A Closed-Form Solution to Local Non-Rigid Structure-from-Motion

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A recent trend in Non-Rigid Structure-from-Motion (NRSfM) is to express local, differential constraints between pairs of images, from which the surface normal at any point can be obtained by solving a system of polynomial equations. Unfortunately, these systems are of high degree with up to five real solutions. Hence, a computationally expensive strategy is required to select a unique solution. Furthermore, they suffer from degeneracies that make the resulting estimates unreliable, without any mechanism to identify this situation.

In this paper, we show that, under widely applicable assumptions, we can derive a new system of equations in terms of the surface normals, whose two solutions can be obtained in closed-form and can easily be disambiguated locally. Our formalism also allows us to assess how reliable the estimated local normals are and to discard them if they are not. Our experiments show that our reconstructions, obtained from two or more views, are significantly more accurate than those of state-of-the-art methods, while also being faster.

1 INTRODUCTION

Reconstructing the 3D shape of deformable objects from monocular image sequences is known as Non-Rigid Structure-from-Motion (NRSfM) and has applications in domains ranging from entertainment [33] to medicine [25]. Early methods relied on low-rank representations of the surfaces [4], [7], [9], [12], [17], [22], [24], [27], [47], while more recent ones exploit local surface properties to derive constraints and can handle larger deformations [10], [11], [19], [45], [48], [49]. Unfortunately, these constraints have to be enforced jointly on the entire set of reconstructed points for a whole sequence. Hence, the computational cost increases non-linearly with the number of images and quickly becomes prohibitive. Furthermore, a globally optimal solution is obtained using an iterative refinement, which requires a reliable initialization that is not always available. Finally, these global methods cannot handle missing data.

In earlier work [35], [36], [37], we have shown that local methods constitute a powerful alternative. Expressing isometry, conformality, or equiareality constraints in terms of differential properties makes the number of local variables remain fixed. Unfortunately, the systems of equations that arise in these computations are bivariate of high degree. They can have up to five real solutions. Therefore, a computationally expensive strategy is required to select a unique solution. Furthermore, they suffer from degeneracies that make the resulting estimates unreliable, without any mechanism to identify this situation.

In this paper, we introduce a new local method. Instead of inferring the depth derivatives, we estimate surface normals. More specifically, given a 2D warp between two images, we consider tangent planes at corresponding points. For each pair of points, we compute the homography relating the two planes and decompose it to compute the normals by solving local differential constraints [35], [36]. This has two solutions, instead of five in our earlier approaches [34]. For each plane, we pick the right one by enforcing an easy-to-compute measure of local smoothness. Furthermore, our formalism lets us assess how well-conditioned the problem was and, hence, how usable the resulting normals are. In other words, we can derive from an image pair a set of reliable normals and discard the others.

We will demonstrate on both synthetic and real data that we outperform state-of-the-art local and global methods at a fraction of the computational cost. Our contribution is therefore an approach to NRSfM that relies on solving in closed form a set of equations relating surface normals at corresponding points. Being entirely local, the computation is both fast and reliable. Although our solution is designed for isometric or conformal deformations, it yields good results for generic ones.

2 RELATED WORK

NRSfM was introduced in [7] and the ill-posedness of the problem was handled by constraining the deformations to lie on a low-dimensional manifold. Later variants introduced additional constraints for efficient low-rank factorization [4], [8], [9], [16], [17], [46] or performed additional optimization [12], [13], [22], [23], [27], [32], [50] to improve the statistical modeling. Learning-based techniques have been used to tune the dimensionality of the
deformation space [15], [21], [38] using a large amount of annotated data for supervision. [30], [42] formulated learning-based techniques in an unsupervised setting to reconstruct from sparse and dense data, respectively. However, this does not overcome the fundamental limitation of approaches relying on low-rank assumptions: they cannot model complex deformations. Furthermore, they do not naturally handle missing data and occlusions, and complex formulations [14] are required to overcome this. As a result, these methods have been limited to objects that deform in a relatively predictable way, such as human faces. Recently, these limitations have been addressed by imposing constraints between corresponding points across images in one of the following ways.

Modeling Global Deformations. Several methods seek to enforce physical properties on the deformation, such as isometry that preserves local distances on the deforming surface. They approximate isometry by inextensibility [11], [19], piece-wise inextensibility [39], [40], [49], local or piece-wise rigidity [10], [24], [45], [48]. A globally optimal solution is then found by jointly solving over all corresponding points. This requires a computationally expensive optimization, which makes this approach impractical for handling large numbers of images. To handle non-isometric surfaces, a mechanics-based approach is proposed in [1], [2], [3], introducing the forces required to compute the resulting shape. In any event, all these methods require an initialization, usually obtained using standard rigid-body reconstruction techniques. Furthermore, they are often inaccurate.

Modeling Local Deformations. In earlier works, we have proposed methods that rely on formulating local deformation constraints in terms of algebraic expressions. This makes it possible to reconstruct each surface point independently by solving algebraic equations, which reduces the computation cost. Being local, these methods inherently handle missing data and occlusions. In [35], we treated surfaces as locally planar (LP) and formulated local isometric constraints using metric tensors and connections representing the rate of change of metric tensors. In [36], we extended this deformation modeling to conformal and equiareal deformations by assuming the deformation to be locally linear (LL). For each pair of images, we obtained two cubic equations in two variables related to local depth derivatives with 9 possible solutions. In practice, up to 5 of them can be real. We found a unique solution by minimizing sum-of-squares of residuals over multiple images. In [34], we proposed two fast solutions to the equations of isometric NRSfM [35]. Using substitution and resultants, we converted the original bivariate equations to univariate ones that can be solved efficiently. However, this comes at the cost of adding phantom solutions that cannot be identified. We picked the solution that yields the smallest residual of the isometry constraints on the entire image set. In [37], we proposed an NRSfM solution for generic deformations. It uses only connections to formulate constraints to enforce surface smoothness.

Table 1 summarizes the characteristics of these local methods.

Table 1: Summary of the local methods we developed in earlier work.

| Method        | Deformation modeling | Variables | Assumptions | Constraints on | Degree | Solution strategy                                      | Unique solution |
|---------------|----------------------|-----------|-------------|---------------|--------|------------------------------------------------------|-----------------|
| [35] Isometry | Depth                | LP        | Metric tensor, Connections | 3               | Sum-of-squares minimisation of bivariates | >>3 images |
| [36] Isometry, Conformality, Equiareality | Depth | LP+LL | Metric tensor, Connections | 3 | Sum-of-squares minimisation of bivariates | >>3 images |
| [37] Diffeomorphism | Depth | LL | Connections | 10 | Reduce to univariates using resultants | >>3 images |
| [34] Isometry | Depth | LP | Metric tensor, Connections | 3 | Reduce to univariates using resultants or substitution | >>3 images |
| Ours          | Isometry, Conformality | Normals | LP+LL | Metric tensor, Connections | 2 | Closed-form solution from univariates | >>2 images using local smoothness |

Fig. 1: A 2-view model for NRSfM. Assuming \( \psi \) to be locally isometric/conformal, our goal is to find \( \phi, \overline{\phi} \) given that \( \eta \) is known.

### 3 Formalism and Assumptions

At the heart of our approach is the fact that the normals at two different instants at a point on a deforming 3D surface can be computed given the point’s projections in two images and a 2D warp between these images, under the sole assumptions of local surface planarity and deformation local linearity. In this section, we first introduce the NRSfM setup we will use in the rest of this paper, which is similar to the one of [36]. We then explain what our assumptions mean and why they are widely applicable. Finally, we formulate the constraints we will use for reconstruction purposes.

#### 3.1 Setup

Fig. 1 depicts our setup when using only two images, \( I \) and \( \overline{I} \), acquired by a calibrated camera. In each one, we denote the deforming surface as \( S \) and \( \overline{S} \), respectively, and model it in terms of functions \( \phi, \overline{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) that associate an image point to a surface point. Let us assume that we are given an image registration function \( \eta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that associates points in the first
image to points in the second. This is often referred to as a warp. In practice, it can be computed using standard image matching techniques, such as optical flow [43, 44] or SIFT [28]. These functions can be composed to create a mapping \( \psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) between 3D surface points seen in the two images. We use a parametric representation of \( \eta \) and \( \phi \) with B-splines [6], which allows us to accurately obtain first- and second-order derivatives of these functions. A finite-difference approach could also be used. Given a point \( \mathbf{x} = (u, v) \) on \( \mathcal{I} \) and its corresponding 3D point \( \mathbf{X} = \phi(\mathbf{x}) \) on \( \mathcal{S} \), we write \( \phi(\mathbf{x}) = \frac{1}{\beta(u, v)} \begin{pmatrix} u & v & 1 \end{pmatrix}^\top \), where \( \beta \) represents the inverse of the depth. The Jacobian of \( \phi \) is given by

\[
\mathbf{J}_\phi = \frac{1}{\beta(u, v)} \begin{pmatrix} 1 - u k_1 & -u k_2 \\
-k_1 & 1 - v k_2 
\end{pmatrix},
\]  

where \( k_1 = \frac{\partial \beta}{\partial u} \) and \( k_2 = \frac{\partial \beta}{\partial v} \). \( \pi, \bar{\pi}, \phi, \bar{\phi}, k_1, \) and \( k_2 \) are defined similarly in \( \mathcal{I} \).

### 3.2 Local Planarity and Linearity

In this work, we assume local planarity of the 3D surfaces and local linearity of the deformations as described in [20, 26]. We now describe these two assumptions and argue that they are weak ones that are generally applicable.

**Surface Local Planarity.** Let \( \mathbf{x}_0 \) be an image point with surface normal \( \mathbf{n} \) at \( \phi(\mathbf{x}_0) \). All points \( \mathbf{x} = (u, v) \) sufficiently close to \( \mathbf{x}_0 \) can be accurately described as lying on the tangent plane. Hence, they satisfy \( \mathbf{n}^\top \phi(\mathbf{x}) + d = 0 \), where \( d \) is a scalar, which we can rewrite as \( \beta = -\mathbf{n}^\top \mathbf{x} \). Therefore, the inverse depth \( \beta \) that appears in Eq. 1 is a linear function of \( \mathbf{x} \) even though \( \phi \) is not. Nevertheless, all higher-order derivatives of \( \phi \) can be expressed in terms of \( \beta \) and its first-order derivatives. This is widely viewed as a weak assumption that applies to most smooth manifolds [26]. For example, our planet is a sphere that can be treated as locally planar.

**Deformation Local Linearity.** According to [20], every non-linear function can be approximated with an infinite number of linear functions. This assumption has been successfully used in shape-matching [31]. We assume the deformation \( \psi \) that relates locally two planes to be smooth enough to be well described locally by its first-order approximation, so that we can ignore its second derivatives. In other words, we use a first-order approximation for the local deformations but a second-order one for the surface depth to allow for globally non-planar shapes. This is a looser set of assumptions than what is normally used in NRSfM. For example, [12, 27] and other low-rank methods assume the deformation space to be small; physics-based methods that use inextensibility [11, 49] or piecewise-rigidity [45, 48] make a much stronger assumption.

Under the assumption of local planarity, we have \( \mathbf{X} \) and \( \bar{\mathbf{X}} \) lying on a planar surface. A generic transformation between these two surfaces, which defines the deformation \( \psi \), can be expressed as \( \mathbf{X} = \mathbf{S} \mathbf{X} + \mathbf{T} \), where \( \mathbf{R} \) and \( \mathbf{T} \) are rotation and translation and \( \mathbf{S} \) is a scaling matrix. If \( \mathbf{S} \) happens to be a purely diagonal matrix with equal entries, \( \psi \) is a planar homography, and the resulting deformation is purely isometric or conformal. Nevertheless, \( \psi \) is linear. Therefore, local planarity of surfaces implies local linearity of deformations. However, the reverse is not true.

### 3.3 Differential Constraints across Images

To express constraints between quantities computed in \( \mathcal{I} \) and \( \mathcal{I} \), we define metric tensors and connections as described in [26].

**Metric Tensors.**

The metric tensors \( \mathbf{g} \) in \( \mathcal{I} \) and \( \mathbf{g} \) in \( \mathcal{I} \) are first-order differential quantities that capture local distances and angles. They can be written as

\[
\mathbf{g} = \mathbf{J}_\phi^\top \mathbf{J}_\phi \quad \text{and} \quad \mathbf{g} = \mathbf{J}_\pi^\top \mathbf{J}_\pi,
\]  

where \( \mathbf{J}_\phi \) and \( \mathbf{J}_\pi \) are local surface jacobians computed according to Eq. 1. These tensors can be used to impose isometry, conformity, and equiareality constraints by forcing the scalars \( k_1 \) and \( k_2 \) of Eq. 1 to satisfy one of the three conditions below:

\[
\begin{align*}
\mathbf{g} &= \mathbf{J}_\pi^\top \mathbf{g} \mathbf{J}_\pi, \quad &\text{Isometry} \\
\mathbf{g} &= \lambda^2 \mathbf{J}_\pi^\top \mathbf{g} \mathbf{J}_\pi, \quad &\text{Conformality} \\
\sqrt{\det(\mathbf{g})} &= \sqrt{\det(\mathbf{J}_\phi^\top \mathbf{g} \mathbf{J}_\phi)}, \quad &\text{Equiareality}
\end{align*}
\]

where \( \mathbf{J}_\eta \) is the Jacobian of the warp \( \eta \).

**Linear Relation between Surface Derivatives.** Given \( \mathbf{J}_\phi \), a local reference frame on the surfaces can be expressed with the column vectors as tangents and their cross product as normal. Connections are second-order differential quantities that express the rate of change of this local frame. Using connections under the assumption of local linearity as stated above, it can be shown [36] that

\[
\begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \end{pmatrix} = \mathbf{J}_\phi^\top \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{J}_\eta \frac{\partial^2 \eta}{\partial \pi \partial \tau},
\]  

where \( \frac{\partial^2 \eta}{\partial \pi \partial \tau} \) are the second-order derivatives of the warp. Solutions to isometric, conformal and equiareal NRSfM can be obtained by solving the metric tensor preservation equations in Eq. 3 under the constraints of Eq. 4.

### 4 Computing Normals from Two Images

In earlier approaches [36], the NRSfM problem was addressed by solving the system of Eq. 3 under the isometry, conformity, and equiareality constraints of Eq. 3 with respect to the variables \( k_1 \) and \( k_2 \) of Eq. 1. Here, we solve this system of equations directly in terms of the surface normals. We will show that, not only can this be done in closed form, but it also allows us to identify degenerate situations that result in unreliable estimates.

**Differentiating the Warp.**

Let us consider a point \( \mathbf{X} = (\pi, \bar{\pi}, \bar{\pi}) \) in \( \mathcal{I} \) and its corresponding point \( (u, v)^\mathcal{I} = \eta(\pi, \bar{\pi}) \) in \( \mathcal{I} \), with corresponding points on surfaces \( \mathbf{X} \) and \( \mathbf{X} \). Assuming the surfaces to be locally planar means that there is a \( 3 \times 3 \) homography matrix \( \mathbf{H} = [h_{ij}]_{1 \leq i, j \leq 3} \) such that \( \mathbf{X} = \lambda \mathbf{H} \mathbf{X} \). Since we assume a perspective projection for the camera, we write

\[
\begin{pmatrix} 1 \\ \bar{\pi} \end{pmatrix} = \frac{1}{\bar{\pi}} \mathbf{H} \mathbf{X} \implies \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\bar{\pi}} \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} \pi \\ 1 \end{pmatrix},
\]

where \( \bar{\pi} = h_{31} \pi + h_{32} \bar{\pi} + h_{33} \). The first- and second-order derivatives of \( \eta \) can be computed as

\[
\begin{align*}
\mathbf{J}_\eta &= \begin{pmatrix} \frac{\partial \eta}{\partial \pi} \\ \frac{\partial \eta}{\partial \bar{\pi}} \end{pmatrix} = \frac{1}{\bar{\pi}} \begin{pmatrix} h_{11} - h_{31} \pi & h_{12} - h_{32} \bar{\pi} \\ h_{21} - h_{31} \pi & h_{22} - h_{32} \bar{\pi} \end{pmatrix}, \\
\frac{\partial^2 \eta}{\partial \pi \partial \bar{\pi}} &= -\mathbf{J}_\eta \begin{pmatrix} 2h_{31} & h_{32} & 0 \\ 0 & h_{31} & 2h_{32} \end{pmatrix}.
\end{align*}
\]
Image Embedding and Local Normal.

The unit normal $n$ at $x$ is the cross product of the columns of the matrix $J_φ$ from Eq. 1. This lets us write

$$n = \frac{1}{\beta^2 \sqrt{\det g}} \begin{pmatrix} k_1 \\ k_2 \\ 1 - u k_1 - v k_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ 1 \end{pmatrix} = \frac{1}{\beta^2 \sqrt{\det g}} \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ 1 \end{pmatrix}.$$  

$$\Rightarrow \begin{pmatrix} k_1 \\ k_2 \\ 1 \end{pmatrix} = \beta^2 \sqrt{\det g} n_x.$$

Given the normal $n$ of Eq. 7, we rewrite the matrix $J_φ$ of Eq. 1 as

$$J_φ = \begin{pmatrix} 0 & u k_1 + v k_2 - 1 & k_2 \\ 0 & -k_1 & 0 \\ -k_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \beta \sqrt{\det g} n_x E.$$

We can now rewrite the differential constraints across images introduced in Section 3.3 in terms of the normals.

Linear Relation between Surface Normals.

Given the $η$ derivatives from Eq. 6, the linear relation of Eq. 4 becomes

$$\begin{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{pmatrix} = J_η \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \frac{1}{\bar{σ}} \begin{pmatrix} h_{31} \\ h_{32} \end{pmatrix},$$

$$\Rightarrow \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = (J_η^{-1} m_1) (k_1) + \frac{1}{\bar{σ}} \begin{pmatrix} h_{31} \\ h_{32} \end{pmatrix}.$$  

Using Eq. 8, we rewrite the above expression as

$$n = \frac{\beta^2}{\bar{σ}} \sqrt{\det g} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix}$$

$$= \frac{\beta^2}{\bar{σ}} \sqrt{\det g} \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix}$$

$$= \frac{\beta^2}{\bar{σ}} \sqrt{\det g} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix} = \frac{\beta^2}{\bar{σ}} \sqrt{\det g} H \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix},$$

which directly relates the two normals.

Metric Tensor.

As shown in Fig. 1, we can write $\bar{φ} = ψ o φ o ψ$. Differentiating this expression and multiplying it by its transpose yields

$$g = J_φ^T J_φ = J_φ^T J_η^T J_η J_φ J_η,$$

Using Eq 9, we write $J_φ J_η = \beta \sqrt{\det g} n_x E J_η$. Given the $η$ derivatives of Eq. 6, we simplify $E J_η$ to $\frac{1}{\bar{σ}} \begin{pmatrix} h_{11} \times x \\ h_{12} \times x \\ h_{13} \times x \end{pmatrix}$, where $h_1$, $h_2$, $h_3$ are the first two columns of the homography matrix $H$, and $\hat{x} = (u \ v \ 1)^T$. By writing $z_1 = n \times (h_1 \times x)$ and $z_2 = n \times (h_2 \times x)$, Eq. 3 reduces to

$$\begin{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{\det g} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix} \end{pmatrix}.$$  

NRSfM from Isometric/Conformal Constraints.

So far, we have expressed the metric preservation conditions in terms of the normals of the two surfaces under consideration. The only unknown left in the system is therefore $n$. We now show that this unknown can in fact be computed in closed form.

Given the multiplicative nature of the cross product, the constraints on the normals of Eq. 11 imply that

$$\frac{\beta^2}{\bar{σ}} \sqrt{\det g} \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix} \end{pmatrix}$$

$$= \frac{\beta^2}{\bar{σ}} \sqrt{\det g} \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix} \end{pmatrix}.$$  

$$= \frac{\beta^2}{\bar{σ}} \sqrt{\det g} H n_x E.$$

Injecting this expression into the isometric/conformal metric tensor preservation relation of Eq. 13 yields

$$z_1 H^{-1} H^{-1} z_1 - z_1 H^{-1} H^{-1} z_2 = \frac{\lambda^2 \beta^2}{\bar{σ}^2} (z_1 z_1 - z_1 z_2),$$

$$\Rightarrow z_1 (H - \frac{\lambda^2 \beta^2}{\bar{σ}^2} H^{-1} I_{3 \times 3}) z_1 = 0, \forall i, j \in \{1, 2\},$$

where $H = H^{-1}$. Assuming $H$ to be normalized, that is, its second singular value to be 1, the relation between a 3D point observed in the two input images is given by $φ(x) = H φ(x)$. Using Eq. 5 yields $\bar{σ} = \beta \bar{σ}$. By writing $z_1 = [n]_x [h_1] x$, the above constraints further simplify to

$$[n]_x^T (H - \lambda^2 I_{3 \times 3}) [n]_x = 0.$$  

Since $H \sim φ$, we divide the above expression by $\lambda^2$ and, with a slight abuse of notation, write $\frac{1}{\lambda} \bar{H}$ as $\bar{H}$. This simplifies the above expressions to

$$[n]_x^T \bar{H} - \lambda^2 I_{3 \times 3} [n]_x = [n]_x^T S [n]_x = 0.$$  

Degenerate Cases. The system of Eq. 18 holds as long as $S$ is a non-null matrix, which means $H \bar{H} \neq I_{3 \times 3}$. Therefore, $H \bar{H}$ should not be an orthogonal matrix, which makes pure rotations and reflections cause degeneracies.

Affine Stability. Under affine imaging conditions, $h_{31} = h_{32} = 0$, and $h_{33} = 1$. In this case, $z_1$ and $S$ remain non-null, and thus the system in Eq. 18 does not become degenerate, and we can still compute the normal.

Solution. The solution to the system in Eq. 18 can be obtained by homography decomposition [29]. We give an overview of the solution here but recommend reading [29] for more detail.

$S = \{s_{ij}\}$ is a symmetric matrix expressed in terms of $\bar{H}$, and $H^-1$. It can be numerically computed using $η$ and image observations $(x, \hat{x})$. Specifically, Eq. 11 gives the closed-form definition $H^T = \begin{pmatrix} I_{2 \times 2} & 0 \\ -x^T & 1 \end{pmatrix} \begin{pmatrix} J_η^T m_1 \\ n \end{pmatrix}$. Let us write $n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$. Since $n_3 \neq 0$, we define $y_1 = \frac{n_1}{n_3}$ and $y_2 = \frac{n_2}{n_3}$ and expand the system in Eq. 18 accordingly. This yields 6 constraints, out of which only 3 are unique. They are given by

$$s_{33} y_2^2 - 2s_{23} y_2 + s_{22} = 0.$$
By solving the first two, we obtain
\[
\begin{align*}
\frac{s_{33}y_1^2 - 2s_{13}y_1 + s_{11}}{s_{33}} &= 0, \\
\frac{s_{22}y_1^2 - 2s_{12}y_1y_2 + s_{11}y_2^2}{s_{33}} &= 0.
\end{align*}
\]
(19)

By solving the first two, we obtain
\[
y_1 = \frac{s_{13} \pm \sqrt{s_{13}^2 - 8s_{33}s_{11}}}{s_{33}}
\]
and \(y_2 = \frac{s_{23} \pm \sqrt{s_{23}^2 - 8s_{33}s_{22}}}{s_{33}}\). We use the third expression to disambiguate the solutions. Ultimately, this gives us closed-form expressions for the two potential solutions for the normal, written as
\[
\mathbf{n}_a = \left( s_{13} + \sqrt{s_{13}^2 - 8s_{33}s_{11}} \right), \frac{s_{23} + \sqrt{s_{23}^2 - 8s_{33}s_{22}}}{s_{33}}, \left( \frac{s_{33}}{s_{33}} \right),
\]
\[
\mathbf{n}_b = \left( s_{13} - \sqrt{s_{13}^2 - 8s_{33}s_{11}} \right), \frac{s_{23} - \sqrt{s_{23}^2 - 8s_{33}s_{22}}}{s_{33}}, \left( \frac{s_{33}}{s_{33}} \right),
\]
where \(s = \text{sign}(s_{23}s_{33} - s_{12}s_{33})\).

Normal Validation. The normals thus obtained must be visible to the camera. Given the analytical normal in Eq. 7, \(\mathbf{n}_a\) and \(\mathbf{n}_b\) are visible if \(\frac{s_{13}}{s_{23} - s_{33}s_{22}} > 0\), i.e., they have a similar orientation towards the camera. We discard the normals that do not meet the visibility constraint.

Normal Selection. Using Eq. 8, the local depth derivatives \((k_1, k_2)\) at \(\mathbf{x}\) are given by \(k_1 = \frac{\mathbf{n}_a}{\mathbf{n}_a^\top + \sqrt{n_a^\top + n_a^2}}\). From the solution in Eq. 20, we thus obtain two possible solutions for the local depth derivatives \((k_{1a}, k_{2a})\) and \((k_{1b}, k_{2b})\). We pick the normal that minimizes the corresponding sum of squares of depth derivatives. That is, we compute the normal \(\mathbf{n}\) as
\[
\mathbf{n} = \left\{ \begin{array}{ll}
\mathbf{n}_a & \text{if } k_{1a}^2 + k_{2a}^2 \leq k_{1b}^2 + k_{2b}^2 \\
\mathbf{n}_b & \text{otherwise}
\end{array} \right.
\]
(21)

Following Eq. 5, \(\mathbf{n}\) is then obtained as \(\mathbf{H}^\top \mathbf{n}\).

Measure of Degeneracy. In degenerate situations, the singular values \((\sigma_1, \sigma_2, \sigma_3)\) of \(\mathbf{H}\) are all one. We use the ratio \(\frac{\sigma_2}{\sigma_3}\) to quantify the degeneracy. Thus, we only reconstruct from \(\tilde{\mathbf{S}}\) if \(\frac{\sigma_2}{\sigma_3} > \tau\), and we set \(\tau = 1.05\).

Surface Reconstruction. We consider a planar surface and bend it to match the normals obtained using the homography decomposition mentioned above, as opposed to [34], [36], [37] which integrate the normals on each surface. The upside of surface bending is that it does not require to set a smoothness parameter, which needs to be tuned for the normal integration. Furthermore, surface bending is much faster than its normal integration counterpart in the presence of dense data. It is also less affected by the noise in the normals corresponding to high-perspective image regions.

5 Normals from Multiple Images

Methods such as those of [34], [36], [37] pick a reference image and formulate reconstruction constraints between it and the other images, which are then solved by solving a least-squares problem over the entire set of images. We use the same strategy, except that we reconstruct from image pairs, one of them being the reference image. Therefore, for \(N\) images, we obtain \(N - 1\) estimates for the reference image and 1 estimate for each of the non-reference images.

More formally, let \(\{ \mathbf{x}_i' \}, i \in [1, M], j \in [1, N]\), be a set of \(N\) point correspondences between \(M\) images. Our goal is to find the 3D point \(\mathbf{X}_i\) and the normal \(\mathbf{n}_i\) corresponding to each \(\mathbf{x}_i'\). Using Eq. 11, we write the local homography for each point correspondence \(\mathbf{H}_{ik}^{j}\) between image pairs \((i, k) \in [1, M], i \neq k\), using the warp \(\eta\). Each local homography \(\mathbf{H}_{ik}^{j}\) is normalized by dividing it by its second singular value. We compute \(\mathbf{H}_{ik}^{j}\) given by the ratio of the first and third singular value, and the normals for each local homography \(\mathbf{H}_{ik}^{j}\) using Eq. 20. We then pick a unique solution using Eq. 21. The solution on the reference and non-reference image is given by \(\mathbf{n}_{ik}^{j}\) and \(\mathbf{n}_{ik}^{j'}\), respectively. For non-degenerate cases, where \(\frac{\sigma_2}{\sigma_3} \geq 1.05\), we compute the normal \(\mathbf{n}_i\) by taking the median of the \(n_{ik}^{j}\)’s computed over \(k\) reference images. We obtain a 3D surface by bending a planar surface to match the obtained normals on each surface. We summarize our complete pipeline in Algorithm 1.

Algorithm 1: Our NRSIM Algorithm

Data: \(x'_i, \mathbf{H}_{ik}^{j}\) and \(\mathbf{H}_{ik}^{j}\)  
Result: \(\mathbf{n}_i\)

\[
\begin{align*}
\sigma_{ik}^2 &= 1.05; \\
\text{for each reference image } k &= [1, M] \text{ do} \\
\text{for each point } j &= [1, N] \text{ do} \\
\text{for images } i &= [1, M], i \neq k \text{ do} \\
\text{if } \mathbf{H}_{ik}^{j} > \sigma_{ik}^2 \text{ then} \\
\text{Compute normals using (20);} \\
\text{Pick a solution } \mathbf{n}_{ik}^{j} \text{ using (21);} \\
\text{Write } \mathbf{n}_{ik}^{j} = (\mathbf{H}_{ik}^{j})^\top \mathbf{n}_{ik}^{j}. \\
\text{else} \\
\text{Set } \mathbf{n}_{ik}^{j}, \mathbf{n}_{ik}^{j'} \text{ to zero;}
\end{align*}
\]
for each point \(j = [1, N] \text{ do} \\
for images \(i = [1, M] \text{ do} \\
\text{Obtain } \mathbf{n}_i \text{ by as the median of the non-zero } \mathbf{n}_{ik}^{j}\text{ s;}
\end{align*}
\]

6 Experiments

We compare our method against state-of-the-art ones on both synthetic and real datasets with available ground truth.

6.1 Datasets

Synthetic Datasets. We created 3 smooth surfaces: a plane, a cylindrical surface and a stretched surface with 400 tracked correspondences, as shown in Figure 2.

Real Datasets from our Previous Work. These include the Paper [41], Rug [35] and Tshirt [10] datasets. Paper comprises 191 images from a video of a deforming sheet of paper with 1500 point correspondences. Rug comprises 159 images from a video of a deforming rug with 3900 point correspondences. Tshirt has 10 wide-baseline images with 85 point correspondences. The correspondences in the Paper dataset were obtained using SIFT with a manual supervision of accuracy and are thus highly accurate. By contrast, those in the Rug dataset were computed using the dense optical flow method of [13] and contain errors due to optical drift and regional mismatches due to the lack of texture. The correspondences in Tshirt are computed manually. The ground truth for Paper and Rug is obtained using kinect, which is very noisy and contains large, inconsistent depth variations. For an apt comparison, we refined the ground truth to obtain smooth surfaces.
Fig. 2: **Reconstructed normals.** A synthetic deforming surface reconstructed in three different frames. The predicted normals are shown in blue and the ground-truth ones in green.

The ground truth for **Tshirt** is computed using rigid reconstruction of each image from multiple views.

**NRSfM Challenge Dataset.** It consists of 5 image sequences depicted by Fig. 5. They feature 5 kinds of non-rigid motions: articulated (piecewise-rigid) with 207 images and 69 point correspondences, balloon (conformal) with 51 images and 211 point correspondences, paper bending (isometric) with 40 images and 153 point correspondences, rubber (elastic) with 40 images and 481 point correspondences, and paper being torn with 432 images and 405 point correspondences. The dataset features images from 6 different camera motions and provides image points captured assuming both a perspective and an orthographic projection. It provides only one ground-truth surface for each of the sequences. The correspondences are sparse and not well-distributed across the images.

**Datasets used by [42].** [42] released the **Paper**, **Tshirt**, **Actor** and **Expressions** datasets, which have been widely used by many physics-based and low-rank constraints based methods. The **Paper** images are the same as the one used by us. [42] uses 60K dense correspondences computed using optical flow [13] and the raw depth data from the kinect is considered as the ground truth.

The **Tshirt** data has 300 images with 70K dense correspondences computed using [13], with the kinect raw depth data as ground truth. To deal with the inconsistent depth variations of the raw kinect data, [42] refines the raw data and focuses on small portions of theses datasets where the inconsistent depth variations are minimal, as shown in Fig 3. **Actor** contains 100 images of a deforming human face with 36K dense correspondences, and **Expressions** includes 384 3D shapes of a deforming human faces with 1000 point correspondences. The ground truth for both these datasets is synthetic. Fig 3 shows some samples.

**Blue Sheet Dataset.** Additionally, we recorded a video sequence featuring a textureless blue sheet deforming isometrically using a Kinect. It comprises 60 images and 7K point correspondences that were tracked using dense optical flow [13]. Optical flow on textureless surfaces is prone to large errors, and the flow we obtained confirms this.

6.2 Baselines and Metrics

We compare our method to local linearity-based diffeomorphic NRSfM **Pa20** [37], jointly solving isometric/conformal NRSfM **Pa19** [36], two fast solutions **Pa21-R** and **Pa21-S** [34] that transform the original constraints to univariate polynomials, which can
be easily solved, and local and piecewise homography decomposition, Ch14 [10] and Va09 [48], respectively. These are methods that, like ours, reconstruct local/piecewise surface normals and integrate them to obtain depth. Note that the solution to isometric NRSfM in [35] is the same as the one in Pa19. Therefore, there is no need for additional comparison.

We report errors in terms of accuracy of the normals $E_n$ and 3D points $E_d$. $E_n$ is computed as the average dot product between ground-truth and computed normals. The normal integration done in the above methods yields a smooth reconstruction by enforcing a local smoothness on the normals. As a consequence, it improves the quality of the reconstructed normals. Therefore, we also report $E_n(s)$, which is the error between the smoothened and the ground-truth normals. $E_d$ is the mean RMSE between the ground-truth and computed 3D points.

We also compare our approach against three of the best global methods, Ch17 [11], Ji17 [19] and Lee16 [27], along with a dense method, An17 [5]. They directly return 3D points. Hence, we only report $E_d$ for these methods.

While comparing on the datasets used by [42], we report $E_d$ as the mean 3D error, as computed in this method. Therefore, $E_d = \frac{1}{N} \sum_t ||P_{recon} - P_{GT}||_2$, where $P_{recon}$ is the obtained reconstruction, $P_{GT}$ is the ground truth and $N$ is the number of
images in the dataset.

In the remainder of this section we will refer to the method described in this paper as Ours.

6.3 Comparative Results

Results on Synthetic Data. Fig. 2 shows the generated surfaces. The performance of all methods is averaged over 10 trials with added Gaussian noise with a 3 pixels standard deviation. As Ours can reconstruct from two images only, we perform both pairwise reconstructions and joint reconstruction from the image triplet available for each surface. We report the results in Table 2. For methods that perform normal integration, we report errors in the normals due to smoothing is huge for Ch14 of both computed and smoothened normals. The improvement for methods that perform normal integration, we report errors in the normals due to smoothing is huge for Ch14 of both computed and smoothened normals. The improvement

method does not return a result because we are not using enough images.

Lee16 and An17

TABLE 2: Synthetic experiments results. ‘X’ indicates that the method does not return a result because we are not using enough images.

| Method | Surfaces 1, 2 | Surfaces 1,3 | Surfaces 1,2,3 |
|--------|---------------|---------------|---------------|
| Lee16  | X X X X       | X X X X       | X X X         |
| An17   | X X X X       | X X X X       | X X X         |
| Va09   | 16.4 12.3 10.2| 24.5 16.7 12.1| 17.3 16.0 9.8 |
| Ch14   | X X X X       | X X X X       | 28.1 24.1 20.2|
| Ch17   | X X X X       | X X X X       | X X 14.5      |
| Ji17   | X X X X       | X X X X       | X X 15.6      |
| Pa19   | X X X X       | X X X X       | 17.4 13.6 4.3 |
| Pa20   | X X X X       | X X X X       | 24.7 15.6 9.3 |
| Pa21-S  | X X X X      | X X X X      | 16.4 13.3 4.2 |
| Pa21-R  | X X X X      | X X X X      | 16.2 13.3 4.2 |
| Ours   | 4.0 3.5 2.1   | 8.3 7.6 4.3  | 9.3 8.4 3.2   |

Lee16 and An17 are not designed to work on wide-baseline data, therefore we did not evaluate them on this dataset.

We report our quantitative results in Table 3, and Figure 4 depict qualitative ones. We outperform all baselines in terms of Ed on the Paper and Rug dataset with partial and full correspondences. On the Tshirt dataset, Ch17 and Ji17 perform better. Crucially, our performance is achieved at a much reduced computational cost by solving a set of equations in closed form, as opposed to invoking a complex solver. As a result, our approach is about 150 times faster than Ch17 on 350 correspondences and can handle thousands whereas Ch17 cannot. Furthermore, our approach is also 50 times faster than Pa19, the counterpart local approach which uses expensive polynomial solvers, because we do not have to derive a complicated formulation to obtain a unique solution for each correspondence.

Table 4 provides a detailed analysis of the run-times of all the methods on 350 and 1500 points. We assume that the input point correspondences and their derivatives are pre-computed. Therefore, the timings only encode the computation of the normals or 3D points. Our approach yields the fastest run-times, seconded by An17. Note, however, that An17 has a parallel implementation and is computationally optimized. By contrast, our approach, as all the other ones, is implemented in Matlab and not optimized for speed.

The relative slowness of the other local method arises from the local normal estimators of Pa19 and Pa20 having to minimize the sum of squares of polynomials, which is expensive even if it has linear complexity. Pa20 is further slowed down by having to transform polynomials into univariate expressions. Pa21-S and Pa21-R obtain analytical solutions but require a fairly expensive disambiguation. By contrast, our local normal estimator is computationally cheap as it has a closed-form solution.

Results on the NRSfM Challenge Dataset. Fig. 5 compares the performance of Ours with that of other methods in terms of Ed, measured in mm, with Best being the one that does best as reported in the benchmark statistics provided on the website. The local methods show a significant performance improvement compared to the other ones. Pa19 uses second-order derivatives of the image registration $\eta$, which can be highly erroneous on this dataset. It uses an expensive polynomial solver, which cannot handle such large noise and fails on a large number of cases. Pa21-S and Pa21-R find an analytical solution to the isometric/conformal NRSfM problem in Pa19, which requires a non-linear refinement to obtain a unique solution; they obtain decent results on this dataset. Pa20 solves NRSfM using diffeomorphic constraints, which uses only first-order derivatives of $\eta$, and is thus less impacted by the sparsity of the data and performs better than Pa21-S and Pa21-R. Ours requires second-order derivatives of the image registration, but it is equipped with a measure to compute the well-conditioning of the data. This lets us identify and discard the non-isometric/non-conformal data and reconstruct from as-isometric(or conformal)-as-possible data. As a result, Ours yields better results than Pa20.

Fig. 5 shows some reconstructions obtained with our method.

Results on the Blue Sheet Dataset and on the Datasets used by [42]. These datasets are large in terms of the number of either point correspondences or images they contain. We compare the performance of Ours with An17, which is designed for reconstructing dense objects, however, it takes several hours to reconstruct. Additionally, we report the performance of our other local methods Pa19, Pa21-S and Pa21-R. In this case, we report
TABLE 3: (left) RMSE results on the datasets used in our previous work. ’X’ indicates that the method does not evaluate normals. ‘—’ indicates that method failed to return a result due to its high computational complexity. (right) Computation times as a function of the number of images and points used.

| Method | Paper (partial) | Rug (partial) | Paper (full) | Rug (full) | Tshirt (full) |
|--------|-----------------|--------------|--------------|------------|--------------|
| Lee16  | X X 21.6        | X X 89.8     | X X 21.9     | X X 90.7   | X X X       |
| An17   | X X 14.7        | X X 60.6     | X X 14.7     | X X 63.7   | X X X       |
| Ch14   | —— —— ——        | —— —— ——     | —— —— ——     | 23.4 16.5 12.6 |
| Va09   | —— —— ——        | —— —— ——     | —— —— ——     | 27.1 16.8 14.6 |
| Ch17   | X X 5.4         | X X 63.5     | —— —— ——     | X X 3.7    |
| Ji17   | X X 5.7         | X X 67.1     | —— —— ——     | X X 5.2    |
| Pa19   | 17.3 10.5 8.3   | 34.5 16.7 52.4| 16.8 8.6 7.2 | 35.8 18.2 54.3| 35.8 17.2 8.9|
| Pa20   | 20.7 19.4 10.2  | 28.1 21.5 46.1| 24.8 19.0 11.3| 29.4 22.1 47.1| 29.4 27.0 13.0|
| Pa21-S | 15.6 9.3 5.9    | 26.6 15.5 40.1| 18.4 8.8 5.3 | 31.0 17.5 43.4| 31.0 17.2 7.1 |
| Pa21-R | 15.3 9.0 5.9    | 26.8 15.7 40.8| 19.8 8.0 5.2 | 32.2 17.8 44.5| 30.2 16.5 7.1 |
| Ours   | 8.9 8.3 3.9     | 18.3 15.5 25.9| 9.1 8.4 4.1  | 19.3 16.3 31.1| 18.4 16.3 6.6 |

TABLE 4: Computation times as a function of the number of images and points used.

| Images | Lee16 | An17 | Va09 | Ch14 | Ch17 | Ji17 | Pa19 | Pa20 | Pa21-S | Pa21-R | Ours |
|--------|-------|------|------|------|------|------|------|------|--------|--------|------|
| 350 points |
| 10     | 17.8  | 10   | 69.4 | 75.3 | 31.3 | 341  | 9.7  | 24.5 | 4.1    | 4.1    | 0.2  |
| 30     | 23.4  | 12   | 3103 | 3407 | 129  | ——   | 12.5 | 32.7 | 9.3    | 9.4    | 0.6  |
| 60     | 45.6  | 19   | ——   | ——   | ——   | ——   | 14.8 | 45.3 | 11.4   | 11.4   | 1.9  |
| 1500 points |
| 10     | 256   | 12   | 1435 | 1256 | 995  | 2532 | 103  | 745  | 15.7   | 15.6   | 0.5  |
| 30     | 987   | 14   | ——   | ——   | 3400 | ——   | 118  | 1205 | 73     | 73     | 2.0  |
| 60     | 1705  | 22   | ——   | ——   | ——   | ——   | 124  | 1807 | 90     | 93     | 5.8  |

Fig. 5: NRSfM challenge dataset and some reconstructions using Ours. Green indicates the ground truth and blue indicates our reconstruction.

7 CONCLUSION

We have proposed an approach to NRSfM that can estimate normals from image pairs given a 2D warp and point correspondences between the two images. It does so in closed form from individual correspondences and is therefore fast. Furthermore, it can estimate if these normals are reliable given the motion from one image to
TABLE 5: Results on the NRSfM challenge datasets.

| Method | Articulated (full) (missing) | Balloon (full) (missing) | Paper (full) (missing) | Stretch (full) (missing) | Tearing (full) (missing) |
|--------|-----------------------------|-------------------------|------------------------|-------------------------|------------------------|
| Lee16  | 10.5                        | 88.4                    | 65.4                   | 70.3                    | 59.5                   |
| An17   | 65.1                        | 73.7                    | 55.2                   | 48.9                    | 64.7                   |
| Vo99   | 58.1                        | 58.9                    | 40.3                   | 50.8                    |                        |
| Ch14   |                            |                         |                        |                         |                        |
| Ch17   | 91.6                        | 75.5                    | 58.0                   | 66.5                    | 63.8                   |
| Pa21   | 21.3                        | 22.0                    | 27.7                   | 31.0                    | 39.0                   |
| Pa21-S  | 26.0                      | 25.2                    | 29.2                   | 30.1                    | 40.4                   |
| Pa21-R  | 25.1                      | 25.6                    | 29.4                   | 29.0                    | 40.5                   |
| Best   | 40.7                        | 46.6                    | 28.0                   | 55.7                    | 39.0                   |
| Ours   | 21.0                        | 22.4                    | 27.6                   | 38.1                    | 28.3                   |

Fig. 6: Blue Sheet dataset. Reconstructed surfaces for two images. The predictions of Ours are shown in blue, of Pa21-R in red, and of An17 in black. Note that our reconstructions are less noisy and match the surface 3D shape much better.

TABLE 6: Performance on dense datasets.

| Method | Blue sheet | Paper | Tshirt | Actor | Expressions |
|--------|------------|-------|-------|-------|-------------|
| An17   | 0.0558     | 0.0448| 0.0276| 0.0010| 0.1352      |
| Si20   | -----      | 0.0332| 0.0309| 0.0181| 0.0260      |
| Pa21-S  | 0.0462     | 0.0552| 0.0402| 0.0077| 0.0180      |
| Pa21-R  | 0.0463     | 0.0533| 0.0399| 0.0075| 0.0180      |
| Pa19   | 0.0479     | 0.0547| 0.0391| 0.0079| 0.0180      |
| Ours   | 0.0404     | 0.0313| 0.0263| 0.0072| 0.0171      |

the next. When they are found to be, our experiments show that they are indeed very accurate. As a result, our method performs well with various deformation types and can reconstruct large and small deformations at a low computational cost. Our next step will be to remove the dependency on expensive methods to compute warps and integrate normals so that a truly real-time application can be developed.

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Fig. 7: Actor and Expressions datasets. Reconstructed surfaces for two images. The predictions of Ours, Pa21-R and of An17 are quite similar.

Fig. 8: Paper and Tshirt datasets. The predictions of Ours are shown in blue, of Pa21-R in red, and of An17 in black. Note that our reconstructions are less noisy and match the surface 3D shape much better.

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