Abstract. Let \( v = v_0 + v_1 \) be a \( \mathbb{Z}_2 \)-graded (super) vector space with an even \( C^\times \)-action and \( \chi \in v_0^\ast \) be a fixed point of the induced action. In this paper we prove an equivariant Darboux-Weinstein theorem for the formal polynomial algebras \( \hat{A} = S[v_0]^{\times} \otimes \Lambda(v_1) \). We also give a quantum version of it. Let \( g = g_0 + g_1 \) be a basic Lie superalgebra and \( e \in g_0 \) be a nilpotent element. We use the equivariant quantum Darboux-Weinstein theorem to give a Poisson geometric realization of the finite \( W \)-superalgebra \( \mathcal{U}(g, e) \) in the sense of Losev. An indirect relation between finite \( W \)-(super)algebras \( \mathcal{U}(g, e) \) and \( \mathcal{U}(g_0, e) \) is presented. Finally we use such a realization to study the finite-dimensional irreducible modules over \( \mathcal{U}(g, e) \).

1. Introduction

1.1. Darboux-Weinstein theorems. In [Wei], the author proves the Darboux-Weinstein (DW) theorem, which states that each Poisson manifold is locally the product of a symplectic manifold and a Poisson manifold having a point where the rank is zero. We recall the formal version of this theorem.

Let \( \hat{A}_0 = S[[V_0]] \) be the formal power series algebra of a vector space \( V_0 \) over \( \mathbb{C} \) and \( \omega :\{\cdot, \cdot\} \) be a continuous Poisson bracket on \( \hat{A}_0 \). The DW theorem says that \( \hat{A}_0 \) can be split as a product

\[
\hat{A}_0 \cong S[[V_0]] \otimes \hat{B}_0
\]

of Poisson algebras (see [Kal] for the proof). Here the Poisson product on the formal power algebra \( S[[V_0]] \) arises naturally from a symplectic subspace \( V_0 \subset V \) on which the restriction of the Poisson bi-vector is non-degenerate, while \( \hat{B}_0 \) is the centralizer of \( V_0 \) in \( \hat{A}_0 \) with respect to the Poisson product (c.f. Example 1.2).

This theorem plays a fundamental role in the theory of symplectic singularities (c.f. [Kal]). In the recent works of Losev on the representations of certain quantum algebras, the quantum DW (qDW) theorem is always a powerful tool. Those quantum algebras include finite \( W \)-algebras, symplectic reflection algebras and rational Cherendick algebras, and arise from the formal quantization of Poisson algebras.

1.1.1. In the present paper we consider an equivariant super DW (esDW) theorem and its quantum version. Then we use it to study the representations of finite \( W \)-superalgebras.
Definition 1.1. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded commutative algebra over the complex number field $\mathbb{C}$. A Poisson bracket $\{\cdot, \cdot\}$ on $A$ is a linear map $A \otimes A \to A$ with

1. $\{\cdot, \cdot\}$ is a Lie superalgebra;
2. $\{f, gh\} = \{f, g\} h + (-1)^{|f||g|} g \{f, h\}$ for all homogenous $f, g, h \in A$.

A Poisson bracket $\{\cdot, \cdot\}$ is called even (resp. odd) if it is an even (resp. odd) linear map between vector superspaces.

Example 1.2. (1) Let $v$ be a vector superspace of dimension $(2n, m)$ equipped with a super symplectic form $\omega$. Then there is a standard Poisson bracket on $S[v]$ given by $\{x, y\} = \omega(x, y)$ for all $x, y \in v$.

(2) Let $g$ be a Lie superalgebra, then $S[g]$ also has a standard Poisson structure given by $\{x, y\} = [x, y]$ for all $x, y \in g$.

Let $v = v_0 + v_1$ be a vector superspace and $A$ be the super symmetric polynomial algebra $S[v] = S[v_0] \otimes \Lambda(v_1)$. For a fixed $\chi \in v_0^*$, let $M_\chi$ be the maximal ideal of $S[v]$ corresponding to $\chi$ i.e. $M_\chi$ is the ideal generated by $x - \chi(x)$ for $x \in v_0 \oplus v_1$. Let $A^\wedge_\chi$ be the completion of $A$ at $\chi$. Namely $A^\wedge_\chi = \varprojlim A/AM_\chi^k$ is the topological algebra with the ideals $AM_\chi^k$ forming a set of fundamental neighborhoods of $0 \in A^\wedge_\chi$. From now on, the formal power series superalgebra $A^\wedge_\chi$ for a fixed $\chi$ is denoted by $\hat{A}$ in the present paper.

Suppose that there is a continuous $\mathbb{C}^\times$-action on $\hat{A}$ which arises from an even linear algebraic $\mathbb{C}^\times$-action on $v$. Let $\chi \in v_0^*$ be a fixed point for the induced action. Assume that there is an integer $k \in \mathbb{Z}$ such that $\{t \cdot f, t \cdot g\} = t^k \{f, g\}$ for all $f, g \in \hat{A}$ and $t \in \mathbb{C}^\times$. Let $V \subset T_\chi^\times(\text{Spec}(A)) = v$ be a subspace such that the restriction of the Poisson bi-vector $\Pi$ (see (2.1) for the definition) on $V$ is non-degenerate. In §2 we prove the following $\mathbb{C}^\times$-equivariant DW theorem for the formal polynomial superalgebra $\hat{A}$ with an even Poisson bracket.

Theorem 1.3 (Equivariant super DW (esDW) theorem). In the above setting, we have the following $\mathbb{C}^\times$-equivariant isomorphism of Poisson superalgebras

$$\phi : S[[V]] \otimes \hat{B} \to \hat{A},$$

where $\hat{B}$ is the centralizer of $V$ in $\hat{A}$ with respect to $\{\cdot, \cdot\}$, which is a Poisson subalgebra of $\hat{A}$, and the restriction of $\phi$ on $\hat{B}$ is the identity map.

We prove the theorem by constructing such an isomorphism explicitly. In the case if $V = v$ and $v$ is pure even, our proof is more direct than the one in [Kal].

In the super setting, there is a significant difference between equivariant and non-equivariant cases.

1.1.2. For an associative superalgebra $A$, denote by $A[[\hbar]]$ the vector space of formal power series of variable $\hbar$ with coefficients in $A$.

Definition 1.4. Let $A$ be Poisson superalgebra, a formal quantization of $A$ is a pair $(A[[\hbar]], *)$, where the star product $*: A[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} A[[\hbar]] \to A[[\hbar]]$ satisfies
(1) * is associative;
(2) for all $Z_2$-homogenous $f, g \in A$,
\[ f * g - fg \in A[[\hbar]]^{2} \quad \text{[2] and} \quad f * g - (-1)^{|f||g|}g * f - \{f, g\}\hbar^{2} \in A[[\hbar]]^{3}. \]

For a Poisson superalgebra $A$ with a $\mathbb{C}^{\times}$-action, by a $\mathbb{C}^{\times}$-equivariant quantization $(A[[\hbar]], *)$ we mean that there is a $\mathbb{C}^{\times}$-action on $A[[\hbar]]$ such that the quotient map $A[[\hbar]] \to A[[\hbar]]/A[[\hbar]]\hbar = A$ is $\mathbb{C}^{\times}$-equivariant.

Example 1.5. (1) Let $(v, \omega)$ be a symplectic superspace. Denote by $(A_{h}(v), \ast)$ the $\mathbb{C}[[\hbar]]$-algebra generated by basis $v_{1}, v_{2}, \ldots, v_{n}$ of $v$ with commutating relations
\[ v_{i} \ast v_{j} - (-1)^{|v_{i}||v_{j}|}v_{j} \ast v_{i} = \omega(v_{i}, v_{j})\hbar^{2} \]
for all $1 \leq i, j \leq n$. Taking $h = 1$, we obtain the Weyl algebra $A(v)$ from $S[v][[\hbar]] \subset A_{h}(v)$. We have the following isomorphism of $\mathbb{C}[[\hbar]]$ spaces
\[ S[v][[\hbar]] \longrightarrow A_{h}(v), \quad v_{i}^{1} \cdots v_{n}^{m} \mapsto v_{1}^{1} \ast \cdots \ast v_{n}^{m}. \]
Under this identification, it is easy to check that $A_{h}(v)$ is a quantization of the Poisson algebra $S[v]$.

(2) For a Lie superalgebra $g$, the Poisson algebra $S[g]$ has a standard quantization $(S[g][[\hbar]], \ast)$, which is the quotient of the formal tensor algebra $T(g)[[\hbar]]$ modulo the two sided ideal generated by
\[ x \otimes y - (-1)^{|x||y|}y \otimes x = [x, y]\hbar^{2} \]
for all the $x, y \in g$. Taking $h = 1$ we can recover the universal enveloping algebra $\mathfrak{U}(g)$ from $S[g][[\hbar]] \subset S[g][[\hbar]]$.

Theorem 1.6 (Super quantum equivariant DW (sqeDW) theorem). Let $\hat{A}, V$ be as in Theorem 1.3 and $(\hat{A}[\hbar], \ast)$ be a $\mathbb{C}^{\times}$-equivariant quantization of $\hat{A}$. Then there exists a $\mathbb{C}^{\times}$-equivariant isomorphism of $\mathbb{C}[[\hbar]]$ algebras
\[ \Phi_{h} : A_{h}(V)^{\circ} \otimes_{\mathbb{C}[[\hbar]]} \hat{B}_{h} \longrightarrow \hat{A}[[\hbar]] \]
Here $\hat{B}_{h}$ is a quantization of $\hat{B}$.

We identify $S[[V]]$ with a Poisson subalgebra of $\hat{A}$ in the proof of Theorem 1.3. Under this identification the isomorphism $\phi$ is given by $\phi(a \otimes b) = ab$ for all $a, b \in \hat{A}$. The isomorphism $\Phi_{h}$ is constructed in a similar way.

1.2. Finite $W$-superalgebra via quantum DW theorem. For a complex reductive Lie algebra $g$ and a nilpotent $e$ in it, one has the finite $W$-algebra $\mathfrak{U}(g, e)$, see [Pr1] for the definition. There are several different but equivalent definitions for the finite $W$-algebras. In [Lo2], the author constructed the finite $W$-algebra from the Fedosov quantization. Via this construction, Losev relates the representation theoretic objects of the $\mathfrak{U}(g)$ and $\mathfrak{U}(g, e)$. Those objects include the prime (primitive) ideals, Harish-Chandra bimodules etc. Using those correspondences, Losev classifies the finite-dimensional modules for classical $W$-algebra $\mathfrak{U}(g, e)$ (c.f. [Lo2]).

1 Usually $\hbar^{2}$ is replaced by $\hbar$, here we need such a slight modification for its application.
1.2.1. In [WZ], [ZS1] the authors study the finite $W$-superalgebras. Let us recall the definition of finite $W$-superalgebras. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a basic Lie superalgebra with an even non-degenerate super-symmetric invariant bilinear form (say Killing form) $(\cdot, \cdot)$. Denote by $\mathcal{U}$ the universal enveloping algebra of $\mathfrak{g}$. Let $e \in \mathfrak{g}_0$ be a nilpotent element and $\chi \in \mathfrak{g}^*$ be the unique element with $\chi(X) = (e, X)$ for all $X \in \mathfrak{g}$. Since the Killing form is even, we have $\chi(x) = 0$ for all $x \in \mathfrak{g}_1$.

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading for $e$, see [Hoy] for the definition. It follows from the property of Killing form that the super symplectic form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}(-1)$ given by

$$\langle x, y \rangle = \chi([x, y])$$

is non-degenerate.

Let $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, choose a Lagrangian subspace (isotropic subspace of maximal dimension) $l$ of $\mathfrak{g}(-1)$ and set $\mathfrak{m} = \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus l$. Let $\mathfrak{g}'$ (resp. $\mathfrak{m}'$) be the subspace of $\mathcal{U}$ consisting of $\{ x - \chi(x) | x \in \mathfrak{g} \}$ (resp. $\mathfrak{m}$). Since $\chi(x) = 0$ for all $x \in \mathfrak{g}_1$, $\mathfrak{g}'$ (resp. $\mathfrak{m}'$) is a $\mathbb{Z}_2$-graded vector space.

Denote by $I_\chi$ the left ideal of $\mathcal{U}$ generated by $\mathfrak{m}'$. Then the finite $W$-superalgebra $\mathcal{U}(\mathfrak{g}, e)$ associate to the nilpotent element $e$ is defined (c.f. [ZS2, §2.2.3]) by

$$\mathcal{U}(\mathfrak{g}, e) := (\mathcal{U}/I_{\chi})^{\text{adm}} = \{ \mathfrak{g} \in \mathcal{U}/I_{\chi} | (a - \chi(a))y \in I_{\chi} \text{ for all } a \in \mathfrak{m} \},$$

which is equal to $(\mathcal{U}/I_{\chi})^{\text{ts}}$. In Section 3 (see Theorem 3.8), we prove that the finite $W$-superalgebra $\mathcal{U}(\mathfrak{g}, e)$ is isomorphic to the super quantum Darboux Weinstein slices $\mathcal{W}_\chi$ (which will often simply denoted by $\mathcal{W}$). This generalizes the non-super results from [Lo2], [Lo1]. We use a similar idea of the proof as in loc. cit.. A difference in our approach is that we use the explicit construction in the proof of Theorem 1.6 but not the theory of Fedosov quantization and invariant theory. As an application we reobtain the PBW base theorem of finite $W$-superalgebras, which is a main result of [ZS1].

1.2.2. Let $\mathcal{U}_{m'}$ be the completion of $\mathcal{U}$ with respect to the fundamental system $\mathcal{U}m^{k}$ i.e. $\mathcal{U}_{m'}^{\wedge} = \lim \mathcal{U}/(\mathcal{U}m^{k})$. It follows from the definition of the good $\mathbb{Z}$-grading that $\text{dim}([g, f]_1)$ is odd if and only if $\text{dim}(g(-1)_1)$ is odd. In this case there exists $\Theta \in g(-1)_1$ with $\langle \Theta, \Theta \rangle = 1$. Let $\bar{m}$ be a maximal isotropic subspace of symplectic superspace $[g, f]$ subject to $\langle \Theta, \bar{m} \rangle = 0$ whenever $\text{dim}([g, f]_1)$ is odd, and $V$ be the subspace $\bar{m}^* \oplus \bar{m}$.

Denote by $\mathbf{A}(V)^\wedge_{\bar{m}} \otimes \mathcal{W}$ the completion of $\mathbf{A}(V) \otimes \mathcal{W}$ with respect to the fundamental system $\mathbf{A}(V) \otimes \mathcal{W}/(\mathbf{A}(V) \otimes \mathcal{W})\bar{m}^{k}$. Using the realization introduced above we prove the following splitting theorem.

**Theorem 1.7.** There exists an isomorphism

$$\Phi : \mathbf{A}(V)^\wedge_{\bar{m}} \otimes \mathcal{W} \cong \mathcal{U}_{m'}^{\wedge}$$

of associative algebras.
1.2.3. As applications of the above splitting theorem, we firstly give an alternative proof of the super Skryabin’s equivalence (c.f. Theorem 4.1) which was stated in [ZS1].

Secondly, we establish a relation between finite $W$-algebras and $W$-superalgebras. For a Lie superalgebra $g = g_0 \oplus g_1$, there is a natural embedding $\mathcal{U}(g_0) \hookrightarrow \mathcal{U}(g)$ of the universal enveloping algebras. Almost all the results on representations of Lie superalgebras rely on this relation. It is very useful to find a similar story for finite $W$-superalgebras. Unfortunately, there is no canonical embedding of $\mathcal{U}(g_0, e)$ into $\mathcal{U}(g, e)$ for general nilpotent $e$. What is a good remedy, in [SS] we show that the $W$-algebra $\mathcal{U}(g_0, e)$ can be embedded into a larger associative algebra $\mathcal{A}_t = Cl(V) \otimes \mathcal{U}(g, e)$ (see Theorems 3.11). This enables $\mathcal{U}(g_0, e)$ to play a role in the representation theory of $\mathcal{U}(g, e)$ as $\mathcal{U}(g_0, e)$ does in the representation theory of $g$ (this will appear in [SXZ]). For an immediate consequence of this fact, see Corollary 3.12. Finally we point out that the associative algebra $\mathcal{A}_t$ is a Clifford algebra over $\mathcal{U}(g, e)$.

1.3. Finite-dimensional representations of finite $W$-algebras. The relation introduced above will give us a powerful tool to investigate representations of finite $W$-superalgebras. In the present work, we give some information on the set of finite-dimensional irreducible $\mathcal{U}(g, e)$-modules by relating two sided ideals of $\mathcal{U}(g, e)$ and $\mathcal{U}$. Denote by $\mathfrak{i}d(\mathcal{A})$ the set of two sided ideals of an associative algebra $\mathcal{A}$. Following [Lo2] in the non-super case, we construct a map $\bullet^1: \mathfrak{i}d(\mathcal{U}) \to \mathfrak{i}d(\mathcal{U}(g, e))$ and $\bullet^1: \mathfrak{i}d(\mathcal{U}) \to \mathfrak{i}d(\mathcal{U}(g, e))$. In the case of semisimple Lie algebra, those correspondences were used to prove the existence of finite-/one-dimensional $\mathcal{U}(g, e)$-modules.

In the case when $e$ is a regular nilpotent element in general linear Lie superalgebra or is a minimal nilpotent element in $g_0$ for arbitrary basic Lie superalgebra $g$, the finite-dimensional representations of $\mathcal{U}(g, e)$ were studied in [BBQ], [BQ] and [ZS2]. In [BBC], the authors proved that irreducible modules over the principal finite $W$-superalgebra $\mathcal{U}(g, e)$ are finite dimensional and classified them by highest weight theory using the triangular decomposition of $\mathcal{U}(g, e)$. In [PS1] the authors considered the finite $W$-algebra $\mathcal{U}(g, e)$ for a basic Lie superalgebra and the queer Lie superalgebra $Q(n)$ associated with regular even nilpotent coadjoint orbits and proved that all irreducible representations of $\mathcal{U}(g, e)$ are finite dimensional. They classified irreducible $\mathcal{U}(g, e)$-modules for $Q(n)$ in [PS3]. For general $W$-superalgebras of type $Q(n)$, see [PS2]. All the representations in loc.cit were constructed somehow from an explicit presentation of $\mathcal{U}(g, e)$. In the present work we study finite-dimensional representations of $\mathcal{U}(g, e)$ by the above mentioned conceptional approach. In the last section we construct a series of finite-dimensional modules via the above correspondences (see Theorem 4.7). On the other hand, we show that for any primitive ideal $J \subset \mathcal{U}$ supported on $\emptyset$, the pre-image of $J$ under the map $\bullet^1$ is finite. Recall that Lexcter establishes a bijection between primitive ideals of $\mathcal{U}(g_0)$ and $\mathcal{U}$ (c.f. [Le]). For further information about the later, see [CM].
Thus we make a step forward to a classification of the finite-dimensional irreducible modules over $\mathfrak{U}(\mathfrak{g}, e)$.

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2. Proofs of theorem 1.3 and 1.6

Let us recall the notion of Poisson bi-vector at first. Especially, $M_\chi$ denotes the maximal ideal of $\hat{S}[v]$ corresponding to $\chi$, this is to say, $M_\chi$ is the ideal generated by $x - \chi(x)$ for $x \in v$. Identify the cotangent space of the formal super scheme $\text{Spec}(\hat{A})$ at the close point $\chi$ with the super space $v$. For a Poisson bracket $\{\cdot, \cdot\}$ on $\hat{A}$, the Poisson bi-vector $\Pi \in \bigwedge^2 v$ associated to it is given by

$$\Pi(df, dg) := \{f, g\} + M_\chi \in \mathbb{C}. \quad (2.1)$$

If $\Pi$ is non-degenerate, we say that the Poisson bracket $\{\cdot, \cdot\}$ is symplectic.

Assume that $\hat{A}$ is equipped with an even continuous Poisson bracket $\{\cdot, \cdot\}$. The Poisson bi-vector $\Pi \in \bigwedge^2 v$ associated to it is given by

$$\Pi(df, dg) := \{f, g\} + M_\chi \in \mathbb{C}. \quad (2.1)$$

If $\Pi$ is non-degenerate, we say that the Poisson bracket $\{\cdot, \cdot\}$ is symplectic.

Lemma 2.1. (i) Suppose that $f, g \in \hat{A}M_\chi$ are even elements with $\{f, g\} = 1 + t$ for some $t \in M_\chi$. Then there exists $g' \in \hat{A}M_\chi^2$ such that $\{f, g + g'\} = 1$.

(ii) Suppose that $f, g \in \hat{A}M_\chi$ are even elements with $\{f, g\} = 1$. Denote by $V_1$ the symplectic vector space spanned by $f$ and $g$ with symplectic form $\omega_1(f, g) = 1$. Then we have the following isomorphism of Poisson algebras

$$\phi_1 : S[[V_1]] \otimes \hat{B}_1 \rightarrow \hat{A}; \quad a \otimes b \mapsto ab,$$

where $\hat{B}_1 = \ker(\text{ad}_f) \cap \ker(\text{ad}_g)$.

Before giving the proof we clarify the notations of $\deg(a)$, $\text{ad}_a$ and $\text{ad}(a)$. Fix a $\mathbb{Z}_2$-homogenous basis $x_1, x_2, \ldots, x_N$ of $v'$ of $x - \chi(x) | x \in v \subset S(v)$. Recall that $A = S[v]$, $\hat{A} = S[v_0]^\chi \otimes \bigwedge(v_1)$. Let $M_\chi$ be the maximal ideal of $\hat{A}$ generated by $x_1, x_2, \ldots, x_N$. For a super monomial $x_1^{i_1} \cdots x_N^{i_N} \in A$, its order is defined by $\sum_{k=1}^Ni_k$. For $a \in \hat{A}$, write $a = \sum_{i=1}^\infty a_i$ where $a_i$ is the sum of the monomials with order $i$. The degree $\deg(a)$ of $a$ is defined to be the minimal $i$ such that $a_i \neq 0$. For any $a \in A$, by $\text{ad}_a$ we mean the adjoint operator $\bullet \mapsto \{a, \bullet\}$ on $A$; for an associative algebra $(A, \circ)$ and $a \in A$, we denote by $\text{ad}(a)$ the super commutator operator $\bullet \mapsto [a, \bullet]$.

Now we are ready to prove Lemma 2.1.
Proof. (i) Suppose \( \deg(t) = r \). By the notation \( o(t) \), we denote an element with higher degree than \( t \). Set
\[
g_1 = g + \sum_{i=1}^{r+1} (-1)^i \frac{1}{i!} g^i \text{ad}_f^{-1}(t).
\]
Since
\[
\{f, g^i \text{ad}_f^{-1}(t)\} = ig^{i-1} \text{ad}_f^{-1}(t) + g^i \text{ad}_f(t) + o(t) \quad \text{for } i = 1, 2, 3, \ldots, r,
\]
we have \( \{f, g_1\} = 1 + t_1 \) for some \( t_1 \) with \( \deg(t_1) > r \). By the same procedure we obtain \( g_k, t_k \) with \( \deg(g_k) > \deg(g_{k-1}) \), \( \deg(t_k) > \deg(t_{k-1}) \) and \( \{f, g_k\} = 1 + t_k \) for all \( k = 2, 3, 4, \ldots \). Since \( \deg(g^i \text{ad}_f(t)) \geq \deg(t) \), the series \( \{g_k\}_{k=1}^{\infty} \) converges in \( \hat{A} \). Set \( g' = \lim_{k \to \infty} g_k - g \). It is easy to check that \( \deg(g') \geq 2 \) and \( \{f, g + g'\} = 1 \). Thus (i) follows.
(ii) For any \( a \in \hat{A} \) observe that
\[
a - \sum_{0 \leq i+j} \infty (-1)^{i+j} \frac{1}{i!j!} g^i f^j \text{ad}_f^i \circ \text{ad}_g^j(a) \in \hat{B}_1.
\]
Making the same observation as that for all \( \text{ad}_f^i \circ \text{ad}_g^j(a) \) in the above, we can see that \( \phi_1 \) is surjective. For \( \sum_{i,j \geq 1} g^i f^j \otimes a_{i,j} \in \ker(\phi_1) \) i.e.
\[
\sum_{i,j \geq 0} g^i f^j \cdot a_{i,j} = 0
\]
acting on both sides by \( \text{ad}_f^i \circ \text{ad}_g^j \), we have \( a_{i,k} = 0 \). So \( \phi_1 \) is also injective. \( \square \)

The following is an analogue of Lemma [2.1] for odd elements.

**Lemma 2.2.**

(i) Suppose that there are odd elements \( f, g \in \hat{A}M_\chi \) with \( \{f, g\} = 1 + t \) for some \( t \in M_\chi \). Then there exists an odd \( h \in \hat{A} \) such that \( \{h, h\} = 1 \).

(ii) For an odd \( h \in \hat{A} \) with \( \{h, h\} = 1 \), we have following isomorphism
\[
\phi : \mathbb{C}h \otimes \hat{B}_1 \rightarrow \hat{A}; \quad a \otimes b \mapsto ab,
\]
of Poisson algebras. Here \( \hat{B}_1 = \ker \text{ad}_h \).

**Proof.** (i) If either \( \{f, f\} \) or \( \{g, g\} \), say \( \{f, f\} \) is invertible. We write \( \{f, f\} = 1 + t_1 \) for some \( t_1 \in M_\chi \) and set \( h = \frac{1}{\sqrt{1+t_1}}f \) (here we interpret \( \sqrt{1+t} \) as its Taylor expansion about \( t = 0 \)). In the other case (namely \( \{f, f\} \) and \( \{g, g\} \in M_\chi \)), we write \( \{f + g, f + g\} = 1 + t_1 \) for some \( t_1 \in M_\chi \) and set \( h = \frac{1}{\sqrt{1+t_1}}(f + g) \). In both cases, we have \( \{h, h\} = 1 \).

For (ii) we only need to prove that the homomorphism \( \phi \) is bijective. Suppose \( f_1 + f_2 h = 0 \) where \( f_1, f_2 \in \hat{B}_1 \). Acting on both sides by \( \text{ad}_h \) we have \( f_2 = 0 \) and hence \( f_1 = 0 \). So the homomorphism \( \phi \) is injective. On the other hand we have
\[
\{h, hf\} = f - h\{h, f\} \quad \text{for all } f \in \hat{A}.
\]
It follows form the definition of even Poisson bracket that \( \{h, hf\}, \{h, f\} \in \ker(\text{ad}_h) \). So \( \phi \) is surjective. \( \square \)
Now we ready to give

**Proof of Theorem 1.3** We process by induction on \( \dim(V) = (2n|m) \). Suppose that \( m = 0 \). Fix a \( \mathbb{C}^\times \)-homogenous basis \( x_1, x_2, \ldots, x_N \) of \( \mathfrak{u}' \). Since the \( \mathbb{C}^\times \)-action is even, those basis are also \( \mathbb{Z}_2 \)-homogenous. By the definition of Poisson bi-vector, there exist even \( x_i \), \( x_j \) with \( \{ x_i, x_j \} = 1 + t \) for some \( t \in M_\chi \). Thus we have already find even homogenous \( f, g \in \mathbb{A}M_\chi \) with \( \{ f, g \} = 1 + t \) for some \( t \in M_\chi \). We can do \( \mathbb{C}^\times \)-homogeneously in the proof of Lemma 2.1 and obtain new \( \mathbb{C}^\times \)-homogenous even elements \( f, g \) with \( \{ f, g \} = 1 \). By Lemma 2.1 we have the following \( \mathbb{C}^\times \)-equivariant isomorphism of Poisson algebras

\[
\phi_1 : S[[V_1]] \otimes \hat{B}_1 \rightarrow \hat{A}; a \otimes b \mapsto ab.
\]

Thus we can complete the proof in this case by induction on \( \dim(V) \).

Suppose that the theorem holds for all \( p < m \). We may find odd \( \mathbb{C}^\times \)-homogenous \( f, g \in \mathbb{A}M_\chi \) such that \( \{ f, g \} = 1 + t \) with \( t \in M_\chi \).

**Case 1:** Either \( \{ f, f \} \) or \( \{ g, g \} \) is invertible. By the same argument as in case 1 of Lemma 2.2 we may find a \( \mathbb{C}^\times \)-homogenous odd \( h \) with \( \{ h, h \} = 1 \). Thus by Lemma 2.2 (ii), we have the following \( \mathbb{C}^\times \)-equivariant isomorphism of Poisson algebras

\[
\phi_1 : \mathbb{C}h \otimes \hat{B}_1 \rightarrow \hat{A}; a \otimes b \mapsto ab.
\]

Now the theorem follows by induction.

**Case 2:** \( \{ f, f \}, \{ g, g \} \in M_\chi \).

We write

\[
\{ f, g \} = 1 + t, \{ s_1, s_1 \} = t_1, \{ g, g \} = t_2 \text{ with } t, t_1, t_2 \in M_\chi.
\]

Set \( f_1 = f - gt_1/2 \). Then we have

\[
\{ f_1, g_1 \} = 1 + t^{(1)}, \{ f_1, f_1 \} = 1 + t^{(1)}, \{ g, g \} = 1 + t_2 \text{ with } t^{(1)}, t_1, t_2 \in M_\chi.
\]

It is straightforward to check that \( \deg(t^{(1)}) > \deg(t_1) \). Construct \( f_k, t^{(k)}_1 \) in the same way for \( k = 2, 3, 4, \ldots \). Take \( f_\infty = \lim_{k \to \infty} f_k \). We do the same procedure to the initial pair \( (f_\infty, g) \) and set \( g_\infty = \lim_{k \to \infty} g_k \). Now replace the \( f \) and \( g \) by \( f_\infty \) and \( g_\infty \) respectively. The new odd elements \( f, g \) are \( \mathbb{C}^\times \)-homogenous and satisfy

\[
\{ f, g \} = 1 + q, \{ f, f \} = 0, \{ g, g \} = 0 \text{ with } q \in M_\chi.
\]

Replacing \( f \) (resp. \( g \)) by \( \frac{f}{\sqrt{1 + q}} \) (resp. \( \frac{g}{\sqrt{1 + q}} \)) for \( i = 1, 2 \), we may assume that \( q = 0 \) in the above equality. From the construction, we see that \( f \) and \( g \) are \( \mathbb{C}^\times \)-homogenous. For \( h^+ = f + g, h^- = f - g \), we have \( \{ h^+, h^+ \} = 1, \{ h^-, h^- \} = 1, \{ h^-, h^+ \} = 0 \). Using Lemma 2.2 twice, we have \( \mathbb{C}^\times \)-equivariant isomorphism

\[
\phi : \hat{B}_1 \otimes \bigwedge (V_1) \rightarrow \hat{A}; a \otimes b \mapsto ab
\]

of Poisson algebras. Here \( V_1 = \mathbb{C}(h^+, h^-) = \mathbb{C}(f, g) \) and \( \hat{B}_1 = \ker(\text{ad}_f) \cap \ker(\text{ad}_g) \). Now we can complete the proof by induction. \( \square \)
Remark 2.3. Specially if \( V = T^*_\chi(\text{Spec}(A))(= \mathfrak{v}) \), the theorem is called formal Darboux theorem. Our proof still valid in case of smooth super manifolds and is more direct and simpler than the one outlined in [Kos].

Proof of Theorem 1.6

We prove the theorem by induction on \( m \). If \( m = 0 \), choose \( \mathbb{C}^x \)-homogenous even elements \( f, g \in M_\chi \) such that \( \{f, g\} = 1 \). So we may write
\[
[f, g] = \hbar^2 + a_2 \hbar^3 + o(\hbar^3)
\]
Where \( a_2 \in \hat{A} \), \( o(\hbar^3) \) means some element in \( \hat{A}[\hbar][\hbar^4] \) and \([f, g]\) is by definition the super commutator \( f \ast g - (-1)^{|f||g|}g \ast f \) for \( f, g \in \hat{A} \). Set
\[
g_1 = g - \sum_{i=1}^{\infty} (-1)^i g^i \text{ad}^{i-1}(f)(a_2) \hbar^2.
\]
We have
\[
[f, g_1] = \hbar^2 + a_2 \hbar^3 + o(\hbar^4)
\]
Define \( g_k \) iteratively for \( k = 2, 3, \ldots \). Replacing \( g \) by \( \lim g_k \), we have \([f, g] = \hbar^2\).

By the similar argument in the proof of Lemma 2.1 (ii), we have the following isomorphism of quantum algebras
\[
\Phi_{1,\hbar} : A_h^{\wedge_0}(V_1) \otimes \hat{B}_{1,\hbar} \rightarrow \hat{A}[\hbar]; a \otimes b \mapsto ab.
\]
(2.2)
where \( V_1 = \mathbb{C}\langle f, g \rangle \) and \( \hat{B}_{1,\hbar} = \ker(\text{ad}(f)) \cap \ker(\text{ad}(g)) \).

For any \( a \in \hat{A}[\hbar] \) we have
\[
a - \sum_{i,j} (-1)^{i+j} \frac{1}{i! j!} f^i g^j \text{ad}(f)^i \text{ad}(g)^j(a) \hbar^{-2(i+j)} \in \ker(\text{ad}(f)) \cap \ker(\text{ad}(g))
\]
(2.3)
Applying same procedure to \( \text{ad}(f)^i \text{ad}(g)^j(a) \) for all \( i, j \in \mathbb{N} \), we have that \( \Phi_{1,\hbar} \) is surjective. By a similar argument as in Lemma (ii) 2.2, we show that \( \Phi_{1,\hbar} \) is injective. Thus in the case of \( m = 0 \), the proof is completed by induction on \( n \).

Now suppose \( m > 0 \).

Case 1: There exists an odd \( \mathbb{C}^x \)-homogenous \( f \) such that \( \{f, f\} = 1 + t \), with \( t \in M_\chi \). By Lemma 2.2 we may assume that \( \{f, f\} = 1 \) and have
\[
[f, f] = \hbar^2 + a_2 \hbar^3 + o(\hbar^3).
\]
Set \( f_1 = f - \frac{1}{2} f \ast a_2 \hbar^2 \). Since \([f, [f, f]] = 0\), we have \([f, a_2] = 0\). This implies
\[
[f_1, f_1] = \hbar^2 + a_2 \hbar^3 + o(\hbar^4).
\]
Define \( f_k \) for \( k = 2, 3, 4 \ldots \) iteratively and replace \( f \) by \( \lim f_k \). For the new \( f \), we have \([f, f] = \hbar^2\). By the same argument as in non-quantum case, we have the following \( \mathbb{C}^x \)-equivariant homomorphism of quantum algebras
\[
\Phi_{1,\hbar} : \hat{B}_{1,\hbar} \otimes \bigwedge (\mathbb{C}f)[[\hbar]] \rightarrow \hat{A}[[\hbar]] \; ; \; a \otimes b \mapsto a \ast b.
\]
Here \( \hat{B}_{1,\hbar} = \ker(\text{ad}(f)) \).
Case 2: Otherwise, by case 2 in the proof of Theorem 1.3, we may find $C^\times$-homogenous $f, g \in A_1$ with
\[\{f, g\} = 1, \{f, f\} = 0, \{f, g\} = 0\]
Using similar arguments as in the proof of Theorem 1.3 and the case 1, we may construct new $f, g \in A[[\hbar]]$ such that
\[\{f, g\} = \hbar^2, \{f, f\} = 0, \{g, g\} = 0\].
Thus we may construct a $C^\times$-equivariant isomorphism of quantum algebras as case 2 in the proof of Theorem 1.3. The proof is completed by induction on $\dim(V) = (2n|m)$. □

3. Realization of W-superalgebras via Darboux-Weinstein theorem

In the next two subsections we recall the results from [Lo2] on the Rees algebra of the filtered associative algebras and completion of $U$. Although those results are given in non-super cases, they are still hold in the super cases.

3.1. Rees algebras. Let $A$ be an associative algebra with an increasing filtration $F_iA, i \in \mathbb{Z}$ such that $\bigcup_{i \in \mathbb{Z}} F_iA = A, \bigcap_{i \in \mathbb{Z}} F_iA = 0$. Let $R_h(A) = \bigoplus_{i \in \mathbb{Z}} F_iA\hbar^i$ and view it as a subalgebra of $A[[\hbar]]$. The algebra $R_h(A)$ is called Rees algebra of $A$. For an ideal $I$ in $C\llbracket\hbar\rbracket$-algebra $B$, define $R_h(I) = \bigoplus_{i \in \mathbb{Z}} (F_i(A) \cap I)\hbar^i$.

Definition 3.1. ([Lo2, Definition 3.2.1]) An ideal $I$ in $C\llbracket\hbar\rbracket$-algebra $B$ is called $\hbar$-saturated if $I = \hbar^{-1} I \cap \mathbb{C}\llbracket\hbar\rbracket$.

Proposition 3.2. ([Lo2, Proposition 3.2.2]) The map $I \mapsto R_h(I)$ establishes a bijection between the set $\text{id}(A)$ and the set of $\hbar$-saturated ideals of $R_h(A)$.

Let $v = v_0 + v_1$ be a superspace with a $C^\times$-action. Suppose that there is a Poisson bracket $\{\cdot, \cdot\}$ on $\hat{A} = S[[v]]$ and $k \in \mathbb{Z}$ with $\{t \cdot f, t \cdot g\} = t^k \{f, g\}$ for all $f, g \in \hat{A}$. Let $(\hat{A}_h = S[[v, \hbar]], \ast)$ be a $C^\times$-equivariant quantization of $\hat{A}$ as in §1.1.2. Assume that $A_h = S[[v, \hbar]]$ form a subalgebra of quantum algebra $S[[v, \hbar]]$, i.e. $A_h$ is closed under the product $\ast$.

For $f, g \in \hat{A}$ write
\[f \ast g = \sum_i D_i(f, g)\hbar^i,\]
where $D_i : \hat{A} \otimes \hat{A} \to \hat{A}$ is a linear map determined by the product $\ast$ in the quantum algebra $\hat{A}[[\hbar]]$.

There is a product $\circ$ on $\hat{A}$ given by
\[f \circ g = \sum_i D_i(f, g).\]
(3.1)

We denote by $\hat{A}$ the algebra from it. Let $A \subset \hat{A}$ be the subalgebra from $A := S(v)$. There is an action of $C^\times$ on the Rees algebra $R_h(A)$ by
\[t \cdot a_h = t^a_i h^i\]
(3.2)
for all $a_i h^i \in F_i(A)h^i$ and $t \in C^\times$. 
Take \( I \in \mathfrak{d}(A) \) and set \( I = \text{gr}(I) \). Let \( \tilde{\mathcal{I}}_h \) be the closure of \( R_h(I) \) with respect to the \( M_{0,h} \)-adic topology on \( \hat{A}_h \).

**Proposition 3.3.** ([Lo2, Proposition 3.2.10])

1. The ideal \( \tilde{\mathcal{I}}_h \) is \( h \)-saturated.
2. The classical part of \( \tilde{\mathcal{I}}_h \) coincides with the closure \( \tilde{I} \) of \( I \) in \( S[[v]] \). Here the classical part of an ideal of \( \hat{A}_h \) means its image under the projection \( \hat{A}_h \to \hat{A} \) by specializing \( h \) to 0.
3. Suppose that the grading on \( v \) is positive. Let \( \mathcal{I}_h \) be a closed \( h \)-saturated, \( \mathbb{C}^\times \)-stable ideal of \( R_h(A) \), then there exists a unique \( I \in \mathfrak{d}(A) \) such that \( \mathcal{I}_h \cap A[h] = R_h(I) \).

### 3.2. Completions of \( \mathcal{U} \)

Let \( \mathcal{U} \) be the universal enveloping algebra of the Lie superalgebra \( g \). Let \( g = \oplus g(i) \) be the good \( \mathbb{Z} \)-grading in [1,2]. The Kazhdan action of \( \mathbb{C}^\times \) on \( g \) is given by \( t \cdot y = t^{i+2}y \) for all \( t \in \mathbb{C}^\times \) and \( y \in g(i) \). Since \( \chi \in g^\ast \) is an element of degree \(-2\), \( \chi \) is a fixed point of the induced \( \mathbb{C}^\times \)-action on \( g^\ast \). Extend the Kazhdan action to \( \mathcal{U} \) canonically.

#### 3.2.1. Let \( A_h = S[g][h] \) (see Example [1.5] for the definition), \( A = S[g] \) and \( \pi : A_h \to A \) be the algebra homomorphism given by taking \( h = 0 \). Let \( M_\chi \) be the maximal ideal of \( A = S[g] \) corresponding to the point \( \chi \in g^\ast_0 \) and \( A^{\wedge^\chi} \) be the completion of \( A \) with respect to the ideal \( M_\chi \). Set

\[
(A_h)^{\wedge^\chi} = \lim_n A_h/M_{\chi,h}^n
\]

where \( M_{\chi,h} \) is the two-sided ideal of \( A_h \) generated by \( \pi^{-1}(M_\chi) \). We extend the Kazhdan action on \( A_h \) to \( (A_h)^{\wedge^\chi} \) by \( t \cdot h = th \). It is easy to check that \( (A_h)^{\wedge^\chi} \) is a \( \mathbb{C}^\times \)-equivariant quantization of the Poisson algebra \( A^{\wedge^\chi} \). We simplify the notations \( A^{\wedge^\chi} \) and \( A_h^{\wedge^\chi} \) by \( A \) and by \( A_h \), respectively.

#### 3.2.2. Set

\[
\tilde{g}_e = \begin{cases} g_e & \text{if dim}(g(-1)) \text{ is even;} \\ g_e + \mathbb{C} \Theta & \text{if dim}(g(-1)) \text{ is odd.} \end{cases}
\] (3.3)

Choose a homogenous (with respect to the Kazhdan action) basis

\[
v_{\pm 1}, v_{\pm 2}, \ldots, v_{\pm n}, v_{2n+1}, \ldots, v_{2n+m}
\]

of \( g' \) such that \( v_1, v_2, \ldots, v_n \) \( (\text{resp. } v_{-1}, v_{-2}, \ldots, v_{-n}) \) form a basis of \( m' \) \( (\text{resp. } (m')^\ast) \) and \( v_{2n+1}, \ldots, v_{2n+m} \) form a basis of \( \tilde{g}'_e \). Let \( d_i \) denote the degree of \( v_i \), \( i \in \{-1, -2, \ldots, -n; 2n + 1, \ldots, 2n + m\} \). Denote by \( A^\circ \) the subalgebra of \( \hat{A} \) consisting of elements \( \sum_{i<n} f_i \) where \( f_i \) is a homogenous power series with degree \( i \) (with respect to the Kazhdan grading). Let \( \mathcal{I}^\circ(k) \) be the ideal of \( A^\circ \) consisting of all those \( a \) satisfying that any monomial in \( a \) has the form \( v_{i_1} \circ \cdots \circ v_i \) with \( v_{i+\ldots+i+1} \in m' \).

We need to introduce some new notion before giving the next lemmas. Generally, for an associative algebra \( A \) with \( \mathbb{C}^\times \)-action, by \( A_{\mathbb{C}^\times-\text{fin}} \) we mean the \( \mathbb{C}^\times \)-local finite part of \( A \), i.e. the sum of all finite-dimensional \( \mathbb{C}^\times \)-stable subspaces \( \underline{A} \subset A \).
Lemma 3.4. (i) The map 
\[ A[h] \to A[h, h^{-1}]; \sum_{i,j} f_i h^j \mapsto \sum_{i,j} f_i h^{i+j} \]
is a $\mathbb{C}^x$-equivariant monomorphism of $\mathbb{C}[h]$-algebras. Its image coincides with $R_h(A)$.

(ii) The map 
\[ S[[\mathfrak{g}, h]]_{\mathbb{C}^x,\text{fin}} \to A^\circ[h, h^{-1}]; \sum_{i,j} f_i h^j \mapsto \sum_{i,j} f_i h^{i+j} \]
is a $\mathbb{C}^x$-equivariant monomorphism of $\mathbb{C}[h]$-algebras. Its image coincides with $R_h(A^\circ)$.

Here $f_i$ is a $\mathbb{C}^x$-homogenous element in $A$ with $t \cdot f_i = t^i f_i$ for all $t \in \mathbb{C}^x$. The $\mathbb{C}^x$-action on the images of the above two maps are given by (3.2).

Lemma 3.5. Let $\tilde{m}$ be a subspace in $A^\circ$, which has a basis in the form of $v_i + u_i$, $i = 1, \cdots, m$, with $u_i \in F_0(A^\circ) \cup g^2A^\circ + F_{-2}(A^\circ)$. Then $A^\circ \tilde{m} \in \mathcal{T}(1)$ and for any $k \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $A^\circ \tilde{m}^l \in \mathcal{T}(k)$.

Lemma 3.6. For the universal algebra $U \subset A^\circ$, the systems $Um^k$ and $\mathcal{J}(k) := \mathcal{T}(k) \cap U$ are compatible, i.e. for any $k \in \mathbb{N}$ there exist $k_1, k_2$ such that $\mathcal{J}(k_1) \subset Um^{k_2}, Um^{k_2} \subset \mathcal{J}(k)$.

The above-mentioned three lemmas are super versions of Lemmas 3.2.5, 3.2.8 and 3.2.9 in [Lo2], which will be used later. We omit the proofs of them, which are exactly the same as in the non-super case.

3.3. Construction of $W_\chi$. Recall that the Poisson bivector $\Pi$ on the closed point $\chi$ of $\text{Spec}(A)$ is given by $(x, y) \mapsto \chi([x, y])$ for all $x, y \in \mathfrak{g}$. It is easy to check that $\Pi$ is non-degenerate on $[g, f] \subset T^*_\chi(\text{Spec}(A))$. Thus $\Pi$ is also non-degenerate on $V = \tilde{m} \oplus \tilde{m}^*$ (see [1.2.2] for the notations). Since there is a $\mathbb{C}^x$-stable decomposition $\mathfrak{g} = V \oplus \tilde{g}_e$ with respect to the Kazhdan action, we have $\hat{B}_h = S[[\tilde{g}_e, h]]$ as vector superspaces. By Theorem 1.6, we have the following isomorphism
\[ \Phi_h : A_{h}^\circ(V) \hat{\otimes}_{\mathbb{C}[h]} \hat{B}_h \to \hat{A}_h \]
of quantum algebras.

Now we set
\[ W_\chi = \frac{(\hat{B}_h)_{\mathbb{C}^x,\text{fin}}}{(h-1)(\hat{B}_h)_{\mathbb{C}^x,\text{fin}}}. \]
We will show that $W_\chi$ is isomorphic to the $W$-superalgebra $U(\mathfrak{g}, e)$. Let us discuss the relation between $W_\chi$ and the transverse slices of super nilpotent orbit $G \cdot \chi$ before doing that. Since there is a $\mathbb{C}^x$-stable decomposition $\mathfrak{g} = V \oplus \tilde{g}_e$ with respect to the Kazhdan action, we have $\hat{B}_h = S[[\tilde{g}_e, h]]$ as vector superspaces. Since all $\mathbb{C}^x$-homogenous elements in $\tilde{g}_e$ have positive Kazhdan grading, we have $(\hat{B}_h)_{\mathbb{C}^x,\text{fin}} = S[\tilde{g}_e, h]$ as vector superspaces. The product $*$ on $(\hat{B}_h)_{\mathbb{C}^x,\text{fin}}$ gives $S[\tilde{g}_e, h]$ a quantum algebra structure. Thus it is clear from the construction that
\( \mathcal{W}_\chi \) is a filtered algebra with \( \text{gr}(\mathcal{W}_\chi) = S[\mathfrak{g}_e] \), if \( \dim \mathfrak{g}(-1)_1 \) is even; and \( \text{gr}(\mathcal{W}_\chi) = S[\mathfrak{g}_e] \otimes \mathbb{C}[\Theta] \) if \( \dim \mathfrak{g}(-1)_1 \) is odd, where \( \mathbb{C}[\Theta] \) is the exterior algebra generated by one element \( \Theta \). This is a PBW theorem for \( \mathcal{W}_\chi \) (compare with theorem 0.1 in [ZS1]). This provides \( S[\mathfrak{g}_e] \) with an even Poisson bracket, and \( \mathcal{W}_\chi \) is a filtered quantization of \( \mathcal{W}_\chi \) of this Poisson algebra (c.f [CG] §1.3).

We will write \( \mathcal{W}_\chi \) as \( \mathcal{W} \) in the sequence for simplicity.

### 3.4. The realization of \( \mathcal{U}(\mathfrak{g}, e) \) via \( \mathcal{W} \).

**Lemma 3.7.** Let \( v_i, i = \pm 1, \pm 2, \ldots, \pm n \) be a basis of the symplectic space \( V \subset \mathfrak{g}' \) with \( (v_i, v_{-j}) = \delta_{i,j} \). Then \( v_i - \Phi^{-1}(v_i) \in F_{d_i}(\hat{A}_h) \cap \mathfrak{g}'^2 \hat{A}_h \) for \( i = \pm 1, \pm 2, \ldots, \pm n \). We also have \( \Phi^{-1}(x) - x \in V \hat{A}_h \) for all \( x \in \mathfrak{g}' \).

**Proof.** Note that \( [v_1, v_{-1}] = h^2 + \mathfrak{g}' \hat{A}_h \). We can modify \( v_1, v_{-1} \) as in the proof of Theorem 1.6 and get new elements \( \bar{v}_1, \bar{v}_{-1} \) with \( [\bar{v}_1, \bar{v}_{-1}] = \hbar^2 \) and \( \bar{v}_i = v_i + u_i \) for some \( u_i \in \mathfrak{g}'^2 \hat{A}_h \) for \( i = 1, -1 \). Set \( V_1 = \mathbb{C}(\bar{v}_1, \bar{v}_{-1}), \hat{B}_1[[\hbar]] = \ker(\text{ad}(\bar{v}_1)) \cap \ker(\text{ad}(\bar{v}_{-1})) \). By Theorem 1.6 we have the following isomorphism

\[
\Phi_{1,\hbar} : \begin{cases} 
S[[V_1]][[\hbar]] \otimes \hat{B}_1[[\hbar]] & \to \hat{A}_h, \quad \text{if } v \in \mathfrak{g}'_0, \\
\wedge(V_1)[[\hbar]] \otimes \hat{B}_1[[\hbar]] & \to \hat{A}_h, \quad \text{if } v \in \mathfrak{g}'_1
\end{cases}
\]

of quantum algebras.

By the arguments in the proof of Theorem 1.6 for \( i = \pm 2, \ldots, \pm n \), there exist \( u_i^{(1)} \in \mathfrak{g}'^2 \hat{A}_h \) such that \( u_i^{(1)} + v_i \in \hat{B}_1[[\hbar]] \). It is easy to check that

\[
[u_i^{(1)} + v_i, u_j^{(1)} + v_j] - \delta_{i,-j}\hbar^2 \in \mathfrak{g}' \hat{A}_h.
\]

Now we can accomplish the proof of the first statement by induction. The second statement follows from the construction of the isomorphism \( \Phi_\hbar \). \( \square \)

Denote by \( \hat{A}_1 \) the associative algebra from the quantum algebra \( \mathcal{A}_h^\otimes(V) \hat{\otimes} \mathbb{C}[[\hbar]] \hat{B}_h \) as in the equation (3.1). Construct \( \mathcal{A}_1^\otimes \) in the same way as \( \mathcal{A}_1^\otimes \), which is actually a subalgebra of \( \hat{A}_1 \) from \( A_1 := S[V] \otimes \mathbb{C}[\mathfrak{g}_e] \). Recall that \( \mathfrak{m} \) is the Lagrangian subspace of \( V \) spanned by \( v_{-1}, \ldots, v_{-n} \). Construct \( \mathcal{J}_1^\otimes(k) \) from \( \mathfrak{m} \) in the same way as \( \mathcal{J}_1^\otimes(k) \) from \( \mathfrak{m}' \) in §3.2.

**Theorem 3.8.** \( \mathcal{W} \) and \( \mathcal{U}(\mathfrak{g}, e) \) are isomorphic as associative algebras.

The proof of the above theorem is basically somewhat as in the non-super case in [Lo2]. We recall it for the reader’s convenience.

**Proof.** By Lemma 3.4 we have the following isomorphisms

\[
(\hat{A}_h)_{\mathcal{C}^{\times-\text{fin}}} \cong \mathcal{R}_h(\mathcal{A}_1^\otimes); \quad \sum_i f_i h^i \mapsto f_i h^{i+j}
\]

and

\[
(\mathcal{A}_h^\otimes(V) \hat{\otimes} \mathbb{C}[[\hbar]]) \hat{B}_h)_{\mathcal{C}^{\times-\text{fin}}} \cong \mathcal{R}_h(\mathcal{A}_1^\otimes); \quad \sum_i f_i h^i \mapsto f_i h^{i+j}
\]
of $\mathbb{C}[h]$ algebras. Thus restricting $\Phi_h$ to the $\mathbb{C}^\times$-local finite part of $\hat{A}_h$, we obtain an isomorphism $\Phi_h^\circ : R_h(\mathcal{A}^\circ) \to R_h(\mathcal{A}^\circ)$ of $\mathbb{C}[h]$-algebras.

Taking $h = 1$, we have a $\mathbb{C}^\times$-equivariant isomorphism $\Phi^\circ : \mathcal{A}^\circ \to \mathcal{A}_1^\circ$ of algebras (with respect to “$\circ$”). Lemmas 3.5 and 3.7 imply $\Phi^\circ(J_1^\circ(1)) \subset J_1^\circ(1)$ and $\Phi^\circ(\mathcal{A}^\circ) = \mathcal{A}_1^\circ$. Hence $\Phi^\circ(J_1^\circ(1)) = J_1^\circ(1)$.

From the definition (1.1), it follows that $U(\hat{g}, e) \cong (\mathcal{A}^\circ/J_1^\circ(1))^{\mathcal{J}^\circ(1)}$. By the construction of $W$, we know that $(\mathcal{A}_1^\circ/J_1^\circ(1))^{\mathcal{J}^\circ(1)} \cong W$.

Thus finally we have

$$U(\hat{g}, e) \cong (\mathcal{A}^\circ/J_1^\circ(1))^{\mathcal{J}^\circ(1)} \cong (\mathcal{A}_1^\circ/J_1^\circ(1))^{\mathcal{J}^\circ(1)} \cong W.$$ 

\[
\square
\]

Proof of Theorem 1.7 By Lemma 3.6 and the proof of Theorem 3.8 the systems $\Phi^\circ(\mathcal{A}(V) \otimes W/(\mathcal{A}(V) \otimes W)\mathfrak{m}^k)$ and $U\mathcal{m}^k$ are compatible. Thus the morphism $\Phi^\circ$ can be extended to an isomorphism

$$\Phi : \mathcal{A}(V)_{\mathfrak{m}}^\wedge \otimes W \cong U_{\mathfrak{m}}^\wedge$$

of topological algebras. \[\square\]

As a corollary to Theorem 3.8 along with the arguments in [338] we reobtain the following result which is a main result of [ZS1].

Corollary 3.9. ([ZS1] Theorem 0.1) Associated with a basic classical Lie superalgebra $\hat{g}$ and a nilpotent element $e \in \hat{g}_0$, the following statements hold

1. $\text{gr} U(\hat{g}, e) \cong S(\hat{g}_e)$ as $\mathbb{C}$-algebras when $\dim \hat{g}(-1)\iota$ is even.

2. $\text{gr} U(\hat{g}, e) \cong S(\hat{g}_e) \otimes \mathbb{C}[\Theta]$ as $\mathbb{C}$-algebras when $\dim \hat{g}(-1)\iota$ is odd.

Remark 3.10. In [ZS3], the authors introduced the so-called refined $W$-algebra $U'(\hat{g}, e)$, which is by definition equal to $(U/I_\lambda)^{ad\mathfrak{m}}$ with $\mathfrak{m}$ being either $\mathfrak{m}$ itself or its one-dimensional extension (the occurrence is dependent on the parity of $\dim \hat{g}(-1)\iota$, taking the former if even, and taking the latter if odd).

They showed that $U'(\hat{g}, e)$ is a subalgebra of $U(\hat{g}, e)$ and satisfies that $\text{gr} U'(\hat{g}, e) \cong S(\hat{g}_e)$. The arguments in the present paper are still available to this refined version of the finite $W$-algebras, by some correspondingly modifying in the construction.

3.5. A relation between finite $W$-superalgebras and $W$-algebras. Let $\hat{g} = \hat{g}_0 \oplus \hat{g}_1$ be a basic Lie superalgebra and $e \in \hat{g}_0$ be a nilpotent. The construction of $W$ in the present paper is inspired by its non-super counterpart given in [Lo4]. We briefly describe it as below.

Let $\hat{A}_{0,h} = S[\hat{g}_0]^\wedge_h$ be the $\mathbb{C}^\times$-equivariant quantization of $\hat{A}_0 = S[\hat{g}_0]^\wedge$ constructed in the same way as in [332]. Restriction of the Poisson bivector on $V_0 = T^*(G_0\cdot\chi)$ is non-degenerate. By the eqDW theorem we have a $\mathbb{C}^\times$-equivariant isomorphism

$$\Phi_{0,h} : \hat{A}_{0,h} \to S[[V_0]]_h \otimes B_{0,h}.$$
of quantum algebras. Here the notations in the above isomorphism have the same meaning as in the previous sections. Now set
\[ W_0^{\mathbb{C}} = \left( \hat{B}_{0,h} \right)_{C^\infty,\text{fin}} / (h - 1)(\hat{B}_{0,h})_{C^\infty,\text{fin}}. \]

By the same argument as in the proof of Theorem 3.8, we have that \( W_0 \) is isomorphic to the finite \( W \)-algebra \( \mathfrak{u}(\mathfrak{g}_0, e) \). The above fact is first proved in [Lo3] relying on the main result of [Lo3], which is based on the Fedosov quantization and algebraic invariant theory. Those theories are not necessary to the arguments in the present paper.

Applying Theorem 1.6 to \( \hat{A}_h = S[\mathfrak{g}]^{h_x} \) and \( V_0 \), we have the following isomorphism
\[ \hat{A}_h \rightarrow S[[V_0]]_h \otimes \hat{C}_h \]
of quantum algebras. Here \( \hat{C}_h \) is defined similarly to \( \hat{B}_h \) in Theorem 1.6. Denote by
\[ A_{\mathbb{C}} = \left( \hat{C}_h \right)_{C^\infty,\text{fin}} / (\hat{C}_h)_{C^\infty,\text{fin}}(h - 1). \]

The following theorem relates the finite \( W \)-algebras and \( W \)-superalgebras.

**Theorem 3.11.**

(i) We have an embedding \( W_0 \hookrightarrow A_{\mathbb{C}} \). The later is finitely generated over the former.

(ii) There exists an isomorphism \( \Psi : \text{Cl}(V_1) \otimes W \rightarrow A_{\mathbb{C}} \) of associative superalgebras. Here \( V_1 \) is the odd part of \( V = \mathfrak{m} \oplus \mathfrak{m}^* \), \( \text{Cl}(V_1) \) is the Clifford algebra from \( (V_1, \chi([\cdot, \cdot])) \).

**Proof.** (i) Let \( v_1, v_{-1}, \ldots, v_l, v_{-l}, \ldots, v_n, v_{-n} \) (resp. \( v_{l+1}, v_{l-1}, \ldots, v_n, v_{-n} \)) be a basis of \( V_0 \) (resp \( V_1 \)) with \( \omega(v_i, v_j) = \delta_{i,j} \). We construct \( \breve{v}_1, \breve{v}_{-1}, \ldots, \breve{v}_n, \breve{v}_{-n} \) as in the proof of Lemma 3.7 by the indicated order. Since we have \( \breve{v}_i \in \hat{B}_{0,h} \) for all \( i = \pm 1, \ldots, \pm l \), both of \( \Phi_h \) and \( \Phi_{0,h} \) may be given by those \( \breve{v}_i \)s as before. Thus we have the first statement of (i). Since \( \breve{v}_i \in \hat{C}_h \) for all \( i = \pm(l + 1), \ldots, \pm n \), we have the following isomorphism
\[ \Psi_h : \text{Cl}(V_1)_h \otimes \hat{B}_h \rightarrow \hat{C}_h \]
of quantum algebras by a similar argument as in the proof of Theorem 1.6. Now the statement (ii) follows from taking \( C^\infty \)-local finite part and specializing \( h \) to 1. Note that \( \text{gr}(A_{\mathbb{C}}) \) is isomorphic to \( S((\mathfrak{g}_0)_e) \otimes \wedge(\mathfrak{g}_1) \) and \( \text{gr}(W_0) \) is isomorphic to \( S((\mathfrak{g}_0)_e) \). Here \( (\mathfrak{g}_0)_e \) is the centralizer of \( e \) in \( \mathfrak{g}_0 \). Thus the last statement of (i) follows.

**Corollary 3.12.** The finite \( W \)-superalgebra \( W \) has finite-dimensional representations. For a regular nilpotent element \( e \in \mathfrak{g}_0 \), any irreducible representation of \( W \) is finite-dimensional.
Proof. The first statement follows from the fact that \( \mathcal{W} \) has finite dimensional modules. Since \( \mathcal{W}_0 \) is isomorphic to the center of \( \mathcal{U}(\mathfrak{g}_0) \), the second statement of Theorem 3.11 (i) implies that of the corollary. \( \square \)

Remark 3.13. The statement (ii) of Theorem 3.11 provides a bijection between the sets of finite-dimensional irreducible modules of \( \mathcal{A}_i \) and \( \mathcal{W} \). For some special \( e \in \mathfrak{g} \), we will give a classification of \( \text{Irr}^{\text{fin}}(\mathcal{W}) \) by using such a correspondence in a forthcoming work [SXZ].

4. On representations of \( \mathcal{W} \)

4.1. Skryabin’s equivalence. We say that a \( \mathcal{U} \) module is Whittaker if \( x \) acts locally nilpotent on \( M \) for all \( x \in \mathfrak{m}' \). For \( Q_\chi = \mathcal{U}/ \mathcal{U}\mathfrak{m}' \), we have the following functors between the category \( \mathcal{C} \) of Whittaker \( \mathcal{U} \)-modules and the category of \( \mathcal{U}(\mathfrak{g}, e) \)-modules.

\[
\begin{align*}
\mathcal{C} & \rightarrow \mathcal{U}(\mathfrak{g}, e)\text{-mod}; \\
M & \mapsto M^{\mathfrak{m}'}, \\
\mathcal{U}(\mathfrak{g}, e)\text{-mod} & \rightarrow \mathcal{C}; \\
N & \mapsto S(N) := Q_\chi \otimes N.
\end{align*}
\]

Theorem 4.1. The functors given above are a pair of quasi-inverse equivalence.

This theorem was first stated in [ZS1]. Following [Lo2] we give an alternative proof to it by using the Poisson realization given above.

Proof. Let \( M \) be a Whittaker \( \mathfrak{g} \) module. Since \( \mathfrak{m}' \) acts locally nilpotent on \( M \), we can view \( M \) as a continuous \( \mathcal{U}^\wedge \mathfrak{m}' \) module. Thus we can view \( M \) as an \( \mathcal{A}_V(\mathcal{W}) \)-module via the isomorphism \( \Phi \). Now we have

\[
M^{\mathfrak{m}'} = M^3 = M^{\Phi^{-1}(3)} = M^{\mathfrak{m}_1} = M^{\bar{\mathfrak{m}}}. \tag{4.1}
\]

Here \( \mathfrak{I} \) (resp. \( \mathfrak{I}_1 \)) is the closure of \( \mathfrak{I} \) (resp. \( \mathfrak{I}_1 \)) in \( \mathcal{U}^\wedge \mathfrak{m} \) (resp. \( \mathcal{A}_V(\mathcal{W}) \)).

For such an \( \mathcal{A}_V(\mathcal{W}) \)-module \( M \), by Lemma 2.1 and its proof we have the following \( \mathcal{A}_V(\mathcal{W}) \)-module isomorphism

\[
S(\bar{\mathfrak{m}}^*) \otimes M^{\bar{\mathfrak{m}}} \longrightarrow M \tag{4.2}
\]

\[
x \otimes m \mapsto x \cdot m \quad \text{for } x \in S(\bar{\mathfrak{m}}^*) \text{ and } m \in M.
\]

It is clear that \( Q_\chi = \mathcal{U}/ \mathcal{U}\mathfrak{m}' = \mathcal{U}^\wedge/ \mathcal{U}\mathfrak{I}_1 \). Hence by (4.1) we have \( \mathcal{W} = Q_\chi^{\bar{\mathfrak{m}}} \). Thus Isomorphism (4.2) implies \( Q_\chi = S(\bar{\mathfrak{m}}^*) \otimes Q_\chi^{\bar{\mathfrak{m}}} = S(\bar{\mathfrak{m}}^*) \otimes \mathcal{W} \).

Finally we have

\[
S(M^{\mathfrak{m}'}) = Q_\chi \otimes_\mathcal{W} M^{\mathfrak{m}} = S(\bar{\mathfrak{m}}^*) \otimes \mathcal{W} \otimes_\mathcal{W} M^{\mathfrak{m}} = M
\]

\[
(Q_\chi \otimes N)^{\mathfrak{m}'} = (S(\bar{\mathfrak{m}}^*) \otimes \mathcal{W} \otimes_\mathcal{W} \otimes N)^{\mathfrak{m}} = N. \quad \square
\]

For an associative algebra \( \mathcal{A} \) and an \( \mathcal{A} \)-module \( M \), denote by \( \text{GK}_\mathcal{A}(M) \) the Gelfand-Kirillov dimension of \( M \). The following theorem gives the behavior of GK dimension under the Skryabin’s equivalence.
Lemma 4.2. For any $\mathcal{W}$-module $N$, we have
\[ \text{GK}_{\mathcal{U}(g)}(S(N)) = \text{GK}_{\mathcal{U}}(S(N)) = \text{GK}_{\mathcal{W}}(N) + \dim(m') \]

Proof. Let $\mathcal{A}, \mathcal{A}_1$ have the same meaning as in Subsection 4.3. Let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ generated by $g_0$. Choose a finite-dimensional subspace $M_0$ generating $\mathcal{A}_0$ module $S(M)$. Then the lemma follows from the simple fact
\[ \lim_{n \to \infty} \frac{\ln \dim((F_nA)M_0)}{\ln n} = \lim_{n \to \infty} \frac{\ln \dim(F_n(A_0)M_0)}{\ln n}, \]
along with the argument in the proof of [Lo2, Proposition 3.3.5]. \[\square\]

4.2. Maps $\bullet^\dagger$ and $\bullet$. For an associative algebra $\mathcal{A}$, we denote by $\text{id}(\mathcal{A})$ the set of its two-sided ideals. An ideal $\mathcal{I} \subset \mathcal{A}$ is said to be primitive if it is the annihilator of an irreducible $\mathcal{A}$ module. The set of primitive ideals of $\mathcal{A}$ is denoted by $\text{Prim}(\mathcal{A})$. Obviously $\text{Prim}(\mathcal{A}) \subset \text{id}(\mathcal{A})$. Let $\text{Prim}^{\cofin}(\mathcal{A})$ be the set of primitive ideals of $\mathcal{A}$ with finite codimension. It is well known that there is a bijection between the set of finite-dimensional irreducible $\mathcal{A}$ modules and $\text{Prim}^{\cofin}(\mathcal{A})$.

From now on, we assume that $g$ is a basic Lie superalgebra of type I. In this section we establish correspondences $\bullet^\dagger$ and $\bullet$ between $\text{id}(\mathcal{W})$ and $\text{id}(\mathcal{U})$ following Losev in the non-super case. Then we study the finite-dimensional irreducible representations of $\mathcal{W}$ by them.

We simplify the notation $\mathcal{A}(V)^\wedge \otimes \mathcal{W}$ by $\mathcal{A}(\mathcal{W})$. Following [Lo2], we construct a map $\text{id}(\mathcal{W}) \to \text{id}(\mathcal{U})$, $\mathcal{I} \mapsto \mathcal{I}^\dagger$ as follows. For $\mathcal{I} \in \text{id}(\mathcal{W})$ define an ideal of $\mathcal{A}(\mathcal{W})$ by
\[ \mathcal{A}(\mathcal{I})^\wedge = \lim(\mathcal{A}(\mathcal{W})(\mathcal{I}) + \mathcal{A}(\mathcal{W})\bar{m}^k)/\mathcal{A}(\mathcal{W})\bar{m}^k. \]

Set $\mathcal{I}^\dagger = \mathcal{U} \cap \Phi(\mathcal{A}(\mathcal{I})^\wedge)$. Now we have the following properties of the map $\bullet^\dagger$.

Theorem 4.3. (i) $(\mathcal{I}_1 \cap \mathcal{I}_2)^\dagger = \mathcal{I}_1^\dagger \cap \mathcal{I}_2^\dagger$, for all $\mathcal{I}_1, \mathcal{I}_2 \in \text{id}(\mathcal{W})$.
(ii) For any $\mathcal{W}$-module $N$, $\text{Ann}(N)^\dagger = \text{Ann}(S(N))$.
(iii) $\iota(\mathcal{I}^\dagger \cap \mathcal{Z}(g)) = \mathcal{I} \cap \iota(\mathcal{Z}(g))$. Here $\mathcal{Z}(g)$ is the center of $\mathcal{U}$ and $\iota$ is the natural inclusion $\mathcal{Z}(g) \hookrightarrow \mathcal{W}$.
(iv) $\mathcal{I}^\dagger$ is prime provided $\mathcal{I}$ is prime. The ideal $\mathcal{I}^\dagger$ is primitive if $\mathcal{I}$ is prime with $\text{codim}_{\mathcal{Z}(\mathcal{W})}(\mathcal{Z}(\mathcal{W}) \cap \mathcal{I}) = 1$.

Proof. We only prove the second statement of (iv). The proofs of the remaining statements are the same as those of [Lo2, Theorem 1.2.2]. For a prime ideal $\mathcal{I} \subset \mathcal{W}$ with $\text{codim}_{\mathcal{Z}(\mathcal{W})}(\mathcal{Z}(\mathcal{W}) \cap \mathcal{I}) = 1$, statement (iii) implies $\text{codim}_{\mathcal{Z}(g)}(\mathcal{I}^\dagger \cap \mathcal{Z}(g)) = 1$. Since we are assuming $g$ is of type I, we can prove the same argument as in [Jan, Proposition 7.3] that any prime ideal $\mathcal{I} \subset \mathcal{U}$ with $\text{codim}_{\mathcal{Z}(\mathcal{U})}(\mathcal{Z}(\mathcal{U}) \cap \mathcal{I}) = 1$ is primitive. \[\square\]

The following is another description of the map $\bullet^\dagger$.

Proposition 4.4. We have $\mathcal{R}_h(\mathcal{I}^\dagger) = \mathcal{R}_h(\mathcal{U}) \cap \Phi^{-1}(S[[V, h]] \otimes \mathcal{I}_h)$. For a given $\mathcal{I} \in \text{id}(\mathcal{W})$, $\mathcal{I}^\dagger$ is uniquely determined by the above property.

We omit the proof of the proposition, which may be given by the same argument as in the proof of [Lo2, Proposition 3.4.1].
4.3. Finite-dimensional representations of $\mathcal{U}(\mathfrak{g}, e)$. In this subsection we construct a series of finite-dimensional $\mathcal{U}(\mathfrak{g}, e)$-modules. Moreover, we prove that by such a construction, all finite-dimensional irreducible representations of $\mathcal{W}$ can be exhausted (see Theorem 4.8). Our another main tool, a map $\bullet_1 : \hat{\mathfrak{g}}(\mathfrak{u}) \mapsto \hat{\mathfrak{g}}(\mathcal{W})$ is defined as follows. For an ideal $I \in \hat{\mathfrak{g}}$, denote by $\mathcal{R}_h(I)$ the closure of $\mathcal{R}_h(I)$ in $S[\mathfrak{g}]_{\mathfrak{g}}^{\wedge}$. Define $I_1$ to be the unique (by Proposition 3.3(3)) ideal in $\mathcal{W}$ such that

$$\mathcal{R}_h(I_1) = \Phi_h^{-1}(\mathcal{I}_h) \cap \mathcal{R}_h(\mathcal{W}).$$

Proposition 4.5. For any $J \in \hat{\mathfrak{g}}(\mathcal{W})$ and $J \in \hat{\mathfrak{g}}(\mathcal{U})$ we have $J \supset (J_1)_{\overline{1}}$ and $J \subset (J_1)_{\overline{1}}$.

Proof. We have

$$\mathcal{R}_h((J_1)_{\overline{1}}) = \Phi_h^{-1}(\mathcal{R}_h(I_1)) \cap \mathcal{R}_h(\mathcal{W}) = \Phi_h^{-1}(S[[V, h]] \otimes \mathcal{R}_h(I)) \cap \mathcal{R}_h(\mathcal{W}) \subset (S[[V, h]] \otimes \mathcal{R}_h(I)) \cap \mathcal{R}_h(\mathcal{W}) = \mathcal{R}_h(I) \cap \mathcal{R}_h(\mathcal{W}) = \mathcal{R}_h(I).$$

Here the first equation follows from Proposition 4.4 and the last one follows from Proposition 3.3(3). Hence we have $(J_1)_{\overline{1}} \subset J$.

$$\mathcal{R}_h((J_1)_{\overline{1}}) = \Phi_h^{-1}(S[[V, h]] \otimes \mathcal{R}_h(I_1)) \cap \mathcal{R}_h(\mathcal{U}) = \Phi_h^{-1}(S[[V, h]] \otimes (\Phi_h^{-1}(\mathcal{R}_h(I)) \cap S[[S, h]]) \cap \mathcal{R}_h(\mathcal{U}) = \mathcal{R}_h(I) \cap \mathcal{R}_h(\mathcal{U}) \supset \mathcal{R}_h(I).$$

Here the third equation follows from the forthcoming fact 4.3. Thus we have $J \subset (J_1)_{\overline{1}}$. 

\[\Box\]

Proposition 4.6. The isomorphism $\Phi_h$ induces an isomorphism between $gr(J_1)$ and $(gr(J) + I(S))/I(S)$, where $I(S)$ is the ideal of $S[\mathfrak{g}]$ generated by $V$. Here "gr" means the associated grading with respect to the Kazdan filtration.

Proof. The two sided $h$-saturated ideal $\Phi_h^{-1}(\mathcal{I}_h)$ is stable to the adjoint action of $S[V, h]^\wedge$. From the proof of Theorem 4.6 we have

$$\Phi_h^{-1}(\mathcal{I}_h) = S[V, h]^\wedge \otimes (S[\tilde{\mathfrak{g}}, h]^\wedge \cap \Phi_h^{-1}(\mathcal{I}_h)).$$

Let $I^\wedge$, $\Phi_h^{-1}(\mathcal{I}_h)^\wedge$ and $I(S)^\wedge$ be the classical part of $\mathcal{R}_h(J_1), \mathcal{R}_h(\mathcal{I}_h)$ and $I(S)$ respectively. The equation (4.3) and the second statement of Lemma 3.7 implies

$$I^\wedge \simeq (\Phi_h^{-1}(\mathcal{I}_h)) \wedge + I(S)^\wedge / I(S)^\wedge.$$

Since $gr(J_1)$ is dense in $I^\wedge$, by the second statement of Lemma 3.7 and the third statement of Proposition 3.3 we have

$$gr(J_1) \simeq (gr(J) + I(S))/I(S).$$
Let $v = v_0 + v_1$ be a superspace and $A$ be a filtered associative algebra with $gr(A) = S[v]$. For a two sided ideal $J \subset A$, we denote by $V(J)$ the maximal spectrum of $S[v_0]/gr(J \cap S[v_0])$. For an $A$ module $M$ with annihilator $J$, the associated variety $AV(M)$ of $M$ is defined to be $V(J)$. The set of the primitive ideals $J \subset U$ with $V(J) = \overline{G_0} \cdot \chi$ is denoted by $Prim_0(U)$. Let $S_0$ be the Slodowy slice of $O$ in $g_0$.

**Theorem 4.7.** For any $J \in Prim_0(U)$, the composition factors of $\mathcal{W}/J^\dagger$ are finite-dimensional irreducible $U(g,e)$-modules. Any finite-dimensional irreducible $U(g,e)$-module is can be obtained in this way.

**Proof.** By Proposition 4.6 we have $gr(J^\dagger) = (gr(J) + I(S))/I(S)$. Thus we have

$$gr(J^\dagger) \cap C[S_0] = (gr(J) \cap S[g_0] + I(S) \cap S[g_0])/I(S) \cap S[g_0].$$

This implies

$$V(gr(J^\dagger)) = S_0 \cap O = \chi.$$ 

Hence $J^\dagger$ has finite codimension in $\mathcal{W}$, i.e. $\mathcal{W}/J^\dagger$ is a finite-dimensional $\mathcal{W}$-module. So we prove the first statement.

Let $M$ be an irreducible $\mathcal{W}$-module and $J \in Prim^{cofin}(\mathcal{W})$ be its annihilator. By Proposition 4.5 we have the following canonical surjective homomorphisms $\mathcal{W}/((J^\dagger)^\dagger) \rightarrow \mathcal{W}/(J) \rightarrow M$. Thus the theorem follows.

**Theorem 4.8.** For any $J \in Prim^{cofin}(\mathcal{W})$, we have $J^\dagger \in Prim_0 U$; for any $J \in Prim_0 U$, there are finitely many $J \in Prim^{cofin}(\mathcal{W})$ with $J = J^\dagger$.

**Proof.** Let $J \in Prim^{cofin} \mathcal{W}$ be the annihilator of a given irreducible $\mathcal{W}$ module $M$. By Theorem 4.3 (ii), we have $J^\dagger = Ann(S(M))$. Since $S(M)$ is a Whittaker module for the pair $(g_0,e)$, we have $\overline{G_0} \cdot \chi \subset V(J^\dagger)$ [Pr2 Theorem 3.1].

Since the associated variety of an irreducible $U(g_0)$ module is the closure of a nilpotent orbit, there exists an irreducible sub-quotient $M^\dagger$ of $S(M)$ with $AV(M^\dagger) = V(J^\dagger)$. For the annihilator $J^\dagger_0 \subset U(g_0)$ of $M^\dagger$, we have $dim(V(J^\dagger_0)) \leq 2GK_{\mathcal{U}(g_0)}(M^\dagger)$ by [Jan] §10.7. By [Mus] Lemma 7.3.3(a) we have $GK_{\mathcal{U}(g_0)}(M^\dagger) \leq GK_{\mathcal{U}(g_0)}(S(M))$. By Lemma 4.2 we have $GK_{\mathcal{U}(g_0)}(S(M)) = dim(m_0)$. The later equals $dim(G_0 \cdot \chi)/2$. Thus finally we have $dim(V(J^\dagger)) \leq dim(G_0 \cdot \chi)$ and hence the first statement follows.

Suppose $J = J^\dagger$ for some $J \in Prim^{cofin} \mathcal{W}$. By Proposition 4.5 we have $J \supset J^\dagger$. By the proof of Theorem 4.7 $J^\dagger$ has finite codimension in $\mathcal{W}$. Thus any prime ideal containing $J^\dagger$ is minimal. So by [Dis] Proposition 3.1.10] the number of primitive ideals $J \in Prim^{cofin} \mathcal{W}$ with $J = J^\dagger$ is finite.

Finally we summarize our results on $\text{Irr}^{\text{fin}}(\mathcal{W})$ in this section. For a Lie superalgebra $g = g_0 \oplus g_1$ of type I, Theorem 4.8 provides us a map

$$Prim^{cofin} \mathcal{W} \rightarrow Prim_0 U; \ J \mapsto J^\dagger$$
with finite fiber. For the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$, Lezter established a bijection

$$\nu : \text{Prim}(\mathcal{U}) \longrightarrow \text{Prim}(\mathcal{U}(\mathfrak{g}_0)).$$

We refer the construction of $\nu$ to [Le, §3.3] or [Mus, §15.2.2]. It can be easily deduced from the construction that $\nu(I)$ is supported on $G_0 \cdot \chi$ if and only if so is $I$. Thus we make a step forward a description of the set $\text{Irr}^{\text{fin}}(W)$.

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