Numerical simulation of sound amplification base on finite difference method

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Abstract. In daily life, we will find that the loudspeaker device is always cone-shaped, because the conical device has a good sound amplification effect. This article mainly explores the influence of the shape and size of the conical device on the loudness of the distant sound by establishing a theoretical model and using the finite difference method for numerical simulation. At the same time, the theoretical numerical simulation method is promoted, so that it can be widely used in more aspects.

Keywords: Conical device; sound propagation; partial differential equations; finite difference method; numerical simulation.

1. Introduction
In daily life, when we hold a conical frame with our hands and speak to others, we will feel that the sound has a significant amplification effect. Many loudspeakers are also made into a cone shape. The size, shape and other parameters of such a conical device have a great influence on the amplification of the sound. Therefore, exploring such effects is helpful to find the optimal parameters of the loudspeaker and has important practical significance.

The current research on the effect of conical devices on sound propagation is mostly from the perspective of energy, which is using parameters such as acoustic impedance to analyze sound amplification [1]. As shown in Figure 1, since a reflecting wall is formed on the side of the device, the sound wave energy is more concentrated in the area 1 along the direction of the cone axis through reflection and more dispersed in the peripheral area 2 deviating from the device. Therefore, the sound in specific direction is amplified. This is helpful to analyze the physical process of sound waves in the device and find out the cause of sound amplification. However, it is difficult to quantitatively describe the sound propagation in the space outside the device from the perspective of energy.
By consulting the literature, there is another mathematical description of the sound field changes in the device as establishing and solving the Webster equation [2]. The Webster equation can be established through selecting the sound pressure $P$ as the physical quantity to be studied and treating the device as a passive sound transmission pipe. The analytical solutions to the equations of different shapes of pipes are different. In this way, the sound pressure changes in the pipe can be quantitatively described.

However, the Webster equation only discusses the internal area of the device. In order to optimize the sound transmission of the device to a distance by analyzing the influence of the device on sound propagation, we also need to quantitatively analyze the sound pressure changes outside the device.

This article focuses on the sound field description of the area outside the device. First, the three-dimensional space can be simplified into a two-dimensional plane due to the rotational symmetry. As shown in Figure 2, the plane is divided into two areas: area I inside the device and area II outside the device. The sound pressure $P$ is selected as the physical quantity to be studied. The sound pressure of area I is described by Webster equation, and then the sound pressure function $P_0$ at the right boundary of area I obtained by the solution is used as a boundary condition of area II. For region II, the partial differential equation of sound pressure $P$ can be derived by the state equation, continuity equation and motion equation of the air volume element. Then, the finite difference method is used to solve the equation numerically, so as to complete the quantitative description of the sound field changes outside the device.

2. Theoretical Analysis
As shown in Figure 2, the area I can be described by Webster equation. The following will mainly focus on analyzing the laws of physics for the air volume element in the area II as well as establishing the partial differential equations for the sound pressure $P$ and its definite conditions.
2.1. Physical Equation of Sound Field

2.1.1. State equation. The acoustic wave velocity \(c\) in the gas has the following relationship with the gas pressure \(P\) and the gas density \(\rho\) [4]:

\[
c^2 = \frac{dP}{d\rho} \tag{1}
\]

Assuming that sound waves propagate in an ideal gas, the state equation of the ideal gas is given by:

\[
P = \frac{mRT}{MV} \tag{2}
\]

So:

\[
P = \frac{RT}{M} \rho \tag{3}
\]

\(M\) is the molar mass of the gas, \(T\) is the gas temperature, and \(R\) is a constant. Regarding the sound propagation in the gas as an isothermal process. It can be known from formula (3) that the pressure \(P\) is proportional to the density \(\rho\), and the proportional coefficient is \(c^2\) from formula (1). So the speed of sound here is regarded as a constant, denoted as \(c_0\).

So the state equation of sound wave is given by:

\[
P = c_0^2 \rho \tag{4}
\]

2.1.2. Continuity equation. The volume element \(dV\) is taken as the research object in the three-dimensional space. Since sound propagation changes the mass in \(dV\), the mass increment per unit time in the three directions \(x\), \(y\), and \(z\) in space can be expressed:

\[
\begin{align*}
dm_x &= -\rho v_x dy dz \\
dm_y &= -\rho v_y dz dx \\
dm_z &= -\rho v_z dx dy
\end{align*} \tag{5}
\]

Summing formula (5) and rewriting it by Gauss's theorem, the total mass increment in the volume element \(dV\) is obtained:

\[
dm = -\nabla \cdot (\rho \vec{v}) dV \tag{6}
\]

And the mass increment per unit time can be expressed:

\[
dm = \frac{\partial \rho}{\partial t} dV \tag{7}
\]

From formula (6) and formula (7), the continuity equation of mass in the volume element can be obtained:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}) \tag{8}
\]

Assuming that the medium gas is uniform-the gas density in the space position can be regarded as a constant, denoted as \(\rho_0\). The formula (8) can be written:

\[
\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \cdot \vec{v} \tag{9}
\]

2.1.3. Motion equation. Due to sound propagation, there is a pressure difference on both sides of the interface along the direction of the sound wave, resulting in the following forces on the gas [4]:

\[
dF = -\nabla P dV \tag{10}
\]

According to Newton's second law:

\[
dF = \rho [\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}] dV \tag{11}
\]

Since it is a uniform gas, the gas velocity should be the same at different positions, so the second term on the right side of the equation (11) is discarded. Then the rewritten formula (11) and formula (10) are combined to obtain the motion equation of the gas in the volume element \(dV\):
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\( \frac{\partial \nu}{\partial t} = -\nabla P \)  \hspace{1cm} (12)

2.2. Sound Wave Equation

2.2.1. Partial differential equations. Combining the three physical equations: formulas (4)(9)(12), making the following substitutions in turn:

First, substituting formula (4) into formula (9) to eliminate the density \( \rho \) and the following formula is obtained:

\[ \frac{1}{c_0^2} \frac{\partial P}{\partial t} = -\rho_0 \nabla \cdot \vec{v} \]  \hspace{1cm} (13)

Then the formula (13) is derived from the time \( t \), and the following formula is obtained:

\[ \frac{1}{c_0^2} \frac{\partial^2 P}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \vec{v}}{\partial t} \]  \hspace{1cm} (14)

Finally, substituting formula (12) into formula (14) to eliminate the velocity \( \vec{v} \), the wave equation of sound propagation in a uniform ideal gas is obtained:

\[ \frac{\partial^2 P}{\partial t^2} - c_0^2 \Delta P = 0 \]  \hspace{1cm} (15)

2.2.2. Definite conditions. As shown in Figure 3, the rectangular coordinate is established in area II.

\[ \frac{\partial^2 P}{\partial t^2} - c_0^2 \Delta P = 0 \]  \hspace{1cm} (15)

Fig. 3 The coordinate system in area II

For region II, a partial differential equation describing the change of the sound pressure has been established, but initial conditions and boundary conditions need to be given to solve it.

For the initial sound pressure at time \( t = 0 \), there is the following description:

\[ P(x, y, 0) = 0 \]  \hspace{1cm} (16)

The sound pressure at the left boundary of area II can be obtained by solving the Webster equation of area I.

By consulting the literature [2], the sound pressure in the one-dimensional duct follows the Webster equation:

\[ \frac{1}{c_0^2} \frac{d^2 P}{dx^2} + \frac{1}{S(x)} \frac{dS(x)}{dx} \frac{dP}{dx} + k_0^2 P(x) = 0 \]

The analytical solution of the Webster equation of the conical duct is given by [2]:

\[ P = \frac{P_0^+ e^{jk_0x} + P_0^- e^{-jk_0x}}{\sqrt{S_0(1 + \frac{x}{r} tan\theta)}} \]  \hspace{1cm} (17)

Then the sound pressure at the left boundary at a distance \( h \) from the right side of the speaker is:

\[ P_0 = \frac{P_0^+ e^{jk_0h} + P_0^- e^{-jk_0h}}{\sqrt{S_0(1 + \frac{h}{r} tan\theta)}} \]  \hspace{1cm} (18)
It can be found that the value of \( P_0 \) will be different when the shape and size of the device are different, which will affect the sound propagation in area II.

So the boundary conditions of area II is:

\[
P(0, y, t) = P_0
\]  

(19)

Thus, a complete mathematical description of the sound propagation in Zone II is obtained:

\[
\begin{cases}
\frac{\partial^2 P}{\partial t^2} - c_0^2 \Delta P = 0 \\
P(x, y, 0) = 0 \\
P(0, y, t) = P_0
\end{cases}
\]  

(20)

2.3. Finite Difference Solution

The finite difference method divides the continuous space in a finite interval into discrete grids. The physical value of each grid point is obtained by iterating the difference equation.

2.3.1. Difference equation. Due to the conical shape of the device, the space at the sound field has rotational symmetry, which can be regarded as a two-dimensional plane rotating around the central axis of the cone.

![Discretization of time and space](image)

Fig. 4 Discretization of time and space

As shown in Figure 4, the two-dimensional space is discretized. Suppose the grid intervals of the \( x \) and \( y \) axes are \( h_x, h_y \), and the time interval is \( \tau \).

Using the explicit difference format to discretize formula (15), the discretized difference equation of the sound field definite solution problem is obtained as follows:

\[
P^{k+1}_{i,j} = AP_{i,j}^k - P^{k-1}_{i,j} + B(P_{i+1,j}^k + P_{i-1,j}^k) + C(P_{i,j+1}^k + P_{i,j-1}^k)
\]  

(21)

Where \( P_{i,j}^k \) represents the sound pressure value at the grid point of the coordinate \((i, j, k)\).

Each coefficient in formula (21) satisfies the following relationship:

\[
\begin{cases}
A = 2(1 - B - C) \\
B = \left(\frac{c_0 \tau}{h_x}\right)^2 \\
C = \left(\frac{c_0 \tau}{h_y}\right)^2
\end{cases}
\]  

(22)

2.3.2. Numerical solution. First, the grid interval \( h_x, h_y, \tau \) needs to be given. Then, taking into account the convergence of the iterative formula (21) of the difference equation, the coefficients \( A, B, C \) in the formula must meet the following conditions [3]:

\[
A = 2(1 - B - C) \geq 0
\]  

(23)
Only when the above conditions are met, formula (21) is convergent and solvable. In addition, the grid interval \( h_x, h_y, \tau \) should be as small as possible, so that the more grid points, the discrete numerical solution can better simulate the continuous real situation.

Then, formula (22) can be used to solve formula (20) numerically through programming. The algorithm steps are as follows:

Step1: Assign \( h_x, h_y, \tau, P, \epsilon \);
Step2: Calculate the sound pressure \( P_1 \) at each grid point by iterative formula (21);
Step3: If \( P_1 \) follows \( |P_1 - P| \leq \epsilon \), go to Step5; otherwise, go to Step4;
Step4: Let \( P = P_1 \), go back to Step2;
Step5: Print \( P_1 \).

3. Calculation Results

3.1. Simulation of Sound Propagation

Using MATLAB software for programming solution, the following sound propagation simulation results can be obtained:

![Fig. 5 Spatial distribution of sound pressure](image)

Figure 5 shows the spatial distribution of the sound pressure generated by the conical device at 2ms, 4ms, 6ms, and 8ms. This result is hard to obtain from the perspective of energy, which shows the superiority of partial differential equations in describing the changing process of the sound field.
3.2. Effect of Device Parameters

The shape and size of the conical device are described by angle $\theta$ and height $H$, as shown in Figure 6.

![Fig. 6 Device parameters](image)

To study the effect of the shape and size of the conical device on sound amplification, the angle $\theta$ is set to $15^\circ, 30^\circ, 45^\circ$, and the height $H$ is set to $4\, cm, 6\, cm, 8\, cm, 10\, cm, 12\, cm$. Then use MATLAB programming to solve the effect of the above-mentioned 15 devices on sound propagation. On the symmetry axis of the device, the sound pressure level is taken at a distance of $15\, cm$ from the sound source, and the calculation result shown in Figure 7.

![Fig. 7 Effect of the shape and size of the device](image)

It can be seen from the figure above that the larger the opening angle $\theta$ and height $H$ of the device, the better the sound amplification effect. In addition, the effect that the sound pressure increases with height decreases as the angle increases.

4. Conclusion

Combining Webster equations and physics equations, a theoretical model of sound propagation is established, and then numerically solved by the finite difference method, and the following conclusions are obtained:

1. The finite difference method can solve the second-order homogeneous linear partial differential equations stably and accurately. The results obtained can well simulate the actual process of sound propagation.
2. The shape of the conical device has a significant effect on sound amplification. The larger the angle $\theta$ of the device, the better the sound amplification effect on the axis of symmetry.
3. The size of the conical device also has a significant effect on sound amplification. The greater the height $H$ of the device is, the better the sound amplification effect on the axis of symmetry will be.
4. The effect that the sound pressure increases with height decreases as the angle increases.
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