ON FORMATION OF SINGULARITY FOR NON-ISENTROPIC NAVIER-STOKES EQUATIONS WITHOUT HEAT-CONDUCTIVITY

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Dedicated to Professor Peter Lax on the occasion of his 90th birthday, with admiration

Abstract. It is known that smooth solutions to the non-isentropic Navier-Stokes equations without heat-conductivity may lose their regularity in finite time in the presence of vacuum. However, in spite of the recent progress on such blowup phenomena, it remains to give a possible blowup mechanism. In this paper, we present a simple continuation principle for such system, which asserts that the concentration of the density or the temperature occurs in finite time for a large class of smooth initial data, which is responsible for the breakdown of classical solutions. It also gives an affirmative answer to a strong version of a problem proposed by J. Nash in 1950s.

1. Introduction. In this paper, we consider the system of partial differential equations for the three-dimensional compressible, and non-isentropic Navier-Stokes equations in the Eulerian coordinates

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div} u + \nabla P = 0, \\
c_v [(\rho \theta)_t + \text{div}(\rho u \theta)] - \kappa \Delta \theta + P \text{div} u = 2\mu |\nabla \theta|^2 + \lambda (\text{div} u)^2,
\end{cases}
\]

(1.1)

where \( t \geq 0 \) is time, \( x \in \Omega \subset \mathbb{R}^3 \) is the spatial coordinate, and \( \rho, u = (u_1, u_2, u_3)^{tr}, \theta, P = R \rho \theta \) (\( R > 0 \)), represent respectively the fluid density, velocity, absolute
temperature, pressure; \( \mathcal{D}(u) \) is the deformation tensor given by
\[
\mathcal{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{tr}).
\]
The constant viscosity coefficients \( \mu \) and \( \lambda \) satisfy the physical restrictions
\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{1.2}
\]
Positive constants \( c_v, \kappa, \) and \( \nu \) are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity.

The compressible Navier-Stokes system (1.1) consists of a set of equations describing compressible viscous heat-conducting flows. Indeed, the equations (1.1)_1, (1.1)_2, and (1.1)_3 respectively describe the conservation of mass, momentum, and energy.

There is a considerable body of literature on the multi-dimensional compressible Navier-Stokes system (1.1) by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [2, 4, 5, 14, 10, 16, 17, 18, 21, 23, 28] and the references cited therein. However, many physically important and mathematically fundamental problems are still open due to the lack of smoothing mechanism and the strong nonlinearity. For example, although the local strong solutions to the compressible Navier-Stokes system (1.1) for general initial data with nonnegative density were respectively obtained by [2], whether the unique local strong solution can exist globally in time is an outstanding challenging open problem in contrast to the isentropic case [14].

In the presence of vacuum, as pointed out by Xin [28], non-isentropic Navier-Stokes equations without heat-conductivity will develop finite time singularity, see also Cho [1] for positive heat conductivity. Moreover, very recently, Xin-Yan [29] further proved that any classical solutions of viscous compressible fluids with or without heat conduction will blow up in finite time, as long as the initial data has an isolated mass group. Their results hold for the whole space and bounded domains, yet the blowup mechanism is not clarified. It is the main purpose if this paper to resolve this key issue. Theorem 1.2 reveals that the concentration of the density or the temperature must be responsible for the loss of regularity in finite time.

Although vacuum will lead to breakdown of smooth solutions in finite time, it is also important to study the mechanism of blowup and structure of possible singularities of general strong (or smooth) solutions to the compressible Navier-Stokes system.

The pioneering work can be traced to Serrin’s criterion [24] on the Leray-Hopf weak solutions to the three-dimensional incompressible Navier-Stokes equations, which can be stated that if a weak solution \( u \) satisfies
\[
\begin{align*}
\|u\|_{L^r(0,T;L^s)} & \leq 1, \\
2s & + 3r \leq 1, \quad 3 < r \leq \infty,
\end{align*}
\] (1.3)
then it is regular.

Recently, Huang-Li-Xin [12] extended the Serrin’s criterion (1.3) to the barotropic compressible Navier-Stokes equations and showed that if \( T^* < \infty \) is the maximal time of existence of a strong (or classical) solution \( (\rho, u) \), then
\[
\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^r(0,T;L^s)} \right) = \infty, \tag{1.4}
\]
with \( r \) and \( s \) as in (1.3). For more information on the blowup criteria of barotropic compressible flow, we refer to [13, 15, 8, 25] and the references therein.
When it comes to the fully compressible Navier-Stokes system (1.1), the problem is more complicated. In [19], Nash proposed a problem on the possible blowup of compressible heat-conductive flows. He wrote “This should give existence, smoothness, and continuation (in time) of flows, conditional on the non-appearance of certain gross type of singularity, such as infinities of temperature or density.”

Under the condition that \( \lambda < 7 \mu \), Fan-Jiang-Ou [3] obtained the following blowup criterion

\[
\lim_{T \to T^*} (\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)}) = \infty.
\]

Later, under just the physical restrictions (1.2), Huang-Li [9] established the following blowup criterion:

\[
\lim_{T \to T^*} (\|\theta\|_{L^2(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)}) = \infty,
\]

where \( \mathcal{D}(u) \) is the deformation tensor.

In the absence of vacuum, Sun-Wang-Zhang [26] established the following blowup criterion for bounded domains with positive heat-conductivity \( \kappa > 0 \) that

\[
\lim_{T \to T^*} (\|\rho, \frac{1}{\rho}\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)}) = \infty,
\]

provided that (1.2) and (1.5) hold true. As a consequence, Nash’s problem is partially verified as [26] can’t rule out the possibility of appearance of vacuum.

Recently, for \( \kappa > 0 \), we [11] establish a blowup criterion allowing initial vacuum, which is independent of temperature, as follows

\[
\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^r(0,T;L^s)}) = \infty,
\]

where \( r, s \) satisfy (1.3).

As a matter of fact, the blowup criterion (1.7) further implies

\[
\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)}) = \infty,
\]

as long as (1.5) holds true. This makes Nash’s problem as an immediately corollary for positive heat-conductivity flows. Our main goal in this paper is to give an affirmative answer to a strong version of Nash’s proposal without heat-conduction.

We will assume that \( \kappa = 0 \), and without loss of generality, take \( c_v = R = 1 \). The system (1.1) is reduced to

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \text{div}u + \nabla P &= 0, \\
P_t + \text{div}(Pu) + P \text{div}u &= 2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div}u)^2.
\end{align*}
\]

The system (1.9) is supplemented with the following initial conditions:

\[
(\rho, u, P)(x, 0) = (\rho_0, u_0, P_0)(x), \quad x \in \mathbb{R}^3,
\]

(\rho, u, P) satisfies the far field condition:

\[
(\rho, u, P)(x, t) \to (0, 0, 0) \text{ as } |x| \to \infty;
\]
Remark 1. Indeed, the definition above also ensures functions vanishing at infinity.

For constant

Theorem 1.2.

and the compatibility conditions:

\begin{align*}
(\text{Strong Solutions})
\end{align*}

Definition 1.1 defined as follows.

In accordance with [2], the strong solutions to the Cauchy problem (1.9)-(1.11) are defined as follows.

\begin{align*}
\text{Notations.} \quad & \text{For } 1 \leq p \leq \infty \text{ and integer } k \geq 0, \text{ the standard homogeneous and inhomogeneous Sobolev spaces in } R^3 \text{ are denoted by:}
\begin{align*}
\{ L^p = L^p(R^3), \quad W^{k,p} = W^{k,p}(R^3), \quad D^{k,p} = \{ u \in L^1_{\text{loc}}(R^3) \mid \nabla^k u \in L^p \}, \nonumber \\
D^0_0 = \{ u \in L^0 \mid \nabla u \in L^2 \}, \quad H^0_0 = L^2 \cap D^0_0, \quad H^k = W^k,2 \nonumber \\
\\}
\end{align*}

Denote by

\begin{align*}
\dot{f} = f_t + u \cdot \nabla f, \quad \int f dx = \int_{R^3} f dx.
\end{align*}

In accordance with [2], the strong solutions to the Cauchy problem (1.9)-(1.11) are defined as follows.

\begin{align*}
\text{Definitio}n 1.1 (\text{Strong Solutions}). \quad (\rho, u, P) \text{ is called a strong solution to (1.9) in } R^3 \times (0, T), \text{ if for some } q_0 > 3,
\begin{align*}
\{ (\rho, P) \geq 0, \quad \rho \in C([0, T]; C_0 \cap H^1 \cap W^{1,q_0}), \quad \rho_t \in C([0, T]; L^{q_0}), \nonumber \\
P \in C([0, T]; H^1 \cap W^{1,q_0}), \quad P_t \in C([0, T]; L^{q_0}) \nonumber \\
u \in C([0, T]; D^0_0 \cap D^{2,2} \cap L^2(0, T; D^{2,q_0})), \nonumber \\
u_t \in L^2(0, T; D^{1,2}), \quad \sqrt{\rho} u_t \in L^\infty(0, T; L^2), \nonumber 
\}\end{align*}

and (\rho, u, P) satisfies both (1.1) almost everywhere in R^3 \times (0, T) and (1.10) almost everywhere in R^3, where C_0 is the completion of W^{1,q} in L^\infty, consists of continuous functions vanishing at infinity.

Remark 1. Indeed, the definition above also ensures

\begin{align*}
\rho, P, u \in C^1([0, T] \times R^3).
\end{align*}

Then the main result in this paper can be stated as follows:

\begin{align*}
\text{Theorem 1.2.} \quad \text{For constant } \tilde{q} \in (3, 6], \text{ assume that } (\rho_0 \geq 0, u_0, P_0 \geq 0) \text{ satisfies}
\begin{align*}
\rho_0 \in C_0 \cap H^1 \cap W^{1,\tilde{q}}, \quad P_0 \in H^1 \cap W^{1,\tilde{q}},
\end{align*}

\begin{align*}
u_0 \in D^0_0 \cap D^{2,2}, \quad \rho_0 |u_0|^2 \in L^1,
\end{align*}

\text{and the compatibility conditions:}

\begin{align*}
- \mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P_0 = \sqrt{\rho_0} g,
\end{align*}

\text{with vector function } g \in (L^2(R^3))^3. \text{ Let } (\rho, u, P) \text{ be the strong solution to the compressible Navier-Stokes system (1.9) in } R^3. \text{ If } T^* < \infty \text{ is the maximal time of existence, then}

\begin{align*}
\lim_{T \to T^*} ((\rho)L^{\infty}(0,T;L^\infty) + ||\theta||L^{\infty}(0,T;L^\infty)) = \infty.
\end{align*}

\text{provided}

\begin{align*}
\mu > 4\lambda.
\end{align*}

A few remarks are in order:

\textbf{Remark 2.} Under the conditions of Theorem 1.2, the local existence of the strong solution was guaranteed in [2]. Thus, the assumption of T^* makes sense.

\textbf{Remark 3.} The main contribution of Theorem 1.2 asserts that Nash’s proposal even holds for zero heat-conductivity flows. In view of (1.15), the formation of singularity is only due to the concentration of either the density or temperature. In this sense, we give an affirmative answer to a strong version of Nash’s problem.
Remark 4. Condition (1.16) is only used in obtaining estimates in Lemma 3.2.

Remark 5. It’s easy to prove a same continuation principle for two-dimensional problem without any restrictions on \( \mu, \lambda \). Since the proof is analogous and simpler, we omit it for simplicity.

We may also investigate the following different boundary conditions.

1. \( \Omega = \mathbb{R}^3 \) and constants \( \tilde{\rho}, \tilde{P} \geq 0 \), \((\rho, u, P)\) satisfies the far field condition either vacuum or non-vacuum:

\[
(\rho, u, P)(x, t) \to (\tilde{\rho}, 0, \tilde{P}) \text{ as } |x| \to \infty;
\]
and initial condition

\[
\begin{align*}
\rho_0 - \tilde{\rho} & \in C_0 \cap H^1 \cap W^{1, \tilde{q}}, \quad P_0 - \tilde{P} \in H^1 \cap W^{1, \tilde{q}} \\
u_0 & \in D_{0}^1 \cap D^{2, 2}, \quad \rho_0 |u_0|^2 \in L^1,
\end{align*}
\]

2. \( \Omega \) is a bounded smooth domain.

\[
u = 0 \quad \text{on} \; \partial \Omega.
\]

For constant \( \tilde{q} \in (3, 6] \), assume that \((\rho_0 \geq 0, u_0, P_0 \geq 0)\) \( u \) satisfies

\[
\rho_0 \in C(\bar{\Omega}) \cap W^{1, \tilde{q}}(\Omega), \quad P_0 \in W^{1, \tilde{q}}(\Omega), \quad u_0 \in H_{0}^1(\Omega) \cap H^2(\Omega).
\]

(3) \( \Omega = T^3 = \mathbb{R}^3/\mathbb{Z}^3 \) and constant \( \tilde{q} \in (3, 6] \), assume that \((\rho_0 \geq 0, u_0, P_0 \geq 0)\) satisfy

\[
\rho_0 \in C(T^3) \cap W^{1, \tilde{q}}(T^3), \quad P_0 \in W^{1, \tilde{q}}(T^3), \quad u_0 \in H^2(T^3).
\]

Our next theorem asserts that Nash’s proposal also holds for different boundary conditions.

**Theorem 1.3.** Let \((\rho, u, P)\) be the strong solution to the full compressible Navier-Stokes system (1.9) together with

\[
(\rho, u, P)(x, 0) = (\rho_0, u_0, P_0), \quad x \in \Omega,
\]
and

1. \((\rho_0, u_0, P_0)\) satisfy (1.17)-(1.18) for \( \Omega = \mathbb{R}^3 \),
2. \((\rho_0, u_0, P_0)\) satisfy (1.19)-(1.20) for bounded domain \( \Omega \),
3. \((\rho_0, u_0, P_0)\) satisfy (1.21) for periodic domain \( \Omega \),

and the compatibility conditions:

\[
- \mu \Delta u_0 - \left( \mu + \lambda \right) \nabla \text{div} u_0 + \nabla P_0 = \sqrt{\rho_0} g,
\]

with vector function \( g \in (L^2(\Omega))^3 \). If \( T^* < \infty \) is the maximal time of existence, then

\[
\lim_{T \to T^*} (\|\rho\|_{L^\infty(0, T; L^\infty)} + \|\theta\|_{L^\infty(0, T; L^\infty)}) = \infty.
\]

provided

\[
\mu > 4\lambda.
\]

Let \((\rho, u, \theta)\) be a strong solution described in Theorem 1.2. Suppose that (1.15) were false, that is,

\[
\lim_{T \to T^*} (\|\rho\|_{L^\infty(0, T; L^\infty)} + \|\theta\|_{L^\infty(0, T; L^\infty)}) = M_0 < +\infty.
\]

One needs to show that

\[
\sup_{0 \leq t \leq T^*} (\|\rho(P)\|_{H^1 \cap W^{1, 4}} + \|\nabla u\|_{H^1}) \leq C.
\]

We now comment on the analysis of this paper.
We would like emphasize that the main merit of this note is the observation that the heat conduction plays no role in ruling out shock-type singularities for the full compressible Navier-Stokes system. Indeed, in the absence of the heat-conduction, though the compressible Navier-Stokes-Fourier system contains more hyperbolic modes, yet we show that it is the concentration of the density or temperature which must be responsible for any finite time formation of singularities, in contrast to the ideal compressible Euler system where shocks must form in general. Such kind of phenomena have been only proved previously for heat conductive flows where the parabolic structure is important.

For the mathematical analysis, we use the basic formulation of the Helmholtz decomposition of the momentum equations in terms of the “effective viscous flux” as first introduced by Hoff[6], Serre[22] and Vaigant-Kazhikhov[27] as in many recent studies on viscous compressible fluids, the energy estimates of the material derivatives of the velocity established first by Hoff in [6], and some recent ideas developed in [7,9,11-13,15,25,26], yet there are some new difficulties which need to be overcome in this note.

First, though the energy equation can be written in the form which is linear in the pressure \( P \), it contains source terms involving \( |\nabla u|^2 \) which is difficult to estimate. Furthermore, the equations become highly nonlinear for estimates of the pressure gradients. Indeed, it seems that in the absence of heat conduction, no useful information concerning the bound of \( ||\nabla P||_{L^2 L^2} \) can be obtained, which is quite different from the case of heat-conductive flows where the bound for \( ||\nabla \theta||_{L^2 L^2} \) is a natural byproduct of the heat dissipation and entropy estimates, which are useful for higher derivatives estimates. Similarly, it is also difficult to estimate \( ||\nabla u||_{L \sim L^2} \). Our approach can be sketched as follows. The first step is to estimate \( ||\nabla u||_{L \sim L^2} \). To this end, the key is, instead of using the linear equation for the pressure, we rewrite the temperature equation in the conservative form for the total energy, \( \rho E = \rho(\theta + \frac{1}{2}|u|^2) \), and use the effective viscous flux as the multiplier for this equation, see (3.23), to derive the desired derivatives estimate (3.19). This will require bounds on \( \int \rho |u|^6 dx \) and \( \int \int |u|^2|\nabla u|^2 dx dt \) which can be obtained from the momentum equations by choosing suitable multipliers as done previously in [6,15].

Next, we estimate \( ||\nabla \rho||_{L \sim L^p} \) and \( ||\nabla P||_{L \sim L^p} \) simultaneously. The strategy is to derive a Log Gronwall type inequality for the density and pressure gradients, as done by previous works[11-12]. However, additional cares are needed to obtain these estimates. Indeed, extra delicate estimates are crucial in deriving the bound for pressure gradients which is mainly due to the source terms involving \( |\nabla u|^2 \). For example,

\[
\frac{d}{dt}||\nabla P||_{L^p} \leq C(1 + ||\nabla u||_{L^\infty})(||\nabla P||_{L^p} + ||\nabla^2 u||_{L^p}) + C||\nabla^2 u||_{L^p}. \tag{1.28}
\]

In order to get the bound for \( ||\nabla P||_{L \sim L^p} \), the multiplication of \( ||\nabla u||_{L^\infty} \) and \( ||\nabla^2 u||_{L^p} \) need to be more carefully treated than previous works. Say, one need to prove that

\[
||\nabla^2 u||_{L^p} \leq C \left( 1 + ||\nabla u||_{L^2}^{1-\alpha} + ||\nabla P||_{L^p} \right) \tag{1.29}
\]

for some \( \alpha \in (0,1) \) and

\[
||\nabla u||_{L^\infty} \leq C(1 + ||\nabla u||_{L^2}^{1-\beta}) \log(e + ||\nabla u||_{L^2} + ||\nabla P||_{L^2}) + C||\nabla u||_{L^2} \tag{1.30}
\]

for some \( \beta \in (0,1) \), which is new in this note.

The rest of the paper is organized as follows: in the next section, we collect some elementary facts and inequalities that will be needed later. The main result,
Lemma 2.1. Assume that the initial data \((\rho_0 \geq 0, u_0, P_0 \geq 0)\) satisfy (1.13)-(1.14). Then there exists a positive time \(T_1 \in (0, \infty)\) and a unique strong solution \((\rho, u, P)\) to the Cauchy problem (1.9)-(1.11) on \(R^3 \times (0, T_1]\).

Next, the following well-known Sobolev inequality will be used later frequently (see [20]).

Lemma 2.2. For \(p \in (1, \infty)\) and \(q \in (3, \infty)\), there exists a generic constant \(C > 0\), which depends only on \(p, q\), such that for \(f \in L^p_0\) and \(g \in L^q \cap D^{1,q}\), we have

\[
\|f\|_{L^p} \leq C\|\nabla f\|_{L^2}, \quad \|g\|_{L^\infty} \leq C\|g\|_{L^p} + C\|\nabla g\|_{L^q}.
\]

The following logarithmic estimate from [12] will be used to estimate \(\|\nabla u\|_{L^\infty}\).

Lemma 2.3. For \(3 < q < \infty\), there is a constant \(C(q)\) such that the following estimate holds for all \(\nabla u \in L^2(R^3) \cap D^{1,q}(R^3)\),

\[
\|\nabla u\|_{L^\infty} \leq C (\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.
\]

Finally, we consider the following Lamé system

\[-\mu \Delta v(x) - (\mu + \lambda)\nabla \text{div} v(x) = f(x), \quad x \in \Omega,
\]

where \(v = (v_1, v_2, v_3)\), \(f = (f_1, f_2, f_3)\), and \(\mu, \lambda\) satisfy (1.2).

Assume that \(\Omega\) is a bounded smooth domain and \(v = 0\) on \(\partial \Omega\).

The following logarithmic estimate for the Lamé system (2.3), which can be found in [11] will be used to estimate \(\|\nabla u\|_{L^\infty}\) and \(\|\nabla \rho\|_{L^2 \cap L^q}\).

Lemma 2.4. Let \(\mu, \lambda\) satisfy (1.2). Assume that \(f = \text{div} g\) where \(g = (g_{kj})_{3 \times 3}\) with \(g_{kj} \in L^2 \cap W^{1,q}\) for \(k, j = 1, \ldots, 3, r \in (1, \infty)\), and \(q \in (3, \infty)\). Then the Lamé system (2.3) together with (2.4) has a unique solution \(v \in H_0^1 \cap W^{2,q}\), and there exists a generic positive constant \(C\) depending only on \(\mu, \lambda, q, r\) and \(\Omega\) such that

\[
\|\nabla v\|_{L^r} \leq C\|g\|_{L^r},
\]

and

\[
\|\nabla v\|_{L^\infty} \leq C (1 + \ln(e + \|\nabla g\|_{L^q}))\|g\|_{L^\infty} + \|g\|_{L^r}.
\]

3. Proof of Theorem 1.2. Throughout the rest of the section, \(C\) will denote a generic constant depending only on \(\rho_0, u_0, \theta_0, T^*, M_0, \lambda, \mu\).

Suppose that the result of Theorem 1.1 does not hold for a particular strong solution. Let \(T^* < \infty\) be the maximal time of existence and let \(M_0\) be as in (1.26), that is,

\[
\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\theta\|_{L^\infty(0,T;L^\infty)}) = M_0 < +\infty.
\]
Our main goal is to show that
\[
\sup_{0 \leq t \leq T^*} (\|(\rho, P)\|_{H^1 \cap W^{1,4}} + \|\nabla u\|_{H^1}) \leq C.
\] (3.2)

We start with the standard energy estimate.

Lemma 3.1. \[
\int \left( \frac{1}{2} \rho |u|^2 + P \right) dt = \int \left( \frac{1}{2} \rho_0 |u_0|^2 + P_0 \right) dx = E_0.
\] (3.3)

Next, a high energy estimate for the velocities holds under the conditions (1.16) and (1.26). This is crucial in deriving the first order derivatives of the velocities where terms involving temperature gradient should be avoided, such as \(\nabla P\). The choice of multiplier of the momentum equations is similar to the one used in Hoff[6].

Lemma 3.2. Under the condition (1.26), as long as \(\mu > 4\lambda\), it holds that
\[
\int \rho(|u|^2 + |u|^6) dt + \int_0^t \int \nabla |u|^2 (1 + |u|^2 + |u|^4) dx dt \leq C
\] (3.4)

Proof. It follows from (1.26) that
\[
\|\rho\|_{L^1 \cap L^\infty} \leq C.
\] (3.5)

Multiplying (1.1)_2 by \(q|u|^{q-2}u\), and integrating over \(\Omega\), one obtains by using lemma 2.1 that
\[
\frac{d}{dt} \int \rho |u|^q dx + \int (q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + \mu(q - 2)|\nabla u|^2]) dx + q(\lambda + \mu)(\nabla |u|^{q-2}) : \text{div} u dx
\]
\[
= q \int P \text{div} (|u|^{q-2}u) dx
\]
\[
\leq C \int \rho \frac{1}{2} |u|^{q-2} |\nabla u| dx
\]
\[
\leq \varepsilon \int |u|^{q-2} |\nabla u|^2 dx + C(\varepsilon) \int \rho |u|^{q-2} dx
\]
\[
= \varepsilon \int |u|^{q-2} |\nabla u|^2 dx + C(\varepsilon) (\int \rho |u|^{q} dx)^{\frac{2}{q-2}}.
\] (3.6)

Noting that \(|\nabla u| \leq |\nabla u|\), one gets that
\[
q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + \mu(q - 2)|\nabla u|^2] + q(\lambda + \mu)(\nabla |u|^{q-2}) : \text{div} u
\]
\[
\geq q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + \mu(q - 2)|\nabla u|^2]
\]
\[
- (\lambda + \mu)(q - 2)|\nabla u| |\text{div} u|
\]
\[
= q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(|\text{div} u| - \frac{(q - 2)}{2} |\nabla u|)^2]
\]
\[
+ q|u|^{q-2}[\mu(q - 2) - \frac{1}{4}(\lambda + \mu)(q - 2)^2]|\nabla u|^2
\]
\[
\geq q|u|^{q-2}[\mu(q - 1) - \frac{1}{4}(\lambda + \mu)(q - 2)^2]|\nabla u|^2
\] (3.7)

Consequently, recall \(q = 6\) and \(\mu > 4\lambda\), the left-hand side of (3.7) is greater than
\[
6(\mu - 4\lambda)|u|^{4}|\nabla u|^2.
\] (3.8)
Hence, taking $\varepsilon$ small enough in (3.6) and Gronwall’s inequality implies
\[
\int \rho |u|^6 dx + \int_0^t \int |\nabla u|^2 |u|^4 dx dt \leq C. \tag{3.9}
\]
Taking $q = 4$ again, one can also prove
\[
\int \rho |u|^4 dx + \int_0^t \int |\nabla u|^2 |u|^2 dx dt \leq C. \tag{3.10}
\]
This completes the proof.

Also, $\theta$ is always non-negative before blowup time $T^*$.

**Lemma 3.3.** As long as $\theta_0 \geq 0$, it holds that
\[
\inf_{(x, t) \in \mathbb{R}^3 \times (0, T^*)} \theta(x, t) \geq 0. \tag{3.11}
\]

**Proof.** The equation for $P$ can be rewritten as
\[
P_t + \text{div}(P'u) + P\text{div}u = F \geq 0. \tag{3.12}
\]
Since $\nabla u \in L^1(0, T; L^{\infty}(\mathbb{R}^3))$, we can always define particle path before blowup time.
\[
\begin{aligned}
\frac{d}{dt}X(x, t) &= u(X(x, t), t), \\
X(x, 0) &= x.
\end{aligned} \tag{3.13}
\]
Consequently, along particle path, one has
\[
\frac{d}{dt}P(X(x, t), t) = -2P\text{div}u + F, \tag{3.14}
\]
which implies
\[
P(X(x, t), t) = \exp(-2 \int_0^t \text{div}uds) \left[ P(0) + \int_0^t \exp(2 \int_0^s \text{div}u d\tau) F ds \right] \geq 0. \tag{3.15}
\]
Hence, $\theta \geq 0$ follows immediately from (3.15).

Before proving Theorem 1.2, we state some a priori estimates under the condition (1.26).

Let $E$ be the specific energy defined by
\[
E = \theta + \frac{|u|^2}{2}. \tag{3.16}
\]
Let $G$ be the effective viscous flux introduced by Hoff[6](see also Serre[22], Vaigant-Kazhikhov[27]), $\omega$ be vorticity given by
\[
G = (2\mu + \lambda)\text{div}u - P, \quad \omega = \text{curl}u. \tag{3.17}
\]
Consequently, the momentum equations can be rewritten as
\[
\rho \dot{u} = \nabla G - \mu \nabla \times \omega. \tag{3.18}
\]
Then, we derive the following crucial estimate on the $L^{\infty}(0, T; L^2)$-norm of $\nabla u$.

The main trick here is to make use of the previous Lemma 3.2 and rewrite the pressure equation in a conservative form of the total energy, thus is quite pleasant to avoid temperature gradient estimates which is a big trouble here.
Lemma 3.4. Under the condition (1.26), it holds that for $0 \leq T < T^*$,
\[
\sup_{0 \leq t \leq T} \int |\nabla u|^2 dx + \int_0^T \int \rho|\dot{u}|^2 dx dt \leq C. \tag{3.19}
\]

Proof. First, multiplying $(1.1)_2$ by $u_t$ and integrating the resulting equation over $\Omega$ show that
\[
\frac{1}{2} \frac{d}{dt} \int (\mu|\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2) \, dx + \int \rho|\dot{u}|^2 dx \\
= \int \rho \dot{u} \cdot (u \cdot \nabla) u \, dx + \int P \text{div} u_t \, dx \\
\leq \frac{1}{4} \int \rho|\ddot{u}|^2 dx + C \int \rho |u|^2 |\nabla u|^2 dx + \frac{d}{dt} \int P \text{div} u \, dx - \int P \text{div} u_t \, dx \tag{3.20}
\]
\[
= \frac{1}{4} \int \rho|\ddot{u}|^2 dx + C \int \rho |u|^2 |\nabla u|^2 dx \\
+ \frac{d}{dt} \int \left( P \text{div} u - \frac{1}{2(2\mu + \lambda)} P^2 \right) \, dx - \frac{1}{2(2\mu + \lambda)} \int P_t G \, dx
\]
Then, we will estimate the last term on the right-hand side of (3.20).

First, it follows from (1.1) that $E$ satisfies
\[
(\rho E)_t + \text{div}(\rho Eu) = \text{div} F \tag{3.21}
\]
with
\[
F = \frac{\mu}{2} \nabla(|u|^2) + \mu u \cdot \nabla u + \lambda \text{div} u - Pu. \tag{3.22}
\]
It follows from (3.16) and (3.21) that
\[
- \int P_t G \, dx = - \int (\rho E)_t G \, dx + \frac{1}{2} \int (\rho |u|^2)_t G \, dx \\
\leq C \int (\rho E |u| + |u||\nabla u|) |\nabla G| \, dx \\
- \frac{1}{2} \int (\text{div}(\rho u)|u|^2 G - 2\rho u \cdot u G) \, dx = \sum_{i=1}^3 I_i.
\]

Cauchy’s and Sobolev’s inequalities yield that
\[
I_1 + I_2 \leq \eta \|\nabla G\|_{L^2}^2 + C(\eta) \int (\rho^2 E^2 |u|^2 + |u|^2 |\nabla u|^2) \, dx \\
\leq \eta \|\nabla G\|_{L^2}^2 + C(\eta) \int (\rho |u|^6 + \rho |u|^2 + |u|^2 |\nabla u|^2) \, dx. \tag{3.24}
\]

Integration by parts and recall $G = (2\mu + \lambda) \text{div} u - P$ also gives
\[
I_3 \leq C \int \rho |u|^3 |\nabla G| \, dx + C \int (\rho |u|^2 |\nabla u| + \rho |u| |\dot{u}|) |G| \, dx \\
\leq C \int \rho |u|^3 |\nabla G| \, dx + C \int (\rho |u|^2 |\nabla u| + \rho |u| |\dot{u}|) (|\nabla u| + |P|) \, dx \\
\leq C(\eta) \int (\rho |u|^6 + \rho |u|^2 + |u|^2 |\nabla u|^2) \, dx + \eta \|\nabla G\|_{L^2}^2 + \eta \int \rho |\dot{u}|^2 \, dx. \tag{3.25}
\]
In view of (3.18),
\[
\dot{\rho}u = \nabla G - \mu \nabla \times \omega. \tag{3.26}
\]
which implies
\[ \|\nabla G\|_2^2 \leq C\|\rho\|_2^2 \leq C \int \rho|\dot{u}|^2 dx. \] (3.27)

Substituting (3.23)-(3.25) and (3.27) into (3.20), one obtains after choosing \( \eta \) suitably small that
\[ \frac{d}{dt} \int \Phi dx + \int \rho|\dot{u}|^2 dx \leq C \int (\rho|u|^6 + \rho|u|^2 + |u|^2|\nabla u|^2) dx, \] (3.28)

where
\[ \Phi = \mu|\nabla u|^2 + (\mu + \lambda)(\text{div}\, u)^2 - 2P\text{div}\, u + (2\mu + \lambda)P^2 \]
satisfies
\[ \Phi \geq \frac{\mu}{2}|\nabla u|^2 - C \int P^2 \, dx. \] (3.29)

Consequently, Gronwall’s inequality together with Lemma 3.2 implies (3.19). This finishes the proof of Lemma 3.4.

Next Lemma deals with \( \nabla \dot{u} \). We follow the calculations due to Hoff[6].

**Lemma 3.5.** Under the condition (1.26), it holds that for \( 0 \leq T < T^* \),
\[ \int \rho|\dot{u}|^2 dx + \int_0^t \int |\nabla \dot{u}|^2 dx \, dt \leq C. \] (3.30)

**Proof.** Make use the fact
\[ \dot{f} = f_t + \text{div}(f \otimes u) - f \text{div}u \] (3.31)
to obtain that
\[ \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t + \text{div}(\nabla P \otimes u) = \mu[\Delta u_t + \text{div}(\Delta u \otimes u)] + (\lambda + \mu)[\nabla \text{div}u_t + \text{div}((\nabla \text{div}u) \otimes u)]. \] (3.32)

Multiplying the above equation by \( \dot{u} \) and integrating over \( \mathbb{R}^3 \) show that
\[ \frac{1}{2} \int \frac{d}{dt} \rho|\dot{u}|^2 \, dx = \int P_t \text{div}\, \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla P + \mu \int \dot{u} \cdot [\Delta u_t + \text{div}(\Delta u \otimes u)] \, dx + (\mu + \lambda) \int \dot{u} \cdot [\nabla \text{div}u_t + \text{div}((\nabla \text{div}u) \otimes u)] \, dx = \sum_{i=1}^{3} N_i. \] (3.33)

First, recalling that
\[ P_t + \text{div}(Pu) + \text{Pdiv}u = 2\mu|\mathcal{D}(u)|^2 + \lambda(\text{div}u)^2 \] (3.34)
One can get after integration by parts and using the above equation that
\begin{align*}
N_1 &= \int P \text{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla P \, dx \\
&= -\int \text{div} \text{div}(P \dot{u}) - \int P \text{div} \text{div} \dot{u} \, dx \\
&\quad + \int (2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2) \text{div} \dot{u} \, dx + \int (u \cdot \nabla \dot{u}) \cdot \nabla P \, dx \\
&\leq \int P (u \cdot \nabla) \text{div} \dot{u} \, dx - \int P (u \cdot \nabla) \text{div} \dot{u} \, dx - \int P \nabla \dot{u} : \nabla u \, dx \\
&\quad + C \left( \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}^2 \right) \\
&\leq C (1 + \|\nabla u\|_{L^2}^4) + \frac{\mu}{8} \|\nabla \dot{u}\|_{L^2}^2 \tag{3.35}
\end{align*}
Integration by parts leads to
\begin{align*}
N_2 &= \mu \int \dot{u}_j [\partial_t \nabla u_j + \text{div}(u \nabla u_j)] \, dx \\
&= -\mu \int (\partial_i \dot{u}_j \partial_t u_{ij} + \Delta u_j u \cdot \nabla \dot{u}_j) \, dx \\
&\quad - \mu \int \left( |\nabla \dot{u}|^2 - \partial_i \dot{u}_j u_k \partial_k \partial_t u_{ij} - \partial_i \dot{u}_j \partial_t u_k u_{ij} + \Delta u_j u \cdot \nabla \dot{u}_j \right) \, dx \\
&\quad - \mu \int \left( |\nabla \dot{u}|^2 - \partial_i \dot{u}_j \partial_t u_{ij} \text{div} u - \partial_i \dot{u}_j \partial_t u_k u_{ij} - \partial_i \dot{u}_j \partial_t u_k \partial_k \dot{u}_j \right) \, dx \\
&\leq -\frac{7\mu}{8} \int |\nabla \dot{u}|^2 \, dx + C \int |\nabla u|^4 \, dx. \tag{3.36}
\end{align*}
Similarly,
\begin{align*}
N_3 &\leq -\frac{7}{8} (\mu + \lambda) \|\text{div} \dot{u}\|_{L^2}^2 + C \int |\nabla u|^4 \, dx. \tag{3.37}
\end{align*}
Substituting (3.35)-(3.37) into (3.33), we obtain after choosing \( \varepsilon \) suitably small that
\begin{align*}
\frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \mu \|\nabla \dot{u}\|_{L^2}^2 \\
&\leq C \int |\nabla u|^4 \, dx + C. \tag{3.38}
\end{align*}
Note that
\begin{align*}
\|\nabla u\|_4^4 &\leq \|\nabla u\|_2 \|\nabla u\|_6^3 \\
&\leq C \|\nabla u\|_6^3 (1 + \|\nabla G\|_2 + \|\nabla \omega\|_2) \tag{3.39}
\end{align*}
Substituting this estimate into (3.38) and once again recalling that
\begin{align*}
\|\nabla u\|_6 &\leq \|G\|_6 + \|\omega\|_6 + C \\
&\leq C \|\rho \dot{u}\|_{L^2} + C \in L^2(0, T), \tag{3.40}
\end{align*}
we conclude by Gronwall’s inequality that
\begin{align*}
\int \rho |\dot{u}|^2 \, dx + \int_0^t \int |\nabla \dot{u}|^2 \, dx \, dt \leq C. \tag{3.41}
\end{align*}
Finally, the following Lemma 3.6 will treat the higher order derivatives of the solutions which are needed to guarantee the extension of local strong solution to be a global one.

Thanks to the previous Lemmas 3.3-3.5, compared to the previous works\cite{9, 11}, we are able to give a more delicate estimate of \(\|\nabla u\|_{L^\infty}\) and \(\|\nabla^2 u\|_{L^p}\) which is the key estimate in the Gronwall type inequality for both density and pressure, see (3.46)-(3.47) below.

**Lemma 3.6.** Under the condition (1.26), it holds that for \(0 \leq T < T^*\),
\[
\sup_{0 \leq t \leq T} (\|\rho, P\|_{H^1 \cap H^{1,4}} + \|\nabla u\|_{H^1}) \leq C. \tag{3.42}
\]

**Proof.** In view of (3.40) and (3.30), one has
\[
\|\nabla u\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2} + C \leq C. \tag{3.43}
\]

For \(2 \leq p \leq \tilde{q} < 6\), direct calculations show that
\[
\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}. \tag{3.44}
\]
Similarly,
\[
\frac{d}{dt} \|\nabla P\|_{L^p} \leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla P\|_{L^p} + \|\nabla^2 u\|_{L^p}) + C\|\nabla^2 u\|_{L^p}. \tag{3.45}
\]
Applying the standard \(L^p\)-estimate to (3.18) gives
\[
\|\nabla G\|_{L^p} + ||\omega||_{L^p} \leq C\|\rho \dot{u}\|_{L^p} \leq C + C\|\nabla \dot{u}\|_{L^2},
\]
which shows
\[
\|G\|_{L^\infty} \leq \|G\|_{L^2}^{\beta} \|\nabla G\|_{L^2}^{1-\beta} \leq C + C\|\nabla \dot{u}\|_{L^2}^{1-\beta}.
\]
for some \(\beta \in (0,1)\).

Applying the standard \(L^p\)-estimate to (1.1) leads to
\[
\|\nabla^2 u\|_{L^p} \leq C\left(\|\rho \dot{u}\|_{L^p} + \|\nabla P\|_{L^p}\right)
\leq C\left(1 + \|\rho \dot{u}\|_{L^2}^{1-\alpha} + \|\nabla P\|_{L^p}\right)
\leq C\left(1 + \|\nabla \dot{u}\|_{L^2}^{1-\alpha} + \|\nabla P\|_{L^p}\right) \tag{3.46}
\]
for some \(\alpha \in (0,1)\), where we have used Lemma 3.5.

This together with Lemma 2.4 gives
\[
\|\nabla u\|_{L^\infty} \leq C(1 + \|\nabla \dot{u}\|_{L^2}^{1-\beta}) \log(e + \|\nabla \dot{u}\|_{L^2} + \|\nabla P\|_{L^q}) + C\|\nabla \dot{u}\|_{L^2}. \tag{3.47}
\]
Substituting (3.46) and (3.47) into (3.44)-(3.45) yields that
\[
f'(t) \leq Cg(t)f(t) \ln f(t), \tag{3.48}
\]
where
\[
f(t) = e + \|\nabla \rho\|_{L^4} + \|\nabla P\|_{L^q}, \quad g(t) = 1 + \|\nabla \dot{u}\|_{L^2}^2.
\]
It thus follows from (3.48), (3.30), and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|(\nabla \rho, \nabla P)\|_{L^q} \leq C, \tag{3.49}
\]
which, combined with (3.47) and (3.30) gives directly that
\[
\int_0^T \|\nabla u\|_{L^2}^2 dt \leq C. \tag{3.50}
\]
Taking $p = 2$ in (3.44), one can get by using (3.50), (3.46) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \| \nabla \rho, \nabla P \|_{L^2} \leq C,
\] (3.51)
which together with (3.46) yields that
\[
\sup_{0 \leq t \leq T} \| \nabla^2 u \|_{L^2} \leq C \sup_{0 \leq t \leq T} (\| \rho \dot{u} \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \nabla P \|_{L^2}) \leq C.
\]
This combined with (3.49), (3.51), and (3.19) finishes the proof of Lemma 3.6. □

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $M_0$ be as in the proof and in view of Lemmas 3.5 and 3.6, one has
\[
\sup_{t \in (0, T^*)} \left( \| (\rho, P) \|_{H^{1 \wedge W^{1,4}}} + \| \nabla u \|_{H^1} + \int \rho |\dot{u}|^2(t, \cdot) dx \right) \leq C(M_0). \tag{3.52}
\]

Let $T_1 = T_1(C(M_0))$ be as in Lemma 2.1. Also, standard arguments yield that $\rho \dot{u} \in C([0, T]; L^2)$, which implies
\[
\rho \dot{u}(x, T^* - T_1/2) = \lim_{t \to (T^* - T_1/2)} \rho \dot{u} \in L^2.
\]

Hence,
\[
-\mu \Delta u - (\mu + \lambda) \nabla \div u + \nabla P |_{t = T^* - T_1/2} = \sqrt{\rho}(x, T^* - T_1/2)g(x)
\]
with
\[
g(x) = \begin{cases} 
\rho^{-1/2}(x, T^*)(\rho \dot{u})(x, T^* - T_1/2), & \text{for } x \in \{ x | \rho(x, T^*) > 0 \}, \\
0, & \text{for } x \in \{ x | \rho(x, T^*) = 0 \},
\end{cases}
\]
satisfying $g \in L^2$ and $\| g \|_{L^2} \leq C(M_0)$ due to (3.52).

Then the solution at time $T^* - T_1/2$ provides initial data satisfying the hypotheses (1.12)-(1.13) and so by Lemma 2.1 a unique solution with this data exists on the interval $[T^* - T_1/2, T^* + T_1/2]$. The uniqueness gives that joining this to the original solution on $[0, T^*)$ gives the required extension on $[0, T^* + T_1/2]$ which contradicts to the definition of $T^*$. We thus finish the proof of Theorem 1.2. □

4. **Outline of Theorem 1.3.** The main idea is quite analogous to section 3.

**Proof.** Case I. $\Omega = \mathbb{R}^3$ with non-vacuum far-field.

It’s sufficient to note that
\[
(\rho - \check{\rho})_t + \div((\rho - \check{\rho})u) + \check{\rho} \div u = 0. \tag{4.1}
\]

Multiplying (4.1) by $\rho - \check{\rho}$ and integrating the resulting equation over $\Omega$, we obtain after using Lemma 3.2 that
\[
(\| \rho - \check{\rho} \|^2_{L^2})'(t) \leq C \| \rho - \check{\rho} \|^2_{L^2} + C \| \nabla u \|^2_{L^2},
\]
Therefore,
\[
\rho - \check{\rho} \in L^\infty L^2. \tag{4.2}
\]

Similarly, one can show
\[
(\rho - \check{\rho}, P - \check{P}) \in L^\infty(0, t; L^1 \cap L^\infty). \tag{4.3}
\]

Then the remaining proof can be done step by step.

Case II. $\Omega = \mathbb{T}^3$. 
We need only to redefine the effective flux $G$ as

$$G = (2\mu + \lambda)\text{div} - P + \bar{P},$$

where

$$\bar{P} = \frac{1}{|T|} \int Pdx.$$ 

(4.5)

For any $q \in [1, \infty)$ and under the condition (1.26), $G$ satisfies

$$\|G\|_{L^q} \leq C\|\nabla u\|_{L^q} + \|P\|_{L^q} + C$$

(4.6)

and

$$\|\nabla G\|_{L^q} + \|\omega\|_{L^q} \leq C\|\rho \dot{u}\|_{L^q}.$$ 

(4.7)

Then the higher regularity of the density, velocities and temperature can be obtained without difficulty.

**Case III.** $\Omega$ is a bounded domain.

Since there is no boundary condition for effective viscous flux. We will outline the proof of Lemma 3.4.

Motivated by [26], we decompose the velocity into two parts. It follows from Lemma 2.4 that for any $t \in [0, T]$, there exists a unique $v(t, \cdot) \in H_0^1 \cap W^{2,\bar{q}}$ satisfying

$$\mu \Delta v + (\mu + \lambda)\nabla \text{div} v = \nabla P,$$

(4.8)

which together with (2.5) yields that

$$\|\nabla v\|_{L^p} \leq C\|P\|_{L^p} \leq C,$$

(4.9)

and that

$$- \int P_t \text{div} vdx = - \int (\mu \nabla v_t \cdot \nabla v + (\mu + \lambda)\text{div} v_t \text{div} v)dx$$

$$= - \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla v|^2 + (\mu + \lambda)(\text{div} v)^2) dx.$$ 

(4.10)

Denoting by

$$w = u - v,$$

(4.11)

we have $w \in H_0^1 \cap W^{2,\bar{q}}$, for a.e. $t \in [0, T]$. Moreover, for a.e. $t \in [0, T]$, $w$ satisfies

$$\mu \Delta w + (\mu + \lambda)\nabla \text{div} w = \rho \dot{u},$$

(4.12)

which together with the standard $L^2$-estimate for elliptic system gives

$$\|\nabla w\|_{L^6} + \|\nabla^2 w\|_{L^2} \leq C\|\rho \dot{u}\|_{L^2}.$$ 

(4.13)

It follows from (3.16) and (3.21) that

$$- \int P_t \text{div} wdx = - \int (\rho E) \text{div} wdx + \int (\rho |u|^2) \text{div} wdx$$

$$\leq C \int (\rho E|u| + |u| |\nabla u|) |\nabla^2 w| dx$$

$$- \frac{1}{2} \int (\text{div}(\rho u)|u|^2 \text{div} w - 2\rho u \cdot u_t \text{div} w) dx = \sum_{i=1}^3 J_i.$$ 

(4.14)

In fact, (4.14) has a same structure of (3.23). Here $\text{div} \omega$ plays a same role as $G$ in (3.23).

We then finish the proof of Theorem 1.3 for bounded domain by adapting a same procedure as Theorem 1.2 with the help of Lemma 2.4. \qed
Acknowledgments. The authors would like to thank the anonymous referee’s careful reading, suggestions and pointing out some typos and problems.

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Received June 2015; revised October 2015.

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