Short-range impurity in the vicinity of a saddle point and the levitation of the 2D delocalized states in a magnetic field.

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Abstract

The effect of a short-range impurity on the transmission through a saddle-point potential for an electron, moving in a strong magnetic field, is studied. It is demonstrated that for a random position of an impurity and random sign of its potential the impurity-induced mixing of the Landau levels diminishes on average the transmission coefficient. This results in an upward shift (levitation) of the energy position of the delocalized state in a smooth potential. The magnitude of the shift is estimated. It increases with decreasing magnetic field $B$ as $B^{-4}$.

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I. INTRODUCTION

The fate of the two-dimensional delocalized states (DS’s) in a magnetic field became recently the subject of intensive experimental and theoretical studies. The goal of these studies is to trace the position of DS’s as magnetic field decreases. More than a decade ago it was predicted that with decreasing magnetic field each DS departs from the center of the Landau level (LL) and floats up in energy (levitation scenario). This prediction was derived from the non-interacting scaling theory of the quantum Hall effect. Since the position of a DS determines the boundary between the insulating and the quantum Hall phases, such a floating implies that within some range of concentrations the electron gas undergoes an insulator-metal-insulator transition with decreasing magnetic field. Later it was argued that this behavior should persist in the presence of interactions. Experimental observation of the insulator-metal-insulator transition was reported by several groups.

It is obvious just from the electron-hole symmetry that the deviation of DS from the center of Landau level is possible only due to the disorder-induced LL mixing. Numerical simulations indeed support the levitation scenario. Analytical theory of the levitation was developed only for the region of magnetic fields where the departure of the DS from the center of LL is relatively small. It was assumed that the random potential is smooth, so that the structure of electronic states is described by the network model of Chalker and Coddington. In this model, delocalization results from the tunneling of an electron through the saddle points of a random potential which are connected by equipotential lines. It was demonstrated that the LL mixing changes on average the transmittancy of a saddle point in such a way that it becomes smaller than 1/2 for the energy at the center of the LL. This means that to achieve the 1/2 average transmittancy, the energy should be shifted upwards, which is equivalent to the levitation. However, in a smooth potential the LL mixing is generally weak since it is associated with the large momentum transfer. Short-range potential is much more effective in this respect. That is why in this paper we consider the situation when a small portion of short-range impurities is present in a sample in addition
to a smooth potential. Within the Chalker–Coddington model the shift in position of the
DS results from the change in the transmittancy of a saddle point due to LL mixing. Then
it is obvious that impurities, responsible for this shift, are those that fall in the vicinity of
saddle points. We assume the sign of the impurity potential to be random and show that
the net effect of such impurities is the reduction of the transmittancy at a given energy, and,
hence, the levitation of DS’s, even if the mixing of LL’s by a smooth potential is neglected.

The plan of the paper is the following. In Sec. II we consider the motion of an electron
in a strong magnetic field and a saddle–point potential with a short–range impurity located
nearby. We derive the system of equations for the motion of the guiding center using the
procedure developed by Fertig and Halperin\textsuperscript{20}. In Sec. III we demonstrate that coupling
of LL’s by an impurity leads to the renormalization of its effective strength. In Sec. IV a
general expression for the transmission coefficient is derived and different limits are analyzed.
In Sec. V the impurity–induced change in the average transmittancy is studied. In Sec. VI the magnitude of levitation of DS’s, which results from the reduction of the average transmittancy is, estimated.

II. HAMILTONIAN

The Hamiltonian we consider has the form

$$\hat{H} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla + \frac{e}{c} A \right)^2 + V_{SP}(x, y) + V(x, y) \equiv \hat{H}_0 + V(x, y),$$

(1)

where

$$V_{SP} = -\frac{m\Omega^2}{2}(x^2 - y^2)$$

(2)

is a potential of the saddle point;

$$V(x, y) = V_0 \ F \left( \sqrt{(x - x_0)^2 + (y - y_0)^2} \right)$$

(3)

is the short range potential of an impurity centered at the point $(x_0, y_0)$. $V_0$ is chosen in
such a way that $F(0) = 1$. We choose a symmetric gauge, $A = \frac{B}{2}(-y, x, 0)$, where $B$ is the
magnetic field. To calculate the transmission coefficient of the saddle point we adopt the approach developed by Fertig and Halperin. It was demonstrated that the Hamiltonian $\hat{H}_0$ can be separated into two parts describing respectively the motion of a guiding center and a cyclotron motion. The unitary transformation which provides this separation is as follows:

\[\begin{align*}
    x &= l\sqrt{2} (\cos \beta X + \sin \beta s), \\
    \frac{\partial}{\partial x} &= \frac{1}{l\sqrt{2}} \left( \cos \beta \frac{\partial}{\partial X} + \sin \beta \frac{\partial}{\partial s} \right), \\
    y &= il\sqrt{2} \left( \sin \beta \frac{\partial}{\partial X} - \cos \beta \frac{\partial}{\partial s} \right), \\
    \frac{\partial}{\partial y} &= -\frac{i}{l\sqrt{2}} (\sin \beta X - \cos \beta s),
\end{align*}\]

where $l = \sqrt{\hbar/m\omega_c}$ is the magnetic length ($\omega_c$ stands for the cyclotron frequency). The new operators satisfy the commutation relations $[X, s] = \left[ \frac{\partial}{\partial X}, \frac{\partial}{\partial s} \right] = 0$, $\left[ \frac{\partial}{\partial X}, X \right] = \left[ \frac{\partial}{\partial s}, s \right] = 1$.

The parameter $\beta$ is determined by the equation

\[\tan(2\beta) = -\frac{1}{2} \left( \frac{L}{l} \right)^4,\]

where $L$ is the characteristic length defined as $L = \sqrt{\hbar/m\Omega}$. The condition that the potential is smooth can be quantitatively expressed as $\omega_c \gg \Omega$, which means that $L \gg l$. Then the solution of Eq.(6), can be written as

\[\beta = -\frac{\pi}{4} + \left( \frac{l}{L} \right)^4 + O \left( \frac{l}{L} \right)^8.\]

As a result of the transformation (4)–(7), the Hamiltonian $\hat{H}_0$ takes the form

\[\hat{H}_0 = E_1 \left( -\frac{\partial^2}{\partial X^2} - X^2 \right) + \frac{1}{2} E_2 \left( -\frac{\partial^2}{\partial s^2} + s^2 \right),\]

where the energies $E_1, E_2$ in the limit $L \gg l$ are given by

\[E_1 = \frac{\hbar \Omega^2}{2\omega_c}, \quad E_2 = \hbar \omega_c.\]

It is seen that the variable $X$ describes the motion of the guiding center whereas $s$ is responsible for the cyclotron motion.

We search for the solution of the Schrödinger equation, $\hat{H}\Psi = E\Psi$, in the form
\[ \psi(x, y) = \sum_n C_n(X) \psi_n(s), \]  
\[ \psi_n(s) = (2n + 1) \psi_n(s). \]

where \( \psi_n \) are the eigenfunctions of the harmonic oscillator

\[ \left( -\frac{\partial^2}{\partial s^2} + s^2 \right) \psi_n(s) = (2n + 1) \psi_n(s). \]

The coefficients \( C_n \) satisfy the system of equations

\[ \frac{d^2 C_n}{dX^2} + (X^2 + \varepsilon_n) C_n = \]

\[ = \frac{V_0}{E_1} \int_{-\infty}^{\infty} ds \, \psi_n(s) F \left( l(X - s) - x_0, -il \left( \frac{\partial}{\partial X} + \frac{\partial}{\partial s} \right) - y_0 \right) \Psi(X, s), \]

with \( \varepsilon_n \) defined as

\[ \varepsilon_n = \frac{E - \hbar \omega_c (n + 1/2)}{E_1}. \]

First, it can be readily seen that, since the function \( F \) is nonzero only when both arguments are smaller than \( a \), where \( a \ll l \) is the radius of the impurity potential, one can take \( \psi_n(s) \) out of the integral at point \( s = X - \frac{x_0}{l} \). We also see that after switching to the \((X, s)\) variables the impurity potential became an operator. Thus, to find its action on the function \( \Psi \) one should perform the Fourier transformation

\[ \Psi(X, s) = \Psi \left( \frac{X + s}{2} + \frac{x_0}{2l}, \frac{X + s}{2} - \frac{x_0}{2l} \right) = \int_{-\infty}^{\infty} dq \, e^{iq \frac{x_0}{l}} \Phi_q, \]

where we again took into account that \( X - s \approx \frac{x_0}{l} \). As a result the differential operator \( \frac{\partial}{\partial X} + \frac{\partial}{\partial s} \) in the argument of \( F \) can be replaced by \( iq \), so that the right-hand side takes the form

\[ \frac{V_0}{E_1} \psi_n(X - x_0/l) \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dq \, e^{iq \frac{x_0}{l}} \Phi_q \, F \left( l(X - s) - x_0, ql - y_0 \right). \]

Now the integration over \( s \) and \( q \) can be easily performed using the short-range character of \( F \). This picks the values \( q = \frac{y_0}{l} \) and \( s = X - \frac{x_0}{l} \) and generates the factor \( \frac{\pi a^2}{l} \). After that it is convenient to express the component \( \Phi_{q_0} \) in terms of \( \Psi \) using the inverse Fourier transformation. Finally we obtain
\[ \frac{d^2 C_n}{dX^2} + (X^2 + \varepsilon_n) C_n = \tilde{V}_0 \lambda e^{iXY_0} \psi_n(X - X_0), \]  
(16)

where we have introduced the notations

\[ \tilde{V}_0 = \frac{1}{E_1} \frac{V_0 d^2}{2 l^2}, \quad X_0 = \frac{x_0}{l}, \quad Y_0 = \frac{y_0}{l}. \]  
(17)

The parameter \( \lambda \) is determined as

\[ \lambda = \int_{-\infty}^{\infty} ds \ e^{-isY_0} \Psi(s, s - X_0) = \sum_n \int_{-\infty}^{\infty} ds \ e^{-isY_0} C_n(s) \psi_n(s - X_0). \]  
(18)

In the second equality we have substituted \( \Psi(s, s - X_0) \) using the representation (10). Eqs. (16), (18) form a closed system of equations describing the motion of the guiding center. In the absence of a short-range impurity, \( V_0 = 0 \), so that (16) reduces to the system of identical Schrödinger equations for a particle in an inverted parabolic potential. The transmission coefficient for this potential is well known:

\[ T_0(\varepsilon_n) = \frac{1}{1 + e^{-\pi\varepsilon_n}}, \]  
(19)

and is independent of the number of LL.

**III. RENORMALIZATION OF THE SCATTERING STRENGTH**

Since we are interested in scattering of an electron with energy close to the center of some LL, say \( n_0 \), by the saddle point potential, we can assume \( \varepsilon_m \approx (n_0 - m) \frac{\hbar \omega_c}{E_1} \) for any \( m \neq n_0 \). The condition \( \omega_c \gg \Omega \) guarantees that \( \hbar \omega_c \gg E_1 \). This means that each of Eqs. (16) with \( m \neq n_0 \) contains a big parameter \( \varepsilon_m \gg 1 \). Then for \( m \neq n_0 \) one can neglect the first two terms in (16) and get the solution

\[ C_n = \lambda \frac{\tilde{V}_0}{\varepsilon_n} e^{iXY_0} \psi_n(X - X_0). \]  
(20)

Substituting this solution into Eq. (18), enables us to express the constant \( \lambda \) in terms of only one unknown function \( C_{n_0}(X) \).
\[ \lambda = \frac{\int_{-\infty}^{\infty} ds \ e^{-isY_0} C_{n_0}(s) \psi_{n_0}(s - X_0)}{1 - \tilde{V}_0 \sum_{m \neq n_0} \frac{1}{\varepsilon_m}}. \] (21)

Once \( \lambda \) is determined, one can write a closed equation for \( C_{n_0} \)

\[ \frac{d^2 C_{n_0}}{dX^2} + (X^2 + \varepsilon_{n_0}) C_{n_0} = \tilde{V}_0^R e^{iXY_0} \psi_{n_0}(X - X_0) \int_{-\infty}^{\infty} ds e^{-isY_0} C_{n_0}(s) \psi_{n_0}(s - X_0), \] (22)

where \( \tilde{V}_0^R \) is defined as

\[ \tilde{V}_0^R = \frac{\tilde{V}_0}{1 + \tilde{V}_0 K} \] (23)

with \( K \) given by

\[ K = -\sum_{m \neq n_0} \frac{1}{\varepsilon_m} \] (24)

We see that the parameter \( K \) describes the effect of all other LL’s on the motion of an electron on the level \( n_0 \). It is apparent from (23) that this effect reduces to the renormalization of the scattering strength \( \tilde{V}_0 \). Important is that, since the impurity is short-range, it causes a strong local mixing of LL’s so that many levels with \( m \neq n_0 \) contribute to the renormalization constant \( K \). Indeed, the terms in the sum (24) fall off with the number of LL as \( \frac{1}{m} \). In the case \( n_0 = 0 \) all the terms in the sum (24) have the same sign, so, strictly speaking, the sum diverges logarithmically. The cutoff value of \( m \) is determined by the following condition.

For large enough numbers of LL’s \( \psi_m(s) \) is a rapidly oscillating function, the typical period of oscillations being \( \sim 1/\sqrt{m} \). Taking \( \psi_m(s) \) out of integral in (12) is justified only if this period is much larger than the dimensionless size of the impurity potential \( a/l \). This leads to the cutoff value \( m_{\text{max}} \sim l^2/a^2 \). Thus we obtain

\[ K \approx \frac{E_1}{\hbar \omega_c} \ln m_{\text{max}} = \frac{2E_1}{\hbar \omega_c} \ln \frac{l}{a}. \] (25)

For \( n_0 \neq 0 \) the sum in (24) contains both positive and negative contributions. Positive terms are those with \( m < n_0 \), while the terms with \( m > n_0 \) are negative. For large enough \( n_0 \) (but much smaller than \( m_{\text{max}} \)) we have

\[ K \approx \frac{E_1}{\hbar \omega_c} \ln \frac{m_{\text{max}}}{n_0} = \frac{E_1}{\hbar \omega_c} \ln \frac{l^2}{a^2n_0}. \] (26)

Thus the magnitude of the renormalization constant \( K \) decreases with the number of LL.
IV. TRANSMISSION COEFFICIENT

In this section we will drop for simplicity the LL index of the transmission coefficient. Without an impurity, the solutions of Eq. (22) are the parabolic cylinder functions: $D_\nu(X\sqrt{2}e^{i\pi/4})$ and $D_\nu(-X\sqrt{2}e^{i\pi/4})$, where $\nu = -\frac{i\varepsilon}{2} - \frac{1}{2}$. Using the asymptotic form of $D_\nu$

$$D_\nu(X) \sim X^\nu e^{-\frac{1}{4}X^2}, \quad X \to \infty, \quad \left( -\frac{3\pi}{4} < \arg X < \frac{3\pi}{4} \right)$$

$$D_\nu(X) \sim X^\nu e^{-\frac{1}{4}X^2} - \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{-i\pi\nu}X^{\nu-1}e^{\frac{1}{2}X^2}, \quad X \to -\infty, \quad \left( -\frac{5\pi}{4} < \arg X < -\frac{\pi}{4} \right),$$

one can see that it is the function $D_\nu(X\sqrt{2}e^{i\pi/4})$ that has the "right" behavior at $X \to \pm\infty$ (no incoming wave at $X \to +\infty$). The amplitude transmission coefficient, $t$, emerges from the comparison of the amplitudes of the transmitted and the incoming waves in (27), (28)

$$t = \left( \sqrt{2}e^{i\pi/4} \right)^{2\nu+1} \frac{\left( -\nu \right)^{\nu+1/2}}{(2\pi)^{1/2}} \frac{\Gamma(-\nu)}{\Gamma\left( \frac{i\varepsilon}{2} + \frac{1}{2} \right)} \frac{1}{2}.$$  

It is easy to see that $|t|^2 = T_0(\varepsilon)$, where $T_0(\varepsilon)$ is given by (19). With the right-hand side in (22) present, it is convenient to search for a solution of Eq. (22) in the form of a linear combination of the functions $D_\nu$

$$C_n(X) = D_\nu(X\sqrt{2}e^{i\pi/4}) + \int_{-\infty}^{\infty} d\varepsilon' \left[ a^+(\varepsilon') D_{\nu'}(X\sqrt{2}e^{i\pi/4}) + a^-(\varepsilon') D_{\nu'}(-X\sqrt{2}e^{i\pi/4}) \right].$$

The expressions for the coefficients $a^\pm$ are obtained by multiplying Eq. (22) by $D_\nu^\prime(\pm X\sqrt{2}e^{i\pi/4})$ and integrating over $X_0$. One has

$$a^\pm(\varepsilon') = \frac{\tilde{V}_0^R}{\sqrt{2}} \frac{J(\Omega(\varepsilon'))^{*}}{\alpha(\varepsilon')(\varepsilon - \varepsilon')},$$

where $I^\pm$ and $J$ are defined as

$$I^\pm(\varepsilon) = \int_{-\infty}^{\infty} dX e^{-i\varepsilon X} \psi_n(X - X_0) D_\nu(\pm X\sqrt{2}e^{i\pi/4}),$$

$$J = \int_{-\infty}^{\infty} ds e^{-i\varepsilon X} C_n(s) \psi_n(s - X_0),$$

and the function $\alpha(\varepsilon)$ is determined by the normalization condition
Using the asymptotics (27), (28) it can be shown that

\[
\alpha(\varepsilon) = 2\pi \left( e^{\frac{\pi\varepsilon}{2}} + e^{-\frac{\pi\varepsilon}{2}} \right) e^{-\frac{\pi\varepsilon}{4}}.
\]  

(35)

Upon substituting (31) into (30), we obtain

\[
C_n(X) = D_\nu \left( X \sqrt{2} e^{i\pi/4} \right) + \frac{\tilde{V}_0^R}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d\varepsilon'}{(\varepsilon - \varepsilon')} \left[ (I^+(\varepsilon'))^* D_{\nu'} \left( X \sqrt{2} e^{i\pi/4} \right) + (I^-(\varepsilon'))^* D_{\nu'} \left( -X \sqrt{2} e^{i\pi/4} \right) \right].
\]  

(36)

For \( X \to \pm \infty \) only the poles contribute to the integrals in Eq. (36). In the limit \( X \to \infty \), \( D_{\nu'}(X \sqrt{2} e^{i\pi/4}) \) is just a transmitted wave, whereas \( D_{\nu'}(-X \sqrt{2} e^{i\pi/4}) \) represents the combinations of waves traveling in both directions. Collecting the components corresponding to the transmission, we obtain the following general expression for the transmission coefficient

\[
T(\varepsilon) = T_0(\varepsilon) \left| 1 + i\pi \sqrt{2} J \tilde{V}_0^R \alpha^{-1}(\varepsilon) \left[ (I^+(\varepsilon))^* + i e^{-\frac{\pi\varepsilon}{4}} (I^-(\varepsilon))^* \right] \right|^2.
\]  

(37)

To find the constant \( J \) one should multiply (36) by \( e^{-iXY_0 \psi_n(X - X_0)} \) and integrate. This will generate \( J \) in the left-hand side; the solution of the resulting equation for \( J \) has the form

\[
J = \frac{I^+(\varepsilon)}{1 - \frac{\tilde{V}_0^R}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d\varepsilon'}{(\varepsilon - \varepsilon')} \alpha^{-1}(\varepsilon') \left[ |I^+(\varepsilon')|^2 + |I^-(\varepsilon')|^2 \right]}.
\]  

(38)

With \( J \) given by (38), the final result for the transmission coefficient reads

\[
T(\varepsilon) = T_0(\varepsilon) \left| 1 + \frac{i\pi \sqrt{2} \tilde{V}_0^R \alpha^{-1}(\varepsilon) \left[ |I^+(\varepsilon)|^2 + i e^{-\frac{\pi\varepsilon}{4}} I^+(\varepsilon)(I^-(\varepsilon))^* \right]}{1 - \frac{\tilde{V}_0^R}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d\varepsilon'}{(\varepsilon - \varepsilon')} \alpha^{-1}(\varepsilon') \left[ |I^+(\varepsilon')|^2 + |I^-(\varepsilon')|^2 \right]} \right|^2.
\]  

(39)

The denominator of (39) can be presented as \( \left( 1 - \tilde{V}_0^R \Sigma(\varepsilon) \right) - i \tilde{V}_0^R \Theta(\varepsilon) \), where the functions \( \Sigma \) and \( \Theta \) are defined as

\[
\Sigma(\varepsilon) = \frac{1}{\sqrt{2}} P \int_{-\infty}^{\infty} \frac{d\varepsilon'}{(\varepsilon - \varepsilon')} \alpha^{-1}(\varepsilon') \left[ |I^+(\varepsilon')|^2 + |I^-(\varepsilon')|^2 \right],
\]  

(40)

\[
\Theta(\varepsilon) = \frac{\pi}{\sqrt{2}} \alpha^{-1}(\varepsilon) \left[ |I^+(\varepsilon)|^2 + |I^-(\varepsilon)|^2 \right],
\]  

(41)
where the symbol $P$ stands for principal part. The meaning of the functions $\Sigma$ and $\Theta$ can be understood in the following way. It is known that without a smooth potential, a short–range impurity pulls out a state from the center of a degenerate LL and forms a bound state. In the presence of a saddle–point potential the energy position of the bound state is determined by the condition: $1 - \tilde{V}_0^R \Sigma(\varepsilon) = 0$. On the other hand, since the potential (2) “bends” the LL, the bound state becomes degenerate with the continuum of states at the LL and, thus, acquires a finite width which is described by the function $\Theta(\varepsilon)$. It is seen that $\Theta(\varepsilon)$ represents the sum of two terms, which correspond to the widths associated with the outcome to the left and to the right, respectively. When the impurity is located exactly at the saddle point ($X_0 = Y_0 = 0$) we have $I^+(\varepsilon) = I^-(\varepsilon)$, so that (39) reduces to

$$T(\varepsilon) = T_0(\varepsilon) \frac{\left(1 - \tilde{V}_0^R \Sigma(\varepsilon) - \tilde{V}_0^R \Theta(\varepsilon) e^{-\frac{\pi}{4} \varepsilon^2}\right)^2}{\left(1 - \tilde{V}_0^R \Sigma(\varepsilon) \right)^2 + (\tilde{V}_0^R \Theta(\varepsilon))^2}.$$  

(42)

For this particular location of the impurity, the functions $\Sigma$ and $\Theta$ are, respectively, odd and even functions of energy. We have evaluated these functions for the lowest LL. Note first that for $n = 0$ and arbitrary $X_0, Y_0$ the integrals $I^+$ and $I^-$ can be expressed in terms of the parabolic cylinder functions if one substitutes into (32) the integral representation of $D_\nu(X)$. Then one gets

$$I^+(\varepsilon, X_0, Y_0) = (2\pi)^{1/4} e^{-\frac{1}{4}(X_0^2 + Y_0^2 + 2iX_0Y_0 + \frac{\pi}{4})} D_\nu\left(e^{\frac{\pi}{4}}(X_0 - iY_0)\right);$$

$$I^-(\varepsilon, X_0, Y_0) = I^+(\varepsilon, -X_0, -Y_0).$$  

(43)

Since $D_\nu(0) = 2^{-(\nu/2+1)} \Gamma(-\frac{\nu}{2})/\Gamma(-\nu)$, where $\Gamma(z)$ is the $\Gamma$–function, we get the following expression for $I^+ = I^-$ at $X_0 = Y_0 = 0$

$$I^+(\varepsilon) = I^-(\varepsilon) = \pi^{-\frac{1}{8}} 2^{\frac{13}{8}} e^{-\frac{\pi}{8}} \frac{\Gamma\left(\frac{i\varepsilon}{4} + \frac{1}{4}\right)}{\Gamma\left(\frac{i\varepsilon}{2} + \frac{1}{2}\right)}.$$  

(44)

Then the functions $\Theta$ and $\Sigma$ take the form

$$\Theta(\varepsilon) = \frac{1}{4\sqrt{2\pi}} \left| \Gamma\left(\frac{i\varepsilon}{4} + \frac{1}{4}\right) \right|^2.$$  

(45)

$$\Sigma(\varepsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\varepsilon' \Theta(\varepsilon') = \frac{1}{2\sqrt{2}} \int_{0}^{\infty} dv \frac{e^{-\frac{v}{4}} \sin\left(\frac{\pi v}{4}\right)}{\sqrt{1 + e^{-v}}}.$$  

(46)
\[ \Theta(\varepsilon) \text{ is shown in Fig. 1 together with the function } \Sigma \text{ calculated numerically. The asymptotic behavior of } \Sigma(\varepsilon) \text{ is as follows: } \Sigma(\varepsilon) = 1.4\varepsilon, \text{ for } |\varepsilon| \ll 1; \Sigma(\varepsilon) = 1/\varepsilon, \text{ for } |\varepsilon| \gg 1. \text{ The energy dependence of the transmission coefficient for different amplitudes of the impurity potential, } \tilde{V}_0^R, \text{ is shown in Fig. 2. At zero energy } \Sigma(\varepsilon) \text{ vanishes, so that (42) simplifies to} \]

\[ T(\varepsilon = 0) = \frac{1}{2} \left( 1 - \frac{\tilde{V}_0^R}{4\sqrt{2\pi}} \left| \frac{\Gamma(\frac{1}{4})}{4\sqrt{2\pi}} \right|^2 \right). \]  

(47)

Fig. 3 shows the transmission coefficient at \( \varepsilon = 0 \) as a function of \( \tilde{V}_0^R \).

V. REDUCTION OF THE AVERAGE TRANSMITTANCY

Without an impurity, \( T_0(\varepsilon) \) obeys the relation: \( T_0(\varepsilon) + T_0(-\varepsilon) = 1 \) which is the manifestation of the symmetry of the saddle point (transmission of an electron with energy \( \varepsilon \) can be viewed as a reflection for an electron with energy \( -\varepsilon \)). Obviously, when the impurity is randomly positioned around the saddle point and the sign of \( \tilde{V}_0^R \) is random, this symmetry relation should hold on average. Indeed, it can be demonstrated that for an arbitrary position of an impurity, \((X_0, Y_0)\), the transmission coefficient satisfies the relation

\[ T(\varepsilon, \tilde{V}_0^R, X_0, Y_0) + T(-\varepsilon, -\tilde{V}_0^R, -Y_0, -X_0) = 1 \]  

(48)

To prove (48), one should rewrite (39) in the form

\[ T(\varepsilon) = T_0(\varepsilon) \left\{ (1 - \tilde{V}_0^R \Sigma(\varepsilon)) + i\tilde{V}_0^R \Pi(\varepsilon) - \tilde{V}_0^R \Lambda(\varepsilon) e^{-\frac{2\pi}{n}} \right\}^2, \]  

(49)

where the functions \( \Pi \) and \( \Lambda \) are defined as

\[ \Pi(\varepsilon) = \frac{\pi}{\sqrt{2}} \alpha^{-1}(\varepsilon) \left[ |I^+(\varepsilon)|^2 - |I^-(\varepsilon)|^2 \right], \]  

\[ \Lambda(\varepsilon) = \pi \sqrt{2} \alpha^{-1}(\varepsilon) \left( I^+(\varepsilon)I^-(\varepsilon) \right)^* \]  

(50)

(51)

The function \( \Pi \) is real and turns to zero for an impurity located at \( X_0 = Y_0 = 0 \); \( \Lambda(\varepsilon) \) is, generally speaking, a complex function. It is obvious that \( \Pi^2 + |\Lambda|^2 = \Theta^2 \). One can verify
that under the transformation \((\varepsilon, X_0, Y_0) \to (-\varepsilon, -Y_0, -X_0)\) the integrals \(I^+\) and \(I^-\) with an accuracy of phase factors are transformed into \(e^{\frac{\pi \varepsilon}{4}}(I^-)^*\) and \(e^{\frac{\pi \varepsilon}{4}}(I^+)^*\) respectively; with the same accuracy the conjugated values \((I^+)^*\) and \((I^-)^*\) are transformed into \(e^{-\frac{\pi \varepsilon}{4}}I^-\) and \(e^{-\frac{\pi \varepsilon}{4}}I^+\). Then the rules of transformation for the functions \(\Sigma, \Theta, \Pi,\) and \(\Lambda\) are as follows: 
\(\Pi \to -\Pi,\ \Sigma \to -\Sigma,\ \Theta \to \Theta,\ \Lambda \to \Lambda.\) Using these properties, the relation (49) can be easily proved. The meaning of this relation is that the transmittancy of the saddle point, averaged over the amplitude \(\tilde{V}_0^R\), position \((X_0, Y_0)\), and energy \(\varepsilon\) with any symmetric distribution function, is equal to \(1/2\). The averaging over \(\varepsilon\) has the meaning of averaging over the background value of the saddle point potential, \(V_{SP}\), (this value was set zero in (2)).

Our main observation is that for a symmetric distribution of the “bare” amplitudes \(\tilde{V}_0\), the distribution of renormalized amplitudes, \(\tilde{V}_0^R\), which are defined by Eq. (23), is asymmetric. The origin of the asymmetry is the LL mixing. Indeed, denote with \(\phi(\tilde{V}_0)\) the distribution function of \(\tilde{V}_0\) (we assume that \(\phi(\tilde{V}_0) = \phi(-\tilde{V}_0)\)). Then using (23) one can easily find the distribution function of \(\tilde{V}_0^R\)

\[
\tilde{\phi}(\tilde{V}_0^R) = \frac{1}{(1 - K\tilde{V}_0^R)^2} \phi \left( \frac{\tilde{V}_0^R}{1 - K\tilde{V}_0^R} \right).
\]  

(52)

The degree of asymmetry of \(\tilde{\phi}\), due to the finite renormalization constant \(K\), is determined by the product \(K\overline{V}\), where \(\overline{V}\) is the width of the distribution function \(\phi\). Consider, for example, the Lorentzian distribution: \(\phi = \pi^{-1}\overline{V}/(\overline{V}_0^2 + \overline{V}^2)\). Then \(\tilde{\phi}\) takes the form

\[
\tilde{\phi}(\tilde{V}_0^R) = \frac{\overline{V}(1 + K^2\overline{V}_0^2)}{\pi \left( (1 + K^2\overline{V}_0^2)\tilde{V}_0^R - K\overline{V}^2 \right)^2 + \overline{V}^2}.
\]

(53)

We see that \(\tilde{\phi}\) is also a Lorentzian, but it is centered at \(\tilde{V}_0^R = K\overline{V}_0^2/(1 + K^2\overline{V}^2)\). For \(K\overline{V} \ll 1\) the shift of the center, \(K\overline{V}_0^2\), is relatively small as compared to the width \(\overline{V}\). However for \(K\overline{V} \gg 1\) we get a narrow peak at \(\tilde{V}_0^R = K^{-1}\) with the width \((K^2\overline{V})^{-1} \ll K^{-1}\). This means that all the impurities, which were with equal probability repulsive and attractive before the renormalization, become effectively repulsive after the renormalization.

The asymmetry in \(\tilde{V}_0^R\) makes the average transmittancy of the saddle point smaller than \(1/2\). To demonstrate it, it is convenient to study the combination \(T(\varepsilon, X_0, Y_0) + \).
\[ T(-\varepsilon, -Y_0, -X_0) \] for the same \( \tilde{V}_0^R \). Moreover, since \( K \ll 1 \), the renormalized impurity strength \( \tilde{V}_0^R = K^{-1} \) is much larger than 1, so that in calculating this combination one can expand (49) with respect to the small parameter \( 1/\tilde{V}_0^R \). Using the transformation rules for the functions \( \Sigma, \Theta, \Pi \) and \( \Lambda \), established above, we obtain

\[ T(\varepsilon, X_0, Y_0) + T(-\varepsilon, -Y_0, -X_0) = 1 - \frac{2}{\tilde{V}_0^R (\Sigma^2 + \Theta^2)^2 \cosh \frac{\pi \varepsilon}{4}} \Delta(\varepsilon, X_0, Y_0), \] (54)

where \( \Delta(\varepsilon, X_0, Y_0) \) is defined as

\[ \Delta(\varepsilon, X_0, Y_0) = (\Theta^2 - \Sigma^2) \text{Re} \Lambda + 2 \Pi \Sigma \text{Im} \Lambda + 2 |\Lambda|^2 \Sigma \sinh \frac{\pi \varepsilon}{2}. \] (55)

We see that it is the sign of the parameter \( \Delta \) that determines the sign of deviation of the average transmittancy from \( 1/2 \).

Consider first the simplest case \( X_0 = Y_0 = 0 \). Then we have \( \Pi = 0 \), \( \text{Im} \Lambda = 0 \) and \( \text{Re} \Lambda = \Theta \). Hence, the parameter \( \Delta \) reduces to

\[ \Delta(\varepsilon, 0, 0) = \Theta \left[ (\Theta^2 - \Sigma^2) + 2 \Theta \Sigma \sinh \frac{\pi \varepsilon}{2} \right]. \] (56)

It follows from (56) that \( \Delta \) is positive if the condition \( \Theta(\varepsilon) > |\Sigma(\varepsilon)| e^{-\frac{|\varepsilon|}{2}} \) is met. Fig. 1 illustrates that this indeed is the case for any energy \( \varepsilon \).

As the displacement \( (X_0, Y_0) \) increases, the parameter \( \Delta \) falls off due to the general decay of the integrals \( I^+, I^- \). However, \( \Delta \) remains positive on average. The easiest way to see it is to consider the limit of large displacements: \( (X_0^2 + Y_0^2) \gg 1 \). In this limit \( I^+ \) and \( I^- \) acquire big phases. This results in a big phase of the function \( \Lambda \propto I^+(I^-)^* \). Consequently, \( \text{Re} \Lambda \) and \( \text{Im} \Lambda \) oscillate rapidly with the change of the impurity position (roughly as \( \sin(X_0^2 + Y_0^2) \) and \( \cos\left(\frac{X_0^2 - Y_0^2}{2}\right) \)). Thus the contributions of the first two terms in (55) average out. When considering the third term, we note that \( |\Lambda|^2 \) depends only on the magnitude of the displacement (it can be shown that \( |\Lambda| \propto (X_0^2 + Y_0^2)^{-\frac{1}{2}} e^{-\frac{X_0^2 + Y_0^2}{4}} \), i.e. \( |\Lambda|^2 \) is constant if \( X_0^2 + Y_0^2 \) is constant. The form of the function \( \Sigma \) depends on the angular position of the impurity when \( X_0^2 + Y_0^2 \) is fixed, but, averaged over the angular position, \( \Sigma \) is an odd function of energy. One can check formally that \( \Sigma(\varepsilon, X_0, Y_0) + \Sigma(\varepsilon, Y_0, X_0) \) changes sign when \( \varepsilon \) changes sign. Therefore,
the product $\Sigma \sinh \frac{\pi \varepsilon}{2}$ is positive on average. Thus we conclude that average $\Delta$ is positive both for small and large displacements, and, consequently, the average transmittance of the saddle point is diminished due to the impurity–induced LL mixing.

VI. CONCLUSION

In order to compensate for the reduction of the average transmittancy, the energy position of the delocalized state in a smooth potential shifts up with respect to the center of LL. Let us estimate this shift (levitation). Denote with $n$ the concentration of the impurities. Clearly, to produce a significant effect on the transmittancy, the impurity should be located within the interval of the order of magnetic length from the center of the saddle point ($X_0 \sim Y_0 \sim 1$). The probability to find such an impurity is $\sim n l^2$. For $X_0 \sim Y_0 \sim 1$ we have $\Sigma \sim \Theta \sim \Delta \sim 1$ in Eq. (54). Thus, the magnitude of the reduction of the average transmittancy is of the order of $(\tilde{V}_0^R)^{-1}$. On the other hand, we have established that if an impurity is strong enough, the LL mixing renormalizes its amplitude to $V_0^R = K^{-1}$. As a result, in the presence of impurities, the delocalized state corresponds to such an energy for which, in the absence of impurities, the average transmittancy of the saddle point exceeds $1/2$ by $\sim n l^2 K$. Since without an impurity the transmission coefficient $T_0$ has the energy scale $E_1$, determined by Eq. (8), we get the following estimate for the magnitude of the levitation $\delta E$

$$\frac{\delta E}{E_1} \sim n l^2 K.$$  \hspace{1cm} (57)

It is instructive to compare the levitation with the spacing, $\hbar \omega_c$, between the LL’s. One gets

$$\frac{\delta E}{\hbar \omega_c} \sim n l^2 K \frac{E_1}{\hbar \omega_c} \sim n l^2 \left( \frac{E_1}{\hbar \omega_c} \right)^2 \ln \frac{l}{a} \sim n l^2 \left( \frac{\Omega}{\omega_c} \right)^4 \ln \frac{l}{a},$$ \hspace{1cm} (58)

where we have used the expression (23) for $K$. In our consideration we have assumed that there is only a single impurity near the the saddle point, which implies that $n l^2 \ll 1$. We have also assumed that $\Omega \ll \omega_c$. Then the above estimate shows that under the assumptions
adopted the relative levitation is small. It also shows that the magnetic field dependence of the levitation is $\delta E \propto B^{-4}$.

In numerical simulations the authors restricted the study to the two lowest LL’s. They found that, due to the LL mixing, the lower DS shifts up whereas the upper DS shifts down in energy (“attraction” of DS’s). Our result is consistent with this observation. Indeed, if only two LL’s are considered, the renormalization constant, $K$, would be positive for the lower level and negative for the upper level.

Note in conclusion, that our main result – the reduction of the transmittancy due to the LL mixing, was derived in the limit of the strong renormalization of the impurity potential. The criterion for that, formulated in the previous Section, is $K \nabla > 1$, $\nabla$ being the typical value of the dimensionless potential $\tilde{V}_0$. Using Eqs. (25) and (17) this criterion can be rewritten in terms of the “bare” amplitude and size of the impurity potential as $V_0 a^2 \ln \frac{1}{a} > \hbar \omega_c l^2 \sim \hbar^2/m$. On the other hand, in our approach we have treated the impurity as point-like and, thus, neglected the change of the electron wave function within the radius $a$. This is justified if the condition $V_0 < \hbar^2/ma^2$ is met. The second condition seems to restrict the validity of the theory to the region $1 > \frac{V_0}{(\hbar^2/ma^2)} > 1/\ln \frac{1}{a}$. It appears, however, that the condition $\frac{V_0}{(\hbar^2/ma^2)} < 1$ is not relevant. We address this question in the Appendix and show that as soon as $V_0 > \frac{\hbar^2}{ma^2} \ln \frac{1}{a}$, it renormalizes to $\frac{\hbar^2}{ma^2} \ln \frac{1}{a}$ (which is equivalent to $\tilde{V}_0^R = K^{-1}$), as we have assumed.

APPENDIX:

Since the presence of the saddle-point potential is not important for the renormalization procedure, we will trace the renormalization for the case when $V_{SP}$ is absent. In this case the effect of an impurity is just the formation of the bound states which split off the LL’s. In the symmetric gauge with an impurity at the origin, the Schrödinger equation allows the separation of variables. The radial wave function, $R(\rho)$, for a zero orbital moment, satisfies the equation
\[
\frac{d^2 R}{d \rho^2} + \frac{1}{\rho} \frac{d R}{d \rho} + \left[ \frac{2m}{\hbar^2} (E - V_0 F(\rho)) - \frac{m^2 \omega_c^2}{4 \hbar^2 \rho^2} \right] R = 0. \tag{A1}
\]

We will analyze the solutions of (A1), corresponding to high LL’s, for which the calculations can be performed semiclassically without invoking the hypergeometric function. Then the Bohr-Sommerfeld condition for the energy levels, \( E_p \), reads

\[
E_p = \hbar \omega_c \left( p + \frac{1}{2} - \frac{\varphi_p}{\pi} \right), \tag{A2}
\]

where \( \varphi_p \) is an additional phase shift acquired at the origin (\( \varphi_p = 0 \) for \( V_0 = 0 \)). This shift should be found by matching the solutions at \( \rho < a \) and at \( \rho > a \). Important is that for large \( p \) the magnetic potential, \( m\omega_c^2 \rho^2 / 8 \), in (A1) comes into play only at large \( \rho \sim lp^{1/2} \). For smaller \( \rho \) one can neglect the magnetic potential. Then the solution of (A1) (which is finite at \( \rho = 0 \)) can be written as

\[
R(\rho) = J_0(q\rho), \quad \rho < a
\]

\[
R(\rho) = \nu_1 J_0(k\rho) + \nu_2 N_0(k\rho), \quad \rho > a, \tag{A3}
\]

where \( J_0 \) and \( N_0 \) are, respectively, the Bessel and the Neumann functions of order zero; \( k = (2mE_p)^{1/2}/\hbar \), and \( q = (2m(E_p - V_0))^{1/2}/\hbar \). The constants \( \nu_1, \nu_2 \) determine the phase shift \( \varphi_p \). Indeed, at \( k\rho \gg 1 \) the functions \( J_0 \) and \( N_0 \) oscillate as \( \sin(k\rho - \frac{\pi}{4}) \) and \( \cos(k\rho - \frac{\pi}{4}) \), so that \( \varphi_p = \arctan(\nu_2/\nu_1) \). The continuity conditions for \( R(\rho) \) and \( dR/d\rho \) at \( \rho = a \) can be written as

\[
J_0(qa) = \nu_1 J_0(ka) + \nu_2 N_0(ka),
\]

\[
qJ_1(qa) = \nu_1 kJ_1(ka) + \nu_2 kN_1(ka). \tag{A4}
\]

Solving this system yields the following expression for \( \varphi_p \)

\[
\tan \varphi_p = \frac{J_0(ka)}{J_0(qa)} \frac{qJ_1(qa) - kJ_1(ka)}{qJ_0(qa)N_0(ka) - kN_1(ka)}. \tag{A5}
\]

The case we are interested in is \( ka \ll 1 \). This allows to use the small–argument asymptotics’ for the functions depending on \( ka \). One gets
\[
\tan \varphi_p = \frac{\pi}{4} \left[ \frac{2qa J_1(qa) - k^2a^2}{1 + qa \frac{J_1(qa)}{J_0(qa)} \ln \frac{2}{\gamma ka}} \right],
\]

where \(\gamma\) is the Euler constant. In the limit of a point-like impurity, considered by Gredeskul and Azbel, we have \(V_0 < \hbar^2/ma^2\) (which is equivalent to \(qa < 1\)), and Eq. (A6) simplifies to

\[
\tan \varphi_p = \frac{\pi ma^2}{2\hbar^2} \left[ \frac{V_0}{1 + \frac{ma^2}{\hbar^2}(V_0 - E_p) \ln \frac{2}{\gamma ka}} \right].
\]

We see that \(V_0\) enters into the phase shift in the same “renormalized” form as in Eq. (23) for \(\tilde{V}_0^R\). If the renormalization is weak, we get \(\varphi_p = -\frac{\pi ma^2}{2\hbar^2} V_0\), which leads to the standard result, \(V_0a^2/2\ell^2\), for the binding energy. As \(V_0\) exceeds \(\hbar^2/ma^2\ln \frac{2}{\gamma ka}\) (but remains smaller than \(\hbar^2/ma^2\)) we get from (A7)

\[
\varphi_p = -\frac{\pi}{2 \ln \frac{2}{\gamma ka}},
\]

so that \(V_0\) disappears from the phase shift and, correspondingly, from the binding energy. This is equivalent to the conclusion that \(\tilde{V}_0^R\) renormalizes to \(K^{-1}\).

Most importantly, Eq. (A8) remains valid when \(V_0\) gets much larger than \(\hbar^2/ma^2\), and the approach, adopted in Sec. II, is not applicable any more. Indeed, as \(qa\) increases, the combination \(qa \frac{J_1(qa)}{J_0(qa)}\) becomes either oscillating (for \(V_0 < 0\)) or monotonously increasing (for \(V_0 > 0\)) function with a typical magnitude much larger than unity. Then the result (A8) immediately follows from (A7).
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FIGURES

FIG. 1. Dimensionless functions $\Theta$ (solid curve) and $\Sigma$ (long-dashed curve) are shown versus the dimensionless energy $\varepsilon$ for the impurity position right at the center of the saddle point. The dotted curve represents the ratio $|\Sigma|e^{-\frac{1}{2\sqrt{2}}}/\Theta$.

FIG. 2. The transmission coefficient as a function of energy for the case $X_0 = Y_0 = 0$ is shown for different values of the renormalized impurity potential $\tilde{V}_0^R$: $\tilde{V}_0^R = -5$ (solid curve), $\tilde{V}_0^R = -3$ (long-dashed curve), and $\tilde{V}_0^R = -1$ (dashed-dotted curve).

FIG. 3. The transmission coefficient at zero energy is shown as a function of the dimensionless impurity potential $v = \tilde{V}_0^R|\Gamma(1/4)|^2/4(2\pi)^{1/2}$.
