Analytical effective one-body formalism for extreme-mass-ratio inspirals with eccentric orbits

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Received 13 January 2021, revised 31 March 2021
Accepted for publication 27 April 2021
Published 21 June 2021

Abstract

Extreme-mass-ratio inspirals (EMRIs) are among the most important sources for future spaceborne gravitational wave detectors. In this kind of system, compact objects usually orbit around central supermassive black holes on complicated trajectories. Usually, these trajectories are approximated as the geodesics of Kerr space-times, and orbital evolution is simulated with the help of the adiabatic approximation. However, this approach omits the influence of the compact object on its background. In this paper, using the effective one-body formalism, we analytically calculate the trajectory of a nonspinning compact object around a massive Kerr black hole in an equatorial eccentric orbit (omitting the orbital inclination) and express the fundamental orbital frequencies in explicit forms. Our formalism includes the first-order corrections for the mass ratio in the conservative orbital motion. Furthermore, we insert the mass-ratio-related terms into the first post-Newtonian energy fluxes. By calculating the gravitational waves using the Teukolsky equations, we quantitatively reveal the influence of the mass of the compact object on the data analysis. We find that the shrinking of geodesic motion by taking small objects as test particles may not be appropriate for the detection of EMRIs.

Keywords: gravitational wave, extreme-mass-ratio inspiral, effective one-body

(Some figures may appear in colour only in the online journal)

1. Introduction

The successful detection of gravitational waves (GWs) by Advanced Laser Interferometer Gravitational Wave Observatory (LIGO) and Virgo [1–6] has announced that the era of GW astronomy is imminent. This kind of ground-based detector observes GWs in the high-frequency band. The Laser Interferometer Space Antenna (LISA), a spaceborne GW detector proposed by Europe and USA [7], and, at the same time, two Chinese space projects Taiji [8] and Tian-Qin [9], are planned to launch after 2030. All these detectors focus on GWs at low frequencies (of about 0.1 mHz to 1 Hz). Extreme-mass-ratio inspirals (EMRIs) composed of compact objects (stellar black holes, neutron stars, white dwarfs, etc.) and supermassive black holes (SMBHs) are expected to be among the most important sources for these spaceborne detectors [10–12].

The signals from EMRIs are usually very weak, but after a year of observation, the signal-to-noise ratio can reach a value that can be detected by matched filtering technology [10]. To detect this type of long-duration signal, the accuracy requirements imposed on waveform templates are very high. Typically, after 10⁵ cycles, the dephasing should be less than a few radians [11, 13]. Nowadays, there are a few EMRI templates, such as the analytic kludge (AK) [14], augmented analytic kludge (AAK) [15], numerical kludge (NK) [16], and exploratory software package of EMRI gravitational-wave (XSPEG) [17] templates, and, more recently, a waveform obtained via multivoice decomposition [18]. All of these treat the small object as a test particle and omit its mass from their conservation dynamics. Some works have included the correction due to the small mass using the effective-one-body (EOB) formalism, but only for circular orbits [19, 20] or for
eccentric orbits with fitted data parameters [21]. There are also intermediate-mass-ratio inspirals (IMRIs) composed of compact stellar objects and intermediate-mass black holes (IMBHs) or IMBHs orbiting SMBHs; the mass ratios of IMRIs are around $10^{-3}$ [22]. In these situations, the mass-ratio correction to the conservative orbital dynamics should be more important.

The EOB formalism, by including the mass-ratio corrections in post-Newtonian (PN) expansions, can accurately describe the dynamic evolution of binary black holes [23, 24], and is widely used to construct waveform templates for LIGO [25–32]. Most of these models have only considered cases with circular orbits. Recently, Hinderer et al reported an analytical eccentric EOB dynamics for Schwarzschild black holes (BHs) [33]. Cao and Han built an eccentric effective-one-body-numerical-relativity (EOBNR) waveform template spanning effective-one-body-numerical-relativity eccentric (SEOBNRE) for spinning black holes [34], but the orbits were not geometrized and the orbital parameters were well defined.

It is well known that the orbits of EMRIs can be highly eccentric [11], and that, in general, the supermassive black hole at the center should be spinning. In this paper, we extend the previous work by Hinderer and Babak to Kerr black holes. To start with, for equatorial eccentric EMRIs (omitting the orbital inclination and the spin of the small compact object), we analytically convert the original EOB dynamic equations into geometric kinetic motion with orbital parameters: the semilatus rectum, $p$, and the eccentricity $e$, together with two phase variables associated with the spatial geometries of the radial and azimuthal motions, denoted by $\xi$ and $\phi$, respectively. Because of the extremely small mass ratio, we omit the spin of the effective small body, and proceed without the complicated spin–spin coupling terms.

An important feature of the dynamics of an extreme-mass-ratio binary system in a bounded equatorial eccentric orbit is that the orbit can be characterized by two frequencies: the radial frequency $\omega_r$ associated with the libration between the apoapsis and the periapsis, and the azimuthal rotational frequency, $\omega_\phi$. Once these two frequencies and orbital parameters have been obtained, one can solve the Teukolsky equations [35] to get accurate waveforms for eccentric EMRIs. The combination of the EOB and Teukolsky-based waveforms has been implemented by one of the authors and was called the ET code [21, 36–41].

This paper is organized as follows. In section 2, we reparameterize the original spinning EOB dynamic description to a geometric formalism in the more efficient re-parameterized terms of $(p, e, \xi, \phi)$. We analytically express the fundamental frequencies using two integrals with the parameter $\xi$. We then focus on the evolution of the orbital parameters, by combining the PN radiation reaction formulas. We also show the waveforms calculated from the Teukolsky equations. In particular, we investigate the influence of the mass ratio on the detection of EMRIs. Section 3 contains our conclusions and an outlook on tasks remaining for future work. Finally, the appendices contain the details of the EOB formalism and the expressions for orbital evolution.

Throughout this paper, we will use the geometric units $G = c = 1$, and the units of time and length is the mass of system $M$, and the units of linear and angular momentum are $\mu$ and $\mu M$ respectively, where $\mu$ is the reduced mass of the effective body.

2. Geometrization of the conservative dynamics in deformed Kerr space-time

2.1. The effective one-body Hamiltonian

The EOB formalism was originally introduced in [23, 24] to describe the evolution of binary systems. We start by considering an EMRI system with a central Kerr black hole $m_1$ and an inspiraling object $m_2$ (we assume it is nonspinning, for simplicity) which is restricted to the equatorial plane of $m_1$ ($m_2 < m_1$). For the moment, we neglect the radiation reaction effects and focus on purely geodesic motion. The conservative orbital dynamics is derived via Hamilton’s equations using the EOB Hamiltonian $H_{\text{EOB}} = M \sqrt{1 + 2\nu(H_{\text{eff}} - 1)}$, where $M = m_1 + m_2$, $\nu = m_2/m_1^2$, $\mu = \nu M$, and $H_{\text{eff}} = H_{\text{eff}}/\mu$. The deformed Kerr metric is given by [42]

\begin{align}
  g^{tt} &= -\frac{\Lambda_r}{\Delta_{\text{eff}}\Sigma}, \\
  g^{rr} &= \frac{\Delta_r}{\Sigma}, \\
  g^{\theta\theta} &= \frac{1}{\Sigma}, \\
  g^{\phi\phi} &= \frac{1}{\Lambda_r} \left( -\frac{\bar{\omega}_{\text{eff}}^2}{\Delta_{\text{eff}}} + \Sigma \right), \\
  g^{\psi\psi} &= \frac{\bar{\omega}_{\text{eff}}}{\Delta_{\text{eff}}} \\
\end{align}

The quantities $\Sigma$, $\Delta_r$, $\Delta_{\text{eff}}$, $\Lambda_r$, and $\bar{\omega}_{\text{eff}}$ in equations (1)–(5) are given by

\begin{align}
  \Sigma &= \frac{r^2}, \\
  \Delta_r &= \frac{r^2}{A(u) + \frac{a^2}{M^2} u^2}, \\
  \Delta_{\text{eff}} &= \Delta_1 D^{-1}(u), \\
  \Lambda_r &= (r^2 + a^2)^2 - a^2 \Delta_r, \\
  \bar{\omega}_{\text{eff}} &= 2a M r + \omega_{1A}^{\text{eff}} \nu M^3 r + \omega_{2A}^{\text{eff}} \nu \frac{Ma^3}{r},
\end{align}

where $a = |S_{\text{Kerr}}|/M$ is the effective Kerr parameter and $u = M/r$. The values of $\omega_{1A}^{\text{eff}}$ and $\omega_{2A}^{\text{eff}}$ are about $-10$ and $20$, respectively, as given by a preliminary comparison of the EOB model with numerical relativity results [43, 44]. The metric potentials $A$ and $D$ for the EOB model are given in appendix A. Though the spin of the small body is zero, its effective spin is nonzero. The effective spins of the particles and the deformed Kerr black hole are [45]
we can omit these three terms for simplicity, and retain sufficient accuracy. The effective spins can then be rewritten as

\begin{equation}
S^* = a M \frac{m_2}{m_1}, \quad S_{Kerr} = S_1 + S_2.
\end{equation}

For \( S_2 = 0 \), it is clear that \( S_{Kerr} = S_1 \) and \( S^* \approx \mathcal{O}(\nu) \). The last three terms in the equation for \( S^* \) are at least \( \mathcal{O}(\nu^{-2}) \) [42, 45, 46]. In this paper, considering the small mass ratio, we can omit these three terms for simplicity, and retain sufficient accuracy. The effective spins can then be rewritten as

\begin{equation}
S^* = a M \frac{m_2}{m_1},
\end{equation}

\begin{equation}
S_{Kerr} = a M.
\end{equation}

The effective Hamiltonian should then describe a spinning test particle in the deformed Kerr metric, and reads

\begin{equation}
H_{eff} = H_{NS} + H_5,
\end{equation}

where \( H_5 \) is the Hamiltonian caused by the spin of the effective particle. In this work, to develop the model step by step, we ignore the spin of the small compact object, though some researchers have argued that the spin is non-negligible for EMRIs [36, 47]. In this situation, considering that the effective spin parameter of the small object \( s \sim \mu a/M \) should be very small for EMRIs, we plan to omit this term in this work, for simplification. In Table 1, the ratios of \( H_5 \) to \( H_{NS} \) for different spin and orbital radii for small mass-ratio (equal or less than \( 10^{-1} \)) circular orbits are listed. We find that the magnitude of this spin Hamiltonian term is a fraction of only about \( 10^{-2} \nu \) of the nonspinning term; even for extreme Kerr parameters and close to the horizon, it is just \( 10^{-1} \nu \), and the relative error in the orbital frequency due to omitting \( H_5 \) is also \( \lesssim 10^{-7} \nu \) (for smaller spin and larger orbital radii, this error will be reduced). Therefore, for the EMRIs described in this paper, we omit this nonlinear term, and will include it in the next work on the topic of comparable binaries.

The most dominant part, \( H_{NS} \), is the Hamiltonian for a nonspinning test particle of mass \( \mu \) in the deformed Kerr metric. Using equation (2.26) of [48], we have

\begin{equation}
H_{NS} = \frac{g^{00}P_\mu}{\sqrt{-g}} + \frac{1}{\sqrt{-g}} \left[ \mu^2 + \left( g^{00} - \frac{(g^{00})^2}{g^{\mu\nu}} \right) P_\mu^2 + g^\mu\nu P_\mu^2 + \frac{Q_4 P_r^4}{\mu^2} \right],
\end{equation}

where \( P_r \) and \( P_\phi \) are the radial and azimuthal angular momenta. The functions \( Q_4 = \frac{2(4 - 3\omega_4 M^2)}{\rho_f} \) and \( Q_4 = \frac{Q_4 P_r^4}{\mu^2} \) [49] represent a non-geodesic term that appears at the third PN order.

The energy of the system is given by

\begin{equation}
E = H_{EOB},
\end{equation}

which implies the relation

\begin{equation}
\dot{H}_{eff}(E) = 1 - \frac{1}{2\nu} \left( \frac{E^2}{M^2} - 1 \right).
\end{equation}

The canonical EOB dynamics without the radiation reaction are

\begin{equation}
\dot{r} = \frac{\partial H_{EOB}}{\partial P_r}, \quad \dot{P}_r = -\frac{\partial H_{EOB}}{\partial r},
\end{equation}

\begin{equation}
\dot{\phi} = \frac{\partial H_{EOB}}{\partial P_\phi}, \quad \dot{P}_\phi = -\frac{\partial H_{EOB}}{\partial \phi} = 0.
\end{equation}

The effective Hamiltonian associated with the deformed Kerr metric has the form

\begin{equation}
\dot{H}_{eff} = \frac{\Delta_0 (\Delta_r(r^2 + \Delta_r P_r^2 + r^2 Q_4 P_r^3) + r^4 P_r^5)}{\Lambda_r} + \Delta_0 (\Delta_r (\Delta_r P_r^2 + \Delta_0 P_\phi^2 + r^2 Q_4 P_r^3) + r^4 P_r^5),
\end{equation}

where we define the reduced momenta \( \hat{P}_r = P_r/\mu \) and \( \hat{P}_\phi = P_\phi/\mu \). Solving equation (18) for \( P_r \), in terms of \( (E, P_\phi, r) \) leads to

\begin{equation}
\dot{r} = \frac{\Delta_0 (\Delta_r (\Delta_r P_r^2 + \Delta_0 P_\phi^2 + r^2 Q_4 P_r^3) + r^4 P_r^5)}{E \sqrt{\Delta_r (\Delta_r (r^2 + \Delta_r P_r^2 + r^2 Q_4 P_r^3) + r^4 P_r^5) + r^4 P_r^5}}.
\end{equation}
Equations (20) are more convenient than the original canonical EOB equations (17), because the dependence on \( \ddot{P}_r \) (19) has been eliminated by the energy \( E \), which is a constant in a conservative system and only changes due to the radiation reaction if GW fluxes are considered.

2.2. Re-parameterization of the constants and equations of motion

The constants of motion and the dynamic equations in the last subsection can be written in terms of the geometrized orbital elements, i.e., the \emph{semi-latus rectum}, \( p \), and the eccentricity, \( e \). This will make the description of the system more intuitive. For an eccentric orbit, periastron and apastron points exist, which can be expressed as

\[
r_1 = \frac{pM}{1 - e}, \quad r_2 = \frac{pM}{1 + e},
\]

where \( r_1, r_2 \) are the turning points of the radial motion, i.e., the apastron and periastron, respectively. By setting the radial equation of motion (20a) equal to zero, i.e., \( \ddot{P}_r = 0 \), we can solve for the two points. Furthermore, by taking \( r_1, r_2 \) into equation (19) to make \( \ddot{P}_r = 0 \) again, we finally obtain the constants of motion (\( E, \ddot{P}_\phi \)) in terms of \((p,e)\)

\[
\frac{E^2}{M^2} = 1 + 2\nu \left( \frac{a_1 \ddot{P}_\phi + \sqrt{b_1 c_1^2 + 1}}{b_1} - 1 \right),
\]

where the coefficients are

\[
a_1 = \frac{(1 - e)^2((1 - e)^2\nu(\omega_1^d + \omega_2^d) + 2p^2)}{p^3 - a^2p^3(A(r_1) - 2)(1 - e)^2},
\]

\[
a_2 = \frac{(1 + e)^3((1 + e)^2\nu(\omega_1^d + \omega_2^d) + 2p^2)}{p^3 - a^2p^3(A(r_2) - 2)(1 + e)^2},
\]

\[
b_1 = \frac{a_1(1 - e)^2 + A(r_1)p^2}{\sqrt{p^2 - a^2(A(r_1) - 2)(1 - e)^2}},
\]

\[
b_2 = \frac{a_2(1 + e)^2 + A(r_2)p^2}{\sqrt{p^2 - a^2(A(r_2) - 2)(1 + e)^2}},
\]

\[
c_1 = \frac{(1 - e)^2}{p^2 - a^2(A(r_1) - 2)(1 - e)^2},
\]

\[
c_2 = \frac{(1 + e)^2}{p^2 - a^2(A(r_2) - 2)(1 + e)^2}.
\]

The above formalism for a Kerr black hole is much more complicated than the Schwarzschild ones described in [33]. Obviously, for the test-particle limit \( \nu \rightarrow 0 \), the above results will return the geodesic motion of a test particle in Kerr space-time.

The description of geodesic motion around BHs is based on the \emph{semi-latus rectum}, \( p \), and the eccentricity, \( e \). This \( E \) can be expressed as

\[
r = \frac{pM}{1 + e \cos \xi},
\]

so that the periastron and apastron correspond to \( \xi = (0, \pi) \mod 2\pi \), respectively. Taking a derivation of equation (24), we can obtain the evolution equation for the phase variable \( \xi \)

\[
\dot{\xi} = \frac{(1 + e \cos \xi)\ddot{r}}{epM \sin \xi} + \frac{\cot \xi}{e \dot{e}} - \frac{1 + e \cos \xi}{e \sin \xi} \ddot{p}.
\]

For a conservative system, \( \dot{p} = \dot{e} = 0 \). However, if we take the radiation reaction of the GWs into account, the rate of change of the energy and the reduced angular momentum are given by

\[
\begin{align}
\frac{dE}{dt} &= \frac{\partial E}{\partial \dot{p}} \dot{p} + \frac{\partial E}{\partial \dot{e}} \dot{e}, \\
\frac{d\dot{P}_\phi}{dt} &= \frac{\partial \dot{P}_\phi}{\partial \dot{p}} \dot{p} + \frac{\partial \dot{P}_\phi}{\partial \dot{e}} \dot{e}.
\end{align}
\]

Using equations (26), we can express the evolution of \((p, e)\) in terms of the energy and angular momentum fluxes of gravitational radiation

\[
\begin{align}
\dot{\nu} &= -c_{\nu \nu} \frac{d\nu}{dt} - c_\nu \frac{dE}{dt} - c_{\nu \phi} \frac{d\dot{P}_\phi}{dt}, \\
\dot{c}_b &= c_b - \frac{\partial C}{\partial b}, \quad \frac{\partial E}{\partial \dot{p}}(\partial \dot{P}_\phi/\partial \dot{e}) - (\partial E/\partial \dot{e})(\partial \dot{P}_\phi/\partial \dot{p}),
\end{align}
\]

where \( C = (E, \ddot{P}_\phi), b = (p, e), \) and the derivatives can be computed from the expressions in equation (22).

The final set of EOB equations of motion with the radiation reaction are equations (27a) together with the evolution of the phases described by equations (25) and (26b), and the radius of motion at an arbitrary moment, as given by equation (24). All the equations of motion are now expressed only in terms of the geometric parameters \((p, e, \xi)\) and the effective Kerr parameter \( a \).
Finally, we can get the coordinates of the effective test particle in terms of $\xi$ only:

$$t(\xi) = \int_0^\xi \frac{1}{p} d\xi, \quad (28a)$$
$$r(\xi) = \frac{pM}{1 + e \cos \xi}, \quad (28b)$$
$$\theta = \frac{\pi}{2}, \quad (28c)$$
$$\phi(\xi) = \int_0^\xi \frac{\dot{\phi}}{p} d\xi, \quad (28d)$$

where $\mathcal{P}(e, p, \xi)$ is the first term on the right-hand side of equation (25)

$$\mathcal{P}(e, p, \xi) = \frac{(1 + e \cos \xi)^2}{e p \sin \xi} \times \frac{\Delta_r(\Delta_p^2 + 2r^2 Q_p \hat{P}_r^2)}{E \sqrt{\Delta_r(\Delta_r^2 + \Delta_p^2 + 2r^2 Q_p \hat{P}_r^2) + r^4 \hat{P}_r^2}}. \quad (28e)$$

2.3. Quantitative influences of the mass ratio on the conservative dynamics

We now calculate the fundamental orbital frequencies in terms of the geometric parameters. This has been done in the test-particle ($\nu = 0$) limit due to the analytical integrals of the geodesic in Kerr space-time [50, 51]. However, in the EOB formalism with the mass-ratio correction, the situation becomes complicated. First, we express the radial frequency $\omega_r$ which reflects the period of the radial motion from the periastron to the apastron and back to the periastron again, and the orbital period $T_r$ can be calculated by taking $\xi$ from 0 to $2\pi$, then

$$\omega_r = \frac{2\pi}{\int_0^{2\pi} \frac{1}{p} d\xi}. \quad (29)$$

In equation (20b), we have already written the variation of $\phi$ with coordinate time $t$. Strictly speaking, the radial motion is the real periodic motion. When the particle passes through the periastron twice, $\Delta \phi$ will be larger than $2\pi$ because of the periastron precession caused by the relativistic effect. We can then define the frequency of the azimuthal motion to be $\Delta \phi/T_r$. Therefore, we can compute the azimuthal frequency from the orbital average of the $\phi$ motion as

$$\omega_\phi = \langle \dot{\phi} \rangle = \frac{\int_0^{2\pi} \frac{\dot{\phi}}{p} d\xi}{\int_0^{2\pi} \frac{1}{p} d\xi}, \quad (30)$$

where $\hat{r}$ is given in equation (20a), and $P_r$, $E$, and $P_\phi$ are expressed in equations (19) and (22). By just replacing $r$ in these equations with $\xi$, we cause the above two integrals to only contain the argument $\xi$, meaning that they can easily be integrated. We do not write down the fully expanded expressions for these two integrals here, as they are direct and trivial.

With the expressions of the two fundamental frequencies at hand, we now investigate how the effective Kerr parameter $a$ and the symmetric mass ratio $\nu$ affect the features of the radial and azimuthal frequencies and compare them with the case of the test-particle limit in conservative dynamics. The effect of the spin parameter on these frequencies for various values of the semilatus rectum, $p$, and the eccentricity, $e$, is shown in figure 1. Obviously, as the semilatus rectum increases, we can say that the mean orbital radius correspondingly increases, leading to a decrease in the periastron precession $\omega_\phi/\omega_r - 1$, which approaches the Newtonian limit $\omega_\phi = \omega_r$. Interestingly, as the spin of the central SMBH increases, the periastron precession decreases. The effect of eccentricity on the precession becomes obvious only when the separation, $p$, and spin, $a$, are both small (see the top three curves in figure 1).

We now consider the radial and azimuthal frequencies’ shifts due to the mass ratio of the binary in the absence of the radiation reaction. It is known that the motion of the test particle has a precise analytical solution. For a binary system, where the mass ratio cannot be omitted, we have given the analytical EOB orbital solution equations (28). In order to observe the impact of the mass ratio on the radial and azimuthal frequencies under different conditions with various orbital parameters, we compare the test-particle results and the EOB results in table 2. The rightmost column of this table shows the relative frequency shift divided by the mass ratio: $\Delta \omega_r/(\omega_r)$, where $\Delta \omega_r/\omega_r$ is the relative difference of the radial/azimuthal frequency between the EOB frequencies ($\omega_{r,0}$, $\omega_{\phi,0}$) and the test-particle frequencies, i.e., $(\omega_r - \omega_{r,0})/\omega_{r,0}$, $(\omega_\phi - \omega_{\phi,0})/\omega_{\phi,0}$.

The frequency shift $\Delta \omega_r/(\omega_r)$ due to the mass ratio is almost independent of the mass ratio itself, and the relative

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.pdf}
\caption{Effect of the spin parameter on periastron precession for a binary with a symmetric mass ratio $\nu = 10^{-3}$. With a fixed spin parameter, an orbit with a bigger semilatus rectum has a smaller periastron precession. For an orbit with the same eccentricity and semilatus rectum, a larger spin parameter induces a smaller periastron precession.}
\end{figure}
Table 2. Relative differences in the orbital frequencies $\omega_r$, $\omega_f$ between the EOB model with small mass ratios and the test-particle approximation.

| $a/M$ | $p/M$ | $e$ | Test-particle | $\nu = 10^{-2}$ | $\nu = 10^{-3}$ | $\nu = 10^{-4}$ | $\nu = 10^{-5}$ | $\nu = 10^{-6}$ | $\frac{\Delta}{\omega}$ (/$/nu$) |
|-------|-------|-----|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------------------------|
| 0.99  | 5     | 0.1 |               | 0.050841032     | 0.052967383     | 0.051054792     | 0.050862411     | 0.050843170     | 0.050841250     | 4.2                     |
| 0.99  | 10    | 0.1 | $\omega$     | 0.081375480     | 0.080796016     | 0.081318545     | 0.081369800     | 0.081374912     | 0.081375431     | 0.70                    |
| 0.99  | 5     | 0.6 | $\omega$     | 0.031648541     | 0.032582060     | 0.031743453     | 0.031658044     | 0.031649491     | 0.031648640     | 3.0                     |
| 0.99  | 20    | 0.6 | $\omega$     | 0.051669258     | 0.050254323     | 0.051523452     | 0.051654638     | 0.051667794     | 0.051669119     | 2.8                     |
| 0.5   | 5     | 0.1 |               | 0.050841032     | 0.052967383     | 0.051054792     | 0.050862411     | 0.050843170     | 0.050841250     | 4.2                     |
|       |       |     | $\omega$     | 0.081375480     | 0.080796016     | 0.081318545     | 0.081369800     | 0.081374912     | 0.081375431     | 0.70                    |
|       |       |     | $\omega$     | 0.031648541     | 0.032582060     | 0.031743453     | 0.031658044     | 0.031649491     | 0.031648640     | 3.0                     |
|       |       |     | $\omega$     | 0.051669258     | 0.050254323     | 0.051523452     | 0.051654638     | 0.051667794     | 0.051669119     | 2.8                     |
|       |       |     | $\omega$     | 0.050841032     | 0.052967383     | 0.051054792     | 0.050862411     | 0.050843170     | 0.050841250     | 4.2                     |
Figure 2. The frequency shifts $\Delta \omega_r$ and $\Delta \omega_f$ versus eccentricity $e$ for various $a$, $\nu$, and $p$. The triangles, solid lines, and points represent $\nu = 10^{-6}$, $10^{-4}$, and $10^{-2}$, respectively.

Figure 3. The frequency shifts $\Delta \omega_r$ and $\Delta \omega_f$ versus the semilatus rectum, $p$, for various $a$, $e$ with $\nu = 10^{-4}$. When $p$ becomes small, the frequency shift grows very rapidly.

Figure 4. The frequency shifts $\Delta \omega_r$ and $\Delta \omega_f$ versus the Kerr parameter $a$ for various $p$, $e$ with $\nu = 10^{-4}$. When $a$ becomes smaller, the frequency shift can become larger, since the orbit approaches the ISBO.
shift $\Delta \omega/\omega$ is in the range of about $[0.1 \nu, 10 \nu]$, based on table 2. This indicates that we must consider the influence of the mass ratio in the conservative dynamics of EMRIs, because the cycles of typical EMRI waves are $\sim 1/\nu$ in the LISA waveband, and if the relative frequency error reaches $\sim \nu$, the dephasing will accumulate to a few radians at the end of the evolution. This may cause a failure to detect EMRIs using the test-particle ($\nu = 0$) model.

Figure 2 illustrates the effect of eccentricity on the radial frequency shift (left panel) and the azimuthal frequency shift (right panel) for various values of the \textit{semilatus rectum}, $p$, and the spin parameter, $a$. It shows that the shifts of $\omega_r$ and $\omega_\phi$ due to the mass ratio have different effects versus eccentricity. The results for different mass ratios are also plotted (triangles, solid lines, and points represent $\nu = 10^{-6}$, $10^{-4}$, and $10^{-2}$, respectively), clearly showing that $\Delta \omega/(\omega \nu)$ is not sensitive to the mass ratio, except for the radial frequency when the trajectory approaches the vicinity of the innermost stable bound orbit (ISBO) and the mass ratio becomes 0.01 (blue lines in the left panel).

Figure 3 demonstrates the effect of the \textit{semilatus rectum}, $p$, on the radial frequency shift (left panel) and the azimuthal frequency shift (right panel) for various values of the spin $a$ and the eccentricity $e$. When $p$ becomes smaller, the frequency shifts become larger. The sudden growth of the frequency shift is due to the orbit approaching the ISBO. Figure 4 illustrates the effect of the BH’s spin on the radial frequency shift (left panel) and the azimuthal frequency shift (right panel) for various values of the \textit{semilatus rectum}, $p$, and the eccentricity, $e$. For the cases of $p = 5$, when $a$ becomes small, the orbits are very close to the ISBO, and the frequency shifts grow very rapidly.

The \textit{semilatus rectum}, $p$, of the ISBO is the separatrix of the bound orbits, and in the test-particle limit, it is given by the analytic expression from equation (24) of [52]

$$p_s = (6 + 2a)M \mp 8a \sqrt{\frac{1 + e}{2e + 6}} + \mathcal{O}(a^2). \quad (31)$$

However, the above equation is approximate for the ISBO, even for test particles. Considering the influence of the small mass on the background, the ISBOs of EMRIs should deviate from the test-particle ISBO. Figure 5 shows the effect of the mass ratio on the separatrix value $p_s(e)$. We can see that the EOB’s ISBO can deviate by as much as 10% from the test-particle limit. In addition, the error of approximate expression (31) becomes large for a rapidly spinning black hole.

### 3. The orbital evolution and waveforms

In this section, we introduce the gravitational wave fluxes of energy and angular momentum described in previous literature, and calculate the gravitational wave strain using the Teukolsky equation, which is a perturbation theory for Kerr black holes [35]. We use the PN radiation reaction formulas which include the mass ratio in the 1PN terms of energy and angular momentum fluxes, and compare the performance of approximate fluxes from the 2PN to the 4PN.

#### 3.1. Radiation fluxes

The analytic 4PN $\mathcal{O}(e^6)$ formula for energy and angular momentum fluxes in Boyer-Lindquist coordinates is given by [50] in terms of the parameter $\nu \equiv \sqrt{1/p}$. For convenience, we use $p$ and new expressions, as follows

$$\langle \mathcal{F} \rangle_{4\text{PN}} = \frac{32\nu^2(1 - e^2)^{3/2}}{5p} \left\{ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right\}$$

$$- \frac{1}{p^2} \left( \frac{1247}{336} + \frac{9181e^2}{672} + \frac{809e^4}{128} - \frac{8609e^6}{5376} \right)$$

$$+ \frac{1}{p^{3/2}} \left[ \pi \left( 4 + \frac{1375e^2}{48} + \frac{3935e^4}{192} + \frac{10007e^6}{9216} \right) \right]$$

$$- \frac{q}{p^2} \left( \frac{73}{12} + \frac{823e^2}{24} + \frac{949e^4}{32} + \frac{491e^6}{192} \right)$$

$$- \frac{1}{p^2} \left( \frac{4471}{9072} + \frac{172157e^2}{2592} \right)$$

$$+ \frac{2764345e^4}{24192} - \frac{3743e^6}{2304}$$

$$- \frac{q^2}{p^2} \left( \frac{33}{16} + \frac{359e^2}{32} + \frac{1465e^4}{128} + \frac{883e^6}{768} \right)$$

$$- \frac{1}{p^{5/2}} \left[ \frac{8191}{672} + \frac{44531e^2}{336} \right]$$
\[
\begin{align*}
&+ \frac{4311389e^4}{43008} - \frac{15670391e^6}{387072} \\
&- \left( \frac{q}{p^2} \right) \frac{3749}{336} + \frac{1759e^2}{56} - \frac{111203e^4}{1344} - \frac{49685e^6}{448} \\
&+ \left( \frac{1}{p^3} \right) \frac{6643739519}{69854400} + \frac{43072561991e^2}{27941760} \\
&+ \frac{919773569303e^4}{27941760} + \frac{308822406727e^6}{186278400} \\
&+ \log(p) \left( \frac{586}{105} + \frac{726e^2}{63} + \frac{553297e^4}{1260} + \frac{187357e^6}{1260} \right) \\
&- \gamma \left( \frac{1712}{105} + \frac{14552e^2}{63} + \frac{553297e^4}{1260} + \frac{187357e^6}{1260} \right) \\
&- \log(p) \left( \frac{3424}{105} + \frac{13696e^2}{315} \right) \\
&+ \frac{12295049e^4}{1260} - \frac{24908851e^6}{252} \\
&- \log(3) \left( \frac{234009e^2}{560} - \frac{2106081e^4}{448} + \frac{86419261e^6}{35840} \right) \\
&+ \pi^2 \left( \frac{16}{3} + \frac{680e^2}{9} + \frac{5171e^4}{36} + \frac{1751e^6}{36} \right) \\
&- q^2 \left( \frac{169}{6} + \frac{4339e^2}{16} + \frac{42271e^4}{96} + \frac{4867907e^6}{2764} \right) \\
&+ q^4 \left( \frac{3419}{168} + \frac{50271e^2}{224} + \frac{340141e^4}{896} + \frac{1013347e^6}{5376} \right) \\
&- \log(5) \left( \frac{5224609375e^6}{193536} \right) \\
&- \frac{1}{p^{7/2}} \left[ \pi \left( \frac{16285}{504} + \frac{22798583e^2}{48384} + \frac{65448785e^4}{48384} \right) \right. \\
&\left. + \frac{41758768871e^6}{41803776} \right] \\
&- q^2 \pi \left( \frac{65}{8} + \frac{2277e^2}{32} + \frac{103229e^4}{768} + \frac{1307875e^6}{18432} \right) \\
&- q^4 \left( \frac{83819}{1296} + \frac{12203083e^2}{18144} \right. \\
&\left. + \frac{3918011e^4}{2268} + \frac{1325729e^6}{1728} \right) \\
&+ q^6 \left( \frac{151}{12} + \frac{6497e^2}{48} + \frac{14041e^4}{64} + \frac{12789e^6}{128} \right) \\
&- \frac{1}{p^3} \left[ \frac{323105549467}{3178375200} + \frac{13084171033763e^2}{3178375200} \right. \\
&\left. + \frac{454079900391707e^4}{25427001600} + \frac{12759433101997e^6}{847566720} \right] \\
&+ \log(p) \left( \frac{232597}{8820} + \frac{831307e^2}{1176} + \frac{25695073e^4}{11760} \right) \\
&+ \frac{14109647e^6}{14112} \\
&- \log(2) \left[ \frac{39931}{294} - \frac{2373293e^2}{1260} + \frac{2273961523e^4}{17640} \right. \\
&\left. - \frac{503388564173e^6}{317520} \right] \\
&+ \log(3) \left( \frac{47385}{1568} - \frac{55105839e^2}{15680} + \frac{3074548023e^4}{71680} \right. \\
&\left. - \frac{107130980133e^6}{1003520} \right] \\
&+ \log(5) \left( \frac{15869140625e^4}{903168} + \frac{10089048828125e^6}{16257024} \right) \\
&- \gamma \left( \frac{232597}{4410} + \frac{831307e^2}{588} + \frac{25695073e^4}{5880} + \frac{14109647e^6}{7056} \right) \\
&+ \pi^2 \left( \frac{1369}{126} + \frac{83045e^2}{252} + \frac{56023e^4}{56} + \frac{327895e^6}{1008} \right) \\
&- q\pi \left( \frac{3883}{168} + \frac{57405e^2}{224} + \frac{15198859e^4}{43008} + \frac{59177891e^6}{64512} \right) \\
&+ q^2 \left( \frac{124091}{9072} + \frac{6211679e^2}{9072} + \frac{32107339e^4}{18144} + \frac{16038995e^6}{10368} \right) \\
&- q^4 \left( \frac{17}{16} + \frac{279e^2}{16} + \frac{5399e^4}{192} + \frac{2513e^6}{192} \right) \\
&\end{align*}
\]
\[
\left( \frac{G^7}{\hbar^2} \right)_{\text{NP}} = \frac{32\nu^2(1 - \nu^2)^{3/2}}{5p^{7/2}} \left\{ 1 + \frac{7}{8} \nu^2 \right\} \\
- \frac{1}{p} \left[ \frac{1247}{336} + \frac{425e^2}{336} - \frac{10751e^4}{2688} \right] \\
+ \frac{1}{p^{3/2}} \left[ \pi \left( \frac{4}{8} + \frac{97e^2}{32} + \frac{49e^4}{4608} \right) \right] \\
- \frac{q}{p^3} \left( \frac{61}{12} + \frac{119e^2}{8} + \frac{183e^4}{32} \right) \\
+ \frac{1}{p^{5/2}} \left[ \frac{44711}{9072} + \frac{302893e^2}{6048} \right] \\
+ \frac{701675e^4}{162661e^6} \\
- \frac{24192}{16128} \\
- q^2 \left( \frac{33}{16} + \frac{95e^2}{16} + \frac{311e^4}{128} \right) \\
- \frac{1}{p^{7/2}} \left[ \pi \left( \frac{8191}{672} + \frac{4836e^2}{1344} \right) \right]
\]
where $q = a/M$ is the dimensionless spin. The averages of the fluxes of the 1PN order, given in terms of the quantities $(e, p)$ and the mass ratio $\nu$ in the EOB gauge without spin, are [33]:

$$\langle \mathcal{F} \rangle = \frac{32 \nu^2 (1 - e^2)^{3/2}}{5p^5} \left\{ 1 + \frac{1247}{336} \frac{5\nu}{4} + e^2 \left( \frac{9181}{672} + \frac{325\nu}{24} \right) - e^4 \left( \frac{809}{128} - \frac{435\nu}{32} \right) - e^6 \left( \frac{8609}{5376} - \frac{185\nu}{192} \right) \right\},$$

$$\langle \mathcal{G} \rangle = \frac{32 \nu^2 (1 - e^2)^{3/2}}{5p^5/2} \left\{ 1 + \frac{7\nu^2}{8} - \frac{1}{p} \left( \frac{1247}{336} + \frac{7\nu}{4} + e^2 \left( \frac{425}{336} + \frac{401\nu}{48} \right) - e^4 \left( \frac{10751}{2688} - \frac{205\nu}{96} \right) \right) \right\}.$$
coordinates at the 1PN order. This may hint that the PN fluxes can be used in the EOB equations for the extreme-mass-ratio limit. Therefore, by combining the PN fluxes of a test particle orbiting a Kerr black hole and the 1PN fluxes of a nonspinning binary with a mass ratio, we derive expressions for energy and angular momentum fluxes containing both spin and mass ratio:

\[ \langle \mathcal{F} \rangle_{\text{2PN}} = \frac{32 \nu^2 (1 - e^2)^{3/2}}{5p^5} \left\{ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right\} - \frac{1}{p} \left\{ \frac{1247}{336} + 5 \nu \frac{9181}{672} + 325 \nu \right\} - e^4 \left( \frac{809}{128} - \frac{435 \nu}{32} - \frac{8609}{5376} - \frac{185 \nu}{192} \right) + \frac{4}{125} \left( \frac{1375 e^2}{48} + \frac{3935 e^4}{9216} \right) - \frac{73}{12} \left( \frac{823 e^2}{24} + \frac{949 e^4}{32} + \frac{491 e^6}{192} \right) - e^4 \left( \frac{33}{16} + \frac{359 e^2}{32} + \frac{1465 e^4}{128} + \frac{883 e^6}{768} \right) \right\}, \tag{36} \]

\[ \langle \mathcal{G}^\nu \rangle_{\text{2PN}} = \frac{32 \nu^2 (1 - e^2)^{3/2}}{5p^5} \left\{ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right\} - \frac{1}{p} \left\{ \frac{1247}{336} + 5 \nu \frac{9181}{672} + 325 \nu \right\} - e^4 \left( \frac{809}{128} - \frac{435 \nu}{32} - \frac{8609}{5376} - \frac{185 \nu}{192} \right) + \frac{4}{125} \left( \frac{1375 e^2}{48} + \frac{3935 e^4}{9216} \right) - \frac{73}{12} \left( \frac{823 e^2}{24} + \frac{949 e^4}{32} + \frac{491 e^6}{192} \right) - e^4 \left( \frac{33}{16} + \frac{359 e^2}{32} + \frac{1465 e^4}{128} + \frac{883 e^6}{768} \right) \right\}, \tag{37} \]

Here, for simplicity, we just write the 2PN formalism. Note that we only add the mass-ratio terms into the 1PN. Indeed, there are \( \nu, \nu^2, \) and \( \nu q \) terms in the second PN that may also be important for the flux calculation. The impact of the second-order self-forces based on the PN fluxes [53] was comprehensively studied by Isoyama et al. [54] for circular orbits with exponential resummation. The next-to-leading tail-induced spin-orbit effects on the fluxes were given in [55]. Compared with these PN fluxes, the above formalism may miss the \( \nu, \nu^2, \) and \( \nu q \) terms. In the next step, we will include these missing terms in our fluxes and improve the orbital evolution. In the adiabatic limit, the evolution of energy and angular momentum is driven by the orbit-averaged radiation reaction forces, so that

\[ \dot{E} = - \langle \mathcal{F} \rangle, \quad \dot{P}_\theta = - \langle \mathcal{G}^\theta \rangle. \tag{38} \]

Using equations (27a) and (16), we obtain the evolution of \( \nu \) and \( \rho \) due to gravitational radiation

\[ \dot{\nu} = \frac{E/M}{(\partial H_{\text{eff}}/\partial p)(\langle \mathcal{F} \rangle/\mu) - (\partial H_{\text{eff}}/\partial \nu)(\langle \mathcal{G}^\nu \rangle/\mu)}, \tag{39a} \]

\[ \dot{\rho} = \frac{E/M}{(\partial H_{\text{eff}}/\partial \nu)(\langle \mathcal{F} \rangle/\mu) - (\partial H_{\text{eff}}/\partial \rho)(\langle \mathcal{G}^\rho \rangle/\mu)}. \tag{39b} \]

The derivatives are computed from the expressions in equation (22). The evolution of the auxiliary phase of radial motion can now be calculated by

\[ \dot{\xi} = \frac{1 + \cos \xi}{\epsilon p M \sin \xi} + \frac{\cos \xi \dot{\epsilon}}{e \sin \xi}. \tag{40} \]

Figure 6 demonstrates the evolution of eccentricity and the semilatus rectum with different orbital parameters and mass ratios. We find that the evolution of \( \nu = 10^{-3} \) deviates from the other two evolutions with smaller mass ratios when the evolution is close to its end. Figure 7 illustrates the evolution of orbits with 2PN (left panels) and 3PN fluxes (right panels). Figure 8 shows the performances of various PN fluxes in orbital evolutions with very extreme spin. Similarly to [51], in the following section, we will use 2PN fluxes to calculate the orbital evolution.

### 3.2. Waveform

In this subsection, we calculate the waveforms by solving the Teukolsky equations [35] to demonstrate the advantages of our analytical EOB solution. Our method is based on frequency-domain decomposition, and has been developed in previous works [21, 36–38], in which the gravitational waveform from an eccentric EMRI with a total mass \( M \) at a distance \( R \), a latitudinal angle \( \Theta \), and an azimuthal angle \( \Phi \) could be written as

\[ h_+ - i h_\times = \frac{2}{R} \sum_{\ell m k} \frac{Z_{\ell m k}^H}{\omega_{\ell m k}^H} \tilde{S}_{\ell m k}^H(\Theta) \phi_{\ell m k}(\Phi), \tag{41} \]

where \( \ell, m, \) and \( k \) are the harmonic numbers and \( \phi_{\ell m k} \equiv \int \omega_{\ell m k}(\dot{\phi}) dt. \tilde{S}_{\ell m k}^H(\Theta) \) denotes spin-weighted spheroidal harmonics that depend on the polar angles \( \Theta \) of the observer’s direction of sight and the direction of orbital angular momentum of the source. \( Z_{\ell m k}^H \) describes the amplitude of each
mode, which can be calculated by the radial component of the Teukolsky equation (see appendix C for details). In this article, we set $\Theta = 0$ (‘face on’), $\Phi = 0$, and $\omega_{\text{inj}}$ is

$$\omega_{\text{inj}} = m\omega_\phi + k\omega_r,$$

(42)

where $\omega_\phi$ and $\omega_r$ denote the orbital frequencies of the radial and azimuthal directions, respectively, which are given in equations (29), (30). Due to our analytical solution for orbits and frequencies given in the previous section, the calculation of the Teukolsky-base waveform becomes very convenient and accurate.

As an example, figure 9 illustrates the numerical waveforms of four evolutionary stages of an EMRI with initial parameters of $p = 20M$, $e = 0.6$ and the dashed line represents evolution with initial parameters of $p = 10M$, $e = 0.6$. The different colors denote different mass ratios.

Matched filtering [56] is widely used in GW detection in the LIGO and Virgo data analyses and will also be used in future spaceborne detectors. We employ this technology to quantitatively analyze the influence of the mass ratio on the EMRI waveforms. Given time series $a(t)$ and $b(t)$, the overlap of the two series is

$$\text{Overlap} = \frac{(a|b)}{\sqrt{(a|a)(b|b)}},$$

(43)

where the inner product between a time series signal $a(t)$ and a template $b(t)$ is

$$ (a|b) = 2\int_0^{\infty} \tilde{a}(f)\tilde{b}(f)^* S_n(f) \, df. $$

(44)

Here, $\tilde{a}(f)$ is the Fourier transform of the time series signal $a(t)$, $\tilde{a}(f)^*$ is the complexity conjugate of $\tilde{a}(f)$, and $S_n(f)$ is the power spectral density of the gravitational wave detectors’ noise. Throughout this paper, the power spectral density is taken to be the LISA noise.
reaction is still included. We employ six EMRIs with mass ratios ranging from $10^{-6}$ to $10^{-3}$. However, for a mass ratio $\sim 10^{-3}$, if the small body is in an extremely relativistic orbit around the central BH, the waveform templates of the test-particle approximation will at least lead to relative errors of the order of the mass ratio in the parameter estimations after a few months of evolution. For a mass ratio of $\sim 10^{-3}$, even for a large orbital separation, the match of the two waveforms rapidly degrades after just two months. We may conclude that the mass-ratio correction in the EMRI waveform model should be important in the detection of this kind of system by LISA. It is worth pointing out that variations of the initial $p$, $e$, and other parameters may still match the true waveform.

The method described in this paper could be useful in the development of an efficient waveform model for mock data analysis for spaceborne projects. Though we have omitted the nonlinear terms in the EOB Hamiltonian here, in extreme mass-ratio cases, we still get the orbital frequency and then the frequencies of the GWs with sufficient accuracy. The frequency is a very sensitive detector of EMRIs. Once we include the effective spin terms, we can expect to extend this model to comparable binaries.

However, the evolution of the orbital parameters in this work may still not be enough. This is due to our use of the PN formalism for GW fluxes. From figure 8, one can see that higher PN orders do not give more convergent results. Of course, one could numerically calculate the energy fluxes accurately using the Teukolsky equation, but this would be computationally expensive. Some researchers have developed analytical or fitting formalisms for the Teukolsky-based fluxes with very high accuracies, but only for circular orbits in the Schwarzschild [57] and Kerr [58] cases. A recent study developed a model that evolved the EMRI orbits precisely and rapidly using the gravitational self-force method [59]. In our next work, we may try to employ this fast orbital evolution.

In addition, a waveform model for the mock data analysis should be efficient, i.e., it should generate the waveforms quickly. For a typical EMRI with a $10^6 M_\odot$ SMBH and a mass ratio of $10^{-5}$, our model takes about 522 seconds to generate one-year waveforms using a single CPU. The majority of the CPU time is used to solve the Teukolsky equation and to sum the harmonic number $k$ from $-1$ to 8. The well-known AK and AAK templates take 229 and 209 seconds, respectively, but the NK template consumes much more CPU time. The XSPC [17], an EMRI template that was also developed by some of us, needs 37.5 minutes to finish the evolution.

There is still some room to speed up our numerical codes. The orbital evolution is very fast, because we just need to integrate the two ordinary differential equations for $p$ and $e$. However, the numerical calculation of the Teukolsky equation (even though we use a semi-analytical algorithm [60]) requires more CPU time. Fortunately, although the frequency needs to be updated at every step, the amplitude of the GW does not. If we had a fitting formula for accurate Teukolsky-base waveforms, the computation time would be greatly reduced. As a conclusion, our model may be a potential candidate for the mock data analysis.

4. Conclusions and outlook

In this paper, based on EOB theory, we described analytical orbital solutions for elliptic EMRIs with spinning black holes. The solutions were derived using the geometric parameters $p$ and $e$ instead of the EOB coordinates and momentum. The fundamental properties of the motion due to the mass ratio and the black hole’s spin were discussed. We also gave the expressions for two orbital frequencies. With this formalism at hand, it was convenient to combine it with the frequency-domain Teukolsky equation, and to generate accurate numerical waveforms. In addition, we expressed the forms of orbital evolution under gravitational radiation. We also inserted the first PN mass-ratio correction into the energy and angular momentum fluxes and showed the effects of fluxes with different PN orders on the orbital evolution.
Using matched filtering, we revealed the influence of the mass ratio on the detection of EMRI GWs. We indicated that for mass ratios \( \nu \lesssim 10^{-5} \), the conservative gravitational self-force of small objects should be considered in the construction of EMRI waveform templates. Considering that the main EMRI waveform models such as AK, AAK, NK, XSPEG, etc., treat the small objects as test particles, our model may represent progress in the development of EMRI templates.

In this work, a few approximations have been used. As we mentioned before, in this model, we temporarily omitted the effective spin of the small object. In the EOB theory, the spin of the effective test particle is \( \sim \mu a/M \), even if the small object does not really rotate. This is why we state that our current model only works for EMRIs, but it is still an improvement compared to the test-particle approximation. The omission of this term will only induce a relative error in Hamiltonians that is at least two orders smaller than the mass ratio. This ensures that our model is accurate for EMRIs. The conservative formalism for eccentric orbits in the EOB frame is the main innovation described in this paper, and can be used as a step towards faithful EMRI models. Furthermore, the current model cannot be used with inclined orbits. We will solve these two problems in our next work.

For the orbital evolution part, the energy and angular momentum fluxes we used are PN approximations from Gair & Glampedakis’s 2PN and 2.5PN formalisms [51] and Fujita’s 3PN and 4PN [57] results. However, the PN formalism of GW fluxes for orbital evolution may not satisfy the requirements for detecting EMRIs. This inaccuracy may be removed by using higher PN expansions with calibration from Teukolsky-based fluxes. We also expect to include the latest self-force result [59], and evolve the orbits more precisely. In addition, in this work, we transfer the original formalism of fluxes given in [57] to new ones with the orbital parameters \( p \) and \( e \). This new formalism is more convenient to use. We also include the 1PN correction of the mass ratio in the fluxes. This is a tiny innovation, and a step towards more faithful EMRI models.

One of the scientific objectives for the study of EMRIs is to detect the space-time geometry of SMBHs. To support this...
objective, an accurate and efficient waveform template is needed. However, this is still a challenge at the moment. The analytical orbital solution that includes the mass ratio and eccentricity given in this paper is more accurate than the test-particle model and more efficient than the original EOB equations. The combination of the analytical orbit and the Teukolsky equation can generate accurate waveforms. The computational efficiency of our model is similar to those of the AAK and AK models. We hope our work is useful in the development of EMRI waveform templates for spaceborne detectors.

Acknowledgments

This work is supported by NSFC No. 117773059, and we also appreciate the anonymous referees’ suggestions concerning our work. This work was also supported by MEXT, the JSPS Leading-edge Research Infrastructure Program, JSPS Grant-in-Aid for Specially Promoted Research 26000005, JSPS Grant-in-Aid for Scientific Research on Innovative Areas 2905: JP17H06358, JP17H06361, and JP17H06364, JSPS Core-to-Core Program A. Advanced Research Networks, JSPS Grant-in-Aid for Scientific Research (S) 17H06133, the joint research program of the Institute for Cosmic Ray Research, the University of Tokyo, and by the Key Research Program of Frontier Sciences, CAS, No. QYZDB-SSW-SYS016.

Appendix A. Log-resummed, calibrated versions of the potential

The log-resummed, calibrated A-potential is given by the expression from appendix A of [61]

$$A = \Delta_5 \nu + \nu \log (\Delta_6) + 3 \partial_4 \partial_4 + 3 \partial_2 \partial_2 + \partial_1 \partial_1,$$

with

$$\Delta_5 = (K\nu - 1)^2 \left[ \frac{64}{5} \log(u) + \left( \frac{1}{5} a^2 (\Delta_1^3 - 3 \Delta_1 \Delta_2 + 3 \Delta_3) + \frac{5(2K\nu - 1)^2}{5(2K\nu - 1)^2} \right) \right].$$

$$\Delta_4 = \left( 1 + 2 \nu \right) \left( \frac{1}{5} a^2 (\Delta_1^2 - 2 \Delta_2) (K\nu - 1)^2 + 3 \Delta_1^2 + \frac{1}{8} a^2 (K\nu - 1)^2 \right) + 6(6a^2 (\Delta_1^2 - \Delta_2) (K\nu - 1)^2 + 12 \Delta_1^2 \partial_2 + 12 \Delta_1 \partial_2 + 12 \Delta_2) + 64(K\nu - 1)(3 \Delta_3 - 47 K\nu + 47)

+ 123 \Delta_2 (K\nu - 1)^2 \right].$$

$$\Delta_3 = -a^2 \Delta_2 (K\nu - 1)^2 + \frac{\Delta_3}{3} + \frac{1}{8} a^2 (K\nu - 1) + \Delta_1 \Delta_2 - 2(K\nu - 1) (\Delta_2 - K\nu + 1),$$

$$\Delta_2 = \frac{1}{2} \left( \Delta_1 (K\nu - 4K\nu + 4) - 2a^2 \Delta_0 (K\nu - 1)^2 \right),$$

$$\Delta_1 = -2(\Delta_0 + K)(K\nu - 1),$$

$$\Delta_0 = K(K\nu - 2),$$

where $K$ is a calibration parameter tuned to numerical relativity simulations whose most recently updated value was determined in equation (4.8) of [62]

$$K = 267.788 \nu^3 - 126.687 \nu^2 + 10.2573 \nu + 1.7336.$$
proves that our equations (39) are correct by going back to the test-particle limit in Kerr space-time. By setting the Kerr parameter $a = 0$ in our formalism, the right panel repeats the orbital evolution shown in FIG.6 of [33] for an equal-mass binary without spin. This shows that our formalism coincides with the Schwarzschild case when $a = 0$ but with a mass ratio $\nu = 0.25$. Combining both results, our formalism and codes for orbital evolution are validated.

Appendix C. The Teukolsky equation

The gravitational perturbation of Kerr space-time is described by the Teukolsky equation, given by the Weyl curvature (complex) scalar $\psi_4$, decomposed in the frequency domain: $\psi_4 = \rho^4 \int_{-\infty}^{\infty} d\omega \sum_{lm} R_{lmw}(r) \Delta_{lm} \phi_k \left( \begin{array}{cc} \omega \tau & \epsilon_r \omega \tau \end{array} \right) e^{i\omega \tau} e^{-i\omega \tau}$. With spin-weighted spheroidal harmonics, $\Delta_{lm}$, obeys [35]:

$$\Delta \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR_{lm}}{dr} \right) - V(r)R_{lmw} = -T_{lmw}(r),$$

where $T_{lmw}(r)$ is the source term, which is connected by the stress-energy tensor of the perturbation source, and the potential is

$$V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda,$$

where $K = (r^2 + a^2)\omega - ma$, $\lambda = E_{lm} + a^2\omega^2 - 2am\omega - 2$ and $\Delta = r^2 - 2Mr + a^2$.

First, we consider the homogeneous Teukolsky equation, where the source term is zero. We can solve it by an analytical expansion, as discussed in [60, 63] and we do not cover its technical details here. The homogeneous Teukolsky equation allows for two independent solutions: $R_{lmw}^H$, which is purely ingoing at the horizon, and $R_{lmw}^\infty$, which is purely outgoing at infinity:

$$R_{lmw}^H = B_{lmw}^H \Delta e^{-i\omega \tau}, \quad r \rightarrow r_e,$$

$$R_{lmw}^\infty = B_{lmw}^\infty \Delta e^{-i\omega \tau} + r^2 B_{lmw}^\infty e^{-i\omega \tau}, \quad r \rightarrow \infty,$$

where $p = \omega - \frac{ma}{3Mr}$, $r_e = M + \sqrt{M^2 - a^2}$ and $r^*$ is the tortoise coordinate related to $r$ by $dr^*/dr = (r^2 + a^2)/\Delta$.

Then, using the homogeneous solutions and appropriate boundary conditions, we can construct a solution to the radial Teukolsky equation with the source term. By imposing a BH boundary condition, i.e. a wave that is purely outgoing at infinity and purely ingoing at the horizon, the radial function is:

$$R_{lmw}^{BH}(r) = \frac{R_{lmw}^\infty(r)}{2i\omega B_{lmw}^\infty D_{lmw}^\infty} \int_r^\infty dr' R_{lmw}^\infty(r') T_{lmw}(r') \Delta(r')^2 +$$

$$\frac{R_{lmw}^H(r)}{2i\omega B_{lmw}^H D_{lmw}^\infty} \int_r^\infty dr' R_{lmw}^H(r') T_{lmw}(r') \Delta(r')^2,$$

The asymptotic behaviors of this solution near the horizon and at infinity are:

$$R_{lmw}^{BH}(r \rightarrow \infty) = Z_{lmw}^H r^3 e^{i\omega r},$$

$$R_{lmw}^{BH}(r \rightarrow r_e) = Z_{lmw}^H \Delta e^{-i\omega r}.$$
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