SOME RELATIONS DEDUCED FROM REGULARIZED DOUBLE SHUFFLE RELATIONS OF MULTIPLE ZETA VALUES

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Abstract. It is conjectured that the regularized double shuffle relations give all algebraic relations among multiple zeta values. Hence all other algebraic relations should be deduced from the regularized double shuffle relations. In this paper, we reformulate and prove some such relations, for example, the weighted sum formula of L. Guo and B. Xie, some evaluation formulas with even arguments and the restricted sum formulas of M. E. Hoffman and their generalizations.

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Date: August 13, 2018.
2010 Mathematics Subject Classification. 11M32.
Key words and phrases. Multiple zeta values, Multiple zeta-star values, Regularized double shuffle relations.
The first author is supported by the National Natural Science Foundation of China (Grant No. 11471245) and Shanghai Natural Science Foundation (grant no. 14ZR1443500).
1. Introduction

For a sequence $k = (k_1, \ldots, k_n)$ of positive integers, we define its weight, depth and height by

$$\text{wt}(k) = k_1 + \cdots + k_n, \quad \text{dep}(k) = n, \quad \text{ht}(k) = \sharp\{l \mid 1 \leq l \leq n, k_l \geq 2\},$$

respectively. If $k_1 > 1$, we call $k$ an admissible index. For such an admissible index $k = (k_1, \ldots, k_n)$, the multiple zeta value and the multiple zeta-star value are real numbers defined respectively by the following convergent series

$$\zeta(k) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

$$\zeta^*(k) = \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

When $n = 1$, both cases degenerate to the Riemann zeta values, which are the special values of the Riemann zeta function at positive integer arguments. The study of these numbers can be traced back to L. Euler \[10\], while the systematic research began with M. E. Hoffman’s paper \[18\] and D. Zagier’s paper \[46\].

It is easy to see that every multiple zeta-star value can be written as a $\mathbb{Z}$-linear combination of multiple zeta values and vice versa as shown in the following simple example

$$\zeta^*(2, 1) = \sum_{m > n \geq 1} \frac{1}{m^2 n} = \sum_{m > n \geq 1} \frac{1}{m^2 n} + \sum_{m = n \geq 1} \frac{1}{m^2 n} = \zeta(2, 1) + \zeta(3).$$

Hence the $\mathbb{Q}$-vector space $\mathcal{Z}$ spanned by all multiple zeta values coincides with that spanned by all multiple zeta-star values. One important thing in the study of multiple zeta values is to understand the algebraic structure of the space $\mathcal{Z}$. A recent theorem of F. Brown \[4\] states that all periods of mixed Tate motives unramified over $\mathbb{Z}$ are $\mathbb{Q}\left[\frac{1}{2\pi i}\right]$-linear combinations of multiple zeta values, and the multiple zeta values indexed by 2 and 3 are linear generators of the $\mathbb{Q}$-vector space $\mathcal{Z}$. As a consequence, one obtains an upper bound for the dimension of the subspace $\mathcal{Z}_k$ of the space $\mathcal{Z}$, which is spanned by all multiple zeta values of weight $k$. More specifically, we have

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k,$$

where the sequence $\{d_k\}$ is defined by $d_0 = 1, d_1 = 0, d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ for all integers $k \geq 3$. This result was first proved by A. B. Goncharov \[15\] and independently by T. Terasoma \[42\]. The well-known dimension conjecture claims that the dimension of the vector space $\mathcal{Z}_k$ is exactly $d_k$, which was proposed by D. Zagier \[46\].

Since $d_k$ is far less than $2^{k-2}$, the number of the admissible indexes of weight $k$, there must be a lot of linear (algebraic) relations among multiple zeta values. Here we recall the regularized double shuffle relations from \[22, 39\], which are conjectured to give all algebraic relations of multiple zeta values.

As a subset of the field $\mathbb{R}$ of real numbers, $\mathcal{Z}$ is not only a subspace but also a subalgebra, which is because a product of two multiple zeta values is a sum of multiple zeta values. There are two different ways to multiple two multiple zeta values. First one is using the infinite series representations as displaying in the
following example:

\[
\zeta(2)\zeta(2) = \left( \sum_{m=1}^{\infty} \frac{1}{m^2} \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\
= \left( \sum_{m>n \geq 1} + \sum_{n>m \geq 1} + \sum_{m=n \geq 1} \right) \frac{1}{m^2 n^2} \\
= 2\zeta(2,2) + \zeta(4).
\]

We call such products stuffle products. Second one is using the so called Drinfel’d iterated integral representations. In fact, for an admissible index \( k = (k_1, \ldots, k_n) \), we have

\[
\zeta(k) = \int_{1 > t_1 > \cdots > t_k > 0} f_1(t_1) \cdots f_k(t_k) dt_1 \cdots dt_k
\]

with

\[
f_i(t) = \begin{cases} 
\frac{1}{t} & \text{if } i = k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_n = k, \\
\frac{1}{t} & \text{otherwise.}
\end{cases}
\] (1.1)

Using the iterated integral representations, we have, for example,

\[
\zeta(2)\zeta(2) = \left( \int_{1 > t_1 > t_2 > 0} \frac{dt_1 dt_2}{t_1(1 - t_2)} \right) \left( \int_{1 > s_1 > s_2 > 0} \frac{ds_1 ds_2}{s_1(1 - s_2)} \right) \\
= \left( \int_{1 > t_1 > t_2 > s_1 > s_2 > 0} + \int_{1 > s_1 > t_1 > t_2 > s_2 > 0} + \int_{1 > t_1 > s_1 > t_2 > s_2 > 0} + \int_{1 > s_1 > t_1 > s_2 > t_2 > 0} + \int_{1 > s_1 > s_2 > t_1 > t_2 > 0} + \int_{1 > s_1 > s_2 > t_1 > s_2 > 0} \right) \frac{dt_1 dt_2 ds_1 ds_2}{t_1(1 - t_2)s_1(1 - s_2)} \\
= 2\zeta(2,2) + 4\zeta(3,1).
\]

We call such products shuffle products.

Combining stuffle products and shuffle products, one can obtain the (finite) double shuffle relations. However, these are not all algebraic relations among multiple zeta values. For example, Euler’s formula \( \zeta(3) = \zeta(2,1) \) can not be deduced from double shuffle relations. Hence we need to consider the divergent multiple zeta values and to take regularization processes. There are also two regularization processes, using the truncated multiple zeta value \( \zeta_N(k) \) defined as a finite sum

\[
\zeta_N(k) = \sum_{N>m_1>\cdots>m_n>0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}
\]

with \( N > 0 \) and the multiple polylogarithms \( \text{Li}_k(z) \) defined as an iterated integral

\[
\text{Li}_k(z) = \int_{z > t_1 > \cdots > t_k > 0} f_1(t_1) \cdots f_k(t_k) dt_1 \cdots dt_k
\]

with \( |z| < 1 \), respectively. Here \( k = (k_1, \ldots, k_n) \) is a sequence of positive integers and the functions \( f_i(t) \) are defined by (1.1). Note that if \( k \) is admissible, we have
the limits
\[ \lim_{N \to \infty} \zeta_N(k) = \zeta(k), \quad \lim_{z \to 1} \operatorname{Li}_k(z) = \zeta(k). \]

We have the asymptotic properties
\[
\zeta_N(k) = P_k(\log N + \gamma) + O(1/N), \quad N \to \infty \quad (\exists J > 0),
\]
\[
\operatorname{Li}_k(z) = Q_k(-\log(1-z)) + O((1-z)^{J''}), \quad z \to 1 \quad (\exists J'' > 0),
\]
where \( P_k(T), Q_k(T) \in \mathbb{R}[T] \) are polynomials and \( \gamma \) is Euler’s constant. Let \( \rho : \mathbb{R}[T] \to \mathbb{R}[T] \) be the \( \mathbb{R} \)-linear map determined by
\[
\rho(e^{Ts}) = \exp \left( \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) s^n \right) e^{Ts}.
\]
Then for any sequence \( k \) of positive integers, we have
\[
Q_k(T) = \rho(P_k(T)),
\]
which deduces relations of multiple zeta values. Then the so called regularized double shuffle relations contain such type relations and the double shuffle relations.

Since the system of the regularized double shuffle relations is a conjectured full relation system of multiple zeta values, all other algebraic relations should be consequences of this system. In this paper, we reformulate and prove some such relations. For example, we reprove the weighted sum formula of L. Guo and B. Xie, give some evaluation formulas with even arguments and discuss the restricted sum formulas of M. E. Hoffman and their generalizations. Besides relations corresponding to multiple zeta values, we also provide the formulas corresponding to multiple zeta-star values.

The paper is organized as follows. In Section 2, we recall the algebraic setting of the regularized double shuffle relations. Besides descriptions corresponding to multiple zeta values, we also consider the descriptions corresponding to multiple zeta-star values. In Section 3, we state some relations which can be deduced from the regularized double shuffle relations. Finally, in Section 4, we briefly recall some other conjectured full relation systems of multiple zeta values.

2. ALGEBRAIC SETTING OF THE REGULARIZED DOUBLE SHUFFLE RELATIONS

In this section, we recall the algebraic setting of the regularized double shuffle relations from [22] for multiple zeta values and from [35] for multiple zeta-star values. Some new materials are supplied.

Let \( K \) be a field of characteristic zero and \( R \) be a commutative \( K \)-algebra with unitary.

2.1. Double shuffle relations. Let \( A = \{x, y\} \) be an alphabet containing two noncommutative letters. Denote by \( A^* \) the set of all words generated by the letters in \( A \), which contains the empty word 1. The subset of nonempty words of \( A^* \) is denoted by \( A^+ \). Hence \( A^+ = A^* - \{1\} \). Let \( \mathfrak{h} = K\langle A \rangle \) be the \( K \)-algebra of noncommutative polynomials on \( x \) and \( y \) with coefficients in \( K \). As a \( K \)-vector space, \( A^* \) is a basis of \( \mathfrak{h} \). We define two subalgebras
\[
\mathfrak{h}^1 = K + \mathfrak{h}y, \quad \mathfrak{h}^0 = K + x\mathfrak{h}y,
\]
which have \( K \)-basis
\[
z_{k_1} \cdots z_{k_n}, \quad n \geq 0, k_1, \ldots, k_n \geq 1
\]
4.
We have defined in \( S \). Then it is easy to see that by \( S \), which imply that we have invertible linear maps \( S \).

We define a new product \( \circledast \) on \( \mathfrak{h} \) by the \( \mathbb{K} \)-bilinearities and the axioms:

- \((S1)\) \( w \circledast 1 = 1 \circledast w = w \), \( (\forall w \in A^*) \);
- \((S2)\) \( au \circledast bv = a(u \circledast bv) + b(au \circledast v) \), \( (\forall a, b \in A, \forall u, v \in A^*) \).

The product \( \circledast \) is commutative and associative, and is called the shuffle product. Under this new product, \( \mathfrak{h} \) becomes a commutative algebra, and \( \mathfrak{h}^1 \) and \( \mathfrak{h}^0 \) are still subalgebras. We call these commutative algebras shuffle algebras, and denote them by \( \mathfrak{h}_w, \mathfrak{h}_w^1 \) and \( \mathfrak{h}_w^0 \), respectively.

Another product \( * \) can be defined on \( \mathfrak{h}^1 \) by the \( \mathbb{K} \)-bilinearities and the axioms:

- \((H1)\) \( w * 1 = 1 * w = w \), \( (\forall w \in A^* \cap \mathfrak{h}^1) \);
- \((H2)\) \( z_k u * z_l v = z_k(u * z_l v) + z_l(z_k u * v) + z_{k+l}(u * v) \), \( (\forall k, l \in \mathbb{N}, \forall u, v \in A^* \cap \mathfrak{h}^1) \).

Here \( \mathbb{N} \) is the set of all positive integers. The product \( * \) is commutative and associative, and is called the stuffle product (also called harmonic product). Under this new product, \( \mathfrak{h}^1 \) becomes a commutative algebra, and \( \mathfrak{h}^0 \) is still a subalgebra. We call these commutative algebras stuffle algebras, and denote them by \( \mathfrak{h}_w^1 \) and \( \mathfrak{h}_w^0 \), respectively.

Let \( \sigma \) be the automorphism of the noncommutative algebra \( \mathfrak{h} \) determined by:

\[
\sigma(x) = x, \quad \sigma(y) = x + y.
\]

We know that the inverse map \( \sigma^{-1} \) is determined by:

\[
\sigma^{-1}(x) = x, \quad \sigma^{-1}(y) = -x + y.
\]

Note that \( \sigma \) and \( \sigma^{-1} \) are also automorphisms of the shuffle algebra \( \mathfrak{h}_w \). We define a \( \mathbb{K} \)-linear map \( S : \mathfrak{h} \rightarrow \mathfrak{h} \), such that \( S(1) = 1 \) and:

\[
S(wa) = \sigma(w)a, \quad (\forall a \in A, \forall w \in A^*).
\]

Then it is easy to see that \( S \) is invertible, and the inverse linear map \( S^{-1} \) is given by:

\[
S^{-1}(wa) = \sigma^{-1}(w)a, \quad (\forall a \in A, \forall w \in A^*).
\]

We have:

\[
S(\mathfrak{h}^1) = \mathfrak{h}^1, \quad S(\mathfrak{h}^0) = \mathfrak{h}^0,
\]

which imply that we have invertible linear maps \( S|_{\mathfrak{h}^1} \) and \( S|_{\mathfrak{h}^0} \). Note that the map \( S \) defined in [35] is in fact \( S|_{\mathfrak{h}^1} \) here (See also [33]).

We define the shuffle product \( \circledast \) associated to multiple zeta-star values on \( \mathfrak{h} \) by the \( \mathbb{K} \)-bilinearities and the axioms:

- \((SS1)\) \( w \circledast 1 = 1 \circledast w = w \), \( (\forall w \in A^*) \);
- \((SS2)\) \( au \circledast bv = a(u \circledast bv) + b(au \circledast v) - \delta(u)\rho(a)bv - \delta(v)\rho(b)au \),
  \( (\forall a, b \in A, \forall u, v \in A^*) \).

where \( \delta : A^* \rightarrow \{0, 1\} \) is a map defined by:

\[
\delta(w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1, \end{cases}
\]

and \( \rho : A \rightarrow \mathfrak{h} \) is a map defined by:

\[
\rho(x) = 0, \quad \rho(y) = x.
\]
Note that in [35], S. Muneta used a different map instead of \( \rho \). By [33], for any \( u, v \in h \), we have

\[
S(u \overline{\ast} v) = S(u) \ast S(v), \quad S^{-1}(u \overline{\ast} v) = S^{-1}(u) \overline{\ast} S^{-1}(v),
\]

which imply that the product \( \overline{\ast} \) is also commutative and associative. Hence we get the commutative algebra \( R_1 \) and its subalgebras \( h_1^\ast \) and \( h_0^\ast \).

As in [35], the stuffle product \( \ast \) associated to multiple zeta-star values on \( h^1 \) is defined by the \( \mathbb{K} \)-bilinearities and the axioms

(SH1) \( w \ast 1 = 1 \ast w = w, \quad (\forall w \in A^* \cap h^1) \);
(SH2) \( z_k \ast z_l v = z_k (u \ast \overline{\ast} z_l v) + z_l (z_k u \ast \overline{\ast} v) - z_{k+l} (u \overline{\ast} v), \quad (\forall k, l \in \mathbb{N}, \forall u, v \in A^* \cap h^1) \).

By [35], for any \( u, v \in h^1 \), we have

\[
S(u \overline{\ast} v) = S(u) \ast S(v), \quad S^{-1}(u \overline{\ast} v) = S^{-1}(u) \ast S^{-1}(v),
\]

which imply that the product \( \overline{\ast} \) is also commutative and associative. Hence we get the commutative algebra \( h_1^\ast \) and its subalgebra \( h_0^\ast \).

Now for any \( \mathbb{K} \)-linear map \( Z_R : h^0 \to R \), we define

\[
Z^*_R = Z_R \circ S : h^0 \to R,
\]

which is also a \( \mathbb{K} \)-linear map. Then for any admissible index \( k = (k_1, \ldots, k_n) \), we define the multiple zeta value and the multiple zeta-star value associated with the map \( Z_R \) respectively by

\[
\zeta_R(k) = Z_R(z_{k_1} \cdots z_{k_n}), \quad \zeta^*_R(k) = Z^*_R(z_{k_1} \cdots z_{k_n}).
\]

The following lemma is easy to prove.

**Lemma 2.1.** For any \( \mathbb{K} \)-linear map \( Z_R : h^0 \to R \), the following two properties are equivalent

(i) \( Z_R : h^0_1 \to R \) and \( Z_R : h^0_1 \to R \) are algebra homomorphisms, that is for any \( u, v \in h^0 \), we have

\[
Z_R(u \overline{\ast} v) = Z_R(u)Z_R(v) = Z_R(u \ast v);
\]

(ii) \( Z_R^* : h^0_1 \to R \) and \( Z_R^* : h^0_1 \to R \) are algebra homomorphisms, that is for any \( u, v \in h^0 \), we have

\[
Z_R^*(u \overline{\ast} v) = Z_R^*(u)Z_R^*(v) = Z_R^*(u \ast v).
\]

If the \( \mathbb{K} \)-linear map \( Z_R : h^0 \to R \) satisfies the equivalent properties in Lemma 2.1, we call the pair \( (R, Z_R) \) or simply the map \( Z_R \) satisfies the (finite) double shuffle relations. Below, if \( Z_R \) satisfies the double shuffle relations, we always assume that \( Z_R(1) = 1 \). Hence if \( k \) is an empty index, we set

\[
\zeta_R(k) = \zeta_R^*(k) = Z_R(1) = 1.
\]

**2.2. Regularized double shuffle relations.** By [40], the algebra \( h^0_1 \) is a polynomial algebra \( h^0_1 = h_0^0[y] \). Then we can define the regularization map \( \text{reg}_m : h^0_1 \to h^0_1 \), such that for any

\[
w = \sum_{i \geq 0} w_i y^{m_i} \in h^1
\]

with \( w_i \in h^0 \), we have \( \text{reg}_m(w) = w_0 \). The map \( \text{reg}_m \) is an algebra homomorphism. Similarly, by [19], the algebra \( h^1_1 \) is also a polynomial algebra \( h^1_1 = h_0^1[y] \).
Finally by the definitions and the properties of the map $\tilde{S}$, which is an algebra homomorphism.

For any $w \in h^1$, we can write $S(w) \in h_1$ as

$$S(w) = \sum_{i \geq 0} w_i \in y^{\infty}i$$

with $w_i \in h^0$. Applying the map $S^{-1}$, we get

$$w = \sum_{i \geq 0} S^{-1}(w_i) \in y^{\infty}i.$$  

Here $S^{-1}(w_i) \in h_1$. Furthermore, it is easy to see that such decomposition is unique. Hence $h^1_{\infty} = h^0_{\infty}[y]$ is also a polynomial algebra, and one can define the regularization map $\text{reg}_{\infty} : h^1_{\infty} \to h^0_{\infty}$, which is an algebra homomorphism. Similarly, $h^1_{\infty} = h^0_{\infty}[y]$, and one can define an algebra homomorphism $\text{reg}_{\infty} : h^1_{\infty} \to h^0_{\infty}$. Note that we have

$$\text{reg}_{\infty} = S^{-1} \circ \text{reg}_{\infty} \circ S, \quad \text{reg}_{\infty} = S^{-1} \circ \text{reg}_{\infty} \circ S.$$  

Here and below, both $S|_{h^1}$ and $S|_{h^0}$ are denoted by $S$.

Now assume that the $\mathbb{K}$-linear map $Z_R : h^0 \to R$ satisfies the double shuffle relations. Then there exist unique algebra homomorphisms

$$Z^\infty_R : h^1_{\infty} \to R[T], \quad Z^*_{\infty} : h^1_{\infty} \to R[T],$$

$$Z^\infty_R : h^1_{\infty} \to R[T], \quad Z^*_{\infty} : h^1_{\infty} \to R[T],$$

such that

$$Z^\infty_R|_{h^0} = Z_R = Z^*_{\infty}|_{h^0}, \quad Z^\infty_R(y) = T = Z^*_{\infty}(y),$$

$$Z^\infty_R|_{h^0} = Z^*_{\infty}|_{h^0}, \quad Z^\infty_R(y) = T = Z^*_{\infty}(y).$$

More precisely, assume that $w \in h^1$ is decomposed as

$$w = \sum_{i \geq 0} w_i \in y^{\infty}i = \sum_{j \geq 0} \tilde{w}_j \in y^{\infty}, \quad (w_i, \tilde{w}_j \in h^0),$$

then we have

$$Z^\infty_R(w) = \sum_{i \geq 0} Z_R(w_i) T^i, \quad Z^*_{\infty}(w) = \sum_{j \geq 0} Z_R(\tilde{w}_j) T^j;$$

and if $u \in h^1$ is decomposed as

$$u = \sum_{i \geq 0} u_i \in y^{\infty}i = \sum_{j \geq 0} \tilde{u}_j \in y^{\infty}, \quad (u_i, \tilde{u}_j \in h^0),$$

then we have

$$Z^\infty_R(u) = \sum_{i \geq 0} Z_R(u_i) T^i, \quad Z^*_{\infty}(u) = \sum_{j \geq 0} Z_R(\tilde{u}_j) T^j.$$

Finally by the definitions and the properties of the map $S$, we get

$$Z^\infty_R = Z^\infty_R \circ S, \quad Z^*_{\infty} = Z^*_{\infty} \circ S;$$

$$Z^\infty_R|_{T=0} = Z_R \circ \text{reg}_{\infty}, \quad Z^*_{\infty}|_{T=0} = Z_R \circ \text{reg}_{\infty};$$

$$Z^\infty_R|_{T=0} = Z_R \circ \text{reg}_{\infty}, \quad Z^*_{\infty}|_{T=0} = Z_R \circ \text{reg}_{\infty}.$$
For any $\mathbb{K}$-linear map $Z_R : h^0 \to R$, let
\[
A_R(u) = \exp \left( \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_R(k) u^n \right) \in R[[u]].
\]

Let $\rho_R : R[T] \to R[T]$ be the $R$-module homomorphism defined by
\[
\rho_R(e^{Tu}) = A_R(u)e^{Tu}.
\]
Note that $\rho_R$ is invertible, and the inverse $\rho_R^{-1}$ is given by
\[
\rho_R^{-1}(e^{Tu}) = A_R(u)^{-1}e^{Tu}.
\]

In [22], K. Ihara, M. Kaneko and D. Zagier showed that one can use derivations to describe the regularized double shuffle relations of multiple zeta values. Recall that a $\mathbb{K}$-linear map $D : h \to h$ is a derivation if it satisfies
\[
D(1) = 0. \text{ To deal with the multiple zeta-star values, we introduce the following definitions.}
\]

**Definition 2.2.** Assume that $D$ and $L$ are $\mathbb{K}$-linear maps on $h$ with $D(1) = 0$ and $L$ invertible. If
\[
D(uv) = (L^{-1} \circ D \circ L)(uv) + uD(v), \quad (\forall u, v \in A^+),
\]
we call $D$ is a left $L$-derivation on $h$. Similarly, if
\[
D(uv) = D(u)v + u(L^{-1} \circ D \circ L)(v), \quad (\forall u, v \in A^+),
\]
we call $D$ is a right $L$-derivation on $h$.

Note that if $D$ is a left $L$-derivation, then for any $a_1, \ldots, a_n \in A$, we have
\[
D(a_1 \cdots a_n) = \sum_{i=1}^{n-1} a_1 \cdots a_{i-1}(L^{-1} \circ D \circ L)(a_i)a_{i+1} \cdots a_n
\]
\[+ a_1 \cdots a_{n-1}D(a_n).
\]

We define a $\mathbb{K}$-linear map $S : h \to h$ such that $S(1) = 1$ and
\[
S(ua) = w\sigma(a), \quad (\forall a \in A, \forall w \in A^*).
\]
Then it is easy to check that $S = S^{-1} \circ \sigma$, which implies that $S$ is invertible with inverse $S^{-1} = \sigma^{-1} \circ S$, and
\[
S^{-1}(wa) = w\sigma^{-1}(a), \quad (\forall a \in A, \forall w \in A^*).
\]

Then the following result holds.

**Lemma 2.3.** Assume $D : h \to h$ is a $\mathbb{K}$-linear map, such that $D(w) \notin \mathbb{K}$ for any $w \in A^+$. Set $D^* = S^{-1} \circ D \circ S$. Then $D$ is a derivation if and only if $D^*$ is a left $S$-derivation.

**Proof.** Assume that $D$ is a derivation on $h$. For any $u, v \in A^+$, we have
\[
D^*(uv) = S^{-1}(D(S(\sigma(u)v))) = S^{-1}(D(\sigma(u))S(v) + \sigma(u)D(S(v))).
\]
Since $S(v), D(S(v)) \notin \mathbb{K}$, we have
\[
D^*(uv) = \sigma^{-1}(D(\sigma(u)))v + uS^{-1}(D(S(v)))
\]
\[= (\sigma^{-1} \circ S \circ D^* \circ S^{-1} \circ \sigma)(u)v + uD^*(v)
\]
\((\tilde{S}^{-1} \circ D^* \circ \tilde{S})(u)v + uD^*(v)\).

Hence \(D^*\) is a left \(\tilde{S}\)-derivation. Similarly, one can prove the oppositive direction. \(\square\)

For any \(\mathbb{K}\)-linear map \(D\) on \(\mathfrak{h}\), let \(\overline{D} = \tau \circ D \circ \tau\). Here \(\tau\) is the antiautomorphism of the noncommutative algebra \(\mathfrak{h}\) determined by
\[
\tau(x) = y, \quad \tau(y) = x.
\]

Then \(\overline{D}\) is also \(\mathbb{K}\)-linear. Note that \(\tau\) is an automorphism of the shuffle algebra \(\mathfrak{h}_m\). We have the following interesting results, although we will not use the \(m\) below.

**Lemma 2.4.** Assume that \(D\) and \(L\) are \(\mathbb{K}\)-linear maps on \(\mathfrak{h}\) with \(L\) invertible. Then

1. \(D\) is a derivation if and only if \(\overline{D}\) is a derivation;
2. \(D\) is a left \(L\)-derivation if and only if \(\overline{D}\) is a right \(\overline{L}\)-derivation.

**Proof.** (1) Assume that \(D\) is a derivation on \(\mathfrak{h}\). For any \(u, v \in \mathfrak{h}\), we have
\[
\overline{D}(uv) = \tau(D(\tau(v)\tau(u))) = \tau(D(\tau(v))\tau(u) + \tau(v)D(\tau(u)))
\]
\[
= u\tau(D(\tau(v))) + \tau(D(\tau(u)))v = u\overline{D}(v) + \overline{D}(u)v.
\]

Hence \(\overline{D}\) is a derivation. On the contrary, using \(D = \overline{D}\) we get the oppositive direction.

(2) Assume that \(D\) is a left \(L\)-derivation on \(\mathfrak{h}\). For any \(u, v \in A^+\), we have
\[
\overline{D}(uv) = \tau((L^{-1} \circ D \circ \tau)(u))\tau(v) + \tau(v)L^{-1}(D(\tau(u)))
\]
\[
= u\tau(D(\tau(v))) + \tau(D(\tau(u)))v = u\overline{D}(v) + \overline{D}(u)v.
\]

Hence \(\overline{D}\) is a right \(\overline{L}\)-derivation. \(\square\)

Now we define the derivations which describe the regularized double shuffle relations. For any positive integer \(n\), the derivation \(\partial_n\) on \(\mathfrak{h}\) is defined by
\[
\partial_n(x) = x(x + y)^{n-1}y, \quad \partial_n(y) = -x(x + y)^{n-1}y.
\]

Set \(\partial_n^* = S^{-1} \circ \partial_n \circ S\). Then \(\partial_n^*\) is a left \(\tilde{S}\)-derivation on \(\mathfrak{h}\), and satisfies
\[
\partial_n^*(x) = xy^n, \quad \partial_n^*(y) = -xy^n.
\]

We need some other maps, which have a closed relationship with Ohno’s relations ([36]). For any nonnegative integer \(m\), the \(\mathbb{K}\)-linear map \(\sigma_m : h^1 \to h^1\) is defined by \(\sigma_m(1) = 1\), and
\[
\sigma_m(z_{k_1} \cdots z_{k_n}) = \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{\varepsilon_1 + \cdots + \varepsilon_n = m} z_{k_1 + \varepsilon_1} \cdots z_{k_n + \varepsilon_n}, \quad (\forall n, k_1, \ldots, k_n \in \mathbb{N}).
\]

Set \(\sigma_m^* = S^{-1} \circ \sigma_m \circ S\) and \(\overline{\sigma_m} = S^{-1} \circ \overline{\sigma_m} \circ \overline{S}\). Finally, we can state the equivalent properties about the regularized double shuffle relations.

**Theorem 2.5.** Assume that the pair \((R, Z_R)\) satisfies the double shuffle relations. Then the following properties are equivalent:

(i) \((Z^R_R - \rho_R \circ Z^R_R)(w_1) = 0\) for all \(w_1 \in h^1\);

(ii) \((Z^R_R - \rho_R \circ Z^R_R)(w_1)|_{T=0} = 0\) for all \(w_1 \in h^1\);
(b) \( (Z \circ \rho_R - \rho_{\tilde{R}} \circ Z_{\tilde{R}})(w_1) |_{\tau=0} = 0 \) for all \( w_1 \in h^1 \);

(iii) \( Z_{\tilde{R}}(w_1 \otimes w_0 - w_1 \ast w_0) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(c) \( Z_{\tilde{R}}(w_1 \otimes w_0 - w_1 \ast w_0) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(iii') \( Z_{\tilde{R}}(w_1 \otimes w_0 - w_1 \ast w_0) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(c') \( Z_{\tilde{R}}(w_1 \otimes w_0 - w_1 \ast w_0) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(iv) \( Z_R(\text{reg}_m(w_1 \otimes w_0 - w_1 \ast w_0)) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(d) \( Z_R(\text{reg}_m(w_1 \otimes w_0 - w_1 \ast w_0)) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(d') \( Z_R(\text{reg}_m(w_1 \otimes w_0 - w_1 \ast w_0)) = 0 \) for all \( w_1 \in h^1 \) and all \( w_0 \in h^0 \);

(v) \( Z_R(\text{reg}_m(y^m \ast w_0)) = 0 \) for all positive integer \( m \) and all \( w_0 \in h^0 \);

(e) \( Z_R(\text{reg}_m(y^m \ast w_0 - y^m \ast w_0)) = 0 \) for all positive integer \( m \) and all \( w_0 \in h^0 \);

(v') \( Z_R(\text{reg}_m(y^m \ast w_0 - y^m \ast w_0)) = 0 \) for all positive integer \( m \) and all \( w_0 \in h^0 \);

(e') \( Z_R(\text{reg}_m(y^m \ast w_0 - y^m \ast w_0)) = 0 \) for all positive integer \( m \) and all \( w_0 \in h^0 \);

(vi) \( Z_R(\partial_m(w_0)) = 0 \) for all positive integer \( n \) and all \( w_0 \in h^0 \);

(f) \( Z_R^{(\partial_m)}(w_0) = 0 \) for all positive integer \( n \) and all \( w_0 \in h^0 \);

(vii) \( Z_R^{(\sigma_m - \sigma_m^\circ)}(w_0) = 0 \) for all nonnegative integer \( m \) and all \( w_0 \in h^0 \);

(g) \( Z_R^{(\sigma_m^\circ - \sigma_m)}(w_0) = 0 \) for all nonnegative integer \( m \) and all \( w_0 \in h^0 \).

**Remark 2.6.** The items (i)-(vii) and (ii')-(v') in Theorem 2.5 were listed in [22, Theorems 2.3]. Here we add some equivalent properties corresponding to the multiple zeta-star values, which are marked by (a), (b), and so on to distinguish with that in [22, Theorems 2.3].

If the \( K \)-linear map \( Z_R : h^0 \rightarrow R \) satisfies the double shuffle relations, and satisfies the equivalent properties in Theorem 2.5, we call the pair \((R, Z_R)\) or simply the map \( Z_R \) satisfies the regularized double shuffle relations. Hence by [22], if we take \( K = \mathbb{Q} \) and \( R = \mathbb{R} \), and define \( \mathbb{Q} \)-linear map \( Z_R = Z : h^0 \rightarrow \mathbb{R} \), such that

\[
Z(z_{k_1}, z_{k_2}, \ldots, z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n), \quad (k_1 \geq 2, k_2, \ldots, k_n \geq 1),
\]

then \((\mathbb{R}, Z)\) satisfies the regularized double shuffle relations. Here \( \mathbb{Q} \) is the field of rational numbers.

To prove Theorem 2.5, we need two lemmas.

**Lemma 2.7.** Assume that the pair \((R, Z_R)\) satisfies the double shuffle relations and \( n \) is a positive integer. If

\[
Z_R(\text{reg}_m(y^m \ast w_0 - y^m \ast w_0)) = 0
\]

for any integer \( m \) with \( 1 \leq m \leq n \) and any \( w_0 \in h^0 \), then

\[
Z_R(\text{reg}_n(y^n w_1 \ast w_2 - y^n w_1 \ast w_2)) = 0
\]

for any integer \( m \) with \( 1 \leq m \leq n \) and any \( w_1, w_2 \in h^0 \), where \( \circ = \ast \) or \( \circ = \ast \).

**Proof.** We prove the case of \( \circ = \ast \), and one can prove the case of \( \circ = \ast \) similarly. We process by induction on \( n \). For \( n = 1 \), note that for any \( w_1, w_2 \in h^0 \), we have

\[
yw_1 - y \ast w_1, yw_1 - y \ast w_1 \in h^0,
\]

and

\[
yw_1 \ast w_2 - yw_1 \ast w_2 = (yw_1 - y \ast w_1) \ast w_2 - (yw_1 - y \ast w_1) \ast w_2
\]

\[
+ y \ast (w_1 \ast w_2) - y \ast (w_1 \ast w_2) + y \ast (w_1 \ast w_2 - w_1 \ast w_2).
\]
Hence we get
\[ Z_R(\text{reg}_{\oplus}(yw_1 \circ w_2 - yw_1 \circ w_2)) \]
\[ = Z_R(yw_1 - y \circ w_1)Z_R(w_2) - Z_R(yw_1 - y \circ w_1)Z_R(w_2) \]
\[ + Z_R(\text{reg}_{\oplus}(w_1 \circ w_2 - y \circ (w_1 \circ w_2))) \]
\[ + Z_R(\text{reg}_{\oplus}(y))Z_R(w_1 \circ w_2 - w_1 \circ w_2) \]
\[ = Z_R(y \circ w_1 - y \circ w_1)Z_R(w_2) = 0. \]

Now assume that \( n > 1 \). For \( w_1, w_2 \in R^0 \), there exist \( u_i \in R^0 \) such that
\[
y^n w_1 \circ w_2 - y^n w_1 \circ w_2 = (y^n w_1 - y^n w_1) \circ w_2 - (y^n w_1 - y^n w_1) \circ w_2 \]
\[ - (y^n \circ w_1 - y^n \circ w_1) \circ w_2 \]
\[ + y^n \circ (w_1 \circ w_2) - y^n \circ (w_1 \circ w_2) \]
\[ + y^n \circ (w_1 \circ w_2 - w_1 \circ w_2) \]
\[ = \sum_{i=0}^{n-1} (y^i u_i \circ w_2 - y^i u_i \circ w_2) - (y^n \circ w_1 - y^n \circ w_1) \circ w_2 \]
\[ + y^n \circ (w_1 \circ w_2 - w_1 \circ w_2) + y^n \circ (w_1 \circ w_2 - w_1 \circ w_2). \]

Hence using the induction hypothesis, we easily get
\[ Z_R(\text{reg}_{\oplus}(y^n w_1 \circ w_2 - y^n w_1 \circ w_2)) = 0, \]
which finishse the proof.

\[ \square \]

**Lemma 2.8.** Assume that the pair \((R, Z_R)\) satisfies the double shuffle relations and \( n \) is a positive integer. If
\[ Z_R(\text{reg}_{\oplus}(S(y^m) \circ w_0 - S(y^m) \circ w_0)) = 0 \]
for any integer \( m \) with \( 1 \leq m \leq n \) and any \( w_0 \in R^0 \), then
\[ Z_R(\text{reg}_{\oplus}(y^m \circ w_1 - y^m \circ w_1)) = 0 \]
for any integer \( m \) with \( 1 \leq m \leq n \) and any \( w_1 \in R^0 \), where \( \oplus = \circ \) or \( \circ \).

**Proof.** We prove the case of \( \oplus = \circ \), and one can prove the case of \( \oplus = * \) similarly.

We process by induction on \( n \). For \( n = 1 \), the result follows from the fact that \( S(y) = y \). Now assume that \( n > 1 \). Since
\[ S(y^n) = (x + y)^{n-1} y = \sum_{i=0}^{n-2} y^i u_i + y^n \]
with \( u_i \in R^0 \), we have
\[ S(y^n) \circ w_1 - S(y^n) \circ w_1 = \sum_{i=0}^{n-2} (y^i u_i \circ w_1 - y^i u_i \circ w_1) + (y^n \circ w_1 - y^n \circ w_1). \]

Using the induction hypothesis, we have
\[ Z_R(\text{reg}_{\oplus}(y^m \circ w_2 - y^m \circ w_2)) = 0 \]
for any integer \( m \) with \( 1 \leq m \leq n - 1 \) and any \( w_2 \in R^0 \). Then by Lemma 2.7, we get
\[ Z_R(\text{reg}_{\oplus}(y^i u_i \circ w_1 - y^i u_i \circ w_1)) = 0 \]
for any $i$ with the condition $0 \leq i \leq n - 2$. Therefore we have

$$Z_R(\text{reg}_W (y^n \ast w_1 - y^n \ast w_1)) = Z_R(\text{reg}_W (S(y^n) \ast w_1 - S(y^n) \ast w_1)) = 0,$$

which finishes the proof. \qed

**Proof of Theorem 2.5.** One can easily show the implications

(i) $\iff$ (a), (ii) $\iff$ (b), (iii) $\iff$ (c), (iii′) $\iff$ (c′),
(iv) $\iff$ (d) $\implies$ (e), (iv′) $\iff$ (d′) $\implies$ (e′),
(vi) $\iff$ (f), (vii) $\iff$ (g).

Hence to finish the proof, we have to show the implications

(e) $\implies$ (v), (e′) $\implies$ (v′).

Here we give the proof of the first one, and one can prove the second one similarly. Assume that (e) holds, then for any positive integer $m$ and any $w_0 \in \mathfrak{g}^0$, we have

$$Z_R(\text{reg}_W (S(y^m) \ast w_0 - S(y^m) \ast w_0)) = 0.$$ 

Hence we get (v) by using Lemma 2.8. \qed

Finally, we recall two useful formulas to finish this section, which will be used later.

**Lemma 2.9 ([21, 22]).** Let $k$ be a positive integer and $t$ be a variable. Then we have

$$\exp_s \left(-\sum_{n=1}^{\infty} \frac{z_{nk} t^n}{n} \right) = \frac{1}{1 + z_k t} \quad (2.1)$$

and

$$\left(\frac{1}{1 + z_k t}\right) \ast S \left(\frac{1}{1 - z_k t}\right) = 1. \quad (2.2)$$

Hence we have the following relations.

**Corollary 2.10.** Assume that $Z_R : \mathfrak{g}_+^0 \rightarrow R$ is an algebra homomorphism. Then for any integer $k \geq 2$, we have

$$\sum_{n=0}^{\infty} \zeta_R(\{k\}^n) t^n = \exp \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta_R(nk) t^n \right) \quad (2.3)$$

and

$$\left(\sum_{n=0}^{\infty} (-1)^n \zeta_R(\{k\}^n) t^n \right) \left(\sum_{n=0}^{\infty} \zeta_R(\{k\}^n) t^n \right) = 1, \quad (2.4)$$

where $t$ is a variable.

Here and below, the notation $\{k_1, \ldots, k_r\}^n$ stands for $k_1, k_r, \ldots, k_1, k_r$.
3. Relations deduced from regularized double shuffle relations

As stated in Section 1, it is conjectured that the regularized double shuffle relations give all algebraic relations among multiple zeta values. Hence one can obtain any algebraic relations from the regularized double shuffle relations. In this section, we reformulated and proved some of these results, which provide some evidences for the conjecture.

Below we always assume that the map \( Z_R : h^0 \to R \) satisfies the regularized double shuffle relations unless otherwise specified.

3.1. Linear double shuffle relations. If \( Z_R : h^0 \to R \) satisfies the double shuffle relations, we certainly obtain the following linear relations

\[
Z_R(w_1 w_2 - w_2 \ast w_2) = 0, \quad (\forall w_1, w_2 \in h^0)
\]

and

\[
Z_R^*(w_1 \overline{w}_2 - w_2 \overline{w}_2) = 0, \quad (\forall w_1, w_2 \in h^0),
\]

which are equivalent and are called linear double shuffle relations.

3.2. Derivation relations. As part of the regularized double shuffle relations, we have the following linear relations

\[
Z_R(\partial_n(w_0)) = 0, \quad (\forall n \in \mathbb{N}, \forall w_0 \in h^0)
\]

and

\[
Z_R^*(\partial_n^*(w_0)) = 0, \quad (\forall n \in \mathbb{N}, \forall w_0 \in h^0),
\]

which are equivalent and are called derivation relations. The derivation relations for multiple zeta values were introduced by K. Ihara, M. Kaneko and D. Zagier in [22].

3.3. Hoffman’s relation. In the algebra \( h^1 \), we have the following formulas.

Lemma 3.1. For any \( w = z_{k_1} \cdots z_{k_n} \in h^1 \) with \( n, k_1, \ldots, k_n \in \mathbb{N} \), we have

\[
\partial_1(w) = y \wedge w - y \ast w = \sum_{i=1}^{n} \sum_{j=0}^{k_i-2} z_{k_1} \cdots z_{k_{i-1}} z_{k_i-j} \overline{z}_{j+1} \overline{z}_{k_{i+1}} \cdots z_{k_n}
\]

\[
- \sum_{i=1}^{n} z_{k_1} \cdots z_{k_{i-1}} z_{k_i+1} \overline{z}_{k_{i+1}} \cdots z_{k_n}
\]

and

\[
\partial_1^*(w) = y \overline{w} - w \overline{w} = \sum_{i=1}^{n} \sum_{j=0}^{k_i-2} z_{k_1} \cdots z_{k_{i-1}} z_{k_i-j} \overline{z}_{j+1} \overline{z}_{k_{i+1}} \cdots z_{k_n}
\]

\[
- \sum_{i=1}^{n} (k_i - 1 + \delta_{n1}) z_{k_1} \cdots z_{k_{i-1}} z_{k_i+1} \overline{z}_{k_{i+1}} \cdots z_{k_n},
\]

where \( \delta_{ij} \) is Kronecker’s delta symbol defined by

\[
\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]
Proof. We have
\[ \partial^*_1(w) = S^{-1}(\partial_1(S(w))) = S^{-1}(y \ast S(w) - y \ast S(w)) \]
\[ = y \ast w - y \ast w. \]
Other equations can be obtained directly from the definitions.

From Lemma 3.1, we get the following Hoffman’s relations
\[ \sum_{i=1}^{n} \zeta_R(k_1, \ldots, k_{i-1}, k_i, k_{i+1}, \ldots, k_n) \]
\[ = \sum_{i=1}^{n} \sum_{j=0}^{k_i-2} \zeta_R(k_1, \ldots, k_{i-1}, k_i - j, j + 1, k_{i+1}, \ldots, k_n) \]
and
\[ \sum_{i=1}^{n} (k_i - 1 + \delta_{n_i}) \zeta^*_R(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n) \]
\[ = \sum_{i=1}^{n} \sum_{j=0}^{k_i-2} \zeta^*_R(k_1, \ldots, k_{i-1}, k_i - j, j + 1, k_{i+1}, \ldots, k_n), \]
where \( n, k_1, \ldots, k_n \in \mathbb{N} \) with \( k_1 \geq 2 \). The Hoffman’s relation for multiple zeta values was first proved by M. E. Hoffman in [18]. While the Hoffman’s relation for multiple zeta-star values was first obtained by S. Muneta in [35] as a consequence of the regularized double shuffle relations.

3.4. Sum formula. For any integers \( k, n \) with the conditions \( k > n \geq 1 \), we set
\[ S(k, n) = x(x^{k-n-1} \ast y^{n-1})y. \]

Lemma 3.2. For any integers \( k, n \) with the conditions \( k > n \geq 1 \), we have
\[ (\sigma_{n-1} - \sigma_{n-1})(z_{k-n+1}) = S(k, n) - z_k; \]
\[ (\sigma^*_n - \sigma^*_{n-1})(z_{k-n+1}) = \sum_{i=1}^{n} (-1)^{n-i} \binom{k-i-1}{n-i} S(k, i) - z_k. \]

Moreover, if \( n \geq 2 \), we have
\[ (-1)^{n-1} \ast \gamma_{n-1} \ast z_{k-n+1} = S(k, n) - S(k, n - 1). \]

Proof. Equation (3.3) is just [22, Proposition 9]. Since \( \tau \) is an automorphism of \( \mathfrak{H}_m \), we have
\[ \sigma_{n-1}(z_{k-n+1}) = \tau(\sigma_{n-1}(xy^{k-n})) = \tau(x(x^{k-n} \ast y^{n-1})y) \]
\[ = x(x^{k-n-1} \ast y^{n-1})y = S(k, n), \]
which deduces equation (3.1). Now since \( \sigma^{-1} \) is an automorphism of \( \mathfrak{H}_m \), we get
\[ \sigma^*_{n-1}(z_{k-n+1}) = S^{-1}(\sigma_{n-1}(z_{k-n+1})) = S^{-1}(x(x^{k-n-1} \ast y^{n-1})y) \]
\[ = x(x^{k-n-1} \ast (-x + y)^{n-1})y \]
\[ = \sum_{i=1}^{n} (-1)^{n-i} x(x^{k-n-1} \ast x^{n-i} \ast y^{i-1})y \]
\[
\sum_{i=1}^{n} (-1)^{n-i} \binom{k-i-1}{n-i} x(x^{k-i-1} y^{i-1}) y,
\]
which implies equation (3.2).

Let \( k \) and \( n \) be integers with \( k > n \geq 1 \). From (3.2), for any positive integer \( m \leq n \), we have
\[
\frac{(\sigma_{m-1} - \sigma^*_{m-1})(z_{k-m+1})}{(k-m-1)(n-m)} = \sum_{i=1}^{m} (-1)^{m-i} \binom{k-i-1}{m-i} S(k,i) - z_k.
\]

Multiplying \( (k-m-1)/(n-m) \) to the above equation and summing for \( m \) from 1 to \( n \), we get
\[
\sum_{m=1}^{n} \frac{(k-m-1)(\sigma_{m-1} - \sigma^*_{m-1})(z_{k-m+1})}{(k-m-1)(n-m)} = S(k,n) - \frac{(k-1)}{(n-1)} z_k.
\] (3.4)

Then by (3.1) or (3.3), we get the sum formula
\[
\sum_{\text{wt}(k) = 0, \text{deg}(k) \geq n} \zeta_R(k) = \zeta_R(k); \quad (3.5)
\]
and by (3.4), we get the sum formula
\[
\sum_{\text{wt}(k) = 0, \text{deg}(k) \geq n} \zeta_R^*(k) = \left(\frac{k-1}{n-1}\right) \zeta_R(k). \quad (3.6)
\]

The sum formula for multiple zeta values was first proved by A. Granville in [16], and it was shown in [22] that one can deduce the sum formula (3.5) from the regularized double shuffle relations. The equivalence of sum formulas of multiple zeta values and of multiple zeta-star values was first proposed by M. E. Hoffman in [18].

3.5. Weighted sum formula of L. Guo and B. Xie. In this subsection, we discuss the weighted sum formula of L. Guo and B. Xie [17]. We provide a much simpler proof of the weighted sum formula, and give the formula corresponding to the multiple zeta-star values.

For positive integers \( n, k_1, \ldots, k_n \), let
\[
C(k_1, \ldots, k_n) = \sum_{j=1}^{n} 2^{k_1 + \cdots + k_j - j} + 2^{k_1 + \cdots + k_n - n} = \sum_{j=1}^{n-1} 2^{k_1 + \cdots + k_j - j} + 2^{k_1 + \cdots + k_n - n+1} \in \mathbb{N}.
\]

**Lemma 3.3** ([17, Theorems 2.4, 2.6]). For any integers \( k, n \) with \( k > n \geq 2 \), we have
\[
\sum_{l: k_l \geq 1, \sum k_l = k} z_{k_1} z_{k_2} \cdots z_{k_{n-1}} = \sum_{k_1 \geq 1, k_2 \geq 2, \sum k_l = k} z_{k_1} \cdots z_{k_n}
\]
\[
+ (n-1) \sum_{k_1 \geq 1, k_1 + \sum k_l = k} z_{k_1} \cdots z_{k_n} + n \sum_{k_1 \geq 1, k_1 + \sum k_l = k} z_{k_1} \cdots z_{k_n}
\]
\[+(k-n) \sum_{k_1, \ldots, k_{n-1} \geq 1} z_{k_1} \cdots z_{k_{n-1}} \]  

(3.7)

and

\[\sum_{i, k_2, \ldots, k_{n} \geq 1} z_i z_{k_1} \cdots z_{k_{n-1}} = \sum_{k_1, \ldots, k_n \geq 1} [C(k_1, \ldots, k_{n-1}) + z_{k_1} \cdots z_k - \sum_{k_{i+1} \geq 1} z_{k_1} \cdots z_{k_i} z_{k_{i+1}} \cdots z_{k_n}]. \]  

(3.8)

Here, if \(n = 2\), we set \(C(k_2, \ldots, k_{n-1}) = 1\).

**Proof.** In [17], L. Guo and B. Xie proved these formulas by induction on \(n\). Here we provide a much simpler proof. For (3.7), we use the formula

\[z_i z_{k_1} \cdots z_{k_{n-1}} = \sum_{i=0}^{n-1} z_{k_1} \cdots z_{k_i} z_{k_{i+1}} \cdots z_{k_{n-1}} + \sum_{i=1}^{n-1} z_{k_1} \cdots z_{k_{i-1}} z_{k_i} z_{k_{i+1}} \cdots z_{k_{n-1}},\]

which is immediate from the combinatorial description of the product \(*\). Hence the left-hand side of (3.7) is

\[\sum_{i, k_2, \ldots, k_{n} \geq 1} z_i z_{k_1} \cdots z_{k_{n-1}} = \sum_{i=0}^{n-1} (k_1 - 1) z_{k_1} \cdots z_{k_{n-1}} + \sum_{i=1}^{n-1} (k_1 - 1) z_{k_1} \cdots z_{k_{n-1}},\]

which is just the right-hand side of (3.7).

For (3.8), we use the formula

\[z_i z_{k_1} \cdots z_{k_{n-1}} = \sum_{i=1}^{n-1} \sum_{\alpha_j \geq 1, \alpha_i = 1} \prod_{j=1}^{i-1} (\alpha_j - 1) (k_j - 1) z_{\alpha_1} \cdots z_{\alpha_i+1} z_{k_{i+1}} \cdots z_{k_{n-1}} + \sum_{\alpha_j \geq 1, \alpha_i = 1} \prod_{j=1}^{n-1} (\alpha_j - 1) z_{\alpha_1} \cdots z_{\alpha_n},\]

\[= \sum_{i=1}^{n-1} \sum_{\alpha_j \geq 1, \alpha_i = 1} \prod_{j=1}^{i-1} (\alpha_j - 1) (k_j - 1) z_{\alpha_1} \cdots z_{\alpha_i+1} z_{k_{i+1}} \cdots z_{k_{n-1}} + \sum_{\alpha_j \geq 1, \alpha_i = 1} \prod_{j=1}^{n-1} (\alpha_j - 1) z_{\alpha_1} \cdots z_{\alpha_n},\]
which is [32, Proposition 2.7] and can be proved by the combinatorial description of the product \( \overline{\alpha} \). Hence the left-hand side of (3.8) is

\[
\sum_{\alpha_1 \geq 1, \alpha_2 \geq 1} \sum_{k_1 \geq 2, k_1 \geq \alpha_2} \left( \alpha_1 - 1 \right) z_{\alpha_1} z_{\alpha_2} z_{k_2} \cdots z_{k_{n-1}}
\]

\[
+ \sum_{i=2}^{n-1} \sum_{\alpha_1 \geq 1, k_i \geq 1} \alpha_i \sum_{k_i \geq \alpha_i + k_{i+1} + \cdots + k_{n-1} = k} \left( \alpha_1 - 1 \right) \frac{i-1}{j-1} \prod_{j=2}^{i-1} \sum_{\alpha_j \geq 1} \left( \alpha_j - 1 \right)
\]

\[
\times \sum_{k_i = \alpha_i + 1}^{\alpha_i + \alpha_i + 1} \left( \alpha_i - 1 \right) z_{\alpha_1} \cdots z_{\alpha_i + 1} z_{k_i + 1} \cdots z_{k_{n-1}}
\]

\[
+ \sum_{\alpha_i \geq 1} \sum_{k_i = 2} \frac{\alpha_i}{k_i} \sum_{j=2}^{\alpha_i} \left( \alpha_1 - 1 \right) \frac{j-1}{k_i - 1} \prod_{j=2}^{i-1} \sum_{k_j = 1} \left( \alpha_j - 1 \right) z_{\alpha_1} \cdots z_{\alpha_i}
\]

\[
= \sum_{k_1 \geq 1, k_2 \geq 2} \left( 2^{k_1-1} - z_{k_1} \cdots z_{k_n} \right) + \sum_{k_1 \geq 1, k_1 = k} \left( 2^{k_1-1} - 1 \right) z_{k_1} \cdots z_{k_n}
\]

\[
+ \sum_{i=2}^{n-1} \sum_{k_i = 1} \left( 2^{k_i + \cdots + k_{i-1} - 2^{k_2 + \cdots + k_i - (i-1)}} - 1 \right) z_{k_1} \cdots z_{k_n}
\]

\[
+ \sum_{k_1 = 1} \left( 2^{k_1 + \cdots + k_{n-1} - (n-1)} - 2^{k_2 + \cdots + k_{n-1} - (n-2)} \right) z_{k_1} \cdots z_{k_n},
\]

which is just the right-hand side of (3.8).

We can obtain the similar formulas for the products \( \overline{z} \) and \( \overline{z} \).

**Lemma 3.4.** For any integers \( k, n \) with \( k > n \geq 2 \), we have

\[
\sum_{k_1 \geq 1, k_2 \geq 2} z_{k_1} \cdots z_{k_n-1} = \sum_{k_1 \geq 1, k_1 = k} z_{k_1} \cdots z_{k_n}
\]

\[
+ (n-1) \sum_{k_1 \geq 1, k_1 = k} z_{k_1} \cdots z_{k_n} + n \sum_{k_1 \geq 1, k_1 = 2, k_2 \geq 2} z_{k_1} \cdots z_{k_n}
\]

\[
- (k-n) \sum_{k_1 \geq 1, k_2 \geq 2} z_{k_1} \cdots z_{k_n-1}
\]

(3.9)

and

\[
\sum_{k_1 \geq 1, k_1 = k} z_{k_1} \cdots z_{k_n-1}
\]

\[
= - \sum_{k_1 \geq 1, k_2 \geq 1} z_{k_1} \cdots z_{k_n} + \sum_{k_1 \geq 1, k_2 \geq 2} z_{k_1} \cdots z_{k_n-1}
\]

\[
+ \sum_{k_1 \geq 1} [C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1})] z_{k_1} \cdots z_{k_n}
\]
\[ - \sum_{k_1 \geq 1, \ldots, k_n = k} [C(k_1, \ldots, k_{n-1}) - C(k_1, \ldots, k_n)] - C(k_1, \ldots, k_{n-2}) + C(k_2, \ldots, k_{n-2})] z_{k_1} \cdots z_{k_{n-1}}. \] (3.10)

Here, for \( n = 2 \), we set \( C(k_1, \ldots, k_{n-2}) = C(k_2, \ldots, k_{n-2}) = 1 - k; \) and for \( n = 3 \), we set \( C(k_2, \ldots, k_{n-2}) = 1 \).

**Proof.** Similar as (3.7), we use the formula

\[ z_l \bar{z} z_{k_1} \cdots z_{k_{n-1}} = \sum_{i=0}^{n-1} z_{k_1} \cdots z_{k_i} z_l z_{k_{i+1}} \cdots z_{k_{n-1}} \]

\[ - \sum_{i=1}^{n-1} z_{k_1} \cdots z_{k_{i-1}} z_{l+k_i} z_{k_{i+1}} \cdots z_{k_{n-1}} \]

to prove (3.9). For (3.10), we use the formula

\[ z_l \bar{z} z_{k_1} \cdots z_{k_{n-1}} = \sum_{i=1}^{n-1} \sum_{\alpha_j \geq 1, \alpha_i = 0, \alpha_i + \alpha_{i+1} = 1} \prod_{j=1}^{i-1} (\alpha_j - 1)(\alpha_i - 1) \]

\[ \times (z_{\alpha_1} \cdots z_{\alpha_{i+1}} z_{k_{i+1}} \cdots z_{k_{n-1}} - z_{\alpha_1} \cdots z_{\alpha_{i-1}} z_{\alpha_i + \alpha_{i+1}} z_{k_{i+1}} \cdots z_{k_{n-1}}) \]

\[ + \sum_{\alpha_i \geq 1, \alpha_i + \alpha_n = 0} \prod_{j=1}^{n-1} (\alpha_j - 1) (z_{\alpha_1} \cdots z_{\alpha_n} - z_{\alpha_1} \cdots z_{\alpha_{n-2}} z_{\alpha_{n-1} + \alpha_n}). \]

Hence similar as (3.8), the left-hand side of (3.10) is

\[ \sum_{k_1 + \cdots + k_n = k} [C(k_1, \ldots, k_{n-1}) - C(k_1, \ldots, k_n)] z_{k_1} \cdots z_{k_n} - \sum_{k_1 \geq 1, k_2 = 1} z_{k_1} \cdots z_{k_n} \]

\[ - \sum_{\alpha_1, k_2 \geq 1} \sum_{k_1 + \cdots + k_{n-1} = k} (\alpha_1 - 1) z_{\alpha_1 + \alpha_2} z_{k_2} \cdots z_{k_{n-1}} \]

\[ - \sum_{i=2}^{n-1} \sum_{k_1 \geq 1} [2^{k_1 + \cdots + k_{i-1} - (i-2)} 2^{k_2 + \cdots + k_{n-1} - (n-2)} z_{k_1} \cdots z_{k_{i-1}} z_{k_{i+1}} z_{k_{i+2}} \cdots z_{k_n}] \]

Denote the last three terms by \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \), respectively. We find \(-\Sigma_1\) is

\[ \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_n = k} 2^{k_1 - 1} z_{k_1} z_{k_2} z_{k_3} \cdots z_{k_n} + \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_n = k} (2^{k_1 - 1} - 1) z_{k_1 + k_2} z_{k_3} \cdots z_{k_n} \]

\[ = \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_n = k} 2^{k_1 - 1} z_{k_1} z_{k_2} z_{k_3} \cdots z_{k_n} - \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_n = k} z_{k_1 + k_2} z_{k_3} \cdots z_{k_n} \]

\[ = \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_{n-1} = k} (2^{k_1 - 1} - 1) z_{k_1} \cdots z_{k_{n-1}} - \sum_{k_1 \geq 1, k_2 \geq 1, k_1 + \cdots + k_{n-1} = k} z_{k_1} \cdots z_{k_{n-1}}. \]
Similarly, \(-\Sigma_2\) is
\[
\sum_{i=2}^{n-1} \sum_{k_j \geq 1} \left[ 2^{k_1 + \cdots + k_{i-1} - i} - 2^{k_2 + \cdots + k_{i-1} - (i-1)} \right] (2^{k_i} - 2) z_{k_1} \cdots z_{k_{n-1}}
\]
\[
= \sum_{i=2}^{n-1} \sum_{k_j \geq 1} \left[ 2^{k_1 + \cdots + k_i - i} - 2^{k_2 + \cdots + k_i - (i-1)} \right] (2^{k_1 + \cdots + k_{i-1} - (i-1)} + 2^{k_2 + \cdots + k_{i-1} - (i-2)}) z_{k_1} \cdots z_{k_{n-1}}.
\]
And finally, \(-\Sigma_3\) is
\[
\sum_{k_1 + \cdots + k_{n-1} = k} \left[ 2^{k_1 + \cdots + k_{n-1} - (n-1)} - 2^{k_2 + \cdots + k_{n-1} - (n-2)} \right] z_{k_1} \cdots z_{k_{n-1}}.
\]
Thus \(-\Sigma_1 - \Sigma_2 - \Sigma_3\) is
\[
\sum_{k_1 + \cdots + k_{n-1} = k} \left[ C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1}) - C(k_1, \ldots, k_{n-2}) \right] z_{k_1} \cdots z_{k_{n-1}} - \sum_{k_j \geq 1, k_1 \geq 2} z_{k_1} \cdots z_{k_{n-1}}.
\]
Now equation (3.10) follows immediately. 

Note that if \(n \geq 3\), for any positive integers \(k_1, \ldots, k_{n-1}\), we have
\[
C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1}) - C(k_1, \ldots, k_{n-2}) + C(k_2, \ldots, k_{n-2})
\]
\[
= 2 \sum_{i=2}^{n-2} (k_i-1)(2^{k_{i-1}} - 1).
\]
From equations (3.7)-(3.10), we immediately get the following result.

**Corollary 3.5.** For any integers \(k, n\) with \(k > n \geq 2\), we have
\[
\sum_{l, k_j \geq 1, k_1 \geq 2} \left( z_l \in z_{k_1} \cdots z_{k_{n-1}} - z_l \notin z_{k_1} \cdots z_{k_{n-1}} \right)
\]
\[
= \sum_{k_j \geq 1, k_1 \geq 2} \left[ C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1}) \right] z_{k_1} \cdots z_{k_n}
\]
\[
- n \sum_{k_j \geq 1, k_1 \geq 2} z_{k_1} \cdots z_{k_n} - (k - n) \sum_{k_j \geq 1, k_1 \geq 2} z_{k_1} \cdots z_{k_{n-1}}
\]
(3.11) 
and
\[
\sum_{l, k_j \geq 1, k_1 \geq 2} \left( z_l \in z_{k_1} \cdots z_{k_{n-1}} - z_l \notin z_{k_1} \cdots z_{k_{n-1}} \right)
\]
\[
= \sum_{k_j \geq 1, k_1 \geq 2} \left[ C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1}) \right] z_{k_1} \cdots z_{k_n}
\]

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which were first discovered by Y. Ohno and W. Zudilin in [17]. Taking 

\[ -C(k_1, \ldots, k_n-2) + C(k_2, \ldots, k_n-2) \] 

and 

\[ -n \sum_{k_1 \geq 1, k_2 \geq 2 \atop k_1 + \cdots + k_n = k} z_{k_1} \cdots z_{k_{n-1}} + (k - n + 1) \sum_{k_1 \geq 1, k_2 \geq 2 \atop k_1 + \cdots + k_{n-1} = k} z_{k_1} \cdots z_{k_{n-1}}. \] (3.12)

Applying the map \( Z_R \) to (3.11) and the map \( Z_R^* \) to (3.12), and using the sum formulas (3.5) and (3.6), we get the weighted sum formulas 

\[ \sum_{k_1 \geq 1, k_2 \geq 2 \atop k_1 + \cdots + k_n = k} [C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-2})] \zeta_R(k_1, \ldots, k_n) = k \zeta_R(k) \] (3.13)

and 

\[ \sum_{k_1 \geq 1, k_2 \geq 2 \atop k_1 + \cdots + k_n = k} [C(k_1, \ldots, k_{n-1}) - C(k_2, \ldots, k_{n-1}) - C(k_1, \ldots, k_{n-2}) + C(k_2, \ldots, k_{n-2})] \zeta_R^*(k_1, \ldots, k_{n-1}) = \binom{k-1}{n-1} \zeta_R(k), \]

where \( k \) and \( n \) are integers with \( k > n \geq 2 \). Note that if \( Z_R = Z \), then (3.13) is [17, Theorem 2.2]. Taking \( n = 2 \) and using the sum formulas, we find for any \( k \geq 3 \),

\[ \sum_{i=2}^{k-1} 2^i \zeta_R(i, k-i) = (k + 1) \zeta_R(k), \]

\[ \sum_{i=2}^{k-1} 2^i \zeta_R^*(i, k-i) = (2^k + k - 3) \zeta_R(k), \]

which were first discovered by Y. Ohno and W. Zudilin in [38].

3.6. Restricted sum formula of M. Eie, W. C. Liaw and Y. L. Ong. We compute explicitly the relations coming from 

\[ Z_R((\sigma_m - \overline{\sigma_m})(w_0)) = 0 \]

for any nonempty word \( w_0 \in \mathfrak{h}^0 \) and any nonnegative integer \( m \).

Lemma 3.6. Let \( w_0 = x^{a_1} y^{b_1} \cdots x^{a_s} y^{b_s} \in \mathfrak{h}^0 \) with \( s, a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathbb{N} \). Then for any nonnegative integer \( m \), we have 

\[ \sigma_m(w_0) = \sum_{\substack{\epsilon_s \geq 0 \\epsilon_1 + \cdots + \epsilon_s = m \\epsilon_1 \cdots \epsilon_s \epsilon_1 \cdots \epsilon_s = m}} x^{a_1 \epsilon_1 + b_1 \epsilon_1} y^{a_2 \epsilon_2 + b_2 \epsilon_2} \cdots x^{a_s \epsilon_s + b_s \epsilon_s} y^{a_1 \epsilon_1} \cdots y^{a_s \epsilon_s} \] (3.14)

and 

\[ \overline{\sigma_m}(w_0) = \sum_{\substack{\epsilon_s \geq 0 \\epsilon_1 + \cdots + \epsilon_s = m \\epsilon_1 \cdots \epsilon_s \epsilon_1 \cdots \epsilon_s = m}} x^{a_1 \epsilon_1 + b_1 \epsilon_1} y^{a_2 \epsilon_2} x^{a_2 \epsilon_2 - 1 + b_2 \epsilon_2} y^{a_3 \epsilon_3} x^{a_3 \epsilon_3} \cdots x^{a_s \epsilon_s - 1 + b_s \epsilon_s} y^{b_s} \] (3.15)

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Proof. Since \( w_0 = z_{a_1+1}z_{b_1-1}z_{a_2+1}z_{b_2-1} \cdots z_{a_s+1}z_{b_s-1} \), we have

\[
\sigma_m(w_0) = \sum_{\varepsilon_1+\cdots+\varepsilon_s = m} \sum_{\varepsilon_1 > 0} \prod_{i=1}^{s} z_{a_i+\varepsilon_i+1}z_{\varepsilon_i+1}z_{\varepsilon_i+1} \cdots z_{\varepsilon_i+\varepsilon_i+1} + 1
\]

\[
= \sum_{\varepsilon_1 > 0} \prod_{i=1}^{s} x^a_i \left( \sum_{\varepsilon_1 > 0} \prod_{i=1}^{s} x^\varepsilon_i y x^\varepsilon_i y \cdots x^\varepsilon_i y \right) y
\]

\[
= \sum_{\varepsilon_1 > 0} \prod_{i=1}^{s} x^a_i (x^\varepsilon_i y z_{b_i-1}) y,
\]

which is (3.14). The equation (3.15) follows from (3.14) and the fact that \( \tau \) is an automorphism of the shuffle algebra \( H_n \).

With the help of (3.14) and (3.15), for any positive integers \( s, a_1, \ldots, a_s, b_1, \ldots, b_s \) and any nonnegative integers \( m \), we obtain the relation

\[
\sum_{\varepsilon_1 > 0} \sum_{\varepsilon_1 > 0} \zeta_R(k_1+1, k_2, \ldots, k_1, p_1+1, \{1\}^{b_1-1}, \ldots, k_s+1, k_s+2, \ldots, k_s, p_s+1, \{1\}^{b_s-1}) = \sum_{\varepsilon_1 > 0} \sum_{\varepsilon_1 > 0} \zeta_R(a_1 + l_1, l_1, \ldots, l_1, l_1, \ldots, l_1, a_s + l_s, l_s, \ldots, l_s),
\]

(3.16)

which is called the vector version of the restricted sum formula in [8, Theorem 3.4.1]. If \( s = 1 \), we get the restricted sum formula ([9]): for any positive integers \( a, b \) and any nonnegative integer \( m \), we have

\[
\sum_{\varepsilon_1 > 0} \zeta_R(k_1+1, k_2, \ldots, k_{m+1}, \{1\}^{b-1}) = \sum_{\varepsilon_1 > 0} \zeta_R(a_1 + l_1, l_1, \ldots, l_1),
\]

which is the sum formula (3.5) in the case of \( b = 1 \).

Now we consider the relations coming from

\[
Z_H^m(\sigma_m^c - \overline{\sigma_m^c})(w_0) = 0
\]

(3.17)

for any nonempty word \( w_0 \in H^0 \) and any nonnegative integer \( m \). Since \( S^{-1} : H^0 \rightarrow H^0 \) is invertible, we can take \( w_0 = S^{-1}(x^{a_1} y^{b_1} \cdots x^{a_s} y^{b_s}) \), where \( s, a_1, b_1, \ldots, a_s, b_s \) are any positive integers.

Lemma 3.7. Let \( w_0 = S^{-1}(x^{a_1} y^{b_1} \cdots x^{a_s} y^{b_s}) \) with \( s, a_1, b_1, \ldots, a_s, b_s \in \mathbb{N} \). Then we have

\[
\sigma_m(w_0) = \sum_{\varepsilon_1 > 0} \sum_{\varepsilon_1 > 0} \prod_{i=1}^{s-1} \left( -1 \right)^{\varepsilon_{i+1} \varepsilon_{i+2}} \prod_{i=1}^{s} \left( \varepsilon_i + b_i - j_i - 1 \right) \\
\times \prod_{i=1}^{s-1} \left[ x^a_i (x^{\varepsilon_i+b_i-j_i-1} y^{b_i}) (-x + y) \right] x^{a_s} (x^{\varepsilon_s+b_s-j_s-1} y^{b_s}) y
\]

(3.18)
we have

Moreover, let $s$ have nonnegative integers $m \geq 0$ and $s \geq 1$, with $s + 1 \leq m$.

Proof. By (3.14), we get

$$
\sigma_m^*(w_0) = \sum_{s \geq 1} R^{-1}(x^{a_1}(x^{b_1} + y^{b_1})y \cdots x^{a_s}(x^{b_s} + y^{b_s})y).
$$

Since $\sigma^{-1}$ is an automorphism of $h_m$, we find $\sigma_m^*(w_0)$ is

$$
\sum_{\substack{a_1, \ldots, a_s, b_1, \ldots, b_s \geq 0 \atop \sum_i a_i + \sum_j b_j = m}} \prod_{i=1}^{s-1} \left( a_i + \varepsilon_i - j_i - 1 \right) \prod_{i=1}^{s} \left( a_i + \varepsilon_i - j_i - 1 \right) \cdot Z_R^* \left( \prod_{i=1}^{s-1} \left( x^{a_i}(x^{b_i} + y^{b_i})y \right) \right)
$$

$$
\sum_{\substack{a_1, \ldots, a_s, b_1, \ldots, b_s \geq 0 \atop \sum_i a_i + \sum_j b_j = m}} \prod_{i=1}^{s-1} \left( a_i + \varepsilon_i - j_i - 1 \right) \prod_{i=1}^{s} \left( a_i + \varepsilon_i - j_i - 1 \right) \cdot Z_R^* \left( \prod_{i=1}^{s-1} \left( x^{a_i}(x^{b_i} + y^{b_i})y \right) \right)
$$

Let $s = 1$. Then for any positive integers $a, b$ and any nonnegative integer $m$, we have

$$
\sum_{j=0}^{b-1} (-1)^j \binom{m+b-j-1}{m} \sum_{k_1, \ldots, k_j+1 = m+b} \zeta_R^*(a+k_1, k_2, \ldots, k_j+1 + \delta_{j0} a)
$$

$$
= \sum_{j=0}^{m} (-1)^{m-j-1} \binom{a+m-j-1}{a-1} \sum_{k_1, \ldots, k_j+1 = a+m} \sum_{n_1, \ldots, n_{j+1} = b} \zeta_R^*(k_1 + 1, k_2, \ldots, k_j, k_{j+1} + n_1 - 1 + \delta_{j0}, n_2, \ldots, n_{j+1}).
$$

Moreover, let $b = 1$ in above formula. Then for any integers $k, n$ with $k > n \geq 1$, we have

$$
\sum_{j=1}^{n} (-1)^{n-j} \binom{k-j-1}{k-n-1} \sum_{wt(k) = k \text{, admissible}} \zeta_R^*(k) = \zeta_R(k).
$$
which is in fact equivalent to the sum formula (3.6).

3.7. Restricted sum formulas of double zeta values. In [13], it was proved by H. Gangl, M. Kaneko and D. Zagier that for any even integer \( k \geq 4 \), we have the following restricted sum formulas

\[
\begin{align*}
\sum_{i=2}^{k-1} \zeta_R(i, k-i) &= \frac{3}{4} \zeta_R(k), \\
\sum_{i=2}^{k-1} \zeta_R(i, k-i) &= \frac{1}{4} \zeta_R(k), \\
\sum_{i=2}^{k-1} \zeta_R^*(i, k-i) &= \frac{2k-1}{4} \zeta_R(k), \\
\sum_{i=2}^{k-1} \zeta_R^*(i, k-i) &= \frac{2k-3}{4} \zeta_R(k).
\end{align*}
\]

To prove the above equations, one can use Euler’s decomposition formulas

\[
\begin{align*}
z_r \cdot z_s &= \sum_{i=r}^{r+s-1} \left( \frac{i-1}{r-1} \right) z_i z_{r+s-i} + \sum_{i=s}^{r+s-1} \left( \frac{i-1}{s-1} \right) z_i z_{r+s-i}, \\
z_r \cdot \bar{z}_s &= \sum_{i=r}^{r+s-1} \left( \frac{i-1}{r-1} \right) z_i z_{r+s-i} + \sum_{i=s}^{r+s-1} \left( \frac{i-1}{s-1} \right) z_i z_{r+s-i} - \binom{r+s}{r} z_{r+s},
\end{align*}
\]

from which we get

\[
\begin{align*}
\sum_{r=1}^{k-1} (-1)^r (z_r \cdot z_{k-r} - z_r \cdot z_{k-r}) &= -2 \sum_{i=2}^{k-1} (-1)^i z_i z_{k-i} + z_k, \\
\sum_{r=1}^{k-1} (-1)^r (z_r \cdot \bar{z}_{k-r} - z_r \cdot \bar{z}_{k-r}) &= -2 \sum_{i=2}^{k-1} (-1)^i z_i z_{k-i} + z_k.
\end{align*}
\]

Therefore we obtain

\[
\begin{align*}
\sum_{i=2}^{k-1} (-1)^i \zeta_R(i, k-i) &= \frac{1}{2} \zeta_R(k), \\
\sum_{i=2}^{k-1} (-1)^i \zeta_R^*(i, k-i) &= \frac{1}{2} \zeta_R(k),
\end{align*}
\]

which imply the above restricted sum formulas together with the sum formulas.

There are also relations for \( \zeta_R(r, s) \) and \( \zeta_R^*(r, s) \) with \( r + s \) odd. For example, as shown in [13], for any odd integer \( k \geq 3 \) and integers \( r, s \) with \( r + s = k \) and \( r, s \geq 2 \), we have

\[
\zeta_R(r, s) = \frac{(-1)^r + 1}{2} \zeta_R(r) \zeta_R(s) - (-1)^r \sum_{i \in k-2} \binom{i-1}{r-1} \zeta_R(i) \zeta_R(k-i)
\]
Moreover, since for odd integer $k \geq 3$, we have

$$2z_{k-1}z_1 = 2(y \ominus z_{k-1} - y \star z_{k-1}) + \sum_{i=2}^{k-2}((-1)^iz_i) \ominus z_{k-i} - z_i \star z_{k-i}) + (k - 1)z_k,$$

we get

$$\zeta_R(k - 1, 1) = \frac{k - 1}{2} \zeta_R(k) - \sum_{2 \leq i \leq k - 2} \zeta_R(i) \zeta_R(k - i),$$

$$\zeta^*_R(k - 1, 1) = \frac{k + 1}{2} \zeta_R(k) - \sum_{2 \leq i \leq k - 2} \zeta_R(i) \zeta_R(k - i).$$

### 3.8. Some evaluation formulas of multiple zeta value.

In this subsection, we give some evaluation formulas, including the formulas for $\zeta_R(2k, \ldots, 2k)$ and $\zeta^*_R(2k, \ldots, 2k)$.

We take one $\lambda$ and then fix it, such that $\lambda^2 = -24\zeta_R(2)$. Hence we have

$$\zeta_R(2) = -\frac{1}{24} \lambda^2.$$ 

Note that when $Z_R = Z$, we may take $\lambda = 2\pi \sqrt{-1}$.

**Theorem 3.8.** Assume $Z_R : \mathfrak{h}^0 \rightarrow R$ satisfies the regularized double shuffle relations, then for any nonnegative integer $n$, we have

$$\zeta_R(\{2\}^n) = (-1)^n \frac{\lambda^{2n}}{4^n(2n + 1)!}. \quad (3.20)$$

To prove Theorem 3.8, we need the following result.

**Lemma 3.9.** Assume $Z_R : \mathfrak{h}^0 \rightarrow R$ satisfies the regularized double shuffle relations, then for any positive integer $n$, we have

$$\sum_{m=0}^{n-1} \zeta_R(\{2\}^m, 4, \{2\}^{n-1-m}) = \frac{2}{3} n(n + 1) \zeta_R(\{2\}^{n+1}). \quad (3.21)$$

**Proof of Theorem 3.8.** We proceed by induction on $n$. It is trivial to verify the result for $n = 0$ and $n = 1$. Now assume that the formula (3.20) is valid for the positive integer $n$. Using the stuffle product, we have

$$\zeta_R(2) \zeta_R(\{2\}^n) = (n + 1) \zeta_R(\{2\}^{n+1}) + \sum_{m=0}^{n-1} \zeta_R(\{2\}^m, 4, \{2\}^{n-1-m}).$$
Hence by (3.21), we get
\[ \zeta_R(2) \zeta_R(\{2\}^n) = \frac{(n+1)(2n+3)}{3} \zeta_R(\{2\}^{n+1}). \]

Then using the induction assumption, we get the formula for \( \zeta_R(\{2\}^{n+1}) \), which finishes the proof. \( \square \)

To prove Lemma 3.9, we need the following two lemmas. The first one is immediately from the definitions of the stuffle products.

**Lemma 3.10.** Let \( a \) and \( b \) be two positive integers. Then for any positive integer \( n \), we have
\[
\sum_{m=0}^{n-1} (-1)^m z_{b+ma} \ast z_a^{n-1-m} = \sum_{m=0}^{n-1} z_a^m z_{b-a}^{n-1-m}
\]
and
\[
\sum_{m=0}^{n-1} z_{b+ma} \ast z_a^{n-1-m} = \sum_{m=0}^{n-1} z_a^m z_{b-a}^{n-1-m}.
\]

The second one will need the restricted sum formula discussed in the last subsection.

**Lemma 3.11.** Assume that \( Z_R : H^0 \rightarrow R \) satisfies the regularized double shuffle relations. Then for any positive integer \( n \), we have
\[
\sum_{m=2}^{n+1} (-1)^m \zeta_R(2m) \zeta_R(\{2\}^{n+1-m}) = \frac{2}{3} n(n+1) \zeta_R(\{2\}^{n+1}). \quad (3.22)
\]

**Proof.** Let \( t \) be a variable and set
\[
F(t) = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta_R(2n) t^n \right),
\]
we get
\[
F'(t) = F(t) \cdot \left( \sum_{n=1}^{\infty} (-1)^{n-1} \zeta_R(2n) t^{n-1} \right),
\]
\[
F''(t) = F(t) \cdot \left\{ \left( \sum_{n=1}^{\infty} (-1)^{n-1} \zeta_R(2n) t^{n-1} \right)^2 + \sum_{n=1}^{\infty} (-1)^{n-1}(n-1) \zeta_R(2n) t^{n-2} \right\}.
\]

Using stuffle product, we get
\[
\left( \sum_{n=1}^{\infty} (-1)^{n-1} \zeta_R(2n) t^{n-1} \right)^2 = \sum_{n=2}^{\infty} (-1)^n \left( \sum_{m+l=n \atop m, l \geq 1} \zeta_R(2m) \zeta_R(2l) \right) t^{n-2}
\]
\[
= \sum_{n=2}^{\infty} (-1)^n \sum_{m+l=n \atop m, l \geq 1} (\zeta_R(2m, 2l) + \zeta_R(2l, 2m) + \zeta_R(2n)) t^{n-2}
\]
\[
= 2 \sum_{n=2}^{\infty} (-1)^n \left( \sum_{m=1}^{n-1} \zeta_R(2m, 2n-2m) \right) t^{n-2} + \sum_{n=2}^{\infty} (-1)^n(n-1) \zeta_R(2n) t^{n-2}.
\]
Then by the restricted sum formula of double zeta values, we find
\[ F''(t) = F(t) \cdot \frac{3}{2} \sum_{n=2}^{\infty} (-1)^n \zeta_R(2n)t^{n-2}. \]

On the other hand, let \( k = 2 \) in (2.3), one has
\[ F(t) = \sum_{n=0}^{\infty} \zeta_R(\{2\}^n)t^n. \]

Therefore
\[ F''(t) = \sum_{n=0}^{\infty} n(n-1)\zeta_R(\{2\}^n)t^{n-2}. \]

Finally we get
\[ \left( \sum_{n=2}^{\infty} (-1)^n \zeta_R(2n)t^{n-2} \right) \left( \sum_{n=0}^{\infty} \zeta_R(\{2\}^n)t^n \right) = \frac{2}{3} \sum_{n=0}^{\infty} n(n-1)\zeta_R(\{2\}^n)t^{n-2}, \]

which finishes the proof.

Now we come to prove Lemma 3.9.

**Proof of Lemma 3.9.** Taking \( a = 2 \) and \( b = 4 \) in Lemma 3.10, we obtain
\[ \sum_{m=0}^{n-1} (-1)^m z_{2(m+2)} \ast z_{2}^{n-1-m} = \sum_{m=0}^{n-1} z_{2}^m z_{4} z_{2}^{n-1-m}. \]

Then equation (3.21) follows from (3.22).

Let
\[ \Gamma_R(s) = \exp \left( \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_R(n)s^n \right) \in R[[s]] \]
be the gamma series associated to the map \( Z_R : h^0 \to R \). Then by Theorem 3.8 and [28, Theorem 4.5], we get the following theorem.

**Theorem 3.12.** Assume that \( Z_R : h^0 \to R \) satisfies the regularized double shuffle relations. Then the gamma series \( \Gamma_R(s) \) satisfies the reflection formula
\[ \Gamma_R(s)\Gamma_R(-s) = \frac{\lambda s}{e^{\lambda s/2} - e^{-\lambda s/2}}. \]

Using [28, Theorem 4.5], we can obtain some other evaluation formulas. Before stating the formulas, we recall some notations.

Let \( a, b, c \) be positive integers with \( a + b = 2c \). For any integers \( m, n \) with \( m \geq 2n \geq 0 \), we denote by \( I_{m,n} \) the set of all indexes obtained from shuffling \((\{a, b\}^n)\) with \((\{c\}^{m-2n})\). For example, we have
\[ I_{m,0} = \{\{c\}^m\}, \quad I_{2n,n} = \{\{a, b\}^n\} \]

and
\[ I_{2n+1,n} = \{\{c, \{a, b\}^n\}, \{a, c, \{a, b\}^{n-1}\}, \{a, b, c, \{a, b\}^{n-1}\}, \ldots, \{a, b\}^n, c\}. \]

Define
\[ T_{m,n} = \sum_{k=(k_1, \ldots, k_m) \in I_{m,n}} z_{k_1} \ldots z_{k_m} \in h^1, \quad (3.23) \]
Finally, let \( \{B_n\} \) be the Bernoulli numbers defined by
\[
\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}.
\]

**Theorem 3.13.** Assume that \( Z_R : h^0 \to R \) satisfies the regularized double shuffle relations, then

1. for any nonnegative integer \( n \), we have
   \[
   \zeta_R(2n) = -\frac{\lambda^{2n}}{2(2n)!} B_{2n},
   \]
   where we set \( \zeta_R(0) = -\frac{1}{2} \);
2. for any positive integer \( n \), we have
   \[
   \zeta^*_R(\{2\}^n) = (2^{1-2n} - 1) \frac{\lambda^{2n}}{(2n)!} B_{2n} = 2(1 - 2^{1-2n}) \zeta_R(2n);
   \]
3. for any positive integer \( k \) and any nonnegative integer \( n \), we have
   \[
   \zeta_R(\{2k\}^n) = C_n^{(k)} \frac{\lambda^{2nk}}{(2nk)!},
   \]
   where \( C_n^{(k)} \in \mathbb{Q} \) is given by \( C_0^{(k)} = 1 \) and the recursive formula
   \[
   C_n^{(k)} = \frac{1}{2n} \sum_{m=1}^{n} (-1)^m \frac{(2nk)!}{2mk} B_{2mk} C_{n-m}^{(k)}, \quad (\forall n \geq 1);
   \]
4. for any positive integer \( k \) and any nonnegative integer \( n \), we have
   \[
   \zeta^*_R(\{2k\}^n) = C_n^{*,(k)} \frac{\lambda^{2nk}}{(2nk)!},
   \]
   where \( C_n^{*,(k)} \in \mathbb{Q} \) is given by \( C_0^{*,(k)} = 1 \) and the recursive formula
   \[
   C_n^{*,(k)} = -\frac{1}{2n} \sum_{m=1}^{n} \left( \frac{2nk}{2mk} \right) B_{2mk} C_{n-m}^{*,(k)}, \quad (\forall n \geq 1);
   \]
5. for any integers \( m, n \) with \( m \geq n \geq 0 \), we have
   \[
   \sum_{m_0 + m_1 + \cdots + m_{2n} = m - 2n} \zeta_R(\{2\}^{m_0}, 3, \{2\}^{m_1}, 1, \{2\}^{m_2}, 3, \{2\}^{m_3}, 1, \ldots, 1, \{2\}^{m_{2n}}) = (-1)^m \frac{2}{4^m(2m + 2)!} \binom{m + 1}{2n} \lambda^{2m}.
   \]
   In particular, for any nonnegative integer \( n \), we have
   \[
   \zeta_R(\{3, 1\}^n) = \frac{\lambda^{4n}}{2^{4n-1}(4n + 2)!}
   \]
   and
   \[
   \zeta_R(2, \{3, 1\}^n) + \zeta_R(3, 2, 1, \{3, 1\}^{n-1}) + \zeta_R(3, 1, 2, \{3, 1\}^{n-1})
   \]
   \[
   = \frac{\lambda^{4n}}{2^{4n-1}(4n + 2)!}.
   \]
\[ + \cdots + \zeta_R(\{3,1\}^n, 2) = -\frac{\lambda^{2n+2}}{4^{2n+1}(4n+3)!}; \]

(6) for any integers \(m, n\) with \(m \geq 2n \geq 0\), we have

\[
\sum_{m_0+m_1+\cdots+m_{2n}=m-2n} \zeta_R(\{2\}^{m_0}, 3, \{2\}^{m_1}, 1, \{2\}^{m_2}, 3, \{2\}^{m_3}, 1, \ldots, 1, \{2\}^{m_{2n-2}}, 3, \{2\}^{m_{2n-1}}, 1, \{2\}^{m_{2n}}) = \sum_{2j+k+u+2n \geq m, k, l, u, v \geq 0, j \geq 2, j \geq 0} (-1)^j \binom{k+l}{k} \binom{u+v}{u} \binom{i+1}{2j+1} \frac{\beta_k \beta_u}{2^{2i-1}(2i+2)!} \lambda^{2m},
\]

where \(\beta_k = \frac{2^{1-2k}-1}{2k} B_{2k}\). In particular, for any nonnegative integer \(n\), we have

\[
\zeta_R^*(\{3,1\}^n) = \sum_{2j+k+u+2n \geq 2n} (-1)^j \frac{\beta_k \beta_u}{2^{4j-1}(4j+2)!} \lambda^{4n}
\]

and

\[
+ \cdots + \zeta_R(\{3,1\}^n, 2) = \sum_{j=0}^{n} \left\{ \frac{1}{4^{2j+1}(4j+3)!} \sum_{k+u=2(n-j)} (-1)^k \beta_k \beta_u \right\} \lambda^{4n+2}.
\]

**Proof.** We get (1), (3) and (5) from Theorem 3.8 and [28, Theorem 4.5]. By (2.4) and Theorem 3.12, we get

\[
\sum_{n=0}^{\infty} \zeta_R(\{2\}^n) s^{2n} = \Gamma_R(s) \Gamma_R(-s) = \frac{\lambda s}{e^{\lambda s/2} - e^{-\lambda s/2}} \tag{3.25}
\]

which deduces (2).

Taking \(a = b = 2k\) in Lemma 3.10, we get

\[
\sum_{m=1}^{n} \zeta_{2mk}(2k)^{n-m} = n \zeta_{2k}(2k)^{n-m},
\]

which implies

\[
\zeta_R(\{2k\}^n) = \frac{1}{n} \sum_{m=1}^{n} \zeta_R(2mk) \zeta_R(\{2k\}^{n-m}).
\]

Hence by (1), we have

\[
\zeta_R^*(\{2k\}^n) = -\frac{1}{2n} \sum_{m=1}^{n} \frac{\lambda^{2mk}}{(2mk)!} B_{2mk} \zeta_R(\{2k\}^{n-m}).
\]

Then we get (4) by induction on \(n\).

Taking \(a = 3, b = 1, c = 2\) in (3.24) and using (2) and (5), we get (6). \(\square\)
Let $n = 0$ in Theorem 3.13, we get

$$\zeta_R(\{2\}^m) = \sum_{i + j + k = m, i, j, k \geq 0} \frac{(2^{1-i} - 1)(2^{1-j} - 1)}{4^{i}(2i + 1)!(2j)!}(2^{1-k} - 1)B_{2j}B_{2k}\lambda^{2m}.$$  

Comparing with the item (2) in Theorem 3.13, we obtain a relation among Bernoulli numbers.

**Corollary 3.14.** For any nonnegative integer $n$, we have

$$\sum_{i + j + k = n, i, j, k \geq 0} \frac{(2^{1-i} - 1)(2^{1-j} - 1)}{4^{i}(2i + 1)!(2j)!}(2^{1-k} - 1)B_{2j}B_{2k} = \frac{2^{1-2n} - 1}{(2n)!}B_{2n}.$$  

In fact, one can use

$$\sum_{n=0}^{\infty} \frac{2^{1-2n} - 1}{(2n)!}B_{2n}t^{2n} = \frac{t}{e^{t/2} - e^{-t/2}},$$  

$$\sum_{n=0}^{\infty} \frac{1}{4^n(2n + 1)!}t^{2n} = \frac{e^{t/2} - e^{-t/2}}{t}$$

to prove the above corollary.

If $Z_R = Z$ and $\lambda = 2\pi\sqrt{-1}$, we get the evaluation formulas for multiple zeta values and multiple zeta-star values:

(i) the item (1) of Theorem 3.13 becomes Euler’s formula

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}\pi^{2n}}{(2n)!}B_{2n};$$

(ii) Theorem 3.8 is ([1, 19])

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n + 1)!};$$

(iii) the item (2) of Theorem 3.13 is ([34, 48])

$$\zeta^*(\{2\}^n) = (-1)^{n}(2 - 2^{2n})\frac{\pi^{2n}}{(2n)!}B_{2n} = 2(1 - 2^{1-2n})\zeta(2n);$$

(iv) the item (3) of Theorem 3.13 becomes a formula of $\zeta(\{2k\}^n)$ ([1, Section 3]);

(v) the item (4) of Theorem 3.13 becomes a formula of $\zeta^*(\{2k\}^n)$;

(vi) the item (5) of Theorem 3.13 contains the formula

$$\zeta(T_{m,n}) = \frac{2\pi^{2m}}{(2m + 1)!} \left(\frac{m + 1}{2n + 1}\right),$$

and formulas of $\zeta(\{3, 1\}^n)$ and $\zeta(T_{2n+1,n})$ ([1, 2, Theorem 1, Theorem 2], [3, Corollary5.1]). Here we take $a = 3, b = 1, c = 2$ in (3.23);

(vii) the item (6) of Theorem 3.13 contains the formulas of $\zeta^*(T_{m,n}), \zeta^*(\{3, 1\}^n)$ and $\zeta^*(T_{2n+1,n})$ ([34, Theorem B],[44, Theorem 1.1]).

The items (3) and (4) of Theorem 3.13 give the formulas of $\zeta_R(\{2k\}^n)$ and $\zeta_R^*(\{2k\}^n)$. While there are constants defined by recursive relations in the formulas. In fact, we have
Lemma 3.16. Let \( t \) be a variable. Then we have

\[
\sum_{m=0}^{\infty} \frac{\lambda^{2mk}}{(2mk)!} B_{2mk} t^{2mk} = \frac{1}{k} \sum_{j=0}^{k-1} \left( \lambda \rho_k^j t e^{\lambda \rho_k^j t} + 1 \right) \frac{1}{2} \left( e^{\lambda \rho_k^j t} - 1 \right).
\]
Proposition 3.17. Assume that $Z_R : h^0 \to R$ satisfies the regularized double shuffle relations. Let $k$ be a positive integer and $t$ be a variable. Then we have

$$\sum_{n=0}^{\infty} (-1)^n \zeta_R(\{2k\}^n) t^{2nk} = \prod_{j=0}^{k-1} \frac{e^\frac{1}{2} \lambda \rho^j_t - e^{-\frac{1}{2} \lambda \rho^j_t}}{\lambda \rho^j_t}.$$ (3.32)

and

$$\sum_{n=0}^{\infty} \zeta_R(\{2k\}^n) t^{2nk} = \prod_{j=0}^{k-1} \frac{\lambda \rho^j_t}{e^\frac{1}{2} \lambda \rho^j_t - e^{-\frac{1}{2} \lambda \rho^j_t}}.$$ (3.33)

Proof. Taking $a = b = 2k$ in Lemma 3.10, we have

$$\sum_{m=1}^{n} (-1)^{m-1} z_{2mk} * z_{2k}^{n-m} = n z_{2k},$$

which induces

$$\sum_{m=1}^{n} (-1)^{m-1} \zeta_R(2mk) \zeta_R(\{2k\}^{n-m}) = n \zeta_R(\{2k\}^{n}).$$

By item (1) of Theorem 3.13, we obtain

$$2n \zeta_R(\{2k\}^n) = \sum_{m=1}^{n} (-1)^{m} \frac{\lambda^{2mk}}{(2mk)!} B_{2mk} \zeta_R(\{2k\}^{n-m}).$$

Therefore we have

$$\sum_{n=0}^{\infty} (-1)^n (2n + 1) \zeta_R(\{2k\}^n) t^{2nk}$$

$$= \sum_{n=0}^{\infty} (-1)^n t^{2nk} \sum_{m=0}^{n} (-1)^m \frac{\lambda^{2mk}}{(2mk)!} B_{2mk} \zeta_R(\{2k\}^{n-m})$$

$$= \sum_{m=0}^{\infty} \frac{\lambda^{2mk}}{(2mk)!} B_{2mk} t^{2mk} \sum_{n=0}^{\infty} (-1)^n \zeta_R(\{2k\}^n) t^{2nk}.$$ Set

$$f(t) = \sum_{n=0}^{\infty} (-1)^n \zeta_R(\{2k\}^n) t^{2nk}.$$

By Lemma 3.16, we get

$$\sum_{j=0}^{k-1} \left( \frac{\lambda \rho^j_t e^{\lambda \rho^j_t} + 1}{2 e^{\lambda \rho^j_t} - 1} \right) f(t) = \sum_{n=0}^{\infty} (-1)^n (2n + 1) k \zeta_R(\{2k\}^n) t^{2nk}$$
Hence \( f(t) \) satisfies the differential equation

\[
f'(t) = \sum_{j=0}^{k-1} \left( \frac{\lambda p_j^k e^{\lambda p_j^k t} + 1}{2 e^{\lambda p_j^k t} - 1} - \frac{1}{t} \right) f(t).
\]

Solving the differential equation, we get

\[
f(t) = C \prod_{j=0}^{k-1} \frac{e^{\lambda p_j^k t} - e^{-\lambda p_j^k t}}{t},
\]

where \( C \) is a constant. Since \( f(0) = 1 \), we get

\[
C = \left( \prod_{j=0}^{k-1} (\lambda p_j^k) \right)^{-1},
\]

which induces (3.32).

We get (3.33) from (2.4) and (3.32). \( \square \)

Note that if \( Z_R = Z \) and \( \lambda = 2\pi \sqrt{-1} \), (3.32) is equivalent to [1, Eq. (34)].

**Proof of Theorem 3.15.** We get the result from the expansions

\[
e^{s^2} - e^{-s^2} = \sum_{n=0}^{\infty} \frac{s^{2n}}{4^n (2n+1)!} s^{2n},
\]

and Proposition 3.17. \( \square \)

### 3.9. Ohno-Zagier relation and some corollaries.

In [29], the first named author proved that the Ohno-Zagier relation ([37]) can be deduced from the regularized double shuffle relations. In fact, set

\[
X_0(k, n, s) = \sum_{\text{wt}(k) = k \text{, dep}(k) = n, \text{let}(k) = s} \zeta_R(k).
\]

Then we have

**Theorem 3.18 ([29]).** Assume that \( Z_R : \mathbb{R} \to \mathbb{R} \) satisfies the regularized double shuffle relations. Let \( u, v, t \) be variables. Then we have

\[
\sum_{k \geq n + s, \alpha + \beta \geq 1} X_0(k, n, s) u^{k-n-s} v^n t^{s-1} = \frac{1}{uv-t} \left\{ 1 - \exp \left( \sum_{n=2}^{\infty} \frac{\zeta_R(n)}{n} (u^n + v^n - \alpha^n - \beta^n) \right) \right\}.
\]

Here \( \alpha \) and \( \beta \) are determined by \( \alpha + \beta = u + v \) and \( \alpha \beta = t \).

Denote the left-hand side of (3.34) by \( G_0(u, v, w) \). Since the right-hand side of (3.34) is symmetric about \( u \) and \( v \), we get
Corollary 3.19 \([23]\)]. Assume that \(Z_R: \mathfrak{h}^0 \to R\) satisfies the regularized double shuffle relations. Then for any integers \(k, n, s\) with \(k \geq n + s\) and \(n \geq s \geq 1\), we have

\[
\text{“duality of } X_0(k, n, s)\text{”} = X_0(k, n, s).
\]

In above, for any \(w \in \mathfrak{h}^0\), the duality of \(Z_R(w)\) is \(Z_R(\tau(w))\).

As in \([37]\), we have some consequences of Theorem 3.18. Let \(u = v = 0\) in Theorem 3.18. Since \(\alpha, \beta = \pm \sqrt{-t}\) in this case, we get

\[
\sum_{n=0}^{\infty} \zeta_R(\{2\}^n) \tau_t^n = \exp \left( \sum_{n=1}^{\infty} \frac{(\zeta_R(2n) \tau_t^n}{n} \right),
\]

which is of course a special case of \((2.3)\).

Let \(t = 0\) in Theorem 3.18. We take \(\alpha = u + v\) and \(\beta = 0\). Then we have the Aomoto-Drinfel’d-Zagier relation

\[
\sum_{m,n=1}^{\infty} \zeta_R(m+1, \{1\}_n^{n-1}) u^m v^n = 1 - \exp \left( \sum_{n=2}^{\infty} \frac{\zeta_R(n)}{n} (u^n + v^n - (u + v)^n) \right).
\]

Finally, let \(u = -v\) in Theorem 3.18. Then we have

\[
G(u, -u, t) = \sum_{k \geq 2s \geq 2} (-1)^s \left( \sum_{\text{wt}(k) = k, \text{det}(k) = s, k \text{ admissible}} (-1)^{\text{dep}(k)} \zeta_R(k) \right) u^{k-2s} t^{s-1}.
\]

On the other hand, since \(\alpha, \beta = \pm \sqrt{-t}\), we have

\[
G_0(u, -u, t) = \frac{1}{u^2 + t} \left\{ \exp \left( \sum_{n=1}^{\infty} \frac{\zeta_R(2n)}{n} (u^{2n} - (-t)^n) \right) - 1 \right\}
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{\zeta_R(2n)}{n} u^{2n} \right) \frac{\exp \left( - \sum_{n=1}^{\infty} \frac{\zeta_R(2n)}{n} (-t)^n \right)}{u^2 + t} \exp \left( - \sum_{n=1}^{\infty} \frac{\zeta_R(2n)}{n} u^{2n} \right)
\]

\[
= \Gamma_R(u) \Gamma_R(-u) \frac{1}{u^2 + t} \left( \frac{1}{\Gamma_R((-t)^{1/2}) \Gamma_R((-t)^{1/2})} - \frac{1}{\Gamma_R(u) \Gamma_R(-u)} \right).
\]

By Theorem 3.12, we have the expansions

\[
\Gamma_R(s) \Gamma_R(-s) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (2^{1-2n} - 1) \lambda^{2n} s^{2n}, \tag{3.35}
\]

and

\[
\frac{1}{\Gamma_R(s) \Gamma_R(-s)} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{2^{2n}(2n+1)!} s^{2n}. \tag{3.36}
\]

By (3.36), we have

\[
\frac{1}{u^2 + t} \left( \frac{1}{\Gamma_R((-t)^{1/2}) \Gamma_R((-t)^{1/2})} - \frac{1}{\Gamma_R(u) \Gamma_R(-u)} \right)
\]

\[
= \frac{1}{u^2 + t} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{2^{2n}(2n+1)!} (-t)^n - u^{2n}.
\]
\[= - \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{2^{2n}(2n+1)!} \sum_{m=1}^{n} u^{2n-2m} (-t)^{m-1} \]

\[= \sum_{n \geq m \geq 1} \frac{\lambda^{2n}}{2^{2n}(2n+1)!} (-1)^{m} u^{2n-2m} t^{m-1}. \]

Hence using (3.35), we find \( G_0(u, -u, t) \) is

\[= \sum_{n \geq m \geq s \geq 1} \frac{\lambda^{2k}}{2^{2k}(2k+1)!} \left( \sum_{n=0}^{k-s} \binom{2k+1}{2n} (2 - 2^{2n}) B_{2n} \right) u^{2k-2s} t^{s-1}. \]

Therefore, for any integers \( k, s \) with \( k \geq s \geq 1 \), we have

\[\sum_{\text{wt}(k) = 2k+1, \text{ht}(k) = s} (-1)^{\text{dep}(k)} \zeta_R(k) = 0\]

and the Le-Murakami relation

\[\sum_{\text{wt}(k) = 2k, \text{ht}(k) = s} (-1)^{\text{dep}(k)} \zeta_R(k) = \frac{\lambda^{2k}}{2^{2k}(2k+1)!} \sum_{n=0}^{k-s} \binom{2k+1}{2n} (2 - 2^{2n}) B_{2n}. \]

Note that the Le-Murakami relation for multiple zeta values was first proved in [27].

3.10. Restricted sum formulas of M. E. Hoffman and generalizations. In [20], M. E. Hoffman considered the restricted sum

\[\sum_{k_1 + \cdots + k_n = k} \zeta(2k_1, \ldots, 2k_n), \]

and used the symmetric functions to give two formulas of this restricted sum. We find that one can translate the formulas in [20] deduced from the properties of symmetric functions to identities in the algebra \( \mathfrak{h} \). Hence it is natural to consider more general sums

\[\sum_{k_1 + \cdots + k_n = k} \zeta(ak_1, \ldots, ak_n), \sum_{k_1 + \cdots + k_n = k} \zeta^*(ak_1, \ldots, ak_n),\]

where \( a \) is a positive integer with \( a > 1 \).

Now let \( a \) be a fixed positive integer. We set

\[E(t) = \sum_{j=0}^{\infty} z_a^j t^j \in \mathfrak{h}^1[[t]],\]

\[H(t) = \sum_{j=0}^{\infty} S(z_a^j) t^j = S(E(t)) \in \mathfrak{h}^1[[t]],\]

\[\overline{H}(t) = \sum_{j=0}^{\infty} S^{-1}(z_a^j) t^j = S^{-1}(E(t)) \in \mathfrak{h}^1[[t]],\]

\[P(t) = \sum_{j=1}^{\infty} z_{aj} t^{j-1} \in \mathfrak{h}^1[[t]].\]

Lemma 3.20. We have
(1)  \( E(-t) \ast H(t) = 1, \overline{H}(-t) \overline{E}(t) = 1; \)
(2)  \( P(t) = -H(t) \ast \frac{d}{dt} E(-t) = E(-t) \ast \frac{d}{dt} H(t); \)
(3)  \( P(t) = -E(t) \ast \frac{d}{dt} \overline{H}(-t) = \overline{H}(-t) \ast \frac{d}{dt} E(t); \)
(4)  \( \frac{d}{dt} E(-t) = -E(-t) \ast P(t), \frac{d}{dt} H(t) = H(t) \ast P(t); \)
(5)  \( \frac{d}{dt} E(t) = E(t) \overline{E}(t) \ast \frac{d}{dt} P(t), \frac{d}{dt} \overline{H}(-t) = -\overline{H}(-t) \overline{P}(t). \)

**Proof.** From (2), we get (1). By (2.1), we have
\[
\exp \left( -\sum_{n=1}^{\infty} \frac{z_n a(t^n)}{n} \right) = E(-t).
\]
Hence we get
\[
E(-t) \ast \left( -\sum_{n=1}^{\infty} \frac{z_n a(t^n)}{n} \right) = \frac{d}{dt} E(-t),
\]
which is
\[
-E(-t) \ast P(t) = \frac{d}{dt} E(-t).
\]
By (1), we have
\[
\frac{d}{dt} E(-t) \ast H(t) = -E(-t) \ast \frac{d}{dt} H(t).
\]
Then we get (2). Applying the map \( S^{-1} \), we get (3) from (2). Finally, we get (4) from (1), (2), and get (5) from (1), (3).

Now for integers \( k, n \) with \( k \geq n \geq 1 \), we define
\[
N_{k,n} = \sum_{k_1 + \cdots + k_n = k} z_{a_{k_1}} \cdots z_{a_{k_n}} \in h^1
\]
and their generating function
\[
F(t, s) = 1 + \sum_{k \geq n \geq 1} N_{k,n} t^k s^n \in h^1[[t, s]],
\]
where \( t, s \) are variables.

Corresponding to [20, Lemma 2.1], we have the following key result.

**Proposition 3.21.** We have
\[
E((s - 1)t) = E(-t) \ast F(t, s), \quad (3.37)
\]
\[
E((s + 1)t) = E(t) \overline{E}(t) \ast F(t, s). \quad (3.38)
\]
Moreover, we have
\[
F(t, s) = E((s - 1)t) \ast H(t) = E((s + 1)t) \overline{H}(-t). \quad (3.39)
\]
In fact, the two expressions in (3.39) for \( F(t, s) \) are equivalent. To see this, we need the following lemma (Comparing with [20, Lemma 2.3] and [14, Lemma 4.1]).

**Lemma 3.22.** For any integers \( k, n \) with \( k \geq n \geq 1 \), we have
\[
S^{-1}(N_{k,n}) = \sum_{i=1}^{n} (-1)^{n-i} \binom{k-i}{k-n} N_{k,i}, \quad (3.40)
\]
and
\[
S(N_{k,n}) = \sum_{i=1}^{n} \binom{k-i}{k-n} N_{k,i}, \quad (3.41)
\]
which in terms of generating functions are in fact

\[
S^{-1}(F(t, s)) = F\left(t(1 - s), \frac{s}{1 - s}\right)
\]

and

\[
S(F(t, s)) = F\left(t(s + 1), \frac{s}{s + 1}\right).
\]

**Proof.** We prove (3.41) by induction on \(n\). It is trivial to prove (3.41) in the case of \(n = 1\). Now assume that \(n > 1\) and \(k \geq n\). Then we have

\[
S(N_{k,n}) = \sum_{k_1 + \cdots + k_n = k \atop k_i \geq 1} (x^{k_1-1}y + x^{k_1})S(z_{k_2} \cdots z_{k_n})
\]

\[
= \sum_{l=1}^{k-n+1} (z_l + x^l)S(N_{k-l,n-1}).
\]

By the induction assumption, we obtain

\[
S(N_{k,n}) = \sum_{l=1}^{k-n+1} \sum_{i=1}^{n-1} \left(\frac{k - l - i}{k - l - n + 1}\right) z_l N_{k-l,i} + \sum_{l=1}^{k-n+1} (z_l + x^l) \sum_{i=1}^{n-1} \left(\frac{k - l - i}{k - l - n + 1}\right) x^l N_{k-l,i}.
\]

Since the first term on the right-hand of the above equation is

\[
\sum_{i=2}^{n} \sum_{k_1 + \cdots + k_i = k \atop k_i \geq 1} \left(\frac{k - k_1 - i + 1}{n - i}\right) z_{k_1} \cdots z_{k_i},
\]

and the second term is

\[
\sum_{l=1}^{n-1} \sum_{i=1}^{k_1 + \cdots + k_i = k \atop k_i \geq 1} \sum_{l=1}^{k_i - 1} \left(\frac{k - l - i}{n - i - 1}\right) z_{k_1} \cdots z_{k_i},
\]

we get (3.41).

One can prove (3.40) similarly, or just from (3.41) by using the binomial inversion formula. Finally, (3.42) follows from (3.40), and (3.43) follows from (3.41). \(\square\)

To prove Proposition 3.21, let us see what equations we need to show in \(h\).

**Lemma 3.23.** Let \(m, n\) be integers with \(m \geq n \geq 1\). Then we have

\[
\sum_{j=0}^{m-n} (-1)^j z_j^j * N_{m-j,n} = (-1)^{m-n} \binom{m}{n} z_a^m
\]

(3.44)

and

\[
\sum_{j=0}^{m-n} z_a^j * N_{m-j,n} = \binom{m}{n} z_a^m.
\]

(3.45)

**Proof.** We prove (3.44), and one can show (3.45) similarly. In fact, (3.45) follows from (3.44), because they are equivalent.
We proceed by induction on the pair \((n, m)\) with the lexicographic ordering. If \(n = 1\), we have to show that for any positive integer \(m\), the equation
\[
\sum_{j=0}^{m-1} (-1)^j z_a^j \ast z_{a(m-j)} = (-1)^m z_a^m
\]
holds. While this is just the first equation in (4) of Lemma 3.20. If \(m = n\), it is trivial that (3.44) holds.

Now assume that \(n > 1\) and \(m > n\). Then we have
\[
\sum_{j=0}^{m-n} (-1)^j z_a^j \ast N_{m-j,n} = N_{m,n} + \sum_{j=1}^{m-n} (-1)^j \sum_{k_1 + \cdots + k_a = m-j \atop k_i \geq 1} z_a(z_a^{-1} \ast z_{ak_1} \cdots z_{ak_a})
\]
\[
+ \sum_{j=1}^{m-n} (-1)^j \sum_{k_1 + \cdots + k_a = m-j \atop k_i \geq 1} z_{ak_1}(z_a^{-1} \ast z_{ak_2} \cdots z_{ak_a})
\]
\[
+ \sum_{j=1}^{m-n} (-1)^j \sum_{k_1 + \cdots + k_a = m-j \atop k_i \geq 1} z_a(k_1+1)(z_a^{-1} \ast z_{ak_2} \cdots z_{ak_a}).
\]

Denote the last three terms in the right-hand side of the above equation by \(\Sigma_1, \Sigma_2\) and \(\Sigma_3\), respectively. By the induction assumption, we get
\[
\Sigma_1 = z_a \sum_{j=0}^{m-1-n} (-1)^j z_a^j \ast N_{m-1-j,n} = (-1)^{m-n} \binom{m-1}{n} z_a^m.
\]

And by the induction assumption, we have
\[
\Sigma_2 = \sum_{l=1}^{m-n} z_{al} \sum_{j=1}^{m-l-(n-1)} (-1)^j z_a^j \ast N_{m-l-j,n-1}
\]
\[
= \sum_{l=1}^{m-n} z_{al} \left\{ (-1)^{m-l-(n-1)} \binom{m-l}{n-1} z_a^{m-l} - N_{m-l,n-1} \right\}
\]
\[
= \sum_{l=1}^{m-n} (-1)^{m-l-(n-1)} \binom{m-l}{n-1} z_{al} z_a^{m-l} - N_{m,n} + z_{a(m-n+1)} z_a^{n-1}.
\]

Finally, using the induction assumption, we find
\[
\Sigma_3 = \sum_{l=2}^{m-n+1} z_{al} \sum_{j=0}^{m-l-(n-1)} (-1)^j z_a^j \ast N_{m-l-j,n-1}
\]
\[
= - \sum_{l=2}^{m-n+1} \binom{m-l}{n-1} z_{al} z_a^{m-l}.
\]

Now (3.44) follows from the above immediately. □

**Proof of Proposition 3.21.** We get (3.37) from (3.44), and get (3.38) from (3.45). Then using Lemma 3.20, we get (3.39).

Corresponding to [20, Corollary 1.3], we have

**Corollary 3.24.** We have
\[
S(\mathcal{F}(t,s)) \ast \mathcal{F}(t,-s) = 1
\]
and
\[ S^{-1}(\mathcal{F}(t, s)) \mathcal{F}(t, -s) = 1. \]

**Proof.** Using Lemma 3.20, Proposition 3.21 and Lemma 3.22, we get the result. \( \square \)

Let \( s = 1 \) and \( s = -1 \) in Proposition 3.21, we get the corresponding result of [20, Corollary 2.2].

**Corollary 3.25.** We have
\[ \mathcal{F}(t, 1) = H(t), \quad \mathcal{F}(t, -1) = \overline{\mathcal{F}}(-t). \]

Hence for any positive integer \( k \), we have
\[
\sum_{n=1}^{k} N_{k, n} = S(z_{a}^{k}), \quad \sum_{n=1}^{k} (-1)^{k+n} N_{k, n} = S^{-1}(z_{a}^{k}).
\]

Moreover, if \( a > 1 \) and \( Z_{R} : \mathfrak{h}^{0} \to R \) is a \( \mathbb{K} \)-linear map, we have
\[
\sum_{n=1}^{k} \sum_{k_{1} + \cdots + k_{n} = k} \zeta_{R}(ak_{1}, \ldots, ak_{n}) = \zeta_{R}^{*}(\{a\}^{k})
\]
and
\[
\sum_{n=1}^{k} (-1)^{n+k} \sum_{k_{1} + \cdots + k_{n} = k} \zeta_{R}^{*}(ak_{1}, \ldots, ak_{n}) = \zeta_{R}(\{a\}^{k}).
\]

Considering the equations in \( \mathfrak{h}^{1} \) obtained from the two expressions in (3.39) of \( \mathcal{F}(t, s) \), we get the following result, which is corresponding to [20, Lemma 2.6].

**Proposition 3.26.** Let \( k, n \) be integers with \( k \geq n \geq 1 \). Then we have
\[
N_{k, n} = \sum_{j=0}^{k-n} (-1)^{k-n-j} \binom{k-j}{n} S(z_{a}^{j}) z_{a}^{k-j}
\]
\[
= \sum_{j=0}^{k-n} (-1)^{j} \binom{k-j}{n} S^{-1}(z_{a}^{j}) z_{a}^{k-j}.
\]

Moreover, if \( a > 1 \) and \( Z_{R} : \mathfrak{h}^{0} \to R \) is an algebra homomorphism, we have
\[
\sum_{k_{1} + \cdots + k_{n} = k} \zeta_{R}(ak_{1}, \ldots, ak_{n}) = \sum_{j=0}^{k-n} (-1)^{k-n-j} \binom{k-j}{n} \zeta_{R}^{*}(\{a\}^{j}) \zeta_{R}(\{a\}^{k-j}),
\]
\[
\sum_{k_{1} + \cdots + k_{n} = k} \zeta_{R}^{*}(ak_{1}, \ldots, ak_{n}) = \sum_{j=0}^{k-n} (-1)^{j} \binom{k-j}{n} \zeta_{R}(\{a\}^{j}) \zeta_{R}^{*}(\{a\}^{k-j}).
\]

If \( a \) is even, using Theorem 3.15, we get the following restricted sum formulas.

**Theorem 3.27.** Let \( m \) be a positive integer and \( Z_{R} : \mathfrak{h}^{0} \to R \) satisfy the regularized double shuffle relations. Then for any integers \( k, n \) with \( k \geq n \geq 1 \), we have
\[
\sum_{k_{1} + \cdots + k_{n} = k} \zeta_{R}(2mk_{1}, \ldots, 2mk_{n}) = (-1)^{n} \sum_{j=0}^{k-n} \binom{k-j}{n} \sum_{l_{0} + \cdots + l_{m-1} = m} \binom{m}{l_{0}, \ldots, l_{m-1}} z_{a}^{l_{0}+\cdots+l_{m-1}} z_{a}^{(k-j)}
\]

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\[
\times \left[ \prod_{i=0}^{m-1} \frac{(2 - 4n_i)B_{2m_i}}{(2n_i)!(2i + 1)!} \right] \sum_{\rho_m}^{m-1} \frac{2i(n_i + l_i)}{\lambda^{2km} 4^{km}},
\]

(3.46)

and

\[
\sum_{k_1 + \cdots + k_n = k, k_i \geq 1} \zeta_R^*(2mk_1, \ldots, 2mk_n) = \sum_{j=0}^{k-n} \left( \frac{2(2k + 1)}{4^j} \right) \binom{k-j}{n} \binom{2k + 1}{2j} (2 - 4^j) B_{2j}
\]

(3.48)

where \( \rho_m = e^{2\pi i m}. \)

When \( Z_R = Z \) and \( \lambda = 2\pi \sqrt{-1}, \) (3.46) is much simpler than [26, Theorem 7.1]. Let \( a = 1 \) in Theorem 3.27, we have

\[
\sum_{k_1 + \cdots + k_n = k, k_i \geq 1} \zeta_R^*(2k_1, \ldots, 2k_n) = \frac{(1)^n \lambda^{2k}}{4^k(2k + 1)!} \sum_{j=0}^{k-n} \binom{k-j}{n} \binom{2k + 1}{2j} (2 - 4^j) B_{2j}
\]

which correspond to [20, Theorem 1.4] and [14, Eq. (23)] in the case of \( Z_R = Z \) and \( \lambda = 2\pi \sqrt{-1}, \) respectively. Let \( a = 2 \) in Theorem 3.27, and use (3.30) in stead of (3.26), we get

\[
\sum_{k_1 + \cdots + k_n = k, k_i \geq 1} \zeta_R(4k_1, \ldots, 4k_n) = (-1)^{k+n} \frac{2\lambda^{4k}}{4^k(4k + 2)!} \sum_{j=0}^{k-n} \binom{k-j}{n} \binom{4k + 2}{4j}
\]

\[
\times \sum_{i=0}^{2j} (-1)^i \binom{4j}{2i} (2 - 4^i)(2 - 4^{2j-i}) B_{2i} B_{4j-2i}
\]

and

\[
\sum_{k_1 + \cdots + k_n = k, k_i \geq 1} \zeta_R^*(4k_1, \ldots, 4k_n) = (-1)^k \frac{2\lambda^{4k}}{4^k(4k + 2)!} \sum_{j=0}^{k-n} \binom{k-j}{n} \binom{4k + 2}{4j}
\]

\[
\times \sum_{i=0}^{2j} (-1)^i \binom{4j}{2i} (2 - 4^i)(2 - 4^{2j-i}) B_{2i} B_{4j-2i}.
\]

When \( Z_R = Z \) and \( \lambda = 2\pi \sqrt{-1}, \) the first equation above is just [14, Eq. (20)]. Hence we give a pure algebraic proof of [14, Conjecture 4.1] (Comparing with [45, Theorem 1.1]).
Remark 3.28. In [5], K.-W. Chen, C.-L. Chung and M. Eie also considered the restricted sum
\[ \sum_{\substack{k_1 + \cdots + k_n = k \geq 1}} \zeta(ak_1, \ldots, ak_n). \]

We would like to state below the corresponding result of [20, Propositions 2.4, 2.5], although it will not be used.

Proposition 3.29. (1) We have
\[ \frac{\partial F}{\partial t}(t, s) + (1 - s) \frac{\partial F}{\partial s}(t, s) = tP(t) * F(t, s) \]
and
\[ t \frac{\partial F}{\partial t}(t, s) - (s + 1) \frac{\partial F}{\partial s}(t, s) = -tP(t) \star F(t, s); \]
(2) For any positive integers \(k, n\) with \(k \geq n + 1\), we have
\[ k - n \sum_{j=1}^{k-n} z_{aj} * N_{k-j,n} = (k - n)N_{k,n} + (n + 1)N_{k,n+1} \]
and
\[ k - n \sum_{j=1}^{k-n} z_{aj} \star N_{k-j,n} = (n - k)N_{k,n} + (n + 1)N_{k,n+1}. \]

Proof. We get (1) from Proposition 3.21 and Lemma 3.20. Then (2) is deduced from (1). In fact, (1) and (2) are equivalent. \(\square\)

Below we set \(a = 2\). Then we get the result corresponding to [20, Theorem 1.2].

Proposition 3.30. If \(a = 2\) and \(Z_R : \mathfrak{h}^0 \to R\) satisfies the regularized double shuffle relations, we have
\[ Z_R(F(t, s)) = \frac{e^{\lambda \sqrt{(1-s)t/2}} - e^{-\lambda \sqrt{(1-s)t/2}}}{\sqrt{1-s} \left( e^{\lambda \sqrt{t/2}} - e^{-\lambda \sqrt{t/2}} \right)}, \]
\[ Z_R^*(F(t, s)) = \frac{e^{\lambda \sqrt{(1+s)t/2}} - e^{-\lambda \sqrt{(1+s)t/2}}}{e^{\lambda \sqrt{(1+s)t/2}} - e^{-\lambda \sqrt{(1+s)t/2}}}. \]

Proof. If \(a = 2\), by (2.4) and (3.25), we have
\[ Z_R(E((s - 1)t)) = \frac{e^{\lambda \sqrt{(1-s)t/2}} - e^{-\lambda \sqrt{(1-s)t/2}}}{\lambda \sqrt{(1-s)t}}, \]
(3.49)\
\[ Z_R(H(t)) = \frac{\lambda \sqrt{t}}{e^{\lambda \sqrt{t/2}} - e^{-\lambda \sqrt{t/2}}}, \]
(3.50)\
\[ Z_R^*(E((s + 1)t)) = \frac{\lambda \sqrt{(1+s)t}}{e^{\lambda \sqrt{(1+s)t/2}} - e^{-\lambda \sqrt{(1+s)t/2}}}, \]
\[ Z_R^*(H(-t)) = \frac{e^{\lambda \sqrt{t/2}} - e^{-\lambda \sqrt{t/2}}}{\lambda \sqrt{t}}. \]

Then we get the result from Proposition 3.31. \(\square\)

Finally, corresponding to [20, Theorem 1.1], we have
**Theorem 3.31.** Let \( k, n \) be integers with \( k \geq n \geq 1 \) and \( Z_R : h^0 \to R \) satisfy the regularized double shuffle relations. Then we have

\[
\zeta_R(2k_1, \ldots, 2k_n) = \sum_{j=0}^{[\frac{n}{2}]} \frac{\lambda^{2j}}{2^{n-j}(2j + 1)!} \binom{2n - 2j - 1}{n} \zeta_R(2k - 2j)
\]

\[
= \frac{1}{2^{2n-2}} \left( \binom{2n-1}{n} \right) \zeta_R(2k) - \sum_{j=1}^{[\frac{n}{2}]} \frac{\left( \binom{2n-2j-1}{n} \right)}{2^{2n-3}(2j + 1)B_{2j}} \zeta_R(2j) \zeta_R(2k - 2j).
\]

One can prove Theorem 3.31 by using (3.48) and [20, Theorem 1.5]. Here we provide a direct proof, which is similar as the proof of [20, Theorem 1.5]. In fact, by (3.49), we have

\[
Z_R(E((s - 1)t)) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}(1 - s)^n t^n}{2^{n}(2n + 1)!}
\]

\[
= \sum_{m=0}^{\infty} (-1)^m \left( \sum_{n \geq m} \binom{n}{m} \frac{\lambda^{2n}}{2^{2n}(2n + 1)!} t^n \right) s^m.
\]

Set

\[
f(t) = \frac{e^{\lambda \sqrt{t}/2} - e^{-\lambda \sqrt{t}/2}}{\lambda \sqrt{t}} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{2^{2n}(2n + 1)!} t^n,
\]

Then by (3.50), we have

\[
Z_R(F(t, s)) = \sum_{m=0}^{\infty} G_m(t) s^m,
\]

where

\[
G_m(t) = (-1)^m \frac{1}{f(t)} \frac{t^m}{m!} \frac{d^m}{dt^m} f(t).
\]

Set

\[
g(t) = e^{\lambda \sqrt{t}/2} + e^{-\lambda \sqrt{t}/2}.
\]

**Lemma 3.32.** For any nonnegative integer \( m \), we have

\[
\frac{(-1)^m 2^m m^m t^m}{m!} \frac{d^m}{dt^m} f(t) = \left( \sum_{j=0}^{[\frac{m}{2}]} \frac{\lambda^{2j}}{(2j)!} \binom{2m - 2j}{m} t^j \right) f(t)
\]

\[- \left( \sum_{j=0}^{[\frac{m+1}{2}]} \frac{\lambda^{2j}}{(2j + 1)!} \binom{2m - 2j - 1}{m} t^j \right) g(t).
\]

**Proof.** We proceed by induction on \( m \). It is trivial to check the case of \( m = 0 \).

Now assume that the formula is valid for \( m \), then by

\[
\frac{d}{dt} f(t) = -\frac{1}{2t} f(t) + \frac{1}{4t} g(t), \quad \frac{d}{dt} g(t) = \frac{\lambda^2}{4} f(t),
\]

we get

\[
\frac{(-1)^{m+1} 2^{m+1} (m+1)^{m+1} (m+1)!}{(m+1)!} \frac{d^{m+1}}{dt^{m+1}} f(t) = A_m f(t) + B_m g(t),
\]
where

\[ A_m = -\frac{4}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j}{m} (j-m) t^j + \frac{2}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j}{m} t^j \]

\[ + \frac{t}{m+1} \sum_{j=0}^{\infty} \lambda^{2j+2} \binom{2m-2j-1}{m} t^j, \]

\[ B_m = \frac{4}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j}{m} (j-m) t^j \]

\[ - \frac{1}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j}{m} t^j. \]

We have

\[ A_m = \frac{2}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j}{m} (2m-2j+1) t^j \]

\[ + \frac{1}{m+1} \sum_{j=1}^{\infty} \frac{\lambda^{2j}}{(2j-1)!} \binom{2m-2j+1}{m} (2m-2j+1) t^j \]

\[ = \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j+1}{m+1} t^j \]

\[ + \frac{1}{m+1} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j+1}{m} (m-2j+1) t^j \]

\[ + \frac{1}{m+1} \sum_{j=1}^{\infty} \frac{\lambda^{2j}}{(2j-1)!} \binom{2m-2j+1}{m} t^j \]

\[ = \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j+1}{m+1} t^j + \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j+1}{m} t^j \]

\[ - \frac{1}{m+1} \sum_{j=1}^{\infty} \frac{\lambda^{2j}}{(2j-1)!} \binom{2m-2j+1}{m} t^j \]

\[ + \frac{1}{m+1} \sum_{j=1}^{\infty} \frac{\lambda^{2j}}{(2j-1)!} \binom{2m-2j+1}{m} t^j \]

\[ = \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \binom{2m-2j+2}{m+1} t^j + C_m, \]
where
\[ C_m = \begin{cases} 0 & \text{if } m \text{ is even}, \\ \frac{\lambda^{m+1} \cdot t^{m+1}}{(m+1)!} & \text{if } m \text{ is odd}, \end{cases} \]
which implies that
\[ A_m = \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{\lambda^{2j}}{(2j)!} \left( \frac{2m - 2j + 2}{m+1} \right) t^j. \]
Similarly, we have
\[ B_m = -\sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{\lambda^{2j}}{(2j+1)!} \left( \frac{2m - 2j + 1}{m+1} \right) t^j. \]
Hence we finish the proof. \( \square \)

By Lemma 3.32, we get
\[ G_m(t) = 1 \frac{\lambda^{2j}}{(2j)!} \left( \frac{2m - 2j}{m} \right) t^j - \frac{1}{2^{2m}} \left( \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{\lambda^{2j}}{(2j+1)!} \left( \frac{2m - 2j + 1}{m+1} \right) t^j \right) g(t) \frac{f(t)}{f(t)}. \]

**Lemma 3.33.** Let \( Z_R : b^0 \rightarrow R \) satisfy the regularized double shuffle relations. Then we have
\[ g(t) = -4 \sum_{n=0}^{\infty} \zeta_R(2n) t^n. \]

**Proof.** Applying the operator \( \frac{d}{dt} \) to both sides of the equation
\[ e^{\lambda \sqrt{t}/2} - e^{-\lambda \sqrt{t}/2} = \lambda \sqrt{t} \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \zeta_R(2n) t^n \right), \]
we get
\[ \frac{\lambda}{4 \sqrt{t}} \left( e^{\lambda \sqrt{t}/2} + e^{-\lambda \sqrt{t}/2} \right) = \left( e^{\lambda \sqrt{t}/2} - e^{-\lambda \sqrt{t}/2} \right) \left( \frac{1}{2t} - \sum_{n=1}^{\infty} \zeta_R(2n) t^{n-1} \right). \]
Then the result follows from \( \zeta_R(0) = -\frac{1}{2} \). \( \square \)

Using Lemma 3.33, we have
\[ G_m(t) = 1 \frac{\lambda^{2j}}{(2j)!} \left( \frac{2m - 2j}{m} \right) t^j + \frac{1}{2^{2m-2}} \sum_{0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor \sum_{k \geq j} \frac{\lambda^{2j}}{(2j+1)!} \left( \frac{2m - 2j - 1}{m} \right) \zeta_R(2k - 2j) t^k. \]

Finally we get
\[ Z_R(F(t, s)) = \sum_{m \geq 0} \frac{1}{2^{2m}} \frac{\lambda^{2j}}{(2j)!} \left( \frac{2m - 2j}{m} \right) t^j s^m. \]
Comparing the coefficients of \( t^k s^n \), we get Theorem 3.31.

### 3.11. Some evidence of Brown-Zagier relation

In [47], D. Zagier proved that for any nonnegative integers \( n \) and \( m \), we have

\[
\zeta(\{2\}^n, 3, \{2\}^m) = \sum_{r=1}^{m+n+1} c_{m,n}^r \zeta(2r + 1) \zeta(\{2\}^{m+n+1-r}),
\]

(3.51)

\[
\zeta^*(\{2\}^n, 3, \{2\}^m) = \sum_{r=1}^{m+n+1} c_{m,n}^{*, r} \zeta(2r + 1) \zeta^*(\{2\}^{m+n+1-r}),
\]

(3.52)

where

\[
c_{m,n}^r = 2(-1)^r \left\{ \binom{2r}{2m + 2} - (1 - 2^{-2r}) \binom{2r}{2n + 1} \right\},
\]

\[
c_{m,n}^{*, r} = -2 \left\{ \binom{2r}{2m} - \delta_{r,n} - (1 - 2^{-2r}) \binom{2r}{2n + 1} \right\}.
\]

The formula (3.51) was used in [4]. The first named author gave a simpler proof of this evaluation formula in [31], and T. Terasoma showed that this formula is a consequence of associator relations in [43]. We give the following conjecture.

**Conjecture 3.34.** Let \( Z_R : h^0 \to R \) satisfy the regularized double shuffle relations. Then for any nonnegative integers \( m, n \), we have

\[
\zeta_R(\{2\}^n, 3, \{2\}^m) = \sum_{r=1}^{m+n+1} c_{m,n}^r \zeta_R(2r + 1) \zeta_R(\{2\}^{m+n+1-r}),
\]

(3.53)

\[
\zeta_R^*(\{2\}^n, 3, \{2\}^m) = \sum_{r=1}^{m+n+1} c_{m,n}^{*, r} \zeta_R(2r + 1) \zeta_R^*(\{2\}^{m+n+1-r}).
\]

(3.54)

In [30], the first named author showed that (3.53) and (3.54) are equivalent. Here we show that under the stuffle product, if summing over \( n + m = k \) for some fixed \( k \) of both sides of (3.53) (resp. (3.54)), we indeed get equal.

**Proposition 3.35.** For any positive integers \( a, b \) and any nonnegative integer \( k \), we have

\[
\sum_{n=m=k \atop n, m \geq 0}^{m+n+1} \sum_{r=1}^{m} c_{m,n}^r z_{b+(r-1)a} * z_a^{m+n+1-r} = \sum_{m=0}^{k} (-1)^m z_{b+ma} * z_a^{k-m} = \sum_{n=0}^{k} z_a^n z_b^k z_a^{-n},
\]

and

\[
\sum_{n=m=k \atop n, m \geq 0}^{m+n+1} \sum_{r=1}^{m} c_{m,n}^{*, r} z_{b+(r-1)a} * z_a^{m+n+1-r} = \sum_{m=0}^{k} z_{b+ma} * z_a^{k-m} = \sum_{n=0}^{k} z_a^n z_b^k z_a^{-n}.
\]

Moreover, if \( Z_R : h^0_a \to R \) is an algebra homomorphism, we have

\[
\sum_{n=m=k \atop n, m \geq 0}^{m+n+1} \sum_{r=1}^{m} c_{m,n}^r \zeta_R(2r + 1) \zeta_R(\{2\}^{m+n+1-r}) = \sum_{n=m=k \atop n, m \geq 0}^{m+n+1} \zeta_R(\{2\}^n, 3, \{2\}^m)
\]

\[
+ \sum_{m>0, n \geq 0}^{m+n+1} \sum_{k \geq j} \frac{1}{2^{2m-2}} \frac{\lambda^j}{(2j+1)!} \binom{2m-2j-1}{m} \zeta_R(2k-2j) t^k s^n.
\]
and
\[ \sum_{n+m=k \atop n, m \geq 0} \sum_{r=1}^{m+n+1} c^r_{m,n} z_R(2k+1)z_R((2)^{m+n+1-r}) = \sum_{n+m=k \atop n, m \geq 0} z_R((2)^n, 3, (2)^m). \]

**Proof.** We have
\[ \sum_{n+m=k \atop n, m \geq 0} \sum_{r=1}^{m+n+1} c^r_{m,n} z_R((r-1)a) * z_R^{m+n+1-r} = \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} z_R((r-1)a) * z_R^{k+1-r} \]
\[ = \sum_{r=1}^{k+1} \left( \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} \right) z_R((r-1)a) * z_R^{k+1-r} \]
and
\[ \sum_{n+m=k \atop n, m \geq 0} \sum_{r=1}^{m+n+1} c^r_{m,n} z_R((r-1)a) * z_R^{m+n+1-r} = \sum_{r=1}^{k+1} \left( \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} \right) z_R((r-1)a) * z_R^{k+1-r}. \]

Now for any integer \( r \) with \( 1 \leq r \leq k + 1 \), we have
\[ \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} = 2(-1)^r \sum_{n+m=k \atop n, m \geq 0} \left( \frac{2h}{2m+2} \right) - 2(-1)^r(1-2^{-2r}) \sum_{n+m=k \atop m, n \geq 0} \left( \frac{2h}{2m+1} \right) \]
\[ = 2(-1)^r \left( \sum_{i=1}^{r} \frac{2r}{2i} \right) - 2(-1)^r(1-2^{-2r}) \sum_{i=1}^{r} \left( \frac{2r}{2i-1} \right) \]
and
\[ \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} = -2 \sum_{i=0}^{r} \left( \frac{2r}{2i} \right) + 2 + 2(1-2^{-2r}) \sum_{i=1}^{r} \left( \frac{2r}{2i-1} \right). \]

Since
\[ 2 \sum_{i=0}^{r} \left( \frac{2r}{2i} \right) = 2^{2r} = 2 \sum_{i=1}^{r} \left( \frac{2r}{2i-1} \right), \]
we get
\[ \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} = (-1)^{r+1}, \quad \sum_{n+m=k \atop n, m \geq 0} c^r_{m,n} = 1. \]

Then with the help of Lemma 3.10 we get the result. \( \square \)

### 4. Further remarks

Besides the regularized double shuffle relations, there are some other systems of relations of multiple zeta values, which are also conjectured to give all algebraic relations. For example, the following relations are known
- associator relations ([7, 11]);
- Kawashima’s relation ([25, 41]);
- Kaneko-Yamamoto relation ([24]).
We recall some relationship between the regularized double shuffle relations and the relations listed above. It was proved in [6, 12] that the regularized double shuffle relations can be deduced from the associator relations. In [24], it was proved that the Kaneko-Yamamoto relations are equivalent to the regularized double shuffle relations, and under the duality formula, the Kawashima relation can be deduced from the Kaneko-Yamamoto relations. Since all the four systems of relations should be equivalent, it is interesting to consider other implication relations among these relations.

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