Opinion Dynamics under Social Pressure

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Abstract—We introduce a new opinion dynamics model where a group of agents holds two kinds of opinions: inherent and declared. Each agent’s inherent opinion is fixed and unobservable by the other agents. At each time step, agents broadcast their declared opinions on a social network, which are governed by the agents’ inherent opinions and social pressure. In particular, we assume that agents may declare opinions that are not aligned with their inherent opinions to conform with their neighbors. This raises the natural question: Can we estimate the agents’ inherent opinions from observations of declared opinions? For example, agents’ inherent opinions may represent their true political alliances (Democrat or Republican), while their declared opinions may model the political inclinations of tweets on social media. In this context, we may seek to predict the election results by observing voters’ tweets, which do not necessarily reflect their political support due to social pressure. We analyze this question in the special case where the underlying social network is a complete graph. We prove that, as long as the population does not include large majorities, estimation of aggregate and individual inherent opinions is possible. On the other hand, large majorities force minorities to lie over time, which makes asymptotic estimation impossible.

Index Terms—Opinion dynamics, social networks, Polya urn process, stochastic approximations.

I. INTRODUCTION

Since its inception in the seminal works of French Jr. [1] and DeGroot [2], the mathematical theory of opinion dynamics has evolved into a very pertinent area of research today. This theory tries to understand and analyze the formation and evolution of opinions in groups of individuals pertaining to some topic of common interest, e.g., political support, climate change, etc. Indeed, opinion dynamics models shed light on several important phenomena, such as, consensus (convergence to a common opinion), rate of convergence, and social learning (i.e., whether the consensus opinion matches the outcome of a centralized learning problem). The vast majority of models that have been studied in the literature share two features. Firstly, the agents are reliable in that they share their opinions truthfully. Secondly, the agents update their opinions over time according to a predefined rule.

However, there is increasing evidence that people do not always share what they truly believe. For example, a recent poll showed that 62% of Americans do not share their true political views out of fear of offending others [3]. Moreover, over the last few decades, psychologists have designed experiments to demonstrate that individuals often conform with the opinions of their peers due to a desire to be accepted by the group, even if the group’s opinion does not align with their true beliefs [4]–[8]. This suggests that individuals have two kinds of opinions: inherent and declared. The former is the individual’s true belief, which is unobservable by other individuals. The latter is what individuals declare in public. More importantly, individuals may lie and declare opinions that are not aligned with their inherent opinion to conform with others. This raises the natural question: Can we estimate individuals’ inherent opinions from observations of their declared opinions?

Answering this question is important in many contexts. For instance, it has been shown that conformity can be catastrophic for legal and political institutions as it deprives society of important information [9]–[11]. In particular, Sunstein argued that a Democratic judge sitting with two Republican judges is more likely to reverse an environmental decision at the behest of an industry challenger [11]. Thus, estimating the amount of compliance under social pressure helps in assessing how freely deliberations are held, which is crucial for the functioning of deliberative democracies and trial juries.

In this paper, we introduce a new opinion dynamics model that involves a group of connected agents with both inherent and declared binary opinions. For example, an agent’s inherent opinion might represent their true political stance (Republican or Democrat), and their declared opinions might convey the political inclinations of their tweets or posts on social media. We assume that an agent’s inherent opinion is fixed and unobservable by the other agents. On the other hand, an agent’s declared opinions are updated over time and broadcast over a social network. Furthermore, each agent’s truth probability, i.e., the probability with which the agent declares their inherent opinion, follows the well-known Bradley-Terry-Luce (BTL) or multinomial logit model [12]–[15]; so, it is proportional to the number of times the agent observes broadcast opinions that agree with their inherent opinion. In other words, an agent gains more confidence to tell the truth when they observe more declared opinions aligned with their inherent one. The goal of this work is to construct an estimator that infers the agents’ inherent opinions based on observations of their (unreliable) declared ones.

A. Contributions

This paper makes the following main contributions:

1) We introduce a new self-reinforcing Polya urn-like model of opinion dynamics where agents hold inherent and
declared binary opinions. The former are unobservable, while the latter are influenced by social pressure and broadcast on a social network.

2) We analyze the setting where the social network is a complete graph. We propose a simple estimator that determines whether there is a large majority of inherent opinions in the population, and when there is no large majority, it asymptotically learns the proportion of agents that have a certain inherent opinion by observing declared opinions over time. The existence of a large majority, however, forces the minority to lie, which makes asymptotic estimation impossible. In this case, our estimator produces a lower bound on the majority’s size.

3) When the population does not have a large majority, we also propose an estimator that learns agents’ individual inherent opinions by observing their declared ones.

B. Related Literature

Since the seminal works of French Jr. [1] and DeGroot [2], a myriad of opinion dynamics models have been introduced in the literature. We do not attempt to survey these models here, and instead only discuss the most important and relevant ones. Perhaps the most renowned (non-Bayesian) opinion dynamics model is the DeGroot learning model [2]. In this model, agents update their beliefs about a topic of interest using a weighted average of their neighbors’ beliefs. By appealing to the theory of Markov chains, it can be shown that consensus can be reached in such a model. Friedkin and Johnsen extended the DeGroot model to address the fact that people often disagree in the real world [16]. They argued that this disagreement was a result of individuals trying to adhere to their initial opinions as much as possible. Specifically, they showed that if agents updated their opinions as a convex combination of their initial opinions and a weighted average of their neighbors’ opinions, disagreement could occur. Peng et al. extended DeGroot and Friedkin and Johnsen’s models to analyze self-appraisal and social power in opinion dynamics models [17] (also see [18]). Recently, Gaitonde et al. have built on Friedkin and Johnsen’s model to introduce a game where an attacker tries to increase opinion discord in a population and a defender intervenes against such attacks [19].

One of the first stochastic opinion dynamics models, known as the voter model, was introduced by Holley and Liggett [20]. This model considers a group of connected agents with binary opinions. At different instants of time, an agent is picked at random and decides whether to flip its opinion or not. The transition rate of the flipping probability is a function of the agent’s and its neighbors’ current opinions. Different instances of voter models in the literature correspond to different transition rate functions. For example, linear voter models assume that the transition rate is a convex combination of the neighbors’ opinions [21, Chapter 5]. The theory of voter models tries to understand the relationship between transition rate functions and possible asymptotic behaviors, e.g., consensus, clustering, coexistence, etc. Other interacting particle systems have also been utilized as stochastic models of opinion dynamics. For instance, Montanari and Saberi proposed a model where the probability of adopting a certain opinion is a softmax function of the number of neighbors that currently hold this opinion, i.e., the adoption probability obeys Glauber dynamics for the Ising model [22].

In a different vein, Hayhoe et al. proposed a model for epidemics or opinion dynamics in [23] that serves as an alternative to the famous susceptible-infected-susceptible infection model [24, Chapter 21]. Specifically, they considered a contagion process on a network where every node has a private Polya urn, but to update this urn at every time step, the node samples from a “super urn” which pools all the neighboring urns together. As a result, this model can account for spatial infections between neighbors. The authors then analyzed the asymptotic behavior of the proposed contagion process using martingale methods.

The aforementioned models assume that agents interact on a fixed network, and several models have been proposed to remedy this. For example, Hanaki et al. suggested a social network model where agents can locally alter the structure of the underlying network [25], and Fazeli and Jadabiae extended the network formation model introduced in [26] to develop an opinion dynamics model with a random network [27]. In the latter model, connections between agents are formed endogenously according to the rule “people who have been visited are more likely to be visited in the future.”

In all the opinion dynamics models discussed above, agents share their opinions truthfully. In contrast, agents may lie about their inherent opinions to conform with their neighbors in our model. Thus, the questions that we ask and address in this work are different from those addressed in the literature. Whereas the classical models analyze the asymptotic behavior of opinions, we try to understand whether estimation of inherent opinions is possible in the presence of unreliable agents who are under social pressure.

C. Notational Preliminaries

We briefly introduce some useful notation for the sequel. We define $\mathbb{Z}_+$ and $\mathbb{R}_+$ to be the sets of non-negative integers and non-negative real numbers, respectively. We let $\mathbb{1}\{\cdot\}$ denote the indicator function, which equals 1 if its input proposition is true, and 0 otherwise. We denote by $\text{bin}(n, p)$ the binomial distribution with parameters $n \in \mathbb{Z}_+,\setminus\{0\}$ and $p \in [0, 1]$, and by $\text{Ber}(p)$ the Bernoulli distribution with parameter $p \in [0, 1]$. Lastly, we let $\mathbb{P}(\cdot)$ and $\mathbb{E}[\cdot]$ be the probability and expectation operators, where the underlying measure is clear from context.

D. Outline

The rest of the paper is organized as follows. We introduce our new opinion dynamics model and our objectives in Section II. In Section III, we analyze the case where the agents interact on a complete graph and have the same inherent opinions. Section IV studies the general complete graph setting. Section V presents our conclusions and future work. Lastly, the proofs of our results are provided in the appendices.
II. FORMAL MODEL AND GOALS

We consider a set of $N$ agents denoted by $V = \{1, \ldots, N\}$. Each agent $v \in V$ has an inherent opinion $\phi_v \in \{0, 1\}$, e.g., their support for the Democratic or Republican party. At each time step $n \in \mathbb{Z}_+$, agent $v$ broadcasts a declared opinion $\psi_{v,n} \in \{0, 1\}$ as follows:

\[
\psi_{v,n} \equiv \begin{cases} 
\phi_v, & \text{with probability } p_{v,n} \\
1 - \phi_v, & \text{with probability } 1 - p_{v,n}.
\end{cases}
\] (1)

According to (1), agent $v$ declares its inherent opinion with probability $p_{v,n} \in (0, 1)$ and lies with probability $1 - p_{v,n}$. The declared opinions $\{\psi_{v,n} : v \in V, n \in \mathbb{Z}_+\}$ are broadcast on a connected undirected graph $G = (V, E)$ (with self-loops, but no multiple edges), where the edge set $E \subseteq V \times V$ encodes the relationship between different vertices or agents. In particular, $(i, j) \in E$ means that agent $i$ can observe the declared opinion of agent $j$. For agent $v \in V$ at time $n \in \mathbb{Z}_+$, the truth probability (probability of broadcasting the inherent opinion) is given by the BTL model [12]–[15]:

\[
p_{v,n} \equiv \frac{a_{v,n}}{a_{v,n} + b_{v,n}},
\] (2)

where the random parameters $a_{v,n}, b_{v,n} \in \mathbb{R}_+$ are updated as follows:

1) At time $n = 0$, $a_{v,0} = b_{v,0} = 1$ for all $v \in V$.
2) At time $n \geq 1$, every agent $v \in V$ observes its neighbors’ declared opinions $\{\psi_{u,n-1} : u \in N_v\}$, where $N_v = \{u \in V : (u, v) \in E\}$ denotes its set of neighbors. Then, it updates its truth probability parameters in a self-reinforcing Pólya urn-like manner [28, Section 2.4]:

\[
a_{v,n} = a_{v,n-1} + \gamma \sum_{u \in N_v} \mathbb{I}\{\psi_{u,n-1} = \phi_v\},
\] (3)

\[
b_{v,n} = b_{v,n-1} + \sum_{u \in N_v} \mathbb{I}\{\psi_{u,n-1} \neq \phi_v\},
\] (4)

where $\gamma > 0$ is an honesty parameter.

Note that $a_{v,0}$ and $b_{v,0}$ can take any strictly positive values without changing our analysis. We set them to be 1 to simplify the presentation of our results.

According to (2), (3), and (4), at every time step, agent $v$ must either declare $\phi_v$ or lie about it. It makes this decision by conforming to its neighbors’ declared opinions. In fact, unwinding (3) and (4) and noting that the dominant terms in $a_{v,n}$ and $b_{v,n}$ are $\gamma \sum_{t=0}^n \sum_{u \in N_v} \mathbb{I}\{\psi_{u,t-1} = \phi_v\}$ and $\sum_{t=0}^n \sum_{u \in N_v} \mathbb{I}\{\psi_{u,t-1} \neq \phi_v\}$ (see Proposition 2), we obtain that the probability of telling the truth at time $n$

\[
\text{probability of telling the truth at time } n \approx \frac{\# \{\text{declared opinions that confirm } v's\}}{\gamma \times \# \{v's\text{ inherent opinion until time } n\}}.
\]

Hence, the more the agent observes a certain opinion, the more it is forced to declare this opinion. To understand the meaning of the honesty parameter $\gamma$, let us consider a scenario where the social pressure on an agent is neutral, i.e., it observes an equal number of 0 and 1 declared opinions. In this case, the ratio of the probability of telling the truth to the probability of lying is $\gamma$. This means that the higher the value of $\gamma$, the more honest the agent. In particular, if $\gamma > 1$ (respectively, $\gamma < 1$), the agent leans towards telling the truth (respectively, lying). We will assume throughout this paper that $\gamma > 1$. We remark that such honesty (or conformity) parameters have been considered in the literature in the setting of Friedkin and Johnsen’s model, cf. [29].

The aforementioned description completely determines a self-reinforcing stochastic process on $G$ along with its initial conditions. This raises the natural question as to whether this process is ergodic, or more generally, whether any information about $\{\phi_v : v \in V\}$ can be inferred from observations of the declared opinions $\mathcal{H}_n \equiv \{\psi_{v,k} : v \in V, 0 \leq k < n\}$ after an arbitrarily long period of time, i.e., as $n \to \infty$. In other words, can one infer the agents’ inherent opinions despite the fact that the agents may lie to conform with their neighbors? Under possible regularity conditions, our objectives are to:

1) Estimate the proportion of agents that have inherent opinions equal to 1, viz.

\[
\Phi \equiv \frac{1}{N} \sum_{u \in V} \phi_u
\] (5)

based on $\mathcal{H}_n$, as $n \to \infty$. Formally, we seek to construct strongly consistent estimators $f_n : \{0, 1\}^{N \times n} \to \{0, 1\}$ such that

\[
\mathbb{P}\left(\lim_{n \to \infty} f_n(\mathcal{H}_n) = \Phi \right) = 1,
\] (6)

where underlying probability law is defined by the random variables $\{\psi_{v,k} : v \in V, k \in \mathbb{Z}_+\}$. In the earlier example where agents’ inherent opinions are their true political alliances and their declared opinions are political inclinations of tweets or posts on social media, the problem of learning $\Phi$ corresponds to predicting election results by observing voters’ declared opinions.

2) Decide all inherent opinions $\{\phi_u : u \in V\}$ from $\mathcal{H}_n$ as $n \to \infty$. Formally, we also seek to construct strongly consistent estimators $h_n : \{0, 1\}^{N \times n} \to \{0, 1\}^N$ such that

\[
\mathbb{P}\left(\lim_{n \to \infty} h_n(\mathcal{H}_n) = \{\phi_v : v \in V\} \right) = 1.
\] (7)

III. ANALYSIS OF COMPLETE GRAPH AND A HOMOGENEOUS POPULATION

In this section, we analyze the special case where the graph $G$ is complete with self-loops, and the agents are homogeneous in the sense that they have equal inherent opinions. Moreover, we assume that the estimator is aware of the population’s homogeneity. So, the estimator’s goal is to decide whether all agents have inherent opinions equal to 0 or 1. Although this is a contrived setting, it affords us several insights about the choice of estimator and the technical tools used to prove consistency that will be very useful in the more general setting.

Without loss of generality, we assume that all the agents have inherent opinion 0, i.e., $\phi_v = 0$ for all $v \in V$. In this case, the dynamics of $a_{v,n}$ and $b_{v,n}$ are independent of the
agent $v$. Hence, dropping the subscript $v$ from $a_{v,n}, b_{v,n}$, and $p_{v,n}$, we may write

$$a_0 = b_0 = 1,$$  

(8)

so that $p_0 = \frac{1}{2}$, and for all $n \geq 1$, $p_n = a_n / (a_n + b_n)$ with

$$a_n = a_{n-1} + \gamma N (1 - \Psi_{n-1}),$$

$$b_n = b_{n-1} + N \Psi_{n-1},$$

(9, 10)

where we define the random variables:

$$\forall n \in \mathbb{Z}_+, \; \Psi_n \triangleq \frac{1}{N} \sum_{u \in V} \psi_{u,n} \in \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\right\}.$$  

(11)

$\Psi_n$ is the fraction of agents that hold a declared opinion of 1 at time $n$.

Consider the sequence of estimators:

$$\forall n \geq 1, \; f_n(\mathcal{H}_n) = \Psi_n \triangleq \frac{1}{n} \sum_{k=0}^{n-1} \Psi_k.$$  

(12)

To prove (6), it suffices to establish that $\Psi_n \to 0$ as $n \to \infty$ almost surely (a.s.). Hence, we now analyze the almost sure convergence of the sequence of random variables $\{\Psi_n : n \geq 1\}$. This is done in three steps. First, we show in Proposition 1 that the truth probability $p_n$ converges a.s. to some random variable $p^\ast$. Second, in Theorem 1, we establish that the limit $p^\ast$ belongs to $\{0, 1\}$, i.e., $p^\ast$ is Bernoulli. Moreover, we prove that the average opinion $\hat{\Psi}_n$ converges a.s. to $1 - p^\ast$. Finally, Theorem 2 proves that $p^\ast$ is indeed equal to 1 a.s., which means that the agents tell the truth in the limit.

For any $n \in \mathbb{Z}_+$, conditioned on $a_n$ and $b_n$, $N \Psi_n \sim \text{bin}(N, 1 - p_n)$ obeys a binomial distribution:

$$\mathbb{P}\left(\frac{\Psi_n}{N} = \frac{j}{N} \mid a_n, b_n\right) = \binom{N}{j} \left(\frac{b_n}{a_n + b_n}\right)^{j} \left(\frac{a_n}{a_n + b_n}\right)^{N-j},$$

(13)

for all $j \in \{0, \ldots, N\}$, because $\psi_{n,u} \sim \text{Ber}(1 - p_n)$ are independently distributed random variables. Together, the initial conditions in (8) and the dynamics defined in (9, 10, 11) characterize the trajectory of $\{(a_n, b_n, \Psi_n) : n \in \mathbb{Z}_+\}$. Observe that the truth parameter $p_n$ is invariant to scaling of $(a_n, b_n)$. So, consider the normalized probability parameters $\alpha_n \triangleq a_n / N$ and $\beta_n \triangleq b_n / N$ for all $n \in \mathbb{Z}_+$, so that $p_n = \alpha_n / (\alpha_n + \beta_n)$. These normalized parameters satisfy the stochastic dynamics:

$$\alpha_n = \alpha_{n-1} + \gamma (1 - \Psi_{n-1}),$$

$$\beta_n = \beta_{n-1} + \Psi_{n-1},$$

(14, 15)

for all $j \in \{0, \ldots, N\}$, with initial conditions $\alpha_0 = \beta_0 = \frac{1}{N}$.

In the remainder of this section, we show that the sequence of estimators $f_n$ converges as all agents have inherent opinion 0 over time. Let $\mathcal{F}_n \triangleq \sigma(\mathcal{H}_n)$ be the smallest $\sigma$-algebra generated by $\mathcal{H}_n$, and $\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{Z}_+\}$ be the corresponding filtration. The ensuing proposition constructs martingales and shows that $p_n$ converges a.s. as $n \to \infty$.

**Proposition 1 (Martingale Property):** The following hold:

1) $\{p_n : n \in \mathbb{Z}_+\}$ is an $\mathcal{F}$-sub-martingale, or equivalently, $\{1 - p_n : n \in \mathbb{Z}_+\}$ is an $\mathcal{F}$-super-martingale.

2) There exists a random variable $p^\ast \in [0, 1]$ such that

$$p^\ast = \lim_{n \to \infty} p_n \text{ a.s.}$$

Proposition 1 is established in Appendix I. While one might intuitively expect something like Proposition 1 to hold due to the close resemblance of $\{p_n : n \in \mathbb{Z}_+\}$ with a Pólya urn process (which is obtained by setting $\gamma = 1$ and $N = 1$), we note that unlike the Pólya urn process, $\{\Psi_n : n \in \mathbb{Z}_+\}$ is not an exchangeable sequence (cf. [30, Sections VII.4 and VII.9], [31, Sections 2.2 and 2.4]). Hence, one cannot apply de Finetti’s theorem to easily deduce the probability law of the limiting random variable. Furthermore, we remark that since $p_0 = \frac{1}{2}$ a.s. and the sub-martingale property in Proposition 1 implies that $\{\mathbb{E}[p_n] : n \in \mathbb{Z}_+\}$ forms a non-decreasing sequence, we obtain that

$$\lim_{n \to \infty} \mathbb{E}[p_n] = \mathbb{E}[p^\ast] \in \left[\frac{1}{2}, 1\right]$$

(17)

using Lebesgue’s dominated convergence theorem [32, Chapter 1, Theorem 4.16]. We next demonstrate using Proposition 1 that the limiting random variable $p^\ast$ in part 2 of Proposition 1 is actually a Bernoulli random variable, and it characterizes the limits of both $\hat{\Psi}_n$ and $\Psi_n$ in appropriate senses.

**Theorem 1 (Convergence to Bernoulli Variable):** The following are true:

1) $p^\ast \in \{0, 1\}$ a.s.

2) $\lim_{n \to \infty} \hat{\Psi}_n = 1 - p^\ast$ a.s.

3) $\Psi_n \converges{\text{in distribution}} 1 - p^\ast$ as $n \to \infty$, i.e., $\Psi_n$ converges weakly (or in distribution) to $1 - p^\ast$.

Theorem 1 is proved in Appendix II. Upon inspecting part 3, it is natural to wonder whether the simpler estimator $\hat{\Psi}_n$ learns the true inherent opinions. In the ensuing proposition, we show that the answer is negative; in particular, $\Psi_n$ does not converge almost surely to $1 - p^\ast$ although part 3 of Theorem 1 establishes the weak convergence.

**Proposition 2 (Non-existence of Almost Sure Limit):**

$\{\Psi_n : n \in \mathbb{Z}_+\}$ does not converge to $1 - p^\ast$ a.s. Specifically, we have

$$\mathbb{P}\left(\lim_{n \to \infty} \Psi_n \in \{0, 1\}\right) = 0,$$

i.e., the probability that $\lim_{n \to \infty} \Psi_n$ exists and belongs to $\{0, 1\}$ is zero.

The proof of Proposition 2 can be found in Appendix III. Finally, in the next theorem, we establish that $p^\ast = 1$ a.s.

**Theorem 2 (Asymptotic Learning I):** The following hold:

1) $p^\ast = 1$ a.s.

2) $\lim_{n \to \infty} \hat{\Psi}_n = 0$ a.s.

Theorem 2 is derived using stochastic approximations in Appendix IV. It conveys that if our estimator $f_n$ is aware that the population is homogeneous, then it recovers $\Phi$ in the sense of (6). In particular, when $\hat{\Psi}_n$ converges to 0 (respectively 1), all agents have inherent opinions equal to 0 (respectively 1).
IV. Analysis of Complete Graph with $\Phi \in [0,1]$

In this section, we analyze the complete graph (with self-loops) case for general $\Phi$. In this setting, the dynamics of $a_{v,n}$ and $b_{v,n}$ depend only on the inherent opinion $\phi_v$ of agent $v$. In particular, agents $v$ with $\phi_v = j \in \{0,1\}$ have common probability parameters $a^0_j$ and $b^0_j$ that evolve as follows:

$$a^0_j = a^0_{n-1} + \gamma N(1 - \Psi_{n-1}),$$
$$a^1_j = a^1_{n-1} + \gamma N \Psi_{n-1},$$
$$b^0_j = b^0_{n-1} + N \Psi_{n-1},$$
$$b^1_j = b^1_{n-1} + N (1 - \Psi_{n-1}),$$

for all $n \geq 1$, with $a^0_0 = b^0_0 = b^1_0 = 1$. Moreover, agents with inherent opinion $j \in \{0,1\}$ tell the truth at time $n \in \mathbb{Z}_+$ with probability

$$p^0_j = \frac{a^0_j}{a^0_j + b^1_j}. \quad (22)$$

We define the normalized probability parameters $\alpha^0_j \triangleq a^0_j/N$ and $\beta^0_j \triangleq b^0_j/N$ for all $n \in \mathbb{Z}_+$ and $j \in \{0,1\}$. Thus, $p^0_n = \alpha^0_j / (\alpha^0_j + \beta^0_j)$ as before. These normalized parameters satisfy the following dynamics:

$$\alpha^0_0 = \alpha^0_{n-1} + \gamma (1 - \Psi_{n-1}),$$
$$\alpha^1_0 = \alpha^1_{n-1} + \gamma N \Psi_{n-1},$$
$$\beta^0_0 = \beta^0_{n-1} + \Psi_{n-1},$$
$$\beta^1_0 = \beta^1_{n-1} + (1 - \Psi_{n-1}).$$

We denote by $V^j$ the set of agents that have inherent opinion $j \in \{0,1\}$, and define $\Psi^n_j \triangleq \sum_{v \in V^j} \phi_v / |V^j|$ for all $n \in \mathbb{Z}_+$ and $j \in \{0,1\}$. Thus, we may write:

$$\Psi_n = (1 - \Phi) \Psi^0_n + \Phi \Psi^1_n,$$  
(27)

$$P\left(\Psi_n = \frac{i}{|V^0|} \mid \alpha^0_0, \beta^0_0\right) = \left(\frac{|V^0|}{i}\right) \left(1 - p^0_n\right)^i \left(p^0_n\right)^{|V^0| - i},$$  
(28)

$$P\left(\Psi_n = \frac{j}{|V^1|} \mid \alpha^1_0, \beta^1_0\right) = \left(\frac{|V^1|}{j}\right) \left(1 - p^1_n\right)^j \left(p^1_n\right)^{|V^1| - j},$$  
(29)

$$\mathbb{E}[\Psi_n \mid \alpha^0_n, \alpha^1_n, \beta^0_n, \beta^1_n] = (1 - \Phi)(1 - p^0_n) + \Phi p^1_n, \quad (30)$$

for all $i \in \{0, \ldots, |V^0|\}$ and $j \in \{0, \ldots, |V^1|\}$, where $\Phi = |V^1|/|V^0|$ is the fraction of agents that have inherent opinion 1.

We can construe (18)-(21) as the equations describing the dynamics of two interacting Pólya urns $u_0$ and $u_1$. At time $n$, $u_0$ contains $a^0_n$ white balls and $b^0_n$ red balls, while $u_1$ contains $a^1_n$ white balls and $b^1_n$ red balls. At the next time step $n+1$, $|V^0|$ balls are chosen randomly (with replacement) from $u_0$ for $j \in \{0,1\}$, and we let $W_n = N (1 - \Psi_n)$ and $R_n = N \Psi_n$ be the total number of white and red balls chosen, respectively. Subsequently, $\gamma W_n$ white balls are added to $u_0$, $\gamma R_n$ white balls are added to $u_1$, $R_n$ red balls are added to $u_0$, and $W_n$ red balls are added to $u_1$. We asymptotic behavior of two such interacting urns using stochastic approximations is challenging. Indeed, the probabilities $p^0_n$ and $p^1_n$ of choosing a white ball satisfy the following equations for all $n \in \mathbb{Z}_+$:

$$p^1_{n+1} - p^0_n = \frac{1}{\alpha^0_n + \beta^0_n + \gamma (1 - \Psi_n) + \Psi_n} (F^0(p^0_n, p^1_n) + c^n)$$  
(31)

$$F^0(x, y) = -(\gamma - 1)(1 - \Phi)x^2 + (\gamma - 1)\Phi xy + \beta^0_n (\gamma (1 - \Psi_n) - p^0_n (\gamma (1 - \Psi_n) + \Psi_n)) - F^0(p^0_n, p^1_n),$$

$$F^1(x, y) = -(\gamma - 1) \Phi y^2 + (\gamma - 1)(1 - \Phi)xy + \alpha^1_n (\gamma \Psi_n - p^1_n (\gamma \Psi_n + 1 - \Psi_n)) - F^1(p^0_n, p^1_n),$$

for $x, y \in [0,1]$. The step sizes $(\alpha^0_n + \beta^0_n + \gamma (1 - \Psi_n) + \Psi_n)^{-1}$ and $(\alpha^1_n + \beta^1_n + \gamma \Psi_n + 1 - \Psi_n)^{-1}$ are random and different, which makes the stochastic approximation of (31)-(32) by a differential equation rather difficult.

So, instead of considering (18)-(21) as the dynamics of two urns with two colors, we perceive them as describing one urn with four colors (white, red, blue, and yellow). Following this interpretation, the model becomes tractable. Indeed, the number of white, red, blue, and yellow balls at time $n$ are $a^0_n$, $b^0_n$, $a^1_n$, and $b^1_n$, respectively. Define the probabilities of drawing a white, red, or blue ball as, respectively:

$$X_n \triangleq \frac{\alpha^0_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n},$$

$$Y_n \triangleq \frac{\beta^0_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n},$$

$$Z_n \triangleq \frac{\alpha^1_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n}. \quad (33) \quad (34) \quad (35)$$

The process $(X_n, Y_n, Z_n)$ satisfies the following equations:

$$X_{n+1} - X_n = \frac{\alpha^0_0 + \gamma (1 - \Psi_n)}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n + \gamma + 1} - \frac{\alpha^0_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n},$$

$$Y_{n+1} - Y_n = \frac{\beta^0_0}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n + \gamma + 1} - \frac{\beta^0_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n},$$

$$Z_{n+1} - Z_n = \frac{\alpha^1_0 + \gamma \Psi_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n + \gamma + 1} - \frac{\alpha^1_n}{\alpha^0_n + \beta^0_n + \alpha^1_n + \beta^1_n},$$

for all $n \in \mathbb{Z}_+$, where

$$\gamma_{n+1} = \frac{1}{(4/N) + n(\gamma + 1)}, \quad (36) \quad (37) \quad (38)$$

$$G^1(x, y, z) = -(\gamma + 1)x +$$
\[
\gamma \left(1 - \Phi \right) \frac{1 - x}{x + y} + \Phi \left(1 - \frac{z}{1 - x - y}\right),
\]
\[U_{n+1}^1 = \gamma (1 - \Psi_n) - (\gamma + 1) X_n - G^1(X_n, Y_n, Z_n),
\]
\[G^2(x, y, z) = - (\gamma + 1) y + x + \left(1 - \Phi \right) \frac{1 - x}{x + y} + \Phi \frac{z}{1 - x - y},
\]
\[U_{n+1}^2 = \Psi_n - (\gamma + 1) Y_n - G^2(X_n, Y_n, Z_n),
\]
\[G^3(x, y, z) = - (\gamma + 1) z + x + \left(1 - \Phi \right) \frac{1 - x}{x + y} + \Phi \frac{z}{1 - x - y},
\]
\[U_{n+1}^3 = \Psi_n - (\gamma + 1) Z_n - G^3(X_n, Y_n, Z_n).
\]
Equations (36)-(38) have a common step size \(\gamma_{n+1}\), which allows us to approximate (36)-(38) using the following ordinary differential equation (ODE):
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = G(x, y, z) \triangleq \begin{pmatrix} G^1(x, y, z) \\ G^2(x, y, z) \\ G^3(x, y, z) \end{pmatrix}.
\]
It should be noted that \(p_0^n = X_n/(X_n + Y_n)\) and \(p_1^n = Z_n/(1 - X_n - Y_n)\). This means that analyzing the stochastic process \((X_n, Y_n, Z_n)\) allows us to deduce the asymptotic behavior of the process of interest \((p_0^n, p_1^n)\).

At first glance, the ODE (46) seems to have singularities at \(x + y = 0\) and \(x + y = 1\). However, since (46) is introduced to approximate the process \((X_n, Y_n, Z_n)\), it is sufficient to show that \(G\) is smooth on the range of \((X_n, Y_n, Z_n)\), and that any solution of (46) that starts in this range remains in the range, i.e., the range of \((X_n, Y_n, Z_n)\) is an invariant set of the ODE. These facts are derived in the following lemma.

**Definition 1 (Invariant Set):** A set \(M \subseteq \mathbb{R}^n\) is an invariant set of the ODE \(dx/dt = f(x)\) with vector field \(f: \mathbb{R}^n \to \mathbb{R}^n\) if for all initial conditions \(x(0) \in M\) and all \(t \in \mathbb{R}_+\), the solution \(x(t)\) of the ODE lives in \(M\), i.e., \(x(t) \in M\).

**Lemma 1 (Well-defined ODE):** The following are true:
1) The stochastic processes \(X_n, Y_n, Z_n\), and \(X_n + Y_n\) belong to \([0, \frac{\gamma}{\gamma + 1}], [0, \max\left(\frac{1}{\gamma + 1}, \frac{1}{\gamma + 1}\right)], [0, \frac{1}{\gamma + 1}]\), and \([\frac{1}{\gamma + 1}, \frac{\gamma}{\gamma + 1}]\) respectively.
2) The compact manifold
\[M \triangleq \left\{ (x, y, z) \in \mathbb{R}^3 : x \in \left[0, \frac{\gamma}{\gamma + 1}\right], z \in \left[0, \frac{\gamma}{\gamma + 1}\right], \right.\]
\[y \in \left[0, \max\left(\frac{1}{\gamma + 1}, \frac{1}{\gamma + 1}\right)\right], x + y \in \left[\frac{1}{\gamma + 1}, \frac{1}{\gamma + 1}\right]\}

is an invariant set of the ODE (46).
3) The vector field \(G\) is complete on \(M\), i.e., the ODE (46) with initial condition \(M\) admits a unique solution on \(\mathbb{R}_+\).

Lemma 1 is proved in Appendix V. In the remainder of this section, we show that the average declared opinion (12) can be used to construct an estimator that learns \(\Phi\) provided that the population does not include large majorities, i.e., \(\Phi\) is bounded away from 0 and 1. This main result will be given in Theorem 4. Furthermore, Corollary 1 will demonstrate that an estimate of \(\Phi\) can be used to estimate the individual inherent opinions of the agents. To this end, in the next lemma, we start by showing that the process \((X_n, Y_n, Z_n)\) converges to the equilibrium points of the ODE (46).

**Lemma 2 (Convergence to Equilibria):** The ensuing statements are true:
1) The set of equilibria \(L\) of (46) is
\[L = \begin{cases} \{l_1, l_2, l_3\}, & \text{if } \gamma \geq \max \left(\frac{\Phi}{1 - \Phi}, \frac{1 - \Phi}{\Phi}\right) \\
\{l_1, l_2\}, & \text{otherwise} \end{cases}
\]
where \(l_1 = \left(0, \frac{1}{\gamma + 1}, \frac{\gamma}{\gamma + 1}\right), l_2 = \left(\frac{\gamma}{\gamma + 1}, 0, 0\right)\), and \(l_3 = \left(\frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma + 1}\right)\).
2) The process \((p_0^n, p_1^n)\) converges a.s. to \((p_0^\infty, p_1^\infty) \in L_p\) as \(n \to \infty\), where
\[L_p = \begin{cases} \{(0, 1), (1, 0)\} & \text{if } \gamma \geq \max \left(\frac{\Phi}{1 - \Phi}, \frac{1 - \Phi}{\Phi}\right) \\
\{(0, 1), (1, 0)\}, & \text{otherwise} \end{cases}
\]
and \(l'_3 = \left(\frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma + 1}\right)\).

Lemma 2 is established in Appendix VI. It conveys that the process \((p_0^n, p_1^n)\) converges to a random variable supported on a finite set. The next theorem shows that this process in fact converges to a deterministic vector, which will be used to estimate \(\Phi\).

**Theorem 3 (Convergence to Deterministic Vector):** The process \((p_0^n, p_1^n)\) converges a.s. to \((p_0^\infty, p_1^\infty)\), where
\[\left(p_0^\infty, p_1^\infty\right) = \begin{cases} l'_3, & \text{if } \gamma \geq \max \left(\frac{\Phi}{1 - \Phi}, \frac{1 - \Phi}{\Phi}\right) \\
(1, 0), & \text{if } \frac{\Phi}{1 - \Phi} < \gamma < \frac{1 - \Phi}{\Phi} \\
(0, 1), & \text{if } \frac{1 - \Phi}{\Phi} < \gamma < \frac{\Phi}{1 - \Phi} \end{cases}
\]
and \(l'_3\) is defined in Lemma 2.

Theorem 3 is derived in Appendix VII. So far, we have illustrated that the process \((p_0^n, p_1^n)\) converges to a deterministic vector \((p_0^\infty, p_1^\infty)\). As mentioned earlier, our estimator will use the quantity (12) to learn \(\Phi\). So, the ensuing lemma relates the asymptotic behavior of (12) to \((p_0^\infty, p_1^\infty)\). It will also be used to infer the individual agents’ inherent opinions \(\{\varphi_v : v \in V\}\).

**Lemma 3 (Individual Opinions I):** The following are true:
1) For any agent \(v \in V\), its declared opinions satisfy
\[\psi_v, \infty \triangleq \lim_{n \to \infty} \frac{1}{n} \mathop{\sum_{k=0}^{n-1}} \psi_{v, k} = \begin{cases} p_0^\infty, & \varphi_v = 1; \\
1 - p_0^\infty, & \varphi_v = 0, \end{cases}
\]
where \((p_0^\infty, p_1^\infty)\) is given by (47).
2) The average declared opinion (12) converges a.s. to
\[\psi_{n, \infty} \triangleq \lim_{n \to \infty} \psi_n = (1 - \Phi)(1 - p_0^\infty) + \Phi p_1^\infty.
\]
The proof of Lemma 3 can be found in Appendix VIII. Now consider the sequence of estimators:
\[\forall n \geq 1, \; g_n(H_n) = \frac{\hat{\Psi}_n(\gamma - 1) + 1}{\gamma + 1},
\]
where \(\hat{\Psi}_n\) is defined in (12). We finally state our main result.

**Theorem 4 (Asymptotic Learning II):** The following hold:
1) \(g_n(H_n)\) converges a.s. to a deterministic quantity \(g_{\infty} \in \left[\frac{1}{\gamma + 1}, \frac{\gamma}{\gamma + 1}\right]\) as \(n \to \infty\).
2) If \( g_\infty = \frac{1}{\gamma + 1} \), then \( \Phi \leq \frac{1}{\gamma + 1} \).

3) If \( g_\infty = \frac{1}{\gamma + 1} \), then \( \Phi \geq \frac{1}{\gamma + 1} \).

4) If \( g_\infty \in (\frac{1}{\gamma + 1}, \frac{1}{\gamma + 1}) \), then \( \Phi = g_\infty \).

Theorem 4 is established in Appendix IX. It demonstrates that if the estimator (50) converges to a value strictly between \( (\gamma + 1)^{-1} \) and \( \gamma(\gamma + 1)^{-1} \), then this value is actually equal to \( \Phi \). In other words, our estimator is consistent provided that the population does not include large majorities. On the other hand, if the estimator converges to \( (\gamma + 1)^{-1} \) or \( \gamma(\gamma + 1)^{-1} \), then it cannot estimate the true value of \( \Phi \). Instead, it gives a lower bound on the size of the majority. In this case, the inconsistency of the estimator (50) is due to the presence of a large majority that forces the minority to lie with probability one in the limit. Indeed, \( g_\infty = (\gamma + 1)^{-1} \) (respectively \( g_\infty = \gamma(\gamma + 1)^{-1} \)) implies that the majority of agents have inherent opinions equal to 0 (respectively 1), which corresponds to \( (p_\infty^0, p_\infty^1) = (1, 0) \) (respectively \( (0, 1) \)). Hence, in the limit, the minority of agents with inherent opinion 1 (respectively 0) lie with probability one.

In Lemma 3, (48) partitions the agents into two mutually exclusive groups. The first includes agents in \( V^1 \) whose average declared opinions over time \( \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k} \) converge a.s. to \( p_\infty^1 \), and the second contains agents in \( V^0 \) whose average declared opinions converge to \( 1-p_\infty^0 \). (Recall that agents in \( V^j \) have inherent opinion \( j \in \{0, 1\} \).) If the population does not include a large majority, then we can estimate \( \Phi \) accurately. As a result, \( p_\infty^0 \) and \( p_\infty^1 \) can be computed explicitly. This means that the individual average declared opinions \( \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k} \) and the global average declared opinion \( \Psi_n \) can be used to estimate the individual inherent opinions \( \{\phi_v : v \in V\} \). This is presented formally in the following corollary, which follows immediately from Theorem 4 and Lemma 3.

**Corollary 1 (Individual Opinions II):** Suppose that \( \Psi_\infty \in (0, 1) \). Then, we have for every \( v \in V \):

1) If \( \psi_{v,\infty} = \frac{\gamma(\gamma - 1)}{\Psi_{\infty}(\gamma - 1)} \), then \( \phi_v = 1 \).

2) If \( \psi_{v,\infty} = 1 - \frac{\gamma(\gamma - 1)}{\Psi_{\infty}(\gamma - 1)} \), then \( \phi_v = 0 \), where \( \psi_{v,\infty} \) is given by (48).

**V. Conclusion and Future Work**

In this paper, we introduced a new opinion dynamics model where a population of unreliable agents hold both inherent and declared opinions. The inherent opinions are unknown, while the possibly untrue declared opinions are broadcast over a social network. In the special case of a complete graph, we propose an estimator to learn the agents’ inherent opinions by observing their declared ones, and prove that our estimator is consistent provided that the population does not include large majorities. Furthermore, when a large majority exists, our estimator gives a lower bound on the size of the majority.

There are several open future directions to pursue. Firstly, one could try to generalize our results to larger (non-binary) opinion alphabets or to other graph topologies, e.g., expander graphs. Secondly, one could also try to develop an estimator for the honesty parameter \( \gamma \), which is assumed to be known in our current model. Finally, since asymptotic estimation in the presence of large majorities appears to be impossible, one could explore learning from transient epochs of the stochastic process using concentration of measure ideas as \( N \to \infty \).

**APPENDIX I
PROOF OF PROPOSITION 1**

**Part 1:** To prove the first part, observe that we have

\[
\mathbb{E}[1 - p_n | F_{n-1}] = \mathbb{E} \left[ \frac{\beta_n}{\alpha_n + \beta_n} \bigg| \alpha_{n-1}, \beta_{n-1} \right] \\
= \mathbb{E} \left[ \frac{\beta_{n-1} + \Psi_{n-1}}{\alpha_{n-1} + \beta_{n-1} + \gamma(1 - \Psi_{n-1}) + \Psi_{n-1}} \bigg| \alpha_{n-1}, \beta_{n-1} \right] \\
\leq \mathbb{E} \left[ \frac{\beta_{n-1} + \Psi_{n-1}}{\alpha_{n-1} + \beta_{n-1} + 1} \bigg| \alpha_{n-1}, \beta_{n-1} \right] \\
= \frac{\beta_{n-1} + 1 - p_{n-1}}{\alpha_{n-1} + \beta_{n-1} + 1} \\
= \frac{(\alpha_{n-1} + \beta_{n-1})(1 - p_n) + 1 - p_n}{\alpha_{n-1} + \beta_{n-1} + 1} \\
= 1 - p_{n-1}
\]

where the first equality follows from the Markovian nature of the dynamics (14), (15), and (16), the second equality follows from (14) and (15), the third inequality holds because \( (\gamma - 1)(1 - \Psi_{n-1}) \geq 0 \) since \( \gamma > 1 \), and the fifth equality holds because \( N\Psi_{n-1} \sim \text{bin}(N, 1 - p_{n-1}) \) given \( \alpha_{n-1} \) and \( \beta_{n-1} \). This establishes part 1. (Note that (51) is met with equality if \( \gamma = 1 \), which turns \( \{p_n : n \in \mathbb{Z}_+\} \) into a martingale.)

**Part 2:** The second part follows from the martingale convergence theorem [32, Chapter 4, Theorem 4.1].

**APPENDIX II
PROOF OF THEOREM 1**

**Part 1:** We will prove that \( p^* = 1 \) a.s. using stochastic approximations in Theorem 2, but for now, we present a more elementary proof of part 1 that will be useful in proving parts 2 and 3. First, by unwinding the recursions in (14) and (15), we obtain

\[
\alpha_n = \frac{1}{N} + \gamma n \left(1 - \hat{\Psi}_{n}\right), \\
\beta_n = \frac{1}{N} + n \hat{\Psi}_{n},
\]

for all \( n \geq 1 \), which implies that

\[
1 - p_n = \frac{1}{N} + \hat{\Psi}_{n} \\
= \frac{1}{N} \left[ n \hat{\Psi}_{n} + \gamma n \left(1 - \Psi_{n}\right)\right] \\
= \frac{1}{N} \left[ \frac{1}{N} + \hat{\Psi}_{n}\right] + \gamma \left(1 - \frac{1}{N}\right)\Psi_{n}. \\
\]

Then, letting \( n \to \infty \), we get using part 2 of Proposition 1 that

\[
1 - p^* = \lim_{n \to \infty} \frac{\hat{\Psi}_{n}}{\gamma - (\gamma - 1)\Psi_{n}} \quad \text{a.s.}
\]

which, after applying the function \( g : [0, 1] \to [0, 1] \), \( g(x) = \frac{\gamma x}{1 + (\gamma - 1)x} \), yields

\[
\lim_{n \to \infty} \hat{\Psi}_{n} = g \left( \lim_{n \to \infty} \frac{\Psi_{n}}{\gamma - (\gamma - 1)\Psi_{n}} \right) \\
= g(1 - p^*) = \frac{\gamma(1 - p^*)}{1 + (\gamma - 1)(1 - p^*)} \quad \text{a.s.}
\]

(52)
where we use the continuity of $g$ (along with the Mann-Wald continuous mapping theorem). Hence, using Lebesgue’s dominated convergence theorem [32, Chapter 1, Theorem 4.16], we have

\[
\lim_{n \to \infty} \mathbb{E}\left[\hat{\Psi}_n\right] = \mathbb{E}[g(1 - p^*)] .
\]  

(53)

On the other hand, since \(N \Psi_n \sim \text{bin}(N, 1 - p_n)\) given \(p_n\) for all \(n \in \mathbb{Z}_+\), we have

\[
\lim_{n \to \infty} \mathbb{E}[\Psi_n] = 1 - \lim_{n \to \infty} \mathbb{E}[p_n] = 1 - \mathbb{E}[p^*] ,
\]

where we utilize (17) (which also follows from Proposition 1). Then, using (12) and Cesàro mean convergence [33, Chapter 3, Exercise 14(a)], we obtain

\[
\lim_{n \to \infty} \mathbb{E}\left[\Psi_n\right] = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\Psi_k] = 1 - \mathbb{E}[p^*] .
\]  

(54)

Combining (53) and (54) produces the key equation:

\[
\mathbb{E}[1 - p^*] = \mathbb{E}[g(1 - p^*)] .
\]  

(55)

Now suppose for the sake of contradiction that \(\mathbb{P}(p^* \in (0, 1)) > 0\). To prove part 1, notice that \(g\) is a strictly increasing and strictly concave function on \([0, 1]\) (since \(\gamma > 1\)); indeed, \(g'(x) = \gamma/(1 + (\gamma - 1)x)^2 > 0\) and \(g''(x) = -2\gamma(\gamma - 1)/(1 + (\gamma - 1)x)^3 < 0\) for all \(x \in [0, 1]\). Since \(g(0) = 0\) and \(g(1) = 1\), we must have \(g(x) > x\) for all \(x \in (0, 1)\). Let us rewrite (55) as follows:

\[
\mathbb{P}(p^* = 0) + \mathbb{E}[1 - p^* | p^* \in (0, 1)] \mathbb{P}(p^* \in (0, 1)) = \mathbb{E}[g(1 - p^*) | p^* \in (0, 1)] \mathbb{P}(p^* \in (0, 1))
\]

which implies that

\[
\mathbb{E}[1 - p^* | p^* \in (0, 1)] = \mathbb{E}[g(1 - p^*) | p^* \in (0, 1)],
\]

where the conditional expectations are well-defined because \(\mathbb{P}(p^* \in (0, 1)) > 0\) (by assumption). Since \(g(x) > x\) for all \(x \in (0, 1)\) and \(\mathbb{P}(1 - p^* \in (0, 1)) > 0\), we must also have (cf. [32, Chapter 1, Exercises 4.25 and 4.32])

\[
\mathbb{E}[1 - p^* | p^* \in (0, 1)] < \mathbb{E}[g(1 - p^*) | p^* \in (0, 1)],
\]

which is a contradiction. Thus, we have established that \(\mathbb{P}(p^* \in (0, 1)) = 0\), or equivalently, \(\mathbb{P}(p^* \in \{0, 1\}) = 1\). This proves part 1.

Part 2: To prove part 2, recall from (52) that \(\lim_{n \to \infty} \hat{\Psi}_n = g(1 - p^*)\) a.s. Since \(1 - p^* \in \{0, 1\}\) a.s. due to part 1, and \(g(0) = 0\) and \(g(1) = 1\), we have \(\lim_{n \to \infty} \hat{\Psi}_n = 1 - p^*\) a.s. This establishes part 2.

Part 3: Finally, to prove the third, observe that for all \(k \in \{0, \ldots, N\}\),

\[
\lim_{n \to \infty} \mathbb{P}(N \Psi_n = k | p_n) = \lim_{n \to \infty} \binom{N}{k} (1 - p_n)^k p_n^{N-k} = \binom{N}{k} (1 - p^*)^k (p^*)^{N-k},
\]

where we use the fact that \(N \Psi_n \sim \text{bin}(N, 1 - p_n)\) given \(p_n\) for all \(n \in \mathbb{Z}_+\). Using the dominated convergence theorem, this produces

\[
\lim_{n \to \infty} \mathbb{P}\left(\Psi_n = \frac{k}{N}\right) = \binom{N}{k} \left\{ \begin{array}{ll} \mathbb{P}(p^* = 1), & k = 0 \\ \mathbb{P}(p^* = N), & k = N \\ 0, & k \in \{1, \ldots, N-1\} \end{array} \right.
\]

where the second equality follows from part 1. Therefore, \(\hat{\Psi}_n\) converges weakly (or in distribution) to \(1 - p^*\) as \(n \to \infty\). This completes the proof.

APPENDIX III
PROOF OF PROPOSITION 2

It suffices to prove that \(\mathbb{P}(\lim_{n \to \infty} \Psi_n \in \{0, 1\}) = 0\), since using part 1 of Theorem 1, this implies that \(\{\Psi_n : n \in \mathbb{Z}_+\}\) does not converge to \(1 - p^*\) a.s. as \(n \to \infty\). To this end, for any \(n_0 \geq n_0\), \(\Lambda_{n_0:n} \triangleq \mathbb{P}(\forall n_0 \leq s \leq n, \Psi_s = 1)\).

Observe that for every \(n \geq n_0\), this quantity satisfies the upper bound:

\[
\Lambda_{n_0:n} = \mathbb{P}(\Psi_n = 1 | \forall n_0 \leq s \leq n - 1, \Psi_s = 1) \Lambda_{n_0:n-1} = \mathbb{P}[\Psi_n = 1 | \alpha_n, \beta_n] | \forall n_0 \leq s \leq n - 1, \Psi_s = 1] \Lambda_{n_0:n-1}
\]

\[
\leq \left(1 + \frac{n - n_0}{\alpha_n + \beta_n} \right)^N \Lambda_{n_0:n-1}
\]

\[
\leq \left(1 + \frac{n - n_0}{\alpha_n + \beta_n} \right)^N
\]

where the second and third equalities hold because \(N \Psi_n \sim \text{bin}(N, 1 - p_n)\) is conditionally independent of \(\{\Psi_s : n_0 \leq s \leq n - 1\}\) given \(\alpha_n\) and \(\beta_n\), the fourth inequality holds because \(\beta_n \leq \beta_0 + n\) (using (15)) and \(\alpha_n + \beta_n \geq \alpha_n + \beta_0 + n\) (using (14), (15), and the fact that \(\gamma > 1\)), and the fifth inequality follows from unwinding the recursion. Letting \(n \to \infty\), we obtain that

\[
\lim_{n \to \infty} \Lambda_{n_0:n} \leq \left(\prod_{k=n_0}^n \left(1 + \frac{k}{\alpha_n + \beta_n} \right)^N \right)
\]

\[
= \left(\exp \left(\sum_{k=n_0}^n \log \left(1 + \frac{k}{\alpha_n + \beta_n} \right)\right) \right)^N
\]

\[
= \left(\exp \left(- \sum_{k=n_0}^n \log \left(1 + \frac{1}{1 + Nk} \right)\right) \right)^N
\]

(56)
lim_{x \to 0} \log(1+x)/x = 1 \text{ (by L'Hôpital’s rule [33, Theorem 5.13])} \text{ and } \sum_{k=0}^{\infty} (1+K)k^{-1} = +\infty \text{ (since the harmonic series diverges [33, Theorem 3.28])}, we have
\[ \sum_{k=n_0}^{\infty} \log \left(1 + \frac{1}{1 + NK} \right) = +\infty \]
using the limit comparison test [34, Theorem 8.21]. Using (56), this yields
\[ \lim_{n \to \infty} \lambda_{n,n} = \lim_{n \to \infty} P(\forall n_0 \leq s \leq n, \Psi_s = 1) = 0 \]
which, by the continuity of $P$, implies
\[ P(\forall s \geq n_0, \Psi_s = 1) = 0 \]
for all sufficiently large $n_0$, $\Psi_s = 1$ is a non-increasing sequence of events with limit
\[ \bigcap_{n \geq n_0} \{ \forall n_0 \leq s \leq n, \Psi_s = 1 \} = \{ \forall s \geq n_0, \Psi_s = 1 \}. \]
Then, since the above equality holds for all $n_0 \in \mathbb{Z}_+$, applying the union bound, we get
\[ P\left( \lim_{s \to \infty} \Psi_s = 1 \right) = P(\exists n_0 \in \mathbb{Z}_+, \forall s \geq n_0, \Psi_s = 1) = 0, \]
where the first equality holds because $N\Psi_s \in \{0, \ldots, N\}$ for all $s \in \mathbb{Z}_+$, which means that $\lim_{n \to \infty} \Psi_s = 1$ if and only if there exists $n_0 \in \mathbb{Z}_+$ such that for all $s \geq n_0$, $\Psi_s = 1$. Similarly, since $p_n \leq (\alpha_0 + \gamma n)/\alpha_0 + (\alpha_0 + \beta + \gamma)$ (using (14) and (15)), we can show that
\[ P\left( \lim_{s \to \infty} \Psi_s = 0 \right) = 0. \]
By the union bound, (57) and (58) yield $P(\lim_{n \to \infty} \Psi_s \in \{0, 1\}) = 0$ as desired. This completes the proof.

**APPENDIX IV**

**PROOF OF THEOREM 2**

Part 1: To prove this, we employ the powerful technique of stochastic approximation. Observe that for any $n \in \mathbb{Z}_+$,
\[ p_{n+1} - p_n = \frac{\alpha_0 + \gamma(1 - \Psi_n)}{\alpha_0 + \beta + \gamma(1 - \Psi_n) + \Psi_n} - \frac{\alpha_0}{\alpha_0 + \beta_0} \]
\[ = \frac{\alpha_0 + \gamma(1 - \Psi_n) - \frac{\alpha_0}{\alpha_0 + \beta_0}(\alpha_0 + \beta + \gamma(1 - \Psi_n) + \Psi_n)}{\alpha_0 + \beta + \gamma(1 - \Psi_n) + \Psi_n} \]
\[ = \frac{\gamma(1 - \Psi_n) - \frac{\alpha_0 + \beta_0}{\alpha_0 + \beta_0 + \gamma(1 - \Psi_n)} - p_n(1 - \Psi_n - p_n)}{\alpha_0 + \beta + \gamma(1 - \Psi_n) + \Psi_n} \]
\[ = \gamma_{n+1} \left( F(p_n) + \xi_{n+1} \right), \]
where we let
\[ F(p) = -\gamma_1 p^2 + (\gamma_1 - 1) p, \text{ for } p \in [0, 1], \]
\[ \xi_{n+1} = \gamma(1 - \Psi_n) - p_n(\gamma(1 - \Psi_n) + \Psi_n) - F(p_n) \]
\[ = \frac{(\gamma(1 - p_n) + p_n(\gamma - 1 - \gamma)\Psi_n)}{\alpha_0 + \beta + \gamma(1 - \Psi_n) + \Psi_n}, \]
and we have $\min_{n \geq 1} \frac{1}{\gamma_{n+1}} \leq \min_{n \geq 1} \frac{1}{\gamma_{n+1} \xi_{n+1}}$. Next, we show that $E[\gamma_{n+1} \xi_{n+1} | F_n] \leq C/n^2$ for some constant $C > 0$. For all sufficiently large $n \in \mathbb{Z}_+$.
\[ E[\gamma_{n+1} \xi_{n+1} | F_n] \]

Part 2: The second part follows from part 1 and part 2 of Theorem 1, which shows that $\Psi_n \to 0 \text{ a.s. as } n \to \infty$.

**APPENDIX V**

**PROOF OF LEMMA 1**

Part 1: The first part follows from the definitions of the processes $X_n$, $Y_n$, and $Z_n$ given by (33)-(35).

Part 2: It is sufficient to show that the vector field $G$ is pointing to the interior of the manifold $M$ at the boundary $\partial M$ of $M$, i.e. $\langle G(x,y,z), n(x,y,z) \rangle \geq 0$ for all $(x,y,z) \in \partial M$. 

where \( n(x, y, z) \) is a normal vector to \( \partial M \) at \( (x, y, z) \) pointing to the interior of \( M \), and \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^3 \). At the boundary \( x + y = \frac{z}{1 + \gamma} \), we have

\[
\langle G(x, y, z), n(x, y, z) \rangle = \langle G(x, y, z), (1, 1, 0) \rangle = -\gamma(1 + 1)(x + y) + \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) + \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) \geq -1 + \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) + \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) = 0.
\]

At the boundary \( x + y = \frac{z}{1 + \gamma} \), we have

\[
\langle G(x, y, z), n(x, y, z) \rangle = \langle G(x, y, z), (1, -1, 0) \rangle = \gamma(1 + 1)(x + y) - \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) - \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) = \gamma - \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) - \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) \geq \gamma - \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) - \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) \geq 0.
\]

At the boundary \( x = 0 \), we have

\[
\langle G(x, y, z), n(x, y, z) \rangle = \langle G(x, y, z), (1, 1, 0) \rangle = \gamma \Phi \left( 1 - \frac{z}{1 - y} \right) \geq 0.
\]

At the boundary \( x = \frac{z}{\gamma + 1} \), we have

\[
\langle G(x, y, z), n(x, y, z) \rangle = \langle G(x, y, z), (1, -1, 0) \rangle = \gamma - \gamma \left( (1 - \Phi_x \frac{x}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) - \gamma \left( (1 - \Phi_y \frac{y}{x + y} + \Phi \left( 1 - \frac{z}{1 - x - y} \right) \right) \geq 0.
\]

The proof for the rest of the boundaries is analogous.

**Part 3:** The third part follows from the fact that the vector field \( G \) is smooth on the compact invariant set \( M \).

**APPENDIX VI**

**Proof of Lemma 2**

**A. Preliminary Definitions**

The following definitions will be used in the proof.

**Definition 2 (Semiflow [36]):** A semiflow \( \Phi \) on a smooth manifold \( M \) is a continuous map \( \Phi : \mathbb{R}_+ \times M \to M \) such that \( \Phi_0(x) = x \) and \( \Phi_{t+s} = \Phi_t \circ \Phi_s \).

**Definition 3 (Pseudotrajectory [36]):** A continuous function \( X : \mathbb{R}_+ \to M \) is an asymptotic pseudotrajectory for a semiflow \( \Phi \) if

\[
\forall T > 0, \lim_{t \to \infty \wedge t < T} \sup_{0 \leq s < t} \| X(t + h) - \Phi_h(X(t)) \| = 0.
\]

**Definition 4 (Transitivity [36]):** A compact invariant set \( L \) of a semiflow \( \Phi \) on \( M \) is internally chain transitive if for all \( \delta > 0 \) and \( T > 0 \), for all \( a, b \in L \), there exists \( y_0, \ldots, y_k \in L \) such that \( \|a - y_0\| \leq \delta, \|\Phi_j( y_j) - y_{j+1}\| \leq \delta \), for \( j = 1, \ldots, k-1 \), and \( y_k = b \). Here, invariant means that \( \Phi_t(L) = L \) for all \( t \geq 0 \).
\((X(t), D(t), E(t))\) be a solution of (59). In the following, we show that \((X(t), D(t), E(t))\) is an asymptotic pseudotrajectory (see Definition 3) for \(\Phi^\delta\). Fix \(T > 0\) and let \(h \in [0, T]\). Then,
\[
\| (X(t+h), D(t+h), E(t+h)) - \Phi^\delta_h (X(t), D(t), E(t)) \|
\leq k_1 \int_0^h \| (X(t+\tau), D(t+\tau), E(t+\tau)) \| d\tau
\leq k_1 \int_0^h \| (X(t+\tau), D(t+\tau), E(t+\tau)) \| d\tau
\]
where the first inequality follows from the fact that \(\Phi^\delta_h (X(t), D(t), E(t)) = (X(\tau), D(\tau+t), E(\tau+t))\) where \(X(\tau)\) is the unique solution of \(dx/d\tau = g_1(x)\) with \(x(t) = X(t)\), the second inequality follows from the Lipschitz continuity of \(g_1\) and \(G_1\) (where \(k_1 > 0\) and \(k_2 > 0\) are the corresponding Lipschitz constants, respectively), and the final inequality follows from the fact that \(D(\tau)\) and \(E(\tau)\) are the unique solutions of \(dx/d\tau = -(\gamma+1)x\) with initial conditions \(D(0)\) and \(E(0)\), respectively. By Grönnwall’s inequality [37], we obtain
\[
\| (X(t+h), D(t+h), E(t+h)) - \Phi^\delta_h (X(t), D(t), E(t)) \| 
\leq k_2 \left( \frac{1}{\gamma} D(0) + E(0) \right) \int_0^h e^{-(\gamma+1)(t+\tau)} d\tau e^{h+1}.
\]
Hence,
\[
\max_{h \in [0,T]} \left\| (X(t+h), D(t+h), E(t+h)) - \Phi^\delta_h (X(t), D(t), E(t)) \right\| 
\leq k_2 \left( \frac{1}{\gamma} D(0) + E(0) \right) e^{h+1} T e^{h+1},
\]
(62)
This implies that the left hand side of (62) converges to 0 as \(t \to \infty\) and proves that \((X(t), D(t), E(t))\) is an asymptotic pseudotrajectory for \(\Phi^\delta\). By [36, Theorem 5.7], the limit points of \((X(t), D(t), E(t))\) is internally chain transitive of \(\Phi^\delta\). But, the only internally chain transitive sets of \(\Phi^\delta\) are their equilibrium points \((\gamma, 1, \gamma)\), \((\gamma, 1, \gamma)\), and \((\gamma, 1, \gamma)\), if \(\gamma \geq \max \left( \frac{\Phi}{\Phi - \Phi}, \frac{1}{\Phi - \Phi} \right)\), or \((0, 1, \gamma)\) and \((\gamma, 1, \gamma)\), if \(\gamma < \max \left( \frac{\Phi}{\Phi - \Phi}, \frac{1}{\Phi - \Phi} \right)\). This was proved for any solution of (59). This implies that any solution of (46) converges to an equilibrium point of (46). Thus, the only internally chain sets of the flow generated by (46) are the equilibrium points of (46). By [36, Proposition 4.1] and [36, Proposition 5.7], \((X_n, Y_n, Z_n)\) converges to an equilibrium point \((X_\infty, Y_\infty, Z_\infty) \in L\). Finally, part 2 follows from \(p_n^0 = X_n/(X_n + Y_n)\) and \(p_n^1 = Z_n/(1 - X_n - Y_n)\).

**APPENDIX VII**

**Proof of Theorem 3**

We only prove the case \(\gamma \geq \max \left( \frac{\Phi}{\Phi - \Phi}, \frac{1}{\Phi - \Phi} \right)\). The proofs of the other cases are analogous. By Lemma 2, it is sufficient to show that \((p_n^0, p_n^1)\) does not converge to \((0, 1)\) or \((1, 0)\). Equivalently, it is sufficient to show that \(X_n\) does not converge to 0 or \(\gamma_1\). By defining the processes \(D_n = X_n + \gamma Y_n\) and \(E_n = X_n + Z_n\), the process \(\tilde{X}_n = x_n + \gamma_1 + \gamma_1 - D_n + \gamma_1 - E_n\) satisfies the following equation:
\[
\tilde{X}_{n+1} - \tilde{X}_n = g_1(\tilde{X}_n) + U_{n+1}^1,
\]
where
\[
U_{n+1}^1 = g_1(X_n) - g_1(\tilde{X}_n) + g_2(X_n, D_n, E_n)
- (\gamma + 1) \left( \frac{\gamma}{1 + \gamma} - D_n + \frac{\gamma}{1 + \gamma} - E_n \right) + U_{n+1}^1.
\]
The processes \(D_n\) and \(E_n\) can be computed explicitly as follows:
\[
D_n - \frac{\gamma}{\gamma_1 + 1} = \frac{\alpha_n^1 + \gamma^2 \alpha_n^0}{\alpha_n^0 + \beta_n^1 + \alpha_n^1 + \beta_n^0} - \frac{\gamma}{\gamma_1 + 1} = \frac{2/n + \gamma}{4/n + \gamma_1 + 1} - \frac{\gamma}{\gamma_1 + 1} = \frac{2/n + \gamma_1}{4/n + \gamma_1 + 1}.
\]
\[
E_n - \frac{\gamma}{\gamma_1 + 1} = \frac{2/n + \gamma}{4/n + \gamma_1 + 1} - \frac{\gamma}{\gamma_1 + 1} = \frac{2/n + \gamma_1}{4/n + \gamma_1 + 1}.
\]
We have
\[
| E \left[ \gamma_{n+1} U_{n+1}^1 | F_n \right] |
= \gamma_{n+1} \left| g_1(X_n) - g_1(\tilde{X}_n) + g_2(X_n, D_n, E_n)
- \left( \gamma + 1 \right) \left( \frac{\gamma}{\gamma_1 + 1} - D_n + \frac{\gamma}{\gamma_1 + 1} - E_n \right) \right|
\leq \gamma_{n+1} (k_1 + \gamma_1) \left( \left| D_n - \frac{\gamma}{\gamma_1 + 1} \right| + \left| E_n - \frac{\gamma}{\gamma_1 + 1} \right| \right)
+ \gamma_{n+1} k_2 \left( \left| D_n - \frac{\gamma}{\gamma_1 + 1} \right| + \left| E_n - \frac{\gamma}{\gamma_1 + 1} \right| \right)
\leq C \gamma_n^2
\]
for some constant \(C > 0\), where the first equality follows from \(E[U_{n+1}^1 | F_n] = 0\) and \(X_n, \tilde{X}_n, D_n, E_n \in F_n\), the second inequality follows from the Lipschitz continuity of \(g_1\) (with Lipschitz constant \(k_1\)) and \(G_1\) (with Lipschitz constant \(k_2\)).
and the definition of \( g_2 \) (61), and the last inequality is a consequence of (63)-(64) and \( \gamma_n + 1 = (4/N) + n(\gamma + 1) \).

The second moment of the noise \( \hat{U}_{n+1} \) satisfies

\[
E \left( (\hat{U}_{n+1})^2 \right) \leq 2 \left( g_1(X_n) - g_1(\hat{X}_n) + g_2(X_n, D_n, E_n) \right. \\
- (\gamma + 1) \left( \frac{\gamma}{\gamma + 1} - D_n + \frac{\gamma}{\gamma + 1} - E_n \right) \right)^2 \\
+ 2E \left( (U_{n+1})^2 \right) \\
\leq C_1 \left( D_n - \frac{\gamma}{\gamma + 1} \right)^2 + E_n - \frac{\gamma}{\gamma + 1} \right)^2 \\
+ 2E \left( (U_{n+1})^2 \right)
\]

for some constant \( C_1 > 0 \). By (41), we obtain

\[
E \left( (U_{n+1})^2 \right) \leq \gamma^2 E \left( (\Psi_n - (1 - \Phi)0 + 0) - \Phi p_n^0 \right)^2 \right) \\
= \gamma^2 E \left( ((1 - \Phi)\Psi_n + \Phi \Psi_n - (1 - \Phi)0 - \Phi p_n^0 \right)^2 \right) \\
= \gamma^2 (1 - \Phi)^2 E \left( \Psi_n^0 - (1 - \Phi)0 - \Phi p_n^0 \right)^2 \right) \\
+ \gamma^2 2\Phi^2 E \left( \Psi_n^0 - p_n^0 \right)^2 \right) \\
= \gamma^2 (1 - \Phi)\Psi_n^0 + \gamma^2 \Phi p_n^0 (1 - \Phi)^0 + 1 \right) \\
= \gamma^2 (1 - \Phi) \frac{X_n}{X_n + Y_n} (1 - \Phi)^0 + 1 \right) \\
+ \gamma^2 2\Phi p_n^0 \left( 1 - \frac{Z_n}{1 - X_n - Y_n} \right) \\
\leq \gamma^2 (1 - \Phi) \frac{X_n}{X_n + Y_n} \left( \frac{X_n}{X_n + Y_n} + \gamma \frac{\gamma + 1}{\gamma + 1} - D_n \right) \\
+ \gamma^2 (1 - \Phi) \frac{X_n}{X_n + Y_n} \left( \frac{X_n}{X_n + Y_n} + \gamma \frac{\gamma + 1}{\gamma + 1} - D_n \right)
\]

for some constant \( C_2 > 0 \), where the second equality follows from (27), the third equality follows from the conditional independence of \( \Psi_n \) and \( \Psi_n \), the fourth equality from \( N/\Psi_n \sim bin(N, 1 - \Phi^0) \) and \( N/\Psi_n \sim bin(N, 1 - \Phi^0) \), and the sixth inequality follows from Lemma 1. Hence,

\[
E \left( (\hat{U}_{n+1})^2 \right) \leq C_3 |\hat{X}_n|
\]

for some constant \( C_3 > 0 \).

Using similar arguments to those used in Proposition 2, one can show that \( \Psi_n \) does not converge to 0 or 1. Hence, \( nX_n \to \infty \) as \( n \to \infty \). We know that \( \hat{X}_n - X_n \to 0 \) as \( n \to \infty \). This implies that \( nX_n \to \infty \) as \( n \to \infty \). We have \( g_1(x)x > 0 \) in the neighborhood of 0. Hence, by [35, Theorem 3], we deduce that \( P(0 \to X_n) = 0 \). We deduce that \( P(\hat{X}_n \to 0) = 0 \), which proves the case \( \gamma \geq \max \left( \frac{\Phi}{1 - \Phi}, \frac{1 - \Phi}{\Phi} \right) \).

**APPENDIX VIII**

**PROOF OF LEMMA 3**

**Part 1:** Consider the case \( \phi_v = 1 \). The proof for the \( \phi_v = 0 \) case is analogous. Define the stochastic process

\[
z_n = \sum_{k=0}^{n-1} (\psi_{v,k} + p_k^1).
\]

The process \( z_n \) is \( F_n \)-adapted. Since \( E[\psi_{v,n} | F_n] = p_n^1 \), we obtain that \( E[z_{n+1} | F_n] = z_n \). Thus, \( z_n \) is an \( F \)-martingale. Furthermore, \( |z_n - z_{n-1}| \leq 2 \). Using the Azuma-Hoeffding inequality [38], we deduce that for all \( \epsilon > 0 \),

\[
P \left( \frac{1}{n} z_n > \epsilon \right) \leq 2 \exp \left( -\frac{\epsilon^2 n^2}{2 \sum_{k=1}^{n} 2^2} \right) = 2 \exp \left( -\frac{\epsilon^2 n}{8} \right).
\]

By the Borel-Cantelli lemma [32, Lemma 2.5], we obtain that the event \( A_n = \{ \frac{1}{n} z_n > \epsilon \} \) occurs finitely often. Thus, \( \lim_{n \to \infty} \frac{1}{n} z_n = 0 \). Observe that \( \frac{1}{n} \sum_{k=0}^{n-1} p_k \) converges to \( p_\infty^1 \) by the dominated convergence theorem (or Cesàro summation). This proves the result.

**Part 2:** We have

\[
\hat{\Psi}_n = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{v \in V} \psi_{v,k} = \frac{1}{n} \sum_{v \in V} \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k}
\]

\[
= \frac{1}{n} \sum_{v \in V} \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k}
\]

Using part 1, we obtain

\[
lim_{n \to \infty} \frac{1}{n} \sum_{v \in V, \phi_v = 0} \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k} = \frac{|V^0|}{N} = (1 - \Phi)(1 - p_\infty^0),
\]

\[
lim_{n \to \infty} \frac{1}{n} \sum_{v \in V, \phi_v = 1} \frac{1}{n} \sum_{k=0}^{n-1} \psi_{v,k} = \frac{|V^1|}{N} = \Phi p_\infty^1,
\]

where \( |V^j| \) is the number of agents that have inherent opinion \( j \). This proves the second part.

**APPENDIX IX**

**PROOF OF THEOREM 4**

**Part 1:** The first part follows from part 2 of Lemma 3.

**Parts 2 and 3:** To prove the second part, note that \( g_\infty = \frac{1}{\gamma + 1} \) if and only if \( \Psi_\infty = 0 \). By Lemma 3, if \( \Psi_\infty = 0 \), then \( (p_\infty^1, p_\infty^0) = (1, 0) \). Then, by Theorem 3, \( \gamma \leq \frac{1}{\gamma + 1} \), which is equivalent to \( \Phi \leq \frac{\Phi}{\gamma + 1} \). The third part is shown similarly.

**Part 4:** We have \( g_\infty \in \left( \frac{1}{\gamma + 1}, \frac{1}{\gamma + 1} \right) \) if and only if \( \Psi_\infty \in (0, 1) \). Lemma 3 implies that \( (p_\infty^1, p_\infty^0) \notin \{(0, 1), (1, 0)\} \). So, Theorem 3 implies that \( (p_\infty^1, p_\infty^0) = \left( \frac{\gamma(1 - \Phi)\psi}{\gamma + 1}, \frac{\gamma(1 - \Phi)\psi}{\gamma + 1} \right) \). Therefore,

\[
\hat{\Psi}_\infty = (1 - \Phi)(1 - p_\infty^0) + \Phi p_\infty^1 = \frac{\gamma(1 - \Phi)\psi}{\gamma + 1},
\]

where the second equality follows from (50). This completes the proof.

**ACKNOWLEDGMENT**

The authors are grateful to Govind Ramnarayan for stimulating discussions about the problem formulation.
