Didactic derivation of the special theory of relativity from the Klein–Gordon equation

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Abstract
We present a didactic derivation of the special theory of relativity in which Lorentz transformations are ‘discovered’ as symmetry transformations of the Klein–Gordon equation. The interpretation of Lorentz boosts as transformations to moving inertial reference frames is not assumed at the start, but it naturally appears at a later stage. The relative velocity \( v \) of two inertial reference frames is defined in terms of the elements of the pertinent Lorentz matrix, and the bound \( |v| < c \) is presented as a simple theorem that follows from the structure of the Lorentz group. The polar decomposition of Lorentz matrices is used to explain noncommutativity and nonassociativity of the relativistic composition (‘addition’) of velocities.

Keywords: Lorentz transformations, Lorentz group, composition of velocities

1. Introduction
The special theory of relativity (STR) has been known for more than a hundred years. Nevertheless, its counter-intuitive predictions about the relativity of basic space–time relations, such as the simultaneity of events or the spatial distances between them, still attract attention. In addition, the resulting kinematics of particle motion is far from obvious: the relativistic composition law for velocities is nonlinear, noncommutative and nonassociative; and a massive particle at rest has the nonvanishing energy equal to \( m_0 c^2 \). Not surprisingly, one can find in the existing literature a whole variety of pedagogical presentations of the STR, including popular ones, or those addressed to physicists, or ones that are strictly mathematical. As the representative examples see [1–3], respectively. Below we propose yet another one. Our motivation for working it out is twofold. First, we would like to emphasize in our approach the fact that the physical origin of Lorentz transformations lies in symmetries of field equations. Historically, Lorentz transformation involved an electromagnetic field in a vacuum and Maxwell equations, but nowadays one should also remember other field
equations. We choose the Klein–Gordon equation for a scalar field as the simplest representative of those equations.

The second part of our motivation is to present a compelling argument for the bound \(|v| < c\) on the relative velocity of inertial reference frames\(^1\). We have not found a satisfactory discussion of it in the literature available to us. Most often it is said that \(|v| < c\) because otherwise \(\sqrt{1 - v^2/c^2}\) would be imaginary. But this argument is not fully convincing—an inquiring student could say that perhaps in the case of super-luminal velocities we should just use another form of Lorentz transformations, e.g., with \(\sqrt{v^2/c^2} - 1\). The solution we propose is to introduce Lorentz transformations first, without mentioning the velocity, and to define the velocity at a later stage. Furthermore, we would like to give a precise answer also to another question about velocities: does the STR allow for super-luminal velocities or not?\(^2\)

In our didactic proposal, we start from the Lorentz group introduced simply as a set of coordinate transformations that leave invariant the form of the Klein–Gordon equation. A priori no physical interpretation of these transformations is assumed. In the second step, their interpretation as transformations between uniformly moving reference frames is deduced. It is suggested by the mathematical form of the Lorentz transformations of Cartesian coordinates in space–time. Also, the relative velocity \(v\) of two inertial reference frames is introduced in this step. It is defined in terms of elements of the Lorentz matrix. The bound \(|v| < c\) appears as a simple mathematical consequence of the structure of the Lorentz group. Apart from fulfilling the didactic tasks mentioned above, our approach shows that the STR is a nice example of a theory that provides its own physical interpretation, as opposed to, for example, quantum mechanics, where various physical interpretations seem to be added by hand (and hotly debated). This aspect can render the presented approach interesting also outside the classroom.

The layout of this article is as follows. In the next section, we introduce the Lorentz group, establish its physical interpretation, define the relative velocity \(v\) of two inertial reference frames, and prove that \(|v| < c\). In section 3, we show that the most general Lorentz transformations involve nothing else than the well-known boosts and rotations, apart from time and space reflections. We also digress on the above-mentioned unusual properties of the relativistic composition of velocities. Section 4 is devoted to a brief discussion of velocities of physical particles. Discussion of our approach is given in section 5.

A word about our notation. We use the Cartesian coordinates \(x^i\) in three-dimensional (3D) space. Together with \(x^0 = ct\), where \(t\) is the time, they provide the Cartesian coordinates \(x^\mu\) in the four-dimensional manifold encompassing the space and time (the space–time). Each concrete choice of such coordinates is called the Cartesian coordinate frame in the space–time and denoted as \((x^\mu)\). Such a coordinate frame is associated with the appropriate coordinate net in the space–time, which is an analogy to the coordinate nets on geographical maps. The Latin indices take values 1, 2, 3, while the Greek ones take 0, 1, 2, 3. Summation over repeated indices is understood. Often we will use the matrix notation. In particular, the space–time coordinates form the one-column matrix

\[
x = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = [x^\mu].
\]

\(^1\) We faced this problem teaching classical electrodynamics to undergraduate students of physics.
The boldface denotes three-element columns of coordinates, e.g.,

\[ \mathbf{u} = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} u' \end{bmatrix}, \]

and \(|\mathbf{u}| = \sqrt{u'u'}\). The numbers \(u'\) we call the coordinates of \(\mathbf{u}\), not components, because we prefer the terminology in which component of a vector is also a vector, e.g.,

\[ \mathbf{u}_1 = u' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

is the first component of \(\mathbf{u}\).

2. Lorentz transformations and the bound \(|v| < c\)

Of course, we have to use certain empirical input about the material world. We think that nowadays one of the best choices is the Klein–Gordon equation

\[ \left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \lambda_c^{-2} \right) \psi (x, t) = 0, \] (1)

where \(c = 299 792 458 \text{ m s}^{-1}, \lambda_c > 0\) is a constant of dimension of length called the Compton wave length, and \(\Delta\) is the 3D Laplacian. The time is measured in seconds and the distance in meters. \(c\) and \(\lambda_c\) are regarded here as phenomenological constants. The Klein–Gordon equation is the cornerstone of field theory and particle physics if we neglect the gravitational interaction. In particular, it appears in classical electrodynamics as the wave equation for electric and magnetic fields—in this case \(\lambda_c^{-2} = 0\). In fact, all field equations of the Standard Model of particle physics are closely related to it. Loosely speaking, the Klein–Gordon equation encodes some of the physics of noninteracting fundamental quantum particles\(^2\). The empirical evidence about the fundamental particles shows that the constant \(c\) is universal, i.e., it is common for all of them. This means that \(c\) characterizes the space–time in which the particles live, rather than the particles themselves. On the other hand, \(\lambda_c\) is not universal, as it varies from particle to particle (\(\lambda_c^{-1}\) is proportional to the rest mass of the particle).

Note that in order to write the Klein–Gordon equation (1) for the free quantum particles, we have picked a Cartesian coordinate frame in the space–time. By definition, this frame is called the inertial one. For the correct description of physical phenomena in non-Cartesian coordinates, such as spherical ones or others, or in a noninertial coordinate frame, equation (1) would have to be replaced by another one having a different mathematical form. In most cases, the constant coefficients in front of derivatives would be replaced by certain nontrivial functions of the coordinates, and some new terms could appear on the left-hand side.

The differential operator

\[ \Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]

\(^2\) Actually, in most cases this equation is derived from more fundamental equations like Maxwell or Dirac equations, but this is not important here.
present in (1) can be written in the Beltrami–Laplace form, namely
\[ \Box = - \frac{1}{\sqrt{-\eta}} \frac{\partial}{\partial x^\mu} \left( \eta^{\mu \nu} \sqrt{-\eta} \frac{\partial}{\partial x^\nu} \right). \]

The coefficient \( \sqrt{-\eta} \), where \( \eta = \text{det} \hat{g} \) is trivial in our case, \( \sqrt{-\eta} = 1 \). The Beltrami–Laplace operator is well known in mathematics, in particular in the differential geometry of manifolds to possess a metric tensor \( \hat{g} = [g_{\mu \nu}] \). In that case, \( -\hat{g} \) in the \( \Box \) operator is replaced by the metric tensor \( \hat{g} \). For us, the important point is that the presence of this operator in the Klein–Gordon equation strongly suggests that the space–time should be endowed with the metric \( \hat{g} \), which in the Cartesian coordinates \( \mu \) has the Minkowski form
\[ \hat{g} = \text{diag}[1, -1, -1, -1] = \left[ \eta_{\mu \nu} \right]. \]

Thus, the Minkowski metric \( \hat{g} \) is the diagonal matrix with the shown entries on its diagonal. We will use also the inverse metric tensor \( \hat{g}^{-1} = [\eta^{\mu \nu}] \). Here, the first index \( \mu \) enumerates the rows, and the second index \( \nu \), the columns. In the Cartesian coordinates \( \hat{g}^{-1} \) numerically coincides with \( \hat{g} \). With such justification, we accept the existence of the metric \( \hat{g} \) as the fundamental mathematical property of the space–time, from now on called the Minkowski space–time.

Now, being aware of the presence of the metric, let us ask about its symmetries, that is, the coordinate transformations that leave its form unchanged. The metric always is a covariant tensor of rank two; hence, the elements of \( \hat{g} \) transform according to the following formula
\[ \eta_{\rho \sigma}' \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\sigma} = \eta_{\mu \nu}. \]
(2)

The Lorentz transformations are defined as the linear transformations of the coordinates
\[ x^\mu = L^\mu_{\nu} x^\nu, \] (3)

that leave the Minkowski metric \( \hat{g} = [\eta_{\mu \nu}] \) invariant\(^3\),
\[ \eta_{\rho \sigma}' = \eta_{\rho \sigma}. \] (4)

By assumption, \( L^\mu_{\nu} \) are real numbers, otherwise formula (3) would give rather strange, hard to interpret, complex time and space coordinates \( x^\mu \). They form the four-by-four matrix \( \hat{L} = [L^\mu_{\nu}] \), called the Lorentz matrix. The first index, \( \mu \), enumerates the rows, and the second index, \( \nu \), the columns.

Formulas (2)–(4) lead to the following condition
\[ \eta_{\rho \sigma} L^\rho_{\mu} L^\sigma_{\nu} = \eta_{\mu \nu}. \] (5)

which is the necessary condition for \( \hat{L} \) to be the Lorentz matrix. One can easily see that it is also the sufficient condition. Note that the invariance of the Minkowski metric tensor means that the new coordinates \( x^\lambda \) also are Cartesian coordinates.

Symmetry of the metric is a very important mathematical characteristic of the space–time. Moreover, it can be utilized in order to generate new solutions of the Klein–Gordon equation. If \( \psi(x^\lambda, t) \) is a solution of this equation, then so are all functions of the form \( \psi'(x^\lambda, t) = \psi(L^\lambda_{\mu} x^\mu, \hat{L}^\mu_{\nu} x^\nu/c). \) This assertion can be checked simply by the differentiation of these new functions as required on the l.h.s. of equation (1) and using identities (9) obtained

\(^3\) The assumption of linearity is in fact superfluous—one can prove that formula (2) and condition (4) imply that \( x^\lambda = L^\lambda_{\mu} x^\mu + a^\lambda \), where \( L^\lambda_{\mu} \) and \( a^\lambda \) are constant, i.e., they do not depend on \( x^\mu \); see \( [4] \). We have put \( a^\mu = 0 \) because we are not interested in translations in space and time.
below. The closely related fact is that the operator \( \square \) does not change its form when expressed by the new coordinates \( x'^\mu \). Let us add that operations that transform one solution into another are called symmetries of the pertinent equation, in this case, of the Klein–Gordon equation.

The matrix form of (3) reads
\[
\hat{L}^T \hat{\eta} \hat{L} = \hat{\eta}. \tag{6}
\]
Here, \( \hat{L}^T \) is the transposed matrix, i.e., \( (\hat{L}^T)_{\mu}^{\nu} = L_{\nu}^{\mu} \). The set of all real, four-by-four matrices obeying condition (6) as a group with the multiplication of group elements given by the matrix product. It is called the full-Lorentz group. Taking the determinant of both sides of condition (6) we see that \( \det \hat{L} = \pm 1 \), hence, \( \hat{L}^{-1} \) exists. It is easy to see that \( \hat{L}^{-1} \) obeys that condition too. Note also that \( \hat{L}^{-1} = \hat{\eta}^{-1} \hat{L}^{-T} \hat{\eta} \). Therefore,
\[
(\hat{L}^{-1})_{\nu}^{\mu} = \eta_{\mu}^{\sigma} L_{\sigma}^{\gamma} \eta_{\gamma \nu} \equiv L_{\nu}^{\mu}. \tag{7}
\]
The notation on the r.h.s. reflects the fact that in the case of tensors \( \hat{\eta}^{-1} \) raises and \( \hat{\eta} \) lowers indices. We use it because it is convenient, even though \( \hat{L} \) is the transformation matrix and not a tensor.

Simple algebraic manipulations on condition (6) give the equivalent form
\[
\hat{\eta}^{-1} = \hat{L} \hat{\eta}^{-1} \hat{L}^T. \tag{8}
\]
which for the matrix elements reads
\[
\eta_{\mu \nu} = L_{\mu}^{\alpha} L_{\nu}^{\beta} \eta_{\alpha \beta}. \tag{9}
\]

Until now, all steps have been purely mathematical ones. As physicists, we are interested in actual performing transformations (3) in the material world. In other words, what is the physical interpretation of these transformations? It turns out that the formulas we have obtained give a tantalizing hint about it.

Let us write (3) in the expanded form,
\[
t' = L_{i}^{0} t + L_{i}^{k} x^{i}/c, \quad x'^{i} = c L_{i}^{0} t + L_{k}^{i} x^{k}. \tag{10}
\]
Condition (9) in the case \( \mu = \nu = 0 \) gives
\[
(L_{0}^{0})^2 = 1 + L_{i}^{0} L_{i}^{0}. \]
Because \( L_{j}^{0} \) are real, we see that \( |L_{0}^{0}| \geq 1 \). This implies that \( L_{0}^{0} \neq 0 \); hence, we may calculate \( t \) from the first formula (10),
\[
t = \frac{1}{L_{0}^{0}} \left( t' - L_{k}^{0} x^{k}/c \right). \]
Inserting it in the second formula (10) gives
\[
x'^{i} = c \frac{L_{i}^{0} t'}{L_{0}^{0}} + \left( L_{k}^{i} - \frac{L_{i}^{0} L_{k}^{0}}{L_{0}^{0}} \right) x^{k}. \tag{11}
\]
This formula shows that a material point that has constant, i.e., time-independent, coordinates \( x' \) moves with the velocity
\[
w'^{i} = \frac{L_{i}^{0}}{L_{0}^{0}} \tag{12}\]
with respect to the new coordinate frame \( (x^k') \). This velocity does not depend on the position in the space given by \( x = [x^i] \). Thus, \( \textbf{w}' = [w^i] \) is the velocity of the frame \( (x^\mu) \) with respect to \( (x^\nu) \). Its coordinates \( w^i \) are given with respect to the frame \( (x^\mu) \) because \( w^i = dx^i/dt' \).

Transformation (10) can easily be inverted by replacing \( \hat{L} \) with \( \hat{L}^{-1} \). Using (7) we obtain

\[
t = L_0^0 t' + L_k^0 x^k/c, \quad x^i = cL_0^i t' + L_k^i x^k.
\]

Next, using the first formula (13), we may express \( t' \) by \( t \) in the second formula,

\[
x^i = cL_0^i t + \left( L_k^i - \frac{L_0^i L_k^0}{L_0^0} \right) x^k.
\]

We see that the velocity of the frame \( (x^\mu) \) with respect to the frame \( (x^\nu) \) is given by the formula

\[
\nu^i = c\frac{L_0^i}{L_0^0},
\]

where the coordinates \( \nu^i \) of the velocity are given in the frame \( (x^\mu) \). We shall see in the next section that the coordinates of the two velocities are related to each other, but in general \( \nu^i \neq -w^i \).

The definitions (12) and (15) give

\[
v^i \nu^i = w^i w^i = c^2 \left( 1 - \frac{1}{(L_0^0)^2} \right).
\]

This formula is obtained with the help of the relations \( L_0^0 L_i^0 = (L_0^0)^2 - 1 \), \( L_0^0 L_0^i = (L_0^0)^2 - 1 \), \( L_0^0 L_0^0 = L_0^0 \). Because \( (L_0^0)^2 \geq 1 \), we see that \( \|w\| = \|v\| < c \). It is clear that this theorem follows essentially from the fundamental formulas (3) and (5).

To summarize, we have found that there is the subluminal velocity \( \nu \) associated with the Lorentz transformation \( \hat{L} \). It is defined by formula (15), and interpreted as the velocity of the Cartesian coordinate frame \( (x^\mu) \) with respect to the original coordinate frame \( (x^\nu) \). In our approach this fact has been found as the result of the analysis of Lorentz transformations (3), and not assumed as in other approaches.

### 3. The full physical content of Lorentz transformations

In this section, we show that the general Lorentz transformation can be composed from the following basic transformations: boost with the velocity \( \nu \), spatial rotation, and time or space reflection. This result is very important because it shows how we can physically perform arbitrary Lorentz transformations, at least in principle. The following operations on the Cartesian coordinate net that defines the original coordinates \( x^\mu \) are allowed, and no others: setting it in a uniform motion with the velocity \( \nu \) such that \( \|\nu\| < c \), rotating the coordinate net around an axis through the origin, flipping directions of the spatial coordinate axes to the opposite ones, and replacing the clock with another one with hands rotating anticlockwise.

As the first step, we just solve (16) for \( L_0^0 \). This gives

\[
L_0^0 = \pm \gamma,
\]
where $\gamma = (1 - v^2c^2)^{-1/2}$. Now formula (15) yields

$$L^0_i = \mp \frac{\gamma^\nu}{c},$$

(18)

where we have raised the index 0 and lowered $i$ using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.

A longer calculation is needed in order to find the elements $L^k_i$ of the Lorentz matrix. Condition (5) with $\mu = i, \nu = k$ reads

$$L^0_i L^0_k - L^0_k L^0_i = -\delta_{ik},$$

where $\delta_{ik}$ is the 3D Kronecker delta. Using (18) we obtain the set of quadratic equations for the matrix elements $L^s_i$:

$$L^s_i L^s_k = \delta_{ik} + \gamma^s \frac{\nu^\nu}{c^2}.$$  (19)

Here, matrix algebra is helpful. Introducing the three-by-three matrix $\hat{I} = [L^i_i]$, and the matrix $\nu \otimes \nu = [\nu^\nu^\nu]$ called the tensor product of $\nu$ with itself, we write equation (19) in the matrix form,

$$\hat{L}^T \hat{I} = I_3 + \gamma^s \frac{\nu \otimes \nu}{c^2},$$

(20)

where $I_3$ is the three-by-three unit matrix. It is clear that if $\hat{I}$ is a solution of this equation, then so are all matrices of the form $R \hat{I}$, where $R$ can be an arbitrary orthogonal three-by-three matrix. This matrix $R$ cancels out on the l.h.s. of equation (20) because $R^T R = I_3$. It represents a spatial rotation around a certain axis through the origin, possibly accompanied by a spatial reflection. Actually, any three-by-three matrix $\hat{I}$ can be written in the form

$$\hat{I} = \hat{R} \hat{h},$$

(21)

where $\hat{h}$ is a symmetric, three-by-three real matrix with non-negative eigenvalues. Formula (21) gives the so-called polar decomposition of $\hat{I}$, e.g., [5]. Such decomposition, usually introduced for complex matrices, generalizes to matrices the well-known decomposition of a complex number $z$ into the product of modulus and the phase factor $z = |z| \exp (\text{Arg} z)$. The matrix $\hat{h}$ is the counterpart of $|z|$. For us, the main advantage of the polar decomposition is that the matrix equation (20) is reduced to the equation

$$\hat{h}^2 = I_3 + \gamma^s \frac{\nu \otimes \nu}{c^2},$$

(22)

which has a unique solution for $\hat{h}$ in the indicated class of matrices. Leaving aside directly solving the corresponding set of quadratic equations for the matrix elements of $\hat{h}$, let us guess that $\hat{h}$ has the form

$$\hat{h} = \alpha I_3 + \beta \frac{\nu \otimes \nu}{c^2},$$

(23)

where $\alpha, \beta$ are constants.  This presupposition comes from the observation that

$$\left( \frac{\nu \otimes \nu}{c^2} \right)^2 = \left( 1 - \frac{1}{\gamma^2} \right) \frac{\nu \otimes \nu}{c^2}.$$  

Inserting the matrix (23) in equation (22) and equating the coefficients in front of the matrices $I_3$ and $\nu \otimes \nu/c^2$, we obtain quadratic equations for $\alpha, \beta$ which are easy to solve. Taking care of signs in order to ensure that $\hat{h}$ has non-negative eigenvalues, we finally obtain
Eigenvectors of the matrix (23) are very simple. If \( \mathbf{v} \neq \mathbf{0} \), it is just \( \mathbf{v} \) with the corresponding eigenvalue \( \alpha + (1 - \gamma^{-2})\beta \), and two vectors orthogonal to \( \mathbf{v} \) and to each other, with the double degenerate eigenvalue equal to \( \alpha \). If \( \mathbf{v} = \mathbf{0} \), we may take as the eigenvectors any three orthonormal vectors, and we have the triple degenerate eigenvalue equal to \( \alpha \). The matrix (24) has the eigenvalues 1, 1, and \( \gamma \).

The elements \( L'_{ij} \) remain to be determined. Condition (9) with \( \mu = i, \nu = 0 \) gives

\[
L'_{i0} = L'_i L^0_k / L^0_0 .
\]

All matrix elements on the r.h.s. of this formula are already known (see (17), (18), (21) and (24)). Using these formulas, we obtain

\[
L'_{i0} = -R^i_p h^p_{\gamma k} \frac{v^k}{c} = -\gamma R^i_p \frac{v^p}{c} .
\]

The second equality follows from the fact that \( \mathbf{v} \) is the eigenvector of \( \hat{h} \) with the eigenvalue \( \gamma \).

Note that

\[
w^{ij} = \mp R^i_k v^k,
\]

which follows from formulas (12), (17) and (25). This relation between the coordinates of the relative velocities reflects the fact that, in general, the two coordinate nets are rotated with respect to each other.

Our findings are summarized by presenting the matrix \( \hat{L} \) as the product of matrices corresponding to time reflection, rotation and spatial reflection, and pure boost:

\[
\hat{L} = \left[ \pm \frac{1}{0} \right] \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \ddagger \end{array} \right] \left[ \begin{array}{cccc} \gamma & -\frac{\gamma^i}{c} \\ -\frac{\gamma^i}{c} & \hat{h}^i_k \end{array} \right] \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \ddagger \end{array} \right] .
\]

Formula (27) essentially gives the polar decomposition of the Lorentz matrix \( \hat{L} \), which can be rewritten in the form

\[
\hat{L} = \hat{O} \hat{H}(\mathbf{v}),
\]

where the matrix

\[
\hat{O} = \left[ \pm \frac{1}{0} \right] \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \ddagger \end{array} \right]
\]

is simultaneously orthogonal, \( \hat{O}^T \hat{O} = I_k \), and Lorentz, \( \hat{O}^T \hat{\eta} \hat{O} = \hat{\eta} \). It turns out that (29) gives the most general form of such matrices. The matrix

\[
\hat{H}(\mathbf{v}) = \left[ \begin{array}{cccc} \gamma & -\frac{\gamma^i}{c} \\ -\frac{\gamma^i}{c} & \hat{h}^i_k \end{array} \right],
\]

often called the Lorentz boost, is the symmetric Lorentz matrix with positive eigenvalues (which are equal to \( \gamma (1 \pm w/c), 1, 1 \)), as required in the polar decomposition. Note that \( \hat{H}(\mathbf{v}) = \hat{H}(-\mathbf{v}) \).
The polar decomposition (28) can be used to explain the generally noncommutative and nonassociative character of the relativistic composition of velocities mentioned in the introduction. This composition law can be obtained from the formula

\[
\hat{H}(v) \hat{H}(u) = \hat{O}(v, u) \hat{H}(w(v, u)).
\] (31)

Calculating the upper row of the matrix \(\hat{H}(v) \hat{H}(u)\), we find

\[
\gamma(w(v, u)) = \left(1 + \frac{vu}{c^2}\right)\gamma(v)\gamma(u),
\] (32)

and

\[
w(v, u) = \frac{1}{1 + vu/c^2} \left[ \frac{1}{\gamma(u)} v + \left(1 + \frac{\gamma(u) vu}{1 + \gamma(u) c^2}\right) u \right].
\] (33)

Let us introduce the notation

\[w(v, u) \equiv v \vdash u.\]

In general,

(a) \(v \vdash u \neq u \vdash v\) (the noncommutativity), and

(b) \((v_1 \vdash v_2) \vdash v_3 \neq v_1 \vdash (v_2 \vdash v_3)\) (the nonassociativity).

Obviously, it is rather inappropriate to refer to such a composition law as to the ‘addition’ of velocities, as sometimes happens in the literature.

The noncommutativity follows from the fact that

\[\hat{H}(v) \hat{H}(u) \neq \hat{H}(u) \hat{H}(v),\]

unless \(v\) is parallel to \(u\)—only in this special case \(v \vdash u = u \vdash v\).

In order to see the nonassociativity, let us consider the identity

\[(\hat{H}(v_1)\hat{H}(v_2))\hat{H}(v_3) = \hat{H}(v_1)(\hat{H}(v_2)\hat{H}(v_3))\]

(the matrix product is associative). Applying formula (31) we obtain

\[\hat{O}(v_1, v_2)\hat{H}(v_1 \vdash v_2)\hat{H}(v_3) = \hat{H}(v_1)\hat{O}(v_2, v_3)\hat{H}(v_2 \vdash v_3),\]

and

\[\hat{O}(v_1, v_2)\hat{O}(v_1 \vdash v_2, v_3)\hat{H}(v_1 \vdash v_2)\hat{H}(v_2 \vdash v_3) = \hat{H}(v_1)\hat{O}(v_2, v_3)\hat{H}(v_2 \vdash v_3),\]

The l.h.s. of this formula gives the polar decomposition of \(\hat{H}(v_1)\hat{H}(v_2)\hat{H}(v_3)\). Now, the point is that the Lorentz matrix \(\hat{H}(v_1)\hat{O}(v_2, v_3)\hat{H}^{-1}(v_3)\) present on the r.h.s. of formula (34) in general and is not orthogonal. This can be seen by checking whether it has the form of (29). If it was orthogonal, the r.h.s. of formula (34) would also constitute the polar decomposition of \(\hat{H}(v_1)\hat{H}(v_2)\hat{H}(v_3)\), and the uniqueness of the decomposition would imply associativity. It is clear that the reason for nonassociativity is the presence of the matrix \(\hat{O}(v_2, v_3)\) in the polar decomposition of the matrix \(\hat{H}(v_2)\hat{H}(v_3)\) or, in other words, the fact that superposition of two Lorentz boosts is not a Lorentz boost, in general.
4. The velocities of material particles

The velocity \( \mathbf{v} \) characterizes the relation of the two Cartesian coordinate systems in the Minkowski space–time, established by formula (3) with \( L^\mu_\nu \) obeying the condition (5).

Velocities of material particles are introduced in a different way. The main notion here is that of trajectory. For example, in the case of a single point particle, \( \mathbf{x}(t) = [x^i(t)] \) in the coordinate frame \((X^\mu)\). In order to avoid mathematical problems with differentiation let us consider only smooth trajectories. The velocity \( \mathbf{u}(t) \) of the particle is then defined as the derivative

\[
\mathbf{u}(t) = \frac{\mathbf{d}\mathbf{x}(t)}{dt}.
\]

This definition does not imply any restriction on the magnitude of \(|\mathbf{u}|\). If there is a restriction, it comes from the physical nature of the particle as observed in experiments. A priori each of the three possibilities \(|\mathbf{u}| < c\), \(|\mathbf{u}| = c\), \(|\mathbf{u}| > c\) is allowed for. We know that there exist particles which move with the velocity \(|\mathbf{u}| < c\). There also exist particles with \(|\mathbf{u}| = c\), namely the photons. Perhaps certain neutrinos also belong to this latter class. Superluminal particles, called tachyons, have not been discovered yet, but there is an extensive theoretical literature on them showing that their existence would not generate dramatic problems, at least on the theory side.

The use of \( c \) as the reference velocity in that classification is not accidental. Precisely with this choice the type of the particle does not depend on the choice of the Cartesian coordinate frame in the Minkowski space–time. In the new coordinates \( \mathbf{x}^i\), related to the previous ones by formula (3), the trajectory of the particle is represented by the functions \( \mathbf{x}^i(t') \), and coordinates of the velocity are given by \( u^i(t') = \frac{\mathbf{d}\mathbf{x}^i(t')}{dt'} \). Simple calculation, in which we use formulas (10), gives the following relation

\[
u^i(t') = \frac{cL^i_0 + L^i_k u^k(t)}{cL^0_0 + L^0_k u^k(t)}.
\]

In consequence,

\[
\mathbf{u}^2 - c^2 = \frac{c^2}{(cL^0_0 + L^0_k u^k)^2}(\mathbf{u}^2 - c^2).
\]

In the derivation of this formula we have used the identities

\[
L^i_k L^j_s = L^j_k L^i_s + \delta_{ks}, \quad L^i_0 L^j_s = L^j_0 L^i_s, \quad L^i_0 L^j_0 = 1 + L^0_0 L^0_0
\]

that follow directly from condition (5). It is clear from (36) that the differences \( \mathbf{u}^2 - c^2 \) and \( \mathbf{u}^2 - c^2 \) always have the same sign.

5. Discussion

The presented derivation of Lorentz transformations exploits the fact that they are tightly connected with symmetries of the class of wave equations, known as relativistic wave equations, and represented here by the Klein–Gordon equation that describes matter on the most fundamental level. The wave equations for the electric and magnetic fields of classical electrodynamics belong to this class, but also many others. For example, the Dirac and Proca equations for particles of spin 1/2 or 1, respectively. The Klein–Gordon equation, along with form and symmetries, is regarded here as the initial phenomenological input from which we start. This definition of the Lorentz transformations has the rather interesting aspect that it does not rely on the popular (in the literature), and artificial (in our belief), picture of two
observers moving with a constant relative velocity. After all, one can think of a world with just one observer which for some reason uses all Cartesian coordinate systems allowed by the symmetry of the Klein–Gordon equation. Moreover, in that picture there is no a priori reason to assume that that velocity can not be arbitrarily large. Of course, uniformly moving coordinate frames appear also in our approach, but only at a later stage, when we ask how to physically accomplish the symmetry transformations. In this sense, we emphasize the symmetry aspect, while the relativity aspect of Lorentz transformations is deduced, in the same way as the bound $|v| < c$.

Another point we would like to emphasize is that we do not identify the constant $c$ with the velocity of light in vacuum. It is introduced rather as the fundamental constant of nature, common for all fundamental particles, including massive ones. The reason is that the STR would be valid also in a theoretical world without the photons and electromagnetic field, and $c$ still would play the fundamental role in such a material world. We think that the fact that group velocity of packets of electromagnetic waves in a vacuum is equal to $c$ should better be derived at a certain stage later, as the physical prediction from Maxwell equations. For this reason, we are not satisfied with presentations of the STR that start from the axiom that the velocity of light in vacuum should be the same in all inertial reference frames. Such approaches put too strong an emphasis on the electromagnetic field. A similar viewpoint was presented already in [6]. The choice of the Klein–Gordon equation (the proxy for field equations of the Standard Model of particle physics) as the starting point seems more natural. It is strongly supported by the tremendous phenomenological success of the Standard Model in the last decades, with its plethora of fundamental fields and particles.

Our introduction to Lorentz transformations surely does not belong to popular physics. Nevertheless, it is rather simple—we mainly use elementary properties of matrices. Especially helpful is the polar decomposition, with which we have provided the simple explanation of the peculiar properties of the composition of velocities. Moreover, we think that the presented approach has other merits, such as conceptual economy, naturalness coming from being based directly on relativistic field equations, and the clear logical status of the bound $|v| < c$.

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