AN ELEMENTARY PROOF OF A FORMULA FOR SYT

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Abstract. The goal of this paper is to present a proof of a formula for the standard Young tableaux numbers that is brief, self-contained, and accessible to a wide audience. We begin by noting the one–one correspondence between the set of SYT of shape \((n_1, \ldots, n_d)\) and the set of Catalan words using \((n_1, \ldots, n_d)\) copies of the letters \(\nu_i\). Next, we establish a relevant algebraic result. Using this result, we then present the inductive proof.

1. Introduction

The generalized multidimensional Catalan number \(C(N = (n_1, \ldots, n_d))\) is the number of words that can be formed using \(n_i\) copies of the letter \(\nu_i\) such that for all integers \(r \leq \sum n_i\) the first \(r\) letters contain at least as many copies of \(\nu_i\) as \(\nu_j\) whenever \(i < j\). We refer to \(N\) as a point of dimension \(d\), and the words described above as \(d\)-dimensional words on \(N\). Recall that:

Theorem 1.1. The number of standard Young tableaux of shape \((n_1, n_2, \ldots, n_d)\), is equal to the generalized multidimensional Catalan number \(C(n_1, n_2, \ldots, n_d)\).

Proof. Let \(Y_N\) be the set of set of standard Young tableaux of shape \(N = (n_1, n_2, \ldots, n_d)\), and \(C_N\) be the set of Catalan words using \(n_i\) copies of the letter \(\nu_i\).

To show \(|Y_N| = |C_N|\), define a bijective map, \(\psi : Y_N \rightarrow C_N\), as follows: Let \(y \in Y_N\). Then \(\psi(y)\) is the word which contains a \(\nu_h\) in its \(k\)th position if and only if \(k\) appears in one of the boxes in the \(h\)th row of \(y\). Hence, \(|Y_N| = |C_N|\). □

2. An Important Preliminary Lemma

Lemma 2.1. Let \(V(t_1, \ldots, t_n)\) denote the Vandermonde polynomial on the indeterminates \(t_i\). Then the following holds:

\[
\sum_{i=1}^{n} \left( x_i \ast V(x_1, \ldots, x_i-1, \ldots, x_n) \right) =
\]

\[
\left( \sum_{i=1}^{n} x_i \right) \ast V(x_1, \ldots, x_n) - \left( \frac{n(n-1)}{2} \right) \ast V(x_1, \ldots, x_n).
\]

Proof. To show the two expressions (2.1) and (2.2) are equal we need to show:

Sub-Lemma 2.1(a) That the terms of highest degree, denote \(h\), are equal. Note \(h = \frac{n(n-1)}{2} + 1\).

Sub-Lemma 2.1(b) that the terms of degree \(h - 1\) are equal, and

Sub-Lemma 2.1(c) that, in both expressions, there are no terms of other degree.
Proof of Sub-Lemma 2.1(a). Let $D_d$ be the operation which returns only the degree $d$ part of an expression. It is apparent that:

$$D_h(2.1) = D_h(2.2) = \left( \sum_{i=1}^{n} x_i \right) \ast V(x_1, \ldots, x_n)$$

\(\Box\)

Proof of Sub-Lemma 2.1(c). First, note that $2.1$ is antisymmetric w.r.t. transposition of variables. Hence if $2.1$ is written as a linear combination of monomials, denote $L(2.1)$, then no nonzero monomial in $L(2.1)$ can have the same degree in any two variables. Hence the lowest degree monomial appearing in $L(2.1)$ is of degree:

$$0 + 1 + 2 + \ldots + n - 1 = \frac{n(n - 1)}{2} = h - 1.$$

So $L(2.1)$ contains no monomials of degree less than $h - 1$ (thus only monomials of degrees $h$ and $h - 1$). \(\Box\)

Proof of Sub-Lemma 2.1(b). From the statements above, any monomial in $L(2.1)$ of degree $h - 1$ must have the form: $P(\sigma_j(x_1, \ldots, x_n))$ where $\sigma_j \in S_n$, and:

$$P(t_1, \ldots, t_n) = t_1^{n-1}t_2^{n-2}\ldots t_n^0.$$

If $c_1$ is the coefficient of $m_1 = x_1^{n-1}x_2^{n-2}\ldots x_n^0$ in $L(2.1)$, then, by antisymmetry, the coefficient $c_j$ of $m_j = P(\sigma_j(x_1, \ldots, x_n))$ is given by $\text{sgn}(\sigma_j) \ast c_1$. In order to find $c_1$, first let $2.1_i$ represent the $i^{th}$ summand in the expression $2.1$, so that:

$$L(2.1) = \sum_{i=1}^{d} L(2.1_i),$$

where $L(2.1_i)$ refers to $2.1_i$ expressed as a linear combination of monomials. If we let $c_{1i}$ be the coefficient of $m_1$ in $L(2.1_i)$, then:

$$c_1 = \sum_{i=1}^{n} c_{1i}.$$

To find $c_{1i}$, recall:

$$2.1_i = x_i \ast V(x_1, \ldots, (x_i - 1), \ldots, x_n)$$

Now, the coefficient of $m_1 = x_1^{n-1}x_2^{n-2}\ldots x_n^0$ in $L(2.1_i)$ is the same as the coefficient of $m'_1 = x_1^{n-1}x_2^{n-2}\ldots x_i^{n-i-1}\ldots x_n^0$ in $V(x_1, \ldots, (x_i - 1), \ldots, x_n)$. To find the latter, we first note that all the factors of $m'_1$ are contained in the same term of the expansion of $V(x_1, \ldots, (x_i - 1), \ldots, x_n)$—namely the term:

$$x_1^{n-1}x_2^{n-2}\ldots(x_i - 1)^{n-i}\ldots x_n^0.$$

Using the binomial theorem we conclude that the above term contains a factor of $m'_1$ with coefficient $-(n - i)$. Hence $c_{1i} = -(n - i)$, whence it follows that:

$$c_1 = \sum_{i=1}^{n} c_{1i} = \sum_{i=1}^{n} -(n - i) = -\left( \frac{n(n - 1)}{2} \right)$$
is the coefficient of \(m_1\) in \(L(2.1)\). Therefore:

\[
D_{h-1}(2.1) = - \left( \frac{n(n-1)}{2} \right) \left( \sum_{j=1}^{n!} sgn(\sigma_j) * P(\sigma_j(x_1, \ldots, x_n)) \right) = - \left( \frac{n(n-1)}{2} \right) * V(x_1, \ldots, x_n) = D_{h-1}(2.2).
\]

Hence, Lemma 2.1 is established.

### 3. Counting Catalan Words

**Theorem 3.1.** The number \(C(n_1, \ldots, n_d)\) (and hence the number of SYT of shape \((n_1, \ldots, n_d)\)) is given by:

\[
(n_1 + \cdots + n_d)! \prod_{1 \leq i < j \leq d} \frac{n_i - n_j + j - i}{n_i + j - i}
\]

whenever \(i < j \rightarrow n_i \geq n_j - j + i\), and by 0 otherwise. (The left-hand factor gives the number of words on \(N\), the right-hand, the probability that a given word is Catalan.)

**Remark 3.2.** Expression (3.1) is equivalent to:

\[
\frac{(n_1 + \cdots + n_d)! \cdot V((n_1 - 1), \ldots, (n_d - d))}{(n_1 + d - 1)! \cdots (n_d + d - d)!}.
\]

**Remark 3.3.** Note that Theorem 3.1 gives the same result using the inputs \(N = (n_1, \ldots, n_d)\) and \(N' = (n_1, \ldots, n_d, 0)\). Further, the count \(C(N)\) and \(C(N')\) are obviously the same. Thus Theorem 3.1 holds on the \((d+1)\)-dimensional point \(N'\) if and only if it holds on the \(d\)-dimensional point \(N\).

**Proof of Theorem 3.2.** We proceed by strong induction on the dimension \(d\). Let \(d = 1\). Then \(N = (n_1)\) for some \(n_1 \in \mathbb{N}\), so there is one Catalan word on \(N\), as predicted by Theorem 3.1.

Now let \(d > 1\). Assume Theorem 3.1 holds for all dimensions \(d' < d\)–we denote this statement IH–1. We will show that IH–1 implies Theorem 3.1 holds for dimension \(d\). To do so we use induction on the sum of the coordinates of \(N\), denote \(k = n_1 + \cdots + n_d\).

**Case 1.** There are no Catalan words on \(N\).

**Subcase 1.1.** The conditions for expression (3.1) are not satisfied.

Theorem 3.1 correctly predicts that there are 0 Catalan words on \(N\).
Subcase 1.2. The conditions for expression (3.1) are satisfied.

Since there are no Catalan words on $N$, it follows that, in particular, the word given by ordering the letters of $N$ by increasing subscripts is not Catalan. Thus for some $i < j$, we have $n_i < n_j$. It follows that for some $h$ s.t. $i \leq h \leq j - 1$, we have $n_h < n_{h+1}$. Now, the assumption of the conditions for (3.1) gives us that $h < h + 1 \rightarrow n_h \geq n_{h+1} - (h + 1) + h$, so that $n_h \geq n_{h+1} - 1$. Combining these two inequalities gives that $n_h = n_{h+1} - 1$. But this means that (3.1) evaluated for $N = (n_1, \ldots, n_d)$ contains a factor of 0 (when the indices in the product in (3.1) are $i = h$ and $j = h + 1$), and is therefore equal to 0 as desired.

Case 2. There is at least one Catalan word on $N(n_1, \ldots, n_d)$.

First, note the following: Since we know there exists a Catalan word on $N$, it follows that $i < j$ implies $n_i \geq n_j$. By the Remark 3.3 and IH–1, we know that if $n_d = 0$, then Theorem 3.1 is valid for $N$. Therefore, suppose $n_d > 0$. From this supposition it follows that all $n_i > 0$.

Next, partition the Catalan words on $N$ by their last letter, $v_b$. Let $N''_b = (n_1, \ldots, n_b - 1, \ldots, n_d)$. Since the coordinates of $N$ satisfy all $n_i > 0$ and that $i < j \rightarrow n_i \geq n_j$, the coordinates of $N''_b$ (denote $(n''_b_1, \ldots, n''_b_m)$) satisfy that all $n''_b_1 \geq 0$, and that $i < j \rightarrow n''_b_1 \geq n''_b_1 - 1$. From the latter it follows that $i < j \rightarrow n''_b_1 \geq n''_b_1 - j + i$, so that $N''_b$ satisfies the conditions of (3.1).

Now we adopt the famous recursion of MacMahon [3] to our situation, to conclude that the number of Catalan words on $N$ is the sum of the number of Catalan words on each $N''_b$. We express this as:

$$C(N) = \sum_{b=1}^{d} C(N''_b) = \sum_{i=1}^{d} C(n_1, \ldots, (n_i - 1), \ldots, n_d).$$

(3.3)

Since the sum of the coordinates of each $N''_b$ is $k - 1$, by IH–2, Theorem 3.1 holds for all $N''_b$. Further, since each $N''_b$ satisfies the conditions of (3.1), this means that each $C(N''_b)$ can be evaluated using (3.1), or, equivalently, using (3.2). Hence, (3.3) becomes

$$C(N) = \sum_{i=1}^{d} \left( \frac{(n_1 + \cdots + n_d - 1)! * V((n_1 - 1), \ldots, (n_i - 1), \ldots, (n_d - d))}{(n_1 + d - 1)! \cdots (n_i + d - i - 1)! \cdots (n_d + d - d)!} \right)$$

$$= \sum_{i=1}^{d} \left( \frac{(n_1 + \cdots + n_d - 1)! * V((n_1 - 1), \ldots, (n_i - 1), \ldots, (n_d - d))}{(n_1 + d - 1)! \cdots (n_i + d - i)! \cdots (n_d + d - d)!} \right)$$

$$= \left( \frac{(n_1 + \cdots + n_d - 1)!}{(n_1 + d - 1)! \cdots (n_d + d - d)!} \right) \sum_{i=1}^{d} ((m_i + d - i) * V((m_1 - 1), \ldots, (m_i - 1), \ldots, (m_d - d))).$$

After making the substitution $m_i = n_i + d - i$ in the summation on the right in the expression above, that summation becomes

$$\sum_{i=1}^{d} \left( m_i * V((m_1 - d), \ldots, (m_i - d - 1), \ldots, (m_d - d)) \right)$$

$$= \sum_{i=1}^{d} \left( m_i * V(m_1, \ldots, (m_i - 1), \ldots, m_d) \right).$$

Now, by Lemma 2.1 this equals
\[
\left( \sum_{i=1}^{d} m_i \right) \ast V(m_1, \ldots, m_d) - \left( \frac{d(d-1)}{2} \right) \ast V(m_1, \ldots, m_d).
\]

After changing back to \( n_i = m_i - d + i \), the expression above becomes
\[
\left( \left( \sum_{i=1}^{d} n_i \right) + \frac{d(d-1)}{2} \right) \ast V((n_1 + d - 1), \ldots, (n_i + d - i), \ldots, (n_d + d - d))
\]
\[
- \left( \frac{d(d-1)}{2} \right) \ast V((n_1 + d - 1), \ldots, (n_i + d - i), \ldots, (n_d + d - d))
\]
\[
= \left( \sum_{i=1}^{d} n_i \right) \ast V((n_1 - 1), \ldots, (n_d - d)).
\]

Thus:
\[
C(N) = \frac{(n_1 + \cdots + n_d - 1)!}{(n_1 + d - 1)! \cdots (n_d + d - d)!} \ast \left( \sum_{i=1}^{d} n_i \right) \ast V((n_1 - 1), \ldots, (n_d - d))
\]
\[
= \frac{(n_1 + \cdots + n_d)! \ast V((n_1 - 1), \ldots, (n_d - d))}{(n_1 + d - 1)! \cdots (n_d + d - d)!},
\]

as predicted by Theorem 3.1.

Thus Theorem 3.1 holds on \( N \). Since \( N \) was an arbitrary point whose coordinates sum to \( k \), it holds on all such points. Hence, by induction, Theorem 3.1 holds on all points, \( M \), whose coordinates sum to some \( k' \geq 0 \), i.e., on any point \( M \).

Dimension \( d \) is no longer assumed

The arguments above show that if Theorem 3.1 holds for all \( d' < d \), then it holds for \( d \). Since it holds for \( d = 1 \), Theorem 3.1 is valid for all dimensions. \( \Box \)

References

[1] Laurent Manivel, *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*, American Mathematical Soc., 2001.

[2] N. J. A. Sloane, On-line Encyclopedia of Integer Sequences, \texttt{http://oeis.org}; Sequence A117506.

[3] P. A. MacMahon, *Combinatory Analysis*, I., Cambridge Univ. Press, 1915.

[4] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.