ON SOME POSITIVE DEFINITE FUNCTIONS

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ABSTRACT. We study the function \((1 - \|x\|)/(1 - \|x\|^r)\), and its reciprocal, on the Euclidean space \(\mathbb{R}^n\), with respect to properties like being positive definite, conditionally positive definite, and infinitely divisible.

1. Introduction

For each \(n \geq 1\), consider the space \(\mathbb{R}^n\) with the Euclidean norm \(\| \cdot \|\). According to a classical theorem going back to Schoenberg [11] and much used in interpolation theory (see, e.g., [8]), the function \(\varphi(x) = \|x\|^r\) on \(\mathbb{R}^n\), for any \(n\), is conditionally negative definite if and only if \(0 \leq r \leq 2\). It follows that if \(r_j, 1 \leq j \leq m\), are real numbers with \(0 \leq r_j \leq 2\), then the function
\[
g(x) = 1 + \|x\|^{r_1} + \cdots + \|x\|^{r_m}
\]
is conditionally negative definite, and by another theorem of Schoenberg, (see the statement \(S5\) in Section 2 below), the function
\[
f(x) = \frac{1}{1 + \|x\|^{r_1} + \cdots + \|x\|^{r_m}}
\]
is infinitely divisible. (A nonnegative function \(f\) is called infinitely divisible if for each \(\alpha > 0\) the function \(f(x)^\alpha\) is positive definite.) We also know that for any \(r > 2\), the function \(\varphi(x) = 1/(1 + \|x\|^r)\) cannot be positive definite. (See, e.g., Corollary 5.5.6 of [2].)

With this motivation we consider the function
\[
f(x) = \frac{1}{1 + \|x\| + \|x\|^2 + \cdots + \|x\|^m}, \quad m \geq 1,
\]
and its reciprocal, and study their properties related to positivity. More generally, we study the function
and its reciprocal. As usual, when \( \|x\| = 1 \) the right-hand side of (4) is interpreted as the limiting value \( 1/r \). This convention will be followed throughout the paper. The function (3) is the special case of (4) when \( r = m + 1 \).

Our main results are the following.

**Theorem 1.1.** Let \( 0 < r \leq 1 \). Then for each \( n \), the function \( f(x) = \frac{1 - \|x\|}{1 - \|x\|^r} \) on \( \mathbb{R}^n \) is conditionally negative definite. As a consequence, the function \( g(x) = \frac{1 - \|x\|^r}{1 - \|x\|} \) is infinitely divisible.

The case \( r \geq 1 \) turns out to be more intricate.

**Theorem 1.2.** Let \( n \) be any natural number. Then the function \( g(x) = \frac{1 - \|x\|^r}{1 - \|x\|} \) on \( \mathbb{R}^n \) is conditionally negative definite if and only if \( 1 \leq r \leq 3 \). As a consequence the function \( f(x) = \frac{1 - \|x\|^r}{1 - \|x\|} \) is infinitely divisible for \( 1 \leq r \leq 3 \).

In the second part of Theorem 1.2 the condition \( 1 \leq r \leq 3 \) is sufficient but not necessary. We will show that the function \( f \) is infinitely divisible for \( 1 \leq r \leq 4 \). On the other hand we show that when \( r = 9 \), \( f \) need not even be positive definite for all \( n \).

In the case \( n = 1 \) we can prove the following theorem.

**Theorem 1.3.** For every \( 1 \leq r < \infty \) the function \( f(x) = \frac{1 - |x|}{1 - |x|^r} \) on \( \mathbb{R} \) is positive definite.

### 2. Some classes of matrices and functions

Let \( A = [a_{ij}] \) be an \( n \times n \) real symmetric matrix. Then \( A \) is said to be **positive semidefinite (psd)** if \( \langle x, Ax \rangle \geq 0 \) for all \( x \in \mathbb{R}^n \), **conditionally positive definite (cpd)** if \( \langle x, Ax \rangle \geq 0 \) for all \( x \in \mathbb{R}^n \) for which \( \sum x_j = 0 \), and **conditionally negative definite (cnd)** if \( -A \) is cpd. If \( a_{ij} \geq 0 \), then for any real number \( r \), we denote by \( A^\circ r \) the \( r \)th Hadamard power of \( A \); i.e., \( A^\circ r = [a^r_{ij}] \). If \( A^\circ r \) is psd for all \( r \geq 0 \), we say that \( A \) is **infinitely divisible**.

Let \( f: \mathbb{R} \to \mathbb{R} \) be a continuous function. We say \( f \) is **positive definite** if for every \( n \), and for every choice of real numbers \( x_1, x_2, \ldots, x_n \), the \( n \times n \) matrix \([f(x_i - x_j)]\) is psd. In the same way, \( f \) is called cpd, cnd,
or infinitely divisible if the matrices \([ f(x_i - x_j) ]\) have the corresponding property.

Next, let \( f \) be a nonnegative \( C^\infty \) function on the positive half line \((0, \infty)\). Then \( f \) is called \textit{completely monotone} if

\[
(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } n \geq 0. \tag{5}
\]

According to a theorem of Bernstein and Widder, \( f \) is completely monotone if and only if it can be represented as

\[
f(x) = \int_0^\infty e^{-tx} \, d\mu(t),
\]

where \( \mu \) is a positive measure. \( f \) is called a \textit{Bernstein function} if its derivative \( f' \) is completely monotone; i.e., if

\[
(-1)^{n-1} f^{(n)}(x) \geq 0 \quad \text{for all } n \geq 1. \tag{6}
\]

Every such function can be expressed as

\[
f(x) = a + bx + \int_0^\infty (1 - e^{-tx}) \, d\mu(t), \tag{7}
\]

where \( a, b \geq 0 \) and \( \mu \) is a measure satisfying the condition \( \int_0^\infty (1 \wedge t) \, d\mu(t) < \infty \). If this measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure, and the associated density \( m(t) \) is a completely monotone function, then we say that \( f \) is a \textit{complete Bernstein function}.

The class of complete Bernstein functions coincides with the class of \textit{Pick functions} (or \textit{operator monotone functions}). Such a function has an analytic continuation to the upper half-plane \( \mathbb{H} \) with the property that \( \text{Im } f(z) \geq 0 \) for all \( z \in \mathbb{H} \). See Theorem 6.2 in [10].

For convenience we record here some basic facts used in our proofs. These can be found in the comprehensive monograph [10], or in the survey paper [11].

\textbf{S1.} A function \( \varphi \) on \((0, \infty)\) is completely monotone, if and only if the function \( f(x) = \varphi(\|x\|^2) \) is continuous and positive definite on \( \mathbb{R}^n \) for every \( n \geq 1 \).

\textbf{S2.} A function \( \varphi \) on \((0, \infty)\) is a Bernstein function if and only if the function \( f(x) = \varphi(\|x\|^2) \) is continuous and cnd on \( \mathbb{R}^n \) for every \( n \geq 1 \).

\textbf{S3.} If \( f \) is a Bernstein function, then \( 1/f \) is completely monotone.
S4. If \( f \) is a Bernstein function, then for each \( 0 < \alpha < 1 \), the functions \( f(x)^\alpha \) and \( f(x^\alpha) \) are also Bernstein. If \( f \) is completely monotone, then \( f(x^\alpha) \) has the same property for \( 0 < \alpha < 1 \).

S5. A function \( f \) on \( \mathbb{R} \) is cnd if and only if \( e^{-tf} \) is positive definite for every \( t > 0 \). Combining this with the Bernstein-Widder theorem, we see that if \( f \) is a nonnegative cnd function and \( \varphi \) is completely monotone, then the composite function \( \varphi \circ f \) is positive definite. In particular, if \( r > 0 \), and we choose \( \varphi(x) = x^{-r} \), we see that the function \( f(x)^{-r} \) is positive definite. In other words \( 1/f \) is infinitely divisible.

3. Proofs and Remarks

Our proof of Theorems 1.1 and 1.2 relies on the following proposition. This is an extension of results of T. Furuta \[5\] and F. Hansen \[6\].

Proposition 3.1. Let \( p, q \) be positive numbers with \( 0 < p \leq 1 \), and \( p \leq q \leq p + 1 \). Then the function \( f(x) = (1 - x^q)/(1 - x^p) \) on the positive half-line is operator monotone.

Proof. The case \( p = q \) is trivial; so assume \( p < q \). It is convenient to use the formula

\[
\frac{1 - x^q}{1 - x^p} = \frac{q}{p} \int_0^1 (\lambda x^p + 1 - \lambda)^{\frac{q-p}{p}} d\lambda,
\]

which can be easily verified. If \( z \) is a complex number with \( \text{Im} \, z > 0 \), then for \( 0 < \lambda < 1 \), the number \( \lambda z^p + 1 - \lambda \) lies in the sector \( \{ w : 0 < \text{Arg} \, w < p\pi \} \). Since \( 0 < \frac{q-p}{p} \leq \frac{1}{p} \), we see that \( (\lambda z^p + 1 - \lambda)^{\frac{q-p}{p}} \) lies in the upper half-plane. This shows that the function represented by (8) is a Pick function.

Now let \( 0 < r \leq 1 \). Choosing \( p = r/2 \) and \( q = 1/2 \), we see from Proposition 3.1 that the function \( \varphi(x) = \frac{1 - x^{1/2}}{1 - x^r} \) is operator monotone. Appealing to fact S2 we obtain Theorem 1.1.

Next let \( 1 \leq r \leq 3 \). Choosing \( p = 1/2 \) and \( q = r/2 \), we see from Proposition 3.1 that the function \( \varphi(x) = \frac{1 - x^{r/2}}{1 - x^{1/r}} \) is operator monotone. Again appealing to S2 we see that the function \( g(x) = \frac{1 - \|x\|^r}{1 - \|x\|} \) is cnd on the Euclidean space \( \mathbb{R}^n \) for every \( n \).

The necessity of the condition \( 1 \leq r \leq 3 \) is brought out by the Lévy-Khinchine formula. A continuous function \( g : \mathbb{R} \to \mathbb{C} \) is cnd if
and only it can be represented as

\[ g(x) = a + ibx + c^2x^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{itx} + \frac{itx}{1+t^2}\right) d\nu(t), \]

where \(a, b, c\) are real numbers, and \(\nu\) is a positive measure on \(\mathbb{R}\setminus\{0\}\) such that \(\int (t^2/(1+t^2))d\nu(t) < \infty\). See \[10\]. It is clear then that \(g(x) = O(x^2)\) at \(\infty\). So, if \(r > 3\), the function \(g(x)\) of Theorem 1.2 cannot be cnd on \(\mathbb{R}\). This proves Theorem 1.2 completely.

Now we show that \(f(x) = \frac{1-\|x\|}{1-\|x\|^r}\) is infinitely divisible for \(1 \leq r \leq 4\). The special case \(r = 4\) is easy. We have

\[ \frac{1-\|x\|}{1-\|x\|^4} = \frac{1}{1 + \|x\| + \|x\|^2 + \|x\|^3} = \frac{1}{1 + \|x\|} \cdot \frac{1}{1 + \|x\|^2}, \]

and we know that both \(\frac{1}{1 + \|x\|}\) and \(\frac{1}{1 + \|x\|^2}\) are infinitely divisible, and therefore so is their product. The general case is handled as follows.

By Proposition 3.1, the function \(\frac{1-\|x\|}{1-\|x\|^r}\) is operator monotone for \(1 \leq r \leq 2\). Repeating our arguments above, we see that \(\frac{1-\|x\|^2}{1-\|x\|^r}\) is an infinitely divisible function for \(1 \leq r \leq 2\). We know that \(\frac{1}{1+\|x\|}\) is infinitely divisible; hence so is the product

\[ \frac{1-\|x\|^2}{1-\|x\|^{2r}} \cdot \frac{1}{1 + \|x\|} = \frac{1-\|x\|}{1-\|x\|^{2r}}, \quad 1 \leq r \leq 2. \]

In other words \(\frac{1-\|x\|}{1-\|x\|^r}\) is infinitely divisible for \(2 \leq r \leq 4\).

We now consider what happens for \(r > 4\). In the special case \(n = 1\), Theorem 1.3 says that this function is at least positive definite for all \(r > 4\). By a theorem of Pólya (see [2], p.151) any continuous, nonnegative, even function on \(\mathbb{R}\) which is convex and monotonically decreasing on \([0, \infty)\) is positive definite. So Theorem 1.3 follows from the following proposition.

**Proposition 3.2.** The function

\[ f(x) = \frac{1-x}{1-x^r}, \quad 1 < r < \infty, \]  

on the positive half-line \((0, \infty)\) is monotonically decreasing and convex.

**Proof.** A calculation shows that

\[ f'(x) = \frac{(1-r)x^{r-1} + r x^{r-1} - 1}{(1-x^r)^2}, \]
and \[
f''(x) = \frac{1}{(1-x^r)^3} \left\{ r(1-r)x^{2r-1} + r(1+r)x^{2r-2} - r(1+r)x^{r-1} - r(1-r)x^{r-2} \right\}.\]

Since \(f''(x)\) is well-defined at 1, the function \(\varphi\) must have a zero of order at least three at 1. On the other hand, by the Descartes rule of signs, (see [9], p.46), \(\varphi(x)\) can have at most three positive zeros. Thus the only zero of \(\varphi\) in \((0, \infty)\) is at the point \(x = 1\).

Next note that when \(x\) is small, the last term of \(\varphi(x)\) is dominant, and therefore \(\varphi(x) > 0\). On the other hand, when \(x\) is large, the first term of \(\varphi(x)\) is dominant, and therefore \(\varphi(x) < 0\). Thus \(\varphi(x)\) is positive if \(x < 1\), and negative if \(x > 1\). This shows that \(f''(x) \geq 0\). Hence \(f\) is convex. Since \(f(0) = 1\), and \(\lim_{x \to \infty} f(x) = 0\), this also shows that \(f\) is monotonically decreasing, a fact which can be easily seen otherwise too.

Does the function \(f\) in (9) have any stronger convexity properties? We have seen that if \(1 \leq r \leq 2\), then the reciprocal of \(f\) is operator monotone. Hence by fact S3, \(f\) is completely monotone for \(1 \leq r \leq 2\). For \(r > 2\), however \(f\) is not even log-convex.

Recall that a nonnegative function \(f\) on \((0, \infty)\) is called log-convex if \(\log f\) is convex. If \(f', f''\) exist, this condition is equivalent to

\[
(f'(x))^2 \leq f(x) f''(x) \quad \text{for all } x. \tag{12}
\]

(See [12], p.485). A completely monotone function is log-convex.

**Proposition 3.3.** The function \(f(x) = \frac{1-x}{1-x^r}\) on \((0, \infty)\) is log-convex if and only if \(1 \leq r \leq 2\).

**Proof.** From the expressions (9), (10) and (11) we see that

\[
f(x)f''(x) - (f'(x))^2 = \frac{\psi(x)}{(1-x^r)^4}, \tag{13}
\]

where

\[
\psi(x) = (r-1)x^{2r} - 2rx^{2r-1} + rx^{2r-2} + (r^2 - r + 2)x^r - 2r(r-1)x^{r-1} - 1 + r(r-1)x^{r-2}. \tag{14}
\]
Using condition (12) we see from (13) that $f$ is log-convex if and only if $\psi(x) \geq 0$ for all $x$. If $r > 2$, it is clear from (14) that $\psi(0) = -1$, and $\psi$ is negative in a neighbourhood of 0. So $f$ is not log-convex.

We have already proved that when $1 < r < 2$, $f$ is completely monotone, and hence log-convex. It is instructive to see how the latter property can be derived easily using the condition (12). It is clear from (13) that $\psi$ must have a zero of order at least 4 at 1. On the other hand, there are just four sign changes in the coefficients on the right-hand side of (14). So by the Descartes rule of signs ([9], p.46) $\psi$ has at most four positive zeros. Thus $\psi$ has only one zero, it is at 1 and has multiplicity four. The coefficients of both $x^{2r}$ and $x^{r-2}$ in (14) are positive. Hence $\psi$ is always nonnegative.

Because of $S1$, the function $f(x) = \frac{1-\|x\|^r}{1-\|x\|^r}$ would be positive definite on $\mathbb{R}^n$ for every $n$, if and only if the function

$$h(x) = \frac{1-x^{1/2}}{1-x^{r/2}},$$

(15)
on $(0, \infty)$ were completely monotone. From $S4$ we see that this would be a consequence of the complete monotonicity of the function $f(x) = \frac{1-x^2}{1-x^r}$; but the latter holds if and only if $1 \leq r \leq 2$. We now show that when $r = 9$, the function $h$ in (15) is not even log convex.

For this we use the fact that $h$ is log convex if and only if

$$h \left( \frac{x+y}{2} \right)^2 \leq h(x)h(y) \quad \text{for all } x, y.$$  (16)

Choose $x = 9/25$, $y = 16/25$. Then $\frac{x+y}{2} = 1/2$. When $r = 9$, the function $h$ in (15) reduces to

$$h(x) = \left( \sum_{j=0}^{8} x^{j/2} \right)^{-1}.$$  

So, the inequality (16) would be true for the chosen values of $x$ and $y$, if we have

$$\sum_{j=0}^{8} \left( \frac{3}{5} \right)^j \sum_{j=0}^{8} \left( \frac{4}{5} \right)^j \leq \left( \sum_{j=0}^{8} \left( \frac{1}{\sqrt{2}} \right)^j \right)^2.$$  

A calculation shows that this is not true as, up to the first decimal place, the left-hand side is 10.7 and the right-hand side is 10.6.

We are left with some natural questions:
1. What is the smallest \( r_0 \) for which the function \( f \) of Theorem 1.2 is not infinitely divisible (or positive definite) for all \( \mathbb{R}^n \)? Our analysis shows that \( 4 < r_0 < 9 \).

2. What is the smallest \( n_0 \) for which there exists some \( r > 4 \), such that this function \( f \) is not positive definite on \( \mathbb{R}^{n_0} \)?

3. Is the function \( f \) in Theorem 1.3 infinitely divisible on \( \mathbb{R}^n \)? By Theorem 10.4 in [12] a sufficient condition for this to be true is log convexity of the function \( \frac{1-x}{1-x^r} \) on \( (0, \infty) \). We have seen that this latter condition holds if and only if \( 1 \leq r \leq 2 \). Note that we have shown by other arguments that \( f \) is infinitely divisible for \( 1 \leq r \leq 4 \).

Several examples of infinitely divisible functions arising in probability theory are listed in [12]. Many more with origins in our study of operator inequalities can be found in [3] and [7]. It was observed already in [4] that the function defined in (2) is infinitely divisible.

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