x}{1/x} - 1 \right) dx
\]
\[
\sim \int_{N}^{\infty} x H^{-1/2} \left( H - \frac{1}{2} \right) dx < \infty,
\]
for each \( H < 1/2 \). Thus we just proved that
\[
\int g(x) dx < \infty
\]
and so that the increment of a fractional Brownian motion with Hurst parameter \( H \in (0, \frac{1}{2}) \) satisfies conditions of Proposition 4.5 and we have that the following statement holds:
Corollary 4.6. Fix \( a < b \) in \( \mathbb{R} \) and, for any \( T > 0 \), consider the integral functional

\[
F_T = \frac{1}{\sqrt{(a - b)T}} \int_{aT}^{bT} \left( f \left( B_{u+1}^H - B_u^H \right) - E \left[ f \left( N \right) \right] \right) du,
\]

where \( B_t^H \) is a fractional Brownian motion with Hurst parameter \( H < 1/2 \). Then

\[
d_{TV} \left( \frac{F_T}{\sqrt{\text{Var} F_T}}, N \right) \leq C \sqrt{T},
\]

where \( N \sim \mathcal{N}(0, 1) \) and \( C \) is a constant that does not depend on \( T \).

Remark 4.5. Our result does not guarantee that \( \lim_{T \to \infty} \text{Var} F_T \) exists. A sufficient condition to have \( \lim_{T \to \infty} \text{Var} F_T \in (0, \infty) \) is that \( f \) is symmetric, see [NPR09, Proposition 6.3].

Remark 4.6. Note that when \( H = \frac{1}{2} \), \( B_t^H \) is a classical Brownian motion and Proposition 4.5 applies. While in the case when \( H > \frac{1}{2} \) our result does not apply.

4.2.2 The Breuer-Major Theorem

Let \( X = \{ X_k : k \in \mathbb{Z} \} \) be a centered stationary Gaussian sequence with unit variance. For all \( \nu \in \mathbb{Z} \), we set \( \rho(\nu) = E[X_0X_\nu] \) and we assume that

\[
\sum_{\nu = -\infty}^{\infty} |\rho(\nu)| < \infty.
\]

Let

\[
F_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f \left( X_k \right) - E \left[ f \left( X_k \right) \right],
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a real function of class \( C^2 \) such that \( E |f(N)| < \infty \) and \( E |f''(N)|^4 < \infty \) when \( N \sim \mathcal{N}(0, 1) \). Thanks to Proposition 7.2.3 from [NPR12], we can express \( F_n \) as a non-linear functional of some isonormal Gaussian process and this is necessary for us in order to apply our results. Indeed, we know that there exists a real separable Hilbert space \( H \), as well as an isonormal Gaussian process over \( H \), written \( \{X(h) : h \in H\} \), satisfying the following property: there exists a set \( \mathcal{E} = \{ \varepsilon_k : k \in \mathbb{Z} \} \subset H \) such that

(i) \( \mathcal{E} \) generates \( H \);

(ii) \( \langle \varepsilon_k, \varepsilon_l \rangle_H = \rho(k - l) \) for every \( k, l \in \mathbb{Z} \);

(iii) \( X_k = X(\varepsilon_k) \) for every \( k \in \mathbb{Z} \).

Hence

\[
F_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f \left( X(\varepsilon_k) \right) - E \left[ f \left( X_k \right) \right].
\]

Moreover, we know that there always exists an isomorphism

\[
i : H \to L^2 (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+) \), dx),
\]

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which implies that $X_k$ can be also written as

$$X_k = W(K_k),$$

where $K_k := i(\varepsilon_k)$ and $W$ is an isonormal Gaussian process over $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$. So here $A = \{1, 2, \ldots, n\}$, $\mu$ is the atomic measure given by $\mu(\cdot) = \sum_{k=1}^{n} \delta_k(\cdot)$ and Proposition 4.4 becomes as follows.

**Proposition 4.7.** Assume that

$$|K_k(x)| \leq g(k - x),$$

where $g$ is such that $\sum_{j=-\infty}^{+\infty} g(j) < \infty$. Then

$$d_{TV}(\frac{F_n}{\sqrt{\text{Var}(F_n)}}, N) \leq \frac{C}{\sqrt{n}},$$

where $N \sim \mathcal{N}(0, 1)$ and $C$ is a constant that does not depend on $n$.

Hence, as $n \to \infty$, we obtain a quantitative central limit theorem.

**Remark 4.7.** The assumptions of Proposition 4.7 trivially holds, as in the continuous case showed in the previous section (see Remark 4.4), for both the case of the increment of a Brownian motion, that is $X_k = B_{k+1} - B_k$, and the case of a discrete centred Ornstein-Uhlenbeck process, namely

$$X_k = \gamma X_{k-1} + \sigma (B_k - B_{k-1}),$$

where $\gamma \in (0, 1)$ and $\sigma \in \mathbb{R}_+$. Indeed, in the latter case, one has that

$$X_k \overset{d}{=} W(\sigma \gamma^{k-1} \lfloor x \rfloor \mathbf{1}_{[0,k)}(x)),$$

where $W$ is an isonormal Gaussian process over $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$ (see [Qin11]). Moreover, it holds for the increment of a fractional Brownian motion, that is $X_k = B_{k+1}^H - B_k^H$, and since the computations are analogous of the ones in the previous section we will not show them here.

We conclude our paper with a last example, involving multi-indexed Gaussian processes.

### 4.3 Infinite dimension: Non-linear functional of a Brownian sheet

As a final application, we use our bound in order to estimate the rate of convergence of a non-linear and positive functional of a Brownian sheet towards a standard Gaussian distribution. A particular instance of this application was firstly studied in [PY04] and then in [NP05], where the authors considered a quadratic functional and presented only qualitative central limit theorems. A first quantitative and exact
CLT, still just in the case of a quadratic functional, was then presented in [NP09], see Remark 4.9.

A Brownian sheet $W$ on $[0,1]^n$ is a centred Gaussian process

$$W = \{W(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in [0,1]^n\}$$

with covariance function

$$E[W(x_1, \ldots, x_n)W(y_1, \ldots, y_n)] = \prod_{i=1}^n (x_i \wedge y_i).$$

Note that the Gaussian space generated by $W$ can be identified with an isonormal Gaussian process over $L^2([0,1]^n, dx_1 \cdots dx_n)$.

Let

$$\hat{F}_\varepsilon := \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} (F_\varepsilon - E[F_\varepsilon]),$$

with

$$F_\varepsilon = \int_{[\varepsilon,1]^n} f \left( \frac{W(x_1, \ldots, x_n)}{\sqrt{x_1 \cdots x_n}} \right) \, d\mu(x_1, \ldots, x_n),$$

where $d\mu(x_1, \ldots, x_n) = \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$ and $f : \mathbb{R} \to \mathbb{R}_+$ is a positive function of class $C^2$ such that, for $N \sim \mathcal{N}(0,1)$, $E[f(N)^2] < \infty$ and $f$ admits the Hermite expansion $f(x) = \sum_{q=0}^{\infty} \frac{a_q}{q!} H_q(x)$.

Remark 4.8. First of all, note that

$$E[F_\varepsilon] = \int_{[\varepsilon,1]^n} E\left[ f \left( \frac{W(x_1, \ldots, x_n)}{\sqrt{x_1 \cdots x_n}} \right) \right] \, d\mu(x_1, \ldots, x_n)$$

$$= \int_{[\varepsilon,1]^n} E[f(N)] \, d\mu(x_1, \ldots, x_n)$$

$$= E[f(N)] \mu([\varepsilon,1]^n) = E[f(N)] \left( \log \frac{1}{\varepsilon} \right)^n \xrightarrow{\varepsilon \to 0} +\infty.$$

Therefore, a modification of Jeulin’s Lemma (see [Jeu80, Lemma 1], as well as [Pec01]) yields that $\lim_{\varepsilon \to 0} F_\varepsilon = +\infty$, a.s.-$\mathbb{P}$. In particular, note that the normalisation constant $(\log \frac{1}{\varepsilon})^{-n/2}$ is chosen in order for $\hat{F}_\varepsilon$ to have the variance converging towards a positive finite constant as $\varepsilon$ goes to zero. Indeed we have that

$$\text{Var}(F_\varepsilon) = E(F_\varepsilon^2) - [E(F_\varepsilon)]^2$$

$$= \int_{[\varepsilon,1]^{2n}} \text{Cov}\left( f\left( \hat{W}(K_x) \right), f\left( \hat{W}(K_y) \right) \right) \, d\mu(x) d\mu(y)$$

$$\xrightarrow{\varepsilon \to 0} \sum_{q=1}^{\infty} \frac{c_q^2}{q!} \left( \int_{[\varepsilon,1]^2} \frac{x \wedge y}{\sqrt{xy}}^q \, d\mu(x) d\mu(y) \right)^n = 2^n \left( \log \frac{1}{\varepsilon} \right)^n \sum_{q=1}^{\infty} \frac{c_q^2}{q!} \frac{2^n}{q^n},$$

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where $K_{(x_1, \ldots, x_n)}(u_1, \ldots, u_n) = \frac{1_{[0, x_1]}(u_1) \cdot \ldots \cdot 1_{[0, x_n]}(u_n)}{\sqrt{x_1 \cdot \ldots \cdot x_n}}$ and $\hat{W}$ is an isonormal Gaussian process over $L^2([0, 1]^n, dx_1 \cdot \ldots \cdot dx_n)$. We finally have to note that,

$$E \left[ f(N)^2 \right] < \infty \quad \Rightarrow \quad \sum_{q=1}^{\infty} \frac{e^{2q}}{q!} q^n < \infty.$$  

**Proposition 4.8.** Assume $E \left| f(N) \right|^2 < \infty$ and $E \left| f''(N) \right|^4 < \infty$, where $N \sim \mathcal{N}(0, E \left[ \hat{F}_{\varepsilon}^2 \right])$, then we have that

$$d_{TV} \left( \hat{F}_{\varepsilon}, N \right) \leq \frac{C_n}{(\log \frac{1}{\varepsilon})^{n/2}},$$

where $C_n$ is a constant that does not depend on $\varepsilon$.

**Proof.** We can write $\hat{F}_{\varepsilon}$ as follows

$$\hat{F}_{\varepsilon} = \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f \left( \frac{\hat{W}(x_1, \ldots, x_n)}{\sqrt{x_1 \cdot \ldots \cdot x_n}} \right) - E \left[ f(N) \right] d\mu(x_1, \ldots, x_n)$$

$$= \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f \left( \hat{W} \left( \frac{1_{[0, x_1]} \cdot \ldots \cdot 1_{[0, x_n]}}{\sqrt{x_1 \cdot \ldots \cdot x_n}} \right) \right) - E \left[ f(N) \right] d\mu(x_1, \ldots, x_n)$$

$$= \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f \left( \hat{W}(K_{(x_1, \ldots, x_n)}) \right) - E \left[ f(N) \right] d\mu(x_1, \ldots, x_n).$$

As a consequence, thanks to the stochastic Fubini theorem (see [Ver12]), we can compute

$$D \hat{F}_{\varepsilon}(t) = \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f' \left( \hat{W}(K_{(x_1, \ldots, x_n)}) \right) \times$$

$$\times K_{(x_1, \ldots, x_n)}(t_1, \ldots, t_n) d\mu(x_1, \ldots, x_n)$$

$$= \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f' \left( \hat{W}(K_x) \right) K_x(t) d\mu(x) \quad (4.9)$$

and

$$D^2 \hat{F}_{\varepsilon}(t, s) = \frac{1}{(\log \frac{1}{\varepsilon})^{n/2}} \int_{[\varepsilon, 1]^n} f'' \left( \hat{W}(K_{(x_1, \ldots, x_n)}) \right) K_{(x_1, \ldots, x_n)}(t_1, \ldots, t_n) \times$$

$$\times K_{(x_1, \ldots, x_n)}(s_1, \ldots, s_n) d\mu(x_1, \ldots, x_n). \quad (4.10)$$

So we have that

$$E \left[ \left( D \hat{F}_{\varepsilon}(t) D \hat{F}_{\varepsilon}(s) \right)^2 \right] =$$

$$= \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} E \left[ \left( \int_{[\varepsilon, 1]^n} f' \left( \hat{W}(K_x) \right) f' \left( \hat{W}(K_y) \right) K_x(t) K_y(s) d\mu(x) d\mu(y) \right)^2 \right]$$
Now, without loss of generality, consider the part of the space $[0, 1]^n$ in which $x_i \leq y_i, \forall i$:

\[
\left( D^2 \tilde{F}_\varepsilon \otimes_1 D^2 \tilde{F}_\varepsilon \right) (t, s) = \\
\frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int_{[\varepsilon, 1]^2n} f'' \left( \tilde{W} \left( K_{(x_1, \ldots, x_n)} \right) \right) f'' \left( \tilde{W} \left( K_{(y_1, \ldots, y_n)} \right) \right) \times \\
\times \left( \int_{[0, 1]^n} K_{(x_1, \ldots, x_n)}(u_1, \ldots, u_n)K_{(y_1, \ldots, y_n)}(u_1, \ldots, u_n) du_1 \cdots du_n \right) \\
\times K_{(x_1, \ldots, x_n)}(t_1, \ldots, t_n)K_{(y_1, \ldots, y_n)}(s_1, \ldots, s_n) \times \\
\times d\mu(x_1, \ldots, x_n)d\mu(y_1, \ldots, y_n) \\
= \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int_{[\varepsilon, 1]^2n} f'' \left( \tilde{W} \left( K_{(x_1, \ldots, x_n)} \right) \right) f'' \left( \tilde{W} \left( K_{(y_1, \ldots, y_n)} \right) \right) \times \\
\times E[ \tilde{W} \left( K_{(x_1, \ldots, x_n)} \right) ] \tilde{W} \left( K_{(y_1, \ldots, y_n)} \right) K_{(x_1, \ldots, x_n)}(t_1, \ldots, t_n) \times \\
\times K_{(y_1, \ldots, y_n)}(s_1, \ldots, s_n) d\mu(x_1, \ldots, x_n)d\mu(y_1, \ldots, y_n) \\
= \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int_{[\varepsilon, 1]^2n} f'' \left( \tilde{W} \left( K_x \right) \right) f'' \left( \tilde{W} \left( K_y \right) \right) \prod_{i=1}^{n} \frac{1}{x_i} \frac{1}{y_i^2} dx_i dy_i
\]

so that (note that since we are treating the case $x_i \leq y_i, \forall i$ this implies that $t_i \leq s_i, \forall i$)

\[
E \left[ \left( D^2 \tilde{F}_\varepsilon \otimes_1 D^2 \tilde{F}_\varepsilon \right) (t, s) \right]^2 = \\
= \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int_{[\varepsilon, 1]^2n} f'' \left( \tilde{W} \left( K_x \right) \right) f'' \left( \tilde{W} \left( K_y \right) \right) f'' \left( \tilde{W} \left( K_w \right) \right) f'' \left( \tilde{W} \left( K_z \right) \right) \times \\
\times \prod_{i=1}^{n} \frac{1}{x_i} \frac{1}{y_i^2} dx_i dy_i \prod_{i=1}^{n} \frac{1}{w_i} \frac{1}{z_i^2} dw_i dz_i
\]

\[
\leq \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int_{[\varepsilon, 1]^2} f''(N)^4 \prod_{i=1}^{n} \frac{1}{x_i} \frac{1}{y_i^2} dx_i dy_i^2 \\
= \frac{1}{(\log \frac{1}{\varepsilon})^{2n}} \int f''(N)^4 	imes
\]

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Remark where (1/cn) (log 1/ε)2n ≤ dKol( ̂Fε, N) ≤ Cn (log 1/ε)n/2,
where, again, cn and Cn are constants that do not depend on ε.

Remark 4.9. In [NP09, Proposition 5.2], the authors obtain an exact rate of convergence in Kolmogorov distance when f(x) = x², that is
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A Appendix

For the rest of this appendix, \(o(1)\) will indicate a generic numerical sequence \(\{\varepsilon_n, n \geq 1\}\), uniquely depending on \(n\) and \(p_n\), whose exact definition might change from one display to another, and such that \(\varepsilon_n \to 0\).

A.1 Proof of Proposition 4.3

The proof is divided into three parts.

**Part 1:** We first establish that

\[
\frac{1}{n^{p-1}} \sum_{(t,s)\in R^0} E \left[ Y_{it_1} \cdots Y_{ip-2k}Y_{ks_1} \cdots Y_{sp-2l} \right] = \\
\frac{e}{n\sqrt{\pi p^k}} (1 + o(1)) \mathbb{1}_{\{i\neq k\}} + \frac{e}{\sqrt{\pi p^k}} (1 + o(1)) \mathbb{1}_{\{i=k\}},
\]

where \(R^0(i,k)\) is some index-set that gives the dominant contribution to \((A.1)\) below. See Lemma \([A.3]\) for more details. This step is the most crucial and involved part of our proof of Proposition 4.3.

**Part 2:** We will then show that, \(\forall i, k\)

\[
E \left[ (A^{p-1})_{ik} \right]^2 = \frac{1}{n^{p-1}} \sum_{s_1,s_2=1, t_1=\ldots,t_{p-2}=1} E \left[ Y_{it_1} \cdots Y_{ip-2k}Y_{ks_1} \cdots Y_{sp-2l} \right],
\]

\[
= \frac{1 + o(1)}{n^{p-1}} \sum_{(t,s)\in R^0(i,k)} E \left[ Y_{it_1} \cdots Y_{ip-2k}Y_{ks_1} \cdots Y_{sp-2l} \right].
\]

See Lemma \([A.4]\) for more details. This step is the most crucial and involved part of our proof of Proposition 4.3.

**Part 3:** Finally, we will show that the expectation in \((4.4)\) satisfies

\[
E \left[ (A^{p-1})_{ik}^2 (A^{p-1})_{lm}^2 \right] = E \left[ (A^{p-1})_{ik}^2 \right] E \left[ (A^{p-1})_{lm}^2 \right] (1 + o(1)).
\]

See Lemma \([A.5]\) for a proof.
Combining the three parts, one finally obtains the result in Proposition 4.3 as follows:

\[
A_2(i, k, l, m) := \frac{4^4 p^4}{n^2} E \left[ \left( (A_{p-1})^2_{ik} \right) \left( (A_{p-1})^2_{lm} \right) \right] = \frac{4^4 p^4}{n^2} \left( 1 + o(1) \right) \tag{by (A.3)}
\]

\[
= \frac{1 + o(1)}{n^{p-1}} \sum_{(t,s) \in R^0(i,k)} E \left[ Y_{i t_1} \cdots Y_{p-2k} Y_{k s_1} \cdots Y_{p-2s} \right] \times \frac{1 + o(1)}{n^{p-1}} \sum_{(u,v) \in R^0(l,m)} E \left[ Y_{i u_1} \cdots Y_{u_{p-2m}} Y_{m v_1} \cdots Y_{v_{p-2v}} \right] \tag{by (A.2)}
\]

\[
= \left\{ \frac{e^2}{\pi} \left[ \frac{1}{n^2 p^3} + o \left( \frac{1}{n^2 p^3} \right) \right] 1_{\{i \neq k, l \neq m\}} + \frac{e^2}{\pi} \left[ \frac{1}{n^2 p^3} + o \left( \frac{1}{n^2 p^3} \right) \right] 1_{\{i = k, l \neq m\}} + \frac{e^2}{\pi} \left[ \frac{1}{p^3} + o \left( \frac{1}{p^3} \right) \right] 1_{\{i = k, l = m\}} \right\}.
\]

An important ingredient for our proof is a theorem by Sinai and Soshnikov, [SS98]. Before our proof, we recall necessary background around this result in the following subsection.

### Around the Sinai and Soshnikov Theorem
This section is based on [SS98, SS99].

#### Definition A.1.
- Let \( V_n := \{1, 2, \ldots, n\} \). A closed path \( P \) of length \( r \) is a vector of the type \((i_0, i_1, \ldots, i_{r-1}, i_0)\), where \( i_0, i_1, \ldots, i_{r-1} \in V_n \) (repetitions are allowed). We will sometimes represent \( P \) in the form
  \[
i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_{r-1} \rightarrow i_0,
\]
  thus identifying \( P \) with an oriented graph with edges
  \[
\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{r-1}, i_0\}.
\]
  The fact that repetitions are allowed within the components of \( P \) implies that such a graph might have subloops of arbitrary length.

- An unordered pair \( \{i, j\} \), where \( i, j \in \{1, 2, \ldots, n\} \), is called an edge of the path \( P \) if \( P \) has either the step \( i \rightarrow j \) or \( j \rightarrow i \).

- From now on, we assume \( r \) to be even, that is \( r = 2s \); indeed \( E(X_{i_0 i_1} X_{i_1 i_2} \cdots X_{i_{r-1} i_0}) \) is non-zero only if each edge in \( P \) appears exactly an even number of times and this can happen only if \( r \) is even. Paths verifying such a property are called even paths.
The $l^{th}$ step $i_{l-1} \to i_l$ of the path $P$ is called marked if the unordered edge \{ $i_{l-1}, i_l$ \} occurs an odd number of times up to the instant $l$ (inclusive). Note that the first step is always marked and that for even paths the number of marked steps equals the number of unmarked ones. Moreover, at each step of even paths, the number of marked steps is greater than or equal to the number of unmarked ones.

A path $P$ will be called simple even path if it contains exactly $\frac{r}{2} + 1$ indexes taking different values from the set of vertices $\{1, 2, \ldots, n\}$.

**Remark A.1.** One can prove that our definition of simple even path coincides with the one presented in [SS98] and with the one of path without self-intersections given in [SS99].

The following result, due to Ya. Sinai and Soshnikov [SS98], is fundamental for our counting arguments.

**Theorem A.1** ([SS98], Proposition 1). The main contribution to the number of all even paths of length $r = 2s$ on the set of $n$ vertices $\{1, \ldots, n\}$ where $r = o(n^{1/2})$ as $n \to \infty$ is given by simple even paths, i.e.

$$\frac{\#_{n,r}(\text{simple even paths})}{\#_{n,r}(\text{even paths})} \to 1.$$  

Moreover,

$$E(Tr A^r) = \frac{n}{\sqrt{\pi r^3}} (1 + o(1)).$$

**Theorem A.2** ([SS98], Theorem 2). Let $p = o(\sqrt{n})$. Then $\text{Var} Tr A^p \leq K$ for all $n$ and $\text{Var} Tr A^p \to \frac{1}{\pi}$ as $n \to \infty$, where $K$ is an absolute finite constant.

Part 1 of the proof

Define the following set

$$\mathcal{R}^0(i, k) = \{(t, s) \in V^{p-2} \times V^{p-2} : i \to t_1 \to \cdots \to t_{p-2} \to k \to s_1 \to \cdots \to s_{p-2} \to i$$

is a simple even path ,

then we have the following result.

**Lemma A.3.** For fixed $i, k$ we have that

$$\frac{1}{n^{p-1}} \sum_{(t, s) \in \mathcal{R}^0(i, k)} E[Y_{it_1} \cdots Y_{t_{p-2} k} Y_{ks_1} \cdots Y_{s_{p-2} i}] \leq$$

$$\leq \frac{e}{n \sqrt{\pi p^3}} (1 + o(1)) \mathbb{1}_{\{i \neq k\}} + \frac{e}{\sqrt{\pi p^3}} (1 + o(1)) \mathbb{1}_{\{i = k\}}.$$  

(A.4)

**Proof.** We have to consider two cases.

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\(i = k\): When \((t, s) \in R^0(i, i)\), the closed paths in \(R^0(i, i)\) take the following form

\[
i \rightarrow t_1 \rightarrow \cdots \rightarrow t_{p-2} \rightarrow i \rightarrow s_1 \rightarrow \cdots \rightarrow s_{p-2} \rightarrow i. \tag{A.5}
\]

In particular, when \(p\) is odd, the number of vertices, different from \(i\), that we can and have to choose in order for \((A.5)\) to be a simple even path is \(p - 1\). Therefore, when \(p\) is odd, the cardinality of \(R^0(i, i)\) is bounded by

\[
\frac{(n - 1)!}{(n - p)! (p - 1)!} \left( \frac{(p - 2)!}{p! (p - 1)!} \right), \tag{A.6}
\]

where the \((2p - 2)\)-th Catalan number counts the ways in which one can go in \((2p - 2)\) steps from \(i\) to \(i\) keeping the number of marked steps always above the number of unmarked ones and in such a way that the total numbers of marked and unmarked steps coincide (see [Fel68, Chapter III]).

Remark A.2. Note that the cardinality of \(R^0(i, i)\) is bounded by and not equal to \((A.6)\), as the Catalan number does not take into account the fact that the path \((A.5)\) has a fixed index in the middle.

Thus we have

\[
\frac{1}{n^{p-1}} \sum_{(t, s) \in R^0(i, i)} E[Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i}] \leq \frac{1}{n^{p-1}} \frac{(n - 1)!}{(n - p)! (p - 1)!} \left( \frac{(2p - 2)!}{p! (p - 1)!} \right)^{p-1} \left( \frac{1}{4} \right)^{p-1} \text{ Stirling normalization} \times \text{ variances}
\]

\[
\approx \frac{1}{n^{p-1}} \sqrt{\frac{2\pi n}{n - (p - 1)}} \frac{e^{-n}}{e^{-(p-1)}} \times \frac{\sqrt{4\pi (p - 1)}}{\sqrt{2\pi (p - 1)}} \frac{e^{-p} \sqrt{2\pi (p - 1)}}{e^{-p+1}} = \frac{1}{e^{p-1}} \left( \frac{n}{n - (p - 1)} \right)^{n-(p-1)} \left( \frac{e}{\sqrt{\pi p^3}} \right) = e^{-p+1} \sqrt{\pi p^3} (1 + o(1)).
\]

When \(p\) is even, \(R^0(i, i)\) is empty. Indeed the number of distinct indexes (different from \(i\)) that we can choose is at most \(p - 2\), and therefore it is not possible to have \(p\) distinct vertices in the path.

\(i \neq k\): A generic element \((t, s) \in R^0(i, k)\), independently of the fact that \(p\) is odd or even, is defined in terms of \(p - 2\) distinct indexes and hence the cardinality of \(R^0(i, k)\) is bounded by

\[
\frac{(n - 2)!}{(n - p)! (p - 2)!} \left( \frac{(2p - 2)!}{p! (p - 1)!} \right).
\]
and, with analogous computations as in the previous case, we have that

\[
\frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{R}^0(i,i)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] \leq \frac{1}{n^{p-1}} \frac{(n-2)!}{(n-p)!} \frac{(p-2)!}{p!} \left( \frac{1}{4} \right)^{p-1} e \sqrt{\pi} p^p (1 + o(1)).
\]

\[\square\]

**Part 2 of the proof**  Let

\[\mathcal{I}(i,k) := \{(t,s) \in V^{p-2} \times V^{p-2}: \text{the following path is even} \}
\]

\[i \to t_1 \to \cdots \to t_{p-2} \to k \to s_1 \to \cdots \to s_{p-2} \to i\]

then

\[
E \left[ (A^{p-1})_{ik} \right] = \frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{I}(i,k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] = \frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{I}(i,k) \setminus \mathcal{R}^0(i,k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] + \frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{R}^0(i,k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right].
\]

**Lemma A.4.** For fixed \(i, k, l, m\) we have that

\[
\frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{I}(i,k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] = 1 + o(1).
\]

**Proof.** Thanks to Lemma [A.3] when \(i = k\), we just have to show that

\[
\frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{I}(i,i) \setminus \mathcal{R}^0(i,i)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] = o \left( \frac{e}{\sqrt{\pi} p^p} \right) \quad (A.7)
\]

and, when \(i \neq k\), we just have to show that

\[
\frac{1}{n^{p-1}} \sum_{(t,s) \in \mathcal{I}(i,k) \setminus \mathcal{R}^0(i,k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] = o \left( \frac{e}{n \sqrt{\pi} p^p} \right) \quad (A.8)
\]

Let us prove first (A.7). For a given \((t,s) \in \mathcal{I}(i,i) \setminus \mathcal{R}^0(i,i)\), one has that in the path

\[i \to t_1 \to \cdots \to t_{p-2} \to i \to s_1 \to \cdots \to s_{p-2} \to i \quad (A.9)\]
either each edge appears exactly twice but it is not simple even, or at least one edge appears more than twice but an even number of times, which implies that the number of distinct indexes in \(A.9\), different from \(i\), is smaller than \(p - 1\) (it can be \(p - 2, p - 3\)...). As a consequence, the cardinality of \(\mathcal{I}(i, i) \setminus \mathcal{R}^0(i, i)\) is bounded by

\[
\sum_{j=1}^{p-2} \frac{(n - 1)!}{(n - 1 - j)! j!} \frac{(2p - 2)!}{p! (p - 1)!} \leq \frac{(2p - 2)!}{p! (p - 1)!} \sum_{j=1}^{p-2} \frac{(n - 1)!}{(n - 1 - j)!} \leq \frac{(2p - 2)!}{p! (p - 1)!} (n - 1 - (p - 2))!.
\]

Hence, using Stirling formula as before, we deduce the estimate

\[
\frac{1}{n^{p-1}} \sum_{(t, s) \in \mathcal{I}(i, i) \setminus \mathcal{R}^0(i, i)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] \leq \frac{1}{n^{p-1}} \frac{(p - 2)! (n - 1)!}{(n - p + 1)! p! (p - 1)!} \left( \frac{1}{4} \right)^{p-1} \text{variances}
\]

\[
= \frac{e \sqrt{p}}{n \sqrt{\pi}} (1 + o(1)) = o \left( \frac{e}{\sqrt{\pi p^3}} \right),
\]

and \(A.7\) is proved.

Let us now prove \(A.8\). For \((t, s) \in \mathcal{I}(i, k) \setminus \mathcal{R}^0(i, k)\), one has that in the path

\[
i \rightarrow t_1 \rightarrow \cdots t_{p-2} \rightarrow k \rightarrow s_1 \rightarrow \cdots \rightarrow s_{p-2} \rightarrow i
\]

(A.10) either each edge appears exactly twice but it is not simple even or at least one edge appears more than twice but an even number of times, which implies that the number of distinct indexes in \(A.10\), different from \(i\) and \(k\), is smaller than \(p - 2\) (it can be \(p - 3, p - 4\)...). It follows that the cardinality of \(\mathcal{I}(i, k) \setminus \mathcal{R}^0(i, k)\) is bounded by

\[
\sum_{j=1}^{p-3} \frac{(n - 2)! j!}{(n - 2 - j)! j! p! (p - 1)!} \frac{(2p - 2)!}{p! (p - 1)!} \sum_{j=1}^{p-3} \frac{(n - 2)!}{(n - 2 - j)!} \leq \frac{(2p - 2)! (p - 3) (n - 2)!}{p! (p - 1)! (n - p + 1)!}.
\]

Hence, with analogous computations, we have that

\[
\frac{1}{n^{p-1}} \sum_{(t, s) \in \mathcal{I}(i, k) \setminus \mathcal{R}^0(i, k)} E \left[ Y_{it_1} \cdots Y_{t_{p-2}k} Y_{ks_1} \cdots Y_{s_{p-2}i} \right] \leq \frac{1}{n^{p-1}} \frac{(p - 3)! (n - 2)!}{(n - p + 1)! p! (p - 1)!} \left( \frac{1}{4} \right)^{p-1} \text{variances}
\]

\[
= \frac{e \sqrt{p}}{n^2 \sqrt{\pi}} (1 + o(1)) = o \left( \frac{e}{n \sqrt{\pi p^3}} \right),
\]

and \(A.8\) is proved too. \(\square\)
Part 3 of the proof  Since

\[ E \left[ (A^{p-1})_{ik} (A^{p-1})_{lm} (A^{p-1})_{ik} (A^{p-1})_{lm} \right] = \]

\[ = \frac{1}{n^{2p-2}} \sum_{t_1, \ldots, t_{p-2}, u_1, \ldots, u_{p-2}, s_1, \ldots, s_{p-2}, v_1, \ldots, v_{p-2}} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right] \quad (A.11) \]

and the \(Y_{ij}\)'s are i.i.d. \(\mathcal{N}(0,1/4)\), when one of the random variables in \(A.11\) appears exactly an odd number of times, the expectation is zero in \(A.11\). Consider the following paths

\[ i \to t_1 \to \cdots \to t_{p-2} \to k \to s_1 \to \cdots \to s_{p-2} \to i \quad (A.12) \]

\[ l \to u_1 \to \cdots \to u_{p-2} \to m \to v_1 \to \cdots \to v_{p-2} \to l \quad (A.13) \]

and define

\[ \mathcal{C} = \{ (t, s, u, v) \in V^{p-2} \times V^{p-2} \times V^{p-2} \times V^{p-2} : \]

\[ \text{each edge in the juxtaposition of } (A.12) \text{ and } (A.13) \text{ appears exactly an even number of times} \}, \]

\[ \mathcal{C}_1 = \{ (t, s, u, v) \in \mathcal{C} : \text{at least one edge in } (A.12) \text{ appears in } (A.13) \} \]

and

\[ \mathcal{C}_2 = \{ (t, s, u, v) \in \mathcal{C} : \text{no edge in } (A.12) \text{ appears in } (A.13) \}. \]

Then, we have that

\[ E \left[ (A^{p-1})_{ik} (A^{p-1})_{lm} (A^{p-1})_{ik} (A^{p-1})_{lm} \right] = \]

\[ = \frac{1}{n^{2p-2}} \sum_{(t, s, u, v) \in \mathcal{C}} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right] \]

\[ = \frac{1}{n^{2p-2}} \sum_{(t, s, u, v) \in \mathcal{C}_1} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right] + \]

\[ + \frac{1}{n^{2p-2}} \sum_{(t, s, u, v) \in \mathcal{C}_2} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} \right] E \left[ Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right]. \]

Lemma A.5. For fixed \(i, k, l, m\) we have that

\[ \frac{1}{n^{2p-2}} \sum_{(t, s, u, v) \in \mathcal{C}} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right] = \]

\[ = \frac{1}{n^{2p-2}} \sum_{(t, s, u, v) \in \mathcal{C}_2} E \left[ Y_{it_1} \cdots Y_{ip-2k} Y_{ks_1} \cdots Y_{sp-2l} \right] \times \]

\[ \times E \left[ Y_{lt_1} \cdots Y_{up-2m} Y_{mv_1} \cdots Y_{vp-2l} \right] \left( 1 + o(1) \right). \]
Proof. From Part 1 and Part 2 we already know that
\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{C}_2} E[Y_{it_1} \cdots Y_{t_{p-k}}Y_{ks_1} \cdots Y_{sp_{-2i}}] E[Y_{lu_1} \cdots Y_{u_{p-2m}}Y_{mv_1} \cdots Y_{v_{p-2l}}] =
\] 
\[
= \frac{e^2}{\pi p^3 n^2} \{i \neq k\} \{l \neq m\} + 2 \frac{e^2}{\pi p^3 n} \{i \neq k\} \{l = m\} + \frac{e^2}{\pi p^3} \{i = k\} \{l = m\}.
\]

As a consequence, we just have to show that, when \(i \neq k\) and \(l \neq m\),
\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{C}_1} E[Y_{it_1} \cdots Y_{t_{p-k}}Y_{ks_1} \cdots Y_{sp_{-2i}}Y_{lu_1} \cdots Y_{u_{p-2m}}Y_{mv_1} \cdots Y_{v_{p-2l}}] =
\] 
\[
= o\left(\frac{e^2}{\pi p^3 n^2}\right);
\]
(A.14)

when \(i \neq k\) and \(l = m\),
\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{C}_1} E[Y_{it_1} \cdots Y_{t_{p-k}}Y_{ks_1} \cdots Y_{sp_{-2i}}Y_{lu_1} \cdots Y_{u_{p-2i}}Y_{lu_1} \cdots Y_{v_{p-2l}}] =
\] 
\[
= o\left(\frac{e^2}{\pi p^3 n}\right);
\]
(A.15)

when \(i = k\) and \(l = m\),
\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{C}_1} E[Y_{it_1} \cdots Y_{t_{p-k}}Y_{is_1} \cdots Y_{sp_{-2i}}Y_{lu_1} \cdots Y_{u_{p-2i}}Y_{lu_1} \cdots Y_{v_{p-2i}}] =
\] 
\[
= o\left(\frac{e^2}{\pi p^3}\right).
\]
(A.16)

Let us start by proving (A.14) and (A.15). When \((t,s,u,v) \in \mathcal{C}_1\) and either \(i \neq k\), \(l \neq m\) or \(i \neq k\), \(l = m\), the maximum number of different indexes that the paths (A.12) and (A.13) can have is 2\(p - 5\). Hence the cardinality of \(\mathcal{C}_1\) is bounded by
\[
\sum_{j=1}^{2p-5} \frac{(n-3)!}{(n-3-j)!j!} \frac{(4p-4)!}{2p-1!(2p-2)!} \leq \frac{(2p-5)(n-3)!}{(n-2p+2)!(2p-1)!} \frac{(4p-4)!}{(2p-2)!}.
\]

Therefore we have that
\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{C}_1} E[Y_{it_1} \cdots Y_{t_{p-k}}Y_{ks_1} \cdots Y_{sp_{-2i}}Y_{lu_1} \cdots Y_{u_{p-2m}}Y_{mv_1} \cdots Y_{v_{p-2l}}] \leq
\] 
\[
\leq \frac{1}{n^{2p-2}} \frac{(n-3)!}{(n-2p+2)!(2p-1)!} \frac{(4p-4)!}{(2p-2)!} \frac{1}{4} 2^{p-2}
\] 
\[
= \frac{e}{n^3 \sqrt{\pi p^3 n^2}} (1 + o(1)) = o\left(\frac{e^2}{\pi p^3 n^2}\right),
\]
and (A.14) and (A.15) are proved. Finally, let us prove (A.16). When \(i = k\), \(l = m\) and \((t,s,u,v) \in \mathcal{C}_1\), the maximum number of different indexes that the paths (A.12) and (A.13) can have is 2\(p - 3\). Hence the cardinality of \(\mathcal{C}_1\) is bounded by
\[
\sum_{j=1}^{2p-3} \frac{(n-2)!}{(n-2-j)!j!} \frac{(4p-4)!}{2p-1!(2p-2)!} \leq \frac{(2p-3)(n-2)!}{(n-2p+3)!(2p-1)!} \frac{(4p-4)!}{(2p-2)!}.
\]
We have therefore that

\[
\frac{1}{n^{2p-2}} \sum_{(t,s,u,v) \in \mathcal{E}_1} E \left[ Y_{it_1} \cdots Y_{i_{p-2}t} Y_{ks_1} \cdots Y_{s_{p-1}t} Y_{i_{p-2}m} Y_{ms_1} \cdots Y_{s_{Q-2}k} \right] \leq \\
\leq \frac{1}{n^{2p-2}} (2p - 3) \frac{(n - 2)!}{(n - 2p + 1)! (2p - 1)! (2p - 2)!} \frac{1}{4} 2^{2p-2} = \\
= \frac{e}{n \sqrt{\pi p^3}} \left(1 + o(1)\right) = o \left(\frac{e^2}{\pi p^3}\right),
\]

and (A.16) is proved. \(\square\)

### A.2 Proof of Proposition 4.2

The proof is divided into three parts.

**Part 1:** We first establish that,

\[
\frac{1}{n^{2p-4}} \sum_{(t,s) \in \mathcal{S}^{(0)}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{S}^{(0)}(i,l;Q_1,Q_2)} E \left[ Y_{it_1} \cdots Y_{i_{p-2}t} Y_{ms_1} \cdots Y_{s_{Q-2}k} Y_{i_{p-1}m} Y_{ms_1} \cdots Y_{s_{Q-2}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{i_{p-2}u} Y_{v_{s_1}} \cdots Y_{v_{s-2}k} \right] = \\
\leq \left\{ \begin{array}{cl}
\frac{e}{2 \sqrt{2 \pi p^3}} \mathbb{1}_{\{i = l\}} + \frac{e}{2 n \sqrt{2 \pi p^3}} \mathbb{1}_{\{i \neq l\}} & \mathbb{1}_{\{Q_1 = 0\}} + \\
+ 2 \left[ \frac{e^2}{\sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{i = l,k = m\}} + \frac{2 e^2}{n \sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{i \neq l,k = m\}} + \\
+ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{i \neq l,k \neq m\}} \right] \mathbb{1}_{\{Q_1, Q_2 \text{ even}, Q_1 \neq 0\}} + \\
+ \left[ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \right] \mathbb{1}_{\{Q_1, Q_2 \text{ odd}\}} \left(1 + o(1)\right), \end{array} \right\
\](A.17)

with \(2Q = Q_1 + Q_2\) and where \(\mathcal{S}^{(0)}(i,l;Q_1,Q_2)\) and \(\mathcal{S}^{(0)}(k,m;Q_1,Q_2)\) are some sets of indices that gives the dominant contribution to (A.18) below. See Lemma A.6 for a detailed proof.

**Part 2:** We will show that

\[
E \left[ (A^{Q_1})_{km} (A^{Q_2})_{mk} \right] E \left[ (A^{2p-4-Q_1})_{il} (A^{2p-4-Q_2})_{il} \right] = \\
= \frac{1}{n^{2p-4}} \sum_{t_1, \ldots, t_{Q-1}} \sum_{s_1, \ldots, s_{Q-1}} \sum_{u_1, \ldots, u_{p-2}} \sum_{v_1, \ldots, v_{p-2}} E \left[ Y_{it_1} \cdots Y_{i_{p-2}t} Y_{ms_1} \cdots Y_{s_{Q-2}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{i_{p-2}u} Y_{v_{s_1}} \cdots Y_{v_{s-2}k} \right] = \\
= \frac{e}{n \sqrt{\pi p^3}} \left(1 + o(1)\right) = o \left(\frac{e^2}{\pi p^3}\right),
\]

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Part 3: Finally, we will prove that

\[ E \left[ (A^{Q_1})_{km} (A^{2p-4-Q_1})_{il} (A^{Q_2})_{km} (A^{2p-4-Q_2})_{il} \right] = E \left[ (A^{Q_1})_{km} (A^{Q_2})_{mk} \right] E \left[ (A^{2p-4-Q_1})_{il} (A^{2p-4-Q_2})_{li} \right] (1 + o(1)). \]  

(A.20)

See Lemma A.8 for more details.

Combining the three parts, one obtains the result in Proposition 4.2 by the following relations:

\[ \mathcal{A}_1(i, k, l, m) := \]

\[ = \frac{4^4 p^4}{n^4} \sum_{Q_1, Q_2=0}^{2p-4} (Q_1 + 1) (Q_2 + 1) E \left[ (A^{Q_1})_{km} (A^{2p-4-Q_1})_{il} (A^{Q_2})_{km} (A^{2p-4-Q_2})_{il} \right] \]

\[ = \frac{4^4 p^4}{n^2} \sum_{Q_1, Q_2=0}^{2p-4} (Q_1 + 1) (Q_2 + 1) E \left[ (A^{Q_1})_{km} (A^{Q_2})_{mk} \right] \times \]

\[ \times E \left[ (A^{2p-4-Q_1})_{il} (A^{2p-4-Q_2})_{li} \right] (1 + o(1)) \quad \text{by (A.20)} \]

\[ \leq \frac{4^4 p^4 (2p-3)^2}{n^2} \sum_{Q_1, Q_2=0}^{2p-4} 1 + o(1) \sum_{(t,a) \in \mathcal{T}^0(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{T}^0(i,l;Q_1,Q_2)} E \left[ Y_{t_{Q_1-1} \ldots t_{Q_1-1}} Y_{u_{Q_2-1} \ldots u_{Q_2-1}} \right] \times \]

\[ \times E \left[ Y_{i_{Q_1-1} \ldots i_{Q_1-1}} \right] \quad \text{by (A.19)} \]

\[ \leq \frac{4^4 p^4 (2p-3)^2}{n^2} \sum_{Q_1, Q_2=0}^{2p-4} \left\{ \frac{e}{2 \sqrt{2 \pi p^3}} \mathbb{I}_{\{i=l\}} + \frac{e}{2 n \sqrt{2 \pi p^3}} \mathbb{I}_{\{i\neq l\}} \right\} \mathbb{I}_{\{Q_1=Q_2=0\}} + \]

\[ \mathbb{I}_{\{Q_1=0\}} + \mathbb{I}_{\{Q_2=0\}} \]
\[ + 2 \left[ \frac{e^2}{\sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{i=l,k=m\}} + \frac{2 e^2}{n \sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{i\neq l,k=m\}} + \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{l\neq k\neq m\}} \right] \mathbb{1}_{\{Q_1,Q_2 \text{ even}, Q_1 \neq 0\}} + \left[ \frac{e^2}{n^2 \sqrt{2 \pi^2 p^3 Q^3}} \mathbb{1}_{\{Q_1,Q_2 \text{ odd}\}} \right] (1 + o(1)) . \quad \text{[by (A.17)]} \]

**Part 1 of the proof** First of all, note that if \( Q_1 \) and \( Q_2 \) have different parity, then

\[ E \left[ (A_{Q_1})_{km} (A_{Q_2})_{mk} \right] = E \left[ (A_{2^{p-4}Q_1})_d (A_{2^{p-4}Q_2})_l \right] = 0. \]

This happens also when \( k \neq m \) and either \( Q_1 \) or \( Q_2 \) is equal to zero. Therefore we have to exclude these cases from our study. In addition, if \( Q_1 = Q_2 = 0 \), then \( \forall i, l, k, m \)

\[ E \left[ (A_{Q_1})_{km} (A_{Q_2})_{mk} \right] E \left[ (A_{2^{p-4}Q_1})_d (A_{2^{p-4}Q_2})_l \right] = E \left[ (A_{2^{p-4}})_d (A_{2^{p-4}})_l \right]. \]

From now on, unless otherwise specified, we will assume that \( Q_1 \) and \( Q_2 \) have the same parity. For fixed \( i, k, l, m, q_1, q_2, q_3, q_4 \) we introduce the following sets

\[ \mathcal{S}^0(i, l; q_1, q_2, q_3, q_4) := \{ (u, v) \in V_{2^{p-5}Q_1} \times V_{2^{p-5}Q_2} : \text{the path} \]

\[ i \rightarrow u_1 \rightarrow \cdots \rightarrow u_{2^{p-4}Q_1-1} \rightarrow l \rightarrow v_1 \rightarrow \cdots \rightarrow v_{2^{p-4}Q_2-1} \rightarrow i \text{ is simple even} \}

and

\[ \mathcal{P}^0(k; m; q_1, q_2, q_3, q_4) := \{ (t, s) \in V_{Q_1-1} \times V_{Q_2-1} : \text{the path} \]

\[ k \rightarrow t_1 \rightarrow \cdots \rightarrow t_{Q_1-1} \rightarrow m \rightarrow s_1 \rightarrow \cdots \rightarrow s_{Q_2-1} \rightarrow k \text{ is simple even} \}. \]

Note that when either \( Q_1 = Q_2 = 0 \) or \( Q_1 = 0 \) and \( k \neq m \) or \( Q_2 = 0 \) and \( k \neq m \) the set \( \mathcal{P}^0(k; m; q_1, q_2, q_3, q_4) \) is empty.

**Lemma A.6.** For fixed \( i, k, l, m \) and \( q_1, q_2, q_3, q_4 \) we have that

\[ \frac{1}{n^{2p-4}} \sum_{(t, s) \in \mathcal{P}^0(k; m; q_1, q_2)} \sum_{(u, v) \in \mathcal{S}^0(i, l; q_1, q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q_1-1}m} Y_{ms_1} \cdots Y_{s_{Q_2-1}k} \right] \times \]

\[ \times E \left[ Y_{iu_1} \cdots Y_{u_{2^{p-5}Q_1-1}l} Y_{lv_1} \cdots Y_{v_{2^{p-5}Q_2-1}i} \right] \leq \]

\[ \leq \left\{ \frac{e}{2 \sqrt{2 \pi p^3}} \mathbb{1}_{\{i=l\}} + \frac{e}{2 n \sqrt{2 \pi p^3}} \mathbb{1}_{\{i\neq l\}} \right\} \mathbb{1}_{\{Q_1=Q_2=0\}} + \]
\[
+ 2 \left[ \frac{e^2}{\sqrt{2 \pi p^3} Q_1^2} \mathbb{1}_{\{i = l, k = m\}} + \frac{2 e^2}{n \sqrt{2 \pi p^3} Q_2^2} \mathbb{1}_{\{i \neq l, k = m\}} + \frac{e^2}{n^2 \sqrt{2 \pi p^3} Q_1^2} \mathbb{1}_{\{i \neq l, k \neq m\}} \right] \mathbb{1}_{\{Q_1, Q_2 \text{ even}, Q_3 \neq 0\}} + \left[ \frac{e^2}{n^2 \sqrt{2 \pi p^3} Q_2^2} \mathbb{1}_{\{Q_1, Q_2 \text{ odd}\}} \right] \left( 1 + o(1) \right),
\]

with \(2Q = Q_1 + Q_2\).

Proof. For the proof of this lemma we have to consider different cases.

- \(Q_1 = Q_2 = 0\):

  \(i = l\): The cardinality of \(\mathcal{X}^0(i, i; 0, 0)\) is bounded by

  \[
  \frac{(n - 1)!}{(n - 2p + 3)! (2p - 4)!} \frac{(2p - 4)!}{(2p - 3)! (2p - 4)!} \frac{(4p - 8)!}{(2p - 3)! (2p - 4)!} \cdot \frac{1}{4} 2^{p-4}
  \]

  \[
  \geq \frac{1}{n^{2p-4}} \sum_{(u, v) \in \mathcal{X}(i, i; 0, 0)} E \left[ Y_{u_1} \ldots Y_{u_{2p-5}} Y_{v_1} \ldots Y_{v_{2p-5}} \right] \leq \frac{1}{n^{2p-4}} \sum_{(u, v) \in \mathcal{X}(i, i; 0, 0)} E \left[ Y_{u_1} \ldots Y_{u_{2p-5}} Y_{v_1} \ldots Y_{v_{2p-5}} \right]
  \]

  \[
  \leq \frac{1}{2 \sqrt{2 \pi p^3}} \left( 1 + o(1) \right).
  \]

- \(i \neq l\): The cardinality of \(\mathcal{X}^0(i, l; 0, 0)\) is bounded by

  \[
  \frac{(n - 2)!}{(n - 2p + 3)! (2p - 5)!} \frac{(2p - 5)!}{(2p - 3)! (2p - 4)!} \frac{(4p - 8)!}{(2p - 3)! (2p - 4)!},
  \]

  while \(\# \mathcal{X}(k, m; 0, 0) = 1\). Hence we have that

  \[
  \frac{1}{n^{2p-4}} \sum_{(u, v) \in \mathcal{X}(i, i; 0, 0)} E \left[ Y_{u_1} \ldots Y_{u_{2p-5}} Y_{v_1} \ldots Y_{v_{2p-5}} \right] \leq \frac{1}{n^{2p-4}} \sum_{(u, v) \in \mathcal{X}(i, i; 0, 0)} E \left[ Y_{u_1} \ldots Y_{u_{2p-5}} Y_{v_1} \ldots Y_{v_{2p-5}} \right]
  \]

  \[
  \leq \frac{1}{2 \sqrt{2 \pi p^3}} \left( 1 + o(1) \right).
  \]

- \(Q_1\) and \(Q_2\) have the same parity and and at least one of the two is non-zero:
- Either $Q_1, Q_2$ are even and $i \neq l, k \neq m$ or $Q_1, Q_2$ are odd and any fixed $i, k, l, m$: Let

$$Q := \frac{Q_1 + Q_2}{2}.$$ 

The cardinality of $\mathcal{F}^0(k, m; Q_1, Q_2)$ is bounded by

$$\frac{(n - 2)!}{(n - Q + 1)! (Q - 1)!} \left( \frac{2Q}{(Q + 1)!} \right),$$

while the cardinality of $\mathcal{F}^0(i, l; Q_1, Q_2)$ is bounded by

$$\frac{(n - 2)!}{(n - 2 - (2p - 5 - Q))! (2p - 5 - Q)!} \left( \frac{4p - 8 - 2Q}{(2p - 3 - Q)! (2p - 4 - Q)!} \right).$$

Hence we have that

$$\frac{1}{n^{2p-4}} \sum_{(u,v) \in \mathcal{F}^0(i,l;Q_1,Q_2)} \sum_{(t,s) \in \mathcal{F}^0(k,m;Q_1,Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q1-1}m} Y_{ms_1} \cdots Y_{s_{Q2-1}k} \right] \times \times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{t_{v_1} \cdots Y_{v_{2p-5-Q_2}}} \right] \leq$$

$$\leq \frac{1}{n^{2p-4}} \frac{(n - 2)!}{(n - Q + 1)! (Q + 1)!} \times \times \frac{(n - 2)!}{(n - 2 - (2p - 5 - Q))! (2p - 5 - Q)!} \left( \frac{4p - 8 - 2Q}{(2p - 3 - Q)! (2p - 4 - Q)!} \right)^{2p-4}

= \frac{e^2}{n^2 \sqrt{2 \pi^2 p^2 Q^3}} (1 + o(1)).$$

- $Q_1, Q_2$ even and $i = l, k = m$: The cardinality of $\mathcal{F}^0(k, k; Q_1, Q_2)$ is bounded by

$$\frac{(n - 1)!}{(n - 1 - Q)! (Q)!} \left( \frac{2Q}{(Q + 1)!} \right),$$

while the cardinality of $\mathcal{F}^0(i, i; Q_1, Q_2)$ is given by

$$\frac{(n - 1)!}{(n - 1 - (2p - 4 - Q))! (2p - 4 - Q)!} \left( \frac{4p - 8 - 2Q}{(2p - 3 - Q)! (2p - 4 - Q)!} \right).$$

Hence we have that

$$\frac{1}{n^{2p-4}} \sum_{(u,v) \in \mathcal{F}^0(i,i;Q_1,Q_2)} \sum_{(t,s) \in \mathcal{F}^0(k,k;Q_1,Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q1-1}m} Y_{ms_1} \cdots Y_{s_{Q2-1}k} \right] \times \times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{t_{v_1} \cdots Y_{v_{2p-5-Q_2}}} \right] \leq$$

$$\leq \frac{1}{n^{2p-4}} \frac{(n - 1)!}{(n - 1 - Q)! (Q + 1)!} \left( \frac{2Q}{(Q + 1)!} \right)^{2p-4}.$$
\[
\frac{(n-1)!}{(n-2p+3+Q)!} \frac{(4p-8-2Q)!}{(2p-3-Q)! (2p-4-Q)!} \left( \frac{1}{4} \right)^{2p-4} \\
\times e^2 \sqrt{2 \pi^2 p^3 Q^3} (1 + o(1)).
\]

\(- Q_1, Q_2 \text{ even and either } i = l, k \neq m \text{ or } i \neq l, k = m: \) The cardinality of \( \mathcal{F}^0(k; m; Q_1, Q_2) \) is bounded by
\[
\frac{(n-3)!}{(n-3-(Q-1))! (Q-1)!} \frac{(2Q)!}{(Q+1)! Q!},
\]
while the cardinality of \( \mathcal{F}^0(i; i; Q_1, Q_2) \) is bounded by
\[
\frac{(n-3)! (2p-4-Q)!}{(n-3-(2p-4-Q))! (2p-4-Q)! (2p-3-Q)! (2p-4-Q)!}.
\]

Hence we have that
\[
\frac{1}{n^{2p-4}} \sum_{(u,v) \in \mathcal{F}^0(i; i; Q_1, Q_2)} E \left[ Y_{kt1} \cdots Y_{tQ_1-1m} Y_{mns1} \cdots Y_{sQ_2-1k} \right] \times
\times E \left[ Y_{in1} \cdots Y_{u2p-5-Q_1} Y_{v1} \cdots Y_{v2p-5-Q_2} \right] \leq
\leq \frac{1}{n^{2p-4}} \frac{(n-3)!}{(n-3-Q+2)! (Q+1)! Q!} \times
\times \frac{(n-3)!}{(n-2p+1+Q)! (2p-3-Q)! (2p-4-Q)!} \left( \frac{1}{4} \right)^{2p-4} \\
\times e^2 \sqrt{2 \pi^2 p^3 Q^3} (1 + o(1)).
\]

The case in which \( i \neq l \) and \( k = m \) can be treated analogously and obviously gives the same result.

\( \square \)

**Part 2 of the proof** For fixed \( i, k, l, m, Q_1, Q_2 \) define the following sets
\[
\mathcal{J}(i; l; Q_1, Q_2) := \\
:= \{ (u, v) \in V^{2p-5-Q_1} \times V^{2p-5-Q_2}: \text{ each edge in the path } \\
i \rightarrow u_1 \rightarrow \cdots \rightarrow u_{2p-4-Q_1-1} \rightarrow l \rightarrow v_1 \rightarrow \cdots \rightarrow v_{2p-4-Q_2-1} \rightarrow i \text{ appears exactly an even number of times } \}.
\]

\[
\mathcal{J}(i; l; Q_1, Q_2) := \\
:= \{ (u, v) \in V^{2p-5-Q_1} \times V^{2p-5-Q_2}: \text{ each edge in the path } \\
i \rightarrow u_1 \rightarrow \cdots \rightarrow u_{2p-4-Q_1-1} \rightarrow l \rightarrow v_1 \rightarrow \cdots \rightarrow v_{2p-4-Q_2-1} \rightarrow i \text{ appears exactly an even number of times } \}.
\]
Lemma A.7. For fixed

\[ i \to u_1 \to \cdots \to u_{2p-4-Q_1-1} \to l \to v_1 \to \cdots \to v_{2p-4-Q_2-1} \to i \]

appears exactly twice \}

and, just for non-zero \( Q_1 \) and \( Q_2 \),

\[
\mathcal{K}(k, m; Q_1, Q_2) := \\
:= \{ (t, s) \in V^{Q_1-1} \times V^{Q_2-1} : \text{each edge in the path} \\
k \to t_1 \to \cdots \to t_{Q_1-1} \to m \to s_1 \to \cdots \to s_{Q_2-1} \to k \\
\text{appears exactly an even number of times } \} ,
\]

\[
\mathcal{T}(k, m; Q_1, Q_2) := \\
:= \{ (t, s) \in V^{Q_1-1} \times V^{Q_2-1} : \text{each edge in the path} \\
k \to t_1 \to \cdots \to t_{Q_1-1} \to m \to s_1 \to \cdots \to s_{Q_2-1} \to k \\
\text{appears exactly twice } \} ,
\]

One has that

\[
E \left[ \left( A^{Q_1} \right)_{km} \left( A^{Q_2} \right)_{mk} \right] E \left[ \left( A^{2p-4-Q_1} \right)_{il} \left( A^{2p-4-Q_2} \right)_{li} \right] = \\
= \frac{1}{n^{2p-4}} \sum_{(t, s) \in \mathcal{K}(k, m; Q_1, Q_2)} \sum_{(u, v) \in \mathcal{T}(i, l; Q_1, Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q_1-1}m} Y_{m_{s_1}} \cdots Y_{s_{Q_2-1}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{l_{v_1}} \cdots Y_{v_{2p-5-Q_2}} \right].
\]

**Lemma A.7.** For fixed \( i, k, l, m \) and \( q_1, q_2, q_3, q_4 \) we have that

\[
\frac{1}{n^{2p-4}} \sum_{(t, s) \in \mathcal{K}(k, m; Q_1, Q_2)} \sum_{(u, v) \in \mathcal{T}(i, l; Q_1, Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q_1-1}m} Y_{m_{s_1}} \cdots Y_{s_{Q_2-1}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{l_{v_1}} \cdots Y_{v_{2p-5-Q_2}} \right] = \\
= \frac{1}{n^{2p-4}} \sum_{(t, s) \in \mathcal{J}(k, m; Q_1, Q_2)} \sum_{(u, v) \in \mathcal{J}(i, l; Q_1, Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q_1-1}m} Y_{m_{s_1}} \cdots Y_{s_{Q_2-1}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{l_{v_1}} \cdots Y_{v_{2p-5-Q_2}} \right] (1 + o(1))
\]

**Proof.** Since we have that

\[
\frac{1}{n^{2p-4}} \sum_{(t, s) \in \mathcal{K}(k, m; Q_1, Q_2)} \sum_{(u, v) \in \mathcal{T}(i, l; Q_1, Q_2)} E \left[ Y_{kt_1} \cdots Y_{t_{Q_1-1}m} Y_{m_{s_1}} \cdots Y_{s_{Q_2-1}k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{l_{v_1}} \cdots Y_{v_{2p-5-Q_2}} \right] =
\]

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\[
= \frac{1}{n^{2p-4}} \sum_{(t,s) \in \mathcal{K} \setminus \mathcal{T}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.21)
\]

\[
+ \frac{2}{n^{2p-4}} \sum_{(t,s) \in \mathcal{K} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.22)
\]

\[
+ \frac{2}{n^{2p-4}} \sum_{(t,s) \in \mathcal{K} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.23)
\]

\[
+ \frac{1}{n^{2p-4}} \sum_{(t,s) \in \mathcal{J} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.24)
\]

\[
+ \frac{2}{n^{2p-4}} \sum_{(t,s) \in \mathcal{J} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.25)
\]

\[
+ \frac{1}{n^{2p-4}} \sum_{(t,s) \in \mathcal{J} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] \quad (A.26)
\]

we have to show that all the sums \((A.21), (A.22), (A.23), (A.24), (A.25)\) are negligible with respect to \((A.26)\). By virtue of Lemma \(A.6\), it suffices to show that, for \(A.21\) one has

\[
\frac{1}{n^{2p-4}} \sum_{(t,s) \in \mathcal{J} \setminus \mathcal{F}(k,m;Q_1,Q_2)} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{P}(i,l;Q_1,Q_2)} \sum_{j=1}^{n} E \left[ Y_{kt_1} \cdots Y_{tQ_1-1m} Y_{ms_1} \cdots Y_{sQ_2-1k} \right] \times \\
\times E \left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}} \right] = \left\{ o \left( \frac{e}{2 \sqrt{2\pi p^3}} \right) 1_{\{i=l\}} + o \left( \frac{e}{2 \sqrt{2\pi p^3}} \right) 1_{\{i \neq l\}} \right\} 1_{\{Q_1=Q_2=0\}} +
\]

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\[
+ 2 \left[ 0 \left( \frac{e^2}{\sqrt{2 \pi^2} p^3 Q^3} \right) \mathbb{1}_{\{i=l,k=m\}} + o \left( \frac{2 e^2}{n \sqrt{2 \pi^2} p^3 Q^3} \right) \mathbb{1}_{\{i\neq l,k=m\}} + \\
+ o \left( \frac{e^2}{n^2 \sqrt{2 \pi^2} p^3 Q^3} \right) \mathbb{1}_{\{i\neq l,k\neq m\}} \right] \mathbb{1}_{\{Q_1, Q_2 \text{ even}, Q_1 \neq 0\}} + \\
+ \left[ o \left( \frac{e^2}{n^2 \sqrt{2 \pi^2} p^3 Q^3} \right) \right] \mathbb{1}_{\{Q_1, Q_2 \text{ odd}\}} \right],
\]

with \(2Q = Q_1 + Q_2\). Let us prove it just for representative cases, since all the others cases are similar. Take for instance \(i = l\) and \(Q_1 = Q_2 = 0\), then

\[
\frac{1}{n^{2p-4}} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{S}(i,i;0,0)} E \left[ Y_{i u_1} \cdots Y_{i u_{2p-5} v_1} \cdots Y_{i v_{2p-5} i} \right]
\]

When \((u,v) \in \mathcal{J} \setminus \mathcal{S}(i,i;0,0)\), in the path

\[
i \rightarrow u_1 \rightarrow \cdots \rightarrow u_{2p-5} \rightarrow i \rightarrow v_1 \rightarrow \cdots \rightarrow v_{2p-5} \rightarrow i
\]

at least one edge appears more than twice but an even number of times, so that the number of distinct indexes in \(\Box\), different from \(i\), is smaller than \(2p - 4\). Therefore, the cardinality of \(\mathcal{J} \setminus \mathcal{S}(i,i;0,0)\) is bounded by

\[
\sum_{j=1}^{2p-5} \frac{(n-1)!}{(n-1-j)!j!} \frac{(4p-8)!}{(2p-3)!(2p-4)!} \leq \frac{(2p-5)(n-1)!}{(n-1-(2p-5))!(2p-3)!(2p-4)!} \cdot
\]

In this case, we will have at most \(2p - 5\) variances and one fourth moment; hence

\[
\frac{1}{n^{2p-4}} \sum_{(u,v) \in \mathcal{J} \setminus \mathcal{S}(i,i;0,0)} E \left[ Y_{i u_1} \cdots Y_{i u_{2p-5} v_1} \cdots Y_{i v_{2p-5} i} \right] \leq \\
\leq \frac{1}{n^{2p-4}} \frac{(2p-5)(n-1)!}{(n-2p+4)!} \frac{(4p-8)!}{(2p-3)!(2p-4)!} \left( \frac{1}{4} \right)^{2p-5} \left( \frac{3}{16} \right)
\]

\[
= \frac{3e}{2n \sqrt{4 \pi p^3}} \left( 1 + o(1) \right) = o \left( \frac{e}{2 \sqrt{2 \pi p^3}} \right),
\]

which is the desired result. All the other cases can be easily proved similarly. \(\square\)

**Part 3 of the proof** Consider the following paths

\[
k \rightarrow t_1 \rightarrow \cdots \rightarrow t_{Q_1-1} \rightarrow m \rightarrow s_1 \rightarrow \cdots \rightarrow s_{Q_2-1} \rightarrow k
\]
\[ i \rightarrow u_1 \rightarrow \cdots \rightarrow u_{2p-4-Q_1} \rightarrow l \rightarrow v_1 \rightarrow \cdots \rightarrow v_{2p-4-Q_2-1} \rightarrow i \quad \text{(A.29)} \]

and define
\[
\mathcal{D} = \left\{ (t, s, u, v) \in V^{Q_1-1} \times V^{Q_2-1} \times V^{2p-5-Q_1} \times V^{2p-5-Q_2} : \right. \]
\[
\text{each edge in (A.28)-(A.29) appears exactly an even number of times} \left. \right\},
\]
\[
\mathcal{D}_1 = \left\{ (t, s, u, v) \in \mathcal{D} : \text{at least one edge in (A.28) appears in (A.29)} \right\},
\]
and
\[
\mathcal{D}_2 = \left\{ (t, s, u, v) \in \mathcal{D} : \text{no edge in (A.28) appears in (A.29)} \right\}.
\]

Then we have that
\[
E\left[ (A^{Q_1})_{km} (A^{2p-4-Q_1})_{il} (A^{Q_2})_{km} (A^{2p-4-Q_2})_{il} \right] = \\
= \frac{1}{n^{2p-4}} \sum_{(t,s,u,l) \in \mathcal{D}} E\left[ Y_{kt} Y_{tQ_{1-1}m} Y_{ms_1} \cdots \right. \\
\cdots Y_{sQ_{2-1}k} Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}l} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}l} \bigg] \\
= \frac{1}{n^{2p-2}} \sum_{(t,s,u,l) \in \mathcal{D}_1} E\left[ Y_{kt} Y_{tQ_{1-1}m} Y_{ms_1} \cdots \right. \\
\cdots Y_{sQ_{2-1}k} Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}l} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}l} \bigg] + \\
+ \frac{1}{n^{2p-2}} \sum_{(t,s,u,l) \in \mathcal{D}_2} E\left[ Y_{kt} Y_{tQ_{1-1}m} Y_{ms_1} \cdots Y_{sQ_{2-1}k} \right] \times \\
\times E\left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}l} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}l} \right].
\]

**Lemma A.8.** For fixed \( i, k, l, m \) and \( q_1, q_2, q_3, q_4 \) we have that
\[
\frac{1}{n^{2p-4}} \sum_{(t,s,u,l) \in \mathcal{D}} E\left[ Y_{kt} Y_{tQ_{1-1}m} Y_{ms_1} \cdots \right. \\
\cdots Y_{sQ_{2-1}k} Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}l} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}l} \bigg] \leq \\
\leq \frac{1}{n^{2p-2}} \sum_{(t,s,u,l) \in \mathcal{D}_2} E\left[ Y_{kt} Y_{tQ_{1-1}m} Y_{ms_1} \cdots Y_{sQ_{2-1}k} \right] \times \\
\times E\left[ Y_{iu_1} \cdots Y_{u_{2p-5-Q_1}l} Y_{lv_1} \cdots Y_{v_{2p-5-Q_2}l} \right] (1 + o(1)).
\]

**Proof.** The proof of the statement is analogous to the proof of Lemma A.5. \( \square \)

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