First Non-Abelian Cohomology of Topological Groups

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Abstract

Let $G$ be a topological group and $A$ a topological $G$-module (not necessarily abelian). In this paper, we define $H^0(G, A)$ and $H^1(G, A)$ and will find a six terms exact cohomology sequence involving $H^0$ and $H^1$. We will extend it to a seven terms exact sequence of cohomology up to dimension two. We find a criterion such that vanishing of $H^1(G, A)$ implies the connectivity of $G$. We show that if $H^1(G, A) = 1$, then all complements of $A$ in the semidirect product $G \rtimes A$ are conjugate. Also as a result, we prove that if $G$ is a compact Hausdorff group and $A$ is a locally compact almost connected Hausdorff group with the trivial maximal compact subgroup then, $H^1(G, A) = 1$.

Keywords: Almost connected group, inflation, maximal compact subgroup, non-abelian cohomology of topological groups, restriction.

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1 Introduction

Let $G$ and $A$ be topological groups. It is said that $A$ is a topological $G$-module, whenever $G$ continuously acts on the left of $A$. For all $g \in G$ and $a \in A$ we denote the action of $g$ on $a$ by $^g a$.

In section 2, We define $H^0(G, A)$ and $H^1(G, A)$.

In section 3, we define the covariant functor $H^i(G, -)$ for $i = 0, 1$ from the category of topological $G$-modules to the category of pointed sets. Also, we define two connecting maps $\delta^0$ and $\delta^1$.

A classical result of Serre [6], asserts that if $G$ is a topological group and $1 \to A \to B \to C \to 1$ a central short exact sequence of discrete $G$-modules then, the sequence $1 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to H^2(G, A)$ is exact.

In section 4, we generalize the above result to the case of arbitrary topological $G$-modules (not necessarily discrete).
We show that if $G$ is a connected group and $A$ a totally disconnected group then, $H^1(G, A) = 1$.

In section 5, we show that if $G$ has an open component (for example $G$ with the finite number of components) and for every discrete (abelian) $G$-module $A$ $H^1(G, A) = 1$ then, $G$ is a connected group.

In section 6, we show that vanishing of $H^1(G, A)$ implies that the complements of $A$ in the (topological) semidirect product $G \ltimes A$, are conjugate.

In section 7, we prove that, if $G$ is a compact Hausdorff group and $A$ a locally compact almost connected Hausdorff group then there exists a $G$-invariant maximal compact subgroup $K$ of $A$ such that the natural map $\iota^*: H^1(G, K) \to H^1(G, A)$ is onto. As a result, if $G$ is compact Hausdorff and $A$ is a locally compact almost connected Hausdorff group with trivial maximal compact subgroup then, $H^1(G, A) = 1$.

All topological groups are arbitrary (not necessarily abelian). We assume that $G$ acts on itself by conjugation. The center of a group $G$ and the set of all continuous homomorphisms of $G$ into $A$ are denoted by $Z(G)$ and $Hom_c(G, A)$, respectively. The topological isomorphism is denoted by “$\simeq$”.

Suppose that $A$ is an abelian topological $G$-module. Take $\tilde{C}^0(G, A) = A$ and for every positive integer $n$, let $\tilde{C}^n(G, A)$ be the set of continuous maps $f : G^n \to A$ with the coboundary map $\delta^n : \tilde{C}^n(G, A) \to \tilde{C}^{n+1}(G, A)$ given by

$$\delta^n f(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

The $n$th cohomology of $G$ with coefficients in $A$ in the sense of Hu [5], is the abelian group

$$H^n(G, A) = \text{Ker} \delta^n / \text{Im} \delta^{n-1}.$$
\[
\beta(g) = a^{-1}\alpha(g)a, \text{ for all } g \in G.
\]

It is easy to show that \(\sim\) is an equivalence relation. Now we define

\[H^1(G, A) = \text{Der}_c(G, A)/\sim.\]

**Notice 2.3.** There exists the trivial continuous derivation \(\alpha_0 : G \to A\) where \(\alpha_0(g) = 1\); Hence, \(H^1(G, A)\) is nonempty. In general, \(H^1(G, A)\) is not a group. Thus, we will view \(H^1(G, A)\) as a pointed set with the basepoint \(\alpha_0\).

Note that \(H^0(G, A)\) is a subgroup of \(A\), so it is a pointed set with the basepoint 1. Also if \(A\) is a Hausdorff group, then, \(H^0(G, A)\) is a closed subgroup of \(A\).

**Remark 2.4.** (i) If \(A\) is an abelian group then, \(H^1(G, A)\) is the first (abelian) group cohomology in the sense of Hu, i.e., it is the group of all continuous derivations of \(G\) into \(A\) reduced modulo the inner derivations. \([5]\)

(ii) If \(A\) is a trivial topological \(G\)-module then, \(H^1(G, A) = \text{Hom}_c(G, A)/\sim.\)

Here \(\alpha \sim \beta\) if \(\exists a \in A\) such that \(\beta(g) = a^{-1}\alpha(g)a, \forall g \in G.\)

(iii) Let \(G\) be a connected group and \(A\) a totally disconnected group then, \(H^1(G, A) = 1.\)

**Proof.** (i) and (ii) are obtained from the definition of \(H^1(G, A).\)

(iii): If \(\alpha \in \text{Der}_c(G, A)\) then, \(\alpha(1) = 1.\) On the other hand \(G\) is a connected group and \(A\) is totally disconnected. So, \(\alpha = \alpha_0.\) Thus, \(H^1(G, A) = 1.\)

**3** \(H^i(G, -)\) as a Functor and the Connecting Map \(\delta^i\) for \(i = 0, 1\)

In this section we define two covariant functors \(H^0(G, -)\) and \(H^1(G, -)\) from the category of topological \(G\)-modules \(G\mathcal{M}\) to the category of pointed sets \(\mathcal{PS}.\)

Furthermore, We will define the connecting maps \(\delta^0\) and \(\delta^1.\)

Let \(A, B\) be topological \(G\)-modules and \(f : A \to B\) a continuous \(G\)-homomorphism. We define \(H^i(G, f) = f^*_i : H^i(G, A) \to H^i(G, B), i = 0, 1,\) as follows:

For \(i = 0,\) take \(f_0^* = f|_{A^G}.\) This gives a homomorphism from \(H^0(G, A)\) to \(H^0(G, B),\) since \(f\) is a homomorphism of \(G\)-modules. So if \(a \in A^G,\) then, \(gf(a) = f(ga) = f(a),\) for each \(g \in G.\) Hence, \(f(a) \in B^G,\) i.e., \(f_0^*\) is well-defined.
For simplicity, we write $\alpha$ instead of $[\alpha] \in H^1(G, A)$.

If $\alpha \in H^1(G, A)$, then, take $f^*_i(\alpha) = f \circ \alpha$. Now if $g, h \in G$, then,

$$f^*_i(\alpha)(gh) = f(\alpha(gh)) = f(\alpha(g) \alpha(h)) = f(\alpha(g)) f(\alpha(h)) = f^*_i(\alpha)(g) f^*_i(\alpha)(h).$$

Thus, $f^*_i(\alpha)$ is a continuous derivation.

Moreover, if $\alpha, \beta \in H^1(G, A)$ are cohomologous then, there is $a \in A$ such that $\beta(g) = a^{-1} \alpha(g)^g a$. Hence, $f(\beta(g)) = f(a)^{-1} f(\alpha(g))^g f(a)$. So, $f^*_i(\alpha) \sim f^*_i(\beta)$.

The fact that $H^1(G, \_)$ is a functor follows from the definition of $f^*_i$, ($i = 0, 1$).

Also $H^0(G, \_)$ is a covariant functor from $G\mathcal{M}$ to the category of topological groups $\mathcal{T}G$.

Suppose that $1 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ is an exact sequence of topological $G$-modules and continuous $G$-homomorphisms such that $\iota$ is an embedding. Thus, we can identify $A$ with $\iota(A)$.

Now we define a coboundary map $\delta^0 : H^0(G, C) \to H^1(G, A)$.

Let $c \in H^0(G, C)$, $b \in B$ with $\pi(b) = c$. Then, we define $\delta^0(c)$ by $\delta^0(c)(g) = b^{-1}g b, \forall g \in G$. It is obvious that $\delta^0(c)$ is a continuous derivation. Let $b' \in B$, $\pi(b') = c$. Then, $b' = b a$ for some $a \in A$. So,

$$(b')^{-1}g b' = a^{-1} b^{-1} g b a = a^{-1} \delta^0(c)(g)^g a.$$  

Thus, the derivation obtained from $b'$ is cohomologous in $A$ to the one obtained from $b$, i.e., $\delta^0$ is well-defined.

Now, suppose that $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ is a central exact sequence of $G$-modules and continuous $G$-homomorphisms such that $\iota$ is a homeomorphic embedding and in addition $\pi$ has a continuous section $s : C \to B$, i.e., $\pi s = Id_C$.

We construct a coboundary map $H^1(G, C) \xrightarrow{\delta^1} H^2(G, A)$. Here $H^2(G, A)$ is defined in the sense of Hu [5]. By assumption $\iota(A) \subset Z(B)$, so, $A$ is an abelian topological $G$-module.

Let $\alpha \in H^1(G, C)$ and $s : C \to B$ be a continuous section for $\pi$. Define $\delta^1(\alpha)$ via $\delta^1(\alpha)(g, h) = s \alpha(g)^g (s \alpha(h))(s \alpha(gh))^{-1}$. It is clear that $\delta^1(\alpha)$ is a continuous map.

We show that $\delta^1(\alpha)$ is a factor set with values in $A$, and independent of the choice of the continuous section $s$. Also $\delta^1$ is well-defined.

Since $\alpha$ is a derivation, we have:

$$\pi(\delta^1(\alpha)(g, h)) = \pi(s \alpha(g)^g (s \alpha(h))(s \alpha(gh))^{-1}) = \alpha(g)^g \alpha(h)(\alpha(gh))^{-1} = 1.$$  

Thus, $\delta^1(\alpha)$ has values in $A$. 

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Next, we show that $\delta^1(\alpha)$ is a factor set, i.e.,

$$
\delta^1(\alpha)(h, k)\delta^1(\alpha)(g, hk) = \delta^1(\alpha)(gh, k)\delta^1(\alpha)(g, h), \ \forall g, h, k \in G. \ \ (3.1)
$$

First we calculate the left hand side of (3.1). For simplicity, take $b_g = s\alpha(g)$, $\forall g \in G$. Since $A \subset Z(B)$, thus,

$$
g\delta^1(\alpha)(h, k)\delta^1(\alpha)(g, hk) = g(b_h b_k b_{hk}^{-1})(b_g g b_{gh} b_{ghk}^{-1}) = b_g g(b_h b_k b_{hk}^{-1}) b_{gh} b_{ghk}^{-1}$$

$$= b_g g(b_h b_k) g b_{hk} b_{ghk}^{-1} = b_g g b_h b_k b_{ghk}^{-1}.
$$

On the other hand,

$$\delta^1(\alpha)(gh, k)\delta^1(\alpha)(g, h) = (b_{gh} g b_k b_{ghk}^{-1})(b_g b_{gh} b_{ghk}^{-1}) = b_g b_{gh} b_h b_{ghk}^{-1}.$$ 

Therefore, $\delta^1(\alpha)$ is a factor set.

Next, we prove that $\delta^1(\alpha)$ is independent of the choice of the continuous section. Suppose that $s$ and $u$ are continuous sections for $\pi$. Take $b_g = s\alpha(g)$ and $b_g' = u\alpha(g)$, for a fixed $\alpha \in Der_c(G, C)$. Since $\pi(b_g') = \alpha(g) = \pi(b_g)$, then, $b_g' = b_g a_g$ for some $a_g \in A$. Obviously the function $\kappa : G \to A$, defined by $\kappa(g) = a_g$, is continuous. Thus, 

$$
(\delta^1)'(\alpha)(g, h) = b_g b_h b_{gh} = b_g \kappa(g) g b_h \kappa(h)(\kappa(gh))^{-1} b_{gh}^{-1}$$

$$= (\kappa(g) g \kappa(h)(\kappa(gh))^{-1})(b_g g b_h b_{gh}^{-1}) = \delta^1(\kappa)(g, h)\delta^1(\alpha)(g, h),$$

where $\delta^1(\kappa)(g, h) = g\kappa(h)(\kappa(gh))^{-1} \kappa(g)$.

The coboundary map $\bar{\delta}^1 : \bar{C}^1(G, A) \to \bar{C}^2(G, A)$ is defined as in [5]. Consequently, $\delta^1(\kappa)$ and $(\delta^1)'(\kappa)$ are cohomologous.

Suppose that $\alpha$ and $\beta$ are cohomologous in $Der_c(G, A)$. Then, there is $c \in C$ such that $\beta(g) = c^{-1}\alpha(g)^g c$, $\forall g \in G$.

Let $s : C \to A$ be a continuous section for $\pi$. Since

$$\pi(s(c^{-1}\alpha(g)^g c)) = \pi(s(c)^{-1} s\alpha(g)^g s(c)),$$

then, there exists a unique $\gamma(g) \in ker\pi = A$ such that

$$\gamma(g)(s(c)^{-1} s\alpha(g)^g s(c)) = s(c^{-1}\alpha(g)^g c).$$

It is clear that the map $\gamma : G \to A$, $g \mapsto \gamma(g)$ is continuous. Therefore,

$$\delta^1(\beta)(g, h) = s\beta(g) g s\beta(h) (s\beta(gh))^{-1}$$

$$= s(c^{-1}\alpha(g)^g c) g s(c^{-1}\alpha(h)^h c) (s(c^{-1}\alpha(gh)^{gh} c))^{-1}$$

$$= \gamma(g)[s(c)^{-1} s\alpha(g)^g s(c)] \gamma(h)(s(c)^{-1} s\alpha(h)^h s(c)) (\gamma(gh)[s(c)^{-1} s\alpha(gh)^{gh} s(c)])^{-1}$$
= \gamma(g)\gamma(gh)^{-1}\gamma(g)[s(c)^{-1}sa(g)gs(c)] = \gamma(g)[s(c)^{-1}sa(gh)gs(c)]^{-1}

= \delta^1(\gamma)(g, h)[s(c)^{-1}sa(g)gs(hs(c)) = \delta^1(\gamma)(g, h)[\delta^1(a)(g, h)].

The last equality is obtained from the fact that \(\delta^1(\alpha, h) \in A \subset Z(B)\) and \(s(c) \in B\). Now, note that \(\delta^1(\alpha)\) is cohomologous to \(\delta^1(\beta)\), when \(\alpha\) is cohomologous to \(\beta\). Thus, \(\delta^1\) is well-defined.

4 A Cohomology Exact Sequence

Let \((X, x_0), (Y, y_0)\) be pointed sets in \(\mathcal{PS}\) and \(f : (X, x_0) \rightarrow (Y, y_0)\) a pointed map, i.e., \(f : X \rightarrow Y\) is a map such that \(f(x_0) = y_0\). For simplicity, we write \(f : X \rightarrow Y\) instead of \(f : (X, x_0) \rightarrow (Y, y_0)\). The kernel of \(f\), denoted by \(\text{Ker}(f)\), is the set of all points of \(X\) that are mapped to the basepoint \(y_0\). A sequence \((X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)\) of pointed sets and pointed maps is called an exact sequence if \(\text{Ker}(g) = \text{Im}(f)\).

**Theorem 4.1.** (i) Let \(\begin{array}{c}
1 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \xrightarrow{\delta} 1
\end{array}\) be a short exact sequence of topological \(G\)-modules and continuous \(G\)-homomorphisms, where \(i\) is homeomorphic embedding. Then, the following is an exact sequence of pointed sets,

\[
\begin{array}{cccc}
0 & \rightarrow & H^0(G, A) & \xrightarrow{i_0^*} \rightarrow \ H^0(G, B) & \xrightarrow{\pi_0^*} \rightarrow \ H^0(G, C) \\
\lower{1ex}\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow \\
\end{array}
\]  

\[
\begin{array}{cccc}
\delta_0^* & \rightarrow & H^1(G, A) & \xrightarrow{i_1^*} \rightarrow \ H^1(G, B) & \xrightarrow{\pi_1^*} \rightarrow \ H^1(G, C) \\
\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow \\
\end{array}
\]

(ii) In addition, if \(i(A) \subset Z(B)\), and \(\pi\) has a continuous section, then

\[
\begin{array}{cccc}
0 & \rightarrow & H^0(G, A) & \xrightarrow{i_0^*} \rightarrow \ H^0(G, B) & \xrightarrow{\pi_0^*} \rightarrow \ H^0(G, C) \\
\lower{1ex}\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow \\
\delta_0^* & \rightarrow & H^1(G, A) & \xrightarrow{i_1^*} \rightarrow \ H^1(G, B) & \xrightarrow{\pi_1^*} \rightarrow \ H^1(G, C) & \xrightarrow{\delta_1^*} \rightarrow \ H^2(G, A) \\
\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow \\
\end{array}
\]

is an exact sequence of pointed sets.

**Proof.** (i): We prove the exactness term by term.

1. Exactness at \(H^0(G, A)\): This is clear, since \(i\) is one to one.

2. Exactness at \(H^0(G, B)\): Since \(\pi_0^*\delta_0^* = (\pi i)_0^* = 1\), then, \(\text{Im}(i_0^*) \subset \text{Ker}(\pi_0^*)\). Now we show that \(\text{Ker}(\pi_0^*) \subset \text{Im}(i_0^*)\). If \(b \in \text{Ker}(\pi_0^*)\), then, \(\pi(b) = 1\)
and \( b \in H^0(G, B) \). There is an \( a \in A \) such that \( \iota(a) = b \). Moreover, \( \iota(ga) = \iota(a) \), \( \forall g \in G \). So, \( ga = a, \forall g \in G \), since \( \iota \) is one to one. Thus, \( a \in H^0(G, A) \). Hence, \( b \in \text{Im}(\iota_b^*) \).

3. Exactness at \( H^0(G, C) \): Take \( c \in \text{Im}(\pi_b^*) \). So, \( c = \pi(b) \) for some \( b \in H^0(G, B) \). Thus, \( \delta^0(c)(g) = b^{-1}g \). Hence, \( \delta^0(c) \sim \alpha_0 \). Conversely, if \( \delta^0(c) \sim \alpha_0 \), then, there is \( a_1 \in A \) such that \( \delta^0(c)(g) = a_1^{-1}a_1, \forall g \in G \). Let \( c = \pi(b) \) for some \( b \in B \). Then, by definition of \( \delta^0(c)(g) \), there is \( a_2 \in A \) such that \( b^{-1}g = a_2^{-1}\delta^0(c)(g)a_2, \forall g \in G \). So, \( b(a_1a_2)^{-1} \in H^0(G, B) \). Since \( \pi_0(b(a_1a_2)^{-1}) = c \), then, \( c \in \text{Im}\pi_b^* \).

4. Exactness at \( H^1(G, A) \): Let \( c \in H^0(G, C) \). Then, there is \( b \in B \) such that \( \pi(b) = c \). So,

\[
i_1^*\delta^0(c)(g) = \iota(\delta^0(c)(g)) = \iota(b^{-1}g) = b^{-1}g.
\]

Consequently, \( i_1^*\delta^0(c) \sim \beta_0 \), where \( \beta_0(g) = 1, \forall g \in G \). Conversely, let \( \alpha \in \text{Ker}i_1^* \). Then, there is \( b \in B \) such that \( \iota\alpha(g) = b^{-1}g, \forall g \in G \). So, \( \pi(b^{-1}g) = 1, \forall g \in G \). Take \( c = \pi(b) \). Hence, \( c \in \text{Ker}i_1^* \). Thus, \( \delta^0(c) \sim \iota(\alpha) = \alpha \).

5. Exactness at \( H^1(G, B) \): Since \( \pi_1^*i_1^* = (\pi\iota)_1^* = 1 \), then, \( \text{Im}\pi_1^* \subset \text{Ker}\pi_1^* \).

Conversely, let \( \beta \in \text{ker}\pi_1^* \). Then, there is \( c \in C \) such that \( \pi\beta(g) = c^{-1}g \), for all \( g \in G \). Let \( b \in B \) and \( c = \pi(b) \). Therefore, \( \pi(\beta(g)) = \pi(b^{-1}g) \), \( \forall g \in G \). On the other hand, the map \( \tau : A \to A, a \mapsto b^{-1}ab \), is a topological isomorphism, because \( A \) is a normal subgroup of \( B \). So, for every \( g \in G \) there is a unique element \( a_g \in G \) such that \( \beta(g) = (b^{-1}a_gb)(b^{-1}g) \). Thus, \( \beta(g) = b^{-1}a_g g b, \forall g \in G \). Hence, \( a_g = b\beta(g)g b^{-1}, \forall g \in G \). Obviously, the map \( \alpha : G \to A \) via \( \alpha(g) = a_g \) is a continuous derivation, and \( i_1^*(\alpha) \sim i_1^*(\beta) = \beta \).

(ii): It is enough to show the exactness at \( H^1(G, C) \). Let \( [\beta] \in H^1(G, B) \) and \( s \) be a continuous section for \( \pi \). Then, there is a continuous map \( z : G \to A \) such that \( s\pi\beta(g) = \beta(g)z(g) \). Thus,

\[
\delta^1(\pi^*_1(\beta))(g, h) = s(\pi\beta(g))g s(\pi\beta(h))(s(\pi\beta(gh)))^{-1} = \beta(g)g \beta(h) \beta(gh)^{-1} \delta^1(z)(g, h) = \delta^1(z)(g, h).
\]

So, \( \text{Im}\pi_1^* \subset \text{Ker}\delta^1 \). Conversely, let \( [\gamma] \in \text{ker}\delta^1 \). Then, there is a continuous function \( \alpha \in \tilde{C}^1(G, A) \) such that \( \delta^1(\gamma) = \tilde{\delta}^1(\alpha) \). Thus,

\[
s\gamma(g)^g s\gamma(h)(s\gamma(gh))^{-1} = g\alpha(h)\alpha(gh)^{-1} g\alpha(h), \forall g, h \in G.
\]

Assume \( \beta(g) = s\gamma(g)\alpha(g)^{-1}, \forall g \in G \). Since \( A \subset Z(B) \), then, \( \beta \) is a continuous derivation from \( G \) to \( B \). Also \( \pi\beta = \gamma \). Hence, \( \pi_1^*([\beta]) = [\gamma] \).

The following two corollaries are immediate consequences of Theorem 4.1.

**Corollary 4.2.** Let \( A \xrightarrow{i} B \xrightarrow{\pi} C \xrightarrow{1} 1 \) be a short exact sequence of discrete \( G \)-modules, and \( G \)-homomorphisms then, there is the exact sequence \( (i) \) of pointed sets.
**Corollary 4.3.** Let \( 0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 1 \) be a central short exact sequence of discrete \( G \)-modules and \( G \)-homomorphisms then, there is the exact sequence (ii) of pointed sets.

**Remark 4.4.** If we restrict ourselves to the discrete coefficients then, Corollary 4.2 and Corollary 4.3 are the same as Proposition 36 and Proposition 43 in [6, Chapter I], respectively.

**Lemma 4.5.** Let \( G \) be a connected group, and \( A \) a totally disconnected abelian topological \( G \)-module. Then, \( H^n(G, A) = 0 \) for every \( n \geq 1 \).

**Proof.** Consider the coboundary maps \( \tilde{\delta}^n : \tilde{C}^n(G, A) \to \tilde{C}^{n+1}(G, A) \). Since \( G \) is connected and \( A \) is totally disconnected then, \( G \) acts trivially on \( A \), and the continuous maps from \( G^n \) into \( A \) are constant. If \( n \) is an even positive integer then, one can see that \( \text{Ker} \tilde{\delta}^n = \tilde{C}^n(G, A) \) and \( \text{Im} \tilde{\delta}^{n-1} = \tilde{C}^n(G, A) \). Thus, \( H^n(G, A) = \frac{\text{Ker} \tilde{\delta}^n}{\text{Im} \tilde{\delta}^{n-1}} = \frac{\tilde{C}^n(G, A)}{\tilde{C}^{n}(G, A)} = 0 \). Now suppose that \( n \) is odd. It is easy to check that \( \text{Ker} \tilde{\delta}^n = 0 \). Consequently, \( H^n(G, A) = 0 \).

**Remark 4.6.** The existence of continuous section in theorem 4.1 is essential.

For example, consider the central short exact sequence of trivial \( S^1 \)-modules:

\[
0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{\pi} S^1 \to 1,
\]

here \( \pi \) is the exponential map, given by \( \pi(t) = e^{2\pi it} \) and \( \iota \) is the inclusion map. This central exact sequence has no continuous section. For if it has a continuous section then by [1, Lemma 3.5], \( \mathbb{R} \) is homeomorphic to \( \mathbb{Z} \times S^1 \). This is a contradiction since \( \mathbb{R} \) is connected but \( \mathbb{Z} \times S^1 \) is disconnected. Thus, \( \text{Hom}_c(S^1, \mathbb{R}) = 0 \). Now by Lemma 4.5, \( H^1(S^1, \mathbb{Z}) = H^1(S^1, \mathbb{R}) = H^2(S^1, \mathbb{Z}) = 0 \), On the other hand, \( H^1(S^1, S^1) = \text{Hom}_c(S^1, S^1) \neq 0 \). Thus, we don’t obtain the exact sequence (4.2).

### 5 Connectivity of Topological Groups

In this section by using the inflation and the restriction maps, we find a necessary and sufficient condition for connectivity of a topological group \( G \).
**Definition 5.1.** Let $A$ be a topological $G$-module and $A'$ a topological $G'$-module. Suppose that $\phi : G' \to G$, $\psi : A \to A'$ are continuous homomorphisms. Then, we call $(\phi, \psi)$ a cocompatible pair if

$$\psi(\phi(g')a) = g'\psi(a), \forall g' \in G', \forall a \in A.$$ 

For example, if $N$ is a subgroup of $G$ and $A$ a topological $G$-module then, $(1, Id_A)$ is a cocompatible pair, where $1 : N \to G$ is the inclusion map and $Id_A$ is the identity map. Also, suppose that $\pi : G \to G/N$ is the natural projection and $j : A^N \to A$ is the inclusion map. Then, $(\pi, j)$ is a cocompatible pair.

Note that a cocompatible pair $(\phi, \psi)$ induces a natural map as follows:

$$\text{Der}_c(G, A) \to \text{Der}_c(G', A')$$

by $\alpha \mapsto \psi \alpha \phi$,

which induces the map:

$$(\phi, \psi)^* : H^1(G, A) \to H^1(G', A')$$

by $[\alpha] \mapsto [\psi \alpha \phi]$.

**Definition 5.2.** Let $N$ be a subgroup of $G$ and $A$ a topological $G$-module. Suppose that $\iota : N \to G$ is the inclusion map. The induced map $(\iota, Id_A)^*$ is called the restriction map and it is denoted by $\text{Res}^1 : H^1(G, A) \to H^1(N, A)$.

**Definition 5.3.** Let $N$ be a normal subgroup of $G$ and $A$ a topological $G$-module. Suppose that $\pi : G \to G/N$ is the natural projection and $j : A^N \to A$ is the inclusion map. The induced map $(\pi, j)^*$ is called the inflation map and it is denoted by $\text{Inf}^1 : H^1(G/N, A^N) \to H^1(G, A)$.

Note that if $A$ is an abelian topological $G$-modules then, $\text{Inf}^1$ and $\text{Res}^1$ are group homomorphisms.

**Lemma 5.4.** Let $A$ be a topological $G$-module, and $N$ a normal subgroup of $G$. Then,

(i) $H^1(N, A)$ is a $G/N$-set. Moreover, if $A$ is an abelian topological $G$-module then, $H^1(N, A)$ is an abelian $G/N$-module.

(ii) $\text{Im Res}^1 \subset H^1(N, A)^{G/N}$.

**Proof.** (i) Since $N$ is a normal subgroup of $G$, then, there is an action of $G$ on $\text{Der}_c(N, A)$ as follows:

For every $g \in G$ we define $g\alpha = \tilde{\alpha}, \forall g \in G$, with $\tilde{\alpha}(n) = g\alpha(g^{-1}n), n \in N$.

In fact, $\tilde{\alpha}$ is continuous and we have:

$$\tilde{\alpha}(mn) = g\alpha(g^{-1}(mn)) = g\alpha(g^{-1}m^{-1}n) = g\alpha(g^{-1}m)n\alpha(g^{-1}n) = \tilde{\alpha}(m)n\tilde{\alpha}(n),$$
whence, \( \tilde{\alpha} \in \text{Der}_c(N, A) \). It is clear that \( g^h\alpha = g^{(h\alpha)} \). Moreover, if \( A \) is an abelian group, it is easy to verify that \( g(\alpha\beta) = g\alpha g\beta \). Now suppose that \( \alpha \sim \beta \). Then, there is an \( a \in A \) with \( \beta(n) = a^{-1}\alpha(n)n a, \forall n \in N \). Thus, for every \( g \in G, n \in N \),

\[
g\beta(g^{-1}n) = g a^{-1}(g\alpha(g^{-1}n))g(\alpha^{-1}n a).
\]

Therefore,

\[
\tilde{\beta}(n) = (ga)^{-1}\tilde{\alpha}(n)(ga), \text{i.e., } \tilde{\alpha} \sim \tilde{\beta}.
\]

Thus, the action of \( G \) on \( \text{Der}_c(N, A) \) induces an action of \( G \) on \( H^1(N, A) \). It is sufficient to show for every \( m \in N, m\alpha \sim \alpha \). In fact, for every \( n \in N \)

\[
m\alpha(m^{-1}n) = m\alpha(m^{-1}nm) = m(\alpha(m^{-1})m^{-1}\alpha(n)m^{-1}n\alpha(m)) = m\alpha(m^{-1})\alpha(n)n\alpha(m) = \alpha(m)^{-1}\alpha(n)n\alpha(m).
\]

Thus, \( \tilde{\alpha} \sim \alpha \).

(ii) By a similar argument as in (i), we have

\[
g\alpha(g^{-1}n) = \alpha(g)^{-1}\alpha(n)n\alpha(g), \forall g \in G, n \in N
\]

whence, \( g^N(\alpha) \sim \alpha, \forall gN \in G/N \).

**Lemma 5.5.** Let \( N \) be a normal subgroup of a topological group \( G \) and \( A \) a topological \( G \)-module. Then, there is an exact sequence

\[
1 \longrightarrow H^1(G/N, A^N) \xrightarrow{\text{Inf}^1} H^1(G, A) \xrightarrow{\text{Res}^1} H^1(N, A)^{G/N}.
\]

**Proof.** The map \( \text{Inf}^1 \) is one to one: If \( \alpha, \beta \in \text{Der}_c(G/N, A^N) \) and \( \text{Inf}^1[\alpha] = \text{Inf}^1[\beta] \), then, \( \alpha\pi \sim \beta\pi \). Thus, there is an \( a \in A \) such that \( \beta\pi(g) = a^{-1}\alpha\pi(g)g\alpha, \forall g \in G \). Hence, \( \beta(gN) = a^{-1}\alpha(gN)g\alpha, \forall gN \in G/N \). On the other hand, if \( g \in G \), then, \( \alpha(gN) = \beta(gN) = 1 \), and hence, \( a \in A^N \). This implies that \( (g\alpha)^n = g\alpha, \forall g \in G \). Consequently, \( \alpha \sim \beta \), i.e., \( \text{Inf}^1 \) is one to one.

Now we show that \( \text{KerRes}^1 = \text{ImInf}^1 \). Since \( \text{Res}^1\text{Inf}^1[\alpha] = [\alpha(\pi\iota)] = 1 \), then, \( \text{ImInf}^1 \subset \text{KerRes}^1 \).

Let \( [\alpha] \in \text{KerRes}^1 \). Then, there is an \( a \in A \) such that \( \alpha(n) = a^{-1}n a, \forall n \in N \). Consider the continuous derivation \( \beta \) with \( \beta(g) = aa^{-1}g\), \( \forall g \in G \). Since \( \beta(n) = 1, \forall n \in N \) then, \( \beta \) induces the continuous derivation \( \gamma : G/N \to A \) via \( \gamma(gN) = \beta(g) \). Also \( \text{Im}\gamma \subset A^N \), since for all \( n \in N \),

\[
n\gamma(gN) = n\beta(g) = \beta(g)g\beta(g^{-1}ng) = \beta(g) = \gamma(gN).
\]

Hence, \( \text{Inf}^1[\gamma] = [\gamma\pi] = [\beta] = [\alpha] \). Consequently, \( \text{KerRes}^1 \subset \text{ImInf}^1 \).
Lemma 5.6. Let $G$ be a topological group and $A$ a topological $G$-module. Suppose that $A$ is totally disconnected and $G_0$ the identity component of $G$. Then, the map

$$H^1(G/G_0, A) \xrightarrow{Inf^1} H^1(G, A)$$

is bijective.

Proof. Since $G_0$ acts trivially on $A$, then, $A^{G_0} = A$. On the other hand, $H^1(G_0, A) = 1$. Thus, by Lemma 5.5, the sequence

$$0 \longrightarrow H^1(G/G_0, A) \xrightarrow{Inf^1} H^1(G, A) \longrightarrow 0$$

is exact.

Theorem 5.7. Let $G$ be a topological group which has an open component. Then, $G$ is connected iff $H^1(G, A) = 1$ for every discrete abelian $G$-module $A$.

Proof. Assume $G$ is a connected group and $A$ a discrete abelian $G$-module. Since every discrete $G$-module $A$ is totally disconnected then, $H^1(G, A) = 1$. Conversely, Suppose that $H^1(G, A) = 1$, for every discrete abelian $G$-module $A$. By Lemma 5.6, $H^1(G/G_0, A) = 1$, for every discrete abelian $G$-module $A$. Since $G/G_0$ is discrete, then, the cohomological dimension of $G/G_0$ is equal to 0 which implies that $G/G_0 = 1$ [4, Chapter VIII], i.e., $G = G_0$.

6 Complements and First Cohomology

Let $G$ and $A$ be topological groups. Suppose that $\chi : G \times A \to A$ is a continuous map such that $\tau_g : A \to A$, defined by $\tau_g(a) = \chi(g, a)$, is a homeomorphic automorphism of $A$ and the map $g \mapsto \tau_g$ is a homomorphism of $G$ into the group of homeomorphic automorphisms, $Aut_h(A)$, of $A$. By $G \ltimes \chi A$ we mean the (topological) semidirect product with the group operation, $(g, a)(h, b) = (gh, \tau_h(a)b)$, and the product topology of $G \times A$. Sometimes for simplicity we denote $G \ltimes \chi A$ by $G \ltimes A$ and view $G$ and $A$ as topological subgroups of $G \ltimes A$ in a natural way. So every element $e$ in $G \ltimes N$ can be written uniquely as $e = gn$ for some $g \in G$ and $n \in N$.

Let $E = G \ltimes N$. A subgroup $X$ of $E$ such that $E \simeq X \ltimes N$ is called a complement of $N$ in $E$. Indeed, any conjugate of $G$ is a complement.

We show that the complements of $N$ in $E$ correspond to continuous derivations from $G$ to $N$. If $X$ is any complement, for every $g \in G$, then, $g^{-1}$ has a unique expression of the form $g^{-1} = xn$ where $x \in X$ and $n \in N$. Define $\alpha_X : G \to N$ by $\alpha_X(g) = n$. Obviously, $\alpha_X(g) = \pi_2|_G(g^{-1})$, where
\( \pi_2 : X \ltimes N \to N \) is given by \( \pi_2(x,n) = n \). Hence, \( \alpha_X \) is continuous. Now if \( g_i \in G \) then, \( g_i^{-1} = x_i n_i \) for some \( x_i \in X \), \( n_i \in N \), \( i = 1, 2 \). We have:

\[
(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1} = x_2 n_2 x_1 n_1 = x_2 x_1^{n_1^{-1} x_1^{-1}} n_2 = x_2 x_1 n_1^{g_1 g_2}.
\]

By definition of \( \alpha \), \( \alpha_X(g_1 g_2) = \alpha_X(g_1)^{g_2} \alpha_X(g_2) \), i.e., \( \alpha_X \in \text{Der}_c(G,N) \).

Conversely, suppose that \( \alpha : G \to N \) is a continuous derivation. Then, \( X_\alpha = \{ \alpha(g)g | g \in G \} \subset E \) is a corresponding complement to \( \alpha \) in \( E \). Obviously, the continuous map \( \kappa : g \mapsto \alpha(g)g \), is a homomorphism. Suppose that \( \pi_1 : G \ltimes N \to G \) is given by \( \pi_1(g,n) = g \). Hence, \( \pi_1|_{X_\alpha} : X_\alpha \to G \) is the inverse of \( \kappa \), since \( \pi_1|_{X_\alpha}(\alpha(g)g) = \pi_1|_{X_\alpha}(g^{\kappa^{-1} n}) = g, \forall g \in G \). Thus, \( X_\alpha \cong G \).

Define the map \( \chi : X_\alpha \times N \to N \) by \( \chi(\alpha(g)g) = g^g n \), for all \( g \in G, n \in N \). Clearly, \( \chi \) is a continuous map. Hence, \( X_\alpha \ltimes N \cong G \ltimes N = E \).

In fact we have proved the following theorem.

**Theorem 6.1.** Let \( G \) be a topological group and \( N \) a topological \( G \)-module. Then, the map \( X \mapsto \alpha_X \) is a bijection from the set of all complements of \( N \) in \( G \ltimes N \) onto \( \text{Der}_c(G,N) \).

**Theorem 6.2.** If \( A \) is a topological \( G \)-module then, there is a map from \( H^1(G,A) \) onto the set of conjugacy classes of complements of \( A \) in \( G \ltimes A \). Moreover, if \( A \) is an abelian group then, this map is one to one.

**Proof.** Suppose that \( X \) and \( Y \) are the complements of \( A \) in \( G \ltimes A \) such that \( \alpha_X \sim \alpha_Y \). Hence, there is \( a \in A \) such that \( \alpha_Y(g) = a^{-1} \alpha_X(g)^a, \forall g \in G \). Thus, for each \( g \in G \), we have \( \alpha_Y(g)g = a^{-1} \alpha_X(g)^a g a = a^{-1} \alpha_X(g) ga \). This implies that \( X = a^{-1} Y \).

Moreover, suppose that \( A \) is an abelian group and \( X \) and \( Y \) are conjugate complements. So, \( X = n^y Y \) for some \( n \in N \). If \( g \in G \), then, \( \alpha(g)g \in X \) where \( \alpha_X \) is a continuous derivation arising from \( X \). Hence, \( \alpha_X(g)g = n^y \) for some \( y \in Y \). Now \( n^y = [n,y]y \), so, \( \alpha_X(g)g = [n,y]y \), which shows that

\[
[n,g] = n g n^{-1} g^{-1} = n (\alpha_X(g)^n y) n^{-1} (\alpha_X(g)^n y)^{-1} = n (\alpha_X(g)) n y n^{-1} y^{-1} (\alpha_X(g))^{-1} = (n \alpha_X(g))(n y n^{-1} y^{-1}) (n^{-1} \alpha_X(g))^{-1} = [n,y] = n^y y^{-1},
\]

because \( A \) is an abelian group. Therefore,

\[
g^{-1} = y^{-1} [n,y]^{-1} \alpha_X(g) = y^{-1} (y^n y^{-1} \alpha_X(g)) = y^{-1} (y \alpha_X(g)^n y^{-1}).
\]

Thus, by definition of \( \alpha_Y \), we get \( \alpha_Y(g) = y \alpha_X(g)^n y^{-1} \). Consequently, \( \alpha_X \sim \alpha_Y \).

As an immediate result, we have the following corollary.
Corollary 6.3. Let $A$ be a topological $G$-module and $H^1(G, A) = 1$. Then, the complements of $A$ in $G \ltimes A$ are conjugate.

7 Vanishing of $H^1(G, A)$

Let $G$ be a compact Hausdorff group and $A$ a topological $G$-module. Suppose that $A$ is an almost connected locally compact Hausdorff group. Then, we prove there exists a $G$-invariant maximal compact subgroup $K$ of $A$, and for every such topological submodule $K$, the natural map $t_1^*: H^1(G, K) \to H^1(G, A)$ is onto. In addition, as a result, If $A$ has trivial maximal compact subgroup then, $H^1(G, A) = 1$.

Recall that $G$ is almost connected if $G/G_0$ is compact where $G_0$ is the connected component of the identity of $G$.

Definition 7.1. An element $g \in G$ is called periodic if it is contained in a compact subgroup of $G$. The set of all periodic elements of $G$ is denoted by $P(G)$.

Definition 7.2. A maximal compact subgroup $K$ of a topological group $G$ is a subgroup $K$ that is a compact space in the subspace topology, and maximal amongst such subgroups.

If a topological group $G$ has a maximal compact subgroup $K$, then, clearly $gKg^{-1}$ is a maximal compact subgroup of $G$ for any $g \in G$. There exist topological groups with maximal compact subgroups and compact subgroups which are not contained in any maximal one [3]. Note that if $G$ is almost connected then, $P(G/G_0) = G/G_0$.

Lemma 7.3. Let $G$ be a locally compact topological group such that $P(G/G_0)$ is a compact subgroup of $G/G_0$, and $K$ a maximal compact subgroup of $G$. Then, any compact subgroup of $G$ can be conjugated into $K$ [3, Theorem 1].

Lemma 7.4. Let $G$ be a compact group and $A$ a topological $G$-module such that $A$ is a locally compact almost connected, and let $C$ be a $G$-invariant compact subgroup of $A$. Then, there exists a $G$-invariant maximal compact subgroup $K$ of $A$ which contains $C$.

Proof. Let $E = G \times A$, be the semidirect product of $A$ and $G$ with respect to the action of $G$ on $A$. Note that topologically $E$ is the product of $A$ and $G$. We first observe that $E/E_0$ is almost connected. Let $A_0, G_0$ and $E_0$ be the components of $A, G$ and $E$, respectively. It is easily seen that $E_0 = A_0 \times G_0$. Also $E/(A_0 \times G_0)$ is homeomorphic to the compact space $A/A_0 \times G/G_0$. Hence, $E/E_0$ is compact. Consequently, $E$ is almost connected. Now, by assumption, $C$ is a $G$-invariant compact subgroup of $A$. Thus, $G \ltimes C$ is a compact subgroup.
of $E$. Since $E$ is almost connected, there exists a maximal compact subgroup $L$ of $E$ which contains $G \ltimes C$. Let $K = L \cap A$. Since $K$ is a closed subspace of $L$, then, $K$ is compact. Also $L$ contains $G$. Thus, $L$ is $G$-invariant. In fact, for every $g \in G$ and every $\ell \in L$, we have $g\ell = g\ell g^{-1} \in L$. This immediately implies that $K$ is $G$-invariant, since $A$ is $G$-invariant. Let $K'$ be a compact subgroup of $G$. By Lemma 7.3, there is $e \in E$ such that $eK'e^{-1} \subset L$. Thus, $eK'e^{-1} \subset L \cap A = K$. But there exist $g \in G$ and $a \in A$ such that $e = ga$. Thus, $aK'a^{-1} \subset g^{-1}Kg = g^{-1}K = K$. Therefore, $K$ is a $G$-invariant maximal compact subgroup of $A$ which contains $C$.

**Theorem 7.5.** Let $G$ be a compact Hausdorff group and $A$ a topological $G$-module. Let $A$ be an almost connected locally compact Hausdorff group. Then, there exists a $G$-invariant maximal compact subgroup $K$ of $A$, and for every such topological submodule $K$, the natural map $\iota_1^* : H^1(G,K) \to H^1(G,A)$ is onto.

**Proof.** By Lemma 7.4, there exists a $G$-invariant maximal compact subgroup $K$ of $A$. Also $G \ltimes K$ is a maximal compact subgroup of $G \ltimes A$ [2, Theorem 1.1]. Let $\alpha : G \to A$ be a continuous derivation. Then, define the continuous homomorphism $\kappa : G \to G \ltimes A$ via $g \mapsto \alpha(g)g$. Since $\kappa$ is a continuous homomorphism then, $\kappa(G)$ is a compact subgroup of $G \ltimes A$. By Lemma 7.3 there is $ag \in G \ltimes A$ such that $(ag)\kappa(G)(ag)^{-1} \subset G \ltimes K, \forall x \in G$. This is equivalent to $(ag)\alpha(x)x(a^{-1})^{-1} \subset G \ltimes K, \forall x \in G$. Hence, for all $x \in G$, $g[a^{-1}a\alpha(x)x(a^{-1})]gxy^{-1} \in G \ltimes K$. Since $K$ is $G$-invariant then, $(g^{-1}a)\alpha(x)x(g^{-1}a^{-1}) \subset K, \forall x \in G$. Now define $\beta : G \to K$ by $\beta(x) = (g^{-1}a)\alpha(x)x(g^{-1}a^{-1}), \forall x \in G$. Hence, $\iota_1^*([\beta]) = [\alpha]$, i.e., $\iota_1^*$ is onto map.

**Corollary 7.6.** Let $G$ be a compact Hausdorff group and $A$ a topological $G$-module. Let $A$ be an almost connected locally compact Hausdorff group with the trivial maximal compact subgroup. Then, $H^1(G,A) = 1$.

**Proof.** It is clear.

**References**

[1] R.C. Alperin and H. Sahleh, Hopf’s formula and the Schur multiplicator for topological groups, *Kyungpook Math. Journal*, 31(1) (1991), 35-71.

[2] J. An, M. Liu and Z. Wang, Nonabelian cohomology of compact lie groups, *J. Lie Theory*, 19(2009), 231-236.

[3] R.W. Bagley and M.R. Peyrovian, A note on compact subgroups of topological groups, *Bull. Austral. Math. Soc.*, 33(1986), 273-278.
[4] K.S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, Springer Verlag, Berlin/ New York, 87(1982).

[5] S.T. Hu, Cohomology theory in topological groups, *Michigan Math. J.*, 1(1) (1952), 11-59.

[6] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin, (1997).