The Beurling-Wintner problem for characteristic functions

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Abstract: This paper concerns a long-standing problem raised by Beurling and Wintner on completeness of the dilation system \( \{ \varphi(kx) : k = 1, 2, \cdots \} \) generated by the odd periodic extension on \( \mathbb{R} \) of any \( \varphi \in L^2[0,1] \). Up to now there has been no explicit description of solutions of the Beurling-Wintner problem even for characteristic functions. We focus on characteristic function \( 1_V \) of an open subset \( V \) of \( (0,1) \) where \( V \) is the union of finitely many intervals with rational endpoints. Using substantially techniques from analytic number theory, we fully solved the Beurling-Wintner problem in most interesting situations and exhibit the explicit form of such \( V \). As a consequence, it yields a complete solution for the rational version of Kozlov’s problem. Moreover, we find that the Beurling-Wintner problem is closely related to the Twin Prime Conjecture and the Sophie Germain Prime Conjecture.

Keywords: Beurling-Wintner problem; Kozlov’s problem; Dirichlet character; Dirichlet series; Twin prime problem.

2010 AMS Subject Classification: 42A65; 47A16; 46E22; 30B50; 11M06.

1 Introduction

Let \( \varphi \) be a function in \( L^2 = L^2[0,1] \) whose odd 2-periodic extension as a function on the real line \( \mathbb{R} \) is still denoted by \( \varphi \). A fundamental unsolved problem in classical analysis is the following.

Beurling-Wintner problem: determine all \( \varphi \) such that linear span of the following periodic dilation system (p.d.s.)

\[ \{ \varphi(kx) : k = 1, 2, \cdots \} \]  

is dense on \( L^2 \).

The problem receives considerable attentions([Win, BM, Bou1, Bou2, Har, Sz, HW, Koz1, Koz2, Koz3, Ah, NGN, HLS, Mi1, Mi2, Mi3, Mi4, Mi5, No, DG, MNS]) over decades since it was first posed by Beurling and Wintner independently in 1940s ([Beu, Win]). Let \( \mathcal{C} \) be the set of functions generating complete periodic dilation systems. Up to now there
has been no explicit description of $\mathcal{C}$ even for characteristic functions, as first considered by Kozlov [Koz3].

**Kozlov’s Problem:** for which $r \in (0, 1]$ is the p.d.s. $\{\mathbf{1}\}$ of the characteristic function $\mathbf{1}_{(0,r)}$ of the open interval $(0, r)$ complete?

Only several partial results on the weaker Kozlov problem has been known. The case $r = 1$ was proved in [Ah] with a long proof. Nikolski proved that $\mathbf{1}_{(0,r)} \in \mathcal{C}$ for $r = \frac{1}{2}, \frac{2}{3}$ and $\mathbf{1}_{(0,r)} \notin \mathcal{C}$ for $r$ in a neighborhood of $\frac{1}{3}$ or $r = \frac{2}{p}$, where $p$ is an odd prime and $q$ is odd with $\sin^2 \frac{q \pi}{2p} < \frac{1}{p+1}$, which were claimed in [Koz3] (also see [Ni3, Ni5]). The proofs were exhibited in Nikolski’s talk [Ni4] in 2018. Moreover, he also proved that the p.d.s. of $\mathbf{1}_{(0,\frac{1}{3})}$ is incomplete.

It is conceivable that any solid progress on the original completeness problem will be highly non-trivial in light of both the problem itself and its connection to widely open unsolved problems in other areas. In fact, the completeness problem was shown to be equivalent to cyclicity of certain functions in the Hardy spaces of Dirichlet series as well as functions over infinite polydisk (see [HLS, Ni1], and we will review some details later in Section 2), and determining cyclic vectors in multi-variate Hardy space is still a longstanding open problem in operator theory. Some evidences suggest probable connections between the completeness problem and number theory. Cyclicity in Hardy spaces constitutes an operator theoretic version of the well-known Baez-Duarte’s reformulation [Ny, BD] of the Riemann Hypothesis (see [No]). In this paper, by virtue of a statement on completeness we present a sufficient condition for both the Twin Prime Conjecture and the conjecture on Sophie Germain primes to be false (see Remark 1.6).

We refer to [HLS, McCa] for a complete solution when the p.d.s. of an $L^2$-function $\varphi$ forms a Riesz basis of $L^2$, which is a special complete system. Also see some closely related works [Da1, Ro1, Ro2, CH1, CH2, HS, BBB1, BBB2, DGMS, QQ].

In this paper, we make it explicit by attacking the completeness problem with a substantial involvement of techniques from analytic number theory. We focus on characteristic function $\mathbf{1}_V$ of an open subset $V$ of $(0, 1)$ where $V$ is the union of finitely many intervals with rational endpoints. In particular, we completely settle the completeness problem for $\mathbf{1}_V$ if $V$ is a single interval:

**Theorem 1.1.** Let $\alpha, \beta$ be two rational numbers with $0 \leq \alpha < \beta \leq 1$ and put $I = (\alpha, \beta)$. Then $\mathbf{1}_I \in \mathcal{C}$ if and only if $I$ is one of the following 10 intervals:

$$(0, 1), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right), \left(0, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, 1\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{5}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{1}{6}, \frac{5}{6}\right).$$

As a consequence, we solve the rational version of Kozlov’s problem as follows:

**Corollary 1.2.** Suppose that $r$ is a rational number in $(0, 1)$. Then the p.d.s. of $\mathbf{1}_{(0,r)}$ is complete if and only if $r = 1, \frac{1}{2}, \frac{2}{3}$. 

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When \( V \) consists of more than one intervals, the situation in general becomes extremely complicated and resisting. However, we find many cases in which the completeness problem can be fully solved by developing new machineries to deal with zeros of Dirichlet series.

To begin with, let \( 0 \leq \frac{s_1}{t_1} < \frac{s'_1}{t'_1} < \cdot \cdot \cdot < \frac{s_N}{t_N} < \frac{s'_N}{t'_N} \leq 1 \) \((N \in \mathbb{N})\) be \(2N\) irreducible fractions.

Set
\[
V = (\frac{s_1}{t_1}, \frac{s'_1}{t'_1}) \cup \cdot \cdot \cdot \cup (\frac{s_N}{t_N}, \frac{s'_N}{t'_N}), \tag{1.2}
\]
and
\[
t_V = \text{lcm}(t_1, t'_1, \cdot \cdot \cdot, t_N, t'_N). \tag{1.3}
\]
Here, the notation “lcm” in (1.3) stands for “the least common multiple”. Then \( V \) is an open subset of \((0, 1)\) with \(N\) component intervals and rational boundary points, and can be rewrote as
\[
V = (\frac{\alpha_1}{t_V}, \frac{\beta_1}{t_V}) \cup \cdot \cdot \cdot \cup (\frac{\alpha_N}{t_V}, \frac{\beta_N}{t_V}), \tag{1.4}
\]
where
\[
\gcd(\alpha_1, \beta_1, \cdot \cdot \cdot, \alpha_l, \beta_l, t_V) = 1.
\]

In this situation Beurling-Wintner problem is to ask that whether one can write out all such \( V \) satisfying \( 1_V \in \mathcal{C} \). This problem is solvable when the least common denominator \( t_V \) is small. We explicitly determine the case \( t_V \leq 6 \) (see Theorem 6.1).

The case when \( t_V \) is a power of some prime is also completely solved. To our surprise, in this case any \( V \) with \( t_V \geq 7 \) and \( 1_V \in \mathcal{C} \) has specific form.

**Theorem 1.3.** Let \( V \) be given as in (1.4). Suppose that \( t_V \geq 7 \) and \( t_V = p^k \) for some prime \( p \) and some \( k \in \mathbb{N} \).

(1) If \( p = 2 \) then \( 1_V \in \mathcal{C} \) if and only if
\[
V = (0, \frac{1}{2^k}) \cup (\frac{2}{2^k}, \frac{3}{2^k}) \cup \cdot \cdot \cdot \cup (\frac{2^k - 2}{2^k}, \frac{2^k - 1}{2^k})
\]
or
\[
V = (\frac{1}{2^k}, \frac{2}{2^k}) \cup (\frac{3}{2^k}, \frac{4}{2^k}) \cup \cdot \cdot \cdot \cup (\frac{2^k - 1}{2^k}, 1).
\]

(2) If \( p \geq 3 \) then \( 1_V \in \mathcal{C} \) if and only if
\[
V = (0, \frac{1}{p^k}) \cup (\frac{2}{p^k}, \frac{3}{p^k}) \cup \cdot \cdot \cdot \cup (\frac{p^k - 1}{p^k}, 1)
\]
or
\[
V = (\frac{1}{p^k}, \frac{2}{p^k}) \cup (\frac{3}{p^k}, \frac{4}{p^k}) \cup \cdot \cdot \cdot \cup (\frac{p^k - 2}{p^k}, \frac{p^k - 1}{p^k}).
\]
Note that Theorem 1.3 fails for \( t_V = 3, 4, 5 \) (see Theorem 6.1).
The “if” part of Theorem 1.3 always holds. Set

\[
V_{t,0} = \begin{cases} 
(0, \frac{1}{t}) \cup (\frac{3}{t}, \frac{4}{t}) \cup \cdots \cup (\frac{t-2}{t}, \frac{t-1}{t}), & t = 2, 4, 6, \ldots; \\
(0, \frac{1}{t}) \cup (\frac{2}{t}, \frac{3}{t}) \cup \cdots \cup (\frac{t-1}{t}, 1), & t = 3, 5, 7, \ldots,
\end{cases}
\]

and

\[
V_{t,1} = \begin{cases} 
(\frac{1}{t}, \frac{2}{t}) \cup (\frac{3}{t}, \frac{4}{t}) \cup \cdots \cup (\frac{t-1}{t}, 1), & t = 2, 4, 6, \ldots; \\
(\frac{1}{t}, \frac{2}{t}) \cup (\frac{2}{t}, \frac{3}{t}) \cup \cdots \cup (\frac{t-2}{t}, \frac{t-1}{t}), & t = 3, 5, 7, \ldots.
\end{cases}
\]

We will show that \( 1_{V_{t,0}}, 1_{V_{t,1}} \in C \) holds for every \( t \geq 2 \) (see Example 3.6). But, more surprisingly, under some mild conditions, the converse remains true.

**Theorem 1.4.** Let \( V \) be given as in (1.4). Suppose that \( t_V \geq 7, 3 \nmid t_V, \) and there exists \( s \in \mathbb{N} \) with \( \gcd(s, t_V) = 1, \) such that \( \frac{s}{t_V} \) is a boundary point of \( V. \) Then \( 1_V \in C \) if and only if \( V = V_{t_V,0} \) or \( V_{t_V,1}. \)

The following two examples illustrate that neither of the two assumptions (“\( 3 \nmid t_V \)” and “there exists \( s \in \mathbb{N} \) with \( \gcd(s, t_V) = 1, \) such that \( \frac{s}{t_V} \) is a boundary point of \( V \)”)) in Theorem 1.4 can be dropped (see Examples 7.1 and 7.2 for details):

1. For \( V = (\frac{1}{15}, \frac{7}{15}) \cup (\frac{14}{15}, \frac{13}{15}) \) and \( W = (\frac{2}{15}, \frac{4}{15}) \cup (\frac{8}{15}, \frac{14}{15}) \) we have \( 1_V, 1_W \in C. \)

2. Suppose that \( p_1, p_2 \) are twin primes \((p_1 = p_2 + 2)\), and set

\[
V = \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \cup \left( \frac{3}{p_1}, \frac{3}{p_2} \right) \cup \cdots \cup \left( \frac{2m-1}{p_1}, \frac{2m-1}{p_2} \right) \cup \cdots \cup \left( \frac{p_1-2}{p_1}, 1 \right)
\]

and

\[
W = \left( 0, \frac{2}{p_1} \right) \cup \left( \frac{2}{p_2}, \frac{4}{p_1} \right) \cup \cdots \cup \left( \frac{2m-2}{p_2}, \frac{2m}{p_1} \right) \cup \cdots \cup \left( \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \right).
\]

Then \( 1_V, 1_W \in C. \)

From Theorem 1.4, it is reasonable to conjecture that most \( V \) with \( 1_V \in C \) has form \( V_{t,0} \) or \( V_{t,1}. \) One may ask that if “exceptions” also have some specific forms. Inspired by the example (2) above, we investigate the case that every denominator of boundary points of \( V \) is a prime. An open subset \( V \) of \((0, 1)\) is called non-degenerated if \((0, 1) \setminus V \) has no isolated points.

**Theorem 1.5.** Let \( V \) be a proper and non-degenerated open subset of \((0, 1)\), and \( \{ \frac{s_i}{t_i} \}_{i=1}^{M} \) be all boundary points of \( V \) in \((0, 1)\), where each \( \frac{s_i}{t_i} \) is an irreducible fraction. Assume that for any \( 1 \leq i \leq M, t_i \) is an odd prime, and not all \( t_i \) are the same. Then \( 1_V \in C \) if and only if there exist two primes \( p_1, p_2 \) such that one of the following situations is true:
(i) \( p_1 = p_2 + 2 \) and
\[
V = \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \cup \left( \frac{3}{p_1}, \frac{3}{p_2} \right) \cup \ldots \cup \left( \frac{2m-1}{p_1}, \frac{2m-1}{p_2} \right) \cup \ldots \cup \left( \frac{p_1-2}{p_1}, 1 \right);
\]

(ii) \( p_1 = p_2 + 2 \) and
\[
V = \left( 0, \frac{2}{p_1} \right) \cup \left( \frac{2}{p_2}, \frac{4}{p_1} \right) \cup \ldots \cup \left( \frac{2m-2}{p_2}, \frac{2m}{p_1} \right) \cup \ldots \cup \left( \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \right);
\]

(iii) \( p_1 = 2p_2 + 1 \) and
\[
V = \left( 0, \frac{2}{p_1} \right) \cup \left( \frac{1}{p_2}, \frac{4}{p_1} \right) \cup \ldots \cup \left( \frac{m-1}{p_2}, \frac{2m}{p_1} \right) \cup \ldots \cup \left( \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \right);
\]

(iv) \( p_1 = 2p_2 - 1 \) and
\[
V = \left( 0, \frac{1}{p_2} \right) \cup \left( \frac{2}{p_1}, \frac{2}{p_2} \right) \cup \ldots \cup \left( \frac{2m-2}{p_1}, \frac{m}{p_2} \right) \cup \ldots \cup \left( \frac{p_1-1}{p_1}, 1 \right).
\]

Remark 1.6. (1) In the above Situations (i), (ii), the number of all boundary points of \( V \) in \((0,1)\), \( M = p_2 \); in the Situations (iii), (iv), the number of component intervals of \( V \), \( N = p_2 \).

(2) A prime \( p \) is called a Sophie Germain prime if \( 2p + 1 \) or \( 2p - 1 \) also is a prime. It remains an unanswered question whether there are infinitely many Sophie Germain primes. Motivated by Theorem 1.1, we conjecture that there are only finitely many sets \( V \) such that each denominator \( t_i \) is an odd prime and \( 1 \in C \). However, from Theorem 1.5 this is equivalent to that the following two statements hold simultaneously:

(i) there are only finitely many pairs of twin primes.
(ii) there are only finitely many Sophie Germain primes.

Therefore, the Beurling-Wintner problem is closely related to the Twin Prime Conjecture and the Sophie Germain Prime Conjecture.

(3) By combining Theorems 1.4 and 1.5, we further completely solve the case when \( t_V \) is a product of two primes not less than 5.

(4) A more general assumption of Theorem 1.3 is presented in Section 8 (see Theorem 8.1).

The above-mentioned theorems will be proved in Sections 6-8. In sections 2,3,4,5 we provide both necessary technical preparation and conceptual guidelines.
As we will explain in Subsection 2.2, a classical treatment to the B-W problem for the characteristic function $1_V$ is invoking a unitary transform, the so-called Beurling-Wintner transform (see (2.5)), which reduces the problem into a cyclic vector problem for multiplier algebra of the Hardy space of Dirichlet series. Via the B-W transform, $1_V$ corresponds to a translation $D(s+1)$ of some Dirichlet series $D$ with periodic coefficients, and notions from the theory of periodic arithmetical functions play a key role in studying the cyclicity of $D(s+1)$, such as Dirichlet characters, Dirichlet-$L$ functions and the finite Fourier transform. More importantly, the result of Saias and Weingartner [SW] decomposes $D$, provided $D(s+1)$ to be cyclic, into a multiplication of some Dirichlet polynomial and some Dirichlet-$L$ function.

This allows us to reduce the problem to the existence of such decomposition and zeros of the Dirichlet polynomial

$$P(s) = \sum_{d \mid q} \frac{(f \ast \mu \psi)(d)}{d^s}$$

in some half-plane, where $q = 2t_V$, $\psi$ is a primitive Dirichlet character (see Subsection 2.1 for the definition), and $\frac{\sqrt{2}}{2\pi n} f$ is the sine-Fourier coefficients of $1_V$, that is,

$$\frac{\sqrt{2}}{2\pi n} f(n) = \int_0^1 1_V \cdot \sqrt{2} \sin(n\pi x) dx, \quad n = 1, 2, \ldots.$$ 

See Subsection 3.1 for details.

As it is expectably very difficult to check decomposability of $D$, our observation is that the finite Fourier transform

$$g(m) = \frac{1}{q} \sum_{n=1}^q f(n) e^{-2\pi i mn/q}, \quad m = 1, 2, \ldots$$

do actually carries the information from boundary points of $V$. More precisely, $g(m)$ coincides with the jump of $1_V$ at the point $\frac{m}{t_V}$. With the aid of the theory of the finite Fourier transform, we can establish new criteria of the B-W problems for $1_V$ by using the information from boundary points of $V$ instead of the sine-Fourier coefficients of $1_V$.

In Section 4, we determine the Dirichlet character $\psi$ appearing in (1.5), which is of independent number theoretic interest. We will show if $1_V \in \mathcal{C}$ then $\psi \equiv 1$ under the assumptions $t_V \geq 7$ and $3 \nmid t_V$ (Theorem 4.1). The key ingredient to the proof of Theorem 4.1 is a result in pure number theory (Lemma 4.2), which merely invokes the regularity of distribution of non-vanishing values of $g$.

In Section 5, we establish a formula for the number of component intervals of $V$ with $1_V \in \mathcal{C}$ by using the Euler totient function $\phi$. As a consequence, we show $t_V \leq 6$ when $V$ is a subinterval of $(0, 1)$ satisfying $1_V \in \mathcal{C}$, which together with Theorem 6.1 immediately gives Theorem 1.1. Moreover, let $\mathcal{V}_n$ ($n \in \mathbb{N}$) denote the set of all non-degenerated open subsets $V$ of $(0, 1)$ with rational boundary points which have at most $n$ component intervals. It will
be shown that there are at most \( \binom{2n}{t(n)+1} \) elements in
\[ \{ V \in \mathcal{V}_n : 1_V \in C \}, \]
where
\[ t(n) = \prod_{p \leq 4n+1} p^{[\log_6 6n]} \).

\section{Preliminaries}

This section consists of two parts. In the first part, we briefly recall some basic materials from analytic number theory. We refer the reader to \cite{Apo, Co, Da2, MV, O} for more details. In the second part, we list some preparatory results on zeros of functions. All these materials will be used throughout this paper.

\subsection{Some elements from analytic number theory}

An arithmetical function is a function defined on the set \( \mathbb{N} \) of positive integers. The Möbius function \( \mu \) and the Euler totient function \( \phi \) are among the most basic arithmetical functions, where
\[ \mu(n) = \begin{cases} 1, & n = 1; \\ (-1)^k, & n \text{ is the product of } k \text{ distinct primes}; \\ 0, & \text{otherwise}, \end{cases} \]
and
\[ \phi(n) = \# \{ m : 1 \leq m \leq n, \gcd(m, n) = 1 \}, \quad n = 1, 2, \ldots. \]

An arithmetical function \( f \) is said to be multiplicative if \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \) for any \( m, n \geq 1 \) with \( \gcd(m, n) = 1 \); \( f \) is said to be completely multiplicative (or totally multiplicative) if \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \) for any \( m, n \geq 1 \). Both \( \mu \) and \( \phi \) are multiplicative.

Dirichlet characters with modulus \( q \) arise from characters of the group \( (\mathbb{Z}/q\mathbb{Z})^\times \) of reduced residue classes mod \( q \). Recall that a character of an abelian group \( G \) is a group homomorphism from \( G \) to the multiplicative group \( \mathbb{C}^\times \) of nonzero complex numbers, and a character is said to be principle if its value is identically 1. Equivalently, Dirichlet characters \( \chi \mod q \) can be also defined to be completely multiplicative and periodic arithmetical functions with period \( q \), such that \( \chi(n) \neq 0 \) if \( \gcd(n, q) = 1 \), while \( \chi(n) = 0 \) if \( \gcd(n, q) > 1 \). Correspondingly, a Dirichlet character is said to be principle if it only takes values 0 or 1.

There are exactly \( \phi(q) \) distinct Dirichlet characters mod \( q \). Basic theory of character groups implies the following important orthogonality relations for Dirichlet characters mod \( q \) (see \cite[Theorem 6.16]{Apo} and \cite[Corollary 4.5]{MV}):
(1) If \( \gcd(m, q) = 1 \), then

\[
\sum_{\chi \mod q} \overline{\chi(n)} \chi(m) = \begin{cases} 
\phi(q), & n \equiv m \pmod{q}; \\
0, & n \not\equiv m \pmod{q}.
\end{cases}
\]  

(2.1)

(2) \( \sum_{n=1}^{q} \chi(n) = \begin{cases} 
\phi(q), & \chi \text{ is principal}; \\
0, & \text{otherwise}.
\end{cases} 
\)  

(2.2)

Suppose that \( \chi \) is a Dirichlet character mod \( q \). We say a Dirichlet character \( \psi \mod d \) induces \( \chi \) if \( d \mid q \) and \( \chi \) admits a decomposition \( \chi = \psi \chi_q \), where \( \chi_q \) is the principle Dirichlet character mod \( q \). The Dirichlet character \( \chi \) is said to be primitive if there is no proper divisor \( d \) of \( q \), such that \( \chi \) is induced by a Dirichlet character mod \( d \). From the definitions, it is readily seen that every Dirichlet character is uniquely induced by a primitive Dirichlet character.

The Legendre symbol \( (n \mid p) \) for an odd prime \( p \) is a primitive Dirichlet character mod \( p \), which is defined as

\[
(n \mid p) = \begin{cases} 
0, & p \mid n; \\
1, & n \equiv m^2 \pmod{p} \text{ for some } 1 \leq m \leq p - 1; \\
-1, & \text{otherwise}.
\end{cases}
\]

For any Dirichlet character \( \chi \mod q \), the Gauss sum associated with \( \chi \) is defined as

\[
\tau(n, \chi) := \sum_{m=1}^{q} \chi(m) e^{2\pi i mn/q}.
\]

The Gauss sum \( \tau(1, \chi) \) is abbreviated as \( \tau(\chi) \). The following formula allows us to calculate \( \tau(n, \chi) \) in terms of the primitive Dirichlet character inducing \( \chi \) [Has, pp. 444-450] (also see [MV, Theorem 9.12]).

**Lemma 2.1.** Let \( \chi \) be a Dirichlet character mod \( q \) induced by the primitive Dirichlet character \( \psi \mod q_0 \). Then for each \( n \geq 1 \),

\[
\tau(n, \chi) = \frac{\phi(q)}{\phi(\frac{q}{n})} \mu(\frac{q}{nq_0}) \psi(\frac{q}{nq_0}) \overline{\psi(\frac{n}{\hat{n}})} \tau(\psi),
\]

where \( \hat{n} = \gcd(n, \frac{q}{q_0}) \). In particular, when \( q_0 \nmid \frac{q}{\gcd(n, q)} \), we have \( \tau(n, \chi) = 0 \) since \( \psi(\frac{n}{\hat{n}}) = 0 \) in this case.

For an arithmetical function \( f \) we put

\[
D_f(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.
\]
Then $D_{fg} = D_f \cdot D_g$ for any arithmetical functions $f, g$, where the Dirichlet convolution $f \ast g$ is given by

$$(f \ast g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right)g(d), \quad n = 1, 2, \ldots.$$ 

If $\chi$ is a Dirichlet character, one defines its Dirichlet $L$-function by

$$L(s, \chi) = D_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s \in \mathbb{C}_1.$$ 

Here for a real number $\sigma$,

$$\mathbb{C}_\sigma := \{ s \in \mathbb{C} : \text{Re } s > \sigma \}.$$ 

By analytic continuation, this function can be extended to a meromorphic function on the whole complex plane. Dirichlet $L$-functions are generalizations of the Riemann zeta-function. The case $\chi \equiv 1$ corresponds to the Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.$$ 

Notice that $L$-function of $\chi$ can be written as an Euler product:

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \chi(p)p^{-s}\right)^{-1}, \quad s \in \mathbb{C}_1.$$ 

Therefore, each Dirichlet $L$-function has no zeros in $\mathbb{C}_1$.

Dirichlet $L$-function was introduced by Dirichlet to prove his celebrated theorem on arithmetic progressions. Dirichlet’s theorem states that for any pair $a, q$ of relatively prime positive integers, the arithmetic progression

$$a, a + q, a + 2q, \ldots$$

contains infinitely many primes.

From now on, we fix a positive integer $q$ in this subsection. Let $H_q$ denote the linear space of all periodic arithmetic functions with period $q$, equipped with the inner product

$$\langle f, g \rangle = \sum_{n=1}^{q} f(n)\overline{g(n)}, \quad f, g \in H_q.$$ 

For each divisor $d$ of $q$, and each Dirichlet character $\chi \mod \frac{q}{d}$, define

$$\xi_{\chi}(n) = \begin{cases} \chi\left(\frac{n}{d}\right), & d \mid n; \\ 0, & d \nmid n. \end{cases}$$ 

It was proved in [CDZ] that there are exactly $q$ functions of the type $\xi_{\chi}$, and these functions forms an orthogonal basis for $H_q$. 

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Let $\mathcal{P}_q$ denote the set of primitive Dirichlet characters whose modulus is a divisor of $q$. For $\psi \in \mathcal{P}_q$ with modulus $q_0$, set $E_{q,\psi}$ to be the subspace of $H_q$ spanned by 
$$ \{ \xi_{\psi, \frac{d}{q_0}} : d \mid \frac{q}{q_0} \}, $$
where $\psi_{\frac{d}{q}}$ is the Dirichlet character mod $\frac{q}{d}$ induced by $\psi$. Then $H_q$ admits an orthogonal decomposition \cite{CDZ} 
$$ H_q = \bigoplus_{\psi \in \mathcal{P}_q} E_{q,\psi}. \quad (2.3) $$
We give a characterization of functions in $E_{q,\psi}$ which will be used in the sequel.

**Lemma 2.2.** Suppose $f \in H_q$ and $\psi$ is a Dirichlet character mod $q_0$ in $\mathcal{P}_q$ (here $q_0 \mid q$). Then the following statements are equivalent:

1. $f \in E_{q,\psi}$;
2. $f(n) = f(\widehat{n})\psi(\frac{\widehat{n}}{q_0})$ for each $n \geq 1$, where $\widehat{n} = \gcd(n, \frac{q}{q_0})$;
3. $(f \ast \mu \psi)(n) = 0$ whenever $n \nmid \frac{q}{q_0}$;
4. $D_f(s) = P(s)L(s, \psi)$ for some Dirichlet polynomial $P$.

Before proving Lemma 2.2, we need an auxiliary lemma. By the fact that the Möbius inversion of a multiplicative arithmetical function is also multiplicative, one has

**Lemma 2.3.** For $q \geq 1$, let $\chi_q$ denote the principle Dirichlet character mod $q$. Then

$$(\mu \ast \chi_q)(n) = \begin{cases} 
\mu(n), & n \mid q; \\
0, & n \nmid q.
\end{cases}$$

**Proof of Lemma 2.2.** Let $\chi_m$ $(m \geq 1)$ denote the principle Dirichlet character mod $m$.

$(1) \Rightarrow (2)$. It suffices to prove the case $f = \xi_\chi$, where $\chi = \psi \chi_{\frac{d}{q}}$ with $d \mid \frac{q}{q_0}$, i.e., $\chi$ is the Dirichlet character mod $\frac{q}{d}$ induced by $\psi$.

Given $n \geq 1$, we want to show $\xi_\chi(n) = \xi_\chi(\widehat{n})\psi(\frac{\widehat{n}}{q_0})$, where $\widehat{n} = \gcd(n, \frac{q}{q_0})$. If $d \nmid n$ then $\xi_\chi(n) = \xi_\chi(\widehat{n}) = 0$. So we may assume $d \mid n$, which immediately implies $d \mid \widehat{n}$ since $d \mid \frac{q}{q_0}$. Therefore, we have

$$ \xi_\chi(n) = \chi(\frac{n}{d}) = \psi(\frac{n}{d})\chi_{\frac{d}{q}}(\frac{n}{d}), $$
$$ \xi_\chi(\widehat{n})\psi(\frac{\widehat{n}}{q_0}) = \chi(\frac{n}{d})\psi(\frac{n}{d}) = \psi(\frac{n}{d})\chi_{\frac{d}{q}}(\frac{n}{d})\psi(\frac{n}{d}) = \psi(\frac{n}{d})\chi_{\frac{d}{q}}(\frac{n}{d})\psi(\frac{n}{d}). $$

It remains to show $\chi_{\frac{d}{q}}(\frac{n}{d}) = \chi_{\frac{d}{q}}(\frac{\widehat{n}}{q_0})$ provided $\gcd(\frac{n}{d}, q_0) = 1$. If $\gcd(\frac{n}{d}, q_0) = 1$ then

$$ \gcd(\frac{n}{d}, \frac{q}{d}) = \gcd(\frac{n}{d}, \frac{q}{dq_0}) = \frac{1}{d} \gcd(n, \frac{q}{q_0}) = \frac{\widehat{n}}{d}, $$
which gives that $\gcd(\frac{n}{d}, \frac{q}{d}) = 1$ if and only if $\gcd(\frac{n}{d}, \frac{q}{d}) = \frac{n}{d} = 1$. This completes the proof.

(2)⇒(3). To reach a contradiction, we assume conversely that $(f \ast \mu \psi)(n) \neq 0$ for some $n \geq 1$ with $n \nmid \frac{n}{d_0}$. From the assumption in (2), we see that

$$(f \ast \mu \psi)(n) = \sum_{k|n} f(k) \mu(\frac{n}{k}) \psi(\frac{n}{k}) = \sum_{k|n} f(\hat{k}) \psi(\frac{n}{k}) \mu(\frac{n}{k}) \psi(\frac{n}{k}). \quad (2.4)$$

Note that $\hat{k}$ runs over all divisors of $\hat{n}$, and $\hat{k} = d$ for some divisor $d$ of $\hat{n}$ if and only if $k = ld$ with $\gcd(l, r_d) = 1$, where $r_d = \frac{q}{d_0}$. By (2.4), we have

$$(f \ast \mu \psi)(n) = \sum_{d|\hat{n}} f(d) \sum_{l|\frac{n}{d} \ (l, r_d) = 1} \psi(l) \mu(\frac{n}{ld}) \psi(\frac{n}{ld})$$

$$(f \ast \mu \psi)(n) = \sum_{d|\hat{n}} f(d) \psi(\frac{n}{d}) \sum_{l|\frac{n}{d}} \mu(\frac{n}{ld}) \chi_{rd}(l)$$

$$(f \ast \mu \psi)(n) = \sum_{d|\hat{n}} f(d) \psi(\frac{n}{d}) (\mu \ast \chi_{rd})(\frac{n}{d}).$$

This together with Lemma 2.3 implies $n \mid \frac{n}{d_0}$.

(3)⇒(4). Recall that $L(s, \psi)$ has no zeros in $\mathbb{C}_1$ and $\frac{1}{L(s,\psi)} = D_{\mu \psi}(s)$ on $\mathbb{C}_1$. Hence

$$P(s) = \frac{D_f(s)}{L(s, \psi)} = D_f(s) D_{\mu \psi}(s) = D_{f \ast \mu \psi}(s) = \sum_{d|\frac{n}{d_0}} \frac{(f \ast \mu \psi)(d)}{d^s}.$$  

In particular, $P$ is a Dirichlet polynomial.

(4)⇒(1). Assume that $D_f(s) = P(s)L(s, \psi)$ for some Dirichlet polynomial $P$. By (2.3), $f$ can be decomposed as $f = \sum_{\chi \in \mathcal{P}_q} f_{\chi}$, where $f_{\chi} \in E_{q, \chi}$. We have shown that for each $\chi \in \mathcal{P}_q$, there exist a Dirichlet polynomial $P_{\chi}$, such that $D_{f_{\chi}}(s) = P_{\chi}(s)L(s, \chi)$, which gives

$$P(s)L(s, \psi) = D_f(s) = \sum_{\chi \in \mathcal{P}_q} D_{f_{\chi}}(s) = \sum_{\chi \in \mathcal{P}_q} P_{\chi}(s)L(s, \chi).$$

Since the Dirichlet $L$-functions of primitive Dirichlet characters are linearly independent over the Dirichlet polynomials [KP Lemma 8.1], it follows that $P_{\psi} = P$ and $P_{\chi} = 0$ for any $\chi \in \mathcal{P}_q$ other than $\psi$. Then we have

$$f = f_{\psi} \in E_{q, \psi}. \quad \square$$

Combining Lemma 2.4 with Lemma 2.2, we immediately see the following.

**Lemma 2.4.** Suppose $\psi \in \mathcal{P}_q$. Then $\tau(\cdot, \chi) \in E_{q, \psi}$, where $\chi$ is the Dirichlet character mod $q$ induced by $\psi$. 

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Proof. It is clear that \( \tau(\cdot, \chi) \in H_q \) by the definition of the Gauss sum. Using Lemma 2.1, one can easily check that \( \tau(n, \chi) = \tau(\hat{n}, \chi)\psi(\frac{\hat{n}}{n}) \) for each \( n \geq 1 \), where \( \hat{n} = \gcd(n, \frac{q}{q_0}) \).

Finally, we introduce some duality between the subspaces \( E_{q, \psi} \) and \( E_{q, \bar{\psi}} \). For each \( \eta \in H_q \), one can define a linear transform \( \Gamma_\eta \) on \( H_q \) by putting

\[
(\Gamma_\eta f)(m) = \sum_{n=1}^{q} \eta(mn) f(n), \quad f \in H_q, n = 1, 2, \ldots.
\]

Then \( \Gamma_\eta(E_{q, \psi}) \subseteq E_{q, \bar{\psi}} \) for any \( \psi \in P_q \) [CDZ Corollary 1]. When \( \eta \) is the exponential function \( \eta(n) = \frac{1}{q} e^{-2\pi in/q} (n \geq 1) \), the transform \( T := \Gamma_\eta \) coincides with the Fourier transform on the group \( \mathbb{Z}/q\mathbb{Z} \) (here functions in \( H_q \) are naturally identified with functions on \( \mathbb{Z}/q\mathbb{Z} \)). Since the Fourier transform \( T \) is invertible and \( \dim E_{q, \psi} = \dim E_{q, \bar{\psi}} \), we further conclude the following result.

Lemma 2.5. If \( f \in H_q \) and \( \psi \in P_q \), then \( f \in E_{q, \psi} \) if and only if \( T f \in E_{q, \bar{\psi}} \).

2.2 The Beurling-Wintner transform and some preparatory results on zeros of functions

Let us first recall the Beurling-Wintner transform. Suppose that \( \varphi \in L^2 \) has Fourier-sine expansion

\[
\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n\pi x), \quad 0 < x < 1.
\]

The Beurling-Wintner transform \( \mathcal{D}\varphi \) of \( \varphi \in L^2 \) is the Dirichlet series

\[
\mathcal{D}\varphi(s) := \sum_{n=1}^{\infty} a_n n^{-s} \tag{2.5}
\]

This transform \( \mathcal{D} \) has the following property,

\[
\mathcal{D}\left( \sum_{k=1}^{K} c_k \varphi(kx) \right) = \left( \sum_{k=1}^{K} c_k k^{-s} \right) \cdot \mathcal{D}\varphi. \tag{2.6}
\]

That is to say, a linear combination of dilations of \( \varphi \) is transformed into multiplication of \( \mathcal{D}\varphi \) by a Dirichlet polynomial.

In fact, the B-W transform \( \mathcal{D} \) is a unitary transform from \( L^2 \) onto the Hardy space of Dirichlet series

\[
\mathcal{H}^2 := \{ D = \sum_{n=1}^{\infty} a_n n^{-s} : \|D\|^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty \},
\]

which is introduced in [HLS].

Writing a complex variable \( s = \sigma + it \), for each Dirichlet series \( D \), there exists a unique \( \sigma_{\alpha}(D) \in [-\infty, +\infty] \) (called the abscissa of absolute convergence of \( D \)) such that if \( \sigma > \sigma_{\alpha}(D) \),
the series $D$ converges absolutely, but not if $\sigma < \sigma_a(D)$. By the Cauchy-Schwarz inequality, for each $D \in \mathcal{H}^2$ one has $\sigma_a(D) \leq \frac{1}{2}$. Thus $D$ defines a holomorphic function in $\mathbb{C}_{\frac{1}{2}}$. It was shown in [HLS] that the set $\mathcal{H}^\infty$ of Dirichlet series that can be extended to bounded holomorphic functions on $\mathbb{C}_0$ coincides with the multiplier algebra of $\mathcal{H}^2$.

The Bohr transform $\mathcal{B}$ appeared much earlier. Let $p_j$ ($j \in \mathbb{N}$) be the $j$-th prime number. In 1913, Bohr noticed that the terms $p_1^{-s}, p_2^{-s}, \cdots$ in Dirichlet series possess some kind of independence [Bo]. By the variable substitution $z_1 = p_1^{-s}, z_2 = p_2^{-s}, \cdots$, a Dirichlet series $D$ is transformed into a power series $\mathcal{B}D$ in infinitely many variables. To be more specific, for any $n \geq 1$ let $n = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ be its prime factorization. Then one can define a finitely supported sequence $\alpha(n)$ for each natural number $n$ by putting

$$\alpha(n) = (\alpha_1, \cdots, \alpha_l, 0, 0, \cdots).$$

Denote the monomial $z_1^{\alpha_1} \cdots z_l^{\alpha_l}$ by $z^{\alpha(n)}$. The Bohr transform of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is defined to be

$$\mathcal{B}(\sum_{n=1}^{\infty} a_n n^{-s}) = \sum_{n=1}^{\infty} a_n z^{\alpha(n)}. \tag{2.7}$$

Set $\mathcal{F} = \mathcal{B}D$ and let $\mathbb{D}_2^\infty$ denote Hilbert’s multidisk

$$\mathbb{D}_2^\infty = \{ z = (z_1, z_2, \cdots) \in l^2 : |z_j| < 1 \text{ for each } j \in \mathbb{N} \}.$$ 

Let $\varphi \in L^2$, and $\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin(n \pi x)$ be its Fourier-sine expansion. Then by the Cauchy-Schwarz inequality, the power series $\mathcal{F}\varphi = \sum_{n=1}^{\infty} a_n z^{\alpha(n)}$ converges pointwise in $\mathbb{D}_2^\infty$.

The Hardy space $H^2(\mathbb{D}_2^\infty)$ over the infinite polydisk $\mathbb{D}_2^\infty$, defined as

$$H^2(\mathbb{D}_2^\infty) := \{ F = \sum_{n=1}^{\infty} a_n z^{\alpha(n)} : \| F \|^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty \},$$

is an analytic function space on $\mathbb{D}_2^\infty$ [Ni1, Ni5]. The multiplier algebra $H^\infty(\mathbb{D}_2^\infty)$ of $H^2(\mathbb{D}_2^\infty)$ is exactly the set of bounded holomorphic functions on $\mathbb{D}_2^\infty$ [Ni1].

The Bohr transform $\mathcal{B}$, restricted on $\mathcal{H}^2$, is a unitary transform from $\mathcal{H}^2$ onto $H^2(\mathbb{D}_2^\infty)$ which has the following properties [HLS]:

1. if $f$ is a multiplier of $\mathcal{H}^2$, then $\mathcal{B}f$ is a multiplier of $H^2(\mathbb{D}_2^\infty)$, and $\mathcal{B}(fh) = \mathcal{B}f \mathcal{B}h$, $h \in \mathcal{H}^2$;

2. the Bohr transform $\mathcal{B}$ establishes an isometric isomorphism from the Banach algebra $\mathcal{H}^\infty$ onto the Banach algebra $H^\infty(\mathbb{D}_2^\infty)$. 

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Following [HLS [N1]], we say that a Dirichlet series $D \in \mathcal{H}^2$ is cyclic if the multiplier invariant subspace generated by $D$ is the whole space $\mathcal{H}^2$, and correspondingly, a function $F \in H^2(\mathbb{D}_2^\infty)$ is cyclic if the multiplier invariant subspace generated by $F$ is the whole space $H^2(\mathbb{D}_2^\infty)$.

The following result, coming from [HLS] (also see [Ni1, Ni5]), translates the B-W problem into the cyclic vector problem in $\mathcal{H}^2$ or $H^2(\mathbb{D}_2^\infty)$, respectively.

**Proposition 2.6.** Suppose $\varphi \in L^2$. Then the following statements are equivalent:

1. the p.d.s. $\{\varphi(kx) : k \in \mathbb{N}\}$ of $\varphi$ is complete;
2. $D\varphi$ is cyclic in $\mathcal{H}^2$;
3. $F\varphi$ is cyclic in $H^2(\mathbb{D}_2^\infty)$.

Beurling first gave a necessary condition for $\varphi$ to be in $C$, which is the following [Beu] (also see [Ni5, Corollary 6.6.3]).

**Lemma 2.7.** If $\varphi \in C$, then $F\varphi$ has no zeros in $\mathbb{D}_2^\infty$.

We also record the following lemma, which is needed in this paper when we consider zeros of Dirichlet series. Note that if $D$ is a Dirichlet series with $\sigma_a(D) \leq 0$, then $D(s + \delta) \in \mathcal{H}^\infty$ for each $\delta > 0$, and thus $BD$ is well-defined on

$$\{(p_1^{-\delta}z_1, p_2^{-\delta}z_2, \cdots) : \text{for all } \delta > 0, \text{and } z = (z_1, z_2, \cdots) \in \mathbb{D}_2^\infty\},$$

which contains the set $\mathbb{D}_0^\infty$, where

$$\mathbb{D}_0^\infty = \{z \in \mathbb{D}_2^\infty : z \text{ has only finitely many nonzero entries}\}.$$

**Lemma 2.8.** Let $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $\sigma_a(D) \leq 0$. Then $BD$ has no zeros in $\mathbb{D}_0^\infty$ if and only if $a_1 \neq 0$ and $D$ has no zeros in $\mathbb{C}_0$.

*Proof.* Set $F = BD$. For the necessity, assume that $F$ has no zeros in $\mathbb{D}_0^\infty$. Let $p_j (j \in \mathbb{N})$ denote the $j$-th prime, and put

$$D_j(s) = \sum_{n=1}^{\infty} a_n \rho_j(n)n^{-s}, j = 1, 2, \cdots,$$

where $\rho_j$ is a completely multiplicative arithmetical functions defined by putting $\rho_j(p_1) = \cdots = \rho_j(p_j) = \rho_j(1) = 1$ and $\rho_j(p_{j+1}) = \rho_j(p_{j+2}) = \cdots = 0$. Then for each $j \in \mathbb{N}$, $\sigma_a(D_j) \leq 0$ and

$$D_j(s) = F(p_1^{-s}, \cdots, p_j^{-s}, 0, 0, \cdots), \quad s \in \mathbb{C}_0,$$
which yields that $D_j$ has no zeros in $\mathbb{C}_0$. Moreover, $\{D_j\}_{j \geq 1}$ converges uniformly to $D$ on each $\mathbb{C}_\sigma$ with $\sigma > 0$. Combining $a_1 = F(0) \neq 0$ with Hurwitz’s theorem shows that $D$ has no zeros in $\mathbb{C}_0$.

Now suppose that $a_1 \neq 0$ and $D$ has no zeros in $\mathbb{C}_0$. To reach a contradiction, assume conversely that $F$ has a zero $w = (w_1, \cdots, w_N, 0, 0, \cdots) \ (N \in \mathbb{N})$ in $\mathbb{D}_0^\infty$. Then one can take some $\delta > 0$ such that $|w_j| < p_j^{\delta} (1 \leq j \leq N)$, which implies that

$$B(D(s + \delta))(z) = F(p_1^{-\delta} z_1, p_2^{-\delta} z_2, \cdots), \quad z \in \mathbb{D}^\infty$$

also has a zero $\tilde{w} = (p_1^{\delta} w_1, \cdots, p_N^{\delta} w_N, 0, 0, \cdots) \in \mathbb{D}_0^\infty$. We conclude that $D(s + \delta)$ is not invertible in the algebra $\mathcal{H}^\infty$, otherwise $\frac{1}{D(s + \delta)} \in \mathcal{H}^\infty$ and since $B$ is multiplicative on $\mathcal{H}^\infty$,

$$1 = B(D(s + \delta))(\tilde{w}) \cdot B\left(\frac{1}{D(s + \delta)}\right)(\tilde{w}) = 0.$$ 

Hence we can find a sequence $\{s_k\}_{k \geq 1}$ in $\mathbb{C}_\delta$ satisfying $D(s_k) \to 0 \ (k \to \infty)$.

Since $D$ is zero-free and bounded on $\mathbb{C}_{\frac{\delta}{2}}$, the functions

$$h_k(s) = D(s + s_k - \delta), \quad k = 1, 2, \cdots$$

constitute a normal family on $\mathbb{C}_{\frac{\delta}{2}}$. Then one can take a subsequence $\{h_{k_i}\}_{i \geq 1}$ converging to a holomorphic function $h$ uniformly on compact subsets of $\mathbb{C}_{\frac{\delta}{2}}$. Since each $h_k$ is zero-free and

$$h(\delta) = \lim_{i \to \infty} h_{k_i}(\delta) = 0,$$

we have $h \equiv 0$ by Hurwitz’s theorem. On the other hand, recall that a Dirichlet series converges uniformly to its constant term when Re $s$ tends to $+\infty$, and hence we can choose $M$ sufficiently large so that $|D(s)| \geq \frac{|a_1|}{2}$ for $s \in \mathbb{C}_M$. This gives that $|h_{k_i}(s)| \geq \frac{|a_1|}{2}$ whenever Re $s > M$, which contradicts with $h \equiv 0$. This shows that $F = BD$ has no zeros in $\mathbb{D}_0^\infty$, completing the proof.

The following result follows immediately.

**Lemma 2.9.** If $D \in \mathcal{H}^2$ and $D$ is cyclic, then $D$ has a nonzero constant term and no zeros in $\mathbb{C}_\sigma$, where $\sigma = \max\{\sigma_d(D), 0\}$.

**Proof.** Assume $D \in \mathcal{H}^2$ is cyclic. Then $D(s + \sigma)$ is also cyclic in $\mathcal{H}^2$, which follows from

$$\inf_{\psi \in \mathcal{H}^\infty} ||\psi D_{\sigma} - 1|| \leq \inf_{\psi \in \mathcal{H}^\infty} ||\psi D_{\sigma} - 1|| \leq \inf_{\psi \in \mathcal{H}^\infty} ||\psi D - 1|| = 0,$$

where $D_{\sigma}(s) = D(s + \sigma)$, $\psi_{\sigma}(s) = \psi(s + \sigma)$. Combining Proposition 2.6 with Lemma 2.7 we see that $BD_{\sigma}$ has no zeros in $\mathbb{D}_0^\infty$. Therefore, Lemma 2.8 yields the desired conclusion. \qed
It was shown in [NGN] that Beurling’s necessary condition (Lemma 2.7) is also sufficient for a finite linear combinations of \(\sin(\pi x), \sin(2\pi x), \ldots\). One can use Lemma 2.8 to get its Dirichlet series version.

**Lemma 2.10.** Let \(P\) be a Dirichlet polynomial. Then \(P\) is cyclic in \(\mathcal{H}^2\) if and only if \(P\) has a nonzero constant term and no zeros in \(\mathbb{C}_0\).

In [SW], Saias and Weingartner considered the functions \(f \in H_q\) that do not belong to any \(E_{q,\psi}\) (namely \(f \notin \bigcup_{\psi \in P_q} E_{q,\psi}\)), and proved that the Dirichlet series \(D_f\) always have zeros in \(\mathbb{C}_1\). Their result can be refined as follows (remark that the equivalence of (2) and (4) has been established in [SW]).

**Proposition 2.11.** Suppose \(f \in H_q\). Then the following statements are equivalent:

1. \(D_f(s+1)\) is cyclic in \(\mathcal{H}^2\);
2. \(f(1) \neq 0\) and \(D_f\) has no zeros in \(\mathbb{C}_1\);
3. \(f(1) \neq 0\), \(f \in E_{q,\psi}\) for some \(\psi \in P_q\), and the Dirichlet polynomial
   \[
   P(s) = \sum_{d|\frac{n}{n_0}} \frac{(f * \mu)(d)}{d^s}
   \]
   has no zeros in \(\mathbb{C}_1\);
4. \(f(1) \neq 0\) and \(D_f(s) = P(s)L(s, \psi)\), where \(\psi \in P_q\) and \(P\) is a Dirichlet polynomial that has no zeros in \(\mathbb{C}_1\).

**Proof.** The implication (1)\(\Rightarrow\) (2) has been established by Lemma 2.9. The implications (2)\(\Rightarrow\) (3)\(\Rightarrow\) (4) immediately follows from the result of Saias and Weingartner and Lemma 2.2. (4)\(\Rightarrow\) (1): Let \(\chi_n\) \((n \geq 1)\) denote the principle Dirichlet character mod \(n\), and \(M\) the multiplier invariant subspace of \(\mathcal{H}^2\) generated by \(Q(s)L(s+1, \psi)\), where \(Q(s) = P(s+1)\). By Lemma 2.10 it suffices to show \(Q \in M\). To this end, we only need to prove \(Q(s)L(s+1, \psi \chi_k) \in M\) for each \(k > q\) since \(L(s+1, \psi \chi_k)\) converges to the constant function 1 in \(\mathcal{H}^2\)-norm as \(k \to \infty\).

For each \(n \geq 1\) put \(\rho_n = \mu * \chi_n\). Then using Lemma 2.3 we see that \(D_{\psi \rho_n}\) \((n \geq 1)\) is a Dirichlet polynomial, and hence \(D_{\psi \rho_n}(s+1) \in \mathcal{H}^\infty\). Since for any \(k > q\),

\[
\psi * \psi \rho_k! = \psi * \mu * \psi \chi_k! = \psi \chi_k!,
\]

we have

\[
Q(s)L(s+1, \psi \chi_k) = Q(s)L(s+1, \psi)D_{\psi \rho_k!}(s+1) \in M.
\]

This completes the proof. \(\square\)
3 Some criteria

In this section, we first give a preliminary criterion (Theorem 3.1 below) to determine when the characteristic function of an open subset of \((0, 1)\) consisting of finitely many component intervals with rational endpoints generates complete periodic dilation system. Then based on Theorem 3.1, we establish new criteria (Theorems 3.9, 3.10 and 3.12) by virtue of information from boundary points of \(V\). These criteria will be used in the subsequent sections.

3.1 A preliminary criterion

To state Theorem 3.1, we need to establish some notations, which will be used throughout the rest of this paper.

Let us recall the notations in Introduction. Suppose

\[
0 \leq \frac{s_1}{t_1} < \frac{s'_1}{t'_1} < \cdots < \frac{s_N}{t_N} < \frac{s'_N}{t'_N} \leq 1,
\]

where \(\frac{s_i}{t_i}, \frac{s'_i}{t'_i} (i = 1, 2, \cdots, N)\) are irreducible fractions. Set

\[
V = \left( \frac{s_1}{t_1}, \frac{s'_1}{t'_1} \right) \cup \cdots \cup \left( \frac{s_N}{t_N}, \frac{s'_N}{t'_N} \right),
\]

\[
t_V = \text{lcm}(t_1, t'_1, \cdots, t_N, t'_N).
\]

For simplicity, we abbreviate \(t_V\) to \(t\) if no confusion is caused. Then we can rewrite \(V\) as

\[
V = \left( \frac{\alpha_1}{t}, \frac{\beta_1}{t} \right) \cup \cdots \cup \left( \frac{\alpha_N}{t}, \frac{\beta_N}{t} \right).
\]

Since \(\{\sqrt{2}\sin(n\pi x) : n \in \mathbb{N}\}\) is a canonical orthonormal basis of \(L^2\), we have

\[
1_V = 2 \sum_{n=1}^{\infty} \left[ \int_0^1 1_V \sin(n\pi x) \, dx \right] \sin(n\pi x).
\]

Set

\[
q = 2t
\]

and

\[
f(n) = 2\pi n \int_0^1 1_V \sin(n\pi x) \, dx = 2\pi n \int_{V} \sin(n\pi x) \, dx, \quad n = 1, 2, \cdots.
\]

Then by (1.4),

\[
f(n) = 2\pi n \sum_{i=1}^{N} \int_{\frac{\alpha_i}{t}}^{\frac{\beta_i}{t}} \sin(n\pi x) \, dx = 2 \sum_{i=1}^{N} \left( \cos \frac{n\pi \alpha_i}{t} - \cos \frac{n\pi \beta_i}{t} \right).
\]
In particular, 
\[ f(1) = 2 \sum_{i=1}^{N} \left( \cos \frac{\pi \alpha_i}{t} - \cos \frac{\pi \beta_i}{t} \right) > 0 \]  
(3.4)
since the function \( \cos x \) is strictly decreasing on \([0, \pi]\). It follows from (3.3) that \( f \in H_q \), i.e., \( f(n + q) = f(n) \) for each \( n \geq 1 \). An application of the B-W transform to \( 1_V = \frac{1}{\pi} \sum_{n=1}^{\infty} f(n) \sin(n \pi x) \) gives
\[ D1_V(s) = \frac{\sqrt{2}}{2\pi} \sum_{n=1}^{\infty} \frac{f(n)}{n^{s+1}} = \frac{\sqrt{2}}{2\pi} D_f(s + 1). \]  
(3.5)
Recall that the Dirichlet series \( D_f \) is defined by
\[ D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \]
It is deduced from (3.5) that the B-W transform \( D1_V \) of \( 1_V \) is a translation of \( \frac{\sqrt{2}}{2\pi} D_f \).

**Theorem 3.1.** Let \( V, q \) and \( f \) be given as in (1.2), (3.1) and (3.2). Then the following statements are equivalent:

1. \( 1_V \in \mathcal{C} \);
2. \( D1_V \) has no zeros in \( \mathbb{C}_0 \);
3. \( F1_V \) has no zeros in \( \mathbb{D}_2^\infty \);
4. there exists some \( \psi \in \mathcal{P}_q \) such that \( f \in E_{q, \psi} \) and the Dirichlet polynomial
\[ P(s) = \sum_{d|q_0} \frac{(f * \mu \psi)(d)}{d^s} \]
has no zeros in \( \mathbb{C}_1 \), where \( q_0 \) is the modulus of \( \psi \).

**Proof.** Since \( D1_V(s) = \frac{\sqrt{2}}{2\pi} D_f(s + 1) \) for some \( f \in H_q \) with \( f(1) \neq 0 \) (see (3.5) and (3.4)), the equivalence of (1), (2) and (4) immediately follows from Proposition 2.11. Moreover, the implication (1)\( \Rightarrow \) (3) is due to Beurling’s necessary condition (Lemma 2.7), and the implication (3)\( \Rightarrow \) (2) follows immediately from Lemma 2.8.

**Remark 3.2.** (1) When \( 1_V \in \mathcal{C} \), the primitive Dirichlet character \( \psi \) in Theorem 3.1 (4) is uniquely determined by the requirement \( f \in E_{q_1, \psi_1} \). In fact, if \( f \in E_{q_1, \psi_1} \) for some
$q_1 \in \mathbb{N}$ and some $\psi_1 \in \mathcal{P}_{q_1}$, then it follows from Lemma 2.2 that there exist Dirichlet polynomials $P$ and $P_1$ such that

$$P_1(s)L(s, \psi_1) = D_f(s) = P(s)L(s, \psi).$$

By the fact that the Dirichlet $L$-functions of primitive Dirichlet characters are linearly independent over the Dirichlet polynomials [KLP, Lemma 8.1], one has $\psi_1 = \psi$.

(2) Beurling’s necessary condition (Lemma 2.7) is also sufficient for the characteristic function $1_V$ of an open subset $V$ of $(0,1)$ consisting of finitely many component intervals with rational endpoint.

Combining Theorem 3.1 with Remark 3.2 (1), one obtains

**Corollary 3.3.** Let $V, q$ and $f$ be given as in (1.2), (3.1) and (3.2). If $f \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$, then $1_V \in \mathcal{C}$ if and only if the Dirichlet polynomial

$$P(s) = \sum_{d \mid q_0} \frac{(f \ast \mu \psi)(d)}{d^s}$$

has no zeros in $\mathbb{C}_1$, where $q_0$ is the modulus of $\psi$.

Since $f(1) \neq 0$ (see (3.4)), we have the following direct consequence of Corollary 3.3

**Corollary 3.4.** Let $V, q$ and $f$ be given as in (1.2), (3.1) and (3.2). If $f \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$, and the modulus of $\psi$ is $q$, then $1_V \in \mathcal{C}$.

We can obtain some examples of open sets $V$ with $1_V \in \mathcal{C}$ by using Theorem 3.1

**Example 3.5.** For $V = (0,1)$, we have

$$f(n) = 2(\cos 0 - \cos(n\pi)) = 2 - 2 \cos(n\pi) = 4\chi_2(n),$$

where $\chi_2$ is the Dirichlet character mod 2. Therefore,

$$D1_V(s) = \frac{\sqrt{2}}{2\pi} D_f(s + 1) = \frac{\sqrt{2}}{\pi} L(s + 1, \chi_2)$$

has no zeros in $\mathbb{C}_0$, forcing $1_V \in \mathcal{C}$.

**Example 3.6.** Set

$$V_{t,0} = \begin{cases} (0, \frac{1}{t}) \cup (\frac{2}{t}, \frac{3}{t}) \cup \cdots \cup (\frac{t-2}{t}, \frac{t-1}{t}), & t = 2, 4, 6, \cdots; \\ (0, \frac{1}{t}) \cup (\frac{1}{t}, \frac{2}{t}) \cup \cdots \cup (\frac{t-1}{t}, 1), & t = 3, 5, 7, \cdots, \end{cases}$$

and

$$V_{t,1} = \begin{cases} (\frac{1}{t}, \frac{2}{t}) \cup (\frac{2}{t}, \frac{3}{t}) \cup \cdots \cup (\frac{t-1}{t}, 1), & t = 2, 4, 6, \cdots; \\ (\frac{1}{t}, \frac{2}{t}) \cup (\frac{2}{t}, \frac{3}{t}) \cup \cdots \cup (\frac{t-2}{t}, \frac{t-1}{t}), & t = 3, 5, 7, \cdots. \end{cases}$$
Then for \( t \geq 2, 1_{V_{t,0}}, 1_{V_{t,1}} \in C \). In fact, set \( \varphi \equiv 1 \) on \((0, 1)\) and extend \( \varphi \) to an odd 2-periodic function on \( \mathbb{R} \), which is still denoted by \( \varphi \). Hence

\[
1_{V_{t,0}}(x) = \frac{\varphi(x) + \varphi(tx)}{2}, \quad 0 < x < 1
\]

and

\[
1_{V_{t,1}}(x) = \frac{\varphi(x) - \varphi(tx)}{2}, \quad 0 < x < 1,
\]

which gives

\[
\mathcal{D}1_{V_{t,0}}(s) = \frac{1 + t^{-s}}{2} \mathcal{D}\varphi(s) = \frac{\sqrt{2}}{\pi}(1 + t^{-s})L(s + 1, \chi_2)
\]

and

\[
\mathcal{D}1_{V_{t,1}}(s) = \frac{1 - t^{-s}}{2} \mathcal{D}\varphi(s) = \frac{\sqrt{2}}{\pi}(1 - t^{-s})L(s + 1, \chi_2).
\]

Obviously, both \( \mathcal{D}1_{V_{t,0}} \) and \( \mathcal{D}1_{V_{t,1}} \) have no zeros in \( \mathbb{C}_0 \).

### 3.2 Some criteria via boundary points of \( V \)

We continue to establish several criteria that are more convenient in some special cases, especially when the least common denominator \( t \) is small (Theorem 6.1). To this end, we represent the function \( f \) in another way.

Let \( J_V(x) \ (x \in \mathbb{R}) \) denote the jump of \( 1_V \) at \( x \):

\[
J_V(x) = 1_V(x^+) - 1_V(x^-) = \lim_{u \to x^+} 1_V(u) - \lim_{v \to x^-} 1_V(v),
\]

where \( 1_V \) is identified with its odd 2-periodic extension on \( \mathbb{R} \). The function \( 1_V \) can be represented as

\[
1_V = \frac{1}{2} J_V(0) 1_{(0, 1)} + \sum_{m=1}^{t-1} J_V(\frac{m}{t}) 1_{(\frac{m}{t}, 1)}.
\]

Then by (3.2),

\[
f(n) = J_V(0) + J_V(1) \cos(n\pi) + 2 \sum_{m=1}^{t-1} J_V(\frac{m}{t}) \cos \frac{mn\pi}{t} = \sum_{m=1}^{q} J_V(\frac{m}{t}) e^{2\pi i mn/q}. \tag{3.6}
\]

Put

\[
g(m) = J_V(\frac{m}{t}), \quad m \in \mathbb{Z}. \tag{3.7}
\]

Then \( g \) coincide with the Fourier transform \( \mathcal{T}f \) of \( f \) since

\[
g(m) = \frac{1}{q} \sum_{n=1}^{q} f(n) e^{-2\pi i mn/q}
\]
by (3.6) (see [Apo, Theorem 8.4], also see Subsection 2.1 for the definition of the transform $T$). Since $f$ is real-valued, if $f \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$ then $\psi$ is necessarily real-valued by Lemma 3.7 below, which yields that $g$ belongs to the same space $E_{q,\psi}$ by Lemma 2.5. In the same way, if $g \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$ then $f \in E_{q,\psi}$.

**Lemma 3.7.** Suppose $\psi \in \mathcal{P}_q$ is of modulus $q_0$, and $f \in E_{q,\psi}$ with $f \not\equiv 0$. If $f$ is real-valued, then $\psi$ is also real-valued.

**Proof.** It suffices to show $\psi(a) \in \mathbb{R}$ for every $a \in \mathbb{N}$ with $\gcd(a,q_0) = 1$. By Dirichlet’s Theorem on arithmetic progressions (see Subsection 2.1), for such $a$ there is a prime $p > q$ such that $p \equiv a \pmod{q_0}$. Choose $m \in \mathbb{N}$ satisfying $f(m) \neq 0$, and put $n = \gcd(m, \frac{q}{q_0})$. Then

\[
\gcd(pm, \frac{q}{q_0}) = \gcd(m, \frac{q}{q_0}) = n,
\]

which gives

\[
f(pm) = f(n)\psi(pm) = f(n)\psi(m)\psi(p) = f(m)\psi(p).
\]

Therefore $\psi(a) = \psi(p) = \frac{f(pm)}{f(m)}$ is a real number. \qed

Intuitively, it is much easier to check that $g$ belongs to such a space other than $f$ since $g(m)$ only takes values $0, 1$ or $-1$ for $1 \leq m \leq t-1$. So one may expect to use the information from boundary points of $V$ instead of the sine-Fourier coefficients of $1_V$. We summarize the above discussion into the following result.

**Corollary 3.8.** Let $V, q, f$ and $g$ be given as in (1.2), (3.1), (3.2) and (3.7). If $f \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$, then $g \in E_{q,\psi}$, and vice versa.

In particular, if $1_V \in \mathcal{C}$ then $g \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$.

We will use Corollary 3.8 to establish Theorems 3.9, 3.10 and 3.12.

**Theorem 3.9.** Let $V, q$ and $g$ be given as in (1.2), (3.1) and (3.7). Suppose that $g \in E_{q,\psi}$ for some $\psi \in \mathcal{P}_q$, and let $q_0$ denote the modulus of $\psi$. We also assume that

1. $\frac{q}{q_0}$ is a product of some distinct primes that cannot divide $q_0$. \\
2. $g(d_0) \neq 0$ for some divisor $d_0$ of $\frac{q}{q_0}$, and $g(d) = 0$ for any other divisor $d$ of $\frac{q}{q_0}$.

Then $1_V \in \mathcal{C}$.

To prove Theorem 3.10 we need the following.
Lemma 3.11. Suppose that \( f \in E_{q,\psi} \) for some \( \psi \in \mathcal{P}_q \), and the modulus of \( \psi \) is \( q_0 \). Then

\[ f(n) = \sum_{d\mid \frac{q}{q_0}} g(d) \tau(n, \psi_\frac{q}{d}), \quad n = 1, 2, \ldots, \]

where \( g = T f \) is the Fourier transform of \( f \), \( \psi_\frac{q}{d} \) is the Dirichlet character mod \( \frac{q}{d} \) induced by \( \psi \).

Proof. By Lemma 2.5 we have \( g \in E_{q,\psi} \), which together with Lemma 2.2 in turn gives

\[ f(n) = \sum_{m=1}^{q} g(m) e^{2\pi i mn/q} = \sum_{m=1}^{q} g(\hat{m}) \overline{\psi(\frac{m}{m})} e^{2\pi i m n/q}. \]

Note that \( \hat{m} = d \) for some divisor \( d \) of \( \frac{q}{q_0} \) if and only if \( m = ld \) with \( \gcd(l, r_d) = 1 \), where \( r_d = \frac{q dq_0}{d_0} \). It follows that

\[ f(n) = \sum_{d\mid \frac{q}{q_0}} g(d) \sum_{l=1}^{\frac{q}{d}} \overline{\psi(l)} e^{2\pi i d l n/q} \]
\[ = \sum_{d\mid \frac{q}{q_0}} g(d) \sum_{l=1}^{\frac{q}{d}} \overline{\psi_\frac{q}{d}(l)} e^{2\pi i d l n/q} \]
\[ = \sum_{d\mid \frac{q}{q_0}} g(d) \sum_{l=1}^{\frac{q}{d}} \psi_\frac{q}{d}(l) e^{2\pi i d l n/q} \]
\[ = \sum_{d\mid \frac{q}{q_0}} g(d) \tau(n, \psi_\frac{q}{d}), \]

where \( \chi_{r_d} \) is the principle Dirichlet character mod \( r_d \). The proof is complete. \( \square \)

Proof of Theorem 3.10. Let \( d_0 \) be the unique divisor of \( \frac{q}{q_0} \) satisfying \( g(d_0) \neq 0 \). Then by Lemma 3.11

\[ f(n) = g(d_0) \tau(n, \psi_\frac{q}{d_0}), \quad n = 1, 2, \ldots. \]  \hfill (3.8)

This together with Lemma 2.4 yields \( f \in E_{\frac{q}{d_0},\psi} \), and thus \( (f * \mu \psi)(n) = 0 \) whenever \( n \nmid \frac{q}{d_0} \) by Lemma 2.2. It suffices to show that

\[ P(s) = \sum_{d\mid \frac{q}{d_0}} (f * \mu \psi)(d)d^{-s} \]

has no zeros in \( \mathbb{C}_1 \). Fix a divisor \( k \) of \( \frac{q}{d_0 q_0} \). Then by Assumption (1),

\[ \gcd(k, \frac{q}{k q_0}) = \gcd(k, q_0) = 1 \]

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and

\[ \mu(k)^2 = \psi(k)^2 = 1. \]

Thus, \( \gcd(k, \frac{q}{k}) = 1 \), \( \phi(k)\phi\left(\frac{q}{kd_0}\right) = \phi\left(\frac{q}{d_0}\right) \), and by Lemma 2.1,

\[ \tau(k, \psi, \frac{q}{d_0}) = \frac{\phi\left(\frac{q}{d_0}\right)}{\phi\left(\frac{q}{kd_0q_0}\right)} \tau\left(\frac{q}{kd_0q_0}\right) = \phi(k)\mu\left(\frac{q}{kd_0q_0}\right)\tau\left(\frac{q}{kd_0q_0}\right). \]

It follows that for any divisor \( l \) of \( \frac{q}{k} \),

\[ \tau(k, \psi, \frac{q}{d_0}) \mu(l)\psi(l) = \phi(k)\tau(\psi)\mu\left(\frac{q}{kd_0q_0}\right)\psi\left(\frac{q}{kd_0q_0}\right) \mu(l)\psi(l) = \phi(k)\tau(\psi)\mu\left(\frac{q}{kd_0q_0}\right)\psi\left(\frac{q}{kd_0q_0}\right) \mu(l)\psi(l)\mu(k)^2\psi(k)^2 \quad (3.9) \]

Therefore,

\[ (f \ast \mu \psi)(d) = \sum_{k \mid d} f(k)\mu\left(\frac{d}{k}\right)\psi\left(\frac{d}{k}\right) = g(d_0) \sum_{k \mid d} \tau(k, \psi, \frac{q}{d_0}) \mu\left(\frac{d}{k}\right)\psi\left(\frac{d}{k}\right) = C \mu(d)\psi(d) \sum_{k \mid d} \phi(k) \]

\[ = Cd\mu(d)\psi(d), \]

where \( C = g(d_0)\tau(\psi)\mu\left(\frac{q}{kd_0q_0}\right)\psi\left(\frac{q}{kd_0q_0}\right) \neq 0 \) (see [Apo, Theorem 8.15]), the second identity follows from (3.8), and the third identity follows from (3.9). This gives

\[ P(s) = C \sum_{d \mid d_0q_0} \mu(d)\psi(d)d^{1-s} = C \prod_{p \mid d_0q_0} (\mu(p)\psi(p) + p^{1-s}) = C \prod_{p \mid d_0q_0} (-\psi(p) + p^{1-s}). \]

It is clear that \( P(s) \neq 0 \) whenever \( \Re s > 1 \). \[ \square \]

To continue, we need to introduce some more notations. For \( q \in \mathbb{N} \), set

\[ M_q(z) = \sum_{d \mid q} \mu\left(\frac{q}{d}\right)z^{\alpha(d)} \]

and

\[ S_q = \{ j \in \mathbb{N} : p_j \mid q \}, \]

where \( p_j \) is the \( j \)-th prime. It is clear that \( M_q \) is a polynomial only involving complex variables \( \{z_j : j \in S_q\} \). Since

\[ \sum_{d \mid q} \mu\left(\frac{q}{d}\right)d^{-s} = q^{-s} \sum_{d \mid q} \mu\left(\frac{q}{d}\right)\left(\frac{q}{d}\right)^s = q^{-s} \prod_{j \in S_q} (1 - p_j^s) = q^{-s} \prod_{j \in S_q} \frac{p_j^{-s} - 1}{p_j^{-s}}. \]
we have
\[ M_q = \mathcal{B}(\sum_{d|q} \mu(q/d) d^{-s}) = \mathcal{B}(q^{-s} \prod_{j \in S_q} \frac{p_j^{-s} - 1}{p_j - 1}). \]

It follows that if \( q (q \geq 2) \) has prime factorization \( q = p_{j_1}^{k_1} \cdots p_{j_l}^{k_l} \), then \( S_q = \{ j_1, \cdots, j_l \} \) and
\[ M_q(z_{j_1}, \cdots, z_{j_l}) = z_{j_1}^{k_1-1}(z_{j_1} - 1) \cdots z_{j_l}^{k_l-1}(z_{j_l} - 1). \tag{3.10} \]

From this, one concludes that \( M_{q_1 q_2} = M_{q_1} M_{q_2} \) whenever \( \gcd(q_1, q_2) = 1 \). The space \( E_{q, \chi_1} \) is abbreviated as \( E_q \), where \( \chi_1 \equiv 1 \).

Lastly, for a nonempty subset \( S \) of \( \mathbb{N} \), define
\[ \mathbb{D}^S = \{ z \in \mathbb{D}_\infty^2 : z_j = 0 \text{ for } j \notin S \}. \]
Then \( \mathbb{D}^S \) is identified with the \( n \)-polydisk in an obvious way, where \( n = \#S \).

**Theorem 3.12.** Let \( V, q \) and \( g \) be given as in (1.2), (3.1) and (3.7). If \( g \in E_q \), then \( 1_V \in \mathcal{C} \) if and only if the polynomial
\[ R = \sum_{d|q} g(q/d) M_d \tag{3.11} \]
has no zeros in \( \mathbb{D}^S_q \).

Ramanujan’s sum \( c_q \) is defined by
\[ c_q(n) := \tau(n, \chi_q) = \sum_{m=1 \atop (m,q)=1}^{q} e^{2\pi i mn/q}, \]
where \( \chi_q \) is the principle Dirichlet character mod \( q \). The following formula will be used to prove Theorem 3.12.

**Lemma 3.13.** For \( q \geq 1 \) we have
\[ (\mu * c_q)(n) = \begin{cases} n\mu(q/n), & n \mid q; \\ 0, & n \nmid q. \end{cases} \]

**Proof.** Fix \( n \in \mathbb{N} \). It follows from [Apo, Theorem 8.6] that for any divisor \( d \) of \( n \),
\[ c_q(d) = \sum_{k | (d, q)} k \mu(q/k). \]
Hence, we have
\[ (\mu * c_q)(n) = \sum_{d|n} c_q(d) \mu(n/d) = \sum_{d|n} \sum_{k | (d, q)} k \mu(q/k) \mu(n/d) = \sum_{k | (n, q)} k \mu(q/k) \sum_{d \in A_k} \mu(n/d), \]

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where \( A_k = \{ d : k | d, \ d | n \} \). Note that for any divisor \( k \) of \( \gcd(n, q) \), \( d \in A_k \) if and only if \( d = ck \) for some integer \( c \) with \( c | \frac{n}{k} \). The sum \( \sum_{d \in A_k} \mu \left( \frac{n}{d} \right) \) in the above equality is equals to

\[
\sum_{c | \frac{n}{k}} \mu \left( \frac{n}{ck} \right) = \begin{cases} 
1, & \frac{n}{k} = 1; \\
0, & \frac{n}{k} > 1.
\end{cases}
\]

If \( n \nmid q \) then \( \gcd(n, q) < n \), and thus \( (\mu * c_q)(n) = 0 \). Now assume \( n | q \). Then

\[
(\mu * c_q)(n) = \sum_{k|n} k \mu \left( \frac{q}{k} \right) \sum_{d \in A_k} \mu \left( \frac{n}{d} \right) = n \mu \left( \frac{q}{n} \right).
\]

This completes the proof. \( \square \)

**Proof of Theorem 3.12.** It suffices to show

\[
\sum_{d|q} \frac{(f * \mu)(d)}{d} z^{\alpha(d)} = B \sum_{d|q} \frac{(f * \mu \psi)(d)}{d^s} = \sum_{d|q} g \left( \frac{q}{d} \right) M_d.
\]

By Lemma 3.11 we have

\[
(f * \mu)(d) = \sum_{k|d} f(k) \mu \left( \frac{d}{k} \right)
\]

\[
= \sum_{k|d} \mu \left( \frac{d}{k} \right) \sum_{c|q} g(c) \tau(k, \chi_{\frac{q}{c}})
\]

\[
= \sum_{c|q} g(c) \sum_{k|d} \mu \left( \frac{d}{k} \right) c_{\frac{q}{c}}(k)
\]

\[
= \sum_{c|q} g(c) (\mu * c_{\frac{q}{c}})(d),
\]

where \( \chi_{\frac{q}{c}} \) is the principle Dirichlet character mod \( \frac{q}{c} \). For a divisor \( c \) of \( q \), Lemma 3.13 gives

\[
(\mu * c_{\frac{q}{c}})(d) = \begin{cases} 
\mu \left( \frac{q}{cd} \right), & c \mid \frac{q}{d}; \\
0, & c \nmid \frac{q}{d}.
\end{cases}
\]

Therefore,

\[
(f * \mu)(d) = \sum_{c|\frac{q}{d}} g(c) (\mu * c_{\frac{q}{c}})(d) = d \sum_{c|\frac{q}{d}} g(c) \mu \left( \frac{q}{cd} \right),
\]

25
which yields

\[
\sum_{d|q} \frac{(f * \mu)(d)}{d} z^{\alpha(d)} = \sum_{c|q} \sum_{d|\frac{q}{c}} g(c) \mu\left(\frac{q}{cd}\right) z^{\alpha(d)} = \sum_{c|q} g(c) M_{\frac{q}{c}} = \sum_{d|q} g\left(\frac{q}{d}\right) M_{d}.
\]

\[\square\]

4 Determining the character \(\psi\)

In this section, we will show that for most \(V\), which is given as in (1.2) and satisfies \(1_V \in C\), the Dirichlet character \(\psi\) determined by \(V\) (see Theorem 3.1 and Remark 3.2) coincides with the Dirichlet character \(\chi_1 \mod 1\), i.e., \(\psi \equiv 1\). Recall that the least common denominator \(t_V\) (given as in (1.3)) is abbreviated as \(t\).

**Theorem 4.1.** Let \(V, q\) and \(g\) be given as in (1.2), (3.1) and (3.7). Assume that one of the following two conditions is satisfied:

1. \(t_0 = 1, 2\) or \(t_0 = p^k\) for some \(k \in \mathbb{N}\) and some prime \(p \equiv 3 \mod 4\), where \(t_0 = \gcd(t_1, t_1', \ldots, t_l, t_l')\).

2. \(t \geq 7\) and \(3 \nmid t\).

If \(1_V \in C\) then \(g \in E_q\), i.e., \(g(m_1) = g(m_2)\) whenever \(\gcd(m_1, q) = \gcd(m_2, q)\).

As a consequence, \(1_V \in C\) if and only if \(g \in E_q\) and the polynomial \(R\) in (3.11) has no zeros in \(D_{S^a}\).

4.1 Proof of Theorem 4.1

The key ingredient to the proof of Theorem 4.1 is the following result that only involves arithmetical functions in the space \(E_{q, \psi}\).

**Lemma 4.2.** Suppose that \(q\) is an even positive integer with \(q \geq 14\) and \(3 \nmid q\), and that \(\psi\) is an even Dirichlet character in \(P_q\). Assume that there is a function \(g \in E_{q, \psi}\), such that

\[\gcd\left(\{m \in \mathbb{Z} : g(m) \neq 0\} \cup \left\{\frac{q}{2}\right\}\right) = 1,
\]

and that the sum function \(S(m) = \sum_{i=1}^{m} g(i)\) of \(g\) only takes values 0 or 1 for \(1 \leq m \leq \frac{q}{2} - 1\). Then \(\psi \equiv 1\).
Due to the length of the proof of Lemma 4.2, we first use Lemma 4.2 and the following four lemmas to prove Theorem 4.1, and then prove Lemma 4.2 in Subsection 4.3.

Lemma 4.3. [Ca, Proposition 2.1.34] Suppose that \( q \) is the product of two relatively prime positive integers \( q_1, q_2 \), and \( \psi \) is a primitive Dirichlet character mod \( q \). Then \( \psi = \psi_1 \psi_2 \), where \( \psi_i \) (\( i = 1, 2 \)) is a primitive Dirichlet characters mod \( q_i \).

Remark that the decomposition \( \psi = \psi_1 \psi_2 \) in Lemma 4.3 is unique due to the Chinese remainder theorem.

Lemma 4.4 below comes from standard analytic number theory, see [Da2, pp. 37-38] for instance.

Lemma 4.4. (1) There are only three real primitive Dirichlet characters mod \( 2^k \) (\( k \in \mathbb{N} \)): the Dirichlet character \( \psi_{4}^{pr} \) mod 4 determined by \( \psi_{4}^{pr}(3) = -1 \); the Dirichlet character \( \psi_{8,1}^{pr} \) mod 8 determined by \( \psi_{8,1}^{pr}(3) = 1 \) and \( \psi_{8,1}^{pr}(5) = \psi_{8,1}^{pr}(7) = -1 \); the Dirichlet character \( \psi_{8,2}^{pr} \) mod 8 determined by \( \psi_{8,2}^{pr}(3) = \psi_{8,2}^{pr}(5) = -1 \) and \( \psi_{8,2}^{pr}(7) = 1 \).

(2) For an odd prime \( p \), the only real primitive Dirichlet character mod \( p \) is the Legendre symbol \( (n | p) \), while if \( k \geq 2 \) there are no real primitive Dirichlet characters mod \( p^k \).

Remark 4.5. In light of Lemma 4.3 and 4.4, for a real primitive Dirichlet character \( \psi \), its modulus \( q_0 \) has form

\[
q_0 = 2^k s,
\]

where \( k = 0, 2 \) or 3, and \( s = 1 \) or \( s = p_1 \cdots p_l \), a product of some distinct primes. Write \( \psi(n) = \psi_1(n)\psi_2(n) \), where \( \psi_1 \equiv 1 \) or \( \psi_1 = \psi_4^{pr}, \psi_{8,1}^{pr}, \psi_{8,2}^{pr} \), and \( \psi_2 \equiv 1 \) or

\[
\psi_2(n) = (n | s) := \prod_{i=1}^{l} (n | p_i).
\]

Since \( \psi_{4}^{pr}(-1) = \psi_{8,1}^{pr}(-1) = -1 \), \( \psi_{8,2}^{pr}(-1) = 1 \), and \( \psi_2(-1) = \pm 1 \) if and only if \( s \equiv \pm 1 \) (mod 4) (see [Ap6, Theorem 9.10]), we conclude that when \( \psi \) is even (i.e., \( \psi(-1) = 1 \)),

(1) \( s \equiv 1 \) (mod 4) if \( k = 0 \) or \( k = 3 \), \( \psi_1 = \psi_{8,2}^{pr} \);

(2) \( s \equiv 3 \) (mod 4) if \( k = 2 \) or \( k = 3 \), \( \psi_1 = \psi_{8,1}^{pr} \).

Lemma 4.6. Suppose \( q = 2t \) (\( t \in \mathbb{N} \)), \( g \in E_{q,\psi} \) for some \( \psi \in \mathcal{P}_q \), and \( \tfrac{a}{b} \) is an irreducible fraction in (0, 1) with \( b \mid t \). Put \( \tilde{a} = \gcd(a, 2) \) and \( \tilde{b} = \gcd(b, 2) \). Then we have

(1) \( q_0 \mid \tilde{b} \tilde{b} \) provided \( g(\tfrac{a}{b}) \neq 0 \), where \( q_0 \) is the modulus of \( \psi \).

(2) \( g(\tfrac{a}{b}) = g(\tfrac{\tilde{a}}{\tilde{b}})\psi(\tfrac{a}{\tilde{a}}) \) and \( |g(\tfrac{a}{b})| = |g(\tfrac{\tilde{a}}{\tilde{b}})| \).
Proof. A direct calculation gives
\[ \gcd\left(\frac{at}{b}, \frac{q}{q_0}\right) = \frac{t}{bq_0} \gcd(aq_0, 2b) = \frac{\tilde{a}t}{bq_0} \gcd\left(\frac{a}{\tilde{a}}, \frac{2}{q_0}b\right) = \frac{\tilde{a}t}{bq_0} \gcd\left(q_0, \frac{2}{a}b\right). \]

Since \( g \in E_{q, \psi} \),
\[ g\left(\frac{at}{b}\right) = g(\gcd\left(\frac{at}{b}, \frac{q}{q_0}\right)) \psi(c), \quad (4.1) \]
where
\[ c = \frac{at}{\gcd\left(\frac{at}{b}, \frac{q}{q_0}\right)} = \frac{aq_0}{\tilde{a} \cdot \gcd\left(q_0, \frac{2}{a}b\right)}. \]

To show (1), assume \( g\left(\frac{at}{b}\right) \neq 0 \). By (4.1), we have \( \psi(c) \neq 0 \), equivalently, \( \gcd(c, q_0) = 1 \). This yields \( q_0 = \gcd\left(q_0, \frac{2}{a}\right) \), forcing \( q_0 \mid \frac{2}{a} \). Since there is no primitive Dirichlet character \( \psi \mod 2 \), Lemma 4.3 further implies \( q_0 \mid b \) if \( b \) is odd. Therefore, we have proved (1).

By (1), if \( q_0 \nmid \tilde{b}b \) then \( g\left(\frac{at}{b}\right) = g\left(\frac{\tilde{a}t}{b}\right) = 0 \). So we only need to prove (2) for the case that \( q_0 \mid \tilde{b}b \). In this case, we have \( q_0 \mid \frac{2}{a} \), and thus \( c = \frac{a}{a} \). Moreover,
\[ \gcd\left(\frac{at}{b}, \frac{q}{q_0}\right) = \frac{at}{c} = \frac{\tilde{a}t}{b}. \]

Hence by (4.1),
\[ g\left(\frac{at}{b}\right) = g\left(\frac{\tilde{a}t}{b}\right) \psi\left(\frac{a}{a}\right). \]

It remains to show \( \gcd(a, q_0) = 1 \), which would imply \( |\psi\left(\frac{a}{a}\right)| = 1 \). When \( b \) is odd, \( q_0 \mid b \); when \( b \) is even, \( q_0 \mid 2b \) and \( a \) is necessarily odd since \( \gcd(a, b) = 1 \). In either case, we have proved \( \gcd(a, q_0) = 1 \). The proof is complete. \( \Box \)

Lemma 4.7. Let \( V \) and \( g \) be given as in (1.2) and (3.7). Then the sum function \( S(m) = \sum_{i=1}^{m} g(i) \) of \( g \) only takes values \(-1\) or \( 1 \) for \( 1 \leq m \leq t - 1 \).
Proof. For \( 1 \leq m \leq t - 1 \),
\[ S(m) = \sum_{i=1}^{m} J_V\left(\frac{i}{t}\right) = 1_V\left(\frac{m}{t}\right) - 1_V(0^+). \]

This proves the lemma. \( \Box \)

Proof of Theorem 4.1. Assume \( 1_V \in \mathcal{C} \). Then by Corollary 3.8, \( g \in E_{q, \psi} \) for some \( \psi \in \mathcal{P}_q \).

Since \( g \) is an even function by its definition, it follows that \( \psi \) is also even, i.e., \( \psi(-1) = 1 \).

Assume (1) holds and list all of the different numbers appearing in \( \{t_i\}_{i=1}^l \cup \{t'_i\}_{i=1}^{l'}: b_1, \ldots, b_n \ (n \in \mathbb{N}) \). Then there exist \( n \) integers \( a_1, \ldots, a_n \), such that for each \( 1 \leq i \leq n \), \( \gcd(a_i, b_i) = 1 \) and \( g\left(\frac{a_i t}{b_i}\right) = J_V\left(\frac{a_i}{b_i}\right) \neq 0 \). By Lemma 4.6 (1), \( q_0 \mid 2b_i \ (1 \leq i \leq n) \), which yields \( q_0 \mid 2t_0 \) since \( t_0 = \gcd(b_1, \ldots, b_n) \). Thus, the characterization of \( q_0 \) in Remark 4.5 forces \( q_0 = 1 \).
Now assume (2) holds. As mentioned in Introduction (see (1.4)), one can rewrite $V$ as

$$V = \left(\frac{\alpha_1}{t}, \frac{\beta_1}{t}\right) \cup \cdots \cup \left(\frac{\alpha_l}{t}, \frac{\beta_l}{t}\right),$$

where $\alpha_1, \beta_1, \ldots, \alpha_l, \beta_l$ are integers. It is clear that

$$\gcd(\alpha_1, \beta_1, \ldots, \alpha_l, \beta_l, t) = 1.$$

Since $t = \frac{q}{2}$ and $g(\alpha_i), g(\beta_i) \neq 0 (1 \leq i \leq l)$, it follows that the greatest common divisor of

$$\{m \in \mathbb{Z} : g(m) \neq 0\} \cup \left\{\frac{q}{2}\right\}$$

is 1. Hence, Lemma 4.7 implies that $(1 - 21_{V(0^+)})g$ satisfies all hypothesis in Lemma 4.2. This completes the proof. □

Remark 4.8. The conclusion $g \in E_q$ is useful in determining the construction of $V$. By Lemma 4.6, for two irreducible fractions $\frac{a_1}{b}, \frac{a_2}{b} \in (0, 1)$ with the same denominator $b$, if $a_1 - a_2$ is even and one of $\frac{a_1}{b}, \frac{a_2}{b}$ is a left (or right) endpoint of some component interval of $V$, then so is the other one.

4.2 Preparatory lemmas

In this subsection, we establish some preparatory lemmas for the proof of Lemma 4.2.

**Lemma 4.9.** (1) For every prime $p$ with $p \geq 7$, there exist two integers $a, b$ with $2 \leq a, b \leq p - 2$, such that $(1 - a^2 \mid p) = 1$ and $(1 - b^2 \mid p) = -1$.

(2) For $a, b \in \mathbb{Z}$ with $5 \nmid ab$, there exists $k \in \{1, 2\}$ such that $((ka)^2 - b^2 \mid 5) = -1$.

**Proof.** (1) It suffices to find two integers $\tilde{a}, \tilde{b}$ with $2 \leq \tilde{a}, \tilde{b} \leq p - 2$, such that

$$(\tilde{a}^2 - 1 \mid p) = 1, \quad (\tilde{b}^2 - 1 \mid p) = -1.$$ 

If $(2 \mid p) = 1$ and $(3 \mid p) = -1$, then

$$(3^2 - 1 \mid p) = (8 \mid p) = (2 \mid p) = 1,$$

$$(2^2 - 1 \mid p) = (3 \mid p) = -1.$$ 

If $(2 \mid p) = -1$ and $(3 \mid p) = 1$, then

$$(2^2 - 1 \mid p) = (3 \mid p) = 1.$$ 

$$(3^2 - 1 \mid p) = (2 \mid p) = -1.$$ 

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If \((2 | p) = (3 | p) = -1\), then
\[(5^2 - 1 | p) = (24 | p) = (6 | p) = 1\]

\[(2^2 - 1 | p) = (3 | p) = -1.\]

Now suppose \((2 | p) = (3 | p) = 1\) and let \(n\) be the smallest positive integer such that \((n | p) = -1\). Then \(n \geq 4\), and
\[((n - 2)^2 - 1 | p) = ((n - 1)(n - 3) | p) = (n - 1 | p)(n - 3 | p) = 1,\]
\[((n - 1)^2 - 1 | p) = (n(n - 2) | p) = (n | p)(n - 2 | p) = -1.\]

This proves (1).

(2) Note that \(a^2 \equiv \pm 1 \pmod{5}\), \(b^2 \equiv \pm 1 \pmod{5}\). Then one can finish the proof by taking \(k = 1\) if \(a^2 \equiv -b^2 \pmod{5}\) or \(k = 2\) if \(a^2 \equiv b^2 \pmod{5}\). □

**Lemma 4.10.** Suppose that the Dirichlet character \(\psi \in \mathcal{P}_q\) with modulus \(q_0\) is even, and \(n \in \mathbb{N}\) with \(n | \frac{q}{q_0}\).

1. Let \(q_0 = q_1q_2\) with \(\gcd(q_1, q_2) = 1\), and \(\psi = \psi_1\psi_2\) be the decomposition of \(\psi\) according to Lemma 4.3. Then for any \(d | q_1\) and any \(k \in \mathbb{N}\), we have
\[\psi((\frac{kq}{dn})^2 - 1) = \psi_1((\frac{kq}{dn})^2 - 1).\]

2. If \(p\) is an odd prime divisor of \(q_0\) and \(p^2 \nmid \frac{q}{n}\), then \(\psi((\frac{kq}{pn})^2 - 1) = 1\) for every \(k \in \mathbb{N}\).

3. If \(q_0\) has a prime divisor \(p\) (\(p \geq 7\)) and \(p^2 \nmid \frac{q}{n}\), then there exist two integers \(k, l\) with \(1 \leq k, l \leq \frac{q-1}{2}\), such that \(\psi((\frac{lk}{pn})^2 - 1) = 1\) and \(\psi((\frac{lk}{pn})^2 - 1) = -1\).

**Proof.** (1) Given \(d | q_1\) and \(k \in \mathbb{N}\), since \(d | q_1\) and \(n | \frac{q}{q_0}\) one has \(dn | \frac{q}{q_0}\), i.e., \(q_2 | \frac{q}{dn}\). This gives
\[\psi((\frac{kq}{pm})^2 - 1) = \psi_2(-1) = \frac{\psi(-1)}{\psi_1(-1)} = \frac{1}{\psi_1(-1)} = \psi_1(-1),\]
and thus
\[\psi((\frac{kq}{dn})^2 - 1) = \psi_1((\frac{kq}{dn})^2 - 1)\psi_2((\frac{kq}{dn})^2 - 1)\]
\[= \psi_1((\frac{kq}{dn})^2 - 1)\psi_1(-1)\]
\[= \psi_1(1 - (\frac{kq}{dn})^2).\]

(2) Since \(\gcd(p, \frac{q}{p}) = 1\) (see Remark 4.5), by (1) we have
\[\psi((\frac{kq}{pm})^2 - 1) = (1 - (\frac{kq}{pm})^2 | p), \quad k \in \mathbb{N}.\]
This proves (2) since \( p \mid \frac{q}{pn} \).

(3) We need the fact that if \( p \nmid m \) then there exists an integer \( k \) with \( 1 \leq k \leq \frac{p-1}{2} \), such that \( m \equiv k (\mod p) \) or
\[
 m \equiv p - k \equiv -k (\mod p),
\]
and thus \( m^2 \equiv k^2 (\mod p) \).

It follows from Lemma 4.9 (1) that there exists an integer \( a \) with \( 2 \leq a \leq p - 2 \), such that \((1 - a^2 \mid p) = 1\). Since \( p \nmid \frac{q}{pn} \), one can choose \( c \in \mathbb{N} \) satisfying \( \frac{ac}{pn} \equiv 1 (\mod p) \). By the above fact, there exists an integers \( k \) with \( 1 \leq k \leq \frac{p-1}{2} \), such that \( k^2 \equiv (ac)^2 (\mod p) \), and therefore
\[
 \left( \frac{q}{pn} \right)^2 \equiv (\frac{ac}{pn})^2 \equiv a^2 (\frac{c}{pn})^2 \equiv a^2 (\mod p).
\]
Hence, similar to the proof of (2), we have
\[
\psi \left( \left( \frac{q}{pn} \right)^2 - 1 \right) = (1 - (\frac{q}{pn})^2 \mid p) = (1 - a^2 \mid p) = 1.
\]

Similarly, one can find an integer \( l \) with \( 1 \leq l \leq \frac{p-1}{2} \), such that \( \psi \left( \left( \frac{q}{pn} \right)^2 - 1 \right) = 1 \).

**Lemma 4.11.**

(1) Suppose \( a, b, c \in \{-1, 0, 1\} \), \( |a + b| \leq 1 \), \( |b + c| \leq 1 \) and \( |a + b + c| \leq 1 \).

Then \( b = -a = -c \) if \( ac = 1 \), while \( b = 0 \) if \( ac = -1 \).

(2) Suppose \( n \in \mathbb{N} \), \( a_j \in \{-1, 1\} \) (\( j = 1, 2, \cdots, n \)), and \( |a_j + a_{j+1}| \leq 1 \) for every \( 1 \leq j \leq n - 1 \). Then \( a_j = (-1)^{j-1} a_1 \).

One can check Lemma 4.11 directly.

**Lemma 4.12.**

Suppose that the sum function \( S(m) = \sum_{i=1}^{m} g(i) \) of an arithmetic function \( g \) only takes values 0 or 1 for \( 1 \leq m \leq M \) \((M \in \mathbb{N})\). For \( 1 \leq m_1 < m_2 \leq M \),
\[
\sum_{i=m_1+1}^{m_2-1} g(i) = -g(m_1) = -g(m_2)
\]
if \( g(m_1)g(m_2) = 1 \), while \( \sum_{i=m_1+1}^{m_2-1} g(i) = 0 \) if \( g(m_1)g(m_2) = -1 \).

**Proof.** Set
\[
\alpha = \sum_{i=m_1+1}^{m_2-1} g(i) = S(m_2 - 1) - S(m_1).
\]
Then \( g(m_1), \alpha, g(m_2) \in \{-1, 0, 1\} \),
\[
|g(m_1) + \alpha| = |S(m_2 - 1) - S(m_1 - 1)| \leq 1,
\]
\[
|\alpha + g(m_2)| = |S(m_2) - S(m_1)| \leq 1,
\]
and
\[
|g(m_1) + \alpha + g(m_2)| = |S(m_2) - S(m_1 - 1)| \leq 1,
\]
here we define \( S(0) = 0 \). Thus Lemma 4.11 (1), applied to \( g(m_1), \alpha \) and \( g(m_2) \), finishes the proof. \( \square \)
4.3 Proof of Lemma 4.2

This subsection is dedicated to the proof of Lemma 4.2. It is equivalent to prove that the modulus $q_0$ of $\psi$ is 1. To reach a contradiction, assume conversely $q_0 > 1$. The proof can be roughly divided into three steps. In the first step, we establish several lemmas as preparation (Lemmas 4.13-4.18). In the second step, we show $q_0 = 5$ (Lemma 4.19). The proof of Lemma 4.2 will be completed in the last step.

Since $q_0 > 1$, by Lemma 4.6 (1),

$$g(q) = g\left(\frac{q}{2}\right) = 0$$

and the following conclusion holds.

**Lemma 4.13.** If $4 \mid q$ then $g\left(\frac{q}{4}\right) = 0$.

Set $\widehat{m} = \gcd(m, \frac{q}{q_0})$ ($m \in \mathbb{Z}$). Since for $1 \leq m \leq \frac{q}{2} - 1$, $S(m)$ only takes values 0 or 1, and

$$g(q - m) = g(\widehat{m})\psi\left(\frac{q - m}{\widehat{m}}\right) = g(\widehat{m})\psi\left(\frac{q}{\widehat{m}} - \frac{m}{\widehat{m}}\right) = g(\widehat{m})\psi\left(\frac{m}{\widehat{m}}\right)\psi(-1) = g(m),$$

it follows that $g$ only takes values 0 or $\pm 1$. In particular, Lemma 3.7 implies that $\psi$ is real-valued.

Take $m_0$ to be the smallest positive integer among those $m$’s satisfying $g(m) \neq 0$. Then $g(m_0) = S(m_0) = 1$. From the identity

$$g(m_0) = g(\widehat{m_0})\psi\left(\frac{m_0}{\widehat{m_0}}\right)$$

we deduce that $m_0 = \widehat{m_0}$, i.e., $m_0 \mid \frac{q}{q_0}$.

Let us first assume that the following inequality holds, and put its proof at the end of this subsection.

**Lemma 4.14.** We have $m_0 \leq \frac{q}{q_0}$.

We also need the following identity. Suppose $d \mid q_0$ with $d \geq 4$, $k \in \mathbb{N}$ with $k < \frac{d}{2}$, and $m \in \mathbb{Z}$. Then we have

$$g\left(\frac{kq}{d} + m\right) = g(\widehat{m})\psi\left(\frac{kq}{dm} + \frac{m}{\widehat{m}}\right).$$

Indeed, since $\frac{q}{q_0} \mid \frac{q}{d}$, $\gcd(\frac{kq}{d} + m, \frac{q}{q_0}) = \gcd(m, \frac{q}{q_0}) = \widehat{m}$, which immediately gives (4.2). As a consequence,

$$g\left(\frac{kq}{d} - m_0\right)g\left(\frac{kq}{d} + m_0\right) = g(m_0)\psi\left(\frac{kq}{dm_0} - 1\right)g(m_0)\psi\left(\frac{kq}{dm_0} + 1\right) = \psi\left(\frac{kq}{dm_0}\right)^2 - 1).$$
and \( g(\frac{kq}{d} + m) = 0 \) whenever \( 1 \leq |m| < m_0 \) (in the case \( m_0 > 1 \)) since \( \hat{m} \leq |m| < m_0 \), which yields
\[
\sum_{m=-m_0+1}^{m_0-1} g(\frac{kq}{d} + m) = g(\frac{kq}{d}) = g(\frac{q}{q_0})\psi(\frac{kq}{d}). \tag{4.4}
\]

Note that
\[
1 \leq \frac{kq}{d} - m_0 < \frac{kq}{d} + m_0 \leq \frac{q}{2} - 1
\]
(since \( m_0 \leq \frac{q}{q_0} \leq \frac{q}{d} \) by Lemma 4.14), and \( \psi(\frac{kq}{d}) \neq 0 \) only when
\[
\frac{q_0}{d} \cdot \gcd(k, d) = \gcd(\frac{kq_0}{d}, q_0) = 1.
\]

With (4.3) and (4.4), invoking Lemma 4.12 we obtain the following three conclusions.

**Lemma 4.15.** If \( d \geq 4 \) is a proper divisor of \( q_0 \), then \( \psi((\frac{kq}{q_0m_0})^2 - 1) \neq 1 \) for \( 1 \leq k < \frac{d}{2} \).

**Lemma 4.16.** If \( \psi((\frac{kq}{q_0m_0})^2 - 1) = 1 \) for some \( 1 \leq k < \frac{q_0}{2} \), then
\[
g(\frac{q}{q_0})\psi(k) = -\psi(\frac{kq}{q_0m_0} + 1),
\]
equivalently,
\[
g(\frac{q}{q_0}) = -\psi(k)\psi(\frac{kq}{q_0m_0} + 1).
\]

**Lemma 4.17.** If \( \psi((\frac{kq}{q_0m_0})^2 - 1) = -1 \) for some \( 1 \leq l < \frac{q_0}{2} \), then \( g(\frac{q}{q_0})\psi(l) = 0 \).

Consider the case that \( q_0 \) is an odd prime. If \( q_0^2 \mid \frac{q}{m_0} \), then \( \psi(\frac{kq}{q_0m_0} + 1) = \psi(1) = 1 \) and \( \psi(\frac{kq}{q_0m_0} - 1) = \psi(-1) = 1 \) for every \( 1 \leq k \leq \frac{q_0 - 1}{2} \) since \( q_0 \mid \frac{q}{q_0m_0} \) and \( \psi \) is even. By Lemma 4.16
\[
\psi(k) = \psi(k)\psi(\frac{kq}{q_0m_0} + 1) = -g(\frac{q}{q_0}) = \psi(1)\psi(\frac{q}{q_0m_0} + 1) = 1
\]
holds for \( 2 \leq k \leq \frac{q_0 - 1}{2} \). This is impossible because it would imply that \( \psi \) does not take value \(-1\) (note that \( \psi(q_0 - k) = \psi(-k) = \psi(k) \)). Hence we further conclude that

**Lemma 4.18.** If \( q_0 \) is an odd prime, then \( q_0^2 \nmid \frac{q}{m_0} \).

We are ready to prove the following conclusion.

**Lemma 4.19.** We have \( q_0 = 5 \) and \( \psi(m) = (m \mid 5) \).

**Proof.** Note that \( 3 \nmid q_0 \). Then by Remark 4.5 it suffices to show that

1. \( q_0 \) cannot be divided by a prime \( p \) with \( p \geq 7 \);
2. \( q_0 \) is odd.
(1) Assume that \( q_0 \) has a prime divisor \( p \) with \( p \geq 7 \). Combining Lemma 4.10 (2) with (3), we see that there exists \( k \in \mathbb{N} \) with \( k \leq \frac{p-1}{2} \) such that \( \psi((\frac{q}{pqm})^2 - 1) = 1 \). Lemma 4.15 shows that \( p \) cannot be a proper divisor of \( q_0 \), i.e., \( q_0 = p \), and then by Lemma 4.18, \( q^2 \nmid \frac{q}{pqm} \). Moreover, it follows from Lemma 4.16 that \( g(\frac{q}{pqm}) \neq 0 \). Again by Lemma 4.10 (3), there exists \( l \in \mathbb{N} \) with \( l \leq \frac{q_0-1}{2} \) such that \( \psi((\frac{q}{pqm})^2 - 1) = -1 \), and thus \( g(\frac{q}{q_0})\psi(l) = 0 \) by Lemma 4.17. This yields \( \psi(l) = 0 \), contradicting with the fact \( \gcd(l, q_0) = 1 \).

(2) Assume that \( q_0 \) is even. Then we can write \( q_0 = q_1 q_2 \), where \( q_1 = 4 \) or 8 and \( q_2 = 1 \) or 5. By Lemma 4.3 for each \( i \in \{1, 2\} \) one can find a primitive Dirichlet characters \( \psi_i \) mod \( q_i \) such that \( \psi = \psi_1 \psi_2 \). Noting that \( \psi_2(-1) = 1 \) whether \( q_2 \) equals 1 or 5, we necessarily have \( \psi_1(-1) = 1 \) since \( \psi(-1) = 1 \), and then \( q_1 = 8, \psi_1 = \psi_{8,2}^{pr} \). By combining Lemma 4.10 (1) with Lemma 4.15, we have

\[
\psi_{8,2}^{pr}(1 - (\frac{q}{4mq})^2) = \psi((\frac{q}{4mq})^2 - 1) \neq 1,
\]

which yields \( 16 \nmid \frac{q}{m} \) (otherwise, \( 4 \mid \frac{q}{4mq} \) and then \( \psi_{8,2}^{pr}(1 - (\frac{q}{4mq})^2) = 1 \)). That is to say, \( \frac{q}{8m} \) is odd, and hence so is \( \frac{q}{q_0 m} \). Thus, for any \( k \in \mathbb{N} \),

\[
\hat{2^k}m_0 = \gcd(2^k m_0, \frac{q}{q_0}) = m_0 \cdot \gcd(2^k, \frac{q}{q_0 m_0}) = m_0.
\]

Note that \( q_2 \mid \frac{q}{q_0} \) for every divisor \( d \) of \( \frac{q}{q_0} \) since \( q_2 = \frac{q_0}{8} \). Then

\[
\psi_2(\frac{q}{8m} + \frac{m}{m}) = \psi_2(\frac{m}{m}) = \psi_2(\frac{|m|}{m}), \quad m \in \mathbb{Z}.
\]

(4.5)

Combining (4.2) with (4.5) one obtains that for any \( m \in \mathbb{Z} \),

\[
g(\frac{q}{8} + m) = g(\hat{m})\psi(\frac{q}{8m} + \frac{m}{m}) = g(\hat{m})\psi_{8,2}^{pr}(\frac{q}{8m} + \frac{m}{m}) = g(\hat{m})\psi_{8,2}^{pr}(\frac{8m}{8m} + \frac{m}{m} = \psi_{8,2}^{pr}(\frac{|m|}{m}).
\]

(4.6)

In particular,

\[
g(\frac{q}{8} - 4m_0)g(\frac{q}{8} + 4m_0) = g(m_0)\psi_{8,2}^{pr}(\frac{q}{8m_0} - 4)\psi_2(4)g(m_0)\psi_{8,2}^{pr}(\frac{q}{8m_0} + 4)\psi_2(4)
\]

\[
\psi_{8,2}^{pr}(\frac{q}{8m_0})^2 - 16
\]

(4.7)

\[
= 1,
\]

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and since \( n^2 \equiv 1 \pmod{8} \) for every odd integer \( n \),
\[
\begin{align*}
g\left(\frac{q}{8} - 2m_0\right)g\left(\frac{q}{8} + 2m_0\right) &= g(m_0)\psi_{8,2}^{pr}\left(\frac{q}{8m_0}\right) - 2)\psi_2(2)g(m_0)\psi_{8,2}^{pr}\left(\frac{q}{8m_0} + 2\right) \\
&= \psi_{8,2}^{pr}\left(\frac{q}{8m_0}\right)^2 - 4) \\
&= \psi_{8,2}^{pr}(-3) \\
&= -1.
\end{align*}
\]

By Lemma 4.12
\[
\sum_{m=-2m_0+1}^{2m_0-1} g\left(\frac{q}{8} + m\right) = 0.
\]

Moreover, it follows from (4.6) that \( g\left(\frac{q}{8} + m\right) \neq 0 \) if and only if \( g(\hat{m}) \neq 0 \), \( \frac{q}{8m} + \frac{m}{m} \) is odd, and \( \gcd\left(\frac{m}{m}, q_2\right) = 1 \). Therefore for \( m \in \mathbb{N} \), \( g\left(\frac{q}{8} + m\right) \) and \( g\left(\frac{q}{8} - m\right) \) are either simultaneously zero or simultaneously nonzero since \( \frac{q}{8m} + \frac{m}{m} \) and \( \frac{q}{8m} - \frac{m}{m} \) are simultaneously odd or even. Put
\[
m_1 = \min\{2m_0 \leq m \leq 4m_0 : g\left(\frac{q}{8} - m\right)g\left(\frac{q}{8} + m\right) = 1\},
\]
where the above set is nonempty since it contains \( 4m_0 \) by (4.7). Then \( m_1 > 2m_0 \) by (4.8) and
\[
g\left(\frac{q}{8} - m\right) + g\left(\frac{q}{8} + m\right) = 0, \quad 2m_0 \leq m < m_1.
\]

It follows from (4.9) that
\[
\sum_{m=-m_1+1}^{m_1-1} g\left(\frac{q}{8} + m\right) = \sum_{m=-2m_0+1}^{2m_0-1} g\left(\frac{q}{8} + m\right) = 0.
\]

However, since \( g\left(\frac{q}{8} - m_1\right)g\left(\frac{q}{8} + m_1\right) = 1 \) we should have
\[
\sum_{m=-m_1+1}^{m_1-1} g\left(\frac{q}{8} + m\right) \neq 0
\]
by Lemma 4.12. This shows that \( q_0 \) cannot be even.

We proceed to complete the proof of Lemma 4.2. It follows from Lemmas 4.14 and 4.19 that \( \frac{q}{5m_0} \) is an integer not less than 5. We also have \( 3 \nmid \frac{q}{5m_0} \) since \( 3 \nmid q \), and \( 5 \nmid \frac{q}{5m_0} \) by Lemma 4.18 forcing \( \frac{q}{5m_0} \in \{7, 8\} \) or \( \frac{q}{5m_0} \geq 11 \). Moreover,
\[
\hat{5m_0} = \gcd(5m_0, \frac{q}{5}) = m_0 \cdot \gcd(5, \frac{q}{5m_0}) = m_0.
\]

By (4.2) we have
\[
g\left(\frac{5q}{5} + m\right) = g(\hat{m})(\frac{kq}{5m} + \frac{m}{m} | 5), \quad m \in \mathbb{Z}, \quad k = 1, 2.
\]
In particular, by (4.10), for \( k \in \{1, 2\} \),
\[
g(\frac{kq}{5} - 5m_0)g(\frac{kq}{5} + 5m_0) = g(m_0)(\frac{kq}{5m_0} - 5 | 5)g(m_0)(\frac{kq}{5m_0} + 5 | 5)
\]
\[
= (\frac{kq}{5m_0})^2 | 5),
\]
\[
= 1. \tag{4.12}
\]

**Claim.** \( 25 | q \) and when \( \frac{q}{5m_0} = 7 \) or 8, one has \( g(\frac{q}{25}) \neq 0 \) and hence for \( k \in \{1, 2\} \),
\[
g(\frac{kq}{5} - \frac{q}{25})g(\frac{kq}{5} + \frac{q}{25}) = g(\frac{q}{25})(5k - 1 | 5)g(\frac{q}{25})(5k + 1 | 5)
\]
\[
= g(\frac{q}{25})^2(-1 | 5)
\]
\[
= 1. \tag{4.13}
\]

For this, we need the following calculations: if \( 25 | q \) then
\[
\hat{kq} \equiv \gcd(\frac{kq}{25}, \frac{q}{5}) = \frac{q}{25} \cdot \gcd(k, 5) = \frac{q}{25}
\]
for \( k \in \mathbb{N} \) with \( 5 \nmid k \). Similarly, if \( 35 | q \) then \( \hat{kq} \equiv \frac{q}{35} \) for \( k \in \mathbb{N} \) with \( 7 \nmid k \); if \( 40 | q \) then \( \hat{kq} \equiv \frac{q}{40} \) for \( k \in \mathbb{N} \) with \( 2 \nmid k \). We also note that if \( g(m) \neq 0 \) then \( \hat{m} \equiv \frac{q}{5} \) and \( \frac{q}{5m} \leq \frac{q}{5m_0} \).

Assume \( \frac{q}{5m_0} = 7 \). For \( m \in \mathbb{N} \) satisfying \( g(m) \neq 0 \), \( \frac{q}{5m} = 1, 2, 4, 5 \) or 7, and thus \( \frac{q}{5} \) divides 4m, 5m or 7m. For \( m \in \mathbb{N} \) with \( \frac{2q}{35} \leq m < \frac{3q}{35} \), we have \( \frac{q}{5} \leq 4m < \frac{2q}{5} \) and \( \frac{2q}{5} < 7m < \frac{3q}{5} \). This shows that there is no integer \( m \) between \( \frac{2q}{35} \) and \( \frac{3q}{35} \) such that \( \frac{q}{5} \) divides 4m or 7m. Since
\[
g(\frac{2q}{35})g(\frac{3q}{35}) = g(\frac{q}{35})(2 | 5)g(\frac{q}{35})(3 | 5) = (6 | 5) = 1,
\]
it follows from Lemma 4.12 that
\[
\sum_{m=\frac{2q}{35}+1}^{\frac{3q}{35}-1} g(m) \neq 0.
\]
That is to say, there is an integer \( m_1 \) between \( \frac{2q}{35} \) and \( \frac{3q}{35} \) such that \( g(m_1) \neq 0 \) and \( \frac{q}{5} \mid 5m_1 \).

Since \( \frac{q}{5} < 5m_1 < \frac{3q}{5} \) one has \( m_1 = \frac{2q}{25} \), which yields 25 \( \mid q \) and \( g(\frac{q}{25}) = g(\frac{2q}{25}) \neq 0 \).

Assume \( \frac{q}{5m_0} = 8 \). For \( m \in \mathbb{N} \) satisfying \( g(m) \neq 0 \), \( \frac{q}{5m} = 1, 2, 4, 5, 7 \) or 8, and thus \( \frac{q}{5} \) divides 5m, 7m or 8m. Consider the sets
\[
A_r = \{ m \in \mathbb{Z} : \frac{9q}{40} < m < \frac{11q}{40}, g(m) \neq 0, \frac{q}{5} \mid rm \}, \quad r = 5, 7, 8.
\]
Since
\[
g(\frac{9q}{40})g(\frac{11q}{40}) = g(\frac{q}{40})(9 | 5)g(\frac{q}{40})(11 | 5) = (99 | 5) = 1,
\]

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it follows from Lemma 4.12 that
\[
\sum_{m \in A_5} g(m) + \sum_{m \in A_7} g(m) + \sum_{m \in A_8} g(m) = \sum_{m = \frac{9q}{35} + 1}^{\frac{11q}{35} - 1} g(m) \neq 0.
\]

By using an argument similar to that in the last paragraph, it suffices to prove
\[
\sum_{m \in A_7} g(m) = \sum_{m \in A_8} g(m) = 0.
\]

Since \(\frac{9q}{5} < 8 \cdot 8 < \frac{11q}{5}\), we see that if \(A_8 \neq \emptyset\) then it has exactly one element \(m = \frac{q}{4}\).

Therefore, \(A_8 = \emptyset\) by Lemma 4.13. On the other hand, since \(\frac{7q}{5} < 7 \cdot 8 < 2 \cdot q\), \(A_7\) contains one or two elements in \(\{\frac{8q}{35}, \frac{9q}{35}\}\), and in either case one has \(35 \mid q\). From the identities
\[
g\left(\frac{8q}{35}\right) = g\left(\frac{q}{35}\right)(8 \mid 5) = -g\left(\frac{q}{35}\right),
\]
\[
g\left(\frac{9q}{35}\right) = g\left(\frac{q}{35}\right)(9 \mid 5) = g\left(\frac{q}{35}\right),
\]

we deduce that \(A_7 = \{\frac{8q}{35}, \frac{9q}{35}\}\), and then
\[
\sum_{m \in A_7} g(m) = g\left(\frac{8q}{35}\right) + g\left(\frac{9q}{35}\right) = -g\left(\frac{q}{35}\right) + g\left(\frac{q}{35}\right) = 0.
\]

This proves the claim.

We have shown that there exists \(m_2 \in \mathbb{N}\) with \(m_2 \leq \min\{5 \cdot m_0, \frac{q}{11}\}\) such that
\[
g\left(\frac{kq}{5} - m_2\right)g\left(\frac{kq}{5} + m_2\right) = 1, \quad k = 1, 2.
\]

Indeed, if \(\frac{m_0}{5 \cdot m_0} = 7\) or 8, one can take \(m_2 = \frac{q}{25}\) by (4.13), while if \(\frac{m_0}{5 \cdot m_0} \geq 11\) one can take \(m_2 = 5 \cdot m_0\) by (4.12). Put
\[
m_3 = \min\{m_0 < m \leq m_2 : g\left(\frac{kq}{5} - m\right)g\left(\frac{kq}{5} + m\right) = 1, k = 1, 2\},
\]
and
\[
m_4 = \max\{m_0 \leq m < m_3 : g(\widehat{m}) \neq 0\}.
\]

Now we will show \(5 \nmid \frac{m_4}{m_4}\) and \(5 \nmid \frac{q}{5 \cdot m_4}\). If \(5 \mid \frac{m_4}{m_4}\) then
\[
m_4 \geq 5 \cdot \widehat{m_4} \geq 5 \cdot m_0 \geq m_2 \geq m_3 > m_4,
\]
which is impossible. A direct calculation gives
\[
g\left(\frac{kq}{5} - m_4\right)g\left(\frac{kq}{5} + m_4\right) = g(\widehat{m_4})(\frac{kq}{5 \cdot m_4} - \frac{m_4}{m_4} \mid 5)g(\widehat{m_4})(\frac{kq}{5 \cdot m_4} + \frac{m_4}{m_4} \mid 5)
\]
\[
= g(\widehat{m_4})^2(\left(\frac{kq}{5 \cdot m_4}\right)^2 - \left(\frac{m_4}{m_4}\right)^2 \mid 5)
\]
\[
= \left(\frac{kq}{5 \cdot m_4}\right)^2 - \left(\frac{m_4}{m_4}\right)^2 \mid 5)
\]

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for \( k \in \{1, 2\} \). This shows that if \( 5 \mid \frac{q}{5m_4} \) then
\[
g\left(\frac{kq}{5} - m_4\right)g\left(\frac{kq}{5} + m_4\right) = \left(-\left(\frac{m_4}{m_4}\right)^2 \mid 5\right) = 1, \quad k = 1, 2,
\]
contradicting with the definition of \( m_3 \). Therefore, \( 5 \mid \frac{m_4}{m_4} \) and \( 5 \mid \frac{q}{5m_4} \).

By Lemma 4.9, there exists \( k_0 \in \{1, 2\} \) such that
\[
g\left(\frac{k_0q}{5} - m_4\right)g\left(\frac{k_0q}{5} + m_4\right) = \left(\left(\frac{k_0q}{5m_4}\right)^2 - \left(\frac{m_4}{m_4}\right)^2 \mid 5\right) = -1. \tag{4.14}
\]
Since \( m_3 \leq m_2 \leq \frac{q}{11} \) and \( q \geq 20 \), one has
\[
1 \leq \frac{q}{5} - \frac{q}{11} - 1 \leq \frac{k_0q}{5} - m_3 - 1 \leq \frac{k_0q}{5} + m_3 \leq \frac{2q}{5} + \frac{q}{11} < \frac{q}{2}.
\]
Moreover, for any integer \( m \) between \( m_4 \) and \( m_3 \) we have \( g(\hat{m}) = 0 \), and thus by (4.11),
\[
g\left(\frac{k_0q}{5} - m\right) = g\left(\frac{k_0q}{5} + m\right) = 0.
\]

It follows that
\[
|g\left(\frac{k_0q}{5} - m_3\right) + g\left(\frac{k_0q}{5} - m_4\right)| = |S\left(\frac{k_0q}{5} - m_4\right) - S\left(\frac{k_0q}{5} - m_3 - 1\right)| \leq 1,
\]
and
\[
|g\left(\frac{k_0q}{5} + m_4\right) + g\left(\frac{k_0q}{5} + m_3\right)| = |S\left(\frac{k_0q}{5} + m_3\right) - S\left(\frac{k_0q}{5} + m_4 - 1\right)| \leq 1.
\]
This together with (4.14) yields \( g(\frac{k_0q}{5} - m_3) = 0 \) or \( g(\frac{k_0q}{5} + m_3) = 0 \), contradicting with the definition of \( m_3 \). Hence we complete the proof of Lemma 4.12.

It remains to prove Lemma 4.14.

**Proof of Lemma 4.14.** Assume conversely \( m_0 > \frac{q}{5q_0} \), which is equivalent to that \( m_0 = \frac{q}{q_0}, \frac{q_0}{2q_0} \) or \( \frac{q_0}{4q_0} \) since \( m_0 \mid \frac{q}{q_0} \) and \( 3 \mid q \). It is clear that each divisor \( d \) of \( \frac{q}{q_0} \) satisfying \( d \geq m_0 \) is necessarily divided by \( m_0 \). Thus, \( m_0 \mid m \) for every \( m \in \mathbb{N} \) with \( g(m) \neq 0 \) (since \( g(\hat{m}) \neq 0 \), the definition of \( m_0 \) gives \( \hat{m} \geq m_0 \)). Combining this with the assumption that
\[
\gcd\left(\{m \in \mathbb{Z} : g(m) \neq 0\} \cup \left\{\frac{q}{2}\right\}\right) = 1,
\]
one has \( \gcd(m_0, \frac{q}{2}) = 1 \). A direct calculation gives
\[
\gcd(m_0, \frac{q}{2}) = \begin{cases} m_0, & \text{if } \frac{m_0}{2}, \frac{q}{m_0} \text{ is even;} \\ \frac{m_0}{2}, & \text{if } \frac{m_0}{2}, \frac{q}{m_0} \text{ is odd.} \end{cases}
\]
Then \( m_0 = 1 \) if \( m_0 = \frac{q_0}{2q_0} \), \( \frac{q_0}{4q_0} \) or \( q_0 \) is even, while \( m_0 = 2 \) if \( m_0 = \frac{q_0}{4q_0} \) and \( q_0 \) is odd. Now we have the following five cases.

**Case 1.** \( m_0 = \frac{q_0}{q_0} \) and \( q_0 \) is even.
In this case, \( m_0 = 1 \) and
\[
g(m) = g(\hat{m})\psi\left(\frac{m}{m}\right) = g(1)\psi(m) = \psi(m), \quad m \in \mathbb{Z}.
\]
Since \( q_0 = q \geq 14 \) and \( 3 \nmid q_0 \), we have \( q_0 = 2^l s \), where \( l = 2 \) or \( 3 \), and \( s \geq 5 \) is odd. Moreover, if \( s = 5 \) then \( \psi(m) = \psi_{\mathbb{Z},2}(m) \cdot (m \mid 5) \) (see Remark 4.5). In this case, one has
\[
\psi(3) = \psi_{\mathbb{Z},2}(3) \cdot (3 \mid 5) = 1,
\]
and then
\[
g(1)g(3) = \psi(1)\psi(3) = 1.
\]
It follows from Lemma 4.12 that \( \psi(2) = g(2) \neq 0 \), which contradicts with the fact \( 2 \mid q_0 \).

From this, we deduce that \( s \geq 7 \), i.e., \( q_0 \) has a prime divisor \( p \geq 7 \). By Lemma 4.10 (2), there exists \( k \in \mathbb{N} \) with \( 1 \leq k \leq \frac{p-1}{2} \), such that
\[
g\left(\frac{kq_0}{p} - 1\right)g\left(\frac{kq_0}{p} + 1\right) = \psi\left(\frac{kq_0}{p} - 1\right)\psi\left(\frac{kq_0}{p} + 1\right) = \psi\left(\frac{(kq_0)^2}{p} - 1\right) = 1.
\]
Again by Lemma 4.12, \( \psi\left(\frac{kq_0}{p}\right) \neq 0 \), which is also a contradiction since
\[
\gcd\left(\frac{kq_0}{p}, q_0\right) \geq \frac{q_0}{p} > 1.
\]

Case 2. \( m_0 = \frac{q}{q_0} \) and \( q_0 \) is odd.

In this case, \( m_0 = 2 \), \( g(1) = 0 \), and
\[
g(m) = g(\hat{m})\psi\left(\frac{m}{m}\right) = \begin{cases} g(1)\psi(m) = 0, \quad m \text{ is odd;} \\ g(2)\psi\left(\frac{m}{2}\right) = \psi\left(\frac{m}{2}\right), \quad m \text{ is even}. \end{cases} \quad (4.15)
\]
Moreover, \( q_0 \geq 13 \) since \( q \geq 14 \) and \( q_0 \equiv 1 \pmod{4} \) (see Remark 4.5). Since \( \psi(1) = \psi(4) = 1 \) and \( |\psi(2)| = |\psi(3)| = 1 \), there exists \( c \in \{1, 2, 3\} \) such that \( \psi(c)\psi(c + 1) = 1 \). Therefore
\[
g(2c)g(2c + 2) = \psi(c)\psi(c + 1) = 1,
\]
forcing \( g(2c + 1) \neq 0 \) by Lemma 4.12. However, this contradicts with (4.15).

Case 3. \( m_0 = \frac{q}{2q_0} \) or \( \frac{q}{4q_0} \), and \( q_0 \) is even.

In this case, \( m_0 = 1 \) and \( 4 \mid q_0 \) (see Remark 4.5). This gives \( 8 \mid q \) since \( \frac{q}{q_0} = 2 \) or \( 4 \), and thus \( \frac{q}{4} \pm 1 \) is odd. Hence \( \frac{q}{4} \pm 1 = 1 \) and \( g(\frac{q}{4} \pm 1) = g(1)\psi(\frac{q}{4} \pm 1) = \psi(\frac{q}{4} \pm 1) \). Since
\[
\psi\left(\frac{q}{4} - 1\right) = \psi\left(1 - \frac{q}{4}\right) = \psi\left(1 - \frac{q}{4} + \frac{q}{2}\right) = \psi\left(\frac{q}{4} + 1\right),
\]
and \( \gcd\left(\frac{q}{4} \pm 1, q_0\right) = 1 \), we have \( g(\frac{q}{4} - 1)g(\frac{q}{4} + 1) = 1 \), forcing \( g(\frac{q}{4}) \neq 0 \) by Lemma 4.12. This contradicts with Lemma 4.13.
Case 4. \( m_0 = \frac{q}{q_0} \) and \( q_0 \) is odd.

In this case, \( m_0 = 1, \frac{q}{q_0} = 2, \) and similar to Case 2, we have \( q_0 \geq 13 \) and

\[
g(m) = \begin{cases} 
\psi(m), & m \text{ is odd;} \\
g(2)\psi\left(\frac{m}{2}\right), & m \text{ is even.}
\end{cases}
\]

Again take \( c \in \{1, 2, 3\} \) satisfying \( \psi(c)\psi(c + 1) = 1 \). Then

\[
g(q_0 - 2c - 2)g(q_0 - 2c) = \psi(q_0 - 2c - 2)\psi(q_0 - 2c) = \psi^2(-2)\psi(c)\psi(c + 1) = 1,
\]

forcing \( g(q_0 - 2c - 1) \neq 0 \) by Lemma 4.12, and thus \( g(2) \neq 0 \) since

\[
g(q_0 - 2c - 1) = g(2)\psi\left(\frac{q_0 - 2c - 1}{2}\right).
\]

Let \( p \) be the smallest odd prime of \( q_0 \). For any \( m \in \mathbb{N} \) with \( m < p \), we have \( g(m) \neq 0 \) since \( \psi(m) \neq 0 \). Moreover,

\[
|g(m) + g(m + 1)| = |S(m + 1) - S(m - 1)| \leq 1
\]

for \( 1 \leq m \leq p - 2 \). It follows from Lemma 4.11 (2) that

\[
g(m) = (-1)^{m-1}g(1) = (-1)^{m-1}, \quad m = 1, 2, \ldots, p - 1.
\]

In particular, the identity \( g(4) = g(2)\psi(2) \) gives \( \psi(2) = 1 \). We further have \( \psi\left(\frac{p+1}{2}\right) = 1 \). In fact, if \( \frac{p+1}{2} \) is odd then \( \psi\left(\frac{p+1}{2}\right) = g\left(\frac{p+1}{2}\right) = 1 \); if \( \frac{p+1}{2} \) is even, then

\[
g\left(\frac{p+1}{2}\right) = g(2)\psi\left(\frac{p+1}{4}\right) = -\psi\left(\frac{p+1}{4}\right),
\]

and thus

\[
\psi\left(\frac{p+1}{2}\right) = \psi(2)\psi\left(\frac{p+1}{4}\right) = -\psi(2)g\left(\frac{p+1}{2}\right) = 1.
\]

It follows from this claim that

\[
g(p + 1) = g(2)\psi\left(\frac{p+1}{2}\right) = -1 = g(p - 1),
\]

forcing \( \psi(p) = g(p) = 1 \) by Lemma 4.12. This contradict with the fact \( \gcd(p, q_0) = p > 1 \).

Case 5. \( m_0 = \frac{q}{q_0} \) and \( q_0 \) is odd.

In this case, \( m_0 = 1, \frac{q}{q_0} = 4 \) and \( q_0 \equiv 1 \pmod{4} \). This yields

\[
\overline{q_0} = \overline{q_0} \pm 2 = \overline{q_0} \pm 4 = 1,
\]

\[
\overline{q_0} - 3 = \overline{q_0} + 1 = 2
\]

and

\[
\overline{q_0} - 1 = \overline{q_0} + 3 = 4.
\]
It follows that
\[
g(q_0 \pm 2) = g(1)\psi(q_0 \pm 2) = \psi(\pm 2) = \psi(2),
g(q_0 \pm 4) = g(1)\psi(q_0 \pm 4) = \psi(\pm 4) = 1,
\]
which shows \(g(q_0 - 3) = g(q_0 + 3)\) by Lemma 4.12. Moreover,
\[
g(q_0 - 3) = g(2)\psi\left(\frac{q_0 - 3}{2}\right) = g(2)\psi\left(\frac{q_0 - 3}{2}\right)\psi(4) = g(2)\psi(2q_0 - 6) = g(2)\psi(6),
\]
and similarly, \(g(q_0 + 1) = g(2)\psi(2), g(q_0 + 3) = g(4)\psi(3)\). Therefore \(g(2)\psi(2)\psi(3) = g(4)\psi(3)\), forcing \(g(q_0 - 1) = g(q_0 + 1)\). Since \(g(q_0 - 2)g(q_0 + 2) = 1\), again by Lemma 4.12 we have \(g(q_0) \neq 0\). However,
\[
g(q_0) = g(\tilde{q}_0)\psi\left(\frac{q_0}{\tilde{q}_0}\right) = g(1)\psi(q_0) = 0.
\]
This completes the proof. \(\square\)

5 The number of component intervals of \(V\)

Using the Euler totient function \(\phi\), we obtain a formula for the number of component intervals of \(V\) with \(1_V \in C\). As a consequence, it will be shown that when \(V\) is a subinterval of \((0, 1)\) that satisfies \(1_V \in C\), the least common denominator \(t\) is not larger than 6.

**Theorem 5.1.** Let \(V, q\) and \(g\) be given as in (1.2), (3.1) and (3.7). Suppose \(1_V \in C\).

1. The set of all boundary points of \(V\) in \((0, 1)\) is

\[
\bigcup_{n \geq 3, J_V(\frac{2}{n}) \neq 0} \left\{ \frac{2m}{n} : 1 \leq m < \frac{n}{2}, \gcd(m, n) = 1 \right\},
\]

and the number of component intervals of \(V\) is

\[
N = \frac{1_V(0^+) + 1_V(1^-)}{2} + \frac{1}{4} \sum_{n \geq 3, J_V(\frac{2}{n}) \neq 0} \phi(n).
\]

2. If \(g \in E_q\) then

\[
2(1_V(0^+) + 1_V(1^-)) + \sum_{n \geq 3, J_V(\frac{2}{n}) = 1} \phi(n) = \sum_{n \geq 3, J_V(\frac{2}{n}) = -1} \phi(n).
\]
Proof. Let $M$ be the number of boundary points of $V$ in $(0, 1)$. Then

$$N = \frac{1}{2}(1_V(0^+) + 1_V(1^-) + M).$$

(5.3)

We also put

$$F_n = \{m : 1 \leq m < \frac{n}{2}, \gcd(m, n) = 1\}, \quad n = 1, 2, \cdots.$$

It is easy to see that to any rational number $r \in (0, 1)$, there corresponds a unique positive integer $n$ with $n \geq 3$, such that

$$r = \frac{m}{n^2}$$

for some $m \in F_n$.

(1) By Lemma 4.6, $|J_V(\frac{2m}{n})| = |J_V(\frac{2}{n})|$ for any relatively prime positive integers $m, n$. Thus for $n \geq 3$ and $m \in F_n$, $\frac{2m}{n}$ is a boundary point of $V$ in $(0, 1)$ if and only if $J_V(\frac{2}{n}) \neq 0$. Therefore, the set of all boundary points of $V$ in $(0, 1)$ coincides with the set given in (5.1), and

$$M = \sum_{n \geq 3} \#F_n = \frac{1}{2} \sum_{n \geq 3} \phi(n).$$

This together with (5.3) gives (5.2).

(2) Assume $g \in E_q$. Then by Lemma 4.6, $J_V(\frac{2m}{n}) = J_V(\frac{2}{n})$ for any relatively prime positive integers $m, n$. Thus for $n \geq 3$ and $m \in F_n$, $\frac{2m}{n}$ is a left endpoint of some component interval of $V$ if and only if $J_V(\frac{2}{n}) = 1$. It follows that

$$N = 1_V(0^+) + \sum_{n \geq 3} \#F_n = 1_V(0^+) + \frac{1}{2} \sum_{n \geq 3} \phi(n),$$

which proves (2) by (1).

We record the following corollary. Setting

$$t(n) = \prod_{p \leq 4n+1} p^{\log_p 6n}, \quad n = 1, 2, \cdots,$$

we have

Corollary 5.2. Let $V$ be given as in (1.2). If $1_V \in C$ then $t \mid t(N)$, i.e., every boundary point of $V$ has form $\frac{s}{t(N)}$ for some integer $s$.

Proof. To reach a contradiction, assume conversely that $V$ has a boundary point $\frac{s_0}{t_0} \in (0, 1)$ with $t_0 \nmid t(N)$, where $s_0, t_0$ is a pair of relatively prime positive integers. Then there is a prime divisor $p$ of $t_0$, such that either $p > 4N + 1$ or $p^{\log_p 6N} | t_0$. In either case, we would have

$$\phi(t_0 \cdot \gcd(t_0, 2)) > 4N.$$  

(5.4)
In fact, if there is a prime divisor \( p \) of \( t_0 \), such that \( p > 4N + 1 \), then
\[
\phi(t_0) \geq \phi(p) = p - 1 > 4N;
\]
if \( 2^{[\log_2 6N]} + 1 | t_0 \), then
\[
\phi(t_0) \geq \phi(2^{[\log_2 6N]} + 1) = 2^{[\log_2 6N]} > 2N;
\]
if there is an odd prime divisor \( p \) of \( t_0 \), such that \( p^{[\log_p 6N]} + 1 | t_0 \), then
\[
\phi(t_0) \geq \phi(p^{[\log_p 6N]} + 1) = p^{[\log_p 6N]}(p - 1) \geq \frac{2}{3} p^{[\log_p 6N]} + 1 > 4N.
\]
Therefore, when \( t_0 \) is odd, \( \phi(t_0) > 4N \); when \( t_0 \) is even, \( \phi(t_0) > N \) and thus \( \phi(2t_0) = 2\phi(t_0) > 4N \).

Applying Proposition 4.6 (2), we see that
\[
\frac{2}{t_0 \cdot \frac{2}{\gcd(s_0, 2)}} = \frac{\gcd(s_0, 2)}{t_0}
\]
is also a boundary point of \( V \). Then by Theorem 5.1 (1) and (5.4),
\[
\phi(t_0) \leq \phi(t_0 \cdot \frac{2}{\gcd(s_0, 2)}) \leq 4N < \phi(t_0 \cdot \gcd(t_0, 2)) \leq \phi(2t_0).
\]
This implies that both \( s_0 \) and \( t_0 \) are even, which contradicts with the assumption that \( s_0 \) and \( t_0 \) are relatively prime. The proof is complete.

Corollary [5.2] leads to an upper bound for open subsets \( V \) with \( 1_V \in \mathcal{C} \) when the number of component intervals of \( V \) is restricted. More precisely, let \( \mathcal{V}_n \ (n \in \mathbb{N}) \) denote the set of all non-degenerated open subsets \( V \) of \((0, 1)\) with rational boundary points which have at most \( n \) component intervals. It follows from Corollary [5.2] that there are at most \( \binom{2n}{l(n)+1} \) elements in
\[
\{ V \in \mathcal{V}_n : 1_V \in \mathcal{C} \}.
\]

We end this section with the following corollary.

**Corollary 5.3.** Let \( V \) be given as in (1.2). If \( N = 1 \) (i.e., \( V \) is a subinterval of \((0, 1)\)) and \( 1_V \in \mathcal{C} \), then \( t \leq 6 \).

**Proof.** Since \( t(1) = 60 \), Corollary [5.2] gives \( t | 60 \). Thus, if \( 4 \nmid t \) and \( 5 \nmid t \) then \( t \mid 6 \).

Assume \( 5 | t \). Then by Theorem 5.1 (1), there exists \( l \in \mathbb{N} \) such that \( 5 | l \) and \( J_V(\frac{\pi}{4}) \neq 0 \).
Moreover,
\[
4 \geq \sum_{n \geq 3, J_V(\frac{\pi}{4}) \neq 0} \phi(n) \geq \phi(l) \geq \phi(5) = 4.
\]
which yields that \( l = 5 \) or 10, and that for \( n \geq 3 \), \( J_V(\frac{2}{n}) \neq 0 \) only when \( n = l \). Therefore, \( t = 5 \leq 6 \).

Assume \( 4 \mid t \). Similarly, there exists \( l' \in \mathbb{N} \) such that \( 8 \mid l' \), \( J_V(\frac{2}{l'}) \neq 0 \) and

\[
4 \geq \sum_{n \geq 3, J_V(\frac{2}{n}) \neq 0} \phi(n) \geq \phi(l') \geq \phi(8) = 4,
\]

which gives \( l' = 8 \) and \( t = 4 \leq 6 \).

\[\square\]

## 6 The case \( t \leq 6 \)

In this section, we explicitly determine \( V \) in case \( t \leq 6 \). This together with Corollary 5.3 gives Theorem 1.1.

**Theorem 6.1.** Let \( V \) be given as in (1.2).

(1) When \( t = 1, 2, 1_{(0,1)}, 1_{(0,\frac{1}{2})}, 1_{(\frac{1}{2},1)} \in C \);

(2) When \( t = 3 \), \( 1_V \in C \) if and only if \( V = (0, \frac{2}{3}), (\frac{1}{3}, 1), (\frac{1}{3}, \frac{2}{3}) \) or \( (0, \frac{1}{3}) \cup (\frac{2}{3}, 1) \);

(3) When \( t = 4 \), \( 1_V \in C \) if and only if \( V = (\frac{1}{4}, \frac{3}{4}), (0, \frac{1}{4}) \cup (\frac{1}{2}, \frac{3}{4}) \) or \( (\frac{1}{4}, \frac{1}{2}) \cup (\frac{3}{4}, 1) \);

(4) When \( t = 5 \), \( 1_V \in C \) if and only if \( V = (\frac{1}{5}, \frac{3}{5}), (\frac{2}{5}, \frac{3}{5}), (\frac{1}{2}, \frac{3}{5}) \cup (\frac{3}{5}, \frac{4}{5}) \) or \( (0, \frac{1}{5}) \cup (\frac{2}{5}, \frac{3}{5}) \cup (\frac{4}{5}, 1) \);

(5) When \( t = 6 \), \( 1_V \in C \) if and only if \( V = (\frac{1}{6}, \frac{5}{6}), (0, \frac{1}{6}) \cup (\frac{1}{3}, \frac{5}{6}), (0, \frac{1}{3}) \cup (\frac{1}{2}, \frac{5}{6}), (0, \frac{1}{2}) \cup (\frac{2}{3}, 1), (\frac{1}{6}, \frac{2}{3}) \cup (\frac{5}{6}, 1), (0, \frac{1}{6}) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{5}{6}, 1) \) or \( (0, \frac{1}{3}) \cup (\frac{1}{2}, \frac{2}{3}) \cup (\frac{5}{6}, 1) \).

To prove Theorem 6.1, we need to establish the following.

**Lemma 6.2.** For any \( z \in \overline{D} \), we have

(1) \(|z^2 - 2z + 3| \geq |z - 1|^k\), \( k = 1, 2 \).

(2) \(|z^2 + z - 3| \geq |z^2 - z - 1|\).

**Proof.** Put \( x = \text{Re} z \) and \( y = \text{Im} z \).

(1) Since

\[
\text{Re} (z - 1)^2 = (x - 1)^2 - y^2 \geq (x - 1)^2 + x^2 - 1 \geq -1,
\]

one has

\[
|z^2 - 2z + 3| = |(z - 1)^2 + 2| \geq |(z - 1)^2| = |z - 1|^2.
\]

When \(|z - 1| \geq 1\),

\[
|z^2 - 2z + 3| \geq |z - 1|^2 \geq |z - 1|;
\]

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when $|z - 1| \leq 1,$

$$|z^2 - 2z + 3| \geq 2 - |z - 1|^2 \geq 1 \geq |z - 1|.$$  

(2) A direct calculation gives

$$|z^2 - z - 3|^2 - |z^2 - z - 1|^2 = (x^2 + x - 3 - y)^2 + (2xy + y)^2 -
[(x^2 - x - 1 - y)^2 + (2xy - y)^2]
= 4(2 - x^2)(1 - x) + 4(x + 1)y^2
\geq 0.$$  

Proof of Theorem 6.1 (1) has been shown in Examples 3.5 and 3.6. For the rest of the proof, suppose $1_V \in C$ and put $g(m) = J_V(m)$ $(m \in \mathbb{Z})$. According to Corollary 3.8, we can assume $g \in E_{q, \psi}$ for some $\psi \in \mathcal{P}_q$, where $q = 2t$. Let $q_0$ denote the modulus of $\psi$. If $q_0 = 1$ then by Theorem 3.12, $1_V \in C$ if and only if the polynomial

$$R = \sum_{d|q} g\left(\frac{q}{d}\right)M_d$$

has no zeros in $D^{S_q}$. Without loss of generality, we may also assume $V \neq V_{t,0}, V_{t,1}$ for $t = 3, 4, 5, 6$. Lastly, we refer to Remark 4.5 for possible values of $q_0$.

(2) The case $t = 3$: $q = 6$ and then $q_0 = 1, \psi \equiv 1$.

By a direct calculation,

$$R(z_1, z_2) = g(6) + g(3)M_2(z_1) + g(2)M_3(z_2) + g(1)M_6(z_1, z_2)
= g(0) + g(3)(z_1 - 1) + g(2)(z_2 - 1) + g(1)(z_1 - 1)(z_2 - 1).$$

There are only 4 possibilities:

(a) $g(0) = 2, g(1) = -1, g(2) = g(3) = 0,$

$$R(z_1, z_2) = 1 + z_1 + z_2 - z_1z_2.$$  

Then $R$ has the zero $(-\frac{1}{2}, -\frac{1}{3})$ in $D^2$.

(b) $g(0) = 2, g(2) = -1, g(1) = g(3) = 0,$

$$R(z_1, z_2) = 3 - z_2.$$  

Then $R$ has no zeros in $D^2$.

(c) $g(1) = 1, g(3) = -2, g(0) = g(2) = 0,$

$$R(z_1, z_2) = (z_1 - 1)(z_2 - 3).$$  

Then $R$ has no zeros in $D^2$.
(d) \( g(2) = 1, g(3) = -2, g(0) = g(1) = 0, \)

\[ R(z_1, z_2) = 1 - 2z_1 + z_2. \]

Then \( R \) has the zero \( (\frac{1}{2}, 0) \) in \( \mathbb{D}^2 \).

Therefore, \( 1_V \in \mathcal{C} \) if and only if \( V = (0, \frac{2}{3}) \) or \( (\frac{1}{3}, 1) \).

(3) The case \( t = 4: q = 8 \) and then \( q_0 = 8, \psi \equiv 1 \) or \( q_0 = 8, \psi = \psi_{8,2}^{\text{pr}} \).

Assume \( q_0 = 1 \). By Remark 4.8, if one of \( \frac{1}{4} \) and \( \frac{3}{4} \) is a left (right) endpoint of some component interval of \( V \), then so is the other one. It follows that \( \frac{1}{2} \) is a boundary point of \( V \), which contradicts with the assumption \( V \neq V_{4,0}, V_{4,1} \). Hence, \( q_0 = q = 8, \)

\[ g(3) = g(1)\psi(3) = -g(1), \]

and by Theorem 4.11 \( g(0) = g(2) = g(4) = 0 \). This together with Theorem 3.9 implies that \( 1_V \in \mathcal{C} \) if and only if \( V = (\frac{1}{4}, \frac{3}{4}) \).

(4) The case \( t = 5: q = 10 \) and then \( q_0 = 1, \psi \equiv 1 \) or \( q_0 = 5, \psi(n) = (n \mid 5) \).

Assume \( q_0 = 1 \). If one of \( \frac{1}{5} \) and \( \frac{3}{5} \) is a left (right) endpoint of some component interval of \( V \), then so is the other one. This also holds for the pair of fractions \( \frac{2}{5}, \frac{4}{5} \). An argument similar to that in (3) leads to a contradiction. Hence, \( q_0 = q = 5 \). It follows that

\[ g(3) = g(1)\psi(3) = -g(1), \]

\[ g(4) = g(2)\psi(2) = -g(2), \]

and by Theorem 4.11 \( g(0) = g(5) = 0 \). This together with Theorem 3.10 implies that \( 1_V \in \mathcal{C} \) if and only if \( V = (\frac{1}{5}, \frac{3}{5}) \) or \( (\frac{2}{5}, \frac{4}{5}) \).

(5) The case \( t = 6: q = 12 \) and then \( q_0 = 1, \psi \equiv 1 \) or \( q_0 = 12, \psi(n) = \psi_{4,2}^{\text{pr}}(n) \cdot (n \mid 3) \).

Assume \( q_0 = 1 \). By a direct calculation,

\[ R(z_1, z_2) = g(12) + g(6)M_2(z_1) + g(4)M_3(z_2) + g(3)M_4(z_1) + g(2)M_6(z_1, z_2) + g(1)M_{12}(z_1, z_2) \]

\[ = g(0) + g(6)(z_1 - 1) + g(4)(z_2 - 1) + g(3)z_1(z_1 - 1) + g(2)(z_1 - 1)(z_2 - 1) + g(1)z_1z_2(z_1 - 1)(z_2 - 1). \]

There are 12 possibilities:

(a) \( g(2) = 1, g(3) = -1, g(0) = g(1) = g(5) = g(4) = g(6) = 0, \)

\[ R(z_1, z_2) = (z_1 - 1)(z_2 - 1). \]

Then \( R \) has the zero \( (-\frac{1}{2}, \frac{1}{2}) \) in \( \mathbb{D}^2 \).
(b) \( g(3) = 1, g(4) = -1, g(0) = g(1) = g(5) = g(2) = g(6) = 0, \)
\[
R(z_1, z_2) = z_1^2 - z_1 - z_2 + 1.
\]

Then \( R \) has the zero \((\frac{1}{2}, \frac{3}{5})\) in \( \mathbb{D}^2 \).

(c) \( g(0) = 2, g(1) = g(5) = -1, g(2) = 1, g(3) = g(4) = g(6) = 0, \)
\[
R(z_1, z_2) = -(z_1 - 1)^2z_2 + z_1^2 - 2z_1 + 3.
\]

Since for any \( z \in \mathbb{D} \), \( z^2 - 2z + 3 \neq 0 \) and \(|z^2 - 2z + 3| \geq |z - 1|^2\) by Lemma [6.2] (1), \( R \) has no zeros in \( \mathbb{D}^2 \).

(d) \( g(0) = 2, g(1) = g(5) = -1, g(2) = 1, g(3) = g(4) = g(6) = 0, \)
\[
R(z_1, z_2) = 2 - z_1(z_1 - 1)(z_2 - 2).
\]

Then \( R \) has the zero \((\lambda, -\frac{1}{2})\) in \( \mathbb{D}^2 \), where \( \lambda \) is a root of \( z^2 - z + \frac{4}{5} \).

(e) \( g(0) = 2, g(1) = g(5) = -1, g(4) = 1, g(2) = g(3) = g(6) = 0, \)
\[
R(z_1, z_2) = 2 - (z_1^2 - z_1 - 1)(z_2 - 1).
\]

Then \( R \) has the zero \((\frac{1}{2}, -\frac{3}{5})\) in \( \mathbb{D}^2 \).

(f) \( g(1) = g(5) = 1, g(2) = -1, g(6) = -2, g(0) = g(3) = g(4) = 0, \)
\[
R(z_1, z_2) = (z_1 - 1)(z_1z_2 - z_1 - z_2 - 1).
\]

Then \( R \) has the zero \((-\frac{1}{2}, -\frac{1}{2})\) in \( \mathbb{D}^2 \).

(g) \( g(1) = g(5) = 1, g(3) = -1, g(6) = -2, g(0) = g(2) = g(4) = 0, \)
\[
R(z_1, z_2) = (z_1 - 1)(z_1z_2 - 2z_1 - 2).
\]

Then \( R \) has the zero \((-\frac{3}{4}, -\frac{2}{3})\) in \( \mathbb{D}^2 \).

(h) \( g(1) = g(5) = 1, g(4) = -1, g(6) = -2, g(0) = g(2) = g(3) = 0, \)
\[
R(z_1, z_2) = (z_1^2 - z_1 - 1)z_2 - (z_1^2 + z_1 - 3).
\]

Since for any \( z \in \mathbb{D} \), \( z^2 - z + 3 \neq 0 \) and \(|z^2 + z - 3| \geq |z^2 - z - 1|\) by Lemma [6.2] (2), \( R \) has no zeros in \( \mathbb{D}^2 \).

(i) \( g(0) = 2, g(2) = -1, g(3) = 1, g(4) = -1, g(1) = g(5) = g(6) = 0, \)
\[
R(z_1, z_2) = 2 + z_1^2 - z_1z_2.
\]

Then \( R \) has no zeros in \( \mathbb{D}^2 \).
(j) $g(2) = 1, g(3) = -1, g(4) = 1, g(6) = -2, g(0) = g(1) = g(5) = 0,$

$$R(z_1, z_2) = 2 - z_1^2 + z_1z_2 - 2z_1.$$ 

Then $R$ has the zero $(\sqrt{3} - 1, 0)$ in $\mathbb{D}^2$.

(k) $g(0) = 2, g(2) = -1, g(3) = 1, g(6) = -2, g(1) = g(5) = g(4) = 0,$

$$R(z_1, z_2) = -(z_1 - 1)z_2 + z_1^2 - 2z_1 + 3.$$ 

Since for any $z \in \mathbb{D}$, $z^2 - 2z + 3 \neq 0$ and $|z^2 - 2z + 3| \geq |z - 1|$ by Lemma 6.2 (1), $R$ has no zeros in $\mathbb{D}^2$.

(l) $g(0) = 2, g(3) = -1, g(4) = 1, g(6) = -2, g(1) = g(5) = g(2) = 0,$

$$R(z_1, z_2) = 3 - z_1^2 - z_1 + z_2.$$ 

Then $R$ has no zeros in $\mathbb{D}^2$.

Therefore, in case $q_0 = 1$, $1_V \in \mathcal{C}$ if and only if $V = (0, \frac{1}{6}) \cup (\frac{1}{3}, \frac{5}{6})$, $(0, \frac{1}{3}) \cup (\frac{1}{2}, \frac{2}{3})$, $(0, \frac{1}{2}) \cup (\frac{1}{3}, 1)$ or $(\frac{1}{6}, \frac{5}{6}) \cup (\frac{5}{6}, 1)$.

Now assume $q_0 = q = 12$. Then

$$g(5) = g(1)\psi(5) = -g(1),$$

and by Theorem 3.9 $g(0) = g(2) = g(3) = g(4) = g(6) = 0$. This together with Theorem 3.9 implies that in case $q_0 = 12$, $1_V \in \mathcal{C}$ if and only if $V = (\frac{1}{6}, \frac{5}{6})$. \hfill \square

7 When $V = V_{t,0}$ or $V_{t,1}$?

In this section, we will prove Theorems 1.3 and 1.4, which are restated below.

**Theorem 1.3.** Let $V$ be given as in (1.2). Suppose that $t \geq 7$ and $t = p^k$ for some prime $p$ and some $k \in \mathbb{N}$. Then $1_V \in \mathcal{C}$ if and only if $V = V_{t,0}$ or $V_{t,1}$.

**Theorem 1.4.** Let $V$ be given as in (1.2). Suppose that $t \geq 7$, $3 \nmid t$, and there exists $s \in \mathbb{N}$ with $\gcd(s, t) = 1$, such that $\frac{s}{t}$ is a boundary point of $V$. Then $1_V \in \mathcal{C}$ if and only if $V = V_{t,0}$ or $V_{t,1}$.

Neither of the two assumptions

(1) $3 \nmid t$;

(2) there exists $s \in \mathbb{N}$ with $\gcd(s, t) = 1$, such that $\frac{s}{t}$ is a boundary point of $V$

in Theorem 1.4 can be dropped. This can be illustrated by the following examples.
Example 7.1. For $V = \left(\frac{1}{15}, \frac{7}{15}\right) \cup \left(\frac{11}{15}, \frac{13}{15}\right)$, $W = \left(\frac{2}{15}, \frac{4}{15}\right) \cup \left(\frac{8}{15}, \frac{14}{15}\right)$ and $m \in \mathbb{Z}$, put $g_V(m) = J_{\psi}(\frac{m}{15})$ and $g_W(m) = J_{\psi}(\frac{m}{15})$. Then $g_V, g_W \in E_{30, \psi}$, where $\psi(n) = (n \mid 5)$. It follows from Theorem 3.10 that $1_V, 1_W \in \mathcal{C}$.

Example 7.2. Suppose that $p_1, p_2$ are twin primes ($p_1 = p_2 + 2$), and set

$$ V = \left(\frac{1}{p_1}, \frac{1}{p_2}\right) \cup \left(\frac{3}{p_1}, \frac{3}{p_2}\right) \cup \cdots \cup \left(\frac{2m-1}{p_1}, \frac{2m-1}{p_2}\right) \cup \cdots \cup \left(\frac{p_1-2}{p_1}, 1\right) $$

and

$$ W = \left(0, \frac{2}{p_1}\right) \cup \left(\frac{2}{p_2}, \frac{4}{p_1}\right) \cup \cdots \cup \left(\frac{2m-2}{p_2}, \frac{2m}{p_1}\right) \cup \cdots \cup \left(\frac{p_2-1}{p_2}, \frac{p_1-1}{p_1}\right). $$

Then $1_V, 1_W \in \mathcal{C}$.

In fact, for $m \in \mathbb{Z}$, put $g_V(m) = J_{\psi}(\frac{m}{p_1 p_2})$ and $g_W(m) = J_{\psi}(\frac{m}{p_1 p_2})$. Then $g_V, g_W \in E_{2p_1 p_2}$. By a direct calculation,

$$ R_V(z) = \sum_{d \mid 2p_1 p_2} g_V(\frac{2p_1 p_2}{d}) M_d(z) $$

$$ = g(p_1 p_2) M_2(z) + g(p_2) M_{2p_1}(z, \lambda) + g(p_1) M_{2p_2}(z, \mu) $$

$$ = -2M_2(z) + M_{2p_1}(z, \lambda) - M_{2p_2}(z, \mu) $$

$$ = M_2(z)(-2 + M_{p_1}(\lambda) - M_{p_2}(\mu)) $$

$$ = (z-1)(-2 + \lambda - \mu), $$

$$ R_W(z) = \sum_{d \mid 2p_1 p_2} g_W(\frac{2p_1 p_2}{d}) M_d(z) $$

$$ = g(2p_1 p_2) + g(2p_2) M_{p_1}(\lambda) + g(2p_1) M_{p_2}(\mu) $$

$$ = 2 - M_{p_1}(\lambda) + M_{p_2}(\mu) $$

$$ = 2 - \lambda + \mu. $$

Both $R_V$ and $R_W$ has no zeros in $\mathbb{D}^{2p_1 p_2}$. It follows from Theorem 3.13 that $1_V, 1_W \in \mathcal{C}$.

Similarly, one can obtain the following two examples.

Example 7.3. Suppose that $p_1, p_2$ are two primes satisfying $p_1 = 2p_2 - 1$, and set

$$ V = \left(0, \frac{1}{p_2}\right) \cup \left(\frac{2}{p_1}, \frac{2}{p_2}\right) \cup \cdots \cup \left(\frac{2m-2}{p_2}, \frac{m}{p_1}\right) \cup \cdots \cup \left(\frac{p_1-1}{p_1}, 1\right). $$

Then $1_V \in \mathcal{C}$.

Example 7.4. Suppose that $p_1, p_2$ are two primes satisfying $p_1 = 2p_2 + 1$, and set

$$ V = \left(0, \frac{2}{p_1}\right) \cup \left(\frac{1}{p_2}, \frac{4}{p_1}\right) \cup \cdots \cup \left(\frac{m-1}{p_2}, \frac{2m}{p_1}\right) \cup \cdots \cup \left(\frac{p_2-1}{p_2}, \frac{p_1-1}{p_1}\right). $$

Then $1_V \in \mathcal{C}$. 

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We first prove Theorem 1.4 and then prove Theorem 1.3.

**Proof of Theorem 1.4.** We only need to prove the necessity. For this, assume \( 1_{V} \in \mathcal{C} \) and for \( m \in \mathbb{Z} \), set \( \hat{m} = \gcd(m, 2t) \), \( g(m) = J_{V}(\frac{m}{t}) \). Moreover, \( g(m_{1}) = g(m_{2}) \) whenever \( \hat{m}_{1} = \hat{m}_{2} \) by Theorem 4.1. In particular, since \( 3 \nmid t \) one has

\[
g(3m) = g(m) = g(\hat{m}), \quad m \in \mathbb{Z}.
\]

It suffices to show \( g(m) = (-1)^{m-1}g(1) \) for \( 1 \leq m \leq t - 1 \). According to Lemma 4.11 (2), this in turn reduces to showing \( g(m) \neq 0 \) for \( 1 \leq m \leq t - 1 \).

Assume conversely that there exists \( m \in \mathbb{N} \) with \( m < t \) such that \( g(m) = 0 \). Take \( m_{0} \) to be the smallest positive integer among those \( m \)'s satisfying \( g(m) = 0 \). We claim that \( 4 \leq m_{0} \leq \frac{t}{2} \) and \( 3 \nmid m_{0} \). On one hand, the identity \( g(\hat{m}_{0}) = g(m_{0}) = 0 \) implies that \( \hat{m}_{0} = m_{0} \), i.e., \( m_{0} \mid 2t \). Since \( m_{0} < t \) and \( 3 \nmid t \), we have \( m_{0} \leq \frac{t}{2} \) and \( 3 \nmid m_{0} \). On the other hand, since

\[
\hat{s}_{0} = \gcd(s_{0}, 2t) = \gcd(s_{0}, 2),
\]

it follows that \( g(\gcd(s_{0}, 2)) = g(s_{0}) \neq 0 \). If \( s_{0} \) is odd, then \( g(3) = g(1) \neq 0 \), forcing \( g(2) \neq 0 \) by Lemma 4.12. If \( s_{0} \) is even, then \( t \) is odd and \( \hat{A} = 2 \). This yields that \( g(4) = g(2) \neq 0 \), and thus \( g(1) = g(3) \neq 0 \) by Lemma 4.12. In either case, \( g(m) \neq 0 \) for \( m \in \{1, 2, 3\} \), which gives \( m_{0} \geq 4 \).

By Lemma 4.11 (2),

\[
g(m) = (-1)^{m-1}g(1), \quad 1 \leq m \leq m_{0} - 1.
\]

(7.1)

From this, we can deduce that \( m_{0} \equiv 1 \pmod{3} \). In fact, if \( m_{0} \equiv 2 \pmod{3} \) then

\[
g(m_{0} + 1) = g\left(\frac{m_{0} + 1}{3}\right) = (-1)^{\frac{m_{0}}{3} - 2}g(1) = (-1)^{\frac{m_{0} - 2}{3}}g(1) = g(m_{0} - 1) \neq 0,
\]

which would yield \( g(m_{0}) \neq 0 \) by Lemma 4.12. Hence,

\[
g(m_{0} + 2) = g\left(\frac{m_{0} + 2}{3}\right) = (-1)^{\frac{m_{0} - 1}{3}}g(1) = (-1)^{\frac{m_{0} - 1}{3}}g(1) = -g(m_{0} - 1),
\]

forcing \( g(m_{0} + 1) = g(m_{0}) + g(m_{0} + 1) = 0 \) by Lemma 4.12. Since \( \hat{m}_{0} + 1 \neq m_{0} \), we also have \( \hat{m}_{0} + 1 = m_{0} + 1 \), i.e., \( m_{0} + 1 \mid 2t \).

Set \( 2u \) to be the even integer in \( \{m_{0}, m_{0} + 1\} \). Then \( u \leq m_{0} - 2 \) since \( m_{0} \geq 4 \). We have shown previously \( u \mid t \) and

\[
g(2u) = 0.
\]

(7.2)

Write \( t = 2^{k}s \) and \( u = 2^{l}v \), where \( k, l \geq 0 \) and \( s, v \) are odd. Put \( u' = 2^{l}v' \), where \( v' = \frac{s + v}{2} \) when \( s \equiv v \pmod{4} \) and \( k > l \); \( v' = \frac{s - v}{2} \) when \( s \not\equiv v \pmod{4} \) or \( k = l \). Then \( 2u' < t - 1 \) and \( 2u' \equiv cu \pmod{s} \) for some \( c \in \{-1, 1\} \), which gives

\[
2u' \pm 1 = \gcd(2u' \pm 1, 2t) = \gcd(2u' \pm 1, s) = \gcd(cu \pm 1, s) = \gcd(u \pm c, s).
\]

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Therefore, $2u' \pm 1 \leq u + 1 \leq m_0 - 1$, and thus (7.1) yields

$$g(2u' \pm 1) = g(2u' \pm 1) = (-1)^{2u'\pm1}g(1) = g(1) \neq 0.$$ 

It follows from Lemma 4.12 that $g(2u') = -g(1) \neq 0$. However, since

$$\text{gcd}(u', t) = 2^l \text{gcd}(v', s) = 2^l \text{gcd}(2v', s) = 2^l \text{gcd}(v, s) = \text{gcd}(u, t),$$

we have

$$\hat{2u'} = 2 \text{gcd}(u', t) = 2 \text{gcd}(u, t) = \hat{2u},$$

and then $g(2u') = g(2u) = 0$ by (7.2). This is a contradiction and completes the proof. \(\Box\)

To prove Theorem 1.3, we need two more lemmas.

Lemma 7.5. Suppose $k \geq 2$ and $g \in E_{2\cdot3^k}$. Assume that either $g(1)$ or $g(2)$ is not equal to 0, and the sum function $S(m) = \sum_{i=1}^{m} g(i)$ of $g$ only takes values 0 or 1 for $1 \leq m \leq 3^k - 1$.

1. If $g(1)g(2) \neq 0$ then $g(m) = (-1)^m - 1$ for every $1 \leq m \leq 3^k - 1$.
2. If $g(1)g(2) = 0$ then $g(3^l) = g(2 \cdot 3^l) = -1$ for every $1 \leq l \leq k - 1$.

Proof. For every $1 \leq l \leq k - 1$, since $g \in E_{2\cdot3^k}$ we have

$$g(3^l - 1) = g(3^l + 1) = g(2)$$

and

$$g(3^l - 2) = g(3^l + 2) = g(1).$$

By Lemma 4.12 if $g(2) \neq 0$ then $g(3^l) = -g(2)$; if $g(2) = 0$ then $g(1) \neq 0$, which gives

$$g(3^l) = g(3^l - 1) + g(3^l) + g(3^l + 1) = -g(1).$$

Similarly,

$$g(2 \cdot 3^l) = \begin{cases} -g(1), & g(1) \neq 0; \\ -g(2), & g(1) = 0, \end{cases} \quad 1 \leq l \leq k - 1.$$

This proves the lemma. \(\square\)

Lemma 7.6. Suppose $k \geq 3$, $0 < \theta_0 \leq \pi$, $0 < r < 2 \cos \frac{\theta_0}{2k-1}$, and $k\pi - \theta_0 \leq \theta \leq k\pi + \theta_0$. Then there exists $z \in \mathbb{D}$, such that $z^k - z^{k-1} = re^{i\theta}$.

Proof. Let $f(z) = z^k - z^{k-1}$. By the continuity of $f$ on $(-1, 0)$, $f(z) = (-1)^k \cdot r = re^{ik\pi}$ for some $z \in (-1, 0)$. We also note that the image of $f$ on $\mathbb{D}$ is symmetric with respect to the real line. So it suffices to consider $\theta \in [k\pi - \theta_0, k\pi)$, and then

$$\frac{\pi}{2} \leq \frac{\theta - \pi}{k - 1} < \frac{2\theta - \pi}{2k - 1} < \pi.$$
Set \( a = \frac{\theta - \pi}{k - 1} \), \( b = \frac{2\theta - \pi}{2k - 1} \) and
\[
\lambda(\alpha) = \frac{\sin(\theta - (k - 1)\alpha)}{\sin[\theta - k\alpha]}, \quad \alpha \in [a, b].
\]

Since for \( a \leq \alpha \leq b \), one has \( \frac{\pi}{2} \leq \theta - k\alpha \leq \pi \) and \( 0 < \theta - (k + 1)\alpha \leq \frac{\pi}{2} \), and thus \( \lambda \) is strictly decreasing on \([a, b]\). It follows that \( 0 < \lambda(\alpha) < 1 \) for \( a < \lambda < b \) since \( \lambda(a) = 1 \) and \( \lambda(b) = 0 \). Put \( z(\alpha) = \lambda(\alpha)e^{i\alpha} \) and \( w(\alpha) = f(z(\alpha)) \). Then for any \( a < \alpha < b \), \( z(\alpha) \in \mathbb{D} \) and
\[
\arg(z(\alpha) - 1) = -\arctan \frac{\lambda(\alpha) \sin \alpha}{1 - \lambda(\alpha) \cos \alpha}
= -\arctan \frac{\sin(\theta - (k - 1)\alpha) \sin \alpha}{\sin[\theta - k\alpha] - \sin(\theta - (k - 1)\alpha) \cos \alpha}
= -\arctan \frac{\sin(\theta - (k - 1)\alpha) \sin \alpha}{-\cos(\theta - (k - 1)\alpha) \sin \alpha}
= \theta - (k - 1)\alpha,
\]
which gives \( \arg w(\alpha) = \theta \). Since \( |w(b)| = 0 \) and
\[
|w(a)| = 2\cos \frac{k\pi - \theta}{2k - 1} \geq 2\cos \frac{\theta_0}{2k - 1} > r,
\]
there exists \( a < \alpha_0 < b \) such that \( |w(\alpha_0)| = r \). Therefore, one can take \( z = z(\alpha_0) \) to meet the requirement.

\[\square\]

**Proof of Theorem 1.3.** According to Theorem 1.4, it remains to prove the case when \( p = 3 \) and \( k \geq 2 \). Moreover, we only need to prove the necessity. For this, assume \( 1_V \in C \) and set \( g(m) = J_V(\frac{m}{3^k}) \) (\( m \in \mathbb{Z} \)). By Theorem 4.1 \( g \in E_{2,3^k} \) and the polynomial
\[
R = \sum_{d|2 \cdot 3^k} g(\frac{2 \cdot 3^k}{d}) M_d
\]
has no zeros in \( \mathbb{D}^{S_{2,3^k}} \). There exists a boundary point of \( V \) being of form \( \frac{s}{3^k} \), where \( s \) is an integer that cannot divided by 3. This implies that either \( g(1) \) or \( g(2) \) is not equal to 0 since
\[
g(\gcd(s, 2)) = g(\gcd(s, 2 \cdot 3^k)) = g(s) = J_V(\frac{s}{3^k}) \neq 0.
\]
Applying Lemma 7.3 (1), one sees that it suffices to show \( g(1)g(2) \neq 0 \).

To reach a contradiction, assume \( g(1)g(2) = 0 \) conversely and set \( a = g(1) + g(2) \). Then exactly one of \( \frac{1}{3^k} \) and \( \frac{2}{3^k} \) is a boundary point of \( V \), forcing
\[
a = J_V(\frac{1}{3^k}) + J_V(\frac{2}{3^k}) = 1 - 21_V(0^+) = 1 - g(0) = 1 - g(2 \cdot 3^k). \quad (7.3)
\]
Similarly, since \( g(3^k-1) = g(2) \) and \( g(3^k - 2) = g(1) \), one has
\[
g(3^k - 2) + g(3^k - 1) = J_{V}(\frac{1}{3^k}) + J_{V}(\frac{2}{3^k}) = -21V(1^-) - 1 = -g(3^k) - 1.
\]
That is to say, \( g(2 \cdot 3^k) = 1 - a \) and \( g(3^k) = -1 - a \). Also from (7.3), we deduce that \( \sum_{i=1}^{m} ag(i) \) only takes values 0 or 1 for \( 1 \leq m \leq 3^k - 1 \). It follow from Lemma 7.5 (2) that
\[
g(3^l) = g(2 \cdot 3^l) = -a
\]
for every \( 1 \leq l \leq k - 1 \). By a direct calculation,
\[
R(z_1, z_2) = g(2 \cdot 3^k) + g(3^k)M_2(z_1) + \sum_{l=1}^{k-1} g(2 \cdot 3^l)M_{3k-l}(z_2)
\]
\[
+ \sum_{l=1}^{k-1} g(3^l)M_2(z_1)M_{3k-l}(z_2) + g(2)M_{3k}(z_2) + g(1)M_2(z_1)M_{3k}(z_2)
\]
\[
= [g(1)z_2^{k} - (g(1) + a)z_2^{k-1} - 1]z_1 + (g(2) - g(1))z_2^{k}
\]
\[
+ (g(1) - g(2))z_2^{k-1} + 2.
\]
This yields that
\[
|g(1)z_2^{k} - (g(1) + a)z_2^{k-1} - 1| \leq |(g(2) - g(1))(z_2^{k} - z_2^{k-1}) + 2|
\]
holds for all \( z \in \overline{D} \).

There are four possibilities:

(a) \( g(1) = 1, \ g(2) = 0, \) and then
\[
|z^{k} - 2z^{k-1} - 1| \leq |z^{k} - z^{k-1} - 2|, \quad z \in \overline{D};
\]

(b) \( g(1) = -1, \ g(2) = 0, \) and then
\[
|z^{k} - 2z^{k-1} + 1| \leq |z^{k} - z^{k-1} + 2|, \quad z \in \overline{D};
\]

(c) \( g(1) = 0, \ g(2) = 1, \) and then
\[
|z^{k-1} + 1| \leq |z^{k} - z^{k-1} + 2|, \quad z \in \overline{D};
\]

(d) \( g(1) = 0, \ g(2) = -1, \) and then
\[
|z^{k-1} - 1| \leq |z^{k} - z^{k-1} - 2|, \quad z \in \overline{D}.
\]
When $k = 2$, the inequalities in (a) and (d) fail for $z = -1$; the inequality in (b) fails for $z = i$; the inequality in (c) fails for $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. For $k \geq 3$, since
\[ 2 \cos \frac{\theta_0}{2k - 1} \geq 2 \cos \frac{\pi}{5} > \frac{3}{2}, \]
letting $\theta_0 = \pi$ in Lemma [7.6], we see that there exist $\lambda, \mu \in \mathbb{D}$, such that
\[ \lambda^k - \lambda^{k-1} = \frac{3}{2}, \quad \mu^k - \mu^{k-1} = -\frac{3}{2}. \]
However, taking $z = \lambda$ in the inequalities in (a) and (d), one has
\[ \frac{1}{2} \geq |\lambda^k - 2\lambda^{k-1} - 1| = |2 - \lambda^k| \geq 2 - |\lambda|^k > 1, \]
\[ \frac{1}{2} \geq |\lambda^{k-1} - 1| = |\lambda^k - \frac{5}{2}| \geq \frac{5}{2} - |\lambda|^k > \frac{3}{2}. \]
That is to say, both of the inequalities in (a) and (d) fail for $z = \lambda$. Similarly, both of the inequalities in (b) and (c) fail for $z = \mu$. The proof is complete. \qed

8 The case that each $t_i$ is a prime

In this section, we will prove the following generalization of Theorem 1.5.

**Theorem 8.1.** Let $V$ be a proper and non-degenerated open subset of $(0,1)$ with finitely many component intervals, and all its boundary points in $(0,1)$ be $\{\frac{a_i}{t_i}\}_{i=1}^M$ $(M \in \mathbb{N})$, where each $\frac{a_i}{t_i}$ is an irreducible fraction. Assume that not all denominators $t_i$ are the same, and

1. for any $1 \leq i \leq M$, $t_i$ is odd, and $t_i$ is not a product of two primes;
2. for any $1 \leq i, j \leq M$, either $t_i = t_j$ or $\gcd(t_i, t_j) = 1$.

Then $1_V \in \mathcal{C}$ if and only if there exist two primes $p_1, p_2$ such that one of the following situations is true:

(i) $p_1 = p_2 + 2$ and
\[ V = \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \cup \left( \frac{3}{p_1}, \frac{3}{p_2} \right) \cup \cdots \cup \left( \frac{2m-1}{p_1}, \frac{2m-1}{p_2} \right) \cup \cdots \cup \left( \frac{p_1-2}{p_1}, 1 \right); \]

(ii) $p_1 = p_2 + 2$ and
\[ V = \left( 0, \frac{2}{p_1} \right) \cup \left( \frac{2}{p_2}, \frac{4}{p_1} \right) \cup \cdots \cup \left( \frac{2m-2}{p_2}, \frac{2m}{p_1} \right) \cup \cdots \cup \left( \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \right); \]

(iii) $p_1 = 2p_2 + 1$ and
\[ V = \left( 0, \frac{2}{p_1} \right) \cup \left( \frac{1}{p_2}, \frac{4}{p_1} \right) \cup \cdots \cup \left( \frac{m-1}{p_2}, \frac{2m}{p_1} \right) \cup \cdots \cup \left( \frac{p_2-1}{p_2}, \frac{p_1-1}{p_1} \right); \]
Recall that if \(8.1\) Preparatory lemmas

Subsection 8.2. \(8.1\) are satisfied. That is to say, Theorem 8.1 is a generalization of Theorem 1.5.

Moreover, \(8.1\) If Lemma 8.3. \(8.1\) Remark 8.2. \(8.1\) if each \(k\) when \(1\)

Suppose \(k\) \(2\) and \(k\) \(3\), \(11\)

Proof. \(8.1\) Since the roots \(8.1\) In particular, \(8.1\) \(1\)

First note that for \(8.1\) \(4\). Now put \(8.1\) \(2\)

\[ V = \left(0, \frac{1}{p_2}\right) \cup \left(\frac{2}{p_1}, \frac{2}{p_2}\right) \cup \cdots \cup \left(\frac{2m-2}{p_1}, \frac{m}{p_2}\right) \cup \cdots \cup \left(\frac{p_1-1}{p_1}, 1\right). \]

Remark 8.2. If each \(t_i\) in Theorem 8.1 is a prime, then both two assumptions in Theorem 8.1 are satisfied. That is to say, Theorem 8.1 is a generalization of Theorem 1.3.

We list some lemmas in Subsection 8.1 as preparation, and then prove Theorem 8.1 in Subsection 8.2.

8.1 Preparatory lemmas

Recall that if \(q (q \geq 2)\) has prime factorization \(q = p_{j_1}^{k_1} \cdots p_{j_l}^{k_l}\), then \(S_q = \{j_1, \cdots, j_l\}\) and

\[ M_q(z_{j_1}, \cdots, z_{j_l}) = z_{j_1}^{k_1-1}(z_{j_1} - 1) \cdots z_{j_l}^{k_l-1}(z_{j_l} - 1). \]

Moreover, \(M_{q_1 q_2} = M_{q_1} M_{q_2}\) whenever \(\gcd(q_1, q_2) = 1\).

Lemma 8.3. If \(q (q \geq 2)\) is neither a prime nor a product of two primes, then

\[ M_q(\mathbb{D}^{S_q}) \supseteq \{re^{i\theta} : 0 < r \leq \frac{9}{5}, \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\} \cup \left\{\frac{2}{5}\right\}. \]

In particular, if \(q (q \geq 2)\) is not a product of two primes, then

\[ M_q(\mathbb{D}^{S_q}) \supseteq \left\{-\frac{9}{5}, 0\right\}, \quad M_q(\overline{\mathbb{D}^{S_q}}) \supseteq \{i-1, -i-1\}. \]

To prove Lemma 8.3 we need an auxiliary result.

Lemma 8.4. \(1\) Suppose \(k \geq 2\) and \(w \in \mathbb{D}\), there exists \(z \in \mathbb{D}\), such that \(z^k - z^{k-1} = w\).

\(2\) Suppose \(k \in \{3, 4\}\), \(0 < r \leq \frac{9}{5}\) and \(\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\). Then there exist \(z, w \in \mathbb{D}\), such that

\((z-1)^k = re^{i\theta} \text{ and } (w-1)^k = \frac{2}{5}\).

Proof. \(1\) Since the roots \(\alpha_1, \alpha_2, \cdots, \alpha_{k+1}\) of \(f(z) = z^k(z-1) - w\) satisfy

\[ |\alpha_1 \alpha_2 \cdots \alpha_{k+1}| = |w| < 1, \]

there exists \(i \in \{1, 2, \cdots, k+1\}\) such that \(\alpha_i \in \mathbb{D}\).

\(2\) First note that for \(\lambda > 0\) and \(\frac{\pi}{12} < \alpha < \frac{3\pi}{12}\), \(|\lambda e^{i\alpha} + 1| < 1\) if and only if \(\lambda < -2 \cos \alpha\). In particular, \(|\left(\frac{3}{5}\right)^{\frac{1}{k}} e^{i\frac{2\pi}{k}} + 1| < 1\). Then we can take \(w = \left(\frac{3}{5}\right)^{\frac{1}{k}} e^{i\frac{2\pi}{k}} + 1\) for \(k = 3\) or \(w = 1 - \left(\frac{3}{5}\right)^{\frac{1}{k}}\) for \(k = 4\). Now put \(\lambda = r^{\frac{1}{k}}, \alpha = \frac{\theta + 2\pi}{k}\) and \(z = \lambda e^{i\alpha} + 1\). Then \((z-1)^k = re^{i\theta}\). Moreover, when \(k = 3, \frac{11\pi}{12} < \alpha < \frac{13\pi}{12}\) and

\[ \lambda \leq \left(\frac{9}{5}\right)^{\frac{1}{3}} < 2 \cos \frac{\pi}{12} < -2 \cos \alpha; \]
when $k = 4$, $\frac{11\pi}{16} < \alpha < \frac{13\pi}{16}$ and

$$\lambda \leq \left(\frac{9}{5}\right)^{\frac{1}{4}} < 2 \cos \frac{5\pi}{16} < -2 \cos \alpha.$$ 

In either case, we have $z \in \mathbb{D}$. \qed

**Proof of Lemma 8.3.** We have the following three cases.

**Case 1.** $q = p^k$ for some prime $p$ and some integer $k$ ($k \geq 3$).

By (3.10), $M_q$ only depends on one variable and $M_q(z) = z^k - z^{k-1}$. Taking $\theta_0 = \pi$ in Lemma 7.6, we have

$$M_q(\mathbb{D}) \supseteq \{z \in \mathbb{C} : |z| < 2 \cos \frac{\pi}{2k-1}\}. \quad (8.1)$$

Note that $2 \cos \frac{\pi}{5} > \frac{2}{5}$ and for $k \geq 4$,

$$2 \cos \frac{\pi}{2k-1} = \frac{2 \cos \frac{\pi}{7}}{2} > \frac{9}{5}.$$ 

It remains to show that when $k = 3$,

$$M_q(\mathbb{D}) \supseteq \{re^{i\theta} : 0 < r \leq \frac{9}{5}, \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}.$$ 

For $k = 3$, taking $\theta_0 = \frac{\pi}{4}$ in Lemma 7.6, we have

$$M_q(\mathbb{D}) \supseteq \{re^{i\theta} : 0 < r < 2 \cos \frac{\pi}{20}, \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}.$$ 

This completes the proof for Case 1.

**Case 2.** $q = p_{j_1}p_{j_2} \cdots p_{j_l}$, where $l \geq 3$ and $j_1 < j_2 < \cdots < j_l$.

By (3.10), $M_q$ depends on $l$ variables $z_{j_1}, z_{j_2}, \ldots, z_{j_l}$ and

$$M_q(z) = (z_{j_1} - 1)(z_{j_2} - 1) \cdots (z_{j_l} - 1).$$

When $l$ is odd, by taking $z_{j_1} = z_{j_2} = z_{j_3}$ and $z_{j_k} = 0$ ($4 \leq k \leq l$), one obtains that

$$M_q(\mathbb{D}^{S_q}) \supseteq \{w : w = (z - 1)^3 \text{ for some } z \in \mathbb{D}\}.$$ 

Similarly, when $l$ is even,

$$M_q(\mathbb{D}^{S_q}) \supseteq \{w : w = (z - 1)^4 \text{ for some } z \in \mathbb{D}\}.$$ 

Hence, in Case 2, the conclusion immediately follows from Lemma 8.4 (2).

**Case 3.** $q$ is not a power of some prime, and $p^2 \mid q$ for some prime $p$.

Take $k \in \mathbb{N}$ such that $p^k \mid q$ and $p^{k+1} \nmid q$, and put $q' = \frac{q}{p^k}$. Then $M_q = M_{p^k}M_{q'}$ and

$$M_q(\mathbb{D}^{S_q}) = M_{p^k}(\mathbb{D}^{S_{p^k}}) \cdot M_{q'}(\mathbb{D}^{S_{q'}}) =: \{\lambda \mu : \lambda \in M_{p^k}(\mathbb{D}^{S_{p^k}}), \mu \in M_{q'}(\mathbb{D}^{S_{q'}})\}.$$ 

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Since $k \geq 2$, by (8.1), $M_{p^k}(\mathbb{D}^{S_{p^k}}) \supseteq \mathbb{D}$. Moreover, letting $I$ denote the interval $(-1,1)$, we have $M_q(\mathbb{D}^{S_q}) \supseteq M_q(I^{S_q})$. It follows that $M_q(\mathbb{D}^{S_q})$ contains $(0,2)$ or $(-2,0)$, and thus

$$M_q(\mathbb{D}^{S_q}) \supseteq \{z \in \mathbb{C} : |z| < 2\}.$$

The proof is complete. \qed

**Lemma 8.5.** If $q$ ($q \geq 2$) is not a product of two primes, then there exists $z \in \overline{\mathbb{D}^{S_q}}$, such that one of the following four inequalities holds:

1. $|M_q(z) - 1 - i| > \sqrt{10}$.
2. $|M_q(z) + 3 + i| < \sqrt{2}$.
3. $|M_q(z) + 3 + i| > \sqrt{10}$.
4. $|M_q(z) - 3 - i| < \sqrt{10}$.

**Proof.** For any complex number $c$ with $|c| > 1$, one can take $z, w \in \overline{D}$ satisfying $|z-c| = |c|+1$ and $|w-c| = |c| - 1$. This fact proves the case when $q$ is a prime.

Now assume that $q$ is also not a prime. By Lemma 8.3, there exist $u, v, w \in \mathbb{D}^{S_q}$ such that $M_q(u) = -\frac{9\sqrt{2}}{10} - \frac{9\sqrt{2}}{10}i$, $M_q(v) = -\frac{27\sqrt{14}}{50} - \frac{27\sqrt{14}}{50}i$ and $M_q(w) = \frac{2}{5}$. The proof is finished by taking $z = u$ in (1), $z = v$ in (2) and $z = w$ in (3) and (4). \qed

**Lemma 8.6.** Suppose that $\frac{1}{2} \leq \lambda < 5$, $q_1, q_2 \geq 2$, $\gcd(q_1, q_2) = 1$ and neither $q_1$ nor $q_2$ is a product of two primes.

1. If $2 - \lambda M_{q_1} + M_{q_2}$ has no zeros in $\mathbb{D}^{S_{q_1} q_2}$, then $q_1$ is a prime.
2. If $M_{q_1} + M_{q_2}$ has no zeros in $\mathbb{D}^{S_{q_1} q_2}$, then both $q_1$ and $q_2$ are primes.

**Proof.** (1) If $q_1$ is not a prime, then by Lemma 8.3 one can take $z^1 = \mathbb{D}^{S_{q_1}}$ and $z^2 = \mathbb{D}^{S_{q_2}}$, such that $M_{q_1}(z^1) = \frac{2}{5}$ and $M_{q_2}(z^2) = \frac{10-2\lambda}{5}$.

(2) If one of $q_1, q_2$, say $q_1$, is not a prime, then by Lemma 8.3 one can take $z^1 = \mathbb{D}^{S_{q_1}}$ and $z^2 = \mathbb{D}^{S_{q_2}}$, such that $M_{q_1}(z^1) = \frac{2}{5}$ and $M_{q_2}(z^2) = -\frac{2}{5}$. \qed

### 8.2 Proof of Theorem 8.1

This section is dedicated to the proof of Theorem 8.1. The sufficiency has been established in Examples 6.2-6.4. For the necessity, assume $1_V \in C$.

To begin, we introduce some notations. We first list all of the different numbers appearing in $\{t_i\}_{i=1}^M$: $d_1, d_2, \ldots, d_n$ ($n \geq 2$). Set $d_0 = 1$, $t = d_1 d_2 \cdots d_n$, $q = 2t$, $g(m) = J_V \left( \frac{m}{d_i} \right)$ ($m \in \mathbb{Z}$) and

$$a_i = J_V \left( \frac{1}{d_i} \right) = g \left( \frac{t}{d_i} \right), \quad b_i = J_V \left( \frac{2}{d_i} \right) = g \left( \frac{q}{d_i} \right), \quad 0 \leq i \leq n.$$
Then by Theorem 4.1, \( g \in E_q \) and the polynomial

\[
R = \sum_{d|q} g\left(\frac{q}{d}\right)M_d = \sum_{i=0}^{n} (a_i M_{2d_i} + b_i M_{d_i})
\]

has no zeros in \( \mathbb{D}^{S_q} \). Since each \( d_i \) is odd, we have

\[
M_{2d_i} = M_2 M_{d_i} = (z_1 - 1) M_{d_i},
\]

and thus

\[
R = \sum_{i=0}^{n} [a_i(z_1 - 1) + b_i] M_{d_i}
\]

\[
= a_0 z_1 + b_0 - a_0 + \sum_{i=1}^{n} (a_i z_1 + b_i - a_i) M_{d_i}.
\]

(8.2)

Set \( \Lambda = \{-1, 0, 1\}^2 \setminus (0, 0) \) and \( I = \{1, 2, \ldots, n\} \). For \( (a, b) \in \Lambda \) put

\[
I_{a,b} = \{i \in I : a_i = a, b_i = b\}, \quad n_{a,b} = \# I_{a,b}.
\]

We also put

\[
e_i = \begin{cases} \frac{2}{d_i}, & i \in I_{0,1} \cup I_{0,-1}; \\ \frac{1}{d_i}, & \text{otherwise}, \end{cases}
\]

\[
f_i = \begin{cases} \frac{2}{d_i}, & i \in I_{1,0} \cup I_{-1,0}; \\ \frac{1}{d_i}, & \text{otherwise}. \end{cases}
\]

Arrange \( \{e_i\}_{i=1}^{n} \) and \( \{f_i\}_{i=1}^{n} \) from small to large, respectively:

\[
e_{i_1} < e_{i_2} < \cdots < e_{i_n}; \quad f_{j_1} < f_{j_2} < \cdots < f_{j_n}.
\]

It is obvious that \( e_{i_1} \) is the smallest boundary point of \( V \) in \( (0, 1) \), and \( 1 - f_{j_1} \) is the largest boundary point of \( V \) in \( (0, 1) \). If \( b_0 = 0 \) then \( g(te_{i_1}) = J_V(e_{i_1}) = 1 \), and thus \( i_1 \in I_{1,0} \cup I_{0,1} \cup I_{1,1} \cup I_{1,-1} \);

if \( b_0 = 2 \) then \( g(te_{i_1}) = J_V(e_{i_1}) = -1 \), and thus \( i_1 \in I_{-1,0} \cup I_{0,-1} \cup I_{-1,1} \cup I_{-1,-1} \).

Similarly, if \( a_0 = 0 \) then \( j_1 \in I_{-1,0} \cup I_{0,-1} \cup I_{1,-1} \cup I_{-1,-1} \);

if \( a_0 = -2 \) then \( j_1 \in I_{1,0} \cup I_{0,1} \cup I_{1,1} \cup I_{-1,1} \).

Following the notations above, we establish several lemmas.
Lemma 8.7. Let $J$ be a nonempty subset of $I$, and put
\[ R_J = a_0 z_1 + b_0 - a_0 + \sum_{i \in J} (a_i z_1 + b_i - a_i) M_{d_i}, \]
\[ R_J' = b_0 + \sum_{i \in J} b_i M_{d_i}, \]
\[ R_J'' = b_0 - 2a_0 + \sum_{i \in J} (b_i - 2a_i) M_{d_i}. \]

Then

1. $R_J$ and $R_J''$ have no zeros in $\mathbb{D}^{S_q}$.
2. $R_J'$ has no zeros in $\mathbb{D}^{S_q}$ provided $b_i \neq 0$ for some $i \in J$.
3. the inequality
   \[ |a_0 + \sum_{i \in J} a_i M_{d_i}| \leq |b_0 - a_0 + \sum_{i \in J} (b_i - a_i) M_{d_i}| \]
   holds on $\overline{\mathbb{D}}^{S_q}$.

To prove Lemma 8.7, we need a multi-dimensional version of Hurwitz's theorem [Na, pp. 80].

Lemma 8.8. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and \{f_k\}_{k \geq 1} a sequence of holomorphic functions on $\Omega$ with each $f_k$ zero-free. If \{f_k\}_{k \geq 1} converges to a function $f$ uniformly on compact subset of $\Omega$, then either $f \equiv 0$ or $f$ has no zeros in $\Omega$.

Proof of Lemma 8.7. Put $S = \bigcup_{i \in J \cup \{0\}} S_d$. Then $S \subseteq S_q$ and $1 \in S$. For each $k \in \mathbb{N}$, we define a polynomial $R_{I,k}$, which depends on variables \{z_j : j \in S\}, by substituting $z_j = 1 - \frac{1}{k}$ in $R$ for every $j \in S_q \setminus S$. We further define two polynomials $R_{I,k}'$, $R_{I,k}''$ by substituting $z_1 = 1 - \frac{1}{k}$ and $z_1 = \frac{1}{k} - 1$ in $R_{I,k}$, respectively. Then for each $k \in \mathbb{N}$, $R_{I,k}$, $R_{I,k}'$ and $R_{I,k}''$ have no zeros in $\mathbb{D}^{S_q}$. Moreover, \{R_{I,k}\}_{k \geq 1}, \{R_{I,k}'\}_{k \geq 1}$ and \{R_{I,k}''\}_{k \geq 1} converge uniformly to $R_J$, $R_J'$ and $R_J''$ on $\overline{\mathbb{D}}^{S_q}$, respectively. Since for any $i \in I$, $a_i z_1 + b_i - a_i \neq 0$ and $b_i - 2a_i \neq 0$, (1) and (2) immediately follows from Lemma 8.8.

To see (3), notice that
\[ R_J = (a_0 + \sum_{i \in J} a_i M_{d_i}) z_1 + b_0 - a_0 + \sum_{i \in J} (b_i - a_i) M_{d_i}. \]
Since each $M_{d_i}$ does not depend on the variable $z_1$, (3) follows from (1).

\[ \square \]

Lemma 8.9. The following identity holds,
\[ a_0 + b_0 + \sum_{i \in I_{-1,0} \cup I_{0,1}} \phi(d_i) + 2 \sum_{i \in I_{1,1}} \phi(d_i) = \sum_{i \in I_{-1,0} \cup I_{0,1}} \phi(d_i) + 2 \sum_{i \in I_{1,1}} \phi(d_i). \]
Proof. Set \( D_+ = \{ d \geq 3 : J_V(\frac{d}{2}) = 1 \} \) and \( D_- = \{ d \geq 3 : J_V(\frac{d}{2}) = -1 \} \). The by Theorem 5.1 we have

\[
a_0 + b_0 + \sum_{d \in D_+} \phi(d) = \sum_{d \in D_-} \phi(d).
\]

If \( d \) is odd then \( d \in D_+ \) if and only if \( d = d_i \) for \( i \in I_{0,1} \cup I_{1,1} \cup I_{-1,1} \). If \( d \) is even then \( d \in D_+ \) if and only if \( d = 2d_i \) for \( i \in I_{1,0} \cup I_{1,1} \cup I_{1,-1} \). That is to say,

\[
D_+ = \{ d_i : i \in I_{0,1} \cup I_{1,1} \cup I_{-1,1} \} \cup \{ 2d_i : i \in I_{1,0} \cup I_{1,1} \cup I_{1,-1} \}.
\]

Similarly,

\[
D_- = \{ d_i : i \in I_{0,-1} \cup I_{-1,-1} \cup I_{-1,1} \} \cup \{ 2d_i : i \in I_{-1,0} \cup I_{-1,-1} \cup I_{-1,1} \}.
\]

Since each \( d_i \) is odd, \( \phi(2d_i) = \phi(2) \phi(d_i) = \phi(d_i) \), and thus

\[
\sum_{d \in D_+} \phi(d) = \sum_{i \in I_{0,1} \cup I_{1,1} \cup I_{-1,1}} \phi(d_i) + 2 \sum_{i \in I_{1,0} \cup I_{1,1} \cup I_{1,-1}} \phi(d_i),
\]

\[
\sum_{d \in D_-} \phi(d) = \sum_{i \in I_{1,0} \cup I_{-1,1}} \phi(d_i) + 2 \sum_{i \in I_{-1,0} \cup I_{-1,-1}} \phi(d_i) + \sum_{i \in I_{1,-1} \cup I_{-1,1}} \phi(d_i).
\]

The proof is complete. \( \square \)

**Lemma 8.10.** (1) If \( i_1, j_1 \in I_{1,1} \cup I_{-1,1} \cup I_{1,-1} \cup I_{-1,-1} \) then \( i_1 = j_1 \).

(2) If \( i_1 \in I_{0,1} \cup I_{0,-1} \) then \( i_1 = j_1 \).

(3) If \( j_1 \in I_{1,0} \cup I_{-1,0} \) then \( i_1 = j_1 \).

**Proof.** (1) If \( i_1, j_1 \in I_{1,1} \cup I_{-1,1} \cup I_{1,-1} \cup I_{-1,-1} \) then

\[
e_{i_1} \leq e_{j_1} = f_{j_1} \leq f_{i_1} = e_{i_1},
\]

forcing \( i_1 = j_1 \).

(2) Note that \( e_i \leq 2f_i \) for every \( i \in I \). If \( i_1 \in I_{0,1} \cup I_{0,-1} \) then

\[
e_{j_1} \geq e_{i_1} = 2f_{i_1} \geq 2f_{j_1} \geq e_{j_1},
\]

forcing \( i_1 = j_1 \).

(3) The proof is similar to that of (2). \( \square \)

**Lemma 8.11.** (1) If \( a_0 = -2, b_0 = 0 \), then \( n_{0,1} = n_{1,1} = n_{-1,1} = 0 \) and \( n_{-1,0} \leq 1 \).

(2) If \( a_0 = 0, b_0 = 2 \), then \( n_{0,1} + n_{1,1} \leq 1, n_{-1,0} = n_{-1,1} = 0 \) and \( n_{0,1} + n_{-1,-1} \leq 1 \).

(3) If \( a_0 = -2, b_0 = 2 \), then \( n_{0,1} + n_{1,1} + n_{-1,1} = 1 \) and \( 2n_{-1,0} + n_{0,1} + 3n_{-1,1} + n_{-1,-1} \leq 3 \).
Proof. Assume $a_0 = -2$ or $b_0 = 2$. We first show the following two inequality:

$$n_{0,1} + n_{1,1} + n_{-1,1} \leq \frac{b_0}{2},$$  \hspace{1cm} (8.3)

$$2n_{-1,0} + n_{0,1} + 3n_{-1,1} + n_{-1,-1} \leq \frac{b_0 - 2a_0}{2}.$$  \hspace{1cm} (8.4)

Put $J_1 = I_{0,1} \cup I_{1,1} \cup I_{-1,1}$ and $J_2 = I_{-1,0} \cup I_{0,1} \cup I_{1,1} \cup I_{-1,-1}$. To prove (8.3), assume that $J_1$ is nonempty without loss of generality. Lemma 8.7 (1), applied to the set $J_1$, implies that

$$a_0 z_1 + b_0 - a_0 + \sum_{i \in I_{0,1}} M_d + z_1 \sum_{i \in I_{1,1}} M_d + (2 - z_1) \sum_{i \in I_{-1,1}} M_d$$

has no zeros in $D^{S_9}$. By Lemma 8.3 for any $-\frac{9}{5} < \lambda < 0$ and any $i \in J$, the polynomial $M_d$ can take value $\lambda$ in $D^{S_9}$, and thus

$$a_0 z_1 + b_0 - a_0 + [n_{0,1} + n_{1,1} z_1 + n_{-1,1} (2 - z_1)] \lambda \neq 0.$$

In particular, by taking $z_1 = 1 - \varepsilon$ ($0 < \varepsilon < 1$), we have

$$b_0 - \varepsilon a_0 + [n_{0,1} + (1 - \varepsilon) n_{1,1} + (1 + \varepsilon) n_{-1,1}] \lambda \neq 0, \quad -\frac{9}{5} \leq \lambda < 0.$$  \hspace{1cm} (8.5)

Since $b_0 - \varepsilon a_0 > 0$, (8.5) is equivalent to the following,

$$\frac{9}{5} [n_{0,1} + (1 - \varepsilon) n_{1,1} + (1 + \varepsilon) n_{-1,1}] < b_0 - \varepsilon a_0.$$

Letting $\varepsilon \to 0$, we further have

$$n_{0,1} + n_{1,1} + n_{-1,1} \leq \frac{5b_0}{9}.$$  \hspace{1cm} (8.6)

This proves (8.3) since $n_{0,1} + n_{1,1} + n_{-1,1}$ is an integer and the integral part of $\frac{5b_0}{9}$ is equal to $\frac{b_0}{2}$.

To prove (8.4), assume that $J_2$ is nonempty without loss of generality. Applying Lemma 8.7 (1) to the set $J_2$, we see that

$$b_0 - 2a_0 + 2 \sum_{i \in I_{-1,0}} M_d + \sum_{i \in I_{0,1}} M_d + 3 \sum_{i \in I_{1,1}} M_d + \sum_{i \in I_{-1,-1}} M_d$$

has no zeros in $D^{S_9}$. Using the argument in the last paragraph, we can obtain

$$2n_{-1,0} + n_{0,1} + 3n_{-1,1} + n_{-1,-1} \leq \frac{5(b_0 - 2a_0)}{9}.$$  \hspace{1cm} (8.7)

This proves (8.4) since the integral part of $\frac{5(b_0 - 2a_0)}{9}$ is equal to $\frac{b_0 - 2a_0}{2}$.

(1) and (2) has been proved by combining (8.3) with (8.4). To complete the proof, assume $a_0 = -2$ and $b_0 = 2$. Then $i_1 \not\in I_{1,0}$ and $j_1 \in J_1 \cup I_{1,0}$. By Lemma 8.10 (3), $j_1 \not\in I_{1,0}$, and thus $J_1$ is nonempty. This proves (3).

\qed
Lemma 8.12. (1) If \( a_0 = -2, b_0 = 0 \), then \( n_{1,0}n_{0,-1} = n_{1,0}n_{-1,-1} = 0 \).

(2) If \( a_0 = 0, b_0 = 2 \), then \( n_{1,0}n_{0,1} = n_{1,0}n_{1,1} = 0 \).

(3) If \( a_0 = -2, b_0 = 2 \), then \( n_{0,1}n_{-1,0} = n_{1,1}n_{-1,0} = 0 \).

Proof. To show (1), assume \( n_{1,0} \geq 1 \) and \( n_{0,-1} + n_{-1,-1} \geq 1 \), and take \( i \in I_{1,0} \) and \( j \in I_{0,-1} \cup I_{-1,-1} \). Applying Lemma 8.7 (3) to \( \{i, j\} \), we see that one of the following two inequalities holds on \( \mathbb{D}^{S_{1}} \):

\[
|2 - M_{d_{i}}| \leq |2 - M_{d_{i}} - M_{d_{j}}|;
\]

\[
|2 - M_{d_{i}} + M_{d_{j}}| \leq |2 - M_{d_{i}}|.
\]

By Lemma 8.3, \( M_{d_{i}} \) can take value \(-1 - i\) in \( \mathbb{D}^{S_{1}} \), and then one of the following two inequalities holds on \( \mathbb{D}^{S_{1}} \):

\[
|M_{d_{j}} - 3 - i| \geq \sqrt{10};
\]

\[
|M_{d_{j}} + 3 + i| \leq \sqrt{10}.
\]

This leads to a contradiction due to Lemma 8.5.

Using Lemma 8.5 one can prove (2) and (3) similarly. \( \square \)

We are ready to prove the necessity in Theorem 8.1, which is due to combining the following two lemmas.

Lemma 8.13. (1) We have \( a_0 = -2 \) or \( b_0 = 2 \).

(2) If \( a_0 = 0, b_0 = 2 \) then \( n_{0,1} = 1 \) or \( n_{1,1} = 1 \).

Lemma 8.14. (1) If \( a_0 = -2, b_0 = 0 \), then there exist two primes \( p_1, p_2 \), such that \( p_1 = p_2 + 2 \) and

\[
V = \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \cup \left( \frac{3}{p_1}, \frac{3}{p_2} \right) \cup \ldots \cup \left( \frac{2m - 1}{p_1}, \frac{2m - 1}{p_2} \right) \cup \ldots \cup \left( \frac{p_1 - 2}{p_1}, 1 \right).
\]

(2) If \( a_0 = 0, b_0 = 2 \) and \( n_{0,1} = 1 \), then there exist two primes \( p_1, p_2 \), such that \( p_1 = p_2 + 2 \) and

\[
V = \left( \frac{2}{p_1}, \frac{4}{p_1} \right) \cup \ldots \cup \left( \frac{2m - 2}{p_2}, \frac{2m}{p_1} \right) \cup \ldots \cup \left( \frac{p_1 - 1}{p_1}, \frac{p_2 - 1}{p_2}, 1 \right).
\]

(3) If \( a_0 = 0, b_0 = 2 \) and \( n_{1,1} = 1 \), then there exist two primes \( p_1, p_2 \), such that \( p_1 = 2p_2 + 1 \) and

\[
V = \left( \frac{2}{p_1}, \frac{4}{p_1} \right) \cup \ldots \cup \left( \frac{m - 1}{p_2}, \frac{2m}{p_1} \right) \cup \ldots \cup \left( \frac{p_2 - 1}{p_2}, \frac{p_1 - 1}{p_1} \right).
\]

(4) If \( a_0 = -2, b_0 = 2 \), then there exist two primes \( p_1, p_2 \), such that \( p_1 = 2p_2 - 1 \) and

\[
V = \left( \frac{1}{p_2}, \frac{2}{p_2} \right) \cup \ldots \cup \left( \frac{m - 2}{p_1}, \frac{m}{p_2} \right) \cup \ldots \cup \left( \frac{p_1 - 1}{p_1}, 1 \right).
\]
Case 1. We have two cases:

In fact, applying Lemma 8.7 (1) to \( \{i, j\} \), we see that \((a_iz_1+b_i-a_i)M_{d_i}+(a_2z_1+b_2-a_j)M_{d_j}\) has no zeros in \( \mathbb{D}^{S_q} \), and hence so does \((b_i-a_i)M_{d_i}+(b_j-a_j)M_{d_j}\).

We first show

\[
n_{1,1} = n_{-1,-1} = 0. \tag{8.6}
\]

Suppose \(n_{1,1} \geq 1\) and take \(i \in I_{1,1}\). Then Lemma 8.7 (1) implies that the polynomial \(R_i = z_1M_{d_i}\) has no zeros in \( \mathbb{D}^{S_q} \), which is impossible. This gives \(n_{1,1} = 0\). Similarly, one has \(n_{-1,-1} = 0\).

As a consequence of (8.6), \(i_1 \in I_{1,0} \cup I_{0,1} \cup I_{1,-1}\) and \(j_1 \in I_{-1,0} \cup I_{0,-1} \cup I_{1,-1}\). It follows from Lemma 8.10 (2) that \(i_1 \notin I_{0,1}\), and then

\[
I_{1,0} \cup I_{1,-1} \neq \emptyset.
\]

We then show

\[
n_{0,1} = n_{-1,0} = n_{-1,1} = 0. \tag{8.7}
\]

Suppose \(i \in I_{1,0} \cup I_{1,-1}\) and \(j \in I_{0,1} \cup I_{-1,0} \cup I_{1,-1}\). Then the claim ensures that \((b_i-a_i)M_{d_i}+(b_j-a_j)M_{d_j}\) has no zeros in \( \mathbb{D}^{S_q} \). This contradicts with Lemma 8.3 since \(b_i-a_i < 0\) and \(b_j-a_j > 0\), and thus we have shown (8.7).

With (8.6) and (8.7), Lemma 8.9 immediately gives

\[
\sum_{i \in I_{1,0}} \phi(d_i) = \sum_{i \in I_{0,-1}} \phi(d_i),
\]

which yields that \(n_{1,0}\) and \(n_{0,-1}\) are either simultaneously zero or simultaneously nonzero. We have two cases: \(n_{1,0} = n_{0,-1} = 0\) or \(n_{1,0}, n_{0,-1} \geq 1\).

Case 1. \(n_{1,0} = n_{0,-1} = 0\).

In this case, \(I = I_{1,-1}\) and

\[
\frac{1}{d_{i_1}} < \frac{1}{d_{i_2}} < \cdots < \frac{1}{d_{i_n}}.
\]

For any \(i, j \in I\) \((i \neq j)\), it follows from the claim that \(-2M_{d_i} - 2M_{d_j}\) has no zeros in \( \mathbb{D}^{S_q} \), and then by Lemma 8.6 (2), each \(d_i\) \((i \in I)\) is a prime. Put \(m_0 = \lfloor \frac{d_{i_1}}{d_{i_2}} \rfloor\), the integral part of \(\frac{d_{i_1}}{d_{i_2}}\). Then \(m_0 \geq 1\) and

\[
\frac{m_0}{d_{i_1}} < \frac{1}{d_{i_2}} < \frac{m_0+1}{d_{i_1}}.
\]

Since \(J_V(\frac{1}{d_{i_1}}) = 1\) and \(J_V(\frac{m_0}{d_{i_1}})J_V(\frac{m_0+1}{d_{i_1}}) = -1\), there exists a irreducible fraction \(\frac{1}{d_{ik}}\) \((2 \leq k \leq n)\), such that

\[
\frac{m_0}{d_{i_1}} < \frac{1}{d_{ik}} < \frac{m_0+1}{d_{i_1}}.
\]
and $J_V(\frac{l}{d_{i_k}}) = -1$. It follows that $l$ is even (since $i_k \in I_{1,-1}$), and thus

\[
\frac{l}{d_{i_k}} \geq \frac{2}{d_{i_2}} > \frac{2m_0}{d_{i_1}} \geq \frac{m_0 + 1}{d_{i_2}} > \frac{l}{d_{i_k}},
\]

which is impossible.

**Case 2.** $n_{1,0}, n_{0,-1} \geq 1$.

Take $i \in I_{1,0}, j \in I_{0,-1}$. By the claim, $-M_i - M_j$ has no zeros in $\mathbb{D}_z$, and then by Lemma 8.6 (2), both $d_i$ and $d_j$ are primes. Applying Lemma 8.7 (3) to $\{i, j\}$, we see that the inequality

\[
|M_{d_i}| \leq |M_{d_i} + M_{d_j}|
\]

holds on $\mathbb{D}_z$. That is to say, $|z - 1| \leq |z + w - 2|$ for $z, w \in \mathbb{D}$. But this fails for $z = i$ and $w = \frac{2+i}{\sqrt{3}}$, and then (1) has been proved.

(2) In the light of Lemma 8.11 (2), it suffices to show $n_{0,1} \geq 1$ or $n_{1,1} \geq 1$. To reach a contradiction, assume conversely $n_{0,1} = n_{1,1} = 0$. Lemma 8.11 (2) also gives $n_{-1,0} = n_{-1,1} = 0$, which forces $i_1 \in I_{0,-1} \cup I_{-1,-1}$ and $j_1 \in I_{0,-1} \cup I_{1,-1} \cup I_{-1,-1}$. By Lemma 8.10 we have $j_1 \not\in I_{1,-1}$, i.e., $j_1 \in I_{0,-1} \cup I_{-1,-1}$. It follows that $J_V(e_{i_2}) = J_V(1 - f_{j_2}) = 1$, and then $i_2 \in I_{1,0} \cup I_{1,-1}$ and $j_2 \in I_{1,0}$. In particular, $i_2 \neq j_1$ and $j_2 \neq i_1$. Since

\[
f_{i_2} \geq f_{j_2} = \frac{2}{d_{j_2}} = 2e_{j_2} \geq 2e_{i_2} = \frac{2}{d_{i_2}} \geq f_{i_2},
\]

we further have $i_2 = j_2$.

We first show $i_1 = j_1 \in I_{0,-1}$. Suppose $i_1 \in I_{-1,-1}$. Since

\[
f_{i_1} = \frac{1}{d_{i_1}} = e_{i_1} < e_{j_2} = \frac{1}{d_{j_2}} < \frac{2}{d_{j_2}} = f_{j_2}, \tag{8.8}
\]

one has $i_1 = j_1$, and thus

\[
J_V(1 - f_{j_1}) = J_V(1 - \frac{1}{d_{j_1}}) = J_V(1 - \frac{2}{d_{j_1}}) = -1.
\]

This implies that there exists some $j \in I \setminus \{j_1\}$, such that $\frac{1}{d_{j_1}} < f_j < \frac{2}{d_{j_1}}$ and $J_V(1 - f_j) = 1$. Therefore,

\[
\frac{2}{d_{j_2}} = f_{j_2} \leq f_j < \frac{2}{d_{j_1}},
\]

which contradicts with (8.8). So we have shown $i_1 \in I_{0,-1}$, and then Lemma 8.10 (2) gives $i_1 = j_1$. It follows that

\[
\frac{1}{d_{i_2}} = e_{i_2} > e_{i_1} = \frac{2}{d_{i_1}}, \tag{8.9}
\]

In particular, $d_{i_1} > 3$. 

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Let \( p \) be the smallest prime satisfying \( p \nmid d_{i_1} \). Then

\[
J_V(1 - f_{j_1}) = J_V(1 - f_{i_1}) = J_V\left(1 - \frac{1}{d_{i_1}}\right) = J_V\left(1 - \frac{p}{d_{i_1}}\right) = -1.
\]

This implies that there exists some \( j \in I \setminus \{j_1\} \), such that \( \frac{1}{d_{i_1}} < f_j < \frac{p}{d_{i_1}} \) and \( J_V(1 - f_j) = 1 \). Therefore,

\[
\frac{2}{d_{i_2}} = \frac{2}{d_{j_2}} = f_{j_2} \leq f_j < \frac{p}{d_{j_1}} = \frac{p}{d_{i_1}}.
\] (8.10)

Combining this with (8.9), one has \( p \geq 5 \), i.e., \( 3 \mid d_{i_1} \). Thus, \( d_{i_1} \) is not a prime (since \( d_{i_1} > 3 \)) and \( 3 \nmid d_{i_2} \).

Now we have \( n_{-1,-1} = 0 \). In fact, if \( i \in I_{-1,-1} \) then applying Lemma 8.7 (1) to \( \{i_1, i\} \), we see that the polynomial \( 2 - M_{d_{i_1}} + M_{d_i} \) has no zeros in \( \mathbb{D}^{S_q} \). This leads to a contradiction due to Lemma 8.6 (1).

From (8.9) and (8.10), we see that

\[
2 < \frac{1}{d_{i_2}} < \frac{3}{d_{i_2}} < \frac{2p}{d_{i_1}}
\]

Note that \( J_V\left(\frac{m}{d_{i_1}}\right) = 0 \) for each \( 2 < m < 2p \), and

\[
J_V(e_{i_2}) = J_V\left(\frac{1}{d_{i_2}}\right) = J_V\left(\frac{3}{d_{i_2}}\right) = 1
\]

since \( 3 \nmid d_{i_2} \). Then \( n \geq 3 \) and there exists some \( i \in I \setminus \{i_1, i_2\} \), such that \( \frac{1}{d_{i_2}} < e_i < \frac{3}{d_{i_2}} \) and \( J_V(e_i) = -1 \). It follows that \( i \in I_{0,-1} \) and

\[
\frac{3}{d_{i_2}} > e_i = \frac{2}{d_{i}} = 2f_i > 2f_{i_2} = \frac{4}{d_{i_2}}.
\]

This is a contradiction. \( \square \)

**Proof of Lemma 8.14** Let \( [x] (x \in \mathbb{R}) \) be the integral part of \( x \).

(1) Assume \( a_0 = -2 \), \( b_0 = 0 \). By Lemma 8.11 (1), we have

\[
n_{0,1} = n_{1,1} = n_{-1,1} = 0,
\] (8.11)

forcing \( j_1 \in I_{1,0} \). Therefore, Lemma 8.10 (3) gives \( i_1 = j_1 \), and Lemma 8.12 gives

\[
n_{0,-1} = n_{-1,-1} = 0.
\] (8.12)

Since \( i_1 \in I_{1,0} \), one has \( J_V(e_{i_2}) = -1 \), which implies \( i_2 \in I_{-1,0} \). Again by Lemma 8.11 (1), \( I_{-1,0} = \{i_2\} \). The inequality

\[
\frac{1}{d_{i_1}} = e_{i_1} < e_{i_2} = \frac{1}{d_{i_2}}
\]

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yields
\[ d_{i_1} \geq d_{i_2} + 2 \quad (8.13) \]
since both \(d_{i_1}\) and \(d_{i_2}\) are odd. Applying Lemma \[8.7\] (1) to \(\{i_1, i_2\}\), we see that the polynomial \(2 - M_{d_{i_1}} + M_{d_{i_2}}\) has no zeros in \(D_{\mathbb{S}_q}\), and then by Lemma \[8.6\] (1), \(d_{i_1}\) is a prime. With \(8.11\) and \(8.12\), Lemma \[8.9\] and \(8.13\) gives
\[ \phi(d_{i_2}) = -2 + \sum_{i \in I_{1,0}} \phi(d_i) \geq -2 + \phi(d_{i_1}) = d_{i_1} - 3 \geq d_{i_2} - 1 \geq \phi(d_{i_2}). \]
This yields that \(d_{i_2}\) is also a prime, \(d_{i_1} = d_{i_2} + 2\) and \(I_{1,0} = \{i_1\}\). Rewrite \(p_k = d_{i_k}\) \((k = 1, 2)\).

Then for each \(1 \leq m \leq \frac{p_1 - 1}{2}\), \(\frac{2m - 1}{p_1}\) is a left endpoint of some component interval of \(V\), and \(\frac{2m + 1}{p_1}\) is a right endpoint of some component interval of \(V\). Moreover, for \(k \in \{1, 2\}\), \(J_V(\frac{2m}{p_k}) = 0 \quad (1 \leq m \leq \frac{p_k - 1}{2})\).

It remains to show \(n = 2\). Assume conversely \(n \geq 3\). Then \(i_k \in I_{1,-1}\) for every \(3 \leq k \leq n\), \(J_V(\frac{1}{d_{i_3}}) = 1\) and
\[ \frac{1}{d_{i_3}} = f_{i_3} > f_{j_1} = \frac{2}{p_1} > \frac{1}{p_2}. \quad (8.14) \]
This gives that
\[ \frac{2m - 1}{p_2} < \frac{1}{d_{i_3}} < \frac{2m + 1}{p_1} \]
for some \(1 \leq m \leq \frac{p_2 - 1}{2}\), and that there is at least one boundary point of \(V\) between \(\frac{1}{d_{i_3}}\) and \(\frac{2m + 1}{p_1}\). If \(\frac{2}{d_{i_3}} > \frac{2m + 1}{p_1}\) then \(n \geq 4\) and \(J_V(e_{i_4}) = -1\). This is impossible since one should have \(i_4 \in I_{1,-1}\). Hence, we have shown
\[ \frac{4m - 2}{p_2} < \frac{2}{d_{i_3}} < \frac{2m + 1}{p_1} = \frac{2m + 1}{p_2 + 2}, \]
forcing \(m = 1\). It follows from \(8.14\) that
\[ \frac{4}{p_1} < \frac{2}{d_{i_3}} < \frac{3}{p_1}, \]
which is a contradiction.

(2) Assume \(a_0 = 0, b_0 = 2\) and \(n_{0,1} = 1\). By Lemma \[8.11\] (2) and Lemma \[8.12\] (2), we have
\[ n_{1,1} = n_{-1,0} = n_{-1,1} = n_{-1,-1} = n_{1,0} = 0, \quad (8.15) \]
forcing \(i_1 \in I_{0,-1}\). It follows from Lemma \[8.10\] (2) that \(i_1 = j_1\), and then \(J_V(1 - f_{j_2}) = 1\). This yields \(I_{0,1} = \{j_2\}\) and
\[ \frac{1}{d_{j_1}} = f_{j_1} < f_{j_2} = \frac{1}{d_{j_2}}. \quad (8.16) \]
Applying Lemma 8.7 (1) to \( \{j_1, j_2\} \), we see that the polynomial \( 2 - M_{d_{j_1}} + M_{d_{j_2}} \) has no zeros in \( \mathbb{D}^S \), and then by Lemma 8.6 (1), \( d_{j_1} \) is a prime. With (8.15), Lemma 8.9 and (8.16) gives

\[
2 + \phi(d_{j_2}) = \sum_{i \in I_{0,-1}} \phi(d_i) \geq \phi(d_{j_1}) = d_{j_1} - 1 \geq d_{j_2} + 1 \geq 2 + \phi(d_{j_2}).
\]

This yields that \( d_{j_2} \) is also a prime, \( d_{j_1} = d_{j_2} + 2 \) and \( I_{0,-1} = \{j_1\} \). Rewrite \( p_k = d_{j_k} \) (\( k = 1, 2 \)). Then for each \( 1 \leq m \leq \frac{p_{k-1}}{2} \), \( \frac{2m}{p_k} \) is a right endpoint of some component interval of \( V \), and \( \frac{2m-2}{p_{k-1}} \) is a left endpoint of some component interval of \( V \). Moreover, for \( k \in \{1, 2\} \), \( J_V \left( \frac{2m-1}{p_k} \right) = 0 \) (\( 1 \leq m \leq \frac{p_{k-1}}{2} \)). By using an argument similar to that in the second paragraph of (1), we can show \( n = 2 \), which proves (2).

(3) Assume \( a_0 = 0 \), \( b_0 = 2 \) and \( n_{1,1} = 1 \). By Lemma 8.11 (2) and Lemma 8.12 (2), we have

\[
n_{0,1} = n_{-1,0} = n_{-1,1} = n_{1,0} = 0,
\]

forcing \( i_1 \in I_{0,-1} \cup I_{-1,-1} \).

**Claim.** \( i_1 \in I_{0,-1} \).

To see this, assume conversely \( i_1 \in I_{-1,-1} \). Then by Lemma 8.10 (1), one necessarily has \( j_1 \notin I_{1,-1} \). This gives \( j_1 \in I_{0,-1} \cup I_{-1,-1} \), and thus \( J_V(1 - f_{j_2}) = 1 \). It follows that \( I_{1,1} = \{j_2\} \) and

\[
\frac{1}{d_{i_1}} = e_{i_1} \leq e_{j_2} = \frac{1}{d_{j_2}}.
\]

Since \( i_1 \neq j_2 \), one has

\[
d_{i_1} \geq d_{j_2} + 2. \tag{8.18}
\]

Applying Lemma 8.7 (2) to \( \{i_1, j_2\} \), we see that the polynomial \( 2 - M_{d_{i_1}} + M_{d_{j_2}} \) has no zeros in \( \mathbb{D}^S \), and then by Lemma 8.6 (1), \( d_{i_1} \) is a prime. With (8.17), Lemma 8.9 and (8.18) gives

\[
2 + 2\phi(d_{j_2}) = \sum_{i \in I_{0,-1}} \phi(d_i) + 2 \sum_{i \in I_{-1,-1}} \phi(d_i) \geq 2\phi(d_{i_1}) = 2d_{i_1} - 2 \geq 2d_{j_2} + 2 \geq 4 + 2\phi(d_{j_2}),
\]

which is a contradiction.

By the claim and Lemma 8.10 (2), we have \( i_1 = j_1 \), and then \( J_V(1 - f_{j_2}) = 1 \). It follows that \( I_{1,1} = \{j_2\} \) and

\[
\frac{2}{d_{j_1}} = e_{j_1} = e_{i_1} \leq e_{j_2} = \frac{1}{d_{j_2}},
\]

which yields

\[
d_{j_1} \geq 2d_{j_2} + 1. \tag{8.19}
\]

Applying Lemma 8.7 (2) to \( \{j_1, j_2\} \), we see that the polynomial \( 2 - M_{d_{j_1}} + M_{d_{j_2}} \) has no zeros in \( \mathbb{D}^S \), and thus by Lemma 8.6 (1), \( d_{j_1} \) is a prime. With (8.17), Lemma 8.9 and (8.19) gives

\[
2 + 2\phi(d_{j_2}) = \sum_{i \in I_{0,-1}} \phi(d_i) + 2 \sum_{i \in I_{-1,-1}} \phi(d_i) \geq \phi(d_{j_1}) = d_{j_1} - 1 \geq 2d_{j_2} + 2 \geq 2 + 2\phi(d_{j_2}).
\]
This yields that $d_{j_2}$ is also a prime, $d_{j_1} = 2d_{j_2} + 1$, $I_{0,-1} = \{j_1\}$ and $n_{-1,-1} = 0$. Rewrite $p_k = d_{j_k}$ ($k = 1, 2$). Then for each $1 \leq m \leq \frac{p_1 - 1}{2}$, $\frac{2m}{p_1}$ is a right endpoint of some component interval of $V$, and $\frac{m - 1}{p_2}$ is a left endpoint of some component interval of $V$. Moreover, $J_V(\frac{2m - 1}{p_1}) = 0$ ($1 \leq m \leq \frac{p_1 - 1}{2}$).

It remains to show $n = 2$. Assume conversely $n \geq 3$. Then $j_k \in I_{1,-1}$ for every $3 \leq k \leq n$, $J_V(1 - \frac{1}{d_{j_3}}) = J_V(\frac{2}{d_{j_3}}) = -1$ and

$$\frac{1}{d_{j_3}} = f_{j_3} > f_{j_2} = \frac{1}{p_2}.$$ 

This gives that

$$\frac{2m}{p_1} < 1 - \frac{1}{d_{j_3}} < \frac{m}{p_2}$$

for some $1 \leq m \leq p_2 - 1$, and that there is at least one boundary point of $V$ between $1 - \frac{1}{d_{j_3}}$ and $\frac{2m}{p_1}$. If $1 - \frac{2}{d_{j_3}} < \frac{2m}{p_1}$ then $n \geq 4$ and $J_V(1 - f_{j_4}) = 1$. This is impossible since one should have $j_4 \in I_{1,-1}$. Hence, we have shown

$$2 - \frac{2m}{p_2} < \frac{2}{d_{j_3}} < 1 - \frac{2m}{p_1} = 1 - \frac{2m}{2p_2 + 1},$$

which leads to a contradiction since $m \leq p_2 - 1$.

(4) Assume $a_0 = -2$, $b_0 = 2$. Then $i_1 \in I_{-1,0} \cup I_{0,-1} \cup I_{-1,1} \cup I_{-1,-1}$, and $j_1 \in I_{1,0} \cup I_{0,1} \cup I_{1,1} \cup I_{-1,1}$. By Lemma \ref{lemma8.10} (2) (3), $i_1 \not\in I_{0,-1}$ and $j_1 \not\in I_{1,0}$. It follows from Lemma \ref{lemma8.11} (3) that there are three possible cases:

(i) $I_{0,1} = \{j_1\}$ and $n_{1,1} = n_{-1,1} = 0$;

(ii) $I_{1,1} = \{j_1\}$ and $n_{0,1} = n_{-1,1} = 0$;

(iii) $I_{-1,1} = \{j_1\}$ and $n_{0,1} = n_{1,1} = 0$.

In either case, we further have $n_{-1,0} = 0$ by Lemma \ref{lemma8.11} (3) and Lemma \ref{lemma8.12} (3). In what follows, we will prove that Cases (ii) and (iii) cannot occur.

In Case (ii), one necessarily has $i_1 \in I_{-1,-1}$. Thus $i_1 = j_1$ by Lemma \ref{lemma8.10} (1), which is impossible.

Now consider Case (iii). In this case, we also have $n_{-1,-1} = 0$ by Lemma \ref{lemma8.11} (3), forcing $i_1 \in I_{-1,1}$. Hence, Lemma \ref{lemma8.10} (1) gives $i_1 = j_1$. Since

$$n_{0,1} = n_{1,1} = n_{-1,0} = n_{-1,-1} = 0,$$

it follows from Lemma \ref{lemma8.9} that

$$\sum_{i \in I_{1,0}} \phi(d_i) = \sum_{i \in I_{0,-1}} \phi(d_i),$$

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and thus \( n_{1,0} \) and \( n_{0,-1} \) are either simultaneously zero or simultaneously nonzero.

First, we assume \( n_{1,0} = n_{0,-1} = 0 \). Then \( i_k \in I_{1,-1} \) for every \( 2 \leq k \leq n \). Set \( m_0 = \frac{d_{i_1}}{d_{i_2}} \).

The inequality
\[ \frac{1}{d_{i_1}} = e_{i_1} < e_{i_2} = \frac{1}{d_{i_2}} \]
gives \( m_0 \geq 1 \). Since
\[ \frac{m_0}{d_{i_1}} < \frac{1}{d_{i_2}} < \frac{m_0 + 1}{d_{i_1}} \]
and \( J_V\left(\frac{1}{d_{i_2}}\right) = 1 \), we see that if both \( m_0 \) and \( m_0 + 1 \) are relatively prime to \( d_{i_1} \), then
\[ J_V\left(\frac{m_0}{d_{i_1}}\right)J_V\left(\frac{m_0 + 1}{d_{i_1}}\right) = -1. \]
This would imply that there exists an irreducible fraction \( \frac{d}{d_k} \) (\( 2 \leq k \leq n \)), such that
\[ \frac{m_0}{d_{i_1}} < \frac{l}{d_{i_k}} < \frac{m_0 + 1}{d_{i_1}} \]
and \( J_V\left(\frac{l}{d_{i_k}}\right) = -1 \). It follows that \( l \) is even (since \( i_k \in I_{1,-1} \)) and
\[ \frac{l}{d_{i_k}} \geq \frac{2}{d_{i_k}} = 2e_{i_k} \geq 2e_{i_2} = \frac{2}{d_{i_2}} \geq \frac{2m_0}{d_{i_1}} \geq \frac{m_0 + 1}{d_{i_1}} > \frac{l}{d_{i_k}}, \]
which is impossible. Hence, \( d_{i_1} \) cannot be a prime, and \( \gcd(m_0(m_0 + 1), d_{i_1}) > 1 \). Let \( p \) be the smallest prime factor of \( d_{i_1} \). Then \( p \leq m_0 + 1 \), \( J_V\left(\frac{p}{d_{i_1}}\right) = J_V\left(\frac{p + 1}{d_{i_1}}\right) = 1 \) and \( J_V\left(\frac{p}{d_{i_1}}\right) = 0 \).

So there exists an irreducible fraction \( \frac{p}{d_{k'}} \) (\( 2 \leq k' \leq n \)), such that
\[ \frac{p - 1}{d_{i_1}} < \frac{p}{d_{i_k'}} < \frac{p + 1}{d_{i_1}} \]
and \( J_V\left(\frac{p}{d_{i_k'}}\right) = -1 \). Similarly, \( l' \) is even and \( \frac{l'}{d_{i_k'}} \geq \frac{2m_0}{d_{i_1}} \), forcing \( p + 1 > 2m_0 \geq 2p - 2 \). This also leads to a contradiction.

We then assume \( n_{1,0}, n_{0,-1} \geq 1 \). Fix \( i \in I_{1,0} \) and \( j \in I_{0,-1} \). Applying Lemma \textit{8.7} (1) to \( \{i, i_1\} \), we see that the polynomial \( 2 - \frac{2}{3}M_{d_i} + M_{d_{i_1}} \) has no zeros in \( \mathbb{D}^{S_4} \), and thus by Lemma \textit{8.6} (1), \( d_i \) is a prime. Similarly, by applying Lemma \textit{8.7} (2) to \( \{j, i_1\} \), we also see that \( d_j \) is a prime. Finally, applying Lemma \textit{8.7} (3) to \( \{i, j, i_1\} \), one has that the inequality
\[ |2 - M_{d_i} + M_{d_{i_1}}| \leq |4 - M_{d_i} - M_{d_j} + 2M_{d_{i_1}}| \]
holds on \( \overline{\mathbb{D}}^{S_4} \). Since both \( d_i \) and \( d_j \) are primes, and by Lemma \textit{8.3}, \( M_{d_{i_1}} \) can take value \(-\frac{2}{3}\) in \( \mathbb{D}^{S_{d_{i_1}}} \), the above inequality yields
\[ \left|\frac{1}{3} - (z - 1)\right| \leq \left|\frac{2}{5} - (z - 1) - (w - 1)\right|, \quad z, w \in \overline{\mathbb{D}}. \]

It is obvious that for any \( z \in \overline{\mathbb{D}} \), one can take \( w \in \overline{\mathbb{D}} \) such that
\[ \left|\frac{2}{5} - (z - 1) - (w - 1)\right| = \left|\frac{12}{5} - z - w\right| = \left|\frac{12}{5} - z\right| - 1. \]
Therefore, we further have
\[ \left| \frac{6}{5} - z \right| + 1 = \left| \frac{1}{5} - (z - 1) \right| + 1 \leq \left| \frac{12}{5} - z \right|, \quad z \in \overline{D}. \]

But this fails for \( z = \frac{5}{8} + i\frac{\sqrt{3}m}{8} \), and thus Case (iii) cannot occur.

We have shown that only Case (i) can occur. Therefore, \( i_1 \in I_{-1,-1} \) and
\[ \frac{1}{d_{j_1}} = f_{j_1} < f_{i_1} = \frac{1}{d_{i_1}} = e_{i_1} < e_{j_1} = \frac{2}{d_{j_1}}. \tag{8.20} \]

Applying Lemma 8.7 (2) to \( \{i_1, j_1\} \), we see that the polynomial \( 2 - M_{d_{i_1}} + M_{d_{j_1}} \) has no zeros in \( \mathbb{D}^{S_q} \), and thus by Lemma 8.6, \( d_{i_1} \) is a prime.

For \( (a, b) \in A \) with \( n_{a,b} \geq 1 \), put \( d_{a,b} = \max\{d_i : i \in I_{a,b}\} \).

**Claim.** If \( n_{1,0} \geq 1 \) then \( n_{1,-1} \geq 1 \) and \( \left\lfloor \frac{d_{i_1}}{d_{i_1-1}} \right\rfloor < 2\left\lfloor \frac{d_{i_1}}{d_{i_1-1}+1} \right\rfloor \).

For this, suppose \( n_{1,0} \geq 1 \) and write \( m_1 = \left\lfloor \frac{d_{i_1}}{d_{i_1-1}} \right\rfloor \). Then for any \( j \in I_{1,0} \),
\[ \frac{1}{d_j} = e_j > e_{i_1} = \frac{1}{d_{i_1}}, \tag{8.21} \]
and
\[ \frac{m}{d_j} \leq 1 - \frac{2}{d_j} \leq 1 - \frac{2}{d_{i_1,0}} \leq 1 - \frac{2m_1}{d_{i_1}}, \quad m = 1, 3, \ldots, d_j - 2. \tag{8.22} \]

(8.21) immediately gives \( m_1 \geq 1 \). To reach a contradiction, assume conversely that \( n_{1,-1} = 0 \) or \( \left\lfloor \frac{d_{i_1}}{d_{i_1-1}} \right\rfloor \geq 2m_1 \) for every \( i \in I_{1,-1} \). Thus, if \( i \in I_{1,-1} \) then
\[ \frac{m}{d_i} \leq 1 - \frac{1}{d_i} \leq 1 - \frac{1}{d_{i_1-1}} \leq 1 - \frac{1}{d_{i_1}} \left\lfloor \frac{d_{i_1}}{d_{i_1-1}} \right\rfloor \leq 1 - \frac{2m_1}{d_{i_1}}, \quad m = 1, 2, \ldots, d_i - 1. \tag{8.23} \]

Now we will show \( J_V(\frac{2m}{d_{j_1}}) = 0 \) or \( \frac{2m}{d_{j_1}} > \frac{m_1+1}{d_{i_1}} \). In fact, if \( J_V(\frac{2m}{d_{j_1}}) \neq 0 \) and \( \frac{2m}{d_{j_1}} < \frac{m_1+1}{d_{i_1}} \), then both \( \frac{1}{d_{i_1,0}} \) and \( \frac{2m}{d_{i_1}} \) belong to the interval \( (\frac{m}{d_{i_1}}, \frac{m_1+1}{d_{i_1}}) \) (see (8.20) and the definition of \( m_1 \)), and \( J_V(\frac{1}{d_{i_1,0}}) = J_V(\frac{2m}{d_{i_1}}) = 1 \). This implies that there exists an irreducible fraction \( \frac{1}{d_\nu} (\nu \in I) \) between \( \frac{1}{d_{i_1,0}} \) and \( \frac{2m}{d_{i_1}} \), such that \( J_V(\frac{1}{d_\nu}) = -1 \), which gives \( \nu \in I_{0,-1} \cup I_{1,-1} \cup (I_{-1,-1} \setminus \{i_1\}) \).

Since
\[ 1 - f_\nu \geq 1 - e_\nu \geq 1 - \frac{l}{d_\nu} > 1 - \frac{m_1+1}{d_{i_1}}, \]
we have
\[ 1 - \frac{m_1'+1}{d_{i_1}} < 1 - f_\nu < 1 - \frac{m_1'}{d_{i_1}} \]
for some \( 0 \leq m_1' \leq m_1 \). Note that \( J_V(1 - \frac{m_1+1}{d_{i_1}}) = J_V(1 - f_\nu) = -1 \) and \( J_V(1 - \frac{m_1'}{d_{i_1}}) < 0 \).

Then there exist two irreducible fractions \( \frac{l_{k}}{d_{\nu_k}} (k = 1, 2) \), such that
\[ 1 - \frac{2m_1}{d_{i_1}} \leq 1 - \frac{m_1'+1}{d_{i_1}} < \frac{l_1}{d_{\nu_1}} < 1 - f_\nu < \frac{l_2}{d_{\nu_2}} < 1 - \frac{m_1'}{d_{i_1}} \]
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and $J_V(\frac{\nu_k}{d_{V_k}}) = 1$ ($k = 1, 2$). This yields that for $k \in \{1, 2\}$, $\nu_k = j_1$ or $\nu_k \in I_{1,0} \cup I_{1,-1}$. By (8.22) and (8.23), $\nu_k \not\in I_{1,0} \cup I_{1,-1}$ ($k = 1, 2$). Hence, $\nu_1 = \nu_2 = j_1$, and thus $\frac{2}{d_{j_1}} < \frac{1}{d_{i_1}}$ (since $l_2 - l_1$ is a positive even integer), contradicting with (8.20). So we have shown that $J_V(\frac{2m_l}{d_{j_1}}) = 0$ or $\frac{2m_l}{d_{j_1}} > \frac{m_1 + 1}{d_{i_1}}$. As a consequence, $m_1 \geq 2$.

For each $1 \leq m \leq 2m_1 - 1$, take an irreducible fraction $\frac{l_m'}{d_{j_m}} \in (1 - \frac{m + 1}{d_{i_1}}, 1 - \frac{m}{d_{i_1}})$ satisfying $J_V(\frac{l_m'}{d_{j_m}}) = 1$. Again by (8.22) and (8.23), for any $1 \leq m \leq 2m_1 - 1$ we have $\nu_m' \not\in I_{1,0} \cup I_{1,-1}$, and then $\nu_m' = j_1$ and $l_m'$ is even. On one hand, since $1 - \frac{1}{d_{j_1}} > 1 - \frac{1}{d_{i_1}}$,

$$l_m' \leq l_1' - 2(m - 1) \leq d_{j_1} - 2m - 1$$

for each $1 \leq m \leq 2m_1 - 1$. On the other hand, since for each $1 \leq m \leq 2m_1 - 1$, $1 - \frac{1}{d_{j_1}} - \frac{l_m'}{d_{j_1}} < \frac{m_1 + 1}{d_{i_1}}$, it follows that

$$d_{j_1} - 1 - l_m' < \frac{d_{j_1}(m + 1)}{d_{i_1}} < 2(m + 1),$$

i.e., $d_{j_1} - 1 - l_m' \leq 2m$. Therefore

$$l_m' = d_{j_1} - 2m - 1, \quad m = 1, 2, \ldots, 2m_1 - 1. \quad (8.24)$$

If $J_V(\frac{2m}{d_{j_1}}) = 0$ then $d_{j_1}$ is not a prime, and $c := \gcd(m_1, d_{j_1}) > 1$. It follows that $J_V(1 - \frac{c}{d_{j_1}}) = 0$, which contradicts with (8.24) since $c \leq m_1 \leq 4m_1 - 1$. If $\frac{2m_1}{d_{j_1}} > \frac{m_1 + 1}{d_{i_1}}$ then by (8.24),

$$\frac{2m_1}{m_1 + 1} > \frac{d_{j_1}}{d_{i_1}} = \frac{d_{j_1}}{2m_1} \cdot \frac{2m_1}{d_{i_1}} > \frac{d_{j_1}}{2m_1} \cdot \frac{d_{j_1} - l_2}{d_{j_1}} = \frac{4m_1 - 1}{2m_1},$$

which is also a contradiction. Thus, the claim has been proved.

We then use the claim to show

$$n_{1,0} = n_{1,-1} = 0. \quad (8.25)$$

Suppose $n_{1,-1} \geq 1$ and write $m_2 = \lfloor \frac{d_{i_1}}{d_{i_{1,-1}}} \rfloor$. For any $i \in I_{1,-1}$,

$$\frac{1}{d_i} = e_i > e_{i_1} = \frac{1}{d_{i_1}},$$

forcing $m_2 \geq 1$. Since

$$1 - \frac{m_2 + 1}{d_{i_1}} < 1 - \frac{1}{d_{i_{1,-1}}} < 1 - \frac{m_2}{d_{i_1}}$$

and

$$J_V(1 - \frac{m_2 + 1}{d_{i_1}}) = J_V(1 - \frac{1}{d_{i_{1,-1}}}) = J_V(1 - \frac{m_2}{d_{i_1}}) = -1,$$

there exist two irreducible fractions $\frac{l_k}{d_{\nu_k}}$ ($k = 1, 2$), such that

$$1 - \frac{m_2 + 1}{d_{i_1}} < \frac{l_1}{d_{\nu_1}} < 1 - \frac{1}{d_{i_{1,-1}}} < \frac{l_2}{d_{\nu_2}} < 1 - \frac{m_2}{d_{i_1}}$$

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and \( J_V(\frac{k}{d_{i_k}}) = 1 \) \((k = 1, 2)\). This yields that for \( k \in \{1, 2\}, \nu_k = j_1 \) or \( \nu_k \in I_{1,0} \cup I_{1,-1} \). By (8.20), \( \nu_1 \) and \( \nu_2 \) cannot be equal to \( j_1 \) simultaneously. Moreover, since for any \( i \in I_{1,-1}, \)
\[
J_V(1 - \frac{1}{d_i}) = -1 \quad \text{and} \quad 1 - \frac{m}{d_i} \leq 1 - \frac{2}{d_{i_1}} < 1 - \frac{2m_2}{d_{i_1}} \leq 1 - \frac{m_2 + 1}{d_{i_1}} < \frac{l_1}{d_{\nu_1}} < \frac{l_2}{d_{\nu_2}}
\]
for every \( m \geq 2 \), we have \( \nu_k \not\in I_{1,-1} \) \((k = 1, 2)\). Therefore, \( \nu_s \in I_{1,0} \) for some \( s \in \{1, 2\} \), and then \( l_s \) is odd. It follows that
\[
\frac{m_2 + 1}{2} \geq \frac{d_{i_1}}{d_{\nu_s}} - l_s = \frac{d_{i_1}}{d_{\nu_s}} \cdot \frac{m_2+1}{d_{\nu_s} \cdot \frac{l_1}{d_{\nu_1}} \geq \frac{d_{i_1}}{d_{\nu_s}}} \geq \frac{d_{i_1}}{d_{1,0}} \geq \left[ \frac{d_{i_1}}{2} \right].
\]
i.e., \( m_2 \geq 2l_{d_{i_1}} \), contradicting with the claim. Hence, we have shown (8.25).

Recall \( I_{0,1} = \{ j_1 \} \) and \( n_{1,1} = n_{-1,1} = n_{-1,0} = 0 \). With (8.25), Lemma 8.9 and (8.20) gives
\[
\phi(d_{j_1}) = \sum_{i \in I_{0,-1}} \phi(d_i) + 2 \sum_{i \in I_{1,-1}} \phi(d_i) \geq 2\phi(d_{i_1}) = 2d_{i_1} - 2 \geq d_{j_1} - 1 \geq \phi(d_{j_1}).
\]
This yields that \( d_{j_2} \) is also a prime, \( d_{j_1} = 2d_{i_1} - 1 \), \( I_{-1,1} = \{ i_1 \} \) and \( n_{0,-1} = 0 \). Rewrite \( p_1 = d_{j_1}, \) \( p_2 = d_{i_1} \). Then for each \( 1 \leq m \leq p_2, \frac{m}{p_2} \) is a right endpoint of some component interval of \( V \), and \( \frac{2m-2}{p_1} \) is a left endpoint of some component interval of \( V \). Moreover, \( J_V(\frac{2m-1}{p_1}) = 0 \) \((1 \leq m \leq \frac{p_2-1}{2})\). The proof is complete. \( \square \)

**Acknowledgement** The authors are partially supported by NNSF of China (12231005 and 12101428).

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