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A primitive derivation and logarithmic
differential forms of Coxeter arrangements

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Abstract

Let $W$ be a finite irreducible real reflection group, which is a Coxeter group. We explicitly construct a basis for the module of differential 1-forms with logarithmic poles along the Coxeter arrangement by using a primitive derivation. As a consequence, we extend the Hodge filtration, indexed by nonnegative integers, into a filtration indexed by all integers. This filtration coincides with the filtration by the order of poles. The results are translated into the derivation case.

1 Introduction and main results

Let $V$ be a Euclidean space of dimension $\ell$. Let $W$ be a finite irreducible reflection group (a Coxeter group) acting on $V$. The Coxeter arrangement $\mathcal{A} = \mathcal{A}(W)$ corresponding to $W$ is the set of reflecting hyperplanes. We use [5] as a general reference for arrangements. For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Their product $Q := \prod_{H \in \mathcal{A}} \alpha_H$, which lies in the symmetric algebra $S := \text{Sym}(V^*)$, is a defining polynomial for $\mathcal{A}$. Let $F := \mathcal{S}(0)$ be the quotient field of $\mathcal{S}$. Let $\Omega_S$ and $\Omega_F$ denote the $\mathcal{S}$-module of regular 1-forms on $V$ and the $F$-vector space of rational 1-forms on $V$ respectively. The action of $W$ on $V$ induces the canonical actions of $W$ on $V^*, S, F, \Omega_S$ and $\Omega_F$, which enable us to consider their $W$-invariant parts. Especially let $R = \mathcal{S}^W$ denote the invariant subring of $\mathcal{S}$.

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In [16], Ziegler introduced the $S$-module of logarithmic 1-forms with poles of order $m (m \in \mathbb{Z}_{\geq 0})$ along $\mathcal{A}$ by
\[
\Omega(\mathcal{A}, m) := \{ \omega \in \Omega_F \mid Q^m \omega$ and $(Q/\alpha_H)^m (d\alpha_H \wedge \omega)$ are both regular for all $H \in \mathcal{A} \}.
\]
Note $\Omega(\mathcal{A}, 0) = \Omega_{S}$. Define the total module of logarithmic 1-forms by
\[
\Omega(\mathcal{A}, \infty) := \bigcup_{m \geq 0} \Omega(\mathcal{A}, m).
\]
In this article we study the total module $\Omega(\mathcal{A}, \infty)$ of logarithmic 1-forms and its $W$-invariant part $\Omega(\mathcal{A}, \infty)^W$ by introducing a geometrically-defined filtration indexed by $\mathbb{Z}$.

Let $P_1, \ldots, P_\ell \in R$ be algebraically independent homogeneous polynomials with $\deg P_1 \leq \cdots \leq \deg P_\ell$, which are called basic invariants, such that $R = \mathbb{R}[P_1, \ldots, P_\ell]$ [3, V.5.3, Theorem 3]. Define the primitive derivation $D := \partial/\partial P_k : F \to F$. Let $T := \{ f \in R \mid Df = 0 \} = \mathbb{R}[P_1, P_2, \ldots, P_{\ell-1}]$. Consider the $T$-linear connection (covariant derivative)
\[
\nabla_D : \Omega_F \to \Omega_F
\]
characterized by $\nabla_D(f \omega) = (Df) \omega + f(\nabla_D \omega)$ ($f \in F, \omega \in \Omega_F$) and $\nabla_D(d\alpha) = 0$ ($\alpha \in V^*$).

In Section 2, using the primitive derivation $D$, we explicitly construct logarithmic 1-forms
\[
\omega^{(m)}_1, \omega^{(m)}_2, \ldots, \omega^{(m)}_{\ell}
\]
for each $m \in \mathbb{Z}$ satisfying $\nabla_D \omega^{(2k+1)}_j = \omega^{(2k-1)}_j$ ($k \in \mathbb{Z}, 1 \leq j \leq \ell$). The 1-forms $\omega^{(m)}_1, \omega^{(m)}_2, \ldots, \omega^{(m)}_{\ell}$ form a basis for the $S$-module $\Omega(\mathcal{A}, -m)$ when $m \leq 0$. Thus it is natural to define $\Omega(\mathcal{A}, -m)$ to be the $S$-module spanned by $\{ \omega^{(m)}_1, \omega^{(m)}_2, \ldots, \omega^{(m)}_{\ell} \}$ for all $m \in \mathbb{Z}$.

The following two main theorems will be proved in Section 2:

**Theorem 1.1**

1. The $R$-module $\Omega(\mathcal{A}, 2k - 1)^W$ is free with a basis $\mathcal{B}_{-k}$ for $k \in \mathbb{Z}$.
2. The $T$-module $\Omega(\mathcal{A}, 2k - 1)^W$ is free with a basis $\bigcup_{p \geq -k} \mathcal{B}_p$ for $k \in \mathbb{Z}$.
3. $\mathcal{B} := \bigcup_{k \in \mathbb{Z}} \mathcal{B}_k$ is a basis for $\Omega(\mathcal{A}, \infty)^W$ as a $T$-module.

**Theorem 1.2**

1. The $\nabla_D$ induces a $T$-linear automorphism $\nabla_D : \Omega(\mathcal{A}, \infty)^W \cong \Omega(\mathcal{A}, \infty)^W$.
2. Define $\mathcal{F}_0 := \bigoplus_{j=1}^\ell T(dP_j), \mathcal{F}_k := \nabla_D^{k} \mathcal{F}_0$ and $\mathcal{F}_{-k} := (\nabla_D^{-1})^{k} \mathcal{F}_0$ ($k > 0$). Then $\Omega(\mathcal{A}, \infty)^W = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k$.
3. $\Omega(\mathcal{A}, 2k - 1)^W = \mathcal{J}^{(-k)}$, where $\mathcal{J}^{(-k)} := \bigoplus_{p \geq -k} \mathcal{F}_p$ for $k \in \mathbb{Z}$.
Let us briefly discuss our results in connection with earlier researches. Let \( \text{Der}_F \) denote the \( F \)-vector space of \( \mathbb{R} \)-linear derivations of \( F \) to itself. It is dual to \( \Omega_F \). The inner product \( I : V \times V \to \mathbb{R} \) induces \( I^* : V^* \times V^* \to \mathbb{R} \), which is canonically extended to a nondegenerate \( F \)-bilinear form \( I^* : \Omega_F \times \Omega_F \to F \). Define an \( F \)-linear isomorphism

\[
I^* : \Omega_F \to \text{Der}_F
\]

by \( I^*(\omega)(f) := I^*(\omega, df) \) (\( f \in F \)). Let \( \mathcal{G}_k := I^*(\mathcal{F}_{k-1}) \) and \( \mathcal{H}^{(k)} := I^*(\mathcal{J}^{(k-1)}) \) for \( k \in \mathbb{Z} \). Thanks to Theorem 1.2, we have commutative diagrams

\[
\cdots \xrightarrow{\nabla_D} \mathcal{F}_1 \xrightarrow{\nabla_D} \mathcal{F}_0 \xrightarrow{\nabla_D} \mathcal{F}_{-1} \xrightarrow{\nabla_D} \mathcal{F}_{-2} \xrightarrow{\nabla_D} \mathcal{F}_{-3} \xrightarrow{\nabla_D} \mathcal{F}_{-4} \xrightarrow{\nabla_D} \cdots
\]

\[
\cdots \xrightarrow{\nabla_D} \mathcal{G}_2 \xrightarrow{\nabla_D} \mathcal{G}_1 \xrightarrow{\nabla_D} \mathcal{G}_0 \xrightarrow{\nabla_D} \mathcal{G}_{-1} \xrightarrow{\nabla_D} \mathcal{G}_{-2} \xrightarrow{\nabla_D} \mathcal{G}_{-3} \xrightarrow{\nabla_D} \cdots,
\]

\[
\cdots \xrightarrow{\nabla_D} \mathcal{J}(1) \xrightarrow{\nabla_D} \mathcal{J}(0) \xrightarrow{\nabla_D} \mathcal{J}(-1) \xrightarrow{\nabla_D} \mathcal{J}(-2) \xrightarrow{\nabla_D} \mathcal{J}(-3) \xrightarrow{\nabla_D} \mathcal{J}(-4) \xrightarrow{\nabla_D} \cdots
\]

\[
\cdots \xrightarrow{\nabla_D} \mathcal{H}(2) \xrightarrow{\nabla_D} \mathcal{H}(1) \xrightarrow{\nabla_D} \mathcal{H}(0) \xrightarrow{\nabla_D} \mathcal{H}(-1) \xrightarrow{\nabla_D} \mathcal{H}(-2) \xrightarrow{\nabla_D} \mathcal{H}(-3) \xrightarrow{\nabla_D} \cdots.
\]

in which every \( \nabla_D \) is a \( T \)-linear isomorphism. The objects in the left halves of the diagrams were introduced by K. Saito who called the decomposition \( \text{Der}_R = \bigoplus_{k \geq 0} \mathcal{G}_k \) the Hodge decomposition and the filtration \( \text{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \cdots \) the Hodge filtration in his groundbreaking work [7, 8]. They are the key to define the flat structure on the orbit space \( V/W \). The flat structure is also called the Frobenius manifold structure from the view point of topological field theory [4].

Our main theorems 1.1 and 1.2 are naturally translated by \( I^* \) into the corresponding results concerning the \( \mathcal{G}_k \)'s and the \( \mathcal{H}^{(k)} \)'s in Section 3. So we extend the Hodge decomposition and Hodge filtration, indexed by nonnegative integers, to the ones indexed by all integers. The Hodge filtration \( \text{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \cdots \) was proved to be equal to the contact-order filtration [13]. On the other hand, Theorem 1.2 (3) asserts that the filtration \( \cdots \supset \mathcal{J}^{(-1)} \supset \mathcal{J}^{(0)} = \Omega_R \), indexed by nonpositive integers, coincides with the pole-order filtration of the \( W \)-invariant part \( \Omega(\mathcal{A}, \infty)^W \) of the total module \( \Omega(\mathcal{A}, \infty) \) of logarithmic 1-forms. This direction of researches is related with a generalized multiplicity \( m : \mathcal{A} \to \mathbb{Z} \) and the associated logarithmic module \( D\Omega(\mathcal{A}, m) \) introduced in [1].

In Section 4, we will give explicit relations of our bases to the bases obtained in [11], [15] and [2].
2 Construction of a basis for $\Omega(\mathcal{A}, \infty)$

Let $x_1, \ldots, x_\ell$ denote a basis for $V^*$ and $P_1, \ldots, P_\ell$ homogeneous basic invariants with $\deg P_1 \leq \cdots \leq \deg P_\ell : S^W = R = \mathbb{R}[P_1, \ldots, P_\ell]$. Let $x := [x_1, \ldots, x_\ell]$ and $P := [P_1, \ldots, P_\ell]$ be the corresponding row vectors. Define $A := [I^*(x_i, x_j)]_{1 \leq i, j \leq \ell} \in \text{GL}_\ell(\mathbb{R})$ and $G := [I^*(dP_i, dP_j)]_{1 \leq i, j \leq \ell} \in M_{\ell, \ell}(R)$. Then $G = J(P)^T AJ(P)$, where $J(P) := \left[ \frac{\partial P_i}{\partial x_j} \right]_{1 \leq i, j \leq \ell}$ is the Jacobian matrix. It is well-known (e.g., [3, V.5.5, Prop. 6]) that $\det J(P) = Q$, where $\hat{=}$ stands for the equality up to a nonzero constant multiple. Let $\text{Der}_R$ be the $R$-module of $R$-linear derivations of $R$ to itself: $\text{Der}_R = \bigoplus_{i=1}^r R (\partial/\partial P_i)$.

Recall the primitive derivation $D = \partial/\partial P_\ell \in \text{Der}_R$ and $T = \ker(D : R \to R) = \mathbb{R}[P_1, \ldots, P_{\ell-1}]$. We will use the notation $D[M] := [D(m_{ij})]_{1 \leq i, j \leq \ell}$ for a matrix $M = [m_{ij}]_{1 \leq i, j \leq \ell} \in M_{\ell, \ell}(F)$. The next Proposition is due to K. Saito [7, (5.1)] [4, Corollary 4.1]:

**Proposition 2.1**

$D[G] \in \text{GL}_\ell(T)$, that is, $D^2[G] = 0$ and $\det D[G] \in \mathbb{R}^\times$.

Now let us give a key definition of this article, which generalizes the matrices introduced in [11, Lemma 3.3].

**Definition 2.2**

The matrices $B = B^{(1)}$ and $B^{(k)} (k \in \mathbb{Z})$ are defined by

$$B := J(P)^T AD[J(P)], \quad B^{(k)} := kB + (k - 1)B^T.$$ 

In particular, $D[G] = B + B^T = B^{(k+1)} - B^{(k)}$ for all $k \in \mathbb{Z}$.

**Lemma 2.3**

$B^{(k)} \in \text{GL}_\ell(T)$ for all $k \in \mathbb{Z}$, that is, $D \left[B^{(k)} \right] = 0$ and $\det B^{(k)} \in \mathbb{R}^\times$.

**Proof.** If $k \geq 1$, then the statement is proved in [11, 3.3 and 3.6] and [13, Lemma 2]. Suppose $k \leq 0$. Since

$$B^{(1-k)} = (1 - k)B + (-k)B^T = -(kB + (k - 1)B^T)^T = -(B^{(k)})^T,$$

we obtain $B^{(k)} = -(B^{(1-k)})^T \in \text{GL}_\ell(T)$ because $1 - k \geq 1$. \hfill $\square$

The following Lemma is in [11, pp. 670, Lemma 3.4 (iii)]:

**Lemma 2.4**

(1) $\det J(D^k[x]) = Q^{-2k}$, where $J(D^k[x]) := [\partial D^k(x_j)/\partial x_i]_{1 \leq i, j \leq \ell}$ $(k \geq 1)$.

(2) $D[J(P)] = -J(D[x])J(P)$ and thus $\det D[J(P)] = Q^{-1}$.
Definition 2.5
Define \( \{R_k\}_{k \in \mathbb{Z}} \subset M_{\ell, \ell}(F) \) by
\[
R_{1-2k} = D^k[J(P)] \quad (k \geq 0),
\]
\[
R_{2k-1} = (-1)^k J(D^k[x])^{-1} D[J(P)] \quad (k \geq 1),
\]
\[
R_{2k} = (-1)^k J(D^k[x])^{-1} \quad (k \geq 0),
\]
\[
R_{-2k} = D^{k+1}[J(P)] D[J(P)]^{-1} \quad (k \geq 0).
\]

In particular, \( R_1 = J(P), \) \( R_0 = I_\ell \) and \( R_{-1} = D[J(P)] \).

The following Proposition is fundamental.

Proposition 2.6
For \( k \in \mathbb{Z} \), we have
1. \( \det R_k = Q^k \),
2. \( R_{2k} = R_{2k-1} D[J(P)]^{-1} = R_{2k-1} B^{-1} J(P)^T A \),
3. \( R_{2k+1} = R_{2k} J(P)(B^{(k+1)})^{-1} B \),
4. \( R_{2k+1} = R_{2k-1} B^{-1} G(B^{(k+1)})^{-1} B \), and
5. \( D[R_{2k+1}] = R_{2k-1} \).

Proof. (2) is immediate from Definition 2.5 because \( B^{-1} J(P)^T A = D[J(P)]^{-1} \).

(4) Let \( k \geq 1 \). Recall the original definition of \( B^{(k)} \) in [11, Lemma 3.3] given by
\[
B^{(k+1)} = -J(P)^T A J(D^{k+1}[x]) J(D^k[x])^{-1} J(P).
\]

Compute
\[
R_{2k-1} R_{2k+1}^{-1} = -D[J(P)]^{-1} J(D^k[x]) J(D^{k+1}[x])^{-1} D[J(P)]
\]
\[
= -D[J(P)]^{-1} A^{-1} J(P)^T J(P)^T A J(P) J(P)^{-1}
\]
\[
J(D^k[x]) J(D^{k+1}[x])^{-1} A^{-1} J(P)^T J(P)^T A D[J(P)]
\]
\[
= B^{-1} G(B^{(k+1)})^{-1} B.
\]

Next we will show that
\[
D^{k+1}[J(P)] = D^k[J(P)] B^{-1} B^{(1-k)} G^{-1} B
\]
for \( k \geq 0 \) by an induction on \( k \). When \( k = 0 \) we have
\[
J(P) B^{-1} B^{(1)} G^{-1} B = J(P) J(P)^{-1} A^{-1} J(P)^{-T} J(P)^T A D[J(P)] = D[J(P)].
\]

Next assume \( k > 0 \). Compute
\[
D^{k+1}[J(P)] = D[D^k[J(P)]] = D[D^{k-1}[J(P)] B^{-1} B^{(2-k)} G^{-1} B]
\]
\[
= D^k[J(P)] B^{-1} B^{(2-k)} G^{-1} B + D^{k-1}[J(P)] B^{-1} B^{(2-k)} D[G^{-1}] B
\]
\[
= D^k[J(P)] B^{-1} \{B^{(2-k)} - D[G]\} G^{-1} B
\]
\[
= D^k[J(P)] B^{-1} B^{(1-k)} G^{-1} B,
\]
where, in the above, we used the induction hypothesis
\[ D^k[J(P)] = D^{k-1}[J(P)]B^{-1}B^{(2-k)}G^{-1}B, \]
a general formula
\[ D[G^{-1}] = -G^{-1}D[G]G^{-1} \]
and
\[ D[G] = B + B^T = B^{(2-k)} - B^{(1-k)}. \]
This implies \( R_{2k-1} = R_{2k+1}B^{-1}B^{(1-k)}G^{-1}B \) which proves (4).

(3) follows from (2) and (4) because \( G = J(P)^T AJ(P) \).

(1) Since \( \det B^{(k)} \in \mathbb{R}^\times \), \( \det J(D^k[x]) \equiv Q^{-2k} \) and \( \det D[J(P)] \equiv Q^{-1} \) by Lemma 2.3 and Lemma 2.4, (1) is proved.

(5) follows from the following computation:
\[
D[R_{2k+1}]B^{-1} = D[R_{2k+1}B^{-1}] = D[R_2B^{-1}G(B^{(k+1)})^{-1}]
\]
\[
= \{ D[R_2B^{-1}G + R_2B^{-1}D[G])(B^{(k+1)})^{-1} \}
\]
\[
= \{ R_2B^{-1}G + R_2B^{-1}(B^{(k+1)} - B^{(k)})\}B^{(k+1)}^{-1}
\]
\[
= R_{2k+1}B^{-1}
\]

Definition 2.7
For \( m \in \mathbb{Z} \) define \( \omega_1^{(m)}, \ldots, \omega_\ell^{(m)} \in \Omega_F \) by
\[
[\omega_1^{(m)}, \ldots, \omega_\ell^{(m)}] := [dx_1, \ldots, dx_\ell]R_m.
\]
When \( m = 2k + 1 \ (k \in \mathbb{Z}) \), let
\[
B_k := \{ \omega_1^{(2k+1)}, \ldots, \omega_\ell^{(2k+1)} \}.
\]

For example, \( \omega_j^{(1)} = dP_j \) for \( 1 \leq j \leq \ell \) and \( B_0 = \{ dP_1, \ldots, dP_\ell \} \) because
\[
[\omega_1^{(1)}, \ldots, \omega_\ell^{(1)}] = [dx_1, \ldots, dx_\ell]J(P) = [dP_1, \ldots, dP_\ell].
\]

Proposition 2.8
The subset
\[
\mathcal{B} := \bigcup_{k \in \mathbb{Z}} B_k = \{ \omega_j^{(2k+1)} \mid 1 \leq j \leq \ell, \ k \in \mathbb{Z} \}
\]
of \( \Omega_F \) is linearly independent over \( T \).
Proof. Assume
\[ \sum_{k \in \mathbb{Z}} [\omega^{(2k+1)}_1, \ldots, \omega^{(2k+1)}_\ell] g^{(2k+1)} = 0 \]
with \( g^{(2k+1)} = [g_1^{(2k+1)}, \ldots, g_\ell^{(2k+1)}]^T \in T^T, k \in \mathbb{Z} \) such that there exist integers \( d \) and \( e \) such that \( d \geq e, g^{(2d+1)} \neq 0, g^{(2e+1)} \neq 0 \) and \( g^{(2k+1)} = 0 \) for all \( k > d \) and \( k < e \). Then
\[ 0 = \sum_{k=e}^{d} [dx_1, \ldots, dx_\ell] R_{2k+1} g^{(2k+1)} \]
implies that
\[ 0 = \sum_{k=e}^{d} R_{2k+1} g^{(2k+1)}. \]
By Proposition 2.6 (4), there exist \((\ell \times \ell)\)-matrices \( H_{2k+1} \) \((e \leq k \leq d)\) such that
\[ R_{2k+1} = R_{2e+1} H_{2k+1} \]
and \( H_{2k+1} \) can be expressed as a product of \((k - e)\) copies of \( G \) and matrices belonging to \( \text{GL}_\ell(T) \). Since \( \det(R_{2e+1}) \neq 0 \) by Proposition 2.6 (1),
\[ 0 = \sum_{k=e}^{d} H_{2k+1} g^{(2k+1)}. \]
Note \( D^{d-e}[H_{2k+1}] = 0 \) \((k < d)\) by Proposition 2.1 and Lemma 2.3. Applying \( D^{d-e} \) to the above, we thus obtain
\[ D^{d-e}[H_{2d+1}] g^{(2d+1)} = 0. \]
Since the matrix \( D^{d-e}[H_{2d+1}] \), which is a product of \((d - e)\) copies of \( D[G] \) and matrices in \( \text{GL}_\ell(T) \), is nondegenerate, we get \( g^{(2d+1)} = 0 \), which is a contradiction. \( \square \)

Proposition 2.9
\[ \nabla_D \omega^{(2k+1)}_j = \omega^{(2k-1)}_j \quad (k \in \mathbb{Z}, 1 \leq j \leq \ell). \]

Proof. By Proposition 2.6 (5) we have
\[ \left[ \nabla_D \omega^{(2k+1)}_1, \ldots, \nabla_D \omega^{(2k+1)}_\ell \right] = [dx_1, \ldots, dx_\ell] D[R_{2k+1}] \]
\[ = [dx_1, \ldots, dx_\ell] R_{2k-1} = \left[ \omega^{(2k-1)}_1, \ldots, \omega^{(2k-1)}_\ell \right]. \quad \square \]
Recall

\[
\Omega(A, \infty) : = \bigcup_{m \geq 0} \Omega(A, m)
= \{ \omega \in \Omega_F \mid Q^m \omega \in \Omega_S \text{ for some } m > 0 \text{ and }
\quad d\alpha_H \wedge \omega \text{ is regular at generic points on } H
\quad \text{for each } H \in \mathcal{A} \}.
\]

Lemma 2.10
\[\nabla D(\Omega(A, m)^W) \subseteq \Omega(A, m+2)^W \text{ for } m > 0.\]

Proof. Choose \( H \in \mathcal{A} \) arbitrarily and fix it. Pick an orthonormal basis \( \alpha_H = x_1, x_2, \ldots, x_\ell \) for \( V^* \). Let \( s = s_H \in W \) be the orthogonal reflection through \( H \). Then \( s(x_1) = -x_1, s(x_i) = x_i \) (\( i \geq 2 \)), \( s(Q) = -Q \). Let

\[
\omega = \sum_{i=1}^{\ell} (f_i/Q^m)dx_i \in \Omega(A, m)^W
\]

with each \( f_i \in S \). Then

\[
\nabla D \omega = \sum_{i=1}^{\ell} D(f_i/Q^m)dx_i
\]

is \( W \)-invariant with poles of order \( m+2 \) at most. The 2-form

\[
(Q/x_1)^m dx_1 \wedge \omega = \sum_{i=2}^{\ell} (f_i/x_1^m)dx_1 \wedge dx_i
\]

is regular because \( \omega \in \Omega(A, m)^W \). Let \( i \geq 2 \). Then \( f_i \in x_1^m S \). This implies that \( g_i := Q^{m+2}D(f_i/Q^m) \in x_1^{m+1}S \). It is enough to show \( g_i \in x_1^{m+2}S \) because

\[
(Q/x_1)^{m+2} dx_1 \wedge \nabla D \omega = \sum_{i=2}^{\ell} (g_i/x_1^{m+2})dx_1 \wedge dx_i.
\]

When \( m \) is odd, we have \( s(g_i) = s(Q^{m+2}D(f_i/Q^m)) = -g_i \). Thus \( g_i \in x_1^{m+2}S \). When \( m \) is even, we have \( s(g_i) = s(Q^{m+2}D(f_i/Q^m)) = g_i \). Thus \( g_i \in x_1^{m+2}S \).

Lemma 2.11
\[\mathcal{B}_{-k} \subset \Omega(A, 2k-1)^W \text{ for } k \geq 1.\]
Proof. We will show by an induction on \( k \). Fix \( 1 \leq j \leq \ell \). Recall \( \omega_j^{(-1)} = \nabla_D dP_j \) by Proposition 2.9. Since \( dP_j \in \Omega(\mathcal{A},0)^W \), we have \( \nabla_D dP_j \in \Omega(\mathcal{A},2)^W \) by Lemma 2.10. On the other hand, \( \nabla_D dP_j \) has poles of order one at most because \( dP_j \) is regular. Thus \( \omega_j^{(-1)} \in \Omega(\mathcal{A},1)^W \). The induction proceeds by Proposition 2.9 and Lemma 2.10.

We extend the definition of \( \Omega(\mathcal{A},m) \) to the case when \( m \) is a negative integer:

\[
\Omega(\mathcal{A}, m) := \bigoplus_{j=1}^{\ell} S \omega_j^{(-m)} \quad (m < 0).
\]

**Theorem 2.12**

\( \Omega(\mathcal{A},m) \) is a free \( S \)-module with a basis \( \omega_1^{(-m)}, \omega_2^{(-m)}, \ldots, \omega_\ell^{(-m)} \) for \( m \in \mathbb{Z} \).

**Proof.** Case 1. When \( m < 0 \) this is nothing but the definition.

Case 2. Let \( m = 2k - 1 \) with \( k \geq 1 \). Recall \( \mathcal{B}_{-k} \subset \Omega(\mathcal{A},2k-1)^W \) from Lemma 2.10 and \( R_{1-2k} = Q^{1-2k} \) by Proposition 2.6 (1). Thus we have

\[
\omega_1^{(-2k+1)} \wedge \omega_2^{(-2k+1)} \wedge \cdots \wedge \omega_\ell^{(-2k+1)} = (\det R_{1-2k}) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell = Q^{1-2k}(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell).
\]

This shows that \( \mathcal{B}_{-k} \) is an \( S \)-basis for \( \Omega(\mathcal{A},2k-1) \) by Saito-Ziegler’s criterion [16, Theorem 11].

Case 3. Let \( m = 2k \) with \( k \geq 0 \). When \( k = 0 \), the assertion is obvious because \( \omega_j^{(0)} = dx_j \) and \( \Omega(\mathcal{A},0) = \Omega_S \). Let \( k \geq 1 \). By Proposition 2.6 (2) we have

\[
\begin{bmatrix}
\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}
\end{bmatrix} = [dx_1, \ldots, dx_\ell] R_{-2k} = [dx_1, \ldots, dx_\ell] R_{-2k-1} B^{-1} J(\mathbf{P})^T A = \begin{bmatrix}
\omega_1^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}
\end{bmatrix} B^{-1} J(\mathbf{P})^T A.
\]

This implies that \( \omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)} \) lie in \( \Omega(\mathcal{A},2k+1) \) by Lemma 2.11. By Proposition 2.6 (3) we have

\[
Q^{2k} R_{-2k} = Q^{2k-1} R_{-2k+1} B^{-1} B^{(-k+1)} Q J(\mathbf{P})^{-1}.
\]

Since both \( Q^{2k-1} R_{-2k+1} \) and \( Q J(\mathbf{P})^{-1} \) belong to \( M_{k,k}(S) \), so does \( Q^{2k} R_{-2k} \).

In other words, the differential forms \( \omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)} \) have poles of order at most \( 2k \) along \( \mathcal{A} \). Since it is easy to see that \( \Omega(\mathcal{A},2k) = \Omega(\mathcal{A},2k+1) \cap (1/Q^{2k}) \Omega_S \), we know that \( \omega_j^{(-2k)} \) belongs to \( \Omega(\mathcal{A},2k) \) for each \( j \). We can apply Saito-Ziegler’s criterion [16, Theorem 11] to conclude that \( \{\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}\} \)
is a basis for $\Omega(A, 2k)$ over $S$ because $\det R_{-2k} = Q^{-2k}$ by Proposition 2.6 (1).

We are now ready to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.**

(1) It is enough to show that $B_{-k}$ spans $\Omega(A, 2k - 1)^W$ over $R$. Express an arbitrary element $\omega \in \Omega(A, 2k - 1)^W$ as

$$\omega = \sum_{j=1}^{\ell} f_j \omega_j^{(-2k+1)}$$

with each $f_j \in S$. For any $s \in W$, get

$$0 = \omega - s(\omega) = \sum_{j=1}^{\ell} [f_j - s(f_j)] \omega_j^{(-2k+1)}.$$ 

Since $B_{-k}$ is linearly independent over $F$, we obtain $f_j \in S^W = R$.

(2) Let $d_j := \deg P_j$ and $m_j := d_j - 1$ for $1 \leq j \leq \ell$. Let $h := d_\ell$ denote the Coxeter number. Define the degree of a homogeneous rational 1-form by

$$\deg(\sum_{i=1}^{\ell} f_i \, dx_i) = d \iff f_i = 0 \text{ or } \deg f_i = d \quad (1 \leq i \leq \ell).$$

Then

$$\deg \omega_j^{(2k+1)} = m_j + kh.$$ 

Recall that $B$ is linearly independent over $T$ by Proposition 2.8. Let $M_{-k}$ denote the free $T$-module spanned by $\bigcup_{p \geq -k} B_p$. Recall that $\Omega(A, 2k - 1)^W$ is a free $R$-module with a basis $B_{-k}$ by (1). If $p \geq -k$, then $R_{2p+1} = R_{-2k+1}H$ with a certain matrix $H \in M_\ell(R)$ because of Proposition 2.6 (3). This implies that $M_{-k} \subseteq \Omega(A, 2k - 1)^W$. Use a Poincaré series argument to prove that they are equal:

$$\text{Poin}(M_{-k}, t) = (1 - t^{d_1})^{-1} \ldots (1 - t^{d_{\ell-1}})^{-1} \sum_{p \geq -k} (t^{m_1+ph} + \ldots + t^{m_{\ell}+ph})$$

$$= (1 - t^{d_1})^{-1} \ldots (1 - t^{d_{\ell}})^{-1} (t^{m_1-kh} + \ldots + t^{m_{\ell}-kh})$$

$$= \text{Poin}(\Omega(A, 2k - 1)^W, t).$$

Therefore $M_{-k} = \Omega(A, 2k - 1)^W$.

(3) Thanks to Proposition 2.8, it is enough to prove that $B$ spans $\Omega(A, \infty)^W$ over $T$. Let $\omega \in \Omega(A, \infty)$. Then $\omega \in \Omega(A, 2k - 1)^W$ for some $k \geq 1$. By
(2) and (3) we conclude that \( \omega \) is a linear combination of \( \bigcup_{p \geq -k} B_p \) with coefficients in \( T \). This shows that \( B \) spans \( \Omega(A, \infty) \) over \( T \). \qed

**Proof of Theorem 1.2 (1).** By Lemma 2.9,

\[
\nabla_D : \Omega(A, \infty)^W \rightarrow \Omega(A, \infty)^W
\]

induces a bijection \( \nabla_D : B \rightarrow B \). Apply Theorem 1.1 (3) to prove that \( \nabla_D \) is a \( T \)-isomorphism. \qed

Let \( \nabla^{-1}_D : \Omega(A, \infty) \rightarrow \Omega(A, \infty) \) denote the inverse \( T \)-isomorphism.

**Definition 2.13**

For \( k \in \mathbb{Z} \), define

\[
\mathcal{F}_0 := \bigoplus_{j=1}^{\ell} T (dP_j), \quad \mathcal{F}_k := \nabla^k_D(\mathcal{F}_0) \ (k > 0), \quad \mathcal{F}_{-k} := (\nabla^{-1}_D)^k(\mathcal{F}_0) \ (k > 0).
\]

Thus \( \nabla_D \) induces a \( T \)-isomorphism \( \nabla_D : \mathcal{F}_k \sim \mathcal{F}_{k-1} \) for each \( k \in \mathbb{Z} \). Since \( \nabla_D \) induces a bijection \( \nabla_D : B_k \rightarrow B_{k-1} \) by Lemma 2.9, each \( \mathcal{F}_k \) is a free \( T \)-module of rank \( \ell \) with a basis \( B_k = \{ \omega_j^{(2k+1)} \mid 1 \leq j \leq \ell \} \).

**Proof of Theorem 1.2 (2) and (3).**

(2) By Theorem 1.1 (3), \( B = \bigcup_{k \in \mathbb{Z}} B_k \) is a basis for \( \Omega(A, \infty)^W \) as a \( T \)-module. On the other hand, each \( \mathcal{F}_k \) has a basis \( B_k \) over \( T \) for each \( k \in \mathbb{Z} \).

(3) By Theorem 1.1 (2), \( \mathcal{J}^{(-k)} = \Omega(A, 2k - 1)^W \). \qed

**Example 2.14**

Let \( A \) be the \( B_2 \) type arrangement defined by \( Q = xy(x + y)(x - y) \) corresponding to the Coxeter group of type \( B_2 \). Then \( P_1 = (x^2 + y^2)/2, \ P_2 = (x^4 + y^4)/4 \) are basic invariants. Then \( T = \mathbb{R}[P_1] \) and \( R = \mathbb{R}[P_1, P_2] \). Let

\[
\omega = (x^4 + y^4)(\frac{dx}{x} + \frac{dy}{y}) \in \Omega(A, 1)^W.
\]

The unique decomposition of \( \omega \) corresponding to the decomposition \( \Omega(A, 1)^W = \mathcal{J}^{(-1)} = \mathcal{F}_{-1} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \ldots \) is explicitly given by:

\[
\omega = -8P_1^3\omega_1^{(-1)} + (8/3)P_1^2\omega_2^{(-1)} - 4P_1\omega_1^{(1)} + 2\omega_2^{(1)} \in \mathcal{F}_{-1} \oplus \mathcal{F}_0
\]

by an easy calculation.
Corollary 2.15
The $\nabla_{D} : \Omega(A, \infty)^{W} \to \Omega(A, \infty)^{W}$ induces an $T$-isomorphism

$$\nabla_{D} : \Omega(A, 2k - 1)^{W} = \mathcal{J}^{(-k)} \xrightarrow{\sim} \mathcal{J}^{(-k-1)} = \Omega(A, 2k + 1)^{W}. $$

Concerning the strictly increasing filtration

$$\ldots \Omega(A, 2k - 1) \subset \Omega(A, 2k) \subset \Omega(A, 2k + 1) \subset \ldots ,$$

the following Proposition asserts the $W$-invariant parts of $\Omega(A, 2k - 1)$ and $\Omega(A, 2k)$ are equal.

Proposition 2.16
$\Omega(A, 2k)^{W} = \Omega(A, 2k - 1)^{W} = \mathcal{J}^{(-k)}$ for $k \in \mathbb{Z}$. In particular, $\Omega_{R} = \Omega_{S}^{W} = \Omega(A, -1)^{W}$.

Proof. It is obvious that $\Omega(A, 2k - 1) \subseteq \Omega(A, 2k)$ because $R^{-2k+1} = R^{-2k}J(P)(B^{(1-k)})^{-1}B$ by Proposition 2.6 (3). Thus $\Omega(A, 2k - 1)^{W} \subseteq \Omega(A, 2k)^{W}$.

Let $\omega = \sum_{j=1}^{\ell} f_{j} \omega_{j}^{(-2k)} \in \Omega(A, 2k)^{W}$ with $f_{j} \in S$. Since

$$(\text{Eq})_{k} \quad \left[\omega_{1}^{(-2k)}, \ldots, \omega_{\ell}^{(-2k)}\right] = \left[\omega_{1}^{(-2k-1)}, \ldots, \omega_{\ell}^{(-2k-1)}\right] D[J(P)]^{-1}$$

by Proposition 2.6 (2), we may express

$$\omega = \sum_{j=1}^{\ell} f_{j} \omega_{j}^{(-2k)} = \sum_{j=1}^{\ell} f_{j} \left( \sum_{i=1}^{\ell} h_{ij} \omega_{i}^{(-2k-1)} \right) = \sum_{i=1}^{\ell} \left( \sum_{j=1}^{\ell} h_{ij} f_{j} \right) \omega_{i}^{(-2k-1)},$$

where $h_{ij}$ is the $(i, j)$-entry of $D[J(P)]^{-1}$. Note that $\omega \in \Omega(A, 2k + 1)^{W}$ and that $\Omega(A, 2k + 1)^{W}$ has a basis $\{\omega_{1}^{(-2k-1)}, \omega_{2}^{(-2k-1)}, \ldots, \omega_{\ell}^{(-2k-1)}\}$ over $R$.

Then we know that $\sum_{j=1}^{\ell} h_{ij} f_{j}$ is $W$-invariant for $1 \leq i \leq \ell$. Applying $(\text{Eq})_{0}$ we have

$$\omega' := \sum_{j=1}^{\ell} f_{j} dx_{j} = \sum_{j=1}^{\ell} f_{j} \omega_{j}^{(0)} = \sum_{j=1}^{\ell} f_{j} \sum_{i=1}^{\ell} h_{ij} \omega_{i}^{(-1)} = \sum_{i=1}^{\ell} \left( \sum_{j=1}^{\ell} h_{ij} f_{j} \right) \omega_{i}^{(-1)} \in \Omega_{S}^{W}.$$  

Recall $\Omega_{S}^{W} = \Omega_{R} = \bigoplus_{i=1}^{\ell} R(dP_{i})$ by [9]. Thus there exist $g_{i} \in R$ $(1 \leq i \leq \ell)$ such that

$$\omega' = \sum_{i=1}^{\ell} g_{i} dP_{i} = \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} g_{i} \left( \partial P_{i} / \partial x_{j} \right) \right) dx_{j}.$$
This implies
\[ f_j = \sum_{i=1}^{\ell} g_i \left( \partial P_i / \partial x_j \right) \quad (1 \leq i \leq \ell). \]

Since
\[ \left[ \omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)} \right] J(P) = \left[ \omega_1^{(-2k+1)}, \ldots, \omega_\ell^{(-2k+1)} \right] B^{-1} B^{(1-k)} \]

by Proposition 2.6 (3), one has
\[ \omega = \sum_{j=1}^{\ell} f_j \omega_j^{(-2k)} = \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} g_i \left( \partial P_i / \partial x_j \right) \right) \omega_j^{(-2k)} \]
\[ = \sum_{i=1}^{\ell} g_i \left( \sum_{j=1}^{\ell} \left( \partial P_i / \partial x_j \right) \omega_j^{(-2k)} \right) \in \bigoplus_{i=1}^{\ell} R \omega_i^{(-2k+1)} = \Omega(A, 2k - 1)^W. \]

This proves \( \Omega(A, 2k)^W \subseteq \Omega(A, 2k - 1)^W. \)

**3 The case of derivations**

Denote \( \partial / \partial x_i \) and \( \partial / \partial P_i \) simply by \( \partial_{x_i} \) and \( \partial_{P_i} \) respectively. Then
\[ \text{Der}_S = \bigoplus_{j=1}^{\ell} S \partial_{x_j}, \quad \text{Der}_R = \bigoplus_{j=1}^{\ell} R \partial_{P_j}, \quad \text{Der}_F = \bigoplus_{j=1}^{\ell} F \partial_{x_j}. \]

In this section we translate the results in the previous section by the \( F \)-isomorphism
\[ I^* : \Omega_F \rightarrow \text{Der}_F \]
defined by \( I^*(\omega)(f) = I^*(\omega, df) \) for \( f \in F \) and \( \omega \in \Omega_F \). Explicitly we can express
\[ I^* \left( \sum_{j=1}^{\ell} f_j \, dx_j \right) = \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} I^*(dx_i, dx_j) f_i \right) \partial_{x_j} \]
for \( f_j \in F \) \( (1 \leq j \leq \ell) \).

**Definition 3.1**

Define \( \eta_j^{(m)} := I^*(\omega_j^{(m)}) \) for \( m \in \mathbb{Z}, \ 1 \leq j \leq \ell. \)

Then
\[ \left[ \eta_1^{(m)}, \ldots, \eta_\ell^{(m)} \right] = [\partial_{x_1}, \ldots, \partial_{x_\ell}] AR_m. \]
In particular,
\[ [\eta_1^{(1)}, \ldots, \eta_\ell^{(1)}] = [\partial x_1, \ldots, \partial x_\ell] AJ(P) = [I^*(dP_1), \ldots, I^*(dP_\ell)], \]
\[ [\eta_1^{(-1)}, \ldots, \eta_\ell^{(-1)}] = [\partial x_1, \ldots, \partial x_\ell] AD[J(P)] = [\partial p_1, \ldots, \partial p_\ell] B. \]

**Definition 3.2**

Define
\[ D(A, m) := \{ \theta \in \text{Der}_S \mid \theta(H) \in S \cdot \alpha_H^m \text{ for all } H \in A \} \]
for \( m \geq 0 \) which is the \( S \)-module of logarithmic derivations along \( A \) of contact order \( m \). When \( m < 0 \) define
\[ D(A, m) := \bigoplus_{1 \leq j \leq \ell} S \eta_j^{(m)}. \]
Lastly define
\[ D(A, -\infty) := \bigcup_{m \in \mathbb{Z}} D(A, m). \]

**Theorem 3.3**

\( D(A, m) \) is a free \( S \)-module with a basis \( \eta_1^{(m)}, \eta_2^{(m)}, \ldots, \eta_\ell^{(m)} \) for \( m \in \mathbb{Z} \).

**Proof.** Case 1. When \( m < 0 \) this is nothing but the definition.

Case 2. Let \( m \geq 0 \). For a canonical contraction \( \langle \ , \ \rangle : \text{Der}_F \times \Omega_F \rightarrow F \), define the \((\ell \times \ell)\)-matrix
\[ Y_m := [\langle \omega_i^{(-m)}, \eta_j^{(m)} \rangle]_{1 \leq i,j \leq \ell} = R_{-m} AR_m \]
for \( m \geq 0 \). Since the two \( S \)-modules \( \Omega(A, m) \) and \( D(A, m) \) are dual each other (see [16]) , it is enough to show that \( \det Y_m \in \text{GL}_\ell(S) \). It follows from the following Proposition 3.6.

**Corollary 3.4**

\( I^*(\Omega(A, m)) = D(A, -m) \) for \( m \in \mathbb{Z} \) and \( I^*(\Omega(A, \infty)) = D(A, -\infty) \).

**Corollary 3.5**

\( \Omega(A, -m) = \{ \omega \in \Omega_S \mid I^*(\omega, d\alpha_H) \in S \cdot \alpha_H^m \text{ for any } H \in A \} \) for \( m > 0 \).

**Proposition 3.6**

1. \( Y_{2k-1} = (-1)^{k+1}B^T(B^{(k)})^{-1}B \in \text{GL}_\ell(T) \) for \( k \in \mathbb{Z} \),
2. \( Y_{2k} = (-1)^kA \in \text{GL}_\ell(\mathbb{R}) \) for \( k \in \mathbb{Z} \).
Proof.
(1) Case 1.1. Let \( m = 2k - 1 \) with \( k \geq 1 \). We prove by an induction on \( k \). When \( k = 1 \),
\[
Y_1 = R^T_1AR_1 = D[J(P)]^T AJ(P) = B^T \in \text{GL}_\ell(T).
\]
Assume that \( k > 1 \) and prove by induction. By using Proposition 2.6 (5) and (4), we obtain
\[
Y_{2k-1} = R^T_{1-2k}AR_{2k-1} = D[R_{3-2k}]^T AR_{2k-3} B^{-1} G(B^{(k)})^{-1} B
\]
\[
= \{D[R^T_{3-2k}AR_{2k-3}] - R^T_{3-2k} D[AR_{2k-3}]\} B^{-1} G(B^{(k)})^{-1} B
\]
\[
= -R^T_{3-2k}AR_{2k-5} B^{-1} G(B^{(k-1)})^{-1} BB^{-1} B^{(k-1)}(B^{(k)})^{-1} B
\]
\[
= -R^T_{3-2k}AR_{2k-3} B^{-1} (B^{(k-1)})^{-1} B
\]
\[
= (-1)^{k+1} B^T (B^{(k-1)})^{-1} BB^{-1} B^{(k-1)}(B^{(k)})^{-1} B
\]
\[
= (-1)^{k+1} B^T (B^{(k)})^{-1} B.
\]
Case 1.2. Next assume that \( m = 2k - 1 \) with \( k \leq 0 \). Recall that
\[
(B^{(1-k)})^T = -kB + (1-k)B^T = -B^{(k)}.
\]
Then
\[
R^T_{1-2k}AR_{2k-1} = (R^T_{2k-1}AR_{1-2k})^T = ((-1)^k B^T (B^{(1-k)})^{-1} B)^T
\]
\[
= (-1)^{k+1} B^T (B^{(k)})^{-1} B.
\]
(2) Apply (1), Proposition 2.6 (2) and (3) to compute
\[
R^T_{1-2k}AR_{2k} = J(P)^{-T} (B^{(1-k)})^T B^T R^T_{2k-1}AR_{2k-1} B^{-1} J(P)^T A
\]
\[
= J(P)^{-T} (B^{(1-k)})^T B^T Y_{2k-1} B^{-1} J(P)^T A = (-1)^k A. \quad \square
\]
Remark. Corollaries 3.4 and 3.5 show that the definitions of \( D(A, m) \) and \( \Omega(A, m) \) for \( m \in \mathbb{Z}_{<0} \) are equivalent to those of \( D\Omega(A, m) \) and \( \Omega D(A, m) \) in [1].

Consider the \( T \)-linear connection (covariant derivative)
\[
\nabla_D : \text{Der}_F \to \text{Der}_F
\]
characterized by \( \nabla_D(fX) = (Df)X + f(\nabla_D X) \) and \( \nabla_D(\partial_{e_j}) = 0 \) for \( f \in F \), \( X \in \text{Der}_F \) and \( 1 \leq j \leq \ell \). Then it is easy to see the diagram
\[
\begin{array}{ccc}
\Omega_F & \xrightarrow{\nabla_D} & \Omega_F \\
\downarrow & & \downarrow \\
\text{Der}_F & \xrightarrow{\nabla_D} & \text{Der}_F
\end{array}
\]
is commutative. In fact

\[
\nabla_D \circ I^* \left( \sum_{j=1}^{\ell} f_j \, dx_j \right) = \nabla_D \left[ \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} I^*(dx_i, dx_j) f_i \right) \, \partial x_j \right]
\]

\[
= \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} I^*(dx_i, dx_j) D(f_i) \right) \partial x_j
\]

\[
= I^* \left( \sum_{j=1}^{\ell} D(f_j) \, dx_j \right) = I^* \circ \nabla_D \left( \sum_{j=1}^{\ell} f_j \, dx_j \right).
\]

Define \( C_k := I^*(B_{k-1}) = \{ \eta_1^{(2k-1)}, \eta_2^{(2k-1)}, \ldots, \eta_\ell^{(2k-1)} \} \) for each \( k \in \mathbb{Z} \). The following Theorems 3.7 and 3.9 can be proved by translating Theorems 1.1 and 1.2 through \( \nabla_D \).

**Theorem 3.7**

1. The \( R \)-module \( D(A, 2k-1)W \) is free with a basis \( C_k \) for \( k \in \mathbb{Z} \).
2. The \( T \)-module \( D(A, 2k-1)W \) is free with a basis \( \bigcup_{p \geq k} C_p \) for \( k \in \mathbb{Z} \).
3. \( C := \bigcup_{k \in \mathbb{Z}} C_k \) is a basis for \( D(A, -\infty)W \) as a \( T \)-module.

**Definition 3.8**

Define

\[
G_k := I^*(\mathcal{F}_{k-1}), \quad \mathcal{H}^{(k)} := I^*(\mathcal{F}^{(k-1)}) \quad (k \in \mathbb{Z}, \ 1 \leq j \leq \ell).
\]

Then

\[
G_k = \bigoplus_{1 \leq j \leq \ell} T \eta_j^{(2k-1)}, \quad \mathcal{H}^{(k)} = \bigoplus_{p \geq k} G_p.
\]

The \( \nabla_D \) induces \( T \)-isomorphisms

\[
\nabla_D : G_{k+1} \rightarrow G_k, \quad \nabla_D : D(A, 2k+1)W \rightarrow D(A, 2k-1)W.
\]

In particular,

\[
G_0 = \bigoplus_{j=1}^{\ell} T \partial P_j, \quad \text{and} \quad \mathcal{H}^{(0)} = \bigoplus_{j=1}^{\ell} R \partial P_j = \text{Der}_R.
\]

**Theorem 3.9**

1. The \( \nabla_D \) induces a \( T \)-linear automorphism \( \nabla_D : D(A, -\infty)W \rightarrow D(A, -\infty)W \).
2. \( D(A, -\infty)W = \bigoplus_{k \in \mathbb{Z}} G_k \).
3. \( D(A, 2k-1)W = \mathcal{H}^{(k)} = \bigoplus_{p \geq k} G_p \) (\( k \in \mathbb{Z} \)).
Remark. The construction of a basis $\eta^{(1)}_1, \ldots, \eta^{(1)}_\ell$ for $D(\mathcal{A}, 1)$ is due to K. Saito [6]. A basis for $D(\mathcal{A}, 2)$ was constructed in [10]. In [11] $D(\mathcal{A}, m)$ was found to be a free $S$-module for all $m \geq 0$ whenever $\mathcal{A}$ is a Coxeter arrangement. Note that it is re-proved in Theorem 3.3 in this article. In [8] K. Saito called the decreasing filtration $\text{Der}_R = H^{(0)} \supset H^{(1)} \supset \ldots$ and the decomposition $\text{Der}_R = D(\mathcal{A}, -1)^W = H^{(0)} = \bigoplus_{p \geq 0} \mathcal{G}_p$ the Hodge filtration and the Hodge decomposition respectively. They are essential to define the flat structure (or equivalently the Frobenius manifold structure in topological field theory) on the orbit space $V/W$. Note that Theorem 3.9 (3), when $k \geq 0$, is the main theorem of [13].

4 Relation among bases for logarithmic forms and derivations

In the previous section we constructed a basis $\{\omega^{(m)}_j\}$ for $\Omega(\mathcal{A}, m)$ and a basis $\{\eta^{(m)}_j\}$ for $D(\mathcal{A}, m)$ for $m \in \mathbb{Z}$. In this section we briefly describe their relations to other bases constructed in the earlier works [11], [15], and [2]. In [11], the following bases for $D(\mathcal{A}, 2k+1)$ and $D(\mathcal{A}, 2k)$ are given:

$$[\xi^{(2k+1)}_1, \ldots, \xi^{(2k+1)}_\ell] := \{\partial_{x_1}, \ldots, \partial_{x_\ell}\}AJ(D^k[x])^{-1}J(P),$$
$$[\xi^{(2k)}_1, \ldots, \xi^{(2k)}_\ell] := \{\partial_{x_1}, \ldots, \partial_{x_\ell}\}AJ(D^k[x])^{-1}.$$

The two bases $\{\eta^{(m)}_j\}$ and $\{\xi^{(m)}_j\}$ are related as follows:

Proposition 4.1

For $k \in \mathbb{Z}_{\geq 0},$

$$[\xi^{(2k+1)}_1, \ldots, \xi^{(2k+1)}_\ell] = (-1)^k[r^{(2k+1)}_1, \ldots, r^{(2k+1)}_\ell]B^{-1}B^{(k+1)},$$
$$[\xi^{(2k)}_1, \ldots, \xi^{(2k)}_\ell] = (-1)^k[r^{(2k)}_1, \ldots, r^{(2k)}_\ell].$$

Proof. The second formula is immediate from Definition 2.5. The following computation proves the first formula:

$$J(D^k[x])^{-1}J(P) = (-1)^{k+1}R_{2k+1}D[J(P)]^{-1}J(D^{k+1}[x])J(D^k[x])^{-1}J(P),$$
$$= (-1)^kR_{2k+1}D[J(P)]^{-1}A^{-1}J(P)^{-1}T B^{(k+1)}$$
$$= (-1)^kR_{2k+1}B^{-1}B^{(k+1)}. \square$$
In [15], the following bases are given:

\[
\begin{align*}
\nabla I \big( dP^1 \big) \nabla_{D}^{-k} \theta_E, & \ldots, \nabla I \big( dP^\ell \big) \nabla_{D}^{-k} \theta_E \quad \text{for } D(A, 2k + 1), \\
\nabla_{\partial x_1} \nabla_{D}^{-k} \theta_E, & \ldots, \nabla_{\partial x_\ell} \nabla_{D}^{-k} \theta_E \quad \text{for } D(A, 2k).
\end{align*}
\]

Here \( \theta_E \) is the Euler derivation. Their relations to \( \{ \eta^{(m)}_j \} \) are given as follows:

**Proposition 4.2**

Let \( k \in \mathbb{Z}_{\geq 0} \). Then

\[
\begin{align*}
\nabla I \big( dP^1 \big) \nabla_{D}^{-k} \theta_E, & \ldots, \nabla I \big( dP^\ell \big) \nabla_{D}^{-k} \theta_E = [\eta^{(2k+1)}_1, \ldots, \eta^{(2k+1)}_\ell] B^{-1} B^{(k+1)}, \\
\nabla_{\partial x_1} \nabla_{D}^{-k} \theta_E, & \ldots, \nabla_{\partial x_\ell} \nabla_{D}^{-k} \theta_E = [\eta^{(2k)}_1, \ldots, \eta^{(2k)}_\ell] A^{-1}.
\end{align*}
\]

**Proof.** By [12, Theorem 1.2.] and [14] one has

\[
\begin{align*}
\nabla I \big( dP^1 \big) \nabla_{D}^{-k} \theta_E, & \ldots, \nabla I \big( dP^\ell \big) \nabla_{D}^{-k} \theta_E = (-1)^{k} [\xi^{(2k+1)}_1, \ldots, \xi^{(2k+1)}_\ell].
\end{align*}
\]

Combining with Proposition 4.1, we have the first relation. For the second one, compute

\[
\begin{align*}
\nabla_{\partial x_1} \nabla_{D}^{-k} \theta_E, & \ldots, \nabla_{\partial x_\ell} \nabla_{D}^{-k} \theta_E A J(P) = \nabla I \big( dP^1 \big) \nabla_{D}^{-k} \theta_E, \ldots, \nabla I \big( dP^\ell \big) \nabla_{D}^{-k} \theta_E \quad \\
& = [\eta^{(2k+1)}_1, \ldots, \eta^{(2k+1)}_\ell] B^{-1} B^{(k+1)} \\
& = [\eta^{(2k)}_1, \ldots, \eta^{(2k)}_\ell] J(P)
\end{align*}
\]

by Proposition 2.6 (3). \( \square \)

Next let us review the bases for \( \Omega(A, m) \) described in [2, Theorem 6]: Let \( k \in \mathbb{Z}_{\geq 0} \) and \( P_1 \) the smallest degree basic invariant. Then

\[
\{ \nabla_{\partial x_1} \nabla^k_{D} dP^1, \ldots, \nabla_{\partial x_\ell} \nabla^k_{D} dP^1 \}
\]
forms a basis for \( \Omega(A, 2k + 1) \) and

\[
\{ \nabla_{\partial x_1} \nabla^k_{D} dP^1, \ldots, \nabla_{\partial x_\ell} \nabla^k_{D} dP^1 \}
\]
forms a basis for \( \Omega(A, 2k) \).

**Proposition 4.3**

Let \( k \geq 0 \). Then

\[
\begin{align*}
\nabla_{\partial x_1} \nabla^k_{D} dP^1, & \ldots, \nabla_{\partial x_\ell} \nabla^k_{D} dP^1 = [\omega^{(-2k-1)}_1, \ldots, \omega^{(-2k-1)}_\ell] B^{-1}, \\
\nabla_{\partial x_1} \nabla^k_{D} dP^1, & \ldots, \nabla_{\partial x_\ell} \nabla^k_{D} dP^1 = [\omega^{(-2k)}_1, \ldots, \omega^{(-2k)}_\ell] A^{-1}.
\end{align*}
\]
Proof. First, note that $[\nabla_D, \nabla_{\partial P_i}]$ is $W$-invariant, hence in $\text{Der}_R$. Since the smallest degree of derivations in $\text{Der}_R$ is $\deg \partial P_i$, it follows that $[\nabla_D, \nabla_{\partial P_i}] = 0$. In other words, $\nabla_{\partial P_i}$ and $\nabla_{\partial P_{i'}}$ commute for all $i$. Hence

$$[\nabla_{\partial P_1} \nabla_D^k dP_1, \ldots, \nabla_{\partial P_\ell} \nabla_D^k dP_1] = \nabla_D^k [\nabla_{\partial P_1} dP_1, \ldots, \nabla_{\partial P_\ell} dP_1].$$

Our proof is an induction on $k$. First assume that $k = 0$. Choose

$$P_1 = \frac{1}{2}[x_1, \ldots, x_\ell] A^{-1} [x_1, \ldots, x_\ell]^T,$$

and

$$dP_1 = [dx_1, \ldots, dx_\ell] A^{-1} [x_1, \ldots, x_\ell]^T.$$

Compute

$$[\nabla_{\partial P_1} dP_1, \ldots, \nabla_{\partial P_\ell} dP_1] = [\nabla_{\partial x_1} dP_1, \ldots, \nabla_{\partial x_\ell} dP_1] J(P)^{-T} B = [dx_1, \ldots, dx_\ell] A^{-1} J(P)^{-T} B = [dx_1, \ldots, dx_\ell] D[J(P)] = [\omega_1^{(-1)}, \ldots, \omega_\ell^{(-1)}].$$

For $k > 0$, apply $\nabla_D^k$ and use the commutativity. Then we have the first relation. For the second relation use Proposition 2.6 (2) to compute:

$$[\nabla_{\partial x_1} \nabla_D^k dP_1, \ldots, \nabla_{\partial x_\ell} \nabla_D^k dP_1] = [\nabla_{\partial P_1} \nabla_D^k dP_1, \ldots, \nabla_{\partial P_\ell} \nabla_D^k dP_1] J(P)^T = [\omega_1^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}] B^{-1} J(P)^T = [dx_1, \ldots, dx_\ell] R_{-2k-1} B^{-1} J(P)^T = [dx_1, \ldots, dx_\ell] R_{-2k} A^{-1} = [\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}] A^{-1}.$$

Remark. If $k < 0$ in Propositions 4.2 and 4.3, then the derivations and 1-forms in the left hand sides are proved to form bases for the logarithmic modules $D\Omega(A, 2k + 1), D\Omega(A, 2k), \Omega D(A, 2k + 1)$ and $\Omega D(A, 2k)$ in [1]. By using the same arguments in the proofs above, we can show that Propositions 4.2 and 4.3 hold true for all integers $k$ in the logarithmic modules $D\Omega(A, m)$ and $\Omega D(A, m)$ with $m : A \rightarrow \mathbb{Z}$.

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