The induced representations of Brauer algebra
and the Clebsch-Gordan coefficients of $SO(n)$

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Abstract

Induced representations of Brauer algebra $D_f(n)$ from $S_{f_1} \times S_{f_2}$ with $f_1 + f_2 = f$ are discussed. The induction coefficients (IDCs) or the outer-product reduction coefficients (ORCs) of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f \leq 4$ up to a normalization factor are derived by using the linear equation method. Weyl tableaus for the corresponding Gel’fand basis of $SO(n)$ are defined. The assimilation method for obtaining CG coefficients of $SO(n)$ in the Gel’fand basis for no modification rule involved couplings from IDCs of Brauer algebra are proposed. Some isoscalar factors of $SO(n) \supset SO(n-1)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0]$ with $\sum_{i=1}^{4} \lambda_i \leq 4$ are tabulated.

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1. Introduction

Clebsch-Gordan Coefficients (CGCs) are of importance in many physical problems. Besides those\(^\text{[1,2]}\) of SO(3) and SO(4), which were discussed extensively and expressed in various forms, Isoscalar Factors (ISFs) of \(SO(n) \supset SO(n-1)\), which can be used to evaluate CGCs of SO(n) in its canonical basis according to the Racah factorization lemma,\(^\text{[3]}\) were derived analytically for the coupling \([l_1, 0] \times [l_2, 0]\) to \([L_1, L_2, 0]\) by using substitution group technique.\(^\text{[4]}\) The ISFS for the coupling \([l_1, l_2, 0] \times [l_3, 0]\) to \([L_1, L_2, 0]\) of some special cases were derived by using group chain transformation method.\(^\text{[5]}\) A special class of multiplicity-free \(O(n) \supset O(n-1)\) isoscalar factors were derived in the Gel’fand basis in\(^\text{[6]}\). Very recently, isoscalar factors of \(O(n) \supset O(n-1)\) for the coupling \([l_1, l_2, l_3, 0] \times [1, 0]\) have been derived by using the irreducible tensor basis method.\(^\text{[7]}\) However, unlike those of SU(n) case, CGCs of SO(n) in its canonical basis or ISFs of \(SO(n) \supset SO(n-1)\), generally, are rank \(n\) dependent, which makes it difficult to derive them for general case analytically. On the other hand, analytical expressions of the CGCs in most cases are very complicated and difficult to be used in applications. Therefore, tables of CGCs or ISFs, if available, are more convenient in practical use.

In this paper, we will outline a procedure for deriving CGCs of \(SO(n)\) in its canonical basis from Induction coefficients (IDCs) of \(S_{f_1} \times S_{f_2} \uparrow D_f(n)\). Brauer algebra \(D_f(n)\), which are similar to the group algebra of symmetric group \(S_f\) related to the decomposition of \(f\)-rank tensors of the general linear group GL(n), are the centralizer algebra of the orthogonal group O(n) or the symplectic group Sp(2m) when \(n = -2m\). More precisely, if \(G\) is the orthogonal group O(n) or the symplectic group Sp(2m), the corresponding centralizer algebra \(B_f(G)\) are quotients of Brauer’s \(D_f(n)\) and \(D_f(-2m)\), respectively. Hence, duality relation between \(D_f(n)\) and O(n) or Sp(2m) is quite the same as the Schur-Weyl duality relation between \(S_f\) and GL(n). Irreducible representations of \(D_f(n)\) in the standard basis, i.e. the basis adapted to the chain \(D_f(n) \supset D_{f-1}(n) \supset \cdots \supset D_2(n)\), have been constructed by using the induced representation and linear equation method,\(^\text{[8]}\) and more elaborately by Leduc and Ram using the so-called ribbon Hopf algebra approach.\(^\text{[9]}\) Racah
coefficients of O(n) and Sp(2m) were successfully derived from Subduction coefficients (SDCs) of $D_f(n)$ by using the Brauer-Schur-Weyl duality relation. A new simple Young diagrammatic method for Kronecker products of O(n) and Sp(2m) was also formulated, which, actually, is based on the induced representation theory of Brauer algebra discussed in this paper.

In Sec. 2, we will briefly review irreps of $D_f(n)$ in the standard basis. Then, induced representations of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f$ will be defined. In Sec. 3, based on the linear equation method, which has been proved effective in evaluating IDCs, SDCs of Hecke algebra and SDCs of Brauer algebra, a procedure for the evaluation of IDCs of $D_f(n)$ will be outlined. In Sec. 4, Weyl tableaux for SO(n) in its canonical basis will be defined. Then, a general procedure for evaluating CGCs of SO(n) in its canonical basis for no modification rule involved couplings will be outlined. Finally, some analytical expressions for the ISFs of $SO(n) \supset SO(n - 1)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \hat{0}]$ with $\sum_{i=1}^4 \lambda_i \leq 4$ will be tabulated in Sec. 5.

2. Brauer algebra and its outer-product basis

Brauer algebra $D_f(n)$ is defined algebraically by $2f - 2$ generators $\{g_1, g_2, \ldots, g_{f-1}, e_1, e_2, \ldots, e_{f-1}\}$ satisfying the following relations:

\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i, \quad |i - j| \geq 2, \]  
\[ (1a) \]
\[ e_i g_i = e_i, \quad e_i g_i e_i = e_i. \]  
\[ (1b) \]

Using the above defined relations and by drawing pictures of link diagrams, one can also derive other useful ones. For example,

\[ e_i e_j = e_j e_i, \quad |i - j| \geq 2, \]
\[ e_i^2 = n e_i, \]

4
\[(g_i - 1)^2(g_i + 1) = 0. \quad (2)\]

We assume that the base field is \( \mathbb{C} \). The star operation, a conjugate linear map \( \dagger \), on \( D_f(n) \) is defined by

\[
g_i^\dagger = g_i, \quad e_i^\dagger = e_i \quad \text{for} \quad i = 1, 2, \ldots, f - 1, \quad (3)
\]

which are necessary in defining orthonormal basis of \( D_f(n) \).

It is easy to see that \( \{g_1, g_2, \cdots, g_{f-1}\} \) generate a subalgebra \( \mathbb{C}S_f \), which is isomorphic to the group algebra of the symmetric group; that is, \( D_f(n) \supset \mathbb{C}S_f \). The properties of \( D_f(n) \) have been discussed in [12-13]. Based on these results, it is known that \( D_f(n) \) is semisimple, i.e., it is a direct sum of full matrix algebra over \( \mathbb{C} \), when \( n \) is not an integer or is an integer with \( n \geq f - 1 \), otherwise \( D_f(n) \) is no longer semisimple. In the following, we assume that \( n \) is an integer with \( n \geq f - 1 \). In this case \( D_f(n) \) is semisimple. Irreducible representations of \( D_f(n) \) can be denoted by a Young diagram with \( f, f - 2, f - 4, \cdots, 1 \) or 0 boxes. An irrep of \( D_f(n) \) with \( f - 2k \) boxes is denoted as \( [\lambda]_{f-2k} \). The branching rule of \( D_f(n) \downarrow D_{f-1}(n) \) is

\[
[\lambda]_{f-2k} = \oplus_{[\mu] \rightarrow [\lambda]} [\mu],
\]

where \( [\mu] \) runs through all the diagrams obtained by removing or (if \( [\lambda] \) contains less than \( f \) boxes) adding a box to \( [\lambda] \). Hence, the basis vectors of \( D_f(n) \) in the standard basis can be denoted by

\[
\begin{pmatrix}
[\lambda]_{f-2k} & D_f(n) \\
[\mu] & D_{f-1}(n) \\
\vdots & \vdots \\
[p] & D_{f-p+1}(n) \\
[\nu] & D_{f-p}(n)
\end{pmatrix}
= \begin{pmatrix}
[\lambda]_{f-2k} \\
[\mu] \\
\vdots \\
[p] \\
Y_{M}^{[\nu]}
\end{pmatrix},
\quad (4)
\]
where \( \nu \) is identical to the same irrep of \( S_{f-p} \), \( Y_M^{[\nu]} \) is a standard Young tableau, and \( M \) can be understood either as the Yamanouchi symbols or indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols. Irreps in the standard basis given by (4) were given in [8] for \( f \leq 5 \). Higher dimensional results can also derived by using the method given in [8] or by using Leduc and Ram’s formulae.

In order to study CGCs of SO(n), we need consider induced representations of Brauer algebra, \( S_{f_1} \times S_{f_2} \uparrow D_f(n) \) with \( f_1 + f_2 = f \), for the outer-products

\[
[\lambda_1] \times [\lambda_2] \uparrow \sum \lambda \{\lambda_1 \lambda_2 \lambda\} [\lambda],
\]

where \( \{\lambda_1 \lambda_2 \lambda\} \) is number of occurrence of irrep \( [\lambda] \) in the outer-product \([\lambda_1] \times [\lambda_2] \). The standard basis vectors of \([\lambda_1]_{f_1} \) and \([\lambda_2]_{f_2} \) for \( D_{f_1}(n) \) and \( D_{f_2}(n) \), which are the same as those for \( S_{f_1} \) and \( S_{f_2} \), can be denoted by \( |Y^{[\lambda_1]}_{m_1}(\omega_1^0)\rangle \), and \( |Y^{[\lambda_2]}_{m_2}(\omega_2^0)\rangle \), respectively, where

\[
(\omega_1^0) = (1, 2, \cdots, f_1), \quad (\omega_2^0) = (f_1 + 1, f_1 + 2, \cdots, f_1 + f_2)
\]

are indices in the standard tableaus \( Y^{[\lambda_1]}_{m_1} \) and \( Y^{[\lambda_2]}_{m_2} \), respectively. The product of the two basis vectors is denoted by

\[
|Y^{[\lambda_1]}_{m_1}, Y^{[\lambda_2]}_{m_2}, (\omega_1^0), (\omega_2^0)\rangle \equiv |Y^{[\lambda_1]}_{m_1}(\omega_1^0)\rangle |Y^{[\lambda_2]}_{m_2}(\omega_2^0)\rangle,
\]

which is called primitive uncoupled basis vector.

When \( n \) is a positive integer, we can use tensor products of the rank-1 unit tensor operator of \( O(n) \) to construct the basis of \( D_f(n) \) in the standard basis explicitly through so-called cabling [8]. In this case the indices \( 1, 2, \cdots, f \) are used to distinguish tensor operators from different spaces. We also need a set of the corresponding indices \( i_1, i_2, \cdots, i_f \) to label the tensor components which can be taken as \( n \) different values, namely

\[
T^{1}_{i_1} T^{2}_{i_2} \cdots T^{f}_{i_f} = T^{i_{12} \cdots f}_{i_{12} \cdots f}.
\]
The actions of $g_i$ and $e_i$ on (8) are given by

$$g_i T_{j_1 j_2 \ldots j_i j_{i+1} \ldots j_f}^1 \rightarrow T_{j_1 j_2 \ldots j_{i} j_{i+1} \ldots j_f}^1,$$

$$e_i T_{j_1 j_2 \ldots j_i j_{i+1} \ldots j_f}^1 = \delta_{j_i j_{i+1}} \sum_j (s) T_{j_1 j_2 \ldots j_{i} j_{i+1} \ldots j_f}^1,$$  \hspace{1cm} (9a)

where the sum, $\sum^{(s)}$, on the right hand means

$$\sum_j^{(s)} T_{j j}^{12} = \sum_{j \notin SO(2)} T_{j j}^{12} - (T_{a_2}^{1} \frac{2}{-a_2} + T_{-a_2}^{1} \frac{2}{-a_2}).$$ \hspace{1cm} (9b)

In order to discuss couplings of $SO(n)$ in the canonical basis, i.e. the basis adapted to $SO(n) \supset SO(n-1) \supset SO(n-2) \supset \cdots \supset SO(2)$, the rank-1 $SO(n)$ tensor components are classified according to the $SO(n) \supset SO(n-1)$ reduction. Namely, the tensor components of $j$ for rank-1 tensor $T_j^{(1)}$ of $SO(n)$ are labeled by $j = \pm \alpha_2, \alpha_3, \cdots \alpha_n$. The minus sign introduced in (9b) are consistent with the Condon-Shortley phase convention$[^1]$ for CGCs of $SO(3)$. We assume that $\{T_{j_1 j_2 \ldots j_f}^{1 \cdot \cdot f}\}$ spans a orthonormal inner product space, namely

$$\left( T_{j' j'_{1} j'_{2} \ldots j'_{f'}}^{1' \cdot \cdot f'}, T_{j_{1} j_{2} \ldots j_{f}}^{1 \cdot \cdot f} \right) = \prod \delta_{j_{i}, j'_{i}}.$$ \hspace{1cm} (10)

Then, the primitive uncoupled basis vectors given by (7) can be expressed in terms of these $T$ operators. For example, $S_1 \times S_1$ basis vector can be expressed as

$$|1> = |1, 2> = T_{i_1 i_2}^{11} T_{i_1 i_2}^{22}.$$ \hspace{1cm} (11)

Other uncoupled basis vectors can be obtained by acting $g_1$, and $e_1$, respectively, on (11).

$$|2> = g_1 |1> = T_{i_1 i_2}^{21} T_{i_1 i_2}^{11}, \quad |3> = e_1 T_{i_1 i_2}^{11} T_{i_1 i_2}^{22} = \delta_{i_1, i_2} \sum_i^{(s)} T_{i}^{11} T_{i}^{22}.$$ \hspace{1cm} (12)

The left coset decomposition of $D_f(n)$ with respect to the subalgebra $S_{f_1} \times S_{f_2}$ is denoted by
$$D_f(n) = \sum_{\omega}^\oplus Q^k_\omega (S_{f_1} \times S_{f_2}),$$

(13)

where the left coset representatives \(\{Q^k_\omega\}\) have two types of operations. One is the order-preserving permutations,

$$Q^k_\omega = 0 \omega = (\omega_1, \omega_2),$$

(14)

where

$$\omega_1 = (a_1, a_2, \ldots, a_{f_1}), \quad \omega_2 = (a_{f_1+1}, a_{f_1+2}, \ldots, a_f)$$

(15)

with \(a_1 < a_2 < \cdots < a_{f_1}, a_{f_1+1} < a_{f_1+2} < \cdots < a_f\), and \(a_i\) represents any one of the numbers 1, 2, \ldots, \(f\). The other, \(\{Q^k_\omega \geq 1\}\), contains \(k\)-time trace contractions between two sets of indices \((\omega_1)\) and \((\omega_2)\). For example, in \(S_2 \times S_1 \uparrow D_3(n)\) for the outer product \([2] \times [1]\), there are six elements in \(\{Q^k_\omega\}\) with

$$\{Q^0_\omega\} = \{1, g_2, g_1 g_2\}, \quad \{Q^1_\omega\} = \{e_2, g_1 e_2, e_1 g_2\}.$$ 

(16)

The ordering of the sequences \((\omega)\) is specified in the following way. If there is no trace contraction, we regard the part \((\omega_1) = (a_1, a_2, \ldots, a_{f_1})\) as a vector of length \(f_1\). If the last nonzero component of the vector \((\omega_1) - (\bar{\omega}_1)\) is less than zero, then we say \((\omega) \leq (\bar{\omega}_1)\). This ordering of \((\omega_1, \omega_2)\) is consistent with that for symmetric groups [chen]. If there is \(k\)-time trace contraction, we regard \(\omega^k\) as vector of length \(k\) with the components \((a_{i_1} a_{i'_1})(a_{i_2} a_{i'_2}) \cdots (a_{i_k} a_{i'_k})\). If the last nonzero component of the vector \(\omega^k - \bar{\omega}^k\) is less than zero, we say \(\omega^k < \bar{\omega}^k\). The total order of \((\omega_1) (\omega_2)\) is specified by \(k = 0, 1, 2, \cdots, \min(f_1, f_2)\), where \((\omega_1), (\omega_2)\) stands for \(k\)-time contractions between indices in \((\omega_1)\) and \((\omega_2)\). For example, in \(S_2 \times S_1 \uparrow D_3(n)\) for the outer-product \([2] \times [1]\), the six elements are arranged as \(\{1, g_2, g_1 g_2, e_1 g_2, g_1 e_2, e_2\}\).
The uncoupled basis vectors needed in construction of the coupled basis vectors of \([\lambda]\) for \(D_f(n)\), are denoted by

\[
Q^k_\omega |Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}; (\omega_1^0), (\omega_2^0) > = |Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}; (\omega_1), (\omega_2) > ,
\]

(17)
The basis vectors of \([\lambda]f_{-2k}\) can thus be expressed in terms of the uncoupled basis vectors given by (17):

\[
|[\lambda]f_{-2k}, \tau; \rho > = \sum_{m_1 \omega k'} C^{[\lambda]f_{-2k} \rho \tau}_{m_1 m_2 k' \omega} Q^k_\omega |Y_{m_1}^{[\lambda_1]}(\omega_1^0), Y_{m_2}^{[\lambda_2]}(\omega_2^0) > ,
\]

(18)
where \(\rho\) is the multiplicity label needed in the outer-product \([\lambda_1]f_1 \times [\lambda_2]f_2 \uparrow [\lambda]f_{-2k}\), \(\tau\) stands for other labels needed for the irrep \([\lambda]f_{-2k}\), \(0 \leq k' \leq k\), and the coefficient

\[
C^{[\lambda]f_{-2k} \rho \tau}_{m_1 m_2 k' \omega}
\]

is \([\lambda_1]f_1 \times [\lambda_2]f_2 \uparrow [\lambda]f_{-2k}\) Induction coefficient (IDC) or the Outer-product reduction coefficient (ORC).

The IDCs satisfy the following orthogonality relation:

\[
\sum_{m_1 m_2 k' \omega m_1' m_2' k' \omega'} C^{[\lambda]f_{-2k} \rho \tau}_{m_1 m_2 k' \omega} C^{[\lambda]f_{-2k} \rho' \tau'}_{m_1' m_2' k' \omega'} \mathcal{N}^{[\lambda_1][\lambda_2]}_{m_1 m_2 k' \omega; m_1' m_2' k' \omega'} = \delta_{\lambda \lambda'} \delta_{\tau \tau'} \delta_{\rho \rho'},
\]

(19)
where \(\mathcal{N}^{[\lambda_1][\lambda_2]}\) is symmetric norm matrix, of which the elements are defined [8] by

\[
\mathcal{N}^{[\lambda_1][\lambda_2]}_{m_1 m_2 k' \omega; m_1' m_2' k' \omega'} = < Y_{m_1}^{[\lambda_1]} Y_{m_2}^{[\lambda_2]}; (\omega_1^0)(\omega_2^0) | Q^{k'}_{\omega'} Y_{m_1'}^{[\lambda_1]} Y_{m_2'}^{[\lambda_2]}; (\omega_1^0)(\omega_2^0) > .
\]

(20)
These matrix elements can easily be calculated by using the algebraic relations of Brauer algebra given by (1) and (2) and those given in [8]. While the coupled basis vectors

\[
|[\lambda]f_{-2k}, \tau; \rho >\]

are orthonormal.

\[
< [\lambda']f_{-2k'}; \tau', \rho' | [\lambda]f_{-2k}; \tau, \rho > = \delta_{\lambda \lambda'} \delta_{\tau \tau'} \delta_{\rho \rho'} \delta_{kk'}.
\]

(21)

3. Evaluation of the IDCs
The linear equation method (LEM) has been proved effective in deriving SDCs and IDCs of Hecke algebra,[14] as well as SDCs of Brauer algebra.[10] The procedure for the evaluation of the IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ is similar to that proposed in [8].

Firstly, applying the operators $R_i$ ($= g_i$ or $e_i$) with $i = 1, 2, \cdots, f_1 + f_2 - 1$ to (18), the left-hand side of (18) becomes

$$\sum_{m_1 m_2 \omega k'} \sum_{\rho' \tau'} C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho' \tau'} < [\lambda] f_{-2k} \rho' \tau' | R_i | [\lambda] f_{-2k} \rho \tau > Q^k_\omega | Y_{m_1}^{[\lambda_1]} Y_{m_2}^{[\lambda_2]}, (\omega^0_1)(\omega^0_2) >. \quad (22)$$

While the right-hand side of (18) becomes

$$\sum_{m_1 m_2 \omega k'} C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho \tau} (R_i Q^k_\omega) | Y_{m_1}^{[\lambda_1]} Y_{m_2}^{[\lambda_2]}, (\omega^0_1)(\omega^0_2) >. \quad (23)$$

Then, combining (22) and (23), we get

$$\sum_{\rho' \tau'} C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho' \tau'} < [\lambda] f_{-2k} \rho' \tau' | R_i | [\lambda] f_{-2k} \rho \tau >= C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho' \tau'} f_i, \quad (24)$$

where $C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho' \tau'} f_i$ is the coefficient in front of $Q^k_\omega | Y_{m_1}^{[\lambda_1]} Y_{m_2}^{[\lambda_2]}, (\omega^0_1)(\omega^0_2) >$ after applying $R_i$ to the right-hand side of (18), and $< [\lambda] f_{-2k} \rho' \tau' | R_i | [\lambda] f_{-2k} \rho \tau >$ is matrix elements of $R_i$ in the standard basis given in (4), which have already been known.[8]

The linear relations or so-called a part of the intertwining relations among the IDCs given by (24) are sufficient to determine these IDCs up to a normalization factor,[8] which can then be calculated by using the orthogonality relation (19). It will be shown that the CGCs of SO(n), expressed in terms of these IDCs, need to be normalized again according to different cases. Therefore, the normalization of these IDCs is not necessary. However, the sign of the normalization factors, which gives overall phase of the IDCs should be chosen beforehand. In our calculation, the overall phase is fixed by requiring that the IDCs with min($\tau$) at first, then with min($m_1$), and smallest indices $\omega$ and $k'$ be positive

$$C_{m_1 m_2 k' \omega}^{[\lambda] f_{-2k} \rho \min(\tau)} > 0 \quad (25)$$
Using the algebraic relations of Brauer algebra, Eq. (24), and irreducible representations of symmetric groups in the standard basis,\[15\] one can obtain all the IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f$. In what follows, we will give a simple example of deriving the IDCs and some basic features of these coefficients.

**Example 1.** Deriving IDCs of $S_1 \times S_1 \uparrow D_2(n)$. The outer product reduction is $[1] \times [1] \uparrow [2] + [1^2] + [0]$. In this case, Eq. (18) can be written as

\[
[2] = \sum_{i=1}^{3} a_i |i>, \quad [1^2] = \sum_{i=1}^{3} b_i |i>, \quad [0] = \sum_{i=1}^{3} c_i |i>, \quad (26a)
\]

where $a_i$, $b_i$, and $c_i$ are the corresponding IDCs, and $|i> (i = 1, 2, 3)$ are the uncoupled basis vectors defined by

\[
|1> = |1, 2>, \quad |2> = g_1 |1>, \quad |3> = e_1 |1>. \quad (26b)
\]

Applying generators $g_1$ and $e_1$, respectively, to (26a), one obtains

\[
a_1 = a_2, \quad a_3 = -\frac{2}{n} a_1,
\]

\[
b_1 = -b_2, \quad b_3 = 0,
\]

\[
c_1 = c_2 = 0, \quad c_3 \neq 0. \quad (26c)
\]

The norm matrix for this case is

\[
\mathcal{N}^{[1][1]} = \begin{pmatrix} 1 & \delta_{i_1 i_2} & \delta_{i_1 i_2} \\ \delta_{i_1 i_2} & 1 & \delta_{i_1 i_2} \\ \delta_{i_1 i_2} & \delta_{i_1 i_2} & n\delta_{i_1 i_2} \end{pmatrix}, \quad (26d)
\]

which can be proved by using (8)-(10). Hence, the coupled basis vectors can now be written as

\[
[2] = a_1 (|1> + |2> - \frac{2}{n} |3>),
\]
\[ ||[1^2]|| = b_1 (|1 > -|2 >), \]
\[ ||[0]|| = c_3 |3 >. \]  \hspace{1cm} (27)

Using the norm matrix (26d), one can check that basis vectors given by (27) are orthogonal. The normalization factors, of which the signs should be chosen according to (25), can easily be obtained by using (21) and (26).

\[ a_1 = \sqrt{\frac{n}{2(n + \delta_{i_1i_2}(n - 2))}}, \quad b_1 = \sqrt{\frac{1}{2}}, \quad c_3 = \sqrt{\frac{1}{n}}. \]  \hspace{1cm} (28)

It can easily be seen that \(|3 >\) is a null vector when \(i_1 \neq i_2\). In this case, (27) becomes outer-product basis vectors of symmetric group \(S_1 \times S_1 \uparrow S_2\). It is clear that the induced representations of \(D_f(n)\) from \(S_{f_1} \times S_{f_2}\) are \(SO(n)\) tensor component dependent. Actually, the normalization of these basis vectors is not necessary with respect to representations of \(D_f(n)\). On the other hand, It can be easily seen from (26)-(28) that normalization factors of the IDC’s are also \(SO(n)\) tensor component dependent. The situation will become more complicated when \(f_1 + f_2 = f \geq 3\). Furthermore, our purpose is to evaluate CGCs of \(O(n)\) from these IDCs. The coupled basis vectors of \(O(n)\) obtained from these IDCs through assimilation need to be normalized again. Therefore, we only list unnormalized IDCs of \(S_{f_1} \times S_{f_2} \uparrow D_f(n)\). The method of how to evaluate \(O(n)\) CGCs from these unnormalized IDCs will be presented in the next section.

In the following, we list unnormalized IDCs of \(S_{f_1} \times S_{f_2} \uparrow D_f(n)\) with \(f_1 + f_2 = f \leq 4\). The signs of the normalization factors given below are all chosen to be positive, which is fixed by our phase convention (25). Only the absolute values of these normalization factors need to be determined according to different \(SO(n)\) tensor components later.

(1) \(D_1(n) \times D_1(n) \uparrow D_2(n)\) for \([1] \times [1] = [2] + [1^2] + [0]\).
\[ |[2]\rangle = a_1 \left( |1> + |2> - \frac{2}{n} |3> \right), \]

\[ |[1^2]\rangle = \sqrt{\frac{1}{2}} \left( |1> - |2> \right), \]

\[ |[0]\rangle = \sqrt{\frac{1}{n}} |3>, \]

where \( |1> = |1, 2>, |2> = g_1 |1>, |3> = e_1 |1>. \)

(2) \( D_2(n) \times D_1(n) \uparrow D_3(n) \) for \( [2] \times [1] = [3] + [21] + [1]. \)

\[ |[3]\rangle = a_1 \left( |1> + |2> + |3> - \frac{2}{n+2} (|4> + |5> + |6>) \right), \]

\[ |[21]_1\rangle = \frac{1}{\sqrt{3}} a_2 \left( 2|1> - |2> - |3> + \frac{1}{n-1} (2|4> - |5> - |6>) \right), \]

\[ |[21]_2\rangle = a_2 \left( |2> - |3> + \frac{1}{n-1} (|5> - |6>) \right), \]

\[ |[1][0]\rangle = a_3 |4>, \]

\[ |[1][2]\rangle = \sqrt{\frac{1}{2(n+2)(n-1)}} a_3 \left( 2|4> - n(|5> + |6>) \right), \]

\[ |[1][1^2]\rangle = \sqrt{\frac{n}{2(n-1)}} a_3 \left( |5> - |6> \right), \]

where \( |1> = |12, 3>, |2> = g_2 |1>, |3> = g_1 g_2 |1>, |4> = e_1 g_2 |1>, |5> = g_1 e_2 |1>, |6> = e_2 |1>. \)

(3) \( D_2(n) \times D_1(n) \uparrow D_3(n) \) for \( [1^2] \times [1] = [1^3] + [21] + [1]. \)
\[ |[1^3]\rangle = \frac{1}{\sqrt{3}} (|1 > |2 > +|3 >), \]
\[ |[21]\rangle = 3a_1 \left( |2 > +|3 > - \frac{1}{n-1} (|4 > +|5 > +|6 >) \right), \]
\[ |[21]\rangle = a_1 \left( 2|1 > +|2 > -|3 > + \frac{3}{n-1} (|5 > -|6 >) \right), \]
\[ |[1][0]\rangle = a_2 |4 >, \]
\[ |[1][2]\rangle = \frac{1}{\sqrt{2(n+2)(n-1)}} a_2 (2|4 > +n(|5 > +|6 >)), \]
\[ |[1][2]\rangle = \sqrt{n/2(n-1)} a_2 (|6 > -|5 >), \]

where \[ |1 >= \frac{1}{2}, 3 >, |2 >= g_2|1 >, |3 >= g_1g_2|1 >, |4 >= e_1g_2|1 >, |5 >= g_1e_2|1 >, |6 >= e_2|1 >. \]

(4) \( D_3(n) \times D_1(n) \uparrow D_4(n) \) for \[ [3] \times [1] = [4] + [31] + [2]. \]

\[ |[4]\rangle = c_1 \left( |1 > +|2 > +|3 > +|4 > - \frac{2}{n+1} (|5 > +|6 > +|7 > +|8 > +|9 > +|10 >) \right), \]
\[ |[31]\rangle = \frac{c_2}{\sqrt{2}} \left( 3|1 > -|2 > -|3 > -|4 > + \frac{2}{n} (|5 > +|6 > -|7 > +|8 > -|9 > -|10 >) \right), \]
\[ |[31]\rangle = c_2 \left( 2|2 > -|3 > -|4 > + \frac{1}{n} (2|5 > -|6 > +|7 > -|8 > +|9 > -2|10 >) \right), \]
\[ |[31]\rangle = \sqrt{3} c_2 \left( |3 > -|4 > + \frac{1}{n} (|6 > +|7 > -|8 > -|9 >) \right), \]
\[ |[2][1][0]\rangle = c_3 |5 >, \]
\[ |[2][1][2]| = \frac{1}{\sqrt{2(n+2)(n-1)}} c_3 (2|5 > -n(|6 > +|8 >)) , \]

\[ |[2][1][1^2]| = \frac{2}{\sqrt{(n+2)(n-1)}} c_3 (|6 > -|8 >) , \]

\[ |[2][3]| = \frac{1}{\sqrt{3n(n+4)}} c_3 \left( \frac{2}{n+2}(|5 > +|6 > +|8 >) - |7 > -|9 > -|10 >) \right) , \]

\[ |[21]_1| = \sqrt{\frac{2(n-1)}{3(n^2-4)}} c_3 \left( 2|10 > - |7 > -|9 > + \frac{1}{n+1}(2|5 > -|6 > -|8 >) \right) , \]

\[ |[21]_2| = \sqrt{\frac{2(n-1)}{(n^2-4)}} c_3 \left( |9 > -|10 > + \frac{1}{n+1}(|6 > -|8 >) \right) , \]

where \(|1 > = |123, 4 >, |2 > = g_3|1 >, |3 > = g_2g_3|1 >, |4 > = g_1g_2g_3|1 >, |5 > = e_1g_2g_3|1 >, |6 > = g_1e_2g_3|1 >, |7 > = g_1g_2e_3|1 >, |8 > = e_2g_3|1 >, |9 > = g_2e_3|1 >, |10 > = e_3|1 >. \]

(5) \( D_3(n) \times D_1(n) \uparrow D_4(n) \) for \([1^3] \times [1] = [1^4] + [211] + [1^2] \).

\[ |[1^4]| = \frac{1}{2} (|1 > -|2 > +|3 > -|4 >) , \]

\[ |[21^2]_1| = \frac{(n-2)}{\sqrt{2}} c_4 \left( 3|1 > +|2 > -|3 > +|4 > - \frac{4}{n-2} (|7 > -|9 > +|10 >) \right) , \]

\[ |[21^2]_2| = (n-2)c_4 \left( 2|2 > +|3 > -|4 > + \frac{1}{n-2} (3|6 > +|7 > -3|8 > -9 > -2|10 >) \right) , \]

\[ |[21^2]_3| = \sqrt{3}(n-2)c_4 \left( 3| > +|4 > - \frac{1}{n-2} (2|5 > +|6 > +|7 > +|8 > +|9 >) \right) , \]

\[ |[1^2][1][0]| = c_5|5 > , \]

\[ |[1^2][1][2]| = \frac{1}{\sqrt{2(n+2)(n-1)}} c_5 (2|5 > +n(|6 > +|8 >)) , \]
\(|[1^2][1][1^2]\rangle = \sqrt{\frac{n}{2(n-1)}} c_5 (|8\rangle - |6\rangle) ,

\(|[1^2][21]\rangle_1 = \sqrt{\frac{n(n-1)}{2(n^2-4)}} c_5 (|7\rangle + |9\rangle + \frac{1}{n-1} (2|5\rangle + |6\rangle + |8\rangle)) ,

\(|[1^2][21]\rangle_2 = -\sqrt{\frac{n(n-1)}{6(n^2-4)}} c_5 (\frac{3}{n-1}(|6\rangle - |8\rangle) + |7\rangle - |9\rangle - 2|10\rangle) ,

\(|[1^2][3]\rangle = \frac{1}{3} \sqrt{\frac{n}{n-2}} c_5 (|7\rangle - |9\rangle + |10\rangle) ,

where |1\rangle = \begin{pmatrix} 1 \\ 2, 4 \\ 3 \end{pmatrix} , |2\rangle = g_3|1\rangle > , |3\rangle = g_2 g_3|1\rangle > , |4\rangle = g_1 g_2 g_3|1\rangle > , |5\rangle = e_1 g_2 g_3|1\rangle > , 

|6\rangle = g_1 e_2 g_3|1\rangle > , |7\rangle = g_1 g_2 e_3|1\rangle > , |8\rangle = e_2 g_3|1\rangle > , |9\rangle = g_2 e_3|1\rangle > , |10\rangle = e_3|1\rangle > .

(6) \(D_3(n) \times D_1(n) \uparrow D_4(n)\) for \([21] \times [1] = [31] + [22] + [211] + [1^2] + [2]\).

\(|[31]\rangle_1 = \frac{n(n+2)}{\sqrt{3}} d_1 (|2\rangle + |3\rangle + |4\rangle + \frac{1}{2n} (|13\rangle + |18\rangle - 2|19\rangle + \sqrt{3} (|14\rangle - |17\rangle)) - \frac{(2n+1)}{n(n+2)} (|9\rangle + |11\rangle + |16\rangle) - \frac{\sqrt{3}}{n(n+2)} (|10\rangle + |12\rangle - 2|15\rangle) ,

\(|[31]\rangle_2 = \frac{\sqrt{6} n(n+2)}{4} d_1 (|1\rangle + \frac{1}{2}|2\rangle > - \frac{1}{6}(|3\rangle + |4\rangle > + \frac{\sqrt{3}}{2} (|7\rangle + |8\rangle > + \frac{(n-4)}{6n(n+2)} (2|9\rangle - |11\rangle - |16\rangle) + \frac{\sqrt{3}(3n+4)}{6n(n+2)} (|12\rangle - |15\rangle - 2|10\rangle) + \frac{\sqrt{3}(n-4)}{6n(n+2)} (|14\rangle - |17\rangle) - \frac{(11n+4)}{6n(n+2)} (|13\rangle + |18\rangle) - \frac{4}{3n} |19\rangle >)

\(|[31]\rangle_3 = \frac{\sqrt{8} n(n+2)}{8} d_1 (|3\rangle - |4\rangle + \sqrt{3} (|7\rangle - |8\rangle) + \frac{\sqrt{3}}{n(n+2)} (|5\rangle + |6\rangle) + \frac{(n-4)}{n(n+2)} (|13\rangle + |11\rangle - |16\rangle - |18\rangle) - \frac{\sqrt{3}(3n+4)}{n(n+2)} (|12\rangle + |15\rangle + |14\rangle + |17\rangle) - \frac{4\sqrt{3}}{n+2} |20\rangle) ,

\(|[21^2]\rangle_1 = \frac{3\sqrt{2}}{4} d_2 (|1\rangle - |2\rangle + \frac{\sqrt{3}}{6} (|7\rangle + |8\rangle - \sqrt{3}|3\rangle + \sqrt{3}|4\rangle > + \frac{\sqrt{3}}{6(n-2)} (2|10\rangle - 2\sqrt{3}|9\rangle > - |12\rangle + |15\rangle + |14\rangle - |17\rangle) + \frac{1}{2(n-2)} (|11\rangle + |16\rangle - |13\rangle - |18\rangle) ,

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\[ |[2^1^2^2]| = \frac{3\sqrt{7}}{4}d_2 \left( |4 > - |3 > + \frac{\sqrt{7}}{9}(6 |5 > - 2 |6 > - |7 > + |8 > + \frac{1}{9(n-2)}(3 |13 > - 3 |18 > + 3\sqrt{3}|12 > + |15 > - 9 |11 > + 9 |16 > - \sqrt{3}|14 > - \sqrt{3}|17 > - 4\sqrt{3}|20 > \right), \]

\[ |[2^1^2^3]| = d_2 \left( |6 > - |7 > + |8 > + \frac{1}{2(n-2)}(|14 > + |17 > - \sqrt{3}(|13 > - |18 > - 2 |20 > \right), \]

\[ |[2^2^1]| = d_3 \left( |5 > - |6 > + \frac{1}{n-2}(|12 > + |15 > - |14 > - |17 > \right), \]

\[ |[2^2^2]| = d_3 \left( |1 > - \frac{1}{n-2} |19 > \right), \]

\[ |[2^1^1^0]| = d_4 \left( |9 > + \sqrt{3} |10 > \right), \]

\[ |[2^1^1^2]| = \sqrt{\frac{2}{(n+2)(n-1)}}d_4 \left( |9 > + \sqrt{3} |10 > - \frac{n}{2}(|11 > + |16 > - \frac{\sqrt{3}}{2}(|12 > - |15 > \right), \]

\[ |[2^1^2^1]| = \sqrt{\frac{n}{2(n-1)}}d_4 \left( |11 > - |16 > + \sqrt{3}(|12 > + |15 > \right), \]

\[ |[2^3^1]| = \frac{2}{\sqrt{3(n+2)(n+4)}}d_4 \left( |9 > + |11 > + |16 > + \sqrt{3}(|10 > + |12 > - |15 > \right) - \frac{n+2}{2}(|13 > + |18 > - 2 |19 > + \sqrt{3}|14 > - \sqrt{3}|17 > \right), \]

\[ |[2^2^1^1]| = \sqrt{\frac{2}{3(n-2)(n-1)}}d_4 \left( |9 > + \sqrt{3} |10 > - \frac{1}{2}(|11 > + |16 > - \frac{\sqrt{3}}{2}(|12 > - |15 > \right) - \frac{n-1}{2}(|13 > + |18 > - \frac{n-1}{2}\sqrt{3}(|14 > - |17 > - 2(n-1)|19 > \right), \]

\[ |[2^2^1^2]| = \sqrt{\frac{1}{2(n-2)(n-1)}}d_4 \left( |11 > - |16 > + \sqrt{3}(|12 > + |15 > - (n-1)(|13 > - |18 > - (n-1)\sqrt{3}(|14 > + |17 > \right), \]

\[ |[1^2^1^0]| = d_5 \left( |9 > - \sqrt{3} |10 > \right), \]
where $|1> = |12, 3, 4>, |2> = g_3|1>, |3> = g_2g_3|1>, |4> = g_1g_2g_3|1>,$
$|5> = |13, 2, 4>, |6> = g_3|1>, |7> = g_2g_3|5>, |8> = g_1g_2g_3|5>, |9> = e_1g_2g_3|1>,$
$|10> = e_1g_2g_3|5>, |11> = g_2e_1g_2g_3|1>, |12> = g_2e_1g_2g_3|5>, |13> = g_3g_2e_1g_2g_3|1>,
|14> = g_3g_2e_1g_2g_3|5>, |15> = e_2g_3|5>, |16> = e_2g_3|1>, |17> = g_3e_2g_3|5>, |18> = g_3e_2g_3|1>, |19> = e_3|1>, |20> = e_3|5>.$

(7) $D_2(n) \times D_2(n) \uparrow D_4(n)$ for $[2] \times [2] = [4] + [31] + [22] + [2] + [1^2] + [0].$

$|[4]> = f_1 (\frac{n+4}{2} (|1> + |2> + |3> + |4> + |5> + |6>) - |13> - |14> - |17> - |18> - |15> - |11> - |9> - |12> - |7> - |8> - |10> - |16> + \frac{2}{n+2}(|21> + |20> + |19>)),$

$|[31]> = \sqrt{\frac{7}{3}} f_2 (|1> + |2> + |3> + |4> - |5> - |6> + \frac{2}{n+2} (3|14> + 3|10 > + 3|8 >$
\[-|17 > -|18 > -|16 > -|12 > -|11 > -|15 > -|13 > -|9 > -|7 >),\]

\[|31)_2 = \sqrt{\frac{1}{3}}f_2 (2|1 > -|2|6 > -|2 > -|3 > +|4 > +|5 > +\frac{1}{n+2}(4|12 > +|7 > +|4|16 > -2|15 > -|2|11 > -|2|13 > -|2|9 > -|2|17 > -|2|18 >)),\]

\[|31)_3 = f_2 (|2 > -|3 > +|4 > -|5 > -\frac{2}{n+2}(|13 > -|9 > +|17 > -|18 > +|15 > -|11 >)),\]

\[|22)_1 = f_3 \left( \frac{2(n-2)}{\sqrt{3}}(|1 > +|6 >) - (n - 2)(|2 > +|3 > +|4 > +|5 >) - |13 > -|9 > -|11 > -|15 > -|14 > -|10 > -|12 > -|16 > +2(|17 > +|18 > +|7 > +|8 >) - \frac{4}{n-1}|19 > +\frac{2}{n-1}(|21 > +|20 >)),\]

\[|22)_2 = f_3 \left( \sqrt{3}(n - 2)(|2 > -|3 > -|4 > +|5 >) - \sqrt{3}(|13 > -|9 > -|15 > +|11 > +|14 > -|10 > +|12 > -|16 >) + \frac{2\sqrt{3}}{n-1}(|21 > -|20 >)),\]

\[|2][1][0] = f_4 (|7 > +|8 > -\frac{2}{n}|19 >)),\]

\[|2][1][2] = \sqrt{\frac{2}{n+2(n-1)}}f_4 (|7 > +|8 > +|21 > +|20 > -\frac{2}{n}|13 > +|9 > +|14 > +|10 > -\frac{2}{n}|19 >)),\]

\[|2][1][2] = \sqrt{\frac{n}{2(n-1)}}f_4 (|9 > -|13 > -|14 > +|10 > +\frac{2}{n}|21 > -|20 >)),\]

\[|2][21]_1 = \sqrt{\frac{2}{3(n-2)(n-1)}}f_4 (|17 > +|8 > +|21 > +|20 > -2|19 > -\frac{n-1}{2}(|15 > +|11 > +|12 > +|16 > +|17 > +|18 >) - \frac{1}{2}(|13 > +|9 > +|14 > +|10 >)),\]

\[|2][21]_2 = \sqrt{\frac{2}{n-2(n-1)}}f_4 (|21 > -|20 > +\frac{n-1}{2}(|15 > -|11 > -|12 > +|16 > -\frac{1}{2}(|13 > -|9 > +|14 > -|10 >)),\]

\[|2][3] = \sqrt{\frac{2}{3(n+2)(n+4)}}f_4 (|13 > +|9 > +|7 > +|10 > +|14 > +|8 >
\[-\frac{n+2}{2}(|17> + |18> + |16> + |12> + |11> + |15>)\],

\[|1^2[1][0]\rangle = f_5(|7> - |8>),\]

\[|1^2[1][2]\rangle = \sqrt{\frac{2}{(n+2)(n-1)}}f_5\left(|7> - |8> - \frac{n}{2}(|13> + |9> - |14> - |10>)\right),\]

\[|1^2[1][2]\rangle = \sqrt{\frac{n}{2(n-1)}}f_5(|9> + |14> - |10> - |13>),\]

\[|1^2[21]_1\rangle = \sqrt{\frac{2n}{(n-4)(n-1)}}f_5\left(|7> - |8> + \frac{n-1}{n}(|11> + |15> - |12>) - |16>)\right),\]

\[|1^2[21]_2\rangle = \sqrt{\frac{3(n-1)}{3(n-2)}}f_5\left(|7> - |18> + \frac{1}{2}(15 > |11> + |12> - |16>)\right),\]

\[|1^2[3]\rangle = \sqrt{\frac{n+2}{3(n-2)}}f_5(|17> - |18> - |15> + |11> - |12> + |16>),\]

\[|0[1][0]\rangle = f_6|9>,\]

\[|0[1][2]\rangle = \frac{2n}{\sqrt{2(n+2)(n-1)}}f_6\left(|20> + |21> + \frac{2(n+2)(n-1)}{n^2}2(n+2)(n-1)|19>\right),\]

\[|0[1][12]\rangle = \frac{2n}{\sqrt{2(n+2)(n-1)}}f_6(|20> - |21>),\]

where \(|1>=|12, 34>, |2>=g_2|1>, |3>=g_1g_2|1>, |4>=g_3g_2|1>, |5>=g_1g_3g_2|1>, |6>=g_2g_1g_3|1>, |7>=e_1g_2|1>, |8>=e_1g_3g_2|1>, |9>=g_1e_2|1>, |10>=g_2g_3e_1g_2|1>, |11>=g_1g_3e_2|1>, |12>=g_2g_1g_3e_2|1>, |13>=e_2|1>, |14>=e_2g_1g_3g_2|1>, |15>=g_3e_2|5>, |16>=g_2g_1e_3g_2|1>, |17>=e_3g_2|1>, |18>=e_3g_1g_2|1>, |19>=e_3e_1g_2|1>, |20>=g_2e_1e_3g_2|1>, |21>=e_2g_1g_3e_2|1>.\]
(8) $D_2(n) \times D_2(n) \uparrow D_4(n)$ for $[2] \times [1^2] = [31] + [211] + [2] + [1^2] + [0]$. 

$[31]_1 = h_1 \left( [1 > +|2 > +|3 > - \frac{2}{n+2}(|9 > +|13 > +|7 > )) \right)$

$[31]_2 = -\sqrt{\frac{2}{3}} h_1 \left( [1 > + |2 > + \frac{1}{2} |3 > - \frac{3}{2} |4 > + |5 > ) + \frac{1}{n+2} (3|8 > + 3 |15 > + 3 |11 > + |13 > + |9 > + |7 > )) \right)$

$[31]_3 = -\sqrt{\frac{3}{2}} h_1 \left( \frac{1}{2} (|2 > - |3 > + |4 > - |5 > - |6 > ) - \frac{1}{n+2} (|17 > - |18 > - |13 > + |9 > - |15 > + |11 > )) \right)$

$[211]_1 = \sqrt{\frac{3}{2}} h_2 \left( 2|7 > - 2|8 > - \sqrt{3}(|13 > + |9 > - |15 > - |11 > ) - |14 > + |10 > + |16 > - |12 > + (n - 2) (2|1 > - |2 > - 3 > + |4 > + |5 > )) \right)$

$[211]_2 = \sqrt{\frac{3}{2}} h_2 \left( 3(|2 > - |3 > ) - |4 > + |5 > + 2|6 > + \frac{1}{n-2} (|15 > - |11 > + |12 > + |16 > ) - 3(|13 > - |9 > + |14 > + |10 > ) - 2(|17 > - |18 > )) \right)$

$[211]_3 = h_2 \left( |4 > - |5 > + |6 > - \frac{1}{n-2} (|17 > - |18 > + |15 > - |11 > + |12 > + |16 > ) \right)$

$[2][1][0] = h_3 \left( |7 > + |8 > - \frac{2}{n} |19 > \right)$,

$[2][1][2] = \sqrt{\frac{2}{(n+2)(n-1)}} h_3 \left( |7 > + |8 > - |21 > + |20 > - \frac{n}{2} (|13 > + |9 > - |14 > + |10 > - \frac{2}{n} |19 > ) \right)$,

$[2][1][1^2] = \sqrt{\frac{n}{2(n-1)}} h_3 \left( |9 > - |13 > + |14 > + |10 > - \frac{2}{n} (|21 > + |20 > ) \right)$,

$[2][21]_1 = \sqrt{\frac{2}{3(n-2)(n-1)}} h_3 \left( |7 > + |8 > + \frac{(n-2)}{n} (|21 > - |20 > + 2|19 > ) - \frac{n-1}{2} (|15 > + |11 > + |12 > + |16 > ) - (n - 1) (|17 > + |18 > ) - \frac{1}{2} (|13 > + |9 > + |14 > + |10 > ) \right)$,

$[2][21]_2 = \sqrt{\frac{n-1}{2(n-2)}} h_3 \left( |15 > - |11 > - |12 > - |16 > + \frac{2(n-2)}{n(n-1)} (|21 > + |20 > ) \right)$
\[-\frac{1}{n-1}(|13 > -|9 > -|14 > -|10 > )\),

\(|2>[3]) = \sqrt{\frac{n+2}{3(n+4)}} h_3 \left( |17 > +|18 > -|15 > -|11 > -|12 > +|16 > + \frac{2(n+4)}{n(n+2)}(|21 > -|19 > -|20 > ) + \frac{2}{n+2}(|13 > +|9 > -|14 > +|7 > +|10 > +|8 > )\),

\(|1^2][1][0]) = h_4 \left( |7 > -|8 > \right),

\(|1^2][1][2]) = \sqrt{\frac{2}{(n+2)(n-1)}} h_4 \left( |7 > -|8 > -\frac{n}{2}(|13 > +|9 > ) + |14 > -|10 > \right),

\(|1^2][1][1^2]) = \sqrt{\frac{n}{2(n-1)}} h_4 \left( |9 > -|14 > -|10 > -|13 > \right),

\(|1^2][21][1]) = \sqrt{\frac{n(n-1)}{2(n-4)}} h_4 \left( |15 > +|11 > -|12 > +|16 > + \frac{2}{n-1}(|7 > -|8 > ) \\
-\frac{1}{n-1}(|13 > +|9 > +|14 > -|10 > )\right),

\(|1^2][21][2]) = \sqrt{\frac{n(n-1)}{6(n-4)}} h_4 \left( |15 > -|11 > +|12 > +|16 > -2|17 > +2|18 > \\
-\frac{3}{n-1}(|13 > -|9 > +|14 > +|10 > )\right),

\(|1^2][21][3]) = -\frac{1}{3} \sqrt{\frac{2n}{3(n-1)}} h_4 \left( |17 > -|18 > +|15 > -|11 > +|12 > +|16 > \right),

\(|0][1][0]) = h_5 |9 > ,

\(|0][1][2]) = \sqrt{\frac{2}{(n+2)(n-1)}} h_5 \left( \frac{n\sqrt{n}}{2}(|21 > -|20 > ) + |19 > \right),

\(|0][1][1^2]) = \sqrt{\frac{n}{2(n-1)}} h_5 \left( |20 > +|21 > \right),

where \(1 := \begin{pmatrix} 12, & 3 \\ 4 \\ \end{pmatrix}, \quad 2 := g_2 |1 >, \quad 3 := g_1 g_2 |1 >, \quad 4 := g_3 g_2 |1 >, \quad 5 := \quad 22
\[ g_1 g_3 g_2 | 1 >, | 6 >= g_2 g_1 g_3 g_2 | 1 >, | 7 >= e_1 g_2 | 1 >, | 8 >= e_1 g_3 g_2 | 1 >, | 9 >= g_1 e_2 | 1 >, | 10 >= g_2 g_3 e_1 g_2 | 1 >, | 11 >= g_1 g_3 e_2 | 1 >, | 12 >= g_2 g_1 g_3 e_2 | 1 >, | 13 >= e_2 | 1 >, | 14 >= e_2 g_1 g_3 g_2 | 1 >, | 15 >= g_3 e_2 | 5 >, | 16 >= g_2 g_1 e_3 g_2 | 1 >, | 17 >= e_3 g_2 | 1 >, | 18 >= e_3 g_1 g_2 | 1 >, | 19 >= e_3 e_1 g_2 | 1 >, | 20 >= g_2 e_1 e_3 g_2 | 1 >, | 21 >= e_2 g_1 g_3 e_2 | 1 >. \]

(9) \[ D_2(n) \times D_2(n) \uparrow D_4(n) \] for \[ | 1^2 \times 1^2 = | 1^4 + 211 + 22 + 2 + 1^2 + 0. \]

\[ |[1^3]| = k_1 (| 1 > - | 2 > + | 3 > + | 4 > + | 6 > - | 5 >) \]

\[ |[211]| = -\sqrt{2} k_2 \left( | 2 > + | 3 > - | 4 > - | 5 > - \frac{1}{n-2} (| 9 > - | 15 > - | 11 > + | 10 > + | 12 > + | 13 > + | 14 > - | 16 > + | 2 > | 7 > - 2 | 8 >) \right) \]

\[ |[211]| = -\sqrt{2} k_2 \left( | 2 | 1 > - | 2 > - | 6 > + | 2 > - | 3 > + | 4 > - | 5 > - \frac{3}{n-2} (| 13 > - | 9 > - | 14 > - | 10 > + | 15 > + | 12 > + | 16 > - | 2 | 17 > + | 2 | 18 > \right) \]

\[ |[211]| = -k_2 \left( | 1 > - | 2 > + | 3 > - | 4 > + | 5 > - | 6 > + \frac{2}{n-2} (| 17 > - | 18 > + | 15 > - | 11 > + | 12 > + | 16 > \right) \]

\[ |[22]| = k_3 \left( (n-2)(| 2 > + | 3 > + | 4 > + | 5 > - | 13 > - | 9 > - | 15 > - | 11 > - | 14 > + | 10 > - | 12 > + | 16 > - 2 (| 17 > + | 18 > + | 7 > + | 8 > + \frac{2}{n-1} (| 2 | 19 > + | 2 | 21 > - | 2 | 20 > \right) \]

\[ |[22]| = k_3 \left( | 2 | 1 > + | 2 > - | 3 > - | 4 > + | 5 > + 2 | 6 > - \frac{3}{n-2} (| 13 > - | 9 > - | 15 > + | 11 > + | 14 > + | 10 > + | 16 > + | 12 > + \frac{6}{(n-2)(n-1)} (| 20 > + | 21 > \right) \]

\[ |[2 | 1 | 0] = k_4 \left( | 7 > + | 8 > - \frac{2}{n} | 19 > \right) \]

\[ |[2 | 1 | 2] = k_4 \left( (7 > + | 8 > - | 21 > + | 20 > + \frac{4}{n} (| 13 > + | 9 > - | 14 > - | 10 > - \frac{2}{n} | 19 > \right) \]

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\begin{align*}
|2[1][1^2]\rangle &= \sqrt{\frac{n}{2(n-1)}} k_1 \left(-|9 > +|13 > +|14 > +|10 > - \frac{2}{n}(|21 > +|20 >)\right), \\
|2[21]_1\rangle &= \sqrt{\frac{1}{6(n-2)(n-1)}} k_4 (|13 > +|9 > +|14 > -|10 > + (n-1)(2|17 > +2|18 > +|15 > +|11 > +|12 > -|16 >) + 2|7 > +2|8 > -2(2|19 > +|21 > -|20 >)), \\
|2[21]_2\rangle &= \sqrt{\frac{1}{2(n-2)(n-1)}} k_4 (|13 > -|9 > +|14 > +|10 > -(n-1)(|15 > -|11 > -|12 > -|16 >) - 2|21 > -2|20 >) \\
|2[3]\rangle &= -\sqrt{\frac{2}{3(n+2)(n+4)}} k_4 (|13 > +|9 > -|7 > +|14 > -|10 > -|8 > -|19 > -|20 > +|21 > + \frac{n^2}{2}(|17 > +|18 > -|15 > -|11 > -|12 > +|16 >)) \\
|1^2[1][0]\rangle &= k_5 (|7 > -|8 >), \\
|1^2[1][2]\rangle &= \frac{n}{\sqrt{2(n^2+2)(n-1)}} k_5 (|13 > +|9 > -|14 > +|10 > + \frac{2}{n}(|7 > -|8 >)), \\
|1^2[1][1^2]\rangle &= \sqrt{\frac{n}{2(n-1)}} k_5 (|13 > -|14 > -|10 > -|9 >), \\
|1^2[21]_1\rangle &= \sqrt{\frac{n}{2(n^2-4)}} k_5 (|13 > +|9 > -|14 > +|10 > -(n-1)(|15 > +|11 > -|12 > +|16 >) + 2(|7 > -|8 >)) \\
|1^2[21]_2\rangle &= \sqrt{\frac{3n}{2(n-1)(n^2-4)}} k_5 (|13 > -|9 > -|14 > -|10 > - \frac{n-1}{3}(|15 > -|11 > +|12 > +|16 > -2|17 > +2|18 >)) \\
|1^2[1^3]\rangle &= \sqrt{\frac{2n(n-1)}{9(n-2)}} k_5 (|17 > -|18 > +|15 > -|11 > +|12 > +|16 >),
\end{align*}
$|0[1]0⟩ = k_0|9⟩,$

$|0[1]2⟩ = \frac{n}{\sqrt{2(n+2)(n-1)}} k_6 (|21⟩ - |20⟩ - \frac{2}{n}|19⟩),$

$|0[1]1^2⟩ = \frac{n}{2(n-1)} k_6 (|20⟩ + |21⟩),$

where $|1⟩ = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $|2⟩ = g_2|1⟩$, $|3⟩ = g_1 g_2|1⟩$, $|4⟩ = g_3 g_2|1⟩$, $|5⟩ = g_1 g_3 g_2|1⟩$, $|6⟩ = g_2 g_1 g_3 g_2|1⟩$, $|7⟩ = e_1 g_2|1⟩$, $|8⟩ = e_1 g_3 g_2|1⟩$, $|9⟩ = g_1 e_2|1⟩$, $|10⟩ = g_2 g_3 e_1 g_2|1⟩$, $|11⟩ = g_1 g_3 e_2|1⟩$, $|12⟩ = g_2 g_1 g_3 e_2|1⟩$, $|13⟩ = e_2|1⟩$, $|14⟩ = e_2 g_1 g_3 e_2|1⟩$, $|15⟩ = g_3 e_2|5⟩$, $|16⟩ = g_2 g_1 e_3 g_2|1⟩$, $|17⟩ = e_3 g_2|1⟩$, $|18⟩ = e_3 g_1 g_2|1⟩$, $|19⟩ = e_3 e_1 g_2|1⟩$, $|20⟩ = g_2 e_1 e_3 g_2|1⟩$, $|21⟩ = e_2 g_1 g_3 e_2|1⟩$.

In the above expressions, the IDCs are determined up to an absolute normalization constant, which can be calculated for different cases by using the corresponding norm matrix as has been done in the previous example. In the next section, we will outline an assimilation method for evaluating CGCs of $O(n)$ in the Gel’fand basis from these IDCs.

4. Evaluating CGCs of $SO(n)$ in its Gel’fand basis

Irreducible representations of $SO(m)$, where $m = 2, 3, \cdots, n$, can be labeled by partitions $[\lambda_1 \lambda_2 \cdots \lambda_m]$ with $h = m/2$ for $m$ even, and $h = (m-1)/2$ for $m$ odd, which satisfy the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \text{ for } m \text{ odd,}$$

$$\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_m| \geq 0 \text{ for } m \text{ even,}$$ (29)

where $\lambda_{im} (i = 1, 2, \cdots, h)$ are all integers because we only discuss tensor representations of $SO(m)$. The partitions of two groups, for example, $SO(2p + 1)$ and $SO(2p)$, in the
canonical chain $SO(n) \supset \cdots \supset SO(2p+1) \supset SO(2p) \supset \cdots \supset SO(2)$ are related by the betweenness conditions
\[
\lambda_{1,2p+1} \geq \lambda_{1,2p} \geq \lambda_{2,2p+1} \geq \cdots \geq \lambda_{p,2p+1} \geq |\lambda_{p,2p}|, \quad (30)
\]
Similar to $U(n)$ case, we can define Weyl tableau for $SO(n)$ in the Gel'fand basis.

\[
W^{[\lambda]} = \begin{pmatrix}
  f_{12} (\pm a_2)'s & f_{13} a_3's & f_{14} a_4's & f_{15} a_5's & f_{16} a_6's & \cdots \\
  f_{24} (\pm a_4)'s & f_{25} a_5's & f_{26} a_6's & \cdots \\
  f_{36} (\pm a_6)'s & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

(31)

where the signs in the front of $a_{2k}$ ($k = 1, 2, \cdots$) should always be the same. They can be taken all positive or all negative. The correspondence between the Weyl tableau and the Gel'fand basis is realized in the following way:

\[
\pm f_{12} = \lambda_{12}, \quad f_{12} + f_{13} = \lambda_{13}, \quad f_{12} + f_{13} + f_{14} = \lambda_{14}, \cdots,
\]
\[
\pm f_{24} = \lambda_{24}, \quad f_{24} + f_{25} = \lambda_{25}, \quad f_{24} + f_{25} + f_{26} = \lambda_{26}, \cdots,
\]
\[
\pm f_{36} = \lambda_{36}, \quad f_{36} + f_{46} = \lambda_{46}, \quad \cdots. \quad (32)
\]

For example, basis vectors of $SO(5) \supset SO(4) \supset SO(3) \supset SO(2)$ can be denoted either by Gel'fand symbol or by Weyl tableau as

\[
\begin{pmatrix}
  [\lambda_{15} \lambda_{25}] \\
  [\lambda_{14} \lambda_{24}] \\
  \lambda_{13} \\
  \lambda_{12}
\end{pmatrix} = \begin{pmatrix}
  f_{12} (\pm a_2)'s & f_{13} a_3's & f_{14} a_4's & f_{15} a_5's \\
  f_{24} (\pm a_4)'s & f_{25} a_5's
\end{pmatrix}, \quad (33)
\]

where
\[ f_{12} = |\lambda_{12}|, \quad f_{13} = \lambda_{13} - |\lambda_{12}|, \quad f_{14} = \lambda_{14} - \lambda_{13}, \quad f_{15} = \lambda_{15} - \lambda_{14}, \]

\[ f_{24} = |\lambda_{24}|, \quad f_{25} = \lambda_{25} - |\lambda_{24}|, \]  

(34)

and signs in the front of \(a_2\) or \(a_4\) should be taken as positive (negative) if \(\lambda_{12} \geq 0\) (\(< 0\)) or \(\lambda_{24} \geq 0\) (\(< 0\)).

An assimilation method for obtaining CGCs of \(SO(n)\) from IDCs of Brauer algebra is the following: Firstly, the one-box representation of \(D_1(n)\) is just rank-1 tensor of \(SO(n)\)

\[ \begin{array}{c} i \\ \hline \end{array} \rightarrow T_{j_i}^i, \]  

(35)

where the index \(i\) is used to indicate that the tensor operator is in the \(i\)-th space, while \(j_i\) is the tensor component, and can be taken as \(n\) different values. Then, an irrep \([\lambda]\) of \(SO(n)\) can be constructed by rank-1 tensors through cabling

\[ T_{i_1}^{1} T_{i_2}^{2} \cdots T_{i_f}^{f} \Rightarrow T_{i_{12}\ldots i_f}^{[\lambda] f}. \]  

(36)

Next, the symmetry properties of \(f\) space indices \(\{1, 2, \cdots, f\}\) and those tensor components \(\{i_1, i_2, \cdots, i_f\}\) are the same, i.e. interchange of \(i\) and \(k\) is the same as interchange of \(j_i\) and \(j_k\). Hence, there is a natural assimilation

\[ i \rightarrow j_i. \]  

(37)

After interchange \(i\) with \(j_i\) in basis vectors of \(S_{f_1} \times S_{f_2} \uparrow D_f(n)\), the resulting basis vectors become the corresponding \(SO(n)\) orthogonal basis vectors in Weyl tableau forms. This fact just reflects the Brauer-Schur-Weyl duality relation between \(D_f(n)\) and \(O(n)\). For example, the basis vector of \(D_1(n) \times D_1(n) \uparrow D_2(n)\) induced representation for the coupling \([1] \times [1] \uparrow [1^2]\) can be expressed as

\[ \left| \begin{array}{c} 1 \\ 2 \\ \hline \end{array} \right\rangle = \sqrt{\frac{1}{2}} \left( \left| \begin{array}{c} 1 \\ 2 \\ \hline \end{array} \right\rangle - \left| \begin{array}{c} 2 \\ 1 \\ \hline \end{array} \right\rangle \right). \]  

(38)
After the assimilation, one get the corresponding orthogonal basis vector of $SO(n) \times SO(n) \rightarrow SO(n)$ coupling with

$$\left| i_1^1, i_2^2 \right> = \sqrt{\frac{1}{2}} \left( \left| i_1^1, i_2^2 \right> - \left| i_2^2, i_1^1 \right> \right). \quad (39)$$

The left hand sides of (38) and (39) are the same, namely

$$\sqrt{\frac{1}{2}} \left( \left| i_1^1, i_2^2 \right> - \left| i_2^2, i_1^1 \right> \right) = \sqrt{\frac{1}{2}} \left( \left| i_1^1, i_2^2 \right> - \left| i_2^2, i_1^1 \right> \right) = \sqrt{\frac{1}{2}} (1 - g_1) |1, 2 > = \sqrt{\frac{1}{2}} (T^{12}_{i_1 i_2} - T^{12}_{i_2 i_1}). \quad (40)$$

The difference is only the space indices are interchanged to the corresponding $SO(n)$ tensor components. Furthermore, such interchange keeps Brauer algebra action $g_i$ or $e_i$ on the uncoupled basis vectors unchanged. While the meaning of (38) and (39) is different. The former gives the basis vector of induced representation of $D_1(n) \times D_1(n) \uparrow D_2(n)$, the latter gives basis vector of $SO(n) \times SO(n) \rightarrow SO(n)$ in the canonical basis. This assimilation can thus be obtained just because of the Brauer-Schur-Weyl duality between $D_f(n)$ and $O(n)$. Furthermore, the phase convention of $SO(n)$ CGCs have already been determined by that of IDCs of Brauer algebra. Therefore, it is not necessary to consider the phase convention for $SO(n)$ CGCs again.

However, according to Lemma 2 of Ref. [11], for the group $O(n)$, where $n = 2l$ or $2l + 1$, the irrep $[\lambda_1, \lambda_2, \cdots, \lambda_p, 0]$ is non-standard if $p > l$. In these cases, modification rules will be needed. In such circumstances, some irregular representations will involved, which cannot be obtained by using the assimilation method. For example, coupled basis vectors of $[1] \times [1] \rightarrow [11]$ or $[1 - 1]$ for $SO(4)$ can not be expressed by uncoupled basis vectors as given by (39) because the subirreps of $SO(4)$ involve the coupling $[1] \times [1] \rightarrow [11]$ for $SO(3)$. The irrep $[11] = [1]$ for $SO(3)$ obviously needs modification rules for $O(n)$. i.e., (39) is only valid for $SO(n)$ for $n \geq 5$. Therefore, we only consider CGCs with no
modification rule needed couplings. Using this assimilation method, one can evaluate CGCs of \( SO(n) \) in its canonical basis with no modification needed couplings from IDCs of \( S_{f_1} \times S_{f_2} \uparrow D_f(n) \). In the following, we will give an example to show how this method works.

**Example 2.** Find \( SO(n) \) CGCs for \([1] \times [1] = [2] + [1^2]^* + [0]\), where \(*\) indicates this irrep is only valid for \( n \geq 5 \).

Step 1. Write the corresponding basis vectors of \( S_{f_1} \times S_{f_2} \uparrow D_2(n) \). Using the results given in the previous section, we have

\[
\begin{pmatrix} 1 & 2 \end{pmatrix} = a_1 (1 + g_1 - \frac{2}{n} e_1) \begin{pmatrix} 1, & 2 \end{pmatrix}, \quad (41a)
\]

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \sqrt{\frac{1}{2}} (1 - g_1) \begin{pmatrix} 1, & 2 \end{pmatrix}, \quad (41b)
\]

\[
|0\rangle = \sqrt{\frac{1}{n}} e_1 \begin{pmatrix} 1, & 2 \end{pmatrix}. \quad (41c)
\]

Step 2. Make assimilations. We need consider three different ways of assimilation in this case.

(a) \( i_1 = \tau \alpha_k, \ i_2 = \alpha_m \ (k < m) \), where \( \tau \) can be taken as \( - \) for \( k = 2 \), and \( \tau = + \) for other cases. Because the tensor indices are different, contraction of \( i_1 \) and \( i_2 \) is zero. Hence, we get

\[
\begin{pmatrix} \tau \alpha_k & \alpha_m \end{pmatrix} = \sqrt{\frac{1}{2}} (\begin{pmatrix} \tau \alpha_k & \alpha_m \end{pmatrix} + \begin{pmatrix} \alpha_m & \tau \alpha_k \end{pmatrix}), \quad (42a)
\]

\[
\begin{pmatrix} \tau \alpha_k \\ \delta \alpha_m \end{pmatrix} = \sqrt{\frac{1}{2}} (\begin{pmatrix} \tau \alpha_k & \alpha_m \end{pmatrix} - \begin{pmatrix} \alpha_m & \tau \alpha_k \end{pmatrix}), \quad (42b)
\]

where the normalization factor \( a_1 = \sqrt{\frac{1}{2}} \). It should be noted that the \( SO(2m) \) tensor indices \( \delta \alpha_{2m} \) for \( n \) even can be taken \(-\alpha_{2m}\) only in the \( m\)-th row in the Weyl tableau.
\(-\alpha_{2m}\) in the \(p\)-th row with \(p < m\) is forbidden according to the definition of \(SO(n)\) Weyl tableau. Hence, \(\delta\alpha_m\) should be replaced by \(\alpha_m\) in the \(p\)-th row with \(p < m\). Thus, \(\delta\) can be taken as \(-\) only for \(m = 4\), and the only possible minus sign of \(\tau\) allowed in the first row is \(k = 2\) case. For example, from (42a), for \(m = 3\) and \(k = 2\), one gets the following CGCs of \(SO(3)\)

\[
\begin{pmatrix} 1 & 1 \\ \pm 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ \pm 1 \end{pmatrix} = \sqrt{\frac{1}{2}} \quad \begin{pmatrix} 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} 2 \\ \pm 1 \end{pmatrix} = \sqrt{\frac{1}{2}}.
\] (43a)

From (42b) for \(k = 3\) and \(m = 5\), one gets the following CGCs for \(SO(5)\)

\[
\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ \pm 1 \end{pmatrix} = \sqrt{\frac{1}{2}}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ \pm 1 \end{pmatrix} = -\sqrt{\frac{1}{2}}.
\] (43b)

For \(m \leq 4\) in (42b) there will be representations involving modification rules, which will not be considered in this paper.

(b) \(i_1 = i_2 = a_n\). In this case, the trace contraction is nonzero. We have

\[
e_1 \biggl| \begin{array}{c} \alpha_n \\ \alpha_n \end{array} \biggr> = \sum_{\mu}^{(s)} \biggl| \begin{array}{c} \alpha_\mu \\ \alpha_\mu \end{array} \biggr>,
\] (44a)

where there are \(n\) terms involved in the sum. It should be understood that the sum on the right hand side of (44a) is shorthand notation, of which the exact expression should be

\[
\sum_{\mu}^{(s)} \biggl| \begin{array}{c} \alpha_\mu \\ \alpha_\mu \end{array} \biggr> = \sum_{\mu \geq 3}^{(s)} \biggl| \begin{array}{c} \alpha_\mu \\ \alpha_\mu \end{array} \biggr> + \sum_{\delta = +, -} \biggl| \begin{array}{c} \delta \alpha_2 \\ -\delta \alpha_2 \end{array} \biggr>. \] (44b)

Thus, we get

\[
\biggl| \begin{array}{c} \alpha_n \\ \alpha_n \end{array} \biggr> = 2a_1 \left( 1 - \frac{1}{n} \right) \biggl| \begin{array}{c} \alpha_n \\ \alpha_n \end{array} \biggr> - \frac{1}{n} \sum_{\mu \neq n}^{(s)} \biggl| \begin{array}{c} \alpha_\mu \\ \alpha_\mu \end{array} \biggr>. \] (45)

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After normalization, (45) becomes

$$\langle \alpha_n | \alpha_n \rangle = \sqrt{\frac{n-1}{n}} \langle \alpha_n | \alpha_n \rangle - \sqrt{\frac{1}{n(n-1)} \sum_{\mu \neq n}^{(s)} \langle \alpha_\mu | \alpha_\mu \rangle}.$$

Similarly, we have

$$\langle 0 | = \sqrt{\frac{1}{n}} \langle \alpha_n | \alpha_n \rangle + \sqrt{\frac{1}{n} \sum_{\mu \neq n}^{(s)} \langle \alpha_\mu | \alpha_\mu \rangle}.$$

(c) $i_1 = i_2 = a_k (2 < k < n)$. The final results in this case are similar to those of (46) and (47).

$$\langle \alpha_k | \alpha_k \rangle = \sqrt{\frac{k-1}{k}} \langle \alpha_k | \alpha_k \rangle - \sqrt{\frac{1}{k(k-1)} \sum_{\mu \neq k}^{(s)} \langle \alpha_\mu | \alpha_\mu \rangle}.$$

$$\langle 0 | = \sqrt{\frac{1}{k}} \langle \alpha_k | \alpha_k \rangle + \sqrt{\frac{1}{k} \sum_{\mu \neq k}^{(s)} \langle \alpha_\mu | \alpha_\mu \rangle}.$$

The corresponding CGCs of $SO(n)$ can now be read off from (42a,b), (46)-(48). When $n = 4$, for example, the $SO(4) \supset SO(3) \supset SO(2)$ CGCs read off from (46) and (47) are

$$\langle \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \rangle = \frac{3}{4}, \quad \langle \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} | \begin{array}{ccc} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{array} \rangle = \frac{1}{12},$$

$$\langle \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} | \begin{array}{ccc} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{array} \rangle = -\frac{1}{12}, \quad \langle \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} | \begin{array}{ccc} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{array} \rangle = \frac{1}{4},$$

$$\langle \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} | \begin{array}{ccc} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{array} \rangle = -\frac{1}{4}. \quad (50)$$

When $n = 4$ and $k = 3$, from (48) and (49), we have
\[ \begin{align*}
\langle [1] [1] | [2] \rangle &= \sqrt{\frac{2}{3}}, \quad \langle [1] [1] | [2] \rangle = \langle [1] [1] | [2] \rangle = \sqrt{\frac{1}{6}}, \quad (51) \\
\langle [1] [1] | [0] \rangle &= \sqrt{\frac{1}{3}}, \quad \langle [1] [1] | [0] \rangle = \langle [1] [1] | [0] \rangle = -\sqrt{\frac{1}{3}}. \quad (52)
\end{align*} \]

where (52) gives CGCs of \( SO(3) \).

Using this method, we have derived CGCs of \( SO(n) \) for the resulting irrep \( [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \hat{0}] \) with \( \sum_{i=1}^{4} \lambda_i \leq 4 \) and no modification rule involved couplings from IDCs of \( S_{f_1} \times S_{f_2} \uparrow D_f(n) \) with \( f_1 + f_2 = f \) given in the previous section. However, the expressions of the CGCs for any \( n \) are too cumbersome to be tabulated. While the Isoscalar factors (ISFs) of \( SO(n) \supset SO(n-1) \) for any \( n \), which can be obtained according to Racah factorization lemma,\(^5\) are concise and easily to be listed in a table. For example, one can easily read off the following ISFs of \( SO(n) \supset SO(n-1) \) with \( n \geq 4 \) from (46):

\[ \begin{align*}
\langle [1] [1] | [2] \rangle &= \sqrt{\frac{n-1}{n}}, \quad \langle [1] [1] | [2] \rangle = -\sqrt{\frac{1}{n}}. \quad (53)
\end{align*} \]

The notation for the ISFs used in (53) is much simpler than that of CGCs. Therefore, we shall only list ISFs of \( SO(n) \supset SO(n-1) \) for \( n \geq 4 \) in the next section.

5. ISFs of \( SO(n) \supset SO(n-1) \)

In this section, we will list some ISFs of \( SO(n) \supset SO(n-1) \) derived by using the assimilation method outlined in Sec. 4. According to Racah factorization lemma, \( SO(n) \) CGCs in the canonical basis \( SO(n) \supset SO(n-1) \supset \cdots \supset SO(2) \) can be expressed as

\[
\begin{pmatrix}
[\lambda_{1n}] & [\lambda_{2n}] & \tau_n [\lambda_n] \\
[\lambda_{1 n-1}] & [\lambda_{2 n-1}] & [\lambda_{n-1}] \\
\cdots & \cdots & \cdots \\
m_{12} & m_{22} & [m_2]
\end{pmatrix} = \sum_{\tau_{n-1}} \begin{pmatrix}
[\lambda_{1 n-1}] & [\lambda_{2 n-1}] & \tau_{n-1} [\lambda_{n-1}]
\end{pmatrix} \times
\]

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\[
\begin{pmatrix}
[\lambda_{1-1}] & [\lambda_{2-n}] & \tau_{n-1} [\lambda_{n-1}] \\
[\lambda_{1-n-2}] & [\lambda_{2-n-2}] & [\lambda_{n-2}] \\
\vdots & \vdots & \vdots \\
m_{12} & m_{22} & [m_2]
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
[\lambda_{1-n}] & [\lambda_{2-n}] & \tau_n [\lambda_n] \\
[\lambda_{1-n-1}] & [\lambda_{2-n-1}] & [\lambda_{n-1}] \\
\vdots & \vdots & \vdots \\
m_{12} & m_{22} & [m_2]
\end{pmatrix}
\]

is \(SO(n)\) CGC,

\[
\left\langle [\lambda_{1-n}] [\lambda_{2-n}] | \tau_n [\lambda_n] \right\rangle
\]

is \(SO(n) \supset SO(n-1)\) ISF, and \(\tau_n\) is the multiplicity label needed in the coupling \([\lambda_{1-n}] \times [\lambda_{2-n}] \downarrow [\lambda_n]\). The ISFs satisfy the following orthogonality conditions

\[
\sum_{\lambda_{1-n-1}\lambda_{2-n-1}} \left\langle [\lambda_{1-n}] [\lambda_{2-n}] | \tau_n [\lambda_n] \right\rangle \left\langle [\lambda_{1-n}] [\lambda_{2-n}] | \tau'_n [\lambda'_n] \right\rangle = \delta_{\lambda_n, \lambda_n'} \delta_{\tau_n, \tau_n'},
\]

\[
\sum_{\tau_n, \lambda_n} \left\langle [\lambda_{1-n}] [\lambda_{2-n}] | \tau_n [\lambda_n] \right\rangle \left\langle [\lambda_{1-n}] [\lambda_{2-n}] | \tau_n [\lambda_n] \right\rangle = \delta_{\lambda'_n, \lambda_{n-1}} \delta_{\lambda_n, \lambda_{n-1}}.
\]

In the following, we list no modification rule involved \(SO(n) \supset SO(n-1)\) ISFs for the coupling \([\lambda_1] \times [\lambda_2]\) with resulting irreps \([\lambda_{1-n}, \lambda_{2-n}, \lambda_{3-n}, \lambda_{4-n}, \hat{0}]\) for \(\sum_i \lambda_{im} \leq 4\), which are obtained from IDCs of \(S_{f_1} \times S_{f_2} \uparrow D_f(n)\) with \(f_1 + f_2 = f \leq 4\).

6. Conclusions
In this paper, induced representations of \( D_f(n) \) from \( S_{f_1} \times S_{f_2} \) with \( f_1 + f_2 = f \) are constructed. The IDCs of \( S_{f_1} \times S_{f_2} \uparrow D_f(n) \) with \( f \leq 4 \) up to a normalization factor are derived by using the linear equation method. It is found that these IDCs are \( SO(n) \) tensor component dependent. Weyl tableaus for the corresponding Gel’fand basis of \( SO(n) \) are defined. The assimilation method for obtaining CGCs of \( SO(n) \) in the Gel’fand basis with no modification rule involved couplings from IDCs of Brauer algebra are proposed, which is based on the Brauer-Schur-Weyl duality relation between \( O(n) \) and Brauer algebra \( D_f(n) \). Isoscalar factors of \( SO(n) \supset SO(n-1) \) for the resulting irrep \([\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0]\) with \( \sum_{i=1}^{4} \lambda_i \leq 4 \) are tabulated. From these tables of ISFs, one can find that there are two types of no modification rule involved ISFs of \( SO(n) \supset SO(n-1) \) or CGCs of \( SO(n) \) in its canonical basis. The type-one ISFs or CGCs are \( n \)-independent, which are the same as ISFs of \( U(n) \supset U(n-1) \) or CGCs of \( U(n) \) in the canonical basis. Therefore, The following type of ISFs of \( U(n) \supset U(n-1) \) are also \( SO(n) \supset SO(n-1) \) ISFs:

\[
\left\langle \begin{array}{c|c|c|c}
[\lambda_1] & [\lambda_2] & \tau_n & [\lambda] \\
[\nu_1] & [\nu_2] & \tau_{n-1} & [\nu] \\
\end{array} \right\rangle
\]

(57)

with \( \sum_i (\lambda_{1i} + \lambda_{2i}) = \sum_i \lambda_i, \sum_j (\nu_{1j} + \nu_{2j}) = \sum_j \nu_j, \) and \( \sum_i \lambda_i - \sum_j \nu_j = 0 \) or 1. Hence, ISFs for \( U(n) \supset U(n-1) \) of this type or CGCs for \( U(n) \) of this type derived previously,[15–17] in which many results are with outer-multiplicity, are also those of \( SO(n) \). From Brauer algebra point of view, there is no trace contraction between \([\lambda_1]\) and \([\lambda_2]\) in \( SO(n) \) couplings in these cases. As a consequence, these coefficients can be derived from the IDCs of \( S_{f_1} \times S_{f_2} \uparrow S_f \), which has already been discussed in [15]. The type-two ISFs of \( SO(n) \supset SO(n-1) \) (CGCs of \( SO(n) \) in the canonical basis) are rank-\( n \) dependent, which can only be obtained from IDCs of \( S_{f_1} \times S_{f_2} \uparrow D_f(n) \). However, the problem concerning how to derive modification rule involved CGCs of \( SO(n) \) from the IDCs of Brauer algebra still needs further study.

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Table 1. ISFs \( \begin{bmatrix} \lambda_1 \\ \nu_1 \\ \nu_2 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \nu_1 \\ \nu_2 \end{bmatrix} \) of \( SO(n) \supset SO(n - 1) \) for \([1] \times [1] = [2] + [1^2]^* + [0] \).

| \( \lambda \) | \( \lambda_1 \) | \( \lambda_2 \) | \( [10] \) | \( [10] \) | \( [10] \) | \( [10] \) | \( [10] \) | \( [10] \) | \( [10] \) |
| \( \nu \) | \( \nu_1 \) | \( \nu_2 \) | \( [10] \) | \( [10] \) | \( [0] \) | \( [0] \) | \( [10] \) | \( [0] \) | \( [0] \) |

| \( [2] \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( [2] \) |

| \( [2] \) | \( 0 \) | \( \sqrt{\frac{1}{2}} \) | \( \sqrt{\frac{1}{2}} \) | \( 0 \) |
| \( [1] \) |

| \( [2] \) | \( \sqrt{\frac{1}{n}} \) | \( 0 \) | \( 0 \) | \( -\sqrt{\frac{n-1}{n}} \) |
| \( [0] \) |

| \( [1^2] \) | \( 0 \) | \( \sqrt{\frac{1}{2}} \) | \( -\sqrt{\frac{1}{2}} \) | \( 0 \) |
| \( [1] \) |

| \( [1^2] \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( [1^2]^* \) |

| \( [0] \) | \( \sqrt{\frac{n-1}{n}} \) | \( 0 \) | \( 0 \) | \( \sqrt{\frac{1}{n}} \) |
| \( [0] \) |

* The corresponding ISFs for this irrep are only valid for \( n \geq 5 \).

** It can be taken as \([11]\) or \([1-1]\) when \( n = 5 \).
Table 2. ISFs \( \left\langle \begin{bmatrix} \lambda_1 \\ \nu_1 \\ \rho_1 \\ [2] \\ [1] \\ [1] \\ [2] \\ [0] \\ [1] \end{bmatrix} \left| \begin{bmatrix} \lambda \\ \nu \\ \rho \\ [2] \\ [1] \end{bmatrix} \right\rangle \) of \( SO(n) \supset SO(n-1) \) for \( [2] \times [1] = [30] + [21]^* + [1] \).

| \[\lambda\] | \[\lambda_1\] | \[\lambda_2\] | \[2\] | \[1\] | \[2\] | \[1\] | \[2\] | \[1\] | \[2\] | \[1\] | \[2\] | \[1\] | \[2\] | \[1\] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \[\nu\] | \[\nu_1\] | \[\nu_2\] | \[2\] | \[1\] | \[2\] | \[0\] | \[1\] | \[1\] | \[0\] | \[0\] | \[1\] | \[0\] | \[0\] |

| \[3\] | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
|\[3\] |
| \[3\] | 0 | \(\sqrt{\frac{1}{3}}\) | \(\sqrt{\frac{2}{3}}\) | 0 | 0 | 0 | 0 |
| \[2\] |
| \[3\] | \(\sqrt{\frac{2(n-2)}{3(n+2)(n-1)}}\) | 0 | 0 | \(\sqrt{\frac{2(n+1)}{3(n+2)}}\) | -\(\sqrt{\frac{n(n+1)}{3(n+2)(n-1)}}\) | 0 |
| \[1\] |
| \[3\] | 0 | 0 | \(\sqrt{\frac{2}{n+2}}\) | 0 | 0 | \(\sqrt{\frac{n}{n+2}}\) |
| \[0\] |
| \[21\] | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \[21]^*\ |
| \[21\] | 0 | \(\sqrt{\frac{2}{3}}\) | -\(\sqrt{\frac{1}{3}}\) | 0 | 0 | 0 | 0 |
| \[2\] |
| \[21\] | \(\sqrt{\frac{n+1}{3(n-1)^2}}\) | 0 | 0 | \(\sqrt{\frac{n-2}{3(n-1)}}\) | \(\sqrt{\frac{2n(n-2)}{3(n-1)^2}}\) | 0 |
| \[1\] |
| \[21\] | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| \[1^2]^*\ |

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\[
\begin{bmatrix}
[1] & \sqrt{\frac{n(n+1)(n-2)}{(n+2)(n-1)^2}} & 0 & 0 & -\sqrt{\frac{n}{(n+2)(n-1)}} & -\sqrt{\frac{2}{(n+2)(n-1)^2}} & 0 \\
[0] & 0 & 0 & \sqrt{\frac{n}{n+2}} & 0 & 0 & -\sqrt{\frac{2}{n+2}}
\end{bmatrix}
\]

* The corresponding ISFs for this irrep are only valid for \( n \geq 5 \). *2 It can be taken as \([21]\) or \([2-1]\) when \( n = 5 \). *3 It can be taken as \([11]\) or \([1-1]\) when \( n = 5 \).
Table 3. ISFs $\left\langle \begin{bmatrix} \lambda_1 \\ \nu_1 \end{bmatrix}, \begin{bmatrix} \lambda_2 \\ \nu_2 \end{bmatrix} \right| \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \rangle$ of $SO(n) \supset SO(n-1)$ for $[30] \times [10] = [40] + [31]^* + [20]$. 

| $\lambda$ | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $\nu$ | 3 | 1 | 3 | 0 | 2 | 1 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |

| 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | $\frac{1}{2}$ | $\frac{\sqrt{3}}{4}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | $\sqrt{\frac{n-1}{2(n+1)(n+4)}}$ | 0 | 0 | $-\sqrt{\frac{n+3}{2(n+4)}}$ | $\sqrt{\frac{(n+2)(n+3)}{2(n+1)(n+4)}}$ | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | $\sqrt{\frac{3(n-2)}{2(n-1)(n+4)}}$ | 0 | 0 | $\sqrt{\frac{3(n+2)}{4(n+4)}}$ | $\sqrt{\frac{(n+1)(n+2)}{4(n-1)(n+4)}}$ | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | $\sqrt{\frac{3}{n+4}}$ | 0 | 0 | $\sqrt{\frac{n+1}{n+4}}$ | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | $\sqrt{\frac{3}{4}}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 |

* The corresponding ISFs for this irrep are only valid for $n \geq 5$. 

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Table 3. (Continued)

| \(\lambda\) / \(\nu\) | [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] [3] [1] |
|------------------------|-------------------------------------------------|
| \(\sqrt{n+3\over 2n(n+1)}\) | 0 | 0 | \(-\sqrt{n-1\over 2n}\) | \(-\sqrt{(n-1)(n+2)\over 2n(n+1)}\) | 0 | 0 | 0 |
| \(\sqrt{n+2\over 2n(n-1)}\) | 0 | 0 | \(\sqrt{n+2\over 2n(n-1)}\) | 0 | 0 | \(\sqrt{n-2\over 4n}\) | \(-\sqrt{3(n+1)(n-2)\over 4n(n-1)}\) | 0 |
| \(\sqrt{n+1\over n+1}\) | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| \(\sqrt{n+1\over n+1}\) | 0 | 0 | 0 | 0 | \(\sqrt{n+1\over n+1}\) | 1 | 0 | 0 | 0 |
| \(\sqrt{(n+2)(n+3)(n-1)\over n(n+1)(n+4)}\) | 0 | 0 | \(\sqrt{n+2\over n(n+4)}\) | \(\sqrt{4\over n(n+1)(n+4)}\) | 0 | 0 | 0 |
| \(\sqrt{(n^2-4)(n+1)\over n(n-1)(n+4)}\) | 0 | 0 | \(\sqrt{n^2-4\over n(n-1)(n+4)}\) | 0 | 0 | \(-\sqrt{2(n+1)\over n(n+4)}\) | \(\sqrt{6\over n(n-1)(n+4)}\) | 0 |
| \(\sqrt{n+1\over n+4}\) | 0 | 0 | 0 | 0 | \(\sqrt{n+1\over n+4}\) | 0 | 0 | \(-\sqrt{3\over n+4}\) | 0 |
* It can be taken as [21] or [2-1] when $n = 5$. ** It can be taken as [11] or [1-1] when $n = 5$
Table 4. ISFs $\langle \begin{bmatrix} \lambda_1 \\ \nu_1 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \nu_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \rangle$ of $SO(n) \supset SO(n-1)$ for $[1^2] \times [1] = [21] + [1^3] + [1]$ for $n \geq 7$.

| $\lambda$ / $\nu$ | $[\lambda_1]$ | $[\lambda_2]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ |
|-----------------|--------------|--------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $[\nu_1]$ $[\nu_2]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ | $[1^2]$ | $[10]$ |

\[
\begin{array}{cccccc}
[21] & 1 & 0 & 0 & 0 & 0 \\
[21] & & & & & \\
[21] & 0 & 0 & 1 & 0 & \\
[2] & & & & & \\
[21] & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 & \\
[1^2] & & & & & \\
[21] & \sqrt{\frac{1}{n-1}} & 0 & 0 & \sqrt{\frac{n-2}{n-1}} & \\
[1] & & & & & \\
[1^3] & 1 & 0 & 0 & 0 & \\
[1^3]^* & & & & & \\
[1^3] & 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & 0 & \\
[1] & & & & & \\
[1] & \sqrt{\frac{n-2}{n-1}} & 0 & 0 & -\sqrt{\frac{n-1}{n-1}} & \\
[1] & & & & & \\
\end{array}
\]
\begin{align*}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
& \begin{bmatrix}
0 & 0 & 1 & 0
\end{bmatrix}
\end{align*}

* It can be taken as [111] or [11-1] when $n = 7$
Table 5. ISFs $\left< \begin{array}{c} \lambda_1 \\ \nu_1 \\ \vdots \\ \lambda \\ \nu \end{array} \right| \begin{array}{c} \lambda_2 \\ \nu_2 \\ \vdots \\ \lambda \\ \nu \end{array} \right>$ of $SO(n) \supset SO(n-1)$ for $[1^3] \times [1] = [211] + [1^4] + [1^2]$ for $n \geq 9$.

| $\lambda$ | $\lambda_1$ | $\lambda_2$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\nu$ | $\nu_1$ | $\nu_2$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ | $[1^3]$ | $[1^0]$ |

| $[211]$ | $1$ | $0$ | $0$ | $0$ |
| $[211]$ |
| $[211]$ | $0$ | $0$ | $1$ | $0$ |
| $[21]$ |
| $[211]$ | $\sqrt{\frac{1}{n-2}}$ | $0$ | $0$ | $\sqrt{\frac{n-3}{n-2}}$
| $[1^2]$ |
| $[211]$ | $0$ | $\frac{\sqrt{3}}{4}$ | $\frac{1}{2}$ | $0$ |
| $[1^3]$ |
| $[1^4]$ | $1$ | $0$ | $0$ | $0$ |
| $[1^4]^*$ |
| $[1^4]$ | $0$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{4}$ | $0$ |
| $[1^3]$ |
| $[1^2]$ | $\sqrt{\frac{n-4}{n-2}}$ | $0$ | $0$ | $-\sqrt{\frac{n-2}{n-2}}$ |
| $[1^2]$ |
* It can be taken as [1111] or [111-1] when $n = 9$
Table 6. ISFs \( \left\langle \begin{array}{cc} \lambda_1 & \lambda_2 \\ \nu_1 & \nu_2 \end{array} \right\rangle \) of \( SO(n) \supset SO(n-1) \) for \([2] \times [1^2] = [31] + [211] + [2] + [1^2] \) for \( n \geq 7 \).

| \lambda | \lambda_1 | \lambda_2 | 2 | [1^2] | 2 | [1^2] | 2 | [1^2] | 2 | [1^2] | 2 | [1^2] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \nu | \nu_1 | \nu_2 | 2 | [1^2] | 2 | [1^2] | 1 | [1^2] | 1 | [1^2] | 0 | [1^2] |

\[
\begin{array}{ccccccccccccc}
[31] & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
[31] & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
[31] & 0 & \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 & 0 & 0 \\
[21] & \sqrt{\frac{1}{n}} & 0 & 0 & \sqrt{\frac{n-1}{n}} & 0 & 0 & 0 \\
[31] & \sqrt{\frac{n-3}{2(n+2)(n-1)}} & 0 & 0 & \sqrt{\frac{n+1}{2(n+2)}} & -\sqrt{\frac{n(n+1)}{2(n+2)(n-1)}} & 0 \\
[1^2] & 0 & \sqrt{\frac{2(n+2)(n-1)}{n(n+1)^2(n-2)^2}} & \sqrt{\frac{2(n-1)^2(n+2)}{n(n+1)(n-2)^2}} & 0 & \sqrt{\frac{(n-1)(n+2)}{(n+1)(n-2)}} & 0 \\
[211] & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
[211]^* & & & & & & & & & & & &
\end{array}
\]
* It can be taken as [211] or [21-1] when $n = 7$
Table 6. (Continued)

| $\lambda$ | $\lambda_1$ | $\lambda_2$ | $[2]$ | $[1^2]$ | $[2]$ | $[1^2]$ | $[2]$ | $[1^2]$ | $[2]$ | $[1^2]$ | $[2]$ |
|-------|-------------|-------------|------|--------|------|--------|------|--------|------|--------|------|
| $\nu$  | $\nu_1$     | $\nu_2$     | $[2]$ | $[1^2]$ | $[2]$ | $[1]$  | $[1^2]$| $[1]$  | $[1]$ | $[0]$  | $[1^2]$|

| 211   | 0 | $\sqrt{\frac{n}{4}}$ | $\frac{1}{2}$ | 0 | 0 | 0 |
| 21    | $\sqrt{\frac{n+1}{2(n-2)(n-1)}}$ | 0 | 0 | $\sqrt{\frac{n-3}{2(n-2)}}$ | $-\sqrt{\frac{n(n-3)}{2(n-2)(n-1)}}$ | 0 |
| 211   | 0 | 0 | 1 | 0 | 0 | 0 |
| 13*   | $\sqrt{\frac{n-1}{n}}$ | 0 | 0 | $\sqrt{\frac{1}{n}}$ | 0 | 0 |
| 2     | 0 | $\sqrt{\frac{(n-2)(n+1)}{2n(n-1)}}$ | $\sqrt{\frac{n-2}{2n}}$ | 0 | 0 | $-\sqrt{\frac{1}{n-1}}$ |
| 2     | 0 | 0 | 0 | 1 | 0 | 0 |
| 0     | $\sqrt{\frac{n(n+1)(n-3)}{(n-1)(n^2-4)}}$ | 0 | 0 | $-\sqrt{\frac{n}{(n+2)(n-2)}}$ | $\sqrt{\frac{4}{(n^2-1)(n-1)}}$ | 0 |
| 12    | 0 | $\sqrt{\frac{n(n+1)}{2(n+2)(n-1)}}$ | $-\sqrt{\frac{n}{2(n+2)}}$ | 0 | 0 | $\sqrt{\frac{n-2}{(n+2)(n-1)}}$ |
* It can be taken as [111] or [11-1] when $n = 7$
Table 7. ISFs \(\begin{bmatrix} \lambda_1 \\ \nu_1 \\ \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \nu_2 \\ \end{bmatrix} \) of \(SO(n) \supset SO(n-1)\) for \([1^2] \times [1^2] = [22] + [21^2]\) 

\[+[1^4]^* + [20] + [1^2] + [0] \text{ for } n \geq 7.\]

| \(\lambda\) | \(\lambda_1\) | \(\lambda_2\) | \([1^2]\) | \([1^2]\) | \([1^2]\) | \([1^2]\) | \([1^2]\) | \([1^2]\) |
|---|---|---|---|---|---|---|---|---|
| \(\nu\) | \(\nu_1\) | \(\nu_2\) | \([1^2]\) | \([1^2]\) | \([1^2]\) | \([1]\) | \([1^2]\) | \([1]\) |

| \(22\) | 1 | 0 | 0 | 0 |
| \(22\) | |
| \(22\) | 0 | \(\sqrt{\frac{1}{2}}\) | \(\sqrt{\frac{1}{2}}\) | 0 |
| \(21\) | |
| \(22\) | \(\sqrt{\frac{1}{n-2}}\) | 0 | 0 | \(\sqrt{\frac{n-3}{n-2}}\) |
| \(2\) | |
| \(21^2\) | 1 | 0 | 0 | 0 |
| \(21^2\) | |
| \(21^2\) | 0 | \(\sqrt{\frac{1}{2}}\) | \(-\sqrt{\frac{1}{2}}\) | 0 |
| \(21\) | |
| \(21^2\) | 0 | \(-\sqrt{\frac{1}{2}}\) | \(-\sqrt{\frac{1}{2}}\) | 0 |
| \(1^2\) | |
| \(21^2\) | \(\sqrt{\frac{1}{n-2}}\) | 0 | 0 | \(\sqrt{\frac{n-3}{n-2}}\) |
\[ [1^4] \]
\[ [1^4]^3 \]

1 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0

* The corresponding ISFs for this irrep are only valid for \( n \geq 9 \). *\(^2\) It can be taken as [211] or [21-1] when \( n = 7 \). *\(^3\) It can be taken as [1111] or [111-1] when \( n = 9 \).
Table 7. (Continued)

| [\lambda] | [\lambda_1] | [\lambda_2] | [1^2] | [1^2] | [1^2] | [1^2] | [1^2] | [1^2] |
|-----------|--------------|-------------|-------|-------|-------|-------|-------|-------|
| /         | [\nu_1] | [\nu_2] | [1^2] | [1^2] | [1^2] | [1^2] | [1^2] | [1^2] |

\[
\begin{array}{cccc}
[1^2] & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\
[1^3] & \sqrt{\frac{n-3}{n-2}} & 0 & 0 & -\sqrt{\frac{1}{n-2}} \\
[2] & \sqrt{\frac{2}{n}} & 0 & 0 & \sqrt{\frac{n-2}{n}} \\
[0] & \sqrt{\frac{n-3}{n-2}} & 0 & 0 & -\sqrt{\frac{1}{n-2}} \\
[1^2] & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\
[1] & \sqrt{\frac{n-2}{n}} & 0 & 0 & -\sqrt{\frac{2}{n}} \\
[0] & \sqrt{\frac{n-2}{n}} & 0 & 0 & -\sqrt{\frac{2}{n}} \\
[0] & \sqrt{\frac{n-2}{n}} & 0 & 0 & -\sqrt{\frac{2}{n}} \\
\end{array}
\]
Table 8. ISFs $\langle \begin{bmatrix} \lambda_1 & \lambda_2 \\ \nu_1 & \nu_2 \end{bmatrix} | \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \rangle$ of $SO(n) \supset SO(n-1)$ for $[21] \times [10] = [31] + [22] + [211] + [20] + [1^2]$ for $n \geq 7$.

| $\lambda$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\nu$     | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ | $[21]$ |

$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{3}{8}} & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{\frac{n-3}{2n(n-2)}} & 0 & 0 & \sqrt{\frac{n-1}{2n}} & \sqrt{\frac{(n-1)^2}{2n(n-2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\n\sqrt{\frac{3(n-3)}{4(n^2-4)}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{3(n+1)}{4(n+2)}} & -\sqrt{\frac{n^2-1}{4(n^2-4)}} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{n-1}{2n(n+2)}} & 0 & -\sqrt{\frac{3(n+1)}{2n(n+2)}} & 0 & 0 & 0 & -\sqrt{\frac{n^2-1}{n(n+2)}} & 0 & 0
\end{bmatrix}$
\[
\begin{bmatrix}
[22] & 0 & \frac{1}{2} & \sqrt{\frac{7}{8}} & 0 & -\sqrt{\frac{7}{8}} & 0 & 0 & 0 \\
[21] & \end{bmatrix}
\]
Table 8. (Continued)

| $\lambda$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ | $[21]$ | $[1]$ |
|-----------|--------|------|-------|------|-------|------|-------|------|-------|------|-------|------|-------|------|
| $\nu$     | $[21]$ | $[1]$ | $[21]$ | $[0]$ | $[2]$ | $[1]$ | $[2]$ | $[0]$ | $[1]^2$ | $[1]$ | $[1]^2$ | $[0]$ | $[1]$ | $[1]$ | $[1]$ | $[0]$ |

| [22]      | $\sqrt{\frac{n-1}{2(n-2)^2}}$ | 0   | 0     | $\sqrt{\frac{n-3}{2(n-2)^2}}$ | 0   | 0     | $-\sqrt{\frac{(n-1)(n-3)}{2(n-2)^2}}$ | 0 |
| [2]       | $\sqrt{\frac{n+1}{4(n-2)^2}}$ | 0   | 0     | 0     | 0     | $-\sqrt{\frac{n-3}{4(n-2)^2}}$ | $\sqrt{\frac{3(n-1)(n-3)}{4(n-2)^2}}$ | 0 |
| [21]^2    | 1     | 0    | 0     | 0     | 0     | 0     | 0     | 0     | 0   |
| [21]^2*   | 0     | $\sqrt{\frac{3}{8}}$ | $-\frac{3}{4}$ | 0 | $-\frac{1}{4}$ | 0 | 0 | 0 |
| [21]      | 0     | 0    | 0     | 0     | 0     | 0     | 0     | 1     | 0   |
| [211]     | 0     | 0    | 0     | 1     | 0     | 0     | 0     | 0     | 0   |
| [13]*2    | 0     | 0    | 0     | $\sqrt{\frac{n+1}{4n}}$ | 0     | $\sqrt{\frac{3(n-1)}{4n}}$ | 0     | 0     | $-\sqrt{\frac{1}{2n}}$ |
| [2]       | 0     | 0    | $\sqrt{\frac{n+1}{4n}}$ | 0     | $\sqrt{\frac{3(n-1)}{4n}}$ | 0     | 0     | 0     | $-\sqrt{\frac{1}{2n}}$ |
| [2]       | 0     | 0    | 0     | 0     | 0     | 0     | 0     | 1     | 0   |

55
\[
\begin{array}{cccccc}
[1^2] & \sqrt{\frac{(n-1)^2(n-3)}{(n+2)(n-2)^2}} & 0 & 0 & 0 & \sqrt{\frac{n-1}{n+1}} & -\sqrt{\frac{3}{(n+2)(n-2)^2}} & 0 \\
[1^2] & 0 & 0 & \sqrt{\frac{3(n+1)}{4(n+2)}} & 0 & -\sqrt{\frac{n-1}{4(n-2)}} & 0 & 0 \\
[1] & 0 & 0 & \sqrt{\frac{3(n+1)}{4(n+2)}} & 0 & 0 & \sqrt{\frac{3}{2(n+2)}} \\
\end{array}
\]

* It can be taken as [211] or [21-1] when \( n = 7 \).

*² It can be taken as [111] or [11-1] when \( n = 7 \)
Table 9. ISFs \( \left( \begin{array}{c|c|c} [\lambda] & [\lambda_1] & [\lambda_2] \\ \hline [\nu] & [\nu_1] & [\nu_2] \end{array} \right) \) of \( SO(n) \supset SO(n-1) \) for \( [20] \times [20] = [40] + [31] + [22] + [20] + [1^2] + [0] \) for \( n \geq 5 \).

\[
\begin{array}{cccccccccc}
\lambda & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\nu & 2 & 2 & 2 & 1 & 2 & 0 & 1 & 2 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
[4] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
[4] & 0 & \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
[3] & \sqrt{\frac{2(n-3)}{3(n+4)(n-1)}} & 0 & -\sqrt{\frac{n(n+3)}{6(n+4)(n-1)}} & 0 & \sqrt{\frac{2(n-3)}{3(n+4)}} & 0 & -\sqrt{\frac{n(n+3)}{6(n+4)(n-1)}} & 0 & 0 \\
[2] & 0 & \sqrt{\frac{n-2}{(n+4)(n-1)}} & 0 & -\sqrt{\frac{n-2}{(n+4)(n-1)}} & 0 & -\sqrt{\frac{n(n+3)}{2(n+1)(n+4)}} & 0 & -\sqrt{\frac{n(n+1)}{2(n-1)(n+4)}} & 0 \\
[1] & \sqrt{\frac{2(n-2)}{(n+4)(n-1)(n+2)}} & 0 & 0 & 0 & \sqrt{\frac{4(n+1)}{(n+2)(n+4)}} & 0 & 0 & 0 & \sqrt{\frac{n^2(n+1)}{(n+4)(n+2)(n-1)}} \\
[0] & 0 & \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
[31] & 0 & \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
[31]^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
[31] & 0 & \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
[3] & 0 & \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
|        | [31]  | [2]                      | [31]  | [1]                      | [31]  | [21]\(^*^2\) |
|--------|------|--------------------------|------|--------------------------|------|--------------|
|        | 0    | 0 \(\sqrt{\frac{1}{2}}\) | 0    | 0 \(\sqrt{\frac{1}{2}}\) | 0    | 0 \(\sqrt{\frac{1}{2}}\) |
|        | 0    | \(\sqrt{\frac{n}{(n+2)(n-1)}}\) | 0    | \(\sqrt{\frac{n}{(n+2)(n-1)}}\) | 0    | -\(\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}\) |
|        | 0    | \(\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}\) | 0    | \(\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}\) | 0    | \(\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}\) |

* It can be taken as [31] or [3-1] when \(n = 5\).

*\(^2\) It can be taken as [21] or [2-1] when \(n = 5\)
Table 9. (Continued)

| \( \lambda \) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \nu \) / | 2 | 2 | 2 | 1 | 2 | 0 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 0 | 0 |

\[
\begin{align*}
\sqrt{\frac{1}{n+2}} & & 0 & & 0 & & 0 & & 0 & & \sqrt{\frac{n+1}{n+2}} & & 0 & & 0 & & 0 & & 0 & & 0 \\
\sqrt{\frac{n+3}{3(n-2)(n-1)}} & & 0 & & 0 & & 0 & & 0 & & \sqrt{\frac{n(n-3)}{3(n-2)(n-1)}} & & 0 & & \sqrt{\frac{n-3}{3(n-2)}} & & 0 & & \sqrt{\frac{n(n-3)}{3(n-2)(n-1)}} & & 0 & & 0 \\
\sqrt{\frac{n(n^2-9)}{(n+4)(n-2)(n-1)}} & & 0 & & -\sqrt{\frac{4}{(n+4)(n-2)(n-1)}} & & 0 & & 0 & & -\sqrt{\frac{n}{(n+4)(n-2)}} & & 0 & & -\sqrt{\frac{4}{(n+4)(n-2)(n-1)}} & & 0 & & 0 \\
\sqrt{\frac{n(n+1)}{2(n+4)(n-1)}} & & 0 & & 0 & & -\sqrt{\frac{n(n+1)}{2(n+4)(n-1)}} & & 0 & & \sqrt{\frac{n-2}{(n-1)(n+4)}} & & 0 & & \sqrt{\frac{n-2}{(n-1)(n+4)}} & & 0 & & 0 \\
\sqrt{\frac{2(n+1)}{(n+4)(n-1)}} & & 0 & & 0 & & 0 & & \sqrt{\frac{n-2}{n+4}} & & 0 & & 0 & & 0 & & -\sqrt{\frac{4(n-2)}{(n+4)(n-1)}} \\
\sqrt{\frac{2+1}{n+2}} & & 0 & & 0 & & 0 & & -\sqrt{\frac{1}{n+2}} & & 0 & & 0 & & 0 & & 0 & & 0
\end{align*}
\]
\[
\begin{bmatrix}
1^2 \\
1 \\
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & \sqrt{\frac{(n+1)(n-2)}{2(n+2)(n-1)}} & 0 & \sqrt{\frac{(n+1)(n-2)}{2(n+2)(n-1)}} & 0 & \sqrt{\frac{n}{(n-1)(n+2)}} & 0 & -\sqrt{\frac{n}{(n-1)(n+2)}} & 0 \\
0 \\
\end{bmatrix}
\]

\*
It can be taken as [11] or [1-1] when \( n = 5 \).

\*
It can be taken as [22] or [2-2] when \( n = 5 \).