Unknotting numbers of 2-spheres in the 4-sphere

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Abstract

We compare two naturally arising notions of ‘unknotting number’ for 2-spheres in the 4-sphere: namely, the minimal number of 1-handle stabilizations needed to obtain an unknotted surface, and the minimal number of Whitney moves required in a regular homotopy to the unknotted 2-sphere. We refer to these invariants as the stabilization number and the Casson–Whitney number of the sphere, respectively. Using both algebraic and geometric techniques, we show that the stabilization number is bounded above by one more than the Casson–Whitney number. We also provide explicit families of spheres for which these invariants are equal, as well as families for which they are distinct. Furthermore, we give additional bounds for both invariants, concrete examples of their non-additivity, and applications to classical unknotting number of 1-knots.

1. Introduction and motivation

This paper compares and relates a slew of algebraic and geometric measures of complexity of 2-knots in the 4-sphere. What has traditionally been called the ‘unknotting number’ of a 2-knot \( K \subset S^4 \), which we call the stabilization number \( u_{st}(K) \), records the minimal number of stabilizations of \( K \) required to obtain a smoothly embedded surface that bounds a solid handlebody \([14]\). This is analogous to the minimal number of 1-dimensional stabilizations (that is, band attachments) of a 1-knot needed to obtain an unlink. This is bounded above by, but is not in general equal to, the classical unknotting number: indeed there are many examples of low-crossing knots for which this inequality is strict.

The classical unknotting number of a 1-knot embedded in the 3-sphere records the minimal number of double points that occur during any regular homotopy to the unknot. The analogue we consider in the 4-dimensional setting is the minimal number of Whitney moves needed in a regular homotopy taking a 2-knot \( K \) to the unknot (double points are introduced/removed by a finger move/Whitney move). We call the minimal number of Whitney moves the Casson–Whitney number \( u_{cw}(K) \) of the knot \( K \), since techniques for manipulating finger moves (the inverse homotopy to a Whitney move) were pioneered by Casson \([6]\). In Section 3, we use recent results of \([33]\) to obtain the following relationship between the stabilization number and the Casson–Whitney number.

**Theorem 1.1.** For any 2-knot \( K \), \( u_{st}(K) \leq u_{cw}(K) + 1 \).

A careful manipulation of simple regular homotopies to the unknot in Section 4 also gives settings in which this inequality is always strict.

**Theorem 1.2.** Any 2-knot \( K \) with \( u_{cw}(K) = 1 \) also has \( u_{st}(K) = 1 \).
Moreover, by considering the effect of finger moves and stabilizations on the fundamental group of the complement, we are able to find examples of 2-knots for which equality of these unknotting invariants does not hold.

**Theorem 1.3.** There are infinitely many 2-knots $K$ with $u_{st}(K) = 1$ and $u_{cw}(K) = 2$.

In Section 4, we also give some special families of 2-knots in which we can bound both the stabilization number and the Casson–Whitney number from above, using explicit geometric constructions. For instance, we find that the fusion number of a ribbon 2-knot is an upper bound for the Casson–Whitney number.

**Theorem 1.4.** For a ribbon 2-knot $K$, $u_{cw}(K) \leq \text{fus}(K)$.

The analogous result for the stabilization number $u_{st}(K) \leq \text{fus}(K)$ is due to Miyazaki [24]. In Section 5, we develop the algebraic Casson–Whitney number $a_{cw}$, a natural lower bound for $u_{cw}$, and prove that for a pair of 2-knots, both admitting a Fox coloring, the Casson–Whitney number of their connected sum must be at least 2. A Fox coloring of a 2-knot is a surjection from the fundamental group of its complement onto a dihedral group, which sends meridians of the 2-knot to reflections. The determinant of a 2-knot is the evaluation of its Alexander ideal $\Delta(K)$ at $t = -1$, and as in the classical case a 2-knot $K$ has a $p$-coloring for prime $p$ if and only if $p$ divides its determinant $\Delta(K)|_{-1}$ [15]. Thus, Casson–Whitney number one 2-knots cannot be factored into a connected sum of two 2-knots, each with non-trivial determinant (cf. the result of Scharlemann that unknotting number one 1-knots are prime [30]).

**Theorem 1.5.** Let $K_1, K_2$ be 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$. Then $u_{cw}(K_1 \# K_2) \geq 2$.

Miyazaki found 2-knots $K_1, K_2$ with $u_{st}(K_i) = 1$ but $u_{st}(K_1 \# K_2) = 1$ as well [24]. Since his examples have non-trivial determinants, these examples together with the above theorem imply Theorem 1.3. The non-additivity of both $u_{st}$ and $u_{cw}$ is discussed in Section 6, where we provide explicit families of 2-knots for which additivity fails by an arbitrarily large amount.

**Theorem 1.6.** For any positive $c, n \in \mathbb{N}$, there exist 2-knots $K_1, \ldots, K_n$ with $u_{st}(K_i) = u_{cw}(K_i) = c$, $c \leq u_{st}(K_1 \# \cdots \# K_n) \leq 2c$, and $c \leq u_{cw}(K_1 \# \cdots \# K_n) \leq 2c$.

In the final section, we suggest some possible directions for further study. Recently, the relationship between two similar invariants $d_{st}$ and $d_{\text{sing}}$ was studied by Singh [33] (the invariant $d_{\text{sing}}$ already appeared as $\mu_{\text{sing}}$ in [16]). His invariants record the minimal ‘width’ of a sequence of stabilizations and destabilizations of a regular homotopy, meaning the maximum number of stabilizations or double points that occur simultaneously. The invariants we consider, on the other hand, record the minimal ‘length’ of a sequence of stabilizations and destabilizations of a regular homotopy, meaning the total number of stabilizations or double points that occur overall. Many of the geometric techniques used in our arguments are inspired by those of Singh, as well as both Gabai [12] and Schneiderman and Teichner [31]. A recent paper of Miller and Powell [23] also studies the stabilization distance between arbitrary surfaces in $S^4$, as well as the related relative setting of properly embedded surfaces in $B^4$. 
All manifolds and maps will be smooth unless stated otherwise. All (ambient) manifolds will also be assumed to be connected, and both manifolds and surfaces will be both compact and orientable.

2. Background

For all definitions below, let $S$ be a smoothly immersed surface in $S^4$. We use the shorthand $\pi S := \pi_1(S^4 - N(S), *)$ for the fundamental group of the complement (of a neighborhood $N(S)$ of $S$), where a basepoint $*$ is understood. This section will mainly be spent analyzing the algebraic impact of the geometric operations we will be interested in.

2.1. The stabilization number

**Definition 2.1.** Suppose $\alpha$ is an arc with interior embedded in $S^4 - S$, whose endpoints lie on the surface $S$. The normal bundle of $\alpha$ in $S^4$ contains an embedded copy of $D^2 \times I$ intersecting $S$ in exactly $D^2 \times \partial I$ such that the surface $(S - (D^2 \times \partial I)) \cup (\partial D^2 \times I)$ is orientable. This resulting surface is called the stabilization of $S$ along $\alpha$. We call $\alpha$ the guiding arc for the stabilization.

Observe that, as suggested by our terminology, the isotopy class of the stabilization depends only on the guiding arc $\alpha$ and not on the choice of sub-bundle $D^2 \times I$ (see Remark 2.5 for a similar discussion and [12, Remark 5.3] for more detail). Since guiding arcs with the same endpoints that are homotopic rel boundary are also isotopic rel boundary in dimension 4, the stabilizations along these arcs are isotopic.

**Definition 2.2.** A smoothly unknotted surface in $S^4$ is a smoothly embedded surface of any genus that bounds a smoothly embedded solid handlebody.

Indeed, since $S^4$ is simply connected, the cores of the 1-handles of any pair of handlebodies of the same genus are isotopic. This can be used to guide an isotopy to show that there is a unique unknotted surface of each genus.

Any closed orientable surface $K$ in the 4-sphere is smoothly isotopic to an unknotted surface after a finite number of stabilizations. To see this, note that such a surface $K$ bounds a smooth embedded 3-manifold $M \subset S^4$ called its Seifert solid which can be built as a handlebody from $K \times I$ by attaching 1-handles to $K \times \{1\}$, followed by 2 and 3-handles. Performing stabilizations to $K$ along the core arcs of the 1-handles of $M$ gives a surface $K'$ that bounds the solid handlebody consisting of the 2 and 3-handles of $M$, and so by definition is unknotted.

**Definition 2.3.** The stabilization number $u_{st}(K)$ of a 2-knot $K$ is the minimal number of 1-handle stabilizations needed to obtain an unknotted surface.

2.2. The Casson–Whitney number

By Smale [34, Theorem D] and Hirsch [13, Theorem 8.3], embedded surfaces in a orientable 4-manifold are homotopic if and only if they are regularly homotopic, that is, homotopic through immersions. Generically, there are only finitely many times during a regular homotopy at which the immersed sphere is not self-transverse — at these times, double points of opposite sign are either introduced or cancelled.

\[\text{With corners smoothed, as in Figure 1.}\]
Definition 2.4. The local model for the regular homotopy removing pairs of double points is called a Whitney move; this homotopy is supported in a regular neighborhood of a Whitney disk $W$. The inverse to this homotopy is called a finger move, which is supported in a regular neighborhood of an arc $\alpha$ whose endpoints lie on the surface, and whose interior is embedded in the complement. We call the arc $\alpha$ a guiding arc for the finger move (the analog in this context of the guiding arc from Definition 2.1). These two homotopies are depicted in Figure 2. Also labeled in the figure are the Whitney arcs $\omega_1$ and $\omega_2$, whose union is the boundary of the Whitney disk $W$. Each Whitney arc connects a pair of double points along a sheet of the immersed surface.

Remark 2.5. To explicitly define a Whitney move in local coordinates requires that the normal disk bundle of the Whitney disk be framed ‘compatibly’ with respect to its boundary on the immersed sphere; refer to [11] as well as Casson’s lectures in [6] for more details. Likewise, a framing of the normal $B^3$-bundle of the guiding arc compatible with the surface at its endpoints is needed to explicitly define a finger move. The choice of framing for the guiding arc will be suppressed, however, since (as with stabilizations) the resulting immersed...
surface up to ambient isotopy is independent of this choice of framing and depends only on
the homotopy class of the arc itself, rel boundary (see the discussion in [12, Remark 5.3] for
instance).

From now on, we shall always consider generic regular homotopies, in the sense that they
are compositions of finger and Whitney moves as in Definition 2.4. In fact, since the guiding
arcs of the finger moves can be isotoped away from the Whitney disks in the ambient 4-
manifold, a deformation of the homotopy (without increasing the number of finger and Whitney
moves) arranges for all of the finger moves to occur first, and simultaneously, followed by
all of the Whitney moves. A more detailed discussion of this folklore fact can be found in
[26, Section 4.1]. We will also always assume that our regular homotopies are of this
form.

**Definition 2.6.** The length of a regular homotopy between surfaces is its total number
of finger moves, or equivalently, Whitney moves. The Casson–Whitney number $u_{cw}(K)$ of a
2-knot $K$ is the minimal length of any regular homotopy from $K$ to the unknot.

In general, finger moves (like stabilizations) depend on the choice of guiding arc up to
homotopy rel boundary. Namely, if two guiding arcs are homotopic and hence isotopic
rel endpoints, then performing finger moves along these arcs results in immersions that
are ambiently isotopic in $S^4$. In particular, it is critical to many of our arguments that
all guiding arcs — and hence finger moves — are isotopic in the complement of the
unknot $U$.

**Definition 2.7.** We call the result of performing $n$ finger moves on the unknot $U$ the
standard immersed sphere with $2n$ double points. Often, we reserve the use of $\Sigma$ to denote
this immersion.

We later observe that there is indeed a unique standard immersed sphere for each $n$, up to
ambient isotopy of $S^4$.

**Definition 2.8.** After the finger moves and before the Whitney moves, any regular
homotopy from a 2-knot $K$ to the unknot $U$ restricts to the standard immersion $\Sigma$.

\[
\begin{array}{c}
\text{2-knot } K \xrightarrow{\text{finger moves}} \text{standard immersion } \Sigma \xrightarrow{\text{Whitney moves}} \text{unknot } U \\
\end{array}
\]

Therefore, a regular homotopy from a 2-knot $K$ to the unknot $U$ is given by two collections
of Whitney disks that pair the double points of the standard immersion: a set of standard
Whitney disks leading to the unknot $U$, and a set of knotted Whitney disks leading to the
knot $K$, as illustrated in Figure 3.

### 2.3. Fundamental group calculations

Below, we describe the effects of finger moves and stabilizations on $\pi_1$ of the complement of a
(possibly immersed) surface $S = S_1 \cup \cdots \cup S_n$. In particular, each move introduces one relation
to $\pi_1$ as stated in the results below. For detailed proofs, we refer to the original sources [5, 6, 20].

Begin by picking a basepoint $*$ in the complement of the link $S$ and a basepoint $*_i \in S_i$ on
each component of $S$. For each $i$, fix an arc $\rho_i$ with interior in $S^4 - S$ connecting the basepoint
$*$ to the basepoint $*_i$. 
Figure 3 (colour online). Decomposing a regular homotopy from a 2-knot $K$ to the unknot $U$. The standard immersed sphere $\Sigma$ obtained after the finger moves (FM) and before Whitney moves (WM) on $K$ is drawn from two different perspectives (middle left and middle right) to show the knotted and standard Whitney disks (red and blue, respectively).

Figure 4 (colour online). A choice of pushoff in gray, giving an element of $\pi S$ corresponding to the guiding arc $\alpha$. Also pictured are unbased meridians $m_i$ for the components $S_i$.

**Definition 2.9.** A meridian of $S$, and more specifically of the component $S_i$, is an element of $\pi S$ that can be represented by a simple closed curve $\gamma : S^1 \to S^4 - S$ bounding a disk in $S^4$ that transversely intersects $S_i$ in a single point.

An orientation of $S$ and the ambient space induces a positive orientation on the meridian. The set of positively oriented meridians of a connected component of a knotted surface forms a conjugacy class of the fundamental group of its complement. That is, $x$ is a meridian of $S_i$ if and only if $x^w := w^{-1}xw$ is as well, for any $w \in \pi S$. If $S$ is connected, this element $w$ may be chosen to lie in the commutator subgroup $(\pi S)'$, a fact which we exploit in Section 5.

**Definition 2.10.** Let $\alpha$ be an arc with interior embedded away from $S$, connecting $*_i$ to $*_j$ for (possibly equal) indices $i, j$. We call this a guiding arc for $S$, as we did without specifying base points in Definitions 2.1 and 2.4. Each push-off of the loop $\rho_i \alpha \rho_j^{-1}$ into the complement of the surface $S$ gives an element $g \in \pi S$ that is said to correspond to the arc $\alpha$, see Figure 4. Note that the element $g$ is well defined (that is, independent of the push-off) up to left multiplication by meridians of $S_i$ and right multiplication by meridians of $S_j$.

From now on, we will always assume that the guiding arcs used for both stabilizations and finger moves are of this form, that is, connecting a basepoint $*_i \in S_i$ to a basepoint $*_j \in S_j$ for
some (possibly equal) indices \( i, j \). We often refer to arcs corresponding to the identity element, as well as stabilizations and finger moves done along such a guiding arc, as ‘trivial’.

**Remark 2.11.** Let \( \alpha \) and \( \beta \) be guiding arcs for \( S \) with the same endpoints \( *_{i} \) on \( S_{i} \) and \( *_{j} \) on \( S_{j} \). Suppose that \( \alpha \) corresponds to an element \( g \in \pi S \) and \( \beta \) corresponds to \( m_{1}^{n_{1}} gm_{2}^{n_{2}} \) for some \( n_{1}, n_{2} \in \mathbb{Z} \) and meridians \( m_{i}, m_{j} \) to \( S_{i}, S_{j} \), respectively. Then, the guiding arcs \( \alpha \) and \( \beta \) are isotopic in the complement of \( S \) rel boundary via a sequence of ‘boundary twists’ as pictured in Figure 5. It follows that the surfaces obtained by either stabilizing or performing finger moves along these arcs are ambiently isotopic.

In particular, it follows from Remark 2.11 that all guiding arcs for the unknot \( U \) are isotopic, since \( \pi U \cong \mathbb{Z} \). So for any \( n > 0 \), there is a unique surface (up to isotopy) resulting from \( n \) stabilizations of \( U \) — namely, the genus \( n \) unknotted surface, as in Definition 2.2. Similarly, the immersed sphere resulting from \( n \) finger moves on \( U \) is ambiently isotopic to the standard immersion with \( 2n \) double points, as in Definition 2.7.

**Lemma 2.12 (Stabilization relation).** Let \( \alpha_{1}, \ldots, \alpha_{k} \) be disjointly embedded guiding arcs along which stabilizing \( S \) gives the surface \( S' \). Then

\[
\pi(S') \cong \pi S \left/ \langle \langle g_{i}^{-1} a_{i} g_{i} b_{1} \rangle \rangle \right.,
\]

where \( a_{i}, b_{1} \) are meridians to the components of \( S \) containing the endpoints of \( \alpha_{i} \) (as in Definition 2.9), and the element \( g_{i} \) corresponds to \( \alpha_{i} \) (as in Definition 2.10).

Refer to Figure 6(a) for a schematic of the set-up in Lemma 2.12. Note that each \( g_{i}^{-1} a_{i} g_{i} \) is also a meridian; hence the relation introduced by stabilizing can also be thought of as one which simply identifies two meridians. We make the following definitions for \( n \)-knots, because we will think about them in reference to 1-knots as well as 2-knots.

**Definition 2.13.** Let \( K \) be an \( n \)-knot. The minimal number of relations of the form \( x = y \), where \( x, y \) are meridians of \( K \), which abelianize the knot group is called the **algebraic stabilization number** \( a_{\text{st}}(K) \) of \( K \).

**Lemma 2.14 (Finger move relation).** Suppose that \( S' \) is the result of performing finger moves on \( S \) along disjointly embedded guiding arcs \( \alpha_{1}, \ldots, \alpha_{k} \). Then

\[
\pi(S') \cong \pi S \left/ \langle \langle [a_{i}, g_{i}^{-1} b_{i} g_{i}] \rangle \rangle \right.,
\]
where $a_i, b_i$ are meridians to the components of $S$ containing the endpoints of $\alpha_i$ (as in Definition 2.9), and the element $g_i$ corresponds to $\alpha_i$ (as in Definition 2.10).

Figure 6(b) gives a schematic of the set-up in Lemma 2.14. Note that while the stabilization relation identifies two meridians, a finger move relation can only make them commute. This discrepancy leads to our result in Section 5 that the stabilization and Casson–Whitney numbers are not equal in general.

**Definition 2.15.** Let $K$ be an $n$-knot. The minimal number of relations of the form $xy = yx$ which abelianize the knot group, where $x, y$ are meridians of $K$, is called the algebraic Casson–Whitney number $a_{cw}(K)$ of $K$.

This minimum gives an algebraic lower bound for $u_{cw}(K)$, since a regular homotopy from a 2-knot $K$ to the unknot starts with a sequence of finger moves on $K$ to the standard immersion $\Sigma$ with $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$; thus the corresponding finger move relations abelianize $\pi K$.

We summarize the results of this section in the following proposition. To our knowledge, these are the sharpest algebraic lower bounds for the unknotting numbers.

**Proposition 2.16.** For any 2-knot $K$, $a_{st}(K) \leq u_{st}(K)$ and $a_{cw}(K) \leq u_{cw}(K)$.

Table 1 gives a glossary of the main invariants that will be referred to.

### Table 1. Overview of main invariants for a 2-knot $K$

| Symbol | Description |
|--------|-------------|
| $\pi K$ | knot group/surface group |
| $\mu(K)$ | meridional rank of $K$ |
| $\mu(K)$ | Nakanishi index |
| $a(K)$ | Ma–Qiu index |
| $u_{st}(K)$ | stabilization number |
| $u_{cw}(K)$ | Casson–Whitney number |
| $a_{st}(K)$ | algebr. stabilization number |
| $a_{cw}(K)$ | algebr. Casson–Whitney number |
| $fus(K)$ | fusion number of a ribbon knot |
| $\pi K$ | fundamental group of knot complement $S^4 - K$ |
| $\mu(K)$ | minimal number of meridians which generate $\pi K$ |
| $\mu(K)$ | minimal size of generating set of Alexander module of $K$ |
| $a(K)$ | minimal size of normal generating set of commutator subgroup $(\pi K)'$ |
| $u_{st}(K)$ | minimal number of 1-handle stabilizations needed to obtain an unknotted surface from $K$ |
| $u_{cw}(K)$ | minimal number of Whitney moves in a regular homotopy from $K$ to the unknot |
| $a_{st}(K)$ | minimal number of 1-handle stabilizations on $K$ needed to obtain a surface with group $\mathbb{Z}$ |
| $a_{cw}(K)$ | minimal number of finger moves on $K$ needed to obtain an immersed 2-knot with group $\mathbb{Z}$ |
| $fus(K)$ | minimal number of fusion tubes in a ribbon presentation for $K$ |
Figure 7 (colour online). In both figures (A) and (B), oriented meridians $x$ and $y$ to the surface are drawn in pink and blue, and the basepoint in green. The gray annuli (immersed in the complement of the surface) are null-homotopies giving the algebraic relations from Lemma 2.12 and Lemma 2.14. On the right, the image of the grey annulus is exactly the Clifford torus around the double point, illustrating that the commutator relation $[x, y] = 1$ holds.

Figure 8 (colour online). Tubing an immersed surface along an arc $\alpha$ (red) that connects oppositely signed double points. Sheets of the surface are drawn in pink and blue; the pink sheet is an arc persisting into the past and future. To tube the double points together, remove a disk in the blue sheet around each double point and add the linking annulus of the guiding arc $\alpha$ as shown on the right.

3. Relating the stabilization and Casson–Whitney numbers

Fix a 2-knot $K \subset S^4$ and let $U \subset S^4$ denote the unknotted 2-sphere. We begin by introducing some terminology needed only in this section.

**Definition 3.1.** Given an immersed surface $\Sigma$ in $S^4$ with algebraically zero double points, and any choice of disjointly embedded arcs on $\Sigma$ pairing double points of opposite sign, there is an associated tubed surface obtained by tubing the double points along these arcs as in Figure 8. Note that this surface is oriented, since the endpoints of each arc are double points of opposite orientation, and that a priori the smooth (and even topological) isotopy class of the resulting surface depends on the arcs along which the tubing is done.

Although we choose not to define it rigorously here, the procedure of ‘tubing’ employed in the definition above is described in Remark 5.3 of [12], as well as in Definition 2.6 of [33]. Indeed, isotopies between associated tubed surfaces are the focus of both papers. To ensure that the associated tubed surfaces that arise in our discussions are isotopic, we will be especially interested in regular homotopies of the following type.

**Definition 3.2.** A regular homotopy of length $n$ from $K$ to $U$ in $S^4$ is called arc-standard if its standard Whitney disks $W_1, \ldots, W_n$ and knotted Whitney disks $V_1, \ldots, V_n$ have at least one Whitney arc in common for each $i$. 


Figure 9 (colour online). ‘Standard pictures’ of the spheres $U$ and $K$ (top right and left) given by Whitney moves on the immersion $\Sigma$, and their isotopic stabilizations $F_U$ and $F_K$, along the red or blue guiding arcs (lower right and left). Note that although the local models of $U$ and $K$ after the Whitney moves look identical, the interiors of the Whitney disks $W_i$ and $V_i$, and hence these portions of $U$ and $K$, may be embedded very differently in $S^4$.

Remark 3.3. It is unknown whether every 2-knot in $S^4$ admits an arc-standard regular homotopy to the unknot, let alone one of minimal length. There are many non-simply connected 4-manifolds containing pairs of 2-spheres between which there is no analog of an arc-standard homotopy. For instance, any pair of spheres related by such a homotopy must have vanishing Freedman–Quinn invariant\footnote{This concordance invariant was defined by Freedman and Quinn [11] in the 1990s, and later corrected by Stong [36]. Schneiderman and Teichner give a nice exposition in [31].} since in this case all double curves of the trace of the homotopy are trivially double covered. However, there are many instances where this does not hold — see, for example, [12, 31], or [32].

With this terminology in place, we state our first result. Although this fact is implied by Singh’s proof of Theorem 1.4 in [33], we state and prove it here in our setting.

Proposition 3.4. If there is a length $n$ arc-standard homotopy from $K$ to $U$, then $K$ can be unknotted with $n$ stabilizations.

Proof. Such a regular homotopy is given by a set of standard Whitney disks $W_1, \ldots, W_n$ and knotted Whitney disks $V_1, \ldots, V_n$ for the standard immersion $\Sigma$ with $2n$ double points, as in Definition 2.7 and Definition 2.8. Since the regular homotopy is arc-standard, by definition for each $i$ the standard and knotted Whitney disks have at least one common Whitney arc $\alpha_i$.

The end of the Whitney homotopy for each Whitney disk $W_i$ and $V_i$ gives a ‘local model’ of the resulting embedded 2-sphere, as illustrated in the top left and right of Figure 9. Stabilizing $K$ along guiding arcs connecting the sheets of $K$ parallel to each knotted Whitney disk gives an embedded genus $n$ surface $F_K$ shown on the bottom left of Figure 9. Likewise, stabilizing the unknot $U$ along guiding arcs connecting the sheets of $U$ parallel to each standard Whitney disk gives a genus $n$ standard surface $F_U$ shown on the bottom right of Figure 9. Both $F_K$ and $F_U$ are isotopic to the associated tubed surface $\Sigma$ stabilized along the common Whitney arcs $\alpha_1, \ldots, \alpha_n$, depicted on the bottom center of Figure 9 (the tube along the Whitney arc $\alpha_i$ is shown in green). Since the surface $F_U$ is a stabilization of the unknot, it follows that $F_U$ and hence $F_K$ is unknotted. $\square$
The following $\pi_1$ calculation is used in the proof of Theorem 1.1 and is very similar to Casson’s proof of Lemma 2.14, see [6]. Before stating the lemma, we establish some necessary notation. For any 4-manifold $X$, suppose that $S \subset X$ is an immersed surface with positive and negative double points $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, respectively. For each $i$, let $\alpha_i \subset S$ be an embedded arc connecting $p_i$ to $q_i$, and let $a_i, b_i \in \pi S$ be positively oriented meridians for the sheets of the double points that do not contain $\alpha_i$. As illustrated in Figure 10, the meridian $a_i$ is constructed by running along $\rho_i$, around the boundary of a disk normal to $S$, and back along $\rho_i^{-1}$ to the base point $\ast$ of $\pi_1 S$. The meridian $b_i$ is constructed analogously, but using the path $\eta_i$. The arc $\alpha_i \subset S$ corresponds to an element $g_i \in \pi S$ given by the composition of paths $\rho_i \alpha_i \eta_i^{-1}$, where $\alpha_i'$ is a push-off of the arc $\alpha_i$ into the normal disk bundle of $S$ — this element is hence well defined only up to twists around the normal disk bundle of $S$ restricted to $\alpha_i$. However, as this indeterminacy does not affect the fundamental group calculations below†, we suppress it from notation.

Lemma 3.5 (Tubing relation). Let $S \subset X^4$ be an immersed surface whose associated tubed surface $S'$ is constructed by tubing together oppositely signed double points $p_i$ and $q_i$ along arcs $\alpha_i \subset S$, as in Definition 3.1. Then,

$$\pi(S') \cong \pi S / \langle \langle g_i^{-1} a_i g_i b_i^{-1} \rangle \rangle$$

for elements $a_i, b_i, g_i \in \pi S$ defined both in the paragraph above and illustrated in Figure 10.

Proof. For each $i$, consider a disk $D_i$ normal to $S$ at an interior point of the arc $\alpha_i$, as in Figure 11. The intersection of this disk with the tubed surface $S'$ then consists of $\partial D_i$, together with the point where $\alpha_i$ intersects $D_i$. Note that the complement of $S' \cup D_1 \cup \cdots \cup D_n$ is homotopy equivalent to the complement of the immersion $S$. So, to compare $\pi(S')$ and $\pi S$, we remove the regular neighborhoods of each disk $D_i$ from $X - S'$ to obtain $X - S$ in two stages: first we delete neighborhoods of embedded arcs $\gamma_i \subset D_i$ connecting $\alpha_i$ to $\partial D_i$, and then delete neighborhoods of the remaining disks $D_i' \subset D_i$ whose boundary circles run around $\partial D_i$ and then forward and back along $\gamma_i$, as in Figure 11.

†This follows since the Clifford tori around the double points $p_i$ and $q_i$ allow the meridians $a_i$ and $b_i$ to commute with twists around $\alpha$. 
Let $S'_i$ denote the union $S' \cup N(\gamma_1) \cup \cdots \cup N(\gamma_n)$. Note that $\pi(S') \cong \pi_1(S^4 - S'_i)$ since each $\gamma_i$ has codimension three. The complement $X - S$ is obtained from $X - S'_i$ by removing regular neighborhoods of the disks $D'_i$. Dually, this implies that the complement $X - S'_i$ is obtained from $X - S$ by attaching $n$ many 2-handles to $X - S$ along the boundaries of disks normal to $D'_i$ in the complement of $S'_i \cup N(D'_i) \cup \cdots \cup N(D'_n)$, which is diffeomorphic to $X - S$. Therefore, $\pi(S')$ is obtained from $\pi S$ by adding $n$ relators — namely the boundaries of these 2-handles. The boundaries of these 2-handles are exactly the small gray circles as shown in Figure 11. We make these into elements of $\pi S$ by pre- and post-composing the circles with the gray arc from the basepoint as shown in Figure 11. These elements are then exactly the elements $g_i^{-1}a_ig_i^{-1} \in \pi S$ as is shown by the homotopy-equivalence in Figure 11.

We conclude the section by proving Theorem 1.1 and Theorem 1.2. Note that Singh gives the analog of Theorem 1.1 for his related invariants $d_{st}$ and $d_{sing}$ in [33, Theorem 1.4], and in fact, our proof of Theorem 1.1 relies on Lemmas 3.1 and 4.1 of his paper. We are unable to provide any examples in which the +1 term is necessary, and so leave this as a question in Section 7; this question is also left open in Singh’s setting.

**Proof of Theorem 1.1.** Take a length $u_{cw}(K)$ regular homotopy from $K$ to $U$, with associated tubed surfaces $F_U$ and $F_K$ constructed as in the proof of Proposition 3.4. Note that since the homotopy is not necessarily arc-standard, the double points of $\Sigma$ are tubed together along different arcs; thus it is unclear whether or not $F_U$ and $F_K$ are isotopic. In his proof of [33, Theorem 1.4], Singh produces a sequence of tubed surfaces $T_1, \ldots, T_m$ for the standard immersed sphere $\Sigma$ such that $T_1 = F_U$, $T_m = F_K$, and such that each consecutive pair $T_i$ and $T_{i+1}$ become isotopic after a single stabilization.

Note that $\pi T_i \cong \mathbb{Z}$ for each $i$. This follows from Lemma 3.5, since each $T_i$ is an associated tubed surface for the standard immersion $\Sigma$ with $\pi \Sigma \cong \mathbb{Z}$. Thus, any stabilization of $T_i$ is isotopic to the stabilization done along the trivial guiding arc. This, combined with the fact that $T_i$ and $T_{i+1}$ become isotopic after a single stabilization, implies that the trivial stabilizations of the tubed surfaces $T_1, \ldots, T_m$ are all pairwise isotopic; in particular, $F_U$ and $F_K$ become isotopic after a single stabilization.

**Remark 3.6.** Note that in the proof of Theorem 1.1 above, it is critical that each ‘intermediate’ tubed surface $T_i$ has $\pi T_i \cong \mathbb{Z}$. It was pointed out to us by Peter Teichner that these surfaces give interesting candidates for ‘exotic’ unknotted surfaces. Although each $T_i$
The automorphism $\tau$ of the domain of the immersion $f: S^2 \looparrowright S^4$ with image $\Sigma$.

Figure 12 (colour online). The automorphism $\tau$ of the domain of the immersion $f: S^2 \looparrowright S^4$ with image $\Sigma$.

becomes smoothly unknotted after a single stabilization, it is unclear whether each $T_i$ is even topologically unknotted (see the discussion in Section 7).

**Proof of Theorem 1.2.** We argue that for $K$ with $u_{\text{CW}}(K) = 1$, there is an arc-standard length one regular homotopy from $K$ to the unknot $U$. It then follows from Proposition 3.4 that $u_{\text{st}}(K) = 1$. Start by letting $\Sigma$ denote the standard immersion with two oppositely signed double points. Fix a parametrization $f: S^2 \looparrowright S^4$ of $\Sigma$ with double point pre-images $p = \{p, p^*\}$ and $n = \{n, n^*\}$. By definition of the Casson–Whitney number, there is a regular homotopy from $\Sigma$ to $K$ consisting of a single Whitney move along a knotted Whitney disk $V$ with pre-image $f^{-1}(\partial V)$ equal to a pair of ‘knotted’ arcs $v, v^* \subset S^2$ with $\partial v = \{p, n\}$ and $\partial v^* = \{p^*, n^*\}$.

We claim that there is a standard Whitney disk $W$ one of whose boundary arcs has pre-image equal to $v$. To see this, let $W$ be any standard Whitney disk with $f^{-1}(\partial W)$ equal to the ‘standard’ arcs $w, w^* \subset S^2$. Consider the map $\tau: S^2 \rightarrow S^2$ given by a braid twist about the points $p \cup p^*$, as in Figure 12, that fixes the arc $\gamma$ connecting $p$ to $p^*$ setwise. Since $\Sigma$ is a standard immersion, the loop $f(\gamma)$ bounds an embedded disk (usually referred to in the literature as an accessory disk) in $S^4$ away from $\Sigma$. It therefore follows from [31, Lemma 3.9]$^\dagger$ that there is an ambient isotopy $\rho: S^4 \times I \rightarrow S^4$ with $\rho_1(\Sigma) = \Sigma$ carrying the standard Whitney disk $W$ to one whose boundary arcs are the image under $\tau$ of those for $W$, as illustrated in Figure 12 as well as [31, Figure 18]. We retain the labels $w, w^*$, and $W$ even after such an isotopy occurs.

The isotopy $\rho$ can be applied once if necessary, as shown in the top row of Figure 13, so that $\partial w = \{p, n\} = \partial v$ for some choice of labeling of the standard arcs. Note that $w$ and $v$ are now isotopic rel the points in $p \cup n$ if and only if the loop $w \cup v$ (with either orientation) is null homologous in the annulus $S^2 - \{p^*, n^*\}$. This can be arranged by applying the isotopy $\rho^2$ as shown in the bottom row of Figure 13 to insert full twists of $w$ around $p^*$. For the standard Whitney disk $W$ with $w = v$, the regular homotopy from $K$ to $U$ consisting of the finger move that is inverse to the Whitney move along the knotted Whitney disk $V$, followed by the Whitney move along $W$, is arc-standard.

4. **Geometric upper bounds**

The Casson–Whitney unknotted number can be bounded from above geometrically, by constructing simple regular homotopies to the unknot. We do this for some well-known families of spheres.

$^\dagger$Although Schneiderman and Teichner are working in a different context, their Lemma 3.9 applies in our case since (as they note in the discussion in Section 3.G.) their isotopy is supported locally, in the neighborhood of an accessory disk.
Figure 13 (colour online). The ambient isotopies $\rho$ and $\rho^2$ from the proof of Theorem 1.2 used to move the standard disk $W$ to one with $w = v$. The result of each isotopy is illustrated from the perspective of the induced maps $\tau$ and $\tau^2$ on the domain of the immersion $f : S^2 \hookrightarrow S^4$ with image $\Sigma$.

Definition 4.1. A ribbon 2-knot is formed from $n$ stabilizations of the $(n + 1)$-component unlink $U_1 \sqcup \cdots \sqcup U_{n+1}$ in $S^4$, as in Figure 14. The minimal number $n$ needed to put a ribbon 2-knot $K$ in this form is called the fusion number of $K$, denoted $\text{fus}(K)$.

Remark 4.2 (Tube map). Satoh proved in [28] that every ribbon 2-knot is the tube of a virtual arc. Essentially, one can use virtual diagrams to make a shorthand picture for a broken
Figure 15 (colour online). A regular homotopy of a ribbon 2-knot $K$, as in Definition 4.1, supported near one component $U_i$ of the unlink and one guiding arc of a stabilization. The various shadings of $K$ suggest its fourth coordinates — so, the red and blue portions of the surface are disjoint from the black ones. The homotopy consists of one finger move followed by one Whitney move, and (thought of from left to right) has the effect of removing a meridian of $U_i$ from the word in $\pi(U_1 \cup \cdots \cup U_{n+1})$ giving the homotopy class of the guiding arc of the stabilization.

Figure 16 (colour online). Miyazaki’s proof that $u_{st}(K) \leq \text{fus}(K)$: The red stabilization of the black unlink is the unknot $U$. Hence, the guiding arcs for the blue stabilizations are isotopic rel boundary to trivial arcs in the complement of $U$.

In this language, changing a virtual crossing to a positive or a negative classical crossing is achieved by a finger move and then a Whitney move on its tube (the analog in this setting of the homotopy in Figure 15). Thus if $K$ is a ribbon 2-knot and $k$ is any virtual arc such that $\text{Tube}(k) = K$, any sequence of crossing changes which unknotts $k$ as a virtual (or welded) arc yields a sequence of finger and Whitney moves which unknotts $K$.

In [24], Miyazaki proved that $u_{st}(K) \leq \text{fus}(K)$ for a ribbon 2-knot $K$. We prove the corresponding statement for the Casson–Whitney number in Theorem 1.4. The proof is inspired by Miyazaki’s, however, the argument is subtler, so we first sketch Miyazaki’s argument.

Let $K$ be a ribbon 2-knot, formed by stabilizing the unlink $U_1 \cup \cdots \cup U_{n+1}$ along guiding arcs connecting consecutive components $U_i$ and $U_{i+1}$, as in Definition 4.1. The ‘obvious’ stabilizations of $K$ which fuse $U_i$ to $U_{i+1}$ as in Figure 16 result in an unknotted surface $K'$: thinking of $K'$ as first formed by attaching this second set of tubes, and then the original tubes defining $K$, produces the same surface $K'$. However, this is clearly unknotted, since the ‘obvious’ tubes result in an unknotted sphere, and so the original tubes are (trivial) stabilizations of an unknotted surface, which must be unknotted.

Although one could perform $\text{fus}(K)$ finger moves to abelianize the group of $K$, the rest of Miyazaki’s argument breaks down in our case: if one thinks of the finger moves as performed on the $n+1$ component unlink, then the group of the complement is not abelian since it takes
\[
\left(\frac{n+1}{2}\right)
\]
finger moves to abelianize the group of the complement of this unlink. To remedy this, we think of all but one of the tubes as already attached and proceed by induction, allowing us to work with two components instead of \(n+1\).

**Theorem 1.4.** For a ribbon 2-knot \(K\), \(u_{cw}(K) \leq \text{fus}(K)\).

**Proof.** Let \(n = \text{fus}(K)\). Then, as in Definition 4.1, the knot \(K\) can be obtained from the unlink \(U = U_1 \sqcup \cdots \sqcup U_{n+1}\) by \(n\) stabilizations along guiding arcs \(\alpha_1, \ldots, \alpha_n\). After an isotopy, we may assume that each \(\alpha_i\) connects \(U_i\) to \(U_{i+1}\) as in Figure 14. Let \(L\) be the 2-component link obtained by stabilizing the unlink only along the guiding arcs \(\alpha_2, \ldots, \alpha_n\). Recall from Section 2, in particular Lemma 2.12, that

\[
\pi L \cong \langle m_1, m_2, \ldots, m_{n+1} | \ m_j^g_j = m_{j+1} \text{ for } 1 < j < n+1 \rangle,
\]

where the \(g_j\) correspond to the guiding arcs \(\alpha_i\) as in Definition 2.10, and \(m_1, \ldots, m_{n+1} \in \pi U\) are meridians of each component \(U_1, \ldots, U_{n+1}\).

Perform \(n\) finger moves to \(L\) along trivial guiding arcs from \(U_{n+1} \rightarrow U_i, i \leq n\) as in Figure 17(b) and call the resulting immersed 2-component link \(S\). By Lemma 2.14, we have made \(m_{n+1}\) commute with \(m_i\) for all \(i < n+1\), therefore by considering the previous relations we see that \(\pi S \cong \mathbb{Z} \oplus \mathbb{Z}\) generated by \(m_1\) and \(m_2\).

Now consider the element \(g_1\) corresponding to the guiding arc \(\alpha_1\) as in Figure 17(b). Since \(\pi S\) is both abelian and generated by \(m_1\) and \(m_2\), by Remark 2.11, \(\alpha_1\) is isotopic to a trivial arc between \(U_1\) and \(U_2\) as shown in Figure 17(c). We can now undo the finger move that intersects \(U_1\) (that is, do the Whitney move) and proceed, by the same reasoning, to straighten out all of the other arcs \(\alpha_i\) to be trivial arcs as in Figure 17(d). Proceeding in this way unknots \(K\) (by trivializing the guiding arcs \(\alpha_1, \ldots, \alpha_n\)) with \(n\) finger moves and \(n\) Whitney moves. \(\square\)
The first examples of non-trivially knotted spheres in $S^3$ are given by twist spun knots $\tau^n(k)$, where $k$ is a 1-knot with unknotting number $u(k) = 1$ and meridional rank $\mu(k) > 2$ (in fact, it is shown in [2] that there exist unknotting number one knots with arbitrarily large meridional rank). Then, we will see that the spun knot $K = \tau^0(k)$, defined below, has $u_{cw}(K) = 1$ by Corollary 4.6. Moreover, the isomorphism between $\pi K$ and $\pi k$ preserves the meridians, and so the meridional ranks of $K$ and $k$ are equal. This gives $u_{cw}(K) < \text{fus}(K)$, since spun knots are ribbon and the meridional rank of any ribbon 2-knot is less than or equal to one more than its fusion number.

A generalization of this family of spheres for which $u_{cw}$ is particularly convenient to analyze is constructed by ‘spinning’ 3-balls containing properly embedded knotted arcs through an open book decomposition of $S^4$.

**Definition 4.4 (Twist spun knots).** Given a 1-knot $k \in S^3$, let $k'$ be the properly embedded knotted arc in the 3-ball whose tubular neighborhood $N(k')$ has complement $B^3 - N(k')$ diffeomorphic to $S^3 - N(k)$. For $n \in \mathbb{Z}$, consider the quotient

$$(B^3, k') \times S^1 \Big/ (r_n, \theta)(x, \theta) \sim (x, 0), \quad x \in \partial B^3, \quad \theta \in [0, 2\pi],$$

where $r_n, \theta : B^3 \to B^3$ denotes the ambient isotopy rotating $B^3$ by an angle of $n\theta$ about an axis with endpoints $\partial k' \subset \partial B^3$. For each $n$, this quotient space is diffeomorphic to $S^4$, and gives an open book decomposition with binding an unknotted 2-sphere $U$ and 3-ball pages $B^3_0$ for all $\theta \in S^1$. The quotient of $k' \times S^1$ is a 2-sphere $\tau^n k \subset S^4$ called the $n$-twist spin of $k$.

Due to Artin [1], the collection of 0-twist spun knots, often simply called ‘spun knots’, were the first examples of non-trivially knotted spheres in $S^4$. Artin proved that the group of the spin knot $\tau^0(k)$ is isomorphic to the group of the classical knot $k$, showing that every 1-knot group is also a 2-knot group.

Twist spinning was introduced by Zeeman in [36] as a generalization of the spinning construction. For $n \neq 0$, Zeeman proved that the resulting twist spin knot is fibered by the $n$-fold cyclic branched cover of $k$. Thus $\tau^\pm k$ is unknotted, for all $k$. Twist spun knots provide a large generalization of spun knots. Cochran proved that any non-trivial twist spun knot $\tau^n k$ with $n \neq 0$ is not ribbon [7], in contrast to spun knots, which are always ribbon.

**Lemma 4.5.** Fix two parallel strands of a 1-knot $k \subset S^3$, and let $k_s$ denote the knot obtained by inserting $s$ full twists into these strands. Then, for any $n \in \mathbb{Z}$, there is a length one regular homotopy between the twist spins $\tau^n(k)$ and $\tau^n(k_s)$.

**Proof.** By performing a finger move on $\tau^n(k_{s+1})$ along the arc $\alpha_{s+1} \subset B^3_0$ as in Figure 19, we obtain an immersed surface $\Sigma_{s+1}$ that is also obtained by a finger move to $\tau^n(k_s)$ along $\alpha'_s \subset B^3_0$, which we will also denote $\Sigma'_s$, so that $\Sigma'_s = \Sigma_{s+1}$. Note that the twist parameter $n$ is unchanged since the twisting can be assumed to occur in a small interval in $S^1$ away from the double points of the immersion, the knotted arc $k_s$ twists $n$ times in both $\Sigma'_s$ and $\Sigma_{s+1}$.

By instead performing a finger move on $\tau^n(k_s)$ along the arc $\alpha_s \subset B^3_0$, we obtain a surface $\Sigma_s$ where $\Sigma_s = \Sigma'_s$ by rotation. Similarly by performing a finger move to $\tau^n(k_{s-1})$ along the arc $\alpha_{s-1} \subset B^3_0$, we obtain a surface $\Sigma'_{s-1}$ with $\Sigma_{s-1} = \Sigma_s$. 

Figure 18 (colour online). A schematic for the spin $\tau^0k$ and the twist spins $\tau^n k$ of a 1-knot $k$. The open book decomposition of $S^4$ from Definition 4.4 has been knocked down one dimension on the left, and so the blue 2-sphere $U$ around which the knotted arc is spun is instead a blue circle (only an arc of which is drawn). Note that viewing a page $B^3_0$ from the ‘opposite side’ reverses its orientation, and therefore also the orientation of $k$.

Figure 19 (colour online). Top row: The intersections of the twist spins $\tau^n(k_{s+1})$, $\tau^n(k_s)$ and $\tau^n(k_{s-1})$ with $B^3_0$ and $B^3_\pi$, as in Definition 4.4. Only the relevant crossing of each cross section is drawn. Bottom row: Schematics of the immersed spheres (compare to the embedded spheres in Figure 18) obtained by doing finger moves along the red guiding arcs in each diagram of the top row. Again, only the relevant crossing of the cross section in each page $B^3_0$ is shown. The isotopy between the two immersions $\Sigma'_s$ and $\Sigma_s$ is via a rotation by $\pi$.

Thus, we have equivalent immersed surfaces

$$\cdots = \Sigma_{s+1} = \Sigma'_s = \Sigma_s = \Sigma'_{s-1} = \cdots$$

so that for any $s, t \in \mathbb{Z}$, the two knots $\tau^n(k_s), \tau^n(k_t)$ are related by a single finger and Whitney move. $\square$

Some implications of Lemma 4.5 are immediate. For instance, as there is a length one regular homotopy between 1-knots related by a single crossing change, we obtain the following corollary.
Corollary 4.6. Let \( k : S^1 \rightarrow S^3 \) be a classical knot. For any twist spin \( \tau^n k \), \( u_{\text{CW}}(\tau^n k) \leq u(k) \), where \( u(k) \) is the classical unknotting number of \( k \).

Although we are not aware of another instance of Corollary 4.6 in the literature, the analogous result for \( u_{\text{st}} \) was proved by Satoh [29], and also follows from Proposition 9 of [4]. Furthermore, Satoh proved in [29] that for any twist spin of a \( b \)-bridge knot, the stabilization number is strictly less than \( b \). When \( b = 2 \), we prove that the same inequality holds for the Casson–Whitney number of any twist spin.

Theorem 4.7. If the twist spin \( \tau^n k \) of a 2-bridge knot \( k \) is not unknotted, then it has \( u_{\text{CW}}(\tau^n k) = 1 \).

Proof. Since \( k \) is 2-bridge, it can be put into normal form [8] with non-zero twist parameters \( a_1, b_1, \ldots, a_m, b_m \) indicating the number of half twists in each region, as in Figure 20. In fact, we may assume that the terms \( a_i \) and \( b_i \) are all even\(^1\). Start by performing a finger move of \( \tau^n k \) along the red guiding arc pictured in the leftmost diagram of Figure 20, at some angle \( \theta \in S^1 \). This results in an immersed sphere \( \Sigma_k \) that we will prove is the standard immersed sphere gotten by one finger move on the unknot, by induction on the number \( m \) of twist region pairs \( a_i, b_i \).

When \( m = 0 \), the knot \( k \) and hence also its \( n \)-twist spin \( \tau^n k \) are unknotted. Therefore, \( \Sigma_k \) is the standard immersion by definition. So, suppose \( m > 1 \). Let \( \hat{k} \) denote the 2-bridge knot with two fewer twist parameters \( a_1, b_1, \ldots, a_{m-1}, b_{m-1} \) and assume as the inductive hypothesis that the immersed sphere gotten by a finger move of the twist spin \( \tau^n \hat{k} \) along the red guiding arc pictured in the rightmost diagram of Figure 20 is ambiently isotopic to the standard immersion \( \Sigma \). Observe that the guiding arc for this finger move is isotopic to the red guiding arc shown in the middle diagram of Figure 20; therefore doing a finger move of \( \tau^n \hat{k} \) along this arc also gives the standard immersion \( \Sigma \).

Now, since the knots \( k \) and \( \hat{k} \) differ only along a single twist region, by Lemma 4.5, the twist spins \( \tau^n k \) and \( \tau^n \hat{k} \) must give ambiently isotopic immersions after one finger move. Indeed, the guiding arcs for the finger moves of \( \tau^n k \) and \( \tau^n \hat{k} \) that are used in the proof of Lemma 4.5 (that is, those from Figure 19) are equal to the red guiding arcs from the middle and leftmost diagrams of Figure 20. It follows that \( \Sigma_k \) is ambiently isotopic to the standard immersion \( \Sigma \), as desired. \( \square \)

Remark 4.8. The quantities in the inequality Corollary 4.6 can be arbitrarily far apart. For instance, each non-trivial \((2, p)\) torus knot \( k_p \) has unknotting number \( |p - 1|/2 \) by [21] and bridge number equal to 2. Therefore, by Theorem 4.7, \( u_{\text{CW}}(\tau^n k_p) = 1 < u(k_p) \) for any non-trivial twist spin of \( k_p \), whenever \( p \geq 5 \).

\(^1\)It was pointed out in [3] that this can be shown using the continued fraction notation for 2-bridge knots, for instance, see [19].
Remark 4.9. Satoh’s proof \[29\] that \( u_{st}(K) \leq b - 1 \) for any twist spin of a \( b \)-bridge knot relies on the fact that 1-handles can be slid over one another when \( b > 2 \). However, this cannot be done with finger moves, making our proof of Theorem 4.7 difficult to extend to knots with higher bridge number.

5. Algebraic lower bounds

In this section, we discuss the algebraic Casson-Whitney number \( a_{cw}(K) \) of a 2-knot \( K \), the minimal number of meridian-commuting relations which abelianize the knot group of \( K \) (see Definition 2.15 for the precise definition). This algebraic invariant is the sharpest lower bound we are aware of for the Casson–Whitney number \( u_{cw} \), and in Section 5.3 we show that it is also a lower bound for the classical unknotting number. It is clear that \( a_{st}(K) \leq a_{cw}(K) \), as stabilization relations identify two meridians, while finger move relations merely force them to commute (see Section 2.3 for a thorough description of the effects of the corresponding geometric operations on the knot group). This subtle difference is used to prove Theorem 1.3, in which we give 2-knots for which \( a_{st}(K) < a_{cw}(K) \) and for which this difference is realized geometrically.

5.1. Previously known results

The minimal number of generators of the Alexander module, called the Nakanishi index \( m(K) \), is a classical lower bound for the unknotting number of 1-knots \[25\]. In \[23, 24\] it is shown that the Nakanishi index is also a lower bound for the stabilization number \( u_{st}(K) \) of 2-knots. Indeed, any set of relators which abelianize the group of a knot must normally generate its commutator subgroup. It is an exercise (cf. \[27, Exercise 7.D.5\]) to show that the images of these relators generate the Alexander module.

A subtler but sharper bound for the classical unknotting number is the Ma-Qiu index \( a(K) \), defined as the minimal number of relations needed to abelianize the knot group \[22\]. In fact, the algebraic stabilization number \( a_{st}(K) \) (Definition 2.13), the minimal number of stabilization relations needed to abelianize the knot group, is also a lower bound for the classical unknotting number. This is evident from the proof of \[22\], and also by combining the inequalities from Proposition 2.16 and Corollary 4.6, applied to the spin of a 1-knot \( k \): \( a_{st}(k) = a_{st}(\tau^0 k) \leq a_{cw}(\tau^0 k) \leq a_{cw}(\tau^0 k) \leq u_{st}(k) \).

As noted in Proposition 2.16, the algebraic stabilization number, \( a_{st}(K) \), is a natural lower bound for the stabilization number \( u_{st}(K) \). This is also studied in \[18\], where it is called the weak unknotting number. In this section, we investigate the algebraic Casson–Whitney number, which is a natural lower bound for the Casson–Whitney number. By Corollary 4.6, it also provides a lower bound for the classical unknotting number via spinning, which as we show in Theorem 1.5 is sharper than the bounds provided by \( a_{st}(K) \) and \( u_{st}(K) \).

We summarize the previously known results regarding these invariants in the proposition below.

**Proposition 5.1** (Kanenobu, Ma-Qiu, Miyazaki, Nakanishi). If \( k \) is a 1-knot, then
\[
m(k) \leq a(k) \leq a_{st}(k) \leq u(k).
\]

If \( K \) is a 2-knot, then
\[
m(K) \leq a(K) \leq a_{st}(K) \leq u_{st}(K).
\]

As pointed out in \[22\], the first inequality above is often strict: the Ma-Qiu index is positive whenever \( \pi K \) is not abelian, but the Alexander module and hence the Nakanishi index can be
zero for non-trivial knots, for example, Alexander polynomial one 1-knots. While \( m(k), a(k) \), and \( \mu(k) \) are known to be non-additive on certain classical knots (see the end of Section 5.2), we are unaware of any classical knots for which \( a_{cw} \) is non-additive. We show in Theorem 6 that is non-additive on certain 2-knots.

5.2. The algebraic Casson–Whitney number

Recall from Section 2.3 that each finger move on a 2-knot \( K \) adds a relation of the form \([x, y] = 1\), where \( x, y \) are meridians of \( K \). As noted after Definition 2.9, \( y \) is equal to \( x^w \) for some \( w \in (\pi K)' \). Therefore, the algebraic Casson–Whitney number \( a_{cw}(K) \) is equal to the minimal number of elements \( w_i \in (\pi K)' \) such that the relations \( \{[x, x^{w_i}] = 1\} \) abelianize \( \pi K \).

These finger move relations are ‘weaker’ than the relations induced by stabilizations, in that every finger move relation is also a stabilization relation. Recall from Definition 2.13 that \( a_{st}(K) \) denotes the minimal number of stabilization relations needed to abelianize the knot group; these relations are of the form \( x = y \), where \( x \) and \( y \) are meridians, or equivalently \( [x, w] = 1 \), where \( w \in (\pi K)' \) and \( y = x^w \). Thus \( a_{st}(K) \) is the minimal number of elements \( w_i \in (\pi K)' \) such that the relations \( \{[x, w_i] = 1\} \) abelianize \( \pi K \). Although \( x^w \) is not in the commutator subgroup, \( x^w = x[x, w] \), so the finger move relation \( [x, x^w] = 1 \) is equivalent to the stabilization relation \( [x, [x, w]] = 1 \), and we see that \( a_{st}(K) \leq a_{cw}(K) \).

On the other hand, an obvious upper bound for \( a_{cw}(K) \) is \( \mu(K) - 1 \), where \( \mu(K) \) is the meridional rank of \( K \): forcing any single meridian to commute with the rest of a generating set of meridians will force that meridian into the center of the group. Since all knot groups are normally generated by any meridian, this abelianizes the group. We summarize the relationships between these invariants below, which are defined for \( n \)-knots because we will later refer to the case \( n = 1 \) as well as our usual case \( n = 2 \) (although these invariants are well defined for all \( n \geq 1 \) because they only depend on the knot group and the information of a meridian).

**Proposition 5.2.** For any \( n \)-knot \( K \),

\[
m(K) \leq a(K) \leq a_{st}(K) \leq a_{cw}(K) \leq \mu(K) - 1.
\]

In Theorem 1.5, we show that the inequality \( a_{st}(K) \leq a_{cw}(K) \) can be strict. In fact, we find infinitely many 2-knots \( K \) with \( a_{st}(K) = \mu_{st}(K) = 1 \) and \( a_{cw}(K) = 2 \), enabling us to prove in Theorem 1.3 that \( \mu_{st}(K) < u_{cw}(K) \) for infinitely many 2-knots \( K \). The last inequality may also be strict, for the same reason pointed out after Remark 4.3.

**Proposition 5.3.** For \( \alpha \in \{a, a_{st}, a_{cw}\} \) and for \( n \)-knots \( K_1 \) and \( K_2 \),

\[
\max\{\alpha(K_1), \alpha(K_2)\} \leq \alpha(K_1 \# K_2) \leq \alpha(K_1) + \alpha(K_2).
\]

**Proof.** The proof is the same in all three cases; we follow Kanenobu in [18] for \( \alpha = a_{st} \). Let \( g_1, \ldots, g_n \) be a minimal set of relators of the required form (depending on \( \alpha \)) which abelianize \( \pi(K_1 \# K_2) \). Let \( \phi \) be the surjection \( \phi: \pi(K_1 \# K_2) \to \pi K_1 \) which sends all meridians of \( K_2 \) to the meridian of amalgamation. Note that \( \pi K_1 / \langle \langle \phi(g_1), \ldots, \phi(g_n) \rangle \rangle \cong \mathbb{Z} \) and that each \( \phi(g_i) \) is a relator of the required form for computing \( \alpha(K_1 E) \). Therefore, \( \alpha(K_1 \# K_2) \geq \alpha(K_1) \).

Repeating the argument for \( K_2 \) obtains the first inequality.

The second is obtained by imposing relations on the group of \( K_1 \# K_2 \) which abelianize \( K_1 \) and \( K_2 \) separately. Since \( \pi(K_1 \# K_2) \cong (\pi K_1 * \pi K_2)/\langle \langle x_{1}^{-1}x_2 \rangle \rangle \), where \( x_i \) are meridians of \( K_i \), these relations abelianize the group of the connected sum. \( \square \)
As a first application of Proposition 5.2, we show that any natural number can occur as the Casson–Whitney number of a 2-knot. We will make use of determinants in the following proposition and in Theorem 1.5, which we introduce now.

The Alexander module of a 2-knot is the first homology of the infinite cyclic cover, viewed as a \( \mathbb{Z}[t^{\pm 1}] \)-module. The determinant of a 2-knot \( K \) is defined in [15] as the positive generator of the evaluation of the Alexander ideal at \( t = -1 \), that is, \( \Delta(K)|_{-1} := n \), where \( n > 0 \) is the generator of the principal ideal \( \{ f(-1) : f(t) \in \Delta(K) \} \subseteq \mathbb{Z} \). Equivalently, it is the order of the \( \mathbb{Z} \)-module induced by setting \( t = -1 \) in the Alexander module. As with classical knots this is always an odd integer, and in [15, Proposition 5.9] it is shown that even twist-spinning preserves the determinant, while odd twist-spins always have determinant 1. The classical fact that a 1-knot \( k \) admits a Fox \( p \)-coloring for prime \( p \) if and only if \( p \) divides the classical determinant \( |\Delta_k(-1)| \), where \( \Delta_k(t) \) is the Alexander polynomial of \( k \), carries over without change to this definition of determinant for non-principal ideals. A Fox \( p \)-coloring of a 2-knot is a surjection from its knot group onto the dihedral group \( D_p \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2 \), which sends meridians of the 2-knot to reflections.

**Proposition 5.4.** Let \( n \in \mathbb{N} \). Then there exists a 2-knot \( K \) with \( u_{cw}(K) = n \).

**Proof.** Let \( J \) be any 2-knot with \( u_{cw}(J) = 1 \) and Nakanishi index \( m(J) = 1 \), for instance, \( J \) could be any even twist-spin of a 2-bridge knot, by Theorem 4.7: 2-bridge knots have non-trivial determinants, which are preserved by even twist-spinning [15]. Therefore the Alexander module of \( J \) is non-trivial, so it must be cyclic since it is a quotient of the original 2-bridge knot’s Alexander module.

Then letting \( K = \#^n J \) obtains the desired result: the Nakanishi index \( m(K) = n \), since the Alexander module of \( K \) is generated by \( n \) elements and surjects onto a vector space of dimension \( n \), so by Proposition 5.2 \( u_{cw}(K) \geq n \). Conversely, \( K \) can be unknotted in \( n \) pairs of finger and Whitney moves by performing the optimal length one regular homotopy for \( J \) on each summand. \( \square \)

Scharlemann proved that unknotting number one knots are prime, that is, if \( K_1 \) and \( K_2 \) are non-trivial classical knots, then the unknotting number of \( K_1 \# K_2 \) is at least 2 [30]. Here we prove a special case of the analogous statement for \( u_{cw} \), which works whenever the 2-knots in question have non-trivial determinants, or equivalently whenever their knot groups admit non-trivial Fox colorings. This reproves the same special case of Scharlemann’s theorem for classical knots, via the bound given by Corollary 4.6. The technical core of our proof is a Freiheitssatz for one-relator quotients of free products of cyclic groups due to Fine, Howie, and Rosenberger [10].

**Theorem** (Fine, Howie, Rosenberger). Suppose \( G = \langle a_1, \ldots, a_n \mid a_1^{e_1}, \ldots, a_n^{e_n}, R^m \rangle \), where \( n \geq 2, m \geq 2, e_i = 0 \) or \( e_i \geq 2 \) for all \( i \), and \( R(a_1, \ldots, a_n) \) is a cyclically reduced word which involves all of \( a_1, \ldots, a_n \). Then the subgroup of \( G \) generated by \( a_1, \ldots, a_{n-1} \) is isomorphic to \( \langle a_1, \ldots, a_{n-1} \mid a_1^{e_1}, \ldots, a_{n-1}^{e_{n-1}} \rangle \).

Their result generalizes the more well-known Freiheitssatz for one-relator groups, a classical result in combinatorial group theory characterizing the torsion in a one-relator group. It is proved by finding explicit representations of these groups into \( \text{PSL}_2(\mathbb{C}) \).

**Theorem 1.5.** Let \( K_1, K_2 \) be 2-knots with determinants \( \Delta(K_i)|_{-1} \neq 1 \). Then \( u_{cw}(K_1 \# K_2) \geq 2 \).
Proof. Let $x_1$ and $x_2$ be meridians of $K_1$ and $K_2$, respectively, and form the connected sum so that $x_1$ and $x_2$ become identified in the group of $K_1 \# K_2$, that is,

$$\pi(K_1 \# K_2) \cong \pi K_1 \ast \pi K_2 / \langle\langle x_1^{-1} x_2 \rangle\rangle.$$  

Denote $x$ as the image of these meridians.

The claim to be proved is that for any $w \in \pi(K_1 \# K_2)$, the relation $[x, x^w] = 1$ does not abelianize $\pi(K_1 \# K_2)$, since then $u_{cw}(K_1 \# K_2) \geq a_{cw}(K_1 \# K_2) \geq 2$.

Let $p_1$ and $p_2$ be prime divisors of $\Delta(K_1)_{-1}$ and $\Delta(K_2)_{-1}$, respectively. Then $K_i$ admits a Fox $p_i$-coloring

$$\phi_i : \pi K_i \rightarrow D_{p_i} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_2 = (z_i, a_i \mid z_i^2 = a_i^{p_i} = 1, za_i z = a_i^{-1})$$

with $x_i$ mapping to $z_i$, the generator of $\mathbb{Z}_2$. The group of the connected sum naturally surjects onto the group

$$G := (z, a_1, a_2 \mid z^2 = a_1^{p_1} = a_2^{p_2} = 1, za_1 z = a_1^{-1}, za_2 z = a_2^{-1})$$

$$\cong (\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}) \rtimes \mathbb{Z}_2,$$

by mimicking the connected sum operation on the dihedral groups. To be explicit, first define

$$\phi_1 \ast \phi_2 : \pi K_1 \ast \pi K_2 \rightarrow D_{p_1} \ast D_{p_2} \text{ in the obvious way. Then } G \text{ can be constructed from } D_{p_1} \ast D_{p_2} \text{ by identifying the images of the meridians: }$$

$$G \cong \frac{D_{p_1} \ast D_{p_2}}{\langle\langle z_1^{-1} z_2 \rangle\rangle}.$$  

We will show that $G/\langle\langle \phi([x, x^w]) \rangle\rangle$ is not abelian, hence $\pi(K_1 \# K_2)/\langle\langle [x, x^w] \rangle\rangle$ is not abelian either.

Note that $\phi(x) = z$ and $\phi([x, x^w]) = [z, z^w]$, where $v = \phi(w)$ is in the commutator subgroup $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$ of $G$. Then $G/\langle\langle [z, z^w] \rangle\rangle$ is the image of the induced homomorphism which we would like to show is non-abelian. We will do this by showing that its commutator subgroup is non-trivial. Let $N = \langle\langle [z, z^w] \rangle\rangle$, the normal closure of $[z, z^w]$ in $G$. As $[z, z^w]$ is a commutator, $N$ is contained in the commutator subgroup $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$ of $G$. The goal now is to show that $(\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2})/N$ is not the trivial group.

Note that $[z, z^w] = z(v^{-1} z v)(v^{-1} z v) = (z v^{-1} z v)^2 = [z, v]^2$. It will be convenient to describe $N$ as the normal closure inside of $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$ of some elements of $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$. Denote $g = [z, v]$. Now, $N$ is the normal subgroup generated by all elements of the form $h^{-1} g^2 h$, where $h \in G$ is arbitrary. Any $h \in G$ can be written as $z^n c$, where $n = 0$ or 1 and $c \in \mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$. Then $h^{-1} g^2 h = c^{-1} z^n g^2 z^n c$. Since $c^{-1} g^2 c$ is already in the normal closure of $g^2$ in $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$, it suffices to consider $n = 1$, that is, $h = z c$. Note that $z g^2 z = (z g z)^2 = (z [z, v] z)^2 = (v^{-1} z v z)^2 = [v, z]^2 = [z, v]^{-2} = (g^2)^{-1}$. Then $c^{-1} z g^2 z c = c^{-1} g^2 c = (c^{-1} g^2 c)^{-1}$, so in fact $N$ is the normal closure in $\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$ of just $g^2$. By the Freiheitssatz, $(\mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2})/\langle\langle g^2 \rangle\rangle$ is non-trivial for any element $g \in \mathbb{Z}_{p_1} \ast \mathbb{Z}_{p_2}$ (we may assume $g$ is cyclically reduced, since this does not change the isomorphism type of the quotient). If $g$ involves only one of the generators $a_1$ or $a_2$, then clearly the other factor survives in the quotient.

Corollary 5.5. Let $k_1$ and $k_2$ be classical knots with determinants $|\Delta_{k_i}(-1)| \neq 1$. Then

$$u_{cw}(\tau^n k_1 \# \tau^m k_2) \geq 2$$

for any even integers $n, m$.

Corollary 5.6. Let $K_1$ and $K_2$ be even twist-spins of 2-bridge knots. Then

$$u_{cw}(K_1 \# K_2) = 2.$$
Proof. Since 2-bridge knots have non-trivial determinants, their even twist spins do as well [15]. Then \( u_{cw}(K_1 \# K_2) \geq 2 \) follows from Theorem 1.5. The reverse inequality follows from Theorem 4.7 and the elementary fact that \( u_{cw}(K_1 \# K_2) \leq u_{cw}(K_1) + u_{cw}(K_2) \).

It is interesting to note that in the case of 2-bridge knots \( k_1, k_2 \), the knot group \( \pi(\tau^2 k_1 \# \tau^2 k_2) \cong (\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}) \times \mathbb{Z} \), where \( p_i = |\Delta_k, (-1)| \), and that the proof of Theorem 1.5 goes through in that setting without the further quotient to \( G \). In fact, \( G \) arises naturally as the group of \( \tau^n k_1 \# \tau^m k_2 \# \mathbb{RP}^2 \), where \( n, m \) are even and \( \mathbb{RP}^2 \) denotes a standard projective plane.

For odd integers \( p, q \in \mathbb{Z} \), let \( K_{p,q} \) denote the spin of \( T(2, p) \# T(2, q) \). Miyazaki proved that \( u_{st}(K_{p,q}) = 1 \), whenever \( q = p + 2, p + 4 \), or \( p + 6 \), when \( \gcd(p, p + 6) = 1 \) [24]. Therefore, \( u_{st} \) fails to be additive in these cases. However, it follows from Corollary 5.6 that \( u_{cw} \) is additive in these cases, and in particular that \( u_{cw}(K_{p,q}) = 2 \). This proves Theorem 1.3.

**Theorem 1.3.** There are infinitely many 2-knots \( K \) with \( u_{st}(K) = 1 \) and \( u_{cw}(K) = 2 \).

5.3. Application to classical unknotting number

As noted at the start of Section 5.1, the Nakanishi index, Ma-Qiu index, and algebraic stabilization number are all previously established lower bounds for the classical unknotting number. In this section, we point out that the algebraic Casson–Whitney number is also a lower bound for the classical unknotting number, which is sharper than the aforementioned invariants in many cases.

Perhaps the most interesting reason to study \( a_{cw} \) as a lower bound for the unknotting number is that the above three invariants all fail to be additive in many simple cases, such as \( T(2, p) \# T(2, q) \) when \( p, q \) are coprime [17]. By Theorem 1.5, \( a_{cw}(T(2, p) \# T(2, q)) = 2 \) for all (odd) \( p, q \). We do not know any cases where \( a_{cw} \) fails to be additive on classical knots, although it seems difficult to prove this is always the case. Still, this poses a potentially interesting avenue to study the classical unknotting number, via a lower bound which comes from four dimensional techniques.

Let \( k \) be a 1-knot. Remembering that spinning preserves the knot group (and its meridians), \( a_{cw}(k) = a_{cw}(\tau^0 k) \). By Proposition 2.16 \( a_{cw}(\tau^0 k) \leq u_{cw}(\tau^0 k) \), and by Corollary 4.6, \( u_{cw}(\tau^0 k) \leq u(k) \). Putting these facts together, we have:

**Proposition 5.7.** For any 1-knot \( k \), \( a_{cw}(k) \leq u(k) \).

As noted in Section 5.2, this reproves a special case of Scharlemann’s theorem that unknotting number one knots are prime [30]. Namely, if \( k_1 \) and \( k_2 \) are classical knots with non-trivial determinants, then \( u(k_1 \# k_2) \geq 2 \).

6. Strong non-additivity of \( u_{st} \) and \( u_{cw} \)

As noted in Section 5.2, Miyazaki was the first to prove that \( u_{st} \) is non-additive. For certain \( p, q \) (see section for precise description) he showed that \( u_{st}(\tau(T(2, p) \# T(2, q))) = 1 \). As pointed out by Kanenobu [18], the Nakanishi index proves that taking iterated connected sums of \( K = \tau(T(2, p) \# T(2, q)) \) has \( u_{st}(\#^n K) = n \), while \( u_{st}(\#^n T(2, p)) + u_{st}(\#^n T(2, q)) = 2n \). This shows the existence of 2-knots \( K_1, K_2 \) with \( u_{st}(K_1) + u_{st}(K_2) - u_{st}(K_1 \# K_2) \) arbitrarily large. In this section, we investigate and prove a stronger version of non-additivity for both the stabilization and Casson–Whitney number. For notational convenience, throughout the section we use \( \alpha \) to denote either \( a_{st} \) or \( a_{cw} \), and \( \nu \) to denote the corresponding \( u_{st} \) or \( u_{cw} \).
Our geometric study of strong non-additivity is inspired by Kanenobu’s work in [18] establishing the non-additivity of $a_{st}$. In particular, for each $n \geq 1$, Kanenobu gave examples of 2-knots $K_1, \ldots, K_n$ with $a_{st}(K_i) = 1$ and $a_{st}(K_1 \# \cdots \# K_n) = 1$.

**Question 6.1 (Kanenobu).** Is $u_{st}(K_1 \# \cdots \# K_n) = 1$ as well?

We generalize Kanenobu’s result for $a_{st}$ and prove a corresponding result for $a_{cw}$. We then prove analogous results for the geometric versions $u_{st}$ and $u_{cw}$, answering Kanenobu’s question in the affirmative at the expense of a small correction factor. In fact, Corollary 6.6 shows that the connected sums $K_1 \# \cdots \# K_n$ in Kanenobu’s original examples have both stabilization number and Casson-Whitney number at most 2.

**Theorem 6.2.** Let $\alpha \in \{a_{st}, a_{cw}\}$. Let $K_1, \ldots, K_n$ be 2-knots with $\alpha(K_i) \leq c$ for some $c \in \mathbb{N}$. Suppose that there exist meridians $x_i \in \pi K_i$ and relatively prime integers $j_i \in \mathbb{Z}$ such that each $x_i^{j_i}$ lies in the center $Z(\pi K_i)$ of the knot group of $K_i$. Then, $\alpha(K_1 \# \cdots \# K_n) \leq c$.

**Proof.** We will prove the case $\alpha = a_{st}$ and $c = 1$ in detail, then point out the changes necessary for the general result.

Since $\alpha(K_i) = 1$, there exists an element $w_i \in (\pi K_i)'$ such that $\pi K_i/\langle\langle [x_i, w_i] \rangle\rangle \cong \mathbb{Z}$. Let $K = K_1 \# \cdots \# K_n$, and let $x = x_1$ be the meridian of amalgamation. We will show that $\pi K/\langle\langle [x, w_1 w_2 \cdots w_n] \rangle\rangle \cong \mathbb{Z}$. For $m \leq n$, let

$$R_m = [x, w_1 w_2 \cdots w_m]$$

and

$$G_m = \pi(K_1 \# \cdots \# K_m) / \langle\langle R_m \rangle\rangle$$

Note that $G_1 \cong \mathbb{Z}$ by assumption; we will show that $G_m \cong G_{m-1}$, so that by induction $G_n \cong \mathbb{Z}$.

Since $j_1$ and $j_2j_3 \cdots j_m$ are coprime, there exist integers $s$ and $t$ so that $sj_1 + tj_2 \cdots j_m = 1$. Note that $x^{sj_1} \in Z(\pi K_1)$ and $x^{sj_1 - 1} = x^{-tj_2 \cdots j_m} \in Z(\pi(K_2 \# \cdots \# K_m))$. The relation $R_m$ is equivalent to $x = (w_1 \cdots w_m)^{-1}x(w_1 \cdots w_m)$. Raising both sides to the $s$th power, we obtain:

$$x^{sj_1} = (w_2 \cdots w_m)^{-1} w_1^{-1} x^{sj_1} w_1 (w_2 \cdots w_m)$$

$$= (w_2 \cdots w_m)^{-1} x^{sj_1} (w_2 \cdots w_m)$$

$$= (w_2 \cdots w_m)^{-1} x(w_2 \cdots w_m) x^{sj_1 - 1},$$

which is equivalent to $x = (w_2 \cdots w_m)^{-1} x(w_2 \cdots w_m)$. We can repeat this procedure until we reach $x = w_1^{-1} x w_m$, or $[x, w_m] = 1$, the relation which abelianizes $\pi K_m$. Since $w_m$ is in the commutator subgroup of $\pi K_m$, it is trivial in the abelianization, so the relation $[x, w_m] = 1$ abelianizes the subgroup of $\pi(K_1 \# \cdots \# K_{m-1})$ corresponding to $\pi K_m$, and the induced relation on $\pi(K_1 \# \cdots \# K_{m-1})$ is $[x, w_1 w_2 \cdots w_{m-1}] = 1:

$$G_m = \pi(K_1 \# \cdots \# K_{m-1}) / \langle\langle [x, w_1 w_2 \cdots w_{m-1}] \rangle\rangle \cong \pi(K_1 \# \cdots \# K_{m-1}) / \langle\langle [x, w_1 w_2 \cdots w_{m-1}] \rangle\rangle = G_{m-1}.$$
Let $R_m = [x, x^{w_m w_{m-1} \cdots w_1}]$ and let $v_i = w_m w_{m-1} \cdots w_i$, so, for example, $v_1 = v_2 w_1$, and choose $s$ and $t$ as before. Let $G_m = \pi(K_1 \# \cdots \# K_m)/\langle\langle R_m \rangle\rangle$. As before, we will show by induction that $G_n \cong G_1$, which is infinite cyclic by assumption. The relation which kills $R_m$, $[x, x^{v_1}] = 1$, is equivalent to $x = (x^{v_1})^{-1} x x^{v_1}$. Raising both sides to the power $sj_1$, we obtain

$$x^{sj_1} = v_1^{-1} x^{-1} v_1 x^{sj_1} v_1^{-1} x v_1$$

$$= v_1^{-1} x^{-1} v_2 w_1 x^{sj_1} v_1^{-1} v_2^{-1} x v_1$$

$$= v_1^{-1} x^{-1} v_2 x^{sj_1} v_2^{-1} x v_1$$

$$= v_1^{-1} x^{-1} v_2 x v_2^{-1} x v_2 x^{sj_1^{-1}} w_1$$

$$= w_1^{-1} x^{-1} v_2 x v_2^{-1} x v_2 x^{-1} w_1 x^{sj_1}.$$  

After canceling the $x^{sj_1}$ terms from both sides, we can further cancel the $w_1$ terms to obtain $x = v_2^{-1} x^{-1} v_2 x v_2^{-1} x v_2$, or $1 = [x, x^{v_2}]$. Repeating this procedure we eventually reach $1 = [x, x^{v_m}] = [x, x^{w_m}]$, the relation which abelianizes $\pi K_m$. Thus

$$G_m = \pi(K_1 \# \cdots \# K_m) / \langle\langle [x, x^{w_m w_{m-1} \cdots w_1}] \rangle\rangle$$

$$\cong \pi(K_1 \# \cdots \# K_{m-1}) / \langle\langle [x, x^{w_{m-1} \cdots w_1}] \rangle\rangle = G_{m-1},$$

and by induction $G_n \cong \mathbb{Z}$. The adaptation to $c > 1$ is the same as in the previous case. \hfill $\square$

**Remark 6.3.** There are many non-trivial examples of 2-knots $K_1, \ldots, K_n$ satisfying the hypotheses of Theorem 1.6. For instance, the technical condition that the $j$th power of a meridian is central is satisfied by any $j$-twist spun knot [36]. Indeed, Kanenobu uses twist-spun knots with coprime twist indices to construct his examples of strong algebraic non-additivity in [18].

Recall Proposition 5.3, which says that for a pair of 2-knots $K_1, K_2$, the algebraic lower bounds satisfy $\max\{\alpha(K_1), \alpha(K_2)\} \leq \alpha(K_1 \# K_2) \leq \alpha(K_1) + \alpha(K_2)$. Kanenobu used his non-additivity result for $a_{\text{st}}$ to prove the following theorem. We note that by Theorem 1.6, his original examples work to prove the following corollary for $\alpha = a_{\text{cw}}$ as well.

**Corollary 6.4 (Kanenobu).** For any positive integers $p_1, \ldots, p_n$ and any integer $q$ with $\max\{p_i\} \leq q \leq p_1 + \cdots + p_n$, there exist 2-knots $K_1, \ldots, K_n$ satisfying:

1. $a_{\text{st}}(K_i) = a_{\text{cw}}(K_i) = p_i$ for all $i$; and
2. $a_{\text{st}}(K_1 \# \cdots \# K_n) = a_{\text{cw}}(K_1 \# \cdots \# K_n) = q$.

While these examples show that the algebraic Casson–Whitney index $a_{\text{cw}}$ is non-additive on general 2-knot groups, we do not know of any classical knot groups for which this is the case. This is in contrast with the algebraic stabilization number $a_{\text{st}}$, which fails to be additive for classical knots by [24] (see the discussion at the end of Section 5). Now, to extend these algebraic results on the non-additivity of $a_{\text{st}}$ and $a_{\text{cw}}$ to their geometric counterparts $u_{\text{st}}$ and $u_{\text{cw}}$, we first relate these invariants through the following lemma.

**Lemma 6.5.** For $1 \leq i \leq n$, let $K_i$ be a 2-knot and let $K = K_1 \# \cdots \# K_n$. If $u_{\text{st}}(K_i) \leq c$ for each $i$, then $u_{\text{st}}(K) \leq c + a_{\text{st}}(K)$. Similarly, if $u_{\text{cw}}(K_i) \leq c$ for each $i$, then $u_{\text{cw}}(K) \leq c + a_{\text{cw}}(K)$. 

Proof. We prove the statement for the stabilization number by induction on the number $n$ of summands. The proof for the Casson–Whitney number is similar. Indeed, it will be convenient for the inductive step to prove a slightly stronger statement: For each $n$, any connected sum $K = K_1 \# \cdots \# K_n$ can be unknotted by first stabilizing at least $a_{st}(K)$ times to obtain a surface $F$ with $\pi F \cong \mathbb{Z}$, and then by stabilizing $c$ times along guiding arcs which are necessarily trivial, since $\pi F$ is cyclic. This statement holds in the case $n = 1$ since the guiding arcs for the trivial stabilizations can be isotoped in the complement of $F$ to be guiding arcs for a collection of $c$ stabilizations that smoothly unknot $K = K_1$. To proceed with the inductive step, we assume that the statement holds when $n = \ell$ and show that it holds when $n = \ell + 1$.

Let $K = K_1 \# \cdots \# K_{\ell + 1}$. Stabilize $K a_{st}(K)$-times to obtain a surface $F$ with $\pi F \cong \mathbb{Z}$, as before, and let $F'$ be the result of (trivially) stabilizing $F$ an additional $c$-times. We claim that $F'$ is unknotted. Since $\pi F \cong \mathbb{Z}$, the guiding arcs for the $c$ trivial stabilizations are isotopic in the complement of $F$ to guiding arcs for a different set of $c$ stabilizations which unknotted the first summand $K_1$. Therefore, $F'$ is the result of trivially stabilizing the surface $F''$ $c$-times, where $F''$ is the surface obtained from $K_2 \# \cdots \# K_{\ell + 1}$ by the stabilizations induced by the original $a_{st}(K)$ stabilizations which produced $F$ from $K$. By the proof of Proposition 5.3 these stabilizations abelianize $\pi(K_2 \# \cdots \# K_{\ell + 1})$, so $\pi(F'') \cong \mathbb{Z}$ and by induction the $c$ trivial stabilizations unknot $F''$. \qed

Our first examples of the non-additivity of the stabilization and Casson–Whitney number now follow as a corollary of Theorem 1.6 and Lemma 6.5.

Corollary 6.6. For $n \geq 1$, consider the $j_i$-twist spins $K_i = \tau^{j_i}k_i$ of classical knots $k_1, \ldots, k_n$, where each $k_i$ is either 2-bridge or has unknotting number one, with pairwise coprime twist indices $j_i \geq 2$. Then,

$$u_{st}(K_i) = u_{cw}(K_i) = 1 \text{ for all } i,$$

$$u_{st}(K_1 \# \cdots \# K_n) \leq 2 \text{ and } u_{cw}(K_1 \# \cdots \# K_n) \leq 2.$$

Proof. First note that by either Corollary 4.6 or Theorem 4.7 (depending on whether the knot $k_i$ is 2-bridge or unknotting number one), $u_{cw}(K_i) = 1$ for each $i$. So, it just remains to show that $u_{st}(K_1 \# \cdots \# K_n) \leq 2$ and $u_{cw}(K_1 \# \cdots \# K_n) \leq 2$. This follows from the previous results of this section. In particular, as noted in Remark 6.3 above, the twist spins $K_i$ have $a_{st}(K_i) = 1$ as well as meridians $x_i \in \pi K_i$ such that $x_i^{j_i} \in \mathbb{Z}(\pi K_i)$. Therefore, these knots satisfy the hypotheses of Theorem 1.6, and so $a_{st}(K) = u_{cw}(K) = 1$ as well. Now Lemma 6.5 applies, and we can conclude that both $u_{st}(K_1 \# \cdots \# K_n), u_{cw}(K_1 \# \cdots \# K_n) \leq 2$, as desired. \qed

Moreover, using a different family of twist spun 2-knots, we formulate the more general non-additivity result featured in the introduction.

Theorem 1.6. For any positive $c, n \in \mathbb{N}$, there exist 2-knots $K_1, \ldots, K_n$ with

$$u_{st}(K_i) = u_{cw}(K_i) = c,$$

$$c \leq u_{st}(K_1 \# \cdots \# K_n) \leq 2c, \text{ and }$$

$$c \leq u_{cw}(K_1 \# \cdots \# K_n) \leq 2c.$$

Proof. Let $\nu \in \{u_{st}, u_{cw}\}$. For the $i$th prime $p_i \in \mathbb{N}$, let $K_i$ be the connected sum of $c$ copies of $\tau^{p_i}T(2, p_i)$, the $p_i$-twist spin of the $(2, p_i)$-torus knot. Since the Alexander module of each summand $\tau^{p_i}T(2, p_i)$ is cyclic, the Nakamura index $m(K_i)$ of the connected sum is equal to $c$. 

Figure 21 (colour online). A schematic for the proof of Lemma 6.5, where \( u_{st}(K_1) = 2 \) and \( a_{st}(K_1 \# \cdots \# K_n) = 1 \). The blue handle abelianizes the group of \( K_1 \# \cdots \# K_n \) and the trivial red handles allow us to inductively unknot each summand.

This matches the upper bound for \( \nu \) given by Theorem 4.7, and so \( \nu(K_i) = c \). Now, each \( K_i \) can also be thought of as a single \( p_i \)-twist spin of the connected sum of \( c \) copies of \( T(2, p_i) \). Therefore \( K = K_1 \# \cdots \# K_n \) is a connected sum of twist-spun knots with coprime twist indices, and so Theorem 1.6 applies to show that \( a_{st}(K) = c \). Then by Lemma 6.5, \( \nu(K) \leq 2c \).

The proof of the next corollary follows from Corollary 4.6, Theorem 4.7, and Lemma 6.5.

**Corollary 6.7.** Let \( n \in \mathbb{N} \) and let \( k_1, \ldots, k_n \) be 1-knots, each either 2-bridge or with unknotting number one. Let \( j_1, \ldots, j_n \) be coprime integers at least 2 and let \( K_i = \tau_{j_i} k_i \). Then \( \nu(K_i) = 1 \) for all \( i \) and \( \nu(K_1 \# \cdots \# K_n) \leq 2 \), where \( \nu \in \{u_{st}, u_{cw}\} \).

### 7. Questions

Here we present some questions that remain.

1. Is \( u_{st} \leq u_{cw}, u_{st} = a_{st}, \) or \( u_{cw} = a_{cw} \)? A 2-knot \( K \) with \( u_{st}(K) > u_{cw}(K) \) or \( u_{st}(K) > a_{st}(K) \) would yield a counterexample to the conjecture that smoothly embedded orientable surfaces in \( S^4 \) with knot group \( \mathbb{Z} \) are smoothly unknotted,\(^\dagger\) since in both cases a surface could be obtained whose complement has cyclic fundamental group but which is not smoothly unknotted. On the other hand, a 2-knot \( K \) with \( u_{cw}(K) > a_{cw}(K) \) would give an immersed 2-sphere \( \Sigma^* \) with \( \pi_1(S^4 - \Sigma^*) \cong \mathbb{Z} \) that is not the result of finger moves on the unknot.

2. Is having a regular homotopy to the unknot where the boundaries of the knotted and standard Whitney disks agree (as in the proof of Theorem 1.4) a characterization of ribbon 2-knots?

3. Given a 2-knot \( K \) in \( S^4 \), are Singh’s invariants \( d_{st}(K) \) and \( d_{sing}(K) \) from [33] ever greater than 1?

4. Does there exist a 2-knot \( K \) such that \( u_{cw}(K) - u_{st}(K) > 1 \)?

5. Are Casson–Whitney number one 2-knots ‘algebraically prime’, that is, if \( K = K_1 \# K_2 \), then at least one summand \( K_1 \) or \( K_2 \) has knot group \( \mathbb{Z} \)?

6. Are pairs of 2-knots in \( S^4 \) always related by an arc-standard regular homotopy? Recall that in the proof of Theorem 1.2, we prove this for 2-knots with a length 1 regular homotopy to the unknot, by starting with a homotopy for which all pairs of knotted and unknotted arcs in the pre-image of the standard immersion have the same endpoints, and then performing ‘standard braid twists’ and isotopies rel endpoints until certain pairs of arcs agree. However, even allowing additional manipulations like ‘slides’ of Whitney disks (as in Figure 4 of [31]),

\(^\dagger\)And in fact, it is only known in genus 0 [11, Theorem 11.7A] or if the genus is \( \geq 3 \) [9] that such surfaces are even topologically unknotted.
such an argument seems to fail for certain initial configurations of Whitney arcs for regular homotopies of higher length, including the one in Figure 22. Thus we ask: can the arcs in Figure 22 actually appear as the pre-images of the knotted and standard Whitney arcs of a length 2 regular homotopy from a 2-knot $K$ with $u_{cw}(K) = 2$ to the unknot?

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