(k,d)–Mean Labeling of Some Family of Trees

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Abstract: Mean labeling of graphs was discussed in [24-25] and the concept of odd mean labeling was introduced in [22]. k–odd mean labeling and (k,d)–odd mean labeling are introduced and discussed in [1,6-8]. k–mean, k–even mean and (k,d)–even mean labeling are introduced and discussed in [9-17]. In this paper, we introduce (k,d)–mean labeling and we have obtained results for some family of trees.

Keywords: (k,d)–mean labeling, (k,d)–mean graph

1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [20]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph $G$.

A graph labeling is an assignment of integers to the vertices or edges of a graph. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Graph labeling was first introduced in the late 1960’s. Many studies in graph labeling refer to Rosa’s research in 1967 [23]. Labeled graphs serve as useful models for a broad range of applications such as X-ray, crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly interesting applications of graph labeling can be found in [2-5].

Mean labeling of graphs was discussed in [24,25]. Vaidya and et al. [28-31] have investigated several new families of mean graphs. Nagarajan and et al. [27] have found some new results on mean graphs.

Ponraj, Jayanthi and Ramya extended the notion of mean labeling to super mean labeling in [21]. Gayathri and Tamilselvi [18-19,26] extended super mean labeling to k–super mean, (k,d)–super mean, k–super edge mean and (k,d)–super edge mean labeling. Manickam and Marudai [22] introduced the concept of odd mean graphs. Gayathri and Amuthavalli [1,6-8] extended this concept to k–odd mean and (k,d)–odd mean graphs. Gayathri and Gopi [9-17] extended this concept to k–mean, k–even mean and (k,d)–even mean graphs.

In this paper, we extend k–mean graphs to (k,d)–mean graphs since there are graphs which are (k,d)–mean for all $k \geq 2$ and $d \geq 2$ but not (k,1)–mean for any $k \geq 1$. Here, we have found (k,d)–mean labeling of some family of trees. Throughout this paper, $k$ and $d$ denote any positive integer greater than or equal to 1.

For brevity, we use (k,d)–ML for (k,d)–mean labeling and (k,d)–MG for (k,d)–mean graph.

2. Main Results

Definition 2.1

A $(p,q)$ graph $G$ is said to have a mean labeling if there is an injective function $f$ from the vertices of $G$ to $\{0,1,2,\ldots,q\}$ such that the induced map $f^*$ defined on $E$ by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection from $E$ to $\{1,2,\ldots,q\}$.

A graph that admits a mean labeling is called a mean graph.

Definition 2.2

A $(p,q)$ graph $G$ is said to have a k–mean labeling if there is an injective function $f$ from the vertices of $G$ to $\{0,1,2,...,k+q-1\}$ such that the induced map $f^*$ defined on $E$ by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection from $E$ to $\{k,k+1,k+2,...,k+q-1\}$. A graph that admits a k–mean labeling is called a k–mean graph.

Observation 2.3

Every 1–mean labeling is a mean labeling.

Definition 2.4

A $(p,q)$ graph $G$ is said to have a (k,d)–mean labeling if there exists an injective function $f$ from the vertices of $G$ to $\{0,1,2,...,k+(q-1)d\}$ such that the induced map $f^*$ defined on $E$ by $f^*(uv) = \frac{f(u)+f(v)}{2}$ is a bijection from $E$ to $\{k,k+d,k+2d,...,k+(q-1)d\}$. A graph that admits a (k,d)–mean labeling is called a (k,d)–mean graph.

Observation 2.5

1) Every (k,1)–mean labeling is a k–mean labeling
2) Every (1,1)–mean labeling is a mean labeling.
Theorem 2.6
The path graph $P_n$ is a $(k,d)$–mean graph for all $k$ and $d$, when $n$ is even.

**Proof**
Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i, v_{i+1} \mid 1 \leq i \leq n-1\}$ be denoted as in the Figure 2.1

\[ \text{Figure 2.1: Ordinary labeling of } P_n \]

First we label the vertices as follows:
Define $f : V \to \{0, 1, 2, \ldots, k + (q-1)d\}$ by
\[ f(v_i) = k + d(i-1) - 1, \text{ if } i \text{ is odd} \]
\[ f(v_i) = k + d(i-2), \text{ if } i \text{ is even} \]

Then the induced edge labels are
\[ f^*(v_i, v_{i+1}) = k + d(i-1), \text{ for } 1 \leq i \leq n-1. \]

The above defined function $f$ provides $(k,d)$–mean labeling of the graph.

So, the path graph $P_n$ is a $(k,d)$–mean graph for all $k$ and $d$, when $n$ is even.

$(k,d)$ mean labeling of $P_n$ for different cases of $k$ and $d$ when $n$ is even are shown in illustration 2.7.

**Illustration 2.7**
(100, 20)–mean labeling of the graph $P_{10}$ is shown in Figure 2.2

\[ \text{Figure 2.2: (100, 20)--ML of } P_{10} \]

(55, 44)–mean labeling of the graph $P_6$ is shown in Figure 2.3

\[ \text{Figure 2.3: (55, 44)--ML of } P_6 \]

(33, 11)–mean labeling of the graph $P_3$ is shown in Figure 2.4

\[ \text{Figure 2.4: (33, 11)--ML of } P_3 \]

(60, 13)–mean labeling of the graph $P_{10}$ is shown in Figure 2.5

\[ \text{Figure 2.5: (60, 13)--ML of } P_{10} \]

**Definition 2.8**
A **comb graph** $P_n^*$ is a tree obtained from a path by attaching exactly one pendant edge to each vertex of the path.

Theorem 2.9
The comb graph $P_n^*$ is a $(k,d)$–mean graph for all $k$ and $d$.

**Proof**
Let $V(P_n^*) = \{u_1, u_2, \ldots, u_{2n}, u'_1, u'_2, \ldots, u'_n\}$ and $E(P_n^*) = \{u_i, u_{i+1}, 1 \leq i \leq n \text{ and } u_i, u_{i+1}, 1 \leq i \leq n-1\}$ be denoted as in the Figure 2.6.

\[ \text{Figure 2.6: Ordinary labeling of } P_n^* \]

First we label the vertices as follows:
Define $f : V \to \{0, 1, 2, \ldots, k + (q-1)d\}$ by
\[ f(u_i) = k \]
\[ f(u'_i) = k - 1 \]
\[ f(u_n) = k + (q-1)d \]
\[ f(u'_n) = k + (q-1)d - 1 \]
\[ f(u_1) = k + 2d(i-1), \text{ for } 2 \leq i \leq n-1 \]
\[ f(u'_1) = k + 2d(i-1) - 1, \text{ for } 2 \leq i \leq n-1 \]

Then the induced edge labels are
\[ f^*(u_i, u_{i+1}) = k + 2d(i-1), \text{ for } 1 \leq i \leq n \]
\[ f^*(u_i, u_{i+1}) = k + 2d(i-1) - 1, \text{ for } 1 \leq i \leq n-1 \]

The above defined function $f$ provides $(k,d)$–mean labeling of the graph. So, the graph $P_n^*$ is a $(k,d)$–mean graph for all $k$ and $d$. $(k,d)$–mean labeling of $P_n^*$ for different cases of $k$ and $d$ are shown in illustration 2.10

**Illustration 2.10**
(4,6)–mean labeling of the graph $P_5^*$ is shown in Figure 2.7

\[ \text{Figure 2.7: (4,6)--ML of } P_5^* \]

(13,3)–mean labeling of the graph $P_5^*$ is shown in Figure 2.8

\[ \text{Figure 2.8: (13,3)--ML of } P_5^* \]

(23,8)–mean labeling of the graph $P_8^*$ is shown in Figure 2.9

\[ \text{Figure 2.9: (23,8)--ML of } P_8^* \]
(34,9)–mean labeling of the graph \( P_{1}^{+} \) is shown in Figure 2.10

Figure 2.10: (34,9)–ML of \( P_{1}^{+} \)

Definition 2.11
A twig is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

Theorem 2.12
The twig graph \( T_{n} \) is a \((k,d)\)–mean graph for all \( k \) and \( d \), when \( n \) is even.

Proof
Let \( V(T_{n}) = \{v_{i}, 0 \leq i \leq n-1, u_{i}, w_{i}, 1 \leq i \leq n-2\} \) and \( E(T_{n}) = \{v_{i}u_{i}, v_{i}w_{i}, 1 \leq i \leq n-2 \text{ and } v_{i}v_{i+1}, 0 \leq i \leq n-2\} \) be denoted as in the Figure 2.11.

Figure 2.11: Ordinary labeling of \( T_{n} \)

First we label the vertices as follows:
Define \( f: V \rightarrow \{0, 1, 2, \ldots, k+(q-1)d\} \) by
\[
\begin{align*}
f(v_{i}) & = \begin{cases} 
k + 3d - 1 & \text{if } i \text{ is even} \\
k + 3d(i - 1) & \text{if } i \text{ is odd} 
\end{cases} \\
f(u_{i}) & = \begin{cases} 
k + d(3i - 4) + 1 & \text{if } i \text{ is even} \\
k + d(3i - 1) & \text{if } i \text{ is odd} 
\end{cases} \\
f(w_{i}) & = \begin{cases} 
k + d(3i - 2) + 1 & \text{if } i \text{ is even} \\
k + d(3i + 1) & \text{if } i \text{ is odd} 
\end{cases}
\end{align*}
\]

Then the induced edge labels are
\[
\begin{align*}
f^*(v_{i}v_{i+1}) & = k + 3d, \quad \text{for } 0 \leq i \leq n-2 \\
f^*(v_{i}u_{i}) & = k + d(3i - 2), \quad \text{for } 1 \leq i \leq n-2 \\
f^*(v_{i}w_{i}) & = k + d(3i - 1), \quad \text{for } 1 \leq i \leq n-2
\end{align*}
\]

The above defined function \( f \) provides \((k,d)\)–mean labeling of the graph.

So, the twig graph \( T_{n} \) is a \((k,d)\)–mean graph for all \( k \) and \( d \), when \( n \) is even.

\((k,d)\)–mean labeling of \( T_{n} \) for different cases of \( k \) and \( d \) when \( n \) is even are shown in illustration 2.13.

Illustration 2.13
(6,6)–mean labeling of the graph \( T_{6} \) is shown in Figure 2.12

Figure 2.12: (6,6)–ML of \( T_{6} \)

(3,2)–mean labeling of the graph \( T_{3} \) is shown in Figure 2.13

Figure 2.13: (3,2)–ML of \( T_{3} \)

The star \( K_{1,n} \) \((n \geq 4)\) is a \((k,d)\)–mean graph for all \( k \geq n-2 \) and for all \( d \) satisfying \((q-1)d \leq k + 1\) except when \( n \) is odd and \( d \) is even.

Proof
Let \( V(K_{1,n}) = \{u, v_{1}, v_{2}, \ldots, v_{n}\} \) and \( E(K_{1,n}) = \{uv_{i}, 1 \leq i \leq n\} \) be denoted as in Figure 2.16.

Figure 2.16: Ordinary labeling of \( K_{1,n} \)

First we label the vertices as follows:
Define $f : V \rightarrow \{0, 1, 2, \ldots, k + (q-1)d\}$ by

Case (1): when $n$ is even

Subcase (i): $d$ is odd
\[
f(u) = k + (q-1)d - 1
\]
\[
f(v_i) = k - (q-1)d + 2i - 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} - 1
\]
\[
f(v_i) = k, \quad \text{for } i = \frac{n}{2}
\]
\[
f(v_i) = k + (q-1)d + (i-n) - 1, \quad \text{for } \frac{n}{2} + 1 \leq i \leq n - 1
\]
\[
f(v_n) = k + (q-1)d
\]

Subcase (ii): $d$ is even
\[
f(u) = k + (q-1)d - 1
\]
\[
f(v_i) = k - (q-1)d + 2i - 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} - 1
\]
\[
f(v_i) = k - 1, \quad \text{for } i = \frac{n}{2}
\]
\[
f(v_i) = k + (q-1)d + (i-n) - 1, \quad \text{for } \frac{n}{2} + 1 \leq i \leq n - 1
\]
\[
f(v_n) = k + (q-1)d
\]

Then the induced edge labels are

\[
f^*(uv_i) = k + d(i-1), \quad \text{for } 1 \leq i \leq n
\]

Case (2): when $n$ is odd

Subcase (i): $d$ is odd
\[
f(u) = k + (q-1)d - 1
\]
\[
f(v_i) = k - (q-1)d + 2i - 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2}
\]
\[
f(v_i) = k + d, \quad \text{for } i = \frac{n+1}{2}
\]
\[
f(v_i) = k + (q-1)d + (i-n) - 1, \quad \text{for } \frac{n+3}{2} \leq i \leq n - 1
\]
\[
f(v_n) = k + (q-1)d
\]

Then the induced edge labels are

\[
f^*(uv_i) = k + d(i-1), \quad \text{for } 1 \leq i \leq n
\]

The above defined function $f$ provides $(k,d)$–mean labeling of the graph.

So, the star $K_{1,n}$ is a $(k,d)$–mean graph for all $k \geq n - 2$ and for all $d$ satisfying $(q-1)d \leq k + 1$ except when $n$ is odd and $d$ is even.

$(k,d)$–mean labeling of $K_{1,n}$ for different cases of $k$ and $d$ except when $n$ is odd and $d$ is even are shown in Illustration 2.15.

Illustration 2.15

$(6,1)$–mean labeling of the graph $K_{1,8}$ is shown in Figure 2.17

$(3,1)$–mean labeling of the graph $K_{1,5}$ is shown in Figure 2.18

$(5,1)$–mean labeling of the graph $K_{1,6}$ is shown in Figure 2.19

$(6,1)$–mean labeling of the graph $K_{1,7}$ is shown in Figure 2.20

Definition 2.16

A bistar $B_{n,r}$ is a tree obtained by joining the center vertices of the copies of $K_{1,n}$ and $K_{1,r}$ with an edge.

Theorem 2.17

The Bistar $B_{n,r} (n \geq 2)$ is a $(k,d)$–mean graph for all $k$ and $d$.

Proof

Let $V(B_{n,r}) = \{u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$

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and $E(B_{n,n}) = \{uv, uu_i, vv_i, 1 \leq i \leq n\}$ be denoted as in the Figure 2.21.

**Figure 2.21:** Ordinary labeling of $B_{n,n}$

First we label the vertices as follows:

Define $f : V \rightarrow \{0, 1, 2, ..., k + (q - 1)d\}$ by

- $f(u) = k + (q - 1)d$
- $f(u_i) = k + 2di - 1$, for $1 \leq i \leq n - 1$
- $f(u_0) = k + (q - 1)d - 1$
- $f(v) = k - 1$
- $f(v_i) = k + 2di - 1$, for $1 \leq i \leq n$

Then the induced edge labels are

- $f^*(uv) = k + d(i - 1)$, for $1 \leq i \leq n$
- $f^*(uu_i) = k + (n + i)d$, for $1 \leq i \leq n$

The above defined function $f$ provides $(k,d)$--mean labeling of the graph.

So, the graph $B_{n,n}$ is a $(k,d)$--mean graph for all $k$ and $d$.

(6,7)--mean labeling of the graph $B_{7,7}$ is shown in Figure 2.22.

**Figure 2.22:** (6,7)--ML of $B_{7,7}$

(3,5)--mean labeling of the graph $B_{5,5}$ is shown in Figure 2.23

**Figure 2.23:** (3,5)--ML of $B_{5,5}$

(5,7)--mean labeling of the graph $B_{6,6}$ is shown in Figure 2.24

**Figure 2.24:** (5,7)--ML of $B_{6,6}$

(4,4)--mean labeling of the graph $B_{8,8}$ is shown in Figure 2.25

**Figure 2.25:** (4,4)--ML of $B_{8,8}$

**Theorem 2.19**

The Bistar $B_{n,n+1}$ ($n \geq 2$) is a $(k,d)$--mean graph for all $k$ and for all $d \leq k + 1$.

**Proof**

Let $V(B_{n,n+1}) = \{u, v, u_1, u_2, ..., u_n, v_1, v_2, ..., v_{n+1}\}$ and $E(B_{n,n+1}) = \{uv, uu_i, 1 \leq i \leq n, vv_i, 1 \leq i \leq n + 1\}$ be denoted as in the Figure 2.26.

**Figure 2.26:** Ordinary labeling of $B_{n,n+1}$

First we label the vertices as follows:

Define $f : V \rightarrow \{0, 1, 2, ..., k + (q - 1)d\}$ by

- $f(u) = k - d + 1$
- $f(u_i) = k + d - 2$
- $f(u_0) = k + d(2i - 1) - 1$, for $2 \leq i \leq n$
- $f(v) = k + (q - 1)d - 1$
- $f(v_i) = k + d(2i - 1)$, for $1 \leq i \leq n + 1$

Then the induced edge labels are

- $f^*(uu_i) = k + d(i - 1)$, for $1 \leq i \leq n$
- $f^*(uv) = k + nd$
- $f^*(vv_i) = k + (n + i)d$, for $1 \leq i \leq n + 1$
The above defined function $f$ provides $(k,d)$--mean labeling of the graph.

So, the graph $B_{n,m+1}$ is a $(k,d)$--mean graph for all $k$ and for all $d \leq k + 1$.

$(k,d)$--mean labeling of $B_{n,m+1}$ for different cases of $k$ and $d \leq k + 1$ are shown in illustration 2.20.

**Illustration 2.20**

$(4,5)$--mean labeling of the graph $B_{5,6}$ is shown in Figure 2.27

![Figure 2.27: (4,5)--ML of B_{5,6}](image1)

$(1,2)$--mean labeling of the graph $B_{3,4}$ is shown in Figure 2.28

![Figure 2.28: (1,2)--ML of B_{3,4}](image2)

$(3,3)$--mean labeling of the graph $B_{4,5}$ is shown in Figure 2.29

![Figure 2.29: (3,3)--ML of B_{4,5}](image3)

$(6,6)$--mean labeling of the graph $B_{6,7}$ is shown in Figure 2.30

![Figure 2.30: (6,6)--ML of B_{6,7}](image4)

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